D-branes on Orbifolds and Topology Change

Tomomi Muto*
{
Department of Physics, Kyoto University
Kyoto 606-01, Japan

Abstract

We consider D-branes on an orbifold $\mathbb{C}^3/\mathbb{Z}_n$ and investigate the moduli space of the D-brane world-volume gauge theory by using toric geometry and gauged linear sigma models. For $n = 11$, we find that there are five phases, which are topologically distinct and connected by flops to each other. We also verify that non-geometric phases are projected out for $n = 7, 9, 11$ cases as expected. Resolutions of non-isolated singularities are also investigated.

*e-mail address: muto@gauge.scphys.kyoto-u.ac.jp
1 Introduction

In string theory, it had been thought that standard concepts of space-time would break down at the scale $\sqrt{\alpha'}$ since it is the scale of probes, i.e. fundamental strings. In [1], however, it was argued that the structure of space-time on sub-stringy scales can be probed by D-branes. Space-time appearing in the D-brane approach has very different features from that probed by fundamental strings. First of all, space-time coordinates are promoted to non-commuting matrices, and usual space-time emerges from moduli space of D-brane world-volume gauge theory. So it is interesting to investigate space-time by using D-branes as probes and compare it with space-time probed by fundamental strings. Investigations toward this direction were made in [2, 3, 4], in which D-branes on orbifolds were studied. In particular, three dimensional orbifolds in [4] serve as local descriptions of singularities in Calabi-Yau manifolds.

Investigations on the moduli space of Calabi-Yau manifolds were made in [6, 7] based on fundamental strings. It was shown that the moduli space has a rich phase structure. It includes topologically distinct Calabi-Yau spaces and non-geometric phases such as orbifold and Landau-Ginzburg phases. Topologically distinct Calabi-Yau phases are connected by flops. A flop is achieved by a sequence of operations: first blowing down some homologically nontrivial cycle $\mathbb{CP}^1$ and then blowing up another $\mathbb{CP}^1$. In this process, Hodge numbers do not change but more subtle topological indices such as intersection numbers among homology cycles change. In the course of the flop, the space becomes singular due to the shrinking of some $\mathbb{CP}^1$. As a conformal field theory, however, this process occurs smoothly. Possibility of the smooth topology change is demonstrated by using mirror symmetry in [6]. In the gauged linear sigma model approach [7], the singularity is avoided by giving non-zero theta angle. As for the non-geometric phases, it is argued that they can also be interpreted geometrically by analytic continuation to Calabi-Yau phases, although part of the Calabi-Yau manifold has been shrunk to string or sub-stringy scales.

In [4], the moduli space of Calabi-Yau manifolds was investigated by using D-branes as probes. They considered D-branes in typeII string theory on orbifolds $\mathbb{C}^3/\mathbb{Z}_3$ and $\mathbb{C}^3/\mathbb{Z}_5$. It gives $U(1)^n$ gauge theory with Fayet-Iliopoulos D-terms coming from twisted sectors of closed strings. As the coefficients of the Fayet-Iliopoulos D-terms change, the moduli space of the gauge theory changes. A priori, it may seem that a rich phase structure arises as in [4, 7]. However it is shown that only Calabi-Yau phases are allowed and non-geometric phases are projected out. This result matches the analytically continued picture of the moduli space in [8]. It is also consistent with the study of the moduli space of Calabi-Yau compactifications in M theory [9], which can be thought of as strong coupling limit of type IIA theory.

To proceed the comparison between the moduli space of Calabi-Yau spaces probed by fundamental strings and that probed by D-branes, it is important to investigate topology

\footnote{In [8], orbifold singularities of Calabi-Yau fourfolds were investigated along the lines of [9].}
changing process in the D-brane approach. In this paper we present an explicit example in which the moduli space includes topologically distinct Calabi-Yau phases connected by flops based on D-brane world-volume gauge theory.

The organization of this paper is as follows. In section 2, we review flops in terms of toric geometry and gauged linear sigma models which are necessary to the analyses in the following sections. In section 3, we explain D-branes on orbifold $\mathbb{C}^3/\mathbb{Z}_n$. In section 4, we first review the work [4], which treats $n = 3, 5$ cases. We then consider $n = 7, 9, 11$ cases and explicitly check that non-geometric phases are projected out. In the $n = 11$ case, we present the model in which there are five topologically distinct phases connected by flops. In section 5, we consider D-brane on orbifolds with non-isolated singularities. Section 6 contains discussion.

2 Topology change in toric geometry

In this section, we review toric varieties and physical realization of toric varieties in terms of gauged linear sigma models emphasizing topology changing process. For details, see [4, 10, 11].

A complex $d$-dimensional toric variety is a space which contains algebraic torus $(\mathbb{C}^*)^d$ as a dense open subset. A toric variety is determined by a combinatorial data $\Delta$ called a fan, so we denote it by $V_\Delta$. A fan $\Delta$ is a collection of strongly convex rational polyhedral cones in $\mathbb{R}^d$ with apex at the origin. To be a fan it must have the property that (1) any two members of the collection intersect in a common face, (2) for each member of $\Delta$ all its faces are also in $\Delta$.

$V_\Delta$ can be expressed in the form $(\mathbb{C}^k - F_\Delta)/(\mathbb{C}^*)^{k-d}$, where $k$, $F_\Delta$ and the action of $(\mathbb{C}^*)^{k-d}$ on $\mathbb{C}^k$ are determined by $\Delta$ as follows.

1. Let $\vec{n}_1, \vec{n}_2, \ldots, \vec{n}_k$ be the integral generators of the one dimensional cones in $\Delta$. Then associate a homogeneous coordinate $p_i$ of $\mathbb{C}^k$ with each vector $\vec{n}_i$.

2. Define $F_\Delta$, a subset of $\mathbb{C}^k$, by

$$F_\Delta = \bigcap_{\sigma \in \Delta} \{(p_1, p_2, \ldots, p_k) \in \mathbb{C}^k; \prod_{\vec{n}_i \notin \sigma} p_i = 0\}. \quad (2.1)$$

Here $\sigma \in \Delta$ means that $\sigma$ is a cone in $\Delta$, and $\vec{n}_i \in \sigma$ means that $\vec{n}_i$ is a generator of some one-dimensional cone in $\sigma$.

3. $k$ vectors $\vec{n}_1, \vec{n}_2, \ldots, \vec{n}_k$ in $\mathbb{R}^d$ satisfy $(k - d)$ relations

$$\sum_{i=1}^{k} Q_i^{(a)} \vec{n}_i = 0 \quad (2.2)$$

with $a = 1, 2, \ldots, k - d$. Then the action of $(\mathbb{C}^*)^{k-d}$ on $p_i$ is defined as

$$p_i \to \lambda_1^{Q_i^{(1)}} \lambda_2^{Q_i^{(2)}} \cdots \lambda_{k-d}^{Q_i^{(k-d)}} p_i, \quad \lambda_a \in \mathbb{C}^*. \quad (2.3)$$
An important point is that a set of vectors \( \{ \vec{n}_1, \ldots, \vec{n}_k \} \) determines the action of \((\mathbb{C}^\ast)^{k-d}\) on \(\mathbb{C}^k\), but does not determine \(F_\Delta\). To determine \(F_\Delta\) we must specify which vectors generate each cone \(\sigma\) in \(\Delta\). The specification is called a triangulation of \(\Delta\). In general there are various triangulations for the same set of vectors \(\{ \vec{n}_1, \ldots, \vec{n}_k \}\). Different triangulations correspond to different \(F_\Delta\) and hence different toric varieties \(V_\Delta\).

Various properties of a toric variety \(V_\Delta\) can be expressed in terms of its fan \(\Delta\). For example, \(V_\Delta\) is compact if and only if \(\Delta\) spans the whole of \(\mathbb{R}^d\).

What we want to investigate is geometry in the neighborhood of singularities in Calabi-Yau manifolds, so we consider cases where \(V_\Delta\) has zero Chern class. For these cases a fan \(\Delta\) must have the following property,

\[
\vec{\mu} \cdot \vec{n}_i = 1 \quad \text{for } \forall i \quad (2.4)
\]

for a certain vector \(\vec{\mu} \in \mathbb{Z}^d\). This condition leads to relations among \((\mathbb{C}^\ast)^{k-d}\) charges,

\[
\sum_i Q_i^{(a)} = 0 \quad (2.5)
\]

due to \((2.2)\). \((2.4)\) also implies that all the points specified by vectors \(\vec{n}_1, \ldots, \vec{n}_k\) lie in a certain hyperplane \(\Gamma\). Thus we can represent a fan \(\Delta\) in terms of the intersection of \(\Delta\) with the hyperplane \(\Gamma\). We call it as a toric diagram.

In the following we consider cases that the fan \(\Delta\) is simplicial, for which \(V_\Delta\) has at most quotient singularities. Simplicial fan is such that each cone \(\sigma\) can be written in the form

\[
\sigma = \mathbb{R}_{\geq 0} \vec{n}_1 + \cdots + \mathbb{R}_{\geq 0} \vec{n}_r \quad (2.6)
\]

for some linearly independent vectors \(\vec{n}_1, \ldots, \vec{n}_r \in \mathbb{Z}^d\). When \(r = d\), we define a volume for simplicial cones to be \(d!\) times the volume of the polyhedron with vertices \(O, \vec{n}_1, \ldots, \vec{n}_d\). Here we take \(\vec{n}_i\) to be the first nonzero lattice point on the ray \(\mathbb{R}_{\geq 0} \vec{n}_i\). \(V_\Delta\) is smooth if every \(d\)-dimensional cone in the fan has volume one. If some cones have volume greater than one, \(V_\Delta\) has quotient singularities. However we can obtain a smooth toric variety by subdividing the cone until each cone has volume one. This procedure corresponds to a blow up of the singularity. An important point is that the resolution of the singularity is not necessarily unique. There are often numerous ways of subdividing the cones in \(\Delta\). Thus there are numerous smooth varieties that can arise from different ways of resolving the singularity. In general these varieties are topologically distinct. For Calabi-Yau threefold such topologically distinct varieties can be connected by flops.

A flop is a transformation of a manifold into a topologically different manifold which replaces some homologically nontrivial cycle \(\mathbb{CP}^1\) with another \(\mathbb{CP}^1\). In toric language, a flop is described by toric diagrams which include figure \(\text{[1]}\). We denote toric varieties

---

\(\text{[2]}\)For fans satisfying \((2.4)\), the volume of a cone can be represented by “area” of intersection of the cone with the hyperplane \(\Gamma\). Unit of the area is defined as the area of the intersection between \(\Gamma\) and a cone with unit volume.
The toric varieties are written as

\[ \overline{\text{CP}} \]

occurs by the replacement of \( \xi \). The most important point is that the inclusion of Fayet-Iliopoulos D-term with \( \xi < 0 \) provides the degrees of freedom which are the counterparts to the possible triangulations of \( \Delta \).

In the gauged linear sigma model description, a flop is described by a model with the following D-flatness condition

\[ |p_0|^2 - |p_1|^2 - |p_2|^2 + |p_3|^2 = \xi. \]  

The \( U(1) \) action on four chiral superfields is \( p_0 \to \lambda p_0, p_1 \to \lambda^{-1} p_1, p_2 \to \lambda^{-1} p_2, p_3 \to \lambda p_3 \). For \( \xi < 0 \), the vacuum moduli space specified by the D-flatness condition do not contain \( \{ p_1 = p_2 = 0 \} \). This condition is equivalent to removing the point set \( F_{\Delta a} \),
description. For $\xi > 0$, the vacuum moduli space do not contain $\{p_0 = p_3 = 0\}$. This condition is equivalent to removing the point set $F_{\Delta_0}$. For both cases $|\xi|$ controls the size of $\mathbb{CP}^1$ associated with the blow up. The flop occurs in passing from $\xi < 0$ to $\xi > 0$ and vice versa. Thus the sign of the Fayet-Iliopoulos parameter specifies the triangulation as noted above. Classically there is a singularity at $\xi = 0$, but as long as theta angle is generic, smooth topology change can occur $\mathbb{F}$.

3 D-branes on $\mathbb{C}^3/\mathbb{Z}_n$

We consider a typeII string theory on an orbifold $\mathbb{C}^3/\mathbb{Z}_n$, with $\mathbb{Z}_n$ a subgroup of $SU(3)$. We begin with several remarks on orbifolds itself. We take complex coordinates for $\mathbb{C}^3$ as $X^\mu$ ($\mu = 1, 2, 3$), and an action of a generator $g$ of $\mathbb{Z}_n$ on $X^\mu$ as

$$g : X^\mu \rightarrow \omega^{a_\mu} X^\mu, \quad \omega = \exp(2\pi i / n)$$

(3.1)

where $(a_1, a_2, a_3)$ are integers which satisfy $a_1 + a_2 + a_3 \simeq 0$, mod $n$. Here we consider cases $a_\mu \neq 0$ since an orbifold with $a_\mu = 0$ for some $\mu$ is a direct product of $\mathbb{C}^2/\mathbb{Z}_n$ and $\mathbb{C}$. Thus orbifolds are labeled by $n$ and $\vec{a} = (a_1, a_2, a_3)$, but different $\vec{a}$'s often give the same orbifold. That is, if there is some integer $k$ which has no common factor with $n$ except one and satisfies the equation

$$(\omega^{a_1}, \omega^{a_2}, \omega^{a_3}) = ((\omega^k)^{a_1'}, (\omega^k)^{a_2'}, (\omega^k)^{a_3'})$$

(3.2)

the orbifold with $\vec{a} = (a_1, a_2, a_3)$ is the same as the orbifold with $\vec{a} = (a_1', a_2', a_3')$. This is because the two cases give the same action of $\mathbb{Z}_n$ on $X^\mu$ if we take all elements $g^l$, ($l = 0, 1, \ldots, n - 1$) into account. It includes a trivial case $a_\mu \rightarrow a_\mu + n$ ($k = 1$). Apart from this, an exchange between $a_\mu$ and $a_\nu$ does not change the geometry since it is merely an exchange between $X^\mu$ and $X^\nu$. We present explicit examples of the equivalence in the following sections.

To describe D-branes on the orbifold $\mathbb{C}^3/\mathbb{Z}_n$, we first consider $n$ D-branes on $\mathbb{C}^3$. It gives four dimensional $N = 4$ supersymmetric $U(n)$ gauge theory. Bosonic field contents of the theory are a gauge field $A_\alpha$, ($\alpha = 0, 1, 2, 3$), and three complex scalar fields $X^\mu$, ($\mu = 1, 2, 3$) where $(X^\mu)^\dagger = X^{\bar{\mu}}$. We then impose invariance under $\mathbb{Z}_n$. It acts on the space-time indices of $\mathbb{C}^3$ as $[3.1]$, and on Chan-Paton indices (which label the D-branes) in the regular representation $\gamma(g)_{ij} = \delta_{ij}\omega^l$. Thus the generator $g$ of $\mathbb{Z}_n$ acts on the gauge field as

$$g : A_\alpha \rightarrow \gamma(g) A_\alpha \gamma(g)^{-1}.$$  

(3.3)

Invariance under $\mathbb{Z}_n$ gives $A_{\alpha ij} = A_{\alpha i} \delta_{ij}$, where $i, j = 0, 1, \ldots, n - 1$, therefore the gauge group $U(n)$ is reduced to $U(1)^n$.

$^3$Note that the resulting theory has $N = 1$ supersymmetry due to the relation $a_1 + a_2 + a_3 \simeq 0$, mod $n$. 

6
For $X^\mu$, $\mathbb{Z}_n$ acts as

$$g : X^\mu \rightarrow \omega^{a\mu} \gamma(g) X^\mu \gamma(g)^{-1}.$$  

(3.4)

Invariance under $\mathbb{Z}_n$ gives

$$X^\mu_{ij} = X^\mu_{ij} \delta_{j,i+a},$$  

(3.5)

which leave $3n$ fields. We denote these fields as

$$x_i = X^1_{i,i+a}, \quad y_i = X^2_{i,i+a}, \quad z_i = X^3_{i,i+a}.$$  

(3.6)

Since all fields have charge zero for diagonal $U(1)$, nontrivial gauge symmetry is $U(1)^{n-1}$.

The field contents can be described by quiver diagrams\cite{2, 3}. It consists of $n$ vertices associated with gauge fields $A_{\alpha i}$ and $3n$ oriented links associated with $X^\mu_{ij}$. The links represent the information on $U(1)^n$ charges of $X^\mu_{ij}$. The equivalence (3.2) except $k = 1$ corresponds to a certain exchange among vertices.

To obtain vacuum moduli space of the theory, we first impose F-flatness conditions. F-flatness conditions give $(2n-2)$ equations

$$x_iz_{i+1} = z_ix_{i+2}, \quad y_iz_{i+2} = z_iy_{i+2},$$  

(3.7)

so the resulting space is $(n+2)$-dimensional, which we denote as $\mathcal{N}$. Then we impose D-flatness conditions and divide by the gauge symmetry. D-flatness conditions give (real) $(n-1)$ equations

$$|x_{i-1}|^2 + |y_{i-2}|^2 + |z_{i-3}|^2 - |x_i|^2 - |y_i|^2 - |z_i|^2 = \zeta_i,$$  

(3.8)

where $i = 1, 2, \ldots, n-1$ and $\zeta_i$ is a coefficient of Fayet-Iliopoulos D-term, which comes from the twisted sector of closed strings. Together with a gauge fixing of $U(1)^{n-1}$ symmetry, we have three dimensional space as expected. This is the vacuum moduli space of the D-brane world-volume gauge theory. We denote it by $\mathcal{M}$.

Note that although F-flatness condition leads to $[X^\mu, X^\nu] = 0$, $[X^\mu, \tilde{X}^\mu]$ does not vanish if $\zeta_i \neq 0$. Thus the vacuum moduli space is embedded nontrivially into a larger non-commuting configuration space.

We express the geometry of the moduli space by using toric method as follows. First, we express $(n+2)$-dimensional space $\mathcal{N}$ by a toric variety $V_{\Delta_N} = (\mathbb{C}^{n+2+k} - F_{\Delta_N})/(\mathbb{C}^*)^k$. To make the translation into a toric description, we use another definition\cite{6} of toric varieties from that described in section 2. First, we introduce $3n$ variables $u_i$ and $(n+2)$ variables $v_i$ as $(u_1, \ldots, u_{3n}) = (x_1, \ldots, x_{n-1}, y_1, \ldots, y_{n-1}, x_0, y_0, z_1, \ldots, z_{n-1}, z_0)$, $(v_1, \ldots, v_{n+2}) = (x_0, y_0, z_1, \ldots, z_{n-1}, z_0)$. Then the solution of the F-flatness conditions is written in the form

$$u_i = \prod_{j=1}^{n+2} v_j^{m_{ij}}$$  

(3.9)
where \( m_{ij} \in \mathbb{Z} \). We can regard \( m_{ij} \) as the \( j \)-th component of the \( i \)-th \((n + 2)\)-dimensional vector \( \vec{m}_i \). These \( 3n \) vectors define a cone \( \hat{\sigma} \) in \( \mathbb{R}^{n+2} \) as

\[
\hat{\sigma} = \{ \vec{m} \in \mathbb{R}^{n+2}; \vec{m} = \sum_{i=1}^{3n} a_i \vec{m}_i, \ a_i > 0 \}.
\]

We now define dual cone \( \sigma \) of \( \hat{\sigma} \) by

\[
\sigma = \{ \vec{n} \in \mathbb{R}^{n+2}; \vec{m} \cdot \vec{n} \geq 0, \ \forall \vec{m} \in \hat{\sigma} \}.
\]

A collection of cones, \( \sigma \) and its faces, defines the fan \( \Delta_N \).

Then we express the three dimensional vacuum moduli space \( \mathcal{M} \) by a toric variety. It is obtained from \( N \) by imposing \((n - 1)\) \( D \)-flatness conditions and \( U(1)^{n-1} \) gauge fixing. It is equivalent to the holomorphic quotient by \((\mathbb{C}^*)^{n-1}\) after removing an appropriate point set. Together with the holomorphic quotient by \((\mathbb{C}^*)^k\) in \( N \), \( \mathcal{M} \) is expressed as \( \mathcal{M} = (\mathbb{C}^{n+2+k} - F_{\Delta_M})/(\mathbb{C}^*)^{k+n-1} \). Note that \((\mathbb{C}^*)^{n-1}\) charges of \( p_0, p_1, \ldots, p_{n+1+k} \) are determined from \( U(1)^{n-1} \) charges of \( (x_i, y_i, z_i) \) through the following relations

\[
u_i = \prod p_j^{\vec{n}_j \cdot \vec{m}_i}.
\]

In this case, the cone \( \hat{\sigma} \) is generated by nine vectors, 

\[
\vec{m}_1 = (1, 0, 1, 0, -1), \quad \vec{m}_2 = (1, 0, 0, 1, -1),
\]

In the following, we take \( a_3 = -1 \) for convenience. This choice is always allowed due to the equivalence \( a_\mu \leftrightarrow a_\nu \).

4 Orbifold resolution and topology change

In this section, we study the vacuum moduli space of D-brane world-volume gauge theory on \( \mathbb{C}^3/\mathbb{Z}_n \) for \( n = 3, 5, 7, 9, 11 \). Although \( n = 3, 5 \) cases were already investigated in [4], we describe the results for these cases rather elaborately to explain the method of analysis.

4.1 D-branes on \( \mathbb{C}^3/\mathbb{Z}_3 \)

For \( \mathbb{C}^3/\mathbb{Z}_3 \), any model is equivalent to the model with \( \vec{a} = (2, 2, -1) \). For example, a model with \( \vec{a} = (1, 1, 1) \) is equivalent to the model with \( \vec{a} = (2, 2, -1) \) as follows,

\[
(\omega^1, \omega^1, \omega^1) = (\omega^{1+3}, \omega^{1+3}, \omega^{1-3}) = ((\omega^3)^2, (\omega^3)^2, (\omega^3)^{-1}).
\]

In this case, the cone \( \hat{\sigma} \) is generated by nine vectors,

\[
\vec{m}_1 = (1, 0, 1, 0, -1), \quad \vec{m}_2 = (1, 0, 0, 1, -1),
\]
\[ \vec{m}_3 = (0, 1, 1, 0, -1), \]
\[ \vec{m}_4 = (0, 1, 0, 1, -1), \]
\[ \vec{m}_5 = (1, 0, 0, 0, 0), \]
\[ \vec{m}_6 = (0, 1, 0, 0, 0), \]
\[ \vec{m}_7 = (0, 0, 1, 0, 0), \]
\[ \vec{m}_8 = (0, 0, 0, 1, 0), \]
\[ \vec{m}_9 = (0, 0, 0, 0, 1). \]

The fan $\Delta_N$ is determined by the condition $\vec{m}_i \cdot \vec{n} \geq 0$ ($i = 1, 2, \ldots, 9$), and its one-dimensional cones are generated by
\[ \vec{n}_0 = (1, 0, 0, 0, 0), \]
\[ \vec{n}_1 = (0, 1, 0, 0, 0), \]
\[ \vec{n}_2 = (0, 0, 1, 0, 0), \]
\[ \vec{n}_3 = (0, 0, 0, 1, 0), \]
\[ \vec{n}_4 = (1, 1, 0, 0, 1), \]
\[ \vec{n}_5 = (0, 0, 1, 1, 1). \]

They satisfy the relation $\vec{n}_0 + \vec{n}_1 - \vec{n}_2 - \vec{n}_3 - \vec{n}_4 + \vec{n}_5 = 0$, so the $C^*$ charges of $p_i$'s are given by
\[ Q_N = \begin{pmatrix} 1 & 1 & -1 & -1 & 1 \end{pmatrix}. \]  
(4.4)

Combining $(C^*)^2$ charges which originate from $U(1)^2$ gauge group of the original D-brane world-volume theory, we have $(C^*)^3$ charges of $p_i$'s as
\[ Q_{tot} = \begin{pmatrix} 1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 \end{pmatrix}. \]  
(4.5)

The resulting toric variety takes the form $(C^6 - F_\Delta) / (C^*)^3$. The charge matrix (4.3) determines six vectors which generate one dimensional cones in $\Delta$.
\[ \vec{n}_0 = (-1, -1, 1), \]
\[ \vec{n}_1 = (0, 1, 1), \]
\[ \vec{n}_2 = (0, 0, 1), \]
\[ \vec{n}_3 = (0, 0, 1), \]
\[ \vec{n}_4 = (0, 0, 1), \]
\[ \vec{n}_5 = (1, 0, 1). \]

Here $\vec{n}_2 = \vec{n}_3 = \vec{n}_4$, so two of the three homogeneous coordinates $p_2, p_3, p_4$ are redundant.

It implies that the toric variety is written in the form
\[ (C^6 - F_\Delta) / (C^*)^3 = ((C^4 - F_{\Delta M}) \times (C^*)^2) / (C^*)^3 = (C^4 - F_{\Delta M}) / C^*. \]  
(4.7)
This is the moduli space $\mathcal{M}$ of the D-brane world-volume gauge theory.

To determine the triangulation of $\Delta_{\mathcal{M}}$, we go to the gauged linear sigma model description. It is a $U(1)^3$ gauge theory with six chiral superfields. The D-flatness conditions are

\begin{align*}
|p_0|^2 + |p_1|^2 - |p_2|^2 - |p_3|^2 - |p_4|^2 + |p_5|^2 &= 0, \\
- |p_2|^2 + |p_3|^2 &= \zeta_1, \\
- |p_3|^2 + |p_4|^2 &= \zeta_2.
\end{align*}

Here $\zeta_1$ and $\zeta_2$ are coefficients of Fayet-Iliopoulos D-term associated to the $U(1)^2$ gauge group of the original D-brane theory. Now we eliminate two of $p_2$, $p_3$ and $p_4$. When $\zeta_1 < 0$ and $\zeta_2 > 0$, $p_2$ and $p_4$ can not vanish, so they take value in $(\mathbb{C}^*)^2$. Then we can eliminate these fields by using D-flatness conditions and $U(1)^2$ gauge symmetry. Thus we have

\begin{align*}
|p_0|^2 + |p_1|^2 - 3|p_2|^2 + |p_5|^2 &= -\zeta_1 + \zeta_2 \equiv \xi.
\end{align*}

It is the D-flatness condition of a $U(1)$ gauged linear sigma model with four chiral superfields with charges

\[ Q_{\mathcal{M}} = \begin{pmatrix} 1 & 1 & -3 & 1 & -\zeta_1 + \zeta_2 \end{pmatrix} \]

Here we include the information on the Fayet-Iliopoulos D-term parameter. As we are considering the region $\zeta_1 < 0$ and $\zeta_2 > 0$, the Fayet-Iliopoulos parameter of the resulting $U(1)$ gauge theory $\xi = -\zeta_1 + \zeta_2$ is positive. Then $(p_0, p_1, p_5) \neq (0, 0, 0)$, which implies $F_\Delta = \{p_0 = p_1 = p_5 = 0\}$ in the toric description. The toric diagram which gives this $F_\Delta$ is figure 2. The diagram consists of three triangles with area one, so the corresponding toric variety is a smooth manifold obtained by blowing up the orbifold singularity of $\mathbb{C}^3/\mathbb{Z}_3$.

In [4], it was also shown that the result does not depend on which set of fields are eliminated out of $p_2$, $p_3$ and $p_4$. Therefore the orbifold phase is projected out.
4.2 D-branes on $C^3/Z_5$

For $C^3/Z_5$, any model is equivalent to the model with $\vec{a} = (3, 3, -1)$. For example, a model with $\vec{a} = (1, 1, 3)$ is equivalent to the model with $\vec{a} = (3, 3, -1)$ as follows,

$$ (\omega^1, \omega^1, \omega^3) = (\omega^{1+5}, \omega^{1+5}, \omega^{3-5}) = ((\omega^2)^3, (\omega^2)^3, (\omega^2)^{-1}) \quad (4.13) $$

The quiver diagram for $\vec{a} = (1, 1, 3)$ and that for $\vec{a} = (3, 3, -1)$ are related by the following permutation $\tau$ of vertices (see figure 3),

$$ \tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 5 & 2 & 4 \end{pmatrix}. \quad (4.14) $$

The moduli space takes the form $(C^3 - F_\Delta)/(C^*)^{10} = (C^5 - F_{\Delta_M})/(C^*)^2$, and the charge matrix is given by

$$ Q_M = \begin{pmatrix} 1 & 1 & -3 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \end{pmatrix}. \quad (4.15) $$

4.3 D-branes on $C^3/Z_7$

There are two types of models for $C^3/Z_7$. One has $\vec{a} = (4, 4, -1)$ and the other has $\vec{a} = (3, 5, -1)$. The quiver diagrams are depicted in figure 3.

For the case of $\vec{a} = (4, 4, -1)$, one can see that $\Delta_N$ is generated by 31 vectors in $R^9$, and hence $N$ takes the form $(C^{31} - F_{\Delta_N})/(C^*)^{22}$. Combining $(C^*)^6$ quotient coming from $U(1)^6$ gauge symmetry of the D-brane theory, $M$ takes the form $(C^{31} - F_\Delta)/(C^*)^{28}$. However
Figure 4: The toric diagram for $n = 5$, $\vec{a} = (3, 3, -1)$.

Figure 5: (a) The quiver diagram for $n = 7$, $\vec{a} = (4, 4, -1)$, (b) the quiver diagram for $n = 7$, $\vec{a} = (3, 5, -1)$.

one can see that 25 vectors out of 31 are redundant. Hence we eliminate 25 coordinates by $(\mathbb{C}^*)^{25}$ action and finally the moduli space takes the form $(\mathbb{C}^6 - F_{\Delta M})/(\mathbb{C}^*)^3$. For one choice of redundant variables, the charge matrix is given by

$$Q_M = \begin{pmatrix}
1 & 1 & -3 & 1 & 0 & 0 & 2\zeta_1 + \zeta_2 + \zeta_3 + \zeta_4 \\
0 & 0 & 1 & -2 & 1 & 0 & \zeta_2 + \zeta_5 \\
0 & 0 & 0 & 1 & -2 & 1 & \zeta_3 + \zeta_6 
\end{pmatrix}.$$ (4.16)

For this choice of redundant variables, Fayet-Iliopoulos parameters must satisfy

$$\zeta_1 > 0, \quad \zeta_2 > 0, \quad \zeta_3 > 0, \quad \zeta_2 + \zeta_5 > 0, \quad \zeta_3 + \zeta_6 > 0, \quad \zeta_1 + \zeta_2 + \zeta_3 + \zeta_4 > 0,$$

$$\zeta_1 + \zeta_2 + \zeta_3 + \zeta_4 + \zeta_5 > 0, \quad \zeta_1 + \zeta_2 + \zeta_3 + \zeta_4 + \zeta_5 + \zeta_6 > 0.$$ (4.17)

Under these conditions, Fayet-Iliopoulos parameters of the resulting $U(1)^3$ gauged linear sigma model satisfy inequalities

$$\xi_1 = 2\zeta_1 + \zeta_2 + \zeta_3 + \zeta_4 > 0,$$

$$\xi_2 = \zeta_2 + \zeta_5 > 0,$$

$$\xi_3 = \zeta_3 + \zeta_6 > 0.$$ (4.18)
These inequalities determine the phase uniquely. The corresponding toric diagram is figure 6.

Figure 6: The toric diagram for $n = 7$, $\vec{a} = (4, 4, -1)$.

For the case of $\vec{a} = (3, 5, -1)$, one can see that $\Delta_N$ is generated by 24 vectors in $\mathbf{R}^9$, and hence $\mathcal{N}$ takes the form $(C^{24} - F_{\Delta_N})/(C^*)^{15}$. Combining the $(C^*)^6$ coming from $U(1)^6$ gauge symmetry of the D-brane theory, $\mathcal{M}$ takes the form $(C^{24} - F_{\Delta})/(C^*)^{21}$. However one can see that 18 vectors out of 24 are redundant. Hence the moduli space takes the form $(C^6 - F_{\Delta_M})/(C^*)^3$. For one choice of redundant variables, the charge matrix is given by

$$Q_{\mathcal{M}} = \begin{pmatrix} 1 & 0 & -2 & 1 & 0 & 0 & \zeta_1 + \zeta_2 + \zeta_3 \\ 0 & 1 & 0 & -2 & 1 & 0 & \zeta_1 + \zeta_4 + \zeta_5 \\ 0 & 0 & 1 & 0 & -2 & 1 & \zeta_2 + \zeta_4 + \zeta_6 \end{pmatrix}.$$ 

(4.19)

It determines eight vectors which generate one dimensional cones for $\Delta_M$,

$$\vec{n}_0 = (1, 2, 1),$$
$$\vec{n}_1 = (2, 0, 1),$$
$$\vec{n}_2 = (1, 1, 1),$$
$$\vec{n}_3 = (1, 0, 1),$$
$$\vec{n}_4 = (0, 0, 1),$$
$$\vec{n}_5 = (-1, -1, 1).$$

(4.20)

For the choice of redundant variables Fayet-Iliopoulos parameters must satisfy

$$\zeta_1 > 0, \quad \zeta_2 > 0, \quad \zeta_4 > 0, \quad \zeta_1 + \zeta_2 + \zeta_3 > 0, \quad \zeta_1 + \zeta_4 + \zeta_5 > 0,$$
$$\zeta_2 + \zeta_4 + \zeta_6 > 0, \quad \zeta_1 + \zeta_2 + \zeta_3 + \zeta_4 + \zeta_5 > 0, \quad \zeta_1 + \zeta_2 + \zeta_3 + \zeta_4 + \zeta_6 > 0,$$
$$\zeta_1 + \zeta_4 + \zeta_5 + \zeta_6 > 0, \quad \zeta_1 + \zeta_2 + \zeta_3 + \zeta_4 + \zeta_5 + \zeta_6 > 0.$$

(4.21)

Under these conditions, Fayet-Iliopoulos parameters of the resulting $U(1)^3$ gauged linear sigma model satisfy inequalities

$$\xi_1 = \zeta_1 + \zeta_2 + \zeta_3 > 0,$$

$$13$$
\[ \xi_2 = \zeta_1 + \zeta_4 + \zeta_5 > 0, \]  
\[ \xi_3 = \zeta_2 + \zeta_4 + \zeta_6 > 0. \]  
(4.22)  

The triangulation of the toric diagram is uniquely determined as in figure 7. Thus only the completely resolved phase is realized.

Here we comment on the correspondence between the classification of models by \( \vec{a} \) and the classification of toric diagrams. In the \( n = 7 \) case, if we allow \( a_\mu = 0 \) for some \( \mu \), there are three types of models, \( \vec{a} = (4, 4, -1), \vec{a} = (3, 5, -1) \) and \( \vec{a} = (1, 6, 0) \). On the other hand, triangles with area seven whose corners lie on the lattice \( \mathbb{Z}^2 \) are also classified into three types; the first is the triangle corresponding to figure 6, the second is that corresponding to figure 3 and the third is that with corners \((0, 0, 1), (0, 1, 1) \) and \((7, 0, 1) \). Here the classification of triangles is defined by the following equivalence relation: two triangles are equivalent if they are related by \( GL(2, \mathbb{Z}) \) transformation with determinant \( \pm 1 \). In fact, there is a relation between \( \vec{a} \) and the triangle which appears in the corresponding toric diagram[12]. That is, if we write three corners of the triangle as \( \vec{l}_\mu = (l_{\mu 1}, l_{\mu 2}, l_{\mu 3}), (\mu = 1, 2, 3) \), the following relations  
\[ \frac{1}{n} \sum_{\mu=1}^{3} a_\mu l_{\mu i} \in \mathbb{Z} \]  
(4.23)  
hold for an appropriate order of \( (a_1, a_2, a_3) \). Note that the triangle corresponding to the \( \vec{a} = (1, 6, 0) \) model have lattice points on its codimension one boundary. This is a characteristic feature to the cases whose singularity is non-isolated (see section 5).

From the examples treated in this paper, it seems that the classification of models on \( C^3/\mathbb{Z}_n \) corresponds to the classification of triangles with area \( n \) (see section 6).
4.4 D-branes on $C^3/\mathbb{Z}_9$

For $C^3/\mathbb{Z}_9$, there are two types of models. One has $\vec{a} = (5, 5, -1)$ and the other has $\vec{a} = (4, 6, -1)$.\footnote{$\vec{a} = (3, 3, 3)$ also satisfies $a_1 + a_2 + a_3 \equiv 0 \mod 9$, but it is nothing but the model on $C^3/\mathbb{Z}_3$ with $\vec{a} = (1, 1, 1)$.}

For the $\vec{a} = (4, 6, -1)$ case, the action of $\mathbb{Z}_9$ on $X^\mu$ is

$$X^1, X^2, X^3 \rightarrow (\omega^{4k} X^1, \omega^{6k} X^2, \omega^{-k} X^3), \quad \omega^9 = 1 \quad (4.24)$$

where $k = 1, \ldots, 9$. For $k = 3$, $(X^1, X^2, X^3) \rightarrow (\omega^3 X^1, X^2, \omega^{-3} X^3)$, which means that there is a non-trivial fixed point $(0, X^2, 0)$ with $X^2 \neq 0$. Therefore the corresponding space has a non-isolated singularity. We discuss such cases in section 5.

For the case of $\vec{a} = (5, 5, -1)$, one can see that $\Delta_N$ is generated by 78 vectors in $\mathbb{R}^{11}$, and hence $N$ takes the form $(C^{78} - F_{\Delta_N})/(C^*)^{67}$. Combining the $(C^*)^8$ quotient coming from $U(1)^8$ gauge symmetry of the D-brane theory, $M$ takes the form $(C^{78} - F_\Delta)/(C^*)^{75}$. However one can see that 71 vectors out of 78 are redundant. Hence the moduli space takes the form $(C^7 - F_{\Delta_M})/(C^*)^4$. For one choice of redundant variables, the charge matrix is given by

$$Q_M = \begin{pmatrix} 1 & 1 & -3 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 1 & -2 \end{pmatrix} \quad (4.25)$$

By a similar analysis to the previous sections we can see that the toric diagram is given by figure 8.

![Toric diagram](image_url)

Figure 8: The toric diagram for $n = 9$, $\vec{a} = (5, 5, -1)$.

4.5 D-branes on $C^3/\mathbb{Z}_{11}$ and topology change

For $C^3/\mathbb{Z}_{11}$, there are two types of models. One has $\vec{a} = (6, 6, -1)$ and the other has $\vec{a} = (5, 7, -1)$.\footnote{$\vec{a} = (3, 3, 3)$ also satisfies $a_1 + a_2 + a_3 \equiv 0 \mod 9$, but it is nothing but the model on $C^3/\mathbb{Z}_3$ with $\vec{a} = (1, 1, 1)$.}
For the case of \( \vec{a} = (6, 6, -1) \), one can see that \( \Delta_\mathcal{N} \) is generated by 201 vectors in \( \mathbb{R}^{13} \), and hence \( \mathcal{N} \) takes the form \( (C^{201} - F_{\Delta_\mathcal{N}})/\langle C^* \rangle^{188} \). Combining the \( \langle C^* \rangle^{10} \) quotient coming from \( U(1)^{10} \) gauge symmetry of the D-brane theory, \( \mathcal{M} \) takes the form \( (C^{201} - F_\Delta)/\langle C^* \rangle^{108} \). However one can see that 193 vectors out of 201 are redundant. Hence the moduli space takes the form \( (C^8 - F_{\Delta_M})/\langle C^* \rangle^5 \) with the charge matrix,

\[
Q_M = \begin{pmatrix}
  1 & 1 & -3 & 1 & 0 & 0 & 0 & 0 \\
  0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 \\
  0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 \\
  0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\
  0 & 0 & 0 & 0 & 0 & 1 & -2 & 1
\end{pmatrix}.
\]

(4.26)

The corresponding toric diagram is figure 9.

![Toric Diagram](image)

**Figure 9:** The toric diagram for \( n = 11, \vec{a} = (6, 6, -1) \).

For the case of \( \vec{a} = (5, 7, -1) \), one can see that \( \Delta_\mathcal{N} \) is generated by 91 vectors in \( \mathbb{R}^{13} \), and hence \( \mathcal{N} \) takes the form \( (C^{91} - F_{\Delta_\mathcal{N}})/\langle C^* \rangle^{78} \). Combining the \( \langle C^* \rangle^{10} \) quotient coming from \( U(1)^{10} \) gauge symmetry of the D-brane theory, \( \mathcal{M} \) takes the form \( (C^{91} - F_\Delta)/\langle C^* \rangle^{88} \). However one can see that 83 vectors out of 91 are redundant. Hence we can eliminate 83 coordinates by \( \langle C^* \rangle^{83} \) action and finally the moduli space takes the form \( (C^8 - F_{\Delta_M})/\langle C^* \rangle^5 \). For one choice of redundant variables, the charge matrix is given by

\[
Q_M = \begin{pmatrix}
  1 & 0 & -2 & 1 & 0 & 0 & 0 & 0 & \zeta_1 + \zeta_2 + \zeta_3 + \zeta_4 + \zeta_5 \\
  0 & 1 & 0 & -2 & 0 & 1 & 0 & 0 & \zeta_1 + \zeta_6 + \zeta_7 \\
  1 & 0 & 0 & 0 & -2 & 0 & 0 & 1 & \zeta_1 + \zeta_4 + \zeta_5 + \zeta_8 \\
  0 & 0 & 0 & 1 & 0 & -2 & 1 & 0 & \zeta_2 + \zeta_3 + \zeta_8 \\
  0 & 0 & 1 & 0 & 0 & 0 & -2 & 1 & \zeta_2 + \zeta_4 + \zeta_6 + \zeta_8 + 2\zeta_9 + \zeta_{10}
\end{pmatrix}.
\]

(4.27)

It determines eight vectors which generate one dimensional cones in \( \Delta_M \),

\[
\vec{n}_0 = (0, 2, 1),
\]

\[
\vec{n}_1 = (3, 0, 1),
\]

16
\[ \vec{n}_2 = (1, 1, 1), \]
\[ \vec{n}_3 = (2, 0, 1), \]
\[ \vec{n}_4 = (0, 1, 1), \]
\[ \vec{n}_5 = (1, 0, 1), \]
\[ \vec{n}_6 = (0, 0, 1), \]
\[ \vec{n}_7 = (-1, -1, 1). \]

Figure 10 represents these eight vectors.

\[ \xi_1 > 0, \quad \zeta_1 > 0, \quad \zeta_2 > 0, \quad \zeta_3 > 0, \quad \zeta_4 > 0, \quad \zeta_5 > 0, \quad \zeta_6 > 0, \quad \zeta_7 > 0, \quad \zeta_8 > 0, \quad \zeta_9 > 0, \quad \zeta_10 > 0, \]
\[ \zeta_1 + \zeta_2 + \zeta_3 + \zeta_4 + \zeta_5 + \zeta_6 + \zeta_7 + \zeta_8 + \zeta_9 + \zeta_{10} > 0, \]
\[ \zeta_1 + \zeta_2 + \zeta_3 + \zeta_4 + \zeta_5 + \zeta_6 + \zeta_7 + \zeta_8 + \zeta_9 > 0, \]
\[ \zeta_1 + \zeta_2 + \zeta_3 + \zeta_4 + \zeta_5 + \zeta_6 + \zeta_7 > 0, \]
\[ \zeta_1 + \zeta_2 + \zeta_3 + \zeta_4 + \zeta_5 + \zeta_6 > 0, \]
\[ \zeta_1 + \zeta_2 + \zeta_3 + \zeta_4 + \zeta_5 + \zeta_6 > 0, \]
\[ \zeta_1 + \zeta_2 + \zeta_3 + \zeta_4 + \zeta_5 + \zeta_6 + \zeta_7 + \zeta_8 + \zeta_9 + \zeta_{10} > 0, \]
\[ \zeta_1 + \zeta_2 + \zeta_3 + \zeta_4 + \zeta_5 + \zeta_6 + \zeta_7 + \zeta_8 + \zeta_9 + \zeta_{10} > 0. \]

Under these conditions, Fayet-Iliopoulos parameters of the resulting $U(1)^5$ gauged linear sigma model satisfy the following inequalities
\[ \xi_1 = \zeta_1 + \zeta_2 + \zeta_3 + \zeta_4 + \zeta_5 > 0, \]
\[ \xi_2 = \xi_1 + \xi_6 + \xi_7 > 0, \]
\[ \xi_3 = \xi_1 + \xi_4 + \xi_5 + \xi_8 > 0, \quad (4.30) \]
\[ \xi_4 = \xi_2 + \xi_3 + \xi_8 > 0, \]
\[ \xi_5 = \xi_2 + \xi_4 + \xi_6 + \xi_8 + 2\xi_9 + \xi_{10} > 0, \]

However these conditions are not enough to determine the phase. To see this we explicitly write down the D-flatness conditions.

\[ |p_0|^2 - 2|p_2|^2 + |p_3|^2 = \xi_1, \]
\[ |p_1|^2 - 2|p_3|^2 + |p_5|^2 = \xi_2, \]
\[ |p_0|^2 - 2|p_4|^2 + |p_6|^2 = \xi_3, \quad (4.31) \]
\[ |p_3|^2 - 2|p_5|^2 + |p_6|^2 = \xi_4, \]
\[ |p_2|^2 - 2|p_6|^2 + |p_7|^2 = \xi_5. \]

Using these equations, we have the following equations

\[ |p_2|^2 - |p_4|^2 - |p_5|^2 + |p_6|^2 = (-\xi_1 + \xi_3 + \xi_4)/2 \equiv \eta_1, \]
\[ |p_3|^2 - |p_2|^2 - |p_5|^2 + |p_4|^2 = (\xi_1 - \xi_3 + \xi_4)/2 \equiv \eta_2, \quad (4.32) \]
\[ |p_0|^2 - |p_2|^2 - |p_4|^2 + |p_3|^2 = (\xi_1 + \xi_3 - \xi_4)/2 \equiv \eta_3, \]
\[ |p_1|^2 - |p_2|^2 - |p_3|^2 + |p_4|^2 = (\xi_1 + 2\xi_2 - \xi_3 + \xi_4)/2 \equiv \eta_4. \]

If \( \eta_i \) can have both positive and negative value, topology change occurs as the sign of \( \eta_i \) changes. In fact, we can see that it occurs as follows. Under the conditions \((4.30)\), \( \eta_i \)'s must satisfy

\[ \eta_1 + \eta_2 = \xi_4 > 0, \]
\[ \eta_1 + \eta_3 = \xi_3 > 0, \]
\[ \eta_2 + \eta_3 = \xi_1 > 0, \]
\[ \eta_3 + \eta_4 = \xi_1 + \xi_2 > 0, \]
\[ \eta_4 - \eta_2 = \xi_2 > 0. \quad (4.33) \]

These inequalities restrict possible sign of \( \eta_i \) and the following five cases, which we call (a), (b), (c), (d) and (e), are allowed.

\[ (\eta_1, \eta_2, \eta_3, \eta_4) \sim \begin{cases} 
(+, +, +, +), & (a) \\
(-, +, +, +), & (b) \\
(+, -, +, +), & (c) \\
(+, +, -, +), & (d) \\
(+, -, +, -). & (e) 
\end{cases} \quad (4.34) \]

We can see that the triangulation of the toric diagram is uniquely determined for each case. The corresponding toric diagrams are depicted in figure [1]. Each triangulation consists of
eleven cones with volume one, thus it represents a phase of completely resolved manifolds with definite topology. The phases (a) and (b) are connected by a flop with respect to the parallelogram with vertices $\vec{n}_2, \vec{n}_4, \vec{n}_5$ and $\vec{n}_6$, and the flop is accomplished by the change of the sign of $\eta_1$. Similarly the phases (a) and (c), (a) and (d), (c) and (e) are connected by a flop controlled by $\eta_2, \eta_3, \eta_4$, respectively.

We can see that there is no triangulation except these five cases which completely resolves the orbifold singularity.

5 D-branes on orbifolds with non-isolated singularities

Orbifolds often have non-isolated singularities as in the $n = 9, \vec{a} = (4, 6, -1)$ case. Consider an orbifold $\mathbb{C}^3/\mathbb{Z}_n$ with $\vec{a} = (a_1, a_2, a_3)$. If $n$ and $a_\mu$ have a common factor except one, the orbifold has a non-isolated fixed point of $\mathbb{Z}_n$ and hence has a non-isolated singularity. Thus there are models with non-isolated singularity if $n$ is not a prime number. Especially orbifold singularities are always non-isolated if $n$ is even. The analyses in the previous sections can be almost straightforwardly applicable to these cases. We have explicitly calculated the moduli space of D-branes on such orbifolds for $n = 4, 6$ cases and $n = 9, \vec{a} = (4, 6, -1)$ case and verified that only the geometric phases are realized.

Topologically distinct phases emerge in the $n = 6, \vec{a} = (2, 5, -1)$ case, in which there are five phases connected by flops (figure 12). We have also found that the moduli space

---

\footnote{In the $n = 2$ case, $a_\mu$ must be zero (mod 2) for some $\mu$, so the orbifold is a direct product of $\mathbb{C}^2/\mathbb{Z}_2$ and $\mathbb{C}$.}
Figure 12: The toric diagrams for the $n = 6$, $\vec{a} = (2, 5, -1)$ model. There are five phases connected by flops.

consists of six blown-up phases connected by flops in the $n = 9$, $\vec{a} = (4, 6, -1)$ case. We can see that for both cases there is no other triangulation which completely resolves the orbifold singularity.

We also note that toric diagrams corresponding to non-isolated cases always have vertices on its codimension one boundary.

6 Discussion

From the analyses we have made in this paper, it seems that there is a correspondence between the classification of models on $\mathbb{C}^3/\mathbb{Z}_n$ by $\vec{a}$ and the classification of triangles with area $n$ (as noted in subsection 4.3), and what phases are allowed for each model is determined according to the method of subdivisions of the triangle with area $n$ into $n$ triangles with area one. (Here vertices of each triangle lie on the lattice $\mathbb{Z}^2$.) It means that only completely blown-up phases are realized as pointed out in [4], and topology change can occur if the triangle with area $n$ includes parallelogram with respect to the lattice $\mathbb{Z}^2$.

It will be interesting to prove the above-mentioned rule on the allowed phases in general.

Acknowledgements

I would like to thank T. Aoyama for valuable discussions.

20
References

[1] M.R. Douglas, D. Kabat, P. Pouliot and S.H. Shenker, *Nucl. Phys.* B485 (1997) 85.
[2] M.R. Douglas and G. Moore, hep-th/9603167.
[3] C.V. Johnson and R.C. Myers, *Phys. Rev.* D55 (1997) 6382.
[4] M.R. Douglas, B.R. Greene and D.R. Morrison, hep-th/9704151.
[5] K. Mohri, hep-th/9707012.
[6] P.S. Aspinwall, B.R. Greene and D.R. Morrison, *Nucl. Phys.* B416 (1994) 414.
[7] E. Witten, *Nucl. Phys.* B403 (1993) 159.
[8] P.S. Aspinwall, B.R. Greene and D.R. Morrison, *Nucl. Phys.* B420 (1994) 184.
[9] E. Witten, *Nucl. Phys.* B471 (1996) 195.
[10] P.S. Aspinwall and B.R. Greene, *Nucl. Phys.* B437 (1995) 205.
[11] D.R. Morrison and M.R. Plesser, *Nucl. Phys.* B440 (1995) 279.
[12] P.S. Aspinwall, hep-th/9403123.