A Talenti Comparison Result for Solutions to Elliptic Problems with Robin Boundary Conditions

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Abstract

Comparison results of Talenti type for elliptic problems with Dirichlet boundary conditions have been widely investigated in recent decades. In this paper, we deal with Robin boundary conditions. Surprisingly, contrary to the Dirichlet case, Robin boundary conditions make the comparison sensitive to the dimension, and while the planar case seems to be completely settled, in higher dimensions some open problems are yet unsolved. © 2023 The Authors.

1 Introduction

Let $\beta$ be a positive parameter, and let $\Omega$ be an open, bounded set of $\mathbb{R}^N$, $N \geq 2$, with Lipschitz boundary. For a given nonnegative (not identically zero) $f \in L^2(\Omega)$ we consider the problem

$$
\begin{cases}
-\Delta u = f & \text{in } \Omega, \\
\frac{\partial u}{\partial \nu} + \beta u = 0 & \text{on } \partial \Omega,
\end{cases}
$$

where $\nu$ denotes the outer unit normal to $\partial \Omega$.

A function $u \in H^1(\Omega)$ is a weak solution to (1.1) if

$$
\int_{\Omega} \nabla u \nabla \phi \, dx + \beta \int_{\partial \Omega} u \phi \, d\mathcal{H}^{n-1} = \int_{\Omega} f \phi \, dx \quad \forall \phi \in H^1(\Omega).
$$

We will establish a comparison principle with the solution to the problem

$$
\begin{cases}
-\Delta v = f^\sharp & \text{in } \Omega^\sharp, \\
\frac{\partial v}{\partial \nu} + \beta v = 0 & \text{on } \partial \Omega^\sharp,
\end{cases}
$$

where $\Omega^\sharp$ denotes the ball, centered at the origin, with the same Lebesgue measure as $\Omega$ and $f^\sharp$ is the decreasing Schwarz rearrangement of $f$. Our main theorems are the following:
**Theorem 1.1.** Let $u$ and $v$ be the solution to Problem (1.1) and to Problem (1.3), respectively. Then we have

\[ \|u\|_{L^p,1}(\Omega) \leq \|v\|_{L^p,1}(\Omega^\#) \quad \text{for all } 0 < p \leq \frac{N}{2N-2}, \]

\[ \|u\|_{L^{2p,2}(\Omega)} \leq \|v\|_{L^{2p,2}(\Omega^\#)} \quad \text{for all } 0 < p \leq \frac{N}{3N-4}. \]

**Remark 1.2.** If $f$ doesn’t have constant sign, the usual way to understand Schwarz rearrangement is by rearranging the modulus ($f^\# = |f|^\#$), and by comparison principle Theorem 1.1 follows at once.

**Theorem 1.3.** Let us assume $f \equiv 1$, and let $u$ and $v$ be the solutions to Problem (1.1) and Problem (1.3), respectively. Then, when $N = 2$, we have

\[ u^\#(x) \leq v(x) \quad x \in \Omega^\#, \]

while, when $N \geq 3$, we have $\|u\|_{L^{p,1}(\Omega)} \leq \|v\|_{L^{p,1}(\Omega^\#)}$ and $\|u\|_{L^{2p,2}(\Omega)} \leq \|v\|_{L^{2p,2}(\Omega^\#)}$ for all $0 < p \leq \frac{N}{N-2}$. We remind the reader that, for $0 < p < \infty$ and $0 < q \leq \infty$, the Lorentz space $L^{p,q}(\Omega)$ consists of all measurable functions $g$ in $\Omega$ such that the quantity

\[ \|g\|_{L^{p,q}(\Omega)} = \left\{ \begin{array}{ll}
 p^{\frac{1}{q}} \left( \int_0^{\infty} t^q \left[ \{ x \in \Omega : |g(x)| > t \} \right]^\frac{q}{p} \frac{dt}{t} \right)^{\frac{1}{q}}, & 0 < q < \infty, \\
 \sup_{t>0} \left( \int_t^{\infty} \left[ \{ x \in \Omega : |g(x)| > s \} \right]^\frac{q}{p} ds \right)^{\frac{1}{q}} & q = \infty.
\] 

is finite. For $p = q$ (see [11]), Lorentz spaces coincide with $L^p$ spaces (since $\|g\|_{L^{p,p}(\Omega)} = \|g\|_{L^p(\Omega)}$). Therefore, in the hypothesis of Theorem 1.1 when $N = 2$ we have $\|u\|_{L^1(\Omega)} \leq \|v\|_{L^1(\Omega^\#)}$ and $\|u\|_{L^2(\Omega)} \leq \|v\|_{L^2(\Omega^\#)}$. Moreover, in the hypothesis of Theorem 1.3 when $N \geq 3$ we have $\|u\|_{L^1(\Omega)} \leq \|v\|_{L^1(\Omega^\#)}$ and $\|u\|_{L^2(\Omega)} \leq \|v\|_{L^2(\Omega^\#)}$.

Here it is important to notice that the proof confines the result to $q = 1$ and $q = 2$, and it is not clear to us whether such a restriction is only technical or deep-seated into the problem.

Comparison results à la Talenti have been widely studied in recent decades, after in his seminal paper [18] Talenti proved that, if $u$ is the solution to

\[ \begin{cases}
 -\Delta u = f & \text{in } \Omega, \\
 u = 0 & \text{on } \partial \Omega,
\end{cases} \]

and $v$ is the solution to

\[ \begin{cases}
 -\Delta v = f^\# & \text{in } \Omega^\#, \\
 v = 0 & \text{on } \partial \Omega^\#,
\end{cases} \]

then $u^\#(x) \leq v(x)$ for all $x$ in $\Omega^\#$. 
It is impossible to make a comprehensive list of all the results developed in the wake of this fundamental achievement. Generalization to semilinear and nonlinear elliptic equations are, for instance, in [2][19], anisotropic elliptic operators are considered for instance in [1], while parabolic equation are handled for instance in [2]. Higher-order operators have been investigated for instance in [4][20]. A survey on Talenti’s technique as well many other references can be found in [12][14][21]. However, to our knowledge, in the literature there are no comparison results related to Talenti techniques concerning Robin boundary conditions. We mention, however, that when \( f = 1 \), it has been proved in [9] with a completely different argument that, if \( u \) and \( v \) are solutions to Problem (1.1) and to Problem (1.3), respectively, then
\[
\|u\|_{L^1(\Omega)} \leq \|v\|_{L^1(\Omega^3)}.
\]

Recently quite a few papers dealing with a priori estimates and Robin boundary conditions have appeared, some of them (see, for instance, [15][16]) simply inspired by our result, others (as for instance [3][17]) generalize our technique to nonlinear contexts.

The paper is organized as follows. In the next section we introduce some basic notations, and recall the notion of decreasing rearrangements as well as some of their basic properties. Section 3 contains the main results, and we provide detailed proofs of Theorems 1.1–1.3. Comparison principles of Talenti type for elliptic equations with Dirichlet boundary conditions are usually obtained deducing some integrodifferential inequality for the distribution function of the solution, which holds as an equality in the spherically symmetric case. Integrating such an inequality leads to sharp estimates for \( L^p \) and/or Lorentz norms of the solution and in the most favorable cases even a sharp pointwise estimate. Arguing in the same way with Robin boundary conditions leads to the nonstandard integrodifferential equation in the statement of Lemma 3.2, which apparently cannot be integrated on the spot because of a nasty boundary integral term. That’s the main reason that Robin boundary conditions have discouraged people for so many years.

In this respect Lemma 3.5 in Section 3 is the first key step to show how to estimate such an extra boundary term. The remaining part of the section consists of a careful integration of the integrodifferential equation obtained in Lemma 3.2 where we apply over and over a special version of the Gronwall lemma (Lemma 3.1) and eventually we get a comparison principle in terms of Lorentz norms.

In Section 4 we show an alternative proof of the Bossel-Daners inequality for planar domains. Finally, in the last section we provide some examples, discuss the optimality of our results, and provide a list of open problems.

## 2 Notation and Preliminaries

The solution \( u \in H^1(\Omega) \) to (1.1) is the unique minimizer of

\[
\min_{w \in H^1(\Omega)} \frac{1}{2} \int_{\Omega} |\nabla w|^2 \, dx + \frac{1}{2} \beta \int_{\partial \Omega} w^2 \, d\mathcal{H}^{n-1} - \int_{\Omega} f \, w \, dx.
\]
By the standard comparison principle $u$ is nonnegative. An easy way to prove this is to rely on uniqueness and observe that any minimizer of (2.1) cannot change sign; otherwise it could be replaced by its modulus (contradicting uniqueness).

For $t \geq 0$ we denote

$$U_t = \{ x \in \Omega : u(x) > t \}, \quad \partial U_t^{\text{int}} = \partial U_t \cap \Omega, \quad \partial U_t^{\text{ext}} = \partial U_t \cap \partial \Omega,$$

and

$$\mu(t) = |U_t|, \quad P_u(t) = \text{Per}(U_t).$$

the Lebesgue measure of $U_t$ and its perimeter in $\mathbb{R}^N$, respectively. Moreover, $\Omega^\sharp$ denotes the ball, centered at the origin, with the same measure as $\Omega$, and $v$ denotes the unique, radial, and decreasing-along-the-radius, solution to Problem (1.3).

Then, using the same notation as above, for $t \geq 0$ we set

$$V_t = \{ x \in \Omega^\sharp : v(x) > t \}, \quad \phi(t) = |V_t|, \quad P_v(t) = \text{Per}(V_t).$$

Since $v$ is radial, positive, and decreasing along the radius then, for $0 \leq t \leq \min_{\Omega^\sharp} v$, $V_t$ coincides with $\Omega^\sharp$, while, for $\min_{\Omega^\sharp} v < t < \max_{\Omega^\sharp} v$, $V_t$ is a ball concentric to $\Omega^\sharp$ and strictly contained in it.

In what follows we denote by $\omega_N$ the measure of the unit ball in $\mathbb{R}^N$.

**Definition 2.1.** Let $h : x \in \Omega \to [0, +\infty]$ be a measurable function; then the decreasing rearrangement $h^*$ of $h$ is defined as follows:

$$h^*(s) = \inf \{ t \geq 0 : |\{ x \in \Omega : |h(x)| > t \}| < s \} \quad s \in [0, \Omega].$$

while the Schwarz rearrangement of $h$ is defined as follows:

$$h^{\#}(x) = h^*(\omega_N |x|^N) \quad x \in \Omega^\sharp.$$

It is easily checked that $h$, $h^*$, and $h^{\#}$ are equidistributed, i.e.,

$$|\{ x \in \Omega : |h(x)| > t \}| = |\{ s \in (0, |\Omega| : h^*(s) > t \}| = |\{ x \in \Omega^\sharp : h^{\#}(x) > t \}|, \quad t \geq 0,$$

and then if $h \in L^p(\Omega)$, $1 \leq p \leq \infty$, then $h^* \in L^p(0, |\Omega|)$, $h^{\#} \in L^p(\Omega^\sharp)$, and

$$\| h^{\#} \|_{L^p(\Omega^\sharp)} = \| h^* \|_{L^p(0, |\Omega|)} = \| h^{\#} \|_{L^p(\Omega^\sharp)}.$$

Moreover, the following inequality, known as the Hardy-Littlewood inequality, holds true:

\begin{equation}
(2.2) \quad \int_{\Omega} |h(x)g(x)|dx \leq \int_0^{\| \Omega \|} h^*(s)g^*(s)ds.
\end{equation}

In the applications of the theory of rearrangements to the study of partial differential equations, one often has to evaluate the integral of a nonnegative function $f \in L^p(\Omega)$, $1 \leq p \leq +\infty$, on the level sets of a measurable function $u$. By (2.2) we get

\begin{equation}
(2.3) \quad \int_{|u(x)| > t} f(x)dx \leq \int_0^{\mu_u(t)} f^*(s)ds.
\end{equation}
Clearly, if we take $f = u \geq 0$ in (2.3) we have

$$
\int_{u(x) > t} u(x) dx = \int_0^{\mu u(t)} u^*(s) ds.
$$

## 3 Proofs of Theorem 1.1 and Theorem 1.3

As a premise to the proof of our main results we remind the reader of the following lemma.

**Lemma 3.1 (Gronwall).** Let $\xi(\tau)$ be a continuously differentiable function satisfying, for some nonnegative constant $C$, the following differential inequality

$$
\tau \xi'(\tau) \leq \xi(\tau) + C \quad \text{for all } \tau \geq \tau_0 > 0.
$$

Then we have

$$
\xi(\tau) \leq \tau \frac{\xi(\tau_0) + C}{\tau_0} - C \quad \text{for all } \tau \geq \tau_0.
$$

(3.1)

$$
\xi'(\tau) \leq \frac{\xi(\tau_0) + C}{\tau_0} \quad \text{for all } \tau \geq \tau_0.
$$

The main ingredient for a comparison result is the following lemma.

**Lemma 3.2.** Let $u$ and $v$ be the solution to (1.1) and (1.3), respectively. For a.e. $t > 0$ we have

$$
\gamma_N \phi(t) \frac{2N-2}{N} = \left( -\phi'(t) + \frac{1}{\beta} \int_{\partial V_t \cap \partial \Omega^1} \frac{1}{u(x)} d\mathcal{H}^{N-1}(x) \right) \int_0^{\phi(t)} f^*(s) ds,
$$

while for almost all $t > 0$ it holds

$$
\gamma_N \mu(t) \frac{2N-2}{N} \leq \left( -\mu'(t) + \frac{1}{\beta} \int_{\partial U^t_{\text{ext}}} \frac{1}{u(x)} d\mathcal{H}^{N-1}(x) \right) \int_0^{\mu(t)} f^*(s) ds.
$$

(3.3)

Here $\gamma_N = N^2 \omega_N^{\frac{2}{N}}$.

**Proof.** Let $t > 0$ and $h > 0$, and let us choose the following test function in (1.2):

$$
\varphi_h(x) = \begin{cases} 
0 & \text{if } 0 < u < t, \\
h & \text{if } u > t + h, \\
u - t & \text{if } t < u < t + h.
\end{cases}
$$

(3.4)
Then,
\[
\int_{U_t \setminus U_{t+h}} |\nabla u|^2 \, dx + \beta h \int_{\partial U_{t+h}^e} u \, dH^{N-1}(x) + \beta \int_{\partial U_{t}^e \setminus \partial U_{t+h}^e} u(u - t) \, dH^{N-1}(x)
\]
\[
= \int_{U_t \setminus U_{t+h}} f(u - t) \, dx + h \int_{U_{t+h}} f \, dx
\]
dividing by \( h \) and letting \( h \) go to 0, using the coarea formula we have that for a.e. \( t > 0 \)
\[
\int_{\partial U_t} g(x) \, dH^{N-1}(x) = \int_{\partial U_t} |\nabla u| \, dH^{N-1}(x) + \beta \int_{\partial U_t^e} u \, dH^{N-1}(x)
\]
\[
= \int_{U_t} f \, dx
\]
where
\[
g(x) = \begin{cases}
|\nabla u| & \text{if } x \in \partial U_t^\text{int}, \\
\beta u & \text{if } x \in \partial U_t^\text{ext}.
\end{cases}
\]
for a.e. \( t > 0 \), using (3.5) and (2.3), we have
\[
P_u^2(t) \leq \left( \int_{\partial U_t} g(x) \, dH^{N-1}(x) \right) \left( \int_{\partial U_t} g(x)^{-1} \, dH^{N-1}(x) \right)
\]
\[
= \left( \int_{\partial U_t} g(x) \, dH^{N-1}(x) \right)
\]
\[
\times \left( \int_{\partial U_t^\text{int}} |\nabla u|^{-1} \, dH^{N-1}(x) + \int_{\partial U_t^\text{ext}} (\beta u)^{-1} \, dH^{N-1}(x) \right)
\]
\[
\leq \int_0^t f^*(s) \, ds \left( -\mu'(t) + \frac{1}{\beta} \int_{\partial U_t^\text{ext}} \frac{1}{u} \, dH^{N-1}(x) \right), \quad t \in [0, \max u].
\]
Then the isoperimetric inequality gives (3.3). If \( v \) solves Problem (1.1), all the previous inequalities hold as equalities, hence (3.2) follows. □

Remark 3.3. We observe that solutions \( u \) and \( v \) to Problem (1.1) and Problem (1.3) always achieve their minima on the boundary of \( \Omega \) and \( \Omega^e \), respectively. From now on we denote by
\[
v_m = \min_{\Omega^e} v, \quad u_m = \min_{\Omega} u.
\]
The following inequality holds true
\[
(3.6) \quad u_m \leq v_m.
\]
In fact, using the weak formulation (1.2) of problems (1.1) and (1.3), with \( \varphi = 1 \),
\[
v_m \text{Per}(\Omega^\varphi) = \int_{\partial \Omega} v(x) d\mathcal{H}^{N-1}(x) = \int_{\Omega} f^2(x) d(x) = \int_{\Omega} f(x) d(x)
\]
\[
= \int_{\partial \Omega} u(x) d\mathcal{H}^{N-1}(x) \geq u_m \text{Per}(\Omega) \geq u_m \text{Per}(\Omega^\varphi).
\]
An important consequence of (3.6) is that
\[
(3.7) \quad \mu(t) \leq \phi(t) = |\Omega| \quad \text{ for all } 0 \leq t \leq v_m,
\]
with strict inequality for some \( 0 \leq t \leq v_m \) unless \( \Omega \) is a ball.

**Remark 3.4.** For the sake of completeness, equation (3.2) can be integrated, and the explicit form of the solution is
\[
v(x) = v_m + \gamma_N^{-1} \int_{\omega_N|x|^N} \frac{\sigma^{\frac{2}{N} - 2}}{d} \left( \int_0^\sigma f^*(s) ds \right) d\sigma,
\]
where \( v_m = \frac{1}{\beta N \omega_N^{\frac{1}{N}} |\Omega|^\frac{1}{N}} \int_0^{[\Omega]} f^*(s) ds. \)

Before proceeding with the proof of the main theorems, it is important to notice that Lemma 3.2 is derived in the wake of Talenti’s original work [18]. With Dirichlet (\( \frac{1}{\beta} = 0 \)) boundary conditions, (3.3) takes the well-known form
\[
\gamma_N \mu(t) \frac{2N - 2}{N} \leq -\mu'(t) \int_0^{\mu(t)} f^*(s) ds,
\]
which can be integrated right away. From a differential inequality to a pointwise inequality, the result is a sharp estimate for the function \( \mu(t) \) (and therefore for \( u \)), where equality is achieved in the radially symmetric case.

Since for Robin boundary conditions \( u \) is nonvanishing on the boundary of \( \Omega \), inequality (3.3) contains a nasty additional term that is difficult to handle. The following lemma allows us to estimate this extra term. However, as we will see later, the employment of such a lemma also naturally enforces the technical restriction on the exponent \( p \) in Theorem 1.1 and Theorem 1.3.

**Lemma 3.5.** For all \( t \geq v_m \) we have
\[
\int_0^t \tau \left( \int_{\partial \mathcal{V} \cap \partial \Omega} \frac{1}{v(x)} d\mathcal{H}^{N-1}(x) \right) d\tau = \frac{\int_0^{[\Omega]} f^*(s) ds}{2\beta},
\]
while
\[
(3.8) \quad \int_0^t \tau \left( \int_{\partial \mathcal{V}^\varphi} \frac{1}{u(x)} d\mathcal{H}^{N-1}(x) \right) d\tau \leq \frac{\int_0^{[\Omega]} f^*(s) ds}{2\beta}.
\]
PROOF. By Fubini’s theorem and using (1.1), we have
\[
\int_0^\infty \tau \left( \int_{\partial U_{\tau}^{ext}} \frac{1}{u(x)} \, d\mathcal{H}^{N-1}(x) \right) \, d\tau = \int_{\partial \Omega} \left( \int_0^{\mu(x)} \frac{\tau}{u(x)} \, d\tau \right) \, d\mathcal{H}^{N-1}(x)
\]
\[
= \int_{\partial \Omega} \frac{u(x)}{2} \, d\mathcal{H}^{N-1}(x) = \frac{\int_{\Omega} f^*(s) \, ds}{2\beta}.
\]
Analogously,
\[
\int_0^\infty \tau \int_{\partial V_{\tau}^{\partial} \cap \partial \Omega^{\partial}} \frac{1}{v(x)} \, d\mathcal{H}^{N-1}(x) \, d\tau = \frac{\int_{\Omega} f^*(s) \, ds}{2\beta}.
\]
Therefore, one trivial inequality for \( t \geq 0 \) is
\[
\int_0^\infty \tau \int_{\partial U_{\tau}^{ext}} \frac{1}{u(x)} \, d\mathcal{H}^{N-1}(x) \, d\tau \leq \int_0^\infty \tau \int_{\partial U_{\tau}^{ext}} \frac{1}{u(x)} \, d\mathcal{H}^{N-1}(x) \, d\tau,
\]
while we observe that for \( t \geq v_m = \min_{\partial \Omega^\partial} v \), then \( \partial V_{t} \cap \partial \Omega^{\partial} = \emptyset \),
\[
\int_0^\infty \tau \int_{\partial V_{t} \cap \partial \Omega^{\partial}} \frac{1}{v(x)} \, d\mathcal{H}^{N-1}(x) \, d\tau = \int_0^\infty \tau \int_{\partial V_{t} \cap \partial \Omega^{\partial}} \frac{1}{v(x)} \, d\mathcal{H}^{N-1}(x) \, d\tau.
\]

Now we are in position to use Lemma 3.2 and Lemma 3.5 together and prove the main theorems.

PROOF OF THEOREM 1.1. Let \( 0 < p \leq \frac{N}{2N-2} \). We start by multiplying (3.3) by \( t \mu(t) \frac{1}{p} \cdot \frac{N-2}{N} \), then we integrate from 0 to some \( \tau \geq v_m \), and we use Lemma 3.5, deducing
\[
\int_0^\tau \gamma_N t \mu(t) \frac{1}{p} \, dt \leq \int_0^\tau -\mu'(t) t \mu(t)^\delta \left( \int_0^{\mu(t)} f^*(s) \, ds \right) \, dt
\]
\[
+ \frac{|\Omega|^{\frac{\delta}{p}}}{2\beta^2} \left( \int_0^{\frac{\delta}{p}} f^*(s) \, ds \right)^2.
\]

Here we have used (3.8) and have set \( \delta = \frac{1}{p} - \frac{2N-2}{N} \geq 0 \).

It is important to notice that (3.9) relies on the fact that \( \mu(t)^\delta \leq |\Omega|^{\delta} \) for which the nonnegativity of the parameter \( \delta \) is fundamental. This is a major technical issue that yields, in the statement of Theorem 1.1, the restriction on \( p \).

We anticipate here that in the proof of Theorem 1.3, where we consider the case \( f = 1 \), the identity \( \int_0^{\mu(t)} f^*(s) \, ds = \mu(t) \) replaces \( \delta \) by \( \delta + 1 \). Then \( \mu(t)^{\delta+1} \leq |\Omega|^{\delta+1} \) provided \( \delta \geq -1 \). This accounts for a wider range of admissible values for \( p \), and eventually justifies the differences in the statements of the two theorems.
Observe also that the first term in (3.9) for $\tau \to \infty$ is (up to a multiplicative constant) the $L_{2p,2}$ Lorentz norm, and most of the upcoming computation aims at estimating the right-hand side. Furthermore, after performing an integration by parts we get rid of $t$, and then the $L_{p,1}$ Lorentz norm also appears naturally. In other words we are confined to $L_{s,q}$ with $q = 1$ and $q = 2$. It is an open issue to understand how and if it is possible to extend the proof to $1 < q < 2$. It seems to us that once (3.9) is on the table, there is not much room left to obtain this.

Now, we continue observing that, even if $\mu(t)$ is not necessarily absolutely continuous, the fact that $\mu(t)$ is a monotone nonincreasing function implies

$$
\int_0^\tau \gamma_N t \mu(t)^{\frac{1}{p}} dt \leq \int_0^\tau \gamma N \mu(t)^{\frac{1}{p}} \left( \int_0^{\mu(t)} f^*(s) ds \right) d\mu(t) + \frac{1}{2v_m} \left( \int_0^{1/\gamma_N} f^*(s) ds \right)^2.
$$

(3.10)

This allows us to integrate by parts both sides of the last inequality, but first we set $F(\ell) = \int_0^\ell w^\delta \left( \int_0^w f^*(s) ds \right) dw$. The nonnegativity of $\delta$ is more than enough to have $F$ well-defined. Hence we have for $\tau \geq v_m$

$$
\tau F(\mu(\tau)) + \tau \int_0^\tau \gamma_N \mu(t)^{\frac{1}{p}} dt \leq \int_0^\tau F(\mu(t)) dt + \int_0^\tau \int_0^{\gamma N \mu(r)^{\frac{1}{p}}} d\gamma N \mu(r)^{\frac{1}{p}} dr dt + \frac{1}{2v_m} \left( \int_0^{1/\gamma_N} f^*(s) ds \right)^2.
$$

Using the same notation of Lemma 3.1 we set

$$
\Gamma(\tau) = \int_0^\tau F(\mu(t)) dt + \int_0^\tau \left( \int_0^{\gamma N \mu(r)^{\frac{1}{p}}} d\gamma N \mu(r)^{\frac{1}{p}} dr \right) dt,
$$

$$
C = \frac{1}{2\beta^2} \left( \int_0^{1/\gamma_N} f^*(s) ds \right)^2, \text{ and } \tau_0 = v_m. \text{ Thereafter we deduce from Lemma 3.1 that}
$$

$$
F(\mu(\tau)) + \int_0^\tau \gamma_N \mu(t)^{\frac{1}{p}} dt \leq \frac{1}{v_m} \left( \int_0^{v_m} F(\mu(t)) dt + \int_0^{v_m} \int_0^{\gamma N \mu(r)^{\frac{1}{p}}} d\gamma N \mu(r)^{\frac{1}{p}} dr dt + \frac{1}{2v_m} \left( \int_0^{1/\gamma_N} f^*(s) ds \right)^2 \right)
$$

$$
\leq F(|\Omega|) + \frac{\gamma_N |\Omega|^{\frac{1}{p}} v_m}{2} + \frac{1}{v_m} \frac{|\Omega|^{\frac{1}{p}}}{2\beta^2} \left( \int_0^{1/\gamma_N} f^*(s) ds \right)^2.
$$

(3.11)

Here the last inequality follows at once from $\mu(t) \leq |\Omega|$. 
Similarly, arguing in the same way with (3.2), we can deduce

\[
F(\phi(t)) + \int_0^\tau \gamma_N \phi(t) \frac{1}{\beta} \, dt \\
= \frac{1}{v_m} \left( \int_0^{v_m} F(\phi(t)) \, dt + \int_0^\tau \int_0^t \gamma_N \phi(r) \frac{1}{\beta} \, dr \, dt \\
+ \frac{|\Omega|^\delta}{2\beta^2} \left( \int_0^{|\Omega|} f^*(s) \, ds \right)^2 \right) \\
= F(|\Omega|) + \frac{\gamma_N |\Omega|^\delta v_m}{2} + \frac{|\Omega|^\delta}{2\beta^2} \left( \int_0^{|\Omega|} f^*(s) \, ds \right)^2.
\]

(3.12)

Taking into account that \( \phi(t) = |\Omega| \) for \( t \leq v_m \), it is possible to make a direct comparison between the right-hand sides (3.11) and (3.12), yielding

\[
F(\phi(t)) + \int_0^\tau \gamma_N \phi(t) \frac{1}{\beta} \geq F(\mu(t)) + \int_0^\tau \gamma_N \mu(t) \frac{1}{\beta}.
\]

Passing to the limit as \( \tau \to \infty \) and using the fact that \( F(0) = 0 \), we get

\[
\int_0^\infty \mu(t) \frac{1}{\beta} \, dt \leq \int_0^\infty \phi(t) \frac{1}{\beta} \, dt,
\]

and hence

\[
\|u\|_{L^{p,1}(\Omega)} \leq \|v\|_{L^{p,1}(\Omega)}.
\]

To conclude the proof, we show that for all \( 0 < p \leq \frac{N}{3N-4} \),

\[
\int_0^\infty t^p \mu(t) \frac{1}{\beta} \, dt \leq \int_0^\infty t^p \phi(t) \frac{1}{\beta} \, dt.
\]

To this aim, we consider the limit as \( \tau \to \infty \) in (3.10) and integrate by parts the first term on the right-hand side, to obtain

\[
\int_0^\infty \gamma_N t^p \mu(t) \frac{1}{\beta} \, dt \leq \int_0^\infty F(\mu(t)) \, dt + \frac{|\Omega|^\delta}{2\beta^2} \left( \int_0^{|\Omega|} f^*(s) \, ds \right)^2.
\]

On the other hand,

\[
\int_0^\infty \gamma_N t^p \phi(t) \frac{1}{\beta} \, dt = \int_0^\infty F(\phi(t)) \, dt + \frac{|\Omega|^\delta}{2\beta^2} \left( \int_0^{|\Omega|} f^*(s) \, ds \right)^2.
\]

Therefore it is enough to show that

\[
(3.13) \quad \int_0^\infty F(\mu(t)) \, dt \leq \int_0^\infty F(\phi(t)) \, dt.
\]
This can be done for instance by multiplying (3.3) by \( t F(\mu(t)) \mu(t)^{-\frac{2N-2}{N}} \) since for \( 0 < p \leq \frac{N}{3N-4} \), the function \( F(\ell) \ell^{-\frac{2N-2}{N}} \) is nondecreasing in \( \ell \). Indeed,

\[
(F(\ell) \ell^{-\frac{2N-2}{N}})^t = \ell^{d-2+\frac{2}{N}} \int_0^\ell f^*(s) \, ds + \left( \frac{2}{N} - 2 \right) \ell^{\frac{2}{N}-3} F(\ell) \\
\geq \ell^{d-2+\frac{2}{N}} \int_0^\ell f^*(s) \, ds \left( \frac{1}{p} - 3 + \frac{4}{N} \right).
\]

Here we have used that \( F(\ell) \ell^{-\frac{2N-2}{N}} \) is nondecreasing in \( \ell \). Indeed,

\[
F(\ell) \ell^{-\frac{2N-2}{N}} \leq \frac{\ell^{d+1}}{\delta + 1} \int_0^\ell f^*(s) \, ds.
\]

An integration from 0 to any \( \tau \geq v_m \) yields

\[
\int_0^\tau \gamma N \tau t F(\mu(t)) \, dt \leq \int_0^\tau -t \mu^{-\frac{2N-2}{N}} F(\mu(t)) \left( \int_0^{\mu(t)} f^*(s) \, ds \right) \, d\mu(t) \\
+ F(|\Omega|) \frac{|\Omega|^{-\frac{2N-2}{N}}}{2\beta^2} \left( \int_0^{\Omega} f^*(s) \, ds \right)^2.
\]

We now set

\[
C = F(|\Omega|) \frac{|\Omega|^{-\frac{2N-2}{N}}}{2\beta^2} \left( \int_0^{\Omega} f^*(s) \, ds \right)^2 \quad \text{and}
\]

\[
H(\ell) = \int_0^\ell \ell^{-\frac{2N-2}{N}} F(\ell) \int_0^\ell f^*(s) \, ds \, dw,
\]

and after an integration by parts on both sides of (3.14), we have

\[
\tau \int_0^\tau \gamma N F(\mu(t)) \, dt + \tau H(\mu(\tau)) \leq \int_0^\tau \int_0^r \gamma N F(\mu(z)) \, dz \, dr + \int_0^\tau H(\mu(t)) \, dt + C.
\]

As before we use Lemma 3.1 with

\[
\xi(\tau) = \int_0^\tau \int_0^r \gamma N F(\mu(z)) \, dz \, dr + \int_0^\tau H(\mu(t)) \, dt \, dt
\]

and \( \tau_0 = v_m \). Thereafter we have

\[
\int_0^\tau \gamma N F(\mu(t)) \, dt + H(\mu(\tau)) \leq \frac{1}{v_m} \left( \int_0^{v_m} \int_0^r \gamma N F(\mu(z)) \, dz \, dr + \int_0^{v_m} H(\mu(t)) \, dt + C \right)
\]

\[
\leq \frac{\gamma N}{2} F(|\Omega|) v_m + H(|\Omega|) + \frac{C}{v_m}.
\]

Since the previous inequality holds as an equality whenever \( \mu \) is replaced by \( \phi \), and since \( \phi(t) = |\Omega| \) for \( t \leq v_m \), we easily deduce

\[
\int_0^\tau \gamma N F(\mu(t)) \, dt + H(\mu(\tau)) \leq \int_0^\tau \gamma N F(\phi(t)) \, dt + H(\phi(\tau))
\]
and for \( \tau \to \infty \) the desired inequality (3.13), which concludes the proof. \( \square \)

**Proof of the Theorem (1.3).** We start with the case \( N = 2 \). We multiply (3.3) by \( t \) and integrate from 0 to \( \tau \) choosing \( \tau \geq v_m \). Taking into account that now
\[
\int_0^{\mu(t)} f^*(s) ds = \mu(t),
\]
we have
\[
2\pi \tau^2 \leq \int_0^{\tau} -\mu'(t) t dt + \frac{|\Omega|}{2\beta^2} \quad \text{for } \tau \geq v_m.
\]
At the same time equality holds true whenever \( \mu \) is replaced by \( \phi \) and therefore
\[
2\pi \tau^2 = \int_0^{\tau} -\phi'(t) t dt + \frac{|\Omega|}{2\beta^2} \quad \text{for } \tau \geq v_m.
\]
Then,
\[
(3.15) \quad \int_0^{\tau} t \left( -d\mu(t) \right) \geq \int_0^{\tau} t \left( -d\phi(t) \right) \quad \tau \geq v_m.
\]
Then an integration by parts gives
\[
(3.16) \quad \mu(\tau) \leq \phi(\tau), \quad \tau \geq v_m.
\]
Since (4.6) is in force, inequality (3.16) follows for \( t \geq 0 \) and the claim is proved.

Now we consider \( N \geq 3 \). Equation (3.3) reads as follows
\[
(3.17) \quad \gamma_N \mu(t)^{\frac{N-2}{N}} \leq \left( -\mu'(t) + \frac{1}{\beta} \int_{\partial U^*_t} \frac{1}{u(x)} dH^{N-1}(x) \right).
\]
Let \( p \leq \frac{N}{N-2} \). We multiply (3.3) by \( t \mu(t)^{\frac{1}{p} - \frac{N-2}{N}} \) and integrate from 0 to some \( \tau \geq v_m \). Then we use Lemma 3.5 to deduce
\[
(3.18) \quad \int_0^{\tau} \gamma_N t \mu(t)^{\frac{1}{p}} dt \leq \int_0^{\tau} -\mu'(t) t \mu(t)^{\eta} dt + \frac{|\Omega|^{\eta+1}}{2\beta^2}.
\]
Here we have used (3.8) and set \( \eta = \frac{1}{p} - \frac{N-2}{N} \), which is nonnegative. Observe that \( \eta \) plays the role of \( \delta + 1 \) defined in the proof of Theorem 1.1 and its nonnegativity is important to deduce (3.18) since \( \mu(t)\eta \leq |\Omega|^{\eta} \) holds true. As before, using the fact that \( \mu(t) \) is a monotone nonincreasing function, we can write
\[
(3.19) \quad \int_0^{\tau} \gamma_N t \mu(t)^{\frac{1}{p}} dt \leq \int_0^{\tau} -t \mu(t)^{\eta} d\mu(t) + \frac{|\Omega|^{\eta+1}}{2\beta^2}.
\]
We set \( G(\ell) = \int_{0}^{\ell} w^\eta = \frac{\ell^{\eta+1}}{\eta+1} \) and after an integration by parts the last inequality reads
\[
\tau G(\mu(\tau)) + \tau \int_0^{\tau} \gamma_N \mu(t)^{\frac{1}{p}} dt
\]
\[
\leq \int_0^{\tau} G(\mu(t)) dt + \int_0^{\tau} \int_0^{\tau} \gamma_N \mu(t)^{\frac{1}{p}} dr dt + \frac{|\Omega|^{\eta+1}}{2\beta^2}.
\]
We can then use Lemma 3.1 with

$$\xi(t) = \int_0^t G(\mu(t))dt + \int_0^t \int_0^t \gamma_N \mu(t) \frac{1}{\beta} d\tau dt,$$

$$C = \frac{\Omega}{2\beta^2} \int_0^1 f^*(s)ds = \frac{\Omega^{\gamma+1}}{2\beta^2}$$

and \(\tau_0 = v_m\) to deduce from Lemma 3.1 that

$$G(\mu(t)) + \int_0^t \gamma_N \mu(t) \frac{1}{\beta} dt$$

\(\leq \frac{1}{v_m} \left( \int_0^{v_m} G(\mu(t))dt + \int_0^{v_m} \int_0^t \gamma_N \mu(r) \frac{1}{\beta} d\tau dt + \frac{\Omega^{\gamma+1}}{2\beta^2} \right)$$

\(\leq G(\Omega) + \frac{\gamma_N}{2} v_m + \frac{C}{v_m}.

Similarly, arguing in the same way as with (3.2), we can deduce

$$G(\phi(t)) + \int_0^t \gamma_N \phi(t) \frac{1}{\beta} dt$$

\(= \frac{1}{v_m} \left( \int_0^{v_m} G(\phi(t))dt + \int_0^{v_m} \int_0^t \gamma_N \phi(r) \frac{1}{\beta} d\tau dt + \frac{\Omega^{\gamma+1}}{2\beta^2} \right)

\(= G(\Omega) + \frac{\gamma_N}{2} v_m + \frac{C}{v_m}.

The last equality holds true since \(\phi(t) = \Omega\) for \(t \leq v_m\). Then it is possible to make a direct comparison between the right-hand sides (3.20) and (3.21), yielding

$$G(\mu(t)) + \int_0^t \gamma_N \mu(t) \frac{1}{\beta} \leq G(\phi(t)) + \int_0^t \gamma_N \phi(t) \frac{1}{\beta}.$$

Passing to the limit as \(t \to \infty\) we get

$$\int_0^{\infty} \mu(t)^{\frac{1}{\beta}} dt \leq \int_0^{\infty} \phi(t)^{\frac{1}{\beta}} dt,$$

and hence

$$\|u\|_{L^{\beta,1}(\Omega)} \leq \|v\|_{L^{\beta,1}(\Omega)}.$$

To conclude the proof we have to show that

$$\int_0^{\infty} t\mu(t)^{\frac{1}{\beta}} dt \leq \int_0^{\infty} t\phi(t)^{\frac{1}{\beta}} dt.$$

To this aim, we consider the limit as \(\tau \to \infty\) in (3.19), and integrating by parts the first term on the right-hand side we obtain

$$\int_0^{\infty} \gamma_N t\mu(t)^{\frac{1}{\beta}} \leq \int_0^{\infty} G(\mu(t))dt + \frac{\Omega^{\gamma+1}}{2\beta^2}.$$
On the other hand,
\[\int_0^\infty \gamma_N t^{\phi(t)} = \int_0^\infty G(\phi(t)) dt + \frac{|\Omega|^{\eta+1}}{2\beta^2}.\]

Therefore it is enough to show that
\[
(3.22) \quad \int_0^\infty G(\mu(t)) dt \leq \int_0^\infty G(\phi(t)) dt.
\]

This can be done, for instance, by multiplying (3.17) by \(t G(\mu(t))\mu(t)^{-\frac{N-2}{N}}\). Since the function \(G(\ell)\ell^{-\frac{N-2}{N}} = \ell^{\eta+2/N}\) is nondecreasing in \(\ell\), an integration from 0 to any \(t \geq v_m\) yields
\[
\int_0^r \gamma_N t G(\mu(t)) dt \leq \int_0^r -t\mu(t)^{-\frac{N-2}{N}} G(\mu(t)) d\mu(t) + G(|\Omega|) \frac{|\Omega|^{\frac{2}{N}}}{2\beta^2}.
\]

We now set \(C = G(|\Omega|) \frac{|\Omega|^{\frac{2}{N}}}{2\beta^2}\) and \(J(\ell) = \int_0^\ell w^{-\frac{N-2}{N}} G(w) dw\), and after an integration by parts on both sides of (3.23), we have
\[
\tau \int_0^r \gamma_N G(\mu(t)) dt + \tau J(\mu(\tau)) \leq \int_0^r \int_0^{\tau^r} \gamma_N G(\mu(z)) dz dr + \int_0^r J(\mu(t)) dt + C.
\]

As before, we use Lemma 3.1 with \(\xi(\tau) = \int_0^r \int_0^{\tau^r} \gamma_N G(\mu(z)) dz dr + \int_0^r J(\mu(t)) dt dt\), and \(\tau_0 = v_m\). Thereafter we deduce that
\[
\int_0^r \gamma_N G(\mu(t)) dt + J(\mu(\tau)) \leq \frac{1}{v_m} \left( \int_0^{v_m} \int_0^{\tau^r} \gamma_N G(\mu(z)) dz dr + \int_0^{v_m} J(\mu(t)) dt + C \right)
\]
\[
\leq \frac{\gamma_N}{2} G(|\Omega|) v_m + J(|\Omega|) + \frac{C}{v_m}.
\]

Since the previous inequality holds as an equality whenever \(\mu\) is replaced by \(\phi\), and since \(\phi(t) = |\Omega|\) when \(t \leq v_m\), we easily infer
\[
\int_0^r \gamma_N G(\mu(t)) dt + J(\mu(\tau)) \leq \int_0^r \gamma_N G(\phi(t)) dt + J(\phi(t)),
\]
and for \(\tau \to \infty\), we get the desired inequality (3.22), which concludes the proof.
\[\square\]
4 The Bossel-Daners Inequality (An Alternative Proof)

We conclude with a remark concerning the first Robin-Laplacian eigenvalue defined by

$$\lambda_{1,\beta}(\Omega) = \min_{w \in H^1(\Omega), w \neq 0} \frac{\int_{\Omega} |\nabla w|^2 \, dx + \beta \int_{\partial\Omega} w^2 \, d\mathcal{H}^{n-1}}{\int_{\Omega} w^2 \, dx}.$$ 

It is known that a Faber-Krahn inequality (namely the so-called Bossel-Daners inequality) for such eigenvalue holds true \(\text{[6–10]}\), that is,

(4.1) \(\lambda_{1,\beta}(\Omega) \geq \lambda_{1,\beta}(\Omega^\sharp).\)

Using an idea contained in \(\text{[13]}\), we have the following:

**Corollary 4.1.** Let \(N = 2\). Then Theorem 1.1 with \(N = 2\) implies (4.1).

**Proof.** Let \(u\) be the first Robin eigenfunction associated to \(\lambda_{1,\beta}(\Omega)\); then \(u\) solves

$$\begin{cases}
\begin{aligned}
-\Delta u &= \lambda_{1,\beta}(\Omega)u & \text{in } \Omega, \\
\frac{\partial u}{\partial v} + \beta u &= 0 & \text{on } \partial\Omega.
\end{aligned}
\end{cases}$$

Denoting by \(z\) the solution to

$$\begin{cases}
\begin{aligned}
-\Delta z &= \lambda_{1,\beta}(\Omega)u^\sharp & \text{in } \Omega^\sharp, \\
\frac{\partial z}{\partial v} + \beta z &= 0 & \text{on } \partial\Omega^\sharp.
\end{aligned}
\end{cases}$$

Theorem 1.1 when \(N = 2\) gives the \(L^2\) comparison principle

$$\int_{\Omega} u^2 = \int_{\Omega^\sharp} (u^\sharp)^2 \leq \int_{\Omega^\sharp} z^2.$$ 

By Cauchy-Schwarz inequality

$$\int_{\Omega^\sharp} u^\sharp z \leq \int_{\Omega^\sharp} z^2.$$ 

Multiplying equation (4) by \(z\) and integrating

$$\lambda_{1,\beta}(\Omega) = \frac{\int_{\Omega^\sharp} |\nabla z|^2 \, dx + \beta \int_{\partial\Omega^\sharp} z^2 \, d\mathcal{H}^{n-1}}{\int_{\Omega^\sharp} u^\sharp z \, dx} \geq \frac{\int_{\Omega^\sharp} |\nabla z|^2 \, dx + \beta \int_{\partial\Omega^\sharp} z^2 \, d\mathcal{H}^{n-1}}{\int_{\Omega^\sharp} z^2 \, dx} \geq \lambda_{1,\beta}(\Omega^\sharp).$$

□
Remark 4.2. We know that the Faber-Krahn inequality (4.1) holds true also for $N > 2$ (see [6,7,10]) and the reader might be unsatisfied by Corollary 4.1 not valid in higher dimensions. But the proof breaks when $N > 2$ because Theorem 1.1 does not grant in general the required $L^2$ comparison principle when $f$ is an arbitrary nonnegative function. However, this does not exclude that an $L^2$ comparison holds true for the very special case $f = \lambda_{1,\beta} u$. We consider this point a major future research direction.

5 Conclusions and Open Problems

We already observed in the Introduction that in the hypothesis of Theorem 1.1 when $N = 2$ we have $\|u\|_{L^1(\Omega)} \leq \|v\|_{L^1(\Omega^d)}$ and $\|u\|_{L^2(\Omega)} \leq \|v\|_{L^2(\Omega^d)}$, and one may ask whether $\|u\|_{L^p(\Omega)} \leq \|v\|_{L^p(\Omega^d)}$ for other values of $p$. We know for sure that for large values of $p$ the answer is No. Example \[1\] serves as a counterexample when $p = \infty$ and $N = 2$. While Example \[2\] serves as a counterexample for $p = 2$ and $N = 3$, nevertheless the following question is still unsolved.

OPEN PROBLEM 1. Under the hypothesis of Theorem \[1\] is $\|u\|_{L^1(\Omega)} \leq \|v\|_{L^1(\Omega^d)}$ even for $N \geq 3$?

Contrary to the classical comparison principle \[13\] for the Poisson equation with Dirichlet boundary conditions, our result in general establishes only comparison in Lorentz spaces. In the hypothesis of Theorem \[1\] when $N = 2$ we have $u^\beta \leq v$ in $\Omega^d$, therefore the following question arises.

OPEN PROBLEM 2 Under the hypothesis of Theorem \[1\] is $u^\beta \leq v$ in $\Omega^d$ even for $N \geq 3$?

The following examples show that the restrictions on the power $p$ in Theorem \[1\] is not just a technical issue. For simplicity we use domains that are the union of two disconnected balls. With just a little effort, by a continuity argument, one can join such balls with narrow (almost zero measure) tubes and make these examples work with a connected domain.

EXAMPLE 1. Consider $\Omega \subset \mathbb{R}^2$ equal to the union of two disjoint disk $D_1$ (centered at the origin) and $D_r$ with radii $1$ and $r$, respectively. We choose $\beta = \frac{1}{2}$ and fix $f = 1$ on $D_1$ and $f = 0$ on $D_r$. Both $u$ and $v$ in Theorem \[1\] can be explicitly computed.

Indeed,

$$u(x, y) = \begin{cases} 
\frac{1 - x^2 - y^2}{4} + 1 & \text{in } D_1, \\
0 & \text{in } D_r,
\end{cases}$$
and
\[v(x, y) = \begin{cases} \frac{1 - x^2 - y^2}{4} + 1 & \text{if } 0 \leq x^2 + y^2 \leq 1, \\ -\frac{1}{4} \log(x^2 + y^2) + A & \text{if } 1 \leq x^2 + y^2 \leq r^2 + 1, \end{cases}\]

where
\[A = \frac{1}{\sqrt{r^2 + 1}} + \frac{1}{4} \log(r^2 + 1).\]

We have \(\|u\|_{L^\infty(\Omega)} - \|v\|_{L^\infty(\Omega^2)} = u(0, 0) - v(0, 0) = \frac{r^2}{4} + o(r^2)\).

**Example 2.** Consider \(\Omega \subset \mathbb{R}^3\) equal to the union of two disjoint balls \(B_1\) (centered at the origin) and \(B_r\), with radii 1 and \(r\), respectively. We choose \(f = 1\) on \(B_1\) and \(f = 0\) on \(B_r\). For \(\beta = \frac{1}{2}\) both \(u\) and \(v\) in Theorem 1.1 have the following simple form:

\[u(x, y) = \begin{cases} \frac{1 - x^2 - y^2 - z^2}{6} + \frac{2}{3} & \text{in } D_1, \\ 0 & \text{in } D_r, \end{cases}\]

and
\[v(x, y) = \begin{cases} \frac{1 - x^2 - y^2 - z^2}{6} + \frac{2}{3} + A & \text{if } 0 \leq x^2 + y^2 + z^2 \leq 1, \\ \frac{1}{3\sqrt{x^2 + y^2 + z^2}} + A - \frac{1}{3} & \text{if } 1 \leq x^2 + y^2 + z^2 \leq (r^3 + 1)^{2/3}, \end{cases}\]

where
\[A = \frac{1}{3} + \frac{2}{3(r^3 + 1)^{2/3}} - \frac{1}{3\sqrt{r^3 + 1}} = \frac{2}{3} - \frac{r^3}{3} + o(r^3).\]

Therefore
\[v(x, y) = \begin{cases} u(x, y, z) - \frac{r^3}{3} + o(r^3) & \text{if } 0 \leq x^2 + y^2 + z^2 \leq 1, \\ \frac{2}{3} + o(r) & \text{if } 1 \leq x^2 + y^2 + z^2 \leq (r^3 + 1)^{2/3}. \end{cases}\]

It follows that
\[\|v\|^2_{L^2(\Omega^2)} = \int_{B_1} \left(u - \frac{r^3}{3}\right)^2 \, dx \, dy \, dz + \frac{16}{27} \pi r^3 + o(r^3)\]

\[= \|u\|^2_{L^2(\Omega)} - \frac{2}{3} r^3 \int_{B_1} u \, dx \, dy \, dz + \frac{16}{27} \pi r^3 + o(r^3)\]

\[= \|u\|^2_{L^2(\Omega)} - \frac{88}{135} \pi r^3 + \frac{16}{27} \pi r^3 + o(r^3),\]

and we have \(\|u\|^2_{L^2(\Omega)} - \|v\|^2_{L^2(\Omega^2)} = \frac{8\pi}{135} r^3 + o(r^3).\)
Acknowledgment. This work was supported by GNAMPA grant 2018 “Aspetti Geometrici delle EDP e Disuguaglianze Funzionali in forma ottimale”. We are grateful to Dorin Bucur for many interesting discussions and for having inspired Examples [1] and [2]. We also appreciated the anonymous reviewer for her/his helpful comments and for many suggestions. Open Access Funding provided by Universita degli Studi di Napoli Federico II within the CRUI-CARE Agreement.

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Received June 2020.