Quantum reference systems: a new framework for quantum mechanics

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Abstract

The new concept of quantum reference systems is introduced, and a corresponding new foundation of nonrelativistic quantum mechanics is given in terms of a set of postulates. The resulting theory gives an explanation for the measurement without assuming an a priori classical background. Schrödinger’s cat paradox and the Einstein-Podolsky-Rosen paradox gain resolution, too. It is also shown that, despite of the violation of Bell’s inequality, quantum mechanics is a local theory.

1 Introduction: the problems

Quantum mechanics proved to be extremely successful in describing natural phenomena at the microscopic (molecular, atomic, subatomic) level. Moreover, there is no evidence that its validity would be limited when it is applied to macroscopic systems. Just oppositely, in several cases it gives excellent results in solid state physics and the discrepancies in other cases may be attributed to computational difficulties rather than to some limitations of the theory itself. Hence there is little doubt concerning the universal validity of quantum mechanics, as long as nonrelativistic phenomena are concerned, and in the present paper we also insist on this opinion.
However, the basic laws of quantum mechanics, especially the superposition principle, seem to contradict sharply to the elementary observations concerning macroscopic bodies. The observations (and even everyday experience) show that macroscopic bodies at a given instant of time have well determined coordinates and momenta (up to a minor inaccuracy required by the uncertainty relations, which is usually much smaller than the resolution of the available experimental techniques). Thus we may assume that macroscopic systems should be quantum mechanically described by such wave functions which are well localized both in coordinate and in momentum space. As an example we may think of a narrow Gaussian wave packet

\[ \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left( -\frac{(x - \bar{x})^2}{4\sigma^2} + \frac{i}{\hbar} \bar{p}(x - \bar{x}) \right) \]  

(1)

describing the initial state of a point particle of large mass in one dimension. Provided that \( \sigma \) is much smaller than the characteristic length scale on which the potential changes considerably and that \( \frac{\hbar}{\sigma} \) is much smaller than the expectation value of the momentum \( \langle \bar{p} \rangle \), the wave packet is and remains to be localized both in momentum and in coordinate space for a long time. Suppose now that \( \Psi_1 \) and \( \Psi_2 \) are such localized states of the same macroscopic system, and they are macroscopically different, e.g. the distance between the centers of mass of the wave functions is one meter. Quantum mechanics, according to the superposition principle, would also allow the state \( \alpha \Psi_1 + \beta \Psi_2 \) to exist, where \( \alpha \) and \( \beta \) are some nonzero complex numbers fulfilling \( |\alpha|^2 + |\beta|^2 = 1 \). Such a state, however, cannot be observed.

Another puzzle can be that wavepackets in quantum mechanics usually spread with time, thus after sufficiently long time no localized states could be observed. This again contradicts the experience.

Thinking of Ehrenfest’s theorem\[2\], one suspects that the answer to both questions can be that the initial wave functions have been localized\[1\], and as the spread of the wave packets is extremely slow for a free particle of big mass (one gram, say), the age of the universe has not been enough for a considerable spreading. But this expectation is wrong! Let alone the question

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1Localization means here simply that the width of the wave function at the given instant of time is small compared to the characteristic classical lengths in coordinate representation and is also small compared to the characteristic momentum in momentum representation. It has nothing to do with Anderson localization.
why the initial conditions have been localized, there are situations, when the
spread of the wave packet is fast even in case of macroscopic systems.

\textit{i, Chaotic systems}

This happens in classically chaotic systems\cite{2}, where wave packets spread
exponentially with time in the semiclassical limit, corresponding to the ex-
ponential separation of nearby classical orbits. Therefore, in case of chaotic
systems the time needed for a spreading to macroscopically observable sizes
can be estimated by

\[ T = \frac{1}{\lambda} \ln \left( \frac{s_{\text{final}}}{s_{\text{initial}}} \right), \quad (2) \]

where $\lambda$ stands for the largest Lyapunov exponent (a classical quantity!)\cite{3},
while the widths of the wave packet in the initial and in the final state are
denoted by $s_{\text{initial}}$ and $s_{\text{final}}$, respectively. Obviously, even for, say, $s_{\text{final}}/s_{\text{initial}} = 10^{15}$ the time $T$ remains to be experimentally accessible. Chaotic systems,
however, seem to be also well localized both in coordinate and in momentum
space (at any given instant of time).

\textit{ii, Quantum measurements}

Another situation, where the explanation relying on appropriate initial
conditions fails, is a typical quantum measurement when there are several
possible outcomes. Suppose, e.g., that the $z$-component of the spin ($\hat{S}_z$) of a
spin-half particle $P$ is measured by a suitable measuring device $M$, e.g., by
a Stern-Gerlach apparatus. If the state of the particle is an eigenstate of $\hat{S}_z$
($|\uparrow\rangle$, say, corresponding to $S_z = +\frac{1}{2}$), then the measuring device goes over
into a state $|m_\uparrow\rangle$ with unit probability. The state $|m_\uparrow\rangle$, describing the
situation when there is a spot on the upper side of the photographic plate of
the device is assumed to fit the experience, i.e., it is well localized both in
coordinate and in momentum representation. The measurement process in
this case may be symbolically written as

\[ |\uparrow\rangle |m_0\rangle \rightarrow |\uparrow\rangle |m_\uparrow\rangle, \quad (3) \]

where $|m_0\rangle$ is the state of the measuring device before the measurement
(no spot on the photographic plate), and $\rightarrow$ is a shorthand notation for the
unitary time evolution during the measurement, which is assumed to fulfill the time dependent Schrödinger equation. If the initial state of the particle has been $| \downarrow >$ corresponding to $S_z = -\frac{1}{2}$, then the measurement process will be (analogously to Eq.(3))

$$| \downarrow > | m_0 > \rightarrow | \downarrow > | m_\downarrow > . \quad (4)$$

Suppose now that the initial state of the particle is not an eigenstate of $\hat{S}_z$, but a superposition of both, $\alpha | \uparrow > + \beta | \downarrow >$. According to the linearity of the Schrödinger equation, the measurement process can be written now as

$$(\alpha | \uparrow > + \beta | \downarrow >) | m_0 > \rightarrow | \Psi > = \alpha | \uparrow > | m_\uparrow > + \beta | \downarrow > | m_\downarrow > . \quad (5)$$

One can see, that quantum mechanics predicts the final state of the compound system $P + M$ to be a superposition of two, macroscopically distinguishable states, and this final state occurs with unit probability. In contrast, in this situation one would actually observe that the state of the measuring device is either $| m_\uparrow >$ or $| m_\downarrow >$, and repeating the measurement under the same circumstances (i.e., with the same initial state), these possibilities would occur randomly, sometimes the first one (with a probability $|\alpha|^2$), sometimes the second one (with a probability $|\beta|^2$). By now it is well known and proven, that this randomness cannot be a consequence of neglecting some local hidden parameters[4],[5] which would allow for a deterministic description at a deeper level[6]. Thus, the above contradiction needs an explanation within quantum mechanics. Note that the apparent absence of macroscopic superpositions is expressed in an extreme form in Schrödinger’s cat paradox[7].

The problems described above (usually mentioned under the heading ‘the interpretation of quantum mechanics’) has been unsolved for a long time, and during the many years a number of different attempts has been made for their solution. We apologize for not reviewing most of these. We do so partly because of lack of space and partly because the present approach is a completely new one. Nevertheless, we shall briefly review the Copenhagen interpretation, as this is the best known and most widely accepted interpretation of quantum mechanics, and also, because the Einstein-Podolsky-Rosen (EPR) paradox relies on this interpretation.

\footnote{Note that nonlocal hidden parameter theories can be constructed[8], however, in the present paper we assume that the principle of locality never breaks down.}
The relation between the present interpretation and the so called decoherence theories will be discussed, too, as the latter are quite popular today and determine the thinking of many experts. The common element is the assumption that Schrödinger’s equation is universally valid, implying that the transition from a pure to a mixed state is a result of the interaction of the system with its environment. The difference is that decoherence theories concentrate mainly on the quantitative description of that transition (i.e., on the effective solution of Schrödinger’s equation written down for a macroscopic system and its environment), while the present interpretation concentrates on the meaning of quantum states, especially, on the meaning of the above mixed state and on its relation to the experience.

We shall also mention Everett’s interpretation\[9\], as it is based on similar assumptions as our approach (e.g., relativity of the states, universality of Schrödinger’s equation), and therefore there is a superficial similarity between the two approaches. This makes necessary to emphasize the rather fundamental differences, clearly distinguishing thus the present approach from Everett’s interpretation. The distinction from two further approaches\[10\],\[11\] will be discussed as well.

At this point the author feels it appropriate to sketch some general features of his approach, in order to prepare the reader for the coming hard trials. The difficulties in understanding will be of conceptual rather than of mathematical nature. The basic idea (cf. Section 3) itself will be neither obvious nor natural, and the related postulates (cf. Section 4) of the theory will look rather complicated and arbitrary. Although some motivation and reasoning will be given beforehand, the actual justification of the whole scheme comes \textit{a posteriori}. It is of worth emphasizing that the correctness of a theory is independent of the emotions rised by its strange-looking concepts. Instead, the relevant questions are whether

1. the theory is free from internal contradictions,

2. its predictions are consistent with experience and

3. with firmly established physical principles.

\[3\]The author confesses that he himself has had a very hard time to get used to the new concepts, too.
As for question 1, the author shall try to convince the reader that in case of the present theory the answer is affirmative. Certainly, no rigorous proof of that will be given.

As for question 2, it will be shown that the theory gives the same theoretical predictions concerning the results of measurements as traditional quantum mechanics, and in this sense the answer is affirmative. There are, however, other aspects as well: if one applies traditional quantum mechanics to macroscopic systems, one gets results which contradict the experience (Schrödinger’s cat paradox). In the present theory no such contradiction appears.

Finally, traditional quantum mechanics precludes any explanation of the classical properties, these are taken for granted. Our approach does not use the concept of a classical background, therefore, if correct, it should lead under realistic conditions to the observed macroscopic properties, like localization in coordinate and in momentum space. The ultimate reason for such a behavior is the interaction of the system with its environment (which is another physical system). As the calculation scheme is well defined, the theory can be checked in numerical experiments. Whether the prescription of the present theory indeed gives a correct result, is not known at the moment and is not discussed in this paper. The question of the emergence of classical properties has been considered by many other authors, but the results and explanations strongly depend on the underlying interpretation. Papers based on the Schmidt decomposition[12] are relevant from the point of view of our approach as well[10],[11],[13],[14].

Concerning question 3 one may tell that a distinctive feature of the present interpretation is that according to it quantum mechanics is consistent with the principle of locality. This may be surprising to many, as it is widely believed that the experimentally proven violation of Bell’s inequality implies that either locality, or realism or inductive inference is violated in Nature. Nevertheless, there is a fourth, evident-looking assumption made at the derivation of Bell’s inequality, namely, that if two different states exist, they can also be compared, without destroying them. The present interpretation shows that if two different states individually exist, it means that each of them may be checked by a suitable measurement without changing that

4This also implies that the present interpretation cannot be considered as a ‘rewording’ of traditional quantum mechanics or of any other previous interpretation.
state[5], but such a measurement will typically change the other state. Therefore, these states, although individually exist, are not comparable. Hence, the violation of Bell’s inequality implies the invalidity of the above fourth assumption rather than that of the principle of locality or realism or inductive inference. One may claim that the answer to question 3 is positive, too.

Some other important features of the present theory are the following:

1. the Schrödinger equation is assumed to be universally valid (if relativistic effects are negligible),

2. measurements are explained in terms of the usual interactions,

3. there is no 'reduction' or 'collapse' of the wave function,

4. the new formulation accommodates the indeterministic nature of quantum mechanics,

5. all the conclusions can and should be drawn by using exclusively the rules of the present theory. Handwaving, intuitive arguments or arguments borrowed from other interpretations are not appropriate.

The paper is organized as follows. In Section 2 we briefly review and criticize the Copenhagen interpretation and in particular the concept of the collapse of the wave function, aimed at solving the problem of the measurement. The motivation for and the essence of the basic new physical assumption of the paper is given in Section 3. We introduce here the concept of the quantum reference systems on which quantum states depend. The differences from the above mentioned other interpretations are discussed here, too. The new physical assumption gives a ‘new freedom’ in the theory, and makes necessary a development of a new formulation of quantum mechanics giving account of the relation among states with respect to different reference systems. This is done in Section 4 such a way that all the well established results of quantum mechanics remain unchanged. In Section 5 Schrödinger’s cat paradox is discussed and it is demonstrated that within the present approach the paradox disappears. In Section 6 we discuss the Einstein-Podolsky-Rosen paradox and show that it gains resolution within the new framework. In Section 7 the violation of Bell’s inequality is analyzed. It is shown that according to

\[5\text{But only one state may be actually checked at the same time!}\]
the present approach correlations between measurements done on separate particles can be attributed exclusively to a previous interaction ('common past') of the particles, thus quantum mechanics is a *local* theory. In the concluding Section 8 we summarize the results and discuss some general features of our approach like unitary covariance of the formulation and the relation of quantum reference systems to the usual ones. A derivation of the so called Schmidt canonical form is given in Appendix A. Appendix B discusses the relation of the present approach to the Copenhagen interpretation, showing that the latter corresponds to a special choice of the quantum reference system.

2 Criticism of the traditional (Copenhagen) solution of the problem

Traditionally, one assumes that 'due to the macroscopic nature' of the system $P + M$ the wave function 'collapses' either (with probability $|\alpha|^2$) to $| \uparrow > | m_\uparrow >$ or (with probability $|\beta|^2$) to $| \downarrow > | m_\downarrow >$ [16]. As is well known, using this assumption one may construct a paradox, discovered by Einstein, Podolsky and Rosen (EPR in a shorthand notation) [17]. These authors considered a situation when two separate particles are described by a correlated wave function (i.e., with a sum of product states, instead of a single product of the states of the particles), due to some interaction having taken place in the past. If a measurement is made on the first particle, and the collapse of the wave function of the whole system takes place, then one concludes that the state of the second particle becomes a definite wave function, and it depends on the kind and result of the measurement, although the second particle does not interact with the measuring device, nor with the first particle during and after the measurement. The original conclusion drawn by EPR was that quantum mechanics did not give a complete description of the physical processes. This conclusion, in view of the developments since, especially of the failure of local hidden variable theories[5], cannot be accepted. The paradox, however, should be somehow resolved. If one wants to maintain the concept of the collapse of the wave function there are essentially two possibilities.
a. It is well known that the EPR paradox does not imply that quantum mechanics contradicts the experiments. It points out the ambiguity of the concept of the wave function. If one does not assume that the wave function corresponds to some state which actually exists (i.e., which would be an 'element of the reality', using EPR's terminology), then no contradiction appears. This leads to the only consistent version of the Copenhagen interpretation: the wave functions have no objective meaning, they are simply calculational tools\[18\]. The aim of the theory is the prediction of the correlations between successive measurements, and the only 'elements of reality' are the results of the measurements.

b. One assumes that the wave functions correspond to some objective states of the systems. Then the EPR paradox implies the existence of an 'instantaneous' interaction, which is faster than the light, violating also causality, although it does not appear directly in experiments.

Experiments actually prove a correlation between separated, noninteracting particles. One would think that this is a consequence of the previous interaction ('common past') between them. If, however, one pursues this line, some inequalities are found for the joint probabilities of the measurements done on the two particles\[3\]. As is known, Bell’s inequality is not always fulfilled by quantum mechanics, moreover, experiments support the quantum mechanical prediction\[19\]. As it has been stressed by d’Espagnat\[20\], the conclusion would be, as long as one is to assume the existence of some objective states (not necessarily described by wave functions)\[6\] that the world is 'nonlocal', as separated particles can 'feel' each other.

The possibility a, although is now widely accepted, does not seem to be satisfactory (among others, in quantum cosmology). It is hard to believe that quantum mechanics is not a model of the physical world and that the only reality is the result of a measurement. One is inclined to reject the collapse of the wave function, rather than confine physics to the laboratories.

As for the possibility b, it does not seem to be acceptable to reject such well proven physical principles, like causality and locality. Nevertheless, it is less obvious how this problem is related to the collapse of the wave function.

\[6\]I.e., using the terminology of Ref.\[20\], when we have to do with a realistically interpretable theory.
One may think that rejecting the existence of the collapse the original EPR argument will not work, but the observed correlations and the violation of Bell’s inequality remain there and may still imply 'nonlocality’. We shall see, however, that the proposal described below solves this difficulty, too, as the derivation of Bell’s inequality contains a hidden assumption, not allowed according to the proposed theory. Thus locality will be restored and the observed correlations will be completely attributed to the interaction having taken place in the past.

3 The basic idea of the proposed solution

In view of the difficulties caused by the collapse of the wave function we reject its existence, but keep all the other parts of quantum mechanics (including Schrödinger’s equation, the expression for the probabilities of the possible results of a measurement etc.). Of course, we are then back at the contradiction presented in Section 1: the quantum mechanical prediction is one definite state, a superposition of two localized states, but experience corresponds to one or the other localized state. As a first remark, let us observe that experience refers to the state of the measuring device \( M \) rather than that of the whole system \( P + M \), thus we have to compare prediction and experience concerning the state of the measuring device. As the measuring device is a subsystem having no own wave function one has to use a reduced density matrix\(^7\) for its description. It is defined by

\[
\hat{\rho}_M = Tr_P (|\Psi\rangle\langle\Psi|),
\]

where \( Tr_P \) stands for the trace operation in the Hilbert space of the particle \( P \). (As the Hilbert space built up of all the possible states of a physical system is uniquely related to that system, here and hereafter we use the term 'the system’s Hilbert space'.) As is known, this object contains all the information necessary to predict the probabilities of the possible results of any measurement done on the subsystem \( M \). Inserting the expression for \( |\Psi\rangle \) (cf. Eq.(5)) into Eq.(3) we get

\[
\hat{\rho}_M = |m_\uparrow\rangle |\alpha|^2 < m_\uparrow | + |m_\downarrow\rangle |\beta|^2 < m_\downarrow |.
\]

\(^7\)Quite rigorously, experience refers to the content of the mind of an observer\(^21\), and what follows will not contradict this view, however, such a rigour which would rise too much philosophical questions is not needed here.
Comparing it with the experience one can see that the actually observed states of the measuring device are just the eigenstates of the reduced density matrix $\hat{\rho}_M$, and their probabilities to occur are the corresponding eigenvalues. Nevertheless, despite of this remarkable finding, we still have the contradiction: the quantum mechanical prediction is the whole reduced density matrix, while only one of its eigenstates is observed. According to our assumptions, we accept that the quantum prediction is correct without any modification (collapse, e.g.), and, of course, we also accept the experience. In order to resolve the apparent contradiction, let us discuss a bit more detail how we have got the quantum prediction and what it means. When calculating $\hat{\rho}_M$, we have used the wave function $|\Psi>$ (cf. Eq.(5)) of the whole system $P+M$. Traditionally a wave function is conceived as the result of a suitable measurement (the preparation), thus one can tell that when calculating $\hat{\rho}_M$ we have actually used the information gained from a measurement done on the whole system $P+M$. In contrast, in case of the experience, a 'measurement', i.e., an observation is done directly on the system $M$. Therefore, the reason for the difference between prediction and experience is that the information used when describing the system $M$ stems from measurements done on different systems. Let us call the system which has been measured (it is $P+M$ in the first case and $M$ in the second case) the quantum reference system. Using this terminology, we may tell that we make a measurement on the quantum reference system $R$, thus we prepare its state $|\psi_R>$ and using this information we calculate the state $\hat{\rho}_S(R) = Tr_{R,S}|\psi_R><\psi_R|$ of a subsystem $S$. We shall call $\hat{\rho}_S(R)$ the state of $S$ with respect to $R$. Obviously $\hat{\rho}_R(R) = |\psi_R><\psi_R|$, thus $|\psi_R>$ may be identified with the state of the system $R$ with respect to itself. For brevity we shall call this the internal state of $R$.

Let us emphasize that up to this point, despite of the new terminology, nothing has been added to usual quantum mechanics.

Let us return now to the question why the state of a system $S$ depends on the choice of the quantum reference system $R$ ($S \subseteq R$). It is certainly very surprising from the classical point of view that the state of a system is found to be different if we measure the system directly or if we measure it together with an environment. Nevertheless, in the spirit of the Copenhagen interpretation one may tell that this is just because in quantum mechanics measurements usually change the states, different measurements lead to different changes, thus the above mentioned difference may be attributed
completely to the different measurements. This argument is, however, not compelling. At this decisive point we leave the traditional framework of quantum mechanics and make the fundamental assumption that the dependence of the states on quantum reference systems is an inherent property of quantum mechanics and not the result of the disturbance due to measurements. At the same time it is also assumed that the states exist even in the absence of any measurement. Our assumption means that one and the same system $A$ is characterized by a multitude of states $\hat{\rho}_A(R_1)$, $\hat{\rho}_A(R_2)$, ..., each referring to a different quantum reference system $R_i$. As for their physical meaning, one can conceive the state $\hat{\rho}_A(R)$ as the description of the system $A$ using the information gained from a suitable measurement performed on $R$. The term 'suitable' means that the measurement does not change the state $\hat{\rho}_R(R)$ (the state of the quantum reference system with respect to itself), otherwise the state $\hat{\rho}_A(R)$ changes due to the measurement. Let us draw the attention of the reader to a subtle point: although one cannot know a priori what such a suitable measurement is, it does not lead to any contradiction if we assume the existence of such a measurement. The possibility of nondisturbing measurements is the expression of realism: the states exist whether we measure it or not, and in principle they can be learned.

Note that the above assumption is a radical departure from the conventional point of view of quantum mechanics. There anything that exists is associated with some actual measurement, thus states themselves are thought to be created by measurements. Obviously, such an approach excludes from the beginning the possibility to describe the measurements in terms of quantum mechanics, as measurements are treated as primary entities and states as the secondary ones. (Even if one considers the measuring device as part of the quantum system, a further measuring device and measurement is needed to describe the states according to the usual Copenhagen interpretation.) In contrast, in our approach the primary concepts are the states, and measurements are derived. As noted above, this is the obvious prerequisite if one wants to describe quantum measurements in terms of quantum mechanics. We shall see that in terms of the states depending on quantum reference systems one can actually interpret the measurements independently of classical mechanics and can establish quantum mechanics consistently, freely from

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8 Of course, these states are unknown for us in the absence of measurements and usually will be changed if this or that measurement is performed.
paradoxes.

Let us summarize the main idea expounded above:

*The dependence of the state of a physical system on quantum reference systems is an inherent property of quantum mechanics. Consequently, a state with respect to the chosen quantum reference system exists even in the absence of measurements, and in principle there exist a suitable measurement which, if performed on the quantum reference system in order to determine this state, does not disturb it.*

This conclusion is the essential new physical assumption upon which the whole theory presented in this paper is based. Let us discuss it in a bit more detail.

The meaning of the quantum reference systems is rather analogous to the classical coordinate systems. Choosing a classical coordinate system means that we imagine what we would experience if we were there. Similarly, choosing a quantum reference system $R$ means that we imagine what we would experience if we did a measurement on $R$ that does not disturb $\hat{\rho}_R(R) = |\psi_R><\psi_R|$. In order to see that such a measurement exists, consider an operator $\hat{A}$ (which acts on the Hilbert space of $R$) whose eigenstates include $|\psi_R>$. The measurement of $\hat{A}$ will not disturb $|\psi_R>$. Nevertheless, there exist important features of the quantum reference systems that are quite unusual from the classical point of view. It is certainly not surprising, that a given system can be characterized by a multitude of states, each referring to a different quantum reference system. One may safely say that all these states exist, just like in case of the classical coordinate systems all the states with respect to different coordinate systems exist. Indeed, in the spirit of Einstein, Podolsky and Rosen we may accept that a state $\hat{\rho}_S(R)$ is an element of the reality if there exists a suitable nondisturbing measurement to it. As we have seen, such a measurement always exists. The nondisturbing measurement may of course be repeated and one gets the same result, as before. Nevertheless, the states defined with respect to different quantum reference systems are not necessarily comparable. What does it mean? We explain it on an explicit example. Let us consider three distinguishable spin-half particles, $A$, $B$ and $C$. Be $R_1 = A + B$ and $R_2 = B + C$. Suppose that the system $A + B + C$ is isolated. Denoting the eigenstates of the $\hat{S}_z$, $\hat{S}_x$ spin-component by $|\uparrow>$, $|\downarrow>$ and $|+>$, $|->$, respectively, be
the state of $A + B + C$ (considering only the spin degrees of freedom)

$$
|\phi> = \alpha (\beta|A, \uparrow> |B, \downarrow> + \gamma^*|A, \downarrow> |B, \uparrow>) |C, + > + \delta (\gamma|A, \uparrow> |B, \downarrow> - \beta^*|A, \downarrow> |B, \uparrow>) |C, - >
= |A, \uparrow> |B, \downarrow> (\alpha \beta |C, + > + \delta \gamma |C, - > )
+ |A, \downarrow> |B, \uparrow> (\alpha \gamma^* |C, + > - \delta \beta^* |C, - > )
$$

(8)

with $|\alpha|^2 + |\delta|^2 = 1$, $|\beta|^2 + |\gamma|^2 = 1$. In order to calculate the states $\hat{\rho}_{R_1}(R_1)$ and $\hat{\rho}_{R_2}(R_2)$ one has to calculate the density matrices $\hat{\rho}_{A+B}(A+B+C)$ and $\hat{\rho}_{B+C}(A+B+C)$, respectively, and has to find the eigenvectors (cf. Section 4, Postulates 4, 6 and Proposition 1). The result is that $\hat{\rho}_{R_1}(R_1)$ is a projector corresponding either to the state vector

$$
|\psi_+> = \beta|A, \uparrow> |B, \downarrow> + \gamma^*|A, \downarrow> |B, \uparrow>
$$

(9)

or to

$$
|\psi_-> = \gamma|A, \uparrow> |B, \downarrow> - \beta^*|A, \downarrow> |B, \uparrow>,
$$

(10)

while $\hat{\rho}_{R_2}(R_2)$ is the projector corresponding either to

$$
\left(|\alpha|^2|\beta|^2 + |\delta|^2|\gamma|^2\right)^{-\frac{1}{2}} |B, \downarrow> (\alpha \beta |C, + > + \delta \gamma |C, - > )
$$

(11)

or to

$$
\left(|\alpha|^2|\gamma|^2 + |\delta|^2|\beta|^2\right)^{-\frac{1}{2}} |B, \uparrow> (\alpha \gamma^* |C, + > - \delta \beta^* |C, - > ) .
$$

(12)

Suppose we make a nondisturbing measurement on $R_1$. The dynamics of this measurement may be written symbolically as

$$
|\psi_+ >|m_0> \rightarrow |\psi_+ > |m_+ > .
$$

(13)

This implies that after the measurement the state of the whole system $A + B + C + M$ is given by

$$
\alpha (\beta|A, \uparrow> |B, \downarrow> + \gamma^*|A, \downarrow> |B, \uparrow>) |m_+ > |C, + > + \delta (\gamma|A, \uparrow> |B, \downarrow> - \beta^*|A, \downarrow> |B, \uparrow>) |m_- > |C, - > .
$$

(14)
Repeating the calculation for $\hat{\rho}_{R_2}(R_2)$ one gets now the result that it is a projector corresponding to one of the four state vectors

\begin{align*}
|B, \downarrow> |C, +> & \quad (15) \\
|B, \downarrow> |C, -> & \quad (16) \\
|B, \uparrow> |C, +> & \quad (17) \\
|B, \uparrow> |C, -> & \quad . \quad (18)
\end{align*}

Comparing these states with Eqs. (11), (12) one can see that the measurement which did not disturb $\hat{\rho}_{R_1}(R_1)$, changed $\hat{\rho}_{R_2}(R_2)$. Therefore, we cannot learn both $\hat{\rho}_{R_1}(R_1)$ and $\hat{\rho}_{R_2}(R_2)$ at the same time without changing at least one of them, so we cannot compare these states. We can learn only one of them, certainly, we may decide, which one. We shall express this surprising property by saying that although both $\hat{\rho}_{R_1}(R_1)$ and $\hat{\rho}_{R_2}(R_2)$ exist, they are not comparable. It is important to understand, that this property does not influence the realism of the theory. Mathematically we may conceive reality as a set containing states and sets of states. If a suitable nondisturbing measurement existed, so that both $\hat{\rho}_{R_1}(R_1)$ and $\hat{\rho}_{R_2}(R_2)$ could be learned at the same time without changing them, it would mean that not only $\hat{\rho}_{R_1}(R_1)$ and $\hat{\rho}_{R_2}(R_2)$ as individual states, but also the set containing both states would be an element of reality. Certainly if two states are elements of the reality, it does not imply automatically that the set of those two states is also an element of the reality. It implies only that the set of the two states is a subset of reality. Indeed, in the above example both $\hat{\rho}_{R_1}(R_1)$ and $\hat{\rho}_{R_2}(R_2)$ are elements of reality as individual states (as there exists for each a suitable nondisturbing measurement), but the set of these states is not an element of reality, as no corresponding nondisturbing measurement exists.

It is interesting to note that there exists a classical analogue of the above noncomparability, i.e., descriptions with respect to different coordinate systems may be noncomparable. It is known that general relativity allows coordinate systems even inside of a black hole. Consider now two different black holes and introduce a coordinate system inside each of them. Probably no one doubts the reality of the descriptions with respect to these coordinate systems. We may indeed check what we may experience with respect to one of these systems. We may freely choose one of the black holes and may

\footnote{Note that the state $\hat{\rho}_{R_1}(R_1)$ does not change, as expected.}
fall into it. Then we see what is inside. However, if we do so we cannot come back and thus automatically prevent ourselves from learning the other blackhole interior. Therefore, analogously to the quantum case, descriptions with respect both to the one and to the other black hole interior exist, but they cannot be observed simultaneously. Of course, we do not want to say that there is any deeper connection between the underlying physics of the quantum case and that of the above classical example. Just oppositely, the mechanisms leading to noncomparability are completely different in the classical and the quantum case. In the quantum case a nondisturbing measurement performed on one system changes the state of the other system. In contrast, in the classical case performing an observation in one coordinate system prevents from performing any observation in the other system. There is no disturbance on the other coordinate system at all.

An important feature of the usual coordinate systems is that properties of a given physical system observed from different coordinate frames are unambiguously related, namely, quantum states with respect to different coordinate systems are connected via suitable unitary transformations. The concept of quantum reference systems is different: usually the relation of states with respect to different quantum reference systems is not one-to-one: in the above example, if the state of $M$ with respect to $P + M$ is $\hat{\rho}_M$, then the state of $M$ with respect to $M$ can be either $|m_r\rangle$, or $|m_i\rangle$. It is of worth mentioning that this feature expresses the indeterminism of quantum mechanics in our approach. Nevertheless, the concept of the usual coordinate frames and that of the quantum reference systems are connected: as we shall see, the former can be considered a special case of the latter. From the mathematical point of view, however, they are rather different: a classical coordinate system is a coordinate system in the mathematical sense, while a quantum reference system is a Hilbert space.

We discuss now the (rather fundamental) differences between our approach and Everett’s interpretation\[9\]. Everett writes about ‘relative states’ of a subsystem, and he defines them with respect to the states of the complementary subsystem. In contrast, in our approach the state of a subsystem is defined with respect to a system which contains the subsystem to be described. This difference leads to a completely different physical picture in the two cases: Everett attributes a physical meaning to the product components of the entangled state of an isolated compound system, all being somehow relevant (parallel worlds), while in the present interpretation these
components do not have separately any physical meaning, only together, and this whole entangled state describes the compound system with respect to itself (i.e., the quantum reference system is the compound system). The state of a subsystem with respect to this quantum reference system is the corresponding reduced density matrix and not some factor of the component product states. The problem why we see only one result at each quantum measurement when the whole wave function of the universe is still present is solved in the present interpretation by the realization that in case of a quantum measurement the measuring device (or the observer), i.e., a subsystem is the relevant quantum reference system. Therefore, the relevant quantum reference system is different than before, and there is no logical necessity that a one-to-one relationship between the descriptions with respect to these different quantum reference systems exists. This remark also shows that the present interpretation does not involve the parallel worlds.

Ideas somewhat similar to ours has been put forward in Ref. [10], where the author writes about Schmidt states as experienced 'subjectively' by the observer. The concept of quantum reference systems is, however, not introduced there, instead, the idea mentioned above is associated with Everett’s interpretation, which differs markedly from our approach.

Another similar interpretation is the so called modal interpretation [11], which states that one of the eigenstates of the reduced density matrix corresponds to an actually existing property of the subsystem. (‘Actually existing’ here means that in an absolute sense, independently of quantum reference systems.) In our approach these states are the states of the subsystem with respect to itself, i.e., they are not relevant with respect to other quantum reference systems. In Section 7 we argue that the modal interpretation implies Bell’s inequality, while our approach does not.

Finally, it is reasonable to discuss the relation of the present approach to the so called decoherence theories [15], [10]. These latter are quite popular today and determine the thinking of many experts. The main idea is that the transition of a macroscopic system $M$ from a pure to a mixed state is due to the interaction between the system $M$ and its environment $E$. The effect is actually a consequence of the time-dependent Schrödinger equation written down for the closed $M + E$ compound system, i.e., decoherence theories assume the universality of the Schrödinger equation, just like the present approach. Explicitly, if initially $M$ is in a pure state $|\varphi>$, and correspondingly the system $M + E$ is in the product state $|\varphi> |\xi>$, the time dependent
Schrödinger equation written down for $M + E$ leads to the unitary time evolution

$$|\phi(t)\rangle = \exp \left( -i \frac{\hat{H}t}{\hbar} \right) (|\varphi\rangle |\xi\rangle)$$

(19)

where the Hamiltonian $\hat{H}$ includes the interaction term between $M$ and $E$. Due to this term $|\phi(t)\rangle$ ceases to be of product form, instead, it becomes a sum of product states,

$$|\phi(t)\rangle = \sum_j c_j(t) |\varphi_j(t)\rangle |\xi_j(t)\rangle,$$

(20)

where $|\varphi_j(t)\rangle$ and $|\xi_j(t)\rangle$ are orthonormed sets of states (at any given instant of time $t$). Note that the form (20), called Schmidt decomposition (cf. Appendix A) does not restrict the generality of the discussion. Calculating the reduced density matrix from $|\phi(t)\rangle$ for the system $M$ one obtains

$$\hat{\rho}_M(t) = \sum_j |\varphi_j(t)\rangle |\xi_j(t)\rangle <\varphi_j(t)||c_j(t)|^2.$$

(21)

The transition from a pure to a mixed state is actually a generic property of interacting quantum systems (even if they are not macroscopic). We also used it already in Eqs.(3)-(7). There the macroscopic system was the measuring device and the role of the environment was played by the spin-half particle. Therefore, up to this point the present approach agrees with that of decoherence theories. The difference between the two approaches lies in the different questions these theories try to answer. The main achievement of decoherence theories is the quantitative description of the transition from a pure to a mixed state, i.e., the actual calculation of $\hat{\rho}_M(t)$ for various, more or less realistic models. This is that part of decoherence theories where one may speak about a consensus in the literature. Concerning the physical meaning of $\hat{\rho}_M(t)$, i.e., its relation to the experience, no such consensus has been achieved. The present approach puts emphasis exactly on that problem. It will be shown that the new idea presented in this section leads to a consistent set of postulates, which offer in principle an answer to all potential problems. The power of this approach will be demonstrated in Sections 5-7, where Schrödinger’s cat paradox and the EPR paradox will be resolved and the violation of Bell’s inequality will be explained without giving up the principle of locality.
4 Rules of the new framework

In this section the new version of the rules of quantum mechanics is formulated, including the connections between states of a system with respect to different reference systems. We shall see that the principle that quantum states depend on the quantum reference systems can be incorporated into quantum mechanics so as to recover its usual results exactly.

Let us denote the physical system to be described by $A$ and the reference system by $R$. The state of $A$ with respect to $R$ will be denoted by $\hat{\rho}_{A}(R)$.

**Postulate 1.** The system $A$ to be described is a subsystem of the reference system $R$.

Note that the reference system may coincide with the system to be described ($A = R$). In such a case we speak about an internal state.

**Definition 1.** $\hat{\rho}_{A}(A)$ is called the internal state of $A$.

**Postulate 2.** The state $\hat{\rho}_{A}(R)$ is a positive definite, Hermitian operator with unit trace, acting on the Hilbert space of $A$.

Note that the unit trace means that we confine our considerations to normalizable states, which is not a serious restriction in nonrelativistic quantum mechanics.

**Postulate 3.** The internal states $\hat{\rho}_{A}(A)$ are always projectors, i.e., $\hat{\rho}_{A}(A) = |\psi_{A} \rangle \langle \psi_{A}|$.

In what follows these projectors will be identified with the corresponding wave functions $|\psi_{A} \rangle$ (as they are uniquely related, apart from a phase factor). According to Postulate 2 the internal states are normalized to unity.

**Postulate 4.** The state of a system $A$ with respect to the reference system $R$ (denoted by $\hat{\rho}_{A}(R)$) is the reduced density matrix of $A$ calculated from the internal state of $R$, i.e.

$$\hat{\rho}_{A}(R) = Tr_{R\setminus A} (\hat{\rho}_{R}(R)) \; ,$$  \hspace{1cm} (22)
where \( R \setminus A \) stands for the subsystem of \( R \) complementer to \( A \).

One can see that Postulate 4 is consistent with Postulate 2 and Postulate 3.

We introduce now the notion of isolated and closed systems.

**Definition 2.** An isolated system is such a system that has not been interacting with the outside world. A closed system is such a system that is not interacting with any other system at the given instant of time (but might have interacted in the past).

An isolated system can be described by a wave function, and, moreover, this wave function always occurs as a factor in the internal state of any broader system. Thus we set

**Postulate 5.** If \( I \) is an isolated system then its state is independent of the reference system \( R \):

\[
\hat{\rho}_I(R) = \hat{\rho}_I(I) .
\]  

(23)

In other terms, Postulate 5 means that the state of an isolated system has an absolute meaning. This is why it can be related to the internal states of its subsystems, as is postulated below.

**Postulate 6.** If the reference system \( R = I \) is an isolated one then the state \( \hat{\rho}_A(I) \) commutes with the internal state \( \hat{\rho}_A(A) \).

This means that the internal state of \( A \) coincides with one of the eigenstates of \( \hat{\rho}_A(I) \). Note that usually there is no one-to-one correspondence between states with respect to different reference systems, thus one cannot tell (knowing \( \hat{\rho}_A(I) \)) which eigenstate corresponds to \( \hat{\rho}_A(A) \)). In what follows, these eigenstates \( |\phi_{A,j} > \) will play an important role. They will be identified with the corresponding projector \( \hat{\pi}_{A,j} = |\phi_{A,j} >> \phi_{A,j}| \), and we shall call them the possible internal states, as they constitute the set of those internal states of \( A \) that are compatible with \( \hat{\rho}_A(I) \). (Of course, at a given
time only one of these states exists *with respect to* $A$.) For further reference, we set

**Definition 3.** The possible internal states are the eigenstates of $\hat{\rho}_A(I)$ provided that the reference system $I$ is an isolated one.

It can happen that some of the eigenvalues of $\hat{\rho}_A(I)$ coincides. Nevertheless, by requiring the continuity of the possible internal states as a function of time (using the usual Hilbert space norm) one can resolve the degeneracy (this degeneracy can remain constantly there if $A$ does not interact with other systems, but then it is an isolated system and the degenerated eigenvalue is zero, which is irrelevant [cf. *Postulate 7* below]). Thus the possible internal states are practically unambiguously defined once $\hat{\rho}_A(I)$ is known.

An important property of the possible internal states is given by the following statement:

**Proposition 1.** If $A$ and $B$ are two disjointed physical systems (i.e., they have no common subsystems) with possible internal states $|\phi_{A,j}\rangle$ and $|\phi_{B,j}\rangle$, respectively, and the joint system $A+B$ is an isolated one, then the internal state of $A+B$ can be written as

$$|\psi_{A+B}\rangle = \sum_j c_j |\phi_{A,j}\rangle |\phi_{B,j}\rangle .$$

(24)

Eq. (24) is called the Schmidt canonical form[13], for a recent explicit application to simple systems see Ref.[14]. It is straightforward to see that the reverse of *Proposition 1* is also true, i.e., once Eq. (24) holds with some orthonormed set of states $|\phi_{A,j}\rangle$ and $|\phi_{B,j}\rangle$ ($j = 1, 2, 3, ..$) then these are the possible internal state of $A$ and $B$, respectively. As for a proof of *Proposition 1*, see Appendix A.

**Postulate 7.** If $I$ is an isolated system, then the probability $P(A,j)$ that the eigenstate $|\phi_{A,j}\rangle$ of $\hat{\rho}_A(I)$ coincides with $\hat{\rho}_A(A)$ is given by the corresponding eigenvalue $\lambda_j$.

Due to the normalization of the internal states these eigenvalues add up to unity.
It is important to emphasize that despite of the probability introduced above, it is not possible to classify reference systems having the same internal state according to the internal states of their subsystems, as the former state exists with respect to the reference system while the latter states exist only with respect to the subsystems. It is possible, however, to learn the internal states of the subsystems via suitable measurements (a quantity whose operator commutes with $\hat{\rho}_A(I)$ should be measured), destroying at the same time the original internal state of $I$. Thus, for an observer before the measurement the internal state of $I$ is known and the internal state of $A$ is known afterwards. Once the former has been recorded, one can speak about the joint probability of the internal state of $I$ and that of $A$. The record, of course, presupposes that a third system interacts with $I$. The probabilities $P(A,j)$ in Postulate 7 are equal to those of the various outcomes of the measurement mentioned above. In order to establish this connection in terms of the rules of the present approach two further postulates are needed.

**Postulate 8.** The result of a measurement is contained unambiguously in the internal state of the measuring device.

**Postulate 9.** If $A$ and $B$ are two disjointed physical systems with possible internal states $|\phi_{A,j}\rangle$ and $|\phi_{B,j}\rangle$, respectively, and both systems are contained in the isolated reference system $I$, then the joint probability that $|\phi_{A,j}\rangle$ coincides with the internal state of $A$ and at the same time $|\phi_{B,k}\rangle$ coincides with the internal state of $B$ ($j,k = 1, 2, 3, \ldots$) is given by

$$P(A,j,B,k) = \text{Tr}_{A+B}(\hat{\pi}_{A,j}\hat{\pi}_{B,k}\hat{\rho}_{A+B}(I)) \quad ,$$

where

$$\hat{\pi}_{A,j} = |\phi_{A,j}\rangle\langle \phi_{A,j}|, \quad \hat{\pi}_{B,k} = |\phi_{B,k}\rangle\langle \phi_{B,k}| \quad .$$

As an example let us consider the situation when $A + B$ is an isolated system. According to Proposition 1 (cf. Eq.(24)) $P(A,j,B,k) = |c_j|^2 \delta_{j,k}$.

Note that it is important that $A$ and $B$ are disjointed systems (this will play an important role when explaining why Bell’s inequality does not hold,

\footnote{This statement corresponds to that made in Section 3. Note that in principle it also applies if the experience of a living being is concerned (cf. Schrödinger’s cat\footnote{\cite{10}} or Wigner’s friend\footnote{\cite{23}}). Indeed, the present approach does not distinguish between physical systems, they can be either microscopic or macroscopic, or even living beings.}
cf. Section 7), as this ensures that the product \( \hat{\pi}_{A,j}\hat{\pi}_{B,k} \) is a single projector acting on the Hilbert space of \( A+B \), which in turn implies (due to the positive definiteness of \( \hat{\rho}_{A+B}(I) \)) that \( P(A, j, B, k) \) is indeed real and nonnegative.

Due to the completeness of the possible internal states \( \sum_k \hat{\pi}_{A,k} = 1 \) holds, therefore \( \sum_k P(A, j, B, k) = P(A, j) \), as expected.

More generally, we can consider the joint probabilities for more than two disjointed physical systems. Then we set

\[
P(A_1, j_1, A_2, j_2, ..., A_n, j_n) = \text{Tr}_{A_1+A_2+...+A_n}[\hat{\pi}_{A_1,j_1}\hat{\pi}_{A_2,j_2}...\hat{\pi}_{A_n,j_n}\hat{\rho}_{A_1+A_2+...+A_n}(I)],
\]

(26)

where \( \hat{\pi}_{A_i,j_i} = |\phi_{A_i,j_i}><\phi_{A_i,j_i}| \).

Note that \( n = 1 \) and \( n = 2 \) correspond to Postulate 7 and Postulate 9, respectively.

Let us discuss now the relation of the probabilities \( P(A, j) \) and \( P(A, j, B, k) \) with the measurements. The term ‘measurement’ means here a usual interaction of the system in question with another system called the measuring device. For simplicity it is assumed that the measurement process has two special properties (note that this simplification does not lead to any essential restriction of the generality of our discussion):

1. before the measurement the measuring device can be considered an isolated system
2. if the initial state of the system is a member of a certain orthonormed set of states \( |\varphi_j> \) then the product state \( |\varphi_j> |m_0> \) evolves during the measurement (according to the time dependent Schrödinger equation) into the product state \( |\varphi_j> |m_j> \), where \( |m_j> (j = 1, 2, ...) \) is an orthonormed set of states in the Hilbert space of the measuring device. (Thus we consider quantum nondemolition measurements [24].)
As a foregoing general remark let us mention that it is not possible to de-

erive the concept of the above probabilities directly from the measurements,

as the interpretation of the latter’s results inevitably needs the postulates

again, where the probabilities are already involved. What we can show is

that the postulates offer a consistent interpretation of the observed relative

frequencies, in the same way as the mathematical probability theory does,

and, moreover, we can reproduce all the usual results of the quantum me-

chanics concerning the probabilities.

Let us consider first the case when just $\hat{\rho}_A(I)$ is measured ($I$ is isolated),

i.e., when the basic dynamics of the measurement process is given by the

relations

$$|\phi_{A,j} > |m_0 > \rightarrow |\phi_{A,j} > |m_j > , (j = 1, 2, ..). \quad (27)$$

Therefore, using Eq. (24) with $B = I \setminus A$ the change of the internal state of

the isolated system $I + M$ ($M$ standing for the measuring device) is given by

$$\left( \sum_j c_j |\phi_{A,j} > |\phi_{B,j} > \right) |m_0 > \rightarrow \sum_j c_j |\phi_{A,j} > |\phi_{B,j} > |m_j > . \quad (28)$$

Using Postulate 4 and Definition 3 one finds that after the measurement the

possible internal states of $A$ are still the $|\phi_{A,j} >$-s, and the possible internal

states of the measuring device are the $|m_j >$-s. Furthermore, Postulate 7 implies that the probability to get the $j$-th results (i.e., the probability

that $|m_j >$ coincides with the internal state of $M$, cf. Postulate 8) is $|c_j|^2$,

which coincides with the probability $P(A, j)$. Using now Postulate 9 we get

$P(A, j, M, k) = |c_j|^2 \delta_{j,k}$, i.e., if the $j$-th result is obtained at the measurement,

then the internal state of $A$ is just $|\phi_{A,j} >$.

Consider now the joint probability $P(A, j, B, k)$ occurring in Postulate 9.

Suppose that we measure simultaneously $\hat{\rho}_A(I)$ and $\hat{\rho}_B(I)$ by two measuring

devices, $M^a$ and $M^b$, respectively. Let $C = I \setminus (A + B)$, then the internal

state of the system $A + B + C + M^a + M^b$ before the measurement is given by

$$\sum_j \sum_k \sum_l c_{j,k,l} |\phi_{A,j} > |m^a_0 > |\phi_{B,k} > |m^b_0 > |\phi_{C,l} > , \quad (29)$$

and after the measurement it is

$$\sum_j \sum_k \sum_l c_{j,k,l} |\phi_{A,j} > |m^a_j > |\phi_{B,k} > |m^b_k > |\phi_{C,l} > . \quad (30)$$
As a consequence of the orthogonality of the states $|m^a_j\rangle$ one concludes that the possible internal states of $A$ are the $|\phi_{A,j}\rangle$-s, and, due to the orthogonality of the states $|m^b_k\rangle$ that the possible internal states of $B$ are the $|\phi_{B,k}\rangle$-s. Therefore, using Eq. (25) we get

$$P(A, j, B, k) = \sum_l |c_{j,k,l}|^2.$$  \hspace{1cm} (31)

Let us calculate now the possible internal states of the combined measuring device $M^a + M^b$. They turn out to be the product states $|m^a_j\rangle |m^b_k\rangle$ ($j, k = 1, 2, \ldots$), and the probability that they coincide with the internal state of $M^a + M^b$ is

$$P(M^a + M^b, (j, k)) = \sum_l |c_{j,k,l}|^2$$  \hspace{1cm} (32)

i.e., it coincides with $P(A, j, B, k)$. Using Postulate 10 with $n = 3$ we can calculate the joint probability $P(A, j, B, k, M^a + M^b, (l, m))$ to get

$$P(A, j, B, k, M^a + M^b, (l, m)) = \sum_l |c_{j,k,l}|^2 \delta_{j,l} \delta_{k,m}.$$  \hspace{1cm} (33)

Thus we can see that if $M^a$ shows the $j$-th result and $M^b$ shows the $k$-th result, then the internal state of $A$ and $B$ has been being $|\phi_{A,j}\rangle$ and $|\phi_{B,k}\rangle$, respectively
\footnote{Strictly speaking, in order to see that the internal states of $A$ and of $B$ have not changed during the measurement, we need Postulate 11 concerning the time evolution, see later.}. It is a general feature that the simultaneous existence of internal states of different systems (which also means that the reference systems are different) can be defined and provided with probabilities if all they can be brought into correlation with the internal state of the same physical system (in the above example this latter has been the combined measuring device), i.e., if they become comparable with respect to that physical system. The condition for that is that the systems in question are disjointed or they are strongly correlated with disjointed systems. As an example we can consider the joint probability of the possible internal states of a system $A$ and that of one of its subsystems $B$. As $B \subset A$, they are not disjointed, but $A$ is strongly correlated with $I \setminus A$, where $I$ is an isolated system containing $A$ (cf. Proposition 1), i.e., if the internal state of $A$ coincides with the $j$-th
possible internal state $|\phi_{A,j}>$, then the internal state of $I \setminus A$ coincides with the corresponding possible internal state $|\phi_{I \setminus A}>$. The systems $B$ and $I \setminus A$ are already disjointed. Therefore, $P(A, j, B, k) = P(I \setminus A, j, B, k)$, the latter probability being already well defined according to Postulate 9. Note that it can be brought to the form

$$P(A, j, B, k) = \langle \phi_{B,k} | \hat{\rho}_B(A, j) | \phi_{B,k} > ,$$

where

$$\hat{\rho}_B(A, j) = Tr_{A \setminus B} (|\phi_{A,j}> < \phi_{A,j}|) .$$

Let us outline now how the probability $P(A, j)$ can be related to the actual relative frequencies (this procedure is quite similar to that used in probability theory). Suppose that we are given $n >> 1$ identical isolated systems $I_i$ ($i = 1, 2, ... n$), all being in the same internal state, except for a translation, and each containing the identical subsystems $A_i$. Let us measure on each $A_i$ the quantity $\hat{\rho}_{A_i}(I)$ ($I = I_1 + I_2 + ... + I_n$) using a measuring device $M_i$, and consider the internal state of the combined device $M_1 + M_2 + ... + M_n$. Due to the independence of the measurements we get the polynomial distribution for the probability that $k_1$ devices show the first result, $k_2$ devices the second result etc. This distribution is for $n >> 1$ sharply peaked near the mean values $\bar{k}_1 = nP(A, 1)$, $\bar{k}_2 = nP(A, 2)$, ....

Up to now we discussed rather special measurements in order to demonstrate the concepts and their consistency. Let us turn now our attention to the question what our approach gives in case of an arbitrary measurement. Consider an isolated system $I = A + B$, and make a measurement on its subsystem $A$. Suppose that the measurement is defined by

$$|\phi_j > |m_0 > \rightarrow |\phi_j > |m_j > ,$$

where the states $|\phi_j > (j = 1, 2, ...)$ are the eigenstates of the measured quantity, and therefore constitute a complete orthonormed set. Using Eqs.(24) and (36) the change of the internal state of the isolated system $A + B + M$ during the measurement can be written as

$$\left( \sum_j c_j |\phi_{A,j} > |\phi_{B,j} > \right) |m_0 >$$

$$\rightarrow \sum_j \sum_k c_j < \varphi_k |\phi_{A,j} > |\varphi_k > |m_k > |\phi_{B,j} > .$$

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Using our postulates, it is easy to show that the possible internal states of $M$ are the $|m_k\rangle$s with the probability

$$ P(M, k) = \sum_j |c_j|^2 < \varphi_k|\phi_{A,j} > |^2 = < \varphi_k|\hat{\rho}_A(I)|\varphi_k > $$

which coincide with the internal state of $M$, which is exactly the same what we get in the usual formulation of quantum mechanics for the occurrence of the $k$-th result of the measurement.

Let us consider now the time evolution of the states. In case of closed systems the usual unitary time evolution is assumed, i.e.,

**Postulate 11.** The internal state $|\psi_C\rangle$ of a closed system $C$ satisfies the time dependent Schrödinger equation

$$ i\hbar \partial_t |\psi_C\rangle = \hat{H} |\psi_C\rangle \quad . \quad (38) $$

Here $\hat{H}$ stands for the Hamiltonian.

As for the time evolution of the internal states of non-closed systems, it can be defined if the initial and the final states can be compared, i.e., if the initial state has been 'recorded' (via suitable interaction with some other system) in order to make it comparable with the final state (they can be compared only if they are related to the internal state of the same system at the same time, cf. the discussion of the joint probabilities above).

To be more explicit, consider a system $A$ and assume that its initial internal state (at time 0) has been determined via the measurement of $\hat{\rho}_A(I)$, where $I$ is an isolated reference system (cf. Eq.(28)). Suppose now that the measuring device $M$ has not interacted with any systems after this measurement. Actually it is of no importance whether $M$ interacted with other systems or not, if yes, we can consider their union with $M$. The important assumption is that $M$ has not interacted with $A$ after the first measurement. Therefore, the internal state $|\psi_M(t)\rangle$ evolves unitarily according to Postulate 11, and it is uniquely related even at a later time $t$ to the initial internal state $|\psi_A(0)\rangle$ of $A$. Therefore, the joint probability $P_t(A, (k|j))$ that the internal state of $A$ coincides at time 0 with its $j$-th possible internal state $|\phi_{A,j}(0)\rangle$ and it coincides at time $t$ with its $k$-th possible internal state $|\phi_{A,k}(t)\rangle$ is given by

$$ P_t(A, (k|j)) = P(M, j, A, k)(t) \quad , \quad (39) $$

thus, it is expressed through a joint probability of type (25).
5 Resolution of Schrödinger’s cat paradox

5.1 The paradox within the framework of the Copenhagen interpretation

The well known thought experiment contains a radioactive nucleus, whose decay triggers through a detector a device which breaks a poison capsule, letting some poisonous gas out, and thus killing a cat. At a given instant of time the state of the nucleus is a superposition of a decayed and a nondecayed state, therefore, due to the linearity of the time dependent Schrödinger equation the state of the whole system is given by

\[ |\alpha|+ > |d_+ > |d > + \beta|− > |d_− > |a > , \]  

where \(|+ >\) and \(|− >\) stand for the decayed and nondecayed state of the nucleus, respectively, \(|d_+ >\) and \(|d_− >\) describe the corresponding states of the detector+device+poison system, finally, \(|d >\) and \(|a >\) stand for the dead and the alive state of the cat, respectively. If one looks at the cat, according to the Copenhagen interpretation the collapse of the wave function takes place, and one finds either that the cat is dead or that it is living. Before looking at it, however, Eq.(40) should hold, which contains both the living and the dead states, so one should conclude that the cat is neither living nor dead, until someone looks at it. This unphysical conclusion is the paradox. We formulate it now in somewhat different form. Physically we expect that the cat is either living or dead. Therefore, even if noone looks at it, its state must be either \(|a >\) or \(|d >\). In contrast, if one calculates the state of the cat, i.e., its reduced density matrix, it is

\[ \hat{\rho}_c = |d > |\alpha|^2 < d| + |a > |\beta|^2 < a| , \]  

which contains both states. One may make the contradiction even sharper, if an observer is also included (here we assume the universal applicability of the quantum mechanical description). Denoting the state of the observer by \(|o_d >\) if he finds the cat dead and by \(|o_a >\) if he finds it alive, the state of the whole system will be

\[ |\alpha|+ > |d_+ > |d > |o_d > + \beta|− > |d_− > |a > |o_a > , \]  

and the state of the observer is

\[ \hat{\rho}_o = |o_d > |\alpha|^2 < o_d| + |o_a > |\beta|^2 < o_a| . \]
There is no doubt that in this situation the observer would experience something definite, i.e., its state must be either $|o_a>$ or $|o_d>$. This contradicts Eq.(43), therefore one may say that the quantum mechanical prediction in this situation contradicts the experience.

5.2 Resolution of the paradox according to the present approach

Our approach specifies what corresponds to the experience of the cat or that of the observer (cf. Postulate 8 and the footnote there): it is the internal state of the cat or that of the observer. Applying Postulate 6, the internal state of the cat can be $|a>$ or $|d>$, and the internal state of the observer can be $|o_a>$ or $|o_d>$. This theoretical prediction is in complete agreement with the actual experience, therefore, according to the present approach no paradox appears.

Note that Postulate 9 implies that if the cat’s internal state is $|d>$ ($|a>$), then the observer’s internal state is $|o_d>$ ($|o_a>$) as physically expected.

6 Resolution of the EPR paradox

6.1 A review of the paradox within the framework of the Copenhagen interpretation

Let us review briefly the EPR paradox first within the framework of the Copenhagen interpretation [17]. For simplicity, the paradox will be discussed in the case of a spin system [25]. Consider a system consisting of two different spin-half fermions, which have interacted in the past but which do not interact any longer. Let us concentrate on the spin degrees of freedom. Suppose that the initial state of this two particle system is given by

$$a|1,\uparrow> - b|1,\downarrow> = |2,\uparrow> - |2,\downarrow>,$$  (44)

where $a$ and $b$ are complex numbers, satisfying

$$|a|^2 + |b|^2 = 1.$$  (45)

The notation $|1,\uparrow>$ stands for such a state of the first particle, where the $z$ component of the spin (denoted by $S_{1z}$) has the definite value $+\frac{1}{2}$. The other
notations have an analogous meaning. Consider now what happens if one performs a measurement on the first particle. Let us consider the situation when one measures $\hat{S}_{1z}'$, where the $z'$ axis is obtained from the $z$ axis by a rotation at an angle $\delta$ around the $x$ axis. Note that in this section all the rotations will be given relative to the first (unprimed) coordinate system. This coordinate system will be called the original one. The initial state of the whole system (including the measuring device) is given by

$$|m_0> = (a|1, \uparrow> |2, \downarrow> - b|1, \downarrow> |2, \uparrow>)$$

where $|m_0>$ stands for the initial state of the measuring device. The time evolution during the measurement can be established by using the relations

$$|m_0> |1, \delta, \uparrow> \rightarrow |m_+> |1, \delta, \uparrow>$$

and

$$|m_0> |1, \delta, \downarrow> \rightarrow |m_-> |1, \delta, \downarrow>$$

where

$$|1, \delta, \uparrow> = \cos \left( \frac{\delta}{2} \right) |1, \uparrow> - \sin \left( \frac{\delta}{2} \right) |1, \downarrow>$$

and

$$|1, \delta, \downarrow> = \sin \left( \frac{\delta}{2} \right) |1, \uparrow> + \cos \left( \frac{\delta}{2} \right) |1, \downarrow>$$

Thus the final state of the whole system is

$$|m_+> |1, \delta, \uparrow> \left( a \cos \left( \frac{\delta}{2} \right) |2, \downarrow> + b \sin \left( \frac{\delta}{2} \right) |2, \uparrow> \right)$$

$$+ |m_-> |1, \delta, \downarrow> \left( a \sin \left( \frac{\delta}{2} \right) |2, \downarrow> - b \cos \left( \frac{\delta}{2} \right) |2, \uparrow> \right).$$

According to the Copenhagen interpretation, the wave function collapses and thus, if $S_{1,z}' = +\frac{1}{2}$ has been measured, the second particle will have the wave function

$$\left( |a|^2 \cos^2 \left( \frac{\delta}{2} \right) + |b|^2 \sin^2 \left( \frac{\delta}{2} \right) \right)^{-\frac{1}{2}} \left( a \cos \left( \frac{\delta}{2} \right) |2, \downarrow> + b \sin \left( \frac{\delta}{2} \right) |2, \uparrow> \right)$$
which is the eigenfunction of the $\hat{S}_{2,z''}$ spin component corresponding to the eigenvalue $-\frac{1}{2}$. Here $z''$ stands for the $z$ axis of a new coordinate system obtained from the original one by a rotation with the Eulerian angles $\alpha, \beta, \gamma$ given by

$$\exp(i\alpha) = \frac{ab}{|ab|} ,$$

(53)

$$\tan \left( \frac{\beta}{2} \right) = \frac{|b|}{|a|} \tan \left( \frac{\delta}{2} \right) ,$$

(54)

$$\exp(-i\gamma) = \frac{b/a}{|b/a|} .$$

(55)

Similarly, if $S_{1,z'} = -\frac{1}{2}$ has been measured, the wave function of the second particle will be

$$\left( |a|^2 \sin^2 \left( \frac{\delta}{2} \right) + |b|^2 \cos^2 \left( \frac{\delta}{2} \right) \right)^{-\frac{1}{2}} \left( a \sin \left( \frac{\delta}{2} \right) |2, \downarrow> - b \cos \left( \frac{\delta}{2} \right) |2, \uparrow> \right)$$

(56)

which is the eigenfunction of the $\hat{S}_{2,z'''}$ spin component corresponding to the eigenvalue $+\frac{1}{2}$. Here $z'''$ stands for the $z$ axis of another coordinate system obtained from the original one by a rotation with the Eulerian angles $\alpha', \beta', \gamma'$ given by

$$\exp(-i\alpha') = \frac{ab}{|ab|} ,$$

(57)

$$\tan \left( \frac{\beta'}{2} \right) = \frac{|a|}{|b|} \tan \left( \frac{\delta}{2} \right) ,$$

(58)

$$\exp(i\gamma') = \frac{a/b}{|a/b|} .$$

(59)

Summing up, having performed the measurement of $\hat{S}_{1,z'}$, if the result is $S_{1,z'} = +\frac{1}{2}$, one can predict for sure that $S_{2,z''} = -\frac{1}{2}$, and if the result is
\( S_{1,z'} = -\frac{1}{2} \), then one can predict for sure that \( S_{2,z'''} = +\frac{1}{2} \). Obviously, the second particle has not been disturbed in any way. According to the EPR argument, if the value of a quantity can be predicted without disturbing the corresponding system, then there must exist an element of the physical reality that corresponds to this quantity.

Therefore, either \( S_{2,z''} \) or \( S_{2,z'''} \) is an element of the reality. But these quantities depend on the measurement done on the first particle, as they depend on \( \delta \). One expects that

STATEMENT I. the elements of the reality attached to the second particle are independent of what have happened with the first particle, as they are separated and do not interact.

Therefore, it does not matter that one can perform on the first particle only one measurement at the same time (cf. the next to last paragraph in [17]). Either \( S_{2,z''} \) or \( S_{2,z'''} \) must be an element of the physical reality for any \( \delta \)-value.

We may set different values for \( \delta \), say

a, \( \delta = 0 \)

and

b, \( \delta = \vartheta \neq 0 \).

In the first case, \( z'' \) and \( z''' \) coincide with the \( z \) axis. Thus we see that either \( S_{2,z} \) and \( S_{2,z''} \) or \( S_{2,z} \) and \( S_{2,z'''} \) are elements of the reality at the same time (here \( z'' \) and \( z''' \) correspond to \( \delta = \vartheta \)).

On the other hand, \( \hat{S}_{2,z} \) commutes neither with \( \hat{S}_{2,z''} \), nor with \( \hat{S}_{2,z'''} \) (except for some special values of \( \vartheta \) that are not of interest now) that implies that these quantities can be elements of the reality at the same time only if

STATEMENT II. the quantum mechanical description is incomplete.

(cf. the alternatives (1) and (2) in [17].) As has been shown, STATEMENT I. implies STATEMENT II.

The first statement is usually identified with the requirement of the locality. Therefore, Einstein, Podolsky and Rosen accept that it is true, thus they conclude that STATEMENT II. is also true, i.e., the quantum mechanical description is incomplete.

Sometimes people reject STATEMENT II. and therefore reject STATEMENT I. as well, saying that according to quantum mechanics, Nature is nonlocal [20].
6.2 Solution of the paradox according to the present approach

Let us consider now what the present approach says. This time we have to consider an isolated system (otherwise most of our postulates do not work), therefore we include the preparation process which prepares the state \( |44\rangle \). The macroscopic preparation device \( M_p \) and the two particle system \( P_1 + P_2 \) together are assumed to constitute initially an isolated system\(^{12}\). The initial internal state of the system \( M_p + P_1 + P_2 \) evolves unitarily (according to the time dependent Schrödinger equation) to an entangled state which can be written, using the Schmidt canonical form (24) as

\[
|\mu, 1, 2\rangle = \sum_j c_j |\mu_j\rangle + \alpha |\mu_p\rangle |\psi_p\rangle
\]

(60)

Here \( |\psi_p\rangle \) stands for the prepared state \( |44\rangle \), the states \( |\psi_j^\perp(1, 2)\rangle \) are orthogonal to it and together constitute an orthonormed set of states. Note that the preparation device is constructed such a manner (mathematically, its Hamiltonian has such a form) that the state \( |\psi_p\rangle \) will be practically the same, while all the other states and coefficients may vary from one preparation process to the other, corresponding to the different initial conditions. The states \( |\mu_j\rangle, |\mu_p\rangle \) of the preparation device also constitute an orthonormed set of states. It is important to note that the coordinate dependence of the state \( |\psi_p\rangle \) (that is suppressed in Eq. (44)) is such that it describes the two particle system entering the apparatus where the measurement will be done. In contrast, the states \( |\psi_j^\perp(1, 2)\rangle \) describe the two particle system when it does not enter the apparatus. This implies mathematically that the dynamics of the measurement done on the first particle (cf. Eqs. (47), (48)) should be completed by the rule

\[
|m_0\rangle |\psi_j^\perp(1, 2)\rangle \rightarrow |m_0\rangle |\psi_j^\perp(1, 2)\rangle ,
\]

(61)

Therefore, the state of the isolated system \( M_p + P_1 + P_2 + M \) after the measurement can be written as

\[
|\mu, 1, 2, m\rangle = |m_0\rangle \left( \sum_j c_j |\mu_j\rangle + |\psi_j^\perp(1, 2)\rangle \right)
\]

\(^{12}\) We may include in \( M_p \) even that part of the environment which interacts with the actual preparation device.
According to the rules given in the previous Section (see Postulate 4 and Definition 3), the possible internal states of the two particle system are

$$|\psi_{\pm}(1,2)\rangle,$$

(63)

and

$$\left(|a|^2 \cos^2 \left(\frac{\delta}{2}\right) + |b|^2 \sin^2 \left(\frac{\delta}{2}\right)\right)^{-\frac{1}{2}} \times |\cos \left(\frac{\delta}{2}\right)|_{1,\delta,\downarrow} + b \sin \left(\frac{\delta}{2}\right)|_{2,\uparrow}.$$

(64)

The states (63) correspond to the case when the preparation was not successful and the first particle did not interact with the measuring device $M$. Certainly, this case is of no interest for us, and we shall concentrate on the states (64), (65). Choosing the reference system $R$ to be the two particle system $P_1 + P_2$, we get that the state of the second particle with respect to the two particle system is

$$\dot{\rho}_{P_2}(P_1 + P_2) = |\xi><\xi|,$$

(66)

where

$$|\xi> = \left(|a|^2 \cos^2 \left(\frac{\delta}{2}\right) + |b|^2 \sin^2 \left(\frac{\delta}{2}\right)\right)^{-\frac{1}{2}} \times \left(a \cos \left(\frac{\delta}{2}\right)\right)|_{1,\delta,\downarrow} + b \sin \left(\frac{\delta}{2}\right)|_{2,\uparrow}.$$

(67)
if the internal state of the two particle system coincides with Eq.(64) and it is
\[ |\xi> = \left( |a|^2 \sin^2 \left( \frac{\delta}{2} \right) + |b|^2 \cos^2 \left( \frac{\delta}{2} \right) \right)^{-\frac{1}{2}} \times \left( a \sin \left( \frac{\delta}{2} \right) |2, \downarrow> - b \cos \left( \frac{\delta}{2} \right) |2, \uparrow> \right), \tag{68} \]
if the internal state of the two particle system coincides with Eq.(65). Comparing Eqs.(67), (68) with Eqs.(52), (56), respectively, we see that the state of the second particle which was derived using the assumption of the collapse of the wave function is actually \( \hat{\rho}_{P_2}(P_1 + P_2) \). For a more general derivation of this result see Appendix B.

In the present approach the states (which depend on the quantum reference systems) represent the elements of the reality\(^1\). The correspondence with the earlier discussion is clear: \( S_{2,z''} \) is said to be an element of the reality if \( |\xi> \) (cf. Eq.(66)) is an eigenstate of \( \hat{S}_{2,z''} \). The paradox can be formulated now that if one rejects STATEMENT II., then one has to reject STATEMENT I. as well, i.e., the state \( \hat{\rho}_{P_2}(P_1 + P_2) \) representing an element of the reality attached to the second particle depends on the measurement done on the first particle. But this state describes \( P_2 \) with respect to the quantum reference system \( P_1 + P_2 \). Therefore we conclude as follows:

The dependence of the state \( \hat{\rho}_{P_2}(P_1 + P_2) \) on the measurement done on the first particle is attributed now to the disturbance of the quantum reference system \( P_1 + P_2 \) rather than to any nonlocal influence on the second particle.

Hence, one can see that in the present approach STATEMENT I. is not necessarily implied by locality. What locality actually implies is that the internal state \( \hat{\rho}_{P_2}(P_2) \) of the second particle must be independent of the measurement done on the first particle, as in this case neither the system to be described, nor the quantum reference system is influenced by the measurement.

---

\(^{13}\)Note that from this point of view the EPR criterion concerning the elements of the reality is only a sufficient one.
Let us show now that it is fulfilled. (As we have already postulated the unitary time evolution of the internal states of closed systems, here we actually check the consistency of the postulates.) The arguments become more transparent if we consider more generally the question whether the internal state of a system $A = P_2$ is influenced by the interaction of another, separated system $B = P_1$ with a further separated system $C = M$, assuming that $A$ and $B$ have interacted before, while $A$ and $C$ not. The composed system $A + B + C$ is assumed to be an isolated one. Under these assumptions the internal state of the system $A + B + C$ can be written as

$$|\psi_{A+B+C}(0)\rangle = \left(\sum_j c_j |\phi_{A,j}\rangle |\phi_{B,j}\rangle\right) |\psi_C\rangle. \quad (69)$$

As before, $|\phi_{A,j}\rangle$ and $|\phi_{B,j}\rangle$ stand for the possible internal states of $A$ and of $B$, respectively, while $|\psi_C\rangle$ is the internal state of $C$ (it is still isolated). According to the assumptions, the time evolution operator $\hat{U}_t$ factorizes as

$$\hat{U}_t = \hat{U}_t(A)\hat{U}_t(B+C), \quad (70)$$

where the arguments correspond to the systems whose Hilbert space is acted upon by the operators. Therefore,

$$|\psi_{A+B+C}(t)\rangle = \sum_j c_j \left(\hat{U}_t(A)|\phi_{A,j}\rangle\right) \left(\hat{U}_t(B+C)|\phi_{B,j}\rangle |\psi_C\rangle\right). \quad (71)$$

As $\hat{U}_t(A)|\phi_{A,j}\rangle$ and $\hat{U}_t(B+C)|\phi_{B,j}\rangle$ are orthonormed set of states in the Hilbert space of $A$ and of $B + C$, respectively, they are the possible internal states of $A$ and of $B + C$, respectively, according to the reverse of Proposition 1. Thus we see that the possible internal states of $A$ are independent of the interaction of $B$ and $C$. According to Postulate 11 the actual internal state of $A$ is independent of it, too. We get the same result if we apply Eq.(39) with the replacement $M = B + C$. Hence, we can say that quantum mechanics fulfills any reasonable requirement of locality.

14For $t = 0$ it follows by assumption, and for later times it is due to the unitarity of the time evolution operators $\hat{U}_t(A), \hat{U}_t(B+C)$. 

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7 Violation of Bell’s inequality

We discuss now the question of the violation of Bell’s inequality[5]. As a brief reminder, let us think again of two spin-half particles in an entangled state, which is due to some previous interaction between them. Suppose that the particles are already separated so much that they can no longer interact with each other. Imagine that we perform spin measurements in different directions on each of the two separated particles. It is natural to assume that any correlation between the results of the measurements performed on the different particles can come only from the previous interaction which created the entangled state. One may also assume that there are some stable properties attached to each system, so that these properties ‘store’ the correlation after the systems have become separated. Using these assumptions one may derive that the correlations cannot be arbitrary but must satisfy a certain inequality, namely

\[ P(\alpha+, \beta+) \leq P(\alpha+, \gamma+) + P(\gamma+, \beta+) \]  

(72)

This is Bell’s inequality. The correlations may be calculated quantum mechanically, and the quantum prediction does not always satisfy Bell’s inequality. Correlations are measurable quantities, and experiments[19] have proved the correctness of the quantum prediction and thus the violation of Bell’s inequality.

Most people seem to believe that the above result implies that separated systems can influence each other (‘nonlocality’). This belief is based on the careful analysis of the above sketched derivation of Bell’s inequality. It turns out that this derivation is completely independent of quantum mechanics, and it is based on a few very fundamental assumptions[20, 21]: realism, inductive inference and Einstein separability. Realism and inductive inference are so important in physics that we certainly do not want to give up them. The usual conclusion is that Einstein separability is violated.

Nevertheless, we maintain that such a conclusion is physically unacceptable. The principle of Einstein separability has served us well in every branch of physics, even in quantum physics (apart from the measurement process),

\[ P(\alpha+, \beta+) \]  

stands for the probability that measuring the spin of the first particle along a direction at an angle \( \alpha \) compared to the \( z \) axis, the result is \( + \frac{1}{2} \) and at the same time measuring the spin of the second particle along a direction at an angle \( \beta \) compared to the \( z \) axis, the result is \( + \frac{1}{2} \).
including the most sophisticated quantum field theories. The only way out can be if there is some further, independent and hidden assumption, which seems to us obvious, but which is not valid in quantum mechanics. Then the violation of Bell’s inequality implies the invalidity of this assumption rather than that of Einstein separability or locality.

In this section we show that it is indeed the case. The hidden, not allowed assumption mentioned above is connected to the fact that in the present theory it may happen that different states (corresponding to different quantum reference systems), although individually exist, cannot be compared (cf. Section 3). In case of the violation of Bell’s inequality it turns out that the states of the measuring devices and those which ‘store’ the correlations are not comparable as any attempt for a comparison changes the correlations themselves. Therefore, the usual picture about stable properties which ‘store’ the correlations and are comparable in principle at any time with anything does not apply, although the correlations may be attributed exclusively to the ‘common past’ (previous interaction) of the particles.

We discuss now quantitatively the situation mentioned above, i.e., when measurements on both particles are performed. Before the measurements the internal state of the isolated system \(P_1 + M_1 + P_2 + M_2\) (\(P_1, P_2\) standing for the particles and \(M_1, M_2\) for the measuring devices, respectively) is given by

\[
\left( \sum_j c_j |\phi_{P_1,j} > |\phi_{P_2,j} > \right) |m_0^{(1)} > |m_0^{(2)} > ,
\]

while it is

\[
\sum_j c_j \hat{U}_t(P_1 + M_1) \left( |\phi_{P_1,j} > |m_0^{(1)} > \right) \hat{U}_t(P_2 + M_2) \left( |\phi_{P_2,j} > |m_0^{(2)} > \right) ,
\]

with a time \(t\) later, i.e. during and after the measurements. In Eqs.\((73)\), \((74)\) the general notation have been used in order to exhibit the mathematical structure. Evidently, \(j\) can take on the value 1 and 2, \(c_1 = a, c_2 = -b\) (\(|a|^2 + |b|^2 = 1\) and

\[
|\phi_{P_1,1} > = |1, \uparrow>
|\phi_{P_1,2} > = |1, \downarrow>
|\phi_{P_2,1} > = |2, \downarrow>
|\phi_{P_2,2} > = |2, \uparrow> .
\]
According to Eq. (74) $P_1 + M_1$ and $P_2 + M_2$ are closed systems, their internal states evolve unitarily and do not influence each other. This time evolution can be derived from the relations

$$|\xi(P_i, j)| > |m_0^{(i)}| \rightarrow |\xi(P_i, j)| > |m_j^{(i)}|,$$

(76)

where $i, j = 1, 2$ and $|\xi(P_i, j)|$ is the $j$-th eigenstate of the spin measured on the $i$-th particle along an axis $z^{(i)}$ which closes an angle $\vartheta_i$ with the original $z$ direction. Denoting the internal state of $P_i$ by $|\psi_{P_i}|$, we get

$$|\psi_{P_i}| > |m_0^{(i)}| \rightarrow \sum_j <\xi(P_i, j)|\psi_{P_i}| > |\xi(P_i, j)| > |m_j^{(i)}|.$$

(77)

As we see, the $i$-th measurement process is completely determined by the initial internal state of the particle $P_i$. Therefore, any correlation between the measurements may only stem from the initial correlation of the internal states of the particles.

Using Eq. (73) the final state (74) may be written as

$$\sum_{j, k} \left( \sum_l c_l <\xi(P_1, j)|\phi_{P_1, l}| <\xi(P_2, k)|\phi_{P_2, l}| > \right)$$

$$\times |m_j^{(1)}| > |m_k^{(2)}| > |\xi(P_1, j)| > |\xi(P_2, k)| >.$$

(78)

According to Postulate 6 and Definition 3 the possible internal states of $M_1$ and $M_2$ are the $|m_j^{(1)}| >$-s and the $|m_k^{(2)}| >$-s, respectively.

The probability to observe the $j$-th result (up or down spin in a chosen direction) in the $i$-th measurement ($i = 1, 2$) is $P(M_i, j) = \sum_l |c_l|^2 <\xi(P_i, j)|\phi_{P_i, l}|^2$. This may be interpreted in conventional terms: $|c_l|^2$ is the probability that $|\psi_{P_i}| = |\phi_{P_i, l}|$, and $| <\xi(P_i, j)|\phi_{P_i, l}| > |^2$ is the conditional probability that one gets the $j$-th result if $|\psi_{P_i}| = |\phi_{P_i, l}|$. Thus we see that the initial internal state of $P_i$ determines the outcome of the $i$-th measurement in the usual probabilistic sense. But doesn’t it mean that the internal states of $P_1$ and $P_2$ play the role of local hidden variables? No, because hidden variables are thought to be comparable with the results of the measurements so that their joint probability may be defined, while in our theory there is no way to define the joint probability $P(P_1, l_1, P_2, l_2, (0); M_1, j, M_2, k, (t))$, i.e., the probability that initially $|\psi_{P_1}| = |\phi_{P_1, l_1}|$ and $|\psi_{P_2}| = |\phi_{P_2, l_2} |$ and
finally $|\psi_{M_1} > = |m_j^{(1)} >$ and $|\psi_{M_2} > = |m_k^{(2)} >$. Intuitively we would write

$$P(P_1, l_1, P_2, l_2, (0); M_1, j, M_2, k, (t)) = |c_1|^2 \delta_{l_1,l_2} \langle \xi(P_1,j) | \phi_{P_1,l_1} > |^2 \langle \xi(P_2,k) | \phi_{P_2,l_2} > |^2;$$

as $|c_1|^2 \delta_{l_1,l_2}$ is the joint probability that $|\psi_{P_1} > = |\phi_{P_1,l} >$ and $|\psi_{P_2} > = |\phi_{P_2,l} >$, and $| \langle \xi(P_i,j) | \phi_{P_i,l_i} > |^2$ is the conditional probability that one gets the $j$-th result in the $i$-th measurement if initially $|\psi_{P_i} > = |\phi_{P_i,l_i} >$ ($i = 1, 2$). Certainly the existence of such a joint probability would immediately imply the validity of Bell’s inequality, thus it is absolutely important to understand why this probability does not exist. Here we mention that the so called modal interpretation [11], which is equivalent with assuming the existence of the internal state of the subsystem in an absolute sense (i.e., independently of quantum reference systems) would lead to Bell’s inequality. Indeed, the absolute existence of the the initial internal states of $P_1$ and of $P_2$ would mean that initially the particle pairs can be conceived as if they were sorted according to the internal state of $P_1$ and of $P_2$, and in each case one can consider the measurements as done independently (due to their separation) on the corresponding internal state, that implies Eq.(79).

Returning to our approach, let us remark, first of all, that using Postulate 9, we may calculate the correlation between the measurements, i.e., the joint probability that $|\psi_{M_1} > = |m_j^{(1)} >$ and $|\psi_{M_2} > = |m_k^{(2)} >$. We obtain

$$P(M_1, j, M_2, k) = \left| \sum_l c_l \langle \xi(P_1,j) | \phi_{P_1,l} > \langle \xi(P_2,k) | \phi_{P_2,l} > \right|^2. \tag{80}$$

This is the usual quantum mechanical expression which violates Bell’s inequality and whose correctness is experimentally proven. Thus our theory gives the correct expression for the correlation. Nevertheless, if the joint probability (79) exists, it leads to

$$P(M_1, j, M_2, k) = \sum_l |c_l|^2 | \langle \xi(P_1,j) | \phi_{P_1,l} > |^2 | \langle \xi(P_2,k) | \phi_{P_2,l} > |^2 \tag{81}$$

which satisfies Bell’s inequality and contradicts Eq.(80).

Let us demonstrate that no such contradiction appears. Evidently, the joint probability $P(P_1, l_1, P_2, l_2, (0); M_1, j, M_2, k, (t))$ can be physically meaningful only if one can compare the initial internal states of $P_1$ and $P_2$ with the
final internal states of $M_1$ and $M_2$ by suitable nondisturbing measurements. If we try to compare the initial internal states of $P_1$ and of $P_2$ with the final internal states of $M_1$ and $M_2$, the first difficulty appears because we want to compare states given at different times. Nevertheless, as the initial internal state of $P_i$ is uniquely related to the final internal state of the system $P_i + M_i$, the joint probability $P(P_1, l_1, P_2, l_2, (0); M_1, j, M_2, k, (t))$ (if exists) coincides with $P(P_1 + M_1, l_1, P_2 + M_2, l_2, M_1, j, M_2, k)$, where all the occurring states are given after the measurements. As the systems $P_1 + M_1$, $P_2 + M_2$, $M_1$, $M_2$ are not disjointed, our Postulates do not provide us with an expression for the joint probability we are seeking for. If we check $|\psi_{M_1} >$ and $|\psi_{M_2} >$ by suitable nondisturbing measurements, we destroy $|\psi_{P_1 + M_1} >$ and $|\psi_{P_2 + M_2} >$, inhibiting any comparison. On the other hand, if we check first $|\psi_{P_1 + M_1} >$ and $|\psi_{P_2 + M_2} >$, then $P(M_1, j, M_2, k)$ changes. In fact, after suitable measurements performed on $P_i + M_i$ (which do not change the internal states of $P_i + M_i$) by further measuring devices $\tilde{M}_i$ we get for the internal state of the whole system

$$
\sum_l c_l \left( \sum_j <\xi(P_1, j)|\phi_{P_1, l} > |\xi(P_1, j) > |m_j^{(1)} > \right) \times \left( \sum_k <\xi(P_2, k)|\phi_{P_2, l} > |\xi(P_2, k) > |m_k^{(2)} > \right) |\tilde{m}_l^{(1)} > |\tilde{m}_l^{(2)} > .
$$

This is exactly the same expression as that we would get if the initial internal states of $P_1$ and $P_2$ were recorded by $\tilde{M}_1$ and $\tilde{M}_2$, respectively. As the systems $M_1$, $M_2$, $\tilde{M}_1$, $\tilde{M}_2$ are disjointed, we may apply Postulate 10 for $n = 4$ and we indeed get for $P(\tilde{M}_1, l_1, \tilde{M}_2, l_2, M_1, j, M_2, k)$ the expression (81). Do we get then a contradiction with Eq.(80)? No, because applying Postulate 9 directly, we get in this case Eq.(82) instead of Eq.(80). Thus we see that the extra measurements have changed the correlations and our theory gives account of this effect consistently.

Note that the extra measurements have not changed the internal state of $P_1$ and of $P_2$, nor have they modified the time evolution of the internal state of $P_1 + M_1$ and of $P_2 + M_2$ during the other two measurements. They have, however, changed the correlation between the measurements. It may seem misterious how this can be, once the measurements themselves have not changed. The explanation is again connected to the dependence of the states on the quantum reference systems. Considering, say, the time evolution of the
internal state of $P_1 + M_1$ during the first measurement (using $M_1$), the quantum reference system is $P_1 + M_1$, whose internal state has been the same initially as before. The probability distribution $P(M_1, j) = \sum_k P(M_1, j, M_2, k)$ is unchanged, too (cf. Eqs. (80), (81)). If one wants to see the correlations, the time evolution of the internal state of the system $P_1 + P_2 + M_1 + M_2$ is needed. But then the quantum reference system is $P_1 + P_2 + M_1 + M_2$. One can see here explicitly the role of the fact that the internal state of $P_1 + P_2 + M_1 + M_2$ usually cannot be reconstructed if the internal state of $P_1 + M_1$ and that of $P_2 + M_2$ are given. When the measurements by $\tilde{M}_1$ and $\tilde{M}_2$ have been previously performed, the internal state of $P_1 + P_2 + M_1 + M_2$ is

$$|\psi_{P_1 + P_2 + M_1 + M_2} > = |\psi_{P_1 + M_1} > |\psi_{P_2 + M_2} >.$$  (83)

In the absence of the extra measurements, however, it is

$$|\psi_{P_1 + P_2 + M_1 + M_2} > = \sum_j c_j |\phi_{P_1 + M_1, j} > |\phi_{P_2 + M_2, j} > \neq |\psi_{P_1 + M_1} > |\psi_{P_2 + M_2} >.$$ (84)

This difference is responsible for the violation of Bell’s inequality, while all the interactions take place locally.

Summarizing, we have seen that the initial internal state of $P_1 (P_2)$ determines the first (second) measurement process, therefore, these states ‘carry’ the initial correlations and ‘transfer’ them to the measuring devices. As the measurement processes do not influence each other, the observed correlations may stem only from the ‘common past’ of the particles. On the other hand, any attempt to compare the initial internal states of $P_1$ and $P_2$ with the results of both measurements changes the correlations, thus a joint probability for the simultaneous existence of these states cannot be defined. This means that the reason for the violation of Bell’s inequality is that the usual derivations always assume that the states (or ‘stable properties’) which carry the initial correlations can be freely compared with the results of the measurements. This comparability is usually thought to be a consequence of realism. According to the present theory, the above assumption goes beyond the requirements of realism and proves to be wrong, because each of the states $|\psi_{P_1 + M_1} >$, $|\psi_{P_2 + M_2} >$, $|\psi_{M_1} >$ and $|\psi_{M_2} >$ exists individually, but they cannot be compared without changing the correlations.
8 Conclusion

A new theoretical framework for nonrelativistic quantum mechanics has been presented. It coincides with the usual one (especially, Schrödinger’s equation is not modified), except for the postulates concerning the measurement. Instead of these latter, a dependence of the states on quantum reference systems (they are themselves physical systems) is postulated. The relations among the different kinds of states have been consistently postulated allowing one to recover all the usual predictions concerning the experiments and also to resolve the well known old paradoxes (Schrödinger’s cat paradox and the EPR paradox). It has also been shown that despite of the violation of Bell’s inequality correlations between separated systems can always be attributed to a previous interaction (‘common past’), i.e., quantum mechanics is a local theory. An important feature of the present approach is that measurements and macroscopic systems do not play a privileged role. Measurements are considered as usual interactions corresponding to some special Hamiltonians. An a priori classical background is also absent. In principle this renders possible to check the validity of the present approach, as the observed macroscopic properties, first of all, the localization of the internal states of macroscopic bodies in both coordinate and momentum space should follow from the theory when using realistic assumptions concerning the relevant energies, structure and interactions. Demonstrating this would be a convincing argument in favour of the present approach. (Note that it would also demonstrate that classical mechanics can be derived from quantum mechanics.) The solution of this question needs, however, a lot of further work and is beyond the scope of the present paper.

Summarizing, the achievements of the theory are the following:

i, it gives a foundation of quantum mechanics independently of classical physics
ii, it is free from inconsistencies
iii, it gives in principle concrete and well defined predictions concerning the possible results of a measurement and their probabilities, provided the initial quantum state of the measuring device and the measured object is given.
iv, it resolves the famous paradoxes (Schrödinger’s cat and EPR), and
v, explains why the violation of Bell’s inequality does not imply nonlocality.

Let us consider now the question how the the dependence on the usual coordinate frames enters the present formalism, or, more generally, how the
canonical transformations appear. To this end, let us remark first that we considered all the time nonrelativistic quantum mechanics, and therefore all the physical systems has been assumed to consist of a given number of particles which never decay. Let us perform some unitary transformations in the Hilbert spaces of all the particles. It is easy to see that the whole formalism presented in Section 3 is covariant with respect to these transformations, i.e, the same relations hold among the transformed quantities as have previously held among the untransformed ones. (Note that the Hamiltonian will not be invariant in general, unless in the special case of symmetry transformations.)

The reason is that the operations occuring in the Postulates (i.e., the trace operation and the calculation of eigenvalues and eigenvectors of density matrices) are themselves covariant. Hence the present approach has the same covariance properties with respect to unitary transformations as traditional quantum mechanics.

As for the link to traditional coordinate systems let us consider some group of geometric symmetry (e.g. the rotation group) and a multitude of isolated systems $I_j$ which will play the role of coordinate systems. Suppose that each $I_j$ has a distinguished subspace in its Hilbert space such a way that each state can be transformed by the application of a suitable group element into a state lying in this subspace, while any state in the subspace leaves the subspace under the application of any group element. The state of $Q$ with respect to the quantum reference system $Q + I_j$ coincides with the internal state of $Q$, as $I_j$ is isolated. This state, however, may be still transformed by any element of the symmetry group to get an equivalent description. Let us choose that group element which transfers the internal state of $I_j$ into its distinguished subspace. The state of $Q$ we get this way is what one can call the description of the quantum system $Q$ with respect to the coordinate system $I_j$.

Finally, let us emphasize, that according to the present theory, there is no collapse of the wave function, there are no parallel worlds, no nonlocal interactions, no corrections contributing to Schrödinger’s equation if it is applied to macroscopic systems, and gravity has nothing to do with quantum measurements. There is, however, a new principle, the dependence of the states on quantum reference systems, which removes inconsistencies from quantum mechanics and which might serve as a guide if one wants to establish

\footnote{As is well known, such groups have unitary representations in Hilbert spaces.}
quantum theory in a new field.

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A Proof of Proposition 1

Let us expand $|\psi_{A+B} >$ (cf. Eq.(24)) in terms of the possible internal states $|\phi_{B,j} >$. (It is possible, as these latter constitute a complete orthonormed set of states.) We get

$$|\psi_{A+B} > = \sum_j d_j |\zeta_j > |\phi_{B,j} >, \quad (85)$$

where $|\zeta_j >$-s are normed states in the Hilbert space of $A$, and $d_j$-s are appropriate coefficients. We prove that the $|\zeta_j >$-s coincide with the possible internal states $|\phi_{A,j} >$. Indeed, calculating $\hat{\rho}_B(A+B)$ (by using Eq.(85) and Postulate 4) one obtains

$$\hat{\rho}_B(A+B) = \sum_j \sum_k d_j^* d_k <\zeta_j|\zeta_k > |\phi_{B,k} >|\phi_{B,j} >. \quad (86)$$

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which must coincide with \( \sum_j |\phi_{B,j} > \lambda_j < \phi_{B,j}| \) (here \( \lambda_j \) stands for the \( j \)-th eigenvalue of \( \hat{\rho}_B(A + B) \)), cf. Definition 3. Therefore, we get

\[
< \zeta_j | \zeta_k > = \delta_{j,k}
\]

and \( |d_j|^2 = \lambda_j \). Calculating now \( \hat{\rho}_A(A + B) \) we get

\[
\hat{\rho}_A(A + B) = \sum_j |\zeta_j > |d_j|^2 < \zeta_j |
\]

which, in view of Eq.(87) shows that \( |\phi_{A,j} >= |\zeta_j >. \)

As a final remark, suppose that \( |\xi_j > \) and \( |\chi_j > \) are arbitrary orthonormed set of states in the Hilbert space of \( A \) and of \( B \), respectively, then expanding Eq.(24) on the basis \( |\xi_l > |\chi_m > (l, m = 1, 2, 3, ...) \) one gets

\[
\gamma_{l,m} = \sum_j c_j \alpha_{j,l} \beta_{j,m}
\]

Here \( \gamma_{l,m} = (< \xi_l | < \chi_m |) |\psi_{A+B} >, \alpha_{j,l} = < \xi_l | \phi_{A,j} >, \) and \( \beta_{j,m} = < \chi_m | \phi_{B,j} >. \) The two latter matrices are obviously unitary ones. If \( l \) and \( m \) is restricted to be smaller than some \( N >> 1 \), then Eq.(89) is the well known singular value decomposition\[12\] of the general complex matrix \( \gamma_{l,m}. \)

B Correspondence between the Copenhagen interpretation and the present approach

In a special case (cf. the discussion following Eq.(65) it has already been shown that the state one obtains when using the idea of the collapse of the wave function corresponds in the present approach to the situation when the reference system is chosen to be the complementer system of the measuring device. We derive here this result in more general terms. Suppose that we are given a microscopic system \( Q \) and a measuring device \( M \). They are initially separated and the compound system \( Q + M \) is isolated all the time. One can see that we are still using an idealized model, although it is well known that quantum states of macroscopic systems are extremely sensitive (due to their enormous level density), thus they cannot be practically isolated. For a correct description one should include in \( M \) a large neighbourhood of the measuring device - this, however, does not modify our conclusion. The
reason is that the result of the measurement is a rather robust property, not sharing the sensitivity of the corresponding quantum state. This is why our idealization is acceptable.

Suppose that the measurement on a subsystem $S$ of $Q$ has been performed. Denoting by $|\xi_{S,j}>$ the eigenstates of the measured quantity, the change of the state of the whole system $Q + M$ can be expressed as

$$|m_0 > \sum_j c_j |\xi_{S,j} > |\chi_{Q\setminus S,j} >$$

$$\rightarrow \sum_j c_j |m_j > |\xi_{S,j} > |\chi_{Q\setminus S,j} >,$$

(90)

where $|m_j >$ refers to the possible internal states of the measuring device (this follows from the orthogonality of the states $|\xi_{S,j} >$, cf. Proposition 1), while $c_j |\chi_{Q\setminus S,j} >$-s are (not necessarily orthogonal) ‘coefficient states’ arising when expanding the initial internal state of $Q$ in terms of the states $|\xi_{S,j} >$. The Copenhagen interpretation tells us that due to the collapse of the wave function only one of the terms of the sum in Eq.(90) survives, implying that the states of $M$, $Q$, $S$, and $Q \setminus S$ are $|m_j >$, $|\xi_{S,j} > |\chi_{Q\setminus S,j} >$, $|\xi_{S,j} >$, and $|\chi_{Q\setminus S,j} >$, respectively. According to the present interpretation, the possible internal states of $Q$ are just the $|\xi_{S,j} > |\chi_{Q\setminus S,j} >$-s (due to the orthogonality of the states $|m_j >$), thus, if the $j$-th possible result has been observed, the states $|\xi_{S,j} > |\chi_{Q\setminus S,j} >$, $|\xi_{S,j} >$, and $|\chi_{Q\setminus S,j} >$ may be identified with $\hat{\rho}_Q(Q)$, $\hat{\rho}_S(Q)$ and $\hat{\rho}_{Q\setminus S}(Q)$, respectively. Note that all the latter states are dyads. In conclusion, we have seen that the states obtained using the concept of the collapse of the wave function correspond in the present approach to a specific choice of the reference system, namely, when the reference system is the whole quantum system $Q$, whose subsystem is subject to the measurement.

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