On moduli spaces of 4- or 5-instanton bundles

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Abstract
We study the scheme of multi-jumping lines of an \(n\)-instanton bundle mainly for \(n \leq 5\). We apply it to prove the irreducibility and smoothness of the moduli space of 5-instanton. Some particular situations with higher \(c_2\) are also studied.

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Introduction
An $n$-instanton is an algebraic rank 2 vector bundle $E$ over $\mathbb{P}_3(\mathcal{O})$ with Chern classes $c_1 = 0$, $c_2 = n$, which is stable (i.e: $h^0E = 0$), and satisfies the natural cohomology property $h^1E(-2) = 0$. Denote by $I_n$ the moduli space of those bundles.

In general, little is known about $I_n$. A classical description of its tangent space gives that every irreducible component of $I_n$ is at least $8n - 3$ dimensional. In fact it has been shown in the early eighties that for $n \leq 4$ the moduli space $I_n$ is irreducible (Cf [H], [E-S], [Ba2]) and smooth (Cf [LeP]).

Define a line $L$ of $\mathbb{P}_3$ to be a jumping line of order $k$ for $E$ (denoted $k$-jumping line) when the restriction of $E$ to $L$ is $\mathcal{O}_L(k) \oplus \mathcal{O}_L(-k)$. Such a line is called a multi-jumping line if $k \geq 2$. It seems difficult to study the moduli space without some results on the schemes of jumping lines. Despite the fact that those schemes have a determinantal structure involving strong properties when $E$ is in some open set of $I_n$, results satisfied by any $n$-instanton are much weaker. Indeed, it seems hopeless to expect more than the fact that any $n$-instanton has a scheme of multi-jumping lines with good codimension in $G$ (i.e is a curve). Surprisingly this statement which was already conjectured in [E-S] is still open for $n > 3$.

In the first 3 parts, we will study this scheme of multi-jumping lines for any $n$-instanton with $n \leq 5$; we will prove the following:

**Proposition 3.3.2** The scheme of multi-jumping lines of any 4- or 5-instanton is a curve in the Grassmann manifold $G$.

The techniques developed here will also apply to some particular families of instanton with higher $c_2$ (Cf 1.2.5 and 4.5.3).

As a consequence of this result, we will recover the results of Barth and of Le Potier about $I_4$ and obtain the following one which is also announced by G.Trautmann and A.Tikhomirov using independent methods.

**Theorem 4.3.2 and 4.3.3** The moduli space of 5-instanton is smooth and irreducible of dimension 37.

The proofs of the irreducibility of $I_n$ for $n \leq 4$ were all different. Our method will be to extend the one of Ellingsrud and Stromme to $n = 4$ and $n = 5$. Using the jumping lines containing a fixed point $P$ of $\mathbb{P}_3$, they constructed a morphism from some open subset $U_P$ of $I_n$ to the moduli space $\Theta$ of semi-table $\theta$-characteristic over plane curves of degree $n$.

The conjecture on the codimension of multi-jumping lines means that the $(U_P)_{P \in \mathbb{P}_3}$ cover $I_n$. Ellingsrud and Stromme described 2 other difficulties to extend their method:

- Understand the images of the morphisms $U_P \rightarrow \Theta$. This however seems only to be a real problem for $n \geq 12$, as when $n \leq 11$ those morphism have a dense image.
- Obtain a description of the fiber of such a morphism.

In fact, the data describing this fiber are satisfying a condition which was empty for $n = 3$, and which need to be understood for higher $n$. An explicit description of the fiber will be given in the $\S 4$ for any $n$, but the key result needed to have a grip on the moduli spaces is to understand when this fiber is singular. This will be understood for $n = 4$, and 5, and then we will have to check that the bundles in the ramification of all the morphism are nevertheless smooth points of $I_n$ to obtain the above theorem. We will need there the study of the family $4.5.4$.

To prove the proposition 3.3.2, we will assume that there is some 4- or 5-instanton $E$ such that its scheme of multi-jumping lines contains some surface. All along the proof, we will try to translate this excess of multi-jumping lines to some properties of $E$ with respect to its restrictions to planes: indeed the problem of the dimension in $\mathbb{P}^3$ of the non-stable planes has first been studied by Barth (Cf [Ba1]) in 1977, and recently extended by Coanda (Cf [Co]). Those results will be fundamental for us, because they will be used at every end of proof, and furthermore, many proofs are inspired by those of Coanda.

In the §1 we will study the restrictions of $E$ to planes, in other words it gives properties of stable and semi stable rank 2 vector bundles over the plane with $c_2 \leq 5$ because the $h^1(E(-2) = 0$ condition implies that $E$ has no unstable planes. For instance, we will give informations on the intersection of the scheme of multi-jumping lines with some $\beta$-plane, or on the bidegree of the hypothetical congruence $S$ of multi-jumping lines.

In the §2 we will take care of the $(k \geq 3)$-jumping lines, to obtain that a general hyperplane section of $S$ can avoid those points. This is the major difficulty for $c_2 = 5$, and it will enable us to prove in the §3 that a general hyperplane section of $S$ has no trisecant. The remaining cases will then be strongly bounded, and we will first get rid of the situation where $S$ is ruled which is easy for $c_2 = 5$, but not for $c_2 = 4$. The case $\deg S = 4$ will also be particular, and the final step will need the residual calculus of 4.4.

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Notations:

In the following, we will denote by $p$ and $q$ the projections from the points/lines incidence variety to $\mathbb{P}_3$ and to the Grassmann manifold of lines of $\mathbb{P}_3$ denoted by $G$. (or to $\mathbb{P}_2$ and $\mathbb{P}^3_2$ in the plane situation). Let’s define by $\tau = p^*\mathcal{O}_{\mathbb{P}_3}(1)$ and $\sigma = q^*\mathcal{O}_G(1)$. When a blow up $\mathbb{P}'_3$ of $\mathbb{P}_3$ along a line will be used, we will denote by $p'$ and $q'$ the projections to $\mathbb{P}_3$ and $\mathbb{P}_1$. 
Let’s call by $\alpha$-plane and $\beta$-plane the 2 dimensional planes included in $G$ which are made of the lines of $\mathbb{P}_3$ containing a same point, or included in a same plane, and finally denote by $\mathbb{P}(F) = Proj(SymF^*)$.

Denote by $M$ the scheme of multi-jumping lines of the $n$-instanton $E$ with $n \leq 5$. We will say that a plane $H$ is a jumping plane if the restriction $E_H$ is semi-stable but not stable.

**Remark** The scheme of multi-jumping lines $M$ of a 4- or 5-instanton $E$ is at most 2 dimensional.

Indeed, if this was not true, any plane would contain infinitely many multi-jumping lines, and we will show in the 1.1.4 that it is not possible in a stable plane. Then every plane would be a jumping plane and according to [Ba1] it is not possible when $n > 1$. □

So we will finally denote by $S$ the hypothetical purely 2-dimensional sub-scheme of $M$ defined by the following sequence:

$$0 \rightarrow J \rightarrow O_M \rightarrow O_S \rightarrow 0$$

where $J$ is the biggest subsheaf of $O_M$ which is at most 1-dimensional.

1 **Some properties of $S$ for $c_2 \leq 5$**

1.1 **A link between trisecant lines to $S$ and jumping plane**

We’d like to study here the lines $d$ of $\mathbb{P}_5$ which meet $M$ in length 3 or more. Those lines have to be included in the Grassmannian $G$, and each of them can be identified to a plane pencil of lines of $\mathbb{P}_3$. Let $h$ be the $\beta$-plane containing $d$, and $H$ the associated plane of $\mathbb{P}_3$. As $R^2q_*E$ is zero, we have an isomorphism between the restriction of $R^1q_*p^*E$ to $h$, and $R^1q_*p^*E_H$, where $E_H$ is the restriction of $E$ to $H$, and also between $R^1q_*p^*E_H$ restricted to a line of $h$ and the sheaf constructed analogously when blowing up the associated point of $H$. (So the projections over $H$ and $\mathbb{P}_1$ will be still denoted by $p$ and $q$).

**Proposition 1.1.1** For $c_2 \leq 5$, let $d$ be a line of $\mathbb{P}_5$ meeting $M$ in a 0 dimensional scheme of length 3 or more not containing jumping lines of order 3 or more, then the plane $H$ associated to $d$ is not a stable plane for $E$.

Let $d$ be such a line, and assume that $H$ is stable (ie that $E_H$ is stable), and blow up $H$ at the point $D$ included in all the lines represented by $d$. The resolution of this blow up is:

$$0 \rightarrow O_{d \times H}(-\tau - \sigma) \rightarrow O_{d \times H} \rightarrow O_{\bar{H}} \rightarrow 0$$

Twisting it by $p^*E_H$ gives using the functor $q_*$: 4
The length induced by the scheme of multi-jumping lines on $d$ and is at least 3 by hypothesis. The sheaf $q,p^*E_H$ is locally free of rank $-h^1E_H + h^1E_H(-1) = 2$ over $d = \mathbb{P}_1$. So it has to split in $\mathcal{O}_d(-a) \oplus \mathcal{O}_d(-b)$ with $a,b > 0$ and $a + b = h^1E_H(-1) - c_1(R^1q_*p^*E_H)$. But $h^1E_H(-1)$ is $c_2$, so it gives already a contradiction for $c_2 \leq 4$.

For $c_2 = 5$, one has necessarily $a = b = 1$, so $h^0(q_*p^*E_H(\sigma))$ which is also $h^0(J_D \otimes E_H(1))$ is 2. Those 2 sections $s_1$ and $s_2$ are thus proportional on the conic $Z_{s_1 \wedge s_2}$ which has to be singular in $D$ because this point is in the first Fitting ideal of $2\mathcal{O}_H \rightarrow E_H(1)$.

If $Z_{s_1 \wedge s_2}$ is made of 2 distinct lines $d_1$ and $d_2$, we have the following exact sequence because both $s_1$ and $s_2$ vanish at $D$:

$$0 \rightarrow 2\mathcal{O}_H \rightarrow E_H(1) \rightarrow \mathcal{O}_{d_1}(\alpha) \oplus \mathcal{O}_{d_2}(\beta) \rightarrow 0$$

Computing Euler-Poincaré characteristics gives $\alpha + \beta = -3$, so one of them is less or equal to $-2$, thus one of the $d_i$ is a $(k \geq 3)$-jumping line containing $D$ which contradicts the hypothesis.

It is the same when $Z_{s_1 \wedge s_2}$ is a double line $d_1$. We have the exact sequence:

$$0 \rightarrow 2\mathcal{O}_H \rightarrow E_H(1) \rightarrow \mathcal{L} \rightarrow 0$$

where $\mathcal{L}$ is a sheaf of rank 1 on the plane double structure of $d_1$ (in fact $\mathcal{L}$ is not locally free at $D$). We have the linking sequence:

$$0 \rightarrow \mathcal{O}_{d_1}(\alpha) \rightarrow \mathcal{L} \rightarrow \mathcal{O}_{d_1}(\beta) \rightarrow 0$$

once again $\alpha + \beta = -3$, so either $h^0\mathcal{O}_{d_1}(\beta) < h^1\mathcal{O}_{d_1}(\alpha)$, or $h^1\mathcal{O}_{d_1}(\beta) \neq 0$, but in both cases $R^1q_*p^*\mathcal{L}(d_1)$ is non zero, so $d_1$ would be a $(k \geq 3)$-jumping line containing $D$ as previously.

We can’t hope such a result in a jumping plane (ie semi-stable but not stable), but we will show in Lemma 1.1.3 that the problems arise only at finitely many points of the plane.

**Lemma 1.1.2** Let $F$ be a bundle over $\mathbb{P}_n$ with $c_1F = 0$. If $F(k)$ has a section of vanishing locus $Z$, then for any $i \geq k - 1$, the scheme of jumping lines of order at least $i + 2$ is isomorphic to the scheme of lines at least $(i + k + 2)$-secant to $Z$.

We deduce from the following sequence using the standard construction:

$$0 \rightarrow \mathcal{O}_{\mathbb{P}_n} \rightarrow F(k) \rightarrow \mathcal{J}_Z(2k) \rightarrow 0$$

an isomorphism $R^1q_*p^*F(i) \simeq R^1q_*p^*\mathcal{J}_Z(k + i)$ when $i \geq k - 1$. And the supports of those sheaves are by definition (Cf [G-P]) the above schemes.
Lemma 1.1.3 For $c_2 \leq 5$, any $\beta$-plane associated to a jumping plane contain only a finite number of lines of $\mathbb{P}_5$ meeting $M$ in a scheme of length 3 or more made only of 2-jumping lines.

If $H$ is a jumping plane, we can take $k = 0$ in lemma [1.1.2] and study the bisecant lines to $Z$. To prove the lemma it is enough to show that for any point $N$ not in $Z$ with no $(k \geq 3)$-jumping lines through $N$, the line $N^v$ is not trisecant to $M$ where $N^v$ is the lines of $H$ containing $N$. So let $N$ be such a point, and study the bisecant lines to $Z$ in $N^v$ by blowing up $H$ at $N$. One has from the standard construction the exact sequence because $N \notin Z$:

$$0 \rightarrow q_* p^* J_Z \rightarrow H^1 J_Z(-1) \otimes O_{N^v}(-1) \rightarrow H^1 J_Z \otimes O_{N^v} \rightarrow R^1 q_* p^* J_Z \rightarrow 0$$

where $q_* p^* J_Z$ is isomorphic to $O_{\mathbb{P}_1}(-a)$ for some $a > 0$ with $a = \deg Z - c_1(R^1 q_* p^* J_Z)$ because there are only a finite number of bisecant to $Z$ through $N$ ($N \notin Z$). So if $Z$ has at least 3 bisecant (with multiplicity) through $N$, then $a \leq 2$ for $c_2 \leq 5$. If $a = 1$, then $Z$ is included in a line containing $N$ which must be a $c_2$-jumping line, and it conflicts with the hypothesis. So only the case $a = 2$, and $c_2 = 5$ is remaining. But then, the section of $q_* p^* J_Z(2\sigma)$ would give a conic $C$ singular in $N$ and containing $Z$. We have the linking sequence where $d_1$ may be equal to $d_2$:

$$0 \rightarrow \mathcal{O}_{d_1}(-1) \rightarrow \mathcal{O}_{C} \rightarrow \mathcal{O}_{d_2} \rightarrow 0$$

As previously $Z$ can’t be included in a line so it cuts $d_2$ and $h^0(J_Z \otimes \mathcal{O}_{d_2}) = 0$. Twisting the previous sequence by $J_Z$ gives when computing Euler-Poincaré’s characteristics $h^1(J_Z \otimes \mathcal{O}_{d_1}(-1)) + h^1(J_Z \otimes \mathcal{O}_{d_2}) = 4$ because $Z$ is included in $C$. So one of the $d_i$ would be a trisecant line to $Z$, thus it would be a 3-jumping line through $N$ which contradicts the hypothesis and gives the lemma.

Proposition 1.1.4 Any plane $H$ containing infinitely many multi-jumping lines is a jumping plane. In that situation, the support of $R^1 q_* p^* E$ induce on the $\beta$-plane associated to $H$ a reduce line with some possible embedded points.

When $c_2 = 5$, there is a 3-jumping line in this pencil.

When $c_2 = 4$ there are 4 embedded points.

Let’s first notice that $H$ can’t be a stable plane. So assume that it is stable, then, for $c_2 \leq 5$, the bundle $E_H(1)$ has a section vanishing on some scheme $Z$ of length at most 6. We showed in the proof of [1.1.3] the link between multi-jumping lines in $H$ and trisecant lines to $Z$, but if $Z$ had infinitely many trisecant lines (necessarily passing through a same point of $Z$), then on the blow up of $H$ at this point, the sheaf $p^* E(\tau - 3x)$ would have a section (where $x = \tau - \sigma$ is the exceptional divisor) which is not possible because it has a negative $c_2$.

So $H$ is a jumping plane and the section of $E_H$ has infinitely many bisecant lines (necessarily containing a same point). Let’s blow up $H$ at this point, then
$p^*E_H$ has a section vanishing 2 times on the exceptional divisor $x$. The residual scheme of 2$x$ is empty when $c_2 = 4$ and a single point when $c_2 = 5$. The line of $H$ containing this point and the point blown up is thus a trisecant line to the vanishing of the section of $E_H$, so it is a 3-jumping line.

Although it is not required by the following, it is interesting to understand more this situation. For example, when $c_2 = 4$ we can understand the scheme structure of multi-jumping lines in this plane. Let $s$ be the section of $E_H$, $Z$ its vanishing locus and $J_Z$ its ideal. In fact $Z$ is the complete intersection of 2 conics singular at the same point. Let $I$ be the point/line incidence variety in $H^v \times H$, and pull back on $I$ the resolution of $J_Z$:

$$R^1q_*O_I(-4\tau) \longrightarrow 2R^1q_*O_I(-2\tau) \longrightarrow R^1q_*p^*J_Z \longrightarrow 0$$

to obtain by relative duality:

$$q_*O_I(2\tau + \sigma)^v \longrightarrow 2q_*O_I(-\sigma) \longrightarrow R^1q_*p^*J_Z \longrightarrow 0$$

As $I = \mathbb{P}(\Omega_{H^v}(2)^v)$ with $0 \rightarrow \Omega_{H^v}(2) \rightarrow 3\Omega_{H^v}(1) \rightarrow \Omega_{H^v}(2) \rightarrow 0$, one has: $q_*\Omega_I(k\tau) = Sym_k(\Omega_{H^v}(2))$, and we are reduced to study the degeneracy locus of a map: $2\Omega_{H^v} \longrightarrow Sym_2(\Omega_{H^v}(2))$. This locus contain a reduced line because through a general point $P$ of $H$ there is no conic singular in $P$ containing $Z$ so $a \geq 3$ in the proof of 1.1.3. So the residual scheme of this line in the degeneracy locus has the good dimension, so we can compute its class with the method of the appendix 4.4, which is simpler here because the excess is a Cartier divisor. So this locus is given by:

$$c_2(Sym_2(\Omega_{H^v}(2))) - h.c_1(Sym_2(\Omega_{H^v}(2))) + h^2 = 4h^2$$

where $h$ is the hyperplane class of $H^v$. The geometric interpretation of this residual locus should be that it is made of twice the lines which occur as doubled lines in the pencil of singular conics containing $Z$. So the ideal of the multi-jumping lines of $E_H$ should look like $(y^2, yx^2(x - 1)^2)$ in $H^v$.

### 1.2 Some preliminary results

In the last section we studied the link between trisecant lines to $S$ and jumping planes. We can notice that some hypothetical bad singularities of $S$ give rise to vector bundles with a line in $\mathbb{P}_3^v$ of jumping planes. Unfortunately, such bundles exist, for example when the instanton has a $c_2$-jumping line (Cf [S]). A well understanding of those bundle would give many shortcuts, particularly when $c_2 = 4$, but the following example shows that some may still be unknown.

**Example 1.2.1** There are 4-instanton with a 2-jumping line such that every plane containing this line is a jumping plane.
Let’s construct this bundle from an elliptic curve $C$ and two degree 2 invertible sheaves $\mathcal{L}$ and $\mathcal{L}'$ such that $(\mathcal{L}^v \otimes \mathcal{L}')^\otimes 4 = \mathcal{O}_C$, where 4 is the smallest integer with this property.

Let $\Sigma \xrightarrow{\pi} C$ be the ruled surface $\mathbb{P}(\mathcal{L} \oplus \mathcal{L}')$ and $\Sigma$ its quartic image in $\mathbb{P}_3 = \mathbb{P}(H^0(\mathcal{L}) \oplus H^0(\mathcal{L}'))$. Let $D = \pi^*(\mathcal{L}^v)^\otimes 4 \otimes \mathcal{O}_\Sigma(4)$. The linear system $|D|$ is $h^0((\mathcal{L}^v)^\otimes 4 \otimes \text{Sym}_4(\mathcal{L} \oplus \mathcal{L}')) - 1$ dimensional which is 1 due to the relation between $\mathcal{L}$ and $\mathcal{L}'$. So define the bundle $E$ by the following exact sequence:

$$0 \rightarrow E(-2) \rightarrow 2\mathcal{O}_{\mathbb{P}_3} \rightarrow \phi^*(\pi^*(\mathcal{L}^v)^\otimes 4 \otimes \mathcal{O}_\Sigma(4)) \rightarrow 0$$

where $\phi$ is the morphism from $\Sigma$ to $\mathbb{P}_3$, and where the dimension of $|D|$ proves that $E$ is an instanton.

Let $s$ and $s'$ be the only sections of $\pi^*\mathcal{L}^v \otimes \mathcal{O}_\Sigma(1)$ and of $\pi^*\mathcal{L}'' \otimes \mathcal{O}_\Sigma(1)$. The linear system $|D|$ is made of curves of equation $\lambda s^4 + \mu s'^4$ with generic element a degree 8 smooth elliptic curve section of $E(2)$. Furthermore, when $\lambda$ or $\mu$ vanishes, this element of $|D|$ provides to the double line of $\Sigma$ associated to $s$ or $s'$ a multiple structure. Every plane containing one of those lines (notes $d$ and $d'$) must be a jumping plane, and every ruling of $\Sigma$ is a 4-secant to those section of $E(2)$, so they are 2-jumping lines. In any plane containing $d$ or $d'$, the jumping section vanishes on a scheme of ideal $(y^2, x(x - 1))$ where $y$ is the equation of $d$ or $d'$. So the lines $d$ or $d'$ must be 2-jumping lines, and those sections of $E(2)$ induce on the exceptional divisor of $\mathbb{P}_3$ blown up in $d$ or $d'$ a curve of bidegree $(2, 2)$.

**Remark 1.2.2** For every $c_2$, if there is a line $\delta$, at least 1-jumping for some instanton $E$ such that $\mathcal{J}^k_\delta \otimes E(k)$ has a section, then $\delta$ is a $k$-jumping line, and for $k \geq 2$ this section is irreducible.

Let’s first notice that if $\mathcal{J}^k_\delta \otimes E(k)$ has a section when $\delta$ is a jumping line, then this section is set-theoretically $\delta$. Indeed, consider the blowing up $\overline{\mathbb{P}_3'}$ of $\mathbb{P}_3$ along $\delta$, and denote by $p'$ et $q'$ the projections of $\overline{\mathbb{P}_3'}$ on $\mathbb{P}_3$ et $\mathbb{P}_1$. The restriction of the section of $p'^*E(k\sigma)$ to the exceptional divisor is a section of $\mathcal{O}_{\mathbb{P}_1 \times \mathbb{P}_1}(k, a) \oplus \mathcal{O}_{\mathbb{P}_1 \times \mathbb{P}_1}(k, -a)$ where $\delta$ is a $a$-jumping line. As $a > 0$, this section is in fact a section of $\mathcal{O}_{\mathbb{P}_1 \times \mathbb{P}_1}(k, a)$, so it has no embedded nor isolated points. So the section of $p'^*E(k\sigma)$ don’t have irreducible components which meet the exceptional divisor, but the section of $\mathcal{J}^k_\delta \otimes E(k)$ has to be connected in $\mathbb{P}_3$ for $k \geq 2$ because $h^1E(-2) = 0$. So this section must be set-theoretically $\delta$ when $k \geq 2$.

On another hand, the class in the Chow ring of $\overline{\mathbb{P}_3'}$ of a divisor of the exceptional $\mathbb{P}_1 \times \mathbb{P}_1$ of bidegree $(k, a)$ is: $k\tau^2 + (a - k)\tau\sigma$, but this curve is included in a section of $p'^*E(k\sigma)$ of class $c_2\tau^2$ which gives $a = k$.

The main problem encountered to classify those bundle comes from the fact that the section of $p'^*E(k\sigma)$ may have a multiple structure not included in the exceptional divisor as in the previous example. So we will not try to classify all those bundle, but we will need the following:
Lemma 1.2.3 When $c_2 = 4$ and $h^0(E(1)) = 0$, if there is a $(k \leq 2)$-jumping line $d$, such that every plane containing $d$ is a jumping one, then $\mathcal{J}_d^{\otimes 2} \otimes E(2)$ has a section.

Still blow up $\mathbb{P}_3$ along $d$, the class of the exceptional divisor $x = \mathbb{P}_1 \times d$ is $\tau - \sigma$, and we have the exact sequence:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}_3 \times \mathbb{P}_1}(-\tau - \sigma) \longrightarrow \mathcal{O}_{\mathbb{P}_3 \times \mathbb{P}_1} \longrightarrow \mathcal{O}_{\mathbb{P}_3'} \longrightarrow 0$$

which gives when twisted by $p^*E(\tau)$ using the functor $q'_*$ the following exact sequence because $h^0(E(1)) = 0$.

$$0 \longrightarrow q'_*p'^*E(\tau) \rightarrow H^1E \otimes \mathcal{O}_{\mathbb{P}_1}(-1) \longrightarrow H^1E(1) \otimes \mathcal{O}_{\mathbb{P}_1} \rightarrow R^1q'_*p'^*E(\tau) \rightarrow 0$$

The sheaf $R^1q_*p'^*E(\tau)$ is locally free of rank 1 over $\mathbb{P}_1$ because every plane containing $d$ is a jumping one and $c_2 = 4$. On another hand, the resolution of $x$ in $\mathbb{P}_3'$ gives the exact sequence:

$$0 \rightarrow p'^*E(\sigma) \rightarrow p'^*E(\tau) \rightarrow p'^*E_x(\tau) \rightarrow 0$$

But $p'^*E_x(\tau) \simeq \mathcal{O}_{\mathbb{P}_3 \times d}(0, -a + 1) \oplus \mathcal{O}_{\mathbb{P}_3 \times d}(0, a + 1)$ with $a \leq 2$, and the map $R^1q'_*p'^*E(\sigma) \rightarrow R^1q'_*p'^*E(\tau)$ is a surjection, so $R^1q'_*p'^*E(\tau) \simeq \mathcal{O}_{\mathbb{P}_3}(b)$ with $b \geq 1$ due to the surjection of $H^1E \otimes \mathcal{O}_{\mathbb{P}_1}$ onto $R^1q_*p^*E$. But $h^1(K) = 0$ so $h^1(q'_*p'^*E(\tau)) \leq 2$ and as $q'_*p'^*E(\tau)$ is locally free of rank 3, we can deduce that $h^0(q'_*p'^*E(\tau + \sigma)) \neq 0$. So we have a section of $\mathcal{J}_d \otimes E(2)$ whose restriction to every plane containing $d$ is proportional to the jumping section because the jumping section don’t have a vanishing locus included in a line ( $d$ is not $c_2$-jumping). So in every plane containing $d$, the section of $\mathcal{J}_dE(2)$ must vanish on a conic which has to be twice $d$, so it gives a section of $\mathcal{J}_d^{\otimes 2}E(2)$.

Example 1.2.4 There are bundles $E$ with $h^1E(-2) \mod 2 = 1$, $c_2 = 4$ having a line congruence of multi-jumping lines.

Take 2 skew lines and put on each of them a quadruple structure made by the complete intersection of 2 quadrics singular along those lines. So the disjoint union of these two elliptic quartics gives a section of $E(2)$ where $E$ has $c_2 = 4$ and $h^1E(-2) \mod 2 = 1$ according to [C]. Every line meeting those 2 lines is 4-secant to this section which gives a $(1, 1)$ congruence of 2-jumping lines.

Lemma 1.2.5 For every $c_2$, the t’Hooft bundles (ie $h^0E(1) \neq 0$) have a 1-dimensional scheme of multi-jumping lines.
We can remark that using deformation theory around such a bundle Brun and Hirshowitz (Cf [B-H]) proved that any instanton in some open subset of the irreducible component of \( I \) containing the t'Hooft bundles has a smooth curve as scheme of multi-jumping lines. This is much stronger but unfortunately the t'Hooft bundles are not in this open set because they always have a 3-jumping line when \( n \geq 3 \). (Take a quadrisecant to the section of \( E(1) \) which is made of \( n + 1 \) disjoint lines Cf [H])

So, let \( s \) be a section of \( E(1) \) with vanishing locus \( Z \). For any \( k \in \mathbb{N} \), according to the 1.1.2 the scheme of \((k+2)\)-jumping lines is isomorphic to the scheme of \((k+3)\)-secant lines to \( Z \). The lemma is thus immediate when the lines making \( Z \) are reduced. If it is not the case, a 2-parameter family of trisecant to \( Z \) may arise from 2 kinds of situations:

a) There is a line \( d \) such that the scheme induce on \( d \) by \( Z \) has a congruence of bisecant lines which also meet another line \( d' \) of \( Z \). This congruence must then be set-theoretically the lines meeting \( d \) and \( d' \), so for any plane \( H \) containing \( d' \), the lines passing through \( d \cap H \) are bisecant to \( Z \) at this point. The line \( d \) would then be equipped by \( Z \) of a multiple structure doubled in every plane containing \( d \), which is impossible because those planes would be unstable.

b) The scheme induce by \( Z \) on some line \( d \) (noted \( Z_d \)) has a 2-dimensional family of trisecant lines which can’t lye in a same plane. Every plane containing \( d \) is a jumping one, because the restriction of the section of \( E(1) \) to this plane is vanishing on \( d \), and has to contain infinitely many multi-jumping lines by hypothesis. As those lines are bisecant to the jumping section which is 0 dimensional, those lines must form a pencil through a point of \( d \) by hypothesis. So \( d \) is a multi-jumping line which contradicts the 1.2.2 because \( s \) is a section of \( J_d \otimes E(1) \).

We can now reformulate using the lemma 1.2.5 the theorem of Coanda (Cf [Co]) in a form which will be used at many times in the following:

**Theorem 1.2.6 (Coanda)** For any \( c_2 \), if an instanton has not a 1-dimensional family of multi-jumping lines, then it has at most a 1-dimensional family of jumping planes.

Indeed, Coanda showed that any bundle with a 2-dimensional family of jumping planes is either a special t’Hooft bundle which has its multi-jumping lines in good dimension according to the 1.2.5, or another kind of bundles which are not an instanton. In fact, the last one are in the ”big family” of Barth-Hulek (Cf [Ba-Hu]).

### 1.3 A bound on the bidegree of \( S \)

Let \( (\alpha, \beta) \) be the bidegree of \( S \), where \( \alpha \) (resp \( \beta \)) is the number of lines of the congruence \( S \) passing through a general point of \( \mathbb{P}_3 \) (ie in a \( \alpha \)-plane), (resp in a general plane of \( \mathbb{P}_3 \) (ie in a \( \beta \)-plane)).
Proposition 1.3.1 For any $c_2$, if an instanton has at most a 2-dimensional scheme of multi-jumping lines, and if $h^0(J_P^c_{c_2-4} \otimes E(c_2-4))$ is zero for a general point of $\mathbb{P}_3$, then $\alpha \leq 2c_2 - 6$.

Similarly, any stable plane $H$ without $(c_2)$- and $(c_2-1)$-jumping lines and such that $h^0(J_P^c \otimes E_H(k))$ is zero when $P$ is general in $H$ for $0 \leq k \leq c_2 - 4$, then the $\beta$-plane $h$ associated to $H$ cuts the scheme of multi-jumping lines in length at most $2c_2 - 6$. So if those hypothesis are satisfied for the general plane then $\beta \leq 2c_2 - 6$.

Remark: The above hypothesis about $h^0(J_P^c \otimes E_H(k))$ are always satisfied when $c_2 = 4$ or 5 because one has in any stable plane $H$ without $(c_2)$- and $(c_2-1)$-jumping lines: $h^0(E_H(1)) \leq 2$. Furthermore, we will show for those $c_2$, under the assumption of the existence of $S$, that the general member of any 2-dimensional family of planes is stable from the Theorem of Coanda stated in the [1.2.6], and don’t contain $(c_2)$- or $(c_2-1)$-jumping lines according to the [2.1.3, 2.1.4] and [2.2.1]

Start first with the bound of $\alpha$, so take an $\alpha$-plane $p$ cutting $S$ in length $\alpha$ and denote by $P$ its associated point of $\mathbb{P}_3$. The standard construction associated to the blow up of $\mathbb{P}_3$ at $P$ gives the following exact sequence above the exceptional divisor $p$:

$$0 \rightarrow q_*p^*E \rightarrow H^1E(-1) \otimes (O_p \oplus O_p(-1)) \rightarrow H^1E \otimes O_p \rightarrow R^1q_*p^*E \rightarrow 0$$

The sheaf $q_*p^*E$ is thus a second local syzygy, so it has a 3-codimensional singular locus hence it is a vector bundle denoted by $F$ in the following. For every instanton, one has $h^1E(-1) = c_2$ and $h^1E = 2c_2 - 2$. So $\alpha$ is given by $\chi(R^1q_*p^*E) = c_2 - 2 + \chi(F)$.

The stability of $E$ implies that $F$ has no sections, but it is also locally free of first Chern class $-c_2$, so $\chi(F) = -h^1F + h^0F(c_2 - 3)$. Take a $\mathbb{P}_1$ in $p$ without any multi-jumping lines. The restriction of this sequence to this $\mathbb{P}_1$ gives an injection of $F_{\mathbb{P}_1}$ in $c_2O_{\mathbb{P}_1} \oplus c_2O_{\mathbb{P}_1}(-1)$, so the bundle $F_{\mathbb{P}_1}$ has to split into $O_{\mathbb{P}_1}(-a) \oplus O_{\mathbb{P}_1}(-b)$ with $a + b = c_2$ because this line avoids the multi-jumping lines by hypothesis. We can bound $h^0F_{\mathbb{P}_1}(c_2 - 3)$ by $c_2 - 4$ except when $a = 0$ or 1 (which may happen, for example if $E$ had infinitely many jumping planes). But in the cases $a = 0$ or 1, one has $h^0F_{\mathbb{P}_1}(c_2 - 3) = c_2 - 2 - a$, and as we can assume that $P$ is not on the vanishing of a section of $E(1)$ because for $c_2 \geq 2$ $h^0E(1) \leq 2$, we have $h^0F(1) = 0$, so there is an injection of $H^0F_{\mathbb{P}_1}(1)$ into $H^1F$. Then in the cases $a = 0$ or 1, one has $h^1F \geq 2 - a$. But $h^0F(c_2 - 4)$ is zero by hypothesis so one has $h^0F(c_2 - 3) \leq h^0F_{\mathbb{P}_1}(c_2 - 3)$ thus $\chi(F)$ is also bounded by $c_2 - 4$ in the cases $a = 0$ or 1, which gives $\alpha \leq 2c_2 - 6$.

Let’s now take care of the bound of $\beta$. Take a stable plane $H$ without $(c_2)$- and $(c_2-1)$-jumping lines, and denote by $\beta'$ the length of the intersection of the
β-plane \( h \) with the scheme of multi-jumping lines \( M \). Consider now the incidence variety \( I \subset H^v \times H \). The resolution of \( I \) twisted by \( p^*E_H \) gives the following exact sequence:

\[
0 \longrightarrow q_*p^*E_H \longrightarrow H^1E_H(-1) \otimes \mathcal{O}_{H^v}(-1) \longrightarrow H^1E_H \otimes \mathcal{O}_{H^v} \longrightarrow R^1q_*p^*E_H \longrightarrow 0
\]

One has \( h^1E_H(-1) = c_2 \) and \( h^1E_H = c_2 - 2 \), so \( \beta' = \chi(R^1q_*p^*E_H) = c_2 - 2 + \chi(F) \), where this time \( F \) is \( q_*p^*E_H \) which is still locally free of rank 2 and \( c_1F = -c_2 \), so we have \( \chi(F) = -h^1F + h^0F(c_2 - 3) \). As \( H \) is stable without \( (c_2) \)- and \( (c_2 - 1) \)-jumping lines the vanishing locus of a section of \( E_H(1) \) is at most on one conic from the \( [1.1.2] \), so we have \( h^0E_H(1) \leq 2 \), and we can take a point \( P \) of \( H \) which is neither in a multi-jumping line nor in the vanishing locus of some section of \( E_H(1) \). The line \( p \subset H^v \) associated to \( P \) don’t contain any multi-jumping lines, so we have an injection: \( 0 \rightarrow F' \longrightarrow c_2\mathcal{O}_p(-1) \) and \( F' = \mathcal{O}_{V_1}(-a) \oplus \mathcal{O}_{V_1}(-b) \) with \( a + b = c_2 \). But \( P \) is not in the vanishing locus of a section of \( E_H(1) \) so \( a \) and \( b \) are at least 2 then \( h^0F'(c_2 - 3) \leq c_2 - 4 \). By hypothesis, \( h^0F_p(k) = 0 \) for \( 0 \leq k \leq c_2 - 4 \), so for those \( k \) we have by induction that all the \( h^0F(k) \) are zero using the sequence of restriction of \( F(k) \) to \( p \). In particular we have \( h^0F(c_2 - 4) = 0 \) so \( h^0F(c_2 - 3) \) is bounded by \( h^0F_p(c_2 - 3) \leq c_2 - 4 \), which gives \( \beta' \leq 2c_2 - 6 \). Furthermore, the general plane is stable according to \([Ba1]\), so if it satisfies the conditions on \( h^0(J^\otimes k \otimes E_H(k)) \) then we have \( \beta = \beta' \leq 2c_2 - 6 \). □

## 2 “No” k-jumping lines on \( S \) when \( k \geq 3 \)

The aim of this section is to prove that \( S \) is made of 2-jumping lines except at a finite number of points. Let’s first take care of some extremal cases:

### 2.1 Some too particular jumping phenomena

We want to prove here that \( S \) under the hypothesis of the existence of \( S \) with \( c_2 \leq 5 \), there is at most a \((c_2 - 1 - k)\)-dimensional family of k-jumping lines for \( 3 \leq k \leq 5 \).

**Lemma 2.1.1** For \( c_2 \leq 5 \) and \( i = 0 \) or 1, the support of \( R^1q_*p^*E(1 + i) \) meets any \( \beta \)-plane \( h \) which doesn’t have \((k \geq 4 + i)\)-jumping lines in a scheme of length at most \( c_2 - 3 - i \).

Let’s first get rid of the case of a stable plane \( H \). For \( c_2 \leq 5 \), we can pick a section of \( E_H(1) \), and note \( Z \) its vanishing locus. One has \( h^1E_H(1) = h^1J_Z(2) \) which is at most 1 because \( \deg Z \leq 6 \) and \( Z \) has no \((5 + i)\)-secant because there is no \((4 + i)\)-jumping lines in \( H \). So \( R^1q_*p^*E_H(1) \) is the cokernel of some map \((c_2 - 2)\mathcal{O}_h(-1) \rightarrow \mathcal{O}_h \), and its support is 0-dimensional according to the \([1.1.4]\), so it has length 0 or 1. Similarly, if there is a 4-jumping line in \( H \), then \( R^1q_*p^*E_H(2) \)
is the cokernel of some map $2\mathcal{O}_h(-1) \to \mathcal{O}_h$, so the support of this sheaf has length 1.

If $H$ is a jumping plane, then denote by $Z$ the vanishing locus of the section of $E_H$. So we have to study the $(3 + i)$-secant lines to $Z$ with $\deg Z = c_2$, so it is the same problem as the study of the $(2 + i)$-jumping lines of a stable bundle over $H$ with $c_2 \leq 4$. So the result for $i = 0$ is a deduced from [1.3.1], and to obtain $i = 1$, remark that we have shown above that there is at most one 3-jumping line in a stable plane when $c_2 \leq 4$, so there is at most one 4-jumping line in any plane when $c_2 \leq 5$. □

So we can obtain the following:

**Lemma 2.1.2** The jumping lines of order at least 3 of a 4- or 5-instanton make an at most 1-dimensional scheme.

Indeed, if $E$ has a line congruence of $(k \geq 3)$-jumping lines of bidegree $(\alpha, \beta)$, then $\alpha \leq 1$ and $\beta \leq 1$ because we have shown above that when $c_2 \leq 5$ there was no two 3-jumping lines in a stable plane, and there is at most a one parameter family of jumping planes from Coanda’s theorem stated in [1.2.6]. But the congruence $(0, 1), (1, 0), (1, 1)$ contain lines (Cf [R]), and for those $c_2$ it is impossible to have a plane pencil of 3-jumping lines according to the [1.1.4].

**Remark 2.1.3** When $c_2 \leq 5$, if $E$ has a $c_2$-jumping line $d$, then its multi-jumping lines are 3-codimensional in $G$.

Let’s first recall that the instanton property imply directly that any plane is semistable. Furthermore, any semi-stable plane containing a $c_2$-jumping line doesn’t contain another multi-jumping line because the vanishing locus $Z$ of a section of $E_H(1)$ or of $E_H$ is included in $d$ because $d$ is $(\deg Z)$-secant to $Z$, and the multi-jumping lines are at least 2-secant to $Z$.

But the lines of $\mathbb{P}_3$ meeting $d$ is a hypersurface of $G$, so it would cut $S$ in infinitely many points and we could find a multi-jumping line in the same plane as $d$ which is impossible.

So we will assume in the following that $E$ has no $(c_2)$-jumping lines.

**Proposition 2.1.4** The assumption of the existence of $S$ implies that $E$ has at most a finite number of 4-jumping lines when $c_2 = 5$.

Assume here that there is a 4-jumping line $q$. Consider the hypersurface of $S$ made of the lines of the congruence meeting $q$ (i.e: $T_q G \cap S$). But, when $c_2 = 5$, any plane $H$ containing $q$ and another multi-jumping line can’t be stable from the [1.1.2], because if $H$ was stable, the vanishing locus of a section of $E_H(1)$ would be a scheme of degree 6 with $q$ as 5-secant line, so it couldn’t have another trisecant. We had also showed in the [1.1.4] that any plane containing $q$ has necessarily a
finite number of multi-jumping lines, so the curve \( T_q G \cap S \) can’t be a plane curve, thus any plane containing \( q \) must be a jumping plane, and if there was infinitely many such lines \( q \), there would exist a 2-dimensional family of jumping planes, and we could conclude with Coanda’s theorem as in the 1.2.6.

2.2 Curves of 3-jumping lines

One wants here to prove that there is no curve of \((k \geq 3)\)-jumping lines lying on \( S \). When \( c_2 = 4 \) this is exactly the method of Coanda, but there are many more problems when \( c_2 = 5 \) because the ruled surface used may have a singular locus. Let’s first consider the case of a curve of 3-jumping lines.

**Proposition 2.2.1** No 4-instanton with \( h^0 E(1) = 0 \) has a curve of jumping lines of order exactly 3.

When \( c_2 = 5 \), if \( E \) has a surface \( S \) of multi-jumping lines, then there is no curve of 3-jumping lines on \( S \).

Assume that there is such a curve of 3-jumping lines (ie without 4-jumping lines), and take \( \Gamma \) be an integral curve made only of 3-jumping lines. Denote by \( \Sigma \) the ruled surface associated to \( \Gamma \), and \( \tilde{\Sigma} \to \Sigma \) its smooth model.

- **When \( c_2 = 4 \)**

  We showed in the 2.1.1 that two 3-jumping lines can’t cut one another, so according to Coanda’s result (Cf [Co]), the curve \( \Gamma \) is either a regulus of some quadric \( Q \) of \( \mathbb{P}_3 \), or the tangent lines to a skew cubic curve, so in both cases, one has \( \Sigma \simeq \mathbb{P}_1 \times \mathbb{P}_1 \). Let’s now apply the method used by Coanda in [Co]. As there is no 4-jumping lines in \( \Gamma \), we have the exact sequence for some divisor \( A \) of degree \( a \) in \( \text{Num}(\Sigma) \):

  \[
  0 \to \mathcal{O}_\tilde{\Sigma}(-A, 3) \to \pi^* E_\Sigma \to \mathcal{O}_\tilde{\Sigma}(A, -3) \to 0
  \]

  which gives \( c_2(\pi^* E_\Sigma) = d.c_2 = 2a3 \), where \( d = \deg \Sigma \) is 2 or 4. So it gives a contradiction because here \( c_2 = 4 \).

- **When \( c_2 = 5 \)**

  We also have such an exact sequence, but this time \( \tilde{\Sigma} = \mathbb{P}_\Gamma(F^\vee) \) where \( \Gamma \) is the normalization of \( \Gamma \), and \( F \) is a rank 2 vector bundle having a section (denote by \( \mathcal{L} \) its cokernel), and such that \( h^0 F(l) = 0 \) for any invertible sheaf \( l \) of negative degree.

  The surjection from \( F \) to \( \mathcal{L} \) gives a section \( C_0 \) of \( \tilde{\Gamma} \) into \( \tilde{\Sigma} \). Let \( e = -\deg F \), the intersection form in \( \text{Num}\tilde{\Sigma} \) is given by \( \begin{pmatrix} 0 & 1 \\ 1 & -e \end{pmatrix} \) in the basis \((f, C_0)\) where \( f \) is the class of a fiber, so we have the relation \( 5d = 6a + 9e \) which proves that \( d \) is a multiple of 3. On another hand, the dual ruled surface \( \Sigma^\vee \) has no triple points.
because there is at most two 3-jumping lines in a same plane from the 2.1.1, so we have the relation from [K]:

\[ (d - 2)(d - 3) - 6g(d - 4) = 0 \]

(1)

because \( \bar{\Sigma} \) and \( \bar{\Sigma}^v \) have same degree and genus. So when \( d = 3 \) we can work as in [Co] because it gives \( g = 0 \), and \( \bar{\Sigma} = IP(O_{IP_1} \oplus O_{IP_1}(1)) \), so \( e = 1 \) and \( a = 1 \), then we have:

\[ \pi^*O_{\Sigma}(1) \sim_{num} O_{\Sigma}(2, 1) \]

and

\[ h^2(\pi^*E_{\Sigma}(-2)) = h^0(O_{IP_1}(-a + 4 + 2g - 2 - e) \otimes Sym_3 F) \]

which imply \( h^2E_{\Sigma}(-2) \geq 1 \) when taking sections in the following sequence where \( C' \) is at most 1-dimensional.

\[
0 \rightarrow E_{\Sigma}(-2) \rightarrow \pi_\ast\pi^*E_{\Sigma}(-2) \rightarrow E_{\Sigma}(-2) \otimes \omega_{C'} \rightarrow 0
\]

The resolution of \( \Sigma \) in \( IP_3 \) twisted by \( E(-2) \) gives then \( h^3E(-5) \geq 1 \), so \( h^0E(1) \geq 1 \) which contradicts the hypothesis. Thus \( d \) is a multiple of 3 and \( d \geq 6 \). We will now need the following 3 lemmas before going on proving the proposition 2.2.1.

Lemma 2.2.2 A generic ruling of \( \Sigma \) can’t meet points of multiplicity 3 or more of \( \Sigma \)

If the opposite was true, there would be a point \( P \) of \( \Sigma \) included in 3 distinct ruling of \( \Sigma \). Those 3 lines would give 3 triples points of the pentic curve representing the jumping lines through \( P \). This pentic must then be reducible, and those triple points have to lie in a same line which contradicts the fact that a plane has at most two 3-jumping lines (Cf 2.1.1).

Lemma 2.2.3 Any generic ruling of \( \Sigma \) meets only reduced components of the double locus of \( \Sigma \)

Assume that the double locus of \( \Sigma \) has a reducible component meeting all the rulings. So any point of this component is a point \( P \) of the double locus where the tangent cone \( C_P\Sigma \) is a double plane, so this plane must contain the 2 rulings passing through \( P \). On another hand, there are pinch points on \( \Sigma \) because the number \( 2d + 4(d - 1) \) (Cf [K]) is not zero when \( d \) is multiple of 3 and solution of \[4]. So let \( P' \) be a pinch point of \( \Sigma \), then there is through \( P' \) a ruling \( d \) of \( \Sigma \) such that the tangent plane to \( \Sigma \) at any point of \( d \) is a same plane \( H \). This means that the tangent space to \( \Gamma \) at \( d \) is included in the \( \beta \)-plane \( h \) associated to \( H \). So \( \Gamma \cap h_{\{d\}} \) has length at least 2. But under the above hypothesis we can find another ruling of \( \Sigma \) in \( H \). Indeed, by assumption \( d \) meets a non reduced component of the double locus in some point \( P \), so there is another ruling \( d' \) of \( \Sigma \) through \( P \) and we showed that \( C_P\Sigma \) was set-theoretically the plane containing \( d \) and \( d' \), but as the tangent space is constant along \( d \), \( H \) has to be included in \( C_P\Sigma \) so \( d' \) is also in \( H \) and \( h \cap \Gamma \) has length at least 3 which contradicts the 2.1.1.
Lemma 2.2.4 A generic ruling of $\Sigma$ meets at least 4 other distinct rulings of $\Sigma$.

a) One have first to prove that a generic ruling of $\Sigma$ meets the double locus in distinct points. According to the 2.2.3, the only case where this could be not satisfied is when the generic ruling of $\Sigma$ is tangent to some component of the double locus. That is the case where $\Sigma$ is a developable surface. The cone situation is immediate for the lemma 2.2.4, so we will first consider the case where $\Sigma$ is a the developable surface of tangent lines to some curve $C$.

For any point $P$ of $C$, the osculating plane $H_{P}$ to $C$ must coincide with the tangent plane to $\Sigma$ at any point of the ruling $d_{P}$ which is tangent to $C$ at $P$. Furthermore, the tangent space to $\Gamma$ at $d_{P}$ is made of the pencil of lines of $H_{P}$ containing $P$, so it is included in the $\beta$-plane associated to $H_{P}$.

We can also remark that for any point $P'$ of $C$ distinct from $P$, the planes $H_{P}$ and $H_{P'}$ are distinct, otherwise we the $\beta$-plane associated would cut $\Gamma$ in length 4 which contradicts the 2.1.1. So we can translate this by the fact that the dual surface $\Sigma^{v}$ is a developable surface of tangent lines to a smooth curve $C^{v}$.

Consider the projection of $C^{v}$ to $\mathbb{P}_{2}$ from a point of $\mathbb{P}_{3}$ not in $\Sigma^{v}$. The degree of $\Gamma$ is also the degree of the dual curve of this projection, which is according to Plücker’s formula $d = 2(\deg C^{v} - 1) + 2g$ because this projection don’t have cusp. So when $\Sigma$ is developable, the case $d = 6, g = 2$ is not possible, then $d \geq 9$. Let $t$ be a general point of $\Gamma$, then the contact between $T_{t}G$ and $\Gamma$ can’t be of order 6 or more, otherwise the osculating plane to $\Gamma$ at $t$ would be included in $G$, so there would exist a $\beta$-plane meeting $G$ in length 3 or more which contradicts the 2.1.1. But the contact between $T_{t}G$ and $\Gamma$ must be even (Cf [Co] lemma 6), so $T_{t}G \cap \Gamma - \{t\}$ is made of $d - 4$ distinct points because in general $t$ is not bitangent to $C$ because $\Gamma$ is reduced. So, when $\Sigma$ is developable a generic ruling of $\Sigma$ meets at least 5 other distinct ones.

b) So we can assume that $\Sigma$ is not developable, and using the 2.2.3, that a generic ruling of $\Sigma$ meets the double locus in distinct points. One have thus to compute the number of such points. But as $\Sigma$ is not developable, in general a point $t$ of $\Gamma$ is such that the intersection of $T_{t}G$ with $\Gamma - \{t\}$ has length $d - 2$, which gives the lemma because $d \geq 6$. $\blacksquare$

We are now ready to continue the proof of 2.2.1. We obtained in the 2.2.4 that through any ruling $t$ of $\Sigma$ there are at least 4 distinct planes containing another ruling of $\Sigma$. So according to the 2.1.1, we have at least 4 jumping planes through $t$. We can assume that there are only a finite number of jumping planes through $t$ when $t$ is general because the bundle $E$ don’t have a 2-dimensional family of jumping planes (1.2.4). Let’s blow up $\mathbb{P}_{3}$ along $t$ and denote by $p'$ and $q'$ the projections over $\mathbb{P}_{3}$ and $\mathbb{P}_{1}$. We have from the standard construction the following sequence:
0 \to q'_* p'^* E(\tau) \to H^1 E \otimes \mathcal{O}_Y(-1) \quad \longrightarrow \quad H^1 E(1) \otimes \mathcal{O}_Y \to R^1 q'_* p'^* E(\tau) \to 0

where \( h^1 E = 8, h^1 E(1) = 7 \), and where \( R^1 q'_* p'^* E(\tau) \) is not locally free at the jumping planes but has rank 1 because in a stable plane \( H \) containing the 3-jumping line \( t \), the 6 points of the vanishing locus of a section of \( E_H(1) \) must lie on a conic because \( t \) is 4-secant to this locus. The sheaf \( q'_* p'^* E(\tau) \) is thus locally free of rank 2, and we have: \( h^0(q'_* p'^* E(\tau)) = 0, h^1(K) = 0, h^1(q'_* p'^* E(\tau)) = h^0(K) \), et \( h^0(R^1 q'_* p'^* E(\tau)) = 7 - h^1(q'_* p'^* E(\tau)). \)

If \( h^1(q'_* p'^* E(\tau)) = 0 \) or 1 then \( q'_* p'^* E(\tau) = \mathcal{O}_Y(-1) \oplus \mathcal{O}_Y(-1 \text{ or } -2) \), so \( [q'_* p'^* E(\tau)](1) \) would have a section and \( J_2 \otimes E(2) \) also. But this phenomena must occurred for a generic \( t \), so we have \( h^0 E(2) \geq 2 \), but the vanishing locus of those sections of \( E(2) \) would then be in a quartic surface, which must contain every 3-jumping line because those lines are 5-secant to those vanishing locus. But it contradicts deg \( \Sigma \geq 6 \).

So we have \( h^0(R^1 q'_* p'^* E(\tau) \leq 5 \). On another hand \( R^2 q'_* p'^* E(\tau) \) is zero from Grauert’s theorem, so we have by base change for all \( y \) in \( P_1 \):

\( R^1 q'_* p'^* E(\tau)_{\mid y} \simeq H^1 E_Y(1) \) where \( Y \) is the plane associated to \( y \). So we have the exact sequence:

\[
0 \longrightarrow \bigoplus_{\text{k times}} \mathcal{O} \longrightarrow R^1 q'_* p'^* E(\tau) \longrightarrow R^1 q'_* p'^* E(\tau)^{\text{vv}} \longrightarrow 0
\]

where \( k \) is the number of jumping planes through \( t \), so \( k \geq 4 \) from the \( 2.2.4 \). But \( R^1 q'_* p'^* E(\tau)^{\text{vv}} \simeq \mathcal{O}_Y(l) \) with \( l \geq 0 \), so necessarily \( l = 0, k = 4 \). Then \( d = 6 \) so \( g = 2 \) from the formula \( \square \). We also have \( h^0(R^1 q'_* p'^* E(1)) = 5 \) and \( q'_* p'^* E(1) \simeq \mathcal{O}_Y(2) \oplus \mathcal{O}_Y(-2) \), so \( h^0 J_2 \otimes E(3) \geq 2 \).

Take \( s \) and \( s' \) 2 sections of \( J_2 \otimes E(3) \). Those sections are proportional on a sextic surface containing the 4 jumping planes \( H_i \) passing through \( t \) because the restriction of \( s \) and \( s' \) to \( H_i \) are multiple of \( t^2 \) and \( E_H(1) \) can’t have a section not proportional to the section of \( E_H \), because there are no 5-jumping lines according to the \( 2.1.3 \). So denote by \( Q \) the quadric surface such that \( Z_{s,s'} = \bigcup_{i=1}^{k} H_i \cup Q \).

First notice that for a general plane \( H \) containing \( t \), the sections \( s \) and \( s' \) gives, after the division by \( t^2 \), 2 sections of \( E_H(1) \) which are proportional on a conic which must be \( Q \cap H \). Furthermore, this conic must contain \( t \) because \( t \) is 4-secant to the vanishing locus of any section of \( E_H(1) \). Denote by \( d_H \) the line such that \( t \cup d_H = Q \cap H \).

Next, assume that \( G \) is drawn on \( S \) as in the hypothesis of \( 2.2.1 \). The curve \( T_G \cap S \) is made of multi-jumping lines meeting \( t \), and is an hyperplane section of \( S \). We want here to understand this curve by cutting it with planes containing \( t \). First remark that there is no irreducible component of \( T_G \cap S \) included in
some β-plane containing \( t \), otherwise its associated plane would be a jumping one according to the [1.1.4], so it would be one of the \( H_i \), but the \( H_i \) contain two 3-jumping lines, so they contain only a finite number of multi-jumping lines using again the [1.1.4]. So the points of \( T_t G \cap S \) which represent lines lying in a stable plane \( H \) containing \( t \) make a dense subset of \( T_t G \cap S \). Now we can notice that in a general plane \( H \) containing \( t \), the only possible multi-jumping lines are trisecant to any vanishing locus of a section of \( E_H(1) \), so they must be \( t \) or \( d_H \). But \( d_H \) is always in \( Q \), so the curve \( T_t G \cap S \) must be the rulings of \( Q \), and it would imply \( \deg S = 2 \), then the curve \( \Gamma \) of degree 6 and genus 2 could not lie on \( S \) because \( h^0\mathcal{O}_\Gamma(1) = 6 \), and it gives the proposition [2.2.1].

We’d like now to enlarge the result [2.2.1] to the case where the previous curve \( \Gamma \) contains 4-jumping lines when \( c_2 = 5 \).

**Proposition 2.2.5** When \( c_2 = 5 \), the surface \( S \) don't contain a curve of jumping lines of order 3 in general with some 4-jumping lines

So we assume that there is such a curve, and let \( \Gamma \) be an irreducible curve of jumping lines of order 3 in general with at least a 4-jumping line \( q \). Denote by \( \Sigma \) the ruled surface associated to \( \Gamma \).

First notice that \( \chi(E(2)) = 0 \), and the existence of \( q \) implies \( h^1E(2) \neq 0 \), so \( E(2) \) has a section \( s \) vanishing on some degree 9 curve \( Z \). Furthermore, any 3-jumping lines is 5-secant to \( Z \). The part of the proof of the [2.2.1] bounding the number of jumping planes containing a general ruling \( t \) of \( \Sigma \) is still valid in this situation, so we still have \( \deg \Gamma \leq 6 \).

a) Now notice that \( \Gamma \) can’t be a conic. In fact this is the worst case because the bundles of the family [4.5.1] have such a conic of 3-jumping lines with four 4-jumping lines. So the required contradiction need to use the existence of \( S \). In that case \( \Sigma \) is a quadric so we have:

\[
0 \longrightarrow \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-a, 3) \longrightarrow E_\Sigma \longrightarrow \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(a, -3) \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-3) \longrightarrow 0
\]

which gives \( 10 = 6a + n \) where \( n \) is the number of 4-jumping lines on \( \Gamma \). So \( a \) would be 0 or 1 then \( h^0E_\Sigma(2) \geq 12 \). The resolution of the quadric twisted by \( E(2) \) gives the sequence:

\[
0 \longrightarrow H^0E(2) \longrightarrow H^0E_\Sigma(2) \longrightarrow H^1E
\]

So \( h^0E(2) \geq 4 \), and we can solve this case with the [4.5.3].

For the other possible degrees of \( \Gamma \), the situation is easier because none of the ruling of \( \Sigma \) can meet \( q \) because in any plane \( H \) containing \( q \), this line is according to the [1.1.2] (deg \( Z - 1 \))-secant to the 0-dimensional vanishing locus \( Z \) of a section of \( E_H \) or of \( E_H(1) \), and any ruling of \( \Sigma \) is at least 3-secant to \( Z \).

So if \( \Sigma \) has a double locus, it must meet \( q \) in a pinch point, so \( q \) is a torsal line. In other words, there is a line \( L \) included in \( G \) meeting \( \Gamma \) in length at least 2
around \( q \). So, the support of the sheaf \( R^1 q_* p^* E(\tau)_L \) must contain twice the point \( q \). So this sheaf can’t be equal to its restriction at the reduced point \( q \), but it has rank 2 at \( q \), so \( R^1 q_* p^* E(\tau)_L \geq 3 \) which is not possible around a 4-jumping line, because the plane associated to \( L \) can’t be stable, hence according to the 1.1.2 it is for \( c_2 = 5 \) the same problem as the study of the 1.1.1 for \( c_2 = 4 \) around a 3-jumping line, and the proof of 1.1.1 was still valid for \( c_2 = 4 \) in this situation.

If \( \Sigma \) has no singular locus, then as it can’t be a quadric, it is the tangential surface of a skew cubic, hence all its lines are torsals, so we can conclude as previously. \( \square \)

We also have the following:

**Corollary 2.2.6** For \( c_2 = 5 \), any integral curve in \( G \) made only of 2-jumping lines has a degree multiple of 4. The surface \( S_{\text{red}} \) has a degree multiple of 4.

Indeed, any integral curve of degree \( d \) made only of 2-jumping lines gives using the previous notations, a ruled surface such that \( 5d = 4a + 4e \). But the results 2.1.3, 2.1.4, 2.2.1, and 2.2.5 imply the general hyperplane section of \( S_{\text{red}} \) don’t contain any \( (k \geq 3) \)-jumping lines.

## 3 Trisecant lines to \( S \) and applications

In the first section we obtained an interpretation of some trisecant lines to \( S \). The key application of this interpretation is the following proposition:

### 3.1 Trisecant lines to the general hyperplane section of \( S \)

**Proposition 3.1.1** When \( c_2 \leq 5 \), the general hyperplane section of \( S_{\text{red}} \) has no trisecant lines.

Indeed, on one hand the generic hyperplane section of \( S_{\text{red}} \) contains no \( (k \geq 3) \)-jumping lines according to the section 2.1.3, and on the other hand, the lines of \( \mathbb{P}_5 \) which are in a hyperplane make a 2-codimensional subscheme of the Grassmann manifold \( G(1, 5) \). So if the general hyperplane section of \( S_{\text{red}} \) had a trisecant line then \( S_{\text{red}} \) would have a 2-parameter family of trisecant lines meeting \( S \) in 2-jumping lines. But any of those trisecant lines would give a jumping plane according to the 1.1.1, and as there is only a finite number of such trisecant lines in a same jumping \( \beta \)-plane according to the 1.1.3, we would have a 2 parameter family of jumping planes which would contradict the 1.2.3. \( \square \)

### 3.2 The cases where \( S \) is ruled

- Let’s first consider the case where \( S \) is a ruled surface when \( c_2 = 5 \).
Every ruling of $S$ must contain a 3-jumping line according to the $\text{[1.1.4]}$, but there is at most a finite number of 3-jumping lines on $S$ from the $\text{[2.2.1]}$ and the $\text{[2.2.5]}$. So $S$ must be a cone with vertex a 3-jumping line $t$, and this cone is in the $\mathbb{P}_4$ constructed by projectivisation of the tangent space $T_tG$. According to the $\text{[1.1.4]}$, any plane contain at most one pencil of multi-jumping lines, so $S_{\text{red}}$ must have bidegree $(1, \beta)$, and the bound on $\beta$ of the $\text{[3.3]}$ and the $\text{[2.2.6]}$ imply that $\beta = 3$. But the projection from $t$ of this cone gives a curve of bidegree $(1, 3)$ in the quadric obtained by projection of $T_tG \cap G$. So there are many planes containing 3 distinct rulings of $S_{\text{red}}$, then the generic hyperplane section of $S_{\text{red}}$ would have trisecant lines which would contradict the $\text{[3.1.1]}$.

• So assume here that $S$ is a ruled surface and that $c_2 = 4$.

Denote by $C$ the reduced curve described by the center of the pencils of lines associated to the rulings of $S$.

- If $\deg C \geq 3$

A general hyperplane $H$ meeting $C$ in distinct points contain a line of the congruence $S$ passing through each point of $C \cap H$. As $\beta \leq 2$ and $\deg C \geq 3$, those lines must be bisecant to $C$, and the other bisecant to $C$ are not in the congruence $S$. Hence $S$ is a join between 2 components of $C_1$ and $C_2$ of $C$ with $\deg C_1 = 1$ because $S$ must be ruled. So every plane $H$ containing $C_1$ contain a pencil of multi-jumping lines as in the $\text{[1.1.4]}$ which is centered at a point of $C_2 \cap H$. So those plane can’t be stable and according to the $\text{[1.2.3]} J^2_C \otimes E(2)$ has a section. Furthermore, this section vanishes at least with multiplicity 4 on $C_1$ and also on $C_2$, but it has degree 8 and must be connected, then $S$ would be a $\beta$-plane which is not possible.

- If $\deg C = 1$

Then $S$ is a cone with vertex the point representing $C$, and once again every plane containing $C$ would contain infinitely many multi-jumping lines, hence from the $\text{[1.2.3]} J^2_C \otimes E(2)$ has a section $s_0$ vanishing at least with multiplicity 4 on $C$ and on the curve $C'$, where $C'$ is the curve obtained by the center of the pencils of multi-jumping lines which are in planes containing $C$. So we must have $\deg C' = \alpha = 1$ and the connexity of the vanishing locus imply that $C = C'$. So, in any plane $H$ containing $C$, the bundle $E_H$ has a section $s_H$ of type $\text{[1.1.4]}$ such that the connected component of $Z_{s_H}$ is on $C$, and any section $\sigma$ of $E_H(1)$ is proportional to $s_H$, otherwise its vanishing locus $Z_\sigma$ would be in the line $Z_{\sigma \wedge s_H}$ which would be a 4-jumping line, but it would contradict the $\text{[1.1.4]}$. Therefore, every section of $J_C \otimes E(2)$ must be a section of $J^2_C \otimes E(2)$.

Furthermore, if we blow up $\mathbb{P}_3$ along $C$, the bundle $q_* p^* E$ has rank 1, so $h^0 J^2_C \otimes E(2) = 1$. This proves that the sections of $E(2)$ have no base curves, and the number of base points is the number of residual points in the intersection.
of the 3 quartics of $H^0J_{Z_0}(4)$ which is 0 from [Fu] p155. So from Bertini’s theorem, there is a smooth section $s$ of $E(2)$, such that $Z_s$ is also connected from $h^1E(-2) = 0$, and $Z_s \cap C = \emptyset$. The quartic surface $Z_{s_0 \wedge s}$ must cut any plane $H$ containing $C$ in twice $C$, and a conic which must be singular in some $P_H \in C$ because it contains $Z_{sh}$. Furthermore, this conic doesn’t contain $C$ because $Z_s \cap C = \emptyset$ and the lines through $P_H$ are 0- or 4-secant to $Z_s$. So this quartic is a ruled surface with $C$ as directrix, but $C$ is in its singular locus so there are rulings of $Z_{s_0 \wedge s}$ through $P_H$ which are not in $H$. So $C$ must be a triple curve of $Z_{s_0 \wedge s}$, which imply that this quartic has a rational basis. But it contradicts the formula of Segre (Cf [G-P2]) applied to $Z_s$ because it is a degree 8 elliptic curve 4-secant to the rulings of $Z_{s_0 \wedge s}$.

- When $\deg C = 2$

The plane $H$ containing the conic $C$ must then contain infinitely many multi-jumping lines which have to contain a same point $O$ according to the 1.1.4. But the lines of the congruence are secant to $C$, so only one of them contain a fixed point $P$ of $H - \{C, O\}$, then $\alpha = 1$. We had already eliminated the cases of the congruences $(1, 1)$ ($\deg C = 1$), and the congruences $(1, 2)$ are the joins between $C$ and a line $d$ cutting $C$ in 1 point (Cf [R]). As previously, every plane containing $d$ would be of type 1.1.4, and the section of $J^2_d \otimes E(2)$ would vanish on $d \cup C$ at least with multiplicity 4, which is not possible because it has degree 8. $\square$

### 3.3 “No” trisecant cases

The aim of this section is to get rid of the remaining cases.

**Lemma 3.3.1** For $c_2 = 4$ and 5, the congruence $S_{\text{red}}$ can’t have degree 4.

The degree 4 surfaces are classified in [S-D], and as we already eliminate the cases of ruled surfaces in §3.2, only the complete intersection of 2 quadrics in $\mathbb{P}_4$ and the Veronese are remaining.

- For $c_2 = 4$

The Veronese has bidegree $(3, 1)$ or $(1, 3)$ so it would contradict the 1.3.1. If $S$ is the complete intersection of $G$ with another quadric and some $\mathbb{P}_4$ then $S$ has bidegree $(2, 2)$ in $G$ (Cf [A-S]). Furthermore, in that case $S$ is locally complete intersection, so we can compute from the 1.4.1 that the scheme of multi-jumping lines $M$ is made of $S$ with a non empty residual curve $C$. There is a 2 dimensional family of $\beta$-planes which meet $C$, so from the 1.2.4 its general element must be a stable plane, then it must cut $S$ in good dimension according to the 1.1.4. But those $\beta$-planes will contain $\beta + 1 = 3$ multi-jumping lines (with multiplicity) which is not possible when $c_2 = 4$ (Cf 1.3.1). We can remark that it is still true when $C$ is drawn on $S$. Indeed, one has the exact sequence:

\[ 21 \]
where $C$ is the support of $\mathcal{L}$. Restrict it to a stable $\beta$-plane $h$ which meet $C$ to have:

$$
\text{Tor}_1(\mathcal{O}_S, \mathcal{O}_h) \rightarrow \mathcal{L} \otimes \mathcal{O}_h \rightarrow \mathcal{O}_{M \cap h} \rightarrow \mathcal{O}_{S \cap h} \rightarrow 0
$$

As $h$ is stable, it cuts $S$ in dimension 0, so $\text{Tor}_1(\mathcal{O}_S, \mathcal{O}_h) = \mathcal{J}_S \cap \mathcal{J}_h / \mathcal{J}_S \mathcal{J}_h = 0$, which still gives length $\mathcal{O}_{M \cap h} \geq 3$.

• When $c_2 = 5$

The complete intersection of 2 quadrics in $\mathbb{P}_4$ is a Del-Pezzo surface isomorphic to $\mathbb{P}_2$ blown up in 5 points embedded by $3L - E_1 - ... - E_5$. According to the §2, the surface $S$ contains at most a finite number of $(k \geq 3)$-jumping lines, so we can find on $S$ some cubic curves made only of 2-jumping lines, which contradicts the §2.2.6. Similarly, in the Veronese case, there would have many conics made only of 2-jumping lines which also contradicts the §2.2.6. □

**Proposition 3.3.2** For $c_2 = 4$ or 5, the scheme of multi-jumping lines of a mathematical instanton is a curve in $G$.

In fact the case $c_2 = 4$ is already done because $S$ can’t have degree 4 (3.3.1), and the bound of its bidegree of the §3.3 imply that $\alpha = 1$ or $\beta = 1$. But those congruences are ruled (Cf [R]) and we can conclude with the §3.2.

For $c_2 = 5$, let’s first prove the following:

**Lemma 3.3.3** The set of points $p$ of $S$ such that $\dim T_p S_{\text{red}} = 4$ is finite.

Assume that there are infinitely many such $p$, then the general hyperplane section contains one of them, so in general, the surface $S_{\text{red}}$ has no trisecant through $p$ (Cf §3.1.1), so the curve $T_p G \cap S_{\text{red}}$ must be an union of lines containing $p$, and as $S$ is not ruled (Cf §3.2), it can happen only at finitely many $p$ which proves the lemma. □

So a general hyperplane section $\Gamma$ of $S_{\text{red}}$ is locally complete intersection with at most finitely many $p$ such that $\dim T_p \Gamma = 2$. According to the §3.1.1, the curve $\Gamma$ has no trisecant, and we can adapt LeBarz’s formula of [LeBa2]. In fact, when projecting $\Gamma$ to $\mathbb{P}_2$, the singularities of $\Gamma$ won’t gives embedded points, so they have to be removed from the contribution of the apparent double points. Hence the number of trisecant to $\Gamma$ is $(d-4)((d-2)(d-3)−6\pi)$ where $\pi$ is the arithmetic genus of $\Gamma$.

So we have from the §3.1.1 and §3.3.1 that $(d-2)(d-3)−6\pi = 0$. Thus $d−1$ is not a multiple of 3, and then $\Gamma$ is locally complete intersection with maximal
genus in \( \mathbb{P}_4 \), because if \( \epsilon \) is the remainder of the division of \( d - 1 \) by 3, and \( m = \left\lfloor \frac{d-1}{3} \right\rfloor \), then \( \frac{(d-2)(d-3)}{6} = \frac{e^2 - 3e + 2}{6} + \frac{3m(m-1)}{2} + m\epsilon \) because \( \epsilon \neq 0 \).

According to the 3.3.1 and the 2.2.6, we have \( \deg \Gamma = 8 \), and as it has maximal genus in \( \mathbb{P}_4 \), it must be a canonical curve, and it can’t be trigonal (ie have a \( g_1^3 \)) because it has no trisecant. So according to Riemann-Roch \( \Gamma \) hence \( S \) must be the complete intersection of 3 quadrics. So \( S \) is the Kummer K-3 surface and has bidegree \( (4, 4) \) (Cf [A-S]). We can compute from the 4.4.1 that the scheme of multi-jumping lines has a residual curve because \( S \) is locally complete intersection, and we construct as in the 3.3.1 when \( c_2 = 4 \) a 2 parameter family of \( \beta \)-planes containing 5 multi-jumping lines (with multiplicity), which contradicts the 1.2.6.

\[ \blacksquare \]

4 Applications to moduli spaces

4.1 The construction of Ellingsrud and Stromme

Let us recall here the construction made in [E-S], and show how the previous results could be used to study \( I_n \).

Let \( N \) be a point such that there exists an instanton \( E \) which has only a one dimensional family of jumping lines through \( N \) which are all of order 1. Denote by \( U_N \) the subscheme of \( I_n \) made of instantons which don’t have a multi-jumping line through \( N \) and which have a non jumping line through \( N \). The result 3.3.2 and Grauert-Mullich’s theorem implies that the \( (U_N)_{N \in \mathbb{P}_3} \) cover \( I_n \) for \( n = 4 \) or 5.

To describe \( U_N \), first blow up \( \mathbb{P}_3 \) at \( N \), denote by \( Y \) the plane parameterizing the lines containing \( N \), \( p \) and \( q \) the projections on \( \mathbb{P}_3 \) and \( Y \), and \( \tau = p^* \mathcal{O}_{\mathbb{P}_3}(1) \), \( \sigma = q^* \mathcal{O}_Y(1) \), and by \( V = H^0(\mathcal{O}_Y(1)) \). Let \( H \) be a \( \mathbb{C} \)-vector space of dimension \( n \). (\( H \) will stand for \( H^2 E(-3) \)). A point \( E \) of \( U_N \) is characterized (after the choice of an isomorphism: \( H \cong H^2 E(-3) \)) by the following data where \( F \) is naturally \( q_* p^* E \) and \( \theta(1) \) is \( R^1 q_* p^* E(-1) \):

\[
\begin{align*}
1) & \quad m \in V \otimes S_2 H^v \\
2) & \quad \text{a surjection:} \quad 2\mathcal{O}_{\mathbb{P}_2(V^v)} \to \theta(2) \to 0 \quad \text{(denote by \( F \) its kernel)} \\
3) & \quad \text{and a surjection:} \quad q^* F \to q^* \theta(\sigma + \tau) \to 0
\end{align*}
\]

In fact, the symmetric map \( m \) is just the restriction to the \( \alpha \)-plane associated to \( N \) of the 2\( n + 2 \) rank element of \( \Lambda^2 H^0(\mathcal{O}_{\mathbb{P}_3}(1)) \otimes S_2 H^v \) of Tjurin, (Cf [T2] or [LeP]) and it can be obtained from the following sequence coming from Beilinson’s spectral sequence.

\[
0 \to H \otimes \mathcal{O}_Y \xrightarrow{m} H^v \otimes \mathcal{O}_Y(1) \to \theta(2) \to 0
\]  

We have also to recall from [E-S] that the data 3) exists only under the condition that the restriction of \( F \) to \( C \) splits. To be more precise, the restriction of the

\[ \blacksquare \]
data 2) to $C$ gives a sequence

$$0 \longrightarrow \theta^v(-1) \longrightarrow F_C \longrightarrow \theta^v(-2) \longrightarrow 0 \quad (3)$$

which has to split for the existence of the data 3). This is the main difficulty of this description of $U_N$. So let’s first globalize data 2).

The group $P = \frac{GLH}{\pm Id}$ acts on $H$ by ‘p.h.p, and it also acts on the previous data according to the exact sequence (2). Furthermore, 2 elements of $U_N$ are isomorphic if and only if they are in the same orbit of this action.

Our reference about properties and existence of the moduli space of those theta characteristic will be [So]. A net of quadrics will be called semi-stable (stable) if and only if, for all non zero totally isotropic (for the net) subspace $L$ we have: $\dim L + \dim L^\perp \leq \dim H < \dim H$. Let $\Theta$ be the quotient $(V \otimes S_2 H^v)^* / P$, and $\Theta^s_f$ be the stable points which represent locally free sheaves, and $\Theta^s_0$ be the dense open subset of $\Theta$ made of isomorphic classes of $\theta$-characteristic which have a smooth support. The space $\Theta$ is irreducible of dimension $\frac{n(n+3)}{2}$, and normal with rational singularities (Cf [So] théorème 0.5), and $\Theta^s_f$ is smooth (Cf [So] théorème 0.4) because when $\theta$ is locally free, the notion of stable nets coincide with the stability of $\theta$ in the sheaf meaning (without the locally free assumption, the last notion is stronger).

By construction, any element of $U_N$ has a non-jumping line through $N$. This implies that the associated net is semi-stable. Furthermore by definition of $U_N$, the $\theta$-characteristic considered are locally free. So we have a morphism:

$$I_n \supset U_N \longrightarrow \Theta_{sf}$$

The key of the construction is to understand the fiber of this morphism. We have thus to globalize the data 2) and then, to show that for $n \leq 11$ this map is dominant, so we will have to understand under what condition the data 3) exists.

Define by $G^\theta = G(2, H^0(\theta(2)))$ the Grassmann manifold of 2-dimensional subspace of $H^0(\theta(2))$, and $G^\theta_0$ be the open subset of $G^\theta$ made of pairs of sections of $\theta(2)$ with disjoint zeros. Denote by $K\theta$ the tautological subbundle of $\mathcal{O}_{G^\theta_0} \otimes H^0(\theta(2))$. The existence condition of the data 3) was already identified in [E-S] §4.2 as the vanishing locus of a map:

$$\delta_\theta : \mathcal{O}_{G^\theta_0} \longrightarrow H^1(\mathcal{O}_{C(1)}) \otimes 2 K_\theta^v$$

obtained by the following way:

We pick from data 2) the exact sequence:

$$0 \longrightarrow F \longrightarrow \mathcal{O}_{P(V^v)} \boxtimes K_\theta \longrightarrow \theta(2) \boxtimes \mathcal{O}_{G^\theta_0} \longrightarrow 0$$

whose restriction to $C \times G^\theta_0$ (where $C$ is $\theta$’s support) gives:
0 → \mathcal{O}_C(1) \boxtimes \mathcal{O}_{G''_\theta} \rightarrow F(\theta(2) \boxtimes \mathcal{O}_{G''_\theta}) \rightarrow \mathcal{O}_C \boxtimes 2\Lambda K_\theta \rightarrow 0

Pushing down this sequence twisted by \mathcal{O}_C \boxtimes 2\Lambda K_\theta with the second projection of \(C \times G''_\theta\) direct image’s functor, we obtain the boundary \(\delta_\theta\).

Denote by \(Z_{\delta_\theta}\) the vanishing locus of \(\delta_\theta\). Then, according to [E-S], the fiber \(U_{N, \theta}\) is the product of \(Z_{\delta_\theta}\) by \(\mathbb{P}_4\).

Let \(\Theta_0\) be the subspace of \(\Theta\) made of isomorphic classes of \(\theta\)-characteristic with smooth support. Ellingsrud and Stromme had also remarked ([E-S] §4.2) that for any \(\theta\) in \(\Theta_0\), the bundle \(\hat{\Lambda} K_\theta\) generates \(\text{Pic} G''_\theta\), and that for \(n \leq 11\) \(\Theta_0\) is included in \(U_N\)’s image. This image is thus irreducible, normal and \(\frac{n(n+3)}{2}\) dimensional.

Furthermore, if we denote by \(U_N^s\) the elements of \(U_N\) which have a stable image in \(\Theta\), we obtain that the image of \(U_N^s\) is irreducible and smooth. So we need to prove the following lemma:

**Lemma 4.1.1** For \(n = 4\) and \(5\), the \(U_N^s\) make a covering of \(I_n\) when \(N\) travels through \(\mathbb{P}_3\).

We have to understand the non stable nets which could occur in the image of \(U_N\). Let’s recall from [So] that \(\theta\) is not stable if there exists a totally isotropic (for the net \(m\) associated to \(\theta\)) subspace \(L \subset H\) such that \(\dim L + \dim L^\perp \geq n\). As there exists a smooth quadric in the net, \(L\) has to be at most 2 dimensional. The case \(\dim L = 1\) has already been avoided in [E-S] §5.10 using Barth’s condition \(\alpha_2\) of [Ba3]. So we have just to study the case \(\dim L = 2\).

If there exists such a 2-dimensional space, then all the quadrics of the net can be written in the following way: \(\sum_{i=1}^3 Y_i \begin{pmatrix} A_i & 0 \\ B_i & rA_i \end{pmatrix}\) where \(A_i\) is a \(2 \times 2\) matrix, and where \(B_i\) is a \(2 \times 2\) matrix in case \(n = 4\) and a \(3 \times 3\) matrix in case \(n = 5\). Their determinant must then be a multiple of \((\det(\sum_{i=1}^3 Y_i A_i))^2\). The curve of jumping lines of any preimage of this net must contain this double conic. Let \(A_N = \begin{pmatrix} a_N & b_N \\ c_N & d_N \end{pmatrix} = \sum_{i=1}^3 Y_i A_i\), and first prove the following lemma before continuing the 4.1.1:

**Lemma 4.1.2** If the conic \(\det(A_N)\) is singular in \(p\), then either \(p\) represents a multi-jumping line or \(N\) is in the vanishing locus of a section of \(E_H\) for some jumping plane \(H\).

The first case is obtained when \(a_N, b_N, c_N, d_N\) represent lines containing a same point \(p\), and the other cases are reduced to the 2 cases where \(a_N, b_N, d_N\) or \(b_N, c_N, d_N\) is a basis of \(V\), because we are allowed to change the basis of \(L\).
Computing the remaining coefficient in this basis, and using that $A_N$ is singular, we are reduced by changing the basis of $L$ to the case where $a_N$ and $c_N$ are proportional. But in this case, $a_N$ divides all the $(n-1) \times (n-1)$ minors except one. It means that the line $a_N$ cuts the multi-jumping lines in length $n-1$. The computation of [1.1.1] also proves that this is not possible in a stable plane, and the [1.1.3] that it is impossible in a jumping plane $H$ where $N$ is not in the vanishing locus of the section of $E_H$, because there is no jumping line of order $n$ through $N$ when $a_N, b_N, d_N$ or $b_N, c_N, d_N$ is a basis of $V$. This proves the lemma 4.1.2.

To obtain the [1.1.1], we have now to prove that there exists no instanton such that for every $N$ in $\mathbb{P}_3$ the net has such a 2 dimensional totally isotropic space $L_N$. If there was such a bundle, then, its hypersurface of jumping line would contain a double quadratic complex. Let $Q$ be this quadratic complex, and $\Sigma$ be the set of $N \in \mathbb{P}_3$ such that the $\alpha$-plane $\alpha_N$ doesn’t cut $Q$ in a smooth conic, and let $\Sigma'$ be the set of singular rays (i.e: the points of $G$ which are singularities of $Q \cap \alpha_N$ for some $N$). As $\Sigma$ is the degeneracy locus of a square matrix, it is at least 2 dimensional. We can show that for any $Q$, the set $\Sigma'$ is at least 2 dimensional too: let $\sum_{i=0}^5 x_i^2 = 0$ and $\sum_{i=0}^5 k_i x_i^2 = 0$ be the equations of $G$ and $Q$, and consider the third quadric $Q' : \sum_{i=0}^5 k_i^2 x_i^2 = 0$, then $G \cap Q \cap Q' \subset \Sigma'$ because $(x_i) \in G \cap Q \cap Q'$ implies that $(k_i x_i)$ is a line meeting the line $(x_i)$ in some point $N$, and for any $(z_i)$ in the $\alpha$-plane $\alpha_N$, the equation $(z_i)$ cut $(k_i x_i)$ which is $\sum_{i=0}^5 k_i x_i z_i = 0$ means also $(z_i) \in T(x_i) Q$.

Then we can conclude using the lemma 4.1.2, proposition 3.3.2, and Coanda’s theorem ([Co]), remarking in the cases where $N$ is on the jumping section of $E_H$ for some jumping-plane $H$, that $E$ can’t be a special t’Hooft bundle, otherwise there would exist a $n$-jumping line through $N$ which as been excluded in the demonstration of 4.1.2.

4.2 Description of the boundary $\delta_\theta$

We’d like now to understand more explicitly the fiber of $Z_{\delta_\theta}$ over some $\theta$. In other words, we need a description of the splitting condition of the restriction of $F$ to $C$. The following will consist of 2 descriptions of this condition. The first one is explicit in function of the net of quadrics, and enables to prove that the fiber $Z_{\delta_\theta}$ is the complete intersection of $G''_\theta$ with some hyperplanes. This could have been conjectured from Ellingsrud and Stromme viewpoint because of the map $\delta$ of the previous subsection, but we can’t conclude directly from this because there is some torsion in $PicG''_\theta$. Furthermore, we will need a second description to understand the ramification of this morphism in term of vector bundle and to get informations on $I_n$.

- First description
Let $K$ be the 2-dimensional vector space generated by 2 sections of $\theta(2)$ with disjoint zeros. Let $F$ be the vector bundle over the exceptional plane $Y = \mathbb{P}(V^v)$ defined by the exact sequence:

$$0 \to F \to K \otimes \mathcal{O}_Y \to \theta(2) \to 0 \quad (4)$$

Let $S$ be the kernel of the map $H^v \otimes \theta(3) \to \theta^2(4)$ obtained when twisting by $\theta(2)$ the exact sequence (2) of the net of quadric.

As $\mathcal{E}xt^1(\theta(2), \mathcal{O}_Y) = \theta(1)$, we can compute the kernel of the restriction of (4) to $C$ by dualizing (4). This yields to $\text{Tor}_1(\theta(2), \theta(2)) = \mathcal{O}_C(1)$. We can now deduce from (2) the exact sequence:

$$0 \to \mathcal{O}_C(1) \to H \otimes \theta(2) \to H^v \otimes \theta(3) \to \theta^2(4) \to 0$$

Taking there global sections, we get the following diagram where $\Sigma$ is by definition the cokernel of $H \otimes H^0(\theta(2)) \to H^v \otimes H^0(\theta(3))$.

On the other hand, we can compute the following diagram displaying the sequence (4) horizontally and the sequence (2) vertically:

We obtain by passing there to global sections the following:
The splitting condition of sequence (3) is just $f(g(H^0(\mathcal{O}_C))) = 0$, where $f$ and $g$ are constructed in the following diagram obtained by taking global sections in the sequence (4) twisted by $\theta(2)$.

So, the condition that the 2-dimensional vector space $K \subset H^0(\theta(2))$ gives an instanton, which is also the vanishing of $H^0(\mathcal{O}_C)$ in $\Sigma$, can be translated if $W$ denote the preimage by $c$ of $g(H^0(\mathcal{O}_C))$ in $H^v \otimes V \otimes K$, by the condition: $a(W) \subset \text{Im}b$.

We will now identify this condition with an explicit map $\beta: \Lambda^2 H^0(\theta(2)) \rightarrow H^1(\mathcal{O}_C(1))$. To construct $\beta$, let’s first consider the Eagon-Northcott complexes associated to the sequence (2) twisted by $-1$. This gives for the second symmetric power the following sequence:

We obtain by dualizing it the 2 following short exact sequences:

$$0 \rightarrow S_2 H \otimes \mathcal{O}_Y \rightarrow A \rightarrow \mathcal{O}_C(1) \rightarrow 0$$
$$0 \rightarrow A \rightarrow H^v(1) \otimes H \rightarrow (\Lambda^2 H)^v(2) \rightarrow 0$$
Let $S_2 H \otimes \mathcal{O}_Y \xrightarrow{d} H^\vee(1) \otimes H$ be the composition of the 2 previous injections. Then $d$ and $e$ are illustrated in the following diagram not exact in the middle:

$$
\begin{array}{ccccccccc}
0 & \rightarrow & S_2 H \otimes \mathcal{O}_Y & \xrightarrow{d} & H \otimes H^\vee(1) & \xrightarrow{e} & (\Lambda H^\vee)(2) & \rightarrow & 0 \\
& & S_2 m & \downarrow & m \otimes 1 & \downarrow & \Lambda & \downarrow & \\
0 & \rightarrow & S_2 (H^\vee(1)) & \rightarrow & H^\vee(1) \otimes H^\vee(1) & \xrightarrow{e} & (\Lambda H^\vee)(2) & \rightarrow & 0
\end{array}
$$

The display of the monad $(d, e)$ made by the first line is:

$$
\begin{array}{cccccc}
0 & \rightarrow & S_2 H \otimes \mathcal{O}_Y & \rightarrow & A & \rightarrow & \mathcal{O}_C(1) & \rightarrow & 0 \\
& & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
0 & \rightarrow & S_2 H \otimes \mathcal{O}_Y & \xrightarrow{d} & H \otimes H^\vee(1) & \rightarrow & B & \rightarrow & 0 \\
& & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
& & (\Lambda H^\vee)(2) & \rightarrow & (\Lambda H^\vee)(2) & \rightarrow & & & \\
& & 0 & \rightarrow & 0 & \rightarrow & & & 
\end{array}
$$

The last column of this display gives when passing to global sections a boundary map $\Lambda H^\vee \otimes S_2 V \overset{b}{\rightarrow} H^1(\mathcal{O}_C(1))$ which vanishes on the image of the map $H \otimes H^\vee \otimes V \xrightarrow{b} \Lambda H^\vee \otimes S_2 V$ obtained from $e$ by taking global sections. Using now the surjection of $(2)$ we obtain the commutative diagram:

$$
\begin{array}{cccccc}
H \otimes H^\vee \otimes V & \xrightarrow{b'} & H^\vee \otimes H^\vee \otimes S_2 V & \rightarrow & H^1(\mathcal{O}_C(1)) & \rightarrow & 0 \\
\downarrow & & & \downarrow & & & \downarrow \\
H \otimes H^0 \theta(2) & \xrightarrow{b} & H^\vee \otimes H^0 \theta(3) & \rightarrow & & & \\
\end{array}
$$

which identify the conditions $a'(W) \subset \text{Im} b'$ and $a(W) \subset \text{Im} b$, where $a'$ is defined in the diagram below. As we had already identified the splitting condition with $a(W) \subset \text{Im} b$, we can from the following diagram construct a map $\beta$ giving the corollary [4.2.1].

$$
\begin{array}{cccccc}
H \otimes H^\vee \otimes V & \xrightarrow{b'} & \Lambda H^\vee \otimes S_2 V & \xrightarrow{\beta_0} & H^1(\mathcal{O}_C(1)) & \rightarrow & 0 \\
\downarrow & \text{c} & \text{d} & \text{e} & \downarrow & \downarrow & \downarrow \\
H \otimes H^\vee \otimes V & \xrightarrow{\beta} & \Lambda (H^\vee \otimes V) & \rightarrow & \Lambda H^0 \theta(2) & \rightarrow & 0
\end{array}
$$

**Corollary 4.2.1** The splitting condition of the sequence $(3)$ is given by the vanishing of a surjective map $\Lambda (H^0 \theta(2)) \xrightarrow{\beta} H^1(\mathcal{O}_C(1))$

This gives the following description of $Z_{\delta_0}$ where we denote by $Z_s$ the vanishing locus of a section $s$ of $\theta(2)$:

$$
Z_{\delta_0} = \mathbb{P}(\{s \wedge s' \mid s, s' \in H^0 \theta(2), \ Z_s \cap Z_{s'} = \emptyset \text{ and } \beta(s \wedge s') = 0\})
$$
Let $E$ be an $n$-instanton, $\theta$ its associated $\theta$-characteristic, and $s, s'$ be 2 sections of $\theta(2)$ associated to $E$. The purpose of this second description is to understand the possible singularities of $Z_{\delta_\theta}$ at $s \wedge s'$ in terms of the vector bundle $E$. This part will be divided in 3 steps. First, we will construct a new map $\beta_E$, then we’ll have to identify the vanishing locus of $\beta$ and $\beta_E$, and in the third step, this will enable us to understand the particular bundles that give singularities of $Z_{\delta_\theta}$.

1) We will construct here a map $\beta_E : H^0 \theta(2) \oplus H^0 \theta(2) \to H^1 \mathcal{O}_C(1)$. Let’s take the exact sequence of restriction to the exceptional divisor $x = \tau - \sigma$ twisted by $E(\tau)$, where $\tau = p^* \mathcal{O}_{\mathbb{P}^3}(1)$ and $\sigma = q^* \mathcal{O}_Y(1)$.

$$0 \to p^* E(\sigma) \to p^* E(\tau) \to p^* E(\tau)|_x \to 0$$

apply it to the functor $q^*$ to get the following exact sequence where we recall that $F = q_* p^* E$, and that the 2 dimensional vector space in the right of the sequence is naturally the fiber of $E$ at $N$.

$$0 \to F(1) \to q_* p^* E(\tau) \to 2 \mathcal{O}_Y \to 0 \quad (5)$$

Let $2H^0 \theta(2) \xrightarrow{\delta_E} H^1 \mathcal{O}(3)$ be the boundary map obtained when taking global sections in (3) twisted by $\theta(2)$. Taking global sections in (3) twisted by $\theta(3)$ gives us a map $H^1 \mathcal{O}(3) \to H^1 \mathcal{O}_C(1)$. Define $\beta_E$ as the composition of $\delta_E$ and this map:

$$\beta_E : 2H^0 \theta(2) \xrightarrow{\delta_E} H^1 \mathcal{O}(3) \to H^1 \mathcal{O}_C(1))$$

2) To link $\beta$ and $\beta_E$, we will prove here that for every surjection $2 \mathcal{O}_Y \xrightarrow{(\sigma, \sigma')} \theta(2) \to 0$ where $\sigma, \sigma'$ are such that $\beta(\sigma \wedge \sigma') = 0$, then $\beta_E(-\sigma', \sigma) = 0$.

For all that, apply to the sequence 5 the functor $\text{Hom}(\ast, \theta(2))$. It gives a map $\text{Hom}(2 \mathcal{O}_Y, \theta(2)) \to \text{Hom}(q_* p^* E(\tau), \theta(2))$, which enable to pull back any $\psi : 2 \mathcal{O}_Y \xrightarrow{\sigma, \sigma'} \theta(2) \to 0$ by a map $q_* p^* E(\tau) \xrightarrow{\psi} \theta(2)$ making the following diagram commutative:

$$0 \to F(1) \to q_* p^* E(\tau) \to 2 \mathcal{O}_{\mathbb{P}^3} \to 0 \quad (6)$$
where $N$ and $F'$ are by definition the kernel of $\psi$ and $\phi$, and where the middle line is the sequence $\mathbf{3}$. Restricting (4) to $C$ gives the 2 following diagrams where the kernel of the restriction of $\psi$ to $C$ is noted $M$:

\[
\begin{array}{ccccccccc}
0 & \to & F_C(1) & \to & q_*p^*E(\tau)_C & \to & 2\mathcal{O}_{\mathbb{F}_2} & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & F_C(1) & \to & M & \to & \theta^\nu(-2) & \to & 0
\end{array}
\]

(7)

\[
\begin{array}{ccccccccc}
0 & \to & F_C(1) & \to & N_C & \to & F'_C & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & F_C(1) & \to & M' & \to & \theta^\nu(-1) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \theta^\nu(-1) & \to & \theta^\nu(-1) & \to & 0
\end{array}
\]

(8)

Construct a map $0 \to \theta^\nu \to M$ by composition of the injection $0 \to \theta^\nu \to F_C(1)$ of sequence $\mathbf{3}$ with the injection of $F_C(1)$ in $M$ of the diagram $\mathbf{8}$. Denote by $M'$ the cokernel of $i$, and by $E'$ the cokernel of the injection $0 \to \theta^\nu \to q_*p^*E(\tau)_C$ obtained by composing $i$ with the injection of $M$ in $q_*p^*E(\tau)_C$ of the diagram $\mathbf{7}$. Now using the fact that we have an injection of the cokernel of $\mathbf{3}$ in $M'$ and in $E'$, we obtain from $\mathbf{9}$ the following diagram:

\[
\begin{array}{ccccccccc}
0 & \to & \theta^\nu(-1) & \to & E' & \to & 2\mathcal{O}_{\mathbb{F}_2} & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \theta^\nu(-1) & \to & M' & \to & \theta^\nu(-2) & \to & 0
\end{array}
\]

(9)
We can now notice that $\beta_E$ is just the boundary obtained when taking global section in the middle line of (2) twisted by $\theta(2)$. As the right column of (2) twisted by $\theta(2)$ gives an injection on $H^0(O_C)$ in $2H^0(\theta(2))$ of image $(-\sigma', \sigma)$, we obtain that $\beta_E(-\sigma', \sigma) = 0$ if and only if $M'$ splits. But $F_C$ split because $\beta(\sigma' \wedge \sigma) = 0$ by hypothesis, so we have after the choice of a section $\alpha$ of (3) the diagram:

\[
\begin{array}{cccccc}
0 & \rightarrow & F_C(1) & \rightarrow & N_C & \rightarrow & F'_C & \rightarrow & 0 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
0 & \rightarrow & \theta^v(-1) & \rightarrow & M & \rightarrow & M' & \rightarrow & 0 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
0 & \rightarrow & \theta^v(1) & \rightarrow & \theta^v(-1) & \rightarrow & 0 & \rightarrow & 0 \\
\end{array}
\]

which proves that $M' \simeq F_C'$, and that $\beta$ and $\beta_E$ have the same vanishing locus. □

3) Identification of $Z_{\delta \theta'}$'s singularities.

Let $E$ be an instanton, and $s \wedge s'$ its associated element of $Z_{\delta \theta'}$. Denote by $K$ the vector space $Vect(s, s') \subset H^0(\theta(2))$.

The bundle $E$ gives a singularity of $G(2, H^0(\theta(2))) \cap \ker \beta$ if and only if the Zariski tangent space to $G(2, H^0(\theta(2)))$ at $s \wedge s'$ is included in one of the hyperplanes given by $\beta$. In other words, it means that $K$ is in the kernel of one of the skew map given by $\beta$. As $H^1(O_C(1)) = S_{n-4} V$, it means that there is some $p \in S_{n-4} V$ such that $K \otimes p$ is in the kernel of $K \otimes S_{n-4} V \xrightarrow{\beta_E} H^0(\theta(2))^\vee$.

But we can understand this kernel explicitly with the following diagram, where the first 2 lines are obtained from the sequences (4) and (3), and the last from sequence (3) twisted by $O_Y(n-4)$:

\[
\begin{array}{cccccc}
0 & \rightarrow & H^1(\theta(-1)) & \rightarrow & H^2 F(-3) & \rightarrow & H^2(2O_C(-3)) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
0 & \rightarrow & H^1(\theta^v(n-4)) & \rightarrow & H^1(F_C(n-3)) & \rightarrow & (H^0(\theta(2))^\vee, 0 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
0 & \rightarrow & H^0(\delta_E) & \rightarrow & H^0(q \star p \star E(n-3) \sigma) & \rightarrow & K \otimes S_{n-4} V \xrightarrow{\delta_E} H^1(F(n-3)) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
0 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & 0 \\
\end{array}
\]

This proves that the maps $\beta_E$ and $\delta_E$ have the same kernel, which is nothing else than the cokernel of $H^0(q \star p \star E(n-3) \sigma) \xrightarrow{\delta} H^0(q \star p \star E(\tau + (n-4) \sigma))$.

**Corollary 4.2.2** The ramification locus of the map $U_n^* \rightarrow \Theta_{1f}$ is made of special t’Hooft bundle for $c_2 = 4$. 32
For \(c_2 = 5\), there are 3 possibilities. If the Zariski tangent space of \(Z_{\delta_0}\) is 15 (respectively 16) dimensional, then this point arise from a t’Hooft bundle (respectively special t’Hooft). If it is 14 dimensional, then the bundle twisted by 2 have 2 sections vanishing at \(N\).

**NB:** For \(c_2 = 5\), a bundle such that \(h^0J_N \otimes E(2) \geq 2\) doesn’t necessarily give a singularity of \(Z_{\delta_0}\).

The result \([4.2.2]\) is clear for \(c_2 = 4\) because we have in this situation \(h^0(q_{*}p^*E(\tau)) \geq 2\).

When \(c_2 = 5\), any singularity of \(Z_{\delta_0}\) is such that \(K \otimes p\) is in the cokernel of \(j\) for some \(p \in V\), so \(h^0(q_{*}p^*E(\tau + \sigma)) \geq 2\). Although this will be enough to prove the smoothness of \(I_5\), we will go further to understand the ramification, and also because it seems it is not enough to obtain the connectivity. The cases of a 15 or 16 dimensional Zariski tangent space, have the following interpretation in term of the cokernel of \(H^0q_{*}p^*E(\sigma) \xrightarrow{j'} H^0q_{*}p^*E(\tau)\):

Take global sections in (5) to obtain like in the diagram (10) a map \(\beta' : K \to H^1(\theta^v(-1) = H^0(\theta(3))^v\) whose kernel is the cokernel of \(j'\). In fact \(\beta\) is the composition of \(\beta'\) with the injection \(H^0(\theta(3))^v \to (H^0(\theta(2)) \otimes V)^v\). But the Zariski tangent space of \(Z_{\delta_0}\) at \(s \wedge s'\) is 15 (resp. 16) dimensional if and only if there is a 2 (resp. 3) dimensional subspace of \(W\) of \(V\) such that the map \(K \otimes H^0(\theta(2)) \otimes W \to \mathbb{C}\) induced by \(\beta\) is zero.

- If \(\dim W = 3\), then \(\beta' = 0\) and thus \(h^0(q_{*}p^*E(\tau)) = 2\).
- If \(\dim W = 2\), then we have to find an element \(s_0 \in K\) such that the map \(\beta'(s_0) : H^0(\theta(3)) \to \mathbb{C}\) is zero. Let \((p, q)\) be a basis of \(W\), and compute the dimension of \(\{H^0(\theta(2)) \otimes p\} \cap \{H^0(\theta(2)) \otimes q\}\) in \(H^0(\theta(3))\). Indeed, as \(H^0(\theta(3)) \otimes K \xrightarrow{j'} \mathbb{C}\) is zero on the image of \(H^0(\theta(2)) \otimes W \otimes K\) in \(H^0(\theta(3)) \otimes K\), if \(\{H^0(\theta(2)) \otimes p\} \cap \{H^0(\theta(2)) \otimes q\}\) is 0 or 1 codimensional in \(H^0(\theta(3))\), then there is such an \(s_0\). So assume that it is not the case, then \(\dim \{H^0(\theta(2)) \otimes p\} \cap \{H^0(\theta(2)) \otimes q\}\) \(\geq 7\), and this intersection would contain 2 elements independent relatively to \(H^0(\theta(1)) \otimes p\). If \(p\) and \(q\) have to cut one another in a point \(P \in C\), and the 4 remaining points of \(q \cap C\) have to be zeros of \(s\) and \(\sigma\). As \(\theta(2)\) is generated by its global sections, those vanishing in a given point are always a hyperplane of \(\theta(2)\). As the intersection of the 5 hyperplanes associated to \(q \cap C\) is \(H^0(\theta(1)) \otimes q\), the intersection of 4 of them can’t be 7 dimensional, which contradicts the hypothesis.

**Remark 4.2.3** For \(c_2 = n\), if there is a section of \(\theta(2)\) which is in the cokernel of \(H^0q_{*}p^*E(\sigma) \xrightarrow{j'} H^0q_{*}p^*E(\tau)\), then this section vanishes on the vertices of the complete \((n + 1)\)-gon (inscribed in the curve of jumping lines) obtained from the lines through \(N\) which are bisecant to the zero locus of the associated section of \(E(1)\).
Let $E$ be an $n$-instanton such that there is a section $s$ of $\theta(2)$ in the cokernel of $j'$. Let $\mathcal{I}$ be the ideal of the vanishing locus of the section of $E(1)$ coming from $s$. The bundle $F = q_*p^*E$ is also $q_*p^*\mathcal{I}(\tau)$, and we can understand $s$ with the following diagram, where the first column is the natural evaluation:

\[
\begin{array}{cccc}
0 & 0 \\
\theta(2) & L \\
0 & O_Y & q_*p^*E(\tau) & q_*p^*\mathcal{I}(2\tau) & 0 \\
F \oplus F(1) & q_*p^*\mathcal{I}(\tau) & q_*p^*(\mathcal{O}_{\mathbb{P}^3}(\tau))
\end{array}
\]

As $R^1q_*p^*\mathcal{I}(2\tau) = 0$ because there are no multi-jumping lines through $N$, we have for any line $D$ containing $N$ a surjection from $q_*p^*\mathcal{I}(2\tau)_d$ onto $H^0(\mathcal{I}_D(2\tau))$ where $\mathcal{I}_D = \mathcal{I} \otimes \mathcal{O}_D$. But the map $q_*p^*\mathcal{I}(\tau)_d \otimes q_*\mathcal{O}_{\mathbb{P}^3}(\tau) \to H^0(\mathcal{I}_D(2\tau))$ has to be zero if $D$ is bisecant to the scheme defined by $\mathcal{I}$, thus the support of $L$ must contain the points corresponding to those lines. Computing the degree, we can conclude that $s$ vanish on the $(n+1)$-gone whose vertices are the projections from $N$ of the previous bisecant.

**Corollary 4.2.4** When $c_2 = 4$ the fiber $Z_{\delta^0}$ is singular if and only if the support of $\theta$ is a Lur"oth quartic. In this situation, the singularity $s \land s'$ is such that the zeros of $s$ and $s'$ give the pencil of complete pentagons inscribed on the quartic.

### 4.3 The normality condition when $c_2 = 5$; applications

We want here to understand when $Z_{\delta^0}$ is not regular in codimension 1. When $c_2 = 5$, the fiber $Z_{\delta^0}$ is an open subset of the Grassmannian $G(2,10)$ cut by 3 hyperplanes. Let $A_i$ be the skew forms associated to those hyperplanes, and $a_i$ be their skew linear maps. The key result of 4.2.2 and 4.2.3 is that any $s$ in the kernel of all the $a_i$ gives a $(n+1)$-gone inscribed in $\theta$'s support. So we want here to prove the following:

**Proposition 4.3.1** When $c_2 = 5$, if $G(2,H^0\theta(2)) \cap \ker \beta$ is singular in codimension 1 or have an excess dimension, then the $a_i$ have at least a 4-dimensional common kernel.

Both singularity in codimension 1 and excess dimension imply that there is a 12 dimensional subscheme $S$ of $G(2,H^0\theta(2)) \cap \ker \beta$ such that the intersection is not transverse at any points of $S$. So, any $s \land s'$ of $S$ is necessarily in some $\ker(\sum_{i=1}^{3} \lambda_i a_i)$, then $S$ is in $\bigcup_{\lambda_i} G(2, \ker(\sum_{i=1}^{3} \lambda_i a_i))$. As $S$ is 12 dimensional, at
least one of those Grassmann manifold is of dimension 10 or more. So this Grassmann manifold has to be at least 12 dimensional, and for this value of \((\lambda_1, \lambda_2, \lambda_3)\) one have \(\dim \ker(\sum_{i=1}^{3} \lambda_i a_i) = 8\) (none of those maps are zero according to the \([4.2.1]\)). We can assume that this occur for \((1,0,0)\), so \(\dim \ker a_1 = 8\). This yields to the following discussion:

a) There is no \((\lambda_1, \lambda_2, \lambda_3) \neq (1,0,0)\) such that \(\dim \ker(\sum_{i=1}^{3} \lambda_i a_i) = 8\). Then the union of those Grassmann manifolds is the union of \(G(2, \ker a_1)\) with something of dimension at most \(8+2\). So we must have \(G(2, \ker a_1) = S\), but it implies that all the \(A_i\) are zero on \(\wedge \ker a_1\). The \(a_i\) have then at least a 4 dimensional common kernel.

b) Otherwise, we can assume that \(\dim \ker a_2 = 8\). If \(a_3\) had also an 8 dimensional kernel, then the \(a_i\) would easily have a 4 dimensional common kernel, so let’s assume that if \(\lambda_3 \neq 0\) then \(\dim \ker(\sum_{i=1}^{3} \lambda_i a_i) \leq 6\). So \(\dim \left( \bigcup_{\lambda_i, \lambda_3 \neq 0} G(2, \ker(\sum_{i=1}^{3} \lambda_i a_i)) \right) \leq 10\), then \(S \subset \bigcup_{\lambda_i} G(2, \ker(\sum_{i=1}^{3} \lambda_i a_i))\). So \(S \cap G(2, \ker a_1)\) is at most 1 codimensional in \(G(2, \ker a_1)\), and it is in the vanishing of \(A_{2, \ker a_1}^{(1)}\) and of \(A_{3, \ker a_1}^{(2)}\). So those 2 skew forms have to be proportional on \(\ker a_1\), then we can assume that \(A_{3, \ker a_1}^{(2)} = 0\) so the \(a_i\) have again a 4 dimensional common kernel as claimed.

**Theorem 4.3.2** The moduli space \(I_5\) is irreducible of dimension 37.

Let’s first show that for a general \(\theta\) in the image of \(U_N^s\), then \(Z_{\delta_{\theta}}\) is irreducible. As \(Z_{\delta_{\theta}}\) is open in \(G_{\theta}^{\prime} \cap \ker \beta\), it is enough to show that \(G_{\theta}^{\prime} \cap \ker \beta\) is irreducible, where \(G_{\theta}^{\prime} = G(2, H^0(\theta(2)))\). But any degeneracy locus of \(3\mathcal{O}_C \xrightarrow{} \mathcal{O}_C(1)\) is connected (Cf [A-C-G-H] p311), and complete intersection. So it satisfies Serre’s condition (S2), and in our situation, the \([4.3.1]\) shows that if it is not regular in codimension 1 (R1) or if it is excess dimensional, then the support of \(\theta\) is a Darboux pentic according to \([4.2.3]\) (in fact it would be 4 times Darboux if this had a sense!). Anyway \(\theta\) can’t be general in \(\Theta\). Then if \(\theta\) is not Darboux, \(G_{\theta}^{\prime} \cap \ker \beta\) is normal, connected and 13-dimensional, so it is irreducible and \(Z_{\delta_{\theta}}\) too.

But we proved in the \([4.1.1]\) that the \((U_N^s)_{N \in \mathbb{P}_3}\) cover \(I_5\). Furthermore, the basis of the fibration \(U_N^s \rightarrow \Theta_{I_5}^{\prime}\) is irreducible and smooth according to [So], and the fiber is \(\mathbb{P}_4 \times Z_{\delta_{\theta}}\) which is irreducible, normal and 17 dimensional when \(\theta\) is not Darboux. Furthermore, for any \(\theta\) in the image of \(U_N^s\), \(Z_{\delta_{\theta}}\) is at most 15 dimensional because \(G_{\theta}^{\prime}\) is not include in a hyperplane. As \(U_N\) is open in \(I_5\), all its irreducible components have dimension at least 37. But the Darboux \(\theta\) are 3 codimensional in \(\Theta_{I_5}^{\prime}\) which is 20 dimensional, so the preimage of the Darboux \(\theta\) is too small to make an irreducible component of \(U_N^s\). Then \(U_N^s\) has to be irreducible, and 37 dimensional. We can conclude using the \([4.1.1]\) that it is also the case for \(I_5\).

**Theorem 4.3.3** The moduli space of mathematical instanton with \(c_2 = 5\) is smooth.
As $U^*_5$ is a fibration over $\Theta^*_f$, which is smooth, any bundle which is not in the ramification of this morphism is a smooth point of $I_5$. So we have to check that a bundle $E$ which is in all the ramifications of $U^*_N \rightarrow \Theta^*_f$ when $N$ fills $\mathbb{P}_3$ is a smooth point of $I_5$. According to the \textit{1.2.2}, we must have $h^0(J_N E(2)) \geq 2$ for every $N$. So $h^0(E(2)) \geq 4$, and either $E$ is a 't Hooft bundle, or $E$ belongs to the family described in the \textit{1.5.4}, and both are smooth points of the moduli space (Cf \textit{4.3.4}).

### 4.4 Residual class

Let $E$ be an $n$-instanton, and assume that $E$ has a 2 parameter family $S$ of multi-jumping lines. Furthermore, we assume here that $S$ is irreducible, and that the residual scheme of $S$ in the scheme of multi-jumping lines $M$, is a curve (may be empty) not drawn on $S$. Eventually assume that $S_{\text{red}}$ is locally complete intersection.

\[
\begin{array}{c}
\mathbb{P}(H^1 E^\vee \otimes \mathcal{O}_G) \\
\begin{array}{c}
\tilde{S}_{\text{red}} \\
S_{\text{red}}
\end{array} \xrightarrow{j} G \\
\begin{array}{c}
\pi \xrightarrow{g} \tilde{G} \\
\xrightarrow{f} G
\end{array}
\end{array}
\]

In order to define and compute the class of $C$ in the Chow ring of $G$ (graduated by the dimension), we will have to work in the blow up $\tilde{G}$ of $G$ along $\tilde{S}_{\text{red}}$. Let $\tilde{S}_{\text{red}}$ be the exceptional divisor, and $x$ be the class of $\tilde{S}_{\text{red}}$ in $A_3 \tilde{G}$, we have:

\[A_k \tilde{G} = (A_k \tilde{S}_{\text{red}} \oplus A_k G)/\alpha(A_k \tilde{S}_{\text{red}})\]

where $\forall y \in A_k \tilde{S}_{\text{red}}$, $\alpha(\gamma) = (c_1(g^* N/O_N(-1) \cap g^* y, -i_* y))$, and where $N$ is the normal bundle of $S_{\text{red}}$ in $G$. Let’s define:

\[\zeta = c_1(g^* O_G(S_{\text{red}}) = c_1(O_N(-1))\]

The multiplicative structure of $A_\ast \tilde{G}$ is given by the following rules:

\[
\left\{
\begin{array}{c}
f^* \gamma \cdot f^* \gamma' = f^* \gamma \gamma' \\
j_* \tilde{\sigma} : j_* \sigma' = j_* (\zeta \tilde{\sigma} \sigma') \\
f^* \gamma \cdot j_* \tilde{\sigma} = j_* ((g^* \gamma) \tilde{\sigma})
\end{array}
\right.
\]

where $\gamma \in A_\ast G$ and $\tilde{\sigma}, \sigma' \in A_\ast \tilde{S}_{\text{red}}$. So we have: $x^2 = j_* \zeta = (g^* c_1 N, -i_* [S_{\text{red}}])$, because $\zeta = -c_1(g^* N/O_N(-1)) + g^* c_1 N$ and similarly $x^3 = j_* \zeta^2 = ((c_1 N)^2 - c_2(N), -i_* (c_1 N))$, and

\[A_\ast \tilde{S}_{\text{red}} = A_\ast S_{\text{red}}[\zeta]/(\zeta^2 - c_1(N) \zeta + c_2(N)).\]

One has the exact sequence: $H^1 E(-1) \otimes Q^\vee \xrightarrow{\sigma} H^1 E \otimes \mathcal{O}_G \rightarrow R^1 q_* p^* E \rightarrow 0$, where $Q$ is the tautological quotient bundle of $H^0(O_{\mathbb{P}_3}(1))^\vee \otimes \mathcal{O}_G$. According to [G-P], the scheme of multi-jumping lines $M$ is the locus where $\sigma$ has rank at most $2n - 2$. Denote by $U$ the universal subbundle of $\mathbb{P}(H^1 E^\vee \otimes \mathcal{O}_G - \tilde{G})$ and $\pi$ the projection on $\tilde{G}$. The degeneracy locus of $f^* \sigma$ is equal to the one of $f^* \sigma^v$, and we can compute $f^*[M]$ in function of the vanishing locus $Z_s$ of some section $s$ of $U^\vee \otimes \pi^*(H^1 E^\vee \otimes \mathcal{O}_G)$ arising from the sequence:

\[0 \rightarrow U \rightarrow \pi^*(H^1 E^\vee \otimes \mathcal{O}_G) \xrightarrow{\pi^* f^* \sigma^v} \pi^*(H^1 E(-1) \otimes f^* Q^v)^v\]
Indeed, one has \( f^* M = \pi^* (Z_s) \), and \( s \) vanishes on the divisor \( \pi^* \tilde{S} \) so it gives a regular section \( s' \) of \( U^v \otimes \pi^* (H^1 E^v \otimes O_G)(-mx) \) where \( S = mS_{red} \) in \( A_s G \). Computing this class as in [Fu] Ex 14.4 gives:

\[
Z_{s'} = \sum_{i=0}^{2n} (-1)^i c_{2n-i}(U^v \otimes \pi^* f^* F)(\pi^* m x)^i, \text{ where } F = H^1 E(-1)^v \otimes Q.
\]

According to Josefiak-Lascoux-Pragacz, [Fu] Ex14.2.2, one has:

\[
\pi^*(c_{2n-i}(U^v \otimes \pi^* f^* F)) = c_{2n-i-(2n-2)+1}(F - H^1 E^v \otimes O_G)) = c_{3-i}(F)
\]

But the Chern polynomial of \( F \) is:

\[
c_Y(F) = 1 + (nt)Y + [\binom{n+1}{2} t^2 - nu] Y^2 + [2 \binom{n+1}{3} tu] Y^3
\]

where \( t \) is the class of a hyperplane section of \( G \), and \( u \) is represented by the lines in a plane. (Let’s recall that the Chow ring of \( G \) is generated by \( t \) and \( u \) with the relations: \( t^3 = 2tu \); \( u^2 = t^2u \). Furthermore, \( c_2 Q = t^2 - u \) and \( [S] = \alpha(t^2 - u) + \beta u \). So we have:

\[
\pi^*(Z_{s'}) = \sum_{i=0}^{2n} c_{3-i}(F)(-mx)^i = \sum_{i=0}^{3} c_{3-i}(F)(-mx)^i
\]

\[
\pi^*(Z_{s'}) = \left( m^3 [c_2 N - (c_1 N)^2] + m^2 (c_1 N.n.i.t) - m.i.\left[\binom{n+1}{2} t^2 - nu, 2 \binom{n+1}{3} tu - nm t.i. S_{red} + m^3 i_* c_1 N\right] \right)
\]

**Example 4.4.1** If \( S \) is a smooth congruence of bidegree \((\alpha, \beta)\), then the residual class is:

\[
\left( \frac{\alpha^2 + \beta^2 - (n^2 - 7n + 13)\alpha - (n^2 - 5n + 13)\beta + 2(2n-12) - 12\chi(O_S)}{2}, 2 \binom{n+1}{3} - (n-3)(\alpha + \beta) + 2\pi - 2\right),
\]

where \( \pi \) is the genus of a hyperplane section of \( S \), and \( \mathcal{P} \) is the class of a point in \( A_0 S \).

NB: the term of the right is still valid when \( S \) is just locally complete intersection, and that is the non vanishing of this term in the required situations that had been used in the 3.3.2.

Let \( c_i \Omega \) be the Chern classes of the cotangent bundle of \( S \). If \( C \) is a general hyperplane section of \( S \), the normal bundle of \( C \) is \( N_C = N_C \oplus O_C(1) \), where \( N \) is still the normal bundle of \( S \). But \( c_1 N_C = c_1 (\Omega^*_C) |_C + c_1 (\omega_C) = 4(\alpha + \beta) + 2\pi - 2 \), so we have: \( t.c_1 N = [3(\alpha + \beta) + 2\pi - 2], \mathcal{P} \).

On the other hand, one has \( c_1 N^2 - c_2 N = [5(\alpha + \beta) + 8(\pi - 1) + c_2 \Omega], \mathcal{P} \), which gives the formula using Hirzebruch-Riemann-Roch theorem and the following relation satisfied by every smooth congruence (Cf [A-S]):

\[
\alpha^2 + \beta^2 = 3(\alpha + \beta) + 4(2\pi - 2) + 2(c_1 \Omega)^2 - 12\chi(O_S)
\]
4.5 The family of $n$-instantons with $h^0(E(2)) \geq 4$

**Proposition 4.5.1** Let $E$ be any $n$-instanton with $n \geq 5$, $h^0(E(1)) = 0$ and $h^0(E(2)) \geq 4$, then $h^0(E(2)) = 4$ and there is a section of $E(2)$ vanishing on the union of 2 curves of arithmetic genus 0 cutting each other in length 2. Furthermore, one of these curves has to be a skew cubic (not necessarily integral), and the other one may be chosen smooth and has bidegree $(1, n)$ in a smooth fixed quadric.

We can construct from the hypothesis $h^0(E(2)) \geq 4$ a map $4\mathcal{O}_{\mathbb{P}^3} \to E(2)$ whose kernel and cokernel will be denoted by $E'(−2)$ and $\mathcal{L}$. So we have the exact sequence:

$$0 \longrightarrow E'(−2) \longrightarrow 4\mathcal{O}_{\mathbb{P}^3} \longrightarrow E(2) \longrightarrow \mathcal{L} \longrightarrow 0$$

The key is that the support of $\mathcal{L}$ has to contain a quadric surface. So we have to eliminate in the following the other cases.

**First case:** $\dim \ supp \mathcal{L} \leq 1$

**Lemma 4.5.2** If $\dim \ supp \mathcal{L} \leq 1$, then we have $c_2(E') = 8 - n - d_\mathcal{L}$, where $d_\mathcal{L}$ is the degree of the sheaf $\mathcal{E}xt^2(\mathcal{L}, \mathcal{O}_{\mathbb{P}^3})$.

NB: In this case $d_\mathcal{L}$ is also the degree of $\mathcal{L}$, but we’d like to keep the definition of $d_\mathcal{L}$ in the other cases.

Let $a$ be such that $E'^*(-a)$ has a section. Choosing one enables to build the following diagram where the middle line is obtained by dualizing the previous exact sequence, and where $X$ is the vanishing locus of the chosen section ($X$ could be empty):

\[
\begin{array}{cccccccc}
0 & 0 & \to & (4-k)\mathcal{O}_{\mathbb{P}^3}(-2) & \to & \mathcal{O}_{\mathbb{P}^3}(a) & \to & B & \to & 0 \\
0 & \to & E(-4) & \to & 4\mathcal{O}_{\mathbb{P}^3}(-2) & \to & E'^* & \to & \mathcal{E}xt^2(\mathcal{L}, \mathcal{O}_{\mathbb{P}^3}(-2)) & \to & 0 \\
0 & \to & k\mathcal{O}_{\mathbb{P}^3}(-2) & \to & \mathcal{J}_X(-a) & \to & A & \to & 0 \\
0 & \to & 0 & \to & 0 & \to & 0 & \to & 0
\end{array}
\] (11)

We can first take $a = -2$. So there arise $k = 3$ quartic surfaces containing $X$, and $B$ has to vanish. The kernel of the last line of (11) is in this case $E(-4)$. Choose 2 of those quartics, and denote by $\Gamma$ their complete intersection, and by $\mathcal{J}_{X|\Gamma}$ the ideal of $X$ in $\Gamma$. The last line of (11) and the resolution of $\Gamma$ give in the following a section $s$ of $E(2)$ whose vanishing locus will be noted $Z$. 

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but the third column of (12) proves that $X$ is linked by the 2 quartics to the support of $J_{X|\Gamma}$, and the last line implies that this support is the union of $Z$ with the 1-dimensional part of $\mathcal{E}xt^2(\mathcal{L},\mathcal{O}_{\mathbf{P}_3})$’s support counted rank of $\mathcal{E}xt^2(\mathcal{L},\mathcal{O}_{\mathbf{P}_3})$ times. So it proves lemma 4.5.2.

We can remark now that the degeneracy class of $4\mathcal{O}_{\mathbf{P}_3} \to E(2)$ is negative when $n \geq 5$, so the support of $\mathcal{L}$ has to contain at least a one dimensional component. So we have from lemma 4.5.2 $c_2 E' \leq 2$.

Let’s use again the diagram (11), but this time with the biggest $a$ such that $E'^v(-a)$ has a section. We will now study the possible cases recalling that $A$’s support is at most 1 dimensional.

- **If $k = 2$**

  Then the kernel of the first line of (11) is $\mathcal{O}_{\mathbf{P}_3}(-a - 4)$, and it is injected into $E(-4)$. So it gives a section of $E(a)$, and by the hypothesis made on $E$ and $a$, we have $a \geq 2$, and then $X$ is empty because $A$’s support is at most 1 dimensional. So the middle column of (11) splits. (in this situation we will say in the following $E'^v$ splits).

- **If $E'^v$ splits**

  Then the map from $B$ to $\mathcal{E}xt^2(\mathcal{L},\mathcal{O}_{\mathbf{P}_3}(-2))$ of (11) is an injection, so $B$’s support has also to be at most 1 dimensional. The only possible cases are thus:

  - $k = 1$, $a = 2$, $X = \emptyset$, but it implies $E'^v = \mathcal{O}_{\mathbf{P}_3}(2) \oplus \mathcal{O}_{\mathbf{P}_3}(-2)$, so it contradicts the independence of the 4 chosen sections of $E(2)$.

  - $k = 2$, then $A$ and $B$’s supports are 1 dimensional of degree $(a - 2)^2 - c_2(E') - a^2$ and $(a + 2)^2$. As $E'^v$ is splited, the degree of $\mathcal{E}xt^2(\mathcal{L},\mathcal{O}_{\mathbf{P}_3})$ is the sum of $A$ and $B$’s degree. Using $c_2 E' \leq 3 - d_L$, we obtain the contradiction: $d_L \geq a^2 + 5 + d_L$.

  - $k = 3$, $a = -2$ but it is impossible because $E'^v$ is reflexive and $c_1(E'^v) = 0$; $c_2(E'^v) \leq 2$, so $E'^v(1)$ must have a section.
So the only remaining case is:

• $k \geq 3, X \neq \emptyset$

The curve $X$ can’t be in 3 independent planes, so $a \geq 0$. In other words, it means that $E'$ is semi-stable.

- If $a = 0$, then $X$ has degree $c_2(E')$, which must be 1 or 2. So there is a plane $H$ having a curve in its intersection with $X$. This intersection is in fact the union of a curve of degree $b = 1$ or 2 with a scheme $X'$ which is the vanishing of a section of $E_H(-b)$. The last line of \((\square)\) has kernel $E(-4)$ because $k \geq 3$, so it gives the following exact sequence:

$$0 \to E_H(-4) \to kO_H(-2) \to J_{X'}(-b) \to A' \to 0$$

But $X'$ has degree $c_2(E') + b^2 \geq 2$, so it lies in only one line, thus those 3 sections have to be proportional. Then we would have $E_H(-4) = 2O_H(-2)$, which is impossible because an instanton doesn’t have unstable planes.

- If $a = -1$, (i.e: $E^\infty$ stable)

Then we have $c_2(E^\infty) > 0$, and on the other hand, $\mathcal{E xt}^2(\mathcal{L}, \mathcal{O}_{\mathbb{F}_3})$ has a non empty 1 dimensional component, so we have $5 \leq n \leq 7$ and $0 < c_2(E^\infty) \leq 2$, and $d_\mathcal{L} \leq 2$. Take this time a general plane $H$ such that $E_H$ and $E_H^\infty$ are stable, and $d_\mathcal{L} = h^0\mathcal{E xt}^2(\mathcal{L}, \mathcal{O}_{\mathbb{F}_3})$. Restrict to $H$ the second line of diagram \((\square)\) to obtain:

$$0 \to E_H(-4) \to 4O_H(-2) \to E_H^\infty \to \mathcal{E xt}^2(\mathcal{L}, \mathcal{O}_{\mathbb{F}_3}(-2))_H \to 0$$

But $h^1E_H^\infty = 0$ because $c_2(E^\infty) \leq 2$, and $h^0N(1) = h^1E_H(-3) = h^0E_H = 0$, so we have $h^0\mathcal{E xt}^2(\mathcal{L}, \mathcal{O}_{\mathbb{F}_3}(-1))_H \geq h^0E_H^\infty(1)$, which is greater or equal to 4, so it contradicts $d_\mathcal{L} \leq 2$.

**Second case:** The support of $\mathcal{L}$ contain a 2 dimensional part which is a plane $H$.

We have the following sequence, where this time $c_1(E^\infty) = -1$.

$$0 \to E'(-2) \to 4O_{\mathbb{F}_3} \to E(2) \to \mathcal{L} \to 0$$
As previously, we obtain the following diagram:

\[
\begin{array}{ccccccc}
0 & & 0 & & 0 & & 0 \\
(4-k)\mathcal{O}_{\mathbb{P}^3}(-2) & \rightarrow & \mathcal{O}_{\mathbb{P}^3}(a) & \rightarrow & B & \rightarrow & 0 \\
0 & \rightarrow & M^\vee(-2) & \rightarrow & 4\mathcal{O}_{\mathbb{P}^3}(-2) & \rightarrow & E^\vee \\
& & \rightarrow & \mathcal{E}xt^2(\mathcal{L},\mathcal{O}_{\mathbb{P}^3}(-2)) & \rightarrow & 0 \\
k\mathcal{O}_{\mathbb{P}^3}(-2) & \rightarrow & \mathcal{J}_X(-a-1) & \rightarrow & A & \rightarrow & 0 \\
0 & & 0 & & 0 & & 0
\end{array}
\]  \tag{13}

Working as in the first case, we prove that the union of a section of \(E^\vee(2)\) with \(\mathcal{E}xt^2(\mathcal{L},\mathcal{O}_{\mathbb{P}^3})\)'s support (counted rank of \(\mathcal{E}xt^2(\mathcal{L},\mathcal{O}_{\mathbb{P}^3})\) times) is linked by 2 cubic surfaces (sections of \(\Lambda(E^\vee(2))\)) to a section of \(M^\vee(3)\). Furthermore, this section is linked by a section of \(E(2)\) to a (possibly empty) curve of \(H\). But this plane curve is of degree at most 2 because \(E_H\) is necessarily semi-stable. Then, \(c_2(M^\vee(3)) = c_2(M^\vee) \geq n + 2\) (\(n \geq 5\)), so we have \(d_\mathcal{L} + c_2(E^\vee) \leq 0\) and \(d_\mathcal{L} \leq 2\).

But as \(c_1(E^\vee) = -1\) and \(c_2(E^\vee) \leq 0\), we must have \(h^0(E^\vee) \neq 0\), we can assume \(a \geq 0\) in diagram (13).

- If \(X = \emptyset\) Then \(E^\vee\) has to split, so \(B\) injects itself in \(\mathcal{E}xt^2(\mathcal{L},\mathcal{O}_{\mathbb{P}^3}(-2))\), thus \(B\)'s support has to be at most 1 dimensional. As \(A\) has also a 1 dimensional support (at most), we have only the following cases:
  - \(k = 1, a = 1\), so \(E^\vee = \mathcal{O}_{\mathbb{P}^3}(-1) \oplus \mathcal{O}_{\mathbb{P}^3}(2)\), but it conflicts with the independence of the 4 chosen sections of \(E(2)\).
  - \(k = 2\), then \(B\) has a 1 dimensional support of degree \((2 + a)^2 \geq 4\), but it conflicts with \(d_\mathcal{L} \leq 2\).

- If \(X \neq \emptyset\)

The facts that \(A\) has an at most 1 dimensional support, and that \(a \geq 0\) imply \(k = 2, X = \mathbb{P}^1, a = 0\). Then \(X\) must have degree \(c_2(E^\vee)\), which contradicts \(c_2(E^\vee) \leq 0\).

The effective situation:

We can conclude from the previous cases that there is a quadric \(Q\) in \(\mathcal{L}\)'s support because it can’t contain a cubic surface as there is no curve in 3 independent planes. So a section \(C\) of \(E(2)\) have a component \(C_2\) in the quadric \(Q\), and another one \(C_1\), which has to be skew because \(h^0E(1) = 0\), and which lies on 3 quadrics. This curve \(C_1\) must then be a cubic curve of arithmetic genus 0.

Let’s remark that \(C_2\) has degree \(n + 1 \geq 6\), so \(Q\) is normal because \(C\) can’t have plane components of degree 3 or more otherwise there would be an unstable planes, which is not possible for an instanton.
One has first to check that $C_1$ and $C_2$ have a 0-dimensional intersection to get the Proposition 4.5.1. The cubics arising when the section of $E(2)$ moves, are the vanishing locus of sections of $M^r(2)$, which is reflexive of rank 2. Take a normal quadric $Q'$ containing $C_1$, then it must contain a pencil of those cubics say $\lambda C_1 + \mu C_1'$. But if the associated pencil of sections of $E(2)$ was such that $C_1 \cap C_2$ and $C_1' \cap C_2'$ were 1 dimensional, then $C_1 \cap C_1'$ would have a 1 dimensional component, because $C_1 \cap C_2$ and $C_1' \cap C_2'$ lies in the fixed curve $Q \cap Q'$, and this 1-dimensional component of $C_1 \cap C_1'$ would be in the singular locus of $Q'$ which contradicts its normality.

So there is a section of $E(2)$ vanishing on $C = C_1 \cup C_2$, where $C_1$ is a skew cubic of arithmetic genus 0 and $\dim C_1 \cap C_2 = 0$. On the other hand, we have $\omega_C = O_C$, and the exact sequence of liaison $(i=1 \text{ or } 2)$:

$$0 \rightarrow \omega_{C_1} \rightarrow O_C \rightarrow O_{C_3-i} \rightarrow 0$$

gives when $i=1$ by restriction to $C_1$ that $C_1 \cap C_2$ has length 2, and when $i=2$ by restriction to $C_2$ that $C_2$ has arithmetic genus 0. But a quadric cone can’t have curves of arithmetic genus 0 and degree greater or equal to 4, so $Q$ is smooth and $C_2$ has bidegree $(1, n)$ in $Q$.

We’d like now to prove the smoothness of a general $C_2$. Denote by $h$ the 3-dimensional subspace of $|C_2|$ induced by the 4 sections of $E(2)$. The base points of $h$ must be in the singular locus of $Q \cup Q'$ for any quadric $Q'$ containing $C_1$, hence those points are on $C_1$. But we showed that $C_1 \cap C_2$ was 0-dimensional, so the set of base points of $h$ is at most finite. Furthermore, if $C_2$ is singular in some point $P$, then it must contain the ruling of bidegree $(0, 1)$ passing through $P$, so this ruling would be a base curve of $h$ if the general curve was singular at $P$. Hence, the generic element of $h$ is smooth and irreducible of bidegree $(1, n)$, and this is the proposition 4.5.1. □

**Proposition 4.5.3** Let $E$ be a $n$-instanton with $n \geq 5$, $h^0(E(1)) = 0$ and $h^0(E(2)) \geq 4$, then $E$ has only a 1-dimensional scheme of multi-jumping lines.

The bundle $E$ belongs by hypothesis to the family Proposition 4.5.1, so denote by $Q$ the quadric which is the support of the cokernel of the map given by the 4 sections to $E(2)$. Let $s$ be a section of $E(2)$ and $Z_s = C_{2,s} \cup C_{1,s}$ its vanishing locus, where $C_{1,s}$ is the rational cubic and $C_{2,s}$ is the curve of bidegree $(1, n)$ in $Q$. If a line $d$ is a multi-jumping line then $E_d(2) = O_d(a + 2) \oplus O_d(-a + 2)$ with $a \geq 2$. So a multi-jumping line $d$ meets $Z_s$ if and only if $d \cap Z_s$ has length at least 4. So if $E$ has a 2 parameter family of multi-jumping lines, then infinitely many of them would be 4-secant to $Z_s$ and up to a change of the section $s$, we can assume that infinitely many are not in $Q$. So those lines are 2-secant to $C_{2,s}$ and 2-secant to $C_{1,s}$ because they are not in $Q$ and $C_{1,s}$ don’t have trisecant. Denote by $\Sigma$ the ruled surface of $\mathbb{P}_3$ made with those lines.

a) If $C_{1,s}$ is smooth.

Then the basis of $\Sigma$ is a curve $\Gamma$ on the Veronese surface of bisecant lines to $C_{1,s}$,
and \( \Sigma \) must contain \( C_2 \) because \( C_2 \) is irreducible. But we have a morphism of degree 2 from \( C_2 \) onto the basis of \( \Sigma \) hence \( \Gamma \) is smooth and rational, so \( \deg \Sigma = 2 \) or \( \deg \Sigma = 4 \), but \( \Sigma \) contains \( C_1 \cup C_2 \) so \( \deg \Sigma \geq 4 \) because \( h^0E(1) = 0 \), and all the quartics containing \( C_1 \cup C_2 \) are some \( Q \cup Q' \), where \( Q' \) is a quadric containing \( C_1 \), so \( \Sigma \) can’t be a quartic containing \( C_1 \cup C_2 \) and this case is impossible.

b) If \( C_{1,s} \) is made of 3 lines containing a same point \( N \) independent of \( s \), then \( C_{2,s} \) must contain \( N \) because \( Z_s \) is locally complete intersection. As \( C_{2,s} \) is smooth at \( N \), any line through \( N \) meets \( Z_s \) in length 2 around \( N \), and \( N \in Q \), so a line through \( N \) don’t meet \( C_{1,s} \) in another point, and it can’t meet \( C_{2,s} \) in 2 points distinct of \( N \), so it can’t be 4-secant to \( C_{2,s} \cap C_{1,s} \). There must then exist a plane \( H \) containing 2 of the lines of \( C_{1,s} \) and infinitely many multi-jumping lines meeting \( C_{2,s} \). As \( h^0E_H(-1) = 0 \), \( C_{2,s} \cap H \) must be 0-dimensional, and there would be a point \( P \) in \( C_{2,s} \cap H \) such that every line of \( H \) through \( P \) is bisecant to \( C_{2,s} \). Hence \( H \) is tangent to \( Q \) at \( P \), but one of the ruling of \( Q \) through \( P \) would be in \( H \) and would be only 1 secant to \( C_{2,s} \) because it has bidegree \( (1, n) \), which contradicts the definition of \( P \). \( \square \)

**Proposition 4.5.4** Let \( E \) be a \( n \)-instanton with \( n \geq 5 \), \( h^0(E(1)) = 0 \) and \( h^0(E(2)) \geq 4 \), then \( E \) is a smooth point of the moduli space \( I_n \).

From classical theory (cf [LeP]), one has to prove that \( h^2(E \otimes E) = 0 \), and if \( C \) is the vanishing locus of a section of \( E(2) \), then \( h^1(E_C(2)) = 0 \) implies this result. As \( E_C(2) \) is just the normal bundle of \( C \), we are considering the problem of the vanishing of \( h^1(N_C) \), so we will solve it as it was done for smoothing questions in [H-H].

Let’s recall from 4.5.1 that \( C = C_1 \cup C_2 \) where the \( C_i \) have zero arithmetic genus, and where \( C_1 \) is a skew cubic and \( C_2 \) is on a smooth quadric \( Q \). We want first to prove the vanishing of \( h^1(N_{C_1}) \). As \( C_2 \) has bidegree \( (1, n) \) on the smooth quadric \( Q \), we have \( h^1(N_{C_2}) = 0 \). (Cf for example the proof of prop 5.4 1 \( \alpha \) in [H-H]). This will be true for \( C_1 \) even if it is 3 concurrent lines. Indeed \( C_1 \) is the vanishing of a section of \( R(1) \) where \( R \) is a rank 2 reflexive sheaf with \( (c_1, c_2, c_3) = (0, 2, 4) \), and the following exact sequences:

\[
0 \to 2\mathcal{O}_{\mathbb{P}_3}(-1) \to 4\mathcal{O}_{\mathbb{P}_3} \to R(1) \to 0 \& 0 \to 2\mathcal{O}_{\mathbb{P}_3}(-3) \to 3\mathcal{O}_{\mathbb{P}_3}(-2) \to \mathcal{J}_{C_1} \to 0
\]

imply that \( h^1(R(1)) = 0 \), \( h^3(\mathcal{J}_C; R(1)) = 0 \) thus \( h^1(R_{C_1}(1)) = 0 \), but \( R_{C_1}(1) = N_{C_1} \).

So both \( h^1N_{C_i} \) are zero, and we can now deduce from it that \( h^1N_C \) is also zero. We have the following sequence of liaison:

\[
0 \to \mathcal{O}_C \to \mathcal{O}_{C_1} \oplus \mathcal{O}_{C_2} \to \mathcal{O}_{C_1 \cap C_2} \to 0
\]

giving when twisted by \( E(2) \):

\[
0 \to N_C \to E_{C_1}(2) \oplus E_{C_2}(2) \to E_{C_1 \cap C_2}(2) \to 0
\]
Furthermore, the inclusion $\mathcal{J}_C \subset \mathcal{J}_C$ gives a map $N_C \to N_C$. Then we have the following exact sequence defining $L_i$ and $M_i$:

$$
\cdots \to N_C \to E_C(2) \to L_i \to 0
$$

As $h^1(N_C) = 0$ implies $h^1(M_i) = 0$, we have $\left\{ \begin{array}{l} h^1(E_C(2)) = 0 \\
H^0(E_C(2)) \to H^0(L_i) \to 0 \end{array} \right.$, which gives $h^1(N_C) = 0$ when taking global sections in the following diagram.

$$
\begin{array}{ccccccccc}
N_C \oplus N_C & \rightarrow & E_C(2) \oplus E_C(2) & \rightarrow & L_1 \oplus L_2 & \rightarrow & 0 \\
0 & \rightarrow & N_C & \rightarrow & E_C(2) \oplus E_C(2) & \rightarrow & E_C(2) \rightarrow 0 \\
0 & & & & & & 0
\end{array}
$$

We can now conclude that $E$ is a smooth point of the moduli space because $N_C = E_C(2)$, and the previous vanishing gives $h^2(E \otimes E) = 0$ using the sequences:

$$
0 \to E(-2) \to E \otimes E \to J_C E(2) \to 0 \text{ and } 0 \to J_C E(2) \to E(2) \to N_C \to 0
$$

4.6 A $\theta$-characteristic on the curve of multi-jumping lines of an $n$-instanton

Assume here that $E$ is an instanton vector bundle with second Chern class $n$. The aim of this part is to study the scheme of multi-jumping lines of $E$ when it satisfies the properties expected from its determinantal structure.

For instance, according to [B-H], the general member of the irreducible component of the moduli space containing the t'Hooft bundles satisfy those properties.

**Proposition 4.6.1** If an $n$-instanton has only $(k \leq 2)$-jumping lines, and if its scheme of multi-jumping lines is a curve $\Gamma$ in $G$, then we have:

$$(R^1 \pi_* \pi^* E)^{\otimes 4} = \mathcal{O}_\Gamma(n); \omega^\bullet_\Gamma = (R^1 \pi_* \pi^* E)^{\otimes 2}(n-4); (\omega^\bullet_\Gamma)^{\otimes 2} = \mathcal{O}_\Gamma(3n-8)$$

where $\omega^\bullet_\Gamma$ is a dualizing sheaf on $\Gamma$.

In fact this result is just an analogous of the proposition 1.5 of [E-S]. Let $K$ and $Q$ be the tautological bundles on $G$ such that we have the sequence:

$$
0 \to K \to H^0(\mathcal{O}_{P^3}(1))^\vee \otimes \mathcal{O}_G \to Q \to 0
$$

The points/lines incidence variety of $P^3$ is $F = P_G(Q^\vee)$, so one has the exact sequence, where $q$ still be the projection from $F$ to $G$: 

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0 \rightarrow \mathcal{O}_F(-\tau) \rightarrow q^*Q^\vee \rightarrow (\omega_{F/G}(\tau))^\vee \rightarrow 0

We'd like here to make a relative (over $G$) Beilinson's construction. So consider the resolution of the diagonal $\Delta$ of $F \times F$:

\[ 0 \rightarrow \mathcal{O}_F(-\tau) \boxtimes_G^\mathcal{O}_{F/G}(\tau) \rightarrow \mathcal{O}_{F \times F} \rightarrow \mathcal{O}_\Delta \rightarrow 0 \]

which gives when twisted by $p^*E(\tau) \boxtimes \mathcal{O}_F$ the following spectral sequence which stops in $E_2^{p,q}$ and ends to $p^*E(\tau)$ in degree 0 and 0 in the other degrees.

\[
\begin{array}{ccc}
q^*(R^1q_*p^*E) \otimes \mathcal{O}_F(\sigma - \tau) & \xrightarrow{d} & q^*(R^1q_*p^*E(\tau)) \\
\downarrow q & & \downarrow p \\
nq^*q_*p^*E \otimes \mathcal{O}_F(\sigma - \tau) & \xrightarrow{d} & nq^*q_*p^*E(\tau)
\end{array}
\]

By assumption $E$ has no $(k \geq 3)$-jumping lines, so $R^1q_*p^*E(\tau) = 0$, and we have the following surjection where $M$ denotes its kernel.

\[ 0 \rightarrow M \rightarrow p^*E(\tau) \rightarrow q^*(R^1q_*p^*E) \otimes \mathcal{O}_F(\sigma - \tau) \rightarrow 0 \]

Denote by $h$ and $K$ the restrictions of $q$ and $q^*(R^1q_*p^*E) \otimes \mathcal{O}_F(\sigma - \tau)$ to $q^{-1}(\Gamma)$. The sheaf $R^1q_*p^*E$ is locally free over $\Gamma$, so $h_*K = R^1q_*p^*E \otimes h_*\mathcal{O}_F(\sigma - \tau) = 0$, and we have: $h_*M_{|q^{-1}(\Gamma)} = h_*((p^*E(\tau)|_{q^{-1}(\Gamma)}) = 0)$. Furthermore, as $K$ is locally free over $\Gamma$ and $\hat{\Lambda}E = 0$, one has $M_{|q^{-1}(\Gamma)} = K^\vee(2\tau)$, so $h_*((p^*E(\tau)|_{q^{-1}(\Gamma)}) = (R^1q_*p^*E)^\vee \otimes h_*\mathcal{O}_F(3\tau - \sigma)$. But it means that $h_*((p^*E(\tau)|_{q^{-1}(\Gamma)}) \otimes R^1q_*p^*E = \mathcal{O}_G(-1) \otimes \text{Sym}_3Q_{\Gamma}$, and using that $h_*((p^*E(\tau)|_{q^{-1}(\Gamma)}) = 0$, we found the relation: $(R^1q_*p^*E)^{\otimes 4} \otimes \text{det}(h_*((p^*E(\tau)|_{q^{-1}(\Gamma)}))) = \mathcal{O}_T(2)$. But in fact $q_*p^*E(\tau)$ is locally free over $G$ by base change, so $\text{det}(h_*((p^*E(\tau)|_{q^{-1}(\Gamma)})) = \mathcal{O}_T(2 - n)$, because $c_1((q_*p^*E(\tau))) = 2 - n$ from Riemann-Roch over $F$. So we have:

$$(R^1q_*p^*E)^{\otimes 4} = \mathcal{O}_T(n)$$

On another hand, the following resolution of $\mathcal{O}_F$ in $G \times \mathbb{P}_3$:

\[ 0 \rightarrow \mathcal{O}_{G \times \mathbb{P}_3}(-\sigma - 2\tau) \rightarrow Q^\vee \otimes \mathcal{O}_{\mathbb{P}_3}(\tau) \rightarrow \mathcal{O}_{G \times \mathbb{P}_3} \rightarrow \mathcal{O}_F \rightarrow 0 \]

gives when twisted by $p^*E$ the following exact sequence deduced from the Leray spectral sequence:

\[ H^1(E(-1)) \otimes Q^\vee \xrightarrow{\phi} H^1(E) \otimes \mathcal{O}_G \rightarrow R^1q_*p^*E \rightarrow 0 \]

The Eagon-Northcott complexes $(E_i)$ associated to $\phi$ gives resolutions of $M_i$, where we have because $\Gamma$ is a curve: $M_0 = \mathcal{O}_T$, $M_1 = R^1q_*p^*E$, $M_i = \text{Sym}_i M_1$, and furthermore:
\[\omega_\Gamma^\circ = M_2 \otimes \omega_G \otimes \det(H^1(E) \otimes \mathcal{O}_G) \otimes (H^1(E(-1)) \otimes Q^v)^v\]

But \(\omega_G = \mathcal{O}_G(-4)\), and \(\det(H^1(E) \otimes \mathcal{O}_G) \otimes (H^1(E(-1)) \otimes Q^v)^v = \mathcal{O}_G(n)\), so we have:

\[\omega_\Gamma^\circ = (R^1q_*p^*E) \otimes (n - 4)\text{ ans } (\omega_\Gamma^\circ)^\otimes 2 = \mathcal{O}_\Gamma(3n - 8)\]

**Remark 4.6.2** \(\theta = (R^1q_*p^*E) \otimes \omega_\Gamma^\circ(2 - n)\) is a \(\theta\)-characteristic of \(\Gamma\).

If \(n = 2n'\), then so is \(\theta = (R^1q_*p^*E)(n' - 2)\).

**Proposition 4.6.3** If the scheme of multi-jumping lines \(\Gamma\) of an \(n\)-instanton is a curve in \(G\) without \((k \geq 3)\)-jumping lines, then:

\[R^1q_*p^*E(-\tau)|_\Gamma \simeq (R^1q_*p^*E) \otimes Q|_\Gamma(-\sigma)\]

Furthermore, if \(\Gamma\) is smooth, then its normal bundle in \(G\) is:

\[N_{\Gamma,G} = \text{Sym}_2q \otimes \omega_\Gamma(3 - n)\]

This proposition is also due to a relative Beilinson’s construction, but this time the resolution of the diagonal is twisted by \(p^*E \boxtimes \mathcal{O}_F\), so we have the spectral sequence:

\[
\begin{array}{cccccc}
& & & q & & \\
& & & \downarrow & & \\
q^*(R^1q_*p^*E(-\tau)) \otimes \mathcal{O}_F(\sigma - \tau) & \xrightarrow{d} & q^*(R^1q_*p^*E) & \xrightarrow{p} & q^*(q_*p^*E) & \\
& & & \downarrow & & \\
q^*(q_*p^*E(-\tau)) \otimes \mathcal{O}_F(\sigma - \tau) & \xrightarrow{d'} & q^*(q_*p^*E) & \xrightarrow{p} & q^*(q_*p^*E) & \\
& & & \downarrow & & \\
& & & 0 & & \\
\end{array}
\]

which ends with \(0 \rightarrow K \rightarrow p^*E \rightarrow N \rightarrow 0\) where \(N\) is the kernel of \(d\) and \(K\) the cokernel of \(d'\). It gives the exact sequences (*):

\[
0 \longrightarrow q^*(q_*p^*E(-\tau)) \otimes \mathcal{O}_F(\sigma - \tau) \longrightarrow q^*(q_*p^*E) \longrightarrow p^*E \longrightarrow N \rightarrow 0
\]

\[
0 \longrightarrow N(\tau - \sigma) \longrightarrow q^*(R^1q_*p^*E(-\tau)) \longrightarrow q^*(R^1q_*p^*E)(\tau - \sigma) \longrightarrow 0
\]

Let’s restrict this last sequence to \(q^{-1}(\Gamma)\), then we can apply the projection formula to \(q^*(R^1q_*p^*E(-\tau))|_{q^{-1}\Gamma}\) and to \(q^*(R^1q_*p^*E)(\tau - \sigma)|_{q^{-1}\Gamma}\) because they are locally free, and we obtain the exact sequence:

\[
R^1q_*p^*E(-\tau)|_\Gamma \longrightarrow R^1q_*p^*E \otimes q_*\mathcal{O}_\Gamma(\tau - \sigma) \longrightarrow R^1q_*N|_{q^{-1}\Gamma}(\tau - \sigma)
\]

On another hand the restriction to \(q^{-1}\Gamma\) of the first sequence of (*) shows that \(R^1q_*K|_{\Gamma}(\tau) = 0\), so that \(R^1q_*N|_{q^{-1}\Gamma}(\tau) \simeq R^1q_*p^*E|_\Gamma(\tau) = 0\) because \(E\) has no \((k \geq 3)\)-jumping lines. So we have a surjection from \(R^1q_*p^*E(-\tau)|_\Gamma\) to \(R^1q_*p^*E \otimes Q(-1)\) which is in fact an isomorphism because those 2 sheaves are locally free of rank 2 on \(\Gamma\). Furthermore, the curve \(\Gamma\) is given by the first Fitting ideal of the following map \(\phi\):
$H^2(E(-3) \otimes \mathcal{O}_G(-1)) \xrightarrow{\phi} H^1(E(-1)) \otimes \mathcal{O}_G \xrightarrow{j} R^1q_*p^*E(-\tau) \rightarrow 0$

where $\phi$ is symmetric with respect to Serre's duality. Assume now that $\Gamma$ is smooth, and denote by $\mathcal{L}_\Gamma$ the restriction of $R^1q_*p^*E(-1)$ to $\Gamma$.

We just need now, to conclude the proof, to recall the results of Tjurin (Cf [T1]) which give a description of the normal bundle. Let's consider the hypernet of quadrics classically associated to an instanton, and denote by $\mathbb{P}^2$ the restriction of $R^1q_*p^*E(-1)$ to $\mathbb{P}$. The hypernet gives an inclusion $\mathbb{P}(\Lambda V) \subset \mathbb{P}(Sym_2H^\vee)$, and denote by $D_2$ the surface made of the quadrics of the hypernet of rank at most $n-2$, assuming here that $G$ and $\mathbb{P}(\Lambda V)$ cut transversally the stratification of $\mathbb{P}(Sym_2H^\vee)$ by the rank of the quadrics. The second symmetric power of the restriction of $j$ to $\Gamma$ gives a surjection $Sym_2H^\vee \otimes \mathcal{O}_\Gamma \xrightarrow{s_2j_\Gamma} Sym_2\mathcal{L}_\Gamma \rightarrow 0$. Denote by $K$ the kernel of $s_2j_\Gamma$, then the fiber $\mathbb{P}(K_q)$ over some $q \in \Gamma$ is just made of the quadrics of $\mathbb{P}(H)$ which contain the singular locus of $q$. This is also according to [T1] or [T3]§2 lemma 1.1, the projective tangent space in $q$ to the locus of rank at most $n-2$ quadrics. The above hypothesis of transversality implies that $\mathbb{P}(\Lambda V \cap K_q)$ is 3 dimensional. Similarly, composing the inclusion $(\Lambda V) \otimes \mathcal{O}_\Gamma \subset Sym_2H^* \otimes \mathcal{O}_\Gamma$ with $s_2j_\Gamma$ gives the sequence:

$$0 \rightarrow K' \rightarrow (\Lambda V) \otimes \mathcal{O}_\Gamma \rightarrow Sym_2\mathcal{L}_\Gamma \rightarrow 0$$

where $\mathbb{P}(K'_q)$ is the projective tangent space to $D_2$ at a point $q$ of $\Gamma$. So we have the commutative diagram of [T1] restricted to $\Gamma$:

where the first two columns are the Euler relative exact sequences over $\Gamma$ and $\mathbb{P}(\Lambda V)$. But $\Gamma = D_2 \cap G$, so we have $N_{\Gamma,G}(-1) = N_{D_2,\mathbb{P}}(-1)|_{\Gamma}$, then $N_{\Gamma,G} \simeq Sym_2(R^1q_*p^*E(-\tau)|_{\Gamma}) \otimes \mathcal{O}_\Gamma(1)$, which gives with the previous results:

$$N_{\Gamma,G} \simeq Sym_2Q \otimes \omega_\Gamma(3-n)$$

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