Vassiliev Invariants and the Loop States in Quantum Gravity

Louis H. Kauffman

Department of Mathematics, Statistics and Computer Science,
University of Illinois at Chicago,
Chicago, Illinois 60607-7045, USA
(email: U10451@uicvm.uic.edu)

to appear in Knots and Quantum Gravity, Oxford U. Press

1 Introduction

The purpose of this paper is to expose properties of Vassiliev invariants by using the simplest of the approaches to the functional integral definition of link invariants. These methods are strong enough to give the top row evaluations of Vassiliev invariants for the classical Lie algebras. They give an insight into the structure of these invariants without using the full perturbation expansion of the integral. One reason for examining the invariants in this light is the possible applications to the loop variables approach to quantum gravity. The same level of handling the functional integral is commonly used in the loop transform for quantum gravity.

The paper is organized as follows. Section 2 is a brief discussion of the nature of the functional integral. Section 3 details the formalism of the functional integral that we shall use, and works out a difference formula for changing crossings in the link invariant and a formula for the change of framing. These are applied to the case of $SU(N)$ gauge group. Section 4 defines the Vassiliev invariants and shows how to formulate them in terms of the functional integral. In particular we derive the specific expressions for the “top rows” of Vassiliev invariants corresponding to the fundamental representation of $SU(N)$. This gives a neat point of view on the results of Bar-Natan, and also gives a picture of the structure of the graphical vertex associated with the Vassiliev invariant. We see that this vertex is not just a transversal intersection of Wilson loops, but rather has the structure of Casimir insertion (up to first order of approximation) coming from the difference formula in the functional integral. This clarifies an issue raised by John Baez in [4]. Section 5 is a quick remark about the loop formalism.
for quantum gravity and its relationships with the invariants studied in the previous sections. This marks the beginning of a study that will be carried out in detail elsewhere.

2 Quantum mechanics and topology

In [47] Edward Witten proposed a formulation of a class of 3-manifold invariants as generalized Feynman integrals taking the form $Z(M)$ where

$$Z(M) = \int dA \exp\left[i k/4\pi S(M, A)\right].$$

Here $M$ denotes a 3-manifold without boundary and $A$ is a gauge field (also called a gauge potential or gauge connection) defined on $M$. The gauge field is a one-form on $M$ with values in a representation of a Lie algebra. The group corresponding to this Lie algebra is said to be the gauge group for this particular field. In this integral the “action” $S(M, A)$ is taken to be the integral over $M$ of the trace of the Chern-Simons three-form $CS = \text{Ad}A + (2/3)A\cdot A\cdot A$. (The product is the wedge product of differential forms.)

Instead of integrating over paths, the integral $Z(M)$ integrates over all gauge fields modulo gauge equivalence (see [2] for a discussion of the definition and meaning of gauge equivalence.) This generalization from paths to fields is characteristic of quantum field theory.

Quantum field theory was designed in order to accomplish the quantization of electromagnetism. In quantum electrodynamics the classical entity is the electromagnetic field. The question posed in this domain is to find the value of an amplitude for starting with one field configuration and ending with another. The analogue of all paths from point $a$ to point $b$ is “all fields from field $A$ to field $B$”.

Witten’s integral $Z(M)$ is, in its form, a typical integral in quantum field theory. In its content $Z(M)$ is highly unusual. The formalism of the integral, and its internal logic supports the existence of a large class of topological invariants of 3-manifolds and associated invariants of knots and links in these manifolds.

The invariants associated with this integral have been given rigorous combinatorial descriptions (see [30, 11, 27, 29, 25]), but questions and conjectures arising from the integral formulation are still outstanding (see [13]).
3 Links and the Wilson loop

We now look at the formalism of the Witten integral and see how it implicates invariants of knots and links corresponding to each classical Lie algebra. We need the Wilson loop. The Wilson loop is an exponentiated version of integrating the gauge field along a loop \( K \) in three space that we take to be an embedding (knot) or a curve with transversal self-intersections. For the purpose of this discussion, the Wilson loop will be denoted by the notation \( \langle K | A \rangle \) to denote the dependence on the loop \( K \) and the field \( A \). It is usually indicated by the symbolism \( \text{tr} \left( P \exp(\oint K A) \right) \).

Thus \( \langle K | A \rangle = \text{tr} \left( P \exp(\oint K A) \right) \). Here the \( P \) denotes path ordered integration. The symbol \( \text{tr} \) denotes the trace of the resulting matrix.

With the help of the Wilson loop functional on knots and links, Witten writes down a functional integral for link invariants in a 3-manifold \( M \):

\[
Z(M, K) = \int dA \exp[(ik/4\pi)S(M, A)]\text{tr} \left( P \exp(\oint K A) \right)
\]

\[
= \int dA \exp[(ik/4\pi)S]\langle K | A \rangle.
\]

Here \( S(M, A) \) is the Chern-Simons action, as in the previous discussion. We abbreviate \( S(M, A) \) as \( S \). Unless otherwise mentioned, the manifold \( M \) will be the three-dimensional sphere \( S^3 \).

An analysis of the formalism of this functional integral reveals quite a bit about its role in knot theory. This analysis depends upon key facts relating the curvature of the gauge field to both the Wilson loop and the Chern-Simons Lagrangian. To this end, let us recall the local coordinate structure of the gauge field \( A(x) \), where \( x \) is a point in three-space. We can write \( A(x) = A^a_k(x)T_a dx^k \) where the index \( a \) ranges from 1 to \( m \) with the Lie algebra basis \( \{T_1, T_2, T_3, \ldots, T_m\} \). The index \( k \) goes from 1 to 3. For each choice of \( a \) and \( k \), \( A^k_a(x) \) is a smooth function defined on three-space. In \( A(x) \) we sum over the values of repeated indices. The Lie algebra generators \( T_a \) are actually matrices corresponding to a given representation of an abstract Lie algebra. We assume some properties of these matrices as follows:

1. \( [T_a, T_b] = if_{abc}T_c \) where \([x, y] = xy - yx\), and \( f_{abc} \) (the matrix of structure constants) is totally antisymmetric. There is summation over repeated indices.
2. \( \text{tr}(T_a T_b) = \delta_{ab}/2 \) where \( \delta_{ab} \) is the Kronecker delta (\( \delta_{ab} = 1 \) if \( a = b \) and zero otherwise).

We also assume some facts about curvature. (The reader may compare with the exposition in \[24\]. But note the difference in conventions on the use of \( i \) in the Wilson loops and curvature definitions.) The first fact is the relation of Wilson loops and curvature for small loops:

**Fact 1.** The result of evaluating a Wilson loop about a very small planar circle around a point \( x \) is proportional to the area enclosed by this circle times the corresponding value of the curvature tensor of the gauge field evaluated at \( x \). The curvature tensor is written \( F_{rs}^a(x) T_a dx^r dy^s \). It is the local coordinate expression of \( dA + AA \).

**Application of Fact 1.** Consider a given Wilson loop \( \langle K | A \rangle \). Ask how its value will change if it is deformed infinitesimally in the neighborhood of a point \( x \) on the loop. Approximate the change according to Fact 1, and regard the point \( x \) as the place of curvature evaluation. Let \( \delta \langle K | A \rangle \) denote the change in the value of the loop. \( \delta \langle K | A \rangle \) is given by the formula

\[
\delta \langle K | A \rangle = dx^r dx^s F_{rs}^a(x) T_a \langle K | A \rangle.
\]

This is the first-order approximation to the change in the Wilson loop.

In this formula it is understood that the Lie algebra matrices \( T_a \) are to be inserted into the Wilson loop at the point \( x \), and that we are summing over repeated indices. This means that each \( T_a \langle K | A \rangle \) is a new Wilson loop obtained from the original loop \( \langle K | A \rangle \) by leaving the form of the loop unchanged, but inserting the matrix \( T_a \) into the loop at the point \( x \). See the figure below.

\[
\langle K | A \rangle \quad T_a \langle K | A \rangle
\]
Remark on Insertion. The Wilson loop is the limit, over partitions of the loop $K$, of products of the matrices $(1 + A(x))$ where $x$ runs over the partition. Thus one can write symbolically,

$$
\langle K | A \rangle = \prod_{x \in K} (1 + A(x)) = \prod_{x \in K} (1 + A^a_k(x)T_a dx^k).
$$

It is understood that a product of matrices around a closed loop connotes the trace of the product. The ordering is forced by the one-dimensional nature of the loop. Insertion of a given matrix into this product at a point on the loop is then a well-defined concept. If $T$ is a given matrix then it is understood that $T\langle K | A \rangle$ denotes the insertion of $T$ into some point of the loop. In the case above, it is understood from context of the formula $ds^r dx^s F^r_{rs}(x) T_a \langle K | A \rangle$ that the insertion is to be performed at the point $x$ indicated in the argument of the curvature.

Remark. The previous remark implies the following formula for the variation of the Wilson loop with respect to the gauge field:

$$
\frac{\delta \langle K | A \rangle}{\delta (A^a_k(x))} = dx^k T_a \langle K | A \rangle.
$$

Varying the Wilson loop with respect to the gauge field results in the insertion of an infinitesimal Lie algebra element into the loop.

Proof:

$$
\frac{\delta \langle K | A \rangle}{\delta (A^a_k(x))} = \frac{\delta}{\delta (A^a_k(x))} \prod_{y \in K} (1 + A^a_k(y) T_a dy^k)
$$

$$
= \left[ \prod_{y < x} (1 + A^a_k(y) T_a dy^k) \right] [T_a dx^k] \left[ \prod_{y > x} (1 + A^a_k(y) T_a dy^k) \right]
$$

$$
= dx^k T_a \langle K | A \rangle.
$$

\hfill \square

Fact 2. The variation of the Chern-Simons action $S$ with respect to the gauge potential at a given point in three-space is related to the values of the
curvature tensor at that point by the following formula (where $\epsilon_{abc}$ is the epsilon symbol for three indices):

$$F^a_{rs}(x) = \epsilon_{rst} \frac{\delta S}{\delta (A^a_t(x))}.$$ 

With these facts at hand we are prepared to determine how the Witten integral behaves under a small deformation of the loop $K$.

**Proposition 1.** (Compare [24].) All statements of equality in this proposition are up to order $(1/k)^2$.

1. Let $Z(K) = Z(K, S^3)$ and let $\delta Z(K)$ denote the change of $Z(K)$ under an infinitesimal change in the loop $K$. Then

$$\delta Z(K) = (4\pi i/k) \int dA \exp[(ik/4\pi)S][\epsilon_{rst}dx^r dy^s dz^t]T_a T_a \langle K | A \rangle.$$ 

The sum is taken over repeated indices, and the insertion is taken of the matrix products $T_a T_a$ at the chosen point $x$ on the loop $K$ that is regarded as the “center” of the deformation. The volume element $[\epsilon_{rst}dx^r dy^s dz^t]$ is taken with regard to the infinitesimal directions of the loop deformation from this point on the original loop.

2. The same formula applies, with a different interpretation, to the case where $x$ is a double point of transversal self intersection of a loop $K$, and the deformation consists in shifting one of the crossing segments perpendicularly to the plane of intersection so that the self-intersection point disappears. In this case, one $T_a$ is inserted into each of the transversal crossing segments so that $T_a T_a \langle K | A \rangle$ denotes a Wilson loop with a self intersection at $x$ and insertions of $T_a$ at $x + \epsilon_1$ and $x + \epsilon_2$, where $\epsilon_1$ and $\epsilon_2$ denote small displacements along the two arcs of $K$ that intersect at $x$. In this case, the volume form is nonzero, with two directions coming from the plane of movement of one arc, and the perpendicular direction is the direction of the other arc.

**Proof:**

$$Z(K) = \int dA \exp[(ik/4\pi)S] \delta \langle K | A \rangle$$
\[
\begin{align*}
&= \int dA \exp[(ik/4\pi)S]d^3r d^3y \gamma^a F^a_{rs}(x)T_a\langle K | A \rangle \quad \text{(Fact 1)} \\
&= \int dA \exp[(ik/4\pi)S]d^3r d^3y \epsilon_{rst} \frac{\delta S}{\delta (A^a_i(x))} T_a\langle K | A \rangle \quad \text{Fact 2) }
&= \int dA \exp[(ik/4\pi)S] \frac{\delta S}{\delta (A^a_i(x))} \epsilon_{rst} d^3r d^3y T_a\langle K | A \rangle \\
&= (-4\pi i/k) \int dA \frac{\delta \{\exp[(ik/4\pi)S]\}}{\delta (A^a_i(x))} \epsilon_{rst} d^3r d^3y T_a\langle K | A \rangle \\
&= (4\pi i/k) \int dA \exp[(ik/4\pi)S] \epsilon_{rst} d^3r d^3y \frac{\delta \{T_a\langle K | A \rangle\}}{\delta (A^a_i(x))} \\
&= (4\pi i/k) \int dA \exp[(ik/4\pi)S] \epsilon_{rst} d^3r d^3y d^3z \gamma^a T_a T_a\langle K | A \rangle
\end{align*}
\]

This completes the formalism of the proof. In the case of part 2, the change of interpretation occurs at the point in the argument when the Wilson loop is differentiated. Differentiating a self intersecting Wilson loop at a point of self intersection is equivalent to differentiating the corresponding product of matrices at a variable that occurs at two points in the product (corresponding to the two places where the loop passes through the point). One of these derivatives gives rise to a term with volume form equal to zero, the other term is the one that is described in part 2. This completes the proof of the proposition. \(\square\)

**Applying Proposition 1.** As the formula of Proposition 1 shows, the integral \(Z(K)\) is unchanged if the movement of the loop does not involve three independent space directions (since \(\epsilon_{rst} d^3r d^3y d^3z\) computes a volume). This means that \(Z(K) = Z(S^3, K)\) is invariant under moves that slide the knot along a plane. In particular, this means that if the knot \(K\) is given in the nearly planar representation of a knot diagram, then \(Z(K)\) is invariant under regular isotopy of this diagram. That is, it is invariant under the Reidemeister moves II and III. We expect more complicated behavior under move I since this deformation does involve three spatial directions. This will be discussed momentarily.

We first determine the difference between \(Z(K_+)\) and \(Z(K_-)\) where \(K_+\)
and $K_-$ denote the knots that differ only by switching a single crossing. We take the given crossing in $K_+$ to be the positive type, and the crossing in $K_-$ to be of negative type.

The strategy for computing this difference is to use $K_{\#}$ as an intermediate, where $K_{\#}$ is the link with a transversal self-crossing replacing the given crossing in $K_+$ or $K_-$. Thus we must consider $\Delta_+ = Z(K_+) - Z(K_{\#})$ and $\Delta_- = Z(K_-) - Z(K_{\#})$. The second part of Proposition 1 applies to each of these differences and gives

$$\Delta_+ = (4\pi i / k) \int dA \exp[(ik/4\pi)S] \epsilon_{rst} dx^r dy^s dz^t T_a T_a \langle K_{\#} | A \rangle$$

where, by the description in Proposition 1, this evaluation is taken along the loop $K_{\#}$ with the singularity and the $T_a T_a$ insertion occurs along the two transversal arcs at the singular point. The sign of the volume element will be opposite for $\Delta_-$ and consequently we have that

$$\Delta_+ + \Delta_- = 0.$$ 

(The volume element $\epsilon_{rst} dx^r dy^s dz^t$ must be given a conventional value in our calculations. There is no reason to assign it different absolute values for the cases of $\Delta_+$ and $\Delta_-$ since they are symmetric except for the sign.)

Therefore $Z(K_+) - Z(K_{\#}) + (Z(K_-) - Z(K_{\#})) = 0$. Hence

$$Z(K_{\#}) = (1/2)(Z(K_+) + Z(K_-)).$$

This result is central to our further calculations. It tells us that the evaluation of a singular Wilson loop can be replaced with the average of the results of resolving the singularity in the two possible ways.
Now we are interested in the difference $Z(K_+ - Z(K_-)$:

$$Z(K_+ - Z(K_-) = \Delta_+ - \Delta_- = 2\Delta_+$$

$$= (8\pi i / k) \int dA \exp[(ik/4\pi)S][\epsilon_{rst}dx^r dy^s dz^t]T_a T_a \langle K_\# | A \rangle.$$

**Volume Convention.** It is useful to make a specific convention about the volume form. We take

$$[\epsilon_{rst}dx^r dy^s dz^t] = \frac{1}{2} \text{ for } \Delta_+ \text{ and } -\frac{1}{2} \text{ for } \Delta_-.$$

Thus

$$Z(K_+ - Z(K_-) = (4\pi i / k) \int dA \exp[(ik/4\pi)S]T_a T_a \langle K_\# | A \rangle.$$

**Integral Notation.** Let $Z(T_a T_a K_\#)$ denote the integral

$$Z(T_a T_a K_\#) = \int dA \exp[(ik/4\pi)S]T_a T_a \langle K_\# | A \rangle.$$

**Difference Formula.** Write the difference formula in abbreviated form

$$Z(K_+ - Z(K_-) = (4\pi i / k)Z(T_a T_a K_\#).$$

This formula is the key to unwrapping many properties of the knot invariants. For diagrammatic work it is convenient to rewrite the difference equation in the form shown below. The crossings denote small parts of otherwise identical larger diagrams, and the Casimir insertion $T_a T_a K_\#$ is indicated with crossed lines entering a disk labelled $C$. 

![Diagram](attachment:diagram.png)
**The Casimir.** The element $\sum_a T_a T_a$ of the universal enveloping algebra is called the Casimir. Its key property is that it is in the center of the algebra. Note that by our conventions $\text{tr}(\sum a T_a T_a) = \sum_a \Delta_{aa}/2 = d/2$ where $d$ is the dimension of the Lie algebra. This implies that an unknotted loop with one singularity and a Casimir insertion will have $Z$-value $d/2$.

In fact, for the classical semi-simple Lie algebras *one can choose a basis so that the Casimir is a diagonal matrix with identical values $(d/2D)$ on its diagonal.* $D$ is the dimension of the representation. We then have the general formula: $Z(T_a T_a K_{#}^{\text{loc}}) = (d/2D)Z(K)$ for any knot $K$. Here $K_{#}^{\text{loc}}$ denotes the singular knot obtained by placing a local self-crossing loop in $K$ as shown below:

Note that $Z(K_{#}^{\text{loc}}) = Z(K)$. (Let the flat loop shrink to nothing. The Wilson loop is still defined on a loop with an isolated cusp and it is equal to the Wilson loop obtained by smoothing that cusp.)

Let $K_{+}^{\text{loc}}$ denote the result of adding a positive local curl to the knot $K$, and $K_{-}^{\text{loc}}$ the result of adding a negative local curl to $K$. 

10
Then by the above discussion and the difference formula, we have

\[ Z(K_{\text{loc}}^+) = Z(K_{\#}) + (2\pi i/k)Z(T_a T_a K_{\#}) \]
\[ = Z(K) + (2\pi i/k)(d/2D)Z(K). \]

Thus,

\[ Z(K_{\text{loc}}^+) = (1 + (\pi i/k)(d/D))Z(K). \]

Similarly,

\[ Z(K_{\text{loc}}^-) = (1 - (\pi i/k)(d/D))Z(K). \]

These calculations show how the difference equation, the Casimir, and properties of Wilson loops determine the framing factors for the knot invariants. In some cases we can use special properties of the Casimir to obtain skein relations for the knot invariant.

For example, in the fundamental representation of the Lie algebra for \( SU(N) \) the Casimir obeys the following equation (see [24, 3]):

\[ \sum_a (T_a)_{ij}(T_a)_{kl} = \left( \frac{1}{2} \right) \delta_{il} \delta_{jk} - \left( \frac{1}{2N} \right) \delta_{ij} \delta_{kl}. \]

Hence

\[ Z(T_a T_a K_{\#}) = \left( \frac{1}{2} \right) Z(K_0) - \left( \frac{1}{2N} \right) Z(K_{\#}) \]

where \( K_0 \) denotes the result of smoothing a crossing as shown below:
Using $Z(K_\#) = (Z(K_+) + Z(K_-))/2$ and the difference identity, we obtain

$$Z(K_+) - Z(K_-) = (4\pi i/k) \left\{ \left( \frac{1}{2} \right) Z(K_0) - \left( \frac{1}{2N} \right) \left[ (Z(K_+) + Z(K_-))/2 \right] \right\}.$$ 

Hence

$$(1 + \pi i/Nk)Z(K_+) - (1 - \pi i/Nk)Z(K_-) = (2\pi i/k)Z(K_0)$$

or

$$e\left( \frac{1}{N} \right) Z(K_+) - e\left( -\frac{1}{N} \right) Z(K_-) = \{ e(1) - e(-1) \} Z(K_0)$$

where $e(x) = \exp((\pi i/k)x)$ taken up to $O(1/k^2)$.

Here $d = N^2 - 1$ and $D = N$, so the framing factor is

$$\alpha = (1 + (\pi i/k)((N^2 - 1)/N)) = (N - (1/N)).$$

Therefore, if $P(K) = \alpha^{-w(K)} Z(K)$ denotes the normalized invariant of ambient isotopy associated with $Z(K)$ (with $w(K)$ the sum of the crossing signs of $K$), then

$$\alpha e(1/N) P(K_+) - \alpha^{-1} e(-1/N) P(K_-) = \{ e(1) - e(-1) \} P(K_0).$$

Hence

$$e(N) P(K_+) - e(-N) P(K_-) = \{ e(1) - e(-1) \} P(K_0).$$

This last equation shows that $P(K)$ is a specialization of the Homfly polynomial for arbitrary $N$, and that for $N = 2$ ($SU(2)$) it is a specialization of the Jones polynomial.

4 Graph invariants and Vassiliev invariants

We now apply this integral formalism to the structure of rigid vertex graph invariants that arise naturally in the context of knot polynomials. If $V(K)$ is a (Laurent polynomial valued, or, more generally, commutative ring valued) invariant of knots, then it can be naturally extended to an invariant of rigid vertex graphs by defining the invariant of graphs in terms of the knot invariant via an “unfolding” of the vertex as indicated below [26]:

$$V(K_\#) = aV(K_+) + bV(K_-) + cV(K_0).$$
Here \( K_\$ \) indicates an embedding with a transversal 4-valent vertex (\$). We use the symbol \$ to distinguish this choice of vertex designation from the previous one involving a self-crossing Wilson loop.

Formally, this means that we define \( V(G) \) for an embedded 4-valent graph \( G \) by taking the sum over \( a^{+}(S)b^{-}(S)c^{0}(S)V(S) \) for all knots \( S \) obtained from \( G \) by replacing a node of \( G \) with either a crossing of positive or negative type, or with a smoothing (denoted 0). It is not hard to see that if \( V(K) \) is an ambient isotopy invariant of knots, then this extension is a rigid vertex isotopy invariant of graphs. In rigid vertex isotopy the cyclic order at the vertex is preserved, so that the vertex behaves like a rigid disk with flexible strings attached to it at specific points.

There is a rich class of graph invariants that can be studied in this manner. The Vassiliev invariants \([5, 6, 44]\) constitute the important special case of these graph invariants where \( a = +1, b = -1 \) and \( c = 0 \). Thus \( V(G) \) is a Vassiliev invariant if

\[
V(K_\$) = V(K_+) - V(K_-).
\]

\( V(G) \) is said to be the finite type \( k \) if \( V(G) = 0 \) whenever \( \#(G) > k \) where \( \#(G) \) denotes the number of 4-valent nodes in the graph \( G \). If \( V \) is the finite type \( k \), then \( V(G) \) is independent of the embedding type of the graph \( G \) when \( G \) has exactly \( k \) nodes. This follows at once from the definition of finite type. The values of \( V(G) \) on all the graphs of \( k \) nodes is called the top row of the invariant \( V \).

For purposes of enumeration it is convenient to use chord diagrams to enumerate and indicate the abstract graphs. A chord diagram consists in an oriented circle with an even number of points marked along it. These points are paired with the pairing indicated by arcs or chords connecting the paired points. See the figure below.
This figure illustrates the process of associating a chord diagram to a given embedded 4-valent graph. Each transversal self intersection in the embedding is matched to a pair of points in the chord diagram.

5 Vassiliev invariants from the functional integral

In order to examine the Vassiliev invariants associated with the functional integral, we must first normalize these invariants to invariants of ambient isotopy, and then consider the structure of the difference between positive and negative crossings. The framed difference formula provides the necessary information for obtaining the top row.

We have shown that \( Z(K^{\text{loc}}_+) = \alpha Z(K) \) with \( \alpha = e(d/D) \). Hence

\[
P(K) = \alpha^{-w(K)} Z(K)
\]
is an ambient isotopy invariant. The equation

\[
Z(K_+) - Z(K_-) = (4\pi i / k) Z(T_a T_a K_#)
\]
implies that if \( w(K_+) = w + 1 \), then we have the ambient isotopy difference formula:

\[
P(K_+) - P(K_-) = \alpha^{-w} (4\pi i / k) \{ Z(T_a T_a K_#) - (d/2D)Z(K_#) \}.
\]

We leave the proof of this formula as an exercise for the reader.

This formula tells us that for the Vassiliev invariant associated with \( P \) we have

\[
P(K_\#) = \alpha^{-w} (4\pi i / k) \{ Z(T_a T_a K_#) - (d/2D)Z(K_#) \}.
\]
Furthermore, if $V_j(K)$ denotes the coefficient of $(4\pi i/k)^j$ in the expansion of $P(K)$ in powers of $(1/k)$, then the ambient difference formula implies that $(1/k)^j$ divides $P(G)$ when $G$ has $j$ or more nodes. Hence $V_j(G) = 0$ if $G$ has more than $j$ nodes. Therefore $V_j(K)$ is a Vassiliev invariant of finite type. (This result was proved by Birman and Lin [6] by different methods and by Bar-Natan [5] by methods equivalent to ours.)

The fascinating thing is that the ambient difference formula, appropriately interpreted, actually tells us how to compute $V_k(G)$ when $G$ has $k$ nodes. Under these circumstances each node undergoes a Casimir insertion, and because the Wilson loop is being evaluated abstractly, independent of the embedding, we insert nothing else into the loop. Thus we take the pairing structure associated with the graph (the so-called chord diagram) and use it as a prescription for obtaining a trace of a sum of products of Lie algebra elements with $T_a$ and $T_a$ inserted for each pair or a simple crossover for the pair multiplied by $(d/2D)$. This yields the graphical evaluation implied by the recursion

$$V(G) = \{ V(T_a T_a G) - (d/2D) V(G) \}. $$

At each stage in the process one node of $G$ disappears or it is replaced by these insertions. After $k$ steps we have a fully inserted sum of abstract Wilson loops, each of which can be evaluated by taking the indicated trace. This result is equivalent to Bar-Natan’s result, but it is very interesting to see how it follows from a minimal approach to the Witten integral.

In particular, it follows from Bar-Natan [5] and Kontsevich [28] that the condition of topological invariance is translated into the fact that the Lie bracket is represented as a commutator and that it is closed with respect to the Lie algebra. Diagrammatically we have:
Since

\[ T_a T_b - T_b T_a = \sum_c i f_{abc} T_c \]

we obtain

This relationship on chord diagrams is the seed of all the topology. In particular, it implies the basic 4-term relation,

\[ \text{Proof:} \]
The presence of this relation on chord diagrams for $V_i(G)$ with $\#(G) = i$ is the basis for the existence of a corresponding Vassiliev invariant. There is not room here to go into more detail about this matter, and so we bring this discussion to a close. Nevertheless, it must be mentioned that this brings us to the core of the main question about Vassiliev invariants: Are there non-trivial Vassiliev invariants of knots and links that cannot be constructed through combinations of Lie algebraic invariants? There are many other open questions in this arena, all circling this basic problem.

6 Quantum gravity—loop states

We now discuss the relationship of Wilson loops and quantum gravity that is forged in the theory of Ashtekar, Rovelli and Smolin [1]. In this theory the metric is expressed in terms of a spin connection $A$, and quantization involves considering wavefunctions $\psi(A)$. Smolin and Rovelli analyze the loop transform $\hat{\psi}(K) = \int dA\psi(A)\langle K|A \rangle$ where $\langle K|A \rangle$ denotes the Wilson loop for the knot or singular embedding $K$. Differential operators on the wavefunction can be referred, via integration by parts, to corresponding statements about the Wilson loop. It turns out that the condition that $\hat{\psi}(K)$ be a knot invariant (without framing dependence) is equivalent to the so-called diffeomorphism constraint [14] for these wave functions. In this way, knots and weaves and their topological invariants become a language for representing a state of quantum gravity.

The main point that we wish to make in the relationship of the Vassiliev invariants to these loop states is that the Vassiliev vertex
is not simply a transverse intersection of Wilson loops. We have seen that it follows from the difference formula that the Vassiliev vertex is much more complex than this—that it involves the Casimir insertion at the transverse intersection up to the first order of approximation, and that this structure can be used to compute the top row of the corresponding Vassiliev invariant. This situation suggests that one should amalgamate the formalism of the Vassiliev invariants with the structure of the Poisson algebras of loops and insertions used in the quantum gravity theory \[34, 33\]. It also suggests taking the formalism of these invariants (in the functional integral form) quite seriously even in the absence of an appropriate measure theory.

One can begin to work backwards, taking the position that invariants that do not ostensibly satisfy the diffeomorphism constraint (due to change of value under framing change) nevertheless still define states of a quantum gravity theory that is a modification of the Ashtekar formulation. This theory can be investigated by working the transform methods backwards—from knots and links to differential operators and differential geometry.

All these remarks are the seeds for another paper. We close here, and ask the reader to stay tuned for further developments.
Acknowledgements

It gives the author pleasure to acknowledge the support of NSF Grant Number DMS 9205277 and the Program for Mathematics and Molecular Biology of the University of California at Berkeley, Berkeley, CA.

References

[1] A. Ashtekar, C. Rovelli, and L. Smolin, Weaving a classical geometry with quantum threads. (Preprint 1992).

[2] M. F. Atiyah, Geometry of Yang-Mills Fields, Accademia Nazionale dei Lincei Scuola Superiore Lezioni Fermiare, Pisa (1979).

[3] M. F. Atiyah, The Geometry and Physics of Knots, Cambridge University Press (1990).

[4] J. Baez, Link invariants of finite type and perturbation theory, Lett. Math. Phys. 26 (1991) 43-51.

[5] D. Bar-Natan, On the Vassiliev knot invariants. (Preprint 1992).

[6] J. Birman and X. S. Lin, Knot polynomials and Vassiliev’s invariants, Invent. Math. to appear.

[7] B. Brügmann, R. Gambini and J. Pullin, Knot invariants as nondegenerate quantum geometries, Phys. Rev. Lett. 68 (1992) 431-434.

[8] L. Crane, Conformal field theory, spin geometry and quantum gravity, Phys. Lett. B259 (1991) 243-248.

[9] L. Crane and D. Yetter, A categorical construction of 4D topological quantum field theories. (Preprint 1993).

[10] P. A. M. Dirac, Principles of Quantum Mechanics, Oxford University Press (1958).

[11] R. Feynman and A. R. Hibbs, Quantum Mechanics and Path Integrals, McGraw Hill (1965).
[12] D. Freed and R. Gompf, Computer calculations of Witten’s 3-manifold invariants, *Comm. Math. Phys.* **41** (1991) 79-117.

[13] S. Garoufalidis, Applications of TQFT to invariants in low dimensional topology. (Preprint 1993).

[14] B. Hasslacher and M. J. Perry, Spin networks are simplicial quantum gravity, *Phys. Lett.* **B103**.

[15] L. C. Jeffrey, On Some Aspects of Chern-Simons Gauge Theory. (Thesis - Oxford (1991)).

[16] V. F. R. Jones, Index for subfactors, *Invent. Math.* **72** (1983) 1-25.

[17] V. F. R. Jones, A polynomial invariant for links via von Neumann algebras, *Bull. Amer. Math. Soc.* **129** (1985) 103-112.

[18] V. F. R. Jones, A new knot polynomial and von Neumann algebras, Notices Amer. Math. Soc. **33** (1986) 219-225.

[19] V. F. R. Jones, Hecke algebra representations of braid groups and link polynomials, *Ann. of Math.* **126** (1987) 335-338.

[20] V. F. R. Jones, On knot invariants related to some statistical mechanics models, *Pacific J. Math.* **137** (1989) 311-334.

[21] L. H. Kauffman, State models and the Jones polynomial, *Topology* **26** (1987) 395-407.

[22] L. H. Kauffman, *On Knots*, Annals of Mathematics Studies Number 115, Princeton University Press (1987).

[23] L. H. Kauffman, Statistical mechanics and the Jones polynomial, *AMS Contemp. Math. Series*, (1989) **78** 263-297.

[24] L. H. Kauffman, *Knots and Physics*, World Scientific Pub. (1991).

[25] L. H. Kauffman and S. Lins, Temperley Lieb Recoupling Theory and Invariants of 3-Manifolds, (to appear as Annals monograph, Princeton University Press).

20
[26] L. H. Kauffman and P. Vogel, Link polynomials and a graphical calculus, *Journal of Knot Theory and Its Ramifications*, 1 (1992) 59-104.

[27] R. Kirby and P. Melvin, On the 3-manifold invariants of Reshetikhin-Turaev for sl(2, C), *Invent. Math.* 105 (1991) 473-545.

[28] M. Kontsevich, Graphs, homotopical algebra and low dimensional topology. (Preprint 1992).

[29] W. B. R. Lickorish, 3-manifolds and the Temperley Lieb Algebra, *Math. Ann.* 290 (1991) 657-670.

[30] C. W. Misner, K. S. Thorne and J. A. Wheeler, *Gravitation*, W. H. Freeman (1973).

[31] H. Ooguri, Discrete and continuum approaches to three-dimensional quantum gravity. (Preprint 1991).

[32] H. Ooguri, Topological lattice models in four dimensions, *Mod. Phys. Lett.* A7 (1992) 2799-2810.

[33] J. Pullin, Knot theory and quantum gravity – a primer. (Preprint 1993).

[34] L. Smolin, Quantum gravity in the self-dual representation, *Contemp. Math.* 71 (1988) 55-97.

[35] N. Y. Reshetikhin and V. Turaev, Ribbon graphs and their invariants derived from quantum groups, *Comm. Math. Phys.* 127 (1990) 1-26.

[36] N. Y. Reshetikhin and V. Turaev, Invariants of three manifolds via link polynomials and quantum groups *Invent. Math.* 103 (1991) 547-597.

[37] L. Rozansky, A large k asymptotics of Witten’s invariant of Seifert manifolds. (Preprint 1993).

[38] T. Stanford, Finite-type invariants of knots, links and graphs. (Preprint 1992).

[39] V. G. Turaev and O. Viro, State sum invariants of 3-manifolds and quantum 6j symbols, *Topology* 31 (1992) 865-902.
[40] V. G. Turaev, Quantum invariants of links and 3-valent graphs in 3-manifolds. (Preprint 1990).

[41] V. G. Turaev and H. Wenzl, Quantum invariants of 3-manifolds associated with classical simple Lie algebras.

[42] V. G. Turaev, Quantum invariants of 3-manifolds and a glimpse of shadow topology. (Preprint 1990).

[43] V. G. Turaev, Topology of Shadows. (Preprint 1992).

[44] V. Vassiliev, Cohomology of knot spaces. In: Theory of Singularities and its Applications, (V. I. Arnold, ed.), Amer. Math. Soc. (1990), pp. 23-69.

[45] K. Walker, On Witten’s 3-Manifold Invariants. (Preprint 1991).

[46] R. Williams and F. Archer, The Turaev-Viro state sum model and 3-dimensional quantum gravity. (Preprint 1991).

[47] E. Witten, Quantum field theory and the Jones polynomial Commun. Math. Phys. 121 351-399 (1989).

[48] E. Witten, Nucl. Phys. B311 (1989), p. 46.