On characterizations of hypersurfaces in a Sasakian space forms with commuting operators

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Abstract
Let $M$ be a real hypersurface in a Sasakian space form $\mathcal{M}(c)$. In this paper, we prove that if $R_lL_U = L_rR_l$ holds on $M$, then $M$ is a Hopf hypersurface, where $R_l$ and $L_U$ denote the Jacobi operator structure and the induced operator from the Lie derivative with respect to the induced normal vector field $U$, respectively. We characterize such the Hopf hypersurfaces of $\mathcal{M}(c)$.

1. Introduction

Let $\mathcal{M}(c)$, be a Sasakian space form equipped with the metric $g$. Let $M$ be a hypersurface of $\mathcal{M}(c)$ where the structural vector field $\xi$ is tangent to $M$. Let $\phi$ and $N$ denote the contact structure and the locally unit normal vector field on $\mathcal{M}(c)$ and $M$, respectively. Then, $-\phi(N) = U$ is a tangent vector field to $M$, which is called the induced normal vector field on $M$. Now, we consider the hypersurface $M$ with the metric structure $(F, g, \xi, \eta, U, u)$, which is induced from the contact metric and the contact structure $\phi$ of $\mathcal{M}(c)$. If the plane spanned by the structure vector field $\xi$ and the induced normal vector field $U$ turns out to be an invariant subspace by $A$, where $A$ is the shape operator of $M$, the hypersurface $M$ is called a Hopf hypersurface (Abedi et al., 2012).

Hypersurfaces in the Sasakian space forms were studied in (Abedi and Ilmakchi, 2016; Abedi & Ilmakchi, 2015).

The induced operator $L_U$ on a hypersurface $M$ of the form $\mathcal{L}_U g$ is defined by $(\mathcal{L}_U g)(X, Y) = g(L_U X, Y)$ for any vector fields $X$ and $Y$ on $M$, where $L_U$ denotes the Lie derivative operator with respect to the induced normal vector field $U$.

In this paper, at first we show that the operator $L_U$ gives $L_U = FA - AF$ on $M$, and the induced normal vector field $U$ is Killing if $L_U = 0$. Indeed, we show that:

**Theorem 1.1.** Let $M$ be a hypersurface in the Sasakian space form $\mathcal{M}(c)$, where the structural vector field $\xi \in T(M)$, and $L_U = 0$ on $M$. Then $M$ is of one of the following two types

- $M$ is locally isometric to $S^n(r_1) \times S^n(r_2)$, $(r_1^2 + r_2^2 = 1)$.
- $M$ is locally the product $M' \times \xi$, where $M'$ is a totally geodesic manifold and $\xi$ is a geodesic curve of $M$.

For the curvature tensor field $R$ on a real hypersurface $M$, we define the Jacobi operator $R_X$ by $R_X = R(\cdot, X)X$ with respect to a unit vector field $X$ (Kim et al., 2014). Also, we obtain:

**Theorem 1.2.** Let $M$ be a hypersurface in the Sasakian space form $\mathcal{M}(c)$, where the structural vector field $\xi \in T(M)$ and $R_lL_U = 0$ on $M$. Then $M$ is one of the hypersurfaces which are listed in the Theorem 1.1.

Finally, we study the hypersurface in the Sasakian space form $\mathcal{M}(c)$ with commuting operators $R_U$ and $L_U$ and get:

**Theorem 1.3.** Let $M$ be a hypersurface in the Sasakian space form $\mathcal{M}(c)$, where the structural vector field $\xi \in T(M)$ and $R_U L_U = L_U R_U$ on $M$. Then $M$ is one of the hypersurfaces which are included in the Theorem 1.1.

2. Preliminaries

An odd dimensional differentiable manifold $\tilde{M}^{2m+1}$ has an almost contact structure if it admits tensor fields $\phi, \xi$...
and \( \eta \) of type \((1,1), (0, 1) \) and \((1, 0)\), respectively, in which satisfy
\[
\eta(\xi) = 1, \quad \phi^2 = -I + \eta \otimes \xi,
\]
where \( I \) denotes the field of identity transformations of the tangent spaces at the all points. These conditions imply that \( \phi \xi = 0 \) and \( \eta \circ \phi = 0 \), where the endomorphism \( \phi \) has the rank of \( 2m \) at the every point in \( \tilde{M}^{2m+1} \). A manifold \( \tilde{M}^{2m+1} \), which is equipped with an almost contact structure \((\phi, \xi, \eta)\), is called an almost contact manifold and denoted by \( (\tilde{M}^{2m+1}, (\phi, \xi, \eta)) \).

Suppose that \( \tilde{M}^{2m+1} \) is a manifold carrying an almost contact structure. The Riemannian metric \( g \) on \( \tilde{M}^{2m+1} \) which satisfies
\[
g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),
\]
for all the vector fields \( X \) and \( Y \), is said to be compatible with the almost contact structure, and \((\phi, \xi, \eta, g)\) is an almost contact metric structure on \( \tilde{M}^{2m+1} \). Note that putting \( Y = \xi \), yields
\[
\eta(X) = g(X, \xi),
\]
for all the vector fields \( X \) tangent to \( \tilde{M}^{2m+1} \), which shows \( \eta \) is the metric dual to the characteristic vector field \( \xi \).

A manifold \( \tilde{M}^{2m+1} \) is said to be a contact manifold if it carries the global one-form \( \eta \) such that
\[
\eta \wedge (d\eta)^m \neq 0,
\]
everywhere on \( M \). The one-form \( \eta \) is called the contact form.

A submanifold \( M \) of a contact manifold \( \tilde{M}^{2m+1} \) tangent to \( \xi \) is called an invariant (resp. anti-invariant) submanifold if \( \phi(T_pM) \subset T_pM \), \( \forall p \in M \) (resp. \( \phi(T_pM) \subset T^1_pM \), \( \forall p \in M \)).

A submanifold \( M \) tangent to the Riemannian contact manifold \( \tilde{M}^{2m+1} \) is called a contact CR-submanifold if there exists a pair of the orthogonal differentiable distributions \( D \) and \( D^\perp \) on \( M \), such that:

1. \( TM = D \oplus D^\perp \oplus \mathbb{R} \xi \), where \( \mathbb{R} \xi \) is the 1-dimensional distribution spanned by \( \xi \);
2. \( D \) is invariant by \( \phi \), i.e., \( \phi(D_p) \subset D_p, \forall p \in M \);
3. \( D^\perp \) is anti-invariant by \( \phi \), i.e., \( \phi(D^\perp_p) \subset T^1_pM, \forall p \in M \).

Let \((\tilde{M}, \phi, \xi, \eta, \tilde{g})\) be a \((2n + 1)\)-dimensional contact manifold such that
\[
\Delta X \xi = -\phi X, \quad \tilde{g}(\Delta X \xi, Y) = -\eta(X)\eta(Y)X,
\]
then \( \tilde{M} \) is called a Sasakian manifold. By taking into account that, the plane section \( \pi \) of \( TM \) is called a \( \phi \) - section if \( \phi \pi_x \subset \pi_x \) for any \( x \in \tilde{M} \), a Sasakian space form is the Sasakian manifold of the constant \( \phi \) - sectional curvature. The Riemannian curvature tensor field \( \tilde{R} \) of the Sasakian space form is given by (Blair, 1976)
\[
\tilde{R}(X, Y)Z = \frac{c + 3}{4} \{ \tilde{g}(Y, Z)X - \tilde{g}(X, Z)Y \} - \frac{c - 1}{4} \{ \eta(Z)\eta(Y)X - \eta(X)\eta(Y)Y \} + \{ \tilde{g}(Y, Z)\eta(X) - \tilde{g}(X, Z)\eta(Y) \} \xi
\]
for any \( X, Y, Z \in \chi(\tilde{M}) \).

3. Hypersurfaces in the Sasakian space form

Let \((M, g)\) be a real hypersurface tangent to \( \xi \) of the Sasakian space form \( \tilde{M}(c) \) and let \( N \) be a unit normal vector field on \( M \). Then, we have
\[
TM = D \oplus D^\perp \oplus \mathbb{R} \xi,
\]
where \( D \) is a \( \phi \)-invariant subspace and \( D^\perp \) is a one-dimensional subspace, that is spanned by \( U = -\phi(N) \), and is the orthogonal component of \( D \).

Moreover, it is clear that \( \phi TM \subset TM \oplus \text{Span}N \). Hence, we have for any tangent vector field \( X \) the following decomposition in the tangent and the normal components:
\[
\phi X = FX + u(X)N. \tag{3.1}
\]
It is easily shown that \( F \) is a skew-symmetric linear endomorphism that acts on \( T_pM \). Since the structure vector field \( \xi \) is tangent to \( M \), \( (3.1) \) implies
\[
F \xi = 0, \quad FU = 0, \quad g(U, X) = u(X), \quad u(\xi) = g(U, \xi) = 0, \quad u(U) = 1. \tag{3.2}
\]
Next, by applying \( \phi \) to \( (3.1) \) and using \( (3.2) \), we also have
\[
F^2X = -X + \eta(X)\xi + u(X)U, \quad u(FX) = 0. \tag{3.3}
\]
We denote by \( \nabla \) and \( \overline{\nabla} \) the Levi-Civita connections on \( \tilde{M} \) and \( M \), respectively. Then the Gauss formula is given by
\[
\overline{\nabla}_XY = \nabla_XY + h(X, Y),
\]
for any vector fields \( X, Y \) tangent to \( M \). Here and in the sequel \( h \) denotes the second fundamental form and \( A \) is the shape operator corresponding to the normal vector field \( N \). Therefore,
\[
\overline{\nabla}_XY = \nabla_XY + g(A\xi, Y)N.
\]

**Definition 3.1.** (Abedi et al., 2012) Let \( A \) be the shape operator of the hypersurface \( M \) in \( \tilde{M}(c) \) and the plane
spanned by \( \{ \xi, U \} \) be an invariant subspace of \( A \). Then, the hypersurface \( M \) is called a Hopf hypersurface of \( \tilde{M}(c) \).

By taking the covariant derivative of both sides of the Equation (3.1) and comparing the tangent and the normal parts, we have

\[
(\nabla_Y F)X = -g(Y, X)\xi + \eta(X)Y - g(AY, X)U + u(X)AY, 
\]

\[ (\nabla_Y u)X = g(FAY, X), \]

\[ \nabla_X U = FAX. \]

Moreover, for any vector fields \( X, Y, Z \) tangent to \( M \) we get by applying the Equation (3.1), the following Gauss and the Codazzi equations hold

\[ R(X, Y)Z = \frac{c+3}{4} \{ g(Y, Z)X - g(X, Z)Y \} \]

\[ -\frac{c-1}{4} \{ \eta(Z)\eta(Y)X - \eta(X)Y \} + [g(Y, Z)\eta(X) - g(X, Z)\eta(Y)]\xi \]

\[ -g(FY, Z)FX + g(FX, Z)FY + 2g(FX, Y)FZ \]

\[ + g(AY, Z)AX - g(AX, Z)AY, \]

\[ (\nabla_X A)Y \]

\[ = \frac{c-1}{4} \{ u(X)FY - u(Y)FX \}
- 2g(FX, Y)U. \]

(3.10)

\[ + u(AX)AX - u(AX)AU \quad (4.1) \]

for any vector field \( X \) on \( M \).

By applying the Equation (3.6), we have \((L_{uG})X, Y) = g((FA - AF)X, Y)\) for any vector fields \( X \) and \( Y \) on \( M \). Hence, the induced operator \( L_U \) from \( L_{uG} \) is given by

\[ L_U X = (FA - AF)X, \]

for any vector field \( X \) on \( M \).

**Lemma 4.1.** Let \( M \) be a hypersurface in the Sasakian space form \( \tilde{M}(c) \), where the structural vector field \( \xi \in T(M) \) and \( L_U = 0 \) on \( M \). Then \( M \) is a Hopf hypersurface.

**Proof.** From the Equation (4.2) we have

\[ FAX = AFX, \]

for any vector field \( X \) on \( M \). Also, by using the Equations (3.2) and (3.8), we get

\[ AU = \xi + aU, \quad a := u(AX). \quad (4.3) \]

Now, with respect to the Equation (3.8), \( M \) is a Hopf hypersurface.

**Lemma 4.2.** Let \( M \) be a hypersurface in the Sasakian space form \( \tilde{M}(c) \), where the structural vector field \( \xi \in T(M) \) and \( L_U = 0 \) on \( M \). Then \( a \) is constant.

**Proof.** By taking the covariant derivative of the Equation (4.3) and using the Equations (3.6) and (3.7), we have

\[ (\nabla_X A)U + A^2 FX = (X\alpha)U + aFAX + FX. \]

Now, we obtain

\[ g((\nabla_X A)Y - (\nabla_Y A)X, U) + 2g(FAX, AY) = (X\alpha)u(Y) - (Y\alpha)u(X) \]

\[ + 2ag(FAX, Y) + 2g(FX, Y). \]

The last equation substitute into the (3.10) and use the (3.2), we verify that

\[ -\frac{c+3}{2} g(FX, Y) + 2g(FX, A^2 Y) \]

\[ = (X\alpha)u(Y) - (Y\alpha)u(X) + 2ag(FX, AY). \quad (4.4) \]

By putting \( X = U \) into the above equation and taking (3.2), we obtain

\[ X\alpha = (U\alpha)u(X), \quad (4.5) \]
in which, by taking derivative and applying (3.5), we get
\[(Y(U\alpha))u(X) - (X(U\alpha))u(Y) - 2(U\alpha)g(FAX, Y) = 0.\]  
(4.6)

Also, put \(X = U\) into the Equation (4.6) and use (3.2), we see
\[X(U\alpha) = (U\alpha)u(X), \]
(4.7)
where, use (4.5) and (4.7) in (4.6), gives for any vector field \(X\) on \(M\), \((U\alpha)FAX = 0\). If \((U\alpha)\neq 0\), we have \(FX = 0\) for any vector field \(X\) on \(M\), which is a contradiction. Hence, \(U\alpha = 0\). From (4.5) for any vector field \(X\) on \(M\) we have \(X\alpha = 0\). Therefore, \(\alpha\) is a constant.

Since \(A\) is self adjoint, \(D\) and \(\text{span}\{\xi, U\}\) are the invariant subspaces under \(A\), there exist the locally orthonormal frames
\[X_1, \ldots, X_{2n-2},\]
and \(\{W_1, W_2\}\) for \(D\) and the space that is spanned by \(\text{span}\{\xi, U\}\), respectively where
\[AX_i = \mu_i X_i, \quad i = 1, \ldots, 2n - 2,\]
\[AW_1 = \gamma_1 W_1, \quad AW_2 = \gamma_2 W_2.\]
We set
\[W_1 = \xi \cos \theta + U \sin \theta, \]
\[W_2 = -\xi \sin \theta + U \cos \theta.\]
for some \(0 < \theta < \frac{\pi}{2}\). Note that \(\xi\) and \(U\) can not be the eigenvectors of \(A\) hence, \(\cos \theta\) and \(\sin \theta\) can not vanish.

Lemma 4.3. (Abedi & Ilmakchi, 2015) Under the above conditions, \(\gamma_1\gamma_2 = -1\).

Lemma 4.4. Let \(M\) be a hypersurface in the Sasakian space form \(\overline{M}(c)\) where the structural vector field \(\xi \in \text{T}(M)\) and \(L_\xi = 0\) on \(M\). Then the shape operator \(A\) has either the constant eigenvalues
\[\gamma_{1,2} = \frac{\alpha \pm \sqrt{\alpha^2 + 4}}{2}\]
of the same multiplicities \(n\), or distinct eigenvalues
\[\gamma_{1,2} = \frac{\alpha \pm \sqrt{\alpha^2 + 4}}{2}, \quad \lambda_{1,2} = \frac{\alpha \pm \sqrt{\alpha^2 + c + 3}}{2}, \quad c \neq 1\]
where the multiplicities of \(\gamma_1, \gamma_2\) and \(\lambda_1, \lambda_2\) are 1 and \(n - 1\), respectively.

Proof. If we denote by \(\lambda\) the eigenvalue corresponding to the eigenvector of \(A\), which is orthogonal to \(U\) and \(\xi\), then from (3.3) and (4.4) that \(\lambda\) satisfies
\[\lambda^2 = -\alpha \lambda - \frac{c + 3}{4} = 0,\]
and consequently with respect to the Lemma 4.2, the shape operator \(A\) has at most four constant eigenvalues
\[\gamma_{1,2} = \frac{\alpha \pm \sqrt{\alpha^2 + 4}}{2}, \quad \lambda_{1,2} = \frac{\alpha \pm \sqrt{\alpha^2 + c + 3}}{2}, \quad c \neq 1.\]
If the shape operator \(A\) has exactly two constant eigenvalues
\[\gamma_{1,2} = \frac{\alpha \pm \sqrt{\alpha^2 + 4}}{2},\]
whose multiplicities are \(n\), because of \(AF = FA\). Similarly, If the shape operator \(A\) has exactly four constant eigenvalues
\[\gamma_{1,2} = \frac{\alpha \pm \sqrt{\alpha^2 + 4}}{2}, \quad \lambda_{1,2} = \frac{\alpha \pm \sqrt{\alpha^2 + c + 3}}{2}, \quad c \neq 1\]
from \(AF = FA\), the multiplicities are 1 and \(n\), respectively.

In the rest of this section, we add the following Theorem.

Theorem 4.5. (Kim & Pak, 2007) Let \(M\) be an \((n + 1)\) – dimensional contact CR – submanifold of \((n - 1)\) contact CR – dimension immersed in a \((2m + 1)\) – unit sphere \(S^{2m+1}\). If, for any vector fields \(X, Y\) tangent to \(M\), the equality
\[h(FX; Y) - h(X; FY) = 0\]
holds on \(M\), where \(h\) and \(F\) denote the second fundamental form and a skew-symmetric endomorphism acting on tangent space of \(M\), respectively. Then \(M\) is a locally isometric to
\[S^{2n_1+1}(r_1) \times S^{2n_2+1}(r_2) \quad (r_1^2 + r_2^2 = 1),\]
for some integers \(n_1, n_2\) with \(n_1 + n_2 = n - 1\).

Combining from the special case and the results of proofs in (Kim & Pak, 2007), we have

Theorem 4.6. Let \(M^{2n}\) be an immersed hypersurface in the unit sphere \(S^{2n+1}\), where \(AF = FA\) and the shape operator \(A\) has exactly two constant eigenvalues. Then \(M\) is a locally isometric to
\[S^{2n_1+1}(r_1) \times S^{2n_1+1}(r_2) \quad (r_1^2 + r_2^2 = 1),\]
for some integers \(n_1, n_2\) with \(n_1 + n_2 = n - 1\).
Now, we suppose that the shape operator $A$ has exactly four constant eigenvalues. Let $Z = \lambda_1 \xi + U$ then $Z^\perp = \xi - \lambda_1 U$. We consider the distribution $D' = D \oplus \text{span}\{Z^\perp\}$ therefore $D'^\perp = \text{span}\{Z\}$.

**Lemma 4.7.** The distribution $D'$ is involutive in $M$.

**Proof.** We show that $D \oplus \text{span}\{Z^\perp\}$ is involutive. Since
$$g(Z, Z^\perp) = 0,$$
so, for any $X \in D$, we have
$$g([X, Z^\perp], Z) = \lambda g([X, \xi], \xi) + g([X, \xi], U)$$
$$- \lambda_1^2 g([X, U], \xi) - \lambda_1 g([X, U], U),$$
moreover, because
$$g([X, \xi], \xi) = 0,$$
then, $g([X, Z^\perp], Z) = 0$ and it implies that $[X, Z^\perp] \in D \oplus \text{span}\{Z^\perp\}$. This shows the distribution $D \oplus \text{span}\{Z^\perp\}$ is involutive in $M$.

Now, we consider the integral submanifold $M$ of the distribution $D'$ in $M$. On the other hand, because $Z$ is an 1-dimensional distribution, thus is involutive. Also, its integral manifold is the integral curve such that $\zeta$ is a geodesic, that is, $\nabla_{\dot{\zeta}} \zeta^\prime = 0$ because of the assumption $\zeta^\prime = Z$.

**Lemma 4.8.** The integral submanifold $M'$ is a totally geodesic in $M$.

**Proof.** Let $TM' = D \oplus \text{span}\{Z^\perp\}$ and $A'$ is the shape operator corresponding to the normal vector field $Z$. If $X \in D$, we have
$$\nabla_X U = \phi AX \quad \text{and} \quad \nabla_X \xi = -\phi X,$$
then $A' = 0$. Moreover, if $X = Z^\perp$, we have
$$A'Z^\perp = -\nabla_{\dot{Z}}^\perp Z = -\nabla_Z^\perp Z + g(AZ^\perp, Z)N = 0,$$
as respect $g(\nabla_{\dot{Z}}^\perp Z, N) = g(AZ^\perp, Z)$, so $A' = 0$.

**Theorem 4.9.** Let $M$ be a hypersurface in the Sasakian space form $\overline{M}(\zeta)$, where the structural vector field $\xi \in T(M)$ and $L_U = 0$ on $M$. If the shape operator $A$ has exactly four constant eigenvalues, then $M$ is locally a product of $M' \times \zeta$, where $M'$ is a totally geodesic manifold and $\zeta$ is a geodesic curve of $M$.

**Proof.** According to the above assumptions, it is sufficient to show that
$$\nabla_{TM'} TM' \subseteq TM', \quad \nabla_Z Z = 0, \quad \nabla_Z TM' \subseteq TM', \quad \nabla_{TM'} Z = 0.$$

Because $\zeta$ is the geodesic curve, so $\nabla_Z Z = 0$. Whereas $M'$ is a totally geodesic manifold in $M$, so $\nabla_{TM'} Z = 0$. On the other hand, we have
$$g(\nabla_{Z}, Z, Z) = g(\nabla_{Z}, \xi - \lambda_1 U, \lambda_1 \xi + U) = 0,$$
and from the Lemma 4.8 for $X, Y$ in $TM'$ we have $g(\nabla_X Y, Z) = 0$. In the other words $\nabla_Z TM' \subseteq TM'$. Also
$$g(\nabla_Z X, Z) = -g(X, \nabla_Z Z) = 0,$$
for all $X$ in $TM'$, so $\nabla_{TM'} TM' \subseteq TM'$. Hence, by the de Rham decomposition theorem (De Rham, 1952), $M$ is a locally isometric to the Riemannian product of the totally geodesic manifold $M'$ and $\zeta$.

**Proof of Theorem 1.1.** The result follows by Theorems 4.6 and 4.9.

**Proof of Theorem 1.2.** From the Equations (3.2) and (3.8) for some vector field $T$ in $D$ we have
$$AU = \xi + \alpha U + \beta T, \quad \alpha := u(AU), \quad \beta := g(T, AU).$$
(4.8)

By taking into the account the Equations (4.1), (4.2) and applying (3.2), (3.8) and (4.8) we obtain
$$0 = g(R_U L_U (X, \xi), \beta g((AF - FA)X, T),$$
for all the vector field $X$ in $M$. This equation shows that either $\beta = 0$ or $AF - FA = 0$. If $\beta = 0$ then $M$ is the Hoph hypersurface and obviously $L_U = 0$. In the case of $AF - FA = 0$, the Equation (4.2) shows that $L_U = 0$. Therefore, the Theorem 1.1 gives the results.

**Proof of Theorem 1.3.** Similarly, as the proof of the Theorem 1.2, we have the Equation (4.8). On the other hand, from the Equations (4.1) and (4.2) by applying (3.2), (3.8) and (4.8), we obtain
$$g(R_U L_U (X, \xi), \beta g((AF - FA)X, T)$$
(4.9)
$$g(L_U R_U (X, \xi), \epsilon) = 0.$$
In this paper we introduce the Characterizations of hypersurfaces in Sasakian space forms with commuting operators. Older, the other author introduces this condition and similar conditions for hypersurfaces in complex space forms. In the following these conditions can be introduced in the other spaces, similar; Kenmotsu space forms or generalized Sasakian space forms.

Acknowledgements
The author is extremely appreciated the referees for their useful remarks to improve the article.

Funding
The author received no direct funding for this research.

Notes on contributor

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