Automorphism Groups of Compact Complex Surfaces: T-Jordan Property, Tits Alternative and Solvability

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Abstract
Let $X$ be a (smooth) compact complex surface. We show that every torsion subgroup of the biholomorphic automorphism group $\text{Aut}(X)$ is virtually nilpotent. Moreover, we study the Tits alternative of $\text{Aut}(X)$ and virtual derived length of virtually solvable subgroups of $\text{Aut}(X)$.

Keywords
Automorphism group · Compact complex surface · Torsion group · Virtually abelian · Virtually nilpotent · Tits alternative · Virtually solvable · Derived length

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1 Introduction

1.1 T-Jordan Property

The below theorem is our starting point.

**Theorem 1.1** ([20]) Let $G$ be a connected Lie group. Then there is a constant $J = J(G)$ such that every torsion subgroup of $G$ contains a (normal) abelian subgroup of index $\leq J$.

Inspired by the *Jordan property* introduced by Popov [28, Definition 2.1], we propose the following generalisation.

**Definition 1.2** A group $G$ is called *T-Jordan* (alternatively, we say that $G$ has the *T-Jordan* property) if there is a constant $J(G)$ such that every torsion subgroup $H$ of $G$ has an abelian subgroup $H_1$ with the index $[H : H_1] \leq J(G)$.
For a compact complex manifold $X$, we denote by $\text{Aut}(X)$ the group of biholomorphic automorphisms. As in [28], we propose the following conjecture.

**Conjecture 1.3** Let $X$ be a compact complex manifold. Then $\text{Aut}(X)$ is $T$-Jordan.

Using the equivariant Kähler model, Meng and the author gave an affirmative answer to Conjecture 1.3 for compact complex manifolds in Fujiki’s class $C$. A compact complex space is in Fujiki’s class $C$ if it is the meromorphic image of a compact Kähler manifold (cf. [11, Definition 1.1]).

However, we only use Theorem 1.4 below in dimension two and the classical result of Fujiki [11, Theorem 4.8] and Lieberman [21, Proposition 2.2] is enough for its proof.

**Theorem 1.4** (cf. [17, Corollary 1.5]) Let $X$ be a compact complex space which is in Fujiki’s class $C$. Then $\text{Aut}(X)$ is $T$-Jordan.

In this paper, we are focussing on smooth compact complex surfaces, especially the non-Kähler (= non-Fujiki’s class $C$) surfaces. We use the following notation.

- Let $\mathcal{S}$ be the set of smooth compact complex surface $X$ in class VII with the algebraic dimension $a(X) = 0$ and the second Betti number $b_2(X) > 0$.
- Let $\mathcal{S}_0 \subseteq \mathcal{S}$ be those minimal surfaces which have no curve.

Here, surfaces of class VII are those smooth compact complex surfaces with the first Betti number $b_1 = 1$ and Kodaira dimension $\kappa = -\infty$ (cf. [18, §7]). By a minimal surface we mean a smooth compact complex surface that does not contain any $(-1)$-curve $C$ (i.e. a smooth rational curve $C$ with self-intersection $C^2 = -1$). It is known that a smooth compact complex surface $X$ is minimal if and only if any bimeromorphic holomorphic map $X \to X'$ to a smooth compact complex surface $X'$ is an isomorphism.

**Proposition 1.5** Let $X$ be a compact complex surface not in $\mathcal{S}_0$. Then $\text{Aut}(X)$ is $T$-Jordan.

Although we are not able to confirm Conjecture 1.3 for surfaces in $\mathcal{S}_0$, we have a slightly weaker result on torsion subgroups of $\text{Aut}(X)$. Recall that a group $G$ is virtually $\mathcal{P}$, if there exists a finite-index subgroup of $G$ that has the property $\mathcal{P}$. We say that a group $G$ is $\mathcal{P}$-by-$\mathcal{Q}$ if $G$ fits into an short exact sequence

$$1 \to K \to G \to H \to 1$$

where $K$ has the property $\mathcal{P}$ and $H$ has the property $\mathcal{Q}$.

**Proposition 1.6** Let $X$ be a smooth compact complex surface in $\mathcal{S}_0$. Let $G \leq \text{Aut}(X)$ be a torsion subgroup. Then $G$ is virtually abelian.

Our main result below is a direct consequence of Propositions 1.5 and 1.6.

**Theorem 1.7** Let $X$ be a smooth compact complex surface. Then any torsion subgroup $G \leq \text{Aut}(X)$ is virtually abelian.
Note that by the global spherical shell (GSS) conjecture, $E_0 = \emptyset$ (cf. [24, Conjecture 1]).

**Remark 1.8** Assume the global spherical shell conjecture. Let $X$ be a smooth compact complex surface. Then $\text{Aut}(X)$ is T-Jordan.

### 1.2 Tits Alternative

In group theory, the *Tits alternative* is an outstanding theorem about the structure of linear groups (cf. [34]). Campana, Wang and Zhang studied the automorphism groups of compact Kähler manifolds and proved a Tits type alternative for such groups (cf. [5, Theorem 1.5]). We generalise their celebrated result to compact complex spaces in Fujiki’s class $C$ and smooth compact complex surfaces.

**Theorem 1.9** Let $X$ be a compact complex space in Fujiki’s class $C$. Then $\text{Aut}(X)$ satisfies the Tits alternative, that is, for any subgroup $G$ of $\text{Aut}(X)$ either it contains a non-abelian free subgroup, or it is virtually solvable, i.e. admits a solvable subgroup of finite index.

For the definition and properties of Enoki surfaces and Inoue–Hirzebruch surfaces, see [10] and [16], respectively.

**Theorem 1.10** Let $X$ be a smooth compact complex surface. Assume that either $X \notin \Xi$, or $X \in \Xi$ but its minimal model is an Enoki or Inoue–Hirzebruch surface. Then $\text{Aut}(X)$ satisfies the Tits alternative.

### 1.3 Virtual Derived Length

Once we have established the Tits type alternative for $\text{Aut}(X)$, it is natural to study the properties of those virtually solvable subgroups.

We have the following result, which gives a uniform upper bound of the virtual derived length (cf. Definition 2.1) for virtually solvable subgroups of $\text{Aut}(X)$.

**Theorem 1.11** Let $X$ be a smooth compact complex surface. Assume that either $X \notin \Xi$, or $X \in \Xi$ but its minimal model is an Enoki or Inoue–Hirzebruch surface. Let $G \leq \text{Aut}(X)$ be a virtually solvable subgroup. Then the virtual derived length $\ell_{\text{vir}}(G) \leq 4$.

**Remark 1.12** (1) Currently, we are not able to prove Theorems 1.10 and 1.11 in full generality for $X \in \Xi$.

(2) The GSS conjecture claims that any minimal surface in $\Xi$ is a Kato surface (cf. [24, Conjecture 1] and [9, Main Theorem]).

(3) Kato surfaces are divided into four classes: Enoki surfaces (including parabolic Inoue surfaces), half Inoue surfaces, Inoue-Hirzebruch surfaces and intermediate surfaces (cf. [33, §3.3.2]).

(4) Fix $b > 0$. The moduli space of framed Enoki surfaces with $b_2 = b$ is an open subspace of the moduli space of framed Kato surfaces with $b_2 = b$ (cf. [8, Remark, pp. 54] and [33, §3.3.4.1]).
(5) When $X$ is a parabolic Inoue surface (which is in $\Xi$), it has been proved that Aut$(X)$ is virtually abelian (cf. [12, Theorem 1.1]). In particular, Aut$(X)$ satisfies the Tits alternative and any subgroup $G$ of Aut$(X)$ is virtually solvable with virtual derived length $\ell_{\text{vir}}(G) \leq 1$.

2 Preliminaries

In this section, we gather some general preliminary results.

**Definition 2.1** Recall that for a group $G$, its $n$-th derived subgroup is defined recursively by

$$G^{(0)} := G \quad \text{and} \quad G^{(n)} := [G^{(n-1)}, G^{(n-1)}], \quad n \in \mathbb{N}.$$ 

By definition, the group $G$ is solvable if and only if $G^{(n)} = 1$ for some non-negative integer $n$. The minimum of such $n$ is the derived length of $G$ (when $G$ is solvable), denoted by $\ell(G)$.

If $G$ is virtually solvable, we define the virtual derived length to be

$$\ell_{\text{vir}}(G) := \min_{G'} \ell(G')$$

where $G'$ runs through all finite-index subgroups of $G$.

A group $G$ has bounded torsion subgroups if there is a constant $T = T(G)$ such that any torsion subgroup $G_1 \leq G$ has order $|G_1| \leq T$.

**Lemma 2.2** Let

$$1 \longrightarrow N \longrightarrow G \longrightarrow H$$

be an exact sequence of groups. Then the following assertions hold:

(1) If $N$ is T-Jordan and $H$ has bounded torsion subgroups, then $G$ is T-Jordan.

(2) Assume further that the sequence is also right exact. If $N$ is a torsion group and $G$ is T-Jordan, then $H$ is T-Jordan.

**Proof** (1) is clear.

(2) Let $H' \leq H$ be a torsion subgroup, and $G' \leq G$ be its preimage. Since $N$ is a torsion group, $G'$ is also a torsion group and hence there exists an abelian subgroup $G'_1 \leq G'$ such that $[G' : G'_1] \leq J(G)$ for some constant $J(G)$. Let $H'_1 = G'_1/(G'_1 \cap N) \leq H'$, which is abelian with $[H' : H'_1] \leq J(G)$. Therefore, $H$ is T-Jordan.

The following two lemmas are adapted from [31, Corollaries 4.2, 4.4]. Note that every element of a torsion group has finite order.
Lemma 2.3 Let $X$ be an irreducible Hausdorff reduced complex space, and let $\Gamma \leq \text{Aut}(X)$ be a torsion subgroup. Suppose that $\Gamma$ has a fixed point $x$ on $X$. Then the natural representation

$$\Gamma \longrightarrow \text{GL}(T_x, X)$$

is faithful.

Lemma 2.4 Let $X$ be an irreducible Hausdorff reduced complex space, and let $\Delta \leq \text{Aut}(X)$ be a subgroup. Suppose that $\Delta$ has a fixed point $x$ on $X$, and let

$$\sigma : \Delta \longrightarrow \text{GL}(T_x, X)$$

be the natural representation. Suppose that there is a normal subgroup $\Gamma \leq \Delta$ with the quotient group $\Delta / \Gamma$ being torsion, such that the restriction $\sigma|_{\Gamma}$ is a group monomorphism. Then $\sigma$ is an embedding as well.

Proof Let $\Delta_0$ be the kernel of $\sigma$. Since $\Delta_0 \cap \Gamma = \{\text{id}\}$ and $\Delta / \Gamma$ is torsion, we see that $\Delta_0$ is also torsion. Thus, $\Delta_0$ is trivial by Lemma 2.3. \(\square\)

Lemma 2.5 Let $G \leq \text{GL}_n(\mathbb{C})$ be a virtually solvable subgroup. Then $\ell_{\text{vir}}(G) \leq n$

Proof By passing to a finite-index subgroup, we may assume that $G$ itself is solvable. Then some subgroup $G_1$ of finite index in $G$ can be put in triangular form (cf. [4, Corollary, pp. 137]). It follows that the derived length of $G_1$ and hence the virtual derived length of $G$ is at most $n$. \(\square\)

Lemma 2.6 Consider the short exact sequence of groups

$$1 \longrightarrow N \longrightarrow G \longrightarrow H \longrightarrow 1.$$

Then

(1) if $N$ is solvable and $H$ is virtually solvable, then $G$ is virtually solvable with $\ell_{\text{vir}}(G) \leq \ell(N) + \ell_{\text{vir}}(H)$.
(2) if $N$ is finite and $H$ is virtually solvable, then $G$ is virtually solvable with $\ell_{\text{vir}}(G) \leq \ell_{\text{vir}}(H) + 1$;
(3) the group $G$ is virtually solvable if and only if both $N$ and $H$ are virtually solvable;
(4) if both $N$ and $H$ satisfy the Tits alternative, then so does $G$.

Proof (1) Let $H' \leq H$ be a solvable subgroup of finite index (with $\ell(H') = \ell_{\text{vir}}(H)$), and let $G' \leq G$ be its preimage. Since $[G : G'] = [G/N : G'/N] = [H : H'] < \infty$, we may replace $G$, $H$ by $G'$, $H'$ and assume that $H$ is solvable. Then it is clear that $G$ is solvable, and $\ell_{\text{vir}}(G) \leq \ell(G) \leq \ell(N) + \ell(H) \leq \ell(N) + \ell_{\text{vir}}(H)$.

(2) As in (1), we may assume that $H$ is solvable. Let $Z_G(N)$ be the centraliser of $N$ in $G$. Since $N$ is finite, $Z_G(N) \leq H$ is of finite index. Note that $Z_G(N) \cap N$ is abelian (in particular, solvable) and $Z_G(N)/(Z_G(N) \cap N) \leq G/N \simeq H$ is solvable.
Hence, $Z_G(N)$ is also solvable with derived length $\ell(Z_G(N)) \leq \ell(H) + 1$. It follows that $\ell_{\text{vir}}(G) \leq \ell_{\text{vir}}(H) + 1$.

(3) The “only if” direction is clear. Suppose that $N$ and $H$ are virtually solvable. As in (2), we may assume that $H$ is solvable. Let $N' \triangleleft N$ be a finite-index solvable (normal) subgroup of minimal index.

Then $N'$ is normal in $G$; otherwise, if $N''$ is another conjugate of $N'$, then $N'N'' \triangleleft N$ is solvable with smaller index, a contradiction to the choice of $N'$. Consider the following short exact sequence

$$1 \rightarrow N/N' \rightarrow G/N' \rightarrow G/N \simeq H \rightarrow 1.$$ 

By assumption, the first term is finite and $H$ is solvable. Thus, $G/N'$ is virtually solvable by (2). Then using the short exact sequence

$$1 \rightarrow N' \rightarrow G \rightarrow G/N' \rightarrow 1,$$

we obtain that $G$ is virtually solvable by (1).

(4) This follows from (3). Note that if $H$ contains a copy of $\mathbb{Z} \ast \mathbb{Z}$, a non-abelian, free group with two generators, then $G$ contains another copy of $\mathbb{Z} \ast \mathbb{Z}$ as well.

\[\square\]

Using the lemma above, we can give a proof of Theorem 1.9.

**Proof of Theorem 1.9** Let $\pi : X' \rightarrow X$ be an Aut($X$)-equivariant resolution of singularities (cf. [2, Theorem 13.2(2)(ii)]) with Aut($X$) lifts to a (unique) subgroup of Aut($X'$) via $\pi$. Note that $X'$ is still in Fujiki’s class $C$. Therefore, we may replace $X$ by $X'$ and assume that $X$ is smooth.

Denote by

$$\text{Aut}_\tau(X) := \{g \in \text{Aut}(X) \mid g^*|_{H^2(X, \mathbb{Q})} = \text{id}\}.$$ 

Then we have the following short exact sequence:

$$1 \rightarrow G_\tau \rightarrow G \rightarrow G|_{H^2(X, \mathbb{Q})} \rightarrow 1$$

where $G_\tau = G \cap \text{Aut}_\tau(X)$. It suffices to prove that $G_\tau$ satisfies the Tits alternative by [34, Theorem 1] and Lemma 2.6 (4). Let $\sigma : \widetilde{X} \rightarrow X$ be a bimeromorphic holomorphic map from a compact Kähler manifold $\widetilde{X}$ such that $\text{Aut}_\tau(X)$ lifts to $\widetilde{X}$ holomorphically via $\sigma$ (cf. [17, Theorem 1.1]). Viewing $G_\tau \leq \text{Aut}_\tau(X)$ as a subgroup of Aut($\widetilde{X}$), we know that $G_\tau$ satisfies the Tits alternative (cf. [5, Theorem 1.5]). Therefore, $G$ satisfies the Tits alternative. \[\square\]

The theorem below is due to [27, §2, Theorem].

**Theorem 2.7** Let $X$ be an Inoue–Hirzebruch surface. Then Aut($X$) is finite. In particular, Aut($X$) satisfies the Tits alternative and every subgroup $G \leq \text{Aut}(X)$ is finite and hence virtually solvable with virtual derived length $\ell_{\text{vir}}(G) = 0$.

The following lemmas will be used in Sect. 8.
Lemma 2.8 (cf. [13, §2, 4]) Let $G$ be a finitely generated group. Then every finite-index subgroup of $G$ contains a finite-index subgroup $G'$ which is characteristic in $G$.

Lemma 2.9 Let $G$ be a solvable group and $U \triangleleft G$ a non-trivial unipotent (normal) subgroup. Then, after replacing $G$ by a finite-index subgroup, there is a subgroup $Z_1 \cong \mathbb{G}_a \leq U$ which is normal in $G$.

Proof Let $Z = Z(U)$ be the centre of $U$ which is non-trivial and normal in $G$. Note that the commutative unipotent group $Z$ is isomorphic to $\mathbb{G}_a^n$, the additive affine algebraic group of dimension $n$ for some $n \geq 1$. Consider the conjugation action of $G$ on $Z$:

$$c_g : x \mapsto g x g^{-1} \text{ for } x \in Z \text{ and } g \in G.$$ 

This conjugation action $c_g$ is linear: for a fixed $x \in Z \cong \mathbb{G}_a^n$, the morphism $z \mapsto f(z) := c_g(z.x) - z.c_g(x)$ is trivial for all integer points $z \in \mathbb{G}_a$, so $f(z) = 0$ for all points $z \in \mathbb{G}_a$.

Therefore, one has an induced a homomorphism $G \rightarrow \text{GL}_n(\mathbb{C})$. Replacing $G$ by a finite-index subgroup, we may assume that the image $G'$ of the solvable group $G$ in $\text{GL}_n(\mathbb{C})$ is of triangular form (cf. [4, Corollary, pp. 137]). Let $Z_1 \cong \mathbb{G}_a$ be an (1-dimensional) eigenspace of $G'$. Then $Z_1$ is stable under the conjugation action of $G$ and hence is normal in $G$. 

3 Equivariant Fibrations

In this section, we will consider smooth compact complex surfaces $X$ with algebraic dimension $a(X) = 1$, or with Kodaira dimension $\kappa(X) = 1$. If $a(X) = 1$, there is a natural elliptic fibration, the algebraic reduction $\pi : X \rightarrow Y$ (cf. [1, VI. (5.1) Proposition]), which is a holomorphic map. Here, by an elliptic fibration, we mean a surjective morphism between two compact complex spaces with connected fibre, such that a general fibre is isomorphic to a smooth elliptic curve. For $g \in \text{Aut}(X)$, the image of a fibre $F$ of $\pi$ under $g$ is another fibre; otherwise the self-intersection number of $g(F) + F$ is positive and hence $X$ is projective (cf. [1, IV, Theorem 6.2]), a contradiction (cf. [1, IV, Corollary 6.5]). Thus, $\pi$ is $\text{Aut}(X)$-equivariant. When $\kappa(X) = 1$, the pluricanonical fibration gives a natural elliptic fibration as well, which is also $\text{Aut}(X)$-equivariant.

In addition to the above, we will also study affine $\mathbb{A}^1$-bundle $\pi : A \rightarrow E$ over an elliptic curve $E$. Note that the fibre of $\pi : A \rightarrow E$ being an affine line $\mathbb{A}^1$ cannot dominate the base elliptic fibration $E$. It follows that $\pi$ is $\text{Aut}(A)$-equivariant.

Let $\pi : X \rightarrow Y$ be a morphism between complex spaces. Let $G$ be a complex Lie group contained in $\text{Aut}(X)$ such that $\pi \circ g = \pi$ for all $g \in G$, i.e. $G \leq \text{Aut}_Y(X)$. For any finite subset $S \subseteq Y$, denote by $G_S := \{g \in G \mid g|_{\pi^{-1}(S)} = \text{id}\}$, i.e. $G_S$ is the kernel of the natural group homomorphism $G \rightarrow \prod_{y \in S} \text{Aut}(X_y)$. Note that any Lie group is second countable and hence has only countably many connected components.

The ideas of Lemma 3.1 and Proposition 3.2 come from Sheng Meng.

Lemma 3.1 Suppose that $\dim G = 0$. Then $G_y$ is trivial for very general $y \in Y$. 

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Proof Let \( Y_g := \{ y \in Y \mid g|_{x_y} = \text{id} \} \). Then \( Y_g \) is a proper closed subspace of \( Y \) for \( g \neq \text{id} \). Let \( Z := \bigcup_{\text{id} \neq g \in G} Y_g \). Then any \( y \in Y \setminus Z \) is a very general choice. Here we use the condition that \( G \) is countable since \( \dim G = 0 \).

A very general finite subset \( S \subseteq Y \) means that every element of \( S \) is very generally chosen in \( Y \).

**Proposition 3.2** For very general finite subset \( S \subseteq Y \), \( G_S \) is trivial.

**Proof** First, we show by induction on \( \dim G \) that \( \dim G_S = 0 \) for very general finite subset \( S \subseteq Y \). When \( \dim G = 0 \), the claim follows from Lemma 3.1. Now assume that \( \dim G \geq 1 \). Denote by \( T := \{ y \in Y \mid G_0|_{X_y} = \text{id} \} \), where \( G_0 \) is the neutral component of \( G \). Note that \( T \subseteq Y \) is a closed subspace. Then for \( y \notin T \), we have \( \dim G_y < \dim G \). By induction, \( \dim(G_y)_S = 0 \) for any \( y \notin T \) and very general finite subset \( S \subseteq Y \) (depending on \( y \)). Note that \( (G_y)_S = G_{\{y\} \cup S} \). So the claim is proved.

Then by Lemma 3.1, we see that \( G_{S \cup \{y\}} = (G_S)_y \) is trivial for very general \( y \in Y \).

**Lemma 3.3** Let \( X \) be either a (primary or secondary) Kodaira surface, or a minimal smooth complex surface of Kodaira dimension 1. Then there is an \( \text{Aut}(X) \)-equivariant elliptic fibration \( \pi : X \longrightarrow Y \) to a smooth curve \( Y \). Also, the natural image of \( \text{Aut}(X) \) in \( \text{Aut}(Y) \) is finite.

**Proof** We may take \( \pi \) to be either the algebraic reduction or the pluricanonical fibration (see the first paragraph of this section). Then our claim follows from [32, Corollary 3.4, Lemma 3.5] and [30, Proposition 1.2].

**Theorem 3.4** Let \( X \) be either a (primary or secondary) Kodaira surface or a minimal smooth complex surface of Kodaira dimension 1. Then \( \text{Aut}(X) \) is virtually abelian. In particular, \( \text{Aut}(X) \) satisfies T-Jordan property, the Tits alternative and every subgroup \( G \leq \text{Aut}(X) \) is virtually abelian and hence virtually solvable with virtual derived length \( \ell_{\text{vir}}(G) \leq 1 \).

**Proof** By Lemma 3.3, there is a short exact sequence

\[
1 \longrightarrow \text{Aut}_Y(X) \longrightarrow \text{Aut}(X) \longrightarrow \Gamma \longrightarrow 1,\]

where \( \pi : X \longrightarrow Y \) is an elliptic fibration and \( \Gamma \) is a finite subgroup of \( \text{Aut}(Y) \). By Proposition 3.2, there exists a finite subset \( S \) of \( Y \) such that the natural map \( \text{Aut}_Y(X) \hookrightarrow \prod_{y \in S} \text{Aut}(X_y) \) is an injection. Since \( X_y \) is an elliptic curve, \( \prod_{y \in S} \text{Aut}(X_y) \) and hence \( \text{Aut}_Y(X) \) are virtually abelian. Then it is clear that \( \text{Aut}(X) \) is virtually abelian.

**Theorem 3.5** Let \( X \in \Xi \) be an Enoki surface. Then \( \text{Aut}(X) \) is virtually solvable (and hence \( \text{Aut}(X) \) satisfies the Tits alternative) and every subgroup \( G \leq \text{Aut}(X) \) is virtually solvable with \( \ell_{\text{vir}}(G) \leq 3 \).

In the proof below, by a cycle of rational curves, we mean a reduced (singular) complex curve with only nodal singularities such that its dual graph is a cycle and each component of its normalisation is a smooth rational curve.

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Proof By [10, Main Theorem], $X$ has a cycle of rational curves $C$ such that $A := X \setminus C$, the complement of $C$ in $X$, is an affine $\mathbb{A}^1$-bundle $\pi : A \rightarrow E$ over an elliptic curve $E$. Since the algebraic dimension $a(X)$ is 0, the surface $X$ contains finitely many (irreducible) curves (cf. [1, IV. Theorem 8.2]). Replacing $\text{Aut}(X)$ by a finite-index subgroup, we may assume that $C$ is $\text{Aut}(X)$-invariant. Then $\text{Aut}(X)$ acts on $A = X \setminus C$ biholomorphically and faithfully and $\pi$ is $\text{Aut}(X)$-equivariant (see the second paragraph of this section).

Consider the short exact sequence

$$1 \rightarrow K \rightarrow \text{Aut}(X) \rightarrow \Gamma \rightarrow 1,$$

where $K \leq \text{Aut}_E(A)$ and $\Gamma \leq \text{Aut}(E)$. Since $E$ is an elliptic curve, $\text{Aut}(E)$ and hence $\Gamma$ are virtually abelian, or equivalently, virtually solvable with virtual derived length $\leq 1$. By Proposition 3.2, there exists a finite subset $S$ of $E$ such that the natural map $K \hookrightarrow \prod_{e \in S} \text{Aut}(A_e)$ is an injection. Since $A_e \cong \mathbb{A}^1$, the group $\prod_{e \in S} \text{Aut}(A_e)$ and hence $K$ are metabelian, or equivalently, solvable with derived length $\leq 2$. By Lemma 2.6 (1), $\text{Aut}(X)$ is virtually solvable with virtual derived length $\leq 3$. It is clear that any subgroup $G \leq \text{Aut}(X)$ is also virtually solvable and its virtual derived length $\ell_{\text{vir}}(G) \leq 3$.

4 Surfaces with Invariant Curves

\textbf{Lemma 4.1} Let $X$ be a smooth compact complex surface. Suppose that there is a finite non-empty $\text{Aut}(X)$-invariant set $\Sigma$ of (irreducible) curves on $X$. Then $\text{Aut}(X)$ is T-Jordan.

\textbf{Proof} Let $C$ be one of the curves from $\Sigma$. Then the group $\text{Aut}(X, C)$ of automorphisms of $X$ that preserves the curve $C$ has finite index in $\text{Aut}(X)$. We only need to show that $\text{Aut}(X, C)$ is T-Jordan.

Assume first that $C$ is singular. Then $\text{Aut}(X, C)$, after replacing by a subgroup of finite index, fixes some point on $C$. Now Lemma 2.3 implies that $\text{Aut}(X, C)$ is embedded into $\text{GL}_2(\mathbb{C})$. Therefore, the group $\text{Aut}(X, C)$ is T-Jordan (cf. Theorem 1.1).

Now we assume that $C$ is smooth and let $G \leq \text{Aut}(X, C)$ be a torsion subgroup. Let $N_{C/X}$ be the normal bundle of $C$ in $X$. Consider the natural group homomorphism

$$N : \text{Aut}(X, C) \rightarrow \text{Aut}(N_{C/X} \rightarrow C)$$

via $\sigma \mapsto N_\sigma$ (cf. [23, Notation 2.2]). By [23, Lemma 2.3], $\ker N \cap G = \{\text{id}\}$. Moreover, by the proof of [23, Theorem 3.1], there is a monomorphism

$$\text{Aut}(N_{C/X} \rightarrow C) \rightarrow \text{Aut}(\mathbb{P}_C(N_{C/X} \oplus \mathcal{O})).$$

So we may view $G$ as a subgroup of $\text{Aut}(\mathbb{P}_C(N_{C/X} \oplus \mathcal{O}))$. Note that $C$ is projective. So $\mathbb{P}_C(N_{C/X} \oplus \mathcal{O})$ is also a projective manifold. Then the theorem follows from Theorem 1.4. \qed

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Corollary 4.2 Let $X$ be a compact complex surface with $a(X) = 0$. Assume that $X$ contains at least one curve. Then $\text{Aut}(X)$ is $T$-Jordan.

Proof It follows from Lemma 4.1, since $X$ contains at most a finite number of (irreducible) curves (cf. [1, IV. Theorem 8.2]).

Proposition 4.3 Let $X$ be a compact complex surface with $e(X) \neq 0$ and $a(X) = 1$. Then $\text{Aut}(X)$ is $T$-Jordan.

Proof Let $\pi : X \rightarrow Y$ be the algebraic reduction (cf. Section 3), so that $Y$ is a smooth curve and $\pi$ is an $\text{Aut}(X)$-equivariant elliptic fibration. Since $e(X) \neq 0$, the fibration $\pi$ has at least one fibre $X_y$ such that $F = (X_y)_{\text{red}}$ is not a smooth elliptic curve by Suzuki’s formula (cf. [1, III, Proposition 11.4]). Note that $\pi$ has only finitely many singular fibres, whose union is $\text{Aut}(X)$-invariant. Then the assertion follows from Lemma 4.1.

5 Hopf Surfaces

In this section, we study the automorphism groups of Hopf surfaces. By definition, a Hopf surface $X$ is a smooth compact complex surface with universal covering being isomorphic to $\mathbb{C}^2 \setminus \{0\}$. Then, $X$ can be obtained from $\mathbb{C}^2 \setminus \{0\}$ as a quotient by a free action of a discrete group $\Gamma \simeq \pi_1(X)$. If $\pi_1(X) \simeq \mathbb{Z}$, we call such $X$ a primary Hopf surface. In this case, after an appropriate choice of coordinates of $\mathbb{C}^2$, the generator of $\Gamma$ has the form

$$ (z, w) \mapsto (az + \lambda w^m, bw), \quad (5.1) $$

where $m \in \mathbb{Z}_{>0}$ and $a, b, \lambda$ are complex constants subject to the restrictions

$$ (a - b^m)\lambda = 0, \quad 0 < |a| \leq |b| < 1. $$

Any Hopf surface is either primary or a finite étale quotient of a primary Hopf surface, where the latter is called the secondary Hopf surface (cf. [19, Theorem 30]). By the above, $\Gamma$ contains $\Lambda \simeq \mathbb{Z}$ as a subgroup of finite index. In particular, $\Gamma$ is virtually solvable. After replacing $\Lambda$ by a suitable subgroup $\Lambda_0 \simeq \mathbb{Z}$, we may assume that $\Lambda$ is a characteristic subgroup of $\Gamma$ (cf. [31, Lemma 2.10]), with the generator of $\Lambda$ having the form Eq.(5.1). Now there is a short exact sequence

$$ 1 \rightarrow \Gamma \rightarrow \widetilde{\text{Aut}}(X) \rightarrow \text{Aut}(X) \rightarrow 1 \quad (5.2) $$

where $\widetilde{\text{Aut}}(X)$ acts on $\mathbb{C}^2 \setminus \{0\}$ biholomorphically. It follows from the Hartogs extension theorem that $\widetilde{\text{Aut}}(X)$ can be extended to biholomorphic actions on $\mathbb{C}^2$ fixing the origin $0 \in \mathbb{C}$. By assumption, $\Lambda \triangleleft \widetilde{\text{Aut}}(X)$ is a normal subgroup.

Theorem 5.1 Let $X$ be a Hopf surface. Then the group $\text{Aut}(X)$ satisfies the Tits alternative.
Proof Assume first that \( X \) is a primary Hopf surface. Then we may identify \( \tilde{\text{Aut}}(X) \) with a subgroup of \( \text{GL}_2(\mathbb{C}) \times \mathbb{C} \) by [25, § 2] and [35, pp. 24]. Since \( \text{GL}_2(\mathbb{C}) \times \mathbb{C} \) satisfies the Tits alternative, so do \( \tilde{\text{Aut}}(X) \) and \( \text{Aut}(X) \) (cf. Lemma 2.6 (4)).

Now let \( X \) be a secondary Hopf surface. Then either \( \tilde{\text{Aut}}(X) \leq \text{GL}_2(\mathbb{C}) \), or \( \tilde{\text{Aut}}(X) \simeq \mathbb{C} \times \mathbb{C}^* \) or \( \tilde{\text{Aut}}(X) \simeq \mathbb{C} \times (\mathbb{C}^*)^2 \) (cf. [22, Theorem 1]). In the first case, \( \tilde{\text{Aut}}(X) \) and hence \( \text{Aut}(X) \) satisfies the Tits alternative as \( \text{GL}_2(\mathbb{C}) \) does; in the latter two cases, \( \tilde{\text{Aut}}(X) \) and hence \( \text{Aut}(X) \) are already solvable. In a word, \( \text{Aut}(X) \) satisfies the Tits alternative.

\[ \square \]

Theorem 5.2 Let \( X \) be a Hopf surface and \( G \leq \text{Aut}(X) \) a virtually solvable subgroup. Then \( \ell_{\text{vir}}(G) \leq 2 \).

Proof Let \( \tilde{G} \leq \tilde{\text{Aut}}(X) \) be the preimage of \( G \). Then \( \tilde{G} \) is virtually solvable by Lemma 2.6 (3) as \( \Gamma \) is virtually solvable. Therefore, \( \ell_{\text{vir}}(G) \leq \ell_{\text{vir}}(\tilde{G}) \leq \ell_{\text{vir}}(\text{Aut}(X)) \leq 2 \) by Lemma 2.5 and proof of Theorem 5.1. \( \square \)

The lemma below is a simple linear algebra, which is taken from [31, Lemma 6.3].

Lemma 5.3 Let

\[ M = \begin{pmatrix} a & \lambda \\ 0 & b \end{pmatrix} \in \text{GL}_2(\mathbb{C}) \]

be an upper triangular matrix, and \( Z \leq \text{GL}_2(\mathbb{C}) \) the centraliser of \( M \). Then the following assertions hold:

(i) If \( a = b \) and \( \lambda = 0 \), then \( Z = \text{GL}_2(\mathbb{C}) \).

(ii) If \( a \neq b \) and \( \lambda = 0 \), then \( Z \simeq (\mathbb{C}^*)^2 \).

(iii) If \( a = b \) and \( \lambda \neq 0 \), then \( Z \simeq \mathbb{C}^* \times \mathbb{C}^+ \).

Theorem 5.4 Let \( X \) be a Hopf surface. Then \( \text{Aut}(X) \) is T-Jordan.

Proof The proof is adapted from [31, Lemma 6.4].

Consider the short exact sequence Eq. (5.2). The image of the generator of \( \Lambda \) is mapped by the natural homomorphism

\[ \sigma : \tilde{\text{Aut}}(X) \rightarrow \text{GL}(T_{0,\mathbb{C}^2}) \simeq \text{GL}_2(\mathbb{C}) \]

to the matrix

\[ M = \begin{pmatrix} a & \lambda \delta^m \\ 0 & b \end{pmatrix} \]

where \( \delta \) is the Kronecker symbol.

Let \( G \leq \text{Aut}(X) \) be a torsion subgroup, and \( \tilde{G} \) its preimage in \( \tilde{\text{Aut}}(X) \). Thus, one has \( G \simeq \tilde{G}/\Gamma \). By Lemma 2.4, \( \sigma|_{\tilde{G}} \) is an embedding since \( \sigma_{\Lambda} \) is a group monomorphism and \( \tilde{G}/\Lambda \) is torsion. Let \( \Omega \) be the normaliser of \( \sigma(\Lambda) \) in \( \text{GL}_2(\mathbb{C}) \). By construction \( \sigma(\tilde{G}) \) is contained in the normaliser of \( \sigma(\Gamma) \) in \( \text{GL}_2(\mathbb{C}) \), which in turn is contained in \( \Omega \) because \( \Lambda \) is a characteristic subgroup of \( \Gamma \). Hence, every torsion subgroup of

\[ \square \]
\textbf{6 Inoue Surfaces}

An Inoue surface \(X\) is a compact complex surface obtained from \(W := \mathbb{H} \times \mathbb{C}\) as a quotient by an infinite discrete group, where \(\mathbb{H}\) is the upper half complex plane. Inoue surfaces are minimal surfaces in class VII, contain no curve, and have the following numerical invariants:

\[ a(X) = 0, \quad b_1(X) = 1, \quad b_2(X) = 0. \]

There are three families of Inoue surfaces: \(S_M\), \(S^{(+)}\), and \(S^{(-)}\) (cf. \([15]\)), and we will study their automorphisms separately.

Since every holomorphic map from \(\mathbb{C}\) to \(\mathbb{H}\) is constant, any automorphism \(u\) of \(W\) has the form

\[ u(w, z) = (s(w), t(w, z)) \]

(6.1)

where

\[ s(w) = \frac{aw + b}{cw + d}, \quad a, b, c, d \in \mathbb{R} \text{ and } ad - bc > 0. \]

We may expand \(t\) in the power series of \(w\) and \(z\) at \((\sqrt{-1}, 0)\):

\[ t(w, z) = \sum_{p \geq 0} C_{p,0}(w - \sqrt{-1})^p + C_{p,1}(w - \sqrt{-1})^p z. \]

\textbf{6.1 Type} \(S_M\)

Let \(M = (m_{i,j}) \in \text{SL}_3(\mathbb{Z})\) be a matrix with eigenvalues \(\alpha, \beta, \overline{\beta}\) such that \(\alpha > 1\) and \(\beta \neq \overline{\beta}\). Take \((a_1, a_2, a_3)^T\) to be a real eigenvector of \(M\) corresponding to \(\alpha\), and \((b_1, b_2, b_3)^T\) an eigenvector corresponding to \(\beta\). Let \(G_M\) be the group of automorphisms of \(W\) generated by
\[ g_0(w, z) = (\alpha w, \beta z), \]
\[ g_i(w, z) = (w + a_i, z + b_i), \quad i = 1, 2, 3, \]

which satisfy these conditions
\[ g_0 g_i g_0^{-1} = g_1^{m_{1,1}} g_2^{m_{1,2}} g_3^{m_{1,3}}, \]
\[ g_i g_j = g_j g_i, \quad i, j = 1, 2, 3. \]

It can be shown that the action of \( G_M \) on \( W \) is free and properly discontinuous. The quotient \( X := W/G_M \) is an Inoue surface of type \( S_M \). Thus, there is a short exact sequence of groups
\[ 1 \longrightarrow G_M \longrightarrow \tilde{\text{Aut}}(X) \longrightarrow \text{Aut}(X) \longrightarrow 1 \]

where \( \tilde{\text{Aut}}(X) \) acts biholomorphically on \( W \), which is the normaliser of \( G_M \) in \( \text{Aut}(W) \).

We will study \( \tilde{\text{Aut}}(X) \) in detail. Assume that \( u \in \tilde{\text{Aut}}(X) \). Since \( G_M \triangleleft \tilde{\text{Aut}}(X) \), we have \( u g u^{-1} \in G_M \) for all \( g \in G_M \). Indeed, we only need to verify this for the generators of \( G_M \).

By the commutative relations of \( G_M \), we may assume that
\[ u g_i u^{-1} = g_1^{n_{i,1}} g_2^{n_{i,2}} g_3^{n_{i,3}} g_0^{k_i}, \quad 0 \leq i \leq 3 \quad (6.2) \]
for some \( k_i \in \mathbb{Z} \), \( n = (n_{0,1}, n_{0,2}, n_{0,3}) \in \mathbb{Z}^3 \) and \( N := (n_{i,j})_{1 \leq i, j \leq 3} \in \text{Mat}_{3 \times 3}(\mathbb{Z}) \).

Then
\[ g_1^{n_{i,1}} g_2^{n_{i,2}} g_3^{n_{i,3}} g_0^{k_i} u(w, z) = (\alpha^{k_i} s(w) + \sum_{j=1}^{3} n_{i,j} a_j, \beta^{k_i} t(w, z) + \sum_{j=1}^{3} n_{i,j} b_j). \]

Note that
\[ u g_0(w, z) = u(\alpha w, \beta z) = (s(\alpha w), t(\alpha w, \beta z)), \]
\[ u g_i(w, z) = u(w + a_i, z + b_i) = (s(w + a_i), t(z + b_i)). \]

Then, Eq. (6.2) implies
\[ s(\alpha w) = \alpha^{k_0} s(w) + \sum_j n_{0,j} a_j, \quad (6.3) \]
\[ t(\alpha w, \beta z) = \beta^{k_0} t(w, z) + \sum_j n_{0,j} b_j, \quad (6.4) \]
\[ s(w + a_i) = \alpha^{k_i} s(w) + \sum_j n_{i,j} a_j, \quad (6.5) \]
\[ t(z + b_i) = \beta^{k_i} t(w, z) + \sum_j n_{i,j} b_j. \quad (6.6) \]
By Eq. (6.3), we have

\[
\begin{align*}
ac(1 - \alpha^{k_0}) &= c^2 \sum_j n_{0,j} a_j, \\
ad(\alpha - \alpha^{k_0}) + bc(1 - \alpha^{k_0+1}) &= cd(1 + \alpha) \sum_j n_{0,j} a_j, \\
b(1 - \alpha^{k_0}) &= d^2 \sum_j n_{0,j} a_j.
\end{align*}
\] (6.7)

If \(cd \neq 0\), then \(a(1 - \alpha^{k_0}) = c \sum_j n_{0,j} a_j\) and \(b(1 - \alpha^{k_0}) = d \sum_j n_{0,j} a_j\) by the first and the third equalities of Eq. (6.7). Using the fact that \(ad - bc > 0\), the second equality of Eq. (6.7) implies \(k_0 = 0\) and \(\sum_j n_{0,j} a_j = 0\), which contradicts the middle equality above. Therefore, either \(c = 0\) or \(d = 0\).

From Eq. (6.5) one deduces that

\[
\begin{align*}
ac(1 - \alpha^{k_i}) &= c^2 \sum_j n_{i,j} a_j, \\
(ac a_i + ad + bc)(1 - \alpha^{k_i}) &= (c^2 a_i + 2cd) \sum_j n_{i,j} a_j, \quad \text{for all } i. \\
(ad - bc \alpha^{k_i})a_i + bd(1 - \alpha^{k_i}) &= (cd a_i + d^2) \sum_j n_{i,j} a_j.
\end{align*}
\] (6.8)

If \(d = 0\), then \(bc \alpha^{k_i} a_i = 0\) for all \(i\) by the last equality of Eq. (6.8). Since \(bc \neq 0\) now and \(\alpha > 1\), the third equality of Eq. (6.8) implies \(a_i = 0\) for all \(i\), which contradicts the assumption that \((a_1, a_2, a_3)\) is an eigenvector of \(M\). Hence, \(c = 0\) and we may rewrite \(s\) as \(s(w) = aw + b\) with \(a > 0\) and \(b \in \mathbb{R}\) (i.e. \(d = 1\)). Then, by Eq. (6.7), \(k_0 = 1\) and \((1 - \alpha)b = \sum_j n_{0,j} a_j\) and Eq. (6.8) gives \(k_i = 0\) and \(a a_i = \sum_j n_{i,j} a_j\).

Comparing the coefficients of \(z\) in Eq. (6.4), we have

\[
\sum_p C_{p,0}(\alpha w - \sqrt{-1})^p = \beta \sum_p C_{p,0}(w - \sqrt{-1})^p + \sum_j n_{0,j} b_j, \\
\beta \sum_p C_{p,1}(\alpha w - \sqrt{-1})^p = \beta \sum_p C_{p,1}(w - \sqrt{-1})^p.
\] (6.9)

It follows that \(C_{p,0} = C_{p,1} = 0\) for all \(p \geq 1\). Thus, we may rewrite \(t\) as \(t(w, z) = Az + B\) for some \(A \neq 0, B \in \mathbb{C}\). Then Eq. (6.9) becomes \((1 - \beta)B = \sum_j n_{0,j} b_j\), and Eq. (6.6) gives \(Ab_i = \sum_j n_{i,j} b_j\).

We conclude that \(u(w, z) = (aw + b, Az + B)\) where \(a, b, A, B\) satisfy

\[
n \cdot \begin{pmatrix} a_1 b_1 \\ a_2 b_2 \\ a_3 b_3 \end{pmatrix} = ((1 - \alpha)b (1 - \beta)B)
\]

and

\[
N \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = A \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}.
\] (6.10)
It follows that $M$ and $N$ are simultaneously diagonalizable. So $M$ and $N$ commute. Moreover, the matrix associated to $u^{-1}$ is $N^{-1} \in \text{Mat}_{3 \times 3}(\mathbb{Z})$, so $\det N = \pm 1$ and $N \in \text{GL}_3(\mathbb{Z})$.

Consider the following subgroups of $\tilde{\text{Aut}}(X)$:

$$K = \left\{ (w, z) \mapsto \left( w + \frac{1}{1 - \alpha} \sum_{j=1}^{3} n_i a_i, z + \frac{1}{1 - \beta} \sum_{j=1}^{3} n_i b_i \right) \mid n_i \in \mathbb{Z} \right\} \cong \mathbb{Z}^3$$

and

$$\Gamma = \left\{ N \in \text{GL}_3(\mathbb{Z}) \mid N \text{ and } M \text{ are simultaneously diagonalisable} \right\},$$

which are abelian groups. By the construction, $\tilde{\text{Aut}}(X) = K \rtimes \Gamma$. Note also that $G_M = G_1 \times G_0$, where $G_1 = \{ g_{n_1}^{a_1} g_{n_2}^{a_2} g_{n_3}^{a_3} \mid n_j \in \mathbb{Z} \} \cong \mathbb{Z}^3$ and $G_0 = \langle g_0 \rangle \cong \mathbb{Z}$. Now we have the following commutative diagram (by the snake lemma):

$$\begin{array}{cccccc}
1 & & 1 & & 1 & \\
\downarrow & & \downarrow & \phi & \downarrow & \\
1 & \to G_1 & \to K & \to F & \to 1 & \\
\downarrow & \downarrow & \downarrow & \downarrow & \\
1 & \to G_M & \to \tilde{\text{Aut}}(X) & \to \text{Aut}(X) & \to 1 & \\
\downarrow & \downarrow & \downarrow & \downarrow & \\
1 & \to G_0 & \to \Gamma & \to \Gamma/G_0 & \to 1 & \\
\downarrow & \downarrow & \downarrow & \downarrow & \\
1 & \to 1 & \to 1 & \to 1 & \\
\end{array}$$

(6.11)

where $\phi$ is given by

$$\phi(n) = n \cdot (I - M) \quad \text{with} \quad n = (n_1, n_2, n_3).$$

Since $I - M$ is invertible, the group $F$ being the cokernel of $\phi$, is finite (and abelian). Then, by the last column of the commutative diagram, $\text{Aut}(X)$ is metabelian. By Eq. (6.11) and Lemma 2.6 (1), we have

**Theorem 6.1** Let $X$ be an Inoue surface of type $S_M$. Then $\text{Aut}(X)$ is solvable. Moreover, for any $G \leq \text{Aut}(X)$, one has $\ell_{\text{vir}}(G) \leq 2$.

Next, we claim that any torsion subgroup of $\Gamma/\mathbb{Z}$ is finite. Let $n_0$ be the largest integer such that, for some $n_0$-th root of $M$, $M^{1/n_0} \in \text{GL}_3(\mathbb{Z})$. Let $N \in \Gamma$ be an element such that its image in $\Gamma/\mathbb{Z}$ is torsion. Then there is a pair of coprime integers $(m, n)$.
such that $N^n = M^m$ and hence $M^{\frac{m}{n}} \in \text{GL}_3(\mathbb{Z})$. Since $m$ and $n$ are coprime, there are integers $a$ and $b$ such that $am + bn = 1$. Then

$$\frac{1}{n} = a \frac{m}{n} + b.$$ 

It follows that $M^{\frac{1}{n}} \in \text{GL}_3(\mathbb{Z})$ and hence $n \leq n_0$. Therefore,

$$\{(m, n) \mid M^{\frac{m}{n}} \in \text{GL}_3(\mathbb{Z}), m, n \text{ coprime}, m < n\}$$

is a finite set. It follows that the set of torsion elements of $\Gamma/\mathbb{Z}$ is finite (with the bound depending on $X$), which proves the claim. This claim and Eq. (6.11) imply

**Theorem 6.2** Let $X$ be an Inoue surface of type $S_M$. Then any torsion subgroup $G \leq \text{Aut}(X)$ is finite. In particular, $\text{Aut}(X)$ is T-Jordan.

### 6.2 Types $S^{(+)}$ and $S^{(-)}$

Since the constructions of Inoue surfaces of type $S^{(+)}$ and $S^{(-)}$ are almost parallel, we will only focus on type $S^{(+)}$ in this subsection. The same argument works for Inoue surfaces of type $S^{(-)}$.

Let $M \in \text{SL}_2(\mathbb{Z})$ be a matrix with two real eigenvalues $\alpha$ and $1/\alpha$ with $\alpha > 1$. Let $(a_1, a_2)^T$ and $(b_1, b_2)^T$ be real eigenvectors of $M$ corresponding to $\alpha$ and $1/\alpha$, respectively, and fix integers $p_1, p_2, r$ ($r \neq 0$) and a complex number $\tau$. Define $(c_1, c_2)^T$ to be the solution of the following equation:

$$(I - M) \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} + \frac{b_1a_2 - b_2a_1}{r} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix},$$

where

$$e_i = \frac{1}{2} m_{i, 1}(m_{i, 1} - 1)a_1b_1 + \frac{1}{2} m_{i, 2}(m_{i, 2} - 1)a_2b_2 + m_{i, 1}m_{i, 2}b_1a_2, \quad i = 1, 2.$$

Let $G_M^{(+)}$ be the group of analytic automorphisms of $W = \mathbb{H} \times \mathbb{C}$ generated by

$$g_0: (w, z) \mapsto (\alpha w, z + \tau),$$
$$g_i: (w, z) \mapsto (w + a_i, z + b_i w + c_i), \quad i = 1, 2,$$
$$g_3: (w, z) \mapsto \left(w, z + \frac{b_1a_2 - b_2a_1}{r}\right).$$

We have the following relations between these generators:

$$g_3g_i = g_ig_3 \quad \text{for } i = 0, 1, 2, \quad g_1^{-1} g_2^{-1} g_1 g_2 = g_3^r,$$
$$g_0 g_j g_0^{-1} = g_1^{m_{j, 1}} g_2^{m_{j, 2}} g_3^{p_j} \quad \text{for } j = 1, 2.$$
The action of $G^{(+)}_M$ is free and properly discontinuous. The quotient space $X:=W/G^{(+)}_M$ is an Inoue surface of type $S^{(+)}$. Note that $G^{(+)}_M \simeq H(r) \rtimes \mathbb{Z}$ as an abstract group, where $H(r) = \langle g_1, g_2, g_3 \mid g_3g_i = g_ig_3, g_1^{-1}g_2^{-1}g_1g_2 = g_3^r \rangle$ and $\mathbb{Z}$ is generated by $g_0$. In fact, the centre $Z(H(r)) \simeq \mathbb{Z}$ is generated by $g_3$ and $H(r)/Z(H(r)) \simeq \mathbb{Z}^2$.

Similarly, there is a short exact sequence of groups

$$1 \longrightarrow G^{(+)}_M \longrightarrow \widetilde{\text{Aut}}(X) \longrightarrow \text{Aut}(X) \longrightarrow 1,$$

where $\widetilde{\text{Aut}}(X) \leq \text{Aut}(W)$ is the normaliser of $G^{(+)}_M$. Now suppose that $u \in \widetilde{\text{Aut}}(X)$ as in Eq. (6.1) and

$$u \in g^n_{1,1} g^n_{1,2} \frac{b_1a_2 - b_2a_1}{r}$$

for some $l_i, k_i \in \mathbb{Z}, n_0 = (n_{0,1}, n_{0,2}), n_3 = (n_{3,1}, n_{3,2}) \in \mathbb{Z}^2$ and $N = (n_i, j)_{i,j=1,2} \in \text{Mat}_{2\times 2}(\mathbb{Z})$. Then,

$$n_i, j a_j, t(w, z) + k_i \tau + l_i \frac{b_1a_2 - b_2a_1}{r}$$

$$+ \left( \sum_j n_i, j b_j \right) \alpha^{k_i}s(w) + \sum_j n_i, j c_j + e_i(n),$$

where

$$e_i(n) = \frac{1}{2} n_i, 1(n_{i,1} - 1)a_1b_1 + \frac{1}{2} n_i, 2(n_{i,2} - 1)a_2b_2 + n_i, 1n_i, 2a_2b_1.$$

As in the previous subsection, Eq. (6.12) implies

$$s(\alpha w) = \alpha^{k_i}s(w) + n_{0,1}a_1 + n_{0,2}a_2, \quad (6.13)$$

$$t(\alpha w, z + t) = t(w, z) + k_i \tau + l_i \frac{b_1a_2 - b_2a_1}{r}$$

$$+ \left( \sum_j n_{0, j}b_j \right) \alpha^{k_i}s(w) + \sum_j n_{0, j}c_j + e_0(n), \quad (6.14)$$

$$s(w + a_i) = \alpha^{k_i}s(w) + n_{i,1}a_1 + n_{i,2}a_2, \quad (6.15)$$

$$t(w + a_i, z + b_iw + c_i) = t(w, z) + k_i \tau + l_i \frac{b_1a_2 - b_2a_1}{r}$$

$$+ \left( \sum_j n_{i, j}b_j \right) \alpha^{k_i}s(w) + \sum_j n_{i, j}c_j + e_i(n), \quad (6.16)$$

$$s(w) = \alpha^{k_3}s(w) + n_{3,1}a_1 + n_{3,2}a_2, \quad (6.17)$$

$$\Box \text{ Springer}$$
\[ t \left( w, z + \frac{b_1a_2 - b_2a_1}{r} \right) = t(w, z) + k_3 \tau + l_3 \frac{b_1a_2 - b_2a_1}{r} + \left( \sum_j n_{3,j}b_j \right) \alpha^k s(w) + \sum_j n_{3,j}c_j + e_3(n). \tag{6.18} \]

By Eq. (6.13), using the assumption that \( \alpha > 1 \), we have

\[
\begin{aligned}
ac(1 - \alpha^{k_0}) &= c^2(n_{0,1}a_1 + n_{0,2}a_2), \\
bd(1 - \alpha^{k_0}) &= d^2(n_{0,1}a_1 + n_{0,2}a_2), \\
adt + b(1 - \alpha^{k_1}) &= (c^2a_i + 2cd)(n_{i,1}a_1 + n_{i,2}a_2),
\end{aligned}
\tag{6.19}
\]

If \( c \neq 0 \), then \( k_0 = -1 \) by the first two equalities of Eq. (6.19) since \( ad - bc > 0 \). Similarly, if \( d \neq 0 \), then one has \( k_0 = 1 \) by the last two equalities of Eq. (6.19). Therefore, either \( c = 0 \) and \( k_0 = 1 \), or \( d = 0 \) and \( k_0 = -1 \).

By Eq. (6.15), we have

\[
\begin{aligned}
ac(1 - \alpha^{k_i}) &= c^2(n_{i,1}a_1 + n_{i,2}a_2), \\
ada + b(1 - \alpha^{k_i}) &= \alpha^{k_i}bca_i + (cd)\alpha^{k_i}bca_i + b(1 - \alpha^{k_i})(n_{i,1}a_1 + n_{i,2}a_2).
\end{aligned}
\tag{6.20}
\]

If \( d = 0 \), we have \( \alpha^{k_i}bca_i = 0 \) for \( i = 1, 2 \) by the last equality of Eq. (6.20). This implies that \( a_i = 0 \) (\( i = 1, 2 \)) since \( \alpha > 1 \) and \( ad - bc > 0 \), which contradicts the fact that \( (a_1, a_2)^T \) is an eigenvector of \( M \). So, \( c = 0 \) and \( k_0 = 1 \). Also, the second equality of Eq. (6.20), which becomes \( ad = \alpha^{k_i}ad_i \), implies that \( k_i = 0 \) for \( i = 1, 2 \).

Now we may rewrite \( s \) as \( s(w) = aw + b \) with \( a > 0 \) and \( b \in \mathbb{R} \). Then Eqs. (6.19) and (6.20) imply that

\[
\begin{aligned}
(1 - \alpha)b &= n_{0,1}a_1 + n_{0,2}a_2, \\
\alpha a_i &= n_{i,1}a_1 + n_{i,2}a_2, \quad i = 1, 2.
\end{aligned}
\tag{6.21}
\]

Comparing the coefficients of \( z \) of Eq. (6.14), we have

\[
\sum_p C_{p,1}(aw - \sqrt{-1})^p = \sum_p C_{p,1}(w - \sqrt{-1})^p,
\]

which implies that \( C_{p,1} = 0 \) for all \( p \geq 1 \). For the \( z \)-constant part of Eq. (6.14), one has the following equality:

\[
\sum_p C_{p,0}(aw - \sqrt{-1})^p + C_{0,1} \tau \\
= \sum_p C_{p,0}(w - \sqrt{-1})^p + (n_{0,1}b_1 + n_{0,2}b_2)aw + \tau.
\]
Similarly, we conclude that $C_{p, 0} = 0$ for $p \geq 2$. Hence, we may rewrite $t$ as $t(w, z) = Aw + B + Cz$ for some $A, B, C \in \mathbb{C}$ with $C \neq 0$. Now comparing the coefficients of $w$ of Eqs. (6.14) and (6.16), we get

$$\left\{ \begin{array}{l}
\left( 1 - \frac{1}{\alpha} \right) a = n_{0, 1}b_1 + n_{0, 2}b_2,
(C - 1) \tau = l_0 b_2 - a_1 b_1 + (n_{0, 1}b_1 + n_{0, 2}b_2)ab + n_{0, 1}c_1 + n_{0, 2}c_2 + e_0(n),
Cb_l = a(n_{i, 1}b_1 + n_{i, 2}b_2),
Aa_l + Cc_l = l_1 b_2 - a_1 b_1 + (n_{i, 1}b_1 + n_{i, 2}b_2)b + n_{i, 1}c_1 + n_{i, 2}c_2 + e_l(n).
\end{array} \right. \quad (6.22)$$

Finally, Eqs. (6.17) and (6.18) simply give that $k_3 = 0, l_3 = C$ and $n_{3, 1} = n_{3, 2} = 0$. Therefore, $N$ has two eigenvalues $a$ and $C/a$ with eigenvectors $(a_1, a_2)^T$ and $(b_1, b_2)^T$, respectively. It is not hard to see that the matrix associated to $u^{-1}$ is $N^{-1}$. Then $\det N = \pm 1$ and hence $N \in GL_2(\mathbb{Z})$. Moreover, by Eqs. (6.21) and (6.22), $u$ (if exists) is determined by $n_0, N$ and $B$.

Let

$$\Gamma := \{ N \in GL_2(\mathbb{Z}) \mid N \text{ and } M \text{ are simultaneously diagonalisable} \},$$

which is an abelian group, and let $\tau : \widetilde{\text{Aut}}(X) \longrightarrow \Gamma$ be the homomorphism (not necessarily surjective) defined by $u \mapsto N$, where $N$ is the matrix associated with $u$ as constructed above. Let $K$ be the kernel of this homomorphism, with image $\Gamma' \leq \Gamma$. It is clear that any automorphism in $K$ has the form

$$(w, z) \longmapsto (w + b, z + Aw + B)$$

for some $b \in \mathbb{R}$ and $A, B \in \mathbb{C}$ satisfying Eqs. (6.21) and (6.22). An explicit calculation gives that the centre $Z(K) \simeq \mathbb{C}$, which is generated by

$$(w, z) \longmapsto (w, z + B).$$

Further, $K/Z(K) \simeq \mathbb{Z} \times \mathbb{Z}$ is generated by the images of

$$(w, z) \longmapsto (w + b, z) \text{ and } (w, z) \longmapsto (w, z + Aw).$$

Combining these, the snake lemma shows the diagrams below are commutative with exact rows and columns.
Note that \( F' \) is abelian and \( F'' \) is finite and metabelian. Consequently, \( F \), as the quotient of \( K \) by \( H(r) \), is virtually abelian and abelian-by-metabelian.

By the above diagrams and Lemma 2.6 (1),

**Theorem 6.3** Let \( X \) be an Inoue surface of type \( S^{(+)} \) or \( S^{(-)} \). Then \( \text{Aut}(X) \) is solvable. Moreover, for any subgroup \( G \leq \text{Aut}(X) \), one has \( \ell_{\text{vir}}(G) \leq 4 \).

Similar to the argument before Theorem 6.2, any torsion subgroup of \( \Gamma'/\mathbb{Z} \) is finite. It follows from the diagram above that each torsion subgroup of \( \text{Aut}(X) \) has an abelian subgroup with bounded index depending on \( X \).

**Theorem 6.4** Let \( X \) be an Inoue surface of type \( S^{(+)} \) or \( S^{(-)} \). Then \( \text{Aut}(X) \) is \( T \)-Jordan.

### 7 Class VII Surfaces with \( b_2 > 0 \)

We will prove Proposition 1.6 in this section.

**Proof of Proposition 1.6** We may assume that \( G \) is an infinite group. By Corollary 4.2, we only need to consider such an \( X \) that does not have any curves. Consider the action of \( \text{Aut}(X) \) on \( H^*(X, \mathbb{Q}) \), and let \( \text{Aut}^*(X) \) be the kernel of \( \text{Aut}(X) \to \text{GL}(H^*(X, \mathbb{Q})) \). Note that the image of \( G \) in \( \text{GL}(H^*(X, \mathbb{Q})) \) is finite (cf. [17, Lemma 5.2]). By passing to a finite-index subgroup, we may assume that \( G \leq \text{Aut}^*(X) \).

Pick \( \text{id} \neq g \in G \), and let \( G' \) be the centraliser of \( \langle g \rangle \) in \( G \). Since \( g \) has finite order, \( [G : G'] \) is finite. Replacing \( G \) by the finite-index subgroup \( G' \), we may assume that \( g \in Z(G) \), the centre of \( G \). Note that the fixed point set \( \text{Fix}(g) \) of \( g \) is finite with cardinality \( |\text{Fix}(g)| = b_2(X) \) (cf. [29, Lemma 5.4]). Consider the action of \( G \) on the finite set \( \text{Fix}(g) \); noting that \( g \) is in the centre of \( G \). Replacing \( G \) by a subgroup of index \( \leq b_2(X)! \), we may assume that \( G \) fixes some point \( x \in \text{Fix}(g) \). Applying Lemma 2.3, the torsion group \( G \) is embedded into \( \text{GL}_2(\mathbb{C}) \). By Theorem 1.1, we have \( G \) is virtually abelian. 

\( \square \)
8 Compact Kähler Surfaces

In this section, we deal with smooth compact surfaces which are Kähler. The idea of this section comes from De-Qi Zhang.

Let $X$ be a compact Kähler manifold. For a subgroup $G$ of $\text{Aut}(X)$, define $G^0 := G \cap \text{Aut}_0(X)$ and denote by $N(G)$ the set of all elements in $G$ with zero entropy.

**Lemma 8.1** Let $X$ be a smooth compact Kähler surface, and $G \leq \text{Aut}(X)$ a virtually solvable subgroup. Then $\ell_{\text{vir}}(G) \leq 1 + \ell_{\text{vir}}(N(G))$.

**Proof** By taking a finite-index subgroup, we may assume that $G$ is a solvable, $N(G)$ is a normal subgroup of $G$ and $G/N(G) \simeq \mathbb{Z}^r$ for some $r = 0, 1$ (cf. [5, Theorem 1.5]). Take a finite-index subgroup $N' \leq N(G)$ such that $\ell(N') = \ell_{\text{vir}}(N(G))$ and $\ell(N'/N^0) \leq 1$ as in [7, Theorem 1.2 and Proposition 2.6].

Note that $N(G)|_{H^2(X, \mathbb{C})}$ is finitely generated (cf. [5, Theorem 2.2(1)]). Since $N'|_{H^2(X, \mathbb{C})}$ has finite index in $N(G)|_{H^2(X, \mathbb{C})}$, after replacing $N'$ by a finite-index subgroup, we may assume that $N'|_{H^2(X, \mathbb{C})}$ is characteristic in $N(G)|_{H^2(X, \mathbb{C})}$ by Lemma 2.8. In particular, $N'|_{H^2(X, \mathbb{C})} \triangleleft G|_{H^2(X, \mathbb{C})}$.

Let $N \triangleleft N(G)$ be the preimage of $N'|_{H^2(X, \mathbb{C})}$. Then $N \triangleleft G$, $N(G)/N$ is finite, $\ell(N) = \ell_{\text{vir}}(N(G))$ and $\ell(N/N^0) \leq 1$. Since $(G/N)/(N(G)/N) \simeq \mathbb{Z}^r$, replacing $G$ by a finite-index subgroup, we may assume $G/N \simeq \mathbb{Z}^r$ (cf. [36, Lemma 2.4]). It follows that

$$\ell_{\text{vir}}(G) \leq \ell(G) \leq \ell(G/N) + \ell(N) \leq 1 + \ell_{\text{vir}}(N(G)).$$

\[ \square \]

Indeed, it follows from the above proof that

$$\ell_{\text{vir}}(G) \leq 1 + \ell(N) \leq 1 + \ell(N/N^0) + \ell(N^0) \leq 2 + \ell(N^0).$$

When $\kappa(X) \geq 0$, $\text{Aut}_0(X)$ is a complex torus (cf. [11, Corollary 5.11]). In particular, $N^0 = N \cap \text{Aut}_0(X)$ is abelian and hence of derived length (at most) 1. Thus, we have

**Proposition 8.2** Let $X$ be a smooth compact Kähler surface with Kodaira dimension $\kappa(X) \geq 0$. Let $G \leq \text{Aut}(X)$ be virtually solvable. Then $\ell_{\text{vir}}(G) \leq 3$.

Suppose that now $X$ is a compact Kähler surface with $\kappa(X) = -\infty$. Then $X$ is projective.

**Proposition 8.3** Let $X$ be a smooth projective surface. Suppose that $G \leq \text{Aut}(X)$ is virtually solvable, and $U \leq \text{Aut}_0(X)$ is non-trivial unipotent with $U \triangleleft G$. Then the derived length $\ell(U) \leq 2$ and after replacing by a finite-index subgroup, we have $G = G^0$.

**Proof** Replacing by closure or finite-index subgroup, we may assume that $G \leq \text{Aut}(X)$ is closed and solvable, $G|_{\text{NS}(X)}$ is connected and closed, and $U \leq \text{Aut}_0(X)$ is closed and unipotent (cf. [26, Proof of Lemma 2.1(2)]).
By Lemma 2.9, after replacing $G$ by a finite-index subgroup, there is a subgroup $Z_1 \cong \mathbb{G}_a \leq U$ which is normal in $G$. Let $X \twoheadrightarrow Y = X/Z_1$ be the quotient map to the cycle space (cf. [11, § 4]). Then $G$ acts biregularly on $Y$ and the map is $G$-equivariant. Replacing the map by the graph and taking equivariant resolutions, we may assume that both $X$ and $Y$ are smooth, and the map $X \twoheadrightarrow Y$ is a morphism, a $\mathbb{P}^1$-fibration, whose general fibre is a curve with dense orbit $Z_1 \cdot x \cong \mathbb{A}^1$ and whose section at infinity is fixed by $G$. Consider the exact sequence

$$1 \rightarrow K \rightarrow U \rightarrow U|_Y \rightarrow 1.$$  

Then both $K$ and $U|_Y$ are unipotent. Now the restriction to generic fibre $K \rightarrow K|_X$ is injective with image contained in a closed unipotent group acting faithfully on $X_k(Y) = \mathbb{P}^1_k$, and hence is abelian. It is clear that either $U|_Y = \{1\}$, or $U|_Y \cong \mathbb{G}_a$ and $Y = \mathbb{P}^1$. Hence $\ell(U) \leq \ell(U|_Y) + 1 \leq 2$.

Note that $G$ fixes a big class $H_Y$ on the curve $Y$. Take the class $H$ to be the sum of the pullback of $H_Y$ and the section at infinity of the $\mathbb{P}^1$-fibration $X \rightarrow Y$. Then $G$ fixes the big class $H$ on $X$ and hence replaced by a finite-index subgroup, $G$ is contained in $\text{Aut}_0(X)$ (cf. [11, Theorem 4.8], [21, Proposition 2.2] and [6, Corollary 2.2]). Thus $G = G^0$.

**Proposition 8.4** Let $G$ be a smooth projective surface. Suppose that $G \leq \text{Aut}(X)$ is virtually solvable. Then $\ell_{\text{vir}}(G) \leq 4$.

**Proof** Replacing by finite-index subgroup, we may assume that $G$ is solvable. First assume that $G = G^0$ (and connected). Let $U$ be the unipotent radical of (the linear part of) $G = G^0$ so that $G/U$ is a semi-torus and hence commutative (cf. [14, Lemma 4]). Thus, $\ell_{\text{vir}}(G) \leq 1 + \ell(U) \leq 3$ by Proposition 8.3.

Now we may assume that $G \neq G^0$ even after replacing $G$ by a finite-index subgroup. Let $G^0_0 \lhd G^0$ be the identity component, Then by Proposition 8.3, the unipotent radical of (the linear part of) $G^0_0$ is trivial and hence $G^0_0$ itself is a semi-torus and hence commutative.

If $G$ is of zero entropy, by [7, Theorem 1.2] one has $\ell(G/G^0) \leq 1$ after replacing $G$ by a finite-index subgroup. Consider the short exact sequence

$$1 \rightarrow G^0/G^0_0 \rightarrow G/G^0_0 \rightarrow G/G^0 \rightarrow 1.$$  

Since the first term is a finite group, its centraliser in $G/G^0_0$ is of finite index (say of index 1, after replacing $G$ by a finite-index subgroup). Thus,

$$\ell(G/G^0_0) \leq 1 + \ell(G/G^0) = 2,$$

$$\ell_{\text{vir}}(G) \leq \ell(G) \leq \ell(G/G^0_0) + \ell(G^0_0) \leq 3.$$  

If $G$ is not of zero entropy, using Lemma 8.1 and the argument above, $\ell_{\text{vir}}(G) \leq 1 + \ell_{\text{vir}}(N(G)) \leq 1 + 3 = 4$. \qed
Table 1  Non-Kähler minimal smooth compact complex surfaces

| Class of the surface $X$ | $\kappa(X)$ | $\alpha(X)$ | $b_1(X)$ | $b_2(X)$ | $e(X)$ |
|--------------------------|-------------|-------------|----------|----------|-------|
| Surfaces of class VII    | $-\infty$  | 0, 1        | 1        | $\geq 0$ | $\geq 0$ |
| Primary Kodaira surfaces | 0           | 1           | 3        | 4        | 0     |
| Secondary Kodaira surfaces | 0         | 1           | 1        | 0        | 0     |
| Properly elliptic surfaces | 1          | 1           |          |          | $\geq 0$ |

9 Summary

In the Kähler setting, Proposition 1.5 follows from Theorem 1.4; Theorem 1.10 has been proved in [5, Theorem 1.5]; Theorem 1.11 follows from Propositions 8.2 and 8.4.

Let $X$ be a smooth compact complex surface, which is not Kähler. In particular, $X$ is not rational or ruled. Thus, there is a unique minimal surface $X'$ bimeromorphic to $X$ such that

$$\text{Aut}(X) \subseteq \text{Bim}(X) \simeq \text{Bim}(X') = \text{Aut}(X').$$

See [31, Proposition 3.5]. It suffices for us to prove Proposition 1.5 and Theorems 1.10 and 1.11 for minimal surfaces (in the non-Kähler setting). Note that non-Kähler minimal surfaces are surfaces of class VII, (primary or secondary) Kodaira surfaces and some properly elliptic surfaces (cf. [1, IV. Theorem 3.1 and VI. Theorem 1.1]). Since every minimal surface of class VII with vanishing $b_2$ is either a Hopf surface or an Inoue surface (cf. [3]) and minimal surface of class VII with algebraic dimension 1 is a Hopf surface (cf. [1, V. Theorem 18.6]), minimal surfaces of class VII not in $\Xi$ are exactly Hopf surfaces or Inoue surfaces.

Then Proposition 1.5 follows from Theorems 3.4, 5.4, 6.2 and 6.4 and Corollary 4.2; Theorem 1.10 follows from Theorems 2.7, 3.4, 3.5, 5.1, 6.1, and 6.3; and Theorem 1.11 follows from Theorems 2.7, 3.4, 3.5, 5.2, 6.1 and 6.3. Finally, Proposition 1.6 has been proved in Sect. 7.

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