When is a pure state of three qubits determined by its single-particle reduced density matrices?

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Abstract
Using techniques from symplectic geometry, we prove that a pure state of three qubits is up to local unitaries uniquely determined by its one-particle reduced density matrices exactly when their ordered spectra belong to the boundary of the so-called Kirwan polytope. Otherwise, the states with given reduced density matrices are parameterized, up to local unitary equivalence, by two real variables. Given inevitable experimental imprecision, this means that already for three qubits a pure quantum state can never be reconstructed from single-particle tomography. We moreover show that the knowledge of the reduced density matrices is always sufficient if one is given the additional promise that the quantum state is not convertible to the Greenberger–Horne–Zeilinger state by stochastic local operations and classical communication, and discuss generalizations of our results to an arbitrary number of qubits.

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Introduction

Finding solutions to Hamilton’s equations for a given system is a standard problem in classical mechanics. The system in question is often invariant with respect to a certain group of symmetries, and when the symmetries are continuous, such as in the case of rotational symmetry, they form a Lie group $K$. The existence of symmetries is inevitably connected to the first integrals of Hamilton’s equations. The celebrated theorem of Arnold [3] states that in the case when $K = T^n$ is an $n$-dimensional torus and $n$ is half the dimension of the phase space $M$, then the system is completely integrable. When the group $K$ is not Abelian, it is still possible to find corresponding first integrals, although they typically do not Poisson commute. For example, when a particle is moving in a potential with rotational symmetry,

4 A phase space is a symplectic manifold $(M, \omega)$, where $\omega$ is a closed nondegenerate 2-form.
so that $K = SO(3)$, then the conserved quantities are the three components of the angular momentum, corresponding to the invariance of the system with respect to infinitesimal rotations about the axes $x$, $y$ and $z$. These infinitesimal rotations generate the Lie algebra $\mathfrak{so}(3)$ of $SO(3)$. There are many possible bases of $\mathfrak{so}(3)$, each corresponding to a different choice of rotation axes and each giving three first integrals. The mathematical object which encodes information about the first integrals for all possible choices of generators of $\mathfrak{so}(3)$ is an equivariant map $\mu : M \to \mathfrak{so}(3)^*$ from the phase space $M$ to the space of linear functionals on the Lie algebra $\mathfrak{so}(3)$. For every infinitesimal symmetry $\xi \in \mathfrak{so}(3)$, one obtains a corresponding first integral by the formula $\mu_\xi (x) = \langle \mu(x), \xi \rangle$, where by $\langle \cdot, \cdot \rangle$ we denote the pairing between linear functionals from $\mathfrak{t}^*$ and vectors in $\mathfrak{t}$. This idea can be generalized to arbitrary Lie groups $K$ and a corresponding map $\mu : M \to \mathfrak{t}^*$ is called a momentum map [20].

Remarkably, momentum maps appear naturally not only in classical but also in quantum mechanics. Indeed, the Hilbert space $\mathcal{H}$ on which a given quantum-mechanical system is modeled can be seen as a phase space if we identify vectors that differ by a global rescaling or a phase factor $e^{i\phi}$. The set of (pure) quantum states is thus isomorphic to the complex projective space $\mathbb{P}(\mathcal{H})$, which is well known to be a symplectic manifold. When the considered system consists of $N$ subsystems$^5$ then the space $\mathcal{H}$ has the additional structure of a tensor product, namely $\mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_N$, where $\mathcal{H}_i$ is the single-particle Hilbert space. The mathematical properties of this structure manifest physically as entanglement, i.e. quantum correlations between subsystems. These correlations are invariant with respect to the local unitary action, i.e. to the action of the Lie group $K = SU(\mathcal{H}_i)^{\times N}$ on $\mathcal{H}$ by the tensor product. Since $K$ preserves the phase-space structure of $\mathbb{P}(\mathcal{H})$, one obtains a momentum map $\mu : \mathbb{P}(\mathcal{H}) \to \mathfrak{t}^*$, and the image $\mu([v])$ of a quantum state $[v] \in \mathbb{P}(\mathcal{H})$ is, up to some unimportant shifting, the collection of its one-body reduced density matrices (see section 2). The matrices $\rho_i$ can be diagonalized by the action of $K$ and their eigenvalues can be ordered, for example, decreasingly. The map which assigns to the state $[v]$ the collection of the ordered spectra of its one-body reduced density matrices will be denoted by $\Psi : \mathbb{P}(\mathcal{H}) \to \mathfrak{t}_+^*$. This definition can be generalized to an arbitrary compact Lie group $K$, with $\mathfrak{t}_+^*$ being a positive Weyl chamber in $\mathfrak{t}$ (see section 1 for mathematical details and section 2 for the case of three qubits).

Crucial for the rest of the paper is a specific geometric feature of the image of a momentum map, namely its convexity. In the early 1980s, the convexity properties of $\mu(M)$ were investigated by Atiyah [4] and Guillemin and Sternberg [18, 19] in the case of the Abelian group $K = T^n$. They proved that $\mu(M)$ is a convex polytope; it is in fact the convex hull of the image of the set of fixed points. For non-Abelian $K$, this is typically no longer true. However, as was shown in [18] for Kähler manifolds and in [26, 27] for general symplectic manifolds, the set $\Psi(M) = \mu(M) \cap \mathfrak{t}_+^*$ is always a convex polytope—the so-called Kirwan polytope or momentum polytope. This fact translates to the observation that the set of possible local eigenvalues, i.e. the eigenvalues of the one-partial reduced density matrices of an arbitrary pure quantum state is a convex polytope. Finding an explicit description of this polytope is in general a difficult problem; it is an instance of the quantum marginal problem or $N$-representability problem in the case of fermions [41, 12]. Following the groundbreaking work of Klyachko [28] on the famous Horn’s problem, which can be formulated as the problem of determining a Kirwan polytope, Berenstein and Sjamaar [5] found a general solution for the case where $M$ is a co-adjoint orbit of a larger group. This was in turn used to solve the one-body quantum marginal problem; namely, sets of inequalities describing the Kirwan polytope $\Psi(\mathbb{P}(\mathcal{H}))$ were given [29, 13, 30]. Intriguingly, the Kirwan polytope can under certain assumptions be also described in terms of coinvariants or, equivalently, in terms of the representations that occur in

$^5$ We assume for simplicity that their Hilbert spaces are of the same dimension.
the ring of polynomial functions on $M$ [40, 7]. In the context of the quantum marginal problem, this has been discovered by Christandl et al [11, 10] and Klyachko [29]. The interplay of these two complementary perspectives can also be seen in [9], where an algorithm has been given for the more general problem of computing the distribution of eigenvalues of the reduced density matrices of a multipartite pure state in $\mathbb{P}(\mathcal{H})$ drawn at random according to the Haar measure; in particular, this gives an alternative solution to the one-body quantum marginal problem.

The above line of work can also be seen as one of the first successful applications of momentum map geometry to the theory of entanglement (see also [31]). In [42], the importance of the momentum map to entanglement was investigated from a different perspective. The authors showed that restricting the map $\Psi$ to different local unitary orbits in $\mathbb{P}(\mathcal{H})$ gives rise to a well-defined purely geometric measure of entanglement. They also pointed out that the properties of the fibers of $\Psi$ are crucial to the solution of the local unitary equivalence problem and gave an algorithm for checking it for three qubits [43]. Geometrically, the problem of local unitary equivalence of two states reduces to determining whether they both belong to the same orbit of the local unitary group $K = SU(\mathcal{H}_1)^{\times N}$. From the physical point of view, this corresponds to checking if one of the states can be obtained from the other one by unitary quantum operations on the subsystems—an important problem in quantum engineering of states aiming at practical applications [33, 34].

In a subsequent paper, the authors analyzed the geometric structure of the fibers of $\Psi$ for two distinguishable particles, two fermions and two bosons [25]. In all these cases, the $K^C$-action is spherical, i.e. the Borel subgroup $B \subset K^C$ has an open orbit in $\mathbb{P}(\mathcal{H})$. By Brion’s theorem [7], this implies that each fiber of $\Psi$ is contained in a single $K$-orbit. That is, in these cases each quantum state is up to local unitaries uniquely determined by the spectra of its reduced density matrices. Moreover, each fiber of $\Psi$ has the structure of a symmetric space [25].

In situations involving larger numbers of particles, e.g., $N$-qubit systems with $N > 2$, the action of $K^C$ is not spherical and Brion’s theorem cannot be applied. The identification of $K$-orbits, i.e. classes of states which can be mutually transformed into each other by local unitary transformations, generically requires more information than is contained in the spectra of the single-particle reduced density matrices.

This paper explores such situations where a simple application of Brion’s theorem is no longer possible. Its main goal is to analyze the set of quantum states which are mapped by $\Psi$ to the same point of the Kirwan polytope $\Psi(M)$ (again, these are quantum states whose reduced density matrices have the same spectra) by studying the fibers of $\Psi$ or, equivalently, the symplectic quotients $M_\alpha = \Psi^{-1}(\alpha)/K$, in the case where the action of $K^C$ is no longer spherical. Specifically, we consider the above-mentioned case of $N$ qubits, where the Kirwan polytope is known explicitly [22, 6]. A detailed analysis is carried out for three qubits and we make some remarks about the general case. Our main tools are the convexity theorem for projective subvarieties [40, 7] as well as general properties of orbit spaces. Specifically, we show that $M_\alpha$ is generically a two-dimensional stratified symplectic space [36]. For the points in the boundary of the Kirwan polytope the situation is very different. We prove that in this case $M_\alpha$ is a single point, i.e. the dimension drops down by 2 compared with the interior. In particular, these results characterize the $K$-orbits which are uniquely determined by the spectra of the reduced density matrices. They may be contrasted with the well-known fact that the two-particle reduced density matrices generically suffice to determine a pure-state of three qubits [35].

The complexified group $G = K^C$ plays its own role in the classification of states of multiparticle quantum systems. Its elements correspond to stochastic local operations with
classical communication (SLOCC); see e.g. [23]. States can be classified by identifying those which belong to the same SLOCC class, i.e. the same $G$-orbit. As in the case of locally unitary equivalent states (orbits of $K$), a detailed analysis of the corresponding Kirwan polytopes is useful in such a classification. We therefore examine the Kirwan polytopes $\Psi(\overline{G,V})$ for the three-qubit SLOCC classes [15], i.e. for the orbit closures of the complexified group $G = K^C$, and describe their mutual relations. In particular, we find that the map $\Psi$ separates $K$-orbits when restricted to the closure of the so-called $W$-class. That is, states from the $W$-class are up to local unitaries characterized by the collection of spectra of their one-qubit reduced density matrices. Our argument generalizes directly to $W$-states of an arbitrary number of qubits.

The paper is organized as follows. In section 1, we recall the notion of a momentum map and state precisely various versions of the convexity theorems. In section 2, we discuss in detail the structure of the fibers of $\Psi$ for three qubits. Throughout the paper we only prove new results and otherwise give references to the literature.

1. Momentum maps

Let $K$ be a compact, connected Lie group acting on a symplectic manifold $(M, \omega)$ by symplectomorphisms, i.e. the action $\Phi_k : M \to M$ of any group element $g \in K$ preserves the symplectic form, $\Phi_k^*\omega = \omega$. Denote the space of smooth functions on $M$ by $\mathcal{F}(M)$.

**Definition 1.** We say that a symplectic action of $K$ on $(M, \omega)$ is Hamiltonian if and only if there exists a momentum map $\mu : M \to \mathfrak{k}^*$, i.e. a map which satisfies the following three conditions.

(i) For any $\xi \in \mathfrak{k}$, the fundamental vector field $\xi(x) = \frac{d}{dt}\bigg|_{t=0} \Phi(e^{t\xi}, x)$ is the Hamiltonian vector field for the Hamilton function $\mu_\xi(x) = \langle \mu(x), \xi \rangle$; i.e. $d\mu_\xi = \omega(\xi, \cdot)$.

(ii) The induced map $\mathfrak{k} \ni \xi \mapsto \mu_\xi \in \mathcal{F}(M)$ is a homomorphism of Lie algebras, i.e.

$$\mu([\xi, \xi'])(x) = [\mu_\xi, \mu_{\xi'}](x).$$

(iii) The map $\mu$ is equivariant, i.e. $\mu(\Phi_k(x)) = \text{Ad}_k^*\mu(x)$, where $\text{Ad}_k^*$ is the co-adjoint action of $K$ on $\mathfrak{k}^*$ defined by $\langle \text{Ad}_k^*\alpha, \xi \rangle = \langle \alpha, \text{Ad}_k^{-1}\xi \rangle$ in terms of the adjoint action $\text{Ad}_k\xi = \frac{d}{dt}\bigg|_{t=0} g e^{t\xi} g^{-1}$ of $K$ on its Lie algebra $\mathfrak{k}$.

For semisimple $K$, hence in particular for $K = SU(H_1)^\times N$, the momentum map $\mu$ is uniquely defined by the above properties [20].

1.1. Convexity properties of the momentum map

We will now assume that $M$ is compact and connected. Let us choose a maximal torus $T \subset K$, with Lie algebra $\mathfrak{t}$, and a positive Weyl chamber $\mathfrak{t}_+^* \subset \mathfrak{t}^*$. Denote by $\Psi : M \to \mu(M) \cap \mathfrak{t}_+^*$ the map which assigns to $x \in M$ the unique point of intersection $\mu(K,x) \cap \mathfrak{t}_+^*$. Then the following convexity results hold.

(i) The image $\Psi(M)$ is a convex polytope, the so-called Kirwan polytope ([18] and [26, 27]).

(ii) The fibers of $\mu$ (and hence the fibers of $\Psi$) are connected [26, 27].

(iii) The map $\Psi$ is an open map onto its image, i.e. for any open subset $U \subset M$ the image $\Psi(U)$ is open in $\Psi(M) = \mu(M) \cap \mathfrak{t}_+^*$ [32].

(iv) The image $\Psi(M_{\text{max}})$ of $M_{\text{max}} = \{x \in M : \dim K.x \text{ is maximal} \}$ is convex [21].

Since $M_{\text{max}}$ is a connected, open and dense subset of $M$ [39], (iv) implies the following.

**Fact 1.** The set $\mu(M_{\text{max}}) \cap \mathfrak{t}_+^*$ is an open, dense, connected, convex subset of $\Psi(M) = \mu(M) \cap \mathfrak{t}_+^*$. In particular, it contains the (relative) interior of the Kirwan polytope.
Denote by $G = K^C$ the complexification of $K$. This is a complex reductive group. Let us assume as in [7] that $M \subset \mathbb{P}(H)$ is a non-singular $G$-invariant irreducible subvariety of the complex projective space associated with a rational $G$-representation $H$, and let us choose a $K$-invariant inner product on $H$. Then $M$ is a symplectic manifold when equipped with the restriction of the Fubini–Study form,
\[
\omega_{[v]}(\hat{A}_{[v]}, \hat{B}_{[v]}) = 2 \text{Im} \frac{\langle A v | B v \rangle}{\langle v | v \rangle} = -i \frac{\langle [A, B] v | v \rangle}{\langle v | v \rangle} \quad \forall A, B \in \mathfrak{u}(H),
\]
where $[v] \in \mathbb{P}(H)$ is the projection of a vector $v \in H$ and $\hat{A}$ is the fundamental vector field generated by the action of $A \in \mathfrak{u}(H)$. More concretely, we can represent the tangent space $T_{[v]}\mathbb{P}(H)$ by $H/Cv \cong (Cv)^\perp$. In this picture, the tangent vector $A_v$ is given by the orthogonal projection of $A_v$ onto $(Cv)^\perp$. $M$ also carries a canonical momentum map $\mu : M \subset \mathbb{P}(H) \to \mathfrak{v}^*$.

\[
\langle \mu([v]), A \rangle = -i \frac{\langle v | A v \rangle}{\langle v | v \rangle} \quad \forall A \in \mathfrak{k},
\]

In this situation, there are further convexity results.

(v) The image $\Psi(G, x)$ is a convex polytope [7, 40].

(vi) The collection of different polytopes $\Psi(G, x)$, where $x$ ranges over $M$, is finite [17].

### 1.2. Fibers of the momentum map

The following two facts characterizing kernel and image of the differential of the momentum map are well-known consequences of the definition [18].

**Fact 2.** The kernel of $d\mu|_x : T_x M \to \mathfrak{v}^*$ is equal to the $\omega$-orthogonal complement of $T_x(K, x)$, i.e. $\text{Ker}(d\mu|_x) = \{Y \in T_x M : \omega(\hat{x}_Y, Y) = 0 \ \forall \xi \in \mathfrak{k}\} = (T_x(K, x))^{\perp_\omega}$.

**Fact 3.** The image of $d\mu|_x : T_x M \to \mathfrak{v}^*$ is equal to the annihilator of $\mathfrak{k}_x$, the Lie algebra of the isotropy subgroup $K_x \subset K$.

These results have immediate consequences for the characterization of the fibers of $\mu$. Let us assume that $d\mu$ is surjective at a single point $x \in M$ (as will be the case in our applications). Using fact 3, we observe that $d\mu$ is surjective if and only if the Lie algebra $\mathfrak{k}_x$ is trivial. This in turn happens if and only if the isotropy subgroup $K_x$ is discrete. Therefore, $d\mu$ is automatically surjective at all points in $M_{\text{max}}$. In particular, the implicit function theorem together with fact 2 implies that for all $x \in M_{\text{max}}$, the tangent space at $x$ of the $\mu$-fiber through $x$ has dimension
\[
\dim T_x(\mu^{-1}(\mu(x))) = \dim (T_x(K, x))^{\perp_\omega} = \dim M - \dim \mathfrak{v}^*,
\]
and hence

**Fact 4.** For the points $x \in M_{\text{max}}$, i.e. for an open, connected and dense subset of $M$, the intersection $\mu^{-1}(\mu(x)) \cap M_{\text{max}}$ of the $\mu$-fiber through $x$ with $M_{\text{max}}$ is a $\dim (T_x(K, x))^{\perp_\omega}$-dimensional manifold.

Understanding the fibers $\mu^{-1}(\alpha)$ away from the regular points in $M_{\text{max}}$ is a more delicate problem. Since each fiber $\mu^{-1}(\alpha)$ is $K_a$-invariant, where $K_a$ is the isotropy subgroup of $\alpha$ with respect to the co-adjoint action, it is convenient to introduce the symplectic quotient $M_a := \mu^{-1}(\alpha)/K_a = \Psi^{-1}(\alpha)/K$. It is in general a stratified symplectic space [36], which is moreover connected by property (ii) in subsection 1.1. The following is then immediate.
The fiber $\mu^{-1}(\alpha)$ intersects a single $K$-orbit if and only if $M_\alpha$ is a single point or, equivalently, if and only if $M_\alpha$ is zero dimensional.

If $\alpha \in \Psi(M_{\max})$, then the maximal-dimensional stratum of $M_\alpha$ is simply $(\mu^{-1}(\alpha) \cap M_{\max})/K_\alpha$ [38]. Since the isotropy subgroup of any point in $M_{\max}$ is discrete,

$$\dim M_\alpha = \dim M - \dim \mathfrak{t}^* - \dim K_\alpha. \quad (4)$$

In other words, we can by mere dimension counting determine whether a $K$-orbit in $M_{\max}$ is uniquely determined by its image under the momentum map.

### 2. Fibers of the momentum map for three qubits

With the momentum map machinery presented in the preceding section we are now well equipped to analyze when a pure state of three qubits is, up to local unitaries, determined by the spectra of its reduced density matrices.

To this end, let $M = \mathcal{P}(\mathcal{H})$ be the projective space of pure states associated with the three-qubit Hilbert space $\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$. The group of local unitaries $K = SU(2)^\otimes 3$ and its complexification $G = SL(2)^\otimes 3$ act on $M$ by the tensor product. The Lie algebra of $K$ is $\mathfrak{k} = \mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2)$, i.e. triples of traceless anti-Hermitian matrices. As described in the preceding section, $M$ is a symplectic manifold with respect to the Fubini–Study form (1) and the $K$-action is Hamiltonian with the canonical momentum map (2).

It is easy to see that under the identification of $\mathfrak{t}^*$ with $\mathfrak{t}$ induced by the trace inner product, the image $\mu(\mathfrak{t}[v])$ is given by the collection of one-qubit reduced density matrices, namely

$$\mu(\mathfrak{t}[v]) = i\left(\rho_1 - \frac{1}{2}I, \rho_2 - \frac{1}{2}I, \rho_3 - \frac{1}{2}I\right), \quad (5)$$

where $I$ is the $2 \times 2$ identity matrix (see, e.g., [42, 9]).

Let us fix the maximal torus $T \subset K$ to be the set of unitary diagonal matrices with determinant equal to 1. Then the Lie algebra $\mathfrak{t}$ is equal to the space of traceless anti-Hermitian diagonal matrices. We choose as the positive Weyl chamber the following set of matrices:

$$\mathfrak{t}_+^* = \{\text{diag}(-i\lambda_1, i\lambda_2), \text{diag}(-i\lambda_2, i\lambda_1), \text{diag}(-i\lambda_3, i\lambda_3) : \lambda_i \geq 0\}. \quad (6)$$

It follows that, up to some rescaling and shifting, the map $\Psi$ sends a pure state $[v]$ to the (ordered) spectra of its reduced density matrices $\rho_1, \rho_2$ and $\rho_3$.

The following theorem describes the Kirwan polytope in terms of inequalities for a general system of $N$ qubits.

**Fact 6 ([22]).** For an $N$-qubit system, the constraints on the one-qubit reduced density matrices $\rho_i$ of a pure state are given by the polygonal inequalities

$$p_i \leq \sum_{j \neq i} \rho_j,$$

where $p_i \leq \frac{1}{2}$ denotes the minimal eigenvalue of $\rho_i$ ($i = 1, \ldots, N$).

The Kirwan polytope for three qubits is shown in figure 1. This polytope has five vertices, nine edges and six faces. As a convex set, it is of course generated by its vertices, which are

$$v_{\text{SEP}} = \begin{cases} \text{diag} \left( -\frac{1}{2}, \frac{1}{2} \right), & \text{diag} \left( -\frac{1}{2}, \frac{1}{2} \right), & \text{diag} \left( -\frac{1}{2}, \frac{1}{2} \right), \end{cases}$$

$$v_{\text{B1}} = \begin{cases} \text{diag} \left( -\frac{1}{2}, \frac{1}{2} \right), & \text{diag}(0,0), \text{diag}(0,0) \end{cases},$$

$$v_{\text{B2}} = \begin{cases} \text{diag}(0,0), & \text{diag} \left( -\frac{1}{2}, \frac{1}{2} \right), & \text{diag}(0,0) \end{cases},$$

$$v_{\text{B3}} = \begin{cases} \text{diag}(0,0), & \text{diag}(0,0), & \text{diag} \left( -\frac{1}{2}, \frac{1}{2} \right) \end{cases},$$

$$v_{\text{GHZ}} = \begin{cases} \text{diag}(0,0), & \text{diag}(0,0), & \text{diag}(0,0) \end{cases}. \quad (7)$$
Figure 1. The Kirwan polytope $\Psi(M) = \mu(M) \cap t^*_\mathfrak{g}$ for three qubits.

Since the polytope is full dimensional, Sard’s theorem [46] implies the existence of a regular point in $M$, i.e. a point with a discrete isotropy subgroup (see also [8]). Hence,

**Lemma 1.** The set $M_{\text{max}} \subset M$ is connected, open and dense and consists of orbits of dimension $\dim K = 9$.

We will now analyze points inside the interior of the Kirwan polytope. Note first that by fact 1 the pre-image of any such point $\alpha$ contains a point $x \in M_{\text{max}}$. Therefore, fact 4 and (3) show that $\mu^{-1}(\alpha) \cap M_{\text{max}}$ is a manifold of dimension

$$\dim(\mu^{-1}(\alpha) \cap M_{\text{max}}) = \dim(T, K, x, \mathfrak{t}^*) = \dim M - \dim \mathfrak{t}^* = 14 - 9 = 5.$$  

Since $K_{\alpha} = T$ for points in the interior of the positive Weyl chamber, this manifold consists of three-dimensional $T$-orbits, and (4) implies that

$$\dim M_{\alpha} = 5 - 3 = 2.$$  

If we replace $\mu^{-1}(x)$ by $\Psi^{-1}(x) = K, \mu^{-1}(x)$, each $T$-orbit by a $K$-orbit and therefore increase the dimension by

$$\dim \Omega_{\alpha} = \dim \frac{K}{K_{\alpha}} = \dim K - \dim T = 6,$$

then the dimension of the corresponding co-adjoint orbit $\Omega_{\alpha} = K, \alpha \subset \mathfrak{p}^*$.

Summing up, we proved

**Theorem 1.** For any point $\alpha$ inside the interior of the Kirwan polytope there exists a point $x \in M_{\text{max}}$ such that $\alpha = \mu(x)$. The manifold $\mu^{-1}(\alpha) \cap M_{\text{max}}$ is five dimensional, consisting of three-dimensional $T$-orbits. Moreover, $\Psi^{-1}(\alpha) \cap M_{\text{max}}$ is an 11-dimensional manifold consisting of 9-dimensional orbits $K, y$ with the property $\mu(K, y) = \Omega_{\alpha}$, and the symplectic quotient $M_{\alpha}$ has dimension 2.

The following is a direct consequence of theorem 1 (cf the discussion at the end of section 1).

**Corollary 1.** The orbits $K, x$ for which $\mu(K, x) \cap t^*_\mathfrak{g}$ belongs to the interior of Kirwan polytope cannot be separated by the momentum map. In other words, a pure state of three qubits whose spectrum is non-degenerate and satisfies the polygonal inequalities with strict inequality is never determined up to local unitaries by the spectra of its reduced density matrices.

What is left is to analyze points in the boundary of the Kirwan polytope. We postpone this to the end of the section (see subsection 2.3).
2.1. The SLOCC classes and their Kirwan polytopes

As mentioned in the introduction, the classification of G-orbits, where $G=K^C$, gives another view on the entanglement properties of states on the composite system. In this subsection, we explicitly compute the Kirwan polytopes $\Psi(G, x)$ for all G-orbit closures and show how they are related to the polytope $\Psi(\mathbb{P}(H))$. Physically, G-orbits correspond to SLOCC classes [15]; hence, our results describe the spectra of the reduced density matrices of the quantum states in each SLOCC entanglement class. The problem of classifying G-orbits in $\mathbb{P}(H)$ is inherently connected to the momentum map geometry as well as to the construction of the so-called Mumford quotient [27, 40]. We explain this connection in [44, 47].

2.1.1. Classification of G-orbits. 

It has been shown in [15] that there are six SLOCC entanglement classes, i.e. G-orbits in $\mathbb{P}(H)$. For convenience of the reader, we list them below and briefly summarize their basic geometric properties.

(i) The G-orbit of the Greenberger–Horne–Zeilinger state, $x_{\text{GHZ}} = \left[ \frac{1}{\sqrt{2}} (|000\rangle + |111\rangle) \right]$ [16]. It is the open dense orbit, hence of (real) dimension $\dim(\mathbb{P}(H)) = 14$.

(ii) The G-orbit of the W-state, $x_w = \left[ \frac{1}{\sqrt{3}} (|100\rangle + |010\rangle + |001\rangle) \right]$. Here, $\dim G.x_w = 12$, while $\dim K.x_w = 8$. For the proof, it is enough to compute the dimension of the tangent space $T_{x_w}(G.x_w)$, which can be represented as the projection of $\text{Span}_{\mathbb{C}}\{Ax_w : A \in g\}$ onto the orthogonal complement of $x_w$. The Lie algebra $g$ is equal to $sl_2(\mathbb{C}) \oplus sl_2(\mathbb{C}) \oplus sl_2(\mathbb{C})$, where $sl_2(\mathbb{C}) = \text{Span}_{\mathbb{C}}\{E_{12}, E_{21}, E_{11} - E_{22}\}$, and $E_{ij}$ is the $2 \times 2$ matrix with a single non-zero entry equal to 1 in the $i$th row and $j$th column. It is easy to see that the following seven vectors
\[
(E_{12} \otimes I \otimes I)x_w = (I \otimes E_{12} \otimes I) x_w = (I \otimes I \otimes E_{12}) x_w \propto |000\rangle,
(E_{21} \otimes I \otimes I)x_w \propto |110\rangle + |101\rangle,
(I \otimes E_{21} \otimes I)x_w \propto |110\rangle + |011\rangle,
(I \otimes I \otimes E_{21})x_w \propto |101\rangle + |011\rangle,
((E_{11} - E_{22}) \otimes I \otimes I)x_w \propto -|100\rangle + |010\rangle + |001\rangle,
(I \otimes (E_{11} - E_{22}) \otimes I)x_w \propto +|100\rangle - |010\rangle + |001\rangle,
(I \otimes I \otimes (E_{11} - E_{22})) x_w \propto +|100\rangle + |010\rangle - |001\rangle
\]
span a complex vector space of dimension 6 after projection onto $x_w^\perp$ (the last three vectors become linearly dependent). We conclude that $\dim G.x_w = 2 \cdot 6 = 12$. Similarly, one shows that $\dim K.x_w = 8$. For future reference we denote
\[
v_w = \mu(x_w) = i \{ \text{diag} \left( -\frac{1}{6}, \frac{1}{6} \right), \text{diag} \left( -\frac{1}{6}, \frac{1}{6} \right), \text{diag} \left( -\frac{1}{6}, \frac{1}{6} \right) \}. \tag{8}
\]

(iii) The G-orbits through the bi-separable Bell states $x_{B1} = \left[ \frac{1}{\sqrt{2}} (|000\rangle + |111\rangle) \right] = \left[ |0\rangle \otimes \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle) \right]$, $x_{B2} = \left[ \frac{1}{\sqrt{2}} (|000\rangle + |101\rangle) \right]$ and $x_{B3} = \left[ \frac{1}{\sqrt{2}} (|000\rangle + |110\rangle) \right]$. In a similar way as for $x_w$, one can show by an explicit computation that $\dim G.x_{Bk} = 8$ and $\dim K.x_{Bk} = 5$.

(iv) The G-orbit of separable states, generated by $x_{\text{SEP}} = |000\rangle$. By Kostant–Sternberg theorem, it is the unique symplectic orbit [42] and $\dim G.x_{\text{SEP}} = \dim K.x_{\text{SEP}} = 9 - 3 = 6$. 

8
2.1.2. Kirwan polytopes of G-orbit closures. Let us write $X_j := G \cdot x_j$ for the closures of these G-orbits. The representatives $x_j$ that we have chosen above satisfy the following property.

**Lemma 2.** We have $\Psi(x_j) = \mu(x_j) = v_j$, where $v_j$ is defined in (7) and (8). Moreover, $\Psi(x_j)$ is the closest point to the origin in the Kirwan polytope $\Psi(X_j)$.

The first claim follows from a simple computation. The fact that $v_j$ is the closest point to the origin of the Kirwan polytope of the corresponding $G$-orbit closure follows from the general theory of [26]: the gradient descent with respect to the norm square of the momentum map is at any point $x$ implemented by the vector field generated by $i\mu(x) \in i\mathfrak{g} \subseteq \mathfrak{g}$, and one easily checks that $i\mu(x)|_{v_j} = 0$. It will be explained in more detail in [44, 47].

We will now describe the Kirwan polytopes of the $G$-orbit closures. Since $G \cdot x_{\text{GHZ}}$ is dense in $\mathbb{P}(\mathcal{H})$, we immediately obtain that $\Psi(x_{\text{GHZ}})$ is equal to the full Kirwan polytope for three qubits as described by fact 6. On the other hand, since $G \cdot x_{\text{SEP}} = K \cdot x_{\text{SEP}}$ is a single $K$-orbit, $\Psi(x_{\text{SEP}})$ is a single point, namely $\Psi(x_{\text{SEP}}) = \{v_{\text{SEP}}\}$. Similarly, one finds that the Kirwan polytopes for the bi-separable Bell states $x_{\text{BB}}$ are equal to the one-dimensional line segments from $v_{\text{BB}}$ to $v_{\text{SEP}}$. Therefore, the only non-trivial task is the computation of the Kirwan polytope for the $W$-state.

**Proposition 1.** The Kirwan polytope $\Psi(X_W)$ is equal to the convex hull of the points $v_{B1}$, $v_{B2}$, $v_{B3}$ and $v_{\text{SEP}}$. In particular, it is of maximal dimension.

**Proof.** Clearly, $\Psi(X_W)$ is a subset of the full Kirwan polytope $\Psi(\mathbb{P}(\mathcal{H}))$. On the other hand, lemma 2 and convexity imply that it is also contained in the half-space through $v_W$ with the normal vector $v_W$. Since the intersection of this half-space with $\Psi(\mathbb{P}(\mathcal{H}))$ is precisely equal to the convex hull of the points $v_{B1}$, $v_{B2}$, $v_{B3}$ and $v_{\text{SEP}}$, we only need to show that these points are contained in the Kirwan polytope.

We will in fact show that the corresponding pre-images $x_j$ are contained in the orbit closure $X_W$. For this, we observe that the action of the complexification $T^C \subseteq G$ of the maximal torus $T \subseteq K$ applied to $x_W$ gives rise to all states of the form

$$[c_1|100⟩ + c_2|010⟩ + c_3|001⟩] \quad (c_j \neq 0).$$

In particular, $x_j \subseteq X_W = G \cdot x_W$ for $j = B1, B2, B3, \text{SEP}$, and the claim follows. \[\Box\]

2.2. Sphericality of the W SLOCC class

In this subsection, we will show that $X_W = \overline{G \cdot x_W}$ is a spherical variety. It will then be followed by Brion’s theorem that every quantum state in $X_W$ is, up to local unitaries, characterized by the collection of the spectra of its one-qubit reduced density matrices. In other words, $\Psi$ separates the $K$-orbits in $X_W$.

We start by clarifying the geometric structure of $X_W$. Note that since $G = K^C$ is a complex reductive group, any orbit closure $X_j$ is a $G$-invariant irreducible subvariety of $\mathbb{P}(\mathcal{H})$; however, these varieties will in general be singular. This is in fact already the case for $X_W$. Indeed, it is known from [31] that

$$X_W = \overline{\mathbb{P}(\mathcal{H}) \setminus (G \cdot x_{\text{GHZ}})} = \{[v] \in \mathbb{P}(\mathcal{H}) : \text{Det}(v) = 0\},$$

where Det is the Cayley hyperdeterminant (the basic invariant for the $G$-representation $\mathcal{H}$), and one readily verifies that the tangent space at $x_{\text{SEP}} \in X_W$ has complex dimension $7 > 6 = \text{dimc} X_W$.

Let us denote the restriction of $\mu : \mathbb{P}(\mathcal{H}) \to \mathfrak{t}^*$ to $X_W$ by $\mu_W : X_W \to \mathfrak{t}^*$. Our aim now is to prove that $\mu_W$ separates the $K$-orbits in $X_W$. To this end, we use the following theorem by Brion (see [7], and also [24]).
Fact 7. Let $G = K^c$ be a connected complex reductive group, $\mathcal{H}$ a rational $G$-representation and $X$ a $G$-invariant irreducible subvariety of $\mathbb{P}(\mathcal{H})$ (cf subsection 1.1). Then the following are equivalent.

(i) $X$ is spherical, i.e. the (every) Borel subgroup $B$ has a Zariski-open orbit in $X$.
(ii) For every $x \in X$, the fiber $\mu^{-1}(\mu(x))$ is contained in a single $K$-orbit, $K.x$.

We will now show that indeed

Proposition 2. The $G$-variety $X_W$ is spherical.

Proof. Consider the Borel subgroup $B$, which consists of the lower triangular matrices in $G$.

Its Lie algebra $\mathfrak{b}$ is equal to $\mathbb{C} \oplus \mathfrak{n}$, where

$$\mathfrak{n} = \text{Span}_\mathbb{C} \{ E_{21} \otimes I \otimes I, I \otimes E_{21} \otimes I, I \otimes I \otimes E_{21} \}.$$

In order to show that $X_W$ is spherical, we have to show that there exists a Zariski-open orbit $B.x$. Since $B.x$ is Zariski-open in its closure, it suffices to show that the closure of $B.x$ is equal to $X_W$. Now, since the closure of $B.x$ is a closed subvariety, it is either equal to $X_W$ or of lower dimension. Therefore, it suffices to show that the dimension of $B.x$ is equal to the dimension of $X_W$. Since $B.x$ is a smooth variety, we can compute its dimension by computing the dimension of the tangent spaces at any point. Hence in order to show that $X_W$ is spherical, it suffices to show that $\dim \mathfrak{t}_G(B.x) = \dim \mathfrak{t}_G X_W = 6$ for any single point $x$ in the $G$-orbit of $x_W$. We will consider the state

$$x = \left[ \frac{1}{\sqrt{2}} (|100\rangle + |010\rangle + |001\rangle + |000\rangle) \right] = \left( \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \otimes I \otimes I \right) \cdot x_W.$$

Indeed, one can easily verify that the tangent vectors generated by $\mathfrak{b}$, i.e. the projection of

$$(E_{21} \otimes I \otimes I)x \propto |110\rangle + |011\rangle + |010\rangle,$$

$$(I \otimes E_{21} \otimes I)x \propto |110\rangle + |011\rangle + |010\rangle,$$

$$(I \otimes I \otimes E_{21})x \propto |101\rangle + |011\rangle + |001\rangle,$$

$$((E_{11} - E_{22}) \otimes I \otimes I)x \propto -|100\rangle + |010\rangle + |001\rangle + |000\rangle,$$

$$(I \otimes (E_{11} - E_{22}) \otimes I)x \propto +|100\rangle - |010\rangle + |001\rangle + |000\rangle,$$

$$(I \otimes I \otimes (E_{11} - E_{22}))x \propto |100\rangle + |010\rangle - |001\rangle + |000\rangle$$

onto the orthogonal complement $x^\perp$, span a complex vector space of dimension $6$. $\square$

The following is now a direct consequence of proposition 2 and fact 7.

Theorem 2. The momentum map $\mu_W$ separates all $K$-orbits inside $X_W$. That is, every quantum state in the $W$ SLOCC class is (up to local unitaries) uniquely determined by the spectra of its reduced density matrices.

Remark 1. In fact, any $W$-type state of $L$ qubits, i.e. any state which is in the SLOCC class of

$$x_W = \left[ \frac{1}{\sqrt{N}} (|10\ldots0\rangle + |01\ldots0\rangle + \cdots + |00\ldots1\rangle) \right] \in \mathbb{P}((\mathbb{C}^2)^{\otimes L}),$$

is up to local unitary equivalence determined by its single-particle density matrices. This can be shown mutatis mutandis as for the case of three qubits (cf [48] for an alternative proof based on linear-algebraic computations, which appeared after the initial version of this paper).

By Brion’s theorem, it is enough to show that the $G$-variety $G.x_W$ is spherical. To this end, we choose the state

$$x = \left( \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \otimes I \cdots \otimes I \right) x_W = \left[ \frac{1}{\sqrt{N+1}} (|0\ldots0\rangle + \sqrt{N} x_W) \right] \in G.x_W$$
Figure 2. The structure of $\mu^{-1}(\Omega_{\alpha})$, where $\alpha$ is in the interior of the polytope $\Psi(X_W)$.

in complete analogy to the proof of proposition 2. It is again a matter of simple calculations to show that the real dimensions of the orbits $G.x$ and $B.x$, where $B$ is a Borel subgroup defined as in the case of three qubits, are equal (to $4L$). Hence $G.x_W$ is a spherical variety.

Remark 2. The fact that $W$-states are, up to local unitaries, uniquely determined by the spectra of their reduced one-particle density matrices can be, in principle, also determined in a purely algebraic manner from the explicit knowledge of invariants separating different orbits of the group of local unitary transformations. Indeed, if we are able to show that for all states of the $W$-class all invariants depend only on the spectra of one-particle density matrices, then the conclusion follows. For the three-qubit case, this can be, in principle, done using results of [1, 2], where explicit forms of the invariants are given. For $W$-states one can, after some calculations, prove that they depend only on one-particle spectra. For more than three qubits, however, this approach quickly becomes intractable: one first needs to explicitly compute all relevant invariants (which has to be done separately for each number of qubits) and then prove that for states of the $W$-type the result depends only on the local spectra. In contrast, the geometric approach presented above leads immediately to the desired result for an arbitrary number of qubits.

Combining proposition 2 with theorem 1 we conclude that each manifold $\mu^{-1}(\Omega_{\alpha}) \cap M_{\max}$, where $\alpha$ is in the interior of the polytope $\Psi(X_W)$, contains a unique orbit $K.\tilde{x}$ with $\tilde{x} \in G.x_W$. We have illustrated the situation in figure 2.

2.3. The boundary of the Kirwan polytope

In our analysis at the beginning of this section we did only consider quantum states that are mapped into the interior of the Kirwan polytope. We will now prove that any pure quantum state of three qubits that is mapped to the boundary of the Kirwan polytope $\Psi(P(H))$ is, up to local unitaries, uniquely determined by the spectra of its one-body reduced density matrices.

Let us therefore consider a quantum state $x \in P(H)$ that is mapped to the boundary of the Kirwan polytope and write

$$\mu(x) = (\text{diag}(i\lambda_1, i\lambda_1), \text{diag}(i\lambda_2, i\lambda_2), \text{diag}(i\lambda_3, i\lambda_3)),$$

(10)
according to (6). Then the minimal eigenvalue \( p_j \) of the reduced density matrix \( \rho_j \) is given by

\[
p_j = \frac{1}{2} - \lambda_j.
\]

We distinguish two cases as follows.

**2.3.1. Non-degenerate case.** We shall first treat the case where the reduced density matrices of \( x \) all have non-degenerate eigenvalue spectrum; that is, \( \mu(x) \) is contained in the interior of the positive Weyl chamber. We may assume by symmetry that \( \mu(x) \) is contained in the face corresponding to the equation \( p_1 = p_2 + p_3 \) (cf Fact 6), i.e.

\[
-\lambda_1 + \lambda_2 + \lambda_3 = \frac{1}{2}.
\]

That is, \( \mu(x) \) is orthogonal to (annihilated by) the Lie algebra element \( i\xi \in k \), where

\[
\xi = \frac{1}{2}(\text{diag}(-1, 1), \text{diag}(1, -1), \text{diag}(1, -1)),
\]

and fact 3 implies that \( C\xi \) is contained in the Lie algebra of the isotropy subgroup of \( x \). It follows that \( x = [v] \) for some eigenvector \( v \in \mathcal{H} \) with eigenvalue \( \frac{1}{2} \). By diagonalizing the action of \( \xi \) on \( \mathcal{H} \) we find that

\[
x = [c_1|000\rangle + c_2|110\rangle + c_3|101\rangle]
\]

for some constants \( c_j \). Any such state is contained in \( X_W \), the closure of the SLOCC class of the \( W \)-state, as can be seen by applying \( |0\rangle \leftrightarrow |1\rangle \) to the first subsystem and comparing with (9). We can therefore use theorem 2 to conclude that \( x \) is determined up to local unitaries by the eigenvalues of its reduced density matrices.

**2.3.2. Degenerate case.** We will now treat the case where at least one of the reduced density matrices is maximally mixed. Without loss of generality, we may assume that \( \mu(x) \) is contained in the face of the Kirwan polytope defined by \( p_1 = \frac{1}{2} \), i.e. \( \lambda_1 = 0 \). Note that we cannot apply the same reasoning as above, since the faces \( \lambda_j = 0 \) arise from intersecting \( \mu(P(H)) \) with the positive Weyl chamber and not from the geometry of the momentum map. By the Schmidt decomposition, and up to a local unitary on the first subsystem,

\[
x = \left[ \frac{1}{\sqrt{2}} (|0\rangle \otimes |\phi\rangle + |1\rangle \otimes |\psi\rangle) \right]
\]

for orthogonal vectors \( \langle \phi|\psi \rangle = 0 \). The two-qubit reduced density matrix \( \rho_{23} \) of any such state is a normalized projector onto the two-dimensional subspace of \( \mathbb{C}^2 \otimes \mathbb{C}^2 \) with basis vectors \( |\phi\rangle \) and \( |\psi\rangle \); conversely, any other choice of orthonormal basis gives rise to a local unitarily equivalent state.

Let us denote by \( G(2, 4) \) the Grassmannian consisting of two-dimensional subspaces \( K \subseteq \mathbb{C}^2 \otimes \mathbb{C}^2 \). Similar to the case of the projective space, we can consider the tensor product action of \( G' = SL(2) \times SL(2) \) and of its maximal compact subgroup \( K' = SU(2) \times SU(2) \). A momentum map for the \( K' \)-action is given by

\[
\mu' : G(2, 4) \ni K \mapsto i \left( \rho_2 - \frac{1}{2}I, \rho_3 - \frac{1}{2}I \right) \in \mathfrak{k}'^*, \quad \mathfrak{k}' = su(2) \oplus su(2),
\]

where \( \rho_2 \) and \( \rho_3 \) denote the reduced density matrices of \( \rho_{23} = \frac{1}{2}P_K \), the normalized projector onto the subspace \( K \). In view of fact 7, it suffices to establish sphericality of this Grassmannian with respect to the action of \( G' \).
It follows that the tangent space $T_K(B'.K)$ of the reduced density matrices of a randomly chosen pure state of three qubits vanishes precisely on the boundary of the Kirwan polytope. Since it is well known that $f(\alpha) = \text{vol} M_\alpha$ for regular points of the momentum map [14, 38], it is reasonable to wonder whether an analogous statement could hold more generally. (The main result of [37] suggests that this might in fact be true.)

### Proposition 3. The $G'$-variety $G(2, 4)$ is spherical.

**Proof.** Let $B'$ denote the Borel subgroup consisting of lower triangular matrices in $G'$. As in the proof of proposition 2, we will show that there exists a point $K \in G(2, 4)$ at which $\dim K T_K(B'.K) = \dim K G(2, 4) = 4$ (noting that $G(2, 4)$ is smooth). It will be convenient to work with coordinates. Let us therefore consider the Plücker embedding

$$G(2, 4) \to \mathbb{P} \left( \bigwedge^2 \mathbb{C}^4 \right), \quad \text{Span}_\mathbb{C}(\{|\psi\rangle\}, \{|\phi\rangle\}) \mapsto \{|\psi\rangle \wedge |\phi\rangle\}.$$

The image of the subspace $K = \text{Span}_\mathbb{C}(|++\rangle, |--\rangle)$, where $|\pm\rangle = \frac{1}{\sqrt{2}} (|0\rangle \pm |1\rangle)$, is $x = [|++\rangle \wedge |--\rangle] = [|00\rangle \wedge |01\rangle + |00\rangle \wedge |10\rangle - |01\rangle \wedge |11\rangle - |10\rangle \wedge |11\rangle\}$.

It follows that the tangent space $T_K(B'.K)$ is spanned by the vectors

$$(E_{11} \otimes I) x \propto |00\rangle \wedge |11\rangle - |01\rangle \wedge |10\rangle,$$

$$(I \otimes E_{21}) x \propto |00\rangle \wedge |11\rangle + |01\rangle \wedge |10\rangle,$$

$$(E_{11} - E_{22}) (E_{11} \otimes I) x \propto |00\rangle \wedge |01\rangle + |10\rangle \wedge |11\rangle,$$

$$(I \otimes (E_{11} - E_{22})) x \propto |00\rangle \wedge |10\rangle + |01\rangle \wedge |11\rangle,$$

which are orthogonal to $x$ and linearly independent of $\mathbb{C}$. We conclude that the tangent space is of complex dimension 4.

In summary, we have proved the following result.

**Theorem 3.** Let $x$ be a pure quantum state of three qubits such that $\mu(x)$ is contained in the boundary of the Kirwan polytope $\Psi(\mathbb{P}(\mathcal{H}))$ (i.e. its eigenvalues satisfy at least one of the inequalities in fact 6 with equality). Then $x$ is up to local unitaries uniquely determined by $\mu(x)$, i.e. by the spectra of its one-body reduced density matrices.

### 3. Summary

In order to determine which pure states of three qubits are up to local unitaries uniquely determined by the spectra of their reduced density matrices, we have analyzed the change of the structure of the fiber $\Psi^{-1}(\alpha)$ as $\alpha$ varies in the Kirwan polytope $\Psi(\mathbb{P}(\mathcal{H}))$. We have shown that $M_\alpha = \Psi^{-1}(\alpha)/K$ is generically a two-dimensional space. For the points in the boundary of the Kirwan polytope, the situation is rather different. We have shown that in this case $\Psi^{-1}(\alpha)/K$ is a single point, i.e. the dimension drops by 2 compared with the points inside the interior. We have therefore identified all $K$-orbits that are uniquely determined by the spectra of the reduced density matrices. In addition, we have examined the Kirwan polytopes $\Psi(G, x'_\alpha)$ for all six three-qubit SLOCC classes (i.e. for the closures of the orbits of the complexified group $G = K^C_\mathbb{C}$) and their mutual relation. In particular, we have proved that states from the so-called $W$ SLOCC class are up to local unitaries separated by $\Psi$, i.e. each $K$-orbit inside $G x'_\alpha$ is characterized by the collection of spectra of the one-qubit reduced density matrices. This statement generalizes in a straightforward way to any number of qubits.

Interestingly, the drop of the dimension of $M_\alpha$ on the boundary of the Kirwan polytope has the following counterpart in [9]. The probability density $f(\alpha)$ of the eigenvalue distribution of the reduced density matrices of a randomly chosen pure state of three qubits vanishes precisely on the boundary of the Kirwan polytope. Since it is well known that $f(\alpha) = \text{vol} M_\alpha$ for regular points of the momentum map [14, 38], it is reasonable to wonder whether an analogous statement could hold more generally. (The main result of [37] suggests that this might in fact be true.)
We also noticed that the polytope $\Psi(X_W)$ associated with the $W$ SLOCC class is of the same dimension as the polytope $\Psi(X_{GHZ})$ corresponding to the GHZ SLOCC class, whereas for bi-separable and separable states the Kirwan polytope is of strictly lower dimension. On the other hand, it is known that for three qubits only the $W$ and GHZ SLOCC classes represent genuinely entangled states. This intriguing relationship suggests that by looking at the polytopes corresponding to different SLOCC classes one can decide to what extent states from this class are entangled. We believe that this is not a coincidence and conjecture that similar phenomena should be present for $N$-qubit systems, where $N > 3$.

The symplectic methods of this paper can also be applied and extended to other problems of quantum entanglement theory. In [44], two of the authors have analyzed from the topological perspective all SLOCC classes of pure states for both distinguishable and indistinguishable particles using geometric invariant theory and momentum map geometry, resulting in a division of all SLOCC classes into physically meaningful groups of families. Based on the above results, they presented in [45] a general algorithm, working for arbitrary systems of distinguishable and indistinguishable particles, for finding the above-mentioned groups of families. The algorithm provides a certain, physically meaningful, parametrization of SLOCC classes by critical sets of the so-called total variance function of a state. Independently, in [47] one of the authors has in collaboration with others combined symplectic geometry and geometric invariant theory in a novel approach to the study of multiparticle entanglement, resulting in a systematic way of classifying multiparticle entanglement that can be witnessed efficiently and robustly in experiments.

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