A NEW RESULT ON THE EXISTENCE AND NON-EXISTENCE OF POSITIVE SOLUTIONS FOR TWO-PARAMETRIC SYSTEMS OF QUASILINEAR ELLIPTIC EQUATIONS

R.L. ALVES

Abstract

This paper is devoted to the existence and non-existence of positive solutions for a \((p, q)\)-Laplacian system with indefinite nonlinearity depending on two parameters \((\lambda, \mu)\). By using the sub-supersolution method together with adaptations of ideas found in [3, 10], we extend some previous result.

2010 Mathematics Subject Classification. 35B09; 35J62; 35B30; 35B38; 35B99.

Key words. Indefinite nonlinearity, sub-supersolutions, extremal curve, multiplicity.

1. Introduction

In this paper we consider the system of equations

\[
\begin{align*}
-\Delta_p u &= \lambda|u|^{p-2}u + \alpha f(x)|u|^\alpha - 2 u \phi \quad \text{in } \Omega, \\
-\Delta_q v &= \mu|v|^{q-2}v + \beta f(x)|v|^\beta - 2 v \psi \quad \text{in } \Omega, \\
(u, v) &\in W^{1,p}_0(\Omega) \times W^{1,q}_0(\Omega), \\
u, v &> 0 \text{ in } \Omega,
\end{align*}
\]

(P_{\lambda, \mu})

where \(\Omega \subset \mathbb{R}^N (N \geq 3)\) is a smooth bounded domain; the function \(f \in L^\infty(\Omega)\) and has indefinite sign, that is \(f^+ = \max\{f, 0\}\) and \(f^- = \min\{f, 0\}\) are not identically zero in \(\Omega\); \(\lambda, \mu \in \mathbb{R}\); \(\Delta_r (r \in \{p, q\})\) is the \(r\)-Laplacian operator; \(1 < p, q < \infty\),

\[
\alpha \geq p, \beta \geq q \quad \text{and} \quad \frac{\alpha}{p} + \frac{\beta}{q} > 1, \quad \frac{\alpha}{p^*} + \frac{\beta}{q^*} < 1, \quad (1.1)
\]

where \(p^*\) and \(q^*\) are the critical Sobolev exponents of \(W^{1,p}_0(\Omega)\) and \(W^{1,q}_0(\Omega)\).

We will use the symbols \((\hat{\lambda}_1, \phi_1)\) and \((\hat{\mu}_1, \psi_1)\) for the first eigenpairs of the operators \(-\Delta_p\) and \(-\Delta_q\) in \(\Omega\) with zero boundary conditions, respectively.

Consider the product space \(E = W^{1,p}_0(\Omega) \times W^{1,q}_0(\Omega)\) equipped with the norm

\[
\|(u, v)\| = \left( \int_{\Omega} |\nabla u|^p \right)^{\frac{1}{p}} + \left( \int_{\Omega} |\nabla v|^q \right)^{\frac{1}{q}} := \|u\|_p + \|v\|_q, \quad (u, v) \in E.
\]

We say that \((u, v) \in E\) is a positive solution of \((P_{\lambda, \mu})\) if \(u > 0, v > 0\) in \(\Omega\) and

\[
\begin{align*}
\int_{\Omega} |\nabla u|^{p-2}\nabla u \nabla \phi + \int_{\Omega} |\nabla v|^{q-2}\nabla v \nabla \psi - \lambda \int_{\Omega} |u|^{p-2}u \phi - \alpha \int_{\Omega} f(x)|u|^\alpha - 2 |v|^\beta u \phi \\
- \mu \int_{\Omega} |v|^{q-2}v \psi - \beta \int_{\Omega} f(x)|v|^\beta - 2 v \psi = 0, \forall (\phi, \psi) \in E.
\end{align*}
\]
Thus, the corresponding energy functional of problem $(P_{\lambda, \mu})$ is defined by

$$I_\sigma(u, v) = \frac{1}{p} \left( \|u\|_p^p - \lambda |u|_p^p \right) + \frac{1}{q} \left( \|v\|_q^q - \mu |v|_q^q \right) - F(u, v),$$

where $\sigma = (\lambda, \mu)$, $F(u, v) = \int_{\Omega} f |u|^\alpha |v|^\beta$ and $\| \cdot \|_p, \| \cdot \|_q$ are the standard $L^p(\Omega)$ and $L^q(\Omega)$ norm. A pair of functions $(u, v) \in E$ is said to be a weak solution of $(P_{\lambda, \mu})$ if $(u, v)$ is a critical point of $I_\sigma$. Let us observe that if $\sigma = (\hat{\lambda}_1, \hat{\mu}_1), \lambda \in \mathbb{R}$ (respectively $\sigma = (\lambda, \hat{\mu}_1), \lambda \in \mathbb{R}$), then $(\epsilon \phi_1, 0), \epsilon \in \mathbb{R}$ (respectively $(0, \epsilon \psi_1), \epsilon \in \mathbb{R}$) is a weak solution of $(P_{\hat{\lambda}_1, \mu})$ (respectively $(P_{\lambda, \hat{\mu}_1})$).

Under suitable assumptions on the function $f$ (see assumptions $(f)_1 -(f)_2$ below), and by using the fibering Method of Pohozaev, problem $(P_{\lambda, \mu})$ has been studied by Bozhkov-Mitidieri [7], Bobkov-Il’yasov [5, 6] and Silva-Macedo [11]. From these works we know that:

- if $(\lambda, \mu) \in (-\infty, \lambda_1] \times (-\infty, \hat{\mu}_1]$, then problem $(P_{\lambda, \mu})$ admits at least one positive solution (see [5, 7]);
- there exist curves $\Gamma* \subset \Pi := \{(\lambda, \mu) : \lambda_1 < \lambda \text{ and } \hat{\mu}_1 < \mu\}$ which determine regions $\Pi_1, \Pi_2 \subset \Pi$ satisfying $\Gamma* \cap \Pi_1 = \emptyset, \Gamma* \cap \Pi_2 = \emptyset$ such that problem $(P_{\lambda, \mu})$ has no positive solution and has at least one according to $(\lambda, \mu)$ belongs to $\Pi_2$ and $\Pi_1$, respectively (see [5, 6, 11])

These results are depicted in the following figure. We observe that they showed the existence of a positive solution only in the blue region.

![Figure 1. $\Pi_1 =$ blue region; $\Pi_2 = \{(\lambda, \mu) : (0, 0) < \Gamma*(t) < (\lambda, \mu)\}$](image)

The main novelty in the present paper is the investigation of the existence and nonexistence of positive solution outside the sets $\Pi_1$ and $\Pi_2$ (see Figure 1). Special attention is paid in finding of the curve $\Gamma \subset \mathbb{R}^2, \Gamma(t) = (\lambda(t), \mu(t)), t \in \mathbb{R}$ called to be the extremal curve such that problem $(P_{\lambda, \mu})$ admits a positive solution if $\lambda < \lambda(t), \mu < \mu(t), t \in \mathbb{R}$ and $(\lambda, \mu) \notin \Lambda_1 \cup \Lambda_2$, while if $\lambda > \lambda(t)$ and $\mu > \mu(t), t \in \mathbb{R}$ then $(P_{\lambda, \mu})$ admits no positive solution (see (1.2) for the definition of the sets $\Lambda_1$ and $\Lambda_2$).
Our proof is based on the super and sub-solution method and some adaptations of the ideas found in the works of the author et al. \[3, 10\] and Cheng-Zhang \[8\]. As in \[7, 5, 6, 11\], throughout this paper we will assume the following hypothesis

\[(f)_1\] \[F(\phi_1, \psi_1) = \int_{\Omega} f|\phi_1|^\alpha|\psi_1|^\beta < 0;\]

\[(f)_2\] Let \(\Omega^0 = \{x \in \Omega : f(x) = 0\}\) and \(\Omega^+ = \{x \in \Omega : f(x) > 0\}\). The measure of \(\Omega^0\) is not zero and the interior of \(\Omega^0 \cup \Omega^+\) is a regular domain. Moreover \(\Omega = int(\Omega^0 \cup \Omega^+)\) contains a connected component which intersects both \(\Omega^0\) and \(\Omega^+\).

Before stating our main results, we introduce the following two sets

\[
\Lambda_1 = \{ (\lambda, \hat{\mu}_1) \in \mathbb{R}^2 : \lambda \leq \hat{\lambda}_1 \} \quad \text{and} \quad \Lambda_2 = \{ (\hat{\lambda}_1, \mu) \in \mathbb{R}^2 : \mu \leq \hat{\mu}_1 \},
\]

(1.2)

Our first result is the following.

**Theorem 1.1.** Assume (1.1) and \((f)_1 - (f)_2\) hold. There exist \(\lambda_*, \mu_* \in \mathbb{R}\) and a continuous simple arc \(\Gamma\), satisfying \(\lim_{t \to +\infty} \Gamma(t) = (\lambda_*, -\infty)\) and \(\lim_{t \to -\infty} \Gamma(t) = (-\infty, \mu_*)\) that separates \(\mathbb{R}^2\) into two disjoint open subsets \(\Theta_1\) and \(\Theta_2\), with \(\Lambda_1 \cup \Lambda_2 \subset \Theta_1\) such that the system \((P_{\lambda, \mu})\) has no positive solution and has at least one according to \((\lambda, \mu)\) belongs to \(\Theta_2\) and \(\Theta_1 \setminus (\Lambda_1 \cup \Lambda_2)\), respectively. Moreover, \(\partial \Theta_1 \setminus (\Lambda_1 \cup \Lambda_2) = \Gamma\).

Theorem 1.1 partially extend the main results in \[3, 5, 11\], because it show the existence of positive solution of \((P_{\lambda, \mu})\) for every \((\lambda, \mu) \in \Theta_1 \setminus (\Lambda_1 \cup \Lambda_2)\), where \(\Upsilon \cap (\mathbb{R}^2 \setminus (\Theta_1 \cup \Gamma)) = \emptyset\) (see (1.6) for the definition of \(\Upsilon\)). In particular, it shows the existence of positive solution of \(\hat{\lambda}_1 < \lambda, \mu < \hat{\mu}_1\) and \(\lambda < \hat{\lambda}_1, \hat{\mu}_1 < \mu\), too. Furthermore, Theorem 1.1 gives us an almost complete description of the set of parameters \((\lambda, \mu)\) such that \((P_{\lambda, \mu})\) admits a positive solution (see Figure. 2). In particular, we would like to point out that the curves \(\Gamma\) and \(\Gamma^*\) are distinct (see Figures 1 and 2).
The next result deals with the existence and multiplicity of positive solutions of \( P_{\lambda,\mu} \) when \( p = q = 2 \) and \( \lambda = \mu \).

**Theorem 1.2.** Let \( p = q = 2 \), \( f \in C(\Omega) \), and assume that (1.1) and \((f)_1 - (f)_2\) hold. Then there exists \( \tau > 0 \) such that

a) for every \( \lambda \in (\hat{\lambda}_1, \tau) \), \( P_{\lambda,\mu} \) admits at least two positive solutions.

b) for \( \lambda = \hat{\lambda}_1 \) and \( \lambda = \tau \), problem \( P_{\lambda,\mu} \) admits at least one positive solution.

This paper is organized as follows. In Section 2, we give some definitions and prove a sub-supersolution theorem. In Section 3 we study some properties of the set of parameters \((\lambda, \mu)\) such that \( P_{\lambda,\mu} \) admits at least one positive solution, and then give the proof of Theorem 1.1. In Section 4, we prove Theorem 1.2.

**Notations.** Throughout this paper, we make use of the following notations.

- The spaces \( \mathbb{R}^N \) are equipped with the Euclidean norm \( \sqrt{x_1^2 + \cdots + x_N^2} \).
- \( B_r(x) \) denotes the ball centered at \( x \in \mathbb{R}^N \) with radius \( r > 0 \).
- the notation \( (a, b) > (c, d) \) means \( a > c \) and \( b > d \). Similarly, \( (a, b) \geq (c, d) \) means \( a \geq c \) and \( b \geq d \) for all \( (a, b), (c, d) \in \mathbb{R}^2 \).
- \( \lim_{|x| \to \infty} (u(x), v(x)) = (\lim_{|x| \to \infty} u(x), \lim_{|x| \to \infty} v(x)) \) for functions \( u, v : \mathbb{R}^N \to \mathbb{R} \).
- If \( A \) is a measurable set in \( \mathbb{R}^N \), we denote by \( \mathcal{L}(A) \) the Lebesgue measure of \( A \).

2. **Sub-supersolution theorem** for \((\lambda, \mu) \notin [-\infty, \hat{\lambda}_1[ \times ]-\infty, \hat{\mu}_1]\)

In this section we will give some definitions and prove a sub-supersolution theorem that will be essential to prove Theorem 1.1.

Let us consider the space \( C^1(\Omega) = \{ u \in C^1(\Omega) : u = 0 \text{ on } \partial \Omega \} \) equipped with the norm \( \|u\|_{C^1} = \max_{x \in \Omega} |u(x)| + \max_{x \in \Omega} |\nabla u(x)| \). If on \( C^1(\Omega) \) we consider the pointwise partial ordering (i.e., \( u \leq v \) if and only if \( u(x) \leq v(x) \) for all \( x \in \Omega \)), which is induced by the positive cone

\[ C^1_0(\Omega)^+ = \{ u \in C^1_0(\Omega) : u \geq 0 \text{ for all } x \in \Omega \} , \]

then this cone has a nonempty interior given by

\[ \text{int } C^1_0(\Omega)^+ = \left\{ u \in C^1_0(\Omega) : u > 0 \text{ for all } x \in \Omega \text{ and } \frac{\partial u}{\partial n}(x) < 0 \text{ for all } x \in \partial \Omega \right\} , \]

where \( n(x) \) is the outward unit normal vector to \( \partial \Omega \) at the point \( x \in \partial \Omega \). It is known that \( \phi_1, \psi_1 \in \text{int } C^1_0(\Omega)^+ \).

The next proposition has been proved in Marano-Papageorgiou (see [9], Proposition 1).

**Proposition 2.1.** If \( u \in \text{int } C^1_0(\Omega)^+ \) then to every \( v \in C^1_0(\Omega)^+ \) there corresponds \( \epsilon_v > 0 \) such that \( u - \epsilon_v v \in C^1_0(\Omega)^+ \).

The following proposition concerning the regularity of the positive solutions will be useful in the sequel. For proof we refer to [6], Appendix 2.

**Proposition 2.2.** Assume (1.1) holds and let \((u, v)\) be a positive solution of \( P_{\lambda,\mu} \). Then \( u, v \in C^1_0(\Omega) \). Moreover, \( u \) and \( v \) satisfy a boundary point maximum principle on \( \partial \Omega \).
As a consequence of Proposition 2.1 and Proposition 2.2 for any positive solution \((u, v)\) of \(\{P_{\lambda, \mu}\}\) one has \(u \geq c_0, \ v \geq \epsilon v_1\) in \(\Omega\) for \(\epsilon > 0\) small enough.

Now we define the notion of sub-supersolution.

**Definition 2.1.** A pair \((\overline{u}, \overline{v}) \in E\) is said to be a supersolution of \(\{P_{\lambda, \mu}\}\) if \(\overline{u}, \overline{v} \in \text{int} \ C_0^1(\Omega)^+\) and
\[
\int_{\Omega} |\nabla \overline{u}|^{p-2} \nabla \overline{u} \nabla \phi - \lambda \int_{\Omega} |\overline{u}|^{p-2} \overline{u} \phi - \alpha \int_{\Omega} f(x)|\overline{u}|^{q-2}|\overline{u}|^{\beta} \phi \geq 0,
\]
\[
\int_{\Omega} |\nabla \overline{v}|^{q-2} \nabla \overline{v} \nabla \psi - \mu \int_{\Omega} |\overline{v}|^{q-2} \overline{v} \psi - \beta \int_{\Omega} f(x)|\overline{v}|^{q-2}|\overline{v}|^{\beta} \psi \geq 0
\]
for any \((\phi, \psi) \in E\) with \(\phi, \psi \geq 0\) in \(\Omega\).

**Definition 2.2.** A pair \((\underline{u}, \underline{v}) \in E\) is said to be a subsolution of \(\{P_{\lambda, \mu}\}\) if
\[
\int_{\Omega} |\nabla \underline{u}|^{p-2} \nabla \underline{u} \nabla \phi - \lambda \int_{\Omega} |\underline{u}|^{p-2} \underline{u} \phi - \alpha \int_{\Omega} f(x)|\underline{u}|^{q-2}|\underline{u}|^{\beta} \phi \leq 0,
\]
\[
\int_{\Omega} |\nabla \underline{v}|^{q-2} \nabla \underline{v} \nabla \psi - \mu \int_{\Omega} |\underline{v}|^{q-2} \underline{v} \psi - \beta \int_{\Omega} f(x)|\underline{v}|^{q-2}|\underline{v}|^{\beta} \psi \leq 0
\]
for any \((\phi, \psi) \in E\) with \(\phi, \psi \geq 0\) in \(\Omega\).

**Remark 2.1.** The functions \(\underline{u}\) and \(\overline{u}\) may not be in \(\text{int} \ C_0^1(\Omega)^+\).

We introduce the notation
\[
X_{\lambda_1, \mu_1} = \{(\lambda, \mu) \in \mathbb{R}^2 : \lambda < \lambda_1, \mu < \mu_1\}.
\]

We are now ready to prove the main result of this section.

**Theorem 2.1.** Let \((\overline{\lambda}, \overline{\mu}) \notin \Lambda_1 \cup \Lambda_2 \cup X_{\lambda_1, \mu_1}\. Assume (1.1) holds and suppose that \(\{P_{\lambda, \mu}\}\) has a supersolution. Then problem \(\{P_{\lambda, \mu}\}\) has a positive solution \((u, v)\) for each \(\sigma = (\lambda, \mu) \in (-\infty, \overline{\lambda}) \times (-\infty, \overline{\mu}) \setminus \left(\Lambda_1 \cup \Lambda_2 \cup X_{\lambda_1, \mu_1}\right)\). Moreover, one has \(I_\sigma(u, v) < 0\).

**Proof.** We first note that \((\underline{0}, \underline{0}) = (0, 0)\) is a subsolution of \(\{P_{\lambda, \mu}\}\) for any \((\lambda, \mu) \in \mathbb{R}^2\). Let \((\overline{u}, \overline{v})\) be a supersolution of \(\{P_{\lambda, \mu}\}\) and \((\lambda, \mu) \in (-\infty, \overline{\lambda}) \times (-\infty, \overline{\mu}) \setminus \left(\Lambda_1 \cup \Lambda_2 \cup X_{\lambda_1, \mu_1}\right)\. Clearly, \((\overline{u}, \overline{v})\) is a supersolution of \(\{P_{\lambda, \mu}\}\) with \(\overline{u} > 0\) and \(\overline{v} > 0\) in \(\Omega\). The solution of \(\{P_{\lambda, \mu}\}\) will be obtained by minimizing the functional \(I_\sigma (\sigma = (\lambda, \mu))\) over the set
\[
M = \{(u, v) \in E : 0 \leq u \leq \overline{u}, \ 0 \leq v \leq \overline{v}\}.
\]

We first observe that \(M\) is convex and closed with respect to the \(E\) topology, hence weakly closed. Furthermore, for all \((u, v) \in M\) we have
\[
I_\sigma(u, v) \geq \frac{1}{p} \left(\|\nabla u\|_p^p - \lambda \delta u_0^p\right) + \frac{1}{q} \left(\|\nabla v\|_q^q - \mu \delta v_0^q\right) - \int_{\Omega} f(x)|\overline{u}|^q |\overline{v}|^{\beta} \geq 0,
\]
which implies that \(I_\sigma\) is coercive on \(M\).

It is easy to check that \(I_\sigma\) is weakly lower semicontinuous on \(M\). Thus, \(I_\sigma\) verifies the hypotheses of Theorem 1.2 in [12]. According to this one, there exists a relative minimizer \((u, v) \in M\) of \(I_\sigma\). We show in what follows that \((u, v)\) is a positive solution of \(\{P_{\lambda, \mu}\}\).
In the sequel, to simplify the notation, we set \( U = (u, v) \),
\[
H_1(x, s, t) = \lambda |s|^{p-2} s + \alpha f(x) |s|^{\alpha-2} t^{\beta} s \quad \text{and} \quad H_2(x, s, t) = \mu |t|^{p-2} t + \beta f(x) |s|^\alpha |t|^{\beta-2} t.
\]

Let \((\varphi, \psi) = \Psi \in E, \epsilon > 0, \) and consider
\[
w^\sigma := (u + \epsilon \varphi - \overline{\sigma})^+ = \max \{0, u + \epsilon \varphi - \overline{\sigma}\},
\]
\[
w_\epsilon := (u + \epsilon \varphi)^- = \max \{0, -(u + \epsilon \varphi)\},
\]
\[
z^\sigma := (v + \epsilon \psi - \overline{\sigma})^+ = \max \{0, v + \epsilon \psi - \overline{\sigma}\},
\]
and
\[
z_\epsilon := (v + \epsilon \psi)^- = \max \{0, -(v + \epsilon \psi)\}.
\]
Set \( \eta_\epsilon := u + \epsilon \varphi - w^\sigma + w_\epsilon \) and \( \nu_\epsilon := v + \epsilon \psi - z^\sigma + z_\epsilon. \) Then \( U_\epsilon := (\eta_\epsilon, \nu_\epsilon) = U + \epsilon \Psi - (w^\sigma, z^\sigma) + (w_\epsilon, z_\epsilon) \in M, \) and by the convexity of \( M \) we get \( U + t(U_\epsilon - U) \in M, \) for all \( 0 < t < 1. \) Since \( U \) minimizes \( I_\sigma \) in \( M, \) this yields
\[
0 \leq \langle I'_\sigma(U), (U_\epsilon - U) \rangle = \epsilon \langle I'_\sigma(U), (\varphi, \psi) \rangle - \langle I'_\sigma(U), (w^\sigma, z^\sigma) \rangle + \langle I'_\sigma(U), (w_\epsilon, z_\epsilon) \rangle,
\]
so that
\[
\langle I'_\sigma(U), (\varphi, \psi) \rangle \geq \frac{1}{\epsilon} \left[ \langle I'_\sigma(U), (w^\sigma, z^\sigma) \rangle - \langle I'_\sigma(U), (w_\epsilon, z_\epsilon) \rangle \right]. \tag{1.3}
\]

Now, since \( \overline{U} = (\overline{\varphi}, \overline{\psi}) \) is a supersolution to \( \{I_{\lambda, \mu}\}, \) we have
\[
\langle I'_\sigma(U), (w^\sigma, z^\sigma) \rangle = \langle I'_\sigma(\overline{U}), (w^\sigma, z^\sigma) \rangle + \langle I'_\sigma(U), (w^\sigma, z^\sigma) \rangle - \langle I'_\sigma(U), (w^\sigma, z_\epsilon) \rangle
\]
\[
\geq \langle I'_\sigma(U), (\overline{U}, (w^\sigma, z^\sigma) \rangle - \langle I'_\sigma(U), (w^\sigma, z_\epsilon) \rangle
\]
\[
= \int_{\Omega_\epsilon} (|\nabla u|^{p-2} \nabla u - |\nabla \overline{\sigma}|^{p-2} \nabla \overline{\sigma}) \nabla (u + \epsilon \varphi - \overline{\sigma})
\]
\[
- \int_{\Omega_\epsilon} [H_1(x, u, v) - H_1(x, \overline{\varphi}, \overline{\psi})] (u + \epsilon \varphi - \overline{\sigma})
\]
\[
+ \int_{\Omega^*} (|\nabla \overline{\psi}|^{q-2} \nabla \overline{\psi} - |\nabla \overline{\sigma}|^{q-2} \nabla \overline{\sigma}) \nabla (v + \epsilon \psi - \overline{\sigma})
\]
\[
- \int_{\Omega^*} [H_2(x, u, v) - H_2(x, \overline{\varphi}, \overline{\psi})] (v + \epsilon \psi - \overline{\sigma})
\]
\[
\geq \epsilon \int_{\Omega_\epsilon} (|\nabla u|^{p-2} \nabla u - |\nabla \overline{\sigma}|^{p-2} \nabla \overline{\sigma}) \nabla \varphi
\]
\[
- \epsilon \int_{\Omega_\epsilon} |H_1(x, u, v) - H_1(x, \overline{\varphi}, \overline{\psi})||\varphi|
\]
\[
+ \epsilon \int_{\Omega^*} (|\nabla \overline{\psi}|^{q-2} \nabla \overline{\psi} - |\nabla \overline{\sigma}|^{q-2} \nabla \overline{\sigma}) \nabla \psi
\]
\[
- \epsilon \int_{\Omega^*} |H_2(x, u, v) - H_2(x, \overline{\varphi}, \overline{\psi})||\psi|,
\]
where we have used the monotonicity of \( -\Delta_p, -\Delta_q \) and
\[
\Omega_\epsilon = \{ x \in \mathbb{R}^N : u + \epsilon \varphi \geq \overline{\sigma} > u \} \quad \text{and} \quad \Omega^* = \{ x \in \mathbb{R}^N : v + \epsilon \psi \geq \overline{\sigma} > v \}.
\]
Note that \( \mathcal{L}(\Omega_\epsilon) \to 0 \) and \( \mathcal{L}(\Omega^*) \to 0 \) as \( \epsilon \to 0. \) Hence, by absolute continuity of the Lebesgue integral, one has
\[
\frac{\langle I'_\sigma(U), (w^\sigma, z^\sigma) \rangle}{\epsilon} \geq o(\epsilon), \quad \text{where} \quad o(\epsilon) \to 0 \quad \text{as} \quad \epsilon \to 0. \tag{1.4}
\]
By using that $(0,0)$ is a subsolution and following similar arguments as done in the proof of (1.3), we get

$$\frac{\langle I'_\sigma(U), (w, z) \rangle}{\epsilon} \leq o(\epsilon), \quad \text{where} \quad o(\epsilon) \to 0 \quad \text{as} \quad \epsilon \to \infty, \quad (1.5)$$

Putting together (1.3), (1.4) and (1.5) we deduce $\langle I'_\sigma(U), \Psi \rangle \geq 0$, for all $\Psi = (\varphi, \psi) \in E$. Reversing the sign of $\Psi$ we find $\langle I'_\sigma(U), \Psi \rangle = 0$, for all $\Psi \in E$, that is, $U = (u, v)$ is a weak solution of $(P_{\lambda, \mu})$.

We claim that $(u, v)$ is a positive solution of $(P_{\lambda, \mu})$. Indeed, one has $u, v \geq 0$ in $\Omega$ and by the assumption either $\lambda > \hat{\lambda}$ or $\mu > \hat{\mu}$. Without loss of generality we can assume $\lambda > \hat{\lambda}$ (the proof in the second case is similar). From Proposition 2.1 and Proposition 2.2 there exists $\epsilon > 0$ such that $(\epsilon \varphi_1, 0) \in M$, and consequently

$$I_\sigma(u, v) \leq I_\sigma(\epsilon \varphi_1, 0) = \frac{1}{p} \left( \|\epsilon \varphi_1\|_p^p - \lambda |\epsilon \varphi_1|^p \right) < \frac{1}{p} \left( \|\epsilon \varphi_1\|_p^p - \hat{\lambda}_1 |\epsilon \varphi_1|^p \right) = 0,$$

that is, $(u, v) \neq (0, 0)$.

We show in what follows that $v \neq 0$. Assume the contrary, namely $v = 0$. Since $(u, v) \neq (0, 0)$, we obtain $u \geq 0$ in $\Omega$, $u \neq 0$ and

$$\int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla \varphi = \lambda |u|^{p-2} u \varphi,$$

for every $\varphi \in W_0^{1,p}(\Omega)$. By standard regularity arguments and the strong maximum principle, we deduce that $u \in C_0^1(\overline{\Omega})$ and $u > 0$ in $\Omega$. Hence $\lambda = \hat{\lambda}_1$ (due to Theorem 5.1 in [2]), but this clearly contradicts the assumption $\lambda > \hat{\lambda}_1$.

Finally we prove that $u \neq 0$. The following two cases may occur:

CASE 1: $\mu \neq \hat{\mu}_1$.

CASE 2: $\mu = \hat{\mu}_1$.

The former follows as done above to reach $v \neq 0$. For the latter, assume by contradiction that $u = 0$. Because $(0, v)$ is a weak solution of $(P_{\lambda, \hat{\mu}_1})$ and $v \neq 0$, we deduce $v = \epsilon \psi_1$ for some $\epsilon > 0$. Thus

$$0 = I_\sigma(0, \epsilon \psi_1) = I_\sigma(0, v) < 0,$$

which is a contradiction.

Therefore, $u \neq 0$ and $v \neq 0$. From Proposition 2.1 and Proposition 2.2 one has $u, v \in int C_0^1(\overline{\Omega})^+$, that is, $(u, v)$ is a positive solution of $(P_{\lambda, \mu})$. This finishes the proof.

\[ \square \]

Remark 2.2. Let us point out that to get $u > 0, v > 0$ in $\Omega$ in Theorem 2.7 it was essential that $(\bar{\lambda}, \bar{\mu}) \notin \Lambda_1 \cup \Lambda_2 \cup X_{\bar{\lambda}_1, \bar{\mu}_1}$ and $\sigma = (\lambda, \mu) \in (-\infty, \bar{\lambda}) \times (-\infty, \bar{\mu}) \backslash \left( \Lambda_1 \cup \Lambda_2 \cup X_{\bar{\lambda}_1, \bar{\mu}_1} \right)$. Indeed, for $(\lambda, \mu)$ outside of this set we are not able to distinguish among the relative minimizer $(u, v) \in M$ of $I_\sigma$ and the weak solutions $(0, 0), (\epsilon \phi_1, 0), \epsilon > 0$ and $(0, \epsilon \psi_1), \epsilon > 0$, in the cases where $(\lambda, \mu)$ belongs to $X_{\bar{\lambda}_1, \bar{\mu}_1}, \Lambda_2$ and $\Lambda_1$, respectively.

Since every positive solution of $(P_{\lambda, \mu})$ is also a supersolution, the following result holds true.

Corollary 2.1. Let $(\bar{\lambda}, \bar{\mu}) \notin \Lambda_1 \cup \Lambda_2 \cup X_{\bar{\lambda}_1, \bar{\mu}_1}$. Assume (1.1) holds and suppose that $(P_{\lambda, \mu})$ has a positive solution. Then problem $(P_{\lambda, \mu})$ has a positive solution $(u, v)$.
for each $\sigma = (\lambda, \mu) \in (-\infty, \Lambda_1] \times (-\infty, \bar{\Omega}) \setminus \left( \Lambda_1 \cup \Lambda_2 \cup X_{\lambda_1, \bar{\mu}_1} \right)$. Moreover, one has $I_\sigma(u, v) < 0$.

3. PROOF OF THEOREM \[1.1\]

Let us denote by

$$\Upsilon = \{ (\lambda, \mu) \in \mathbb{R}^2 : (P_{\lambda, \mu}) \text{ admits at least one positive solution} \},$$

$$\Upsilon := \text{the closure of } \Upsilon,$$

$$\text{int}(\Upsilon) := \text{the interior of } \Upsilon,$$

$$\partial(\text{int}(\Upsilon)) := \text{the boundary of } \text{int}(\Upsilon),$$

$$d(\text{int}(\Upsilon)) := \text{the derived set of } \text{int}(\Upsilon),$$

$$\overline{\text{int}(\Upsilon)} := \text{the closure of } \text{int}(\Upsilon).$$

We will use the symbols $\hat{\lambda}_1(\tilde{\Omega})$ and $\hat{\mu}_1(\tilde{\Omega})$ for the first eigenvalues of the operators $-\Delta_p$ and $-\Delta_q$ in $\tilde{\Omega}$ with zero boundary conditions, respectively. From now on we assume (1.1) and (f)$_2$ hold.

The main purpose of this section is to study the properties of the set $\Upsilon$ and prove Theorem \[1.1\]. From [7, 5, 6, 11] we deduce the following properties:

(\Upsilon)_1 \quad X_{\hat{\lambda}_1, \hat{\mu}_1} \subset \text{int}(\Upsilon),

(\Upsilon)_2 \quad \text{there exists } \epsilon > 0 \text{ such that } [\lambda_1, \lambda_1 + \epsilon] \times [\mu_1, \mu_1 + \epsilon] \subset \text{int}(\Upsilon),

(\Upsilon)_3 \quad \text{as a consequence of } (\Upsilon)_1 - (\Upsilon)_2 \text{ one has } \Upsilon \neq \emptyset, \text{int}(\Upsilon) \neq \emptyset \text{ and } \text{int}(\Upsilon) \cap \{ (\lambda, \mu) \in \mathbb{R}^2 : \lambda \leq \hat{\lambda}_1 \text{ and } \mu \leq \hat{\mu}_1 \} \neq \emptyset.

Now, we find an upper bound for $\Upsilon$.

**Lemma 3.1.** If $(P_{\lambda, \mu})$ has a positive solution, then $\lambda \leq \hat{\lambda}_1(\tilde{\Omega})$ and $\mu \leq \hat{\mu}_1(\tilde{\Omega})$.

**Proof.** Suppose that $(P_{\lambda, \mu})$ admits a positive solution $(u, v)$ and let $\varphi \in C_0^\infty(\tilde{\Omega})$ with $\varphi \geq 0$. From Proposition \[2.2\] we have $u, v \in \text{int } C_0^1(\tilde{\Omega})$. Hence, by Picone’s identity (see [4]),

$$\int_{\tilde{\Omega}} |\nabla \varphi|^p - \int_{\Omega} |\nabla \varphi|^p \leq \int_{\tilde{\Omega}} |\nabla \varphi|^p \geq \int_{\tilde{\Omega}} \left( |\nabla \varphi|^p - \frac{\lambda u^p}{\varphi^p} \right) \geq 0$$

and consequently, from the equation satisfied by $u$ and (f)$_2$,

$$\int_{\tilde{\Omega}} |\nabla \varphi|^p = \int_{\tilde{\Omega}} |\nabla \varphi|^p \geq \int_{\tilde{\Omega}} \left( \lambda |u|^{p-2}u + \alpha f(x)|u|^{\alpha-2}|v|^\beta \right) \varphi^p / u^{p-1}$$

$$= \int_{\tilde{\Omega}} \left( \lambda u^{p-1} + \alpha f(x)|u|^{\alpha-2}|v|^\beta \right) \varphi^p / u^{p-1}$$

$$\geq \lambda \int_{\tilde{\Omega}} u^{p-1} \varphi^p / u^{p-1},$$

that is

$$\int_{\tilde{\Omega}} |\nabla \varphi|^p \geq \lambda \int_{\tilde{\Omega}} \varphi^p.$$

Taking the infimum with respect to $\varphi$ yields $\hat{\lambda}_1(\tilde{\Omega}) \geq \lambda$. Similarly, one can show that $\mu \leq \hat{\mu}_1(\tilde{\Omega})$. The proof of the lemma is complete. □
As a consequence of Lemma 3.1 we get
\[ \text{int}(\Upsilon) \subset \Upsilon \subset (-\infty, \lambda_1(\hat{\Omega})) \times (-\infty, \mu_1(\hat{\Omega})). \]

Next, we define a family of straight lines
\[ L(t) = \{ (\lambda, \lambda - t) : \lambda \in \mathbb{R} \}, \quad t \in \mathbb{R} \]
and
\[ \lambda(t) = \sup \left\{ \lambda : (\lambda, \lambda - t) \in \text{int}(\Upsilon) \right\}, \quad \mu(t) = \lambda(t) - t \quad \text{and} \quad \Gamma(t) = (\lambda(t), \mu(t)). \]

By Lemma 3.1 one has the estimates
\[
\begin{cases}
\lambda(t) \leq \lambda_1(\hat{\Omega}), & \text{for every } t \in \mathbb{R}, \\
\mu(t) \leq \mu_1(\hat{\Omega}), & \text{for every } t \in \mathbb{R}, \\
\mu(t) \leq \lambda_1(\hat{\Omega}) - t, & \text{for every } t \in \mathbb{R}.
\end{cases}
\]

Thus, the functions \( \lambda(t) \) and \( \mu(t) \) are well defined, and consequently \( \Gamma(t) \) is well defined.

In order to apply Corollary 2.1 and Theorem 2.1 we will obtain estimates from below for \( \Gamma \) in the next lemma.

**Lemma 3.2.** There exists \( \theta \in \mathbb{R} \) such that \( \lambda(t) > \lambda_1 \) for \( t > \theta \) and \( \mu(t) > \mu_1 \) for \( t \leq \theta \).

**Proof.** By \( \Upsilon_3 \) we can find \( (\lambda, \mu) \in \Upsilon \) with \( \lambda > \lambda_1 \) and \( \mu > \mu_1 \). Let us set \( \theta = \lambda - \mu \).

Then, it is easy to see that
\[
L(t) \cap \partial ((-\infty, \lambda] \times (-\infty, \mu]) = \begin{cases} (\lambda, \lambda - t) & \text{if } t > \theta, \\ (\mu + t, \mu) & \text{if } t \leq \theta, \end{cases}
\]
and hence, by Corollary 2.1
\[
\lambda_1 < \lambda \leq \lambda(t) \text{ if } t > \theta
\]
and
\[
\mu_1 < \mu \leq \lambda(t) - t = \mu(t) \text{ if } t \leq \theta.
\]

This concludes the proof.

Furthermore, we have the following lemma.

**Lemma 3.3.** \( \Gamma(t) \in \partial(\text{int}(\Upsilon)) \) for every \( t \in \mathbb{R} \).

**Proof.** For any \( t \in \mathbb{R} \) given, by the definition of \( \lambda(t) \) there exists a sequence \( \{ (\lambda_k, \mu_k) \} \subset L(t) \cap \text{int}(\Upsilon) \) which converge to \( (\lambda(t), \mu) \), for some \( \mu \in \mathbb{R} \). Now, by the definition of \( L(t) \) and this convergence, we have \( \mu_k = \lambda_k - t = \lim_{k \to \infty} \mu_k = \lambda(t) - t = \mu(t) \). Hence, \( (\lambda(t), \mu(t)) = (\lambda(t), \mu) \in \text{int}(\Upsilon) \), that is, \( \Gamma(t) \in \text{int}(\Upsilon) \). We claim that \( \Gamma(t) \notin \text{int}(\Upsilon) \). Indeed, if \( \Gamma(t) \in \text{int}(\Upsilon) \), then there would be a \( r > 0 \) such that \( B_r(\Gamma(t)) \subset \text{int}(\Upsilon) \). Since \( B_r(\Gamma(t)) \cap L(t) \neq \emptyset \) and \( f(\lambda) = \lambda - t \) is an increasing function, there exists \( \lambda > \lambda(t) \) such that \( (\lambda, \lambda - t) \in B_r(\Gamma(t)) \) and so \( (\lambda, \lambda - t) \in \text{int}(\Upsilon) \), which is a contradiction with the definition of \( \lambda(t) \). Therefore \( \Gamma(t) \in \partial(\text{int}(\Upsilon)) \). This concludes the proof of the lemma.

\( \square \)
In the next lemmas, we prove the main properties of $\Gamma$ and $\Upsilon$.

**Lemma 3.4.** The following conclusions hold true:

- $\lambda(t)$ is monotone nondecreasing and $\mu(t)$ is monotone nonincreasing,
- $\Gamma : \mathbb{R} \rightarrow \mathbb{R}^2$ is a continuous function,
- $\lim_{t \to +\infty} \Gamma(t) = (\lambda_*, -\infty)$ and $\lim_{t \to -\infty} \Gamma(t) = (-\infty, \mu_*)$,
- $\Gamma(t)$ is injective,
- $\partial(\text{int}(\Upsilon)) \setminus (\Lambda_1 \cup \Lambda_2) = \{ \Gamma(t) : t \in \mathbb{R} \}$, \hspace{1cm} (1.7)
- $\text{int}(\Upsilon) = \bigcup_{t \in \mathbb{R}} \{ (\lambda, \mu) \in L(t) : (\lambda, \mu) \leq \Gamma(t) \}$. \hspace{1cm} (1.8)

**Proof.** Firstly let us prove $a)$. Let us suppose by contradiction that $\lambda(t)$ is not monotone nondecreasing. Then there exist $t, s \in \mathbb{R}$, with $t < s$ and $\lambda(t) > \lambda(s)$. Thus, $\lambda(t) > \lambda(s)$ and $\mu(t) > \mu(s)$. Let $\lambda$ such that $\lambda(s) < \lambda < \lambda(t)$. Since $\mu(s) = \lambda(s) - s < \lambda - s < \lambda - t < \lambda(t) - t = \mu(t)$, it follows from Lemma 3.2, Theorem 2.1, and definition of $\Gamma(t)$, which is a supersolution of $(P_{\lambda,\lambda})$. So, Lemma 3.2 and Theorem 2.1 imply that system $(P_{\lambda,\lambda})$ admits a solution $(\tilde{u}, \tilde{v})$, which lead us to conclude that $\lambda(s) < \lambda \leq \lambda(s)$, but this is a contradiction. Similarly, we deduce the monotonicity of $\mu(t)$.

Now, let us to prove $b)$. Let $s, t \in \mathbb{R}$. Without loss of generality, we can suppose that $s < t$. Therefore, combining the definition of $\Gamma(t)$ with the monotone nondecrease of $\lambda(t)$ and the monotone nonincrease of $\mu(t)$, one has

$$|\Gamma(s) - \Gamma(t)| \leq |\lambda(s) - \lambda(t)| + |\mu(s) - \mu(t)| = -\lambda(s) + \lambda(t) + \lambda(s) - s - \lambda(t) + t,$$

namely,

$$|\Gamma(s) - \Gamma(t)| \leq |t - s|, \forall s, t \in \mathbb{R}. \hspace{1cm} (1.9)$$

This means that $\Gamma$ is continuous.

Let us prove the first statement of item $c)$. From $\lambda(t) \leq \lambda_1(\hat{\Omega})$ for every $t \in \mathbb{R}$ we obtain $\mu(t) = \lambda(t) - t \leq \lambda_1(\hat{\Omega}) - t$ for every $t \in \mathbb{R}$, passing to the limit as $t \to +\infty$, we have $\lim_{t \to +\infty} \mu(t) = -\infty$. This, item $a)$ and definition of $\lambda_*$ imply that $\lim_{t \to +\infty} \Gamma(t) = (\lambda_*, -\infty)$.

To prove the second statement, we note that $\mu(t) \leq \mu_1(\hat{\Omega})$ for every $t \in \mathbb{R}$ implies $\lambda(t) \leq \lambda_1(\hat{\Omega}) + t$ for every $t \in \mathbb{R}$, and passing to the limit as $t \to -\infty$, we have $\lim_{t \to -\infty} \lambda(t) = -\infty$. This, item $a)$ and definition of $\mu_*$ imply that $\lim_{t \to -\infty} \Gamma(t) = (-\infty, \mu_*)$. This completes the proof of the item $c)$. Now, let us prove $d)$. If $\Gamma(t) = \Gamma(s)$, then $\lambda(t) = \lambda(s)$ and $\lambda(t) - t = \lambda(s) - s$ that implies $t = s$. Therefore, $\Gamma$ is injective and this completes the proof of $d)$. Proof of $e)$. It follows from Lemmas 3.3 and 3.2 that

$$\{ \Gamma(t) : t \in \mathbb{R} \} \subset \partial(\text{int}(\Upsilon)) \setminus (\Lambda_1 \cup \Lambda_2)$$
and so, to complete the proof, it suffices to show
\[
\partial(\text{int}(\mathcal{Y})) \setminus (\Lambda_1 \cup \Lambda_2) \subset \{ \Gamma(t) : t \in \mathbb{R} \}.
\]

To do this, by letting
\[
(a, b) \in \partial(\text{int}(\mathcal{Y})) \setminus (\Lambda_1 \cup \Lambda_2),
\] we have that
\[
(a, b) \in L(t_0)
\]
for \( t_0 = a - b \), whence together with (1.10), we obtain \((a, b) \in L(t_0) \cap \text{int}(\mathcal{Y}) \). Moreover, by definition of \( \lambda(t_0) \), we have that \( a \leq \lambda(t_0) \). Therefore,
\[
\{(a, b), (\lambda(t_0), \mu(t_0))\} \subset L(t_0) \text{ and } a \leq \lambda(t_0).
\]

We are going to proof that \( a = \lambda(t_0) \). If \( a < \lambda(t_0) \), then \( b < \mu(t_0) \). By definition of \( \Gamma(t_0) \), there exists \( \{(\lambda_k, \mu_k)\} \subset \text{int}(\mathcal{Y}) \) such that \( \lambda_k \rightarrow \lambda(t_0) \) and \( \mu_k \rightarrow \mu(t_0) \) with \( k \rightarrow +\infty \). Hence, there exists \( k_0 \in \mathbb{N} \) such that
\[
a < \lambda_{k_0} < \lambda(t_0) \text{ and } b < \mu_{k_0} < \mu(t_0),
\]
which implies, together with Lemma 3.2 and Theorem 2.1 that
\[
(a, b) \in (-\infty, 0] \times (-\infty, 0] \times (-\infty, \hat{\lambda}_1] \times (-\infty, \hat{\mu}_1] \subset \text{int}(\mathcal{Y}),
\]
that is, \((a, b) \in \text{int}(\mathcal{Y})\), but this is a contradiction with (1.10). So
\[
(a, b) = (\lambda(t_0), \mu(t_0)) \in \{ \Gamma(t) : t \in \mathbb{R} \}
\]
that shows (1.3). This finishes the proof of \( e \).

Proof of \( f \). By definition, for any
\[
(a, b) \in \bigcup_{t \in \mathbb{R}} \{(\lambda, \mu) : (\lambda, \mu) \leq \Gamma(t)\}
\]
given, there exists a \( t \in \mathbb{R} \) such that
\[
(a, b) \in L(t), \ a \leq \lambda(t) \text{ and } b \leq \mu(t). \tag{1.11}
\]

In view of Lemma 3.2 \((\lambda(t), \mu(t)) \in L(t) \cap \partial(\text{int}(\mathcal{Y}))\). Let \((\lambda, \mu) < (\lambda(t), \mu(t))\). So, by Lemma 3.2 there exists \((\kappa, \xi) \in \text{int}(\mathcal{Y})\) such that \((\lambda, \mu) < (\kappa, \xi)\) and either \( \hat{\lambda}_1 < \kappa \) or \( \hat{\mu}_1 < \xi \), which implies by Theorem 2.1 that \((\lambda, \mu) \in (\text{int}(\mathcal{Y}) \cup \Lambda_1 \cup \Lambda_2) \subset \text{int}(\mathcal{Y})\). Therefore \((-\infty, \lambda(t)] \times (-\infty, \mu(t)] \subset \text{int}(\mathcal{Y})\) and by (1.11) we have \((a, b) \in \text{int}(\mathcal{Y})\). This means that
\[
\bigcup_{t \in \mathbb{R}} \{(\lambda, \mu) : (\lambda, \mu) \leq \Gamma(t)\} \subset \text{int}(\mathcal{Y}). \tag{1.12}
\]

To end the proof, we claim that
\[
\text{int}(\mathcal{Y}) \subset \bigcup_{t \in \mathbb{R}} \{(\lambda, \mu) \in L(t) : (\lambda, \mu) \leq \Gamma(t)\}. \tag{1.13}
\]

Indeed, for any \((a, b) \in \text{int}(\mathcal{Y})\), we obtain \((a, b) = (a, a - (a - b)) \in L(t)\), where \( t = a - b \) and so \((a, b) \in L(t) \cap \text{int}(\mathcal{Y})\). By the definitions of \( \lambda(t) \) and \( \mu(t) \), we have \( a \leq \lambda(t) \) and \( b \leq \mu(t) \). Hence, \((a, b) \in L(t), a \leq \lambda(t) \) and \( b \leq \mu(t), \) that is,
\[
(a, b) \in \bigcup_{t \in \mathbb{R}} \{(\lambda, \mu) \in L(t) : (\lambda, \mu) \leq \Gamma(t)\}.
\]
Thus, the claim follows. Combining (1.12) and (1.13) we get (1.8). The proof of lemma is now complete.
Lemma 3.5. The following properties hold true:

a) \( \partial(\Omega) = \partial(\text{int}(\Omega)) \).

b) \( \overline{\text{int}(\Omega)} = \text{int}(\overline{\Omega}) \).

Proof. a) It is well known that \( \partial(\text{int}(\Omega)) \subset \partial(\Omega) \). Then it is sufficient to show that \( \partial(\text{int}(\Omega)) \subset \partial(\Omega) \). Indeed, let \( (\lambda, \mu) \in \partial(\Omega) \) be arbitrary and \( r > 0 \). If \( (\kappa, \xi) \in \Omega \cap B_r((\lambda, \mu)) \), then by Corollary 2.1 and \( (\Omega)_1 \) one has \( (\kappa, \xi) \in \text{int}(\Omega) \) and as a consequence, because \( B_r((\lambda, \mu)) \) is an open set, there exists \( (\tilde{\kappa}, \tilde{\xi}) \in \text{int}(\Omega) \cap B_r((\lambda, \mu)). \) Thus, \( \text{int}(\Omega) \cap B_r((\lambda, \mu)) = \emptyset \). On the other hand, since \( \Omega^c \subset (\text{int}(\Omega))^c \) and \( B_r((\lambda, \mu) \cap \Omega^c \neq \emptyset \) we get \( (\lambda, \mu) \cap (\text{int}(\Omega))^c \neq \emptyset \). Because \( r > 0 \) was arbitrary, we conclude that \( (\lambda, \mu) \in \partial(\text{int}(\Omega)) \) and \( \partial(\Omega) \subset \partial(\text{int}(\Omega)) \).

b) Obviously \( \text{int}(\Omega) \subset \overline{\Omega} \). Moreover, we proceed as in the proof of a) to deduce that \( \text{int}(\Omega) \cap B_r((\lambda, \mu)) = \emptyset \) for every \( (\lambda, \mu) \in \overline{\Omega} \) and \( r > 0 \), and as a result we have \( \overline{\text{int}(\Omega)} = \text{int}(\overline{\Omega}) \). The proof of lemma is now complete. □

The following properties hold true:

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1: Let us define

\[ \Theta_1 = \text{int}(\Omega) \text{ and } \Theta_2 = \mathbb{R}^2 \setminus \overline{\Omega}. \]

By definition, system \( (P_{\lambda, \mu}) \) has at least one positive solution for every \( (\lambda, \mu) \in \Theta_1 \), and in particular \( \Theta_1 \setminus (\Lambda_1 \cup \Lambda_2) \) shares the same property. On the other hand, since \( \Omega \subset \overline{\Omega} \), we infer that system \( (P_{\lambda, \mu}) \) has no positive solution for every \( (\lambda, \mu) \in \Theta_2 \). Moreover, from Lemma 3.5 (c) one has \( \partial\Theta_1 \setminus (\Lambda_1 \cup \Lambda_2) = \Gamma \). This concludes the proof of the theorem.

Remark 3.1. We do not know whether \( \Lambda_1 \cup \Lambda_2 \subset \overline{\Omega} \). If this is true, from Lemma 3.4 (e) and Lemma 3.5 (a) - b) one has \( \partial(\Omega) = \partial(\text{int}(\Omega)) = \Gamma \). In Theorem 1.2 we will show that if \( p = q = 2 \), then Problem \( (P_{\lambda, \mu}) \) admits at least one positive solution, and hence \( (\Lambda_1 \cup \Lambda_2) \cap \Omega \neq \emptyset \).

Remark 3.2. Also, we do not know whether \( \Gamma \subset \Omega \). If this is true, note that from Lemma 3.5 (a) - b) one has \( \Omega = \text{int}(\overline{\Omega}) \). In particular, \( \overline{\Omega} \) is a closed set.

4. Proof of Theorem 1.2

In this section we will prove Theorem 1.2. Let us assume \( p = q = 2 \), \( 2 < \alpha + \beta < 2^* \) and \( f \in C(\overline{\Omega}) \). Consider the problem

\[
\begin{cases}
-\Delta u = \lambda u + f(x)|u|^{\alpha+\beta-2}u & \text{in } \Omega, \\
\quad u > 0 & \text{in } \Omega \\
\quad u = 0 & \text{on } \partial \Omega,
\end{cases}
\]

\((P_{\lambda})\)

Let \( (f)_1 - (f)_2 \) hold. From Alama-Tarantello [1] there exists \( \tau > \hat{\lambda}_1 \) such that:

a) for every \( \lambda \in (\hat{\lambda}_1, \tau) \), \((P_{\lambda})\) admits at least two positive solutions.

b) For \( \lambda = \hat{\lambda}_1 \) and \( \lambda = \tau \), problem \((P_{\lambda})\) admits at least one positive solution.

c) For \( \lambda > \tau \) problem \((P_{\lambda})\) does not admit any positive solutions.
By using this result we have.

Proof of Theorem 1.2
a) Let \( u_\lambda \) and \( v_\lambda \) be two distinct positive solutions of \((P_\lambda)\). It is easy to see that \((u_\lambda, u_\lambda)\) is a positive solution of the problem

\[
\begin{aligned}
-\Delta u_\lambda &= \lambda u_\lambda + f(x)|u_\lambda|^{\alpha-2}u_\lambda \quad \text{in } \Omega, \\
-\Delta u_\lambda &= \lambda u_\lambda + f(x)|u_\lambda|^{\alpha-2}u_\lambda \quad \text{in } \Omega,
\end{aligned}
\]

\((Q_\lambda)\)

Multiplying the first equation of \((Q_\lambda)\) by \( t \) and the second equation by \( s \), where \( t, s > 0 \), we get (in the weak sense)

\[
\begin{aligned}
-\Delta (tu_\lambda) &= \lambda(tu_\lambda) + t^{2-\alpha}s^{-\beta}f(x)|tu_\lambda|^{\alpha-2}|su_\lambda|^{\beta}tu_\lambda \quad \text{in } \Omega, \\
-\Delta (su_\lambda) &= \lambda(su_\lambda) + t^{-\alpha}s^{2-\beta}f(x)|tu_\lambda|^{\alpha}|su_\lambda|^{\beta-2}su_\lambda \quad \text{in } \Omega,
\end{aligned}
\]

\( tu_\lambda, su_\lambda > 0 \) in \( \Omega \),

\( tu_\lambda = su_\lambda = 0 \) on \( \Omega \).

Choosing \( t, s > 0 \) such that

\[ t^{2-\alpha}s^{-\beta} = \alpha \quad \text{and} \quad t^{-\alpha}s^{2-\beta} = \beta \]

we find

\[ t = (\alpha)^{\frac{\beta-2}{d}}(\beta)^{-\frac{\alpha}{d}}, \quad s = (\alpha)^{-\frac{\alpha}{d}}(\beta)^{\frac{\alpha-2}{d}}, \]

where \( d = \frac{\alpha}{2} + \frac{\beta}{2} - 1 \neq 0 \), by the assumption. Hence \((\tilde{u}_\lambda, \tilde{u}_\lambda) = (tu_\lambda, su_\lambda)\) is a positive solution of \((P_{\lambda, \lambda})\). Similarly, we can prove that \((\tilde{v}_\lambda, \tilde{v}_\lambda) = (tv_\lambda, sv_\lambda)\) is a positive solution of \((P_{\lambda, \lambda})\) with \((\tilde{v}_\lambda, \tilde{v}_\lambda) \neq (\tilde{u}_\lambda, \tilde{u}_\lambda)\). This completes the proof of a).

Arguing as in the proof of a) we can prove b). This finishes the proof of Theorem 1.2.

References

[1] S. Alama, G. Tarantello, On semilinear elliptic equations with indefinite nonlinearities, Calc. Var. 1 (1993) 439-475.
[2] An Lê, Eigenvalue problems for the p-Laplacian, Nonlinear Anal., 64 (2006), 1057-1099.
[3] R.L. Alves, C.O. Alves, C.A. Santos, Extremal curves for existence of positive solutions for multi-parameter elliptic systems in \( \Omega \), Milan J. Math. 88, 1-33 (2020). https://doi.org/10.1007/s00032-019-00305-3.
[4] W. Allegretto, Y. Huang, A Picone’s Identity for the p-Laplacian and applications. Nonlinear Anal. 32(7), 819-830 (1998).
[5] V. Bobkov, Y. Il’yasov, Asymptotic behaviour of branches for ground states of elliptic systems, Electron. J. Differential Equations (2013), No. 212, 21.
[6] V. Bobkov, Y. Il’yasov, Maximal existence domains of positive solutions for two-parametric systems of elliptic equations, Complex Var. Elliptic Equ. 61 (2016), no. 5, 587-607.
[7] Y. Bozhkov, E. Mitidieri, Existence of multiple solutions for quasilinear systems via fibering method, J. Differential Equations 190 (2003) 239-267.
[8] X. Cheng, Z. Zhang, Positive solutions for a class of multi-parameter elliptic systems. Nonlinear Anal. Real World Appl. 14, 1551-1562 (2013).
[9] S.A. Marano, N.S. Papageorgiou, Positive solutions to a Dirichlet problem with p-Laplacian and concave-convex nonlinearity depending on a parameter. Commun. Pure Appl. Anal. 2013, 12, 815-829.
[10] C.A. Santos, R.L. Alves, M. Reis, J. Zhou, Maximal domains of the \((\lambda, \mu)\)-parameters to existence of entire positive solutions for singular quasilinear elliptic systems. J. Fixed Point Theory Appl. 22, 54 (2020). https://doi.org/10.1007/s11784-020-00783-8.
[11] K. Silva, A. Macedo, *On the extremal parameters curves of a quasilinear elliptic system of differential equation*, Nonlinear Differential Equations and Applications, (2018) 25:36. https://doi.org/10.1007/s00030-018-0527-5.

[12] M. Struwe, *Variational Methods*, Springer, Fourth Edition, 2007.

Ricardo Lima Alves  
Departamento de Matemática  
Universidade de Brasília  
70910-900 Brasília  
DF - Brasil  
e-mail: ricardoalveslima8@gmail.com