Primordial Entropy Production and $\Lambda$-driven Inflation from Quantum Einstein Gravity*  

A. Bonanno$^{1,2}$ and M. Reuter$^3$  

1INAF-Catania Astrophysical Observatory, Via S.Sofia 78, 95123 Catania, Italy.  
2INFN, Via S.Sofia 64, 95123 Catania, Italy.  
3Institute of Physics, University of Mainz, Staudingerweg 7, D-55099 Mainz, Germany  

Abstract. We review recent work on renormalization group (RG) improved cosmologies based upon a RG trajectory of Quantum Einstein Gravity (QEG) with realistic parameter values. In particular we argue that QEG effects can account for the entire entropy of the present Universe in the massless sector and give rise to a phase of inflationary expansion. This phase is a pure quantum effect and requires no classical inflaton field.

1. Introduction

After the introduction of the effective average action and its functional renormalization group equation for gravity [1] detailed investigations of the nonperturbative renormalization group (RG) behavior of Quantum Einstein Gravity have become possible [1–16]. The exact RG equation underlying this approach defines a Wilsonian RG flow on a theory space which consists of all diffeomorphism invariant functionals of the metric $g_{\mu\nu}$. The approach turned out to be an ideal setting for investigating the asymptotic safety scenario in gravity [17, 18] and, in fact, substantial evidence was found for the nonperturbative renormalizability of Quantum Einstein Gravity. The theory emerging from this construction (“QEG”) is not a quantization of classical general relativity. Instead, its bare action corresponds to a nontrivial fixed point of the RG flow and is a prediction therefore. The effective average action [1, 20–22] has crucial advantages as compared to other continuum implementations of the Wilson RG, in particular it is closely related to the standard effective action and defines a family of effective field theories $\{\Gamma_k[g_{\mu\nu}], 0 \leq k < \infty\}$ labeled by the coarse graining scale $k$. The latter property opens the door to a rather direct extraction of physical information from the RG flow, at least in single-scale cases: If the physical process or phenomenon under consideration involves only a single typical momentum scale $p_0$ it can be described by a tree-level evaluation of $\Gamma_k[g_{\mu\nu}]$, with $k = p_0$. The precision which can be achieved by this effective field theory description depends on the size of the fluctuations relative to the mean values. If they are large, or if more than one scale is involved, it might be necessary to go beyond the tree analysis. The RG flow of the effective average action, obtained by different truncations of theory space, has been the basis of various investigations of “RG improved” black hole and cosmological spacetimes [31–41]. We shall discuss some aspects of this method below.

A special class of RG trajectories obtained from QEG in the Einstein-Hilbert approximation [1], namely those of the “Type IIIa” [4], possess all the qualitative properties one would expect

* Talk given by M.R. at ICGC-07, Pune, India, Dec.17-21, 2007.
from the RG trajectory describing gravitational phenomena in the real Universe we live in. In particular they can have a long classical regime and a small, positive cosmological constant in the infrared (IR). Determining its parameters from observations, one finds [27] that, according to this particular QEG trajectory, the running cosmological constant $\Lambda(k)$ changes by about 120 orders of magnitude between $k$-values of the order of the Planck mass and macroscopic scales, while the running Newton constant $G(k)$ has no strong $k$-dependence in this regime. For $k > m_{Pl}$, the non-Gaussian fixed point (NGFP) which is responsible for the renormalizability of QEG controls their scale dependence. In the deep ultraviolet ($k \to \infty$), $\Lambda(k)$ diverges and $G(k)$ approaches zero.

An immediate question is whether there is any experimental or observational evidence that would hint at this enormous scale dependence of the gravitational parameters, the cosmological constant in particular. Clearly the natural place to search for such phenomena is cosmology. Even though it is always difficult to give a precise physical interpretation to the RG scale $k$ is is fairly certain that any sensible identification of $k$ in terms of cosmological quantities will lead to a $k$ which decreases during the expansion of the Universe. As a consequence, $\Lambda(k)$ will also decrease as the Universe expands. Already the purely qualitative assumption of a positive and decreasing cosmological constant supplies an interesting hint as to which phenomena might reflect a possible $\Lambda$-running. To make the argument as simple as possible,

![Figure 1](image.png)

**Figure 1.** The left panel shows the quasi-Newtonian potential corresponding to de Sitter space. The curve moves upward as the cosmological constant decreases. On the right panel the “realistic” RG trajectory of Section 3 is shown.

let us first consider a Universe without matter, but with a positive $\Lambda$. Assuming maximal symmetry, this is nothing but de Sitter space, of course. In static coordinates its metric is $ds^2 = -(1 + 2\Phi_N(r))dt^2 + (1 + 2\Phi_N(r))^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$ with $\Phi_N(r) = -\frac{1}{6} \Lambda r^2$. In the weak field and slow motion limit $\Phi_N$ has the interpretation of a Newtonian potential, with a correspondingly simple physical interpretation. The left panel of Fig.1 shows $\Phi_N$ as a function of $r$; for $\Lambda > 0$ it is an upside-down parabola. Point particles in this spacetime, symbolized by the black dot in Fig.1, “roll down the hill” and are rapidly driven away from the origin and from any other particle. Now assume that the magnitude of $|\Lambda|$ is slowly (“adiabatically”) decreased. This will cause the potential $\Phi_N(r)$ to move upward as a whole, its slope decreases. So the change in $\Lambda$ increases the particle’s potential energy. This is the simplest way of understanding that a positive decreasing cosmological constant has the effect of “pumping” energy into the matter degrees of freedom. More realistically one will describe the matter system in a hydrodynamics or quantum field theory language and one will include its back reaction onto the metric. But the basic conclusion, namely that a slow decrease of a positive $\Lambda$ transfers energy into the matter system, will remain true.
We are thus led to suspect that, because of the decreasing cosmological constant, there is a continuous inflow of energy into the cosmological fluid contained in an expanding Universe. It will “heat up” the fluid or, more exactly, lead to a slower decrease of the temperature than in standard cosmology. Furthermore, by elementary thermodynamics, it will increase the entropy of the fluid. If during the time \( dt \) an amount of heat \( dQ > 0 \) is transferred into a volume \( V \) at the temperature \( T \) the entropy changes by an amount \( dS = dQ/T > 0 \). To be as conservative (i.e., close to standard cosmology) as possible, we assume that this process is reversible. If not, \( dS \) is even larger.

In standard Friedmann-Robertson-Walker (FRW) cosmology the expansion is adiabatic, the entropy (within a comoving volume) is constant. It has always been somewhat puzzling therefore where the huge amount of entropy contained in the present Universe comes from. Presumably it is dominated by the CMBR photons which contribute an amount of about \( 10^{88} \) to the entropy within the present Hubble sphere. (We use units such that \( k_B = 1 \).) In fact, if it is really true that no entropy is produced during the expansion then the Universe would have had an entropy of at least \( 10^{88} \) immediately after the initial singularity which for various reasons seems quite unnatural. In scenarios which invoke a “tunneling from nothing”, for instance, spacetime was “born” in a pure quantum state, so the very early Universe is expected to have essentially no entropy. Usually it is argued that the entropy present today is the result of some sort of “coarse graining” which, however, typically is not considered an active part of the cosmological dynamics in the sense that it would have an impact on the time evolution of the metric, say.

In [27] we argued that in principle the entire entropy of the massless fields in the present universe can be understood as arising from the mechanism described above. If energy can be exchanged freely between the cosmological constant and the matter degrees of freedom, the entropy observed today is obtained precisely if the initial entropy at the “big bang” vanishes.

There is another, more direct potential consequence of a decreasing positive cosmological constant, namely a period of automatic inflation during the very first stages of the cosmological evolution. It is not surprising, of course, that a positive \( \Lambda \) can cause an accelerated expansion, but in the classical context the problem with a \( \Lambda \)-driven inflation is that it would never terminate once it has started. In popular models of scalar driven inflation this problem is circumvented by designing the inflaton potential in such a way that it gives rise to a vanishing vacuum energy after a period of “slow roll”.

As we are going to review in these notes generic RG cosmologies based upon the QEG trajectories have an era of \( \Lambda \)-driven inflation immediately after the big bang which ends automatically as a consequence of the RG running of \( \Lambda(k) \). Once the scale \( k \) drops significantly below \( \mpl \), the accelerated expansion ends because the vacuum energy density \( \rho_\Lambda \) is already too small to compete with the matter density. Clearly this is a very attractive scenario: neither to trigger inflation nor to stop it one needs any ad hoc ingredients such as an inflaton field or a special potential. It suffices to include the leading quantum effects in the gravity + matter system. Furthermore, asymptotic safety offers a natural mechanism for the quantum mechanical generation of primordial density perturbations, the seeds of cosmological structure formations.

In these notes we review a concrete investigation along these lines. For further details we refer to [25, 27].

2. The improved Einstein equations

The computational setting of our investigation [27] are the RG improved Einstein equations: By means of a suitable cutoff identification \( k = k(t) \) we turn the scale dependence of \( G(k) \) and \( \Lambda(k) \) into a time dependence, and then substitute the resulting \( G(t) \equiv G(k(t)) \) and \( \Lambda(t) \equiv \Lambda(k(t)) \) into the Einstein equations \( G_{\mu\nu} = -\Lambda(t)g_{\mu\nu} + 8\pi G(t)T_{\mu\nu} \). We specialize \( g_{\mu\nu} \).
to describe a spatially flat \((K = 0)\) Robertson-Walker metric with scale factor \(a(t)\), and we take
\(T_{\mu\nu} = \text{diag}[-\rho, p, p, p]\) to be the energy momentum tensor of an ideal fluid with equation of state \(p = w\rho\) where \(w > -1\) is constant. Then the improved Einstein equation boils down to the modified Friedmann equation and a continuity equation:

\[
H^2 = \frac{8\pi}{3} G(t) \rho + \frac{1}{3} \Lambda(t) \tag{2.1a}
\]

\[
\dot{\rho} + 3H(\rho + p) = -\frac{\dot{\Lambda} + 8\pi \rho \dot{G}}{8\pi G} \tag{2.1b}
\]

The modified continuity equation (2.1b) is the integrability condition for the improved Einstein equation implied by Bianchi’s identity, \(D^\mu [\Lambda(t) g_{\mu\nu} + 8\pi G(t) T_{\mu\nu}] = 0\). It describes the energy exchange between the matter and gravitational degrees of freedom (geometry). For later use let us note that upon defining the critical density \(\rho_{\text{crit}}(t) \equiv \frac{3}{H^2} \frac{H(t)^2}{8\pi G(t)}\) and the relative densities \(\Omega_M \equiv \rho/\rho_{\text{crit}}\) and \(\Omega_\Lambda = \rho_\Lambda/\rho_{\text{crit}}\) the modified Friedmann equation (2.1a) can be written as \(\Omega_M(t) + \Omega_\Lambda(t) = 1\).

3. RG trajectory with realistic parameter values

Before we start solving the modified field equations let us briefly review how the type IIIa trajectories of the Einstein-Hilbert truncation can be matched against the observational data [24]. This analysis is fairly robust and clearcut; it does not involve the NGFP. All that is needed is the RG flow linearized about the Gaussian fixed point (GFP) which is located at \(g = \lambda = 0\). In its vicinity one has [1] \(\Lambda(k) = \Lambda_0 + \nu G k^2 + \cdots\) and \(G(k) = G + \cdots\). Or, in terms of
the dimensionless couplings, \( \lambda(k) = \Lambda_0/k^2 + \nu G k^2 + \cdots \), \( g(k) = G k^2 + \cdots \). In the linear regime of the GFP, \( \Lambda \) displays a running \( \propto k^4 \) and \( G \) is approximately constant. Here \( \nu \) is a positive constant of order unity \([1, 4]\), \( \nu \equiv -\Phi^2(0) \). These equations are valid if \( \lambda(k) \ll 1 \) and \( g(k) \ll 1 \). They describe a 2-parameter family of RG trajectories labeled by the pair \((\Lambda_0, G)\). It will prove convenient to use an alternative labeling \((\lambda, \nu)\) the pair \((\lambda, \nu)\) is convenient to introduce the abbreviation \( \lambda \equiv \lambda_T/2 \nu \). When parameterized by the pair \((\lambda_T, k_T)\) the trajectories assume the form

\[
\Lambda(k) = \frac{1}{2} \lambda_T k_T^2 \left[ 1 + (k/k_T)^4 \right] \equiv \Lambda_0 \left[ 1 + (k/k_T)^4 \right] \tag{3.1}
\]

\[
G(k) = \frac{\lambda_T}{2 \nu k_T^2} \equiv \frac{g_T}{k_T^2}
\]

or, in dimensionless form,

\[
\lambda(k) = \frac{1}{2} \lambda_T \left( \frac{k}{k_T} \right)^2 \equiv \left( \frac{k}{k_T} \right)^2, \quad g(k) = \frac{g_T}{k_T^2} \tag{3.2}
\]

As for the interpretation of the new variables, it is clear that \( \lambda_T \equiv \lambda(k \equiv k_T) \) and \( g_T \equiv g(k = k_T) \), while \( k_T \) is the scale at which \( \beta_\lambda \) (but not \( \beta_g \)) vanishes according to the linearized running: \( \beta_\lambda(k_T) \equiv kd\lambda(k)/dk|_{k=k_T} = 0 \). Thus we see that \((g_T, \lambda_T)\) are the coordinates of the turning point \( T \) of the type IIIa trajectory considered, and \( k_T \) is the scale at which it is passed. It is convenient to refer to the “RG time” \( \tau \) to this scale: \( \tau(k) \equiv \ln(k/k_T) \). Hence \( \tau > 0 \) \((\tau < 0)\) corresponds to the “UV regime” \((\text{“IR regime”})\) where \( k > k_T \) \((k < k_T)\).

Let us now hypothesize that, within a certain range of \( k \)-values, the RG trajectory realized in Nature can be approximated by \((3.2)\). In order to determine its parameters \((\Lambda_0, G)\) or \((\lambda_T, k_T)\) we must perform a measurement of \( G \) and \( \Lambda \). If we interpret the observed values \( G_{\text{observed}} = m_{\text{Pl}}^2 \), \( m_{\text{Pl}} \approx 1.2 \times 10^{19} \text{ GeV} \), and \( \Lambda_{\text{observed}} = 3 \Omega_0 H_0^2 \approx 10^{-120} m_{\text{Pl}}^2 \) as the running \( G(k) \) and \( \Lambda(k) \) evaluated at a scale \( k \ll k_T \), then we get from \((3.2)\) that \( \Lambda_0 = \Lambda_{\text{observed}} \) and \( G = G_{\text{observed}} \). Using the definitions of \( \lambda_T \) and \( k_T \) along with \( \nu = O(1) \) this leads to the order-of-magnitude estimates \( g_T \approx \lambda_T \approx 10^{-60} \) and \( k_T \approx 10^{-30} m_{\text{Pl}} \approx (10^{-3}\text{ cm})^{-1} \). Because of the tiny values of \( g_T \) and \( \lambda_T \) the turning point lies in the linear regime of the GFP.

Up to this point we discussed only that segment of the “trajectory realized in Nature” which lies inside the linear regime of the GFP. The complete RG trajectory obtains by continuing this segment with the flow equation both into the IR and into the UV, where it ultimately spirals into the NGFP. While the UV-continuation is possible within the Einstein-Hilbert truncation, this approximation breaks down in the IR when \( \lambda(k) \) approaches \( 1/2 \). Interestingly enough, this happens near \( k = H_0 \), the present Hubble scale. The right panel of Fig.1 shows a schematic sketch of the complete trajectory on the \( g-\lambda \)-plane and Fig.2 displays the resulting \( k \)-dependence of \( G \) and \( \Lambda \).

4. Primordial entropy generation

Let us return to the modified continuity equation \((2.1b)\). After multiplication by \( a^3 \) it reads

\[
[\dot{\rho} + 3H(\rho + p)] a^3 = \bar{\mathcal{P}}(t) \tag{4.1}
\]

where we defined

\[
\bar{\mathcal{P}} = -\left( \frac{\dot{\Lambda} + 8\pi \rho \dot{G}}{8\pi G} \right) a^3 \tag{4.2}
\]
Without assuming any particular equation of state eq. (4.1) can be rewritten as

$$\frac{d}{dt}(\rho a^3) + p \frac{d}{dt}(a^3) = \tilde{\mathcal{P}}(t)$$

(4.3)

The interpretation of this equation is as follows. Let us consider a unit coordinate, i.e. comoving volume in the Robertson-Walker spacetime. Its corresponding proper volume is $V = a^3$ and its energy contents is $U = \rho a^3$. The rate of change of these quantities is subject to (4.3):

$$\frac{dU}{dt} + p \frac{dV}{dt} = \tilde{\mathcal{P}}(t)$$

(4.4)

In classical cosmology where $\tilde{\mathcal{P}} \equiv 0$ this equation together with the standard thermodynamic relation $dU + pdV = TdS$ is used to conclude that the expansion of the Universe is adiabatic, i.e. the entropy inside a comoving volume does not change as the Universe expands, $dS/dt = 0$.

Here and in the following we write $S \equiv sa^3$ for the entropy carried by the matter inside a unit comoving volume and $s$ for the corresponding proper entropy density.

When $\Lambda$ and $G$ are time dependent, $\tilde{\mathcal{P}}$ is nonzero and we interpret (4.4) as describing the process of energy (or “heat”) exchange between the scalar fields $\Lambda$ and $G$ and the ordinary matter. This interaction causes $S$ to change:

$$T \frac{dS}{dt} = T \frac{d}{dt}(sa^3) = \tilde{\mathcal{P}}(t)$$

(4.5)

The actual rate of change of the comoving entropy is

$$\frac{dS}{dt} = \frac{d}{dt}(sa^3) = \mathcal{P}(t)$$

(4.6)

where

$$\mathcal{P} \equiv \tilde{\mathcal{P}}/T$$

(4.7)

If $T$ is known as a function of $t$ we can integrate (4.5) to obtain $S = S(t)$. In the RG improved cosmologies the entropy production rate per comoving volume

$$\mathcal{P}(t) = -\left[\frac{\dot{\Lambda} + 8\pi \rho \dot{G}}{8\pi G}\right] a^3 T$$

(4.8)

is nonzero because the gravitational “constants” $\Lambda$ and $G$ have acquired a time dependence.

Clearly we can convert the heat exchanged, $Tds$, to an entropy change only if the dependence of the temperature $T$ on the other thermodynamical quantities, in particular $\rho$ and $p$ is known. For this reason we shall now make the following assumption about the matter system and its (non-equilibrium!) thermodynamics:

The matter system is assumed to consist of $n_{\text{eff}} = n_b + \frac{7}{8} n_f$ species of effectively massless degrees of freedom which all have the same temperature $T$. The equation of state is $p = \rho/3$, i.e. $w = 1/3$, and $\rho$ depends on $T$ as

$$\rho(T) = \kappa^4 T^4, \quad \kappa \equiv (\pi^2 n_{\text{eff}}/30)^{1/4}$$

(4.9)

No assumption is made about the relation $s = s(T)$.

The first assumption, radiation dominance and equal temperature, is plausible since we shall find that there is no significant entropy production any more once $H(t)$ has dropped substantially below $m_{\text{Pl}}$. The second assumption, eq. (4.9), amounts to the hypothesis that the injection of energy into the matter system disturbs its equilibrium only very weakly. The approximation is
that the *equilibrium* relations among $\rho$, $p$, and $T$ are still valid in the non-equilibrium situation of a cosmology with entropy production [33].

By inserting $p = \rho/3$ and (4.19) into the modified continuity equation the entropy production rate can be seen to be a total time derivative: $\mathcal{S}(t) = \frac{4}{3}G\kappa a^3 \rho^{3/4}$. Therefore we can immediately integrate (4.6) and obtain $S(t) = \frac{4}{3}G\kappa a^3 \rho^{3/4} + S_c$, $s(t) = \frac{4}{3}G\rho(t)^{3/4} + \frac{S_c}{a(t)^3}$. Here $S_c$ is a constant of integration. In terms of $T$, using (4.9) again,

$$s(t) = \frac{2\pi^2}{45} n_{\text{eff}} T(t)^3 + \frac{S_c}{a(t)^3} \quad (4.10)$$

The final result (4.10) is very remarkable for at least two reasons. First, for $S_c = 0$, eq.(4.10) has exactly the form valid for radiation *in equilibrium*. Note that we did not postulate this relationship, only the $\rho(T)$–law was assumed. The equilibrium formula $s \propto T^3$ was derived from the cosmological equations, i.e. the modified conservation law. This result makes the hypothesis “non-adiabatic, but as little as possible” selfconsistent.

Second, if $\lim_{t \to 0} a(t)\rho(t)^{1/4} = 0$, which is actually the case for the most interesting class of cosmologies we shall find, then $S(t \to 0) = S_c$. As we mentioned in the introduction, the most plausible initial value of $S$ is $S = 0$ which means a vanishing constant of integration $S_c$ here. But then, with $S_c = 0$ the *entire* entropy carried by the massless degrees of freedom is due to the RG running. So it indeed seems to be true that the entropy of the CMBR photons we observe today is due to a coarse graining but, unexpectedly, not a coarse graining of the matter degrees of freedom but rather of the gravitational ones which determine the background spacetime the photons propagate on.

5. Solving the RG improved Einstein Equations

In [27] we solved the improved Einstein equations (2.1a, 2.1b) for the trajectory with realistic parameter values which was discussed in Section 3. The solutions were determined by applying the algorithm described at the end of Section 2. Having fixed the RG trajectory, there exists a 1-parameter family of solutions $(H(t), \rho(t))$. This parameter is conveniently chosen to be the relative vacuum energy density in the fixed point regime, $\Omega^*_\Lambda$.

The very early part of the cosmology can be described analytically. For $k \to \infty$ the trajectory approaches the NGFP, $(g, \lambda) \to (g_s, \lambda_s)$, so that $G(k) = g_s/k^2$ and $\Lambda(k) = \Lambda_s k^2$. In this case the differential equation can be solved analytically, with the result

$$H(t) = \alpha/ t, \quad \rho(t) = A t^\alpha, \quad \alpha = \left[ \frac{1}{2} (3 + 3w) (1 - \Omega^*_\Lambda) \right]^{-1} \quad (5.1)$$

and $\rho(t) = \frac{\rho_0}{t^4}, G(t) = \frac{\dot{\rho} t^2}{2}, \Lambda(t) = \frac{\dot{\Lambda}}{t^2}$. Here $A, \dot{\rho}, \dot{G},$ and $\dot{\Lambda}$ are positive constants. They depend on $\Omega^*_\Lambda$, which assumes values in the interval $(0, 1)$. If $\alpha > 1$ the deceleration parameter $q = \alpha^{-1} - 1$ is negative and the Universe is in a phase of *power law inflation*. Furthermore, it has *no particle horizon* if $\alpha \geq 1$, but does have a horizon of radius $d_H = t/(1 - \alpha)$ if $\alpha < 1$. In the case of $w = 1/3$ this means that there is a horizon for $\Omega^*_\Lambda < 1/2$, but none if $\Omega^*_\Lambda \geq 1/2$.

If $w = 1/3$, the above discussion of entropy generation applies. The corresponding production rate reads $\mathcal{S}(t) = 4\kappa (\alpha - 1) A^3 \rho^{3/4} t^{3\alpha - 4}$. For the entropy per unit comoving volume we find, if $\alpha \neq 1$, $S(t) = S_c + \frac{4\kappa}{3} A^3 \rho^{3/4} t^{3(\alpha - 1)}$, and the corresponding proper entropy density is $s(t) = \frac{S_c}{A^3 t^{3\alpha}} + \frac{4\kappa}{3} \frac{\rho^{3/4}}{t^{3\alpha - 4}}$. For the discussion of the entropy we must distinguish 3 qualitatively different cases.

(a) **The case** $\alpha > 1$, i.e. $1/2 < \Omega^*_\Lambda < 1$: Here $\mathcal{S}(t) > 0$ so that the entropy and energy content of the matter system increases with time. By eq.(4.8), $\mathcal{S} > 0$ implies $\dot{\Lambda} + 8\pi\rho\dot{G} < 0$. Since $\dot{\Lambda} < 0$ but $\dot{G} > 0$ in the NGFP regime, the energy exchange is predominantly due to the decrease of $\Lambda$ while the increase of $G$ is subdominant in this respect.
The comoving entropy $S(t)$ has a finite limit for $t \to 0$, $S(t \to 0) = S_c$, and $S(t)$ grows monotonically for $t > 0$. If $S_c = 0$, which would be the most natural value in view of the discussion in the introduction, all of the entropy carried by the matter fields is due to the energy injection from $\Lambda$.

(b) The case $\alpha < 1$, i.e. $0 < \Omega^*_\Lambda < 1/2$: Here $\mathcal{P}(t) < 0$ so that the energy and entropy of matter decreases. Since $\mathcal{P} < 0$ amounts to $\dot{\Lambda} + 8\pi\rho\dot{G} > 0$, the dominant physical effect is the increase of $G$ with time, the counteracting decrease of $\Lambda$ is less important. The comoving entropy starts out from an infinitely positive value at the initial singularity, $S(t \to 0) \to +\infty$. This case is unphysical probably.

(c) The case $\alpha = 1$, i.e. $\Omega^*_\Lambda = 1/2$: Here $\mathcal{P}(t) \equiv 0$, $S(t) = \text{const}$. The effect of a decreasing $\Lambda$ and increasing $G$ cancel exactly.

At lower scales the RG trajectory leaves the NGFP and very rapidly “crosses over” to the GFP. This is most clearly seen in the behavior of the anomalous dimension $\eta_N(k) \equiv k\partial_k \ln G(k)$ which quickly changes from its NGFP value $\eta_* = -2$ to the classical $\eta_N = 0$. This transition happens near $k \approx m_{\text{Pl}}$ or, since $k(t) \approx H(t)$, near a cosmological “transition” time $t_{\text{tr}}$ defined by the condition $k(t_{\text{tr}}) = \xi H(t_{\text{tr}}) = m_{\text{Pl}}$. (Recall that $\xi = O(1)$). The complete solution to the improved equations can be found with numerical methods only. It proves convenient to use logarithmic variables normalized with respect to their respective values at the turning point. Besides the "RG time" $\tau \equiv \ln(k/k_T)$, we use $x \equiv \ln(a/a_T)$, $y \equiv \ln(t/t_T)$, and $U \equiv \ln(H/H_T)$.

Summarizing the numerical results one can say that for any value of $\Omega^*_\Lambda$ the UV cosmologies consist of two scaling regimes and a relatively sharp crossover region near $k, H \approx m_{\text{Pl}}$ corresponding to $x \approx -34.5$ which connects them. At higher $k$-scales the fixed point approximation is valid, at lower scales one has a classical FRW cosmology in which $\Lambda$ can be neglected.

As an example, Fig.3 shows the crossover cosmology with $\Omega^*_\Lambda = 0.98$ and $w = 1/3$. The entropy production rate $\mathcal{P}$ is maximum at $t_{\text{tr}}$ and quickly goes to zero for $t > t_{\text{tr}}$; it is non-zero for all $t < t_{\text{tr}}$. By varying the $\Omega^*_\Lambda$-value one can check that the early cosmology is indeed described by the NGFP solution (5.1). For the logarithmic $H$ vs. $a$- plot, for instance, it predicts $U = -2(1 - \Omega^*_\Lambda)x$ for $x < -34.4$. The left part of the plot in Fig.3a and its

**Figure 2.** The dimensionful quantities $\Lambda(k)$ and $G(k)$ for the RG trajectory with realistic parameter values.
counterparts with different values of $\Omega^*_{\Lambda}$ indeed comply with this relation. If $\Omega^*_{\Lambda} \in (1/2, 1)$ we have $\alpha = (2 - 2\Omega^*_{\Lambda})^{-1} > 1$ and $a(t) \propto t^\alpha$ describes a phase of accelerated power law inflation.

When $\Omega^*_{\Lambda} > 1/2$ the slope of $\mathcal{U}(x) = -2(1 - \Omega^*_{\Lambda})x$ decreases and finally vanishes at $\Omega^*_{\Lambda} = 1$. This limiting case corresponds to a constant Hubble parameter, i.e. to de Sitter space. For values of $\Omega^*_{\Lambda}$ smaller than, but close to 1 this de Sitter limit is approximated by an expansion $a \propto t^\alpha$ with a very large exponent $\alpha$.

The phase of power law inflation automatically comes to a halt once the RG running has reduced $\Lambda$ to a value where the resulting vacuum energy density no longer can overwhelm the matter energy density.

6. Inflation in the fixed point regime

Next we discuss in more detail the epoch of power law inflation which is realized in the NGFP regime if $\Omega^*_{\Lambda} > 1/2$. Since the transition from the fixed point to the classical FRW regime is rather sharp it will be sufficient to approximate the RG improved UV cosmologies by the following caricature: For $0 < t < t_{tr}$, the scale factor behaves as $a(t) \propto t^\alpha$, $\alpha > 1$. Here $\alpha = (2 - 2\Omega^*_{\Lambda})^{-1}$ since $w = 1/3$ will be assumed. Thereafter, for $t > t_{tr}$, we have a classical,
entirely matter-driven expansion $a(t) \propto t^{1/2}$.

6.1. Transition time and apparent initial singularity

The transition time $t_{tr}$ is dictated by the RG trajectory. It leaves the asymptotic scaling regime near $k \approx m_{Pl}$. Hence $H(t_{tr}) \approx m_{Pl}$ and since $\xi = O(1)$ and $H(t) = \alpha/t$ we find the estimate

$$t_{tr} = \alpha t_{Pl}$$

Here, as always, the Planck mass, time, and length are defined in terms of the value of Newton’s constant in the classical regime: $t_{Pl} = \ell_{Pl} = M_{Pl}^{-1} = G_{1/2}^0 = G_{\text{observed}}^{1/2}$. Let us now assume that $\Omega_{\Lambda}^*$ is very close to 1 so that $\alpha$ is large: $\alpha \gg 1$. Then (6.1) implies that the transition takes place at a cosmological time which is much later than the Planck time. At the transition the Hubble parameter is of order $m_{Pl}$, but the cosmological time is in general not of the order of $t_{Pl}$. Stated differently, the “Planck time” is not the time at which $H$ and the related physical quantities assume Planckian values. The Planck time as defined above is well within the NGFP regime: $t_{Pl} = t_{tr}/\alpha \ll t_{tr}$.

At $t = t_{tr}$ the NGFP solution is to be matched continuously with a FRW cosmology (with vanishing cosmological constant). We may use the classical formula $a \propto \sqrt{t}$ for the scale factor, but we must shift the time axis on the classical side such that $a$, $H$, and then as a result of (2.1a) also $\rho$ are continuous at $t_{tr}$. Therefore $a(t) \propto (t - t_{as})^{1/2}$ and $H(t) = \frac{1}{2} (t - t_{as})^{-1}$ for $t > t_{tr}$. Equating this Hubble parameter at $t = t_{tr}$ to $H(t) = \alpha/t$, valid in the NGFP regime, we find that the shift $t_{as}$ must be chosen as $t_{as} = (\alpha - \frac{1}{2}) t_{Pl} = (1 - \frac{1}{2\alpha}) t_{tr} < t_{tr}$. Here the subscript 'as' stands for “apparent singularity”. This is to indicate that if one continues the classical cosmology to times $t < t_{tr}$, it has an initial singularity (“big bang”) at $t = t_{as}$. Since, however, the FRW solution is not valid there nothing special happens at $t_{as}$; the true initial singularity is located at $t = 0$ in the NGFP regime. (See Fig. 4.)

6.2. Crossing the Hubble radius

In the NGFP regime $0 < t < t_{tr}$ the Hubble radius $\ell_H(t) \equiv 1/H(t)$, i.e. $\ell_H(t) = t/\alpha$, increases linearly with time but, for $\alpha \gg 1$, with a very small slope. At the transition, the slope jumps from $1/\alpha$ to the value 2 since $H = 1/(2t)$ and $\ell_H = 2t$ in the FRW regime. This behavior is sketched in Fig. 4.

Let us consider some structure of comoving length $\Delta x$, a single wavelength of a density perturbation, for instance. The corresponding physical, i.e. proper length is $L(t) = a(t) \Delta x$ then. In the NGFP regime it has the time dependence $L(t) = (t/t_{tr})^\alpha L(t_{tr})$. The ratio of $L(t)$ and the Hubble radius evolves according to $L(t)/\ell_H(t) = (t/t_{tr})^{\alpha-1} L(t_{tr})/\ell_H(t_{tr})$. For $\alpha > 1$, i.e. $\Omega_{\Lambda}^* > 1/2$, the proper length of any object grows faster than the Hubble radius. So objects which are of “sub-Hubble” size at early times can cross the Hubble radius and become “super-Hubble” at later times, see Fig. 4.

Let us focus on a structure which, at $t = t_{tr}$, is $e^N$ times larger than the Hubble radius. Before the transition we have $L(t)/\ell_H(t) = e^N (t/t_{tr})^{\alpha-1}$. Assuming $e^N > 1$, there exists a time $t_N < t_{tr}$ at which $L(t_N) = \ell_H(t_N)$ so that the structure considered “crosses” the Hubble radius at the time $t_N$. It is given by

$$t_N = t_{tr} \exp \left( - \frac{N}{\alpha - 1} \right)$$

(6.2)

What is remarkable about this result is that, even with rather moderate values of $\alpha$, one can easily “inflate” structures to a size which is by many $e$-folds larger than the Hubble radius during a very short time interval at the end of the NGFP epoch.

Let us illustrate this phenomenon by means of an example, namely the choice $\Omega_{\Lambda}^* = 0.98$ used in Fig. 3. Corresponding to 98% vacuum and 2% matter energy density in the NGFP regime,
Figure 4. Shown is the proper length $L$ and the Hubble radius as a function of time. The NGFP and FRW cosmologies are valid for $t < t_{tr}$ and $t > t_{tr}$, respectively. The classical cosmology has an apparent initial singularity at $t_{as}$ outside its domain of validity. Structures of size $e^N \ell_{Pl}$ at $t_{tr}$ cross the Hubble radius at $t_N$, a time which can be larger than the Planck time.

this value is still “generic” in the sense that $\Omega_\Lambda^*$ is not fine tuned to equal unity with a precision of many decimal places. It leads to the exponent $\alpha = 25$, the transition time $t_{tr} = 25 \ell_{Pl}$, and $t_{as} = 24.5 \ell_{Pl}$.

The largest structures in the present Universe, evolved backward in time by the classical equations to the point where $H = m_{Pl}$, have a size of about $e^{60} \ell_{Pl}$ there. We can use (6.2) with $N = 60$ to find the time $t_{60}$ at which those structures crossed the Hubble radius. With $\alpha = 25$ the result is $t_{60} = 2.05 \ell_{Pl} = t_{tr}/12.2$. Remarkably, $t_{60}$ is smaller than $t_{tr}$ by one order of magnitude only. As a consequence, the physical conditions prevailing at the time of the crossing are not overly “exotic” yet. The Hubble parameter, for instance, is only one order of magnitude larger than at the transition: $H(t_{60}) \approx 12m_{Pl}$. The same is true for the temperature; one can show that $T(t_{60}) \approx 12T(t_{tr})$ where $T(t_{tr})$ is of the order of $m_{Pl}$. Note that $t_{60}$ is larger than $t_{Pl}$.

6.3. Primordial density fluctuations

QEG offers a natural mechanism for generating primordial fluctuations during the NGFP epoch. They have a scale free spectrum with a spectral index close to $n = 1$. This mechanism is at the very heart of the “asymptotic safety” underlying the nonperturbative renormalizability of QEG. A detailed discussion of this mechanism is beyond the scope of the present review; the reader is referred to [5, 25, 27]. Suffice it to say that the quantum mechanical generation of the primordial fluctuations happens on sub-Hubble distance scales. However, thanks to the inflationary NGFP era the modes relevant to cosmological structure formation were indeed smaller than the Hubble radius at a sufficiently early time, for $t < t_{60}$, say. (See the $L(t)$ curve in Fig.4.)

7. Conclusions

We advocated the point of view that the scale dependence of the gravitational parameters has an impact on the physics of the Universe we live in and we tried to identify known features of the Universe which could possibly be due to this scale dependence. We proposed three possible candidates for such features: the entropy carried by the radiation which fills the Universe today, a period of automatic, $\Lambda$-driven inflation that requires no ad hoc inflaton, and the primordial density perturbations. While there is clearly no direct observational evidence for inflation it can
explain super-Hubble sized perturbations. For further details we refer to [27].

7.1. Acknowledgments
M. R. would like to thank the organizers of ICGC-07 for their cordial hospitality at Pune and for creating a stimulating scientific atmosphere.

7.2. Reference list
[1] M. Reuter, Phys. Rev. D 57 (1998) 971 and hep-th/9605030
[2] D. Dou and R. Percacci, Class. Quant. Grav. 15 (1998) 3449.
[3] O. Lauscher and M. Reuter, Phys. Rev. D 65 (2002) 025013 and hep-th/0108040
[4] M. Reuter and F. Saueressig, Phys. Rev. D 65 (2002) 065016 and hep-th/0110054.
[5] O. Lauscher and M. Reuter, Phys. Rev. D 66 (2002) 025026 and hep-th/0205062
[6] O. Lauscher and M. Reuter, Class. Quant. Grav. 19 (2002) 483 and hep-th/0110021
[7] O. Lauscher and M. Reuter, Int. J. Mod. Phys. A 17 (2002) 993 and hep-th/0112089
[8] M. Reuter and F. Saueressig, Phys. Rev. D 66 (2002) 125001 and hep-th/0206145
[9] W. Souma, Prog. Theor. Phys. 102 (1999) 181.
[10] R. Percacci and D. Perini, Phys. Rev. D 67 (2003) 081503; Phys. Rev. D 68 (2003) 044018; D. Perini, Nucl. Phys. Proc. Suppl. 127 C (2004) 185.
[11] A. Codello and R. Percacci, Phys. Rev. Lett. 97 (2006) 221301; A. Codello, R. Percacci and C. Rahmede, hep-th/0705.1769; P. Machado and F. Saueressig, arXiv: 0712.0445 [hep-th].
[12] D. Litim, Phys. Rev. Lett. 92 (2004) 201301; hep-th/0606044; P. Fischer and D. Litim, hep-th/0622003
[13] A. Bonanno and M. Reuter, JHEP 02 (2005) 035 and hep-th/0410191
[14] For a review see: M. Reuter and F. Saueressig, arXiv: 0708.1317 [hep-th].
[15] For a review see: O. Lauscher and M. Reuter, in Quantum Gravity, B. Fauser, J. Tolksdorf and E. Zeidler (Eds.), Birkhäuser, Basel (2007) and hep-th/0511260. For a general introduction see: C. Kiefer, Quantum Gravity, Second Edition, Oxford, 2007.
[16] R. Percacci and D. Perini, Class. Quant. Grav. 21 (2004) 5035 and hep-th/0401071.
[17] S. Weinberg in General Relativity, an Einstein Centenary Survey, S. W. Hawking and W. Israel (Eds.), Cambridge University Press (1979); S. Weinberg, hep-th/9702027
[18] For a review see: M. Niedermaier and M. Reuter, Living Reviews in Relativity, 9 (2006) 5.
[19] P. Forgács and M. Niedermaier, hep-th/0207028; M. Niedermaier, JHEP 12 (2002) 066; Nucl. Phys. B 673 (2003) 131; preprint gr-qc/0610018.
[20] C. Wetterich, Phys. Lett. B 301 (1993) 90.
[21] M. Reuter and C. Wetterich, Nucl. Phys. B 417 (1994) 181, Nucl. Phys. B 427 (1994) 291, Nucl. Phys. B 391 (1993) 147, Nucl. Phys. B 408 (1993) 91; M. Reuter, Phys. Rev. D 53 (1996) 4430, Mod. Phys. Lett. A 12 (1997) 2777.
[22] For a review see: J. Berges, N. Tetradis and C. Wetterich, Phys. Rep. 363 (2002) 223; C. Wetterich, Int. J. Mod. Phys. A 16 (2001) 1951; H. Gies, hep-ph/0611146.
[23] O. Lauscher and M. Reuter, JHEP 10 (2005) 050 and hep-th/0508202.
[24] A. Bonanno and M. Reuter, Phys. Rev. D 71 (2006) 083005 and hep-th/0602159; Phys. Rev. D 60 (1999) 084011 and gr-qc/0811026; M. Reuter and E. Tuiran, hep-th/0612037.
[25] A. Bonanno and M. Reuter, Phys. Rev. D 65 (2002) 043508 and hep-th/0106133.
[26] M. Reuter and F. Saueressig, JCAP 09 (2005) 012 and hep-th/0507167.
[27] A. Bonanno and M. Reuter, JCAP 08 (2007) 024 and arXiv: 0706.0174 [hep-th].
[28] A. Bonanno and M. Reuter, Phys. Lett. B 527 (2002) 9 and astro-ph/0106488; Int. J. Mod. Phys. D 13 (2004) 107 and astro-ph/0210472.
[29] E. Bentivegna, A. Bonanno and M. Reuter, JCAP 01 (2004) 001 and astro-ph/0303150.
[30] A. Bonanno, G. Esposito and C. Rubano, Gen. Rel. Grav. 35 (2003) 1899; Class. Quant. Grav. 21 (2004) 5005; A. Bonanno, G. Esposito, C. Rubano and P. Scudellaro, Class. Quant. Grav. 23 (2006) 3103; preprint gr-qc/0609012.
[31] M. Reuter and H. Weyer, Phys. Rev. D 69 (2004) 104022 and hep-th/0311196; M. Reuter and H. Weyer, Phys. Rev. D 70 (2004) 124028 and hep-th/0410117; M. Reuter and H. Weyer, JCAP 12 (2004) 001 and hep-th/0410119.
[32] F. Girelli, S. Liberati, R. Percacci, C. Rahmede, gr-qc/0607030.
[33] J.A.S. Lima, Phys. Rev. D54 (1996) 2571; J.A.S. Lima, Gen. Rel. Grav. 29 (1997) 805.