Ground state solution for a class of modified nonlinear fourth-order elliptic equation with sign-changing unbounded potential

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Abstract

We are concerned on the fourth-order elliptic equation

\[(P_\lambda) \begin{cases} \Delta^2 u - \Delta u + V(x)u - \lambda \Delta[\rho(u^2)]\rho'(u^2)u = f(u) \text{ in } \mathbb{R}^N, \\ u \in W^{2,2}(\mathbb{R}^N), \end{cases} \]

where \( \Delta^2 = \Delta(\Delta) \) is the biharmonic operator, \( 3 \leq N \leq 6 \), the radially symmetric potential \( V \) may change sign and \( \inf_{\mathbb{R}^N} V(x) = -\infty \) is allowed. If \( f \) satisfies a type of nonquadracity and monotonicity conditions and \( \rho \) is a suitable smooth function, we prove, via variational approach, the existence of a radially symmetric nontrivial ground state solution \( u_\lambda \) for problem \((P_\lambda)\) for all \( \lambda \geq 0 \).

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1 Introduction and statement of the main result

The study of existence of a standing wave solution for quasilinear Schrödinger equation of the form

\[
\frac{i \partial z}{\partial t} = -\Delta z + W(x)z - l(x, z) - k[\Delta \rho(|z|^2)]\rho'(|z|^2)z,
\]

where \( z : \mathbb{R} \times \mathbb{R}^N \to \mathbb{C} \), \( W : \mathbb{R}^N \to \mathbb{R} \) is a given potential, \( k \) is a real constant and \( l \) and \( \rho \) are real functions, is related with several phenomena in physics. For instance, if \( k \neq 0 \)

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and for specific functions $l, \rho$, equations (1) appears in plasma physics and fluid mechanics [15, 16, 19, 21], in the classical and quantum theory of Heisenberg ferromagnet and magnons [27, 30], in dissipative quantum mechanics [13], in laser theory [2, 28] and in condensed matter theory [23]. The particular case $\rho(s) = s$ occurs in theory of superfluids (see [15, 16, 20] and the references in [17]). The case $\rho(s) = (1 + s^{1/2})$ appears in the self-channeling of a high-power ultra short laser in matter (see [8, 9]).

If we want to find standing wave solutions for (1), we take $z(t, x) := \exp(-iEt)u(x)$ with $E \in \mathbb{R}$ and $u : \mathbb{R}^N \to \mathbb{R}$ a function, which leads to consider the following elliptic equation

(2) 
$$-\Delta u + V(x)u - k\Delta(\rho(u^2))\rho'(u^2)u = f_1(x, u) \quad \text{in} \ \mathbb{R}^N.$$ 

In the last sixteen years, many authors has dedicated to study problem (2) via various methods. To the case $\rho(s) = s$, we refer to [25], where $V$ is radially symmetric and might change sign. In [35], under the condition that $V$ is a continuous function satisfying $\inf_{\mathbb{R}^N} V(x) > -\infty$, the authors apply the dual approach and the mountain pass theorem to obtain infinitely many solutions of the nonautonomous problem (2) with $f_1(x, u)$ odd in the variable $u$. There are many works concerning problem (2) with $V$ satisfying $\inf_{\mathbb{R}^N} V(x) > 0$ and $f_1$ a subcritical or critical nonlinearity (see, for example, [7, 11, 20, 26, 29, 32]). The zero mass potential case $V \equiv 0$ was studied in [6] and the vanishing potential case was considered in [1]. We stress that the authors in [7] used the Nehari method, showing that the infimum for the energy functional associated to problem (2) on the Nehari set is achieved at some nontrivial solution; while, in [24], the authors considered $\rho(s) = (1 + s^2)^{1/2}$ and, exploring properties of the Pohozaev manifold, they proved the existence of a nontrivial positive solution.

The following fourth-order elliptic equation, in its turn,

(3) 
$$\Delta^2 u - \Delta u + V(x)u = f_2(x, u) \quad \text{in} \ \mathbb{R}^N,$$

where $\Delta^2(u) = \Delta(\Delta u)$, has become extremely relevant after Lazer and Mckenna [18] propose to study periodic oscillations and traveling waves in a suspension bridge by the following Dirichlet problem

$$\left\{ \begin{array}{l} 
\Delta^2 u + c\Delta u = h(x, u) \quad \text{in} \ \Omega, \\
|u|_{\partial\Omega} = |\Delta|_{\partial\Omega} = 0,
\end{array} \right.$$ 

where $\Omega \subset \mathbb{R}^N$ is a bounded domain and $c \in \mathbb{R}$. Equations (11) models several phenomena in physics, as static deflection of an elastic plate in a fluid and communication satellites, which the reader can find, for example, in [5, 14, 34] and in its references.

The problem (3), on unbounded domain, has also attracted interest in recent years. In [36], the authors studied the asymptotically linear case and used a variant version of the mountain pass theorem to obtain the existence of a ground state solution. The superlinear case was investigated in [33], where the symmetric mountain pass theorem was employed to guarantee that infinitely many nontrivial solutions exist. In [3], the authors established the existence of two solutions for problem (3) without the term $\Delta u$ and the nonlinearity $f_2$ involving critical growth.
In every paper about problems (2) and (3) that we mentioned until here, the condition \( \inf_{\mathbb{R}^N} V(x) > -\infty \) was essential to overcome the well known lack of compactness due to unbounded domains.

When \( \inf_{\mathbb{R}^N} V(x) = -\infty \) is allowed, very little is found in the literature. On problem (2), we refer the reader [22], where the authors considered \( \rho(s) = s \) and employed the mountain pass theorem and interaction with the limit problem to obtain the existence of a nontrivial solution. On problem (3), up to our knowledge, there is no result considering this condition on potential \( V \). Since we allow this condition occurs (see hypothesis \((V)\) below), this is the first article that takes it into account.

Our main goal is to show the existence of a stationary solution of a linear combination between problems (2) and (3) (see [4]), namely,

\[
(P_\lambda) \quad \left\{ \begin{array}{l}
\Delta^2 u - \Delta u + V(x)u - \lambda \Delta [\rho(u^2)]\rho'(u^2)u = f(u) \text{ in } \mathbb{R}^N, \\
u \in W^{2,2}(\mathbb{R}^N),
\end{array} \right.
\]

with \( 3 \leq N \leq 6, \lambda \geq 0, V \) satisfying a technical hypothesis on its negative part and possessing a asymptotic positive limit, the nondecreasing nonlinearity \( f \) has a subcritical growth and the function \( \rho \) is smooth and satisfies some delicate conditions.

Moreover, unlike the authors in [6, 24] and others, we avoid the use of any change of variables, and for this reason we have to restrict the dimension \( N \) to deal with the term \( \Delta [\rho(u^2)]\rho'(u^2)u \) in the dual approach. To face compactness issues, we will assume that \( V \) is a radially symmetric potential, which guarantees some compact embeddings of a correct Sobolev space. We decide to work with a general function \( \rho \) since our work includes, along others examples, two important one: \( \rho(s) = s \) and \( \rho(s) = (1 + s)^{1/2} \).

Hereafter, for \( r \geq 1 \), let us denote \( |\cdot|_r \) the usual norm in \( L^r(\mathbb{R}^N) \). Also, we will equip the space \( W^{2,2}(\mathbb{R}^N) \) with the norm

\[
||u||^2 = \int_{\mathbb{R}^N} (|\Delta u|^2 + |\nabla u|^2 + u^2)dx,
\]

which turns \((W^{2,2}(\mathbb{R}^N), ||\cdot||)\) a Hilbert space. On the potential \( V : \mathbb{R}^N \to \mathbb{R} \), writing \( V(x) = V^+(x) - V^-(x) \), where \( V^\pm(x) := \max\{\pm V(x), 0\} \), we impose the following conditions.

\((V_1)\) \( V(x) = V(|x|) \) for all \( x \in \mathbb{R}^N \) and \( \lim_{|x| \to +\infty} V(x) = V_\infty > 0 \);

\((V_2)\) Let \( 2^* = \frac{2N}{N-2} \). If \( S \) is the best constant to the Sobolev embedding \( D^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N) \), namely,

\[
S = \inf_{u \in D^{1,2}\setminus\{0\}} \frac{\|\nabla u\|_2^2}{\|u\|_2^{2^*}},
\]

then

\[
|V^-|_{N/2} < S.
\]

The nonlinearity \( f : \mathbb{R} \to \mathbb{R} \) belongs to \( C^2(\mathbb{R}, \mathbb{R}) \) and satisfies:
\((f_1)\) \(f(0) = f'(0) = 0.\)

\((f_2)\) \(f'(t)t^2 - f(t)t \geq \delta |t|^p\) for all \(t \in \mathbb{R}\) and for some \(\delta > 0\) and \(4 < p < 2_+ := \frac{2N}{N - 4}\) if \(N > 4\) or \(p > 4\) if \(N = 3.\)

\((f_3)\) If \(F(t) = \int_0^t f(s)ds\), then \(\frac{1}{4}f(t)t - F(t) \geq 0\) for all \(t \in \mathbb{R}.\)

\((f_4)\) \(\lim_{t \to +\infty} \frac{f(t)}{t^{p-1}} = m > 0.\)

On function \(\rho : [0, +\infty) \to \mathbb{R}\), denoting by \(\rho^{(i)}\) the \(i\)-th derivative of \(\rho\), we will assume the following.

\((\rho_1)\) \(\rho\) is continuous and belongs to \(C^4((0, +\infty), \mathbb{R}).\)

\((\rho_2)\) \(|\rho^{(i)}(s)| \leq C_i\) for all \(s \in [0, +\infty)\) and for some \(C_i > 0\), where \(i \in \{1, 2, 3, 4\}\).

\((\rho_3)\) \(|s\rho'(s)\rho''(s)| \leq C_5\) for all \(s \in [0, +\infty)\) and for some \(C_5 > 0.\)

\((\rho_4)\) \(\rho'(s) \geq -\sqrt{2}\rho''(s)s \geq 0\) for all \(s \in [0, +\infty).\)

\((\rho_5)\) \(2\rho''(s) + \rho'''(s)s \leq 0\) for all \(s \in [0, +\infty).\)

Though the assumptions \((\rho_1) - (\rho_5)\) are very technical, each of them is necessary to state and prove our main result obtained in this paper.

**Lemma 1.** The following functions \(\rho : [0, +\infty) \to \mathbb{R}\) satisfy the assumptions \((\rho_1) - (\rho_5)\):

\[\begin{align*}
\text{a.} & \quad \rho(s) = a + bs \text{ with } a, b \geq 0; \\
\text{b.} & \quad \rho(s) = (1 + s)^{1/2}; \\
\text{c.} & \quad \rho(s) = a + bs + (1 + s)^{1/2}; \\
\text{d.} & \quad \rho(s) = (1 + s)^{\alpha} \text{ with } 1 - 1/\sqrt{2} \leq \alpha \leq 1.
\end{align*}\]

**Proof.** This is a straightforward calculation. \(\square\)

**Remark 1.** Hypothesis \((V_2)\) allows potential \(V\) to change signal and, in addition, to occur \(\inf_{\mathbb{R}^N} V(x) = -\infty.\)

**Remark 2.** Condition \((f_2)\) ensures that \(0 < t \mapsto f(t)/t\) is increasing (and so \(0 < t \mapsto f(t)\) as well). However, we will need this assumption only one time in the format presented in \((f_2)\). Hypothesis \((f_4)\), in its turn, shows that \(f(t) \to +\infty\) as \(t \to +\infty\) and, consequently,

\[\lim_{t \to +\infty} \frac{F(t)}{t^p} = \frac{m}{p} > 0.\]

Thus, since \(4 < p\), necessarily

\[\lim_{t \to +\infty} \frac{F(t)}{t^4} = +\infty.\]
Remark 3. Hypothesis (ρ4) implies that \( \rho'(s) \geq -\sqrt{2}\rho''(s)s \geq -\rho''(s)s \geq 0 \).

In the sequel, we set precisely up the result obtained.

Theorem 1. Under conditions \((V_1),(V_2),(f_1)-(f_4)\) and \((\rho_1)-(\rho_5)\), problem \((P_\lambda)\) has a nontrivial radially symmetric ground state solution \(u_\lambda \in W^{2,2}(\mathbb{R}^N)\).

2 Preliminaries and variational framework

A direct consequence from conditions \((f_1)\) and \((f_4)\) is: for all \(\varepsilon > 0\), there is a constant \(C_\varepsilon > 0\) such that, if \(s \in \mathbb{R}\), then

\[
|f(s)| \leq \varepsilon|s| + C_\varepsilon|s|^p \quad \text{and} \quad |F(s)| \leq \varepsilon|s|^2 + C_\varepsilon|s|^{p+1}.
\]

To tackle the kind of problem \((P_\lambda)\) via variational approach, some difficulties appear naturally. The first that we point our is, since \(V^-\) may not be a \(L^\infty(\mathbb{R}^N)\)-function, we need to show that the integral \(\int_{\mathbb{R}^N} V(x) u^2 dx\) is finite for all \(u \in W^{2,2}(\mathbb{R}^N)\). The following result proves it and one more important fact.

Lemma 2. Conditions \((V_1)\) and \((V_2)\) imply that the quadratic form

\[
\forall u \mapsto ||u||_V^2 := \int_{\mathbb{R}^N} (|\Delta u|^2 + |\nabla u|^2 + V(x) u^2)dx
\]

defines a norm in \(W^{2,2}(\mathbb{R}^N)\), which is equivalent to the norm \(\| \cdot \|\).

Proof. Let \(u \in W^{2,2}(\mathbb{R}^N)\). By Hölder and Gagliardo-Nirenberg inequalities, one has

\[
\int_{\mathbb{R}^N} V^-(x) u^2 dx \leq |V^-|_{N/2}|u|_{2^*}^2 \leq \frac{|V^-|_{N/2}}{S} \int_{\mathbb{R}^N} |\nabla u|^2 dx.
\]

This implies that

\[
\int_{\mathbb{R}^N} (|\Delta u|^2 + |\nabla u|^2 + V(x) u^2) dx \geq \int_{\mathbb{R}^N} (|\Delta u|^2 + \left(1 - \frac{|V^-|_{N/2}}{S}\right) |\nabla u|^2 + V^+(x) u^2) dx.
\]

\[
\geq \left(1 - \frac{|V^-|_{N/2}}{S}\right) \int_{\mathbb{R}^N} (|\Delta u|^2 + |\nabla u|^2 + V^+(x) u^2) dx
\]

Once we clearly have

\[
\int_{\mathbb{R}^N} (|\Delta u|^2 + |\nabla u|^2 + V(x) u^2) dx \leq \int_{\mathbb{R}^N} (|\Delta u|^2 + |\nabla u|^2 + V^+(x) u^2) dx,
\]

in view of \((6)\), it is enough to show that the function

\[
\forall u \mapsto \int_{\mathbb{R}^N} (|\Delta u|^2 + |\nabla u|^2 + V^+(x) u^2) dx
\]

defines an equivalent norm in \(W^{2,2}(\mathbb{R}^N)\). The arguments in the proof of the Lemma 2.1 in \[12\] may be applied to get this. The lemma is proved. \(\square\)
The next result shows another difficulty found and how to overcome it. The variational approach to problem (P\_λ) is guaranteed as well.

**Proposition 1.** Under assumptions (V\_1), (V\_2), (f\_1), (f\_2) and (ρ\_1) – (ρ\_3), the Euler-Lagrange functional I\_λ : W^{2,2}(\mathbb{R}^N) \to \mathbb{R} associated to problem (P\_λ) is given by

\[
I\_λ(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\Delta u|^2 + |∇ u|^2 + V(x)u^2)dx + \frac{λ}{4} \int_{\mathbb{R}^N} |∇ ρ(u^2)|^2dx - \int_{\mathbb{R}^N} F(u)dx.
\]

Moreover, I\_λ \in C^1(W^{2,2}(\mathbb{R}^N), \mathbb{R}) and

\[
I\_λ(u) = \int_{\mathbb{R}^N} (Δ u Δ ϕ + ∇ u ∇ ϕ + V(x)u ϕ)dx + λ \int_{\mathbb{R}^N} (ρ(u^2)) ∙ (∇(ρ'(u^2))uϕ)dx - \int_{\mathbb{R}^N} f(ϕ)dx.
\]

for all ϕ ∈ W^{2,2}(\mathbb{R}^N).

**Proof.** Let us concern only with the term Φ(u) := \frac{1}{4} \int_{\mathbb{R}^N} |∇ ρ(u^2)|^2dx. Since

\[
W^{2,2}(\mathbb{R}^N) \subset W^{1,r}(\mathbb{R}^N), \text{ for } 2 \leq r \leq 2^* \quad \text{and} \quad W^{2,2}(\mathbb{R}^N) \subset L^s(\mathbb{R}^N), \text{ for } 2 \leq s \leq 2^*,
\]

we have from hypothesis (ρ\_2) that, if u ∈ W^{2,2}(\mathbb{R}^N), then

\[
\frac{1}{4} \int_{\mathbb{R}^N} |∇ ρ(u^2)|^2dx = \int_{\mathbb{R}^N} [ρ'(u^2)]^2u^2 |∇ u|^2dx \leq C^2 \int_{\mathbb{R}^N} u^2 |∇ u|^2dx,
\]

what implies by using Hölder inequality with exponents p = 3 and q = p/(p - 1) = 3/2 that

\[
\frac{1}{4} \int_{\mathbb{R}^N} |∇ ρ(u^2)|^2dx \leq C^2 \left( \int_{\mathbb{R}^N} u^6dx \right)^{1/3} \left( \int_{\mathbb{R}^N} |∇ u|^3dx \right)^{2/3} < +\infty.
\]

So, Φ is well defined in W^{2,2}(\mathbb{R}^N). Supposing, now, that u is a classical solution for (P\_λ), i.e., u ∈ C^4(\mathbb{R}^N, \mathbb{R}) satisfies pointwise (P\_λ), consider ϕ ∈ C^∞(\mathbb{R}^N, \mathbb{R}) and note that, by Divergence Theorem applied to the vector field V = ∇(ρ(u^2))ρ'(u^2)uϕ, one has

\[
\int_{B_R} \text{div}[∇(ρ(u^2))ρ'(u^2)uϕ]dx = \int_{B_R} \text{div} Vdx = \int_{\partial B_R} \text{V} ∙ \eta ds = 0,
\]

where B\_R is the ball centered in 0 with radius R > 0 large enough so that supp ϕ ⊂ B\_R and η(x) is the normal vector in x to ∂B\_R, boundary of B\_R. We may conclude that

\[
\int_{\mathbb{R}^N} \text{div}[∇(ρ(u^2))ρ'(u^2)uϕ]dx = 0, \text{ i.e.,}
\]

\[
\int_{\mathbb{R}^N} -Δ[ρ(u^2)]ρ'(u^2)uϕdx = \int_{\mathbb{R}^N} ∇(ρ(u^2)) ∙ (∇(ρ'(u^2))uϕ)dx.
\]
A simple but wearing calculation shows that $\Phi'(u)\varphi = \int_{\mathbb{R}^N} \nabla (\rho(u^2)) \cdot \nabla (\rho'(u^2)u\varphi)\,dx$ for all $\varphi \in C_0^\infty(\mathbb{R}^N, \mathbb{R})$. Let us check that $\Phi'$ is continuous. Consider $u_n \to u$ in $W^{2,2}(\mathbb{R}^N)$ and note that, if $\varphi \in W^{2,2}(\mathbb{R}^N)$ with $\|\varphi\| \leq 1$, one has

$$|\Phi'(u_n)\varphi - \Phi(u)\varphi| = \left| \int_{\mathbb{R}^N} \nabla (\rho(u_n^2)) \cdot \nabla (\rho'(u_n^2)u_n\varphi)\,dx - \int_{\mathbb{R}^N} \nabla (\rho(u^2)) \cdot \nabla (\rho'(u^2)u\varphi)\,dx \right|$$

$$= \left| \int_{\mathbb{R}^N} \nabla (\rho(u_n^2)) \cdot \nabla (\rho'(u_n^2)u_n\varphi)\,dx - \int_{\mathbb{R}^N} \nabla (\rho(u^2)) \cdot \nabla (\rho'(u^2)u\varphi)\,dx \right|$$

$$+ \int_{\mathbb{R}^N} 2[\rho'(u_n^2)]^2u_n|\nabla u_n|^2\varphi\,dx - \int_{\mathbb{R}^N} 2[\rho'(u^2)]^2u|\nabla u|^2\varphi\,dx$$

$$+ \int_{\mathbb{R}^N} 2[\rho'(u_n^2)]^2u_n\nabla u_n\nabla \varphi\,dx - \int_{\mathbb{R}^N} 2[\rho'(u^2)]^2u\nabla u\nabla \varphi\,dx.$$

We shall estimate each one of the three differences in (8). For the first one, set the functions $g_n := u_n^2 \rho'(u_n^2)\rho''(u_n^2)$ and $g_0 := u^2 \rho'(u^2)\rho''(u^2)$. Thus, if we call

$$\mathcal{I}_n^{(1)} \varphi = \int_{\mathbb{R}^N} 4u_n^3 \rho'(u_n^2)\rho''(u_n^2)|\nabla u_n|^2\varphi\,dx - \int_{\mathbb{R}^N} 4u^3 \rho'(u^2)\rho''(u^2)|\nabla u|^2\varphi\,dx,$$

we have by Hölder inequality and hypothesis $(\rho_3)$ that

$$|\mathcal{I}_n^{(1)} \varphi| = 4\left| \int_{\mathbb{R}^N} u_ng_n|\nabla u|^2\varphi\,dx - \int_{\mathbb{R}^N} u_0g_0|\nabla u|^2\varphi\,dx \right|$$

$$\leq 4\int_{\mathbb{R}^N} |u_ng_n| |\nabla u|^2|\varphi|\,dx + 4\int_{\mathbb{R}^N} |u_ng_n - u_0g_0| |\nabla u|^2|\varphi|\,dx$$

$$\leq 4C_5 \left( \int_{\mathbb{R}^N} |u_n|^3|\varphi|^3\,dx \right)^{1/3} \left( \int_{\mathbb{R}^N} (|\nabla u_n| + |\nabla u|)^3\,dx \right)^{1/3} \left( \int_{\mathbb{R}^N} |\nabla u_n - \nabla u|^3\,dx \right)^{1/3}$$

$$+ 4 \left( \int_{\mathbb{R}^N} |u_ng_n - u_0g_0|^6\,dx \right)^{1/6} \left( \int_{\mathbb{R}^N} |\varphi|^6\,dx \right)^{1/6} \left( \int_{\mathbb{R}^N} |\nabla u|^3\,dx \right)^{2/3}. \quad (9)$$

From $(\rho_1)$ and the convergence $u_n(x) \to u(x)$ a.e. $x \in \mathbb{R}^N$, one has $g_n(x) \to g_0(x)$ a.e. $x \in \mathbb{R}^N$. By using this, condition $(\rho_3)$ and the embeddings in (7), we may apply Lebesgue Theorem to obtain

$$\int_{\mathbb{R}^N} |u_ng_n - u_0g_0|^6\,dx \to 0$$
as $n \to +\infty$. This and the embeddings in (7) one more time transform (9) in

$$|\mathcal{I}_n^{(1)} \varphi| \leq C \left( \int_{\mathbb{R}^N} |\nabla u_n - \nabla u|^3\,dx \right)^{1/3} \left( \int_{\mathbb{R}^N} |\varphi|^6\,dx \right)^{1/6}$$

$$+ C \left( \int_{\mathbb{R}^N} |u_ng_n - u_0g_0|^6\,dx \right)^{1/6} \left( \int_{\mathbb{R}^N} |\varphi|^6\,dx \right)^{1/6} \leq o_n(1)$$

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for some $C > 0$ and uniformly on $||\varphi|| \leq 1$ as $n \to +\infty$. Similar arguments may be done to prove that the other two differences in $\mathfrak{S}$ also goes to zero uniformly on $||\varphi|| \leq 1$ as $n \to +\infty$. This shows that $\Phi'$ is continuous and, consequently, $\Phi$ is a $C^1$ functional, as we wished to prove. 

By Lemma 2 and Proposition 1 $I_\lambda$ is a $C^1$-functional and, for each $u \in W^{2,2}(\mathbb{R}^N)$, we may write

$$I_\lambda(u) = \frac{1}{2} ||u||^2_V + \frac{\lambda}{4} \int_{\mathbb{R}^N} |\nabla \rho(u^2)|^2 dx - \int_{\mathbb{R}^N} F(u) dx.$$ 

We mean that a function $u \in W^{2,2}(\mathbb{R}^N)$ is a weak solution for problem $(P_\lambda)$ if, for all $\varphi \in W^{2,2}(\mathbb{R}^N)$, it holds $I_\lambda'(u)\varphi = 0$. Thus, nontrivial weak solutions for problem $(P_\lambda)$ are critical points of functional $I_\lambda$ and all of them are contained in the Nehari set

$$\mathcal{N} = \{ u \in W^{2,2}(\mathbb{R}^N) \setminus \{0\}; \ I_\lambda'(u)u = 0 \}.$$ 

To prove Theorem 1 we will use the standard arguments of minimization on Nehari manifold, showing that the infimum $m := \inf_{u \in \mathcal{N}} I_\lambda(u)$ is well defined and is achieved at some nontrivial solution for problem $(P_\lambda)$.

As already said, in this problem, the lack of compactness, when we work on unbounded domain as $\mathbb{R}^N$, will be overcome by the symmetry of the problem. We restrict our functional $I_\lambda$ on the subspace contained in $W^{2,2}(\mathbb{R}^N)$ of the radially symmetric functions, i.e., the space $\mathcal{H}^{2,2}_{rad}(\mathbb{R}^N)$ of the functions $u \in W^{2,2}(\mathbb{R}^N)$ satisfying $u(x) = u(|x|)$ for all $x \in \mathbb{R}^N$. Since the properties of this space are used only in the end of this work, we decide to develop all the survey considering the whole space $W^{2,2}(\mathbb{R}^N)$ and, as soon as these properties are needed, we point out this fact.

### 3 Minimization arguments and proof of Theorem 1

We wish minimize the functional $I_\lambda$ on the set $\mathcal{N}$. For this, we present some property of the Nehari set.

**Lemma 3.** If $u \in W^{2,2}(\mathbb{R}^N) \setminus \{0\}$, then there exists $t_u > 0$ such that $t_u u \in \mathcal{N}$. In particular, $\mathcal{N} \neq \emptyset$.

**Proof.** For $t \geq 0$, consider the $C^1$-function $g(t) = I_\lambda(tu)$. Then, we see from hypothesis $(f_1)$ that

$$g(t) \geq \frac{t^2}{2} \left( ||u||^2_V - \int_{\mathbb{R}^N} F(tu) \frac{dx}{t^2} \right) = \frac{t^2}{2} \left( ||u||^2_V - o_t(1) \right),$$

as $t \to 0^+$. Therefore, for small values of $t > 0$, $g(t) > 0$. On the other side, from conditions $(f_3)$ and $(p_2)$, one has

$$g(t) \leq \frac{t^4}{2} \left( o_t(1) + C_1 \int_{\mathbb{R}^N} u^2 |\nabla u|^2 dx - \int_{\mathbb{R}^N} F(tu) dx \right) - \int_{\mathbb{R}^N} (F(tu) / (tu)^4) u^4 dx \to -\infty,$$ (10)

for some $C > 0$ and uniformly on $||\varphi|| \leq 1$ as $n \to +\infty$.
as \( t \to +\infty \). Thus, \( g \) assume a maximum, say \( t_u > 0 \), where \( g'(t_u) = I_\lambda'(t_u)u = 0 \), that is, \( t_u u \in \mathcal{N} \), as we wished. \( \square \)

**Lemma 4.** The set \( \mathcal{N} \) is a \( C^1 \)-manifold.

**Proof.** Let \( J_\lambda(u) = I_\lambda'(u)u \) for \( u \in W^{2,2}(\mathbb{R}^N) \). Recalling that, for all \( u \in \mathcal{N} \),

\[
I_\lambda'(u)u = ||u||_{V'}^2 + \lambda \int_{\mathbb{R}^N} \nabla \rho(u^2) \cdot \nabla (\rho'(u^2)u^2) dx - \int_{\mathbb{R}^N} f(u)udx = 0,
\]

and since

\[
\lambda \int_{\mathbb{R}^N} \nabla \rho(u^2) \cdot \nabla (\rho'(u^2)u^2) dx = 4\lambda \int_{\mathbb{R}^N} \rho'(u^2)\rho''(u^2)u^4|\nabla u|^2 dx + \lambda \int_{\mathbb{R}^N} |\nabla \rho(u^2)|^2 dx,
\]

we may write

\[
J_\lambda(u) = I_\lambda'(u)u = ||u||_{V'}^2 + 4\lambda \int_{\mathbb{R}^N} \rho'(u^2)\rho''(u^2)u^4|\nabla u|^2 dx + \lambda \int_{\mathbb{R}^N} |\nabla \rho(u^2)|^2 dx - \int_{\mathbb{R}^N} f(u)udx = 0.
\]

(11)

Hence, for all \( u \in \mathcal{N} \), we have

\[
J_\lambda'(u)u = 2||u||_{V'}^2 + 4\lambda \left[ 2 \int_{\mathbb{R}^N} \rho''(u^2)\rho''(u^2)u^6|\nabla u|^2 dx + 2 \int_{\mathbb{R}^N} \rho'(u^2)\rho''(u^2)u^6|\nabla u|^2 dx \\
+ 4 \int_{\mathbb{R}^N} \rho'(u^2)\rho''(u^2)u^4|\nabla u|^2 dx + 2 \int_{\mathbb{R}^N} \rho'(u^2)\rho''(u^2)u^4|\nabla u|^2 dx \\
+ 4\lambda \int_{\mathbb{R}^N} \rho'(u^2)\rho''(u^2)u^4|\nabla u|^2 dx + \lambda \int_{\mathbb{R}^N} |\nabla \rho(u^2)|^2 dx - \int_{\mathbb{R}^N} (f'(u)u^2 + f(u)u) dx.
\]

(12)

Let us reorganize some terms. By hypothesis \((\rho_4)\), we obtain two inequality, namely,

\[
\int_{\mathbb{R}^N} 2[\rho''(u^2)u^2]^2 u^2 |\nabla u|^2 dx \leq \int_{\mathbb{R}^N} [\rho'(u^2)]^2 u^2 |\nabla u|^2 dx = \frac{1}{4} \int_{\mathbb{R}^N} |\nabla \rho(u^2)|^2 dx
\]

and, since \( \rho''(t) \leq 0 \) and \( \rho'(t) \geq 0 \),

\[
2 \int_{\mathbb{R}^N} \rho'(u^2)\rho''(u^2)u^4|\nabla u|^2 dx \leq \int_{\mathbb{R}^N} \rho'(u^2)\rho''(u^2)u^4|\nabla u|^2 dx.
\]

(14)

Thus, by (13) and (14), we get from (12) that

\[
J_\lambda'(u)u \leq 2||u||_{V'}^2 + 4\lambda \left[ \frac{1}{4} \int_{\mathbb{R}^N} |\nabla \rho(u^2)|^2 dx + \int_{\mathbb{R}^N} \rho'(u^2)\rho''(u^2)u^4|\nabla u|^2 dx \\
+ 2 \int_{\mathbb{R}^N} \rho'(u^2)u^4|\nabla u|^2 (2\rho''(u^2) + \rho'''(u^2)u^2) dx \\
+ 4\lambda \int_{\mathbb{R}^N} \rho'(u^2)\rho''(u^2)u^4|\nabla u|^2 dx + \lambda \int_{\mathbb{R}^N} |\nabla \rho(u^2)|^2 dx \\
- \int_{\mathbb{R}^N} (f'(u)u^2 + f(u)u) dx.
\]

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Now, by using condition \((\rho_5)\) and \((11)\), one has
\[
J_\lambda'(u)u \leq 2\|u\|_{V^p}^2 + 8\lambda \int_{\mathbb{R}^N} \rho'(u^2)\rho''(u^2)u^4 |\nabla u|^2 dx + 2\lambda \int_{\mathbb{R}^N} |\nabla \rho(u^2)|^2 dx \\
- \int_{\mathbb{R}^N} (f'(u)u^2 + f(u)u) dx
\]
\[= 2 \int_{\mathbb{R}^N} f(u) u dx - \int_{\mathbb{R}^N} (f'(u)u^2 + f(u)u) dx \]
\[= \int_{\mathbb{R}^N} (f(u)u - f'(u)u^2) dx < 0,
\]
where we used hypothesis \((f_2)\) and the fact that \(u \neq 0\). Since \(J\) is a \(C^1\)-functional (see Remark 4 below), we apply the Implicit Function Theorem to guarantee that \(\mathcal{N}\) is a \(C^1\)-manifold. The lemma is proved.

**Lemma 5.** There hold \(\|u\|_{V^p}^2 > c_0\) for all \(u \in \mathcal{N}\), for some \(c_0 > 0\), and \(m = \inf_{\mathcal{N}} I_\lambda(u) > 0\).

**Proof.** For any \(u \in \mathcal{N}\), we have
\[
\|u\|_{V^p}^2 + 4\int_{\mathbb{R}^N} \rho'(u^2)\rho''(u^2)u^4 |\nabla u|^2 dx + \lambda \int_{\mathbb{R}^N} |\nabla \rho(u^2)|^2 dx - \int_{\mathbb{R}^N} f(u) u dx = 0,
\]
what yields by \((\rho_4)\) (see Remark 3)
\[
\|u\|_{V^p}^2 \leq \|u\|_{V^p}^2 + 4\lambda \int_{\mathbb{R}^N} \rho'(u^2)u^2 |\nabla u|^2 [\rho''(u^2)u^2 + \rho'(u^2)] dx = \int_{\mathbb{R}^N} f(u) u dx.
\]
From (5), for all \(\varepsilon > 0\), there is a positive constant \(C = C(\varepsilon)\) such that
\[
\|u\|_{V^p}^2 \leq \int_{\mathbb{R}^N} f(u) u dx \leq \varepsilon \|u\|_{V^p}^2 + C\|u\|^{p+1}_{V^p},
\]
where we used the continuous embedding \(W^{2,2}(\mathbb{R}^N) \hookrightarrow L^s(\mathbb{R}^N)\) for \(s = 2\) and \(s = p + 1\). So, choosing \(\varepsilon\) adequately, we can find a positive constant \(c_0 > 0\) satisfying \(0 < c_0 \leq \|u\|_{V^p}^2\), what proves the first part of the lemma. For the second one, note that, from hypotheses \((f_3)\) and \((\rho_4)\),
\[
I_\lambda(u) = I_\lambda(u) - \frac{1}{4} I_\lambda'(u)u = \frac{1}{4} \|u\|_{V^p}^2 - \lambda \int_{\mathbb{R}^N} \rho'(u^2)\rho''(u^2)u^4 |\nabla u|^2 dx \\
+ \int_{\mathbb{R}^N} \left( \frac{1}{4} f(u) u - F(u) \right) dx \geq \frac{1}{4} \|u\|_{V^p}^2,
\]
which, applying the first part of the lemma, provides \(I_\lambda(u) \geq \frac{c_0}{4} > 0\), and concludes the proof.
Despite there is a minimizing sequence in \( \mathcal{N} \) for functional \( I_\lambda \), it can not be a sequence that converges weakly for a solution \( u_0 \in \mathcal{N} \) for problem \((P_\lambda)\). In the next result, we will show the existence of an appropriate minimizing sequence for our purpose.

**Lemma 6.** There exists a \((PS)_m\)-sequence \((u_n) \subset \mathcal{N}\) for functional \(I_\lambda\), i.e., a sequence \((u_n) \subset \mathcal{N}\) satisfying

\[
I_\lambda(u_n) \to m \quad \text{and} \quad I'_\lambda(u_n) \to 0
\]

as \( n \to +\infty \).

**Proof.** We will apply Ekeland’s Principle (see Theorem 8.5 in [31]). Since \( f \in C^2(\mathbb{R}, \mathbb{R}) \) and by hypotheses on function \( \rho \), it is possible to show that functional \( J_\lambda(u) = I'_\lambda(u)u \) belongs to \( C^2(W^{2,2}(\mathbb{R}^N), \mathbb{R}) \) (see Remark [4] after this proof). By Lemma [4], \( J'_\lambda(u) \neq 0 \) for all \( u \in \mathcal{N} \).

In view of \( I_\lambda \in C^1(W^{2,2}(\mathbb{R}^N), \mathbb{R}) \), by Ekeland’s Principle, there exist a sequence \((u_n) \subset \mathcal{N}\) and a sequence \((\lambda_n) \subset \mathbb{R}\) such that

\[
I_\lambda(u_n) \to m \quad \text{and} \quad (I'_\lambda(u_n) + \lambda_n J'_\lambda(u_n)) \to 0
\]

as \( n \to +\infty \). From the inequality

\[
m + o_n(1) = I_\lambda(u_n) = I_\lambda(u_n) - \frac{1}{4}I'_\lambda(u_n)u_n \geq \frac{1}{4}||u_n||^2_{V},
\]

it follows that \((u_n)\) is bounded and, consequently,

\[
o_n(1) = ||I'_\lambda(u_n) + \lambda_n J'_\lambda(u_n)|| \geq \frac{|I'_\lambda(u_n)u_n + \lambda_n J'_\lambda(u_n)u_n|}{||u_n||} \geq C|\lambda_n J'_\lambda(u_n)u_n|,
\]

for some \( C > 0 \). Observing the proof of Lemma [4] (specifically (15)) together with hypothesis \((f_2)\), it yields

\[
|J'_\lambda(u_n)u_n| \geq \int_{\mathbb{R}^N} (f'(u_n)u_n^2 - f(u_n)u_n)dx \geq \delta |u_n|_{p+1}^{p+1}.
\]

We may apply the first part of Lemma [5] the boundness of \((u_n)\) and the inequalities

\[
c_0 \leq ||u_n||^2_{V} \leq \int_{\mathbb{R}^N} f(u_n)u_n dx \leq \varepsilon |u_n|_{2}^2 + C\varepsilon |u_n|_{p+1}^{p+1}
\]

to guarantee the existence of a positive constant \( C > 0 \) such that

\[
|u_n|_{p+1}^{p+1} \geq C > 0.
\]

Thus, \( |J'_\lambda(u_n)u_n| \geq \delta C > 0 \) and, substituting this in (17), we have necessarily \( \lambda_n \to 0 \) as \( n \to +\infty \). Finally, the arguments contained in proof of Proposition [4] may be used to ensure that, since \((u_n)\) is bounded, so \(|J'_\lambda(u_n)\varphi|\) is bounded uniformly on \( ||\varphi||_{V} \leq 1 \). To see this, it is enough apply similar inequalities as in (9) together conditions \((p_1)\) and \((p_2)\) in the expression of \(J'_\lambda(u_n)\varphi\). Therefore, if \( ||\varphi||_{V} \leq 1 \), then

\[
o_n(1) = ||I'_\lambda(u_n) + \lambda_n J'_\lambda(u_n)|| \geq |I'_\lambda(u_n)\varphi + \lambda_n J'_\lambda(u_n)\varphi| \geq |I'_\lambda(u_n)\varphi| + o_n(1),
\]

and consequently \( I'_\lambda(u_n) \to 0 \) as \( n \to +\infty \). The proof is completed. \( \square \)
Remark 4. In the proof of the Lemmas 4 and 6, we naturally used that $J$ is a $C^1$-functional (in the end of the proof of the Lemma 4) and that $J$ is a $C^2$-functional (in the proof of the Lemma 4). We decide to omit the proofs of these facts, because they are just a tedious but elementary calculations, in which the assumptions on functions $\rho$ and $f$ are employed and several arguments as in proof of the Proposition 4 are done.

We are now able to prove our main result. Before it, see that every result already done until here remains true if we change $W^{2,2}(\mathbb{R}^N)$ by $H^2_{\text{rad}}(\mathbb{R}^N)$. Because of this, in the next proof, we consider $I_\lambda|_{H^2_{\text{rad}}(\mathbb{R}^N)}$. Also, note that every critical point in $H^2_{\text{rad}}(\mathbb{R}^N)$ for the functional $I_\lambda$ is also a critical point in $W^{2,2}(\mathbb{R}^N)$ for the same functional. This is a principle of symmetric criticality for reflexive spaces due to de Morais Filho, do Ó and Souto (see [10], Section 3, Proposition 3.1).

Proof of Theorem 1: From Lemma 3 consider $(u_n) \subset \mathcal{N}$ a $(PS)_m$ sequence for functional $I_\lambda$, i.e., $I_\lambda(u_n) \to m$ and $I'_\lambda(u_n) \to 0$ as $n \to +\infty$. As already done before, we have that $(u_n)$ is bounded in $H^2_{\text{rad}}(\mathbb{R}^N)$ and, hence, up to a subsequence, we get the existence of $u_\lambda \in H^2_{\text{rad}}(\mathbb{R}^N)$ such that the weak convergence $u_n \rightharpoonup u_\lambda$ holds as $n \to +\infty$. By Sobolev embeddings and assumptions on functions $\rho$ and $f$, we obtain $I'_\lambda(u_\lambda)\varphi = 0$ for all $\varphi \in C_0^\infty(\mathbb{R}^N)$. For density argument, we have $I'_\lambda(u_\lambda) = 0$. Since $H^2_{\text{rad}}(\mathbb{R}^N)$ is compactly embedded in $L^{p+1}(\mathbb{R}^N)$ (see Remark 5 below), it follows from (3), Lemma 5, (16) and the boundedness of $|u_n|_2$ that

$$0 < c_0 \leq \frac{1}{\lambda} \int_{\mathbb{R}^N} f(u_n)u_n dx \leq \varepsilon C + C|u_\lambda|_{p+1}^p + o_n(1),$$

for some $C > 0$, what guarantees that $u_\lambda \neq 0$. Consequently, $u_\lambda \in \mathcal{N}$ is a nontrivial weak solution for problem $(P_\lambda)$. Let us show that it is also a ground state solution. For this, by the weak convergence $u_n \rightharpoonup u_\lambda$, the convergences $u_n(x) \to u_\lambda(x)$ and $|\nabla u_n(x)| \to |\nabla u_\lambda(x)|$ a.e. $x \in \mathbb{R}^N$ and Fatou’s Lemma, hypothesis $(f_3)$ and $(\rho_4)$ may be applied to obtain, up to a subsequence,

$$m \leq I_\lambda(u_\lambda) = I_\lambda(u_\lambda) - \frac{1}{4} I'_\lambda(u_\lambda)u_\lambda = \frac{1}{4} |u_\lambda|_V^2 - \lambda \int_{\mathbb{R}^N} \rho'(u_\lambda^2)\rho''(u_\lambda^2)u_\lambda^4 |\nabla u_\lambda|^2 dx + \int_{\mathbb{R}^N} \left( \frac{1}{4} f(u_\lambda) - F(u_\lambda) \right) dx$$

$$\leq \liminf_{n \to +\infty} \left[ \frac{1}{4} |u_n|_V^2 - \lambda \int_{\mathbb{R}^N} \rho'(u_n^2)\rho''(u_n^2)u_n^4 |\nabla u_n|^2 dx + \int_{\mathbb{R}^N} \left( \frac{1}{4} f(u_n) - F(u_n) \right) dx \right]$$

$$= \liminf_{n \to +\infty} \left( I_\lambda(u_n) - \frac{1}{4} I'_\lambda(u_n)u_n \right) = m,$$

that is, $I_\lambda(u_\lambda) = m$ and the proof of the theorem is completed.

Remark 5. The proof of the compact embedding $H^2_{\text{rad}}(\mathbb{R}^N) \hookrightarrow L^s(\mathbb{R}^N)$ for $2 < s < 2^*$, follows directly the same steps contained in section 1.5 from [37], where the reader will find the proof of the well known compact embedding $H^1_{\text{rad}}(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N)$ for $2 < r < 2^*$. The main ingredients are the usual Sobolev embeddings and Lion’s Lemma.
If \( \lambda = 0 \), the method applied in this article works for all \( N \geq 3 \), and the problem

\[
(P_0) \quad \begin{cases}
\Delta^2 u - \Delta u + V(x)u = f(u) \text{ in } \mathbb{R}^N, \\
u \in W^{2,2}(\mathbb{R}^N),
\end{cases}
\]

weakening the hypotheses on the nonlinearity \( f \) adequately, has a nontrivial radially symmetric ground state solution \( u_0 \in H^2_{\text{rad}}(\mathbb{R}^N) \).

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