We explore the covariance of redshift-space matter power spectra after a standard density-field reconstruction. We derive perturbative formula of the covariance at the tree-level order and find that the amplitude of the off-diagonal components from the trispectrum decreases by reconstruction. Using a large set of $N$-body simulations, we also find the similar reduction of the off-diagonal components of the covariance and thereby the signal-to-noise ratio (S/N) of the post-reconstructed (post-rec) power spectra significantly increases compared to the pre-reconstructed spectra. This indicates that the information leaking to higher-order statistics come back to the two-point statistics by reconstruction. Interestingly, the post-rec spectra have higher S/N than the linear spectrum with Gaussian covariance when the scale of reconstruction characterized with the smoothing scale of the shift field is below $\sim 10h^{-1}\text{Mpc}$ where the trispectrum becomes negative. We demonstrate that the error of the growth rate estimated from the monopole and quadrupole components of the redshift-space matter power spectra significantly improves by reconstruction. We also find a similar improvement of the growth rate even when taking into account the super-sample covariance, while the reconstruction cannot correct for the field variation of the super-sample modes.

I. INTRODUCTION

A biggest challenge in the modern cosmology is the mystery of dark matter and dark energy[e.g., 1]. Large-scale structure traced by galaxies is one of powerful cosmological probes to study the properties of the dark components. The baryonic acoustic oscillation (BAO) imprinted on galaxy distributions is a powerful cosmological probe to study the expansion history of the Universe [2–24]. Bulk motion of galaxies associated with the growth of the large-scale structure generates the anisotropy in the redshift-space galaxy distribution, i.e., the redshift-space distortion, which probes the growth rate of the large-scale structure and is useful to test General Relativity and modified gravity models [e.g., 25–31]. The full shape of the power spectrum also has fruitful information of cosmology [e.g., 32]. Upcoming spectroscopic galaxy surveys such as PFS [33], DESI [34], HETDEX [35], Euclid [36] and WFIRST [37] are expected to do precise cosmological studies to clarify the nature of dark matter and dark energy.

In the linear perturbation theory, different wavelength modes of the fluctuations of matter density field grow independently. Since the gravitational growth of the large-scale structure is a nonlinear process, different modes are coupled with each other, which makes a precise cosmological analysis difficult. For example, the BAO signal in the large-scale structure is degraded and the BAO scale is biased at later times [e.g., 38]. The perturbation theory breaks down in the nonlinear regime and thus precise analytical prediction is difficult at small scale. The two-point statistics such as the power spectrum characterize the whole statistical properties in Gaussian fields. However, non-Gaussianity increases in the evolved matter density field and thereby the two-point statistics no longer fully carry the cosmological information in observational data. The information content of the power spectrum are shown to be saturated on nonlinear scale [39–41]. This indicates that the cosmological information leaks to higher-order statistics beyond two-point statistics, which makes our cosmological analysis more complicated.

Density-field reconstruction aims for recovering the initial or linearly evolved density field. A standard BAO reconstruction method proposed by [42] shifts mass particles or galaxies toward their initial Lagrangian positions to recover the original BAO signal. The shift field is estimated using the inverse Zeldovich approximation [43] from the observed (evolved) density field after smoothing small-scale power. The reconstruction effectively undoes the bulk motion and thereby the BAO signal is successfully recovered [44–48]. The BAO reconstruction method has been applied to various galaxy surveys [18–24, 49]. It was also shown that the correlation of the reconstructed matter density field with the initial density field extends to more nonlinear scale [47, 50, 56]. Ref. [57] derived the 1-loop perturbative formula of the reconstructed matter...
II. PERTURBATIVE FORMULA OF COVARIANCE OF MATTER POWER SPECTRA IN REDSHIFT SPACE

In this section, we derive the covariance of the monopole ($\ell = 0$) and quadrupole ($\ell = 2$) components of the redshift-space matter power spectrum in a perturbative approach.

The covariance can be generally decomposed into the Gaussian and the non-Gaussian parts as

\[ \text{Cov} = \text{Cov}^{(G)} + \text{Cov}^{(NG)}. \]

(1)

When neglecting the convolution with the survey geometry, the Gaussian part is given by

\[ \text{Cov}^{(G)}_{\ell \ell'}(k_i, k_j) = \frac{(2\ell + 1)(2\ell' + 1)}{2} \int_{-1}^{1} d\mu \mathcal{L}_\ell(\mu_k) \mathcal{L}_{\ell'}(\mu_{k'}) \frac{2}{V} \frac{(2\pi)^3}{V_k} \delta_{ij} \left[ P^z(k_i) + \frac{1}{n} \right] \left[ P^z(k_j) + \frac{1}{n} \right], \]

(2)

where $\mu_k$ is the cosine angle between $k$ and the line-of-sight direction, $\mathcal{L}_\ell$ is the $\ell$-th Legendre polynomial, i.e., $\mathcal{L}_0 = 1$ and $\mathcal{L}_2 = (3\mu^2 - 1)/2$, $V$ is the sample volume, and $n$ is the number density of mass particles. The volume of $k$-binning shell $V_k$ is approximated as $4\pi k^3 \Delta k$ where the binning width $\Delta k$ is much smaller the mean wavenumber of $i$-th bin $k_i$. The redshift-space linear matter power spectrum $P^z(k)$ is given by

\[ P^z(k) = Z_1(k)^2 P_L(k), \]

(3)

where $P_L(k)$ is the linear matter power spectrum in real space and $Z_1$ is the first-order Eulerian kernel in redshift space

\[ Z_1(k) = 1 + f \mu_k^2, \]

(4)

with the linear growth rate $f = d \ln D/d \ln a$ defined as the logarithmic derivative of the linear growth factor.

Next we derive the non-Gaussian covariance at tree level. We do not take into account the higher-order covariance such as the one-loop covariance for simplicity. The one-loop covariance becomes important at higher $k$, however, the shot noise usually dominates the error at higher $k$ in actual observed data. In section VI, we take into account the super-sample covariance (SSC).

The tree-level covariance comes from the tree-level term of the trispectrum of matter power spectra given by

\[ T^{(\text{tree})}(k_1, k_2, k_3, k_4) = 4[Z_2(k_{12}, -k_1)Z_2(k_{12}, k_3)P_L(k_{12}) \times Z_1(k_1)P_L(k_1)Z_1(k_3)P_L(k_3) + (11 \text{ perms.})] + 6[Z_3(k_1, k_2, k_3)Z_1(k_1)P_L(k_1) \times Z_1(k_2)P_L(k_2)Z_1(k_3)P_L(k_3) + (3 \text{ perms.})] \]

(5)

where $Z_n$ is the $n$-th order Eulerian perturbation kernel of the matter density field in the redshift space [69, 70].

Throughout this paper, we assume a flat $\Lambda$CDM model with the best-fit values of Planck TT,TE,EE+lowP in 2015, i.e., $\Omega_b = 0.0492$, $\Omega_m = 0.3156$, $h = 0.6727$, $n_s = 0.9645$, and $\sigma_8 = 0.831$ [67].

The covariance of the redshift-space matter power spectra after the field is reconstructed. The covariance of matter power spectra has been investigated both from the perturbation theory and from $N$-body simulations [3, 39, 40, 59–65]. Different Fourier modes grow independently in the linear level and thereby the covariance of the power spectrum is diagonal. Mode coupling associated with the nonlinear gravity generates off-diagonal components in the covariance, which leads to the saturation of the information content of the power spectrum. Since the reconstruction effectively linearizes the field by partially removing mode-coupling effect, it is expected that the covariance of the power spectrum is also more diagonalized. Here we first explore how the reconstruction alters the properties of the covariance and affects the information content of the matter power spectrum by using the perturbation theory and also using a large set of $N$-body simulations. We also investigate the impact on the growth rate measurement when using the covariance matrix of reconstructed spectra. The mode coupling between the small-scale modes and the large-scale modes beyond survey size, known as ‘beat coupling’ [59] or ‘super-sample covariance (SSC)’ [60, 66], also has a significant contribution to the covariance. We consider the effect of SSC on the reconstructed covariance.

This paper is organized as follows: in Section II, we derive a tree-level perturbative formula of the covariance of monopole and quadrupole components of matter power spectra after reconstructing the field. We see how the off-diagonal components of their covariance are changed by reconstruction with different smoothing scales. In Section II, we also numerically estimate the covariance using a large set of $N$-body simulations to see the behavior of the covariance of reconstructed spectra. In Section IV, we evaluate the S/N of the reconstructed power spectra to discuss how much the information content is recovered. In Section V, we study the impact on the growth rate measurement by using the covariance of reconstructed spectra. We also consider the effect of the super-sample covariance on our results in Section VI. Section VII is devoted to the summary and conclusions.
The tree-level covariance of the multipole power spectra is written as \[39\] \[65\]

\[
\text{Cov}^{(\text{tree})}(k_i, k_j) = \frac{1}{V} \int_{k_{ij}} \int_{k_{ij}'} T^{(\text{tree})}(\mathbf{k}, -\mathbf{k}', -\mathbf{k}')
\]

\[
= \frac{1}{V} \int_{k_{ij}} \int_{k_{ij}'} \left[ 12Z_3(k, -\mathbf{k}, \mathbf{k}')Z_1(k')^2Z_1(k')P_L(k')^2P_L(k')^2L(k' - k) + 8Z_2(k - \mathbf{k}', \mathbf{k}')Z_2(k')^2P_L(k')^2P_L(k' - k) \right.
\]

\[
+ 8Z_2(\mathbf{k} - \mathbf{k}', \mathbf{k}')Z_2(\mathbf{k}')^2P_L(\mathbf{k}')^2P_L(\mathbf{k}' - \mathbf{k}) \times Z_1(k)P_L(k)Z_1(k')P_L(k') + (\mathbf{k} \leftrightarrow \mathbf{k}'), \tag{6}
\]

The integral denotes

\[
\int_{k_{ij}} = \int_{k_{ij}} \frac{dk}{V_{k_i}} (2\ell + 1) L_\ell(\mu_k). \tag{7}
\]

The tree-level covariance after reconstruction can be obtained by replacing \( Z_n \) with the kernel of post-recon spectra \( Z^{(\text{rec})}_n \) derived by \[58\]. The first-order kernel is not changed by the reconstruction

\[
Z_1^{(\text{rec})}(k) = Z_1(k). \tag{8}
\]

The relation of the 2nd and 3rd-order post-recon kernels to the pre-recon kernels are given as \[58\]

\[
Z_2^{(\text{rec})}(k_1, k_2) = Z_2(k_1, k_2)
+ \frac{1}{2} \left[ (\mathbf{k} \cdot S^{(1)}(k_1))(k_2 \cdot L^{(1)}(k_2)) + (\mathbf{k} \cdot S^{(2)}(k_2))(k_1 \cdot L^{(1)}(k_1)) \right], \tag{9}
\]

and

\[
Z_3^{(\text{rec})}(k_1, k_2, k_3) = Z_3(k_1, k_2, k_3)
+ \frac{1}{6} \left[ 2(\mathbf{k} \cdot S^{(1)}(k_1))Z_2(k_2, k_3)
+ (\mathbf{k} \cdot S^{(2)}(k_2))(k_3 \cdot L^{(1)}(k_3))
+ (\mathbf{k} \cdot S^{(2)}(k_1, k_2))(k_3 \cdot L^{(1)}(k_3)) \right.
\]

\[
+ (\text{2 perms.}) \right], \tag{10}
\]

where \( \mathbf{k} = k_1 + \ldots + k_n \) in the \( n \)-th order kernel. In the above equations, \( L^{(n)} \) represents the \( n \)-th order Lagrangian kernel in redshift space, which is related to the same order of Lagrangian kernel in real space \( \mathbf{L}^{(n)} \) as \[70\]

\[
\mathbf{L}^{(n)} = \mathbf{R}^{(n)}\mathbf{L}^{(n)}. \tag{11}
\]

The redshift-space distortion tensor at \( n \)-th order \( \mathbf{R}^{(n)} \) is given by

\[
\mathbf{R}^{(n)}_{ij} = \delta_{ij} + n f\hat{z}_i\hat{z}_j, \tag{12}
\]

where \( \delta_{ij} \) is the Kronecker delta and \( \hat{z}_i \) the \( i \)-th component of the unit vector in the line-of-sight direction. In a standard reconstruction \[12\], the shift field is estimated from the smoothed density field using the inverse Zeldovich approximation. The \( n \)-th order kernel of the shift field \( S^{(n)} \) is then given by

\[
S^{(n)}(k_1, ..., k_n) = -n!W(k)\mathbf{L}^{(1)}(k)Z_n(k_1, ..., k_n), \tag{13}
\]

where \( W(k) \) is the smoothing kernel and we adopt Gaussian kernel, i.e., \( W(k) = \exp(-k^2 R_s^2/2) \) with different smoothing scales \( R_s = 5h^{-1}\text{Mpc}, 10h^{-1}\text{Mpc} \) and \( 20h^{-1}\text{Mpc} \).

Figure 1 compares the tree-level non-Gaussian covariance of the monopole spectra before and after the reconstruction. The plotted covariance is normalized with their Gaussian components, i.e.,
\[ \text{Cov}_{(\text{tree})}(k_i, k_j)/[\text{Cov}_{(G)}(k_i, k_j) \text{Cov}_{(G)}(k_j, k_j)]^{1/2} \]. The off-diagonal components have positive values for the pre-recon spectra, which means that the different modes are positively correlated by the mode coupling of gravity. After reconstruction, we find that the positive correlation decreases and becomes negative at \( R_s < \) less than 10h\(^{-1}\)Mpc. This comes from that the values of the tree-level trispectra shifts from positive to negative by replacing the perturbative kernels with the reconstructed one. This is related to our previous finding that the amplitudes of the one-loop terms of the power spectrum given by the same perturbative kernels decrease after the reconstruction [58].

### III. N-BODY SIMULATIONS

We measure the covariance of the multipole components of matter power spectra over an ensemble of dark matter N-body simulations as follows:

\[
\text{Cov}_{\ell \ell'} = \frac{1}{N_{\text{real}} - 1} \sum_{i=1}^{N_{\text{real}}} [P_{\ell,i}(k)-P_{\ell}(k)][P_{\ell',i}(k')-P_{\ell'}(k')],
\]

where \( N_{\text{real}} \) is the number of realizations and \( N_{\text{bin}} \) is the total number of \( k \) bins of monopole and quadrupole components. The binning width of \( k \) is uniform with 30 \( \alpha H_0 \) Mpc. We perform N-body simulations using a publicly available code \texttt{Gadget-2} [71]. The initial distribution of mass particles is based on the 2LPT code [72] with Gaussian initial conditions at the input redshift of 31. The initial linear power spectrum is computed by CAMB [74]. We adopt 4000 realizations with 512\(^3\) mass particles in a cubic box with a side length of 500h\(^{-1}\)Mpc and two output redshifts of \( z = 0 \) and \( z = 1.02 \).

The N-body particles are assigned to 512\(^3\) grid cells to calculate the density contrast, and then perform the Fourier transform [75] to measure the multipole components of the power spectra \( P_{\ell} \). The reconstructed density field is computed as follows [42]:

- The shift field for the reconstruction is computed from the smoothed redshift-space mass density field using the inverse Zeldovich approximation, i.e., \( \delta(k) = -k^2 \delta_m(z)(k; R_s) \) where a Gaussian smoothing filter \( W(k; R_s) = \exp(-k^2R_s^2/2) \) at different smoothing scales \( R_s = 20h^{-1}\)Mpc, 10h\(^{-1}\)Mpc and 5h\(^{-1}\)Mpc. Note that we leave the reconstructed field anisotropic on large scales to constrain the growth rate from the anisotropy due to the redshift-space distortion.

- Each mass particle is displaced following the above shift field at the position interpolated from the shift field at neighboring grids with the clouds-in-cell scheme.

- Random particles are also displaced using the same shift vector field in the same manner as the mass particles.

- Reconstructed density field is obtained by the displaced random field subtracted from the displaced data field as \( \delta_{\text{rec}}(x) = \delta_m(x) - \delta_{\text{rec}}(x) \).

Figure 2 shows the correlation matrix of the monopole spectra computed from the simulations at fixed \( k_i = 0.085h/\text{Mpc} \) and 0.175h/\text{Mpc} at \( z = 1.02 \) and 0. Pre-reconstruction spectra are positively correlated among different modes and thus the off-diagonal components are positive [3] [59]. We find that the off-diagonal components substantially decrease to be nearly zero by reconstruction with \( R_s = 10h^{-1}\)Mpc. At \( R_s = 5h^{-1}\)Mpc, the off-diagonal components become negative values. This behavior is qualitatively consistent with the perturbation theory shown in the previous section.

### IV. SIGNAL-TO-NOISE RATIO OF MULTIPLE POWER SPECTRA

In this section, we evaluate the information content of redshift-space matter power spectra from the following signal-to-noise ratio (S/N):

\[
(S/N)^2 = \sum_{\ell, \ell'} \sum_{i,j} P_{\ell}(k_i)(\text{Cov})_{ij\ell\ell}^{-1} P_{\ell}(k_j).
\]

The multipole spectra \( P_{\ell} \) and their covariance are directly estimated from the simulations. The inverse covariance matrix is computed by multiplying a so-called Hartlap factor \( \alpha = (N_{\text{real}} - N_{\text{bin}} - 2)/(N_{\text{real}} - 1) \) with the inverse of the covariance matrix (eq. [14] [76]).

Figure 3 compares the S/N of pre-reconstruction spectra and post-reconstruction spectra with different \( R_s \) as a function of the maximum wavenumber \( k_{\text{max}} \). We find that the post-reconstruction spectra have higher S/N than the pre-recon spectra. The improvement is larger at higher \( k \). For example, the S/N of the post-recon spectra with \( R_s = 10h^{-1}\)Mpc is improved by 7% (\( k_{\text{max}} = 0.1h/\text{Mpc} \)) and 30% (\( k_{\text{max}} = 0.2h/\text{Mpc} \)) at \( z = 1.02 \) relative to the pre-recon spectra. The improvement is more significant at \( z = 0 \): 18% (\( k_{\text{max}} = 0.1h/\text{Mpc} \)) and 69% (\( k_{\text{max}} = 0.2h/\text{Mpc} \)). Since the diagonal components of the covariance matrix is dominated by the Gaussian terms, the improvement of the S/N mainly comes from the decrement of the off-diagonal components as shown in Figure 1 and 2.

We also plot the S/N estimated from the linear spectra and the Gaussian covariance as a reference. Lower panels focus on the differences of S/N from the linear Gaussian one. Interestingly, it is found that the S/N of the post-reconstruction spectra at \( R_s = 10h^{-1}\)Mpc and 5h\(^{-1}\)Mpc are comparable to or higher than the linear
FIG. 2. Correlation coefficients of the monopole components of the matter power spectrum with fixed $k_i = 0.085h/Mpc$ (upper) and $0.175h/Mpc$ (lower) at $z = 1.02$ (left) and $z = 0$ (right). Different symbols denote the results for pre-rec and post-recon spectra with different $R_s$. Off-diagonal components of post-recon spectra significantly decreases and have negative values at $R_s$ less than $10h^{-1}$Mpc, which is consistent with the behavior of the perturbation theory in Figure 1.

Gaussian one. The similar trend is found from the perturbative predictions where the linear spectra and the tree-level covariance (eq.6) are applied to calculate the S/N (eq.15), though the agreement of the perturbation with the numerical results is limited to be at $k \leq 0.1h/Mpc$. The S/N from the perturbation rapidly increase at high $k$ because the determinant of tree-level covariance diverges \cite{40}. As shown in Figure 1 and 2, the off-diagonal components become negative at $R_s \approx 10h^{-1}$Mpc and thereby the S/N of the post-reconstruction spectra becomes higher than the linear Gaussian one.

Information of nonlinear growth of structure can be normally captured by higher-order statistics beyond two-point statistics. The reconstruction returns the information leaking to higher-order statistics back to the two-point statistics. The return is larger at smaller $R_s$ where smaller scales can be reconstructed and thereby the S/N increases at smaller $R_s$. The growth information is however buried on strongly nonlinear regime and thus the increment of S/N from $R_s = 10h^{-1}$Mpc to $R_s = 5h^{-1}$Mpc at $z = 0$ is limited.
FIG. 3. Comparison of the signal-to-noise ratios (S/N) of the sum of the monopole and quadrupole components of redshift-space matter power spectra before reconstruction and after reconstruction with $R_s = 20h^{-1}$Mpc, $10h^{-1}$Mpc and $5h^{-1}$Mpc at $z = 1.02$ (left) and $z = 0$ (right). For comparison, we plot the S/N of the linear power spectrum using the analytical Gaussian covariance $\text{Cov}^{(\text{GA})}$ with dotted lines. We find that the post-recon spectra has a better S/N than pre-reconstruction one and also that the post-reconstruction spectra with $R_s = 10h^{-1}$Mpc and $5h^{-1}$Mpc have higher S/N than the Gaussian one. Lower panels focus on the differences of the S/N for the linear Gaussian one. For comparison, the S/N of the linear power spectra using the tree-level perturbative covariance $\text{Cov}^{(\text{GN})} + \text{Cov}^{(\text{tree})}$ are plotted with dashed lines and show the similar behavior.

V. IMPACTS ON GROWTH RATE MEASUREMENTS

In this section we explore if the estimates of cosmological parameters are improved by reconstruction, particularly focusing on the growth rate. We evaluate the error of the growth rate including the systematics when using the 1-loop perturbative formulae as a theoretical modeling of the matter power spectra. The likelihood is estimated as follows:

$$\mathcal{L} \propto \exp \left( -\frac{\chi^2}{2} \right),$$

$$\chi^2(p_i) = \sum_{k_i \leq k_{\text{max}}} \sum_{k_j \leq k_{\text{max}}} 0.2 \sum_{\ell \ell'} [P_{\ell}^{\text{theory}}(k_i; p_i) - P_{\ell}^{\text{sim}}(k_i)] \mathbf{Cov}^{-1}(\ell \ell')(k_i, k_j) [P_{\ell}^{\text{theory}}(k_j; p_i) - P_{\ell}^{\text{sim}}(k_j)],$$

For the theoretical model, we adopt the 1-loop perturbative formulae derived in our previous work $[58]$ with the lowest-order counterterms $\alpha_\ell(\ell = 0, 2)$ multiplied with $k^2$ times the linear power spectrum $P_{\ell}^{\text{linear}}$

$$P_{\ell}^{\text{theory}} = P_{\ell}^{\text{1-loop}} + \alpha_\ell k^2 P_{\ell}^{\text{linear}}(k).$$

The set of free parameters $p_i$ is the growth rate $f$ and two counterterms $\alpha_\ell$ and other cosmological parameters are fixed for simplicity. In the theoretical covariance of $P_\ell$ is again estimated from 4000 realizations of N-body with the survey volume $V = (500h^{-1}\text{Mpc})^3$ and the number density $n \sim 1/(h^{-1}\text{Mpc})^{-3}$, which fully takes into account the mode coupling between different bins of $k$. Note that in our previous paper $[58]$, we assumed the Gaussian covariance with different volumes and number density for simplicity to estimate the impact of the growth rate measurement. We estimate the posterior distribution using a nested sampling algorithm multinest $[17]$, implemented in Monte Python $[18]$.

Figure 4 and 5 compare the monopole and quadrupole spectra from N-body simulations with the 1-loop perturbation theory at $z = 1.02$ and 0. We adopt the best-fit values of $\alpha_\ell$ by fitting the spectrum out to $k = 0.2h/\text{Mpc}$. Each panel shows the results of pre-recon (upper-left) and post-recon spectra with different $R_s = 20h^{-1}\text{Mpc}$ (upper-right), $10h^{-1}\text{Mpc}$ (lower-left) and $5h^{-1}\text{Mpc}$ (lower-right). We find that the fitting to the simulated spectrum is best for the post-recon spectra with $R_s = 20h^{-1}\text{Mpc}$, while the post-recon spectra with $R_s = 5h^{-1}\text{Mpc}$ is the worst fitting. More quantitatively speaking, the minimum chi-squared val-
FIG. 4. Monopole and quadruple components of the matter power spectrum from N-body simulations (filled circles) before reconstruction (upper-left) and after reconstruction with $R_s = 20h^{-1}\text{Mpc}$ (upper-right), $10h^{-1}\text{Mpc}$ (lower-left) and $5h^{-1}\text{Mpc}$ (lower-right). Error-bars denote the $1\sigma$ sample variance from 4000 N-body simulations where each volume is $(500h^{-1}\text{Mpc})^3$. For comparison, the 1-loop perturbative formula are plotted with solid lines using the best-fit values of the lowest-order counterterms $\alpha_\ell$ ($\ell = 0, 2$) up to $k_{\text{max}} = 0.2h/\text{Mpc}$. The linear power spectra are also plotted with dashed lines. All of the plotted spectra are divided with the no-wiggle spectra. Small panels show the differences between the simulated spectra and the 1-loop perturbative formula with the best-fit values of $\alpha_\ell$. The output redshift is $z = 1.02$. 
ues are 0.76 (pre-rec), 0.09 (post-rec with 20h⁻¹Mpc), 0.88 (post-rec with 10h⁻¹Mpc), and 3.6 (post-rec with 5h⁻¹Mpc) at z = 1.02 and 8.9 (pre-rec), 0.7 (post-rec with 20h⁻¹Mpc), 4.7 (post-rec with 10h⁻¹Mpc), and 27 (post-rec with 5h⁻¹Mpc). Ref. [58] showed that the reconstruction partially suppresses the nonlinearity of the gravitational growth and thereby the perturbation works at higher k. However, when R_s is too small, the shift
FIG. 6. Linear growth rate relative to the input value obtained by fitting the 1-loop perturbative formulae of the monopole and quadrupole power spectra to the simulated spectra with the maximum wavenumber $k_{\text{max}}$ varied (eq. [16]). The counterterms $\alpha_{\ell}$ ($\ell = 0$ and 2) are freely fitted, while other cosmological parameters are fixed. Different plots show the pre-recon spectra (x-shaped crosses) and post-recon spectra with $R_s = 20h^{-1}\text{Mpc}$ (circles), $10h^{-1}\text{Mpc}$ (triangles) and $5h^{-1}\text{Mpc}$ (+-shaped crosses) at $z = 1.02$ (left panels) and $z = 0$ (right panels). Covariance of the multipole power spectra is estimated from 4000 realizations of $N$-body results with $(500h^{-1}\text{Mpc})^3$ volume. The error-bars denote the 1σ uncertainty.

FIG. 7. Same as Figure 3 but for the S/N of pre-recon and post-recon spectra with $R_s = 10h^{-1}\text{Mpc}$ and the super-sample covariance (SSC) is included in the simulated covariance. The predictions from the 1-loop perturbative formulae are also estimated from the covariance including SSC, i.e., $\text{Cov}^{(GA)} + \text{Cov}^{(\text{tree})} + \text{Cov}^{(\text{SSC})}$.
field estimated from the evolved density field becomes more nonlinear and thereby the perturbation does not work well. This is consistent with our previous work in reals-space matter clustering [57].

Figure 8 shows the best-fit values of $f$ against the input value $f_{\text{input}}$ and its 1 sigma error. We find that the value of $f/f_{\text{input}}$ is consistent with unity up to $k_{\text{max}} \sim 0.3h/\text{Mpc}$ at $z = 1.02$ and $k_{\text{max}} \sim 0.2h/\text{Mpc}$ at $z = 0$ for both pre-rec and post-rec spectra. Note that, since the 1-loop approximation does not work at such high $k_{\text{max}}$, the input value of $f$ can be recovered by chance. It is found that the statistical error decreases after reconstruction over all range of $k$. For example, the error decrements of $f$ from the post-rec spectra relative to that from the pre-rec spectra at $z = 1.02$ are 11% ($R_s = 20h^{-1}\text{Mpc}$), 20% ($R_s = 10h^{-1}\text{Mpc}$), and 17% ($R_s = 5h^{-1}\text{Mpc}$) at $k_{\text{max}} = 0.2h^{-1}\text{Mpc}$ and 13% ($R_s = 20h^{-1}\text{Mpc}$), 33% ($R_s = 10h^{-1}\text{Mpc}$) and 33% ($R_s = 5h^{-1}\text{Mpc}$) at $k_{\text{max}} = 0.3h^{-1}\text{Mpc}$. The error decrements at $z = 0$ are 18% ($R_s = 20h^{-1}\text{Mpc}$), 33% ($R_s = 10h^{-1}\text{Mpc}$), and 22% ($R_s = 5h^{-1}\text{Mpc}$) when $k_{\text{max}} = 0.2h^{-1}\text{Mpc}$. We find that the error of $f$ is significantly improved and the error improvement is almost maximized around $R_s \sim 10h^{-1}\text{Mpc}$ where the covariance is almost diagonal by reconstruction. Table 1 summarizes the minimum chi-squared values and the reduction of statistical errors for pre-reconstruction and post-reconstruction with three different smoothing scales.

![Image of graphs showing $z = 1.02$ and $z = 0$ with SSC](image)

FIG. 8. Same as Figure 6 but for the SSC effect is included. Here we fix the smoothing scale $R_s$ for the post-rec spectra $10h^{-1}\text{Mpc}$.

| $z$ | $\chi^2_{\text{min}}$ | 20h$^{-1}\text{Mpc}$ | 10h$^{-1}\text{Mpc}$ | 5h$^{-1}\text{Mpc}$ |
|-----|----------------|-----------------|----------------|----------------|
| $z = 1$ | 0.76 | 0.09 | 0.88 | 3.6 |
| $z = 0$ | 8.9 | 0.7 | 4.7 | 27 |

| $k_{\text{max}}[h/\text{Mpc}]$ | pre-rec | 20h$^{-1}\text{Mpc}$ | 10h$^{-1}\text{Mpc}$ | 5h$^{-1}\text{Mpc}$ |
|----------------|---------|----------------|----------------|----------------|
| $z = 1$ | 0.2h$^{-1}\text{Mpc}$ | - | 11% | 20(13)% | 17% |
| $z = 0$ | 0.3h$^{-1}\text{Mpc}$ | - | 13% | 33(30)% | 33% |
| $z = 0$ | 0.2h$^{-1}\text{Mpc}$ | - | 18% | 33(28)% | 22% |

TABLE I. A summary of the minimum chi-squared values and the reduction of statistical errors for pre-reconstruction and post-reconstruction with three different smoothing scales $R_s$. For the reduction of statistical errors at $z = 1$, the results with two different $k_{\text{max}}$ are shown. The error reduction for $R_s = 10h^{-1}\text{Mpc}$ shown in the bracket is the case with the super-sample covariance.

VI. SUPER-SAMPLE COVARIANCE

The super-sample covariance (SSC) comes from the mixing between the long-wavelength modes beyond the survey window and the short-wavelength modes inside the survey area. The response of the power spectrum to the change in background density $\delta_b$ is given as [60, 66].

$$\text{Cov}_{\ell\ell'}^{\text{SSC}} = \sigma_b^2 \frac{\partial P_s(k')}{\partial \delta_b} \frac{\partial P_s(k)}{\partial \delta_b}$$

(19)
where the variance of $\delta_b$ in the survey window is defined as

$$\sigma_b^2 = \frac{1}{V_2} \int \frac{dq}{(2\pi)^3} |\hat{W}(q)|^2 P_L(q),$$  \hspace{1cm} (20)

with the Fourier transform of the survey mask field $W(x)$ given as $\hat{W}(q)$. The response of the multipole power spectrum to $\delta_b$ is given by \[79\]

$$\frac{\partial \ln P_\ell(k)}{\partial \delta_b} = G_\ell + D_\ell \frac{d \ln k^3 P_\ell(k)}{d \ln k},$$  \hspace{1cm} (21)

where the first term is the growth modulation by the background density, which is also known as beat coupling (BC) \[79\], and the second term is the dilation effect that comes from the change of the local expansion rate depending on the background density \[60\]. Here we neglect the response of the background tide for simplicity. The growth and dilation term for $\ell = 0$ and 2 are given in Table 1 and 2 of \[79\] as

$$G_0 = \frac{68}{21} (1 + f) + \frac{164}{105} f^2 + \frac{4}{15} f^3$$\hspace{1cm}(22)

$$G_2 = \frac{122}{21} f + \frac{65}{14} f^2 + \frac{58}{63} f^3 - \frac{3}{5} f + \frac{2}{3} f^2,$$\hspace{1cm}(23)

and

$$D_0 = -\frac{1}{3} (1 + f) + \frac{1}{2} f^2 + \frac{1}{3} f^3$$\hspace{1cm}(24)

$$D_2 = -\frac{2}{3} f + \frac{4}{15} f^2 + \frac{10}{63} f^3 - \frac{3}{5} f + \frac{2}{3} f^2,$$\hspace{1cm}(25)

where the linear bias is set to be unity here. The density fluctuation in a given survey is defined against the mean within the survey window rather than the global mean and thereby the normalization of the power spectrum is altered as \[60\]

$$P_\ell^s(k) = \frac{P_\ell(k)}{(1 + \delta_b^{(z)})^2},$$  \hspace{1cm} (26)

where $\delta_b^{(z)}$ is the background density in redshift space and thereby the response is changed to

$$\frac{\partial \ln P_\ell(k)}{\partial \delta_b} \rightarrow \frac{\partial \ln P_\ell^s(k)}{\partial \delta_b} \simeq \frac{\partial \ln P_\ell(k)}{\partial \delta_b} - \left( 2 + \frac{2}{3} f \right).$$  \hspace{1cm} (27)

The non-Gaussian covariance is computed as the sum of the tree-level term (eq.\[6\]) and the SSC as

$$\text{Cov}^{(NG)} = \text{Cov}^{(tree)} + \text{Cov}^{(SSC)}.$$  \hspace{1cm} (28)

In section \[11\] we numerically compute the covariance from the ensemble average over $N$-body simulations with the volume of $(500h^{-1}\text{Mpc})^3$, however, the fluctuations beyond the boxsize are not taken into account. We compute the covariance including SSC by extracting sub-boxes with the volume of $(500h^{-1}\text{Mpc})^3$ from the 8 realizations of large simulation boxes with the volume of $(4h^{-1}\text{Gpc})^3$ containing 4096$^3$ particles. The total number of subboxes becomes $8 \times (4h^{-1}\text{Gpc}/500h^{-1}\text{Mpc})^3 = 4096$. The mean density field is computed in each subbox and the shift field for reconstruction is computed from the smoothed density field using particle data in each subbox. The reconstructed density field is also computed in each subbox data including mass particles shifted from neighboring subboxes. Strictly speaking, the fluctuation beyond the large simulation boxsize $4h^{-1}\text{Gpc}$ is not included in the covariance, however, the SSC is dominated by the fluctuations below this size. For the purpose of comparison with the perturbation theory, however, we integrate $k$ from $2\pi/(4h^{-1}\text{Gpc})$ in the calculation of $\sigma_b$ (eq.\[20\]).

Figure 7 shows the comparison of the S/N for pre-recon and post-recon spectra with $R_s = 10h^{-1}\text{Mpc}$. We find that both S/N decrease when including SSC. Since the reconstruction is performed within the survey area, the bulk motion of super-sample modes cannot be corrected by the reconstruction. The post-recon spectra, however, have still higher S/N than the pre-recon spectra. The reconstruction improves S/N by 5% ($k_{\text{max}} = 0.1h/\text{Mpc}$) and by 19% ($k_{\text{max}} = 0.2h/\text{Mpc}$) at $z = 1.02$ and by 14% ($k_{\text{max}} = 0.1h/\text{Mpc}$) and by 40% ($k_{\text{max}} = 0.2h/\text{Mpc}$) at $z = 0$. The perturbation formulae also show the consistent results with the numerical one and they quantitatively agree up to $k \sim 0.1h/\text{Mpc}$.

Figure 8 shows the impact on the growth rate measurements when including SSC. The input value of growth rate are again recovered up to $k_{\text{max}} \sim 0.3h/\text{Mpc}$ at $z = 1.02$ and $k_{\text{max}} \sim 0.2h/\text{Mpc}$ at $z = 0$ for both pre-recon and post-recon spectra. The improvements of the error by the reconstruction with $R_s = 10h^{-1}\text{Mpc}$ are 13% when $k_{\text{max}} = 0.2h/\text{Mpc}$ and 30% when $k_{\text{max}} = 0.3h/\text{Mpc}$ at $z = 1.02$ and 28% when $k_{\text{max}} = 0.2h/\text{Mpc}$ at $z = 0$, which are comparable to the improvement without SSC.

VII. SUMMARY AND CONCLUSIONS

We investigated the covariance of the redshift-space matter power spectra after a standard density-field reconstruction that is commonly used in the BAO analysis. We derived the perturbative formula of the covariance of the multipole components of the power spectra at tree level. We find that the positive off-diagonal components of the covariance from the tree-level trispectra decrease after the reconstruction and have negative values at the smoothing scale of the shift field $R_s$ less than $\sim 10h^{-1}\text{Mpc}$. We also computed the covariance of the multipole power spectra directly from a large set of $N$-body simulations. We find the significant decrease of the off-diagonal components and the behavior is consistent with the perturbation theory. In consequence, the information content of the post-recon power spectra evaluated with the signal-to-noise ratio (S/N) of their monopole and quadrupole components significantly increase compared to the pre-recon power spectra. Inter-
demonstrated that the reconstruction significantly re-
structured spectra. We also studied the super-sample
covariance effect both from perturbative and numerical
approaches. We find that the S/N reduces even after
the reconstruction because the reconstruction performs
within the survey area and thus the bulk motion of the
super-sample modes cannot be corrected by reconstruc-
tion. Even when the SSC is included, the post-recon
spectra still have higher S/N than the pre-recon spectra.

We find that the tree-level perturbative approach is
limited to describe the simulated covariance at \( k \leq 0.1h/Mpc \). This indicates that higher-order mode cou-
pling needs to be taken into account to describe the co-
variance more accurately. There are several works to
describe mode couplings at higher \( k \) based on the effec-
tive field theory [80], the response approach [81], and also
semi-analytical models [82][81]. It may be interesting to
apply these methods to describe the covariance of recon-
structed spectra.

Recovery of cosmological information in the two-point
statistics makes the cosmological analysis simpler. We
 demonstrated that the reconstruction significantly re-
duced the error of growth rate inferred from the redshift-
space power spectrum. So far the reconstruction has
been mainly applied to the BAO analysis due to the lack
of theoretical understandings of the reconstructed spec-
trum. Since it is found that the error of the full shape
of the power spectrum is improved, it is interesting to
investigate how the other cosmological parameters are
improved by using the information of the full shape of
power spectra after reconstruction. We also have to take
into account the galaxy bias and the shot noise as well as
various observational effects such as survey geometry to
apply the actual observational data [e.g., 65]. We leave
this for future work.

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