Notes on the post-Newtonian limit of the massive Brans–Dicke theory

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Abstract
We consider the post-Newtonian limit of the massive Brans–Dicke theory and make some notes about the post-Newtonian limit of the case $\omega = 0$. This case is dynamically equivalent to the metric $f(R)$ theory. It is known that this theory can be compatible with the solar system tests if the Chameleon mechanism occurs. Also, it is known that this mechanism is because of the nonlinearity in the field equations produced by the largeness of the local curvature relative to the background curvature. Thus, the linearization of the field equations breaks down. On the other hand, we know that the Chameleon mechanism exists when a coupling between the matter and the scalar field exists. In the Jordan frame of the Brans–Dicke theory, we have no such coupling. But in the Einstein frame, this theory behaves like a Chameleon scalar field. By confining ourselves to the case $\omega = 0$, we show that ‘Chameleon-like’ behaviour can exist also in the Jordan frame, but it has an important difference compared with the Chameleon mechanism. Also we show that the conditions which lead to the existence of a ‘Chameleon-like’ mechanism are consistent with the conditions in the post-Newtonian limit which correspond to a heavy scalar field at the cosmological scale and a small effective cosmological constant. Thus, one can linearize field equations to the post-Newtonian order, and this linearization has no contradiction with the existence of ‘Chameleon-like’ behaviour.

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1. Introduction

The dynamics of the massive Brans–Dicke (BD) theory can be determined by the following action:

$$S = \frac{1}{2k^2} \int d^4x \sqrt{-g} \left[ \phi R - \frac{\omega}{\phi} \partial_\mu \phi \partial^\mu \phi - V(\phi) \right] + S_m[g, \psi].$$

(1)
where \( k^2 = 8\pi G \) and \( S_m \) is the matter action. In the original BD theory, the mass term \( V(\phi) \) is zero and so the scalar field is massless. The post-Newtonian (PN) limit of the massless BD theory has been investigated in [1]. It is known that the massless BD theory cannot always produce the corresponding GR case in the limit \( \omega \to \infty \). And there exist some exact solutions that does not go over to the Einstein’s theory in this limit [2]. Banerjee et al [3] argued that this is due to the zero trace of the energy-momentum tensor of matter. But even for a non-zero trace of the energy-momentum tensor, the limit of \( \omega \to \infty \) does not lead to GR necessarily [4]. Whether the BD theory behaves like GR at \( \omega \to \infty \) or not depends on the validity of the main assumption of the PN formalism. For example, in the PN limit we assume that the perturbation of the scalar field due to the local gravitating system under consideration is very small compared to its cosmological background value. If this assumption fails, then this theory will not behave as GR in the PN limit. The situation is more complex in the massive BD theory even if the coupling constant \( \omega \) is zero. The weak field limit of this case is the subject of controversy. More specifically, several authors claimed that the metric \( f(\mathcal{R}) \) theory (which is equivalent to the massive BD theory with \( \omega = 0 \)) is characterized by an ill-defined Newtonian regime [5].

It is known that the Jordan frame of BD theory is related to the Einstein frame via a conformal transformation. In the Einstein frame, there exists a conformal coupling between the matter and the scalar field which leads to an interaction between them [2]. Hence, this theory in the Einstein frame can be considered as an interacting quintessence model[6]. On the other hand, this interaction leads to a density-dependent scalar field mass, i.e. the associated mass of the scalar field can change with environment [7]. This behaviour is the reason for naming such a theory the Chameleon theory [7]. The mass of the scalar field can be very large at the dense places, and so the scalar field interaction with the matter can strongly be suppressed. Also, under the condition named the thin-shell condition [7], the scalar field outside the source is produced only with a very thin shell of matter near the surface of the source. In the other words, the interior part of the source has no contribution to the generation of the scalar field.

An important question arises here. We know that some ambiguity exists for the physical equivalence of the Einstein frame and the Jordan frame. Can we conclude the existence of the Chameleon mechanism in the Jordan frame from its existence in the Einstein frame? We expect that one should see this behaviour directly in the Jordan frame without switching to the Einstein frame. Also it is natural to expect the trace of this behaviour in the PN limit. However, it is claimed in the literature that when the Chameleon mechanism occurs, linearization of the field equations (which is necessary in the PN limit) breaks down and the behaviour of the scalar field is governed by non-linear dynamics[8]. But, by working directly in the Jordan frame, we show that there is no Chameleon mechanism in the form as in [7] but a different version of it which we named as ‘Chameleon-like’. We verify that linearization of the field equations to the PN order has no contradiction with the existence of the ‘Chameleon-like’ mechanism, and the required conditions for the existence of this mechanism are consistent with the required conditions in the PN limit for the viability in the solar system.

The outline of this paper is as follows. In section 2 we review the PN limit (expansion) of the massive BD theory and look for nonlinear terms responsible for the Chameleon mechanism. However, we will show that such terms do not exist. In section 3 we confine ourselves to the \( \omega = 0 \) case and show that the ‘Chameleon-like’ behaviour can exist in this theory. Finally, we compare the result of the PN limit and Chameleon-like behaviour and show that they are consistent with each other. Also, we clarify the range of validity of the PN
expansion and show that this expansion does not break down when the Chameleon mechanism occurs.

2. Post-Newtonian limit of the massive BD theory

In order to find some solutions of the field equations in the PN approximation, we expand the equations of motion around the background values of the metric and the scalar field. More specifically, we will use $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, $g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$, $\phi(x,t) = \phi_0(t) + \phi(x,t)$ and $V(\phi) = V_0 + \phi V_0' + \phi^2 V_0''/2 + \cdots$. It is worth mentioning that we are working in the PN coordinate system [9]. The PN limit of any scalar-tensor theory requires a knowledge of $g_{00}$ to $O(4)$, $g_{0j}$ to $O(3)$, $g_{ij}$ to $O(2)$ and $\phi$ to $O(4)$. Note that in the PN limit, we take $v^2 \sim U \sim O(2)$ where $v$ is the characteristic velocity of particles and $U$ is the Newtonian gravitational potential. Thus, we look for the solutions of the field equations in the form of a Taylor expansion as

$$
\begin{align*}
g_{00} &\simeq -1 + h_{00}^{(2)} + h_{00}^{(4)}, \\
g_{0j} &\simeq h_{0j}^{(3)}, \\
g_{ij} &\simeq \delta_{ij} + h_{ij}^{(2)}, \\
\phi &\simeq \phi_0 + \phi^{(2)} + \phi^{(4)}.
\end{align*}
$$

(2)

Variation of the action (1) with respect to $g_{\mu\nu}$ and $\phi$ will yield the following field equations respectively:

$$
\begin{align*}
R_{\mu\nu} &= \frac{k^2}{\phi} \left( T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right) + \frac{\omega}{\phi^2} \partial_{\mu} \phi \partial_{\nu} \phi + \frac{1}{\phi} \partial_{\mu} \partial_{\nu} \phi + \frac{1}{2\phi} g_{\mu\nu} \Box \phi + V(\phi), \\
\Box \phi - \frac{dV_{\text{eff}}}{d\phi} &= \frac{k^2}{3 + 2\omega} T, \\
\frac{dV_{\text{eff}}}{d\phi} &= \frac{1}{3 + 2\omega} \left( \phi \frac{dV}{d\phi} - 2V \right).
\end{align*}
$$

(3) (4) (5)

The full derivation of the PN limit of the massive BD theory can be found in [10]. Let us just review in brief the results of [10] which are necessary for our work. The field equation (4) in the PN approximation is

$$
[\nabla^2 - m_0^2] \psi^{(2)}(x, t) = -\frac{k^2 \rho}{3 + 2\omega},
$$

(6)

where $m_0^2$ is the mass associated with the scalar field at the cosmological scales given by

$$
m_0^2 = \frac{\phi_0 V_0' - V_0}{3 + 2\omega}
$$

(7)

in which the prime denotes differentiation with respect to $\phi$. In the Solar system the expressions of $h_{00}^{(2)}$ and $h_{ij}^{(2)}$ take the following forms far from the source:

$$
\begin{align*}
h_{00}^{(2)} &= 2 \frac{G_{\text{eff}} M_{\odot}}{r} + \frac{\Lambda_{\text{eff}}}{3} r^2, \\
h_{ij}^{(2)} &= \delta_{ij} \left[ 2\gamma \frac{G_{\text{eff}} M_{\odot}}{r} - \frac{\Lambda_{\text{eff}}}{3} r^2 \right],
\end{align*}
$$

(8)

where $\Lambda_{\text{eff}} = V_0/2\phi_0$ and

$$
G_{\text{eff}} = \frac{k^2}{8\pi \phi_0} \left( 1 + \frac{e^{-m_0 r}}{3 + 2\omega} \right).
$$

(9)
Note that in the derivation of equation (8), we have used the assumption $m_0^2 > 0$. Using equation (8), the PPN parameter $\gamma$ is given by

$$
\gamma = \frac{h^{(2)}_{ii}}{h^{(2)}_{00}} = \frac{3 + 2\omega - e^{-m_0 r}}{3 + 2\omega + e^{-m_0 r}}.
$$

(10)

See also [11] in which a special potential has been investigated in the PN limit. If the scalar field is very light, then the parameter $\gamma$ is space independent, and as we expect, the massive BD theory behaves like massless BD theory, i.e. this parameter takes the form $\gamma = (1 + \omega)/(2 + \omega)$.

We are interested in the case $\omega = 0$ which, as mentioned before, is dynamically equivalent to the metric $f(R)$ theory. In this case, if the scalar field is very light, then $\gamma = 0.5$, which is in patent disagreement with observation since $\gamma_{ob} \simeq 1$ [12]. However, if

$$
m_0^2 L^2 \gg 1,
$$

(11)

where $L$ represents a typical experimental length scale (below the planetary scale), then the scalar field is heavy and its interaction is short range and it will be hidden from the local experiments. In this case, $\gamma$ is near to 1. On the other hand, the cosmological constant term $(V_0/6\phi_0)r^2$ must be very small. Otherwise, this term will change the gravitational dynamics of local systems such as solar system. Thus, for having an observationally acceptable theory (at least in the solar system) the following condition should be satisfied too:

$$
\frac{V_0}{\phi_0} \frac{r^2}{L^2} \ll 1,
$$

(12)

where $L_d$ is a length scale of the same order or larger than the solar system. Thus, the massive BD theory with $\omega = 0$ can be compatible with the solar system tests if both conditions (11) and (12) are satisfied. Note that this conclusion is also true for the metric $f(R)$ theory [10].

Some notes are in order here. It is claimed in the literature that the Chameleon behaviour is due to the nonlinear terms in the field equations[8]. So, we have to reconsider the PN limit with more care. Maybe there exists some nonlinear terms in the PN expansion which have been forgotten in the above considerations. The main assumptions on the scalar field in obtaining the PN limit are $|\phi^{(2)}| \ll 1$ and $\phi \simeq \phi_0 + \phi^{(3)} + \phi^{(4)}$. Let us examine the first assumption. Can we use this assumption in the gravitational systems such as solar system? In fact, $\phi^{(2)}$ is the contribution of the scalar field due to the local gravitating system. A comparison with the results of [13] may be useful here. In order to find the weak field limit of the metric $f(R)$ theory, Chiba et al [13] have expanded the Ricci scalar as $R = R_0 + R_1(r)$, where $R_0$ is the background curvature and $R_1(r)$ is the perturbation produced by a given source. In order to linearize the field equations, they have used $R_1(r)/R_0 \ll 1$. However, as we will discuss in the next section, this assumption is experimentally unacceptable near the Earth or at the solar system and therefore leads to a wrong weak field limit. Similarly, in the PN limit we linearize the field equation of $\phi$ using the condition $|\phi^{(2)}| \ll 1$. If, for an arbitrary system, $\phi^{(2)}$ is the same order or greater than $\phi_0$, then it is straightforward to show that the PN expansion fails, and we will not be able to use this approximation for that system. In this paper we do not want to find under which conditions this assumption is applicable, but we will show that when the thin-shell condition is satisfied, then $|\phi^{(2)}| \ll 1$ is also satisfied. Thus we expect that the physical results of the Chameleon-like behaviour and the PN limit be similar.

Now, consider the second assumption. We know that the odd-order terms $O(1), O(3)$ cannot exist in the $g_{00}$ and $g_{ij}$ components of metric. In fact, conservation of rest mass prevents terms of order $O(1)$, and conservation of energy in the Newtonian limit prevents terms of order $O(3)$. Appearance of other odd-order terms is theory dependent, for example, $O(5)$ cannot exist in GR (because of conservation of energy in the PN limit) but $O(7)$ can exist.
Also, we find the appropriate expansion of the scalar field from its field equation. For example, consider the field equation of the $\phi$ field in the original BD theory:

$$
\Box \phi = \frac{8\pi}{3 + 2\omega} T.
$$

(13)

By expanding the rhs for a perfect fluid to the fourth order and using an appropriate gauge condition, we can easily show that only even-order terms can appear in the expansion of $\phi$ (i.e. $\phi \simeq \phi_0 + \phi^{(2)} + \phi^{(4)}$) [14]. However, in the massive BD theory, the situation is not so trivial. In this case, consider equation (4) and assume the following expansion for the scalar field $\phi$:

$$
\phi \simeq \phi_0 + \phi^{(1)} + \phi^{(2)} + \phi^{(3)} + \phi^{(4)} + \ldots.
$$

(14)

It is an easy job to verify that if $n$ is even (odd), then $\Box \phi^{(n)}$ contains only even (odd)-order terms. Also, we know

$$
g^{\mu\nu}T^\nu_{\mu\nu} \simeq \eta^{\gamma\sigma}(h_{\mu\nu,\gamma} - \frac{1}{2}h_{\gamma\nu,\mu}) + O(5)
$$

(15)

and will use the following gauge conditions:

$$
h_{\mu\nu,0} + \frac{1}{2}h_{\gamma\nu,0} = \mathcal{R}_0, \quad h_{j,\nu} - \frac{1}{2}h_{\nu,j} = \mathcal{R}_j.
$$

(16)

Note that for our purpose it is not necessary to fix the explicit form of $\mathcal{R}_0$ and $\mathcal{R}_j$. However, we should keep in our mind that $\mathcal{R}_0$ is of order $O(3)$ and $\mathcal{R}_j$ is of order $O(2)$. Using these gauge conditions and arranging the terms with the same orders in (4), we obtain

$$
(\nabla^2 - m_0^2)\phi^{(1)} = 0,
$$

$$
(\nabla^2 - m_0^2)\phi^{(2)} = -\frac{k^2\rho}{3 + 2\omega} + \frac{\phi_0 V_0^{\mu\nu}}{2(3 + 2\omega)}[\phi^{(1)}]^2,
$$

$$
(\nabla^2 - m_0^2)\phi^{(3)} = \left(\partial_i\partial_0 + h^{ij}\partial_j\partial_i + \mathcal{R}_j\partial_i\right)\phi^{(1)}
$$

$$
+ \frac{1}{3 + 2\omega} \left[ \phi_0 V_0^{\mu\nu}\phi^{(1)}[\phi^{(1)}]^2 + \frac{\phi_0 V_0^{\mu\nu} + V_0^{\mu\nu}}{6}[\phi^{(1)}]^3 \right],
$$

(17)

$$
\nabla^2 \phi^{(4)} = \frac{8\pi GT^{(4)}}{3 + 2\omega} (\partial_i\partial_0 + h^{ij}\partial_j\partial_i + \mathcal{R}_j\partial_j)\phi^{(2)}.
$$

Thus, it is obvious that the equation of motion of $\phi$, unlike the massless BD theory, allows the odd-order terms. Also, it is interesting that there exists a nonlinear term in the equation of $\phi^{(2)}$. If $\phi^{(1)} \neq 0$, then this term will change the solutions of the metric components and also the scalar field itself. But, we should take into account the metric field equations too. If one writes the 0-0 component of (3) to the second order, then there will be a term with the first order which should be zero separately:

$$
(\nabla^2 + V_0^0)\phi^{(1)} = 0.
$$

(18)

Similarly, from the $i-j$ component of (3), we have $\partial_i\partial_j\phi^{(1)} = 0$ and from its 0 – $j$ component $\partial_0\partial_j\phi^{(1)} = 0$. Thus, the metric field equations force $\phi^{(1)}$ to be zero. This conclusion is also true for $\phi^{(3)}$, and one can check it by writing equation (3) to the higher orders. As a final result of this section, one can be sure that the PN expansion obtained in [10] is complete, and there is no nonlinear term in the field equations. So, we should see the trace of Chameleon mechanism in this PN limit. In the next section, we show that the ‘Chameleon-like’ behaviour can exist in this theory, and then we show how this mechanism and the PN limit are connected.
3. Chameleon-like behaviour in the massive BD theory with $\omega = 0$

In this section we show that the Chameleon-like behaviour can exist in the massive BD theory with a vanishing coupling constant. Our main purpose is to address the question: Can we see the Chameleon mechanism in the PN limit?

As we discussed in the introduction, for Chameleon scalar fields the associated mass is dependent on the matter density of the environment, and so the scalar field is short range in the dense places and is long range in the low densities such as cosmos [7]. The main reason for such behaviour is that in these theories the scalar field interacts directly with matter particles through a conformal coupling of the form $e^{\phi/M_{pl}}$. On the other hand, the BD theory in the Einstein frame behaves like an interacting quintessence with $\beta = -1/\sqrt{6+4\omega}$ [2].

More specifically, when the coupling constant is zero, the conformal coupling takes the form $e^{-\phi/\sqrt{6M_{pl}}}$, i.e. $\beta = -1/\sqrt{6}$. Thus, we expect that in the Einstein frame, the massive BD theory behaves as a Chameleon theory; however, is it true in the Jordan frame? It is worth stressing that the physical results can be completely different in these frames, and we should be very careful in interpreting the meaning of them. Although, it is not yet clear that which one of these frames are really physical, but it is usually convenient to work in the Einstein frame which is mathematically simpler and then convey the results to the Jordan frame. For more details, see [2] chapter 2.

As we mentioned before, the metric $f(R)$ theory is dynamically equivalent to the massive BD theory. In the context of this theory, several authors claimed that an ill-defined behaviour in the Newtonian limit exists ($\gamma = 0.5$) [5]. This conclusion can be inferred directly from analogy with the BD theory ($\omega = 0$). However, using the Chameleon mechanism in the Einstein frame, Faulkner et al [15] have shown that the metric $f(R)$ gravity can lead to $\gamma \sim 1$. They have transformed their results to the Jordan frame.

Here, using exactly the same method as in [16], we want to explore the Chameleon behaviour and the thin-shell effect in the Jordan frame of the massive BD theory with $\omega = 0$. Assuming that the pressure is zero ($p = 0$), one can rewrite equation (4) as

$$3 \Box \phi + 2V - (\phi - 1) \frac{dV}{d\phi} = R - k^2 \rho.$$  (19)

Note that $R = \frac{dV}{d\phi}$. Now, if $\phi \sim 1$ and $\frac{|V|}{V'} \ll 1$, then the above equation can be written as

$$3 \Box \phi \simeq R - k^2 \rho.$$  (20)

Thus, this theory will behave like GR if $3 |\Box \phi| \ll k^2 \rho$. In the weak field limit, where one can use $\Box \sim \nabla^2$, this condition can be expressed as follows:

$$\lambda_{\phi}^2 \left|_{R = k^2 \rho} \right. \nabla^2 \rho \ll \rho,$$  (21)

where $\lambda_{\phi} = 1/m_0$ is the Compton wavelength of the field. This condition is the Compton condition. In order to interpret this condition, consider a source with its density changing slowly with radius, say $\rho \sim r^\epsilon$ (this is the case for the Earth and the Sun). Then the Compton condition can be written as

$$\lambda_{\phi} \left|_{R = k^2 \rho} \right. \frac{\partial \rho}{\partial r} \ll \rho.$$  (22)

Thus, if $\Delta r$ is a length scale in which the density changes concretely ($\Delta \rho \sim \rho$), then we can infer that the density changes on the length scales that are much longer than the Compton wavelength. Physically it means that the scalar field interaction will suppress on scales larger than $\lambda_{\phi}$. 

6
As in [16], in order to find the corresponding thin-shell condition [7], we take the static spherically symmetric metric around a given source at the origin as follows:

\[ ds^2 = -[1 - 2A(r) + 2B(r)] dt^2 + [1 + 2A(r)](dr^2 + r^2 d\Omega). \]

(23)

We will assume that \(|A(r)| \ll 1\) and \(|B(r)| \ll 1\) near the source and inside it. By taking into account these assumptions and the definition of the Ricci tensor, one can verify that

\[ \nabla^2 (A + B) \simeq -\frac{1}{2} R \]

(24)

\[ \nabla^2 B \simeq -\frac{1}{2} \left( R^0_0 + \frac{R}{2} \right). \]

(25)

By substituting \( R^0_0 \) from the 0–0 component of equation (3) into (25), we obtain

\[ \nabla^2 B(r) \simeq -\frac{1}{3} \left( R - k^2 \rho \right) - \frac{V}{12\phi}. \]

(26)

On the other hand, by using the conditions \( \phi \simeq 1 \) and \(|\phi'|| \ll 1\), we obtain

\[ \nabla^2 B(r) \simeq -\frac{1}{3} (R - k^2 \rho), \quad \nabla^2 A(r) \simeq -\frac{1}{6} (R - k^2 \rho) - \frac{1}{2} k^2 \rho. \]

(27)

And the field equation of the scalar field takes the form

\[ \nabla^2 \phi \simeq -\frac{1}{3} (R - k^2 \rho) = -\nabla^2 B(r). \]

(28)

Thus, assuming that \( B(r) \) is finite at the origin, the solution of \( B(r) \) and the scalar field are related as

\[ B(r) = \phi_0 - \phi(r) = -\Delta \phi(r), \]

(29)

where, as before, \( \phi_0 \) is the background value of the scalar field. Following [16], it is convenient to define an effective mass as

\[ m_{\text{eff}} = \int \left( \rho(r') - \frac{R(r')}{k^2} \right) dv'. \]

(30)

Thus, the solutions of \( A \) and \( B \) become

\[ B(r) = -\frac{2G}{3} \int \frac{\rho(r') - \frac{k^2 \rho}{4}}{[r - r']} dv' \simeq -\frac{2G m_{\text{eff}}}{3r}, \]

\[ A(r) \simeq -\frac{Gm}{r} - \frac{Gm_{\text{eff}}}{3r}, \]

where \( m = \int \rho(r') dv' \). Finally, the PPN parameter \( \gamma \) is found to be

\[ \gamma = \frac{A}{A - B} \simeq \frac{3m + \frac{1}{2} m_{\text{eff}}}{3m + m_{\text{eff}}}. \]

(32)

It is clear form this equation that if \( m_{\text{eff}} \ll m \), then \( \gamma \rightarrow 1 \). An important result may be obtained from (31),

\[ |B(r)| \leq \frac{2G}{3} \int \frac{\rho(r') dv'}{|r - r'|} = \frac{2}{3} \Phi_N(r), \]

(33)

where \( \Phi_N(r) \) is the Newtonian potential. Now, using equation (29) we obtain

\[ |\Delta \phi| \leq \frac{2}{3} \Phi_N(r). \]

(34)

This condition sets an upper bound on the difference between the values of the scalar field from the interior to the exterior of the source. Note that if \(|\Delta \phi| \ll \frac{2}{3} \Phi_N(r)\), then \( m_{\text{eff}} \ll m \), so
$R \to k^2 \rho$, and we will recover the GR results. This condition is called the thin-shell condition. It is easy to convert this condition to the corresponding Einstein frame version. Then it will be exactly the thin-shell condition obtained in [7] with $\beta = -1/\sqrt{6}$. As has been discussed in [7, 16], if this condition is satisfied, then the exterior field is generated only by the thin shell near the surface of the spherical source. However, it is noteworthy that we have named this behaviour 'Chameleon-like' here because it is different from what is known in the original Chameleon theories [7]. In fact, in the Jordan frame there is no direct interaction between the matter and the scalar field, and it looks like that we have strongly restricted the scalar field evolution such that it behaves like GR. Also, we will show that if the thin-shell condition is satisfied, then the scalar field needs to be heavy at the cosmological scales. But, we know that in the original Chameleon theory [7], the thin-shell condition does not make any restriction on the interaction range of the scalar field far away from the source. It is important to stress that although in the Jordan frame there is no coupling between the matter and the scalar field, the scalar field couples directly to the Ricci curvature and consequently mass of the scalar field can depend to the curvature. Thus, we expect something similar to the Chameleon effect in the Jordan frame. However, this 'Chameleon-like' behaviour ensures us that the massive BD theories with a vanishing coupling constant (or equivalently metric $f(R)$ theory) can be compatible with the solar system tests. Also, in the cosmological considerations it can be different and distinguishable from $\Lambda$ cold dark matter ($\Lambda$CDM). For example, phantom division can occur in it [16].

Now, we are in a position to answer the main question of our paper. Is it allowable to linearize the scalar field equation when the Chameleon-like behaviour exists? From the thin-shell condition and the PN expansion of the scalar field, we have

$$|\phi^{(2)}| \ll \Phi_N(r).$$

(35)

On the other hand, for having the Chameleon-like behaviour, $\phi \sim 1$; hence, by using (35), we obtain

$$\left| \frac{\phi^{(2)}}{\phi_0} \right| \ll 1.$$

(36)

Thus, as discussed before, the PN limit is applicable. In the following, we show that the second necessary condition for the ‘Chameleon-like’ behaviour, i.e. $|V'/V| \gg 1$, forces the scalar field to be heavy at the cosmological scales. To do this, we write the Ricci scalar as follows:

$$R(r) = \frac{dV}{d\phi}.$$  

(37)

Therefore, the background curvature scalar is $R_0 = V'$. Regarding that $\phi_0$ is the minimum of the effective potential $V_{\text{eff}}$ and also defining $\eta(\phi)$ and $M^2$ as

$$\eta(\phi) = \frac{V'(\phi)}{V(\phi)}, \quad M^2 = \frac{m_0^2}{H_0^2 \phi_0^2},$$

(38)

we have

$$\eta_0 = \eta(\phi_0) = \frac{2}{\phi_0},$$

$$\eta'_0 = -\frac{2}{\phi_0^2} + \frac{M^2}{2},$$

$$\eta''_0 = \frac{4}{\phi_0^3} - \frac{3M^2}{\phi_0} + \frac{2V^{(3)}_0}{\phi_0^2 V_0^2}.$$
The equation of the scalar field (4). We rewrite it as follows:

\[
\eta_0^{(3)} = \frac{12}{\phi_0} + \frac{18M^2}{\phi_0^2} - \frac{16V_0^{(3)}}{\phi_0 V_0} - \frac{3M^4}{4} + \frac{2V_0^{(4)}}{\phi_0 V_0},
\]

\[
\eta_0^{(4)} = \frac{48}{\phi_0^3} - \frac{120M^2}{\phi_0^3} + \frac{120V_0^{(5)}}{\phi_0 V_0^2} + \frac{15M^4}{\phi_0^2} - \frac{20V_0^{(4)}}{\phi_0 V_0} - \frac{10M^2 V_0^{(3)}}{\phi_0 V_0} + \frac{2V_0^{(5)}}{\phi_0 V_0}. \tag{39}
\]

Expanding \(\eta(\phi)\) in powers of \(\Delta \phi\) and using (39), we obtain

\[
\eta(\phi) = \frac{V'}{V} \simeq \frac{2}{\phi_0} + \frac{M^2}{2} \Delta \phi + \frac{2V_0^{(3)}}{\phi_0 V_0} \frac{\Delta \phi^2}{2!} + \frac{2V_0^{(4)}}{\phi_0 V_0} \frac{\Delta \phi^3}{3!} + \frac{2V_0^{(5)}}{\phi_0 V_0} \frac{\Delta \phi^4}{4!} + O(\Delta \phi^5).
\]

Note that we have expanded \(\eta(\phi)\) to the higher orders of \(\Delta \phi\) because when the Chameleon-like behaviour occurs, the difference between \(|\eta(\phi)|\) \(\gg 1\) and \(\eta_0 \sim 2\) is very large. First assume that \(M^2 \ll 1\), then one can rewrite \(\eta(\phi)\) as

\[
\eta(\phi) \simeq \frac{2}{\phi_0} \left( 1 + \frac{V_0^{(3)}}{V_0} \frac{\Delta \phi^2}{2!} + \frac{V_0^{(4)}}{V_0} \frac{\Delta \phi^3}{3!} + \frac{V_0^{(5)}}{V_0} \frac{\Delta \phi^4}{4!} + \cdots \right). \tag{40}
\]

For \(|\eta(\phi)| \gg 1\), it is necessary that

\[
\left| \frac{V_0^{(3)}}{V_0} \right| \left| \frac{\Delta \phi^2}{2!} \right| + \left| \frac{V_0^{(4)}}{V_0} \right| \left| \frac{\Delta \phi^3}{3!} \right| + \left| \frac{V_0^{(5)}}{V_0} \right| \left| \frac{\Delta \phi^4}{4!} \right| \gg 1. \tag{42}
\]

There are many cases for which this inequality is satisfied. For example, assume that these terms are of the same order, i.e.

\[
\left| \frac{V_0^{(4)}}{V_0^{(3)}} \right| \sim \left| \frac{V_0^{(5)}}{V_0^{(4)}} \right| \simeq (\Delta \phi^{-1}) \gg 1. \tag{43}
\]

Thus it is, in principle, possible to satisfy \(|\eta(\phi)| \gg 1\), while \(m_0^2/H_0^2 \ll 1\). However, some notes are in order here.

Does \(m_0^2/H_0^2 \ll 1\) guarantee that the scalar field is long range? Consider the field equation of the scalar field (4). We rewrite it as follows:

\[
\Box \phi = -m_0^2(\phi - \phi_0) + \frac{\phi_0 V_0^{(5)}}{6} (\phi - \phi_0)^2 + \cdots = -\frac{k^2 \rho}{3}. \tag{44}
\]

If the second term is larger than the self-interacting terms, then the solution of \(\phi\) for an isolated gravitating system contains the Yukawa-like term \(e^{-m_0 \phi}/r\). So, the magnitude of \(m_0\) is directly related to the range of the scalar field in the sense that if the scalar field is light (heavy), then it is long (short) range. On the other hand, if the second term is negligible compared with the third term, even for small \(\Delta \phi\), then the dynamics of the scalar field is dominated by the third term, and there is not necessarily a Yukawa-like solution containing \(m_0\) in it. In the other words, in this case, the lightness of the scalar field dose not necessarily mean the long rangeness of it. But for the Chameleon scalar fields [7], the lightness (heaviness) of the scalar field implies the long (short) rangeness of it. And of course, this is one of the main goals of that theory.

The second problem for the Chameleon-like behaviour by the light scalar fields is that we encounter with non-smooth varying potentials, see (43). Thus, it is very unlikely to produce the Chameleon-like behaviour by the light scalar fields \((m_0^2/H_0^2 \ll 1)\).
Now, assume that $M^2 \gg 1$. In this case one can rewrite (40) as
\[
\eta(\phi) \simeq \frac{2}{\phi_0} + \frac{M^2}{2} \Delta \phi + \frac{2}{\phi_0} \left( \frac{V_0^{(3)}(\phi)}{V_0} \frac{\Delta \phi^2}{2!} + \frac{V_0^{(4)}(\phi)}{V_0} \frac{\Delta \phi^3}{3!} + \frac{V_0^{(5)}(\phi)}{V_0} \frac{\Delta \phi^4}{4!} \right) - \frac{10 M^2 V_0^{(3)}}{\phi_0 V_0} + O(\Delta \phi^5).
\]
(45)

It is clear that without any restrictive conditions such as (43) on the form of the potential and only with the assumption that $m_0^2 / H_0^2 \gg 1$, we can produce $|\eta(\phi)| \gg 1$. In this case, one can write the Ricci scalar as follows:
\[
\frac{R(r)}{R_0} \simeq 1 + \frac{m_0^2}{4H_0^2} \psi^{(2)}(r) + O(\Delta \phi^2),
\]
(46)

where $H_0$ is the Hubble parameter. Experimentally we know that $\Delta R / R_0 = \frac{R - R_0}{R_0} \gg 1$ in the solar system (it is obvious in GR). For example, on the Earth $\frac{\Delta R}{R_0} \sim O\left(\rho_{\text{Air}} / H_0^2 M_{\text{pl}}^2\right) \sim 10^{27}$, and in the solar system we can estimate this ratio using the local dark matter density which yields $\frac{\Delta R}{R_0} \sim O\left(\rho_{\text{DM}} / H_0^2 M_{\text{pl}}^2\right) \sim 10^6$. Consequently, we can again infer from (46) that
\[
\frac{m_0^2}{H_0^2} \gg 1,
\]
(47)

which can be consistent with condition (11). This condition shows that the Compton wavelength is very smaller than the Hubble length scale. This means that the scalar field is heavy in the cosmological scales. For example, near the Earth, $\Delta R / R_0 \sim 10^{27}$. By assuming that $L \sim R_{\text{Earth}}$ and $\phi^{(2)} \sim 10^{-14}$ (for this assumption, we have used (35)) and using equations (35) and (46), it is easy to show that $m_0^2 L^2 \sim 10^5$. It is clear from (46) that if the scalar field is very light in the cosmological scales, then a gross violation of experiment will occur. We saw this result also in the PN limit where $\gamma$ was equal to 0.5 if $m_0^2 L^2 \ll 1$.

Thus, it seems that the Chameleon-like behaviour can only occur for the heavy scalar fields ($m_0^2 / H_0^2 \gg 1$). And this is completely against the Chameleon scalar fields [7]. As we mentioned before, the Chameleon scalar fields are light and free at the cosmological scales.

Also since $\phi_0$ is the minimum of the effective potential, we can write
\[
\frac{V_0}{\phi_0} = \frac{V_0'}{\phi_0} = \frac{k^2 \rho_0}{2},
\]
(48)

where $\rho_0 \sim 10^{-29} h^2 \text{gm cm}^{-3}$ is the current matter density of the universe. By multiplying this equation with $L_L^2$, $L_L$ is a length scale of the same order of the solar system, i.e. $L_L \sim 10^{15}$ cm, we can rewrite (48) as
\[
\frac{V_0}{\phi_0} L_L^2 \ll 1,
\]
(49)

which is equivalent to condition (12) which came from the PN limit considerations.

So, we see that the trace of Chameleon-like behaviour is clear in the PN limit as we expected. As a final result, the PN limit of the massive BD theory (with $\omega = 0$) is applicable for gravitational systems such as the solar system where the Chameleon-like behaviour can occur. And one can linearize the field equations to the PN order, and the result of this expansion and the Chameleon behaviour is similar. As we mentioned before the Chameleon-like behaviour which exists in the Jordan frame is different from the original Chameleon theory [7]. On the other hand, we showed that the required conditions for this mechanism and those coming from PN limit considerations for having a viable theory in the solar system are consistent with each other.
4. Conclusion

In this paper, we have shown that although the field equation of the scalar field in the massive BD theory allows the existence of the odd-order terms in the PN expansion of the scalar field, but they are all zero. This means that there are no nonlinear terms in the field equations. This is an interesting result because unlike the claim that exists in the literature, one can linearize the field equation to the PN order and also see the Chameleon-like behaviour. It is worth mentioning that our conclusion does not mean that the nonlinearity in the expansion of the Ricci scalar, coming from the largeness of the local Ricci scalar relative to the cosmological background Ricci scalar, is not important. This means that the smallness of $\phi^{(2)}$ relative to $\phi_0$ does not necessarily imply $\Delta R/R_0 \ll 1$, and one should note that the condition $\Delta R/R_0 \gg 1$ has been saved in obtaining the PN approximation. For the massive BD theory for which the coupling constant is zero, we showed that the Chameleon-like behaviour can exist if $\phi \simeq 1$, $|V/V'| \ll 1$ and if the thin-shell condition $|\phi^{(2)}| \ll \Phi_N(r)$ is satisfied, and more importantly this mechanism is different from the Chameleon mechanism. In fact, if the scalar field is forced to be heavy at the solar system through the Chameleon-like behaviour, then it has to also be heavy at the cosmological scales. And this is completely against the nature of the Chameleon scalar fields. Also, we showed that the results of the PN limit and Chameleon-like behaviour are similar, and when the Chameleon-like behaviour exists, the perturbation of $\phi$ due to the local system is much smaller than the cosmological background value. Thus, we can linearize the field equations to the PN order.

We finish the conclusion by two examples which confirm the consistency of the PN consideration and the Chameleon-like behaviour. Consider the model $f(R) \sim R^{1+\delta}$. This model can be consistent with the solar system observations if $\delta \sim -1.23 \pm 2.05 \times 10^{-17}$ [17] (albeit we have taken into account the term $(1 - 1/\sqrt{1 - e^2})^{-1}$ which is absent in equation (83) of [17]); thus, the Chameleon mechanism occurs for this model. On the other hand, from (46) and the amount of $\Delta R/R_0$ in the solar system, we can write

$$\frac{m_0^2}{H_0^2} > 10^{12}. \quad (50)$$

By using this condition for the above model, we reach that $\delta < 10^{-12}$, $\delta \sim 10^{-12}$ is sufficient for being consistent with the light deflection experiments, but for consistency with other observations such as the perihelion precession of Mercury, a smaller amount of $\delta$ is needed. So, condition (50) leads to a right bound on $\delta$ in agreement with the result of [17]. For another example, consider the model $f(R) \sim R - \lambda R_c (\frac{R}{R_c})^p$ [18]. The bound on $p$ coming from the Chameleon effect is $p < 10^{-10}$ [19]. On the other hand, using equation (50) one can easily verify that $p < 10^{-12}$ which is consistent with the previous bound.

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References

[1] Nutku Y 1969 Astrophys. J. 155 999–1007
Nordtvedt K 1970 Astrophys. J. 161 1059–67
Wagoner R V 1970 Phys. Rev. D 1 3209

[2] Faraoni V 2004 Cosmology in Scalar-Tensor Gravity (Dordrecht: Kluwer)
[3] Banerjee N and Sen S 1997 Phys. Rev. D 56 1334
[4] Bhadra A and Nadi K K 2001 Phys. Rev. D 64 087501
[5] Chiba T 2003 Phys. Lett. B 575 1
Soussa M E and Woodard R P 2004 Gen. Rel. Grav. 36 855
Olmo G J 2005 Phys. Rev. Lett. 95 261102
Erickcek A L, Smith T L and Kamionkowski M 2006 Phys. Rev. D 74 121501
[6] Amendola L 2000 Phys. Rev. D 62 043511
[7] Khoury J and Weltman A 2004 Phys. Rev. D 69 044026
Khoury J and Weltman A 2004 Phys. Rev. Lett. 93 171104
[8] De Felice A and Tsujikawa S 2010 Living Rev. Rel. 13 3
Navarro I and van Acoleyen K 2007 J. Cosmol. Astropart. Phys. JCAP02(2007)022
[9] Will C M 1993 Theory and Experiment in Gravitational Physics (Cambridge: Cambridge University Press)
[10] Olmo G J 2005 Phys. Rev. D 72 083505
[11] Perivolaropoulos L 2010 Phys. Rev. D 81 047501
[12] Bertotti B, Iess L and Tortora P 2003 Nature 425 374
[13] Chiba T, Smith T L and Erickcek A L 2007 Phys. Rev. D 75 124014
[14] Weinberg S 1972 Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity
(New York: Wiley)
[15] Fulkner T, Tegmark M, Bunn E F and Mao Y 2007 Phys. Rev. D 76 063505
[16] Hu W and Sawicki I 2007 Phys. Rev. D 76 064004
[17] Clifton T and Barrow J D 2005 Phys. Rev. D 72 103005
[18] Li B and Barrow J D 2007 Phys. Rev. D 75 084010
[19] Capozziello S and Tsujikawa S 2008 Phys. Rev. D 77 107501