Variationally complete actions on compact symmetric spaces

Claudio Gorodski* and Gudlaugur Thorbergsson

November 13, 2018

Abstract

We prove that an isometric action of a compact Lie group on a compact symmetric space is variationally complete if and only if it is hyperpolar.

1 Introduction

The main result of this paper is the following theorem.

Theorem An isometric action of a compact Lie group on a compact symmetric space is variationally complete if and only if it is hyperpolar.

Variationally complete actions were introduced by Bott in [1] (see also [2]). Let $G$ be a compact Lie group acting on a complete Riemannian manifold $M$ by isometries. A geodesic $\gamma$ in $M$ is called $G$-transversal if it is orthogonal to the $G$-orbit through $\gamma(t)$ for every $t$. One can show that a geodesic $\gamma$ is $G$-transversal if there is a point $t_0$ such that $\dot{\gamma}(t_0)$ is orthogonal to $G\gamma(t_0)$. A Jacobi field along a geodesic in $M$ is called $G$-transversal if it is the variational vector field of a variation through $G$-transversal geodesics. The action of $G$ on $M$ is called variationally complete if every $G$-transversal Jacobi field $J$ in $M$ that is tangent to the $G$-orbits at two different parameter values is the restriction of a Killing field on $M$ induced by the $G$-action. It is proved in [2] on p. 974 that instead of requiring tangency at two different points in the definition of variational completeness it is equivalent to require tangency at one point and vanishing at another point.

Conlon considered in [3] actions of a Lie group $G$ on a complete Riemannian manifold $M$ with the property that there is a connected submanifold $\Sigma$ of $M$ that meets all orbits of $G$ in such a way that the intersections between $\Sigma$ and the orbits of $G$ are all orthogonal. Such a submanifold is called a section and an action admitting a section is called polar. Notice that neither do we assume as Conlon in [3] that $\Sigma$ is closed nor do we assume that it is properly embedded as is usually required in the recent literature on the subject. It is easy to see that a section $\Sigma$ is totally geodesic in $M$. An action admitting a section that is flat in the induced metric is called hyperpolar. Conlon proved in [3] that a hyperpolar action of a compact Lie group $G$ on a complete Riemannian manifold $M$ with a section $\Sigma$ is variationally complete.

*Partially supported by CNPq and FAPESP.
group on a complete Riemannian manifold is variationally complete. Notice that he does not use in his proof that the sections of the action are closed. His result therefore implies one direction of the main theorem of this paper.

In the case of Euclidean spaces, hyperpolar representations were classified by Dadok in [4]. As a consequence of his classification he obtained that a hyperpolar representation of a compact Lie group is orbit equivalent to the isotropy representation of a symmetric space. (We recall that two isometric actions are said to be orbit equivalent if there is an isometry between the action spaces under which the orbits of the two actions correspond.)

On the other hand, we classified variationally complete representations in [6]. As a consequence of our classification we obtained that a variationally complete representation of a compact Lie group is hyperpolar. Di Scala and Olmos gave in [3] a very short, simple proof of this result. It follows that a representation is hyperpolar if and only if it is variationally complete.

Next we discuss the case of compact symmetric spaces. There are two important classes of hyperpolar actions on them. It is clear that a cohomogeneity one action on a compact Lie group on a compact symmetric space is hyperpolar. Hermann constructed in [9] a class of examples of variationally complete actions on compact symmetric spaces which in fact turned out to be hyperpolar. Namely, if $K_1$ and $K_2$ are two symmetric subgroups of the same compact Lie group $G$, then the action of $K_1$ on $G/K_2$ is hyperpolar and so is the action of $K_1 \times K_2$ on $G$. Kollross [10] classified hyperpolar actions on compact irreducible symmetric spaces. It follows from his classification that, in the irreducible case, all hyperpolar actions belong to either one of these classes.

Finally, we add that the concepts of hyperpolarity and variational completeness admit natural extensions in the context of proper Fredholm actions of Hilbert-Lie groups on Hilbert spaces (see [11] and [12]). We shall make use of them in the course of our proof.

2 The proof of the theorem

Conlon proved in [3] that hyperpolar actions are variationally complete as we pointed out in the introduction. In this section we will prove the converse of his result for actions on compact symmetric spaces. Our strategy is as follows. First we reduce to the case of a symmetric space of compact type. Next we lift the action on the symmetric space to a variationally complete action of a path group on a Hilbert space. Then we use an argument similar to that in [5] to show that the lifted action is hyperpolar. Finally, it is easy to see that a section for the action on the Hilbert space induces a flat section for the original action.

Let $M$ be a compact Riemannian symmetric space. We identify $M$ with the coset space $G/K$, where $G$ is the connected component of the group of all isometries of $M$ and $K$ is the isotropy subgroup of a chosen base point. Let $H$ be a compact Lie group acting by isometries on $M$. An action of a group is variationally complete (resp. hyperpolar) if and only if the same is true for the restriction of that action to the connected component of the group. This follows from the definition in the case of variationally complete actions and is easy in the case of hyperpolar actions, see Proposition 2.4 in [7]. We can thus assume that $H$ is a connected, closed subgroup of $G$.

Let $\hat{M}$ be a compact Riemannian covering space of $M$ which splits as a product of a
torus $T^k$ and a symmetric space $N$ of compact type. Let $\hat{G}$ be the connected component of the group of all isometries of $\hat{M}$. Then $\hat{G}$ is a covering group of $G$ and we let $\hat{H}$ denote the subgroup of $\hat{G}$ which is the connected component of the inverse image of $H$. Notice that the action of $\hat{H}$ on $\hat{M}$ is variationally complete if the one of $H$ on $M$ is so. In this case, the restriction $\sigma_2$ of the action of $\hat{H}$ to $N$ is clearly still variationally complete. We will prove below that a variationally complete action on a symmetric space of compact type is hyperpolar. Since the restriction $\sigma_1$ of the action of $\hat{H}$ to $T^k$ is orbit equivalent to the action of a torus on $T^k$ and therefore hyperpolar, it follows that the product action $\sigma_1 \times \sigma_2$ on $T^k \times N = \hat{M}$ is hyperpolar. The action of $\hat{H}$ on $\hat{M}$ is therefore also hyperpolar since it is orbit equivalent to the product action of $\sigma_1 \times \sigma_2$ on $\hat{M}$, which can be seen by arguments similar to those in the proof of Theorem 4 (ii) in [3] or of Proposition 3.4 (d) in [6]. Now we can simply project a section of the $\hat{H}$-action in $\hat{M}$ down to $M$ to see that the action of $H$ on $M$ is hyperpolar.

It follows from the discussion in the previous paragraph that from now on we can restrict to the case where $M$ is of compact type. In this case the group $G$ is semisimple. There is an involution $\sigma$ of $G$ such that $K$ is open in the fixed point set of $\sigma$. Now the Lie algebra of $G$ decomposes into the $\pm 1$-eigenspaces of $d\sigma$, $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, where $\mathfrak{k}$ is the Lie algebra of $K$, and the Riemannian metric in $M$ is the $G$-invariant metric induced by some $\text{Ad}_G$-invariant inner product on $\mathfrak{g}$. The group $H$ is a closed subgroup of $G$ so that it acts on $M$ by left translations, and $H \times K$ acts on $G$ by $(h, k) \cdot g = h g k^{-1}$, where $h \in H$, $k \in K$ and $g \in G$. Clearly, the projection $\pi : G \to G/K$ is an equivariant Riemannian submersion.

**Lemma 1** If the action of $H$ on $M$ is variationally complete, then the action of $H \times K$ on $G$ is also variationally complete.

**Proof.** Let $\gamma$ be an $H \times K$-transversal geodesic in $G$ defined on $[0, 1]$ and let $J$ be an $H \times K$-transversal Jacobi field along $\gamma$ such that $J(0) = 0$ and $J(1)$ is tangent to the $H \times K$-orbit through $\gamma(1)$. We must show that $J$ is the restriction along $\gamma$ of an $H \times K$-Killing field.

We first identify the tangent and normal spaces to the $H \times K$-orbits in $G$. Let $\mathfrak{h}$ be the Lie algebra of $H$. Then $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$ and we denote by $\mathfrak{h}^\perp$ its orthogonal complement in $\mathfrak{g}$ with respect to the $\text{Ad}_G$-invariant inner product on $\mathfrak{g}$. Note that $[\mathfrak{h}, \mathfrak{h}^\perp] \subseteq \mathfrak{h}^\perp$ and that $\mathfrak{k}$ and $\mathfrak{p}$ are also mutually orthogonal in $\mathfrak{g}$. Given $a \in G$, the tangent space to $H \cdot a \cdot K$ at the point $a$ is $\mathfrak{h} \cdot a + a \cdot \mathfrak{k}$. Therefore the left translates under $a^{-1}$ of the tangent and normal spaces of $H \cdot a \cdot K$ at $a$ are respectively $\mathfrak{k}^a := \mathfrak{k} + \text{Ad}_{a^{-1}} \mathfrak{h}$ and $\mathfrak{m}^a := \mathfrak{p} \cap \text{Ad}_{a^{-1}} \mathfrak{h}^\perp$. It will be convenient to set $\mathfrak{q}^a := \mathfrak{k} \cap \text{Ad}_{a^{-1}} \mathfrak{h}$.

Let $\gamma(0) = a \in G$. Then $\gamma(t) = ae^{tX}$ for $X \in \mathfrak{m}^a$. Since $J(0) = 0$, setting $\gamma_a(t) = ae^{t(X + \alpha Y)}$, where $Y \in \mathfrak{m}^a$ and $Y = a^{-1} \cdot J'(0)$, defines a variation of $\gamma = \gamma_0$ through $H \times K$-transversal geodesics which induces $J$. Since $\pi : G \to G/K$ is an equivariant Riemannian submersion and each $\gamma_a$ is a horizontal curve with respect to $\pi$, we have that $\{\pi \gamma_a\}$ is a variation of $\tilde{\gamma} = \pi \gamma$ through $H$-transversal geodesics in $M$. Moreover, the associated $H$-transversal Jacobi field $\tilde{J}$ along $\tilde{\gamma}$ satisfies $\tilde{J}(0) = 0$, $\tilde{J}(0) = a \cdot \pi_{K} Y$, and $\tilde{J}(1)$ is tangent to the $H$-orbit through $\pi(ae^{X})$. It follows by variational completeness of the action of $H$ on $M$ that $\tilde{J}$ is the restriction along $\tilde{\gamma}$ of an $H$-Killing field on $M$, namely $\tilde{J}(t) = Z \cdot \tilde{\gamma}(t)$ for some
where \( g \), that is, each one of them induces parallel transport along the geodesic \( s \mapsto \pi(e^{sX}) \). Denote by \( \nabla \) the Levi-Civita connection of \( M \). Then we have

\[
\bar{a}^{-1} \cdot J'(0) = \bar{a}^{-1} \cdot \nabla_{a^{-1}X} J = \nabla_{\pi^*X}(a^{-1} \cdot \bar{J}) = \left. \frac{d}{dt} \right|_{t=0} \pi^*_a \text{Ad}(ae^{tX})Z = \pi^*_a [\text{Ad}_{a^{-1}}Z, X].
\]

We deduce that \( \bar{J}'(0) = a \cdot \pi^*_a [\text{Ad}_{a^{-1}}Z, X] \). But we already know that \( \bar{J}'(0) = a \cdot \pi^*_a Y \).

Since \( Y \in m^a \subset p \) and \([\text{Ad}_{a^{-1}}Z, X] \in [q^a, m^a] \subset m^a \subset p\), this implies \( Y = [\text{Ad}_{a^{-1}}Z, X] \).

Finally, consider the \( H \times K \)-Killing field in \( G \) given by

\[
U \cdot y = \frac{\partial}{\partial \alpha} \bigg|_{\alpha=0} e^{\alpha Z} ye^{-\alpha \text{Ad}_{a^{-1}}Z}.
\]

Then

\[
U \cdot \gamma(t) = \frac{\partial}{\partial \alpha} \bigg|_{\alpha=0} e^{\alpha Z} ae^{tX}a^{-1}e^{-\alpha Z}a.
\]

Therefore \( U \cdot \gamma(0) = 0 \) and

\[
\left. \frac{d}{dt} \right|_{t=0} U \cdot \gamma(t) = \frac{\partial}{\partial \alpha} \bigg|_{\alpha=0} \frac{\partial}{\partial t} \bigg|_{t=0} e^{\alpha Z} ae^{tX}a^{-1}e^{-\alpha Z}a
\]

\[
= \left( \frac{\partial}{\partial \alpha} \bigg|_{\alpha=0} \text{Ad}_{a^\alpha X} \right) \cdot a
\]

\[
= [Z, \text{Ad}_{aX}] \cdot a.
\]

Hence \( a^{-1} \cdot \left. \frac{d}{dt} \right|_{t=0} U \cdot \gamma(t) = [\text{Ad}_{a^{-1}}Z, \text{Ad}_{aX}] = Y \). It follows that \( J(t) = U \cdot \gamma(t) \). \( \square \)

Let \( V = L^2([0, 1]; g) \) denote the Hilbert space of \( L^2 \)-integrable paths \( u : [0, 1] \to g \), and let \( \hat{G} = H^1([0, 1]; G) \) denote the Hilbert-Lie group of \( H^1 \)-paths in \( G \) parametrized on \([0, 1]\) (the elements of \( \hat{G} \) are the absolutely continuous paths \( g : [0, 1] \to G \) whose derivative is square integrable). We have that \( \hat{G} \) acts by affine isometries on \( V \): \( g \ast u = gug^{-1} - g'g^{-1} \), where \( g \in \hat{G} \) and \( u \in V \). Let \( \mathcal{P}(G, H \times K) \) denote the subgroup of all paths \( g \in \hat{G} \) such that \((g(0), g(1)) \in H \times K \). Let \( \varphi : V \to \hat{G} \) be the parallel transport map defined by \( \varphi(u) = h(1) \), where \( h \in \hat{G} \) is the unique solution of \( h^{-1}h' = u \), \( h(0) = 1 \). Then it is known that (see [11, 12, 13]):

(a) the action of \( \hat{G} \) on \( V \) is proper and Fredholm;
(b) \( \varphi(g \ast u) = g(0)\varphi(u)g(1)^{-1} \), where \( g \in \hat{G} \) and \( u \in V \);
(c) \( \mathcal{P}(G, H \times K) \ast u = \varphi^{-1}((H \times K) \cdot \varphi(u)) \);
(d) the action of $\mathcal{P}(G, 1 \times G)$ on $V$ is simply transitive;

(e) $\varphi : V \to G$ is a Riemannian submersion and a principal $\Omega_1(G) := \mathcal{P}(G, 1 \times 1)$-bundle;

(f) the horizontal distribution $\mathcal{H}$ of $\varphi$ is given by $\mathcal{H}(u) = \{\text{Ad}_g \dot{Y} : Y \in \mathfrak{g}\}$, where $\dot{Y}$ denotes the constant path with value $Y$ and $g$ is the unique element of $\mathcal{P}(G, 1 \times G)$ which satisfies $u = g \ast \hat{0}$;

(g) the tangent space to the orbit through $\dot{Y}$ at $\dot{Y}$ is

$$\{[\xi, \dot{Y}] - \xi' : \xi \in H^1([0, 1], \mathfrak{g}), \xi(0) = \mathfrak{h}, \xi(1) = \mathfrak{v}\}.$$

**Lemma 2** If the action of $H \times K$ on $G$ is variationally complete, then the action of $\mathcal{P}(G, H \times K)$ on $V$ is also variationally complete.

Proof. Let $\gamma$ be a transversal geodesic in $V$ defined on $[0, 1]$ and let $J$ be a transversal Jacobi field along $\gamma$ such that $J(0) = 0$ and $J(1)$ is tangent to the orbit through $\gamma(1)$. We must show that $J$ is the restriction along $\gamma$ of a Killing field induced by the $\mathcal{P}(G, H \times K)$-action.

Let $\gamma(0) = u \in V$. Since $\mathcal{P}(G, H \times K) \supset \Omega_1(G)$, we have that the normal space to the orbit through $u$ is contained in the horizontal subspace $\mathcal{H}(u)$. Let $g$ be the unique element of $\mathcal{P}(G, 1 \times G)$ which satisfies $u = g \ast \hat{0}$. Now $\gamma(t) = u + t\text{Ad}_g \dot{X}$ for some $X \in \mathfrak{g}$. Since $\gamma(0) = 0$, setting $\gamma_a(t) = u + t\text{Ad}_g (\dot{X} + \alpha \dot{Y}) = g \ast t(\dot{X} + \alpha Y)$, where $J'(0) = \text{Ad}_g \dot{Y}$ for some $Y \in \mathfrak{g}$, defines a variation of $\gamma' = \gamma_0$ through transversal geodesics which induces $J$. Since $\varphi : V \to G$ is an equivariant Riemannian submersion with respect to the homomorphism $h \in \mathcal{P}(G, H \times K) \mapsto (h(0), h(1)) \in H \times K$, and each $\gamma_a$ is a horizontal curve with respect to $\varphi$, we have that $\{\varphi_\gamma\}$ is a variation of $\gamma' = \varphi_\gamma$ through transversal geodesics in $G$. Moreover, the associated transversal Jacobi field $\ddot{J}$ along $\gamma'$ satisfies $\ddot{J}(0) = 0$, $\ddot{J}'(0) = d\varphi_u(\text{Ad}_g \dot{Y}) = Y \cdot a$, where $a = (g(1))^{-1} = \varphi(u)$, and $\ddot{J}(1)$ is tangent to the orbit through $\gamma(1)$. It follows by variational completeness of the action of $H \times K$ on $G$ that $\ddot{J}$ is the restriction along $\gamma$ of an $H \times K$-Killing field on $G$, namely $\ddot{J}(t) = Z \cdot \dddot{\gamma}(t) - \dddot{\gamma}(t) \cdot W$ for some $Z \in \mathfrak{h}$, $W \in \mathfrak{k}$.

Note that $\dddot{\gamma}(t) = e^{tx}a$. Therefore we can write $\dddot{J}(t) = e^{tx} \cdot (\text{Ad}_{e^{-tx}Z - \text{Ad}_a W}) \cdot a$. It follows that $0 = \dddot{J}(0) = (Z - \text{Ad}_a W) \cdot a$, so that $W = \text{Ad}_{a^{-1}}Z \in \mathfrak{k} \cap \text{Ad}_{a^{-1}}\mathfrak{h} = \mathfrak{q}^a$ and $\dddot{J}(t) = e^{tx} \cdot (\text{Ad}_{e^{-tx}Z - Z} \cdot a).$ Then we have that $\dot{Y} = \dddot{J}(0) = [Z, X] \cdot a,$ which gives $\dot{Y} = [Z, X]$.

Finally, consider the one-parameter subgroup $\{g_a\}$ of $\mathcal{P}(G, H \times K)$ given by $g_a = ge^{az} g^{-1}$. Note that $g_a(0) = e^{az} \in H$ and $g_a(1) = a^{-1} e^{az} a \in K$, so that $g_a$ is well defined. We next show that the Killing field induced by $\{g_a\}$ coincides with $J$ along $\gamma$. It suffices to compare their initial values. Since $g_a \ast \gamma = (ge^{az}) \ast t\dot{X}$, we have

$$ge^{az} \ast \dot{0} = -(ge^{az})' (ge^{az})^{-1} = -g' g^{-1} = g \ast \dot{0} = u.$$
and
\[
\frac{\partial}{\partial t} \bigg|_{t=0} \frac{\partial}{\partial \alpha} \bigg|_{\alpha=0} g e^{\alpha z^*} t \dot{X} = \frac{\partial}{\partial \alpha} \bigg|_{\alpha=0} \frac{\partial}{\partial t} \bigg|_{t=0} \text{Ad}_{g e^{\alpha z}} \dot{X} + u \\
= \frac{\partial}{\partial \alpha} \bigg|_{\alpha=0} \text{Ad}_{g e^{\alpha z}} \dot{X} \\
= \text{Ad}_q [Z, X] \\
= \text{Ad}_q \dot{Y}.
\]

This completes the proof. □

**Lemma 3** If the action of \( \mathcal{P}(G, H \times K) \) on \( V \) is variationally complete, then it is hyperpolar.

**Proof.** Let \( N_0 \) in \( V \) be an orbit which is the preimage of a principal orbit of \( H \times K \) in \( G \) and \( \gamma \) a geodesic starting orthogonally to \( N_0 \) in \( p \), i.e., \( \xi = \gamma'(0) \) is in the normal space \( \nu_p(N_0) \). Let \( N_1 \) be the orbit through \( q = \gamma(1) \) and assume that also \( N_1 \) is the preimage of a principal orbit in \( G \). It follows from variational completeness that \( \gamma(1) \) is not a focal point of \( N_0 \) along \( \gamma \). We will prove that the tangent spaces of \( N_0 \) at \( \gamma(0) \) and of \( N_1 \) at \( \gamma(1) \) coincide if considered to be affine subspaces of \( V \). It follows that the normal spaces of \( N_0 \) at \( \gamma(0) \) and of \( N_1 \) at \( \gamma(1) \) coincide which obviously implies that the action is hyperpolar.

Let \( E_p \subset T_p N_0 \) be the direct sum of the eigenspaces of the Weingarten operator \( A_\xi^{N_0} \) corresponding to the nonvanishing eigenvalues and define a corresponding subspace \( E_q \subset T_q N_1 \) with respect to the Weingarten map \( A^{N_1}_\gamma \). Similarly let \( Z_p \) be the zero eigenspace in \( T_p N_0 \) and \( Z_q \) the zero eigenspace in \( T_q N_1 \). We will first prove that \( E_p = E_q \) using an argument from [3]. Then we will prove that \( Z_p = Z_q \). It will follow that \( T_p N_0 = T_q N_1 \) finishing the proof.

Let \( X \in E_p \) be an eigenvector corresponding to the nonvanishing eigenvalue \( \lambda \) of \( A_\xi^{N_0} \). Then \( J(t) = (1 - \lambda t)X \) is a transversal Jacobi field along \( \gamma \) that is tangent to the orbit \( N_0 \) and vanishes in \( t = \frac{1}{\lambda} \). Variational completeness now implies that this Jacobi field is induced by the action. Therefore \( J(t) \) is tangent to the orbit through \( \gamma(t) \) for every \( t \). Since \( \gamma(1) \) is not a focal point of \( N_0 \) along \( \gamma \), we have that \( 1 - \lambda \neq 0 \) and then \( X \) lies in \( T_q N_1 \). It is now clear have that \( X \) is an eigenvalue of \( A_\xi^{N_1} \) corresponding to a nonzero eigenvalue. This proves that \( E_p \subset E_q \). Analogous arguments imply \( E_q \subset E_p \) proving \( E_p = E_q \).

It is left to prove that \( Z_p = Z_q \). Let \( X \in Z_p \). Let \( \{ \psi_\alpha \} \) be a one-parameter subgroup of \( \mathcal{P}(G, H \times K) \) such that
\[
X = \left. \frac{d}{d\alpha} \right|_{\alpha=0} \psi_\alpha p.
\]
The variation \( \{ \psi_\alpha \gamma \} \) induces a Jacobi field \( J \) along \( \gamma \) with \( J(0) = X \). We first show that \( J(t) = X + t\eta \) where \( \eta \in \nu_p(N_0) \). In fact
\[
J'(0) = \left. \frac{d}{d\alpha} \right|_{\alpha=0} \psi_{\alpha \gamma} \xi = -A_\xi X + \nabla_{\alpha \delta}^\perp \psi_{\alpha\delta} \xi \big|_{\alpha=0} = \nabla_{\alpha \delta}^\perp \psi_{\alpha\delta} \xi \big|_{\alpha=0}
\]
which shows that \( J'(0) \) lies in \( \nu_p(N_0) \) as we wanted to prove.

Our next goal is to show that \( \eta = 0 \). Assume that \( \eta \neq 0 \). We know that \( X + \eta \in T_q N_1 \). Then \( \eta \notin \nu_q(N_1) \) since \( \langle X + \eta, \eta \rangle = ||\eta||^2 \neq 0 \). This implies that \( X + \eta \notin Z_q \) since otherwise
\[ J'(1) = \eta \] would be a normal vector. Let \( Y \in E_q \) be the image of \( X + \eta \) under the orthogonal projection along \( Z_q \) into \( E_q \). We have that \( Y \neq 0 \) or, which is the same, \( \langle Y, X + \eta \rangle \neq 0 \). We know that \( \langle Y, X \rangle = 0 \) since \( E_p = E_q \). We also have that \( \langle Y, \eta \rangle = 0 \) since \( Y \) is a tangent vector of \( N_0 \) in \( p \) and \( \eta \) is a normal vector. Hence we also get \( \langle Y, X + \eta \rangle \neq 0 \) which is a contradiction. This proves that \( \eta = 0 \). We therefore have that \( J(t) = X \) which implies that \( X \in T_q N_1 \) and it is clear that \( X \in Z_q \). The proof that \( Z_q \subset Z_p \) is analogous. Hence \( Z_p = Z_q \). \( \square \)

With the following lemma we finish the proof of the theorem in the introduction.

**Lemma 4** If the action of \( \mathcal{P}(G, H \times K) \) on \( V \) is hyperpolar, then the action of \( H \) on \( M \) is also hyperpolar.

**Proof.** Let \( \Sigma \) be a section of the action of \( \mathcal{P}(G, H \times K) \) on \( V \) and let \( A \) be the image of \( \Sigma \) under \( \varphi \) in \( G \). We will show that \( A \) is a flat section of the action of \( H \times K \) on \( G \). Since \( \Sigma \) is horizontal with respect to the Riemannian submersion \( \varphi \), we have that \( \varphi|_\Sigma : \Sigma \to G \) is an isometric immersion. Property (c) before Lemma 2 implies that \( d\varphi_u(T_u \Sigma) \) is perpendicular to the \( H \times K \)-orbit through \( \varphi(u) \) in \( G \) for all \( u \in V \). It follows that \( A \) is a submanifold in \( G \) which meets all orbits perpendicularly. Moreover \( A \) is flat since \( \varphi|_\Sigma \) is an isometric immersion. This finishes the proof that the action of \( H \times K \) on \( G \) is hyperpolar. Similar arguments show that \( \pi(A) \) is a flat section of the action of \( H \) on \( M \). \( \square \)

It is interesting to remark that it follows from the results in section 2 of [8] that, in the case where the metric in \( G \) is induced from the Killing form of \( g \), we have that \( A \) is a torus, and hence \( A \) and \( \pi(A) \) are properly embedded sections. In the case where the metric in \( G \) is not induced from the Killing form of \( g \), we do not know of any example of a hyperpolar action with a section that is not properly embedded. Here we have assumed the symmetric space \( M \) to be of compact type. On the other hand, if the symmetric space \( M \) is a torus, a hyperpolar action can have a section that is not properly embedded. To see this let \( T^k \) be a proper subtorus of \( M \) that we think of as a Lie group. The orbits of \( T^k \) are the cosets of \( T^k \), and the image \( \Sigma \) under the exponential map of the normal space of \( T^k \) in \( M \) at the identity element is clearly a section. If \( M \) is not a rational torus, the subtorus \( T^k \) can be chosen so that \( \Sigma \) is not properly embedded, see section 2 in [8].

**References**

[1] R. Bott, *An application of the Morse theory to the topology of Lie groups*, Bull. Soc. Math. France **84** (1956), 251–281.

[2] R. Bott and H. Samelson, *Applications of the theory of Morse to symmetric spaces*, Amer. J. Math. **80** (1958), 964–1029, correction in Amer. J. Math. **83** (1961), 207–208.

[3] L. Conlon, *Variational completeness and \( K \)-transversal domains*, J. Differential Geom. **5** (1971), 135–147.

[4] J. Dadok, *Polar actions induced by actions of compact Lie groups*, Trans. Amer. Math. Soc. **288** (1985), 125–137.
[5] A. J. Di Scala and C. Olmos, *Variationally complete representations are polar*, Proc. Amer. Math. Soc. **129** (2001), 3445–3446.

[6] C. Gorodski and G. Thorbergsson, *Representations of compact Lie groups and the osculating spaces of their orbits*, Preprint, University of Cologne, 2000 (also E-print math. DG/0203196).

[7] E. Heintze, R. S. Palais, C.-L. Terng, and G. Thorbergsson, *Hyperpolar actions and k-flat homogeneous spaces*, J. Reine Angew. Math. **454** (1994), 163–179.

[8] ——, *Hyperpolar actions on symmetric spaces*, Geometry, Topology, and Physics for Raoul Bott (S. T. Yau, ed.), Conf. Proc. Lecture Notes Geom. Topology, IV, International Press, Cambridge, MA, 1995, pp. 214–245.

[9] R. Hermann, *Variational completeness for compact symmetric spaces*, Proc. Amer. Math. Soc. **11** (1960), 544–546.

[10] A. Kollross, *A classification of hyperpolar and cohomogeneity one actions*, Trans. Amer. Math. Soc. **354** (2002), 571–612.

[11] C.-L. Terng, *Proper Fredholm submanifolds of Hilbert space*, J. Differential Geom. **29** (1989), 9–47.

[12] ——, *Polar actions on Hilbert space*, J. Geom. Anal. **5** (1995), 129–150.

[13] C.-L. Terng and G. Thorbergsson, *Submanifold geometry in symmetric spaces*, J. Differential Geom. **42** (1995), 665–718.