Hodge decompositions and Poincaré duality models

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We use Hodge decompositions to construct Poincaré duality models and improve results of Lambrechts & Stanley in the simply-connected case. The main idea is the construction of a certain oriented extension of the Sullivan minimal model which admits a Hodge decomposition.

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1 Introduction

A Poincaré DGA of degree $n$ (:= PDGA) is a non-negatively graded unital CDGA $V$ with finite-dimensional homology $H(V)$ equipped with an orientation $o^H : H(V) \to \mathbb{R}$ in degree $n$ such that the induced pairing $\langle h_1, h_2 \rangle^H := o^H(h_1 \wedge h_2)$ for $h_1, h_2 \in H(V)$ is non-degenerate. Such $H(V)$ is called a Poincaré duality algebra (:= PD-algebra). A morphism of PDGA’s $V_1$ and $V_2$ (:= PDGA-morphism) is a DGA-morphism $f : V_1 \to V_2$

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such that the induced map \( f_\ast : H(V_1) \to H(V_2) \) preserves the orientation. A \textit{differential Poincaré duality algebra} of degree \( n \) (\( = \) dPD-algebra) is a finite-dimensional non-negatively graded unital CDGA \( M \) equipped with an orientation \( o : M \to \mathbb{R} \) in degree \( n \) such that the induced pairing \( \langle v_1, v_2 \rangle := o(v_1 \wedge v_2) \) for \( v_1, v_2 \in M \) is non-degenerate. A \textit{Poincaré duality model} (\( = \) PD-model) of a PDGA \( V \) is a dPD-algebra \( M \) such that there is an isomorphism of PDGA-quasi-isomorphisms connecting \( M \) to \( V \).

It was shown in [LS08] that a PD-model of a PDGA \( V \) exists provided that \( H^0(V) \cong \mathbb{R} \) and \( H^1(V) = 0 \). Moreover, [LS08, Theorem 7.1] asserts the following. Let \( M_1 \) and \( M_2 \) be two PD-models of \( V \) and suppose that the following holds:

1. \( n \geq 7 \),
2. \( H^0(V) \cong \mathbb{R} \), \( H^1(V) = H^2(V) = H^3(V) = 0 \) and
3. \( M^0_i \cong \mathbb{R} \), \( M^1_i = M^2_i = 0 \) for \( i \in \{1, 2\} \).

Then there is a third PD-model \( M_3 \) and DGA quasi-isomorphisms \( M_1 \to M_3 \leftarrow M_2 \). Such statement can be called the “weak uniqueness” of PD-models in contrast to the “strict uniqueness” of Sullivan minimal models. In order to obtain \( M \), they take a Sullivan minimal model \( W := \Lambda U \to V \), which is of finite type due to \( H^0(V) \cong \mathbb{R} \) and \( H^1(V) = 0 \), and construct a certain extension \( \hat{W} \) which is quasi-isomorphic to \( W \) via the inclusion \( W \hookrightarrow \hat{W} \) and oriented such that the canonical projection

\[
\pi_Q : \hat{W} \to \hat{W}/\hat{W}^\perp := Q(\hat{W}), \quad \text{where } \hat{W}^\perp := \{ w \in \hat{W} \mid w \perp \hat{W} \},
\]

is a quasi-isomorphism. It is then easy to see that \( V \leftarrow \hat{W} \to Q(\hat{W}) := M \) is a PD-model of \( V \). The extension \( \hat{W} \) is constructed from \( W \) by adding new generators which commute with \( W \) and “kill”, in the homological sense, generators of the group \( H(W^\perp) \), called the “orphans”, which obstruct \( W \to Q(W) \) from being a quasi-isomorphism.

In [Fio+19], it was observed that for an oriented DGA \( A \), the vanishing of \( H(A^+) \) implies that \( A \) admits a \textit{Hodge decomposition}, i.e., that there exists a complement \( H \) of \( \text{im } d \) in \( \ker d \), called the \textit{harmonic subspace}, and a complement \( C \) of \( \ker d \) in \( A \), called the \textit{coexact part}, such that \( C \perp H \perp C \). They say that \( A \) is of \textit{Hodge type} if it admits a Hodge decomposition \( A = H \oplus dC \oplus C \). In fact, \( A \) is of Hodge type if and only if \( H(A^+) = 0 \) provided that \( Q(A) \) is of finite type. Based on this observation, we present a different construction of an oriented extension \( \hat{W} \) of \( W \) which is aimed at obtaining a Hodge decomposition of \( \hat{W} \) rather than at \( H(W^\perp) = 0 \). The obstruction for the existence of a Hodge decomposition are elements \( c \in C \) which are not perpendicular to \( C \) and which we call “non-degenerates”. We get rid of them by adding their “exact partners”, i.e., exact elements \( dw \) such that \( \langle c, \cdot \rangle = \langle dw, \cdot \rangle \) holds on \( C \). Then we take the coexact part generated by \( c - dw \) instead.

The following is Proposition 4.3 in the text.

\textbf{Theorem 1.1.} Let \( W \) be a PDGA of degree \( n \) which is connected, simply-connected and of finite type. Then it admits an oriented extension \( \hat{W} \) of Hodge type which is connected, of finite type and retracts onto \( W \). If \( W \) is oriented, then we can achieve
that the inclusion \( \iota : W \rightarrow \hat{W} \) is orientation preserving. If \( n \geq 5 \), then we can achieve that \( \hat{W} \) is simply-connected.

Using \( \hat{W} \) from Theorem 1.1 in the proofs of [LS08] and taking care of the orientation so that we get PDGA-quasi-isomorphisms, we obtain the following improvement of [LS08, Theorem 1.1] and [LS08, Theorem 7.1]. It is a combination of Proposition 5.4 and Remark 5.5 in the text.

**Theorem 1.2.** A PDGA \( V \) with \( H^0(V) \simeq \mathbb{R} \) and \( H^1(V) = 0 \) admits a connected and simply-connected PD-model \( M \) in the form

\[
\begin{array}{ccc}
\mathcal{U} & \xleftarrow{\iota_3} & \mathcal{M}_3 \\
V & \xleftarrow{\iota_2} & M_2 \\
\xrightarrow{\iota_1} & \mathcal{M}_1 & \\
\end{array}
\]

where \( \hat{\mathcal{U}} \) is an oriented extension of Hodge type of a Sullivan minimal model \( \mathcal{U} \rightarrow V \) which is connected, simply-connected and of finite type. If \( M_1 \) and \( M_2 \) are two connected and simply-connected PD-models of \( V \) and if \( H^2(V) = 0 \), then there is a connected and simply-connected PD-model \( M_3 \) together with injective orientation preserving PDGA-quasi-isomorphisms

\[
\begin{array}{ccc}
\mathcal{U} & \xleftarrow{\iota_3} & \mathcal{M}_3 \\
V & \xleftarrow{\iota_2} & M_2 \\
\xrightarrow{\iota_1} & \mathcal{M}_1 & \\
\end{array}
\]

The assumption \( H^2(V) = 0 \) is necessary for the statement to hold for all \( V \).

We conjecture that the theorem holds even if \( H^2(V) \neq 0 \) provided that one does not insist on \( M_3 \) being simply-connected.

Another notion from [Fio+19] is that of a small subalgebra \( S_{H,C}(V) \) for a Hodge decomposition \( V = H \oplus dC \oplus C \). By definition, it is the smallest dg-subalgebra which admits a Hodge decomposition with harmonic subspace \( H \). If \( H^0(V) \simeq \mathbb{R} \) and \( H^1(V) = 0 \), then \( S_{H,C}(V) \) is of finite type and \( V \leftrightarrow S_{H,C}(V) \rightarrow Q(S_{H,C}(V)) \) provides a canonical PD-model of \( V \). This is particularly useful if \( M \) is an oriented closed \( n \)-manifold and \( V \) its de Rham algebra \( \Omega(M) \) oriented by \( \int_M : \Omega(M) \rightarrow \mathbb{R} \). Given a Riemannian metric \( g \) on \( M \), there is a canonical Riemannian Hodge decomposition \( \Omega(M) = H \oplus d\Omega(M) \oplus d^*\Omega(M) \), where \( d^* = \pm d^* \) is the Hodge codifferential and \( H = \{ \omega \in \Omega(M) \mid d\omega = d^*\omega = 0 \} \) the space of harmonic forms. We denote the corresponding small subalgebra by \( S_g(\Omega(M)) \). If \( M \) is connected and \( H^1_{dR}(M) = 0 \), one obtains a canonical PD-model \( M_g := Q(S_g(\Omega(M))) \). We do not know how \( M_g \) depends on \( g \) but we can give an example of an artificial Hodge decomposition \( \Omega(SU(6)) = H \oplus dC \oplus C \) such that \( Q(S_{H,C}(\Omega(SU(6)))) \neq M_g \) for a given biinvariant Riemannian metric \( g \) on \( SU(6) \). We also show that de Rham complex of the connected sum
$\#^7\text{CP}^2$ does not admit a PD-model $\mathcal{M}$ with just one direct PDGA-quasi-isomorphism between $\mathcal{M}$ and $\Omega(\#^7\text{CP}^2)$.

Our interest in PD-models stems from the study of the $\text{IBL}_{\infty}$-chain-model of equivariant Chas-Sullivan string topology of $\mathcal{M}$ proposed in [CFL15] (IBL stands for involutive bi-Lie algebra). The space of cyclic cochains of a dPD-algebra carries a canonical dIBL-structure with the Hochschild differential. To obtain the $\text{IBL}_{\infty}$-chain-model, one starts with the canonical IBL-structure on cyclic cochains of $H_{\text{dR}}(\mathcal{M})$ and “corrects it” by twisting with a Maurer-Cartan element constructed from Feynman integrals associated to trivalent ribbon graphs with a propagator coming from a version of Chern-Simons theory on $\mathcal{M}$. The Maurer-Cartan element can be interpreted as an effective action and the twisting can be seen as a homotopy transfer of an ill-defined dIBL-structure on cyclic cochains of $\Omega(\mathcal{M})$, which is a PDGA but not a dPD-algebra, to a well-defined $\text{IBL}_{\infty}$-structure on cyclic cochains of $H_{\text{dR}}(\mathcal{M})$. A detailed proof that this construction gives a chain model of string topology is being prepared by K. Cieliebak and E. Volkov.

The problem of finding a propagator and computing integrals explicitly is notoriously hard except for the case of $\mathcal{M} = S^n$ with $n \neq 2$ which was done by the author in [Haj19]. Moreover, we proved vanishing results for the Maurer-Cartan element suggesting that if $H^1_{\text{dR}}(\mathcal{M}) = 0$, then the $\text{IBL}_{\infty}$-chain model depends only on the PDGA-structure of $\Omega(\mathcal{M})$. In this case, we proposed to consider the canonical dIBL-structure on cyclic cochains of the PD-model $\mathcal{M}_g$ instead. We call this approach the algebraic approach and the approach based on the Chern-Simons Maurer-Cartan element the geometric approach. We conjectured that if $H^1_{\text{dR}}(\mathcal{M}) = 0$, then the resulting $\text{IBL}_{\infty}$-chain-models from the algebraic and the geometric approach are $\text{IBL}_{\infty}$-homotopy equivalent. Zigzags of connected and simply-connected PDGA’s might be particularly useful because our vanishing results for the Maurer-Cartan element apply.

The author has recently become aware of [NW19], where Poincaré duality models are also used in the context of $\text{IBL}_{\infty}$-algebras and a proof of the string topology conjecture from [CFL15] is proposed.

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2 Notation and conventions

We always work over the field $\mathbb{R}$. We work in the category of $\mathbb{Z}$-graded vector spaces $V = \bigoplus_{i \in \mathbb{Z}} V^i$. We say that $W$ is a subspace of $V$ if it is a direct sum of subspaces $W^i \subset V^i$ for all $i \in \mathbb{Z}$. We say that $U$ is a complement of $W$ in $V$ if it is a direct
Algebraic prerequisites

Let \((V, d)\) be a cochain complex. An orientation \(o\) in degree \(n \in \mathbb{N}_0\) is a map \(o : V \to \mathbb{R}\) of degree \(-n\) such that \(o \circ d = 0\) and \(o \neq 0\). The triple \((V, d, o)\) is called an oriented cochain complex. An orientation \(o\) on \(V\) induces a canonical orientation \(o^H\) on \(H(V)\) such that \(o^H([v]) = o(v)\) for all \(v \in \ker d\). An orientation \(o^H\) on \(H(V)\) can always be extended to an orientation \(o\) on \(V\). The extension is unique if \(d^n = 0\). A cyclic structure of degree \(n \in \mathbb{N}_0\) on \((V, d)\) is a bilinear form \(\langle \cdot, \cdot \rangle : V \otimes V \to \mathbb{R}\) of degree \(-n\) such that for all \(v_1, v_2 \in V\), the following holds:

\[
\langle v_1, v_2 \rangle = (-1)^{\deg v_1 \deg v_2} \langle v_2, v_1 \rangle, \tag{1}
\]

\[
\langle dv_1, v_2 \rangle = (-1)^{1+\deg v_1 \deg v_2} \langle dv_2, v_1 \rangle. \tag{2}
\]

A cyclic structure \(\langle \cdot, \cdot \rangle\) on \(V\) induces a canonical cyclic structure \(\langle \cdot, \cdot \rangle^H\) on \(H(V)\) such that \(\langle [v_1], [v_2] \rangle^H = \langle v_1, v_2 \rangle\) for all \(v_1, v_2 \in \ker d\). A cyclic structure on a DGA \((V, d, \wedge)\) is a cyclic structure \(\langle \cdot, \cdot \rangle\) on \((V, d)\) such that for all \(v_1, v_2, v_3 \in V\), the following holds:

\[
\langle v_1 \wedge v_2, v_3 \rangle = (-1)^{\deg v_3 (\deg v_1 + \deg v_2)} \langle v_3 \wedge v_1, v_2 \rangle. \tag{3}
\]

We remark that any bilinear form on a unital DGA which satisfies (2) and (3) satisfies automatically (1).

Lemma 3.1. (a) A non-trivial cyclic structure \(\langle \cdot, \cdot \rangle\) on a unital DGA \((V, d, \wedge)\) induces a canonical orientation \(o\) on \(V\) such that \(o(v) = \langle v, 1 \rangle\) for all \(v \in V\).

(b) An orientation \(o\) on a CDGA \((V, d, \wedge)\) induces a canonical cyclic structure \(\langle \cdot, \cdot \rangle\) on \((V, d, \wedge)\) such that \(\langle v_1, v_2 \rangle = o(v_1 \wedge v_2)\) for all \(v_1, v_2 \in V\).

Proof. (a) For all \(v \in V\), it holds

\[
o(dv) = \langle dv, 1 \rangle = -\langle d1, v \rangle = 0.
\]
Because $\langle \cdot, \cdot \rangle \neq 0$, there are $v_1, v_2 \in V$ such that $\langle v_1, v_2 \rangle \neq 0$. Then

$$o(v_1 \wedge v_2) = \langle v_1 \wedge v_2, 1 \rangle = \langle 1 \wedge v_1, v_2 \rangle = \langle v_1, v_2 \rangle \neq 0.$$  

(b) For all $v_1, v_2 \in V$, it holds

$$\langle v_1, v_2 \rangle = o(v_1 \wedge v_2) = (-1)^{\deg v_1 \deg v_2} o(v_2 \wedge v_1) = (-1)^{\deg v_1 \deg v_2} \langle v_2, v_1 \rangle$$

and

$$\langle dv_1, v_2 \rangle = o(dv_1 \wedge v_2)$$

$$= o(d(v_1 \wedge v_2) - (-1)^{\deg v_1} v_1 \wedge dv_2)$$

$$= (-1)^{1 + \deg v_1} o(v_1 \wedge dv_2)$$

$$= (-1)^{1 + \deg v_1 \deg v_2} o(dv_2 \wedge v_1)$$

$$= (-1)^{1 + \deg v_1 \deg v_2} \langle dv_2, v_1 \rangle.$$

For all $v_1, v_2, v_3 \in V$, it holds

$$\langle v_1 \wedge v_2, v_3 \rangle = o(v_1 \wedge v_2 \wedge v_3)$$

$$= (-1)^{\deg v_3 (\deg v_1 + \deg v_2)} o(v_3 \wedge v_1 \wedge v_2)$$

$$= (-1)^{\deg v_3 (\deg v_1 + \deg v_2)} \langle v_3 \wedge v_1, v_2 \rangle.$$

This finishes the proof. 

---

Let $\langle \cdot, \cdot \rangle$ be a cyclic structure on a cochain complex $(V, d)$. A complement $\mathcal{H}$ of $dV$ in $\ker d$ is called a harmonic subspace and a complement $C$ of $\ker d$ in $V$ a coexact part. If $C \perp C \oplus \mathcal{H}$, then $V = \mathcal{H} \oplus dC \oplus C$ is called a Hodge decomposition of $V$. We say that $(V, d, \langle \cdot, \cdot \rangle)$ is of Hodge type if it admits a Hodge decomposition. In this case, for any harmonic subspace $\mathcal{H}$, there is a coexact part $C$ such that $V = \mathcal{H} \oplus dC \oplus C$ is a Hodge decomposition; see [Fio+19, Remark 2.6].

**Lemma 3.2** ([CFL15, Lemma 11.1]). Let $\langle \cdot, \cdot \rangle$ be a cyclic structure on a cochain complex $(V, d)$. Suppose that $V$ is of finite type, i.e., it holds $\dim V^i < \infty$ for all $i \in \mathbb{Z}$, and that $\langle \cdot, \cdot \rangle$ is non-degenerate. Then $V$ is of Hodge type.

The degenerate subspace is defined by

$$V^\perp := \{ v \in V \mid v \perp V \}.$$

If $\langle \cdot, \cdot \rangle$ is a cyclic structure on a DGA $(V, d, \wedge)$, then $V^\perp$ is a dg-ideal. The non-degenerate quotient is defined by

$$Q(V) := V/V^\perp.$$

It is a DGA with a canonical cyclic structure $\langle \cdot, \cdot \rangle^Q$ such that the canonical projection
Let \( \langle \cdot , \cdot \rangle \) be a cyclic structure on a cochain complex \( (V , d) \). Then the following holds:

(a) If \( V \) is of Hodge type and \( \langle \cdot , \cdot \rangle^H \) non-degenerate, then \( V^\perp \) is acyclic.

(b) If \( V^\perp \) is acyclic and \( \mathcal{Q}(V) \) of finite type, then \( V \) is of Hodge type.

Proof. (a) Consider a Hodge decomposition \( V = \mathcal{H} \oplus dC \oplus C \). Let \( v \in V^\perp \) be such that \( dv = 0 \). If \( v \notin dC \), then \( [v] \neq 0 \) in \( H(V) \) and the non-degeneracy of \( \langle \cdot , \cdot \rangle^H \) implies that there is an \( h \in \mathcal{H} \) such that \( \langle v , h \rangle \neq 0 \). This contradicts \( v \in V^\perp \). Therefore, there is a \( w \in C \) such that \( v = dw \). For any \( h \in \mathcal{H} \) and \( c_1 , c_2 \in C \), we have

\[
\langle h + dc_1 + c_2 , w \rangle = \langle dc_1 , w \rangle = \pm \langle dw , c_1 \rangle = \pm \langle v , c_1 \rangle = 0.
\]

Therefore, it holds \( w \in V^\perp \).

(b) Because \( V^\perp \subseteq V \) is an acyclic subcomplex and we work over \( \mathbb{R} \), there exists a complementary subcomplex \( W \subseteq V \) (this elementary fact is proven in [Hai19, Lemma 6.1.12]). The restriction of \( \langle \cdot , \cdot \rangle \) to \( W \) is non-degenerate and since \( W \cong \mathcal{Q}(V) \) is of finite type, Lemma 5.2 asserts the existence of a Hodge decomposition \( W = \mathcal{H} \oplus dC' \oplus C' \). Let \( C'' \subseteq V^\perp \) be a complement of \( dV^\perp \) in \( V^\perp \). Set \( C := C' \oplus C'' \). It is easy to see that \( V = \mathcal{H} \oplus dC \oplus C \) is a Hodge decomposition.

Consider a Hodge decomposition \( V = \mathcal{H} \oplus dC \oplus C \) for a cyclic structure \( \langle \cdot , \cdot \rangle \) on a cochain complex \( (V , d) \). The standard Hodge homotopy is a map \( \mathcal{P} : V \to V \) of degree \(-1\) defined by

\[
\mathcal{P}(v) := \begin{cases} 
-c & \text{if } v = dc \text{ for some } c \in C, \\
0 & \text{for } v \in \mathcal{H} \oplus C.
\end{cases}
\]

If \( \pi^H : V \to \mathcal{H} \) denotes the canonical projection, then it holds

\[
[\mathcal{P} , d] = \mathcal{P} \circ d + d \circ \mathcal{P} = \pi^H - \mathbb{1}.
\]

Suppose that \( \langle \cdot , \cdot \rangle \) is a cyclic structure on a DGA \( (V , d , \wedge) \) and \( V = \mathcal{H} \oplus dC \oplus C \) is a Hodge decomposition. The smallest dg-subalgebra \( W \subseteq V \) such that \( \mathcal{H} \subseteq W \) and \( \mathcal{P}(W) \subseteq W \) is called the small subalgebra and is denoted by \( S_{\mathcal{H} , C}(V) \). It is easy to see that \( S = \mathcal{H} \oplus dS \oplus \mathcal{P}(S) \) is a Hodge decomposition of \( S := S_{\mathcal{H} , C}(V) \).

Lemma 3.4. Let \( \langle \cdot , \cdot \rangle \) be a cyclic structure on a DGA \( (V , d , \wedge) \). Let \( V = \mathcal{H} \oplus dC \oplus C \) be a Hodge decomposition and \( S := S_{\mathcal{H} , C}(V) \) the corresponding small subalgebra. Then the following holds:

(a) The vector space \( S \) is generated by Kontsevich-Soibelman-like evaluations of rooted binary trees with \( l \geq 1 \) leaves labeled with elements of \( \mathcal{H} \), internal nodes
labeled with $\land$ and edges labeled with either $\parallel$ or $P$.

(b) If $V$ is unital and $H(V)$ is connected, simply-connected and of finite type, then so is $S$.

Proof. (a) Denote by $\mathcal{T}$ the set of representatives of isomorphism classes of labeled trees and by $\langle \mathcal{T} \rangle$ the vector space generated by their evaluations. It holds $\langle \mathcal{T} \rangle \subset S$. As for $\supset$, it is enough to check that for any $T_1, T_2 \in \mathcal{T}$, it holds $T_1 \land T_2, P(T_1) \in \langle \mathcal{T} \rangle$. The product $T_1 \land T_2$ corresponds to the evaluation of the tree obtained by grafting $T_1$ and $T_2$ at the root, and hence it lies in $\langle \mathcal{T} \rangle$. As for $dT_1$, resp. $P(T_1)$, we imagine $d$, resp. $P$ being applied to the root of $T_1$ and use the Leibnitz identity and (4) to propagate them to the leaves. We also use that $P \circ P = 0$ and $dH = P(H) = 0$. The claim follows.

(b) Suppose that $H$ is of finite type and it holds $H^0 = \text{span}\{ 1 \}$ and $H^1 = 0$. Let $T \in \mathcal{T}$ be a labeled tree with $l$ leaves. Contracting the leaves labeled with $1$, we can assume that every harmonic form at a leaf has degree at least 2. Any rooted binary tree with $l$ leaves has $2l - 1$ edges. We can assume that the $l$ edges adjacent to the leaves are labeled with $\parallel$ since otherwise the evaluation of $T$ vanishes due to $P(H) = 0$. The least degree is achieved if the remaining edges are labeled with $P$. Therefore, the total degree $D$ satisfies $D \geq 2l - (l - 1) = l + 1$. It follows that $S$ is connected, simply-connected and of finite type. □

A differential Poincaré duality algebra (=: dPD-algebra) of degree $n \in \mathbb{N}_0$ is a non-negatively graded unital CDGA $(V, d, \land)$ of finite dimension equipped with an orientation $o$ in degree $n$ such that the induced cyclic structure $\langle v_1, v_2 \rangle = o(v_1 \land v_2)$ is non-degenerate. If $d = 0$, we call $V$ a Poincaré duality algebra (=: PD-algebra). A Poincaré DGA (=: PDGA) of degree $n \in \mathbb{N}_0$ is a non-negatively graded unital CDGA $(V, d, \land)$ whose homology $H(V)$ is equipped with an orientation $o^H$ making it into a Poincaré duality algebra of degree $n$. An oriented PDGA is a PDGA $V$ together with an orientation $o : V \to \mathbb{R}$ which induces the given orientation $o^H$ on $H(V)$. A dPD-algebra $V$ is of Hodge type by Lemma 3.2 and it is easy to see that the induced cyclic structure $\langle \cdot, \cdot \rangle^H$ on $H(V)$ is non-degenerate. Therefore, a dPD-algebra is canonically an oriented PDGA. Given PDGA’s $V_1$ and $V_2$ of the same degree, a PDGA-morphism $f : V_1 \to V_2$ is a DGA-morphism such that the induced map $f_* : H(V_1) \to H(V_2)$ preserves the pairing.

**Lemma 3.5.** Let $V_1$ and $V_2$ be dPD-algebras of degree $n$, and let $f : V_1 \to V_2$ be a PDGA-quasi-isomorphism. Then $f$ preserves $\langle \cdot, \cdot \rangle$ and is thus injective.
Proof. For $v_1, v_2 \in V$ with $\deg v_1 + \deg v_2 = n$, it holds

$$
\langle f(v_1), f(v_2) \rangle = \langle 1 \land f(v_1), f(v_2) \rangle = \langle f(v_1) \land f(v_2), 1 \rangle = \langle [f(v_1 \land v_2)], [1] \rangle^H \tag{\ast}
$$

where $[\cdot]$ denotes the cohomology class. The non-degeneracy of $\langle \cdot, \cdot \rangle$ and the fact that $V_1$ and $V_2$ are non-negatively graded imply $V_1^{n+1} = V_2^{n+1} = 0$. Therefore, we have $df(v_1 \land v_2) = d(v_1 \land v_2) = 0$, and $(\ast)$ hold.

\[\Box\]

4 Existence of extension of Hodge type

Let $\langle \cdot, \cdot \rangle$ be a cyclic structure of degree $n$ on a cochain complex $(V, d)$. If $H$ is a harmonic subspace and $C$ a coexact part such that $H \perp C$, then the decomposition $V = H \oplus dC \oplus C$ is called $H$-orthogonal. Suppose that $H(V)$ is of finite type and the induced cyclic structure $\langle \cdot, \cdot \rangle^H$ on $H(V)$ is non-degenerate. For any harmonic subspace $H$, there is the orthogonal projection $\pi^\perp : V \to H$ such that $\langle v, h \rangle = \langle \pi^\perp(v), h \rangle$ for all $v \in V$ and $h \in H$. Given any coexact part $C$, we can set

$$
C' := \{ c - \pi^\perp(c) \mid c \in C \} \tag{5}
$$

and obtain an $H$-orthogonal decomposition $V = H \oplus dC' \oplus C'$.

**Lemma 4.1.** Let $\langle \cdot, \cdot \rangle$ be a cyclic structure of degree $n$ on a cochain complex $(V, d)$ such that $V$ admits an $H$-orthogonal decomposition $V = H \oplus dC \oplus C$. Denote

$$
C^\perp := \{ c \in C \mid c \perp C \}
$$

and suppose that there is a complement $E$ of $C^\perp$ in $C$ and a map

$$
\rho : \bigoplus_{i \geq \lceil n/2 \rceil} E^i \longrightarrow dC
$$

such that for all $i \geq \lceil n/2 \rceil$, $e_1 \in E^{n-i}$, $e_2 \in E^i$ and $c \in C^{\perp n-i}$, the following holds:

$$
\langle e_1, \rho(e_2) \rangle = \langle e_1, e_2 \rangle, \tag{6}
$$

$$
\langle e, \rho(e_2) \rangle = 0. \tag{7}
$$
If \( n = 2k \) for some \( k \in \mathbb{N}_0 \), we suppose additionally that \( \dim E^k < \infty \). Then there is a coexact part \( \tilde{C} \) such that \( V = \mathcal{H} \oplus \mathcal{C} \oplus \tilde{C} \) is a Hodge decomposition.

**Proof.** Let \( \kappa : C \to dC \) be a map satisfying the following:

\[
\kappa(v) := \begin{cases} 
\rho(v) & \text{if } v \in E^i \text{ for } i > \left\lceil \frac{n}{2} \right\rceil, \\
0 & \text{if } v \in C^{\perp} \text{ for } i \geq \left\lceil \frac{n}{2} \right\rceil, \\
0 & \text{if } v \in C^i \text{ for } i < \left\lfloor \frac{n}{2} \right\rfloor.
\end{cases}
\]

The case of \( v \in E^k \) if \( n = 2k \) for some \( k \in \mathbb{N}_0 \) is specified as follows. The restriction of \( \langle \cdot, \cdot \rangle \) to \( E^k \) is non-degenerate. If \( k \) is even, then \( \langle \cdot, \cdot \rangle : E^k \otimes E^k \to \mathbb{R} \) is an inner product, and because \( \dim E^k < \infty \) by the assumption, there is an orthonormal basis \( \eta_1, \ldots, \eta_m \) of \( E^k \) for some \( m \in \mathbb{N} \). Set

\[
\kappa(\eta_i) := \frac{1}{2}\rho(\eta_i) \quad \text{for all } i \in \{1, \ldots, m\}. \tag{8}
\]

If \( k \) is odd, then \( \langle \cdot, \cdot \rangle : E^k \otimes E^k \to \mathbb{R} \) is a symplectic form, and there is a symplectic basis \( \eta_1, \theta_1, \ldots, \eta_m, \theta_m \) of \( E^k \) for some \( m \in \mathbb{N} \). We use the convention \( \langle \theta_i, \eta_j \rangle = \delta_{ij} \) for \( i, j = 1, \ldots, m \). Set

\[
\kappa(\eta_i) := \rho(\eta_i) \quad \text{and} \quad \kappa(\theta_i) := 0 \quad \text{for all } i \in \{1, \ldots, m\}. \tag{9}
\]

Let \( \tilde{C} := \{ c - \kappa(c) \mid c \in C \} \).

It is again a coexact part perpendicular to \( \mathcal{H} \). Let \( c_1, c_2 \in C \) with \( \deg c_1 + \deg c_2 = n \) and \( \deg c_1 \leq \deg c_2 \). Write \( c_1 = c_1^+ + e_1 \) and \( c_2 = c_2^+ + e_2 \) for \( c_1^+, c_2^+ \in C^\perp \) and \( e_1, e_2 \in E \). Then the following holds:

\[
\begin{align*}
\langle c_1 - \kappa(c_1), c_2 - \kappa(c_2) \rangle &= \langle c_1, c_2 \rangle - \langle \kappa(c_1), c_2 \rangle - \langle c_1, \kappa(c_2) \rangle \\
&= \langle e_1, e_2 \rangle - \langle \kappa(e_1), e_2 \rangle - \langle e_1, \kappa(e_2) \rangle \\
&= \langle e_1, e_2 \rangle - \langle \kappa(e_1), e_2 \rangle - \langle e_1, \kappa(e_2) \rangle \\
&\quad - \langle \kappa(e_1), c_2^+ \rangle - \langle e_1^+, \kappa(e_2) \rangle. \tag{8*}
\end{align*}
\]

It holds \((**)=0\) because of \((7)\). As for \((*)\), if \( \deg c_1 < \deg c_2 \), then

\[
(*) = \langle e_1, e_2 \rangle - \langle e_1, \kappa(e_2) \rangle = 0
\]

by \((8)\). If \( \deg c_1 = \deg c_2 = k \) and \( k \) is even, we plug in the orthonormal basis and
use (8) to get the following:
\[
e_1 = \eta_i, \ e_2 = \eta_j : \quad (\ast) = \langle \eta_i, \eta_j \rangle - \langle \kappa(\eta_i), \eta_j \rangle - \langle \eta_i, \kappa(\eta_j) \rangle
\]
\[
= \langle \eta_i, \eta_j \rangle - \langle \eta_j, \kappa(\eta_i) \rangle - \langle \eta_i, \kappa(\eta_j) \rangle
\]
\[
= \langle \eta_i, \eta_j \rangle - \frac{1}{2} \langle \eta_j, \rho(\eta_i) \rangle - \frac{1}{2} \langle \eta_i, \rho(\eta_j) \rangle
\]
\[
= \langle \eta_i, \eta_j \rangle - \frac{1}{2} \langle \eta_j, \eta_i \rangle - \frac{1}{2} \langle \eta_i, \eta_j \rangle = 0.
\]
If \( k \) is odd, we plug in the symplectic basis and use (9) to get the following:
\[
e_1 = \eta_i, \ e_2 = \eta_j : \quad (\ast) = \langle \eta_j, \kappa(\eta_i) \rangle - \langle \eta_i, \kappa(\eta_j) \rangle
\]
\[
= \langle \eta_j, \eta_i \rangle - \langle \eta_i, \kappa(\eta_j) \rangle = 0,
\]
\[
e_1 = \theta_i, \ e_2 = \eta_j : \quad (\ast) = \langle \theta_i, \eta_j \rangle - \langle \eta_i, \kappa(\eta_j) \rangle
\]
\[
= \langle \theta_i, \eta_j \rangle - \langle \theta_i, \eta_j \rangle = 0,
\]
\[
e_1 = \theta_i, \ e_2 = \theta_j : \quad (\ast) = 0.
\]
This shows that \( \tilde{C} \perp \tilde{C} \).
\]

**Lemma 4.2.** Let \((V, d, \wedge, o)\) be an oriented PDGA of degree \( n \geq 1 \). Suppose that \( V \)

is connected, of finite type and that the induced cyclic structure \( \langle \cdot, \cdot \rangle^H \) on \( H(V) \) is non-degenerate.

(a) Suppose that for some \( l > \left\lceil \frac{n}{2} \right\rceil \), there is an \( \mathcal{H} \)-orthogonal decomposition \( V = H \oplus dC \oplus C \), a complement \( E \) of \( C^\perp \) in \( C \) and a map
\[
\rho : E^{[n/2]} \oplus \cdots \oplus E^{l-1} \rightarrow dC
\]
for which (8) and (9) hold for all \( i \in \{ \left\lceil \frac{n}{2} \right\rceil, \ldots, l-1 \} \). Then we can tensor \( V \) with an acyclic Sullivan DGA with finitely many generators in degrees \( l-1 \) and \( l \) and obtain a connected PDGA \((\tilde{V}, \tilde{d}, \tilde{\wedge})\) with an orientation \( \tilde{o} : \tilde{V} \rightarrow \mathbb{R} \) extending \( o \) over the inclusion \( V \hookrightarrow \tilde{V} \), an \( \mathcal{H} \)-orthogonal decomposition \( \tilde{V} = \tilde{H} \oplus \tilde{dC} \oplus \tilde{C} \), a complement \( \tilde{E} \) of \( C^\perp \) in \( \tilde{C} \) and a map
\[
\tilde{\rho} : \tilde{E}^{[n/2]} \oplus \cdots \oplus \tilde{E}^{l} \rightarrow \tilde{dC}
\]
for which (8) and (9) hold for all \( i \in \{ \left\lceil \frac{n}{2} \right\rceil, \ldots, l-1 \} \).
(b) If \( n \geq 3 \) and \( V = H \oplus dC \oplus C \) is a non-degenerate decomposition satisfying
\[
V^1 \wedge C^{n-l} \subset H^{n-l+1} \oplus dC^{n-l} \oplus C^{n-l+1},
\]
then (a) holds also for \( l = \left\lceil \frac{n}{2} \right\rceil \). That means that we can construct \((\tilde{V}, \tilde{d}, \tilde{\wedge}), \tilde{o}, \tilde{H}, \tilde{C}, \tilde{E}, \tilde{\rho} \) such that (8) and (9) hold for \( i = \left\lceil \frac{n}{2} \right\rceil \).
Proof. (a) Set \( m := \dim E^l \) and consider the (non-minimal) Sullivan DGA
\[
\Lambda := \Lambda(w_1, \ldots, w_m, z_1, \ldots, z_m) \quad \text{with} \quad \deg w_i = l - 1, \deg z_i = l, \, dw_i = dz_i = 0, \\
\quad \text{and} \, \, d w_i = z_i \, \, \text{for all} \, \, i \in \{1, \ldots, m\}.
\]

Let \( \hat{V} := V \otimes \Lambda \). We will denote the internal \( \otimes \) by \( \wedge \). Note that this is the final \( \hat{V} \) if \( V^1 = 0 \).

Construction of \( \hat{o} \): It holds
\[
\Lambda = \bigoplus_{k=0}^{\infty} \Lambda_k \quad \text{with} \quad \Lambda_k := \bigoplus_{r, m \geq 0 \atop r + m = k} \Lambda_r w \otimes \Lambda_m z,
\]
where \( \Lambda_r w \) and \( \Lambda_m z \) are the vector spaces generated by monomials \( w_I = w_{i_1} \ldots w_{i_r} \) and \( z_J = z_{j_1} \ldots z_{j_m} \) for multiindices \( I = \{i_1, \ldots, i_r\} \) and \( J = \{j_1, \ldots, j_m\} \), respectively.

It holds
\[
V = H \oplus dE \oplus dC_{\perp} \oplus E \oplus C_{\perp}.
\]

Consider the direct sum decomposition of \( \hat{V} \) obtained from (11) and (12) using the distributivity of \( \wedge \) and \( \oplus \). Let \( \xi_1, \ldots, \xi_m \) be a basis of \( E^l \) and \( \xi_1^*, \ldots, \xi_m^* \) its dual basis in \( E^{n-l} \). Define a map \( \hat{o} : \hat{V} \to \mathbb{R} \) by 0 on the complement of \( V \oplus E^{n-l} \wedge \Lambda_1 z \oplus dE^{n-l} \wedge \Lambda_1 w \) in \( \hat{V} \) and by
\[
\hat{o}(v) := o(v) \quad \text{for all} \, \, v \in V, \\
\hat{o}(\xi^i \wedge z_j) := o(\xi^i \wedge \xi_j) \quad \text{and} \\
\hat{o}(d \xi^i \wedge w_j) := (-1)^{\deg \xi^i + 1} o(\xi^i \wedge \xi_j) \quad \text{for all} \, \, i, j \in \{1, \ldots, m\}.
\]

In order to show that \( \hat{o} \) is an orientation, we must check that \( \hat{o} \neq 0 \) and \( \hat{o} \circ d = 0 \).

The first condition is clear because \( \hat{o} \) restricts to \( o \) on \( V \). As for the second condition, denote
\[
\hat{\nu}_k := V \wedge \Lambda_k \quad \text{for} \, \, k \geq 0.
\]

It holds \( \hat{V} = \bigoplus_{k=0}^{\infty} \hat{\nu}_k \) and \( \hat{d} \hat{\nu}_k \subset \hat{\nu}_k \) for all \( k \geq 0 \). Thus, \( \hat{d} \hat{V} = \bigoplus_{k=0}^{\infty} \hat{d} \hat{\nu}_k \). By the definition of \( \hat{o} \), it holds \( \hat{d} \hat{\nu}_0 = dV \subset \ker o \subset \ker \hat{o} \) and \( \bigoplus_{k=2}^{\infty} \hat{d} \hat{\nu}_k \subset \bigoplus_{k=2}^{\infty} \hat{\nu}_k \subset \ker \hat{o} \).

As for \( \hat{d} \hat{\nu}_1 \), we write
\[
\hat{\nu}_1 = \text{span}\{v \wedge w_j, v \wedge z_j \mid v \in V, j = 1, \ldots, m\}
\]
and compute
\[
\hat{d} \hat{\nu}_1 = \text{span}\{ \hat{d}(v \wedge w_j), \hat{d}(v \wedge z_j) \mid v \in V, j = 1, \ldots, m\} = \text{span}\{ dv \wedge w_j + (-1)^{\deg v} v \wedge z_j \mid v \in V, j = 1, \ldots, m\}.
\]
We write \( v \in V^{n-l} \) as \( v = h + dc + c^\perp + \sum_{i=1}^m \alpha_i \xi^i \) for \( h \in H^{n-l} \), \( c \in C^{n-l-1} \), \( c^\perp \in C^\perp \) and \( \alpha_i \in \mathbb{R} \), and compute for every \( j = 1, \ldots, m \) the following:

\[
\hat{o}(dv \wedge w_j) = \hat{o}(dc^\perp \wedge w_j + \sum_{i=1}^m \alpha_i d\xi^i \wedge w_j)
\]

\[
= \sum_{i=1}^m \alpha_i \hat{o}(d\xi^i \wedge w_j)
\]

\[
= \sum_{i=1}^m (-1)^{\deg \xi^i + 1} \alpha_i \hat{o}(\xi^i \wedge \xi_j)
\]

\[
= (-1)^{n-l+1} \sum_{i=1}^m \alpha_i \hat{o}(\xi^i \wedge z_j)
\]

\[
= (-1)^{n-l+1} \hat{o}\left((h + dc + c^\perp) \wedge z_j + \sum_{i=1}^m \alpha_i \xi^i \wedge z_j\right)
\]

\[
= (-1)^{\deg v + 1} \hat{o}(v \wedge z_j).
\]

Consequently, it holds \( \hat{d} \hat{V}_1 \subset \ker \hat{o} \).

**Construction of \( \hat{H} \):** It holds \( \hat{H}(\Lambda) = 0 \), and hence \( H(\hat{V}) \simeq H(V) \otimes H(\Lambda) = H(V) \) by Künneth’s formula. Because \( H \subset \ker \hat{d} \), \( \hat{H} \cap \im \hat{d} = 0 \) and \( \dim(\hat{H}) = \dim H(V) = \dim H'(V) \) for every \( i \geq 0 \), \( H \) is a harmonic subspace. Therefore,

\[
\hat{H} := H
\]

is a harmonic subspace in \( \hat{V} \).

**Construction of \( \hat{C} \):** First, because the inclusion \( V \hookrightarrow \hat{V} \) onto \( \hat{V}_0 \) is a quasi-isomorphism and \( H(\hat{V}) \cong \bigoplus_{k \geq 0} H(V_k) \), it holds \( H(V_k) = 0 \) for all \( k \geq 1 \). For \( k \geq 2 \), let \( \hat{C}_k \subset \hat{V}_k \) be an arbitrary complement of \( \ker \hat{d} \cap \hat{V}_k = \hat{d} \hat{V}_k \) in \( \hat{V}_k \). Set

\[
\hat{C} := C \oplus (V \wedge \Lambda_1 w)' \oplus \bigoplus_{k \geq 2} \hat{C}_k,
\]

where \( (V \wedge \Lambda_1 w)' \) is defined similarly as \( (\hat{V}_1 \wedge \Lambda_1 w)' \). The fact that \( (V \wedge \Lambda_1 w)' \), or equivalently \( V \wedge \Lambda_1 w \), is a complement of \( \ker \hat{d} \cap \hat{V}_1 = \hat{d} \hat{V}_1 \) in \( \hat{V}_1 \) follows from \( (13) \). Clearly, \( \hat{C} \perp \hat{H} \).

**Degreewise description of \( \hat{C} \) and \( \hat{C}^\perp \):** For all \( n-l \leq i \leq l \), it holds

\[
\hat{C}^i = \begin{cases} 
C^{n-l} & \text{for } i = n-l, \\
C^i & \text{for } n-l < i < l-1, \\
C^{l-1} \oplus \Lambda_1 w & \text{for } i = l-1, \\
C^l \oplus (V^1 \wedge \Lambda_1 w)' & \text{for } i = l.
\end{cases}
\]

(14)
Then this is the assumption of this lemma, and so we can repeat the process of adding the exact partners to non-degnerates for $\hat{E}$ of $\hat{C}$ satisfying $E \subset \hat{E} \subset C \oplus (V \wedge \Lambda_1 w)'$.

Comparing (14) to (15), we see that $\hat{E}^i = E^i$ for $n - l < i \leq l - 1$. As for $\hat{E}^l$ and $\hat{E}^{n-1}$, suppose that

$$V^1 \wedge \Lambda_1 w \perp C^{l-1}.$$  

(16)

Then $\hat{C}^{l-1} = C^{l-1}$, and hence $E^{n-1} = E^{n-1}$. The restriction of $\langle \cdot, \cdot \rangle$ to $\hat{E}^{n-1} \oplus \hat{E}^l$ is non-degenerate in both variables, and hence it induces an isomorphism $\hat{E}^l \simeq \hat{E}^{n-1}$. It holds $E^{n-1} \simeq E^l$ by the assumptions, and thus $\hat{E}^l = E^l$ for dimensional reasons. Suppose that (16) does not hold. We set $\rho := \rho : \hat{E}^{[n/2]} \oplus \cdots \oplus \hat{E}^{l-1} = E^{[n/2]} \oplus \cdots \oplus E^{l-1} \rightarrow \hat{\delta} \hat{C}$ and see that $\hat{V}$, $\hat{C}$, $\hat{E}$ and $\rho$ satisfy (6) and (7) for all $i \in \{[\frac{n}{2}], \ldots, l - 1\}$. This is the assumption of this lemma, and so we can repeat the process of adding the process of adding exact partners to non-degnerates for $\hat{V}$. Because

$$\dim \hat{C}^{l-1} < \dim C^{l-1},$$

(17)

condition (16) will be satisfied after finitely many steps.

Construction of $\hat{\rho}$: We define $\rho : \hat{E}^{[\frac{n}{2}]} \oplus \cdots \oplus \hat{E}^{l-1} \rightarrow \hat{\delta} \hat{C}$ by $\hat{\rho} := \rho$ on $\hat{E}^{[\frac{n}{2}]} \oplus \cdots \oplus \hat{E}^{l-1}$ and by

$$\rho(z_i) := z_i \quad \text{for all } i = 1, \ldots, m.$$  

The validity of (6) and (7) for $i \in \{[\frac{n}{2}], \ldots, l\}$ follows from the construction.

(b) The proof of (a) works until (14) and (15). For $n = 2l - 1$, it holds

$$\hat{C}^{l-1} = \hat{C}^{n-1} = C^{n-1} \oplus \Lambda_1 w,$$

$$\hat{C}^l = C^l \oplus (V^1 \wedge \Lambda_1 w)' \oplus (\Lambda_2 w)^l,$$

and for $n = 2l$, we have

$$\hat{C}^l = \hat{C}^{n-1} = C^{n-1} \oplus (V^1 \wedge \Lambda_1 w)' \oplus (\Lambda_2 w)^l.$$  

Note that $(\Lambda_2 w)^l = 0$ unless $l = 2$ and $n \in \{3, 4\}$. We can not achieve (16) via repeated extension because (17) must not hold. Instead, (16) is implied by the assumption (10)
and the construction of \( \hat{\omega} \). For \( n = 2l - 1 \), we have
\[
\hat{C}^{n-1} = C^{l-1} \oplus \Lambda_1 w,
\]
and for \( n = 2l \), it holds
\[
\hat{C}^{n-1} = C^{n-1} \oplus (V^1 \wedge \Lambda_1 w)' \oplus (\Lambda_2 w)' .
\]
The proof of (a) goes on because \( \hat{E}^{n-1} = E^{n-1} \).
\[\square\]

Notice that the problem with \( n = 0 \) in (a) and with \( n = 1, 2 \) in (b) is that \( \lfloor \frac{n}{2} \rfloor = 1 \), and hence \( w_i \) would land in degree 0.

We say that a PDGA \( \hat{V} \) is an extension of a PDGA \( V \) if there is an injective PDGA-quasi-isomorphism \( \iota : V \hookrightarrow \hat{V} \). We say that an extension \( \hat{V} \) retracts onto \( V \) if there is a PDGA-quasi-isomorphism \( p : \hat{V} \rightarrow V \) such that \( p \circ \iota = 1 \).

**Proposition 4.3.** Let \( V \) be a PDGA of degree \( n \) which is connected, simply-connected and of finite type. Then it admits an oriented extension \( \hat{V} \) of Hodge type which is connected, of finite type and retracts onto \( V \). If \( \hat{V} \) is oriented, then we can achieve that the inclusion \( \iota : V \hookrightarrow \hat{V} \) is orientation preserving. If \( n \geq 5 \), then we can achieve that \( \hat{V} \) is simply-connected.

**Proof.** For \( n = 0 \), any connected PDGA is of Hodge type. For \( n = 1 \), there is no simply-connected PDGA. For \( n = 2 \), if \( V = \mathcal{H} \oplus dC \oplus C \) is any \( \mathcal{H} \)-orthogonal decomposition, then \( V^0 = \text{span}\{1\} \), \( V^1 = 0 \), \( V^2 = \mathcal{H}^2 \oplus C^2 \) by the assumptions, and hence it is a Hodge decomposition. Suppose that \( n \geq 3 \). Pick an \( \mathcal{H} \)-orthogonal decomposition \( V = \mathcal{H} \oplus dC \oplus C \) and apply (b) of Lemma 4.2 to get \( \hat{V}, \mathcal{H}, \hat{C}, \hat{E} \) and \( \hat{\rho} : \hat{E}^{[n/2]} \rightarrow d\hat{C} \).

Condition (10) is trivially satisfied because \( V^1 = 0 \). Notice that for \( n \in \{3, 4\} \), \( w_i \) are added in degree 1. A recursive application of (a) of Lemma 4.2 for \( l = \lfloor \frac{n}{2} \rfloor + 1 \), \( \ldots, n \) gives \( \hat{V}, \hat{H}, \hat{C}, \hat{E} \) and \( \hat{\rho} : \hat{E}^{[n/2]} \oplus \cdots \oplus \hat{E}^n \rightarrow d\hat{C} \). Lemma 4.1 implies that \( \hat{V} \) is of Hodge type. Because \( \hat{V} \) arose from \( V \) by repeated tensoring with acyclic Sullivan DGA's, it retracts onto \( V \).
\[\square\]

## 5 Poincaré duality models

We say that a dPD-algebra \( \mathcal{M} \) is a Poincaré duality model (= PD-model) of a PDGA \( V \) if there are PDGA's \( Z_1, \ldots, Z_k \) for some \( k \in \mathbb{N} \) and a zig-zag of PDGA-quasi-isomorphisms
\[
V \leftarrow Z_1 \rightarrow Z_2 \leftarrow \cdots \rightarrow Z_{k-1} \leftarrow Z_k \rightarrow \mathcal{M} .
\]
Such zig-zag is called a weak homotopy equivalence of PDGA's. Let \((V, d, \wedge, o)\) be an oriented PDGA of Hodge type such that \( \mathcal{H}^0(V) = \text{span}\{1\} \) and \( \mathcal{H}^1(V) = 0 \). Given a Hodge decomposition \( V = \mathcal{H} \oplus dC \oplus C \), the small subalgebra \( \mathcal{S}_{\mathcal{H}, C}(V) \) is connected,
simply-connected and of finite type by Lemma 3.4. Because $S_{H,C}(V)$ is of Hodge type, Lemma 3.3 implies that the canonical projection $S_{H,C}(V) \to Q(S_{H,C}(V))$ is a quasi-isomorphism. Therefore, we obtain the following PD-model of $V$:

$$S_{H,C}(V) \to Q(S_{H,C}(V)) =: M,$$

The following is our version of [LS08, Theorem 1.1].

**Proposition 5.1.** A PDGA $V$ which satisfies $H^0(V) = \text{span}\{1\}$ and $H^1(V) = 0$ admits a connected and simply-connected PD-model $M$ in the form

$$\Lambda U \to V \quad Q(\Lambda U) =: M,$$

where $\Lambda U$ is an oriented extension of Hodge type of a Sullivan minimal model $\Lambda U \to V$ which is connected, simply-connected, and of finite type.

**Proof.** For $n = 0$, the inclusion $\text{span}\{1\} \hookrightarrow V$ is a PD-model. For $n = 1$, no PDGA satisfies the assumptions. For $n \geq 2$, a Sullivan minimal model $\Lambda U \to V$ exists, is connected, simply-connected, of finite type, and it holds $d(\Lambda U)^2 = 0$; see [FOT08, Theorem 2.24]. Its homology inherits an orientation from $H(V)$, and we extend it arbitrarily to $\Lambda U$. If we denote $W := \Lambda U$, then the following holds for an $H$-orthogonal decomposition $W = H \oplus dC \oplus C$ for $n \in \{2, 3, 4\}$:

- $n = 2$: $W^2 = H^2$, $W^1 = 0$, $W^0 = \text{span}\{1\}$
- $n = 3$: $W^3 = H^3$, $W^2 = 0$, $W^1 = 0$, $W^0 = \text{span}\{1\}$
- $n = 4$: $W^4 = H^4 \oplus dC^3 \oplus C^4$, $W^3 = C^3$, $W^2 = \Lambda U^2$, $W^1 = 0$, $W^0 = \text{span}\{1\}$.

Therefore, $W = H \oplus dC \oplus C$ is automatically a Hodge decomposition, and thus we can take $\Lambda U = \Lambda U$. For $n \geq 5$, apply Proposition 4.3 to $\Lambda U$ to obtain an oriented extension of Hodge type $\Lambda U$ which is connected, simply-connected, of finite type and retracts onto $\Lambda U$. Lemma 3.3 asserts that the canonical projection $\Lambda U \to Q(\Lambda U)$ is an orientation preserving quasi-isomorphism.

The next remark shows that the zig-zag (19) is optimal in the sense that it is not always possible to shorten it.
Remark 5.2. The de Rham algebra $(\Omega, d, \wedge)$ of the oriented simply-connected closed 4-manifold $#^7\mathbb{CP}^2$, where $#$ denotes the connected sum, oriented by the integration $\int_{#^7\mathbb{CP}^2} : \Omega \to \mathbb{R}$ does not admit a PD-model $M$ with just one arrow $h : M \to \Omega$ or $h' : \Omega \to M$. The computation in the proof of Lemma 3.5 shows that $h$ and $h'$ must preserve the cyclic structure and thus be injective. This is not possible for $h'$ as $\dim \Omega = \infty$ and $\dim M < \infty$. As for $h$, we can assume that $M \subset \Omega$. Let $H \subset M$ be an arbitrary harmonic subspace in $M$. The cohomology ring $H(\Omega)$ can be computed with the help of the Mayer-Vietoris sequence. One obtains $H(\Omega) = \text{span}\{1\} \oplus \text{span}\{k_0, \ldots, k_6\} \oplus \text{span}\{v\}$, where $\deg v = 4$, $\deg k_i = 2$ and it holds $k_i \wedge k_j = 0$ if $i \neq j$ and $k_i \wedge k_i = \pm v$ for all $i, j \in \{0, \ldots, 6\}$. The sign depends on the orientation of the corresponding $\mathbb{CP}^2$-factor. We normalize $\langle 1, v \rangle = 1$. Because $H \simeq H(\Omega)$, it follows that $H = \text{span}\{1\} \oplus \text{span}\{k_0, \ldots, k_6\} \oplus \text{span}\{v\}$ for some closed $v \in \Omega^4$ and $k_i \in \Omega^2$ such that $[v] = v$ and $[k_i] = k_i$ for all $i \in \{0, \ldots, 6\}$.

In general, it holds $k_i \wedge k_j = d\eta_{ij}$ and $k_i \wedge k_i = v + d\xi_i$ for some $\eta_{ij}, \xi_i \in M^3$. If $f \in C^\infty(#^7\mathbb{CP}^2)$ is not a constant, then $f^k$ ($k \in \mathbb{N}_0$) are linearly independent over $\mathbb{R}$, and because $\dim M < \infty$, it must hold $M^0 = \text{span}\{1\}$. By Poincaré duality, it holds $M^3 = \text{span}\{V\}$. Therefore, for all $p \in #^7\mathbb{CP}^2$, the following relations must hold:

\[
\begin{align*}
k_i(p) \wedge k_j(p) &= 0, \\
k_i(p) \wedge k_i(p) &= \pm v(p).
\end{align*}
\]

Given $\lambda_i \in \mathbb{R}$ for $i \in \{0, \ldots, 6\}$, then

\[
\left(\sum_{i=0}^{6} \lambda_i k_i(p)\right) \wedge k_j(p) = \pm \lambda_j v(p).
\]

Therefore, $v(p) \neq 0$ implies that $k_0(p), \ldots, k_6(p)$ are linearly independent. However, this is impossible because

\[
\dim(\Lambda^2 T^* \#^7\mathbb{CP}^2) = \binom{4}{2} = 6 < 7.
\]

This finishes the argument.

The next remark shows that two PD-models of the same PDGA must not be isomorphic even if they arise as in (18).

Remark 5.3. We will show that the de Rham algebra $(\Omega, d, \wedge)$ of the compact simply-connected 35-dimensional Lie group $SU(6)$ oriented by the integration $\int_{SU(6)} : \Omega \to \mathbb{R}$
We claim that the integral can be made non-zero by a choice of $\xi$ which exists by [Fio+19, Remark 2.6]. Lemma 3.4 implies that the following elements is precisely the subspace of harmonic forms $\eta$ differential forms $H(\Omega)$ small subalgebras corresponding to different Hodge decompositions. The cohomology ring $H(\Omega)$ is freely generated by single elements in degrees $3, 5, \ldots, 11$; see [MT91, Corollary 3.11]. There exist a bi-invariant Riemannian metric $g$ and bi-invariant differential forms $\eta_3, \eta_5, \ldots, \eta_{11} \in \Omega$ in the corresponding degrees such that the free algebra

$$H_1 := \Lambda(\eta_3, \ldots, \eta_{11})$$

is precisely the subspace of harmonic forms $\{\omega \in \Omega \mid d\omega = d^*\omega = 0\}$ for the metric $g$; see [FOT08, Chapter 1]. For $\xi_6 \in \Omega^6$ and $\xi_8 \in \Omega^8$, which are going to be specified later, consider

$$\eta_7 := \eta_7 + d\xi_6 \quad \text{and} \quad \eta_9 := \eta_9 + d\xi_8.$$ 

Let $H_2$ be the vector space obtained from $H_1$ by replacing $\eta_7$ and $\eta_9$ with $\eta_7'$ and $\eta_9'$, respectively. We emphasize that we replace just the vectors and not their products; e.g., $\eta_7' \wedge \eta_9'$ might not be. The small subalgebra $H_1$ of the Riemannian Hodge decomposition $\Omega = H_1 \oplus d\Omega \oplus d^*\Omega$ clearly satisfies $H_1 = H_1$. Let $H_2$ be the small subalgebra corresponding to a Hodge decomposition $V = H_2 \oplus dC \oplus C$ which exists by [Fio+19, Remark 2.6]. Lemma 3.4 implies that the following elements in degrees 15, resp. 20 must be contained in $H_2$ ($P_2$ denotes the standard Hodge homotopy for the second Hodge decomposition):

$$y := P_2(\eta_7' \wedge \eta_9' - \eta_7 \wedge \eta_9) = P_2(d(\xi_6 \wedge \eta_9 - \eta_7 \wedge \xi_8 + \xi_6 \wedge d\xi_8)),$$

$$z := \eta_9' \wedge \eta_{11} - \eta_9 \wedge \eta_{11} = d(\xi_8 \wedge \eta_{11}).$$

Using Stokes’ theorem and $d \circ P_2 = -\pi_{\text{im}} d$, we get

$$\langle y, z \rangle = \int_{SU(6)} P_2(d(\xi_6 \wedge \eta_9)) \wedge d(\xi_8 \wedge \eta_{11})$$

$$= -\int_{SU(6)} d(\xi_6 \wedge \eta_9 - \eta_7 \wedge \xi_8 + \xi_6 \wedge d\xi_8) \wedge \xi_8 \wedge \eta_{11}$$

$$= -\int_{SU(6)} d\xi_6 \wedge \eta_9 \wedge \xi_8 \wedge \eta_{11} - \int_{SU(6)} \frac{1}{2} d((\eta_7 + d\xi_6) \wedge \xi_8 \wedge \eta_{11})$$

$$= -\int_{SU(6)} d\xi_6 \wedge \xi_8 \wedge \eta_9 \wedge \eta_{11}.$$ 

We claim that the integral can be made non-zero by a choice of $\xi_6$ and $\xi_8$. Because $[\eta_9 \wedge \eta_{11}] \neq 0$ in $H(\Omega)$, there is a $p \in SU(6)$ such that $\eta_9(p) \wedge \eta_{11}(p) \neq 0$. Pick local coordinates $(x^i)$ around $p$ such that $x^i(p) \neq 0$ for all $i \in \{1, \ldots, 35\}$. Write $\eta_9(p) \wedge \eta_{11}(p) = \sum_{I \subseteq \{1, \ldots, 35\}, |I| = 20} \alpha_I \, dx^I$ and suppose that $\alpha_{I_0} \neq 0$ for some $I_0 \subseteq \{1, \ldots, 35\}$ with $|I_0| = 20$. Consider the complement $J$ of $I_0$ in $\{1, \ldots, 35\}$ and write $J = J_1 \cup J_2$ for some $J_1 = \{J_{11}, \ldots, J_{17}\}$ and $J_2 = \{J_{21}, \ldots, J_{28}\}$. Locally, set

$$\xi_6 := x^{J_{11}} \, dx^{J_{11} \setminus \{J_{11}\}} \quad \text{and} \quad \xi_8 := dx^{J_2}.$$
Multiplication with a bump function which equals 1 on a neighborhood of \( p \) and 0 on a neighborhood of the complement of the coordinate chart gives globally defined forms on \( SU(6) \). By the construction, the integrand \( d\xi_6 \wedge \xi_8 \wedge \eta_9 \wedge \eta_{11} \) is non-zero around \( p \). Because the \( \langle \cdot, \cdot \rangle \) is non-degenerate, there is a function \( f \in C^\infty(SU(6)) \) such that

\[
\langle f, d\xi_6 \wedge \xi_8 \wedge \eta_9 \wedge \eta_{11} \rangle = \int_{SU(6)} f d\xi_6 \wedge \xi_8 \wedge \eta_9 \wedge \eta_{11} \neq 0.
\]

Replacing \( \xi_8 \) with \( f \xi_8 \) makes \( \langle y, z \rangle \) non-zero. Consider the canonical projection \( \pi^\mathbb{Q} : \mathbb{S}_2 \to Q(\mathbb{S}_2) \). Because \( \langle y, z \rangle \neq 0 \), the element \( \pi^\mathbb{Q}(z) \in Q(\mathbb{S}_2)^{20} \) is non-zero. Because \( \pi^\mathbb{Q} \) is a chain map and \( z \) exact, \( \pi^\mathbb{Q}(z) \) is exact too. Because \( \pi^\mathbb{Q} \) is a quasi-isomorphism and \( \eta_9 \wedge \eta_{11} \) generates non-trivial homology, \( \pi^\mathbb{Q}(\eta_9 \wedge \eta_{11}) \) does that too. It follows that the vectors \( \pi^\mathbb{Q}(z) \) and \( \pi^\mathbb{Q}(\eta_9 \wedge \eta_{11}) \) can not be multiples of each other, and hence \( \dim Q(\mathbb{S}_2)^{20} \geq 2 \). On the other hand, it holds \( Q(\mathbb{S}_1)^{20} = H^1_1 = \langle \eta_9 \wedge \eta_{11} \rangle \). This shows that \( Q(\mathbb{S}_1) \) and \( Q(\mathbb{S}_2) \) can not be isomorphic.

Instead of “strict uniqueness”, we have “weak uniqueness” of PD-models. The following proposition is our version of [LS08, Theorem 7.1].

**Proposition 5.4.** Let \( V_1 \) and \( V_2 \) be connected and simply-connected dPD-algebras of degree \( n \) which are weakly homotopy equivalent to PDGA’s. If \( H^2(V_1) = H^2(V_2) = 0 \), then there is a connected and simply-connected dPD-algebra \( V_3 \) and injective orientation preserving PDGA-quasi-isomorphisms

\[
\begin{array}{c}
V_1 \\
h_1 \downarrow \\
V_3 \\
\downarrow h_2 \\
V_2.
\end{array}
\]

**Proof.** For \( n = 0 \), it holds \( V_1 = V_2 = \text{span}\{1\} \). For \( n = 1 \), no simply-connected dPD-algebra exists. For each \( n \in \{2, 3, 4\} \), it follows from \( V_1 = V_2 = \text{span}\{1\}, V_1^1 = V_2^1 = 0 \) and Poincaré duality that \( V_1 = H(V_1) \) and \( V_2 = H(V_2) \). Given a zig-zag of PDGA-quasi-isomorphisms

\[
V_1 \leftarrow Z_1 \rightarrow Z_2 \leftarrow Z_3 \rightarrow Z_4 \leftarrow \cdots \leftarrow Z_k \rightarrow V_2,
\]

let \( h : H(V_1) \rightarrow H(V_2) \) be the induced map on homology. Then \( V_3 := H(V_2), h_1 := h \) and \( h_2 := 1 \) does the job. Suppose that \( n \geq 5 \) and consider a Sullivan minimal model \( \Lambda U \rightarrow Z_2 \) of the DGA \( Z_2 \) from the zig-zag \((20)\). Because \( H^0(Z_2) \simeq \mathbb{R} \) and \( H^1(Z_2) = H^2(Z_2) = 0 \), the inductive construction of the Sullivan minimal model shows that \( U^1 = U^2 = 0 \). The Lifting Lemma [FOT08, Lemma 2.15] asserts the existence of DGA-quasi-isomorphisms \( \Lambda U \rightarrow Z_1 \) and \( \Lambda U \rightarrow Z_3 \) such that the following diagram

```
commutes up to homotopy of DGA’s:

\[
\begin{array}{ccc}
\Lambda U & \rightarrow & \Lambda U \\
\downarrow & & \downarrow \\
Z_1 & \rightarrow & Z_2 \\
\downarrow & & \downarrow \\
Z_2 & \rightarrow & Z_3 \\
\downarrow & & \downarrow \\
Z_3 & \rightarrow & Z_4
\end{array}
\]

The diagram commutes strictly at the level of homology, and hence there is an orientation on \(H(\Lambda U)\) such that the three vertical arrows become PDGA-quasi-isomorphisms. Hence, we can replace the segment \(V_1 \leftarrow Z_1 \rightarrow Z_2 \leftarrow Z_3 \rightarrow Z_4\) in (20) by the shorter segment \(V_1 \leftarrow \Lambda U \rightarrow Z_4\). Repeating this process, (20) can be shortened to

\[
\begin{array}{ccc}
\Lambda U & \rightarrow & \Lambda U \\
\downarrow & & \downarrow \\
V_1 & \rightarrow & V_2
\end{array}
\]

where \(H(\Lambda U)\) is oriented, i.e., \(\Lambda U\) is a PDGA, and \(f_1\) and \(f_2\) are PDGA-quasi-isomorphisms. We will now closely follow the proof of [LS08, Theorem 7.1]. The only difference is that we take care of orientations and use the oriented extension of Hodge type from Proposition 4.3 instead of the extension obtained by killing the orphans.

Consider the relative Sullivan minimal model of the multiplication \(\mu : \Lambda U \otimes \Lambda U \rightarrow \Lambda U\). By [FOT08, Example 2.48], it is given by the commutative diagram

\[
\begin{array}{ccc}
\Lambda U & \rightarrow & \Lambda U \otimes \Lambda U \\
\downarrow & & \downarrow \\
M & = & \Lambda U \otimes \Lambda U \otimes \Lambda(sU),
\end{array}
\]

where \(\iota\) is the inclusion into the first two factors, which is a cofibration, \(p\) is a surjective quasi-isomorphism and \(s\) denotes the suspension defined by \((sU)^i = U^{i+1}\). For completeness, given \(u \in U\), the differential \(D\) on \(M\) satisfies \(D(u \otimes 1 \otimes 1) = du \otimes 1 \otimes 1\), \(D(1 \otimes u \otimes 1) = 1 \otimes du \otimes 1\) and \(D(1 \otimes 1 \otimes su) = u \otimes 1 \otimes 1 \otimes u \otimes 1 + \gamma_u\) for a decomposable element \(\gamma_u \in M\). We orient \(H(M)\) such that \(p\) becomes a PDGA-quasi-isomorphism. For \(i \in \{1, 2\}\), consider the commutative diagram

\[
\begin{array}{ccc}
\Lambda U & \rightarrow & \Lambda U \otimes \Lambda U \\
\downarrow & & \downarrow \\
\Lambda U \otimes \Lambda U & \rightarrow & \Lambda U \otimes \Lambda U \\
\downarrow & & \downarrow \\
M & \rightarrow & M \otimes_{\Lambda U \otimes \Lambda U} (V_1 \otimes V_2) =: W,
\end{array}
\]

where \(\iota_i : \Lambda U \rightarrow \Lambda U \otimes \Lambda U\) and \(\eta_i : V_1 \rightarrow V_1 \otimes V_2\) are the canonical inclusions into the
i-th factor and the lower square with maps \( f \) and \( \eta \) is a pushout diagram; see [Men15, Example 1.4]. It is well-known that the model category of non-negatively graded unital CDGA’s is proper, and hence pushouts along cofibrations preserve quasi-isomorphisms. Therefore, \( f : M \to W \) is a DGA-quasi-isomorphism. We orient \( H(W) \) such that \( W \) becomes a PDGA and \( f \) a PDGA-quasi-isomorphism. At the level of algebras, it holds

\[
W \simeq V_1 \otimes V_2 \otimes \Lambda(sU).
\]

From \( V_0^1 = V_0^2 = \text{span}\{1\}, V_1^1 = V_1^2 = 0 \) and \( U^1 = U^2 = 0 \) it follows that \( W \) is connected and simply-connected. It is also of finite type because \( V_1, V_2 \) and \( \Lambda(sU) \) are. Proposition 4.3 for \( n \geq 5 \) applies and gives an oriented extension of Hodge type \( \hat{W} \) which is connected, simply-connected and of finite type. Consider the composition \( \pi : W \hookrightarrow \hat{W} \to \mathcal{Q}(\hat{W}) \), where the first map is the inclusion and the second map the canonical projection \( \pi : W \to \mathcal{Q}(\hat{W}) \). It is a PDGA-quasi-isomorphism by Lemma 3.5. For \( i \in \{1,2\} \), we define \( h_i := \pi \circ \eta \circ \eta_i : V_i \to \mathcal{Q}(\hat{W}) =: V_3 \) and compute using

\[
(\eta \circ \eta_i) \circ f_i = f \circ (\iota \circ \iota_1) \quad \text{and} \quad p \circ (\iota \circ \iota_1) = (p \circ \iota) \circ \iota_1 = \mu \circ \iota_1 = \mathbb{I}
\]

that

\[
(h_i)^\ast = \pi^\ast \circ f^\ast \circ (\iota \circ \iota_1)^\ast \circ (f_i)^{-1} = \pi^\ast \circ f^\ast \circ p^\ast \circ (f_i)^{-1}.
\]

Therefore, \( h_1 \) and \( h_2 \) are PDGA-quasi-isomorphisms and since \( V_1, V_2, V_3 \) are dPD-algebras, Lemma 3.5 implies that they are orientation preserving inclusions.

The next remark shows that the assumption \( H^2(V_1) = H^2(V_2) = 0 \) cannot be omitted.

**Remark 5.5.** We will construct dPD-algebras \( V_1 \) and \( V_2 \) satisfying all assumptions of Proposition 5.4 except for \( H^2(V_1) = H^2(V_2) = 0 \) and show that there is no simply-connected dPD-algebra \( V_3 \) which admits PDGA-quasi-isomorphisms \( V_1 \to V_3 \leftarrow V_2 \). Consider the minimal Sullivan DGA

\[
\Lambda := \Lambda(a, b, c) \quad \text{with} \quad \deg a = 2, \ \deg b = 3, \ \deg c = 5,
\]

\[
da = dc = 0, \ \db = a^2.
\]

It holds

\[
H(\Lambda) = \text{span}\{[1], [a], [c], [ac]\}.
\]

We set \( V_1 := H(\Lambda) \) and equip it with an orientation \( \circ : V_1 \to \mathbb{R} \) satisfying \( \circ([ac]) = 1 \). This makes \( V_1 \) into a connected and simply-connected dPD-algebra. Consider the
following unital CDGA:

\[ V_2 := \text{span}\{1, k, w, z, l, v\} \quad \text{with} \quad \deg k = 2, \deg w = 3, \deg z = 4, \deg l = 5, \deg v = 7 \]

\[ k \land l = v, \quad k \land k = z, \quad z \land w = v, \quad k \land w = l, \]

\[ dk = dz = dl = dv = 0, \quad dw = z. \]

We equip \( V_2 \) with an orientation \( o : V_2 \to \mathbb{R} \) satisfying \( o(v) = 1 \). This makes \( V_2 \) into a connected and simply-connected dPD-algebra. The following morphisms of algebras are PDGA-quasi-isomorphisms:

\[
\begin{align*}
f_1 : \Lambda & \longrightarrow V_1 \\
a & \mapsto [a] \\
b & \mapsto 0 \\
c & \mapsto [c]
\end{align*}
\[
\begin{align*}
f_2 : \Lambda & \longrightarrow V_2 \\
a & \mapsto k \\
b & \mapsto w \\
c & \mapsto l.
\end{align*}
\]

Therefore, \( V_1 \) and \( V_2 \) are weakly homotopy equivalent as PDGA’s. Suppose that there is a dPD-algebra \( V_3 \) and PDGA-quasi-isomorphism \( h_1 : V_1 \to V_3 \) and \( h_2 : V_2 \to V_3 \). By Lemma 3.5 \( h_1 \) and \( h_2 \) are injective and orientation preserving. The images \( h_1([a]) \) and \( h_2(k) \) are closed elements in \( V_3^2 \) and because \( [a] \land [a] = 0 \) and \( k \land k = z \neq 0 \), they can not be multiples of each other. Because \( H^2(V_3) \cong \mathbb{R} \), non-zero elements in \( V_3^2 \) whose differentials kill the additional closed elements in \( V_3^2 \) must exist.

\[\triangleright\]

**Conjecture 5.6.** Proposition 5.4 holds even if \( H^2(V_1) = H^2(V_2) \neq 0 \) if \( V_3 \) is not-required to be simply-connected.

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