DUALITY FOR COHOMOLOGY OF CURVES WITH COEFFICIENTS IN ABELIAN VARIETIES

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Abstract. In this paper, we formulate and prove a duality for cohomology of curves over perfect fields of positive characteristic with coefficients in Néron models of abelian varieties. This is a global function field version of the author’s previous work on local duality and Grothendieck’s duality conjecture. It generalizes the perfectness of the Cassels–Tate pairing in the finite base field case. The proof uses the local duality mentioned above, Artin–Milne’s global finite flat duality, the nondegeneracy of the height pairing and finiteness of crystalline cohomology. All these ingredients are organized under the formalism of the rational étale site developed earlier.

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§1. Introduction

1.1 Aim of the paper

Let $X$ be a proper smooth geometrically connected curve over a perfect field $k$ of characteristic $p > 0$. Let $A$ be an abelian variety over the function field $K$ of $X$, with dual $A^\vee$. We denote the Néron models of $A$ and $A^\vee$ over $X$ by $\mathcal{A}$ and $\mathcal{A}^\vee$, respectively. Let $\mathcal{A}_0^\vee$ be the maximal open subgroup scheme of $\mathcal{A}^\vee$ with connected fibers. We view these group schemes over $X$ as representable sheaves on $X$ in the flat topology and hence in the étale topology. In this paper, we formulate and prove a duality that relates the two étale cohomology complexes

$$R\Gamma(X, \mathcal{A}), \quad R\Gamma(X, \mathcal{A}_0^\vee)$$

to each other. ($R\Gamma(X, \mathcal{A})$ is a complex whose $n$th cohomology is $H^n(X, \mathcal{A})$.)

This is the global function field version of the author’s work [Suz14] on duality for cohomology of local fields with coefficients in abelian varieties that solved Grothendieck’s duality conjecture [Gro72, IX, Conjecture 1.3]. The complexes above are not viewed as complexes of mere abelian groups, but as complexes of sheaves on the ind-rational pro-étale site $\text{Spec} k^{\text{indrat}}_{\text{proet}}$ defined in [Suz14]. In particular, this duality treats the structure of $H^n(X, \mathcal{A})$ as the perfection (inverse limit along Frobenii) of a smooth group scheme over the base field $k$, which is related to Artin–Milne’s duality [AM76] for cohomology of $X$ with coefficients in finite flat group schemes. The object $H^1(X, \mathcal{A})$ has unipotent connected part (which is $p$-power torsion) and is the sheafified version of the Tate–Shafarevich group of $A/K$, which we call the Tate–Shafarevich scheme. See Proposition 3.2.10 and the preceding paragraph for a more precise (but a bit subtle) relationship to Tate–Shafarevich groups. The above duality generalizes the perfectness of
the Cassels–Tate pairing in the finite base field case. The duality for the part
\( H^0(X, A) = A(K) \) includes the nondegeneracy of the height pairing. When
showing the finite-dimensionality of the Tate–Shafarevich scheme, we will
use the finiteness of crystalline cohomology of proper smooth surfaces over \( k \).

Our duality extends a partial result of Milne [Mil06, III, Theorem 11.6] on
duality \( T_p \Gamma(K, A^\vee) \leftrightarrow H^2(X, A) \). He also pointed out that the part killed
by \( p \) (not the whole \( p \)-primary part) of the Tate–Shafarevich group can
be infinite when \( k \) is algebraically closed [Mil06, III, Remark 9.9]. This
phenomenon has been studied by Vvedenskii ([Vve81] for example), which,
in our formulation, can be explained by the connected part of the Tate–
Shafarevich scheme. It appears from [Vve78, Remark 7] that Vvedenskii at
least once imagined a possibility to construct a duality theory similar to
ours.

The most interesting part of this duality theory lies in \( p \)-torsion. The
prime-to-\( p \) part is essentially classical (cf. [Ray95]). When \( k \) has zero
characteristic, which is out of scope of this paper, one should probably take
(nonsemistable) degenerations of Hodge structures into account.

When the first version of this paper was uploaded to arXiv, Česnavičius
pointed the author to the preprint [DH18] by Demarche–Harari, which was
written independently at almost the same time. They develop compact
support flat cohomology, a key technical tool that we also develop, in a
way very similar to us. The setting and the details are different. For more
details, see Section 1.4.

1.2 Statement of the main theorem

Now we formulate our duality. Let \( \text{Ab}(X_{\text{fppf}}) \) and \( \text{Ab}(k_{\text{proet}}^{\text{indrat}}) \) be the
categories of sheaves of abelian groups on the fppf site \( X_{\text{fppf}} \) and the ind-
rational pro-étale site \( \text{Spec } k_{\text{proet}}^{\text{indrat}} \) [Suz14, Section 2.1], respectively. We
define a left exact functor
\[
\Gamma(X, \cdot) : \text{Ab}(X_{\text{fppf}}) \to \text{Ab}(k_{\text{proet}}^{\text{indrat}})
\]
by sending an fppf sheaf \( F \) on \( X \) to the pro-étale sheafification of the
presheaf \( k' \mapsto F(X \times_k k') \) on \( \text{Spec } k_{\text{proet}}^{\text{indrat}} \), where \( k' \) runs through ind-rational
\( k \)-algebras. Denote its right derived functor by
\[
R\Gamma(X, \cdot) : D(X_{\text{fppf}}) \to D(k_{\text{proet}}^{\text{indrat}})
\]
and set \( H^n(X, \cdot) = H^n R\Gamma(X, \cdot) \). The complex of sheaves \( R\Gamma(X, A) \) on
\( \text{Spec } k_{\text{proet}}^{\text{indrat}} \) is the main object of study. Denote the derived sheaf-Hom
functor for $\text{Spec } k^\text{indrat}_\text{proet}$ by $R\text{Hom}_{k^\text{indrat}_\text{proet}}$. For $G \in D(k^\text{indrat}_\text{proet})$, its Serre dual \cite[Section 2.4]{Suz14} is defined by

$$G^\text{SD} = R\text{Hom}_{k^\text{indrat}_\text{proet}}(G, \mathbb{Z}).$$

The Poincaré biextension $\mathcal{A}_0^\vee \otimes^L \mathcal{A} \to \mathbb{G}_m[1]$ as a morphism in $D(X_{fppf})$, the cup product and the degree map of the Picard scheme induce morphisms

$$R\Gamma(X, \mathcal{A}_0^\vee) \otimes^L R\Gamma(X, \mathcal{A}) \to R\Gamma(X, \mathbb{G}_m)[1] \to H^1(X, \mathbb{G}_m) \to \mathbb{Z}$$

in $D(k^\text{indrat}_\text{proet})$. Hence we have a morphism

$$R\Gamma(X, \mathcal{A}) \to R\Gamma(X, \mathcal{A}_0^\vee)^\text{SD}.$$

Its Serre dual

$$(1.2.1) \quad R\Gamma(X, \mathcal{A}_0^\vee)^{\text{SDSD}} \to R\Gamma(X, \mathcal{A})^\text{SD}$$

is our duality morphism. Let

$$V = VH^1(X, \mathcal{A}_0^\vee)_{\text{div}} = (T^1H^1(X, \mathcal{A}_0^\vee)_{\text{div}}) \otimes \mathbb{Q}$$

be the rational Tate module of the maximal divisible subsheaf of $H^1(X, \mathcal{A}_0^\vee)$.

**Theorem A.** There exist canonical morphisms

$$R\Gamma(X, \mathcal{A})^\text{SD} \to V \to R\Gamma(X, \mathcal{A}_0^\vee)^{\text{SDSD}}[1]$$

such that the triangle

$$V[-1] \to R\Gamma(X, \mathcal{A}_0^\vee)^{\text{SDSD}} \to R\Gamma(X, \mathcal{A})^\text{SD} \to V$$

is distinguished in $D(k^\text{indrat}_\text{proet})$.

Both of the objects $R\Gamma(X, \mathcal{A}_0^\vee)^{\text{SDSD}}$ and $R\Gamma(X, \mathcal{A})^\text{SD}$ are concentrated in degrees $-1, 0, 1, 2$, while $VH^1(X, \mathcal{A}_0^\vee)_{\text{div}}$ has no connected part, is uniquely divisible and concentrated in the single degree zero. Hence our duality morphism is “close” to be an isomorphism. The divisible part of the usual Tate–Shafarevich group, when $k$ is finite, is conjectured to be zero. But our Tate–Shafarevich scheme, $H^1(X, \mathcal{A})$, has nonzero divisible part in general, which might be called the space of “transcendental cycles with coefficients in the Néron model.”
Concrete consequences of this theorem will be explained at Section 3.4. We will also give a version for cohomology of dense open subschemes of $X$ in Section 4.1 and explain the link of Theorem A to the known duality theory in the finite base field case in Section 4.2. A very small amount of explicit calculations is given in the course of Remark 3.4.2.

A small remark is that the first cohomology classifies torsors or principal bundles. The Tate–Shafarevich scheme $H^1(X, A)$ might alternatively be called the moduli of $G$-bundles on $X$ and denoted by $\text{Bun}_G(X)$, where $G = A$. We do not pursue this viewpoint, merely mentioning that our crucial point is to evaluate functors on ind-rational $k$-algebras only.

1.3 Outline of proof

Here is a rough outline of the proof. The first deep input is Artin–Milne’s global duality [AM76, Corollary (4.9)] for a finite flat group scheme $N$ over $X$:

$$R\Gamma(X, N^{CD}) \leftrightarrow R\Gamma(X, N),$$

where $N^{CD}$ is the Cartier dual of $N$, the cohomology is taken in the fppf topology and we ignored the shift of degrees for simplicity. Artin–Milne uses the étale site of all perfect $k$-schemes and we will bring their result to our site $\text{Spec} k^{\text{indrat}}_{\text{proet}}$ by restriction and pro-étale sheafification. At each closed point $x \in X$, we have Bester’s local finite flat duality [Bes78, Theorem 3.1] in the form stated in [Suz14, Theorem (5.2.1.2)]:

$$R\Gamma_x(\hat{O}_x, N^{CD}) \leftrightarrow R\Gamma_x(\hat{O}_x, N),$$

where $\hat{O}_x$ is the completed local ring of $X$ at $x$, and the fppf cohomology functor $R\Gamma(\hat{O}_x, \cdot)$ and its version with support $R\Gamma_x(\hat{O}_x, \cdot)$ with values in $D(k^{\text{indrat}}_{\text{proet}})$ are as defined in [Suz14, Section 3.3]. ([Suz14, Theorem (5.2.1.2)] fits better in the setting of the present paper as it is a derived categorical and sheaf-theoretic version of the duality isomorphism of profinite abelian groups in [Bes78, Theorem 3.1]. Also [Suz14, Theorem (5.2.1.2)] corrects some inaccuracies in Bester’s paper; see [Suz14, Remark 5.2.1.5].) Hence we can pass from $X$ to a dense open subscheme $U$ of $X$:

$$R\Gamma(U, N^{CD}) \leftrightarrow R\Gamma_c(U, N),$$

where $R\Gamma_c(U, \cdot)$ is the fppf cohomology with compact support with values in $D(k^{\text{indrat}}_{\text{proet}})$ that we define and study in this paper. The fppf cohomology with compact support, as usual abelian groups, is already defined in
We need some clarification about the definition given in [Mil06, III, Section 0], more than just bringing it to $D(k_{\text{proet}})$. This point was simultaneously found by Demarche and Harari [DH18]; see Section 1.4 for the details. The duality (1.3.1) implies, taking $U$ small enough so that $A$ has good reduction over $U$, a duality

$$R\Gamma(U, A_0^\vee[n]) \leftrightarrow R\Gamma_c(U, A[n])$$

for any $n \geq 1$. At each $x \in X$, we have local duality for abelian varieties [Suz14, Remark (4.2.10)]

$$R\Gamma_x(\hat{O}_x, A_0^\vee) \leftrightarrow R\Gamma(\hat{O}_x, A).$$

Hence we can pass from $U$ to $X$ and take the limit in $n$:

$$R \lim_{\leftarrow n} (R\Gamma(X, A_0^\vee) \otimes^L \mathbb{Z}/n\mathbb{Z}) \leftrightarrow R\Gamma(X, A) \otimes^L \mathbb{Q}/\mathbb{Z}. \tag{1.3.2}$$

So far we have treated essentially torsion objects. To pass to the desired integral statement

$$R\Gamma(X, A_0^\vee) \leftrightarrow R\Gamma(X, A)$$

up to $VH^1(X, A_0^\vee)_{\text{div}}$, we need to study the structure of $H^n(X, A)$ for all $n$. (Of course we need the double dual SDSD for the precise statement.) The Lang–Néron theorem (the arbitrary base field version of the Mordell–Weil theorem) shows that the group of rational points $\Gamma(X, A)$ is an étale group with finitely generated group of geometric points extended by an abelian variety over $k$, the $K/k$-trace of $A$. The part

$$(\pi_0\Gamma(X, A_0^\vee))/\text{torsion} \leftrightarrow (\pi_0\Gamma(X, A))/\text{torsion}$$

is given by the height pairing.

For higher cohomology, the duality theorems used so far and finiteness statements therein give some information. But this information does not exclude the possibility that $H^1(X, A)$ has connected divisible unipotent ind-algebraic part (such as the direct limit of the groups of Witt vectors $\lim_{\rightarrow n} W_n$) since $H^2_x(\hat{O}_x, A)$ does have such part. We need here a global finiteness result to eliminate this possibility. That is, we compare $H^1(X, A)$ with the Brauer group of a proper smooth surface over $k$ up to some discrepancy of bounded torsion (i.e., discrepancy killed by multiplication by some positive integer) and hence with the part $F = p$ of the second rational crystalline cohomology
of the surface. This part does not have connected part by finiteness of crystalline cohomology, which proves the desired property of \( H^1(X, A) \). Dualizing, this in turn shows that \( H^2(X, A) \) has no connected part.

Then we can pass to the rational statement

\[
R \lim_{\longleftarrow n} R\Gamma(X, A_0^n) \leftrightarrow R\Gamma(X, A) \otimes \mathbb{Q},
\]

where the derived inverse limit is for multiplication by positive integers \( n \). The nondegeneracy of the height pairing shows that this rational duality pairing is perfect up to \( \varphi(H^1(X, A_0^n))_{\text{div}} \). Then we can finally pass to the desired integral statement, proving the main theorem.

Two remarks are in order. The reader may have an impression from the above outline that the duality theorem of this paper is closely related to duality for crystalline cohomology and Milne’s flat duality for surfaces [Mil76]. Probably it will likely be possible to reduce our theorem to those duality theories when \( A \) is the Jacobian of a proper smooth curve over \( K \) with a \( K \)-rational point or has semistable reduction everywhere. But there are differences between these special cases and the general case, coming from isogenies of \( p \)-power degrees and wild ramification. Without our formulation, it could be difficult to formulate a duality statement that is invariant under isogenies and/or admits Galois descent.

The local duality [Suz14] used above is where the site \( \text{Spec} \mathbb{K}^{\text{indrat}} \) plays a crucial role (see [Suz14, Section 1.2] for a little more details). This should be compared with Milne’s approach [Mil06, III, Section 11] that uses the étale site of all perfect \( k \)-schemes.

1.4 Organization

Up to the point (1.3.2) above, we will know that \( H^n(X, A) \) for \( n \geq 1 \) is an ind-algebraic group of so-called “cofinite type.” This is a basic finiteness condition for ind-algebraic groups. Section 2.1 develops the notion of cofinite type objects in the setting of ind-categories of general artinian abelian categories. Section 2.2 studies cofinite type objects in the special case of the category of ind-algebraic groups. Section 2.3 briefly reviews the ind-rational pro-étale site and study some derived limits such as the one \( R \lim_{\longleftarrow n} \) mentioned above.

In Section 2.5 (Section 2.4 to be mentioned soon), we basically review the local duality [Suz14]. Additionally, we improve and simplify the formulation. We eliminate the relative fppt site (\( \text{Spec} K_{\text{fppt}}^{k^\text{indrat}} \) in the notation of [Suz14, Section 3.3]) and instead formulate the same duality result
using the usual fppf site for local fields. We can still define the key notion of the structure morphism of a local field, which targets the ind-rational étale site of the residue field. This newer version is not, however, a morphism of sites, but only a “premorphism of sites,” which does not necessarily have an exact pullback functor. A premorphism of sites is not really new and nothing but a morphism of topologies in the terminology of [Art62, Definition 2.4.2]. (We change the terminology since the direction of morphisms is somewhat confusing.) Its definition is recalled and some basic site-theoretic propositions are given in the preceding Section 2.4. The comparison between the two formulations is given in Appendix A. We do use the local duality results of [Suz14] and do not reprove them.

As explained above, we will need fppf cohomology of curves with compact support $R\Gamma_c(U, \cdot)$ as a complex of sheaves on $\text{Spec } k^{\text{indrat}}$. We will develop this machinery in Section 2.7. There are actually two versions of compact support fppf cohomology currently known, as explained in [Mil06, III, Remark 0.6(b)]. Their difference is whether we use Henselian local rings $\mathcal{O}_x^h$ or their completions $\hat{\mathcal{O}}_x$ for the local components. The latter version is what we need in the duality theory of this paper. But this latter version has a problem about covariant functoriality in $U$ coming from the difference between two versions of local cohomology with support $R\Gamma_x(\mathcal{O}_x^h, \cdot)$ and $R\Gamma_x(\hat{\mathcal{O}}_x, \cdot)$. We will see this problem in Remark 2.7.9. The difference between the two versions of local cohomology with support vanishes if the coefficient sheaf is representable by a smooth or finite flat group scheme, by a Greenberg approximation argument. However, we need various mapping cone constructions in order to establish basics about compact support cohomology including covariant functoriality, which forces us to work on the level of complexes and, consequently, to explicitly take injective resolutions everywhere. For nonrepresentable sheaves, the difference between the two versions of local cohomology with support is unavoidable. Hence we need to keep all track of this difference from the beginning to build the theory. Some part of this consideration is purely local, which is the content of the preceding Section 2.6. After these subsections, we can and will ignore the difference since we are interested in smooth or finite flat group schemes only.

Let us mention here that the same problem of compact support fppf cohomology is realized and solved by Demarche and Harari [DH18] independently at almost the same time. Their setting is over a finite base field $k = \mathbb{F}_q$, cohomology is viewed as usual complexes of abelian groups, not sheaves, and coefficient sheaves are affine group schemes for the most part.
Their method to establish covariant functoriality of compact support fppf cohomology is very similar to our method in Section 2.6–2.7.

Back to this paper, Section 3.1 proves (1.3.1). Section 3.2 begins by proving (1.3.2) (or the statement before limit in \( n \)). Various consequences and further structural results on \( \mathbf{H}^n(X, \mathcal{A}) \) are given: the structure of \( \Gamma(X, \mathcal{A}) \) (Proposition 3.2.3); the structure of \( \mathbf{H}^1(X, \mathcal{A}) \) and its relation to the Tate–Shafarevich group (Proposition 3.2.3, 3.2.10, 3.2.12); how the duality pairing interacts with Néron component groups at the closed points of \( X \) (Proposition 3.2.9); and the relation to the height pairing (Proposition 3.2.13).

The properties of \( R\Gamma(X, \mathcal{A}) \), \( R\Gamma(X, \mathcal{A}_0^\vee) \) and the pairing between them proven up to this point, together with the preliminaries in Section 2, allow us to deduce the main theorem as a result of formal calculations. This is done in Section 3.3. The outline of proof above basically explains that any choice of the mapping cone of the duality morphism (1.2.1) is concentrated in degree zero with cohomology isomorphic to \( V\mathbf{H}^1(X, \mathcal{A}_0^\vee)_{\text{div}} \). In addition, we heavily use derived limit arguments, the structure of \( \mathbf{H}^n(X, \mathcal{A}) \) and the nondegeneracy of the height pairing, to show that the mapping cone can be taken canonically.

Section 3.4 just collects all the results obtained so far to state them as a single theorem. This finishes the proof of Theorem A. A relation to Milne’s result [Mil06, Theorem 11.6] is also explained. The intersection of the connected part and the divisible part of the Tate–Shafarevich scheme \( \mathbf{H}^1(X, \mathcal{A}) \) is an interesting but difficult finite étale \( p \)-group. We briefly explain this in Remark 3.4.2.

Section 4.1 gives a duality for open \( U \subset X \):

\[
R\Gamma(U, \mathcal{A}^\vee) \leftrightarrow R\Gamma_c(U, \mathcal{A})
\]

up to the same obstruction \( V\mathbf{H}^1(X, \mathcal{A}_0^\vee)_{\text{div}} \). Again, it requires some work to ensure that the mapping cone of this duality morphism can be taken canonically.

Applying the derived global section \( R\Gamma(k, \cdot) \) to the complex of sheaves \( R\Gamma(X, \mathcal{A}) \) recovers the usual complex of abelian groups \( R\Gamma(X, \mathcal{A}) \). In a nonderived categorical language, this means that there is a spectral sequence

\[
E_2^{ij} = H^i(k, \mathbf{H}^j(X, \mathcal{A})) \Rightarrow H^{i+j}(X, \mathcal{A}).
\]

In Section 4.2, when the base field \( k \) is finite, we apply \( R\Gamma(k, \cdot) \) to translate our duality theorem into the classical duality theorems. We explain
the relations to the finiteness of Tate–Shafarevich groups, Kato–Trihan’s arithmetic cohomology [KT03], the Cassels–Tate pairing and the Weil-étale cohomology $R\Gamma(X_W, A)$.

**Notation.** We fix two universes $U_0$ and $U$ such that $\mathbb{N} \in U_0 \in U$. The categories of $U$-small sets and $U$-small abelian groups are denoted by $\text{Set}$ and $\text{Ab}$, respectively. For an abelian category $\mathcal{A}$, the category of complexes is denoted by $\text{Ch}(\mathcal{A})$. Its homotopy category is $K(\mathcal{A})$ and derived category $D(\mathcal{A})$. We denote the full subcategory of bounded, bounded below and bounded above complexes by $D^b(\mathcal{A}), D^+(\mathcal{A})$ and $D^-(\mathcal{A})$, respectively. Similar notation applies to $\text{Ch}(\mathcal{A})$ and $K(\mathcal{A})$. The mapping cone of a morphism $A \to B$ in $\text{Ch}(\mathcal{A})$ is denoted by $[A \to B]$. If $A \to B$ is a morphism in a triangulated category together with a certain canonical choice of a mapping cone, then this mapping cone is also denoted by $[A \to B]$ by abuse of notation. If we say $A \to B \to C$ is a distinguished triangle in a triangulated category, we implicitly assume that a morphism $C \to A[1]$ to the shift of $A$ is given, and the triangle $A \to B \to C \to A[1]$ is distinguished.

§2. Site-theoretic foundations and local duality

2.1 Ind-objects of cofinite type

Let $p$ be a prime number. Let $\mathcal{A}$ be an artinian ($U_0$-)small abelian category such that any object of $\mathcal{A}$ is killed by some power of $p$. Note that no nonzero object of $\mathcal{A}$ is divisible, which will be used frequently later on. Denote its ind-category by $\mathcal{I}\mathcal{A}$ (or more precisely, the $U_0$-indcategory, where index sets are $U_0$-small). The category $\mathcal{I}\mathcal{A}$ is abelian by [KS06, Theorem 8.6.5(i)]. Note that objects of $\mathcal{A}$ need not be artinian in $\mathcal{I}\mathcal{A}$. For instance, the additive algebraic group scheme $G_a$ over a field of characteristic $p$ contains the ind-infinite-étale subgroup $\overline{\mathbb{F}_p}$. We say that an object $A \in \mathcal{I}\mathcal{A}$ is of cofinite type if the $p^n$-torsion part $A[p^n]$ is in $\mathcal{A}$ for all $n \geq 0$. In this case, the equality $A = \lim \rightarrow A[p^n]$ gives a presentation of $A$ as a filtered direct limit of objects of $\mathcal{A}$. Denote by $\mathcal{I}\mathcal{I}\mathcal{A}$ the full subcategory of $\mathcal{I}\mathcal{A}$ consisting of objects of cofinite type.

**Proposition 2.1.1.** For any $A \in \mathcal{I}\mathcal{I}\mathcal{A}$, we have $A/p^n A \in \mathcal{A}$ for all $n \geq 0$.

**Proof.** Let $A \in \mathcal{I}\mathcal{I}\mathcal{A}$. To show $A/p^n A \in \mathcal{A}$, it is enough to show this for $n = 1$. The object $A/pA$ is the union of the increasing sequence of subobjects $A[p^n]/p(A[p^{n+1}])$ each in $\mathcal{A}$. It is enough to show that this sequence stabilizes, or $A[p^{m-1}] + p(A[p^{m+1}]) = A[p^m]$ for large $m$. This is
equivalent that the decreasing sequence
\[
\ldots \hookrightarrow A[p^{m+1}]/A[p^m] \overset{p}{\hookrightarrow} A[p^m]/A[p^{m-1}] \hookrightarrow \ldots
\]
stabilizes. It indeed does since each object in the sequence belongs to the artinian category $\mathcal{A}$.

**Proposition 2.1.2.** The category $\mathcal{I}^f \mathcal{A}$ is an abelian subcategory of $\mathcal{I} \mathcal{A}$.

**Proof.** Let $f: A \to B$ be a morphism in $\mathcal{I}^f \mathcal{A}$. Then Ker($f$)[$p^n$] for any $n$ is the kernel of the restriction $A[p^n] \to B[p^n]$ of $f$. Hence Ker($f$)[$p^n$] $\in \mathcal{A}$, thus Ker($f$) $\in \mathcal{I}^f \mathcal{A}$. Therefore in the exact sequence $0 \to$ Ker($f$) $\to A$ $\to$ Im($f$) $\to 0$, the kernel and cokernel of $p^n$ on the first two terms are in $\mathcal{A}$. Hence Im($f$) $\in \mathcal{I}^f \mathcal{A}$. The same argument for $0 \to$ Im($f$) $\to B$ $\to$ Coker($f$) $\to 0$ implies that Coker($f$) $\in \mathcal{I}^f \mathcal{A}$. Hence $\mathcal{I}^f \mathcal{A}$ is an abelian subcategory of $\mathcal{I} \mathcal{A}$.

**Proposition 2.1.3.** For any $A \in \mathcal{I}^f \mathcal{A}$, the decreasing sequence
\[
\ldots \hookrightarrow p^2A \hookrightarrow pA \hookrightarrow A
\]
stabilizes.

**Proof.** For any $n$, we have $p^n: A/(A[p^n] + pA) \cong p^nA/p^{n+1}A$. The isomorphism
\[
A/pA \cong \varprojlim_n (A[p^n] + pA)/pA
\]
in $\mathcal{I} \mathcal{A}$ factors through $(A[p^n] + pA)/pA$ for some $n$ since $A/pA \in \mathcal{A}$. Hence $A/pA \cong (A[p^n] + pA)/pA$ for all large $n$. For such $n$, we have $A = A[p^n] + pA$ and $p^{n+1}A = p^nA$. Hence the sequence stabilizes.

For $A \in \mathcal{I}^f \mathcal{A}$, we define
\[
A_{\text{div}} = \bigcap_n p^nA \quad (= p^nA \text{ for some } n),
\]
\[
A_{/\text{div}} = A/A_{\text{div}}.
\]

**Proposition 2.1.4.** If $A \in \mathcal{I}^f \mathcal{A}$, then $A_{\text{div}} \in \mathcal{I}^f \mathcal{A}$ is divisible and $A_{/\text{div}} \in \mathcal{A}$. The sequence
\[
0 \to A_{\text{div}} \to A \to A_{/\text{div}} \to 0
\]
is exact. The object $A_{\text{div}}$ is the largest divisible subobject of $A$. The object $A_{/\text{div}}$ is the largest quotient of $A$ that belongs to $\mathcal{A}$.

**Proof.** Obvious.
Proposition 2.1.5. The category $\mathcal{I}^f \mathcal{A}$ is artinian.

Proof. By the previous proposition, an object of $\mathcal{A}$ is artinian in $\mathcal{I}^f \mathcal{A}$ since an object of $\mathcal{I}^f \mathcal{A}$ embeddable into an object of $\mathcal{A}$ does not have divisible part and hence itself is in $\mathcal{A}$. It is enough to show that a decreasing sequence $A_0 \supset A_1 \supset \cdots$ of divisible objects in $\mathcal{I}^f \mathcal{A}$ stabilizes. Since $A_n[p] \in \mathcal{A}$ is artinian for any $n$, there is some $m$ such that $A_m[p] = A_{m+1}[p] = \cdots$. From the exact sequence

$$0 \to A_n[p] \to A_n[p^2] \xrightarrow{p} A_n[p] \to 0,$$

we know that $A_m[p^2] = A_{m+1}[p^2] = \cdots$ for the same $m$. Inductively, we have $A_m = A_{m+1} = \cdots$. Denote the procategory of $\mathcal{A}$ by $\mathcal{P}\mathcal{A}$ and the ind-category of $\mathcal{P}\mathcal{A}$ by $\mathcal{I}\mathcal{P}\mathcal{A}$. For $A \in \mathcal{I}^f \mathcal{A}$, we define

$$TA = \lim_{\leftarrow n} A[p^n] \in \mathcal{P}\mathcal{A},$$

$$VA = \lim_{\rightarrow m} TA \in \mathcal{I}\mathcal{P}\mathcal{A},$$

where the direct limit in the second definition is for multiplication by $p^m$. We call $TA$ the Tate module of $A$ and $VA$ the rational Tate module of $A$. For example, if $A$ is $\mathbb{Q}_p/\mathbb{Z}_p$ in the category of torsion abelian groups, then $TA = \mathbb{Z}_p$ in the category of profinite abelian groups and $VA = \mathbb{Q}_p$ as an ind-object of profinite abelian groups. For general $\mathcal{A}$, if $A \in \mathcal{A}$, then the system $\{A[p^n]\}$ that defines $TA$ has essentially zero transition morphisms, hence $TA = 0$. For each $m$, we consider the morphism $TA \to A$ in $\mathcal{I}\mathcal{P}\mathcal{A}$ given by $(a_n)_n \mapsto a_m$. They form a morphism $VA \to A$ in $\mathcal{I}\mathcal{P}\mathcal{A}$.

Proposition 2.1.6. The functor $A \mapsto VA$ from $\mathcal{I}^f \mathcal{A}$ to $\mathcal{I}\mathcal{P}\mathcal{A}$ is exact.

Proof. Let $0 \to A \to B \to C \to 0$ be exact in $\mathcal{I}^f \mathcal{A}$. For any $n$, we have an exact sequence

$$0 \to A[p^n] \to B[p^n] \to C[p^n] \to A/p^n A \to B/p^n B \to C/p^n C \to 0$$

in $\mathcal{A}$. The inverse limit in $n$ in $\mathcal{P}\mathcal{A}$ gives an exact sequence

$$0 \to TA \to TB \to TC \to A_{/\text{div}} \to B_{/\text{div}} \to C_{/\text{div}} \to 0$$

in $\mathcal{P}\mathcal{A}$. (Note that Mittag-Leffler conditions are not relevant here since filtered inverse limits in a procategory are exact by definition.) Since each
(·)/\text{div} \in \mathcal{A} is killed by multiplication by some power of \( p \), the direct limit in multiplication by \( m \) gives the desired exact sequence \( 0 \to VA \to VB \to VC \to 0 \) in \( \text{IP}_A \).

**Proposition 2.1.7.** Let \( A \in I^f\mathcal{A} \). Then we have an exact sequence

\[ 0 \to TA \xrightarrow{p^n} TA \to A_{\text{div}} \cap A[p^n] \to 0 \]

in \( \text{P}_A \) for any \( n \geq 0 \) and an exact sequence

\[ 0 \to TA \to VA \to A_{\text{div}} \to 0 \]

in \( \text{IP}_A \). We have \( TA = T(A_{\text{div}}) \) and \( VA = V(A_{\text{div}}) \).

**Proof.** The multiplication-by-\( p^n \) map gives an exact sequence \( 0 \to A[p^n] \to A \to p^n A \to 0 \). We have \( T(A[p^n]) = 0 \) and \( A[p^n]/\text{div} = A[p^n] \) since \( A[p^n] \in \mathcal{A} \) has no divisible part. Hence the long exact sequence in the proof of the previous proposition for this sequence is

\[ 0 \to TA \to T(p^n A) \to A[p^n] \to A_{\text{div}} \to (p^n A)_{\text{div}} \to 0. \]

in \( \text{P}_A \). Since \( A/p^n A \in \mathcal{A} \), we have \( T(p^n A) = TA \). From this, we get the first exact sequence. The direct limit in \( n \) gives the second exact sequence. We have \( T(A_{\text{div}}) = 0 \) since \( A_{\text{div}} \in \mathcal{A} \). It follows that \( TA = T(A_{\text{div}}) \) and \( VA = V(A_{\text{div}}) \).

Next, let \( \mathcal{A} \) be an artinian abelian category such that any object is killed by multiplication by some positive integer (not necessarily a prime power). For each prime \( p \), denote by \( \mathcal{A}_p \) the full subcategory of \( \mathcal{A} \) of objects killed by a power of \( p \). This is an artinian abelian category. Any object of \( \mathcal{A} \) can be canonically decomposed as \( \mathcal{A} = \bigoplus_p \mathcal{A}_p \), where each summand \( \mathcal{A}_p \) belongs to \( \mathcal{A}_p \) and \( \mathcal{A}_p = 0 \) for almost all \( p \). We call \( \mathcal{A}_p \) the \( p \)-primary part of \( \mathcal{A} \). Consider the ind-category \( I\mathcal{A} \). Any object of \( I\mathcal{A} \) can be canonically decomposed as \( \mathcal{A} = \bigoplus_p \mathcal{A}_p \) (a filtered direct limit of finite partial sums), where each summand \( \mathcal{A}_p \) belongs to \( I\mathcal{A}_p := I(\mathcal{A}_p) \).

We say that an object \( A \in I\mathcal{A} \) is of **cofinite type** if the \( n \)-torsion part \( A[n] \) is in \( \mathcal{A} \) for all \( n \geq 1 \). In this case, the equality \( A = \lim \mathcal{A}[n] \) gives a presentation of \( A \) as a filtered direct limit of objects of \( \mathcal{A} \). Denote by \( I^f\mathcal{A} \) the full subcategory of \( I\mathcal{A} \) consisting of objects of cofinite type. An object \( A \in I\mathcal{A} \) is of cofinite type if and only if \( \mathcal{A}_p \) is of cofinite type for all \( p \). Note that \( I^f\mathcal{A} \) is not necessarily artinian. For instance, the object \( \bigoplus_p \mathbb{Z}/p\mathbb{Z} \) in
the category of torsion abelian groups of cofinite type is not artinian. For $A \in \mathcal{I} \mathcal{A}$, we define

$$A_{\text{div}} = \bigoplus_p (A_p)_{\text{div}} \in \mathcal{I} \mathcal{A}, \quad A/_{\text{div}} = A/A_{\text{div}} \in \mathcal{I} \mathcal{A},$$

where $(A_p)_{\text{div}} \in \mathcal{I} \mathcal{A}_p$ is previously defined. We also define

$$TA = \lim_{\leftarrow n} A[n] \in \mathcal{P} \mathcal{A}, \quad VA = \lim_{\rightarrow m} TA \in \mathcal{I} \mathcal{P} \mathcal{A},$$

where the inverse limit in the first definition is over multiplication by $n \geq 1$ and the direct limit in the second is over multiplication by $m \geq 1$. Also define $T_pA = T(A_p)$, $V_pA = V(A_p)$.

**Proposition 2.1.8.**

1. For any $A \in \mathcal{I} \mathcal{F} \mathcal{A}$ and $n \geq 1$, we have $A/nA \in \mathcal{A}$.
2. The category $\mathcal{I} \mathcal{F} \mathcal{A}$ is an abelian subcategory of $\mathcal{I} \mathcal{A}$.
3. If $A \in \mathcal{I} \mathcal{F} \mathcal{A}$, then $A_{\text{div}}$ is the largest divisible subobject of $A$, and $A/_{\text{div}}$ is the largest quotient of $A$ such that $(A/_{\text{div}}) p$ belongs to $\mathcal{A}_p$ for all $p$.

The sequence

$$0 \to A_{\text{div}} \to A \to A/_{\text{div}} \to 0$$

is an exact sequence in $\mathcal{I} \mathcal{F} \mathcal{A}$.

4. Let $A \in \mathcal{I} \mathcal{F} \mathcal{A}$. Then we have exact sequences

$$0 \to TA \xrightarrow{n} TA \to A_{\text{div}} \cap A[n] \to 0$$

in $\mathcal{P} \mathcal{A}$ for any $n \geq 1$ and

$$0 \to TA \to VA \to A_{\text{div}} \to 0$$

in $\mathcal{I} \mathcal{P} \mathcal{A}$. We have $TA = T(A_{\text{div}}) = \prod_p T_pA$ and $VA = V(A_{\text{div}})$.

**Proof.** All the statements are reduced to the $p$-primary parts for primes $p$, which have already been proved.

Contrary to the $p$-primary case, $A/_{\text{div}}$ might not be in $\mathcal{A}$ for general $A \in \mathcal{I} \mathcal{F} \mathcal{A}$. An example is $\bigoplus_p \mathbb{Z}/p\mathbb{Z}$. This is related to the Mittag-Leffler condition. Here is a general fact.
Proposition 2.1.9. Let $B$ be an object of any abelian category $\mathcal{B}$. Consider the inverse system in $\mathcal{B}$ given by multiplication maps by positive integers on $B$. This system is Mittag-Leffler if and only if $B$ has a divisible subobject $B'$ such that $B/B'$ is killed by multiplication by some positive integer. In this case, $B'$ is the maximal divisible subobject of $B$, and we say that $B$ is divisibly ML.

Proof. Elementary.

Proposition 2.1.10. Let $A \in \mathcal{I}'\mathcal{A}$. Then $A$ is divisibly ML if and only if $A/\text{div} \in \mathcal{A}$ if and only if $A_p$ is divisible for almost all $p$.

Proof. We have

$$A/\text{div} = \bigoplus_p A_p / \bigoplus_p (A_p)/\text{div}.$$ 

The result follows from this.

Note that divisibly ML objects in $\mathcal{I}'\mathcal{A}$ do not form an abelian subcategory. For instance, the direct sum of the exact sequences $0 \to \mathbb{Z}/p\mathbb{Z} \to \mathbb{Q}_p/\mathbb{Z}_p \to \mathbb{Q}/\mathbb{Z} \to 0$ for primes $p$ gives an exact sequence $0 \to \bigoplus_p \mathbb{Z}/p\mathbb{Z} \to \mathbb{Q}/\mathbb{Z} \to 0$, where the last two terms are divisible but the first term is not divisibly ML.

2.2 Ind-algebraic groups of cofinite type

Let $k \in U_0$ be a perfect field of characteristic $p > 0$. We quickly recall some notation about perfections of algebraic groups from [Suz14, Section 2.1]. Let Alg$/k$ be the category of quasi-algebraic groups over $k$ (commutative, as assumed throughout the paper) in the sense of Serre [Ser60]. Recall that a quasi-algebraic group is the perfection (inverse limit along Frobenii) of a (not necessarily connected) algebraic group [Ser60, Section 1.2, Définition 2; Section 1.4, Proposition 10]. For example, the perfection of the additive group Spec $k[x]$ is Spec $k[x, x^{1/p}, x^{1/p^2}, \ldots]$, which we simply call the additive group and denote by $G_a$ by abuse of notation. We say that a quasi-algebraic group is a unipotent group, a torus or a (semi-)abelian variety if it is the perfection of such a group. We call an object of the procategory $\text{PAlg}$/k a proalgebraic group, an object of the ind-category $\text{IAlg}$/k an ind-algebraic group and an object of the ind-procategory $\text{IPAlg}$/k = $\text{I}(\text{PAlg}$/k) an ind-proalgebraic group. Let Alg$/k$ be the full subcategory of Alg$/k$ consisting of groups whose identity component is
unipotent. Let $\text{LAlg}/k$ be the category of perfections of smooth group schemes over $k$. A finitely generated étale group is an étale group with a finitely generated group of geometric points. A lattice is a finitely generated étale group with no torsion. Let $\text{FEt}/k \subset \text{FGEt}/k \subset \text{Et}/k$ be the categories of finite étale groups, finitely generated étale groups, étale groups, respectively, over $k$. For any prime $l$ (possibly equal to $p$), we denote the full subcategory of $\text{FEt}/k$ of groups of $l$-power order by $\text{FEt}^l/k$.

For $A \in \text{LAlg}/k$, we denote its identity component by $A_0 \in \text{Alg}/k$ and set $\pi_0(A) = A/A_0 \in \text{Et}/k$ ([DG70, II, Section 5, Proposition 1.8] plus perfection, noting that perfection does not change the underlying topological space). The endofunctor $A \mapsto A_0$ on $\text{Alg}/k$ extends to endofunctors on $\text{PIAlg}/k$, $\text{IAlg}/k$ and $\text{IPAlg}/k$, still denoted by $A \mapsto A_0$. We say that $A \in \text{IPAlg}/k$ is connected if $A = A_0$. For $A \in \text{IPAlg}/k$, we define $\pi_0(A) = A/A_0 \in \text{IPFEt}/k$.

The category $\text{Alg}_{\text{uc}}/k$ is an artinian abelian category such that any object is killed by multiplication by some positive integer. Hence we can apply the results and notation in the previous subsection to $A = \text{Alg}_{\text{uc}}/k$.

**Proposition 2.2.1.** For any $A \in \text{IFAlg}_{\text{uc}}/k$, we have $A_0, \pi_0(A) \in \text{IFAlg}_{\text{uc}}/k$.

**Proof.** We may assume that $A$ is unipotent. Consider the sequence

$$
0 \to A_0[p^n] \to A[p^n] \to \pi_0(A)[p^n] \to A_0/p^nA_0 \to A/p^nA \to \pi_0(A)/p^n\pi_0(A) \to 0
$$

for any $n \geq 0$. Since $A[p^n]$ is quasi-algebraic and $\pi_0(A)[p^n]$ is étale, the image of $A[p^n] \to \pi_0(A)[p^n]$ is finite and hence $A_0[p^n]$ is quasi-algebraic. Therefore $A_0 \in \text{IFAlg}_{\text{uc}}/k$ and so $A_0/p^nA_0$ is quasi-algebraic by Proposition 2.1.1. This implies that $\pi_0(A)[p^n]$ is quasi-algebraic (i.e., finite étale) and $\pi_0(A) \in \text{IFAlg}_{\text{uc}}/k$.

We define $\text{IFAlg}_{\text{uc}}/k$ to be the full subcategory of $\text{IFAlg}_{\text{uc}}/k$ of objects whose identity component is quasi-algebraic (i.e., belongs to $\text{Alg}_{\text{uc}}/k$). Equivalently, an object $A \in \text{IFAlg}_{\text{uc}}/k$ is an extension of a torsion étale group of cofinite type ($= \pi_0(A)$) by a connected unipotent quasi-algebraic group ($= A_0$). Such objects can naturally be viewed as objects of $\text{LAlg}/k$; see [Suz13, Section 2.1, Footnote 3].

**Proposition 2.2.2.** Let $A \in \text{IFAlg}_{\text{uc}}/k$. Then $(A_0)_{\text{div}} = (A_{\text{div}})_0$. We denote these isomorphic objects by $A_{0\text{div}}$. 
Proof. We may assume that $A$ is unipotent. It is enough to show that $(A_{0})_{\text{div}}$ is connected and $(A_{\text{div}})_{0}$ is divisible. Take a power $p^{n}$ of $p$ that kills $(A_{0})_{\text{div}} \in \text{Alg}_{uc}/k$. Then multiplication by $p^{n}$ gives a surjection $A_{0} \twoheadrightarrow (A_{0})_{\text{div}}$. Hence $(A_{0})_{\text{div}}$ is connected. On the other hand, the exact sequence $0 \rightarrow (A_{\text{div}})_{0} \rightarrow A_{\text{div}} \rightarrow \pi_{0}(A_{\text{div}}) \rightarrow 0$ and the snake lemma gives a surjection $\pi_{0}(A_{\text{div}})[p] \twoheadrightarrow (A_{\text{div}})_{0}/p((A_{\text{div}})_{0})$. This implies that $(A_{\text{div}})_{0}/p((A_{\text{div}})_{0})$ is étale and connected, hence zero. Thus $(A_{\text{div}})_{0}$ is divisible.

An example of a connected divisible group in $\text{If}_{\text{Alg}_{uc}}/k$ is the direct limit $\lim_{\rightarrow} W_{n}$ of (perfections of) groups of $p$-typical Witt vectors of finite length. We have $T \lim_{\rightarrow} W_{n} = \lim_{\leftarrow} W_{n}$, which is the group $W$ of $p$-typical Witt vectors of infinite length.

**Proposition 2.2.3.** Let $A \in \text{If}_{\text{Alg}_{uc}}/k$. Then the following are equivalent.

1. $A \in \text{L}_{\text{If}} \text{Alg}_{uc}/k$.
2. $A_{0,\text{div}} = 0$.
3. $V_{p}A \in \text{IPFE}_{t}/k$. (Recall that $V_{p}A$ is defined in Section 2.1.)
4. $V_{p}A(k')$ as a functor on algebraically closed fields $k'$ over $k$ is constant.

Proof. We may assume that $A$ is unipotent. The equivalence between (1) and (2) follows from the previous proposition. We know that $V_{p}A = V_{p}(A_{\text{div}})$ surjects onto $A_{\text{div}}$. Hence $(V_{p}A)_{0} = (V_{p}(A_{\text{div}}))_{0}$ surjects onto $A_{0,\text{div}}$. This shows that (3) is equivalent to (2). Obviously (3) implies (4).

We show that (4) implies (2). We may assume that $k = \overline{k}$ and $A$ is divisible. Then the assumption implies that $V_{p}A(k')$ as a functor on arbitrary perfect fields $k'$ over $k$ is constant. The surjection $V_{p}A \twoheadrightarrow A$ implies that $A(k')$ as a functor on perfect fields $k'$ over $k$ is constant. This implies that $A_{0}(k')$ as a functor on perfect fields $k'$ over $k$ is constant. Hence we may further assume that $A$ is connected. We then want to show that $A = 0$ if $A(k')$ as a functor on perfect fields $k'$ over $k$ is constant. Let $A_{n} = (A[p^{n}])_{0} \in \text{Alg}_{uc}/k$. Then $A = \lim_{\rightarrow} A_{n}$. It follows that the generic point of $A_{n}$ for any $n$ maps to a $k$-value point of $A$ and hence to a $k$-value point of $A_{m}$ for some $m \geq n$. This means that the injective morphism $A_{n} \hookrightarrow A_{m}$ of quasi-algebraic groups is generically constant. Therefore $A_{n} = 0$ for any $n$ and $A = 0$.

**Proposition 2.2.4.** The category $\text{L}_{\text{If}} \text{Alg}_{uc}/k$ is an abelian subcategory of $\text{If}_{\text{Alg}_{uc}}/k$.
Proof. This follows from the equivalence between (1) and (3) (or (4)) of the previous proposition.

Proposition 2.2.5. Let $A \in \text{I} / \text{Alg}_{uc}/k$ and define $A_{0 \cap \text{div}} := A_0 \cap A_{\text{div}}$. Then $A_{0 \cap \text{div}}/A_{0 \text{div}} \in \text{FEt}_p/k$. We have exact sequences

$$
0 \to A_{0 \cap \text{div}} \to A_0 \to (A_{/\text{div}})_0 \to 0,
0 \to A_{0 \cap \text{div}} \to A_{\text{div}} \to (\pi_0 A)_{\text{div}} \to 0.
$$

Proof. Since $A_0/A_{\text{div}} = (A_0)_{/\text{div}}$ is quasi-algebraic unipotent, it is killed by some power $p^n$ of $p$. Hence $A_{0 \cap \text{div}}/A_{\text{div}}$ is a subgroup of the $p^n$-torsion part of $A_{\text{div}}/A_{0 \text{div}} = \pi_0(A_{\text{div}})$. But $\pi_0(A_{\text{div}})$ is a torsion étale group of cofinite type. Hence its $p^n$-torsion part is finite. Therefore $A_{0 \cap \text{div}}/A_{0 \text{div}}$ is finite. The exactness of the sequences is clear.

The group $A_{0 \cap \text{div}}$ in the proposition is in general nonzero even if $A \in \text{I} / \text{Alg}_{uc}/k$. It is $\mathbb{Z}/p\mathbb{Z}$ if $A$ is the cokernel of the diagonal embedding of $\mathbb{Z}/p\mathbb{Z}$ into $G_a \oplus (\mathbb{Q}_p/\mathbb{Z}_p)$. In particular, $A_{0 \cap \text{div}}$ can be neither connected nor divisible. On the other hand, $A_{\text{div}}$ is always connected and divisible.

Proposition 2.2.6. For any $A \in \text{I} / \text{Alg}_{uc}/k$, we have $\pi_0(A)_{/\text{div}} = \pi_0(A_{/\text{div}})$. Denote these isomorphic objects by $\pi_0 A_{/\text{div}}$. The kernel of $A \to \pi_0 A_{/\text{div}}$ is $A_{0 + \text{div}} := A_0 + A_{\text{div}}$.

Proof. Obvious.

For example, if $A = G_a \oplus \mathbb{Q}/\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$, then $A_{0 + \text{div}} = G_a \oplus \mathbb{Q}/\mathbb{Z}$ and $\pi_0 A_{\text{div}} = \mathbb{Z}/p\mathbb{Z}$. To see a more nontrivial example, one may work out the subgroup $A$ of $\lim_{\rightarrow} W_n = \{ (\ldots, x_{-2}, x_{-1}, x_0) \mid x_n = 0 \text{ for almost all } n \}$ defined by $(F - 1)x_n = (F - 1)^2 x_{-1} = 0$ for all $n \leq -2$, where $F$ is the Frobenius. Then $A_0 \cong G_a$ is defined by $x_n = 0$ for all $n \leq -1$, $A_{\text{div}} \cong \mathbb{Q}_p/\mathbb{Z}_p$ by $(F - 1)x_n = 0$ for all $n \leq 0$, $A_{0 + \text{div}}$ by $(F - 1)x_n = 0$ for all $n \leq -1$, and $\pi_0 A_{\text{div}} \cong \mathbb{Z}/p\mathbb{Z}$.

Here is a connection diagram between the named subgroups of $A \in \text{I} / \text{Alg}_{uc}/k$:

$$
0 \xrightarrow{\text{conn div}} A_{0 \text{div}} \xrightarrow{\text{FEt}_p} A_{0 \cap \text{div}} \xrightarrow{\text{ét div}} A_{\text{div}} \\
\text{conn alg} \downarrow \quad \downarrow \quad \downarrow \text{conn alg} \\
A_0 \xrightarrow{\text{ét div}} A_{0 + \text{div}} \oplus \text{FEt}_l \longrightarrow A
$$
All the arrows are inclusions. The label of an arrow means that the subquotient there is of that type. For example, \( A_{0+\text{div}}/A_{\text{div}} \) is connected quasi-algebraic. The symbol \( \bigoplus_l \text{F}	ext{Et}_l \) means that the (sub)quotient there is a direct sum of finite étale \( l \)-primary groups for primes \( l \) (with \( l = p \) allowed), which is finite if and only if \( A \) is divisibly ML. The upper and lower sides of the square have the same subquotient, and the left and right sides of the square have the same subquotient. The two step subquotient \( A_0/A_{0\text{div}} \) is connected quasi-algebraic and, similarly, \( A_{\text{div}}/A_{0\text{div}} \) is étale divisible. In particular, we have \( \pi_0(A_{0+\text{div}}) = \pi_0(A)_{\text{div}} \) and \( (A_{0+\text{div}})/_{\text{div}} = (A/_{\text{div}}) \). Later we will see in Proposition 3.2.3 that the cohomology object \( G \) in any positive degree of the complex \( R\Gamma(X, \mathcal{A}) \) mentioned in Introduction belongs to \( \text{I}^f\text{Alg}_{\text{uc}}/k \). With some more effort, we will see in Theorem 3.4.1 that \( G_{0\text{div}} = 0 \) (i.e., \( G \in \text{I}^f\text{Alg}_{\text{uc}}/k \)) and \( G/G_{0+\text{div}} \) is finite (i.e., \( G \) is divisibly ML).

### 2.3 The ind-rational pro-étale site and some derived limits

We quickly recall some definitions and notation about the ind-rational (pro-)étale site from [Suz14, Section 2.1]. We say that a \( k \)-algebra is rational if it is a finite direct product of perfections (direct limit along Frobenii) of finitely generated fields over \( k \), and ind-rational if it is a filtered union of rational \( k \)-subalgebras. We denote the category of rational (resp. ind-rational) \( k \)-algebras with \( k \)-algebra homomorphisms by \( k^{\text{rat}} \) (resp. \( k^{\text{indrat}} \)). (More precisely, we only consider \( \mathcal{U}_0 \)-small ind-rational \( k \)-algebras. Hence \( k^{\text{indrat}} \) is a \( \mathcal{U} \)-small \( \mathcal{U}_0 \)-category.) We can endow \( k^{\text{indrat}} \) with the étale topology. That is, an étale covering of \( k' \in k^{\text{indrat}} \) is a finite family \( \{k'_i\} \) of étale \( k' \)-algebras such that the product \( \prod k'_i \) is faithfully flat over \( k' \). The resulting site is the ind-rational étale site \( \text{Spec} k^{\text{indrat}}_{\text{et}} \). We can also endow \( k^{\text{indrat}} \) with the pro-étale topology of Bhatt and Scholze [BS15]. That is, a pro-étale covering of \( k' \in k^{\text{indrat}} \) is a finite family \( \{k'_i\} \) of \( k' \)-algebras such that each \( k'_i \) is a filtered direct limit of étale \( k' \)-algebras and the product \( \prod k'_i \) is faithfully flat over \( k' \). The resulting site is the ind-rational pro-étale site \( \text{Spec} k^{\text{indrat}}_{\text{proet}} \). The category of sheaves of sets (resp. abelian groups) on \( \text{Spec} k^{\text{indrat}}_{\text{proet}} \) is denoted by \( \text{Set}(k^{\text{indrat}}_{\text{proet}}) \) (resp. \( \text{Ab}(k^{\text{indrat}}_{\text{proet}}) \)). (Here, the target categories \( \text{Set} \) and \( \text{Ab} \) for sheaves are the categories of \( \mathcal{U} \)-small sets and abelian groups.) The category of complexes in \( \text{Ab}(k^{\text{indrat}}_{\text{proet}}) \) is denoted by \( \text{Ch}(k^{\text{indrat}}_{\text{proet}}) \), its homotopy category by \( K(k^{\text{indrat}}_{\text{proet}}) \) and its derived category by \( D(k^{\text{indrat}}_{\text{proet}}) \). See [KS06, Chapter 18] for details about unbounded derived categories of sheaves on sites. The cohomology of \( k' \in k^{\text{indrat}} \)
with coefficients in \( G \in D(k_{\text{proet}}) \) is denoted by \( R\Gamma(k_{\text{proet}}', G) \), with \( n \)th cohomology \( H^n(k_{\text{proet}}', G) \) when \( G \in \text{Ab}(k_{\text{proet}}) \). The sheaf-Hom, \( n \)th sheaf-Ext and derived sheaf-Hom functors are denoted by \( \text{Hom}_{k_{\text{proet}}} \), \( \text{Ext}_{k_{\text{proet}}}^n \) and \( R\text{Hom}_{k_{\text{proet}}} \), respectively. Similar notation applies to \( \text{Spec} k_{\text{et}} \).

For \( G \in D(k_{\text{proet}}) \), we define its Serre dual \([\text{Suz14}, \text{Section 2.4}]\) to be \( G^{SD} = R\text{Hom}_{k_{\text{proet}}}(G, \mathbb{Z}) \).

See the list of examples of Serre duals in the paragraph after the proof of \([\text{Suz14}, \text{Proposition (2.4.1)}]\). A key example there is \( G_a^{SD} \sim G_a [-2] \), which comes from \( G_a \sim \to \text{Ext}_{k_{\text{proet}}}^1(G_a, \mathbb{Z}/p\mathbb{Z}) \) defined by sending \( c \in G_a \) to the pullback of the Artin–Schreier extension class \( 0 \to \mathbb{Z}/p\mathbb{Z} \to G_a \to G_a \to 0 \) by the multiplication-by-\( c \) map \( G_a \to G_a \) to the final term. In particular, \( G_a \) is Serre reflexive. We say that \( G \) is Serre reflexive if the canonical morphism \( G \to G^{SDSD} \) is an isomorphism. For \( G \in \text{Ab}(k_{\text{proet}}) \), we denote its torsion part by \( G_{tor} \) and set \( G / G_{tor} = G / G_{tor} \).

The natural Yoneda functor \( \text{Alg}/k \to \text{Ab}(k_{\text{proet}}) \) extends to an additive functor \( \text{IPAlg}/k \to \text{Ab}(k_{\text{proet}}) \). (Here, as before, we consider the \( \mathcal{U}_0 \)-procategory of \( \text{Alg}/k \) and the \( \mathcal{U}_0 \)-indcategory of the \( \mathcal{U}_0 \)-category \( \text{PAlg}/k \).) This functor is exact by \([\text{Suz14}, \text{Proposition (2.1.2)(e)}]\), which is fully faithful and induces a fully faithful embedding \( D^b(\text{IPAlg}/k) \to D(k_{\text{proet}}) \) by \([\text{Suz14}, \text{Proposition (2.3.4)}]\). If \( G \in \text{IPAlg}/k \) is an extension of an object of \( \text{IALg}_{uc}/k \) by an object of \( \text{PAlg}/k \), then we denote \( G^{SD'} = \text{Ext}_{k_{\text{proet}}}^1(G, \mathbb{Q}/\mathbb{Z}) \).

By \([\text{Suz14}, \text{Proposition (2.4.1)(b)}]\), we have \( G^{SD} = G^{SD'} [-2] \). The same proposition shows that if \( G \) is connected unipotent proalgebraic or ind-algebraic, then \( G^{SD'} \) is connected unipotent ind-algebraic or proalgebraic, respectively, and \( G \sim G^{SDSD'} \). In particular, \( G \) is Serre reflexive. By \([\text{Suz14}, \text{Proposition (2.4.1)(b)}]\), if \( G \) is semiabelian, then \( G^{SD'} \) is the Pontryagin dual of the Tate module \( TG \). If \( G \) is not necessarily connected, then we denote \( G_0^{SD'} = (G_0)^{SD'} \). If \( G \) is an extension of a torsion étale group by a profinite-étale group, then its Pontryagin dual is denoted by \( G^{PD} \), which can be given by \( \text{Hom}_{k_{\text{proet}}} (G, \mathbb{Q}/\mathbb{Z}) \), or by \( G^{SD}[1] \) by \([\text{Suz14}, \text{Proposition (2.4.1)(b)}]\).
For example, if $G = \mathbb{Q}_p$, then $G^{\text{PD}} \cong \mathbb{Q}_p$. If the torsion part of $G \in \text{Ab}(k^{\text{indrat}}_{\text{proet}})$ is such a group, then we denote $G^{\text{PD}}_{\text{tor}} = (G_{\text{tor}})^{\text{PD}}$. If $G$ is a lattice over $k$, then its dual lattice is denoted by $G^{\text{LD}}$, which can be given by $\text{Hom}_{k^{\text{indrat}}_{\text{proet}}}(G, \mathbb{Z})$. If the torsion-free quotient of $G \in \text{Ab}(k^{\text{indrat}}_{\text{proet}})$ is such a group, then we denote $G^{\text{LD}}_{/\text{tor}} = (G_{/\text{tor}})^{\text{LD}}$.

For an abelian ($\mathcal{U}$-)category, we say that it has exact products if the product $\prod_{\lambda \in \Lambda} A_{\lambda}$ of any family of objects $\{A_{\lambda}\}$ (with $\Lambda \in \mathcal{U}$) exists and for any family of surjections $\{A_{\lambda} \twoheadrightarrow B_{\lambda}\}_{\lambda \in \Lambda}$ (with $\Lambda \in \mathcal{U}$), the morphism $\prod_{\lambda} A_{\lambda} \rightarrow \prod_{\lambda} B_{\lambda}$ is surjective.

**Proposition 2.3.1.** The category $\text{Ab}(k^{\text{indrat}}_{\text{proet}})$ has enough projectives and exact products. For every sequence

$$\cdots \xrightarrow{\varepsilon_3} A_3 \xrightarrow{\varepsilon_2} A_2 \xrightarrow{\varepsilon_1} A_1$$

in $\text{Ab}(k^{\text{indrat}}_{\text{proet}})$, its derived inverse limit is represented by the complex $\prod_n A_n \rightarrow \prod_n A_n$ in degrees 0 and 1, where the morphism sends $(a_n)_n$ to $(a_n - \varphi_n(a_{n+1}))_n$. If the sequence is Mittag-Leffler, then its derived inverse limit is zero in positive degrees.

**Proof.** For any $k' \in k^{\text{indrat}}$, there exists an ind-étale faithfully flat homomorphism $k' \rightarrow k''$ to a w-contractible ring $k''$ [BS15, Definition 2.4.1] by [BS15, Lemma 2.4.9]. An ind-étale algebra over an ind-rational $k$-algebra is ind-rational over $k$ by [Suz13, Proposition 2.1.2], so $k'' \in k^{\text{indrat}}$. By the definition of w-contractibility, we know that the sheaf of free abelian groups $\mathbb{Z}[	ext{Spec } k'']$ generated by the representable sheaf of sets $\text{Spec } k''$ is a projective object of $\text{Ab}(k^{\text{indrat}}_{\text{proet}})$. The natural morphism $\mathbb{Z}[	ext{Spec } k''] \rightarrow \mathbb{Z}[	ext{Spec } k']$ is surjective. Since any object of $\text{Ab}(k^{\text{indrat}}_{\text{proet}})$ is a quotient of a direct sum (with index set $\in \mathcal{U}$) of objects of the form $\mathbb{Z}[	ext{Spec } k']$ with $k' \in k^{\text{indrat}}$, it follows that $\text{Ab}(k^{\text{indrat}}_{\text{proet}})$ has enough projectives.

The category $\text{Ab}(k^{\text{indrat}}_{\text{proet}})$ has products. Having enough projectives, it has exact products. The stated complex calculates the derived limit by [Nee01, Remark A.3.6] or [Roo06, Lemma 2.2]. The statement about Mittag-Leffler sequences follows from [Roo06] or [Nee01, Lemma A.3.15]. ☐

Let $\text{P Ab}(k^{\text{indrat}}_{\text{proet}})$ be the procategory of $\text{Ab}(k^{\text{indrat}}_{\text{proet}})$. (Here we consider the $\mathcal{U}$-procategory since $\text{Ab}(k^{\text{indrat}}_{\text{proet}})$ is a $\mathcal{U}$-category.) A filtered inverse system $\{A_{\lambda}\}_{\lambda}$ in $\text{Ab}(k^{\text{indrat}}_{\text{proet}})$ as an object of $\text{P Ab}(k^{\text{indrat}}_{\text{proet}})$ is denoted by $\lim_{\lambda} A_{\lambda}$. The category $\text{Ab}(k^{\text{indrat}}_{\text{proet}})$ is a Grothendieck category and hence has enough injectives. Therefore the procategory $\text{P Ab}(k^{\text{indrat}}_{\text{proet}})$ also
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has enough injectives by [KS06, Exercise 13.6(ii)]. As P Ab(k_{proet}^{indrat}) has exact products (and exact filtered inverse limits), it follows from [KS06, Theorem 14.4.4] that any additive functor from P Ab(k_{proet}^{indrat}) to an abelian category admits a right derived functor on (unbounded) derived categories, which can be calculated by K-injective (or homotopically injective) complexes in P Ab(k_{proet}^{indrat}). In particular, the inverse limit functor \( \lim: P \text{Ab}(k_{proet}^{indrat}) \to \text{Ab}(k_{proet}^{indrat}) \) admits a right derived functor \( R \lim: D(P \text{Ab}(k_{proet}^{indrat})) \to D(k_{proet}^{indrat}) \). For a more detailed treatment of \( R \lim \), see [KS06, Proposition 13.3.15, Corollary 13.3.16, Example 13.3.17(i)].

For a sheaf \( A \in \text{Ab}(k_{proet}^{indrat}) \), the multiplication maps by integers \( n \geq 1 \) yield a pro-object \( \lim_n A \) with index set \( \mathbb{Z}_{\geq 1} \) ordered by divisibility. Note that there is a cofinal map \( \sigma \) from the set of positive integers with the usual ordering \( \leq \) to the set of positive integers with the divisibility ordering. For example, we can take \( \sigma(n) \) to be the \( n \)th power of the product of the first \( n \) primes. Hence the pro-object \( \lim_n A \) is isomorphic to the sequence \( \cdots \to A_{\sigma(2)} \to A_{\sigma(1)} \). The assignment \( A \mapsto \lim_n A \) defines an exact functor \( \text{Ab}(k_{proet}^{indrat}) \to \text{PAb}(k_{proet}^{indrat}) \) and hence a triangulated functor \( D(k_{proet}^{indrat}) \to D(\text{PAb}(k_{proet}^{indrat})) \). By composing it with \( R \lim \), we have a triangulated endofunctor on \( D(k_{proet}^{indrat}) \), which we denote by \( A \mapsto R \lim_n A \). The objects \( \lim_n A \) and \( R \lim_n A \) are uniquely divisible. We have a projection \( \lim_n A \to A \) to the \( n = 1 \) term in \( D(\text{PAb}(k_{proet}^{indrat})) \). This induces a morphism \( R \lim_n A \to A \) in \( D(k_{proet}^{indrat}) \). It is an isomorphism if and only if \( A \) is uniquely divisible. We have \( R \lim_n A = 0 \) if \( A \) is killed by multiplication by some positive integer.

For \( A \in \text{Ab}(k_{proet}^{indrat}) \), we define \( A_{\text{div}} \) to be the image of the natural morphism \( \lim_n A \to A \). It is the maximal divisible subsheaf of \( A \). We denote \( A/_{\text{div}} = A/A_{\text{div}} \). We define

\[
TA = \lim_n A[n], \quad VA = (TA) \otimes \mathbb{Q}, \quad \hat{A} = \lim_n (A \otimes \mathbb{Z}/n\mathbb{Z}),
\]

where \( A[n] \) is the \( n \)-torsion part of \( A \). If \( A \in I^f \text{Alg}_{uc}/k \), then Section 2.1 gives an object \( TA \in \text{PAlg}_{uc}/k \), while the above definition gives an object

\[\text{1}^{\text{For comparison of derived functors of inverse limits defined for pro-objects and for inverse systems, see [Pro99, Corollary 7.3.7] for example. This reference assumes that the abelian category in question has exact products. This assumption is satisfied for our category \( \text{Ab}(k_{proet}^{indrat}) \) by the proposition above.}\]

TA ∈ Ab(k^\text{indrat}_{\text{proet}}). They are compatible under the Yoneda functor PAlg_{uc}/k ↪ Ab(k^\text{indrat}_{\text{proet}}). A similar relation holds for VA. We denote TA_{\text{div}} = T(A_{\text{div}}), VA_{\text{div}} = V(A_{\text{div}}). If A_{\text{div}} is a torsion étale group, we denote A_{\text{div}}^{\text{PD}} = (A_{\text{div}})^{\text{PD}}. The notation \(A^\wedge_{\text{div}}\) means \((A_{\text{div}})^\wedge\). For A ∈ FGEt/k, we denote \(A^{\text{LD}_{\text{tor}}} = (A^{\text{LD}})^{\wedge}_{\text{tor}}\). (The general rule here is that we read the subscript first and then the superscript, from the inside to the outside. Sticking to one such convention reduces excessive usage of parentheses. But we are not strictly consistent with this rule, as the ind-rational pro-étale site Spec \(k^\text{indrat}_{\text{proet}}\) or the open subscheme \(A_0^\vee \subset A^\vee\) of the Néron model shows.)

**Proposition 2.3.2.** For any \(A ∈ Ab(k^\text{indrat}_{\text{proet}})\), we have \(R^n \lim_{\leftarrow} A = 0\) for \(m \geq 2\). If \(A\) is divisibly ML (Proposition 2.1.9), then \(R \lim_{\leftarrow} A = \lim_{\leftarrow} A\) and \(A_{\text{div}} \sim \hat{A}\). If \(A_{\text{div}}\) is torsion, then \(V A \sim \lim_{\leftarrow} A\).

**Proof.** As above, represent the pro-object \(\text{"lim"}_n A\) as a sequence \(\cdots \to A_{\sigma(2)} \to A_{\sigma(1)}\). Then the vanishing of the derived limit in degree \(\geq 2\) follows from the previous proposition. If \(A\) is divisibly ML, then \(R \lim_{\leftarrow} A = \lim_{\leftarrow} A\) by the previous proposition. We then have \(\hat{A} = (A_{\text{div}})^\wedge = A_{\text{div}}\). For general \(A\), we have an exact sequence \(0 \to TA \to \lim_{\leftarrow} A \to A_{\text{div}} \to 0\). Tensoring by \(\mathbb{Q}\), we have an exact sequence \(0 \to VA \to \lim_{\leftarrow} A \to A_{\text{div}} \otimes \mathbb{Q} \to 0\). Hence \(VA = \lim_{\leftarrow} A\) if \(A_{\text{div}} \otimes \mathbb{Q} = 0\).

**Proposition 2.3.3.** Let \(A ∈ I^f\text{Alg}_{uc}/k\) be divisibly ML. Then the morphism \(VA \to A\) induces an isomorphism \(R \lim_{\leftarrow} A = VA\) in \(D(k^\text{indrat}_{\text{proet}})\).

**Proof.** Recall that any object of \(\text{Alg}_{uc}/k\) is killed by multiplication by some positive integer. Hence \(A_{\text{div}}\) is a torsion sheaf over \(\text{Spec } k^\text{indrat}_{\text{proet}}\). (Recall that this means \(A(k')\) is a torsion abelian group for any \(k' ∈ k^\text{indrat}\). See [AGV73, IX, Définition 1.1, Proposition 1.2(iii)].) Now the result follows from the previous proposition.

We denote \(A^\infty = \hat{\mathbb{Z}} \otimes \mathbb{Q} \in \text{IPFEt}/k\) (which is the adele ring of \(\mathbb{Q}\) without components of infinite places). We have \(A^\infty = V(\mathbb{Q}/\mathbb{Z})\). The assumption that \(A\) is divisibly ML in the above proposition cannot be dropped. For instance, applying \(R \lim_n\) to the exact sequence \(0 \to \bigoplus_p \mathbb{Z}/p\mathbb{Z} \to \mathbb{Q}/\mathbb{Z} \to \mathbb{Q}/\mathbb{Z} \to 0\), we know that \(R \lim_n \bigoplus_p \mathbb{Z}/p\mathbb{Z}\) is isomorphic to \([A^\infty \to A^\infty][-1]\), or \([\hat{\mathbb{Z}} \to \hat{\mathbb{Z}}][-1] \otimes \mathbb{Q}\), or \((\prod_p \mathbb{Z}/p\mathbb{Z})/(\prod_p \mathbb{Z}/p\mathbb{Z})[-1]\).

**Proposition 2.3.4.** Let \(A ∈ \text{FGEt}/k\). Then \(R \lim_{\leftarrow} A = A \otimes_{\mathbb{A}^\infty/\mathbb{Q}} \mathbb{A}^\infty/\mathbb{Q}[-1]\) in \(D(k^\text{indrat}_{\text{proet}})\).
Proof. We may assume that $A$ is torsion-free. We have an exact sequence $0 \to A \to A \otimes \mathbb{Q} \to A \otimes \mathbb{Q}/\mathbb{Z} \to 0$. Applying $R \lim_{\leftarrow n}$ and using the previous proposition, we obtain a distinguished triangle $R \lim_{\leftarrow n} A \to A \otimes \mathbb{Q} \to A \otimes A^\infty$. From this, the result follows.

For $A \in D(k_{proet}^{indrat})$, there is a canonical choice of a mapping cone of the above defined morphism $\lim_{\leftarrow n} A \to A$. We denote this cone by $\lim_{\leftarrow n} (A \otimes^L \mathbb{Z}/n\mathbb{Z})$. It is an object of $D(P \text{Ab}(k_{proet}^{indrat}))$. (This construction can also be understood more conceptually using the derived functor of the procategory extension of the two variable functor $\otimes$, at least when $A$ is assumed to be bounded above.)

**Proposition 2.3.5.** Let $A \in D(k_{proet}^{indrat})$. The distinguished triangle

$$\lim_{\leftarrow n} A \to A \to \lim_{\leftarrow n} (A \otimes^L \mathbb{Z}/n\mathbb{Z})$$

induces an exact sequence

$$0 \to \lim_{\leftarrow n} (H^i(A) \otimes \mathbb{Z}/n\mathbb{Z}) \to H^i(\lim_{\leftarrow n} (A \otimes^L \mathbb{Z}/n\mathbb{Z})) \to \lim_{\leftarrow n} H^{i+1}(A)[n] \to 0$$

in $P \text{Ab}(k_{proet}^{indrat})$ for any $i \in \mathbb{Z}$.

Proof. Obvious.

**Proposition 2.3.6.** If a morphism $A \to B$ in $D(k_{proet}^{indrat})$ induces an isomorphism

$$A \otimes^L \mathbb{Z}/n\mathbb{Z} \cong B \otimes^L \mathbb{Z}/n\mathbb{Z}$$

for any $n \geq 1$, then it also induces an isomorphism

$$\lim_{\leftarrow n} (A \otimes^L \mathbb{Z}/n\mathbb{Z}) \cong \lim_{\leftarrow n} (B \otimes^L \mathbb{Z}/n\mathbb{Z}).$$

Proof. This follows from the previous proposition.

If $A \in D(k_{proet}^{indrat})$, we can apply $R \lim_{\leftarrow n}$ to $\lim_{\leftarrow n} (A \otimes^L \mathbb{Z}/n\mathbb{Z})$. We denote the result by

$$R \lim_{\leftarrow n} (A \otimes^L \mathbb{Z}/n\mathbb{Z}) \in D(k_{proet}^{indrat}).$$

**Proposition 2.3.7.** For $A \in D(k_{proet}^{indrat})$, we have a distinguished triangle

$$R \lim_{\leftarrow n} A \to A \to R \lim_{\leftarrow n} (A \otimes^L \mathbb{Z}/n\mathbb{Z}).$$
If a morphism $A \to B$ in $D(k_{\text{proet}}^\text{indrat})$ induces an isomorphism

$$A \otimes L Z/nZ \sim B \otimes L Z/nZ$$

for any $n \geq 1$, then it also induces an isomorphism

$$R \lim \limits_n (A \otimes L Z/nZ) \sim R \lim \limits_n (B \otimes L Z/nZ).$$

Proof. This follows from the previous proposition. \qed

**Proposition 2.3.8.** Let $A \in D(k_{\text{proet}}^\text{indrat})$. Assume that

$$R \lim \limits_n (H^j(A) \otimes Z/nZ) = R \lim \limits_n (H^j(A)[n]) = 0$$

for any $i \geq 1$ and $j \in \mathbb{Z}$. Then we have an exact sequence

$$0 \to H^i(A)^{\wedge} \to H^iR \lim \limits_n (A \otimes L Z/nZ) \to TH^{i+1}(A) \to 0, $$

for all $i \in \mathbb{Z}$.

Proof. By Proposition 2.3.5, we know that applying $R \lim \limits_n$ to $H^j$ \textit{“}lim\textit{”} \textit{n} $(A \otimes L Z/nZ)$ results zero for any $i \geq 1$ and $j \in \mathbb{Z}$. Hence the $i$th cohomology of $R \lim \limits_n (A \otimes L Z/nZ)$ is given by applying \textit{lim} to the $i$th cohomology of \textit{“}lim\textit{”} \textit{n} $(A \otimes L Z/nZ)$ for any $i \in \mathbb{Z}$. Applying \textit{lim} to the exact sequence in Proposition 2.3.5, we get the required exact sequence. \qed

**Proposition 2.3.9.** For any $A \in D(k_{\text{proet}}^\text{indrat})$, the natural morphism

$$\left(R \lim \limits_n (A \otimes L Z/nZ)\right) \otimes Q \to R \lim \limits_n (A \otimes L Q/Z)$$

is an isomorphism. Denote these isomorphic objects by $A \hat{\otimes} A^\infty$. We have a natural distinguished triangle

$$R \lim \limits_n A \to A \otimes Q \to A \hat{\otimes} A^\infty.$$

The assignment $A \mapsto A \hat{\otimes} A^\infty$ is a triangulated endofunctor on $D(k_{\text{proet}}^\text{indrat})$ that sends $\mathbb{Z}$ to $A^\infty$. 
**Proof.** We have a distinguished triangle

\[ R \lim_{n} A \to A \to R \lim_{n} (A \otimes^L \mathbb{Z}/n\mathbb{Z}) \]

and hence a distinguished triangle

\[ R \lim_{n} A \to A \otimes Q \to \left( R \lim_{n} (A \otimes^L \mathbb{Z}/n\mathbb{Z}) \right) \otimes Q. \]

Also we have a distinguished triangle

\[ A \to A \otimes Q \to A \otimes^L \mathbb{Q}/\mathbb{Z} \]

and hence a distinguished triangle

\[ R \lim_{n} A \to A \otimes Q \to R \lim_{n} (A \otimes^L \mathbb{Q}/\mathbb{Z}). \]

Comparing the two triangles, we get the result.

The boundedness conditions in the following two propositions might be unnecessary.

**Proposition 2.3.10.** For \( A \in D^+(k_{\text{indrat}}^{\text{proet}}) \), there is a canonical isomorphism

\[ R \lim_{n} (A \otimes^L \mathbb{Z}/n\mathbb{Z}) = R \text{Hom}_k(\mathbb{Q}/\mathbb{Z}, A)[1]. \]

**Proof.** We have

\[ R \text{Hom}_k(\mathbb{Q}, A) = R \lim_{n} R \text{Hom}_k(\mathbb{Z}, A) = R \lim_{n} A \]

by the proof of [Suz14, Proposition (2.2.3)] (which needs the bounded below condition). Comparing the two distinguished triangles

\[ R \lim_{n} A \to A \to R \lim_{n} (A \otimes^L \mathbb{Z}/n\mathbb{Z}), \]

\[ R \text{Hom}_k(\mathbb{Q}, A) \to R \text{Hom}_k(\mathbb{Z}, A) \to R \text{Hom}_k(\mathbb{Q}/\mathbb{Z}, A)[1], \]

we get the result.

**Proposition 2.3.11.** For \( A \in D^-(k_{\text{indrat}}^{\text{proet}}) \) and \( B \in D^+(k_{\text{indrat}}^{\text{proet}}) \), there is a canonical isomorphism

\[ R \lim_{n}(R \text{Hom}_k(A, B) \otimes^L \mathbb{Z}/n\mathbb{Z}) = R \text{Hom}_k(A \otimes^L \mathbb{Q}/\mathbb{Z}, B)[1]. \]
Proof. We know that $R \mathbf{Hom}_k(A, B)$ is bounded below. By the previous proposition, the left-hand side can be written as

$$R \mathbf{Hom}_k(\mathbb{Q}/\mathbb{Z}, R \mathbf{Hom}_k(A, B))[1].$$

This is also the right-hand side by the derived tensor-Hom adjunction. □

2.4 Premorphisms of sites

For a site $S$, we denote its category of sheaves of sets (resp. abelian groups) by Set($S$) (resp. Ab($S$)). The category of complexes in Ab($S$) is Ch($S$), its homotopy category $K(S)$ and derived category $D(S)$. We say $A \in$ Ch($S$) is $K$-limp if $R\Gamma(X, A) = \Gamma(X, A)$ for any object $X \in S$, where $\Gamma$ in the right-hand side is applied termwise. See also [Spa88, Corollary 5.17] and [Sch17, Appendix A]. This is not a “K”-version of limp sheaf as defined in [Sta18, Tag 072Y] since $X$ is not allowed to be an arbitrary sheaf of sets but only a representable one. $K$-injectives are $K$-limp.

The following three propositions are essentially well known, at least for bounded below complexes of sheaves or just sheaves; see [Mil80, III, Sections 1, 2].

**Proposition 2.4.1.** Let $S$ be a site and $A, J \in$ Ch($S$) complexes such that $J$ is $K$-injective. Then the total complex of the sheaf-Hom double complex $\mathbf{Hom}_S(A, J)$ is $K$-limp.

**Proof.** Note that $\mathbf{Hom}_S(A, J) = R \mathbf{Hom}_S(A, J)$. By [KS06, Proposition 18.6.6], we have $R\Gamma(X, R \mathbf{Hom}_S(A, J)) = R \mathbf{Hom}_{S/X}(A, J)$ for any $X \in S$, where $S/X$ is the localization of $S$ at $X$. Hence

$$R\Gamma(X, \mathbf{Hom}_S(A, J)) = R \mathbf{Hom}_{S/X}(A, J)$$

$$= \mathbf{Hom}_{S/X}(A, J) = \Gamma(X, \mathbf{Hom}_S(A, J)).$$ □

Let $f^{-1}$ be a functor from the underlying category of a site $S$ to the underlying category of another site $S'$. The right composition with $f^{-1}$ defines a functor from the category of presheaves of sets on $S'$ to the category of presheaves of sets on $S$, which is the pushforward functor for presheaves. If this functor sends sheaves to sheaves, $f^{-1}$ is called a continuous functor in the terminology of [AGV72, III, Definition 1.1]. We say that $f^{-1}$ then defines a continuous map of sites $f : S' \to S$. The pushforward functor Set($S'$) $\to$ Set($S$) or Ab($S'$) $\to$ Ab($S$) is denoted by $f_*$. The left adjoint to $f_*$ : Set($S'$) $\to$ Set($S$) exists by [AGV72, III, Proposition 1.2], which we
denote by \( f^{*_{\text{set}}} : \text{Set}(S) \to \text{Set}(S') \). The left adjoint to \( f_* : \text{Ab}(S') \to \text{Ab}(S) \) exists by [AGV72, III, Proposition 1.7], which we denote by \( f^* : \text{Set}(S) \to \text{Set}(S') \). They are called pullback functors. The continuous map \( f : S' \to S \) is called a morphism of sites if the pullback \( f^{*_{\text{set}}} : \text{Set}(S) \to \text{Set}(S') \) for sheaves of sets is exact. In this case, the pullback for sheaves of abelian groups \( f^* : \text{Ab}(S) \to \text{Ab}(S') \) and \( f^{*_{\text{set}}} \) are compatible with forgetting group structures by [AGV72, III, Proposition 1.7, 4]), so we do not have to distinguish \( f^* \) and \( f^{*_{\text{set}}} \). In general, we use \( f^* \) to mean the pullback functor for sheaves of abelian groups.

**Proposition 2.4.2.** Let \( f : S' \to S \) be a continuous map of sites. For any \( K\)-limp complex \( A' \) in \( \text{Ab}(S') \), we have \( Rf_*A' = f_*A' \) in \( D(S) \).

**Proof.** Let \( A' \sim \sim I' \) be a \( K \)-injective replacement. Let \( B' \) be the mapping cone of \( A' \to I' \), which is an exact complex and hence \( Rf_*B' = 0 \) in \( D(S) \). For any \( X' \in S' \), consider the distinguished triangle \( \Gamma(X', A') \to \Gamma(X', I') \to \Gamma(X', B') \). In \( D(\text{Ab}) \), we have \( \Gamma(X', A') = R\Gamma(X', A') \) by assumption and \( \Gamma(X', I') = R\Gamma(X', I') \) by the \( K \)-injectivity of \( I' \). Hence \( \Gamma(X', B') = R\Gamma(X', B') \), which is zero in \( D(\text{Ab}) \) since \( B' \) is exact. This means that \( B' \) is also exact as a complex of presheaves. Hence \( f_* B' \) is exact. Consider the distinguished triangle \( f_*A' \to f_*I' \to f_*B' \). We have \( f_*I' = Rf_*I' \) and \( f_*B' = Rf_*B' = 0 \), so \( f_*A' = Rf_*A' \).

If \( S \) and \( S' \) are sites defined by pretopologies, and if \( f^{-1} \) is a functor from the underlying category of \( S \) to that of \( S' \) that sends coverings to coverings and \( f^{-1}(Y \times_X Z) = f^{-1}(Y) \times_{f^{-1}(X)} f^{-1}(Z) \) whenever \( Y \to X \) appears in a covering family, then \( f^{-1} \) is called a morphism of topologies in the terminology of [Art62, Definition 2.4.2] and defines a continuous map \( f : S' \to S \). We call such a continuous map a premorphism of sites. In this case, \( f_* \) sends acyclic sheaves (i.e., those \( A' \) with \( H^n(X', A') = 0 \) for any object \( X' \) of \( S' \) and \( n \geq 1 \)) to acyclic sheaves and hence induces the Leray spectral sequence \( R\Gamma(X, Rf_*A') = R\Gamma(f^{-1}(X), A') \) for any \( X \in S \) and \( A' \in \text{Ab}(S') \) [Art62, Section II.4]. Here is a slight generalization.

**Proposition 2.4.3.** Let \( f : S' \to S \) be a premorphism of sites defined by pretopologies. Then \( f_* \) sends \( K \)-limp complexes to \( K \)-limp complexes. For any \( X \in S \) and \( A' \in D(S') \), we have an isomorphism

\[
R\Gamma(X, Rf_*A') = R\Gamma(f^{-1}X, A'),
\]

where \( f^{-1} : S \to S' \) is the underlying functor of \( f \).
Proof. By [Suz13, Lemma 3.7.2], we know that $f^*: \text{Ab}(S) \to \text{Ab}(S')$ admits a left derived functor $Lf^*: D(S) \to D(S')$, which is left adjoint to $Rf_*: D(S') \to D(S)$ and sends the sheaf $\mathbb{Z}[X]$ of free abelian groups generated by any $X \in S$ to $\mathbb{Z}[f^{-1}X]$. For any $A' \in \text{Ch}(S')$ and $X \in S$, we have

$$R\Gamma(X, Rf_*A') = R\text{Hom}_S(\mathbb{Z}[X], Rf_*A') = R\text{Hom}_{S'}(Lf^*\mathbb{Z}[X], A')$$

$$= R\text{Hom}_{S'}(\mathbb{Z}[f^{-1}X], A') = R\Gamma(f^{-1}X, A').$$

If $A'$ is K-limp, then $Rf_*A' = f_*A'$ by Proposition 2.4.2 and $R\Gamma(f^{-1}X, A') = \Gamma(f^{-1}X, A')$. Hence $R\Gamma(X, f_*A') = \Gamma(f^{-1}X, A') = \Gamma(X, f_*A')$, and so $f_*A'$ is K-limp.

2.5 Local duality without relative sites

Let $\hat{K}_x$ be a complete discrete valuation field with perfect residue field $k_x \in \mathcal{U}_0$ of characteristic $p > 0$. The ring of integers is denoted by $\hat{\mathcal{O}}_x$ with maximal ideal $\mathfrak{p}_x$. By the fppf site $\text{Spec } \hat{\mathcal{O}}_x\text{,fppf}$ of $\hat{\mathcal{O}}_x$, we mean the category of $(\mathcal{U}_0\text{-small})$ $\hat{\mathcal{O}}_x$-algebras endowed with the fppf topology. The same applies to the fppf site $\text{Spec } \hat{K}_x\text{,fppf}$ of $\hat{K}_x$. Sheaves take values in Set or Ab (of $\mathcal{U}$-small sets or abelian groups). The Hom functor and the sheaf-Hom functor for $\text{Spec } \hat{\mathcal{O}}_x\text{,fppf}$ are denoted by $\text{Hom}_{\hat{\mathcal{O}}_x}$ and $\text{Hom}_{\hat{\mathcal{O}}_x}$, respectively. Their derived functors are denoted by $\text{Ext}^n_{\hat{\mathcal{O}}_x}$, $\text{Ext}^n_{\hat{\mathcal{O}}_x}$, $R\text{Hom}_{\hat{\mathcal{O}}_x}$, $R\text{Hom}_{\hat{\mathcal{O}}_x}$.

Similar notation applies to $\text{Spec } \hat{K}_x\text{,fppf}$. We denote $k_x^{\text{indrat}} = (k_x)^{\text{indrat}}$, $k_x^{\text{indrat}} = \text{Spec}(k_x)^{\text{indrat}}$, and $k_x^{\text{indrat}} = \text{Spec}(k_x)^{\text{indrat}}$.

The ring $\hat{\mathcal{O}}_x$ has a natural structure of a $W(k_x)$-algebra of profinite length, where $W$ denotes the ring of $p$-typical Witt vectors of infinite length. In equal characteristic, this structure factors through $k_x$ so that $\hat{\mathcal{O}}_x$ is profinite length over $k_x$. In mixed characteristic, it is finite free over $W(k_x)$. For $k'_x \in k_x^{\text{indrat}}$, we define

$$\hat{\mathcal{O}}_x(k'_x) = W(k'_x) \otimes W(k_x) \hat{\mathcal{O}}_x = \lim_{\leftarrow n}(W_n(k'_x) \otimes W_n(k_x) \hat{\mathcal{O}}_x/\mathfrak{p}_x^n),$$

$$\hat{K}_x(k'_x) = \hat{\mathcal{O}}_x(k'_x) \otimes_{\hat{\mathcal{O}}_x} \hat{K}_x.$$

Proposition 2.5.1. The functors $k'_x \mapsto \hat{\mathcal{O}}_x(k'_x), \hat{K}_x(k'_x)$ define premorphisms of sites

$$\pi_{\hat{\mathcal{O}}_x}: \text{Spec } \hat{\mathcal{O}}_x\text{,fppf} \to \text{Spec } k_x^{\text{indrat}}\text{,et},$$

$$\pi_{\hat{K}_x}: \text{Spec } \hat{K}_x\text{,fppf} \to \text{Spec } k_x^{\text{indrat}}\text{,et}.$$
Proof. From the proof of [Suz13, Proposition 2.3.1], we know that if $k'_x \to k''_x$ is a (faithfully flat) étale homomorphism in $k^{\text{indrat}}_x$, then $\hat{O}_x(k'_x) \to \hat{O}_x(k''_x)$ is (faithfully flat) étale, and if $k''''_x \in k^{\text{indrat}}_x$ is another object, then $\hat{O}_x(k''''_x) \otimes \hat{O}_x(k'_x) \hat{O}_x(k''_x)$ is isomorphic to $\hat{O}_x(k''''_x \otimes k'_x k''_x)$. Hence $\hat{\pi}_x$ is a premorphism of sites. So is $\hat{\pi}_x$. □

The scheme morphism $j : \text{Spec } \hat{K}_x \hookrightarrow \text{Spec } \hat{O}_x$ defines a morphism of sites

$$j : \text{Spec } \hat{K}_{x, \text{ff}} \to \text{Spec } \hat{O}_{x, \text{ff}}.$$ 

Its pullback functor $j^*$ is the restriction functor for sheaves from $\hat{O}_x$ to $\hat{K}_x$. This restriction functor is frequently omitted by abuse of notation, so a sheaf (or a complex of sheaves) $F$ on $\text{Spec } \hat{O}_{x, \text{ff}}$ restricted to $\text{Spec } \hat{K}_{x, \text{ff}}$ will frequently be written by just $F$. More generally, if $f : Z \to Y$ is a morphism of schemes, then the restriction $f^*F$ of a sheaf (or a complex of sheaves) $F$ on $Y_{\text{ff}}$ will frequently be written by just $F$. But the important point is that $f$ is a localization morphism [AGV72, III, Section 5]. Recall from [AGV72, III, Section 5] that for a site $S$ and its object $V$, the category $S/V$ of pairs $(W, g)$ ($W$ an object of $S$ and $g : W \to V$ a morphism in $S$) is equipped with the natural induced topology. The functor $j_V : (W, g) \mapsto W$ defines a continuous map of sites $S \to S/V$ whose pushforward functor $\text{Set}(S) \to \text{Set}(S/V)$ is the restriction functor. In our situation, by $f$ being a localization morphism, we mean that $f : Z \to Y$ belongs to the underlying category of $Y_{\text{ff}}$ and $Z_{\text{ff}}$ can be identified with $Y_{\text{ff}}/Z$. Hence $f^*$ admits an exact left adjoint $f_! : \text{Ab}(Z_{\text{ff}}) \to \text{Ab}(Y_{\text{ff}})$ by [AGV72, IV, Proposition 11.3.1]. Hence $f^*$ send $(K)$-injectives to $(K)$-injectives. This will be needed to pass from sheaf or complex level statements to derived categorical statements.

We have the pushforward functors

$$(\pi'_x)_* : \text{Ab}(\hat{O}_{x, \text{ff}}) \to \text{Ab}(k^{\text{indrat}}_{x,\text{et}}),$$

$$(\pi'_x)_* : \text{Ab}(\hat{K}_{x, \text{ff}}) \to \text{Ab}(k^{\text{indrat}}_{x,\text{et}})$$

by $\pi'_x$ and $\pi'_{\hat{K}_x}$. Let $\text{Ab}(k^{\text{indrat}}_{x,\text{et}}) \to \text{Ab}(k^{\text{indrat}}_{x,\text{proet}})$ be the pro-étale sheafification functor. The composite functors of these functors are denoted by

$$\Gamma(\hat{O}_x, \cdot) : \text{Ab}(\hat{O}_{x, \text{ff}}) \to \text{Ab}(k^{\text{indrat}}_{x,\text{proet}}),$$

$$\Gamma(\hat{K}_x, \cdot) : \text{Ab}(\hat{K}_{x, \text{ff}}) \to \text{Ab}(k^{\text{indrat}}_{x,\text{proet}}).$$
We have the right derived functors

\[ R\Gamma(\hat{\mathcal{O}}_x, \cdot) : D(\hat{\mathcal{O}}_{x, \text{fppf}}) \to D(\text{k}_{x, \text{proet}}), \]
\[ R\Gamma(\hat{K}_x, \cdot) : D(\hat{K}_{x, \text{fppf}}) \to D(\text{k}_{x, \text{proet}}). \]

Since sheafification is exact, \( R\Gamma(\hat{\mathcal{O}}_x, \cdot) \) is the composite of \( R(\pi_{\hat{\mathcal{O}}_x})_* \) and the pro-étale sheafification functor \( D(\text{k}_{x, \text{et}}) \to D(\text{k}_{x, \text{proet}}) \). The same is true for \( R\Gamma(\hat{K}_x, \cdot) \). We denote \( H^n(\hat{\mathcal{O}}_x, \cdot) = H^n R\Gamma(\hat{\mathcal{O}}_x, \cdot) \) and similarly \( H^n(\hat{K}_x, \cdot) \). By Proposition A.1, these functors are compatible with the functors \( R\tilde{\Gamma}(\hat{\mathcal{O}}_x, \cdot) \) and \( R\tilde{\Gamma}(\hat{K}_x, \cdot) \) in the notation of [Suz14, paragraph before Proposition (3.3.8)].

The pro-étale sheafification in general makes it hard to calculate the derived global section \( R\Gamma(k'_x, \text{proet}, \cdot) \) of the objects \( R\Gamma(\hat{\mathcal{O}}_x, G) \) and \( R\Gamma(\hat{K}_x, G) \) at each \( k'_x \in \text{k}_{x, \text{indrat}} \) for a general complex \( G \). In general, it is not clear whether the natural morphisms

\[ R\Gamma(\hat{\mathcal{O}}_x, G) \to R\Gamma(k'_x, \text{proet}, R\Gamma(\hat{\mathcal{O}}_x, G)) \]
\[ R\Gamma(\hat{K}_x, G) \to R\Gamma(k'_x, \text{proet}, R\Gamma(\hat{K}_x, G)) \]

are isomorphisms or not. The first (resp. second) morphism is an isomorphism if \( R\Gamma(\hat{\mathcal{O}}_x, G) \) (resp. \( R\Gamma(\hat{K}_x, G) \)) is P-acyclic in the sense of [Suz14, Section 2.4]. This condition is satisfied for \( G \) a smooth group scheme or a finite flat group scheme by [Suz14, Propositions (3.4.2), (3.4.3)]. This is why the notion of P-acyclicity is introduced in [Suz14, Section 2.4]. The following proposition shows we can eliminate P-acyclicity from [Suz14] to a certain extent.

**Proposition 2.5.2.** Let \( G \in D(\hat{\mathcal{O}}_{x, \text{fppf}}) \). For any \( w \)-contractible \( k'_x \in \text{k}_{x, \text{indrat}} \) ([BS15, Definition 2.4.1], which includes any algebraically closed field over \( k_x \)), we have

\[ R\Gamma(k'_x, \text{proet}, R\Gamma(\hat{\mathcal{O}}_x, G)) = R\Gamma(\hat{\mathcal{O}}_x(k'_x), G). \]

A similar relation holds with \( \hat{\mathcal{O}}_x \) replaced by \( \hat{K}_x \).

**Proof.** We have

\[ R\Gamma(k'_x, \text{et}, R(\pi_{\hat{\mathcal{O}}_x})_* G) = R\Gamma(\hat{\mathcal{O}}_x(k'_x), G) \]

by Proposition 2.4.3. Hence it is enough to show that \( R\Gamma(k'_x, \text{proet}, \tilde{F}) = R\Gamma(k'_x, \text{et}, F) \) for any \( F \in D(k_{x, \text{et}}) \), where \( \tilde{F} \) is the pro-étale sheafification.
of $F$. The section functor $\Gamma(k'_x, \cdot)$ is exact on $\text{Ab}(k'_{x, \text{proet}})$ and $\text{Ab}(k'_{x, \text{et}})$ since $k'_x$ is $w$-contractible. Hence it is enough to show that $\Gamma(k'_x, \widetilde{F}) = \Gamma(k'_x, F)$ for any $F \in \text{Ab}(k'_{x, \text{et}})$. Sheafification is given by applying the zeroth Čech cohomology presheaf functor $\check{H}^0$ twice [Mil80, III, Remark 2.2(c)]. Any pro-étale covering of a $w$-contractible scheme has a section by definition and the Čech complex for a covering with section is null-homotopic. Hence for any presheaf $F$ on Spec $k'_{x, \text{proet}}$, we have

$$\Gamma(k'_x, \check{H}^0 \check{H}^0 (F)) = \Gamma(k'_x, \check{H}^0 (F)) = \Gamma(k'_x, F).$$

We define a functor by the mapping fiber construction

$$\Gamma_x(\hat{O}_x, \cdot) := [\Gamma(\hat{O}_x, \cdot) \to \Gamma(\hat{K}_x, j^* \cdot)][-1]: \text{Ch}(\hat{O}_x, \text{fppf}) \to \text{Ch}(k'_{x, \text{proet}})$$

on the category of complexes of sheaves. This is an additive functor that commutes with the translation functors, that is, a functor of additive categories with translation in the terminology of [KS06, Definition 10.1.1]. It induces a functor on the homotopy categories by [KS06, Proposition 11.2.9]. We have its right derived functor

$$R\Gamma_x(\hat{O}_x, \cdot) = [R\Gamma(\hat{O}_x, \cdot) \to R\Gamma(\hat{K}_x, j^* \cdot)][-1]: D(\hat{O}_x, \text{fppf}) \to D(k'_{x, \text{proet}})$$

by [KS06, Theorem 14.3.1(vi)]. We set $H^n_x(\hat{O}_x, \cdot) = H^n R\Gamma_x(\hat{O}_x, \cdot)$. Note that $H^n_x \neq \Gamma_x$ in this definition. (It can be shown that they define the same right derived functors by the same method as [Suz14, Proposition (3.3.3)].) By Proposition A.2, this is compatible with the functor denoted by $R\Gamma_x(\hat{O}_x, \cdot)$ in [Suz14, paragraph before Proposition (3.3.8)]. We frequently omit the restriction functor $j^*$ from this type of formulas by abuse of notation.

Let $A, B \in \text{Ch}(\hat{O}_x, \text{fppf})$. For any $k'_x \in k'_{x, \text{indrat}}$, the functoriality of $\Gamma_x(\hat{O}_x, \cdot)$ gives a morphism

$$\text{Hom}_{\hat{O}_x(k'_x)}(A, B) \to \text{Hom}_{k'_{x, \text{proet}}/k'_x}(\Gamma_x(\hat{O}_x, A), \Gamma_x(\hat{O}_x, B))$$

in $\text{Ch}(\text{Ab})$, where $\text{Hom}_{k'_{x, \text{proet}}/k'_x}$ is the Hom functor for the category of sheaves on the localization of Spec $k'_{x, \text{proet}}$ at $k'_x$. This is functorial on $k'_x$, so we have a morphism

$$\Gamma(\hat{O}_x, \text{Hom}_{\hat{O}_x}(A, B)) \to \text{Hom}_{k'_{x, \text{proet}}}(\Gamma_x(\hat{O}_x, A), \Gamma_x(\hat{O}_x, B)).$$
in $\text{Ch}(k_{\text{indrat},\text{proet}})$. The right derived functor of the left-hand side (as a functor on $A, B$) is

$$R\Gamma(\hat{O}_x, R\text{Hom}_{\hat{O}_x}(A, B))$$

by Propositions 2.4.1, 2.4.2 and the theorem on derived functors of composition [KS06, Proposition 10.3.5(ii)]. Hence we have a morphism

$$(2.5.2) \quad R\Gamma(\hat{O}_x, R\text{Hom}_{\hat{O}_x}(A, B)) \to R\text{Hom}_{k_{\text{indrat},\text{proet}}} (R\Gamma_x(\hat{O}_x, A), R\Gamma_x(\hat{O}_x, B))$$

in $D(k_{\text{indrat},\text{proet}})$, functorial on $A, B \in D(\hat{O}_x,\text{fppf})$, by the universal property of the right derived functor. (Note that the right-hand side of this is not the right derived functor of the right-hand side of (2.5.1). The problem is that $\pi_{\hat{O}_x}$ is only a premorphism of sites and hence its pushforward functor might not send (K-)injectives to (K-)injectives. Sheafification also might not send (K-)injectives to (K-)injectives.) Similarly, we have a morphism

$$(2.5.3) \quad \Gamma_x(\hat{O}_x, \text{Hom}_{\hat{O}_x}(A, B)) \to \text{Hom}_{k_{\text{indrat},\text{proet}}} (\Gamma(\hat{O}_x, A), \Gamma(\hat{O}_x, B))$$

in $\text{Ch}(k_{\text{indrat},\text{proet}})$, functorial on $A, B \in \text{Ch}(\hat{O}_x,\text{fppf})$. Deriving, we have a morphism

$$(2.5.4) \quad R\Gamma_x(\hat{O}_x, R\text{Hom}_{\hat{O}_x}(A, B)) \to R\text{Hom}_{k_{\text{indrat},\text{proet}}} (R\Gamma(\hat{O}_x, A), R\Gamma(\hat{O}_x, B))$$

in $D(k_{\text{indrat},\text{proet}})$, functorial on $A, B \in D(\hat{O}_x,\text{fppf})$. Also we have a morphism

$$(2.5.5) \quad \Gamma(\hat{K}_x, \text{Hom}_{\hat{K}_x}(A, B)) \to \text{Hom}_{k_{\text{indrat},\text{proet}}} (\Gamma(\hat{K}_x, A), \Gamma(\hat{K}_x, B))$$

in $\text{Ch}(k_{\text{indrat},\text{proet}})$, functorial on $A, B \in \text{Ch}(\hat{K}_x,\text{fppf})$. Deriving, we have a morphism

$$(2.5.6) \quad R\Gamma(\hat{K}_x, R\text{Hom}_{\hat{K}_x}(A, B)) \to R\text{Hom}_{k_{\text{indrat},\text{proet}}} (R\Gamma(\hat{K}_x, A), R\Gamma(\hat{K}_x, B))$$

in $D(k_{\text{indrat},\text{proet}})$, functorial on $A, B \in D(\hat{K}_x,\text{fppf})$. These morphisms are compatible with the morphisms in [Suz14, Proposition (3.3.8)] by Proposition A.3.

With the comparison results in Appendix A, we can translate the results of [Suz14] to our setting. In particular, by [Suz14, Propositions (3.4.2)(a), (3.4.3)(c); Section 4.1], any term in the localization triangle

$$R\Gamma(\hat{O}_x, G_m) \to R\Gamma(\hat{K}_x, G_m) \to R\Gamma_x(\hat{O}_x, G_m)[1]$$
is concentrated in degree zero, where we have an exact sequence

\[ 0 \to \hat{\mathcal{O}}_x^\times \to \hat{\mathcal{K}}_x^\times \to \mathbb{Z} \to 0. \]

We call the (iso)morphisms

(2.5.7) \[ R\Gamma(\hat{\mathcal{K}}_x, \mathbb{G}_m) \to R\Gamma(\hat{\mathcal{O}}_x, \mathbb{G}_m)[1] = \mathbb{Z} \]

the trace (iso)morphisms (at \( x \)).

Let \( A \) be an abelian variety over \( \hat{\mathcal{K}}_x \) with dual \( A^\vee \). By the Barsotti–Weil formula [Oor66, Chapter III, Theorem (18.1)], we have a canonical isomorphism \( A^\vee \cong \mathbf{Ext}^1_{\hat{\mathcal{K}}_x}(A, \mathbb{G}_m) \) defined by the Poincaré bundle. With \( \text{Hom}_{\hat{\mathcal{K}}_x}(A, \mathbb{G}_m) = 0 \), we have a canonical morphism \( A^\vee \to R\text{Hom}_{\hat{\mathcal{K}}_x}(A, \mathbb{G}_m)[1] \) in \( D(\hat{\mathcal{K}}_x, \text{fppf}) \). Hence we have a morphism

\[ R\Gamma(\hat{\mathcal{K}}_x, A^\vee) \to R\Gamma(\hat{\mathcal{K}}_x, R\text{Hom}_{\hat{\mathcal{K}}_x}(A, \mathbb{G}_m))[1] \]

\[ \to R\text{Hom}_{k^{\text{indrat}}, \text{proet}}(R\Gamma(\hat{\mathcal{K}}_x, A), R\Gamma(\hat{\mathcal{K}}_x, \mathbb{G}_m))[1] \]

\[ \to R\text{Hom}_{k^{\text{indrat}}, \text{proet}}(R\Gamma(\hat{\mathcal{K}}_x, A), \mathbb{Z})[1] = R\Gamma(\hat{\mathcal{K}}_x, A)_{\text{SD}_x}[1] \]

by (2.5.6) and (2.5.7), where \( \text{SD}_x = R\text{Hom}_{k^{\text{indrat}}, \text{proet}}(\cdot, \mathbb{Z}) \). Its Serre dual, when \( A \) is replaced by \( A^\vee \), is denoted by

\[ \theta_A : R\Gamma(\hat{\mathcal{K}}_x, A^\vee)_{\text{SD}_x} \to R\Gamma(\hat{\mathcal{K}}_x, A)_{\text{SD}_x}[1]. \]

These morphisms agree with the morphisms defined in [Suz14, Section 4.1]. Hence [Suz14, Theorem (4.1.2)] implies that \( \theta_A \) is an isomorphism.

**Proposition 2.5.3.** Let \( A \) be an abelian variety over \( \hat{\mathcal{K}}_x \) with Néron model \( A \). Let \( A_0 \) be the maximal open subgroup scheme of \( A \) with connected fibers and \( A_x \) the special fiber of \( A \) over \( x = \text{Spec} \ k_x \). Then \( R\Gamma(\hat{\mathcal{O}}_x, A) \) is concentrated in degree 0; \( R\Gamma(\hat{\mathcal{O}}_x, A_0) \) in degree 0; \( R\Gamma_x(\hat{\mathcal{O}}_x, A) \) in degree 2; and \( R\Gamma_x(\hat{\mathcal{O}}_x, A_0) \) in degrees 1, 2. We have

\[ \Gamma(\hat{\mathcal{O}}_x, A) = \Gamma(\hat{\mathcal{K}}_x, A) \in \text{PAlg}/k_x, \quad \Gamma(\hat{\mathcal{O}}_x, A_0) = \Gamma(\hat{\mathcal{K}}_x, A)_0, \]

\[ H^1_{\text{et}}(\hat{\mathcal{O}}_x, A_0) = \pi_0(\mathcal{A}_x) \in \text{FEt}/k_x, \]

\[ H^2_{\text{et}}(\hat{\mathcal{O}}_x, A) = H^2_{\text{et}}(\hat{\mathcal{O}}_x, A_0) = H^1(\hat{\mathcal{K}}_x, A) \in \text{IAlg}_{\text{uc}}/k_x. \]

The isomorphic groups in the third line are divisible.
Proof. The group $H^1(\hat{K}_x, A)$ is divisible since $H^2(\hat{K}_x, A[n]) = 0$ for any $n \geq 1$ by [Suz14, Proposition (3.4.3)(b)]. The statements about $R\Gamma(\hat{O}_x, A)$, $R\Gamma(\hat{O}_x, A_0)$ follow from [Suz14, Propositions (3.4.2)(a), (3.4.3)(d)]. These propositions, the localization triangle $R\Gamma_x(\hat{O}_x, A) \to R\Gamma(\hat{O}_x, A) \to R\Gamma(\hat{K}_x, A)$ and the similar triangle for $A_0$ imply the rest of the statements.

We recall [Suz14, Remark (4.2.10)]. Let $A$ and $A^\nu$ be the Néron models of $A$ and $A^\nu$, respectively. Let $A_0^\nu$ be the maximal open subgroup scheme of $A^\nu$ with connected fibers. The Poincaré biextension $A^\nu \otimes^L A \to G_m[1]$ as a morphism in $D(\hat{K}_x, fppf)$ canonically extends to a biextension $A_0^\nu \otimes^L A \to G_m[1]$ as a morphism in $D(\hat{O}_x, fppf)$ by [Gro72, IX, 1.4.3]. Hence we have a morphism $A_0^\nu \to R\text{Hom}_{k_{\text{proj}}} A, G_m[1])$. With the functoriality morphism (2.5.3) and the trace isomorphism (2.5.7), we have morphisms

$$R\Gamma_x(\hat{O}_x, A)_{SD_x} \to R\text{Hom}_{k_{\text{proj}}} \left( R\Gamma(\hat{O}_x, A), R\Gamma_x(\hat{O}_x, G_m[1]) \right)$$

$$= R\text{Hom}_{k_{\text{proj}}} \left( R\Gamma(\hat{O}_x, A), \mathbb{Z} \right) = R\Gamma(\hat{O}_x, A)_{SD_x}.$$

**Proposition 2.5.4.** The diagram

$$
\begin{array}{ccc}
R\Gamma(\hat{O}_x, A)_{SD_x} & \longrightarrow & R\Gamma(\hat{K}_x, A)_{SD_x} \\
\downarrow & & \downarrow \\
R\Gamma_x(\hat{O}_x, A)_{SD_x} & \longrightarrow & R\Gamma_x(\hat{K}_x, A)_{SD_x}[1]
\end{array}
$$

is a morphism of distinguished triangles. Applying $SD_xSD_x$, the induced diagram

$$
\begin{array}{ccc}
R\Gamma(\hat{O}_x, A_{SD_x}SD_x) & \longrightarrow & R\Gamma(\hat{K}_x, A_{SD_x}SD_x) \\
\downarrow & & \downarrow \theta_{A} \\
R\Gamma_x(\hat{O}_x, A_{SD_x}) & \longrightarrow & R\Gamma_x(\hat{K}_x, A_{SD_x}[1])
\end{array}
$$

is an isomorphism of distinguished triangles.

Proof. By [Suz14, Proposition (3.3.8)], we know that the first diagram is a morphism of distinguished triangles. The terms in the first diagram can be identified with the terms in the first diagram of [Suz14, Proposition (4.2.3)]. The uniqueness stated in [Suz14, Proposition (4.2.3)] shows that
the morphisms in the first diagram here and the morphisms in the first diagram of [Suz14, Proposition (4.2.3)] are equal. Hence the second diagram is an isomorphism of distinguished triangles by [Suz14, Proposition (4.2.7), Theorem (4.1.2)].

Assume that $k_x$ is a finite extension of another perfect field $k$. We have a finite étale morphism $f_x: \text{Spec } k_x \to \text{Spec } k$. This induces a morphism of sites

$$f_x: \text{Spec } k_{x, \text{proet}} \to \text{Spec } k_{\text{proet}}$$

Its pushforward functor is the Weil restriction functor $\text{Res}_{k_x/k}$, which is exact [BS15, Lemma 6.1.17]. We denote the composites of $\Gamma(\hat{O}_x, \cdot)$, $R\Gamma(\hat{O}_x, \cdot)$ and $(f_x)_* = \text{Res}_{k_x/k}$ by

$$\Gamma(\hat{O}_x/k, \cdot): \text{Ab}(\hat{O}_{x, \text{fppf}}) \to \text{Ab}(k_{\text{proet}}),$$

$$R\Gamma(\hat{O}_x/k, \cdot): D(\hat{O}_{x, \text{fppf}}) \to D(k_{\text{proet}}),$$

respectively, with cohomology $H^n(\hat{O}_x/k, \cdot) = H^n R\Gamma(\hat{O}_x/k, \cdot)$. Similar functors

$$\Gamma(\hat{K}_x/k, \cdot): \text{Ab}(\hat{K}_{x, \text{fppf}}) \to \text{Ab}(k_{\text{proet}}),$$

$$\Gamma_x(\hat{O}_x/k, \cdot): \text{Ch}(\hat{O}_{x, \text{fppf}}) \to \text{Ch}(k_{\text{proet}})$$

and their derived functor are defined. We have

$$(f_x)_* R\text{Hom}_{k_{x, \text{proet}}} (\cdot, \mathbb{Z}) = R\text{Hom}_{k_{\text{proet}}} ((f_x)_*(\cdot), \mathbb{Z})$$

by the duality for finite étale morphisms [Mil80, V, Proposition 1.13] (which is for the étale topology; the same proof works for the pro-étale topology).

In other words, we have

$$\text{Res}_{k_x/k} \circ \text{SD} = \text{SD} \circ \text{Res}_{k_x/k},$$

where $\text{SD} = R\text{Hom}_{k_{\text{proet}}} (\cdot, \mathbb{Z})$ as before. Hence applying $\text{Res}_{k_x/k}$ to the duality statements above over $k_x$ defines some new duality statements over $k$. Doing this for $\theta_A$ for instance defines a new isomorphism

$$R\Gamma(\hat{K}_x/k, A^\vee)^{\text{SD} \text{ SD}} \cong R\Gamma(\hat{K}_x/k, A)^{\text{SD}}$$

in $D(k_{\text{proet}})$. (Here we are not using any specific property of our duality isomorphism. We are using nothing but the obvious fact that if a morphism
C \to D$ in $D(k^{\text{indrat}}_{x,\text{proet}})$ is an isomorphism, then the induced morphism $\text{Res}_{k_x/k} C \to \text{Res}_{k_x/k} D$ is an isomorphism.) More explicitly, this morphism comes from the morphisms

$$R\Gamma(\hat{K}_x/k, A^V) \to R\Gamma(\hat{K}_x/k, R\text{Hom}_{\hat{K}_x}(A, G_m))[1]$$
$$\to R\text{Hom}_{k^{\text{indrat}}_{\text{proet}}}(R\Gamma(\hat{K}_x/k, A), R\Gamma(\hat{K}_x/k, G_m))[1]$$
$$\to R\text{Hom}_{k^{\text{indrat}}_{\text{proet}}}(R\Gamma(\hat{K}_x/k, A), \mathbb{Z})[1] = R\Gamma(\hat{K}_x/k, A)^{\text{SD}}[1]$$

The trace morphism in this situation used here is

$$(2.5.8) R\Gamma(\hat{K}_x/k, G_m) \to R\Gamma(\hat{O}_x/k, G_m)[1] = \text{Res}_{k_x/k} \hat{K}_x^\times/\hat{O}_x^\times = \text{Res}_{k_x/k} \mathbb{Z} \to \mathbb{Z},$$

where the last morphism is

$$\mathbb{Z}[\text{Hom}_k(k_x, \overline{k})] \cong \mathbb{Z}[k_x:k] \xrightarrow{\text{sum}} \mathbb{Z}$$
on geometric points.

### 2.6 Henselizations and completions

Let $\mathcal{O}_x^h$ be an excellent Henselian discrete valuation ring of equal characteristic $p > 0$ with perfect residue field $k_x (\in U_0)$. We denote the maximal ideal by $p_x^h$ and the fraction field by $K_x^h$. The corresponding objects after completion are denoted by $\hat{\mathcal{O}}_x, \hat{p}_x$ and $\hat{K}_x$.

We make a variant of the constructions in the previous subsection using Henselizations instead of completions. For $k'_x \in k^{\text{indrat}}_x$, the Henselization of the (nonlocal) ring $k'_x \otimes_{k_x} \mathcal{O}_x^h$ at the ideal $k'_x \otimes_{k_x} p_x^h$ [Ray70b, Chapter XI, Section 2] is denoted by $(k'_x \otimes_{k_x} \mathcal{O}_x^h)^h$. We define

$$\mathcal{O}_x^h(k'_x) = (k'_x \otimes_{k_x} \mathcal{O}_x^h)^h, \quad K_x^h(k'_x) = \mathcal{O}_x^h(k'_x) \otimes_{\mathcal{O}_x^h} K_x^h.$$

The functors $\mathcal{O}_x^h$ and $K_x^h$ define premorphisms of sites

$$\pi_{\mathcal{O}_x^h} : \text{Spec} \mathcal{O}_{x,\text{fppf}}^h \to \text{Spec} k^{\text{indrat}}_{x,\text{proet}}, \quad \pi_{K_x^h} : \text{Spec} K_{x,\text{fppf}}^h \to \text{Spec} k^{\text{indrat}}_{x,\text{et}}.$$

The pro-étale sheafifications of their pushforward functors are denoted by

$$\Gamma(\mathcal{O}_x^h, \cdot) : \text{Ab}(\mathcal{O}_{x,\text{fppf}}^h) \to \text{Ab}(k^{\text{indrat}}_{x,\text{proet}}),$$
$$\Gamma(K_x^h, \cdot) : \text{Ab}(K_{x,\text{fppf}}^h) \to \text{Ab}(k^{\text{indrat}}_{x,\text{proet}}).$$
We set
\[ \Gamma_x(O^h_x, \cdot) = [\Gamma(O^h_x, \cdot), \rightarrow \Gamma(K^h_x, j^* \cdot)][-1]: \]
\[ \text{Ch}(O^h_{x,\text{fpf}}) \rightarrow \text{Ch}(k^\text{indrat}_x, \text{proet}), \]
where \( j: \text{Spec } K^h_x \rightarrow \text{Spec } O^h_x \) is the natural morphism inducing a morphism on the fppf sites. We have their right derived functors
\[ R \Gamma(O^h_x, \cdot): D(O^h_{x,\text{fpf}}) \rightarrow D(k^\text{indrat}_x, \text{proet}), \]
\[ R \Gamma(K^h_x, \cdot): D(K^h_{x,\text{fpf}}) \rightarrow D(k^\text{indrat}_x, \text{proet}), \]
\[ R \Gamma_x(O^h_x, \cdot) = [R \Gamma(O^h_x, \cdot) \rightarrow R \Gamma(K^h_x, j^* \cdot)][-1]: \]
\[ D(O^h_{x,\text{fpf}}) \rightarrow D(k^\text{indrat}_x, \text{proet}). \]

Again, we frequently omit the restriction functor \( j^* \) by abuse of notation.

We say that a sheaf \( F \in \text{Set}(O^h_{x,\text{fpf}}) \) is locally of finite presentation if it commutes with filtered direct limits as a functor on the category of \( O^h_x \)-algebras.

**Proposition 2.6.1.** For any sheaf \( A \in \text{Ab}(O^h_{x,\text{fpf}}) \) locally of finite presentation and \( n \geq 0 \), the sheaf \( R^n(\pi_{O^h_x})_* A \in \text{Ab}(k^\text{indrat}_x, \text{et}) \) is locally of finite presentation. In particular, we have \( H^n(O^h_x, A) = R^n(\pi_{O^h_x})_* A \). That is, \( H^n(O^h_x, A) \) is the étale sheafification of the presheaf
\[ k'_x \mapsto H^n(O^h_x(k'_x), A). \]
A similar statement holds when \( O^h_x \) is replaced by \( K^h_x \).

**Proof.** The sheaf \( O^h_x \) is locally of finite presentation by the construction of Henselization. Let \( A \in \text{Ab}(O^h_{x,\text{fpf}}) \) be locally of finite presentation. Let \( k'_x = \bigcup k'_{x,\lambda} \in k^\text{indrat}_x \) with \( k'_{x,\lambda} \in k^\text{rat}_x \). Then for any \( n \geq 0 \), we have
\[ H^n(O^h_x(k'_x), A) = \lim_{\lambda} H^n(O^h_x(k'_{x,\lambda}), A). \]
By sheafification, we know that \( R^n(\pi_{O^h_x})_* A \) is locally of finite presentation. The same proof works for \( K^h_x \).
where $f: \text{Spec} \, \hat{O}_x \to \text{Spec} \, \mathcal{O}_x^h$ is the natural scheme morphism inducing a morphism on the fppf sites. We call its right derived functor

$$Ro_x(\mathcal{O}_x^h, \cdot) = [R\Gamma_x(\mathcal{O}_x^h, \cdot) \to R\Gamma_x(\hat{O}_x, f^* \cdot)]$$

the obstruction for cohomological approximation (at $x$). Define $D(\mathcal{O}_x^h, fppf)_{ca}$ to be the kernel of the functor $Ro_x(\mathcal{O}_x^h, \cdot)$, that is, the full subcategory of $D(\mathcal{O}_x^h, fppf)$ consisting of objects $A$ with $Ro_x(\mathcal{O}_x^h, A) = 0$, or

$$R\Gamma_x(\mathcal{O}_x^h, A) = R\Gamma_x(\hat{O}_x, f^* A).$$

Such an object $A$ is said to satisfy cohomological approximation.

**Proposition 2.6.2.** Any smooth group scheme or finite flat group scheme over $\mathcal{O}_x^h$ satisfies cohomological approximation.

**Proof.** Any finite flat group scheme $N$ is a closed subgroup scheme of some smooth affine group scheme $G$ by [Bég81, Proposition 2.2.1]. The fppf quotient $H = G/N$ is a smooth affine group scheme by descent. We have a distinguished triangle $Ro_x(\mathcal{O}_x^h, N) \to Ro_x(\mathcal{O}_x^h, G) \to Ro_x(\mathcal{O}_x^h, H)$. If two of the terms are zero, then so is the other. Hence the finite flat case is reduced to the smooth case.

Assume that $A$ is a smooth group scheme over $\mathcal{O}_x^h$. Let $N \subset A$ be the schematic closure of the identity section of $A \times_{\mathcal{O}_x^h} \hat{K}_x$. Then $N$ is an étale group scheme over $\mathcal{O}_x^h$ with trivial generic fiber and $A/N$ is a separated smooth group scheme over $\mathcal{O}_x^h$ by [Ray70a, Proposition 3.3.5]. The both objects $R\Gamma_x(\mathcal{O}_x^h, N)$ and $R\Gamma_x(\hat{O}_x, N)$ are isomorphic to $N \times_{\mathcal{O}_x^h} k_x$.

Hence we may assume that $A$ is smooth separated. Then we have $H^0_x(\hat{O}_x, A) = 0$. Also $H^n(\hat{O}_x, A) = 0$ for $n \geq 1$ by [Suz14, Proposition (3.4.2)(a)]. Hence

$$H^1_x(\hat{O}_x, A) = \Gamma(\hat{K}_x, A)/\Gamma(\hat{O}_x, A),$$

$$H^n_x(\hat{O}_x, A) = H^{n-1}(\hat{K}_x, A)$$

for $n \geq 2$. These sheaves are locally of finite presentation (even before pro-étale sheafification) by the Greenberg approximation argument [Suz14, Propositions (3.2.8), (3.2.9)]. On the other hand, the sheaves $H^n(\mathcal{O}_x^h, A)$, $H^n(\hat{K}_x, A)$ and thus $H^n_x(\mathcal{O}_x^h, A)$ are locally of finite presentation for $n \geq 0$ by the previous proposition. Therefore it is enough to show that the morphism

$$R\Gamma_x(\mathcal{O}_x^h, A) \to R\Gamma_x(\hat{O}_x, A)$$
is an isomorphism when $k$ is algebraically closed by [Suz14, the second paragraph after Proposition (2.4.1)]. The statements to prove are

\[
\Gamma(K^h_x, A)/\Gamma(\hat{K}^h_x, A) = \Gamma(\hat{O}^h_x, A)/\Gamma(\hat{O}_x, A),
\]

\[
H^n(K^h_x, A) = H^n(\hat{K}^h_x, A), \quad n \geq 1.
\]

They can be proven in the same way as the Greenberg approximation argument [Suz14, Propositions (3.2.8), (3.2.9)] (or equivalently, by the same argument as [Mil06, I, Remark 3.10] combined with [Čes15, Proposition 3.5(b)]). \qed

Perhaps any group scheme locally of finite type over $\mathcal{O}^h_x$ might satisfy cohomological approximation since the Greenberg approximation itself holds in this generality. We do not pursue this point. But see [DH18, Lemma 2.6, Remark 2.7] for a related result.

If $k_x$ is a finite extension of another perfect field $k$ and $f: \text{Spec } k_x \to \text{Spec } k$ is the natural morphism, then the composite of $\Gamma(\mathcal{O}^h_x, \cdot)$ and $f_*$ is denoted by

\[
\Gamma(\mathcal{O}^h_x/k, \cdot): \text{Ch}(\mathcal{O}^h_x,\text{fppf}) \to \text{Ch}(k_{\text{proet}}^\text{indrat})
\]

with right derived functor

\[
R\Gamma(\mathcal{O}^h_x/k, \cdot): D(\mathcal{O}^h_x,\text{fppf}) \to D(k_{\text{proet}}^\text{indrat}).
\]

Similar notation applies to other objects, defining $\Gamma(K^h_x/k, \cdot)$, $\Gamma_x(\mathcal{O}^h_x/k, \cdot)$, $o_x(\mathcal{O}^h_x/k, \cdot)$ and their derived functors.

2.7 The fppf site of a curve over the rational étale site of the base

Let $U$ be a smooth geometrically connected curve over a perfect field $k \in U_0$ of characteristic $p > 0$, with smooth compactification $X$ and function field $K$. By the fppf site $U_{\text{fppf}}$ of $U$, we mean the category of ($U_0$-small) $U$-schemes endowed with the fpff topology. For $k' \in k_{\text{indrat}}$, we denote $U_{k'} = U \times_k k'$. The functor sending $k' \in k_{\text{indrat}}$ to $U_{k'}$ defines a premorphism of sites

\[
\pi_U: U_{\text{fppf}} \to \text{Spec } k_{\text{et}}^\text{indrat}.
\]

We define a left exact functor

\[
\Gamma(U, \cdot): \text{Ab}(U_{\text{fppf}}) \to \text{Ab}(k_{\text{proet}}^\text{indrat})
\]
by the composite of the pushforward functor $\pi_{U*}$ and the pro-étale sheafification. We have its right derived functor

$$R\Gamma(U, \cdot): D(U_{\text{fppf}}) \to D(k_{\text{proet}}^{\text{indrat}}),$$

with cohomologies $H^n = H^n R\Gamma$. For any $A \in \text{Ab}(U_{\text{fppf}})$ and $n \geq 0$, the sheaf $H^n(U, A)$ on $\text{Spec} k_{\text{proet}}^{\text{indrat}}$ is the pro-étale sheafification of the presheaf

$$k' \mapsto H^n(U_{k'}, A).$$

Let $\text{Spec} k_{\text{et}}^{\text{perf}}$ be the category of $(U_0$-small) perfect $k$-schemes endowed with the étale topology. The structure morphism $U \to \text{Spec} k$ induces a morphism of sites

$$\pi_U^{\text{perf}}: U_{\text{fppf}} \to \text{Spec} k_{\text{et}}^{\text{perf}},$$

which is defined by the functor that sends a perfect $k$-scheme $S$ to $U \times_k S$. In [AM76, (3.1)], this is denoted by $\pi^{f,p}$.

**Proposition 2.7.1.** Let

$$f: \text{Spec} k_{\text{et}}^{\text{perf}} \to \text{Spec} k_{\text{et}}^{\text{indrat}}$$

be the premorphism of sites defined by the identity functor. Then we have $\pi_U = f \circ \pi_U^{\text{perf}}$. The functor $f_*$ is exact. The functors $\Gamma(U, \cdot)$ and $R\Gamma(U, \cdot)$ are the pro-étale sheafifications of $f_*\pi_U^{\text{perf}}$ and $f_*R\pi_U^{\text{perf}}$, respectively.

**Proof.** Obvious. □

In this sense, our constructions are pro-étale sheafifications of restrictions (from all perfect schemes to only ind-rational algebras) of the constructions in [AM76]. In the next section, we will translate results in [AM76] to our setting in this way.

For a closed point $x \in X$, let $k_x$, $\mathcal{O}_x^h$, $\hat{O}_x$, $K_x^h$, $\hat{K}_x$ be the residue field, the Henselian local ring, the completed local ring, their fraction fields, respectively, at $x$. The results and notation in the previous two subsections apply to $\hat{O}_x$ and $\mathcal{O}_x^h$.

Assume $x \in U$. For any $k' \in k_{\text{indrat}}$, we have $\mathcal{O}_x^h(k' \otimes_k k_x) = (k' \otimes_k \mathcal{O}_x^h)^h$. Hence the morphism $\text{Spec} \mathcal{O}_x^h \to U$ induces a morphism $\text{Spec} \mathcal{O}_x^h(k' \otimes_k k_x) \to U_{k'}$. This induces a homomorphism

$$\Gamma(U_{k'}, A) \to \Gamma(\mathcal{O}_x^h(k' \otimes_k k_x), A).$$
for any $A \in \text{Ab}(U_{\text{fppf}})$. Thus we have a morphism
\[ \Gamma(U, \cdot) \to \Gamma(\mathcal{O}_x^h/k, \cdot) \]
of left exact functors $\text{Ab}(U_{\text{fppf}}) \to \text{Ab}(k_{\text{proet}}^{\text{indrat}})$ and a morphism
\[ R\Gamma(U, \cdot) \to R\Gamma(\mathcal{O}_x^h/k, \cdot) \]
of triangulated functors $D(U_{\text{fppf}}) \to D(k_{\text{proet}}^{\text{indrat}})$.

Assume $x \not\in U$. For any $k' \in k_{\text{proet}}^{\text{indrat}}$, we similarly have a morphism $\text{Spec } K^h_x(k' \otimes_k k_x) \to U_{k'}$. This induces a homomorphism
\[ \Gamma(U_{k'}, A) \to \Gamma(K^h_x(k' \otimes_k k_x), A) \]
for any $A \in \text{Ab}(U_{\text{fppf}})$. Thus we have a morphism
\[ \Gamma(U, \cdot) \to \Gamma(K^h_x/k, \cdot) \]
of left exact functors $\text{Ab}(U_{\text{fppf}}) \to \text{Ab}(k_{\text{proet}}^{\text{indrat}})$ and a morphism
\[ R\Gamma(U, \cdot) \to R\Gamma(K^h_x/k, \cdot) \]
of triangulated functors $D(U_{\text{fppf}}) \to D(k_{\text{proet}}^{\text{indrat}})$.

For a dense open subscheme $V \subset U$, we have a morphism
\[ \Gamma(U, \cdot) \to \Gamma(V, \cdot) \]
of left exact functors $\text{Ab}(U_{\text{fppf}}) \to \text{Ab}(k_{\text{proet}}^{\text{indrat}})$ and a morphism
\[ R\Gamma(U, \cdot) \to R\Gamma(V, \cdot) \]
of triangulated functors $D(U_{\text{fppf}}) \to D(k_{\text{proet}}^{\text{indrat}})$. Let $Z = U \setminus V$, which is a finite set of closed points of $X$. We define a functor of additive categories with translation by
\[ \Gamma_Z(U, \cdot) = [\Gamma(U, \cdot) \to \Gamma(V, \cdot)][-1]: \text{Ch}(U_{\text{fppf}}) \to \text{Ch}(k_{\text{proet}}^{\text{indrat}}). \]
We have its right derived functor
\[ R\Gamma_Z(U, \cdot) = [R\Gamma(U, \cdot) \to R\Gamma(V, \cdot)][-1]: D(U_{\text{fppf}}) \to D(k_{\text{proet}}^{\text{indrat}}). \]
When $Z = \{x\}$, these are also denoted by $\Gamma_x(U, \cdot)$ and $R\Gamma_x(U, \cdot)$. 
Proposition 2.7.2. Let \( V \subset U \) be a dense open subscheme and set \( Z = U \setminus V \). The diagram

\[
\begin{array}{ccc}
\Gamma(U, \cdot) & \longrightarrow & \Gamma(V, \cdot) \\
\downarrow & & \downarrow \\
\bigoplus_{x \in U \setminus V} \Gamma(O_x^h/k, \cdot) & \longrightarrow & \bigoplus_{x \in U \setminus V} \Gamma(K_x^h/k, \cdot)
\end{array}
\]

of left exact functors \( \text{Ab}(U_{\text{fppf}}) \to \text{Ab}(k^\text{indrat}) \) is commutative. The induced morphism

\[
R\Gamma_Z(U, \cdot) \to \bigoplus_{x \in U \setminus V} R\Gamma_x(O_x^h/k, \cdot)
\]

of triangulated functors \( D(U_{\text{fppf}}) \to D(k^\text{indrat}) \) is an isomorphism.

Proof. The first statement about the commutativity is obvious. For the second statement, it is enough to treat the case that \( Z \) is a singleton \( \{x\} \).

We want to show that the morphism

\[
R\Gamma_x(U, \cdot) \to R\Gamma_x(O_x^h/k, \cdot)
\]

is an isomorphism. Let \( f_x : \text{Spec} \, k_x \to \text{Spec} \, k \) be the natural morphism. The above morphism is the pro-étale sheafification of

\[
[R\pi_{U*} \to R\pi_{V*}][-1] \to f_x[R(\pi_{O_x^h})_* \to R(\pi_{K_x^h})_*][-1].
\]

Applying \( R\Gamma(k', \cdot) \) for \( k' \in k^\text{indrat} \) to this morphism before sheafification, we have a morphism

\[
R\Gamma_x(U_{k'}, \cdot) \to R\Gamma_x((k' \otimes_k O_x^h)^h, \cdot)
\]

where the left-hand (resp. right-hand) side is the fppf cohomology of \( U_{k'} = U \times_k k' \) (resp. \( (k' \otimes_k O_x^h)^h \)) with support on \( x \times_k k' \) (resp. the ideal generated by \( k' \otimes_k p_x^h \)) [Mil06, III, Proposition 0.3]. It is enough to show that this morphism is an isomorphism for any \( k' \). We may assume that \( k' \in k^\text{rat} \) since cohomology and henselization commute with filtered inverse limits of schemes. Then \( k' \) is a finite product of perfect fields. Hence we may assume that \( k' \) is a perfect field. Replacing \( k \) by \( k' \), we may assume that \( k' = k \). Hence we are reduced to showing that

\[
R\Gamma_x(U, \cdot) \sim R\Gamma_x(O_x^h, \cdot)
\]
on $D(U_{fppf})$. Since Spec $O^h_x$ is a filtered inverse limit of affine étale $U$-schemes and cohomology commutes with such limits, we can push them forward to the étale sites. The statement to prove is thus
\[ R\Gamma_x(U_{et}, \cdot) \sim R\Gamma_x(O^h_{x,et}, \cdot) \]
on $D(U_{et})$. This is the excision isomorphism of étale cohomology [Mil80, III, Corollary 1.28].

We define a functor of additive categories with translation by
\[ \Gamma_c(U, \cdot) = \left[ \Gamma(U, \cdot) \to \bigoplus_{x \in U} \Gamma(\hat{K}_x/k, \cdot) \right][-1]: \]
\[ \text{Ch}(U_{fppf}) \to \text{Ch}(k^{\text{proet}}), \]
where the sum is over all $x \in X \setminus U$. We have its right derived functor
\[ R\Gamma_c(U, \cdot) = \left[ R\Gamma(U, \cdot) \to \bigoplus_{x \notin U} R\Gamma(\hat{K}_x/k, \cdot) \right][-1]: \]
\[ D(U_{fppf}) \to D(k^{\text{proet}}). \]
Here we are working with unbounded complexes, making the definition of compact support cohomology more involved than [DH18]. This is important in view of the definition of the pairing (2.7.2) below (which uses derived tensor product $\otimes^L$).

For another dense open subscheme $V \subset U$, unfortunately there is no obvious morphism from $R\Gamma_c(V, \cdot)$ to $R\Gamma_c(U, \cdot)$, and no natural distinguished triangle
\[ R\Gamma_c(V, A) \to R\Gamma_c(U, A) \to \bigoplus_{x \in U \setminus V} R\Gamma(\hat{O}_x/k, A), \]
unless $A \in D(U_{fppf})$ satisfies cohomological approximation at all $x \in U \setminus V$ (see Propositions 2.7.3 and 2.7.4 below). We need to define a variant of $R\Gamma_c(V, \cdot)$ that does admit a natural morphism to $R\Gamma_c(U, \cdot)$. We define a functor $\text{Ch}(U_{fppf}) \to \text{Ch}(k^{\text{proet}})$ of additive categories with translation by
\[ \Gamma_c(V, U, \cdot) = \left[ \Gamma_c(U, \cdot) \to \bigoplus_{x \in U \setminus V} \Gamma(\hat{O}_x/k, \cdot) \right][-1] \]
\[ = \left[ \Gamma(U, \cdot) \to \bigoplus_{x \notin U} \Gamma(\hat{K}_x/k, \cdot) \oplus \bigoplus_{x \in U \setminus V} \Gamma(\hat{O}_x/k, \cdot) \right][-1]. \]
Here we used the fact that for any two morphisms $A, B \to C$ of complexes in an additive category, we have natural isomorphisms of complexes
\[
[A \oplus B \to C] \cong [A \to [B \to C]] \cong [B \to [A \to C]],
\]
or dually, for any two morphisms $A \to B, C$ of complexes in an additive category, we have natural isomorphisms of (mapping fiber) complexes
\[
[A \to B \oplus C][-1] \cong [[A \to B][-1] \to C][-1] \cong [[A \to C][-1] \to B][-1].
\]

We have the right derived functor $D(U_{fpf}) \to D(\mathcal{U}_{proet})$
\[
R\Gamma_c(V, U, \cdot) = \left[ R\Gamma(U, \cdot) \to \bigoplus_{x \in U \setminus V} R\Gamma(\hat{O}_x/k, \cdot) \right][-1]
\]
\[
= \left[ R\Gamma(U, \cdot) \to \bigoplus_{x \notin U} R\Gamma(\hat{K}_x/k, \cdot) \oplus \bigoplus_{x \in U \setminus V} R\Gamma(\hat{O}_x/k, \cdot) \right][-1].
\]

By definition, we have a distinguished triangle
\[
R\Gamma_c(V, U, \cdot) \to R\Gamma_c(U, \cdot) \to \bigoplus_{x \in U \setminus V} R\Gamma(\hat{O}_x/k, \cdot).
\]

By this, we mean the values of these functors at any object form a distinguished triangle.

We also need to define a variant of $R\Gamma_c(U, \cdot)$. Define
\[
\Gamma_c(U, V, \cdot) = \left[ \Gamma(V, \cdot) \to \bigoplus_{x \notin U} \Gamma(\hat{K}_x/k, \cdot) \oplus \bigoplus_{x \in U \setminus V} \Gamma_x(\hat{O}_x/k, \cdot)[1] \right][-1],
\]
where the morphism from $\Gamma(V, \cdot)$ to $\Gamma_x(\hat{O}_x/k, \cdot)[1]$ is the composite
\[
\Gamma(V, \cdot) \to \Gamma(K^h_x/k, \cdot) \to \Gamma(\hat{K}_x/k, \cdot) \to \Gamma_x(\hat{O}_x/k, \cdot)[1].
\]

Its derived functor is
\[
R\Gamma_c(U, V, \cdot) = \left[ R\Gamma(V, \cdot) \to \bigoplus_{x \notin U} R\Gamma(\hat{K}_x/k, \cdot) \oplus \bigoplus_{x \in U \setminus V} R\Gamma_x(\hat{O}_x/k, \cdot)[1] \right][-1].
\]

For any $x \in U \setminus V$, the inclusion into the second summand defines morphisms
\[
\Gamma_x(\hat{O}_x/k, \cdot) \to \Gamma_c(U, V, \cdot)
\]
and

\[(2.7.1) \quad R\Gamma_x(\hat{O}_x/k, \cdot) \to R\Gamma_c(U, V, \cdot).\]

The last morphism will be used later to connect trace morphisms in the local and global situations.

Let \(Z = U \setminus V\). Define

\[o_Z(U, \cdot) = \left[\Gamma_Z(U, \cdot) \to \bigoplus_{x \in U \setminus V} \Gamma_x(\hat{O}_x/k, \cdot)\right],\]

with derived functor

\[R o_Z(U, \cdot) = \left[R\Gamma_Z(U, \cdot) \to \bigoplus_{x \in U \setminus V} R\Gamma_x(\hat{O}_x/k, \cdot)\right].\]

We explain the notation to be used in the next proposition. A commutative diagram of distinguished triangles in a triangulated category

\[
\begin{array}{cccc}
A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & A[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & A'[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
A'' & \longrightarrow & B'' & \longrightarrow & C'' & \longrightarrow & A''[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
A[1] & \longrightarrow & B[1] & \longrightarrow & C[1] & \longrightarrow & A[2]
\end{array}
\]

means a commutative diagram all of whose rows and columns are distinguished triangles (where the right lower square is actually “anticommutative” [KS06, Diagram (10.5.5)], but we largely ignore commutative vs. anticommutative issues, which is especially harmless if \(A\) or \(C''\) is zero for example). As usual, we will hide the shifted terms \(B[1], A''[1], A[2]\) etc. for brevity and mention the remaining \(3 \times 3\) diagram as a commutative diagram of distinguished triangles. A commutative diagram of distinguished triangles
of triangulated functors

\[
\begin{array}{ccc}
F & \longrightarrow & G & \longrightarrow & H \\
\downarrow & & \downarrow & & \downarrow \\
F' & \longrightarrow & G' & \longrightarrow & H' \\
\downarrow & & \downarrow & & \downarrow \\
F'' & \longrightarrow & G'' & \longrightarrow & H''
\end{array}
\]

means a commutative diagram of morphisms of triangulated functors whose values at any object form a commutative diagram of distinguished triangles in the above sense.

**Proposition 2.7.3.** Let \( V \subset U \) be a dense open subscheme and set \( Z = U \setminus V \). The natural morphisms form a commutative diagram

\[
\begin{array}{ccc}
\Gamma_c(V, U, \cdot) & \longrightarrow & \Gamma_c(U, \cdot) & \longrightarrow & \bigoplus_{x \in U \setminus V} \Gamma(\hat{O}_x/k, \cdot) \\
\downarrow & & \downarrow & & \downarrow \\
\Gamma_c(V, \cdot) & \longrightarrow & \Gamma_c(U, V, \cdot) & \longrightarrow & \bigoplus_{x \in U \setminus V} \Gamma(\hat{O}_x/k, \cdot) \\
\downarrow & & \downarrow & & \downarrow \\
o_Z(U, \cdot) & \longrightarrow & o_Z(U, \cdot) & \longrightarrow & 0
\end{array}
\]

of distinguished triangles of triangulated functors \( K(U_{\text{fppf}}) \to K(k_{\text{proet}}^{\text{indrat}}) \). With the isomorphism

\[
Ro_Z(U, \cdot) \sim \bigoplus_{x \in U \setminus V} Ro_x(O^{h}_x/k, \cdot)
\]

coming from the previous proposition, we have a canonical commutative diagram

\[
\begin{array}{ccc}
RT\Gamma_c(V, U, \cdot) & \longrightarrow & RT\Gamma_c(U, \cdot) & \longrightarrow & \bigoplus_{x \in U \setminus V} RT\Gamma(\hat{O}_x/k, \cdot) \\
\downarrow & & \downarrow & & \downarrow \\
RT\Gamma_c(V, \cdot) & \longrightarrow & RT\Gamma_c(U, V, \cdot) & \longrightarrow & \bigoplus_{x \in U \setminus V} RT\Gamma(\hat{O}_x/k, \cdot) \\
\downarrow & & \downarrow & & \downarrow \\
\bigoplus_{x \in U \setminus V} Ro_x(O^{h}_x/k, \cdot) & \longrightarrow & \bigoplus_{x \in U \setminus V} Ro_x(O^{h}_x/k, \cdot) & \longrightarrow & 0
\end{array}
\]

of distinguished triangles of triangulated functors \( D(U_{\text{fppf}}) \to D(k_{\text{proet}}^{\text{indrat}}) \).
Proof. For \( A \in \text{Ch}(U_{\text{fppf}}) \), let

\[
C = \Gamma(U, A), \quad D = \Gamma(V, A), \quad E = \bigoplus_{x \in U} \Gamma(\hat{K}_x/k, A),
\]

\[
C' = \bigoplus_{x \in U \setminus V} \Gamma(\hat{O}_x/k, A), \quad D' = \bigoplus_{x \in U \setminus V} \Gamma(\hat{K}_x/k, A).
\]

These are objects of \( \text{Ch}(k_{\text{indrat}}) \). We have a natural commutative diagram

\[
\begin{array}{ccc}
C & \longrightarrow & D \\
\downarrow & & \downarrow \\
C' & \longrightarrow & D'
\end{array}
\]

in \( \text{Ch}(k_{\text{indrat}}) \). The value at \( A \) of the first diagram shifted by one can be written as

\[
\begin{array}{ccc}
[C \to E \oplus C'] & \longrightarrow & [C \to E] \\
\downarrow & & \downarrow \\
[D \to E \oplus D'] & \longrightarrow & [D \to E \oplus [C' \to D']] \\
\downarrow & & \downarrow \\
[[C \to D] \to [C' \to D']] & \longrightarrow & [ [C \to D] \to [C' \to D'] ] \longrightarrow 0
\end{array}
\]

(which is actually a \( 4 \times 4 \) diagram as we are omitting the shifted terms). It is routine to check that this diagram is commutative up to homotopy. The second diagram results from the first.

We say that an object \( A \in D(U_{\text{fppf}}) \) satisfies cohomological approximation if \( R\rho_x(\mathcal{O}^h_x, A) = 0 \) for any \( x \in U \). We denote by \( D(U_{\text{fppf}})_{\text{ca}} \) the full subcategory of \( D(U_{\text{fppf}}) \) consisting of objects satisfying cohomological approximation. It is a triangulated subcategory.

**Proposition 2.7.4.** On \( D(U_{\text{fppf}})_{\text{ca}} \), we have isomorphisms

\[
R\Gamma_c(V, U, \cdot) = R\Gamma_c(V, \cdot), \quad R\Gamma_c(U, V, \cdot) = R\Gamma_c(U, \cdot),
\]

a distinguished triangle

\[
R\Gamma_c(V, \cdot) \to R\Gamma_c(U, \cdot) \to \bigoplus_{x \in U \setminus V} R\Gamma(\hat{O}_x/k, \cdot)
\]
and a morphism
\[ R\Gamma_x(\hat{O}_x/k, \cdot) \to R\Gamma_c(U, \cdot) \]
for any \( x \in U \) compatible with (2.7.1).

**Proof.** Obvious from the previous proposition. \( \square \)

**Proposition 2.7.5.** If \( A \) is a smooth group scheme or a finite flat group scheme over \( U \), then \( A \in D(U_{fppf}) \).

**Proof.** This follows from Proposition 2.6.2. \( \square \)

Let \( B, C \in D(U_{fppf}) \). To simplify the notation, we denote \( R\text{Hom}_{\hat{O}_x} \) for any \( x \in U \) by \([\cdot, \cdot]_{\hat{O}_x}\). Denote similarly \( R\text{Hom}_U \) by \([\cdot, \cdot]_U\) and \( R\text{Hom}_{\text{indrat} \text{proet}} \) by \([\cdot, \cdot]_k\). A similar construction to (2.5.1) defines a morphism
\[ R\Gamma(U, [B, C]_U) \to [R\Gamma_c(U, B), R\Gamma_c(U, C)]_k. \]
This is equivalent to a morphism
\[ (2.7.2) \]
\[ R\Gamma(U, A) \otimes^L R\Gamma_c(U, B) \to R\Gamma_c(U, A \otimes^L B) \]
via the derived tensor-Hom adjunction [KS06, Theorem 18.6.4(vii)] and the change of variables \([B, C]_U \sim A \) and \( A \otimes^L B \sim C \).

For each \( x \in U \setminus V \), we have natural morphisms
\[
R\Gamma_x(\hat{O}_x/k, [B, C]_{\hat{O}_x}) \to R\Gamma_x(\hat{O}_x/k, [B, C]_{\hat{O}_x})
\]
\[
\to [R\Gamma(\hat{O}_x/k, B), R\Gamma_x(\hat{O}_x/k, C)]_k \to [R\Gamma(\hat{O}_x/k, B), R\Gamma_c(U, V, C)]_k
\]
using the morphisms (2.5.3) and (2.7.1). Using the morphisms in Proposition 2.7.3, we have morphisms
\[
R\Gamma(U, [B, C]_U) \to [R\Gamma_c(U, B), R\Gamma_c(U, C)]_k
\]
\[ \to [R\Gamma_c(U, B), R\Gamma_c(U, V, C)]_k; \]
\[
R\Gamma(V, [B, C]_V) \to [R\Gamma_c(V, B), R\Gamma_c(V, C)]_k
\]
\[ \to [R\Gamma_c(V, U, B), R\Gamma_c(U, V, C)]_k. \]

**Proposition 2.7.6.** The above morphisms give a morphism from the distinguished triangles
\[
\bigoplus_{x \in U \setminus V} R\Gamma_x(\hat{O}_x/k, [B, C]_{\hat{O}_x}) \to R\Gamma(U, [B, C]_U) \to R\Gamma(V, [B, C]_V)
\]
to the distinguished triangle
\[
\bigoplus_{x \in U \setminus V} \left[ R\Gamma(\hat{\mathcal{O}}_x/k, B), R\Gamma_c(U, V, C) \right]_k \to \left[ R\Gamma_c(U, B), R\Gamma_c(U, V, C) \right]_k
\to \left[ R\Gamma_c(V, U, B), R\Gamma_c(U, V, C) \right]_k.
\]

Proof. We only prove the commutativity of the square
\[
R\Gamma_x(\mathcal{O}_x^b/k, [B, C]_{\mathcal{O}_x^b}) \longrightarrow R\Gamma(U, [B, C]_U)
\]
\[
\downarrow
\]
\[
\left[ R\Gamma(\hat{\mathcal{O}}_x/k, B), R\Gamma_c(U, V, C) \right]_k \longrightarrow \left[ R\Gamma_c(U, B), R\Gamma_c(U, V, C) \right]_k
\]
for \( x \in U \setminus V \). There are two more squares whose commutativity has to be proven. They can be treated similarly, so we omit their treatment. It is enough to show that the diagram
\[
\Gamma_x(\mathcal{O}_x^b/k, [B, C]_{\mathcal{O}_x^b}) \longrightarrow \Gamma(U, [B, C]_U)
\]
\[
\downarrow
\]
\[
\left[ \Gamma(\hat{\mathcal{O}}_x/k, B), \Gamma_c(U, V, C) \right]_k^c \longrightarrow \left[ \Gamma_c(U, B), \Gamma(U, V, C) \right]_k^c
\]
in \( K(k_{indrat}^{proet}) \) for \( x \in U \setminus V \) and \( B, C \in K(U_{fppf}) \) is commutative, where \([\cdot, \cdot]_k^c\) is the sheaf-Hom complex functor \( \text{Hom}_k^{indrat} \). By the tensor-Hom adjunction, it is enough to show that the morphism
\[
\Gamma_x(\mathcal{O}_x^b/k, A) \otimes \Gamma_c(U, B) \to \Gamma_x(\mathcal{O}_x^b/k, A) \otimes \Gamma(\hat{\mathcal{O}}_x/k, B)
\]
\[\text{(2.7.3)}\]
\[
\to \Gamma_x(\hat{\mathcal{O}}_x/k, A \otimes B) \to \Gamma_c(U, V, A \otimes B)
\]
and the morphism
\[
\Gamma(U, A) \otimes \Gamma_c(U, B) \to \Gamma_c(U, A \otimes B) \to \Gamma_c(U, V, A \otimes B)
\]
in \( \text{Ch}(k_{indrat}^{proet}) \) are compatible up to homotopy via
\[
\Gamma_x(\mathcal{O}_x^b/k, A) \xrightarrow{\sim} \Gamma_x(U, A) \to \Gamma(U, A)
\]
(where \( \sim \) is a quasi-isomorphism). It is routine to check that the morphism \(\text{(2.7.3)}\), the morphism
\[
\Gamma_x(U, A) \otimes \Gamma_c(U, B) \to \Gamma_x(U, A) \otimes \Gamma(U, B) \to \Gamma_x(U, A \otimes B)
\to \Gamma_x(\hat{\mathcal{O}}_x/k, A \otimes B) \to \Gamma_c(U, V, A \otimes B)
\]
and the morphism

\[ \Gamma_c(U, A) \otimes \Gamma_c(U, B) \to \Gamma_c(U, A) \otimes \Gamma(U, B) \]

\[ \to \Gamma_c(U, A \otimes B) \to \Gamma_c(U, V, A \otimes B) \]

are all compatible (without homotopy) via

\[ \Gamma_x(\mathcal{O}_x/h/k, A) \xrightarrow{\sim} \Gamma_x(U, A) \to \Gamma_c(U, A). \]

Hence we need to show that the diagram

\[
\begin{array}{ccc}
\Gamma_c(U, A) \otimes \Gamma_c(U, B) & \to & \Gamma(U, A) \otimes \Gamma_c(U, B) \\
\downarrow & & \downarrow \\
\Gamma_c(U, A) \otimes \Gamma(U, B) & \to & \Gamma_c(U, A \otimes B)
\end{array}
\]

in \( \text{Ch}(k_{\text{proet}}^{\text{indrat}}) \) is commutative up to homotopy. The left upper term is the total complex of the three term complex in degrees 0, 1, 2 of double complexes

\[ \Gamma(U, A) \otimes \Gamma(U, B) \]

\[ \to \Gamma(U, A) \otimes \bigoplus_{x \not\in U} \Gamma(\hat{K}_x/k, B) \oplus \bigoplus_{x \not\in U} \Gamma(\hat{K}_x/k, A) \otimes \Gamma(U, B) \]

\[ \to \bigoplus_{x \not\in U} \Gamma(\hat{K}_x/k, A) \otimes \bigoplus_{x \not\in U} \Gamma(\hat{K}_x/k, B). \]

The right lower term is the total complex of the two term complex in degrees 0, 1 of double complexes

\[ \Gamma(U, A \otimes B) \to \bigoplus_{x \not\in U} \Gamma(\hat{K}_x/k, A \otimes B). \]

The required homotopy is given by the projection to the diagonal

\[ \bigoplus_{x \not\in U} \Gamma(\hat{K}_x/k, A) \otimes \bigoplus_{x \not\in U} \Gamma(\hat{K}_x/k, B) \to \bigoplus_{x \not\in U} \Gamma(\hat{K}_x/k, A \otimes B) \]

in degree 2 and zero in other degrees.

Proposition 2.7.7. Let \( A, B, C \in D(U_{\text{fppf}})^{\text{ca}}. \) Let \( A \to R \text{Hom}_U(B, C), \) or equivalently, \( A \otimes^L B \to C, \) be a morphism in \( D(U_{\text{fppf}}). \) Then the
morphism in Proposition 2.7.6 induces a morphism of distinguished triangles from
\[
\bigoplus_{x \in U \setminus V} R\Gamma_x(\hat{O}_x/k, A) \to R\Gamma(U, A) \to R\Gamma(V, A)
\]
to
\[
\bigoplus_{x \in U \setminus V} [R\Gamma(\hat{O}_x/k, B), R\Gamma_c(U, C)]_k \to [R\Gamma_c(U, B), R\Gamma_c(U, C)]_k
\]
\[
\to [R\Gamma_c(V, B), R\Gamma_c(U, C)]_k.
\]

**Proof.** This follows from Proposition 2.7.4. \(\square\)

We say that a sheaf \(F \in \text{Set}(U_{fppf})\) is locally of finite presentation if \(F(\varprojlim U_\lambda) = \varprojlim F(U_\lambda)\) for any filtered inverse system \(\{U_\lambda\}\) of quasi-compact quasi-separated \(U\)-schemes with affine transition morphisms.

**Proposition 2.7.8.** For any sheaf \(A \in \text{Ab}(U_{fppf})\) locally of finite presentation and \(n \geq 0\), the sheaf \(R^n\pi_{U*}A \in \text{Ab}(k^{\text{indrat}}_{\proet})\) is locally of finite presentation. In particular, we have \(H^n(U, A) = R^n\pi_{U*}A\). That is, \(H^n(U, A)\) is the étale sheafification of the presheaf
\[
k' \mapsto H^n(U_{k'}, A).
\]

**Proof.** The functor \(k' \mapsto U_{k'}\) from the opposite category of \(k^{\text{indrat}}\) to the category of \(U\)-schemes commutes with filtered inverse limits. The same proof as Proposition 2.6.1 works. \(\square\)

In particular, in the situation of this proposition, we have isomorphisms
\[
R\Gamma(k^{\text{proet}}, R\Gamma(U, A)) = R\Gamma(k^{\text{et}}, R\pi_{U*}A) = R\Gamma(U, A)
\]
and a spectral sequence
\[
E_2^{ij} = H^i(k^{\text{proet}}, H^j(U, A)) \Rightarrow H^{i+j}(U, A).
\]

**Remark 2.7.9.** Proposition 2.6.2 may be false if \(A\) is replaced by a general fppf sheaf over \(O^h_x\). In fact, whenever \(O^h_x\) is not complete, there exists an fppf sheaf \(A\) over \(O^h_x\) such that the map \(H^0_x(O^h_x, A) \to H^0_x(\hat{O}_x, A)\) is not surjective, where \(H^0_x(O^h_x, A)\) is the kernel of \(\Gamma(O^h_x, A) \to \Gamma(K^h_x, A)\) and \(H^0_x(\hat{O}_x, A)\) is defined similarly. This shows that the distinguished triangle in Proposition 2.7.4 and a similar long exact sequence stated in
[Mil06, III, Remark 0.6(b)] do not exist for general sheaves. Coefficients in smooth group schemes and finite flat group schemes as stated in Proposition 2.7.5 are sufficient in this paper and in [Mil06, III].

To give an example of such an fppf sheaf, consider the two ring homomorphisms $\mathcal{O}_x^h \hookrightarrow \mathcal{O}_x^h[t] \twoheadrightarrow k_x$, where the first one is the inclusion and the second is the map $t \mapsto 0$ followed by the reduction map. Denote the morphisms $\text{Spec } k_x, \text{fppf} \to \text{Spec } \mathcal{O}_x^h[t], \text{fppf} \to \text{Spec } \mathcal{O}_x^h, \text{fppf}$ induced on the fppf sites by $i$ and $f$, respectively. Then a desired counterexample is given by $A = f_! i_* \mathbb{Z}$, where $f_!$ is the left adjoint of $f^*$ [Mil80, II, Remark 3.18].

Indeed, let $f^\text{pre}_!$ be the left adjoint of the presheaf pullback functor by $f$ and set $A^\text{pre} = f^\text{pre}_! i_* \mathbb{Z}$. Denote $\hat{\mathcal{O}}_x \otimes \mathcal{O}_x^h \hat{\mathcal{O}}_x$ by $\hat{\mathcal{O}}_x^{\otimes 2}$. Then $\Gamma(\hat{\mathcal{O}}_x, A^\text{pre})$ and $\Gamma(\hat{\mathcal{O}}_x^{\otimes 2}, A^\text{pre})$ are the free abelian groups generated by the sets $\hat{p}_x = p^h_x \hat{\mathcal{O}}_x$ and $p^h_x \hat{\mathcal{O}}_x^{\otimes 2}$, respectively. Consider the element of $\Gamma(\hat{\mathcal{O}}_x, A^\text{pre})$ corresponding to any element $c$ of $\hat{p}_x$ not in $\mathcal{O}_x^h$. Since $c \otimes 1$ and $1 \otimes c$ are distinct in $\hat{\mathcal{O}}_x^{\otimes 2}$, these elements do not glue in $\Gamma(\hat{\mathcal{O}}_x^{\otimes 2}, A^\text{pre})$. They do not even after replacing $\hat{\mathcal{O}}_x^{\otimes 2}$ by its fppf cover since $f^\text{pre}_!$ sends separated presheaves to separated presheaves by construction. Hence the corresponding element of $\Gamma(\hat{\mathcal{O}}_x, A) = H^0_x(\hat{\mathcal{O}}_x, A)$ does not come from an element of $\Gamma(\mathcal{O}_x^h, A) = H^0_x(\mathcal{O}_x^h, A)$.

§3. Global duality and its proof

In the rest of this paper, let $X$ be a proper smooth geometrically connected curve over a perfect field $k \in \mathcal{U}_0$ of characteristic $p > 0$ with function field $K$. We continue the notation in Section 2.7. We denote $\text{Ab}(k, \text{indrat})$ by $\text{Ab}(k)$ and $D(k, \text{proet})$ by $D(k)$. We denote $\text{Hom}_k$ by $\text{Hom}_k$ and use the notation $\text{Ext}_k^n$ and $R \text{Hom}_k$ similarly. Exact sequences and distinguished triangles of objects over $k$ are always considered in $\text{Ab}(k)$ or $D(k)$ unless otherwise noted.

Let $A$ be an abelian variety over $K$ with Néron model $\mathcal{A}$ over $X$. (Actually Sections 3.1 and 3.3 do not use $A$.) The maximal open subgroup scheme of $\mathcal{A}$ with connected fibers is denoted by $\mathcal{A}_0$. The fiber of $\mathcal{A}$ over any closed point $x = \text{Spec } k_x$ of $X$ is denoted by $\mathcal{A}_x$. The dual of $A$ is denoted by $A^\vee$, with Néron model $\mathcal{A}^\vee$ and the corresponding subscheme $\mathcal{A}_0^\vee$ and the fibers $\mathcal{A}_x^\vee$.

3.1 Duality for finite flat coefficients over open curves

By Proposition 2.7.8, the sheaf $\mathbf{H}^n(X, G_m) \in \text{Ab}(k^\text{indrat})$ for any $n$ is locally of finite presentation and the étale sheafification of the presheaf $k' \mapsto H^n(X_{k'}, G_m)$.
Let \( \text{Pic}_X = \text{Pic}_{X/k} \) be the perfection of the Picard scheme of \( X \) over \( k \). This represents the sheaf \( R^{1}\pi_{X/\text{et}}\text{perf} G_m \) on \( \text{Spec } k_{\text{et}}^{\text{perf}} \) and hence the sheaf \( H^1(X, G_m) \) on \( \text{Spec } k_{\text{indrat}}^{\text{proet}} \).

**Proposition 3.1.1.** We have \( \Gamma(X, G_m) = G_m \), \( H^1(X, G_m) = \text{Pic}_X \) and \( H^n(X, G_m) = 0 \) for \( n \geq 2 \).

**Proof.** The statements for \( n = 0, 1 \) are obvious. For \( n \geq 2 \), it is a classical fact \([\text{Mil80}, \text{III, Example 2.22, Case (d)}]\) that \( H^n(X_{k'}, G_m) = 0 \) for any algebraically closed fields \( k' \) over \( k \). This implies \( H^n(X, G_m) = 0 \) since this sheaf is locally of finite presentation.

Let \( U \subset X \) be a dense open subscheme. The smooth group scheme \( G_m \in \text{Ab}(U_{\text{fppf}}) \) satisfies cohomological approximation by Proposition 2.7.5. Hence by Proposition 2.7.4 and the above proposition, we have morphisms

\[
R\Gamma_c(U, G_m) \to R\Gamma(X, G_m) \to \text{Pic}_X[-1]^{\text{deg}} \to \mathbb{Z}[-1].
\]

We call the composite the (global) trace morphism.

**Proposition 3.1.2.** The global trace morphism \( R\Gamma_c(U, G_m) \to \mathbb{Z}[-1] \) and the local trace morphism \( R\Gamma_x(\hat{O}_x/k, G_m) \to \mathbb{Z}[-1] \) at \( x \in U \) in (2.5.8) are compatible under the morphism \( R\Gamma_x(\hat{O}_x/k, G_m) \to R\Gamma_c(U, G_m) \) in Proposition 2.7.4.

**Proof.** We are comparing the degree morphism \( \text{Pic}_X \to \mathbb{Z} \) and the valuation morphism \( \text{Res}_{k_s/k} K_x^\times/\hat{O}_x^\times \to \mathbb{Z} \). It is enough to compare them on \( \bar{k} \)-points since \( K_x^\times/\hat{O}_x^\times \cong \mathbb{Z} \) is étale. Then the comparison is between the abstract group homomorphisms \( \text{Pic}(X_{\bar{k}}) \to \mathbb{Z} \) and \( \bigoplus_{x \to x} K_x^\times/\hat{O}_x^\times \to \mathbb{Z} \), where the sum is over all \( \bar{k} \)-points of \( X_{\bar{k}} \) lying over \( x \). This is obvious.

Tensoring \( \mathbb{Z}/p\mathbb{Z}[-1] \), the global trace morphism induces a morphism

\[
R\Gamma(X, \mu_p) \to \mathbb{Z}/p\mathbb{Z}[-2].
\]

Propositions 2.7.1 and 2.7.8 allow us to translate the results of \([\text{AM76}]\) to our setting. The kernel of the morphism

\[
H^1(X, \Omega_X^1) \xrightarrow{C-1} H^1(X, \Omega_X^1), \quad \text{or } \quad G_a \xrightarrow{F-1} G_a
\]

identifies \( H^2(X, \mu_p) \) as \( \mathbb{Z}/p\mathbb{Z} \) as explained in \([\text{AM76}, \text{Introduction}]\), where \( C \) is the Cartier operator and \( F \) is the Frobenius. This gives a morphism
$R\Gamma(X, \mu_p) \to \mathbb{Z}/p\mathbb{Z}$. This is equal to the above trace morphism since we have a commutative diagram

\[
\begin{array}{ccccccccc}
H^1(X, G_m) & \longrightarrow & H^1(X, G_m) & \longrightarrow & H^1(X, \Omega^1_X) & \longrightarrow & H^1(X, \Omega^1_X) \\
\downarrow \text{deg} & & \downarrow \text{deg} & & \downarrow \text{Res} & & \downarrow \text{Res} \\
\mathbb{Z} & \longrightarrow & \mathbb{Z} & \longrightarrow & G_a & \longrightarrow & G_a
\end{array}
\]

where $\text{Res}$ denotes the residue map.

We need the following result of Milne [Mil06, III, Theorem 11.1], which is the generalization for open curves of the corresponding result of Artin and Milne [AM76]. Since [Suz14, Section 5.2] replaces Bester’s local finite flat duality, we give a proof based on [Suz14, Theorem (5.2.1.2)] for clarity.

**Theorem 3.1.3.** Let $U \subset X$ be a dense open subscheme and $N$ a finite flat group scheme over $U$. Then $R\Gamma(U, N) \in D^b(I\text{Alg}_{\text{uc}})$ and $R\Gamma_c(U, N) \in D^b(P\text{Alg}_{\text{uc}})$, which are both concentrated in degrees 0, 1, 2. Consider the morphism

$R\Gamma(U, N^{\text{CD}}) \otimes^L R\Gamma_c(U, N) \to R\Gamma_c(U, G_m) \to \mathbb{Z}[-1]$ given by the cup product morphism (2.7.2) and the trace morphism (3.1.1). The induced morphism

$R\Gamma(U, N^{\text{CD}}) \to R\Gamma_c(U, N)^{\text{SD}}[-1]$ in $D(k)$ is an isomorphism.

**Proof.** The proof proceeds by dévissage.

*Step 1:* If $U = X$, then the theorem in this case is [AM76, (0.3), (4.9)].

*Step 2:* Let $V \subset U$ be a dense open subscheme. The theorem is true for $N$ over $U$ if and only if so is for $N$ over $V$. Indeed, the groups $N, N^{\text{CD}}$ and $G_m$ satisfy cohomological approximation by Proposition 2.7.5. The natural morphism $N^{\text{CD}} \to R\text{Hom}_U(N, G_m)$, the trace morphism $R\Gamma_c(U, G_m) \to \mathbb{Z}[-1]$ and the morphism in Proposition 2.7.7 give a morphism of distinguished triangles

\[
\bigoplus_{x \in U \setminus V} R\Gamma_x(\hat{O}_x/k, N^{\text{CD}}) \longrightarrow R\Gamma(U, N^{\text{CD}}) \longrightarrow R\Gamma(V, N^{\text{CD}})
\]

\[
\bigoplus_{x \in U \setminus V} R\Gamma(\hat{O}_x/k, N)^{\text{SD}}[-1] \longrightarrow R\Gamma_c(U, N)^{\text{SD}}[-1] \longrightarrow R\Gamma_c(V, N)^{\text{SD}}[-1]
\]
We know that $R\Gamma_x(\hat{\mathcal{O}}_x/k, N^{CD})$ is in $D^b(I\text{Alg}_{uc}/k)$ concentrated in degree 2 and $R\Gamma(\hat{\mathcal{O}}_x/k, N)$ is in $D^b(P\text{Alg}_{uc}/k)$ concentrated in degrees 0, 1 by [Suz14, Proposition (3.4.2)(b), Proposition (3.4.6)]. The left vertical morphism is an isomorphism by Bester’s duality [Suz14, Theorem (5.2.1.2)]. Hence the theorem for $N$ over $U$ and the theorem for $N$ over $V$ are equivalent.

**Step 3:** If $0 \to N' \to N \to N'' \to 0$ is an exact sequence of finite flat group schemes over $U$ and the theorem is true for $N'$ and $N''$, then so is for $N$.

**Step 4:** The general case. The group $N_K = N \times_X K$ over $K$ has a filtration by finite flat subgroup schemes whose each successive subquotient or its dual is of height one. A finite flat group scheme over $K$ of height one or with dual of height one extends to $X$ as a finite flat group scheme by [Mil06, III, Propositions B.4, B.5]. By spreading out, we know that $N_V = N \times_X V$ over some dense open $V \subset U$ has a filtration by finite flat subgroup schemes whose successive subquotients are finite flat and extendable to $X$ as finite flat group schemes. Hence the previous three steps imply the theorem.

For how the above theorem is related to the corresponding duality statement [Mil06, III, Theorem 8.2] in the finite base field case, see Remark 4.2.11 below.

**3.2 Mod $n$ duality for Néron models and preliminary calculations**

Let $A$ be as in the beginning of this section. The Poincaré biextension $A^\vee \otimes^L A \to \mathbf{G}_m[1]$ as a morphism in $D(K_{fppf})$ canonically extends to a biextension $A_0^\vee \otimes^L A \to \mathbf{G}_m[1]$ as a morphism in $D(X_{fppf})$ by [Gro72, IX, 1.4.3]. With this morphism, the cup product morphism (2.7.2) and the trace morphism (3.1.1), we have morphisms

($3.2.1$) \[ R\Gamma(X, A_0^\vee) \otimes^L R\Gamma(X, A) \to R\Gamma(X, \mathbf{G}_m[1]) \to \mathbb{Z}. \]

**Proposition 3.2.1.** Let $n \geq 1$. Then $R\Gamma(X, A) \otimes^L \mathbb{Z}/n\mathbb{Z}$ and $R\Gamma(X, A_0) \otimes^L \mathbb{Z}/n\mathbb{Z}$ are both in $D^b(\text{Alg}_{uc})$ concentrated in degrees $-1, 0, 1$. Consider the morphism (3.2.1) derived-tensored with $\mathbb{Z}/n\mathbb{Z}$:

$$ (R\Gamma(X, A_0^\vee) \otimes^L \mathbb{Z}/n\mathbb{Z}) \otimes^L (R\Gamma(X, A) \otimes^L \mathbb{Z}/n\mathbb{Z}) \to \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}[1], $$

where the last morphism is the connecting morphism for the short exact sequence $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$. The induced morphism

$$ R\Gamma(X, A_0^\vee) \otimes^L \mathbb{Z}/n\mathbb{Z} \to (R\Gamma(X, A) \otimes^L \mathbb{Z}/n\mathbb{Z})^{SD}[1] $$

in $D(k)$ is an isomorphism.
Proof. We denote \((\cdot) \otimes^L \mathbb{Z}/n\mathbb{Z}\) by \((\cdot)_n\) to simplify the notation. Take a dense open subscheme \(U \subset X\) over which \(A\) is an abelian scheme. We have a morphism of distinguished triangles

\[
\bigoplus_{x \notin U} R\Gamma_x(\hat{\mathcal{O}}_x/k, A^n) \longrightarrow R\Gamma(X, A^n) \longrightarrow R\Gamma(U, A^n)
\]

by the same method as the proof of the previous theorem. The right vertical morphism is an isomorphism by the previous proposition since \((A^\vee \otimes \mathbb{Z}/n\mathbb{Z})[-1]\) over \(U\) is the \(n\)-torsion part of \(A\), which is finite flat over \(U\). The left vertical morphism is an isomorphism by Proposition 2.5.4. Hence so is the middle vertical morphism. The object \(R\Gamma_x(\hat{\mathcal{O}}_x/k, A^n)\) is in \(D^b(I\text{Alg}_{\text{uc}}/k)\) concentrated in degrees 0, 1 and the object \(R\Gamma(\hat{\mathcal{O}}_x/k, A)\) is in \(D^b(P\text{Alg}_{\text{uc}}/k)\) concentrated in degrees \(-1, 0\) by Proposition 2.5.3.

The objects \(R\Gamma(U, A^n)\) and \(R\Gamma_c(U, A^n)\) are in \(D^b(I\text{Alg}_{\text{uc}}/k)\) and in \(D^b(P\text{Alg}_{\text{uc}}/k)\), respectively, and concentrated in degrees \(-1, 0, 1\) by the previous proposition. Hence the same is true for \(R\Gamma(X, A^n)\) and \(R\Gamma(X, A)\). Therefore they are in both \(D^b(P\text{Alg}_{\text{uc}}/k)\) and \(D^b(I\text{Alg}_{\text{uc}}/k)\), hence in \(D^b(I\text{Alg}_{\text{uc}}/k)\).

In the next proposition, we consider not necessarily perfect group schemes over \(k\). But we will soon apply perfection.

**Proposition 3.2.2.** Let \(\text{Res}_{X/k} A\) be the Weil restriction of \(A\) as a functor on the category of (not necessarily perfect) \(k\)-schemes. Let \(\text{Tr}_{K/k} A\) be the \(K/k\)-trace of \(A\) ([Lan83, VIII, Theorem 8], [Con06, Definition 6.1]), which is an abelian variety over \(k\) with canonical \(K\)-morphism \((\text{Tr}_{K/k} A)_K \to A\). Let \(\text{Tr}_{K/k} A \to \text{Res}_{X/k} A\) be the \(k\)-morphism induced by the \(X\)-morphism \((\text{Tr}_{K/k} A)_X \to A\), which itself is induced by \((\text{Tr}_{K/k} A)_K \to A\).

Then \(\text{Res}_{X/k} A\) is represented by a group scheme locally of finite type over \(k\). The morphism \(\text{Tr}_{K/k} A \to (\text{Res}_{X/k} A)_0\) to the identity component has finite infinitesimal kernel and cokernel. The étale \(k\)-group \(\pi_0 \text{Res}_{X/k} A\) of connected components has finitely generated group of geometric points.

**Proof.** Since \(X\) is proper over \(k\) and \(A\) is quasi-projective over \(X\) by [BLR90, 6.4/1], the result on existence of Hilbert schemes [Gro95,
Section 4.c] shows that Res\(_{X/k} A\) is a group scheme locally of finite type over \(k\). The kernel of \((\text{Tr}\_K/k A)\_K \to A\) is a finite infinitesimal \(K\)-group by [Con06, Theorem 6.12]. It follows that \(\text{Tr}\_K/k A \to \text{Res}\_K/k A\) is injective on geometric points. Hence its kernel is a finite infinitesimal \(k\)-group. The group of \(k\)-points of the cokernel of \(\text{Tr}\_K/k A \to \text{Res}\_K/k A\) is \(A(X_k) / (\text{Tr}\_K/k A)(\bar{k})\), which is finitely generated by the Lang–Néron theorem [Con06, Theorem 7.1]. This implies that the connected group scheme \((\text{Res}\_X/k A)_0 / \text{Tr}\_K/k A\) of finite type over \(k\) has finitely generated group of geometric points. It follows that \((\text{Res}\_X/k A)_0 / \text{Tr}\_K/k A\) has trivial reduced part and hence is finite infinitesimal. Hence \(\pi_0 \text{Res}\_X/k A\) has finitely generated group of geometric points.

**Proposition 3.2.3.** The sheaf \(\Gamma(X, A)\) is an extension of a finitely generated étale group by an abelian variety over \(k\). For \(n \geq 1\), we have \(H^n(X, A) \in \text{I}^{f\text{Alg}_{\text{loc}}/k}\). The group \(H^2(X, A)\) is divisible. For \(n \geq 3\), we have \(H^n(X, A) = 0\).

**Proof.** The statement about \(\Gamma(X, A)\) follows from the previous proposition since it has the same values as \(\text{Res}\_X/k A\) on ind-rational \(k\)-algebras and hence is represented by the perfection of \(\text{Res}\_X/k A\).

We show that the sheaf \(H^n(X, A)\) for any \(n \geq 1\) is torsion. By Proposition 2.7.8, it is an étale sheafification of the presheaf \(k' \mapsto H^n(X_{k'}, A)\) and locally of finite presentation. Hence it is enough to show that the abstract group \(H^n(X_{k'}, A)\) for \(n \geq 1\) is torsion for any algebraically closed field \(k'\) over \(k\). Assume that \(k = \bar{k}\). For any dense open \(U \subset X\), by applying \(R\Gamma(k, \cdot)\) to the isomorphism in Proposition 2.7.2, we have a distinguished triangle

\[
\bigoplus_{x \not\in U} R\Gamma_x(\hat{O}_x, A) \to R\Gamma(X, A) \to R\Gamma(U, A).
\]

Taking the limit in smaller and smaller \(U\), we have a distinguished triangle

\[
\bigoplus_{x \in X} R\Gamma_x(\hat{O}_x, A) \to R\Gamma(X, A) \to R\Gamma(K, A).
\]

We have \(R\Gamma_x(\hat{O}_x, A) = H^1(\hat{K}_x, A)[-2]\) by Proposition 2.5.3. Since Galois cohomology groups are torsion in positive degrees, it follows that \(H^n(X, A)\) is torsion for any \(n \geq 1\). The same is true if \(k\) is replaced by any algebraically closed field over \(k\). Hence \(H^n(X, A)\) for \(n \geq 1\) is torsion.
Let $m \geq 1$. Denote $C_m = R\Gamma(X, \mathcal{A}) \otimes^L \mathbb{Z}/m\mathbb{Z}$. For any $n \in \mathbb{Z}$, we have an exact sequence

$$0 \to H^{n-1}(X, \mathcal{A}) \otimes \mathbb{Z}/m\mathbb{Z} \to H^{n-1}(C_m) \to H^n(X, \mathcal{A})[m] \to 0,$$

where $[m]$ denotes the $m$-torsion part. We have $H^{n-1}(C_m) \in \text{Alg}_{uc}/k$ by Proposition 3.2.1. We have $\Gamma(X, \mathcal{A}) \otimes \mathbb{Z}/m\mathbb{Z} \in \text{F} \text{E} \text{t}/k$ by what we have shown above about the structure of $\Gamma(X, \mathcal{A})$. Therefore $H^1(X, \mathcal{A})[m] \in \text{Alg}_{uc}/k$. Since $m$ is arbitrary, the torsionness shown above then shows that $H^1(X, \mathcal{A}) \in \text{I} \text{f} \text{Alg}_{uc}/k$. Hence $H^2(X, \mathcal{A}) \in \text{I} \text{f} \text{Alg}_{uc}/k$. Since $H^{n-1}(C_m) = 0$ for $n \geq 3$ by Proposition 3.2.1, we know that $H^2(X, \mathcal{A})$ is divisible and $H^n(X, \mathcal{A}) = 0$ for $n \geq 3$.

For each closed point $x \in X$, we regard the component group $\pi_0(\mathcal{A}_x)$ to be an étale group over $k_x$. Let $i_x: x \hookrightarrow X$ be the inclusion, where we identified $x = \text{Spec } k_x$. We have an exact sequence

$$0 \to \mathcal{A}_0 \to \mathcal{A} \to \bigoplus_x i_x \ast \pi_0(\mathcal{A}_x) \to 0$$

in $\text{Ab}(X_{fppf})$. The sheaf $i_x \ast \pi_0(\mathcal{A}_x)$ for any $x$ is an étale scheme over $X$ if the étale group $\pi_0(\mathcal{A}_x)$ over $k_x$ is constant and an étale algebraic space over $X$ in general by [Ray70a, Proposition (3.3.6.1)].

**Proposition 3.2.4.** The above sequence induces a distinguished triangle

$$R\Gamma(X, \mathcal{A}_0) \to R\Gamma(X, \mathcal{A}) \to \bigoplus_x \text{Res}_{k_x/k} \pi_0(\mathcal{A}_x),$$

where $\text{Res}_{k_x/k}$ denotes the Weil restriction functor. In particular, we have an exact sequence

$$0 \to \Gamma(X, \mathcal{A}_0) \to \Gamma(X, \mathcal{A}) \to \bigoplus_x \text{Res}_{k_x/k} \pi_0(\mathcal{A}_x)$$

$$\to H^1(X, \mathcal{A}_0) \to H^1(X, \mathcal{A}) \to 0$$

and an isomorphism $H^2(X, \mathcal{A}_0) = H^2(X, \mathcal{A})$. The group $\bigoplus_x \text{Res}_{k_x/k} \pi_0(\mathcal{A}_x)$ is finite étale over $k$.

**Proof.** Obvious.
Let $\mathcal{G}_m$ be the Néron (lft) model of $\mathbf{G}_m$ over $X$ [BLR90, 10.1/5]. It fits in the canonical exact sequence

$$0 \to \mathcal{G}_m \to \mathcal{G}_m \to \bigoplus_x i_x \ast \mathbb{Z} \to 0$$

of smooth group schemes over $X$, where the sum is over all closed points $x \in X$. At each $x$, we have Grothendieck’s pairing [Gro72, IX, 1.2.1]

$$\pi_0(\mathcal{A}_x^\vee) \times \pi_0(\mathcal{A}_x) \to \mathbb{Q}/\mathbb{Z}$$

over $k_x$. Combining these two, in $D(X_{\text{fppf}})$, we have morphisms

$$\left( \bigoplus_x i_x \ast \pi_0(\mathcal{A}_x^\vee) \right) \otimes^L \left( \bigoplus_x i_x \ast \pi_0(\mathcal{A}_x) \right) \to \bigoplus_x i_x \ast \mathbb{Q}/\mathbb{Z} \to \bigoplus_x i_x \ast \mathbb{Z}[1] \to \mathcal{G}_m[2].$$

Denote $\Phi_{A,X} = \bigoplus_x i_x \ast \pi_0(\mathcal{A}_x)$ and $\Phi_{A^\vee,X}$ similarly. The above defines a morphism

$$\Phi_{A^\vee,X} \otimes^L \Phi_{A,X} \to \mathcal{G}_m[2].$$

Recall again that we have morphisms

$$\mathcal{A}_0^\vee \otimes^L \mathcal{A} \to \mathcal{G}_m[1], \quad \mathcal{A}^\vee \otimes^L \mathcal{A}_0 \to \mathcal{G}_m[1]$$

defined by the canonical extensions of the Poincaré biextension.

**Proposition 3.2.5.** The above morphisms

$$\begin{array}{ccc}
\mathcal{A}_0^\vee & \longrightarrow & \mathcal{A}^\vee \\
\downarrow & & \downarrow \\
R \text{Hom}_X(\mathcal{A}, \mathcal{G}_m)[1] & \longrightarrow & R \text{Hom}_X(\mathcal{A}_0, \mathcal{G}_m)[1] \\
\downarrow & & \downarrow \\
R \text{Hom}_X(\Phi_{A^\vee,X}, \mathcal{G}_m)[2]
\end{array}$$

form a morphism of distinguished triangles in $D(X_{\text{fppf}})$, where the horizontal triangles are the natural ones.

To prove this, we need some notation and three lemmas. Let $X_{\text{sm}}$ be the smooth site of $X$, that is, the category of smooth schemes over $X$ with $X$-scheme morphisms endowed with the étale (or smooth) topology. Denote the sheaf-Hom functor for $X_{\text{sm}}$ by $\text{Hom}_{X_{\text{sm}}}$ and $\text{Ext}_{X_{\text{sm}}}^n$, $R \text{Hom}_{X_{\text{sm}}}$ similarly (while $\text{Hom}_X$ is still the sheaf-Hom for $X_{\text{fppf}}$).
Lemma 3.2.6. To prove Proposition 3.2.5, it is enough to show the modified statement in \( D(X_{\text{sm}}) \) where \( R \text{Hom}_X \) is replaced by \( R \text{Hom}_{X_{\text{sm}}} \).

Proof. Let \( f: X_{\text{fppf}} \to X_{\text{sm}} \) be the premorphism of sites defined by the identity functor. By [Suz13, Lemma 3.7.2], the pullback functor \( f^*: \text{Ab}(X_{\text{sm}}) \to \text{Ab}(X_{\text{fppf}}) \) admits a left derived functor \( Lf^*: D(X_{\text{sm}}) \to D(X_{\text{fppf}}) \), which is left adjoint to \( Rf_*: D(X_{\text{fppf}}) \to D(X_{\text{sm}}) \) and satisfies \( L_n f^* Z[Y] = 0 \) for any smooth \( X \)-scheme \( Y \) and \( n \geq 1 \) (where \( Z[Y] \) is the sheaf of free abelian groups generated by the representable sheaf of sets \( Y \)). By [Suz18, Proposition 4.2], we have \( Lf^* G = G \) for any smooth group algebraic space \( G \) over \( X \). Also \( Rf_* G = G \) since the fppf cohomology with coefficients in a smooth group algebraic space agrees with the étale cohomology [Mil80, III, Remark 3.11(b)]. Therefore if \( H \) is another smooth group algebraic space over \( X \), then

\[
Rf_* R \text{Hom}_X (G, H) = Rf_* R \text{Hom}_X (Lf^* G, H) = R \text{Hom}_{X_{\text{sm}}} (G, Rf_* H) = R \text{Hom}_{X_{\text{sm}}} (G, H)
\]

by the derived tensor-Hom adjunction [Suz18, Proposition 3.1(1)], which is applicable to our situation since the category of smooth schemes over \( X \) has finite products. Applying \( Rf_* \) to the diagram in the statement, we know that the modified statement implies the original statement.

Lemma 3.2.7. Let \( x \in X \) be a closed point and \( N \) a finite étale group over \( k_x \) with Pontryagin dual \( N^{PD} \). Let \( i_x: x_{\text{sm}} \to X_{\text{sm}} \) be the premorphism of sites defined by the inclusion \( i_x: x \hookrightarrow X \). Then the truncation \( \tau_{\leq 2} \) of \( R \text{Hom}_{X_{\text{sm}}}(i_{x*} N, G_m) \) in degrees \( \leq 2 \) is canonically isomorphic to \( i_{x*} N^{PD}[-2] \).

Proof. Let \( j_x: U = X \setminus \{x\} \hookrightarrow X \) and denote its extension-by-zero functor by \( j_x!: D(U_{\text{sm}}) \to D(X_{\text{sm}}) \). We have a distinguished triangle

\[
R \text{Hom}_{X_{\text{sm}}}(i_{x*} N, G_m) \to R \text{Hom}_{X_{\text{sm}}}(N, G_m) \to R \text{Hom}_{X_{\text{sm}}}(j_x! N, G_m)
\]

in \( D(X_{\text{sm}}) \) (where \( N \) is base-changed to \( X \)). We have

\[
R \text{Hom}_{X_{\text{sm}}}(N, G_m) = N^{PD} \otimes^L G_m[-1],
\]

\[
R \text{Hom}_{X_{\text{sm}}}(j_x! N, G_m) = Rj_{x*} R \text{Hom}_{U_{\text{sm}}}(N, G_m) = Rj_{x*} (N^{PD} \otimes^L G_m)[-1] = N^{PD} \otimes^L Rj_{x*} G_m[-1].
\]
Hence

\[ R \text{Hom}_{X_{\text{sm}}}(i_\ast N, G_m) = N^{PD} \otimes^L [G_m \to Rj_\ast G_m][-2]. \]

By definition, \( j_\ast G_m \) is the Néron model over \( X \) of \( G_m \) over \( U \). Hence it fits in the exact sequence

\[ 0 \to G_m \to j_\ast G_m \to i_\ast \mathbb{Z} \to 0. \]

We have \( R^1 j_\ast G_m = 0 \) by the proof of [Mil06, III, Lemma C.10]. Hence

\[ \tau \leq 2 R \text{Hom}_{X_{\text{sm}}}(i_\ast N, G_m) = N^{PD} \otimes^L i_\ast \mathbb{Z}[-2] = i_\ast N^{PD}[-2]. \]

**Lemma 3.2.8.** We have

\[ \text{Hom}_{X_{\text{sm}}}(A, G_m) = \text{Hom}_{X_{\text{sm}}}(A_0, G_m) = 0. \]

**Proof.** Any morphism from \( A \) or \( A_0 \) to \( G_m \) over any smooth scheme over \( X \) is generically zero and hence zero. This implies the result.

Now we prove Proposition 3.2.5.

**Proof of Proposition 3.2.5.** The commutativity of the left square in the diagram in the statement is easy to see. To see the commutativity of the right square, the above three lemmas show that it is enough to check the commutativity of the diagram

\[ \xymatrix{ \mathcal{A}^\vee \ar[r] & \Phi_{A^\vee,X} \\ \text{Ext}^1_{X_{\text{sm}}}(A_0, G_m) \ar[r] & \Phi_{A,X}^{PD} \ar[u] } \]

where we denoted

\[ \Phi_{A,X}^{PD} = \bigoplus_x i_\ast(\pi_0(\mathcal{A}_x)^{PD}) \]

Any morphism \( \mathcal{A}^\vee \to \Phi_{A,X}^{PD} \) is determined by its values at \( \mathcal{O}^{sh}_x \) for all closed points \( x \in X \), where \( \mathcal{O}^{sh}_x \) is the strict henselization of \( X \) at \( x \). Hence it is enough to show that the diagram

\[ \xymatrix{ \mathcal{A}^\vee(\mathcal{O}^{sh}_x) \ar[r] & \pi_0(\mathcal{A}^\vee_x)(\overline{k}_x) \\ \text{Ext}^1_{\mathcal{O}^{sh}_x}(A_0, G_m) \ar[r] & \text{Ext}^2_{\mathcal{O}^{sh}_x}(i_\ast \pi_0(\mathcal{A}_x), G_m) } \]
is commutative, where \( \overline{k_x} \) is the algebraic closure of \( k_x \). Let \( K_x^{sh} \) be the fraction field of \( \mathcal{O}_x^{sh} \). Let \( \xi: 0 \to G_m \to H \to A \to 0 \) be an extension as an element of
\[
A^\vee(\mathcal{O}_x^{sh}) = A^\vee(K_x^{sh}) = \text{Ext}^1_{K_x^{sh},\text{sm}}(A, G_m),
\]
where the second isomorphism is the Barsotti–Weil formula [Oor66, Chapter III, Theorem (18.1)]. Let \( H \) be the Néron (lft) model over \( \mathcal{O}_x^{sh} \) of \( H \) and \( H_0 \) the maximal open subgroup scheme with connected fibers. The image of \( \xi \) under the left vertical morphism is the extension \( 0 \to G_m \to H_0 \to A_0 \to 0 \) (which is exact since \( R^1 j_{x*} G_m = 0 \) and \( H_0 \) is of finite type). Its image under the lower vertical morphism is the extension
\[
\eta_1: 0 \to G_m \to H_0 \to A \to i_x*\pi_0(\mathcal{A}_x) \to 0,
\]
given by composing it with \( 0 \to A_0 \to A \to i_x*\pi_0(\mathcal{A}_x) \to 0 \). On the other hand, we have an exact sequence
\[
0 \to i_x*\mathbb{Z} \to i_x*\pi_0(\mathcal{H}_x) \to i_x*\pi_0(\mathcal{A}_x) \to 0.
\]
The image of the extension \( \xi \) under the upper horizontal morphism followed by the right vertical morphism is the extension
\[
\eta_2: 0 \to G_m \to \mathcal{G}_m \to i_x*\pi_0(\mathcal{H}_x) \to i_x*\pi_0(\mathcal{A}_x) \to 0,
\]
given by the composite with \( 0 \to \mathcal{G}_m \to \mathcal{G}_m \to i_x*\mathbb{Z} \to 0 \). We need to show that \( \eta_1 \) and \( \eta_2 \) are equivalent. Denote by \( \mathcal{H}' \) the inverse image of \( A_0 \) by \( H \to A \). Consider the extension
\[
\eta_3: 0 \to G_m \to \mathcal{H}_0 \times_{A_0} \mathcal{H}' \to \mathcal{H} \to i_x*\pi_0(\mathcal{A}_x) \to 0,
\]
where the first morphism (from \( G_m \) to \( \mathcal{H}_0 \times_{A_0} \mathcal{H}' \)) is the inclusion into the first factor, the second the projection onto to the second factor and the third the natural morphism. We have a morphism \( \eta_3 \to \eta_1 \) of extensions, where \( \mathcal{H}_0 \times_{A_0} \mathcal{H}' \to \mathcal{H}_0 \) is the first projection. We also have a morphism \( \eta_3 \to \eta_1 \), where \( \mathcal{H}_0 \times_{A_0} \mathcal{H}' \to \mathcal{G}_m \) is the subtraction map \((a, b) \mapsto a - b\). Therefore \( \eta_1 \) and \( \eta_2 \) are equivalent. This proves that the right square in the statement is commutative.

We finally show that the hidden square
\[
\begin{array}{ccc}
\Phi_{A^\vee, X} & \longrightarrow & A_0^\vee[1] \\
\downarrow & & \downarrow \\
R \text{Hom}_X(\Phi_{A, X}, G_m)[2] & \longrightarrow & R \text{Hom}_X(A, G_m)[2]
\end{array}
\]
is commutative. Interchanging the variables, this is equivalent to showing that the diagram
\[
\begin{array}{ccc}
A & \longrightarrow & R\text{Hom}_X(A_0, G_m)[1] \\
\downarrow & & \downarrow \\
\Phi_{A,X} & \longrightarrow & R\text{Hom}_X(\Phi_{A^\vee,X}, G_m)[2]
\end{array}
\]
is commutative. This diagram is the same, up to replacing \(A\) by \(A^\vee\), as the diagram whose commutativity has just been proved.

**Proposition 3.2.9.** The diagram in the previous proposition, after applying \(R\Gamma(X, \cdot)\), the cup product morphism (2.7.2) and the trace morphism (3.1.1), induce a morphism of distinguished triangles
\[
\begin{array}{ccc}
R\Gamma(X, A_0^\vee) & \longrightarrow & R\Gamma(X, A^\vee) \\
\downarrow & & \downarrow \\
R\Gamma(X, A)^{SD} & \longrightarrow & R\Gamma(X, A_0)^{SD}
\end{array}
\]
in \(D(k)\). The right vertical morphism is the sum over \(x \in X\) of the Weil restrictions of Grothendieck’s pairings, which is an isomorphism [Suz14, Theorem C].

**Proof.** The existence of the stated morphism of distinguished triangle is self-explanatory. To show the description of the right vertical morphism, it is enough to show that the morphism \(\bigoplus_x i_x^* Z \to G_m[1]\) after applying \(R\Gamma(X, \cdot)\) can be identified with the summation map \(\bigoplus_x \text{Res}_{k_x/k} \pi_0(A_x^\vee)\). The group \(\bigoplus_x i_x^* Z\) can be identified with the sheaf of divisors on \(X\) and the sequence \(0 \to G_m \to G_m \to \bigoplus_x i_x^* Z \to 0\) can be identified with the divisor exact sequence. Hence the composite
\[
\Gamma \left( X, \bigoplus_x \text{Res}_{k_x/k} Z \right) \to H^1(X, G_m) \to Z
\]
of the connecting morphism of the divisor exact sequence and the degree map is the summation map.

If \(k\) is algebraically closed or finite, we denote the Tate–Shafarevich group of \(A\) over \(K\) by \(\text{III}(A/K)\), which is the kernel of the natural homomorphism from \(H^1(K, A)\) to the direct sum of \(H^1(K_x, A)\) over the closed points \(x \in X\).
If \( k \) is algebraically closed, we also call \( \mathcal{X}(A/X) \) the Tate–Shafarevich group of \( A \) over \( X \) and denote it by \( \Sha(A/X) \). If \( k \) is a general perfect field with algebraic closure \( \overline{k} \), then the group \( \Sha(A_{\overline{k}}/X_{\overline{k}}) \) has a natural action of \( G_k = \text{Gal}(\overline{k}/k) \). Let \( \Sha(A_{\overline{k}}/X_{\overline{k}})^{G_k} \) be the \( G_k \)-invariant part, which is independent of the choice of an algebraic closure \( \overline{k} \). We use similar notation when \( k \) is replaced by any perfect field \( k' \overline{k} \). Consider the functor \( k' \mapsto \Sha(A_{\overline{k}}/X_{\overline{k}})^{G_k} \) on perfect fields \( k' \overline{k} \), which commutes with filtered direct limits. (Note that \( \Sha(A_{\overline{k}}/X_{\overline{k}})^{G_k} \) cannot be written as \( \Sha(A_{\overline{k}}/X_{\overline{k}}) \); the latter does not even make sense if \( k' \overline{k} \) is not algebraic over \( k \) since the ring \( K_{\overline{k}} \otimes_k k' \) in this case is not a field. The scheme \( X_{\overline{k}} \) has much more closed points than \( X_k \), which significantly affect the definition of \( \Sha(A_{\overline{k}}/X_{\overline{k}}) \).) The above functor uniquely extends to a functor \( k' \mapsto \Sha(A_{\overline{k}}/X_{\overline{k}})^{G_k} \) by abuse of notation. It is obviously a sheaf for the \( \acute{e}tale \) topology. It is moreover a sheaf for the \( pro-\acute{e}tale \) topology since it commutes with filtered direct limits.

**Proposition 3.2.10.** The sheaf \( \mathbf{H}^1(X,A) \) on \( \text{Spec } k' \) is canonically isomorphic to the sheaf \( k' \mapsto \Sha(A_{\overline{k}}/X_{\overline{k}})^{G_k} \).

**Proof.** The sheaf \( \mathbf{H}^1(X,A) \) is locally of finite presentation as seen before. It is enough to show that the group of \( k' \)-valued points of \( \mathbf{H}^1(X,A) \) is canonically isomorphic to \( \Sha(A_{\overline{k}}/X_{\overline{k}}) \) for any algebraically closed field \( k' \) over \( k \). The former group is \( H^1(X_{\overline{k}},A) \). That it is canonically isomorphic to the latter is \( \text{[Mil06, Lemma 11.5]} \).

**Proposition 3.2.11.** The group \( V_p \mathbf{H}^1(X_{k'}/A) \) as a functor on algebraically closed fields \( k' \) over \( k \) is constant.

**Proof.** We may assume that \( k = \overline{k} \). By \( \text{[Kat99, Theorem 11]} \), there exist a proper smooth geometrically connected curve \( C \) over \( K \) having a \( K \)-rational point and a surjective homomorphism \( J \rightarrow A \) from the Jacobian \( J \) of \( C \) over \( K \). By Poincaré complete reducibility, there exists a homomorphism \( A \rightarrow J \) over \( K \) such that the composite \( A \rightarrow J \rightarrow A \) is multiplication by some positive integer \( m \). Let \( \mathcal{J} \) be the Néron model of \( J \) over \( X \). Then we have homomorphisms \( A \rightarrow \mathcal{J} \rightarrow A \) over \( X \) whose composite is multiplication by \( m \). Therefore \( V_p \mathbf{H}^1(X,A) \) is a direct factor of \( V_p \mathbf{H}^1(X,\mathcal{J}) \). Hence it is enough to show that \( V_p \mathbf{H}^1(X,\mathcal{J}) \) does not depend on the algebraically
closed base field $k$. Let $\mathcal{C}/X$ be a proper flat regular model of $C/K$ [Lip78]. By [Gro66, Section 4.6], there exists a canonical isomorphism $H^1(X, \mathcal{F}) \cong H^2(\mathcal{C}, \mathbb{G}_m)$. Hence it is enough to show that $V_p H^2(\mathcal{C}, \mathbb{G}_m)$ does not depend on the algebraically closed base field $k$. Note that $H^2(\mathcal{C}, \mathbb{G}_m)$ is the Brauer group of the proper smooth surface $\mathcal{C}$ over $k$.

By [Ill79, II, (5.8.5)], we have a canonical exact sequence

$$0 \to \text{NS}(\mathcal{C}) \otimes \mathbb{Q}_p \to H^2(\mathcal{C}, \mathbb{Q}_p(1)) \to V_p H^2(\mathcal{C}, \mathbb{G}_m) \to 0,$$

where $\text{NS}$ denotes the Néron–Severi group and the middle term is

$$\left(\lim_{\leftarrow n} H^2(\mathcal{C}, \mu_{p^n})\right) \otimes \mathbb{Q}.$$

The group $\text{NS}(\mathcal{C})$ does not depend on $k$. By [Ill79, II, Theorem 5.5.3], there exists a canonical exact sequence

$$0 \to H^2(\mathcal{C}, \mathbb{Q}_p(1)) \to H^2_{\text{crys}}(\mathcal{C}/W(k)) \otimes \mathbb{Q} \to H^2_{\text{crys}}(\mathcal{C}/W(k)) \otimes \mathbb{Q} \to 0,$$

where the last two groups are the rational crystalline cohomology. They are finite-dimensional over $W(k)[1/p]$. Hence $H^2(\mathcal{C}, \mathbb{Q}_p(1))$ is finite-dimensional over $\mathbb{Q}_p$ whose dimension does not depend on $k$. Therefore $V_p H^2(\mathcal{C}, \mathbb{G}_m)$ does not depend on $k$. This finishes the proof.

**Proposition 3.2.12.** The group $H^1(X, \mathcal{A})$ is in $L^f \text{Alg}_{\text{Snc}}/k$.

**Proof.** This follows from the previous propositions, Proposition 3.2.3 and Proposition 2.2.3.

**Proposition 3.2.13.** The morphism (3.2.1) induces a morphism

$$\Gamma(X, \mathcal{A}_0^\vee) \otimes \Gamma(X, \mathcal{A}) \to \mathbb{Z}.$$

This agrees with the height pairing [MB85, III, Section 3].

**Proof.** This morphism is equal to the morphism

$$\Gamma(X, \mathcal{A}_0^\vee) \otimes \Gamma(X, \mathcal{A}) \to H^1(X, \mathbb{G}_m) \to \mathbb{Z}.$$

Hence a more explicit description can be given as follows. Let $P$ be the extension of the Poincaré bundle to $\mathcal{A}_0^\vee \times_X \mathcal{A}$. Let $f: X \to \mathcal{A}_0^\vee$ and $g: X \to \mathcal{A}$ be sections over $X$. By pulling back $P$ by $f \times g: X \to \mathcal{A}_0^\vee \times_X \mathcal{A}$, we have a line bundle on $X$. Its degree is the value of the pairing at $(f, g)$. This pairing is equal to the height pairing by [MB85, III, Section 3].
Proposition 3.2.14. Consider the morphism
\[ \pi_0(\Gamma(X, A^0_0))_{/\text{tor}} \times \pi_0(\Gamma(X, A))_{/\text{tor}} \to \mathbb{Z} \]
coming from (3.2.1). The induced morphism
\[ \pi_0(\Gamma(X, A^0_0))_{/\text{tor}} \to \pi_0(\Gamma(X, A))_{/\text{tor}}^{LD} \]
is injective with finite cokernel.

Proof. This follows from the previous proposition and the nondegeneracy of the height pairing [Con06, Theorem 9.15].

We summarize the results obtained so far.

Proposition 3.2.15. Let \( C = R\Gamma(X, A) \) and \( D = R\Gamma(X, A^0_0) \), which are objects of \( D(k) \). The sheaf \( H^0C \) is an extension of a finitely generated étale group by an abelian variety; \( H^1C \in \mathcal{L}^f\text{Alg}/k; H^2C \in \mathcal{I}^f\text{Alg}/k \), which is divisible; and \( H^nC = 0 \) for other values of \( n \). The same are true for \( D \).

There is a canonical pairing \( C \otimes^L D \to \mathbb{Z} \) in \( D(k) \). The induced morphism
\[ (\pi_0H^0D)_{/\text{tor}} \to (\pi_0H^0C)_{/\text{tor}}^{LD} \]
is injective with finite cokernel. For any \( n \geq 1 \), the induced morphism
\[ D \otimes^L \mathbb{Z}/n\mathbb{Z} \to (C \otimes^L \mathbb{Z}/n\mathbb{Z})^{SD}[1] \]
is an isomorphism.

3.3 Formal steps toward duality for Néron models

Throughout this subsection, we fix two objects \( C, D \in D(k) \) and a morphism \( C \otimes^L D \to \mathbb{Z} \), and assume the following:

1. The sheaf \( H^0C \) is an extension of a finitely generated étale group by an abelian variety; \( H^1C \in \mathcal{L}^f\text{Alg}/k; H^2C \in \mathcal{I}^f\text{Alg}/k \), which is divisible; and \( H^nC = 0 \) for other values of \( n \).
2. The same are true for \( D \).
3. The morphism
\[ (\pi_0H^0D)_{/\text{tor}} \to (\pi_0H^0C)_{/\text{tor}}^{LD} \]
induced from \( C \otimes^L D \to \mathbb{Z} \) is injective and its cokernel \( \delta_{\text{Height}} \) is finite.
(4) For any $n \geq 1$, the induced morphism

$$\Psi^L \mathbb{Z}/n\mathbb{Z} \to (C \otimes^L \mathbb{Z}/n\mathbb{Z})^{SD}[1]$$

is an isomorphism.

It follows from (1) that $C \otimes^L \mathbb{Z}/n\mathbb{Z} \in D^b(\text{Alg}_{uc}/k)$ for any $n \geq 1$. The same is true for $D$. Hence the isomorphism in (4) belongs to $D^b(\text{Alg}_{uc}/k)$.

The goal of this subsection is to prove that there exist canonical morphisms

$$C^{SD} \to VH^1 D \to D^{SDSD}[1]$$

such that the triangle

$$VH^1 D[-1] \to D^{SDSD} \to C^{SD} \to VH^1 D$$

is distinguished (Proposition 3.3.19). In order to prove this, we first extract as much information as possible from the limit in $n$ of the isomorphism (4) (Proposition 3.3.1 to 3.3.7). Then we describe each cohomology object of $D^{SDSD}$ and $C^{SD}$ (Proposition 3.3.8). They are concentrated in degrees $-1, 0, 1, 2$. With these two steps and (3), we can show that there exist canonical distinguished triangles

$$D^{SDSD} \otimes \mathbb{Q} \to C^{SD} \otimes \mathbb{Q} \to VH^1 D,$$

$$R \lim_n D^{SDSD} \to R \lim_n C^{SD} \to VH^1 D$$

(Proposition 3.3.11 to 3.3.13 for the first triangle and Propositions 3.3.15 to 3.3.17 for the second). These are easier to establish than the integral statement since $D^{SDSD} \otimes \mathbb{Q}$ and $C^{SD} \otimes \mathbb{Q}$ are concentrated in degrees $-1, 0$, and $R \lim_n D^{SDSD}$ and $R \lim_n C^{SD}$ are concentrated in degrees $1, 2$. Observe that these ranges of degrees have no intersection. This disjointness puts a strong restriction on possible choices of a mapping cone of $D^{SDSD} \to C^{SD}$. With this, we can get the desired canonical distinguished triangle.

We begin with taking limits:

**Proposition 3.3.1.** The morphism $D \to C^{SD}$ induces an isomorphism

$$R \lim_n (D \otimes^L \mathbb{Z}/n\mathbb{Z}) \to R \lim_n (C^{SD} \otimes^L \mathbb{Z}/n\mathbb{Z}).$$
Proof. For any $n \geq 1$, we have
\[ C^{SD} \otimes^L \mathbb{Z}/n\mathbb{Z} = (C \otimes^L \mathbb{Z}/n\mathbb{Z})^{SD}[1]. \]
Hence Proposition 2.3.6 implies the result.

To calculate the right-hand side, it is easier to write it with torsion objects only:

**Proposition 3.3.2.** We have
\[ R \lim_n (R \text{Hom}_k(C, \mathbb{Z}) \otimes^L \mathbb{Z}/n\mathbb{Z}) = R \text{Hom}_k(C \otimes^L \mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z}). \]
Hence the isomorphism in the previous proposition can also be written as
\[ R \lim_n (D \otimes^L \mathbb{Z}/n\mathbb{Z}) \simto R \text{Hom}_k(C \otimes^L \mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z}). \]

Proof. We have
\[ R \text{Hom}_k(C \otimes^L \mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) = R \lim_n R \text{Hom}_k(C \otimes^L \mathbb{Z}/n\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) \]
by [Suz14, Proposition (2.2.3)] (or its proof) in the notation therein. For any $n \geq 1$, we have
\[ R \text{Hom}_k(C \otimes^L \mathbb{Z}/n\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) = R \text{Hom}_k(C \otimes^L \mathbb{Z}/n\mathbb{Z}, \mathbb{Z}[1]) = R \text{Hom}_k(C, \mathbb{Z}) \otimes^L \mathbb{Z}/n\mathbb{Z} \]
since derived-tensoring $\mathbb{Z}/n\mathbb{Z}$ kills uniquely divisibles. The result follows by taking the limit.

**Proposition 3.3.3.** The groups $H^2C, H^2D \in \mathcal{V}Alg/k$ are étale. The groups $H^1C, H^1D \in \mathcal{V}Alg/k$ are divisibly ML. The isomorphism in Proposition 3.3.2 yields, on cohomology, the following duality pairings and morphisms:

1. Pontryagin duality between $T(H^0C)_0 \in \text{PFEt}/k$ and $H^2D \in \text{IFEt}/k$.
2. Pontryagin duality between $T(H^0D)_0 \in \text{PFEt}/k$ and $H^2C \in \text{IFEt}/k$.
3. Pontryagin duality between $(\pi_0H^0C)_{\text{tor}}, \pi_0(H^1D)_{\text{div}} \in \text{FEt}/k$.
4. Pontryagin duality between $(\pi_0H^0D)_{\text{tor}}, \pi_0(H^1C)_{\text{div}} \in \text{FEt}/k$.
5. An injection $(\pi_0H^1C)_{\text{PD}} \hookrightarrow T(H^1D)_{\text{div}}$ in $\text{PFEt}/k$ whose cokernel $\delta_{\text{Tran}}$ is finite.
6. A surjection $(H^1C)^{\text{SD}} \twoheadrightarrow ((H^1D)_{\text{div}})_0$ of connected unipotent quasi-algebraic groups whose kernel $\delta_{\text{CT}}$ is finite.
7. An exact sequence $0 \rightarrow \delta_{\text{Tran}} \rightarrow \delta_{\text{Height}} \rightarrow \delta_{\text{CT}} \rightarrow 0$ in $\text{FEt}/k$. 
The suggestive subscripts CT and Tran will be explained after Theorem 3.4.1. The directions of the morphisms in (5) and (6) may look wrong, but they are indeed correct.

**Proof.** We set
\[ X = C \otimes L \mathbb{Q}/\mathbb{Z}, \quad Y = R \text{Hom}_k(X, \mathbb{Q}/\mathbb{Z}), \quad Z = R \lim_{\leftarrow n} (D \otimes L \mathbb{Z}/n\mathbb{Z}). \]

We are going to write down the effects of the isomorphism \( Z \xrightarrow{\sim} Y \) on their cohomology objects. For any \( i \), we have
\[
0 \to (H^iC) \otimes \mathbb{Q}/\mathbb{Z} \to H^iX \to (H^{i+1}C)_{\text{tor}} \to 0, \\
0 \to (H^{-i+1}X)^{\text{SD}'}_0 \to H^iY \to \pi_0(H^{-i}X)^{\text{PD}} \to 0, \\
0 \to (H^iD)^{\wedge} \to H^iZ \to T(H^{i+1}D) \to 0,
\]
where the second line comes from [Suz14, Proposition (2.4.1)(b)] and [Mil06, III, Theorem 0.14], and the third line comes from Proposition 2.3.8 and [Suz14, Proposition (2.1.2)(f)]. We have \( H^{-1}X = (H^0C)_{\text{tor}} \). From the structure of \( H^0C \) and the torsionness of \( H^1C \) and \( H^2C \), we have
\[
0 \to T(H^0C)_0 \otimes \mathbb{Q}/\mathbb{Z} \to H^{-1}X \to (\pi_0H^0C)_{\text{tor}} \to 0, \\
0 \to \pi_0(H^0C)_{\text{tor}} \otimes \mathbb{Q}/\mathbb{Z} \to H^0X \to H^1C \to 0, \\
H^1X = H^2C,
\]
and \( H^iX = 0 \) for other values of \( i \). In particular, \( H^{-1}X \) has trivial identity component. Using these, we have
\[
H^{-1}Y = \pi_0(H^2C)^{\text{PD}}, \\
0 \to (H^2C)^{\text{SD}}_0 \to H^0Y \to \pi_0(H^0X)^{\text{PD}} \to 0, \\
0 \to (H^0X)^{\text{SD}'}_0 \to H^1Y \to (H^{-1}X)^{\text{PD}} \to 0,
\]
and \( H^iY = 0 \) for other values of \( i \). On the other hand, we know that \( TH^0D = T(H^0D)_0 \) and \( (H^0D)^{\wedge} = \pi_0(H^0D)^{\wedge} \) from the structure of \( H^0D \) and that \( (H^2D)^{\wedge} = 0 \) from the divisibility of \( H^2D \). Hence
\[
H^{-1}Z = T(H^0D)_0, \\
0 \to \pi_0(H^0D)^{\wedge} \to H^0Z \to T(H^1D)_{\text{div}} \to 0, \\
0 \to (H^1D)^{\wedge} \to H^1Z \to TH^2D \to 0,
\]
and \( H^iZ = 0 \) for other values of \( i \).
Since $H^1D \in L^f \text{Alg}_{uc}/k$, we know that $H^0Z \cong H^0Y$ has trivial identity component. Hence so does $H^2C$. Thus $H^2C$ is étale. Since $H^1C \in L^f \text{Alg}_{uc}/k$, we know that the identity component of $H^0X$ is quasi-algebraic. Hence so is $H^1Y \cong H^1Z$, and so is $TH^2D$. Therefore $H^2D$ is étale.

Hence $H^iZ \cong H^iY$ has trivial identity component for $i \neq 1$. Comparing $H^{-1}$, $H^0$, $(H^1)_0$, $\pi_0H^1$ of $Y$ and $Z$, we have

\[ T(H^0D)_0 \cong (H^2C)^{PD}, \]
\[ 0 \to (\pi_0H^0D)^\wedge \to (\pi_0H^0X)^{PD} \to T(H^1D)_{\text{div}} \to 0, \]
\[ ((H^1D)^\wedge)_0 \cong (H^0X)^{SD'}, \]
\[ 0 \to \pi_0((H^1D)^\wedge) \to (H^{-1}X)^{PD} \to TH^2D \to 0, \]

respectively. We have (2) from the first line. Comparing the last line with the exact sequence for $H^{-1}X$ given above and using the finiteness of $(\pi_0H^0C)_{\text{tor}}$, we obtain (1), (3) and that $H^1D$ is divisibly ML. From the second line, we have $(\pi_0H^0X)_{\text{div}} = (\pi_0H^0D)_{\text{tor}}$, which is finite. On the other hand, the exact sequence for $H^0X$ given above gives $(\pi_0H^0X)_{\text{div}} = (\pi_0H^1C)_{\text{div}}$. Hence we obtain (4) and that $H^1C$ is divisibly ML. By Proposition 2.3.2, we have $(H^1D)^\wedge = (H^1D)_{\text{div}}$.

From the second line, we have

\[ 0 \to (\pi_0H^0D)^{\wedge}_{\text{tor}} \to (\pi_0H^0X)^{PD}_{\text{div}} \to T(H^1D)_{\text{div}} \to 0, \]

where $(\pi_0H^0X)^{PD}_{\text{div}} = ((\pi_0H^0X)_{\text{div}})^{PD}$. Back to the relation between $X$ and $C$, we have

\[ 0 \to (\pi_0H^0C)_{\text{tor}} \otimes \mathbb{Q}/\mathbb{Z} \to (H^0X)_{0+\text{div}} \to (H^1C)_{0+\text{div}} \to 0. \]

The long exact sequence of $\text{Ext}_k^\cdot(\cdot, \mathbb{Q}/\mathbb{Z})$ for this short exact sequence and $((H^1D)_{\text{div}})_0 \cong (H^0X)^{SD'}$ yields

\[ 0 \to (\pi_0H^1C)^{PD}_{\text{div}} \to (\pi_0H^0X)^{PD}_{\text{div}} \to (\pi_0H^0C)^{LD^\wedge}_{\text{tor}} \]
\[ \to (H^1C)^{SD'}_0 \to ((H^1D)_{\text{div}})_0 \to 0. \]

We will apply the lemma below to the two exact sequences (3.3.1) and (3.3.2). The composite

\[ (\pi_0H^0D)^{\wedge}_{\text{tor}} \hookrightarrow (\pi_0H^0X)^{PD}_{\text{div}} \to (\pi_0H^0C)^{LD^\wedge}_{\text{tor}}. \]
is the completion of the injective morphism $(\pi_0 H^0 D)_{/\text{tor}} \hookrightarrow (\pi_0 H^0 C)_{/\text{tor}}^{\text{LD}}$ with finite cokernel $\delta_{\text{Height}}$. Hence we can apply the lemma, yielding (5), (6) and (7).

**Lemma 3.3.4.** Let

$$0 \rightarrow F \rightarrow W \rightarrow E' \rightarrow 0,$$

$$0 \rightarrow E \rightarrow W \rightarrow F' \rightarrow G \rightarrow G' \rightarrow 0$$

be exact sequences in an abelian category such that the composite $F \rightarrow W \rightarrow F'$ is injective. Then the composite $E \rightarrow W \rightarrow E'$ is injective, and we have an exact sequence

$$0 \rightarrow \text{Coker}(E \hookrightarrow E') \rightarrow \text{Coker}(F \hookrightarrow F') \rightarrow \text{Ker}(G \twoheadrightarrow G') \rightarrow 0.$$ 

**Proof.** Elementary.

If the groups $(H^1 C)_{0\cap \text{div}}, (H^1 D)_{0\cap \text{div}}$ (which are in general finite étale $p$-groups by Proposition 2.2.5) are zero, then the morphisms in (5), (6) are simplified to more symmetric expressions

$$(H^1 C)^{\text{PD}} \rightarrow T(H^1 D)_{\text{div}}, \quad (H^1 C)_{0}^{\text{SD'}} \rightarrow (H^1 D)_{0}.$$ 

We do not assume these conditions. See Remark 3.4.2 for a little more details about this point.

**Proposition 3.3.5.** The cohomology objects $H^n$ of the complex $R \varprojlim_n (D \otimes^L \mathbb{Z}/n\mathbb{Z})$ are described as follows:

$$H^{-1} = T(H^0 D)_0$$

$$0 \rightarrow (\pi_0 H^0 D) \rightarrow H^0 \rightarrow T(H^1 D)_{\text{div}} \rightarrow 0,$$

and $H^n = 0$ for $n \neq -1, 0, 1$. The cohomology objects $H^n$ of the complex $R \varprojlim_n (C^{\text{SD}} \otimes^L \mathbb{Z}/n\mathbb{Z})$ are described as follows:

$$H^{-1} = (H^2 C)^{\text{PD}},$$

$$0 \rightarrow (\pi_0 H^1 C)^{\text{PD}} \rightarrow H^0 \rightarrow (\pi_0 H^0 C)_{/\text{tor}} \rightarrow \delta_{\text{CT}} \rightarrow 0,$$

and $H^n = 0$ for $n \neq -1, 0, 1$. 


Proof. This is mostly given in the proof of Proposition 3.3.3. We show here only the second line for the second complex. Similar to (3.3.2), we have an exact sequence

\[ 0 \to (\pi_0 H^1 C)^{PD} \to (\pi_0 H^0 X)^{PD} \to (\pi_0 H^0 C)^{\text{LD}^\wedge}_{/\text{tor}} \]
\[ \to (H^1 C)_0^{\text{SD}'} \to ((H^1 D)/\text{div})_0 \to 0. \]

The $H^0$ of the second complex in the statement is $(\pi_0 H^0 X)^{PD}$. The kernel of the last surjection is $\delta_{CT}$. Hence we get the desired exact sequence.

Tensoring $\mathbb{Q}$ to the isomorphism in Proposition 3.3.1, we have an isomorphism

\[ (3.3.3) \quad D \hat{\otimes} \mathbb{A}^\infty \simeq C^{\text{SD}} \otimes \mathbb{A}^\infty, \]

where $\mathbb{A}^\infty = \hat{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q} \in \text{IPF}\text{Et}/k$ as in Section 2.3.

Proposition 3.3.6. The cohomology objects $H^n$ of the complex $D \otimes \mathbb{A}^\infty$ are described as follows:

\[ H^{-1} = V(H^0 D)_0, \]
\[ 0 \to (\pi_0 H^0 D)_{/\text{tor}} \otimes \mathbb{A}^\infty \to H^0 \to V(H^1 D)_{\text{div}} \to 0, \]
\[ H^1 = VH^2 D, \]

and $H^n = 0$ for other degrees. The cohomology objects $H^n$ of the complex $C^{\text{SD}} \otimes \mathbb{A}^\infty$ are described as follows:

\[ H^{-1} = (V H^2 C)^{PD}, \]
\[ 0 \to (V(H^1 C)_{\text{div}})^{PD} \to H^0 \to (\pi_0 H^0 C)^{\text{LD}^\wedge}_{/\text{tor}} \otimes \mathbb{A}^\infty \to 0, \]
\[ H^1 = (V(H^0 C)_0)^{PD}, \]

and $H^n = 0$ for other degrees.

Proof. This follows from the previous proposition.

Proposition 3.3.7. The isomorphism (3.3.3) induces Pontryagin duality between $V(H^0 D)_0$ and $V H^2 C$ and between $V H^2 D$ and $V(H^0 C)_0$. In the previous proposition, both of the sequence for $H^0$ of $D \hat{\otimes} \mathbb{A}^\infty$ and the sequence for $H^0$ of $C \hat{\otimes} \mathbb{A}^\infty$ canonically split. The parts $(\pi_0 H^0 D)_{/\text{tor}} \otimes \mathbb{A}^\infty$ and $(\pi_0 H^0 C)_{/\text{tor}} \otimes \mathbb{A}^\infty$ are Pontryagin dual to each other. The parts $V(H^1 D)_{\text{div}}$ and $V(H^1 C)_{\text{div}}$ are Pontryagin dual to each other.
Proof. The induced morphism

$$(\pi_0 H^0 D)_{/\text{tor}} \otimes \mathbb{A}_\infty \to (\pi_0 H^0 C)_{/\text{tor}} \otimes \mathbb{A}_\infty$$

is an isomorphism. The rest follows from this.

Proposition 3.3.8. The cohomology objects $H^n$ of the complex $D^{SDSD}$ are described as follows:

$$H^{-1} = T(H^0 D)_0, \quad H^0 = \pi_0 H^0 D,$$

$$H^1 = H^1 D, \quad H^2 = H^2 D,$$

and $H^n = 0$ for other values of $n$. The cohomology objects $H^n$ of the complex $C^{SD}$ are described as follows:

$$H^{-1} = (H^2 C)^{\text{PD}},$$

$$0 \to (\pi_0 H^1 C)^{\text{PD}} \to H^0 \to (\pi_0 H^0 C)_{/\text{tor}}^{\text{LD}}$$

$$\to (H^1 C)_0^{\text{SY}} \to H^1 \to (\pi_0 H^0 C)_{/\text{tor}}^{\text{PD}} \to 0,$$

$$H^2 = (T(H^0 C)_0)^{\text{PD}},$$

and $H^n = 0$ for other values of $n$.

Proof. First we treat $D^{SDSD}$. The cohomology objects of the mapping cone

$$[(H^0 D)_0 \to D]$$

are in $\text{FGEt}/k$ in degree 0, in $\text{IAlg}_{uc}/k$ in degrees 1, 2 and zero in other degrees. Each of them is Serre reflexive by [Suz14, Proposition (2.4.1)(b)]. Hence this mapping cone itself is Serre reflexive. On the other hand, the double Serre dual of the abelian variety $(H^0 C)_0$ over $k$ is its Tate module placed in degree $-1$ by [Suz14, Proposition (2.4.1)(c)]. Therefore we have a distinguished triangle

$$T(H^0 D)_0[1] \to D^{SDSD} \to [(H^0 D)_0 \to D].$$

The desired description follows from this.

For $C^{SD}$, consider the hyperext spectral sequence

$$E_2^{ij} = \text{Ext}_k^i(H^{-j} C, \mathbb{Z}) \Longrightarrow H^{i+j} R \text{Hom}_k(C, \mathbb{Z}).$$
For any object $E$ of $\text{IAlg}/k$ or $\text{FGEt}/k$, we have

\[
\text{Hom}_k(E, \mathbb{Z}) = (\pi_0 E)^{\text{LD}}_{/\text{tor}}, \quad \text{Ext}^1_k(E, \mathbb{Z}) = (\pi_0 E)^{\text{PD}}_{/\text{tor}},
\]

\[
\text{Ext}^2_k(E, \mathbb{Z}) = E_0^{\text{SD}'}, \quad \text{Ext}^3_k(E, \mathbb{Z}) = 0
\]

by [Suz14, Proposition (2.4.1)(a), (b)]. We have

\[
(H^1 C)/\text{tor} = (H^2 C)/\text{tor} = (H^2 C)_0 = 0
\]

by Proposition 3.3.3. Hence the $E_2^{ij}$-term of the above spectral sequence is zero unless $(i, j)$ is $(-1, -2), (-1, -1), (0, 0), (2, -1)$ or $(2, 0)$. We have

\[
(H^0 C)_0^{\text{SD}'} = (T(H^0 C)_0)^{\text{PD}}
\]

by [Suz14, Proposition (2.4.1)(c)]. Therefore the $E_2$-sheet gives the desired description.

**Proposition 3.3.9.** The morphism

\[
(\pi_0 H^0 C)^{\text{LD}}_{/\text{tor}} \to (H^1 C)_0^{\text{SD}'}
\]

in the previous proposition and the morphism

\[
(\pi_0 H^0 C)^{\text{LD}^\wedge}_{/\text{tor}} \to (H^1 C)_0^{\text{SD}'}
\]

in (the proof of) Proposition 3.3.5 are compatible under the natural morphism $(\cdot) \to (\cdot)^\wedge$.

**Proof.** The first morphism is the differential

\[
(3.3.4) \quad \text{Hom}_k(H^0 C, \mathbb{Z}) \to \text{Ext}_k^2(H^1 C, \mathbb{Z})
\]

of the spectral sequence with $E_2^{ij} = \text{Ext}_k^i(H^{-j} C, \mathbb{Z})$. The second is the connecting morphism

\[
\text{Hom}_k((H^0 C) \otimes \mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) \to \text{Ext}_k^1(H^1 C, \mathbb{Q}/\mathbb{Z})
\]

for the short exact sequence

\[
(3.3.5) \quad 0 \to (H^0 C) \otimes \mathbb{Q}/\mathbb{Z} \to H^0(C \otimes^L \mathbb{Q}/\mathbb{Z}) \to H^1 C \to 0.
\]

We want to show that these morphisms are compatible under the natural morphism

\[
\text{Hom}_k(H^0 C, \mathbb{Z}) \to \text{Hom}_k((H^0 C) \otimes \mathbb{Q}/\mathbb{Z), \mathbb{Q}/\mathbb{Z})}
\]
and the isomorphism
\[ \text{Ext}_k^1(H^1C, \mathbb{Q}/\mathbb{Z}) = \text{Ext}_k^2(H^1C, \mathbb{Z}). \]

By general nonsense, (3.3.4) comes from the (shifted) connecting morphism \((H^1C)(-2) \to H^0C\) of the truncation distinguished triangle \(H^0C \to \tau_{\leq 1}C \to (H^1C)[-1]\). This triangle induces a distinguished triangle
\[ (H^0C) \otimes^L \mathbb{Q}/\mathbb{Z} \to (\tau_{\leq 1}C) \otimes^L \mathbb{Q}/\mathbb{Z} \to (H^1C) \otimes^L \mathbb{Q}/\mathbb{Z}[-1]. \]

Taking \(H^0\), we have an exact sequence
\[ 0 \to (H^0C) \otimes \mathbb{Q}/\mathbb{Z} \to H^0((\tau_{\leq 1}C) \otimes^L \mathbb{Q}/\mathbb{Z}) \to H^1C \to 0, \]
which recovers (3.3.5). From these observations, we can finish the comparison by applying \(\text{Ext}_k(\cdot, \mathbb{Z})\).

**Proposition 3.3.10.** We have
\[ R \lim_n (C \otimes^L \mathbb{Z}/n\mathbb{Z}) = R \lim_n (C^{\text{SDSD}} \otimes^L \mathbb{Z}/n\mathbb{Z}), \]
\[ C \otimes A^\infty = C^{\text{SDSD}} \otimes A^\infty. \]
The same is true for \(D\).

**Proof.** By Proposition 3.3.8, we have a distinguished triangle
\[ V(H^0C)_0[1] \to C \to C^{\text{SDSD}}. \]
We have
\[ R \lim_n (V(H^0C)_0 \otimes \mathbb{Z}/n\mathbb{Z}) = 0 \]
since \(V(H^0C)_0\) is uniquely divisible. This implies the result.

**Proposition 3.3.11.** The cohomology objects \(H^n\) of the complex \(D^{\text{SDSD}} \otimes \mathbb{Q}\) are described as follows:
\[ H^{-1} = V(H^0D)_0, \quad H^0 = (\pi_0H^0D)_{/\text{tor}} \otimes \mathbb{Q}, \]
and \(H^n = 0\) for other values of \(n\). The cohomology objects \(H^n\) of the complex \(C^{\text{SD}} \otimes \mathbb{Q}\) are described as follows:
\[ H^{-1} = (VH^2C)^{\text{PD}}, \quad 0 \to (VH^1C)^{\text{PD}} \to H^0 \to (\pi_0H^0C)_{/\text{tor}} \otimes \mathbb{Q} \to 0, \]
and \(H^n = 0\) for other values of \(n\).
Proof. Tensor $\mathbb{Q}$ with the isomorphisms and exact sequences in Proposition 3.3.8.

**Proposition 3.3.12.** The morphism

$$D^{SDSD} \otimes \mathbb{Q} \to C^{SD} \otimes \mathbb{Q}$$

induces an injection onto a direct summand in degree 0 with cokernel $(VH^1C)^{PD}$ and isomorphisms on cohomology in other degrees.

Proof. From degree zero, we have a morphism

$$\left(\pi_0H^0D\right)_{tor} \otimes \mathbb{Q} \to \left(\pi_0H^0C\right)_{tor}^{LD} \otimes \mathbb{Q}.$$ 

This is an isomorphism since $\delta_{\text{Height}}$ is finite. Together with the previous proposition, the statement about degree zero follows. For degree $-1$, consider the commutative diagram

$$
\begin{array}{ccc}
D^{SDSD} \otimes \mathbb{Q} & \longrightarrow & C^{SD} \otimes \mathbb{Q} \\
\downarrow & & \downarrow \\
D^{SDSD} \hat{\otimes} A_\infty & \longrightarrow & C^{SD} \hat{\otimes} A_\infty
\end{array}
$$

The vertical morphisms induce isomorphisms in degree $-1$ by the previous proposition and Proposition 3.3.7. The last mentioned proposition also shows that the lower horizontal morphism induces an isomorphism in degree $-1$. Hence so is the upper morphism.

In particular, we have two canonical morphisms

$$D^{SDSD} \otimes \mathbb{Q} \to C^{SD} \otimes \mathbb{Q} \to (VH^1C)^{PD}.$$

**Proposition 3.3.13.** There exists a unique morphism

$$(VH^1C)^{PD}[-1] \to D^{SDSD} \otimes \mathbb{Q}$$

such that the resulting triangle

$$(VH^1C)^{PD}[-1] \to D^{SDSD} \otimes \mathbb{Q} \to C^{SD} \otimes \mathbb{Q} \to (VH^1C)^{PD}$$

is distinguished.

Proof. The existence is clear from the previous proposition. The uniqueness follows from the following general lemma below.
Lemma 3.3.14. If \( E \to F \to G \) are two morphisms in the derived category of an abelian category such that \( E \) (resp. \( G \)) is concentrated in nonpositive (resp. nonnegative) degrees and if there exists a morphism \( G[-1] \to E \) that yields a distinguished triangle \( G[-1] \to E \to F \to G \), then such a morphism \( G[-1] \to E \) is unique.

Proof. Let \( f, g : G[-1] \to E \) be two morphisms such that the two triangles \( G[-1] \to E \to F \to G \) are both distinguished. By an axiom of triangulated category, there exists an automorphism \( h \) on \( G \) such that the diagram
\[
\begin{array}{ccc}
E & \longrightarrow & F \\
\| & & \| \\
E & \longrightarrow & F \\
\end{array}
\begin{array}{ccc}
f \downarrow h & & \downarrow g \\
& & \\
0 & & 0
\end{array}
\begin{array}{ccc}
& & \\
\| & & \| \\
& & \\
E & \longrightarrow & F \\
\end{array}
\begin{array}{ccc}
g \downarrow h^{-1} & & \downarrow 0 \\
& & \\
& & \\
E & \longrightarrow & F \\
\end{array}
\begin{array}{ccc}
& & \\
& & \\
& & \\
E & \longrightarrow & F \\
\end{array}
\begin{array}{ccc}
& & \\
& & \\
& & \\
G & \longrightarrow & E[1]
\end{array}
\]
is commutative. Hence the diagram
\[
\begin{array}{ccc}
E & \longrightarrow & F \\
\downarrow 0 & & \downarrow 0 \\
E & \longrightarrow & F \\
\end{array}
\begin{array}{ccc}
f \downarrow h^{-1} & & \downarrow 0 \\
& & \\
& & \\
E & \longrightarrow & F \\
\end{array}
\begin{array}{ccc}
g \downarrow 0 & & \downarrow 0 \\
& & \\
& & \\
E & \longrightarrow & F \\
\end{array}
\begin{array}{ccc}
& & \\
& & \\
& & \\
G & \longrightarrow & E[1]
\end{array}
\]
is commutative. Therefore there exists a morphism \( r : E[1] \to G \) such that \( h - 1 = r \circ f \). But the assumptions on \( E \) and \( G \) imply that \( \text{Hom}(E[1], G) = 0 \). Hence \( r = 0 \), \( h = 1 \) and thus \( f = g \). \( \square \)

Proposition 3.3.15. The cohomology objects \( H^n \) of the complex \( R \lim \leftarrow_n D^{SDSD} \) are described as follows:

\[
0 \to (\pi_0 H^0 D)_{/\text{tor}} \otimes \mathbb{A}^\infty/Q \to H^1 \to VH^1 D \to 0, \quad H^2 = VH^2 D,
\]

and \( H^n = 0 \) for other values of \( n \). The cohomology objects \( H^n \) of the complex \( R \lim \leftarrow_n C^{SD} \) are described as follows:

\[
H^1 = (\pi_0 H^0 C)^{LD}_{/\text{tor}} \otimes \mathbb{A}^\infty/Q, \quad H^2 = (V(H^0 C)_0)^{PD},
\]

and \( H^n = 0 \) for other values of \( n \).

Proof. For \( R \lim \leftarrow_n D^{SDSD} \), apply \( R \lim \leftarrow_n \) to the groups in Proposition 3.3.8 and use Propositions 2.3.3 and 2.3.4.
For $R \lim_{\leftarrow n} C^{SD}$, we have
\[
R \lim_{\leftarrow n} R \text{Hom}_k(C, \mathbb{Z}) = R \text{Hom}_k(C \otimes \mathbb{Q}, \mathbb{Z}) = R \lim_{\leftarrow n} R \text{Hom}_k(H^0C, \mathbb{Z})
\]
by [Suz14, Proposition (2.3.3)(c)] and the torsionness result of higher cohomology in Proposition 3.3.3. The cohomology objects $H^n$ of the complex $R \text{Hom}_k(H^0C, \mathbb{Z})$ are
\[
H^0 = (\pi_0H^0C)^{LD}/\text{tor}, \quad H^1 = (\pi_0H^0C)^{PD}/\text{tor}, \quad H^2 = (T(H^0C)_0)^{PD}
\]
by the same argument as the proof of Proposition 3.3.8. Applying $R \lim_{\leftarrow n}$ to these groups and using Propositions 2.3.3 and 2.3.4, we get the desired description.

**Proposition 3.3.16.** The morphism
\[
R \lim_{\leftarrow n} D^{SDSD} \to R \lim_{\leftarrow n} C^{SD}
\]
is a surjection onto a direct summand on cohomology in degree 1 with kernel $VH^1D$ and an isomorphism on cohomology in any other degree.

**Proof.** The same argument as the proof of Proposition 3.3.12 works, this time using the isomorphism
\[
(\pi_0H^0D)_{/\text{tor}} \otimes A^\infty/Q \sim (\pi_0H^0C)_{/\text{tor}} \otimes A^\infty/Q
\]
and the commutative diagram
\[
\begin{array}{ccc}
D^{SDSD} \otimes A^\infty & \longrightarrow & C^{SD} \otimes A^\infty \\
\downarrow & & \downarrow \\
R \lim_{\leftarrow n} D^{SDSD}[1] & \longrightarrow & R \lim_{\leftarrow n} C^{SD}[1]
\end{array}
\]
In particular, we have two canonical morphisms
\[
VH^1D[-1] \to R \lim_{\leftarrow n} D^{SDSD} \to R \lim_{\leftarrow n} C^{SD}
\]
Proposition 3.3.17. There exists a unique morphism

\[ R \lim_n C^{SD} \to VH^1D \]

such that the resulting triangle

\[ VH^1D[-1] \to R \lim_n D^{SDSD} \to R \lim_n C^{SD} \to VH^1D \]

is distinguished.

Proof. Similar to the proof of Proposition 3.3.13.

Proposition 3.3.18. Consider the natural commutative diagram

\[
\begin{array}{ccc}
R \lim_n D^{SDSD} & \longrightarrow & R \lim_n C^{SD} \\
\downarrow & & \downarrow \\
D^{SDSD} \otimes Q & \longrightarrow & C^{SD} \otimes Q \\
\downarrow & & \downarrow \\
(H^1C)^{PD} & \longrightarrow & (VH^1C)^{PD}
\end{array}
\]

where the rows are distinguished. The isomorphism \( VH^1D \cong (VH^1C)^{PD} \) in Proposition 3.3.7 is the unique morphism that completes the above diagram into a morphism of distinguished triangles.

Proof. A morphism \( VH^1D \to (VH^1C)^{PD} \) with the required property exists by an axiom of triangulated categories. We need to show that such a morphism has to be the isomorphism of Proposition 3.3.7.

Consider the morphism of distinguished triangles

\[
\begin{array}{ccc}
R \lim_n D^{SDSD} & \longrightarrow & D^{SDSD} \otimes Q \\
\downarrow & & \downarrow \\
R \lim_n C^{SD} & \longrightarrow & C^{SD} \otimes Q \\
\downarrow & & \downarrow \\
(C^{SD} \otimes A^\infty) & \longrightarrow & (C^{SD} \otimes A^\infty)
\end{array}
\]

coming from Propositions 2.3.9, 3.3.7 and 3.3.10. Denote the upper triangle by \( E \to F \to G \) and the lower by \( E' \to F' \to G' \). By Propositions 3.3.6, 3.3.11 and 3.3.15, we have a commutative diagram with exact rows

\[
\begin{array}{cccc}
0 & \longrightarrow & H^0F & \longrightarrow & H^0G & \longrightarrow & H^1E & \longrightarrow & 0 \\
\downarrow & & \downarrow l & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^0F' & \longrightarrow & H^0G' & \longrightarrow & H^1E' & \longrightarrow & 0
\end{array}
\]
Hence any morphism $VH^1 D \to (VH^1 C)^{PD}$ with the required property has to be the connecting morphism

$$\text{Ker}(H^1 E \to H^1 E') \to \text{Coker}(H^0 F \to H^0 F')$$

of the snake lemma for this diagram. Using Propositions 3.3.7, 3.3.12 and 3.3.16, we can see that this is indeed the isomorphism of Proposition 3.3.7.

By Propositions 3.3.13 and 3.3.17, we have morphisms

$$C^{SD} \to C^{SD} \otimes \mathbb{Q} \to (VH^1 C)^{PD},$$

(3.3.7) $$VH^1 D[-1] \to R \lim_{\leftarrow n} D^{SDSD} \to D^{SDSD}.$$ Together with the isomorphism $(VH^1 C)^{PD} \cong VH^1 D$ of Proposition 3.3.7, we have a triangle

$$VH^1 D[-1] \to D^{SDSD} \to C^{SD} \to VH^1 D$$

**Proposition 3.3.19.** The above triangle is distinguished.

**Proof.** Let $W$ be any mapping cone of the morphism $D^{SDSD} \to C^{SD}.$ Consider the commutative diagram of distinguished triangles

$$\begin{array}{c}
R \lim_{\leftarrow n} D^{SDSD} & \longrightarrow & R \lim_{\leftarrow n} C^{SD} & \longrightarrow & R \lim_{\leftarrow n} W \\
\downarrow & & \downarrow & & \downarrow \\
D^{SDSD} & \longrightarrow & C^{SD} & \longrightarrow & W \\
\downarrow & & \downarrow & & \downarrow \\
R \lim_{\leftarrow n} (D^{SDSD} \otimes^L \mathbb{Z}/n\mathbb{Z}) & \longrightarrow & R \lim_{\leftarrow n} (C^{SD} \otimes^L \mathbb{Z}/n\mathbb{Z}) & \longrightarrow & R \lim_{\leftarrow n} (W \otimes^L \mathbb{Z}/n\mathbb{Z})
\end{array}$$

The left lower horizontal morphism is an isomorphism by Propositions 3.3.1 and 3.3.10. The (canonical choice of a) mapping fiber of the left upper horizontal morphism is $VH^1 D[-1]$ by Proposition 3.3.17. Therefore $W$ can actually be taken as $VH^1 D,$ that is, there exists a morphism of distinguished triangle

$$\begin{array}{c}
VH^1 D[-1] & \longrightarrow & R \lim_{\leftarrow n} D^{SDSD} & \longrightarrow & R \lim_{\leftarrow n} C^{SD} \\
\| & & \downarrow & & \downarrow \\
VH^1 D[-1] & \longrightarrow & D^{SDSD} & \longrightarrow & C^{SD}
\end{array}$$
The lower horizontal morphism in the left square has to be the morphism (3.3.7). For this choice of $W$, we show that the connecting morphism $C^{SD} \to VH^1D$ for the lower triangle is equal to the morphism (3.3.6) composed with the isomorphism $(VH^1C)^{PD} \cong VH^1D$ of Proposition 3.3.7. Consider the commutative diagram

\[
\begin{array}{c}
D^{SDSD} \longrightarrow C^{SD} \longrightarrow VH^1D \\
\downarrow \quad \downarrow \\
D^{SDSD} \otimes \mathbb{Q} \longrightarrow C^{SD} \otimes \mathbb{Q} \longrightarrow (VH^1C)^{PD}
\end{array}
\]

By an axiom of triangulated categories, there exists a morphism $f: VH^1D \to (VH^1C)^{PD}$ that completes the diagram into a morphism of distinguished triangles. Hence we have a morphism of distinguished triangles

\[
\begin{array}{c}
R \lim_{\xrightarrow{n}} D^{SDSD} \longrightarrow R \lim_{\xrightarrow{n}} C^{SD} \longrightarrow VH^1D \\
\downarrow \quad \downarrow f \\
D^{SDSD} \otimes \mathbb{Q} \longrightarrow C^{SD} \otimes \mathbb{Q} \longrightarrow (VH^1C)^{PD}
\end{array}
\]

Applying Proposition 3.3.18 to this diagram, we know that $f$ has to be the isomorphism given by Proposition 3.3.7. Hence the morphism $C^{SD} \to (VH^1C)^{PD}$ has to come from the morphism (3.3.6).

Suppose that we have two other objects $C', D' \in D(k)$ and a morphism $C' \otimes^L D' \to \mathbb{Z}$ satisfying the same assumptions (1)–(4) listed at the beginning of this subsection. Suppose also that we have a perfect pairing $C'' \times D'' \to \mathbb{Q}/\mathbb{Z}$ of finite étale groups over $k$. Suppose finally that we have two distinguished triangles $C' \to C \to C''$ and $D \to D' \to D''$ such that the morphisms

\[
\begin{array}{c}
D \longrightarrow D' \longrightarrow D'' \\
\downarrow \quad \downarrow \quad \downarrow l \\
C^{SD} \longrightarrow C'^{SD} \longrightarrow C''^{PD}
\end{array}
\]

form a morphism of distinguished triangles.
Proposition 3.3.20. The morphism $D \to D'$ induces an isomorphism $VH^1 D \cong VH^1 D'$. The diagram

$$
\begin{array}{ccc}
D^{SD} & \longrightarrow & D'^{SD} \\
\downarrow & & \downarrow \\
C^{SD} & \longrightarrow & C'^{SD} \\
\downarrow & & \downarrow \\
VH^1 D & \sim & VH^1 D'
\end{array}
$$

is a commutative diagram of distinguished triangles.

Proof. Since $C''$ and $D''$ are finite, the morphism $H^1 D \to H^1 D'$ is surjective with finite kernel. Hence $VH^1 D \cong VH^1 D'$. For the commutativity, the only part to check is hidden in the above diagram: the composite $VH^1 D'[\{-1\}] \to D'^{SD} \to D''$ is zero; and the composite $C'^{PD}[\{-1\}] \to C^{SD} \to VH^1 D$ is zero. These are obvious since there are no nonzero morphisms between finite groups and uniquely divisible groups with any shift. \[ \square \]

3.4 Main theorem

Now we apply the results of the previous subsection to $C = R\Gamma(X, \mathcal{A})$ and $D = R\Gamma(X, \mathcal{A}_\vee')$. Statement (5) in the following theorem proves Theorem A in Introduction.

Theorem 3.4.1.

(1) We have $H^n(X, \mathcal{A}) \in \text{LAlg}/k$ for any $n$ and $H^n(X, \mathcal{A}) = 0$ for $n \neq 0, 1, 2$.

(2)

- $\Gamma(X, \mathcal{A})_0$ is an abelian variety.
- $\pi_0 \Gamma(X, \mathcal{A}) \in \text{FGEt}/k$.
- $H^1(X, \mathcal{A})_0$ is unipotent quasi-algebraic.
- $\pi_0 H^1(X, \mathcal{A}) \in \text{Et}/k$ is torsion of cofinite type.
- $H^1(X, \mathcal{A})_{\text{div}} \in \text{Et}/k$ is torsion of cofinite type.
- $H^1(X, \mathcal{A})_{\text{div}} \in \text{Alg}_{\text{uc}}/k$.
- $H^1(X, \mathcal{A}) \in \text{Et}/k$ is divisible torsion of cofinite type.

(3) The same statements as (1) and (2) hold with $\mathcal{A}$ replaced by $\mathcal{A}_\vee'$.

(4) The group of $k'$-valued points of $H^1(X, \mathcal{A})$ for $k' \in \k^\text{indrat}$ is given by $\Pi(\mathcal{A}_{\overline{k'}}/X_{\overline{k'}})^{G_{k'}}$ in the notation of the paragraph before Proposition 3.2.10.
(5) There exists a canonical distinguished triangle
\[ \mathbb{R}
\Gamma(X, \mathcal{A}_0^\vee)^{\text{SDSD}} \to \mathbb{R}
\Gamma(X, \mathcal{A})^{\text{SD}} \to V \mathcal{H}^1(X, \mathcal{A}_0^\vee)_{\text{div}}. \]

The first morphism is induced from the morphism \(3.2.1\).

(6) This distinguished triangle induces, on cohomology, the following duality pairings and morphisms:

(a) Pontryagin duality between \(T(\Gamma(X, \mathcal{A}_0^\vee)_0)\) and \(H^2(X, \mathcal{A})\).
(b) Pontryagin duality between \(T(\Gamma(X, \mathcal{A})_0)\) and \(H^2(X, \mathcal{A}^\vee)\).
(c) Pontryagin duality between \((\pi_0 \Gamma(X, \mathcal{A}))_{\text{tor}}\) and \(\pi_0 \mathcal{H}^1(X, \mathcal{A}_0^\vee)_{/\text{div}}\).
(d) Pontryagin duality between \((\pi_0 \Gamma(X, \mathcal{A}^\vee))_{\text{tor}}\) and \(\pi_0 \mathcal{H}^1(X, \mathcal{A})_{/\text{div}}\).
(e) An injection \(\pi_0(\Gamma(X, \mathcal{A}_0^\vee))_{/\text{tor}} \hookrightarrow \pi_0(\Gamma(X, \mathcal{A}))_{/\text{tor}}\) whose cokernel \(\delta_{\text{Height}}\) is finite étale.
(f) An injection \((\pi_0 \mathcal{H}^1(X, \mathcal{A}))_{/\text{div}} \hookrightarrow TH^1(X, \mathcal{A}_0^\vee)_{/\text{div}}\) whose cokernel \(\delta_{\text{Tran}}\) is finite étale.
(g) A surjection \(H^1(X, \mathcal{A})^{\text{SD}} \twoheadrightarrow (H^1(X, \mathcal{A}^\vee)_{/\text{div})_0}\) whose kernel \(\delta_{\text{CT}}\) is finite étale.
(h) An exact sequence \(0 \to \delta_{\text{Tran}} \to \delta_{\text{Height}} \to \delta_{\text{CT}} \to 0\).

In particular, the morphism \(6f\) induces a Pontryagin duality between \(V \mathcal{H}^1(X, \mathcal{A})_{\text{div}}\) and \(V \mathcal{H}^1(X, \mathcal{A}_0^\vee)_{\text{div}}\).

(7) The morphism \(6e\) agrees with the height pairing.

(8) The distinguished triangle \(5\) and the corresponding triangle with \(\mathcal{A}, \mathcal{A}^\vee\) switched fit in the commutative diagram of distinguished triangles

\[
\begin{array}{ccc}
\mathbb{R}
\Gamma(X, \mathcal{A}_0^\vee)^{\text{SDSD}} & \to & \mathbb{R}
\Gamma(X, \mathcal{A})^{\text{SD}} \\
\downarrow & & \downarrow \\
\mathbb{R}
\Gamma(X, \mathcal{A}^\vee)^{\text{SDSD}} & \to & \mathbb{R}
\Gamma(X, \mathcal{A}_0)^{\text{SD}} \\
\downarrow & & \downarrow \\
\bigoplus_x \text{Res}_{k_x/k} \pi_0(\mathcal{A}_x^\vee) & \sim & \bigoplus_x \text{Res}_{k_x/k} \pi_0(\mathcal{A}_x)^{\text{PD}} \\
\end{array}
\]

\[ \mathbb{R}
\Gamma(X, \mathcal{A}_0^\vee)^{\text{SDSD}} \to \mathbb{R}
\Gamma(X, \mathcal{A})^{\text{SD}} \to V \mathcal{H}^1(X, \mathcal{A}_0^\vee)_{\text{div}}. \]

The left lower horizontal morphism is the sum over all closed points \(x \in X\) of the Weil restrictions of Grothendieck’s pairings, which is an isomorphism \([\text{Suz14, Theorem C}]\).

Proof. (1) and (2) follow from Propositions 3.2.15 and 3.3.3. (3) follows from Proposition 3.2.4. (4) is Proposition 3.2.10. (5) follows from Proposition 3.3.19. (6) follows from Propositions 3.3.3 and 3.2.15. (7) is Proposition 3.2.13. (8) follows from Propositions 3.2.9 and 3.3.20.
Several comments are in order. We consider (6g) as a geometric analogue of the Cassels–Tate pairing in view of Proposition 4.2.8, whence the symbol $\delta_{\text{CT}}$. The divisible part of the Tate–Shafarevich group when $k$ is algebraically closed is of transcendental nature, whence the symbol $\delta_{\text{Tan}}$. Confusingly, the morphisms in (6f) and (6g) are from the duals, not to.

The exact sequence (6h) is mysterious. It came from applying the formal procedure of Lemma 3.3.4 to the two exact sequences (3.3.1) and (3.3.2) (where one should note that the central term $H^0X$ in the notation there is the “Selmer scheme” explained below). The morphism $\delta_{\text{Height}} \to \delta_{\text{CT}}$ is induced by the morphism

$$\pi_0(\Gamma(X, A))^{1D}/_{\text{tor}} \to H^1(X, A)^{0}_{0}$$

of Propositions 3.3.8 and 3.3.9 (i.e., the latter morphism factors through $\delta_{\text{Height}} \to \delta_{\text{CT}}$). This morphism is analogous to Artin’s period map [Art74] for supersingular K3 surfaces.

Milne [Mil06, III, paragraph before Theorem 11.6] made the hypothesis that the sheaf $R^2\pi_X^\text{perf}A$ on $\text{Spec } k_{\text{perf}}^\text{et}$ has no connected part. This sheaf restricted to $\text{Spec } k_{\text{indrat}}^\text{et}$ is $H^2(X, A)$ by Proposition 2.7.1. Hence the result $H^2(X, A) \in \text{Et}/k$ in (2) above says that his hypothesis is true at least “birationally.”

Also [Mil06, III, Theorem 11.6] (plus the fact that $H^2(K \otimes \overline{k}, N) = 0$ for finite flat $N$ and hence $H^2(K \otimes \overline{k}, A) = 0$) says that $H^2(X, A)$ injects into the Pontryagin dual of $T(\Gamma(X, A_0^\vee)_0)$. Thus (6a) above shows that this injection is actually bijective.

We have an exact sequence

$$0 \to \pi_0(\Gamma(X, A))^{1D}/_{\text{tor}} \otimes \mathbb{Q}/\mathbb{Z} \to H^0(R\Gamma(X, A) \otimes^L \mathbb{Q}/\mathbb{Z}) \to H^1(X, A) \to 0$$

in $L^f/\text{Alg}_{\text{uc}}/k$. The first term is the Mordell–Weil group tensored with $\mathbb{Q}/\mathbb{Z}$. In Introduction, we called the last term the Tate–Shafarevich scheme. This terminology is justified by (4). Along this line, the middle term might be called the Selmer scheme.

**Remark 3.4.2.** As we saw after Lemma 3.3.4, if the finite étale $p$-groups $H^1(X, A)_{0,\text{div}}$ and $H^1(X, A_0^\vee)_{0,\text{div}}$ (intersection of connected part and divisible part) are zero, then the morphisms in (6f) and (6g) are simplified to more symmetric expressions

$$H^1(X, A)_{\text{div}}^{PD} \hookrightarrow TH^1(X, A_0^\vee)_{\text{div}} \quad \text{and} \quad H^1(X, A)_{0}^{SD'} \to H^1(X, A_0^\vee).$$
It is not clear whether these vanishing conditions are always satisfied or not. To see cases where the conditions are indeed satisfied, let $A$ be the Jacobian of a proper smooth geometrically connected curve $C$ over $K$ with a $K$-rational point (in particular, $A \cong A^\vee$) and $C$ a proper flat regular model over $X$ of $C$. As we saw in the proof of Proposition 3.2.11, there exists a canonical isomorphism $H^1(X, A) \cong H^2(C, G_m)$. Let $H^2(C, G_m)$ be the (pro-)étale sheafification of the presheaf $k' \in k^{\text{indrat}} \mapsto H^2(C_{k'}, G_m)$, which is locally of finite presentation. Then the above isomorphism extends to an isomorphism $H^1(X, A) \cong H^2(C, G_m)$. Note that $H^1(X, \mathcal{A}_0)$ surjects onto $H^1(X, A)$ with finite kernel by Proposition 3.2.4.

If $C$ is an Artin supersingular K3 surface (hence an elliptic fibered over $X$), then [Art74, (4.2), (4.4)] shows that $C$ is Shioda supersingular (i.e., satisfies the Tate conjecture) and $H^2(C, G_m) \cong G_a$. In particular, $H^2(C, G_m)_{\text{ét}} = 0$. If $C$ is an Artin nonsupersingular K3 surface, then [MR15, Proposition 4.7, Lemma 2.1] applied to the de Rham–Witt complex of $C$ with $r = 1$, $j = 3$ together with [Ill79, II, (5.7.6), Section 7.2(a)] (or [Yui86, Proposition (4.4)]) shows that $H^2(C, G_m)$ is étale. Hence again $H^2(C, G_m)_{\text{ét}} = 0$. In this case, [Ill79, II, Proposition 5.2] shows that the dimension of $V_p H^2(C, G_m)$ over $\mathbb{Q}_p$ is $22 - 2h - \rho$, where 22 is the second Betti number of $C$, $h$ the height of the formal Brauer group of $C$ and $\rho$ the geometric Picard number of $C$. As soon as the inequality $\rho \leq 22 - 2h$ is strict, the group $H^2(C, G_m)_{\text{div}}$ is nonzero.

Similar arguments apply to the case where $C$ is an abelian surface, showing that either $H^2(C, G_m)_{\text{ét}} = 0$ (nonsupersingular case) or $H^2(C, G_m)_{\text{div}} = 0$ (supersingular case). Similarly, we know that $H^2(C, G_m)_{\text{div}} = 0$ whenever the geometric Picard number of $C$ is equal to the second Betti number of $C$. This includes the cases of rational surfaces, ruled surface, Enriques surfaces ([BM76, Theorem 4]; we drop the condition that $C$ has a $K$-rational point) and quasi-elliptic surfaces ([BM76, Proposition 12]; we drop the condition that $C$ is smooth over $K$). If $C$ is the product of $X$ with another curve $Y$ such that $X$ is a supersingular elliptic curve and $Y$ is a genus two curve whose Jacobian is absolutely simple with $p$-rank one (see [HOMNS11, Section 8 Examples; Corollary 3] for the existence of such a curve; again we drop the condition that $C$ has a $K$-rational point), then a calculation similar to the above shows that $H^2(C, G_m)_{\text{ét}} \cong G_a$ and $T_p H^2(C, G_m)_{\text{div}}$ has rank four. It is not clear whether $H^2(C, G_m)_{\text{ét}}$ is trivial or not in this case.

If there is a case where $H^1(X, \mathcal{A})_{\text{ét}}$ is nonzero, it might mean that supersingularity and nonsupersingularity are somehow “mixed up” in $A$. 
(or $\mathcal{C}$) in an interesting way. Perhaps it could be the Tate conjecture that does not allow such a mixing phenomenon.

§4. Some more results

Let $X$ and $A$ be as in the beginning of the previous section.

4.1 Duality for Néron models over open curves

In this subsection, we set

$$V = V\mathbf{H}^1(X, A^\vee_0)_{\text{div}} = (V\mathbf{H}^1(X, A)_{\text{div}})^{\text{PD}},$$

the last isomorphism coming from Theorem 3.4.1 (6). Hence (5) of the same theorem can be written as the canonical distinguished triangle

$$R\Gamma(X, A^\vee_0)^{\text{SDSD}} \to R\Gamma(X, A)^{\text{SD}} \to V.$$

Let $U \subset X$ be a dense open subscheme. We have a pairing

$$R\Gamma(U, A^\vee_0) \otimes^L R\Gamma_c(U, A) \to R\Gamma_c(U, G_m)[1] \to \mathbb{Z},$$

where the last morphism is the trace morphism (3.1.1). This induces a morphism

$$R\Gamma(U, A^\vee_0) \to R\Gamma_c(U, A)^{\text{SD}}.$$

Proposition 4.1.1. We have a canonical morphism between canonical distinguished triangles

$$\bigoplus_{x \notin U} R\Gamma_x(\hat{O}_x, A^\vee_0) \longrightarrow R\Gamma(X, A^\vee_0) \longrightarrow R\Gamma(U, A^\vee_0)$$

$$\bigoplus_{x \notin U} R\Gamma(\hat{O}_x, A)^{\text{SD}} \longrightarrow R\Gamma(X, A)^{\text{SD}} \longrightarrow R\Gamma_c(U, A)^{\text{SD}}$$

The left vertical morphism is an isomorphism.

Proof. This follows from the same method as the proof of Proposition 3.2.1.

Proposition 4.1.2. The morphisms

$$R\Gamma(U, A^\vee_0) \otimes^L \mathbb{Q}/\mathbb{Z} \to R\Gamma(U, A^\vee_0)^{\text{SDSD}} \otimes^L \mathbb{Q}/\mathbb{Z}$$

$$\to R\Gamma_c(U, A)^{\text{SD}} \otimes^L \mathbb{Q}/\mathbb{Z}$$

are isomorphisms.
Proof. We show that the first morphism is an isomorphism. We have

\[ R\Gamma(X, A_0^\vee) \otimes L \mathbb{Q}/\mathbb{Z} \xrightarrow{\sim} R\Gamma(X, A_0^\vee)^{SD SD} \otimes L \mathbb{Q}/\mathbb{Z} \]

by the same proof as Proposition 3.3.10. We know that \( R\Gamma_x(\hat{O}_x, A_0^\vee) \in D^b(I\text{Alg}_{\text{uc}}) \) is Serre reflexive for any \( x \) by Proposition 2.5.3 and [Suz14, Proposition (2.4.1)(b)]. Hence

\[ R\Gamma_x(\hat{O}_x, A_0^\vee) \otimes L \mathbb{Q}/\mathbb{Z} \xrightarrow{\sim} R\Gamma_x(\hat{O}_x, A_0^\vee)^{SD SD} \otimes L \mathbb{Q}/\mathbb{Z}. \]

Therefore the previous proposition gives the result.

The composite morphism has already been shown to be an isomorphism essentially in the proof of Proposition 3.2.1.

Proposition 4.1.3. We have

\[ R\Gamma(X, A_0^\vee)^{SD SD} \otimes \mathbb{Q} \xrightarrow{\sim} R\Gamma(U, A_0^\vee)^{SD SD} \otimes \mathbb{Q}, \]

\[ R\Gamma(X, A)^{SD} \otimes \mathbb{Q} \xrightarrow{\sim} R\Gamma_c(U, A)^{SD} \otimes \mathbb{Q}. \]

They induce a canonical isomorphism between distinguished triangles

\[
\begin{array}{cccc}
R\Gamma(X, A_0^\vee)^{SD SD} \otimes \mathbb{Q} & \longrightarrow & R\Gamma(X, A)^{SD} \otimes \mathbb{Q} & \longrightarrow & V \\
\downarrow & & \downarrow & & \\
R\Gamma(U, A_0^\vee)^{SD SD} \otimes \mathbb{Q} & \longrightarrow & R\Gamma_c(U, A)^{SD} \otimes \mathbb{Q} & \longrightarrow & V
\end{array}
\]

Proof. An object of \( I\text{Alg}_{\text{uc}}/k \) is torsion since a unipotent quasi-algebraic group in positive characteristic is torsion. Hence Propositions 2.5.3 and 2.5.4 show that \( R\Gamma_x(\hat{O}_x/k, A_0^\vee) \) and \( R\Gamma(\hat{O}_x/k, A)^{SD} \) for any \( x \) are killed after tensored with \( \mathbb{Q} \). This implies the result.

From this, we have canonical morphisms

\[ R\Gamma_c(U, A)^{SD} \rightarrow R\Gamma_c(U, A)^{SD} \otimes \mathbb{Q} \rightarrow V \]

\[ \rightarrow R\Gamma(X, A_0^\vee)^{SD SD}[1] \rightarrow R\Gamma(U, A_0^\vee)^{SD SD}[1]. \]

Here is the duality for \( A \) over \( U \):

Proposition 4.1.4. Consider the triangle

\[ V[-1] \rightarrow R\Gamma(U, A_0^\vee)^{SD SD} \rightarrow R\Gamma_c(U, A)^{SD} \rightarrow V \]
coming from the above morphisms. This triangle is distinguished. The diagram

\[
\begin{array}{ccc}
R\Gamma(X, A^\vee_0)^{SDSD} & \longrightarrow & R\Gamma(X, A)^{SD} \\
\downarrow & & \downarrow \\
R\Gamma(U, A^\vee_0)^{SDSD} & \longrightarrow & R\Gamma_c(U, A)^{SD}
\end{array}
\]

is a morphism of distinguished triangles.

**Proof.** Choose a mapping cone distinguished triangle

(4.1.1) \[ R\Gamma(U, A^\vee_0)^{SDSD} \rightarrow R\Gamma_c(U, A)^{SD} \rightarrow E. \]

Applying \((\cdot) \otimes Q\) and \((\cdot) \otimes^{L} Q/Z\), we get a commutative diagram of distinguished triangles

\[
\begin{array}{ccc}
R\Gamma(U, A^\vee_0)^{SDSD} & \longrightarrow & R\Gamma_c(U, A)^{SD} \\
\downarrow & & \downarrow \\
R\Gamma(U, A^\vee_0)^{SDSD} \otimes Q & \longrightarrow & R\Gamma_c(U, A)^{SD} \otimes Q \\
\downarrow & & \downarrow \\
R\Gamma(U, A^\vee_0)^{SDSD} \otimes^{L} Q/Z & \longrightarrow & R\Gamma_c(U, A)^{SD} \otimes^{L} Q/Z \\
\end{array}
\]

We have \(E \otimes^{L} Q/Z = 0\) by Proposition 4.1.2. Hence \(E \cong E \otimes Q\) and \(E\) is uniquely divisible. By an axiom of triangulated categories, the distinguished triangle in the middle row is isomorphic to the distinguished triangle

\[ R\Gamma(U, A^\vee_0)^{SDSD} \otimes Q \rightarrow R\Gamma_c(U, A)^{SD} \otimes Q \rightarrow V \]

in the previous proposition. Therefore the distinguished triangle (4.1.1) can be chosen so that \(E = V\) and the diagram

\[
\begin{array}{ccc}
R\Gamma(U, A^\vee_0)^{SDSD} & \longrightarrow & R\Gamma_c(U, A)^{SD} \\
\downarrow & & \downarrow \\
R\Gamma(U, A^\vee_0)^{SDSD} \otimes Q & \longrightarrow & R\Gamma_c(U, A)^{SD} \otimes Q \\
\end{array}
\]

is a morphism of distinguished triangles. By an axiom of triangulated categories, the right half of the commutative diagram in Proposition 4.1.1
can be extended to a morphism of distinguished triangles
\[
\begin{array}{ccc}
R\Gamma(X, A_0^\vee)^{SDSD} & \longrightarrow & R\Gamma(X, A)^{SD} \\
\downarrow & & \downarrow \\
R\Gamma(U, A_0^\vee)^{SDSD} & \longrightarrow & R\Gamma_c(U, A)^{SD}
\end{array}
\]

Tensoring $\mathbb{Q}$ has to recover the isomorphism of distinguished triangles in the previous proposition. Hence the right vertical morphism is the identity. Thus we have a morphism of distinguished triangles
\[
\begin{array}{ccc}
V[-1] & \longrightarrow & R\Gamma(X, A_0^\vee)^{SDSD} \\
\downarrow & & \downarrow \\
V[-1] & \longrightarrow & R\Gamma(U, A_0^\vee)^{SDSD}
\end{array}
\]

With these diagrams, the result follows.

### 4.2 Link to the finite base field case

In this section, we assume that the base field $k$ is the finite field $\mathbb{F}_q$ of $q$ elements. The main reference about the finite base field case of duality for abelian varieties is [Mil06, II, Section 5; III, Section 9]. As in [Suz14, Section 10], let $k_{\text{prozar}}$ be the full subcategory of the category of $k$-algebras consisting of filtered unions of finite products of copies of $k$. For $k' \in k_{\text{prozar}}$, we say that a finite family $\{k'_i\}$ of $k'$-algebras is a covering if $\prod k'_i$ is faithfully flat over $k'$. This defines a topology on $k_{\text{prozar}}$.

We denote the resulting site by $\text{Spec } k_{\text{prozar}}$. The identity functor defines a morphism of sites $\text{Spec } k_{\text{indrat}} \rightarrow \text{Spec } k_{\text{prozar}}$. By abuse of notation, we denote its pushforward functor $D(k_{\text{indrat}}) \rightarrow D(k_{\text{prozar}})$ by $R\Gamma(k_{\text{proet}}, \cdot)$, with cohomologies $H^n(k_{\text{proet}}, \cdot)$ and $H^0 = \Gamma$. If $C \in \text{Ab}(k_{\text{indrat}})$ is locally of finite presentation, then $H^n(k_{\text{proet}}, C)$ is a constant sheaf for any $n$.

We will first relate $R \text{Hom}_k(\cdot, \mathbb{Q}/\mathbb{Z})$ ($= R \text{Hom}_{k_{\text{indrat}}}(\cdot, \mathbb{Q}/\mathbb{Z})$) to $R \text{Hom}_{k_{\text{prozar}}}(\cdot, \mathbb{Q}/\mathbb{Z})$. For any $C \in D(k_{\text{indrat}})$, we have morphisms
\[
R\Gamma(k_{\text{proet}}, k_{\text{proet}}(C, \mathbb{Q}/\mathbb{Z})) \\
\rightarrow R \text{Hom}_{k_{\text{prozar}}}(R\Gamma(k_{\text{proet}}, C), R\Gamma(k_{\text{proet}}, \mathbb{Q}/\mathbb{Z})) \\
\rightarrow R \text{Hom}_{k_{\text{prozar}}}(R\Gamma(k_{\text{proet}}, C), \mathbb{Q}/\mathbb{Z})[-1],
\]

where the last morphism comes from $H^1(k, \mathbb{Q}/\mathbb{Z}) = \mathbb{Q}/\mathbb{Z}$ and $H^n(k, \mathbb{Q}/\mathbb{Z}) = 0$ for $n \geq 2$. 


Proposition 4.2.1. If $C \in \text{IPAlg}/k$, then the above morphism

$$R \Gamma(k_{\text{proet}}, R \text{Hom}_k(C, \mathbb{Q}/\mathbb{Z})) \to R \text{Hom}_{k_{\text{prozar}}}(R \Gamma(k_{\text{proet}}, C), \mathbb{Q}/\mathbb{Z})[-1]$$

is an isomorphism.

Proof. The same proof as [Suz14, Proposition (10.4)] works. $\square$

We denote $(\cdot)_{\text{PD}} = R \text{Hom}_{k_{\text{prozar}}}(\cdot, \mathbb{Q}/\mathbb{Z})$. If $C \in \text{Ab}(k_{\text{indrat}}_{\text{proet}})$ is an extension of a torsion (constant) group by a profinite group, then $C_{\text{PD}} = \text{Hom}_{k_{\text{prozar}}}(C, \mathbb{Q}/\mathbb{Z})$ is the usual Pontryagin dual. It follows that if $C = (\prod \mathbb{Z}/l\mathbb{Z})/(\bigoplus \mathbb{Z}/l\mathbb{Z})$, where $l$ ranges over all primes, then $C_{\text{PD}} \cong C[-1]$.

By abuse of notation, we denote the composite of

$$R \Gamma(X, \cdot): D(X_{\text{fppf}}) \to D(k_{\text{proet}})$$

and

$$R \Gamma(k_{\text{proet}}, \cdot): D(k_{\text{proet}}) \to D(k_{\text{prozar}})$$

by $R \Gamma(X, \cdot)$, with cohomologies $H^n(X, \cdot)$ and $H^0 = \Gamma$. If $F \in \text{Ab}(X_{\text{fppf}})$ is locally of finite presentation, then $H^n(X, F)$ is a constant sheaf for any $n$.

Now let $F$ be $A$, the Néron model of an abelian variety $A$ as before. We will deduce the duality for $R \Gamma(X, A)$ from Proposition 3.3.2. The group $\Gamma(X, A)$ is the Mordell–Weil group of $A$, which is finitely generated. We have an exact sequence

$$0 \to \Pi(A/K) \to H^1(X, A) \to \bigoplus_x H^1(k_x, \pi_0(A_x))$$

by [Mil06, III, Proposition 9.2]. In particular, $H^1(X, A)$ is a torsion group of cofinite type,

$$H^1(X, A)_{\text{div}} = \Pi(A/K)_{\text{div}},$$

and the sequence

$$0 \to \Pi(A/K)_{\text{div}} \to H^1(X, A)_{\text{div}} \to \bigoplus_x H^1(k_x, \pi_0(A_x))$$

is exact. It seems that the finiteness of $\Pi(A/K)_{\text{div}}$ is not known unconditionally in general. Hence we cannot assume that $H^1(X, A)$ is divisibly ML. By [Mil06, II, Proposition 5.1(a); III, Lemma 7.10(c)], the group $H^2(X, A)$ is torsion and $H^n(X, A) = 0$ for $n \geq 3$. 
**Proposition 4.2.2.** Applying $R\Gamma(k_{proet}, \cdot)$ to the second isomorphism in Proposition 3.3.2 gives an isomorphism

$$R\lim_{\leftarrow n}(R\Gamma(X, A_0^\vee) \otimes L \mathbb{Z}/n\mathbb{Z}) \sim (R\Gamma(X, A) \otimes L \mathbb{Q}/\mathbb{Z})^{PD}[-1].$$

**Proof.** This follows from the previous proposition. □

The morphism in this proposition can alternatively be defined by the morphisms

$$R\Gamma(X, A_0^\vee) \otimes L R\Gamma(X, A) \to R\Gamma(X, G_m)[1] \to H^3(X, G_m)[-2] \cong \mathbb{Q}/\mathbb{Z}[-2].$$

The proposition says that this is a perfect pairing up to uniquely divisibles. This more or less recovers [Mil06, III, Theorem 9.4]. We will state an integral version using Weil-étale cohomology at the end of this subsection.

We will express the conjectural finiteness of $\text{III}(A/K)$ in terms of $R\Gamma(X, A)$. Let $TH^1(X, A_0^\vee)_{\text{div}}$ be the profinite Tate module of $H^1(X, A_0^\vee)_{\text{div}}$. Let $VH^1(X, A_0^\vee)_{\text{div}}$ be $(TH^1(X, A_0^\vee)_{\text{div}}) \otimes \mathbb{Q}$. We need the following variant of [Mil06, III, Theorem 9.4(b)] (cf. [KT03, Proposition 2.4(3)]), which we deduce from the above proposition.

**Proposition 4.2.3.** In $H^0$, the isomorphism in Proposition 4.2.2 induces an exact sequence

$$0 \to \Gamma(X, A_0^\vee)^{\wedge} \to H^2(X, A)^{PD} \to TH^1(X, A_0^\vee)_{\text{div}} \to 0.$$

In particular, $H^2(X, A)$ is divisibly ML and of cofinite type.

**Proof.** Let $C$ be the isomorphic object in Proposition 4.2.2. We have a distinguished triangle

$$R\lim_{\leftarrow n}\left((\tau_{\leq 1} R\Gamma(X, A_0^\vee) \otimes L \mathbb{Z}/n\mathbb{Z})\right) \to C \to R\lim_{\leftarrow n}(H^2(X, A_0^\vee) \otimes L \mathbb{Z}/n\mathbb{Z})[-2].$$

The third term is concentrated in degrees $\geq 1$. Hence the first and second terms have the same $H^0$. Since $\Gamma(X, A)$ is finitely generated and $H^1(X, A)$ is torsion of cofinite type, we can apply Proposition 2.3.8 (or its version for Spec $k_{prozar}$) to obtain an exact sequence

$$0 \to \Gamma(X, A_0^\vee)^{\wedge} \to H^0 C \to TH^1(X, A_0^\vee)_{\text{div}} \to 0.$$
On the other hand, the cohomology objects of \( R\Gamma(X, \mathcal{A}) \otimes^L \mathbb{Q}/\mathbb{Z} \) are torsion (constant) groups, with \( H^1 \) given by \( H^2(X, \mathcal{A}) \). Hence \( H^0C = H^2(X, \mathcal{A})^{PD} \). This gives the stated exact sequence. Since \( \Gamma(X, \mathcal{A}) \) is finitely generated and \( H^1(X, \mathcal{A}) \) is of cofinite type, it follows that \( H^2(X, \mathcal{A}) \) is divisibly ML and of cofinite type.

We will describe \( H^n(k_{proet}, VH^1(X, \mathcal{A})_{div}) \) for each \( n \) in terms of \( H^1(X, \mathcal{A}) \). Note that for a finite \'{e}tale group \( C \) over \( k \), the object \( R\Gamma(k_{proet}, C) \) is the mapping fiber of the endomorphism \( F - 1 \) on \( C \), where \( F \) is the \( q \)th power Frobenius morphism. Taking limits, we know that the same is true for an ind-profinite-\'{e}tale group \( C \). Since \( H^1(X, \mathcal{A})_{div} \) is a torsion \'{e}tale group of cofinite type by Theorem 3.4.1 (2), we know that \( R\Gamma(k_{proet}, VH^1(X, \mathcal{A})_{div}) \) is the mapping fiber of \( F - 1 \) on \( VH^1(X, \mathcal{A})_{div} \).

**Proposition 4.2.4.** The natural morphism

\[
(4.2.1)\quad VH^1(X, \mathcal{A})_{div} \to \Gamma(k_{proet}, VH^1(X, \mathcal{A})_{div})
\]

is an isomorphism. There exists a canonical exact sequence

\[
(4.2.2)\quad 0 \to H^1(X, \mathcal{A})_{div}^\wedge \to H^1(k_{proet}, VH^1(X, \mathcal{A})_{div}) \to (VH^1(X, \mathcal{A}_{\text{div}})_{div})^{PD} \to 0.
\]

*A priori*, \( H^1(X, \mathcal{A})_{div} \) might contain a subgroup isomorphic to \( \bigoplus_l \mathbb{Z}/l\mathbb{Z} \) for example. In this case, the term \( H^1(X, \mathcal{A})_{div}^\wedge / H^1(X, \mathcal{A})_{div} \) above would contain \( (\prod_l \mathbb{Z}/l\mathbb{Z})/(\bigoplus_l \mathbb{Z}/l\mathbb{Z}) \).

**Proof.** We have

\[
R\Gamma(X, \mathcal{A}) \hat{\otimes} \mathbb{A}^\infty = R\Gamma(k_{proet}, R\Gamma(X, \mathcal{A}) \hat{\otimes} \mathbb{A}^\infty).
\]

Denote these isomorphic objects by \( C \). We first calculate \( H^nC \) through the left-hand side. Recall that \( \Gamma(X, \mathcal{A}) \) is finitely generated, \( H^1(X, \mathcal{A}) \) is torsion of cofinite type, and \( H^2(X, \mathcal{A}) \) is torsion, divisibly ML and of cofinite type (Proposition 4.2.3). Hence Proposition 2.3.8 gives exact sequences

\[
(4.2.3)\quad 0 \to \Gamma(X, \mathcal{A}) \otimes \mathbb{A}^\infty \to H^0C \to VH^1(X, \mathcal{A})_{div} \to 0,
\]

\[
0 \to H^1(X, \mathcal{A})^\wedge \otimes \mathbb{Q} \to H^1C \to VH^2(X, \mathcal{A})_{div} \to 0
\]

and \( H^nC = 0 \) for \( n \neq 0, 1 \). We have

\[
H^1(X, \mathcal{A})^\wedge \otimes \mathbb{Q} = \frac{H^1(X, \mathcal{A})_{div}^\wedge}{H^1(X, \mathcal{A})_{div}}.
\]
Next we calculate $H^nC$ through the right-hand side. By Proposition 3.3.6 and Theorem 3.4.1 (2), (6a), the $H^{-1}$ (resp. $H^1$) of $R\Gamma(X, A) \otimes \mathbb{A}^\infty$ is the rational Tate module (resp. the Pontryagin dual of the rational Tate module) of an abelian variety over $k$. For any abelian variety $B$ over $k$, we have $R\Gamma(k_{proet}, B) = B(k)$ by Lang’s theorem, and $B(k)$ is finite. Hence $R\Gamma(k_{proet}, VB) = R\Gamma(k_{proet}, (VB)^{PD}) = 0$. Therefore

$C = R\Gamma\left(k_{proet}, H^0\left(R\Gamma(X, A) \otimes \mathbb{A}^\infty\right)\right)$.

By Propositions 3.3.6 and 3.3.7, we have a split exact sequence

$$0 \to \pi_0(\Gamma(X, A)) \otimes \mathbb{A}^\infty \to H^0\left(R\Gamma(X, A) \otimes \mathbb{A}^\infty\right) \to VH^1(X, A)_{\text{div}} \to 0$$

in $\text{Ab}(k_{proet}^{\text{indrat}})$. Since $\Gamma(X, A)_0$ is the perfection of an abelian variety over $k$, the group $H^n(k_{proet}, \Gamma(X, A)_0)$ is zero for $n \geq 1$ by Lang’s theorem and finite for $n = 0$. Therefore $H^n(k_{proet}, \Gamma(X, A))$ is $\Gamma(X, A)$ if $n = 0$ and $H^n(k_{proet}, \pi_0(\Gamma(X, A)))$ if $n \geq 1$, which is zero if $n \geq 2$. Since $\pi_0(\Gamma(X, A)) \in \text{FGEt}/k$, the Frobenius action on $\pi_0(\Gamma(X, A)) \otimes \mathbb{Q}$ and hence on $\pi_0(\Gamma(X, A)) \otimes \mathbb{A}^\infty$ are semisimple. Therefore its invariant part and coinvariant part agree. Hence

$$H^n(k_{proet}, \pi_0(\Gamma(X, A)) \otimes \mathbb{A}^\infty) = \begin{cases} \Gamma(X, A) \otimes \mathbb{A}^\infty & \text{if } n = 0, 1, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore we have split exact sequences

$$(4.2.4) \quad 0 \to \Gamma(X, A) \otimes \mathbb{A}^\infty \to H^0C \to \Gamma(k_{proet}, VH^1(X, A)_{\text{div}}) \to 0,
0 \to \Gamma(X, A) \otimes \mathbb{A}^\infty \to H^1C \to H^1(k_{proet}, VH^1(X, A)_{\text{div}}) \to 0,$$

and $H^nC = 0$ for $n \neq 0, 1$.

Comparing the first exact sequences in (4.2.3) and (4.2.4), we know that (4.2.1) is an isomorphism. By Proposition 4.2.3, we have an exact sequence

$$0 \to (VH^1(X, A_0^\vee)_{\text{div}})^{PD} \to VH^2(X, A)_{\text{div}} \to (\Gamma(X, A_0^\vee) \otimes \mathbb{A}^\infty)^{PD} \to 0.$$
The composite map from $\Gamma(X, A) \otimes A^\infty$ to $(\Gamma(X, A^\vee_0) \otimes A^\infty)^{\text{PD}}$ is given by the height pairing tensored with $A^\infty$, which is an isomorphism. Therefore the composite map from $D$ to $H^1(k_{\text{proet}}, VH^1(X, A_{\text{div}}))$ is an isomorphism. This gives (4.2.2) since $D$ is an extension of $(VH^1(X, A^\vee_0)_{\text{div}})^{\text{PD}}$ by $H^1(X, A)^{\wedge}_{\text{div}}/H^1(X, A)/_{\text{div}}$.

**Proposition 4.2.5.** The following are equivalent:

- $\text{III}(A/K)$ is finite.
- $R\Gamma(k_{\text{proet}}, VH^1(X, A_{\text{div}}))$ is zero.

*Proof.* The finiteness of $\text{III}(A/K)$ is equivalent to the finiteness of $H^1(X, A)$, which itself is equivalent to

$$VH^1(X, A)_{\text{div}} = \frac{H^1(X, A)^{\wedge}_{\text{div}}}{H^1(X, A)/_{\text{div}}} = 0.$$ 

It is also equivalent to the finiteness of $\text{III}(A^\vee/K)$ by [Mil06, I, Remark 6.14(c)]. Therefore the stated equivalence follows from the previous proposition.

Note that if $\text{III}(A/K)$ is finite, then the full BSD conjecture for $A$ is true by the result of Kato–Trihan [KT03].

Recall from [KT03, Section 2.2] that a complex, called the arithmetic cohomology and denoted by $R\Gamma_{ar}$, is defined to be the mapping fiber of the morphism

$$R\Gamma(U, A_{\text{tor}}) \oplus \bigoplus_{x \not\in U} (\Gamma(\hat{K}_x, A) \otimes L \mathbb{Q}/\mathbb{Z})[-1] \to \bigoplus_{x \not\in U} R\Gamma(\hat{K}_x, A_{\text{tor}}),$$

where $U$ is an open subscheme of $X$ over which $A$ has good reduction. Here we understand the functor $R\Gamma(\hat{K}_x, \cdot)$ as the composite of $R\Gamma(\hat{K}_x, \cdot)$ and $R\Gamma(k_{\text{prozar}}, \cdot)$, and $\Gamma = H^0 R\Gamma$. To be clear, we denote this complex by $R\Gamma_{ar,A}$ and its cohomology objects by $H^n_{ar,A}$. This mapping fiber is defined on the level of complexes, briefly explained in [KT03, Section 2.1]. As they wrote, “how to use complexes is evident and so we do not explain it.” We follow the same style in the rest of this subsection.

We similarly have the mapping cone of the morphism

$$R\Gamma(X, A) \otimes L \mathbb{Q}/\mathbb{Z} \to \left( \prod_{x \in X} \tau_{\geq 1} R\Gamma(\hat{O}_x, A) \right) \otimes L \mathbb{Q}/\mathbb{Z}.$$
We have 
\[ \tau \geq 1 \Gamma(\hat{O}_x, A) = H^1(k_x, \pi_0(A_x))[-1]. \]
Hence we have the mapping cone
\[ \left[ R\Gamma(X, A) \otimes^L \mathbb{Q}/\mathbb{Z} \to \bigoplus_x H^1(k_x, \pi_0(A_x)) \right]. \]
Also we have the mapping cone of the morphism
\[ \left( \bigoplus_{x \in X} \tau \leq 1 R\Gamma_x(\hat{O}_x, A_0) \right) \otimes^L \mathbb{Q}/\mathbb{Z} \to R\Gamma(X, A_0) \otimes^L \mathbb{Q}/\mathbb{Z}. \]
We have 
\[ \tau \leq 1 R\Gamma_x(\hat{O}_x, A_0) = \Gamma(k_x, \pi_0(A_x))[-1]. \]
Hence we have the mapping cone
\[ \left[ \bigoplus_x \Gamma(k_x, \pi_0(A_x)) \to R\Gamma(X, A_0) \otimes^L \mathbb{Q}/\mathbb{Z} \right]. \]

**Proposition 4.2.6.** There exist canonical isomorphisms
\[ R\Gamma_{ar, A} \cong \left[ R\Gamma(U, A) \otimes^L \mathbb{Q}/\mathbb{Z} \to \bigoplus_{x \notin U} H^1(k_x, \pi_0(A_x)) \right] [-2] \]
\[ \cong \left[ \bigoplus_x \Gamma(k_x, \pi_0(A_x)) \to R\Gamma(X, A_0) \otimes^L \mathbb{Q}/\mathbb{Z} \right] [-1]. \]

**Proof.** We first show the first isomorphism. The mapping cone of
\[ \Gamma(k_x, A) \otimes^L \mathbb{Q}/\mathbb{Z}[-1] \to R\Gamma(\hat{K}_x, A_{tor}) \]
is \( H^1(K_x, A) \otimes^L \mathbb{Q}/\mathbb{Z}[-2] \). Hence
\[ R\Gamma_{ar, A} = \left[ R\Gamma(U, A) \otimes^L \mathbb{Q}/\mathbb{Z} \to \bigoplus_{x \notin U} H^1(k_x, A) \otimes^L \mathbb{Q}/\mathbb{Z}[-1] \right] [-2] \]
The morphism
\[ R\Gamma(X, A) \to R\Gamma(U, A) \]
and the morphism
\[ \bigoplus_{x \not\in U} H^1(\hat{O}_x, A)[-1] \to \bigoplus_{x \not\in U} H^1(\hat{K}_x, A)[-1] \]
have the same mapping cone \( \bigoplus_{x \not\in U} R\Gamma_x(\hat{O}_x, A)[1] \). This gives the desired result.

For the second isomorphism, use the distinguished triangle
\[ R\Gamma(X, A_0) \to R\Gamma(X, A) \to \bigoplus_x R\Gamma(k_x, \pi_0(A_x)). \]

**Proposition 4.2.7.** Consider the diagram
\[
\begin{array}{ccc}
\bigoplus_x \Gamma(k_x, \pi_0(A_x^\vee))[-1] & \to & R\lim_n (R\Gamma(X, A_0^\vee) \otimes^L \mathbb{Z}/n\mathbb{Z}) \\
\downarrow i & & \downarrow i \\
\bigoplus_x H^1(k_x, \pi_0(A_x))^{PD}[-1] & \to & (R\Gamma(X, A) \otimes^L \mathbb{Q}/\mathbb{Z})^{PD}[-1]
\end{array}
\]
where the horizontal morphisms come from the morphisms in the previous proposition (with \( R\lim_n (\cdot \otimes^L \mathbb{Z}/n\mathbb{Z}) \) applied), the left vertical one from Grothendieck’s pairing and the right vertical one from Proposition 4.2.2. This diagram is commutative and canonically induces an isomorphism
\[ R\lim_n (R\Gamma_{ar, A}^\vee \otimes^L \mathbb{Z}/n\mathbb{Z}) \cong (R\Gamma_{ar, A})^{PD}[-2] \]
on the mapping cones.

**Proof.** Consider the morphism
\[ \bigoplus_x \tau_{\leq 1} R\Gamma_x(\hat{O}_x, A_0^\vee) \to R\Gamma(X, A_0^\vee). \]
We have a natural morphism from this morphism to the morphism
\[ \bigoplus_x \tau_{\leq 1} R\Gamma_x(\hat{O}_x, R\text{Hom}_{\hat{O}_x}(A^\vee, G_m))[1] \to R\Gamma(X, R\text{Hom}_{\hat{O}_x}(A^\vee, G_m))[1] \]
(i.e., have a commutative square whose upper and lower sides are these morphisms). Using the functoriality/cup product morphisms similar to \((2.5.4)\) and \((2.7.2)\) and the morphism \( R\Gamma_x(\hat{O}_x, G_m) \to R\Gamma(X, G_m) \) we have
a morphism from this morphism to the morphism
\[
\bigoplus_x R \text{Hom}_{\text{prozar}}(R\Gamma(\hat{O}_x, A), \tau \geq 3 R\Gamma(X, G_m))[1]
\]
\[
\rightarrow R \text{Hom}_{\text{prozar}}(R\Gamma(X, A), \tau \geq 3 R\Gamma(X, G_m))[1].
\]
We have \(\tau \geq 3 R\Gamma(X, G_m) = \mathbb{Q}/\mathbb{Z}[-3]\). Hence the mapping cone of this morphism can be written as
\[
R \text{Hom}_{\text{prozar}}\left([R\Gamma(X, A) \rightarrow \prod_x \tau \geq 1 R\Gamma(\hat{O}_x, A)], \mathbb{Q}/\mathbb{Z}\right)[-1].
\]
These are functorial on the level of complexes. Hence we have a morphism
\[
\left[\bigoplus_x \tau \leq 1 R\Gamma_x(\hat{O}_x, A_0^\vee) \rightarrow R\Gamma(X, A_0^\vee)\right]
\]
\[
\rightarrow R \text{Hom}_{\text{prozar}}\left([R\Gamma(X, A) \rightarrow \prod_x \tau \geq 1 R\Gamma(\hat{O}_x, A)], \mathbb{Q}/\mathbb{Z}\right)[-1].
\]
Applying \(R \lim \leftarrow_n (\cdot \otimes L \mathbb{Z}/n\mathbb{Z})\), we get the result.

This form of duality on \(R\Gamma_{ar, A}\) is also given in [TV17, Corollary 4.11, Remark 4.13].

For a torsion group \(C\) of cofinite type, we call \((C/\text{div})^{\text{PD}}\) the essentially torsion part of the profinite group \(C^{\text{PD}}\), which is the product over all primes \(l\) of the torsion part of the \(l\)-adic component of \(C^{\text{PD}}\). For example, \(\prod_l \mathbb{Z}/l\mathbb{Z}\) is the essentially torsion part of itself. By [KT03, Proposition 2.4], we know that \(H^1_{ar, A}\) is isomorphic to the Selmer group of \(A\). Hence the isomorphism in the above proposition in degree 1 on the essentially torsion parts gives a perfect pairing
\[
\text{III}(A^\vee/K)^{\text{div}} \times \text{III}(A/K)^{\text{div}} \rightarrow \mathbb{Q}/\mathbb{Z}
\]
between a profinite group and a torsion group. For any prime \(l\), this pairing on \(l\)-primary parts is a pairing between finite \(l\)-groups.

**Proposition 4.2.8.** This pairing agrees with the Cassels–Tate pairing.

**Proof.** The isomorphism in the proposition gives a Pontryagin duality between the exact sequences
\[
\bigoplus_x \Gamma(k_x, \pi_0(A_x^\vee)) \rightarrow H^1(X, A_0^\vee)^{\text{div}} \rightarrow (H^1_{ar, A_0^\vee})^{\text{div}} \rightarrow 0
\]
and

\[ 0 \to (H^1_{\text{ar}, A})_{/\text{div}}^\wedge \to H^1(X, A)_{/\text{div}}^\wedge \to \bigoplus_x H^1(k_x, \pi_0(A_x)). \]

Hence the proof of [Mil06, III, Corollary 9.5] shows that the paring induced on the third term of the first exact sequence and the first term of the second exact sequence is the Cassels–Tate pairing.

We briefly recall the triangulated functor

\[ R\Gamma(k_W, \cdot): D(k^{\text{indrat}}_{\text{proet}}) \to D(k_{\text{prozar}}) \]

defined in [Suz14, Section 10]. Let \( F \) be the \( q \)th power Frobenius morphism for any \( k \)-algebra, which induces an action on any object of \( \text{Ab}(k^{\text{indrat}}_{\text{proet}}) \).

The functor \( k' \mapsto k' \otimes_k \bar{k} \) defines a premorphism of sites \( f: \text{Spec } k^{\text{indrat}}_{\text{proet}} \to \text{Spec } k_{\text{prozar}}. \) Then

\[ R\Gamma(k_W, C) = f_*[C \overset{F-1}{\to} C][-1] \]

for \( C \in D(k^{\text{indrat}}_{\text{proet}}) \).

We define

\[ R\Gamma(X_W, \cdot) = R\Gamma(k_W, R\Gamma(X, \cdot)): D(X_{\text{fppf}}) \to D(k_{\text{prozar}}). \]

We denote \( (\cdot)^{\text{LD}} = R\text{Hom}_{k_{\text{prozar}}} (\cdot, \mathbb{Z}). \)

**Proposition 4.2.9.** We have

\[ R\Gamma(k_W, R\Gamma(X, A_0)^{\text{SDSD}}) = R\Gamma(X_W, A_0^\vee), \]

\[ R\Gamma(k_W, R\Gamma(X, A)^{\text{SD}}) = R\Gamma(X_W, A)_{\text{LD}}[-1], \]

\[ R\Gamma(k_W, V\mathbb{H}^1(X, A)) = R\Gamma(k_{\text{proet}}, V\mathbb{H}^1(X, A)). \]

**Proof.** For the first isomorphism, note that the mapping fiber of

\[ R\Gamma(X, A_0^\vee) \to R\Gamma(X, A_0^\vee)^{\text{SDSD}} \]

is concentrated in degree zero whose cohomology is \( \lim_n \Gamma(X, A_0^\vee)_0 \). Since the endomorphism \( F - 1 \) on the abelian variety \( \Gamma(X, A_0^\vee)_0 \) is surjective with finite kernel by Lang’s theorem, we know that \( \lim_n \Gamma(X, A_0^\vee)_0 \) is killed by applying \( R\Gamma(k_W, \cdot) \). This gives the desired result.

The second isomorphism follows from

\[ R\Gamma(k_W, R\Gamma(X, A)^{\text{SD}}) = R\Gamma(k_W, R\Gamma(X, A))_{\text{LD}}[-1], \]
which is [Suz14, Proposition (10.4)]. For the third, we already saw in the proof of Proposition 4.2.5 that $R\Gamma(k_{\text{proet}}, V\mathcal{H}^1(X, \mathcal{A}))$ is the mapping fiber of $F - 1$.

Here is a Weil-étale analogue of Proposition 4.2.2.

**Proposition 4.2.10.** Applying $R\Gamma(k_{W}, \cdot)$ to the distinguished triangle in Theorem 3.4.1, we have a canonical distinguished triangle

$$R\Gamma(X_{W}, \mathcal{A}^\vee_0) \to R\Gamma(X_{W}, \mathcal{A})^{\text{LD}}[-1] \to R\Gamma(k_{W}, V\mathcal{H}^1(X, \mathcal{A}^\vee_0)).$$

The third term is zero if and only if $\mathcal{I}I(A/K)$ is finite.

**Proof.** This follows from the previous proposition and Proposition 4.2.5.

If $\mathcal{I}I(A/K)$ is finite, then the cohomology objects of $R\Gamma(X_{W}, \mathcal{A})$ and $R\Gamma(X_{W}, \mathcal{A}^\vee_0)$ are finitely generated abelian groups, and the above proposition gives a duality between these objects via $R\text{Hom}(\cdot, \mathbb{Z})$.

We can also define a Weil-étale version $R\Gamma_{ar, A, W}$ of $R\Gamma_{ar, A}$ by

$$R\Gamma_{ar, A, W} = \left[ R\Gamma(X_{W}, \mathcal{A}) \to \bigoplus_x H^1(k_x, \pi_0(\mathcal{A}_x))[-1] \right] [-1]$$

$$\cong \left[ \bigoplus_x \Gamma(x, \pi_0(\mathcal{A}_x))[-1] \to R\Gamma(X_{W}, \mathcal{A}_0) \right]$$

and prove the existence of a canonical distinguished triangle

$$R\Gamma_{ar, A^\vee, W} \to R\Gamma_{ar, A, W}^{\text{LD}}[-1] \to R\Gamma(k_{W}, V\mathcal{H}^1(X, \mathcal{A}^\vee_0)).$$

**Remark 4.2.11.** We can also apply $R\Gamma(k_{\text{proet}}, \cdot)$ to Theorem 3.1.3 to recover the duality $R\Gamma(U, N) \leftrightarrow R\Gamma_c(U, N)$ for a finite flat group scheme $N$ over an open curve $U$ over $k = \mathbb{F}_q$ stated in [Mil06, III, Theorem 8.2], including the topological group structures on the relevant cohomology groups. Define $R\Gamma_c(U, N) \in D(k_{\text{prozar}})$ by $R\Gamma(k_{\text{proet}}, R\Gamma_c(U, N)) = R\Gamma(k_{W}, R\Gamma_c(U, N))$. Since $R\Gamma_c(U, N) \in D^b(P\text{Alg}_{\text{fnc}}/k)$ by Theorem 3.1.3, we know that $R\Gamma_c(U, N) \in D^b(P\text{Fin})$ by [Suz14, Proposition (10.3)(b)], where Fin is the category of finite abelian groups. Therefore each cohomology object $H^n_c(U, N) := H^n R\Gamma_c(U, N)$ is a profinite group. Now applying $R\Gamma(k_{\text{proet}}, \cdot)$ to Theorem 3.1.3 and using Proposition 4.2.1, we obtain an
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isomorphism

$$R\Gamma(U, N) \cong R\Gamma_c(U, N)^{PD}[-3]$$

in $D(k_{\text{prozar}})$. Hence we have a Pontryagin duality $H^n(U, N) \leftrightarrow H^{3-n}_c(U, N)$ for any $n$ between the torsion group and the profinite group, which recovers [Mil06, III, Theorem 8.2].

A similar remark applies to Proposition 4.1.4 and [Mil06, III, Theorem 9.4].

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**Appendix A. Comparison of local duality with and without relative sites**

We continue the notation of Section 2.5. We recall the relative fppf site $\text{Spec} \hat{O}_{x, \text{fppf}}/k_{x, \text{et}}^{\text{indrat}}$ of $\hat{O}_x$ over $k_x$ from [Suz13, Definition 2.3.2] and [Suz14, Section 3.3]. (Here, again, we abbreviate $(k_x)^{\text{indrat}}$ by $k_x^{\text{indrat}}$ and $(\text{Spec } k_x)^{\text{indrat}}$ by $\text{Spec } k_x^{\text{indrat}}$.) The category $\hat{O}_x/k_x^{\text{indrat}}$ is the category of
pairs \((S, k_S)\), where \(k_S \in k_x^{\text{indrat}}\) and \(S\) a \(\hat{\mathcal{O}}_x(k_S)\)-algebra. (In \cite[Section 2.3]{Suz13} and \cite[Section 3.3]{Suz14}, \(S\) is required to be finitely presented over \(\hat{\mathcal{O}}_x(k')\). The generalization to arbitrary algebras here does not change cohomology theory and Ext groups/sheaves as long as we treat sheaves representable by schemes locally of finite type over \(\hat{\mathcal{O}}_x\).) A morphism \((S, k_S) \to (S', k_{S'})\) consists of a \(k_x\)-algebra homomorphism \(k_S \to k_{S'}\) and an \(\hat{\mathcal{O}}_x(k_S)\)-algebra homomorphism \(S \to S'\), with composition defined in the obvious way. A finite family of morphisms \\{(\(S, k_S\) \to (S_i, k_{S_i})\}\ is called an fppf/étale covering if each \(k_S \to k_{S_i}\) is étale and \(S \to \prod_i S_i\) faithfully flat. This defines a site, which we denote by \(\text{Spec} \hat{\mathcal{O}}_{x, \text{fppf}}/k_x^{\text{indrat}}\). Its category of sheaves is denoted by \(\text{Ab}(\hat{\mathcal{O}}_{x, \text{fppf}}/k_x^{\text{indrat}})\). The categories \(\text{Ch}(\hat{\mathcal{O}}_{x, \text{fppf}}/k_x^{\text{indrat}}), \text{D}(\hat{\mathcal{O}}_{x, \text{fppf}}/k_x^{\text{indrat}})\) and the functors \(\text{Hom}_{\hat{\mathcal{O}}_{x, \text{fppf}}/k_x^{\text{indrat}}}\), \(R \text{Hom}_{\hat{\mathcal{O}}_{x, \text{fppf}}/k_x^{\text{indrat}}}\) are defined in a similar way as Section 2.5.

The functors
\[
\begin{align*}
\pi_{\hat{\mathcal{O}}_x}': \text{Spec} \hat{\mathcal{O}}_{x, \text{fppf}}/k_x^{\text{indrat}} & \to \text{Spec} k_{x, \text{et}}^{\text{indrat}}, \\
\pi_{\hat{\mathcal{K}}_x}': \text{Spec} \hat{\mathcal{K}}_{x, \text{fppf}}/k_x^{\text{indrat}} & \to \text{Spec} k_{x, \text{et}}^{\text{indrat}}, \\
j': \text{Spec} \hat{\mathcal{K}}_{x, \text{fppf}}/k_x^{\text{indrat}} & \to \text{Spec} \hat{\mathcal{O}}_{x, \text{fppf}}/k_x^{\text{indrat}}
\end{align*}
\]
define morphisms of sites
\[
\pi_{\hat{\mathcal{O}}_x}' = \pi_{\hat{\mathcal{O}}_x} \circ j'.
\]
We denote the composite of the derived pushforward
\[
R(\pi_{\hat{\mathcal{O}}_x}')_\ast: \text{D}(\hat{\mathcal{O}}_{x, \text{fppf}}/k_x^{\text{indrat}}) \to \text{D}(k_{x, \text{et}}^{\text{indrat}})
\]
and the pro-étale sheafification \(\text{D}(k_x^{\text{indrat}}) \to \text{D}(k_{x, \text{proet}})\) by
\[
R\Gamma'(\hat{\mathcal{O}}_x, \cdot): \text{D}(\hat{\mathcal{O}}_{x, \text{fppf}}/k_x^{\text{et}}) \to \text{D}(k_{x, \text{proet}}^{\text{indrat}}).
\]
Similarly, we denote the composite of the derived pushforward
\[
R(\pi_{\hat{\mathcal{K}}_x}')_\ast: \text{D}(\hat{\mathcal{K}}_{x, \text{fppf}}/k_x^{\text{indrat}}) \to \text{D}(k_{x, \text{et}}^{\text{indrat}})
\]
and the pro-étale sheafification \(\text{D}(k_x^{\text{indrat}}) \to \text{D}(k_{x, \text{proet}})\) by
\[
R\Gamma'(\hat{\mathcal{K}}_x, \cdot): \text{D}(\hat{\mathcal{K}}_{x, \text{fppf}}/k_x^{\text{indrat}}) \to \text{D}(k_{x, \text{proet}}^{\text{indrat}}).
\]
In [Suz14, Section 3.3, paragraph before Proposition (3.3.8)], these functors were denoted by \( \hat{\Gamma}(\mathcal{O}_x, \cdot) \) and \( \hat{\Gamma}(\mathcal{K}_x, \cdot) \), respectively.

We compare the above morphisms of sites and the premorphisms of sites in Section 2.5. The functor sending \((S, k_S) \in \hat{\mathcal{O}}_x / k^{\text{indrat}}\) to \(S\) defines a premorphism of sites

\[
\bar{f}: \text{Spec } \hat{\mathcal{O}}_x, \text{fppf} \to \text{Spec } \hat{\mathcal{O}}_x, \text{fppf} / k^{\text{indrat}}.
\]

Similarly, the functor sending \((S, k_S) \in \hat{\mathcal{K}}_x / k^{\text{indrat}}\) to \(S\) defines a premorphism of sites

\[
f: \text{Spec } \hat{\mathcal{K}}_x, \text{fppf} \to \text{Spec } \hat{\mathcal{K}}_x, \text{fppf} / k^{\text{indrat}}.
\]

The pushforward functors \(\bar{f}_*\) and \(f_*\) are exact. They send the sheaf representable by a \(\hat{\mathcal{O}}_x\)-algebra (resp. \(\hat{\mathcal{K}}_x\)-algebra) \(S\) to the sheaf representable by \((S, k)\). Hence a scheme over \(\hat{\mathcal{O}}_x\) (resp. \(\hat{\mathcal{K}}_x\)) regarded as a sheaf on \(\text{Spec } \hat{\mathcal{O}}_x, \text{fppf} / k^{\text{indrat}}\) mentioned in [Suz13, Section 2.3] is nothing but its image by \(\bar{f}_*\) (resp. \(f_*\)).

**Proposition A.1.** We have \(\pi_{\hat{\mathcal{O}}_x} = \pi'_{\hat{\mathcal{O}}_x} \circ \bar{f}\) and \(\pi_{\hat{\mathcal{K}}_x} = \pi'_{\hat{\mathcal{K}}_x} \circ f\), and

\[
R\Gamma(\hat{\mathcal{O}}_x, \cdot) = R\Gamma'(\hat{\mathcal{O}}_x, \bar{f}_* \cdot), \quad R\Gamma(\hat{\mathcal{K}}_x, \cdot) = R\Gamma'(\hat{\mathcal{O}}_x, f_* \cdot).
\]

In particular, for a group scheme \(A\) over \(\hat{\mathcal{O}}_x\) or \(\hat{\mathcal{K}}_x\), we have

\[
R\Gamma(\hat{\mathcal{O}}_x, A) = R\Gamma'(\hat{\mathcal{O}}_x, A), \quad R\Gamma(\hat{\mathcal{K}}_x, A) = R\Gamma'(\hat{\mathcal{O}}_x, A).
\]

**Proof.** The composite \(\pi'_{\hat{\mathcal{O}}_x} \circ \bar{f}\) is defined by the composite of the functors \(k_x' \mapsto (\hat{\mathcal{O}}_x(k_x'), k_x') \mapsto \hat{\mathcal{O}}_x(k_x')\) which defines \(\pi_{\hat{\mathcal{O}}_x}\). Hence \(\pi_{\hat{\mathcal{O}}_x} = \pi'_{\hat{\mathcal{O}}_x} \circ \bar{f}\). Similar for \(\pi_{\hat{\mathcal{K}}_x} = \pi'_{\hat{\mathcal{K}}_x} \circ f\). The equalities of the derived pushforwards follow from Propositions 2.4.2, 2.4.3 and the theorem on derived functors of composition [KS06, Proposition 10.3.5(ii)].

**Proposition A.2.** The mapping fiber functor

\[
\Gamma'_x(\mathcal{O}_x, \cdot) := [\Gamma'(\mathcal{O}_x, \cdot) \to \Gamma'(\mathcal{K}_x, j^* \cdot)][-1]:
\]

\[
\text{Ch}(\hat{\mathcal{O}}_x, \text{fppf} / k^{\text{indrat}}) \to \text{Ch}(k^{\text{indrat}}_{\text{proet}})
\]

and the functor

\[
\Gamma_x(\mathcal{O}_x, \cdot) := [\Gamma(\mathcal{O}_x, \cdot) \to \Gamma(\mathcal{K}_x, j^* \cdot)][-1]:
\]

\[
\text{Ch}(\hat{\mathcal{O}}_x, \text{fppf}) \to \text{Ch}(k^{\text{indrat}}_{\text{proet}})
\]
are compatible under the functor\[ \bar{f}_s : \text{Ch}(\hat{O}_{x,\text{ffppf}}) \to \text{Ch}(\hat{O}_{x,\text{ffppf}/k_{\text{indrat}}}). \]

The right derived functors\[ R\Gamma'_x(\mathcal{O}_x, \cdot) = [R\Gamma'(\mathcal{O}_x, \cdot) \to R\Gamma'(K_x, j'^* \cdot)][-1]: \]
\[ D(\hat{O}_{x,\text{ffppf}}/k_{\text{indrat}}) \to D(k_{\text{indrat}}), \]
and\[ R\Gamma_x(\mathcal{O}_x, \cdot) = [R\Gamma(\mathcal{O}_x, \cdot) \to R\Gamma(K_x, j^* \cdot)][-1]: \]
\[ D(\hat{O}_{x,\text{ffppf}}) \to D(k_{\text{indrat}}), \]
are also compatible under the functor\[ \bar{f}_s : D(\hat{O}_{x,\text{ffppf}}) \to D(\hat{O}_{x,\text{ffppf}/k_{\text{indrat}}}). \]

Proof. We have a commutative diagram of premorphisms of sites\[
\begin{array}{ccc}
\text{Spec} \hat{K}_{x,\text{ffppf}} & \xrightarrow{f} & \text{Spec} \hat{K}_{x,\text{ffppf}/k_{\text{et}}} \\
\downarrow j & & \downarrow j' \\
\text{Spec} \hat{O}_{x,\text{ffppf}} & \xrightarrow{\bar{f}} & \text{Spec} \hat{O}_{x,\text{ffppf}/k_{\text{et}}} \\
\end{array}
\]
Hence the first statement follows from Proposition A.1. We know that \( j \) and \( j' \) are localization morphisms [AGV72, III, Section 5]. Hence their pullback functors admit exact left adjoints by [AGV72, IV, Proposition 11.3.1] and hence send K-injectives to K-injectives. With Proposition 2.4.2, we know that the first statement implies the second.

By [Suz14, Proposition (3.3.8)], the functor \( R\Gamma'_x(\mathcal{O}_x, \cdot) \) is isomorphic to the functor denoted by \( R\hat{\Gamma}_x(\mathcal{O}_x, \cdot) \) in [Suz14, paragraph before Proposition (3.3.8)].

We can define the morphism of functoriality of \( \bar{f}_s \)
\[ \bar{f}_s R\text{Hom}_{\mathcal{O}_x}(A, B) \to R\text{Hom}_{\mathcal{O}_x,\text{ffppf}/k_{\text{et}}}(\bar{f}_s A, \bar{f}_s B) \]
in \( D(\mathcal{O}_{x,\text{ffppf}}/k_{\text{et}}) \), functorial on \( A, B \in D(\mathcal{O}_{x,\text{ffppf}}) \), in a way similar to the definition of (2.5.2). Hence we have a morphism\[ R\Gamma(\mathcal{O}_x, R\text{Hom}_{\mathcal{O}_x}(A, B)) = R\Gamma'(\mathcal{O}_x, \bar{f}_s R\text{Hom}_{\mathcal{O}_x}(A, B)) \]
\[ \to R\Gamma'(\mathcal{O}_x, R\text{Hom}_{\mathcal{O}_x,\text{ffppf}/k_{\text{et}}}(\bar{f}_s A, \bar{f}_s B)). \]
There is a similar morphism of functoriality of \( f_* \).
Proposition A.3. Let \( A, B \in D(\mathcal{O}_{x, \text{fppf}}) \). Under the above morphism, the morphism (2.5.2)

\[
R\Gamma(\mathcal{O}_x, R\text{Hom}_{\mathcal{O}_x}(A, B)) \\
\rightarrow R\text{Hom}_{\mathcal{O}_{x, \text{proet}}} (R\Gamma_x(\mathcal{O}_x, A), R\Gamma_x(\mathcal{O}_x, B))
\]

and the morphism

\[
R\Gamma'(\mathcal{O}_x, R\text{Hom}_{\mathcal{O}_{x, \text{fppf}}/k_{\text{indrat}}} (\bar{f}_*A, \bar{f}_*B)) \\
\rightarrow R\text{Hom}_{\mathcal{O}_{x, \text{proet}}} (R\Gamma'_x(\mathcal{O}_x, \bar{f}_*A), R\Gamma'_x(\mathcal{O}_x, \bar{f}_*B))
\]

in [Suz14, Proposition (3.3.8)] are compatible under the isomorphisms in the previous two propositions. Similar compatibilities hold for the morphisms (2.5.4), (2.5.6) and other morphisms in [Suz14, Proposition (3.3.8)].

Proof. This reduces to a corresponding nonderived statement about morphisms between

\[
\Gamma(\mathcal{O}_x, \text{Hom}_{\mathcal{O}_x}(A, B)), \\
\Gamma'(\mathcal{O}_x, \text{Hom}_{\mathcal{O}_{x, \text{fppf}}/k_{\text{indrat}}} (\bar{f}_*A, \bar{f}_*B)), \\
\text{Hom}_{\mathcal{O}_{x, \text{proet}}} (\Gamma_x(\mathcal{O}_x, A), \Gamma_x(\mathcal{O}_x, B))
\]

on the level of complexes of sheaves. Evaluate these complexes at each \( k'_x \in k_{x, \text{indrat}} \) and then check the compatibility.

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