GENERALIZED PRIME IDEALS OF THE RINGS $C(X,Y)$ AND THE QUASI-COMPONENTS OF $X$

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Dedicated in the memory of Dr. Kwangil Koh

Abstract. In the set of continuous functions $C(X,Y)$ where $Y$ has a topology close to being discrete, there is an equivalence relation on $X$ which characterizes the quasi-components of $X$. If $Y$ satisfies weak algebraic conditions with a single binary operation then a stable set of functions forms an object generalizing an ideal of a ring. Calling such sets ideals there is a concept of a prime ideal. The ideal of functions vanishing on a quasi-component are prime ideals of $C(X,Y)$. If $Y$ has a zero set that is open then these prime ideals are min-max implying that when $Y$ is a ring all of the prime ideals of $C(X,Y)$ are of this form and min-max. However this is a study of $C(X,Y)$ and its ideals beginning with a few algebraic hypothesis on $Y$ and adding to them as needed. So there are conditions when a prime ideal is minimal, when the set of quasi-components is in bijective correspondence with the set of prime ideals of functions which vanish on them, when $C(X,Y)$ can not be like a local ring, when the prime (nill) radical is trivial, and when the corresponding $\text{Spect}(C(X,Y))$ is a disconnected topological space. Example of results are; if a quasi-component of $X$ is open then the prime ideal of functions vanishing on it is minimal independent of the zero set of $Y$ being open; if the zero set of $Y$ is open then the set of all ideals of functions vanishing on quasi-components is the set of minimal prime ideals irrespective of any quasi-component being open in $X$. These results imply the same for the ideals of the ring $C(X,Y)$ when $Y$ is a ring. The techniques developed give a method of generating unlimited number of rings with prescribed sets of prime ideals and minimal prime ideals.

Let $X$ and $Y$ be a topological spaces and $C(X,Y)$ the set of all continuous functions from $X$ to $Y$. Assume that $X$ and $Y$ both are of cardinality two or greater, $\|X\| > 1$ and $\|Y\| > 1$. Let $\mathbb{Z}$, $\mathbb{Z}_n$, $\mathbb{Q}$, and $\mathbb{R}$ be respectively the integers, the integers mod $n$, the rational numbers, and the reals. The question whether there is a relationship between the connectivity of $X$
and the ring C(X,Y), and the observation that the rings C(X,Z) and C(X,Q) need not be isomorphic, motivated this study.

**Theorem 1.** If Y has a continuous binary operation denoted by · which may satisfy additional algebraic axioms, then the point-wise composition of f and g, f · g, defined by (f · g)(x) = f(x) · g(x), is a continuous binary operation on C(X,Y) satisfying those axioms. When appropriate we will have additional unitary and binary operations which must be continuous.

**Proof.** For example the continuity of the defined binary operation in C(X,Y) would follow because the product map <f, g> (x) = (f(x), g(x)) from X to Y×Y is continuous and the binary operation in Y which is a map from Y×Y to Y is continuous. □

**Note.** As continuity of any operation in Y is not always needed and is evident when used, reference to continuity will usually be omitted. Further assume the least number of algebraic operations and properties that are needed. As a consequence most of the results do not use associativity, commutativity, or distributivity.

For want of a peer the presentation here is idiosyncratic. It follows the conversation that the author was having with himself as it was being written and revised. Consequently employing elementary ideas from several areas, the development is often tedious and elementary. Principally these areas are algebra, algebraic geometry, analysis, and point set topology but alas not algebraic topology.

If Y is an algebraic structure then C(X,Y) has a similar algebraic structure with the usual operations on functions induced from Y. We will be interested in a particular equivalence relation on X relative to its quasi-connectivity as a topological space and how this relates to the prime ideals of C(X,Y) when Y has special topological and algebraic properties. The equivalence classes relate to the ideals of another ring of continuous functions isomorphic to C(X,Y) which is easier to study. These ideals have a geometric origin which can be manipulated. The prime ideals of these C(X,Y) are of fundamental importance. However to develop a clear insight into the origin of these ideals it was necessary to assume a strong
topology on \( Y \) ("almost discrete") and start with weak algebraic properties such as a magma that need not be associative or commutative.

First, for any equivalence relation on \( X \), let \( \Pi \) denote the set of equivalence classes, and \( X/\Pi \) the topological quotient space. Use \([x]\) for a point in \( X/\Pi \) and for its corresponding subset of \( X \). For practical purposes use \( \Pi \) for both \( \Pi \) and \( X/\Pi \).

Further consider

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{p} & & \downarrow{c} \\
\Pi & & 
\end{array}
\]

where \( p \) is the projection, \( p(x) = p([x]) = [x] \), and \( c \in C(\Pi, Y) \). Set \( Fcn(\Pi, Y) = \{ h : \Pi \rightarrow Y \} \), continuity not assumed.

\textbf{Lemma 1.} If \( Y \) is a magma then \( C(X,Y) \) is a magma. If \( f \) and \( g \) are constant on an equivalence class of \( X \) then \( f \cdot g \) is constant on that equivalence class. This also holds for induced unitary operations on \( C(X,Y) \).

\textbf{Lemma 2.} If the functions of \( C(X,Y) \) are constant on the equivalence classes of \( X \) and if \( Y \) is a magma there is an injective magma homorphism \( G : C(X,Y) \rightarrow Fcn(\Pi, Y) \) which is a bijection if the equivalence classes are open in \( X \).

\textit{Proof.} Define \( G(f) \) by \( G(f)([x]) = f(x) \). That \( G \) is an injective algebraic homomorphism follows directly from its definition.

Suppose that the \([x]\) are open in \( X \). To see the surjectivity let \( h : \Pi \rightarrow Y \) and define \( f \) by \( f(x) = h([x]) \). To see that \( f \) is continuous let \( U \subset Y \) be open. The set of points of \( \Pi \) that \( G(f) \) map into \( U \) correspond to the union in \( X \) of the equivalence classes that \( f \) map into \( U \). As the equivalence classes are open so is this union. \( \square \)
Let $M(\Pi)$ be the free Magma generated by $\Pi$. Consider

$$
\begin{array}{c}
\Pi \\
\downarrow \quad \quad i \\
M(\Pi) \\
\downarrow \quad \quad \quad \quad \Downarrow
\end{array}
$$

and the universal defining property of $M(\Pi)$.

**Lemma 3.** If $i^* : \text{Hom}(M(\Pi), Y) \to \text{Fcn}(\Pi, Y)$ is the morphism induced by $i$ then it is an isomorphism.

**Theorem 2.** If the functions of $C(X,Y)$ are constant on equivalence classes and $Y$ is a magma then $(i^*)^{-1} \circ G : C(X,Y) \to \text{Hom} (M(\Pi), Y)$ is an injective representation of $C(X,Y)$ as a sub-magma of the magma of homorphisms of the free Magma into $Y$.

This result extends to $C(X,Y)$ as a group or ring.

**Note.** Throughout the paper we will expropriate the language of ring theory although most results require only one binary operation, $\cdot$. A second binary operation, associativity, commutativity, or distributivity are not usually assumed being introduced in stages. Modified definitions are given and appropriate hypothesis are stated as needed. If $Y$ has a unit for $\cdot$ (one sided or two sided identity element), denoted 1, then $C(X,Y)$ has a unit of same type. Denote this unit by $\text{Id}$ or when convenient use $I$.

The topological information of $X$ and algebraic information of $C(X,Y)$ that we wish to relate is carried by $\Pi$ and $C(\Pi,Y)$. The projection $p$ of $X$ to $\Pi$ is clear. To compare $C(X,Y)$ and $C(\Pi,Y)$ where $X$ and $Y$ are topological spaces and the functions of $C(X,Y)$ are constant on equivalence classes, consider the following definitions.

**Definition 1.** Define $G : C(X,Y) \to C(\Pi,Y)$ by $G(f) = f \circ p^{-1}$ and $H : C(\Pi,Y) \to C(X,Y)$ by $H(\hat{f}) = \hat{f} \circ p$ where $\hat{f} \in C(\Pi,Y)$.

Note: We are not assuming that equivalence classes are open in $X$. Also remember that $V$ is open in $\Pi$ if and only if $p^{-1}(V)$ is open in $X$. 
Lemma 4. \( H(\hat{f}) \) is a continuous function. As the functions of \( C(X,Y) \) are constant on equivalence classes, \( G(f) = f \circ p^{-1} = \hat{f} \) is a continuous function.

Proof. First \( H(\hat{f}) \) is a continuous function by its definition. Second \( G(f) = \hat{f} \) is a function as \( f \circ p^{-1}(p(x)) = f(x) \). To establish continuity let \( U \) is open in \( Y \). We need to show that \( [G(f)]^{-1}(U) = (\hat{f})^{-1}(U) = (f \circ p^{-1})^{-1}(U) \) is open and that \( p^{-1}((f \circ p^{-1})^{-1}(U)) \) is open. This follows as \( x \in p^{-1}((f \circ p^{-1})^{-1}(U)) \) iff \( p(x) \in (f \circ p^{-1})^{-1}(U) \) iff \( (f \circ p^{-1})(p(x)) \in U \) iff \( f(p^{-1}(p(x))) \in U \) iff \( f([x]) \in U \) iff \( f(x) \in U \) iff \( x \in f^{-1}(U) \).

Henceforth assume that the functions of \( C(X,Y) \) are constant on the equivalence classes, \([x]\). This is a condition that clearly holds if the equivalence classes of \( X \) are connected sets and \( Y \) has the discrete topology. Note \( C(Z,Z_2) \) is extensively known.

Lemma 5. For these sets of continuous functions both \( H \) and \( G \) are bijections which are inverses.

Proof. That they are injective is straight forward. To see the surjection, for an \( f \) such that \( G(f) = \hat{f} \) use \( f = \hat{f} \circ p \) and for \( \hat{g} \) such that \( H(\hat{g}) = g \) use \( \hat{g} = g \circ p^{-1} \).

For the next theorem assume that \( Y \) has a continuous binary operation as in Theorem 1. Also introduce operations in \( C(X,Y) \) and \( C(\Pi,Y) \) as in Theorem 1. The operation need not be associative or commutative.

Theorem 3. If \( Y \) has a continuous binary or unitary operation then \( C(X,Y) \) and \( C(\Pi,Y) \) are corresponding isomorphic structures with \( H \) and \( G \) inverse isomorphisms.

Proof. The proof is straight forward as these are function spaces keeping in mind that functions in \( C(X,Y) \) are constant on equivalent classes. However the proof is tedious.

The following observations through the Comment are about relations in general and will be used later. Let \( X \) and \( Y \) be sets and \( F(X,Y) \) the functions from \( X \) to \( Y \).

Definition 2. Let \( J \subseteq F(X,Y) \) and define \( x \equiv_J y \) iff for all \( f \in J, f(x) = f(y) \). Denote the equivalence class by \([x]_J\) and the corresponding partition of \( X \) by \( \Pi_J \). Also define \( x \equiv y \)
iff for all \( f \in F(X,Y) \), \( f(x) = f(y) \) that is when \( J = F(X,Y) \). Denote this class by \([x]\) and the correspond partition of \( X \) by \( \Pi \).

**Lemma 6.** If \( \emptyset \neq J \subseteq A \subseteq F(X,Y) \) then \([x] \subseteq [x]_A \subseteq [x]_J\).

In diffidence to our intention we now assume that \( Y \) has a unique element denoted \( 0 \). For convenience call this element zero which could be any fixed element. Later it will become a null element and then an algebraic zero.

**Definition 3.** Let \( \emptyset \neq A \subseteq F(X,Y) \) and \( b \in Y \). Define the \( b \) set of \( A \) to be \( V(A,b) = \{x \mid \forall f \in A, f(x) = b\} = \bigcap_{f \in A} f^{-1}(b) \). If \( Y \) has the element \( 0 \), let \( V(A) = V(A,0) = \bigcap_{f \in A} f^{-1}(0) \) be called the zero set of \( A \). Let \( V(f) = V(\{f\}) \). Note in the case of \( C(X,Y) \) if \( \{b\} \) is closed in \( Y \) then \( V(A,b) \) is closed in \( X \).

**Lemma 7.** \( \emptyset \neq J \subseteq A \subseteq F(X,Y) = F \) then \( V(F,b) \subseteq V(A,b) \subseteq V(J,b) \).

**Definition 4.** If \( b \in Y \), \( \emptyset \neq U \subseteq X \), and \( \emptyset \neq J \subseteq F(X,Y) \) define \( I(U,b)_J = \{f \mid f \in J, f(U) = b\} \). Let \( I(U,b) = I(U,b)_F \) and if \( Y \) has a zero element as above, denote \( I(U,0)_J \) by \( I(U)_J \) and \( I(U)_F \) by \( I(U) \).

**Lemma 8.** Via definition \( I(U,b)_J \subseteq J \) and so \( I(U)_J \subseteq J \).

**Lemma 9.** \( U \subseteq V(I(U,b)_J, b) \) so \( U \subseteq V(I(U,0)_J, 0) \) and therefore \( U \subseteq V(I(U)) \).

*Proof.* If \( x \in U \) then \( f \in I(U,b)_J \) implies that \( f(x) = b \), that is \( \forall f \in I(U,b)_J, f(x) = b \), so \( x \in V(I(U,b)_J, b) \). \( \Box \)

**Lemma 10.** If \( J \subseteq A \) then \( I(U,b)_J \subseteq I(U,b)_A \subseteq I(U,b)_F \).

**Lemma 11.** \( J \subseteq I(V(J,b), b) \) and therefore \( J \subseteq I(V(J)) \).

**Lemma 12.** If \( U_1 \subseteq U_2 \subseteq X \) then \( I(X,b)_J \subseteq I(U_2,b)_J \subseteq I(U_1,b)_J \).

**Corollary.** If \( \emptyset \neq J \subseteq A \subseteq F(X,Y) \) and \( B \subseteq F(X,Y) \) then \( I([J],b)_B \subseteq I([A],b)_B \subseteq I([x],b)_B \).
Theorem 4. If for every \( f \) in \( J \), \( f(x) = b \) then \( V(J, b) = [x]_J \)

Proof. \( V(J, b) = \bigcap_{f \in J} f^{-1}(b) = \{ y \mid \forall f \in J, f(y) = f(x) = b \} = [x]_J \) where \( f(x) = b \). \( \square \)

Corollary. If for every \( f \in J \), \( f(x) = b \), and if \( A \subseteq F(X,Y) \) then \( I(V(J, b), c)_A = I([x]_J, c)_A \). In particular if \( b = c = 0 \) then \( I(V(J)) = I([x]_J) \subseteq I([x]) \).

Observation: Let \( F(X,Y) = A_1 \supseteq A_2 \supseteq \ldots \supseteq A_n \). Then \( [x] = [x]_{A_1} \subseteq [x]_{A_2} \subseteq \ldots \subseteq [x]_{A_n} \). Thus if \( \{x\} \neq \emptyset \) and \( [x]_{A_i} = \{x\} \) then they are all equal for \( j \leq i \). Moreover \( I([x]_{A_1}, b)_J \supseteq I([x]_{A_2}, b)_J \supseteq \ldots \supseteq I([x]_{A_n}, b)_J \). Finally recall \( C(\Pi, Y) \xrightarrow{H} C(X,Y) \) is an isomorphism. If functions are continuous on these equivalence classes of \( A_n \) then by Theorem 3, \( C(\Pi, Y) \) and all of the \( C(\Pi_{A_i}, Y) \) are algebraic isomorphic. We could consider \( (X/\Pi_1)/\Pi_2 \), but not at this time.

Note. Use \( U^c \) for \( X \setminus U \) for any set \( U \subseteq X \). \( U^c \) is the complement of \( U \) in \( X \).

Now proceed with \( X \) as a topological space.

Definition 5. The quasi-component of \( x \in X \) is the intersection of all clopen (closed and open) subsets of \( X \) which contain \( x \), [5], page 246. The quasi-component of \( x \) is denoted \( Q_x \).

Corollary. If \( y \in Q_x \) then \( Q_x = Q_y \) otherwise there is a clopen set \( U \) containing \( y \) that does not contain \( x \) but then \( U^c \) is clopen, \( x \in U^c \) and \( y \notin U^c \). Quasi-components are disjoint and clopen sets are unions of quasi-components.

Note. A quasi component must be closed but need not be open. For example consider \( X = \{0, 1, 1/2, 1/3, \ldots, 1/n, \ldots\} \subseteq \mathbb{R} \) with the topology induced from \( \mathbb{R} \). Zero is a non-open quasi-component.

Hence forth unless otherwise indicated \( Y \) is a topological space such that for any two distinct points there is a clopen set containing one of the points but not the other. Such a space is \( T_2 \) and totally separated. Hence every point is a quasi-component, [10] page 32. The rational numbers with the subspace topology inherited from the reals is a prime example.
Note Y need not be zero dimensional that is it may not have a clopen set basis, for examples see [10].

We are now interested in the equivalence relation $x \cong y$.

**Note.** It is easy to create topological spaces, $X$, with any number of components and quasi-components with different topological properties. Then choosing $Y$ with algebraic properties, we can obtain algebraic structures on $C(X,Y)$ which are different from $Y$ and which reflect the quasi-components as situated in $X$. Passing to $C(\Pi,Y)$ will facilitate proofs.

**Theorem 5.** The elements, $[x]$, of $\Pi$, the equivalence classes, are precisely the quasi-components of $X$. That is $[x] = Q_x$ for all $x \in X$.

**Proof.** If $Q_x \nsubseteq [x]$ then there is an $f \in C(X,Y)$ that is not constant on $Q_x$. So there exist a $y \in Q_x$ such that $f(x) \neq f(y)$. Let $U \subseteq Y$ be a clopen set about $f(x)$ such that $f(y) \notin U$. Then $f^{-1}(U)$ is a clopen set about $x$ not containing $y$. This is a contradiction to the definition of $Q_x$. Therefore $Q_x \subseteq [x]$.

On the other hand, let $y \in [x]$ and let $V$ be a clopen set about $x$. If there is only one $V$ then $V = X$ and there is nothing to prove. Choose an $a$ and $b$ in $Y$ such that $a \neq b$. The characteristic function $\chi(z) = \begin{cases} b & \text{if } z \in V \\ a & \text{if } z \notin V \end{cases}$ is continuous on $X$ with $\chi(x) = b$. Since $y \in [x]$ it follows $\chi(y) = b$ and hence $y \in V$. Thus $[x]$ is contained in any clopen set containing $x$ and thus $[x] \subseteq Q_x$. \qed

**Note.** This is a new characterization of quasi-components.

**Corollary.** Functions are constant on the quasi-components.

**Corollary.** If $U$ is clopen and $x \in X$ then either $U \supseteq [x]$ or $U \cap [x] = \emptyset$.

**Proof.** Either $[x] \subseteq U$ or $[x] \nsubseteq U$. If $[x] \nsubseteq U$ then $x \notin U$. So $x \in X \setminus U = U^c$ which is clopen. Thus $U \cap [x] = \emptyset$. \qed
COMMENT: If we assume $Y$ has a binary operation with a zero $0$ and unit $1$ then quasi-components can be characterized using a slightly different equivalence relation as follows. Consider the set of characteristic functions in $C(X,Y)$ of the form $\chi_u$ which is $0$ on $U$ and $1$ on $U^c$ where $U$ is clopen. Define $x \sim_c y$ if and only if for all $\chi_u \in C(X,Y)$, $\chi_u(x) = \chi_u(y)$. Denoting the equivalence class of $x$ by $[x]_\chi$ an easy adaptation of Theorem 5 shows that $Q_x = [x]_\chi$.

This establishes that the two equivalence classes are equal, that is $[x] = [x]_\chi$. The set of $\chi_u$ is closed under multiplication in $C(X,Y)$ but need not be closed under addition if $Y$ is ring. Ultimately we are interested in $C(X,Y)$ as a ring although multiplication will dominate throughout most the development without addition or using associativity or commutativity for an operation.

Convention: When $Y$ has a zero $0$ and unit $1$, $\chi_u$ will denote a continuous characteristic function as in the comment and $\chi_{u^c}$ analogously. Let $\chi_x$ denote $\chi_{\{x\}}$. Different characteristic functions when needed will be defined.

**Note.** As components are subsets of quasi-components, functions that are constant on a component are constant on its corresponding quasi-component. Hence the interest in the quasi-components of $X$.

**Lemma 13.** Any two quasi-components can be separated by a clopen set.

**Proof.** Let $[x]$ and $[y]$ be two distinct quasi-components. So there exist clopen sets $U$ and $V$ such that $U \supseteq [x]$, $V \supseteq [y]$, $x \notin V$ and $y \notin U$. By the corollary $[x] \cap V = \phi$ and $[y] \cap U = \phi$. So it follows that $[x] \subseteq U \cap V^c$ and $[y] \subseteq V \cap U^c$. These are the desired clopen sets. □

**Corollary.** Quasi-components are disjoint and every clopen set is a union of quasi-components.
Now returning to $C(\Pi, Y)$ and the paragraph before Lemma 1) consider

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{p} & & \downarrow{c_f} \\
\Pi & & \\
\end{array}
\]

where $p$ is the projection and the function $c_f$ is defined in the natural way as follows.

**Definition 6.** Let $z_x = p([x])$ and set $c_f(z_x) = f(p^{-1}(z_x))$ and again $\Pi$ denotes $X/\Pi$.

**Lemma 14.** $c_f$ is continuous.

*Proof.* If $U$ is open in $Y$ we need to show that $c_f^{-1}(U)$ is open in $\Pi$, that is that $p^{-1}((c_f^{-1}(U)))$ is open in $X$. First note that $f^{-1}(U)$ is open and is the union of all of the equivalence classes of the $x$ such that $f(x) \in U$, that is $f([x]) = f(x) \in U$. The elements of $\Pi$ that $c_f = f \circ p^{-1}$ maps into $U$ correspond to the union of the quasi-components of $X$ that $f$ maps into $U$. This union is $f^{-1}(U)$ and is $p^{-1}(c^{-1}(U))$ that is $f^{-1}(U) = p^{-1}(c^{-1}(U))$. Therefore $c_f^{-1}(U)$ is open. \qed

**Lemma 15.** If $U$ is clopen in $X$ then $U = p^{-1}p(U)$.

*Proof.* $x \in U$ iff $[x] \subset U$ iff $p(x) = p([x]) \in p(U)$ iff $x \in [x] \subset p^{-1}p(U)$. \qed

**Corollary.** If $W$ is clopen in $\Pi$ then $p^{-1}(W)$ is clopen in $X$ and conversely if $U$ is clopen in $X$ then $p(U)$ clopen in $\Pi$. Thus the intersection of clopen sets defining $[x]$ in $X$ correspond to the intersection of clopen sets containing $p(x)=z$ in $\Pi$. That is the quasi-components are in bijective correspondence.

**Theorem 6.** Each point of $\Pi$ is a quasi-component of $\Pi$. Thus $\Pi$ is totally separated and hence $T_2$, [10], pg. 32.

COMMENT: If $X = \{x\}$ is a singleton, then $C(X, Y)$, $C(\Pi, Y)$, and $Y$ are in bijective correspondence and $C(X, Y)$ and $C(\Pi, Y)$ are naturally algebraically isomorphic to $Y$. However
this occurs in other special cases. Although $X = \{x\}$ has been excluded, $\Pi$ is a single point when $X$ is connected. Thus $C(X,Y)$, $C(\Pi,Y)$, and $Y$ would be in bijective correspondence. For an example where $C(X,Y)$ and $Y$ are different consider $C(Z, Z_2)$. Here $Z$ is topologically $Z/\Pi$ and the cardinalities of $C(Z, Z_2)$ and $Z_2$ do not match.

**Note.** We will not consider $C(\Pi/\Pi, Y)$ as the intersection of clopen sets defining $[x]$ corresponds to the intersection of clopen sets defining $p([x]) = z_x$ which correspond to the intersection of the clopen sets defining $z_x = p(z_x) \in C(\Pi/\Pi, Y)$. Also we know that $C(X,Y)$, $C(\Pi,Y)$, and $C(\Pi/\Pi,Y)$ are algebraically isomorphic. Moreover $\Pi$ and $\Pi/\Pi$ are topologically isomorphic.

Based on the development, this note, and Theorems 3, 5, and 6, it is convenient to replace $\Pi$ by a topological space $Z$ whose points are quasi-component. Denote this topology by $\mathcal{T}$ which is totally separated, [10] pg. 32 and thus $T_2$ and Urysohn. Hence forth we consider $C(Z,Y)$ and let the elements of $C(Z,Y)$ be denoted by $f$.

Example: If $Z_2$ is chosen for $Z$ and $Z_3$ for $Y$ then $\|Z_2\| = 2$, $\|Z_3\| = 3$, and $\|C(Z_2, Z_3)\| = 9$, none of which are isomorphic as sets, topologies, or algebraic structures.

**Note.** For examples we could start with an $X$ and use the topology $\mathcal{T}$ on $Z = X/\Pi = \Pi$ inherited from $X$ by $p$. For convenience use $Z$ for $Z$ with this induced topology, $\mathcal{T}$. Later we compare other topologies for $Z$ denoted by $\mathcal{T}_1$ and $\mathcal{T}_3$.

**Note.** Despite the similarities of the topologies of $Z$ and $Y$, $C(Z,Y)$ can be very different from $Y$. For examples take $Y$ as $Z_n$ and take $Z$ as any discrete space of any cardinality greater than one.

We will now assume that $Y$ has a continuous binary operation denoted by $\cdot$ with a left, right, or two sided null element, $0$ (denoting the corresponding null function in $C(Z,Y)$ by $\Theta$). We can use either of the three null elements unless stated otherwise. For convenience use $0 \cdot a = 0$ for all $a$ in $Y$. When needed the unit for $Y$ is denoted $1$ and the unit for $C(Z,Y)$ by
Id or I. Properties such as associativity or commutatively are still not assumed. The 0 and 1 in Y and hence in C(Z,Y) are not unique unless \( \cdot \) is commutative. Also use the notation \((f \cdot g)(z) = f(z) \cdot g(z)\) in C(Z,Y).

**Corollary.** If \( Y \) has a unit then any characteristic function, \( \chi \), zero on a set and one on its complement is idempotent that is \( \chi \cdot \chi = \chi \) where \( \chi \) need not be in \( C(Z,Y) \). The characteristic function of the zero set is the Id and of \( Z \) is \( \Theta \).

Recall that for set functions the next two lemmas hold.

**Lemma 16.** If \( B_1 \subset B_2 \) then \( f^{-1}(B_1) \subset f^{-1}(B_2) \). So if \( b \in B \) then \( f^{-1}(b) \subset f^{-1}(B) \).

**Lemma 17.** Suppose \( f^{-1}(b) \) is not empty and \( b = \cap B \) then \( f^{-1}(b) = \cap f^{-1}(B) \).

**Proof.** The inverse of an intersection is the intersection of the inverses. Thus \( f^{-1}(b) = f^{-1}(\cap B) = \cap f^{-1}(B) \).

From Definition 3 select:

**Definition 7.** \( B \subset Z \) is a zero set iff \( B = V(f) = f^{-1}(0) \) for some \( f \in C(Z,Y) \).

**Lemma 18.** These sets are closed as \( \{0\} \) is closed in \( Y \).

**Lemma 19.** \( V(f) \cup V(g) \subseteq V(f \cdot g) \)

**Note.** Observe that a clopen set \( U \) is the zero set of the characteristic function., \( \chi_u(z) = \begin{cases} 0 & \text{if } z \in U \\ a & \text{if } z \notin U \end{cases} \) where \( a \neq 0 \).

**Lemma 20.** A zero set, \( V(f) \), is an intersection of clopen sets.

**Proof.** In \( Y \), \( \{0\} \) is the intersection of all clopen sets containing it. As the inverse of a clopen set is clopen and as the inverse of an intersection is the intersection of the inverses, \( f^{-1}(0) \) is in an intersection of clopen sets. The f image of a point in this intersection must be 0 otherwise it can be separated from 0 by clopen sets. □
**Theorem 7.** Each point \( z \in Z \) is the intersection of all zero sets containing it.

*Proof.* As each \( z \in Z \) is a quasi-component and as the intersection of an intersection is just the intersection of the underlying sets, the intersection of the zero sets at \( z \) contain \( \{z\} \) as a quasi-component. If the two sets are not equal then there is a clopen set containing \( z \) which does not contain all of the intersection of zero sets. But this clopen set is the zero set of a continuous characteristic function. \( \square \)

**Corollary.** Each quasi-component \( z \in Z \) is both the intersection of all zero sets and of all clopen sets containing it.

**Note.** However \( \{z\} \) need not be a zero set. For an example in the real line with the induced topology let \( X = \{0, 1, 1/2, \ldots, 1/n, \ldots\} \). Then \( Z = X = \{z_0, z_1, z_2, \ldots\} \). Here \( \{z_0\} \) is a quasi-component which is closed but not open. Let \( Y \) be a space such as \( Z_2 \) where \( \{0\} \) is clopen. Then for any continuous function \( f, V(f) \neq \{z_0\} \)

Now from Definition 3 select:

**Definition 8.** Let \( S \subseteq C(Z, Y) \). We say \( V(S) = \{z \in Z : f(z) = 0 \text{ for all } f \in S\} = \bigcap_{f \in S} f^{-1}(0) = \bigcap_{f \in S} V(f) \) is the zero set of \( S \).

The \( V(S) \) are closed as each \( V(f) \) is closed. Moreover every clopen set is the zero set of continuous characteristic functions. The characteristic functions zero on \( U \) and otherwise a will be denoted \( \chi_{u,a} \). When \( Y \) has a unit a will be 1 and write \( \chi_u \).

**Corollary.** \( V(\emptyset) = Z \).

**Corollary.** AS \( \|Z\| \geq 2 \), \( V(C(Z, Y)) = \emptyset \).

**Lemma 21.** If \( J \subseteq C(Z, Y) \) and \( S \subseteq J \) then \( V(J) \subseteq V(S) \). If \( Y \) has an additional associative operation denoted by addition and \( S \) generates \( J \) in the sense that if \( f \in J \) then \( f = \sum_{i=1}^{n} g_i \cdot f_i \) where \( f_i \in S \) and \( g_i \in C(Z, Y) \), then \( V(S) = V(J) \).
However this additional operation is still not needed.

**Definition 9.** A right zero-divisor is any element $y_1 \in Y$ such that $y \cdot y_1 = 0$ for some $y \in Y$, $y \neq 0$. For convenience assume $y_1 \neq 0$. An element $y \in Y$ is nilpotent iff $y^n = 0$ for some integer $n$ and idempotent iff $y \cdot y = y$. Analogously define left and two sided zero divisors, nilpotent, and idempotent elements for $Y$ an for $C(Z,Y)$. In the nilpotent case, · must be associative.

**Lemma 22.** If $Y$ has no right (left) divisors of zero then $V(f_1) \cup V(f_2) = V(f_1 \cdot f_2)$.

**Note.** This result suggest a natural topology for $Z$ to be compared with $\mathcal{F}$.

**Theorem 8.** If $Y$ has no right (left) divisors of zero then the zero sets of $Z$ in the sense of Definition 8 form a Zariski topology for $Z$ having the $V(S)$ as the topology of closed sets, [9] pg.s 6 and 7. Denote this topology by $\mathcal{T}_3$. When $\mathcal{T}_3$ is present assume no divisors of zero.

**Lemma 23.** A set $U \subseteq Z$ open in $\mathcal{T}_3$ is open in the topology $\mathcal{F}$ of $Z$. So $\mathcal{T}_3 \subseteq \mathcal{F}$.

**Proof.** As $U$ is open in $\mathcal{T}_3$ it is the complement of a zero sets of $\mathcal{F}$ and hence open in $\mathcal{F}$. Alternately said the closed sets, $V(f)$, of $Z$ in $\mathcal{F}$ generate $\mathcal{T}_3$. Thus $\mathcal{T}_3$ maybe courser than $\mathcal{F}$. 

**Note.** As the clopen sets of $Z$ need not form a basis for the topology $\mathcal{F}$ of $Z$. Let $\mathcal{T}_1$ be the topology which has these clopen sets as an open set basis, [7] pg 47, T11. By duality as the clopen sets are also closed they form a closed basis for the $\mathcal{T}_1$ topology. Clearly $\mathcal{T}_1 \subseteq \mathcal{T}$.

**Lemma 24.** $\mathcal{T}_1 \subseteq \mathcal{T}_3$.

**Proof.** Observe $U$ is a basic open set of in $\mathcal{T}_1$ if and only if $U$ is clopen in $\mathcal{F}$ if and only if $U^c$ is clopen in $\mathcal{F}$. Thus $U^c$ is a zero set with respect to $\mathcal{F}$ which implies $U^c$ is a basic closed set in $\mathcal{T}_3$ which implies $U = (U^c)^c$ is open in $\mathcal{T}_3$. Therefore $\mathcal{T}_1 \subseteq \mathcal{T}_3$.

**Lemma 25.** $\mathcal{T}_3 \subseteq \mathcal{T}_1$. 

14
Proof. Let $U \in \mathcal{T}_3$ be an open set of $\mathcal{T}_3$ and thus equal to $(V(S))^c$ for some $S \subset C(Z,Y)$. Now $(V(S))^c = (\bigcap_{f \in S} f^{-1}(0))^c = \bigcup_{f \in S} (f^{-1}(0))^c$. Recall $\{0\} \in Y$ is a quasi component and $\{0\} = \bigcap\{K : K \text{ clopen and } 0 \in K\}$. So $f^{-1}(0) = f^{-1}(\bigcap\{K : K \text{ clopen and } 0 \in K\}) = \bigcap\{f^{-1}(K) : K \text{ clopen and } 0 \in K\}$. Thus $(f^{-1}(0))^c = (\bigcap\{f^{-1}(K) : K \text{ clopen and } 0 \in K\})^c = \bigcup\{(f^{-1}(K))^c : K \text{ clopen and } 0 \in K\}$. Now as $K$ is clopen $f^{-1}(K)$ is clopen implying $(f^{-1}(K))^c$ is clopen which implies that $U = \bigcup_{f \in S} (f^{-1}(0))^c$ is open in $\mathcal{T}_1$. (For an easier conceptual proof use a sub-basis argument.)

**Theorem 9.** $\mathcal{T}_1 = \mathcal{T}_3 \subseteq \mathcal{T}$.

**Corollary.** The quasi-components of $\mathcal{T}_1$ and $\mathcal{T}_3$ are the same.

**Corollary.** If open sets of $\mathcal{T}$ contain clopen sets then $\mathcal{T} = \mathcal{T}_1 = \mathcal{T}_3$ as then $\mathcal{T} \subseteq \mathcal{T}_1$.

**Lemma 26.** A set $U$ is clopen in $\mathcal{T}$ if and only if it is clopen in $\mathcal{T}_1$.

**Proof.** Let $U$ and hence $U^c$ be clopen in $\mathcal{T}$. Then they are both open in $\mathcal{T}_1$ and being complements are both closed, therefore clopen in $\mathcal{T}_1$.

Let $U$ be clopen in $\mathcal{T}_1$. By Theorem 9 it is clopen in $\mathcal{T}$. \qed

**Theorem 10.** The quasi-components of $\mathcal{T}$, $\mathcal{T}_1$, and $\mathcal{T}_3$ are the same, namely $\{z\}$ where $z \in Z$. Moreover their sets of clopen quasi-components are the same.

As $\mathcal{T}$ is The original topology for $Z$ we denote $(Z,\mathcal{T})$ by $Z$ and continue to use $C(Z,Y)$ for $C((Z,\mathcal{T}),Y)$.

**Lemma 27.** If $Y$ has a clopen base for its topology then $C(Z,Y) = C((Z,\mathcal{T}_1),Y)$.

**Proof.** Let $f \in C(Z,Y)$. If $V$ is open in $Y$ it is a union of clopen sets of $Y$. Therefore $f^{-1}(V)$ is a union of clopen sets of $\mathcal{T}$. Thus it is open in $\mathcal{T}_1$ as it is a union of open sets of $\mathcal{T}_1$. Therefore $f \in C((Z,\mathcal{T}_1),Y)$.

Let $f \in C((Z,\mathcal{T}_1),Y)$ and let $V$ be open in $Y$. Then $f^{-1}(V)$ is open in $(Z,\mathcal{T}_1)$. By Theorem 9 it is open in $\mathcal{T}$ and thus $f \in C(Z,Y)$. \qed
Lemma 28. Moreover $C(Z, Y), C((Z, \mathcal{T}_1), Y)$, and $C((Z, \mathcal{T}_5), Y)$ are the same as sets.

Theorem 11. If $Y$ has an open base topology then $C(Z, Y) = C((Z, \mathcal{T}_1), Y) = C((Z, \mathcal{T}_5), Y)$ are algebraically isomorphic with the same continuous functions, cf. Theorem 1.

Note. Although $\mathcal{T}$ for want of evidence in some cases could be stronger than $\mathcal{T}_1$ in what follows $\mathcal{T}$ is used. It may be that in any case $C((Z, Y)$ and $C((Z, \mathcal{T}_1), Y)$ are algebraic isomorphic and that $\mathcal{T}_1$ has for an open basis the clopen sets of $\mathcal{T}$.

For convenience we will drop the modifiers right or left as most of the proofs use characteristic functions $\chi$.

Theorem 12. Although $Y$ may not have a divisor of zero if $Z$ has two or more elements then $C(Z, Y)$ has a zero-divisor. If $Y$ has nilpotent elements so does $C(Z, Y)$.

Proof. Let $U$ be a clopen set in $Z$ containing one point but not another. Then choose $a \in Y$, $a \neq 0$ and define $\psi_{u,a}(z) = \begin{cases} a & \text{if } z \in U \\ 0 & \text{if } z \notin U \end{cases}$ and $\chi_{u,a}(z) = \begin{cases} 0 & \text{if } z \in U \\ a & \text{if } z \notin U \end{cases}$. Note $\psi_{u,a}$ and $\chi_{u,a}$ are continuous, neither are the zero function, and $\Theta = \chi_{u,a} \cdot \psi_{u,a}$. Therefore $C(Z, Y)$ has a zero divisor.

If $a \in Y$ is nilpotent then $\chi_{u,a}$ is nilpotent. \hfill $\square$

Corollary. For every $a \neq 0, a \in Y$ and for every clopen set $U \subseteq Z$, $\chi_{u,a}$ is a distinct divisor of zero. This says that the cardinality of the set of divisors of zero exceeds that of $Z$, $\mathcal{T}$, and $Y$. Mutatis Mutandis for nilpotent elements when $\cdot$ in $Y$ is associative.

Corollary. If $C(Z, Y)$ has no zero divisors then $Z$ consist of a single point and therefore $C(Z, Y)$ and $Y$ are algebraically isomorphic.

Corollary. If $f \in C(Z, Y)$ is nilpotent then for every $z \in Z$, $f(z)$ is nilpotent as is $\chi_{\emptyset, f(z)}$.

Note. $\mathbb{Z}_3$ has no zero divisors but $C(\{z_1, z_2\}, \mathbb{Z}_3)$ does.

Adapting Definition 4 for $C(Z, Y)$ were $Y$ has the annihilator, $0$: 
Definition 10. For $U \subset Z$ let $I(U)$ denote those functions that annihilates $U$ that is $I(U) = \{f : f(z) = 0 \text{ for all } z \in U\}$. Use $I(z)$ for $I(\{z\})$.

Corollary. If $U_1 \subseteq U_2$, then $I(U_2) \subseteq I(U_1)$.

Corollary. $\{z\} = V(I(z))$.

Proof. If $z_1 \in V(I(z))$ and $z_1 \neq z$, then they can be separated by clopen sets. \hfill \square

Corollary. If $1 \in Y$ and if $U$ is clopen then $I(U) = (\chi_u)$ where $f \in (\chi_u)$ gives $f = f \cdot \chi_u$.

Corollary. If $U_1$ is clopen, $U_2 \neq \emptyset$, and $U_1 \subsetneq U_2$ then $I(U_1) \neq I(U_2)$

Corollary. If $z_1 \neq z_2$ then $I(z_1) \neq I(z_2)$ and neither is a subset of the other.

Proof. This follows as $z_1$ and $z_2$ can be separated by disjoint clopen sets. \hfill \square

Corollary. If $z \in U$ but $\{z\} \neq U$ then $I(z) \neq I(U)$ although $I(U) \subset I(z)$.

Proof. If $\{z\} \neq U$ there exists $y \in U, y \neq z$. So there is a clopen set $U_z$ such that $y \notin U_z$. Therefore there is a characteristic function zero on $U_z$ and thus on $z$ but not on $y$, so not on $U$. \hfill \square

Lemma 29. If $U_1$ is clopen and $U_1^c \cap U_2 \neq \emptyset$ where $U_2$ need not be clopen, then $I(U_1) \neq I(U_2)$.

Proof. As there is a characteristic function $\chi_{U_1}$ which does not vanish on $U_1^c$ and therefore not on $U_2$, $I(U_1) \neq I(U_2)$. \hfill \square

Definition 11. Call a subset $I$ of $C(Z,Y)$ which contains the zero function, $\Theta$, and which is closed with respect to the operation $\cdot$, a right (left) multiplicative ideal if $f \in C(Z,Y)$ and $g \in I$ then $f \cdot g \in I, (g \cdot f \in I)$. A right (left) ideal $I \neq C(Z,Y)$ is prime if $f \cdot g \in I$ implies $f \in I$ or $g \in I$. Keep in mind that $\chi_u \cdot \chi_{u^c} = \chi_{u^c} \cdot \chi_u = \Theta$. If $I$ is generated by $f$, $I = (f) = \{g \cdot f | g \in C(Z,Y)\}$, it is called principal.
Recall as the algebraic properties and definitions for \( C(Z,Y) \) are to be defined from those of \( Y \) we assume the weakest properties of interest when not specified. The \( I(z) \) although ideals in the weak multiplicative sense will be called ideals. However the development is such that the results will apply after additional operations for \( Y \) are assumed and finally ring ideals are obtained.

**Corollary.** By the third Corollary to Definition 10, if \( 1 \in Y \) and if \( U \) is clopen then \( I(U) = (\chi_u) \) the principal ideal generated by \( \chi_u \).

**Corollary.** \( (\Theta) \) is not prime as for any clopen \( U \), \( \chi_{u,a} \cdot \chi_{u,c,a} = \Theta \).

**Corollary.** If \( I \) is a prime ideal then there is an \( f \in I \) such that \( V(f) \neq \emptyset \).

**Proof.** As the cardinality of \( Z \) in at least 2, let \( U \) be a clopen set containing one point but not another. Then \( \Theta = \chi_{u,a} \cdot \chi_{u,c,a} \) and thus \( \chi_{u,a} \) or \( \chi_{u,c,a} \) is in I neither of which has an empty zero set. \( \square \)

**Lemma 30.** For any \( U \), \( I(U) \) is an ideal. If \( Y \) has no divisors of zero then \( I(z) \) is prime.

**Proof.** To see that \( I(z) \) is prime let \( f \cdot g \in I(z) \). So \( f(z) \cdot g(z) = 0 \) and hence as \( Y \) has no divisors of zero \( f(z)=0 \) or \( g(z)=0 \). This says \( f \) or \( g \) is in \( I(z) \). \( \square \)

Example: The multiplication of \( Z_4 \) has divisors of zero. Choosing the discrete topology on \( Z_4 \) and on \( Z = \{ z_1, z_2 \} \), a set of two elements, the ideals \( I(z_1) \) and \( I(z_2) \) are not prime ideals of \( C(Z,Z_4) \).

**Theorem 13.** If \( I \) is an ideal of functions vanishing on two or more points, ie. \( \|V(I)\| \geq 2 \) then \( I \) is not prime.

**Proof.** In \( Z \) let \( z_1 \) and \( z_2 \) be two such distinct points of \( V(I) \). By Lemma 13, \( z_1 \) and \( z_2 \) can be separated by two clopen sets \( U_1 \) and \( U_2 \) respectively with characteristic functions \( \chi_{u_1} \) and \( \chi_{u_2} \) such that \( \chi_{u_1} \cdot \chi_{u_2} = \Theta \in I \) but neither zero on \( \{ z_1, z_2 \} \). \( \square \)
Corollary. If $I(U)$ is a prime ideal and $U \neq \emptyset$ then $U$ is a singleton. Also observe that $I(\emptyset) = C(Z,Y)$.

Proof. The claim for $I(\emptyset)$ is from first order logic. □

Corollary. If $J \subseteq I$, both ideals, and $\|V(I)\| \geq 2$ then $J$ is not prime.

Proof. As $V(I) \subseteq V(J)$, $2 \leq \|V(I)\| \leq \|V(J)\|$. □

Theorem 14. If $Y$ has no divisors of zero then $I(U)$ is a prime ideal if and only if $U$ is a singleton that is there exist a $z$ such that $U = \{z\}$.

Proof. Use the Corollary to Theorem 13 and Lemma 30. □

Corollary. If $Y$ has no divisors of zero then the intersection of the prime ideals is trivial.

Lemma 31. If $Y$ has no divisors of zero and its binary operation is associative, then $C(Z,Y)$ has no nontrivial nilpotent elements.

Proof. Suppose $f \in C(Z,Y)$, $f \neq \Theta$, and $f^n = \theta$ for some $n$. Thus if $f(z) = a \neq 0$, then $f^n(z) = a^n = a^{n-1} \cdot a = \theta(z) = 0$. □

Corollary. If $Y$ is a ring and has no divisors of zero then the prime (nill) radical of $C(Z,Y)$ is trivial and the prime spectrum of $C(Z,Y)$ is disconnected.

Proof. The intersection of all $I(z)$ is \{\Theta\}, the only nilpotent element. If $U$ is clopen note $\chi_u \cdot \chi_u = \chi_u$ and use [1] pg 13, exercise 22 (i) and (iii) for the disconnectivity. □

Now enlarge the investigation not only to obtain a better grasp of the ideals but to examine possible directions of interest.

Lemma 32. If $I$ is a proper ideal then:

1) If $U$ and $U_1$ are clopen, $U \subseteq U_1$, and $\chi_{u,a} \in I$ then $\chi_{u_1,a} \in I$.

2) For every $U$ clopen and $I$ prime, either $\chi_{u,a} \in I$ or $\chi_{u^c,a} \in I$. Thus every prime ideal has for every clopen set $U$, a divisor of zero , either $\chi_{u,a}$ or $\chi_{u^c,a}$. These are idempotents when $1 \in Y$ and $a=1$. 

19
Clearly if U is clopen and \( z \in u \) then \( \chi_{u,1} = \chi_u \in I(z) \).

More generally:

**Lemma 33.** For an ideal I if \( f \in I \) then for any clopen U, \( f \cdot \chi_{u,a} \in I \) and \( f \cdot \chi_{u^c,a} \in I \).

**Theorem 15.** If \( 1 \in Y \) then every nontrivial ideal I has a function f such that \( f \neq \Theta \), \( U=V(f) \) is clopen, and \( \emptyset \neq V(f) \neq Z \).

**Proof.** Choose \( g \in I \) such that \( g \neq \Theta \). Thus \( V(g) \neq Z \). If \( V(g) = \emptyset \) let \( U \neq Z \) be clopen. Then \( f = g \cdot \chi_u \in I \) and \( U=V(f) \).

If \( V(g) \neq \emptyset \) as \( g \neq \Theta \) for \( y \in \text{Range}(g) \setminus \{0\} \), choose W clopen such that \( 0 \in W \) and \( y \notin W \). Then for \( U = g^{-1}(W) \), \( f = g \cdot \chi_u \in I \), \( V(g) \subseteq V(f) \neq Z \), and \( V(f)=U \) is clopen. \( \square \)

**Note.** Two operations in \( C(Z,Y) \) are used in Lemma 35, Theorems 16 and 18, and used from Definition 12 on although not always needed. For convenience call \( f \cdot g \) multiplication and when \( f+g \) is introduced and call it addition. As before let \( \Theta \) be zero and I or Id be the unit. When the + operation is used the previous results are not effected by this additional operation. For example if \( Y \) is a ring then \( I(z) \) is a ring ideal. For expediency a ring was used in the Corollary to Lemma 31 although the results is more general. In what follows when addition is present distribution may not be needed.

**Corollary.** If \( Y \) has addition \( \chi_u + \chi_{u^c} = \text{Id} \). Note we need only that \( 1 \in Y \) and that \( 1+0=1 \).

**Lemma 34.** If \( J \subset I(z) \) and \( V(J) \neq V(I(z)) = \{z\} \) then J is not prime.

**Proof.** This is a Corollary to Theorem 13. \( \square \)

**Lemma 35.** If \( C(Z,Y) \) has addition with a unit, if \( \{z\} \) is clopen, and if \( J \subsetneq I(z) \neq C(Z,Y) \) is an ideal then J is not prime.

**Proof.** As \( z \) is clopen \( I(z) = (\chi_z) \). If J is prime so \( \Theta = \chi_{z^c} \cdot \chi_z \in J \) then \( \chi_z \) or \( \chi_{z^c} \) is in J. If \( \chi_z \in J \) then \( J \supsetneq (\chi_z) = I(z) \) contrary to \( J \subsetneq I(z) \). If \( \chi_{z^c} \in J \) then \( \text{Id} = \chi_z + \chi_{z^c} \in I(z) \) contrary to \( I(z) \neq C(Z,Y) \). \( \square \)
Theorem 16. If $Y$ has no divisors of zero, has addition with a unit, and if $\{z\}$ is clopen, then $I(z)$ is a minimal prime ideal.

Lemma 36. From the third Corollary to Definition 10, if $1 \in Y$, and, if $\{z\}$ is open, that is isolated, then $I(z) = (\chi_z)$ is principal and $\chi_z$ is an idempotent and a divisor of zero.

Corollary. If $I(z) = (f)$ is principal and $\{z\}$ is open then $(\chi_z) = (f)$.

Corollary. If $I(z)$ is not $(\chi_z)$ then the quasi-component $\{z\}$ is not open.

Lemma 37. If $f \neq \Theta$ and $V(f)$ is clopen then $f$ is a divisor of zero.

Proof. If $V(f)$ is clopen then $V^c(f)$ is clopen. Thus $\chi_{V^c(f)} \cdot f = \Theta$. □

Lemma 38. If $V(f) \supseteq \{z\}$, then $I(z) \supseteq (f)$.

Lemma 39. If $V(f) \supseteq V(\chi_u) = U$ then $(f) \subseteq (\chi_u)$.

Lemma 40. In general $V(f) = V((f))$.

Proof. As $f \in (f)$ then $V((f)) \subseteq V(f)$. Suppose $z \in V(f)$ and recall for any $h \in (f)$, $h = g \cdot f$ for some $g$. So $h(z) = 0$. Thus for all $h \in (f)$, $h(z) = 0$ which says $z \in \bigcap_{h \in (f)} h^{-1}(0) = V((f))$. □

Corollary. If $I(z) = (f)$ then $\{z\} = V(f)$.

Proof. As $V(h \cdot f) \supseteq V(h) \bigcup V(f)$, the second Corollary to Definition 10 says, $\{z\} = V(I(z)) = V((f)) = V(f)$. □

Note. Recall if $1 \in Y$ a prime ideal has a divisor of zero that is an idempotent for every clopen subset $U \subseteq Z$, either $\chi_u$ or $\chi_{V^c}$. Now consider divisors of zero in general. Lemmas 41 and 42 are independent of the topologies of $Z$ and $Y$ and holds if $Y$ is a magma with no "divisors of zero."
Lemma 41. If \( f \) is a divisor of zero and \( g \cdot f = \Theta, g \neq \Theta, \) and if \( Y \) has no divisors of zero then \( V^c(f) \cap V^c(g) = \emptyset. \)

Proof. If \( z \in V^c(f) \cap V^c(g) \) then \( f(z) \neq 0 \neq g(z) \) but \( g \cdot f = \Theta \) says \( g(z) \cdot f(z) = 0 \) contrary to \( Y \) has no divisors of zero. \( \square \)

Lemma 42. If \( f \) is a divisor of zero and \( Y \) and \( g \) as above then \( Z = V^c(f) \cup V^c(g) \), disjoint.

Proof. For \( z \in Z, g(z) \cdot f(z) = \Theta(z) = 0. \) As \( Y \) has no divisors of zero at least one but by the previous Lemma not both \( f(z) \) or \( g(z) \) is zero. \( \square \)

Theorem 17. If \( f, g, \) and \( Y \) are as above then \( Z = V^c(f) \cup V^c(g) = V(f) \cup V(g) \) are disjoint unions and \( V(f), V(g), V^c(f) \) and \( V^c(g) \) are all clopen.

Corollary. If \( Y \) has no divisors of zero then \( f \) is a divisor of zero iff \( V(f) \neq Z \) is clopen in which case \( (f) \subseteq (\chi_{v(f)}). \)

Note. When \( Y \) has no divisor of zero and \( V(f) = \{z\} \) then \( f \) is a divisor of zero iff \( z \) is isolated in which case \( I(z) = (\chi_z) \supseteq (f). \) This is a consequence of the Corollary and Lemma 39.

Corollary. If \( Y \) has no divisor of zero and \( \{z\} \) is not open then:

1) \( f \) is not a divisor of zero or \( V(f) \neq \{z\} \) in which case \( I(z) \neq (f). \)
2) if \( f \) is a divisor of zero then \( V(f) \neq \{z\} \) in which case \( I(z) \neq (f). \)
3) if \( V(f) = \{z\} \) then \( f \) is not a divisor of zero.

Theorem 18. If \( I_1 \) and \( I_2 \) are ideals, \( I_2 \neq C(Z,Y), 1 \in Y, Y \) has addition, \( I_1 \) prime, \( I_1 \subseteq I_2, \) and \( U \) clopen then \( \chi_U \in I_1 \) if and only if \( \chi_U \in I_2. \)

Proof. Note \( \chi_\emptyset = Id. \) So suffices to consider \( \chi_U \in I_2. \) If \( \chi_U \notin I_1 \) then \( \chi_{UC} \in I_1. \) So \( \chi_U + \chi_{UC} = Id \in I_2 \) contrary to \( I_2 \neq C(Z,Y). \) \( \square \)

Corollary. If \( I_1 \cap I_2 \) is prime then \( \chi_U \in I_1 \) if and only if \( (\chi_U) \subseteq I_1 \cap I_2. \)
It is important to note if \( Y \) has a unit and the two operations then even without distributivity of the two operations, \( \chi_u + \chi_u^c = Id \).

Note. The sequent of results suggest that there are binary set operations on the set of prime ideals, the set of clopen sets, and the set of characteristic functions. These operations then satisfy further properties. First recall that if \( I \) is a prime ideal then \( \chi_{u,a} \) or \( \chi_{u^c,a} \) is in \( I \). However although stated in terms of primality, it is not needed in many of the following cases yielding results for ideals in general. The results from Definition 12 through the Summary Note following Lemma 55 are related to possible convergence structures in \( C(Z,Y) \) and may not be of algebraic interest. However some of the set function techniques are used later.

Now assume \( 1 \in Y \). If and when two operations are used and it is easy to see if additional hypotheses are needed, they will not be stated except when they add to the flow of the narrative.

Definition 12. Let \( \Phi \) be the set of all ideals and let \( P \) be the set of all prime ideals together with \((\theta)\). Define for each clopen \( U \),

\[
\Phi_u = \{ I \in \Phi | \chi_u \in I \},
\]
\[
P_u = \{ I \in P | \chi_u \in I \} = \{ I \in P | (\chi_u) \subseteq I \}, \text{ and thus}
\]
\[
P_{uc} = \{ I \in P | \chi_u^c \in I \} = \{ I \in P | (\chi_u^c) \subseteq I \}.
\]

Lemma 43. 1) \( \chi_z = \Theta \) and \( \chi_z \in \bigcap P \),

2) \( \chi_{\emptyset} = Id \) as \( \chi_{\emptyset}(z) \) is 1 if \( z \notin \emptyset \) and 0 otherwise.

3) \( \Phi_z = \Phi \) and \( P_z = P \) as \( \chi_z = \Theta \in I \) for all \( I \).

4) \( \Phi_\emptyset = P_\emptyset = \{ C(Z,Y) \} \) as \( \chi_\emptyset = Id \).

5) For any clopen \( U \), \( C(Z,Y) \in P_u \).

Lemma 44. If \( I_1 \) and \( I_2 \) are ideals, \( I_2 \neq C(Z,Y) \), and \( I_1 \subseteq I_2 \) then \( I_1 \in P_u \) if and only if \( I_2 \in \Phi_u \). Thus if \( I_2 \) is also prime, \( I_1 \in P_u \) if and only if \( I_2 \in P_u \).
Lemma 45. For each non empty clopen set $U$, $U \neq Z$, $\mathcal{P} = \mathcal{P}_u \cup \mathcal{P}_{u^c}$ and $\mathcal{P}_u \cap \mathcal{P}_{u^c} = \emptyset$.

Proof. At lease one but not both $\chi_u$ and $\chi_{u^c}$ can belong to $I$. \qed

Lemma 46. If $U$ and $W$ are clopen then $\mathcal{P}_u \subseteq \mathcal{P}_u \cup \mathcal{P}_u \cup \mathcal{P}_{u^c} \cap \mathcal{P}_{u^c}$. (also for $\Phi_U$)

Proof. Let $I \in \mathcal{P}_u$. It follows that $\chi_w \cdot \chi_u = \chi_u \cup w$ and $\chi_{w^c} \cdot \chi_w$ are in $I$. \qed

Corollary. If $U_1 \subseteq U_2$ are clopen then $\mathcal{P}_{u_1} \subseteq \mathcal{P}_{u_2} \cup \mathcal{P}_{u_1} \cup u_2$ and $\mathcal{P}_{u_1} \subseteq \mathcal{P}_{u_2}$ as $\chi_{u_2} = \chi_{u_2} \cdot \chi_{u_1}$. So if $U_2^c \subseteq U_1^c$ then $\mathcal{P}_{u_2} \subseteq \mathcal{P}_{u_1}$. (also for $\Phi_u$).

Corollary. Let $U_1$ and $U_2$ be clopen:

1) Recall if $U_1 \subseteq U_2$ then $\mathcal{P}_{u_1} \subseteq \mathcal{P}_{u_2}$.
2) $\mathcal{P}_{u_1} \cup u_2 \supseteq \mathcal{P}_{u_1} \cup \mathcal{P}_{u_2}$.
3) $\mathcal{P}_{u_1} \cap u_2 \subseteq \mathcal{P}_{u_1} \cap \mathcal{P}_{u_2} \subseteq \mathcal{P}_{u_1} \cup u_2$.
4) If $U_1 \cap U_2 = \emptyset$ then $\mathcal{P}_{u_1} \cap \mathcal{P}_{u_2} = \{C(X,Y)\}$.

(corollary also holds for $\Phi_u$).

Proof. For 3) use the preceding corollary and 2) of this corollary.

For 4), if $U_1 \cap U_2 = \emptyset$ then $U_1^c \supseteq U_2$. Let $I \in \mathcal{P}_{u_1} \cap \mathcal{P}_{u_2}$ so that $\chi_{u_1}$ and $\chi_{u_2}$ belong to $I$. Then $\chi_{u_1} = \chi_{u_1} \cdot \chi_{u_2} \in I$. Consequently both $\chi_{u_1}$ and $\chi_{u_2}$ belong to $I$. Thus $I=C(Z,Y)$ and $\mathcal{P}_{u_1} \cap \mathcal{P}_{u_2} = \{C(Z,Y)\}$. \qed

Definition 13. Let $\mathcal{U} = \{U|U is clopen\}$.

Define for $I \in \mathcal{P}$,

$\mathcal{U}_I = \{U|\chi_u \in I\} = \{U|(\chi_u) \subseteq I\}$ and

$\mathcal{U}_I = \{U|\chi_{u^c} \in I\}$.

Lemma 47. 1) $\mathcal{U}_I(\Theta) = \{U|\chi_u \in (\Theta)\} = \{Z\}$ as $\chi_Z = \Theta$.
2) $\mathcal{U}_{C(Z,Y)} = \mathcal{U}_{C(Z,Y)} = \mathcal{U}$.
3) $\mathcal{U}_{\Theta} = \{\emptyset\}$. 

24
4) \( \cup_{(\Theta)} \cup_{(\Theta)}^c = \{\emptyset, Z\} \neq \cup \).

5) If \( I_1 \) and \( I_2 \) are in \( \mathcal{P} \) and \( I_1 \subseteq I_2 \) then \( \cup_{I_1} \subseteq \cup_{I_2} \) and if \( I_2 \neq C(Z,Y) \) then \( \cup_{I_1} = \cup_{I_2} \).

**Lemma 48.** \( I \in \mathcal{P}_u \) if and only if \( U \in \cup_I \).

**Corollary.** \( (\Theta) \in \mathcal{P}_Z \) if and only if \( Z \in \cup_{(\Theta)} \).

**Lemma 49.** If \( I \in \mathcal{P} \setminus \{C(Z,Y)\} \) then \( \cup = \cup_I \cup_{I^c} \) and \( \cup_I \cap \cup_{I^c} = \emptyset \).

**Lemma 50.** If \( I_1 \cap I_2 \) is prime, then \( \cup_{I_1 \cap I_2} = \cup_{I_1} \cap \cup_{I_2} \).

**Proof.** \( U \in \cup_{I_1 \cap I_2} \) if and only if \( U \in \cup_{I_1} \) and \( U \in \cup_{I_2} \) if \( \chi_u \in I_1 \) and \( \chi_u \in I_2 \) if \( \chi_u \in I_1 \cap I_2 \) if \( I_1 \cap I_2 \in \mathcal{P}_u \) if \( U \in \cup_{I_1 \cap I_2} \). \( \square \)

**Lemma 51.** If \( I_1 \cup I_2 \) is a prime ideal, then \( \cup_{I_1 \cup I_2} = \cup_{I_1} \cup I_2 \).

**Proof.** \( U \in \cup_{I_1 \cup I_2} \) if and only if \( U \in \cup_{I_1} \) or \( U \in \cup_{I_2} \) if and only if \( \chi_u \in I_1 \) or \( \chi_u \in I_2 \) if \( \chi_u \in I_1 \cup I_2 \). \( \square \)

**Definition 14.** Let \( \mathcal{X} = \{\chi_u| \text{U is clopen}\} \). For \( I \in \mathcal{P} \) define \( \mathcal{X}_I = \{\chi_u| \chi_u \in I\} \subseteq I \) and
\[
\mathcal{X}_I^c = \{\chi_u| \chi_u^c \in I\}.
\]

**Lemma 52.**
1) \( \mathcal{X}_{(\Theta)} = \{\chi_u| \chi_u \in (\Theta)\} = \{\Theta\} \) and \( \mathcal{X}_{C(Z,Y)} = \mathcal{X} \).
2) If \( I_1 \) and \( I_2 \) are in \( \mathcal{P} \) and \( I_1 \subseteq I_2 \) then \( \mathcal{X}_{I_1} \subseteq \mathcal{X}_{I_2} \) and if \( I_2 \neq C(Z,Y) \) then \( \mathcal{X}_{I_1} = \mathcal{X}_{I_2} \).
3) \( \chi_u \in \mathcal{X}_I \) if and only if \( U \in \cup_I \) if and only if \( I \in \mathcal{P}_u \).

**Corollary.** \( \chi_z = \Theta \in \mathcal{X}_{(\Theta)} \) iff \( Z \in \cup_{(\Theta)} \) iff \( (\Theta) \in \mathcal{P}_Z = \mathcal{P} \) and \( \chi_{\emptyset} \in \mathcal{X}_{C(Z,Y)} \) if and only if \( \emptyset \in \cup_{C(Z,Y)} \) if and only if \( C(Z,Y) \in \mathcal{P}_\emptyset \).

**Lemma 53.** \( \mathcal{X} = \mathcal{X}_I \cup \mathcal{X}_I^c \) and \( \mathcal{X}_I \cap \mathcal{X}_I^c = \emptyset \).

**Lemma 54.** If \( I_1, I_2, \) and \( I_1 \cap I_2 \) are in \( \mathcal{P} \) (or \( I_1 \) and \( I_2 \) are in \( \Phi \) necessary changes made) then \( \mathcal{X}_{I_1} \cap \mathcal{X}_{I_2} = \mathcal{X}_{I_2} \cap \mathcal{I}_2 \).
Lemma 55. If $I_1$, $I_2$, and $I_1 \cup I_2$ are in $\mathcal{P}$ (or $\Phi$), then $\mathcal{X}_{I_1} \cup \mathcal{X}_{I_2} = \mathcal{X}_{I_1 \cup I_2}$.

Note. In summary: Let $U_1$ and $U_2$ be clopen.
A) $\mathcal{P}_\emptyset = \{C(Z,Y)\}; \mathcal{P}_Z = \mathcal{P}$; If $U_1 \subseteq U_2$, then $\mathcal{P}_{u_1} \subseteq \mathcal{P}_{u_2}$; $\mathcal{P}_{u_1 \cap u_2} \subseteq \mathcal{P}_{u_1} \cap \mathcal{P}_{u_2}$; and $\mathcal{P}_{u_1} \cup \mathcal{P}_{u_2} \subseteq \mathcal{P}_{u_1 \cup u_2}$.
B) $\mathcal{U}_{(\emptyset)} = \{Z\}; \mathcal{U}_{C(Z,Y)} = \mathcal{U}$; if $I_1$ and $I_2$ are in $\mathcal{P}$ and $I_1 \subseteq I_2$, then $\mathcal{U}_{I_1} \subseteq \mathcal{U}_{I_2}$ and if $I_2 \neq C(Z,Y)$, then $\mathcal{U}_{I_1} = \mathcal{U}_{I_2}$; if $I_1 \cap I_2$ is prime then $\mathcal{U}_{I_1 \cap I_2} = \mathcal{U}_{I_1} \cap \mathcal{U}_{I_2}$; and if $I_1 \cup I_2$ is a prime ideal then $\mathcal{U}_{I_1 \cup I_2} = \mathcal{U}_{I_1} \cup \mathcal{U}_{I_2}$.
C) $\mathcal{X}_{(\emptyset)} = \{\emptyset\}; \mathcal{X}_{C(Z,Y)} = \mathcal{X}$; if $I_1$ and $I_2$ are in $\mathcal{P}$ and $I_1 \subseteq I_2$ then, $\mathcal{X}_{I_1} \subseteq \mathcal{X}_{I_2}$ and if $I_2 \neq C(Z,Y)$ then $\mathcal{X}_{I_1} = \mathcal{X}_{I_2}$; if $I_1 \cap I_2$ is prime then, $\mathcal{X}_{I_1 \cap I_2} = \mathcal{X}_{I_1} \cap \mathcal{X}_{I_2}$; and if $I_1 \cup I_2$ is a prime ideal then $\mathcal{X}_{I_1 \cup I_2} = \mathcal{X}_{I_1} \cup \mathcal{X}_{I_2}$.

Note. By Zorn’s lemma every prime ideal of $C(X,Y)$ with the one operation, $\cdot$, contains a minimal prime ideal with respect to $\cdot$, follow the proof of Lemma 2, in [3] pg 73.

Theorem 19. If $J$ is prime and $V(J) \neq \emptyset$ then there is a unique $z$ such that $V(J) = \{z\}$ and $J \subseteq I(z)$.

Proof. If $V(J) \neq \emptyset$ then $\| V(J) \| \geq 1$. If $\| V(J) \| > 1$ then $J$ is not prime. So $\| V(J) \| = 1$ that is there exist an unique $z$ such that $V(J) = \{z\}$ and thus for all $f \in J$, $f(z) = 0$. \(\square\)

Corollary. If $\{z\} = V(J)$ is open then $J = I(z) = (\chi_z)$. That is a prime ideal $J$ with a non empty zero set must be of the form $I(z)$ where $\{z\} = V(J)$ even if $Y$ has a divisor of zero.

Proof. Note $\chi_z$ or $\chi_z \epsilon$ is in $J$. So $\chi_z$ must be in $J$. If $f \in I(z)$ then $\{z\} \subseteq V(f)$. Therfore $f = f \cdot \chi_z \in J$. \(\square\)

Theorem 20. If $J \subseteq \neq I$ where $I$ is an ideal and $J$ is a prime prime ideal, and $I \neq C(Z,Y)$, then for every clopen $U$, one but not both of $\chi_u$ and $\chi_u \epsilon$ is in $I$.

Proof. As $J \subseteq I$ and as one but not both is in $J$ then this is true for $I$ by Theorem 18. \(\square\)
Note. Let $Z$ is a discrete space of three points and $Y$ be $\mathbb{Z}_2$. Then for example the ideals of functions vanishing on two points are not prime.

Lemma 56. If $Y$ is a division ring and $f(z) \neq 0$ for all $x \in Z$ then $g(z) = \frac{1}{f(z)} \in C(Z,Y)$.

Proof. Let $U$ be open in $Y$ and $0 \notin U$ and let $U^{-1} = \{1/u : u \in U \}$. Then $g^{-1}(U) = \{z : f(z) \in U^{-1}\} = f^{-1}(U^{-1})$ which is open in $Z$. □

Lemma 57. If $Y$ is a division ring, $I$ a proper function ideal of $C(Z,Y)$ then for each $f \in I$ there exist $z \in Z$ such that $f(z) = 0$ otherwise $f$ is invertible and thus $I = (1)$.

Corollary. If $Y$ is a division ring and $I = (f) = \{h \cdot f\}$ for all $h \in C(Z,Y)$ which is a proper ideal then as there exist an $z$ such that $f(z) = 0$, $(f) \subset I(z)$.

Recall: 1) Every non empty subset of $Z$ is a set of quasi-components.

2) As $H : C(X,Y) \rightarrow C(\Pi,Y)$ is an algebraic isomorphism, $H$ maps ideals to ideals of the same algebraic kind and vice versa.

Moreover:

Lemma 58. If $I(x)$ is an ideal in $C(X,Y)$ corresponding to $[x]$ and $z = [x]$ then $H(I(x)) = I(z)$.

Proof. Let $f \in I(x)$, that is $f(x) = 0$. Now as $f \in C(X,Y)$, $H(f)(z) = f \circ p^{-1}(z) = f([x]) = f(x) = 0$. Thus $H(f) \in I(z)$. That is $H(I(x)) \subseteq I(z)$. Now suppose $\hat{f} \in I(z)$ that is $\hat{f}(z) = 0$. As $\hat{f} \in C(\Pi,Y)$, $G(\hat{f}) \in C(X,Y)$ and $G(\hat{f})(x) = (\hat{f} \circ p)(x) = \hat{f}(z) = 0$, that is $G(\hat{f}) \in I(x)$. Then $H(G(\hat{f})) = \hat{f}$, that is $\hat{f} \in H(I(x))$ and so $I(z) \subseteq H(I(x))$. □

It is now appropriate to consider set and algebraic observations about $\chi_u, \mathfrak{X}, \mathfrak{X}_I, \text{and } \mathfrak{X}$ where $\mathfrak{X}$ is yet to be defined.

Lemma 59. The following are already given or evident where $U, U_1, \text{and } U_2$ are clopen and $I \subset C(Z,Y)$ is an ideal. When needed $1+0=0+1=1$.

1) $\chi_u \cdot \chi_u = \chi_u$, that is $\chi_u$ is idempotent and if $Y$ has addition, then $\chi_u + \chi_u^c = \text{Id}$. 27
2) $\chi_{u_1} \cdot \chi_{u_2} = \chi_{(u_1 \cup u_2)}$ and if $\chi_{u_1}$ or $\chi_{u_2}$ is in $I$ then $\chi_{(u_1 \cup u_2)} = \chi_{u_1} \cdot \chi_{u_2}$ is in $I$.

3) If $Y$ has addition and $1+1=0$, then $\chi_{u_1} + \chi_{u_2} = \chi_{((u_1 \cap u_2) \cup (u_1 \cup u_2))}$ and $\chi_u + \chi_u = \chi = \emptyset$.

4) If $U_1 \subset U_2$ then $\chi_{u_2} = \chi_{u_2} \cdot \chi_{u_1}$.

5) If $U_1 \neq U_2$ then $\chi_{u_1} \neq \chi_{u_2}$.

6) $U_1 \cap U_2 = V(\chi_{(u_1 \cap u_2)})$.

7) $U_1 \cup U_2 = V(\chi_{u_1}) \cup V(\chi_{u_2}) = V(\chi_{u_1} \cdot \chi_{u_2})$.

8) If $I$ is a prime ideal then for all clopen $U$, $\chi_u$ or $\chi_{u^c}$ is in $I$.

9) If $U \subseteq U_1$, and $\chi_u \in I$ then $\chi_{u_1} \in I$.

10) If $I \neq C(Z,Y)$ is a prime ideal, $U_1 \cap U_2 = \emptyset$, and $\chi_{u_1} \in I$ then $\chi_{u_2} \notin I$.

11) If in $Y$, $1+1=0$, if $I$ is closed under addition (or if $\chi_{(u_1 \cap u_2)} \in I$), and if $\chi_{u_1}$ and $\chi_{u_2}$ are in $I$ then $\chi_{u_1} + \chi_{u_2} = \chi_{((u_1 \cap u_2) \cup (u_1 \cup u_2))} \in I$.

12) If $\emptyset \neq U \neq Z$ then $(\chi_u) \cap (\chi_{u^c}) = (\emptyset)$ and $(\chi_u) + (\chi_{u^c}) = C(Z,Y)$.

13) $Id = \chi_{\emptyset}$ and for any ideal $I$, $\Theta = \chi = I$ and thus $X = I \neq \emptyset$.

14) $X_I$ is multiplicatively closed.

15) If $\chi_u \in X_I$ then $\chi_w \cdot \chi_u = \chi_{w \cup u} \in X_I$.

16) From 11, if $Y$ and $I$ are closed under addition (or if $\chi_{(v \cap u)} \in I$), $1+1=0$, and $\chi_u$ and $\chi_v$ are in $X_I$ then $\chi_u + \chi_w \in X_I$.

17) If $\chi_u \cdot \chi_w = \chi_{w \cup u} \in X_I$, then $\chi_u$ or $\chi_w$ is in $X_I$.

18) If the operations are distributive, $\chi_u \in I$, and $\chi_u + \chi_w = \Theta$ then $\chi_w \in I$.

19) If $\chi_u \cdot \chi_w$ and $\chi_v + \chi_u$ are in $I$ and $1+1=0$ then $V \cap U \neq \emptyset$.

Proof. For 10). We can use $U_2 \subseteq U_1^c$ which implies that $\chi_{(u_1^c)} = \chi_{(u_1^c)} \cdot \chi_{u_2}$. If $\chi_{u_2} \in I$ then $\chi_{(u_1^c)} \in I$ and therefore $\chi_{u_1} + \chi_{(u_1^c)} = Id \in I$ which is a contradiction.

For 18). So $\chi_u \cdot \chi_{u^c} + \chi_w \cdot \chi_{u^c} = \chi_{u^c} \cdot \Theta$ which says $\theta + \chi_w \cdot \chi_{u^c} = \theta$ implying that $\chi_w \in I$.

For 19). Note $\chi_{(v \cup u)} = \chi_v \cdot \chi_u \in I$ implies that $\chi_{(v \cup u)^c} \notin I$. Also observe as $1+1=0$, $\chi_v + \chi_u = \chi_{((v \cap u) \cup (v \cup u))}$ and $\chi_{(v \cup u)} \cdot \chi_{(v \cup u)} \in I$. Together these imply that $\chi_{(v \cap u)} \in I$. Now if $V \cap U = \emptyset$ then $\chi_{(v \cap u)} = Id \in I$. □
Theorem 21. If \( \cdot \) in \( Y \) is closed, associative, and commutative then \( X = (X, \Theta, +, \cdot) \) has the properties that \( \cdot \) is closed, associative, and commutative and is such that every element is idempotent. Moreover by 3), if \( Y \) has addition that is closed, associative, commutative, and \( 1+1=0 \) then + is closed, associative, commutative, and \( \chi_U + \chi_U = \Theta \), that is \( \chi_U \) is \( \chi_U \). If the operations in \( Y \) are distributive then \( X \) is distributive. In this case \( X \) is a subring of \( C(Z,Y) \) isomorphic to \( C(Z, \mathbb{Z}_2) \).

Theorem 22. By 14), \( X_I \) is multiplicative closed, by 15) it is an ideal, and by 17) it is prime. Now by 16), if \( 1+1=0 \) in \( Y \), if \( Y \) is a ring, and if \( I \) is an ideal then \( X_I \) is a prime ring idea in \( X \) and a subring of \( C(Z,Y) \).

Now consider the set of \( X_I \).

Definition 15. Set \( X = \{X_I | I \in \mathcal{P}\} \) and set \( \mathcal{D} = \{\Theta\} = \{\chi_Z\} = X_{(\Theta)} \).

Lemma 60. When appropriate, define addition and multiplication in \( X \) in the natural way.

1) If \( I_1 \subseteq I_2 \) then \( X_{I_1} \subseteq X_{I_2} \).
2) If \( I_1 \cap I_2 \in \mathcal{P} \) then \( X_{I_1} \cap X_{I_2} = X_{I_1 \cap I_2} \).
3) If \( I_1 \cup I_2 \in \mathcal{P} \) then \( X_{I_1} \cup X_{I_2} = X_{I_1 \cup I_2} \).
4) \( X_I + \mathcal{D} = X_I + X_{(\Theta)} = X_I \).
5) \( X_I \cdot \mathcal{D} = \mathcal{D} \).
6) If \( 1+1=0 \) in \( Y \), \( Y \) is a ring, and \( I_1 \) and \( I_2 \) are ideals then by Theorem 22, \( X_{I_1} + X_{I_2} \) and \( X_{I_1} \cdot X_{I_2} \) are ideals in \( X \) but need not be in \( X \).
7) If \( 1+1=0 \) in \( Y \), \( Y \) a ring, and \( I_1 \) and \( I_2 \) are ideals then \( X_{I_1} \) and \( X_{I_2} \) are prime ideals, \( X_{I_1} + X_{I_2} \) is in \( X \), and in \( X \) whenever \( I_1 + I_2 \) is prime.
8) If \( 1+1=0 \) in \( Y \) then \( X_{I_1} + X_{I_2} \subseteq X_{I_1 + I_2} \) as in general \( X_I \subseteq I \).
9) If \( 1+1=0 \) in \( Y \), \( Y \) is a ring, and \( I_1 \) and \( I_2 \) are ideals then \( X_{I_1} \cdot X_{I_2} \) is in \( X \) and in \( X \) whenever \( I_1 \cap I_2 \) is prime.

Proof. For 7). From Theorem 22, \( X_{I_1} \) and \( X_{I_2} \) are ideals of \( X \) and thus their sum is an ideal of \( X \). To show this ideal is in \( X \) let \( \chi_{u_1} \in X_{I_1} \) and \( \chi_{u_2} \in X_{I_2} \). By Lemma 59, No. 11)
\(\chi_{u_1} + \chi_{u_2} = \chi_{u_1} \in I_1 + I_2.\) For 9). First \(\mathfrak{X}_{I_1} \cdot \mathfrak{X}_{I_2}\) is an ideal. Suppose \(\chi_u \cdot \chi_w \in \mathfrak{X}_{I_1} \cdot \mathfrak{X}_{I_2}\). Then \(\chi_u \cdot \chi_w\) is a finite sum of terms of the form \(\chi_{u_i} \cdot \chi_{w_i}\) where \(\chi_{u_i} \in I_1\) and \(\chi_{w_i} \in I_2\). Thus \(\chi_{u_i} \cdot \chi_{w_i} \in I_1 \cap I_2\) and consequently \(\chi_u \cdot \chi_w \in I_1 \cap I_2\). □

**Note.** As \(\mathfrak{X}\) does not in general have algebraic properties like \(\mathfrak{X}\) when \(1+1=0\) in \(Y\) and because \(\chi_u + \chi_u = \text{Id}\) and \(\chi_u \cdot \chi_w = \Theta\) in \(\mathfrak{X}\), the following definition is of interest when \(I_1 + I_2\) and \(I_1 \cdot I_2\) are present.

**Definition 16.** In a structure \(R\) like a ring, possibly weaker, \(f \in R\) has has a complement \(f^c \in R\) iff \(f + f^c = \text{Id}\) and \(f \cdot f^c = \Theta\). \(R\) has complements iff every element has a complement.

**Corollary.** The only ideal of \(R\) that contains an element and its complement is \(R\). Thus if \(I\) is a proper ideal then \(f\) and \(f^c\) are not both in \(I\).

**Corollary.** If \(f\) has a complement then it and its complement are idempotent. Thus if \(R\) has complements it is Boolean.

**Corollary.** If \(I_1 + I_2 = R\) where \(I_1\) and \(I_2\) are proper ideals and \(f \in R\) has a complement, then \(f\) is in one of the ideals and its complement is in the other.

**Corollary.** If \(f\) has an additive inverse and a complement then its complement is unique (addition associative and commutative).

**Theorem 23.** If \(R\) has complements, \(I_1\) is a prime ideal, \(I_2\) an ideal, and \(I_1 + I_2 \neq R\), then \(I_1 + I_2\) is a prime ideal.

**Proof.** As \(I_1 + I_2\) is an ideal let \(f = f_1 \cdot f_2 \in I_1 + I_2\). As \(I_1\) is prime and \(f \cdot f^c = \Theta\), \(f\) or \(f^c\) is in \(I_1\). If \(f^c \in I_1\) then \(f^c + f = (f^c + \Theta) + f = \text{Id} \in I_1 + I_2 = R\). Thus \(f = f_1 \cdot f_2 \in I_1\) and consequently \(f_1\) or \(f_2\) is in \(I_1\) from which it follows that \(f_1 + \Theta\) or \(f_2 + \Theta\) is in \(I_1 + I_2\). □

**Lemma 61.** If the two operations are associative and distributive, if \(f_1\) and \(f_2\) are multiplicative commuting idempotents, and if their product has an additive inverse then \((f_1 \cdot f_2)^c = Id - f_1 \cdot f_2\)
Note that for characteristic functions $\chi_u^c = \chi_{u^c}$.

**Corollary.** If $\chi_{u_1}$ and $\chi_{u_2}$ are in $C(Z,Y)$ or $X$ and $Y$ is a ring then $(\chi_{u_1} \cdot \chi_{u_2})^c = (\chi_{u_1 \cup u_2})^c = Id - \chi_{u_1 \cup u_2} = \chi_{(u_1 \cup u_2)^c}$. Consequently if $1+1=0$ then $\chi_{u_1} + \chi_{u_2} = \chi_{u_1 \cap u_2} - \chi_{u_1 \cup u_2}$.

**Lemma 62.** In $C(Z,\mathbb{Z}_2)$ or $X$ if $I_1$ and $I_2$ prime ideals and $\chi_u \cdot \chi_w \in I_1 \cdot I_2$ then $\chi_u^c \cdot \chi_w^c - (\chi_u \cdot \chi_w)^c = \chi_{u \cap w} - \chi_{u \cup w}^c$.

**Proof.** Observe $(\chi_u \cdot \chi_w)^c = Id - \chi_{u \cup w}$ and $\chi_u^c \cdot \chi_w^c = \chi_{u^c \cap w^c} = Id - (\chi_{u^c \cup w^c})^c = Id - \chi_{u \cap w}^c$ and subtract the first from the second. □

**Note.** If any point of $Y$ is open then all points of $Y$ are open and hence $Y$ has the discrete topology. We will use the term $\{0\}$ is open to indicate that $Y$ is discrete. Also in what follows $Y$ need not be a ring that is $\cdot$ will usually suffice. Thus again the ideals need not be ring ideals.

Here it is convenient to state a stronger form of Theorem 16.

**Theorem 24.** If $\{z\}$ is clopen in $Z$, $J$ a prime ideal, and $J \subseteq I(z)$ then $J = I(z)$. Thus if $\{z\}$ is open in $Z$, $I(z)$ is a minimal prime as it contains a prime ideal.

**Proof.** As $\chi_z \cdot \chi_{z^c} = 0$, $\chi_z \in J$. Thus if $f \in I(z)$ then $f = f \cdot \chi_z \in J$. □

**Theorem 25.** If $\{0\}$ is clopen in $Y$, $J$ is a prime ideal, and $J \subseteq I(z)$ then $J = I(z)$. Thus if $\{0\}$ is open in $Y$, the $I(z)$ are the min-max prime ideal.

**Proof.** For $f \in I(z)$ choose $U = f^{-1}(0)$. Then $\chi_U \in J$ and consequently $f = f \cdot \chi_U \in J$. □

**Lemma 63.** If $\{z\}$ is clopen then $I(z) = (\chi_z)$ is principal and thus $\chi_U \in I(z)$ for every clopen $U$ containing $z$.

**Proof.** Use Lemma 38 and Lemma 59, No.4. Note that $\chi_z$ is continuous independent of whether $\{0\} \subseteq Y$ is open or not. □
Note. If $\chi$ is a characteristic function and $W \subset Y$ is clopen such that $0 \in W$ and $1 \notin W$ then $\chi^{-1}(0) = \chi^{-1}(W)$ is clopen although $\{0\} \subset Z$ need not be open. Likewise $\chi^{-1}(1)$ is clopen.

Lemma 64. If $(\chi_u)$ is neither $(\Theta)$ or $C(Z,Y)$ then neither is $(\chi_{w^c})$ and if $\cdot$ is commutative and associative then $(\chi_u) \cap (\chi_{w^c}) = (\Theta)$.

Proof. If $(\chi_{w^c}) = C(Z,Y) = (\chi_{Z^c}) = (\chi_{Z^c})$ then $(\chi_u) = (\chi_Z) = (\Theta)$.
If $(\chi_{w^c}) = (\Theta) = (\chi_Z)$ then $(\chi_u) = (\chi_{Z^c}) = (\chi_{Z^c}) = (Id) = C(Z,Y)$.
Suppose $\chi_w \in (\chi_u) \cap (\chi_{w^c})$ and thus $\chi_w = \chi_{w_1} \cdot \chi_u = \chi_{w_2} \cdot \chi_{w^c}$. As $\chi_w = \chi_w \cdot \chi_w$ form the composite product to show $\chi_w = \Theta$. □

Theorem 26. If $\cdot$ is commutative and associative, if $J$ be a prime ideal, and if $J \neq C(Z,Y)$ or $(\Theta)$, then let $\chi_1 \in J$ and set $U_1 = \chi_1^{-1}(0) \neq Z$ which is clopen. If $U$ is clopen and $U \cap U_1 \neq \emptyset$ then $\chi_u \in J$.

Proof. Let $\chi_{u_1} = \chi_1 \in (\chi_{u_1})$. Cases: (a) $U_1 \subseteq U$, (b) $U \subseteq U_1$, and (c) $U_1 \cap U$ is in both.
(a) Note, $\chi_u = \chi_u \cdot \chi_{u_1} \in (\chi_{u_1}) \subseteq J$.
(b) If $U \subseteq U_1$ then $U_1^c \subseteq U^c$ which implies $\chi_{w^c} = \chi_{w^c} \cdot \chi_{u_1^c} \in (\chi_{u_1^c})$. Thus $\chi_u \notin (\chi_{u_1^c})$ and thus $\chi_u \in (\chi_{u_1}) \subseteq J$.
(c) As $U \cap U_1$ is clopen and as $U \cap U_1 \subseteq U_1$ by (b) $\chi_u U_1 \subseteq J$. Now as $U \cap U_1 \subseteq U$ we have from (a), $\chi_u \in J$. □

Lemma 65. Assuming that $\cdot$ is commutative and associative let $J$ be a prime ideal, and $\chi_{u_1} \in J$. If $U$ is clopen and if $z \in U \cap U_1$ then $\chi_u \in J \cap I(z)$.

Theorem 27. If $\cdot$ is commutative and associative, if $J$ is a prime ideal, and if $\{0\} \subseteq Y$ is open then there exist a $\{z\}$ such that $I(z) \subseteq J$. Note $\{z\} \subset Z$ need not be open.

Proof. Let $U$ be clopen. As $\chi_u$ or $\chi_{w^c}$ is in $J$ with out loss of generality call it $\chi_{u_1}$. Choose $z \in U_1$ and $f \in I(z)$. Let $U_f = f^{-1}(0) = V(f)$ which is clopen. Then $z \in U_1 \cap U_f$. Thus by Theorem 26, $\chi_f \in J$ and therefore $f = f \cdot \chi_f \in J$. Consequently $I(z) \subseteq J$. □
**Theorem 28.** If \( \cdot \) is associative and commutative, if \( J \) is a prime ideal, if \( V(J) \neq \emptyset \), and if \( \{0\} \) is open in \( Y \) then there exist a unique \( z \) such that \( J = I(z) \). In this case the \( I(z) \) are minimal prime ideals. Hence if \( \{z\} \) is open the ideal is principal (Lemma 63).

**Proof.** By Theorem 19 there exist a unique \( z_1 \) such that \( J \subseteq I(z_1) \) and by Theorem 27 there is a \( \{z_2\} \) such that \( I(z_2) \subseteq J \). Thus as \( I(z_2) \subseteq J \subseteq I(z_1) \), \( z_1 = z_2 \) by Corollary 5 of Definition 10. \( \square \)

**Corollary.** If \( \cdot \) is associative and commutative, \( J \) is prime, \( \{0\} \) is open in \( Y \), and \( J \neq I(z) \) for any \( z \) then \( V(J) = \emptyset \) and \( J \nsubseteq I(z) \).

**Note.** The functions of \( C(Z, \mathbb{Z}_2) \) are the characteristic functions and the constant functions \( \Theta = \chi_\emptyset \) and \( Id = \chi_\emptyset \). The characteristic functions other than \( \Theta \) and \( Id \) are surjective. For \( \chi \in C(Z, \mathbb{Z}_2) \) let \( U = \chi^{-1}(0) \) so that \( \chi = \chi_U \).

As a quasi-component of \( X \) or \( Z \) is defined as the intersection of the clopen sets containing the point \( x \) or \( z = [x] \) and each quasi-component, \( [x] \) or \( [z] \), corresponds to \( I(x) \) or \( I(z) \) and hence to the characteristic functions of the clopen sets containing \( x \) or \( z \), it is natural to check the relationship between \( C(X,Y) \), \( C(\Pi,Y) \), \( C(Z,Y) \), and \( C(Z, \mathbb{Z}_2) \) where \( \mathbb{Z}_2 \) has the discrete topology. The classical example is \( C(Z, \mathbb{Z}_2) \) which easily generalizes to \( C(Z, \mathbb{Z}_2) \).

What is important (regardless of the topology of \( X \)) because of continuity, the functions of \( C(X, \mathbb{Z}_2) \) are of the form \( \chi_U \) where \( U \) is a clopen set. In Theorems 29 and 30 assume \( Z \) has the discrete topology and \( Y \) has multiplication and addition with 0 and 1 with 0 open. Consequently in \( C(Z,Y) \), \( \chi_u + \chi_{uc} = Id \).

**Theorem 29.** If \( I \) and \( J \) are proper prime ideals neither can be a subset of the other.

**Proof.** For ideals as subsets, if \( \chi_u \) is in one it must be in the other for \( \chi_{uc} \) is excluded from both. \( \square \)

**Corollary.** All proper prime ideals are min-max.
**Note.** It is known that (for rings) as $\chi_u^n = \chi_u$ the prime ideals are max.

**Theorem 30.** All proper prime ideals are of the form $I(z)$.

**Proof.** Let $J$ be a proper prime ideal and $\chi_u \in J$ which implies $(\chi_u) \subseteq J$ and $U \neq \emptyset$. Now choose $z \in U$ so that $(\chi_u) \subseteq I(z)$. As $\{z\}$ is clopen $I(z) = (\chi_z)$. Note $\chi_u : \chi_z \in (\chi_u) \cap (\chi_z)$. Consequently $\chi_u \in (\chi_z)$ which says $(\chi_u) \subseteq (\chi_z)$ and $\chi_z \in (\chi_u)$ which says $(\chi_z) \subseteq (\chi_u)$. Thus $(\chi_u) = (\chi_z) = I(z) \subseteq J$. Therefore, $I(z)$ being a prime ideal, $J = I(z)$. \qed

**Note.** Theorems 29 and 30 establishes that the proper prime ideals of $C(\mathbb{Z}, \mathbb{Z}_2)$ are all min-max and of the form $I(z)$.

*To return to the comparison of the function spaces, assume $Z$ has a possibly weaker $\mathcal{T}$ topology and $Y$ has the clopen base topology and $\cdot$ with a 0 and 1.*

In consonant with the comment following Theorem 5) on page 8, now set $[x]_\chi = [x]_2$ where $[x]_2$ is defined:

**Definition 17.** $[x]_2 = \{y \mid \forall \chi \in C(Z, \mathbb{Z}_2), \chi(y) = \chi(x)\}$ and $\Pi_2 = \{[x]_2\}$.

Note that $Q_x = [x] = [x]_2$, (pg 8), so $\Pi = \Pi_2 = Z$ and consequently $C(X, Y) = C(\Pi, Y) = C(\Pi_2, Y) = C(Z, Y)$ and $C(X, \mathbb{Z}_2) = C(\Pi_2, \mathbb{Z}_2) = C(Z, \mathbb{Z}_2)$.

First there there is natural injection of $C(Z, \mathbb{Z}_2)$ into $C(Z, Y)$ if $Y$ has a binary operation with at least a null element, 0 and unitary element, 1.

**Definition 18.** Define $j : \mathbb{Z}_2 \hookrightarrow Y$ by $j(0) = 0$ and $j(1) = 1$ and define $\mathbb{J} : C(Z, \mathbb{Z}_2) \rightarrow C(Z, Y)$ by $J(\chi) = j \circ \chi$.

**Note.** In the present context $\mathbb{J}$ embeds $C(Z, \mathbb{Z}_2)$ in $C(Z, Y)$ as $\mathcal{X}$. If $Y$ is a ring such that $1 + 1 = 0$ then $\mathbb{J}$ embeds $C(Z, \mathbb{Z}_2)$ in $C(Z, Y)$ as the subring $\mathcal{X}$, by Theorem 21.

**Theorem 31.** Pulling together results $C(\Pi, \mathbb{Z}_2) \overset{G}{\rightarrow} C(X, \mathbb{Z}_2) \overset{J}{\rightarrow} C(X, Y) \overset{H}{\rightarrow} C(\Pi, Y)$ where the reader can choose the algebraic setting.
Example: $C(\Pi, Z_2)$ and $C(\Pi, Y)$ need not be of the same cardinality and thus neither algebraically or topologically isomorphic. Let $X = Y = Z$ and $Z_2$ all have the discrete topology. Then $C(Z, Z) = Z$ and $C(Z, Z_2) = Z^{Z_2}$. So $\| C(Z, Z) \| = \aleph^\aleph$ and $\| C(Z, Z_2) \| = \aleph_2^\aleph$. Now observe $\aleph_2^\aleph = 2^{\aleph_0} = \aleph_1^\aleph$. Therefore $C(Z, Z)$ and $C(Z, Z_2)$ are not isomorphic in any set based category.

In the other direction there is an interesting algebraic projection of $C(Z, Y)$ onto $C(Z, Z_2)$ when $Y$ has the two binary operation with a unique open null element and unit so that $0 \cdot y = 0$ and $1 \cdot y = y$. Recall that the functions of $C(Z, Z_2)$ are characteristic functions where $\chi_Z = \Theta$ and $\chi_0 = Id$.

**Definition 19.** Define $l : Y \to Z_2$ by $l(y) = 0$ if $y = 0$ and $1$ if $y \neq 0$.

**Lemma 66.** $l$ is continuous as $\{0\}$ is open in $Y$. If $Y$ has no divisors of zero, then $l$ is a surjective multiplicative homomorphism where $l(y_1 \cdot y_2) = l(y_1) \cdot l(y_2)$.

**Definition 20.** If $\{0\}$ is open in $Y$ define $L : C(Z, Y) \to C(Z, Z_2)$ by $\chi_f = L(f) = l \circ f$ so that $\chi_f(z) = L(f)(z) = (l \circ f)(z) = l(f(z))$.

**Note.** First $L(Id) = Id$ and $L(\Theta) = \Theta$. Second if $f^{-1}(0) = g^{-1}(0) = U$ or $\emptyset$ then $L(f) = L(g)$.

**Theorem 32.** If $Y$ has no divisors of zero and $\{0\}$ is open in $Y$ then $L : C(Z, Y) \to C(Z, Z_2)$ is a surjective multiplicative homomorphism.

**Proof.** First $(L(f_1 \cdot f_2))(z) = (l \circ (f_1 \cdot f_2))(z) = l((f_1 \cdot f_2)(z)) = l(f_1(z) \cdot f_2(z)) = l(f_1) \cdot l(f_2(z))$. To show surjective let $\chi \in C(Z, Z_2)$, then $\chi^{-1}(0)$ and $\chi^{-1}(1)$ are clopen sets in $Z$. So any function from $Z$ to $Y$ that is zero on $\chi^{-1}(0)$ and one on $\chi^{-1}(1)$ is continuous and has as its $L$ image $\chi$. □

**Lemma 67.** There is a one to one correspondence between the clopen sets of $Z$ and the functions of $C(Z, Z_2)$ where if $U$ is clopen in $Z$ then $\chi_U$ is in $C(Z, Z_2)$ and if $\chi$ is in $C(Z, Z_2)$ then $U = \chi^{-1}(0)$ where $\chi_U = \chi$. 35
Note. $I_2(z)$ in $C(Z, Z_2)$ is the prime ideal of functions that map $z$ to $0$. By Theorems 27) and 28) as all functions of $C(Z, Z)$ are of the form $\chi_u$, $U$ clopen, the $I_2(z)$ are the min-max prime ideals of $C(Z, Z_2)$. This suggest.

**Theorem 33.** If $\cdot$ is associative and commutative and $\{0\}$ is open in $Y$ then all proper prime ideals of $C(Z, Y)$ are min-max and of the form $I(z)$, (For $CX, Y$) of the form $I(Q_x)$ where $Q_x$ is the quasi-component of $x \in X$).

**Proof.** By Theorem 27) if $J$ is a proper prime ideal then $V(J) \neq \emptyset$. Thus by Theorem 28) there is a unique $z$ such that $J = I(z)$. Now if there as a proper prime ideal $K \subset J$ then there is a $z'$ such that $K = I(z') \subset I(z)$ implying $K$ is not prime for its zeros set would have at least two elements. Analogues if $J \subset K$ then $J$ could not be prime.

**Note.** Again in $C(Z, Y)$ if $\{0\} \subset Y$ is open and $\cdot$ is associative and commutative, then the $I(z)$ are the mini-max prime ideals.

**Theorem 34.** If $Y$ has no nontrivial left zero-divisors (or $\{0\}$ is open in $Y$) then the intersection of all prime ideals is zero. We say the prime radical (or nil radical) is zero although all that is used is that $Y$ has only a single operation with a left zero.

**Proof.** As the intersection of the $I(z)$ over all $z$ in $Z$ is $(\Theta)$, Corollary to Theorem 14 (or Theorem 33).

**Note.** There is a one-to one correspondence between ideals $I(z)$ and the points of $Z$ and hence the quasi-components $X$. Consequently if $Y$ is a ring and therefore $C(X, Y)$ is a ring, there is such a one-to-one correspondence between quasi-components of $X$ and the prime ring ideals, $I([x])$ of $C(X, Y)$ when $\{0\}$ is open in $Y$.

**Lemma 68.** If $Z$ has the discrete topology then $C(Z, Y)$ isomorphic to the categorical product of $Y$ indexed on $Z$ in the category of $Y$. 36
EXAMPLES: Let $Y$ be a ring with unit, $Z$ a set with the discrete topology, and $n = \| Z \| < \aleph_0$. Let $Z = \{ z_i : i = 0, 1, ..., n \}$. Define $\chi_i(z) = \begin{cases} 1 & \text{if } z = z_i \\ 0 & \text{if } z \neq z_i \end{cases}$. So $\chi_i$ is in $C(Z,Y)$. For each $i$ set $f_i = \chi_i \cdot f$. Then $f(z) = \sum_{i=0}^{n} (\chi_i \cdot f_i)(z)$ for any $z$ and $f \in C(Z,Y)$. If $I$ is an ideal and $f \in I$ then $\chi_i \cdot f \in I$. Thus the elements of $I$ have a representation of this form. All such rings should be well known. If $\| Z \| \geq \aleph_0$, we still have categorical products for which the same results hold with the Axiom of Choice as $C(Z,Y)$ is a function space. All of these rings have the prime radical zero and the $I(z)$ are minimal prime ideals.

If however not all of the points of $Z$ are clopen then the ring $C(Z,Y)$ is no longer a product. Consider for an example all points clopen except one.

Aside: For a ring of functions the set of ring ideals of the form $I(U)$ where $U$ is a clopen set form a complete modular lattice using intersection and addition.

Now assume that $Y$ will be a ring and that the objects being discussed are ring objects.

**Theorem 35.** If $I$ is a proper ideal, $f \in I$ such that $f \neq \Theta$, and $U$ is a clopen set then the principal ideals generated by $f \cdot \chi_u$ and $f \cdot (1 - \chi_u)$ are ideals of $I$.

**Theorem 36.** If $z$ is not open then for every clopen set $U$ containing $z$ there is a $z_u \in U$ such that $z_u \neq z$. $z$ is the unique cluster point of $\{ z_u \}$ where $\{ z \} = \cap \{ U : U \text{ is clopen and } z \in U \}$.

**Proof.** As $z$ is not open, $\{ z \} \neq U$. Suppose $z_1 \neq z$ is another cluster point of $\{ z_u \}$. But $z$ and $z_1$ can be separated by clopen sets. □

**Lemma 69.** If $U$ is clopen then $I(U) = (\chi_u)$ and $I(U^c) = (\chi_{u^c}) = (1 - \chi_u)$. Therefore $I(U)$ and $I(U^c)$ are coprime (comaximal). Here $U^c = Z \setminus U$ is the Z complement of $U$.

**Corollary.** If $z$ is open then $I(z) = (\chi_z)$ and $I(z^c) = (\chi_{z^c})$ are co-prime.

**Lemma 70.** If $Y$ has no divisors of zero then $V((f)) = V(f)$, cf. Lemma 40.

**Proof.** If $g \in (f)$ then $g = h \cdot f$ for some $h \in C(Z,Y)$. So $V(g) = V(h \cdot f) = V(h) \cup V(f)$ which implies $V((f)) = V(f)$. □
Lemma 71. If \( f \in I(z) \) and \( f^{-1}(0) \neq \{z\} \) then \( I(z) \neq \langle f \rangle \) although \( \langle f \rangle \subset I(z) \).

Proof. By Lemma 40 or 70, \( V((f)) = V(f) = f^{-1}(0) \neq \{z\} = V(I(z)) \) so \( f \neq I(z) \). \qed

Lemma 72. \( I(\cap \{U : z \in U\}) = I(z) \).

Lemma 73. \( \cap \{I(A_\alpha) : \alpha \in \Lambda\} = I(\cup \{A_\alpha : \alpha \in \Lambda\}) \).

Corollary. For any set \( U \), \( \cap \{I(z) : z \in U\} = I(U) \). Hence \( \cap \{I(z) : z \in Z\} = I(Z) = (\Theta) \).

A Summary:

\( C(Z,Y) \) is not a local ring Lemma 59, no. 1 and [1], pg. 11 exercise 12. Spect\( (C(Z,Y)) \) is disconnected topological space, [1], pg. 14, exercise 22. If \( Y \) has no one or two sided divisors of zero (eg. an integral domain), then the prime radical (nullradical) of \( C(Z,Y) \) is trivial, Corollary to Theorem 14.

Now to continue, from proposition 1.11 [ 1 ] page 8, we obtain.

Lemma 74. Let \( Y \) be an integral domain.

a). If an ideal \( I \subset \bigcup_{1 \leq i \leq n} I(z_i) \) then \( I \subseteq I(z_i) \) some \( i \).

b). Given \( \{z_i\}_{i=1}^{n} \) and \( I(\bigcup_{i} \{z_i\}) = \bigcap_{i} I(z_i) \subset p \). If \( p \) is prime then \( I(z_i) \subseteq p \) for some \( i \). If \( p = \bigcap_{i} I(z_i) \), then \( p = I(z_i) \) for some \( i, 1 \leq i \leq n \).

Theorem 37. Let \( I \) be a proper ideal, \( f \in I \) such that \( f \neq 0 \), and \( U \) a clopen set. Then the principal ideals \( \langle f \cdot \chi_u \rangle \) and \( \langle f \cdot \chi_{uc} \rangle \) are sub ideals of \( I \).

Lemma 75. If \( f \) is a function, \( U \) a clopen set, \( I \) an ideal, and if \( f \cdot \chi_u \) and \( f \cdot \chi_{uc} \) are both elements of \( I \), then \( f = f \cdot \chi_u + f \cdot \chi_{uc} \) is an element of \( I \).

Theorem 38. If \( I \) is a prime ideal, \( f \notin I \), and \( U \) a clopen set, then one and at most one of \( \langle f \cdot \chi_u \rangle \) and \( \langle f \cdot \chi_{uc} \rangle \) is a principal sub ideal of \( I \).

Proof. At least one of \( \chi_u \) and \( \chi_{uc} \) is in \( I \). \qed

38
Lemma 76. Let $I_1, I_2, \ldots, I_n$ be prime ideals and $I$ an ideal such that $I \not\subset I_i$ and $I \not= I_i$ for $i = 1, 2, \ldots, n$, then there exists $a \in I$ such that $a \notin I_i$ for $i = 1, 2, \ldots, n$. (This is well known to ring theorists)

Theorem 39. If $Y$ is a division ring and $Z = \bigcup_{i=1}^n \{z_i\}$ then the $I(z_i)$ are the maximal ideals. Hence $C(Z,Y)$ is semi-local, definition on pg. 4 of [1].

Proof. Note every $z$ is clopen. Let $I$ be a maximal ideal such that $I \not= I(z_i)$ for any $i = 1, 2, \ldots, n$. Therefore for each $i$ there exist $f_i$ in $I$ such that $f_i(z_i) \neq 0$. Thus $\hat{f}_i(\xi) = \begin{cases} f_i(\xi) & \text{if } \xi = z_i \\ 0 & \text{if } \xi \neq z_i \end{cases}$ is continuous and in $I$ as it is $f_i$ times a characteristic function one on $z_i$. Therefore $f(z) = \sum_{i=1}^n \hat{f}_i(z) \neq 0$ is continuous in $I$ and not zero for all $z \in Z$. Hence $I$ is the ring $C(Z,Y)$. \[
\]

Note. By Theorems 24 and 25, if $I(z)$ is not an minimal prime ideal then $\{z\}$ is not open in $Z$ and $\{0\}$ is not open in $Y$. Significantly if $\{0\}$ is open in $Y$ then all of the $I(z)$ are minimal. This suggest interesting examples. Let $Z$ be any set supporting a topology where all points are clopen except a distinguished point denoted $d$ which is closed but not open. Denote this topological space by $Z_c$. Let $Z_d$ denote the topological space consisting of the same set with the discrete topological. Let $\iota : Z_d \rightarrow Z_c$ be the set identity function, $\iota(z) = z$, which is continuous. Define $\mathfrak{I} : C(Z_c,Y) \rightarrow C(Z_d,Y)$ by $\mathfrak{I}(f) = f \circ \iota$. $\mathfrak{I}$ embeds $C(Z_c,Y)$ algebraically isomorphically into $C(Z_d,Y)$ but not onto as $\chi_d \in C(Z_d,Y)$ and $\chi_d \notin C(Z_c,Y)$. The ideals of $C(Z_c,Y)$ do not map to ideals in $C(Z_d,Y)$. Let $I_c$ denote an ideal in $C(Z_c,Y)$ and $I_d$ an ideal in $C(Z_d,Y)$. Note $\mathfrak{I}(I_c(z)) \subset I_d(z)$ for some $I_d(z)$ and $\mathfrak{I}(I_c(d)) \neq I_d(d) = (\chi_d)$ as $\chi_d \notin I_c(d)$. $\mathfrak{I}(I_d(d))$ is not prime although $\{d\}$ is open in $Z_d$ and $I_d(d)$ is a minimal prime ideal. All ideals of both rings of functions that vanish at a point are minimal prime except $I_c(d)$ which is prime in $C(Z_c,Y)$ but may not be minimal unless $\{0\}$ is open in $Y$. Note $\mathfrak{I}(I_c(d))$ is not prime which is a sub ideal of the minimal prime ideal $I_d(d)$ in $C(Z_d,Y)$.

For example if $Y = \mathbb{Z}_2$, then $I_c(d)$ is a minimal prime ideal in $C(Z_c,\mathbb{Z}_2)$. But in this example there is no function $f$ such that $V(f) = f^{-1}(0) = d$. See 6) below.
Note. Considerations to use for examples:

Let Z be a topological space whose points are quasi-components. See Theorem 5 and the second paragraph of the Note following the Comment after Theorem 6.

Let Y be a $T_0$ topological space with a basis of clopen sets. See paragraph before the second Note following Definition 5.

Consider $C(Z,Y)$. In the text the algebraic conditions on Y are successively stronger. For the examples assume that Y is a ring although this can be weakened.

1) A point $z \in Z$ is isolated iff there exists an open point $\{y\} \subset Y$ and a function $f \in C(Z,Y)$ such that $f^{-1}(y) = x$.

2) By Theorem 14 if Y has no divisors of zero the $I(z)$ are prime ideals.

3) By Theorem 16 or 24, if $\{ z \}$ is open then $I(z)$ is a minimal prime ideal.

4) By Theorems 14, 27, and 28 if $\{ 0 \}$ is open in Y then for every $z$, $I(z)$ is a minimal prime ideal and there are no other minimal prime ideals.

5) By Lemma 68, if Z has the discrete topology then $C(Z,Y)$ is algebraically isomorphic to the categorical product of Y indexed on Z in the category of Y.

6) To complement 4) above consider $C(Z,Y)$ where $Z = \{ 0 \} \cup \{ 1/n | n = 1, 2,... \}$ has the topology such that all points of Z are open except $\{ 0 \}$ which is closed but not open. Choose the rationales for Y with the topology such that $\{ 0 \}$ is open. Then the function $f = \{(0,0)\} \cup \{(y,y) | y = 1/n, n = 1, 2,... \}$ is not continuous. The functions of $I(0)$ must be eventually zero.

The following observation and question indicates that our results are not the strongest possible, perhaps foolish, or that a critical example supporting this development is difficult to construct.

Let $\{ z_0 \}$ be a quasi-component of Z possibly not open. Assume $\{ 0 \}$ in Y is not open and let $\mathfrak{U} = \{ U : z_0 \in U and U open \}$, so $\bigcap \mathfrak{U} = \{ z_0 \}$. If $f \in I(z_0)$ then $f = \bigcup \{ f \cdot \chi_U : U \in \mathfrak{U} \}$. Suppose $J \subseteq I(z_0)$ is a prime ideal. Then $f \cdot \chi_U \in J$ but $V(f)$ need not be copen. Can this imply that $f \in J$, that is $J = I(z_0)$. If $\{ z_0 \}$ is clopen, $J = I(z_0)$ but if $\{ z_0 \}$ is not open $I(z_0)$ may not be a minimal prime ideal.
Examples:
As a first example consider $X = \{(0,0)\} \cup \{(0,1)\} \cup \{(1/n, y) : 0 \leq y \leq 1 \text{ and } n \in \mathbb{N}\}$, $\mathbb{N}$ the positive integers, as a subset of the plane. Then the union of the components $\{(0,0)\}$ and $\{(0,1)\}$ is the quasi-component $\{(0,0), (0,1)\}$.

Now in general each point of a space lies in a component, the maximal connected set containing the point, and lies in its quasi-component, an intersection of clopen sets, which need not be connected. Thus quasi-components are a union of components.

Open components are quasi-components which are thus clopen. But quasi-components need not be open. If components are open then they are quasi-components of $X$.

Let the components of $X$ be open. Thus $X/\Pi$ has the discrete topology. If the set $\{[x]\}$ has the discrete topology then $C(X,Y)$, $C(Z,Y)$, $C(X/\Pi,Y)$, and $C(\{[x]\},Y)$ are isomorphic. Assuming Well Ordering, if $\kappa$ is the cardinality of $\{[x]\}$ and $\kappa$ has the discrete topology then these rings are isomorphic with $C(\kappa,Y)$ which is a product.

If $X$ has only one quasi-component, $[x]$, that is not a component then the components in the complement of $[x]$ are clopen quasi-components. If $C(\kappa,Y)$ is as above then there is an injective homorphism of $C(X,Y)$ strictly embedding it into $C(\kappa,Y)$. As $[x]$ is the intersection of clopen sets and is not itself open, each of these clopen sets defining $[x]$ must intersect a quasi-component other than $[x]$. Using these observations identify $C(\kappa,Y)$ with all sequences from $\kappa$ to $Y$ and $C(X,Y)$ with those that are continuous at $[x]$. As above consider again $X = \{(0,0)\} \cup \{(0,1)\} \cup \{(1/n, y) : 0 \leq y \leq 1 \text{ and } n \in \mathbb{N}\}$, $\mathbb{N}$ the positive integers, as a subset of the plane and $Y = \mathbb{Z}_2$, the ring of 0 and 1. Here $C(\kappa,\mathbb{Z}_2)$ is equinumerous with the set of functions from $\mathbb{N}$ to $\mathbb{Z}_2$ of cardinality $2^{\aleph_0}$ which is larger than $\aleph_0$. As each functions in $C(X,\mathbb{Z}_2)$ is eventually constant $C(X,\mathbb{Z}_2)$ injects into the set of all finite sequences of $\mathbb{N}$ into $\mathbb{Z}_2$ which is of cardinality less than or equal to $\aleph_0$. Thus the cardinality of $C(\kappa,\mathbb{Z}_2)$ is strictly larger than that of $C(X,\mathbb{Z}_2)$. Consequently they can not be algebraically isomorphic.

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