Non-commutative $U(1)$ Super-Yang–Mills Theory:
Perturbative Self-Energy Corrections

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Abstract. The quantization of the non-commutative $\mathcal{N} = 1$, $U(1)$ super-Yang–Mills action is performed in the superfield formalism. We calculate the one-loop corrections to the self-energy of the vector superfield. Although the power-counting theorem predicts quadratic ultraviolet and infrared divergences, there are actually only logarithmic UV and IR divergences, which is a crucial feature of non-commutative supersymmetric field theories.

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1 Introduction

We know that the concept of space-time as a differentiable manifold cannot be reasonably applied to extremely short distances [1]. Simple heuristic arguments show that it is impossible to locate a particle with arbitrarily small uncertainty [2]. An interesting concept in order to replace standard differential geometry is non-commutative geometry pioneered by Connes [3, 4]. Non-commutative geometry can be regarded as an extension of the principles of quantum mechanics to geometry itself: space-time coordinates become non-commutative operators.

The general strategy in non-commutative geometry is to generalize the mathematical structures encountered in ordinary physics. Standard quantum field theories deal with problems of interactions at short distances. Quantum field theory (QFT) on spaces with different short-distance structure may therefore show interesting features. Since singularities in standard QFT are a consequence of point-like interactions, there has been hope that ‘smearing out the points’ [5] avoids these UV divergences. However, it was first noticed by Filk [6] that divergences are not avoided on non-commutative \( \mathbb{R}^4 \). This raised the question of whether the QFT is renormalizable, or not.

Scalar field theories were investigated in [7, 8], where a crucial feature of non-commutative field theories appears—the UV/IR mixing. On the one-loop level the question of renormalizability was affirmed for Yang–Mills theory on non-commutative \( \mathbb{R}^4 \) [9, 10, 11] and the non-commutative 4-torus [12] as well as for supersymmetric Yang–Mills theory in (2+1) dimensions, with space being the non-commutative 2-torus [13]. QED on non-commutative \( \mathbb{R}^4 \) was treated in [14, 15] and BF–Yang–Mills theory in [16]. The Chern–Simons model on non-commutative space was treated in [17], see also [18].

Concerning supersymmetry, also a deformation of the anticommutator of the fermionic superspace coordinates was considered [19], but this deformation is not compatible with supertranslations and chiral fields. At the component level, renormalizability of the Wess–Zumino model to all orders of perturbation theory was shown in [20]. A superspace formulation (at the classical level) of the Wess–Zumino model and of super-Yang–Mills theory was given in [21]. Eventually, renormalizability of the Wess–Zumino model in the non-commutative superspace formalism was established in [22]. Non-commutative \( \mathcal{N} = 1,2 \) super-Yang–Mills theories were studied by Zanon in [23], using the background field method, with the result that at one loop there are only logarithmic divergences in the self-energy. This is remarkable because the power-counting theorem predicts quadratic divergences for \( \mathcal{N} = 1 \) super-Yang–Mills theory, which would lead, according to the power-counting analysis of non-commutative field theories by Chepelev and Roiban [24], to non-renormalizability on non-commutative space-time. The lowering of the degree of divergence from quadratic to logarithmic seems to be governed by non-renormalization theorems, see [25].

In this paper we reinvestigate the question of UV/IR mixing in non-commutative \( \mathcal{N} = 1 \) super-Yang–Mills theory, where we work in the non-commutative superfield formalism [22]. It turns out that the one-loop self-energy of the superfield suffer indeed only from logarithmic IR divergences. UV divergences are multiplicatively renormalizable as usual.
Assuming that this behaviour continues to all orders, non-commutative $\mathcal{N} = 1$ super-Yang–Mills theory would be renormalizable, according to [24], provided that commutants-type divergences are absent.

Therefore, non-commutativity does not spoil the cancellation of quadratic and linear divergences in supersymmetric theories, as stated already in the literature [26, 20, 27, 23, 28, 29, 30].

On the other hand, non-commutative non-supersymmetric theories suffer from quadratic (linear) IR divergences which would prevent renormalizability at higher loop order. Possible ways out could be hard non-commutative loops resummation [31] or the use of field redefinitions [32].

The paper is organized as follows: Section 2 presents the Moyal product applied to superfields, while section 3 treats the action of our model. In section 4 the Legendre transformation and the perturbative expansion are performed and, after a short power-counting argument given in section 5 the self-energy of the vector superfield is calculated at the one-loop level (section 6). Appendices contain some calculations and conventions.

2 Moyal Product for Superfields

We consider a non-commutative ($\mathcal{N} = 1$) superspace characterized by the algebra

$$[\hat{x}^\mu, \hat{x}^\nu] = i \Theta^{\mu\nu},$$

where $\Theta^{\mu\nu}$ is an antisymmetric, constant and real matrix. We do not deform the anti-commuting coordinates $\theta_\alpha$ and $\bar{\theta}^\alpha$, i.e. we assume

$$\{\theta_\alpha, \theta_\beta\} = \{\bar{\theta}^\alpha, \bar{\theta}^\beta\} = \{\hat{x}^\mu, \theta_\alpha\} = \{\hat{x}^\mu, \bar{\theta}^\alpha\} = 0.$$  

(2)

The non-commutative algebra is represented on an ordinary manifold by the Moyal product [6]. The Moyal product of two vector superfields can be written as [22]

$$(\phi \star \phi')(x, \theta_1, \bar{\theta}_1) = \int dP V_2 dP V_3 \delta_{V}(1, 2) \delta_{V}(1, 3) \times \phi(p_2, \theta_2) \phi'(p_3, \theta_3, \bar{\theta}_3) e^{-i(p_2 + p_3) x} e^{-ip_2 \wedge p_3}.$$  

(3)

The Moyal product has the important property

$$\int dV_1 (\phi \star \phi')(1) = \int dV_1 (\phi' \star \phi)(1) = \int dV_1 \phi(1) \phi'(1).$$

(4)

This implies in particular that one can perform cyclic rotations of the fields under the integral.

For definiteness we have used vector superfields in (3) and (4). Of course, one can easily write down analogous formulae for (anti-)chiral superfields.
3 The Action

For simplicity we choose the gauge group $U(1)$. We introduce a vector superfield $\phi$ whose
gauge transformation is given by [21]:
\[
(e^{\phi'})_* = (e^{i\Lambda})_* \ast (e^{\phi})_* \ast (e^{i\Lambda})_*,
\]
with a chiral superfield $\Lambda$ (gauge parameter). With the help of the Baker–Campbell–Hausdorff formula we obtain the infinitesimal gauge transformation of $\phi$ itself:
\[
\phi' = \phi + i(\Lambda - \bar{\Lambda}) + i2[\phi, \Lambda + \bar{\Lambda}]_* + \frac{i}{12}[\phi, [\phi, \Lambda - \bar{\Lambda}]_*]_* + \ldots,
\]
where the dots denote terms that contain three or more powers of $\phi$. The gauge-invariant NCSYM action is given by
\[
S_{\text{inv}} = -\frac{1}{128g^2} \int dS W^\alpha W_\alpha,
\]
with
\[
W_\alpha := D^2 ((e^{-\phi})_* \ast D_\alpha (e^{\phi})_*).
\]
We perform a Taylor expansion of the integrand,
\[
S_{\text{inv}} = -\frac{1}{128g^2} \int dV \left[ -\phi D^\alpha \bar{D}^2 D_\alpha \phi - (\bar{D}^2 D^\alpha \phi)[\phi, D_\alpha \phi]_* \\
- \frac{1}{3}[\phi, \bar{D}^2 D^\alpha \phi][\phi, D_\alpha \phi]_* + \frac{1}{4}[\phi, D^\alpha \phi]_* \bar{D}^2[\phi, D_\alpha \phi]_* + \mathcal{O}(\phi^5) \right].
\]
In order to prepare the quantization, we introduce a chiral superfield $B$ (multiplier field) and two chiral anticommuting superfields $c_+$ (ghost) and $c_-$ (antighost). The BRS transformations are given by:
\[
s\phi = c_+ - \bar{c}_+ + \frac{1}{2}[\phi, c_+ + \bar{c}_+]_* + \frac{1}{12}[\phi, [\phi, c_+ - \bar{c}_+]_*]_* + \ldots
\]
\[
=: Q_s(\phi, c_+),
\]
\[
s c_+ = -c_+ \ast c_+,
\]
\[
s c_- = B,
\]
\[
s B = 0,
\]
\[
s \bar{B} = 0.
\]
Now we can write down the BRS-invariant total action:
\[
S_{\text{tot}} = S_{\text{inv}} + S_{gf} + S_{\phi\pi},
\]
where $S_{\text{inv}}$ is given by [33] and the gauge fixing and the Faddeev–Popov terms are given by [33]:
\[
S_{gf} = -\frac{1}{128} \int dV (B + \bar{B})\phi,
\]
\[
S_{\phi\pi} = \frac{1}{128} \int dV (c_- + \bar{c}_-)Q_s(\phi, c_+).
\]
Using (10), the Faddeev–Popov term can be rewritten as

\[
S_{\phi \pi} = \frac{1}{128} \int dV \left( \bar{c}_- c_+ - c_- \bar{c}_+ \right.
+ \left( c_- + \bar{c}_+ \right) \left( \frac{1}{2} \left[ \phi, c_+ + \bar{c}_+ \right] + \frac{1}{12} \left[ \phi, \left[ \phi, c_+ - \bar{c}_+ \right] \right] + \ldots \right).
\]

(14)

Again the dots denote terms with three or more powers of \( \phi \).

In the following we will also include a mass term in the total action,

\[
S_{mass} = \frac{1}{16g^2} \int dV M^2 \phi^2,
\]

(15)
in order to avoid an IR divergence in the propagator of the vector superfield.

4 Generating Functionals

The generating functional of connected Green’s functions for the free theory can be obtained from the bilinear part \( S_{bil} \) of \( S_{tot} + S_{mass} \) via a Legendre transformation:

\[
Z_{bil}^c = S_{bil} + \int dV J \phi + \int dS \left( J_B B + \eta_- c_+ + \eta_+ c_- \right)
+ \int d\bar{S} \left( \bar{J}_B \bar{B} + \bar{\eta}_- \bar{c}_+ + \bar{\eta}_+ \bar{c}_- \right)
= S_{bil} + \int dP_V J_{-p} \bar{\phi}_p + \int dP_S \left( J_{B, -p} \bar{B}_p + \bar{\eta}_{-p} \bar{c}_{+, p} + \bar{\eta}_{+, p} \bar{\bar{c}}_{-, p} \right)
+ \int dP_{\bar{S}} \left( \bar{J}_{B, -p} \bar{B}_p + \bar{\eta}_{-p} \bar{\bar{c}}_{+, p} + \bar{\eta}_{+, p} \bar{\bar{\bar{c}}}_{-, p} \right),
\]

(16)

where \( \phi, B, \bar{B}, c_\pm \) and \( \bar{c}_\pm \) are replaced by the inverse solutions of

\[
\frac{\delta V S_{bil}}{\delta V \phi_{-p}} = -J_p,
\quad \frac{\delta S S_{bil}}{\delta S B_{-p}} = -J_{B,p},
\quad \frac{\delta S S_{bil}}{\delta S \bar{B}_{-p}} = -\bar{J}_{B,p},
\quad \frac{\delta S S_{bil}}{\delta S \bar{c}_{+, -p}} = \bar{\eta}_{+, p},
\quad \frac{\delta S S_{bil}}{\delta S \bar{\bar{c}}_{+, -p}} = \bar{\eta}_{+, p}.
\]

(17)

This leads to

\[
Z_{bil}^c = \int dP_{V1} \int dP_{V2} \frac{1}{2} \bar{J}_{-p_1} \Delta \phi \left( 1, 2 \right) J_{-p_2} + \int dP_{V1} dP_{S2} \bar{J}_{-p_1} \Delta \phi_B \left( 1, 2 \right) \bar{J}_{B,-p_2}
+ \int dP_{V1} dP_{S2} \bar{\eta}_{-p_1} \Delta B \left( 1, 2 \right) \bar{J}_{-p_2} + \int dP_{S1} dP_{S2} \bar{J}_{-p_1} \Delta B_B \left( 1, 2 \right) \bar{J}_{B,-p_2}
- \int dP_{\bar{S}1} dP_{\bar{S}2} \bar{\eta}_{-p_1} \Delta \bar{c}_{+, -} \left( 1, 2 \right) \bar{\eta}_{+, -p_2} - \int dP_{\bar{S}1} dP_{\bar{S}2} \bar{\eta}_{-p_1} \Delta \bar{\bar{c}}_{-, -} \left( 1, 2 \right) \bar{\eta}_{-, -p_2},
\]

(18)
where the propagators are given by

\[ \Delta_{\phi}(1, 2) = -g^2(2\pi)^4\delta^4(p_1 + p_2)e^{i p_2 \cdot \theta_2 \cdot \Phi_1 - \theta_1 \cdot \Phi_2} \frac{1}{p_2^2 (p_2^2 - M^2)}; \]

\[ \Delta_{\phi B}(1, 2) = 8(2\pi)^4\delta^4(p_1 + p_2)e^{i p_2 \cdot \theta_2 \cdot \Phi_1 - \theta_1 \cdot \Phi_2} \left( \frac{1 - \theta_2^1 \theta_2^2}{p_2^2} \right) + \frac{1}{4} \theta_2^2 \theta_2^2 p_2^2, \]

\[ \Delta_{\phi \bar{B}}(1, 2) = 8(2\pi)^4\delta^4(p_1 + p_2)e^{i p_2 \cdot \theta_2 \cdot \Phi_1 - \theta_1 \cdot \Phi_2} \left( \frac{1 + \theta_2^1 \theta_2^2}{p_2^2} \right) + \frac{1}{4} \theta_2^2 \theta_2^2 p_2^2, \]

\[ \Delta_{\bar{c} + c}(1, 2) = \Delta_{\bar{c} - c}(1, 2) = \Delta_{\phi B}(1, 2), \]

\[ \Delta_{BB}(1, 2) = \frac{16M^2}{g^2} \Delta_{\phi B}. \] (19)

The generating functional of general Green’s functions is given by

\[ Z = \mathcal{N}^{-1} e^{\frac{i}{\hbar} \Gamma_{int} e^{\phi} Z_{bd}}, \] (20)

where \( \mathcal{N} \) is a normalization factor such that \( Z[0] = 1 \) and

\[ \Gamma_{int} = \frac{1}{4!} \left( \frac{\hbar}{i} \right)^4 \int dP_{V1} dP_{V2} dP_{V3} dP_{V4} \Gamma_{\phi^4}(1, 2, 3, 4) \frac{\delta^4}{\delta V_{-p_1} \delta V_{-p_2} \delta V_{-p_3} \delta V_{-p_4}} \]

\[ + \frac{1}{3!} \left( \frac{\hbar}{i} \right)^3 \int dP_{V1} dP_{V2} dP_{V3} \Gamma_{\phi^3}(1, 2, 3) \frac{\delta^3}{\delta V_{-p_1} \delta V_{-p_2} \delta V_{-p_3} \delta V_{-p_4}} \]

\[ + \left( \frac{\hbar}{i} \right)^3 \int dP_{V1} dP_{V2} dP_{V3} \Gamma_{c_{-c} \phi}(1, 2, 3) \frac{\delta^2}{\delta S \delta \Phi V_{-p_1} \delta S \delta \Phi V_{-p_2} \delta V_{-p_3} \delta V_{-p_4}} \]

\[ + \left( \frac{\hbar}{i} \right)^3 \int dP_{V1} dP_{V2} dP_{V3} \Gamma_{c_{-c} \phi}(1, 2, 3) \frac{\delta^2}{\delta \Phi V_{-p_1} \delta \Phi V_{-p_2} \delta S \delta \Phi V_{-p_3} \delta S \delta \Phi V_{-p_4}} \]

\[ + \left( \frac{\hbar}{i} \right)^3 \int dP_{V1} dP_{V2} dP_{V3} \Gamma_{\bar{c} + \bar{c} \phi}(1, 2, 3) \frac{\delta^2}{\delta \Phi V_{-p_1} \delta \Phi V_{-p_2} \delta \Phi V_{-p_3} \delta \Phi V_{-p_4}} \]

\[ + \frac{1}{2!} \left( \frac{\hbar}{i} \right)^4 \int dP_{V1} dP_{V2} dP_{V3} dP_{V4} \Gamma_{c_{-c} \phi^2}(1, 2, 3, 4) \frac{\delta^3}{\delta S \delta \Phi V_{-p_1} \delta S \delta \Phi V_{-p_2} \delta S \delta \Phi V_{-p_3} \delta S \delta \Phi V_{-p_4}} \]

\[ + \frac{1}{2!} \left( \frac{\hbar}{i} \right)^4 \int dP_{V1} dP_{V2} dP_{V3} dP_{V4} \Gamma_{\bar{c} + \bar{c} \phi^2}(1, 2, 3, 4) \frac{\delta^3}{\delta S \delta \Phi V_{-p_1} \delta S \delta \Phi V_{-p_2} \delta S \delta \Phi V_{-p_3} \delta S \delta \Phi V_{-p_4}} \]

\[ + \frac{1}{2!} \left( \frac{\hbar}{i} \right)^4 \int dP_{V1} dP_{V2} dP_{V3} dP_{V4} \Gamma_{c_{-c} \phi^2}(1, 2, 3, 4) \frac{\delta^3}{\delta S \delta \Phi V_{-p_1} \delta S \delta \Phi V_{-p_2} \delta S \delta \Phi V_{-p_3} \delta S \delta \Phi V_{-p_4}} \]

\[ + \frac{1}{2!} \left( \frac{\hbar}{i} \right)^4 \int dP_{V1} dP_{V2} dP_{V3} dP_{V4} \Gamma_{\bar{c} + \bar{c} \phi^2}(1, 2, 3, 4) \frac{\delta^3}{\delta S \delta \Phi V_{-p_1} \delta S \delta \Phi V_{-p_2} \delta S \delta \Phi V_{-p_3} \delta S \delta \Phi V_{-p_4}} \] (21)
which is given by looked up in appendix A. The final results for these vertex functions (22) are rather complicated and can be necessary vertices for the calculation of the one-loop self-energy part of the vector superfield defined in terms of the vectorial measure (14). One should notice here that these are all (anti-)chiral measure (16), whereas interactions between ghosts and vector superfields are in the ghost sector is due to our convention that source terms for the ghosts involve the (anti-)chiral derivatives have been performed. The mixture of (anti-)chiral and vectorial field derivatives in the ghost sector is due to our convention that source terms for the ghosts involve the (anti-)chiral measure [16], whereas interactions between ghosts and vector superfields are defined in terms of the vectorial measure [11]. One should notice here that these are all necessary vertices for the calculation of the one-loop self-energy part of the vector superfield. The final results for these vertex functions (22) are rather complicated and can be looked up in appendix A.

Furthermore, we mention the generating functional of connected Green’s functions, which is given by

$$Z^c = \frac{\hbar}{i} \ln Z.$$  

(23)

5 Power-Counting

We note that, apart from the exponentials and the $\theta$-factors in the numerator, the vector field propagators are of order $\frac{1}{(p^2)^2}$ and the ghost propagators of order $\frac{1}{p^2}$. We consider the exponentials and the $\theta$-factors. From the invariance of Green’s functions with respect to translations and supersymmetry transformations one finds that a one-particle irreducible Green’s function in momentum space can always be written as [33]:

$$\tilde{\Gamma}(1,\ldots,n) = \frac{\delta^n \Gamma[\phi_i]}{\delta \phi(p_1) \ldots \delta \phi(p_n)} = (2\pi)^4 \delta^4(\sum_{j=1}^{n} p_j) e^{-\sum_{i=2}^{n} p_i \mu (\theta_i \sigma^\mu \theta_i^\dagger - \theta_i^\dagger \sigma^\mu \theta_i)} \tilde{f}(-p_2,\ldots,-p_n, \theta_{i1}, \theta_{i1}).$$  

(24)

This general structure is true in particular for propagators and vertices. Thus, the general structure of the integrand of the superspace integral corresponding to an arbitrary Feynman graph is

$$I = \exp \left( - \sum_{i \leq j} \sum_{\tau} l_{ij,\tau,\mu} (\theta_i \sigma^\mu \theta_j - \theta_j^\dagger \sigma^\mu \theta_i) \right) \prod_{i \leq j} \prod_{\tau} \tilde{f}_{ij,\tau}(-l_{ij,\tau}, \theta_{ij}, \theta_{ij}).$$  

(25)
Here, \( l_{ij,\tau} \) is the momentum running from point \( i \) to point \( j \), and \( \tau \) counts momenta running between the same pair of points. We have chosen a basis for \( (l_{ij,\tau}) = (p, k) \), where \( p \) and \( k \) are the external and internal momenta, respectively. With momentum conservation,

\[
\sum_{j,\tau} l_{ij,\tau} = p_i,
\]

we find

\[
E(p, k, \theta_i, \bar{\theta}_i) = E(p, k, \theta_{i1}, \bar{\theta}_{i1}) - \sum_i (\theta_i \sigma^\mu \bar{\theta}_i - \theta_{i1} \sigma^\mu \bar{\theta}_{i1}) p_{i,\mu}.
\]

(27)

Therefore, the exponentials appearing in the formulae for the propagators and vertices can be rewritten as

\[
e^{k_\mu (\theta_i \sigma^\mu \bar{\theta}_j - \theta_j \sigma^\mu \bar{\theta}_i)} \Rightarrow e^{k_\mu (\theta_{i1} \sigma^\mu \bar{\theta}_{j1} - \theta_{j1} \sigma^\mu \bar{\theta}_{i1})},
\]

(28)

if and only if \( k \) is an internal momentum. A Taylor expansion of the exponentials shows that from the \( \theta \)-factors and the exponential we will at most get terms like \( \theta_{ij}^2 \bar{\theta}_{ij} k^2 \). The highest power of \( k^2 \) that can appear is just the number of independent differences \( \theta_{ij} \) (with \( j = 1 \) in the calculation above) that can be constructed, which is exactly \( n - 1 \), \( n \) being the number of vertices. So we find for the superficial divergence degree of an 1PI-graph

\[
d(\Gamma) = 4L - 2G - 4V + 2(n_G + n_V - 1) + 2n_V.
\]

(29)

Here, \( L \) is the number of loop integrations, \( G \) and \( V \) are the numbers of ghost and vector superfield propagators, respectively, \( n_G \) and \( n_V \) count the ghost-vector superfield and the pure-vector superfield vertices. The last term, \( 2n_V \), has to be included in (29) because of the four covariant derivatives that appear in the parts of the Lagrangian corresponding to the three and four vertices of the vector superfield.

Using the topological relation \( L = G + V - n_G - n_V + 1 \) and charge conservation for the ghost fields, \( 2n_G = 2G + N_G \) (\( N_G \) being the number of external ghost fields), we find

\[
d(\Gamma) = 2 - N_G.
\]

(30)

For the vector superfield self-energy (\( N_G = 0 \)), this means a superficial degree of divergence of \( 2 \).

### 6 Self-Energy of the Vector Superfield

The following Feynman graphs contribute to the self-energy of the vector superfield at the one-loop level (continuous lines denote vector superfield propagators, dotted lines ghost
propagators): 

\[
I_1 : \begin{array}{c}
    & 1 & 4 & 3 & 2 \\
\end{array}
\]

\[
I_2 : \begin{array}{c}
    1 & 4_+ & 3_+ & 2 \\
\end{array}
\]

\[
I_3 : \begin{array}{c}
    & 4_+ & 3_+ & 2 \\
\end{array}
\]

\[
I_4 : \begin{array}{c}
    & 1 & 4 & 5 & 2 \\
\end{array}
\]

\[
I_5 : \begin{array}{c}
    1 & 4_+ & 5_+ & 2 \\
\end{array}
\]

\[
I_6 : \begin{array}{c}
    & 1 & 4_+ & 5_+ & 6_+ \\
\end{array}
\]

\[
I_7 : \begin{array}{c}
    & 4_+ & 5_+ & 6_+ \\
\end{array}
\]

\[
I_8 : \begin{array}{c}
    & 4_+ & 5_+ & 6_+ \\
\end{array}
\]

From the generating functional (23) we obtain the following integrals corresponding to these graphs (after amputation of the external lines):

\[
I_1 = \frac{\hbar}{2i} \int dP_3 dP_4 \Gamma_{\phi^4}(3, 4, 1, 2) \Delta_{\phi^2}(3, 4),
\]

\[
I_2 = -\frac{\hbar}{i} \int dP_3 dP_4 \Gamma_{\bar{c}_- c_+ \phi^2}(3, 4, 1, 2) \Delta_{\bar{c}_- c_+}(3, 4),
\]

\[
I_3 = -\frac{\hbar}{i} \int dP_3 dP_4 \Gamma_{\bar{c}_- \bar{c}_+ \phi^2}(3, 4, 1, 2) \Delta_{\bar{c}_- c_+}(3, 4),
\]

\[
I_4 = \frac{\hbar}{2i} \int dP_3 dP_4 dP_5 dP_6 \Gamma_{\phi^3}(3, 4, 1) \Delta_{\phi^2}(3, 6) \Gamma_{\phi^2}(5, 6, 2) \Delta_{\phi^2}(5, 4),
\]

\[
I_5 = -\frac{\hbar}{i} \int dP_3 dP_4 dP_5 dP_6 \Gamma_{c_- c_+ \phi}(3, 4, 1) \Delta_{\bar{c}_- \bar{c}_+}(3, 6) \Gamma_{\bar{c}_- \bar{c}_+ \phi}(5, 6, 2) \Delta_{\bar{c}_- \bar{c}_+}(5, 4),
\]

\[
I_6 = -\frac{\hbar}{i} \int dP_3 dP_4 dP_5 dP_6 \Gamma_{\bar{c}_- \bar{c}_+ \phi}(3, 4, 1) \Delta_{\bar{c}_- \bar{c}_+}(3, 6) \Gamma_{\bar{c}_- \bar{c}_+ \phi}(5, 6, 2) \Delta_{\bar{c}_- \bar{c}_+}(5, 4),
\]

\[
I_7 = -\frac{\hbar}{i} \int dP_3 dP_4 dP_5 dP_6 \Gamma_{\bar{c}_- \bar{c}_+ \phi}(3, 4, 1) \Delta_{\bar{c}_- \bar{c}_+}(3, 6) \Gamma_{\bar{c}_- \bar{c}_+ \phi}(5, 6, 2) \Delta_{\bar{c}_- \bar{c}_+}(5, 4),
\]

\[
I_8 = -\frac{\hbar}{i} \int dP_3 dP_4 dP_5 dP_6 \Gamma_{\bar{c}_- \bar{c}_+ \phi}(3, 4, 1) \Delta_{\bar{c}_- \bar{c}_+}(3, 6) \Gamma_{\bar{c}_- \bar{c}_+ \phi}(5, 6, 2) \Delta_{\bar{c}_- \bar{c}_+}(5, 4).
\]

(Note that $\Delta_{\bar{c}_- \bar{c}_+}(3, 6) = -\Delta_{\bar{c}_+ \bar{c}_-}(3, 6, 3)$.) We now insert the explicit expressions for the propagators and vertices into the eight integrals in (32). After some lengthy simplifications of the integrands (see appendix A) we arrive at

\[
I_1 = \frac{\hbar}{128i} \delta^4(p_1 + p_2) e^{-p_1,\mu(\theta_1 \sigma^\nu \bar{\theta}_2 - \theta_2 \sigma^\nu \bar{\theta}_1)} \int d^4k \sin^2(p_1 \wedge k) \frac{1}{k^2(k^2 - M^2)}
\]

8
\[
\times \left( -\frac{2}{3} + \frac{1}{6} \theta_{12}^2 \bar{b}_{12}^2 (2k^2 + p_1^2) \right),
\]
(33)

\[
I_2 = \frac{\hbar}{384i} \delta^4(p_1 + p_2) e^{-p_{1,\mu}(\theta_1 \sigma^\mu \theta_2 - \theta_2 \sigma^\mu \theta_1)} \int d^4k \sin^2(p_1 \wedge k) \frac{1}{k^2} \bar{b}_{12}^2 \bar{b}_{12}^2,
\]
(34)

\[
I_3 = \frac{\hbar}{384i} \delta^4(p_1 + p_2) e^{-p_{1,\mu}(\theta_1 \sigma^\mu \theta_2 - \theta_2 \sigma^\mu \theta_1)} \int d^4k \sin^2(p_1 \wedge k) \frac{1}{k^2} \bar{b}_{12}^2 \bar{b}_{12}^2,
\]
(35)

\[
I_4 = \frac{\hbar}{128i} \delta^4(p_1 + p_2) e^{-p_{1,\mu}(\theta_1 \sigma^\mu \theta_2 - \theta_2 \sigma^\mu \theta_1)} \int d^4k \sin^2(p_1 \wedge k)
\times \frac{1}{k^2(k^2 - M^2)(k + p_1)^2((k + p_1)^2 - M^2)}
\times \left( -4(k^2)^2 - 8k^2(kp_1) - 5k^2p_1^2 - p_1^2(kp_1) - \frac{1}{4} \bar{b}_{12}^2 \bar{b}_{12}^2 \left( -4(k^2)^2 p_1^2 
+ 8k^2(kp_1)^2 + 8(kp_1)^2p_1^2 + (kp_1)(p_1^2)^2 - 3(p_1^2)^2k^2 \right) \right),
\]
(36)

\[
I_5 = \frac{\hbar}{256i} \delta^4(p_1 + p_2) e^{-p_{1,\mu}(\theta_1 \sigma^\mu \theta_2 - \theta_2 \sigma^\mu \theta_1)} \left( 1 - (\theta_1 \sigma^\rho \theta_1)_{p_1,\rho} + \frac{1}{4} \bar{b}_{12}^2 \bar{b}_{12}^2 \right)
\times \int d^4k \sin^2(p_1 \wedge k) \frac{1}{k^2(p_1 + k)^2},
\]
(37)

\[
I_6 = \frac{\hbar}{256i} \delta^4(p_1 + p_2) e^{-p_{1,\mu}(\theta_1 \sigma^\mu \theta_2 - \theta_2 \sigma^\mu \theta_1)} \left( 1 + (\theta_1 \sigma^\rho \theta_1)_{p_1,\rho} + \frac{1}{4} \bar{b}_{12}^2 \bar{b}_{12}^2 \right)
\times \int d^4k \sin^2(p_1 \wedge k) \frac{1}{k^2(p_1 + k)^2},
\]
(38)

\[
I_7 = -\frac{\hbar}{256i} \delta^4(p_1 + p_2) e^{-p_{1,\mu}(\theta_1 \sigma^\mu \theta_2 - \theta_2 \sigma^\mu \theta_1)} \int d^4k \sin^2(p_1 \wedge k) \frac{1}{k^2(p_1 + k)^2}
\times \left( 1 - (\theta_1 \sigma^\rho \theta_1)(p_1 + 2k)_{\rho} + \theta_{12}^2 \bar{b}_{12}^2 \left( \frac{1}{4} p_1^2 + p_1k + k^2 \right) \right),
\]
(39)

\[
I_8 = -\frac{\hbar}{256i} \delta^4(p_1 + p_2) e^{-p_{1,\mu}(\theta_1 \sigma^\mu \theta_2 - \theta_2 \sigma^\mu \theta_1)} \int d^4k \sin^2(p_1 \wedge k) \frac{1}{k^2(p_1 + k)^2}
\times \left( 1 + (\theta_1 \sigma^\rho \theta_1)(p_1 + 2k)_{\rho} + \theta_{12}^2 \bar{b}_{12}^2 \left( \frac{1}{4} p_1^2 + p_1k + k^2 \right) \right).
\]
(40)

This gives up to terms in the integrand, which evaluate for \( M \neq 0 \) to finite quantities

\[
\sum_{i=1}^{8} I_i = \frac{\hbar}{128i} \delta^4(p_1 + p_2) e^{-p_{1,\mu}(\theta_1 \sigma^\mu \theta_2 - \theta_2 \sigma^\mu \theta_1)} \int d^4k \frac{\sin^2(p_1 \wedge k)}{k^2(k + p_1)^2}
\times \left( -\frac{14}{3} - \frac{1}{4} \theta_{12}^2 \bar{b}_{12}^2 \left( -4kp_1 - \frac{26}{3} p_1^2 + \frac{8(kp_1)^2}{k^2 - M^2} \right) \right).
\]
(41)

As usual we write \( \sin^2(p_1 \wedge k) = \frac{1}{2} - \frac{1}{2} \cos(2p_1 \wedge k) \) and refer to the part corresponding to \( \frac{1}{2} \) as ‘planar’ and the part corresponding to \( \frac{1}{2} \cos(2p_1 \wedge k) \) as ‘non-planar’. The planar
part of \((41)\) is UV-divergent and evaluated in dimensional or analytic regularization to

\[
\left(\sum_{i=1}^{8} I_i\right)_{\text{planar}}^{\text{reg}} = -\frac{7}{3} \frac{h \pi^2}{128 e} \delta^4 (p_1 + p_2) e^{-p_{1,\mu}(\theta_1 \sigma_1 \bar{\theta}_1 - \theta_2 \sigma_2 \bar{\theta}_2)} \left(1 - \frac{1}{4} \theta_{12}^2 \bar{\theta}_{12}^2 p_1^2 \right) + \mathcal{O}(1). \quad (42)
\]

This means that the divergence in the planar part of the self-energy is transversal, so that it can be removed by multiplicative renormalization. Because of the oscillating integrand, the non-planar part of \((41)\) turns out to be finite for \(p_1 \neq 0\) and is evaluated to (with \(\tilde{p}_1^\mu := \Theta^{\mu \nu} p_{1,\nu}\)):

\[
\left(\sum_{i=1}^{8} I_i\right)_{\text{non-planar}} = \frac{7}{3} \frac{h \pi^2}{64} \delta^4 (p_1 + p_2) e^{-p_{1,\mu}(\theta_1 \sigma_1 \bar{\theta}_1 - \theta_2 \sigma_2 \bar{\theta}_2)} \times \left(1 - \frac{1}{4} \theta_{12}^2 \bar{\theta}_{12}^2 p_1^2 \right) \int_0^1 dx K_0 \left(\sqrt{x(1-x)p_1^2 \tilde{p}_1^2}\right) + \mathcal{O}(1), \quad (43)
\]

where \(\mathcal{O}(1)\) in \((43)\) collects terms that are regular for \(p_1 \to 0\), and \(K_0(y) = \log 2 - \log y - \gamma + \mathcal{O}(y^2)\) is the modified Bessel function of second kind (with Euler’s \(\gamma = 0.577...\)). Thus, the non-planar part of the self-energy is only logarithmically divergent for \(p_1 \to 0\), as expected for a supersymmetric theory, despite the fact that it is non-commutative.

7 Conclusions

We have computed the one-loop self-energy of the vector superfield in non-commutative \(\mathcal{N} = 1, U(1)\) super-Yang–Mills theory in the superfield formulation, where our results can be summarized in the following way:

- UV divergences can be multiplicatively renormalized in the usual way.
- IR divergences are only logarithmic and do not spoil renormalizability.

Because there are only logarithmic divergences, a non-commutative field theory is, according to [24], power-counting renormalizable (assuming there is no problem with commutants). This would imply that non-commutative \(\mathcal{N} = 1, U(1)\) super-Yang–Mills theory is renormalizable, because supersymmetry avoids quadratic (and linear) IR divergences.

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A Calculations

A.1 $\phi^4$-vertex

Let us compute the $\phi^4$-vertex given in (22), $\Gamma_{\phi^4}(1, 2, 3, 4) = \frac{\delta_{V}\phi^4}{\delta V\phi_1 \delta V\phi_2 \delta V\phi_3 \delta V\phi_4}$, where $S_{\phi^4}$ denotes the part of the total action which is quartic in $\phi$:

$$
S_{\phi^4} = -\frac{1}{128 g^2} \int dV \left( \frac{1}{4} [\tilde{D}^2 \phi, D^\alpha \phi]_\star - \frac{1}{12} [\phi, \tilde{D}^2 D^\alpha \phi]_\star + \frac{1}{2} [\tilde{D}_\alpha \phi, \tilde{D}^\alpha D^\alpha \phi]_\star \right) [\phi, D_\alpha \phi]_\star.
$$

(A.1)

Using the definition of the Moyal product (3) and integrating by parts several times, this expression can be written as

$$
S_{\phi^4} = -\frac{1}{256 g^2} \int dP_{V5} dP_{V6} dP_{V7} dP_{V8} (2\pi)^4 \delta^4 (p_5 + p_6 + p_7 + p_8) e^{-i p_5 \wedge p_6 - i p_7 \wedge p_8}
$$

$$
\times \tilde{\phi}_{p_5} \tilde{\phi}_{p_6} \tilde{\phi}_{p_7} \tilde{\phi}_{p_8} \mathcal{V}(5, 6, 7, 8) \delta V(5, 6) \delta V(5, 7) \delta V(5, 8),
$$

(A.2)

where the differential operator $\mathcal{V}$ is given by

$$
\mathcal{V}(5, 6, 7, 8)
$$

$$
= \left( \tilde{D}^2_{5, -p_5} - \tilde{D}^\alpha_{7, -p_7} \right) D_{9, 6, -p_6} \left[ \frac{1}{2} \tilde{D}^2_{6, -p_6} - \frac{1}{6} \tilde{D}^\alpha_{6, -p_6} \right] + \left( \tilde{D}^\alpha_{7, -p_7} - \tilde{D}^8_{8, -p_8} \right) D_{9, 5, -p_5} \left[ \frac{1}{2} \tilde{D}^2_{6, -p_6} - \frac{1}{6} \tilde{D}^\alpha_{6, -p_6} \right].
$$

(A.3)

We notice that $\mathcal{V}$ has the following symmetry properties:

$$
\mathcal{V}(5, 6, 7, 8) = -\mathcal{V}(6, 5, 7, 8) = -\mathcal{V}(5, 6, 8, 7) = \mathcal{V}(6, 5, 8, 7).
$$

(A.4)

Using these symmetries we can write $\Gamma_{\phi^4}$ as

$$
\Gamma_{\phi^4}(1, 2, 3, 4) = \frac{1}{64 g^2} (2\pi)^4 \delta^4 (p_1 + p_2 + p_3 + p_4)
$$

$$
\times \left[ \sin(p_1 \wedge p_2) \sin(p_3 \wedge p_4) (\mathcal{V}(1, 2, 3, 4) + \mathcal{V}(3, 4, 1, 2)) + \sin(p_1 \wedge p_3) \sin(p_4 \wedge p_2) (\mathcal{V}(1, 3, 4, 2) + \mathcal{V}(4, 2, 1, 3)) + \sin(p_1 \wedge p_4) \sin(p_2 \wedge p_3) (\mathcal{V}(1, 4, 2, 3) + \mathcal{V}(2, 3, 1, 4)) \right]
$$

$$
\times \tilde{\delta}_V(1, 2) \tilde{\delta}_V(1, 3) \tilde{\delta}_V(1, 4).
$$

(A.5)

In order to simplify this expression we have to evaluate terms like

$$
A(1, 2, 3, 4) := \tilde{D}^\alpha_{4, -p_4} \tilde{D}_{\alpha, 2, -p_2} \tilde{D}_{\alpha, 2, -p_2} \tilde{D}^\alpha_{1, -p_1} \tilde{\delta}_V(1, 2) \tilde{\delta}_V(1, 3) \tilde{\delta}_V(1, 4).
$$

(A.6)
This can be done most easily if we first insert exponentials in the following way:

\[
A(1, 2, 3, 4) = \left( \tilde{D}_4^\alpha_{-p_4} e^{p_{4,\mu}(\theta_3 \sigma^\nu \tilde{\theta}_1 - \theta_4 \sigma^\nu \tilde{\theta}_3)} \tilde{\delta}_V(4, 3) \right) \times \left( \tilde{D}_{\alpha, 2, -p_2} \tilde{D}_{\alpha, 2, -p_2} e^{p_{2,\mu}(\theta_3 \sigma^\nu \tilde{\theta}_2 - \theta_3 \sigma^\nu \tilde{\theta}_2)} \tilde{\delta}_V(2, 3) \right) \left( \tilde{D}_{\alpha, 1, -p_1} e^{p_{1, \nu}(\theta_3 \sigma^\nu \tilde{\theta}_1 - \theta_1 \sigma^\nu \tilde{\theta}_3)} \tilde{\delta}_V(1, 3) \right), \tag{A.7}
\]
which is allowed because of $\theta^3 = \tilde{\theta}^3 = 0$. We now use the identities

\[
\tilde{D}_{\alpha, i, p} e^{p_{\mu}(\theta_i \sigma^\nu \tilde{\theta}_j - j \sigma^\nu \tilde{\theta}_j)} = e^{p_{\mu}(\theta_i \sigma^\nu \tilde{\theta}_j - j \sigma^\nu \tilde{\theta}_j)} \tilde{D}_{\alpha, i, j, p}, \tag{A.8}
\]
\[
\tilde{D}_{\alpha, i, p} e^{p_{\mu}(\theta_i \sigma^\nu \tilde{\theta}_j - j \sigma^\nu \tilde{\theta}_j)} = e^{p_{\mu}(\theta_i \sigma^\nu \tilde{\theta}_j - j \sigma^\nu \tilde{\theta}_j)} \tilde{D}_{\alpha, i, j, p}, \tag{A.9}
\]
which can easily be verified, leading to

\[
A(1, 2, 3, 4) = e^{E_{\alpha}^{(4)}} \left( \tilde{D}_{4, -p_4} \tilde{\delta}_V(4, 3) \right) \left( \tilde{D}_{\alpha, 2, -p_2} \tilde{D}_{\alpha, 2, -p_2} \tilde{\delta}_V(2, 3) \right) \left( \tilde{D}_{\alpha, 1, -p_1} \tilde{\delta}_V(1, 3) \right), \tag{A.10}
\]
where we have used the notation (B.11). We can readily evaluate the covariant derivatives of the delta functions, which give

\[
A(1, 2, 3, 4) = \frac{1}{256} e^{E_{\alpha}^{(4)}} \theta_{43}^{\alpha} \tilde{\theta}_{13}^{\alpha} (\theta_{23, \alpha} \theta_{23, \beta} + \frac{1}{2} (\sigma^\mu \tilde{\theta}_{23})_\alpha p_{2, \mu} \tilde{\theta}_{23, \beta} \tilde{\theta}_{23}) \theta_{23}^{\alpha} \theta_{13}^{\beta}. \tag{A.11}
\]

With the help of the Fierz identity (C.2), this expression can be rewritten as

\[
A(1, 2, 3, 4) = \frac{1}{256} e^{E_{\alpha}^{(4)}} p_{\alpha} (\theta_{23} \sigma^\nu \tilde{\theta}_{23})(\theta_{43} \tilde{\theta}_{23}) \tilde{\theta}_{23}^{\alpha} (\tilde{\theta}_{23} \tilde{\theta}_{13}) \theta_{13}^{\alpha}. \tag{A.12}
\]

By applying this procedure to all the terms in (A.9), we arrive at

\[
\Gamma_{\phi^4}(1, 2, 3, 4) = \frac{1}{64 g^2} (2\pi)^4 \delta^4(p_1 + p_2 + p_3 + p_4)
\]
\[
\times \left[ \sin(p_1 \wedge p_2) \sin(p_3 \wedge p_4) \left( F(1, 2, 3, 4) - F(2, 1, 3, 4) - F(1, 2, 4, 3)
+ F(2, 1, 4, 3) + F(3, 4, 1, 2) - F(4, 3, 1, 2) - F(3, 4, 2, 1) + F(4, 3, 2, 1) \right)
+ \sin(p_1 \wedge p_3) \sin(p_4 \wedge p_2) \left( F(1, 3, 4, 2) - F(3, 1, 4, 2) - F(1, 3, 2, 4)
+ F(3, 1, 2, 4) + F(4, 2, 1, 3) - F(2, 4, 1, 3) - F(4, 2, 3, 1) + F(2, 4, 3, 1) \right)
+ \sin(p_1 \wedge p_4) \sin(p_2 \wedge p_3) \left( F(1, 4, 2, 3) - F(4, 1, 2, 3) - F(1, 4, 3, 2)
+ F(4, 1, 3, 2) + F(2, 3, 1, 4) - F(3, 2, 1, 4) - F(2, 3, 4, 1) + F(3, 2, 4, 1) \right) \right], \tag{A.13}
\]
where $F$ is given by

\[
F(i, j, k, l) = \frac{1}{256} e^{E_{\alpha}^{(4)}} p_{\alpha} (\theta_{jk} \sigma^\nu \tilde{\theta}_{jk})(\theta_{lk} \tilde{\theta}_{lk}) \tilde{\theta}_{lk}^2 (\tilde{\theta}_{lj} \tilde{\theta}_{lk}^2)^2 \left( -\frac{1}{2} \tilde{\theta}_{lj}^2 + \frac{2}{3} \tilde{\theta}_{lk}^2 \right). \tag{A.14}
\]
A.2 The other vertices

The other vertices can be evaluated in a similar way as the $\phi^4$-vertex. Since the calculations are much easier than in the case of the $\phi^4$-vertex, we only give the results:

\[ \Gamma_{\phi^3}(1, 2, 3) = -\frac{i}{64g^2}(2\pi)^4\delta^4(p_1 + p_2 + p_3)\sin(p_2 \wedge p_3) \]
\[ \times \epsilon_{p_2,\mu}(\theta_1 \sigma^\mu \theta_2 - \theta_2 \sigma^\mu \theta_1) + p_{3,\mu}(\theta_1 \sigma^\mu \theta_3 - \theta_3 \sigma^\mu \theta_1) \]
\[ \times \left( \frac{1}{8}(\theta_{21}^2 - \theta_{31}^2)(\theta_2 \theta_3) - \frac{1}{8}(\theta_{21} \theta_{31})(\theta_{21}^2 - \theta_{31}^2) - \frac{1}{16}\theta_{21} \theta_{31}^2 \right) \]
\[ + \frac{1}{16}\theta_{21}^2 \theta_{31}^2 - \frac{1}{4}(p_{2,\rho} \theta_{21} \sigma^\rho \theta_{21} - p_{3,\rho} \theta_{31} \sigma^\rho \theta_{31})(\theta_{21} \theta_{31})(\theta_{21} \theta_{31}) \right), \]
\[ (A.15) \]

\[ \Gamma_{c_{-c+}\phi^2}(1, 2, 3, 4) = -\frac{1}{384}(2\pi)^4\delta^4(p_1 + p_2 + p_3 + p_4)\tilde{\delta}_V(1, 2)\tilde{\delta}_V(1, 3)\tilde{\delta}_V(1, 4) \]
\[ \times \left( \sin(p_1 \wedge p_4) + \sin(p_1 \wedge p_3) + \sin(p_2 \wedge p_4) \right), \]
\[ (A.16) \]

\[ \Gamma_{\bar{c}_{-c+}\phi^2}(1, 2, 3, 4) = -\Gamma_{c_{-c+}\phi^2}(1, 2, 3, 4) = \Gamma_{c_{-c+}\phi}(1, 2, 3, 4), \]
\[ (A.17) \]

\[ \Gamma_{c_{-c-}\phi}(1, 2, 3) = -\frac{i}{128}\tilde{\delta}_V(1, 2)\tilde{\delta}_V(1, 3)(2\pi)^4\delta^4(p_1 + p_2 + p_3)\sin(p_2 \wedge p_3), \]
\[ (A.18) \]

\[ \Gamma_{\bar{c}_{-c-}\phi}(1, 2, 3) = \Gamma_{c_{-c-}\phi}(1, 2, 3) = \Gamma_{c_{-c+}\phi}(1, 2, 3). \]
\[ (A.19) \]

A.3 The integrals

As an example, we show how to compute $I_7$ in (32). Inserting the formulae for the propagators and vertices we obtain

\[ I_7 = -\frac{\hbar}{i} \int dP_{V3}dP_{V4}dP_{V5}dP_{V6} \left( -\frac{i}{128} \right) \tilde{\delta}_V(3, 1)\tilde{\delta}_V(4, 1)(2\pi)^4\delta^4(p_3 + p_4 + p_1) \]
\[ \times \sin(p_4 \wedge p_1)(-8)(2\pi)^4\delta^4(p_6 + p_3)\epsilon_{p_3,\mu}(\theta_{36} \sigma^\mu \theta_{36} - \theta_{36} \sigma^\mu \theta_{36}) \]
\[ \times \frac{1 - (\theta_{36} \sigma^\rho \theta_{36}) p_{3,\rho} + \frac{1}{4}\theta_{36}^2 \theta_{36} \theta_{36}}{p_3^2} \left( -\frac{i}{128} \right) \tilde{\delta}_V(5, 2)\tilde{\delta}_V(6, 2)(2\pi)^4\delta^4(p_5 + p_6 + p_2) \]
\[ \times \sin(p_6 \wedge p_2)(-8)(2\pi)^4\delta^4(p_4 + p_5)\epsilon_{p_5,\mu}(\theta_{54} \sigma^\mu \theta_{54} - \theta_{54} \sigma^\mu \theta_{54}) \frac{1 - (\theta_{54} \sigma^\rho \theta_{54}) p_{5,\rho} + \frac{1}{4}\theta_{54}^2 \theta_{54} \theta_{54}}{p_5^2} \]
\[ = -\frac{\hbar}{256i}^4\delta(p_1 + p_2) \int d^4k \, \partial_3^2 \partial_5^2 \ldots \partial_6^2 \partial_6 \left[ \sin^2(p_1 \wedge k)\tilde{\delta}_V(3, 1)\tilde{\delta}_V(4, 1) \right. \]
\[ \times \tilde{\delta}_V(5, 2)\tilde{\delta}_V(6, 2) \epsilon_{k,\mu}(\theta_{36} \sigma^\mu \theta_{36} - \theta_{36} \sigma^\mu \theta_{36}) \epsilon_{(p_1 + k)\nu}(\theta_{54} \sigma^\nu \theta_{54} - \theta_{54} \sigma^\nu \theta_{54}) \frac{1}{k^2(p_1 + k)^2} \]
\[ \times \left. \left( 1 - (\theta_{36} \sigma^\rho \theta_{36}) k_\rho + \frac{1}{4}k^2 \theta_{36}^2 \theta_{36} \right) \right] \left( 1 - (\theta_{54} \sigma^\rho \theta_{54}) (p_1 + k)_{\rho} + \frac{1}{4}(p_1 + k)^2 \theta_{54}^2 \theta_{54} \right). \]
\[ (A.20) \]
short-hand notation for \( \frac{\partial}{\partial \theta_{i,\alpha}} \frac{\partial}{\partial \bar{\theta}_{i,\dot{\alpha}}} \). In the last three lines of (A.20) we can replace \( \theta_3 \) and \( \theta_4 \) with \( \theta_1 \), and \( \theta_5 \) and \( \theta_6 \) with \( \theta_2 \), because of the superspace delta functions that are contained in the integrand. Thus, we obtain

\[
I_7 = -\frac{\hbar}{256i} \delta^4(p_1 + p_2) e^{-p_{1,\mu}(\theta_1^{\sigma} \theta_2^{\dot{\sigma} - \theta_2^{\sigma} \theta_1^{\dot{\sigma}})} \int d^4k \sin^2(p_1 \wedge k) \\
\times \left[ \partial_3^2 \partial_3^2 \ldots \partial_6^2 \partial_6^2 \tilde{\Delta}(3, 1) \tilde{\Delta}(4, 1) \tilde{\Delta}(5, 2) \tilde{\Delta}(6, 2) \right] \frac{1}{k^2(p_1 + k)^2} \\
\times \left( 1 - (\theta_1^{\sigma} \bar{\theta}_{12}) k_{\rho} + \frac{1}{4} k^2 \theta_{12}^2 \theta_{12}^2 \right) \left( 1 - (\theta_1^{\sigma} \bar{\theta}_{12})(p_1 + k)_{\lambda} + \frac{1}{4}(p_1 + k)^2 \theta_{12}^2 \right).
\]

(A.21)

The term between the square brackets of this expression simply gives a factor 1. Applying the identity (C.11) in the last two lines of (A.21) we arrive at

\[
I_7 = -\frac{\hbar}{256i} \delta^4(p_1 + p_2) e^{-p_{1,\mu}(\theta_1^{\sigma} \theta_2^{\dot{\sigma} - \theta_2^{\sigma} \theta_1^{\dot{\sigma}})} \int d^4k \sin^2(p_1 \wedge k) \frac{1}{k^2(p_1 + k)^2} \\
\times \left( 1 - (\theta_1^{\sigma} \bar{\theta}_{12})(p_1 + 2k)_{\rho} + \theta_{12}^2 \theta_{12}^2 \left( \frac{1}{4} p_1^2 + p_1 k + k^2 \right) \right).
\]

(A.22)

B Notations and Conventions

We frequently use the definition

\[
\frac{1}{2} k_j p_j G^{ij} := k \wedge p.
\]

(B.1)

The Fourier transform of a field is

\[
\tilde{\phi}(x) = \int \frac{d^4p}{(2\pi)^4} \tilde{\phi}(p) e^{-ipx}.
\]

(B.2)

The covariant derivatives are defined as

\[
D_{\alpha} = \partial_{\alpha} - i(\sigma^{\mu} \theta_{\alpha}) \partial_{\mu}, \quad D_{\dot{\alpha}} = -\bar{\partial}_{\dot{\alpha}} + i(\theta^{\sigma}_{\dot{\alpha}}) \partial_{\mu}.
\]

(B.3)

In momentum space the covariant derivatives read

\[
\tilde{D}_{\alpha,p} = \partial_{\alpha} - p_{\mu} \sigma^{\mu}_{aa} \theta^{a}, \quad \tilde{D}_{\dot{\alpha},p} = -\bar{\partial}_{\dot{\alpha}} + p_{\mu} \sigma^{\mu}_{\dot{\alpha}a} \theta^{a}.
\]

(B.4)

We use the following definitions concerning Grassmann-valued objects,

\[
\chi \eta := \chi^{\alpha} \eta_{\alpha} = -\eta_{\alpha} \chi^{\alpha} = \eta^{\dot{\alpha}} \chi_{\dot{\alpha}} = \eta \chi, \quad \bar{\chi} \bar{\eta} := \bar{\chi}_{\dot{\alpha}} \bar{\eta}^{\dot{\alpha}} = \bar{\eta} \bar{\chi}.
\]

(B.5)

We have the integration measures,

\[
\int dV := \int d^4x D^2 \bar{D}^2, \quad \int dS := \int d^4x D^2, \quad \int d\bar{S} := \int d^4x \bar{D}^2.
\]

(B.6)
The integration over $x$ cancels the total divergence parts of the covariant derivatives. Therefore, in momentum space we have to define

$$\int dP_V := \int \frac{d^4p}{(2\pi)^4} \tilde{D}_{p \to 0}^2 \tilde{D}_{p \to 0}^2,$$

$$\int dP_S := \int \frac{d^4p}{(2\pi)^4} \tilde{D}_{p \to 0}^2,$$

$$\int dP := \int \frac{d^4p}{(2\pi)^4} \tilde{D}_{p \to 0}^2.$$ (B.7)

The delta functions and their Fourier transforms are given by

$$\delta_V(1, 2) = \frac{1}{16} \theta_{i2}^2 \theta_{j2}^2 \delta^4(x_1 - x_2), \quad \tilde{\delta}_V(1, 2) = \frac{1}{16} \theta_{i2}^2 \theta_{j2}^2,$$

$$\delta_S(1, 2) = -\frac{1}{4} \theta_{i2}^2 \delta^4(x_1 - x_2), \quad \tilde{\delta}_S(1, 2) = -\frac{1}{4} \theta_{i2}^2,$$

$$\delta_{\bar{S}}(1, 2) = -\frac{1}{4} \bar{\theta}_{i2}^2 \delta^4(x_1 - x_2), \quad \tilde{\delta}_{\bar{S}}(1, 2) = -\frac{1}{4} \bar{\theta}_{i2}^2.$$ (B.8)

Here $\theta_{ij}^\alpha := \theta_i^\alpha - \theta_j^\alpha$. We use functional derivation in superspace:

$$\frac{\delta_V \phi_i}{\delta_V \phi_j} = \delta^4(x_i - x_j) \delta_V(i, j).$$ (B.9)

In momentum space we get an extra factor $(2\pi)^4$:

$$\frac{\delta_V \tilde{\phi}_p}{\delta_V \tilde{\phi}_{p_j}} = (2\pi)^4 \delta^4(p_i - p_j) \tilde{\delta}_V(i, j).$$ (B.10)

Finally, we use the definition

$$E^{(n)}_i := \sum_{j=1,\ldots,n, j \neq i} p_{j,\mu} \left( \theta_i \sigma^{\mu} \bar{\theta}_j - \bar{\theta}_j \sigma^{\mu} \bar{\theta}_i \right).$$ (B.11)

C Useful Formulae

These helpful formulae are used throughout our calculations:

$$\theta_{ji} \theta_{ki} = \frac{1}{2} (\theta_{ji}^2 + \theta_{ki}^2 - \theta_{jk}^2), \quad (\theta \sigma^{\mu} \bar{\theta})(\theta \sigma^{\nu} \bar{\theta}) = \frac{1}{2} \eta^{\mu\nu} \theta^2 \bar{\theta}^2,$$ (C.1)

$$\theta_{\alpha}(\theta \psi) = -\frac{1}{2} \theta^2 \psi, \quad \bar{\theta}_{\dot{\alpha}}(\bar{\theta} \bar{\psi}) = -\frac{1}{2} \bar{\theta}^2 \bar{\psi},$$ (C.2)

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