Adequacy of Effective Born for electroweak effects
and TauSpinner algorithms for LEP, Tevatron, HL-LHC and FCC simulated samples.

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ABSTRACT

Matching and comparing the measurements of the past and future experiments calls for consistency checks of calculations used for their interpretation. On the other hand, new calculation schemes of the field theory can be beneficial for precision, even if they may obscure comparisons with earlier results. Over the years concepts of Improved Born, Effective Born as well as of effective couplings, in particular of $\sin^2 \theta_W^{\text{eff}}$ mixing angle for electroweak interactions, evolved.

In our discussion we use four DIZET electroweak library versions; that of today and of the last 30 years. They were used for phenomenology of practically all HEP accelerator experiments. Versions differ by incremental updates of certain types of corrections which became available with time. We rely on the codes published and archived with KKMC Monte Carlo program for $e^+e^- \rightarrow f \bar{f} n(\gamma)$. All these versions became recently available for TauSpinner algorithm of simulated event reweighting as well. Such reweighting can be performed after events are generated and stored in the data files. To this end DIZET is first invoked, and its results are used. Documentation of TauSpinner upgrade, to version 2.1.0, and of its arrangement for semi-automated EW effects benchmark plots are provided. Some details of the tool which can be used to complete simulation sample with electroweak effects are given.

Focus is however on the numerical results, on the different approximations introduced in Improved Born to obtain Effective Born, suited better to match with QCD corrections. The $\tau$ lepton polarization $P_\tau$, forward backward asymmetry $A_{FB}$ and parton level total cross section $\sigma_{tot}$ are used to monitor size of electroweak effects and effective $\sin^2 \theta_W^{\text{eff}}$ picture limitations for precision physics. Collected results enumerate: (i) feasibility of Effective Born approximation and $\sin^2 \theta_W^{\text{eff}}$, (ii) differences between versions of electroweak libraries and (iii) parametric ambiguities due to e.g. $m_{\tau}$ or $\Delta \alpha_W^{(5)}(s)$. These results can be considered as examples only, but allow to evaluate adequacy of Effective Born with respect of Improved Born (definitions are addressed too).

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1 Introduction

One of precision high energy physics great achievements in electroweak sector measurements is establishing that quantum field theory can be indeed used to calculate matching with measurement predictions [1,2]. To handle results and interpretation, concept of idealized observables was very useful [3]. Over many years the effective electroweak mixing angle $\sin^2 \theta_W^{\text{eff}}$ (of the process $e^+e^- \rightarrow Z/\gamma^* \rightarrow f\bar{f}$, where $f$ denote leptons or quarks) was a prime candidate to evaluate sensitivity of observations of the electroweak sectors and in fact established itself as the idealized observable [2]. This universal quantity is not of the principal nature and to remain useful, confirmation that at the requested precision level necessary approximation holds is needed. Already in the past, limits for the validity of the assumptions made were investigated. Let us point to early reference [4], where dependence on flavour and flavour dependent $\sin^2 \theta_W^{\text{eff}}$ were elaborated.

In the calculation scheme used at LEP times, EW corrections calculated at one-loop level, were improved with selected, dominant higher order terms and embedded in Improved Born Approximation [5]. This formulation was a cornerstone for LEP measurements [1]. The Effective Born means redefinition of the coupling constants to the values incorporating dominant contributions from the higher order (loop) corrections (at fixed energy, e.g. $\sqrt{s}$ or $M_Z$). Such redefinition usually does not break properties for factorizing out calculation of strong interaction effects. In contrary, Improved Born where couplings are accompanied with energy and angle dependent complex form-factors if used directly, can easily damage strong interaction gauge invariance and enforce necessity to simultaneous calculation of the electroweak and QCD effects, without conveniences of factorization. The effective coupling definitions rely on properties of electroweak loop corrections, which could invalidate the concept of Effective Born and $\sin^2 \theta_W^{\text{eff}}$ definition as well. The concept evolved over time, let us recall some references [6-8,10]. In fact, approach variants, controversies and differences in conventions can be now easily identified and avoided. One should keep in mind that such conventions were used in data analyzes of the past. With the improving measurements precision, one has to readdress validity of assumptions and approximations necessary for definitions and usefulness of Effective and Improved Born.

To address the above points one need to investigate first if Improved or Effective Born can be separated with sufficient precision from complete calculations including strong interactions. References [9,10] were devoted to study how strong interaction separates in LHC processes of $W$ and $Z$ boson production and decay. The study was also useful for validation of TauSpinner event reweighting algorithm [11,12] in its implementation of electroweak effects [13]. Similar evaluations were completed in the past for $e^+e^-$ collisions in context of Monte Carlo generators [14,15,16] and for semi-analytical calculations in [5]. One of the important numerical assumption was that in all these applications, numerical differences between Improved and Effective Born are, around $Z$ pole, not too large. The one loop genuine weak corrections were usually calculated for all these projects with the help of DIZET library [17]. Important was inclusion of some higher order strong interaction or QED effects. One has always to check if necessary higher order contributions to weak loop effects can be (and are) introduced into Effective or Improved Born and thus partly re-summed as well. For DIZET archivization and evolution see Ref. [18] and references therein.

Obviously, whenever precision is expected to improve, assumptions behind calculations need to be revisited. The TauSpinner algorithms can be helpful in that respect and used to evaluate if for a given observable some classes of the corrections are necessary or can be ignored. Independently if old calculations are sufficient or if the new one may be necessary, it is useful to establish which of the effects need to be taken into account.

We focus on discussion/evaluation of: (i) suitability of Effective versus Improved Born approximation (also usefulness of $\sin^2 \theta_W^{\text{eff}}$), (ii) differences between results of DIZET library versions in use over the last 30 years, (iii) ambiguities due to parametric uncertainties or due to some (sometimes) missing contributions. We explain minor but useful for studies of electroweak effects TauSpinner extensions with respect to Ref. [13] too.

In the scope of the paper, we present numerical results, either from semi-analytical calculations (predominantly for $e^+e^- \rightarrow l^+l^-$ processes) or from reweighting of the Monte Carlo event samples (predominantly for $pp \rightarrow Z/\gamma^* \rightarrow l^+l^-$ processes at 8 TeV center of mass collisions). In the second case we use Powheg Monte Carlo $Z + \text{jet}$ events as described in Refs. [9,13].

Section 2 recalls basic definitions necessary for the introduction of the Improved Born, practical comments on $\sin^2 \theta_W^{\text{eff}}$ definitions are provided. Simplifications enabling introduction of Effective Born are explained as well. The purpose of Section 3 is to recall first numerical results on non-electroweak effects which are instrumental for the concept of Effective Born and effective couplings. Both Improved Born and Effective Born require interpolation of the $2 \rightarrow 2$ kinematics and evaluation of its scattering angle. This requires careful optimization in presence of QED/QCD initial and final state emissions, see references above, which discuss the issue. In this context redefinition of couplings is convenient as such couplings can be usually used in strong interactions amplitudes without complication of gauge dependence restoration. Alternatively re-weighting to Improved Born can be advocated following e.g. TauSpinner solution of Ref. [13]. The particular choice will need to be decided by the user precision requirements. That is why we do not give any guidelines here. Definition of simplified test observables, used all over the paper, are provided.

Section 4 is devoted for comparison of Improved Born and Effective Born. In particular results useful to evaluate precision of Effective Born and $\sin^2 \theta_W^{\text{eff}}$ approximations with respective of Improved Born are provided. Most of the numerical
2 Improved Born and electroweak form-factors.

In Improved Born Approximation, the complete $O(\alpha)$ EW corrections, supplemented by selected higher order terms, are handled with form-factor corrections, dependent on $(s,t)$, multiplying couplings and propagators of the usual Born expressions. Let us continue with definition of Improved Born, we use in TauSpinner. It was detailed already in Ref. [13] but we will recall it with Eq. (1) for the process $e^+e^- \rightarrow f\bar{f}$. The formula can be used also in the case when initial and final state are interchanged. The $z$ component of the fermion isospin $T_3^{e,f}$, electroweak mixing angle $\sin^2\theta_W$, $c_W^2 = 1 - s_W^2$ and electric charge $q_{e,f}$ are used, as usual, for the coupling constants calculation. The Mandelstam variables $s = (p_e + p_{\bar{e}})^2$ and $t = (p_f - p_{\bar{f}})^2$ are used for the kinematical dependance. Fermi coupling $G_F$, QED coupling constant $\alpha$, the Z boson mass $M_Z$ and width $\Gamma_Z$ complete basic notations. Definition of electroweak form-factors $K_{e}(s,t), K_{f}(s,t), K_{ef}(s,t)$, $\ell = e, \mu, \tau$, the $\rho_{f}(s,t)$, and photon vacuum polarization $\Pi_{\gamma\gamma}(s)$, are as used in [13]. It is important that they are only weakly dependent on $t$, and the $s$ dependence is not sizable as well.

$$ME_{\text{Born+EW}} = \mathcal{N}_T \left[ [\bar{u}^{\rho}\gamma_\mu \gamma_5 u] \cdot (q_e \cdot q_f) \cdot (v_e \cdot v_f) \cdot \chi_T(s) \right] + [\bar{u}^{\rho}\gamma_\mu \gamma_5 u] \cdot (v_e \cdot v_f) + [\bar{u}^{\rho}\gamma_\mu \gamma_5 u] \cdot (a_e \cdot a_f) \cdot Z_{V_{hi}} \cdot \chi_Z(s) \right],$$

$$v_e = (2 \cdot T_3^e - q_e \cdot s_W^2 \cdot K_f(s,t))/\Delta,$n$$v_f = (2 \cdot T_3^f - q_f \cdot s_W^2 \cdot K_s(s,t))/\Delta,$$n$$a_e = (2 \cdot T_3^e)/\Delta,$$n$$a_f = (2 \cdot T_3^f)/\Delta,$$n$$\chi_Z(s) = \frac{G_F \cdot s}{\sqrt{2} \cdot 8 \pi \cdot \alpha \cdot M_Z^2} \cdot \frac{s}{s - M_Z^2 + i \cdot \Gamma_Z \cdot s / M_Z},$$n$$\Gamma_{V_{hi}} = \frac{1}{2 \cdot (1 + \Pi_{\gamma\gamma}(s))},$$n$$Z_{V_{hi}} = \rho_{f}(s,t),$$n$$\chi_T(s) = 1,$n

$$v_{ve} = \frac{1}{v_e \cdot v_f} \cdot \left[ (2 \cdot T_3^e) \cdot (2 \cdot T_3^f) - q_e \cdot s_W^2 \cdot K_f(s,t) - 4 \cdot q_f \cdot s_W^2 \cdot K_e(s,t) \right] \cdot \frac{1}{\Delta^2}.$$n

In the formula $u$, $v$ stand for spinors - fermions wave functions and $\mathcal{N}_T$ normalization factor which is convention dependent (e.g. for wave functions normalization).

The formula (1) is not the only possibility for implementation of EW corrections. Genuine electroweak corrections can be, under some conditions, combined with the ones of QED or strong interactions. For example in KKMC implementation, Improved

\footnote{At the LO EW $K_{e}(s,t) = K_{f}(s,t) = 0$, $\Pi_{\gamma\gamma}(s) = 0$ and $\frac{GM_Z^2}{2 \sin^2 \theta_W} = 1$. We use $s_W^2 = 1 - M_W^2 / M_Z^2$ for the on-mass-shell definition, while $\sin^2 \theta_W$ is used for the effective value corresponding to the ratio of couplings at the Z-pole $\bar{v}_W^\ell / \bar{a}_W^\ell = 1 - 4 q_f \sin^2 \theta_W^{\ell \ell}$.}
or Effective Born approximation is not used, electroweak form-factors are installed into spin amplitudes directly\(^2\). Care of the gauge cancellation was essential for that. The “running” \( \Gamma_Z \) for \( \chi_Z(s) \) propagator is used in Eq. (1) as was commonly the case for LEP physics, but less so for LHC oriented MC’s. Another possibility is to use Effective Born (not exact at one loop level, it will be discussed in Section 3.3). The main idea is to simplify formula (1). In particular, include in \( \sin^2 \theta_W^{eff} \) the bulk effect of \( K_e(s,t) \) and \( K_f(s,t) \) form-factors present in front of \( s_W^2 \), and vacuum polarization corrections \( \Pi_{eff} \) into redefinition of \( \alpha \). In fact a real part or a module, and all calculated at the Z pole. It is also possible to think of \( \sin^2 \theta_W^{eff} \) as the best result of the fit to the data. At certain precision level, such distinction may start to play a role and one should be aware of alternatives to avoid misunderstandings.

Definitions of form-factors follows the one of the DIZET library and are as used for spin amplitudes in KKMC Monte Carlo as well. For convenience they are used in Improved Born, formula (1), of TauSpinner too. Details of EW scheme, initialization parameters are provided in Appendix A in Fig. 1) electroweak form-factors calculated with DIZET library are drawn.

Disadvantage of the Improved Born with respect to the Effective one, is that it is not immediate to merge its formulation into calculation for strong interactions. The formula (1) can be used as a starting point to explain relation between Improved Born and Effective Born, where \( s \) and \( t \) dependent form-factors are avoided. For the latter one, form-factors are often set to unity, but may be replaced with the real constants without compromising strong interaction calculations. One should keep in mind that numerical values for e.g. \( s_W^2 \) and \( \alpha \) need then to be chosen differently, also to accumulate dominant numerical contributions of the loop corrections. Technical details for the arrangements used in the programs are collected in Table 12 of the Appendix C.

### 3 Factorization requirements for Born amplitudes

In general case, but especially for lepton pair production at LHC, definition and use of the quark \( 2 \to 2 \) Born level scattering as a building block for phenomenology picture, may seem to be always resulting with rather crude and difficult to control approximation. Let us recall arguments, why it is not necessarily the case.

Properties of Born level spin amplitudes, lead to features necessary for its factorization from complete formulae. Quality of such separation is of a decisive importance. Already long time ago [20], even in the presence of hard bremsstrahlung photons, part of amplitudes which corresponds to the Born level distribution, were identified and separated out. This was studied in the context of hadronic kinematic configuration of \( pp \) LHC physics as well [6][10] and for configurations with up to two high \( p_T \) jets. These factorization properties were expected, thanks to e.g. results of [24]. They require that the Born cross section is described by spherical harmonics of the second order. Indeed, Born cross-section, for \( f \bar{f} \to Z/\gamma^* \to \ell^+\ell^- \) reads (azimuthal angle dependence can be avoided with appropriate choice of the reference frame):

\[
\frac{d \sigma^{\text{Born}}}{d \cos \theta}(s, \cos \theta, p) = (1 + \cos^2 \theta) F_0(s) + 2 \cos \theta F_1(s) - p(1 + \cos^2 \theta) F_2(s) + 2 \cos \theta F_3(s)
\]

where \( p \) denotes polarization of the outgoing leptons, \( \theta \) an angle between incoming quark and outgoing lepton in the rest frame of outgoing leptons. For general orientation of the reference frame all second order spherical harmonics in \( \theta \) and \( \phi \) angles appear. Second order spherical harmonics are sufficient also when transverse spin effects are taken into account.

The \( F_i \) read:

\[
F_0(s) = \frac{\pi \alpha^2}{2 s}[\chi_f(s)\chi_f^*(s)] + 2 \cdot \chi_f(s) Re \chi_Z(s) q_f q_v v_f v_e + |\chi_Z(s)|^2 (v_f^2 + a_f^2)(v_e^2 + a_e^2),
\]

\[
F_1(s) = \frac{\pi \alpha^2}{2 s}[2 \chi_f(s) Re \chi_Z(s) q_f q_v v_f v_e + |\chi_Z(s)|^2 2 v_f a_f 2 v_e a_e],
\]

\[
F_2(s) = \frac{\pi \alpha^2}{2 s}[2 \chi_f(s) Re \chi_Z(s) q_f q_v v_f v_e + |\chi_Z(s)|^2 (v_f^2 + a_f^2)2 v_e a_e],
\]

\[
F_3(s) = \frac{\pi \alpha^2}{2 s}[2 \chi_f(s) Re \chi_Z(s) q_f q_v v_f v_e + |\chi_Z(s)|^2 (v_f^2 + a_f^2)2 v_e a_e],
\]

Unfortunately with \( s, t \) dependent electroweak form-factors of Eq. (1), assumption on spherical harmonics decomposition of the second order only, does not hold. Inevitable approximation needs to be re-checked if it matches the required precision.\(^2\)

\(^2\)That could be disastrous, as gauge cancellations would be broken. Spinor techniques of Kleiss-Stirling are exploited for KKMC Monte Carlo of \( e^+e^- \to l^+l^- \gamma \gamma^* \) processes, where second order QED matrix element and coherent exclusive exponentiation is used [16][15]. Nonphysical huge contributions proportional even to \( \sim 1/m_Z^2 \) could appear. However this is not the case as contributions to Yennie Frautchi Suura spin amplitude level \( \hat{p}_0, \hat{p}_1, ... \) terms are calculated explicitly and gauge cancellations are explicitly performed, before electroweak form-factors installation.

One needs to keep in mind this difficulty when other applications are developed. Even in KKMC case this needed to be watched after, in the context of e.g. IFI interference contribution, where cancellation of real and virtual corrections play a role and may be obscured by energy angular dependence of form-factors. Formally of higher order, such mismatches could substantially impact conclusions if attention was not paid.
Figure 1: Plots from new examples/Dizet-example directory of TauSpinner: Real parts of the $\rho_{e,up}$, $\rho_{e}$, $\rho_{up}$ and $\rho_{e,up}$. EW form-factors of $ee \to Z \to u\bar{u}$ process, as a function of $\sqrt{s}$ and for the few values of $\cos \theta$. Note, that $\rho_{e,up}$ depends on the flavour of outgoing quarks. On the last plot (bottom line) the $\alpha(s)/\alpha(0) = \Gamma_{11}$ is also shown.
dependence integrated over, without (left-hand plot) and with (right-hand plot) box corrections. The DIZET 6.45 using be evaluated. For example, figure where $e^+e^-$, or parton level, total cross sections and asymmetries, like introduced later Fig. 4 can be used. The $\sin^2 \theta_W^{\text{eff}}$ supposedly represents typical observable and/or coupling constant. That is why Fig. 2 of $\sqrt{s}$ and flavour dependent $\sin^2 \theta_W^{\text{eff}}$, provides a hint on the size of the effect as well. For $t$-dependence electroweak boxes contribute with correction of isospin dependent sign. On the other hand, the $\sin^2 \theta_W^{\text{eff}}$ variations remain below $20 \cdot 10^{-5}$ in the range of $M_Z \pm 5 \text{GeV}$. The $t$ dependence originating from $WW$ and $ZZ$ boxes becomes sizable once $s$ approaches $4M_W^2$, the $W$-pair production threshold.

Note that even if for a given observable, the nearby of the $Z$ peak dominates for LEP (or even FCC), this may not be the case for LHC or linear colliders. There off the $Z$ peak contributions are larger due to PDF or beamstrahlung spread. This needs to be kept in mind, also in context of the $\sin^2 \theta_W^{\text{eff}}$ interpretation as universal idealized observable. Let us recall that off the $Z$ peak form-factors dependence on flavour and scattering angle increase, see Fig. 1. It is therefore of interest to validate range where form-factors, angle, and energy dependence can be safely ignored and thus Effective Born used.

### 3.1 Improved Born Approximation and Effective Born

In principle it is not possible to absorb fully effects of electroweak form-factors of Eq. (1) (which are all complex and angle/energy dependent), into re-scaling of constants. In particular introduction of $\sin^2 \theta_W^{\text{eff}}$ is prone to complications and ambiguities (see footnote1 earlier in the text). Let us now recall details of Effective Born amplitude definition, which differs, from formula (1): the form-factors are replaced by effective coupling constants (see also Table 12 of Appendix C).

\[
ME_{\text{Born-eff}} = \mathcal{N} \left\{ \begin{array}{c}
[\bar{q}^\mu q^\nu \bar{\psi}^\alpha u^\beta] \cdot (q_e \cdot q_f) \cdot \Gamma_{V11} \cdot \chi_f(s) \\
+ [\bar{u}^\mu u^\nu \bar{\psi}^\beta v^\alpha] \cdot (v_e \cdot v_f \cdot v_{e_f}) \cdot \bar{u}^\mu u^\nu \bar{\psi}^\beta v^\alpha \cdot (v_e \cdot a_f) \\
+ \bar{u}^\mu u^\nu \bar{\psi}^\beta v^\alpha \cdot (a_e \cdot v_f) \cdot \bar{u}^\mu u^\nu \bar{\psi}^\beta v^\alpha \cdot (a_e \cdot a_f) \cdot Z_{V11} \cdot \chi_Z(s) \end{array} \right\}
\]
\[ \nu_e = \frac{(2 \cdot T_3^e - 4 \cdot q_e \cdot s_{W}^2)}{\Delta} \]
\[ \nu_f = \frac{(2 \cdot T_3^f - 4 \cdot q_f \cdot s_{W}^2)}{\Delta} \]
\[ \alpha_e = \frac{(2 \cdot T_3^e)}{\Delta}, \quad \Delta = 4s_w c_w \]
\[ \alpha_f = \frac{(2 \cdot T_3^f)}{\Delta}, \quad \chi z(s) = \frac{\mu^2 \cdot M^2 \cdot \Delta^2 \cdot s}{\sqrt{2} \cdot 8 \pi \cdot \alpha \cdot (s - M_Z^2 + i \cdot \Gamma_Z \cdot s/M_Z)}. \]
\[ \Gamma_{\nu i} = 1, \quad Z_{\nu i} = \text{Re} \rho_{\nu i}(M_Z^2), \quad \chi \ell (s) = 1, \]
\[ \nu_{\nu_{ef}} = 1, \quad s_{W} = (1 - c_w^2) = \sin^2 \theta_{W}^{eff}(M_Z^2), \]
\[ \alpha = \alpha(M_Z^2) = \frac{\alpha(0)}{2 - (1 - \text{Re} \rho_{\nu i}(M_Z^2))}. \]

Note that \( \sin^2 \theta_{W}^{eff}(M_Z^2), \rho_{\nu i}(M_Z^2) \) and \( \alpha(M_Z^2) \) are now used. They absorb dominant parts of EW corrections; EW form-factors and vacuum polarization corrections. This useful approximation may take into account bulk of the electroweak effects, and couplings of fixed values are used. There is some level of ambiguity in the numerical values. The best match to Improved Born, should correspond to the values predicted by these calculations. In particular \( \sin^2 \theta_{W}^{eff}(M_Z^2) \) = \( \text{Re}K(M_Z^2, -M_Z^2/2)\bar{s}_{W}^2 \), \( \bar{s}_{W}^2 = 1 - M_Z^2/M_Z^2 \), where \( M_W \) is a calculated quantity including EW corrections. The \( s = M_Z^2, t = -M_Z^2/2 \), correspond to Born level, scattering angle \( \theta = 0 \). Alternatively, one could use best measured values \([11][12]\) and not rely on the EW calculations. This may be of importance for calculations which are focused e.g. on strong interactions. Complications due to mixed EW and strong interaction loops in Feynman diagrams i.e. gauge dependence non-cancellation cannot be avoided. The price is precision limitation with respect to Improved Born. Such an approach was used for the previous Tauola/TauSpinner ME implementation, with \( \sin^2 \theta_{W}^{eff}(M_Z^2) \), \( \alpha(M_Z^2) \) as measured at LEP [12] and \( \rho_{\nu i} = 1 \) for simplification.

To monitor EW effects we use \( e^+e^- (q\bar{q}) \) total cross section \( \sigma^{tot} \), forward-backward asymmetry \( A_{FB} \) and \( \tau \) lepton polarization \( P_{\tau} \), as a function of \( \sqrt{s} \).

The \( Z/\gamma^* \)-boson \( e^+e^- (q\bar{q}) \) cross section \( \sigma^{tot} \): in the EW LO, the \( e^+e^- (q\bar{q}) \) cross section \( \sigma^{tot} \) depends only on coupling constants and two parameters \( (M_Z, \Gamma_z) \). The effect on \( \sigma^{tot} \) from EW loop corrections are due to corrections to the propagators: vacuum polarization corrections (running \( \alpha \) and \( \rho \) form-factor, causing change in relative contributions of the \( Z \) and \( \gamma \); and change of the \( Z \)-boson vector to axial coupling ratio \( \sin^2 \theta_{W}^{eff} \). They affect not only \( s \)-dependence but normalization of the cross-section too.

The forward-backward asymmetry \( A_{FB} = \frac{\sigma(R) - \sigma(L)}{\sigma(R) + \sigma(L)} \), is defined in a standard way. For \( e^+e^- \) collision, an angle \( \theta \) between incoming particle and outgoing lepton is taken. For \( pp \) collision \( \text{Collins-Soper frame} \) [22] is used for angle \( \theta \) definition. The asymmetry varies strongly with \( \sqrt{s} \) around the \( Z \) peak, because it is proportional to product of small vector \( v_i \) and \( v_f \) couplings of incoming parton and outgoing lepton. The product is specially small for \( e^+e^- \) initial state. That is why, off the peak, \( s \)-channel \( Z \)-photon exchange interference, quickly become sizable.

The \( \tau \) polarization \( P_{\tau} = \frac{\sigma(R) - \sigma(L)}{\sigma(R) + \sigma(L)} \), where \( \sigma(R/L) \) denote cross section for production of right/left hand polarized \( \tau \), is of interest in itself as it offers independent data-point for precision electroweak sector measurements. It is of convenience, because, in the first approximation, it is linearly proportional to \( \sin^2 \theta_{W}^{eff} \), thus it is useful for discussion of systematic ambiguities. The systematic errors for this measurement differ from that of \( \sigma^{tot} \) or \( A_{FB} \). Predominantly because \( P_{\tau} \) is not measured directly, but through distribution of \( \tau \) decay products only. These points were recently recalled in [23]. On the other hand, relation between \( P_{\tau} \), \( Z \) couplings and \( \sin^2 \theta_{W}^{eff} \) is not affected and generally is of the same nature like for \( A_{FB} \). That is why this data point is particularly suitable for discussion with \( e^+e^- \) semi-analytical results.

Test observables and \( \sin^2 \theta_{W}^{eff} \)

It is worth to point that of \( e^+e^- \) scattering results, the ones for \( P_{\tau} \) are particularly convenient in discussion of \( \sin^2 \theta_{W}^{eff} \). This is because \( P_{\tau} \) is the linearly proportional to small vector \( Z \)-lepton coupling, thus to \( \sin^2 \theta_{W}^{eff} \) itself. Also, the \( P_{\tau} \) varies with energy in the vicinity of the \( Z \) peak relatively slowly. To a good approximation, as one can easily deduce form formula [4] any variation \( \delta \) of measured \( P_{\tau} \) at the \( Z \) peak, translates into \( \frac{1}{8} \delta \) shift of \( \sin^2 \theta_{W}^{eff} \), quite independently of the flavour of incoming state. This holds not only for \( e^+e^- \) but for incoming quarks too. For \( A_{FB} \) and \( \sigma^{tot} \) similar relations can be obtained, then \( \rho \) and \( \alpha \) dependence would need to be taken into the picture. The initial state flavour and much stronger energy dependence would lead to multitude of cases. That is why we will use \( P_{\tau} \) as an example to discuss suitability of \( \sin^2 \theta_{W}^{eff} \) picture and its limitations.

\[ \text{At the Z peak } P_{\tau} \simeq \frac{2\delta}{\alpha_e}. \]
Fig. for the $e^+ e^-$ case is shown for the start of numerical comparisons of Improved Born and Effective Born (TAUOLA/LEP initialization as specified later and installed in Tauola distribution) is shown. Differences are not large, but possibly not always satisfactory for precision physics. In fact TAUOLA/LEP Effective Born becomes insufficient for high precision measurements, especially of hadron colliders (where off the Z peak contributions, contrary to the FCC, can not be minimized/excluded by the fixed colliding quark energies).

Figure 3: The $\sigma_{tot}(s), A_{FB}(s)$ and $P_t(s)$ (left side, top, center and bottom plots respectively) of TauSpinner calculation with Effective Born (Tauola/LEP as installed in Tauola since December 2019). In the right side plots results of Improved Born calculations with Electroweak form-factors from DIZET 6.45 are compared with those of the left side plots. Note the differences depicted in the right side plots are enhanced in part because input parameters of Tauola/LEP initialization, see Table 1 for details. The $M_z$ and $\Gamma_Z$ are slightly different than those used for Improved Born, causing sizable twinkle in the top right plot.


4 From Improved Born to Effective Born: numerical results.

Let us now attempt to identify those Effective Born simplifications which are of numerical consequences and those which are important from the theoretical perspective, but hopefully not so much numerically. The results will be now compared with respect to Improved Born results, the most precise ones. We keep all input parameters as of Improved Born but gradually simplify EW correcting terms.

It is helpful to test and to understand impact of simplification steps from Improved → Effective Born, on $\sigma^{\text{tot}}, A_{FB}$ and $P_t$ and for all the elementary processes: $e^+e^- \to \tau^+\tau^-$ and $u\bar{u}(d\bar{d}) \to \tau^+\tau^-$. We concentrate on $e^+e^- \to \tau^+\tau^-$ process and choose for Figs. 4 and 5 the energy range important for the measurement of the $Z$ boson couplings that is $M_Z \pm 5\text{GeV}$. The $e^+e^-$ case is simpler to present and conclusion would not differ much if instead quark level processes would be used. For the pp collision, parton distribution functions would make discussion obscured and if dropped out, unrealistic. For reference Improved Born results we use DIZET 6.45.

Approximation as of green points of Fig.4

in this first step of approximation introduced into Eq. $\gamma\gamma$ the $s$ and $t$ dependent form-factors are replaced with their values at the $Z$ peak and for the scattering angle $\cos\theta = 0$ (marked “complex of $Z$ peak”).

One can see, that if constant complex couplings calculated at the $Z$ peak instead of $(s,t)$ dependent form-factors are used, in the range of $M_Z \pm 5\text{GeV}$ the $P_t, A_{FB}$ and $\sigma^{\text{tot}}$ departs from the exact result at the peak, respectively by up to $8 \cdot 10^{-5}, 45 \cdot 10^{-5}$ and $40 \cdot 10^{-5}$. These largest differences are at the edge of the range, where cross section is already about a factor 20 smaller than at the peak.

If in addition $\nu\nu_{\text{ef}}$ was set to 1, additional changes were marginal, that is why in the Figure case of $\nu\nu_{\text{ef}} = 1$ is not presented. Then the mixing term is avoided and effective couplings are attributed to incoming outgoing flavour suffice.

Approximation as of blue triangles in Fig.4

marked “real $\nu$, $\nu\nu=1$”, the second step is to neglect imaginary parts of the vector couplings to $Z$. They are about factor of 100 smaller than the real parts. Now, the differences became larger, respectively up to $166 \cdot 10^{-5}, 50 \cdot 10^{-5}$ and $32 \cdot 10^{-5}$. That means non-negligible degradation for $P_t$, corresponding, in the language of $\sin^2\theta_w^{\text{eff}}$ to $21 \cdot 10^{-5}$ prediction ambiguity.

Approximations as of red triangles and yellow stars in Fig.4

marked respectively “real $\Pi_{\tau\mu}$” and “LEP2005 style”, the role of imaginary part of $\Pi_{\tau\mu}$ requires special attention, particularly for $A_{FB}$. For red triangles, with respect to previous case, imaginary parts of $\Pi_{\tau\mu}$ and $\rho_{\ell/f}$ are set to zero, whereas for yellow stars, imaginary part of $\rho_{\ell/f}$ is set to zero only. The differences for “real $\Pi_{\tau\mu}$” (“LEP 2005 style”) are respectively $152 \cdot 10^{-5}, 188 \cdot 10^{-5}, 32 \cdot 10^{-5}, (172 \cdot 10^{-5}, 60 \cdot 10^{-5}, 30 \cdot 10^{-5})$. Numerical effect of these imaginary parts, which can not be easily absorbed in redefinition of the couplings, need to be kept in mind.

With “LEP2005 style” parametrization we still do not address more subtle LEP time choices used in data analysis. In particular, of parametrisations used to separate contributions from s-channel exchange of $Z$ boson and virtual photon exchange interfering background. In practice, in “LEP2005 style” variant, we use formula $\gamma\gamma$ but with $K_{\ell/f}(s,t) \to \text{Re}K_{\ell/f}(M_Z^2, -M_Z^2/2), K_{\ell/f}(s,t) \to 1, \rho_{\ell/f} \to \text{Re}\rho_{\ell/f}(M_Z^2)$, that translates into use of flavour dependent $\sin^2\theta_w^{\text{eff}}$ of $(M_Z^2)$. For $\alpha(M_Z^2)$ the replacement $\Pi_{\ell\mu}(s) \to \Pi_{\ell\mu}(M_Z^2)$ with complex value is used. Now the purpose is to evaluate numerical consequences of $\alpha(s)$ imaginary part. But it is also similar to what was used at a time of final precision data analysis of all LEP collaborations combined.

Motivations of the choices are discussed in Ref. 7. Numerical impact is presented in Table 19 and is discussed in Section 5.4 of that reference, see also Section 5.4.4 of 2.

Figure 4 is accompanied with extensive Table 14 of Appendix C. In total, results of eleven initializations variants are used for the Table. Of those, four are used in the figure. The variants, with gradually introduced simplifications to Improved Born EW corrections, were chosen. Most of the results were obtained with semi-analytic scripts of TauSpinner package, described in Appendix B. Details of the initializations are depicted in Table 13.

Our main observations:

(i) The $\Pi_{\tau\mu}$ imaginary part, formally contributing at higher orders, was included in calculations for final LEP time data analysis. Its impact is largest for $A_{FB}$ whereas for $P_t$ imaginary parts of $\nu_s, \nu_t$ couplings are more important. (ii) The form-factors replacement with constant effective couplings is numerically less important than when their imaginary parts are dropped. Also, closer to the $Z$ peak one goes, the smaller is disturbing for Effective Born picture, contribution of photon exchange. The same is true for complex part of $\rho_{\ell/f}$. In overall, numerical impact on observables is not universal and distinct sets of effective couplings might be needed for each of our test observables to match best result of Improved Born.

4.1 The $\nu_0$, $\nu_1$, $\nu_2$ variants of Effective Born.

The formulae for Improved Born Eq. $\gamma\gamma$ and for Effective Born Eq. 4 differ with minor, but numerically important details. We will evaluate numerical effects again with the help of options in TauSpinner explained in Appendix C in particular in Table 15.

One can ask the question how close one can approach Improved Born results, with the effective ones, without breaking
Table 1: The EW parameters used for: the EW LO Born in α(0) scheme, and for variants of effective Born. The $G_{\mu} = 1.1663887 \cdot 10^{-5}$ GeV$^{-2}$, $M_Z = 91.1876$ GeV ($M_Z = 91.1887$ GeV for TAUOLA/LEP) and $\mathcal{K}_f, \mathcal{K}_e, \mathcal{K}_{\ell f} = 1$.

| Effective Born | EW LO α(0) scheme | Effective Born | Effective Born | Effective Born |
|---------------|--------------------|---------------|---------------|---------------|
| TAUOLA/LEP   | $\alpha = 1/128.6667471 \quad s_W^2 = 0.23152$ | $\alpha = 1/137.03599 \quad s_W^2 = 0.21215$ | $\alpha = 1/128.9503022 \quad s_W^2 = 0.231499 \quad \rho_{eff} = 1.0$ | $\alpha = 1/128.9503022 \quad s_W^2 = 0.231499 \quad \rho_{up} = 1.005$ |

This hints that Effective Born may work better for pp collisions than for $e^+e^-$, because of smaller charge for quarks than for leptons.

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4In Appendix, Table 14, two versions of Improved Born are used, with weak boxes included and not. This is important for large energy range. For $e^+e^-$ we concentrate mostly on the region of Z-pole where impact of electroweak boxes is marginal. We demonstrate quantitative impact of the (s,t) dependence which cannot be absorbed into effective couplings.

5If Effective Born v0 would be used, but with LO EW parameters the shifts on our tests observables with respect to Improved Born, would be about a factor 100 larger.

6This hints that Effective Born may work better for pp collisions than for $e^+e^-$, because of smaller charge for quarks than for leptons.
Figure 4: Left side plots, Improved Born level and vicinity of the Z peak: $\sigma^{\text{tot}}$ (top), $A_{FB}$ (middle) and $P_{t}$ (bottom). Right side plots enumerate, with ratios or differences the effects of simplifications with respect to Improved Born results. Green points: instead of form-factors their constant values calculated at $s=M_{Z}^{2}$, $t=-M_{Z}^{2}/2$ are used. Blue triangles: as for green ones, but in addition $\nu_{\ell f}$ = 1 and only real parts of $\nu_{e}$, $\nu_{f}$ are used. Red triangles: as in blue triangles, but only real parts of $\Pi_{\gamma\gamma}$ and $\rho_{\ell f}$ are taken into account. Yellow stars: with respect to red triangles imaginary parts of $\Pi_{\gamma\gamma}$ are switched back on.
Figure 5: Left side plots, Improved Born in the vicinity of the $Z$ peak: $\sigma^{tot}$ (top), $A_{FB}$ (middle) and $P_\tau$ (bottom), as in Fig. Right side plots enumerate, with ratios or differences the effects of Effective Born simplifications with respect to Improved Born. Green points: Effective Born $v0$, Blue triangles: Effective Born $v1$ Red (rotated) triangles: Effective Born $v2$. 
Table 2: EW corrections to cross-sections $\sigma^{tot}$ in the specified mass windows. DIZET 6.45 form-factors and running width was used in re-weighting of LHC $pp \rightarrow Zj; \ Z \rightarrow l^+l^−$ events simulated at 8 TeV. From the first two lines magnitude of EW corrections with respect to lowest order, $\alpha(0)$ scheme can be read off. Following three lines demonstrate precision of Effective Born variants with respect to Improved Born.

| Corrections to cross-section | $89 < m_{ee} < 93$ GeV | $81 < m_{ee} < 101$ GeV |
|-----------------------------|--------------------------|--------------------------|
| $\sigma^{tot}$(Improved Born, no boxes)/$\sigma$(EW LO $\alpha(0)$) | 0.96505 | 0.96626 |
| $\sigma^{tot}$(Improved Born, with boxes)/$\sigma$(EW LO $\alpha(0)$) | 0.96510 | 0.96631 |
| $\sigma^{tot}$(Eff. Born v0)/$\sigma$(Improved Born, with boxes) | 1.01142 | 1.01135 |
| $\sigma^{tot}$(Eff. Born v1)/$\sigma$(Improved Born, with boxes) | 1.00130 | 1.00132 |
| $\sigma^{tot}$(Eff. Born v2)/$\sigma$(Improved Born, with boxes) | 0.99989 | 0.99987 |

complexity is required. Use of numerically adapted Eq. (4) constants, which originally in Eq. (1) were multiplied by form-factors, does not suffice. For high precision, picture of effective couplings is not universal: appropriate for $A_{FB}$ choice may be not optimal for $P_l$.

4.2 Case of $pp \rightarrow ll$ processes at LHC

Let us now discuss properties of these benchmark observables distributions and how numerically significant is change from Improved Born to the Effective Born approximation in $pp$ case. In contrary to $e^+e^-$ when semi analytic methods were applied, now simulated event sample and TauSpriin reweighting is used.

In Fig. 6 (top-left) distributions of generated and EW corrected $Z$-line-shape (through $\sigma^{tot}$) are shown for the $pp$ collision case. EW weight is calculated using $\cos^\theta$ definition of the scattering angle as defined in [13]. On the logarithmic scale difference is barely visible. In the following plots of the same Figure we study it in more details. The ratios of the $Z$ line-shape distributions with gradually introduced EW corrections are shown. We intend to evaluate the size of complete Improved Born predictions with respect to variants of Effective Born. That is why, for reference predictions (denominator of the weights) the following: (i) EW LO $\alpha(0)$ (top-right plot), (ii) Effective Born $v0$ (bottom-left plot) and (iii) Effective Born $v2$ (bottom right plot), are used. For numerators, Improved Born of form-factors without/with box diagram contributions are used. At the $Z$-pole, complete EW corrections of Improved Born give for $\sigma$ about 0.01% different results from the one of Effective Born $v2$. It demonstrates that if for events generation EW LO matrix element is used with effective variant $v2$ parametrization, the size of missing EW effects will be significantly reduced.

Similar conclusions can be drawn from Table 2 where numerical impact of EW corrections on the normalization i.e. ratios of the $pp$ cross-sections integrated in the range $81 < m_{ee} < 101$ GeV and $89 < m_{ee} < 93$ GeV are given. Total EW correction for EW LO $\alpha(0)$ cross section is about 0.035, while for the Effective Born $v0$ it is of about 0.01 and for Effective Born $v2$ is of about 0.0001. The main improvement of $v2$ with respect to $v0$ is thanks to $\rho_{eff} \neq 1$ introduced already for $v1$.

Let us now turn our attention to the EW corrections for the forward backward asymmetry $A_{FB}$. Again for the $pp \rightarrow Z/\gamma \rightarrow l^+l^-$ process, energy range from 60 to 150 GeV was chosen which is of interest for electroweak effects. As in case of cross section, shape and size of the correction depend on whether box exchange diagrams are included in the Improved Born result, or not. In top-left plot of Fig. 7 the $A_{FB}$ distribution, as generated (EW LO) and superimposed with EW corrected result is shown. The points for the two cases are practically indistinguishable. Further three plots of the Figure, with the difference $\Delta A_{FB} = A_{FB}^{\text{Eff. Born}} - A_{FB}^{\text{Improved Born}}$ provide details. For the reference $A_{FB}^{\text{Eff. Born}}$, the three versions of the Effective Born detailed in Table 1 are used again: (i) EW LO $\alpha(0)$, (ii) $v0$ and (iii) $v2$. The EW corrections for $A_{FB}$ of EW LO Born with $\alpha(0)$ scheme, integrated around the $Z$-pole, necessary to reproduce Improved Born result are -0.03514. The Effective Born $v0$ reproduces Improved Born up to $\Delta A_{FB}$ of about -0.0004, while the Effective Born $v2$ up to -0.0002. The $v2$ variant is again better, by a factor of two, than the $v0$ one.

All that points to limitation of real constants Effective Born and its parametrization with $\sin^2 \theta_w^{\text{eff}}(M_Z)$ at about $20 \cdot 10^{-5}$ or so, even if $\alpha(M_Z)$ and $\rho_{f}(M_Z)$ is used. Note that even if $P_l$ is not particularly suitable for $pp$ collision measurements, it weakly depends on the production process and that is why it is suitable for numerical $\sin^2 \theta_w^{\text{eff}}(M_Z)$ ambiguities evaluation in general case. That is why previous subsection results are of the relevance for $pp$ too. It is worth to note that they also point to $20 \cdot 10^{-5}$ as an ultimate precision tag.
Figure 6: Top-left: Z line-shape distribution as generated with Powheg+MiNLO (blue triangles) and after reweighting introducing all EW corrections discussed (red triangles). The points are barely distinguishable. Ratios of Improved Born results (with and without EW boxes) to Effective Born in: (i) EW LO $\alpha(0)$ scheme are given in top-right, (ii) in bottom-left to Effective Born $v_0$ and (iii) in bottom-right plots to Effective Born $v_2$.

Table 3: The difference in forward-backward asymmetry, $\Delta A_{FB}$, in the specified mass windows. DZET 6.45 form-factors and running width was used in re-weighting of LHC $pp \rightarrow Zj; Z \rightarrow l^+l^-$ events simulated at 8 TeV. From the first two lines magnitude of EW corrections with respect to lowest order, $\alpha(0)$ scheme. Following three lines demonstrate precision of Effective Born variants with respect to Improved Born.

| Corrections to $A_{FB}$                              | $89 < m_{ee} < 93$ GeV | $81 < m_{ee} < 101$ GeV |
|-----------------------------------------------------|------------------------|------------------------|
| $A_{FB}$ (Improved Born, no boxes) - $A_{FB}$ (EW LO $\alpha(0)$) | -0.03491               | -0.03515               |
| $A_{FB}$ (Improved Born, with boxes) - $A_{FB}$ (EW LO $\alpha(0)$) | -0.03489               | -0.03514               |
| $A_{FB}(\text{Eff. Born } v_0) - A_{FB}(\text{Improved Born, with boxes})$ | -0.00039               | -0.00042               |
| $A_{FB}(\text{Eff. Born } v_1) - A_{FB}(\text{Improved Born, with boxes})$ | -0.00042               | -0.00042               |
| $A_{FB}(\text{Eff. Born } v_2) - A_{FB}(\text{Improved Born, with boxes})$ | -0.00022               | -0.00024               |
Figure 7: Top-left: the $A_{FB}$ distribution as generated in Powheg+MiNLO sample (blue triangles) and after reweighting introducing all EW corrections (red triangles). The two choices are barely distinguishable. The differences $\Delta A_{FB} = A_{FB} - A_{FB}^{ref}$ of Improved Born results (with and without EW boxes) to Effective Born in: (i) EW LO $\alpha(0)$ scheme are given in top-right, (ii) in bottom-left to Effective Born $v0$ and and (iii) in bottom-right plots to Effective Born $v2$. 
5 Electroweak corrections in TauSpinner: library versions and initializations

In the present section we address impact of DIZET library variants which have by now a life-time of more than three decades. The versions of the DIZET electroweak correction library, which are used in our numerical discussions are presented briefly in Appendix [13] details are given in Ref. [18]. Specification of initializations are collected in Appendix [A.1]. One may wonder if the last version would not suffice. However, availability of the software used for the solutions of legacy measurements is of some value. That is why, in Ref. [18] several versions of the present and past electroweak DIZET library are collected. On the other hand, archived with [14] less popular calculations of the past, will not receive our attention.

Each of the four versions of DIZET library of electroweak effects comes with wealth of options which may be activated with their input flags. These options can be used to evaluate importance of the particular improvement introduced over the years. The graphical programs to monitor the changes are available in TAUOLA/TauSpinner/examples directory. The tau-reweight-test.cxx can be used to demonstrate how events can be corrected with the weight representing improvement from TAUOLA Effective Born of its constant couplings to the one of Improved Born of formula (1) with form-factors interpolated from the text files with tables prepared with KKMC interface to DIZET.

The default for anomalous Born function, introduced for the first time in Ref. [24], is not anymore dummy but is now the one of EW Improved Born, which uses the EW form-factors tables (if available). New sub-directory Dizet-example collects programs and scripts for form-factors graphic representation. Plots of form-factors can be drawn, as a function of energy, scattering angle and flavour of incoming partons (it can be electron-positron pair as well). The integrated over angle partonic cross section $\sigma^{tot}$, $A_{FB}$ and $P_{t}$ can be graphically presented. Comparison plots can be prepared, either with the help of FFdrawDwa.C script to compare results with EW form-factors obtained with variants of DIZET initialization or with FFdraw.C to compare Improved Born and Effective Born of the choice as implemented in TAUOLA package. For technical details see Appendix [B] An example results for comparison of Effective Born as encapsulated in TauSpinner/Tauola (version of December 2019) defaults and Improved Born with electroweak form-factors of DIZET 6.45 were shown in Fig. [3]. Note the minor discontinuity of right hand side plots at 30 GeV. It is smaller than calculation precision and its origin is in limited granularity used for tabulation. One should note, that differences between semi-analytical results obtained from Improved Born and Effective Born even in case when detailed tuning of parameters is not performed is not large, from the perspective of many applications. We concentrate when presenting comparison results on $e^+e^-\rightarrow \tau^+\tau^-$ production process as its phenomenology represents LEP time reference to present day projects, in particular for LHC measurements. Also, parton level cross sections, necessary for LHC phenomenology, are obscured by hadronic interaction effects, thus are more complex for interpretation and require simultaneous evaluation of hadronic interactions.

Numerical results, as in previous section, are monitored with $\sin^2\theta_W^{eff}$, $P_t$, $A_{FB}$ and parton level $\sigma^{tot}$. In Fig. [8] and Table [4] it is shown how results depend on the library version. The presentation in Table [4] includes predictions on $\alpha(M_Z^2)$, $M_W$, $\Delta r$, and $\sin^2\theta_W^{eff}$. Further results are delegated to Appendices.

By inspection of Table [4] one can conclude that the choice of electroweak library variant is not of great importance, unless precision better than $20 \times 10^{-5}$ on $\sin^2\theta_W^{eff}$ is required. Even if precision requirements are not very demanding one should keep in mind, that below 40 GeV, in older versions of DIZET hadronic part of $\Pi_{eff}(s)$ was set to zero, see also Fig. [3] for minor discontinuity at 30 GeV due to edge of tabulation zones. Further details on the impact of change of options/flags of DIZET 6.45 are collected in Tables [8][9] of Appendix [A.2].

5.1 Parametric uncertainties

The precision of the electroweak calculations depends not only on the EW scheme used for the calculations (see e.g. [26][27], but also on imposed set of input parameters and corresponding parametric uncertainties. Parametric uncertainties are defined as the ones due to ambiguities of electroweak calculation inputs, such as $m_t$, $M_W$ or $\Delta\alpha^{(5)}(s)$. That is the reason, why precision of these input parameters (taken from measurements), specially $M_W$ or $\sigma^{tot}_{\tau^+\tau^-}\text{-shad}(s)$ (for $\alpha_{\text{QED}}(M_Z^2)$), is of importance. For clarity of the presentation this topic is covered only in Appendices [A.2][A.3], in particular in Tables [8][9][10][11]. We show how some phenomenologically sound quantities depend on initialization ambiguities for $\alpha(0)$ EW scheme used in DIZET library. In particular how they depend on: (i) distinct $\Delta\alpha^{(5)}(s)$ parametrization, (ii) uncertainty from changing $\Delta\alpha^{(5)}(M_Z^2)$ by $\pm 0.0001$, (iii) uncertainty due to top mass change by $\pm 0.5$ GeV. The estimated total parametric uncertainty for EW $\alpha(0)$ scheme used for $\sin^2\theta_W^{eff}(M_Z^2)$ is about 0.00005.

To summarize, these example results can be useful by themselves but they represent precondition for choice of the electroweak schemes and their inputs. Large parametric uncertainties could indicate that some schemes are not optimal. That is why, they contribute to evaluation of reliability for the Effective Born (effective couplings) concept too.
Figure 8: Comparison of $\sigma_{\text{tot}}(m_{\tau^+\tau^-})$, $A_{FB}(m_{\tau^+\tau^-})$ and $P_{\tau}(m_{\tau^+\tau^-})$, obtained from TauSpinner calculations of Improved Born and electroweak tables calculated with the DIZET libraries. As a reference, version 6.42 improved with photon vacuum polarization of Ref. [25] is used. Note, that interface of photon vacuum polarization of DIZET 6.42 and 6.21 prevented its calculation below 40 GeV. This, and other minor parameter variation in particular of $M_Z$ and $\Gamma_Z$, lead to below precision requirements, bumps on the plots which do not need to be investigated now, and even for the precision tests of the SM at LHC. On the other hand, proper adjustment for the effects important at (and around) $Z$ peak, need to be performed. The corresponding EW corrected $M_W$, $\sin^2\theta_W^{\text{eff}}$ and $\alpha$ at Z-pole are collected in Table 4. Similar results can be obtained from TauSpinner for the quark level Effective Born and Improved Born predictions.

Table 4: Predictions for different versions of DIZET 6.XX explained in Appendix A. The $\Delta r$, $\Delta r_{\text{rem}}$ represent corrections to $M_W$ calculations, see Eq. (7), the $s_W^2 = 1 - \frac{M_Z^2}{M_W^2}$.

| Parameter                        | DIZET 6.21 | DIZET 6.42 CPC | DIZET 6.42 (Jeg. 2017) | DIZET 6.45 |
|----------------------------------|------------|----------------|------------------------|------------|
| $\alpha(M_Z^2)$                 | 0.007759954| 0.007759954    | 0.0077549256           | 0.0077549256 |
| $\frac{1}{\alpha(M_Z^2)}$       | 128.86674175| 128.86674175  | 128.95030206           | 128.950302056 |
| $M_W$ (GeV)                      | 80.3560012 | 80.3535973     | 80.3621285             | 80.3589358 |
| $\Delta r$                       | 0.03676619 | 0.03690875     | 0.03640232             | 0.03633354 |
| $\Delta r_{\text{rem}}$         | 0.01168031 | 0.01168001     | 0.01168106             | 0.01168393 |
| $s_W^2$                          | 0.22345780 | 0.22350426     | 0.2233937              | 0.22340108 |
| $\sin^2\theta_W^{\text{eff repon}}(M_Z^2)$ | 0.23173519 | 0.23174233     | 0.23157947             | 0.23149900 |
| $\sin^2\theta_W^{\text{eff up–quark}}(M_Z^2)$ | 0.23162861 | 0.23174233     | 0.23147298             | 0.23139248 |
| $\sin^2\theta_W^{\text{eff down–quark}}(M_Z^2)$ | 0.23150149 | 0.23174233     | 0.23134599             | 0.23126543 |
6 Summary

One loop electroweak corrections play important role in the precision tests of the Standard Model. At the same time, other effects, related to special classes of higher order corrections had to be taken into account. That is the reason, why special libraries of electroweak corrections were developed, maintained and gradually improved. Over the last 30 years DIZET library was established as a prominent one. Special role was played by the so-called EW $\alpha$($0$) scheme. This scheme and DIZET library found large spectrum of applications, not only in phenomenology of LEP $e^+e^-$ collisions but $p\bar{p}$ and $pp$ Tevatron and LHC experiments as well. An attempt to archive distinct versions of electroweak libraries is provided in [18]. These results and methods need to be reproducible at the time of future FCC or similar experiments. Negligible differences of the past, may play important role for future higher precision projects. From our investigations, see Subsection 4.1, 4.2, we can conclude that effective couplings approach can be useful for $\sin^2 \theta_{W}^{eff}$ precision up to about $20 \cdot 10^{-5}$. Beyond that, Improved Born without simplifications is needed.

In principle, DIZET rely on one loop calculation, but it is supplemented with dominant higher order terms. Presented implementation of TauSpinner weights enable discussion of particular classes of higher order effects. In the present paper we have explained how numerical impact of some effects of electroweak results can be imprinted into broad spectrum of simulation samples where electroweak loop effects are missing, or impact of their initialization is to be studied. We have installed into TauSpinner library useful for that purpose algorithms. Example numerical results are focused on center of mass system energy dependence of total cross section, forward-backward asymmetry of leptons and $\tau$ lepton polarization. Those were studied for $e^+e^-$ and $pp$ collisions.

New TauSpinner algorithms have potential to improve e.g. electroweak effects in simulation samples obtained from programs predominantly of strong interactions. We have shown results of re-weighting with different level of sophistication for implementation of EW corrections. Let us point that from the perspective of forthcoming efforts on higher order high precision electroweak calculations, TauSpinner/DIZET algorithms may be used as a set of methods for evaluation which contributions (and to which order) need to be taken into account to attain requested precision level.

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The electromagnetic coupling evolves from Thomson limit and for by renormalisation group arguments, correct generalisation is to compute higher order corrections, see more discussion in [5].

The Sirlin’s parameter \( \Delta r \) is calculated analytically. Both \( \Delta \alpha(\mu) \) and \( \Delta \alpha(Z) \) involve re-summation and higher order corrections. Since this term implicitly depends on \( M_W \) and \( M_Z \) iterative procedure is needed. The re-summation term in formula (7) is not formally justified by renormalisation group arguments, correct generalization is to compute higher order corrections, see more discussion in [5].

The hadronic vacuum polarization correction is contained in the quantity denoted as \( \Delta \alpha_h^{(S)}(M_Z) \), which is treated as one of the input parameters. It can be either computed from quark masses or, preferably, fitted to experimental low energy \( e^+e^- \rightarrow hadrons \) data [31]. The leptonic loop correction \( \Delta \alpha_l(M_Z) \) is calculated analytically. Both \( \Delta \alpha_h^{(S)}(M_Z) \) and \( \Delta \alpha_l(M_Z) \) are...
significant, respectively about 0.0275762 and 0.0314976, the remaining terms are rather marginal, respectively about $-5 \cdot 10^{-5}$ and $-1 \cdot 10^{-5}$.

In the OMS renormalisation scheme the weak mixing angle is defined uniquely through the gauge-boson masses:

$$\sin^2 \theta_W = \sin^2 \theta_W = 1 - \frac{M_W^2}{M_Z^2}. \tag{9}$$

With this scheme, measuring $\sin^2 \theta_W$ would be equivalent to indirect measurement of $M_W^2$ through the relation \(9\).

### A.1 Initialization flags and input parameters

The recommended sets of flags are quite stable since 1995, new options consider updated parametrisations of the vacuum polarization hadronic corrections $\Delta \alpha_{\text{had}}^{(5)}$ (flag \texttt{IHVP}), updated calculations for two loop fermionic corrections (flag \texttt{IAMT4}) and updated three-loop corrections (flag \texttt{IAFMT}).

In Table 5 we collected information on the initialization flags recommended for different versions of \texttt{DIZET 6.XX}. For detailed information about meaning of the individual flags see \texttt{DIZET 6.XX} documentations [17, 5, 32]. Let us here just explain those, for which recommended values have changed since version \texttt{DIZET 6.21}:

- **Switch for Hadronic vacuum polarization corrections $\Delta \alpha_{\text{had}}^{(5)}$:***
  - \texttt{IHVP} = 1 parametrization of [33]
  - \texttt{IHVP} = 5 parametrization of [31]

- **Switch for re-summation of the leading $O(G_f m_f^2)$ electroweak corrections:**
  - \texttt{IAMT4} = 4 with two-loop sub-leading corrections and re-summation [34, 35, 36, 37]
  - \texttt{IAMT4} = 5 with fermionic two-loop corrections to $M_W$ [38, 39, 40]
  - \texttt{IAMT4} = 6 with complete two-loop corrections to $M_W$ [41] and fermionic two-loop corrections to $\sin^2 \theta_W$ [42]
  - \texttt{IAMT4} = 7 with complete two-loop corrections to $\sin^2 \theta_W$ lep and $\sin^2 \theta_W$ fb [43, 44]
  - \texttt{IAMT4} = 8 with complete two-loop corrections to $\sin^2 \theta_W$ lep [45]

- **Switch for three-loop corrections $O(\alpha^3)$ to the electroweak $\rho$ parameter:**
  - \texttt{IAFMT} = 1 corrections $O(G_n m_n^2 \alpha_n^2)$ included [46]
  - \texttt{IAFMT} = 2 corrections $O(G_n m_n^2 \alpha_n^2)$, $O(G_n M_Z^2 \alpha_n^2 + \log(m_t^2))$ included \texttt{IAFMT} = 3 corrections $O(G_n m_n^2 \alpha_n^2)$, $O(G_n M_Z^2 \alpha_n^2 + \log(m_t^2))$ and $O(G_n M_Z^2 / m_t^2 \alpha_n^2)$ included

Since LEP time physics measurements evolved, and as a consequence initialization parameters as well. For the recent status summary see last edition by Particle Data Group [47]. The Higgs boson has been discovered at LHC and its mass measured with precision of 25 MeV [48]. The W boson mass in known at LHC with precision better than 18 MeV [49] and the top mass is known with precision much better than 1 GeV [50].

In Tables 6 and 7 we collected initialization parameters: masses and couplings, used of the paper numerical evaluation. The exact values of some of them, which serve as benchmark values for different comparisons, has been chosen as such to be fully compatible with the ongoing studies of the LHC EW Working Group [51].

### A.2 Numerical results

The \texttt{DIZET} library, when invoked, provide tabulated (s,t) dependent form-factors. It calculates also $M_W$, Stirling parameter $\Delta_r$, $\Delta_r \text{rem}$ and flavour dependent $\sin^2 \theta_W$ at Z peak. In Table 4 we have collected numerical results on predicted masses and couplings, including EW corrections. Those values come directly as control printout from \texttt{DIZET 6.XX} code. In total, evolution of the implemented EW corrections, lead to shift in the predicted $M_W$ by $+3$ MeV, on-shell $s_W^2$ by $-0.00005$ and $\sin^2 \theta_W$ lepton by $-0.00020$. Let us comment on this evolution:

- The change in $\alpha(M_Z^2)$ is due to improvements in the theoretical predictions and experimental low-energy measurements over last 25 years, and following update in the used parametrisation from [33] to [31].

- The $\Delta_r$ and $\Delta_r \text{rem}$, which are displayed separately, represent gauge invariant corrections to $M_W$ calculation as shown in formulas (5) and (7). The $\Delta_r$ is affected by options used for calculating $\Delta r_L$, which depends on the flag \texttt{AMT4} used. It also depends on the parametrisation of $\Delta \alpha(M_Z)$. The sensitivity of the $\Delta r \text{rem}$ to all changed introduced between v6.21 and v6.45 is almost negligible.

- As a consequence of different predicted $M_W$, the on-shell $s_W^2$ has evolved as well.
Table 5: DIZET initialization flags: different versions defaults.

| Input NPAR() | Internal flag | DIZET 6.21 Defaults in [17] | DIZET 6.42 Defaults in [32] | DIZET 6.45 | Comments |
|--------------|---------------|------------------------------|-------------------------------|------------|----------|
| NPAR(1)      | IHVP          | 1                            | 1                             | 5          | $\Delta \alpha_{\text{head}}$ param. from [31] in v6.45 |
| NPAR(2)      | IAMT4         | 4                            | 4                             | 8          | New development in v6.42, v6.45 |
| NPAR(3)      | IQCD          | 3                            | 3                             | 3          | $M_W$ calculated with formula [5] |
| NPAR(4)      | IMOMS         | 1                            | 1                             | 1          | Not used since v6.21 |
| NPAR(5)      | IMASS         | 0                            | 0                             | 0          |         |
| NPAR(6)      | ISCRE         | 0                            | 0                             | 0          |         |
| NPAR(7)      | IALEM         | 3                            | 3                             | 3          |         |
| NPAR(8)      | IMASK         | 0                            | 0                             | 0          |         |
| NPAR(9)      | ISCAL         | 0                            | 0                             | 0          |         |
| NPAR(10)     | IBARB         | 2                            | 2                             | 2          |         |
| NPAR(11)     | IFTJR         | 1                            | 1                             | 1          |         |
| NPAR(12)     | IFACR         | 0                            | 0                             | 0          |         |
| NPAR(13)     | IFACT         | 0                            | 0                             | 0          |         |
| NPAR(14)     | IHIGS         | 0                            | 0                             | 0          |         |
| NPAR(15)     | IAMFT         | 1                            | 3                             | 3          |         |
| NPAR(16)     | IEWLC         | 1                            | 1                             | 1          |         |
| NPAR(17)     | ICZAK         | 1                            | 1                             | 1          |         |
| NPAR(18)     | IHIG2         | 1                            | 1                             | 1          |         |
| NPAR(19)     | IALE2         | 3                            | 3                             | 3          |         |
| NPAR(20)     | IGREF         | 2                            | 2                             | 2          |         |
| NPAR(21)     | IDDZZ         | 1                            | 1                             | 1          |         |
| NPAR(22)     | IAMW2         | 0                            | 0                             | 0          |         |
| NPAR(23)     | ISFSR         | 1                            | 1                             | 1          |         |
| NPAR(24)     | IDMWW         | 0                            | 0                             | 0          |         |
| NPAR(25)     | IDSWW         | 0                            | 0                             | 0          |         |
Table 6: The EW parameters used at tree-level EW, with on-mass-shell definition (LEP convention).

| Parameter \( M_Z, \Gamma_Z, \Gamma_W, 1/\alpha, \alpha, G_\mu, M_W, s_W^2, \alpha_s(M_Z) \) | Value |
|---|---|
| \( M_Z \) (GeV) | 91.1876 |
| \( \Gamma_Z \) (GeV) | 2.4952 |
| \( \Gamma_W \) (GeV) | 2.085 |
| \( 1/\alpha \) | 137.035999139 |
| \( \alpha \) | 0.007297353 |
| \( G_\mu \) (GeV\(^{-2}\)) | \( 1.1663787 \cdot 10^{-5} \) |
| \( M_W \) (GeV) | 80.93886 |
| \( s_W^2 \) | 0.2121517 |
| \( \alpha_s(M_Z) \) | 0.12017890 |

Table 7: Values of fermions and Higgs boson masses used for calculating EW corrections.

| Parameter | Mass (GeV) | Description |
|---|---|---|
| \( m_e \) | 5.1099907e-4 | mass of electron |
| \( m_\mu \) | 0.1056583 | mass of muon |
| \( m_\tau \) | 1.7770500 | mass of tau |
| \( m_u \) | 0.0620000 | mass of up-quark |
| \( m_d \) | 0.0830000 | mass of down-quark |
| \( m_c \) | 1.5000000 | mass of charm-quark |
| \( m_s \) | 0.2150000 | mass of strange-quark |
| \( m_b \) | 4.7000000 | mass of bottom-quark |
| \( m_t \) | 173.0 | mass of top quark |
| \( m_H \) | 125.0 | mass of Higgs boson |
A.3 Parametric uncertainties on $\sin^2 \theta^\text{eff}_W (M_Z^2)$ predictions

We have studied dominant parametric uncertainties from $\Delta \alpha_h^{(5)}(M_Z^2)$ and $m_t$ for $\sin^2 \theta^\text{eff}_W (M_Z^2)$ prediction. Recent detailed discussion on the parametric uncertainties of SM parameters can be found in [52]. Both components of $\sin^2 \theta^\text{eff}_W (M_Z^2)$ = $\text{Re} \mathcal{F}(M_Z^2, -M_Z^2, \frac{M_Z^2}{2}) \cdot s_W^2$ definition are sensitive to parametric uncertainties.

- In Table 8 we show impact of changing $\Delta \alpha_h^{(5)}(M_Z^2) \pm 0.0001$, which is the uncertainty of the parametrization of $\Delta \alpha_h^{(5)}$. The resulting uncertainty on $\sin^2 \theta^\text{eff}_W (M_Z^2)$ is of $\pm 0.000035$.
- In Table 9 we show impact of changing $m_t \pm 0.5$ GeV, which is roughly the anticipated uncertainty of the measurements at LHC [50]. The resulting uncertainty on $\sin^2 \theta^\text{eff}_W (M_Z^2)$ is of $\pm 0.000016$.

The total parametric uncertainty, added in quadrature, on $\sin^2 \theta^\text{eff}_W (M_Z^2)$ is about $\pm 0.00005$. 

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Table 8: The DIZET 6.45 predictions for two different parametrisations of $\Delta \alpha_h^{(5)}(M_Z^2)$. Other flags as in Table 5.

| Parameter | $\Delta \alpha_h^{(5)}(M_Z^2) = 0.0280398$ (param. Jegerlehner 1995) | $\Delta \alpha_h^{(5)}(M_Z^2) = 0.0275762$ (param. Jegerlehner 2017) | $\Delta$ |
|-----------|--------------------------|--------------------------|---------|
| $\alpha(M_Z^2)$ | $0.0077587482$ | $0.0077549256$ | $0.00000326$ |
| $1/\alpha(M_Z^2)$ | $128.88676996$ | $128.95030224$ | $-0.003536$ |
| $M_W$ (GeV) | $80.350538$ | $80.358936$ | $0.008398$ |
| $\Delta r$ | $0.03690783$ | $0.03640338$ | $-0.000465$ |
| $\Delta r_{\text{rem}}$ | $0.01168001$ | $0.01167960$ | $-0.000001$ |
| $s_W^2$ | $0.22356339$ | $0.22340108$ | $-0.000161$ |
| $\sin^2 \theta^\text{eff}_W^{\text{lepton}}(M_Z^2)$ | $0.23166087$ | $0.23149900$ | $-0.000287$ |
| $\sin^2 \theta^\text{eff}_W^{\text{up-quark}}(M_Z^2)$ | $0.23155425$ | $0.23139248$ | $-0.000177$ |
| $\sin^2 \theta^\text{eff}_W^{\text{down-quark}}(M_Z^2)$ | $0.23147290$ | $0.23126543$ | $-0.000247$ |

Table 9: The DIZET 6.45 predictions with improved treatment of two-loop corrections. Other flags as in Table 5.

| Parameter | AMT4 = 4 | AMT4 = 8 | $\Delta$ |
|-----------|----------|----------|---------|
| $\alpha(M_Z^2)$ | $0.0077549256113$ | $0.0077549256002$ | $0.0000000111$ |
| $1/\alpha(M_Z^2)$ | $128.895030224$ | $128.95030224$ | $-0.000276$ |
| $M_W$ (GeV) | $80.350538$ | $80.358936$ | $0.008398$ |
| $\Delta r$ | $0.03690783$ | $0.03640338$ | $-0.000465$ |
| $\Delta r_{\text{rem}}$ | $0.01168001$ | $0.01167960$ | $-0.000001$ |
| $s_W^2$ | $0.22356339$ | $0.22340108$ | $-0.000161$ |
| $\sin^2 \theta^\text{eff}_W^{\text{lepton}}(M_Z^2)$ | $0.23166087$ | $0.23147290$ | $-0.000287$ |
| $\sin^2 \theta^\text{eff}_W^{\text{up-quark}}(M_Z^2)$ | $0.23155425$ | $0.23139248$ | $-0.000177$ |
| $\sin^2 \theta^\text{eff}_W^{\text{down-quark}}(M_Z^2)$ | $0.23147290$ | $0.23126543$ | $-0.000247$ |

- Evolution of $\sin^2 \theta^\text{eff}_W$, illustrated with Fig. 2, comes from changing $s_W^2$ and $\mathcal{F}(s,t)$ form-factors. It impact $P^7, A_{FB}$, $\sigma^{tot}$ too.

In Table 8 we document impact of changing only parametization of $\Delta \alpha_h^{(5)}$, with other parameters and flags unchanged. Dominant effect comes from EW corrections to $M_W$, which shifts its value by +8.4 MeV, reflected in change of $s_W^2$ by -0.00016.

The impact on the form-factors is less significant and final shift in the $\sin^2 \theta^\text{eff}_W^{\text{lepton}}(M_Z^2)$ is of -0.00023.

In Table 9 we document impact of changing only two-loop corrections to $M_W$, with other parameters and flags unchanged. The resulting shift on $M_W$ is smaller, -2.9 MeV only, resulting in +0.00006 shift on $s_W^2$ and, while multiplied with form-factors which also have changed, correspondingly in -0.00008 shift on $\sin^2 \theta^\text{eff}_W^{\text{lepton}}(M_Z^2)$. 
Table 10: The DIZET 6.45 predictions: uncertainty from changing $\Delta \alpha_{h}^{(5)}(M_{Z}^{2}) = 0.0275762$ (param. [31]), by ± 0.0001.

| Parameter | $\Delta \alpha_{h}^{(5)}(M_{Z}^{2}) - 0.0001$ | $\Delta \alpha_{h}^{(5)}(M_{Z}^{2}) = 0.0275762$ | $\Delta \alpha_{h}^{(5)}(M_{Z}^{2}) + 0.0001$ | $\Delta/2$ |
|-----------|----------------------------------|-----------------------------------|----------------------------------|--------|
| $\alpha(M_{Z}^{2})$ | 0.0077541016 | 0.0077549256 | 0.0077557498 | |
| $1/\alpha(M_{Z}^{2})$ | 128.96400565 | 128.95030224 | 128.93659846 | |
| $M_{W}$ (GeV) | 80.360747 | 80.358936 | 80.357124 | 1.8 MeV |
| $\Delta r$ | 0.03629414 | 0.03640338 | 0.03651261 | |
| $\Delta r_{rem}$ | 0.01167983 | 0.01167960 | 0.01167938 | |
| $s_{W}^{2}$ | 0.22336607 | 0.22340108 | 0.22343610 | 0.000035 |
| $\sin^{2} \theta_{W}^{eff \text{ lepton}}(M_{Z}^{2})$ | 0.23146409 | 0.23149900 | 0.23153392 | 0.000035 |
| $\sin^{2} \theta_{W}^{eff \text{ up-quark}}(M_{Z}^{2})$ | 0.23135758 | 0.23139248 | 0.23142737 | 0.000035 |
| $\sin^{2} \theta_{W}^{eff \text{ down-quark}}(M_{Z}^{2})$ | 0.23123057 | 0.23126543 | 0.23130029 | 0.000035 |

Table 11: The DIZET 6.45 predictions: uncertainty from changing top-quark mass $m_{t} = 173.0$ GeV by ± 0.5 GeV.

| Parameter | $m_{t} - 0.5$ GeV | $m_{t} = 173.0$ GeV | $m_{t} + 0.5$ GeV | $\Delta/2$ |
|-----------|------------------|------------------|------------------|--------|
| $\alpha(M_{Z}^{2})$ | 0.0077549221 | 0.0077549256 | 0.0077549291 | |
| $1/\alpha(M_{Z}^{2})$ | 128.95036003 | 128.95030224 | 128.95024461 | |
| $M_{W}$ (GeV) | 80.355935 | 80.358936 | 80.361941 | 3 MeV |
| $\Delta r$ | 0.03658500 | 0.03640338 | 0.03622132 | |
| $\Delta r_{rem}$ | 0.01167011 | 0.01167960 | 0.01168907 | |
| $s_{W}^{2}$ | 0.22345908 | 0.22340108 | 0.22334300 | 0.000035 |
| $\sin^{2} \theta_{W}^{eff \text{ lepton}}(M_{Z}^{2})$ | 0.23140736 | 0.23139248 | 0.23137758 | 0.000016 |
| $\sin^{2} \theta_{W}^{eff \text{ up-quark}}(M_{Z}^{2})$ | 0.23135758 | 0.23139248 | 0.23137758 | 0.000016 |
| $\sin^{2} \theta_{W}^{eff \text{ down-quark}}(M_{Z}^{2})$ | 0.23123057 | 0.23126543 | 0.23130029 | 0.000016 |
B Technical documentation of upgrades for TauSpinner electroweak re-weighting code

In TauSpinner the Improved Born of \([1]\) is coded as default for its nonSMBorn function. If form-factors are available, look-up tables present, then they will be used for re-weighting algorithm. At default, it will be then assumed\([1]\) that sample was generated with Effective Born Tauola/LEP variant, see Table\([4]\). A wealth of options is available for electroweak form-factors calculation, see Section A. Choice of options simplifying Improved Born to the cases closer, or to effective Born itself, are listed in Table 13.

To monitor in a quicker manner look-up tables with form-factors, root scripts FFdraw.C and FFdrawDwa.C are provided, see Appendix[5]. These scripts provide semi-analytical results for Born level cross section, \(A_{FB}\) and \(A_{pw}\) as a function of centre-of-mass energy, for incoming \(e^+e^- (or \ u\bar{u}, d\bar{d})\) pairs. Two versions can be compared; e.g. EW improved Born with default Effective Tauola/LEP Born of TauSpinner, or of two variants of EW-initialization.

In the distribution tar-ball \([3]\) of Tauola/TauSpinner there are two example main programs, which demonstrate how re-weighting of EW effects is implemented (on the LHE and HepMC event formats) and how analytically these effects can be monitored. These programs, respectively  that are prepared. These parameters are used for Born level cross-section, \(\sigma\) calculation, see Table 5 and Section E. Choice of options simplifying Improved Born to the cases closer, or to effective Born itself, are listed in Table 13.

| Table 13 Choice of options simplifying Improved Born to the cases closer, or to effective Born itself. |
|---------------------------------------------------------------|
| Option | Description |
|-------|-------------|
| A | Default setting for simplifying Born to cases closer |
| B | Effective Born setting |
| C | Asymptotic Born setting |

In the distribution tar-ball \([3]\) of Tauola/TauSpinner there are two example main programs, which demonstrate how re-weighting of EW effects is implemented (on the LHE and HepMC event formats) and how analytically these effects can be monitored. These programs, respectively tau-reweight-test.cxx and Dizet-example/table-parsing-test.cxx need explanation because they evolved with time and more initialization options were introduced. In particular, to evaluate numerical differences between Effective Born and EW Improved Born as well as of the intermediate variants; the steering flag for the variants is keyGSW. Note, that to re-weight, program needs to flip for each event, initialization between two variants and save some intermediate results to avoid massive recalculations. The semi-analytical table-parsing-test.cxx is obviously much faster. It is useful for study of small effects and not only to checks of the correctness for electroweak form-factors tabulation.

Let us collect technical details for these programs and dependencies between routines:

- Dizet-example/table-parsing-test.cxx. This semi-analytic program is fast, does not require continuous interchanging between variants of initialization and is suitable to study small variation for predictions due to initialization fine tuning. It can be set for incoming \(e^+e^- (or \ u\bar{u}, d\bar{d})\) pairs. It is obviously unable to tackle experimental selection.

- For tau-reweight-test.cxx event re-weighting is demonstrated (LHE or HepMC format can be used) and for each event, two variants of Born are calculated. That is why flag keyGSW can not be set once; method calculateWeightFromParticlesH() is executed twice and ratio of the result is used. At the beginning EWanomInit (defined locally in demo) is called and defines initialization variant with the help of ExtraEWparamsSet() method. Next, for each event in the loop, EWreInit (it is expected to be adjusted by the user) re-initializes keyGSW and other parameters, with the, otherwise dummy, call on sigbornsdelt. One should note that sigbornsdelt call in tau-reweight-test.cxx returns dummy variable. The call is to pass keyGSW only. Returned value is used in Dizet-example/table-parsing-test.cxx though.

- In all the code of src/ew_born.cxx and src/tau_reweight_lib.cxx the keyGSW is not used. Corresponding routines are EW variants independent.

The src/initwsw.f is the file where routines for calculating Effective- and/or EW Improved-Born are placed. In the src/nonSM.cxx file, the method default_nonSM_Born2() resides. The class datamember variable m_keyGSW is used for keyGSW, it is accessed with the help of the keyGSWGet(keyGSW) method; options are collected in Table 13. In Table 12 names of variables corresponding to the form-factors, introduced in Eq. (1) of Improved Born, which are also used in KKMC and in Dizet-example scripts of TauSpinner, are explained. These form-factors are used when t_bornnew_ calculation is invoked.

From the point of view of electroweak calculations most of the variants are set and stored in EWtables.cxx. In particular class/file variable m_keyGSW is stored among other variables of initialization. To access or modify ExtraEWparamsGet and ExtraEWparamsSet are prepared. These parameters used in initEWff passed with ExtraEWparamsGet in particular keyGSW is passed to initwkswelt. In the EWtables.cxx, the code for sigbornswdelt and Asnbornswdelt are stored. Through these methods keyGSW is passed to t_bornnew_. The Asnbornswdelt; a near clone of sigbornswdelt, is used in drawing scripts for asymmetries. It is important to note, that the functions t_born and t_bornnew which are used in tauola universal interface \([5]\) and in TauSpinner, are normalized to lowest order Born cross section, of photon exchange only and mass effects excluded. That means \(\frac{d\sigma}{d\cos\theta} (s, \cos\theta, \rho)\) is multiplied by \(\frac{\alpha}{\pi s} (\alpha = \alpha_{QED}(0))\). Thus, Born expression used in the weight calculations should be weighted using the effective value \(\alpha_{QED}(0)\) of the coupling constant.
Table 12: Electroweak form-factors of formula (1) calculated by DIZET and provided by KKMC tabulation code. Names as used in: formula (1), KKMC tabulation code and in Dizet-example are collected. In Effective Born Eq. (4) numerical constants replace form-factors of Improved Born (1). Depending on Effective Born version (see Table 10) $\sin^2 \theta_W$ of Improved Born is replaced by $\sin^2 \theta_{\text{eff}}$ or by flavour dependent variants $\sin^2 \theta_{\text{eff}}^{\uparrow/\downarrow}$, the $\rho_{\ell f}(s,t)$ is then replaced with constant $\rho_{\ell f}$. Similarly $\alpha(0)$ is replaced with $\alpha(M_Z)$

| Form-factor | in KKMC | in Dizet example | in Effective Born |
|-------------|---------|------------------|------------------|
| $\rho_{\ell f}(s,t)$ | GSW(1) | FF1 | 1 |
| $v_1$ | 1.005 | |
| $v_2$ | 1.005403 (up) | |
| $v_{\text{bornnew}}$ | 1.005889 (down) | |
| $K_e(s,t)$ | GSW(2) | FF2 | 1 |
| $K_f(s,t)$ | GSW(3) | FF3 | 1 |
| $K_{\ell f}(s,t)$ | GSW(4) | FF4 | 1 |
| $\Pi_{\ell f}(s)$ | GSW(6) | FF6 | 1 |

The approach $q^2 p_f^2 (1 + \cos^2 \theta)$ at low energies. This condition, defines normalization for the EW-improved (or other e.g. non SM variant) of user provided Born$^2$. This is important if weights are used to relate cross sections obtained from $t_{\text{born}}$ and $t_{\text{bornnew}}$. It is of no importance if only spin weight or weight based solely on user provided Born is used. This reference $t_{\text{born}}$ of Effective Born (inherited from Tauola) can be easily modified/replaced by the user, as well as is the case with independently initialized $t_{\text{bornnew}}$ of Improved Born.

In TauSpinner option to improve precision of generated MC events and reweight from “fixed” to “running” width propagator, see Appendix [13] for explanation, is available. For that TauSpinner initialization with KEYGSW=11, KEYGSW=10 and KEYGSW=2 is prepared. For KEYGSW=11, KEYGSW=12 and KEYGSW=13 results corresponding to Effective Born, variants $v_0, v_1, v_2$ respectively can be obtained, see Table [13]. Let us explain now meaning of all other sigbornswdelt(mode,ID, s, cc, SWeff, DeltSQ, DeltV, Gmu, alfinv, AMZ00, GAM00, KEYGSW) input parameters. mode can be set to 0 or 1. In the second case SWeff, AMZ00, GAM00 for $\sin^2 \theta_{\text{eff}}^{\uparrow/\downarrow}, M_Z, \Gamma_Z$ will be overwritten with values stored in electroweak tables calculated with the DIZET library. ID=0,1,2 denotes that calculation is performed respectively for outgoing lepton or down/up quark pair. The s, cc denote Mandelstam variable and scattering angle. Anomalous coupling $\delta_{\text{SW}}$ and $\delta_{\nu}$ of [13] Appendix B, are initialized with DeltSQ, DeltV respectively, finally also $G_F, 1/\alpha$ with Gmu, alfinv. The Asnbornswdelt() feature the same set of input parameters, but returns difference for cross section of forward and backward hemispheres instead of the sum.

For important technical details, README files and comments in the code of the distribution tar-ball [53] can be helpful.

C Initialization of variants for EW Improved Born

In the previous appendix we have completed presentation of some easy to activate in TauSpinner options. In many cases, it is sufficient to change some well defined keys and/or input parameters such as Z boson mass or some couplings. This of course shifts the results. The $\sigma^{\text{eff}}, A_{FB}$ and $P_{l}$ predictions at the Z-pole are collected in Table [14] for incoming $e^+e^-$, up or down quarks.

One can see that some options e.g. KEYGSW=0,2,4,10 are prepared for technical tests, rather than for evaluation of physics ambiguities, while other options are more useful. All these options are useful for test of particular parts of electroweak predictions obtained with a given version of electroweak form-factors. This supplements discussed earlier options of electroweak form-factors initialization, see Appendix [A.1] and Table [5]. Further, of more historical nature tests, with electroweak form-factors calculated with older versions of electroweak DIZET library presented in Appendix [4] are collected in Fig. [8] and Table. [4] Results of the present Appendix are to supplement discussion of reliability and limitation of the Effective Born variants as compared with Improved Born, in general and in the context of particular applications.

The Effective Born, variants $v_0, v_1, v_2$, require change of input parameters. Then, mode=0 and KEYGSW at 11, 12 or 13 should be respectively set. One should notice some shifts of Table [14] results with respect to the ones presented e.g. in Tables [2][3]. Note, that we do not average over energy ranges and incoming quark flavours now.

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$^1$The method for non SM Born can be replaced with the pointer to the one of user choice.
Table 13: Initialization variants for non-standard Born of quark level Drell Yan $2 \rightarrow 2$ processes. It can be used to impose with the event weight electroweak loop effects on event samples. Variants are steered by the keyGSW parameter. Corresponding code is stored in: (A) - INITWKSDELt, (B) - T\_BORNNEW and (C) - EW\_tables\_cxx. Fixed, running and fixed re-scaled $\Gamma_Z$ correspond respectively to Eqs. (10), (11) and (12). Further combination of options can be set by the simple re-coding. The initialization of effective Tauola Born is not affected by these options. It is performed elsewhere. For keyGSW=11,12,13, when mode=0 results of Effective Born variants $v_0$, $v_1$, $v_2$ can be obtained.

| KEYGSW | A: VVCor | B: propagator | C: form-factors |
|--------|----------|----------------|-----------------|
| 0      | 1        | photon propagator off, fixed $\Gamma_Z$ | all FF1=1 |
| 1      | on       | running $\Gamma_Z$, fixed $\Gamma_Z$ | all FF1 from EW tables |
| 2      |           | all FF1=1      |                 |
| 3      | 1        | running $\Gamma_Z$ | all FF1=1 but FF6= $\Pi_{\gamma\gamma}(M_Z^2)$ |
| 4      | 1        | running $\Gamma_Z$ | all FF1=1 but FF6, FF1= $\rho_{\gamma\gamma}(M_Z^2, -M_Z^2/2), \Pi_{\gamma\gamma}(M_Z^2)$ |
| 5      | 1        | running $\Gamma_Z$ | all FF1 from EW tables calculated at $(M_Z^2, -M_Z^2/2)$ |
| 10     | 1        | fixed $\Gamma_Z$, re-scaled running $\Gamma_Z$ | all FF1=1 |
| 11     | 1        | running $\Gamma_Z$ | FF1, can be used for Effective Born $v_0$ |
| 12     | 1        | running $\Gamma_Z$ | FF1 set as for Effective Born $v_1$ |
| 13     | 1        | running $\Gamma_Z$ | FF1 set as for Effective Born $v_2$ |

Table 14: Numerical results for initialization variants as explained in Table 13. Numerical results for $v_0$, $v_1$, $v_2$ are also provided, then in addition to keyGSW=11,12 or 13, input parameters for sigbornswdelt(), AsNbornswdelt() need to be adjusted and mod=0. The $\alpha(M_Z^2)/\alpha(0)$ factors entering cross sections normalization are dropped out from the $\sigma^{\text{tot}}$ ratios.

| KEYGSW | \(\frac{\sigma^{\text{tot}}_{\gamma\gamma}(M_Z^2)}{\sigma^{\text{tot}}_{\gamma\gamma}\text{input}(M_Z^2)}\) | \(P_{\gamma\gamma}(\tau\tau)(M_Z^2)\) | \(\frac{\sigma^{\text{tot}}_{\gamma\gamma}(M_Z^2)}{\sigma^{\text{tot}}_{\gamma\gamma}\text{input}(M_Z^2)}\) | \(A_{FB}^{\gamma\gamma\gamma\gamma}(M_Z^2)\) | \(A_{FB}^{\gamma\gamma\gamma\gamma}(M_Z^2)\) | \(P_{\gamma\gamma}(\tau\tau)(M_Z^2)\) | \(A_{FB}^{\gamma\gamma\gamma\gamma}(M_Z^2)\) |
|--------|-----------------------------------------------|---------------------------------|-----------------------------------------------|-----------------------------------|-----------------------------------|---------------------------------|-----------------------------------|
| 0      | 0.989939                                      | 0.1463264                       | 0.0161546                                      | 0.1463264                       | 0.0161546                                      | 0.1463264                       | 0.0161546                                      |
| 1      | 1.000000                                      | 0.149616                        | 0.0177039                                      | 0.149616                        | 0.0177039                                      | 0.149616                        | 0.0177039                                      |
| 2      | 1.003736                                      | 0.2093134                       | 0.0303312                                      | 0.2093134                       | 0.0303312                                      | 0.2093134                       | 0.0303312                                      |
| 3      | 1.001438                                      | 0.2094436                       | 0.0349617                                      | 0.2094436                       | 0.0349617                                      | 0.2094436                       | 0.0349617                                      |
| 4      | 1.011159                                      | 0.209759                        | 0.0344227                                      | 0.209759                        | 0.0344227                                      | 0.209759                        | 0.0344227                                      |
| 5      | 1.000001                                      | 0.149558                        | 0.0176505                                      | 0.149558                        | 0.0176505                                      | 0.149558                        | 0.0176505                                      |
| 10     | 1.002227                                      | 0.2093154                       | 0.0303312                                      | 0.2093154                       | 0.0303312                                      | 0.2093154                       | 0.0303312                                      |
| 11     | 1.007366                                      | 0.2093134                       | 0.0303312                                      | 0.2093134                       | 0.0303312                                      | 0.2093134                       | 0.0303312                                      |
| 11 v0  | 0.989934                                      | 0.1463264                       | 0.0161546                                      | 0.1463264                       | 0.0161546                                      | 0.1463264                       | 0.0161546                                      |
| 12 v1  | 0.9999098                                     | 0.1463351                       | 0.0161556                                      | 0.1463351                       | 0.0161556                                      | 0.1463351                       | 0.0161556                                      |
| 13 v2  | 0.9999098                                     | 0.1463351                       | 0.0161556                                      | 0.1463351                       | 0.0161556                                      | 0.1463351                       | 0.0161556                                      |

27
D The $s$ dependent Z-boson width

In formula (11) for the definition of Z propagator running width is used:

$$\chi(s) = \frac{1}{s-M_Z^2 + i\Gamma_Z s/M_Z}$$

The form-factors of eq. (11) are calculated for the on mass-shell (nominal) value of $M_Z$. The introduction of so-called $s$-dependent width is equivalent to partial resummation to higher orders of dominant loop correction: the boson $s$ dependent self-energy. In fact such resummation, running Z width, was used in many analyses of LEP I era.

However, in Monte Carlos and strong interaction calculations of LHC era, the Z propagator of constant width is often used:

$$\chi'(s) = \frac{1}{s-M_Z^2 + i\Gamma_Z' M_Z}.$$ (11)

One can ask the question, how analytic forms of (10) and (11) translate to each other. In fact, this well known translation is known at least since Ref. [55] published more than 30 years ago, but let us readdress it for the reference again. From Eq. (10), we obtain Eq. (11) if the following redefinitions are used

$$\chi(s) = \frac{1}{s(1+\Gamma_Z/M_Z) - M_Z^2}$$

$$= \frac{(1-i\Gamma_Z/M_Z)}{s(1+i\Gamma_Z/M_Z) - M_Z^2(1-i\Gamma_Z/M_Z)}$$

$$= \frac{(1-i\Gamma_Z/M_Z)}{(1+i\Gamma_Z/M_Z)^2} \frac{1}{s-M_Z^2 + M_Z^2/s + i\Gamma_Z M_Z}$$

$$= \frac{N_s}{s-M_Z^2 + i\Gamma_Z M_Z}$$

$$M_Z' = \frac{M_Z}{\sqrt{1+\Gamma_Z^2/M_Z^2}}$$

$$\Gamma_Z' = \frac{\Gamma_Z}{\sqrt{1+\Gamma_Z^2/M_Z^2}}$$

$$N_Z' = \frac{(1-i\Gamma_Z/M_Z)}{(1+i\Gamma_Z^2/M_Z^2)} = \frac{(1-i\Gamma_Z^2/M_Z^2)}{(1+\Gamma_Z^2/M_Z^2)}.$$ (12)

The $s$-dependent width in Z propagator translates into mass and width shift and introduction of the complex factor in front of the constant width Z propagator. This last point is possibly least trivial as it effectively means redefinition of Z coupling. That is why it can not be understood as parameter re-scaling. It points to present in higher order relations between vacuum polarization and vertex. Most of the changes are due to the term $\Gamma_Z^2/M_Z^2$ except of the overall phase which result from $1-i\Gamma_Z/M_Z$ factor and which change the $\gamma-Z$ interference. The shift in $M_Z$ is by about 34 MeV downwards, and the shift in $\Gamma_Z$ by 1 MeV, due to the reparametrisation of the Z-boson propagator.

In Figure 9 shown is comparison of the cross-sections and $A_{fb}$, between different implementations of $\chi(s)$. Dashed line of reference corresponds to using formula (10). Green line to complete formula (12). Red line corresponds to formula (12) but without $N_Z'$, scaling and blue line to formula (11), with nominal $M_Z$ and $\Gamma_Z$.

It is common for LHC MC generators to use formula (11) for Z propagator, with $M_Z$ and $\Gamma_Z$ of nominal, on-mass-shell values. Numerically this is better approximation than with shifted $M_Z$ and $\Gamma_Z$ but $N'_Z$ missing. This observation is true both for EW LO and EW corrected calculations; for cross-section and $A_{fb}$. Quantitative estimates are collected in Tables 15 and 16.

Note that when options of “running $\Gamma_Z$” and “fixed $\Gamma'_Z$” are compared, the same EW corrections in both cases tuned to “running $\Gamma_Z$” convention are used. It is beyond the scope of the paper to investigate, how NLO+HO corrections, calculated with the fixed width/ pole mass convention, gradually mitigate (as they should) discrepancy observed at EW LO level between Eq. (10) and Eq. (12) definition of Z propagator without $N'_Z$ included.

E Versions of DIZET library

In the KKMC distribution tar-ball, explained in Ref. [56], the code for calculation of electroweak corrections is stored in directory KK-all/dizet. The program stored in that directory calculates electroweak form-factors and writes them into ASCII.
Table 15: Ratio of the cross-sections ($\sigma$), calculated with different form of $Z$-boson propagator and integrated over outgoing lepton pair mass windows. It is shown for EW LO and EW NLO+HO predictions of $O(\alpha(0))$ EW scheme for $pp$ collisions at 8 TeV center of mass energy, while EW NLO+HO corrections are tuned to running $\Gamma_Z$ convention.

| $\sigma$(Fixed)/$\sigma$(Running) | $m_{ee}$ ranges (in GeV): | 90.5 – 91.5 | 89 – 93 | 60 – 81 | 81 – 101 | 101 – 150 |
|----------------------------------|---------------------------|-------------|---------|--------|---------|---------|
| at EW LO:                        |                           |             |         |        |         |         |
| with $M_Z, \Gamma_Z$ shift, no $N_Z'$ | 1.00087                  | 1.00087     | 1.00062 | 1.00086| 1.00071 |
| no $M_Z, \Gamma_Z$ shift, no $N_Z'$ | 0.99620                  | 1.00074     | 0.99716 | 0.99977| 1.00392 |
| at EW NLO+HO:                    |                           |             |         |        |         |         |
| with $M_Z, \Gamma_Z$ shift, no $N_Z'$ | 1.00113                  | 1.00085     | 1.00043 | 1.00083| 1.00075 |
| no $M_Z, \Gamma_Z$ shift, no $N_Z'$ | 0.99746                  | 1.00122     | 0.99719 | 1.00013| 1.00392 |

Table 16: Difference of $A_{fb}$ calculated for different form of $Z$-boson propagator and for integrated outgoing lepton pair mass windows. It is shown for EW LO and EW NLO+HO predictions of $O(\alpha(0))$ EW scheme for $pp$ collisions at 8 TeV center of mass energy, while EW NLO+HO corrections are tuned to running $\Gamma_Z$ convention.

| $A_{fb}$ (Running) - $A_{fb}$ (Fixed) | $m_{ee}$ ranges (in GeV): | 90.5 – 91.5 | 89 – 93 | 60 – 81 | 81 – 101 | 101 – 150 |
|--------------------------------------|---------------------------|-------------|---------|--------|---------|---------|
| at EW LO:                            |                           |             |         |        |         |         |
| with $M_Z, \Gamma_Z$ shift, no $N_Z'$ | -0.00048                 | -0.00047    | -0.00047| -0.00047| -0.00030|
| no $M_Z, \Gamma_Z$ shift, no $N_Z'$  | -0.00006                 | -0.00026    | -0.00012| -0.00040| -0.00005|
| at EW NLO+HO:                        |                           |             |         |        |         |         |
| with $M_Z, \Gamma_Z$ shift, no $N_Z'$ | -0.00053                 | -0.00053    | -0.00052| -0.00053| -0.00024|
| no $M_Z, \Gamma_Z$ shift, no $N_Z'$  | -0.00007                 | -0.00030    | -0.00026| -0.00048| -0.00004|

Figure 9: Ratio of the cross-sections (left) and $\Delta A_{fb}$ (right) for for EW LO but different form of $Z$-boson propagator, see text. The MC $pp \rightarrow Zj; Z \rightarrow l^+l^-$ events were used for estimations.
format text files. To change the version of the form-factors, requires simply use of these tables calculated with different version of library or with different set of initialization parameters. In parallel to program stored in KK-all/dizet which calculates electroweak form-factors as stored with Ref. [57] the following new ones, dizet-6.42-cpc, dizet-6.42 and dizet-6.45 were prepared and can be used following old instruction of KKMC documentation. There was only one change introduced, convenient for table reading by the TauSpinner package [13], the table write format for electroweak form-factors was modified to KK-all/dizet/BornV.h and if tables are used with KKMC the original one of KK-all/bornv/BornV.h file should be re-installed. This is true for all mentioned above DIZET 6.xx variants.

• DIZET 6.21 (dizet) is distributed with KKMC through CPC electroweak corrections and in particular vacuum polarization was not updated for backup compatibility. This version of library is documented in [17]. Note that upgrades of EW corrections within LEP experiments were not always well documented. This is in particular true for the photon vacuum polarization $\Pi_{\gamma\gamma}(s)$.

• DIZET 6.42 (dizet-6.42-cpc) as in published ZFITTER [32]. This is the last published/archived version of DIZET code. Note that as a default $\Pi_{\gamma\gamma}$ from ref [33] is still mentioned but obviously it was upgraded for the final versions of LEP data analysis of Ref. [2].

• DIZET 6.42 (dizet-6.42) with $\Pi_{\gamma\gamma}$ updated to hadr5n17_compact.f of Ref. [25]. Parametrization taken from author web page, dated Oct 8 02:19:56 2017.

• DIZET 6.45 (dizet-6.45) VERSION 6.45 (30 Aug. 2019) with the vacuum polarization code and fermionic two loops corrections, AMT4 flag upgraded by DIZET authors themselves.

Comparison of results for these versions are given in Fig. 8 and in Table 4.