Boundary layer problems for the two-dimensional inhomogeneous incompressible magnetohydrodynamics equations

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Received: 22 September 2020 / Accepted: 22 February 2021 / Published online: 3 April 2021
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Abstract
In this paper, we study the well-posedness of boundary layer problems for the inhomogeneous incompressible magnetohydrodynamics (MHD) equations, which are derived from the two-dimensional density-dependent incompressible MHD equations. Under the assumption that initial tangential magnetic field is not zero and density is a small perturbation of the outer constant flow in supernorm, the local-in-time existence and uniqueness of inhomogeneous incompressible MHD boundary layer equations are established in weighted conormal Sobolev space by energy method.

Mathematics Subject Classification 76D10 · 35M33 · 35Q35

Contents

1 Introduction and main result ....................................... 2
2 Difficulties and outline of our approach ................................. 8
3 A priori estimate ............................................. 10
   3.1 Weighted $\mathcal{H}_m$-estimates with conormal derivative .......... 11
   3.2 Weighted $\mathcal{H}_m$-estimates only on tangential derivative ....... 22
   3.3 Weighted $\mathcal{H}_m^{-1}$-estimates for normal derivative .......... 33
   3.4 $L^\infty$-estimates ............................................ 40
   3.5 Proof of Theorem 3.1 ........................................ 46
1 Introduction and main result

In this paper, we consider the boundary layer problems in the small viscosity and resistivity limit for the two-dimensional inhomogeneous incompressible Magnetohydrodynamics (MHD) equation in a period domain $\Omega := \{(x, y) : x \in \mathbb{T}, y \in \mathbb{R}^+\}$:

$$
\begin{cases}
\partial_t \rho^\varepsilon + \text{div}(\rho^\varepsilon \mathbf{u}^\varepsilon) = 0, \\
\rho^\varepsilon \partial_t \mathbf{u}^\varepsilon + \rho^\varepsilon (\mathbf{u}^\varepsilon \cdot \nabla) \mathbf{u}^\varepsilon - \mu \varepsilon \Delta \mathbf{u}^\varepsilon + \nabla p^\varepsilon = (\mathbf{h}^\varepsilon \cdot \nabla) \mathbf{h}^\varepsilon, \\
\partial_t \mathbf{h}^\varepsilon - \nabla \times (\mathbf{u}^\varepsilon \times \mathbf{h}^\varepsilon) - \kappa \varepsilon \Delta \mathbf{h}^\varepsilon = 0, \\
\text{div} \mathbf{u}^\varepsilon = 0, \quad \text{div} \mathbf{h}^\varepsilon = 0.
\end{cases}
$$

(1.1)

Here, we assume the viscosity and resistivity coefficients have the same order of a small parameter $\varepsilon$. The unknown functions $\rho^\varepsilon$ denotes the density of fluid, $\mathbf{u}^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon)$ denotes the velocity vector, $\mathbf{h}^\varepsilon = (h_1^\varepsilon, h_2^\varepsilon)$ denotes the magnetic field, and $p^\varepsilon = \widetilde{p}^\varepsilon + \frac{|\mathbf{h}^\varepsilon|^2}{2}$ represents the total pressure with $\widetilde{p}^\varepsilon$ the pressure of fluid. This system (1.1) can be used as model to describe a viscous fluid that is incompressible but has nonconstant density, and hence, it is much more complex than the classical incompressible MHD equation with constant density.

To complete the system (1.1), the boundary conditions are given by

$$u_1^\varepsilon|_{y=0} = u_2^\varepsilon|_{y=0} = 0, \quad \partial_y h_1^\varepsilon|_{y=0} = h_2^\varepsilon|_{y=0} = 0. \quad (1.2)$$

As the parameter $\varepsilon$ tends to zero in the system (1.1), we obtain the following system formally

$$
\begin{cases}
\partial_t \rho^0 + \text{div}(\rho^0 \mathbf{u}^0) = 0, \\
\rho^0 \partial_t \mathbf{u}^0 + \rho^0 (\mathbf{u}^0 \cdot \nabla) \mathbf{u}^0 + \nabla p^0 = (\mathbf{h}^0 \cdot \nabla) \mathbf{h}^0, \\
\partial_t \mathbf{h}^0 - \nabla \times (\mathbf{u}^0 \times \mathbf{h}^0) = 0, \\
\text{div} \mathbf{u}^0 = 0, \quad \text{div} \mathbf{h}^0 = 0.
\end{cases}
$$

which is the inhomogeneous incompressible ideal MHD system with the unknown function $(\rho^0, \mathbf{u}^0, \mathbf{h}^0)$. To find out the terms in (1.1) whose contributions are essential for the boundary layer, we use the same scaling as the one used in [31,39]

$$t = t, \quad x = x, \quad \tilde{y} = \varepsilon^{-\frac{1}{2}} y,$$

and set

$$
\rho(t, x, \tilde{y}) = \rho^\varepsilon(t, x, y), \quad p(t, x, \tilde{y}) = p^\varepsilon(t, x, y), \\
u_1(t, x, \tilde{y}) = u_1^\varepsilon(t, x, y), \quad u_2(t, x, \tilde{y}) = \varepsilon^{-\frac{1}{2}} u_2^\varepsilon(t, x, y), \\
h_1(t, x, \tilde{y}) = h_1^\varepsilon(t, x, y), \quad h_2(t, x, \tilde{y}) = \varepsilon^{-\frac{1}{2}} h_2^\varepsilon(t, x, y),
$$
then the system (1.1), after taking the leading order, is reduced to

\[
\begin{align*}
\partial_t \rho + u_1 \partial_x \rho + u_2 \partial_y \rho &= 0, \\
\rho \partial_t u_1 + \rho u_1 \partial_x u_1 + \rho u_2 \partial_y u_1 - \mu \partial_y^2 u_1 + \partial_x p &= h_1 \partial_x h_1 + h_2 \partial_y h_1, \\
\partial_y p &= 0, \\
\partial_t h_1 + \partial_y (u_2 h_1 - u_1 h_2) &= \kappa \partial_y^2 h_1, \\
\partial_t h_2 - \partial_x (u_2 h_1 - u_1 h_2) &= \kappa \partial_x^2 h_2, \\
\partial_x u_1 + \partial_y u_2 &= 0, \quad \partial_x h_1 + \partial_y h_2 = 0,
\end{align*}
\]

(1.3)

where \((t, x, y) \in [0, T] \times \Omega\), here we have replaced \(\tilde{y}\) by \(y\) for simplicity of notations. Indeed, the nonlinear boundary layer system (1.3) becomes the classical well-known unsteady boundary layer system if the density becomes constant and magnetic field vanishes (cf. [45]).

The third equation of system (1.3) implies that the leading order of boundary layers for the total pressure \(p^\varepsilon(t, x, y)\) is invariant across the boundary layer, and should be matched to the outflow pressure \(P(t, x)\) on top of boundary layer, that is, the trace of pressure of idea MHD flow. Hence, we obtain

\[
p(t, x, y) \equiv P(t, x).
\]

Furthermore, the density \(\rho(t, x, y)\), tangential component \(u_1(t, x, y)\) of velocity field, \(h_1(t, x, y)\) of magnetic field, should match the outflow density \(\theta(t, x)\), tangential velocity \(U(t, x)\) and tangential magnetic field \(H(t, x)\), on the top of boundary layer, that is

\[
\rho(t, x, y) \to \theta(t, x), \quad u_1(t, x, y) \to U(t, x), \quad h_1(t, x, y) \to H(t, x), \text{ as } y \to +\infty,
\]

where \(\theta(t, x)\), \(U(t, x)\) and \(H(t, x)\) are the trace of density, tangential velocity and tangential magnetic field respectively. Then, we have the following matching conditions:

\[
\partial_t \theta + U \partial_x \theta = 0, \quad \theta \partial_t U + \theta U \partial_x U + \partial_x P = H \partial_x H, \quad \partial_t H + U \partial_x H - H \partial_x U = 0.
\]

(1.4)

Moreover, by virtue of the boundary condition (1.2), one attains the following boundary condition

\[
u_1 \big|_{y=0} = u_2 \big|_{y=0} = \partial_y h_1 \big|_{y=0} = h_2 \big|_{y=0} = 0.
\]

(1.5)

In this paper, we consider the outer flow \((\theta, U, H) = (1, 1, 1)\), which implies the pressure \(p\) being a constant. On the other hand, it is noted that the fifth equation of (1.3) is a direct consequences of the fourth equation of (1.3). Hence, we only need to study the following initial boundary value problem for the inhomogeneous incompressible MHD boundary layer equation

\[
\begin{align*}
\partial_t \rho + u_1 \partial_x \rho + u_2 \partial_y \rho &= 0, \\
\rho \partial_t u_1 + \rho u_1 \partial_x u_1 + \rho u_2 \partial_y u_1 - \mu \partial_y^2 u_1 &= h_1 \partial_x h_1 + h_2 \partial_y h_1, \\
\partial_y h_1 + \partial_y (u_2 h_1 - u_1 h_2) - \kappa \partial_y^2 h_1 &= 0, \\
\partial_x u_1 + \partial_y u_2 &= 0, \quad \partial_x h_1 + \partial_y h_2 = 0,
\end{align*}
\]

(1.6)

where the density \(\rho := \rho(t, x, y)\), velocity field \((u_1, u_2) := (u_1(t, x, y), u_2(t, x, y))\), the magnetic field \((h_1, h_2) := (h_1(t, x, y), h_2(t, x, y))\) are unknown functions. The boundary
conditions for equation (1.6) are given by

\[
\begin{aligned}
\left\{\begin{array}{l}
u|y=0 = \nu|y=0 = \partial_y h_1|y=0 = h_2|y=0 = 0, \\
\lim_{y \to +\infty} \rho(t, x, y) = \lim_{y \to +\infty} u_1(t, x, y) = \lim_{y \to +\infty} h_1(t, x, y) = 1.
\end{array}\right.
\]
\]

\hspace{1cm} (1.7)

Let us first introduce some weighted Sobolev spaces for later use. For any \( l \in \mathbb{R} \), denote by \( L^2_l(\Omega) \) the weighted Lebesgue space with respect to the spatial variables:

\[
L^2_l(\Omega) := \{ f(x, y) : \Omega \to \mathbb{R}, \| f \|_{L^2_l(\Omega)}^2 := \int_{\Omega} |f(x, y)|^2 dx dy < +\infty, \langle y \rangle := 1 + y, \}\]

and denote the weighted \( L^\infty_l(\Omega) \) Lebesgue space by

\[
L^\infty_l(\Omega) := \{ f(x, y) : \Omega \to \mathbb{R}, \| f \|_{L^\infty_l(\Omega)} := \text{esssup}_{(x, y) \in \Omega} |f(x, y)| < +\infty, \langle y \rangle := 1 + y. \}
\]

To define the conormal Sobolev spaces, we will use the notation: \( Z_1 = \partial_x, Z_2 = \varphi(y) \partial_y \), where the function \( \varphi(y) := \frac{y}{1+y} \). Then, we can define the conormal Sobolev spaces as follows:

\[
H^{m,l}_{\text{co}} := \{ f \in L^2_l(\Omega) | Z^l f \in L^2_l(\Omega), |I| \leq m \},
\]

where \( I = (I_1, I_2) \) and \( Z^l = Z_1^{I_1} Z_2^{I_2} \). We also use the notations

\[
\| u \|_{m,l}^2 = \sum_{|\alpha| \leq m} \| Z^\alpha u \|_{L^2_l(\Omega)}^2, \quad \| u \|_{m,\infty}^2 = \sum_{|\alpha| \leq m} \| Z^\alpha u \|_{L^\infty_l(\Omega)}^2.
\]

It is easy to check that

\[
Z_i Z_j = Z_j Z_i, \quad j, k = 1, 2,
\]

and

\[
\partial_y Z_1 = Z_1 \partial_y, \quad \partial_y Z_2 \neq Z_2 \partial_y.
\]

For later use and notational convenience, set \( Z_\tau = (\partial_t, Z_1) \) and \( Z^\alpha = Z_\tau^{\alpha_1} Z_2^{\alpha_2} = \partial_t^{\alpha_1} Z_1^{\alpha_1} Z_2^{\alpha_2} \), where \( \alpha, \alpha_1, \alpha_2 \) are the differential multi-indices with \( \alpha = (\alpha_1, \alpha_2), \alpha_1 = (\alpha_{11}, \alpha_{12}) \), and we also use the notations

\[
\| f(t) \|_{H^m_t}^2 = \sum_{|\alpha| \leq m} \| Z^\alpha f(t) \|_{L^2_t(\Omega)}^2, \quad \| f(t) \|_{H^\infty_t}^2 = \sum_{|\alpha| \leq m} \| Z^\alpha f(t) \|_{L^\infty_t(\Omega)}^2
\]

for any smooth space-time function \( f(x, t) \). We also use

\[
\| f(t) \|_{H^m_{t,\text{fin}}}^2 = \sum_{|\alpha| \leq m} \| Z^\alpha f(t) \|_{L^2_t(\Omega)}^2, \quad \| f(t) \|_{H^\infty_{t,\text{fin}}}^2 = \sum_{|\alpha| \leq m} \| Z^\alpha f(t) \|_{L^\infty_t(\Omega)}^2.
\]

Finally, we define the functional space \( B^m_l(T) \) for a pair of function \((\rho, u_1, h_1) = (\rho, u_1, h_1)(x, y, t) \) as follows:

\[
B^m_l(T) = \{(\rho - 1, u_1 - 1, h_1 - 1) \in L^\infty([0, T]; L^2_l(\Omega)) : \\
\text{esssup}_{0 \leq t \leq T} \| (\rho, u_1, h_1)(t) \|_{B^m_l} < +\infty \},
\]

\hspace{1cm} (1.8)

where the norm \( \| \cdot \|_{B^m_l} := \| \cdot \|_{B^m_l} + \| \cdot \|_{B^m_l} \) is given by

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\begin{equation}
\| (\rho, u_1, h_1)(t) \|_{B^m_i} := \| (\rho - 1, u_1 - 1, h_1 - 1)(t) \|_{H^m_i}^2 + \| \partial_y (\rho, u_1, h_1)(t) \|_{H^{m-1}_i}^2 \\
+ \| \partial_y \rho(t) \|_{H^1_{l_i}}^2, \tag{1.9}
\end{equation}

and
\begin{equation}
\| (\rho, u_1, h_1)(t) \|_{\bar{B}^m_i} := \sum_{i=0}^{m-1} \| \partial_t^i (\partial_x \rho, \partial_y \rho, \partial_x u_1, \partial_x h_1)(t) \|_{H^m_i}^2 \\
+ \sum_{i=0}^{m-1} \| \partial_t^i \partial_y (\partial_x \rho, \partial_y \rho, \partial_x u_1, \partial_x h_1)(t) \|_{H^{m-1}_i}^2 \\
+ \sum_{i=0}^{m-1} \| \partial_t^i \partial_y (\partial_x \rho, \partial_y \rho, \partial_x u_1, \partial_x h_1)(t) \|_{H^{m-1}_i}^2. \tag{1.10}
\end{equation}

In the present article, we supplement the MHD boundary layer equation (1.6) with the initial data
\begin{equation}
(\rho, u_1, h_1)(0, x, y) = (\rho_0, u_{10}, h_{10})(x, y), \tag{1.11}
\end{equation}
satisfying
\begin{equation}
0 < m_0 \leq \rho_0 \leq M_0 < +\infty, \tag{1.12}
\end{equation}
and
\begin{equation}
\| (\rho_0, u_{10}, h_{10}) \|_{B^m_i} \leq C_0 < +\infty, \tag{1.13}
\end{equation}
where \(m_0, M_0, C_0 > 0\) are positive constants and the time derivatives of initial data in (1.13) are defined through the MHD boundary layer equations (1.6). Hence, we set
\begin{equation}
B^{m,l}_{BL,ap} = \{ (\rho - 1, u_1 - 1, h_1 - 1) \in H^m_i | \partial_t (\rho, u_1, h_1), k = 1, \ldots, m \}
\end{equation}
are defined through Eq. (1.6) \tag{1.14}
and
\begin{equation}
\bar{B}^{m,l}_{BL} = \text{the closure of } B^{m,l}_{BL,ap} \text{ in the norm } \| \cdot \|_{B^m_i}. \tag{1.15}
\end{equation}

Now, we can state the main results with respect to the well-posedness theory for the inhomogeneous incompressible MHD boundary layer equations (1.6)–(1.7) in this paper as follows.

**Theorem 1.1** (Main Theorem) Let \(m \geq 5\) be an integer and \(l \geq 2\) be a real number. Assume the initial data \((\rho_0, u_{10}, h_{10}) \in B^{m,l}_{BL}\) given in (1.15) and satisfying (1.12) and (1.13). Moreover, there exists a small constant \(\delta_0 > 0\) such that
\begin{equation}
h_{10}(x, y) \geq 2\delta_0, \text{ for all } (x, y) \in \Omega, \tag{1.16}
\end{equation}
and
\begin{equation}
\| \rho_0 - 1 \|_{L^\infty_0(\Omega)} \leq \frac{2l - 1}{16} \delta_0, \| \partial_y u_{10} \|_{L^\infty_0(\Omega)} \leq (2\delta_0)^{-1}. \tag{1.17}
\end{equation}

Then, there exist a positive time \(0 < T_0 = T_0(\mu, \kappa, m, l, \delta_0, \| (\rho_0, u_{10}, h_{10}) \|_{B^m_i}, \| (\rho_0, u_{10}, h_{10}) \|_{\bar{B}^m_i})\) and a unique solution \((\rho, u_1, u_2, h_1, h_2)\) to the initial boundary value
problem (1.6)–(1.7), such that
\[
\sup_{0 \leq t \leq T_0} \left\{ \|(\rho - 1, u_1 - 1, h_1 - 1)(t)\|_{H^m}^2 + \|\partial_y (\rho, u_1, h_1)(t)\|_{H^{m-1}}^2 + \|\partial_y \rho(t)\|_{H^1,\infty}^2 \right\}
\]
\[+ \int_0^{T_0} \|\partial_y (\sqrt{\mu} u_1, \sqrt{\kappa} h_1)(t)\|_{H^m}^2 \, dt \]
\[+ \int_0^{T_0} \|\partial_y^2 (\sqrt{\mu} u_1, \sqrt{\kappa} h_1)(t)\|_{H^{m-1}}^2 \, dt \leq \widehat{C}_0 < +\infty,\]

where \(\widehat{C}_0\) depends only on \(l, \delta_0,\) and \(\| (\rho_0, u_{10}, h_{10})\|_{E^{\rho}}\).

**Remark 1.1** Note that we choose the initial data with higher regularity and conormal Sobolev space as our basic space since we construct the approximation system (2.1) to establish the well-posedness for the MHD boundary layer system (1.6).

**Remark 1.2** Note that the approach for the well-posedness result in Theorem 1.1 can be generalized to study the nonlinear problem (1.6) with a non-trivial Euler outflow \((U, H)\) satisfying the Eq. (1.4).

**Remark 1.3** We should point out that the initial condition (1.17) is not required when the incompressible magnetohydrodynamics flows are the case of homogeneous (cf. Remark 3.2). In other words, the local-in-time well-posedness of boundary layer system (1.6) with any large initial data can be obtained only under the condition (1.16) when the density is constant.

We now review some related works to the problem studied in this paper. The MHD system (1.1) is a combination of the inhomogeneous incompressible Navier–Stokes equations of fluid dynamics and Maxwell’s equations of electromagnetism. Since the study for MHD system has been along with that for Navier–Stokes one, let us recall some results about the incompressible inhomogeneous Navier–Stokes equations. If the initial density is bounded away from zero, Kazhikov [24] proved that: the inhomogeneous incompressible Navier–Stokes equations have at least one global weak solutions in the energy space. This result can be generalized to case of initial data with vacuum (cf. [33,46]). Choe and Kim [8] proved the existence and uniqueness of local strong solutions to the initial value problem or the initial boundary value problem even though the initial vacuum exists. Recently, the large-time decay and stability to any given global smooth solutions of the 3D incompressible inhomogeneous Navier–Stokes equations were obtained [1]. Let’s go back to the MHD system (1.1), it is known that Gerbeau and Le Bris [13] (see also Desjardins and Le Bris [11]) established the global existence of weak solutions of finite energy in the whole space or in the torus. The global existence of strong solution with small initial data in some Besov spaces was considered by Abidi and Paicu [2]. Recently, Gui [17] has shown that the 2D incompressible inhomogeneous magnetohydrodynamics system with a constant viscosity is globally well-posed for a generic family of the variations of the initial data and an inhomogeneous electrical conductivity.

When magnetic field vanishes, the MHD system (1.1) turns to be the classical well-known incompressible Navier–Stokes equations if the density being constant. As the viscosity \(\varepsilon\) tends to zero, the Navier–Stokes equations will become the Euler equations. There are lots of literatures on the uniform bounds and the vanishing viscosity limit for the Navier–Stokes equations without boundaries [9,10,23,34]. The time of existence \(T^\varepsilon\) always depend on the viscosity coefficient when the boundary appears. It is difficult to prove that the existence of time stays bounded away from zero. However, for the domain with some special types of
Navier-slip boundary conditions, some uniform $H^3$ (or $W^{2-p}$, with $p$ large enough) estimates and a uniform time of existence have recently been established [5,6,48]. This uniform control in some limited regularity Sobolev spaces can be obtained because these special boundary conditions gives arise to the disappearance of main part of boundary layer. For the three dimensional domain with smooth boundary, Masmoudi and Rousset [35] recently obtained conormal uniform estimates for the incompressible Navier–Stokes equations with Naiver-slip type boundary condition. Furthermore, they also applied the compact argument to establish the convergence of the viscous solution to the inviscid one. This result was generalized to the compressible flow [47], which also shown that the boundary layers for density must be weaker than the one of velocity.

The vanishing viscosity limit of the incompressible Navier–Stokes equations that, in a bounded domain with Dirichlet boundary condition, is an important problem in both physics and mathematics. This is due to the formation of a boundary layer, where the solution undergoes a sharp transition from a solution of the Euler system to the zero non-slip boundary condition on boundary of the Navier–Stokes system. This boundary layer satisfies the Prandtl system formally. Indeed, Prandtl [40] derived the Prandtl equations for boundary layer from the incompressible Navier–Stokes equations with non-slip boundary condition. The first systematic work in rigorous mathematics was obtained by Oleinik [37,38], in which she established the local-in-time well-posedness of the Prandtl equations in dimension two by applying the Crocco transformation under the monotonicity condition on the tangential velocity field in the normal direction to the boundary. For more extensional mathematical results, the interested readers can refer to the classical book finished by Oleinik and Samokhin [39]. By taking care of the cancelation in the convection term to overcome the loss of derivative in the tangential direction of velocity, the researchers in [3] and [36] independently used the simply energy method to establish well-posedness theory for the two-dimensional Prandtl equations in the framework of Sobolev space. For more results in this direction, the interested readers can refer to [7,14–16,22,25,26,30,51,52] and references therein.

Under the influence of electro-magnetic field, the system of magnetohydrodynamics (denoted by MHD) is a fundamental system to describe the movement of electrically conducting fluid, for example plasmas and liquid metals (cf. [4]). On one hand, Gérard-Varet and Prestipino [12] provided a systematic derivation of boundary layer models in magnetohydrodynamics, through an asymptotic analysis of the incompressible MHD system. Furthermore, they also performed some stability analysis for the boundary layer system, and emphasized the stabilizing effect of the magnetic field. On the other hand, if both the hydrodynamic Reynolds numbers and magnetic Reynolds numbers tend to infinity at the same rate, the local-in-time well-posedness of boundary layer system was obtained [31] if there exists a small constant $\delta_0$ such that

$$\langle y \rangle^{i+1} \partial_y^i (\mu_{10}, h_{10})(x, y) \leq (2\delta_0)^{-1}, \text{ for } i = 1, 2, (x, y) \in \Omega,$$

and (1.16) holds. In other words, the local in time well-posedness of MHD boundary layer system holds true under the condition that the initial tangential magnetic field is not zero instead of the monotonicity condition on the tangential velocity field. Recently there are many mathematical results on two dimensional MHD boundary layer system, for the ill-posedness results [28,29], almost global existence [27], global-in-time stability of MHD boundary layer in the Prandtl-Hartmann Regime [49], lifespan of solution with analytic perturbation of general shear flow [50], and local-in-time well-posedness for compressible MHD boundary layer [21].

Finally, we point out that it is an outstanding open problem to rigorously justify the validity of expansion in the inviscid limit. On one hand, Sammartino and Caflisch [43,44]
obtained the well-posedness in the framework of analytic functions without the monotonicity condition on the velocity field and justified the boundary layer expansion for the unsteady incompressible Navier–Stokes equations. Furthermore, Guo and Nguyen [18] concerned nonlinear ill-posedness of the Prandtl equation and an invalidity of asymptotic boundary layer expansion of incompressible fluid flow near a solid boundary. Furthermore, they also shown that the asymptotic boundary layer expansion was not valid for nonmonotonic shear layer flow in Sobolev spaces and verified that Oleinik’s monotonic solutions were well-posed. For the incompressible steady Navier–Stokes equations, Guo and Nguyen [19] justified the boundary layer expansion for the flow with a non-slip boundary condition on a moving plate. This result has been extended to the case of a rotating disk and to the case of nonshear Euler flows ([41,42]). Recently, Guo and Iyer [20] studied the boundary layer expansion for the small viscous flows with the classical no slip boundary conditions or on the static plate. As the magnetic field appears, the Prandtl ansatz boundary layer expansion for the unsteady MHD system was justified [32] when no-slip boundary and perfect conducting boundary conditions are imposed on velocity field and magnetic field respectively.

The rest of this paper is organized as follows. In Sect. 2, we explain the main difficulty and our approach to establish the local-in-time well-posedness theory for the Prandtl type Eq. (1.6). In Sect. 3, one establishes the a priori estimates for the nonlinear problem (3.2). The local-in-time existence and uniqueness of equation (1.6) in Weighted conormal Sobolev space are given in Sect. 4. Finally, some useful inequalities and important equivalent relations will be stated in Appendices A and B.

Before we proceed, let us comment on our notations. Throughout this paper, all constants $C$ may be different in different lines. Subscript(s) of a constant illustrates the dependence of the constant, for example, $C_s$ is a constant depending on $s$ only. Denote by $\partial_y^{-1}$ the inverse of the derivative $\partial_y$, i.e., $(\partial_y^{-1} f)(y) := \int_0^y f(z)dz$. Moreover, we also use the notation $[A, B] = AB - BA$, to denote the commutator between $A$ and $B$. Finally, $\mathcal{P}_i(\cdot, \cdot)$ stands for a polynomial function independent of $\epsilon$, and the index $i$ denote it changing from line to line.

## 2 Difficulties and outline of our approach

The main goal of this section is to explain main difficulties of proving Theorem 1.1 as well as our strategies for overcoming them. In order to solve the Prandtl type Eq. (1.6) in certain $H^m$ Sobolev space, the main difficulty comes from the vertical velocity $u_2 = -\partial_y^{-1}\partial_x u_1$ (and vertical magnetic field $h_2 = -\partial_y^{-1}\partial_x h_1$) creating a loss of $x-$derivative, so the standard energy estimate can not apply directly.

The main idea of establishing the well-posedness of inhomogeneous incompressible MHD boundary layer equations (1.6)–(1.7) is to apply the so-called vanishing viscosity and nonlinear cancelation methods (see [31,36]). To this end, we consider the following approximate problem:

\[
\begin{align*}
\partial_t \rho^\epsilon + u_1^\epsilon \partial_x \rho^\epsilon + u_2^\epsilon \partial_y \rho^\epsilon - \epsilon \partial_x^2 \rho^\epsilon - \epsilon \partial_y^2 \rho^\epsilon &= -\epsilon \partial_x r_1 - \epsilon \partial_y r_2, \\
\rho^\epsilon \partial_t u_1^\epsilon + \rho^\epsilon u_1^\epsilon \partial_x u_1^\epsilon + \rho^\epsilon u_2^\epsilon \partial_y u_1^\epsilon - \epsilon \partial_x^2 u_1^\epsilon - \mu \partial_y^2 u_1^\epsilon &= h_2^\epsilon \partial_x h_1^\epsilon + h_2^\epsilon \partial_y h_1^\epsilon - \epsilon \partial_x r_u, \\
\partial_t h_1^\epsilon + \partial_y (u_2^\epsilon h_1^\epsilon - u_1^\epsilon h_2^\epsilon) - \epsilon \partial_x^2 h_1^\epsilon - \kappa \partial_y^2 h_1^\epsilon &= -\epsilon \partial_x r_h, \\
\partial_x u_1^\epsilon + \partial_y u_2^\epsilon &= 0, \quad \partial_x h_1^\epsilon + \partial_y h_2^\epsilon = 0.
\end{align*}
\]
for any parameter $\epsilon > 0$. Here the functions $r_1, r_2, r_u$ and $r_h$ are defined by

$$(r_1, r_2, r_u, r_h)(t, x, y) = \sum_{i=0}^{m-1} i! \partial_t^i \partial_x^i \partial_y^i (\partial_x \rho, \partial_y \rho, \partial_x u_1, \partial_y h_1)(0, x, y),$$

(2.2)

which gives that by direct calculation

$$\partial_t^i (\rho^\epsilon, u_1^\epsilon, h_1^\epsilon)(0, x, y) = \partial_t^i (\rho, u_1, h_1)(0, x, y), \quad 0 \leq i \leq m.$$  (2.3)

To complete the system (2.1), the boundary conditions are given by

$$\begin{align*}
\partial_y \rho^\epsilon \big|_{y=0} &= u_1^\epsilon \big|_{y=0} = u_2^\epsilon \big|_{y=0} = \partial_y h_1^\epsilon \big|_{y=0} = 0, \\
\lim_{y \to \pm \infty} \rho^\epsilon (t, x, y) &= \lim_{y \to \pm \infty} u_1^\epsilon (t, x, y) = \lim_{y \to \pm \infty} h_1^\epsilon (t, x, y) = 1.
\end{align*}$$

(2.4)

Since the local-in-time existence and uniqueness of regularized Eqs. (2.1)–(2.4) can be obtained easily in $H^m$ Sobolev space for any $\epsilon > 0$, we hope that the solutions $(\rho^\epsilon, u_1^\epsilon, u_2^\epsilon, h_1^\epsilon, h_2^\epsilon)$ of regularized equation (2.1) will converge to the solution $(\rho, u_1, u_2, h_1, h_2)$ of original Prandtl Eq. (1.6) as $\epsilon$ tends to zero. To this end, we need to get the uniform a priori estimates of solution $(\rho^\epsilon, u_1^\epsilon, u_2^\epsilon, h_1^\epsilon, h_2^\epsilon)$ in an existence time independent of $\epsilon$. Although the idea of local-in-times well-posedness of MHD boundary layer equation, which only needs that the background tangential magnetic field has a lower positive bound instead of monotonicity assumption on the tangential velocity, comes from the recent result [31], we have to overcome some essential difficulties when density of fluid changes from a constant to unknown quantity.

First of all, we should work on the conormal Sobolev space to obtain some energy estimates independent of small coefficient $\epsilon$ since there is boundary condition for the first equation of (2.1). To control the vertical velocity $u_2^\epsilon = -\partial_y^{-1} \partial_x u_1^\epsilon$ by the horizontal velocity $u_1^\epsilon$, we need to apply the Hardy type inequality by adding a weight $(1 + y)^1$. Since the conormal derivative $Z_2 = \phi(y) \partial_y$ does not communicate with the normal derivative $\partial_y$, we need to choose the Sobolev space with suitable weight (actually taking $l \geq 2$) to close the energy estimate, which is the first novelty in our paper.

Similar to the Prandtl equation, the main difficulty in the analysis on the system (2.1) in the Sobolev framework is the loss of $x$–derivative in the vertical components $u_2^\epsilon$ and $h_2^\epsilon$ appearing in the terms $u_2^\epsilon \partial_y \rho^\epsilon, \rho^\epsilon \partial_x u_2^\epsilon, u_1^\epsilon h_2^\epsilon \partial_y, h_2^\epsilon \partial_x u_2^\epsilon$ and $u_2^\epsilon \partial_y h_2^\epsilon, h_2^\epsilon \partial_x u_1^\epsilon$ in the first, second and third equations of (2.1), respectively. Motivated by the recent result [31], we construct some quantities $(q_0^\epsilon, u_m^\epsilon, h_m^\epsilon)$ (see the definitions (3.51), (3.54), (3.48) respectively) to avoid the loss of $x$ derivative and obtain the estimate for $(q_0^\epsilon, u_m^\epsilon, h_m^\epsilon)$ in $L^2_\infty$–norm independent of $\epsilon$ by the energy method. To establish the relation between the quantities $(q_m^\epsilon, u_m^\epsilon, h_m^\epsilon)$ and $\partial_m^\epsilon (\rho^\epsilon, u_1^\epsilon, h_1^\epsilon)$, we need to control the quantity $\partial_y (\rho^\epsilon, u_1^\epsilon, h_1^\epsilon)$ in $L^\infty_\infty$–norm. To this end, we apply the low order tangential derivative estimate $E_{m,l}(t)$ (see the definition (3.11)) to control the quantity $\partial_y (u_1^\epsilon, h_1^\epsilon)$ in $L^\infty_\infty$–norm, which can be achieved by the Sobolev embedding inequality. Then, it is easy to get the almost equivalent relation $X_{m,l}(t) \sim Y_{m,l}(t)$ (see the definitions in (3.56) and (B.10) respectively). This is the second novelty in our paper, and it avoids some assumptions on the initial data conditions in (1.19) required in [31] for the MHD boundary layer equation with constant density.
3 A priori estimate

In this section, we will establish a priori estimates (independent of $\epsilon$), which are crucial to prove the Theorem 1.1. First of all, let us define

$$\varrho^\epsilon := \rho^\epsilon - 1, \quad u^\epsilon := u_1^\epsilon - 1 + e^{-y}, \quad v^\epsilon := u_2^\epsilon, \quad h^\epsilon := h_1^\epsilon - 1,$$

(3.1)

then it follows from equation (2.1) that

$$\begin{align*}
\partial_t \varrho^\epsilon + (u^\epsilon + 1 - e^{-y}) \partial_x \varrho^\epsilon + v^\epsilon \partial_y \varrho^\epsilon - \epsilon \partial^2_x \varrho^\epsilon - \epsilon \partial^2_\varrho^\epsilon = -\epsilon \partial_x r_1 - \epsilon \partial_y r_2,
\rho^\epsilon \partial_t u^\epsilon + \rho^\epsilon (u^\epsilon + 1 - e^{-y}) \partial_x u^\epsilon + \rho^\epsilon v^\epsilon \partial_y u^\epsilon + \rho^\epsilon v^\epsilon e^{-y} = \epsilon \partial^2_x u^\epsilon + \mu \partial^2_\varrho^\epsilon + (h^\epsilon + 1) \partial_x h^\epsilon + g^\epsilon \partial_y h^\epsilon - \epsilon \partial_x r_1 - \mu e^{-y},
\partial_t h^\epsilon + (u^\epsilon + 1 - e^{-y}) \partial_x h^\epsilon + v^\epsilon \partial_y h^\epsilon - \epsilon \partial^2_x h^\epsilon = (h^\epsilon + 1) \partial_x h^\epsilon + g^\epsilon \partial_y h^\epsilon - \epsilon \partial_x r_1 - \epsilon \partial_y r_1, \quad \partial_x u^\epsilon + \partial_y v^\epsilon = 0, \quad \partial_x h^\epsilon + \partial_y g^\epsilon = 0, \\
(q^\epsilon, u^\epsilon, h^\epsilon)|_{t=0} := (q_0^\epsilon, u_0^\epsilon, h_0^\epsilon),
\end{align*}$$

(3.2)

with the boundary conditions

$$\begin{align*}
\partial_y q^\epsilon |_{y=0} = u^\epsilon |_{y=0} = v^\epsilon |_{y=0} = \partial_y h^\epsilon |_{y=0} = g^\epsilon |_{y=0} = 0, \\
\lim_{y \to +\infty} q^\epsilon = \lim_{y \to +\infty} u^\epsilon = \lim_{y \to +\infty} h^\epsilon = 0.
\end{align*}$$

(3.3)

Due to the relation (3.1), we can get the relation between two initial data as follows

$$q_0^\epsilon = \rho_0 - 1, \quad u_0^\epsilon = u_{10} - 1 + e^{-y}, \quad h_0^\epsilon = h_{10} - 1,$$

(3.4)

and hence, we have the estimates:

$$\|q_0^\epsilon, u_0^\epsilon, h_0^\epsilon\|_{\mathcal{H}_1^{m}}^2 + \|\partial_y (q_0^\epsilon, u_0^\epsilon, h_0^\epsilon)\|_{\mathcal{H}_1^{m-1}}^2 + \|\partial_y \partial_y q_0^\epsilon\|_{\mathcal{H}_1^{\infty}}^2 \leq C (1 + \|(\rho_0, u_{10}, h_{10})\|_{\mathcal{B}_1^{m}}),$$

and

$$\|q_0^\epsilon, u_0^\epsilon, h_0^\epsilon\|_{\mathcal{H}_1^{m}}^2 + \|\partial_y (q_0^\epsilon, u_0^\epsilon, h_0^\epsilon)\|_{\mathcal{H}_1^{m-1}}^2 + \|\partial_y \partial_y q_0^\epsilon\|_{\mathcal{H}_1^{\infty}}^2 \leq C \|(\rho_0, u_{10}, h_{10})\|_{\mathcal{B}_1^{m}}^2.$$
Theorem 3.1 (a priori estimates) Let \( m \geq 5 \) be an integer, \( l \geq 2 \) be a real number and \( \epsilon \in (0, 1) \), and \( (\varrho^\epsilon, u^\epsilon, v^\epsilon, h^\epsilon, g^\epsilon) \) be sufficiently smooth solution, defined on \([0, T^\epsilon]\), to the regularized MHD boundary layer equations (3.2)–(3.3). The initial data \((\varrho_0^\epsilon, u_0^\epsilon, h_0^\epsilon)\) is defined by \((\rho_0, u_{10}, h_{10})\) given in Theorem 1.1 through the relation (3.4). Then, there exists a time \( T_a = T_a(\mu, \kappa, m, \delta_0) \) independent of \( \epsilon \) such the following a priori estimates hold true for all \( t \in [0, \min(T_a, T^\epsilon)] \):

\[
\Theta_{m,l}(\varrho^\epsilon, u^\epsilon, h^\epsilon)(t) \leq 2C_l \mathcal{P}_0(\delta_0^{-1}, \|(\rho_0, u_{10}, h_{10})\|_{\mathcal{E}_l^m}),
\]

and

\[
\|\partial_y (u^\epsilon - e^{-y})\|_{L_1^\infty(\Omega)} \leq \delta_0^{-1}, \quad \|\varrho^\epsilon(t, \cdot, \cdot)\|_{L_0^\infty(\Omega)} \leq \frac{3(2l - 1)}{32} \delta_0^2, \quad h^\epsilon(t, x, y) + 1 \geq \delta_0,
\]

for all \((t, x, y) \in [0, \min(T_a, T^\epsilon)] \times \Omega\).

Remark 3.1 When the parameter \( \epsilon = 0 \), the regularized Prandtl type Eq. (3.2) become the original Prandtl type Eq. (1.6), and hence Theorem 3.1 also provides a priori estimates for the original MHD boundary layer equation (1.6).

Throughout this section, for any small constant \( \delta \), we assume that the following a priori assumptions:

\[
h^\epsilon(t, x, y) + 1 \geq \delta, \tag{3.8}
\]

and

\[
\|\varrho^\epsilon(t)\|_{L_0^\infty(\Omega)} \leq \frac{2l - 1}{2} \delta^2, \quad \|\partial_y (u^\epsilon - e^{-y})\|_{L_1^\infty(\Omega)} \leq \delta^{-1}, \tag{3.9}
\]

hold on for any \((t, x, y) \in [0, T^\epsilon] \times \Omega\). Thanks to the smallness of \( \delta \), we find

\[
\frac{1}{2} \leq \rho^\epsilon(t, x, y) \leq \frac{3}{2} \tag{3.10}
\]

for \((t, x, y) \in [0, T^\epsilon] \times \Omega\).

### 3.1 Weighted \( \mathcal{H}_1^m \)-estimates with conormal derivative

In this subsection, we will derive the weighted estimates for the quantities \( Z^\alpha_1 Z^\alpha_2 (\varrho^\epsilon, u^\epsilon, h^\epsilon) \) with \(|\alpha_1| + |\alpha_2| = m, |\alpha_1| \leq m - 1\). This goal is easy to reach by the standard energy method because one order tangential derivative loss is allowed. For notational convenience, we denote

\[
\mathcal{E}_{m,l}(t) := \sum_{|\alpha| \leq m} \|Z_\alpha (\varrho^\epsilon, u^\epsilon, h^\epsilon)(t)\|_{L_1^\infty(\Omega)}^2, \tag{3.11}
\]

and

\[
Q(t) := \sup_{0 \leq s \leq t} \left\{ \|Z_t \varrho^\epsilon(s)\|_{L_0^\infty(\Omega)}^2 + \|(u^\epsilon, h^\epsilon)(s)\|_{\mathcal{H}_{1,\infty}}^2 + \|(v^\epsilon, g^\epsilon)(s)\|_{\mathcal{H}_{1,\infty}}^2 + \|\partial_y \varrho^\epsilon, \partial_y u^\epsilon, \partial_y h^\epsilon, \frac{v^\epsilon}{\varphi}(s)\|_{\mathcal{H}_{1,\infty}}^2 \right\}. \tag{3.12}
\]
Proposition 3.2 Let \((q^\varepsilon, u^\varepsilon, v^\varepsilon, h^\varepsilon, g^\varepsilon)\) be sufficiently smooth solution, defined on \([0, T]\), to the Eqs. (3.2)–(3.3). Then, it holds true

\[
\sup_{0 \leq t \leq \tau} E_{m,l}(\tau) + \sum_{0 \leq |\alpha| \leq m} \epsilon \int_0^t \| \partial_x Z^\alpha (q^\varepsilon, u^\varepsilon, h^\varepsilon) \|^2_{L^2_t(\Omega)} d\tau \\
+ \sum_{0 \leq |\alpha| \leq m} \int_0^t \| (\sqrt{\varepsilon} \partial_y Z^\alpha q^\varepsilon, \sqrt{\mu} \partial_y Z^\alpha u^\varepsilon, \sqrt{\kappa} \partial_y Z^\alpha h^\varepsilon) \|^2_{L^2_t(\Omega)} d\tau \\
\leq C \|(q_0^\varepsilon, u_0^\varepsilon, h_0^\varepsilon)\|^2_{\mathcal{M}_l^m} + C \| (\rho_0, u_{10}, h_{10}) \|_{\mathcal{B}_l^m} \\
+ C_{\mu,k,m,l}(1 + Q^2(t)) \int_0^t (1 + \| (q^\varepsilon, u^\varepsilon, h^\varepsilon) \|^2_{\mathcal{M}_l^m} + \| \partial_y (q^\varepsilon, u^\varepsilon, h^\varepsilon) \|^2_{\mathcal{M}_l^{m-1}}) d\tau.
\]

The Proposition 3.2 will be proved in Lemma 3.4. Now, we give the proof for the case \(m = 0\).

Lemma 3.3 For smooth solution \((q^\varepsilon, u^\varepsilon, v^\varepsilon, h^\varepsilon, g^\varepsilon)\) of the Eqs. (3.2)–(3.3), then it holds true

\[
\sup_{\tau \in [0, t]} \| (q^\varepsilon, u^\varepsilon, h^\varepsilon)(\tau) \|^2_{L^2_t(\Omega)} + \epsilon \int_0^t \| \partial_x (q^\varepsilon, u^\varepsilon, h^\varepsilon) \|^2_{L^2_t(\Omega)} d\tau \\
+ \int_0^t \| \partial_y (\sqrt{\varepsilon} q^\varepsilon, \sqrt{\mu} u^\varepsilon, \sqrt{\kappa} h^\varepsilon) \|^2_{L^2_t(\Omega)} d\tau \\
\leq C \|(q_0^\varepsilon, u_0^\varepsilon, h_0^\varepsilon)\|^2_{L^2_t(\Omega)} + C \int_0^t \| (r_1, r_2, r_u, r_h) \|^2_{L^2_t(\Omega)} d\tau \\
+ C_{\mu,k,l}(1 + Q(t)) \int_0^t (1 + \| (q^\varepsilon, u^\varepsilon, h^\varepsilon) \|^2_{\mathcal{M}_l^1}) d\tau.
\]

Proof First of all, multiplying (3.2) by \((q^\varepsilon)\)2\(u^\varepsilon\), integrating over \(\Omega\) and integrating by parts, we find

\[
\frac{d}{dt} \frac{1}{2} \int_{\Omega} (q^\varepsilon)^2 |u^\varepsilon|^2 dxdy + \epsilon \int_{\Omega} (q^\varepsilon)^2 |\partial_x u^\varepsilon|^2 dxdy + \mu \int_{\Omega} (q^\varepsilon)^2 |\partial_y u^\varepsilon|^2 dxdy \\
= \frac{1}{2} \int_{\Omega} (q^\varepsilon)^2 |u^\varepsilon|^2 (\partial_t \rho^\varepsilon + (u^\varepsilon + 1 - e^{-\gamma}) \partial_x \rho^\varepsilon + v^\varepsilon \partial_y \rho^\varepsilon + v^\varepsilon \partial_x \rho^\varepsilon + v^\varepsilon \partial_y \rho^\varepsilon + v^\varepsilon \partial_x \rho^\varepsilon + v^\varepsilon \partial_y \rho^\varepsilon) dxdy \\
+ \int_{\Omega} \rho^\varepsilon v^\varepsilon e^{-\gamma} \cdot (q^\varepsilon)^2 u^\varepsilon dxdy \\
- \int_{\Omega} \rho^\varepsilon v^\varepsilon e^{-\gamma} \cdot (q^\varepsilon)^2 u^\varepsilon dxdy + \int_{\Omega} [(h^\varepsilon + 1) \partial_x h^\varepsilon + g^\varepsilon \partial_y h^\varepsilon] \cdot (q^\varepsilon)^2 u^\varepsilon dxdy \\
+ \int_{\Omega} (-e \partial_x r_u - \mu e^{-\gamma}) \cdot (q^\varepsilon)^2 u^\varepsilon dxdy,
\]

where we have used the boundary condition (3.3) and the divergence-free condition (3.2)4. By routine checking, we have, after using the definition of \(Q(t)\) in (3.12),

\[
\left| \int_{\Omega} (q^\varepsilon)^2 |u^\varepsilon|^2 (\partial_t \rho^\varepsilon + (u^\varepsilon + 1 - e^{-\gamma}) \partial_x \rho^\varepsilon + v^\varepsilon \partial_y \rho^\varepsilon + v^\varepsilon \partial_y \rho^\varepsilon) dxdy \right| \leq C(1 + Q(t)) \| u^\varepsilon \|^2_{L^2(\Omega)}.
\]

By virtue of estimate for density (3.10), we get

\[
\left| \int_{\Omega} (q^\varepsilon)^{2-1} \rho^\varepsilon v^\varepsilon |u^\varepsilon|^2 dxdy \right| \leq C \| (q^\varepsilon)^{-1} v^\varepsilon \|^2_{L^\infty(\Omega)} \| u^\varepsilon \|^2_{L^2(\Omega)}.
\]

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Using the Hölder and Cauchy–Schwartz inequalities, it follows
\[
\left| \mu \int_\Omega \langle y \rangle^{2l-1} u^\varepsilon \cdot \partial_\tau u^\varepsilon \, dx \right| \leq \frac{\mu}{4} \int_\Omega \langle y \rangle^{2l} |\partial_\tau u^\varepsilon|^2 \, dx + C_\mu \int_\Omega \langle y \rangle^{2l-1} |u^\varepsilon|^2 \, dx,
\]
and
\[
\left| \int_\Omega (-\varepsilon \partial_\tau u^\varepsilon - \mu e^{-\varepsilon x}) \cdot \langle y \rangle^{2l} u^\varepsilon \, dx \right| \leq \frac{1}{2} \varepsilon \|\partial_\tau u^\varepsilon\|_{L^2_\varepsilon(\Omega)}^2 + C \left( 1 + \|r^\varepsilon u^\varepsilon \|_{L^2_\varepsilon(\Omega)}^2 + \|u^\varepsilon\|_{L^2_\varepsilon(\Omega)}^2 \right).
\]
Applying the divergence-free condition (3.2), Hölder and Hardy inequalities, we get
\[
\int_\Omega \rho^\varepsilon v^\varepsilon e^{-y} \cdot \langle y \rangle^{2l} u^\varepsilon \, dx \leq C \|v^\varepsilon\|_{L^2_{\varepsilon}(\Omega)} \|u^\varepsilon\|_{L^2_{\varepsilon}(\Omega)} \leq C_I \|u^\varepsilon\|_{\mathcal{H}^1_{\varepsilon}}^2.
\]
Integrating by part and applying the divergence-free condition (3.2), we find
\[
\int_\Omega \left( (h^\varepsilon + 1) \partial_\tau h^\varepsilon + g^\varepsilon \partial_y h^\varepsilon \right) \cdot \langle y \rangle^{2l} u^\varepsilon \, dx
\]
\[
= -\int_\Omega \langle y \rangle^{2l} \partial_\tau h^\varepsilon \cdot h^\varepsilon \, dx - \int_\Omega \langle y \rangle^{2l} h^\varepsilon (h^\varepsilon + 1) \partial_\tau u^\varepsilon \, dx
\]
\[
- \int_\Omega \langle y \rangle^{2l} \partial_y g^\varepsilon u^\varepsilon \, dx - 2l \int_\Omega \langle y \rangle^{2l-1} g^\varepsilon u^\varepsilon \, dx
\]
\[
\leq -\int_\Omega \langle y \rangle^{2l} h^\varepsilon (h^\varepsilon + 1) \partial_\tau u^\varepsilon \, dx - \int_\Omega \langle y \rangle^{2l} h^\varepsilon g^\varepsilon \partial_y u^\varepsilon \, dx
\]
\[
+ C \|\langle y \rangle^{-1} g^\varepsilon \|_{L^\infty_{\varepsilon}(\Omega)} \|u^\varepsilon\|_{L^2_{\varepsilon}(\Omega)} \|h^\varepsilon\|_{L^2_{\varepsilon}(\Omega)}.
\]
Substituting the above estimates into (3.14), and integrating the resulting inequality over [0, t], we obtain
\[
\|\sqrt{\rho^\varepsilon u^\varepsilon(t)}\|_{L^2_{\varepsilon}(\Omega)}^2 + \epsilon \int_0^t \|\partial_\tau u^\varepsilon\|_{L^2_{\varepsilon}(\Omega)}^2 \, d\tau + \mu \int_0^t \|\partial_y u^\varepsilon\|_{L^2_{\varepsilon}(\Omega)}^2 \, d\tau
\]
\[
+ \int_0^t \int_\Omega \langle y \rangle^{2l} h^\varepsilon \cdot ((h^\varepsilon + 1) \partial_\tau u^\varepsilon + g^\varepsilon \partial_y u^\varepsilon) \, dx \, d\tau
\]
\[
\leq \|\sqrt{\rho^\varepsilon u^\varepsilon(0)}\|_{L^2_{\varepsilon}(\Omega)}^2 + C \int_0^t \|r^\varepsilon u^\varepsilon\|_{L^2_{\varepsilon}(\Omega)}^2 \, d\tau
\]
\[
+ C_{\mu, I} (1 + Q(t)) \int_0^t (1 + \|u^\varepsilon\|_{\mathcal{H}^1_{\varepsilon}}^2 + \|h^\varepsilon\|_{L^2_{\varepsilon}(\Omega)}^2) \, d\tau.
\]
Similarly, based on the Eqs. (3.2) and (3.2), it follows directly
\[
\|h^\varepsilon(t)\|_{L^2_{\varepsilon}(\Omega)}^2 + \epsilon \int_0^t \|\partial_\tau h^\varepsilon\|_{L^2_{\varepsilon}(\Omega)}^2 \, d\tau + \kappa \int_0^t \|\partial_y h^\varepsilon\|_{L^2_{\varepsilon}(\Omega)}^2 \, d\tau
\]
\[
- \int_0^t \int_\Omega \langle y \rangle^{2l} h^\varepsilon \cdot [(h^\varepsilon + 1) \partial_\tau u^\varepsilon + g^\varepsilon \partial_y u^\varepsilon] \, dx \, d\tau
\]
\[
\leq \|h^\varepsilon(0)\|_{L^2_{\varepsilon}(\Omega)}^2 + C \int_0^t \|r^\varepsilon h^\varepsilon\|_{L^2_{\varepsilon}(\Omega)}^2 \, d\tau + C_{\kappa, I} (1 + Q(t)) \int_0^t (1 + \|h^\varepsilon\|_{\mathcal{H}^1_{\varepsilon}}^2) \, d\tau.
\]
and

$$\|\varphi^e\|_{L^2_t(\Omega)}^2 + \epsilon \int_0^t \|\partial_x \varphi^e, \partial_y \varphi^e\|_{L^2_t(\Omega)}^2 d\tau \leq \|\varphi_0^e\|_{L^2_t(\Omega)}^2 + C \int_0^t \|(r_1, r_2)\|_{L^2_t(\Omega)}^2 d\tau$$

$$+ C(1 + Q(t)) \int_0^t \|\varphi^e\|_{L^2_t(\Omega)}^2 d\tau.$$  

Therefore, we collect above estimates to complete the proof of Lemma 3.3.  

Now, we establish the following estimate:

**Lemma 3.4** For smooth solution \((\varphi^e, u^e, v^e, h^e, g^e)\) of the Eqs. (3.2)–(3.3), then it holds true

$$\sup_{0 \leq \tau \leq T} E_{m, l}(\tau) + \sum_{|\alpha| \leq m-1} \int_0^T \|\partial_x \varphi^\alpha (\varphi^e, u^e, h^e)\|_{L^2_t(\Omega)}^2 d\tau$$

$$+ \sum_{|\alpha| \leq m-1} \int_0^T \|\varphi^\alpha (\varphi^e, u^e, h^e)\|_{L^2_t(\Omega)}^2 d\tau$$

$$\leq C \|(\varphi_0^e, u_0^e, h_0^e)\|_{H^m_{\Omega}}^2 + C \int_0^T \|(r_1, r_2, r_u, r_h)\|_{H^m_{\Omega}}^2 d\tau$$

$$+ C_{\mu, \alpha, m, l} (1 + Q^2(t)) \int_0^T \|\varphi^e\|_{H^m_{\Omega}}^2 + \|\varphi (\varphi^e, u^e, h^e)\|_{H^{m-1}_{\Omega}}^2 d\tau.$$  

**Proof** The case \(m = 0\) has been proven in the Lemma 3.3. Then, we give the proof for the case \(m \geq 1\).

Step 1: Applying the differential operator \(\partial^\alpha (1 \leq |\alpha| \leq m, |\alpha_1| \leq m-1)\) to the Eq. (3.2)2, we can obtain the evolution equation for \(\partial^\alpha u^e\):

$$\rho^e \partial_t \partial^\alpha u^e + \rho^e (u^e + 1 - e^{-\gamma}) \partial_x \partial^\alpha u^e + \rho^e \partial_y \partial^\alpha u^e - \epsilon \partial_x^2 \partial^\alpha u^e - \mu \partial^\alpha \partial_y^2 u^e$$

$$= (h^e + 1) \partial_x \partial^\alpha h^e + g^e \partial_y \partial^\alpha h^e + Z^\alpha (-\epsilon \partial_x r_u - \mu e^{-\gamma})$$

$$+ C_{11}^\alpha + C_{12}^\alpha + C_{13}^\alpha + C_{14}^\alpha + C_{15}^\alpha + C_{16}^\alpha,$$

where \(C_{11}^\alpha (i = 1, \ldots, 6)\) are defined by

$$C_{11}^\alpha = -[\partial^\alpha, \rho^e \partial_t]u^e, \quad C_{12}^\alpha = -[\partial^\alpha, \rho^e (u^e + 1 - e^{-\gamma}) \partial_x]u^e, \quad C_{13}^\alpha = -[\partial^\alpha, \rho^e \partial_y]u^e,$$

$$C_{14}^\alpha = [\partial^\alpha, (h^e + 1) \partial_x]h^e, \quad C_{15}^\alpha = [\partial^\alpha, g^e \partial_y]h^e, \quad C_{16}^\alpha = -\partial^\alpha (\rho^e v^e e^{-\gamma}).$$

Multiplying the Eq. (3.16) by \(<y>^{2l} \partial^\alpha u^e\), integrating over \(\Omega\) and applying the boundary condition (3.3), we have

$$\frac{d}{dt} \frac{1}{2} \int_\Omega \langle y \rangle^{2l} \rho^e |Z^\alpha u^e|^2 dxdy + \epsilon \int_\Omega \langle y \rangle^{2l} |\partial_x Z^\alpha u^e|^2 dxdy$$

$$= \frac{1}{2} \int_\Omega \langle y \rangle^{2l} |Z^\alpha u^e|^2 (\partial_t \rho^e + (u^e + 1 - e^{-\gamma}) \partial_x \rho^e + v^e \partial_y \rho^e) dxdy$$

$$+ l \int_\Omega \langle y \rangle^{2l-1} \rho^e v^e |Z^\alpha u^e|^2 dxdy + \mu \int_\Omega \partial^2_y Z^\alpha \partial^\alpha u^e \cdot \langle y \rangle^{2l} Z^\alpha u^e dxdy$$

$$+ \int_\Omega \langle h^e + 1 \rangle \partial_x \partial^\alpha h^e + g^e \partial_y \partial^\alpha h^e \cdot \langle y \rangle^{2l} Z^\alpha u^e dxdy$$

$$+ \int_\Omega Z^\alpha (-\epsilon \partial_x r_u - \mu e^{-\gamma}) \cdot \langle y \rangle^{2l} Z^\alpha u^e dxdy + \sum_{i=1}^6 C_{1i}^\alpha \cdot \langle y \rangle^{2l} Z^\alpha u^e dxdy.$$
By routine checking, it follows that
\[
\left| \int_{\Omega} \langle y \rangle^{2l} \left| \mathcal{Z}^\alpha u^\epsilon \right|^2 (\partial_t \rho^\epsilon + (u^\epsilon + 1 - e^{-y}) \partial_y \rho^\epsilon + v^\epsilon \partial_y \rho^\epsilon) \, dxdy \right| \\
\leq C(1 + Q(t)) \| \mathcal{Z}^\alpha u^\epsilon \|_{L_t^2(\Omega)}^2,
\] (3.18)
and
\[
\left| \int_{\Omega} \langle y \rangle^{2l-1} \rho^\epsilon v^\epsilon \left| \mathcal{Z}^\alpha u^\epsilon \right|^2 \, dxdy \right| \leq C \| \langle y \rangle^{-1} v^\epsilon \|_{L_t^\infty(\Omega)} \| \mathcal{Z}^\alpha u^\epsilon \|_{L_t^2(\Omega)}^2.
\] (3.19)

The integration by part with respect to \( y \) variable yields directly
\[
\int_{\Omega} \mathcal{Z}^\alpha \partial_y^2 u^\epsilon \cdot \langle y \rangle^{2l} \mathcal{Z}^\alpha u^\epsilon \, dxdy \\
= \int_T \mathcal{Z}^\alpha \partial_y u^\epsilon \cdot \mathcal{Z}^\alpha u^\epsilon \big|_{y=0} \, dx - 2l \int_{\Omega} \mathcal{Z}^\alpha \partial_y u^\epsilon \cdot \langle y \rangle^{2l-1} \mathcal{Z}^\alpha u^\epsilon \, dxdy \\
- \int_{\Omega} \mathcal{Z}^\alpha \partial_y u^\epsilon \cdot \langle y \rangle^{2l} \partial_y \mathcal{Z}^\alpha u^\epsilon \, dxdy + \int_{\Omega} [\mathcal{Z}^\alpha, \partial_y] \partial_y u^\epsilon \cdot \langle y \rangle^{2l} \mathcal{Z}^\alpha u^\epsilon \, dxdy
\] (3.20)
\[
= \int_T \mathcal{Z}^\alpha \partial_y u^\epsilon \cdot \mathcal{Z}^\alpha u^\epsilon \big|_{y=0} \, dx - 2l \int_{\Omega} \mathcal{Z}^\alpha \partial_y u^\epsilon \cdot \langle y \rangle^{2l-1} \mathcal{Z}^\alpha u^\epsilon \, dxdy \\
- \int_{\Omega} \langle y \rangle^{2l} \partial_y \mathcal{Z}^\alpha u^\epsilon \, dxdy - \int_{\Omega} [\mathcal{Z}^\alpha, \partial_y] u^\epsilon \cdot \langle y \rangle^{2l} \partial_y \mathcal{Z}^\alpha u^\epsilon \, dxdy
\]
\[
+ \int_{\Omega} [\mathcal{Z}^\alpha, \partial_y] \partial_y u^\epsilon \cdot \langle y \rangle^{2l} \mathcal{Z}^\alpha u^\epsilon \, dxdy.
\]
If \( \alpha_2 = 0 \), the boundary condition (3.3) implies \( \mathcal{Z}^\alpha u^\epsilon \big|_{y=0} = 0 \). If \( \alpha_2 \neq 0 \), we apply the property of \( \varphi \), which vanishes on the boundary, to get \( \mathcal{Z}^\alpha u^\epsilon \big|_{y=0} = 0 \), and hence
\[
\int_T \mathcal{Z}^\alpha \partial_y u^\epsilon \cdot \mathcal{Z}^\alpha u^\epsilon \big|_{y=0} \, dx = 0.
\] (3.21)

Next, we deal with term involving \([\mathcal{Z}^\alpha, \partial_y]\). It is worth noting that the operator \( \mathcal{Z}_\tau = (\partial_t, \partial_x) \) communicates with \( \partial_y \), we obtain \([\mathcal{Z}^\alpha, \partial_y] u^\epsilon = 0 \) for \( \alpha_2 = 0 \). By direct computation, we find for \( \alpha_2 \neq 0 \)
\[
[Z^\alpha_2, \partial_y] u^\epsilon = - \sum_{1 \leq k \leq \alpha_2} C_{\alpha_2, k} \partial_y Z^{k-1} \varphi \cdot Z_2^{2-k} \partial_y u^\epsilon.
\]
This and the Hölder inequality yield directly
\[
\left| \int_{\Omega} [\mathcal{Z}^\alpha, \partial_y] u^\epsilon \cdot \langle y \rangle^{2l} \partial_y \mathcal{Z}^\alpha u^\epsilon \, dxdy \right| \leq \frac{1}{4} \| \partial_y \mathcal{Z}^\alpha u^\epsilon \|_{L_t^2(\Omega)}^2 + C_m \| \partial_y u^\epsilon \|_{H_t^{m-1}}^2.
\] (3.22)

Using the relation (3.1) and Cauchy–Schwartz inequality, we obtain
\[
2l \int_{\Omega} \mathcal{Z}^\alpha \partial_y u^\epsilon \cdot \langle y \rangle^{2l-1} \mathcal{Z}^\alpha u^\epsilon \, dxdy \\
= 2l \int_{\Omega} \left( [\mathcal{Z}^\alpha, \partial_y] u^\epsilon \right) \cdot \langle y \rangle^{2l-1} \mathcal{Z}^\alpha u^\epsilon \, dxdy
\] (3.23)
\[
\leq \frac{1}{4} \| \partial_y \mathcal{Z}^\alpha u^\epsilon \|_{L_t^2(\Omega)}^2 + C_m, l (\| [\mathcal{Z}^\alpha, \partial_y] u^\epsilon \|_{L_t^2(\Omega)} + \| \mathcal{Z}^\alpha u^\epsilon \|_{L_t^2(\Omega)}^2)
\]
\[
\leq \frac{1}{4} \| \partial_y \mathcal{Z}^\alpha u^\epsilon \|_{L_t^2(\Omega)}^2 + C_m, l (\| u^\epsilon \|_{H_t^{m-1}}^2 + \| \partial_y u^\epsilon \|_{H_t^{m-1}}^2).
\]
Next, we deal with the term \( \int_\Omega [Z^\alpha, \partial_y] \partial_y u^\epsilon \cdot \langle y \rangle^{2l} Z^\alpha u^\epsilon \, dxdy \). For a smooth function \( f \), it follows

\[
[Z^\alpha, \partial_y] f = \sum_{\beta_2 \neq 0, \beta_2 + \gamma_2 = \alpha_2} C_{\beta_1, \gamma_2} \varphi Z_2^{\beta_2} \left( \frac{1}{\varphi} \right) \partial_y Z_2^\gamma Z_2^{\alpha_1} f. \tag{3.24}
\]

By virtue of the definition of \( \varphi \), it holds by computing directly

\[
|\varphi Z_2^{\beta_2} \left( \frac{1}{\varphi} \right) | \leq C, \quad |\partial_y \left\{ \varphi Z_2^{\beta_2} \left( \frac{1}{\varphi} \right) \right\} | \leq C. \tag{3.25}
\]

For \( \beta_2 \neq 0 \) and \( \beta_2 + \gamma_2 = \alpha_2 \), the integration by part yields immediately

\[
\int_\Omega \varphi Z_2^{\beta_2} \left( \frac{1}{\varphi} \right) \partial_y Z_2^\gamma Z_2^{\alpha_1} \partial_y u^\epsilon \cdot \langle y \rangle^{2l} Z^\alpha u^\epsilon \, dxdy = - \int_\Omega \varphi Z_2^{\beta_2} \left( \frac{1}{\varphi} \right) Z_2^\gamma Z_2^{\alpha_1} \partial_y u^\epsilon \cdot \langle y \rangle^{2l} \partial_y Z^\alpha u^\epsilon \, dxdy

- 2l \int_\Omega \varphi Z_2^{\beta_2} \left( \frac{1}{\varphi} \right) Z_2^\gamma Z_2^{\alpha_1} \partial_y u^\epsilon \cdot \langle y \rangle^{2l-1} Z^\alpha u^\epsilon \, dxdy

- \int_\Omega \partial_y \left\{ \varphi Z_2^{\beta_2} \left( \frac{1}{\varphi} \right) \right\} Z_2^\gamma Z_2^{\alpha_1} \partial_y u^\epsilon \cdot \langle y \rangle^{2l} Z^\alpha u^\epsilon \, dxdy, \tag{3.26}
\]

where the boundary term in the above equality vanishes since the quantity \( Z^\alpha u^\epsilon \mid_{y=0} = 0 \). Then, applying the relation (3.24), estimate (3.25), Hölder and Cauchy inequalities, we obtain

\[
\int_\Omega \varphi Z_2^{\beta_2} \left( \frac{1}{\varphi} \right) \partial_y Z_2^\gamma Z_2^{\alpha_1} \partial_y u^\epsilon \cdot \langle y \rangle^{2l} Z^\alpha u^\epsilon \, dxdy \leq \frac{1}{4} \int \langle y \rangle^{2l} |\partial_y Z^\alpha u^\epsilon|^2 \, dxdy

+ C_{m,l}(\|u^\epsilon\|^2_{H^m} + \|\partial_y u^\epsilon\|^2_{H^{m-1}}),
\]

and hence

\[
\left| \int_\Omega [Z^\alpha, \partial_y] \partial_y u^\epsilon \cdot \langle y \rangle^{2l} Z^\alpha u^\epsilon \, dxdy \right| \leq \frac{1}{4} \int \langle y \rangle^{2l} |\partial_y Z^\alpha u^\epsilon|^2 \, dxdy

+ C_{m,l}(\|u^\epsilon\|^2_{H^m} + \|\partial_y u^\epsilon\|^2_{H^{m-1}}). \tag{3.27}
\]

Plugging the estimates (3.21), (3.22), (3.23) and (3.27) into (3.20), we conclude

\[
\mu \int_\Omega Z^\alpha \partial_y^2 u^\epsilon \cdot \langle y \rangle^{2l} Z^\alpha u^\epsilon \, dxdy \leq -\frac{1}{2} \mu \|\partial_y Z^\alpha u^\epsilon\|^2_{L^2(\Omega)} + C_{\mu,m,l}(\|u^\epsilon\|^2_{H^m} + \|\partial_y u^\epsilon\|^2_{H^{m-1}}). \tag{3.28}
\]

Integrating by part, applying the boundary condition (3.3) and divergence-free condition (3.2)_4, we find

\[
\int_\Omega \{(h^\epsilon + 1) \partial_x Z^\alpha h^\epsilon + g^\epsilon \partial_y Z^\alpha h^\epsilon\} \cdot \langle y \rangle^{2l} Z^\alpha u^\epsilon \, dxdy

= - \int_\Omega \langle y \rangle^{2l} \left( \partial_x h^\epsilon + \partial_y g^\epsilon \right) Z^\alpha u^\epsilon \cdot Z^\alpha h^\epsilon \, dxdy

- \int_\Omega \{(h^\epsilon + 1) \partial_x Z^\alpha u^\epsilon + g^\epsilon \partial_y Z^\alpha u^\epsilon\} \cdot \langle y \rangle^{2l} Z^\alpha h^\epsilon \, dxdy

- 2l \int_\Omega \langle y \rangle^{2l-1} g^\epsilon Z^\alpha u^\epsilon \cdot Z^\alpha h^\epsilon \, dxdy.
\]
For the moment we can substitute the estimate (3.32) into inequality (3.31), and hence

\[
\int\Omega \langle (h^\epsilon + 1) \partial_x Z^\alpha u^\epsilon + g^\epsilon \partial_y Z^\alpha u^\epsilon \rangle \cdot \langle y \rangle^{2l} Z^\alpha h^\epsilon \, dx
dy
+ C l \| (y)^{-1} g^\epsilon \|_{L_0^\infty(\Omega)} \| Z^\alpha u^\epsilon \|_{L_l^2(\Omega)} \| Z^\alpha h^\epsilon \|_{L_l^2(\Omega)},
\]

(3.29)

Using the Hölder and Cauchy inequalities, it follows

\[
\left| \int\Omega Z^\alpha (-\epsilon \partial_x r_\alpha - \mu e^{-\gamma}) \cdot \langle y \rangle^{2l} Z^\alpha u^\epsilon \, dx
dy \right|
\leq \frac{1}{2} \epsilon \| \partial_x Z^\alpha u^\epsilon \|_{L_l^2(\Omega)}^2 + C \left( 1 + \| Z^\alpha r_\alpha \|_{L_l^2(\Omega)}^2 + \| Z^\alpha u^\epsilon \|_{L_l^2(\Omega)}^2 \right).
\]

(3.30)

Substituting the estimates (3.18), (3.19), (3.20), (3.28), (3.29) and (3.30) into the equality (3.17), and integrating over [0, t], we conclude

\[
\int\Omega \langle y \rangle^{2l} \rho^\epsilon |Z^\alpha u^\epsilon|^2 \, dx
dy + \int_0^t \int\Omega \langle y \rangle^{2l} (\epsilon |\partial_x Z^\alpha u^\epsilon|^2 + \mu |\partial_y Z^\alpha u^\epsilon|^2) \, dx
dy \, dt
+ \int_0^t \int\Omega ((h^\epsilon + 1) \partial_x Z^\alpha u^\epsilon + g^\epsilon \partial_y Z^\alpha u^\epsilon) \cdot \langle y \rangle^{2l} Z^\alpha h^\epsilon \, dx
dy \, dt
\leq \int\Omega \langle y \rangle^{2l} \rho_0^\epsilon |Z^\alpha u_0^\epsilon|^2 \, dx
dy + C \int_0^t \| Z^\alpha r_\alpha \|_{L_l^2(\Omega)}^2 \, dt + \sum_{i=1}^6 \int_0^t \| C_{1i}^\alpha \|_{L_l^2(\Omega)}^2 \, dt
+ C_{\mu,m,l}(1 + Q(t)) \int_0^t (1 + \| (u^\epsilon, h^\epsilon) \|_{\mathcal{T}_{l+1}^m}^2 + \| \partial_y u^\epsilon \|_{\mathcal{T}_{l+1}^m}^2) \, dt.
\]

(3.31)

Now, we claim the following estimate, which will be shown later:

\[
\sum_{i=1}^6 \int_0^t \| C_{1i}^\alpha \|_{L_l^2(\Omega)}^2 \, dt \leq C_{m,l}(1 + Q^2(t)) \int_0^t (\| (g^\epsilon, u^\epsilon, h^\epsilon) \|_{\mathcal{T}_{l+1}^m}^2
+ \| (\partial_y u^\epsilon, \partial_y h^\epsilon) \|_{\mathcal{T}_{l+1}^{m-1}}^2) \, dt.
\]

(3.32)

For the moment we can substitute the estimate (3.32) into inequality (3.31), and hence

\[
\int\Omega \langle y \rangle^{2l} \rho^\epsilon |Z^\alpha u^\epsilon|^2 \, dx
dy + \int_0^t \int\Omega \langle y \rangle^{2l} (\epsilon |\partial_x Z^\alpha u^\epsilon|^2 + \mu |\partial_y Z^\alpha u^\epsilon|^2) \, dx
dy \, dt
+ \int_0^t \int\Omega ((h^\epsilon + 1) \partial_x Z^\alpha u^\epsilon + g^\epsilon \partial_y Z^\alpha u^\epsilon) \cdot \langle y \rangle^{2l} Z^\alpha h^\epsilon \, dx
dy \, dt
\leq \int\Omega \langle y \rangle^{2l} \rho_0^\epsilon |Z^\alpha u_0^\epsilon|^2 \, dx
dy + C \int_0^t \| Z^\alpha r_\alpha \|_{L_l^2(\Omega)}^2 \, dt
+ C_{\mu,m,l}(1 + Q^2(t)) \int_0^t (1 + \| (g^\epsilon, u^\epsilon, h^\epsilon) \|_{\mathcal{T}_{l+1}^m}^2 + \| (\partial_y u^\epsilon, \partial_y h^\epsilon) \|_{\mathcal{T}_{l+1}^{m-1}}^2) \, dt.
\]

(3.33)
Step 2: Applying operator $Z^\alpha$ to the first equation of (3.2), multiplying by $\langle y \rangle^{2l} Z^\alpha q^\epsilon$ and integrating over $\Omega$, we find

$$
\frac{1}{2} \frac{d}{dt} \int_\Omega \langle y \rangle^{2l} |Z^\alpha q^\epsilon|^2 \, dx dy + \epsilon \int_\Omega \langle y \rangle^{2l} |\partial_x Z^\alpha q^\epsilon|^2 \, dx dy
= \epsilon \int_\Omega Z^\alpha \partial_y^2 q^\epsilon \cdot \langle y \rangle^{2l} Z^\alpha q^\epsilon \, dx dy + l \int_\Omega \langle y \rangle^{2l-1} v^\epsilon |Z^\alpha q^\epsilon|^2 \, dx dy
+ \int_\Omega (C^\alpha_{21} + C^\alpha_{22}) \cdot \langle y \rangle^{2l} Z^\alpha q^\epsilon \, dx dy
- \epsilon \int_\Omega Z^\alpha (\partial_x r_1 + \partial_y r_2) \cdot \langle y \rangle^{2l} Z^\alpha q^\epsilon \, dx dy,
$$

(3.34)

where $C^\alpha_{21}$ and $C^\alpha_{22}$ are defined by

$$
C^\alpha_{21} = -[Z^\alpha, (u^\epsilon + 1 - e^{-y}) \partial_x] q^\epsilon, \quad C^\alpha_{22} = -[Z^\alpha, v^\epsilon \partial_y] q^\epsilon.
$$

Similar to the equality (3.20), the integration by part gives

$$
\epsilon \int_\Omega Z^\alpha \partial_y^2 q^\epsilon \cdot \langle y \rangle^{2l} Z^\alpha q^\epsilon \, dx dy
= \epsilon \int_\Omega Z^\alpha \partial_y q^\epsilon \cdot Z^\alpha q^\epsilon \big|_{y=0} \, dx - \epsilon \int_\Omega \langle y \rangle^{2l} [Z^\alpha, \partial_y] q^\epsilon \cdot \partial_y Z^\alpha q^\epsilon \, dx dy
- \epsilon \int_\Omega \langle y \rangle^{2l} \partial_y Z^\alpha q^\epsilon \cdot Z^\alpha q^\epsilon \, dx dy - 2l \epsilon \int_\Omega \langle y \rangle^{2l-1} Z^\alpha \partial_y q^\epsilon \cdot Z^\alpha q^\epsilon \, dx dy
+ \epsilon \int_\Omega [Z^\alpha, \partial_y] \partial_y q^\epsilon \cdot \langle y \rangle^{2l} Z^\alpha q^\epsilon \, dx dy.
$$

(3.35)

If $\alpha_2 = 0$, the boundary condition $\partial_y q^\epsilon|_{y=0} = 0$ implies $Z^\alpha \partial_y q^\epsilon|_{y=0} = 0$. If $\alpha_2 \neq 0$, we get from the definition of $Z^\alpha = \varphi(y) \partial_y = \frac{y}{y+1} \partial_y$ that $Z^\alpha \partial_y q^\epsilon|_{y=0} = 0$, and hence

$$
\int_\Omega Z^\alpha \partial_y q^\epsilon \cdot Z^\alpha q^\epsilon |_{y=0} \, dx = 0.
$$

The other terms on the right hand side of (3.35) can take the idea as estimate (3.27), we conclude

$$
\epsilon \int_\Omega Z^\alpha \partial_y^2 q^\epsilon \cdot \langle y \rangle^{2l} Z^\alpha q^\epsilon \, dx dy \leq -\frac{3}{4} \epsilon \int_\Omega \langle y \rangle^{2l} |\partial_y Z^\alpha q^\epsilon|^2 \, dx dy + C_{m,l} (\|q^\epsilon\|_{H^l_{0,\Omega}}^2 + \|\partial_y q^\epsilon\|_{H^{l-1}_{0,\Omega}}^2).
$$

(3.36)

By routine checking, it follows

$$
\left| \int_\Omega \langle y \rangle^{2l-1} v^\epsilon |Z^\alpha q^\epsilon|^2 \, dx dy \right| \leq C \|\langle y \rangle^{-1} v^\epsilon\|_{L^{\infty}_{0,\Omega}} \|Z^\alpha q^\epsilon\|_{L^2_{\Omega}}^2.
$$

(3.37)

Applying the integration by part and Cauchy inequality, we obtain

$$
\epsilon \int_\Omega Z^\alpha (\partial_x r_1 + \partial_y r_2) \cdot \langle y \rangle^{2l} Z^\alpha q^\epsilon \, dx dy
\leq \frac{\epsilon}{2} \|\langle x \rangle Z^\alpha q^\epsilon, \partial_y Z^\alpha q^\epsilon\|_{L^2_{\Omega}}^2 + C (\|Z^\alpha r_1\|_{L^2_{\Omega}}^2 + \|r_2\|_{H^m_{0,\Omega}}^2) + C_l \|Z^\alpha q^\epsilon\|_{L^2_{\Omega}}^2.
$$

(3.38)
Substituting estimates (3.36), (3.37) and (3.38) into (3.34), integrating over \([0, t]\), we get

\[
\int_{\Omega} (y)^{2l} |Z^a \varphi|^2 \, dxdy + \epsilon \int_{0}^{t} \int_{\Omega} (y)^{2l} \left( |\partial_x Z^a \varphi|^2 + |\partial_y Z^a \varphi|^2 \right) \, dxdy \, dt \\
\leq \int_{\Omega} (y)^{2l} |Z^a \varphi_0|^2 \, dxdy + C \int_{0}^{t} (r_1, r_2)^{2} \tau_{l_{m_t}} \, dt + \int_{0}^{t} \| (C_{21}, C_{22}) \|^2_{L^2_{h_t}(\Omega)} \, dt \tag{3.39}
\]

\[
+ C_{m,t} (1 + \| \langle y \rangle^{-1} v^\epsilon \|_{L_0^\infty(\Omega)}^2) \int_{0}^{t} \| (\varphi^\epsilon, u^\epsilon) \|^2_{\tau_{l_{m_t}}} \, dt.
\]

Similar to the claim estimate (3.32), we can justify the estimate

\[
\left| \int_{0}^{t} \| (C_{21}, C_{22}) \|^2_{L^2_{h_t}(\Omega)} \, dt \right| \leq C_{m,t} (1 + Q(t)) \int_{0}^{t} \| (\varphi^\epsilon, u^\epsilon) \|^2_{\tau_{l_{m_t}}} + \| \partial_y \varphi^\epsilon \|^2_{\tau_{l_{m_t}^{-1}}} \, dt.
\]

This and inequality (3.39) yield directly

\[
\int_{\Omega} (y)^{2l} |Z^a \varphi|^2 \, dxdy + \epsilon \int_{0}^{t} \int_{\Omega} (y)^{2l} \left( |\partial_x Z^a \varphi|^2 + |\partial_y Z^a \varphi|^2 \right) \, dxdy \, dt \\
\leq \int_{\Omega} (y)^{2l} |Z^a \varphi_0|^2 \, dxdy + C \int_{0}^{t} (r_1, r_2)^{2} \tau_{l_{m_t}} \, dt + C_{m,t} (1 + Q(t)) \tag{3.40}
\]

\[
+ \int_{0}^{t} \| (\varphi^\epsilon, u^\epsilon) \|^2_{\tau_{l_{m_t}}} + \| \partial_y \varphi^\epsilon \|^2_{\tau_{l_{m_t}^{-1}}} \, dt.
\]

Step 3: Applying operator \( Z^a \) to the third equation of (3.2), multiplying by \((y)^{2l} Z^a h^\epsilon\) and integrating over \(\Omega\), it follows

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} (y)^{2l} |Z^a h^\epsilon|^2 \, dxdy + \epsilon \int_{\Omega} (y)^{2l} \left( |\partial_x Z^a h^\epsilon|^2 + |\partial_y Z^a h^\epsilon|^2 \right) \, dxdy \\
- \kappa \int_{\Omega} Z^a \partial_y h^\epsilon \cdot (y)^{2l} Z^a h^\epsilon \, dxdy \\
= l \int_{\Omega} (y)^{2l-1} v^\epsilon |Z^a h^\epsilon|^2 \, dxdy + \int_{\Omega} \{ (h^\epsilon + 1) \partial_x Z^a u^\epsilon + g^\epsilon \partial_y Z^a u^\epsilon \} \cdot (y)^{2l} Z^a h^\epsilon \, dxdy \\
+ \int_{\Omega} Z^a (g^\epsilon e^{-v} - \epsilon \partial_t r_h) \cdot (y)^{2l} Z^a h^\epsilon \, dxdy + \int_{\Omega} (C_{31}^a + C_{32}^a + C_{33}^a + C_{34}^a) \cdot (y)^{2l} Z^a h^\epsilon \, dxdy,
\]

where \(C_{3i}^a (i = 1, 2, 3, 4)\) are defined by

\[
C_{31}^a = -[Z^a, (u^\epsilon + 1 - e^{-v}) \partial_x] h^\epsilon, \quad C_{32}^a = -[Z^a, v^\epsilon \partial_y] h^\epsilon, \\
C_{33}^a = [Z^a, (h^\epsilon + 1) \partial_x] u^\epsilon, \quad C_{34}^a = [Z^a, g^\epsilon \partial_y] (u^\epsilon - e^{-v}).
\]

Following the idea as the estimate (3.40), we can verify the following estimate

\[
\int_{\Omega} (y)^{2l} |Z^a h^\epsilon|^2 \, dxdy + \epsilon \int_{0}^{t} \int_{\Omega} (y)^{2l} \left( |\partial_x Z^a h^\epsilon|^2 + \kappa |\partial_y Z^a h^\epsilon|^2 \right) \, dxdy \, dt \\
- \int_{0}^{t} \int_{\Omega} \{ (h^\epsilon + 1) \partial_x Z^a u^\epsilon + g^\epsilon \partial_y Z^a u^\epsilon \} \cdot (y)^{2l} Z^a h^\epsilon \, dxdy \, dt \\
\leq \int_{\Omega} (y)^{2l} |Z^a \varphi_0|^2 \, dxdy + C \int_{0}^{t} \| Z^a r_h \|^2_{L^2_{h_t}(\Omega)} \, dt + C_{k,m,l} (1 + Q(t)).
\]
Similarly, we conclude the following estimates
\[
\int_0^t \left( \| u^\epsilon, h^\epsilon \|_{H^m_{\tau t}}^2 + \| \partial_y u^\epsilon, \partial_y h^\epsilon \|_{H^m_{\tau t-1}}^2 \right) d\tau,
\]
which, together with the estimates (3.33) and (3.40), completes the proof of lemma after taking the summation over all $|\alpha| \leq m$ and $|\alpha_1| \leq m - 1$.

**Proof of claim estimate (3.32)** Now we give the estimate for $\int_0^t \| C_{1i}^{\alpha} \|^2_{L^2_\tau(\Omega)} d\tau (i = 1, \ldots, 6)$. By virtue of the Moser type inequality (A.6), we find
\[
\int_0^t \| C_{11}^{\alpha} \|^2_{L^2_\tau(\Omega)} d\tau \leq C_m \sum_{|\beta| \geq 1, \beta + \gamma = \alpha} \int_0^t \| Z^\beta \varrho^\epsilon \cdot Z^\gamma \partial_t u^\epsilon \|^2_{L^2_\tau(\Omega)} d\tau
\]
\[
\leq C_m \| Z^E_i \varrho^\epsilon \|^2_{L^\infty_0(\Omega)} \int_0^t \| \partial_t u^\epsilon \|^2_{H^m_{\tau t-1}} d\tau + C_m \| \partial_t u^\epsilon \|^2_{L^\infty_0(\Omega)}
\]
\[
\int_0^t \| Z^E_i \varrho^\epsilon \|^2_{H^m_{\tau t-1}} d\tau
\]
\[
\leq C_m \| (Z^E_i \varrho^\epsilon, \partial_t u^\epsilon) \|^2_{L^\infty_0(\Omega)} \int_0^t \| (\varrho^\epsilon, u^\epsilon) \|^2_{H^m_{\tau t}} d\tau.
\]

Similarly, we conclude the following estimates
\[
\int_0^t \| C_{12}^{\alpha} \|^2_{L^2_\tau(\Omega)} d\tau \leq C_m (1 + \| u^\epsilon, Z^E_i \varrho^\epsilon, Z^E_i u^\epsilon \|_{L^\infty_0(\Omega)}) \int_0^t (1 + \| (\varrho^\epsilon, u^\epsilon) \|^2_{\tau t}) d\tau,
\]
and
\[
\int_0^t \| C_{14}^{\alpha} \|^2_{L^2_\tau(\Omega)} d\tau \leq C_m \| Z^E_i \varrho^\epsilon \|^2_{L^\infty_0(\Omega)} \int_0^t \| h^\epsilon \|^2_{H^m_{\tau t}} d\tau.
\]

Finally, we deal with the term $\int_0^t \| C_{13}^{\alpha} \|^2_{L^2_\tau(\Omega)} d\tau$. By direct computation, it is easy to check that
\[
C_{13}^{\alpha} = \sum_{|\beta| \geq 1, \beta + \gamma = \alpha} C_{\beta, \gamma} Z^\beta (\varrho^\epsilon v^\epsilon) Z^\gamma \partial_y u^\epsilon + \rho^\epsilon v^\epsilon [Z^u, \partial_y] u^\epsilon.
\]

Using the estimate (3.10), we get
\[
\int_0^t \| \rho^\epsilon v^\epsilon [Z^u, \partial_y] u^\epsilon \|^2_{L^2_\tau(\Omega)} d\tau \leq C_m \| v^\epsilon \|^2_{L^\infty_0(\Omega)} \int_0^t \| \partial_y u^\epsilon \|^2_{H^m_{\tau t-1}} d\tau. \tag{3.41}
\]

In order to control the velocity $v^\epsilon$, the idea is to apply the Hardy inequality and divergence-free condition to transform into the velocity $\partial_y u^\epsilon$ in some weighted Sobolev norm. By virtue of $|\alpha_1| \leq m - 1$ and $|\alpha_1| + \alpha_2 = m$, it follows $\alpha_2 \geq 1$, and hence we can get $\gamma_2 \geq 1$ if $\beta_2 = 0$. Thus it follows from the Moser type inequality (A.6) that
\[
\sum_{|\beta| \geq 1, \beta + \gamma = \alpha} \int_0^t \| Z^\beta (\rho^\epsilon v^\epsilon) Z^\gamma \partial_y u^\epsilon \|^2_{L^2_\tau(\Omega)} d\tau
\]
\[
\leq C_m \| Z^E_i (\rho^\epsilon v^\epsilon) \|^2_{L^\infty_0(\Omega)} \int_0^t \| Z_2 \partial_y u^\epsilon \|^2_{H^m_{\tau t-1}} d\tau
\]
\[
+ C_m \| Z_2 \partial_y u^\epsilon \|^2_{L^\infty_0(\Omega)} \int_0^t \| Z^E_i (\rho^\epsilon v^\epsilon) \|^2_{H^m_{\tau t-1}} d\tau. \tag{3.42}
\]
Using the divergence-free condition, Hardy inequality and Moser type inequality (A.6), we get
\[ \int_0^t \left\| Z^\varepsilon_t (\rho^\varepsilon v^\varepsilon) \right\|_{H^m_{m-1}}^2 d\tau \leq C_m \left\| \rho^\varepsilon \right\|_{L^\infty_0 (\Omega)}^2 \int_0^t \left\| v^\varepsilon \right\|_{H^m_{m-1}}^2 d\tau + C_m \left\| v^\varepsilon \right\|_{L^\infty_0 (\Omega)}^2 \int_0^t \left\| \varepsilon^\varepsilon \right\|_{H^m_{m-1}}^2 d\tau \]
\[ \leq C_{m,l} \int_0^t \left\| \partial_y v^\varepsilon \right\|_{H^m_{m-1}}^2 d\tau + C_m \left\| v^\varepsilon \right\|_{L^\infty_0 (\Omega)}^2 \int_0^t \left\| \varepsilon^\varepsilon \right\|_{H^m_{m-1}}^2 d\tau \]
\[ \leq C_{m,l} (1 + \left\| v^\varepsilon \right\|_{L^\infty_0 (\Omega)}, \int_0^t \left\| (q^\varepsilon, u^\varepsilon) \right\|_{H^m_{m-1}}^2 d\tau. \]

This and the inequality (3.42) yield directly
\[ \sum_{|\beta| \geq 1, \beta + \gamma = \alpha} \int_0^t \left\| Z^\beta (\rho^\varepsilon v^\varepsilon) Z^\gamma \partial_y u^\varepsilon \right\|_{L^2_{L^2}}^2 (\Omega) d\tau \leq C_{m,l} (1 + Q^2 (t)) \]

(3.43)

If \( \beta_2 \geq 1 \), we get after using the Moser type inequality (A.6) that
\[ \sum_{|\beta| \geq 1, \beta + \gamma = \alpha} \int_0^t \left\| Z^\beta (\rho^\varepsilon v^\varepsilon) Z^\gamma \partial_y u^\varepsilon \right\|_{L^2_{L^2}}^2 (\Omega) d\tau \]
\[ \leq C_m \left\| Z_2 (\rho^\varepsilon v^\varepsilon) \right\|_{L^\infty_0 (\Omega)}^2 \int_0^t \left\| \partial_y u^\varepsilon \right\|_{H^m_{m-1}}^2 d\tau + C_m \left\| \partial_y u^\varepsilon \right\|_{L^\infty_0 (\Omega), \int_0^t \left\| Z_2 (\rho^\varepsilon v^\varepsilon) \right\|_{H^m_{m-1}}^2 d\tau. \]

(3.44)

In view of the fact \( Z_2 (\rho^\varepsilon v^\varepsilon) = Z_2 \rho^\varepsilon v^\varepsilon + \rho^\varepsilon Z_2 v^\varepsilon \), we apply divergence-free condition, Hardy and Moser type inequalities to get
\[ \int_0^t \left\| Z_2 (\rho^\varepsilon v^\varepsilon) \right\|_{H^m_{m-1}}^2 d\tau \leq C_m \left\| (Z_2 \rho^\varepsilon, v^\varepsilon) \right\|_{L^\infty_0 (\Omega), \int_0^t \left\| \partial_y v^\varepsilon \right\|_{H^m_{m-1}}^2 d\tau + C_m \left\| \partial_y v^\varepsilon \right\|_{L^\infty_0 (\Omega), \int_0^t \left\| \partial_y v^\varepsilon \right\|_{H^m_{m-1}}^2 d\tau \]
\[ \leq C_{m,l} (1 + \left\| v^\varepsilon, \partial_y u^\varepsilon, Z_2 \rho^\varepsilon \right\|_{L^\infty_0 (\Omega), \int_0^t \left\| (q^\varepsilon, u^\varepsilon) \right\|_{H^m_{m-1}}^2 d\tau. \]

which, along with (3.44), gives directly
\[ \sum_{|\beta| \geq 1, \beta + \gamma = \alpha} \int_0^t \left\| Z^\beta (\rho^\varepsilon v^\varepsilon) Z^\gamma \partial_y u^\varepsilon \right\|_{L^2_{L^2}}^2 (\Omega) d\tau \]
\[ \leq C_{m,l} (1 + Q^2 (t)) \int_0^t \left\| (q^\varepsilon, u^\varepsilon) \right\|_{H^m_{m-1}}^2 d\tau. \]

(3.45)

The combination of the estimates (3.41), (3.43) and (3.45) yields directly
\[ \int_0^t \left\| C_{13}^\varepsilon \right\|_{L^2_{L^2}}^2 (\Omega) d\tau \leq C_{m,l} (1 + Q^2 (t)) \int_0^t \left\| (q^\varepsilon, u^\varepsilon) \right\|_{H^m_{m-1}}^2 d\tau. \]

Similarly, by routine checking, we may conclude that
\[ \int_0^t \left( \left\| C_{14}^\varepsilon \right\|_{L^2_{L^2}}^2 (\Omega) + \left\| C_{15}^\varepsilon \right\|_{L^2_{L^2}}^2 (\Omega) + \left\| C_{16}^\varepsilon \right\|_{L^2_{L^2}}^2 (\Omega) \right) d\tau. \]
\[\leq C_{m, l}(1 + Q(t)) \int_0^t (\|Q^\epsilon, u^\epsilon, h^\epsilon\|_{H^m}^2 + \|\partial_y h^\epsilon\|_{H^m}^2) d\tau.\]

Therefore, we complete the proof of the claim estimate (3.32). \(\square\)

### 3.2 Weighted \(H^m\)-estimates only on tangential derivative

In this subsection, we hope to establish the estimate for the quantity \(Z_{\tau}^{\alpha_1}(Q^\epsilon, u^\epsilon, h^\epsilon)\) with \(|\alpha_1| = m\). However, there is an essential difficulty to achieve this goal since the terms \(u^\epsilon \partial_y Q^\epsilon\), \(\rho^\epsilon u^\epsilon \partial_y (u^\epsilon - e^{-y}) - g^\epsilon \partial_y h^\epsilon\) and \(u^\epsilon \partial_y h^\epsilon - g^\epsilon \partial_y (u^\epsilon - e^{-y})\) will create the loss of one derivative in the tangential variable \(x\). In other words, \(\rho^\epsilon = -\partial_x^{-1} \partial_x \mu \) and \(g^\epsilon = -\partial_y^{-1} \partial_y h^\epsilon\), by the divergence-free condition (3.2)_4, create a loss of \(x\)-derivative that prevents us to apply the standard energy estimates. To overcome this essential difficulty, we take the strategy of the recent interesting result [31] that only needs that the background tangential magnetic field has a lower positive bound instead of monotonicity assumption on the tangential velocity.

However, due to the density being a unknown function instead of a constant, we need to take some new ideas to deal with the terms \(u^\epsilon \partial_y Q^\epsilon\) and \(\rho^\epsilon u^\epsilon \partial_y (u^\epsilon - e^{-y})\).

First of all, applying \(Z_{\tau}^{\alpha_1}(|\alpha_1| = m)\) differential operator to the Eq. (3.2)_3, we find

\[
\begin{aligned}
\{\partial_t + (u^\epsilon + 1 - e^{-y}) \partial_x + u^\epsilon \partial_y - e \partial_x^2 + \kappa \partial_y^2\}(Z_{\tau}^{\alpha_1} h^\epsilon) + Z_{\tau}^{\alpha_1} u^\epsilon \partial_y h^\epsilon &= (h^\epsilon + 1) \partial_x Z_{\tau}^{\alpha_1} u^\epsilon + g^\epsilon \partial_y Z_{\tau}^{\alpha_1} u^\epsilon + Z_{\tau}^{\alpha_1} g^\epsilon \partial_y (u^\epsilon - e^{-y}) - \epsilon Z_{\tau}^{\alpha_1} \partial_x r_h + f_h, \\
\end{aligned}
\]

where the function \(f_h\) is defined by

\[
f_h = -[Z_{\tau}^{\alpha_1}(u^\epsilon + 1 - e^{-y}) \partial_x] h^\epsilon + [Z_{\tau}^{\alpha_1}, (h^\epsilon + 1) \partial_x] u^\epsilon - \sum_{\beta_1 + \gamma_1 = \alpha_1, \beta_1 \neq 0, \beta_1 \neq \alpha_1} C_{\beta_1, \gamma_1} Z_{\tau}^{\beta_1} u^\epsilon Z_{\tau}^{\gamma_1} \partial_y h^\epsilon + \sum_{\beta_1 + \gamma_1 = \alpha_1, \beta_1 \neq 0, \beta_1 \neq \alpha_1} C_{\beta_1, \gamma_1} Z_{\tau}^{\beta_1} g^\epsilon Z_{\tau}^{\gamma_1} \partial_y u^\epsilon.
\]

To eliminate the difficult term \(Z_{\tau}^{\alpha_1} v^\epsilon \partial_y h^\epsilon\), following the idea as in [31], we introduce the stream function \(\psi^\epsilon\) satisfying

\[
\partial_y \psi^\epsilon = h^\epsilon, \quad \partial_x \psi^\epsilon = -g^\epsilon, \quad \psi^\epsilon|_{y=0} = 0.
\]

Then, we can deduce from the Eq. (3.2)_3 and boundary condition (3.3) that

\[
\partial_t \psi^\epsilon + (u^\epsilon + 1 - e^{-y}) \partial_x \psi^\epsilon + v^\epsilon(\partial_y \psi^\epsilon + 1) - e \partial_x^2 \psi^\epsilon - \kappa \partial_y^2 \psi^\epsilon = -\epsilon \partial_y^{-1} \partial_x r_h.
\]

Applying \(Z_{\tau}^{\alpha_1}(|\alpha_1| = m)\) differential operator to above equation, it follows

\[
\begin{aligned}
\{\partial_t + (u^\epsilon + 1 - e^{-y}) \partial_x + u^\epsilon \partial_y - e \partial_x^2 + \kappa \partial_y^2\} Z_{\tau}^{\alpha_1} \psi^\epsilon + Z_{\tau}^{\alpha_1} v^\epsilon (h^\epsilon + 1) &= -\epsilon \partial_y^{-1} Z_{\tau}^{\alpha_1} \partial_x r_h + f_\psi, \\
\end{aligned}
\]

where the function \(f_\psi\) is defined by

\[
f_\psi = -[Z_{\tau}^{\alpha_1}(u^\epsilon + 1 - e^{-y}) \partial_x] \psi^\epsilon - \sum_{\beta_1 + \gamma_1 = \alpha_1, \beta_1 \neq 0, \beta_1 \neq \alpha_1} C_{\beta_1, \gamma_1} Z_{\tau}^{\beta_1} v^\epsilon Z_{\tau}^{\gamma_1} \partial_y \psi^\epsilon.
\]

Set \(\eta_h := \frac{\partial_x h^\epsilon}{h^\epsilon + 1}\) and define the quantity

\[
h_m^\epsilon := Z_{\tau}^{\alpha_1} h^\epsilon - \eta_h Z_{\tau}^{\alpha_1} \psi^\epsilon,
\]

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then multiplying the Eq. (3.47) by $\eta_h$ and substituting the Eq. (3.46), the difficult term $Z_{t}^{\alpha_1}v^e \partial_x h^e$ in (3.46) can be eliminated. Hence, we get the evolution equation for the quantity $h_m^e$ as

$$
\partial_t h_m^e + (u^e + 1 - e^{-y})\partial_x h_m^e + v^e \partial_y h_m^e - \epsilon \partial_x^2 h_m^e - \kappa \partial_y^2 h_m^e \\
- (h^e + 1)\partial_x Z_{t}^{\alpha_1}u^e - g^e \partial_y Z_{t}^{\alpha_1}u^e - Z_{t}^{\alpha_1}g^e \partial_y (u^e - e^{-y}) \\
= -\epsilon Z_{t}^{\alpha_1}\partial_x r_h + \epsilon \eta_h \partial_y^{-1}Z_{t}^{\alpha_1} \partial_x r_h + f_h - \eta_h f \psi + 2\epsilon \partial_x \eta_h \partial_y Z_{t}^{\alpha_1} \psi^e \\
+ 2\kappa \partial_x \eta_h \partial_y Z_{t}^{\alpha_1} \psi^e \\
+ Z_{t}^{\alpha_1} \psi^e (\partial_t + (u^e + 1 - e^{-y})\partial_x + v^e \partial_y - \epsilon \partial_x^2 - \kappa \partial_y^2) \eta_h.
$$

(3.49)

Similarly, after applying $Z_{t}^{\alpha_1}([\alpha_1] = m)$ operator to the first equation of (3.2), we get

$$
\{\partial_t + (u^e + 1 - e^{-y})\partial_x + v^e \partial_y - \epsilon \partial_x^2 - \epsilon \partial_y^2\}(Z_{t}^{\alpha_1} \xi^e) + Z_{t}^{\alpha_1}v^e \partial_y \xi^e \\
= -\epsilon Z_{t}^{\alpha_1} \partial_x r_1 - \epsilon Z_{t}^{\alpha_1} \partial_y r_2 + f_\rho,
$$

(3.50)

where the function $f_\rho$ is defined by

$$
f_\rho = -[Z_{t}^{\alpha_1}, (u^e + 1 - e^{-y})\partial_x] \xi^e - \sum_{\beta_1 + \gamma_1 = \alpha_1 \atop \beta_1 \neq 0, \beta_1 \neq \alpha_1} C_{\beta_1, \gamma_1} Z_{t}^{\beta_1}v^e Z_{t}^{\gamma_1} \partial_y \xi^e.
$$

Set $\eta_\rho := \frac{\partial_t \xi^e}{h^e + 1}$ and define

$$
\xi_m^e := Z_{t}^{\alpha_1} \xi^e - \eta_\rho Z_{t}^{\alpha_1} \psi^e,
$$

(3.51)

we multiply the Eq. (3.47) by $\eta_\rho$ and substitute to the Eq. (3.50), and hence, the evolution for the quantity $\xi_m^e$ can be stated as follows

$$
\partial_t \xi_m^e + (u^e + 1 - e^{-y})\partial_x \xi_m^e + v^e \partial_y \xi_m^e - \epsilon \partial_x^2 \xi_m^e - \epsilon \partial_y^2 \xi_m^e \\
= f_\rho - \eta_\rho f \psi + 2\epsilon \partial_x \eta_\rho \partial_y Z_{t}^{\alpha_1} \psi^e - \kappa \eta_\rho \partial_y Z_{t}^{\alpha_1} \psi^e + \epsilon \partial_y^2 (\eta_\rho Z_{t}^{\alpha_1} \psi^e) \\
+ Z_{t}^{\alpha_1} \psi^e (\partial_t + (u^e + 1 - e^{-y})\partial_x + v^e \partial_y - \epsilon \partial_x^2) \eta_\rho \\
- \epsilon Z_{t}^{\alpha_1} (\partial_x r_1 + \partial_y r_2) + \epsilon \eta_\rho \partial_y^{-1} Z_{t}^{\alpha_1} \partial_x r_h.
$$

(3.52)

Finally, applying $Z_{t}^{\alpha_1}([\alpha_1] = m)$ differential operator to the Eq. (3.2)_2, we get

$$
\{\rho^e \partial_t + \rho^e (u^e + 1 - e^{-y})\partial_x + \rho^e v^e \partial_y - \epsilon \partial_x^2 - \mu \partial_y^2\}(Z_{t}^{\alpha_1}u^e) \\
+ \rho^e Z_{t}^{\alpha_1} v^e \partial_y (u^e - e^{-y}) \\
= (h^e + 1)\partial_x Z_{t}^{\alpha_1}h^e + g^e \partial_y Z_{t}^{\alpha_1}h^e + Z_{t}^{\alpha_1}g^e \partial_y h^e - \epsilon Z_{t}^{\alpha_1} \partial_x r_u + f_u,
$$

(3.53)

where the function $f_u$ is defined by

$$
f_u = -[Z_{t}^{\alpha_1}, \rho^e \partial_t] u^e - [Z_{t}^{\alpha_1}, \rho^e (u^e + 1 - e^{-y})\partial_x] u^e + [Z_{t}^{\alpha_1}, (h^e + 1)\partial_x] h^e \\
- \sum_{\beta_1 + \gamma_1 = \alpha_1 \atop \beta_1 \neq 0} C_{\beta_1, \gamma_1} Z_{t}^{\beta_1} \rho^e Z_{t}^{\gamma_1} \partial_y (u^e - e^{-y}) \\
- \sum_{\beta_1 + \gamma_1 = \alpha_1 \atop \beta_1 \neq 0, \beta_1 \neq \alpha_1} C_{\beta_1, \gamma_1} Z_{t}^{\beta_1} \rho^e Z_{t}^{\gamma_1} \partial_y u^e + \sum_{\beta_1 + \gamma_1 = \alpha_1 \atop \beta_1 \neq 0, \beta_1 \neq \alpha_1} C_{\beta_1, \gamma_1} Z_{t}^{\beta_1} g^e Z_{t}^{\gamma_1} \partial_y h^e.
$$
Set $\eta_u := \frac{\partial_y (\varrho \epsilon - e^{-y})}{h_{\epsilon} + 1}$, multiplying the Eq. (3.47) by $\rho \epsilon \eta_u$, and substituting to the Eq. (3.53), we find the quantity

$$u_{m}^\epsilon := Z_{\tau}^{a_1} u_{m}^\epsilon - \eta_u Z_{\tau}^{a_1} \psi_{\epsilon}$$

(3.54)

satisfying the evolution as follows

$$\begin{align*}
\rho \epsilon \partial_t u_{m}^\epsilon + \rho \epsilon (u_{m}^\epsilon + 1 - e^{-y}) \partial_x u_{m}^\epsilon + \rho \epsilon v_{\epsilon} \partial_y u_{m}^\epsilon - \epsilon \partial_x^2 u_{m}^\epsilon - \mu \partial_y^2 u_{m}^\epsilon
&- (h_{\epsilon} + 1) \partial_x Z_{\tau}^{a_1} h_{\epsilon} - g \epsilon \partial_y Z_{\tau}^{a_1} h_{\epsilon} - Z_{\tau}^{a_1} g \epsilon \partial_y h_{\epsilon}
= f_u - \rho \epsilon \eta_u f_{\psi} - Z_{\tau}^{a_1} \psi_{\epsilon} (\rho \epsilon \partial_t + \rho \epsilon (u_{m}^\epsilon + 1 - e^{-y}) \partial_x + \rho \epsilon v_{\epsilon} \partial_y) \eta_u
&- \epsilon (\rho \epsilon - 1) \eta_u \partial_x^2 Z_{\tau}^{a_1} \psi_{\epsilon} - \kappa \rho \epsilon \eta_u \partial_y^2 Z_{\tau}^{a_1} \psi_{\epsilon} + 2 \epsilon \partial_x \eta_u \partial_x Z_{\tau}^{a_1} \psi_{\epsilon}
&+ \epsilon \partial_x^2 \eta_u Z_{\tau}^{a_1} \psi_{\epsilon} + \mu \partial_y^2 (\eta_u Z_{\tau}^{a_1} \psi_{\epsilon}) - \epsilon Z_{\tau}^{a_1} \partial_x r_u + \epsilon \rho \epsilon \eta_u \partial_y Z_{\tau}^{a_1} \partial_x r_h.
\end{align*}$$

(3.55)

Let us define the functional:

$$X_{m,l}(t) := 1 + \mathcal{E}_{m,l}(t) + \| (\varrho_{m\epsilon}, u_{m\epsilon}, h_{m\epsilon}) (t) \|_{L^2(\Omega)}^2 + \| \partial_y (\varrho_{m\epsilon}, u_{m\epsilon}, h_{m\epsilon}) (t) \|_{L^2(\Omega)}^2$$

$$+ \| \partial_y \varrho_{m\epsilon} (t) \|_{H^1(\Omega)}^2.$$

(3.56)

where $\mathcal{E}_{m,l}(\tau)$ is defined by (3.11). Then, we will establish the following estimate in this subsection.

**Proposition 3.5** Let $(\varrho_{\epsilon}, u_{\epsilon \epsilon}, v_{\epsilon}, h_{\epsilon}, g_{\epsilon})$ be sufficiently smooth solution, defined on $[0, T^\epsilon]$, to the Eqs. (3.2)–(3.3). Under the assumptions of conditions (3.8) and (3.9), it holds true

$$\begin{align*}
\sup_{\tau \in [0, t]} \| (\varrho_{\epsilon m}, u_{\epsilon m}, h_{\epsilon m}) (\tau) \|_{L^2(\Omega)}^2 + \epsilon \int_0^t \| (\partial_x \varrho_{\epsilon m}, \partial_y u_{\epsilon m}, \partial_x h_{\epsilon m}) (\tau) \|_{L^2(\Omega)}^2 d\tau
&+ \epsilon \int_0^t \| (\partial_y \varrho_{\epsilon m} (\tau) \|_{L^2(\Omega)}^2 + \mu \| \partial_y h_{\epsilon m} (\tau) \|_{L^2(\Omega)}^2 d\tau
\leq C \| (\varrho_{\epsilon m}, u_{\epsilon m}, h_{\epsilon m}) (0) \|_{L^2(\Omega)}^2 + C t \| (\rho_0, u_10, h_10) \|_{L^2(\Omega)}^2 + C_{\mu, \kappa, m,l, \delta^{-6}} (1 + Q^3(t)) \int_0^t X_{m,l}(\tau) d\tau.
\end{align*}$$

First of all, we establish the weighted $L^2$–estimate for the quantity $\varrho_{\epsilon m}$.

**Lemma 3.6** Let $(\varrho_{\epsilon}, u_{\epsilon \epsilon}, v_{\epsilon}, h_{\epsilon}, g_{\epsilon})$ be sufficiently smooth solution, defined on $[0, T^\epsilon]$, to the Eqs. (3.2)–(3.3). Under the assumption of condition (3.8), then we have the following estimate for $0 < \delta_1 < 1$:

$$\begin{align*}
\sup_{\tau \in [0, t]} \| \varrho_{\epsilon m} (\tau) \|_{L^2(\Omega)}^2 + \epsilon \int_0^t \| \varrho_{\epsilon m} (\tau) \|_{L^2(\Omega)}^2 + \| \partial_y \varrho_{\epsilon m} \|_{L^2(\Omega)}^2 d\tau
\leq C \| \varrho_{\epsilon m} (0) \|_{L^2(\Omega)}^2 + C \int_0^t \| (r_1, r_2, r_h) \|_{H^1(\Omega)}^4 d\tau
+ C_{\delta_1} \int_0^t (\epsilon \| \partial_x h_{\epsilon m} \|_{L^2(\Omega)}^2 + \kappa \| \partial_y h_{\epsilon m} \|_{L^2(\Omega)}^2) d\tau
+ C_{\kappa, m,l, \delta^{-4}} (1 + Q^2(t)) \int_0^t (\mathcal{E}_{m,l}(\tau) + \| (\varrho_{\epsilon m}, u_{\epsilon m}, h_{\epsilon m}) (\tau) \|_{L^2(\Omega)}^2
+ \| \partial_y \varrho_{\epsilon m} \|_{H^1(\Omega)}^2) d\tau.
\end{align*}$$
Proof  Due to the fact $\partial_y \phi^\varepsilon|_{y=0}$ and $\psi^\varepsilon|_{y=0} = 0$, it follows $\partial_y h_{\rho}|_{y=0} = 0$. Then, multiplying the Eq. (3.52) by $\langle \phi^\varepsilon \rangle Z_{\varepsilon}^{\alpha_1}$, integrating over $\Omega$ and integrating by part, we get

$$
\frac{d}{dt} \int_{\Omega} \langle \phi^\varepsilon \rangle^2 |\phi^\varepsilon_m|^2 dxdy + \varepsilon \int_{\Omega} \langle \phi^\varepsilon \rangle Z_{\varepsilon}^{\alpha_1} \cdot \partial_x \phi^\varepsilon_m dxdy + \varepsilon \int_{\Omega} \langle \phi^\varepsilon \rangle Z_{\varepsilon}^{\alpha_1} \cdot \partial_y \phi^\varepsilon_m dxdy
$$

$$
= l \int_{\Omega} \langle \phi^\varepsilon \rangle^{2l-1} v^\varepsilon |\phi^\varepsilon_m|^2 dxdy - 2l \int_{\Omega} \langle \phi^\varepsilon \rangle^{2l} \partial_y \phi^\varepsilon_m \cdot \phi^\varepsilon_m dxdy + 2\varepsilon \int_{\Omega} \langle \phi^\varepsilon \rangle Z_{\varepsilon}^{\alpha_1} \cdot \partial_x \phi^\varepsilon_m dxdy
$$

$$
+ \int_{\Omega} \partial_2 \phi_{\varepsilon} \cdot (\partial_t + (u^\varepsilon + 1 - e^{-\gamma}) \partial_x + v^\varepsilon \partial_y) \partial_y \partial_x \phi^\varepsilon_m dxdy
$$

$$
+ \int_{\Omega} \phi_{\varepsilon} \cdot (\partial_t + (u^\varepsilon + 1 - e^{-\gamma}) \partial_x + v^\varepsilon \partial_y) \partial_y \partial_x \phi^\varepsilon_m dxdy
$$

$$
+ \int_{\Omega} (f_{\rho} - \eta_{\rho} \phi_{\psi} - \epsilon Z_{\varepsilon}^{\alpha_1} (\partial_x r_1 + \partial_y r_2) + \epsilon \eta_{\rho} \partial_2^{-1} Z_{\varepsilon}^{\alpha_1} \partial_x r_h) \cdot \langle \phi^\varepsilon \rangle^{2l} \phi^\varepsilon_m dxdy.
$$

Using the Hölder and Cauchy inequalities, it follows

$$
\left| \int_{\Omega} \langle \phi^\varepsilon \rangle^{2l-1} v^\varepsilon |\phi^\varepsilon_m|^2 dxdy - 2l \int_{\Omega} \langle \phi^\varepsilon \rangle^{2l} \partial_y |\phi^\varepsilon_m|^2 dxdy \right|
$$

$$
\leq C_l \| v^\varepsilon \|_{L^{\infty}_\Omega} \| |\phi^\varepsilon_m|^2 \|_{L^1_\Omega} + 2l \| |\phi^\varepsilon_m|^2 \|_{L^1_\Omega} \| |\phi^\varepsilon_m|^2 \|_{L^1_\Omega} + \frac{1}{8} \|\partial_y \phi^\varepsilon_m\|^2_{L^2_\Omega} + C_l (1 + \| v^\varepsilon \|_{L^1_\Omega}^2) \|\phi^\varepsilon_m\|^2_{L^2_\Omega},
$$

and

$$
\left| \int_{\Omega} \partial_2 \phi_{\varepsilon} \cdot \partial_x \phi^\varepsilon_m dxdy \right| \leq \| \partial_2 \phi_{\varepsilon} \|_{L^1_\Omega} \| \partial_x \phi^\varepsilon_m \|_{L^2_\Omega}.
$$

By virtue of the fact $\partial_y h_{\rho} = \frac{1}{h^\varepsilon + 1}(\partial_{x\gamma} \phi^\varepsilon - \frac{\partial_{x\varepsilon} \partial_{y\varepsilon} h_{\gamma}}{h^\varepsilon + 1})$, we get

$$
\| \partial_y h_{\rho} \|_{L^1_\Omega} \leq \| \partial_{x\gamma} \phi^\varepsilon \|_{L^1_\Omega} + \| \partial_x \phi^\varepsilon \|_{L^1_\Omega} + \| \partial_y \phi^\varepsilon \|_{L^1_\Omega} + \| \partial_x h_{\gamma} \|_{L^1_\Omega},
$$

where we have used the fact $h^\varepsilon + 1 \geq \delta$. This and inequalities (B.3) and (3.58) give

$$
\left| \epsilon \int_{\Omega} \partial_2 \phi_{\varepsilon} \cdot \partial_x \phi^\varepsilon_m dxdy \right| \leq \delta \epsilon \| \partial_x h^\varepsilon_m \|^2_{L^1_\Omega} + C_l \delta^{-4} (1 + \| Q(t) \|_{L^1_\Omega}^2) \| (\phi^\varepsilon_m, h^\varepsilon_m)\|^2_{L^2_\Omega}.
$$

Using the Hölder and Cauchy inequalities, it follows

$$
\left| \kappa \int_{\Omega} \partial_2 \phi_{\varepsilon} \cdot \partial_x \phi^\varepsilon_m dxdy \right| \leq \delta \kappa \| \partial_x \phi^\varepsilon_m \|^2_{L^1_\Omega} + C_k \delta^{-2} \| Q(t) \|_{L^1_\Omega} \| (\phi^\varepsilon_m, h^\varepsilon_m)\|^2_{L^2_\Omega},
$$

which, together with the estimate (B.4), yields directly

$$
\left| \kappa \int_{\Omega} \partial_2 \phi_{\varepsilon} \cdot \partial_x \phi^\varepsilon_m dxdy \right| \leq \delta \kappa \| \partial_x h^\varepsilon_m \|^2_{L^1_\Omega} + C_k \delta^{-2} \| Q(t) \|_{L^1_\Omega} \| (\phi^\varepsilon_m, h^\varepsilon_m)\|^2_{L^2_\Omega}.
$$

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In view of $\partial_y Q^e |_{y=0} = 0$ and $\psi^e |_{y=0} = 0$, it is easy to justify the fact $\partial_y (\eta_\rho Z^{a_1}_\tau \psi^e) |_{y=0} = 0$, and hence, the integration by parts directly

$$
\epsilon \int_\Omega \partial_y^2 (\eta_\rho Z^{a_1}_\tau \psi^e) \cdot (y)^{2l} \psi^e_m dx dy
= -\epsilon \int_\Omega \partial_y \eta_\rho Z^{a_1}_\tau \psi^e \cdot (2l (y)^{2l-1} \psi^e_m + (y)^{2l} \partial_y \psi^e_m) dx dy
- \epsilon \int_\Omega \eta_\rho Z^{a_1}_\tau h^e \cdot (2l (y)^{2l-1} \psi^e_m + (y)^{2l} \partial_y \psi^e_m) dx dy.
$$

It follows from the Hölder inequality that

$$
|\epsilon \int_\Omega \partial_y \eta_\rho Z^{a_1}_\tau \psi^e \cdot (2l (y)^{2l-1} \psi^e_m + (y)^{2l} \partial_y \psi^e_m) dx dy|
\leq \|\partial_y \eta_\rho Z^{a_1}_\tau \psi^e\|_{L^2_{\tau-1}(\Omega)} \|\psi^e_m\|_{L^2_\tau(\Omega)} + \|\partial_y \eta_\rho Z^{a_1}_\tau \psi^e\|_{L^2_\tau(\Omega)} \|\partial_y \psi^e_m\|_{L^2_\tau(\Omega)}.
$$

(3.59)

Thanks to the relation $\partial_y \eta_\rho = \frac{1}{h^n+1} \{ \partial_y^2 Q^e - \frac{\partial_y Q^e h^e}{h^n+1} \}$, we get

$$
\|\partial_y \eta_\rho Z^{a_1}_\tau \psi^e\|_{L^2_\tau(\Omega)}
\leq (\| Z_2 \partial_y Q^e \|_{L^\infty_\tau(\Omega)} + \delta^{-1} \| Z_2 Q^e \|_{L^\infty_\tau(\Omega)} \| \partial_y h^e \|_{L^\infty_\tau(\Omega)}) \| \frac{1}{\varphi(y)} Z^{a_1}_\tau \psi^e \|_{L^2_{\tau-1}(\Omega)} h^e + 1 \|_{L^2_{\tau-1}(\Omega)} \| \partial_y Q^e \|_{L^\infty(\Omega)} \| \partial_y h^e \|_{L^\infty(\Omega)} \| \partial_y \psi^e_m \|_{L^2_\tau(\Omega)}.
$$

(3.60)

where, in the last inequality, we have used the following estimate

$$
\| \frac{1}{\varphi(y)} Z^{a_1}_\tau \psi^e \|_{L^2_{\tau-1}(\Omega)} \leq C I \delta^{-1} \| h^e \|_{L^2_\tau(\Omega)}.
$$

Combining (3.59) with (3.60), we conclude that

$$
|\epsilon \int_\Omega \partial_y \eta_\rho Z^{a_1}_\tau \psi^e \cdot (2l (y)^{2l-1} \psi^e_m + (y)^{2l} \partial_y \psi^e_m) dx dy|
\leq \frac{1}{8} \epsilon \|\partial_y \psi^e_m\|_{L^2_\tau(\Omega)}^2 + C I \delta^{-4} Q^2(t) \|\psi^e_m, h^e_m\|_{L^2_\tau(\Omega)}^2.
$$

(3.61)

Using the Cauchy inequality and estimate (B.2), we show

$$
|\epsilon \int_\Omega \eta_\rho Z^{a_1}_\tau h^e \cdot (2l (y)^{2l-1} \psi^e_m + (y)^{2l} \partial_y \psi^e_m) dx dy
\leq \frac{1}{8} \epsilon \|\partial_y \psi^e_m\|_{L^2_\tau(\Omega)}^2 + C I \delta^{-4} (1 + \|\partial_y Q^e\|_{L^\infty_\tau(\Omega)}^4 + \|\partial_y h^e\|_{L^\infty_\tau(\Omega)}^4) \|\psi^e_m, h^e_m\|_{L^2_\tau(\Omega)}^2.
$$

This and the inequality (3.61) give

$$
|\epsilon \int_\Omega \partial_y^2 (\eta_\rho Z^{a_1}_\tau \psi^e) \cdot (y)^{2l} \psi^e_m dx dy| \leq \frac{1}{4} \epsilon \|\partial_y \psi^e_m\|_{L^2_\tau(\Omega)}^2
+ C I \delta^{-4} (1 + Q^2(t)) \|\psi^e_m, h^e_m\|_{L^2_\tau(\Omega)}^2.
$$

(3.62)

The integration by part with respect to $x$ variable yields immediately

$$
\epsilon \int_\Omega Z^{a_1}_\tau \psi^e \partial_x^2 \eta_\rho \cdot (y)^{2l} \psi^e_m dx dy = \epsilon \int_\Omega (y)^{2l} \partial_x \eta_\rho (Z^{a_1}_\tau \psi^e \partial_x \psi^e_m + \partial_x Z^{a_1}_\tau \psi^e \psi^e_m) dx dy.
$$
By virtue of the Hölder inequality, we get

\[
\left| \varepsilon \int_{\Omega} \langle y \rangle^{2L} \partial_x \eta_\rho Z_{\tau}^{\alpha_l} \psi^e \partial_x \varphi_m^e \, dx \, dy \right| \leq \varepsilon \| \partial_x \eta_\rho Z_{\tau}^{\alpha_l} \psi^e \|_{L^2_\text{loc}(\Omega)} \| \partial_x \varphi_m^e \|_{L^2_\text{loc}(\Omega)}.
\] (3.63)

Due to the fact \( \partial_x \eta_\rho = \frac{1}{h^e+1} \left[ \partial_x^2 \psi^e - \frac{\partial_x \psi^e \partial_x h^e}{h^e+1} \right] \), it follows

\[
\| \partial_x \eta_\rho Z_{\tau}^{\alpha_l} \psi^e \|_{L^2_\text{loc}(\Omega)} \leq \left( \| \partial_x^2 \psi^e \|_{L^2_\text{loc}(\Omega)} + \eta_{\text{v}} \eta_{\text{h}} \| \partial_x \psi^e \|_{L^2_\text{loc}(\Omega)} \right) \| Z_{\tau}^{\alpha_l} \psi^e \|_{L^2_\text{loc}(\Omega)} \leq C_l \eta_{\text{v}} \eta_{\text{h}} \| \partial_x \psi^e \|_{L^2_\text{loc}(\Omega)}
\]

where we have used the estimate (B.1) in the last inequality. This and inequality (3.63) yield directly

\[
\left| \varepsilon \int_{\Omega} \langle y \rangle^{2L} \partial_x \eta_\rho Z_{\tau}^{\alpha_l} \psi^e \partial_x \varphi_m^e \, dx \, dy \right| \leq \frac{1}{8} \varepsilon \| \partial_x \varphi_m^e \|_{L^2_\text{loc}(\Omega)} + C_l \eta_{\text{v}} \eta_{\text{h}} \| \partial_x \psi^e \|_{L^2_\text{loc}(\Omega)}.
\] (3.64)

Similarly, it is easy to justify that

\[
\left| \varepsilon \int_{\Omega} \langle y \rangle^{2L} \partial_x \eta_\rho Z_{\tau}^{\alpha_l} \psi^e \varphi_m^e \, dx \, dy \right| \leq \frac{1}{8} \varepsilon \| \partial_x \varphi_m^e \|_{L^2_\text{loc}(\Omega)} + \eta_{\text{v}} \eta_{\text{h}} \| \partial_x \psi^e \|_{L^2_\text{loc}(\Omega)}
\]

which, along with (3.64), yields directly

\[
\left| \varepsilon \int_{\Omega} Z_{\tau}^{\alpha_l} \psi^e \partial_x^2 \eta_\rho \cdot \langle y \rangle^{2L} \varphi_m^e \, dx \, dy \right| \leq \varepsilon \left( \frac{1}{8} \| \partial_x \varphi_m^e \|_{L^2_\text{loc}(\Omega)} + \eta_{\text{v}} \eta_{\text{h}} \| \partial_x \psi^e \|_{L^2_\text{loc}(\Omega)} \right) + C_l \eta_{\text{v}} \eta_{\text{h}} \| \partial_x \psi^e \|_{L^2_\text{loc}(\Omega)}.
\] (3.65)

Using the Hölder inequality and estimate (B.1), we get

\[
\left| \int_{\Omega} Z_{\tau}^{\alpha_l} \psi^e \left( \partial_t + (u^e + 1 - e^{-y}) \partial_x + v^e \partial_y \right) \eta_\rho \cdot \langle y \rangle^{2L} \varphi_m^e \, dx \, dy \right| \leq C_l \eta_{\text{v}} \eta_{\text{h}} \| \partial_x \psi^e \|_{L^2_\text{loc}(\Omega)}.
\]

The application of Hölder and Cauchy inequalities gives directly

\[
\left| \int_{\Omega} (f_\rho - \eta_\rho f_\psi) \cdot \langle y \rangle^{2L} \varphi_m^e \, dx \, dy \right| \leq C \left( \| f_\rho \|_{L^2_\text{loc}(\Omega)} + \| f_\psi \|_{L^2_\text{loc}(\Omega)} \right)
\]

Integrating by part and applying the Cauchy inequality, it follows

\[
\left| \int_{\Omega} \left( -e Z_{\tau}^{\alpha_l} \left( \partial_x r_1 + \partial_y r_2 \right) + \varepsilon \eta_\rho \partial_y^{-1} Z_{\tau}^{\alpha_l} \partial_x \eta_\rho \right) \varphi_m^e \, dx \, dy \right| \leq \frac{\varepsilon}{8} \| \partial_x \varphi_m^e, \partial_y \varphi_m^e \|_{L^2_\text{loc}(\Omega)}^2 + \| Z_{\tau}^{\alpha_l} (r_1, r_2, r_h) \|_{L^2_\text{loc}(\Omega)}^4 + C_l \eta_{\text{v}} \eta_{\text{h}} \| \partial_x \psi^e \|_{L^2_\text{loc}(\Omega)}.
\]
Combining the above estimates of terms for the right hand side of (3.57), and integrating the resulting inequality over $[0, t]$, we get

$$
\| \varepsilon^e_m(t) \|^2_{L^2_1(\Omega)} + \frac{1}{2} \int_0^t (\| \partial_x \varepsilon^e_m \|^2_{L^2_1(\Omega)} + \| \partial_y \varepsilon^e_m \|^2_{L^2_1(\Omega)}) dt \\
\leq \| \varepsilon^e_m(0) \|^2_{L^2_1(\Omega)} + \frac{1}{2} \int_0^t (\| \partial_x \varepsilon^e_m \|^2_{L^2_1(\Omega)} + \kappa \| \partial_y \varepsilon^e_m \|^2_{L^2_1(\Omega)}) dt \\
+ \int_0^t \| Z^{\alpha_1}_{\varepsilon}(r_1, r_2, r_h) \|^4_{L^2_1(\Omega)} dt \\
+ C \int_0^t (\| f_{\rho} \|^2_{L^2_1(\Omega)} + \| f_\psi \|^2_{L^2_{-1}(\Omega)}) dt + C_{\kappa, \delta^{-4}} (1 + Q^2(t)) \\
\int_0^t (1 + \| (\varepsilon^e_m, \h^e_m) \|^2_{L^2_1(\Omega)}) dt.
$$

On the other hand, applying the Moser type inequality (A.6) to get

$$
\int_0^t (\| f_{\rho} \|^2_{L^2_1(\Omega)} + \| f_\psi \|^2_{L^2_{-1}(\Omega)}) dt \leq C_{m, l} Q(t) \int_0^t (\| (\varepsilon^e, \h^e, \h^e) \|^2_{\gamma^m} + \| \partial_y \varepsilon^e \|^2_{\gamma^m_{-1}}) dt,
$$

which along with the estimate (B.13) completes the proof of lemma.  

Next, we establish the estimate for the quantities $u^e_m$ and $h^e_m$. Indeed, it is easy to check that

$$
-(h^e + 1) \partial_x Z^{\alpha_1}_{\varepsilon} h^e - g^e \partial_y Z^{\alpha_1}_{\varepsilon} h^e - Z^{\alpha_1}_{\varepsilon} g^e \partial_y h^e \\
= -(h^e + 1) \partial_x h^e_m - g^e \partial_y h^e_m - (h^e + 1) \partial_x \eta_h Z^{\alpha_1}_{\varepsilon} \psi - g^e \partial_y \eta_h Z^{\alpha_1}_{\varepsilon} \psi \\
- g^e \eta_h Z^{\alpha_1}_{\varepsilon} h^e,
$$

and

$$
-(h^e + 1) \partial_x Z^{\alpha_1}_{\varepsilon} u^e - g^e \partial_y Z^{\alpha_1}_{\varepsilon} u^e - Z^{\alpha_1}_{\varepsilon} g^e \partial_y (u^e - e^{-y}) \\
= -(h^e + 1) \partial_x u^e_m - g^e \partial_y u^e_m - (h^e + 1) \partial_x \eta_u Z^{\alpha_1}_{\varepsilon} \psi - g^e \partial_y \eta_u Z^{\alpha_1}_{\varepsilon} \psi - g^e \eta_u Z^{\alpha_1}_{\varepsilon} h^e,
$$

which were first observed in [31]. Then, we have the following estimates:

**Lemma 3.7** Let $(\varepsilon^e, \h^e, \eta^e, \eta^e, \eta^e, g^e)$ be sufficiently smooth solution, defined on $[0, T^e]$, to the Eqs. (3.2)–(3.3). Under the assumption of conditions (3.8) and (3.9), it holds true

$$
\sup_{0 \leq t \leq T^e} \| (u^e_m, h^e_m) (t) \|^2_{L^2_1(\Omega)} + \int_0^t \| \sqrt{e} \partial_x (u^e_m, h^e_m) \|^2_{L^2_1(\Omega)} dt \\
+ \int_0^t \| \partial_y (\sqrt{e} u^e_m, \sqrt{e} h^e_m) \|^2_{L^2_1(\Omega)} dt \\
\leq C \| (u^e_m, h^e_m)(0) \|^2_{L^2_1(\Omega)} + C \int_0^t \| (r_u, r_h) \|^4_{\gamma^m_{1,2,3}} dt \\
+ C_{\mu, k, m, l} \delta^{-6} (1 + Q^3(t)) \int_0^t \varepsilon_m, l(t) dt \\
+ C_{\mu, k, m, l} \delta^{-6} (1 + Q^3(t)) \int_0^t \| (\varepsilon^e_m, u^e_m, h^e_m) \|^2_{L^2_1(\Omega)} dt \\
+ \| (\partial_y u^e, \partial_y h^e) \|^2_{\gamma^m_{-1}} dt.
$$
By routine checking, we may show that
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} \langle y \rangle^{2l} \rho^e |u_m^e|^2 dxdy + \epsilon \int_{\Omega} \langle y \rangle^{2l} |\partial_x u_m^e|^2 dxdy + \mu \int_{\Omega} \langle y \rangle^{2l} |\partial_y u_m^e|^2 dxdy \\
+ \int_{\Omega} \langle y \rangle^{2l} (h^e + 1) \partial_x u_m^e \cdot h_m^e dxdy + \int_{\Omega} \langle y \rangle^{2l} g^e \partial_y u_m^e \cdot h_m^e dxdy = \sum_{i=1}^{9} I_i.
\]

where \( I_i (i = 1, \ldots, 9) \) are defined by

\[
I_1 = \frac{1}{2} \int_{\Omega} \langle y \rangle^{2l} |u_m^e|^2 (\partial_t \rho^e + (u^e + 1 - e^{-\gamma}) \partial_x \rho^e + v^e \partial_y \rho^e) dxdy, \\
I_2 = \int_{\Omega} \langle y \rangle^{2l-1} \rho^e v^e |u_m^e|^2 dxdy, \quad I_3 = -2l \int_{\Omega} \langle y \rangle^{2l-1} (\mu \partial_y u_m^e + g^e h_m^e) \cdot u_m^e dxdy, \\
I_4 = \int_{\Omega} \langle -h^e + 1 \rangle \partial_x \eta h Z_t^{a_1} \psi - g^e \partial_y \eta h Z_t^{a_1} \psi Z_t^{a_1} \psi - g^e \partial_y Z_t^{a_1} \psi \cdot \langle y \rangle^{2l} u_m^e dxdy, \\
I_5 = \int_{\Omega} \langle -Z_t^{a_1} \psi (\rho^e \partial_t + \rho^e (u^e + 1 - e^{-\gamma}) \partial_x + \rho^e v^e \partial_y) \eta u \cdot \langle y \rangle^{2l} u_m^e dxdy, \\
I_6 = \int_{\Omega} \langle f_u - \rho^e \eta u f \psi - \kappa \rho^e \eta u \partial_x Z_t^{a_1} \psi \cdot \langle y \rangle^{2l} u_m^e dxdy, \\
I_7 = \int_{\Omega} \langle e \partial_r \eta u Z_t^{a_1} \psi + \mu \partial_y (\eta u Z_t^{a_1} \psi) \cdot \langle y \rangle^{2l} u_m^e dxdy, \\
I_8 = \int_{\Omega} \langle 1 - \rho^e \rangle \eta u \partial_x Z_t^{a_1} \psi \cdot \langle y \rangle^{2l} u_m^e dxdy, \\
I_9 = \int_{\Omega} \langle -e \partial_r \eta u \partial_x r u + e \rho^e \eta \partial_y Z_t^{a_1} \partial_x r \eta u \cdot \langle y \rangle^{2l} u_m^e dxdy.
\]

By routine checking, we may show that
\[
|I_1| + |I_2| \leq C_l (1 + \|u^e, v^e, \partial_t g^e, \partial_x g^e, \partial_y g^e\|_{L^\infty_t(\Omega)}) \|u_m^e\|_{L^2_t(\Omega)}^2.
\]

By virtue of the Hölder and Cauchy inequalities, we find
\[
|I_3| \leq \frac{1}{8} \mu \|\partial_y u_m^e\|_{L^2_t(\Omega)}^2 + C_{l, l} (1 + \|g^e\|_{L^\infty_t(\Omega)}^2) (\|u_m^e\|_{L^2_t(\Omega)}^2 + \|h_m^e\|_{L^2_t(\Omega)}^2).
\]

Deal with the term \( I_4 \). By virtue of \( h^e + 1 \geq \delta \) and estimate (B.2), we apply the Hölder inequality to get
\[
\left| \int_{\Omega} g^e \eta h Z_t^{a_1} \psi \cdot \langle y \rangle^{2l} u_m^e dxdy \right| \\
\leq x \delta^{-1} \|g^e\|_{L^\infty_t(\Omega)} \|\partial_y h^e\|_{L^\infty_t(\Omega)} \|Z_t^{a_1} \psi\|_{L^2_t(\Omega)} \|u_m^e\|_{L^2_t(\Omega)} \\
\leq C_l \delta^{-2} (\|g^e\|_{L^\infty_t(\Omega)}^2 + \|\partial_y h^e\|_{L^\infty_t(\Omega)}^2) (\|u_m^e\|_{L^2_t(\Omega)}^2 + \|h_m^e\|_{L^2_t(\Omega)}^2).
\]

where \( I_i (i = 1, \ldots, 9) \) are defined by
Similarly, we also get that
\[
\left| \int_\Omega (h^\varepsilon - 1) \partial_x \eta h Z_t^{\alpha_1} \psi^\varepsilon \cdot (y)^2 u_m^\varepsilon \, dx \, dy \right| \leq (1 + \|h^\varepsilon\|_{L^\infty_0(\Omega)}) \|h^\varepsilon + 1\|_{L^\infty_0(\Omega)} \|u_m^\varepsilon\|_{L^2_0(\Omega)}
\]
Similarly, we also have
\[
\left| \int_\Omega g^\varepsilon \partial_y \eta h Z_t^{\alpha_1} \psi^\varepsilon \cdot (y)^2 u_m^\varepsilon \, dx \, dy \right| \leq C \delta^{-2}(1 + Q^2(t)) \|(u_m^\varepsilon, h_m^\varepsilon)\|^2_{L_0^{2}(\Omega)}.
\]
This, along with inequalities (3.67) and (3.68), yields directly
\[
|I_4| \leq C \delta^{-2}(1 + Q^2(t)) \|(u_m^\varepsilon, h_m^\varepsilon)\|^2_{L_0^{2}(\Omega)}.
\]
Similarly, we also get that
\[
|I_5| \leq C \delta^{-2}(1 + Q^2(t)) \|(u_m^\varepsilon, h_m^\varepsilon)\|^2_{L_0^{2}(\Omega)}.
\]
Deal with the term \(I_6\). By virtue of the Hölder and Cauchy inequalities, we get
\[
\left| \int_\Omega (f_u - \rho^\varepsilon \eta u \psi) \cdot (y)^2 u_m^\varepsilon \, dx \, dy \right| \leq C \|f_u\|^2_{L_0^{2}(\Omega)} + \|f \psi\|^2_{L_0^{2}(\Omega)}
\]
\[
+C (1 + \|\eta u\|^2_{L^\infty_0(\Omega)}) \|u_m^\varepsilon\|^2_{L_0^{2}(\Omega)}.
\]
Similar to the estimate (3.67), it is easy to check that
\[
\kappa \left| \int_\Omega \rho^\varepsilon \eta u \partial^2_y Z_t^{\alpha_1} \psi^\varepsilon \cdot (y)^2 u_m^\varepsilon \, dx \, dy \right|
\leq C \kappa \delta^{-1} \|\partial_y u^\varepsilon\|^2_{L^\infty_0(\Omega)} \|\partial_y Z_t^{\alpha_1} h^\varepsilon\|_{L_0^{2}(\Omega)} \|u_m^\varepsilon\|_{L_0^{2}(\Omega)}
\leq \delta_2 \kappa \|\partial_y h_m^\varepsilon\|^2_{L_0^{2}(\Omega)}
\]
\[
+C \kappa \delta^{-2}(\|Z_2 \partial_y h^\varepsilon\|^2_{L^\infty_0(\Omega)} + \|\partial y u^\varepsilon, \partial y h^\varepsilon\|^2_{L^\infty_0(\Omega)}) \|(u_m^\varepsilon, h_m^\varepsilon)\|^2_{L_0^{2}(\Omega)},
\]
where we have used the estimate (B.4) in the last inequality. Similarly, we also have
\[
|2 \varepsilon \left| \int_\Omega \partial_y \eta \partial_y Z_t^{\alpha_1} \psi^\varepsilon \cdot (y)^2 u_m^\varepsilon \, dx \, dy \right|
\leq 2 \varepsilon \|\partial^2_{xy} u^\varepsilon\|^2_{L^\infty_0(\Omega)} + \|\partial y u^\varepsilon\|^2_{L^\infty_0(\Omega)} \|\partial_y h^\varepsilon\|^2_{L^\infty_0(\Omega)} \|u_m^\varepsilon\|^2_{L^\infty_0(\Omega)}
\]
\[
\leq \delta_1 \varepsilon \|\partial_y h_m^\varepsilon\|^2_{L_0^{2}(\Omega)} + C \delta^{-4}(1 + \|(\partial y u^\varepsilon, \partial y h^\varepsilon)\|^4_{L^\infty_0(\Omega)})
\]
\[
+C \|\partial_y h^\varepsilon\|^4_{L^\infty_0(\Omega)} \|(u_m^\varepsilon, h_m^\varepsilon)\|^2_{L_0^{2}(\Omega)}.
\]
This, along with inequalities (3.69) and (3.70), yields directly
\[
|I_6| \leq \delta_2 \kappa \|\partial_y h_m^\varepsilon\|^2_{L_0^{2}(\Omega)} + \delta_1 \varepsilon \|\partial_y h_m^\varepsilon\|^2_{L_0^{2}(\Omega)}
\]
\[
+C (\|f_u\|^2_{L_0^{2}(\Omega)} + \|f \psi\|^2_{L_0^{2}(\Omega)})
\]
\[
+C \kappa \delta^{-4}(1 + Q^2(t)) \|(u_m^\varepsilon, h_m^\varepsilon)\|^2_{L_0^{2}(\Omega)}.
\]
Now, we give the estimate for the term $I_7$. Similar to the estimates (3.62) and (3.65), we can obtain
\[
\left| \epsilon \int_{\Omega} \partial_x^2 \eta \partial_t \psi \cdot (y)^{2l} u_m^\epsilon \, dx \, dy \right| \leq \frac{1}{8} \epsilon \| \partial_x u_m^\epsilon \|_{L^2_1(\Omega)}^2 + C \delta^{-4} \left( 1 + Q^2(t) \right) \| (u_m^\epsilon, h_m^\epsilon) \|_{L^2_1(\Omega)}^2,
\]
and
\[
\left| \epsilon \int_{\Omega} \partial_x^2 (\eta \partial_t \psi \cdot (y)^{2l} u_m^\epsilon \, dx \, dy \right| \leq \frac{1}{8} \epsilon \| \partial_y u_m^\epsilon \|_{L^2_1(\Omega)}^2 + C \mu \delta^{-4} \left( 1 + Q^2(t) \right) \| (u_m^\epsilon, h_m^\epsilon) \|_{L^2_1(\Omega)}^2,
\]
and hence, it follows
\[
|I_7| \leq \frac{1}{8} \left( \epsilon \| \partial_x u_m^\epsilon \|_{L^2_1(\Omega)}^2 + \mu \| \partial_y u_m^\epsilon \|_{L^2_1(\Omega)}^2 \right) + C \mu \delta^{-4} \left( 1 + Q^2(t) \right) \| (u_m^\epsilon, h_m^\epsilon) \|_{L^2_1(\Omega)}^2.
\]
Finally, we deal with the term $I_8$. Indeed, the integration by part with respect to $x$ yields directly
\[
I_8 = \epsilon \int_{\Omega} (y)^{2l} \partial_x \eta \partial_t \psi \cdot \partial_x \partial_t \psi \, dx \, dy,
\]
(3.71)
\[
+ \epsilon \int_{\Omega} (y)^{2l} \partial_x \eta \partial_t u_m^\epsilon \cdot \partial_x \partial_t \psi \, dx \, dy
\]
\[
+ \epsilon \int_{\Omega} (y)^{2l} \partial_x \eta \partial_t u_m^\epsilon \cdot \partial_x \partial_t \psi \, dx \, dy = I_{81} + I_{82} + I_{83}.
\]
By virtue of the estimate (B.3), Hölder and Cauchy equalities, we find
\[
I_{81} \leq \epsilon \| \partial_x \psi \|_{L^\infty_0(\Omega)} \| \partial_y (u^\epsilon - e^{-\gamma}) \|_{L^\infty_1(\Omega)} \| u_m^\epsilon \|_{L^2_1(\Omega)} \| \partial_x \psi \|_{L^2_1(\Omega)} \| \frac{\partial_x \partial_t \psi}{h + 1} \|_{L^2_{-1}(\Omega)} \leq \delta_2 \epsilon \| \partial_x h_m^\epsilon \|_{L^2_1(\Omega)}^2 + C \delta^{-4} \left( 1 + \| \partial_y u_m^\epsilon \|_{L^\infty_1(\Omega)}^4 \right)
\]
(3.72)
\[
+ \| \partial_x \psi \|_{L^\infty_1(\Omega)}^4 \| u_m^\epsilon \|_{L^2_1(\Omega)}^2 \| h_m^\epsilon \|_{L^2_1(\Omega)}^2.
\]
Similar to the estimate (B.3), it is easy to justify
\[
I_{83} \leq \delta_2 \epsilon \| \partial_x h_m^\epsilon \|_{L^2_1(\Omega)}^2 + C \delta^{-4} \left( 1 + \| \partial_x \psi \|_{L^\infty_1(\Omega)}^4 \right)
\]
(3.73)
\[
+ \| \partial_y h_m^\epsilon \|_{L^\infty_1(\Omega)} \| u_m^\epsilon \|_{L^2_1(\Omega)} \| h_m^\epsilon \|_{L^2_1(\Omega)}^2.
\]
Using the Hölder inequality and the estimate (B.3), it follows
\[
I_{82} \leq \epsilon \| \psi \|_{L^\infty_0(\Omega)} \| \partial_y (u^\epsilon - e^{-\gamma}) \|_{L^\infty_1(\Omega)} \| \partial_x u_m^\epsilon \|_{L^2_1(\Omega)} \| \frac{\partial_x \psi}{h + 1} \|_{L^2_{-1}(\Omega)} \leq \frac{4 \delta^{-1}}{2l - 1} \| \psi \|_{L^\infty_0(\Omega)} \| \partial_x h_m^\epsilon \|_{L^\infty_1(\Omega)} \| \partial_y (u^\epsilon - e^{-\gamma}) \|_{L^\infty_1(\Omega)} \| \partial_x u_m^\epsilon \|_{L^2_1(\Omega)} \| h_m^\epsilon \|_{L^2_1(\Omega)}^2
\]
\[
+ \frac{2 \delta^{-1}}{2l - 1} \| \psi \|_{L^\infty_0(\Omega)} \| \partial_y (u^\epsilon - e^{-\gamma}) \|_{L^\infty_1(\Omega)} \| \partial_x u_m^\epsilon \|_{L^2_1(\Omega)} \| \partial_x h_m^\epsilon \|_{L^2_1(\Omega)} \| h_m^\epsilon \|_{L^2_1(\Omega)} \leq 2 \epsilon \| \partial_x h_m^\epsilon \|_{L^\infty_1(\Omega)} \| \partial_x u_m^\epsilon \|_{L^2_1(\Omega)} \| h_m^\epsilon \|_{L^2_1(\Omega)} + \epsilon \| \partial_x u_m^\epsilon \|_{L^2_1(\Omega)} \| \partial_x h_m^\epsilon \|_{L^2_1(\Omega)}.
Then, substituting the estimates (3.72), (3.73) and (3.74) into (3.71), we get

\[ I_{82} \leq \left( \frac{1}{2} + \delta_2 \right) \epsilon \| \partial_x h_m^e \|_{L^2_\Omega}^2 + 2 \epsilon \| \partial_x u_m^e \|_{L^2_\Omega}^2 + C \| \partial_x h^e_m \|_{L^2_\Omega}^2 \| h_m^e \|_{L^2_\Omega}^2. \] (3.74)

Then, substituting the estimates (3.72), (3.73) and (3.74) into (3.71), we get

\[ |I_8| \leq \left( \frac{1}{2} + 3\delta_2 \right) \epsilon \| \partial_x h_m^e \|_{L^2_\Omega}^2 + \frac{1}{2} \epsilon \| \partial_x u_m^e \|_{L^2_\Omega}^2 + C \| \partial_x h^e_m \|_{L^2_\Omega}^2 \| h_m^e \|_{L^2_\Omega}^2. \]

Finally, integrating by part and applying the Cauchy inequality, we get

\[ |I_9| \leq \frac{\epsilon}{8} \| \partial_x u_m^e \|_{L^2_\Omega}^2 + \| Z_t^{\alpha_1} (r_u, r_h) \|_{L^2_\Omega}^4 + C \delta^4 (1 + Q^2(t)) \| u_m^e, h_m^e \|_{L^2_\Omega}^2. \]

Then, substituting the estimates of \( I_1 \) through \( I_9 \) into the equality (3.66), and integrating the resulting inequality over \([0, t]\), we get that

\[
\| \sqrt{\epsilon^2 u_m^\mu (t)} \|_{L^2_\Omega}^2 + \frac{\epsilon}{2} \int_0^t \| \partial_x u_m^e \|_{L^2_\Omega}^2 \, d\tau + \mu \frac{\epsilon}{2} \int_0^t \| \partial_y u_m^e \|_{L^2_\Omega}^2 \, d\tau \\
+ \int_0^t \int_\Omega \langle y \rangle^{\alpha_1} [(h^e + 1) \partial_x u_m^e \cdot h_m^e + g^e \partial_y u_m^e \cdot h_m^e] \, dxdy \, d\tau \\
\leq \| \sqrt{\epsilon^2 u_m^\mu (0)} \|_{L^2_\Omega}^2 + \left( \frac{1}{2} + 3\delta_2 \right) \epsilon \int_0^t \| \partial_x h_m^e \|_{L^2_\Omega}^2 \, d\tau + \delta_2 \kappa \int_0^t \| \partial_y h_m^e \|_{L^2_\Omega}^2 \, d\tau \\
+ C \int_0^t \| Z_t^{\alpha_1} (r_u, r_h) \|_{L^2_\Omega}^4 \, d\tau + C \int_0^t \| f_u \|_{L^2_\Omega}^2 + \| f_\psi \|_{L^2_\Omega}^2 \, d\tau \\
+ C_{m,k} \delta^4 (1 + Q^2(t)) \int_0^t \| (\partial^2_y \psi^e, h_m^e) \|_{L^2_\Omega}^2 \, d\tau.
\] (3.75)

Applying the Moser type inequality (A.6), it is easy to justify

\[ \int_0^t \| f_u \|_{L^2_\Omega}^2 \, d\tau \leq C_{m,l} (1 + Q^2(t)) \int_0^t \| (\psi^e, u^e, h^e) \|_{\mathcal{H}_T^m}^2 + \| (\partial_y u^e, \partial_y h^e) \|_{\mathcal{H}_T^{m-1}}^2 \, d\tau. \] (3.76)

Similarly, thanks to the Eq. (3.49), it is easy to obtain the estimate

\[
\| h_m^e (t) \|_{L^2_\Omega}^2 + \frac{3}{4} \epsilon \int_0^t \| \partial_y h_m^e \|_{L^2_\Omega}^2 \, d\tau + \frac{3}{4} \kappa \int_0^t \| \partial_y h_m^e \|_{L^2_\Omega}^2 \, d\tau \\
- \int_\Omega \langle y \rangle^{\alpha_1} [(h^e + 1) \partial_x u_m^e \cdot h_m^e + g^e \partial_y u_m^e \cdot h_m^e] \, dxdy \, d\tau \\
\leq \| h_m^e (0) \|_{L^2_\Omega}^2 + C \int_0^t \| Z_t^{\alpha_1} (r_u, r_h) \|_{L^2_\Omega}^4 \, d\tau + C_{m,k,l} \delta^{-6} (1 + Q^3(t)) \int_0^t \mathcal{E}_{m,l} \, d\tau \\
+ C_{m,k,l} \delta^{-6} (1 + Q^3(t)) \int_0^t \| (\partial^2_y \psi^e, h_m^e) \|_{L^2_\Omega}^2 + \| (\partial_y u^e, \partial_y h^e) \|_{\mathcal{H}_T^{m-1}}^2 \, d\tau.
\] (3.77)

Therefore, combining the estimates (3.75)–(3.77) with (B.13), and choosing \( \delta_2 \) small enough, we complete the proof of lemma. \( \square \)
Therefore, combining the estimates in Lemmas 3.6 and 3.7 and choosing the constant $\delta_1$ small enough, then we complete the proof of Proposition 3.5.

**Remark 3.2** To deal with the term $\epsilon \int_\Omega (1 - \rho^\epsilon) \eta \partial_x^2 Z \psi^\epsilon \cdot \langle y \rangle^{2l} u_m dx dy$ (i.e., the term $I_8$ on the right handside of equality (3.66)), we require the assumption of condition (3.9). In other words, the condition (3.9) is not required for the homogeneous flow (i.e., $\rho^\epsilon \equiv 1$) since this difficult term will disappear.

### 3.3 Weighted $\mathcal{H}_1^{m-1}$-estimates for normal derivative

In this subsection, we shall provide estimate of $\| (\partial_y Q^\epsilon, \partial_y u^\epsilon, \partial_y h^\epsilon) \|_{\mathcal{H}_1^{m-1}}$, which will be given as follows:

**Proposition 3.8** Let $(Q^\epsilon, u^\epsilon, v^\epsilon, h^\epsilon, g^\epsilon)$ be sufficiently smooth solution, defined on $[0, T^\epsilon]$, to the Eqs. (3.2)–(3.3). Under the assumption of condition (3.8), it holds true

\[
\sup_{0 \leq \tau \leq t} \| (\partial_y Q^\epsilon, \partial_y u^\epsilon, \partial_y h^\epsilon)(\tau) \|_{\mathcal{H}_1^{m-1}}^2 + \epsilon \int_0^t \| \partial_x (\partial_y Q^\epsilon, \partial_y u^\epsilon, \partial_y h^\epsilon) \|^2_{\mathcal{H}_1^{m-1}} d\tau + \int_0^t (\epsilon \| \partial_y^2 Q^\epsilon \|^2_{\mathcal{H}_1^{m-1}} + \mu \| \partial_y^2 u^\epsilon \|^2_{\mathcal{H}_1^{m-1}} + \kappa \| \partial_y^2 h^\epsilon \|^2_{\mathcal{H}_1^{m-1}}) d\tau \\ \leq C \| (\partial_y Q_0, \partial_y u_0, \partial_y h_0) \|^2_{\mathcal{H}_1^{m-1}} + C_{\mu, \kappa, l} (\rho_0, u_{10}, h_{10}) \| B \|_{\mathcal{H}_1^{m-1}} + C_{\mu, \kappa, m, l} \delta^{-2} (1 + Q^3(t)) \int_0^t X_{m, l}(\tau) d\tau.
\]

First of all, we establish the estimate for the quantity $\partial_y Q^\epsilon$ in $\mathcal{H}_1^{m-1}$ norm. To this end, differentiating the density equation (3.2)\_1 with respect to $y$ variable, we get the evolution equation for $\partial_y Q^\epsilon$:

\[
(\partial_t + (u^\epsilon + 1 - e^{-\gamma}) \partial_x + v^\epsilon \partial_y - \epsilon \partial_x^2 - \epsilon \partial_y^2) \partial_y Q^\epsilon = f_1,
\]

where the function $f_1$ is defined by

\[
f_1 := -\epsilon \partial_y (\partial_x r_1 + \partial_y r_2) - (\partial_y u^\epsilon + e^{-\gamma}) \partial_x Q^\epsilon + \partial_x u^\epsilon \partial_y Q^\epsilon.
\]

**Lemma 3.9** For smooth solution $(Q^\epsilon, u^\epsilon, v^\epsilon, h^\epsilon, g^\epsilon)$ of the Eqs. (3.2)–(3.3), then we have

\[
\sup_{\tau \in [0, t]} \| \partial_y Q^\epsilon(\tau) \|^2_{\mathcal{H}_1^{m-1}} + \epsilon \int_0^t \| \partial_{xy} Q^\epsilon \|^2_{\mathcal{H}_1^{m-1}} + \| \partial_y^2 Q^\epsilon \|^2_{\mathcal{H}_1^{m-1}} d\tau \\ \leq \| \partial_y Q_0 \|^2_{\mathcal{H}_1^{m-1}} + \int_0^t \| \partial_y (r_1, r_2) \|^2_{\mathcal{H}_1^{m-1}} d\tau + C_{m, l} (1 + Q(t)) \int_0^t (1 + \| (Q^\epsilon, u^\epsilon) \|^2_{\mathcal{H}_1^{m}} + \| \partial_y (Q^\epsilon, u^\epsilon) \|^2_{\mathcal{H}_1^{m-1}}) d\tau.
\]

**Proof** We will give the proof of estimate (3.9) by induction. First of all, multiplying (3.78) by $\langle y \rangle^{2l} \partial_y Q^\epsilon$, integrating over $\Omega$ and integrating by part with respect to $x$ variable, we find

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} \langle y \rangle^{2l} |\partial_y Q^\epsilon|^2 dxdy + \epsilon \int_{\Omega} \langle y \rangle^{2l} |\partial_x \partial_y Q^\epsilon|^2 dxdy \\
= \epsilon \int_{\Omega} \partial_y^3 Q^\epsilon \cdot \langle y \rangle^{2l} \partial_y Q^\epsilon dxdy + l \int_{\Omega} \langle y \rangle^{2l-1} v^\epsilon |\partial_y Q^\epsilon|^2 dxdy \\
+ \int_{\Omega} f_1 \cdot \langle y \rangle^{2l} \partial_y Q^\epsilon dxdy.
\]
Integrating by part and applying the boundary condition \( \partial_y \varphi^e \vert_{y=0} = 0 \), we get
\[
\epsilon \int_\Omega \partial_y^3 \varphi^e \cdot (\langle y \rangle)^{2l} \partial_y \varphi^e \, dx \, dy
= \epsilon \int_\Omega \partial_y^2 \varphi^e \cdot (\langle y \rangle)^{2l} \partial_y \varphi^e \, dx \, dy
- \epsilon \int_\Omega \partial_y \langle (\langle y \rangle)^{2l} \partial_y \varphi^e \rangle \cdot \partial_y \varphi^e \, dx \, dy
= -\epsilon \int_\Omega \langle (\langle y \rangle)^{2l} \partial_y \varphi^e \rangle^2 \, dx \, dy - 2l \epsilon \int_\Omega \langle (\langle y \rangle)^{2l-1} \partial_y \varphi^e \cdot \partial_y \varphi^e \rangle \, dx \, dy
\leq -\frac{1}{2} \epsilon \int_\Omega \langle (\langle y \rangle)^{2l} \partial_y \varphi^e \rangle^2 \, dx \, dy + C_l \| \partial_y \varphi^e \|^2_{L_{-1}^2(\Omega)},
\]
where we have used the Hölder and Cauchy inequalities in the last inequality. Thus we get
\[
\text{Obliviously, this inequality implies the estimate (3.9) holds for } m = 1. \text{ To prove the general case, assume that (3.9) has been proven for } k \leq m - 2, \text{ we need to prove that it also holds true for } k = m - 1. \text{ Applying the operator } \mathcal{Z}^\alpha \text{ for } |\alpha| = m - 1 \text{ to the Eq. (3.78), we get}
\[
(\partial_t + (u^e + 1 - e^{-\gamma}) \partial_x + v^e \partial_y) \mathcal{Z}^\alpha \partial_y \varphi^e - \epsilon \mathcal{Z}^\alpha \varphi^e_x = \mathcal{Z}^\alpha f_{1} + C_{41} + C_{42}, \quad (3.79)
\]
where \( C_{4i} (i = 1, 2) \) are defined by
\[
C_{41} = -[\mathcal{Z}^\alpha, (u^e + 1 - e^{-\gamma}) \partial_x + v^e \partial_y] \partial_y \varphi^e, \quad C_{42} = -[\mathcal{Z}^\alpha, v^e \partial_y] \partial_y \varphi^e.
\]
Multiplying the Eq. (3.79) by \( (\langle y \rangle)^{2l} \mathcal{Z}^\alpha \partial_y \varphi^e \), integrating over \( \Omega \times [0, t] \), and integrating by part with respect to \( x \) variable, we find
\[
\frac{1}{2} \int_\Omega (\langle y \rangle)^{2l} |\mathcal{Z}^\alpha \partial_y \varphi^e|^2 \, dx \, dy + \epsilon \int_0^t \int_\Omega (\langle y \rangle)^{2l} |\partial_x \mathcal{Z}^\alpha \partial_y \varphi^e|^2 \, dx \, dy \, dt
= \frac{1}{2} \int_\Omega (\langle y \rangle)^{2l} |\mathcal{Z}^\alpha \partial_y \varphi^e|^2 \, dx \, dy + I_{21} + I_{22} + I_{23} + I_{24} + I_{25}, \quad (3.80)
\]
where the term $I_2(i = 1, \ldots, 5)$ are defined by

$$I_{21} = \epsilon \int_0^t \int_\Omega Z^\alpha \partial^3 \phi^\epsilon \cdot (y) Z^\alpha \partial_y \phi^\epsilon \, dx dy \, d\tau, \quad I_{22} = \int_0^t \int_\Omega Z^\alpha f_1 \cdot (y) Z^\alpha \partial_y \phi^\epsilon \, dx dy \, d\tau,$$

$$I_{23} = l \int_0^t \int_\Omega (y)^{2l-1} v^\epsilon |Z^\alpha \partial_y \phi^\epsilon|^2 \, dx dy \, d\tau, \quad I_{24} = \int_0^t \int_\Omega C_{41} \cdot (y) Z^\alpha \partial_y \phi^\epsilon \, dx dy \, d\tau,$$

$$I_{25} = \int_0^t \int_\Omega C_{42} \cdot (y) Z^\alpha \partial_y \phi^\epsilon \, dx dy \, d\tau.$$

Similar to the estimate (3.36), we can get

$$I_{21} \leq -\frac{1}{2} \epsilon \int_0^t \int_\Omega (y)^{2l} |\partial_y Z^\alpha \partial_y \phi^\epsilon|^2 \, dx dy \, d\tau + C_{m.l} \int_0^t (\epsilon \|\partial_y \phi^\epsilon\|^2_{\mathcal{H}^{m-1}} + \epsilon \|\partial_y \phi^\epsilon\|^2_{\mathcal{H}^{m-2}}) \, d\tau.$$

It is easy to justify

$$|I_{23}| \leq C_l \|v^\epsilon\|^2_{L^\infty_0(\Omega)} \int_0^t \|\partial_y \phi^\epsilon\|^2_{\mathcal{H}^{m-1}} \, d\tau.$$

Applying the Moser type inequality (A.6), we conclude

$$|I_{22}| \leq \frac{1}{4} \epsilon \int_0^t (\|\partial_x Z^\alpha \partial_y \phi^\epsilon\|^2_{L^1_\tau(\Omega)} + \|\partial_y Z^\alpha \partial_y \phi^\epsilon\|^2_{L^1_\tau(\Omega)}) \, d\tau + \int_0^t \|\partial_y (r_1, r_2)\|^2_{\mathcal{H}^{m-1}} \, d\tau + C_{m.l} (1 + Q(t)) \int_0^t (1 + \|Q^\epsilon\|_{\mathcal{H}^{m}} + \|\partial_y Q^\epsilon\|_{\mathcal{H}^{m}} + \|\partial_y u^\epsilon\|_{\mathcal{H}^{m}}) \, d\tau,$$

and

$$|I_{24}| \leq C_m (1 + Q(t)) \int_0^t (1 + \|u^\epsilon\|_{\mathcal{H}^{m}} + \|\partial_y Q^\epsilon\|_{\mathcal{H}^{m}}) \, d\tau.$$

Finally, we deal with the term $I_{25}$. It follows from the Hölder inequality that

$$|I_{25}| \leq \int_0^t \|C_{42}\|_{L^1_\tau(\Omega)} \|Z^\alpha \partial_y \phi^\epsilon\|_{L^1_\tau(\Omega)} \, d\tau. \quad (3.81)$$

It is easy to check that

$$[Z^\alpha, v^\epsilon \partial_y] \phi^\epsilon = [Z^\alpha, v^\epsilon] \partial_y^2 \phi^\epsilon + v^\epsilon [Z^\alpha, \partial_y] \partial_y \phi^\epsilon.$$

Since the coefficient $\epsilon$ of the quantity $\partial_y^2 \phi^\epsilon$ in (3.2.1) is sufficiently small, it is not expected to establish a estimate which is uniform in $\epsilon$ for $\|\partial_y^2 \phi^\epsilon\|_{L^\infty_0(\Omega)}$ or $\|\partial_y^2 \phi^\epsilon\|_{\mathcal{H}^{m-1}}$. Hence, we first write

$$[Z^\alpha, \partial_y] \partial_y \phi^\epsilon = \sum_{\beta_1 \neq 0, \beta_2 + \gamma_2 = \alpha_2} C_{\beta_2, \gamma_2} Z_2^{\beta_2} \left(\frac{1}{\varphi}\right) Z_2^{\gamma_2 + 1} \partial_y \phi^\epsilon,$$

and get

$$\int_0^t \|v^\epsilon Z_2^{\beta_2} \left(\frac{1}{\varphi}\right) Z_2^{\gamma_2 + 1} Z^\alpha \partial_y \phi^\epsilon\|^2_{L^1_\tau(\Omega)} \, d\tau \leq C \frac{v^\epsilon}{\varphi} \|v^\epsilon\|^2_{L^\infty_0(\Omega)} \int_0^t \|\partial_y \phi^\epsilon\|^2_{\mathcal{H}^{m-1}} \, d\tau. \quad (3.82)$$
where we have used the estimate (3.25) in the last inequality. Similarly, we have
\[ [\mathcal{Z}^a, \partial_y] \partial_y Z^e = \sum_{|\beta + \gamma| \leq m-1, |\gamma| \leq m-2} C_{\beta, \gamma} \mathcal{Z}^\beta \left( \varphi \right) \mathcal{Z}^\gamma \left( Z_2 \partial_y Z^e \right). \]

If \( \beta = 0 \), it is easy to verify
\[
\int_0^t \left\| \frac{\nu e}{\varphi} \mathcal{Z}^\gamma \left( Z_2 \partial_y Z^e \right) \right\|_{L_t^2(\Omega)}^2 d\tau \leq \left\| \frac{\nu e}{\varphi} \right\|_{L_t^\infty(\Omega)}^2 \int_0^t \left\| \partial_y Z^e \right\|_{\mathcal{H}_t^{m-1}}^2 d\tau.
\]

If \( \beta \neq 0 \), the application of Moser type inequality (A.6) yields directly
\[
\int_0^t \left\| Z^\beta \left( \frac{\nu e}{\varphi} \right) \mathcal{Z}^\gamma \left( Z_2 \partial_y Z^e \right) \right\|_{L_t^2(\Omega)}^2 d\tau \leq C \left\| Z^E_1 \left( \frac{\nu e}{\varphi} \right) \right\|_{L_t^\infty(\Omega)}^2 \int_0^t \left\| Z_2 \partial_y Z^e \right\|_{\mathcal{H}_t^{m-1}}^2 d\tau
\]
\[ + C \left\| Z_2 \partial_y Z^e \right\|_{L_t^\infty(\Omega)}^2 \int_0^t \left\| Z^E_1 \left( \frac{\nu e}{\varphi} \right) \right\|_{L_t^\infty(\Omega)}^2 d\tau. \tag{3.83} \]

Using the Hardy inequality and divergence-free condition of velocity in (3.2)4, we get
\[
\left\| Z^E_1 \left( \frac{\nu e}{\varphi} \right) \right\|_{\mathcal{H}_t^{m-2}}^2 \leq \frac{\nu e}{\varphi} \left\| \partial_y u^e \right\|_{\mathcal{H}_t^{m-1}}^2 \leq C_1 \left\| \partial_y u^e \right\|_{\mathcal{H}_t^{m-1}}^2 \leq C_t \left\| u^e \right\|_{\mathcal{H}_t^{m}},
\]
which, together with (3.83), yields directly
\[
\int_0^t \left\| Z^\beta \left( \frac{\nu e}{\varphi} \right) \mathcal{Z}^\gamma \left( Z_2 \partial_y Z^e \right) \right\|_{L_t^2(\Omega)}^2 d\tau \leq C_t \left( Z^E_1 \left( \frac{\nu e}{\varphi} \right), Z_2 \partial_y Z^e \right) \left\| L_t^\infty(\Omega) \right\|_{L_t^\infty(\Omega)}^2 d\tau.
\]
\[
\int_0^t \left( \left\| u^e \right\|_{\mathcal{H}_t^{m}}^2 + \left\| \partial_y Z^e \right\|_{\mathcal{H}_t^{m-1}}^2 \right) d\tau.
\]

This and inequality (3.82) give directly
\[
|I_{25}| \leq C_{m,t} Q(t) \int_0^t \left( \left\| u^e \right\|_{\mathcal{H}_t^{m}}^2 + \left\| \partial_y Z^e \right\|_{\mathcal{H}_t^{m-1}}^2 \right) d\tau.
\]

Therefore, substituting the estimates of \( I_{21} \) through \( I_{25} \) into (3.80) and using the induction assumption to eliminate the term \( \epsilon \int_0^t \left\| \partial_y u^e \right\|_{\mathcal{H}_t^{m-2}}^2 d\tau \), then the proof of this lemma is completed.

Next, we establish the estimate for \( \left\| \partial_y u^e \right\|_{\mathcal{H}_t^{m-1}} \). Although \( \partial_y u^e \) does not vanish on the boundary, we can take \(-\partial_y u^e \) as the test function thanks to the coefficient \( \mu > 0 \) in (3.2)2.

**Lemma 3.10** For smooth solution \((q^e, u^e, v^e, h^e, g^e)\) of the Eqs. (3.2)–(3.3), then it holds true
\[
\sup_{0 \leq \tau \leq t} \left\| \partial_y u^e (\tau) \right\|_{\mathcal{H}_t^{m-1}}^2 + \int_0^t (\epsilon \left\| \partial_{xy} u^e \right\|_{\mathcal{H}_t^{m-1}}^2 + \mu \left\| \partial_y u^e \right\|_{\mathcal{H}_t^{m-1}}^2 d\tau \leq \left\| \partial_y u_0^e \right\|_{\mathcal{H}_t^{m-1}}^2 + C_\mu \int_0^t \left( \left\| \partial_y r_{u} \right\|_{\mathcal{H}_t^{m-1}}^2 d\tau + C_{s,m,t} (1 + Q^2(t)) \right.
\]
\[
\int_0^t \left( 1 + \left\| (q^e, u^e, h^e) \right\|_{\mathcal{H}_t^{m}}^2 + \left\| (\partial_y u^e, \partial_y h^e) \right\|_{\mathcal{H}_t^{m-1}}^2 \right) d\tau.
\]
**Proof** First of all, multiplying the Eq. (3.2)2 by $-\langle y \rangle 2\partial_y^2 u^\varepsilon$ and integrating over $\Omega$, we find

$$\int_{\Omega} \left( -\rho^\varepsilon \partial_t u^\varepsilon + \varepsilon \partial_x^2 u^\varepsilon + \mu \partial_y^2 u^\varepsilon \right) \cdot \langle y \rangle 2\partial_y^2 u^\varepsilon \, dx \, dy$$

$$= \int_{\Omega} \left( \varepsilon \partial_x r_u + \mu e^{-\gamma} \right) \cdot \langle y \rangle 2\partial_y^2 u^\varepsilon \, dx \, dy - \int_{\Omega} f_2 \cdot \langle y \rangle 2\partial_y^2 u^\varepsilon \, dx \, dy,$$

where $f_2$ is defined by

$$f_2 := -\rho^\varepsilon (u^\varepsilon + 1 - e^{-\gamma}) \partial_x u^\varepsilon - \rho^\varepsilon v^\varepsilon \partial_y u^\varepsilon - \rho^\varepsilon v^\varepsilon \partial_y u^\varepsilon + (h^\varepsilon + 1) \partial_x h^\varepsilon + g^\varepsilon \partial_y h^\varepsilon.$$

Integrating by part and applying the boundary condition $u^\varepsilon|_{y=0} = 0$, we get

$$\epsilon \int_{\Omega} \partial_y^2 u^\varepsilon \cdot \langle y \rangle 2\partial_y^2 u^\varepsilon \, dx \, dy$$

$$= \epsilon \int_{\Omega} \partial_x^2 u^\varepsilon \cdot \partial_y u^\varepsilon \mid_{y=0} \, dx - \epsilon \int_{\Omega} \partial_x^2 \partial_y u^\varepsilon \cdot \langle y \rangle 2\partial_y u^\varepsilon \, dx \, dy$$

$$- 2\epsilon \int_{\Omega} \partial_x^2 u^\varepsilon \cdot \langle y \rangle 2\partial_y u^\varepsilon \, dx \, dy$$

$$= \epsilon \int_{\Omega} \langle y \rangle 2\partial_x \partial_y u^\varepsilon \partial_y u^\varepsilon \, dx \, dy + 2\epsilon \int_{\Omega} \langle y \rangle 2\partial_x \partial_y u^\varepsilon \partial_x \partial_y u^\varepsilon \, dx \, dy.$$
This implies the estimate (3.10) holds for $m = 1$. To prove the general case, let us assume that (3.10) has been proven for $k \leq m - 2$, we need to prove it also holds for $k = m - 1$. Applying $Z^{\alpha}(\alpha = m - 1)$ operator to the second equation of (3.2), multiplying the resulting equation by $-\langle y \rangle 2l \partial_y Z^{\alpha} \partial_y u^\epsilon$ and integrating over $\Omega$, we find

$$\int_{\Omega} (-\rho^\epsilon \partial_t Z^{\alpha} u^\epsilon + \epsilon \partial_x^2 Z^{\alpha} u^\epsilon + \mu Z^{\alpha} \partial_y^2 u^\epsilon) \cdot \langle y \rangle 2l \partial_y Z^{\alpha} \partial_y u^\epsilon dy = \int \mathbb{I} Z^{\alpha}(\rho \partial_t u^\epsilon + \langle y \rangle 2l \partial_y Z^{\alpha} \partial_y u^\epsilon) dy + \int \mathbb{I} Z^{\alpha}(-f_2 + \epsilon \partial_x r_u + \mu e^{-y}) (3.84)$$

\cdot \langle y \rangle 2l \partial_y Z^{\alpha} \partial_y u^\epsilon dy.

In view of the boundary condition $u^\epsilon|_{y=0} = 0$ and the definition of $\varphi(y)$, we can justify that $Z^{\alpha} u^\epsilon|_{y=0} = 0$. Then, integrating by part and applying the fact $Z^{\alpha} u^\epsilon|_{y=0} = 0$, one arrives at

$$\epsilon \int \partial_x^2 Z^{\alpha} u^\epsilon \cdot \langle y \rangle 2l \partial_y Z^{\alpha} \partial_y u^\epsilon dy$$

$$= \epsilon \int \partial_x^2 Z^{\alpha} u^\epsilon \cdot \langle y \rangle 2l \partial_y Z^{\alpha} \partial_y u^\epsilon|_{y=0} + \epsilon \int \langle y \rangle 2l \partial_y^2 Z^{\alpha} u^\epsilon \cdot Z^{\alpha} \partial_y u^\epsilon dy$$

$$- 2\epsilon \int \langle y \rangle 2l \partial_x^2 Z^{\alpha} u^\epsilon \cdot Z^{\alpha} \partial_y u^\epsilon dy$$

$$= \epsilon \int \langle y \rangle 2l \partial_x Z^{\alpha} \partial_y u^\epsilon |y^2 dy + \epsilon \int \langle y \rangle 2l |Z^{\alpha}, \partial_y| \partial_x^2 u^\epsilon \cdot Z^{\alpha} \partial_y u^\epsilon dy$$

$$+ 2\epsilon \int \langle y \rangle 2l \partial_x Z^{\alpha} u^\epsilon \cdot \partial_x Z^{\alpha} \partial_y u^\epsilon dy$$

$$\geq \frac{1}{2} \epsilon \int \langle y \rangle 2l \partial_x Z^{\alpha} \partial_y u^\epsilon |y^2 dy - C(\epsilon \| \partial_x \partial_y u^\epsilon \|_{L^2_\mu}^2 + \| Z^{\alpha} \partial_x u^\epsilon \|_{L^2_\mu}^2).$$

By virtue of the Cauchy–Schwarz inequality, it is easy to justify that

$$\mu \int \partial_x^2 Z^{\alpha} \partial_y^2 u^\epsilon \cdot \langle y \rangle 2l \partial_y Z^{\alpha} \partial_y u^\epsilon dy$$

$$= \mu \int \langle y \rangle 2l |\partial_x Z^{\alpha} \partial_y u^\epsilon |^2 dy + \mu \int \langle y \rangle 2l |Z^{\alpha}, \partial_y| \partial_x Z^{\alpha} \partial_y u^\epsilon dy$$

$$\geq \frac{1}{2} \mu \int \langle y \rangle 2l |\partial_x Z^{\alpha} \partial_y u^\epsilon |^2 dy - C\| Z^{\alpha}, \partial_y \| \partial_x u^\epsilon \|_{L^2_\mu}^2$$

$$\geq \frac{1}{2} \mu \int \langle y \rangle 2l |\partial_x Z^{\alpha} \partial_y u^\epsilon |^2 dy - C\| \partial_x^2 u^\epsilon \|_{H^1_\mu}^2.$$

Integrating by part and applying the boundary condition $Z^{\alpha} u^\epsilon|_{y=0} = 0$, we get

$$- \int \rho^\epsilon \partial_t Z^{\alpha} u^\epsilon \cdot \langle y \rangle 2l \partial_y Z^{\alpha} \partial_y u^\epsilon dy = \frac{d}{dt} \int \langle y \rangle 2l \rho^\epsilon |Z^{\alpha} \partial_y u^\epsilon |^2 dy + II,$$

where $II$ is defined by

$$II := -\frac{1}{2} \int \langle y \rangle 2l \partial_t \rho^\epsilon |Z^{\alpha} \partial_y u^\epsilon |^2 dy - \int \langle y \rangle 2l \rho^\epsilon |Z^{\alpha}, \partial_y| \partial_t u^\epsilon \cdot Z^{\alpha} \partial_y u^\epsilon dy$$

$$+ \int \langle y \rangle 2l \partial_x \rho^\epsilon \partial_t Z^{\alpha} u^\epsilon \cdot Z^{\alpha} \partial_y u^\epsilon dy + 2\int \langle y \rangle 2l \rho^\epsilon \partial_x \partial_t Z^{\alpha} u^\epsilon \cdot Z^{\alpha} \partial_y u^\epsilon dy,$$
which can be estimated as follows

$$|I| \leq C_1 \| (\partial_t \rho^\varepsilon, \partial_y \rho^\varepsilon) \|_{L^\infty_t(\Omega)} (\| u^\varepsilon \|_{L^2_t}^2 + \| \partial_y u^\varepsilon \|_{L^2_t}^2).$$

Using the Hölder and Cauchy inequalities, it follows

$$\left| \int_{\Omega} [Z^\alpha, \rho^\varepsilon] \partial_t u^\varepsilon \cdot (y)^2 \partial_y Z^\alpha \partial_y u^\varepsilon \, dx \, dy \right|$$

$$\leq \frac{\mu}{4} \| (y)^2 \partial_y Z^\alpha \partial_y u^\varepsilon \|_{L^2_t(\Omega)}^2 + C \mu \| [Z^\alpha, \rho^\varepsilon] \partial_t u^\varepsilon \|_{L^2_t(\Omega)}^2,$$

and

$$\left| \int_{\Omega} Z^\alpha (f_2 + \varepsilon \partial_x r_u + \mu e^{-\varepsilon}) \cdot (y)^2 \partial_y Z^\alpha \partial_y u^\varepsilon \, dx \, dy \right|$$

$$\leq \frac{\mu}{4} \| (y)^2 \partial_y Z^\alpha \partial_y u^\varepsilon \|_{L^2_t(\Omega)}^2 + C \mu \left( 1 + \| Z^\alpha r_u \|_{L^2_t(\Omega)}^2 + \| Z^\alpha f_2 \|_{L^2_t(\Omega)}^2 \right).$$

Substituting the above estimates into (3.84), and integrating the inequality over \([0, t]\), we get

$$\int_{\Omega} \langle y \rangle \partial_y Z^\alpha \partial_y u^\varepsilon \rangle \, dx \, dy + \int_0^t \int_{\Omega} \langle y \rangle \partial_x \partial_y Z^\alpha \partial_y u^\varepsilon \rangle \, dx \, dy \, d\tau$$

$$\leq \int_{\Omega} \langle y \rangle \partial_y Z^\alpha \partial_y u_0 \rangle \, dx \, dy + C \int_0^t \| \partial_y \partial_y u^\varepsilon \|_{H^m_t(\Omega)}^2 \, d\tau + C \mu \int_0^t \| \partial_y^2 u^\varepsilon \|_{H^m_t(\Omega)}^2 \, d\tau$$

$$+ C \mu \int_0^t \| \partial_y Z^\alpha \partial_y u^\varepsilon \|_{L^2_t(\Omega)}^2 \, d\tau + C \mu \int_0^t \| \partial_y Z^\alpha \partial_y u^\varepsilon \|_{L^2_t(\Omega)}^2 \, d\tau$$

$$+ C \mu \left( 1 + \| \partial_y \rho^\varepsilon \|_{L^\infty_t(\Omega)} \right) \int_0^t \| u^\varepsilon \|_{H^m_t}^2 + \| \partial_y u^\varepsilon \|_{H^m_t}^2 \, d\tau.$$

By virtue of the Cauchy and Morse type inequality (A.6), we get

$$\int_{\Omega} \| \partial_y Z^\alpha \partial_y u^\varepsilon \|_{L^2_t(\Omega)}^2 \, d\tau \leq C (1 + Q^2(t)) \int_0^t \| (\rho^\varepsilon, u^\varepsilon, h^\varepsilon) \|_{H^m_t}^2 + \| (\partial_y u^\varepsilon, \partial_y h^\varepsilon) \|_{H^m_t}^2 \, d\tau.$$

Similarly, by routine checking, we may show

$$\int_0^t \| Z^\alpha f_2 \|_{L^2_t(\Omega)}^2 \, d\tau \leq C (1 + Q^2(t)) \int_0^t \| (\rho^\varepsilon, u^\varepsilon, h^\varepsilon) \|_{H^m_t}^2 + \| (\partial_y u^\varepsilon, \partial_y h^\varepsilon) \|_{H^m_t}^2 \, d\tau.$$

Thus, it is easy to justify the following estimate for \(|\alpha| = m - 1\)

$$\sup_{0 \leq \tau \leq t} \int_{\Omega} \langle y \rangle \partial_y Z^\alpha \partial_y u^\varepsilon \rangle \, dx \, dy + \int_0^t \int_{\Omega} \langle y \rangle \partial_x \partial_y Z^\alpha \partial_y u^\varepsilon \rangle \, dx \, dy \, d\tau$$

$$\leq \int_{\Omega} \langle y \rangle \partial_y Z^\alpha \partial_y u_0 \rangle \, dx \, dy + C \int_0^t \| \partial_y \partial_y u^\varepsilon \|_{H^m_t(\Omega)}^2 \, d\tau + C \mu \int_0^t \| \partial_y^2 u^\varepsilon \|_{H^m_t(\Omega)}^2 \, d\tau$$

$$+ C \mu \int_0^t \| \partial_y Z^\alpha \partial_y u^\varepsilon \|_{L^2_t(\Omega)}^2 \, d\tau + C \mu \left( 1 + Q^2(t) \right) \int_0^t \| (\rho^\varepsilon, u^\varepsilon, h^\varepsilon) \|_{H^m_t}^2 + \| (\partial_y u^\varepsilon, \partial_y h^\varepsilon) \|_{H^m_t}^2 \, d\tau.$$
Since the terms \( \epsilon \int_0^t \| \partial_x \partial_y u^\epsilon \|_{H^m_l}^2 d\tau \) and \( \mu \int_0^t \| \partial_x^2 u^\epsilon \|_{H^m_l}^2 d\tau \) in (3.85) can be obtained by induction, we complete the proof of this lemma.

Similarly, we can obtain the following estimates for the quantity \( \| \partial_y h^\epsilon \|_{H^{m-1}} \).

**Lemma 3.11** For smooth solution \((q^\epsilon, u^\epsilon, v^\epsilon, h^\epsilon, g^\epsilon)\) of the Eqs. (3.2)–(3.3), then it holds true

\[
\sup_{0 \leq \tau \leq t} \| \partial_y h^\epsilon \|_{H^{m-1}}^2 + \epsilon \int_0^t \| \partial_x y h^\epsilon \|_{H^{m-1}}^2 d\tau + \kappa \int_0^t \| \partial_y^2 h^\epsilon \|_{H^{m-1}}^2 d\tau \\
\leq \| \partial_y h_0^\epsilon \|_{H^{m-1}}^2 + C_\kappa \int_0^t \| \partial_x r^\epsilon \|_{H^{m-1}}^2 d\tau + C_{\kappa, m, l} (1 + Q(t))
\]

Finally, we give the proof for the estimate in Proposition 3.8. Indeed, we recall the estimate (see (B.13)) as follows

\[
\| Z^\epsilon_t (q^\epsilon, u^\epsilon, h^\epsilon) \|_{L^2_l(\Omega)}^2 \leq C_l \delta^{-2} (1 + \| \partial_y (q^\epsilon, u^\epsilon, h^\epsilon) \|_{L^\infty(\Omega)}^2) \| (\rho_m, u_m, h_m) \|_{L^2_l(\Omega)}^2
\]

which, together with the estimates in Lemmas 3.9, 3.10 and 3.11, completes the proof of estimate in Proposition 3.8.

### 3.4 \(L^\infty\)-estimates

To close the estimate, we need to control the \(L^\infty\)–norm of \((\rho^\epsilon, u^\epsilon, v^\epsilon, h^\epsilon, g^\epsilon)\) in \(Q(t)\). Then, we have

**Proposition 3.12** Let \((q^\epsilon, u^\epsilon, v^\epsilon, h^\epsilon, g^\epsilon)\) be sufficiently smooth solution, defined on \([0, T^\epsilon]\), to the Eqs. (3.2)–(3.3), then we have the following estimates:

\[
Q(t) \leq C (1 + \| (\rho_0, u_{10}, h_{10}) \|_{E^m_l} + t \| (\rho_0, u_{10}, h_{10}) \|_{E^m_l} + X^3_{m, l}(t)),
\]

and

\[
\| \partial_y q^\epsilon(t) \|_{H^{l, \infty}}^2 \leq C (\| \partial_y q_0^\epsilon \|_{H^{l, \infty}}^2 + \| (q_0^\epsilon, u_0^\epsilon, h_0^\epsilon) \|_{H^{l, 0}}^2) + C_t \| (\rho_0, u_{10}, h_{10}) \|_{E^m_l}^2
\]

\[
+ C (1 + Q(t)) \int_0^t X^6_{m, l}(\tau) d\tau,
\]

for \(m \geq 5, l \geq 2\).

We point out that the Proposition 3.12 will be proved in Lemmas 3.13 and 3.14. First of all, due to the coefficients \(\mu > 0\) and \(\kappa > 0\), we can apply the Sobolev inequality, and Eqs. (3.2) to establish the estimates as follows.

**Lemma 3.13** Let \((q^\epsilon, u^\epsilon, v^\epsilon, h^\epsilon, g^\epsilon)\) be sufficiently smooth solution, defined on \([0, T^\epsilon]\), to the Eqs. (3.2)–(3.3), then we have the following estimates:

\[
\| Z T q^\epsilon(t) \|_{L^\infty_0(\Omega)} + \| (u^\epsilon, h^\epsilon(t)) \|_{H^{l, \infty}_{T, \text{tan}}} \leq C (E_{3, 0}^{\frac{1}{2}}(t) + \| \partial_y (q^\epsilon, u^\epsilon, h^\epsilon(t)) \|_{H^{l, 0}}), \quad (3.86)
\]

\[
\| (v^\epsilon, g^\epsilon(t)) \|_{H^{l, \infty}_{T, \text{tan}}} \leq C E_{4, 2}^{\frac{1}{2}}(t), \quad (3.87)
\]
Thus we get the estimate (3.86). Using the Hardy inequality and Sobolev inequality (A.3), we get

\[ \| \partial_y u^e \|_{H^1_0(\Omega)} \leq C(\| u^e \|_{L^2(\Omega)} + \| \partial_x u^e \|_{L^2(\Omega)} + \| \partial_y u^e \|_{L^2(\Omega)} + \| \partial_{xy} u^e \|_{L^2(\Omega)}) \]

Similarly, we obtain the estimate (3.86). Using the Hardy inequality and Sobolev inequality (A.3), we get

\[ \| u^e \|_{L^\infty(\Omega)} \leq C(\| u^e \|_{L^2(\Omega)} + \| \partial_x u^e \|_{L^2(\Omega)} + \| \partial_y u^e \|_{L^2(\Omega)} + \| \partial_{xy} u^e \|_{L^2(\Omega)}) \]

Similarly, it is easy to justify

\[ \| h^e \|_{L^\infty(\Omega)} \leq C(\| u^e \|_{L^2(\Omega)} + \| \partial_y u^e \|_{H^1_0(\Omega)}) \]

Thus we obtain the estimate (3.86). Using the Hardy inequality and Sobolev inequality (A.3), we get

\[ \| \nu^e \|_{L^\infty(\Omega)} \leq C(\| u^e \|_{L^2(\Omega)} + \| \partial_x u^e \|_{L^2(\Omega)} + \| \partial_y u^e \|_{L^2(\Omega)} + \| \partial_{xy} u^e \|_{L^2(\Omega)}) \]

Similarly, it is easy to justify

\[ \| \nu^e \|_{L^\infty(\Omega)} \leq C(\| u^e \|_{L^2(\Omega)} + \| \partial_x u^e \|_{L^2(\Omega)} + \| \partial_y u^e \|_{L^2(\Omega)} + \| \partial_{xy} u^e \|_{L^2(\Omega)}) \]

Thus we obtain the estimate (3.86). Using the Hardy inequality and Sobolev inequality (A.3), we get

\[ \| \nu^e \|_{L^\infty(\Omega)} \leq C(\| u^e \|_{L^2(\Omega)} + \| \partial_x u^e \|_{L^2(\Omega)} + \| \partial_y u^e \|_{L^2(\Omega)} + \| \partial_{xy} u^e \|_{L^2(\Omega)}) \]

Similarly, it is easy to justify

\[ \| \nu^e \|_{L^\infty(\Omega)} \leq C(\| u^e \|_{L^2(\Omega)} + \| \partial_x u^e \|_{L^2(\Omega)} + \| \partial_y u^e \|_{L^2(\Omega)} + \| \partial_{xy} u^e \|_{L^2(\Omega)}) \]

Thus we can conclude the estimate

\[ \| \partial_y u^e \|_{L^\infty(\Omega)} \leq C(\| u^e \|_{L^2(\Omega)} + \| \partial_x u^e \|_{L^2(\Omega)} + \| \partial_y u^e \|_{L^2(\Omega)} + \| \partial_{xy} u^e \|_{L^2(\Omega)}) \]

Proof By virtue of the Sobolev inequality (A.3) and the definition of \( E_{\nu,t}(t) \) (see (3.11)), then we get

\[ \| \partial_y u^e \|_{H^1(\Omega)} \leq C(\| u^e \|_{L^2(\Omega)} + \| \partial_x u^e \|_{L^2(\Omega)} + \| \partial_y u^e \|_{L^2(\Omega)} + \| \partial_{xy} u^e \|_{L^2(\Omega)}) \]

Similarly, it is easy to justify

\[ \| h^e \|_{H^1(\Omega)} \leq C(\| u^e \|_{L^2(\Omega)} + \| \partial_y u^e \|_{H^1(\Omega)}) \]

Thus we obtain the estimate (3.86). Using the Hardy inequality and Sobolev inequality (A.3), we get

\[ \| u^e \|_{L^\infty(\Omega)} \leq C(\| u^e \|_{L^2(\Omega)} + \| \partial_x u^e \|_{L^2(\Omega)} + \| \partial_y u^e \|_{L^2(\Omega)} + \| \partial_{xy} u^e \|_{L^2(\Omega)}) \]

Similarly, it is easy to justify

\[ \| h^e \|_{L^\infty(\Omega)} \leq C(\| u^e \|_{L^2(\Omega)} + \| \partial_y u^e \|_{H^1(\Omega)}) \]

Thus we obtain the estimate (3.86). Using the Hardy inequality and Sobolev inequality (A.3), we get

\[ \| u^e \|_{L^\infty(\Omega)} \leq C(\| u^e \|_{L^2(\Omega)} + \| \partial_x u^e \|_{L^2(\Omega)} + \| \partial_y u^e \|_{L^2(\Omega)} + \| \partial_{xy} u^e \|_{L^2(\Omega)}) \]

Similarly, it is easy to justify

\[ \| h^e \|_{L^\infty(\Omega)} \leq C(\| u^e \|_{L^2(\Omega)} + \| \partial_y u^e \|_{H^1(\Omega)}) \]

Thus we obtain the estimate (3.86). Using the Hardy inequality and Sobolev inequality (A.3), we get

\[ \| u^e \|_{L^\infty(\Omega)} \leq C(\| u^e \|_{L^2(\Omega)} + \| \partial_x u^e \|_{L^2(\Omega)} + \| \partial_y u^e \|_{L^2(\Omega)} + \| \partial_{xy} u^e \|_{L^2(\Omega)}) \]

Similarly, it is easy to justify

\[ \| h^e \|_{L^\infty(\Omega)} \leq C(\| u^e \|_{L^2(\Omega)} + \| \partial_y u^e \|_{H^1(\Omega)}) \]

Thus we obtain the estimate (3.86). Using the Hardy inequality and Sobolev inequality (A.3), we get

\[ \| u^e \|_{L^\infty(\Omega)} \leq C(\| u^e \|_{L^2(\Omega)} + \| \partial_x u^e \|_{L^2(\Omega)} + \| \partial_y u^e \|_{L^2(\Omega)} + \| \partial_{xy} u^e \|_{L^2(\Omega)}) \]
By virtue of the definition of \( r_u \) in (2.2), we get for \( m \geq 4, l \geq 1 \) that
\[
\| (\partial_x r_u, \partial_x^2 r_u) \|_{L^2(\Omega)} \leq \| (\rho_0, u_{10}, h_{10}) \|_{\mathcal{B}^m_1}^\frac{1}{2} + C t \| (\rho_0, u_{10}, h_{10}) \|_{\mathcal{B}^m_1}^\frac{1}{2},
\]
which, together with the estimate (3.92), yields directly
\[
\| \partial_y u^\varepsilon \|_{L^\infty(\Omega)} \leq C (1 + \| (\rho_0, u_{10}, h_{10}) \|_{\mathcal{B}^m_1}^\frac{1}{2} + C t \| (\rho_0, u_{10}, h_{10}) \|_{\mathcal{B}^m_1}^\frac{1}{2}) + \mathcal{E}_{4,1}(t) + \| \partial_y (\varphi^\varepsilon, u^\varepsilon, h^\varepsilon) \|^3_{\mathcal{T}^3_1}. \tag{3.93}
\]
Similarly, we get for \( m \geq 4, l \geq 1 \)
\[
\| \partial_y h^\varepsilon \|_{L^\infty(\Omega)} \leq C (1 + \| (\rho_0, u_{10}, h_{10}) \|_{\mathcal{B}^m_1}^\frac{1}{2} + C t \| (\rho_0, u_{10}, h_{10}) \|_{\mathcal{B}^m_1}^\frac{1}{2}) + \mathcal{E}_{4,1}(t) + \| \partial_y (u^\varepsilon, h^\varepsilon) \|^2_{\mathcal{T}^2_1}. \tag{3.94}
\]
and for \( m \geq 5, l \geq 1 \)
\[
\| \hat{Z} : \partial_y u^\varepsilon \|_{L^\infty(\Omega)} \leq C (1 + \| (\rho_0, u_{10}, h_{10}) \|_{\mathcal{B}^m_1}^\frac{1}{2} + C t \| (\rho_0, u_{10}, h_{10}) \|_{\mathcal{B}^m_1}^\frac{1}{2}) + \mathcal{E}_{5,1}(t) + \| \partial_y (u^\varepsilon, h^\varepsilon) \|^2_{\mathcal{T}^2_1}. \tag{3.95}
\]
Then, the combination of estimates (3.93), (3.94) and (3.95) yields (3.89).

Finally, we give the estimate for the quantity \( \| \frac{v^\varepsilon}{\varphi} \|_{\mathcal{T}^1_1, \infty} \). Since vertical velocity \( v^\varepsilon \) vanishes on the boundary (i.e., \( v^\varepsilon \mid_{\gamma=0} = 0 \)), we get \( \langle \gamma \rangle^2 \| v^\varepsilon \mid_{\mathcal{L}^2(\Omega)} + \| \partial_y v^\varepsilon \|_{L^2(\Omega)} \). Using Sobolev inequality (A.3) and divergence-free condition (2.2)4, it follows
\[
\| \frac{v^\varepsilon}{\varphi} \|_{L^\infty(\Omega)} \leq C (\| v^\varepsilon \|_{L^\infty(\Omega)} + \| \partial_y v^\varepsilon \|_{L^2(\Omega)}) \leq C (\mathcal{E}_{3,2}^\frac{1}{2}(t) + \| \partial_y u^\varepsilon \|_{\mathcal{T}^2_1}). \tag{3.96}
\]
Similarly, we also get that
\[
\| \hat{Z} : \left( \frac{v^\varepsilon}{\varphi} \right) \|_{L^\infty(\Omega)}^2 \leq C \left( \mathcal{E}_{4,2}^\frac{1}{2}(t) + \| \partial_y u^\varepsilon \|_{\mathcal{T}^2_1} \right). \tag{3.97}
\]
By virtue of the fact \( \partial_y (\frac{1}{\varphi}) = -\frac{1}{\varphi^2} \), we get after using the divergence-free condition (2.2)4
\[
\| \hat{Z}_2 \left( \frac{v^\varepsilon}{\varphi} \right) \|_{L^\infty(\Omega)} \leq C \left( \| y \partial_y \left( \frac{1}{\varphi} \right) v^\varepsilon \|_{L^\infty(\Omega)} + \| \partial_y v^\varepsilon \|_{L^\infty(\Omega)} \right) \leq C \left( \| v^\varepsilon \|_{L^\infty(\Omega)} + \| \partial_x u^\varepsilon \|_{L^\infty(\Omega)} \right) \leq C \| \partial_x u^\varepsilon \|_{L^\infty(\Omega)},
\]
where we have used the fact \( |v^\varepsilon| \leq y |\partial_y v^\varepsilon|_{L^\infty(\Omega)} \) in the last inequality. Using the above inequality and Sobolev inequality (A.3), we conclude
\[
\| \langle \gamma \rangle \hat{Z}_2 \left( \frac{v^\varepsilon}{\varphi} \right) \|_{L^\infty_0(\Omega)} \leq C \left( \mathcal{E}_{3,1}^\frac{1}{2}(t) + \| \partial_y u^\varepsilon \|_{\mathcal{T}^2_1} \right).
\]
which, together with the estimates (3.96) and (3.97), yields directly
\[ \| v^\varepsilon \|_{\mathcal{H}^1_1} \leq C \left( E_{4,2}^\varepsilon (t) + \| \partial_y u^\varepsilon \|_{\mathcal{H}^2_1} \right). \]

Therefore, we complete the proof of Lemma 3.13. \( \square \)

By virtue of the estimates (3.86)–(3.88) in Lemma 3.13, then \( Q(t) \) can be controlled as follows:
\[ Q(t) \leq C (1 + \| (\rho_0, u_{10}, h_{10}) \|_{\mathcal{E}^m_i} + t \| (\rho_0, u_{10}, h_{10}) \|_{\mathcal{E}^m_i} + X_{m,1}^3(t)) \]
for \( m \geq 5, l \geq 1 \). To close the estimate, we still need to establish the estimate for the quantity \( \| \partial_y Q^\varepsilon(t) \|_{\mathcal{H}^1_1} \). Since the quantity \( y^2 \partial_y Q^\varepsilon \) does not communicate with the diffusive term, this prevents us from applying the maximum principle of transport-diffusion equation. We should point out that Masmoudi and Rousset [35] have applied some estimates of one dimensional Fokker-Planck type equation to achieve this target. However, we can only apply the \( L^\infty \)–estimate of heat equation (see estimate (A.7)) to achieve this goal since \( \partial_y v^\varepsilon \) vanishes on the boundary due to \( u^\varepsilon |_{y=0} = 0 \) and divergence-free condition.

**Lemma 3.14** Let \( (q^\varepsilon, u^\varepsilon, v^\varepsilon, h^\varepsilon, g^\varepsilon) \) be sufficiently smooth solution, defined on \([0, T^\varepsilon]\), to the Eq. (3.2). Then, it holds true
\[ \| \partial_q Q^\varepsilon(t) \|_{\mathcal{H}^1_1} \leq C (\| \partial_q Q^\varepsilon_0 \|_{\mathcal{H}^1_1} + \| Q^\varepsilon(0, u_{10}, h_{10}) \|_{\mathcal{H}^3_1} + C t \| (\rho_0, u_{10}, h_{10}) \|_{\mathcal{E}^m_i}^2) \]
\[ + C (1 + Q(t)) \int_0^t X_{m,1}^6(\tau)d\tau, \]
for \( m \geq 5, l \geq 2 \).

**Proof** By virtue of the Sobolev inequality (A.3), it is easy to justify that
\[ \| \partial_y Q^\varepsilon \|_{\mathcal{H}^1_1} \leq \| \partial_y Q^\varepsilon \|_{L^\infty_0(\Omega)} + \| Z_2 Q^\varepsilon \|_{L^\infty_0(\Omega)} + \| Z_{1,2} \partial_y Q^\varepsilon \|_{L^\infty_0(\Omega)} + \| \partial_{y^2} Q^\varepsilon \|_{L^\infty_0(\Omega)} \]
\[ \leq C (\| (\partial_y Q^\varepsilon, Z_{1,2} \partial_y Q^\varepsilon, y \partial_{y^2} Q^\varepsilon) \|_{L^\infty_0(\Omega)} + E_{3,1}(t) + \| \partial_y Q^\varepsilon \|_{\mathcal{H}^1_1}^2). \]

(3.98)

First of all, since the quantity \( \partial_y Q^\varepsilon \) satisfies the evolution equation (3.78), we may apply the maximum principle of transport-diffusion equation (3.78) to get
\[ \| \partial_y Q^\varepsilon \|_{L^\infty_0(\Omega)} \leq \| \partial_y Q^\varepsilon_0 \|_{L^\infty_0(\Omega)} + \int_0^t \| f_1 \|_{L^\infty_0(\Omega)}d\tau, \]
where \( f_1 \) is defined in (3.3). Thus we apply the Cauchy–Schwartz inequality to get
\[ \| \partial_y Q^\varepsilon \|_{L^\infty_0(\Omega)} \leq 2 \| \partial_y Q^\varepsilon_0 \|_{L^\infty_0(\Omega)} + 2 t \int_0^t \| f_1 \|_{L^\infty_0(\Omega)}d\tau \]
\[ \leq 2 \| \partial_y Q^\varepsilon_0 \|_{L^\infty_0(\Omega)} + 2 t \int_0^t \| \partial_y r_1, \partial_y r_2 \|_{L^\infty_0(\Omega)}d\tau \]
\[ + C t \int_0^t (1 + E_{4,0}^2(\tau) + \| \partial_y Q^\varepsilon, u^\varepsilon, h^\varepsilon \|_{\mathcal{H}^1_1}^2 + \| \partial_y Q^\varepsilon \|_{L^\infty_0(\Omega)}^4) \]
\[ \leq (1 + E_{4,0}^2(\tau) + \| \partial_y Q^\varepsilon, u^\varepsilon, h^\varepsilon \|_{\mathcal{H}^1_1}^2 + \| \partial_y Q^\varepsilon \|_{L^\infty_0(\Omega)}^4) \]
\[ + C t \int_0^t (1 + E_{4,0}^2(\tau) + \| \partial_y Q^\varepsilon, u^\varepsilon, h^\varepsilon \|_{\mathcal{H}^1_1}^2 + \| \partial_y Q^\varepsilon \|_{L^\infty_0(\Omega)}^4) \]
\[
(\partial_t + (u^e + 1 - e^{-y})\partial_x + v^e \partial_y - \epsilon \partial_x^2 - \epsilon \partial_y^2)Z_t^e \partial_y q^e
= Z_t^e f_1 - Z_t^e u^e \partial_{xy} q^e - Z_t^e v^e \partial_y^2 q^e,
\]
and hence it follows from the maximum principle and Cauchy–Schwartz inequality that
\[
\|Z_t^e \partial_y q^e\|_{L_t^\infty (\Omega)}^2 \leq 2\|Z_t^e \partial_y q^e\|_{L_t^\infty (\Omega)}^2 + 2t \int_0^t \| (Z_t^e f_1, Z_t^e u^e \partial_{xy} q^e, Z_t^e v^e \partial_y^2 q^e) \|_{L_t^\infty (\Omega)}^2 d\tau.
\]
Since the vertical velocity \(v^e\) vanishes on the boundary (i.e., \(v^e|_{y=0} = 0\)), we conclude
\[
\|Z_t^e u^e \partial_{xy} q^e\|_{L_t^\infty (\Omega)}^2 + \|Z_t^e v^e \partial_y^2 q^e\|_{L_t^\infty (\Omega)}^2 \leq \|Z_t^e u^e \partial_{xy} q^e\|_{L_t^\infty (\Omega)}^2 + \|Z_t^e v^e \partial_y^2 q^e\|_{L_t^\infty (\Omega)}^2 \leq C (E_{5,1}^2(t) + \|\partial_y u^e\|_{H_{\gamma}^1}^4 + \|\partial_y q^e\|_{H_{\gamma}^1}^4). (3.100)
\]
Thus we obtain the estimate
\[
\|Z_t^e \partial_y q^e\|_{L_t^\infty (\Omega)}^2 \leq 2\|Z_t^e \partial_y q^e\|_{L_t^\infty (\Omega)}^2 + 2t \int_0^t (\|Z_t^e \partial_y (\partial_x r_1, \partial_y r_2)\|_{L_t^\infty (\Omega)}^2 + \|r_u\|_{H_t^4}^4) d\tau + C t \int_0^t (1 + E_{5,1}^2(t) + \|\partial_y (q^e, u^e, h^e)\|_{H_t^4}^4 + \|\partial_y q^e\|_{H_{\gamma}^1}^4) d\tau.
\]
Finally, we deal with the term \(\|y \partial_y^2 q^e\|_{L_t^\infty (\Omega)}\). Let \(\chi(y)\) be a smooth compactly supported function which takes the value one in the vicinity of 0 and is supported in \([0, 1]\), and hence, we get
\[
\partial_y q^e = \chi(y) \partial_y q^e + (1 - \chi(y)) \partial_y q^e \triangleq q^b + q^{int},
\]
where \(q^b\) is compactly supported in \(y\) and \(q^{int}\) is supported away from the boundary.

Since \(H_{\gamma}^0\) norm is equivalent to the usual \(H^m\) norm if the function is support away from the boundary, we apply the Sobolev inequality (A.3) to get
\[
\|y \partial_y q^{int}\|_{L_t^\infty (\Omega)} \leq C \|\partial_y q^e\|_{H_{\gamma}^1}. \quad (3.101)
\]
On the other hand, due to the Eq. (3.78), we can get the evolution equation for \(q^b\):
\[
(\partial_t - \epsilon \partial_y^2)q^b = \epsilon \chi \partial_y^2 q^e + \chi f_1 + R_1 + R_2, \quad (3.102)
\]
where \(R_i (i = 1, 2)\) are defined by
\[
R_1 := -\epsilon \chi'' \partial_y q^e - 2\epsilon \chi' \partial_y^2 q^e, \quad R_2 := -\chi (u^e + 1 - e^{-y}) \partial_{xy} q^e - \chi v^e \partial_y^2 q^e.
\]
Applying the estimate (A.7) to the Eq. (3.102), it follows
\[
\|y \partial_y q^b\|_{L_t^\infty (\Omega)}^2 \leq C (\|q^b_0\|_{L_t^\infty (\Omega)}^2 + \|y \partial_y q^b_0\|_{L_t^\infty (\Omega)}^2) \\
+ C \epsilon^2 \int_0^t \| (\partial_y^2 q^e, y \partial_y (\chi \partial_y^2 q^e)) \|_{L_t^\infty (\Omega)}^2 d\tau \quad (3.103)
\]
\[
+ C \int_0^t \| (\chi f_1, y \partial_y (\chi f_1), R_1, y \partial_y R_1, R_2, y \partial_y R_2) \|_{L_t^\infty (\Omega)}^2 d\tau.
\]
In view of the definition of \(\chi\) and the Sobolev inequality, we find
\[
|\epsilon^2 \int_0^t \| (\chi \partial_y^2 q^e, y \partial_y (\chi \partial_y^2 q^e)) \|_{L_t^\infty (\Omega)}^2 d\tau|
\]
\[
\leq C \epsilon^2 \int_0^t \| \partial_y \psi^\epsilon \|_{H^4_0}^2 \, d\tau + C \epsilon^2 \int_0^t \| \psi^\epsilon \|_{H^4_0}^2 \, d\tau,
\]

and
\[
\| R_1 \|_{L^\infty_0(\Omega)}^2 + \| y \partial_y R_1 \|_{L^\infty_0(\Omega)}^2 \leq C \| \psi^\epsilon \|_{H^4_0}^2.
\]

Using the \( L^\infty \)-estimates in Lemma 3.13, we conclude
\[
\| (\chi f_1, y \partial_y (\chi f_1)) \|_{L^\infty_0(\Omega)}^2 \leq \| (\partial_{xy} r_1, \partial_y^2 r_2, Z_2 \partial_{xy} r_1, Z_2 \partial_y^2 r_2) \|_{L^\infty_0(\Omega)}^2 + \| r_u \|_{H^4_0(\Omega)}^4 + C (1 + \| \psi^\epsilon \|_{H^4_0}^4 + \| \psi^\epsilon \|_{H^4_{1,\infty}}^4).
\]

By virtue of the definition \( \chi \), it follows
\[
\|
\chi' y [(u^\epsilon + 1 - e^{-\gamma}) \partial_{xy} \psi^\epsilon + v^\epsilon \partial_y^2 \psi^\epsilon] + \chi y [(\partial_y (u^\epsilon + 1 - e^{-\gamma}) \partial_{xy} \psi^\epsilon + \partial_y v^\epsilon \partial_y^2 \psi^\epsilon)] + \chi (u^\epsilon + 1 - e^{-\gamma}) y \partial_{xy} \psi^\epsilon + \chi v^\epsilon y \partial_y^3 \psi^\epsilon.
\]

By routine checking, we may check that
\[
y \partial_y R_2 = \chi' y [(u^\epsilon + 1 - e^{-\gamma}) \partial_{xy} \psi^\epsilon + v^\epsilon \partial_y^2 \psi^\epsilon] + \chi y [(\partial_y (u^\epsilon + 1 - e^{-\gamma}) \partial_{xy} \psi^\epsilon + \partial_y v^\epsilon \partial_y^2 \psi^\epsilon)] + \chi (u^\epsilon + 1 - e^{-\gamma}) y \partial_{xy} \psi^\epsilon + \chi v^\epsilon y \partial_y^3 \psi^\epsilon.
\]

Since the velocity \( u^\epsilon \) vanishes on the boundary, the application of Taylor formula yields immediately
\[
\| \chi (u^\epsilon + 1 - e^{-\gamma}) y \partial_{xyy} \psi^\epsilon \|_{L^\infty_0(\Omega)} \leq (1 + \| \partial_y u^\epsilon \|_{L^\infty_0(\Omega)}^2) \| \chi y \partial_{xyy} \psi^\epsilon \|_{L^\infty_0(\Omega)}^2 \leq C (1 + \| \partial_y u^\epsilon \|_{L^\infty_0(\Omega)}^2) \| \varphi(y) Z_2 \partial_{xy} \psi^\epsilon \|_{L^\infty_0(\Omega)}^2,
\]

where we have used the fact that \( y \) is equivalent to \( \frac{y}{1 + y} \) if \( y \in [0, c_0] \). Using the fact \( u^\epsilon |_{y=0} = 0 \) and \( \partial_x u^\epsilon + \partial_y v^\epsilon = 0 \), we have \( \partial_y v^\epsilon |_{y=0} = 0 \), and hence, the Taylor formula implies for \( \xi \in [0, y] \)
\[
v^\epsilon(t, x, y) = v^\epsilon(t, x, 0) + y \partial_y v^\epsilon(t, x, 0) + \frac{1}{2} y \partial_y^2 v^\epsilon(t, x, \xi) = \frac{1}{2} y \partial_y^2 v^\epsilon(t, x, \xi),
\]

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where we have used the fact $v^\varepsilon|_{y=0} = \partial_y v^\varepsilon|_{y=0} = 0$. Thus, it follows
\[
\|x v^\varepsilon y \partial_y^3 v^\varepsilon\|_{L^\infty_0}(\Omega) \leq C \|\partial_y^2 v^\varepsilon\|_{L^\infty_0}(\Omega) \|x y \partial_y^3 v^\varepsilon\|_{L^\infty_0}(\Omega) \\
\leq C \|\partial_{x,y} u^\varepsilon\|_{L^\infty_0}(\Omega) \|\varphi(y) Z_2^2 \partial_y Q^\varepsilon\|_{L^\infty_0}(\Omega).
\]
(3.111)
Then the combination of estimates (3.110), (3.111) and Sobolev inequality (A.3) yields directly
\[
\|x (u^\varepsilon + 1 - e^{-y}) y \partial_{x,y}^2 v^\varepsilon + x v^\varepsilon y \partial_y^3 v^\varepsilon\|_{L^\infty_0}(\Omega) \leq C(1 + \|r_u\|_{H^2_0} + \varepsilon_{x,y}^0(t) + \|\partial_y (Q^\varepsilon, u^\varepsilon, h^\varepsilon)\|_{H^2_0})
\]
which, together with the estimates (3.108) and (3.109), yields directly
\[
\|y \partial_y R_2\|_{L^\infty_0}(\Omega) \leq C(1 + \|r_u\|_{H^2_0} + \varepsilon_{x,y}^0(t) + \|\partial_y (Q^\varepsilon, u^\varepsilon, h^\varepsilon)\|_{H^2_0})
\]
(3.112)
Then, we can get from the estimates (3.104)–(3.107) and (3.112) that
\[
\|y \partial_y v^\varepsilon\|_{L^\infty_0}(\Omega) \leq C(\|\varphi\|_{L^\infty_0}(\Omega) + \|y \partial_y \varphi\|_{L^\infty_0}(\Omega)) \\
+ C \int_0^t \|\partial_{x,y} r_1, \partial_y^2 r_2, Z_2 \partial_{x,y} r_1, Z_2 \partial_y^2 r_2\|_{L^\infty_0}(\Omega) \|\omega\|_{L^\infty_0}(\Omega) d\tau \\
+ C \int_0^t \|r_u\|_{H^2_0} d\tau + C \int_0^t (1 + \varepsilon_{x,y}^0(t) + \|\partial_y (Q^\varepsilon, u^\varepsilon, h^\varepsilon)\|_{H^2_0})
\]
which, together with the estimate (3.101), yields directly
\[
\|y \partial_y^2 \varphi\|_{L^\infty_0}(\Omega) \leq C(\|\partial_y \varphi\|_{L^\infty_0}(\Omega) + \|Z_2 \partial_{x,y} \varphi\|_{L^\infty_0}(\Omega)) + C \|\partial_y Q^\varepsilon\|_{H^2_0} \\
+ C \int_0^t (\|r_u\|_{H^2_0} + \|\partial_{x,y} r_1, \partial_y^2 r_2, Z_2 \partial_{x,y} r_1, Z_2 \partial_y^2 r_2\|_{L^\infty_0}(\Omega)) d\tau \\
+ C \int_0^t (1 + \varepsilon_{x,y}^0(t) + \|\partial_y (Q^\varepsilon, u^\varepsilon, h^\varepsilon)\|_{H^2_0}) + \|\partial_y Q^\varepsilon\|_{H^2_0} d\tau.
\]
(3.113)
Therefore, substituting the estimates (3.99), (3.100) and (3.113) into (3.98), we complete the proof of lemma.

\[\square\]

### 3.5 Proof of Theorem 3.1

Based on the estimates obtained so far, we can complete the proof of Theorem 3.1 in this subsection. First of all, we give the proof for the estimate (3.6). For two parameters $R$ and $\delta$, which will be defined later, we define
\[
T^*_{\varepsilon} := \sup \left\{ T \in [0, 1] \mid \Theta_{M,1}(t) \leq R, \quad \|\partial_y (u^\varepsilon - e^{-y})\|_{L^\infty_0}(\Omega) \leq \delta^{-1}, \right. \\
\left. \|Q^\varepsilon(t)\|_{L^\infty_0}(\Omega) \leq \frac{2l - 1}{2} \delta^2, h^\varepsilon(t, x, y) + 1 \geq \delta, \forall t \in [0, T], (x, y) \in \Omega \right\}.
\]
Now, we write

$$\mathcal{N}_{m,l}(t) := \sup_{0 \leq s \leq t} \left\{ 1 + \mathcal{E}_{m,l}(s) + \| (q^e_m, u^e_m, h^e_m)(s) \|_{L^2_t(\Omega)}^2 ight\}
+ \| (\partial_y q^e_m, \partial_y u^e_m, \partial_y h^e_m)(s) \|_{H^{-1}}^2 + \| \partial_y q^e_m(s) \|_{H^{-1}}^2
+ \int_0^t \| \partial_y^2 (\sqrt{\epsilon} q^e_m, \sqrt{\mu} u^e_m, \sqrt{\kappa} h^e_m) \|_{H^{-1}}^2 d\tau + \epsilon \int_0^t \| \partial_y \phi^e_m(s, u^e_m, h^e_m) \|_{H^{-1}}^2 d\tau
+ \int_0^t (D^m_x + D^m_y)(\tau) d\tau,$$

(3.114)

where $D^m_x(t)$ and $D^m_y(t)$ are defined by

$$D^m_x(t) := \sum_{0 \leq |\alpha| \leq m} \epsilon \| \partial_x^\alpha \phi^e_m(t) \|^2_{L^2_t(\Omega)} + \epsilon \| \partial_x \phi^e_m(t) \|^2_{L^2_t(\Omega)}.$$

(3.115)

and

$$D^m_y(t) := \sum_{0 \leq |\alpha| \leq m} \| \partial_y (\sqrt{\epsilon} \phi^e_m, \sqrt{\mu} \phi^e_m, \sqrt{\kappa} \phi^e_m(t) \|^2_{L^2_t(\Omega)}
+ \| \partial_x (\sqrt{\epsilon} \phi^e_m, \sqrt{\mu} \phi^e_m, \sqrt{\kappa} \phi^e_m(t) \|^2_{L^2_t(\Omega)}.$$

(3.116)

From the estimates in Propositions 3.2, 3.5, 3.8, 3.12, we may conclude for $T_1 \leq T^e$ that

$$\mathcal{N}_{m,l}(t) \leq C \delta^{-2}(1 + \| (\rho_0, u_{10}, h_{10}) \|_{B^m_t}^5 + C_{\mu,k} \| (\rho_0, u_{10}, h_{10}) \|_{B^m_t}^5
+C_{\mu,k,m,l} \delta^{-12} t \mathcal{N}_{m,l}^{12}(t), t \in [0, T_1].$$

(3.117)

On the other hand, recall the almost equivalently relations (see Lemma B.2)

$$\Theta_{m,l}(t) \leq C \| (\rho_0, u_{10}, h_{10}) \|_{B^m_t}^4 + Ct \| (\rho_0, u_{10}, h_{10}) \|_{B^m_t}^4 + C_I \delta^{-8} \Theta_{m,l}(t), \forall t \in [0, T_1].$$

(3.118)

and

$$\mathcal{N}_{m,l}(t) \leq C \| (\rho_0, u_{10}, h_{10}) \|_{B^m_t}^4 + Ct \| (\rho_0, u_{10}, h_{10}) \|_{B^m_t}^4 + C_{\mu,k} \delta^{-8} \Theta_{m,l}(t), \forall t \in [0, T_1],$$

(3.119)

and hence, we may deduce from the estimates (3.117), (3.118) and (3.119) that

$$\Theta_{m,l}(T_1) \leq C_I \mathcal{P}_0(\delta^{-1}, \| (\rho_0, u_{10}, h_{10}) \|_{B^m_t}^5) + C_{\mu,k,m,l} T_1 \mathcal{P}_1(\delta^{-1}, \| (\rho_0, u_{10}, h_{10}) \|_{B^m_t}^5)
+C_{\mu,k,m,l} T_1 \mathcal{P}_2(\delta^{-1}, \| (\rho_0, u_{10}, h_{10}) \|_{B^m_t}^5) + C_{\mu,k,m,l} T_1 \mathcal{P}_3(\delta^{-1}, R).$$

Choose constant $\delta = \frac{\delta_0}{2}$ and $R = 4C_I \mathcal{P}_0(\delta_0^{-1}, \| (\rho_0, u_{10}, h_{10}) \|_{B^m_t}^5)$, we obtain

$$\Theta_{m,l}(T_1) \leq C_I \mathcal{P}_0(\delta_0^{-1}, \| (\rho_0, u_{10}, h_{10}) \|_{B^m_t}^5) + C_{\mu,k,m,l} T_1 \mathcal{P}_4(\delta_0^{-1}, \| (\rho_0, u_{10}, h_{10}) \|_{B^m_t}^5)
+C_{\mu,k,m,l} T_1 \mathcal{P}_5(\delta_0^{-1}, \| (\rho_0, u_{10}, h_{10}) \|_{B^m_t}^5)
+C_{\mu,k,m,l} T_1 \mathcal{P}_6(\delta_0^{-1}, \| (\rho_0, u_{10}, h_{10}) \|_{B^m_t}^5).$$
Choosing the time $T_1 = \min\left\{ \frac{\overline{C}_0}{\mathcal{P}_6(\delta_0^{-1}, \|(\rho_0, u_{10}, h_{10})\|_{\mathcal{B}_1^m})}, \frac{\overline{C}_0}{\mathcal{P}_5(\delta_0^{-1}, \|(\rho_0, u_{10}, h_{10})\|_{\mathcal{B}_1^m})} \right\}$, and hence, it follows

$$\Theta_{m,l}(T_1) \leq 2C_1\mathcal{P}_0(\delta_0^{-1}, \|(\rho_0, u_{10}, h_{10})\|_{\mathcal{B}_1^m}) = \frac{R}{2}.\]$$

Here the constant $\overline{C}_0 := \frac{\mathcal{P}_6(\delta_0^{-1}, \|(\rho_0, u_{10}, h_{10})\|_{\mathcal{B}_1^m})}{2\epsilon_{\mu,\kappa,l}}$. For any smooth function $W(t, x, y)$, it is easy to justify

$$W(t, x, y) = W(0, x, y) + \int_0^t \partial_s W(s, x, y) ds. \tag{3.120}$$

Using the relation (3.120) and the Sobolev inequality (A.3), we get

$$h^\epsilon(t, x, y) + 1 \geq h_0^\epsilon(x, y) + 1 - Ct \sup_{0 \leq s \leq t} \|(h^\epsilon, \partial_y h^\epsilon)\|_{\mathcal{H}_0^2} \geq 2\delta_0 - 2C_1t \frac{\mathcal{P}_0(\delta_0^{-1}, \|(\rho_0, u_{10}, h_{10})\|_{\mathcal{B}_1^m})}{\mathcal{P}_6(\delta_0^{-1}, \|(\rho_0, u_{10}, h_{10})\|_{\mathcal{B}_1^m})}.$$

Choose $T_2 = \min\{T_1, \frac{\delta_0}{2C_1\mathcal{P}_0(\delta_0^{-1}, \|(\rho_0, u_{10}, h_{10})\|_{\mathcal{B}_1^m})}\}$, it follows

$$h^\epsilon(t, x, y) + 1 \geq \delta_0 = 2\delta, \quad \text{for all } (t, x, y) \in [0, T_2] \times \Omega.$$

Similarly, we get from the relation (3.120) and the estimate (3.92) that

$$\|\partial_y(u^\epsilon - e^{-\gamma})(t)\|_{L_1^\infty(\Omega)} \leq \|\partial_y(u_0^\epsilon - e^{-\gamma})(t)\|_{L_1^\infty(\Omega)} + t \sup_{0 \leq s \leq t} \|\partial_y \partial_s u^\epsilon(s)\|_{L_1^\infty(\Omega)} \leq (2\delta_0)^{-1} + C(t + \|\rho_0, u_{10}, h_{10}\|_{\mathcal{B}_1^m})$$

Choosing $T_3 = \min\{T_2, \frac{1}{2C_1\mathcal{P}_0(\delta_0^{-1}, \|(\rho_0, u_{10}, h_{10})\|_{\mathcal{B}_1^m})}\}$, and hence

$$\|\partial_y(u^\epsilon - e^{-\gamma})(t)\|_{L_1^\infty(\Omega)} \leq \delta_0^{-1} = (2\delta)^{-1}, \quad \text{for all } t \in [0, T_3].$$

Finally, from the relation (3.120) and the Sobolev inequality (A.3), we find

$$\|q^\epsilon(t)\|_{L_0^\infty(\Omega)} \leq \|q_0^\epsilon\|_{L_0^\infty(\Omega)} + Ct \sup_{0 \leq s \leq t} \|(q^\epsilon, \partial_y q^\epsilon)(s)\|_{\mathcal{H}_0^2} \leq \frac{2l - 1}{16}\delta_0^2 + 2C_1t \frac{\mathcal{P}_0(\delta_0^{-1}, \|(\rho_0, u_{10}, h_{10})\|_{\mathcal{B}_1^m})}{\mathcal{P}_6(\delta_0^{-1}, \|(\rho_0, u_{10}, h_{10})\|_{\mathcal{B}_1^m})}.$$

Choose $T_4 = \min\{T_3, \frac{2l - 1}{64C_1\mathcal{P}_0(\delta_0^{-1}, \|(\rho_0, u_{10}, h_{10})\|_{\mathcal{B}_1^m})}\}$, we obtain

$$\|q^\epsilon(t)\|_{L_0^\infty(\Omega)} \leq \frac{3(2l - 1)}{32}\delta_0^2 = \frac{3(2l - 1)}{8}\delta^2.$$

for all $t \in [0, T_4]$. Obviously, we conclude that there exists $T_4 > 0$ depending only on $\mu, \kappa, m, l, \delta_0$ and the initial data (hence independent of parameter $\epsilon$) such that for $T \leq \min\{T_4, T^\epsilon\}$, the estimates (3.6) and (3.7) hold on. Of course, it holds that $T_4 \leq T^\epsilon$. Indeed otherwise, our criterion about the continuation of the solution would contradict the definition of $T^\epsilon$. Then, taking $T_0 = T_4$, we obtain the estimate (3.7) and close the a priori assumptions (3.8) and (3.9). Therefore, the proof of Theorem 3.1 is completed.
4 Local-in-time existence and uniqueness

In this section, we will establish the local-in-time existence and uniqueness of solution to the inhomogeneous incompressible MHD boundary layer equations (1.6)–(1.7).

4.1 Existence for the MHD boundary layer system

We shall use the a priori estimates obtained to prove local in time existence result. For \( m \geq 5 \) and \( l \geq 2 \), consider initial data such that \( \| (\rho_0, u_{10}, h_{10}) \|_{B^{m,l}_B} \leq C_0 < +\infty \). For such initial data, we are not aware of a local well-posedness result for the Eqs. (3.2)–(3.3). Since \((\rho_0, u_{10}, h_{10}) \in B^{m,l}_{BL,BL,C} \), there exists a sequence of smooth approximate initial data \((\rho_0^n, u_{10}^n, h_{10}^n) \in B^{m,l}_{BL,BL,C} (\sigma \text{ being a regularization parameter})\), which have enough spatial regularity so that the time derivatives at the initial time can be defined by the Eq. (1.6) and boundary compatibility condition are satisfied. Then, it follows to get a positive time \( T_{e,\sigma} > 0 \) (\( T_{e,\sigma} \) depends on \( \epsilon, \sigma \), and the initial data) for which a solution \((\rho^{e,\sigma}, u^{e,\sigma}, h^{e,\sigma}) \) exists in Sobolev space \( H_{m1}^{4m}(\Omega) \) and \((u^{e,\sigma}, g^{e,\sigma}) \) exists in Sobolev space \( H_{m1-1}^{4m}(\Omega) \) respectively. Applying the a priori estimates given in Theorem 3.1 to \((\rho^{e,\sigma}, u^{e,\sigma}, h^{e,\sigma}) \), we obtain a uniform time \( T_a > 0 \) and a constant \( C_1 \) (independent of \( \epsilon \) and \( \sigma \)) such that it holds true

\[
\Theta_{m,l}(\rho^{e,\sigma}, u^{e,\sigma}, h^{e,\sigma})(t) \leq C_1,
\]

\[
\| u^{e,\sigma} \|_{L^\infty(\Omega)} \leq \frac{2l - 1}{2} \delta_0^{-1},
\]

\[
\| \partial_y(u^{e,\sigma} - e^{-y}) \|_{L^\infty(\Omega)} \leq \delta_0^{-1},
\]

and

\[
h^{e,\sigma}(t, x, y) + 1 \geq \delta_0,
\]

(4.1)

where \( t \in [0, T_0], T_0 := \min(T_a, T_{e,\sigma}) \). Based on the uniform estimates for \((\rho^{e,\sigma}, u^{e,\sigma}, h^{e,\sigma}) \), one can pass the limit \( \epsilon \to 0^+ \) and \( \sigma \to 0^+ \) to get a strong solution \((\rho, u, h)\) satisfying (1.6) by using a strong compactness arguments. Indeed, it follows from (4.1) that \((\rho^{e,\sigma}, u^{e,\sigma}, h^{e,\sigma}) \) is bounded uniformly in \( L^\infty([0, T_0]; H_{co}^{m-1}) \), while \( \partial_y(u^{e,\sigma}, u^{e,\sigma}, h^{e,\sigma}) \) is bounded uniformly in \( L^\infty([0, T_0]; H_{co}^{m-1}) \), and \( \partial_y \rho^{e,\sigma}, u^{e,\sigma}, h^{e,\sigma} \) is bounded uniformly in \( L^\infty([0, T_0]; H_{co}^{m-1}) \). Then, it follows from a strong compactness argument that \((\rho^{e,\sigma}, u^{e,\sigma}, h^{e,\sigma}) \) is compact in \( C([0, T_0]; H_{co,loc}^{m-1}) \). Due to \( \kappa > 0 \), it is easy to check that \( h^{e,\sigma} \) is compact in \( C([0, T_0]; H_{loc}^2) \).

In particular, there exist sequences \( \epsilon_n, \sigma_n \to 0^+ \) and \((\rho, u, h) \in C([0, T_0]; H_{co,loc}^{m-1}) \) such that

\[
(\rho^{e_n,\sigma_n}, u^{e_n,\sigma_n}, h^{e_n,\sigma_n}) \to (\rho, u, h) \text{ in } C([0, T_0]; H_{co,loc}^{m-1}) \text{ as } \epsilon^n, \sigma^n \to 0^+,
\]

and

\[
h^{e_n,\sigma_n} \to h \text{ in } C([0, T_0]; H_{loc}^2) \text{ as } \epsilon^n, \sigma^n \to 0^+.
\]

Furthermore, we apply the Sobolev inequality to get

\[
\sup_{0 \leq \tau \leq t} \| (\partial_y^{-1} \partial_x u^{e_n,\sigma_n} - \partial_y^{-1} \partial_x u)(\tau) \|_{L^\infty(\Omega)} \leq C \sup_{0 \leq \tau \leq t} \| \partial_y^{-1} \partial_x (u^{e_n,\sigma_n} - u)(\tau) \|_{H_{co,loc}^{1-1/2}}^{1/2} \| \partial_x (u^{e_n,\sigma_n} - u)(\tau) \|_{H_{co,loc}^{1+1/2}}^{1/2} \leq C \sup_{0 \leq \tau \leq t} \| (u^{e_n,\sigma_n} - u)(\tau) \|_{H_{co,loc}^2} \to 0, \text{ as } \epsilon^n, \sigma^n \to 0^+.
\]
Hence we denote \( v(t, x, y) = - \int_0^y \partial_x u(t, x, \xi) d\xi \), which satisfies the divergence-free condition \( \partial_t u + \partial_y v = 0 \). Similarly, we denote \( g(t, x, y) = - \int_0^y h(t, x, \xi) d\xi \), which satisfies

\[
\sup_{0 \leq t \leq T} \| (g^{e_n, \sigma_n} - g)(\tau) \|_{L^\infty_0(\Omega)} \leq C \sup_{0 \leq t \leq T} \| (h^{e_n, \sigma_n} - h)(\tau) \|_{H^1_{0,\text{loc}}} \to 0, \quad \text{as } e^n, \sigma^n \to 0^+.
\]

By routine checking, we may show that \((\rho, \rho_1, u_1, u_2, h_1, h_2) := (\rho + 1, u + 1 - e^{-\varphi}, v, h + 1, g)\) is a solution of the original MHD boundary layer system (1.6). Finally, applying the lower semicontinuity of norms to the bound (4.1), one obtains the estimate (1.18) for the solution \((\rho, u_1, h_1)\). Since \(h^{e_n, \sigma_n}\) converges uniformly to \(h\), then we can get \(h_1 \geq \delta\) from (4.2).

### 4.2 Uniqueness for the MHD boundary layer system

In this subsection, we will show the uniqueness of solution to the MHD boundary layer equations (1.6)–(1.7). Let \((\rho_1, u_1, v_1, h_1, g_1)\) and \((\rho_2, u_2, v_2, h_2, g_2)\) be two solutions in the existence time \([0, T_0]\), constructed in the previous subsection, with respect to the initial data \((\rho_1^0, u_1^0, h_1^0)\) and \((\rho_2^0, u_2^0, h_2^0)\) respectively.

Let us set

\[
(\bar{\rho}, \bar{u}, \bar{v}, \bar{\varphi}, \bar{g}) := (\rho_1 - \rho_2, u_1 - u_2, v_1 - v_2, h_1 - h_2, g_1 - g_2),
\]

then they satisfy the following evolution

\[
\begin{aligned}
\partial_t \bar{\rho} + u_1 \partial_x \bar{\rho} + v_1 \partial_y \bar{\rho} + \bar{u} \partial_x \rho_2 + \bar{v} \partial_y \rho_2 &= 0, \\
\rho_1 \partial_t \bar{u} + \rho_1 u_1 \partial_x \bar{u} + v_1 \partial_x u_2 + \rho_1 \bar{u} \partial_x u_2 + \rho_1 \bar{v} \partial_x u_2 - \mu \partial_y^2 \bar{u} &= 0, \\
\partial_t \bar{v} + u_1 \partial_x \bar{v} + v_1 \partial_y \bar{v} + \bar{u} \partial_x v_2 + \bar{v} \partial_x v_2 - \kappa \partial_y^2 \bar{v} &= 0,
\end{aligned}
\]

with the boundary condition and initial data

\[
(\bar{u}, \bar{v}, \partial_y \bar{\varphi}, \bar{g})|_{y=0} = 0, \quad \lim_{y \to -\infty} (\bar{\rho}, \bar{u}, \bar{\varphi}, \bar{g}) = 0, \quad (\bar{\rho}, \bar{u}, \bar{\varphi})|_{t=0} = 0.
\]

Here we assume the two solutions \((\rho_1, u_1, v_1, h_1, g_1)\) and \((\rho_2, u_2, v_2, h_2, g_2)\) have the same initial data \((\rho_1^0, u_1^0, h_1^0) = (\rho_2^0, u_2^0, h_2^0)\). Denote by \(\bar{\varphi} := \partial_y^{-1} \bar{\varphi} = \partial_y^{-1}(h_1 - h_2)\), it follows

\[
\partial_t \bar{\varphi} + u_1 \partial_x \bar{\varphi} + v_1 \partial_y \bar{\varphi} - \bar{u} g_2 + \bar{v} h_2 - \kappa \partial_y^2 \bar{\varphi} = 0.
\]

Define \(\eta_1 := \frac{\partial \rho_2}{h_2}, \eta_2 := \frac{\partial u_2}{h_2}, \eta_3 := \frac{\partial h_2}{h_2}\), and introduce the new quantities:

\[
\hat{\rho} := \bar{\rho} - \eta_1 \bar{\varphi}, \quad \hat{u} := \bar{u} - \eta_2 \bar{\varphi}, \quad \hat{h} := \bar{h} - \eta_3 \bar{\varphi}.
\]

Next, we can obtain that through direct calculation, \((\hat{\rho}, \hat{u}, \hat{h})\) satisfies the following initial boundary value problem:

\[
\begin{aligned}
\partial_t \hat{\rho} + u_1 \partial_x \hat{\rho} + v_1 \partial_y \hat{\rho} &= -\kappa \eta_1 \partial_y \hat{h} - a_{11} \bar{u} - a_{12} \bar{\varphi} - a_{13} \bar{\varphi}, \\
\rho_1 \partial_t \hat{u} + \rho_1 u_1 \partial_x \hat{u} + \rho_1 v_1 \partial_x \hat{u} - h_2 \partial_x \hat{h} - g_1 \partial_y \hat{h} - \mu \partial_y^2 \hat{u} &= -\kappa \rho_1 \eta_2 \partial_y \hat{h} + \mu \partial_y (\partial_y \eta_2 \bar{\varphi} + \eta_2 \bar{\varphi}) - a_{21} \bar{\varphi} - a_{22} \bar{u} - a_{23} \bar{\varphi} - a_{24} \bar{\varphi}, \\
\partial_t \hat{h} + u_1 \partial_x \hat{h} + v_1 \partial_y \hat{h} - h_2 \partial_x \hat{u} - g_1 \partial_y \hat{u} - \kappa \partial_y^2 \hat{h} &= -\kappa \eta_3 \partial_y \hat{h} - a_{31} \bar{u} - a_{32} \bar{\varphi} - a_{33} \bar{\varphi},
\end{aligned}
\]
Thus we can apply the standard energy method for the Eq. (4.5) to establish the following estimate, which we omit the proof for brevity of presentation.

**Proposition 4.1** Let \((ρ_1, u_1, v_1, h_1, g_1)\) and \((ρ_2, u_2, v_2, h_2, g_2)\) be two solutions of MHD boundary layer equations (1.6)–(1.7) with the same initial data, and satisfying the estimate (1.18) respectively. Then, there exists a positive constant

\[
C = C(T_δ, \| (ρ_1, u_1, h_1)(t) \|_{β^m}^m, \| (ρ_2, u_2, h_2)(t) \|_{β^m}^m) > 0,
\]

such that the quantity \((\hat{ρ}, \hat{u}, \hat{h})\) given by (4.4) satisfies

\[
\| (\hat{ρ}, \hat{u}, \hat{h})(t) \|_{L^2}^2 + \int_0^t \| \nabla_y (\sqrt{μ} \hat{u}, \sqrt{κ} \hat{h})(τ) \|_{L^2}^2 dτ \leq C \int_0^t \| (\hat{ρ}, \hat{u}, \hat{h})(τ) \|_{L^2}^2 dτ. \tag{4.6}
\]

Then, we can prove the uniqueness of the solution to (1.6)–(1.7) as follows.

**Proof of Uniqueness** Applying Gronwall’s lemma to the estimate (4.6), we obtain \((\hat{ρ}, \hat{u}, \hat{h}) ≡ 0\). Then, we substitute \(\hat{h} ≡ 0\) into equality \(\hat{V}_{h_2} = \hat{ρ}^{-1}(\hat{h}/h_2)\) to get \(\hat{V} ≡ 0\). From the definition (4.4) we get \((\hat{ρ}, \hat{u}, h_1) ≡ (ρ_2, u_2, h_2)\) due to the fact \((\hat{ρ}, \hat{u}, \hat{h}) ≡ 0\) and \(\hat{V} ≡ 0\). Finally, it follows from the divergence-free condition and the boundary condition \(\hat{V}|_{y=0} = 0\) that \(V = -\hat{ρ}^{-1}\hat{ρ}^{-1} \nabla\hat{u} = 0\), which implies the fact \(v_1 ≡ v_2\). Similarly, it holds true \(g_1 ≡ g_2\).

Therefore, we complete the proof of uniqueness of the solution for the MHD boundary layer equations (1.6)–(1.7) completely.

**Acknowledgements** Jincheng Gao’s research was partially supported by NNSF of China (11801586) and Natural Science Foundation of Guangdong Province of China (2020B1515310004). Daiwen Huang’s research was partially supported by NNSF of China (11971067, 11631008, 11771183). Zheng-an Yao’s research was partially supported by NNSF of China (11971496, 12026244).

**Appendix A: Calculus inequalities**

In this appendix, we will introduce some basic inequalities that are used frequently in this paper. First of all, we introduce the following Hardy type inequality, which can refer to [36].
Lemma A.1 Let the proper function \( f : \mathbb{T} \times \mathbb{R}^+ \rightarrow \mathbb{R} \), and satisfies \( f(x, y)|_{y=0} = 0 \) and \( \lim_{y \to +\infty} f(x, y) = 0 \). If \( \lambda > -\frac{1}{2} \), then it holds true
\[
\|f\|_{L^2_x(T \times \mathbb{R}^+)} \leq \frac{2}{2\lambda + 1} \|\partial_y f\|_{L^2_{x+1}(T \times \mathbb{R}^+)}.
\]
(A.1)

Next, we will state the following Sobolev-type inequality.

Lemma A.2 Let the proper function \( f : \mathbb{T} \times \mathbb{R}^+ \rightarrow \mathbb{R} \), and satisfies \( \lim_{y \to +\infty} f(x, y) = 0 \). Then there exists a universal constant \( C > 0 \) such that
\[
\|f\|_{L^\infty_x(T \times \mathbb{R}^+)} \leq C(\|\partial_y f\|_{L^2_0(T \times \mathbb{R}^+)} + \|\partial^2_{xy} f\|_{L^2_0(T \times \mathbb{R}^+)} + \|\partial_x f\|_{L^2_0(T \times \mathbb{R}^+)} + \|\partial_y f\|_{L^2_0(T \times \mathbb{R}^+)} + \|\partial_{xy} f\|_{L^2_0(T \times \mathbb{R}^+)})
\]
or equivalently
\[
\|f\|_{L^\infty_x(T \times \mathbb{R}^+)} \leq C(\|f\|_{L^2_0(T \times \mathbb{R}^+)} + \|\partial_x f\|_{L^2_0(T \times \mathbb{R}^+)} + \|\partial_y f\|_{L^2_0(T \times \mathbb{R}^+)} + \|\partial_{xy} f\|_{L^2_0(T \times \mathbb{R}^+)}).
\]
(A.2)

Proof Indeed, the estimate (A.3) follows directly from estimate (A.2) and the Cauchy–Schwartz inequality. Hence, we only give the proof for the estimate (A.2). On one hand, thanks to the one-dimensional Sobolev inequality for the \( y \)-variable, we get
\[
|f(x, y)|^2 \leq C \left( \int_0^\infty |\partial_\xi f(x, \xi)|^2 d\xi \right)^{\frac{1}{2}} \left( \int_0^\infty |f(x, \xi)|^2 d\xi \right)^{\frac{1}{2}}.
\]
(A.4)

On the other hand, we apply the following one-dimensional Sobolev inequality for \( x \)-variable to get
\[
|f(x, y)|^2 \leq C(\|f(y)\|^2_{L^2_0(T)} + \|\partial_x f(y)\|^2_{L^2_0(T)}), \quad |\partial_y f(x, y)|^2 \leq C(\|\partial_y f(y)\|^2_{L^2_0(T)} + \|\partial_{xy} f(y)\|^2_{L^2_0(T)}).
\]
(A.5)

Therefore, substituting the estimate (A.5) into (A.4), we complete the proof of estimate (A.2).

Now we will state the Moser type inequality as follow:

Lemma A.3 Denote \( \Omega := \mathbb{T} \times \mathbb{R}^+ \), let the proper functions \( f(t, x, y) : \mathbb{R}^+ \times \Omega \rightarrow \mathbb{R} \) and \( g(t, x, y) : \mathbb{R}^+ \times \Omega \rightarrow \mathbb{R} \). Then, there exists a constant \( C_m > 0 \) such that
\[
\int_0^t \|\nabla^\beta Z^\gamma g\|^2_{L^2_{x,y}}(\tau) d\tau \leq C_m(\|g\|^4_{L^\infty_{x,y}} \int_0^t \|\partial_t f\|^2_{L^2_{x,y}} d\tau + \|g\|^2_{L^\infty_{x,y}} \int_0^t \|\partial_{tt} f\|^2_{L^2_{x,y}} d\tau),
\]
(A.6)

where \( |\beta + \gamma| = m \) and \( l_1 + l_2 = l \).

Proof For any \( p \geq 2 \), due to the relation \( |Z_2 f|^p = Z_2(f Z_2 f |Z_2 f|^{p-2}) - (p - 1) f Z_2^2 f |Z_2 f|^{p-2} \), we find
\[
\int_{\mathbb{R}^+} \langle y \rangle^{l_1 p} |Z_2 f|^p dy = \int_{\mathbb{R}^+} \langle y \rangle^{l_1 p} Z_2(f Z_2 f |Z_2 f|^{p-2}) d\tau - (p - 1)
\]
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\[ \int_{\mathbb{R}^+} \langle y \rangle^{\theta_1} f \, Z_2^2 \, f \, |Z_2 f|^{p-2} \, dy. \]

Integrating by part and applying the Hölder inequality, we find for \( 0 \leq \theta \leq 1 \) and \( 0 \leq \theta_1 \leq \frac{\theta}{2} \) that

\[
\| \langle y \rangle^{\theta_1} Z_2 f \|_{L_p^q(\mathbb{R}^+)} \leq C_p \int_{\mathbb{R}^+} \langle y \rangle^{\theta_1-p-1} |f| (|Z_2 f| + |Z_2^2 f|) |Z_2 f|^{p-2} \, dy \\
\leq C_p \| \langle y \rangle^{\theta_1} Z_2 f \|_{L_0^p(\mathbb{R}^+)} \| \langle y \rangle^{\theta_1} f \|_{L_0^q(\mathbb{R}^+)} \| \langle y \rangle^{(2\theta-\theta_1)t} (|Z_2 f|, |Z_2^2 f|) \|_{L_0^q(\mathbb{R}^+)},
\]

and hence, it follows

\[
\| \langle y \rangle^{\theta_1} Z_2 f \|_{L_0^p(\mathbb{R}^+)}^2 \leq C_p \| \langle y \rangle^{\theta_1} f \|_{L_0^q(\mathbb{R}^+)} \sum_{1 \leq k \leq 2} \| \langle y \rangle^{(2\theta-\theta_1)t} Z_2^k f \|_{L_0^q(\mathbb{R}^+)},
\]

Here \( \frac{1}{q} + \frac{1}{r} = \frac{2}{p} \). Integrating with respect to \( t \) and \( x \) variables, and applying Hölder inequality, we get

\[
\| \langle y \rangle^{\theta_1} Z_2 f \|_{L_p^q(Q_T)}^2 \leq C_p \| \langle y \rangle^{\theta_1} f \|_{L_0^q(Q_T)} \sum_{1 \leq k \leq 2} \| \langle y \rangle^{(2\theta-\theta_1)t} Z_2^k f \|_{L^r(Q_T)}.
\]

Here \( Q_T = \Omega \times [0, T] \). Similarly, it is easy to justify for \( i = 0, 1, \)

\[
\| \langle y \rangle^{\theta_1} Z_i f \|_{L_0^p(Q_T)}^2 \leq C_p \| \langle y \rangle^{\theta_1} f \|_{L_0^q(Q_T)} \sum_{1 \leq k \leq 2} \| \langle y \rangle^{(2\theta-\theta_1)t} Z_i^k f \|_{L_0^q(Q_T)}.
\]

Here \( \frac{1}{q} + \frac{1}{r} = \frac{2}{p} \). By multiple application of the above inequality, we get(proof by induction)

\[
\| \langle y \rangle^{(\alpha|\beta|-(\alpha|\gamma|+1)\theta_1)l} Z^\alpha f \|_{L_0^{q_1}(Q_T)} \leq C_p \| \langle y \rangle^{\theta_1} f \|_{L_0^{q_1}(Q_T)} \sum_{1 \leq |\beta| \leq m} \| \langle y \rangle^{(m\theta-(m+1)\alpha)l} Z^\beta f \|_{L_0^{q_1}(Q_T)},
\]

where \( \frac{1}{p_i} = \frac{1}{q_1} (1 - \frac{|\alpha|}{m}) + \frac{|\alpha|}{r_1 m} \) and \( 1 \leq |\alpha| \leq m - 1 \). Then, we get for \( |\beta| + |\gamma| = m \) that

\[
\| \langle y \rangle^{\gamma} Z^\beta f \, Z^\gamma \|_{L_0^{q_1}(Q_T)} \leq C \| \langle y \rangle^{\frac{|\beta|}{m} l + (1 - \frac{|\beta|}{m})} \|_{L_0^{q_1}(Q_T)} \leq C_m \| \langle y \rangle^{l l} f \|_{L_0^{q_1}(Q_T)} \sum_{1 \leq |\beta| \leq m} \| \langle y \rangle^{l |\beta|} Z^\beta f \|_{L_0^{q_1}(Q_T)} \times \| \langle y \rangle^{l |\gamma|} g \|_{L_0^{q_1}(Q_T)} \sum_{1 \leq |\gamma| \leq m} \| \langle y \rangle^{l |\gamma|} Z^\gamma g \|_{L_0^{q_1}(Q_T)}.
\]

Therefore, we complete the proof of this lemma. ☐

Finally, we establish the following \( L^\infty \)–estimate with weight for the heat equation.
Lemma A.4  For the heat equation \( \partial_t F(t, x) - \epsilon \partial_x^2 F(t, x) = G(t, x), \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^+; \) with the boundary condition \( F(t, x)|_{x=0} = 0 \) and initial data \( F(t, x)|_{t=0} = F_0. \) Then, it holds true
\[
\|x \partial_x F\|_{L^0_0(\mathbb{R}^+)} \leq C (\|F_0\|_{L^\infty(\mathbb{R}^+)} + \|x \partial_x F_0\|_{L^0_0(\mathbb{R}^+)} + C \int_0^t (\|G\|_{L^\infty(\mathbb{R}^+)} + \|x \partial_x G\|_{L^0_0(\mathbb{R}^+)} )d\tau, \quad (A.7)
\]
where \( C \) is a constant independent of the parameter \( \epsilon. \)

**Proof**  First of all, let us consider the heat equation
\[
\partial_t H(t, x) - \epsilon \partial_x^2 H(t, x) = 0, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^+; \quad (A.8)
\]
with the initial data and boundary condition
\[
H(t, x)|_{t=0} = H_0(x), \quad x \in \mathbb{R}^+, \quad H(t, x)|_{x=0} = 0, \quad t \in \mathbb{R}^+.
\]
In order to transform the problem (A.8) into a problem in the whole space, let us define \( \tilde{H}(t, x) \) by
\[
\tilde{H}(t, x) = H(t, x), \quad x > 0; \quad \tilde{H}(t, x) = -H(t, -x), \quad x < 0,
\]
and define the initial data \( \tilde{H}_0(x) \) by
\[
\tilde{H}_0(x) = H_0(x), \quad x > 0; \quad \tilde{H}_0(x) = -H_0(-x), \quad x < 0.
\]
It is easy to justify that the function \( \tilde{H}(t, x) \) solves the following evolution equation
\[
\partial_t \tilde{H}(t, x) - \epsilon \partial_x^2 \tilde{H}(t, x) = 0, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}; \quad \tilde{H}(t, x)|_{t=0} = \tilde{H}_0(x), \quad x \in \mathbb{R}. \quad (A.9)
\]
Define \( S(t, x) = \frac{1}{\sqrt{4\pi \epsilon t}} e^{-\frac{|x|^2}{4\epsilon t}}, \) then the solution of evolution (A.9) can be expressed as
\[
\tilde{H}(t, x) = \int_{\mathbb{R}} \tilde{H}_0(\xi) S(t, x - \xi) d\xi, \quad (A.10)
\]
which implies directly
\[
x \partial_x \tilde{H}(t, x) = \int_{\mathbb{R}} \tilde{H}_0(\xi) x \partial_x S(t, x - \xi) d\xi.
\]
In view of the relation \( x \partial_x S(t, x - \xi) = (x - \xi) \partial_\xi S(t, x - \xi) + \xi \partial_x S(t, x - \xi), \) we get
\[
x \partial_x \tilde{H}(t, x) = \int_{\mathbb{R}} \tilde{H}_0(\xi) (x - \xi) \partial_\xi S(t, x - \xi) d\xi + \int_{\mathbb{R}} \tilde{H}_0(\xi) \xi \partial_x S(t, x - \xi) d\xi.
\]
Due to \( \int_{\mathbb{R}} |(x - \xi) \partial_\xi S(t, x - \xi)| d\xi \leq C, \) it follows
\[
\left| \int_{\mathbb{R}} \tilde{H}_0(\xi) (x - \xi) \partial_\xi S(t, x - \xi) d\xi \right| \leq C \|\tilde{H}_0\|_{L^\infty_0(\mathbb{R})}.
\]
Using the equality \( \partial_\xi S(t, x - \xi) = -\partial_\xi S(t, x - \xi), \) the integration by part yields directly
\[
\left| \int_{\mathbb{R}} \tilde{H}_0(\xi) \xi \partial_x S(t, x - \xi) d\xi \right| \leq C (\|\tilde{H}_0\|_{L^\infty_0(\mathbb{R})} + \|x \partial_x \tilde{H}_0\|_{L^\infty_0(\mathbb{R})}).
\]
and hence, we get
\[ \|x \partial_x H(t, x)\|_{L_0^\infty(\mathbb{R}^+)} \leq \|x \partial_x \tilde{H}(t, x)\|_{L_0^\infty(\mathbb{R})} \leq C\|\tilde{H}_0, x \partial_x \tilde{H}_0\|_{L_0^\infty(\mathbb{R})} \]
\[ \leq C(\|H_0\|_{L_0^\infty(\mathbb{R}^+)} + \|x \partial_x H_0\|_{L_0^\infty(\mathbb{R}^+)}) . \]
This, along with representation (A.10) and the well-known Duhamel formula, we complete the proof of this lemma.

Appendix B: Almost equivalence of weighted norms

In this subsection we will use the quantity \( h^\epsilon_m \) in weighted norm, \( h^\epsilon \) and its derivatives in \( L^\infty \) norm to control the quantities \( Z_t^\alpha \) \( h^\epsilon \) and \( Z_t^\alpha \psi^\epsilon \) in weighted norm. To derive these estimates, we shall apply the Lemma A.1, which has been introduced previously in Appendix A.

**Lemma B.1** Let the stream function \( \psi^\epsilon(t, x, y) \) satisfies \( \partial_y \psi^\epsilon = h^\epsilon, \partial_x \psi^\epsilon = -g^\epsilon, \psi^\epsilon|_{y=0} = 0 \). There exists a constant \( \delta \in (0, 1) \), such that \( h^\epsilon(t, x, y) = 1 \geq \delta, \forall (t, x, y) \in [0, T] \times \Omega \).

Then, for \( l \geq 1 \) and \( |\alpha_1| = m \), we have the following estimates:

\[ \| Z_t^\alpha \psi^\epsilon \|_{L^2_1(\Omega)} = 2 \frac{\delta^{-1}}{2l - 1} \| h^\epsilon_m \|_{L^2_1(\Omega)}, \]
\[ \| Z_t^\alpha h^\epsilon \|_{L^2_1(\Omega)} = 2 \frac{\delta^{-1}}{2l - 1} \| \partial_x h^\epsilon \|_{L^\infty_1(\Omega)} \| h^\epsilon_m \|_{L^2_1(\Omega)}, \]
\[ \| \partial_x Z_t^\alpha h^\epsilon \|_{L^2_1(\Omega)} = 2 \frac{\delta^{-1}}{2l - 1} \| \partial_x h^\epsilon_m \|_{L^2_1(\Omega)} + \delta \frac{\delta^{-1}}{2l - 1} \| \partial_x h^\epsilon \|_{L^\infty_1(\Omega)} \| h^\epsilon_m \|_{L^2_1(\Omega)}, \]
\[ \| \partial_y Z_t^\alpha h^\epsilon \|_{L^2_1(\Omega)} = 2 \frac{\delta^{-1}}{2l - 1} \| \partial_y h^\epsilon_m \|_{L^2_1(\Omega)} + C_l \delta^{-1} \| \partial_y h^\epsilon \|_{L^\infty_1(\Omega)} \| h^\epsilon_m \|_{L^2_1(\Omega)}, \]

where the constant \( C_l \) depends only on \( l \).

**Proof** (i) By virtue of the definition \( h^\epsilon_m = Z_t^\alpha h^\epsilon - \frac{\partial_x h^\epsilon}{h^\epsilon + 1} Z_t^\alpha \psi^\epsilon \), it is easy to obtain \( h^\epsilon_m = (h^\epsilon + 1) \partial_y \left( \frac{Z_t^\alpha \psi^\epsilon}{h^\epsilon + 1} \right) \). Integrating over \([0, y] \) and applying the boundary condition \( \psi^\epsilon|_{y=0} = 0 \), we have

\[ \frac{Z_t^\alpha \psi^\epsilon}{h^\epsilon + 1} = \int_0^y \frac{h^\epsilon_m}{h^\epsilon + 1} d\xi, \]

and along with the Hardy inequality (A.1), yields directly

\[ \| \frac{Z_t^\alpha \psi^\epsilon}{h^\epsilon + 1} \|_{L^2_1(\Omega)} \leq \frac{2}{2l - 1} \| h^\epsilon_m \|_{L^2_1(\Omega)} \]
\[ \leq 2 \frac{\delta^{-1}}{2l - 1} \| h^\epsilon_m \|_{L^2_1(\Omega)}, \]

where we have used the fact \( h^\epsilon + 1 \geq \delta \) in the last inequality.

(ii) In view of the relation \( Z_t^\alpha \psi^\epsilon = h^\epsilon_m + \frac{\partial_x h^\epsilon}{h^\epsilon + 1} Z_t^\alpha \psi^\epsilon \), we get

\[ \| Z_t^\alpha h^\epsilon \|_{L^2_1(\Omega)} \leq \| h^\epsilon_m \|_{L^2_1(\Omega)} + \| \partial_y h^\epsilon \|_{L^\infty_1(\Omega)} \left( \frac{Z_t^\alpha \psi^\epsilon}{h^\epsilon + 1} \right) \]
\[ \| \frac{Z_t^\alpha \psi^\epsilon}{h^\epsilon + 1} \|_{L^2_1(\Omega)}, \]
which, together with estimate (B.6), yields directly
\[
\|Z_{\tau}^{\alpha_1}h^\epsilon\|_{L_2^0(\Omega)} \leq \|h_m^\epsilon\|_{L_2^0(\Omega)} + \frac{2\delta^{-1}}{2l-1}\|\partial_\gamma h^\epsilon\|_{L_0^\infty(\Omega)}\|h_m^\epsilon\|_{L_2^0(\Omega)},
\]
(B.7)

(iii) Differentiating the equality \(Z_{\tau}^{\alpha_1} \psi^\epsilon = (h^\epsilon + 1) \int_0^y \frac{h_m^\epsilon}{h^\epsilon + 1}d\xi\) with respect to \(x\) variable, we find
\[
\partial_x Z_{\tau}^{\alpha_1} \psi^\epsilon = \partial_x h^\epsilon \int_0^y \frac{h_m^\epsilon}{h^\epsilon + 1}d\xi + (h^\epsilon + 1) \int_0^y \frac{\partial_x h_m^\epsilon}{h^\epsilon + 1}d\xi - (h^\epsilon + 1) \int_0^y \frac{h_m^\epsilon \partial_x h^\epsilon}{(h^\epsilon + 1)^2}d\xi,
\]
which, implies that
\[
\frac{\partial_x Z_{\tau}^{\alpha_1} \psi^\epsilon}{h^\epsilon + 1} = \frac{\partial_x h^\epsilon}{h^\epsilon + 1} + \int_0^y \frac{h_m^\epsilon}{h^\epsilon + 1}d\xi + \int_0^y \frac{\partial_x h_m^\epsilon}{h^\epsilon + 1}d\xi - \int_0^y \frac{h_m^\epsilon \partial_x h^\epsilon}{(h^\epsilon + 1)^2}d\xi,
\]
and hence, we apply the Hardy inequality (A.1) and \(h^\epsilon + 1 \geq \delta\) to get
\[
\left\|\frac{\partial_x Z_{\tau}^{\alpha_1} \psi^\epsilon}{h^\epsilon + 1}\right\|_{L_2^0(\Omega)} \leq \frac{4}{2l-1}\left\|\frac{\partial_x h^\epsilon}{h^\epsilon + 1}\right\|_{L_0^\infty(\Omega)}\|h_m^\epsilon\|_{L_2^0(\Omega)} + \frac{2\delta^{-2}}{2l-1}\|\partial_\gamma h^\epsilon\|_{L_0^\infty(\Omega)}\|h_m^\epsilon\|_{L_2^0(\Omega)},
\]
(B.8)

(iv) Differentiating the equality \(Z_{\tau}^{\alpha_1}h^\epsilon = h_m^\epsilon + \frac{\partial_y h^\epsilon}{h^\epsilon + 1}Z_{\tau}^{\alpha_1} \psi^\epsilon\) with the \(y\) variable, it follows
\[
\partial_y Z_{\tau}^{\alpha_1}h^\epsilon = \partial_y h_m^\epsilon + \frac{\partial_y h^\epsilon}{h^\epsilon + 1} Z_{\tau}^{\alpha_1} \psi^\epsilon + \partial_y h^\epsilon \partial_y \left(\frac{Z_{\tau}^{\alpha_1} \psi^\epsilon}{h^\epsilon + 1}\right),
\]
which, together with the relation (B.5), yields
\[
\partial_y Z_{\tau}^{\alpha_1}h^\epsilon = \partial_y h_m^\epsilon + \frac{\partial_y h^\epsilon}{h^\epsilon + 1} \int_0^y \frac{h_m^\epsilon}{h^\epsilon + 1}d\xi + \eta_h h_m^\epsilon
\]
\[
= \partial_y h_m^\epsilon + Z_2 \eta_h \int_0^y \frac{h_m^\epsilon}{h^\epsilon + 1}d\xi + \eta_h h_m^\epsilon,
\]
where \(\varphi(y) = \frac{y}{1+y}\) and \(\eta_h = \frac{\partial_y h^\epsilon}{h^\epsilon + 1}\). The application of Hardy inequality (A.1) and \(h^\epsilon + 1 \geq \delta\) yields
\[
\left\|\partial_y Z_{\tau}^{\alpha_1}h^\epsilon\right\|_{L_2^0(\Omega)} \leq \|\partial_y h_m^\epsilon\|_{L_2^0(\Omega)} + \frac{2\delta^{-1}}{2l-1}\|Z_2 \partial_y h^\epsilon\|_{L_0^\infty(\Omega)}\|h_m^\epsilon\|_{L_2^0(\Omega)} + \frac{\delta^{-1}}{2l-1}\|\partial_\gamma h^\epsilon\|_{L_0^\infty(\Omega)}\|h_m^\epsilon\|_{L_2^0(\Omega)} + \frac{\delta^{-1}}{2l-1}\|Z_2 \partial_y h^\epsilon\|_{L_0^\infty(\Omega)}\|h_m^\epsilon\|_{L_2^0(\Omega)},
\]
(B.9)

Therefore, the estimates (B.6)–(B.9) imply the estimates (B.1)–(B.4).

Let us define
\[
Y_{\gamma,1}(t) := 1 + \|Q^\epsilon, u^\epsilon, h^\epsilon\|_{L_1^0} + \|\partial_\gamma Q^\epsilon, u^\epsilon, h^\epsilon\|_{L_1^0} + \|\partial_\gamma Q^\epsilon\|_{L_1^0},
\]
and hence we will establish the following almost equivalent relation.
Lemma B.2 Let \((q^e, u^e, v^e, h^e, g^e)\) be sufficiently smooth solution, defined on \([0, T^e]\), to the regularized MHD boundary layer equations (3.2)-(3.3). There exists a constant \(\delta \in (0, 1)\), such that \(h^e(t, x, y) + 1 \geq \delta, \forall (t, x, y) \in [0, T] \times \Omega\). Then, for \(m \geq 4\) and \(l \geq 1\), it holds true

\[
\Theta_{m,l}(t) \leq C \| (\rho_0, u_{10}, h_{10}) \|^4 + C \| (\rho_0, u_{10}, h_{10}) \|^4 + C \| h_{10} \|^2, \quad (B.11)
\]

and

\[
N_{m,l}(t) \leq C \| (\rho_0, u_{10}, h_{10}) \|^4 + C \| (\rho_0, u_{10}, h_{10}) \|^4 + C \| h_{10} \|^2, \quad (B.12)
\]

where \(\Theta_{m,l}(t)\) and \(N_{m,l}(t)\) are defined in (3.5) and (3.114) respectively.

Proof By virtue of the definition \(\tilde{q}_m^e = Z_{\tau}^{\alpha_1} q^e - \frac{\partial_y q^e}{h^e + 1} Z_{\tau}^{\alpha_1} \psi^e\) and the estimate (B.1), we find

\[
\| Z_{\tau}^{\alpha_1} q^e \|^2_{L^2(\Omega)} \leq \| \tilde{q}_m^e \|^2_{L^2(\Omega)} + \| \partial_y q^e \|^2_{L^2(\Omega)} \leq \| \tilde{q}_m^e \|^2_{L^2(\Omega)} + C \delta^{-2} (1 + \| \partial_y q^e \|^2_{L^2(\Omega)}).
\]

Similarly, we can obtain for \(|\alpha_1| = m\) that

\[
\| Z_{\tau}^{\alpha_1} (u^e, h^e) \|^2_{L^2(\Omega)} \leq \| (u_m^e, h_m^e) \|^2_{L^2(\Omega)} + C \delta^{-2} (1 + \| \partial_y (u^e, h^e) \|^2_{L^2(\Omega)}).
\]

The combination of the above two estimates yields directly

\[
\| Z_{\tau}^{\alpha_1} (q^e, u^e, h^e) \|^2_{L^2(\Omega)} \leq C \delta^{-2} (1 + \| \partial_y (q^e, u^e, h^e) \|^2_{L^2(\Omega)}), \quad (B.13)
\]

and hence, we have for \(m \geq 4, l \geq 1\)

\[
\| Z_{\tau}^{\alpha_1} (q^e, u^e, h^e) \|^2_{L^2(\Omega)} \leq C (\| u_m^e \|^4_{L^\infty(\Omega)} + \| h_m^e \|^4_{L^\infty(\Omega)} + \| \partial_y q^e \|^4_{L^\infty(\Omega)}) \quad (B.14)
\]

Due to the definition of \(X_{m,l}(t)\) and \(Y_{m,l}(t)\) in (3.56) and (B.10) respectively, we get from (B.14) that

\[
Y_{m,l}(t) \leq C (\| \rho_0, u_{10}, h_{10} \|^2 + C \| \rho_0, u_{10}, h_{10} \|^2 + C \delta^{-4} (1 + X_{m,l}(t)). \quad (B.15)
\]

On the other hand, by virtue of the definition \(q_m^e(t)\) and the estimate (B.1), we find

\[
\| q_m^e(t) \|^2_{L^2(\Omega)} \leq \| Z_{\tau}^{\alpha_1} q^e(t) \|^2_{L^2(\Omega)} + \| \partial_y q^e(t) \|^2_{L^\infty(\Omega)} + \| \partial_y \psi^e \|^2_{L^\infty(\Omega)} \leq \| Z_{\tau}^{\alpha_1} q^e(t) \|^2_{L^2(\Omega)} + C \delta^{-2} (1 + \| \partial_y q^e(t) \|^2_{L^\infty(\Omega)} + \| Z_{\tau}^{\alpha_1} h^e(t) \|^2_{L^2(\Omega)}),
\]

and hence, we also have

\[
\| (u_m^e, h_m^e(t)) \|^2_{L^2(\Omega)} \leq \| Z_{\tau}^{\alpha_1} (u^e, h^e(t)) \|^2_{L^2(\Omega)} + C \delta^{-2} (1 + \| \partial_y (u^e, h^e(t)) \|^2_{L^\infty(\Omega)} + \| Z_{\tau}^{\alpha_1} h^e(t) \|^2_{L^2(\Omega)}).
\]

Then, the combination of the above estimates yields directly
\[
\| (\varrho_m^e, u_m^e, h_m^e)(t) \|^2_{L^2_t} \leq C \| (\varrho_0, u_{10}, h_{10}) \|^2_{B^m_t} + C t \| (\varrho_0, u_{10}, h_{10}) \|^2_{B^m_t} + C \delta^{-4} (1 + Y_{m,l}^6(t)), \tag{B.16}
\]

where \( m \geq 4, l \geq 1 \). According to the definition of \( X_{m,l}(t) \) and \( Y_{m,l}(t) \), we get from (B.16) that

\[
X_{m,l}(t) \leq C \| (\varrho_0, u_{10}, h_{10}) \|^2_{B^m_t} + C t \| (\varrho_0, u_{10}, h_{10}) \|^2_{B^m_t} + C \delta^{-4} (1 + X_{m,l}^6(t)). \tag{B.17}
\]

Next, by virtue of the definition \( \varrho_m^e(t) \) and estimate (3.89), we find

\[
\| \partial_x Z_{\tau} \varrho_m^e \|^2_{L^2_t(\Omega)} \leq \| \partial_x \varrho_m^e \|^2_{L^2_t(\Omega)} + C \delta^{-2} \| \partial_y \varrho_m^e \|^2_{L^2_t(\Omega)} \| \partial_x h_m^e \|^2_{L^2_t(\Omega)} + C \delta^{-4} (\| \partial_{xy} \varrho_m^e \|^2_{L^2_t(\Omega)} + \| \partial_y \varrho_m^e \|^2_{L^2_t(\Omega)} \| \partial_x h_m^e \|^2_{L^2_t(\Omega)} \leq \| \partial_x \varrho_m^e \|^2_{L^2_t(\Omega)} + C \delta^{-4} X_{m,l}(t) \| \partial_x h_m^e \|^2_{L^2_t(\Omega)} + C \delta^{-4} (1 + X_{m,l}^3(t)).
\]

Similarly, by routine checking, we may conclude that

\[
\| \partial_x Z_{\tau} (u^e, h^e) \|^2_{L^2_t(\Omega)} \leq C \| (\varrho_0, u_{10}, h_{10}) \|^2_{B^m_t} + t \| (\varrho_0, u_{10}, h_{10}) \|^2_{B^m_t} + C \delta^{-8} (1 + X_{m,l}^6(t))
\]

\[
+ X_{m,l}(t) \| \partial_x h_m^e \|^2_{L^2_t(\Omega)} + \| \partial_x (u^e, h^e) \|^2_{L^2_t(\Omega)},
\]

for \( m \geq 5, l \geq 1 \), and hence it follows

\[
\epsilon \| \partial_x (Q^e, u^e, h^e) \|^2_{L_t^2} \leq C \| (\varrho_0, u_{10}, h_{10}) \|^2_{B^m_t} + t \| (\varrho_0, u_{10}, h_{10}) \|^2_{B^m_t} + X_{m,l}(t) D_{X}^{m,l}(t) \tag{B.18}
\]

\[
+ C \| (\varrho_0, u_{10}, h_{10}) \|^2_{B^m_t} + t \| (\varrho_0, u_{10}, h_{10}) \|^2_{B^m_t} + C \delta^{-8} (1 + X_{m,l}^6(t)),
\]

where \( D_X^{m,l}(t) \) is defined in (3.115). By virtue of the definition \( \varrho_m^e(t) \) and estimate (3.86), we get

\[
\| \partial_x \varrho_m^e \|^2_{L^2_t(\Omega)} \leq \| \partial_x Z_{\tau} \varrho_m^e \|^2_{L^2_t(\Omega)} + C \delta^{-2} \| \partial_y \varrho_m^e \|^2_{L^2_t(\Omega)} \| \partial_x h_m^e \|^2_{L^2_t(\Omega)} + C \delta^{-4} (\| \partial_{xy} \varrho_m^e \|^2_{L^2_t(\Omega)} + \| \partial_y \varrho_m^e \|^2_{L^2_t(\Omega)} \| \partial_x h_m^e \|^2_{L^2_t(\Omega)} \leq \| \partial_x Z_{\tau} \varrho_m^e \|^2_{L^2_t(\Omega)} + C \delta^{-2} X_{m,l}(t) \| \partial_x h_m^e \|^2_{L^2_t(\Omega)} + C \delta^{-4} (1 + X_{m,l}^3(t)).
\]

Similarly, by routine checking, we may conclude that

\[
\| \partial_x (u^e, h^e) \|^2_{L^2_t(\Omega)} \leq C \delta^{-2} (1 + \| (\varrho_0, u_{10}, h_{10}) \|_{B^m_t} + t \| (\varrho_0, u_{10}, h_{10}) \|_{B^m_t} + Y_{m,l}(t) \| \partial_x h_m^e \|^2_{L^2_t(\Omega)}
\]

\[
+ C \| (\varrho_0, u_{10}, h_{10}) \|^2_{B^m_t} + t \| (\varrho_0, u_{10}, h_{10}) \|^2_{B^m_t} + C \delta^{-8} (1 + Y_{m,l}^6(t))
\]

\[
+ \| \partial_x Z_{\tau} (u^e, h^e) \|^2_{L^2_t(\Omega)},
\]

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and hence, it follows

\[ D^m_{x}(t) \leq C \delta^{-2}(1 + \| (\rho_0, u_{10}, h_{10}) \|^2_{B^m_l} + t \| (\rho_0, u_{10}, h_{10}) \|^2_{B^m_l}) + Y^3_m(t) e \| \partial_x (\mathcal{G}^{e} , u^{e} , h^{e}) \|^2_{L^m_l} + C \| (\rho_0, u_{10}, h_{10}) \|^2_{B^m_l} + Y^6_{m,l}(t)) \]

(B.19)

where $D^m_{x}(t)$ is defined in (3.115). Similarly, we can justify the estimates

\[ \| \partial_y (\sqrt{\mathcal{G}^{e} , \sqrt{\mu} u^{e} , \sqrt{\kappa} h^{e}) \|^2_{B^m_l} \leq C \| (\rho_0, u_{10}, h_{10}) \|^4_{B^m_l} + t \| (\rho_0, u_{10}, h_{10}) \|^4_{B^m_l} \]

(B.20)

and

\[ D^m_{y}(t) \leq C \| (\rho_0, u_{10}, h_{10}) \|^4_{B^m_l} + t \| (\rho_0, u_{10}, h_{10}) \|^4_{B^m_l} \]

(B.21)

Therefore, the combination of estimates (B.15), (B.17), (B.18), (B.19), (B.20) and (B.21) can establish the estimates (B.11) and (B.12).

\[ \square \]

References

1. Abidi, H., Gui, G.L., Zhang, P.: On the decay and stability of global solutions to the 3D inhomogeneous Navier–Stokes equations. Commun. Pure Appl. Math. 64(6), 832–881 (2011)
2. Abidi, H., Paicu, M.: Global existence for the magnetohydrodynamic system in critical spaces. Proc. R. Soc. Edinb. Sect. A 138(3), 447–476 (2008)
3. Alexander, R., Wang, Y.G., Xu, C.J., Yang, T.: Well-posedness of the Prandtl equation in Sobolev spaces. J. Am. Math. Soc. 28(3), 745–784 (2015)
4. Alfvén, H.: Existence of electromagnetic-hydrodynamic waves. Nature 150, 405–406 (1942)
5. Beirão da Veiga, H.: Vorticity and regularity for flows under the Navier boundary condition. Commun. Pure Appl. Anal. 5(4), 907–916 (2006)
6. Beirão da Veiga, H., Crispo, F.: Concerning the $W^{k,p}$-inviscid limit for 3-d flows under a slip boundary condition. J. Math. Fluid Mech. 13(1), 117–135 (2011)
7. Chen, D.X., Wang, Y.X., Zhang, Z.F.: Well-posedness of the linearized Prandtl equation around a non-monotonic shear flow. Ann. Inst. H. Poincaré Anal. Non Linéaire 35(4), 1119–1142 (2018)
8. Choe, H.J., Kim, H.: Strong solutions of the Navier–Stokes equations for nonhomogeneous incompressible fluids. Commun. Part. Differ. Equ. 28(5–6), 1183–1201 (2003)
9. Constantin, P.: Note on loss of regularity for solutions of the 3-D incompressible Euler and related equations. Commun. Math. Phys. 104(2), 311–326 (1986)
10. Constantin, P., Foias, C.: Navier–Stokes Equation. University of Chicago Press, Chicago (1988)
11. Desjardins, B., Le Bris, C.: Remarks on a nonhomogeneous model of magnetohydrodynamics. Differ. Integral Equ. 11(3), 377–394 (1998)
12. Gérard-Varet, D., Prestipino, M.: Formal derivation and stability analysis of boundary layer models in MHD. Z. Angew. Math. Phys. 68(3), 76 (2017)
13. Gerbeau, J.F., Le Bris, C.: Existence of solution for a density-dependent magnetohydrodynamic equation. Adv. Differ. Equ. 2(3), 427–452 (1997)
14. Gie, G.M., Jung, C.Y., Temam, R.: Recent progresses in boundary layer theory. Discrete Cont. Dyn. Syst. 36(5), 2521–2586 (2016)
15. Grenier, E., Guo, Y., Nguyen, T.: Spectral stability of Prandtl boundary layers: an overview. Analysis (Berlin) 35(4), 343–355 (2015)
16. Grenier, E., Guo, Y., Nguyen, T.: Spectral instability of characteristic boundary layer flows. Duke Math. J. 165(16), 3085–3146 (2016)
17. Gui, G.L.: Global well-posedness of the two-dimensional incompressible magnetohydrodynamics system with variable density and electrical conductivity. J. Funct. Anal. 267(5), 1488–1539 (2014)
18. Guo, Y., Nguyen, T.: A note on Prandtl boundary layers. Commun. Pure Appl. Math. 64(10), 1416–1438 (2011)
19. Guo, Y., Nguyen, T.: Prandtl boundary layer expansions of steady Navier–Stokes flows over a moving plate. Ann. PDE 3(1), 10 (2017)
20. Guo, Y., Iyer, S.: Validity of steady Prandtl layer expansions. arXiv:1805.05891
21. Huang, Y.T., Liu, C.J., Yang, T.: Local-in-time well-posedness for compressible MHD boundary layer. J. Differ. Equ. 266(6), 2978–3013 (2019)
22. Ignatova, M., Vicol, V.: Almost global existence for the Prandtl boundary layer equations. Arch. Ration. Mech. Anal. 220(2), 809–848 (2016)
23. Kato, T.: Nonstationary flows of viscous and ideal fluids in $\mathbb{R}^3$. J. Funct. Anal. 9, 296–305 (1972)
24. Kazhikov, A.V.: Solvability of the initial-boundary value problem for the equations of the motion of an inhomogeneous viscous incompressible fluid. Dokl. Akad. Nauk SSSR 216, 1008–1010 (1974). (in Russian)
25. Li, W.X., Wu, D., Xu, C.J.: Gevrey class smoothing effect for the Prandtl equation. SIAM J. Math. Anal. 48(3), 1672–1726 (2016)
26. Li, W.X., Yang, T.: Well-posedness in Gevrey function spaces for the Prandtl equations with non-degenerate critical points. J. Eur. Math. Soc. 22(3), 717–775 (2020)
27. Lin, X.Y., Zhang, T.: Almost global existence for 2D magnetohydrodynamics boundary layer system. Math. Methods Appl. Sci. 41(17), 7530–7553 (2018)
28. Liu, C.J., Wang, D.H., Xie, F., Yang, T.: Magnetic effects on the solvability of 2D MHD boundary layer equations without resistivity in Sobolev spaces. J. Funct. Anal. 279(7), 45 (2020)
29. Liu, C.J., Xie, F., Yang, T.: A note on the ill-posedness of shear flow for the MHD boundary layer equations. Sci. China Math. 61(11), 2065–2078 (2018)
30. Liu, C.J., Yang, T.: Ill-posedness of the Prandtl equations in Sobolev spaces around a shear flow with general decay. J. Math. Pures Appl. (9) 108(2), 150–162 (2017)
31. Liu, C.J., Xie, F., Yang, T.: MHD boundary layers theory in Sobolev spaces without monotonicity. I. Well-posedness theory. Commun. Pure Appl. Math. 72(1), 63–121 (2019)
32. Liu, C.J., Xie, F., Yang, T.: Justification of Prandtl Ansatz for MHD boundary layer. SIAM J. Math. Anal. 51(3), 2748–2791 (2019)
33. Lions, P.L.: Mathematical Topics in Fluid Mechanics, vol. 1, Incompressible Models Oxford Lecture Series in Mathematics and Applications, vol. 3. Oxford University Press, New York (1996)
34. Masmoudi, N.: Remarks about the inviscid limit of the Navier–Stokes system. Commun. Math. Phys. 270(3), 777–788 (2007)
35. Masmoudi, N., Rousset, F.: Uniform Regularity for the Navier–Stokes equation with Navier boundary condition. Arch. Ration. Mech. Anal. 203(2), 529–575 (2012)
36. Masmoudi, N., Wong, T.K.: Local-in-time existence and uniqueness of solutions to the Prandtl equations by energy methods. Commun. Pure Appl. Math. 68(10), 1683–1741 (2015)
37. Oleinik, O.A.: On the system of Prandtl equations in boundary-layer theory. Dokl. Akad. Nauk SSSR 150, 28–31 (1963)
38. Oleinik, O.A.: On the mathematical theory of boundary layer for an unsteady flow of incompressible fluid, Prikl. Mat. Meh. 30, 801–821 (Russian); translated as J. Appl. Math. Mech. 30, 951–974 (1966)
39. Oleinik, O.A., Samokhin, V.N.: Mathematical Models in Boundary Layers Theory, Applied Mathematics and Mathematical Computation, vol. 15. Chapman & Hall/CRC, Boca Raton (1999)
40. Prandtl, L.: Uber flüssigkeits-bewegung bei sehr kleiner reibung. Verhandlungen des III. Internationlen Mathematiker Kongresses, Heidelberg, Teubner, Leipzig, pp. 484–491 (1904)
41. Iyer, S.: Steady Prandtl boundary layer expansions over a rotating disk. Arch. Ration. Mech. Anal. 224(2), 421–469 (2017)
42. Iyer, S.: Steady Prandtl layers over a moving boundary: nonshear Euler flows. SIAM J. Math. Anal. 51(3), 1657–1695 (2019)
43. Sammartino, M., Caflisch, R.E.: Zero viscosity limit for analytic solutions, of the Navier–Stokes equation on a half-space. I. Existence for Euler and Prandtl equations. Commun. Math. Phys. 192(2), 433–461 (1998)
44. Sammartino, M., Caflisch, R.E.: Zero viscosity limit for analytic solutions of the Navier–Stokes equation on a half-space. II. Construction of the Navier–Stokes solution. Commun. Math. Phys. 192(2), 463–491 (1998)
45. Schlichting, H., Gersten, K.: Unsteady Boundary Layers. In: Boundary-Layer Theory. Springer, Berlin, Heidelberg (2017)
46. Simon, J.: Nonhomogeneous viscous incompressible fluids: existence of velocity, density, and pressure. SIAM J. Math. Anal. 21(5), 1093–1117 (1990)
47. Wang, Y., Xin, Z.P., Yong, Y.: Uniform regularity and vanishing viscosity limit for the compressible Navier–Stokes with general Navier-Slip boundary conditions in three-dimensional domains. SIAM J. Math. Anal. 47(6), 4123–4191 (2015)
48. Xiao, Y.L., Xin, Z.P.: On 3D Lagrangian Navier–Stokes α model with a class of vorticity-slip boundary conditions. J. Math. Fluid Mech. 15(2), 215–247 (2013)
49. Xie, F., Yang, T.: Global-in-time stability of 2D MHD boundary layer in the Prandtl-Hartmann regime. SIAM J. Math. Anal. 50(6), 5749–5760 (2018)
50. Xie, F., Yang, T.: Lifespan of solutions to MHD boundary layer equations with analytic perturbation of general shear flow. Acta Math. Appl. Sin. Engl. Ser. 35(1), 209–229 (2019)
51. Xu, C.J., Zhang, X.: Long time well-posedness of Prandtl equations in Sobolev space. J. Differ. Equ. 263(12), 8749–8803 (2017)
52. Zhang, P., Zhang, Z.F.: Long time well-posedness of Prandtl system with small and analytic initial data. J. Funct. Anal. 270(7), 2591–2615 (2016)

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