Variable Order, Directional $\mathcal{H}^2$-Matrices for Helmholtz Problems with Complex Frequency

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March 8, 2019

Abstract
The sparse approximation of high-frequency Helmholtz-type integral operators has many important physical applications such as problems in wave propagation and wave scattering. The discrete system matrices are huge and densely populated; hence their sparse approximation is of outstanding importance. In our paper we will generalize the directional $\mathcal{H}^2$-matrix techniques from the “pure” Helmholtz operator $Lu = -\Delta u + \zeta^2 u$ with $\zeta = -ik$, $k \in \mathbb{R}$, to general complex frequencies $\zeta \in \mathbb{C}$ with $\text{Re}\zeta > 0$. In this case, the fundamental solution decreases exponentially for large arguments. We will develop a new admissibility condition which contains $\text{Re}\zeta$ in an explicit way and introduce the approximation of the integral kernel function on admissible blocks in terms of frequency-dependent directional expansion functions. We develop an error analysis which is explicit with respect to the expansion order and with respect to $\text{Re}\zeta$ and $\text{Im}\zeta$. This allows to choose the variable expansion order in a quasi-optimal way depending on $\text{Re}\zeta$ but independent of, possibly large, $\text{Im}\zeta$. The complexity analysis is explicit with respect to $\text{Re}\zeta$ and $\text{Im}\zeta$ and shows how higher values of $\text{Re}\zeta$ reduce the complexity. In certain cases, it even turns out that the discrete matrix can be replaced by its nearfield part.

Numerical experiments illustrate the sharpness of the derived estimates and the efficiency of our sparse approximation.

Keywords: Helmholtz equation in lossy media, hierarchical matrices, boundary integral operator

Mathematics Subject Classification (2000): 35J05, 65D05, 65N38, 41A10, 65N12

1 Introduction
The numerical simulation of many physical problems involves the solution of large linear systems as a partial step in the overall algorithm. For real-world applications the dimension

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of the system is huge, e.g., of order $10^6 - 10^{10}$, which rules out exact elimination methods based, e.g., on Gauss or Cholesky decompositions. Instead, iterative solvers are employed which require a matrix-vector multiplication in each iteration step. If non-local (integral) operators are involved the system matrices are fully populated and a) the computation of the entries of the system matrix and b) the matrix-vector multiplications typically are the bottlenecks in the solution algorithms.

Since the mid 1980ies, the development of compression algorithms for densely populated matrices related to the numerical discretization of non-local operators has become an important topic in numerical analysis and scientific computing. The fast multipole method has been developed in [40] for evaluating discrete Coulomb potentials. Panel-clustering methods have been introduced first for collocation methods (see [22], [23]) and were extended to Galerkin methods in [44], [24]. The idea of cluster methods have been generalized to a more algebraic setting and led to the hierarchical $\mathcal{H}$-matrices (see [19], [20]). A second hierarchy has been introduced in [21], [6] and the resulting matrices are denoted as $\mathcal{H}^2$ matrices. Most of these methods are restricted to non-oscillatory elliptic problems. For highly oscillatory Helmholtz problems, compression algorithms for the arising non-local integral operators have been developed since the early 1990ies, among them are high-frequency, fast multipole methods [41], [4], [3], [26], butterfly schemes [17], [13], [9], and directional methods [11], [15], [38], [5], [10], [7]. For a comparison of these methods we refer to [10] and [9].

The existing literature is mostly concerned with the “pure” Helmholtz problem, i.e., the operator $L_\zeta u := -\Delta u + \zeta^2 u$ for purely imaginary frequency $\zeta \in i\mathbb{R}$ (exceptions are the papers [4], [3], [26]). In our paper we consider more general frequencies $\zeta \in \mathbb{C}$ with $\text{Re}\ \zeta \geq 0$ and recall different important applications where such frequencies occur. The important difference to the pure Helmholtz problem is that the fundamental solution exhibits an exponential decay for $\text{Re}\ \zeta > 0$ for large arguments. We generalize the directional $\mathcal{H}^2$-matrix approach in [10] to complex frequencies and introduce new admissibility conditions which contain the real part of the complex frequency in an explicit way. We introduce the directional approximation of the integral kernel function on admissible blocks and derive estimates for the approximation error which allow to select the control parameters in a quasi-optimal way. It turns out that a variable expansion order (depending on $\text{Re}\ \zeta$) for different blocks is advantageous compared to a fixed approximation order. In fact, it turns out that for $\text{Re}\ \zeta \gtrsim h_G^{-1} \log(h_G)$ (where $h_G$ is a characteristic mesh width of the underlying boundary element mesh $\mathcal{G}$) the discrete boundary element matrices can be replaced by its nearfield part.

The error estimates on the admissible blocks hold uniformly with respect to high oscillations. This has impact to the complexity analysis – the compression rates benefit from a) the admissibility conditions which are explicit in $\text{Re}\ \zeta$, b) from the variable-order expansion, and mostly from c) the frequency-explicit error estimates which allow to set substantial parts of the system matrix to zero for large enough $\text{Re}\ \zeta$. In our numerical experiments ([36]), we have applied our new admissibility condition and compression method to the BEM matrix for the acoustic single layer potential. As an illustration we depict in Figure 1 the dependence of the sparsity pattern on the real and imaginary part of the wave number $\zeta$. As predicted by our analysis the compression becomes stronger if the ratio $\text{Re}\ \zeta/\text{Im}\ \zeta$ increases.

The paper is structured as follows. In the next section, we will describe three kinds of application where non-local Helmholtz-type integral operators arise for general complex frequencies. In Section 3 we formulate the directional $\mathcal{H}^2$-matrix method with variable rank for general complex frequencies. First, we introduce our new admissibility conditions and then formulate the method in an algorithmic way. In Section 4 we estimate the error for the
Figure 1: Sparsity pattern of the BEM matrices for the boundary integral operator of the acoustic single layer potential. Non-admissible blocks are marked by red and admissible blocks by blue. The ratio \( \text{Re} \zeta / \text{Im} \zeta \) increases from left to right and from top to bottom. On the left top, we consider the pure Helmholtz problem, i.e., \( \text{Im} \zeta = 0 \) while the ratios for the others are given by 1, 2, 3.
original integral kernel function being replaced by the directional, variable order expansion on admissible matrix blocks. The admissibility conditions along the error estimates form the basis for the complexity analysis which is presented in Section 5. Finally, in Section 6 we report on the results of numerical experiments which demonstrate the sharpness of our estimates.

2 Setting

In this section we will introduce three types of applications which lead to Helmholtz-type equations at complex frequencies $\zeta \in \mathbb{C}$, $\text{Re} \zeta \geq 0$.

2.1 Helmholtz Equation with Decay

Time harmonic wave propagation with decay arises in many applications such as, e.g., in viscoelastodynamics for materials with damping (see, e.g., [1]), in electromagnetics for wave propagation in lossy media (see, e.g., [25]), and in non-linear optics (see, e.g., [42]). In the simplest case such problems are modelled by a Helmholtz equation with complex wave number.

2.1.1 Variational Formulation

Let $\Omega^- \subset \mathbb{R}^3$ be a bounded Lipschitz domain with boundary $\Gamma$ and $\Omega^+ := \mathbb{R}^3 \backslash \overline{\Omega}$ its unbounded complement. For $\sigma \in \mathbb{R}$, let

$$C_{\geq \sigma} := \{ \zeta \in \mathbb{C} \mid \text{Re} \zeta \geq \sigma \}. $$

Let the bilinear form $\langle \cdot, \cdot \rangle : C^3 \times C^3 \to \mathbb{C}$ be defined by $\langle x, y \rangle = \sum_{j=1}^3 x_j y_j$ so that the Euclidean norm is given by $||z|| = \langle z, \overline{z} \rangle^{1/2}$. For a complex frequency $\zeta \in C_{\geq 0}$ and $\Omega \in \{ \Omega^-, \Omega^+ \}$ we consider the Helmholtz equation subject to Dirichlet boundary conditions

$$-\Delta u + \zeta^2 u = 0 \quad \text{in } \Omega,$$

$$u = g_D \quad \text{on } \Gamma. \quad \text{(2.1)}$$

If $\Omega = \Omega^+$ we also impose decay conditions at infinity

$$\left| \frac{\partial w}{\partial r} + \zeta w \right| = C r^{-1} \quad \text{for } r := ||x|| \to \infty. \quad \text{(2.2)}$$

Here, $\partial_r$ denote the derivative in radial direction. We introduce the acoustic Newton potential $G(\zeta, \cdot)$ by

$$G(\zeta, z) := \frac{\text{e}^{-\zeta ||z||}}{4\pi ||z||}. \quad \text{(2.3)}$$

For the solution of (2.1) we employ an ansatz as an acoustic single layer potential

$$(S(\zeta) \varphi)(x) := \int_{\Gamma} G(\zeta, y - x) \varphi(y) \, d\Gamma_y \quad \forall x \in \Omega. \quad \text{(2.4)}$$

To determine the unknown boundary density $\varphi : \Gamma \to \mathbb{C}$ we employ the Dirichlet boundary condition and the continuity of the single layer operator up to the boundary. Let

$$(V(\zeta) \varphi)(x) := \int_{\Gamma} G(\zeta, y - x) \varphi(y) \, d\Gamma_y \quad \forall x \in \Gamma. \quad \text{(2.4)}$$
Then, the strong formulation for the unknown density \( \varphi \) is given by
\[
V(\zeta) \varphi = g_D \quad \text{on } \Gamma.
\] (2.5)

For the analysis of the boundary integral equation and its Galerkin discretization it is convenient to introduce the variational formulation. The Sobolev spaces \( H^s(\Gamma) \), \( s \geq 0 \), are defined in the usual way (see, e.g., [18] or [35]) and the spaces with negative order \( s < 0 \) by duality. The norm is denoted by \( \| \cdot \|_{H^s(\Gamma)} \). The variational formulation of (2.5) is as follows: For given \( g_D \in H^{1/2}(\Gamma) \) find \( \varphi \in H^{-1/2}(\Gamma) \) such that
\[
a_\zeta(\varphi, \psi) := (V(\zeta) \varphi, \psi) = (g_D, \psi) \quad \forall \psi \in H^{-1/2}(\Gamma). \tag{2.6}
\]

Here \((\cdot, \cdot)\) denotes the continuous extension of the \( L^2(\Gamma) \) scalar product (with complex conjugation on the second argument) to the anti-dual pairing on \( H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma) \), i.e.,
\[
(V(\zeta) \varphi, \psi) = \int_\Gamma \int_\Gamma G(\zeta, y - x) \varphi(y) \overline{\psi(x)} d\Gamma_y d\Gamma_x.
\]

**Remark 2.1** Existence and uniqueness results for the solution of the continuous problem (2.6) for the case \( \Re \zeta > 0 \) are proved in [2]. For \( \zeta \in i\mathbb{R} \) it is well known that the operator \( V(\zeta) \) is not invertible for discrete spurious frequencies. In this case, stabilized formulations (Brakhage-Werner or those proposed in [12]) cure this problem. We emphasize that \( V(\zeta) \) appears in the stabilized formulations and its sparse representation is still required.

**2.1.2 Galerkin Discretization**

We consider the discretization of (2.6) by a Galerkin boundary element method. For a systematic introduction of boundary element methods we refer, e.g., to the monograph [43, Chap. 4]. Let \( \mathcal{G} = \{ \tau_i : 1 \leq i \leq M \} \) denote a surface mesh of \( \partial\Omega \), consisting of affine or possibly curved triangles (called panels in this context). As a convention the triangles are (relatively) closed sets. For simplicity we assume that the boundary element mesh \( \mathcal{G} \) does not contain hanging nodes, more precisely, that two non-identical triangles \( \tau, \ell \in \mathcal{G} \) either have a positive distance or their intersection is either a common edge or a common vertex. For any \( \tau \in \mathcal{G} \), there is a bijective element map \( \chi_\tau : \hat{\tau} \to \tau \) which maps the reference element \( \hat{\tau} := \text{conv} \{ (i_0^1), (i_0^2), (i_1^1) \} \) to the surface panel \( \tau \); we assume that this mapping is affine if \( \tau \) is a plane triangle with straight edges. In any case we assume that a common side \( E \) of two adjacent triangles \( \tau, \ell \in \mathcal{G} \), are parametrized by \( \chi_\tau, \chi_\ell \) “in the same way”, i.e., \( \chi^{-1}_\tau(x) = \gamma \circ \chi^{-1}_\ell(x) \) for all \( x \in E \) and a suitable affine mapping \( \gamma : \hat{\tau} \to \hat{\ell} \).

The finite-dimensional boundary element space of polynomial degree \( p \in \mathbb{N}_0 \) and smoothness degree \( m \in \{-1, 0\} \) sub-ordinate to \( \mathcal{G} \) is given by
\[
S^{p,m}_\mathcal{G} := \{ u \in L^1(\Gamma) \mid \forall \tau \in \mathcal{G} \quad u|_\tau \circ \chi_\tau \in \mathbb{P}_p \} \cap H^{m+1}(\Gamma),
\]
where \( \hat{\tau} \) denotes the interior of \( \tau \). If no confusion is possible, we suppress the indices \( p, m, \mathcal{G} \) and write \( S \) short for \( S^{p,m}_\mathcal{G} \). The standard Lagrange nodal basis is denoted by \( b_i, i \in \mathcal{I} := \{1, \ldots, n\} \), and depends as well on \( p, m, \mathcal{G} \). Finally we have
\[
S^{p,m}_\mathcal{G} = \text{span} \{ b_i : 1 \leq i \leq n \} \subset H^{-1/2}(\partial\Omega). \tag{2.7}
\]
The maximal mesh width is denoted by
\[ h_G := \max \{ h_{\tau} : \tau \in G \} \quad \text{with} \quad h_{\tau} := \text{diam} \ \tau. \]

For the mesh we define the shape-regularity constants \( c_{sr} \) and \( C_{sr} \) by
\[
c_{sr} := \min_{\tau \in G} \frac{\text{volume} \ \tau}{h_{\tau}^2} \quad \text{and} \quad C_{sr} := \max_{\tau \in G} \frac{\text{volume} \ \tau}{h_{\tau}^2}. \tag{2.8}
\]

Another mesh parameter is
\[
h_{\text{min}} := \min \{ \text{dist} (\tau, \tau') : \forall \tau \in G \ \forall \tau' \in G \ \text{with} \ \tau \cap \tau' = \emptyset \}. \tag{2.9}
\]

We say that a boundary element mesh is quasi-uniform if there exists a constant \( 0 < C_{qu} = O(1) \) such that
\[
h_G \leq C_{qu} h_{\text{min}}. \tag{2.10}
\]

The number of panels is of the same order as the dimension of \( S_{G}^{p,m} \): there exist constants \( C_{loc,p}, C_p \) only depending on the local polynomial degree \( p \) such that for \( m = -1, 0 \)
\[
\max_{i \in I} \sharp \{ j \in I : \text{supp} \ \text{b}_j \cap \text{supp} \ \text{b}_i \} \leq C_{loc,p} \quad \text{and} \quad C_{p}^m \leq \dim S_{G}^{p,m} = n. \tag{2.11}
\]

The Galerkin discretization of equation (2.6) is given by seeking functions \( \varphi_S \in S \) such that
\[
a_{\zeta} (\varphi_S, \psi) = (g_D, \psi) \quad \forall \psi \in S \tag{2.12}
\]
with \( a_{\zeta} (\cdot, \cdot) \) as in (2.6). By using the basis \( \text{b}_i \) we obtain a representation of this equation as a system of linear equations. Let \( K (\zeta) = (K_{i,j} (\zeta))^{n}_{i,j=1} \in \mathbb{C}^{n \times n} \) and \( \mathbf{r} = (r_i)^n_{i=1} \) be defined by
\[
K_{i,j} (\zeta) := (V (\zeta) \text{b}_j, \text{b}_i) \quad \text{and} \quad r_i := (g_D, \text{b}_i), \quad 1 \leq i, j \leq n. \tag{2.13}
\]

The solution \( \phi = (\varphi_i)^n_{i=1} \) of the linear system
\[
K (\zeta) \phi = \mathbf{r}
\]
is then equivalent to the solution of (2.12) via
\[
\varphi_S = \sum_{i=1}^{n} \varphi_i \text{b}_i. \tag{2.14}
\]

**Remark 2.2**

1. From \cite{2} it follows that (2.12) is well posed for \( \text{Re} \ \zeta > 0 \) and, as a consequence, the matrix \( K (\zeta) \) is invertible (cf. Rem. 2.1).

2. The matrix \( K (\zeta) \) is fully populated containing, in general, \( n^2 \) non-zero entries. This is a major bottleneck in a numerical realization of the boundary element method.

\[\text{For a measurable subset } \omega \subset \Gamma \text{ we denote by } |\omega| \text{ the area measure of } \omega.\]
3. Céa’s lemma can be applied to derive error estimates for the solution $\varphi_S$. By using the results in [28, Proof of Prop. 16] we obtain the quasi-optimal, frequency-explicit error estimate
\[
\|\varphi - \varphi_S\|_{H^{-1/2}(\Gamma)} \leq C \frac{|\zeta|^3}{(\text{Re} \zeta)^2} \inf_{\psi \in S} \|\varphi - \psi\|_{H^{-1/2}(\Gamma)}.
\]

4. The theory deteriorates as $\text{Re} \zeta \to 0$ and this is not an artifact. It is well known that the operator $V(\zeta)$ for purely imaginary wave number $\zeta$ is not injective for certain values of $\zeta \in i \mathbb{R}$. Instead of (2.4) one often employs a combined double layer/single layer ansatz, see, e.g., [37], [12], and the resulting boundary integral equation becomes well posed for all purely imaginary frequencies.

2.2 Convolution Quadrature

The linear homogeneous space-time wave equation can be transformed to space-time boundary integral equations with retarded potentials. A popular method for solving these equations is the convolution quadrature (CQ) introduced in [33], [34]. To circumvent the condition for the CQ that the time steps must be constant, the gCQ method has been introduced in [29], [31], [32] to allow for variable time steps. This method involves the numerical approximation of a contour integral of the form
\[
\frac{\Delta_j}{2\pi i} \int_C \frac{V(\zeta)}{\prod_{\ell=j+1}^N (1 - \Delta \zeta)} d\zeta,
\]
where $C$ is a circle in the complex plane with midpoint $R > 0$ and radius $R$. The $j$-th time step is denoted by $\Delta_j$. For the numerical evaluation of this contour integral, a quadrature method has been proposed in [30] which is of the form
\[
\frac{\Delta_j}{2\pi i} \sum_{\ell=1}^{N_Q} w_{\ell} \frac{V(\zeta_{\ell})}{\prod_{\ell=j+1}^N (1 - \Delta \zeta_{\ell})}.
\]
Since the radius $R$ is large (proportional to the reciprocal minimal time step) also the number of quadrature points is large and the numerical realization requires the boundary element discretization of the operator $V(\zeta)$ at all quadrature points on the contour $C$. Hence, also for this application one needs a sparse approximation of the boundary element matrices $K(\zeta)$ for complex frequencies. We emphasize that for this application also the case of non-resolved frequencies arises at certain quadrature points, i.e., the standard resolution condition $h_G |\text{Im} \zeta| \lesssim 1$ is violated. The analysis of our sparse approximation nicely reflects this fact: in certain cases (related to the magnitude of $\text{Re} \zeta$) only the nearfield part of the system matrix has to be generated but not underresolved oscillations in the farfield (cf. Remark 5.1).

The need for the evaluation of such contour integrals also appears for the original convolution quadrature method with constant time stepping since non-local Helmholtz-type boundary element matrices have to be assembled in many contour quadrature points. For the CQ method the resulting system matrix is of block Toeplitz form and FFT-type techniques can be employed to reduce the complexity with respect to the number of time points $N$ from $O(N^2)$ to almost $O(N)$ (up to logarithmic terms). The combination of the FFT techniques in time and sparse matrix techniques in space is far from trivial. In [4], [3] such a fast multipole algorithm is introduced and numerical experiments demonstrate the almost linear complexity (up to logarithmic terms) with respect to the total number $Nn$ of unknowns; the generalization
to general complex frequencies of this fast multipole method is presented and analyzed in [26]. The spatial compression algorithm for the Helmholtz-type boundary element matrices is based on the high-frequency multipole method which goes back to [41] and is different from the directional $H^2$ matrices (for a comparison of these compression methods we refer to [10, §1]). We expect that our directional $H^2$-matrix compression algorithm has the potential to be used within the fast method described in [4], [3] and has the advantage that a fully developed accuracy analysis is available to select the control parameters in a quasi-optimal way.

2.3 Limiting Absorbing Principle

The transformation of the time-space linear wave equation $(\partial_t^2 - \Delta) u = f$ to the frequency domain by a time periodic ansatz leads to the Helmholtz equation of the form (2.1) with purely imaginary frequency $\zeta \in i \mathbb{R}$. This equation is not solvable if $-\zeta^2$ is an eigenvalue of the (negative) Laplacian with Dirichlet boundary conditions. For theoretical as well as for practical reasons (see, e.g., [39], [27]) it can be useful to “add some absorption” to this equation and to consider the equation

$$-\Delta u + (\zeta^2 - i \varepsilon) u = 0 \quad \text{in } \Omega,$$

$$u = g_D \quad \text{on } \Gamma$$

for a small positive parameter $\varepsilon$. This equation is solvable for all frequencies $\zeta \in i \mathbb{R}$ and one can employ the Galerkin boundary element for its discretization. The discretization matrix is given by $K(\tilde{\zeta})$ (cf. (2.13)) for the choice $\tilde{\zeta} = \sqrt{\zeta^2 + i \varepsilon^2} = a + ib$ with $a := \sqrt{\zeta^2 + \varepsilon^2/4} > 0$ and $b := -\varepsilon/(2a)$. This implies $\tilde{\zeta} \in \mathbb{C}_{>0}$. Hence, also in this case sparse matrix techniques which are applicable to $K(\zeta)$, $\zeta \in \mathbb{C}_{>0}$ are important for the Helmholtz equation with artificially added absorption.

3 Directional $H^2$ Matrices for Helmholtz Equations with Decay

Directional $H^2$ matrices have been introduced in [11], [15], [38], [5], [10], [7] for the high-frequency Helmholtz problems for purely imaginary frequency.

3.1 Directionally Admissible Partitionings

We generalize this method to general complex frequencies and analyze its accuracy and complexity explicitly with respect to the real and imaginary part of the wave number. To formulate the algorithm we first introduce some notation.

As a basis for the boundary element space we have chosen Lagrange basis functions $b_i$ which have local support $\omega_i := \text{supp } b_i$. We collect the set of indices $1 \leq i \leq n$ in the set $\mathcal{I}$ so that $\# \mathcal{I} = n$.

**Definition 3.1** For a given set of degrees of freedom $\mathcal{I}$, the cluster tree $\mathcal{T}_\mathcal{I}$ is a labeled tree which satisfies:

1. the label $\hat{\ell}$ of each node $t \in \mathcal{T}_\mathcal{I}$ is a subset of the index set $\mathcal{I}$,
2. the root \( r \in \mathcal{T}_I \) of the tree is assigned \( \hat{r} = I \);

3. for all \( t \in \mathcal{T}_I \) there exists

(a) either a set of nodes sons \((t)\) denoted by sons of \( t \) which satisfies: \( \hat{t} = \bigcup \hat{s} \) and, for all \( t_1, t_2 \in \text{sons} (t) \), it holds either \( t_1 = t_2 \) or \( \hat{t}_1 \cap \hat{t}_2 = \emptyset \).

(b) or \( t \) is called a leaf. The set of leaf clusters is

\( \mathcal{L}_I := \{ t \in \mathcal{T}_I : t \text{ is a leaf}\} \).

Vice versa \( t = \text{father} (t') \) is the father of \( t' \in \text{sons} (t) \);

4. With each cluster, an axis-parallel bounding box \( B_t \) is associated which satisfies

\[
\omega_t := \bigcup_{i \in \hat{i}} \omega_i \subset B_t. \tag{3.1}
\]

The center \( M_t \) of a cluster \( t \) is defined as the barycenter of \( B_t \).

The level of a cluster is given recursively by level \((r) := 0 \) and level \((t') := \text{level} (t) + 1 \) for all \( t \in \mathcal{T}_I \setminus \mathcal{L}_I \) and \( t' \in \text{sons} (t) \). The depth of a cluster tree is depth \((\mathcal{T}_I) := \max \{ \text{level} (t) : t \in \mathcal{T}_I \}\) and \( \mathcal{T}_t := \{ t \in \mathcal{T}_I \mid \text{level} (t) = t \} \). The maximal cluster diameter of level \( t \) is

\[
\delta_t := \max \{ \text{diam} B_t : t \in \mathcal{T}_t \}. \tag{3.2}
\]

A natural choice for the bounding box \( B_t \) is the minimal box such that \((3.1)\) holds but we do not restrict to this choice. However, we assume that there exist positive constants \( c_{\text{vol}}, C_{\text{vol}} \) such that

\[
c_{\text{vol}} |\omega_t| \leq \text{diam}^2 B_t \leq C_{\text{vol}} |\omega_t|. \tag{3.3}
\]

Algorithms for building cluster trees from index sets \( I \) corresponding to boundary element basis functions can be found, e.g., in [45], [20].

The clusters allow via the geometric correspondence \((3.1)\) to identify pairs of regions \( B_t, B_s \subset \Omega \) – associated to pairs of clusters \((t, s)\) – where the kernel function can be approximated by a separable expansion:

\[
G (\zeta, y - x) \approx \sum_{k=1}^K \sum_{\mu=1}^K \gamma_{\nu, \mu} \Phi^\nu (x) \Psi^\mu (y) \quad \forall (x, y) \in B_t \times B_s,
\]

i.e., an expansion where \( x \) and \( y \) appear only in a factorized way. The number \( k \) is denoted as the rank of the separable expansion. To identify these regions we employ an admissibility condition which will be introduced next. It will turn out from our analysis that for a pair \((t, s)\) of admissible clusters the kernel function can be approximated by a separable expansion.

**Definition 3.2 (directional admissibility condition for complex frequencies)** For \( \eta = (\eta_i)_{i=1}^3 \in \mathbb{R}_{>0}^3 \) a pair of clusters \( t, s \in \mathcal{T}_I \) and a direction \( c \in \mathbb{S}_2 \) are \( \eta \)-admissible with respect to a complex frequency \( \zeta \in \mathbb{C}_{>0} \) if they satisfy the following three conditions:

\[
\begin{align*}
|\text{Im} \zeta| \frac{M_t - M_s}{\|M_t - M_s\| - c} &\leq \eta_1 \max \{ \text{diam} (B_t), \text{diam} (B_s) \}, \tag{3.4a} \\
\max \{ \text{diam} (B_t), \text{diam} (B_s) \} &\leq \eta_2 \text{dist} (B_t, B_s), \tag{3.4b} \\
|\text{Im} \zeta| \max \{ \text{diam}^2 (B_t), \text{diam}^2 (B_s) \} &\leq \max \{ \eta_2, \eta_3 (\text{Re} \zeta) \} \text{dist} (B_t, B_s) \text{dist} (B_t, B_s). \tag{3.4c}
\end{align*}
\]
The algorithm for generating a minimal partition $P$ of $I \times I$ by admissible and non-
admissible blocks is of divide-and-conquer type.

**Remark 3.3** We will need the minimal distance between the clusters of admissible blocks and set

$$\delta_{\min} := \min \{ \text{dist} (B_t, B_s) : (t, s) \text{ is admissible} \} .$$

We have by (3.4)

$$\delta_{\min} \geq \frac{1}{\eta_2} \min \{ \text{diam} (B_t) : t \in T_I \} \geq \frac{h_{\min}}{\eta_2}$$

with $h_{\min}$ as in (2.4).

**Algorithm 3.4** The minimal, $\eta$-admissible block partitioning $P$ of $I \times I$ is obtained as the result of the recursive procedure divide($(r, r)$, $\emptyset$) defined by (see [23])

```plaintext```
procedure divide($b, P$);
begin {notation: $b = (t, s)$ for $t, s \in T_I$}
   if sons ($t$) = $\emptyset$ or sons ($s$) = $\emptyset$ then begin
      sons ($b$) $\leftarrow \emptyset$;
      $P \leftarrow P \cup \{b\}$
   end else if ($b$ is admissible) then begin
      sons ($b$) $\leftarrow \emptyset$;
      $P \leftarrow P \cup \{b\}$
   end else for $t' \in$ sons ($t$), $s' \in$ sons ($s$) do begin
      $b' \leftarrow (t', s')$;
      divide($b', P$)
   end end
```

We split the covering $P = P_{\text{near}} \cup P_{\text{far}}$ with

$$P_{\text{far}} := \{ b \in P \mid b \text{ is admissible} \} \quad \text{and} \quad P_{\text{near}} := P \setminus P_{\text{far}}. \quad (3.5)$$

For a cluster $t \in T_I$, we define the set of left and right partners by

$$
\begin{align*}
P_{\text{left}}(t) &:= \{ s : (s, t) \in P_{\text{near}} \}, & P_{\text{right}}(t) &:= \{ s : (t, s) \in P_{\text{near}} \}, \\
P_{\text{left}}(t) &:= \{ s : (s, t) \in P_{\text{far}} \}, & P_{\text{right}}(t) &:= \{ s : (t, s) \in P_{\text{far}} \}, \\
P_{\text{left}}(t) &:= P_{\text{left}}(t) \cup P_{\text{left}}^\text{fat}(t), & P_{\text{right}}(t) &:= P_{\text{right}}(t) \cup P_{\text{right}}^\text{fat}(t)
\end{align*}
$$

**Remark 3.5** We have not assumed that $T_I$ is a balanced tree\(^2\). However, the definition of the sons of a block $b = (t, s)$ imply that each block $b = (t, s) \in P$ consists of clusters $t, s \in T_I$ which have the same level in the cluster tree level $(t) = \text{level} (s)$ and we set level $(b) := \text{level} (t)$.

As a consequence, we have depth $P = \text{depth} T_I$.

\(^2\)A balanced tree is a tree where $\mathcal{L}_I = T_I$ for $L := \text{depth}(T_I)$.
3.2 Approximation of the Kernel Function

Next we explain the approximation of the kernel function $G(\zeta, \cdot)$ for the single layer boundary integral operator (cf. (2.3)) for complex frequencies. For some unit vector $c \in S^2$ we write

$$G(\zeta, z) := e^{-\|z\|} = e^{-i(\text{Im}\zeta)(z, c)} G_c(\zeta, z)$$

with

$$G_c(\zeta, z) := e^{-\|z\|} e^{-i(\text{Im}\zeta)(\|z\| - (z, c))}.$$  \hfill (3.7)

Let $b = (t, s)$ be an admissible block. We approximate this kernel function on $B_t \times B_s$ by

$$\tilde{G}_b(\zeta, \cdot) := e^{-i(\text{Im}\zeta)(\cdot, c)} \mathcal{I}_b(G_c(\zeta, \cdot)),$$

where $\mathcal{I}_b$ denotes the tensor Čebyshev interpolation on $B_t \times B_s$ with polynomials of maximal degree $m$. The degree $m$ as well as the direction $c$ depend on the block $b$, i.e., $m = m(b)$ and $c = c(b)$. The choice of $c(b)$ will be explained next. From the error analysis/admissibility condition it follows that an ideal choice is

$$c = \frac{M_t - M_s}{\|M_t - M_s\|}.$$  \hfill (3.8)

However, for efficiency reasons we restrict the number of possible choices of directions to a finite set $D_\ell \subset S^2$ which will depend on the level $\ell$ of a block. Recall the definition of the maximal cluster diameter $\delta_\ell$ on level $\ell$ (cf. (3.2)). The finite set $D_\ell \subset S^2$ has to satisfy by (3.4a)

$$|\text{Im}\zeta| \sup_{e \in S^2} \inf_{c \in D_\ell} \|e - c\| \leq \frac{\eta_h}{\delta_\ell},$$

which guarantees that, for any block $b$ which satisfies (3.4b) and (3.4c), there exists a direction $c(b) \in D_{\text{level}(b)}$ such that (3.4b) is also satisfied and the block is admissible. There are various methods to construct such sets of directions. Here, we choose the construction as explained in [8, Rem. 3]. Since $\delta_\ell$ is smaller for finer levels ($\ell$ large) we may conclude that the cardinality of the set $D_\ell$ increases for larger blocks.

Next we explain the choice of $m$. Let $b = (t, s)$ be an admissible block. In Section 4 we will prove the estimate

$$\left\| G(\zeta, \cdot) - \tilde{G}_b(\zeta, \cdot) \right\|_{\infty, B_t - B_s} \leq \frac{C_0 e^{-\sigma \|B_t, B_s\|}}{4\pi \text{dist}(B_t, B_s)} \rho_0^{-m}$$

for some $\sigma, C_0 > 0$ and $\rho_0 > 1$. Here $B_t - B_s := \{y - x \mid (x, y) \in B_t \times B_s\}$. If we aim for a constant error $\varepsilon$ on each block $b$, the condition

$$\frac{C_0 e^{-\sigma \|B_t, B_s\|}}{4\pi \text{dist}(B_t, B_s)} \rho_0^{-m} \leq \varepsilon$$

The bilinear form $\langle \cdot, \cdot \rangle : \mathbb{C}^r \times \mathbb{C}^r \to \mathbb{C}$ is defined by $\langle x, y \rangle = \sum_{j=1}^r x_j y_j$.

4To simplify the calculations we restrict to $0 < \varepsilon \leq \min \left\{e^{-1}, \frac{b_{\min}}{q_2} \right\}$ so that $\frac{1}{\delta} \geq \log \frac{1}{\varepsilon}$ for all admissible blocks $b = (t, s)$ (cf. Rem. 3.3).
leads to a dependence of $m$ on $\Re \zeta$, on the block $b = (t, s)$ and on $\varepsilon$ of the form

$$
\tilde{m}_b := \begin{cases} 
\left[ c_0 \log \frac{1}{\varepsilon} - \tilde{\sigma} (\Re \zeta) \dist (B_t, B_s) \right] & \text{if } c_0 \log \frac{1}{\varepsilon} \geq \tilde{\sigma} (\Re \zeta) \dist (B_t, B_s), \\
-1 & \text{otherwise}
\end{cases}
$$

(3.9)

for positive constants $c_0, \tilde{\sigma} > 0$. As a convention, $\tilde{m}_b = -1$ means that the kernel function is replaced by the zero function on this block. To keep the algorithm simple we will restrict to a single expansion order per level $\ell$ by setting

$$
\tilde{m}_\ell := \max \{ \tilde{m}_b : b \in \mathcal{P}_{\text{far}} \text{ with } \text{level} (b) = \ell \}.
$$

In order to get the second (functional) hierarchy (besides the geometric cluster hierarchy) which allows to represent the expansion on larger clusters by an expansion on smaller clusters it is necessary that the sequence of expansion orders $(m_\ell)_{\ell=0}^{\text{depth}(\mathcal{T})}$ is increasing towards the leaves. This is guaranteed by the recursive definition

$$
m (b) := m_\ell := \begin{cases} 
\tilde{m}_0 & \ell = 0, \\
\max \{ m_{\ell-1}, \tilde{m}_\ell \} & \ell \geq 1
\end{cases} \quad \forall b \in \mathcal{P}_{\text{far}} \text{ with } \text{level} (b) = \ell.
$$

(3.10)

We pass the approximation orders from the admissible blocks onto the clusters via (cf. Remark [3.5])

$$
m_\ell := m_{\text{level}(t)}.
$$

The (first) approximation of the kernel function on an admissible block $b = (t, s)$ with expansion order $m = m (b)$ and $c = c (b)$ can be written in the form

$$
\tilde{G}_b (\zeta, y - x) := \sum_{\mu, \nu \in \mathbb{N}_t} \gamma_{\mu, \nu, c}^0 (\zeta) \tilde{\Phi}_{\mu, c}^\ell (\zeta, x) \overline{\Phi_{\nu, c}^s (\zeta, y)},
$$

where we employ the following notation: The index set $\mathbb{N}_t$ is given by

$$
\mathbb{N}_t := \{ \mu \in \mathbb{N}_0^3 \mid 0 \leq \mu_i \leq m_\ell, \quad 1 \leq i \leq 3 \}
$$

(3.11)

and we denote by

$$
k_{\text{level}(t)} := k_\ell := \sharp \mathbb{N}_t
$$

the rank of the expansion for $t \in \mathcal{T}_{\text{level}(t)}$. Note that $\mathbb{N}_t = \emptyset$ for $m_\ell = -1$ so that $k_\ell = 0$. Let $\hat{\xi}_{i,m}$, $0 \leq i \leq m$, denote the Čebyšev nodal points on the unit interval $(-1, 1)$ and let $\hat{L}_{i,m}$ be the corresponding Lagrange polynomials. The tensor versions are given, for $\mu \in \mathbb{N}_0^3$, $0 \leq \mu_i \leq m$, by $\tilde{\xi}_{\mu,m} := \left( \hat{\xi}_{\mu_1,m}, \hat{\xi}_{\mu_2,m}, \hat{\xi}_{\mu_3,m} \right)^T$ and $\tilde{L}_{\mu,m} (x) = \prod_{\ell=1}^3 \tilde{L}_{\mu_{\ell},m} (x_\ell)$. For a box $B_t$, let $\theta_t$ denote an affine pullback to the cube $(-1, 1)^3$. Then, the tensorized Čebyšev nodal points of order $m_\ell$ scaled to the sides of $B_t$ are given by $\xi_{\mu,t} := \theta_t \left( \tilde{\xi}_{\mu,m} \right)$ and $L_{\mu}^t := \tilde{L}_{\mu,m} \circ \theta_t^{-1}$, for all $\mu \in \mathbb{N}_t$. The expansion functions are given by

$$
\tilde{\Phi}_{\mu,c}^\ell (\zeta, \cdot) := e^{(\text{Im} \zeta) (\cdot, c)} L_{\mu}^t \quad \forall t \in \mathcal{T}_t, \quad \forall \mu \in \mathbb{N}_t, \quad \forall c \in \mathcal{D}_{\text{level}(t)}
$$

and the expansion coefficients for $b = (t, s)$ by

$$
\gamma_{\mu, \nu, c(b)}^{(t,s)} (\zeta) := G_c (b) \left( \zeta, \xi_{\mu,t} - \xi_{\nu,s} \right) \quad \forall b = (t, s) \in \mathcal{P}_{\text{far}}, \quad \forall \mu, \nu \in \mathbb{N}_t.
$$

(3.12)
Although this approximation will be slightly modified we introduce the (first) approximate matrix representation of the sesquilinear form $a (\cdot, \cdot) : S \times S \to \mathbb{C}$ (cf. (2.12)). Let $\varphi, \psi \in S$ denote some boundary element functions with basis representation

$$
\varphi = \sum_{i=1}^{n} \varphi_i b_i \quad \text{and} \quad \psi = \sum_{i=1}^{n} \psi_i b_i.
$$

(3.13)

The coefficients are collected in $\Phi = (\varphi_i)_{i=1}^{n}$ and $\psi = (\psi_i)_{i=1}^{n}$. We define the (sparse) matrix $K_{\text{near}} = (K_{i,j}^{\text{near}})_{i,j=1}^{n} \in \mathbb{C}^{n \times n}$ by

$$
K_{i,j}^{\text{near}} := \begin{cases} 
K_{i,j} & \text{if } (i, j) \in (t, s) \text{ for some block } (t, s) \in P_{\text{near}}, \\
0 & \text{otherwise}.
\end{cases}
$$

(3.14)

Then

$$
a_{\zeta} (\varphi, \psi) \approx \langle K^{\text{near}} \varphi, \overline{\psi} \rangle + \sum_{b = (t, s) \in P_{\text{near}}} \sum_{\mu, \nu \in N_b} \gamma_{\mu, \nu} b (\zeta) J_{\mu, \nu}^{t} (\zeta, \psi, \bar{J}_{\nu}^{t} (\zeta, \varphi))
$$

where $c = c (b)$ and the farfield coefficients are given by

$$
J_{\mu, \nu}^{t} (\zeta, \psi) := \sum_{i \in \ell} \overline{\psi_i} \int_{\Gamma} \Phi_{\mu, \nu}^{t} (\zeta, x) b_i (x) d\Gamma_x.
$$

Since the expansion orders $m_\ell$ are monotonously increasing we can express a Lagrange basis $L_{\mu}^{t}$ via the Lagrange basis on $t' \in \text{sons } (t)$ :

$$
L_{\mu}^{t} = \sum_{\nu \in N_{t'}} q_{\nu, \mu} L_{\nu}^{t'}
$$

(3.15)

with the transfer coefficients $q_{\nu, \mu} = L_{\mu}^{t'} (\xi)$. We cannot expect that a direction $c \in D_\ell$ is also contained in the set $D_{\ell+1}$, hence we assign to $c$ the direction $c' = \text{sd} (c) \in D_{\ell+1}$ which has a minimal Euclidean distance. This leads to the recursive definition of the final expansion functions

$$
\Phi_{\mu, c}^{t} (\xi, \cdot) := e^{-i (\text{Im} \zeta) \cdot c} L_{\mu}^{t} \quad \forall t \in T_\ell, \quad \forall \mu \in N_\ell, \quad \forall c \in D_{\text{level}(t)}
$$

and for $t \in T_{\ell} \setminus T_{\ell}$ and $t' \in \text{sons } (t)$ we set $c' := \text{sd} (c)$ and

$$
\Phi_{\mu, c}^{t} (\xi, \cdot) := e^{i (\text{Im} \zeta) \cdot c'} \sum_{\nu \in N_{t'}} q_{\nu, \mu} L_{\nu}^{t'}.
$$

(3.16)

This, in turn, motivates the definition of the farfield coefficients corresponding to $\Phi_{\mu, c}^{t}$ by

$$
J_{\mu, c}^{t} (\xi, \psi) := \sum_{i \in \ell} \overline{\psi_i} \int_{\Gamma} \Phi_{\mu, c}^{t} (\xi, x) b_i (x) d\Gamma_x.
$$

The relation (3.16) allows for an hierarchical computation of these coefficients. First we compute the basis farfield coefficients

$$
J_{\mu, c}^{t} (\xi, b_j) := \int_{\Gamma} \Phi_{\mu, c}^{t} (\xi, x) b_j (x) d\Gamma_x \quad \forall t \in T_{\ell} \quad \forall j \in \hat{t} \quad \forall c \in D_{\text{level}(t)} \quad \forall \mu \in N_\ell.
$$

(3.17)
Then, for a boundary element function $\psi$ as in (3.13) we determine

$$J_{t,\mu,c}^{(t)}(\zeta, \psi) = \sum_{i \in \hat{t}} \overline{\psi_i} J_{t,\mu,c}^{(i)}(\zeta, b_i) \quad \forall t \in \mathcal{L} \quad \forall c \in \mathcal{D}_{\text{level}(t)} \quad \forall \mu \in \mathbb{N}_t$$

and recursively

$$J_{t,\mu,c}^{(t)}(\zeta, \psi) := \sum_{t' \in \text{sons}(t)} \sum_{\nu \in \mathbb{N}_{t'}} q_{t',\mu,\nu} J_{t',\mu,c}^{(t')}(\zeta, \psi) \quad \forall t \in T \setminus \mathcal{L} \quad \forall c \in \mathcal{D}_{\text{level}(t)} \quad \forall \mu \in \mathbb{N}_t$$

by using the hierarchical tree structure.

Once, the farfield coefficients are $J_{t,\mu,c}^{(t)}$ computed, the final approximation of the sesquilinear form $a_{\zeta}(\cdot, \cdot)$ can be evaluated

$$a_{\zeta}(\varphi, \psi) \approx a_{\zeta}^{\mathcal{D}H^2}(\varphi, \psi) := \langle K^{\text{near}} \varphi, \psi \rangle \quad (3.18)$$

$$+ \sum_{b=(t,s) \in \mathcal{P}_{\text{far}}} \sum_{\mu,\nu \in \mathbb{N}_t} \gamma_{b,\mu,\nu} J_{t,\mu,c(b)}^{(t)}(\zeta, \psi) J_{s,\nu,c(b)}^{(s)}(\zeta, \varphi).$$

The algorithmic formulation of an approximate matrix vector multiplication, i.e., the computation of $(a_{\zeta}(\varphi, b_i))_{i=1}^n$ can be derived from (3.18) and the details are in the literature, e.g., in [44], [45], [43], and for our concrete application, e.g., in [7].

**Remark 3.6** We will prove in Sections 4 and 5 that the compression algorithm presented in this section results in a sparse $\mathcal{D}H^2$-matrix approximation and analyze how to choose the control parameters in order to satisfy a prescribed accuracy for this perturbation. However, numerical experiments show that the rank of this approximation may be larger than necessary. In [10], [8] a recompression algorithm is presented for the pure Helmholtz problem ($\zeta \in i \mathbb{R}$) which further compresses an already sparse $\mathcal{D}H^2$-matrix. We do not elaborate this issue here since the recompression algorithm in [10], [8] can be applied verbatim to the case of general complex frequencies and results in nearly optimal storage requirements. This recompression algorithm on top of our $\mathcal{D}H^2$-matrix approximation will be employed for our numerical experiments in Section 6.

### 4 Analysis

In this section, we will investigate the accuracy of the directional $\mathcal{H}^2$ approximation for the acoustic single layer potential for general complex frequencies $z \in \mathbb{C}_{>0}$ by generalizing the results in [10].

The key role is played by derivative-free interpolation estimates which go back to [14]. Let $(t, s)$ denote an admissible block. For $x \in B_t$ and $y \in B_s$, let $z = y - x$ and $r = \|z\|$. The kernel function of the acoustic single layer potential for the complex frequency $\zeta \in \mathbb{C}_{>0}$, is given by

$$G(\zeta, z) := \frac{e^{-\zeta r}}{4\pi r} = e^{-i(\text{Im}\zeta)(z,c)} G_c(z)$$

with

$$G_c(z) := e^{-(\text{Re}\zeta)r} \frac{e^{-i(\text{Im}\zeta)(r-(z,c))}}{4\pi r}$$
for some unit vector $c \in \mathbb{R}^3$, $\|c\| = 1$. We approximate this function on $B_t \times B_s$ by

$$
\tilde{G}_{t,s}(z) := e^{-i(\text{Im}\zeta)(z,c) I_{t \times s}(G_c)},
$$

where $I_{t \times s}$ denote the tensor Čebyšev interpolation on $B_t \times B_s$ with polynomials of maximal degree $m$.

For the error analysis, we modify the theory as in [10] and present the relevant statements in the following.

**Lemma 4.1** Let $t$, $s$, and $c$ satisfy the conditions (3.4a), (3.4c). Let $d, p \in \mathbb{R}^3$ be vectors satisfying

$$
\|p\| \leq \max\{\text{diam}(B_t), \text{diam}(B_s)\},
$$

$$
d - \tau p \in B_t - B_s = \{x - y : (x, y) \in B_t \times B_s\} \ \forall \tau \in [-1, 1].
$$

Then, we have

$$
\left\| \frac{d - \tau p}{\|d - \tau p\|} - c \right\| \leq \frac{\eta_1 + \max\{\eta_2, \eta_3 \text{Re}\zeta\text{ dist}(B_t, B_s)\}}{|\text{Im}\zeta| \max\{\text{diam}(B_t), \text{diam}(B_s)\}} \ \forall \tau \in [-1, 1].
$$

**Proof.** We only sketch the minor modifications in the proof of [10, Lemma 3.9] for our modified admissibility condition (cf. (3.4)). It holds

$$
\|d - \tau p\| \geq \text{dist}(B_t, B_s) \geq \frac{|\text{Im}\zeta| q^2}{\hat{\eta}},
$$

$$
\|M_t - M_s\| \geq \text{dist}(B_t, B_s) \geq \frac{|\text{Im}\zeta| q^2}{\hat{\eta}}.
$$

with $q := \max\{\text{diam}(B_t), \text{diam}(B_s)\}$ and $\hat{\eta} := \max\{\eta_2, \eta_3 \text{Re}\zeta\text{ dist}(B_t, B_s)\}$. Hence,

$$
\left\| \frac{d - \tau p}{\|d - \tau p\|} - \frac{M_t - M_s}{\|M_t - M_s\|} \right\| \leq \frac{\hat{\eta}}{|\text{Im}\zeta| q},
$$

and

$$
\left\| \frac{d - \tau p}{\|d - \tau p\|} - c \right\| \leq \left\| \frac{d - \tau p}{\|d - \tau p\|} - \frac{M_t - M_s}{\|M_t - M_s\|} \right\| + \left\| \frac{M_t - M_s}{\|M_t - M_s\|} - c \right\| \leq \frac{\hat{\eta}}{|\text{Im}\zeta| q} + \frac{\eta_1}{|\text{Im}\zeta| q}.
$$

To formulate the main theorem for the interpolation error, we introduce first some constants. Let $\eta_1, \eta_2, \eta_3$ denote the positive control parameters for the admissibility conditions (3.4). We assume that $0 < \eta_3 < 1$ holds and set $\sigma := 1 - \eta_3 > 0$. The Lebesgue constant for the univariate Čebyšev interpolation is denoted by $\Lambda_m$ and we recall the well-known estimate $\Lambda_m \leq \frac{2}{\pi} \log (m + 1) + 1$. Let

$$
\rho_0 := 1 + \hat{\beta} \quad \text{and} \quad \hat{\beta} := \min\left\{1, \left(\sqrt{\frac{3}{2}} - 1\right) \frac{2}{\eta_2} \frac{2 (1 - \eta_3)}{\eta_3^2 (2 \sqrt{6} + 5)} \right\}.
$$
We set
\[ \alpha := \frac{\left(\sqrt{\hat{\beta}^2 + 1} + \hat{\beta}\right)}{\left(\hat{\beta} + 1\right)} \]  
(4.3)
\[
C_1 := e^m \sup \left\{ \frac{8 (\Lambda_m + 1)}{\left(\rho_0 - 1\right) \alpha^{m/2}} : m \in \mathbb{N} \right\}, \quad (4.4)
\]
\[
C_0 := \sup \left\{ \frac{6 \Lambda_m^5}{\alpha^{m/2}} C_1 : m \in \mathbb{N} \right\} \quad (4.5)
\]
and observe that \( C_0, C_1 \) are bounded since the Lebesgue constant grows only logarithmically in \( m \) and it is easy to see that \( \alpha > 1 \).

**Theorem 4.2** Let \( c \in \mathbb{R}^3 \) and let the block \( b = (t, s) \) satisfy the \( \eta \)-admissibility conditions (3.4) for some \( \eta_3 \in (0, 1) \). Then,
\[
\|G - \tilde{G}_{t,s}\|_{\infty,t \times s} \leq C_0 e^{-\sigma(\text{Re } \zeta)} \frac{\rho_0^m}{4\pi \text{ dist } (B_t, B_s)} \rho_0 \quad \text{for } \sigma := \frac{1 - \eta_3}{2}
\]
where \( \rho_0 > 1 \) depends on \( \eta_2 \) and \( \eta_3 \).

**Proof.** We introduce the function \( G_{dp} : [-1, 1] \rightarrow \mathbb{C} \) by
\[
G_{dp} (x) := e^{-(\text{Re } \zeta) \|d - xp\|} \frac{\exp (-i (\text{Im } \zeta) \left(\|d - xp\| - (d - xp, c)\right))}{4\pi \|d - xp\|}.
\]
and first prove an error estimate for the univariate Čebyšev interpolation of this function. Lemma 4.1 leads to the estimate
\[
\xi := \max \left\{ \left\| \frac{d - \gamma p}{\|d - \gamma p\|} - c \right\| : \gamma \in [-1, 1] \right\} \leq \frac{\eta_1 \max \{\eta_2, \eta_3 \text{ (Re } \zeta) \text{ dist } (B_t, B_s)\}}{\text{Im } \zeta \max \{\text{diam } (B_t), \text{diam } (B_s)\}}.
\]
Note that the admissibility conditions imply that (4.1) holds. Hence, (4.1.1) yields
\[
\delta := \inf \{\|d - \gamma p\| : \gamma \in [-1, 1]\} \geq \text{dist } (B_t, B_s).
\]
The combination with (4.1.1) leads to
\[
\lambda := \frac{\delta}{\|p\|} \geq \frac{2 \text{ dist } (B_t, B_s)}{\max \{\text{diam } (B_t), \text{diam } (B_s)\}} \geq 2/\eta_2 > 0 \quad \text{i.e.,} \quad 1/\lambda \leq \eta_2/2. \quad (4.6)
\]
To estimate the interpolation error for the function \( G_{dp} \) we first derive a bound for the modulus of \( G_{dp} \) in a complex neighborhood of \([-1, 1]\). We set
\[
\beta := \min \left\{ 1, \left(\sqrt{\frac{3}{2}} - 1\right), \frac{\lambda (1 - \eta_3)}{\eta_2 (2\sqrt{6} + 5)} \right\} \geq \frac{\lambda^{\eta_2/2}}{2}
\]
with \( \hat{\beta} \) as in (4.2) and define \( U_{\beta} := \{z \in \mathbb{C} \mid \text{dist } (z, [-1, 1]) \leq \beta\} \). The unique analytic continuation of the square root function \( \sqrt{\cdot} : \mathbb{R}_{>0} \rightarrow \mathbb{R}_{>0} \) to \( \mathbb{C} \setminus \mathbb{R}_{\leq 0} \) is given by
\[
\sqrt{z} = \sqrt{|z|} \frac{z + |z|}{|z| + |z|} \quad \forall z \in \mathbb{C} \setminus \mathbb{R}_{\leq 0}.
\]
The analytic continuation of the function $x \to \|d - xp\|^2$ is then denoted by

$$n_{dp}(z) := \sqrt{\langle d - zp, d - zp \rangle}.$$ 

The modulus of $G_{dp}$ can be estimated by

$$\sup_{z \in U_\beta} |G_{dp}(z)| \leq w(\beta) \chi_+(\beta) \chi_-(\beta).$$

with $w(\beta) := \sup_{z \in U_\beta} e^{-(\Re \zeta) \Re(n_{dp}(z))}$ and

$$\chi_+(\beta) := \sup_{z \in U_\beta} \exp(-i(\Im \zeta)(n_{dp}(z) - \langle d - zp, c \rangle)),$$

$$\chi_-(\beta) := \sup_{z \in U_\beta} \frac{1}{4\pi n_{dp}(z)}.$$ 

For $\chi_+(\beta)$ we obtain $\chi_+(\beta) \leq \exp(\gamma(\beta))$, where the exponent $\gamma(\beta)$ can be estimated by (cf. the proof of Lemma 3.8 in [10])

$$\gamma(\beta) = |\Im \zeta| \|p\| \beta \left( \xi + \frac{\beta}{2(\lambda - \beta)} \right)$$

$$\leq |\Im \zeta| \|p\| \beta \eta_1 + \max \{\eta_2, \eta_3(\Re \zeta) \text{ dist } (B_t, B_s)\} + \frac{|\Im \zeta| \|p\| \beta^2}{2(\lambda - \beta)}$$

$$\leq \frac{\beta}{2} \left( \eta_1 + \max \{\eta_2, \eta_3(\Re \zeta) \text{ dist } (B_t, B_s)\} \right) + \frac{|\Im \zeta| \|p\| \beta^2}{2\lambda(1 - \beta/\lambda)}.$$ 

We obtain with our new parabolic admissibility condition (3.4c)

$$\frac{|\Im \zeta| \|p\| \beta^2}{2\lambda(1 - \beta/\lambda)} \leq \frac{|\Im \zeta| \|p\| \beta^2}{4(1 - \beta/\lambda)} \text{ dist } (B_t, B_s)$$

$$\leq \frac{|\Im \zeta| \max \{\text{diam } (B_t), \text{diam } (B_s)\} \beta^2}{8(1 - \beta/\lambda) \text{ dist } (B_t, B_s)}$$

$$\leq \frac{\beta}{2} \max \{\eta_2, \eta_3(\Re \zeta) \text{ dist } (B_t, B_s)\} \frac{\beta}{4(1 - \beta/\lambda)}.$$ 

Since $\beta \leq \min \{1, \frac{3\lambda}{4}\}$, we have derived

$$\gamma(\beta) \leq \frac{\beta}{2} \left( \eta_1 + \max \{\eta_2, \eta_3(\Re \zeta) \text{ dist } (B_t, B_s)\} \right) + \frac{\beta}{2} \left( \max \{\eta_2, \eta_3(\Re \zeta) \text{ dist } (B_t, B_s)\} \right)$$

$$\leq \eta_1 + \max \{\eta_2, \eta_3(\Re \zeta) \text{ dist } (B_t, B_s)\}.$$ 

The estimate

$$\chi_-(\beta) \leq \frac{1}{4\pi \inf_{z \in U_\beta} |n_{dp}(z)|} \leq \frac{1}{4\pi \text{ dist } (B_t, B_s)(1 - \beta/\lambda)}$$

follows as in the proof of [10] Lemma 3.6. We employ $\beta < \frac{3\lambda}{4}$ so that

$$\chi_-(\beta) \leq \frac{1}{\pi \text{ dist } (B_t, B_s)}.$$ 

Next, we estimate the term $w(\beta)$. For $z \in U_\beta$, we choose $x_z \in [-1, 1]$ such that

$$\min_{x \in [-1, 1]} |z - x| = |z - x_z|.$$
Furthermore, a triangle inequality leads to

\[
\text{Re} n_{dp}(z) \geq \|d - x_z p\| - |n_{dp}(z) - n_{dp}(x_z)|.
\]

We set \( \psi := d - z p \) and \( \varphi = d - x_z p \) so that

\[
n_{dp}(z) - n_{dp}(x_z) = \sqrt{\langle \psi, \psi \rangle} - \|\varphi\| = \|\psi\| \frac{\langle \psi, \psi \rangle + \|\psi\|^2}{\|\psi\|} - \|\varphi\|
\]

\[
= (\|\psi\| - \|\varphi\|) \frac{\langle \psi, \psi \rangle + \|\psi\|^2}{\|\psi\| + \|\psi\|^2} + \|\varphi\| \left( \frac{\langle \psi, \psi \rangle + \|\psi\|^2}{\|\psi\| + \|\psi\|^2} - 1 \right).
\]

This leads to the estimate

\[
|n_{dp}(z) - n_{dp}(x_z)| \leq \|\psi\| - \|\varphi\| + \|\varphi\| \left( \frac{\langle \psi, \psi \rangle + \|\psi\|^2}{\|\psi\| + \|\psi\|^2} - 1 \right).
\]

We know \( |z - x_z| \leq \beta \) so that

\[
\|\psi\| - \|\varphi\| \leq \|p\| |z - x_z| \leq \beta \|p\|. \tag{4.7}
\]

Furthermore, a triangle inequality leads to

\[
\left| \frac{\langle \psi, \psi \rangle + \|\psi\|^2}{\|\psi\|^2} - 1 \right| = \left| \frac{\langle \psi, \psi \rangle + \|\psi\|^2}{\|\psi\|^2} - \frac{\langle \varphi, \varphi \rangle + \|\varphi\|^2}{\|\varphi\|^2} \right|
\]

\[
\leq \left| \frac{\langle \psi, \psi \rangle - \langle \varphi, \varphi \rangle + \|\varphi\|^2 - \|\varphi\|^2}{\|\psi\|^2} \right|
\]

\[
+ \left| \frac{\langle \varphi, \varphi \rangle + \|\varphi\|^2}{\|\psi\|^2} \|\psi\| - \|\varphi\| \right| \left( \|\varphi\| + \|\varphi\|^2 \right) - \|\psi\| + \|\psi\|^2 \right|.
\]

The inequalities

\[
|\langle \psi, \psi \rangle| \leq \|\psi\|^2 \leq (\lambda + \beta)^2 \|p\|^2,
\]

\[
\|\varphi\|^2 \leq \lambda^2 \|p\|^2,
\]

\[
|\langle \varphi, \varphi \rangle + \|\varphi\|^2| = 2 \|\varphi\|^2 \geq 2 \lambda^2 \|p\|^2,
\]

\[
\|\psi\|^2 - \|\varphi\|^2 \leq (\|\psi\| + \|\varphi\|) (\|\psi\| - \|\varphi\|) \leq \|p\|^2 (2\lambda + \beta),
\]

\[
|\langle \psi, \psi \rangle - \langle \varphi, \varphi \rangle| \leq (\|\psi\| + \|\varphi\|) \|\psi - \varphi\| \leq \|p\|^2 (2\lambda + \beta) \beta
\]

are derived by the reasoning: the first one follows by the same arguments as in the proof of Lemma 3.6 in [10], the second one from the definition of \( \zeta \), the third one from the second one, the last two inequalities from (4.7). This leads to

\[
|\langle \psi, \psi \rangle + \|\psi\|^2| = |\langle \varphi, \varphi \rangle + \|\varphi\|^2| - |\langle \psi, \psi \rangle + \|\psi\|^2 - \langle \varphi, \varphi \rangle + \|\varphi\|^2| \geq 2 \|p\|^2 (\lambda^2 - (2\lambda + \beta) \beta).
\]

The combination of these estimates leads to

\[
|n_{dp}(z) - n_{dp}(x_z)| \leq \beta \|p\| \left( 1 + 2 \frac{(2\lambda + \beta) \lambda}{(\lambda^2 - (2\lambda + \beta) \beta)} \right).
\]

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Now we use $\beta \leq \left( \frac{1}{2} \right)^{1/2} - 1$ to obtain

$$|n_{dp}(z) - n_{dp}(x)| \leq \left( 2\sqrt{6} + 5 \right) \beta \|p\|.$$ 

The estimate $\|p\| \leq \frac{1}{2} \max \{\text{diam}(B_t), \text{diam}(B_s)\} \leq \frac{m}{2} \text{dist}(B_t, B_s)$ leads to

$$\text{Re} n_{dp}(z) \geq \|d - x_p\| - |n_{dp}(z) - n_{dp}(x)| \geq \left( \lambda - \frac{\eta_2}{2} \left( 2\sqrt{6} + 5 \right) \beta \right) \|p\|.$$ 

Since $\beta \leq \frac{\lambda(1-m)}{\eta_2(2\sqrt{6}+5)}$ we have proved that

$$\text{Re} n_{dp}(z) \geq \frac{\eta_3 + 1}{2} \lambda \|p\| \geq \frac{1 + \eta_3}{2} \text{dist}(B_t, B_s)$$

holds. The combination of the estimates for $w(\beta), \chi(\beta)$ leads to

$$\sup_{z \in U_{\beta}} |G_{dp}(z)| \leq \exp \left( \eta_1 - \frac{1-m}{2} \text{Re} \zeta \text{dist}(B_t, B_s) \right) \frac{\eta_3}{\pi} \text{dist}(B_t, B_s)$$

Note that $U_{\beta}$ contains the “Bernstein ellipse”:

$$\overline{D}_\rho := \left\{ z = x + iy : x, y \in \mathbb{R}, \left( \frac{2x}{\rho + 1/\rho} \right)^2 + \left( \frac{2y}{\rho - 1/\rho} \right)^2 \right\} < 1$$

for

$$\rho := \sqrt{\beta^2 + 1 + \beta} \geq \sqrt{\beta^2 + 1 + \hat{\beta}} > \hat{\beta} + 1 = \rho_0.$$ 

We have $\rho \geq \alpha \rho_0$ and $\alpha > 1$ (cf. [4.3]). We know, e.g., from [10, Lemma 3.11], that there exists $q \in \mathbb{R}_m$ such that

$$\|G_{dp} - q\|_{\infty,[-1,1]} \leq \frac{2}{\rho_0 - 1} \rho^{-m} \exp \left( \eta_1 - \frac{1-m}{2} \text{Re} \zeta \text{dist}(B_t, B_s) \right) \frac{\eta_3}{\pi} \text{dist}(B_t, B_s).$$

By employing the Lebesgue constant $\Lambda_m$ for the Čebyšev interpolation we conclude that

$$\|G_{dp} - J(G_{dp})\|_{\infty,[-1,1]} \leq C_1 \frac{\alpha^{-m/2} e^{-\sigma(\text{Re} \zeta) \text{dist}(B_t, B_s)}}{4\pi \text{dist}(B_t, B_s)} \rho_0^{-m}. $$

The same arguments as in the proof of [10, Corollary 3.14] now finishes the proof. 

By using the local error estimate we obtain the following consistency estimate.

**Theorem 4.3** Let (2.8), (2.10), (2.11) be satisfied. Let $c \in \mathbb{R}^3$ and let the block $b = (t, s)$ satisfy the $\eta$-admissibility conditions (3.4) for some $0 < \eta_3 < 1$. Then, there exist constants $C_{\text{cons}} > 0$ and $\sigma_2 > 0$ such that the consistency estimate for the approximate sesquilinear form $a^\mathcal{O}H^2_\zeta$ (cf. [3.18]) holds

$$\left| a^\mathcal{O}_\zeta(\varphi, \psi) - a^\mathcal{O}H^2_\zeta(\varphi, \psi) \right| \leq C_{\text{cons}} C_1 \frac{\rho_0^{-m} e^{-\sigma(\text{Re} \zeta)\eta_0}}{h^2_0} \|\varphi\|_{L^2(\Gamma)} \|\psi\|_{L^2(\Gamma)}.$$ 

with

$$C_\Gamma := \int_{\Gamma} \int_{\Gamma} \frac{1}{\|x - y\|} d\Gamma_y d\Gamma_x.$$
Proof. Let \( \varphi, \psi \in S \) with coefficient vectors \( (\varphi_j)_{j=1}^n, (\psi_j)_{j=1}^n \) in their basis representations (cf. (3.13)). Then the difference \( e_\zeta := a_\zeta - a_\zeta^2 \) satisfies
\[
|e_\zeta(\varphi, \psi)| \leq \sum_{b=(t,s) \in \mathcal{P}_{t,s}} \left| \sum_{i \in I} \sum_{j \in J} \psi_i \varphi_j \int_{\omega_i} \int_{\omega_s} E(\zeta, x - y) b_i(x) b_j(y) \, d\Gamma_y \, d\Gamma_x \right|
\]
for
\[
E(\zeta, x - y) := G(\zeta, x - y) - \sum_{\mu, \nu \in \mathbb{N}_i} \gamma_{\mu, \nu}(\zeta) \Phi_{\mu, \nu}(\zeta, x) \Phi_{\mu, \nu}(\zeta, y).
\]
Using the local error estimate (Thm. 4.2) we obtain
\[
|e_\zeta(\varphi, \psi)| \leq \frac{C_0 \rho_0^{-m}}{4\pi} \|\varphi\|_{L^\infty(\Gamma)} \|\psi\|_{L^\infty(\Gamma)} \sum_{b=(t,s) \in \mathcal{P}_{t,s}} \int_{\omega_i} \int_{\omega_s} \frac{e^{-\sigma_1(Re \zeta) \text{dist}(B_t, B_s)}}{\text{dist}(B_t, B_s)} \, d\Gamma_y \, d\Gamma_x.
\]
For all \( (x, y) \in \omega_i \times \omega_s \), the standard admissibility condition (cf. (3.4b)) implies
\[
\text{dist}(B_t, B_s) \geq \|x - y\| - \text{diam} B_t - \text{diam} B_s \geq \|x - y\| - 2\eta_2 \text{dist}(B_t, B_s)
\]
and, in turn,
\[
\text{dist}(B_t, B_s) \geq \frac{1}{1 + 2\eta_2} \|x - y\|.
\]
We employ Remark 3.3 to get
\[
\min \{\text{dist}(B_t, B_s) : (t, s) \text{ is admissible}\} \geq \frac{h_{\min}}{\eta_2} \geq \frac{h_\zeta}{C_{\text{qu}} \eta_2}.
\]
Thus, for \( \sigma_1 := \sigma/(1 + 2\eta_2) \) it holds
\[
|e_\zeta(\varphi, \psi)| \leq \frac{C_0 (1 + 2\eta_2) \rho_0^{-m}}{4\pi} \|\varphi\|_{L^\infty(\Gamma)} \|\psi\|_{L^\infty(\Gamma)} \int_{\Gamma} \int_{\Gamma} \frac{e^{-\sigma_1 |x - y|}}{|x - y|} \, d\Gamma_y \, d\Gamma_x \|
\]
so that for \( \sigma_2 := \sigma_1/C_{\text{qu}} \eta_2 \)
\[
|e_\zeta(\varphi, \psi)| \leq \frac{C_0 (1 + 2\eta_2) \rho_0^{-m} e^{-\sigma_2(Re \zeta) h_\zeta}}{4\pi} C_{\Gamma} \|\varphi\|_{L^\infty(\Gamma)} \|\psi\|_{L^\infty(\Gamma)}.
\]
The shape regularity and quasi-uniformity of the mesh implies (cf. [43, §4.4]) that there exists a constant \( C_{\text{inv}} \) such that
\[
\|\varphi\|_{L^\infty(\Gamma)} \leq C_{\text{inv}} h_\Gamma^{-1} \|\varphi\|_{L^2(\Gamma)}
\]
so that the assertion follows for \( C_{\text{cons}} := \frac{C_0 (1 + 2\eta_2)}{4\pi} C_{\Gamma} C_{\text{inv}}^2 \). \( \blacksquare \)

5 Complexity

In this section, we will estimate the complexity of the fast directional \( H^2 \)-matrix approach for acoustic boundary integral operators with complex frequency. In [7], the complexity was analyzed for purely complex frequencies \( \zeta \in i\mathbb{R} \). Here we generalize this theory to general complex frequencies \( \zeta \) by taking into account the modified admissibility condition (3.4c). We will derive explicit complexity estimates with respect to \( \text{Re} \zeta \) and \( \text{Im} \zeta \).
5.1 Storage Requirements

Remark 5.1 (tridiagonal case) Let the boundary element mesh be quasi-uniform and shape regular and the constants \( c_0, \tilde{\sigma} \) as in (3.9). Then the condition

\[
\text{Re} \, \zeta > c_1 \sqrt{n} \log \frac{1}{\epsilon} \quad \text{for} \quad c_1 := \frac{\tilde{C} c_0 \eta_2}{\tilde{\sigma}} \quad \text{and} \quad \tilde{C} := \sqrt{\frac{C_{\text{sr}}}{C_p}} |\Gamma| C_{\text{qu}}
\]

implies that \( m(b) = -1 \) for all blocks \( b \in \mathcal{P}_{\text{far}} \). As a consequence, the boundary element matrix \( K(\zeta) \) can be replaced by its part, where the kernel function is singular, i.e., \( K(\zeta) \approx K^0(\zeta) = (K^0_{i,j}(\zeta))_{i,j=1}^n \) with

\[
K^0_{i,j}(\zeta) := \begin{cases} K_{i,j}(\zeta) & \text{if } \text{dist}(\omega_i, \omega_j) = 0, \\ 0 & \text{otherwise} \end{cases}
\]

and \( \omega_i = \text{supp} b_i \).

Proof. From Remark 3.3 we obtain \( \text{dist}(B_t, B_s) \geq h_{\min} \eta_2 \) so that the following implication holds (cf. (3.9))

\[
\frac{c_0 \eta_2}{\tilde{\sigma}} \log \frac{1}{\epsilon} < (\text{Re} \, \zeta) h_{\min} \quad \Rightarrow \quad \tilde{m}_b = -1
\]

(5.1)

For a quasi-uniform and shape regular boundary element mesh it holds

\[
|\Gamma| = \sum_{\tau \in \mathcal{G}} |\tau| \leq C_{\text{sr}} \sum_{\tau \in \mathcal{G}} h_\tau^2 \leq C_{\text{sr}} h_G^2 |\mathcal{G}| \leq \frac{C_{\text{sr}} C_{\text{qu}}}{C_p} h_{\min}^2 n
\]

(5.2)

so that \( h_{\min} \geq \tilde{C} n^{-1/2} \). The combination with (5.1) leads to \( \tilde{m}_b = -1 \) for all \( b \in \mathcal{P}_{\text{far}} \). The definition of \( m_b \) (cf. (3.10)) then finishes the proof. \( \blacksquare \)

Remark 5.2 (sectorial case) For \( \zeta \in \mathbb{C}_{>0} \) with \( |\text{Im} \, \zeta| \leq \alpha \text{Re} \, \zeta \) and some \( \alpha > 0 \) the condition

\[
\max \{ \text{diam} (B_t), \text{diam} (B_s) \} \leq \sqrt{\frac{\eta_2}{\alpha}} \text{dist} (B_t, B_s)
\]

is stronger than the condition (3.4c) and can be absorbed into the condition (3.4b) by adjusting \( \eta_2 \leftarrow \eta_2' := \min \{ \eta_2, \sqrt{\frac{\eta_2}{\alpha}} \} \). Since condition (3.4b) is understood as a condition on the number and choice of directions in \( \mathcal{D}_\ell \), the number of elements in the minimal partition \( \mathcal{P} \) of \( I \times I \) is bounded by the number of elements in a partition \( \mathcal{P}' \) where conditions (3.4b) and (3.4c) are replaced by the condition \( \max \{ \text{diam} (B_t), \text{diam} (B_s) \} \leq \eta_2' \text{dist} (B_t, B_s) \). This is the standard admissibility condition for the Laplacian and estimates of the form \( \#\mathcal{P} \leq C n \) are well known (see, e.g., [43, 45]).

Next, we will estimate the number of elements in the minimal partition \( \mathcal{P} \) for the case \( \zeta \in \mathbb{C}_{>0} \) with

\[
|\text{Im} \, \zeta| > \alpha \text{Re} \, \zeta \quad \text{for} \quad \alpha \text{ as in Rem. 5.2}
\]

Note that this condition implies that \( |\zeta| \) and \( |\text{Im} \, \zeta| \) are “equivalent”:

\[
|\text{Im} \, \zeta| \leq |\zeta| \leq \sqrt{1 + \alpha^{-2}} |\text{Im} \, \zeta|.
\]

The theory in this Section is a generalization of the one in [7, Section 5], adapted to our new admissibility condition (3.4c) and our goal is to derive estimates which are explicit in
all relevant parameters, in particular, with respect to $\text{Im} \, \zeta$, $\text{Re} \, \zeta$, $n$, and certain geometric parameters which we will introduce next.

As in [7] we assume that there exist a reference box $B_t$ for each level $\ell$ and constants $\rho_{\text{ref}} > 1, C_{s_b}, C_{s_n} \geq 1, c_{\text{ref}}, C_{b_b}, C_{b_n}, C_{c_v}, C_{c_n} > 0$ such that:

$$\exists d_t \in \mathbb{R}^3 \text{ s.t. } B_t = B_t + d_t, \text{ for all clusters } t \in \mathcal{T}_{\ell}^{(t)}, \tag{5.3}$$

$$\text{diam}(B_t) \leq C_{s_b} \text{diam}(B_{t'}), \text{ for all } t \in \mathcal{T}_{\ell}, t' \in \text{sons}(t), \tag{5.4}$$

$$\# \text{ sons}(t) \leq C_{s_n}, \# \text{ sons}(t) \neq 1, \text{ for all } t \in \mathcal{T}_{\ell}, \tag{5.5}$$

$$|\Gamma \cap \mathcal{B}(x, r)| \leq C_{b_b} r^2, \text{ for all } x \in \mathbb{R}^3, r \geq 0, \tag{5.6}$$

$$\text{diam}^2(B_t) \leq C_{b_b} |B_t \cap \Gamma|, \text{ for all } t \in \mathcal{T}_{\ell}, \tag{5.7}$$

$$\# \{ t \in \mathcal{T}_{\ell}^{(t)} : x \in B_t \} \leq C_{c_v}, \text{ for all } x \in \Omega, \ell \in \mathbb{N}_0, \tag{5.8}$$

$$C_{c_v}^{-1} (k_L + 1) \leq \# \ell \leq C_{c_v}(k_L + 1), \text{ for all leaves } t \in \mathcal{L}_{\ell} \cap \mathcal{T}_{\ell}, \tag{5.9}$$

$$c_{\text{ref}} \rho_{\text{ref}}^{-\ell} \delta_L \leq \delta_{\ell \ell}, \forall 0 \leq \ell \leq L := \text{depth} \mathcal{T}_{\ell} \text{ (with } \delta_{\ell} \text{ as in } (3.2)). \tag{5.10}$$

**Lemma 5.3 (Sparsity)** Let (5.3-5.8) hold. For every cluster $t \in \mathcal{T}_{\ell}$, the sets $\mathcal{P}_{\text{left}}(t)$, $\mathcal{P}_{\text{right}}(t)$ as in (3.0) satisfies

$$\max \{ \| \mathcal{P}_{\text{left}}(t) \|, \| \mathcal{P}_{\text{right}}(t) \| \} \leq \hat{C}_{sp} R_t^2 \tag{5.11}$$

with $\hat{C}_{sp} := C_{s_n} C_{s_b}^2 C_{b_b} C_{c_v} C_{b_b}$ and

$$R_t := \frac{3}{2} + \max \left\{ \frac{1}{\eta_2}, r_t \right\} \text{ and } r_t := \min \left\{ \frac{|\text{Im} \, \zeta| |\text{diam} B_t|}{\eta_2}, \sqrt{\frac{|\text{Im} \, \zeta|}{\eta_3 |\text{Re} \, \zeta|}} \right\}. \tag{5.12}$$

**Proof.** We prove the estimate only for $\| \mathcal{P}_{\text{right}}(t) \|$ while the proof for $\| \mathcal{P}_{\text{left}}(t) \|$ follows verbatim.

Let $t \in \mathcal{T}_{\ell}$ and $s \in \mathcal{P}^{\text{near}}_{\text{right}}(t)$ (cf. (3.0)). Since $t$ and $s$ belong to the same tree level we have $\max \{ \text{diam } B_t, \text{diam } B_s \} = \text{diam } B_t$. Then, for any $z \in B_s$ the estimate

$$\| z - M_t \| \leq \text{diam } B_s + \text{dist } (B_t, B_s) + \frac{1}{2} \text{diam } B_t \tag{5.13}$$

holds. Since the block $(t,s)$ is non-admissible one of the conditions (3.4b), (3.4c) must be violated.

**Case 1.** Let condition (3.4b) be violated. Then,

$$\| z - M_t \| < \left( \frac{3}{2} + \frac{1}{\eta_2} \right) \text{diam } B_t.$$

**Case 2.** Let condition (3.4b) be violated but condition (3.4c) be valid. We set $\lambda_{t,s} := \max \left\{ 1, \frac{\eta_2 |\text{Re} \, \zeta| \text{dist}(B_t, B_s)}{\eta_2} \right\}$ and obtain by combining (3.4b) with the negation of (3.4c)

$$|\text{Im} \, \zeta| (\text{diam } B_t)^2 > \lambda_{t,s} \eta_2 \text{ dist } (B_t, B_s) \geq \lambda_{t,s} \text{ diam } B_t \tag{5.14}$$

so that $|\text{Im} \, \zeta| \text{ diam } B_t > \lambda_{t,s}$. The left inequality in (5.14) can be split into

$$\text{dist } (B_t, B_s) < \frac{|\text{Im} \, \zeta|}{\eta_2} (\text{diam } B_t)^2$$

$$\text{dist } (B_t, B_s) < \sqrt{\frac{|\text{Im} \, \zeta|}{\eta_3 |\text{Re} \, \zeta|}} \text{ diam } B_t$$

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so that
\[ \text{dist} (B_t, B_s) < r_t \text{ diam } B_t \]
for \( r_t \) as in (5.12). The combination of this with (5.13) leads to
\[ \|z - M_t\| \leq \left( r_t + \frac{3}{2} \right) \text{ diam } B_t. \]
The distance estimates in Case 1 and Case 2 lead to
\[ \|z - M_t\| < \min \left\{ \left( \frac{3}{2} + \frac{1}{\eta_2} \right) \text{ diam } B_t, \left( r_t + \frac{3}{2} \right) \text{ diam } B_t \right\} \quad \forall z \in B_s \]
for non-admissible pairs of clusters and allow for an estimate of the cardinality. The cluster \( s \in P_{\text{near}}(t) \) is contained in a ball with center \( M_t \) and radius \( R_t \) as in (5.12). By the same arguments as in the proof of [7, Lem. 2] we obtain
\[ \# P_{\text{near}}(t) \leq C_{\text{bb}} C_{\text{cov}} C_{\text{bp}} R_t^2. \]

It remains to estimate the cardinality of \( \# P_{\text{right}}(t) \). If \( t \) is the root of \( T \) we have \( P_{\text{right}}(t) = P_{\text{left}}(t) = \{ t \} \) and the cardinalities equal 1. For \( t \in T \) being not the root we denote by \( t^+ \) the father of \( t \) which, by construction, is non-admissible. Hence,
\[ \# P_{\text{right}}(t) \leq C_{\text{sn}} \# P_{\text{near}}(t^+) \leq C_{\text{sn}} C_{\text{bb}} C_{\text{cov}} C_{\text{bp}} R_t^2. \]
The final estimate follows from
\[ r_{t^+} \leq \min \left\{ C_{\text{sb}} \frac{\text{Im } \zeta}{\eta_2} \text{ diam } B_t, \sqrt{\frac{\text{Im } \zeta}{\eta_3 \text{ Re } \zeta}} \right\} \leq C_{\text{sb}} r_t. \]

\textbf{Corollary 5.4 (Nearfield sparsity with resolution condition)} For the boundary element discretization we assume (2.8), (2.10), (2.11). Let (5.3-5.8) hold and assume the resolution condition
\[ |\text{Im } \zeta| \text{ diam } B_t \leq C_{\text{res}} \quad \forall t \in L_T. \quad (5.15) \]
Then, there exists a constant \( C_{\text{near}}^t \) such that cardinality of the near field \( P_{\text{near}} \) as in (3.5) can be estimated
\[ \# P_{\text{near}} \leq C_{\text{near}}^t \frac{n}{R_L + 1}. \]

\textbf{Proof.} Let \( b = (t, s) \in P_{\text{near}} \). Then, \( t \) is a leaf or \( s \) is a leaf. In the first case, it holds \( s \in P_{\text{right}}(t) \) and in the second \( t \in P_{\text{left}}(s) \). Let \( t \) be a leaf. Then we employ (5.15) to obtain
\[ r_t \leq \frac{C_{\text{res}}}{\eta_2} \quad \text{and} \quad R_t \leq \frac{3}{2} + \frac{\max \{C_{\text{res}}, 1\}}{\eta_2}. \quad (5.16) \]
The combination with (5.11) leads to
\[ P_{\text{near}}(t) \leq \hat{C}_{\text{np}} R_t^2. \]
The proof of $P_{\text{near}}(s) \leq \hat{C}_{sp} R_{sp}^2$ in case that $s$ is a leaf is verbatim. Hence,

$$\max \{#P_{\text{near}}(t), #P_{\text{right}}(t)\} \leq \hat{C}_{sp} R_{sp}^2,$$

and

$$#P_{\text{near}} \leq \sum_{t \in L_I} \left(#P_{\text{left}}(t) + #P_{\text{right}}(t)\right) \leq 2\hat{C}_{sp} \sum_{t \in L_I} R_{sp}^2 \leq 2\hat{C}_{sp} \left(\frac{3}{2} + \frac{\max \{C_{res}, 1\}}{\eta_2}\right)^2 (\#L_I).$$

For $\#L_I$, we obtain

$$n = \#I = \sum_{t \in L_I} \hat{t} \geq C_{rs}^{-1} (k_L + 1) \#L_I$$

and the assertion follows with

$$C_{\text{near}}^t := 2\hat{C}_{sp} \left(\frac{3}{2} + \frac{\max \{C_{res}, 1\}}{\eta_2}\right)^2 C_{rs}.$$

The next estimate of the number of clusters is proven in [7, Lem. 3] and carries over to our case, since it does not involve the admissibility conditions (3.4).

**Lemma 5.5 (Clusters)** Let (5.3), (5.5), (5.7), (5.8) and (5.9) hold. Then

$$\#T_{\ell} \leq C_{lv} \left|\Gamma\right| \frac{\text{diam}^2(B_{\ell})}{n}, \quad \text{for all } \ell \in \mathbb{N}_0 (5.18)$$

with $C_{lv} := \max\{C_{bb} C_{ov}, 2C_{rs}\}$.

Next we estimate the cardinality of the cluster basis. For each $t \in T_{\ell}$ and $c \in D_{\ell}$ we define

$$P_{\text{far}}(t, c) := \{s \in T_I \mid b = (t, s) \in P_{\text{adm}} \land c(b) = c\}$$

and observe that

$$\bigcup_{c \in D_{\ell}} P_{\text{far}}(t, c) = P_{\text{far}}(t)$$

holds.

**Lemma 5.6 (Block and cluster sums)** Let the set of directions be constructed according to [8, Rem. 3]. Under assumptions (2.11), (3.3), (5.3), (5.5 - 5.9), (5.10) and assuming that $L := \text{depth}(T_I) > 0$, there exists a constant $C_{\sharp}$ such that

$$\sum_{t \in T_{\ell}} \sum_{c \in D_{\ell}} #P_{\text{right}}(t, c) \leq C_{\sharp} \left(\frac{n}{\eta_2^2 (k_L + 1)} + \min \left\{ (1 + L) \left(\frac{|\text{Im} \zeta|}{\eta_2} \right)^2, \frac{n}{\eta_3 (k_L + 1)} \frac{|\text{Re} \zeta|}{\eta_3} \right\}\right)^2. (5.19)$$

There exists $C_{\text{bfc}} > 0$ such that the total number of basis farfield coefficients (cf. (3.17)) is bounded from above by

$$C_{\text{bfc}} k_L (n + k_L (\text{Im} \zeta)^2)$$

The total number of expansion coefficients (cf. (3.12)) is bounded from above by

$$C_{\sharp} k_L \left(\frac{n}{\eta_2^2} + \min \left\{ k_L \left(1 + L\right) \left(\frac{|\text{Im} \zeta|}{\eta_2} \right)^2, \frac{n}{\eta_3} \frac{|\text{Re} \zeta|}{\eta_3} \right\}\right). (5.21)$$
Proof. Part 1. Estimate of the total number of blocks

We follow the arguments in the proof of [8, Lem. 8]. Let $L := \text{depth} (\cal{I}_T)$. The combination of (5.18) with (5.11) leads to

$$
\sum_{t \in \cal{I}_T} \sum_{c \in D_{\text{level}(t)}} \sharp P_{\text{right}} (t, c) \leq \hat{C}_{sp} \sum_{t \in \cal{I}_T} R_t^2 \leq \hat{C}_{sp} C_{lv} |\Gamma| \sum_{\ell = 0}^L \frac{R_{\ell}^2}{\text{diam}^2 B_{\ell}}
$$

with

$$
R_{\ell}^2 \leq 2 \left( \left( \frac{3}{2} + \frac{1}{\eta_2} \right)^2 \right)
$$

and we obtain

$$
\sum_{t \in \cal{I}_T} \sum_{c \in D_{\text{level}(t)}} \sharp P_{\text{right}} (t, c) \leq 2 \hat{C}_{sp} C_{lv} |\Gamma| \left( \sum_{\ell = 0}^L \left( \frac{\frac{3}{2} + \frac{1}{\eta_2}}{\text{diam}^2 B_{\ell}} \right)^2 + \sum_{\ell = 0}^L \min \left\{ \frac{|\text{Im } \zeta|}{\eta_2}, \frac{|\text{Im } \zeta|}{\eta_3 (\text{Re } \zeta)} \right\} \right).
$$

We have for all $0 \leq \ell \leq L$ and $t \in \cal{I}_T \cap \cal{I}_T$ the estimate

$$
\text{diam}^2 B_{\ell} \geq c_{\text{vol}} |\omega_t| \geq c_{sr} \frac{\# t}{C_{\text{loc},p} + 1} \delta_{\text{min}}^2 \geq \frac{c_{sr}}{C_{\text{qu}} C_{\text{loc},p} + 1} \frac{\# t}{h_{\text{diam}}} \geq \frac{c_{sr}}{C_{\text{qu}} C_{\text{loc},p} + 1} \frac{\# t}{\text{vol} |\Gamma|} \frac{\# \cal{I} |k_L + 1}{n}
$$

with

$$
c_{\cal{L}} := \frac{c_{sr}}{C_{\text{qu}} C_{\text{rs}} C_{\text{loc},p} + 1}.
$$

We use (5.10) and get by a geometric sum argument

$$
\sum_{\ell = 0}^L \frac{1}{\text{diam}^2 B_{\ell}} \leq \frac{c_{\text{ref}}}{\rho_{\text{ref}} - 1} \frac{1}{\delta_L} \leq \frac{c_B}{|\Gamma|} \frac{n}{k_L + 1} \text{ with } c_B := \frac{c_{\text{ref}}^2}{c_{\cal{L}} (\rho_{\text{ref}} - 1)}.
$$

We end up with the estimate

$$
\sum_{t \in \cal{I}_T} \sum_{c \in D_{\text{level}(t)}} \sharp P_{\text{right}} (t, c) \leq 2 \hat{C}_{sp} C_{lv} \left( \left( \frac{3}{2} + \frac{1}{\eta_2} \right)^2 \frac{c_B n}{k_L + 1} + \min \left\{ |\Gamma| (L + 1) \left( \frac{|\text{Im } \zeta|}{\eta_2} \right)^2, c_B \frac{|\text{Im } \zeta|}{\eta_3 (\text{Re } \zeta)} \frac{n}{k_L + 1} \right\} \right).
$$

Hence, (5.19) follows with

$$
C_z := 2 \hat{C}_{sp} C_{lv} \max \left\{ \left( \frac{3}{2} \eta_2 + 1 \right)^2 c_B, |\Gamma| \right\}.
$$

Part 2. Estimate of the total number of basis farfield coefficients $J_{\mu,e} (\zeta, b_j)$. 25
From [8, (16)] we conclude that the construction of the set of directions as in [8, Rem. 3] implies
\[ \#D_{\ell} \leq C_{di} (1 + (\text{Im} \, \zeta)^2 \text{diam}^2 B_{\ell}) \quad \forall 0 \leq \ell \leq L := \text{depth } T_I \]
for some constant \( C_{di} > 0 \). Furthermore, we have
\[ \# \{ j \in \ell \} \leq C_{loc,p} \# \ell. \]

The coefficients \( J_{\mu,c} (\zeta, b_j) \) must be computed only for leaves \( t \in L_I \) and we obtain
\[
\sum_{t \in L_I} \sum_{\mu \in \mathcal{N}_t} \sum_{c \in \mathcal{D} \mu} \sum_{j \in \ell} 1 \leq C_{loc,p} \sum_{t \in L_I} \sum_{\mu \in \mathcal{N}_t} \sum_{c \in \mathcal{D} \mu} \# \ell \\
\leq C_{loc,p} C_{di} \sum_{t \in L_I} k_t (\# \ell) (1 + (\text{Im} \, \zeta)^2 \text{diam}^2 B_{\text{level}(t)}) \\
\leq 2C_{loc,p} C_{di} k_L \left( n + C_{rs} k_L (\text{Im} \, \zeta)^2 \sum_{t \in L_I} \text{diam}^2 B_{\text{level}(t)} \right).
\]

It holds
\[
\sum_{t \in L_I} \text{diam}^2 B_{\ell} \leq C_{vol} \sum_{t \in L_I} |\omega_t| \leq C_{vol} C_p |\Gamma|.
\]

This allows to estimate
\[
\sum_{t \in L_I} \sum_{\mu \in \mathcal{N}_t} \sum_{c \in \mathcal{D} \ell} \sum_{j \in \ell} 1 \leq 2C_{loc,p} C_{di} k_L \left( n + C_{rs} C_{vol} C_p |\Gamma| k_L (\text{Im} \, \zeta)^2 \right)
\]
from which \((5.20)\) follows with
\[ C_{\text{tec}} := 2C_{loc,p} C_{di} \max \{ 1, C_{rs} C_{vol} C_p |\Gamma| \}. \]

**Part 3.** Estimate the total number of expansion coefficients \( \gamma_{\mu,b}^{\ell, \nu,(\ell)} (\zeta) \). We obtain the bound (cf. \((3.11)\))
\[
\sum_{b=(t,s) \in \mathcal{P}_{\text{far}}} (\# \mathcal{N}_t) (\# \mathcal{N}_s) = \sum_{t=0}^L k_t^2 \sum_{t \in \mathcal{T}_t} \# \mathcal{P}_{\text{far}} \text{right} (t) \quad \text{Lem. \(5.3\)} \leq \sum_{t=0}^L k_t^2 \sum_{t \in \mathcal{T}_t} R_t^2 \leq k_L^2 \hat{C}_{sp} \sum_{t \in \mathcal{T}_2} R_t^2.
\]

We may argue as in Part 1 to get the assertion. ■

**Lemma 5.7 (Nearfield matrix)** Let the set of directions be constructed according to [8, Rem. 3]. Under assumptions \((2.11), (3.3), (5.3), (5.5-5.9), (5.10)\) and assuming that \( L := \text{depth } (T_I) > 0 \), there exists a constant \( C_{\text{near}} \) such that the number of non-zero nearfield matrix entries is bounded from above by
\[
C_{\text{near}} (k_L + 1) \left( \frac{n}{\eta_2} + \min \left\{ (1 + L) (k_L + 1) \left( \frac{|\text{Im} \, \zeta|}{\eta_2} \right)^2, \frac{n}{\eta_3} \frac{|\text{Im} \, \zeta|}{\text{Re} \, \zeta} \right\} \right).
\]
Proof. Let \((t, s) \in \mathcal{P}_{\text{near}}\). This implies that \(t\) or \(s\) belongs to \(\mathcal{L}_{\ell}\) and we assume that \(t \in \mathcal{L}_{\ell} \cap \mathcal{T}_{t}\) for some \(0 \leq \ell \leq L\). From (5.9) we know that \(\hat{t} \leq C_{rs}(k_{L} + 1)\). The construction of the block partition implies that also \(s \in \mathcal{T}_{t}\). Then, we conclude as in the second inequality of (5.23)

\[
\hat{s} \leq \frac{\omega_{s}}{c_{sr} h_{\min}^{2}} \leq \frac{\omega_{t}}{c_{sr} h_{\min}^{2}} \leq C_{vol} \frac{w_{t}}{c_{vol} c_{sr} h_{\min}^{2}} (C_{loc,p} + 1)
\]

(5.24)

with \(C_{\text{level}} := \frac{C_{\text{qu}}^{2} C_{vol} (C_{loc,p} + 1)}{c_{vol} c_{sr}}\). This allows to estimate the number of non-zero entries in the nearfield matrix by

\[
\hat{\mathcal{K}}_{i,j}^{} \{ s \in \mathcal{P}_{\text{near}} \mid \mathcal{K}_{i,j}^{\text{near}}(s) \neq 0 \} \leq \sum_{t \in \mathcal{L}_{\ell}} \sum_{s \in \mathcal{P}_{\text{near}}(t)} (\hat{s})(\hat{t}) \text{Lem. 5.7} \leq C_{\text{near}}(k_{L} + 1) \left( \frac{n}{\eta_{2}^{2}} + \min \left( (1 + L)(k_{L} + 1) \left( \frac{|\text{Im} \zeta|}{\eta_{2}} \right)^{2}, \frac{n |\text{Im} \zeta|}{\eta_{3} |\text{Re} \zeta|} \right) \right)
\]

for \(C_{\text{near}} := 2C_{\text{level}} C_{rs}^{2}\).

Corollary 5.8 (Nearfield matrix with resolution condition) Let the assumptions of Lemma 5.7 be satisfied and assume the resolution condition

\[
|\text{Im} \zeta| \text{diam } B_{t} \leq C_{\text{res}} \forall t \in \mathcal{L}_{\ell}.
\]

Then, the number of non-zero nearfield matrix entries is bounded from above by

\[
C_{\text{level}} C_{rs}^{2} C_{\text{near}}^{2}(k_{L} + 1) n.
\]

Proof. We employ Corollary 5.3 to estimate the number of non-zero nearfield matrix entries from above by

\[
C_{rs}(k_{L} + 1) C_{\text{near}}^{2} \frac{n}{k_{L} + 1} \max_{t \in \mathcal{L}_{\ell}} \{ \hat{s} \mid s \in (\mathcal{P}_{\text{near}}(t) \cup \mathcal{P}_{\text{right}}(t)) \}
\]

(5.24)

\[
\leq C_{rs}^{2} C_{\text{near}}^{2} C_{\text{level}}(k_{L} + 1) n.
\]

\[
\square
\]

5.2 Computational Complexity

In Section 5.1 we have described the quantities which have to be computed and stored for the directional \(H^{2}\) matrix representation for the acoustic single layer operator with complex frequency and estimated their cardinalities. For the computational complexity the effort for generating these quantities has to be taken into account.
Remark 5.9 (Transfer matrices) Since each cluster in $T \setminus L$ has at least two sons, it follows by a geometric sum argument that $\# T \leq 2 (\# L)$. Hence, the number of the $q_{\nu, \mu, \nu}$ in (3.15) can be bounded from above by

$$\textstyle (\# T) (k_L + 1)^2 \leq 2 (\# L) (k_L + 1)^2 \leq 2 C_{rs} n (k_L + 1).$$

It requires the evaluation of the tensorized Čebyšev polynomials for the clusters $t$ at the Čebyšev nodes of their sons. For the efficient evaluation of Čebyšev polynomials we refer, e.g., to [40], and denote the computational complexity for computing all transfer matrices by

$$2 C_{\text{Cheby}} C_{rs} n (k_L + 1),$$

where $C_{\text{Cheby}}$ depends algebraically on the expansion order while an explicit estimate depends on the chosen evaluation method. We do not elaborate on this issue here but refer to [40] instead.

Remark 5.10 (Expansion coefficients) The number of the expansion coefficients $\gamma^{b}_{\mu, \mu, \nu, c}(\zeta)$ can be bounded by (5.21) and we distinguish between three scenarios:

1. Sectorial case $|\text{Im}\zeta| \leq \alpha$. Then the cardinality is of order $O(n k_L)$,

2. In the high frequency case, i.e., $\text{Re}\zeta = O(1)$ and if the resolution condition (5.15) is satisfied, the cardinality is of order $O((1 + L) k^2 L n)$. This shows that the asymptotic complexity of our algorithm with the new admissibility condition is the same as the algorithm for purely imaginary wave numbers considered in [7, Thm. 1].

3. In the high frequency case, i.e., $\text{Re}\zeta = O(1)$ and if the resolution condition (5.15) is violated we have $O((1 + L) k^2 L |\text{Im}\zeta|^2)$.

The evaluation of (3.12) per coefficient has a computational cost of $O(1)$.

Remark 5.11 (Basic farfield coefficients) The number of basic farfield coefficients $J^{b}_{\mu, c}$ is estimated in (5.20). It requires the integration of the expansion function $\Phi^{b}_{\mu, c}$ multiplied by basis functions. Although exact integration is feasible on plane triangles we recommend to use tensor Gauss rules on triangles which are transformed to squares by simplex coordinates. The order depends on the degree $m_t$ of the Čebyšev polynomials on the leaves $t \in L$ and the resolution condition: if the resolution condition (5.13) is satisfied the plane wave in the integrand is non-oscillatory on the panels and does not cause an increase of the required quadrature order. If the condition is violated the number of Gauss points has to take into account the wave number. Alternatively, more specialized quadrature methods could be employed for highly oscillatory integrals (see, e.g., [36]). We do not discuss this issue here in detail but assume that there exists a constant $C_{qb}$ such that the computational complexity to compute all basis farfield coefficients is bounded by $\mathcal{O} \left( C_{qb} C_{\text{ffc}} k_L (n + k_L |\text{Im}\zeta|^2) \right)$.

Remark 5.12 (Nearfield matrix) The number of non-zero nearfield matrix entries is estimated in Lemma 5.7 and Corollary 5.8. Typically, numerical quadrature is employed to approximate the integrals in (2.13) on $\text{supp} b_i \times \text{supp} b_j$. To take into account the singularity of the kernel functions, we recommend to use the quadrature rules described in [47], [10], [43], [44].
where the computational effort per integral behaves proportionally to \((\log \varepsilon)^4\). As in Remark 5.11, the number of quadrature points has to take into account the wave number only in the case that the resolution condition (5.15) is violated or quadrature techniques for highly oscillatory integrals should be applied. We denote the computational complexity per non-zero nearfield matrix entry by \(C_Q\).

1. **Sectorial case:** The computational complexity in this case is of order \(O(C_Q nk_L)\).

2. **Non-sectorial case, resolution condition satisfied.** Then, Corollary 5.8 implies that the computational cost is of order \(O(C_Q nk_L)\).

3. **Non-sectorial case, resolution condition violated.** Then, the computational cost can be estimated by using Lemma 5.7 by \(O(C_Q k_L^2 |\text{Im} \zeta|^2)\).

**Remark 5.13** We have not taken into account the computational cost of the re-compression algorithm because this would be a repetition of Section 5 in [7] and Section 4 in [8].

## 6 Numerical Experiments

In order to illustrate how our theoretical results compare to practical experiments, we consider the three-dimensional unit sphere \(\Gamma = \{x \in \mathbb{R}^3 : \|x\|_2 = 1\}\), approximated by regularly refining the eight triangular faces of the double pyramid \(\{x \in \mathbb{R}^3 : |x_1| + |x_2| + |x_3| = 1\}\) and projecting the resulting vertices to the sphere \(\Gamma\). This yields a surface mesh with \(n \in \mathbb{N}\) triangles.

We approximate the single-layer potential matrix

\[
K_{i,j}(\zeta) := (V(\zeta) b_j, b_i)
\]

for piecewise constant basis functions \((b_i)_{i=1}^n\) on the surface triangles.

Our approximation scheme uses a constant number \(m\) of interpolation points per coordinate for a total of \(m^3\) points for a bounding box, and the admissibility conditions (3.4) with \(\eta_1 = 10, \eta_2 = 2\) and \(\eta_3 = 1/2\).

In all experiments we rely on algebraic recompression [8] to reduce the storage requirements without significantly changing the approximation error or the number of blocks.

In a first experiment, we compare the pure Helmholtz case \(\zeta = \alpha i\) with the damped case \(\zeta = \alpha + \alpha i\), where \(\alpha = \sqrt{n/128}\) guarantees \(ah_{\mathcal{G}} \approx 0.6\), i.e., approximately ten mesh elements per wavelength. Figure 2 shows the number of blocks \#\(\mathcal{P}\) per degree of freedom for the purely imaginary case (labeled “Imaginary”) and mixed case (labeled “Complex”).

Since we are using a logarithmic scale for the matrix dimension \(n\), Figure 2 suggests that the number of blocks grows like \(O(n \log n)\) in the pure Helmholtz case, but only like \(O(n)\) for the Helmholtz case with decay, in accordance with our theoretical results.

Next we consider the dependence of the matrix approximation error, estimated in the spectral norm by a number of steps of the power iteration for the self-adjoint matrix \((K(\zeta) - \tilde{K}(\zeta))^*(K(\zeta) - \tilde{K}(\zeta))\), on the interpolation order \(m\). Standard polynomial interpolation theory predicts that the asymptotic rate of convergence should be the same for all matrix dimensions \(n\), while the total error also depends on the mesh parameter. Figure 3 uses a logarithmic scale for the relative spectral error on the vertical axis and a linear scale for the
Figure 2: Number of blocks per degree of freedom depending on the matrix dimension $n$. 

\[
\begin{array}{cccccc}
 n & \alpha & \zeta = \alpha + \alpha i & \#P & \#P/n & \zeta = \alpha i \\
 2048 & 4 & 2389 & 1.17 & 10965 & 5.35 \\
 4608 & 6 & 12125 & 2.63 & 30941 & 6.71 \\
 8192 & 8 & 19733 & 2.41 & 81109 & 9.90 \\
 18432 & 12 & 53525 & 2.90 & 240917 & 13.07 \\
 32768 & 16 & 114949 & 3.51 & 517621 & 15.80 \\
 73728 & 24 & 268773 & 3.65 & 1457349 & 19.77 \\
 131072 & 32 & 498725 & 3.80 & 2908053 & 22.19 \\
 294912 & 48 & 1145093 & 3.88 & 7419477 & 25.16 \\
\end{array}
\]
interpolation order on the horizontal. We can observe the expected exponential convergence, and we can also see that the relative error grows slowly as \( n \) increases. This latter effect can be contributed to the fact that our error estimate contains the factor \( 1/\text{dist}(B_t, B_s) \) that grows like \( 1/h_G \) as the mesh is refined.

We can conclude that in the mixed case \( \zeta = \alpha + \alpha \imath \) the number of blocks is \( O(n) \), while the error shows stable convergence. Essentially the performance of the algorithm is comparable to standard interpolation for the Laplace kernel.

In the context of our analysis, the dependence of the complexity and the accuracy on the damping factor \( \Re \zeta \) is of particular interest. To take a closer look, we fix \( n = 32768 \), \( \alpha = 16 \), and consider \( \zeta = \nu + \alpha \imath \) with \( \nu \in \{0, 2, \ldots, 24\} \). The corresponding block numbers are shown in Figure 4.

Based on our theoretical results we expect that the number of blocks is proportional to \( n + \min \left\{ |\Im \zeta|^2 \log n, \frac{|\Im \zeta|}{\Re \zeta} \right\} \), and comparing the numerical results with the function

\[ \nu \mapsto 40000 + \frac{1250000}{\nu + 1} \]  

suggests that the prediction is quite sharp for larger values of \( \nu \).

Of course we are also interested in the dependence of the interpolation error on the real part \( \nu = \Re \zeta \) of \( \zeta \). The relative spectral errors for \( \nu \in \{0, 4, \ldots, 24\} \) and interpolation orders \( m \in \{3, \ldots, 8\} \) are given in Figure 5.

We can see that the rates of convergence are similar for the different values of \( \nu \), while the relative errors decay as \( \nu \) grows.

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Figure 4: Number of blocks depending on the size of the real part Re$\zeta$. The values of $a$ and $b$ in the fit are as in (6.1): $a = 40000$ and $b = 1250000$.

Figure 5: Spectral error versus Re$\zeta$ for different interpolation orders.
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