Unconditional stability and error analysis of an Euler IMEX-SAV scheme for the micropolar Navier-Stokes equations

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Abstract
In this paper, we consider numerical approximations for solving the micropolar Navier-Stokes (MNS) equations, that couples the Navier-Stokes equations and the angular momentum equation together. By combining the scalar auxiliary variable (SAV) approach for the convective terms and some subtle implicit-explicit (IMEX) treatments for the coupling terms, we propose a decoupled, linear and unconditionally energy stable scheme for this system. We further derive rigorous error estimates for the velocity, pressure and angular velocity in two dimensions without any condition on the time step. Numerical examples are presented to verify the theoretical findings and show the performances of the scheme.

Keywords: micropolar Navier-Stokes equations; implicit-explicit schemes; energy stability; error estimates, scalar auxiliary variable

1. Introduction
Let Ω be a convex polygonal/polyhedral with boundary Γ := ∂Ω in \( \mathbb{R}^d \), \( d = 2, 3 \). In this paper, we consider numerical approximation of the following MNS equations:

\begin{align}
\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} - (\nu + \nu_r)\Delta \mathbf{u} + \nabla p - 2\nu_r \nabla \times \mathbf{w} &= 0 \quad \text{in } \Omega \times J, \\
\nabla \cdot \mathbf{u} &= 0 \quad \text{in } \Omega \times J, \\
\mathbf{j} \mathbf{w}_t + \mathbf{j} \mathbf{u} \cdot \nabla \mathbf{w} - (c_a + c_d) \Delta \mathbf{w} - (c_0 + c_d - c_a) \nabla \nabla \cdot \mathbf{w} + 4\nu_r \mathbf{w} - 2\nu_r \nabla \times \mathbf{u} &= 0 \quad \text{in } \Omega \times J,
\end{align}

with boundary and initial conditions

\[ \mathbf{u} = 0, \quad \mathbf{w} = 0 \quad \text{on } \Gamma \times J, \]
\[ \mathbf{u}(x, 0) = \mathbf{u}^0(x), \quad \mathbf{w}(x, 0) = \mathbf{w}^0(x) \quad \text{in } \Omega, \]

where \( T > 0 \) is the final time, \( J = (0, T) \), \( (\mathbf{u}, p, \mathbf{w}) \) represent the the linear velocity, pressure and angular velocity. All the material constants \( j, \nu, \nu_r, c_a, c_d \) and \( c_0 \) are the kinematic viscosity which are assumed to be constant, positive and satisfy \( c_0 + c_d - c_a > 0 \). Moreover, \( \nu \) is the usual Newtonian viscosity, and \( \nu_r \) is the microrotation viscosity. In order to simplify notation, we will set

\[ \nu_0 = \nu + \nu_r, \quad c_1 = c_a + c_d, \quad c_2 = c_0 + c_d - c_a. \]

Furthermore, there is a slight difference in two and three dimensions. Namely, if \( d = 2 \), we assume that the velocity component in the \( z \)-direction is zero and the angular velocity is parallel to the \( z \)-axis \cite{1}. That is,

\[ \mathbf{u} = (u_1, u_2, 0), \quad \mathbf{w} = (0, 0, w). \]

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The MNS equations were first introduced by Eringen [2] to describe the evolution of an incompressible fluid whose material particles possess both translational and rotational motions. The novelties of this system are to reflect the effects of microstructure on the fluid via a microscopic dissipative evolution equations for the angular momentum. This model is often used to describe the motion of blood, certain lubricants, liquid crystals, ferrofluids, and some polymeric fluids [1, 3, 4]. Given the significant role it played in the microfluids, numerical solving of the MNS system has drawn a considerable amount of attention. A penalty projection method is proposed and optimal error estimates are proved in [5]. In [6], Nochetto et al. proposed and analyzed first-order and second-order semi-implicit fully-discrete schemes. These schemes decouple linear velocity computation from angular velocity computation, while being energy-stable. Later, Salgado further adopted the fractional time stepping technique to decouple the computation of pressure and velocity and proved the rigorous error estimates in [7]. In these works, the nonlinear terms are treated either implicitly or semi-implicitly so that one needs to solve a nonlinear system or a linear system with variable coefficients at each time step. It is desirable to treat the nonlinear term explicitly while maintaining energy stability. With such treatment, the schemes only require the solution of linear system with constant coefficients upon discretization, which are very efficient.

In recent years, SAV based schemes have attracted much attention due to their efficiency, flexibility and accuracy. The main idea is to introduce auxiliary variables to preserve the property of energy decay. Several classes of energy stable numerical schemes have been developed for many dissipative systems, like gradient flows [8, 9, 10, 11], NS equations [12, 13, 14], magnetohydrodynamic equations [15, 16, 17] and Cahn-Hilliard-Navier-Stokes equations [18, 19, 20]. In particular, Shen et al. [21] proposed a new class of efficient IMEX BDFk(1 ≤ k ≤ 5) schemes combined with a SAV approach for general dissipative systems. The distinct advantages are that their higher-order versions are also unconditionally energy stable and only require solving one decoupled linear system with constant coefficients at each time step.

For the MNS equations considered in this article, the energy structure is an inequality rather than an equality like many dissipative systems. This fact makes the energy-equality based approaches [8, 21] fail. Thus, it is not trivial to construct efficient SAV schemes for such systems. The aim of this work is to extend the approach proposed in [13] to the MNS equations. Our main contributions are three-folds:

1. We propose a decoupled, linear and first-order scheme for the MNS equations by combining the SAV approach for the convective terms and some subtle IMEX treatments for the coupling terms. The scheme only requires solving a sequence of differential equations with constant coefficients at each time step so it is very efficient and easy to implement.
2. We establish rigorous unconditional energy stability and error analysis for the proposed scheme in two dimensions.
3. We provide some numerical experiments to confirm the predictions of the theory and demonstrate the efficiency of the scheme.

Compared to the Navier-Stokes equations, the error analysis for the MNS equations is much more involved due to the coupling terms. It is remarked that the present idea can be applies to the Boussinesq equations and ferrohydrodynamics equations.

The rest of this paper is organized as follows. In Section 2, we introduce some notations and present the energy estimate fo the MNS equations. In Section 3, we propose the Euler IMEX-SAV scheme and prove the unconditional stability. In Section 4, we carry out a rigorous error analysis for the proposed scheme in two dimensions. In Section 5, we present some numerical experiments. In Section 6, we conclude with a few remarks.

2. Preliminaries

We start by introducing some notations and spaces. As usual, the inner product and norm in $L^2(\Omega)$ are denoted by $(\cdot, \cdot)$ and $\| \cdot \|$, respectively. Let $W^{m,p}(\Omega)$ stand for the standard Sobolev spaces equipped with the standard Sobolev norms $\| \cdot \|_{m,p}$. For $p = 2$, we write $H^m(\Omega)$ for $W^{m,2}(\Omega)$ and its corresponding norm is $\| \cdot \|_m$. For a given Sobolev space $X$, we write $L^q(0,T;X)$ for the Bochner space. Throughout the paper,
we use $C$ to denote generic positive constants independent of the discretization parameters, which may take different values at different places.

For convenience, we introduce some notations for function spaces

$$X := H^1_0(\Omega), \quad V := \{v \in X : \nabla \cdot v = 0\}.$$ 

The following equation for the curl operator will be repeatedly used in our analysis

$$(\nabla \times w, u) = (w, \nabla \times u), \quad \forall u, w \in X.$$ 

Moreover, we recall that the following orthogonal decomposition of $X$,

$$\|\nabla u\|^2 = \|\nabla \times u\|^2 + \|\nabla \cdot u\|^2, \quad \forall u \in X,$$

which implies

$$\|\nabla \times u\| \leq \|\nabla u\|, \quad \|\nabla \cdot u\| \leq \|\nabla u\|, \quad \forall u \in X.$$ 

To deal with the convection terms in (1a) and (1c), we define the following trilinear form,

$$b(u, v, w) = (u \cdot \nabla v, w).$$

It is easy to see that the trilinear form $b(\cdot, \cdot, \cdot)$ is a skew-symmetric with respect to its last two arguments,

$$b(u, v, w) = -b(u, w, v), \quad \forall u \in V, \quad v, w \in X,$$

and

$$b(u, v, v) = 0, \quad \forall u \in V, \quad v \in X.$$ 

To end this section, we give the basic formal energy estimates for the model (1). By taking the $L^2$-inner product of (1a) with $u$, using the integration by parts and (1b), we get

$$\frac{1}{2} \frac{d}{dt} \|u\|^2 + \nu \|\nabla u\|^2 + (u \cdot \nabla u, u) = 2\nu_r (\nabla \times u, u).$$

Taking the $L^2$-inner product of (1c) with $w$, and using the integration by parts, we have

$$\frac{j}{2} \frac{d}{dt} \|w\|^2 + c_1 \|\nabla w\|^2 + j (u \cdot \nabla w, w) + c_2 \|\nabla \cdot w\|^2 + 4\nu_r \|w\|^2 = 2\nu_r (\nabla \times u, w).$$

Adding both ensuing equations and using (4), we obtain

$$\frac{d}{dt} \left( \frac{1}{2} \|u\|^2 + \frac{j}{2} \|w\|^2 \right) + \nu \|\nabla u\|^2 + c_1 \|\nabla w\|^2 + c_2 \|\nabla \cdot w\|^2 + 4\nu_r \|w\|^2 = 2\nu_r (\nabla \times u, w).$$

(5)

Invoking with the Cauchy-Schwarz inequality, Young inequality and (2), the right hand side of (5) can be estimated as

$$4\nu_r (\nabla \times u, w) \leq 4\nu_r \|\nabla \times u\| \|w\| \leq \nu_r \|\nabla \times u\|^2 + 4\nu_r \|w\|^2 \leq \nu_r \|\nabla u\|^2 + 4\nu_r \|w\|^2.$$ 

(6)

Inserting (6) into (5), we have

$$\frac{d}{dt} \left( \frac{1}{2} \|u\|^2 + \frac{j}{2} \|w\|^2 \right) + \nu \|\nabla u\|^2 + c_1 \|\nabla w\|^2 + c_2 \|\nabla \cdot w\|^2 \leq 0.$$

Note that the energy dissipation law for the MNS equations is an inequality rather than an equality, which is different from the one for many other systems. The main reason is that the coupling terms can not be canceled automatically in the process of deriving the energy estimates. We further find that the nonlinear terms do not contribute to the energy due to the skew-symmetric property in the above proof to obtain the law of energy dissipation. The unique "zero-energy-contribution" property will be used to design efficient numerical schemes.
3. Numerical scheme

In this section, we propose a Euler IMEX scheme based on the SAV approach for the MNS equations and show that it is unconditionally energy stable.

Inspired by the recent works [13], we introduce a scalar auxiliary variable

\[ q(t) := \exp\left(-\frac{t}{T}\right). \]  

(7)

Noticing that \( q(t)/\exp\left(-\frac{t}{T}\right) = 1 \), we reformulate (1a) and (1c) into the equivalent forms as follows,

\[ u_t + \frac{q}{\exp\left(-\frac{t}{T}\right)} u \cdot \nabla u - \nu_0 \Delta u + \nabla p - 2\nu_r \nabla \times w = 0, \]  

(8)

and

\[ jw_t + \frac{q}{\exp\left(-\frac{t}{T}\right)} u \cdot \nabla w - c_1 \Delta w - c_2 \nabla \nabla \cdot w + 4\nu_r w - 2\nu_r \nabla \times u = 0. \]  

(9)

Differentiating (7) and using (4), we have

\[ \frac{dq}{dt} = -\frac{1}{T}q + \frac{1}{\exp\left(-\frac{t}{T}\right)} ((u \cdot \nabla u, u) + j (u \cdot \nabla w, w)). \]  

(10)

The last term in this equation is added to balance the nonlinear terms in (8) and (9) in the discretized case. Combining (8)-(10), we recast the original MNS equations as:

\[ u_t + \frac{q}{\exp\left(-\frac{t}{T}\right)} u \cdot \nabla u - \nu_0 \Delta u + \nabla p - 2\nu_r \nabla \times w = 0, \]  

(11a)

\[ \nabla \cdot u = 0, \]  

(11b)

\[ jw_t + \frac{q}{\exp\left(-\frac{t}{T}\right)} u \cdot \nabla w - c_1 \Delta w - c_2 \nabla \nabla \cdot w + 4\nu_r w - 2\nu_r \nabla \times u = 0, \]  

(11c)

\[ \frac{dq}{dt} + \frac{1}{T}q - \frac{1}{\exp\left(-\frac{t}{T}\right)} ((u \cdot \nabla u, u) + j (u \cdot \nabla w, w)) = 0. \]  

(11d)

It is clear that provided with \( q(0) = 1 \), the exact solution of (11d) is given by (7). Therefore, the above system is equivalent to the original system. Note that the SAV \( q(t) \) is related to the nonlinear part of the free energy in the original SAV approach. However, the SAV \( q(t) \) in this paper is purely artificial, which will allow us to construct unconditional energy stable schemes with fully explicit treatment of the nonlinear terms.

**Theorem 3.1.** The expanded system (11) admits the following energy estimate,

\[ \frac{d}{dt} \left( \frac{1}{2} ||u||^2 + \frac{j}{2} ||w||^2 + \frac{1}{2} ||q||^2 \right) + \nu ||\nabla u||^2 + c_1 ||\nabla w||^2 + c_2 ||\nabla \cdot w||^2 + \frac{1}{T}||q||^2 \leq 0. \]  

(12)

**Proof.** Taking the \( L^2 \)-inner product of (11a) with \( u \), using the integration by parts and (11b), we get

\[ \frac{1}{2} \frac{d}{dt} ||u||^2 + \nu_0 ||\nabla u||^2 + \frac{q}{\exp\left(-\frac{t}{T}\right)} (u \cdot \nabla u, u) = 2\nu_r (\nabla \times u, w). \]  

(13)

Taking the \( L^2 \)-inner product of (11c) with \( w \), and using the integration by parts, we have

\[ \frac{j}{2} \frac{d}{dt} ||w||^2 + c_1 ||\nabla w||^2 + \frac{q}{\exp\left(-\frac{t}{T}\right)} (u \cdot \nabla w, w) + c_2 ||\nabla \cdot w||^2 + 4\nu_r ||w||^2 = 2\nu_r (\nabla \times u, w). \]  

(14)
Multiplying (11d) with \( q \), we obtain
\[
\frac{1}{2} \frac{d}{dt} \abs{q}^2 + \frac{1}{T} \abs{q}^2 - \frac{q}{\exp\left(-\frac{t}{T}\right)} \left( (u \cdot \nabla u, u) + j (u \cdot \nabla w, w) \right) = 0. \tag{15}
\]

By combining (13)-(15), we derive
\[
\frac{d}{dt} \left( \frac{1}{2} \|u\|^2 + \frac{j}{2} ||w||^2 + \frac{1}{2} \abs{q}^2 \right) + \nu_0 \|\nabla u\|^2 + c_1 \|\nabla w\|^2 + c_2 \|\nabla \cdot w\|^2 + 4 \nu_r \|w\|^2 + \frac{1}{T} \|q\|^2 = 4 \nu_r (\nabla \times u, w). \tag{16}
\]

We finish the proof by using the estimate (6).

**Remark 3.1.** In this paper, the scalar auxiliary variable is only a time-dependent function \( q(t) = \exp(-t/T) \) not a energy-related function. With this treatment, the algebraic equation for the scalar auxiliary variable is linear and unsolvent. Moreover, the ordinary differential equation for \( q(t) \) is linear and dissipative, which makes our error estimates easier. In fact, the scalar auxiliary variable of this type admits a general form, \( q(t) = C_{q,0} \exp\left(-C_{q,1} t/T\right) \) with \( C_{q,0} \neq 0 \) and \( C_{q,1} \geq 0 \). We refer to [22] for more details about this extension.

### 3.1. The SAV scheme

Let \( \{t_n = n \tau : n = 0, 1, \ldots, N\}, \tau = T/N, \) be an equidistant partition of the time interval \([0, T]\). We denote \((\cdot)^n\) as the variable \((\cdot)\) at time step \( n \). For any function \( v \), define
\[
\delta_t u^{n+1} = \frac{u^{n+1} - u^n}{\tau}.
\]

Combining the backward Euler method and some delicate implicit/explicit treatments for coupling terms, we propose a first-order SAV scheme for solving the system (11) as follows. Given the initial conditions \( u^0, w^0 \) and \( q^0 \), compute \((u^{n+1}, p^{n+1}, w^{n+1}, q^{n+1}), n = 0, 1, \ldots, N - 1 \) by
\[
\delta_t u^{n+1} + \frac{q^{n+1}}{\exp\left(-\frac{t^{n+1}}{T}\right)} u \cdot \nabla u^n - \nu_0 \Delta u^{n+1} + \nabla p^{n+1} = 2 \nu_r \nabla \times w^n, \tag{17a}
\]
\[
\text{div} u^{n+1} = 0, \tag{17b}
\]
\[
\delta_t w^{n+1} + \frac{q^{n+1}}{\exp\left(-\frac{t^{n+1}}{T}\right)} u \cdot \nabla w^n - c_1 \Delta w^{n+1} - c_2 \nabla \nabla \cdot w^{n+1} + 4 \nu_r w^{n+1} = 2 \nu_r \nabla \times u^{n+1} \tag{17c}
\]
\[
\delta_t q^{n+1} + \frac{1}{T} q^{n+1} - \frac{1}{\exp\left(-\frac{t^{n+1}}{T}\right)} \left( (u^n \cdot \nabla u^n, u^{n+1}) + j (u^n \cdot \nabla w^n, w^{n+1}) \right) = 0. \tag{17d}
\]

Before giving further stability estimates, we first elaborate on how to implement the proposed scheme efficiently. Since the auxiliary variable \( q(t) \) is a scalar number rather than a field function, we can solve the nonlocally coupled scheme in a decoupled fashion. Denote
\[
S^{n+1} := q^{n+1} \exp\left(\frac{(t^{n+1})}{T}\right). \tag{18}
\]

We rewrite the first three equations in (17) into
\[
\delta_t u^{n+1} - \nu_0 \Delta u^{n+1} + \nabla p^{n+1} = 2 \nu_r \nabla \times w^n - S^{n+1} u^n \cdot \nabla u^n, \tag{19a}
\]
\[
\text{div} u^{n+1} = 0, \tag{19b}
\]
\[
\delta_t w^{n+1} - c_1 \Delta w^{n+1} - c_2 \nabla \nabla \cdot w^{n+1} + 4 \nu_r w^{n+1} - 2 \nu_r \nabla \times u^{n+1} = -j S^{n+1} (u^n \cdot \nabla) w^n. \tag{19c}
\]
Barring the unknown scalar number $S^{n+1}$, they are linear equations with respect to $u^{n+1}$ and $w^{n+1}$. Inspired by the work in [14], we define two field functions $(u_1^{n+1}, p_1^{n+1}, w_1^{n+1})$, $i = 1, 2$, as solutions to the following two problems:

\[
\frac{u_1^{n+1} - u^n}{\tau} - \nu_0 \Delta u_1^{n+1} + \nabla p_1^{n+1} = 2 \nu_r \nabla \times w^n, \quad \text{div} u_1^{n+1} = 0, \tag{19}
\]

\[
\frac{w_1^{n+1} - w^n}{\tau} - c_1 \Delta w_1^{n+1} - c_2 \nabla \cdot w_1^{n+1} + 4 \nu_r w_1^{n+1} - 2 \nu_r \nabla \times u_1^{n+1} = 0. \tag{20}
\]

and

\[
\frac{u_2^{n+1}}{\tau} - \nu_0 \Delta u_2^{n+1} + \nabla p_2^{n+1} = -u^n \cdot \nabla u^n, \quad \text{div} u_2^{n+1} = 0, \tag{22}
\]

\[
\frac{w_2^{n+1}}{\tau} - c_1 \Delta w_2^{n+1} - c_2 \nabla \cdot w_2^{n+1} + 4 \nu_r w_2^{n+1} - 2 \nu_r \nabla \times u_2^{n+1} = -j(u^n \cdot \nabla)w^n. \tag{24}
\]

Then it is straightforward to verify that the solution of $(u, p, w)$ to the scheme (17) is given by

\[
u_0 \Delta u_1^{n+1} + \nabla p_1^{n+1} = 2 \nu_r \nabla \times w^n, \quad \text{div} u_1^{n+1} = 0, \tag{25}
\]

\[
u_0 \Delta u_2^{n+1} + \nabla p_2^{n+1} = -u^n \cdot \nabla u^n, \quad \text{div} u_2^{n+1} = 0, \tag{26}
\]

\[
u_0 \Delta u_1^{n+1} + \nabla p_1^{n+1} = 2 \nu_r \nabla \times w^n, \quad \text{div} u_1^{n+1} = 0, \tag{27}
\]

where $S^{n+1}$ is to be determined. Inserting (25)-(27) into (17d), we have

\[
\left( \frac{\tau + T}{\tau T} - \exp \left( \frac{2I^{n+1}}{T} \right) A_2 \right) \exp \left( -\frac{In^{n+1}}{T} \right) S^{n+1} = \exp \left( \frac{In^{n+1}}{T} \right) A_1 + \frac{1}{\tau} q^n, \tag{28}
\]

where $A_i = (u^n \cdot \nabla u^n, u_i^{n+1}) + j(u^n \cdot \nabla w^n, w_i^{n+1})$, $i = 1, 2$. From (22)-(24), we know that the coefficient of (28) is positive,

\[
\frac{\tau + T}{\tau T} - \exp \left( \frac{2I^{n+1}}{T} \right) A_2 = \left( \frac{\tau + T}{\tau T} + \exp \left( \frac{2I^{n+1}}{T} \right) \right) \left( \frac{\|u_2^{n+1}\|^2 + j\|w_2^{n+1}\|^2}{\tau} + \nu_0 \|\nabla u_2^{n+1}\|^2 + c_1 \|\nabla w_2^{n+1}\|^2 \right)
\]

\[
+ c_2 \|\nabla \cdot w_2^{n+1}\|^2 + 4 \nu_r \|w_2^{n+1}\|^2 - 2 \nu_r (\nabla \times u_2^{n+1}, w_2^{n+1}) .
\]

\[
\geq \left( \frac{\tau + T}{\tau T} + \exp \left( \frac{2I^{n+1}}{T} \right) \right) \left( \frac{\|u_2^{n+1}\|^2 + j\|w_2^{n+1}\|^2}{\tau} + \nu \|\nabla u_2^{n+1}\|^2 + c_1 \|\nabla w_2^{n+1}\|^2 \right)
\]

\[
+ c_2 \|\nabla \cdot w_2^{n+1}\|^2 + 3 \nu_r \|w_2^{n+1}\|^2 .
\]

This implies the existence and uniqueness of $S^{n+1}$. Thus, we arrive at the final solution algorithm. It involves the following three steps:

1. Solve $(u_i^{n+1}, p_i^{n+1}, w_i^{n+1})$, $i = 1, 2$ by two substeps:
   (a) Solve equations (19)-(20) and (22)-(23) for $(u_i^{n+1}, p_i^{n+1})$, $i = 1, 2$;
   (b) Solve equations (21) and (24) for $w_i^{n+1}$, $i = 1, 2$.

2. Solve equation (28) for $S^{n+1}$.

3. Compute $(u^{n+1}, p^{n+1}, w^{n+1})$ by (25)-(27), compute $q^{n+1}$ by (18).
To conclude, we only need to solve two generalized Stokes equations, and two elliptic equations with constant coefficients plus a purely linear algebraic equation at each time step. Therefore, the scheme is quite efficient in the implementation.

**Remark 3.2.** In this paper, we decouple the NS equations and the angular momentum equation by time-lagging of the angular velocity in (17a). Therefore, we can first solve the NS equations and then solve the angular momentum equation in succession. One can further lag the linear velocity in (17c) to solve the NS equations and the angular momentum equation in parallel. The corresponding theoretical analysis is much similar to the one for the proposed scheme, we leave it for the interested readers.

**Remark 3.3.** From the previous discussions, we can see that the SAV $q(t)$ can help us to design an unconditionally stable scheme. Meanwhile, it can decompose the discrete equations into some sub-equations that can be solved efficiently. In addition, it can also provide a practical strategy of adaptive time-stepping [11, 21]. Generally speaking, when $S^{n+1} = q^{n+1} \exp(T^{n+1}/T)$ deviates from 1, the time step $\tau$ needs to be refined in order to maintain the accuracy. When $S^{n+1}$ stays close to 1, the time step $\tau$ can be relaxed. The detailed mechanism of the variable time step is an interesting work for future research.

### 3.2. Energy stability

The unconditionally energy stability of the scheme is established in the following theorem.

**Theorem 3.2.** The scheme (17) is unconditionally stable in the sense that

$$E^{n+1} - E^n \leq -\tau \nu \left( \| \nabla u^{n+1} \|^2 - \tau c_1 \| \nabla w^{n+1} \|^2 - \tau c_2 \| \nabla \cdot w^{n+1} \|^2 - \frac{\tau}{T} |q^{n+1}|^2 \right) \forall n \geq 0. \quad (29)$$

where

$$E^n = \frac{1}{2} \| u^n \|^2 + \frac{\tau + 4\nu}{2} \| w^n \|^2 + \frac{1}{2} |q^n|^2.$$

**Proof.** Taking the $L^2$-inner product of (17a) with $u^{n+1}$, using the identity

$$(a - b, a) = \frac{1}{2} (|a|^2 - |b|^2 + |a - b|^2), \quad (30)$$

and (17b), we get

$$\frac{\| u^{n+1} \|^2 - \| u^n \|^2 + \| u^{n+1} - u^n \|^2}{2\tau} + \nu_0 \| \nabla u^{n+1} \|^2 = 2\nu_r (\nabla \times w^n, u^{n+1}) - \frac{q^{n+1}}{\exp(-\tau^n/T)} (u^n \cdot \nabla u^n, u^{n+1}). \quad (31)$$

Taking the $L^2$-inner product of (17c) with $w^{n+1}$ and using the identity (30) again, we obtain

$$\frac{j (\| w^{n+1} \|^2 - \| w^n \|^2 + \| w^{n+1} - w^n \|^2}{2\tau} + c_1 \| \nabla w^{n+1} \|^2 + c_2 \| \nabla \cdot w^{n+1} \|^2 + 4\nu_r \| w^{n+1} \|^2 = 2\nu_r (\nabla \times u^{n+1}, w^{n+1}) - j \frac{q^{n+1}}{\exp(-\tau^n/T)} (u^n \cdot \nabla w^n, w^{n+1}). \quad (32)$$

Multiplying (17d) by $q^{n+1}$ and using the identity (30) again, we have

$$\frac{|q^{n+1}|^2 - |q^n|^2 + |q^{n+1} - q^n|^2}{2\tau} = \frac{q^{n+1}}{\exp(-\tau^n/T)} \left( (u^n \cdot \nabla u^n, u^{n+1}) + j (u^n \cdot \nabla w^n, w^{n+1}) \right). \quad (33)$$

By taking the summations of (31)-(33), we get

$$\frac{\| u^{n+1} \|^2 - \| u^n \|^2 + \| u^{n+1} - u^n \|^2}{2\tau} + \frac{\| w^{n+1} \|^2 - \| w^n \|^2 + \| w^{n+1} - w^n \|^2}{2\tau} = \frac{q^{n+1}}{\exp(-\tau^n/T)} \left( (u^n \cdot \nabla u^n, u^{n+1}) + j (u^n \cdot \nabla w^n, w^{n+1}) \right).$$
Using Cauchy-Schwarz inequality and Young inequality, we derive the right hand side of (34) has the following estimate,

\[
\begin{align*}
2\nu_r (\nabla \times w^n, u^{n+1}) + 2\nu_r (\nabla \times u^{n+1}, w^{n+1}) &\leq 2\nu_r \| \nabla \times u^{n+1} \| \| w^n \| + 2\nu_r \| \nabla \times u^{n+1} \| \| w^{n+1} \|
\leq \nu_r \| \nabla u^{n+1} \|^2 + 2\nu_r \| w^n \|^2 + 2\nu_r \| w^{n+1} \|^2.
\end{align*}
\] (35)

Plugging (35) into (34), we gain the required estimate. The proof is thus complete. \qed

We observe that the discrete energy dissipation law (29) is an approximation of the continuous energy dissipation law (12).

**Corollary 3.1** (Stability). Let \((u^n, w^n, p^n, q^n), n \geq 0\) solve (17). Then it satisfies the following stability estimate for any \(m \geq 0\),

\[
\begin{align*}
\| u^m \|^2 &+ (j + 4\tau \nu_r) \| w^m \|^2 + |q^m|^2 \\
&+ 2\tau \sum_{n=0}^{m} \left( \nu \| \nabla u^{n+1} \|^2 + c_1 \| \nabla w^{n+1} \|^2 + c_2 \| \nabla \cdot w^n \|^2 + \frac{1}{T} |q^{n+1}|^2 \right)
\leq \| u^0 \|^2 + (j + 4\tau \nu_r) \| w^0 \|^2 + |q^0|^2.
\end{align*}
\] (36)

**Proof.** By Theorem 3.2, summing up inequality (29) from \(n = 0\) to \(m - 1\), we obtain the stable bound. \qed

Based on this corollary, we can easily obtain the following uniform bounds for any \(m \geq 0\),

\[
\begin{align*}
\| u^m \|^2 &+ j \| w^m \|^2 + |q^m|^2 \leq k_1, \\
\tau \sum_{n=0}^{m} \left( \nu \| \nabla u^n \|^2 + c_1 \| \nabla w^n \|^2 + c_2 \| \nabla \cdot w^n \|^2 + \frac{1}{T} |q^n|^2 \right) &\leq k_2,
\end{align*}
\] (37)

where the constants \(k_i \ (i = 1, 2)\) are independent of \(\tau\).

### 4. Error Analysis

In this section, we give a rigorous error analysis for the scheme (17) in two dimensions. We emphasize that while the scheme can be used in three dimensions, the error analysis cannot be readily extended to three dimensions due to some technical issues. Thus, we set \(d = 2\) in this section.

Denote the following error functions

\[
e_{u}^n = u^n - u(t^n), \quad e_{p}^n = p^n - p(t^n), \quad e_{w}^n = w^n - w(t^n), \quad e_{q}^n = q^n - q(t^n).
\]

Subtracting (11) at \(t^{n+1}\) from (17), and noticing \(q(t^{n+1}) = \exp(-t^{n+1}/T)\), we get the following error equations

\[
\begin{align*}
\delta_t e_{u}^{n+1} + \left( q^{n+1} \exp \left( \frac{t^{n+1}}{T} \right) u^n \cdot \nabla u^n - u(t^{n+1}) \cdot \nabla u(t^{n+1}) \right) \\
- \nu_0 \Delta e_{u}^{n+1} + \nabla e_{p}^{n+1} - 2\nu_r (\nabla \times w^n - \nabla \times w(t^{n+1})) &\equiv R_{u}^{n+1}, \tag{38a}
\n\nabla \cdot e_{u}^{n+1} &\equiv 0, \tag{38b}
\end{align*}
\]
Lemma 4.1. Moreover, for Lemma 4.2.

\[ \delta_t e_{q}^{n+1} + \frac{1}{T} e_{q}^{n+1} - \exp \left( -\frac{t^{n+1}}{T} \right) \left( \frac{u^n \cdot \nabla u^n}{t^n} - \left( u(t^{n+1}) \cdot \nabla u(t^{n+1}) \right) \right) \]

\[ - c_1 \Delta_s e_{w}^{n+1} - c_2 \nabla \cdot e_{w}^{n+1} + 4 \nu e_w^{n+1} - 2 \nu \nabla x e_u^{n+1} = R_w^{n+1}, \]

where \( R_w^{n+1} = 1 \) and the truncation errors,

\[ R_{u}^{n+1} := \frac{1}{T} \int_{t^n}^{t^{n+1}} (s-t^n) u_{tt}(s) ds, R_{w}^{n+1} := \frac{1}{T} \int_{t^n}^{t^{n+1}} (s-t^n) w_{tt}(s) ds, R_{q}^{n+1} := \frac{1}{T} \int_{t^n}^{t^{n+1}} (s-t^n) q_{tt}(s) ds. \]

Let \( P \) be the orthogonal projector in \( L^2(\Omega) \) onto \( V \), we define the Stokes operator \( A \) by

\[ Au = -P \Delta u, \quad \forall u \in D(A) = H^2(\Omega) \cap X. \]

The following estimates for the trilinear form \( b(\cdot, \cdot, \cdot) \) will be used in our analysis [13, 15, 23, 24].

Lemma 4.1. The following estimates of the trilinear form hold

\[ b(u,v,w) \leq C_{b,0} \|\nabla u\| \|\nabla v\| \|\nabla w\|, \quad \forall u, v, w \in X, \]

\[ b(u,v,w) \leq C_{b,1} \|u\| \|v\|_2 \|\nabla w\|, \quad \forall u, w \in X, \quad v \in X \cap H^2(\Omega), \]

\[ b(u,v,w) \leq C_{b,2} \|u\|_2 \|v\| \|\nabla w\|, \quad \forall v, w \in X, \quad u \in V \cap H^2(\Omega), \]

\[ b(u,v,w) \leq C_{b,3} \|\nabla u\| \|v\| \|\nabla w\|_2, \quad \forall u \in V, \quad v \in X, \quad w \in X \cap H^2(\Omega), \]

\[ b(u,v,w) \leq C_{b,4} \|\nabla u\| \|v\| \|\nabla w\|_2, \quad \forall u, v \in X, \quad w \in X \cap H^2(\Omega), \]

\[ b(u,v,w) \leq C_{b,5} \|\nabla u\| \|v\|_2 \|\nabla w\|, \quad \forall u, w \in X, \quad v \in X \cap H^2(\Omega). \]

Moreover, for \( d = 2 \), we have

\[ b(u,v,w) \leq C_{b,6} \|\nabla u\|^{1/2} \|u\|^{1/2} \|\nabla v\|^{1/2} \|v\|^{1/2} \|\nabla w\|, \quad \forall u \in V, \quad v, w \in X, \]

\[ b(u,v,w) \leq C_{b,7} \|\nabla u\|^{1/2} \|u\|^{1/2} \|\nabla v\|^{1/2} \|A v\|^{1/2} \|w\|, \quad \forall v \in X \cap H^2(\Omega), \quad u, w \in X, \]

\[ b(u,v,w) \leq C_{b,8} \|u\|^{1/2} \|A u\|^{1/2} \|\nabla v\| \|w\|, \quad \forall u \in X \cap H^2(\Omega), \quad v, w \in X. \]

The following discrete version of the Gronwall lemma will be frequently used [23, 25, 26].

Lemma 4.2. Let \( a_n, b_n, c_n, \) and \( d_n \) be four non-negative sequences satisfying

\[ a_m + \tau \sum_{n=1}^{m} b_n \leq \tau \sum_{n=0}^{m-1} a_n d_n + \tau \sum_{n=0}^{m-1} c_n + C, \quad m \geq 1, \]

where \( C \) and \( \tau \) are two positive constants. Then

\[ a_m + \tau \sum_{n=1}^{m} b_n \leq \exp \left( \tau \sum_{n=0}^{m-1} d_n \right) \left( \tau \sum_{n=0}^{m-1} c_n + C \right), \quad m \geq 1. \]

4.1. Error estimates for the velocity and angular velocity

In this subsection, we derive the following error estimates for the velocity \( u \) and angular velocity \( w \).
For the second term on the right hand side of (50), we have

\[ \|e_n^{m+1}\|^2 + \nu \epsilon \sum_{n=0}^{m} \|\nabla e_n^{m+1}\|^2 + (j + 4\nu \tau) \|e_n^{m+1}\|^2 + c_1 \epsilon \sum_{n=0}^{m} \|\nabla e_n^{m+1}\|^2 + 2c_2 \tau \sum_{n=0}^{m} \|\nabla \cdot e_n^{m+1}\|^2 + |e_n^{m+1}|^2 + \frac{\tau}{\epsilon} \sum_{n=0}^{m} |e_n^{m+1}|^2 \leq C \tau^2, \quad \forall 0 \leq m \leq N - 1. \] (48)

The proof of the above theorem will be carried out with a sequence of lemmas below.

**Lemma 4.3.** Under the assumptions of Theorem 4.1, we have

\[ \|e_n^{m+1}\|^2 - \|e_n^{m}\|^2 + \|e_n^{m+1} - e_n^{m}\|^2 + \frac{\nu + \nu \tau}{2} \|\nabla e_n^{m+1}\|^2 \leq \exp \left( \frac{t_n^{m+1}}{T} \right) e_n^{m+1} \left( u^n \cdot \nabla u^n, e_n^{m+1} \right) + 2\nu \epsilon \|e_n^{m+1}\|^2 + C \left( \|u(t^n)\|^2 + \|u(t^{m+1})\|^2 + \|\nabla e_u^n\|^2 \right) \|e_n^{m+1}\|^2 + C \tau \int_{t_n}^{t_n^{m+1}} \|u_t(t)\|^2 ds + C \tau \int_{t_n}^{t_n^{m+1}} \|u_t(s)\|^2 ds, \quad \forall 0 \leq n \leq N - 1. \] (49)

**Proof.** Taking the inner product of (38a) with \( e_n^{m+1} \) and using (38b), we obtain

\[ \|e_n^{m+1}\|^2 - \|e_n^{m}\|^2 + \|e_n^{m+1} - e_n^{m}\|^2 + \frac{\nu + \nu \tau}{2} \|\nabla e_n^{m+1}\|^2 = \left( R_n^{m+1}, e_n^{m+1} \right) + \left( u(t^{m+1}) \cdot \nabla u(t^{m+1}) - q^{m+1} \exp \left( \frac{t^{m+1}}{T} \right) u^n \cdot \nabla u^n, e_n^{m+1} \right) + 2\nu \epsilon (\nabla \times u^n - \nabla \times u(t^{m+1}), e_n^{m+1}). \] (50)

For the first term on the right hand side of (50), we get

\[ \left( R_n^{m+1}, e_n^{m+1} \right) \leq \frac{\nu}{8} \|\nabla e_n^{m+1}\|^2 + C \tau \int_{t_n}^{t_n^{m+1}} \|u_t(s)\|^2 ds. \] (51)

For the second term on the right hand side of (50), we have

\[ \left( u(t^{m+1}) \cdot \nabla u(t^{m+1}) - q^{m+1} \exp \left( \frac{t^{m+1}}{T} \right) u^n \cdot \nabla u^n, e_n^{m+1} \right) = \left( (u(t^{m+1}) - u^n) \cdot \nabla u(t^{m+1}, e_n^{m+1}) + (u^n \cdot \nabla u(t^{m+1}) - u^n), e_n^{m+1} \right) - \exp \left( \frac{t^{m+1}}{T} \right) q^{m+1} (u^n \cdot \nabla u^n, e_n^{m+1}). \] (52)

Using Cauchy-Schwarz inequality and (40), the first term on the right hand side of (52) can be bounded by

\[ \left( (u(t^{m+1}) - u^n) \cdot \nabla u(t^{m+1}, e_n^{m+1}) \right. \leq C \|u(t^{m+1}) - u^n\| \|u(t^{m+1})\|_2 \|\nabla e_n^{m+1}\| \leq \frac{\nu}{8} \|\nabla e_n^{m+1}\|^2 + C \|u(t^{m+1}) - u^n\|^2 \|u(t^{m+1})\|^2 \]
Lemma 4.4. Under the assumptions of Theorem 4.1, we have

\[
\frac{\nu}{8} \left\| \nabla e_{u}^{n+1} \right\|^2 + C \left\| u(t^{n+1}) \right\|^2_2 \left\| e_{u}^{n+1} \right\|^2 + C \tau \left\| u(t^{n+1}) \right\|^2_2 \int_{t^n}^{t^{n+1}} \left\| u_{t}(s) \right\|^2 ds.
\] (53)

Similarly, using Cauchy-Schwarz inequality, (40)-(45) and Young inequality, the second term on the right hand side of (52) can be estimated as follows,

\[
(u \cdot \nabla (u(t^{n+1}) - u^n), e_{u}^{n+1}) = (u \cdot \nabla (u(t^{n+1}) - u(t^n)), e_{u}^{n+1}) - (e_{u}^n \cdot \nabla u(t^n), e_{u}^{n+1}) - (u(t^n) \cdot \nabla e_{u}^n, e_{u}^{n+1})
\]

\[
\leq C_{b,1} \left\| u^n \right\| \left\| u(t^{n+1}) - u(t^n) \right\|_2 \left\| \nabla e_{u}^{n+1} \right\| + C_{b,6} \left\| \nabla e_{u}^{n} \right\|^{1/2} \left\| e_{u}^{n} \right\|^{1/2} \left\| \nabla e_{u}^{n+1} \right\| + C_{b,2} \left\| u(t^n) \right\|_2 \left\| e_{u}^{n} \right\| \left\| \nabla e_{u}^{n+1} \right\|
\]

\[
\leq \frac{\nu}{8} \left\| \nabla e_{u}^{n+1} \right\|^2 + C \left( \left\| u(t^n) \right\|^2_2 + \left\| \nabla e_{u}^{n} \right\|^2 \right) \left\| e_{u}^{n} \right\|^2 + C \left\| u^n \right\| \int_{t^n}^{t^{n+1}} \left\| u_{t}(s) \right\|^2 ds.
\] (54)

For the last term on the right hand side of (50), we invoke with Cauchy-Schwarz inequality, Young inequality and (2) to get

\[
2\nu \tau \left( \nabla \times w^n - \nabla \times w(t^{n+1}), e_{u}^{n+1} \right)
\]

\[
= 2\nu \tau \left( w^n - w(t^{n+1}), \nabla \times e_{u}^{n+1} \right)
\]

\[
= 2\nu \tau \left( e_{w}^n \cdot \nabla \times e_{u}^{n+1} \right) - 2\nu \tau \left( w(t^{n+1}) - w(t^n), \nabla \times e_{u}^{n+1} \right)
\]

\[
\leq 2\nu \tau \left\| e_{w}^n \right\| \left\| \nabla \times e_{u}^{n+1} \right\| + 2\nu \tau \left\| \int_{t^n}^{t^{n+1}} w_{t}(s) ds \right\| \left\| \nabla \times e_{u}^{n+1} \right\|
\]

\[
\leq \left( \frac{1}{2} + \frac{\nu}{8} \right) \left\| \nabla e_{u}^{n+1} \right\|^2 + 2\nu \tau \left\| e_{w}^{n} \right\|^2 + C \tau \int_{t^n}^{t^{n+1}} \left\| w_{t}(s) \right\|^2 ds.
\] (55)

Finally, combining (50) with (51)-(55) leads to the desired result. □

Next, we derive an estimate for the angular velocity errors in $L^2$-norm.

Lemma 4.4. Under the assumptions of Theorem 4.1, we have

\[
j \left\| e_{w}^{n+1} \right\|^2 - \left\| e_{w}^{n} \right\|^2 + \left\| e_{w}^{n+1} - e_{w}^{n} \right\|^2 \leq \frac{c_{1}}{2} \left\| \nabla e_{w}^{n+1} \right\|^2 + c_{2} \left\| \nabla \cdot e_{w}^{n+1} \right\|^2 + 2\nu \tau \left\| e_{w}^{n+1} \right\|^2
\]

\[
\leq \frac{\nu}{2} \left\| \nabla e_{u}^{n+1} \right\|^2 - j \exp \left( \frac{t^{n+1}}{T} \right) e_{q}^{n+1} (u^n \cdot \nabla u^n, e_{w}^{n+1})
\]

\[
+ C \left( \left\| w(t^{n+1}) \right\|^2_2 + \left\| \nabla e_{w}^{n} \right\|^2 \right) \left\| e_{w}^{n} \right\|^2 + C \left( \left\| u(t^n) \right\|^2_2 + \left\| \nabla e_{w}^{n} \right\|^2 \right) \left\| e_{w}^{n} \right\|^2
\]

\[
+ C \tau \int_{t^n}^{t^{n+1}} \left\| w_{t}(s) \right\|^2_2 ds + C \tau \left\| w(t^{n+1}) \right\|^2_2 \int_{t^n}^{t^{n+1}} \left\| w_{t}(s) \right\|^2_2 ds
\]

\[
+ C \tau \left\| u^n \right\| \int_{t^n}^{t^{n+1}} \left\| w_{t}(s) \right\|^2_2 ds, \quad \forall 0 \leq n \leq N - 1.
\] (56)

Proof. Taking the inner product of (38c) with $e_{w}^{n+1}$, we obtain

\[
j \left\| e_{w}^{n+1} \right\|^2 - \left\| e_{w}^{n} \right\|^2 + \left\| e_{w}^{n+1} - e_{w}^{n} \right\|^2 \leq \frac{c_{1}}{2} \left\| \nabla e_{w}^{n+1} \right\|^2 + c_{2} \left\| \nabla \cdot e_{w}^{n+1} \right\|^2 + 4\nu \tau \left\| e_{w}^{n+1} \right\|^2
\]
For the last term on the right hand side of (57), we obtain
\[
(R_{w}^{n+1}, e_{w}^{n+1}) = \frac{c_{1}}{6} \left\| \nabla e_{w}^{n+1} \right\|^{2} + C \tau \int_{t^{n}}^{t^{n+1}} \| w_{t}(s) \|^{2} ds.
\] (58)

The second term on the right hand side of (57) can be estimated as follows by using the similar procedure in (52). Thus, we recast as
\[
\frac{1}{2} \left( u(t^{n+1}) \cdot \nabla w(t^{n+1}) - q^{n} \exp \left( \frac{t^{n+1}}{\tau} \right) (u^{n} \cdot \nabla w^{n}) , e_{w}^{n+1} \right) - j \exp \left( \frac{t^{n+1}}{\tau} \right) e_{q}^{n+1} (u^{n} \cdot \nabla w^{n}, e_{w}^{n+1}).
\] (59)

For the first term on the right hand side of (59), similar to (53), we have
\[
\frac{1}{2} \left( u(t^{n+1}) - u^{n} \right) \cdot \nabla w(t^{n+1}), e_{w}^{n+1} \right)
\leq j C_{b,1} \left\| u(t^{n+1}) - u^{n} \right\| \left\| w(t^{n+1}) \right\| \left\| \nabla e_{w}^{n+1} \right\|
\leq \frac{c_{1}}{6} \left\| \nabla e_{w}^{n+1} \right\|^{2} + C \left\| u(t^{n+1}) - u^{n} \right\|^{2} \left\| w(t^{n+1}) \right\|^{2}
\leq \frac{c_{1}}{6} \left\| \nabla e_{w}^{n+1} \right\|^{2} + C \left\| w(t^{n+1}) \right\|^{2} \left\| e_{w}^{n} \right\|^{2} + C \tau \left\| w(t^{n+1}) \right\|^{2} \int_{t^{n}}^{t^{n+1}} \left\| w_{t}(s) \right\|^{2} ds.\] (60)

For the second term on the right hand side of (59), similar to (54), we get
\[
\frac{1}{2} \left( u^{n} \cdot \nabla w(t^{n+1}) - w^{n} \right) e_{w}^{n+1} \right)
\leq j \left( u^{n} \cdot \nabla w(t^{n+1}) - w^{n} \right) e_{w}^{n+1} \right) - j \left( u^{n} \cdot \nabla e_{w}^{n+1} \right) - j \left( u(t^{n}) \cdot \nabla e_{w}^{n+1} \right)
\leq j C_{b,1} \left\| u^{n} \right\| \left\| w(t^{n+1}) \right\| \left\| \nabla e_{w}^{n+1} \right\|
\leq \frac{c_{1}}{6} \left\| \nabla e_{w}^{n+1} \right\|^{2} + C \left\| w(t^{n+1}) \right\|^{2} \left\| e_{w}^{n} \right\|^{2} + C \tau \left\| w(t^{n+1}) \right\|^{2} \int_{t^{n}}^{t^{n+1}} \left\| w_{t}(s) \right\|^{2} ds.\] (61)

For the last term on the right hand side of (57), we obtain
\[
(2 \nu \nabla \times e_{w}^{n+1}, e_{w}^{n+1}) \leq 2 \nu \left\| \nabla \times e_{w}^{n+1} \right\| \left\| e_{w}^{n+1} \right\| \leq \frac{\nu \tau}{2} \left\| \nabla e_{w}^{n+1} \right\|^{2} + 2 \nu \left\| e_{w}^{n+1} \right\|^{2}.\] (62)

Combining (57) with (58)-(62) leads to the desired result.

Now we turn to estimate the error for the auxiliary variable \( q \).
Lemma 4.5. Under the assumptions of Theorem 4.1, we have
\[
\frac{|e_{q+1}^n|^2 - |e_q^n|^2 + |e_q^n - e_q^n|^2}{2T} + \frac{1}{T} |e_{q+1}^n|^2 \\
\leq \exp\left(\frac{n+1}{T}\right) e_{q+1}^n (u_n \cdot \nabla u_n, e_{u+1}) + j \exp\left(\frac{T+1}{T}\right) e_{q+1}^n (u_n \cdot \nabla w_n, e_{w+1}) \\
+ \frac{1}{4k_2} \|\nabla u_n\|^2 |e_{q+1}^n|^2 + C |e_u^n|^2 \|w(t^n+1)\|_2^2 \\
+ C \left(\|u(t^{n+1})\|_2^2 + \|\nabla u(t^{n+1})\|_2^2 + \|w(t^{n+1})\|_2^2 \right) \|u_n\|^2 \\
+ C T \int_{t^n}^{n+1} |q(s)|^2 ds + C T \|w(t^{n+1})\|_2^2 \int_{t^n}^{n+1} \|w_n(s)\|^2 ds \\
+ C T \left(\|u(t^{n+1})\|_2^2 + \|\nabla u(t^{n+1})\|_2^2 + \|w(t^{n+1})\|_2^2 \right) \int_{t^n}^{n+1} \|u_n(s)\|^2 ds, \forall 0 \leq n \leq N - 1, (63)
\]
where \(k_2\) is defined by (37).

Proof. Multiplying both sides of (38d) by \(e_{q+1}^n\) gives
\[
\frac{|e_{q+1}^n|^2 - |e_q^n|^2 + |e_q^n - e_q^n|^2}{2T} + \frac{1}{T} |e_{q+1}^n|^2 \\
= R_{q+1}^n e_{q+1}^n + \exp\left(\frac{T+1}{T}\right) e_q^n \left( (u_n \cdot \nabla u_n, u^{n+1}) - (u(t^{n+1}) \cdot \nabla u(t^{n+1}), u(t^{n+1})) \right) \\
+ j \exp\left(\frac{T+1}{T}\right) e_{q+1}^n \left( (u_n \cdot \nabla w_n, w^{n+1}) - (u(t^{n+1}) \cdot \nabla w(t^{n+1}), w(t^{n+1})) \right). (64)
\]

The first term on the right hand side of (64) can be estimated as follows
\[
R_{q+1}^n e_{q+1}^n \leq \frac{1}{6T} |e_{q+1}^n|^2 + C T \int_{t^n}^{n+1} |q(s)|^2 ds. (65)
\]

The second term on the right hand side of (64) can be recast as
\[
\exp\left(\frac{T+1}{T}\right) e_q^n \left( (u_n \cdot \nabla u_n, u^{n+1}) - (u(t^{n+1}) \cdot \nabla u(t^{n+1}), u(t^{n+1})) \right) \\
= \exp\left(\frac{T+1}{T}\right) e_q^n \left( u_n \cdot \nabla u_n, e_{u+1}^n \right) + \exp\left(\frac{T+1}{T}\right) e_q^n \left( u_n \cdot \nabla (u_n - u(t^{n+1}), u(t^{n+1})) \right) \\
+ \exp\left(\frac{T+1}{T}\right) e_q^n \left( (u_n - u(t^{n+1})) \cdot \nabla u(t^{n+1}), u(t^{n+1}) \right). (66)
\]

Using (42) and (37), the second term on the right hand side of (66) is bounded by
\[
\exp\left(\frac{T+1}{T}\right) e_q^n \left( u_n \cdot \nabla (u_n - u(t^{n+1}), u(t^{n+1})) \right) \\
\leq \exp(1) C_{p,3} |e_{q+1}^n| \|\nabla u_n\| \|u_n - u(t^{n+1})\|_2 \|u(t^{n+1})\|_2 \\
\leq C \|\nabla u_n\| \|u(t^{n+1}) - u(t^n) - e_u^n\| \|u(t^{n+1})\|_2 \|e_{q+1}^n\| \\
\leq \frac{1}{8k_2} \|\nabla u_n\|^2 |e_{q+1}^n|^2 + C |e_u^n|^2 \|u(t^{n+1})\|_2^2 + C T \|u(t^{n+1})\|_2^2 \int_{t^n}^{n+1} \|u_n(s)\|^2 ds, (67)
\]

13
where $k_2$ is given by (37). In a same manner, the third term on the right hand side of (66) can be bounded by
\[
\exp \left( \frac{t_n^{n+1}}{T} \right) e_q^{n+1} \left( (u^n - u^n) \cdot \nabla u(t^{n+1}), u(t^{n+1}) \right) \\
\leq \exp(1) C_{\delta,4} \left\| u(t^{n+1}) - u^n \right\| \left\| \nabla u(t^{n+1}) \right\| \left\| u(t^{n+1}) \right\|_2 \left| e_q^{n+1} \right| \\
\leq C \left\| u(t^{n+1}) - u(t^n) \right\| - e_u^n \left\| \nabla u(t^{n+1}) \right\| \left\| u(t^{n+1}) \right\|_2 \left| e_q^{n+1} \right| \\
\leq \frac{1}{8\tau} | e_q^{n+1} |^2 + C \left\| \nabla u(t^{n+1}) \right\|^2 \left\| u(t^{n+1}) \right\|^2_2 \int_{t_n}^{t_{n+1}} \| u_s(s) \|^2 ds.
\]
(68)

Using the similar procedure in (66), the last term on the right hand side of (64) can be rewritten as
\[
\begin{align*}
&\quad \textstyle j \exp \left( \frac{t_n^{n+1}}{T} \right) e_q^{n+1} \left( (u^n \cdot \nabla w^n, w^{n+1}) - (u(t^{n+1}) \cdot \nabla w(t^{n+1}), w(t^{n+1})) \right) \\
&= j \exp \left( \frac{t_n^{n+1}}{T} \right) e_q^{n+1} \left( (u^n \cdot \nabla w^n, e^{n+1}_w) + j \exp \left( \frac{t_n^{n+1}}{T} \right) e_q^{n+1} \left( (u^n \cdot \nabla (w^n - w(t^{n+1})), w(t^{n+1})) \right) \\
&+ j \exp \left( \frac{t_n^{n+1}}{T} \right) e_q^{n+1} \left( (u^n - u(t^{n+1})) \cdot \nabla w(t^{n+1}), w(t^{n+1}) \right). \tag{69}
\end{align*}
\]

Similar to (67), the first term on the right hand side of (69) can be estimated by
\[
\begin{align*}
&\quad \textstyle j \exp \left( \frac{t_n^{n+1}}{T} \right) e_q^{n+1} \left( (u^n \cdot \nabla w^n, w(t^{n+1})) \right) \\
&\leq j \exp(1) C_{\delta,3} | e_q^{n+1} | \left\| \nabla u^n \right\| \left\| w^n - w(t^{n+1}) \right\| \left\| w(t^{n+1}) \right\|_2 \\
&\leq C \left\| \nabla u^n \right\| \left\| w(t^{n+1}) - w(t^n) \right\| - e_w^n \left\| w(t^{n+1}) \right\|_2 \left| e_q^{n+1} \right| \\
&\leq \frac{1}{8k_2} \left\| \nabla u^n \right\|^2 | e_q^{n+1} |^2 + C \left\| e_w^n \right\|^2 \left\| w(t^{n+1}) \right\|^2_2 \int_{t_n}^{t_{n+1}} \| w_s(s) \|^2 ds,
\end{align*}
\]
(70)

where $k_2$ is defined by (37). For the second term on the right hand side of (69), similar to (67), we have
\[
\begin{align*}
&\quad \textstyle j \exp \left( \frac{t_n^{n+1}}{T} \right) e_q^{n+1} \left( (u^n - u(t^{n+1})) \cdot \nabla w(t^{n+1}), w(t^{n+1}) \right) \\
&\leq j \exp(1) C_{\delta,4} \left\| u(t^{n+1}) - u^n \right\| \left\| \nabla w(t^{n+1}) \right\| \left\| w(t^{n+1}) \right\|_2 \left| e_q^{n+1} \right| \\
&\leq C \left\| u(t^{n+1}) - u(t^n) \right\| - e_u^n \left\| \nabla w(t^{n+1}) \right\| \left\| w(t^{n+1}) \right\|_2 \left| e_q^{n+1} \right| \\
&\leq \frac{1}{6\tau} | e_q^{n+1} |^2 + C \left\| \nabla w(t^{n+1}) \right\|^2 \left\| w(t^{n+1}) \right\|^2_2 \int_{t_n}^{t_{n+1}} \| u_s(s) \|^2 ds.
\end{align*}
\]
(71)

Combining (64) with (65)-(71) yields the desired result.

Now we are in the position to prove Theorem 4.1 by using Lemmas 4.3-4.5.

**Proof.** Summing up (49), (56), (63) and using (36), we have
\[
\begin{align*}
&\quad \| e^{n+1}_u \|^2 - \| e^n_u \|^2 + \| e^{n+1}_w - e^n_w \|^2 \leq \frac{\nu}{2\tau} \| \nabla e^{n+1}_w \|^2 + \frac{j}{2\tau} \| e^{n+1}_w \|^2 - \| e^n_w \|^2 + \| e^{n+1}_w - e^n_w \|^2
\end{align*}
\]
Multiplying (72) by 2 and using (73), we get

\[ \frac{c_1}{2} \| \nabla e_{q+1} \|^2 + C \left( 1 + \| \nabla e_{q+1} \|^2 + \| \nabla e_{w} \|^2 \right) \leq \tau \sum_{n=0}^{m^*} \| \nabla e_{q+1} \|^2 , \]

for all \( t \in [0,T]\).
Now turning to (72), multiply it by 2τ and sum up over n from 0 to m, and using (73), we derive
\[
\|e^{m+1}_u\|^2 + \nu \sum_{n=0}^{m} \|\nabla e^{n+1}_u\|^2 + (j + 4\nu \tau) \|e^{m+1}_w\|^2 + c_1 \tau \sum_{n=0}^{m} \|\nabla e^{n+1}_w\|^2
\]
\[
+ 2c_2 \tau \sum_{n=0}^{m} \|\nabla e^{n+1}_q\|^2 + |c_q^{m+1}|^2 + \tau \sum_{n=0}^{m} |c_q^{n+1}|^2
\]
\[
\leq \frac{\tau}{2k_2} \sum_{n=0}^{m} \|\nabla u^n\|^2 |e^{m+1}_u| + C \tau \sum_{n=0}^{m} \left(1 + \|\nabla e^n_u\|^2\right) \|e^n_u\|^2 + C \tau \sum_{n=0}^{m} \left(1 + \|\nabla e^n_w\|^2\right) \|e^n_w\|^2
\]
\[
+ C \nu \sum_{n=0}^{m+1} \left(\|u_n(s)\|^2 + \|w_n(s)\|^2 + \|u_{n+1}(s)\|^2 + |q_{n+1}(s)|^2 + \|w_{n+1}(s)\|^2\right) ds.
\]
(75)

We deduce form (37) again that
\[
\tau \sum_{n=0}^{m} \left(1 + \|\nabla e^n_u\|^2\right) \leq C, \quad \tau \sum_{n=0}^{m} \left(1 + \|\nabla e^n_w\|^2\right) \leq C.
\]

Using (75) and the discrete Gronwall inequality in Lemma 4.2 to (76), we complete the proof. \qed

4.2. Error estimates for the pressure

The section is devoted to prove the error estimate for the pressure. For this end, we first establish the estimate on \(\delta_t e^{n+1}_u\).

**Lemma 4.6.** Assume the exact solution satisfies \(u \in H^2(0,T; H^2(\Omega)) \cap H^1(0,T; H^2(\Omega)) \cap L^\infty(0,T; H^2(\Omega))\) and \(w \in H^2(0,T; H^{-1}(\Omega)) \cap H^1(0,T; H^2(\Omega)) \cap L^\infty(0,T; H^2(\Omega))\), then we have the following error estimate for \(m \geq 0\)
\[
\|\nabla e^{m+1}_u\|^2 + \tau \sum_{n=0}^{m} \|\delta_t e^{n+1}_u\|^2 + \nu \tau \sum_{n=0}^{m} \|A e^{n+1}_u\|^2 \leq C \tau^2.
\]
(77)

**Proof.** First of all, in virtue of (48), we have
\[
\|\nabla e^{n+1}_u\|^2 \leq \tau^{-1} \left(\tau \sum_{n=0}^{m} \|\nabla e^{n+1}_u\|^2\right) \leq C \tau, \quad \|\nabla e^{n+1}_u\|^2 \leq \tau^{-1} \left(\tau \sum_{n=0}^{m} \|\nabla e^{n+1}_u\|^2\right) \leq C \tau.
\]
Hence, there holds that
\[
\|\nabla u^{n+1}\| \leq \|\nabla e^{n+1}_u\| + \|\nabla u^{(n+1)}\| \leq C \left(\tau^{1/2} + \|\nabla u^{(n+1)}\|\right).
\]
(78)

Taking the inner product of (8a) with \(A e^{n+1}_u + \delta_t e^{n+1}_u\), we obtain
\[
(1 + \nu) \left(\frac{\|\nabla e^{n+1}_u\|^2 - \|\nabla e^n_u\|^2}{2\tau} + \|\nabla e^{n+1}_u - \nabla e^n_u\|^2 + \|\delta_t e^{n+1}_u\|^2 + \nu \|A e^{n+1}_u\|^2
\]
\[
= (R^{n+1}, A e^{n+1}_u + \delta_t e^{n+1}_u) + (2\nu, (\nabla \times u^{n+1}) \cdot \nabla e^{n+1}_u, A e^{n+1}_u + \delta_t e^{n+1}_u)
\]
\[
+ \left(u^{n+1}, \nabla u^{(n+1)} - q^{n+1} \exp \left(\frac{T^{n+1}}{T}\right) u^n, \nabla u^n, A e^{n+1}_u + \delta_t e^{n+1}_u\right).
\]
(79)
For the first term on the right hand side of (79), we get

\[
(R_{u}^{n+1}, Ae_{u}^{n+1} + \delta e_{u}^{n+1}) \leq \frac{1}{12} \|\delta e_{u}^{n+1}\|^2 + \frac{\nu_{0}}{24} \|Ae_{u}^{n+1}\|^2 + C\tau \int_{t_{n}}^{t_{n+1}} \|u_{t}(s)\|^2 ds.
\]  

(80)

For the second term on the right hand side of (79), we get

\[
(2\nu_{r}(\nabla \times u^{n} - \nabla \times w(t^{n+1})), Ae_{u}^{n+1} + \delta e_{u}^{n+1})
= 2\nu_{r}(\nabla \times e_{u}^{0}, Ae_{u}^{n+1} + \delta e_{u}^{n+1}) - 2\nu_{r}(\nabla \times (w(t^{n+1}) - w(t_{n})), Ae_{u}^{n+1} + \delta e_{u}^{n+1})
\leq \frac{1}{6} \|\delta e_{u}^{n+1}\|^2 + \frac{\nu_{0}}{12} \|Ae_{u}^{n+1}\|^2 + C\tau \int_{t_{n}}^{t_{n+1}} \|\nabla w_{t}(s)\|^2 ds
\]

For the last term on the right hand side of (79), we have

\[
\left( u(t^{n+1}) \cdot \nabla u(t^{n+1}) - q^{n+1} \exp \left( \frac{t^{n+1}}{T} \right) u^{n} \cdot \nabla u^{n}, Ae_{u}^{n+1} + \delta e_{u}^{n+1} \right)
= -e_{q}^{n+1} \exp \left( \frac{t^{n+1}}{T} \right) \left( (u^{n} \cdot \nabla) u^{n}, Ae_{u}^{n+1} + \delta e_{u}^{n+1} \right) + \left( (u(t^{n+1}) - u^{n}) \cdot \nabla (u(t^{n+1}), Ae_{u}^{n+1} + \delta e_{u}^{n+1}) \right) + \left( u^{n} \cdot \nabla (u(t^{n+1}) - u^{n}), Ae_{u}^{n+1} + \delta e_{u}^{n+1} \right). \quad (81)
\]

Using (46), (44) and (78), the first term on the right hand side of (81) can be bounded by

\[
-e_{q}^{n+1} \exp \left( \frac{t^{n+1}}{T} \right) \left( u^{n} \cdot \nabla u^{n}, Ae_{u}^{n+1} + \delta e_{u}^{n+1} \right)
= -e_{q}^{n+1} \exp \left( \frac{t^{n+1}}{T} \right) \left( u^{n} \cdot \nabla e_{u}^{0}, Ae_{u}^{n+1} + \delta e_{u}^{n+1} \right) - e_{q}^{n+1} \exp \left( \frac{t^{n+1}}{T} \right) \left( (u^{n} \cdot \nabla u(t^{n}), Ae_{u}^{n+1} + \delta e_{u}^{n+1}) \right)
\leq C_{b,7} e_{q}^{n+1} \|u^{n}\|^{1/2} \|\nabla u^{n}\|^{1/2} \|\nabla e_{u}^{n}\|^{1/2} \|Ae_{u}^{n+1} + \delta e_{u}^{n+1}\|^{1/2} + C_{b,5} e_{q}^{n+1} \|\nabla u^{n}\| \|u(t^{n})\|_{2} \|Ae_{u}^{n+1} + \delta e_{u}^{n+1}\|
\leq \frac{1}{12} \|\delta e_{u}^{n+1}\|^2 + \frac{\nu_{0}}{24} \|Ae_{u}^{n+1}\|^2 + \frac{\nu_{0}}{8} \|\nabla e_{u}^{n}\|^2
+ C \left( \tau + \|\nabla u(t^{n})\|^{2} \right) \|\nabla e_{u}^{n}\|^{2} + C \left( \tau + \|\nabla u(t^{n})\|^{2} \right) \|u(t^{n})\|_{2} \|e_{u}^{n+1}\|^{2}. \quad (82)
\]

Similarly, the second term on the right hand side of (81) can be estimated by

\[
\left( (u(t^{n+1}) - u^{n}) \cdot \nabla (u(t^{n+1}), Ae_{u}^{n+1} + \delta e_{u}^{n+1}) \right)
\leq C_{b,5} \|\nabla u(t^{n+1}) - \nabla u^{n}\| \|u(t^{n+1})\|_{2} \|\nabla e_{u}^{n}\|^{1/2} \|Ae_{u}^{n+1} + \delta e_{u}^{n+1}\|
\leq \frac{1}{12} \|\delta e_{u}^{n+1}\|^2 + \frac{\nu_{0}}{24} \|Ae_{u}^{n+1}\|^2 + C \|u(t^{n+1})\|_{2} \|\nabla e_{u}^{n}\|^{2} + C \|u(t^{n+1})\|_{2} \tau \int_{t_{n}}^{t_{n+1}} \|\nabla u_{t}(s)\|^{2} ds. \quad (83)
\]

The last term on the right hand side of (81) can be bounded by

\[
\left( u^{n} \cdot \nabla (u(t^{n+1}) - u^{n}), Ae_{u}^{n+1} + \delta e_{u}^{n+1} \right)
= \left( u^{n} \cdot \nabla (u(t^{n+1}) - u(t^{n})), Ae_{u}^{n+1} + \delta e_{u}^{n+1} \right) - \left( u^{n} \cdot \nabla e_{u}^{0}, Ae_{u}^{n+1} + \delta e_{u}^{n+1} \right)
\leq C_{b,5} \|\nabla u^{n}\| \|u(t^{n+1}) - u(t^{n})\|_{2} \|\nabla e_{u}^{n}\|^{1/2} \|Ae_{u}^{n+1} + \delta e_{u}^{n+1}\|
+ C_{b,7} \|u^{n}\|^{1/2} \|\nabla u^{n}\|^{1/2} \|\nabla e_{u}^{n}\|^{1/2} \|Ae_{u}^{n+1} + \delta e_{u}^{n+1}\|
\leq \frac{1}{12} \|\delta e_{u}^{n+1}\|^2 + \frac{\nu_{0}}{24} \|Ae_{u}^{n+1}\|^2 + C \left( \tau + \|\nabla u(t^{n})\|^{2} \right) \|\nabla e_{u}^{n}\|^{2}
+ \frac{\nu_{0}}{8} \|\nabla e_{u}^{n}\|^2 + C \tau \left( \tau + \|\nabla u(t^{n})\|^{2} \right) \int_{t_{n}}^{t_{n+1}} \|u_{t}(s)\|_{2}^{2} ds. \quad (84)
\]
Combining (79) with (80)-(84), we have

\[
(1 + \nu_0) \|\nabla e^{n+1}_u\|^2 - \|\nabla e^n_u\|^2 + \frac{1}{2} \|\delta e^{n+1}_u\|^2 + \frac{3
u_0}{4} \|Ae^{n+1}_u\|^2 \\
\leq \frac{\nu_0}{4} \|Ae^n_u\|^2 + C \left( 1 + \tau + \|\nabla (u(t^n))\|^2 \right) \left( \|\nabla e^n_u\|^2 + |e^{n+1}_q|^2 \right) + C \|\nabla e^n_w\|^2 \\
+ C\tau \int_t^{t+1} \left( \|u_t(s)\|^2 ds + \|\nabla u_t(s)\|^2 + \|u_{tt}(s)\|^2 + \|\nabla w_t(s)\|^2 \right) ds.
\]

(85)

Multiplying (85) by 2\tau and summing over \(n\) from 0 to \(m\), and applying the discrete Gronwall inequality in Lemma 4.2, we obtain

\[
\|\nabla e^{m+1}_u\|^2 + \tau \sum_{n=0}^m \|\delta e^{n+1}_u\|^2 + \nu_0 \tau \sum_{n=0}^m \|Ae^{n+1}_u\|^2 \\
\leq C \left( 1 + \tau + \|\nabla (u(t^n))\|^2 \right) \tau \sum_{n=0}^m \left( \|\nabla e^n_u\|^2 + |e^{n+1}_q|^2 \right) + C\tau^2.
\]

(86)

Combining the above estimate with Theorem 4.1, we obtain the desired result. \(\square\)

We are now in position to prove the error estimates for the pressure.

**Theorem 4.2.** Assume the exact solution satisfies \(u \in H^2(0, T; H^2(\Omega)) \cap H^1(0, T; H^2(\Omega)) \cap L^\infty(0, T; H^2(\Omega))\) and \(w \in H^2(0, T; H^{-1}(\Omega)) \cap H^1(0, T; H^2(\Omega)) \cap L^\infty(0, T; H^2(\Omega))\), then we have the following error estimate for \(m \geq 0\)

\[
\tau \sum_{n=0}^m \|e^{n+1}_p\|^2 \leq C\tau^2.
\]

(87)

**Proof.** Taking the inner product of (38a) with \(v \in \mathbf{X}\), we obtain

\[
(\nabla e^{n+1}_p, v) = - (\delta e^{n+1}_u, v) - \nu_0 (\nabla e^{n+1}_u, \nabla v) + (R^{n+1}_u, v) + 2\nu_r (\nabla \times w^n - \nabla \times w(t^{n+1}), v) \\
+ \left( (u(t^{n+1}) \cdot \nabla) u(t^{n+1}) - q^{n+1} \exp \left( \frac{t^{n+1}}{T} \right) (u^n \cdot \nabla) u^n, v \right).
\]

(88)

Note that we only need to estimate the last two terms on the right hand side of (88). For all \(v \in \mathbf{X}\), we use Cauchy-Schwarz inequality, Young inequality and (2) to estimate the fourth term as

\[
2\nu_r (\nabla \times w^n - \nabla \times w(t^{n+1}), v) = 2\nu_r (w^n - w(t^{n+1}), \nabla \times v) \\
= 2\nu_r (e^n_w, \nabla \times v) - 2\nu_r (w(t^{n+1}) - w(t_n), \nabla \times v) \\
\leq 2\nu_r \|e^n_w\| \|\nabla v\| + 2\nu_r \left( \int_t^{t+1} \|w_t(s)\| ds \right) \|\nabla v\|. \\
\leq C \left( \|e^n_w\|^2 + \left( \int_t^{t+1} \|w_t(s)\| ds \right) \|\nabla v\| \right).
\]

(89)

Invoking with (39) and (78), we have

\[
\exp(\frac{t^{n+1}}{T}) (g(t^{n+1})(u(t^{n+1}) \cdot \nabla) u(t^{n+1}) - q^{n+1}(u^n \cdot \nabla) u^n, v) \\
= \left( (u(t^{n+1}) - u^n) \cdot \nabla u(t^{n+1}), v \right) - e_q^{n+1} \exp \left( \frac{t^{n+1}}{T} \right) \left( (u^n \cdot \nabla) u^n, v \right) + (u^n \cdot \nabla (u(t^{n+1}) - u^n), v) \\
\leq C_{b_0} \|\nabla u(t^{n+1}) - \nabla u^n\| \|\nabla u(t^{n+1})\| \|\nabla v\| + C_{b_0} \exp(1)|e_q^{n+1}| \|\nabla u^n\| \|\nabla u^n\| \|\nabla v\|
\]

18
\[
+ C_{b,0} \| \nabla u^n \| \| \nabla (u(t^{n+1}) - u^n) \| \| \nabla v \|
\leq C \left( \| \nabla e_n^{n+1} \| + \int_{t^n}^{t^{n+1}} \| \nabla u_s(s) \| ds + | e_q^{n+1} | \right) \| \nabla v \|.
\]
(90)

Using Theorem 4.1, Lemma 4.6 and the inf-sup condition
\[
\| e_p^{n+1} \| \leq \sup_{v \in X} \frac{(\nabla e_p^{n+1}, v)}{\| v \|},
\]
(91)
we get
\[
\tau \sum_{n=0}^{m} \| e_p^{n+1} \|^2 \leq C \tau \sum_{n=0}^{m} \left( \| \delta t \varepsilon_{n+1}^{n+1} \|^2 + \nu_0 \| \nabla e_u^{n+1} \|^2 + \| \nabla e_u^n \|^2 + | e_q^{n+1} |^2 + \| e_w^n \|^2 \right)
+ C \tau^2 \int_0^{t_m+n+1} \left( \| \nabla u_s(s) \|^2 + \| w_s(s) \|^2 \right) ds
\leq C \tau^2.
\]
The proof is complete.

Remark 4.1. In this paper, we only consider the first-order scheme for MNS equations. However, the SAV approach is known as a useful tool to design high-order schemes. What prevents us from studying the second-order scheme is that the SAV approach only used for the convective terms and the coupling terms are needed to deal with some subtle IMEX treatments. Undoubtedly, one can get a second-order scheme based on the second-order backward difference formula by using the SAV approach for the convective terms and making full-implicit treatments for the coupling terms. Although the obtained scheme is second-order and unconditionally energy stable, it is fully decoupled and thus one needs to solve large linear systems at each time step. Alternatively, one can also design a second-order scheme with a decoupled structure by replacing the full-implicit treatments with some subtle IMEX treatments for the coupling terms in [6, 7]. By using the arguments in this paper, it is easy to give the stability and error estimates for the resulting scheme. But it is a pity that the scheme is only conditionally energy stable in the sense that the energy stability only holds with a time step restriction.

5. Numerical experiments

In this section, we provide some numerical examples to verify the theoretical findings of the proposed scheme. In all examples below, the spatial discretization is based on mixed finite element method. To be more specific, we use the \( P_2 \) element for the angular velocity, the inf-sup stable \( P_2/P_1 \) element for the velocity and pressure. We always fix the mesh size with \( h = 1/150 \) so that the spatial discretization error is negligible compared to the time discretization error. The numerical experiments are carried out using the finite element software FreeFem++ [27]. Without further specified, we take \( j = 1, c_1 = 2 \) and \( c_2 = 1 \).

Example 5.1 (Convergence test). In this example, the computational domain is taken as \( \Omega = (0, 1)^2 \) and the final time is chosen as \( T = 1 \). Consider the following solution to the MNS equations with external forces \( f \) and \( g \) in the momentum equation and angular momentum equation,
\[
u = (\sin t \sin^2(\pi x) \sin(2\pi y), -\sin t \sin(2\pi x) \sin^2(\pi y)), \quad p = \sin t \sin(\pi x) \sin(\pi y), \quad w = \sin t \sin^2(\pi x) \sin^2(\pi y).
\]

In Tables 1-3, we present the numerical results for \( \nu = \nu_r = 1, 0.01 \) and 0.001. From these tables, we observe that the scheme achieve the expected convergence rates in time, which are consistent with the error estimates in Theorems 4.1 and 4.2.
Table 1: Errors and convergence rates for the MNS equations with \( \nu = \nu_r = 1.0 \).

| \( \tau \) | \( \| e_{N\nu}^\tau \| \) | \( \| \nabla e_{N\nu}^\tau \| \) | \( \| e_{Np}^\tau \| \) | \( \| e_{Nw}^\tau \| \) | \( \| e_{Nq}^\tau \| \) |
|-------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 0.2   | 5.23e-3(—)      | 3.78e-3(—)      | 5.07e-2(—)      | 1.60e-3(—)      | 7.67e-3(—)      | 3.40e-2(—)      |
| 0.1   | 2.47e-3(1.08)    | 1.79e-2(1.08)   | 2.29e-2(1.15)   | 7.86e-4(1.03)   | 3.75e-3(1.03)   | 1.77e-2(0.94)   |
| 0.05  | 1.20e-3(1.04)    | 8.71e-3(1.04)   | 1.07e-2(1.09)   | 3.89e-4(1.02)   | 1.85e-3(1.01)   | 9.01e-3(0.97)   |
| 0.025 | 5.91e-4(1.02)    | 4.31e-3(1.02)   | 5.16e-3(1.05)   | 3.89e-4(1.02)   | 1.85e-3(1.01)   | 9.01e-3(0.97)   |

Table 2: Errors and convergence rates for the MNS equations with \( \nu = \nu_r = 0.1 \).

| \( \tau \) | \( \| e_{N\nu}^\tau \| \) | \( \| \nabla e_{N\nu}^\tau \| \) | \( \| e_{Np}^\tau \| \) | \( \| e_{Nw}^\tau \| \) | \( \| e_{Nq}^\tau \| \) |
|-------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 0.2   | 9.34e-3(—)      | 6.78e-2(—)      | 4.98e-2(—)      | 8.89e-4(—)      | 4.03e-3(—)      |
| 0.1   | 4.64e-3(1.01)    | 3.39e-2(1.01)   | 2.27e-2(1.13)   | 4.54e-4(0.97)   | 2.07e-3(0.97)   |
| 0.05  | 2.31e-3(1.01)    | 1.69e-2(1.01)   | 1.07e-2(1.09)   | 2.29e-4(0.99)   | 1.05e-3(0.97)   |
| 0.025 | 1.15e-3(1.00)    | 8.40e-3(1.00)   | 5.18e-3(1.05)   | 1.15e-4(0.99)   | 5.52e-4(0.94)   |

Example 5.2 (Stability test). This example is to test the energy stability of the proposed scheme. For this end, we take the domain as \( \Omega = (0,1)^2 \) and set the initial conditions for \( u \) and \( w \) to be

\[
 u^0 = (x^2 - 1) y(y - 1)(2y - 1), \quad w^0 = \sin(\pi x) \sin(\pi y).
\]

We carry out the numerical experiments with different physical parameters and different time-steps. Figure 1 presents the time evolutions of the discrete energy with the final time \( T = 5.0 \). We observe that all energy curves decay monotonically, which numerically confirms that our scheme is unconditionally energy stable.

Example 5.3 (Stirring of a Passive Scalar). This example is to compute a realistic example about the stirring of a passive scalar. To simulate this example, we supplement the MNS equations (1) with the following convection equation:

\[
 \phi_t + \mathbf{u} \cdot \nabla \phi = 0, \quad (92)
\]

where \( \mathbf{u} \) is the velocity from the MNS equations (1), \( \phi \) denotes the passive scalar whose value does not affect the flow. Since there is no diffusion in (92), thus mixing depends only on the flow pattern. The computational domain is set by \( \Omega = (-1,1)^2 \), the time-step is chosen as \( \tau = 0.01 \) and the final time is taken as \( T = 25.0 \). The angular momentum equation is supplemented by an external force \( g = 25 (x - 1) \), and the initial conditions are described by \( u^0 = 0, w^0 = 0 \) and

\[
 \phi(x,0) = \begin{cases} 
 1, & x_2 < 0, \\
 0, & x_2 \geq 0.
\end{cases}
\]

The profile of the initial scalar is shown in Figure 2. It is remarked that the MNS equations and the convection equation are not solved simultaneously in our implementation. At every time-step, we first solve the MNS equations using the proposed schemes, and then we solve the convection equation (92) by using characteristics-Galerkin method with \( P_2 \) finite element.

Table 3: Errors and convergence rates for the MNS equations with \( \nu = \nu_r = 0.01 \).

| \( \tau \) | \( \| e_{N\nu}^\tau \| \) | \( \| \nabla e_{N\nu}^\tau \| \) | \( \| e_{Np}^\tau \| \) | \( \| e_{Nw}^\tau \| \) | \( \| e_{Nq}^\tau \| \) |
|-------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 0.2   | 2.41e-2(—)      | 2.06e-1(—)      | 5.54e-2(—)      | 7.59e-4(—)      | 3.42e-3(—)      | 3.40e-2(—)      |
| 0.1   | 1.21e-2(1.00)    | 1.01e-1(1.03)   | 2.69e-2(1.04)   | 3.90e-4(0.96)   | 1.77e-3(0.96)   | 1.77e-2(0.94)   |
| 0.05  | 6.05e-3(1.00)    | 4.97e-2(1.02)   | 1.31e-2(1.03)   | 1.97e-4(0.98)   | 9.07e-4(0.97)   | 9.01e-3(0.97)   |
| 0.025 | 3.03e-3(1.00)    | 2.47e-2(1.01)   | 6.51e-3(1.02)   | 9.92e-4(0.99)   | 4.83e-4(0.91)   | 4.55e-3(0.99)   |

20
(a) $\nu = \nu_r = 0.1$

(b) $\nu = \nu_r = 0.01$

Figure 1: Time evolution of the energy with different time step sizes.

Figure 2: Initial profile of the scalar $\phi^0(x)$. 
The evolutions of $\phi$ with different values of $\nu = \nu_r$ are shown in Figures 3-5. Since a linear velocity is generated by the applied torque, the scalar begins to convect by the flow and we can observe the evolution of the variable $\phi$. We can also see that the mixing of $\nu = \nu_r = 0.1$ is most fast one, and $\nu = \nu_r = 0.001$ is most slow one. The obtained results coincide well with those discussed in [6, 7].

6. Concluding remarks

In this paper, we propose and analyze a first-order discretization scheme in time for the MNS equations. The scheme is based on the SAV approach for the convective terms and some subtle implicit-explicit treatments for the coupling terms. The attractive points of this scheme are it is decoupled, linear, unconditionally energy stable and easy to implement. We further derive rigorous error estimates in the two-dimensional case without any condition on the time step. Some numerical experiments are given to confirm the theoretical findings and show the performances of the scheme. In the further, the error estimates in three dimensions and the high order SAV schemes will be considered.

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Figure 3: Mixing of a convected passive scalar $\phi$ by means of an applied torque with $\nu = \nu_r = 1$. 
Figure 4: Mixing of a convected passive scalar $\phi$ by means of an applied torque with $\nu = \nu_r = 0.1$. 
Figure 5: Mixing of a convected passive scalar $\phi$ by means of an applied torque with $\nu = \nu_r = 0.01$. 
Figure 6: Mixing of a convected passive scalar $\phi$ by means of an applied torque with $\nu = \nu_r = 0.001$. 
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