GLOBAL ERROR BOUNDS FOR THE TENSOR
COMPLEMENTARITY PROBLEM WITH A P-TENSOR

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1. Introduction. Given $M \in \mathbb{R}^{n \times n}$ and $q \in \mathbb{R}^n$, the linear complementarity problem \cite{7}, denoted by the LCP($M, q$), is to find a vector $x \in \mathbb{R}^n$ such that

$$x \geq 0, \quad Mx + q \geq 0, \quad \text{and} \quad x^\top (Mx + q) = 0.$$ 

The LCP ($M, q$) has been studied extensively due to its wide applications in bimatrix game, the contact problems, the free boundary problem for journal bearing, the network equilibrium problem, etc. \cite{2, 7, 22}.

For the LCP ($M, q$), one of important issues is to study the related error bound, which is an inequality that bounds the distance from vectors in a test set $T \subseteq \mathbb{R}^n$ to the solution set of the LCP ($M, q$), denoted by $S$, in terms of some residual function. Recalled that a nonnegative valued function $r : S \cup T \rightarrow \mathbb{R}_+$ is said to be a residual function for the LCP ($M, q$) if it satisfies the property that $r(x) = 0$ if and only if $x \in S$. An error bound for the LCP ($M, q$) in term of $r$ is a pair of inequalities of the form

$$c_1 r_1(x)^{\gamma_1} \leq \text{dist}(x, S) \leq c_2 r_2(x)^{\gamma_2} \quad \text{for all} \quad x \in T,$$

for some positive constants $c_1, c_2, \gamma_1$ and $\gamma_2$, where $r_1$ and $r_2$ are two residual functions for the LCP ($M, q$), and $\text{dist}(x, S)$ is the distance from the vector $x$ to the set $S$. If $T = \mathbb{R}^n$, then (1) is called a global error bound for the LCP ($M, q$).

The error bound has been studied extensively for the LCP ($M, q$). For example, in 1990, Mathias and Pang \cite{21} established error bounds for the LCP($M, q$) with a $P$-matrix; while Luo, Mangasarian, Ren, Solodov \cite{19} discussed error bounds...
for the LCP($M, q$) with a nondegenerate matrix. Then, the perturbation bounds of the LCP($M, q$) were obtained by Chen and Xiang [5, 6]. The error bounds for the LCP($M, q$) were given by Li and Zheng [18]. Recently, a series of literatures studied the error bounds for the LCP($M, q$) with $P$-matrices and the computation of those error bounds was proposed by Chen and Xiang [5, 6], the $H$-matrix by Chen, Li, Wu, Vong [4], the $SB$-matrix by Dai, Li, Lu [9], and the $B$-matrix by García-Esnaola and Peña [12], Li, Gan and Yang [17], and Gao and Li [11].

Given a vector $q \in \mathbb{R}^n$ and an $m$th-order $n$-dimensional tensor $\mathcal{A} = (a_{i_1 \cdots i_m})$ with $a_{i_1 \cdots i_m} \in \mathbb{R}$ for all $i_j \in \{1, 2, \cdots, n\}$ and $j \in \{1, 2, \cdots, m\}$, the tensor complementarity problem, denoted by the TCP($\mathcal{A}, q$), is to find a vector $x \in \mathbb{R}^n$ such that

$$x \geq 0, \quad \mathcal{A} x^{m-1} + q \geq 0, \quad \text{and} \quad x^\top (\mathcal{A} x^{m-1} + q) = 0,$$

where $\mathcal{A} x^{m-1} \in \mathbb{R}^n$ with

$$(\mathcal{A} x^{m-1})_i := \sum_{i_2 \cdots i_m = 1}^n a_{i_1 i_2 \cdots i_m} x_{i_2} \cdots x_{i_m} \quad \text{for all} \ i \in \{1, 2, \cdots, n\}.$$

This class of complementarity problems was used firstly by Song and Qi [24]. It is obvious that the TCP($\mathcal{A}, q$) is a nature extension of the LCP($M, q$).

By using special properties of several classes of structured tensors [23], various properties of the solution set of the TCP($\mathcal{A}, q$) have been well studied, including the nonempty compactness of the solution set by Che, Qi and Wei [3], Song and Qi [25, 26], Song and Yu [28], Gowda, Luo, Qi and Xiu [13], Luo, Qi and Xiu [20], and Wang, Huang and Bai [29]; the existence of solution by Huang, Suo and Wang [16], and Song and Qi [27]; the global uniqueness and solvability by Bai, Huang and Wang [1]; and the topological properties of the solution set and stability of the TCP($\mathcal{A}, q$) by Yu, Ling, He and Qi [30]. More recently, the strict feasibility of the TCP($\mathcal{A}, q$) was discussed with the help of $S$-tensor by Guo, Zheng and Huang [14]. In particular, Song and Qi [26] and Song and Yu [28] gave estimations of upper and lower bounds of the solution set of the TCP($\mathcal{A}, q$) with $\mathcal{A}$ being a $P$-tensor and a strictly semi-positive tensor, respectively. In addition, some applications of the TCP($\mathcal{A}, q$) were also given (see, for example, [15]).

Since the TCP($\mathcal{A}, q$) is a generalization of the LCP($M, q$) and the error bound for the LCP($M, q$) has been studied extensively, a nature question is whether can we extend some results of the error bound for the LCP($M, q$) to the TCP($\mathcal{A}, q$) or not? We will give some answers to this question in this paper. We will give two results of the global error bound for the TCP($\mathcal{A}, q$) with $\mathcal{A}$ being a $P$-tensor. After recalling some basic concepts and related results in Section 2, we give our main results in Section 3. The final conclusions are given in Section 4.

Throughout this paper, we assume that $m$ and $n$ are two positive integers with $m \geq 3$ and $n \geq 2$ unless otherwise stated; and use $T_{m,n}$ to denote the set of all real $m$th-order $n$-dimensional tensors. For any positive integer $n$, we denote $[n] := \{1, 2, \cdots, n\}$.

2. Preliminaries. In this section, we recall some basic definitions and related facts, which are useful for our later discussions.
Definition 2.1. [24] A tensor $\mathcal{A} \in \mathbb{T}_{m,n}$ is said to be a $P$-tensor if and only if for each $x \in \mathbb{R}^n \setminus \{0\}$, there exists an index $i \in [n]$ such that
\[ x_i(\mathcal{A}x^{m-1})_i > 0. \]

Yuan and You [31] obtained the following property for $P$-tensor.

Lemma 2.2. [31] There does not exist an odd order $P$-tensor.

Bai, Huang and Wang showed the following result in [1].

Lemma 2.3. [1] For any given $q \in \mathbb{R}^n$ and a $P$-tensor $\mathcal{A} \in \mathbb{T}_{m,n}$, the solution set of the TCP($\mathcal{A}, q$) is nonempty and compact.

Definition 2.4. Let mapping $F : K \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n$. Then, $F$ is said to be a $P$-function if for all pairs of distinct vectors $x$ and $y$ in $K$,
\[ \max_{i \in [n]} (x_i - y_i)(F_i(x) - F_i(y)) > 0. \]

Clearly, $\mathcal{A}$ is a $P$-tensor if the mapping $\mathcal{A}x^{m-1} + q$ with any given $q \in \mathbb{R}^n$ is a $P$-function.

Definition 2.5. [1] Let $\mathcal{A} = (a_{i_1 \cdots i_m}) \in \mathbb{T}_{m,n}$. Then, $\mathcal{A}$ is said to be a strong $P$-tensor if $F(x) = \mathcal{A}x^{m-1} + q$ for any given $q \in \mathbb{R}^n$ is a $P$-function.

It is easy to check from Definitions 2.1 and 2.5 that each strong $P$-tensor is a $P$-tensor.

Lemma 2.6. [1] Suppose that $\mathcal{A} \in \mathbb{T}_{m,n}$ is a strong $P$-tensor, then for any $q \in \mathbb{R}^n$, the TCP($\mathcal{A}, q$) has a unique solution.

For any $M = (m_{ij}) \in \mathbb{R}^{n \times n}$,
\[ \|M\|_\infty := \max_{i \in [n]} \sum_{j=1}^{n} |m_{ij}|. \]

Similarly, for any $\mathcal{A} \in \mathbb{T}_{m,n}$,
\[ ||\mathcal{A}||_\infty := \max_{i \in [n]} \sum_{i_2, \ldots, i_m=1}^{n} |a_{i_1 \cdots i_m}|. \]

Recall that an operator $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is called positively homogeneous if and only if $T(tx) = tT(x)$ for each $t > 0$ and all $x \in \mathbb{R}^n$. Song and Qi defined two positively homogeneous operators in [24]:

- Let $\mathcal{A} \in \mathbb{T}_{m,n}$. For any $x \in \mathbb{R}^n$, define an operator $T_{\mathcal{A}} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ by
\[ T_{\mathcal{A}}x := \begin{cases} \|x\|_2^{2-m} \mathcal{A}x^{m-1}, & x \neq 0, \\ 0, & x = 0. \end{cases} \]  
(2)

- When $m$ is even, for any $x \in \mathbb{R}^n$, define another operator $F_{\mathcal{A}} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ by
\[ F_{\mathcal{A}}x := (\mathcal{A}x^{m-1})^{\frac{1}{m-1}} \]
where $x^{\frac{1}{m-1}} = (x_1^{\frac{1}{m-1}}, x_2^{\frac{1}{m-1}}, \ldots, x_n^{\frac{1}{m-1}})^\top$.  
(3)
In [21], Mathias and Pang defined an important quantity for a $P$ matrix $M$:

$$\alpha(M) := \min_{||x||_\infty = 1} \max_{i \in [n]} x_i (Mx)_i.$$  

As its generalizations, Song and Qi [24] introduced two quantities:

$$\alpha(T_A) := \min_{||x||_\infty = 1} \max_{i \in [n]} x_i (T_A x)_i$$  

for any positive integer $m$; and

$$\alpha(F_A) := \min_{||x||_\infty = 1} \max_{i \in [n]} x_i (F_A x)_i$$  

when $m$ is even. From Lemma 2.2, $P$-tensors are all even order, then $\alpha(T_A)$ and $\alpha(F_A)$ are both well defined for any $P$-tensor.

Song and Qi obtained two necessary and sufficient conditions for a $P$-tensor in terms of $\alpha(T_A)$ and $\alpha(F_A)$.

**Lemma 2.7.** [24] Let $A \in T_{m,n}$. Then,

(i): $A$ is a $P$-tensor if and only if $\alpha(T_A) > 0$;

(ii): when $m$ is even, $A$ is a $P$-tensor if and only if $\alpha(F_A) > 0$.

Since every strong $P$-tensor is a $P$-tensor, the following corollary is obvious.

**Corollary 1.** Let $A \in T_{m,n}$. Then,

(i): $\alpha(T_A) > 0$ if $A$ is a strong $P$-tensor;

(ii): $\alpha(F_A) > 0$ if $A$ is a strong $P$-tensor.

In [21], Mathias and Pang obtained the following global error bound for the $LCP(M, q)$.

**Theorem 2.8.** Let $M$ be an $n \times n$ $P$-matrix. Let $\hat{x}$ denote the unique solution of the $LCP(M, q)$ and $u$ be an arbitrary $n$-vector. Then,

$$\frac{1}{1 + ||M||_\infty} \left| \min \{u, Mu + q\}\right|_\infty \leq ||u - \hat{x}||_\infty \leq \frac{1 + ||M||_\infty}{\alpha(M)} \min \{u, Mu + q\}\right|_\infty.$$

In the next section, we extend such a result to the $TCP(A, q)$.

### 3. Error bounds for the $TCP(A, q)$

In this section, we investigate the global error bound for the $TCP(A, q)$.

From Lemma 2.3, it follows that the solution set of the $TCP(A, q)$, denoted by $S$,

$$S := \{x \in \mathbb{R}^n | x \geq 0, A x^{m-1} + q \geq 0, \text{ and } x^\top (A x^{m-1} + q) = 0\}$$  

is nonempty and compact when $A \in T_{m,n}$ is a $P$-tensor. Therefore, for any $u \in \mathbb{R}^n$, there exists a vector $\hat{x} \in S$ such that

$$\text{dist}(u, S) = ||u - \hat{x}||_\infty,$$

where $\text{dist}(u, S) := \min_{x \in S} ||u - x||_\infty$. Furthermore, we define

$$r_{\hat{x}}(u) := ||v_{\hat{x}}(u)||_\infty$$

with

$$v_{\hat{x}}(u) := \min \left\{ u, \{A(u - \hat{x})^{m-1}\mid |\frac{A}{m-1}\} + (A \hat{x}^{m-1} + q)\right\}, \forall u \in \mathbb{R}^n.$$  

Then, we have the following result.
Theorem 3.1. Given $q \in \mathbb{R}^n$ and $\mathcal{A} \in \mathbb{T}_{m,n}$ with $\mathcal{A}$ being a $P$-tensor. For any $u \in \mathbb{R}^n$, let $\hat{x}$ be given by (7). Suppose that $r_2(\cdot)$ is defined by (8). Then,

$$r_2(u) = 0 \iff u = \hat{x}.$$  

Proof. For simplicity, we use $v$ to replace $v_\circ(u)$ in the following. It is obvious from (8) that we only need to show that

$$v = 0 \iff u = \hat{x}.$$  

We first show sufficiency. Since $\hat{x}$ is a solution to the TCP($\mathcal{A}, q$), it follows that $\min\{\hat{x}_i, \mathcal{A}\hat{x}^{m-1} + q\} = 0$, which means that

$$\hat{x}_i \geq 0, (\mathcal{A}\hat{x}^{m-1} + q)_i \geq 0, \hat{x}_i(\mathcal{A}\hat{x}^{m-1} + q)_i = 0, \forall i \in [n].$$  

This is equivalent to

$$\hat{x}_i \geq 0, (\mathcal{A}\hat{x}^{m-1} + q)_i^{\frac{1}{m-1}} \geq 0, \hat{x}_i(\mathcal{A}\hat{x}^{m-1} + q)_i^{\frac{1}{m-1}} = 0, \forall i \in [n],$$  

i.e.,

$$\min\left\{\hat{x}_i, (\mathcal{A}\hat{x}^{m-1} + q)_i^{\frac{1}{m-1}}\right\} = 0.$$  

This, together with (9) and the condition that $u = \hat{x}$, implies that $v = 0$.  

Next, we show necessity. Since $v = 0$, it follows from (9) that

$$v_i = u_i = 0 \quad \text{or} \quad v_i = [\mathcal{A}(u - \hat{x})^{m-1}]_i^{\frac{1}{m-1}} + (\mathcal{A}\hat{x}^{m-1} + q)_i^{\frac{1}{m-1}} = 0, \forall i \in [n].$$  

We divide the proof into the following two cases:

(C1) Suppose that $v_i = u_i = 0$, then

$$[\mathcal{A}(u - \hat{x})^{m-1}]_i^{\frac{1}{m-1}} + (\mathcal{A}\hat{x}^{m-1} + q)_i^{\frac{1}{m-1}} \geq v_i = 0.$$  

Since $\hat{x}$ is a solution of the TCP($\mathcal{A}, q$), then

$$\hat{x}_i \geq 0, (\mathcal{A}\hat{x}^{m-1} + q)_i \geq 0, \quad \text{and} \quad \hat{x}_i(\mathcal{A}\hat{x}^{m-1} + q)_i = 0$$  

i.e.,

$$\hat{x}_i = 0 \quad \text{or} \quad (\mathcal{A}\hat{x}^{m-1} + q)_i = 0.$$  

\[ \blacktriangleright \] If $\hat{x}_i = 0$, then $u_i - \hat{x}_i = 0$; and hence, $(u - \hat{x})_i[\mathcal{A}(u - \hat{x})^{m-1}]_i = 0$.  

\[ \blacktriangleright \] Otherwise, $(\mathcal{A}\hat{x}^{m-1} + q)_i = 0$, then

$$[\mathcal{A}(u - \hat{x})^{m-1}]_i^{\frac{1}{m-1}} = [\mathcal{A}(u - \hat{x})^{m-1}]_i^{\frac{1}{m-1}} + (\mathcal{A}\hat{x}^{m-1} + q)_i^{\frac{1}{m-1}} \geq 0.$$  

Thereby, $[\mathcal{A}(u - \hat{x})^{m-1}]_i \geq 0$. But, $u_i - \hat{x}_i = -\hat{x}_i \leq 0$, thus we have that $(u - \hat{x})_i[\mathcal{A}(u - \hat{x})^{m-1}]_i \leq 0$.  

(C2) Suppose that $v_i = [\mathcal{A}(u - \hat{x})^{m-1}]_i^{\frac{1}{m-1}} + (\mathcal{A}\hat{x}^{m-1} + q)_i^{\frac{1}{m-1}} = 0$, then $u_i \geq v_i = 0$. Since $\hat{x}$ is a solution of the TCP($\mathcal{A}, q$), then

$$\hat{x}_i \geq 0, (\mathcal{A}\hat{x}^{m-1} + q)_i \geq 0, \quad \hat{x}_i = 0 \quad \text{or} \quad (\mathcal{A}\hat{x}^{m-1} + q)_i = 0.$$  

\[ \blacktriangleright \] If $\hat{x}_i = 0$, then $(\mathcal{A}\hat{x}^{m-1} + q)_i \geq 0$. Since

$$[\mathcal{A}(u - \hat{x})^{m-1}]_i^{\frac{1}{m-1}} + (\mathcal{A}\hat{x}^{m-1} + q)_i^{\frac{1}{m-1}} = 0,$$

it follows that $[\mathcal{A}(u - \hat{x})^{m-1}]_i^{\frac{1}{m-1}} \leq 0$; and so $[\mathcal{A}(u - \hat{x})^{m-1}]_i \leq 0$. However, $u_i - \hat{x}_i = u_i \geq 0$, therefore $(u - \hat{x})_i[\mathcal{A}(u - \hat{x})^{m-1}]_i \leq 0$.  

\[ \blacktriangleright \] Otherwise, $(\mathcal{A}\hat{x}^{m-1} + q)_i = 0$, and then

$$[\mathcal{A}(u - \hat{x})^{m-1}]_i^{\frac{1}{m-1}} = [\mathcal{A}(u - \hat{x})^{m-1}]_i^{\frac{1}{m-1}} + (\mathcal{A}\hat{x}^{m-1} + q)_i^{\frac{1}{m-1}} = 0.$$  

So, $[\mathcal{A}(u - \hat{x})^{m-1}]_i = 0$. Therefore, $(u - \hat{x})_i[\mathcal{A}(u - \hat{x})^{m-1}]_i = 0$.  


In summary, we obtain that
\[ (u - \hat{x})_i |\mathcal{A}(u - \hat{x})^{m-1}|_i \leq 0, \quad \forall \ i \in [n]. \tag{10} \]
However, since \( \mathcal{A} \) is a \( P \)-tensor, it follows that for any \( x \in \mathbb{R}^n \setminus \{0\} \), there exists an index \( i \in [n] \) such that \( x_i (\mathcal{A}x^{m-1})_i > 0 \). Thus, if \( u - \hat{x} \in \mathbb{R}^n \setminus \{0\} \), then we can find some index \( i \in [n] \) such that
\[ (u - \hat{x})_i |\mathcal{A}(u - \hat{x})^{m-1}|_i > 0, \]
which contradicts (10). Therefore, \( u - \hat{x} \notin \mathbb{R}^n \setminus \{0\} \), i.e., \( u - \hat{x} = 0 \).

The proof is complete. \( \square \)

Since \( \hat{x} \) is a solution to the TCP(\( \mathcal{A}, q \)), Theorem 3.1 demonstrates that \( r_z(\cdot) \) is a residual function for the TCP (\( \mathcal{A}, q \)). In the following, we give a global error bound for the TCP(\( \mathcal{A}, q \)) in terms of the residual function \( r_z(\cdot) \) defined by (8) and the quantity \( \alpha(F_{\mathcal{A}}) \) defined by (5).

**Theorem 3.2.** Given \( q \in \mathbb{R}^n \), \( \mathcal{A} \in \mathbb{T}_{m,n} \) with \( \mathcal{A} \) being a \( P \)-tensor and \( \alpha(F_{\mathcal{A}}) \) is defined by (5). For any \( u \in \mathbb{R}^n \), let \( \hat{x} \) be given by (7). Suppose that \( r_z(\cdot) \) is defined by (8). Then, for any \( u \in \mathbb{R}^n \),
\[
\frac{1}{1 + ||\mathcal{A}||_\infty^{m-1}} |r_z(u)| \leq \frac{1 + ||\mathcal{A}||_\infty^{-1}}{\alpha(F_{\mathcal{A}})} \cdot |r_z(u)|.
\]

*Proof.* For simplicity, we use \( v \) to denote \( v_z(u) \) in the following. Then, we need to show that
\[
\frac{1}{1 + ||\mathcal{A}||_\infty^{m-1}} ||v|| \leq ||u - \hat{x}|| \leq \frac{1 + ||\mathcal{A}||_\infty^{-1}}{\alpha(F_{\mathcal{A}})} ||v||.
\tag{11}
\]
We divide the proof into the following two parts.

Part 1. We show that the inequality on the right-hand side of (11) holds.

Let \( \omega = (\mathcal{A}\hat{x}^{m-1} + q)^{1/m-1} \). Since \( \hat{x} \) is a solution to the TCP(\( \mathcal{A}, q \)), it is easy to see that
\[ \omega \geq 0, \quad \hat{x} \geq 0, \quad \hat{x}^\top \omega = 0. \]
Noting that \( v = \min \{u, [\mathcal{A}(u - \hat{x})^{m-1}]^{1/m-1} + (\mathcal{A}\hat{x}^{m-1} + q)^{1/m-1} \} \), if we let \( y = u - v \) and \( z = [\mathcal{A}(u - \hat{x})^{m-1}]^{1/m-1} + (\mathcal{A}\hat{x}^{m-1} + q)^{1/m-1} - v \), then
\[ y \geq 0, \quad z \geq 0, \quad y^\top z = 0. \]
Thus, for any \( i \in [n] \),
\[
0 \geq -x_i z_i - y_i \omega_i = y_i z_i - x_i z_i - y_i \omega_i + x_i \omega_i = (y - \hat{x})_i (z - \omega)_i = (u - v - \hat{x})_i (\mathcal{A}(u - \hat{x})^{m-1})_i^{1/m-1} - v)_i = (u - \hat{x})_i [\mathcal{A}(u - \hat{x})^{m-1}]_i^{1/m-1} - v_i [\mathcal{A}(u - \hat{x})^{m-1}]_i^{1/m-1} - (u - \hat{x})_i v_i + v_i^2 \geq (u - \hat{x})_i [\mathcal{A}(u - \hat{x})^{m-1}]_i^{1/m-1} - v_i [\mathcal{A}(u - \hat{x})^{m-1}]_i^{1/m-1} - (u - \hat{x})_i v_i \geq (u - \hat{x})_i [\mathcal{A}(u - \hat{x})^{m-1}]_i^{1/m-1} - ||v||_\infty ||\mathcal{A}(u - \hat{x})^{m-1}||_\infty - ||v||_\infty ||u - \hat{x}||_\infty, \]
which implies that for any \( i \in [n] \),
\[
(u - \hat{x})_i [\mathcal{A}(u - \hat{x})^{m-1}]_i^{1/m-1} \leq ||v||_\infty ||\mathcal{A}(u - \hat{x})^{m-1}||_\infty + ||v||_\infty ||u - \hat{x}||_\infty.
\]
Thus,
\[
\max_{i \in [n]} (u - \hat{x})_i [\mathcal{A}(u - \hat{x})^{m-1}]_i \leq \|v\|_\infty \|\mathcal{A}(u - \hat{x})^{m-1}\|_\infty \frac{1}{m-1} + \|v\|_\infty \|u - \hat{x}\|_\infty. \tag{12}
\]

Since
\[
\|\mathcal{A}(u - \hat{x})^{m-1}\|_\infty = \left( \max_{i \in [n]} \left| (\mathcal{A}(u - \hat{x})^{m-1})_i \right| \right)^{\frac{1}{m-1}} = \left( \max_{i \in [n]} \left| \sum_{i_2, \ldots, i_m=1}^n a_{i_2 \ldots i_m} (u - \hat{x})_{i_2} \cdots (u - \hat{x})_{i_m} \right| \right)^{\frac{1}{m-1}} \leq \left( \max_{i \in [n]} \sum_{i_2, \ldots, i_m=1}^n \left| a_{i_2 \ldots i_m} \right| \|u - \hat{x}\|^{m-1}_\infty \right)^{\frac{1}{m-1}} = \left( \|\mathcal{A}\|_\infty \|u - \hat{x}\|^{m-1}_\infty \right)^{\frac{1}{m-1}} \|u - \hat{x}\|_\infty,
\]
we further obtain that
\[
\max_{i \in [n]} (u - \hat{x})_i [\mathcal{A}(u - \hat{x})^{m-1}]_i \leq \|v\|_\infty \|\mathcal{A}\|_\infty \frac{1}{m-1} \|u - \hat{x}\|_\infty + \|v\|_\infty \|u - \hat{x}\|_\infty. \tag{13}
\]

In addition, it follows from the definition of $\alpha(F_{\mathcal{A}})$ (i.e., (5)) that
\[
\alpha(F_{\mathcal{A}}) \|x\|_\infty \leq \max_{i \in [n]} x_i(F_{\mathcal{A}}x)_i = \max_{i \in [n]} x_i(\mathcal{A}x^{m-1})_i \frac{1}{m-1}, \quad \forall x \in \mathbb{R}^n.
\]
Then,
\[
\alpha(F_{\mathcal{A}}) \|u - \hat{x}\|_\infty \leq \max_{i \in [n]} (u - \hat{x})_i [\mathcal{A}(u - \hat{x})^{m-1}]_i \frac{1}{m-1}. \tag{14}
\]

Therefore, combining (13) and (14), we have that
\[
\alpha(F_{\mathcal{A}}) \|u - \hat{x}\|_\infty \leq \left( \|\mathcal{A}\|_\infty \frac{1}{m-1} \|u - \hat{x}\|_\infty + \|v\|_\infty \|u - \hat{x}\|_\infty \right)^{\frac{1}{m-1}} \|u - \hat{x}\|_\infty. \tag{15}
\]

Furthermore, since $\mathcal{A}$ is a $P$-tensor, it follows from Lemma 2.7 that $\alpha(F_{\mathcal{A}}) > 0$. Thus, from (15) we obtain that
\[
\|u - \hat{x}\|_\infty \leq \frac{1 + \|\mathcal{A}\|_\infty \|v\|_\infty}{\alpha(F_{\mathcal{A}})} \|v\|_\infty. \tag{16}
\]

Part 2. We show that the inequality on the left-hand side of (11) holds. We consider the following two cases:

**C1** Suppose that $v_i > 0$ for an arbitrarily given $i \in [n]$. Then,
\[
0 < v_i \leq u_i \quad \text{and} \quad 0 < v_i \leq [\mathcal{A}(u - \hat{x})^{m-1}]_i \frac{1}{m-1} + (\mathcal{A}\hat{x}^{m-1} + q)_i \frac{1}{m-1}.
\]
Since $\hat{x}$ is a solution of the TCP($\mathcal{A}$, $q$), it follows that $\hat{x}_i = 0$ or $(\mathcal{A}\hat{x}^{m-1} + q)_i = 0$.

- ♦ If $\hat{x}_i = 0$, then $|v_i| = v_i \leq u_i - \hat{x}_i \leq \|u - \hat{x}\|_\infty$.
- ♦ If $(\mathcal{A}\hat{x}^{m-1} + q)_i = 0$, then
\[
|v_i| = v_i \leq [\mathcal{A}(u - \hat{x})^{m-1}]_i \frac{1}{m-1} + (\mathcal{A}\hat{x}^{m-1} + q)_i \frac{1}{m-1} - (\mathcal{A}\hat{x}^{m-1} + q)_i \frac{1}{m-1} = [\mathcal{A}(u - \hat{x})^{m-1}]_i \frac{1}{m-1} \leq \|\mathcal{A}\|_\infty \|u - \hat{x}\|_\infty.
\]
Thus, we conclude that

\[ |v_i| \leq (1 + \|\mathcal{A}\|^\frac{1}{m-1}) ||u - \hat{x}||_\infty, \quad \forall v_i > 0. \tag{17} \]

**(C2)** Suppose that \( v_i \leq 0 \) for an arbitrarily given \( i \in [n] \). Then,

\[ v_i = u_i \quad \text{or} \quad v_i = [\mathcal{A}(u - \hat{x})]_{i}^{n-1} \frac{1}{i} + (\mathcal{A}\hat{x}^{m-1} + q)_{i}^{1 \frac{1}{m-1}}. \]

Since \( \hat{x} \) is a solution of the TCP(\( \mathcal{A}, q \)), it follows that \( \hat{x}_i \geq 0 \) and \( (\mathcal{A}\hat{x}^{m-1} + q)_{i} \geq 0 \).

\[ \star \quad \text{If} \quad 0 \geq v_i = u_i, \quad \text{then} \quad 0 \leq -v_i = -u_i \leq -u_i + \hat{x}_i \leq ||u - \hat{x}||_\infty, \quad \text{i.e.,} \]

\[ |v_i| \leq ||u - \hat{x}||_\infty. \]

\[ \star \quad \text{If} \quad 0 \geq v_i = [\mathcal{A}(u - \hat{x})]_{i}^{n-1} \frac{1}{i} + (\mathcal{A}\hat{x}^{m-1} + q)_{i}^{1 \frac{1}{m-1}}, \quad \text{then} \]

\[ 0 \leq -v_i \]

\[ = -[(\mathcal{A}(u - \hat{x})]_{i}^{n-1} \frac{1}{i} + (\mathcal{A}\hat{x}^{m-1} + q)_{i}^{1 \frac{1}{m-1}}] \]

\[ \leq -[(\mathcal{A}(u - \hat{x})]_{i}^{n-1} \frac{1}{i} + (\mathcal{A}\hat{x}^{m-1} + q)_{i}^{1 \frac{1}{m-1}}] + (\mathcal{A}\hat{x}^{m-1} + q)_{i}^{1 \frac{1}{m-1}} \]

\[ \leq ||\mathcal{A}||_\infty^{\frac{1}{m-1}}||u - \hat{x}||_\infty. \]

Therefore,

\[ |v_i| \leq (1 + ||\mathcal{A}||_\infty^{\frac{1}{m-1}})||u - \hat{x}||_\infty, \quad \forall v_i \leq 0. \tag{18} \]

Thus, it follows from (17) and (18) that

\[ \frac{||v||_\infty}{1 + ||\mathcal{A}||_\infty^{\frac{1}{m-1}}} \leq ||u - \hat{x}||_\infty. \tag{19} \]

Now, by Part 1 and Part 2 (i.e., combining (16) and (19)) we obtain that (11) holds, which completes the proof. \( \square \)

From Lemma 2.6 we know that the TCP(\( \mathcal{A}, q \)) has a unique solution if the involved tensor \( \mathcal{A} \) is a strong \( P \) tensor. Since every strong \( P \) tensor is a \( P \)-tensor, from Theorem 3.2 we can obtain the following result immediately.

**Corollary 2.** Given \( q \in \mathbb{R}^n \) and \( \mathcal{A} \in \mathbb{T}_{m,n} \) with \( \mathcal{A} \) being a strong \( P \)-tensor. Let \( \alpha(F_{\mathcal{A}}) \) be defined by (5). For any \( u \in \mathbb{R}^n \), let \( \hat{x} \) be the unique solution of the TCP(\( \mathcal{A}, q \)). Suppose that \( r_{\hat{x}}(\cdot) \) is defined by (8). Then, \( \text{for any } u \in \mathbb{R}^n \),

\[ \frac{1}{1 + ||\mathcal{A}||_\infty^{\frac{1}{m-1}}} r_{\hat{x}}(u) \leq ||u - \hat{x}||_\infty \leq \frac{1 + ||\mathcal{A}||_\infty^{\frac{1}{m-1}}}{\alpha(F_{\mathcal{A}})} r_{\hat{x}}(u). \]

Up to now, we have obtained a global error bound for the TCP(\( \mathcal{A}, q \)) with the help of the quantity \( \alpha(F_{\mathcal{A}}) \). In the following, we investigate the global error bound for the TCP(\( \mathcal{A}, q \)) with the help of the quantity \( \alpha(T_{\mathcal{A}}) \). For this purpose, we define the following functions:

\[ r_{\hat{x}}^1(u) := \min \left\{ \frac{||v'_x(u)||_\infty^{\frac{1}{m-1}}}{(1 + ||\mathcal{A}||_\infty^{\frac{1}{m-1}})^{\frac{1}{m-1}}}, \frac{||v'_x(u)||_\infty}{1 + ||\mathcal{A}||_\infty} \right\}, \]

\[ r_{\hat{x}}^2(u) := \max \left\{ \frac{||v'_x(u)||_\infty(1 + ||\mathcal{A}||_\infty^{\frac{1}{m-1}})^{\frac{1}{m-1}}}{\alpha(T_{\mathcal{A}})^{\frac{1}{m-1}}}, \frac{||v'_x(u)||_\infty}{\alpha(T_{\mathcal{A}})^{\frac{1}{m-1}}} \right\}, \tag{20} \]
where
\[
v'_2(u) := \min \{ u, \mathcal{A}(u - \hat{x})^{m-1} + \mathcal{A}\hat{x}^{m-1} + q \}, \quad \forall u \in \mathbb{R}^n,
\] (21)
and the quantity \( \alpha(T_{\mathcal{A}}) \) is defined by (4).

Similar to the proof of Theorem 3.1, we can obtain the following result.

**Theorem 3.3.** Given \( q \in \mathbb{R}^n \) and \( \mathcal{A} \in \mathbb{T}_{m,n} \) with \( \mathcal{A} \) being a P-tensor. For any \( u \in \mathbb{R}^n \), let \( \hat{x} \) be given by (7). Suppose that \( r'_1(\cdot) \) and \( r'_2(\cdot) \) are defined by (20). Then,
\[
r'_2(u) = 0 \iff u = \hat{x} \iff r'_2(u) = 0.
\]

Theorem 3.3 demonstrates that both \( r'_1(\cdot) \) and \( r'_2(\cdot) \) are residual functions for the TCP(\( \mathcal{A}, q \)). In the following, by using these two residual functions, we investigate the global error bound for the TCP(\( \mathcal{A}, q \)).

**Theorem 3.4.** Given \( q \in \mathbb{R}^n \) and \( \mathcal{A} \in \mathbb{T}_{m,n} \) with \( \mathcal{A} \) being a P-tensor. For any \( u \in \mathbb{R}^n \), let \( \hat{x} \) be given by (7). Suppose that \( r'_1(\cdot) \) and \( r'_2(\cdot) \) are defined by (20). Then, for any \( u \in \mathbb{R}^n \),
\[
r'_1(u) \leq ||u - \hat{x}||_\infty \leq r'_2(u).
\]

**Proof.** For simplicity, we use \( v' \) to replace \( v'_2(u) \) in the following. It is obvious that we only need to show that
\[
||u - \hat{x}||_\infty \leq \max \left\{ \frac{||v'||_\infty (1 + ||\mathcal{A}||_\infty)}{\alpha(T_{\mathcal{A}})}, \frac{||v'||_\infty^{\frac{1}{m-1}} (1 + ||\mathcal{A}||_\infty)^{\frac{1}{m-1}}}{\alpha(T_{\mathcal{A}})^{\frac{1}{m-1}}} \right\}.
\] (22)
and
\[
||u - \hat{x}||_\infty \geq \min \left\{ \frac{||v'||_\infty^{\frac{1}{m-1}} (1 + ||\mathcal{A}||_\infty)^{\frac{1}{m-1}}}{1 + ||\mathcal{A}||_\infty}, \frac{||v'||_\infty}{1 + ||\mathcal{A}||_\infty} \right\},
\] (23)
where \( v' := \min \{ u, \mathcal{A}(u - \hat{x})^{m-1} + \mathcal{A}\hat{x}^{m-1} + q \} \) for any \( u \in \mathbb{R}^n \).

We first show that (22) holds.

Let \( \omega' = \mathcal{A}\hat{x}^{m-1} + q. \) Since \( \hat{x} \) is a solution to the TCP(\( \mathcal{A}, q \)), it is easy to see that
\[
\omega' \geq 0, \quad \hat{x} \geq 0, \quad \hat{x}^T\omega' = 0.
\]

Noting that \( v' = \min\{u, \mathcal{A}(u - \hat{x})^{m-1} + \mathcal{A}\hat{x}^{m-1} + q\} \), if we let \( y' = u - v' \) and \( z' = \mathcal{A}(u - \hat{x})^{m-1} + \mathcal{A}\hat{x}^{m-1} + q - v' \), then
\[
y' \geq 0, \quad z' \geq 0, \quad (y')^Tz' = 0.
\]

Thus, for any \( i \in [n] \),
\[
0 \geq -\hat{x}_i z'_i - y'_i \omega'_i = y'_i z'_i - \hat{x}_i z'_i - y'_i \omega'_i + \hat{x}_i \omega'_i = (y' - \hat{x})_i (z' - \omega')_i = (u - v' - \hat{x})_i [\mathcal{A}(u - \hat{x})^{m-1} - v'],
\]
\[
= (u - \hat{x})_i [\mathcal{A}(u - \hat{x})^{m-1} - v'[\mathcal{A}(u - \hat{x})^{m-1} - v']]_i - (u - \hat{x})_i v'_i + v'_i^2 \geq (u - \hat{x})_i [\mathcal{A}(u - \hat{x})^{m-1} - v'[\mathcal{A}(u - \hat{x})^{m-1} - v']]_i - (u - \hat{x})_i v'_i \geq (u - \hat{x})_i [\mathcal{A}(u - \hat{x})^{m-1}]_i - ||v'||_\infty \mathcal{A}(u - \hat{x})^{m-1} ||_\infty - ||v'||_\infty ||u - \hat{x}||_\infty,
\]
which implies that for any \( i \in [n] \),
\[
(u - \hat{x})_i [\mathcal{A}(u - \hat{x})^{m-1}]_i \leq ||v'||_\infty \mathcal{A}(u - \hat{x})^{m-1} ||_\infty + ||v'||_\infty ||u - \hat{x}||_\infty.
\]
Thus,
\[\max_{i \in [n]} (u - \hat{x})_i |\alpha(u - \hat{x})^{m-1}|_i \leq ||v'||_\infty ||\alpha(u - \hat{x})^{m-1}||_\infty + ||v'||_\infty ||u - \hat{x}||_\infty. \quad (24)\]

In addition, it follows from the definition of \(\alpha(T_{\mathcal{A}})\) (i.e., (4)) and \(||x||_2 \geq ||x||_\infty\) that for any \(x \in \mathbb{R}^n\),
\[\alpha(T_{\mathcal{A}})||x||^2_\infty \leq \max_{i \in [n]} x_i(T_{\mathcal{A}}x)_i \]
\[= \max_{i \in [n]} x_i(||x||^{2-m}_{2} \alpha x^{m-1})_i \]
\[= ||x||^{2-m}_{2} \max_{i \in [n]} x_i(\alpha x^{m-1})_i \]
\[\leq ||x||^{2-m}_{\infty} \max_{i \in [n]} x_i(\alpha x^{m-1})_i \]

i.e.,
\[\alpha(T_{\mathcal{A}})||x||^m_\infty \leq \max_{i \in [n]} x_i(\alpha x^{m-1})_i, \quad \forall x \in \mathbb{R}^n.\]

Hence,
\[\alpha(T_{\mathcal{A}})||u - \hat{x}||^m_\infty \leq \max_{i \in [n]} (u - \hat{x})_i |\alpha(u - \hat{x})^{m-1}|_i. \quad (25)\]

So, we derive from (24) and (25) that
\[\alpha(T_{\mathcal{A}})||u - \hat{x}||^m_\infty \leq ||v'||_\infty ||\alpha(u - \hat{x})^{m-1}||_\infty + ||v'||_\infty ||u - \hat{x}||_\infty. \quad (26)\]

It is not difficult to see that
\[||\alpha(u - \hat{x})^{m-1}||_\infty = \max_{i \in [n]} (|\alpha(u - \hat{x})^{m-1}|_i) \]
\[\leq \max_{i \in [n]} \sum_{i_2, \ldots, i_m=1}^n a_{i_2 \cdots i_m} (u - \hat{x})_{i_2} \cdots (u - \hat{x})_{i_m} \]
\[\leq \max_{i \in [n]} \sum_{i_2, \ldots, i_m=1}^n |a_{i_2 \cdots i_m}| ||u - \hat{x}||^{m-1}_{\infty} \]
\[= ||\alpha||_{\infty} ||u - \hat{x}||^{m-1}_{\infty}, \]

which, together with (26), implies that
\[\alpha(T_{\mathcal{A}})||u - \hat{x}||^m_\infty \leq ||v'||_\infty ||\alpha||_{\infty} ||u - \hat{x}||^{m-1}_{\infty} + ||v'||_\infty ||u - \hat{x}||_\infty. \quad (27)\]

Furthermore, we consider the following two cases:

- Suppose that ||u - \hat{x}||_\infty \geq 1. Then, ||u - \hat{x}||_\infty \leq ||u - \hat{x}||^{m-1}_{\infty} (m \geq 2).

From (27) it follows that
\[\alpha(T_{\mathcal{A}})||u - \hat{x}||^m_\infty \leq ||v'||_\infty ||\alpha||_{\infty} ||u - \hat{x}||^{m-1}_{\infty} + ||v'||_\infty ||u - \hat{x}||^{m-1}_{\infty} + ||v'||_\infty ||u - \hat{x}||_\infty. \]

Thus,
\[\alpha(T_{\mathcal{A}})||u - \hat{x}||_\infty \leq ||v'||_\infty ||\alpha||_{\infty} + ||v'||_\infty = ||v'||_\infty (1 + ||\alpha||_{\infty}). \quad (28)\]

Since \(\alpha\) is a \(P\)-tensor, it follows from Lemma 2.7 that \(\alpha(T_{\mathcal{A}}) > 0\). Thus,
\[||u - \hat{x}||_\infty \leq \frac{||v'||_\infty (1 + ||\alpha||_{\infty})}{\alpha(T_{\mathcal{A}})}. \quad (29)\]

- Suppose that ||u - \hat{x}||_\infty < 1. Then, ||u - \hat{x}||_\infty \geq ||u - \hat{x}||^{m-1}_{\infty} (m \geq 2).

From (27) it follows that
\[\alpha(T_{\mathcal{A}})||u - \hat{x}||^m_\infty \leq ||v'||_\infty ||\alpha||_{\infty} ||u - \hat{x}||^{m-1}_{\infty} + ||v'||_\infty ||u - \hat{x}||^{m-1}_{\infty} + ||v'||_\infty ||u - \hat{x}||_\infty. \]

\[\leq ||v'||_\infty ||\alpha||_{\infty} ||u - \hat{x}||^{m-1}_{\infty} + ||v'||_\infty ||u - \hat{x}||^{m-1}_{\infty} + ||v'||_\infty ||u - \hat{x}||_\infty. \]
Thus,
\[
\alpha(T_{\mathcal{A}})||u - \hat{x}||_\infty^{m-1} \leq ||v'||_\infty||\mathcal{A}||_\infty + ||v'||_\infty = ||v'||_\infty(1 + ||\mathcal{A}||_\infty).
\] (30)
Since \(\mathcal{A}\) is a P-tensor, it follows from Lemma 2.7 that \(\alpha(T_{\mathcal{A}}) > 0\). Thus,
\[
||u - \hat{x}||_\infty \leq \frac{||v'||_\infty^{1/m}}{\alpha(T_{\mathcal{A}})}(1 + ||\mathcal{A}||_\infty)^{\frac{1}{m-1}}.
\] (31)
Combining (29) with (31), we obtain that (22) holds.

Now, we show that (23) holds. We first consider the following two cases:

**C1** Suppose that \(v'_i \leq 0\) for an arbitrarily given \(i \in [n]\). Then,
\[
v'_i = u_i \quad \text{or} \quad v'_i = [(\mathcal{A}(u - \hat{x})^{m-1} + \mathcal{A}\hat{x}^{m-1} + q)_i].
\]
Since \(\hat{x}\) is a solution of the TCP(\(\mathcal{A}\), \(q\)), it follows that \(\hat{x}_i \geq 0\) and \((\mathcal{A}\hat{x}^{m-1} + q)_i \geq 0\).

\* If \(0 \geq v'_i = u_i\), then \(0 \leq v'_i = -u_i \leq -u_i + \hat{x}_i \leq ||u - \hat{x}||_\infty\), i.e.,
\[
|v'_i| \leq ||u - \hat{x}||_\infty.
\]
\* If \(0 \geq v'_i = [\mathcal{A}(u - \hat{x})^{m-1} + \mathcal{A}\hat{x}^{m-1} + q)_i\), then
\[
0 \leq -v'_i = -[\mathcal{A}(u - \hat{x})^{m-1} + \mathcal{A}\hat{x}^{m-1} + q)_i
\]
\[
\leq [\mathcal{A}(u - \hat{x})^{m-1} + \mathcal{A}\hat{x}^{m-1} + q)_i + (\mathcal{A}\hat{x}^{m-1} + q)_i
\]
\[
\leq ||\mathcal{A}||_\infty||u - \hat{x}||_\infty^{m-1}.
\]

Thus, we conclude that
\[
|v'_i| \leq ||u - \hat{x}||_\infty + ||\mathcal{A}||_\infty||u - \hat{x}||_\infty^{m-1}, \forall v_i \leq 0.
\] (32)

**C2** Suppose that \(v'_i > 0\) for an arbitrarily given \(i \in [n]\). Then,
\[
v'_i \leq u_i \quad \text{and} \quad v'_i = [\mathcal{A}(u - \hat{x})^{m-1} + \mathcal{A}\hat{x}^{m-1} + q)_i].
\]
Since \(\hat{x}\) is a solution of the TCP(\(\mathcal{A}\), \(q\)), then \(\hat{x}_i = 0\) or \((\mathcal{A}\hat{x}^{m-1} + q)_i = 0\).

\* If \(\hat{x}_i = 0\), then
\[
|v'_i| = v'_i \leq u_i - \hat{x}_i \leq ||u - \hat{x}||_\infty.
\]
\* If \((\mathcal{A}\hat{x}^{m-1} + q)_i = 0\), then
\[
|v'_i| = v'_i \leq [\mathcal{A}(u - \hat{x})^{m-1} + \mathcal{A}\hat{x}^{m-1} + q)_i - (\mathcal{A}\hat{x}^{m-1} + q)_i
\]
\[
\leq ||\mathcal{A}||_\infty||u - \hat{x}||_\infty^{m-1}.
\]

Therefore,
\[
|v'_i| \leq ||u - \hat{x}||_\infty + ||\mathcal{A}||_\infty||u - \hat{x}||_\infty^{m-1}, \forall v_i > 0.
\] (33)

By cases (C1) and (C2) (i.e., combining (32) and (33)), we have that
\[
||v'||_\infty \leq ||u - \hat{x}||_\infty + ||\mathcal{A}||_\infty||u - \hat{x}||_\infty^{m-1}.
\] (34)

Furthermore, if \(||u - \hat{x}||_\infty \geq 1\), then \(||u - \hat{x}||_\infty \leq ||u - \hat{x}||_\infty^{m-1}(m \geq 2)\), and hence, (34) becomes
\[
||v'||_\infty \leq ||u - \hat{x}||_\infty^{m-1} + ||\mathcal{A}||_\infty||u - \hat{x}||_\infty^{m-1} = (1 + ||\mathcal{A}||_\infty)||u - \hat{x}||_\infty^{m-1},
\]
which implies that
\[
||u - \hat{x}||_\infty \geq \frac{||v'||_\infty^{1/m}}{(1 + ||\mathcal{A}||_\infty)^{\frac{1}{m-1}}};
\] (35)
and if \( \|u - \hat{x}\|_\infty < 1 \), then \( \|u - \hat{x}\|_\infty \geq \|u - \hat{x}\|_\infty^{m-1} (m \geq 2) \), and hence, (34) becomes
\[
\|v'\|_\infty \leq \|u - \hat{x}\|_\infty + \|\mathcal{A}\|_\infty \|u - \hat{x}\|_\infty = (1 + \|\mathcal{A}\|_\infty) \|u - \hat{x}\|_\infty,
\]
which implies that
\[
\|u - \hat{x}\|_\infty \geq \frac{\|v'\|_\infty}{1 + \|\mathcal{A}\|_\infty}.
\]
(36)

Therefore, by combining (35) and (36), we obtain that (23) holds. The proof is complete. \( \square \)

From Theorem 3.4 and Lemma 2.6 we can obtain the following result immediately.

**Corollary 3.** Given \( q \in \mathbb{R}^n \) and \( \mathcal{A} \in \mathbb{T}_{m,n} \) with \( \mathcal{A} \) being a strong \( P \)-tensor. Let \( \hat{x} \) be the unique solution to the TCP(\( \mathcal{A}, q \)). Suppose that \( r_1^\Delta(\cdot) \) and \( r_2^\Delta(\cdot) \) are defined by (20). Then, for any given \( u \in \mathbb{R}^n \),
\[
r_1^\Delta(u) \leq \|u - \hat{x}\|_\infty \leq r_2^\Delta(u).
\]

We have obtained two global error bound results for the TCP(\( \mathcal{A}, q \)). What is the relationship between the obtained results and some known results for the LCP(\( M, q \))?

**Remark 1.** When \( m = 2 \), the tensor \( \mathcal{A} \in \mathbb{T}_{m,n} \) reduces to a matrix, denoted by \( M \). Then, for any \( x \in \mathbb{R}^n \), we have that \( F_{\mathcal{A}}x := (\mathcal{A}x^m - 1)^{\frac{1}{m-1}} = Mx \), and hence,
\[
\alpha(F_{\mathcal{A}}) = \min_{\|x\|_\infty = 1} \max_{i \in [n]} x_i (F_{\mathcal{A}}x)_i = \min_{\|x\|_\infty = 1} \max_{i \in [n]} x_i (Mx)_i = \alpha(M).
\]
In addition, it follows from (8) that
\[
\begin{align*}
v_\Delta(u) &= \min \{u, \mathcal{A}(u - \hat{x})^m + (\mathcal{A}\hat{x}^m + q)^{\frac{1}{m-1}}\} \\
&= \min \{u, M(u - \hat{x}) + M\hat{x} + q\} \\
&= \min \{u, Mu + q\},
\end{align*}
\]
which implies that the residual function
\[
r_\Delta(u) = \|v_\Delta(u)\|_\infty = \|\min \{u, Mu + q\}\|_\infty.
\]
Thus, the result obtained in Theorem 3.2 coincides with the one in Theorem 2.8.

Similarly, when \( m = 2 \), the result obtained in Theorem 3.4 also reduces to the one in Theorem 2.8.

4. **Conclusions.** In this paper, we introduced three residual functions for the TCP(\( \mathcal{A}, q \)), by which we obtained two results of the global error bound for the TCP(\( \mathcal{A}, q \)) with a \( P \)-tensor in terms of two quantities \( \alpha(F_{\mathcal{A}}) \) and \( \alpha(T_{\mathcal{A}}) \). The results we obtained reduce to the one achieved by Mathias and Pang [21] when the TCP(\( \mathcal{A}, q \)) reduces to the LCP(\( M, q \)).

It is well known that the error bound has important application in iterative methods for solving the related optimization problem. How to apply the obtained error bounds to the convergence analysis of iterative methods for solving the TCP(\( \mathcal{A}, q \))? This is a further issue to be studied.

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