ALL TIGHT DESCRIPTIONS OF 3-PATHS IN PLANE GRAPHS WITH GIRTH AT LEAST 8

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Abstract. Lebesgue (1940) proved that every plane graph with minimum degree $\delta$ at least 3 and girth $g$ (the length of a shortest cycle) at least 5 has a path on three vertices (3-path) of degree 3 each. A description is tight if no its parameter can be strengthened, and no triplet dropped.

Borodin et al. (2013) gave a tight description of 3-paths in plane graphs with $\delta \geq 3$ and $g \geq 3$, and another tight description was given by Borodin, Ivanova and Kostochka in 2017.

In 2015, we gave seven tight descriptions of 3-paths when $\delta \geq 3$ and $g \geq 4$. Furthermore, we proved that this set of tight descriptions is complete, which was a result of a new type in the structural theory of plane graphs. Also, we characterized (2018) all one-term tight descriptions if $\delta \geq 3$ and $g \geq 3$. The problem of producing all tight descriptions for $g \geq 3$ remains widely open even for $\delta \geq 3$.

Recently, eleven tight descriptions of 3-paths were obtained for plane graphs with $\delta = 2$ and $g \geq 4$ by Jendrol’, Macekov´a, Montassier, and Soták, four of which descriptions are for $g \geq 9$. In 2018, Aksenov, Borodin and Ivanova proved nine new tight descriptions of 3-paths for $\delta = 2$ and $g \geq 9$ and showed that no other tight descriptions exist.

The purpose of this note is to give a complete list of tight descriptions of 3-paths in the plane graphs with $\delta = 2$ and $g \geq 8$.

Keywords: Plane graph, structure properties, tight description, 3-path, minimum degree, height, weight, girth.
1. Introduction

Throughout the paper, $G$ is a plane graph. Let $\delta(G)$ be the minimum vertex degree and let $w_k(G)$ be the minimum degree-sum of a path on $k$ vertices in $G$. We will drop the argument when $G$ is clear from context. The degree of a vertex $v$ or a face $f$, that is the number of edges incident with $v$ or $f$, is denoted by $d(v)$ or $d(f)$, respectively. A $k$-vertex is a vertex $v$ with $d(v) = k$. By $k^+$ or $k^-$ we denote any integer not smaller or not greater than $k$, respectively. Hence, a $k^+$-vertex $v$ satisfies $d(v) \geq k$, etc. An edge $uv$ is an $(i,j)$-edge if $d(u) \leq i$ and $d(v) \leq j$. A path $uvw$ is a path of type $(i,j,k)$ or $(i,j,k)$-path if $d(u) \leq i$, $d(v) \leq j$, and $d(w) \leq k$.

Already in 1904, Wernicke [29] proved that every $G$ with $\delta = 5$ has a $(5,6)$-edge, and Franklin [17] strengthened this to the existence of at least two $6^-$-neighbors of a $5^-$-vertex; this implies that $w_3 \leq 17$, which bound is sharp. Recently, we proved [8] that there is also a $(5,6,6)$-path, which is tight, and there are no tight descriptions of 3-paths for $\delta = 5$ other than $\{(6,5,6)\}$ and $\{(5,6,6)\}$. (A description of paths is tight if no its parameter can be strengthened and no term dropped).

In 1996, Jendrol’ and Madaras [25] ensured for $\delta = 5$ that $w_3 \leq 23$, which is sharp. Recently, we found [9] the first tight description of 4-paths for $\delta = 5$, and then Batueva, Borodin and Ivanova [4] found all remaining nine such tight descriptions.

It follows from Lebesgue’s [28] results in 1940 that each $G$ with $\delta \geq 3$ satisfies $w_2 \leq 14$. For 3-connected plane graphs, Kotzig [27] proved a precise result: $w_2 \leq 13$.

In 1972, Erdős (see [18]) conjectured that Kotzig’s bound $w_2 \leq 13$ holds for all plane graphs with $\delta \geq 3$. Barnette (see [18]) announced to have proved this conjecture, but the proof has never appeared in print. The first published proof of Erdős’ conjecture is due to Borodin [5]. More generally, Borodin [6] proved that every $G$ with $\delta \geq 3$ contains a $(3,10)$-, or $(4,7)$-, or $(5,6)$-edge, which description is tight.

In 1993, Ando, Iwasaki, Kaneo [3] proved that every 3-connected $G$ satisfies $w_3 \leq 21$, which is sharp due to the Jendrol’ construction in [19]. This was refined by Borodin [7] in 1997 as follows: every 3-connected $G$ has: (i) either $w_3 \leq 18$ or a vertex of degree $\leq 15$ adjacent to two 3-vertices, and (ii) either $w_3 \leq 17$ or $w_2 \leq 7$. Here, the bounds $w_3 \leq 21$ and $w_3 \leq 17$ were known to be tight long ago, and the sharpness of $w_3 \leq 18$ was recently confirmed by Borodin et al. [14].

In 1997, Jendrol’ [20] gave an approximate description of 3-paths: every $G$ with $\delta \geq 3$ and $g \geq 3$ has a 3-path of one of the following types: $(10,3,10)$, $(7,4,7)$, $(6,5,6)$, $(3,4,15)$, $(3,6,11)$, $(3,8,5)$, $(3,10,3)$, $(4,4,11)$, $(4,5,7)$, or $(4,7,5)$.

In 2013, Borodin et al. [14] gave the first tight description of 3-paths: every $G$ with $\delta \geq 3$ and $g \geq 3$ has a 3-path of one of the following types: $(3,4,11)$, $(3,7,5)$, $(3,10,4)$, $(3,15,3)$, $(4,4,9)$, $(6,4,8)$, $(7,4,7)$, $(6,5,6)$. Another similar tight description for $\delta \geq 3$ and $g \geq 3$ was given by Borodin, Ivanova and Kostochka [15].

In 2015, we [10] gave seven tight descriptions of 3-paths when $\delta \geq 3$ and $g \geq 4$. Furthermore, we proved that this set of descriptions is complete, which was a result of a new type in the structural theory of plane graphs. Also, we [12] characterized all one-term tight descriptions if $\delta \geq 3$ and $g \geq 3$. The problem of producing all tight descriptions for $g \geq 3$ remains widely open even for $\delta \geq 3$. Other results on $k$-paths with $k \geq 3$ and $\delta \geq 3$ can be found in surveys Borodin–Ivanova [11], Cranston–West [16] and Jendrol’–Voss [26].
Aksenov, Borodin and Ivanova [1] proved precise upper bounds for $w_3$ in several natural classes of plane graphs with $\delta = 2$ and $5 \leq g \leq 7$ and disproved a conjecture by Jendrol’ and Maceková [21] concerning the case $g = 5$.

Recently, eleven tight descriptions of 3-paths were obtained for $\delta = 2$ and $g \geq 4$ by Jendrol’, Maceková, Montassier, and Soták [21–24], four of which descriptions are for $g \geq 9$ (for details, see Theorems 1 and 2 below).

Aksenov, Borodin and Ivanova [2] gave the following complete list of tight descriptions of 3-paths centered at 2-vertices whenever $g \geq 9$.

**Theorem 1** ([2]). There exist precisely these tight descriptions of 3-paths in plane graphs with minimum degree 2 and girth $g$ at least 9:

(A) $g \geq 16$: $\{(2, 2, 2)\}$ (folklore);

(B) $11 \leq g \leq 15$:

$\{(2, 2, 3)\}$ ([22]) and

$\{(2, 3, 2)\}$;

(C) $g = 10$:

$\{(2, 2, 3), (2, 3, 2)\}$ ([21], the tightness shown in [22])

$\{(2, 4, 2)\}$ ([22]),

$\{(2, 3, 3), (2, 2, 4), (3, 2, 3)\}$, and

$\{(3, 2, 4)\}$;

(D) $g = 9$:

$\{(2, 2, 5), (2, 3, 2)\}$ ([23]),

$\{(2, 5, 2), (2, 2, 3)\}$, $\{(2, 2, 5), (3, 2, 3)\}$, $\{(2, 5, 3)\}$, $\{(2, 3, 5)\}$, and

$\{(3, 2, 5)\}$.

In [13] we described all tight descriptions of 3-paths centered at 2-vertices whenever $g \geq 6$.

The purpose of this note is to completely resolve the case $g \geq 8$, as follows.

**Theorem 2.** There exist these and only these five tight descriptions of 3-paths in plane graphs with minimum degree 2 and girth at least 8:

(i) $\{(2, 2, 5), (2, 3, 2)\}$ ([22]);

(ii) $\{(2, 5, 2), (2, 2, 3)\}$;

(iii) $\{(3, 2, 5)\}$ ([13]);

(iv) $\{(2, 3, 5)\}$;

(v) $\{(2, 5, 3)\}$.

2. **Proving Theorem 2**

Since $\{(2, 2, 5), (2, 3, 2)\}$ is a tight description ([22]), it follows that $\{(2, 3, 5)\}$ is also a description, since both a $(2, 2, 5)$-path and a $(2, 3, 2)$-path are also $(2, 3, 5)$-paths. So, we first prove that $\{(2, 5, 2), (2, 2, 3)\}$ is a description. This implies that $\{(2, 5, 3)\}$ is also a description, since both a $(2, 5, 2)$-path and a $(2, 2, 3)$-path are also $(2, 5, 3)$-paths.

Next, we show that all descriptions $\{(2, 5, 2), (2, 2, 3)\}$, $\{(2, 3, 5)\}$ and $\{(2, 5, 3)\}$ are tight.

Finally, we show that there are no tight descriptions for $g \geq 8$ other than those five listed in Theorem 2.
2.1. **Proving that** \{(2,5,2),(2,2,3)\} **is a description.** We need the following refinement of the already mentioned Borodin’s [6] tight description \{(3,10),(4,7),(5,6)\} of 2-paths (that is, edges) in plane graphs with \(\delta \geq 3\).

**Lemma 1** ([13]). Every plane graph with minimum degree at least 3 has at least one of the following:

(a) a 3-face incident with a (3,10)-, or (4,7)-, or (5,6)-edge;
(b) a 4-face incident either with two 3-vertices and another 5-vertex or with a 3-vertex, two 4-vertices and the forth vertex of degree at most 5;
(c) a 5-face incident with four 3-vertices and the fifth vertex of degree at most 5, where all parameters are best possible.

Suppose on the contrary that \(G\) does not obey the description \{(2,5,2),(2,2,3)\}.

We consider the graph \(G^*\) with \(\delta(G^*) \geq 3\) obtained from \(G\) by contracting all 2-vertices and look at its 5-vertices \(f^*\) implied by Lemma 1. Note that \(g(G^*) \geq 3\) since \(g(G) \geq 8\) and due to the absence \((2,2,2)\)-paths in \(G\).

If \(f^* = v_1v_2v_3\) with \(d(v_i) \leq 5\), then then the boundary \(\partial(f)\) of the pre-image \(f\) of \(f^*\) in \(G\) must have at most two 2-vertices at the pair of edges since \(g(G) \geq 8\), which is impossible due to the absence \((2,2,3)\)-paths in \(G\). On the other hand, both edges \(v_1v_2\) and \(v_1v_3\) cannot contain a 2-vertices due to the absence of \((2,5,2)\)-paths in \(G\). This implies \(d(f) \leq 3 + 2 + 1 < 8\), a contradiction.

Now suppose \(d(f^*) = v_1v_2v_3v_4\). It follows from Lemma 1(b) that \(f^*\) must have a 3-vertex, say \(v_1\), opposite to a 5-vertex, \(v_3\), in \(\partial(f)\). It is not hard to see that to avoid \((2,5,2)\)-paths and \((2,2,3)\)-paths, our \(f\) can have at most one 2-vertex at the pair of edges \(v_1v_2, v_1v_4\) and at most two 2-vertices at \(v_2v_3, v_3v_4\). However, then \(d(f) \leq 4 + 1 + 1 < 8\), a contradiction.

Finally, suppose \(d(f^*) = v_1 \ldots v_5\) with \(d(v_1) = \ldots = d(v_4) = 3\) and \(d(v_5) \leq 5\). Now at most one 2-vertex may be put on every edge in \(\partial(f)\) and, moreover, at most one 2-vertex can appear at any two consecutive edges in \(\partial(f)\). This implies \(d(f) \leq 5 + 2 \times 1 < 8\), a contradiction.

2.2. **Proving the tightness of** \{(2,5,2),(2,2,3)\}, \{(2,3,5)\} **and** \{(2,5,3)\}. To show the tightness of \{(2,5,2),(2,2,3)\}, \{(2,3,5)\} and \{(2,5,3)\}, we have to prove that neither \{(2,4,2), (2,2,3)\} nor \{(2,5,2),(2,2,3)\} is a description. To reject the former option, it suffices to put two 2-vertices on every edge of the icosahedron and note that the graph \(H_1\) obtained has girth 9 but no \((4,4,4)\)-paths, since each its 3-path goes through 5-vertex.

To reject the latter, we replace the graph obtained in Jendrol’ et al. (see Fig. 5 in [22]). Take concentric cycles \(W_8 = w_1 \ldots w_8, XY_{16} = x_1y_1 \ldots x_8y_8, Z_8 = z_1 \ldots z_8\), and add a path with two internal 2-vertices between \(w_i\) to \(x_i\) and also between \(y_i\) and \(z_i\) whenever \(1 \leq i \leq 8\). It remains to observe that \(H_2\) obtained has no \((2,5,2)\)-paths (and, in particular, no \((2,2,2)\)-paths and \(g(H_2) = 8\)).

The tightness of \{(2,3,5)\} follows similarly from the same graphs \(H_1\) and \(H_2\). Indeed, we cannot strengthen \{(2,5,3)\} to \{(2,4,3)\}, since each 3-path in \(H_1\) goes through 5-vertex, and to \{(2,5,2)\}, since \(H_2\) has no vertex adjacent to two 2-vertices.

Finally, to see the tightness of \{(2,3,5)\}, it suffices to observe that \{(2,2,3)\} is invalid due to \(H_1\), while \{(2,2,5)\} fails to describe the graph \(H_3\) that is a result of deleting all edges joining 10-vertices to 10-vertices from the triangulation obtained from the icosahedron by putting a 3-vertex into each face and joining it to the
boundary vertices of this face. Note that $H_3$ is bipartite, with every edge joining a 3-vertex to a 5-vertex.

2.3. Proving the non-existence of tight descriptions other than those five in Theorem 2. Suppose $D = \{(x_1, y_1, z_1), \ldots, (x_k, y_k, z_k)\}$ is a tight description of 3-paths in plane graphs with $\delta = 2$ and $g \geq 8$. By symmetry, we can assume that $x_i \leq z_i$ whenever $1 \leq i \leq k$. It follows from the graph $H_1$ above that $D$ must have an entry, say $y_1$ or $z_1$, not smaller than 5.

Case 1. $y_1 \geq 5$. Now $(x_1, y_1, z_1) = (2, 5^+, 2)$ since $\{(2, 5, 3)\}$ is already a tight description. It follows from $H_2$ (which has no $(2, 5, 2)$-paths) that, say, $(x_2, y_2, z_2) = (2^+, 2^+, 3^+)$. Since $\{(2, 5, 2), (2, 2, 3)\}$ is known to be a tight description, it follows that $D = \{(2, 5, 2), (2, 2, 3)\}$.

Case 2. $z_1 \geq 5$. Now $(x_1, y_1, z_1) = (2, 2, 5^+)$ since $\{(3, 2, 5)\}$ and $\{(2, 3, 5)\}$ are tight descriptions. Let $H_4$ be a graph obtained from the dodecahedron by putting a 2-vertex on each edge. Clearly, $H_4$ satisfies $g(H_4) = 10$ and has only $(3, 2, 5)$- and $(2, 3, 2)$-paths. This implies that $D$ must have a term, say $(x_2, y_2, z_2)$, such that either $(x_2, y_2, z_2) = (3^+, 2^+, 3^+)$ or $(x_2, y_2, z_2) = (3^+, 2^+, 3^+)$.

In the first case, we have $D = \{(2, 2, 5), (3, 2, 3)\}$ since $\{(2, 2, 5), (3, 2, 3)\}$ is known to be a tight description. In the second case, we similarly deduce that $D = \{(2, 2, 5), (2, 3, 2)\}$, as desired.

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