Ground state energy of the $\delta$-Bose and Fermi gas at weak coupling from double extrapolation

Sylvain Prolhac
Laboratoire de Physique Théorique, IRSAMC, UPS, Université de Toulouse, CNRS, Toulouse, France
E-mail: sylvain.prolhac@irsamc.ups-tlse.fr

Received 14 October 2016, revised 20 December 2016
Accepted for publication 3 February 2017
Published 7 March 2017

Abstract
We consider the ground state energy of the Lieb–Liniger gas with $\delta$ interaction in the weak coupling regime $\gamma \to 0$. For bosons with repulsive interaction, previous studies gave the expansion $e_B(\gamma) \approx 4\gamma^{3/2}/3\pi + (1/6 - 1/\pi^2)\gamma^2$. Using a numerical solution of the Lieb–Liniger integral equation discretized with $M$ points and finite strength $\gamma$ of the interaction, we obtain very accurate numerics for the next orders after extrapolation on $M$ and $\gamma$. The coefficient of $\gamma^{5/2}$ in the expansion is found to be approximately equal to $-0.001 587 699 865 505 944 989 29$, accurate within all digits shown. This value is supported by a numerical solution of the Bethe equations with $N$ particles, followed by extrapolation on $N$ and $\gamma$. It was identified as $(3\zeta(3)/8 - 1/2)/\pi^3$ by G Lang. The next two coefficients are also guessed from the numerics. For balanced spin $1/2$ fermions with attractive interaction, the best result so far for the ground state energy has been $e_F(\gamma) \approx \pi^2/12 - \gamma/2 + 3\gamma^3/6$. An analogue double extrapolation scheme leads to the value $-\zeta(3)/\pi^3$ for the coefficient of $\gamma^3$.

Keywords: $\delta$-Bose gas, $\delta$-Fermi gas, ground state energy, Richardson extrapolation

(Some figures may appear in colour only in the online journal)
1. Introduction

The Lieb–Liniger model [1] is a one-dimensional quantum integrable model describing $N$ particles with contact interaction. The Hamiltonian of the model is

$$H_N = -\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + 2c \sum_{1 \leq j < k \leq N} \delta(x_j - x_k).$$

(1)

Particles are assumed to move in a circle of length $L$, thus $x_j \equiv x_j + L$. Motivated by recent mathematical results [2–4] in the weak coupling regime $|c| \ll 1$, we focus on two distinct cases: bosons with repulsive interaction, and spin $1/2$ fermions with attractive interaction. For bosons, the Lieb–Liniger model has found recent applications in the context of ultra-cold atomic gases [5–8]. For fermions, the model has been studied in the context of the Bose–Einstein condensation (BEC) of Bardeen–Cooper–Schrieffer (BCS) bound pairs [9, 10].

The Lieb–Liniger model is exactly solvable by the Bethe ansatz. For bosons, the ground state energy can be written as $E_B = \sum_{j=1}^{N} q_j^2$, with the $q_j$’s solution of the Bethe equations

$$i\gamma_j - \frac{1}{L} \sum_{k=1}^{N} \log \left( \frac{q_j - q_k + ic}{q_j - q_k - ic} \right) = \frac{2\pi}{L} \left( j - \frac{N + 1}{2} \right).$$

(2)

At large $L$, $N$ with a fixed density of particles $\rho = N/L$, the rescaled eigenvalue $e_B(\gamma) = E_B/\rho^2N$ with attractive interaction $c > 0$ is known to depend on $L$, $N$ and $c$ only through the reduced coupling constant $\gamma = c/\rho$. One then has $e_B(\gamma) = \frac{\gamma^2}{\kappa_B} \int_{-1}^{1} dy \gamma^2 g_B(y)$, where $g_B$ is the solution of the Lieb–Liniger integral equation [1]

$$f_B(x) = \frac{1}{2\pi} + \frac{\kappa_B}{\pi} \int_{-1}^{1} dy \frac{f_B(y)}{(y-x)^2 + \kappa_B^2},$$

(3)

and $\kappa_B/\gamma = \int_{-1}^{1} dy f_B(y)$.

For spin $1/2$ fermions, a nested version of the Bethe ansatz has to be used [11] in order to compute the ground state energy $E_F$. At large $L$, $N$ in the sector with a total spin of zero, expressions similar to the ones in the bosonic case exist for the rescaled eigenvalue $e_F(\gamma) = \frac{\gamma^2}{4} + E_F/\rho^2N$ with repulsive interaction $c < 0$. In terms of the reduced coupling constant $\gamma = |c|/\rho$, one has $e_F(\gamma) = \frac{\gamma^2}{\kappa_F} \int_{-1}^{1} dy \gamma^2 g_F(y)$, where $g_F$ is the solution of the Gaudin integral equation [12]

$$f_F(x) = \frac{2}{\pi} - \frac{\kappa_F}{\pi} \int_{-1}^{1} dy \frac{f_F(y)}{(y-x)^2 + \kappa_F^2},$$

(4)

and $\kappa_F/\gamma = \int_{-1}^{1} dy f_F(y)$.

At strong coupling $\gamma \gg 1$, the integral equation (3) can be solved by inverting the integral operator $\mathcal{K}_B$ [2], as the convergent expansion $f_B(x) = \sum_{m=0}^{\infty} (\mathcal{K}_B \cdot g)(x)$ with $(\mathcal{K} \cdot g)(x) = \frac{\kappa_B}{\pi} \int_{-1}^{1} \frac{d\gamma}{(y-x)^2 + \kappa_B^2}$. Since $(\mathcal{K}_B \cdot \frac{1}{2\pi})(x)$ is of order $\kappa^{-m}$ when $\kappa \to \infty$, the large $\kappa$ expansion of $f_B(x)$ can be obtained systematically using a computer algebra system by truncating the sum over $m$ and performing explicitly the polynomial integrations resulting from the large $\kappa$ expansion of the kernel. This leads to a large $\gamma$ expansion of $e_B(\gamma)$, where the coefficient of $\gamma^{-m}$, $m \gg 1$, is a polynomial in $\pi^2$ of order $[m/2]$. The beginning of the expansion
is \( e_B(\gamma) \approx \frac{\pi^2}{3} - \frac{4\pi^2}{3\gamma^2} + \frac{4\pi^2}{\gamma^4} + \ldots \), see figure 1 for a plot of the expansion up to order \( \gamma^{-30} \), and \([13, 14]\) for a recent study of the structure of the large \( \gamma \) expansion. Similarly, in the fermionic case, one finds \( e_F(\gamma) \approx \frac{\pi^2}{48} + \frac{\pi^2}{48\gamma^2} + \frac{\pi^2}{64\gamma^4} + \ldots \) instead, see figure 2 for a plot of the expansion up to order \( \gamma^{-30} \), and \([15, 16]\) for studies of the large \( \gamma \) expansion in relation to the BCS–BEC crossover.

Weak coupling \( \gamma \ll 1 \) is much harder. In the case of bosons with repulsive interaction, analytical and numerical studies point to an expansion in powers of \( \gamma^{1/2} \), known exactly only up to order \( \gamma^2 \) \([2, 17, 18]\):

\[
e_B(\gamma) \approx -\frac{4\gamma^{3/2}}{3\pi} + \left( \frac{1}{6} - \frac{1}{\pi^2} \right) \gamma^2 + a\gamma^{5/2} + b\gamma^3 + c\gamma^{7/2} + \ldots.
\]

Previous numerical evaluations provided the values \( a \approx 0.001 \), \( b \approx 0.0018 \) \([17]\) and \( a \approx -0.001 \), \( b \approx 0.00588 \) \([20]\). In the spin 1/2 Fermi case with attractive interaction, analytical studies indicate an expansion with integer powers of \( \gamma \), known exactly only up to order \( \gamma^2 \) \([3, 4, 21]\):

\[
e_F(\gamma) \approx \frac{\pi^2}{12} - \frac{\gamma}{2} + \frac{\gamma^2}{6} + d\gamma^3 + \ldots.
\]

Our main results are very precise numerical values for \( a, b, c \) and \( d \) obtained using the implementation of the Richardson extrapolation \([22]\), using balanced rational functions, known as the Bulirsch–Stoer method \([23, 24]\). This method allows us to extract the limit \( h_0 = h(0) \) of a function \( h(z) \) with a series expansion \( h(z) \approx \sum_{k=0}^{\infty} h_k e^{ik} \) near \( z = 0 \), knowing only a few values \( h(z_n) \) and \( 0 < z_n < \ldots < z \) with high numerical precision. The method
provides a numerically stable evaluation at \( z = 0 \) of the unique interpolating rational function 
\[
R(z) = \frac{P(z)}{Q(z)}
\]
with the \( P \) and \( Q \) polynomials of respective degrees \( -\frac{n}{12} \) and \( \frac{n}{2} \), such that
\[
\sum_{i=0}^{m} e^i(\gamma) = R(z)
\]
is reduced to the extrapolated value \( R(0) \) by iteratively building the triangle \( h^p_m \) with the recursion formula
\[
h^p_m = h^p_{m+1} + \frac{h^p_{m+1} - h^p_m}{(\frac{z_{m+1}}{z_{m+1} + 1})^{1/\theta} - 1},
\]
and boundary conditions \( h^0_0 = 0 \), \( m = 1, \ldots, n + 1 \) and \( h^0_m = h(z_{m}) \), \( m = 1, \ldots, n \), leading to \( h^0_0 = R(0) \). A useful estimator for the order of magnitude of the error is
\[
\epsilon = \left| h^n_1 - h^{n-1}_1 \right| + \left| h^n_2 - h^{n-1}_2 \right| + \left| h^{n-1}_1 - h^{n-1}_2 \right|.
\]

The Richardson extrapolation can, for instance, provide very precise solutions of ordinary differential equations using the modified midpoint approximation scheme \[25\] with 2\( M \) steps, which is known to produce an expansion in \( 1/M \) with the exponent \( \theta = 2 \). We believe that the method is also very useful for extracting the thermodynamic limit of observables in quantum integrable models, where one typically finds clean expansions in the inverse of the system size with simple, easy to guess exponents (see \[26\] for an application to the asymmetric exclusion process, which is an exactly solvable lattice model with classical hard-core particles hopping in a preferred direction).

In this paper, we studied the ground state energy of the \( \delta \)-Bose gas at weak coupling using a numerical solution of either the Bethe equations (2), see section 2, or the integral equation (3), see section 3. Using the Richardson extrapolation, we were able to check the beginning of the
Table 1. The result of the extrapolation for the ground state energy $e_m(\gamma) \approx \sum_{m=0}^{\infty} e_m^B \gamma^{m+1/2}$ of the $\delta$-Bose gas at weak coupling from a numerical solution of the Bethe equations for a finite number of particles. The values are truncated at the estimation of the error $\varepsilon$, see below (7). For $m = 0, \ldots, 7$ they are accurate to the number of digits displayed. For $m \geq 8$ they are accurate only to the number of digits underlined.

| $m$ | $e_m^B = \lim_{\gamma \to 0} \frac{1}{\gamma^{m/2}} \left( e_m(\gamma) - \sum_{l=0}^{m-1} e_l^B \gamma^{l+1/2} \right)$ | Relative error from the exact value |
|-----|-------------------------------------------------|----------------------------------|
| 0   | $1.000 \, 000 \, 000 \, 000 \, 000 \, 000 \, 000 \approx 1$ | $< 10^{-20}$ |
| 1   | $-0.424 \, 413 \, 181 \, 578 \, 387 \, 562 \, 05 \approx -4/3\pi$ | $< 10^{-20}$ |
| 2   | $0.065 \, 345 \, 483 \, 024 \, 328 \, 895 \, 2 \approx 1/6 - 1/\pi^2$ | $< 10^{-19}$ |
| 3   | $-0.001 \, 587 \, 699 \, 865 \, 505 \, 944 \, 99 \approx a$ | $< 10^{-18}$ |
| 4   | $-0.000 \, 168 \, 460 \, 187 \, 827 \, 739 \, 0 \approx b$ | $< 10^{-16}$ |
| 5   | $-0.000 \, 020 \, 864 \, 973 \, 358 \, 402 \approx c$ | $< 10^{-15}$ |
| 6   | $-3.163 \, 214 \, 218 \, 537 \times 10^{-6}$ | |
| 7   | $-6.106 \, 860 \, 595 \, 67 \times 10^{-7}$ | |
| 8   | $-1.484 \, 034 \, 672 \, 650 \times 10^{-7}$ | |
| 9   | $-4.346 \, 096 \, 246 \, 46 \times 10^{-8}$ | |
| 10  | $-1.474 \, 253 \, 312 \times 10^{-8}$ | |

expansion (5) with very high precision by subtracting the first terms and dividing them by the appropriate power of $\gamma$, see tables 1 and 2. Our best result for $a$, coming from the method with the integral equation, is

$$a \approx -0.001 \, 587 \, 699 \, 865 \, 505 \, 944 \, 989 \, 29.$$  (8)

The numerical value has been truncated at the estimator for the error $\varepsilon$. Experience with several other extrapolation calculations, where exact values were also available for comparison, indicates that it is very likely that all the digits displayed in (8) are correct, except the last one, which may be off by 1 due to rounding. The approximate value (8) was then identified as

$$a = \left( \frac{3\zeta(3)}{8} - \frac{1}{2} \right) \frac{1}{\pi^3}$$  (9)

by Lang [27] shortly after the first version of the paper appeared on the arXiv.

The coefficient $b$ of $e(\gamma)$ can similarly be obtained by the extrapolation method after subtracting (5) with the exact value (9). We find

$$b \approx -0.000 \, 168 \, 460 \, 187 \, 827 \, 739 \, 035 \, 45,$$  (10)

which is roughly in agreement with the result $b \approx -0.000 \, 171$ of [20]. By analogy with (9), we sought an exact expression for $\pi^2 b$ as a linear combination of 1, $\zeta(2) = \pi^2/6$, $\zeta(3)$ and $\zeta(4) = \pi^4/90$, with rational coefficients using an integer relations algorithm. We found

$$b = \left( \frac{\zeta(3)}{8} - \frac{1}{6} \right) \frac{1}{\pi^3},$$  (11)

which is in perfect agreement with the numerical value. Iterating again, the next term is numerically equal to

$$c \approx -0.000 \, 020 \, 864 \, 973 \, 358 \, 401 \, 7408,$$  (12)

for which an exact value is presumably

J. Phys. A: Math. Theor. 50 (2017) 144001
A natural conjecture is that $\pi^m$ times the coefficient of $\gamma^{1 + m/2}$ is a linear combination of 1, $\zeta(2)$, ..., $\zeta(m)$ with rational coefficients. In this regard, the absence of even zetas for $a$, $b$, and $c$ is slightly odd.

Numerical values of the next coefficients of $e_B(\gamma)$ can be obtained by iterating the procedure further. It seems, however, difficult to guess the other exact numbers as the precision of the numerical values decreases at each step while the requirement for finding exact numbers increases due to the number of zetas involved in the linear combination. We note that using non-exact values in the subtraction quickly degrades the quality of both the extrapolated result and the estimation for the error (which is typically underestimated) as numerical errors accumulate.

We also applied the extrapolation procedure in the Fermi case, starting with the integral equation (4). Again, we were able to check (6) with high precision, see table 3. We found the numerical value for the coefficient $d$

$$d \approx -0.0123402948369572,$$

which agrees perfectly with the exact number

$$d = -\frac{\zeta(3)}{\pi^2}.$$  

The numerics for the next coefficients are displayed in table 3. We were not able to determine any other exact values for them, due to the relatively small number of digits known. A smaller number of coefficients are accessible with the same amount of computational effort in the Fermi case compared to the Bose case. This is a consequence of the fact that the expansion is in integer powers of $\gamma$ for the Fermi gas instead of $\gamma^{1/2}$ for the Bose gas, and thus requires more accurate values before extrapolating on $\gamma$.  

**Table 2.** The result of the double extrapolation for the ground state energy $e_B(\gamma) \approx \sum_{m=0}^{\infty} e_m \gamma^{1 + m/2}$ of the repulsive $\delta$-Bose gas at weak coupling from a numerical solution of the Lieb–Liniger integral equation (3). The values are truncated at the estimation of the error $\varepsilon$, see below (7). For $m = 0, \ldots, 6$ they are accurate to the number of digits displayed. For $m \geq 7$ they are roughly accurate to the number of digits underlined.

| $m$ | $e_m^B$ | Relative error from the exact value |
|-----|---------|-----------------------------------|
| 0   | 1.000 000 000 000 000 000 000 001 ≈ 1 | $< 10^{-24}$ |
| 1   | -0.424 413 181 578 387 562 050 4 ≈ $-4/3\pi$ | $< 10^{-22}$ |
| 2   | 0.065 345 483 024 328 895 222 79 ≈ $1/6 - 1/\pi^2$ | $< 10^{-21}$ |
| 3   | -0.001 587 699 865 505 944 989 29 ≈ $a$ | $< 10^{-20}$ |
| 4   | -0.000 168 460 187 827 739 035 45 ≈ $b$ | $< 10^{-18}$ |
| 5   | -0.000 020 864 973 358 401 740 8 ≈ $c$ | $< 10^{-18}$ |
| 6   | -3.163 214 218 537 366 62 × 10^{-6} | |
| 7   | -6.106 860 595 675 02 × 10^{-7} | |
| 8   | -1.484 034 672 618 67 × 10^{-7} | |
| 9   | -4.346 096 253 710 042 × 10^{-8} | |
| 10  | -1.474 253 194 402 432 2 × 10^{-8} | |

$$c = \left( -\frac{45\zeta(5)}{1024} + \frac{15\zeta(3)}{256} - \frac{1}{32} \right) \frac{1}{\pi^2}. \quad (13)$$
Motivation for the present work came from the similar numerical extrapolations performed in [26] for the spectral gap of a weakly asymmetric exclusion process, which is an exactly solvable lattice model featuring classical hard-core particles hopping in a slightly preferred direction between the neighbouring sites of a one-dimensional lattice. In this case too, it was possible to guess the exact expressions for the coefficients from the extrapolated numerical values.

2. Double extrapolation from the Bethe equations at a finite number of particles

In this section, we provide independent support for the numerical values (8), (10) and (12) obtained with higher precision in section 3, by solving the Bethe equations (2) numerically and extrapolating on the number of particles.

Using Newton’s method, we obtain numerical values of the Bethe roots \( q_j \) and \( j = 1, \ldots, N \) satisfying the Bethe equations (2) by solving iteratively the linear system

\[
\sum_{k=1}^{N} (q_k^{\text{new}} - q_k) \left[ i \delta_{j,k} + \frac{1}{L} \left( \frac{1}{q_j - q_k + i c} - \frac{1}{q_j - q_k - i c} \right) \right] = \frac{1}{L} \sum_{\ell=1}^{N} \left( \frac{1}{q_j - q_\ell + i c} - \frac{1}{q_j - q_\ell - i c} \right) \delta_{j,k}
\]

for \( q_k^{\text{new}} \) and \( k = 1, \ldots, N \). We start with the known solution for \( Lc \to 0 \) with fixed \( N \), where \( q_j \) and \( j = 1, \ldots, N \) are given in terms of the roots of the \( N \)th Hermite polynomial: \( H_N(\sqrt{2c}q_j/L) = 0 \) with \( H_N(x) = (-1)^N e^{x^2} \partial_N^N e^{-x^2} \). The calculations were done at density \( \rho = 1 \), with 1000 significant digits, for all integers \( N \) between 1 and \( N_{\text{max}}(\gamma) \), and for the 100 values \( \gamma = 1, 0.99, \ldots, 0.02, 0.01 \). The chosen integers \( N_{\text{max}}(\gamma) \) increase from 77 to 308 as \( \gamma \) decreases from 1 to 0.01, in order to maintain a comparable precision for all \( \gamma \) after the extrapolation on the number of particles.

Table 3. The result of the double extrapolation for the ground state energy \( e^\gamma_{\text{new}} = \lim_{\gamma \to 0} \frac{1}{\pi} (\epsilon_\gamma(\gamma) - \sum_{j=0}^{m} \frac{1}{j!} \gamma^j) \) from a numerical solution of the Lieb–Liniger integral equation (4).

| \( m \) | \( e^\gamma_{\text{new}} \) | Relative error from the exact value |
|-------|-----------------|----------------------------------|
| 0     | 0.822 467 033 424 113 22 \( \approx \pi^2/12 \) | \( < 10^{-18} \) |
| 1     | -0.500 000 000 000 000 \( \approx -1/2 \) | \( < 10^{-18} \) |
| 2     | 0.166 666 666 666 666 \( \approx 1/6 \) | \( < 10^{-17} \) |
| 3     | -0.012 340 294 836 957 \( \approx -\zeta(3)/\pi^4 \) | \( < 10^{-15} \) |
| 4     | -0.001 875 499 919 064 |                    |
| 5     | -0.000 380 055 743 439 |                    |
| 6     | -0.000 121 746 636 000 |                    |
For each value of $\gamma$, the limit $N \to \infty$ is reached by extrapolating on $1/N$ with the exponent $\theta = 2$, chosen among simple values to minimize the estimation for the error. The limit $\gamma \to 0$ is then obtained by extrapolating on $\gamma$, with the exponent $\theta = 1/2$ imposed by the form of the weak coupling expansion. The results are summarized in table 1. Perfect agreement is found with the known analytical results.

3. Double extrapolation from the integral equations

In this section, we explain the computations leading to (8), (10) and (12) starting with the solution of a discretized version of the Lieb–Liniger integral equation (3) and extrapolating on the number of points of discretization. We also treat the fermionic case from the Gaudin integral equation (4).

In the bosonic case, we discretize the Lieb–Liniger integral equation (3) for several values of $\kappa$ by replacing the function $f_0$ by the list of $M$ values $f(x_m)$ with $x_m = -1 + 2(m - 1)/M$ and $m = 1, \ldots, M$. The integral in (3) is replaced by a trapezoidal approximation. The solution of the discretized integral equation is then computed with 1000 digits by solving for $f(x_m)$ and $\kappa_{\text{max}}$ for each value of $\kappa$. Extrapolating on $M$ with exponent $\theta = 1$ leads to accurate values for $\gamma_B$ with $\gamma$, given implicitly in terms of $\kappa_{\text{max}}$.

This process is repeated with $M = 2, 3, \ldots, M_{\text{max}}(\kappa)$ for each value of $\kappa$. Extrapolating on $M$ with exponent $\theta = 1$ leads to accurate values for $\gamma_B(\gamma)$ with $\gamma$, given implicitly in terms of $\kappa_{\text{max}}$. This is done for the 100 values $\kappa = 1, 0.99, \ldots, 0.02, 0.01$, with $M_{\text{max}}(\kappa)$ increasing from 71 to 803 as $\kappa \to 0$ (which corresponds to $\gamma \to 0$). Extrapolation on $\kappa$ with exponent $\theta = 1/2$ then leads to the results in table 2. Again, perfect agreement is found with known analytical results. The weak coupling expansion is plotted from table 2 in figure 1.

A similar double extrapolation scheme is used for the Fermi gas starting from the integral equation (4). The extrapolation on $\kappa$ is done for the 100 values $\kappa = 1, 0.99, \ldots, 0.02, 0.01$, with $M_{\text{max}}(\kappa)$ increasing from 67 to 771 as $\kappa \to 0$. The biggest difference is that the exponent $\theta = 1$ is used for the extrapolation on $\kappa$ due to the form of the expansion (6). The results, given in table 3, are in complete agreement with (6). The weak coupling expansion is plotted from table 3 in figure 2.

4. Conclusions

In this paper, we computed accurate values for the coefficients of the ground state energies $\gamma_B(\gamma)$ and $\gamma_F(\gamma)$ of the $\delta$-Bose gas and spin 1/2 $\delta$-Fermi gas in the weak coupling expansion using the Richardson extrapolation. The efficiency of the extrapolation method for this
problem, measured by the very small value for the estimator of the error, is a sign that $\epsilon_B(\gamma)$ and $\epsilon_F(\gamma)$ have clean expansions, respectively, in powers of $\gamma^{1/2}$ and $\gamma$. In particular, it is a good indication for the absence of a sub-leading logarithm in the expansions, since logarithms are known to completely spoil the precision of the Richardson extrapolation. This is in agreement with the exact calculations [2, 4], showing the cancellation of logarithms appearing in intermediate quantities when computing the first few terms of the weak coupling expansion of the ground state energy, but a proof at all orders is still missing.

The very precise numerical results of this paper are another example of the spectacular effectiveness of the Richardson extrapolation. For integrable models, it can provide very accurate numerics in the thermodynamic limit knowing only relatively few finite size values, thanks to the existence of clean expansions with simple exponents combined with efficient ways to compute accurate finite size values for a moderately large number of degrees of freedom. In some cases, such precise numerics even allow us to guess the exact corresponding numbers.

Acknowledgments

I thank G Lang for allowing me to use his conjecture for the exact value of $a$ and C Tracy for suggesting I apply the Richardson extrapolation to the Fermi case too. I also thank the hospitality of H Spohn at TU München, where part of this paper was written.

References

[1] Lieb E H and Liniger W 1963 Exact analysis of an interacting Bose gas. I. The general solution and the ground state Phys. Rev. 130 1605–16
[2] Tracy C A and Widom H 2016 On the ground state energy of the $\delta$-function Bose gas J. Phys. A: Math. Theor. 49 294001
[3] Tracy C A and Widom H 2016 On the ground state energy of the delta-function Fermi gas J. Math. Phys. 57 103301
[4] Tracy C A and Widom H 2016 On the ground state energy of the delta-function Fermi gas II: further asymptotics (arXiv:1609.07793)
[5] Bloch I, Dalibard J and Zwerger W 2008 Many-body physics with ultracold gases Rev. Mod. Phys. 80 885
[6] Cazalilla M A, Citro R, Giamarchi T, Orignac E and Rigol M 2011 One dimensional bosons: from condensed matter systems to ultracold gases Rev. Mod. Phys. 83 1405
[7] Yu-Zhu J, Yang-Yang C and Xi-Wen G 2015 Understanding many-body physics in one dimension from the Lieb–Liniger model Chin. Phys. B 24 050311
[8] Batchelor M T and Foerster A 2016 Yang–Baxter integrable models in experiments: from condensed matter to ultracold atoms J. Phys. A: Math. Theor. 49 173001
[9] Giorgini S, Pitaevskii L P and Stringari S 2008 Theory of ultracold atomic Fermi gases Rev. Mod. Phys. 80 1215
[10] Guan X-W, Batchelor M T and Lee C 2013 Fermi gases in one dimension: from Bethe ansatz to experiments Rev. Mod. Phys. 85 1633
[11] Yang C N 1967 Some exact results for the many-body problem in one dimension with repulsive delta-function interaction Phys. Rev. Lett. 19 1312
[12] Gaudin M 1967 Un système à une dimension de fermions en interaction Phys. Lett. A 24 55–6
[13] Ristivojevic Z 2014 Excitation spectrum of the Lieb–Liniger model Phys. Rev. Lett. 113 015301
[14] Lang G, Hekking F and Minguzzi A 2016 Ground-state energy and excitation spectrum of the Lieb–Liniger model: accurate analytical results and conjectures about the exact solution (arXiv:1609.08865)
[15] Fuchs J N, Recati A and Zwerger W 2004 Exactly solvable model of the BCS–BEC crossover Phys. Rev. Lett. 93 090408
[16] Zhou L, Xu C-Y and Ma Y-L 2012 Exact studies of ground and excited states of one-dimensional δ-interacting Fermi gases in the BCS–BEC crossover J. Stat. Mech. L03002
[17] Takahashi M 1975 On the validity of collective variable description of Bose systems Prog. Theor. Phys. 53 386–99
[18] Popov V N 1977 Theory of one-dimensional Bose gas with point interaction Theor. Math. Phys. 30 222–6
[19] Lee D K 1974 Ground state of a one-dimensional many-boson system Phys. Rev. A 9 1760
[20] Emig T and Kardar M 2001 Probability distributions of line lattices in random media from the 1D Bose gas Nucl. Phys. B 604 479–510
[21] Iida T and Wadati M 2007 Exact analysis of a δ-function spin-1/2 attractive Fermi gas with arbitrary polarization J. Stat. Mech. P06011
[22] Richardson L F 1927 The deferred approach to the limit Phil. Trans. R. Soc. A 226 636–46
[23] Bulirsch R and Stoer J 1991 Introduction to Numerical Analysis (New York: Springer)
[24] Henkel M and Schütz G M 1988 Finite-lattice extrapolation algorithms J. Phys. A: Math. Gen. 21 2617–33
[25] Gragg W B 1965 On extrapolation algorithms for ordinary initial value problems J. Soc. Ind. Appl. Math. Ser. B 2 384–403
[26] Prolhac S 2016 Extrapolation methods and Bethe ansatz for the asymmetric exclusion process J. Phys. A: Math. Theor. 49 454002
[27] Lang G 2016 private communication