Delay-range-dependent Robust Stability for Uncertain Singular Systems with Interval Time-varying Delays

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Abstract In this paper, we consider the delay-range-dependent robust stability problem of uncertain singular system with interval time-varying delays. Stability criteria, which guarantee the concerned singular system is regular, free and stable, is derived in terms of linear matrix inequalities (LMIs) by introducing some identical equations with integral inequality approach (IIA) and using the Leibniz-Newton formula to reduce the conservatism of the results. The upper bound of time-delay can be obtained by using the modified generalized eigenvalue minimization problem (GEVP) technique such that the system can be stabilized for all time-delays. Finally, numerical examples show the method presented in our paper is more effective and less conservative than the existing ones.

Keywords Singular Time Delay Systems, Integral Inequality Approach (IIA), Delay- Dependence, Linear Matrix Inequality (LMI)

1. Introduction

Engineering processes often involve both nonlinear and time-delay models. In many physical and biological phenomena, the rate of variation in the system state depends on past states. Time delay phenomena were first discovered in biological systems and were later found in many engineering systems, such as mechanical transmissions, fluid transmissions, metallurgical processes, and networked control systems. They are often a source of instability and poor control performance. Therefore, many efforts have been made for the stability problems of various delayed systems [1-2, 4-14, 16-31]. Moreover, because of unavoidable factors, such as modeling error, external perturbation and parameter fluctuation, the time delay systems certainly involve uncertainties such as perturbations and component variations, which will change the stability of time delay systems. And in recent years, the stability analysis issues for time delay systems in the presence of parameter uncertainties perturbations have stirred some initial research attention [4, 5, 8, 12, 16, 19, 20, 22-25, 31].

Singular systems have found numerous practical applications: e.g., engineering systems, social systems, economic systems, biological systems. Unlike classical state space representation via a set of ordinary differential equations, singular system can be viewed as a composite formed by several interconnected systems with two layers: dynamic property described by differential equation and interconnection property expressed by algebraic equation. Therefore, some results on the stability of singular systems are achieved [1, 5-8, 12-18, 20-25, 27-30] and the references therein. The existing stability criteria for singular time-delay systems can be classified into two types: delay-independent [5, 24, 25, 30] and delay-dependent [1, 6-8, 12-13, 16-19, 21-23, 26, 28-29, 31]. Generally, delay-dependent conditions are less conservative than the delay-independent ones, especially when the time delay is small. For singular systems with delays, several kinds of simple Lyapunov–Krasovskii functionals, i.e. functionals parameterized with constant matrices, have been proposed, which lead to different levels of conservatism due to the different model transformations and the bounding techniques for some cross-terms [7, 8]. A tighter bounding for cross-terms can reduce the conservatism. However, there are no obvious ways to obtain less conservative results, even if one is willing to expend more computational effort on the problem, and to find a tighter bound for the cross-terms. This is the serious limitation for these criteria. To overcome this limitation, one has to find some more general Lyapunov-Krasovskii functional (LKF) for handling the delay-range-dependent robust stability problem for singular systems. To the best of our knowledge, this delay-range-dependent robust stability problem has not been fully investigated for singular systems with time-varying delay, which motivates the present study.

On the other hand, the range of time-varying delay systems considered in [17, 18, 20, 27] is from 0 to an upper bound. In practice, a time-varying interval delay is often encountered, that is, the range of delay varies in an interval for which the lower bound is not restricted to 0. In this case,
the stability criteria for time-varying delay systems in [17, 18, 20, 27] are conservative because they do not take into account the information of the lower bound of delay. To the best of authors’ knowledge, there have been few results on the delay-range-dependent robust stability of the singular systems with time-varying interval delays, which remain as an interesting research topic.

Motivated by the above discussions, we propose the improved delay-range-dependent robust stability for singular systems with time-varying interval delay and uncertainties. First, by defining a novel Lyapunov function, a delay-range-dependent stability criterion for the nominal singular time-delay system is established in terms of LMIs. Next, less conservative result is obtained by considering some useful terms when estimating the upper bound of the derivative of Lyapunov functional and introducing the additional terms into the proposed Lyapunov functional, which includes the information of the range. The maximum allowable value of the time delay can be obtained by solving a set of linear matrix inequalities (LMIs) and the additional terms into the proposed Lyapunov functional, which includes the information of the range. The maximum allowable value of the time delay can be obtained by solving a set of linear matrix inequalities (LMIs) and the modified generalized eigenvalue minimization problem (GEVP) algorithm. Finally, Numerical examples demonstrate that the results obtained in this paper are effective and are a significant improvement over previous ones.

2. Stability Description and Preliminaries

Consider the following uncertain singular system with an interval time-varying delay as follows:

\[ E \dot{x}(t) = (A + \Delta A(t))x(t) + (B + \Delta B(t))x(t - h(t)), \quad t > 0, \]  
(1a)

with the initial condition

\[ x(t) = \phi(t), \quad t \in [-h, 0], \]  
(1b)

where \( x(t) \in \mathbb{R}^n \) is the state vector of the system; \( A, B \in \mathbb{R}^{n \times n} \) are constant matrices; The matrix \( E \in \mathbb{R}^{n \times n} \) maybe singular, without loss generality, we suppose \( \text{rank}(E) = r \leq n \); \( \phi(t) \) is a continuously real-valued initial function on \([-h, 0]\). \( h(t) \) is a time-varying delay which satisfies:

\[ h_1 \leq h(t) \leq h_2, \quad \dot{h}(t) \leq h_d < 1, \quad \forall t \geq 0, \]  
(2)

where \( h_1 \) and \( h_2 \) are the lower and upper delay bounds, respectively, \( h_1, h_2, h_d \) are constants and \( 0 \leq h_1 < h_2 \).

Time-varying parametric uncertainties \( \Delta A(t) \) and \( \Delta B(t) \) are assumed to be of the following form:

\[ \begin{bmatrix} \Delta A(t) \\ \Delta B(t) \end{bmatrix} = MF(t) \begin{bmatrix} N_e \\ N_s \end{bmatrix}, \]  
(3)

where \( M, N_e, \) and \( N_s \) are known real constant matrices with appropriate dimensions, and \( F(t) \) is an unknown, real, and possibly time-varying matrix with Lebesgue-measurable elements satisfying

\[ F^T(t)F(t) \leq I, \quad \forall t. \]  
(4)

The main objective is to find the range of \( h_1 \leq h(t) \leq h_2 \) and guarantee stability for the uncertain singular time-varying delay systems (1). Here, definitions fundamental lemmas are reviewed.

Definition 1 [3]: The pair \( (E, A) \) is said to be regular if \( \text{det}(sE - A) \) is not identically zero.

Definition 2 [3]: The pair \( (E, A) \) is said to be impulse free if \( \text{deg}(\text{det}(sE - A)) = \text{rank} E \).

Definition 3 [3]: For a given scalar \( \bar{h} > 0 \), the singular time-varying delay system (1) is said to be regular and impulse free for any constant time delay \( h \) satisfying \( 0 \leq h \leq \bar{h} \), it the pairs \( (E, A) \) and \( (E, A + B) \) are regular and impulse free.

Remark 1: The regularity and the absence of impulses of the pair \( (E, A) \) ensures the system (1) with time delay \( h \neq 0 \) to be regular and impulse free, while the fact that the pair \( (E, A + B) \) is regular and impulse free ensures the system (1) with time delay \( h = 0 \) to be regular and impulse free.

Lemma 1 [15]: The singular system \( E\dot{x}(t) = Ax(t) \) is regular, impulse free, and stable, if and only if there exists a matrix \( P \) such that:

\[ A^TP + PA < 0, \]  
(5a)

with the following constraint

\[ P^TE = E^TP \geq 0. \]  
(5b)

Lemma 2 [17]: For any positive semi-definite matrices:

\[ X = \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{12}^T & X_{22} & X_{23} \\ X_{13}^T & X_{23} & X_{33} \end{bmatrix} \geq 0, \]  
(6a)

the following integral inequality holds

\[ \int_{h_1}^{h_2} \dot{x}^T(s) X \dot{x}(s) ds \leq \int_{h_1}^{h_2} \begin{bmatrix} x^T(t) & x^T(t - h(t)) \end{bmatrix} \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^T & X_{22} \end{bmatrix} \begin{bmatrix} x(t) \\ x(t - h(t)) \end{bmatrix} ds. \]  
(6b)

Lemma 3 [2]. The following matrix inequality:

\[ \begin{bmatrix} Q(x) \\ S'(x) \end{bmatrix} < 0, \]  
(7a)

where \( Q(x) = Q'(x) \), \( R(x) = R'(x) \) and \( S(x) \) depend affine on \( x \), is equivalent to

\[ R(x) < 0, \]  
(7b)

\[ Q(x) < 0, \]  
(7c)
and

$$Q(x) - S(x)R^{-1}(x)S^T(x) < 0. \quad (7d)$$

**Lemma 4[2].** Given matrices $Q = Q^T, D, E$, and $R = R^T > 0$ of appropriate dimensions,

$$Q + DF(t)E + E^T F^*(t)D^* < 0, \quad (8a)$$

for all $F(t)$ satisfying $F^*(t)F(t) \leq I$, if and only if there exists some $e > 0$ such that

$$Q + eDD^* + e^{-1}EE < 0. \quad (8b)$$

In this paper, a new Lyapunov functional is constructed, which contains the information of the lower bound of delay $h_1$ and upper bound $h_2$.

The nominal unforced time delay singular system (1) can be written as follows:

$$\dot{E}\dot{x}(t) = Ax(t) + Bx(t-h(t)). \quad (9)$$

The following Theorem 1 presents a delay-range- dependent result in terms of LMIs and expresses the relationships between the terms of the Leibniz–Newton formula.

**Theorem 1:** For three given positive scalars $h_1, h_2$, and $h_3$, the time-delay singular system (9) is asymptotically stable if there exist matrices $P = P^T > 0, Q = Q^T > 0, R_j = R_j^T > 0 (j = 1, 2, 3)$, $S$ with appropriate dimensions, positive semi-definite matrices:

$$X = \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ 0 & X_{22} & X_{23} \\ 0 & 0 & X_{33} \end{bmatrix} \geq 0, \quad \text{and} \quad Y = \begin{bmatrix} Y_{11} & Y_{12} & Y_{13} \\ 0 & Y_{22} & Y_{23} \\ 0 & 0 & Y_{33} \end{bmatrix} \geq 0$$

such that the following LMIs hold:

$$\Xi = \begin{bmatrix} \Xi_{11} & \Xi_{12} & 0 & 0 & \Xi_{14} & \Xi_{16} \\ 0 & \Xi_{22} & \Xi_{23} & \Xi_{24} & \Xi_{25} & \Xi_{26} \\ 0 & 0 & \Xi_{33} & 0 & 0 & 0 \\ \Xi_{41} & 0 & 0 & \Xi_{44} & 0 & 0 \\ \Xi_{51} & 0 & 0 & 0 & \Xi_{55} & 0 \\ \Xi_{61} & 0 & 0 & 0 & 0 & \Xi_{66} \end{bmatrix} < 0, \quad (10a)$$

$$E^T(R_1 - X_{13})E \geq 0, \quad (10b)$$

$$E^T(R_2 - Y_{13})E \geq 0, \quad (10c)$$

with the following constraint

$$P^T = E^TP \geq 0. \quad (10d)$$

where $Z \in R^{n \times r}$ is any matrix with full column rank and satisfies $E^Z = 0$ and

$$\Xi_{11} = A^TP + PA + A^T S + S^T A + Q_1 + Q_2 + Q_3$$

$$\Xi_{12} = PB + S^T B + E^T(h_1 X_{12} - X_{13} + X_{23})E,$$

$$\Xi_{15} = h_1 A^T R_1, \Xi_{16} = h_2 A^T R_2,$$

$$\Xi_{22} = -(1 - h_3)Q_1 + E^T(h_2 X_{11} + X_{13} + h_1 X_{22} - X_{23} - X_{13}^T$$

$$+ h_2 Y_{22} - Y_{13} + h_2 Y_{11} + Y_{13} + Y_{13}^T)E,$$

$$\Xi_{33} = E^T(h_2 Y_{22} - Y_{13} + h_2 Y_{11} + Y_{13} + Y_{13}^T)E,$$

$$\Xi_{23} = E^T(h_2 X_{12} - X_{13} + X_{13} + h_1 X_{22} - X_{23} - Y_{13}^T),$$

$$\Xi_{25} = -Q_3 + E^T(h_2 Y_{11} + Y_{13} + Y_{13}^T)E,$$

$$\Xi_{44} = -Q_2 + E^T(h_2 Y_{22} - Y_{13} - Y_{13} + h_2 X_{22} - X_{23} - X_{13}^T),$$

$$\Xi_{55} = h_1 B^T R_1, \Xi_{66} = h_2 B^T R_2.$$
and Lemma 2, we obtain:
\[ (20) \]
and rank as
\[
\leq -\int_{-h}^{0} \chi^T(s) E Y_3 \dot{E} x(s) ds - \int_{-h}^{0} \chi^T(s) E X_3 \dot{E} x(s) ds,
\]
and
\[
\leq -\int_{-h}^{0} \chi^T(s) E Y_3 \dot{E} x(s) ds \leq (t - h(t)) [\dot{h}_2 Y_1 + \dot{Y}_1 + Y_1] x(t - h(t)) + x^T(t - h(t)) [\dot{h}_2 Y_2 + \dot{Y}_2 + Y_2] x(t - h(t)) + x^T(t) [\dot{h}_2 Y_2 - Y_2] x(t - h(t)) + x^T(t - h(t))[h_2 Y_2 - Y_2] x(t - h(t)) \]

With the operator for the term \( x^T(t) E R \dot{E} x(t) \) as follows:
\[ \dot{h}_2 R_1 + h_2 R_2 ) E x(t) \]

To obtain an easily solvable LMI, we introduce the matrix
\[ Z \in R^{m \times r} \]

Substituting the above equations (13)-(20) into (12) yields
\[ \dot{Y}(x) \leq \xi^T(t) \Omega \xi(t) - \int_{-h}^{0} \xi^T(s) E R (X_3 - X_3) E \dot{E} x(s) ds - \int_{-h}^{0} \chi^T(s) E R (Y_3 - Y_3) \dot{E} x(s) ds \]
and
\[
\xi(t) = \begin{bmatrix} x^T(t) & x^T(t - h(t)) & x^T(h(t)) & x^T(h(t)) \end{bmatrix}
\]
and
\[
\Omega = \begin{bmatrix}
\Omega_{11} & \Omega_{12} & 0 & 0 \\
\Omega_{12} & \Omega_{22} & \Omega_{23} & \Omega_{24} \\
0 & \Omega_{33} & \Omega_{34} & 0 \\
0 & \Omega_{44} & 0 & \Omega_{44}
\end{bmatrix}.
\]

Similarly, we have
\[ \leq (1 - h_2) \dot{Q}_2 + E (h_2 X_1 - X_1 + X_1^T) E + A^T (h_2 R_1 + h_2 R_2) A,
\]
\[ \Omega_{11} = A^T P + PA + A^T Z S^T + S Z^T A + Q_1 + Q_2 + Q_3 + P Z B + S Z^T B + E^T (h_2 X_1 - X_1 + X_1^T) E + A^T (h_2 R_1 + h_2 R_2) A,
\]
\[ \Omega_{12} = \Omega_{21} = 0, \]
\[ \Omega_{22} = \Omega_{33} = \Omega_{44} = 0, \]
\[ \Omega_{12} = \Omega_{21} = 0, \]
\[ \Omega_{33} = Q_1 + E^T (h_2 Y_1 - Y_1 + Y_1^T) E, \]
\[ \Omega_{44} = Q_2 + E^T (h_2 Y_2 - Y_2 + Y_2^T) E + h_2 Y_2 - Y_2 + Y_2^T, \]
\[ \Omega_{33} = Q_3 + E^T (h_2 Y_2 - Y_2 + Y_2^T) E + h_2 Y_2 - Y_2 + Y_2^T, \]
\[ \Omega_{44} = Q_4 + E^T (h_2 Y_2 - Y_2 + Y_2^T) E + h_2 Y_2 - Y_2 + Y_2^T, \]
\[ \Omega_{33} = Q_3 + E^T (h_2 Y_2 - Y_2 + Y_2^T) E + h_2 Y_2 - Y_2 + Y_2^T, \]
\[ \Omega_{44} = Q_4 + E^T (h_2 Y_2 - Y_2 + Y_2^T) E + h_2 Y_2 - Y_2 + Y_2^T, \]
\[ \Omega_{33} = Q_3 + E^T (h_2 Y_2 - Y_2 + Y_2^T) E + h_2 Y_2 - Y_2 + Y_2^T, \]
\[ \Omega_{44} = Q_4 + E^T (h_2 Y_2 - Y_2 + Y_2^T) E + h_2 Y_2 - Y_2 + Y_2^T, \]
\[ \Omega_{33} = Q_3 + E^T (h_2 Y_2 - Y_2 + Y_2^T) E + h_2 Y_2 - Y_2 + Y_2^T, \]
\[ \Omega_{44} = Q_4 + E^T (h_2 Y_2 - Y_2 + Y_2^T) E + h_2 Y_2 - Y_2 + Y_2^T, \]
\[ \Omega_{33} = Q_3 + E^T (h_2 Y_2 - Y_2 + Y_2^T) E + h_2 Y_2 - Y_2 + Y_2^T, \]
\[ \Omega_{44} = Q_4 + E^T (h_2 Y_2 - Y_2 + Y_2^T) E + h_2 Y_2 - Y_2 + Y_2^T, \]
\[ \Omega_{33} = Q_3 + E^T (h_2 Y_2 - Y_2 + Y_2^T) E + h_2 Y_2 - Y_2 + Y_2^T, \]
\[ \Omega_{44} = Q_4 + E^T (h_2 Y_2 - Y_2 + Y_2^T) E + h_2 Y_2 - Y_2 + Y_2^T, \]
\[ \Omega_{33} = Q_3 + E^T (h_2 Y_2 - Y_2 + Y_2^T) E + h_2 Y_2 - Y_2 + Y_2^T, \]
\[ \Omega_{44} = Q_4 + E^T (h_2 Y_2 - Y_2 + Y_2^T) E + h_2 Y_2 - Y_2 + Y_2^T, \]
\[ \Omega_{33} = Q_3 + E^T (h_2 Y_2 - Y_2 + Y_2^T) E + h_2 Y_2 - Y_2 + Y_2^T, \]
\[ \Omega_{44} = Q_4 + E^T (h_2 Y_2 - Y_2 + Y_2^T) E + h_2 Y_2 - Y_2 + Y_2^T, \]
\[ \Omega_{33} = Q_3 + E^T (h_2 Y_2 - Y_2 + Y_2^T) E + h_2 Y_2 - Y_2 + Y_2^T, \]
\[ \Omega_{44} = Q_4 + E^T (h_2 Y_2 - Y_2 + Y_2^T) E + h_2 Y_2 - Y_2 + Y_2^T, \]
complement of Lemma 2 we can get $\Omega < 0$, $E'(R_1 - X_{33})E \geq 0$, $E'(R_2 - Y_{33}) \geq 0$ for any $\xi(t) \neq 0$. Therefore, the interval time-varying delay system (9) is asymptotically stable if (10) is true. Thus, the proof is complete.

3. Robust for Uncertain Singular Interval Time-varying Delay System

In the section, extending Theorem 1 to uncertain singular system (1) with interval time-varying delays yields the following Theorem 2.

Theorem 2: For three given positive scalars $h_1, h_2$, and $h_3$, the uncertain singular time-varying delay system (1) is asymptotically stable if there exist symmetry positive-definite matrices $\eta > 0$, $\zeta > 0$, $\eta > 0$ and matrix $S$ with appropriate dimensions, positive semi-definite matrices

$$X = \begin{bmatrix} X_{11} & X_{12} & \ldots & X_{13} \\ X_{21} & X_{22} & \ldots & X_{23} \\ \vdots & \vdots & \ddots & \vdots \\ X_{13} & X_{23} & \ldots & X_{33} \end{bmatrix} \geq 0,$$

and

$$Y = \begin{bmatrix} Y_{11} & Y_{12} & \ldots & Y_{13} \\ Y_{21} & Y_{22} & \ldots & Y_{23} \\ \vdots & \vdots & \ddots & \vdots \\ Y_{13} & Y_{23} & \ldots & Y_{33} \end{bmatrix} \geq 0$$

such that the following LMIs hold:

$$\Xi = \begin{bmatrix} \Xi_{11} & \Xi_{12} & \ldots & \Xi_{15} \\ \Xi_{12} & \Xi_{22} & \ldots & \Xi_{25} \\ \vdots & \vdots & \ddots & \vdots \\ \Xi_{55} & \Xi_{56} & \ldots & \Xi_{66} \\ \Xi_{56} & \Xi_{66} & \ldots & \Xi_{66} \\ \Xi_{66} & 0 & \ldots & 0 \end{bmatrix} \geq 0,$$

$$\Xi_{11} + \varepsilon N_{11}^T N_{11} \geq 0,$$

$$\Xi_{12} + \varepsilon N_{12}^T N_{12} \geq 0,$$

$$\Xi_{16} + \varepsilon N_{16}^T N_{16} \geq 0,$$

$$\Xi_{55} + \varepsilon N_{55}^T N_{55} \geq 0,$$

where $\Xi_{11} = \Xi_{11} + \varepsilon N_{11}^T N_{11}$, $\Xi_{12} = \Xi_{12} + \varepsilon N_{12}^T N_{12}$, $\Xi_{16} = \Xi_{16} + \varepsilon N_{16}^T N_{16}$, $\Xi_{55} = \Xi_{55} + \varepsilon N_{55}^T N_{55}$, $\Xi_{66} = \Xi_{66} + \varepsilon N_{66}^T N_{66}$, $\varepsilon > 0$, $i < j \leq 6$ are defined in (10).

It is, incidentally, worth noting that the singular uncertain time-varying delay system (1) is asymptotically stable, that is, the uncertain parts of the nominal system can be tolerated within allowable time delay $h_1 \leq h(t) \leq h_2$.

**Proof:** Replacing $A$ and $B$ in (10) with $A + MF(t) N_{11}$ and $B + MF(t) N_{22}$, respectively, we apply Lemma 4 [2] for system (1) is equivalent to the following condition:

$$\Xi + \Gamma_2 F(t)\Gamma_1 + \Gamma_2 F(t)\Gamma_2^T < 0,$$

where $\Xi = \begin{bmatrix} PM & 0 & 0 & h_1 R M & h_3 R M \\ 0 & N_{11} & 0 & 0 & 0 \end{bmatrix}$ and

$$\Gamma_1 = \begin{bmatrix} N_{11} & N_{33} & 0 & 0 & 0 \end{bmatrix}.$$

By Lemma 4 [2], a sufficient condition guaranteeing (10) for system (1) is that there exists a positive number $\varepsilon > 0$ such that

$$\Xi + \varepsilon^{-1} \Gamma_2^T \Gamma_1 + \varepsilon \Gamma_2 F(t) \Gamma_2 < 0.$$

Applying the Schur complement shows that (24) is equivalent to (22a). This completes the proof.

When $h_1 = 0$, Theorem 1 reduces to the following Corollary 1.

**Corollary 1:** For a given scalar $h > 0, h > 0$, the time-delay singular system (9) is asymptotically stable if there exist symmetric positive-definite matrices $\eta > 0$, $\zeta > 0$, $\eta > 0$ and matrix $S$ with appropriate dimensions and a positive semi-definite matrix

$$X = \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \\ \vdots & \vdots & \ddots \\ X_{13} & X_{23} & X_{33} \end{bmatrix} \geq 0$$

such that the following LMIs hold:

$$\Xi = \begin{bmatrix} \Xi_{11} & \Xi_{12} & \ldots & \Xi_{15} \\ \Xi_{12} & \Xi_{22} & \ldots & \Xi_{25} \\ \vdots & \vdots & \ddots & \vdots \\ \Xi_{55} & \Xi_{56} & \ldots & \Xi_{66} \\ \Xi_{66} & 0 & \ldots & 0 \end{bmatrix} \geq 0,$$

$$\Xi_{11} + \varepsilon N_{11}^T N_{11} \geq 0,$$

$$\Xi_{12} + \varepsilon N_{12}^T N_{12} \geq 0,$$

$$\Xi_{16} + \varepsilon N_{16}^T N_{16} \geq 0,$$

$$\Xi_{55} + \varepsilon N_{55}^T N_{55} \geq 0,$$

$$\Xi_{66} + \varepsilon N_{66}^T N_{66} \geq 0,$$

where $\Xi_{11} = \Xi_{11} + \varepsilon N_{11}^T N_{11}$, $\Xi_{12} = \Xi_{12} + \varepsilon N_{12}^T N_{12}$, $\Xi_{16} = \Xi_{16} + \varepsilon N_{16}^T N_{16}$, $\Xi_{55} = \Xi_{55} + \varepsilon N_{55}^T N_{55}$, $\Xi_{66} = \Xi_{66} + \varepsilon N_{66}^T N_{66}$, $\varepsilon > 0$, $i < j \leq 6$ are defined in (10).

Based on that, a convex optimization problem is formulated to find the bound on the allowable delay time $h$ which maintains the delay-dependent stability of the time delay system (9).
Proof: Consider the singular time-varying delay system (9), using the Lyapunov-Krasovskii functional candidate in the following form, we can rewrite as

\[
V(x) = x^T(t)PEx(t) + \int_{\tau_{\theta}(t)}^{t} x^T(s)Qx(s)ds + \int_{\tau_{\theta}(t)}^{t} x^T(s)E^TREx(s)dsd\theta
\]  

(26)

Similar to the above analysis, one can get that \( \dot{V}(x) < 0 \) holds if \( \Psi < 0 \). Thus, the proof is completed.

Now, extending Corollary 1 to time delay singular uncertain system (1) with time-varying structured uncertainties yields the following Corollary2.

Corollary 2: For given scalar \( h > 0, h_d > 0 \), the time-delay singular system (1) is asymptotically stable if there exist symmetric positive-definite matrices \( P = P^T > 0, Q = Q^T > 0, R = R^T > 0, \varepsilon > 0 \), and matrix \( S \) of appropriate dimensions and a positive semi-definite matrix \( \Psi \) such that the following LMIs hold:

\[
P^T E' = E' P \geq 0, \quad \Psi = \begin{bmatrix}
\Psi_{11} + \varepsilon N_1^TN_a & \Psi_{12} + \varepsilon N_2^TN_b & \Psi_{13} & PM \\
\Psi_{12}^T + \varepsilon N_2^TN_a & \Psi_{22} + \varepsilon N_2^TN_b & \Psi_{23} & 0 \\
\Psi_{13} & \Psi_{23} & \Psi_{33} & hRM \\
M^TP & 0 & hM^TR & -\varepsilon I
\end{bmatrix} < 0
\]  

(27b)

and

\[
E'(R - X_a)E \geq 0,
\]  

(27c)

where \( Z \) follows the same definition as that in Theorem 1, and \( \Psi(i, j = 1, 2, 3; i < j \leq 3) \) are defined in (25).

Remark 1: It is interesting to note that \( h, h_d \) appear linearly in (10a) and (22a). Thus a generalized eigenvalue problem as defined in [2] can be formulated to solve the minimum acceptable \( 1/h \) (or \( 1/h_d \)) and therefore the maximum \( h \) (or \( h_d \)) to maintain robust stability as judged by these conditions.

In this way, our optimization problem becomes a standard generalized eigenvalue problem, then which can be solved using GEVP technique. From this discussion, we have the following Remark 2.

Remark 2: Theorem 2 provides delay-dependent asymptotic stability criteria for the time-varying delay singular systems (1) in terms of solvability of LMIs [2]. Based on them, we can obtain the maximum allowable delay bound (MADB) \( h_1 \leq h(t) \leq h_2 \), such that (1) is stable by solving the following convex optimization problem:

\[
\begin{align*}
\text{Maximize} & \quad h_2 \\
\text{Subject to} & \quad \text{Theorem 2}
\end{align*}
\]  

(28)

Inequality (28) is a convex optimization problem and can be obtained efficiently using the MATLAB LMI Toolbox.

To show usefulness of our result, let us consider the following numerical examples.

Table 1. MADBs \( h_2 \) with given \( h_1 \) for different \( h_d \) in example 1

| \( h_d \) | 0.1 | 0.3 | 0.5 | 0.7 |
|---|---|---|---|---|
| \( h_1 \) | 0.1 | [16] 0.5348 | 0.5318 | 0.5305 | 0.5300 |
| Corollary1 | 1.0484 | 1.0271 | 1.0213 | 1.0202 |
| \( h_1 \) | 0.2 | [16] 0.5517 | 0.5414 | 0.5411 | 0.5410 |
| Corollary1 | 1.0485 | 1.0285 | 1.0241 | 1.0231 |
| \( h_1 \) | 0.3 | [16] 0.5519 | 0.5421 | 0.5420 | 0.5411 |
| Corollary1 | 1.0487 | 1.0300 | 1.0268 | 1.0259 |
| \( h_1 \) | 0.4 | [16] 0.5750 | 0.5690 | 0.5688 | 0.5684 |
| Corollary1 | 1.0488 | 1.0318 | 1.0295 | 1.0286 |
| \( h_1 \) | 0.5 | [16] 0.6186 | 0.6154 | 0.6151 | 0.6150 |
| Corollary1 | 1.0490 | 1.0337 | 1.0321 | 1.0312 |
| \( h_1 \) | 0.6 | [16] 0.6776 | 0.6759 | 0.6758 | 0.6755 |
| Corollary1 | 1.0493 | 1.0358 | 1.0347 | 1.0338 |
| \( h_1 \) | 0.7 | [16] 0.7464 | 0.7460 | 0.7459 | 0.7458 |
| Corollary1 | 1.0495 | 1.0381 | 1.0372 | 1.0364 |
| \( h_1 \) | 0.8 | [16] 0.8252 | 0.8250 | 0.8248 | 0.8245 |
| Corollary1 | 1.0498 | 1.0404 | 1.0396 | 1.0387 |
| \( h_1 \) | 0.9 | [16] 0.9122 | 0.9121 | 0.9120 | 0.9118 |
| Corollary1 | 1.0502 | 1.0431 | 1.0426 | 1.0417 |

4. Illustrative Examples

Example 1: Consider the following time delay singular systems:

\[
E\dot{x}(t) = Ax(t) + Bx(t - h(t))
\]  

(29)

where

\[
E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, A = \begin{bmatrix} 0.5 & 0 \\ 0 & -1 \end{bmatrix}, B = \begin{bmatrix} -1.1 & 1 \\ 0 & 0.5 \end{bmatrix}
\]

Find the delay-range \( 0 < h < b(t) < h_d \) to guarantee the above system (29) to be asymptotically stable.

Solution: Choosing \( Z = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \) and taking the parameters \( h_1 = 0.1 \) and \( h_d = 0 \) we get the Theorem 1 remains feasible.
for any delay time \( [h_1, h_2] \in [0.1, 1.0641] \). Comparison with other results is illustrated in Table 1. From Table 1, one can see that our results for this example give larger upper bounds of time-delay than the ones in [16]. By setting \( h_1 = 0 \) and different \( h_2 \), the upper bounds on the time delay from Corollary 1 are shown in Table 2, in which “−” means that the results are not applicable to the corresponding cases. For comparison, the Table 2 also lists the upper bounds obtained from the criteria in [1, 6-8, 17, 19, 20, 22, 23, 26, 28, 29, 31]. It can be seen that our methods are less conservative. The simulation of the system (26) for \( h = 1.06 \) is depicted in Fig.1 with the initial state \([-1 \ 1]^T\).

Table 2. MADBs \( h_2 \) for various \( h_2 \) in example 1

| Methods       | \( h_2 \) | 0 | 0.3 | 0.75 |
|---------------|-----------|---|-----|------|
| [1, 28]       |           | - | -   | -    |
| [29]          | 0.5567    | - | -   | -    |
| [8]           | 0.8707    | - | -   | -    |
| [6]           | 0.9091    | - | -   | -    |
| [7]           | 0.9689    | - | -   | -    |
| [26]          | 1.0423    | - | -   | -    |
| [17, 19, 22, 23, 31] | 1.0660 | - | -   | -    |
| [20]          | 1.0660    | 1.0130 | 0.6496 |
| Corollary 1   | 1.0660    | 1.0263 | 1.0164 |

Example 2: For convenience of comparison, consider the following time-delay system [4, 6, 7, 21].

\[ E \dot{x}(t) = Ax(t) + Bx(t - h(t)) \]  

where \( E = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}, A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, B = \begin{bmatrix} 1 & -1 \end{bmatrix} \).

Solution: In this example, we choose \( Z = \begin{bmatrix} 0 & 1 \end{bmatrix} \). For \( h_1 = 0 \) and \( h_2 = 0 \) the system (30), we are able to find a feasible solution for the set of LMIs for any \([h_1, h_2] \in [0, 1.1547]\). This means that the maximum allowable delay bound (MADB) under which the system is uniformly asymptotically stable is 1.1547. Table 3 lists the maximum allowable delay bound (MADB) as judged by the criteria in [4, 6, 7, 21]. We can see from this table that there is still room for reducing the conservativeness by comparing with the numerical solution, but we know that with fewer matrix variables the stability results obtained in Corollary 1 is less conservative than the one in [4, 6, 7] and the same as [21]. Using these data, a simulation program has been written in Matlab. As Fig. 2 shows, the simulation of the above system (30) for \( h_2 = 1.1547 \) with the initial state \([-1 \ 1]^T\).

Table 3. Comparison of MADBs \( h_2 \) in example 2 for different methods and number of variables

| Methods       | \( h_2 \) | Number of variables |
|---------------|-----------|---------------------|
| [6]           | <1        | 4                   |
| [7]           | 1.150     | 9                   |
| [4]           | 1.1547    | 7                   |
| [21]          | 1.1547    | 5                   |
| Corollary 1   | 1.1547    | 5                   |

Example 3: Consider system (1) with respect to uncertainty (3) as follows:
Delay-range-dependent Robust Stability for Uncertain Singular Systems with Interval Time-varying Delays

\[ Ex(t) = (A + \Delta A(t))x(t) + (B + \Delta B(t))x(t - h(t)), \]

where

\[
E = \begin{bmatrix} 1.0 & 0 \\ 0 & 0 \end{bmatrix}, A = \begin{bmatrix} 0.4 & 0.2 \\ 0 & -1.0 \end{bmatrix}, B = \begin{bmatrix} -1 & 1.0 \\ 0 & 0.3 \end{bmatrix},
\]
\[
M = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, N_s = \begin{bmatrix} -0.05 & 0 \\ 0 & 0.05 \end{bmatrix}, N_s = \begin{bmatrix} 0.05 & 0 \\ 0 & -0.05 \end{bmatrix}
\]

Solution: Choosing \( Z = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \) and taking the various parameters \( h_2 \), the computed maximum upper bounds of \( h_2 \), which guarantee the system (31) is regular, impulse free and robustly asymptotically stable for given bounds of \( h_1 \), are listed in Table 4. Table 4 indicates that the time delay \( h_2 \) decreases when the \( h_\sigma \) increases. We claim that the sharpness of the upper bound delay time \( h_2 \) is negatively correlated with \( h_\sigma \). For \( h_1 = 0 \) and \( h_\sigma = 0 \) the system (31), we are able to find a feasible solution for the set of LMIs for any \([h_1, h_\sigma] \subseteq [0, 1.1782] \). This means that the maximum allowable delay bound (MADB) under which the system is uniformly asymptotically stable is 1.1782. Fig. 3 shows, the simulation of the above system (31) for \( h_2 = 1.17 \) with the initial state \([-1 \ 1]^T \).

Table 4. MADBs \( h_2 \) with given \( h_1 \) for varying \( h_\sigma \) in example 3 (Theorem 2)

| \( h_1 \) | \( h_\sigma \) | 0.3 | 0.6 |
|------|-----|-----|-----|
| 0.3  | 1.1816 | 1.1395 | 1.1255 |
| 0.6  | 1.1816 | 1.1435 | 1.1342 |

Figure 3. The simulation of the example 3 for \( h = 1.17 \) sec

5. Conclusion

In this paper, the improved delay-range-dependent robust stability criterion for uncertain singular time-delay systems with time-varying interval delays has been investigated. By defining a novel Lyapunov function, a delay-range-dependent stability criterion is established in terms of LMIs, which guarantees the nominal singular time-delay systems to be regular, impulse free and asymptotically stable. Less conservative result is obtained by considering some useful terms when estimating the upper bound of the derivative of Lyapunov functional and introducing the additional terms into the proposed Lyapunov function which includes the information of the range. The robust stability problem is also investigated and the obtained results are expressed in terms of strict LMIs, which can be easily solved by using a convex optimization algorithm. By comparing our results with others through numerical examples, it has been shown that the derived criterion is less conservative than those in the literature.

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