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pour les fonctions L de formes automorphes.

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Chapter 1

Introduction and statement of results

1.1 Linnik’s original problem

In view of Dirichlet’s theorem that there are infinitely many primes in the arithmetic progression \( n \equiv l \pmod{q} \) with \( (q, l) = 1 \), it is a natural question how big the least prime is, denoted by \( P(q, l) \), in this arithmetic progression. Linnik [30] [31] proved that there is an absolute constant \( \ell > 0 \) such that

\[
P(q, l) \ll q^\ell,
\]

and this constant \( \ell \) was named after him. Since then, a number of authors have established numerical values for Linnik’s constant \( \ell \), while the best result known is \( \ell = 5.5 \) by Heath-Brown [9]. We remark that these results depend on, among other things, numerical estimates concerning zero-free regions and the Deuring-Heilbronn phenomenon of Dirichlet \( L \)-functions. Under the Generalized Riemann Hypothesis (GRH in brief) for Dirichlet \( L \)-functions, the above bounds can be improved to

\[
P(q, l) \ll \varphi^2(q)(\log q)^2.
\]

(1.1)

The conjectured bound is

\[
P(q, l) \ll_{\varepsilon} q^{1+\varepsilon}
\]

(1.2)

for arbitrary \( \varepsilon > 0 \), and this is a consequence of GRH and another conjecture concerning the universality of the distribution of nontrivial zeros for Dirichlet \( L \)-functions. The
conjectured bound (1.2) is the best possible save the \( \varepsilon \) in the exponent. In fact, a trivial lower bound for \( P(q,l) \) is

\[
\max_l P(q,l) \geq \{1 + o(1)\} \varphi(q) \log q.
\] (1.3)

Linnik’s problem is a rich resource for further mathematical thoughts, and there are a number of problems that can be formulated in a similar manner.

### 1.2 A Linnik-type problem for classical modular forms

Let \( f \) be a normalized Hecke eigenform that is a new form of level \( N \) of even integral weight \( k \) on \( \Gamma_0(N) \). Recall that normalized means that the first Fourier coefficient \( \lambda_f(1) = 1 \), and new form means that \( N \) is the exact level of \( f \), and in this case the Fourier coefficients are equal to the Hecke eigenvalues. It also follows that, for this \( f \), its Fourier coefficients \( \{\lambda_f(n)\}_{n=1}^{\infty} \) are real. Applying a classical theorem of Landau, one shows that the sequence \( \{\lambda_f(n)\}_{n=1}^{\infty} \) must have infinitely many sign changes, i.e. there are infinitely many \( n \) such that \( \lambda_f(n) > 0 \), and there are infinitely many \( n \) such that \( \lambda_f(n) < 0 \). In view of this result, a reasonable question to ask is:

Is it possible to obtain a bound on the first sign change, say, in terms of \( k \) and \( N \)?

This question is similar to Linnik’s problem in nature, and it is named as Linnik-type in this thesis. In general, this seems to be a difficult question.

In a very special case, this has been considered in Siegel [53]; but, in general, developments have been achieved only quite recently. In the case \( N = 1 \), sign changes of the \( \lambda_f(p) \) where \( p \) goes over primes have been considered by Ram Murty [42]. Kohnen and Sengupta [26] have shown that the first sign change of \( \lambda_f(n) \) happens for some \( n \) with

\[
n \ll kN \exp \left( c \sqrt{\frac{\log N}{\log \log(3N)}} \right) (\log k)^{27}, \quad (n,N) = 1,
\] (1.4)

where \( c > 2 \) is a constant and the \( \ll \) constant is absolute. Note that it is natural to assume that \( (n,N) = 1 \), since the eigenvalues \( \lambda_f(p) \) with \( p | N \) are explicitly known by the Atkin-Lehner theory. Recently, Iwaniec, Kohnen, and Sengupta [13] proved that there is some \( n \) with

\[
n \ll (k^2N)^{29/60}, \quad (n,N) = 1,
\] (1.5)
1.2 A Linnik-type problem for classical modular forms

such that $\lambda_f(n) < 0$.

This result is sharp indeed; to see this, let us point out that the convexity bound

$$L(1/2 + it, f) \ll (k^2 N)^{1/4 + \varepsilon}$$

(1.6)

of the automorphic $L$-function $L(1/2 + it, f)$ gives, instead of (1.5), the weaker bound

$$n \ll (k^2 N)^{1/2 + \varepsilon}, \quad (n, N) = 1.$$  (1.7)

The uniform subconvexity bound

$$L(1/2 + it, f) \ll (k^2 N)^{29/120}$$

(1.8)

would prove (1.5), but no result of this quality is known. The best known uniform subconvexity bound like (1.8) is due to Michel and Venkatesh [38], which states that

$$L(1/2 + it, f) \ll (k^2 N)^{1/4 - \delta},$$

(1.9)

where $\delta$ is some positive constant not specified. Iwaniec, Kohnen, and Sengupta [13] manage to establish (1.5) without appealing to (1.8); instead, they use the arithmetic properties of $\lambda_f(n)$, the Ramanujan conjecture proved by Deligne, and sieve methods.

A more precise question to ask is: how long is the sequence of Hecke eigenvalues that keep the same sign. To measure the length of the sequences, define

$$N^+_f(x) = \sum_{\substack{n \leq x, (n,N)=1 \\lambda_f(n)>0}} 1,$$

and define $N^-_f(x)$ similarly by replacing the condition $\lambda_f(n) > 0$ under the summation by $\lambda_f(n) < 0$. Kohnen, Lau, and Shparlinski [25] prove that, if $f$ is a new form, then

$$N^\pm_f(x) \gg_f \frac{x}{\log^{1/2} x},$$

(1.11)

where the implied constant depends on the form $f$. Recently, Wu [55] reduces the 17 in the logarithmic exponent to $1 - 1/\sqrt{3}$, as an simple application of his estimates on power sums of Hecke eigenvalues. Still more recently, Lau and Wu [29] manage to completely get rid of the logarithmic factor in (1.11), getting

$$N^\pm_f(x) \gg_f x,$$

(1.12)

where the implied constant depends on the form $f$. Obviously, this is the best possible result concerning the order of magnitude of $x$.

These materials form our Chapter 2.
1.3 A Linnik-type problem for Maass forms

In Chapter 3, we go on to study Linnik-type problem for Maass forms. Let $f$ be a normalized Maass eigenform that is a new form of level $N$ on $\Gamma_0(N)$. Then, similarly, its Fourier coefficients $\{\lambda_f(n)\}_{n=1}^{\infty}$ are real. Applying Landau’s theorem as in the holomorphic case, one shows that the sequence $\{\lambda_f(n)\}_{n=1}^{\infty}$ must have infinitely many sign changes. Therefore, one may formulate a Linnik-type problem for this normalized Maass eigenform $f$.

In this direction, we prove the following theorem.

**Theorem 3.10.** Let $f$ be a normalized Maass new form of level $N$ and Laplace eigenvalue $1/4 + \nu^2$. Then there is some $n$ satisfying

$$n \ll ((3 + |\nu|)^2 N)^{1/2 - \delta}, \quad (n, N) = 1,$$

(1.13)

such that $\lambda_f(n) < 0$, where $\delta$ is a positive absolute constant.

This is proved by using, among other things, the uniform subconvexity bound (1.9) of Michel and Venkatesh [38], and this explains why we are not able to get an acceptable numerical value for $\delta$. However, the bound (1.9) alone is not enough to establish (1.13); some combinatorial and analytic arguments are also necessary for (1.13). The method in Iwaniec, Kohnen, and Sengupta [13] does not work here; one of the reasons is that for Maass forms the Ramanujan conjecture is still open, and hence the sieves in [13] do not apply.

1.4 A Linnik-type problem for automorphic $L$-functions

The Linnik-type problem considered before can be further generalized to that for automorphic $L$-functions. This is done in Chapter 4.

To each irreducible unitary cuspidal representation $\pi = \otimes \pi_p$ of $GL_m(\mathbb{A}_\mathbb{Q})$, one can attach a global $L$-function $L(s, \pi)$, as in Godement and Jacquet [17], and Jacquet and Shalika [20]. For $\sigma = \Re s > 1$, $L(s, \pi)$ is defined by products of local factors

$$L(s, \pi) = \prod_{p < \infty} L_p(s, \pi_p),$$

(1.14)

where

$$L_p(s, \pi_p) = \prod_{j=1}^{m} \left(1 - \frac{\alpha_p(p, j)}{p^s}\right)^{-1};$$

(1.15)
1.4 A Linnik-type problem for automorphic $L$-functions

the complete $L$-function $\Phi(s, \pi)$ is defined by

$$\Phi(s, \pi) = L_\infty(s, \pi_\infty)L(s, \pi),$$  \hspace{1cm} (1.16)

where

$$L_\infty(s, \pi_\infty) = \prod_{j=1}^{m} \Gamma_R(s + \mu_\pi(j))$$  \hspace{1cm} (1.17)

is the Archimedean local factor. Here

$$\Gamma_R(s) = \pi^{-s/2} \Gamma \left( \frac{s}{2} \right),$$  \hspace{1cm} (1.18)

and $\{\alpha_\pi(p,j)\}_{j=1}^{m}$ and $\{\mu_\pi(j)\}_{j=1}^{m}$ are complex numbers associated with $\pi_p$ and $\pi_\infty$, respectively, according to the Langlands correspondence. The case $m = 1$ is classical; for $m \geq 2$, $\Phi(s, \pi)$ is entire and satisfies a functional equation.

It is known from Jacquet and Shalika [20] that the Euler product for $L(s, \pi)$ in (1.14) converges absolutely for $\sigma > 1$. Thus, in the half-plane $\sigma > 1$, we may write

$$L(s, \pi) = \sum_{n=1}^{\infty} \frac{\lambda_\pi(n)}{n^s},$$  \hspace{1cm} (1.19)

where

$$\lambda_\pi(n) = \prod_{p^\nu \| n} \left\{ \sum_{\nu_1 + \cdots + \nu_m = \nu} \alpha_\pi(p, 1)^{\nu_1} \cdots \alpha_\pi(p, m)^{\nu_m} \right\}.$$  \hspace{1cm} (1.20)

In particular,

$$\lambda_\pi(1) = 1, \quad \lambda_\pi(p) = \alpha_\pi(p, 1) + \cdots + \alpha_\pi(p, m).$$  \hspace{1cm} (1.21)

It also follows from work of Shahidi [49], [50], [51], and [52] that the complete $L$-function $\Phi(s, \pi)$ has an analytic continuation to the whole complex plane and satisfies the functional equation

$$\Phi(s, \pi) = \varepsilon(s, \pi)\Phi(1 - s, \tilde{\pi})$$

with

$$\varepsilon(s, \pi) = \varepsilon_\pi N_\pi^{1/2 - s},$$

where $N_\pi \geq 1$ is an integer called the arithmetic conductor of $\pi$, $\varepsilon_\pi$ is the root number satisfying $|\varepsilon_\pi| = 1$, and $\tilde{\pi}$ is the representation contragredient to $\pi$.

By an argument similar to the case of holomorphic forms or Maass forms, it is possible to establish the following theorem of infinite sign changes.
**Theorem 4.13.** Let $m \geq 2$ be an integer and let $\pi$ be an irreducible unitary cuspidal representation for $GL_m(\mathbb{A}_Q)$ such that $\lambda_\pi(n)$ is real for all $n \geq 1$. Then the sequence $\{\lambda_f(n)\}_{n=1}^{\infty}$ has infinitely many sign changes, i.e. there are infinitely many $n$ such that $\lambda_f(n) > 0$, and there are infinitely many $n$ such that $\lambda_f(n) < 0$.

Iwaniec and Sarnak [15] introduced the analytic conductor of $\pi$. It is a function over the reals given by

$$Q_\pi(t) = N_\pi \prod_{j=1}^{m} (3 + |t + \mu_\pi(j)|), \quad (1.22)$$

which puts together all the important parameters for $\pi$. The quantity

$$Q_\pi = Q_\pi(0) = N_\pi \prod_{j=1}^{m} (3 + |\mu_\pi(j)|) \quad (1.23)$$

is also important, and it is named the conductor of $\pi$.

We may therefore formulate a Linnik-type problem for $\{\lambda_f(n)\}_{n=1}^{\infty}$, and the first sign change is measured by the conductor $Q_\pi$ of $\pi$. Our result in this direction is as follows.

**Theorem 4.15.** Let $m \geq 2$ be an integer and let $\pi$ be an irreducible unitary cuspidal representation for $GL_m(\mathbb{A}_Q)$. If $\lambda_\pi(n)$ is real for all $n \geq 1$, then there is some $n$ satisfying

$$n \ll Q_\pi^{m/2+\varepsilon} \quad (1.24)$$

such that $\lambda_\pi(n) < 0$. The constant implied in (1.24) depends only on $m$ and $\varepsilon$. In particular, the result is true for any self-contragredient representation $\pi$.

Proof of this theorem is quite different from that of Theorem 3.10. One of the most principal difficulties is that there is no relation of Hecke-type in this current general case of $\pi$ being irreducible unitary cuspidal representation, as in classic modular forms or Maass form cases (see (2.5) and (3.11)). These difficulties are overcome by, among other things, new analytic properties of $L(s, \pi)$ due to Harcos [8], an inequality due to Brumley [1], as well as important combinatorial properties of the sequence $\{\lambda_f(n)\}_{n=1}^{\infty}$, established in Lemma 4.12. Clearly this elegant inequality is of independent interest and we believe that it will find other applications.
1.5 Automorphic prime number theorem and a problem of Linnik’s type

To each irreducible unitary cuspidal representation $\pi = \otimes \pi_p$ of $GL_m(\mathbb{A}_Q)$, one can attach a global $L$-function $L(s, \pi)$ as in §1.4. Then, one can link $L(s, \pi)$ with primes by taking logarithmic differentiation in (1.15), so that for $\sigma > 1$,

$$\frac{d}{ds} \log L(s, \pi) = -\sum_{n=1}^{\infty} \frac{\Lambda(n)a_\pi(n)}{n^s},$$

where $\Lambda(n)$ is the von Mangoldt function, and

$$a_\pi(p^k) = \sum_{j=1}^{m} \alpha_\pi(p^j)^k.$$  \hfill (1.26)

The prime number theorem for $L(s, \pi)$ concerns the asymptotic behavior of the counting function

$$\psi(x, \pi) = \sum_{n \leq x} \Lambda(n)a_\pi(n),$$

and a special case of it asserts that, if $\pi$ is an irreducible unitary cuspidal representation of $GL_m(\mathbb{A}_Q)$ with $m \geq 2$, then

$$\psi(x, \pi) \ll \sqrt{Q_\pi} \cdot x \cdot \exp \left( -\frac{c}{2m^4} \sqrt{\log x} \right)$$

for some absolute positive constant $c$, where the implied constant is absolute. In Iwaniec and Kowalski [14], Theorem 5.13, a prime number theorem is proved for general $L$-functions satisfying necessary axioms, from which (1.27) follows as a consequence.

In this chapter, we first investigate the influence of GRH on $\psi(x, \pi)$. It is known that, under GRH, (1.27) can be improved to

$$\psi(x, \pi) \ll x^{1/2} \log^2(Q_\pi x),$$

where the implied constant depends at most on $m$. But better results are desirable. In this direction, we establish the following results.

**Theorem 5.1.** Let $m \geq 2$ be an integer and let $\pi$ be an irreducible unitary cuspidal representation of $GL_m(\mathbb{A}_Q)$. Assume GRH for $L(s, \pi)$. Then we have

$$\psi(x, \pi) \ll x^{1/2} \log^2(Q_\pi \log x)$$
for \( x \geq 2 \), except on a set \( E \) of finite logarithmic measure, i.e.

\[
\int_E \frac{dx}{x} < \infty.
\]

The constant implied in the \( \ll \)-symbol depends at most on \( m \).

**Theorem 5.2.** Let \( m \geq 2 \) be an integer and let \( \pi \) be an irreducible unitary cuspidal representation of \( \text{GL}_m(\mathbb{A}_Q) \). Assume GRH for \( L(s,\pi) \). Then

\[
\int_2^X \frac{|\psi(x,\pi)|^2 dx}{x} \ll X \log^2 Q_\pi.
\]

The constant implied in the \( \ll \)-symbol depends at most on \( m \).

Gallagher [6] was the first to establish a result like Theorem 5.1, in the classical case \( m = 1 \) for the Riemann zeta-function. He proved that, under the Riemann Hypothesis for the classical zeta-function,

\[
\psi(x) := \sum_{n \leq x} \Lambda(n) = x + O(x^{1/2} (\log \log x)^2)
\]

for \( x \geq 2 \), except on a set of finite logarithmic measure, and hence made improvement on the classical estimate error term \( O(x^{1/2} \log^2 x) \) of von Koch [27]. In the same paper, Gallagher [6] also gave short proofs of Cramér’s conditional estimate (see [2] [3])

\[
\int_2^X (\psi(x) - x)^2 \frac{dx}{x} \ll X.
\]

Gallagher’s proofs of the above results make crucial use of his lemma in [7], which is now named after him.

Our Theorems 5.1-5.2 generalize the above classical results to the prime counting function \( \psi(s,\pi) \) attached to irreducible unitary cuspidal representations \( \pi \) of \( \text{GL}_m(\mathbb{A}_Q) \) with \( m \geq 2 \). Our proofs combine the approach of Gallagher with recent results of Liu and Ye ([32], [33]) on the distribution of zeros of Rankin-Selberg automorphic \( L \)-functions.

The above Theorem 5.2 states that, under GRH, \(|\psi(x,\pi)|\) is of size \( x^{1/2} \log Q_\pi \) on average. This can be compared with the next theorem, which gives the unconditional Omega result that \(|\psi(x,\pi)|\) should not be of order lower than \( x^{1/2-\varepsilon} \).

**Theorem 5.3.** Let \( m \geq 2 \) be an integer and let \( \pi \) be an irreducible unitary cuspidal representation of \( \text{GL}_m(\mathbb{A}_Q) \), and \( \varepsilon > 0 \) arbitrary. Unconditionally,

\[
\psi(x,\pi) = \Omega(x^{1/2-\varepsilon}),
\]
1.6 Selberg’s normal density theorem for automorphic $L$-functions

where the implied constant depends at most on $m$ and $\varepsilon$. More precisely, there exists an increasing sequence $\{x_n\}_{n=1}^{\infty}$ tending to infinity such that

$$\lim_{n \to \infty} \frac{|\psi(x_n, \pi)|}{x_n^{1/2 - \varepsilon}} > 0. \quad (1.29)$$

Note that the sequence $\{x_n\}_{n=1}^{\infty}$ and the limit in (1.29) may depend on $\pi$. This result generalizes that for the Riemann zeta-function. It is possible to get better Omega results like those in Chapter V of Ingham [10]. We remark that, unlike the classical case, in Theorems 5.1-5.3 we do not have the main term $x$. This is because $L(s, \pi)$ is entire when $m \geq 2$, while $\zeta(s)$ has a simple pole at $s = 1$ with residue 1.

Connecting with Linnik’s problem for automorphic $L$-functions considered in Chapter 4, we consider a Linnik-type problem in the sequence $\{a_{\pi}(n)\Lambda(n)\}_{n=1}^{\infty}$, defined as in (1.25) and (1.26). These are the coefficients in the Dirichlet series expansion for $-L'(s, \pi)$ with $\sigma > 1$. As a consequence of Theorem 4.15, we establish in this direction the following theorem.

**Theorem 5.12.** Let $m \geq 2$ be an integer and let $\pi$ be an irreducible unitary cuspidal representation for $GL_m(\mathbb{A}_Q)$. If all $a_{\pi}(n)\Lambda(n)$ are real, then $\{a_{\pi}(n)\Lambda(n)\}_{n=1}^{\infty}$ changes sign at some $n$ satisfying

$$n \ll Q_m^{m/2 + \varepsilon}. \quad (1.30)$$

The constant implied in (1.30) depends only on $m$ and $\varepsilon$. In particular, the result is true for any self-contragredient representation $\pi$.

These are the materials in Chapter 5.

1.6 Selberg’s normal density theorem for automorphic $L$-functions

Under the Riemann Hypothesis for the Riemann zeta-function, i.e. in the case of $m = 1$, Selberg [47] proved that

$$\int_1^X \{\psi(x + h(x)) - \psi(x) - h(x)\}^2 dx = o(h(X)^2 X) \quad (1.31)$$

for any increasing functions $h(x) \leq x$ with

$$\frac{h(x)}{\log^2 x} \to \infty,$$
where as usual,
\[ \psi(x) = \sum_{n \leq x} \Lambda(n). \]

In Chapter 6, we prove an analogue of this in the case of automorphic \( L \)-functions.

**Theorem 6.1.** Let \( m \geq 2 \) be an integer and let \( \pi \) be an irreducible unitary cuspidal representation of \( \text{GL}_m(\mathbb{A}_\mathbb{Q}) \). Assume GRH for \( L(s, \pi) \). We have
\[
\int_1^X |\psi(x + h(x), \pi) - \psi(x, \pi)|^2 dx = o(h(X)^2 X),
\]
for any increasing functions \( h(x) \leq x \) satisfying
\[
\frac{h(x)}{\log^2(Q\pi x)} \to \infty.
\]

Our Theorem 6.1 generalizes Selberg’s result to cases when \( m \geq 2 \). It also improves an earlier result of the author [44] that \( (1.32) \) holds for \( h(x) \leq x \) satisfying
\[
\frac{h(x)}{x^\theta \log^2(Q\pi x)} \to \infty,
\]
where \( \theta \) is the bound towards the GRC as explained in Lemma 4.8. The main new idea is a delicate application of Kowalski-Iwaniec’s mean value estimate (cf. Lemma 5.9). We also need an explicit formula established in Chapter 5 in a more precise form.

Unconditionally, Theorem 6.1 would hold for \( h(x) = x^\beta \) with some constant \( 0 < \beta < 1 \). The exact value of \( \beta \) depends on two main ingredients: a satisfactory zero-density estimate for the \( L \)-function \( L(s, \pi) \), and a zero-free region for \( L(s, \pi) \) of Littlewood’s or Vinogradov’s type.
Chapter 2

Classical modular forms and a Linnik-type problem

In this chapter, we will review the concept and some basic properties of classical modular forms. These properties will be used in later chapters of the thesis. The reader is referred to Iwaniec [11] for a detailed treatment of these materials.

2.1 Classical modular forms

Let
\[ SL_2(\mathbb{Z}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, \; ad - bc = \pm 1 \right\} \]
be the modular group. We restrict our attention to the Hecke congruence subgroup of level \( N \), which is
\[ \Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : N|c \right\}, \]
where \( N \) is a positive integer. In this convention, \( \Gamma_0(1) = SL_2(\mathbb{Z}) \), and the index of \( \Gamma_0(N) \) in the modular group is
\[ \nu(N) = [\Gamma_0(1) : \Gamma_0(N)] = N \prod_{p|N} \left( 1 + \frac{1}{p} \right). \]

The group \( \Gamma_0(N) \) acts on the upper half-plane
\[ \mathbb{H} = \{ z : z = x + iy, \; y > 0 \} \]
by
\[ \gamma z = \frac{az + b}{cz + d}, \quad \gamma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \Gamma_0(N). \]

Let \( k \) be a positive integer. The space of cusp forms of weight \( k \) and level \( N \) is denoted by \( S_k(\Gamma_0(N)) \); it is a finite-dimensional Hilbert space with respect to the inner product
\[ \langle f, g \rangle = \int_{\Gamma \backslash \mathbb{H}} f(z) \overline{g(z)} y^k \frac{dx\,dy}{y^2}, \]
where\[ d\mu := \frac{dx\,dy}{y^2} \]
is the invariant measure on \( \mathbb{H} \).

The Hecke operators \( \{T_n\}_{n=1}^{\infty} \) are defined by
\[ (T_n f)(z) = \frac{1}{\sqrt{n}} \sum_{ad=n} (\frac{a}{d})^{k/2} \sum_{b(\mod d)} f\left(\frac{az + b}{d}\right). \tag{2.1} \]

It follows that \( T_m \) and \( T_n \) commute, and for every \( n \), \( T_n \) is also self-adjoint on \( S_k(\Gamma_0(N)) \), i.e.
\[ \langle T_n f, g \rangle = \langle f, T_n g \rangle, \quad (n, N) = 1. \]

Let \( \mathcal{F} = \{f\} \) be an orthonormal basis of \( S_k(\Gamma_0(N)) \). We can assume that every \( f \in \mathcal{F} \) is an eigenfunction for all Hecke operators \( T_n \) with \( (n, N) = 1 \); i.e. there exist complex numbers \( \lambda_f(n) \), such that
\[ T_n f = \lambda_f(n) f, \quad (n, N) = 1. \tag{2.2} \]

The eigenvalues \( \lambda_f(n) \) are related to the Fourier coefficients of \( f(z) \) in such a way that the Fourier series expansion of \( f(z) \) now takes the form
\[ f(z) = \sum_{n=1}^{\infty} a_f(n)n^{(k-1)/2}e(nz), \tag{2.3} \]
with
\[ a_f(n) = a_f(1)\lambda_f(n), \quad (n, N) = 1. \tag{2.4} \]

Note that if \( a_f(1) = 0 \), then all \( a_f(n) = 0 \) for \( (n, N) = 1 \). Here we have used the standard notation \( e(t) := e^{2\pi it} \quad (t \in \mathbb{R}) \).
2.2 Classical automorphic \(L\)-functions

Lemma 2.1. Let \(f\) be an eigenfunction for all Hecke operators \(T_n\) with \((n,N) = 1\), and \(\lambda_f(n)\) be as in (2.2). Then

(i) The Hecke eigenvalues \(\{\lambda_f(n)\}_{n=1}^{\infty}\) are real;

(ii) The Hecke eigenvalues are multiplicative in the following sense:

\[
\lambda_f(m)\lambda_f(n) = \sum_{d|(m,n)} \lambda_f \left( \frac{mn}{d^2} \right), \quad (n,N) = 1.
\]

In particular

\[
\lambda_f(p)^2 = \lambda_f(p^2) + 1, \quad (p,N) = 1. \tag{2.5}
\]

Unfortunately, we cannot deduce from (2.4) that \(a_f(n) \neq 0\) because the condition does not allow us to control all the coefficients in (2.3). However, for new forms, the following result is true.

Lemma 2.2. If \(f\) is a new form, then (2.2) holds for all \(n\). The first coefficient in the Fourier expansion (2.3) does not vanish, so one can normalize \(f\) by setting \(a_f(1) = 1\). In this case, \(a_f(n) = \lambda_f(n)\) for all \(n\), and hence the Fourier expansion of \(f\) takes the form

\[
f(z) = \sum_{n=1}^{\infty} \lambda_f(n)n^{(k-1)/2}e(nz). \tag{2.6}
\]

Lemmas 2.1 and 2.2 will be used several times later.

2.2 Classical automorphic \(L\)-functions

We begin with a cusp form which has Fourier expansion as in (2.3) and (2.4). Define, for \(\sigma > 1\),

\[
L(s,f) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s}. \tag{2.7}
\]

The so-called complete \(L\)-function is defined as

\[
\Phi(s,f) = \pi^{-s} \Gamma \left( \frac{s + (k-1)/2}{2} \right) \Gamma \left( \frac{s + (k+1)/2}{2} \right) L(s,f). \tag{2.8}
\]
This complete $L$-function satisfies the functional equation

$$
\Phi(s, f) = \varepsilon_f N^{1/2-s} \Phi(1-s,\bar{f}),
$$

where $\varepsilon_f$ is a complex number of modulus 1. For any new form $f$, we have the following Euler product for $\Phi(s, f)$.

**Lemma 2.3.** If $f$ is a new form, then the functional equation takes the form

$$
\Phi(s, f) = \varepsilon_f N^{1/2-s} \Phi(1-s,\bar{f}), \quad s \in \mathbb{C}.
$$

For $\sigma > 1$, the function $L(s, f)$ admits the Euler product

$$
L(s, f) = \prod_p \left( 1 - \frac{\lambda_f(p)}{p^s} + \frac{\chi_0^0(p)}{p^{2s}} \right)^{-1},
$$

where $\chi_0^0$ is the principal character modulo $N$.

By Lemmas 2.1 and 2.2, all eigenvalues $\lambda_f(n)$ of a new form $f$ are real. This explains why on the right-hand side of (2.10) we write $\Phi(1-s, f)$ instead of $\Phi(1-s, \bar{f})$.

We may further factor the Hecke polynomial in (2.11) into

$$
1 - \frac{\lambda_f(p)}{p^s} + \frac{\chi_0^0(p)}{p^{2s}} = \left( 1 - \frac{\alpha_f(p)}{p^s} \right) \left( 1 - \frac{\beta_f(p)}{p^s} \right),
$$

where

$$
\begin{cases}
\alpha_f(p) + \beta_f(p) = \lambda_f(p), \\
\alpha_f(p)\beta_f(p) = \chi_0^0(p).
\end{cases}
$$

Recall that the Ramanujan conjecture asserts that

$$
|\alpha_f(p)| = |\beta_f(p)| = 1, \quad (p, N) = 1,
$$

which has been proved by Deligne [5]. For $\sigma > 1$, the symmetric square $L$-function is defined by

$$
L(s, \text{sym}^2 f) = L(2s, \chi_0^0) \sum_{n=1}^{\infty} \frac{\lambda_f(n^2)}{n^s}.
$$

The Euler product of $L(s, \text{sym}^2 f)$ takes the form that, for $\sigma > 1$,

$$
L(s, \text{sym}^2 f) := \prod_p \left( 1 - \frac{\alpha_f(p)\alpha_f(p)}{p^s} \right)^{-1} \left( 1 - \frac{\alpha_f(p)\beta_f(p)}{p^s} \right)^{-1}
\times \left( 1 - \frac{\beta_f(p)\beta_f(p)}{p^s} \right)^{-1}.
$$
2.3 Infinite sign changes of Fourier coefficients

Now suppose \( g \) is a new form of level \( N' \) and weight \( k' \). The Rankin-Selberg \( L \)-function of \( f \) and \( g \) is defined as

\[
L(s, f \otimes g) = L(2s, \chi_N \chi_{N'}) \sum_{n=1}^{\infty} \frac{\lambda_f(n)\lambda_g(n)}{n^s},
\]

if \([N,N']\) is square-free. For \( \sigma > 1 \), the Euler product of \( L(s, f \otimes g) \) takes the form

\[
L(s, f \otimes g) = \prod_{p} \left(1 - \frac{\alpha_f(p)\alpha_g(p)}{p^s}\right)^{-1} \left(1 - \frac{\alpha_f(p)\beta_g(p)}{p^s}\right)^{-1} \times \left(1 - \frac{\beta_f(p)\alpha_g(p)}{p^s}\right)^{-1} \left(1 - \frac{\beta_f(p)\beta_g(p)}{p^s}\right)^{-1}.
\]

These properties are important in our later argument.

2.3 Infinite sign changes of Fourier coefficients

The result in the following seems well known, but we are not able to give a precise reference where it appeared first. As a substitute, we refer to the paper [23] for an extension to quite general subgroups of \( SL_2(\mathbb{R}) \) and a discussion of related topics.

**Proposition 2.4.** (Knopp-Kohlen-Pribitkin [23]). Let \( f \) be a non-zero cusp form of even integral weight \( k \) on \( \Gamma_0(N) \), and suppose that its Fourier coefficients \( a_f(n) \) are real for all \( n \geq 1 \). Then the sequence

\[
\{a_f(n)\}_{n=1}^{\infty}
\]

has infinitely many sign changes, i.e. there are infinitely many \( n \) such that \( a_f(n) > 0 \), and there are infinitely many \( n \) such that \( a_f(n) < 0 \).

It follows from Lemmas 2.1 and 2.2 that, for any new form \( f \), all its eigenvalues \( \lambda_f(n) \) are real. We therefore arrive at the following corollary.

**Corollary 2.5.** Let \( f \) be a normalized Hecke eigenform that is a new form of level \( N \) of even integral weight \( k \) on \( \Gamma_0(N) \). Then the sequence

\[
\{\lambda_f(n)\}_{n=1}^{\infty}
\]

has infinitely many sign changes, i.e. there are infinitely many \( n \) such that \( \lambda_f(n) > 0 \), and there are infinitely many \( n \) such that \( \lambda_f(n) < 0 \).
A more precise question to ask is: how long is the sequence of Hecke eigenvalues that keep the same sign. To measure the length of the sequences, define

\[ N_f^+(x) = \sum_{\substack{n \leq x, (n,N)=1 \ \lambda_f(n)>0}} 1, \]  

and define \( N_f^-(x) \) similarly by replacing the condition \( \lambda_f(n) > 0 \) under the summation by \( \lambda_f(n) < 0 \). Kohlen, Lau, and Shparlinski [25] prove that, if \( f \) is a new form, then

\[ N_f^\pm(x) \gg f \frac{x}{\log^{1/\ell} x}, \]  

where the implied constant depends on the form \( f \). Recently, Wu [55] reduces the 17 in the logarithmic exponent to \( 1 - 1/\sqrt{3} \), as an simple application of his estimates on power sums of Hecke eigenvalues. Still more recently, Lau and Wu [29] manage to completely get rid of the logarithmic factor in (2.18).

**Proposition 2.6.** (Lau-Wu [29]). Let \( f \) be a normalized Hecke eigenform that is a new form of level \( N \) of even integral weight \( k \) on \( \Gamma_0(N) \), and let \( N_f^\pm(x) \) be as in (2.17). Then

\[ N_f^\pm(x) \gg f \frac{x}{\log^{1/\ell} x}, \]  

where the implied constant depends on the form \( f \).

Obviously, this is the best possible result concerning the order of magnitude of \( x \). The proof applies, among other things, the \( B \)-free number method. It is also remarked in [29] that their method works well in other cases, such as forms of half-integral weight.

### 2.4 A Linnik-type problem: the first sign change of Fourier coefficients

#### 2.4.1 Linnik’s original problem

In view of Dirichlet’s theorem that there are infinitely many primes in the arithmetic progression \( n \equiv l \pmod{q} \) with \( (q,l) = 1 \), it is a natural question how big the least prime is, denoted by \( P(q,l) \), in this arithmetic progression. Linnik [30] [31] proved that there is an absolute constant \( \ell > 0 \) such that

\[ P(q,l) \ll q^\ell, \]
and this constant $\ell$ was named after him. Since then, a number of authors have established numerical values for Linnik’s constant $\ell$, while the best result known is $\ell = 5.5$ by Heath-Brown [9]. We remark that these results depend on, among other things, numerical estimates concerning zero-free regions and the Deuring-Heilbronn phenomenon of Dirichlet $L$-functions. Under GRH for Dirichlet $L$-functions, the above bounds can be improved to

$$P(q, l) \ll \varphi^2(q)(\log q)^2. \quad (2.20)$$

The conjectured bound is

$$P(q, l) \ll_\varepsilon q^{1+\varepsilon} \quad (2.21)$$

for arbitrary $\varepsilon > 0$, and this is a consequence of GRH and another conjecture concerning the universality of the distribution of nontrivial zeros for Dirichlet $L$-functions, as shown in Liu and Ye [34]. The conjectured bound (2.21) is the best possible save the $\varepsilon$ in the exponent. In fact, a trivial lower bound for $P(q, l)$ is

$$\max_l P(q, l) \geq \{1 + o(1)\} \varphi(q) \log q. \quad (2.22)$$

The reader is referred to [9] for a survey of results concerning Linnik’s problem.

### 2.4.2 A Linnik-type problem and a classical result of Siegel

According to the results in the previous section, a reasonable question to ask is: *Is it possible to obtain a bound on the first sign change, say, in terms of $k$ and $N$?* In general, this seems to be a difficult question. For a survey of results in this direction, see Kohnen [24].

If $f \neq 0$, then, by the valence formula for modular forms, the orders of zeros of $f$ on the compactified Riemann surface

$$X_0(N) = \Gamma_0(N) \setminus \mathbb{H} \cup \mathbb{P}^1(Q)$$

sum up to $k \frac{12}{12}[\Gamma_0(1) : \Gamma_0(N)]$. Hence there exists a number $n$ in the range

$$1 \leq n \leq \frac{k}{12}[\Gamma_0(1) : \Gamma_0(N)]$$

such that $a_f(n) \neq 0$. Now if we are optimistic, then we can expect a sign change in the range

$$1 \leq n \leq \frac{k}{12}[\Gamma_0(1) : \Gamma_0(N)] + 1.$$
In a very special case, this indeed follows from work of Siegel [53]. To formulate the result, suppose \( k \geq 4 \) and denote by \( d_k \) the dimension of the space \( M_k(\Gamma_0(1)) \) of modular forms of weight \( k \) on \( \Gamma_0(1) \). Recall that \( d_k \) satisfies the formula

\[
d_k = \begin{cases} 
\lfloor k/12 \rfloor & k \equiv 2 \pmod{12}, \\
\lfloor k/12 \rfloor + 1 & \text{otherwise}.
\end{cases}
\]

Then Siegel showed that, for each \( f \in M_k(\Gamma_0(1)) \), there are explicitly computable rational numbers \( \{c_n\}_{n=0}^{d_k} \) depending on \( k \), such that

\[
\sum_{n=0}^{d_k} c_n a_f(n) = 0.
\]

Siegel’s explicit expression for \( c_n \) implies that, for \( k \equiv 2 \pmod{4} \), all the \( c_n \) are strictly positive. Since a cusp form of weight \( k \) on \( \Gamma_0(1) \) is determined by its Fourier coefficients \( \{a_f(n)\}_{n=0}^{d_k-1} \), we conclude immediately that, under the assumption \( k \equiv 2 \pmod{4} \), there must be a sign change of \( a_f(n) \) in the range \( 1 \leq n \leq d_k \). Thus, using the formula for \( d_k \) above, one sees that the above optimistic expectation is justified in this special case.

Unfortunately, when \( k \equiv 0 \pmod{4} \) or if \( N > 1 \), Siegel’s argument does not work any longer, and therefore we need other ideas.

### 2.4.3 Recent developments and comments

In this subsection, we will focus on the case that \( f \) is a normalized Hecke eigenform that is a new form of level \( N \). Recall that normalized means that \( a_f(n) = 1 \), and new form means that \( N \) is the exact level of \( f \), and in this case the Fourier coefficients are equal to the Hecke eigenvalues.

In the case \( N = 1 \), sign changes of the \( \lambda_f(p) \) where \( p \) goes over primes have been considered by Ram Murty [42]. Kohnen and Sengupta [26] have shown that the first sign of \( \lambda_f(n) \) happens for some \( n \) with

\[
n \ll kN \exp \left( c \sqrt{\frac{\log N}{\log \log(3N)}} \right) (\log k)^{27}, \quad (n, N) = 1,
\]

where \( c > 2 \) is a constant and the \( \ll \)-constant is absolute. Note that it is natural to assume that \( (n, N) = 1 \), since the eigenvalues \( \lambda_f(p) \) with \( p|N \) are explicitly known by the Atkin-Lehner theory.
Recently, Iwaniec, Kohnen, and Sengupta [13] established the following result.

**Proposition 2.7.** (Iwaniec-Kohnen-Sengupta [13]). Let $f$ be a normalized Hecke eigenform of integral weight $k$ and level $N$ that is a new form. Then there is some $n$ satisfying
\[ n \ll (k^2N)^{29/60}, \quad (n, N) = 1, \] (2.24)
such that $\lambda_f(n) < 0$.

The convexity bound
\[ L(1/2 + it, f) \ll (k^2N)^{1/4 + \varepsilon} \] (2.25)
gives, instead of (2.24), the weaker bound
\[ n \ll (k^2N)^{1/2 + \varepsilon}, \quad (n, N) = 1. \] (2.26)

The uniform subconvexity bound
\[ L(1/2 + it, f) \ll (k^2N)^{29/120} \] (2.27)
would prove Proposition 2.7, but no result of this quality is known. The best known uniform subconvexity bound like (2.27) is due to Michel and Venkatesh [38], which states that
\[ L(1/2 + it, f) \ll (k^2N)^{1/4 - \delta}, \] (2.28)
where $\delta$ is some positive constant not specified. Iwaniec, Kohnen, and Sengupta [13] manage to establish (2.24) without appealing to (2.27); the key steps in [13] are the following:

- The identity (2.5), i.e.
\[ \lambda_f(p^2) = \lambda_f(p) + 1, \quad (p, N) = 1; \]

- The Ramanujan conjecture (2.12) proved by Deligne, i.e.
\[ |\lambda_f(p)| \leq 2, \quad (p, N) = 1; \]

- Sieve methods.

Of course, the proof of Proposition 2.7 is much more involved. In particular, to carry out the sieves, one still needs the identity $\lambda_f(p^2) = \lambda_f(p) + 1$ several times, and needs the fact $|\lambda_f(p)| \leq 2$ in a more crucial way. We will not get into these details, but just would like to point out that, the approach does not work for those automorphic forms $f$, whose $\lambda_f(n)$ do not satisfy the above two properties.
Classical modular forms and a Linnik-type problem
Chapter 3

A Linnik-type problem for Maass forms

3.1 The spectral theory of Maass forms

In this section, we introduce the notion and basic facts from the theory of Maass forms of weight \( k = 0 \) in the context of the Hecke congruence subgroup \( \Gamma_0(N) \). Philosophically, there is no essential difference from the theory of classical modular forms, except for the existence of a continuous spectrum in the space of Maass forms. A good monograph on this topic is Iwaniec [12]. But one has to admit that some mature methods, which are quite useful in the case of holomorphic forms, do not work in the current situation. Linnik-type problem for Maass forms is such an example, as will be explained in this chapter.

3.1.1 The spectral decomposition: preliminary

A function \( f : \mathbb{H} \to \mathbb{C} \) is said to be automorphic with respect to \( \Gamma_0(N) \) if

\[
f(\gamma z) = f(z), \quad \text{for all } \gamma \in \Gamma_0(N).
\]

Therefore, \( f \) lives on \( \Gamma_0(N) \backslash \mathbb{H} \). We denote the space of such functions by \( \mathcal{A}(\Gamma_0(N) \backslash \mathbb{H}) \). Our objective is to extend automorphic functions into automorphic forms subject to suitable growth condition. The main results hold in the Hilbert space

\[
\mathcal{L}(\Gamma_0(N) \backslash \mathbb{H}) = \{ f \in \mathcal{A}(\Gamma_0(N) \backslash \mathbb{H}) : \|f\| < \infty \}
\]
with respect to the inner product
\[ \langle f, g \rangle = \int_{\Gamma_0(N) \setminus \mathbb{H}} f(z) \overline{g}(z) \frac{dx \, dy}{y^2}. \]

Recall that the standard Laplace operator on the complex plane \( \mathbb{C} \) is defined by
\[ \Delta^e = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}; \]
but on the upper half-plane \( \mathbb{H} \), we should use the non-Euclidean Laplace operator
\[ \Delta = -y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right). \]

This non-Euclidean Laplace operator acts in the dense subspace of smooth functions in \( \mathcal{L}(\Gamma_0(N) \setminus \mathbb{H}) \) such that \( f \) and \( \Delta f \) are both bounded, i.e.
\[ \mathcal{D}(\Gamma_0(N) \setminus \mathbb{H}) = \{ f \in \mathcal{A}(\Gamma_0(N) \setminus \mathbb{H}) : f, \Delta f \text{ smooth and bounded} \}. \]

It is proved that \( \mathcal{D}(\Gamma_0(N) \setminus \mathbb{H}) \) is dense in \( \mathcal{L}(\Gamma_0(N) \setminus \mathbb{H}) \), and \( \Delta \) is positive semi-definite and symmetric on \( \mathcal{D}(\Gamma_0(N) \setminus \mathbb{H}) \). By Friedrich’s theorem in functional analysis, \( \Delta \) has a unique self-adjoint extension to \( \mathcal{L}(\Gamma_0(N) \setminus \mathbb{H}) \).

**Lemma 3.1.** (i) Let \( \Lambda = s(1-s) \) be the eigenvalue of an eigenfunction \( f \in \mathcal{D}(\Gamma_0(N) \setminus \mathbb{H}) \). Then \( \Lambda \) is real and non-negative, i.e. either \( s = 1/2 + it \) with \( t \in \mathbb{R} \), or \( 0 < s < 1 \).

(ii) On \( \mathcal{L}(\Gamma_0(N) \setminus \mathbb{H}) \), the non-Euclidean Laplace operator \( \Delta \) is positive semi-definite and self-adjoint.

With the above self-adjoint extension, one can show that the non-Euclidean Laplace operator \( \Delta \) has the spectral decomposition
\[ \mathcal{L}(\Gamma_0(N) \setminus \mathbb{H}) = \mathbb{C} \oplus \mathcal{C}(\Gamma_0(N) \setminus \mathbb{H}) \oplus \mathcal{E}(\Gamma_0(N) \setminus \mathbb{H}). \]

Here \( \mathbb{C} \) is the space of constant functions, \( \mathcal{C}(\Gamma_0(N) \setminus \mathbb{H}) \) the space of cusp forms, and \( \mathcal{E}(\Gamma_0(N) \setminus \mathbb{H}) \) the space spanned by incomplete Eisenstein series.

### 3.1.2 The discrete spectrum

The structure of the space \( \mathcal{C}(\Gamma_0(N) \setminus \mathbb{H}) \), the space of cusp forms, is characterized by the following result.
3.1 The spectral theory of Maass forms

Lemma 3.2. The automorphic Laplacian $\Delta$ has a purely point spectrum on $C(\Gamma_0(N) \backslash \mathbb{H})$, i.e. the space $C(\Gamma_0(N) \backslash \mathbb{H})$ is spanned by cusp forms. The eigenvalues are

$$0 = \Lambda_0 < \Lambda_1 \leq \Lambda_2 \leq \ldots \to \infty,$$

and the eigenspaces have finite dimension. For any complete orthonormal system of cusp forms $\{u_j\}_{j=1}^\infty$, every $f \in C(\Gamma_0(N) \backslash \mathbb{H})$ has the expansion

$$f(z) = \sum_{j=1}^\infty \langle f, u_j \rangle u_j(z),$$

converging in the norm topology. If $f \in C(\Gamma_0(N) \backslash \mathbb{H}) \cap D(\Gamma_0(N) \backslash \mathbb{H})$, then the series converges absolutely and uniformly on compacta.

Let

$$\mathcal{U} = \{u_j\}_{j=1}^\infty$$

be an orthonormal basis of $C(\Gamma_0(N) \backslash \mathbb{H})$ which are eigenfunctions of $\Delta$, say

$$\Delta u_j = \Lambda_j u_j,$$

with

$$\Lambda_j = s_j(1 - s_j) = \frac{1}{4} + \nu_j^2, \quad s_j = \frac{1}{2} + i\nu_j. \quad (3.1)$$

Note that here the $\nu_j$ in (3.1) is not necessarily real. Any $u_j$ has the Fourier expansion

$$u_j(z) = \sum_{n \neq 0} \rho_j(n) W_s(nz), \quad (3.2)$$

where $W_s(z)$ is the Whittaker function given by

$$W_s(z) = 2|y|^{1/2} K_{s-1/2}(2\pi |y|) \cos(x),$$

and $K_s(y)$ is the $K$-Bessel function. Note that

$$W_s(z) \sim e(z), \quad y \to \infty.$$ 

The automorphic forms $u_j(z)$ are called Maass cusp forms. Sometimes, we write $f$ for Maass cusp forms with Laplace eigenvalue

$$\Lambda = s(1 - s) = \frac{1}{4} + \nu_f^2,$$

and in this case, the Fourier expansion of $f$ takes the form

$$f(z) = \sum_{n \neq 0} \rho_f(n) W_s(nz). \quad (3.3)$$

Compare this with (3.2).
3.1.3 Antiholomorphic involution

Let \( \iota : \mathbb{H} \rightarrow \mathbb{H} \) be the antiholomorphic involution

\[ \iota(x + iy) = -x + iy. \]

If \( f \) is an eigenfunction of \( \Delta \), and

\[ f(z) = \sum_{n \neq 0} \rho_f(n) W_s(nz), \]  

(3.4)

then \( f \circ \iota \) is an eigenfunction with the same eigenvalue. Since \( \iota^2 = 1 \), its eigenvalues are \( \pm 1 \). We may therefore diagonalize the Maass cusp forms with respect to \( \iota \). If \( f \circ \iota = f \), we call \( f \) even. In this case

\[ \rho_f(n) = \rho_f(-n). \]

If \( f \circ \iota = -f \), then we call \( f \) odd, and we have

\[ \rho_f(n) = -\rho_f(-n). \]

3.1.4 The continuous spectrum

On the other hand, in the space \( \mathcal{E}(\Gamma_0(N) \backslash \mathbb{H}) \), the spectrum turns out to be continuous. The spectral resolution of \( \Delta \) in \( \mathcal{E}(\Gamma_0(N) \backslash \mathbb{H}) \) follows from the analytic continuation of the Eisenstein series. The eigenpacket of the continuous spectrum consists of the Eisenstein series \( E_a(z,s) \) on the line \( \sigma = 1/2 \) (analytically continued). These are defined for every cusp \( a \) by

\[ E_a(z,s) = \sum_{\gamma \in \Gamma_a \backslash \Gamma_0(N)} (3\sigma^{-1}_a \gamma z)^s \]

if \( \sigma > 1 \), and by analytic continuation for all \( s \in \mathbb{C} \). Here \( \Gamma_a \) is the stability group of \( a \) and \( a \in SL_2(\mathbb{R}) \) is such that

\[ \sigma_a \infty = a, \quad \sigma_a^{-1} \Gamma_a = \Gamma_{\infty}. \]

The scaling matrix \( \sigma_a \) of \( a \) is only determined up to a translation from the right; however the Eisenstein series does not depend on the choice of \( \sigma_a \), not even on the choice of a cusp in the equivalent class. The Fourier expansion of \( E_a(z,s) \) is similar to that of a cusp form; precisely,

\[ E_a(z,s) = \varphi_a y^s + \varphi_a(s) y^{1-s} + \sum_{n \neq 0} \varphi_a(n,s) W_s(nz), \]
where \( \varphi_a = 1 \) if \( a \sim \infty \), and \( \varphi_a = 0 \) otherwise.

**Lemma 3.3.** The space \( \mathcal{E}(\Gamma_0(N) \backslash \mathbb{H}) \) of incomplete Eisenstein series splits orthogonally into \( \Delta \)-invariant subspaces

\[
\mathcal{E}(\Gamma_0(N) \backslash \mathbb{H}) = \mathcal{R}(\Gamma_0(N) \backslash \mathbb{H}) \oplus_a \mathcal{E}_a(\Gamma_0(N) \backslash \mathbb{H}).
\]

The spectrum of \( \Delta \) in \( \mathcal{R}(\Gamma_0(N) \backslash \mathbb{H}) \) is discrete; it consists of a finite number of points \( \Lambda_j \) with

\[
\Lambda_j \in [0, 1/4).
\]

The spectrum of \( \Delta \) on \( \mathcal{E}_a(\Gamma_0(N) \backslash \mathbb{H}) \) is absolutely continuous; it covers the segment

\[
[1/4, +\infty)
\]

uniformly with multiplicity 1. Every \( f \in \mathcal{E}(\Gamma_0(N) \backslash \mathbb{H}) \) has the expansion

\[
f(z) = \sum_j \langle f, u_j \rangle u_j(z) + \sum_a \frac{1}{4\pi} \int_{-\infty}^{\infty} \langle f, E_a(\cdot, 1/2 + it) \rangle E_a(z, 1/2 + it) dt,
\]

which converges in the norm topology. If \( f \in \mathcal{E}(\Gamma_0(N) \backslash \mathbb{H}) \cap \mathcal{D}(\Gamma_0(N) \backslash \mathbb{H}) \), then the series converges pointwise absolutely and uniformly on compacta.

### 3.1.5 The spectral decomposition: conclusion

Combining Lemmas 3.2-3.3, one gets the spectral decomposition of the whole space \( \mathcal{L}(\Gamma_0(N) \backslash \mathbb{H}) \),

\[
f(z) = \sum_{j=0}^{\infty} \langle f, u_j \rangle u_j(z) + \sum_j \langle f, u_j \rangle u_j(z)
\]

\[
+ \sum_a \frac{1}{4\pi} \int_{-\infty}^{\infty} \langle f, E_a(\cdot, 1/2 + it) \rangle E_a(z, 1/2 + it) dt.
\]

This structure is one of the basics for later arguments.

### 3.2 Hecke theory for Maass forms

For \( n \geq 1 \), define

\[
(T_n f)(z) = \frac{1}{\sqrt{n}} \sum_{ad=n, b \mod d} f \left( \frac{az+b}{d} \right);
\]

(3.6)
nevertheless, only those $T_n$ with $(n,N) = 1$ are interesting. We first examine the action
of $T_n$ on a Maass cusp form $u_j$. For $u_j$ as in (3.1) and (3.2), write

$$u_j(z) = \sum_{m \neq 0} \rho_j(m) W_{s_j}(mz).$$

(3.7)

Then one computes that

$$(T_n u_j)(z) = \sum_{m \neq 0} t_n(m) W_{s_j}(mz),$$

with

$$t_n(m) = \sum_{d|(m,n)} \rho_j\left(\frac{mn}{d^2}\right).$$

It follows that

$$T_m T_n = \sum_{d|(m,n)} T_{mn/d^2},$$

so that in particular $T_m$ and $T_n$ commute. Moreover, the Hecke operators commute
with the non-Euclidean Laplace operator $\Delta$. For every $n$, $T_n$ is also self-adjoint on
$L(\Gamma_0(N) \setminus \mathbb{H})$, i.e.

$$\langle T_n f, g \rangle = \langle f, T_n g \rangle, \quad (n,N) = 1.$$  

Therefore, in the space $L(\Gamma_0(N) \setminus \mathbb{H})$ of cusp forms, an orthonormal basis \( \{u_j\}_{j=1}^\infty \)
can be chosen which consists of simultaneous eigenfunctions for all $T_n$, i.e.

$$T_n u_j(z) = \lambda_j(n) u_j(z), \quad j \geq 1, \quad (n,N) = 1,$$

(3.8)

where $\lambda_j(n)$ is the eigenvalue of $T_n$ for $u_j(z)$. Up to a constant, $\lambda_j(n)$ and the Fourier
coefficient $\rho_j(n)$ are equal. More precisely,

$$\lambda_j(n) \rho_j(1) = \rho_j(n), \quad \text{for all } (n,N) = 1, \quad j \geq 1.$$

(3.9)

Note that if $\rho_j(1) = 0$, then all $\rho_j(n) = 0$ for $(n,N) = 1$.

**Lemma 3.4.** Let $\mathcal{U} = \{u_j\}_{j=1}^\infty$ be an orthonormal basis consisting of simultaneous
eigenfunctions for all $T_n$. Fix a $j$, and let $\{\lambda_j(n)\}_{n=1}^\infty$ be the sequence of eigenvalues for
all $T_n$ as in (3.8).

(i) The Hecke eigenvalues $\{\lambda_j(n)\}_{n=1}^\infty$ are real;

(ii) The Hecke eigenvalues are multiplicative in the following sense:

$$\lambda_j(m) \lambda_j(n) = \sum_{d|(m,n)} \lambda_j\left(\frac{mn}{d^2}\right), \quad (n,N) = 1.$$
3.3 Automorphic $L$-functions for Maass forms

and

$$\lambda_j(m)\lambda_j(p) = \lambda_j(mp), \quad p|N.$$ \hfill (3.10)

It follows that

$$\lambda_j(p)^2 = \lambda_j(p^2) + 1, \quad (p, N) = 1.$$ \hfill (3.11)

As in the case of classical modular forms, we cannot deduce from (3.9) that $\rho_j(n) \not= 0$. Thus, we need to work with the new forms for the same reason.

**Lemma 3.5.** If $u_j$ is a new form, then (3.8) holds for all $n$. The first coefficient in the Fourier expansion (3.7) does not vanish, so one can normalize $u_j$ by setting $\rho_j(1) = 1$. In this case, $\rho_j(n) = \lambda_j(n)$ for all $n$, and hence

$$u_j(z) = \sum_{n \not= 0} \lambda_j(n)W_{h_j}(nz).$$ \hfill (3.12)

The Eisenstein series $E_\infty(z, 1/2 + it)$ is an eigenfunction of all the Hecke operators $T_n, (n, N) = 1$, with eigenvalues $\eta_t(n)$, i.e.

$$T_n E_\infty(z, 1/2 + it) = \eta_t(n)E_\infty(z, 1/2 + it), \quad (n, N) = 1, \quad t \in \mathbb{R},$$ \hfill (3.13)

where

$$\eta_t(n) = \sum_{ad=n} \left(\frac{a}{d}\right)^it.$$ \hfill (3.14)

We recall that $\eta_0(n)$ reduces to the classical divisor function $\tau(n)$.

### 3.3 Automorphic $L$-functions for Maass forms

To a Maass new form $f$ as in (3.3) with Laplace eigenvalue $1/4 + \nu^2$, we may attach an automorphic $L$-function

$$L(s, f) = \sum_{n=1}^{\infty} \frac{\lambda_j(n)}{n^s}, \quad \sigma > 1,$$ \hfill (3.15)
as in §2.2. For the Maass case, the complete $L$-function is defined as
\[
\Phi(s, f) = \pi^{-s} \Gamma \left( \frac{s + \epsilon - 1/2 + \nu}{2} \right) \Gamma \left( \frac{s + \epsilon + 1/2 - \nu}{2} \right) L(s, f),
\] (3.16)
where $\epsilon$ is the eigenvalue of $\iota$ introduced in §3.1.3. The complete $L$-function satisfies the functional equation
\[
\Phi(s, f) = \varepsilon_f N^{1/2-s} \Phi(1-s, \overline{f}),
\]
where $\varepsilon_f$ is a complex number of modulus 1. For any new form $f$, we have the following Euler product for $\Phi(s, f)$.

**Lemma 3.6.** If $f$ is a new form, then the functional equation takes the form
\[
\Phi(s, f) = \varepsilon_f N^{1/2-s} \Phi(1-s, f),
\] (3.17)
and, for $\sigma > 1$, the function $L(s, f)$ admits the Euler product
\[
L(s, f) = \prod_p \left( 1 - \frac{\lambda_f(p)}{p^s} + \frac{\chi^0_N(p)}{p^{2s}} \right)^{-1},
\] (3.18)
where $\chi^0_N$ is the principal character modulo $N$.

By Lemma 3.5, all eigenvalues $\lambda_f(n)$ of a new form $f$ are real. This explains why on the right-hand side of (3.17) we have $\Phi(1-s, f)$ instead of $\Phi(1-s, \overline{f})$. We may further factor the Hecke polynomial in (3.18) into
\[
1 - \frac{\lambda_f(p)}{p^s} + \frac{\chi^0_N(p)}{p^{2s}} = \left( 1 - \frac{\alpha_f(p)}{p^s} \right) \left( 1 - \frac{\beta_f(p)}{p^s} \right),
\]
where
\[
\alpha_f(p) + \beta_f(p) = \lambda_f(p), \quad \alpha_f(p)\beta_f(p) = \chi^0_N(p).
\]
The Generalized Ramanujan’s Conjecture (GRC in brief) in this case asserts that
\[
|\alpha_f(p)| = |\beta_f(p)| = 1, \quad (p, N) = 1;
\] (3.19)
this is still open, and the strongest bound towards the above conjecture is that of Kim and Sarnak [22]:
\[
|\alpha_f(p)| \leq p^{7/64}, \quad |\beta_f(p)| \leq p^{7/64}, \quad (p, N) = 1.
\] (3.20)
The symmetric square $L$-function is defined by
\[
L(s, \text{sym}^2 f) = L(2s, \chi^0_N) \sum_{n=1}^\infty \frac{\lambda_f(n^2)}{n^s}.
\] (3.21)
Now suppose $g$ is a Maass new form of level $N'$ and weight $k'$. The Rankin-Selberg $L$-function of $f$ and $g$ is defined as

$$L(s, f \otimes g) = L(2s, \chi_N^0 \chi_{N'}^0) \sum_{n=1}^{\infty} \frac{\lambda_f(n)\lambda_g(n)}{n^{s}},$$

(3.22)

if $[N, N']$ is square-free. For $\sigma > 1$, the Euler product of $L(s, f \otimes g)$ takes the form

$$L(s, f \otimes g) = \prod_p \left(1 - \frac{\alpha_f(p)\alpha_g(p)}{p^s}\right)^{-1} \left(1 - \frac{\alpha_f(p)\beta_g(p)}{p^s}\right)^{-1}$$

$$\times \left(1 - \frac{\beta_f(p)\alpha_g(p)}{p^s}\right)^{-1} \left(1 - \frac{\beta_f(p)\beta_g(p)}{p^s}\right)^{-1}. \quad (3.23)$$

The following subconvexity bound is a new result of Michel and Venkatesh [38].

**Lemma 3.7.** (Michel-Venkatesh [38]). Let $f$ be a non-zero Maass new form on $\Gamma_0(N)$ with Laplace eigenvalue $1/4 + \nu^2$. Then

$$L(1/2 + it, f) \ll \{(1 + |t + \nu|)^2N\}^{1/4-\delta/2}, \quad (3.24)$$

where $\delta$ is a positive absolute constant.

Subconvexity bounds for any one of the three aspects $\nu, N,$ or $t$ have been studied extensively in the literature, but uniform subconvexity bound is only known of the shape (3.24), where $\delta > 0$ is not specified. See Michel [37] for a survey in this direction, and Michel and Venkatesh [38] for recent developments.

### 3.4 Infinite sign changes of Fourier coefficients of Maass forms

It is pointed out at the end of [23] that similar results hold for Maass forms. Therefore, we have the following general result.

**Proposition 3.8.** (Knopp-Kohmen-Pribitkin [23]). Let $f$ be a non-zero Maass cusp form on $\Gamma_0(N)$ with Fourier expansion (3.4), and suppose that its Fourier coefficients $\rho_f(n)$ are real for all $n \geq 1$. Then the sequence

$$\{\rho_f(n)\}_{n=1}^{\infty}$$
has infinitely many sign changes, i.e. there are infinitely many $n$ such that $\rho_f(n) > 0$, and there are infinitely many $n$ such that $\rho_f(n) < 0$.

Since all eigenvalues $\lambda_f(n)$ of a new form $f$ are real, we arrive at the following corollary.

**Corollary 3.9.** Let $f$ be a normalized Maass eigenform that is a new form of level $N$ on $\Gamma_0(N)$. Then the sequence $\{\lambda_f(n)\}_{n=1}^{\infty}$ has infinitely many sign changes, i.e. there are infinitely many $n$ such that $\lambda_f(n) > 0$, and there are infinitely many $n$ such that $\lambda_f(n) < 0$.

For Maass forms, one may also ask the more precise question how long is the sequence of $\{\lambda_f(n)\}_{n=1}^{\infty}$ that keep the same sign. Like in (2.17), one may also introduce

$$N^+_f(x) = \sum_{\substack{n \leq x, (n,N)=1 \\lambda_f(n) > 0}} 1,$$

and define $N^-_f(x)$ similarly by replacing the condition $\lambda_f(n) > 0$ under the summation by $\lambda_f(n) < 0$. It is possible to establish results similar to those in Kohmen-Lau-Shparlinski [25], in Wu [55], or even in Lau-Wu [29]. But this will carry us too far, and we prefer to do it elsewhere at a later stage.

### 3.5 A Linnik-type problem for Maass forms

As in the case of holomorphic eigenforms, one may also formulate Linnik’s problem for Maass eigenforms. That is:

*For a Maass eigenform $f$, is it possible to obtain a bound on the first sign change of $\lambda_f(n)$, say, in terms of $N$ and the Laplace eigenvalue of $f$?*

There seems no result in this direction. Our result in the following produces one.

**Theorem 3.10.** Let $f$ be a normalized Maass new form of level $N$ and Laplace eigenvalue $1/4 + \nu^2$. Then there is some $n$ satisfying

$$n \ll ((3 + |\nu|)^2 N)^{1/2-\delta}, \quad (n,N) = 1,$$

such that $\lambda_f(n) < 0$, where $\delta$ is a positive absolute constant.

We need Perron’s formula in the following form, the proof of which can be found in standard text books on analytic number theory.
3.5 A Linnik-type problem for Maass forms

Let \( \sigma_a \) be the abscissa of absolute convergence for the Dirichlet series

\[
F(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s},
\]

where \( \{a_n\}_{n=1}^{\infty} \) is a sequence of complex numbers, and \( s = \sigma + it \in \mathbb{C} \) a complex variable. Perron’s formula expresses a partial sum of the coefficients \( a_n \) in terms of \( F(s) \).

**Lemma 3.11.** (Perron’s formula). Define

\[
A(x) = \max_{x/2 < n \leq 3x/2} |a_n|, \quad B(\sigma) = \sum_{n=1}^{\infty} \frac{|a_n|}{n^\sigma}
\]

for \( \sigma > \sigma_a \). Let \( \ell \) be a non-negative integer, \( x \geq 2 \), and \( \|x\| \) denote the distance between \( x \) and the nearest integer. Then, for \( b > \sigma_a \) and \( T \geq 2 \),

\[
\sum_{n \leq x} a_n \left( \log \frac{x}{n} \right)^\ell = \frac{1}{2\pi i} \int_{b-iT}^{b+iT} F(s) \frac{x^s}{s^{\ell+1}} ds + O \left( \frac{x A(x) \log^{\ell+1} x}{T} \right) + O \left( \frac{x^b B(b) \log^\ell x}{T} \right) + O \left\{ A(x) \min \left( 1, \frac{x}{T \|x\|} \right) \log^\ell x \right\}.
\]

In particular, for \( b > \sigma_a \),

\[
\sum_{n \leq x} a_n \left( \log \frac{x}{n} \right)^\ell = \frac{1}{2\pi i} \int_{b-i\infty}^{b+i\infty} F(s) \frac{x^s}{s^{\ell+1}} ds.
\]

Now we are in a position to establish Theorem 3.10.

**Proof of Theorem 3.10.** The idea is to consider the sum

\[
S(x) := \sum_{n \leq x \atop (n, N) = 1} \lambda_f(n) \log \frac{x}{n},
\]

assuming that

\[
\lambda_f(n) \geq 0 \quad \text{for } n \leq x \text{ and } (n, N) = 1.
\]
The desired result will follow from upper and lower bound estimates for $S(x)$.

To get an upper bound for $S(x)$, we apply Perron’s formula (3.30) with $\ell = 1$ to the Dirichlet series (3.15), getting

$$\sum_{n \leq x} \lambda_f(n) \log \frac{x}{n} = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} L(s, f) \frac{x^s}{s^2} ds.$$ 

Moving the contour to the vertical line $\sigma = 1/2$, where we apply the Michel-Venkatesh bound (3.24) for $L(s, f)$, we obtain

$$\sum_{n \leq x} \lambda_f(n) \log \frac{x}{n} = \frac{1}{2\pi i} \int_{1/2-i\infty}^{1/2+i\infty} L(s, f) \frac{x^s}{s^2} ds,$$

$$\ll \int_{-\infty}^{\infty} \{(1 + |t + \nu|)^2 N\}^{1/4-\delta/2} \frac{x^{1/2}}{|t|^2 + 1} dt,$$

$$\ll ((3 + |\nu|)^2 N)^{1/4-\delta/2} x^{1/2}.$$ 

To recover an estimate for $S(x)$ from the above result, we introduce the condition $(n, N) = 1$ by means of the Möbius inversion formula, which gives

$$S(x) = \sum_{d|N} \mu(d) \sum_{dm \leq x} \lambda_f(dm) \log \frac{x}{dm}.$$ 

Since $d|N$, we may apply the multiplicity property (3.10), which states that in the current situation

$$\lambda_f(dm) = \lambda_f(d) \lambda_f(m).$$ 

It follows that

$$S(x) = \sum_{d|N} \mu(d) \lambda_f(d) \sum_{dm \leq x} \lambda_f(m) \log \frac{x}{dm},$$

$$\ll \sum_{d|N} |\mu(d)| \lambda_f(d) \sum_{m \leq x/d} \lambda_f(m) \log \frac{x/d}{m},$$

$$\ll ((3 + |\nu|)^2 N)^{1/4-\delta/2} x^{1/2} \sum_{d|N} \frac{|\lambda_f(d)|}{d^{1/2}}.$$ 

The Kim-Sarnak bound states that $\lambda_f(d) \ll d^{7/64+\varepsilon}$, and therefore,

$$\sum_{d|N} \frac{|\lambda_f(d)|}{d^{1/2}} \ll \tau(N) \ll N^{\varepsilon},$$
here we note that the trivial bound $\lambda_f(d) \ll d^{1/2+\varepsilon}$ works equally well. Consequently, we conclude that

$$S(x) \ll ((3 + |\nu|)^2 N)^{1/4-\delta/2} x^{1/2}. \quad (3.32)$$

To get a lower bound for $S(x)$ under the assumption (3.31), we first get rid of the weight $\log(x/n)$ in a simple way:

$$S(x) \gg \sum_{n \leq x/2} \sum_{(n,N)=1} \lambda_f(n).$$

We now restrict the summation to $n = pq$, where $p$ and $q$ are primes satisfying

$$p \leq \sqrt{x/2}, \quad q \leq \sqrt{x/2}, \quad (p,N) = 1, \quad (q,N) = 1,$$

and use the formulae

$$\begin{cases}
\lambda_f(pq) = \lambda_f(p)\lambda_f(q) & \text{if } p \neq q, \ (p,N) = 1, \ (q,N) = 1, \\
\lambda_f(p^2) = \lambda_f(p)^2 - 1 & \text{if } p = q, \ (p,N) = 1.
\end{cases}$$

We get

$$S(x) \gg \sum_{p \leq \sqrt{x/2}} \sum_{q \leq \sqrt{x/2}} \sum_{(p,N)=1} \lambda_f(pq)$$

$$= \left\{ \sum_{p \leq \sqrt{x/2}} \lambda_f(p) \right\}^2 - \sum_{p \leq \sqrt{x/2}} 1.$$

Recalling the assumption (3.31), we have $\lambda_f(p^2) \geq 0$ for $p \leq \sqrt{x/2}$ and $(p,N) = 1$, and therefore,

$$\lambda_f(p)^2 = \lambda_f(p^2) + 1 \geq 1,$$

that is $\lambda_f(p) \geq 1$. It follows from this and the prime number theorem that

$$S(x) \gg \left\{ \sum_{p \leq \sqrt{x/2}} 1 \right\}^2 - \left\{ \sum_{p \leq \sqrt{x/2}} 1 \right\}$$

$$\gg \frac{x}{\log^2 x}. \quad (3.33)$$
Comparing (3.33) with (3.32), we get
\[ \frac{x}{\log^2 x} \ll S(x) \ll ((3 + |\nu|)^2 N)^{1/4-\delta/2} x^{1/2}, \]
that is
\[ x \ll ((3 + |\nu|)^2 N)^{1/2-\delta+\varepsilon}. \]
This proves the theorem. \qed
Chapter 4

A Linnik-type problem for automorphic $L$-functions

4.1 Automorphic $L$-functions: concepts and properties

To each irreducible unitary cuspidal representation $\pi = \otimes \pi_p$ of $GL_m(\mathbb{A}_\mathbb{Q})$, one can attach a global $L$-function $L(s, \pi)$, as in Godement and Jacquet [17], and Jacquet and Shalika [20]. For $\sigma = \Re s > 1$, $L(s, \pi)$ is defined by products of local factors

$$L(s, \pi) = \prod_{p < \infty} L_p(s, \pi_p),$$

where

$$L_p(s, \pi_p) = \prod_{j=1}^{m} \left(1 - \frac{o_{\pi}(p, j)}{p^s}\right)^{-1};$$

the complete $L$-function $\Phi(s, \pi)$ is defined by

$$\Phi(s, \pi) = L_{\infty}(s, \pi_{\infty})L(s, \pi),$$

where

$$L_{\infty}(s, \pi_{\infty}) = \prod_{j=1}^{m} \Gamma_{\mathbb{R}}(s + \mu_{\pi}(j))$$
A Linnik-type problem for automorphic $L$-functions

is the Archimedean local factor. Here

$$\Gamma_\mathbb{R}(s) = \pi^{-s/2} \Gamma\left(\frac{s}{2}\right),$$

and $\{\alpha_\pi(p, j)\}^m_{j=1}$ and $\{\mu_\pi(j)\}^m_{j=1}$ are complex numbers associated with $\pi_p$ and $\pi_\mathcal{I}$, respectively, according to the Langlands correspondence. The case $m = 1$ is classical; for $m \geq 2$, $\Phi(s, \pi)$ is entire and satisfies a functional equation.

We review briefly some properties of the automorphic $L$-functions $L(s, \pi)$ and $\Phi(s, \pi)$, which we will use for our proofs.

**Lemma 4.1.** (Jacquet-Shalika [20]). The Euler product for $L(s, \pi)$ in (4.1) converges absolutely for $\sigma > 1$.

Thus, in the half-plane $\sigma > 1$, we may write

$$L(s, \pi) = \sum_{n=1}^\infty \frac{\lambda_\pi(n)}{n^s},$$

(4.6)

where

$$\lambda_\pi(n) = \prod_{p^\nu \| n} \left\{ \sum_{\nu_1 + \cdots + \nu_m = \nu} \alpha_\pi(p, 1)^{\nu_1} \cdots \alpha_\pi(p, m)^{\nu_m} \right\}.$$

(4.7)

In particular,

$$\lambda_\pi(1) = 1, \quad \lambda_\pi(p) = \alpha_\pi(p, 1) + \cdots + \alpha_\pi(p, m).$$

(4.8)

**Lemma 4.2.** (Shahidi [49], [50], [51], and [52]). The complete $L$-function $\Phi(s, \pi)$ has an analytic continuation to the whole complex plane and satisfies the functional equation

$$\Phi(s, \pi) = \varepsilon(s, \pi) \Phi(1 - s, \tilde{\pi})$$

with

$$\varepsilon(s, \pi) = \varepsilon_\pi N_\pi^{1/2-s},$$

where $N_\pi \geq 1$ is an integer called the arithmetic conductor of $\pi$, $\varepsilon_\pi$ is the root number satisfying $|\varepsilon_\pi| = 1$, and $\tilde{\pi}$ is the representation contragredient to $\pi$.

If $\tilde{\pi}$ is the representation contragredient to $\pi$, then we have

$$\{\alpha_{\tilde{\pi}}(p, j)\}^m_{j=1} = \{\alpha_\pi(p, j)\}^m_{j=1}$$

(4.9)

and

$$\{\mu_{\tilde{\pi}}(j)\}^m_{j=1} = \{\mu_\pi(j)\}^m_{j=1}.$$
4.1 Automorphic $L$-functions: concepts and properties

It follows from these and (4.7) that

$$\lambda_{\tilde{\pi}}(n) = \lambda_{\pi}(n). \quad (4.11)$$

Therefore, if $\pi$ is self-contragredient, i.e. $\pi = \tilde{\pi}$, then (4.11) states that

$$\lambda_{\pi}(n) = \lambda_{\pi}(n), \quad (4.12)$$

which means that $\lambda_{\pi}(n)$ is real.

**Lemma 4.3.** (Godement-Jacquet [17], and Jacquet-Shalika [20]). The function $\Phi(s, \pi)$ is entire, and bounded in vertical strips with finite width.

**Lemma 4.4.** (Gelbart-Shahidi [16]). The function $\Phi(s, \pi)$ is of order one.

**Lemma 4.5.** (Jacquet-Shalika [20], and Shahidi [49]). The function $\Phi(s, \pi)$ and $L(s, \pi)$ are non-zero in the half-plane $\sigma \geq 1$.

Iwaniec and Sarnak [15] introduced the analytic conductor of $\pi$. It is a function over the reals given by

$$Q_{\pi}(t) = N_{\pi} \prod_{j=1}^{m} (3 + |t + \mu_{\pi}(j)|), \quad (4.13)$$

which puts together all the important parameters for $\pi$. The quantity

$$Q_{\pi} := Q_{\pi}(0) = N_{\pi} \prod_{j=1}^{m} (3 + |\mu_{\pi}(j)|) \quad (4.14)$$

is also important, and it is named as the conductor of $\pi$.

The next lemma is about the distribution of zeros of the function $L(s, \pi)$.

**Lemma 4.6.** All the non-trivial zeros of $\Phi(s, \pi)$ are in the critical strip $0 \leq \sigma \leq 1$. Let $N(T, \pi)$ be the number of nontrivial zeros within the rectangular

$$0 \leq \sigma \leq 1, \ |t| \leq T.$$  

Then

$$N(T, \pi) \ll T \log(Q_{\pi}T),$$

and

$$N(T + 1, \pi) - N(T, \pi) \ll \log(Q_{\pi}T).$$

For proof of this, one is referred to Liu and Ye [32], Lemma 4.3, or Iwaniec and Kowalski [14], Theorem 5.8.
4.2 Three conjectures in the theory of automorphic $L$-functions

It is said in the previous section that all the non-trivial zeros of $\Phi(s, \pi)$ are in the critical strip $0 \leq \sigma \leq 1$, while GRH for $L(s, \pi)$ predicts that they should actually lie on the vertical line $\sigma = 1/2$.

**Generalized Riemann Hypothesis.** *All the zeros of $\Phi(s, \pi)$ have their real parts equal to $1/2$.*

Upper bounds for $L(s, \pi)$ on the critical line $\sigma = 1/2$ is of great importance, and the most optimistic conjecture in this direction can be stated as follows in terms of the analytic conductor defined in (4.13).

**Generalized Lindelöf Hypothesis.** The estimate

$$L\left(\frac{1}{2} + it, \pi\right) \ll Q_\pi(t)^\varepsilon$$

is true for arbitrary $\varepsilon > 0$.

The following result is unconditional. Its proof is based on the fact that the Rankin-Selberg $L$-function $L(s, \pi \otimes \pi')$ exists, where $\pi'$ is another irreducible unitary cuspidal representation. It may happen that $\pi = \pi'$.

**Lemma 4.7.** (Harcos [8]). *Let $\varepsilon > 0$ be arbitrary, and $0 < \sigma < 1$. Then we have the upper bound estimate*

$$L(\sigma + it, \pi) \ll \varepsilon Q_\pi(t)^{1-\sigma+\varepsilon}. \quad (4.15)$$

Taking $\sigma = 1/2$, Lemma 4.7 gives

$$L\left(\frac{1}{2} + it, \pi\right) \ll \varepsilon Q_\pi(t)^{\frac{1}{2}+\varepsilon}.$$  

This is called the convexity bound of $L(s, \pi)$, and it should be emphasized that it is uniform in all parameters. Subconvexity bounds have been established for some $L(s, \pi)$ for some aspects, only when $m = 1, 2, 3, 4, 8$; moreover, when $m \geq 2$, the existing subconvexity bounds are not uniform in all parameters, except the recent uniform result of Michel and Venkatesh [38] for $GL_2$. See Michel [37] for a comprehensive survey in this direction.
4.2 Three conjectures in the theory of automorphic $L$-functions

Good bounds for the local parameters

$$\{\alpha_\pi(p,j)\}_{j=1}^m, \quad \{\mu_\pi(p,j)\}_{j=1}^m$$

are of fundamental importance for the study of automorphic $L$-functions. By the Rankin-Selberg method, one shows that, for all $p$,

$$|\alpha_\pi(p,j)| \leq p^{1/2}, \quad \Re \mu_\pi(j) \leq \frac{1}{2};$$

(4.16)

moreover, for any unramified place,

$$p^{-1/2} \leq |\alpha_\pi(p,j)| \leq p^{1/2}, \quad |\Re \mu_\pi(j)| \leq \frac{1}{2};$$

(4.17)

The bounds (4.16) and (4.17) are called trivial bounds and are hence of little use. The Generalized Ramanujan Conjecture (GRC in brief) asserts that the $1/2$ in (4.17) can be reduced to $0$.

**Generalized Ramanujan Conjecture.** With $\alpha_\pi(p,j)$ and $\mu_\pi(j)$ defined as above,

$$\begin{cases} 
|\alpha_\pi(p,j)| = 1 & \text{if } \pi \text{ is unramified at } p, \\
|\Re \mu_\pi(j)| = 0 & \text{if } \pi \text{ is unramified at } \infty.
\end{cases}$$

The following lemma gives bounds toward the GRC.

**Lemma 4.8.** (Luo-Rudnick-Sarnak [36]). There is a constant $0 \leq \theta < 1/2$, such that

$$\begin{cases} 
|\alpha_\pi(p,j)| \leq p^\theta & \text{if } \pi \text{ is unramified at } p, \\
|\Re \mu_\pi(j)| \leq \theta & \text{if } \pi \text{ is unramified at } \infty.
\end{cases}$$

Actually,

$$\theta = \frac{1}{2} - \frac{1}{m^2 + 1}$$

(4.18)

is acceptable.

This $\theta$ will be used in the next two chapters.
4.3 Hecke $L$-functions as automorphic $L$-functions

It should be pointed out that the Hecke $L$-functions defined in (2.7) and (3.15) are special examples of automorphic $L$-functions. If $\pi$ corresponds to holomorphic new form $f$ with weight $k$ and level $N$, then the conductor is

$$Q_\pi \asymp k^2 N.$$  

If $\pi$ corresponds to Maass new form $f$ with Laplace eigenvalue $1/4 + \nu^2$ and level $N$, then

$$Q_\pi \asymp (3 + |\nu|)^2 N.$$  

In view of these, it is easy to re-state Theorem 3.10 in terms of their conductors $Q_\pi$.

4.4 Rankin-Selberg $L$-functions

Let $\pi$ and $\pi'$ be two irreducible unitary cuspidal representations for $GL_m(\mathbb{A}_\mathbb{Q})$ and $GL_{m'}(\mathbb{A}_\mathbb{Q})$, respectively. The theory for the Rankin-Selberg type $L$-functions $L(s, \pi \otimes \pi')$ was initiated by Rankin [45] and Selberg [48] in the case of classical modular forms. For general automorphic representations, the corresponding theory was initiated and developed in several papers by Jacquet, Piestesi-Shapiro, and Shalika [18] [20] [21], and completed in works of Shahidi [49] [50] [51] [52], Moeglin and Waldspurger [39], and Gelbart and Shahidi [16]. Let $\pi$ and $\pi'$ be as above. When $\sigma > 1$,

$$L(s, \pi \otimes \pi') = \prod_{p<\infty} L_p(s, \pi_p \otimes \pi'_p)$$  \hspace{1cm} (4.19)

with

$$L_p(s, \pi_p \otimes \pi'_p) = \prod_{j=1}^{mm'} \left(1 - \frac{\alpha_{\pi \otimes \pi'}(p, j)}{p^s}\right)^{-1}.$$  

Then the Rankin-Selberg $L$-function $L(s, \pi \otimes \pi')$ is a Dirichlet series

$$L(s, \pi \otimes \pi') = \sum_{n=1}^{\infty} \frac{\lambda_{\pi \otimes \pi'}(n)}{n^s}$$  \hspace{1cm} (4.20)

which is proved to be absolutely convergent for $\sigma > 1$. At the infinite place,

$$L_\infty(s, \pi_\infty \otimes \pi'_\infty) = \prod_{j=1}^{mm'} \Gamma_R(s - \mu_{\pi \otimes \pi'}(j)).$$
Moreover, at places $v$ where $\pi_v$ is unramified, $L_v(s, \pi_v \otimes \pi'_v)$ has the following explicit expression

$$L_p(s, \pi_p \otimes \pi'_p) = \prod_{j=1}^{m} \prod_{j'=1}^{m'} \left(1 - \frac{\alpha_{\pi}(p,j)\alpha_{\pi'}(p,j')}{p^s}\right)^{-1}$$

(4.21)

at $v = p$ a finite place, and at the infinite place $v = \infty$,

$$L_{\infty}(s, \pi_{\infty} \otimes \pi'_{\infty}) = \prod_{j=1}^{m} \prod_{j'=1}^{m'} \Gamma_{\mathbb{R}}(s - \mu_{\pi}(j) - \mu_{\pi'}(j')).$$

(4.22)

The complete $L$-function

$$\Phi(s, \pi \otimes \pi') = L_{\infty}(s, \pi_{\infty} \otimes \pi'_{\infty})L(s, \pi \otimes \pi')$$

satisfies a functional equation, and has properties similar to those stated in the lemmas in §4.1. For simplicity, we do not list all these properties of $L(s, \pi \otimes \pi')$ in detail, but just point out some main differences between the Rankin-Selberg $L$-function $L(s, \pi \otimes \pi')$ and the single $L$-function $L(s, \pi)$:

- The $\Phi(s, \pi \otimes \pi')$ has a meromorphic continuation to $\mathbb{C}$;
- $\Phi(s, \pi \otimes \pi')$ is entire if $\pi$ and $\pi'$ are not twisted equivalent, i.e. $\pi' \neq \tilde{\pi} \otimes |\det|^{-it}$ for any $t \in \mathbb{R}$;
- if $\pi' = \tilde{\pi} \otimes |\det|^{-it}$ for some $t \in \mathbb{R}$, then $L(s, \pi \otimes \pi')$ has only a simple pole at $s = 1 + it$; in particular, the function $L(s, \pi \otimes \tilde{\pi})$ has only a simple pole at $s = 1$.

For $\sigma > 1$, the Euler product of $L(s, \pi \otimes \pi')$ takes the form

$$L(s, \pi \otimes \pi') = \prod_{p} \prod_{j=1}^{m} \prod_{j'=1}^{m'} \left(1 - \frac{\alpha_{\pi}(p,j)\alpha_{\pi'}(p,j')}{p^s}\right)^{-1}.$$  

(4.23)

The following result gives information for the Dirichlet coefficients $\lambda_{\pi \otimes \tilde{\pi}}(n)$ for $L(s, \pi \otimes \tilde{\pi})$; for a proof of this, see Lemma A.1 in Rudnick and Sarnak [46].

**Lemma 4.9.** (Rudnick-Sarnak [46]). Let $\pi$ an irreducible unitary cuspidal representation for $GL_m(A_{\mathbb{Q}})$. Specifying $\pi' = \tilde{\pi}$ in (4.20), and write, for $\sigma > 1$,

$$L(s, \pi \otimes \tilde{\pi}) = \sum_{n=1}^{\infty} \frac{\lambda_{\pi \otimes \tilde{\pi}}(n)}{n^s}.$$  

(4.24)
A Linnik-type problem for automorphic L-functions

Then

\[ \lambda_{\pi \otimes \tilde{\pi}}(n) \geq 0, \quad \text{for all } n \geq 1. \]

Other relations among the Dirichlet coefficients of Rankin-Selberg L-functions \( L(s, \pi \otimes \tilde{\pi}) \) will also be necessary. We reserve the next section for this purpose.

### 4.5 Coefficients of L-functions and Rankin-Selberg L-functions

We need some general lemmas, which will be applied later to Dirichlet coefficients of L-functions \( L(s, \pi) \), or those for Rankin-Selberg L-functions \( L(s, \pi \otimes \tilde{\pi}) \). The first general result is due to Brumley [1], and established by the theory of symmetric algebra.

**Lemma 4.10.** (Brumley [1]). For \( m \) complex numbers \( \{\alpha_j\}_{j=1}^m \), define the coefficients \( b_n \) by

\[
\sum_{n=0}^{\infty} b_n X^n = \prod_{j=1}^{m} \prod_{j'=1}^{m} (1 - \alpha_j \bar{\alpha}_{j'} X)^{-1}. 
\]

If \( \alpha_1 \cdots \alpha_m = 1 \), then we have \( b_m \geq 1 \). In particular for any irreducible unitary cuspidal representation of \( GL_m(\mathbb{A}_\mathbb{Q}) \) and any prime \( p \) such that \( \pi_p \) is unramified, we have

\[ \lambda_{\pi \otimes \tilde{\pi}}(p^m) \geq 1, \]

where \( \lambda_{\pi \otimes \tilde{\pi}}(n) \) is defined by (4.24).

The second general lemma is due to Lü [35]. I am very grateful for his kindness in allowing my reproduction of his proof below.

**Lemma 4.11.** (Lü [35]). For \( m \) complex numbers \( \{\alpha_j\}_{j=1}^m \), define the coefficients \( \ell_n \) by

\[
\sum_{n=0}^{\infty} \ell_n X^n = \prod_{j=1}^{m} (1 - \alpha_j X)^{-1}. 
\] (4.25)

Also, for \( n \geq 1 \), define

\[ a_n = \alpha_1^n + \cdots + \alpha_m^n. \] (4.26)

Then we have, for any \( n \geq 1 \),

\[ n\ell_n = a_1 \ell_{n-1} + a_2 \ell_{n-2} + \cdots + a_{n-1} \ell_1 + a_n. \] (4.27)
4.5 Coefficients of $L$-functions and Rankin-Selberg $L$-functions

Proof. Differentiating (4.25), we get

\[ \sum_{n=1}^{\infty} n \ell_n X^{n-1} = \sum_{i=1}^{m} \alpha_i (1 - \alpha_i X)^{-1} \prod_{j=1}^{m} (1 - \alpha_j X)^{-1} \]

\[ = \left( \sum_{i=1}^{m} \alpha_i (1 - \alpha_i X)^{-1} \right) \prod_{j=1}^{m} (1 - \alpha_j X)^{-1}. \quad (4.28) \]

By expanding \((1 - \alpha_i X)^{-1}\) and using the definition (4.26), the quantity within the last braces in (4.28) can be written as

\[ \sum_{i=1}^{m} \alpha_i (1 - \alpha_i X)^{-1} = \sum_{i=1}^{m} \alpha_i \left( \sum_{u=0}^{\infty} \alpha_i^u X^u \right) \]

\[ = \sum_{u=0}^{\infty} \sum_{i=1}^{m} \alpha_i^u X^u \]

\[ = \sum_{u=0}^{\infty} a_{u+1} X^u. \]

From this and (4.25), one sees that the right-hand side in (4.28) becomes

\[ \left( \sum_{u=0}^{\infty} a_{u+1} X^u \right) \left( \sum_{v=0}^{\infty} \ell_v X^v \right) = \sum_{n=0}^{\infty} \left( \sum_{u+v=n, u \geq 0, v \geq 0} a_{u+1} \ell_v \right) X^n \]

\[ = \sum_{n=0}^{\infty} \left( \sum_{u+v=n+1, u \geq 1, v \geq 0} a_u \ell_v \right) X^n \]

\[ = \sum_{n=1}^{\infty} \left( \sum_{u+v=n, u \geq 1, v \geq 0} a_u \ell_v \right) X^{n-1}. \]

Comparing this with the left-hand side of (4.28), we get, for all \(n \geq 1,\)

\[ n \ell_n = \sum_{u+v=n, u \geq 1, v \geq 0} a_u \ell_v, \]
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which is exactly the assertion (4.27) of the lemma. □

Applying the above two lemmas, we get the following consequence, which is very important in establishing the the main result Theorem 4.15 of this chapter.

**Lemma 4.12.** Let \( m \geq 2 \) be an integer and let \( \pi \) be an irreducible unitary cuspidal representation of \( \text{GL}_m(\mathbb{A}_Q) \). For any prime \( p \) such that \( \pi_p \) is unramified, we have

\[
|\lambda_\pi(p^m)| + |\lambda_\pi(p^{m-1})| + \cdots + |\lambda_\pi(p)| \geq \frac{1}{m},
\]

where \( \lambda_\pi(n) \) is as in (4.6) and (4.7).

**Proof.** The proof is divided into three steps, for a clear presentation. The first two steps deal with the Rankin-Selberg L-function \( L(s, \pi \otimes \tilde{\pi}) \) and the automorphic L-function \( L(s, \pi) \), respectively, and the third is saved for the final argument.

**First step.** Let \( \{\alpha_\pi(p, j)\}_{j=1}^m \) be the set of Satake parameters for \( \pi_p \); we may write \( \alpha_\pi(p, j) = \alpha_j \) for simplicity. Then (4.21) becomes

\[
L_p(s, \pi_p \otimes \tilde{\pi}_p) = \prod_{j=1}^m \prod_{j'=1}^m \left( 1 - \frac{\alpha_j \alpha_{j'}}{p^s} \right)^{-1} = \prod_{\ell=1}^M \left( 1 - \frac{\beta_\ell}{p^s} \right)^{-1},
\]

(4.29)

where we have written \( M = m^2 \) and

\[
\{\beta_\ell\}_{\ell=1}^M = \left\{ \alpha_j \alpha_{j'} \right\}_{1 \leq j \leq m, 1 \leq j' \leq m}.
\]

(4.30)

Therefore, (4.21) and (4.24) give, for \( \sigma > 1 \),

\[
L_p(s, \pi_p \otimes \tilde{\pi}_p) = \prod_{\ell=1}^M \left( 1 - \frac{\beta_\ell}{p^s} \right)^{-1} = \sum_{n=0}^{\infty} \frac{\lambda_{\pi \otimes \tilde{\pi}}(p^n)}{p^{ns}},
\]

This is of the form (4.25), if we make the change of variables

\[
p^{-s} = X, \quad \lambda_{\pi \otimes \tilde{\pi}}(p^n) = \ell_n.
\]

Thus, Lemma 4.11 gives, for all \( n \geq 1 \),

\[
n \lambda_{\pi \otimes \tilde{\pi}}(p^n) = a_{\pi \otimes \tilde{\pi}}(p) \lambda_{\pi \otimes \tilde{\pi}}(p^{n-1}) + a_{\pi \otimes \tilde{\pi}}(p^2) \lambda_{\pi \otimes \tilde{\pi}}(p^{n-2}) + \cdots + a_{\pi \otimes \tilde{\pi}}(p^{n-1}) \lambda_{\pi \otimes \tilde{\pi}}(p) + a_{\pi \otimes \tilde{\pi}}(p^n),
\]

(4.31)
where, because of (4.26), we have written

\[ a_{\pi \otimes \tilde{\pi}}(p^n) = \beta_1^n + \cdots + \beta_M^n. \] (4.32)

Now we write

\[ a_{\pi}(p^n) = \alpha_1^n + \cdots + \alpha_m^n; \] (4.33)

then we have from (4.32) and (4.30) that

\[ a_{\pi \otimes \tilde{\pi}}(p^n) = \sum_{j=1}^{m} \sum_{j'=1}^{m} (\alpha_j \alpha_{j'})^n = |a_{\pi}(p^n)|^2 \geq 0. \] (4.34)

Inserting (4.34) into (4.31), we get

\[ n\lambda_{\pi \otimes \tilde{\pi}}(p^n) = |a_{\pi}(p)|^2\lambda_{\pi \otimes \tilde{\pi}}(p^{n-1}) + |a_{\pi}(p^2)|^2\lambda_{\pi \otimes \tilde{\pi}}(p^{n-2}) + \cdots \]
\[ + |a_{\pi}(p^{n-1})|^2\lambda_{\pi \otimes \tilde{\pi}}(p) + |a_{\pi}(p^n)|^2, \] (4.35)

which holds for all \( n \geq 1 \). From it, we can deduce, by a simple induction on the integer \( n \), that \( \lambda_{\pi \otimes \tilde{\pi}}(p^n) \geq 0 \) for all \( n \geq 1 \). This gives another proof of Lemma 4.9 due to Rudnick-Sarnak [46], for those primes \( p \) such that \( \pi_p \) is unramified.

**Second step.** Let \( \{\alpha_j\}_{j=1}^m \) be as before. Then (4.2) becomes

\[ L_p(s, \pi_p) = \prod_{j=1}^{m} \left( 1 - \frac{\alpha_j}{p^s} \right)^{-1}. \]

Therefore, (4.6) gives, for \( \sigma > 1 \),

\[ L_p(s, \pi_p) = \prod_{j=1}^{m} \left( 1 - \frac{\alpha_j}{p^s} \right)^{-1} = \sum_{n=0}^{\infty} \frac{\lambda_{\pi}(p^n)}{p^{ns}}, \]

with \( \lambda_{\pi}(n) \) defined by (4.7). On changing variables

\[ p^{-s} = X, \quad \lambda_{\pi}(p^n) = \ell_n, \]

the above is again of the form (4.25), and Lemma 4.11 gives, for all \( n \geq 1 \),

\[ n\lambda_{\pi}(p^n) = a_{\pi}(p)\lambda_{\pi}(p^{n-1}) + a_{\pi}(p^2)\lambda_{\pi}(p^{n-2}) + \cdots \]
\[ + a_{\pi}(p^{n-1})\lambda_{\pi}(p) + a_{\pi}(p^n), \] (4.36)
where \( a_\pi(p^n) \) is as in (4.33), as suggested by (4.26).

**Third step.** Taking \( n = m \) in (4.35), we have

\[
m\lambda_{\pi \otimes \pi}(p^m) = |a_\pi(p)|^2 \lambda_{\pi \otimes \pi}(p^{m-1}) + |a_\pi(p^2)|^2 \lambda_{\pi \otimes \pi}(p^{m-2}) + \cdots + |a_\pi(p^{m-1})|^2 \lambda_{\pi \otimes \pi}(p) + |a_\pi(p^m)|^2.
\]  

(4.37)

Brumley’s lemma (Lemma 4.10) now asserts that \( \lambda_{\pi \otimes \pi}(p^m) \geq 1 \), and therefore, the left-hand side in (4.37) is bounded from below by \( m\lambda_{\pi \otimes \pi}(p^m) \geq m \). As we have seen that \( \lambda_{\pi \otimes \pi}(p^n) \geq 0 \) for all \( n \geq 1 \), each term on the right-hand side of (4.37) is non-negative. These observations will be used later.

Before going further, we make a claim:

**Claim C.** There exists a positive integer \( j \) with \( 1 \leq j \leq m \) such that

\[
|a_\pi(p^j)|^2 \geq 1.
\]  

(4.38)

We suppose that Claim C is not true, and establish a contradiction. Since Claim C is not true, we must have

\[
|a_\pi(p^n)|^2 < 1
\]  

(4.39)

for all \( 1 \leq n \leq m \). It thus follows from (4.37) and (4.39) with \( n = 1 \) that

\[
\lambda_{\pi \otimes \pi}(p) = |a_\pi(p)|^2 < 1.
\]  

(4.40)

We may also apply (4.37) and (4.39) with \( n = 2 \), and we get from (4.40) that

\[
2\lambda_{\pi \otimes \pi}(p^2) = |a_\pi(p)|^2 \lambda_{\pi \otimes \pi}(p) + |a_\pi(p^2)|^2 < 1 + 1 = 2,
\]

that is \( \lambda_{\pi \otimes \pi}(p^2) < 1 \). By induction on the \( n \) in (4.35), we can prove that \( \lambda_{\pi \otimes \pi}(p^n) < 1 \) for all \( 1 \leq n \leq m \). In particular, \( \lambda_{\pi \otimes \pi}(p^m) < 1 \). This contradicts Brumley’s lemma (Lemma 4.10), which asserts that

\[
\lambda_{\pi \otimes \pi}(p^m) \geq 1
\]

in the present situation. Hence, Claim C is proved.
4.6 Sign changes of $\lambda_{\pi}(n)$

If $n_0$ is one of the integers in Claim C so that (4.38) holds, then, by (4.34), one has $|a_{\pi}(p^{n_0})|^2 \geq 1$, and consequently,

$$|a_{\pi}(p^{n_0})| \geq 1. \quad (4.41)$$

Now let $n_0$ with $1 \leq n_0 \leq m$ be the smallest integer such that (4.41) holds. It follows that $|a_{\pi}(p^j)| < 1$, for all $1 \leq j < n_0$. Applying (4.36) with $n = n_0$, we deduce that

$$n_0|\lambda_{\pi}(p^{n_0})| = |a_{\pi}(p)\lambda_{\pi}(p^{n_0-1}) + \cdots + a_{\pi}(p^{n_0-1})\lambda_{\pi}(p) + a_{\pi}(p^{n_0})|$$

$$> -|\lambda_{\pi}(p^{n_0-1})| - \cdots - |\lambda_{\pi}(p)| + 1.$$

This implies that

$$m\{|\lambda_{\pi}(p^m)| + \cdots + |\lambda_{\pi}(p)|\} \geq m\{|\lambda_{\pi}(p^{n_0})| + \cdots + |\lambda_{\pi}(p)|\}$$

$$\geq n_0|\lambda_{\pi}(p^{n_0})| + \cdots + |\lambda_{\pi}(p)|$$

$$> 1,$$

and the lemma follows. \qed

4.6 Sign changes of $\lambda_{\pi}(n)$

Working similarly as in Knopp, Kohnen, and Pribitkin [23], we can prove

**Theorem 4.13.** Let $m \geq 2$ be an integer and let $\pi$ be an irreducible unitary cuspidal representation for $GL_m(\mathbb{A}_\mathbb{Q})$ such that $\lambda_{\pi}(n)$ are real for all $n \geq 1$. Then the sequence $\{\lambda_f(n)\}_{n=1}^{\infty}$ has infinitely many sign changes, i.e. there are infinitely many $n$ such that $\lambda_f(n) > 0$, and there are infinitely many $n$ such that $\lambda_f(n) < 0$.

**Corollary 4.14.** Let $m \geq 2$ be an integer and let $\pi$ be an irreducible unitary cuspidal representation for $GL_m(\mathbb{A}_\mathbb{Q})$ such that it is self-contragredient. Then $\lambda_{\pi}(n)$ are real for all $n \geq 1$, and the sequence $\{\lambda_f(n)\}_{n=1}^{\infty}$ has infinitely many sign changes, i.e. there are infinitely many $n$ such that $\lambda_f(n) > 0$, and there are infinitely many $n$ such that $\lambda_f(n) < 0$.

4.7 A Linnik-type problem for automorphic $L$-functions

Now we state our main result in this chapter.
Theorem 4.15. Let $m \geq 2$ be an integer and let $\pi$ be an irreducible unitary cuspidal representation for $GL_m(\mathbb{A}_Q)$. If $\lambda_\pi(n)$ are real for all $n \geq 1$, then there is some $n$ satisfying

$$n \ll Q^{m/2+\varepsilon}$$

such that $\lambda_\pi(n) < 0$. The constant implied in (4.42) depends only on $m$ and $\varepsilon$. In particular, the result is true for any self-contragredient representation $\pi$.

Proof. Still, let us start with the sum

$$S(x) := \sum_{n \leq x} \lambda_\pi(n) \left( \log \frac{x}{n} \right)^\ell,$$

assuming that

$$\lambda_\pi(n) \geq 0, \quad \text{for } n \leq x.$$  \hfill (4.43)

Here $\ell$ is a positive integer that will be decided later. The desired result will follow from upper and lower bound estimates for $S(x)$.

To get an upper bound for $S(x)$, we apply Perron’s formula (3.30) to the Dirichlet series (4.6), getting

$$S(x) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} L(s, \pi) \frac{x^s}{s^{\ell+1}} ds.$$

Moving the contour to the vertical line $\sigma = \varepsilon$ with $\varepsilon$ being an arbitrarily small positive constant, and applying Harcos’ convexity bound (4.15) for $L(s, \pi)$, we obtain

$$S(x) = \frac{1}{2\pi i} \int_{\varepsilon-i\infty}^{\varepsilon+i\infty} L(s, \pi) \frac{x^s}{s^{\ell+1}} ds \ll \varepsilon \int_{-\infty}^{\infty} Q_\pi(t)^{1/2+\varepsilon} \frac{x^\varepsilon}{(|t| + \varepsilon)^{\ell+1}} dt.$$

The analytic conductor $Q_\pi(t)$ is bounded by $|t|^m$ in the $t$-aspect. Thus, if we take $\ell = m$, then the above estimate becomes

$$S(x) \ll_{\ell, m, \varepsilon} Q^{1/2+\varepsilon} x^\varepsilon \int_{-\infty}^{\infty} \frac{(|t| + 1)^m}{(|t| + \varepsilon)^{\ell+1}} dt \ll_{m, \varepsilon} Q^{1/2+\varepsilon} x^\varepsilon. \quad \hfill (4.44)$$

This is the desired upper bound for $S(x)$. 

To get a lower bound for $S(x)$, we apply Lemma 4.12, from which we have

$$S(x) \geq (\log 2)^\ell \sum_{n \leq x/2} \lambda_\pi(n) \sum_{(n, N_\pi) = 1} \lambda_\pi(p) \geq \left( \log 2 \right)^\ell \sum_{p \leq (x/2)^{1/m}} \{ \lambda_\pi(p^m) + \lambda_\pi(p^{m-1}) + \cdots + \lambda_\pi(p) \} \gg \ell, m \sum_{p \leq (x/2)^{1/m}} \frac{1}{p} \gg \ell, m \frac{(x/2)^{1/m}}{\log(x/2)} - \log N_\pi.$$ 

Without loss of generality, we may suppose

$$x \geq C \log^{m+1} Q_\pi; \quad (4.45)$$

where $C$ is a large absolute constant. This requirement is very mild in view of the assertion of the theorem. \qed
A Linnik-type problem for automorphic $L$-functions
Chapter 5

The prime number theorem for automorphic $L$-functions

5.1 The automorphic prime number theorem

To each irreducible unitary cuspidal representation $\pi = \otimes \pi_p$ of $GL_m(\mathbb{A}_Q)$, one can attach a global $L$-function $L(s, \pi)$ as in §4.1. Then, one can link $L(s, \pi)$ with primes by taking logarithmic differentiation in (4.2), so that for $\sigma > 1$,

$$\frac{d}{ds} \log L(s, \pi) = - \sum_{n=1}^{\infty} \frac{\Lambda(n)a_\pi(n)}{n^s},$$

where $\Lambda(n)$ is the von Mangoldt function, and

$$a_\pi(p^k) = \sum_{j=1}^{m} \alpha_\pi(p, j)^k.$$  \hspace{1cm} (5.2)

It will be important later that this is the same as that defined in (4.33). The prime number theorem for $L(s, \pi)$ concerns the asymptotic behavior of the counting function

$$\psi(x, \pi) = \sum_{n \leq x} \Lambda(n)a_\pi(n),$$

and a special case of it asserts that, if $\pi$ is an irreducible unitary cuspidal representation of $GL_m(\mathbb{A}_Q)$ with $m \geq 2$, then

$$\psi(x, \pi) \ll \sqrt{Q_\pi} \cdot x \cdot \exp \left( - \frac{c}{2m^2} \sqrt{\log x} \right).$$  \hspace{1cm} (5.3)
The prime number theorem for automorphic $L$-functions

for some absolute positive constant $c$, where the implied constant is absolute. In Iwaniec and Kowalski [14], Theorem 5.13, a prime number theorem is proved for general $L$-functions satisfying necessary axioms, from which (5.3) follows as a consequence.

In this chapter, we investigate the influence of GRH on $\psi(x, \pi)$. It is known that, under GRH, (5.3) can be improved to

$$\psi(x, \pi) \ll x^{1/2} \log^2(Q_\pi x),$$

but better results are desirable. For a proof of (5.4), see e.g. §5.3. In this direction, we establish the following results.

**Theorem 5.1.** Let $m \geq 2$ be an integer and let $\pi$ be an irreducible unitary cuspidal representation of $GL_m(\mathbb{A}_Q)$. Assume GRH for $L(s, \pi)$. Then we have

$$\psi(x, \pi) \ll x^{1/2} \log^2(Q_\pi \log x)$$

for $x \geq 2$, except on a set $E$ of finite logarithmic measure, i.e.

$$\int_E \frac{dx}{x} < \infty.$$

The constant implied in the $\ll$-symbol depends at most on $m$.

Theorem 5.1 tells that, except on a set of finite logarithm measure $E$, (5.4) can be improved to (5.5). The next two theorems say that $\psi(x, \pi)$ behaves somehow like $x^{1/2}$.

**Theorem 5.2.** Let $m \geq 2$ be an integer and let $\pi$ be an irreducible unitary cuspidal representation of $GL_m(\mathbb{A}_Q)$. Assume GRH for $L(s, \pi)$. Then

$$\int_2^X \left| \psi(x, \pi) \right|^2 \frac{dx}{x} \ll X \log^2 Q_\pi.$$ 

The constant implied in the $\ll$-symbol depends at most on $m$.

Gallagher [6] was the first to establish a result like Theorem 5.1, in the classical case $m = 1$ for the Riemann zeta-function. He proved that, under the Riemann Hypothesis for the classical zeta-function,

$$\psi(x) := \sum_{n \leq x} \Lambda(n) = x + O(x^{1/2}(\log \log x)^2)$$

for $x \geq 2$, except on a set of finite logarithmic measure, and hence made improvement on the classical estimate error term $O(x^{1/2} \log^2 x)$ of von Koch [27]. In the same paper, Gallagher [6] also gave short proofs of Cramér’s conditional estimate (see [2] and [3])

$$\int_2^X (\psi(x) - x)^2 \frac{dx}{x} \ll X.$$
Gallagher’s proofs of the above results make crucial use of his lemma in [7], which is now named after him.

Our Theorems 5.1-5.2 generalize the above classical results to the prime counting function \( \psi(s, \pi) \) attached to irreducible unitary cuspidal representations \( \pi \) of \( GL_m(\mathbb{A}_\mathbb{Q}) \) with \( m \geq 2 \). Our proofs combine the approach of Gallagher with recent results of Liu and Ye ([32], [33]) on the distribution of zeros of Rankin-Selberg automorphic \( L \)-functions.

The above Theorem 5.2 states that, under GRH, \( |\psi(x, \pi)| \) is of size \( x^{1/2} \log Q_\pi \) on average. This can be compared with the next theorem, which gives the unconditional Omega result that \( |\psi(x, \pi)| \) should not be of order lower than \( x^{1/2-\varepsilon} \).

**Theorem 5.3.** Let \( m \geq 2 \) be an integer and let \( \pi \) be an irreducible unitary cuspidal representation of \( GL_m(\mathbb{A}_\mathbb{Q}) \), and \( \varepsilon > 0 \) arbitrary. Unconditionally,

\[
\psi(x, \pi) = \Omega(x^{1/2-\varepsilon}),
\]

where the constant implied in the \( \Omega \)-symbol depends at most on \( m \) and \( \varepsilon \). More precisely, there exists an increasing sequence \( \{x_n\}_{n=1}^\infty \) tending to infinity such that

\[
\lim_{n \to \infty} \frac{|\psi(x_n, \pi)|}{x_n^{1/2-\varepsilon}} > 0.
\]

Note that the sequence \( \{x_n\}_{n=1}^\infty \) and the limit in (5.6) may depend on \( \pi \) and \( \varepsilon \). This result generalizes that for the Riemann zeta-function. It is possible to get better Omega results like those in Chapter V of Ingham [10]. We remark that, unlike the classical case, in Theorems 5.1-5.3 we do not have the main term \( x \). This is because \( L(s, \pi) \) is entire when \( m \geq 2 \), while \( \zeta(s) \) has a simple pole at \( s = 1 \) with residue 1.

## 5.2 Preliminaries

We need some preliminaries to establish the main results.

**Lemma 5.4.** Let \( \pi \) be an irreducible unitary cuspidal representation of \( GL_m(\mathbb{A}_\mathbb{Q}) \) with \( m \geq 2 \). Then

\[
\frac{d}{ds} \log L(s, \pi) = C + \sum_{\rho} \left( \frac{1}{s - \rho} + \frac{1}{\rho} \right) + \sum_{j=1}^{m} \frac{1}{s + \mu_\pi(j)} + \sum_{j=1}^{m} \sum_{n=1}^{\infty} \left( \frac{1}{2n + s + \mu_\pi(j) - \frac{1}{2n}} \right),
\]

where

\[
\mu_\pi(j) = \sum_{\nu} \mu_\pi(\nu) d_{\pi, \nu}.
\]
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where $C$ is a constant depending on $\pi$. The set of all trivial zeros of $L(s, \pi)$ is

$$\{ \mu : \mu = -2n - \mu_\pi(j), \quad n = 0, 1, 2, \ldots; \quad j = 1, \ldots, m \}.$$  

**Proof.** Since $\Phi(s, \pi)$ is of order one (Lemma 4.4), we have (see e.g. Davenport [4], Chapter 11)

$$\Phi(s, \pi) = e^{A+Bs} \prod_{\rho} \left( 1 - \frac{s}{\rho} \right) e^{s/\rho},$$

where $A, B$ are constants depending on $\pi$. Taking logarithmic derivative, we get

$$\frac{d}{ds} \log \Phi(s, \pi) = B + \sum_{\rho} \left( \frac{1}{s-\rho} + \frac{1}{\rho} \right), \quad (5.7)$$

where we set $\log 1 = 0$. By the definition of $\Phi(s, \pi)$,

$$\frac{d}{ds} \log \Phi(s, \pi) = \frac{d}{ds} \log L_{\infty}(s, \pi_{\infty}) + \frac{d}{ds} \log L(s, \pi). \quad (5.8)$$

Applying

$$\frac{d}{ds} \log \Gamma(s) = -\frac{1}{s} - \gamma - \sum_{n=1}^{\infty} \left( \frac{1}{n+s} - \frac{1}{n} \right),$$

where $\gamma$ is Euler’s constant, we have

$$\frac{d}{ds} \log L_{\infty}(s, \pi_{\infty}) = \sum_{j=1}^{m} \frac{d}{ds} \log \pi^{-(s+\mu_\pi(j))/2} + \sum_{j=1}^{m} \frac{d}{ds} \log \Gamma \left( \frac{s + \mu_\pi(j)}{2} \right)$$

$$= -\frac{m}{2} (\log \pi + \gamma) - \sum_{j=1}^{m} \frac{1}{s + \mu_\pi(j)}$$

$$- \sum_{j=1}^{m} \sum_{n=1}^{\infty} \left( \frac{1}{2n + s + \mu_\pi(j)} - \frac{1}{2n} \right).$$

Inserting this and (5.7) into (5.8), we get the lemma. \hfill \Box

Consider the poles of

$$L(1-s, \pi_{\infty}) = \pi^{-ms/2} \prod_{j=1}^{m} \Gamma \left( \frac{1-s + \mu_\pi(j)}{2} \right). \quad (5.9)$$
5.2 Preliminaries

These poles are easily to be seen as

\[ \{ P_{n,j} = 2n + 1 + \mu_\pi(j) : n = 0, 1, 2, \ldots, j = 1, \ldots, m \}. \]

As in [32], we let \( C(m) \) denote the complex plane with the discs

\[ \{ s - P_{n,j} < \frac{1}{8m}, n = 0, 1, 2, \ldots, j = 1, \ldots, m \} \]

excluded. Thus, for any \( s \in C(m) \), the quantity

\[ \frac{1 - s + \mu_\pi(j)}{2} \]

is away from all poles of \( \Gamma(s) \) by at least \( 1/(16m) \). Now we give a remark about the structure of \( C(m) \). For \( j = 1, \ldots, m \), denote by \( \beta(j) \) the fractional part of \( \Re\mu_\pi(j) \). In addition we let \( \beta(0) = 0 \) and \( \beta(m+1) = 1 \). Then all \( \beta(j) \in [0, 1] \), and hence there exist \( \beta(j_1), \beta(j_2) \) such that \( \beta(j_2) - \beta(j_1) \geq 1/(3m) \) and there is no \( \beta(j) \) lying between \( \beta(j_1) \) and \( \beta(j_2) \). It follows that the strip

\[ S_0 = \{ s : \beta(j_1) + 1/(8m) \leq \Re s \leq \beta(j_2) - 1/(8m) \} \]

is contained in \( C(m) \). Consequently, for all \( n = 0, 1, 2, \ldots \), the strips

\[ S_n = \left\{ s : -n + \beta(j_1) + \frac{1}{8m} \leq \Re s \leq -n + \beta(j_2) - \frac{1}{8m} \right\} \quad (5.10) \]

are subsets of \( C(m) \). In [32], §4, Liu and Ye studied distribution of zeros of the Rankin-Selberg \( L \)-function \( L(s, \pi \otimes \pi') \), where \( \pi \) and \( \pi' \) are irreducible unitary cuspidal representations of \( GL_m(\mathbb{A}_\mathbb{Q}) \) and \( GL_{m'}(\mathbb{A}_\mathbb{Q}) \), respectively. This structure of \( C(m) \) is a special case of the \( C(m, m') \) in [32], §4.

The following Lemma 5.5(i) and (ii) are Lemma 4.3(d) and Lemma 4.4 of [32], respectively.

**Lemma 5.5.** Let \( \pi \) be an irreducible unitary cuspidal representation of \( GL_m(\mathbb{A}_\mathbb{Q}) \) with \( m \geq 2 \). Then

(i) For \( |T| > 2 \), there exists \( \tau \) with \( T \leq \tau \leq T + 1 \) such that when \( -2 \leq \sigma \leq 2 \),

\[ \frac{d}{ds} \log L(\sigma \pm i\tau, \pi) \ll \log^2(Q_\pi|\tau|). \]

(ii) If \( s \) is in some strip \( S_n \) as in (5.10) with \( n \leq -2 \), then

\[ \frac{d}{ds} \log L(s, \pi) \ll 1. \]
5.3 An explicit formula

The explicit formula given in Theorem 5.6 below is unconditional; it requires neither GRH nor GRC.

**Theorem 5.6.** Let \( m \geq 2 \) be an integer and let \( \pi \) be an irreducible unitary cuspidal representation of \( \text{GL}_m(\mathbb{A}_\mathbb{Q}) \). Then, for \( x \geq 2 \) and \( T \geq 2 \),

\[
\psi(x, \pi) = -\sum_{|\gamma| \leq T} x^\rho + O\left( \frac{x}{T^{1/4}} \frac{x^{1+\theta}}{T^{1/2}} \log(Q_\pi x) \right) + O(x^\theta \log x) + O\left( \frac{x \log^2(Q_\pi x)}{T^{1/2}} \right),
\]

where \( \theta \) is as in Lemma 4.8.

We will establish Theorem 5.6 at the end of this section. Explicit formulas of different forms were established by Moreno ([40], [41]); under GRC, explicit formulas for general \( L \)-functions were proved in (5.53) of Iwaniec and Kowalski [14].

Using Theorem 5.6, we will by the way give a proof of the prime number theorem for \( \psi(x, \pi) \) under GRH.

**Corollary 5.7.** Let \( m \geq 2 \) be an integer and let \( \pi \) be an irreducible unitary cuspidal representation of \( \text{GL}_m(\mathbb{A}_\mathbb{Q}) \), and assume GRH for \( L(s, \pi) \). Then (5.4) holds.

**Proof.** Theorem 5.6 with \( T = x^8 \) gives

\[
\psi(x, \pi) = -\sum_{|\gamma| \leq T} x^\rho + O(x^\theta \log x) + O\left( \frac{x \log^2(Q_\pi x)}{x} \right).
\]

By Lemma 4.8, the error term is acceptable in (5.4). Under GRH for \( L(s, \pi) \), we have \( \rho = 1/2 + i\gamma \) in the formula above, and therefore, by partial summation and Lemma 4.6,

\[
\sum_{|\gamma| \leq T} x^\rho \ll x^{1/2} \left( \sum_{|\gamma| \leq 1} 1 + \sum_{1 \leq |\gamma| \leq T} \frac{1}{|\gamma|} \right) \ll x^{1/2} \left( \log Q_\pi + \int_1^T \frac{1}{t} dN(t, \pi) \right) \ll x^{1/2} \log^2(Q_\pi T).
\]

This proves (5.4). \( \square \)
5.3 An explicit formula

The following form of Perron’s formula will be needed in the proof of Theorem 5.6.

**Lemma 5.8.** (Perron’s formula). Under the assumption of Lemma 3.11, we have, for \( b > \sigma, x \geq 2, T \geq 2, \)
\[
\sum_{n \leq x} a_n = \frac{1}{2\pi i} \int_{b-iT}^{b+iT} F(s) \frac{x^s}{s} ds + O\left( \sum_{|n-x| \leq x/\sqrt{T}} |a_n| + \frac{x^B(b)}{\sqrt{T}} \right). \tag{5.11}
\]

A key feature of Lemma 5.8 is that individual upper bound for \( a_n \) does not appear on the right-hand side, and this makes Theorem 5.6, and hence Theorems 5.1-5.3, independent of GR C. Perron’s formula of this nature was successfully applied in classical cases where \( a_n \) is not bounded by 1. The specific form of Perron’s formula in Lemma 5.8, though not optimal, must have been known to the expert for some time. It follows from Tenenbaum [54], Theorem II.2.2, for example. See also Liu and Ye [33] for a proof and some applications to automorphic \( L \)-functions.

**Lemma 5.9.** (Iwaniec-Kowalski [14]). Let \( \pi \) be an irreducible unitary cuspidal representation of \( GL_m(\mathbb{A}_\mathbb{Q}) \) with \( m \geq 2 \). Then
\[
\sum_{n \leq x} |\Lambda(n)a_\pi(n)|^2 \ll m^2 x \log^2(Q_\pi x),
\]
where the implied constant is absolute.

This is (5.48) in [14], and proved in the lower part on page 110 of [14].

**Proof of Theorem 5.6.** In view of (4.6) and Lemma 4.1, we can apply Lemma 5.8 with \( \sigma = 1, b = 1 + 1/\log x, \) and
\[
F(s) = \frac{d}{ds} \log L(s, \pi) = -\sum_{n=1}^{\infty} \frac{\Lambda(n)a_\pi(n)}{n^s},
\]
that is \( a_n = -\Lambda(n)a_\pi(n). \)

To estimate the first \( O \)-term in (5.11), we let \( 0 < y \leq x, \) and consider
\[
\sum_{x < n \leq x+y} |\Lambda(n)a_\pi(n)| \ll \left\{ \sum_{n \leq 2x} |\Lambda(n)a_\pi(n)|^2 \right\}^{1/2} \left\{ \sum_{x < n \leq x+y} 1 \right\}^{1/2} \ll \sqrt{x(y+1)} \log(Q_\pi x).
\]
On the other hand, by (5.2) and the bound toward GR C in Lemma 4.8, for \( n = p^k, \)
\[
|a_\pi(n)| = |a_\pi(p^k)| \leq \sum_{j=1}^{m} |\alpha_\pi(p, j)|^k \leq mp^{k\theta} \leq mn^{\theta}.
\]
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Therefore, trivially,

$$\sum_{x < n \leq x + y} |\Lambda(n)a_{\pi}(n)| \ll x^\theta(y + 1) \log x.$$ 

Hence,

$$\sum_{|n-x| \leq x/\sqrt{T}} |\Lambda(n)a_{\pi}(n)|$$

$$\ll \min\left\{ \sqrt{x\left(\frac{x}{T^{1/2}} + 1\right)} \log(Q_{\pi}x), \ x^\theta \left(\frac{x}{T^{1/2}} + 1\right) \log x \right\}$$

$$\ll \min\left( \frac{x}{T^{1/4}}, \frac{x^{1+\theta}}{T^{1/2}} \right) \log(Q_{\pi}x) + x^\theta \log x. \quad (5.12)$$

In the last step, we have considered the two cases $T \leq x^2$ and $T > x^2$ separately. The other $O$-term in (5.11) depends on $B(\sigma)$. For $\sigma > 1$, Cauchy's inequality gives

$$B(\sigma) = \sum_{n=1}^{\infty} \frac{|\Lambda(n)a_{\pi}(n)|}{n^\sigma} \ll \left\{ \sum_{n=1}^{\infty} \frac{|\Lambda(n)a_{\pi}(n)|^2}{n^\sigma} \right\}^{1/2} \left\{ \sum_{n=1}^{\infty} \frac{1}{n^\sigma} \right\}^{1/2}.$$ 

By Lemma 5.9,

$$\sum_{n=1}^{\infty} \frac{|\Lambda(n)a_{\pi}(n)|^2}{n^\sigma} = \int_{1}^{\infty} \frac{1}{u^\sigma} d\left\{ \sum_{n \leq u} |\Lambda(n)a_{\pi}(n)|^2 \right\}$$

$$\ll \log^2 Q_{\pi} + \int_{1}^{\infty} \frac{\log^2(Q_{\pi}u)}{u^\sigma} du$$

$$\ll \frac{\log^2 Q_{\pi}}{\sigma - 1} + \frac{1}{(\sigma - 1)^3}.$$ 

Similarly but more easily, we have

$$\sum_{n=1}^{\infty} \frac{1}{n^\sigma} \ll \frac{1}{\sigma - 1},$$

and consequently,

$$B(\sigma) \ll \frac{\log Q_{\pi}}{\sigma - 1} + \frac{1}{(\sigma - 1)^2}.$$ 

Therefore, the second $O$-term in (5.11) is

$$\ll \frac{x(\log x)(\log Q_{\pi}x)}{\sqrt{T}}. \quad (5.13)$$
Inserting (5.13) and (5.12) into (5.11), we get

\[
\sum_{n \leq x} \Lambda(n) a_{\pi}(n) = \frac{1}{2\pi i} \int_{b-iT}^{b+iT} \left\{ -\frac{L'}{L}(s, \pi) \right\} \frac{x^s}{s} ds
\]

\[+ O \left\{ \min \left( \frac{x}{T^{1/4}}, \frac{x^{1+\theta}}{T^{1/2}} \right) \log(Q_{\pi} x) \right\}
\]

\[+ O \left( \frac{x(\log x)(\log Q_{\pi} x)}{\sqrt{T}} \right) + O(x^\theta \log x). \]  

(5.14)

Next, we shall shift the contour of integration to the left. Choose \( a \) with \(-2 < a < -1\) such that the vertical line \( \sigma = a \) is contained in the strip \( S_{-2} \subset \mathbb{C}(m) \); this is guaranteed by the structure of \( \mathbb{C}(m) \). Without loss of generality, let \( T > 0 \) be a large number such that \( T \) and \(-T\) can be taken as the \( \tau \) in Lemma 5.5(i). Now we consider the contour \( C_1 \cup C_2 \cup C_3 \) with

\[
C_1 : \quad b \geq \sigma \geq a, \quad t = -T;
\]

\[
C_2 : \quad \sigma = a, \quad -T \leq t \leq T;
\]

\[
C_3 : \quad a \leq \sigma \leq b, \quad t = T.
\]

By Lemma 5.4, certain nontrivial zeros \( \rho = \beta + i\gamma \) and trivial zeros \( \mu = \lambda + i\nu \) of \( L(s, \pi) \), as well as \( s = 0 \) are passed by the shifting of the contour. Computing the residues by Lemma 5.4, we have

\[
\frac{1}{2\pi i} \int_{b-iT}^{b+iT} \left\{ -\frac{L'}{L}(s, \pi) \right\} \frac{x^s}{s} ds = \frac{1}{2\pi i} \left( \int_{C_1} + \int_{C_2} + \int_{C_3} \right)
\]

\[= \sum_{\gamma \leq T} \frac{x^\rho}{\rho} - \sum_{a < -\lambda < b} \frac{x^{-\mu}}{-\mu} - \frac{L'(0, \pi)}{L(0, \pi)}. \]  

(5.15)

The integral on \( C_1 \) can be estimated by Lemma 5.5(i) as

\[
\frac{1}{2\pi i} \int_{C_1} \left\{ -\frac{L'}{L}(s, \pi) \right\} \frac{x^s}{s} ds \ll \int_a^b \log^2(Q_{\pi} T) \frac{x^\sigma}{T} d\sigma
\]

\[\ll \frac{x \log^2(Q_{\pi} T)}{T}, \]

and the same upper bound also holds for the integral on \( C_3 \). By Lemma 5.5(ii), then

\[
\frac{1}{2\pi i} \int_{C_2} \left\{ -\frac{L'}{L}(s, \pi) \right\} \frac{x^s}{s} ds \ll \int_{-T}^T \frac{x^a}{|t| + 1} dt
\]

\[\ll x^a \log T. \]
To bound the contribution from the trivial zeros $\mu = \lambda + i\nu$, we apply Lemma 4.8, so that

$$\sum_{a < -\lambda < b \atop |\nu| \leq T} \frac{x^{-\mu}}{\mu} \ll x^\theta,$$

where we have used the fact that there are finite number of trivial zeros $\mu = \lambda + i\nu$ with $a < -\lambda < b, |\nu| \leq T$. Therefore, (5.15) becomes

$$\frac{1}{2\pi i} \int_{b-iT}^{b+iT} \left\{ -\frac{L'}{L}(s,\pi) \right\} \frac{x^s}{s} ds = -\sum_{|\gamma| \leq T} \frac{x^\rho}{\rho} + O(x^\theta) + O\left(\frac{x \log^2(Q \pi T)}{T}\right) + \left(\frac{\log T}{x}\right).$$

Theorem 5.6 then follows from this and (5.14). \qed

### 5.4 Proof of an almost result

The following lemma is necessary for Theorem 5.1.

**Lemma 5.10.** Let $m \geq 2$ be an integer and let $\pi$ be an irreducible unitary cuspidal representation of $GL_m(\mathbb{A}_\mathbb{Q})$. Assuming GRH for $L(s,\pi)$, we have

$$\int_X^{eX} \left| \sum_{T < |\gamma| \leq X^4} \frac{x^\rho}{\rho} \right|^2 dx \ll \frac{\log^2(Q \pi T)}{T},$$

for $4 \leq T \leq X^4$.

To prove Lemma 5.10, we need the following lemma of Gallagher [7].

**Lemma 5.11.** (Gallagher [7]). Let

$$S(u) = \sum_{\nu} c(\nu)e^{2\pi i u \nu}$$

be absolutely convergent, where the coefficients $c(\nu) \in \mathbb{C}$, and the frequencies of $\nu$ run over an arbitrary sequence of real numbers. Then

$$\int_{-U}^{U} |S(u)|^2 du \ll \frac{1}{U^2} \int_{-\infty}^{\infty} \left| \sum_{x < \nu \leq x+1/U} c(\nu) \right|^2 dx.$$
Proof of Lemma 5.10. In the integral in (5.16), we change variables \( x = Xe^{2\pi u} \). By GRH, we have \( \rho = 1/2 + i\gamma \), and therefore

\[
\int_{X}^{eX} \left| \sum_{T<|\gamma|\leq X^4} \frac{x^\rho}{\rho} \right|^2 \frac{dx}{x^2} = 2\pi \int_{0}^{1/(2\pi)} \left| \sum_{T<|\gamma|\leq X^4} \frac{X^\gamma}{\rho} e^{2\pi i\gamma u} \right|^2 du \ll \int_{0}^{1} \left| \sum_{T<|\gamma|\leq X^4} \frac{X^\gamma}{\rho} e^{2\pi i\gamma u} \right|^2 du. \tag{5.17}
\]

By Gallagher’s lemma, the last integral can be estimated as

\[
\ll \int_{-\infty}^{\infty} \left| \sum_{T<|\gamma|\leq X^4, \ t<\gamma\leq t+1} \frac{1}{|\rho|} \right|^2 dt.
\]

In the last integral, \( t \) should satisfy either \( T - 1 \leq t \leq X^4 \) or \( -X^4 - 1 \leq t \leq -T \). By this and Lemma 4.6,

\[
\int_{-\infty}^{\infty} \left\{ \sum_{T<|\gamma|\leq X^4, \ t<\gamma\leq t+1} \frac{1}{|\rho|} \right\}^2 dt \ll \int_{T-1}^{X^4+1} \left\{ \sum_{t<\gamma\leq t+1} \frac{1}{|\rho|} \right\}^2 dt \\
\ll \int_{T-1}^{X^4+1} \frac{\log^2(Q\pi t)}{t^2} dt \\
\ll \frac{\log^2(Q\pi T)}{T}.
\]

This proves the lemma. \( \square \)

Now a proof of Theorem 5.1 is immediate.

Proof of Theorem 5.1. Let \( 2 \leq X \leq x \ll X \), and take \( T = X^4 \) in the explicit formula (Theorem 5.6). Then

\[
\psi(x, \pi) = - \sum_{|\gamma|\leq X^4} \frac{x^\rho}{\rho} + O(x^\rho \log x) + O\left(\frac{\log^2(Q\pi x)}{x}\right). \tag{5.18}
\]
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Note that (5.18) holds unconditionally; while on GRH, the sum then runs over the nontrivial zeros $\rho = 1/2 + i\gamma$ of $L(s, \pi)$ with $|\gamma|$ up to $X^4$.

To prove Theorem 5.1, we split the sum over $|\gamma| \leq T$, with $2 \leq T \leq X^4$ a parameter that will be specified later.

First, we have

$$\sum_{|\gamma| \leq T} \frac{x^\rho}{\rho} \ll x^{1/2} \sum_{|\gamma| \leq T} \frac{1}{|\rho|} \ll x^{1/2} \left\{ \log Q_\pi + \int_1^T \frac{1}{t} dN(t, \pi) \right\} \ll x^{1/2} \log^2(Q_\pi T). \quad (5.19)$$

This inequality asserts that, if $T$ is small, then the contribution to (5.18) from $|\gamma| \leq T$ is also small.

However, the contribution to (5.18) from $T < |\gamma| \leq X^4$ is not always small; we shall show that it is usually small. To this end, define

$$E(X) = \left\{ x \in [X, eX] : \left| \sum_{T < |\gamma| \leq X^4} \frac{x^\rho}{\rho} \right| \geq x^{1/2} \log^2(Q_\pi \log x) \right\}.$$

From this and Lemma 5.10, we deduce that

$$\log^4(Q_\pi \log X) \int_{E(X)} \frac{dx}{x} \ll \int_{E(X)} \left| \sum_{T < |\gamma| \leq X^4} \frac{x^\rho}{\rho} \right|^2 \frac{dx}{x^2} \ll \frac{\log^2(Q_\pi T)}{T},$$

and hence

$$\int_{E(X)} \frac{dx}{x} \ll \frac{\log^2(Q_\pi T)}{T \log^4(Q_\pi \log X)}. \quad (5.20)$$

Now specify

$$T = \log X,$$

and insert (5.19) into (5.18). Then we see from (5.20) that

$$\psi(x, \pi) \ll x^{1/2} \log^2(Q_\pi \log x)$$
5.5 Mean value and Omega estimates for $\psi(x, \pi)$

holds on the interval $[X, eX]$ except on the set $E(X)$ of logarithmic measure

$$\ll \frac{1}{T \log^2(Q\pi T)} \ll \frac{1}{T \log^2 T}.$$ 

By choosing $X = e^T$ with $T = 2, 3, \ldots$, the total logarithmic measure of the exceptional set $E$ will be

$$\ll \sum_{T=2}^{\infty} \frac{1}{T \log^2 T} < \infty,$$

and Theorem 5.1 follows. □

5.5 Mean value and Omega estimates for $\psi(x, \pi)$

In this section, we prove Theorems 5.2 and 5.3.

Proof of Theorem 5.2. By the explicit formula (5.18),

$$\int_{X}^{eX} |\psi(x, \pi)|^2 \frac{dx}{x^2} \ll \int_{X}^{eX} \left| \sum_{|\gamma| \leq X^4} \frac{x^\rho}{\rho} \right|^2 \frac{dx}{x^2} + \int_{X}^{eX} x^{2\theta} \log^2(Q\pi x) \frac{dx}{x^2} + \int_{X}^{eX} \log^2(Q\pi x) \frac{dx}{x}.$$  (5.21)

The last two integrals are $\ll \log^2 Q\pi$. To estimate the first integral on the right-hand side, we take $T = 4$ in Lemma 5.10, getting

$$\int_{X}^{eX} \left| \sum_{|\gamma| \leq X^4} \frac{x^\rho}{\rho} \right|^2 \frac{dx}{x^2} \ll \log^2 Q\pi + \int_{X}^{eX} \left| \sum_{4 < |\gamma| \leq X^4} \frac{x^\rho}{\rho} \right|^2 \frac{dx}{x^2}$$

$$\ll \log^2 Q\pi. \quad (5.22)$$

$$\ll \log^2 Q\pi. \quad (5.23)$$

Therefore, (5.21) becomes

$$\int_{X}^{eX} |\psi(x, \pi)|^2 \frac{dx}{x^2} \ll \log^2 Q\pi,$$

(5.24)

and consequently,

$$\int_{X}^{eX} |\psi(x, \pi)|^2 \frac{dx}{x^2} \ll X \log^2 Q\pi.$$
A splitting-up argument then yields
\[
\int_2^X |\psi(x, \pi)| \frac{dx}{x} = \int_{X/e}^X |\psi(x, \pi)| \frac{dx}{x} + \int_{X/e^2}^{X/e} |\psi(x, \pi)|^2 \frac{dx}{x} + \cdots \\
\ll \frac{X \log^2 Q_\pi}{e} + X \log^2 Q_\pi + \cdots \\
\ll X \log^2 Q_\pi.
\]
This proves Theorem 5.2. \(\square\)

**Proof of Theorem 5.3.** To prove the assertion of the theorem, we assume
\[
\psi(x, \pi) \ll x^\alpha, \tag{5.25}
\]
where \(\alpha\) is some positive absolute constant, and the \(\ll\)-constant depends at most on \(m\) and \(\alpha\). We establish the assertion by finding a contradiction whenever \(\alpha < 1/2\).

Applying partial summation to (5.1), we get from the definition of \(\psi(x, \pi)\) that, for \(\sigma > 1\),
\[
-\frac{L'(s, \pi)}{L(s, \pi)} = s \int_1^\infty \psi(x, \pi) \frac{x}{x^{s+1}} dx. \tag{5.26}
\]
Inserting (5.25), we have
\[
\frac{\psi(x, \pi)}{x^{s+1}} \ll \frac{x^\alpha}{x^{\sigma+1}} \ll \frac{1}{x^{\sigma+1}-\alpha}.
\]
Therefore, for \(\sigma > \alpha + \varepsilon\) with \(\varepsilon > 0\) arbitrary, the integral on the right-hand side of (5.26) is uniformly convergent, and so represents a regular function in the half-plane \(\sigma > \alpha\). It follows from (5.26) that \(L(s, \pi)\) cannot have a zero in the half-plane \(\sigma > \alpha\). This will lead to a contradiction if \(\alpha < 1/2\). \(\square\)

### 5.6 A Linnik-type problem for \(\{a_\pi(n)\Lambda(n)\}_{n=1}^\infty\)

To each irreducible unitary cuspidal representation \(\pi = \otimes \pi_p\) of \(GL_m(\mathbb{A}_\mathbb{Q})\), one can attach a global \(L\)-function \(L(s, \pi)\) as in §4.1. Taking logarithmic differentiation in (4.2), one gets (5.1) with Dirichlet coefficients
\[
\{a_\pi(n)\Lambda(n)\}_{n=1}^\infty,
\]
5.6 A Linnik-type problem for \( \{a_\pi(n)\Lambda(n)\}_{n=1}^\infty \)

where \( \Lambda(n) \) is the von Mangoldt function, and \( a_\pi(p^k) \) as in (5.2), i.e.

\[
a_\pi(p^k) = \sum_{j=1}^m \alpha_\pi(p,j)^k.
\]

(5.27)

If \( \pi \) is self-contragredient, then (4.9) states that

\[
\{\alpha_\pi(p,j)\}_{j=1}^m \overset{m}{=}_{j=1} \{\alpha_\pi(p,j)\}_{j=1}^m,
\]

and hence, by (5.27),

\[
a_\pi(p^k) = a_\pi(p^k),
\]

(5.28)

which means that \( a_\pi(n)\Lambda(n) \) are real for all \( n \geq 1 \).

The purpose of this section is to establish the following Linnik-type theorem for the sequence \( \{a_\pi(n)\Lambda(n)\}_{n=1}^\infty \). Actually, this is a corollary to Theorem 4.15 and Lü’s lemma (Lemma 4.11).

**Theorem 5.12.** Let \( m \geq 2 \) be an integer and let \( \pi \) be an irreducible unitary cuspidal representation for \( \text{GL}_m(k_Q) \). If all \( a_\pi(n)\Lambda(n) \) are real, then \( \{a_\pi(n)\Lambda(n)\}_{n=1}^\infty \) changes sign at some \( n \) satisfying

\[
n \ll Q_\pi^{m/2 + \varepsilon}.
\]

(5.29)

The constant implied in (5.29) depends only on \( m \) and \( \varepsilon \). In particular, the result is true for any self-contragredient representation \( \pi \).

**Proof.** If we abbreviate \( \alpha_\pi(p,j) \) to \( \alpha_j \), then (5.27) takes the form

\[
a_\pi(p^k) = \sum_{j=1}^m \alpha_j^k.
\]

(5.30)

The key observation is that (5.30) is exactly the same as that defined in (4.33), and Lemma 4.11 is applicable. Thus, as in (4.36), we have, for all \( k \geq 1 \),

\[
k\lambda_\pi(p^k) = a_\pi(p)\lambda_\pi(p^{k-1}) + a_\pi(p^2)\lambda_\pi(p^{k-2}) + \cdots
\]

\[
+ a_\pi(p^{k-1})\lambda_\pi(p) + a_\pi(p^k),
\]

(5.31)

where \( \lambda_\pi(p^k) \) is as in (4.7). By induction on \( k \), we show that if \( a_\pi(p^k) \geq 0 \) for all \( k \leq K \), then \( \lambda_\pi(p^k) \geq 0 \) also for all \( k \leq K \).
Now we invoke Theorem 4.15, to deduce that there is an \( n \) with

\[
n \ll Q_\pi^{m/2+\epsilon}
\]

such that \( \lambda_\pi(n) < 0 \). By (4.7), we see that \( \lambda_\pi(n) \) is multiplicative with respect to \( n \), and therefore there must be a power \( p_0^{k_0} \) of a prime \( p_0 \) with

\[
p_0^{k_0} \ll Q_\pi^{m/2+\epsilon}
\]

such that \( \lambda_\pi(p_0^{k_0}) < 0 \). Thus, there must be some \( k_1 \leq k_0 \) such that \( a_\pi(p_0^{k_1}) < 0 \). This proves the theorem.

\[ \square \]
Chapter 6

Selberg’s normal density theorem for automorphic $L$-functions

6.1 Selberg’s normal density theorem

Write, as usual,

$$\psi(x) = \sum_{n \leq x} \Lambda(n).$$

It is known that, under the Riemann Hypothesis for the zeta-function,

$$\psi(x) = x + O(x^{1/2} \log^2 x).$$

From this, a result for primes in short intervals of the form $[x, x + y]$ will follow.

Selberg [47] studied the normal density of primes in short interval. Under the Riemann Hypothesis for the zeta-function, i.e. in the case of $m = 1$, Selberg [47] proved that

$$\int_1^X \{\psi(x + h(x)) - \psi(x) - h(x)\}^2 dx = o(h(X)^2 X)$$  \hspace{1cm} (6.1)

for any increasing functions $h(x) \leq x$ with

$$\frac{h(x)}{\log^2 x} \to \infty.$$

In Chapter 6, we prove an analogue of this in the case of automorphic $L$-functions attached to irreducible unitary cuspidal representation $\pi = \otimes_p \pi_p$ of $GL_m(\mathbb{A}_\mathbb{Q})$. 

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6.2 Selberg’s normal density theorem for automorphic $L$-functions

To each irreducible unitary cuspidal representation $\pi = \otimes \pi_p$ of $GL_m(\mathbb{A}_\mathbb{Q})$, one can attach a global $L$-function $L(s, \pi)$ as in §4.1. Let notation be as in the previous chapter. The prime number theorem for $L(s, \pi)$ concerns the asymptotic behavior of the counting function

$$\psi(x, \pi) = \sum_{n \leq x} \Lambda(n)a_\pi(n),$$

and a special case of it asserts that, if $\pi$ is an irreducible unitary cuspidal representation of $GL_m(\mathbb{A}_\mathbb{Q})$ with $m \geq 2$, then

$$\psi(x, \pi) \ll \sqrt{Q_\pi} \cdot x \cdot \exp\left(-\frac{c}{2m} \sqrt{\log x}\right)$$

(6.2)

for some absolute positive constant $c$, where the implied constant is absolute. In Iwaniec and Kowalski [14], Theorem 5.13, a prime number theorem is proved for general $L$-functions satisfying necessary axioms, from which (6.2) follows as a consequence. Under GRH, (6.2) can be improved to

$$\psi(x, \pi) \ll x^{1/2}\log^2(Q_\pi x).$$

(6.3)

It follows from (6.3) that, under GRH,

$$\psi(x + h(x), \pi) - \psi(x, \pi) = o(h(x))$$

(6.4)

for increasing functions $h(x) \leq x$ satisfying

$$\frac{h(x)}{x^{1/2}\log^2(Q_\pi x)} \to \infty.$$

In view of

$$\psi(x + h(x), \pi) - \psi(x, \pi) = \sum_{x < n \leq x + h(x)} \Lambda(n)a_\pi(n),$$

(6.4) describes oscillation of the coefficients $a_\pi(p)$ in the short intervals $x < p \leq x + h(x)$.

In this Chapter, we will show that (6.4) holds on average for even shorter $h(x)$; see Theorem 6.1 below.
Theorem 6.1. Let $m \geq 2$ be an integer and let $\pi$ be an irreducible unitary cuspidal representation of $GL_m(\mathbb{A}_Q)$. Assume GRH for $L(s, \pi)$. We have

$$\int_1^X |\psi(x + h(x), \pi) - \psi(x, \pi)|^2 dx = o(h(X)^2 X), \quad (6.5)$$

for any increasing functions $h(x) \leq x$ satisfying

$$\frac{h(x)}{\log^2(Q\pi x)} \to \infty.$$

Our Theorem 6.1 generalizes Selberg’s result to cases when $m \geq 2$. It also improves an earlier result of the author [44] that (6.5) holds for $h(x) \leq x$ satisfying

$$\frac{h(x)}{x^\theta \log^2(Q\pi x)} \to \infty,$$

where $\theta$ is the bound towards the GRC as explained in Lemma 4.8. The main new idea is a delicate application of Kowalski-Iwaniec’s mean value estimate (cf. Lemma 5.9). We also need an explicit formula established in Chapter 5 in a more precise form.

Unconditionally, Theorem 6.1 would hold for $h(x) = x^\beta$ with some constant $0 < \beta < 1$. The exact value of $\beta$ depends on two main ingredients: a satisfactory zero-density estimate for the $L$-function $L(s, \pi)$, and a zero-free region for $L(s, \pi)$ of Littlewood’s or Vinogradov’s type.

6.3 Proof of the theorem

We prove Theorem 6.1 in this section. The main tools are the explicit formula in Theorem 5.6 and Lemma 5.10 which is established under GRH.

Proof of Theorem 6.1. The proof of Theorem 5.6 actually gives an alternative form of the explicit formula. Let, as in the proof of Theorem 5.6,

$$-2 < a < -1, \quad b = 1 + \frac{1}{\log x}.$$

Then the proof of Theorem 5.6 actually gives

$$\psi(x, \pi) = - \sum_{|\gamma| \leq T} \frac{x^\rho}{\rho} - \sum_{|\nu| \leq T} \frac{x^{-\mu}}{-\mu} + O\left\{ \sum_{|n-x| \leq x/\sqrt{T}} |\Lambda(n)\alpha_x(n)| \right\}$$

$$+ O\left( \frac{x \log^2(Q\pi x)}{T^{1/2}} \right) + O\left( \frac{\log T}{x} \right), \quad (6.6)$$
Selberg’s normal density theorem for automorphic L-functions

where \( \theta \) is as in Lemma 4.8, and \( \mu \) goes over the trivial zeros \( \mu = \lambda + i\nu \) of \( L(s, \pi) \).

Let \( 100 \leq X \leq x \leq eX \), and take \( T = X^4 \) in the explicit formula (6.6). Since the length of the interval \( (x - x/X^2, x + x/X^2] \) is

\[
\frac{2x}{X^2} \leq \frac{1}{10},
\]

this interval contains at most one integer; we denote this possible integer by \( n_x \). It follows that

\[
\sum_{|n-x| \leq x/X^2} |\Lambda(n)a_{\pi}(n)| = |\Lambda(n_x)a_{\pi}(n_x)|,
\]

and hence (6.6) becomes

\[
\psi(x, \pi) = -\sum_{|\gamma| \leq X^4} \frac{x^\rho}{\rho} - \sum_{a<-\lambda<b, |\nu| \leq X^4} \frac{x^{-\mu}}{-\mu} + O\{|\Lambda(n_x)a_{\pi}(n_x)| + 1\}
\]

\[+ O\left( \frac{\log^2(Q_{\pi}x)}{x} \right), \quad (6.7)\]

Note that (6.7) holds unconditionally. On GRH, the sum then runs over the non-trivial zeros \( \rho = 1/2 + i\gamma \) of \( L(s, \pi) \) with \( |\gamma| \) up to \( X^4 \). It follows that

\[
\psi(x + h, \pi) - \psi(x, \pi)
\]

\[= -\sum_{|\gamma| \leq X^4} \frac{(x + h)^\rho - x^\rho}{\rho} - \sum_{a<-\lambda<b, |\nu| \leq X^4} \frac{(x + h)^{-\mu} - x^{-\mu}}{-\mu}
\]

\[+ O\{|\Lambda(n_{x+h})a_{\pi}(n_{x+h})| + |\Lambda(n_x)a_{\pi}(n_x)| + 1\} + O\left( \frac{\log^2(Q_{\pi}x)}{x} \right)
\]

\[= A + B + C + O\left( \frac{\log^2(Q_{\pi}x)}{x} \right), \quad (6.8)\]

say. We start our proof of Theorem 6.1 by estimating the mean value of (6.8) within \( X \leq x \leq eX \), while \( h \) (\( \leq eX \)) is the length of the interval under consideration. We are interested in how short \( h \) can be.

We start from \( A \). In \( A \), we split the sum over \( |\gamma| \) at \( T \), with \( 4 \leq T \leq X^4 \) a parameter that will be specified later, and denote

\[
S_1(y, \pi) = \sum_{|\gamma| \leq T} y^{i\gamma},
\]
6.3 Proof of the theorem

and

\[ S_2(y, \pi) = \sum_{T<|\gamma| \leq X^4} \frac{y^{i\gamma}}{\rho}. \]

Then GRH asserts that

\[
A = \left\{ -\sum_{|\gamma| \leq T} -\sum_{T<|\gamma| \leq X^4} \right\} \frac{(x+h)^\rho - x^\rho}{\rho}
\]

\[= - \int_x^{x+h} S_1(y, \pi) \frac{dy}{y^{1/2}} + x^{1/2} S_2(x, \pi) - (x+h)^{1/2} S_2(x+h, \pi) \]

\[=: A_1 + A_2 + A_3, \]

say. Hence,

\[
\int_X^{eX} |A|^2 dx \ll \int_X^{eX} |A_1|^2 dx + \int_X^{eX} |A_2|^2 dx + \int_X^{eX} |A_3|^2 dx. \quad (6.9)
\]

By Cauchy’s inequality,

\[
|A_1|^2 \ll \int_x^{x+h} |S_1(y, \pi)|^2 \frac{dy}{y} \int_x^{x+h} 1^2 dy
\]

\[= h \int_x^{x+h} |S_1(y, \pi)|^2 \frac{dy}{y}. \]

We note that the upper bound for \( h \) is \( eX \). Therefore, the contribution from \( |A_1|^2 \) is estimated as

\[
\int_X^{eX} |A_1|^2 dx \ll \int_X^{eX} h \int_x^{x+h} |S_1(y, \pi)|^2 \frac{dy}{y} dx
\]

\[\ll \int_X^{2eX} h^2 |S_1(y, \pi)|^2 \frac{dy}{y}
\]

\[= h^2 \int_X^{2eX} \sum_{|\gamma| \leq T} y^{i\gamma} \frac{dy}{y}
\]

\[= h^2 \int_0^{\log(2e)} \left| \sum_{|\gamma| \leq T} e^{i\gamma(u+\log X)} \right|^2 du, \]
where in the last equality we have changed variables \( y = e^{u + \log X} \). An application of Gallagher’s lemma and Lemma 4.6 to the last integral now leads to

\[
\int_0^{\log(2e)} \left| \sum_{|\gamma| \leq T} e^{i\gamma(u + \log X)} \right|^2 du \ll \int_{-\infty}^\infty \left| \sum_{|\gamma| \leq T \atop t < \gamma \leq t + 1} 1 \right|^2 dt \\
\ll \int_0^T \left\{ \sum_{t < \gamma \leq t + 1} 1 \right\}^2 dt \\
\ll T \log^2 (Q\pi T).
\]

Thus, we have

\[
\int_X^{eX} |A_1|^2 dx \ll h^2 T \log^2 (Q\pi T). \tag{6.10}
\]

The contribution from \( |A_2|^2 \) can be estimated as

\[
\int_X^{eX} |A_2|^2 dx \ll X^2 \int_X^{eX} |S_2(x, \pi)|^2 \frac{dx}{x} \\
= X^2 \int_X^{eX} \left| \sum_{T < |\gamma| \leq X^4} \frac{x^{i\gamma}}{\rho} \right|^2 \frac{dx}{x} \\
\ll \frac{X^2 \log^2 (Q\pi T)}{T}, \tag{6.11}
\]

as a consequence of Lemma 5.10 and GRH. Similarly, after taking \( x + h = y \), we have

\[
\int_X^{eX} |A_3|^2 dx \ll \frac{X^2 \log^2 (Q\pi T)}{T}. \tag{6.12}
\]

We conclude from (6.9)-(6.12) that

\[
\int_X^{eX} |A|^2 dx \ll h^2 T \log^2 (Q\pi T) + \frac{X^2 \log^2 (Q\pi T)}{T}. \tag{6.13}
\]
6.3 Proof of the theorem

For the mean-value of \(|B|^2\), we apply Lemma 4.8, to get

\[
\int_X e^x |B|^2 \, dx = \int_X e^x \left| \sum_{a < -\lambda < b \atop \nu \leq X^4} (x + h) - \mu - x - \mu \right|^2 \, dx
\]

\[\ll \int_X \left\{ \sum_{a < -\lambda < b \atop \nu \leq X^4} x^{-\lambda-1} h \right\}^2 \, dx \int_X (x^{\theta-1} h)^2 \, dx
\]

\[\ll X^{2\theta-1} h^2 \ll h^2. \tag{6.14}\]

It remains to estimate the contribution of \(|C|^2\). We have

\[
\int_X e^x |C|^2 \, dx = \int_X e^x \left\{ |\Lambda(n_{x+h}) a_{\pi}(n_{x+h})| + |\Lambda(n_{x}) a_{\pi}(n_{x})| + 1 \right\}^2 \, dx
\]

\[\ll \int_X e^x \left\{ |\Lambda(n_{x+h}) a_{\pi}(n_{x+h})|^2 + |\Lambda(n_{x}) a_{\pi}(n_{x})|^2 \right\} \, dx + X
\]

\[\ll \sum_{j=\lfloor x \rfloor}^{\lfloor e^x \rfloor} \int_j^{j+1} \left\{ |\Lambda(n_{x+h}) a_{\pi}(n_{x+h})|^2 + |\Lambda(n_{x}) a_{\pi}(n_{x})|^2 \right\} \, dx + X.
\]

Since \(h(x)\) is increasing and \(h(x) \leq x\), we have trivially, for \(j \leq x \leq j + 1\), that

\[j - 1 \leq n_{x+h(x)} \leq 2(j+2), \quad j - 1 \leq n_{x} \leq j + 2.
\]

Thus,

\[
\int_X e^x |C|^2 \, dx \ll \sum_{j=\lfloor x \rfloor}^{\lfloor e^x \rfloor} |\Lambda(j) a_{\pi}(j)|^2 + X
\]

\[\ll X \log^2(Q_{\pi} X), \tag{6.15}\]

by applying Lemma 5.9.

Finally we apply (6.13), (6.14), and (6.15) to (6.8), getting

\[
\int_X e^x |\psi(x + h, \pi) - \psi(x, \pi)|^2 \, dx \ll h^2 T \log^2(Q_{\pi} T) + \frac{X^2 \log^2(Q_{\pi} T)}{T}
\]

\[+ X \log^2(Q_{\pi} X) + \frac{\log^4(Q_{\pi} X)}{X}. \tag{6.16}\]

Now we specify the parameter \(T\) by taking

\[h^2 T \log^2(Q_{\pi} T) = \frac{X^2 \log^2(Q_{\pi} T)}{T},\]
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i.e., taking $T = X/h$. Therefore, the right-hand side of (6.16) becomes

$$hX \log^2 (Q_\pi X) + X \log^2 (Q_\pi X) + \frac{\log^4 (Q_\pi X)}{X},$$

which is of order $o(h^2 X)$ as $h \lesssim eX$ and

$$\frac{h}{\log^2 (Q_\pi X)} \to \infty.$$ 

Thus for such $h$, we have

$$\int_X^{eX} |\psi(x + h, \pi) - \psi(x, \pi)|^2 dx = o(h^2 X). \quad (6.17)$$

In general, let $h = h(x)$ be an increasing function of $x$ satisfying $h(x) \lesssim x$ and

$$\frac{h(x)}{\log^2 (Q_\pi x)} \to \infty.$$ 

Then (6.17) gives

$$\int_{X/e}^X |\psi(x + h(x), \pi) - \psi(x, \pi)|^2 dx \ll \int_{X/e}^X |\psi(x + h(X), \pi) - \psi(x, \pi)|^2 dx$$

$$= o \left( h(X)^2 \frac{X}{e} \right).$$

A splitting-up argument then gives

$$\int_1^X |\psi(x + h(x), \pi) - \psi(x, \pi)|^2 x = o(h(X)^2 X),$$

and hence our Theorem 6.1. \qed
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