The Novel Symmetry Constraint and Binary Nonlinearization of the Super Generalized Broer-Kaup Hierarchy with Self-consistent Sources and Conservation Laws

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Abstract: The super generalized Broer-Kaup (gBK) hierarchy and its super Hamiltonian structure are established based on a loop super Lie algebra and super-trace identity. Then the self-consistent sources, the conservation laws, the novel symmetry constraint and the binary nonlinearization of the super gBK hierarchy are generated, respectively. In addition, the integrals of motion required for Liouville integrability are explicitly given.

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1 Introduction

It is known that super integrable systems provide interesting and important models in the supersymmetry theory. Supersymmetry is originated in 1970s when physicists have proposed simple models with supersymmetric colors in string models and mathematical physics respectively. After that, Wess and Zumino \cite{1} applied supersymmetry to the four-dimensional spacetime. Unfortunately, the supersymmetry partners of any particle have not been found so far, and it is generally believed that this symmetry is spontaneous rupture. In order to unify two kinds

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of particles with different spin and statistical properties—Boson and Fermion, theoretical physicists proposed the concept of hyperspace in the study of unified field theory and quantum field theory. Inspired by this, mathematicians developed the super analysis, the hypergeometric and the super algebra.

Due to the importance of supersymmetry in physics (especially in the exploration of the relationship between supersymmetric conformal field and chord theory), which has captured great attention for the work of super integrable systems associated with Lie super algebra [2], a multitude of classical integrable equations have been extended to be the super completely integrable equations (see Refs [3-11] and references therein). Among those, Hu [12, 13] and Ma [14] has made a great work. In 1990, Hu proposed the super-trace identity in his Ph.D. thesis [12], which is an effective tool to constructing super Hamiltonian structures of super integrable systems. In 2008, Ma given a systematic proof of the super-trace identity and the super double Hamiltonian structure of many super integrable equations is established by using of the super-trace identity (see Refs [14] and references therein).

Soliton equation with self-consistent sources is an important part in soliton theory. They are relevant to some problems related to hydrodynamics, solid state physics, plasma physics, and they are also usually used to describe interactions between different solitary waves, such as the NLS equation with self-consistent sources can describe the propagation of solitary waves in the medium of resonance and non resonant media. It can also describe the interaction between high frequency static wave and ion acoustic wave in plasma [15], the KP equation with self-consistent sources description the interaction of between short wave and length wave spread in the X-Y plane [16], the KdV equation with self-consistent sources description of the interaction of the interaction of plasma high-frequency wave packet weight and a low frequency wave packets [17]. And the conservation laws is also an important part in soliton theory. An infinite number of conservation laws for KdV equation were first discovered by Miura [18] et al. in 1968, and then lots of methods have been developed to find them. This may be mainly due to the contributions of Wadati and others [19-21]. Conservation laws also play an important role in mathematics and engineering as well. Many papers dealing with symmetries and conservation laws were presented. The direct construction method of multipliers for the conservation laws was presented [22]. Thus, the study of integrable equations with self-consistent sources and conservation laws has received much attention. Recently, an army of classical integrable equations to be the super integrable equations, by using symmetry constraints, and the super integrable system with self-consistent sources and conservation laws of the super integrable system are constructed(see Refs[22-30] and references therein).

In recent 10 years, He et al. applied the binary nonlinearization method to the super integrable systems (see Refs [30-35] and references therein) to the construction of finite dimensional super integrable systems from super soliton equations by using symmetry constraints. It is well...
known that a crucial idea in carrying out symmetry constraints is the nonlinearization of Lax pairs for soliton hierarchies. The nonlinearization of Lax pairs can be classified into monononlinearization, which is proposed by Cao [36] and binary nonlinearization, which is proposed by Ma [21,36-39] and has attracted a lot of interest recently [40].

In 2013, Zhang, Han and Tam [41] making use of a Lie algebra and Tu-Ma scheme obtained a new generalized Broer-Kaup (gBK) equation, where the spatial spectral problem is given by

\[ \phi_x = M \phi, \quad M = \begin{pmatrix} -\lambda + \frac{v}{2} & 1 \\ -2w - 2 & \lambda - \frac{v}{2} \end{pmatrix}, \]

where \( v \) and \( w \) are both scalar potentials, \( \lambda \) is the spectral parameter, and in [41], they have presented two kinds of Darboux transformations, the bilinear presentation, the bilinear Bäcklund transformation and the new Lax pair of the gBK equation, respectively, by employing the Bell polynomials. In this paper, we consider the conservation laws, self-consistent sources and the binary nonlinearization of the super gBK hierarchy under the novel symmetry constraint. In addition, under the symmetry constraint, the \( n \)-th flow for a super gBK hierarchy is decomposed into two super finite-dimensional integrable Hamilton systems over the super symmetric manifold.

Organization of this paper. In the next section, we construct the super gBK hierarchy and its super Hamiltonian structure based on a loop super Lie algebra and super-trace identity. In Section 3, we construct the super gBK hierarchy with self-consistent sources. And conservation laws of the super gBK hierarchy are constructed in Section 4. In Section 5, we obtain a symmetry constraint for the potential of the super gBK hierarchy. Then in Section 6, we apply the binary nonlinearization to the super gBK equation hierarchy using a symmetry constraint for the potential of the super gBK hierarchy and obtain a super finite-dimensional integrable Hamiltonian system on the super symmetry manifold, whose integrals of motion are explicitly given. And some conclusions are given in the last Section.

2 The super gBK hierarchy

In this section, we shall construct a super gBK hierarchy starting from a Lie super-algebra. We consider the following spatial spectral problem

\[ \phi_x = M \phi, \quad M = \begin{pmatrix} -\lambda + \frac{v}{2} & 1 & \alpha \\ -2w - 2 & \lambda - \frac{v}{2} & \beta \\ \beta & -\alpha & 0 \end{pmatrix}, \quad \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}, \quad u = \begin{pmatrix} v \\ w \\ \alpha \\ \beta \end{pmatrix}, \]

where \( \lambda \) is the spectral parameter, \( v \) and \( w \) are even potentials, and \( \alpha \) and \( \beta \) are odd potentials.
And associated with the Lie superalgebra $sl(2,1)$. Its basis is

$$
e_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$
e_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \quad e_5 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$  

where $e_1, e_2, e_3$ are even elements and $e_4, e_5$ are odd ones, $[,]$ and $[,]_+$ denote the commutator and the anticommutator, satisfy the following operational relations:

$$[e_1, e_2] = -2e_2, [e_1, e_3] = 2e_3, [e_2, e_3] = -e_1,$$

$$[e_5, e_1] = [e_2, e_4] = e_5, [e_3, e_4] = [e_2, e_5] = 0, [e_3, e_5] = [e_1, e_4] = e_4,$$

$$[e_4, e_4]_+ = -2e_3, [e_5, e_5]_+ = 2e_2, [e_4, e_5]_+ = [e_5, e_4]_+ = e_1. \quad (2.2)$$

Then corresponding loop super algebra is given by

$$sl(2,1) = sl(2,1) \otimes \mathbb{C}[\lambda, \lambda^{-1}],$$

where $\mathbb{C}[\lambda, \lambda^{-1}]$ presents the set of Laurent polynomials in $\lambda$ over the complex number set $\mathbb{C}$.

The corresponding (anti)commutative relations are given as

$$[e_i \lambda^m, e_j \lambda^n] = [e_i, e_j] \lambda^{m+n}, \forall e_i, e_j \in sl(2,1).$$

From the Tu format, we setting

$$N = \begin{pmatrix} A & B & \rho \\ -2B + 2C & -A & \delta \\ \delta & -\rho & 0 \end{pmatrix} = \sum_{m \geq 0} \begin{pmatrix} a_m & b_m & \rho_m \\ -2b_m + 2c_m & -a_m & \delta_m \\ \delta_m & -\rho_m & 0 \end{pmatrix} \lambda^{-m},$$

the corresponding $A, B, C$ are even elements and $\rho, \delta$ are odd elements, if we want to get the super integrable system, we solve the stationary zero curvature equation at first

$$N_x = [M, N]. \quad (2.4)$$

Substituting $M, N$ into Eq.$(2.4)$ and comparing the coefficients of $\lambda^{-m}(m \geq 0)$, we have

$$\begin{cases} (a_{m+1}, -2b_{m+1}, -2\delta_{m+1}, 2\rho_{m+1})^T = \mathcal{L}(a_m, -2b_m, -2\delta_m, 2\rho_m)^T, \\ a_m = \partial^{-1}(2wb_m + 2c_m + \alpha\delta_m + \beta\rho_m). \end{cases} \quad (2.5)$$
Where the recursion operator $\mathcal{L}$ has the following form

$$
\mathcal{L} = \begin{pmatrix}
\frac{1}{2} \partial^{-1} v \partial + \frac{1}{2} \partial & \frac{1}{2} \partial^{-1} w \partial + \frac{1}{2} w + 1 & -\frac{1}{2} \partial^{-1} \alpha \partial + \frac{1}{4} \alpha & -\frac{1}{2} \partial^{-1} \beta \partial - \frac{1}{4} \beta \\
2 & -\frac{1}{2} \partial + \frac{1}{2} v & 0 & \alpha \\
2 \beta - 2 \alpha \partial & -2 \alpha (w + 1) & \partial + \frac{1}{2} v & \alpha \beta - (2 w + 2) \\
-2 \alpha & \beta & -1 & -\partial + \frac{1}{2} v
\end{pmatrix}.
$$

(2.6)

For a given initial value $a_0 = k_0 \neq 0, b_0 = c_0 = \rho_0 = \delta_0 = 0$, the $a_j, b_j, c_j, \rho_j, \delta_j (j \geq 1)$ can be calculated by the recursion relation (2.5). Here we list the several values

$$
\begin{align*}
&a_1 = 0, b_1 = -k_0, c_1 = k_0 w, \rho_1 = -k_0 \alpha, \delta_1 = -k_0 \beta, a_2 = k_0 w - k_0 \alpha \beta, \\
b_2 = -k_0 v, c_2 = \frac{1}{2} k_0 w, \beta_2 = k_0 \alpha x - \frac{1}{2} k_0 v \alpha, \delta_2 = -k_0 \beta x - \frac{1}{2} k_0 v \beta, \\
a_3 = k_0 (\frac{1}{4} w x + v + w v - v \alpha \beta + \alpha x \beta - \alpha \beta x), b_3 = k_0 (\frac{1}{4} v x - w - \frac{1}{4} v^2 - \alpha \alpha x + \alpha \beta), \\
c_3 = \frac{1}{4} k_0 (w x (w v)_x + v w v + w v^2) + \frac{1}{2} k_0 (v x + 2 w^2) - k_0 (w \alpha \beta + \alpha \alpha x - \frac{1}{2} \beta \beta x), \\
\rho_3 = k_0 (-\alpha x + v \alpha x + \frac{1}{2} \alpha v x - \frac{1}{4} \alpha v^2 - \alpha w - \beta x), \\
\delta_3 = k_0 (-\beta x - v \beta x - \frac{1}{2} \beta v x - \frac{1}{4} \beta v^2 - \beta w + (2 w + 2) \alpha x + w x \alpha).
\end{align*}
$$

Then, consider the auxiliary spectral problem associated with the spectral problem (2.1)

$$
\phi_t = N^{(n)} \phi
$$

where

$$
N^{(n)} = N^{(n)}_x + \Delta_n = \sum_{m=0}^n \left( \begin{array}{ccc}
a_m & b_m & \rho_m \\
-2 b_m + 2 c_m & -a_m & \delta_m \\
\delta_m & -\rho_m & 0
\end{array} \right) \lambda^{n-m} + \left( \begin{array}{ccc}
b_{n+1} & 0 & 0 \\
0 & -b_{n+1} & 0 \\
0 & 0 & 0
\end{array} \right).
$$

(2.7)

with $\Delta_n$ being the modification term, substituting Eq. (2.7) into the zero curvature equation

$$
U_t = N^{(n)}_x + [U, N^{(n)}] = 0,
$$

(2.8)

we can obtain the following super gBK hierarchy

$$
\begin{align*}
\begin{pmatrix}
v \\
w \\
\alpha \\
\beta
\end{pmatrix}_t = & \begin{pmatrix}
v \\
w \\
\alpha \\
\beta
\end{pmatrix}
\begin{pmatrix}
2 b_{n+1,x} \\
a_{n+1,x} + \alpha \delta_{n+1} + \beta \rho_{n+1} \\
\alpha \delta_{n+1} + \rho_{n+1} \\
\alpha \delta_{n+1} + \delta_{n+1}
\end{pmatrix} = J
\begin{pmatrix}
a_{n+1} \\
-2 b_{n+1} \\
-2 \delta_{n+1} \\
2 \rho_{n+1}
\end{pmatrix}.
\end{align*}
$$

(2.9)

where the super-Hamiltonian operator $J$ is given by

$$
J = \begin{pmatrix}
0 & -\partial & 0 & 0 \\
-\partial & 0 & -\frac{1}{2} \alpha & \frac{1}{2} \beta \\
0 & -\frac{1}{2} \alpha & 0 & -\frac{1}{2} \\
0 & \frac{1}{2} \beta & -\frac{1}{2} & 0
\end{pmatrix}.
$$

(2.10)
Taking \( k_0 = 2, n = 2, t_2 = t \), the Eq.\((2.9)\) can be reduced to the super gBK equation
\[
\begin{align*}
  v_t &= v_{xx} - 2vv_x - 4w_x - 4\alpha_x + 4\alpha_x^2 + 4\alpha_x\beta + 4\beta_x, \\
  w_t &= -w_{xx} - 2(wv)_x - 2v_x - 2(w + 2\alpha_x + 2(w + 2\alpha_x) - (2w\beta + \frac{1}{2}\beta v^2)(1 + \alpha) - 2\beta_x, \\
  \alpha_t &= 2\alpha_x - 2\alpha_x + 2\alpha_x + 2\alpha_x + 2\alpha_x + \frac{3}{2}\beta_x v_x, \\
  \beta_t &= -2\beta_x + 2\beta_x + 2\beta + 2\alpha_x + 2\alpha_x - \frac{3}{2}\beta v_x.
\end{align*}
\] (2.11)
whose Lax pair are \( M \) and \( N^{(2)} \), \( N^{(2)} \) has the following form
\[
\begin{pmatrix}
  2\lambda - v & -2\lambda - v & 2\lambda + 2\alpha_x - v \\
  4(1 + w)v + 2v(1 + w) - 2\lambda^2 - \frac{1}{2}(v_x - v^2)^2 & 2\alpha_x - 2\beta_x - v\beta \\
  -2\beta - 2\beta_x - v\beta & -2\lambda - 2\alpha_x + v\alpha & 0
\end{pmatrix}
\] (2.12)
when \( \beta = \alpha = 0 \), Eq.\((2.8)\) just reduces to the gBK equation
\[
\begin{align*}
  v_t &= v_{xx} - 2vv_x - 4w_x, \\
  w_t &= -w_{xx} - 2(wv)_x - 2v_x.
\end{align*}
\] (2.13)

Next, we use the super trace identity, which proposed by Hu in \([13]\) and rigorously proved by Ma et al. in ref.\([14]\):
\[
\frac{\delta}{\delta u} \int Str(N\partial M/\partial \lambda) dx = (\lambda^\gamma \partial / \partial \lambda^\gamma) Str(N\partial M/\partial u),
\] (2.14)
where \( Str \) denotes the super trace. It is not difficult to find that
\[
\begin{align*}
  Str(N\partial M/\partial \lambda) &= -2A, Str(N\partial M/\partial v) = A, Str(N\partial M/\partial w) = -2B, \\
  Str(N\partial M/\partial \alpha) &= -2\delta, Str(N\partial M/\partial \beta) = 2\rho,
\end{align*}
\] (2.15)
substituting Eq.\((2.15)\) into Eq.\((2.14)\), and comparing the coefficient of \( \lambda^{-n-1} \) of both sides of Eq.\((2.14)\), we have
\[
\begin{pmatrix}
  \frac{\delta}{\delta v} \\
  \frac{\delta}{\delta w} \\
  \frac{\delta}{\delta \alpha} \\
  \frac{\delta}{\delta \beta}
\end{pmatrix} \int -2a_{n+1} dx = (\gamma - n) \begin{pmatrix} a_n \\ -2b_n \\ -2\delta_n \\ 2\rho_n \end{pmatrix}.
\] (2.16)
To fix the constant \( \gamma \), we set \( n = 1 \) in \((2.16)\) and find that \( \gamma = 0 \), Thus we have
\[
\frac{\delta}{\delta u} \tilde{H}_n = \begin{pmatrix} a_n \\ -2b_n \\ -2\delta_n \\ 2\rho_n \end{pmatrix}, \tilde{H}_n = \int \frac{2a_{n+1}}{n} dx, n \geq 0,
\] (2.17)
specially, by making use of the recursive relationship (2.5), the super gBK equation hierarchy (2.9) possesses the following super bi-Hamiltonian structure

\[ u_{t_n} = J \begin{pmatrix} a_{n+1} \\ -2b_{n+1} \\ -2\delta_{n+1} \\ 2\rho_{n+1} \end{pmatrix} = J \mathcal{L} \begin{pmatrix} a_n \\ -2b_n \\ -2\delta_n \\ 2\rho_n \end{pmatrix} = J \mathcal{L} \frac{\delta \tilde{H}_n}{\delta u}, n \geq 0, \tag{2.18} \]

where the second super-Hamiltonian operator \( R \) is given by

\[ R = J \mathcal{L} = \begin{pmatrix} -2\partial & \frac{1}{2}\partial^2 - \frac{1}{2}\partial v & 0 & \partial \alpha \\ -\frac{1}{2}v \partial - \frac{1}{2}\partial v & -w \partial - \frac{1}{2}\partial w - \partial & R_1 & R_2 \\ 0 & \frac{1}{2} \alpha \partial - \frac{1}{4} v \alpha - \frac{1}{2} \beta & \frac{1}{2} & \frac{1}{2} \beta \partial - \frac{1}{4} v \\ \alpha \partial & -\frac{1}{2} \beta \partial + (w+1)\alpha + \frac{1}{4} v \beta & -\frac{1}{2} \tilde{\partial} - \frac{1}{4} v & (w+1) - \alpha \beta \end{pmatrix} \tag{2.19} \]

with

\[ R_1 = \frac{1}{4} \partial \alpha - \frac{1}{4} v \alpha - \frac{1}{2} \beta, \quad R_2 = \frac{1}{4} \partial \beta + (w+1)\alpha + \frac{1}{4} v \beta. \]

### 3 The super gBK hierarchy with self-consistent sources

In this part, we will construct the super gBK hierarchy with self-consistent sources. At the super-isospectral problem

\[ \phi_x = M\phi, \quad \phi_t = N\phi. \tag{3.1} \]

Let \( \lambda = \lambda_j \), the spectral vector corresponding \( \phi \) remember to \( \phi_j \), we obtain the the linear system as following

\[ \begin{pmatrix} \phi_{1j} \\ \phi_{2j} \\ \phi_{3j} \end{pmatrix}_x = M_j \begin{pmatrix} \phi_{1j} \\ \phi_{2j} \\ \phi_{3j} \end{pmatrix}, \quad \begin{pmatrix} \phi_{1j} \\ \phi_{2j} \\ \phi_{3j} \end{pmatrix}_t = N_j \begin{pmatrix} \phi_{1j} \\ \phi_{2j} \\ \phi_{3j} \end{pmatrix}, \tag{3.2} \]

where \( M_j = M|_{\lambda=\lambda_j}, N_j = N|_{\lambda=\lambda_j}, j = 1,2,...N \). By

\[ \frac{\delta \tilde{H}_n}{\delta u} = \sum_{j=1}^{N} \frac{\delta \lambda_j}{\delta u} = \sum_{j=1}^{N} \begin{pmatrix} \text{Str}(\Psi \frac{\delta M}{\delta v}) \\ \text{Str}(\Psi \frac{\delta M}{\delta w}) \\ \text{Str}(\Psi \frac{\delta M}{\delta \alpha}) \end{pmatrix} = \begin{pmatrix} < \Phi_1, \Phi_2 > \\ 2 < \Phi_1, \Phi_1 > \\ -2 < \Phi_2, \Phi_3 > \\ 2 < \Phi_1, \Phi_3 > \end{pmatrix}, \tag{3.3} \]
where \( \Phi_j = (\phi_{j1}, \cdots, \phi_{jN})^T, j = 1, 2, 3. \) So the super gBK hierarchy with self-consistent sources is proposed

\[
\begin{pmatrix}
    v \\
    w \\
    \alpha \\
    \beta
\end{pmatrix}_t = J
\begin{pmatrix}
    a_n \\
    -2b_n \\
    -2\delta_n \\
    2\rho_n
\end{pmatrix}
+ J
\begin{pmatrix}
    \Phi_1, \Phi_2 > \\
    2 \Phi_1, \Phi_1 > \\
    -2 \Phi_2, \Phi_3 > \\
    2 \Phi_1, \Phi_3 >
\end{pmatrix},
\]

(3.4)

where

\[
\begin{aligned}
\phi_{1j,x} &= \lambda \phi_{1j} + (w - \frac{1}{2}v)\phi_{2j} + \alpha \phi_{3j}, \\
\phi_{2j,x} &= 2v\phi_{1j} - \lambda \phi_{2j} + \beta \phi_{3j}, \\
\phi_{3j,x} &= \beta \phi_{1j} - \alpha \phi_{2j}.
\end{aligned}
\]

(3.5)

For \( n = 2, \) we obtain the super gBK hierarchy with self-consistent sources

\[
\begin{align*}
v_{t2} &= v_{xx} - 2vv_x - 4w_x - 4\alpha \alpha_{xx} + 4\alpha \beta_x + 4\alpha \beta_x - 2\beta \sum_{j=1}^{N} \phi_{1j}^2, \\
w_{t2} &= -w_{xx} - 2(wv)_x - 2v_x + 2(2w + 2)\alpha_x + 2w_x\alpha - (2w\beta + \frac{1}{2}\beta v^2)(1 + \alpha) - 2\beta \beta_x \\
\alpha_{t2} &= 2\alpha_{xx} - 2v\alpha_x + 2\beta_x - \frac{1}{2}\alpha v_x - \alpha \sum_{j=1}^{N} \phi_{1j}^2 - \sum_{j=1}^{N} \phi_{1j}\phi_{3j}, \\
\beta_{t2} &= -2\beta_{xx} - 2v\beta_x + 2\beta_\alpha \alpha_x + 2(2w + 2)\alpha_x + 2v_x - \frac{3}{2}\beta v_x + \beta \sum_{j=1}^{N} \phi_{1j}^2 + \sum_{j=1}^{N} \phi_{2j}\phi_{3j}.
\end{align*}
\]

4 Conservation laws for the super gBK hierarchy

In the following, we will construct conservation laws of the super gBK hierarchy. Introducing two variables

\[
F = \frac{\phi_2}{\phi_1}, \quad G = \frac{\phi_3}{\phi_1}.
\]

(4.1)

So, we have

\[
\begin{align*}
F_x &= -2w - 2 + (2\lambda - v)F + \beta G - F^2 - \alpha FG, \\
G_x &= \beta - \alpha F + (\lambda - \frac{1}{2}v)G - GF - \alpha G^2.
\end{align*}
\]

(4.2)
We expand $F, G$ in powers of $\lambda^{-1}$ as follows

$$F = \sum_{j=1}^{\infty} f_j \lambda^{-j}, \quad G = \sum_{j=1}^{\infty} g_j \lambda^{-j}. \quad \text{(4.4)}$$

Substituting Eq. (4.4) into Eq. (4.2), (4.3) and comparing the coefficients of the same power of $\lambda$, we obtain

$$\lambda^0: \quad f_1 = 1 + w, \quad g_1 = -\beta,$$

$$\lambda^{-1}: \quad f_2 = \frac{1}{2} f_{1x} + \frac{1}{2} v f_1 - \frac{1}{2} \beta g_1 = \frac{1}{2} w_x + \frac{1}{2} v(1 + w),$$

$$g_2 = g_{1x} + \alpha f_1 + \frac{1}{2} v g_1 = -\beta x + \alpha(1 + w) - \frac{1}{2} v \beta,$$

$$\lambda^{-2}: \quad f_3 = \frac{1}{2} f_{2x} + \frac{1}{2} v f_2 - \frac{1}{2} \beta g_2 + \frac{1}{2} f_1^2 + \frac{1}{2} \alpha f_1 g_1$$

$$= \frac{1}{4} w_{xx} + \frac{1}{8} w_x^2 + \frac{1}{2} v w_x + \frac{1}{4} (1 + w)(w_x v + v_x) + \frac{1}{8} v^2 (1 + w)(3 + w) + \frac{1}{2} \beta \beta_x,$$

$$g_3 = g_{2x} + \alpha f_2 + \frac{1}{2} v g_2 + f_1 g_1 + \alpha g_1^2$$

$$= -\beta_{xx} - v \alpha_x + (\alpha_x + \alpha v - \beta)(1 + w) + \frac{3}{2} \alpha w_x - \frac{1}{2} v_x \beta - \frac{1}{4} v^2 \beta.$$

and the recursion formulas for $f_n$ and $g_n$ are given

$$f_{n+1} = \frac{1}{2} f_{nx} + \frac{1}{2} v f_n - \frac{1}{2} \beta g_n + \frac{1}{2} \sum_{l=1}^{n} f_l f_{n-l} + \frac{1}{2} \alpha \sum_{l=1}^{n} f_l g_{n-l}, \quad \text{(4.5)}$$

$$g_{n+1} = g_{nx} + \alpha f_n + \frac{1}{2} v g_n + \sum_{l=1}^{n} f_l g_{n-l}. \quad \text{(4.6)}$$

Because of linear spectral problems Eq. (3.1)

$$(ln \phi_1)_x = -\lambda + \frac{1}{2} v + F + \alpha G, \quad \text{(4.7)}$$

$$(ln \phi_1)_t = A + BF + \rho G. \quad \text{(4.8)}$$

It is easy to calculate that

$$\frac{\partial}{\partial t} ((-\lambda + \frac{1}{2} v + F + \alpha G) = \frac{\partial}{\partial x} (A + BF + \rho G). \quad \text{(4.9)}$$

where

$$A = k_0(\lambda^2 + w - \alpha \beta), \quad B = -k_0(\lambda + \frac{1}{2} v), \quad \rho = -k_0(\alpha \lambda - \alpha_x + \frac{1}{2} v \alpha). \quad \text{(4.10)}$$

In order to obtain the conservation laws for super integrable hierarchy, we define

$$\sigma = -\lambda + \frac{1}{2} v + F + \alpha G, \quad \theta = A + BF + \rho G.$$
Then the Eq. (4.9) can be rewritten as \( \sigma_t = \theta_x \), which is just the formal definition of conservation laws. We expand \( \sigma \) and \( \theta \) as series in powers of \( \lambda \) with the coefficients, which are called conserved densities and fluxes respectively

\[
\sigma = -\lambda + \frac{1}{2}v + \sum_{j=1}^{\infty} \sigma_j \lambda^{-j},
\]

\[
\theta = k_0 \lambda^2 - 2 + \sum_{j=1}^{\infty} \theta_j \lambda^{-j}.
\] (4.11)

The first of the conservation of density and flow

\[
\sigma_1 = f_1 + \alpha g_1 = w + 1 - \alpha \beta,
\]

\[
\theta_1 = k_0[-f_2 - \frac{1}{2}vf_1 - \alpha g_2 + (\alpha_x - \frac{1}{2}v\alpha)g_1]
\]

\[
= k_0[-\frac{1}{2}w_x - v(1 + w) + \alpha \beta - \alpha_x \beta + v\alpha \beta].
\]

With the help of Eq. (4.9), (4.10), (4.11), the recursion relation for \( \sigma_n \) and \( \theta_n \) are given

\[
\sigma_n = f_n + \alpha g_n,
\]

\[
\theta_n = k_0[-f_{n+1} - \frac{1}{2}vf_n - \alpha g_{n+1} + (\alpha_x - \frac{1}{2}v\alpha)g_n].
\]

where \( f_n \) and \( g_n \) can be calculated from Eq. (4.5) and Eq. (4.6).

5. The novel symmetry constraint

In order to compute a symmetry constraint, we consider the spectral problem in Eq.(2.1) and its adjoint spectral problem

\[
\phi_x = \begin{pmatrix} \lambda - \frac{v}{2} & 2w + 2 & \beta \\ -1 & -\lambda + \frac{v}{2} & -\alpha \\ -\alpha & -\beta & 0 \end{pmatrix} \psi, \quad \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}. \] (5.1)

where “St ” means the super transposition. The following result is a general formula for the variational derivative with respect to the potential \( u \) (see [21] for the classical case).

Lemma 5.1 (see [21, 32]) Let \( M(u, \lambda) \) be an even matrix of order \( m+n \) depending on \( u, u_x, u_{xx}, \ldots \) and a parameter \( \lambda \). Suppose that \( \phi = (\phi_e, \phi_o)^T \) and \( \psi = (\psi_e, \psi_o)^T \) satisfy the spectral problem and the adjoint spectral problem

\[
\phi_x = M(u, \lambda)\phi, \quad \psi_x = -M(u, \lambda)^{St}\psi, \] (5.2)
where \( \phi_e = (\phi_1, \cdots, \phi_m) \) and \( \psi_e = (\psi_1, \cdots, \psi_m) \) are even eigenfunctions, and \( \phi_o = (\phi_{m+1}, \cdots, \phi_{m+n}) \) and \( \psi_o = (\psi_{m+1}, \cdots, \psi_{m+n}) \) are odd eigenfunctions. Then the variational derivative of the parameter \( \lambda \) with respect to the potential \( u \) is given by

\[
\frac{\delta \lambda}{\delta u} = \frac{(\psi_e, (-1)^p(u)\psi_o)(\partial M/\partial u)\phi}{-\int \psi^T(\partial M/\partial u)\phi dx},
\]

where we denote

\[
p(v) = \begin{cases} 0, & v \text{ is an even variable,} \\ 1, & v \text{ is an odd variable.} \end{cases}
\]

By Lemma 5.1, it’s easy to get the variational derivative of the spectral parameter \( \lambda \) with respect to the potential \( u \)

\[
\frac{\delta \lambda}{\delta u} = \left( \begin{array}{c} \frac{\delta \lambda}{\delta v} \\ \frac{\delta \lambda}{\delta w} \\ \frac{\delta \lambda}{\delta \alpha} \\ \frac{\delta \lambda}{\delta \beta} \end{array} \right) = \frac{1}{E} \left( \begin{array}{c} \frac{1}{2}(\psi_1 \phi_1 - \psi_2 \phi_2) \\ -2\psi_2 \phi_1 \\ \psi_3 \phi_2 + \psi_1 \phi_3 \\ \psi_2 \phi_3 = -\psi_3 \phi_1 \end{array} \right),
\]

where \( E = \int_{-\infty}^{\infty} (\psi_1 \phi_1 - \psi_2 \phi_2) dx \). When zero boundary conditions \( \lim_{|x| \to \infty} \phi = \lim_{|x| \to \infty} \psi = 0 \) are imposed, we can verify a simple characteristic property of the variational derivative

\[
\mathcal{L} \frac{\delta \lambda}{\delta u} = \lambda \frac{\delta \lambda}{\delta u},
\]

where \( \mathcal{L} \) is defined in (2.6). Consider the spatial system

\[
\begin{aligned}
\begin{cases}
\phi_{1j,x} = \begin{pmatrix} -\lambda + \frac{v}{2} & 1 & \alpha \\ -2w - 2 & \lambda - \frac{v}{2} & \beta \\ \beta & -\alpha & 0 \end{pmatrix} \phi_{1j}, \\
\phi_{2j,x} = \begin{pmatrix} \lambda - \frac{v}{2} & 2w + 2 & \beta \\ -1 & -\lambda + \frac{v}{2} & -\alpha \\ -\alpha & -\beta & 0 \end{pmatrix} \phi_{2j}, \\
\psi_{1j,x} = \begin{pmatrix} -\lambda + \frac{v}{2} & 1 & \alpha \\ -2w - 2 & \lambda - \frac{v}{2} & \beta \\ \beta & -\alpha & 0 \end{pmatrix} \psi_{1j}, \\
\psi_{2j,x} = \begin{pmatrix} \lambda - \frac{v}{2} & 2w + 2 & \beta \\ -1 & -\lambda + \frac{v}{2} & -\alpha \\ -\alpha & -\beta & 0 \end{pmatrix} \psi_{2j}.
\end{cases}
\end{aligned}
\]
and the temporal system

\[
\begin{pmatrix}
\phi_{1j} \\
\phi_{2j} \\
\phi_{3j}
\end{pmatrix}_t = 
\begin{pmatrix}
\sum_{i=0}^{n} a_i \lambda_j^{n-i} + b_{n+1} & \sum_{i=0}^{n} b_i \lambda_j^{n-i} & \sum_{i=0}^{n} \rho_i \lambda_j^{n-i} \\
\sum_{i=0}^{n} (-2b_i + 2c_i) \lambda_j^{n-i} & -\sum_{i=0}^{n} a_i \lambda_j^{n-i} - b_{n+1} & \sum_{i=0}^{n} \delta_i \lambda_j^{n-i} \\
\sum_{i=0}^{n} \delta_i \lambda_j^{n-i} & -\sum_{i=0}^{n} \rho_i \lambda_j^{n-i} & 0
\end{pmatrix}
\begin{pmatrix}
\phi_{1j} \\
\phi_{2j} \\
\phi_{3j}
\end{pmatrix},
\]

where \(\{\lambda_j, j = 1, 2, \cdots, N\}\) are \(N\) distinct eigenparameters, \(\{\phi_j\}\) and \(\{\psi_j\}\) are corresponding eigenfunctions and adjoint eigenfunctions, \(1 \leq j \leq N\). Now for Eq.\((5.7)\) and Eq.\((5.8)\), we have the following symmetry constraints:

\[
\frac{\delta \bar{H}_k}{\delta u} = \sum_{j=1}^{N} \gamma_j \frac{\delta \lambda_j}{\delta u}, \quad k \geq 0,
\]

where letting \(\gamma_j = E_j = \int_{-\infty}^{\infty} (\phi_{1j} \psi_{1j} - \phi_{2j} \psi_{2j}) dx\). If letting initial value \(a_0 = k_0 = 1\), and seek \(k = 1\), we have the following novel symmetry constraint

\[
\begin{cases}
w - \alpha \beta = \frac{1}{2} (\langle \Psi_1, \Phi_1 \rangle - \langle \Psi_2, \Phi_2 \rangle), \\
v = -2(\langle \Psi_2, \Phi_1 \rangle), \\
-2\beta x - v \beta = \langle \Psi_3, \Phi_2 \rangle + \langle \Psi_1, \Phi_3 \rangle, \\
-2\alpha x + v \alpha = \langle \Psi_2, \Phi_3 \rangle - \langle \Psi_3, \Phi_1 \rangle.
\end{cases}
\]

where we use the following notation

\[
\Psi_i = (\psi_{i1}, \cdots, \psi_{iN})^T, \quad \Phi_i = (\phi_{i1}, \cdots, \phi_{iN})^T, \quad i = 1, 2, 3,
\]

and \(\langle \cdot, \cdot \rangle\) denotes the standard inner product of the Euclidian space \(R^N\). We find that the even potentials \(v\) and \(w\) can be explicitly expressed by eigenfunctions, but the odd potentials \(\alpha\) and \(\beta\) cannot. So the symmetry constraint \((5.10)\) is called a novel constraint.
6 The Binary nonlinearization

In this part, we introduce the following new independent odd variables as a result of the odd potentials $\alpha$ and $\beta$ cannot be explicitly expressed by eigenfunctions

$$
\phi_{N+1} = \alpha, \quad \psi_{N+1} = 2\beta.
$$

(6.1)

Considering the new variables of Eq. (6.1) and substituting Eq. (5.10) in to Eq. (5.7), we have the following finite-dimensional super system

$$
\begin{align*}
\phi_{1j,x} &= (-\lambda_j - \langle \psi_2, \Phi_1 \rangle)\phi_{1j} + \phi_{2j} + \phi_{N+1}\phi_{3j}, \\
\phi_{2j,x} &= (-\langle \psi_1, \Phi_1 \rangle - \langle \psi_2, \Phi_2 \rangle + \phi_{N+1}\psi_{N+1} + 2)\phi_{1j} + (\lambda_j + \langle \psi_2, \Phi_1 \rangle)\phi_{2j} + \frac{1}{2}\psi_{N+1}\phi_{3j}, \\
\phi_{3j,x} &= \frac{1}{2}\psi_{N+1}\phi_{1j} - \phi_{N+1}\phi_{2j}, \\
\phi_{N+1,x} &= -\frac{1}{2}(\langle \psi_2, \Phi_3 \rangle - \langle \psi_3, \Phi_1 \rangle) - \langle \psi_2, \Phi_1 \rangle\phi_{N+1}, \\
\psi_{1,j,x} &= (\lambda_j + \langle \psi_2, \Phi_1 \rangle)\psi_{1,j} + (\langle \psi_1, \Phi_1 \rangle - \langle \psi_2, \Phi_2 \rangle + \phi_{N+1}\psi_{N+1} + 2)\psi_{2,j} + \frac{1}{2}\psi_{N+1}\psi_{3,j}, \\
\psi_{2j,x} &= -\psi_{1j} + (-\lambda_j - \langle \psi_2, \Phi_1 \rangle)\psi_{2j} - \phi_{N+1}\psi_{3j}, \\
\psi_{3j,x} &= -\phi_{N+1}\psi_{1j} - \frac{1}{2}\psi_{N+1}\psi_{2j}, \\
\psi_{N+1,x} &= -(\langle \psi_3, \Phi_2 \rangle + \langle \psi_1, \Phi_3 \rangle) + \langle \psi_2, \Phi_1 \rangle\psi_{N+1}.
\end{align*}
$$

(6.2)

where $\Lambda = \text{diag}(\lambda_1, \lambda_2, \cdots, \lambda_N)$, obviously, the system (6.2) can be written to the following Hamiltonian form

$$
\begin{align*}
\Phi_{1,x} = \frac{\partial H_1}{\partial \Phi_1}, \quad \Phi_{2,x} = \frac{\partial H_1}{\partial \Phi_2}, \quad \Phi_{3,x} = \frac{\partial H_1}{\partial \Phi_3}, \quad \Phi_{N+1,x} = \frac{\partial H_1}{\partial \Phi_{N+1}}, \\
\Psi_{1,x} = -\frac{\partial H_1}{\partial \Psi_1}, \quad \Psi_{2,x} = -\frac{\partial H_1}{\partial \Psi_2}, \quad \Psi_{3,x} = -\frac{\partial H_1}{\partial \Psi_3}, \quad \Psi_{N+1,x} = -\frac{\partial H_1}{\partial \Psi_{N+1}}.
\end{align*}
$$

(6.3)

where

$$
H_1 = -\langle \Lambda \Phi_1, \Phi_1 \rangle + \langle \Lambda \Phi_2, \Phi_2 \rangle - 2\langle \Phi_2, \Phi_1 \rangle + \langle \Phi_1, \Phi_2 \rangle \\
- \langle \Phi_2, \Phi_1 \rangle(\langle \psi_1, \Phi_1 \rangle - \langle \psi_2, \Phi_2 \rangle) - \phi_{N+1}\psi_{N+1}(\langle \psi_2, \Phi_1 \rangle) \\
+ \phi_{N+1}(\langle \psi_3, \Phi_2 \rangle + \langle \psi_1, \Phi_3 \rangle) + \frac{1}{2}\psi_{N+1}(\langle \psi_2, \Phi_3 \rangle - \langle \psi_3, \Phi_1 \rangle).
$$

As for the $t_2$-part, substituting the symmetry constraint (5.10) into system (5.8), we obtain the following finite-dimensional system

$$
\begin{align*}
\phi_{1j,t_2} &= (\lambda^2 + \frac{1}{4}(\bar{v}_x - \bar{v})^2 - \bar{\alpha}\bar{\alpha}_x)\phi_{1j} + (-\lambda - \frac{1}{2}\bar{v})\phi_{2j} + (-\bar{\alpha}\bar{\lambda} + \bar{\alpha}_x - \frac{1}{2}\bar{v}\bar{\alpha})\phi_{3j}, \\
\phi_{2j,t_2} &= (2(1 + \bar{w})\lambda + \bar{w}_x + \bar{v}(1 + \bar{w}))\phi_{1j} + (-\lambda^2 - \frac{1}{4}(\bar{v}_x - \bar{v})^2 - \bar{\alpha}\bar{\alpha}_x)\phi_{2j} \\
&+ (-\bar{\beta}\bar{\lambda} - \bar{\beta}_x - \frac{1}{2}\bar{v}\bar{\beta})\phi_{3j}, \\
\phi_{3j,t_2} &= (-\bar{\beta}\bar{\lambda} - \bar{\beta}_x - \frac{1}{2}\bar{v}\bar{\beta})\phi_{1j} + (\bar{\alpha}\bar{\lambda} + \bar{\alpha}_x - \frac{1}{2}\bar{v}\bar{\alpha})\phi_{2j}, \\
\psi_{1,j,t_2} &= (-\lambda^2 - \frac{1}{4}(\bar{v}_x - \bar{v})^2 + \bar{\alpha}\bar{\alpha}_x)\psi_{1,j} + (-2(1 + \bar{w})\lambda + \bar{w}_x - \bar{v}(1 + \bar{w}))\psi_{2j} \\
&+ (-\bar{\beta}\bar{\lambda} - \bar{\beta}_x - \frac{1}{2}\bar{v}\bar{\beta})\psi_{3j}, \\
\psi_{2j,t_2} &= (\lambda + \frac{1}{2}\bar{v})\psi_{1j} + (\lambda^2 + \frac{1}{4}(\bar{v}_x - \bar{v})^2 - \bar{\alpha}\bar{\alpha}_x)\psi_{2j} + (\bar{\alpha}\bar{\lambda} - \bar{\alpha}_x - \frac{1}{2}\bar{v}\bar{\alpha})\psi_{3j}, \\
\psi_{3j,t_2} &= (\bar{\alpha}\bar{\lambda} - \bar{\alpha}_x - \frac{1}{2}\bar{v}\bar{\alpha})\psi_{1j} + (\bar{\beta}\bar{\lambda} + \bar{\beta}_x + \frac{1}{2}\bar{v}\bar{\beta})\psi_{2j},
\end{align*}
$$

(6.4)
where \( \tilde{v}, \tilde{w}, \tilde{\alpha}, \tilde{\beta} \) denote the functions \( v, w, \alpha, \beta \) defined by the symmetry constraint given in Eq. (5.10) and \( \tilde{v}_x, \tilde{w}_x, \tilde{\alpha}_x, \tilde{\beta}_x \) are given by the following identities:

\[
\tilde{v}_x = 4\langle \Lambda \Psi_2, \Phi_1 \rangle + 4\langle \Psi_2, \Phi_1 \rangle^2 + 2(\langle \Psi_1, \Phi_1 \rangle - \langle \Psi_2, \Phi_2 \rangle) - 2\phi_{N+1}(\langle \Psi_2, \Phi_3 \rangle - \langle \Psi_3, \Phi_1 \rangle),
\]

\[
\tilde{w}_x = \langle \Psi_1, \Phi_2 \rangle + (\langle \Psi_1, \Phi_1 \rangle - \langle \Psi_2, \Phi_2 \rangle + \phi_{N+1}\psi_{N+1}) + 2\langle \Psi_2, \Phi_1 \rangle,
\]

\[
\tilde{\alpha}_x = -\frac{1}{2}(\langle \Psi_2, \Phi_3 \rangle - \langle \Psi_3, \Phi_1 \rangle) - \langle \Psi_2, \Phi_1 \rangle\phi_{N+1},
\]

\[
\tilde{\beta}_x = -\frac{1}{2}(\langle \Psi_3, \Phi_2 \rangle + \langle \Psi_1, \Phi_3 \rangle) + \frac{1}{2}\langle \Psi_2, \Phi_1 \rangle\psi_{N+1}.
\]

Considering the new variables in Eq. (6.1) and the symmetry constraint given in Eq. (5.10), the above finite-dimensional super system given in Eq. (6.4) becomes the following finite-dimensional system

\[
\begin{align*}
\phi_{1j,t_2} &= \left[\lambda^2 + \langle \Lambda \Psi_2, \Phi_1 \rangle + \frac{1}{2}(\langle \Psi_1, \Phi_1 \rangle - \langle \Psi_2, \Phi_2 \rangle)\right]\phi_{1j} \\
&\quad + (-\lambda + \langle \Psi_2, \Phi_1 \rangle)\phi_{2j} + [-\phi_{N+1}\lambda - \frac{1}{2}(\langle \Psi_2, \Phi_3 \rangle - \langle \Psi_3, \Phi_1 \rangle)]\phi_{3j}, \\
\phi_{2j,t_2} &= \left[2 + (\langle \Psi_1, \Phi_1 \rangle - \langle \Psi_2, \Phi_2 \rangle) + \phi_{N+1}\psi_{N+1}\right]\lambda + \langle \Psi_1, \Phi_2 \rangle\phi_{1j} \\
&\quad + [-\lambda^2 - \langle \Lambda \Psi_2, \Phi_1 \rangle - \frac{1}{2}(\langle \Psi_1, \Phi_1 \rangle - \langle \Psi_2, \Phi_2 \rangle)]\phi_{2j} \\
&\quad + \left[-\frac{1}{2}\psi_{N+1}\lambda + \frac{1}{2}(\langle \Psi_3, \Phi_2 \rangle + \langle \Psi_4, \Phi_3 \rangle)\right]\phi_{3j}, \\
\phi_{3j,t_2} &= [-\frac{1}{2}\psi_{N+1}\lambda + \frac{1}{2}(\langle \Psi_3, \Phi_2 \rangle + \langle \Psi_4, \Phi_3 \rangle)]\phi_{1j} + \phi_{N+1}\lambda + \frac{1}{2}(\langle \Psi_2, \Phi_3 \rangle - \langle \Psi_3, \Phi_1 \rangle)\phi_{2j}, \\
\phi_{N+1,t_2} &= -\frac{1}{2}(\langle \Lambda \Psi_2, \Phi_3 \rangle - \langle \Lambda \Psi_3, \Phi_1 \rangle) - \phi_{N+1}\langle \Lambda \Psi_2, \Phi_1 \rangle, \\
\psi_{1j,t_2} &= [-\lambda^2 - \langle \Lambda \Psi_2, \Phi_1 \rangle - \frac{1}{2}(\langle \Psi_1, \Phi_1 \rangle - \langle \Psi_2, \Phi_2 \rangle)]\psi_{1j} \\
&\quad + \left[-2 - (\langle \Psi_1, \Phi_1 \rangle + \langle \Psi_2, \Phi_2 \rangle) + \phi_{N+1}\psi_{N+1}\right]\lambda - \langle \Psi_1, \Phi_2 \rangle\psi_{2j} \\
&\quad + \left[-\frac{1}{2}\psi_{N+1}\lambda + \frac{1}{2}(\langle \Psi_3, \Phi_2 \rangle + \langle \Psi_4, \Phi_3 \rangle)\right]\psi_{3j}, \\
\psi_{2j,t_2} &= (\lambda - \langle \Psi_2, \Phi_1 \rangle)\psi_{1j} + \langle \lambda^2 + \langle \Lambda \Psi_2, \Phi_1 \rangle + \frac{1}{2}(\langle \Psi_1, \Phi_1 \rangle - \langle \Psi_2, \Phi_2 \rangle)\rangle\psi_{2j} \\
&\quad + \left[\phi_{N+1}\lambda + \frac{1}{2}(\langle \Psi_2, \Phi_3 \rangle - \langle \Psi_3, \Phi_1 \rangle)\right]\psi_{3j}, \\
\psi_{3j,t_2} &= \phi_{N+1}\lambda + \frac{1}{2}(\langle \Psi_3, \Phi_2 \rangle - \langle \Psi_3, \Phi_1 \rangle)\psi_{1j} + \phi_{N+1}\lambda - \frac{1}{2}(\langle \Psi_3, \Phi_2 \rangle + \langle \Psi_1, \Phi_3 \rangle)\psi_{2j}, \\
\psi_{N+1,t_2} &= -(\langle \Lambda \Psi_2, \Phi_3 \rangle + \langle \Lambda \Psi_1, \Phi_3 \rangle) + \psi_{N+1}\langle \Lambda \Psi_2, \Phi_1 \rangle.
\end{align*}
\]

By a direct but tedious calculation, the finite-dimensional system (6.5) become to the following Hamiltonian form

\[
\begin{align*}
\Phi_{1,t_2} &= \frac{\partial H_2}{\partial \Phi_2}, \quad \Phi_{2,t_2} = \frac{\partial H_2}{\partial \Phi_3}, \quad \Phi_{3,t_2} = \frac{\partial H_2}{\partial \Phi_4}, \quad \Phi_{N+1,t_2} = \frac{\partial H_2}{\partial \Phi_{N+1}}, \\
\Psi_{1,t_2} &= -\frac{\partial H_2}{\partial \Phi_1}, \quad \Psi_{2,t_2} = -\frac{\partial H_2}{\partial \Phi_2}, \quad \Psi_{3,t_2} = \frac{\partial H_2}{\partial \Phi_3}, \quad \Psi_{N+1,t_2} = \frac{\partial H_2}{\partial \Phi_{N+1}},
\end{align*}
\]

where the Hamilton function is

\[
H_2 = \langle \Lambda^2 \Psi_1, \Phi_1 \rangle - \langle \Lambda^2 \Psi_2, \Phi_2 \rangle + \langle \Lambda \Psi_2, \Phi_1 \rangle(\langle \Psi_1, \Phi_1 \rangle - \langle \Psi_2, \Phi_2 \rangle) + 2\langle \Lambda \Psi_2, \Phi_1 \rangle \\
- \langle \Lambda \Psi_1, \Phi_2 \rangle + \langle \Psi_2, \Phi_1 \rangle(\langle \Psi_1, \Phi_1 \rangle - \langle \Psi_2, \Phi_2 \rangle) - \phi_{N+1}(\langle \Lambda \Psi_1, \Phi_3 \rangle + \langle \Lambda \Psi_3, \Phi_2 \rangle) - \frac{1}{2}(\langle \Psi_2, \Phi_3 \rangle \\
- \langle \Psi_3, \Phi_1 \rangle)(\langle \Psi_3, \Phi_2 \rangle + \langle \Psi_1, \Phi_3 \rangle) + \frac{1}{2}(\langle \Lambda \Psi_2, \Phi_3 \rangle - \langle \Lambda \Psi_3, \Phi_1 \rangle)\psi_{N+1}
\]
\[ + \phi_{N+1} \psi_{N+1} \langle \Lambda \Psi_2, \Phi_1 \rangle + \frac{1}{4} (\langle \Psi_1, \Phi_1 \rangle - \langle \Psi_2, \Phi_2 \rangle)^2. \]

In the following, we prove that for any \( n \geq 2 \), the super system given in Eq. (6.8) can be nonlinearized and furthermore, the obtained nonlinearized system is a finite-dimensional super-Hamiltonian system. Therefore, making use of Eq. (6.6) and the recursion relation (2.3), yields

\[
\begin{align*}
\tilde{a}_{m+1} &= \frac{1}{2} (\langle \Lambda^{m-1} \Psi_1, \Phi_1 \rangle - \langle \Lambda^{m-1} \Psi_2, \Phi_2 \rangle), \quad m \geq 1, \\
\tilde{b}_{m+1} &= \langle \Lambda^{m-1} \Psi_2, \Phi_1 \rangle, \quad m \geq 1, \\
\tilde{c}_{m+1} &= \langle \Lambda^{m-1} \Psi_2, \Phi_1 \rangle + \frac{1}{2} \langle \Lambda^{m-1} \Psi_1, \Phi_2 \rangle, \quad m \geq 1, \\
\tilde{\rho}_{m+1} &= -\frac{1}{2} (\langle \Lambda^{m-1} \Psi_2, \Phi_3 \rangle - \langle \Lambda^{m-1} \Psi_3, \Phi_1 \rangle), \quad m \geq 1, \\
\tilde{\delta}_{m+1} &= \frac{1}{2} (\langle \Lambda^{m-1} \Psi_3, \Phi_2 \rangle + \langle \Lambda^{m-1} \Psi_1, \Phi_3 \rangle), \quad m \geq 1,
\end{align*}
\]

where \( \Lambda = \text{diag}(\lambda_1, \lambda_2, \cdots, \lambda_N) \). Substituting Eq. (6.7) into Eq. (6.8), we have

\[
\left\{ \begin{array}{l}
\phi_{1j,t_n} = \left( \sum_{i=0}^n \tilde{a}_i \lambda_j^{n-i} + \tilde{b}_{n+1} \right) \phi_{1j} + \left( \sum_{i=0}^n \tilde{b}_i \lambda_j^{n-i} \right) \phi_{2j} + \left( \sum_{i=0}^n \tilde{\rho}_i \lambda_j^{n-i} \right) \phi_{3j}, \\
\phi_{2j,t_n} = \left( \sum_{i=0}^n (-2\tilde{b}_i + 2\tilde{c}_i) \lambda_j^{n-i} \right) \phi_{1j} + \left( \sum_{i=0}^n \tilde{a}_i \lambda_j^{n-i} - \tilde{b}_{n+1} \right) \phi_{2j} + \left( \sum_{i=0}^n \tilde{\delta}_i \lambda_j^{n-i} \right) \phi_{3j}, \\
\phi_{3j,t_n} = \left( \sum_{i=0}^n \tilde{d}_i \lambda_j^{n-i} \right) \phi_{1j} - \left( \sum_{i=0}^n \tilde{\rho}_i \lambda_j^{n-i} \right) \phi_{2j}, \\
\phi_{N+1,t_n} = \frac{1}{2} (\langle \Lambda^{n-1} \Psi_2, \Phi_3 \rangle - \langle \Lambda^{n-1} \Psi_3, \Phi_1 \rangle) - \phi_{N+1} \langle \Lambda^{n-1} \Psi_2, \Phi_1 \rangle, \\
\psi_{1j,t_n} = \left( \sum_{i=0}^n \tilde{a}_i \lambda_j^{n-i} \right) \psi_{1j} + \left( \sum_{i=0}^n \tilde{b}_i \lambda_j^{n-i} + \tilde{b}_{n+1} \right) \psi_{2j} + \left( \sum_{i=0}^n \tilde{\delta}_i \lambda_j^{n-i} \right) \psi_{3j}, \\
\psi_{2j,t_n} = \left( \sum_{i=0}^n \tilde{b}_i \lambda_j^{n-i} \right) \psi_{1j} + \left( \sum_{i=0}^n \tilde{a}_i \lambda_j^{n-i} + \tilde{b}_{n+1} \right) \psi_{2j} + \left( \sum_{i=0}^n \tilde{\rho}_i \lambda_j^{n-i} \right) \psi_{3j}, \\
\psi_{3j,t_n} = \left( \sum_{i=0}^n \tilde{d}_i \lambda_j^{n-i} \right) \psi_{1j} - \left( \sum_{i=0}^n \tilde{\delta}_i \lambda_j^{n-i} \right) \psi_{2j}, \\
\psi_{N+1,t_n} = -\langle \Lambda^{n-1} \Psi_3, \Phi_2 \rangle + \langle \Lambda^{n-1} \Psi_1, \Phi_3 \rangle + \psi_{N+1} \langle \Lambda^{n-1} \Psi_2, \Phi_1 \rangle.
\end{array} \right.
\]

Next, we show that the nonlinearized super system given in Eq. (6.8) is a finite-dimensional super-Hamiltonian system. Under the constraint (5.10), the identity \( \langle N \rangle_x = [\tilde{M}, \tilde{N}] \) and \( (\tilde{N}^2)_x = [\tilde{M}, \tilde{N}^2] \) are still satisfied, we have

\[
F_x = \left( \frac{1}{2} \text{Str} \tilde{N}^2 \right)_x = \frac{d}{dx} \left( \bar{a}^2 - 2\bar{b}^2 + 2\bar{b} \bar{c} + 2\bar{p} \bar{d} \right) = 0.
\]

The identity indicates that \( F \) is a generating function of integrals of motion for the nonlinearized spatial systems (6.2). Let \( F = \sum_{n \geq 0} F_n \lambda^{-n} \), and we obtain the following formulas:

\[
F_m = \sum_{i=0}^m (\bar{a}_i \bar{a}_{m-i} - 2\bar{b}_i \bar{b}_{m-i} + 2\bar{b}_i \bar{c}_{m-i} + 2\bar{p}_i \bar{d}_{m-i}).
\]
Assume that $\tilde{a}_0 = 1, \tilde{b}_0 = \tilde{c}_0 = \tilde{\rho}_0 = \tilde{\delta}_0 = 0$ and from the above relations, we find

$$F_0 = 1, \quad F_1 = 2\tilde{a}_1 = 0, \quad F_2 = -2,$$

$$F_3 = \langle \Lambda \Psi_1, \Phi_1 \rangle - \langle \Lambda \Psi_2, \Phi_2 \rangle + 4\langle \Lambda^2 \Psi_2, \Phi_1 \rangle - 2\langle \Psi_2, \Phi_1 \rangle - \langle \Psi_1, \Phi_2 \rangle + \langle \Psi_2, \Phi_1 \rangle - \langle \Psi_2, \Phi_2 \rangle + \phi_{N+1}\psi_{N+1} - \phi_{N+1}(\langle \Lambda \Psi_1, \Phi_3 \rangle + \langle \Lambda \Psi_3, \Phi_2 \rangle) - \frac{1}{2}(\langle \Psi_2, \Phi_3 \rangle - \langle \Psi_3, \Phi_2 \rangle) + \phi_{N+1}\psi_{N+1}\langle \Lambda \Psi_2, \Phi_1 \rangle + \frac{1}{4}(\langle \Psi_1, \Phi_1 \rangle - \langle \Psi_2, \Phi_2 \rangle)^2.$$

$$F_m = \sum_{i=2}^{m-2} (\tilde{a}_i\tilde{a}_{m-i} - 2\tilde{b}_i\tilde{b}_{m-i} + 2\tilde{b}_i\tilde{c}_{m-i} + 2\tilde{\rho}_i\tilde{d}_{m-i} + 2\tilde{\rho}_0\tilde{a}_m - 4\tilde{b}_1\tilde{b}_{m-1} + 2\tilde{b}_1\tilde{c}_{m-1} + 2\tilde{b}_{m-1}\tilde{c}_1 + 2\tilde{\rho}_{m-1}\tilde{d}_1) = \langle \Lambda^{m-2} \Psi_1, \Phi_1 \rangle - \langle \Lambda^{m-2} \Psi_2, \Phi_2 \rangle + 2\langle \Lambda^{m-3} \Psi_2, \Phi_1 \rangle - \langle \Lambda^{m-3} \Psi_1, \Phi_2 \rangle + \langle \Lambda^{m-3} \Psi_2, \Phi_1 \rangle(\langle \psi_1, \Phi_1 \rangle - \langle \psi_2, \Phi_2 \rangle + \phi_{N+1}\psi_{N+1}) - \phi_{N+1}(\langle \Lambda^{m-3} \Psi_3, \Phi_2 \rangle + \langle \Lambda^{m-3} \Psi_1, \Phi_3 \rangle + \frac{1}{2}\psi_{N+1}(\langle \Lambda^{m-3} \Psi_2, \Phi_3 \rangle - \langle \Lambda^{m-3} \Psi_3, \Phi_1 \rangle) + \sum_{i=2}^{m-2} \frac{1}{4}(\langle \Lambda^{i-2} \Psi_1, \Phi_1 \rangle - \langle \Lambda^{i-2} \Psi_2, \Phi_2 \rangle)(\langle \Lambda^{m-i-2} \Psi_1, \Phi_1 \rangle - \langle \Lambda^{m-i-2} \Psi_2, \Phi_2 \rangle) + \langle \Lambda^{i-2} \Psi_2, \Phi_1 \rangle\langle \Lambda^{m-i-2} \Psi_1, \Phi_2 \rangle - \langle \Lambda^{i-2} \Psi_2, \Phi_3 \rangle + \langle \Lambda^{i-2} \Psi_1, \Phi_1 \rangle)\right].$$

The constrained temporal part of the super gBK equation hierarchy \((5.8)\) can be rewritten to the following Hamilton form

$$\begin{align*}
\Phi_{1,t_n} &= \frac{\partial F_{n+2}}{\partial \psi_1}, \quad \Phi_{2,t_n} = \frac{\partial F_{n+2}}{\partial \psi_2}, \quad \Phi_{3,t_n} = \frac{\partial F_{n+2}}{\partial \psi_3}, \quad \phi_{N+1,t_n} = \frac{\partial F_{n+2}}{\partial \phi_{N+1}}, \\
\psi_{1,t_n} &= -\frac{\partial F_{n+2}}{\partial \psi_1}, \quad \psi_{2,t_n} = -\frac{\partial F_{n+2}}{\partial \psi_2}, \quad \psi_{3,t_n} = -\frac{\partial F_{n+2}}{\partial \psi_3}, \quad \psi_{N+1,t_n} = -\frac{\partial F_{n+2}}{\partial \phi_{N+1}},
\end{align*}$$

\[(6.11)\]

i.e., for any \(n\), the constrained temporal part of the super gBK equation hierarchy \((5.8)\) is the finite-dimensional super Hamilton hierarchy. As an calculation example, we have the following equation:

$$\Phi_{1,t_n} = \sum_{i=1}^{n} \tilde{a}_i\Lambda^{n-i} + \tilde{b}_{n+1}\phi_1 + \sum_{i=1}^{n} \tilde{b}_i\Lambda^{n-i}\phi_2 + \sum_{i=1}^{n} \tilde{\rho}_i\Lambda^{n-i}\phi_3 = \Lambda^n\phi_1 + \langle \Lambda^{n-1} \psi_2, \phi_1 \rangle - \Lambda^{n-1}\phi_2 - \phi_{N+1}\Lambda^{n-1}\phi_3 - \frac{1}{2} \sum_{i=2}^{n} (\langle \Lambda^{i-2} \psi_1, \phi_1 \rangle - \langle \Lambda^{i-2} \psi_2, \phi_2 \rangle)\Lambda^{n-i}\phi_1$$

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\[ + \sum_{i=2}^{n} \langle \Lambda^{i-2} \Psi_2, \Phi_1 \rangle \Lambda^{n-i} \Phi_2 - \frac{1}{2} \sum_{i=2}^{n} \langle \langle \Lambda^{i-2} \Psi_2, \Phi_3 \rangle - \langle \Lambda^{i-2} \Psi_3, \Phi_1 \rangle \rangle \Lambda^{n-i} \Phi_3 \]
\[ + \langle \langle \Lambda^{i-1} \Psi_2, \Phi_3 \rangle - \langle \Lambda^{i-1} \Psi_3, \Phi_1 \rangle \rangle \Lambda^{n-i} \Phi_3 = \frac{\partial F_{n+2}}{\partial \Psi_1}. \]

It is not difficult to see that \( F_n(n \geq 0) \) are also integrals of motion for the temporal system (5.8), i.e.,
\[ \{ F_m, F_{n+2} \} = \frac{\partial}{\partial t_n} F_m = 0. \quad m, n \geq 0, \] (6.12)
where the Poisson bracket is defined by
\[ \{ f, g \} = \sum_{i=1}^{3} \sum_{i=1}^{N} \left( \frac{\partial f}{\partial \phi_{ij}} \frac{\partial g}{\partial \psi_{ij}} - (-1)^{p(\phi_{ij})p(\psi_{ij})} \frac{\partial f}{\partial \psi_{ij}} \frac{\partial g}{\partial \phi_{ij}} \right) + \frac{\partial f}{\partial \phi_{N+1}} \frac{\partial g}{\partial \psi_{N+1}} + \frac{\partial f}{\partial \psi_{N+1}} \frac{\partial g}{\partial \phi_{N+1}}. \] (6.13)

It is natural for us to set
\[ f_k = \phi_{1k} \psi_{1k} + \phi_{2k} \psi_{2k} + \phi_{3k} \psi_{3k}, \quad 1 \leq k \leq N, \] (6.14)
and verify they are also integrals of motion of the constrained systems (6.2) and (5.8). In the same way with (6.1), we can prove the independence of \( \{ f_k \}_{k=1}^{N}, \{ F_k \}_{k=2}^{2N+3} \). So the following theorem holds true.

**Theorem 6.1** Both the spatial systems (5.7) and the temporal systems (5.8) under the symmetry constraint (5.10) become completely finite-dimensional integrable Hamiltonian systems in the Liouville sense.

## 7 Conclusions and discussions

Starting from Lie super algebras, we may get super equation hierarchy. With the help of variational identity, the Hamiltonian structure can also be presented. Based on Lie super algebra, the self-consistent sources of super gBK hierarchy can be obtained. It enriched the content of self-consistent sources of super soliton hierarchy. In addition, we also get the conservation laws of the super gBK hierarchy. Finally, we have applied the binary nonlinearization method to the super gBK hierarchy by the symmetry constraint (5.10). It provides a new and systematic way to construct a finite-dimensional super Hamiltonian system. The methods in this study can be applied to other super soliton hierarchy to get more super hierarchies with self-consistent. And under constraints of this form, the nonlinearization of the other super soliton hierarchy will be studied in our future work.
Remark 7.1 With the development of soliton theory, super integrable systems associated with Lie super algebra have been paid growing attention, many classical integrable equations have been extended to be the super completely integrable equations. However, according to the essays now available, scholars studied either the self-consistent sources and the conservation laws or the symmetry constraint and the binary nonlinearization of super integrable systems. So far, only author aired one of his essays [30] in which he studied both theories mentioned above. And in this paper, there is a difference is that the super generalized Broer-Kaup equation hierarchy have a novel symmetry constraint compared with paper [30].

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