ON THE GL(⟨n⟩)-MODULE STRUCTURE OF LIE NILPOTENT ASSOCIATIVE RELATIVELY FREE ALGEBRAS

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ABSTRACT. Let $K⟨X⟩$ denote the free associative algebra generated by a set $X = \{x_1, \ldots, x_n\}$ over a field $K$ of characteristic 0. Let $I_p$, for $p \geq 2$, denote the two-sided ideal in $K⟨X⟩$ generated by all commutators of the form $[u_1, \ldots, u_p]$, where $u_1, \ldots, u_p \in K⟨X⟩$. We discuss the $GL(n, K)$-module structure of the quotient $K⟨X⟩/I_p+1$ for all $p \geq 1$ under the standard diagonal action. We give a bound on the values of partitions $\lambda$ such that the irreducible $GL(n, K)$-module $V_\lambda$ appears in the decomposition of $K⟨X⟩/I_p+1$ as a $GL(n, K)$-module. As an application, we take $K = \mathbb{C}$ and we consider the algebra of invariants $(\mathbb{C}⟨X⟩/I_p+1)^G$ for $G = SL(n, \mathbb{C})$, $O(n, \mathbb{C})$, $SO(n, \mathbb{C})$, or $Sp(2s, \mathbb{C})$ (for $n = 2s$). By a theorem of Domokos and Drensky, $(\mathbb{C}⟨X⟩/I_p+1)^G$ is finitely generated. We give an upper bound on the degree of generators of $(\mathbb{C}⟨X⟩/I_p+1)^G$ in a minimal generating set. In a similar way, we consider also the algebra of invariants $(\mathbb{C}⟨X⟩/I_p+1)^G$, where $G = UT(n, \mathbb{C})$, and give an upper bound on the degree of generators in a minimal generating set. These results provide useful information about the invariants in $\mathbb{C}⟨X⟩^G$ from the point of view of Classical Invariant Theory. In particular, for all $G$ as above we give a criterion when a $G$-invariant of $\mathbb{C}⟨X⟩$ belongs to $I_p$.

1. INTRODUCTION

Let $V$ be a finite dimensional vector space over a field $K$ of characteristic zero and let $X = \{x_1, \ldots, x_n\}$ be a basis for $V$. We consider the free associative algebra $K⟨X⟩$ generated by $X$, which can be naturally identified with the tensor algebra $T(V)$ of $V$. We define inductively left-normed long commutators in $K⟨X⟩$ in the usual way: $[u_1, u_2] = u_1u_2 - u_2u_1$ and for $i \geq 3$, we have $[u_1, \ldots, u_i] = ([u_1, \ldots, u_{i-1}], u_i]$, where $u_1, \ldots, u_i$ are arbitrary elements from $K⟨X⟩$. The commutator $[u_1, \ldots, u_i]$ is called a commutator of length $i$ or simply an $i$-commutator. Furthermore, we call a commutator pure if it is of the form $[x_{i_1}, \ldots, x_{i_j}]$ for some $x_{i_1}, \ldots, x_{i_j} \in X$.

The lower central series of $K⟨X⟩$ is the descending filtration

$$L_1 \supseteq L_2 \supseteq \cdots \supseteq L_p \supseteq \cdots,$$

defined inductively by $L_1 = K⟨X⟩$ and $L_p = [L_{p-1}, K⟨X⟩]$ for $p \geq 2$. Then $L_p$ is the Lie ideal in $K⟨X⟩$ generated by all commutators of length $p$. For $p \geq 2$, let $I_p = K⟨X⟩ \cdot L_p$ denote the two-sided associative ideal in $K⟨X⟩$ generated by $L_p$. In other words, $I_p$ is the $T$-ideal generated by the commutator $[x_1, \ldots, x_p]$. Following the traditions in the theory of groups and Lie algebras we denote by $\mathfrak{N}_p$
The group $\text{GL}(n) := \text{GL}(V, K)$ acts on the vector space $V$ with basis $X$ and this action is extended in a natural way to the diagonal action of $\text{GL}(n)$ on $X$. Since for any $p$, $I_p$ is a $\text{GL}(n)$-submodule of $K(X)$, the algebra $F_n(\mathfrak{g}_p)$ is also a $\text{GL}(n)$-module. The $\text{GL}(n)$-module structure of $F_n(\mathfrak{g}_p)$ is known for $p = 1, 2, 3, 4$ (see [18] for $p = 3$ and [17] for $p = 4$). Our first goal is to give some results on the $\text{GL}(n)$-module structure of $F_n(\mathfrak{g}_p)$ for any $p$. More precisely, let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{Z}_{\geq 0}^n$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$ be a non-negative integer partition and let $V_\lambda$ denote the irreducible $\text{GL}(n)$-module corresponding to the partition $\lambda$. In Section 3 we give a bound on the values of $\lambda$ such that $V_\lambda$ appears in the decomposition of $F_n(\mathfrak{g}_p)$ as a $\text{GL}(n)$-module. The main results in this direction are Theorem 3.3 and Corollary 3.4. The proofs are based on several known results on inclusions of products of commutators, which are recalled in Section 2 and one new result on inclusions of products of commutators, which is stated in Section 8. In the end of Section 3 we consider separately the case $n = 3$. As an application of our results on the $\text{GL}(n)$-module structure of $F_n(\mathfrak{g}_p)$, in Section 4 we set $K = \mathbb{C}$ and consider the algebra of invariants $F_n(\mathfrak{g}_p)^G$ where $G$ is one of the classical complex groups $\text{SL}(n) := \text{SL}(V, \mathbb{C})$, $\text{O}(n) := \text{O}(V, \mathbb{C})$, $\text{SO}(n) := \text{SO}(V, \mathbb{C})$, and $\text{Sp}(2s) := \text{Sp}(V, \mathbb{C})$ (the last in the case $n = 2s$). By a theorem of Domokos and Drensky (3, 8), $F_n(\mathfrak{g}_p)^G$ is finitely generated. In Section 4 we give an upper bound on the degree of the generators of $F_n(\mathfrak{g}_p)^G$ in a minimal generating set (Corollary 4.4). The advantage of our upper bound is that it is very explicit. We also give a general form for the Hilbert series $H(F_n(\mathfrak{g}_p)^G, t)$, where $G$ is again one of $\text{SL}(n)$, $\text{O}(n)$, $\text{SO}(n)$, and $\text{Sp}(2s)$, and show that $F_n(\mathfrak{g}_p)^{\text{SL}(n)}$ and $F_n(\mathfrak{g}_p)^{\text{Sp}(2s)}$ are finite-dimensional algebras. In a similar way we consider also the algebra of invariants $F_n(\mathfrak{g}_p)^G$, where $G = \text{UT}(n) := \text{UT}(n, \mathbb{C})$ is the unitriangular group, i.e., the subgroup of $\text{GL}(n)$ consisting of upper triangular matrices with 1’s on the diagonal. It is known (see [7]) that $F_n(\mathfrak{g}_p)^{\text{UT}(n)}$ is finitely generated for any $p \geq 1$. In Section 4 again, we give an upper bound on the degree of the generators in a minimal generating set (Corollary 4.7). The results from Section 4 are then translated in the language of Classical Invariant Theory in Section 5. In particular, for each $G$ as above, we give a criterion when a $G$-invariant of $T(V)$ belongs to $I_p$.

2. Preliminaries

In this section we summarize several results on inclusions of products of commutators, which will be extensively used in the next sections. We state all results for a field of characteristic 0, even though most of the results hold for more general fields.

The oldest result in this direction is the following lemma due to Jennings from 1947.

**Lemma 2.1 ([14]).** Let $p \geq 3$. Then the product of any two $p$-commutators is an element of $I_{p+1}$.
Proof. Consider the product $[c_1, x][c_2, y]$ where $c_1$ and $c_2$ are arbitrary $p - 1$-commutators (i.e., $c_1$ and $c_2$ belong to $I_{p-1}$) and $x, y \in K \langle X \rangle$. We will show that $[c_1, x][c_2, y]$ can be expressed using elements of $I_{p+1}$. We consider the expression

$$
[c_2, c_1 y, x] = -[[c_1 y, c_2], x] = -[c_1[y, c_2], x] - [[c_1, c_2]y, x] = c_1[c_2, y, x] + [c_1, x][c_2, y] - [c_1, c_2][y, x] - [c_1, c_2, x]y.
$$

Hence,

$$
[c_1, x][c_2, y] = [c_2, c_1 y, x] + [c_1, c_2][y, x] + [c_1, c_2, x]y - c_1[c_2, y, x].
$$

Since $[c_1, c_2] \in L_{2p-2} \subseteq I_{2p-2}$ and for $p \geq 3, I_{2p-2} \subseteq I_{p+1}$ the lemma follows. \(\square\)

The next result improves considerably Lemma 2.1 and Lemma 2.2 (but not Lemma 2.3). It was first proved by Latyshev in 1965 and then was independently rediscovered by Gupta and Levin in 1983.

**Theorem 2.4** ([13], [12]). For all $m_1$ and $m_2$

$$
I_{m_1}I_{m_2} \subseteq I_{m_1 + m_2 - 2}.
$$

When one of the integers $m_1$ and $m_2$ is odd, an even stronger inclusion holds. The following theorem was proved several times with different restrictions on the characteristic of $K$. Here we cite only two of the sources. The most general result (i.e., for most general fields $K$) was proved in [5].

**Theorem 2.5** ([1], [5]). If $m_1$ or $m_2$ is odd then

$$
I_{m_1}I_{m_2} \subseteq I_{m_1 + m_2 - 1}.
$$
The proof of Theorem 2.5 given in [1] is based on the following statement, which is of interest of its own.

**Theorem 2.6 ([1])**

\[ [L_{m_1}, I_{m_2}] \subseteq L_{m_1+m_2}, \]

whenever \( m_2 \) is odd.

The next statement is derived from Theorems 2.4 and 2.5 and can be found in [3] and [5].

**Theorem 2.7.** For every \( l \geq 2 \) and for every choice of positive integers \( m_1, \ldots, m_l \) one of the following holds:

(i) If \( k \) indices among \( m_1, \ldots, m_l \) are odd with \( 0 \leq k < l \) then

\[ I_{m_1} \cdots I_{m_l} \subseteq I_{m_1+\cdots+m_l-2l+k+2}. \]

The number \( m_1 + \cdots + m_l - 2l + k + 2 \) is always even.

(ii) If all indices are odd then

\[ I_{m_1} \cdots I_{m_l} \subseteq I_{m_1+\cdots+m_l-l+1}. \]

The number \( m_1 + \cdots + m_l - l + 1 \) is always odd.

In the end of this section, let us fix \( n = 3 \), i.e., let us consider the free associative algebra \( K \langle x_1, x_2, x_3 \rangle \). Then the following result holds:

**Theorem 2.8 ([16]).** Consider the algebra \( K \langle x_1, x_2, x_3 \rangle \). Then

\[ I_{m_1} I_{m_2} \subseteq I_{m_1+m_2-1}, \]

for all integers \( m_1, m_2 \geq 2 \).

It is worth pointing out, that the restriction \( n = 3 \) in Theorem 2.8 is essential. It was proved in [4] that in general for even integers \( m_1 \) and \( m_2 \) we have

\[ I_{m_1} I_{m_2} \not\subseteq I_{m_1+m_2-1}. \]

3. The GL(\( n \))-module structure of \( F_n(\mathfrak{N}_p) \)

In this section we discuss the GL(\( n \))-module structure of \( F_n(\mathfrak{N}_p) \) for any \( p \geq 1 \). First we introduce some notations. Let \( B_n \) denote the subalgebra of \( K \langle X \rangle \) generated by all pure commutators \([x_{i_1}, \ldots, x_{i_r}]\) with \( r \geq 2 \). The elements of \( B_n \) are called proper polynomials. Clearly, \( B_n \) is a GL(\( n \))-submodule of \( K \langle X \rangle \). Let \( \varphi_p : K \langle X \rangle \to F_n(\mathfrak{N}_p) \) denote the natural surjective K-algebra homomorphism. Then \( \varphi_p(B_n) \) is a GL(\( n \))-submodule of \( F_n(\mathfrak{N}_p) \). By a theorem of Drensky ([9]), for any \( p \geq 1 \) we have the following decomposition as GL(\( n \))-modules:

\[
F_n(\mathfrak{N}_p) \cong_{GL(n)} S(V) \otimes \varphi_p(B_n),
\]

where again \( S(V) \) denotes the symmetric algebra of \( V \).

We start by proving the following lemma, which improves considerably Lemma 2.3 from Section 2.

**Lemma 3.1.** Let \( m_1 \) and \( m_2 \) be even integers. Let \([c_1, y] \in L_{m_1} \) and \([c_2, y] \in L_{m_2} \), where \( y \) is an arbitrary element from \( K \langle X \rangle \). Then,

\[ [c_1, y][c_2, y] \in I_{m_1+m_2-1}. \]
Proof. From Equation (1) we have that
\[
[c_1, y][c_2, y] = [c_2, c_1 y, y] + [c_1, c_2][y, y] + [c_1, c_2, y]y - c_1 [c_2, y, y].
\]
We need to show that all non-zero terms on the right side of the above equation belong to \(I_{m_1 + m_2 - 1}\). This clearly holds for \([c_1, c_2, y]y\). Consider the term \([c_1, c_2, y]\). \(c_1\) is a commutator of odd length \(m_1 - 1\), whereas \([c_2, y]\) is a commutator of odd length \(m_2 + 1\). By Theorem 2.5, \([c_2, y]\) \(\in I_{m_1 + m_2 - 1}\). It remains to consider the term \([c_1, c_1 y]\). We have that \([c_2, c_2 y, y] \in I_{m_2 - 1}\) and \(c_1 y \in I_{m_1 - 1}\). By Theorem 2.3
\[
[c_2, c_1 y] \in I_{m_1 + m_2 - 2}.
\]
This completes the proof.

\(\square\)

Corollary 3.2. Let \(m_1, \ldots, m_i\) be even integers, let \(y \in K\langle X\rangle\), and let \([c_1, y] \in L_{m_i}\) for \(i = 1, \ldots, l\). Then
\[
[c_1, y] \cdots [c_l, y] \in I_{m_1 + \cdots + m_{l-1} + 1}.
\]

Proof. If \(l\) is even, for \(k = 1, \ldots, \frac{l}{2}\), Lemma 3.1 implies that \([c_{2k-1}, y][c_{2k}, y] \in I_{m_{2k-1} + m_{2k} - 1}\). Moreover, the number \(m_{2k-1} + m_{2k} - 1\) is odd. Let us denote it by \(j_k\). Therefore, by Theorem 2.7(ii) we obtain that
\[
([c_1, y][c_2, y]) \cdots ([c_{l-1}, y][c_l, y]) \in I_{j_1} \cdots I_{j_{l/2}} \subseteq I_{j_1 + \cdots + j_{l/2} - l/2 + 1} = I_{m_1 + \cdots + m_{l-1} + 1}.
\]

Next, if \(l\) is odd, we can apply the above considerations for \(l - 1\). We obtain that
\[
[c_1, y] \cdots [c_{l-1}, y] \in I_{m_1 + \cdots + m_{l-1} - l + 2}.
\]

Furthermore, \(m_1 + \cdots + m_{l-1} - l + 2\) is an odd number. Hence, by Theorem 2.6
\[
[c_1, y] \cdots [c_{l-1}, y][c_l, y] \in I_{m_1 + \cdots + m_{l-1} - l + 2} = I_{m_1 + \cdots + m_{l-1} + 1}.
\]

\(\square\)

We now give the following estimate for the GL(n)-module structure of \(\varphi_p(B_n)\) for any \(p \geq 1\).

Theorem 3.3. Let \(p \geq 1\). Suppose that
\[
\varphi_p(B_n) \cong_{\text{GL}(n)} \bigoplus_{\lambda} k\lambda V_\lambda,
\]
where again \(V_\lambda\) denotes the irreducible GL(n)-module corresponding to the non-negative integer partition \(\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)\). Then \(\lambda_1 \leq p - 1\).

Proof. For \(p = 1\) the statement is trivially true. Therefore, we give a proof for the case \(p \geq 2\). The algebra \(K\langle X\rangle\) has a \(\mathbb{Z}_{\geq 0}\)-grading given by the weight space decomposition as a GL(n)-module. More precisely,
\[
K\langle X\rangle = \bigoplus_{\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}_{\geq 0}^n} W(\mu),
\]
where \(W(\mu)\) is the weight space corresponding to the weight \(\mu\). This means that, if \(g = \text{diag}(\xi_1, \ldots, \xi_n) \in \text{GL}(n)\) and \(w \in W(\mu)\), then
\[
g(w) = \xi_1^{\mu_1} \cdots \xi_n^{\mu_n} w.
\]
Therefore, a basis for \(W(\mu)\) is given by the monomial \(x_1^{\mu_1} \cdots x_n^{\mu_n}\) and all its distinct permutations.
Consequently, $B_n$ and $\varphi_p(B_n)$ also have $\mathbb{Z}_{\geq 0}$-gradings given respectively by
\[
B_n = \bigoplus_{\mu=(\mu_1,\ldots,\mu_n) \in \mathbb{Z}_{\geq 0}^n} W(\mu) \cap B_n;
\]
\[
\varphi_p(B_n) = \bigoplus_{\mu=(\mu_1,\ldots,\mu_n) \in \mathbb{Z}_{\geq 0}^n} \varphi_p(W(\mu) \cap B_n).
\]

Notice that $\varphi_p(W(\mu) \cap B_n)$ consists of sums of products of pure commutators such that the variable $x_1$ appears in each product exactly $\mu_i$ times for all $i = 1, \ldots, n$. Consider now a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n)$ such that $V_{\lambda}$ appears in the decomposition of $\varphi_p(B_n)$. Therefore, $\varphi_p(W(\lambda) \cap B_n) \neq \emptyset$ and hence $\varphi_p(B_n)$ contains an element $w$ such that $w$ is a sum of products of pure commutators and the variable $x_1$ appears in each product exactly $\lambda_i$ times. We will estimate the maximal number of appearances of $x_1$ in an element of $\varphi_p(B_n)$. For brevity, let us denote this maximal number by $N$.

Let $u$ be a product of pure commutators, such that the number of appearances of $x_1$ in $u$ is equal to $N$. We notice the following property, which was also considered in [3]. If $u \in I_k$ for some $2 \leq k \leq p$ and if we permute the commutators which participate in $u$ then we obtain again a product of pure commutators which belongs to $I_k$. This follows from the fact that for all $i, j \geq 1$, $[L_i, L_j] \subseteq L_{i+j}$. Therefore, without loss of generality, we may consider that $u = u_1u_2u_3$, such that the following conditions are fulfilled: $u_1$ is a product of $l_1$ pure commutators of lengths respectively $m_1, \ldots, m_{l_1}$, such that each $m_i$ is odd; $u_2$ is a product of $l_2$ pure commutators of lengths respectively $n_1, \ldots, n_{l_2}$, such that all $n_i$ are even and the last position in each commutator is equal to $x_1$; $u_3$ is a product of $l_3$ pure commutators of lengths respectively $r_1, \ldots, r_{l_3}$, such that all $r_i$ are even and the last position in each commutator is different from $x_1$. Let us denote the maximal number of appearances of $x_1$ in $u_i$ by $N_i$, for $i = 1, 2, 3$. We need to estimate $N_1, N_2$, and $N_3$.

By Theorem 2.7(ii), $u_1 \in I_{m_1+\cdots+m_{l_1}-l_1+1}$. We set $p_1 = m_1 + \cdots + m_{l_1} - l_1 + 1$. The maximal number of appearances of $x_1$ in $u_1$ is $N_1 = m_1 + \cdots + m_{l_1} - l_1$ (since in a pure $m_i$-commutator the variable $x_1$ can appear at most $m_i - 1$ times). Therefore, $N_1 = p_1 - 1$.

Next, we consider the element $u_2$. By Corollary 3.2, $u_2 \in I_{n_1+\cdots+n_{l_2}-l_2+1}$. Similarly as above, we set $p_2 = n_1 + \cdots + n_{l_2} - l_2 + 1$ and we obtain that $N_2 = p_2 - 1$.

Finally, we take $u_3$. By Theorem 2.7(i), we have that $u_3 \in I_{r_1+\cdots+r_{l_3}-2l_3+2}$. We set $p_3 = r_1 + \cdots + r_{l_3} - 2l_3 + 2$. Since in each of the commutators in $u_3$ the last position is different from $x_1$ we have that the maximal number of appearances of $x_1$ in $u_3$ is $N_3 = r_1 + \cdots + r_{l_3} - 2l_3$. Hence, $N_3 = p_3 - 2$.

In short, the above implies that
\[
u = u_1u_2u_3 \in I_{p_1}I_{p_2}I_{p_3} \subseteq I_p.
\]

Notice that $p_1$ is always an odd number, whereas $p_3$ is always an even number. Therefore, Theorem 2.5 implies that
\[
I_{p_1}I_{p_2} \subseteq I_{p_1+p_2-1}.
\]

Hence, by Theorem 2.4, we obtain that
\[
I_{p_1}I_{p_2}I_{p_3} \subseteq I_{p_1+p_2-1}I_{p_3} \subseteq I_{p_1+p_2+p_3-3} \subseteq I_p.
\]
Therefore, \( p_1 + p_2 + p_3 - 3 \leq p \). Thus, \( N = N_1 + N_2 + N_3 = p_1 - 1 + p_2 - 1 + p_3 - 2 \leq p - 1 \). This proves the statement. \( \square \)

Theorem 3.3 together with Equation \( (2) \) lead to the following corollary.

**Corollary 3.4.** Let \( p \geq 1 \). Suppose that

\[
F_n(\mathcal{N}_p) \cong_{\text{GL}(n)} \bigoplus_\lambda m_\lambda V_\lambda,
\]

where again \( V_\lambda \) denotes the irreducible \( \text{GL}(n) \)-module corresponding to the non-negative integer partition \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \). Then \( \lambda_2 \leq p - 1 \).

**Proof:** For \( p = 1 \) the statement is again trivially true. For \( p \geq 2 \), the proof is a standard task on branching rules. We apply Pieri’s branching rule (also known as Young’s rule) to Equation \( (2) \) and then we use Theorem 3.3. One may read the description of Pieri’s rule in e.g. [11]. \( \square \)

We notice that the upper bound on \( \lambda_1 \) which is obtained in Theorem 3.3 is exact for all known cases. More precisely, using the decomposition of \( \varphi_p(B_n) \) over \( \text{GL}(n) \), which is known for \( p = 2, 3, 4 \), we see that for \( p = 2 \), \( \lambda_1 \leq 1 = p - 1 \) and for \( p = 3 \), \( \lambda_1 \leq 2 = p - 1 \). For \( p = 4 \), we have that \( \lambda_1 \leq 3 = p - 1 \).

For \( p = 5 \), Theorem 3.3 gives that \( \lambda_1 \leq 4 \).

In the end of the section, we would like to consider separately the case \( n = 3 \), i.e., we take the free associative algebra \( K \langle x_1, x_2, x_3 \rangle \) and the relatively free algebra \( F_3(\mathcal{N}_p) \) of rank 3 in the variety \( \mathcal{N}_p \). Then, we have the stronger theorem on inclusions of products of commutators (Theorem 3.3) which leads to a much shorter proof of Theorem 3.3. For completeness of the exposition, we present this proof below.

**Shorter proof of Theorem 3.3 for \( n = 3 \).**

Consider a partition \( \lambda \) such that \( V_\lambda \) appears in the decomposition of \( \varphi_p(B_3) \) as a \( \text{GL}(3) \)-module. Therefore, as in the general proof of Theorem 3.3 \( \varphi_p(B_3) \) contains an element \( u \) such that \( u \) is a product of pure commutators and the variable \( x_1 \) appears in \( u \) exactly \( \lambda_1 \) times. We will again estimate the maximal number of appearances of the variable \( x_1 \) in \( u \). By Theorem 3.3 for all integers \( m_1, \ldots, m_l \geq 2 \)

\[
I_{m_1}I_{m_2} \cdots I_{m_l} \subseteq I_{m_1 + m_2 + \cdots + m_l - l + 1}.
\]

Hence, \( u \) is at most a product of an \( m_1 \)-commutator, \( m_2 \)-commutator, \ldots, and \( m_l \)-commutator such that \( m_1 + \cdots + m_l - l + 1 = p \). Hence \( m_1 + \cdots + m_l = p + l - 1 \). The number of appearances of \( x_1 \) in such a product of pure commutators \( u \) is at most \( p - 1 \). This proves the statement.

4. **The algebra of invariants** \( F_n(\mathcal{N}_p)^G \) for \( G = \text{SL}(n), \text{O}(n), \text{SO}(n), \text{Sp}(2s), \) and \( \text{UT}(n) \)

In this section we set \( K = \mathbb{C} \). Using results from [2] and [10], we will obtain some bounds on the generators of the algebra of invariants \( F_n(\mathcal{N}_p)^G \) for \( G = \text{SL}(n), \text{O}(n), \text{SO}(n), \text{Sp}(2s), \) and \( \text{UT}(n) \).
Proposition 4.1. Let \( p \geq 1 \) and let \( G = \text{SL}(n) \) or \( G = \text{Sp}(2s) \) (for \( n = 2s \)). Then, the algebra of invariants \( F_n(\mathfrak{g}_p)^G \) is finite-dimensional. More precisely, the Hilbert series \( H(F_n(\mathfrak{g}_p)^G,t) \) is a polynomial and it holds that

\[ \deg H(F_n(\mathfrak{g}_p)^G,t) \leq n(p-1), \]

where in the case \( G = \text{Sp}(2s) \) we have that \( n = 2s \).

Proof. We denote by \( F_n^{(i)}(\mathfrak{g}_p) \) the homogeneous component of total degree \( i \) in \( F_n(\mathfrak{g}_p) \). Then we can write the decomposition of \( F_n(\mathfrak{g}_p) \) as a \( \text{GL}(n) \)-module in the following way:

\[ F_n(\mathfrak{g}_p) = \bigoplus_{i \geq 0} F_n^{(i)}(\mathfrak{g}_p) \cong_{\text{GL}(n)} \bigoplus_{i \geq 0} \bigoplus_{\lambda} m_{i,\lambda} V_{\lambda}, \]

where \( |\lambda| = i \) and, by Corollary 3.4, \( \lambda_2 \leq p - 1 \).

Consider first the case when \( G = \text{Sp}(2s) \). Theorem 2.2 from [10] implies that

\[ H(F_{2s}(\mathfrak{g}_p)^{\text{Sp}(2s)},t) = \sum_{i \geq 0} \left( \sum_{\lambda} m_{i,\lambda} \right) t^i, \]

where the second sum runs over partitions \( \lambda \) with \( \lambda_1 = \lambda_2 = \cdots = \lambda_{2s-1} = \lambda_{2s} \) such that \( |\lambda| = i \) and \( \lambda_2 \leq p - 1 \). The largest partition \( \lambda \) which satisfies these conditions is \( \lambda_1 = \lambda_2 = \cdots = \lambda_{2s-1} = \lambda_{2s} = p - 1 \). Therefore, the highest power of \( t \) which can occur in the Hilbert series \( H(F_{2s}(\mathfrak{g}_p)^{\text{Sp}(2s)},t) \) is \( i = \lambda_1 + \cdots + \lambda_{2s} = 2s(p-1) \). This proves the statement for \( G = \text{Sp}(2s) \).

Next, let \( G = \text{SL}(n) \). The results from [2] together with Corollary 3.4 imply that

\[ H(F_n(\mathfrak{g}_p)^{\text{SL}(n)},t) = \sum_{i \geq 0} \left( \sum_{\lambda} m_{i,\lambda} \right) t^i, \]

where the second sum runs over partitions \( \lambda \) with \( \lambda_1 = \lambda_2 = \cdots = \lambda_n \) such that \( |\lambda| = i \) and \( \lambda_2 \leq p - 1 \). Hence, the largest partition \( \lambda \) which satisfies these conditions is again \( \lambda_1 = \lambda_2 = \cdots = \lambda_n = p - 1 \). Therefore, the highest power of \( t \) which can occur in the Hilbert series \( H(F_n(\mathfrak{g}_p)^{\text{SL}(n)},t) \) is again \( i = n(p-1) \). This completes the proof. \( \square \)

Proposition 4.2. Let \( p \geq 1 \) and let \( G = \text{O}(n) \). Then,

\[ H(F_n(\mathfrak{g}_p)^{\text{O}(n)},t) = \frac{p(t)}{1-t^2}, \]

where \( p(t) \) is a polynomial such that

\[ \deg p(t) \leq \begin{cases} n(p-1) & \text{if } p-1 \text{ is even} \\ n(p-2) & \text{if } p-1 \text{ is odd}. \end{cases} \]

Proof. We proceed as in the proof of the previous proposition. We write the decomposition of \( F_n(\mathfrak{g}_p) \) as a \( \text{GL}(n) \)-module in the following way:

\[ F_n(\mathfrak{g}_p) = \bigoplus_{i \geq 0} F_n^{(i)}(\mathfrak{g}_p) \cong_{\text{GL}(n)} \bigoplus_{i \geq 0} \bigoplus_{\lambda} m_{i,\lambda} V_{\lambda}, \]

where again \( |\lambda| = i \) and \( \lambda_2 \leq p - 1 \). Then Theorem 2.2 from [10] implies that

\[ H(F_n(\mathfrak{g}_p)^{\text{O}(n)},t) = \sum_{i \geq 0} \left( \sum_{\lambda} m_{i,\lambda} \right) t^i, \]
where the second sum runs over even partitions $\lambda$ such that $|\lambda| = i$ and $\lambda_2 \leq p - 1$.

If a partition $\lambda' = (\lambda'_1, \lambda'_2, \ldots, \lambda'_n)$ satisfies the above conditions and appears in the decomposition of $F_n(\mathfrak{gl}_p)^{O(n)}$, then due to Equation (2) all partitions of the form $(\lambda'_1 + 2k, \lambda'_2, \ldots, \lambda'_n)$ for $k = 1, 2, \ldots$ also appear in the decomposition of $F_n(\mathfrak{gl}_p)^{O(n)}$. Thus, the following term appears in $H(F_n(\mathfrak{gl}_p)^{O(n)}, t)$:

$$t^{\lambda'}(1 + t^2 + t^4 + \ldots) = \frac{t^{\lambda'}}{1 - t^2}.$$

We can write all terms in $H(F_n(\mathfrak{gl}_p)^{O(n)}, t)$ in this way for a suitable $\lambda'$. Theorem 3.3 implies that when $p - 1$ is even, the largest suitable $\lambda'$ that satisfies the above conditions is $\lambda' = (p - 1, p - 1, \ldots, p - 1)$. Similarly, when $p - 1$ is odd, the largest suitable $\lambda'$ that satisfies the above conditions is $\lambda' = (p - 2, p - 2, \ldots, p - 2)$. This proves the statement.

In the same way we can prove the following proposition.

**Proposition 4.3.** Let $p \geq 1$ and let $G = SO(n)$. Then,

$$H(F_n(\mathfrak{gl}_p)^{SO(n)}, t) = \frac{p(t)}{1 - t^2},$$

where $p(t)$ is a polynomial such that

$$\deg p(t) \leq n(p - 1).$$

By a theorem of Domokos and Drensky ([6]), the algebra of invariants $F_n(\mathfrak{gl}_p)^G$ is finitely generated for any reductive subgroup $G$ of $GL(n)$. Using the above results we can give an explicit upper bound for the degree of the generators in a minimal generating set of $F_n(\mathfrak{gl}_p)^G$ for $G = SL(n)$, $O(n)$, $SO(n)$, or $Sp(2s)$. First, we introduce the following notation. Given a graded algebra $A = \bigoplus_{i \geq 0} A^i$, we denote by $\beta(A)$ the minimal non-negative integer $i$ such that $A$ is generated by homogeneous elements of degree at most $i$. We write $\beta(A) = \infty$ if there is no such $i$.

**Corollary 4.4.** Let $G$ be one of $SL(n)$ or $Sp(2s)$. Then

$$\beta(F_n(\mathfrak{gl}_p)^G) \leq n(p - 1), \text{ for all } p \geq 1.$$

For $G = O(n)$ we have that

$$\beta(F_n(\mathfrak{gl}_p)^{O(n)}) \leq \begin{cases} n(p - 1) & \text{if } p - 1 \text{ is even and } p > 1 \\ n(p - 2) & \text{if } p - 1 \text{ is odd and } p > 2 \\ 2 & \text{if } p = 1, 2. \end{cases}$$

For $G = SO(n)$ we have that

$$\beta(F_n(\mathfrak{gl}_p)^{SO(n)}) \leq \begin{cases} n(p - 1) & \text{if } p > 1 \\ 2 & \text{if } p = 1. \end{cases}$$

**Proof.** By Proposition 4.1, the statement is clear for $G = SL(n)$ and $G = Sp(2s)$. Hence, we consider the case of $G$ being one of $O(n)$ and $SO(n)$. By Propositions 4.2 and 4.3,

$$H(F_n(\mathfrak{gl}_p)^G, t) = \frac{p(t)}{1 - t^2} = p(t)(1 + t^2 + t^4 + \cdots + t^{2k} + \cdots).$$

$F_n(\mathfrak{gl}_p)^G$ has one free generator of degree 2, namely $u = x_1^2 + \cdots + x_n^2$. Therefore, if $\{f_1, \ldots, f_l\}$ is a basis for the homogeneous part $\bigoplus_{i=0}^{\deg p(t)} F_n^{(i)}(\mathfrak{gl}_p)^G$ then
\{f_1, \ldots, f_l, u\} is a generating set (not necessarily minimal) for the whole $F_n(\mathfrak{m}_p)^G$. This proves the statement. \qed

**Remark 4.5.** In [8], Domokos and Drensky give a general bound for $\beta(F_n(\mathfrak{m}_p)^G)$ for any reductive group $G$ as a special case of a more general statement. The advantage of the bound that we find in Corollary 4.4 is that it is very explicit and can be used in explicit computations.

In a similar way we can consider the algebra of invariants $F_n(\mathfrak{m}_p)^{\text{UT}(n)}$, where $\text{UT}(n) := \text{UT}(n, \mathbb{C})$ denotes the unitriangular group, i.e., the subgroup of $\text{GL}(n)$ consisting of upper triangular matrices with 1’s on the diagonal. $\text{UT}(n)$ is a maximal unipotent subgroup of $\text{GL}(n)$ and of $\text{SL}(n)$. Using results from [2] we prove the following proposition.

**Proposition 4.6.** Let $p \geq 1$ and let $G = \text{UT}(n)$. Then,

$$H(F_n(\mathfrak{m}_p)^{\text{UT}(n)}, t) = \frac{p(t)}{1-t},$$

where $p(t)$ is a polynomial such that $\deg p(t) \leq n(p-1)$.

**Proof.** We proceed as before. We write the decomposition of $F_n(\mathfrak{m}_p)$ as a $\text{GL}(n)$-module in the following way:

$$F_n(\mathfrak{m}_p) = \bigoplus_{i \geq 0} F_n(i) (\mathfrak{m}_p) \cong \bigoplus_{i \geq 0} \bigoplus_{\lambda} m_{i,\lambda} V_{\lambda},$$

where again $|\lambda| = i$ and $\lambda_2 \leq p-1$. Each module $V_{\lambda}$ has a one-dimensional $\text{UT}(n)$-invariant subspace generated by any highest weight vector. Therefore (see also [2]),

$$H(F_n(\mathfrak{m}_p)^{\text{UT}(n)}, t) = \sum_{i \geq 0} \left( \sum_{\lambda} m_{i,\lambda} \right) t^i,$$

where the second sum runs over partitions $\lambda$ such that $|\lambda| = i$ and $\lambda_2 \leq p-1$.

Then, as in the proof of Proposition 4.2 we notice the following. If a partition $\lambda' = (\lambda'_1, \lambda'_2, \ldots, \lambda'_n)$ satisfies the above conditions and appears in the decomposition of $F_n(\mathfrak{m}_p)^{\text{UT}(n)}$, then due to Equation 3 all partitions of the form $(\lambda'_1 + k, \lambda'_2, \ldots, \lambda'_n)$ for $k = 1, 2, \ldots$ also appear in the decomposition of $F_n(\mathfrak{m}_p)^{\text{UT}(n)}$. Thus, the following term appears in $H(F_n(\mathfrak{m}_p)^{\text{UT}(n)}, t)$:

$$tl^{(|\lambda'| \lambda')} (1 + t + t^2 + \cdots) = \frac{t^{(|\lambda'| \lambda')}}{1-t}.$$

We can write all terms in $H(F_n(\mathfrak{m}_p)^{\text{UT}(n)}, t)$ in this way for a suitable $\lambda'$. Theorem 3.3 implies that the largest suitable $\lambda'$ that satisfies the above conditions is $\lambda' = (p-1, p-1, \ldots, p-1)$. This proves the statement. \qed

The algebra $F_n(\mathfrak{m}_p)$ is left Noetherian, hence Theorem 7.2 from [7] implies that the algebra of invariants $F_n(\mathfrak{m}_p)^{\text{UT}(n)}$ is finitely generated. Using Proposition 4.6 we can give an upper bound on the degree of the generators of $F_n(\mathfrak{m}_p)^{\text{UT}(n)}$ in a minimal generating set.
Corollary 4.7. For the algebra of invariants $F_n(\mathfrak{g}_p)^{UT(n)}$ it holds that

$$\beta(F_n(\mathfrak{g}_p)^{UT(n)}) \leq \left\{ \begin{array}{ll} n(p-1) & \text{if } p > 1 \\ 1 & \text{if } p = 1. \end{array} \right.$$ 

Proof. We notice that $x_1$ is a free generator of $F_n(\mathfrak{g}_p)^{UT(n)}$ of degree 1. Furthermore, by Proposition 4.6

$$H(F_n(\mathfrak{g}_p)^{UT(n)}, t) = \frac{p(t)}{1-t} = p(t)(1 + t + t^2 + t^3 + \ldots).$$

Therefore, as in the proof of Corollary 4.7, if $\{f_1, \ldots, f_t\}$ is a basis for the homogeneous part $\bigoplus_{i=0}^{\deg p(t)} F_n(\mathfrak{g}_p)^{UT(n)}$ then $\{f_1, \ldots, f_t, x_1\}$ is a generating set (not necessarily minimal) for the whole $F_n(\mathfrak{g}_p)^{UT(n)}$. This proves the statement. \(\square\)

Remark 4.8. Notice that for $p = 2$, using results from [10], we can compute explicitly $\beta(F_n(\mathfrak{g}_p)^{G})$ for $G = SL(n)$, $O(n)$, $SO(n)$, $Sp(2s)$, and $UT(n)$. We see that the bounds obtained in Corollaries 4.4 and 4.7 for $p = 2$ are exact when $G = SL(n)$, $O(n)$, $SO(n)$, and $UT(n)$. Namely, $\beta(F_n(\mathfrak{g}_2)^{SL(n)}) = \beta(F_n(\mathfrak{g}_2)^{SO(n)}) = \beta(F_n(\mathfrak{g}_2)^{UT(n)}) = n = n(p-1)$ and $\beta(F_n(\mathfrak{g}_2)^{O(n)}) = 2$. For $G = Sp(2s)$, we obtain $\beta(F_n(\mathfrak{g}_2)^{Sp(2s)}) = 2 < 2s(p-1)$ for $s > 1$, i.e., when $G = Sp(2s)$ the bound from Corollary 4.4 is not exact for $p = 2$.

5. Applications to Classical Invariant Theory

In this section we will reformulate Corollaries 4.4 and 4.7 in the language of Classical Invariant Theory. We set again $G = C$. First we recall a classical result due to Weyl involving the algebra of invariants $C \langle X \rangle^G = T(V)^G$, where $G$ is one of $O(n)$ or $Sp(2s)$. In order to state this result, we follow the notations and formulations in [13]. Let $\langle \cdot, \cdot \rangle$ denote the non-degenerate symmetric (in the case of $G = O(n)$) or skew symmetric (in the case of $G = Sp(2s)$) form on $V$ preserved by the action of $G$. Using the natural identification of $V^*$ with $V$ we obtain that the form $\langle \cdot, \cdot \rangle$ is an invariant for $G$ in $V \otimes V = V^{\otimes 2}$. Following [13], we denote this invariant by $\theta_2$. For $k = 2j$ we may choose an isomorphism

$$i_k : V^{\otimes k} \cong \otimes^{j} V^{\otimes 2}.$$ 

Specifying $i_k$ amounts to pairing off the factors of $V^{\otimes k}$. Then, in $\otimes^{j} V^{\otimes 2}$ we have the invariant $\otimes^{j} \theta_2$. Hence, in $V^{\otimes k}$ we have the invariant $\theta_k = i^{-1} (\otimes^{j} \theta_2)$. The symmetric group $S_k$ acts on $V^{\otimes k}$ by permuting the factors. Then the following result of Weyl holds:

Theorem 5.1 ([13]). Let $G$ be one of $O(n)$ or $Sp(2s)$. If $k$ is odd, there are no $G$-invariants in $V^{\otimes k}$. If $k$ is even, the translates of $\theta_k$ by $S_k$ span the $G$-invariants in $V^{\otimes k}$.

An analogous result can be found in the literature for $G = SL(n)$.

Propositions 4.1, 4.2, and 4.3 and Corollaries 4.4 and 4.7 lead to the following further characterization of the $G$-invariants in $T(V)$. We include also the cases $G = SL(n)$, $SO(n)$ and $UT(n)$ in our considerations.

Theorem 5.2. Let $G = SL(n)$ or $G = Sp(2s)$ (for $n = 2s$). For any $p \geq 1$, if $f$ is a $G$-invariant in $T(V)$ of degree greater than $n(p-1)$ then $f \in I_{p+1}$.

Theorem 5.3. Let $G = O(n)$.
(i) If \( f \) is a \( G \)-invariant in \( T(V) \) of degree greater than 2 then either \( f \in I_3 \) or \( f \) is equal to a power of \( \theta_2 \).

(ii) For any \( p \geq 3 \) such that \( p - 1 \) is an even number, if \( f \) is a \( G \)-invariant in \( T(V) \) of degree greater than \( n(p - 1) \) then either \( f \in I_{p+1} \) or \( f \) can be expressed by invariants of smaller degree.

(iii) For any \( p \geq 3 \) such that \( p - 1 \) is an odd number, if \( f \) is a \( G \)-invariant in \( T(V) \) of degree greater than \( n(p - 2) \) then either \( f \in I_{p+1} \) or \( f \) can be expressed by invariants of smaller degree.

**Theorem 5.4.** Let \( G = \text{SO}(n) \).

(i) If \( f \) is a \( G \)-invariant in \( T(V) \) of degree greater than 2 then either \( f \in I_2 \) or \( f \) can be expressed by invariants of smaller degree.

(ii) For any \( p \geq 2 \), if \( f \) is a \( G \)-invariant in \( T(V) \) of degree greater than \( n(p - 1) \) then either \( f \in I_{p+1} \) or \( f \) can be expressed by invariants of smaller degree.

**Theorem 5.5.** Let \( G = \text{UT}(n) \).

(i) If \( f \) is a \( G \)-invariant in \( T(V) \) of degree greater than 1 then either \( f \in I_2 \) or \( f \) can be expressed by invariants of smaller degree.

(ii) For any \( p \geq 2 \), if \( f \) is a \( G \)-invariant in \( T(V) \) of degree greater than \( n(p - 1) \) then either \( f \in I_{p+1} \) or \( f \) can be expressed by invariants of smaller degree.

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