Convergence rate of the finite element approximation for extremizers of Sobolev inequalities on 2D convex domains

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Abstract
We investigate a FEM-based numerical scheme approximating extremal functions of the Sobolev inequalities. The main result of this paper shows that if the domain is polygonal and convex in $\mathbb{R}^2$, then the convergence rate of a finite element solution to an exact extremal function is $O(h^2)$ in the $L^2$ norm, and it is $O(h)$ in the $H^1$ norm, where $h$ denotes the mesh size of a triangulation of the domain.

Keywords Finite element method · Extremizers of Sobolev inequalities · Lane–Emden equation

Mathematics Subject Classification Primary 65N30 · 65N12 · 35J60

1 Introduction
Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, where $N \geq 2$. In this paper, we are concerned with extremal functions for the Sobolev inequality

$$C(\Omega, p)\|u\|_{L^p(\Omega)} \leq \|\nabla u\|_{L^2(\Omega)},$$
where

\[
\begin{align*}
2 < p < \infty \quad & \text{if } N = 2 \\
2 < p < \frac{2N}{N-2} \quad & \text{if } N \geq 3,
\end{align*}
\]

and

\[
C(\Omega, p) = \inf \left\{ \frac{\|\nabla u\|_{L^2(\Omega)}}{\|u\|_{L^p(\Omega)}} \mid u \in H^1_0(\Omega), u \neq 0 \right\} (1.1)
\]
denotes the best constant. It is well known that the constant \( C(\Omega, p) \) is attained by a positive function \( U_0 = U_{0;\Omega, p} \) solving the semi-linear elliptic equation

\[
-\Delta u = |u|^{p-2}u \quad \text{in } \Omega, \quad u \in H^1_0(\Omega). (1.2)
\]

This equation is so called the Lane–Emden equation originated from the astrophysics. The aim of this paper is to obtain a sharp convergence rate for a numerical scheme to approximate the minimizer \( U_0 \). This work is motivated by Tanaka et al. [16], where a convergence estimate for the best constant \( C(\Omega, p) \) is established on a two-dimensional bounded convex domain. More precisely, they obtained a criterion that once we have an approximate function to a positive solution to (1.2), we can achieve a sharp estimate of \( C(\Omega, p) \). They computed an approximate solution to (1.2) for \( \Omega = (0, 1)^2 \) using Legendre polynomials, whose approximate error is computed in a posteriori manner. The main result of this paper provides a priori bound for the finite element approximation to (1.1) and (1.2).

Now, we fix a polygonal convex domain \( \Omega \subset \mathbb{R}^2 \) and arbitrary \( p \in (2, \infty) \). For \( h > 0 \), let \( \{T_h\} \) be a family of regular triangulations of \( \Omega \) (we refer to [2] for the definition and properties of the triangulations). The finite element space \( V_h \subset H^1_0(\Omega) \) is given by

\[
V_h = \left\{ v \in H^1_0(\Omega) \mid v \text{ is a polynomial of degree } \leq 1 \text{ on each } T \in T_h \right\}.
\]

On the finite-dimensional space \( V_h \), we formulate the following minimization problem similar to (1.1),

\[
C_h(\Omega, p) = \min \left\{ \frac{\|\nabla \phi_h\|_{L^2}}{\|\phi_h\|_{L^p}} \mid \phi_h \in V_h, \phi_h \neq 0 \right\}. (1.3)
\]

Since \( V_h \) has a finite dimension, there exists a minimizer \( U_h \) for the problem (1.3). Moreover, it follows from the Lagrange multiplier theorem that there exists a constant \( \lambda_h > 0 \) such that

\[
\int_{\Omega} \nabla U_h \cdot \nabla \phi_h \, dx = \lambda_h \int_{\Omega} |U_h|^{p-2}U_h \phi_h \, dx \quad \forall \phi_h \in V_h. (1.4)
\]
Here, we may assume that $\lambda_h = 1$, replacing $U_h$ by $(\|\nabla U_h\|^2_{L^2}/\|U_h\|^p_{L^p})^{\frac{1}{p}} U_h$.

Our main result establishes convergence of a finite element solution $U_h$ to an exact extremal function $U_0$, as well as its rate of convergence.

**Theorem 1.1** Assume that $\Omega \subset \mathbb{R}^2$ is a bounded convex domain with a polygonal boundary, and $p > 2$. Let $\{U_h\}$ be a family of minimizers of the problem (1.3) with $\lambda_h = 1$ in (1.4), and let $U_0 \in H^1_0(\Omega)$ be the (unique) positive minimizer of the problem (1.1), which solves (1.2). Then the following statements hold true:

(i) For any sequence $\{h_n\} \to 0$, $\{U_{h_n}\}$ converges to either $U_0$ or $-U_0$ in $H^1_0(\Omega)$ by choosing a subsequence.

(ii) There exists a universal constant $C > 0$ such that for any sequences $\{h_n\} \to 0$ and $\{U_{h_n}\} \to U_0$, the following inequality holds

\[ \|U_{h_n} - U_0\|_{L^2} \leq C h_n^2 \quad \text{and} \quad \|U_{h_n} - U_0\|_{H^1} \leq C h_n. \]  

(iii) The $L^\infty$ norm of $U_h$ is uniformly bounded, i.e., there exists a universal constant $C > 0$ such that

\[ \|U_h\|_{L^\infty(\Omega)} \leq C. \]

Related to the above result, it is worth to recall that there has been research to develop a numerical scheme to find the solutions to the nonlinear problem (1.2) (see [5,7,12] and references therein). The scheme based on the mountain pass principle was developed by Choi-McKenna [5] to find a minimizer, and it was extended by Li-Zhou [12] to find multiple solutions. In [7], Faou and Jézquel established the exponential convergence rate for the normalized gradient algorithm for the nonlinear Schrödinger equation. Nevertheless, to the authors’ best knowledge, there is no result on the convergence estimate between the solution to (1.2) and the finite element solution of the discrete problem (1.4).

Theorem 1.1 first provides a rate of convergence to an extremal function on a two-dimensional convex polygon. Its proof crucially relies on the non-degeneracy property (for the definition, we refer to Proposition A.3) of the minimizer. In our previous work [6], the non-degeneracy have been employed in the study of the nonrelativistic limit of ground states for the nonlinear pseudo-relativistic equations. However, several things must be modified to fit in the FEM setting. We mention that the proofs for (i), (iii) of Theorem 1.1 do not rely on the non-degeneracy property, and it would be possible extend those results to the three dimensional case without serious difficulty. Also, the result (ii) of Theorem 1.1 can be extended to the three dimensional case once one has a non-degeneracy result to a least energy solution to (1.2).

The remainder of the paper is organized as follows. Section 2 is devoted to proving the $H^1$ convergence of an approximate solution $U_h$. In Sects. 3 and 4, we obtain the convergence rates of $U_h$ in $H^1$ and $L^2$ respectively. In Sect. 5, we prove the uniform $L^\infty$ boundedness of $U_h$. In Sect. 6, we show that our analytic results are consistent with the real numerical implementation. In the “Appendix”, we collect useful analytic tools that are frequently invoked in the preceding sections.
2 Convergence of $U_h$ in $H^1_0$ space

In this section, we prove the $H^1$ convergence of $U_h$ through several steps. We recall that

$$C(\Omega, p) = \min_{v \in C_c^\infty(\Omega, 0)} \frac{\|v\|_{H^1_0(\Omega)}}{\|v\|_{L^p(\Omega)}}$$

and

$$C_h(\Omega, p) = \min_{v \in V_h \setminus \{0\}} \frac{\|v\|_{H^1_0(\Omega)}}{\|v\|_{L^p(\Omega)}},$$

where we imposed the norm $\|\nabla \cdot \|_{L^2(\Omega)}$ on $H^1_0(\Omega)$. We simply denote $C(\Omega, p)$ and $C_h(\Omega, p)$ by $C_0$ and $C_h$ respectively.

**Step 1** The value $C_h$ converges to $C_0$ as $h \to 0$.

**Proof** Since $V_h \subset H^1_0(\Omega)$, one has $C_0 \leq C_h$. From Propositions A.1 and A.2, we can choose some $\psi_h \in V_h$ that satisfies $\|U_0 - \psi_h\|_{H^1_0(\Omega)} \leq C_h$ for some $C > 0$ independent of $h$. Then we see that for small $h > 0$,

$$C_0 = \frac{\|U_0\|_{H^1_0(\Omega)}}{\|U_0\|_{L^p(\Omega)}} \geq \frac{\|\psi_h\|_{H^1_0(\Omega)} - C_h}{\|\psi_h\|_{L^p(\Omega)} + C_h}.$$

By the Sobolev inequality, $\|U_0 - \psi_h\|_{L^p(\Omega)} \leq C\|U_0 - \psi_h\|_{H^1_0(\Omega)} \leq C h$. Hence, there exist values $h_0 > 0$ and $\Lambda > 0$ such that for all $h \in (0, h_0)$ the values $\|\psi_h\|_{L^p(\Omega)}$ and $\|\psi_h\|_{H^1_0(\Omega)}$ belong to $(\Lambda, \frac{1}{\Lambda})$. Given this, the above inequality implies that for a constant $C > 0$ independent of $h \in (0, h_0)$ we have

$$C_0 \geq \frac{\|\psi_h\|_{H^1_0(\Omega)} - C h}{\|\psi_h\|_{L^p(\Omega)} + C h} \geq C_h - C h,$$

which combined with $C_0 \leq C_h$ shows that $\lim_{h \to 0} C_h = C_0$. \qed

**Step 2** For any sequence $\{h_n\} \to 0$, $\{U_{h_n}\}$ converges in $H^1_0(\Omega)$ to a nonzero function $W_0 \in H^1_0(\Omega)$ passing to a subsequence.

**Proof** With Step 1, note that for small $h > 0$,

$$C_0 = \frac{\|U_h\|_{H^1_0(\Omega)}}{\|U_h\|_{L^p(\Omega)}} \leq C_0 + 1. \quad (2.1)$$

By setting $\phi_h = U_h$ in (1.4), we obtain

$$\int_\Omega |\nabla U_h|^2 dx = \int_\Omega U_h^p dx. \quad (2.2)$$

Combining this equation with (2.1), we obtain that for small $h > 0$,

$$C_0 < \|U_h\|_{L^p(\Omega)}^{\frac{p-1}{2}} \|U_h\|_{H^1_0(\Omega)}^{\frac{1}{2}} < C_0 + 1. \quad (2.3)$$
The second inequality of (2.3) and the compactness of the embedding $H^1_0 \hookrightarrow L^p$ say that for any $\{h_n\} \to 0$, $\{U_{h_n}\}$ weakly converges to some $W_0$ in $H^1_0$, and strongly converges in $L^p$ passing to a subsequence. From the first inequality in (2.3), we deduce that $W_0$ is nonzero. Moreover, Proposition A.1 indicates that there exists a sequence $\psi_{h_n} \in V_{h_n}$ such that $\|W_0 - \psi_{h_n}\|_{H^1_0} = o(1)$ so one has

$$\|\nabla W_0\|^2_{L^2} = \lim_{n \to \infty} \int_{\Omega} \nabla U_{h_n} \cdot \nabla W_0 \, dx$$

$$= \lim_{n \to \infty} \left( \int_{\Omega} \nabla U_{h_n} \cdot \nabla \psi_{h_n} \, dx + \int_{\Omega} \nabla U_{h_n} \cdot \nabla (W_0 - \psi_{h_n}) \, dx \right)$$

$$= \lim_{n \to \infty} \left( \int_{\Omega} |U_{h_n}|^{p-2} U_{h_n} \psi_{h_n} \, dx + o(1) \right)$$

$$= \lim_{n \to \infty} \left( \int_{\Omega} |U_{h_n}|^{p-2} U_{h_n} W_0 \, dx + o(1) \right) = \|W_0\|^p_{L^p}. \tag{2.4}$$

Then, the equality (2.2) implies that

$$\|\nabla U_{h_n}\|^2_{L^2} = \lim_{n \to \infty} \|U_{h_n}\|^p_{L^p} = \|W_0\|^p_{L^p} = \|\nabla W_0\|^2_{L^2}.$$ 

Considering this equality and the weak convergence of $\{U_{h_n}\}$ to $W_0$, we conclude that the sequence $\{U_{h_n}\}$ strongly converges to $W_0$ in $H^1_0(\Omega)$. □

**Step 3** The function $W_0$ is either $U_0$ or $-U_0$.

**Proof** Fix an arbitrary $\psi \in H^1_0(\Omega)$. Then, taking $\psi_{h_n} \in V_{h_n}$, which satisfies $\|\psi - \psi_{h_n}\|_{H^1_0} = o(1)$ and using the same arguments in (2.4), we can deduce

$$\int_{\Omega} \nabla W_0 \cdot \nabla \psi \, dx = \int_{\Omega} |W_0|^{p-2} W_0 \psi \, dx,$$

which implies that $W_0$ is a weak solution of (1.2). Since $\{U_{h_n}\} \to W_0$ in $H^1_0$ and $C_{h_n} \to C_0$, we see that

$$C_0 = \frac{\|W_0\|_{H^1_0(\Omega)}}{\|W_0\|_{L^p(\Omega)}}$$

so $W_0$ is also a minimizer of the problem (1.1). From Proposition A.3, we conclude that $W_0$ is either $U_0$ or $-U_0$. □

### 3 $H^1$ error estimates

In this section, we compute a sharp $H^1$ convergence rate for $U_h$. Choose a sequence $\{h_n\} \to 0$ and a sequence of minimizers $\{U_{h_n}\} \subset V_{h_n}$ of (1.3) with $h = h_n$ such that $\lambda_{h_n} = 1$ in (1.4) and $U_{h_n} \to U_0$ in $H^1_0(\Omega)$, where $U_0$ is a unique positive solution of
(1.2). For notational simplicity, we denote \( h_n \) by just \( h \). We divide the proof into several steps. The following elementary estimates will be frequently invoked throughout this section.

**Lemma 3.1** For \( p > 2 \), there exists \( C > 0 \) independent of \( a, b \) such that

\[
|b|^{p-2}b - |a|^{p-2}a \leq C(|b|^{p-2} + |a|^{p-2})|b - a|
\]

and

\[
|b|^{p-2}b - |a|^{p-2}a - (p - 1)|a|^{p-2}(b - a) \leq \begin{cases} C(|b|^{p-3} + |a|^{p-3})|b - a|^2 & \text{if } p \geq 3, \\ C|b - a|^{p-1} & \text{if } 2 < p < 3. \end{cases}
\]

**Step 1** There exists a constant \( C > 0 \) independent of \( h \) such that

\[
\int_{\Omega} |\nabla (U_h - U_0)|^2 - (p - 1)U_0^{p-2}(U_h - U_0)^2 \, dx \\
\leq Ch\|U_h - U_0\|_{H^1_0(\Omega)} + C\|U_h - U_0\|_{H^1_0(\Omega)}^{\min(3,p)} \tag{3.1}
\]

**Proof** We recall that

\[
\begin{aligned}
\int_{\Omega} \nabla U_0 \cdot \nabla \phi \, dx &= \int_{\Omega} U_0^{p-1} \phi \, dx \quad \forall \ \phi \in H^1_0(\Omega), \\
\int_{\Omega} \nabla U_h \cdot \nabla \phi_h \, dx &= \int_{\Omega} |U_h|^{p-2}U_h \phi_h \, dx \quad \forall \ \phi_h \in V_h.
\end{aligned}
\tag{3.2}
\]

Then for all \( \phi_h \in V_h \),

\[
\int_{\Omega} \nabla (U_h - U_0) \cdot \nabla \phi_h \, dx = \int_{\Omega} (|U_h|^{p-2}U_h - U_0^{p-1})\phi_h \, dx. \tag{3.3}
\]

Propositions A.1 and A.2 indicates that there exists \( \psi_h \in V_h \) such that \( \|\psi_h - U_0\|_{H^1_0} \leq Ch \), where \( C \) only depends on \( \Omega \) and \( U_0 \). Since \( U_h \to U \) in \( H^1_0(\Omega) \), we may assume \( \|U_h - U_0\|_{H^1_0(\Omega)} \leq 1 \). Choosing \( \phi_h = U_h - \psi_h \) and using (3.3), we obtain that

\[
\int_{\Omega} \nabla (U_h - U_0) \cdot \nabla (U_h - U_0) \, dx - \int_{\Omega} (|U_h|^{p-2}U_h - U_0^{p-1})(U_h - U_0) \, dx \\
= \int_{\Omega} \nabla (U_h - U_0) \cdot \nabla (\psi_h - U_0) \, dx - \int_{\Omega} (|U_h|^{p-2}U_h - U_0^{p-1})(\psi_h - U_0) \, dx. \tag{3.4}
\]
Using Lemma 3.1, Hölder inequality and Sobolev embedding, we see that

\[
\left| \int_{\Omega} \nabla(U_h - U_0) \cdot \nabla(\psi_h - U_0) \, dx - \int_{\Omega} (|U_h|^{p-2}U_h - U_0^{p-1})(\psi_h - U_0) \, dx \right|
\]
\[
\leq \| \nabla(U_h - U_0) \|_{L^2} \| \nabla(\psi_h - U_0) \|_{L^2}
\]
\[
+ C \int_{\Omega} (|U_h|^{p-2} + U_0^{p-2})|U_h - U_0| |\psi_h - U_0| \, dx
\]
\[
\leq C h \| U_h - U_0 \|_{H^1_0} + C (\| U_h \|_{L^p}^{p-2} + \| U_0 \|_{L^p}^{p-2}) \| U_h - U_0 \|_{L^p} \| \psi_h - U_0 \|_{L^p}
\]
\[
\leq C h \| U_h - U_0 \|_{H^1_0}.
\]

Define

\[
I := \int_{\Omega} (|U_h|^{p-2}U_h - U_0^{p-1})(U_h - U_0) \, dx - \int_{\Omega} (p-1)U_0^{p-2}(U_h - U_0)^2 \, dx.
\]

Now, Lemma 3.1 implies that \( I \) satisfies the following condition: if \( p \geq 3 \), then

\[
|I| \leq C \int_{\Omega} (|U_h|^{p-3} + U_0^{p-3})|U_h - U_0|^3 \, dx
\]
\[
\leq C (\| U_h \|_{H^1_0}^{p-3} + \| U_0 \|_{H^1_0}^{p-3}) \| U_h - U_0 \|_{H^1_0}^3 \leq C \| U_h - U_0 \|_{H^1_0}^3,
\]

and if \( 2 < p < 3 \), then

\[
|I| \leq C \int_{\Omega} |U_h - U_0|^p \, dx \leq C \| U_h - U_0 \|_{H^1_0}^p.
\]

Inserting this inequality and (3.5) into (3.4), we find

\[
\int_{\Omega} |\nabla(U_h - U_0)|^2 - (p-1)U_0^{p-2}(U_h - U_0)^2 \, dx
\]
\[
\leq C h \| U_h - U_0 \|_{H^1} + C \| U_h - U_0 \|_{H^1_{\min(3,p)}}^\frac{1}{p} + C \| U_h - U_0 \|_{H^1_{\min(3,p)}}^\frac{1}{p},
\]

which shows the proof.

\[ \text{Step 2} \] There exists a constant \( C > 0 \) independent of \( h \) such that

\[
\| U_h - U_0 \|_{H^1_0(\Omega)} \leq C h.
\]

**Proof** We decompose the difference \( U_h - U_0 \) as the sum of the tangential part to \( U_0 \) and the orthogonal part to \( U_0 \). In other words, we choose a constant \( \lambda_h \in \mathbb{R} \) and a function \( v_h \in H^1_0(\Omega) \) such that

\[
U_h - U_0 = v_h + \lambda_h U_0 \quad \text{and} \quad \langle v_h, U_0 \rangle_{H^1_0} = 0.
\]

\( \Box \) Springer
We observe that

$$0 = \langle v_h, U_0 \rangle_{H_0^1} = \int_{\Omega} \nabla v_h \cdot \nabla U_0 \, dx = \int_{\Omega} v_h (-\Delta U_0) \, dx = \int_{\Omega} v_h U_0^{p^{-1}} \, dx.$$  

(3.7)

Since \( \|v_h\|_{H^1}^2 + \lambda_h^2 \|U_0\|_{H^1}^2 = \|U_h - U_0\|_{H^1}^2 \to 0 \), we see that \( \|v_h\|_{H^1}, \lambda_h \to 0 \). In particular, we may assume that \( \|v_h\|_{H^1} < 1, |\lambda_h| < 1 \).

We insert (3.6) into the left hand side of (3.1) and use (3.7) to obtain

$$\int_{\Omega} |\nabla (v_h + \lambda_h U_0)|^2 - (p - 1) U_0^{p-2} (v_h + \lambda_h U_0) \, dx$$

$$= \int_{\Omega} |\nabla v_h|^2 - (p - 1) U_0^{p-2} v_h^2 \, dx + \lambda_h^2 \int_{\Omega} |\nabla U_0|^2 - (p - 1) U_0^p \, dx$$

(3.8)

Then combining (3.1), (3.8) and Proposition A.3, we get

$$\int_{\Omega} |\nabla v_h|^2 \, dx \leq C \int_{\Omega} |\nabla v_h|^2 - (p - 1) U_0^{p-2} v_h^2 \, dx$$

$$\leq C \lambda_h^2 + C h \|U_h - U_0\|_{H_0^1}^2 + C \|U_h - U_0\|_{H_0^1}^\text{min\{3,p\}}.$$ 

Thus, using Young’s inequality, we have

$$\|v_h\|_{H_0^1}^2 \leq C \lambda_h^2 + C h \left( \|v_h\|_{H_0^1} + \lambda_h \right) + C \left( \|v_h\|_{H_0^1}^\text{min\{3,p\}} + \lambda_h^\text{min\{3,p\}} \right)$$

$$\leq C \lambda_h^2 + \frac{1}{2} \left( \|v_h\|_{H_0^1}^2 + \lambda_h^2 \right) + C \left( \|v_h\|_{H_0^1}^\text{min\{3,p\}} + \lambda_h^\text{min\{3,p\}} \right) + C h^2,$$

which can be simplified as

$$\|v_h\|_{H_0^1}^2 \leq C \left( \lambda_h^2 + \|v_h\|_{H_0^1}^\text{min\{3,p\}} + h^2 \right)$$  

(3.9)

Meanwhile, the second equality of (3.2) is written as follows: for all \( \phi_h \in V_h \),

$$\int_{\Omega} \nabla ((1 + \lambda_h) U_0 + v_h) \cdot \nabla \phi_h \, dx$$

$$= \int_{\Omega} |(1 + \lambda_h) U_0 + v_h|^{p-2} ((1 + \lambda_h) U_0 + v_h) \phi_h \, dx$$  

(3.10)
Again, we take $\phi_h \in V_h$ such that $\|U_0 - \phi_h\|_{H^1_0(\Omega)} \leq C h$. Then arguing similarly as in Step 1, one has

$$\int_\Omega \nabla((1 + \lambda_h)U_0 + v_h) \cdot \nabla \phi_h \, dx$$

$$= (1 + \lambda_h) \int_\Omega \nabla U_0 \cdot \nabla U_0 \, dx + \int_\Omega \nabla((1 + \lambda_h)U_0 + v_h) \cdot \nabla(\phi_h - U_0) \, dx$$

$$= (1 + \lambda_h) \int_\Omega |\nabla U_0|^2 \, dx + O(h) \tag{3.11}$$

and

$$\int_\Omega |(1 + \lambda_h)U_0 + v_h|^{p-2}((1 + \lambda_h)U_0 + v_h)\phi_h \, dx$$

$$= \int_\Omega |(1 + \lambda_h)U_0 + v_h|^{p-2}((1 + \lambda_h)U_0 + v_h)U_0 \, dx + O(h)$$

$$= (1 + \lambda_h)^{p-1} \int_\Omega U_0^p \, dx + (p - 1)(1 + \lambda_h)^{p-2} \int_\Omega U_0^{p-1} v_h \, dx + II + O(h) \tag{3.12}$$

$$= (1 + \lambda_h)^{p-1} \int_\Omega U_0^p \, dx + II + O(h),$$

where we defined

$$II := \int_\Omega |(1 + \lambda_h)U_0 + v_h|^{p-2}((1 + \lambda_h)U_0 + v_h)U_0 \, dx$$

$$- (1 + \lambda_h)^{p-1} \int_\Omega U_0^p \, dx - (p - 1)(1 + \lambda_h)^{p-2} \int_\Omega U_0^{p-1} v_h \, dx$$

$$= \int_\Omega |(1 + \lambda_h)U_0 + v_h|^{p-2}((1 + \lambda_h)U_0 + v_h)U_0 \, dx - (1 + \lambda_h)^{p-1} \int_\Omega U_0^p \, dx.$$

Here the second equality holds by (3.7). Inserting the estimates (3.11) and (3.12) in (3.10), we obtain the estimate

$$\left|(1 + \lambda_h) \int_\Omega |\nabla U_0|^2 \, dx - (1 + \lambda_h)^{p-1} \int_\Omega U_0^p \, dx\right| \leq |II| + O(h). \tag{3.13}$$

Subsequently, using Lemma 3.1 again, we see that

$$|II| \leq C \int_\Omega (U_0^{p-3} + |v_h|^{p-3})U_0 v_h^2 \, dx$$

$$\leq C (\|U_0\|_{L^p}^{p-3} + \|v_h\|_{L^p}^{p-3}) \|U_0\|_{L^p}^p \|v_h\|_{L^p}^2 \leq C \|v_h\|_{H^1_0}^2 \tag{3.14}$$

if $p \geq 3$ and

$$|II| \leq C \int_\Omega U_0 v_h^{p-1} \, dx \leq C \|U_0\|_{L^p}^p \|v_h\|_{L^p}^{p-1} \leq C \|v_h\|_{H^1_0}^{p-1} \tag{3.15}$$
if $2 < p < 3$. Combining (3.13)–(3.15), we have

$$
\left| \left( (1 + \lambda_h)^{p-1} - (1 + \lambda_h) \right) \int_{\Omega} |\nabla U_0|^2 \, dx \right|
\leq (1 + \lambda_h)^{p-1} \int_{\Omega} U_0^p \, dx - (1 + \lambda_h) \int_{\Omega} |\nabla U_0|^2 \, dx
\leq C(h + \|v_h\|_{H_0^1}^{\min\{p-1,2\}}),
$$

which simplifies to

$$(1 + \lambda_h)^{p-2} - 1 \leq C(h + \|v_h\|_{H_0^1}^{\min\{p-1,2\}}).$$

By invoking the mean value theorem, there exists $\xi_h$ between 0 and $\lambda_h$ such that

$$(1 + \lambda_h)^{p-2} - 1 = (p - 2)(1 + \xi_h)^{p-3}\lambda_h,$$

which indicates that

$$|\lambda_h| \leq \frac{C}{(p - 2)(1 + \xi_h)^{p-3}}(h + \|v_h\|_{H_0^1}^{\min\{p-1,2\}}) \leq C(h + \|v_h\|_{H_0^1}^{\min\{p-1,2\}}),$$

because $\xi_h \to 0$.

Combining this inequality with (3.9), we arrive at the following estimate

$$\|v_h\|_{H_0^1}^2 \leq C \left( \lambda_h^2 + \|v_h\|_{H_0^1}^{\min\{3,p\}} + h^2 \right)
\leq C \left( h^2 + \|v_h\|_{H_0^1}^{\min\{2(p-1),4\}} + \|v_h\|_{H_0^1}^{\min\{3,p\}} + h^2 \right),$$

Since $\|v_h\|_{H_0^1} \to 0$ and $p > 2$, this inequality shows that

$$\|v_h\|_{H_0^1}^2 \leq C h^2 \quad \text{and} \quad \lambda_h^2 \leq C h^2.$$

Thus we finally conclude that

$$\|U_h - U_0\|_{H_0^1}^2 = \|v_h\|_{H_0^1}^2 + \lambda_h^2 \leq C h^2.$$

This completes the proof. \hfill \Box

### 4 $L^2$ error estimates

In this section, we prove the $L^2$ error estimate for $U_h$ by performing a duality argument. We select a sequence $\{h_n\} \to 0$ and a sequence of minimizers $\{U_{h_n}\} \subset V_{h_n}$ of (1.3)
with \( h = h_n \) such that \( \lambda h_n = 1 \) in (1.4) and \( U_{h_n} \to U_0 \) in \( H^1_0(\Omega) \), where \( U_0 \) is a unique positive solution of (1.2). Similar to the analysis in the previous section, we denote \( h_n \) simply by \( h \). Consider the linear operator \( \mathcal{L} : H^2(\Omega) \to L^2(\Omega) \) defined by

\[
\mathcal{L} := -\Delta - (p - 1)U_0^{p-2},
\]

which is the linearized operator of the Eq. (1.2) at \( U_0 \). We prepare the following lemma.

**Lemma 4.1** For given data \( f \in L^2(\Omega) \), there exists a unique solution \( w \in H^1_0(\Omega) \cap H^2(\Omega) \) of the problem

\[
\mathcal{L}[w] = f \quad \text{in } \Omega, \quad w = 0 \quad \text{on } \partial\Omega. \tag{4.1}
\]

such that the following estimate holds for some \( C > 0 \) independent of \( f \):

\[
\|w\|_{H^2(\Omega)} \leq C \|f\|_{L^2(\Omega)}. \tag{4.2}
\]

**Proof** By Proposition A.3, the operator \( \mathcal{L} \) has no kernel element; thus by the Fredholm alternative theory, there exists a unique solution \( w \in H^1_0(\Omega) \cap H^2(\Omega) \) of the problem (4.1). We multiply the Eq. (4.1) by \( U_0 \) and integrate by parts to obtain

\[
\int_\Omega \nabla w \cdot \nabla U_0 \, dx = \int_\Omega (p - 1)wU_0^{p-1} \, dx + \int_\Omega fU_0 \, dx
\]

\[
= \int_\Omega (p - 1)w(-\Delta U_0) \, dx + \int_\Omega fU_0 \, dx
\]

\[
= (p - 1) \int_\Omega \nabla w \cdot \nabla U_0 \, dx + \int_\Omega fU_0 \, dx.
\]

Then, we have

\[
\langle w, U_0 \rangle_{H^1_0} = \int_\Omega \nabla w \cdot \nabla U_0 \, dx = \frac{1}{2 - p} \int_\Omega fU_0 \, dx \tag{4.3}
\]

Now we consider the orthogonal decomposition of \( w \) by \( w = v + \lambda U_0 \) such that \( \langle v, U_0 \rangle_{H^1_0} = 0 \); consequently \( \langle v, \mathcal{L}[U_0] \rangle_{L^2} = 0 \) holds. Then, (4.3) leads to

\[
|\lambda| = \left| \frac{\langle w, U_0 \rangle_{H^1_0}}{\|U_0\|_{H^1_0}^2} \right| \leq C \|f\|_{L^2}. \tag{4.4}
\]
Meanwhile, after multiplying (4.1) by \( w \) we use the decomposition of \( w \) and Proposition A.3 to obtain
\[
\int_{\Omega} f(v + \lambda U_0) \, dx = \int_{\Omega} \mathcal{L}[v + \lambda U_0](v + \lambda U_0) \, dx \\
= \int_{\Omega} \mathcal{L}[v]v \, dx + 2\lambda \int_{\Omega} \mathcal{L}[U_0]v \, dx + \lambda^2 \int_{\Omega} \mathcal{L}[U_0]U_0 \, dx \\
\geq C\|v\|_{H_0^1}^2 + (2 - p)\lambda^2 \int_{\Omega} U_0^{p-2} \, dx.
\]
Combining this equation with (4.4), we have from the Young’s inequality that
\[
\|v\|_{H_0^1}^2 \leq (\|v\|_{L^2} + C|\lambda|)\|f\|_{L^2} + C|\lambda|^2 \\
\leq \frac{1}{2}\|v\|_{L^2}^2 + C\|f\|_{L^2}^2,
\]
which shows that \( \|v\|_{H_0^1} \leq C\|f\|_{L^2} \) by the Sobolev embedding. Since \( \|w\|_{H_0^1} = \|v\|_{H_0^1}^2 + \lambda^2\|U_0\|_{H_0^1} \), we also get \( \|w\|_{H_0^1} \leq C\|f\|_{L^2} \). Considering the following equation
\[
-\Delta w = (p - 1)U_0^{p-2}w + f
\]
and invoking Proposition A.2, we finally have
\[
\|w\|_{H^2} \leq C(\|(p - 1)U_0^{p-2}w\|_{L^2} + \|f\|_{L^2}) \leq C\|f\|_{L^2}.
\]
This completes the proof.

Now we begin the proof of the \( L^2 \) error estimate of (1.5). Let \( w_h \in H^2 \) be a unique solution of the problem
\[
\mathcal{L}[w] = U_h - U_0 \text{ in } \Omega, \quad w = 0 \text{ on } \partial\Omega
\]
such that the estimate \( \|w_h\|_{H^2} \leq C\|U_h - U_0\|_{L^2} \) holds true. Then one has
\[
\int_{\Omega} (U_h - U_0)^2 \, dx = \int_{\Omega} \mathcal{L}[w_h](U_h - U_0) \, dx \\
= \int_{\Omega} \nabla w_h \cdot \nabla(U_h - U_0) \, dx - (p - 1) \int_{\Omega} w_h U_0^{p-2}(U_h - U_0) \, dx
\]
(4.5)
Take \( \phi_h \in V_h \), which satisfies \( \|\phi_h - w_h\|_{H_0^1} \leq Ch\|w_h\|_{H^2} \). Then one must have
\[
\int_{\Omega} \nabla(U_h - U_0) \cdot \nabla \phi_h \, dx = \int_{\Omega} (|U_h|^{p-2}U_h - U_0^{p-1})\phi_h \, dx.
\]
Combining this equation with (4.5), and using Lemma 3.1 and the $H^1$ convergence rate of $U_h$, which was obtained in the previous section, we obtain

$$\int_{\Omega} (U_h - U_0)^2 \, dx = \int_{\Omega} \nabla(w_h - \phi_h) \cdot \nabla(U_h - U_0) \, dx$$

$$- (p - 1) \int_{\Omega} (w_h - \phi_h) U_h^{p-2}(U_h - U_0) \, dx$$

$$+ \int_{\Omega} \left( |U_h|^{p-2}U_h - U_0^{p-1} - (p - 1)U_0^{p-2}(U_h - U_0) \right) \phi_h \, dx$$

$$\leq ||w_h - \phi_h||_{H^1_0}||U_h - U_0||_{H^1_0} + ||w_h - \phi_h||_{L^p}||U_0||_{L^p}^{p-2}||U_h - U_0||_{L^p}$$

$$+ \left\{ \begin{array}{ll}
C(||U_h||_{L^p}^{p-3} + ||U_0||_{L^p}^{p-3})||U_h - U_0||_{L^p}^2||\phi_h||_{L^p} & \text{if } p \geq 3, \\
C||U_h - U_0||_{L^2}^{p-1}||\phi_h||_{L^\frac{2}{p}} & \text{if } 2 < p < 3,
\end{array} \right.$$ 

$$\leq \left\{ \begin{array}{ll}
Ch||w_h - \phi_h||_{H^1_0} + Ch^2||\phi_h||_{H^1_0} & \text{if } p \geq 3, \\
Ch||w_h - \phi_h||_{H^1_0} + C||U_h - U_0||_{L^2}^{p-1}||\phi_h||_{H^1_0} & \text{if } 2 < p < 3.
\end{array} \right.$$ 

Because $||\phi_h - w_h||_{H^1_0} \leq Ch||w_h||_{H^2}$, we see that $||\phi_h - U_h||_{H^1_0} \leq C||w_h||_{H^2}$; consequently, using the estimate $||w_h||_{H^2} \leq C||U_h - U_0||_{L^2}$ from (4.2), one has

$$\int_{\Omega} (U_h - U_0)^2 \, dx \leq \left\{ \begin{array}{ll}
Ch^2||U_h - U_0||_{L^2} & \text{if } p \geq 3, \\
Ch^2||U_h - U_0||_{L^2} + C||U_h - U_0||_{L^2}^p & \text{if } 2 < p < 3.
\end{array} \right.$$ 

Then, we see that the desired $L^2$ convergence rate is obtained in all cases.

**5 The uniform $L^\infty$ estimate**

This section is devoted to prove the uniform $L^\infty$ estimate of $U_h$, i.e., (ii) in Theorem 1.1. We recall that

$$\int_{\Omega} (\nabla U_h \cdot \nabla \phi_h) \, dx = \int_{\Omega} |U_h|^{p-2}U_h \phi_h \, dx \ \forall \ \phi_h \in V_h.$$ 

We define $v_h \in H^1_0(\Omega)$ as the unique solution of

$$-\Delta v = |U_h|^{p-2}U_h \text{ in } \Omega, \quad v \in H^1_0(\Omega).$$

In particular, $v_h$ satisfies

$$\int_{\Omega} \nabla v_h \cdot \nabla \phi \, dx = \int_{\Omega} |U_h|^{p-2}U_h \phi \, dx, \quad \forall \ \phi \in H^1_0(\Omega).$$

Then one must have

$$\int_{\Omega} \nabla(U_h - v_h) \cdot \nabla \phi \, dx = 0 \ \forall \ \phi \in V_h.$$
which implies that \( U_h \) is the \( H^1 \) projection of \( v_h \) to the finite element space \( V_h \). Thus we have from Proposition A.1 that

\[
\|U_h\|_{W^{1,q}(\Omega)} \leq C_q \|v_h\|_{W^{1,q}(\Omega)} \tag{5.1}
\]

for any \( q \geq 2 \), as long as the right hand side is finite. Let \( G(x, y) \) denote the Green function of \(-\Delta\) on \( \Omega \) with the Dirichlet boundary condition. Then \( v_h \) is given by

\[
v_h(x) = \int_\Omega G(x, y)|U_h|^{p-2}U_h(y)dy.
\]

Since we have the following uniform gradient estimate of Green function [8,10]:

\[
|\nabla_x G(x, y)| \leq C \frac{1}{|x - y|},
\]

the Hardy-Littlewood-Sobolev inequality (see e.g. page 354 in [15]) implies that

\[
\|v_h\|_{W^{1,q}(\Omega)} \leq C \|U_h\|_{L^{p-1}}\|
\]

for any \( q > r > 1 \) that satisfies \( \frac{1}{r} - \frac{1}{q} = \frac{1}{2} \). Let us choose \( r = 3/2 \) and \( q = 6 \). Then,

\[
\|v_h\|_{W^{1.6}} \leq C \|U_h\|_{L^{3/2}} = \|U_h\|_{L^{3(p-1)/2}} \leq \|U_h\|_{H^1_0}.
\]

We combine this inequality with (5.1) and use the Sobolev embedding to conclude that

\[
\|U_h\|_{L^\infty} \leq C \|U_h\|_{H^1_0}.
\]

This completes the proof.

6 Numerical results

In the numerical implementation, we computed the approximate solutions in the case of \( p = 4 \) and \( p = 11 \) with \( n = 2 \) and \( \Omega = (0, 1)^2 \). Since we do not have an exact solution, we computed the decrease of the relative errors \( \|u_h - u_{h/2}\|_{L^2(\Omega)} \) and \( \|u_h - u_{h/2}\|_{H^1} \), where \( h \) is the length of the triangle. We computed with \( h \) given by \( h_j = 2^{-j} \) for \( 1 \leq j \leq 7 \), and for each \( 2 \leq j \leq 7 \), we calculated the ratios \( R^0_j \) and \( R^1_j \) given by

\[
R^0_j = \log_2 \left( \frac{\|u_{h_j} - u_{h_{j+1}}\|_{L^2}}{\|u_{h_{j-1}} - u_{h_j}\|_{L^2}} \right) \quad \text{and} \quad R^1_j = \log_2 \left( \frac{\|u_{h_{j+1}} - u_{h_{j+1}}\|_{H^1}}{\|u_{h_{j-1}} - u_{h_j}\|_{H^1}} \right).
\]
To obtain the numerical solution for the nonlinear problem, we modify the combination of the gradient descent method and the normalization used for computation of the ground states to the nonlinear Schrödinger equations (see e.g. [1,7]).

Precisely, we iterate a combination of the gradient descent method and the $L^{p+1}(\Omega)$ normalization: first, we fix an initial data $u_0 \in L^{p+1}(\Omega)$; then, we iterate the following two steps:

1. We select a small value $\eta > 0$. Then we consider the gradient descent of the energy function $E(u)$, i.e.,

$$\nabla E(u) = u - (-\Delta)^{-1}(|u|^{p-1}u)$$  \hfill (6.1)

and substitute $u \rightarrow u - \eta \nabla E(u)$.

2. Next we normalize the $L^{p+1}(\Omega)$-norm as

$$u \rightarrow \frac{u}{\|u\|_{L^{p+1}(\Omega)}}.$$  \hfill (6.2)

In the above, to obtain the function $w = (-\Delta)^{-1}(|u|^{p-1}u)$, we computed the approximate function $w_h \in V_h$ such that

$$\int_{\Omega} \nabla w_h \cdot \nabla \phi \, dx = \int_{\Omega} \phi |u|^{p-1}u(x) \, dx \quad \forall \phi \in V_h.$$  \hfill (6.3)

We chose the domain $\Omega = (0, 1) \times (0, 1)$ and took the initial data $u_0 \in V_h$ so that $u_0 = 1$ on all interior nodes, and $u_0 = 0$ on the boundary nodes. Then, we normalized the $L^{p+1}(\Omega)$ of $u_0$. For the iteration, we took $\eta = 0.2$ and iterated the above two steps for 60 times. We examined two cases $p = 4$ and $p = 11$ in Tables 1 and 2. The results for both cases show that the $L^2$ error is approximately of order $O(h^2)$ and $H^1$ error is approximately of order $O(h)$ as shown in Theorem 1.1.

Figure 1 shows the solutions for $p = 4$ computed with mesh sizes $1/2^3, 1/2^4, 1/2^5,$ and $1/2^6$. Table 1 shows the error of $L^2$ and $H^1$ with the ratios for the case of $p = 4$, and Table 2 shows the corresponding errors and ratios for the case of $p = 11$.

| $h_j$  | $\|u_h - u_{h/2}\|_{L^2}$ | Rate ($R^0_j$) | $\|u_h - u_{h/2}\|_{H^1}$ | Rate ($R^1_j$) |
|-------|-----------------------------|-----------------|-----------------------------|-----------------|
| $2^{-1}$ | 4.4888E−01                   |                | 2.6864E+00                  |                |
| $2^{-2}$ | 8.6927E−02                   | 2.37           | 1.0537E+00                  | 1.35           |
| $2^{-3}$ | 2.2559E−02                   | 1.95           | 5.0503E−01                  | 1.06           |
| $2^{-4}$ | 5.9548E−03                   | 1.92           | 2.4947E−01                  | 1.02           |
| $2^{-5}$ | 1.5506E−03                   | 1.94           | 1.2429E−01                  | 1.01           |
| $2^{-6}$ | 4.0921E−04                   | 1.92           | 5.2089E−02                  | 1.00           |
| $2^{-7}$ | 1.1299E−04                   | 1.86           | 3.1038E−02                  | 1.00           |
Table 2  \( L^2 \) and \( H^1 \) errors for the case of \( p = 11 \)

| \( h_j \) | \( \| u_h - u_{h/2} \|_{L^2} \) | Rate \((R_0^j)\) | \( \| u_h - u_{h/2} \|_{H^1} \) | Rate \((R_1^j)\) |
|-------|-------------|----------|-------------------|----------|
| \( 2^{-1} \) | 6.2791E−01 | – | 3.2719E+00 | – |
| \( 2^{-2} \) | 1.6580E−01 | 1.92 | 1.1568E+00 | 1.50 |
| \( 2^{-3} \) | 6.7736E−02 | 1.29 | 6.7815E−01 | 0.77 |
| \( 2^{-4} \) | 2.6271E−02 | 1.37 | 3.9475E−01 | 0.78 |
| \( 2^{-5} \) | 8.7800E−03 | 1.57 | 2.0637E−01 | 0.94 |
| \( 2^{-6} \) | 2.5323E−03 | 1.81 | 1.0334E−02 | 1.00 |
| \( 2^{-7} \) | 6.7250E−04 | 1.91 | 5.1566E−02 | 1.00 |

Fig. 1  Approximate solutions for \( p = 4 \)

It would be interesting to prove the convergence of the above iteration scheme. We refer to [7] where the authors considered a similar iteration scheme in the context of the nonlinear Schrodinger equations and proved that the scheme converges to a fixed point with an exponential rate. They also proved that the choice of \( \eta > 0 \) does not depend on the size of the spatial discretization. Also, the numerical simulation indicates that the energy, given by

\[
E(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2(x) \, dx - \frac{1}{p + 1} \int_{\Omega} |u|^{p+1}(x) \, dx,
\]

decreases as the iteration proceeds. It would be challenging but interesting to prove the energy decreasing property of the scheme (Fig. 2).

6.1 Further numerical simulations

In this subsection, we supplement a couple of numerical simulations related to our result.

Problem 1. The error estimates for the nonconvex domain.

It would be interesting to extend our result to non-convex domains. There are two issues related to this problem. The first issue is that the non-degeneracy of the solution is not known to hold for general non-convex domain \( \Omega \). The second issue is that the solution \( u \) to (1.2) may not belong to \( H^2(\Omega) \) when \( \Omega \) is non-convex (see [4,9]).
Fig. 2 The energy computed at iteration step $1 \leq j \leq 200$ for $p = 4$ and $h = 1/16$

Table 3 $L^2$ and $H^1$ errors for the case of $p = 4$ for non-convex domain

| $h_j$  | $\|u_h - u_{h/2}\|_{L^2}$ | Rate ($R^0_j$) | $\|u_h - u_{h/2}\|_{H^1}$ | Rate ($R^1_j$) |
|--------|-----------------|---------------|-----------------|---------------|
| $2^{-1}$ | $1.4501E-00$ | $-$ | $1.4075E+00$ | $-$ |
| $2^{-2}$ | $4.5978E-01$ | $1.66$ | $8.2638E-01$ | $0.77$ |
| $2^{-3}$ | $2.7687E-01$ | $0.73$ | $4.6763E-01$ | $0.82$ |
| $2^{-4}$ | $1.1017E-01$ | $1.33$ | $2.2463E-01$ | $1.02$ |
| $2^{-5}$ | $4.5277E-02$ | $1.28$ | $1.2386E-01$ | $0.86$ |
| $2^{-6}$ | $2.0250E-02$ | $1.16$ | $7.7075E-02$ | $0.68$ |

We tested the domain $\Omega = (0, 2) \times (0, 2) - (1, 2) \times (0, 1)$ with $p = 4$. The result shows that both the $H^1$ and $L^2$ norms decrease slowly than the convex case $\Omega = (0,1)^2$ (Table 3).

**Problem 2.** The dependence of the error on the exponent $p$.

We test the relative error as the exponent $p$ increases. It is known that the least energy solution $u$ to (1.2) blows up as the exponent $p$ diverges to $\infty$. It would be interesting to settle a sharp convergence estimate depending on the exponent $p$. Concerning this problem, we give a result of numerical experiment. With $\Omega = (0,1)^2$ and $h = 1/16$, we computed $\|u_h - u_{h/2}\|_{L^2}$ and $\|u_h - u_{h/2}\|_{H^1}$ for $p = 20 \times j$ with $1 \leq j \leq 13$. The result in Table 4 shows that the growth of the error is sublinear in $p$.

To establish the error estimate depending on $p$, one may need to use the result of [13,14] where the authors proved that $p \int_\Omega |u_p|^{p+1}(x)dx$ converges to $8\pi e$ as $p$ goes to infinity, where $u_p$ denotes the least energy solution to (1.2).
| $p$  | 20  | 40  | 60  | 80  | 100 | 120 | 140 | 160 | 180 | 200 | 220 | 240 | 260 |
|------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| $10^3 \|u_h - u_{h/2}\|_{L^2}$ | 2.53 | 5.77 | 6.34 | 6.56 | 6.74 | 6.87 | 6.97 | 7.03 | 7.08 | 7.11 | 7.14 | 7.17 | 7.19 |
| $10^3 \|u_h - u_{h/2}\|_{H^1}$ | 1.63 | 2.88 | 3.18 | 3.34 | 3.43 | 3.48 | 3.51 | 3.53 | 3.55 | 3.56 | 3.56 | 3.57 | 3.58 |
Acknowledgements The authors are very grateful to the anonymous referees for their valuable comments and suggestions. This research of the first author was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (NRF-2017R1C1B5076348). This research of the second author was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (NRF-2017R1C1B1008215). This research of the third author was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (NRF-2017R1D1A1A09000768).

Appendix A: Analytic tools

In the appendix, we arrange the necessary auxiliary tools to handle the analytic issues that arise when we prove our main results.

**Proposition A.1** ([2,3]) For any $u \in H^1_0(\Omega)$, we define $P_h(u)$ by the $H^1_0(\Omega)$-projection of $u$ onto $V_h$. In other words, $P_h(u)$ is a unique element in $V_h$ that satisfies

$$\int_{\Omega} \nabla u \cdot \nabla \phi_h \, dx = \int_{\Omega} \nabla P_h(u) \cdot \nabla \phi_h \, dx \quad \text{for all } \phi_h \in V_h.$$

Then the following estimates hold:

$$\|u - P_h(u)\|_{H^1_0(\Omega)} = o(1), \quad \text{and}$$

$$\|u - P_h(u)\|_{L^2(\Omega)} = O(h)\|u\|_{H^1_0(\Omega)} \quad \text{as } h \to 0.$$

If $u \in H^1_0(\Omega) \cap H^2(\Omega)$ the following estimates hold:

$$\|u - P_h(u)\|_{H^1(\Omega)} = O(h)\|u\|_{H^2(\Omega)} \quad \text{and} \quad \|u - P_h(u)\|_{L^2(\Omega)} = O(h^2)\|u\|_{H^2(\Omega)} \quad \text{as } h \to 0.$$

If $u \in W^{1,q}_0(\Omega)$ for some $q \geq 2$, the following estimate holds:

$$\|P_h(u)\|_{W^{1,q}(\Omega)} \leq C\|u\|_{W^{1,q}(\Omega)}$$

for some $C > 0$ independent of $h$ (refer to [Theorem 8.5.3] in [3]).

**Proposition A.2** ([9]) Let $\Omega \subset \mathbb{R}^2$ be a bounded convex domain with a polygonal boundary. For a given $f \in L^2(\Omega)$, let $u \in H^1_0(\Omega)$ be a weak solution of the problem

$$-\Delta u = f \quad \text{in } \Omega, \quad u \in H^1_0(\Omega)$$

Then $u$ belongs to $H^2(\Omega)$, and there exists a constant $C > 0$ such that

$$\|u\|_{H^2(\Omega)} \leq C\|f\|_{L^2(\Omega)}.$$
Proposition A.3 \([\{6,11\}]\) Let \(\Omega \subset \mathbb{R}^2\) be a bounded convex domain and \(p \in (2, \infty)\). Let \(U\) be a minimizer of the problem
\[
C(\Omega, p) = \inf \left\{ \frac{\|\nabla u\|_{L^2(\Omega)}}{\|u\|_{L^p(\Omega)}} \mid u \in H^1_0(\Omega), u \neq 0 \right\}
\]
satisfying
\[
-\Delta u = |u|^{p-2}u \text{ in } \Omega. \tag{A.1}
\]
Then, the following holds:

(i) \(U\) is sign definite and unique up to a sign.

(ii) \(U\) is non-degenerate. In other words, the linearized equation of \((A.1)\) at \(U\), i.e.,
\[
\Delta \phi + (p-1)U^{p-2}\phi = 0 \text{ in } \Omega, \quad \phi \in H^1_0(\Omega)
\]
admits only the trivial solution.

(iii) The following inequality
\[
\int_{\Omega} |\nabla \phi|^2 - (p-1)U^{p-2}\phi^2 \, dx \geq C \int_{\Omega} |\nabla \phi|^2 \, dx \tag{A.2}
\]
holds true for any \(\phi \in H^1_0(\Omega)\) that satisfies \(\langle \phi, U \rangle_{H^1_0(\Omega)} = 0\) and some \(C > 0\) independent of \(\phi\).

Remark A.4 The statements (i) and (ii) are proved in [11]. The statement (iii) is a natural consequence of (ii). We refer to [6] for the rigorous arguments of the proof.

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