Abstract

In this paper we study the integrability of a class of nonlinear non
autonomous quad graph equations compatible around the cube introduced
by Boll. We show that all these equations possess three point generalized
symmetries which are subcases of either the Yamilov discretization of
the Krichever–Novikov equation or of its non autonomous extension. We
also prove that all those symmetries are integrable as pass the algebraic
entropy test.

1 Introduction

In 1983 Ravil I. Yamilov [20] classified all differential difference equations of the
class $u_n = f(u_{n-1}, u_n, u_{n+1})$ using the generalized symmetry method. From
the generalized symmetry method one obtains integrability conditions which
allow to check whether a given equation is integrable. Moreover in many cases
these conditions enable us to classify equations, i.e. to obtain complete lists of
integrable equations belonging to a certain class. As integrability conditions are
only necessary conditions for the existence of generalized symmetries and/or
conservation laws, one then has to prove that the equations of the resulting
list really possess generalized symmetries and conservation laws of sufficiently
high order. One mainly construct them using Miura-type transformations and
master symmetries, proving the existence of Lax pairs [21, 22]. The result of
Yamilov classification, up to Miura transformation, is the Toda equation and
the so called Yamilov discretization of the Kriechver Novikov equation (YdKN),
a differential difference equation depending on 6 arbitrary coefficients:

$$\frac{dq_k}{dt} = \frac{A(q_k)q_{k+1}q_{k-1} + B(q_k)(q_{k+1} + q_{k-1}) + C(q_k)}{q_{k+1} - q_{k-1}},$$

where:

$$A(q_k) = aq_k^2 + 2bq_k + c,$$

(1)

(2a)

(2b)

(2c)

(2d)
The integrability of (1) is proven by the existence of point symmetries [13] and of a master symmetries [22] from which one is able to construct an infinite hierarchy of generalized symmetries. The problem of finding the Bäcklund transformation and Lax pair in the general case seems to be still open.

In [15] the authors constructed a set of five conditions necessary for the existence of generalized symmetries for a class of differential-difference equations depending only on nearest neighbouring interaction. They used the conditions to propose the integrability of the following non autonomous generalization of the YdKN:

\[
\frac{dq_k}{dt} = A_k(q_k)q_{k+1}q_{k-1} + B_k(q_k)(q_{k+1} + q_{k-1}) + C_k(q_k),
\]

where the now \(k\)-dependent coefficients are given by:

\[
A_k(q_k) = aq_k^2 + 2b_kq_k + c_k,
\]

\[
B_k(q_k) = b_{k+1}q_k^2 + dq_k + e_{k+1},
\]

\[
C_k(q_k) = c_{k+1}q_k^2 + 2e_kq_k + f,
\]

with \(b_k\), \(c_k\) and \(e_k\) 2-periodic functions. Eq. (3) has conservation laws of second and third order and two generalized local symmetries of order \(i\) and \(i+1\), with \(i < 4\).

It was proved in [12] that the three point symmetries of the equations belonging to the so-called ABS classification [1], found systematically in [10], are all particular cases of (1). Here in this note we will show that the three point generalized symmetries of all the equations coming from the classification of Boll [2,4–6], which extends the ABS one [1], are all particular cases of the YdKN or the non autonomous YdKN. In particular we will present the symmetries of all the classes of equations \(H^4\) and \(H^6\), noting that the symmetries of the rhombic \(H^4\) were found firstly in [19]. For the remaining classes of equations namely the trapezoidal \(H^4\) and the \(H^6\) equations, which were found to be linearizable in [9], this is the first time that their generalized symmetries are presented. Furthermore we also present a new suggestion for the integrability of the non-autonomous YdKN (3) based on the algebraic entropy test and use the same criterion to prove the integrability of the other non autonomous equations of the \(H^4\) and \(H^6\) classes.

In Section 2 we present the three point generalized symmetries of the \(H^4\) and \(H^6\) classes and identify them with subcases of the YdKN or of its non autonomous extension. In Section 3 we compute the algebraic entropy for the non autonomous YdKN and its subcases obtained before while in Section 4 we present some brief conclusions.

2 Three point generalized symmetries and their identification

In this Section we consider the various classes of equation coming from the classification of Boll [2,4–6] as presented in [9], show their symmetries and show
the identification of the fluxes of such symmetries with the YdKN and its non autonomous extension.

2.1 Rhombic $H^4$ equations

Once written on the $\mathbb{Z}_2^{(n,m)}$ lattice, according to [19], the three equations belonging to this class have the form:

\begin{align*}
r H_1^n: & \quad (u_{n,m} - u_{n+1,m+1}) (u_{n+1,m} - u_{n,m+1}) - (\alpha - \beta) \\
& \quad + \varepsilon (\alpha - \beta) (F^{(+)}_{n+m} u_{n+1,m} u_{n,m+1} + F^{(-)}_{n+m} u_{n,m} u_{n+1,m+1}) = 0, \\
\end{align*}

\begin{align*}
r H_2^n: & \quad (u_{n,m} - u_{n+1,m+1}) \left( u_{n+1,m} - u_{n,m+1} \right) + \\
& \quad + (\beta - \alpha) (u_{n,m} + u_{n+1,m} + u_{n,m+1} + u_{n+1,m+1}) - \alpha^2 + \beta^2 \\
& \quad - \varepsilon (\beta - \alpha)^3 - \varepsilon (\beta - \alpha) \left( 2F^{(-)}_{n+m} u_{n,m} + 2F^{(+)}_{n+m} u_{n+1,m} + \alpha + \beta \right) \\
& \quad \cdot \left( 2F^{(-)}_{n+m} u_{n+1,m+1} + 2F^{(+)}_{n+m} u_{n,m+1} + \alpha + \beta \right) = 0, \\
r H_3^n: & \quad \alpha (u_{n,m} u_{n+1,m} + u_{n+1,m} u_{n+1,m+1}) \\
& \quad - \beta (u_{n,m} u_{n+1,m+1} + u_{n+1,m} u_{n+1,m+1}) + (\alpha^2 - \beta^2) \delta \\
& \quad - \varepsilon (\alpha^2 - \beta^2) \left( F^{(+)}_{n+m} u_{n+1,m} u_{n,m+1} + F^{(-)}_{n+m} u_{n,m} u_{n+1,m+1} \right) = 0,
\end{align*}

where

\[ F_{n,m}^\pm = \frac{1 \pm (-1)^k}{2}, \quad k \in \mathbb{Z}. \]  \hspace{1cm} (6)

Their generalized symmetries [19] are given by:

\begin{align*}
\hat{X}_n H_1^n & = \frac{1 - \varepsilon}{u_{n+1,m} - u_{n-1,m}} \left( F^{(+)}_{n+m} u_{n+1,m} u_{n,m+1} + F^{(-)}_{n+m} u_{n,m} \right) \partial_{u_{n,m}}, \\
\hat{X}_m H_1^n & = \frac{1 - \varepsilon}{u_{n+1,m} - u_{n-1,m}} \left( F^{(+)}_{n+m} u_{n+1,m} u_{n,m+1} + F^{(-)}_{n+m} u_{n,m} \right) \partial_{u_{n,m}}, \\
\hat{X}_n H_2^n & = \left[ \frac{1 - 4\varepsilon \alpha F^{(-)}_{n+m}}{u_{n+1,m} - u_{n-1,m}} (u_{n+1,m} + u_{n-1,m}) - 4\varepsilon F^{(+)\prime}_{n+m} u_{n+1,m} u_{n-1,m} \right] \\
& \quad + \frac{2\alpha - 4\varepsilon \alpha^2 - 4\varepsilon F^{(-)}_{n+m} u_{n,m}^2 + (1 - 4\varepsilon \alpha F^{(-)}_{n+m}) u_{n,m}}{u_{n+1,m} - u_{n-1,m}} \partial_{u_{n,m}}, \\
\hat{X}_m H_2^n & = \left[ \frac{1 - 4\varepsilon \beta F^{(-)}_{n+m}}{u_{n+1,m} - u_{n-1,m}} (u_{n+1,m} + u_{n-1,m}) - 4\varepsilon F^{(+)\prime}_{n+m} u_{n+1,m} u_{n-1,m} \right] \\
& \quad + \frac{2\beta - 4\varepsilon \beta^2 - 4\varepsilon F^{(-)}_{n+m} u_{n,m}^2 + (1 - 4\varepsilon \beta F^{(-)}_{n+m}) u_{n,m}}{u_{n+1,m} - u_{n-1,m}} \partial_{u_{n,m}}, \quad \text{(7c,d)}
\end{align*}
2.2 Trapezoidal coefficients of (3) in Table 1. Identification is in this paper made explicit by showing the appropriate values of the coefficients in the corresponding cases of the non autonomous YdKN equation (3). Such identification is in this paper made explicit by showing the appropriate values of the coefficients of (3) in Table 1.

Table 1: Identification of the coefficients in the symmetries of the rhombic equations with those of the non autonomous YdKN equation.

| Eq. | k | a | b_k | c_k | d | e_k | f |
|-----|---|---|-----|-----|---|-----|---|
| \( r H_1^x \) | n | 0 | 0 | \(-\varepsilon F_{n+m}^{(+)}\) | 0 | 0 | 1 |
| \( m \) | 0 | 0 | \(-\varepsilon F_{n+m}^{(+)}\) | 0 | 0 | 1 |
| \( r H_2^x \) | n | 0 | 0 | \(-4\varepsilon F_{n+m}^{(+)\prime}\) | 1 | \(-4\varepsilon\alpha F_{n+m}^{(-)}\) | \(2\alpha - 4\varepsilon\alpha^2\) |
| \( m \) | 0 | 0 | \(-4\varepsilon F_{n+m}^{(+)\prime}\) | 1 | \(-4\varepsilon\beta F_{n+m}^{(-)}\) | \(2\beta - 4\varepsilon\beta^2\) |
| \( r H_3^x \) | n | 0 | 0 | \(-\epsilon F_{n+m}^{(+)}\) | \(\frac{1}{2}\) | 0 | \(\delta\alpha\) |
| \( m \) | 0 | 0 | \(-\epsilon F_{n+m}^{(+)}\) | \(\frac{1}{2}\) | 0 | \(\delta\beta\) |

As stated in [19] the fluxes of the symmetries are readily identified with the corresponding cases of the non autonomous YdKN equation [3]. Such identification is in this paper made explicit by showing the appropriate values of the coefficients of (3) in Table 1.

2.2 Trapezoidal \( H^4 \) equations

We now consider the trapezoidal \( H^4 \) equations, which appeared in [6] and whose non-autonomous form was given in [9]:

\[ tH_1: \quad (u_{n,m} - u_{n+1,m})(u_{n+1,m} - u_{n+1,m+1}) - \alpha_2 = 0, \quad (8a) \]

\[ tH_2: \quad (u_{n,m} - u_{n+1,m})(u_{n,m+1} - u_{n+1,m+1}) + \alpha_2 \left( u_{n,m} + u_{n+1,m} + u_{n,m+1} + u_{n+1,m+1} \right) + \frac{\epsilon \alpha_2}{2} \left( 2F_m^{(+)} u_{n,m+1} + 2\alpha_3 + \alpha_2 \right) \left( 2F_m^{(+)} u_{n+1,m+1} + 2\alpha_3 + \alpha_2 \right) + \frac{\epsilon \alpha_2}{2} \left( 2F_m^{(-)} u_{n,m} + 2\alpha_3 + \alpha_2 \right) \left( 2F_m^{(-)} u_{n+1,m} + 2\alpha_3 + \alpha_2 \right) + (\alpha_3 + \alpha_2)^2 - \alpha_3^2 - 2\varepsilon\alpha_2\alpha_3 (\alpha_3 + \alpha_2) = 0 \]
We can easily calculate the three points generalized symmetries of $tH_2^+$ \([8a]\) and of $tH_3^+$ \([8b]\):

\[
\begin{align*}
\bar{X}_{\bar{H}_2^+}^3 &= \left[ \frac{(u_{n,m} + \varepsilon \alpha_2 F_m^{(+)})(u_{n+1,m} + u_{n-1,m}) - u_{n+1,m}u_{n-1,m}}{u_{n+1,m} - u_{n-1,m}} \right] \partial_{u_{n,m}}, \\
\bar{X}_{\bar{H}_3^+}^3 &= \left[ \frac{1}{2} - \varepsilon (\alpha_2 + \alpha_3) F_m^{(+)} \right] \left( u_{n,m+1} + u_{n,m-1} \right) - \varepsilon F_m^{(+)} u_{n,m+1}u_{n,m-1} \\
&= \left[ \frac{1}{2} - \varepsilon (\alpha_2 + \alpha_3) F_m^{(+)} \right] \left( u_{n,m+1} + u_{n,m-1} \right) - \varepsilon F_m^{(+)} u_{n,m+1}u_{n,m-1} \\
\bar{X}_{\bar{H}_3^+}^3 &= \left[ \frac{1}{2} \alpha_2 \alpha_3 (u_{n,m+1} + u_{n,m-1}) - \alpha_2^2 u_{n+1,m}u_{n-1,m} \right] \partial_{u_{n,m}}, \\
\bar{X}_{\bar{H}_3^+}^3 &= \left[ \frac{1}{2} \alpha_2 \alpha_3 (u_{n,m+1} + u_{n,m-1}) - \alpha_2^2 u_{n+1,m}u_{n-1,m} \right] \partial_{u_{n,m}}, \\
\bar{X}_{\bar{H}_3^+}^3 &= \left[ \frac{1}{2} \alpha_2 \alpha_3 (u_{n,m+1} + u_{n,m-1}) - \alpha_2^2 u_{n+1,m}u_{n-1,m} \right] \partial_{u_{n,m}}, \\
\bar{X}_{\bar{H}_3^+}^3 &= \left[ \frac{1}{2} \alpha_2 \alpha_3 (u_{n,m+1} + u_{n,m-1}) - \alpha_2^2 u_{n+1,m}u_{n-1,m} \right] \partial_{u_{n,m}}, \\
\bar{X}_{\bar{H}_3^+}^3 &= \left[ \frac{1}{2} \alpha_2 \alpha_3 (u_{n,m+1} + u_{n,m-1}) - \alpha_2^2 u_{n+1,m}u_{n-1,m} \right] \partial_{u_{n,m}},
\end{align*}
\]

The symmetries in the $n$ and $m$ directions and the linearizations of the $tH_2^+$ equation \([8a]\) have been presented in \([10]\). Their peculiarity is that they are determined by two arbitrary functions of one continuous variable and one lattice index and by arbitrary functions of the lattice indices. This is the first time that we find a lattice equation whose generalized symmetries depend on arbitrary functions. Almost surely this peculiarity is related to the very specific way in which $tH_2^+$ is linearizable. Here we present only the sub-cases which are related to the YdKN equation in its autonomous or non autonomous form.

The general symmetry in the $n$ direction is:
arbitrary parameter. When
\[ B_n(t) = \frac{s^2 t}{(s - t + 1)(s + t)} \]
\[ \hat{X}_n^{H^1} = F_m^{(+)} + \frac{\varepsilon^2 \alpha^2}{u_n+1,m - u_n,m} \]
\[ X_n^{H^1} = F_m^{(+)} \left( B_m \left( \frac{u_{n+1,m} - u_{n,m+1}}{1 + s^2 u_{n+1,m} u_{n,m+1}} \right) + \kappa_m \right) \]
\[ + F_m^{(-)} \left( 1 + s^2 u_n,m \right) \left( C_m \left( u_{n+1,m} - u_{n,m+1} \right) + \lambda_m \right) \partial_{u_{n,m}} \]
\[ \hat{X}_n^{H^1} = F_m^{(+)} \left( \frac{u_{n+1,m} - u_{n,m+1}}{u_{n+1,m} - u_{n,m+1}} \right) + F_m^{(-)} \left( \frac{1 + s^2 u_n,m}{u_{n+1,m} - u_{n,m+1}} \right) \partial_{u_{n,m}} \]
\[ \text{where } B_n(x), \gamma_m \text{ and } \delta_m \text{ are generic functions of their arguments and } \alpha \text{ is an arbitrary parameter. When } B_n(x) = -1/x, \alpha = \gamma_m = \delta_m = 0, \text{ we get} \]
\[ X_n^{H^1} = F_m^{(+)} \left( B_m \left( \frac{u_{n+1,m} - u_{n,m+1}}{1 + s^2 u_{n+1,m} u_{n,m+1}} \right) + \kappa_m \right) \]
\[ + F_m^{(-)} \left( 1 + s^2 u_n,m \right) \left( C_m \left( u_{n+1,m} - u_{n,m+1} \right) + \lambda_m \right) \partial_{u_{n,m}} \]
\[ \text{The general symmetry in the } m \text{ direction is:} \]
\[ \hat{X}_m^{H^1} = [F_m^{(+)} \left( \frac{u_{n+1,m} - u_{n,m+1}}{u_{n+1,m} - u_{n,m+1}} \right) + F_m^{(-)} \left( \frac{1 + s^2 u_n,m}{u_{n+1,m} - u_{n,m+1}} \right) \partial_{u_{n,m}} \]
\[ \text{Let us notice that the symmetries } [9, 11] \text{ in the } n \text{ direction are sub-cases of the original YdKN equation. As } F_m^{(+)} \text{ depends on the other lattice index, it can be treated like a parameter which is either 0 or 1.} \]
\[ \text{The explicit identification of the coefficients of the symmetries } [9, 11] \text{ and 13] is shown in Table 2.} \]

2.3 \( H^6 \) equations

In this subsection we consider the equations of the family \( H^6 \) introduced in [5,6].

We shall present their non autonomous form on the lattice \( Z^2_{(n,m)} \) as given in [9, 10].

\[ 1D_2: \]
\[ \left( F_{n+1,m}^{(-)} - \delta_1 F_n^{(+)} F_m^{(-)} + \delta_2 F_n^{(+)} F_m^{(-)} \right) u_{n,m} \]
\[ + \left( F_{n+1,m}^{(+)} - \delta_1 F_n^{(-)} F_m^{(+)} + \delta_2 F_n^{(-)} F_m^{(+)} \right) u_{n+1,m} \]
\[ + \left( F_{n+1,m}^{(+)} - \delta_1 F_n^{(-)} F_m^{(+)} + \delta_2 F_n^{(-)} F_m^{(+)} \right) u_{n,m+1} \]
\[ + \delta_1 \left( F_m^{(-)} u_{n,m} u_{n+1,m} + F_m^{(+)} u_{n,m+1} u_{n+1,m} \right) \]
\[ + \delta_2 \left( F_m^{(-)} u_{n,m} u_{n+1,m} + F_m^{(+)} u_{n,m+1} u_{n+1,m} \right) = 0, \]
Table 2: Identification of the coefficients in the symmetries of the trapezoidal $H^4$ equations with those of the YdKN equation. In the direction $n$ the YdKN is autonomous while in the $m$ direction is non autonomous. Here the symmetries of $H_1^+$ in the $m$ direction are the subcase (13) of (12) while those in the $n$ direction are the subcase (11) of (10).
\[ 2D_2: \quad \left( F_{m}^{(-)} - \delta_1 F_{n}^{(+) + \delta_2 F_{m}^{(+)} \right) u_{n,m} \] 
\[ + \left( F_{m}^{(-)} - \delta_1 F_{n}^{(-)} F_{m}^{(+)} + \delta_2 F_{n}^{(-)} F_{m}^{(+)} - \delta_1 \lambda F_{n}^{(-)} F_{m}^{(+)} \right) u_{n+1,m} \] 
\[ + \left( F_{m}^{(-)} - \delta_1 F_{n}^{(-)} F_{m}^{(+)} + \delta_2 F_{n}^{(-)} F_{m}^{(+)} - \delta_1 \lambda F_{n}^{(-)} F_{m}^{(+)} \right) u_{n,m+1} \] 
\[ + \left( F_{m}^{(-)} - \delta_1 F_{n}^{(-)} F_{m}^{(+)} + \delta_2 F_{n}^{(-)} F_{m}^{(+)} - \delta_1 \lambda F_{n}^{(-)} F_{m}^{(+)} \right) u_{n+1,m+1} \] 
\[ + \lambda_1 \left( F_{n}^{(-)} u_{n,m} u_{n+1,m+1} + F_{m}^{(+)} u_{n,m} + \delta_1 \delta_2 \lambda = 0, \right. \] 
\[ 3D_2: \quad \left( F_{m}^{(-)} - \delta_1 F_{n}^{(-)} F_{m}^{(+)} + \delta_2 F_{n}^{(-)} F_{m}^{(+)} - \delta_1 \lambda F_{n}^{(-)} F_{m}^{(+)} \right) u_{n,m} \] 
\[ + \left( F_{m}^{(-)} - \delta_1 F_{n}^{(-)} F_{m}^{(+)} + \delta_2 F_{n}^{(-)} F_{m}^{(+)} - \delta_1 \lambda F_{n}^{(-)} F_{m}^{(+)} \right) u_{n+1,m} \] 
\[ + \left( F_{m}^{(-)} - \delta_1 F_{n}^{(-)} F_{m}^{(+)} + \delta_2 F_{n}^{(-)} F_{m}^{(+)} - \delta_1 \lambda F_{n}^{(-)} F_{m}^{(+)} \right) u_{n,m+1} \] 
\[ + \left( F_{m}^{(-)} - \delta_1 F_{n}^{(-)} F_{m}^{(+)} + \delta_2 F_{n}^{(-)} F_{m}^{(+)} - \delta_1 \lambda F_{n}^{(-)} F_{m}^{(+)} \right) u_{n+1,m+1} \] 
\[ + \lambda \left( F_{n}^{(-)} u_{n,m} u_{n+1,m+1} + F_{m}^{(+)} u_{n,m} + \delta_1 \delta_2 \lambda = 0, \right) \] 
\[ D_3: \quad F_{m}^{(+) u_{n,m} + F_{m}^{(+) u_{n+1,m} + F_{n}^{(+) u_{n,m}+1}} \] 
\[ + F_{n}^{(+) u_{n+1,m+1} + F_{n}^{(+) u_{n+1,m}+1} \] 
\[ + F_{n}^{(+) u_{n+1,m}+1 u_{n+1,m}+1} \] 
\[ + F_{n}^{(+) u_{n+1,m}+1 u_{n+1,m}+1} \] 
\[ + F_{n}^{(+) u_{n+1,m}+1 u_{n+1,m}+1} = 0, \] 
\[ 1D_4: \quad \delta_1 \left( F_{n}^{(+) u_{n,m} u_{n+1,m}+1} \right) \] 
\[ + \delta_2 \left( F_{n}^{(+) u_{n,m} u_{n+1,m}+1} \right) \] 
\[ + u_{n,m} u_{n+1,m}+1 + u_{n+1,m} u_{n,m} + \delta_3 = 0, \] 
\[ 2D_4: \quad \delta_1 \left( F_{n}^{(+) u_{n,m} u_{n+1,m}+1} \right) \] 
\[ + \delta_2 \left( F_{n}^{(+) u_{n,m} u_{n+1,m}+1} \right) \] 
\[ + u_{n,m} u_{n+1,m}+1 + u_{n+1,m} u_{n,m} + \delta_3 = 0. \] 

The three forms of the equation \( D_3 \) \( \text{[14]} \), which we will collectively call \( iD_2 \) assuming \( i \) in \( \{1,2,3\} \), possess the following three points generalized symmetries in the \( n \) direction and three points generalized symmetries in the \( m \)
direction:

\[ \hat{X}^{i, D_2} = \frac{\left( F_{n}^{(+)} - \delta_1 F_{n}^{(+)} F_{m}^{(1)} \right) (u_{n+1,m} + u_{n-1,m})}{u_{n+1,m} - u_{n-1,m}} + \frac{\left( F_{n}^{(-)} - \delta_1 F_{n}^{(+)} F_{m}^{(1)} - \delta_1 \delta_2 \right) F_{n}^{(1)} F_{m}^{(-)} u_{n-1,m}}{u_{n+1,m} - u_{n-1,m}} + \frac{\left( F_{n+1}^{(+)} - \delta_1 F_{n+1}^{(1)} F_{m}^{(+)} \right) u_{n,m} + \delta_2 F_{n}^{(-)}}{u_{n+1,m} - u_{n-1,m}} \partial_{u_{n,m}}, \]  

(15a)

\[ \hat{X}^{i, D_2} = \frac{\delta_1 F_{n}^{(-)} F_{m}^{(+)} u_{n,m+1} u_{n,m} - F_{n+1}^{(-)} (u_{n,m+1} + u_{n,m-1})}{u_{n+1,m} - u_{n-1,m}} + \frac{\delta_1 F_{n}^{(+)} u_{n+1,m} + \delta_1 \delta_2 F_{n}^{(1)} F_{m}^{(-)} u_{n,m-1} + \delta_1 F_{n}^{(-)} F_{m}^{(+)} u_{n,m}}{u_{n+1,m} - u_{n-1,m}} + \frac{\left( F_{n+1}^{(+)} + \delta_1 \left( F_{n+1}^{(+)} F_{m}^{(-)} - F_{n}^{(-)} F_{m}^{(-)} \right) + \delta_1 \delta_2 F_{n}^{(-)} F_{m}^{(-)} \right) u_{n,m}}{u_{n+1,m} - u_{n-1,m}} + \frac{\delta_2 (\delta_1 - 1) F_{n}^{(-)} \partial_{u_{n,m}}}{u_{n+1,m} - u_{n-1,m}} \]  

(15b)

\[ \hat{X}^{i, D_2} = \left[ \frac{\left( F_{n}^{(-)} F_{m}^{(-)} \delta_1 + F_{n}^{(-)} F_{m}^{(-)} \delta_1 \delta_2 - F_{n}^{(-)} F_{m}^{(-)} \right) u_{n+1,m} + \left( F_{n}^{(+)} F_{m}^{(1)} \delta_1 - F_{n}^{(+)} F_{m}^{(-)} \right) u_{n-1,m}}{u_{n+1,m} - u_{n-1,m}} \]  

\[ + \frac{\delta_1 F_{n+1}^{(-)} F_{m}^{(-)} u_{n,m} - \left( \delta_1 - 1 \right) F_{m}^{(-)} u_{n,m}}{u_{n+1,m} - u_{n-1,m}} \left] \partial_{u_{n,m}}, \right. \]  

(15c)

\[ \hat{X}^{i, D_2} = \left[ \frac{F_{n}^{(-)} F_{m}^{(-)} \delta_1 u_{n,m+1} u_{n,m} + \left( \delta_1 \delta_2 F_{n}^{(-)} F_{m}^{(-)} + F_{n}^{(+)} F_{m}^{(-)} \right) u_{n,m+1}}{u_{n+1,m} - u_{n-1,m}} + \]  

\[ + \frac{\delta_1 F_{n+1}^{(1)} F_{m}^{(+)} + F_{n}^{(-)} F_{m}^{(-)} - \delta_1 F_{n}^{(-)} F_{m}^{(-)} \) u_{n,m-1}}{u_{n+1,m} - u_{n-1,m}} + \]  

\[ \delta_1 F_{n}^{(-)} F_{m}^{(+)} u_{n,m} + \left( F_{n}^{(-)} F_{m}^{(1)} + \left( \delta_2 - 1 \right) F_{n}^{(-)} F_{m}^{(+)} + F_{m}^{(1)} \right) u_{n,m} + \]  

\[ \frac{\delta_2 (1 - \delta_2) F_{n}^{(-)} - \delta_1 \lambda F_{n}^{(+)} \partial_{u_{n,m}}}{u_{n,m+1} - u_{n,m-1}} \]  

(15d)
\[ \hat{X}_{n}^{2} = \left( \delta_{1} F_{n}^{(+)} F_{m}^{(-)} + \delta_{1} \delta_{2} F_{n}^{(+)} F_{m}^{(-)} - F_{n}^{(+)} F_{m}^{(+) -} \right) u_{n+1,m} + \frac{\delta_{1} F_{n}^{(-1)} F_{m}^{(-)} \delta_{1} - F_{n}^{(-)} F_{m}^{(-)}}{u_{n+1,m} - u_{n-1,m}} + \frac{\delta_{1} F_{n}^{(-1)} F_{m}^{(-)} \delta_{2} + F_{n}^{(-)} \delta_{1} - F_{n}^{(-)} F_{m}^{(-)}}{u_{n+1,m} - u_{n-1,m}} + \left( \frac{\delta_{1} F_{n}^{(-1)} F_{m}^{(-)} \delta_{2} + F_{n}^{(-)} \delta_{1} - F_{n}^{(-)} F_{m}^{(-)}}{u_{n+1,m} - u_{n-1,m}} \right) \partial u_{n,m}, \]

\[ \hat{X}_{m}^{2} = \left[ (1 - \delta_{1} - \delta_{1} \delta_{2}) F_{n}^{(+)} F_{m}^{(-)} u_{n,m+1} + \frac{\delta_{1} F_{n}^{(-1)} F_{m}^{(-)} \delta_{1} - F_{n}^{(-)} F_{m}^{(-)}}{u_{n,m+1} - u_{n,m-1}} + \frac{\delta_{1} F_{n}^{(-1)} F_{m}^{(-)} \delta_{2} + F_{n}^{(-)} \delta_{1} - F_{n}^{(-)} F_{m}^{(-)}}{u_{n,m+1} - u_{n,m-1}} + \left( \frac{\delta_{1} F_{n}^{(-1)} F_{m}^{(-)} \delta_{2} + F_{n}^{(-)} \delta_{1} - F_{n}^{(-)} F_{m}^{(-)}}{u_{n,m+1} - u_{n,m-1}} \right) \partial u_{n,m} - \frac{\lambda \delta_{1} (1 - \delta_{1} - \delta_{1} \delta_{2}) F_{n}^{(+)} F_{m}^{(-)}}{u_{n,m+1} - u_{n,m-1}} \partial u_{n,m}. \]

It can be readily proved that these symmetries are not non autonomous YdKN equations \([3]\), however the equations \( \hat{D}_{2} \) possess also the following point symmetries:

\[ \hat{Y}_{1}^{1, D_{2}} = \left( F_{n}^{(+)} F_{m}^{(+)} + F_{n}^{(+)} F_{m}^{(-)} + F_{n}^{(-)} F_{m}^{(-)} \right) u_{n,m} \partial u_{n,m}, \]

\[ \hat{Y}_{2}^{1, D_{2}} = \left[ \delta_{1} F_{n}^{(+)} F_{m}^{(-)} + \left[ 1 - \delta_{1} (1 + \delta_{2}) \right] F_{n}^{(-)} F_{m}^{(+)} + F_{n}^{(+)} F_{m}^{(-)} \right] \partial u_{n,m}, \]

\[ \hat{Y}_{1}^{2, D_{2}} = \left( F_{n}^{(+)} F_{m}^{(+)} + F_{n}^{(+)} F_{m}^{(-)} + F_{n}^{(-)} F_{m}^{(-)} \right) u_{n,m}, \]

\[ \hat{Y}_{2}^{2, D_{2}} = \left[ \delta_{1} F_{n}^{(+)} F_{m}^{(-)} + \left[ 1 - \delta_{1} (1 + \delta_{2}) \right] F_{n}^{(-)} F_{m}^{(+)} \right] \partial u_{n,m}, \]

As the symmetries \([15]\) are not in the form of the YdKN equation \([3]\), we may look for a linear combination:

\[ \hat{Z}_{j}^{D_{2}} = \hat{X}_{j}^{D_{2}} + K_{1} \hat{Y}_{1}^{D_{2}} + K_{2} \hat{Y}_{2}^{D_{2}}, \quad j = n, m; \quad i = 1, 2, 3 \]

(17)

such that the resulting symmetries of equations \( \hat{D}_{2} \) will be in the form \([3]\). Indeed it turns out that this is the case and the resulting identification with the proper constants \( K_{1} \) and \( K_{2} \) is displayed in Table \([3]\). The fact that the \( \hat{D}_{2} \) equations admit point symmetries and generalized symmetries makes them a unique case among the equations of Boll classification.
\[
\begin{align*}
E_1 & \quad k \quad a \quad b_k \quad c_k \quad d \quad e_k \\
1D_2 & \quad n \ 0 \ 0 \ 0 \ 0 \ 0 \ \frac{1}{2}[\delta_1(1 + \delta_2) - 1]F_n^{(+)}F_m^{(+)} + \frac{1}{2}F_n^{(-)}F_m^{(+)} - \frac{1}{2}F_n^{(-)}F_m^{(-)} = -\delta_2F_m^{(-)} \quad 0 \ -1/2 \\
m & \quad 0 \ 0 \ -F_n^{(-)}F_m^{(+)\delta_1} \ 0 \ 0 \ \frac{1}{2}(\delta_1(1 - \delta_2 - 1)F_n^{(-)}F_m^{(-)} - \frac{1}{2}F_n^{(+)}F_m^{(+)} - \frac{1}{2}\delta_1F_n^{(-)}F_m^{(+)} = \delta_2(\delta_1 - 1)F_n^{(-)} \quad 0 \ -1/2 \\
2D_2 & \quad n \ 0 \ 0 \ 0 \ 0 \ \frac{1}{2}[1 - \delta_1(1 + \delta_2)]F_n^{(+)}F_m^{(-)} + \frac{1}{2}F_n^{(-)}F_m^{(-)} - \frac{1}{2}\delta_1F_n^{(-)}F_m^{(+)} = (\delta_1 - 1)F_m^{(+)} \quad 0 \ -1/2 \\
m & \quad 0 \ 0 \ -\delta_1F_n^{(-)}F_m^{(-)} \ 0 \ 0 \ \frac{1}{2}[\delta_1(1 - \delta_2 - 1)F_n^{(-)}F_m^{(+)} - \frac{1}{2}F_n^{(+)}F_m^{(+)} - \frac{1}{2}\delta_1F_n^{(-)}F_m^{(+)} = \delta_2[\delta_1 - 1]F_n^{(-)} + \lambda\delta_1F_n^{(+)} \quad 0 \ -1/2 \\
3D_2 & \quad n \ 0 \ 0 \ 0 \ 0 \ \frac{1}{2}[\delta_1(1 + \delta_2) - 1]F_n^{(-)}F_m^{(-)} + \frac{1}{2}F_n^{(+)}F_m^{(+)} + \frac{1}{2}\delta_1F_n^{(-)}F_m^{(+)} = (1 - \delta_1)F_m^{(+)} \quad 0 \ 1/2 \\
m & \quad 0 \ 0 \ 0 \ 0 \ \frac{1}{2}[\delta_1(1 - \delta_2 - 1)F_n^{(+)}F_m^{(+)} - \frac{1}{2}F_n^{(+)}F_m^{(+)} + \frac{1}{2}\delta_1F_n^{(-)}F_m^{(+)} = \delta_1\lambda[-\delta_1(1 + \delta_2)]F_n^{(+)} - \delta_2F_n^{(-)} \quad 0 \ 1/2 
\end{align*}
\]

Table 3: Identification of the coefficients of the symmetries of the \(iD_2\) equations and value of the constants \(K_1\) and \(K_2\) in (17) in order to obtain non autonomous YdKN equations.
The $D_3$ equation (14d) admits only the following three points generalized symmetries:

\[
\tilde{X}^{D_3}_n = \left[ \frac{F_n^{(+)} F_m^{(+)} u_{n+1,m} u_{n-1,m} + \frac{1}{2} \left( F_m^{(-)} - F_n^{(-)} F_m^{(+)} \right) u_{n,m} (u_{n+1,m} + u_{n-1,m})}{u_{n+1,m} - u_{n-1,m}} + \frac{F_n^{(-)} F_m^{(+)} u_{n,m}^2 + \left( F_n^{(-)} - F_n^{(+)} F_m^{(+)} \right) u_{n,m}}{u_{n+1,m} - u_{n-1,m}} \right] \partial_{u_{n,m}},
\]

(18a)

\[
\tilde{X}^{D_3}_m = \left[ \frac{F_n^{(+)} F_m^{(+)}}{u_{n,m+1} u_{n,m-1}} + \frac{1}{2} \left( F_n^{(-)} - F_n^{(+) F_m^{(-)} F_n^{(+)}} \right) u_{n,m} (u_{n+1,m+1} + u_{n,m-1}) + \frac{F_n^{(-)} F_m^{(+)} u_{n,m}^2 + \left( F_n^{(-)} - F_n^{(+) F_m^{(+)} \right) u_{n,m}}{u_{n+1,m+1} - u_{n,m-1}} \right] \partial_{u_{n,m}},
\]

(18b)

and no point symmetries. Also the two forms of $D_4$ possess only the following three point generalized symmetries:

\[
\tilde{X}^{D_4}_n = \left[ \frac{-\delta_1 F_n^{(+)}}{u_{n+1,m} u_{n-1,m} - \frac{1}{2} u_{n,m} (u_{n+1,m} + u_{n-1,m}) + \frac{1}{2} u_{n,m}^2 + \left( F_n^{(-)} - F_n^{(+) F_m^{(+)}} \right) u_{n,m} \partial_{u_{n,m}},
\]

(18c)

\[
\tilde{X}^{D_4}_m = \left[ \frac{F_m^{(-)} u_{n,m+1} u_{n,m-1} + \frac{1}{2} u_{n,m}^2 (u_{n+1,m+1} + u_{n+1,m-1}) + \frac{1}{2} u_{n,m}^2 + \left( F_m^{(+) - F_m^{(+) F_n^{(+)}} \right) u_{n,m} \partial_{u_{n,m}},
\]

(18d)

\[
\tilde{X}^{D_4}_m = \left[ \frac{-\delta_2 F_n^{(+)}}{u_{n+1,m} u_{n-1,m} - \frac{1}{2} u_{n,m} (u_{n+1,m} + u_{n-1,m}) + \frac{1}{2} u_{n,m}^2 + \left( F_n^{(-)} - F_n^{(+) F_m^{(+) \right) u_{n,m} \partial_{u_{n,m},}\right.}
\]

(18e)

\[
\tilde{X}^{D_4}_n = \left[ \frac{-\delta_2 F_n^{(+)}}{u_{n+1,m} u_{n-1,m} - \frac{1}{2} u_{n,m} (u_{n+1,m} + u_{n-1,m}) + \frac{1}{2} u_{n,m}^2 + \left( F_n^{(-)} - F_n^{(+) F_m^{(+) \right) u_{n,m} \partial_{u_{n,m}},}\right.}
\]

(18f)

Note that the equation $D_3$ (14d) is invariant under the exchange $n \leftrightarrow m$ so the symmetry $\tilde{X}^{D_3}_m$ (15b) can be obtained from the symmetry $\tilde{X}^{D_3}_n$ (15b) performing such exchange.
\[ \hat{X}^2_{D_4} = \begin{bmatrix} \delta_2 F_n^{(+)} u_{n,m+1} u_{n,m-1} + \frac{1}{2} u_{n,m} (u_{n,m+1} + u_{n,m-1}) \\ u_{n,m+1} - u_{n,m-1} \end{bmatrix} + \]

and no point symmetries. Again the fluxes of the symmetries can be readily identified with some specific form of non autonomous YdKN equations and the explicit form of the coefficients are shown in Table 4.

3 Algebraic entropy for the non autonomous YdKN equation and its subcases.

In the previous section we saw that the fluxes of all the generalized three point symmetries of the \( H^4 \) and \( H^6 \) equations are eventually related either to the YdKN or to the non autonomous YdKN equation. It was remarked in the introduction that the non autonomous YdKN equation passes the necessary condition for the integrability which is only an indication of the integrability of such class of equation. In this paper we have shown that they are symmetries of the \( H^4 \) and \( H^6 \) equations. Here we give a further evidence that the non-autonomous YdKN might be an integrable differential-difference equations based on the algebraic entropy test.

We recall briefly how to compute the algebraic entropy in the case of differential-difference equations of the form \( \frac{du}{dt} = f_n \left( u_{n+1}, u_n, u_{n-1} \right) \). First of all we assume that the equation is solvable for \( u_{n+1} \) uniquely. This is a condition on \( f_n \). Then, starting from \( n = 0 \), we compute \( u_1 \) by substituting the initial conditions:

\[ \left\{ \frac{d^k u_0}{dt^k}, \frac{d^k u_0}{dt^k} \right\} \rightarrow \infty, \quad k = 0. \]  

Knowing \( u_1 \) we can then calculate \( u_2 \) and so on. We define the degree of the iterate at the \( l \)-th step as the maximum between the degree of the numerator and of the numerator of \( u_l \) in the initial conditions. A great simplification in the explicit calculations is obtained if instead of a generic initial condition one parametrizes the curve of the initial condition rationally using the variable \( t \):

\[ u_{-1} = \frac{A_{-1} t + B_{-1}}{A t + B}, \quad u_0 = \frac{A_0 t + B_0}{A t + B}, \]

and then compute the degrees in \( t \). Calculating \( N \) iterates, for a sufficiently large positive integer \( N \), and constructing the generating function one can calculate the algebraic entropy without calculating the entire sequence. For more details on how the method is implemented see [8].

We look for the sequence of degrees of the iterate map for the non autonomous YdKN equation and its particular cases found in the previous section. We find for all the cases, except the symmetries of \( iH_1^2 \), the symmetries \( iH_1^1 \) for \( iH_1^1 \) and the symmetries of the \( iD_2 \) equations, the following values:

\[ 1, 1, 3, 7, 13, 21, 31, 43, 57, 73, 91, 111, 133, 157 \ldots \]
| Eq. | k  | α  | b_k | c_k          | d                          | e_k                                                    | f                          |
|-----|----|----|-----|--------------|----------------------------|--------------------------------------------------------|----------------------------|
| D_3 | n  | 0  | 0   | F_n F_m^+   | 0                          | 1/2 (F_n F_m^- + F_n F_m^- - F_n F_m^+)                 | 0                          |
|     | m  | 0  | 0   | F_n F_m^+   | 0                          | 1/2 (F_n F_m^- + F_n F_m^- - F_n F_m^+)                 | 0                          |
| 1D_4| n  | 0  | 0   | -δ_1 (F_n F_m^+ + F_n F_m^-) | -1/2                       | 0                                                      | δ_2 δ_3 F_m^+               |
|     | m  | 0  | 0   | δ_2 (F_n F_m^+ + F_n F_m^-)  | 1/2                       | 0                                                      | -δ_1 δ_3 F_n^+              |
| 2D_4| n  | 0  | 0   | -F_n F_m^+ δ_1 δ_2         | 1/2                       | 0                                                      | δ_3                        |
|     | m  | 0  | 0   | δ_2 (F_n F_m^+ + F_n F_m^-) | 1/2                       | 0                                                      | -δ_1 δ_3 F_n^+              |

Table 4: Identification of the coefficients of the symmetries for D_3, 1D_4 and 2D_4 with those of a non autonomous YdKN.
This sequence has the following generating function:

$$g(z) = \frac{1 - 2z + 3z^2}{(1 - z)^2}, \quad (22)$$

which gives the following quadratic fit for the sequence (21):

$$d_l = l(l - 1) + 1. \quad (23)$$

For the symmetry in the $n$ direction (5a) of the equation $iH_1^k$ we have the somehow different situation when the sequence growth is different according to the even or odd values of the $m$ variable:

$$m = 2k \quad 1, 1, 3, 7, 10, 17, 23, 33, 42, 55, 67, 83, 98, 117 \ldots \quad (24a)$$

$$m = 2k + 1 \quad 1, 1, 3, 4, 9, 13, 21, 28, 39, 49, 63, 76, 93, 109 \ldots \quad (24b)$$

These sequences have the following generating functions and asymptotic fits:

$m = 2k$,

$$g(z) = \frac{2z^5 - 3z^4 + 3z^3 + z^2 - z + 1}{(1 - z)^3(z + 1)}, \quad (25a)$$

$$d_l = \frac{3}{4}l^2 - \frac{5(-1)^l - 21}{8},$$

$m = 2k + 1$,

$$g(z) = \frac{(z^2 + z + 1)(2z^2 - 2z + 1)}{(1 - z)^3(z + 1)}, \quad (25b)$$

$$d_l = \frac{3}{4}l^2 - \frac{3}{2}l - \frac{5(-1)^l - 19}{8}.$$ 

The symmetry in the $m$ direction (7a) of the equation $iH_1^k$ has the same behaviour by exchanging $m$ with $n$ in formulae (24-25). The $n$ directional symmetry of the equation $iH_1^k$ has almost the same growth for $m$ odd as (24b, 25b) however it is worthwhile to mention that the fit $d_l = \frac{3}{4}l^2 - \frac{5(-1)^l - 19}{8}$ presents a term $l(-1)^l$, new in this kind of results. For $m$ even we have the same growth as (21). The $m$ directional symmetry of the equation $iH_1^k$ has the same growth as the even one of $iH_1^k$ (24a, 25a). For the symmetries (17) of the $iD_2$ equations we have different growth according to the even or odd values of the $m$ or $n$ variables and similar sequences or slightly lower than in the case of equations $H_1^k$, however always corresponding to a quadratic asymptotic fit.

This shows that the whole family of the non autonomous YdKN is integrable according to the algebraic entropy test.

For completeness let us just mention that the symmetries (14) of the $iD_2$ equations have a sequence growth of the same order than those considered above, i.e. quadratic growth and thus null entropy.

### 4 Conclusions

In this note we constructed the symmetries of the equations belonging to the Boll classification [5, 6] and showed that they are integrable (by the algebraic entropy test) and related to particular cases of the non autonomous YdKN equation [3, 15]. This was already known for the rhombic $H^4$ equations [19].
Klein symmetry: proposing such a generalization. A first possibility is to generalize the original autonomous generalization of the
then (a reparametrization of) $Q$ a solution of (29) exists, i.e. one is able to express the $p$ with coefficients given by (29) is a three point generalized symmetry of
the Klein symmetry.
The connection formulae between the coefficient of $Q$ and here we show the explicit identification of the symmetries obtained in that paper with the coefficients of the non autonomous YdKN equation.
We finally note that, as was proved in [12] for the YdKN [1], no equation belonging to the Boll classification has a symmetry which corresponds to the general non autonomous YdKN equation [3]. In all the cases of the Boll classification one has $a = b_k = 0$.
In [18] it was shown that the $Q_V$ equation introduced by Viallet, possessing the Klein symmetry,

$$Q_V : p_1^2 + p_2 \, (u_{n,m} + u_{n,m+1} + u_{n+1,m} + u_{n+1,m+1}) +$$
$$+ p_{2,1} \, (u_{n,m}u_{n+1,m} + u_{n,m+1}u_{n+1,m+1}) +$$
$$+ p_{2,2} \, (u_{n,m}u_{n,m+1} + u_{n+1,m}u_{n+1,m+1}) +$$
$$+ p_{2,0} \, (u_{n,m}u_{n+1,m+1} + u_{n,m+1}u_{n+1,m}) +$$
$$+ p_1 \, (u_{n,m}u_{n+1,m} + u_{n,m}u_{n+1,m+1}) +$$
$$+ u_{n,m}u_{n,m+1}u_{n+1,m+1} + u_{n,m+1}u_{n+1,m}u_{n+1,m+1} +$$
$$+ p_0 \, u_{n,m}u_{n,m+1}u_{n+1,m}u_{n+1,m+1} = 0,$$

admits a symmetry of the form of the YdKN:

$$\hat X_n = \frac{h}{u_{n+1,m} - u_{n-1,m}} - \frac{1}{2} \partial_{u_{n+1,m}} h. \quad (27)$$

where:

$$h(u_{n,m}, u_{n+1,m}; p_1, p_{2,1}, p_2, p_4) = Q_V \partial_{u_{n,m+1}} \partial_{u_{n+1,m+1}} Q_V + (28) - (\partial_{u_{n,m+1}} Q_V) \left( \partial_{u_{n+1,m+1}} Q_V \right).$$

The connection formulae between the coefficient of $Q_V$ and the YdKN [1] is:

$$a = p_1^2 - p_{2,1} p_0, \quad b = \frac{1}{2} [p_1 (p_{2,0} + p_{2,2} - p_{2,1} - p_0 p_4), \quad (29)$$
$$c = p_{2,0} p_{2,2} - p_3 p_1 \quad d = \frac{1}{2} [p_{2,2} - p_{2,1} + p_{2,2} - p_0 p_4],$$
$$e = \frac{1}{2} [p_3 (p_{2,2} - p_{2,1} + p_{2,0}) - p_1 p_4], \quad f = p_3 - p_{2,1} p_4.$$

This is a set of coupled nonlinear algebraic equations between the 7 parameters $p_i$ of $Q_V$ and the 6 ones $(a, \cdots, f)$ of the YdKN. Eq. (29) tells us that the YdKN with coefficients given by (29) is a three point generalized symmetry of $Q_V$. If a solution of (29) exists, i.e. one is able to express the $p_i$ in term of $(a, \cdots, f)$, then (a reparametrization of) $Q_V$ turns out to be a Bäcklund transformation of the YdKN [11].

From the results obtained in this paper one is lead to conjecture a non autonomous generalization of the $Q_V$ equation. We have many possible ways of proposing such a generalization. A first possibility is to generalize the original Klein symmetry:

$$Q (u_{n+1,m}, u_{n,m}, u_{n+1,m+1}, u_{n,m+1}; (-1)^n, (-1)^m) = (30)$$
$$\tau Q (u_{n,m}, u_{n+1,m}, u_{n,m+1}, u_{n+1,m+1}; (-1)^n, (-1)^m),$$
$$Q (u_{n,m+1}, u_{n+1,m+1}, u_{n,m}, u_{n,m+1}; (-1)^n, (-1)^m) =$$
\[ \tau' Q \left( u_{n,m}, u_{n+1,m}, u_{n,m+1}, u_{n+1,m+1}; (-1)^n, (-1)^m \right), \]

where \((\tau, \tau') = \pm 1\) and \(Q(x, u, y, z; (-1)^n, (-1)^m)\) is a multilinear function of its arguments with nonautonomous coefficients in the form of 2-periodic functions in \(n\) and \(m\), i.e., of the form \(\alpha + \beta (-1)^n + \gamma (-1)^m + \delta (-1)^{n+m}\), with \(\alpha, \beta, \gamma\), and \(\delta\) constants. This discrete symmetry is shared by all of the Boll systems and in the autonomous case reduces to the the usual Klein symmetry.

A second possibility is to ask the function \(Q(x, u, y, z; (-1)^n, (-1)^m)\) to respect a strict Klein symmetry just as in (26). Choosing the coefficients for example as

\[
\begin{align*}
p_0 &= 1 + (-1)^n, \\
p_1 &= (-1)^n, \\
p_2,1 &= -1 + (-1)^n, \\
p_2,2 &= (-1)^n, \\
p_2,0 &= 1 + 2(-1)^n, \\
p_3 &= 1 + (-1)^n, \\
p_4 &= 4 + 2(-1)^n,
\end{align*}
\]

provide a non autonomous YdKN. In this case, performing the algebraic entropy test the equation turns out to be integrable. Its generalized symmetries, however, are not necessarily in the form of a non autonomous YdKN equation. A different non autonomous choice of the coefficients of (26), such that (29) is satisfied for the coefficients of the non autonomous YdKN, gives, by the algebraic entropy test, a non integrable equation.

The proof of the existence of a non autonomous generalization of \(Q_V\) together with the derivation of an effective Bäcklund transformation and Lax pair for the YdKN and its non autonomous counterpart is work in progress.

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