On a type Sobolev inequality and its applications.

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Abstract

Assume $\| \cdot \|$ is a norm on $\mathbb{R}^n$ and $\| \cdot \|_*$ its dual. In this paper we consider the closed ball $T := B_{\| \cdot \|}(0, r)$, $r > 0$. Suppose $\varphi$ is an Orlicz function and $\psi$ its conjugate, we prove that for arbitrary $A, B > 0$ and for each Lipschitz function $f$ on $T$ the following inequality holds

$$\sup_{s, t \in T} |f(s) - f(t)| \leq 6AB\left( \int_0^r \psi\left( \frac{1}{A^{n-1}} \varepsilon^{n-1} \right) d\varepsilon + \frac{1}{n|B_{\| \cdot \|}(0, 1)|} \int_T \varphi\left( \frac{1}{B_{\| \cdot \|}} \|\nabla f(u)\|_* \right) du \right),$$

where $| \cdot |$ is the standard Lebesgue measure on $\mathbb{R}^n$. This is a strengthening of the Sobolev inequality obtained in the proof of Theorem 5.1 by M. Talagrand [9]. We use the inequality to state for a given concave, strictly increasing function $\eta : \mathbb{R}_+ \to \mathbb{R}$, with $\eta(0) = 0$, the necessary and sufficient condition on $\varphi$ so that each separable process $X(t), t \in T$ which satisfies

$$\|X(s) - X(t)\|_\varphi \leq \eta(\|s - t\|), \text{ for } s, t \in T$$

is a.s. sample bounded.

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1 Introduction

Let $\| \cdot \|$ be a norm on $\mathbb{R}^n$. We denote by $B_{\| \cdot \|}(x, r)$ the closed ball with the center at $x$ and the radius $r$ with respect to the metric given by $\| \cdot \|$, i.e.

$$B_{\| \cdot \|}(x, r) := \{ y \in \mathbb{R}^n : \| x - y \| \leq r \}.$$

Let $\langle \cdot, \cdot \rangle$ be the canonical scalar product (that is $\langle u, v \rangle := \sum_{i=1}^{n} u_i v_i$, for $u, v \in \mathbb{R}^n$) and $\| \cdot \|_*$ the dual norm, i.e.

$$\| v \|_* := \sup_{u \in B_{\| \cdot \|}(0, 1)} |\langle u, v \rangle|, \text{ for } v \in \mathbb{R}^n.$$

In this paper we consider the closed ball $T := B_{\| \cdot \|}(0, r), r > 0$. We say that $\varphi : \mathbb{R}_+ \to \mathbb{R}$ is an Orlicz function if it is convex, strictly increasing, $\varphi(0) = 0$ and also $\lim_{x \to 0} \varphi(x)/x = 0$, $\lim_{x \to \infty} \varphi(x)/x = \infty$. For each Orlicz function $\varphi$ we define its conjugate

$$\psi(x) := \sup_{y \geq 0} (xy - \varphi(y)), \text{ for } x \geq 0.$$

This $\psi$ is also an Orlicz function. Moreover, it is well known that $\varphi$ is the conjugate function for $\psi$, namely $\varphi(x) = \sup_{y \geq 0} (xy - \psi(y))$. The definition implies the Young inequality

$$xy \leq \varphi(x) + \psi(y), \text{ for } x, y \geq 0. \quad (1)$$

From now on we assume that $\varphi, \psi$ are conjugate Orlicz functions.

In the paper we prove the following Sobolev type inequality and give its applications to the theory of stochastic processes.

**Theorem 1** For each $A, B > 0$ and for each Lipschitz function $f$ on $T$ the following inequality holds

$$\sup_{s,t \in T} |f(s) - f(t)| \leq 6AB \left( \int_0^r \psi\left( \frac{1}{A\varepsilon^{n-1}} \right) \varepsilon^{n-1} d\varepsilon + \frac{1}{n|B_{\| \cdot \|}(0, 1)|} \int_T \varphi\left( \frac{1}{B} \| \nabla f(u) \|_* \right) du \right),$$

where $| \cdot |$ is the standard Lebesgue measure on $\mathbb{R}^n$.

The above inequality is a generalization of Talagrand’s result, who obtained such inequality in the proof of Theorem 5.1, [9] when $\|s-t\| = \sup_{i=1}^{n} |s_i - t_i|$. 

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Since for each \( s, t \in \mathbb{R}^n \) we have \( s \in B_{\| \cdot \|}(t, \| s - t \|) \), the above theorem implies some regularity on \( f \). Namely, for \( \varphi(x) \equiv x^p/p \), \( p > n \) we obtain the following classical result (which can be deduced from Lemma 7.16 in [7] by using Holder inequality).

**Corollary 1** Suppose \( p > n \), then for each Lipschitz function \( f \) on \( \mathbb{R}^n \) the following inequality holds

\[
\sup_{s, t \in \mathbb{R}^n} \frac{|f(s) - f(t)|}{\|s - t\|^{1 - \frac{n}{p}}} \leq \frac{6(p-1)^{1-\frac{1}{p}}}{(n|B|\|0, 1\|))^{1/p}} \left( \int_{\mathbb{R}^n} \|\nabla f(u)\|_p^p du \right)^{1/p}.
\]

**Proof.** The conjugate function for \( \varphi(x) \equiv x^p/p \) is \( \psi(x) \equiv x^{q}/q \), where \( 1/p + 1/q = 1 \). Due to Theorem 1 for each \( A, B > 0 \) we have

\[
|f(s) - f(t)| \leq 6AB \left( \int_0^{\|s-t\|} q^{-1} \left( \frac{1}{A \varepsilon^{n-1}} \right)^q \varepsilon^{n-1} d\varepsilon + \frac{1}{n|B|\|0, 1\|} \int_{B_{\| \cdot \|}(t, \|s-t\|)} p^{-1}\left( \frac{1}{B} \|\nabla f(u)\|_p^p du \right) \right).
\]

We take

\[
A = \left( \int_0^{\|s-t\|} \varepsilon^{(1-q)(n-1)} d\varepsilon \right)^{1/q}, \quad B = \left( \frac{1}{n|B|\|0, 1\|} \int_{\mathbb{R}^n} (\|\nabla f(u)\|_p^p du)^{1/p} \right).
\]

Consequently \( (T = B_{\| \cdot \|}(0, r)) \)

\[
|f(s) - f(t)| \leq 6AB = \frac{6(p-1)^{1/q} \|s - t\|^{1 - \frac{n}{p}}}{(n|B|\|0, 1\|))^{1/p}} \left( \int_{T} \|\nabla f(u)\|_p^p du \right)^{1/p}.
\]

It completes the proof. \[\blacksquare\]

Another immediate consequence of Theorem 1 is the sufficient condition for embedding of the Sobolev space \( W_0^{1, \varphi}(T) \) into \( L_\infty(T) \).

**Corollary 2** If for some \( A > 0 \) we have \( \int_0^T \psi\left( \frac{1}{A \varepsilon^{n-1}} \right) \varepsilon^{n-1} d\varepsilon < \infty \) then the space \( W_0^{1, \varphi}(T) \) embeds into \( L_\infty(T) \).

The result can be deduced from the part II of Theorem 1.1. in the paper by A. Cianchi [3].

To explain applications to stochastic processes we need some definitions. Let \( (K, d) \) be a compact metric space. Denote by \( \mathcal{B}(K) \) the space of Borel bounded functions on \( K \), by \( C(K) \) the Banach space of continuous functions.
on $K$, with sup-norm and by $\text{Lip}(K)$ the space of Lipschitz functions on $T$ with the norm
\[
\|f\|_{\text{Lip}} := \sup_{s \neq t} \frac{|f(s) - f(t)|}{d(s, t)} + D(K)^{-1}\|f\|_{\infty},
\]
where $D(K) := \sup\{d(s, t) : s, t \in K\}$ (the diameter of $K$). Let $\mathcal{P}(K)$ be the set of all Borel probability measures on $K$. For each $\nu \in \mathcal{P}(K)$, $f \in \mathcal{B}(K)$ and $A \in \mathcal{B}(K)$ (with $\nu(A) > 0$) we denote
\[
\int_A f(u)\nu(du) := \frac{1}{\nu(A)} \int_A f(u)\nu(du).
\]
Suppose $X$ is a random variable, we define the Luxemburg norm $\|X\|_{\varphi} := \inf\{c > 0 : \mathbb{E}\varphi\left(\frac{|X|}{c}\right) \leq 1\}$. For a fixed probability space the Banach space $L_{\varphi}$ consists of all random variables for which $\|X\|_{\varphi} < \infty$.

In this paper we consider only separable processes (for the definition see Introduction in the book by Ledoux-Talagrand [8]). For each separable $X(t), t \in K$ we have the following equality
\[
\mathbb{E} \sup_{s, t \in K} |X(s) - X(t)| = \sup_{F \subset K} \mathbb{E} \sup_{s, t \in F} |X(s) - X(t)|,
\]
where the supremum is taken over all finite subsets of $K$. Let us impose the Lipschitz condition on increments of our processes, that is for each $X(t), t \in K$ we assume that
\[
\sup_{s, t \in K} \mathbb{E}\varphi\left(\frac{|X(s) - X(t)|}{d(s, t)}\right) \leq 1.
\]
This condition can be rewritten in terms of Luxemburg norms
\[
\|X(s) - X(t)\|_{\varphi} \leq d(s, t), \text{ for } s, t \in K.
\]
In the theory of stochastic processes a lot of effort has been put in finding criteria for boundedness or continuity of stochastic processes. In most of the cases they are of the following Kolmogorov type: some assumptions on the Orlicz function $\varphi$ and the metric space $(K, d)$ are given so that for each separable process $X$ on $K$ the condition (3) implies that $X$ is bounded a.s.

It is not difficult to prove that under the same assumptions on $\varphi$ and $K$ the two conditions are equivalent:
1. each separable process \( X(t), t \in K \) which satisfies (3) is a.s. bounded.

2. there exists a universal constant \( S < \infty \) such that for each process \( X \)
   
   \[ E \sup_{s,t \in K} |X(s) - X(t)| \leq S. \]  

   The minimal constant \( S \) is denoted by \( S(K, d, \varphi) \). For a proof of this statement we refer to M. Talagrand [9], Theorem 2.3.

**Remark 1** In terms of absolutely summing operators each of the above implications is equivalent to the fact that the injection operator \( J : \text{Lip}(K) \rightarrow C(K) \) is \((\varphi, 1)\) absolutely summing, in the sense of P. Assouad, see [7].

By far the strongest criteria for finiteness of \( S(K, d, \varphi) \) were obtained using the concept of majorizing measures which was introduced by X. Fernique in early 70. It served him and M. Talagrand to characterize bounded Gaussian processes. To explain briefly the concept we introduce the following definitions.

For \( t \in K \) and \( \varepsilon \geq 0 \), we denote by \( B(t, \varepsilon) \), \( S(t, \varepsilon) \) respectively the closed ball and the sphere with the center at \( x \) and the radius \( \varepsilon \) with respect to the metric \( d \), i.e.

\[ B(t, \varepsilon) := \{ s \in K : d(s, t) \leq \varepsilon \}, \quad S(t, \varepsilon) := \{ s \in K : d(s, t) = \varepsilon \}. \]

We say that \( m \in \mathcal{P}(K) \) is a majorizing measure (with respect to \( \varphi \) and \( d \)) if

\[ \mathcal{M}(m, \varphi) := \sup_{t \in K} \int_0^{D(K)} \varphi^{-1}\left(\frac{1}{m(B(t, \varepsilon))}\right) d\varepsilon < \infty. \]

X. Fernique [5], [6] proved that if \( \varphi \) has the exponential growth then the existence of a majorizing measure is the necessary and sufficient condition for the quantity \( S(K, d, \varphi) \) to be finite. Generalizing results of Fernique, Talagrand and others, the author [2] succeeded in proving that for each Orlicz function \( \varphi \) the existence of a majorizing measure is always the sufficient condition for \( S(K, d, \varphi) < \infty \). However, as it will be seen in the next chapters, the existence of a majorizing measure is not always necessary for finiteness of \( S(K, d, \varphi) \). So it is still the open problem to characterize \((K, d)\) and \( \varphi \) for which all processes satisfying (3) are a.s. sample bounded.
This problem was studied in depth by M. Talagrand, [9]. He managed to find such a characterization of \( \varphi; (K, d) \) in two particular, but important for applications, cases. Namely when \( d \) is the Euclidean distance on \( \mathbb{R}^n \) and \( K \) is a ball in \( \mathbb{R}^n \) and the other case when \( K = [-1, 1] \) and the distance \( d \) is given by \( d(x, y) = \eta(|x - y|) \) where \( \eta \) is a concave, strictly increasing function with \( \eta(0) = 0 \). Generalizing his ideas and using Theorem 5.2 we find the characterization in the case when \( K = T = B_\|\cdot\|_1(0, r) \) and \( d(x, y) = \eta(\|x - y\|) \) (\( \eta \) is concave, strictly increasing, with \( \eta(0) = 0 \)).

By the definition \( B(t, \varepsilon) = B_\|\cdot\|_1(t, \eta^{-1}(\varepsilon)) \cap T \). Let \( \lambda \) be a normalized Lebesgue measure on \( T \), that is \( \lambda(A) = \frac{|A|}{|T|} \), for each \( A \in \mathcal{B}(T) \), where \( |\cdot| \) is the standard Lebesgue measure on \( \mathbb{R}^n \). Note that \( \lambda(B(t, \varepsilon)) \leq \frac{\eta^{-1}(\varepsilon)^n}{r^n} \) and \( \lambda(B(t, \varepsilon)) = \frac{\eta^{-1}(\varepsilon)^n}{r^n} \) if \( B_\|\cdot\|_1(t, \eta^{-1}(\varepsilon)) \subseteq T \).

The function \( \eta(y)/y \) is positive and decreasing. We assume that \( \eta'(0) = \infty \) (the case of finite derivative will be considered later). Following M. Talagrand [9] (Theorem 5.2) we introduce a sequence \((r_k)_{k \geq 0}\). Let \( r_0 = \eta(r) \), for \( k \geq 0 \) we define

\[
 r_{k+1} := \inf \{ \varepsilon \geq 0 : r_k \leq 2\varepsilon \text{ or } \frac{\varepsilon}{\eta^{-1}(\varepsilon)} \leq 2 \frac{r_k}{\eta^{-1}(r_k)} \}.
\]

The sequence \((r_k)_{k \geq 0}\) decreases to 0, since \( r_{k+1} \leq \frac{r_k}{2} \). The assumption \( \eta'(0) = \infty \) guarantees that \( r_k > 0 \). There are two possibilities

\[
r_k = 2r_{k+1} \text{ or } \frac{r_{k+1}}{\eta^{-1}(r_{k+1})} = 2 \frac{r_k}{\eta^{-1}(r_k)}.
\]

Denote by \( I \) the set of \( k \geq 0 \) for which the first possibility holds, and the rest by \( J \). Let us notice that necessarily

\[
2r_{k+1} \leq r_k, \quad 2 \frac{r_k}{\eta^{-1}(r_k)} \leq \frac{r_{k+1}}{\eta^{-1}(r_{k+1})} \quad (5).
\]

For \( k \geq 0 \) we define \( S_k \) as a number which satisfies the equation

\[
\int_{r_{k+1}}^{r_k} \frac{\lambda(B(0, \varepsilon))}{\varepsilon} \psi\left(\frac{\varepsilon}{S_k \lambda(B(0, \varepsilon))}\right) d\varepsilon = \int_{r_{k+1}}^{r_k} \frac{\eta^{-1}(\varepsilon)^n}{r^n \varepsilon} \psi\left(\frac{r^n \varepsilon}{S_k \eta^{-1}(\varepsilon)^n}\right) d\varepsilon = 1.
\]

If \( \eta'(0) < \infty \), then there exists \( m \geq 0 \) such that \( r_m > 0 \) and \( r_{m+1} = 0 \). That means

\[
\frac{r_m}{\eta^{-1}(r_m)} \leq \frac{\varepsilon}{\eta^{-1}(\varepsilon)} \leq 2 \frac{r_m}{\eta^{-1}(r_m)}, \quad \text{for } 0 < \varepsilon \leq r_m.
\]

We define \( S_m \) as the infimum over \( c > 0 \) such that

\[
\int_{0}^{r_m} \frac{\lambda(B(0, \varepsilon))}{\varepsilon} \psi\left(\frac{\varepsilon}{c \lambda(B(0, \varepsilon))}\right) d\varepsilon = \int_{0}^{r_m} \frac{\eta^{-1}(\varepsilon)^n}{r^n \varepsilon} \psi\left(\frac{r^n \varepsilon}{c \eta^{-1}(\varepsilon)^n}\right) d\varepsilon \leq 1.
\]

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For simplicity we define also $S_k := 0$, for $k > m$ (in this case).

Let us state the main result of the paper.

**Theorem 2** The following inequality holds

$$K^{-1} \sum_{k \geq 0} S_k \leq S(T, d, \varphi) \leq K \sum_{k \geq 0} S_k,$$

where the constant $K$ depends only on $n$.

In fact we show that $\sum_{k \geq 0} S_k \leq 3(n + 2) S(T, d, \varphi)$ and $S(T, d, \varphi) \leq (a + bn^2) \sum_{k \geq 0} S_k$ for $k \geq 0$, where $a, b$ are universal constants.

**Corollary 3** Let $\varphi(x) \equiv x^p/p$, $p > 1$ and $\eta(x) = x^\alpha$, $0 < \alpha \leq 1$. Then $S(T, d, \varphi) < \infty$ if and only if $n > p \alpha$.

In the case of $T = B_{\|\cdot\|}(0, 1)$ and $d(s, t) = \|s - t\|$ (that is $\eta(x) \equiv x$) we have $m = 0$ in the above construction. Thus $S(T, d, \varphi)$ is comparable with $S_0$, where $S_0$ is such that

$$\int_0^1 \varepsilon^{n-1} \psi\left(\frac{1}{S_0 \varepsilon^{n-1}}\right) d\varepsilon = 1,$$

up to a constant which depends only on $n$. When $d$ is the Euclidean distance this corollary was proved by M. Talagrand (Theorem 5.1, [9]).

In the case of $T = [-1, 1]$, $\eta'(0) = \infty$ it can be observed that for some universal $C > 0$

$$C^{-1} r_k \varphi^{-1}\left(\frac{1}{\eta^{-1}(r_k)}\right) \leq S_k \leq Cr_k \varphi^{-1}\left(\frac{1}{\eta^{-1}(r_k)}\right), \text{ for } k \geq 0.$$

Consequently $S(T, d, \varphi)$ is comparable with $\sum_{k \geq 0} r_k \varphi^{-1}\left(\frac{1}{\eta^{-1}(r_k)}\right)$. The result was obtained by M. Talagrand (Theorem 5.2, [9]).

## 2 Preliminary results

We remind that $\varphi, \psi$ are conjugate Orlicz functions.

**Lemma 1** Following inequalities hold:

$$\varphi\left(\frac{\psi(x)}{x}\right) \leq \psi(x) \leq \varphi\left(\frac{2\psi(x)}{x}\right), \text{ for } x \geq 0;$$

$$\psi\left(\frac{\varphi(x)}{x}\right) \leq \varphi(x) \leq \psi\left(\frac{2\varphi(x)}{x}\right), \text{ for } x \geq 0.$$

(7)

In the symmetric version we can write

$$x \leq \varphi^{-1}(x) \psi^{-1}(x) \leq 2x, \text{ for } x \geq 0.$$
Proof. Fix $x \geq 0$. By the Young inequality we obtain
\[ 2\psi(x) = \frac{2\psi(x)}{x} x \leq \psi(x) + \varphi\left(\frac{2\psi(x)}{x}\right). \]
Hence $\psi(x) \leq \varphi\left(\frac{2\psi(x)}{x}\right)$. To prove the right-hand side of (7) let us notice that since $\varphi(x) = \sup_{y \geq 0}(xy - \psi(y))$, we have for some $y \geq 0$
\[ \varphi\left(\frac{\psi(x)}{x}\right) = \frac{\psi(x)}{x} - \psi(y). \]
It remains to prove that
\[ \frac{\psi(x)}{x} - \psi(y) \leq \psi(x). \tag{8} \]
If $x \leq y$, then the convexity of $\psi$ gives $\frac{\psi(x)}{x} \leq \psi(y)$. If $x > y$, then $\frac{\psi(x)}{x} \leq \psi(x)$. It yields (8), consequently
\[ \varphi\left(\frac{\psi(x)}{x}\right) \leq \psi(x) \leq \varphi\left(\frac{2\psi(x)}{x}\right). \]
It completes the proof.

Lemma 2 Functions $\varphi, \psi$ have following properties:

1. functions $x\varphi(1/x)$, $x\psi(1/x)$ are convex, decreasing;
2. functions $x\varphi^{-1}(1/x)$, $x\psi^{-1}(1/x)$ are concave, increasing.

Proof. It is enough to prove the result for $\varphi$. By the definition $\varphi(x) = \sup_{y \geq 0}(yx - \psi(y))$, so $x\varphi(1/x) = \sup_{y \geq 0}(y - x\psi(y))$. The supremum of convex functions is a convex function, the supremum of decreasing functions is a decreasing function.

Similarly we observe that $\varphi^{-1}(x) = \inf_{y \geq 0} \frac{x + \psi(y)}{y}$. Hence $x\varphi^{-1}(1/x) = \inf_{y \geq 0} \frac{1 + x\psi(y)}{y}$. The infimum of concave functions is a concave function, the infimum of increasing functions is an increasing function.
3 Proof of Theorem 1

Proof of Theorem 1. Fix points \( t = (t_i)_{i=1}^n, s = (s_i)_{i=1}^n \in T \). Let \( g \) be a smooth function on \( \mathbb{R}^n \). We define \( F_t : T \times [0, r] \to T \) by the formula

\[
F_t(u, \varepsilon) = (1 - \varepsilon t) + \varepsilon \frac{t}{r}.
\]

We have

\[
r^n \int_T g(u) du = \int_0^r \frac{\partial}{\partial \varepsilon} (\int_T g(F_t(u, \varepsilon)) \varepsilon^n du) d\varepsilon.
\]

(9)

It can be easily verified that

\[
\frac{\partial}{\partial \varepsilon} g(F_t(u, \varepsilon)) = r^{-1} \sum_{i=1}^n (u_i - t_i) \frac{\partial}{\partial x_i} g(F_t(u, \varepsilon)) = \varepsilon^{-1} \sum_{i=1}^n (u_i - t_i) \frac{\partial}{\partial u_i} g(F_t(u, \varepsilon)).
\]

Hence the following equation holds

\[
\frac{\partial}{\partial \varepsilon} (g(F_t(u, \varepsilon)) \varepsilon^n) = n \varepsilon^{-1} g(F_t(u, \varepsilon)) + \varepsilon^{-1} \sum_{i=1}^n (u_i - t_i) \frac{\partial}{\partial u_i} g(F_t(u, \varepsilon)),
\]

which yields

\[
\frac{\partial}{\partial \varepsilon} (g(F_t(u, \varepsilon)) \varepsilon^n) = \varepsilon^{-1} \sum_{i=1}^n \frac{\partial}{\partial u_i} (g(F_t(u, \varepsilon))(u_i - t_i)).
\]

Applying the generalized Green-Gauss theorem (see Theorem 4.5.6 in [4]) which holds for Lipschitz boundaries, we get

\[
\int_T \sum_{i=1}^n \frac{\partial}{\partial u_i} (g(F_t(u, \varepsilon))(u_i - t_i)) du = \int_{\partial T} g(F_t(u, \varepsilon))(u - t, n(u)) \sigma_{\partial T}(du),
\]

where \( \sigma_{\partial T} \) is the Lebesgue measure on the manifold \( \partial T \), and \( n(u) \) the normal vector to the boundary in \( u \in \partial T \), such that \( \langle n(u), n(u) \rangle = 1 \) \( n(u) \) is well defined \( \sigma_{\partial T}\text{-a.s.} \). Let us notice that the convexity of \( T \) yields \( \langle u - t, n(u) \rangle \geq 0 \). Denoting \( \sigma_t(u) := \langle u - t, n(u) \sigma_{\partial T}(du) \), we obtain due to (9)

\[
r^n \int_T g(u) du = \int_0^r \int_{\partial T} g(F_t(u, \varepsilon))(u - t, n(u)) \sigma_{\partial T}(du) d\varepsilon.
\]

By the standard approximation this equality can be easily generalized to any Borel, bounded function \( g \) on \( T \). We verify also that \( n|T| = \sigma_t(\partial T) \) (consider \( g \equiv 1 \)), consequently for each \( g \in \mathfrak{B}(T) \)

\[
n^{-1} r^n \int_T g(u) du = \int_0^r \int_{\partial T} g(F_t(u, \varepsilon)) \varepsilon^{n-1} \sigma_t(du) d\varepsilon.
\]

(10)
We define \( \partial F_t : \partial T \times [0, 1] \to T \) by \( \partial F_t = F_t|\partial T \times [0, 1] \). The equation (10) implies
\[
|A|/|T| = \sigma_t \otimes d(\varepsilon/\tau)^n((\partial F_t)^{-1}(A)), \text{ for } A \in \mathcal{B}(T).
\]

Let \( a_t : [0, r] \to \mathbb{R} \) denotes
\[
a_t(\varepsilon) := \int\nabla f(T(\varepsilon))d(\varepsilon/\tau) = \int\nabla f((1 - \varepsilon/r)T + \varepsilon u)\sigma_t(du).
\]
Clearly \( a_t \) satisfies Lipschitz condition (because \( f \) does) and
\[
a_t(0) = f(t), \quad a_t(r) = \int\nabla f(\sigma_t(du)).
\]
Since \( f \) is Lipschitz, there exists bounded \( \nabla f, |\cdot| \)-a.s. on \( T \). We check that if \( f \) is differentiable in \( F_t(\varepsilon) \) then
\[
\frac{\partial}{\partial \varepsilon} f(F_t(\varepsilon)) = r^{-1}\langle \nabla f(F_t(\varepsilon)), u - t \rangle.
\]
By the Fubini theorem and (11) we have that \( f(F_t) \) is differentiable \( d\varepsilon \)-a.s. for \( \sigma_t(du) \)-almost all \( u \in \partial T \). Consequently \( d\varepsilon \)-a.s. there holds
\[
a_t'(\varepsilon) = r^{-1}\int\langle \nabla f(F_t(\varepsilon)), u - t \rangle\sigma_t(du).
\]
We have
\[
|a_t'(\varepsilon)| = r^{-1}\left| \int\langle \nabla f(F_t(\varepsilon)), u - t \rangle\sigma_t(du) \right| \leq \\
\leq r^{-1}\int\|u - t\||\nabla f(F_t(\varepsilon))\|_\sigma t(du).
\]
Clearly \( \|u - t\| \leq 2r \). Observe that \( b_t : [0, r] \to \mathbb{R} \) is \( d\varepsilon \)-a.s. well defined by the formula
\[
b_t(\varepsilon) := \int\|\nabla f(F_t(\varepsilon))\|_\sigma t(du).
\]
Hence \( |a_t'(\varepsilon)| \leq 2b_t(\varepsilon) \), \( d\varepsilon \)-a.s. By the Jensen inequality and (10) for each \( B > 0 \) we obtain
\[
\int_0^r \varphi(\frac{1}{B}b_t(\varepsilon))\varepsilon^{n-1}d\varepsilon \leq \int\nabla f(\sigma_t(du))\|\nabla f(F_t(\varepsilon))\|_\sigma t(du) = \\
= n^{-1}r^n\int\varphi(\frac{1}{B}\|\nabla f(\varepsilon/d\varepsilon)\|_\sigma t(du).
The Young inequality (1) gives
\[
\frac{b_t(\varepsilon)}{AB\varepsilon^{n-1}} \leq \psi\left(\frac{1}{A\varepsilon^{n-1}}\right) + \varphi\left(\frac{1}{B} b_t(\varepsilon)\right).
\]
Since \(a_t\) is Lipschitz we get
\[
|f(t) - \int_{\partial T} f(u)\sigma_t(u)| = |a_t(0) - a_t(r)| = \left| \int_0^r a_t'(\varepsilon)d\varepsilon \right| \leq 2 \int_0^r b_t(\varepsilon)d\varepsilon.
\]
Thus
\[
|f(t) - \int_{\partial T} f(u)\sigma_t(u)| \leq 2AB\left(\int_0^1 \psi\left(\frac{1}{A\varepsilon^{n-1}}\right)\varepsilon^{n-1}d\varepsilon + n^{-1}r^n \int_T \varphi\left(\frac{1}{B} \|\nabla f(u)\|_s\right)du\right). \tag{12}
\]
Again due to the generalized Green-Gauss theorem (this version holds for Lipschitz functions and Lipschitz boundaries) we obtain
\[
|\int_{\partial T} f(u)\sigma_t(du) - \int_{\partial T} f(u)\sigma_s(du)| = \frac{1}{n|T|} \int_{\partial T} f(u)(s - t, n(u))\sigma_{\partial T}(du) = n^{-1} \int_T (\nabla f(u), s - t)du \leq 2rn^{-1} \int_T \|\nabla f(u)\|_s du.
\]
By the Young inequality and since \(y\psi(1/y)\) is decreasing we obtain
\[
\|\nabla f(u)\|_s \leq \psi\left(\frac{1}{A\varepsilon^{n-1}}\right) + \varphi\left(\frac{1}{B} \|\nabla f(u)\|_s\right) \leq n \int_0^r \psi\left(\frac{1}{A\varepsilon^{n-1}}\right)\varepsilon^{n-1}d\varepsilon + \varphi\left(\frac{1}{B} \|\nabla f(u)\|_s\right).
\]
It follows that
\[
|\int_{\partial T} f(u)\sigma_t(du) - \int_{\partial T} f(u)\sigma_s(du)| \leq 2AB\left(\int_0^r \psi\left(\frac{1}{A\varepsilon^{n-1}}\right)\varepsilon^{n-1}d\varepsilon + n^{-1}r^n \int_T \varphi\left(\frac{1}{B} \|\nabla f(u)\|_s\right)du\right). \tag{13}
\]
By the definition \(|T| = |B(0, r)| = r^n|B|\|\cdot\|((0, 1)|. Inequalities (12), (13) yield
\[
|f(s) - f(t)| \leq |f(t) - \int_{\partial T} f(u)\sigma_t(u)| + |f(s) - \int_{\partial T} f(u)\sigma_s(u)| + |\int_{\partial T} f(u)\sigma_t(du) - \int_{\partial T} f(u)\sigma_s(du)| \leq 6AB\left(\int_0^r \psi\left(\frac{1}{A\varepsilon^{n-1}}\right)\varepsilon^{n-1}d\varepsilon + \frac{1}{n|B|\|\cdot\|((0, 1)|} \int_T \varphi\left(\frac{1}{B} \|\nabla f(u)\|_s\right)du\right).
\]
\[\blacksquare\]
Remark 2 Let $\sigma_{\partial T}$ be the Lebesgue measure on the manifold $\partial T$ and $n(u)$ the normal vector to the boundary in $u \in \partial T$ such that $(n(u), n(u)) = 1$ ($n(u)$ is well defined $\sigma_{\partial T}$-a.s.). For each $t \in T$ and measure $\sigma_t(du) = (u - t, n(u))\sigma_{\partial T}(du)$ the equality $\sigma_t(\partial T) = n|T| = nr^n|B_{\|\cdot\|}(0, 1)|$ holds.

Corollary 4 For each $t \in T$, there holds

$$|f(t) - \int_T f(u)du| \leq 6AB\left(\int_0^r \psi\left(\frac{1}{A^\epsilon\alpha-1}\right)\epsilon^{n-1}d\epsilon + \frac{1}{n|B_{\|\cdot\|}(0, 1)|} \int_T \varphi\left(\frac{1}{B}\|\nabla f(u)\|_\geq\right)du\right).$$

Proof. This observation is obvious. Theorem 3 yields

$$|f(t) - f(u)| \leq 6AB\left(\int_0^r \psi\left(\frac{1}{A^\epsilon\alpha-1}\right)\epsilon^{n-1}d\epsilon + \frac{1}{n|B_{\|\cdot\|}(0, 1)|} \int_T \varphi\left(\frac{1}{B}\|\nabla f(u)\|_\geq\right)du\right).$$

Integrating both parts and using $|f(t) - \int_T f(u)du| \leq \int_T |f(t) - f(u)|du$ we obtain the corollary.

\[\square\]

4 Construction of the optimal process

We assume that $\eta'(0) = \infty$. In this section we prove the left-hand side of (6) in Theorem 2.

Proof of the left-hand side of (6). We define a stochastic process on a probability space $(T, B(T), \lambda)$ by the formula

$$X(t, \omega) := \int_{d(\omega, 0)}^{d(\omega, t)} g(\epsilon)d\epsilon, \quad \text{for } t \in T, \omega \in T,$$

where $g(\epsilon)$ is a positive function, integrable on each interval $[\delta, \eta(r)]$, $\delta > 0$ and such that $g(\epsilon) = 0$, for $\epsilon > \eta(r)$. Let us notice that the process $X$ is separable. Suppose we have shown that

$$E_l\varphi\left(\frac{X(s) - X(t)}{d(s, t)}\right) = \int_T \varphi\left(\frac{X(s, \omega) - X(t, \omega)}{d(s, t)}\right)\lambda(d\omega) \leq 1,$$

then the process $X(t), t \in T$ satisfies the condition (3). Since

$$X(\omega, \omega) = -\int_0^{d(\omega, 0)} g(\epsilon)d\epsilon, \quad X\left(\frac{\|\omega\| - r}{\|\omega\|}, \omega\right) = \int_{d(\omega, 0)}^{\eta(r)} g(\epsilon)d\epsilon,$$
we have \(\sup_{s,t\in T} |X(s, \omega) - X(t, \omega)| = \int_0^{\eta(r)} g(\varepsilon)d\varepsilon\). Due to the definition of \(S(T, d, \varphi)\) it proves that
\[
\int_0^{\eta(r)} g(\varepsilon)d\varepsilon \leq E \sup_{s,t\in T} |X(s) - X(t)| \leq S(T, d, \varphi).
\] (14)

The convexity of \(\varphi\), \(\varphi(0) = 0\) and the Jensen inequality imply
\[
\int_T \varphi(\frac{|X(s, \omega) - X(t, \omega)|}{d(s, t)}) \lambda(d\omega) \leq
\leq \int_T \varphi(\frac{|d(s, \omega) - d(t, \omega)|}{d(s, t)}) \int_{d(s, \omega)}^{d(t, \omega)} g(\varepsilon)d\varepsilon \lambda(d\omega) \leq
\leq \int_T \frac{|d(s, \omega) - d(t, \omega)|}{d(s, t)} \int_{d(s, \omega)}^{d(t, \omega)} \varphi(g(\varepsilon))d\varepsilon \lambda(d\omega) =
= \frac{1}{d(s, t)} \int_T | \int_{d(s, \omega)}^{d(t, \omega)} \varphi(g(\varepsilon))d\varepsilon | \lambda(d\omega).
\]

The Fubini theorem yields
\[
\int_T | \int_{d(s, \omega)}^{d(t, \omega)} \varphi(g(\varepsilon))d\varepsilon | \lambda(d\omega) = \int_0^{\eta(r)} \varphi(g(\varepsilon)) \lambda(B(s, \varepsilon)\Delta B(t, \varepsilon))d\varepsilon,
\] where \(\Delta\) is the symmetric set difference. Observe that if \(d(s, t) \geq \varepsilon\), then
\[
\lambda(B(s, \varepsilon)\Delta B(t, \varepsilon)) \leq \lambda(B(s, \varepsilon)) + \lambda(B(t, \varepsilon)) \leq 2\frac{\eta^{-1}(\varepsilon)^n}{r^n}.
\]

From the other hand if \(\varepsilon \geq d(s, t)\), then
\[
B_{\|s + t/2\|, \eta^{-1}(\varepsilon) - \frac{1}{2}\|s - t\|} \subset B_{\|s\|, \eta^{-1}(\varepsilon)} \cap B_{\|t\|, \eta^{-1}(\varepsilon)},
\]
and thus
\[
\frac{|B(s, \varepsilon)\Delta B(t, \varepsilon)|}{|B(0, \eta(r))|} \leq 2(\frac{\eta^{-1}(\varepsilon)^n}{r^n} - (\frac{\eta^{-1}(\varepsilon) - \frac{1}{2}\|s - t\|)^n}{r^n}) \leq
\leq n\|s - t\| \frac{\eta^{-1}(\varepsilon)^{n-1}}{r^n}.
\]

Hence, for \(\varepsilon \geq d(s, t)\) we have \(\lambda(B(s, \varepsilon)\Delta B(t, \varepsilon)) \leq n\|s - t\| \frac{\eta^{-1}(\varepsilon)^{n-1}}{r^n}\). Consequently if \(d(s, t) \geq \eta(r)\) then
\[
\int_T \varphi(\frac{|X(s, \omega) - X(t, \omega)|}{d(s, t)}) \mu(d\omega) \leq \frac{2}{d(s, t)} \int_0^{\eta(r)} \frac{\eta^{-1}(\varepsilon)^n}{r^n} \varphi(g(\varepsilon))d\varepsilon \leq
\leq \frac{2}{\eta(r)} \int_0^{\eta(r)} \frac{\eta^{-1}(\varepsilon)^n}{r^n} \varphi(g(\varepsilon))d\varepsilon
\] (15)
and if \( d(s, t) \leq \eta(r) \), then
\[
\int_T \varphi\left( \frac{|X(s, \omega) - X(t, \omega)|}{d(s, t)} \right) \lambda(d\omega) \leq \frac{2}{d(s, t)} \int_0^{d(s, t)} \frac{\eta^{-1}(\varepsilon)^n}{r^n} \varphi(g(\varepsilon)) d\varepsilon + \\
+ \frac{\|s - t\|}{d(s, t)} \int_{d(s, t)}^{\eta(r)} \frac{\eta^{-1}(\varepsilon)^{n-1}}{r^n} \varphi(g(\varepsilon)) d\varepsilon.
\]

(16)
The construction of \( g \) is as follows
\[
g(\varepsilon) := K^{-1} \frac{S_k \eta^{-1}(\varepsilon)^n}{r^n \varepsilon} \psi\left( \frac{r^n \varepsilon}{S_k \eta^{-1}(\varepsilon)^n} \right), \quad \text{for } r_{k+1} < \varepsilon \leq r_k,
\]
where the constant \( K \geq 1 \) we choose later. From the convexity of \( \varphi \) and Lemma 1 we deduce
\[
\varphi(g(\varepsilon)) \leq K^{-1} \psi\left( \frac{r^n \varepsilon}{S_k \eta^{-1}(\varepsilon)^n} \right), \quad \text{for } r_{k+1} < \varepsilon \leq r_k, \quad k \geq 0.
\]

(17)
We show that the process \( X \) satisfies the condition \( 3 \) for such \( g \).
First we assume that \( d(s, t) \leq \eta(r) = r_0 \). Hence there exists \( m \) such that \( r_{m+1} < d(s, t) \leq r_m \). Consider \( k > m \), the definition of \( S_k \) and (17) yield
\[
\int_{r_k+1}^{r_k} \frac{\eta^{-1}(\varepsilon)^n}{r^n} \varphi(g(\varepsilon)) d\varepsilon \leq K^{-1} r_k \int_{r_{k+1}}^{r_k} \frac{\eta^{-1}(\varepsilon)^n}{r^n} \psi\left( \frac{r^n \varepsilon}{S_k \eta^{-1}(\varepsilon)^n} \right) d\varepsilon \leq K^{-1} r_k.
\]
Similarly we obtain
\[
\int_{r_{m+1}}^{d(s, t)} \frac{\eta^{-1}(\varepsilon)^n}{r^n} \varphi(g(\varepsilon)) d\varepsilon \leq K^{-1} d(s, t).
\]
Since (5) gives \( 2r_{k+1} \leq r_k \), it is clear that \( r_k \leq 2^{-k+m+1} d(s, t) \), for \( k > m \).
Applying the above inequalities, we get
\[
\begin{align*}
\frac{2}{d(s, t)} & \int_0^{d(s, t)} \frac{\eta^{-1}(\varepsilon)^n}{r^n} \varphi(g(\varepsilon)) d\varepsilon \leq 2K^{-1}(1 + \sum_{k>m} 2^{-k+m+1}) = 6K^{-1}. 
\end{align*}
\]

(18)
It remains to find the estimation for the second integral in (16). Consider \( 0 \leq k < m \). By (17), the definition of \( S_k \) and since \( \eta^{-1}(y)/y \) is increasing, we have
\[
\begin{align*}
\int_{r_{k+1}}^{r_k} \frac{\eta^{-1}(\varepsilon)^{n-1}}{r^n} \varphi(g(\varepsilon)) d\varepsilon & \leq nK^{-1} \int_{r_{k+1}}^{r_k} \frac{\eta^{-1}(\varepsilon)^{n-1}}{r^n} \psi\left( \frac{r^n \varepsilon}{S_k \eta^{-1}(\varepsilon)^n} \right) d\varepsilon \\
& \leq nK^{-1} \frac{r_{k+1}}{\eta^{-1}(r_{k+1})} \int_{r_{k+1}}^{r_k} \frac{\eta^{-1}(\varepsilon)^n}{r^n} \psi\left( \frac{r^n \varepsilon}{S_k \eta^{-1}(\varepsilon)^n} \right) d\varepsilon = \\
& = nK^{-1} \frac{r_{k+1}}{\eta^{-1}(r_{k+1})}.
\end{align*}
\]

(19)
In the same way, we prove
\[ \int_{d(s,t)}^{r_m} \frac{n \eta^{-1}(\varepsilon)^n}{r^n} \varphi(g(\varepsilon))d\varepsilon \leq nK^{-1} \frac{d(s,t)}{\|s-t\|}. \] (20)

Let us notice that (5) gives
\[ \|s-t\| \frac{r_{k+1}}{d(s,t) \eta^{-1}(r_{k+1})} \leq 2^{-m+k+1}, \text{ for } 0 \leq k < m. \]

Inequalities (19) and (20) imply
\[ \|s-t\| \int_{d(s,t)}^{\eta(r)} \frac{n \eta^{-1}(\varepsilon)^n}{r^n} \varphi(g(\varepsilon))d\varepsilon \leq nK^{-1}(1 + \sum_{k=0}^{m-1} 2^{-m+k+1}) \leq 3nK^{-1}. \] (21)

If we plug estimations (18), (21) into (16), we obtain
\[ \mathbb{E}\varphi\left(\frac{|X(s) - X(t)|}{d(s,t)}\right) \leq 3K^{-1}(2 + n). \]

The second case is when \( d(s,t) \geq \eta(r). \) We use (18) to get
\[ \frac{2}{\eta(r)} \int_{0}^{\eta(r)} \frac{\eta^{-1}(\varepsilon)^n}{r^n} \varphi(g(\varepsilon))d\varepsilon \leq 6K^{-1}. \]

The above inequality and (15) imply
\[ \mathbb{E}\varphi\left(\frac{|X(s) - X(t)|}{d(s,t)}\right) \leq 6K^{-1} \leq 3K^{-1}(2 + n). \]

Therefore for (3) we need that \( K := 3(2 + n). \)

By the definition of \( g \) and numbers \( S_k \) it is clear that for all \( k \geq 0 \) we have \( \int_{r_{k+1}}^{r_k} g(\varepsilon)d\varepsilon = K^{-1}S_k. \) Consequently (13) gives \( \sum_{k=0}^{\infty} S_k \leq KS(T, d, \varphi). \)

The theorem is proved with the constant \( K = 3(2 + n). \)

\[ \blacksquare \]

5 Some basic tools

Before we prove the right-hand side of (6) we establish some helpful results. We start from proving a fact which allows us to consider processes with finite number of different Lipschitz paths.
Lemma 3. Fix any point \( t_0 \in T \). Let \( F \subset T \) be a finite set. For each process \( X(t), t \in T \) which satisfies the condition (3) there exists a sequence of processes \((Y_k)_{k \geq 1}\) which satisfy (3), have finite number of different Lipschitz trajectories and such that

\[
\lim_{k \to \infty} Y_k(t) = X(t) - X(t_0), \quad \text{a.s. and in } L_1, \quad \text{for } t \in F. \tag{22}
\]

In particular (22) implies

\[
\lim_{k \to \infty} E \sup_{s,t \in F} |Y_k(s) - Y_k(t)| = E \sup_{s,t \in F} |X(s) - X(t)|.
\]

Proof. A process \( X(t), t \in T \) is defined on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\). Denote \( Y(t) := X(t) - X(t_0) \). It is clear that \( Y(t), t \in T \) satisfies condition (3) and moreover \( \|Y(t)\|_\varphi \leq d(t, t_0) \), what implies \( E|Y(t)| < \infty \), for \( t \in T \).

First we assume that \( F \) is a finite \( \sigma \)-algebra. Due to (3) we have

\[
|Y(s, \omega) - Y(t, \omega)| \leq d(s, t) \varphi^{-1}(1/\mathbb{P}(A)), \quad \text{for } s, t \in T, \omega \in A,
\]

where \( A \) is an atom in \( \mathcal{F} \). Hence the process \( Y \) has \( \mathbb{P} \)-a.s. finite number of different Lipschitz trajectories.

In the general case we use the fact that \( F \) is a finite set. There exists an increasing sequence of finite \( \sigma \)-algebras \((\mathcal{F}_k)_{k \geq 1}\) which sum generates \( \sigma(Y(t) : t \in F) \). Notice that \( E|Y(t)| < \infty \), for \( t \in T \) (since \( \|Y(t)\|_\varphi < \infty \)), thus we can define \( Y_k(t) := E(Y(t)|\mathcal{F}_k), t \in T \). By the Jensen inequality we get

\[
E\varphi \left( \frac{|Y_k(s) - Y_k(t)|}{d(s, t)} \right) \leq E\varphi \left( \frac{|Y(s) - Y(t)|}{d(s, t)} \right) \leq 1, \quad \text{for } s, t \in T.
\]

The process \( Y_k \) satisfies (3), hence \( \mathbb{P} \)-a.s. it has finite number of different Lipschitz trajectories. Modifying \( Y_k \) on the set of measure 0 we may assume that \( Y_k \) has finite number of different Lipschitz trajectories. Clearly \( Y_k(t) \to Y(t) \) \( \mathbb{P} \)-a.s., for \( t \in F \). Since \( E|Y(t)| < \infty \), the convergence is also in \( L_1 \).

Next step is to prove some approximation on numbers \( S_k \).

Lemma 4. There holds:

\[
\frac{1}{4} r_k \varphi^{-1} \left( \frac{2 r^n}{\eta^{-1}(r_k)^n} \right) \leq S_k, \quad \text{for } k \geq 0;
\]

\[
S_k \leq r_{k+1} \varphi^{-1} \left( \frac{r^n}{\eta^{-1}(r_{k+1})^n} \right), \quad \text{for } k \in I.
\]
Proof. Due to [3] we know that \( r_k - r_{k+1} \geq \frac{1}{2} r_k \). Lemma [2] follows that \( y\psi(1/y) \) is decreasing. Thus, for \( k \geq 0 \) we have

\[
1 = \int_{r_{k+1}}^{r_k} \frac{\eta^{-1}(\varepsilon)^n \psi\left(\frac{r^n \varepsilon}{S_k \eta^{-1}(\varepsilon)^n}\right) d\varepsilon}{r^n \varepsilon} \geq \int_{r_{k+1}}^{r_k} \frac{\eta^{-1}(r_k)^n \psi\left(\frac{r^n r_k}{S_k \eta^{-1}(r_k)^n}\right) d\varepsilon}{r^n \varepsilon} \geq (r_k - r_{k+1}) \frac{\eta^{-1}(r_k)^n \psi\left(\frac{r^n r_k}{S_k \eta^{-1}(r_k)^n}\right)}{2r^n} \psi\left(\frac{r^n r_k}{S_k \eta^{-1}(r_k)^n}\right).
\]

That means

\[
\psi^{-1}\left(\frac{2r^n}{\eta^{-1}(r_k)^n}\right) \geq \frac{r^n r_k}{S_k \eta^{-1}(r_k)^n}.
\]

By Lemma [1] (that is by the inequality \( \varphi^{-1}(y)\psi^{-1}(y) \leq 2y \)) we obtain

\[
S_k \geq \frac{1}{4} r_k \varphi^{-1}\left(\frac{2r^n}{\eta^{-1}(r_k)^n}\right).
\]

We prove the second inequality. Since \( y\psi(1/y) \) is decreasing and \( r_k - r_{k+1} = r_{k+1}, \) for \( k \in I, \) then

\[
1 = \int_{r_{k+1}}^{r_k} \frac{\eta^{-1}(\varepsilon)^n \psi\left(\frac{r^n \varepsilon}{S_k \eta^{-1}(\varepsilon)^n}\right) d\varepsilon}{r^n \varepsilon} \leq r_{k+1} \frac{\eta^{-1}(r_{k+1})^n \psi\left(\frac{r^n r_{k+1}}{S_k \eta^{-1}(r_{k+1})^n}\right)}{r^n} \psi\left(\frac{r^n r_{k+1}}{S_k \eta^{-1}(r_{k+1})^n}\right).
\]

Hence

\[
\psi^{-1}\left(\frac{r^n}{\eta^{-1}(r_{k+1})^n}\right) \leq \frac{r^n r_{k+1}}{S_k \eta^{-1}(r_{k+1})^n}.
\]

Again, using Lemma [1] (the inequality \( y \leq \varphi^{-1}(y) \psi^{-1}(y) \)), we get

\[
S_k \leq r_{k+1} \varphi^{-1}\left(\frac{r^n}{\eta^{-1}(r_{k+1})^n}\right).
\]

Let us remind that \( \lambda \) is the normalized Lebesgue measure on \( T \). For \( 0 < \varepsilon \leq \eta(r) \), we denote

\[
B_\varepsilon(t) := B\left((1 - \frac{\eta^{-1}(\varepsilon)}{r})t, \varepsilon\right), \quad S_\varepsilon(t) := S\left((1 - \frac{\eta^{-1}(\varepsilon)}{r})t, \varepsilon\right).
\]

Observe that \( B_\varepsilon(t) = B_{\| \|}(\{1 - \frac{\eta^{-1}(\varepsilon)}{r}\} t, \eta^{-1}(\varepsilon)) \subset T \), hence \( \lambda(B_\varepsilon(t)) = \frac{\eta^{-1}(\varepsilon)^n}{r^n} \). For each \( f \in C(T) \) we define \( f_\varepsilon(t) := \int_{B_\varepsilon(t)} f(u) \lambda(du) \).

Let us assume that \( 0 < \varepsilon \leq \eta(r) \). We denote by \( \sigma_{t, \varepsilon} \) the Lebesgue measure on the manifold \( S_\varepsilon(t) \). For each \( e \in \mathbb{R}^n \) we define

\[
\Delta_{t, \varepsilon}^e := \{ u \in S_\varepsilon(t) : \langle e, n(u) \rangle \geq 0 \},
\]

\[
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\]
where \( n(u) \) is the normal vector in \( u \in S_{\varepsilon}(t) \), such that \( \langle n(u), n(u) \rangle = 1 \) (\( n(u) \) is well defined \( \sigma_{t,\varepsilon} \)-a.s.). Observe that for \( e \in \mathbb{R}^n \) and \( f \in C(T) \)

\[
\lim_{h \to 0} \frac{1}{h} \int_{B_e(t+he) \setminus B_e(t)} f(u) du = \lim_{h \to 0} \frac{1}{h} \int_{B_e(t) \setminus B_e(t-he)} f(u) du = (1 - \eta^{-1}(\varepsilon) \frac{\varepsilon}{r}) \int_{\Delta_{t,\varepsilon}^e} f(u) \langle e, n(u) \rangle \sigma_{t,\varepsilon}(du).
\]

(23)

Let \( \sigma_{t,\varepsilon}^\varepsilon \) denotes the positive measure on \( \Delta_{t,\varepsilon}^e \) given by the formula \( \sigma_{t,\varepsilon}^\varepsilon(du) := \langle e, n(u) \rangle \sigma_{t,\varepsilon}(du) \). Notice that if \( f \in \text{Lip}(T) \), then there exists \( \nabla f \), \( \| \cdot \| \)-a.s.

**Lemma 5** Fix \( 0 < \varepsilon \leq \eta(r) \). For each \( e \in \mathbb{R}^n \), \( \| e \| = 1 \) and \( f \in \text{Lip}(T) \) the following equality holds \( \| \cdot \| \)-a.s. on \( T \)

\[
\langle \nabla f_\varepsilon(t), e \rangle = (1 - \eta^{-1}(\varepsilon) \frac{\varepsilon}{r}) \beta(\varepsilon) \int_{\Delta_{t,\varepsilon}^e} \int_{\Delta_{t,\varepsilon}^e} \frac{f(u) - f(v)}{2\eta^{-1}(\varepsilon)} \sigma_{t,\varepsilon}^\varepsilon(du) \sigma_{t,\varepsilon}^\varepsilon(dv),
\]

where \( \beta(\varepsilon) \leq n. \)

**Proof.** First we assume \( h > 0 \). Observe that

\[
f_\varepsilon(t + he) - f_\varepsilon(t) = \int_{B_e(t+he) \setminus B_e(t)} f(u) \lambda(du) - \int_{B_e(t) \setminus B_e(t-he)} f(u) \lambda(du) = \frac{1}{|B_e(t)|} \int_{B_e(t+he) \setminus B_e(t)} f(u) du - \int_{B_e(t) \setminus B_e(t-he)} f(u) du.
\]

(24)

By (23) we obtain

\[
\lim_{h \to 0} \frac{1}{h} \int_{B_e(t+he) \setminus B_e(t)} f(u) du = (1 - \eta^{-1}(\varepsilon) \frac{\varepsilon}{r}) \int_{\Delta_{t,\varepsilon}^e} f(u) \sigma_{t,\varepsilon}^\varepsilon(du).
\]

(25)

Let us define \( \beta(\varepsilon) \) by the following formula

\[
\beta(\varepsilon) := \frac{2\eta^{-1}(\varepsilon) \sigma_{t,\varepsilon}^\varepsilon(\Delta_{t,\varepsilon}^e)}{|B_e(t)|} = \frac{2\eta^{-1}(\varepsilon) \int_{\Delta_{t,\varepsilon}^e} \langle e, n(u) \rangle \sigma_{t,\varepsilon}(du)}{|B_e(t)|}.
\]

The homogeneity and symmetry imply

\[
\beta(\varepsilon) = \frac{\int_{\Delta_{0,\eta(r)}^e} \langle re, n(u) \rangle \sigma_{0,\eta(r)}(du)}{|B_{\eta(r)}(0)|} + \frac{\int_{\Delta_{0,\eta(r)}^e} \langle -re, n(u) \rangle \sigma_{0,\eta(r)}(du)}{|B_{\eta(r)}(0)|}.
\]

Due to Remark [2], for each \( t \in B_{\eta(r)}(0) = B(0, \eta(r)) \) we have

\[
\int_{B_{\eta(r)}(0)} \langle u - t, n(u) \rangle \sigma_{0,\eta(r)}(du) = n|B_{\eta ||(0, r)}| = n|B_{\eta(r)}(0)|.
\]

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Applying this equality for \( t = -re, t = re \) and \( t = 0 \), we get
\[
\beta(e) = \int_{\Delta_{0,\eta(x)}^e} \langle u + re, n(u) \rangle \sigma_{0,\eta(x)}(du) + \int_{\Delta_{0,\eta(x)}^{-e}} \langle u - re, n(u) \rangle \sigma_{0,\eta(x)}(du) - \frac{\int_{\Delta_{0,\eta(x)}^e} \langle u, n(u) \rangle \sigma_{0,\eta(x)}(du)}{|B_{\eta(x)}(0)|} - \frac{\int_{\Delta_{0,\eta(x)}^{-e}} \langle u, n(u) \rangle \sigma_{0,\eta(x)}(du)}{|B_{\eta(x)}(0)|} - \frac{\int_{\Delta_{0,\eta(x)}^{-e}} \langle u, n(u) \rangle \sigma_{0,\eta(x)}(du)}{|B_{\eta(x)}(0)|} \leq (n + n - n) = n.
\]
We have used here the fact that \( \sigma_{0,\eta(x)}(S_{\eta(x)}(0) \setminus (\Delta_{0,\eta(x)}^e \cup \Delta_{0,\eta(x)}^{-e})) = 0 \).

Observe that by (23)
\[
\lim_{h \to +0} \frac{1}{h} \int_{B_{\eta(x)}(t) \setminus B_{\eta(x)}(t + he)} f(u) \lambda(du) = (1 - \frac{\eta^{-1}(\varepsilon)}{r}) \int_{\Delta_{t,\varepsilon}^e} f(u) \sigma_{t,\varepsilon}(du).
\tag{26}
\]
Moreover \( \beta(e) = \beta(-e) \), hence applying (24), (25) and (26) we obtain
\[
\lim_{h \to +0} \frac{1}{h} (f_{\varepsilon}(t + he) - f_{\varepsilon}(t)) = (1 - \frac{\eta^{-1}(\varepsilon)}{r}) \beta(e) \int_{\Delta_{t,\varepsilon}^e} \int_{\Delta_{t,\varepsilon}^{-e}} \frac{f(u) - f(v)}{2\eta^{-1}(\varepsilon)} \sigma_{t,\varepsilon}(du) \sigma_{t,\varepsilon}(dv).
\]
The case of \( h < 0 \) can be treated in the similar way.

6 The estimation from above

We assume that \( \eta'(0) = \infty \). In this section we prove the right-hand side of (6) in Theorem 2.

Proof of the right-hand side of (6). Denote \( B_k(x) := B_{r_k}(x) \), for \( x \in T \).
Let us notice that
\[
\lambda(B_k(x)) = \lambda(B_{r_k}(x)) = \frac{\eta^{-1}(r_k)^n}{r^n}, \text{ for } x \in T.
\tag{27}
\]
For each \( k \geq 0 \) we define a linear operator \( S_k : C(T) \to C(T) \) by the formula
\[
S_kf(x) := f_{r_k}(x) = \int_{B_k(x)} f(u) \lambda(du), \text{ for } x \in T.
\]
If \( f, g \in C(T), k \geq 0 \), then such properties can be easily derived:

1. \( S_k1 = 1; \)
2. if \( f \leq g \), then \( S_k f \leq S_k g \) and so \( |S_k f| \leq S_k |f| \);

3. \( S_0 f = \int_T f(u) \lambda(du) \) hence \( S_k S_0 f = S_0 f \);

4. \( \lim_{k \to \infty} S_k f(x) = f(x) \).

Observe that if \( f \in \text{Lip}(T) \), then \( S_k f \) is also Lipschitz and thus differentiable \(|\cdot|\text{-a.s.} \). Fix \( m > 0 \). For \( 0 \leq k \leq m \), we define operators \( L_k, R_k, T_k : C(T) \to C(T) \). We put \( L_m = Id, R_m = Id \) and for \( k < m \)

\[
L_k := L_{k+1}S_{k+1}, \quad R_k := R_{k+1}, \quad \text{if} \ k \in I;
\]

\[
L_k := L_{k+1}, \quad R_k := S_{k+1}R_{k+1}, \quad \text{if} \ k \in J.
\]

Denote also \( T_k := L_k S_k R_k \), for \( 0 \leq k \leq m \). Properties of \( S_k \) imply corresponding properties of operators \( L_k, R_k, T_k \). For \( 0 \leq k \leq m \) and \( f, g \in C(T) \), we have:

1. \( L_k 1 = 1 \), and if \( f \leq g \), then \( L_k f \leq L_k g \);

2. if \( f \in \text{Lip}(T) \), then \( R_k f \) is \(|\cdot|\text{-a.s.} \) differentiable on \( T \);

3. the function \( T_0 f \) is constant;

4. there holds \( |T_m f(t) - T_0 f(t)| \leq \sum_{k=0}^{m-1} |T_k f(t) - T_{k+1} f(t)| \).

Fix \( f \in \text{Lip}(T) \) and points \( s, t \in T \). We will analyse \( |T_{k+1} f(t) - T_k f(t)| \).

There are two cases. Either \( k \in I \) or \( k \in J \). In fact we use two different methods.

**Case 1.** Fix \( k \in I, k < m \). By the definition, we have

\[
T_{k+1} f(t) - T_k f(t) = L_{k+1} S_{k+1} (Id - S_k) R_{k+1} f(t).
\]

Clearly \( R_k = R_{k+1} \), for \( k \in I \). Denote \( g := R_k f \), it can be easily checked

\[
|S_{k+1} (Id - S_k) g(w)| \leq \int_{B_{k+1}(w)} \int_{B_k(u)} |g(u) - g(v)| \lambda(du) \lambda(du).
\]

For each Orlicz function \( \varphi \) there holds

\[
x \leq 1 + \frac{\varphi(xy)}{\varphi(y)}, \quad \text{for} \ x \geq 0, y > 0.
\]

Thus

\[
\frac{|g(u) - g(v)|}{10 r_k \varphi^{-1}(\frac{1}{\lambda(B_{k+1}(w))})} \leq 1 + \lambda(B_{k+1}(w)) \varphi(\frac{|g(u) - g(v)|}{10 r_k}).
\]
Consequently, for \( u \in B_{k+1}(w) \) the inequality holds

\[
|g(u) - g(v)| \leq 10r_k \varphi^{-1}(\frac{1}{\lambda(B_{k+1}(w))})(1 + \lambda(B_{k+1}(w))\varphi(\frac{|g(u) - g(v)|}{10r_k})).
\]

Hence

\[
|S_{k+1}(Id - S_k)g(w)| \leq 10r_k \varphi^{-1}(\frac{r^n}{\eta^{-1}(r_{k+1})^n})(1 + \int_{B_k(w)} \varphi(\frac{|g(u) - g(v)|}{10r_k})\lambda(dv)\lambda(du)).
\]

Let us notice that Lemma 4 and the equality \( r_k = 2r_{k+1} \) yield

\[
\frac{1}{8}r_k \varphi^{-1}(\frac{r^n}{\eta^{-1}(r_{k+1})^n}) \leq \frac{1}{4}r_{k+1} \varphi^{-1}(\frac{2r^n}{\eta^{-1}(r_{k+1})^n}) \leq S_{k+1}.
\]

Take \( K_1 := 80 \). Using Property 1 of \( L_{k+1} \), we obtain

\[
|T_{k+1}f(t) - T_kf(t)| \leq K_1 S_{k+1}(1 + \int_{B_k(w)} \varphi(\frac{|R_kf(u) - R_kf(v)|}{10r_k})\lambda(dv)\lambda(du)). \tag{28}
\]

**Case 2.** Fix \( k \in J, k < m \). It is clear that

\[
T_{k+1}f(t) - T_kf(t) = L_{k+1}(Id - S_k)S_{k+1}R_{k+1}f(t).
\]

For \( k \in J \), the equality \( R_k = S_{k+1}R_{k+1} \) holds. Denote \( g := R_kf \), by the definition we get

\[
|(Id - S_k)g(w)| = |g(w) - \int_{B_k(w)} g(u)du|.
\]

Property 2 of \( R_k \) gives that \( g \) is Lipschitz. Moreover \( w \in B_k(w) \), thus due to Corollary 4 we obtain the crucial inequality. For any \( A, B > 0 \)

\[
|g(w) - \int_{B_k(w)} g(u)du| \leq 6AB(\int_0^{\eta^{-1}(r_k)} \psi(\frac{1}{A^{n-1}})\varepsilon^{n-1}d\varepsilon + \frac{1}{n|B||\|0,1\|} \int_{B_k(w)} \varphi(\frac{1}{B}||\nabla g(u)\|_*)du). \tag{29}
\]

We have used here that \( B_k(w) = B|\|((1 - \frac{r^{-1}(r_k)}{2})w, \eta^{-1}(r_k)) \). It remains to choose constants \( A \) and \( B \). For \( k \in J \), we have \( 2 - \frac{r_k}{\eta^{-1}(r_k)} = \frac{r_{k+1}}{\eta^{-1}(r_{k+1})} = \), so we can take

\[
B := 10\beta \frac{r_k}{\eta^{-1}(r_k)} = 5\beta \frac{r_{k+1}}{\eta^{-1}(r_{k+1})},
\]
where $\beta > 0$ we choose later. Finding suitable $A$ is more difficult. First we observe that

$$\int_{0}^{\eta^{-1}(r_k)} \psi\left(\frac{1}{A\varepsilon^{n-1}}\right)\varepsilon^{n-1}d\varepsilon = \int_{0}^{\eta^{-1}(r_k)} \psi\left(\frac{1}{A\varepsilon^{n-1}}\right)\varepsilon^{n-1}d\varepsilon + \int_{0}^{2n^{-1}(r_{k+1})} \psi\left(\frac{1}{A\varepsilon^{n-1}}\right)\varepsilon^{n-1}d\varepsilon. \quad (30)$$

Replacing $\varepsilon = \frac{\eta^{-1}(r_k)}{r_k} \varepsilon'$ and applying $2 \frac{r_k}{\eta^{-1}(r_k)} = \frac{r_{k+1}}{\eta^{-1}(r_{k+1})}$, we get

$$\int_{2n^{-1}(r_{k+1})}^{\eta^{-1}(r_k)} \psi\left(\frac{1}{A\varepsilon^{n-1}}\right)\varepsilon^{n-1}d\varepsilon = \int_{r_{k+1}}^{r_k} \psi\left(\frac{\eta^{-1}(r_k)}{A^{n-1}(r_k)^{n-1}}\right) r_k \varepsilon^{n-1}d\varepsilon. \quad (31)$$

Since $\eta^{-1}(y)/y$ is increasing, $\eta^{-1}(\varepsilon)^n \leq \eta^{-1}(r_k)\varepsilon^n$, for $r_{k+1} < \varepsilon \leq r_k$. Due to Lemma \[2\] the function $\psi(1/y)$ is decreasing, so

$$\int_{2n^{-1}(r_{k+1})}^{\eta^{-1}(r_k)} \psi\left(\frac{1}{A\varepsilon^{n-1}}\right)\varepsilon^{n-1}d\varepsilon \leq \int_{r_{k+1}}^{r_k} \psi\left(\frac{\eta^{-1}(r_k)}{r_k} \varepsilon\right) \frac{\eta^{-1}(\varepsilon)^n}{\varepsilon} d\varepsilon. \quad (31)$$

We use different estimation for the second integral in (31). Put $\varepsilon = 2\eta^{-1}(\varepsilon')$, there holds

$$\int_{0}^{2n^{-1}(r_{k+1})} \psi\left(\frac{1}{A\varepsilon^{n-1}}\right)\varepsilon^{n-1}d\varepsilon =$$

$$= 2^n \int_{0}^{r_{k+1}} \psi\left(\frac{1}{A^{2n-1}\eta^{-1}(\varepsilon)e^{n-1}}\right) \eta^{-1}(\varepsilon)^n \eta^{-1}(\varepsilon') d\varepsilon. \quad (32)$$

We can assume that $\eta^{-1}(\varepsilon')$ is right continuous with left limits. Since $\eta^{-1}(\varepsilon)$ is a convex function, the inequality $\frac{\eta^{-1}(\varepsilon)}{\varepsilon} \leq \eta^{-1}(\varepsilon')$ holds. By the convexity of $\psi$ we get

$$\psi\left(\frac{1}{A^{2n-1}\eta^{-1}(\varepsilon)e^{n-1}}\right) \leq \frac{\eta^{-1}(\varepsilon)}{\eta^{-1}(\varepsilon')\varepsilon} \psi\left(\frac{\eta^{-1}(\varepsilon')\varepsilon}{2n^{-1}\eta^{-1}(\varepsilon)e^{n-1}}\right) \leq \frac{\eta^{-1}(\varepsilon)}{2n^{-1}\eta^{-1}(\varepsilon)e^{n-1}} \psi\left(\frac{\eta^{-1}(\varepsilon)}{A\eta^{-1}(\varepsilon)e^{n-1}}\right).$$

Consequently

$$\int_{0}^{2n^{-1}(r_{k+1})} \psi\left(\frac{1}{A\varepsilon^{n-1}}\right)\varepsilon^{n-1}d\varepsilon \leq 2 \int_{0}^{r_{k+1}} \psi\left(\frac{\eta^{-1}(\varepsilon')\varepsilon}{A}\right) \frac{\eta^{-1}(\varepsilon)^n}{\varepsilon} d\varepsilon. \quad (32)$$

The derivative $\eta^{-1}(\varepsilon')$ can be controlled on the interval $[0, r_{k+1}]$. Indeed the convexity of $\eta^{-1}$ implies

$$\eta^{-1}(\varepsilon') \leq \frac{\eta^{-1}(2\varepsilon) - \eta^{-1}(\varepsilon)}{\varepsilon} \leq \frac{\eta^{-1}(2\varepsilon)}{\varepsilon}. \quad 22$$
Using (5), we obtain \( \eta^{-1}(\varepsilon') \leq 2^{k+2-i} \eta^{-1}(r_k) \frac{r_k}{r_{k+1}} \), for \( r_{i+1} \leq \varepsilon \leq r_i \) and \( i > k \).

Plugging this estimation into (32), we obtain

\[
\int_0^{2\eta^{-1}(r_{k+1})} \psi(\frac{1}{A\varepsilon^{n-1}}) \varepsilon^{n-1} d\varepsilon \leq 2 \sum_{i=k+1}^{\infty} \int_{r_i}^{r_{i+1}} \psi(2^{k-i}\eta^{-1}(r_k) \frac{4\varepsilon}{A\eta^{-1}(\varepsilon)^n}) \frac{\eta^{-1}(\varepsilon)^n}{\varepsilon} d\varepsilon. \tag{33}
\]

We define constant \( A \) by the formula

\[
A := \frac{m\eta^{-1}(r_k)}{r^n r_k}(S_k + 4 \sum_{i>k} \alpha^{k-i} S_i),
\]

where \( \alpha \) must satisfy the condition \( 1 < \alpha < 2 \). The definition of \( S_k \) and the convexity of \( \psi \) give

\[
\int_{r_{k+1}}^{r_k} \psi(\eta^{-1}(r_k) \frac{\varepsilon}{r_k} \frac{\eta^{-1}(\varepsilon)^n}{\varepsilon}) d\varepsilon \leq \int_{r_{k+1}}^{r_k} \psi(\frac{\eta^{-1}(\varepsilon)^n}{nS_k \eta^{-1}(\varepsilon)^n}) \frac{\eta^{-1}(\varepsilon)^n}{\varepsilon} d\varepsilon \leq n^{-1} r^n. \tag{34}
\]

In the same way, for \( i > k \) we prove

\[
\int_{r_{i+1}}^{r_i} \psi(2^{k-i}\eta^{-1}(r_k) \frac{4\varepsilon}{A\eta^{-1}(\varepsilon)^n}) \frac{\eta^{-1}(\varepsilon)^n}{\varepsilon} d\varepsilon \leq \int_{r_{i+1}}^{r_i} \psi(\frac{\alpha^{i-k}}{2}) \frac{\eta^{-1}(\varepsilon)^n}{nS_i \eta^{-1}(\varepsilon)^n} \frac{\eta^{-1}(\varepsilon)^n}{\varepsilon} d\varepsilon \leq n^{-1} \frac{\alpha^{i-k}}{2} \int_{r_{i+1}}^{r_i} \psi(\frac{\eta^{-1}(\varepsilon)^n}{nS_i \eta^{-1}(\varepsilon)^n}) \frac{\eta^{-1}(\varepsilon)^n}{\varepsilon} d\varepsilon = n^{-1} r^n \frac{\alpha^{i-k}}{2}. \tag{35}
\]

Inequalities (30), (31), (33), (34) and (35) yield

\[
\int_0^{\eta^{-1}(r_k)} \psi(\frac{1}{A\varepsilon^{n-1}}) \varepsilon^{n-1} d\varepsilon \leq n^{-1} r^n (1 + 2 \sum_{i>k} \frac{\alpha^{i-k}}{2}) = n^{-1} r^n \frac{2 + \alpha}{2 - \alpha}.
\]

Denote \( S_k' := S_k + 4 \sum_{i>k+1} \alpha^{k-i} S_i \) and \( K_2 := 60 \beta \). By (29) we obtain

\[
\| (Id - S_k) g(\omega) \| \leq K_2 S_k' \frac{2 + \alpha}{2 - \alpha} + \frac{nr^{-n}}{nB_{\|\cdot\|}(0, 1)} \int_{B_k(\omega)} \varphi(\frac{\eta^{-1}(r_{k+1})}{5\beta r_{k+1}} \|\nabla g(\omega)\|) d\mu \leq K_2 S_k' \frac{2 + \alpha}{2 - \alpha} + \int_T \varphi(\frac{\eta^{-1}(r_{k+1})}{5\beta r_{k+1}} \|\nabla g(\omega)\|) \lambda(du),
\]

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where we have used the fact that $|B|_\|((0,1)| = r^{-n}|T|$. Property 1 of $L_{k+1}$ implies

$$|T_{k+1}f(t) - T_kf(t)| \leq K_2S_k(2 + \frac{\alpha}{2} - \alpha) + \frac{1}{T} \int \varphi\left(\frac{\eta^{-1}(r_{k+1})}{5\beta r_{k+1}} \|\nabla R_kf(u)\|_*\right)\lambda(du).$$

**The last part.** Estimations in cases $k \in I$ and $k \in J$ give

$$|T_mf(t) - T_0f(t)| \leq K_1 \sum_{k \in I, k < m} S_{k+1}(1 + \int_T \int_{B_k(u)} \varphi\left(\frac{|R_kf(u) - R_kf(v)|}{10r_k}\right)\lambda(du)) +$$

$$+ K_2 \sum_{k \in I, k < m} S_k(2 + \frac{\alpha}{2} - \alpha) + \frac{1}{T} \int \varphi\left(\frac{\eta^{-1}(r_{k+1})}{5\beta r_{k+1}} \|\nabla R_kf(u)\|_*\right)\lambda(du).$$

Property 3 of operators $T_k$ gives that $T_0f$ is a constant function. Hence

$$|S_mf(s) - S_mf(t)| = |T_mf(s) - T_mf(t)| \leq$$

$$\leq 2K_1 \sum_{k \in I, k < m} S_{k+1}(1 + \int_T \int_{B_k(u)} \varphi\left(\frac{|R_kf(u) - R_kf(v)|}{10r_k}\right)\lambda(du)) +$$

$$+ 2K_2 \sum_{k \in I, k < m} S_k(2 + \frac{\alpha}{2} - \alpha) + \frac{1}{T} \int \varphi\left(\frac{\eta^{-1}(r_{k+1})}{5\beta r_{k+1}} \|\nabla R_kf(u)\|_*\right)\lambda(du). \quad (36)$$

Let us notice that $B_k(u) \subset B(u, 2r_k)$. We use the above inequality to prove that for each process $X(t)$, $t \in T$ which satisfies (3) the inequality holds

$$\mathbb{E} \sup_{s,t \in T} |X(s) - X(t)| \leq K \sum_{k \geq 0} S_k,$$

where the constant $K$ depends only on $n$. Due to the remark (2) and Lemma 3 we can assume that a process $X(t)$, $t \in T$ has finite number of different Lipschitz trajectories.

**Lemma 6** Let $0 \leq k < m$. For each $u, v \in T$, where $d(u, v) \leq 2r_k$, we have

$$\mathbb{E} \varphi\left(\frac{|R_kX(u) - R_kX(v)|}{10r_k}\right) \leq 1.$$

**Proof.** We denote $X_k := R_kX$. If $R_k = Id$, then the condition (3) implies the lemma. Otherwise there exists $N > 0$ and a sequence $k = k_0 < k_1 < \ldots < k_N \leq m$ such that $X_k = R_kX = S_{k_1}S_{k_2}\ldots S_{k_N}X$. For simplicity we
denote \( u_{k_0} := u, \, v_{k_0} := v \). We obtain the following equalities:

\[
X_k(u) = \int_{B_{k_1}(u_{k_0})} \int_{B_{k_2}(u_{k_1})} \ldots \int_{B_{k_N}(u_{k_{N-1}})} X(u_{k_N}) \lambda(du_{k_N}) \ldots \lambda(du_{k_1});
\]

\[
X_k(v) = \int_{B_{k_1}(v_{k_0})} \int_{B_{k_2}(v_{k_1})} \ldots \int_{B_{k_N}(v_{k_{N-1}})} X(v_{k_N}) \lambda(dv_{k_N}) \ldots \lambda(dv_{k_1}).
\]

Take \( u_{k_{i+1}} \in B_{k_i}(u_{k_i}) \subseteq B(u_{k_i}, 2r_{k_i}) \). By the triangle inequality and \([3]\) we have

\[
d(u_{k_0}, u_{k_N}) \leq \sum_{i=0}^{N-1} d(u_{k_i}, u_{k_{i+1}}) \leq 2 \sum_{i=0}^{N-1} r_{k_i} \leq 4r_{k_0}, \quad d(v_{k_0}, v_{k_N}) \leq 4r_{k_0}.
\]

That means, for some probability measures \( \nu_u, \nu_v \) with supports respectively in \( B(u, 4r_k), B(v, 4r_k) \) the equalities hold

\[
X_k(u) = \int_{B(u,4r_k)} X(w) \nu_u(dw), \quad X_k(v) = \int_{B(v,4r_k)} X(z) \nu_v(dz).
\]

By the assumption \( d(u, v) \leq 2r_k \), hence \( d(w, z) \leq 10r_k \). The Jensen inequality, the Fubini theorem and \([3]\) yield

\[
\mathbb{E}_\varphi\left(\frac{|X_k(u) - X_k(v)|}{10r_k}\right) \leq \int_{B(u,4r_k)} \int_{B(v,4r_k)} \mathbb{E}_\varphi\left(\frac{|X(w) - X(z)|}{d(w, z)}\right) \nu_u(dw) \nu_v(dz) \leq 1.
\]

The Auerbach lemma (for the proof see \([10]\) - Lemma 11, II.E.) gives that there exists biorthogonal system \( \{(b_i, b_i^*)\} \) in the space \( \mathbb{R}^n \times \mathbb{R}^n \) such that \( \|b_i\| = 1, \|b_i^*\|_* = 1 \). Consequently for each \( v \in \mathbb{R}^n \)

\[
\|v\|_* = \sup_{u \in B_{\|\cdot\|}(0,1)} |\langle v, u \rangle| \leq \sup_{u \in B_{\|\cdot\|}(0,1)} \sum_{i=1}^{n} |\langle v, b_i \rangle| |\langle b_i^*, u \rangle| \leq \sum_{i=1}^{n} |\langle v, b_i \rangle|. \tag{37}
\]

**Lemma 7** For \(|\cdot|\) almost all \( t \in T \) the following inequality holds

\[
\mathbb{E}_\varphi\left(\frac{\eta^{-1}(r_{k+1})}{5\beta r_{k+1}} \|\nabla R_k X(t)\|_*\right) \leq 1, \quad 0 \leq k < m, k \in J,
\]

where \( \beta := \sum_{i=1}^{n} \beta(b_i) \) (\( \beta_i \) was defined in Lemma \([3]\)). Since \( \beta_i \leq n \), thus \( \beta \leq n^2 \).
Proof. We put \( X_k := R_k X \). The definition gives that \( X_k = S_{k+1} R_{k+1} X = (X_{k+1})_{r_k+1} \), for \( k \in J \). Applying (37) and Lemma 5 we obtain that for \(| \cdot |\)-almost all \( t \in T \) there holds

\[
\frac{\eta^{-1}(r_{k+1})}{5\beta r_{k+1}} \| \nabla X_k(t) \|_* \leq \sum_{i=1}^{n} \frac{1}{\beta} \frac{\eta^{-1}(r_{k+1})}{5r_{k+1}} |(\nabla (X_{k+1})_{r_{k+1}}(t), b_i)| \leq
\]

\[
\sum_{i=1}^{n} \frac{\beta(b_i)}{\beta} \int_{\Delta_i^+} \int_{\Delta_i^-} \frac{|X_{k+1}(u) - X_{k+1}(v)|}{10r_{k+1}} \sigma^i_1(du) \sigma^i_2(dv),
\]

where \( \sigma^i_1(du) = \sigma^{b_i}_{t, r_{k+1}}(du) \), \( \sigma^i_2(dv) = \sigma^{-b_i}_{t, r_{k+1}}(dv) \). The Jensen inequality yields

\[
\varphi \left( \frac{\eta^{-1}(r_{k+1})}{5\beta r_{k+1}} \| \nabla X_k(t) \|_* \right) \leq \sum_{i=1}^{n} \frac{\beta(b_i)}{\beta} \int_{\Delta_i^+} \int_{\Delta_i^-} \varphi \left( \frac{|X_{k+1}(u) - X_{k+1}(v)|}{10r_{k+1}} \right) \sigma^i_1(du) \sigma^i_2(dv).
\]

Notice that \( d(u, v) \leq 2r_{k+1} \) for \( u \in \Delta^+_{t, r_{k+1}} \), \( v \in \Delta^-_{t, r_{k+1}} \). The Fubini theorem and Lemma 6 imply

\[
E \varphi \left( \frac{\eta^{-1}(r_{k+1})}{5\beta r_{k+1}} \| \nabla X_k(t) \|_* \right) \leq \sum_{i=1}^{n} \frac{\beta(b_i)}{\beta} = 1.
\]

By the Fubini theorem, Lemma 6, Lemma 7 and (36) we obtain

\[
E \sup_{s, t \in T} |S_m X(s) - S_m X(t)| \leq 4K_1 \sum_{k \in I, k < m} S_{k+1} + 2K_2(1 + \frac{2 + \alpha}{2 - \alpha}) \sum_{k \in I, k < m} S'_k.
\]

Let us notice that \( \sum_{k \in I, k < m} S'_k \leq (1 + \frac{1}{\alpha - 1}) \sum_{k \geq 0} S_k \). We have proved that for some constant \( K \) which depends only \( n \) the following inequality holds

\[
E \sup_{s, t \in T} |S_m X(s) - S_m X(t)| \leq K \sum_{k \geq 0} S_k.
\]

Since \( \lim_{m \to \infty} S_m X(t) = X(t) \), thus due to the Fatou lemma

\[
E \sup_{s, t \in T} |X(s) - X(t)| \leq \liminf_{m \to \infty} E \sup_{s, t \in T} |S_m X(s) - S_m X(t)| \leq K \sum_{k \geq 0} S_k.
\]

It ends the proof of the theorem.
7 The case of $\eta'(0) < \infty$

We assume that $\eta'(0) < \infty$. Let us remind that there exists $m \geq 0$ such that $r_m > 0$ and $r_{m+1} = 0$. We have defined $S_m$ as the infimum over all $c > 0$ such that

$$\int_0^{r_m} \frac{\lambda(B(0,\varepsilon))}{\varepsilon} \varphi\left(\frac{\varepsilon}{c\lambda(B(0,\varepsilon))}\right)d\varepsilon = \int_0^{r_m} \frac{\eta^{-1}(\varepsilon)^n}{r^n \varepsilon} \varphi\left(\frac{r^n \varepsilon}{c\eta^{-1}(\varepsilon)^n}\right)d\varepsilon \leq 1$$

and $S_k = 0$, for $k > m$.

**Proof of Theorem 2 in the case of $\eta'(0) < \infty$.** We follow the proof of Theorem 2 in the case of $\eta'(0) = \infty$. The only difference is when $S_m = \infty$.

For all $0 < \delta \leq r_m/2$ we denote by $S_m(\delta)$ numbers such that

$$\int_0^{r_m} \frac{\lambda(B(0,\varepsilon))}{\varepsilon} \varphi\left(\frac{\varepsilon}{S_m^{\delta}\lambda(B(0,\varepsilon))}\right)d\varepsilon = \int_0^{r_m} \frac{\eta^{-1}(\varepsilon)^n}{r^n \varepsilon} \varphi\left(\frac{r^n \varepsilon}{S_m^{\delta}\eta^{-1}(\varepsilon)^n}\right)d\varepsilon = 1.$$

Since $S_m = \infty$ we have $\lim_{\delta \to \infty} S_m(\delta) = \infty$. The proof of the left-hand side of (3) in Theorem 2 implies $S_m(\delta) \leq 3(n+2)S(T,d,\varphi)$. Hence $S(T,d,\varphi) = \infty$. It ends the proof.

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