Some new results on inner product quasilinear spaces

Hacer Bozkurt¹* and Yılmaz Yılmaz²

Abstract: In this article, we research on the properties of the floor of an element taken from an inner product quasilinear space. We prove some theorems related to this new concept. Further, we try to explore some new results in quasilinear functional analysis. Also, some examples have been given which provide an important information about the properties of floor of an inner product quasilinear space.

Subjects: Advanced Mathematics; Analysis-Mathematics; Functional Analysis; Mathematics & Statistics; Pure Mathematics; Science

Keywords: quasilinear space; inner product quasilinear space; Hilbert quasilinear space; orthogonality; orthonormality

AMS subject classifications: 46C05; 46C07; 46C15; 46C50; 97H50

1. Introduction
Aseev (1986) introduced the theory of quasilinear space (briefly, QLSs) which is generalization of classical linear spaces. He used the partial order relation when he defined the quasilinear spaces and so he can give consistent counterparts of results in linear spaces. Further, he also described the convergence of sequences and norm in quasilinear space. This work has inspired a lot of authors to introduce new results on multivalued mappings, fuzzy quasilinear operators and set-valued analysis (Lakshmikantham, Gnana Bhaskar, & Vasundhara Devi, 2006; Rojas-Medar, Jiménez-Gamerob, Chalco-Canoa, & Viera-Brandão, 2005).

ABOUT THE AUTHORS
Hacer Bozkurt received MSc from Sakarya University, and is currently a PhD scholar at İnönü University. Her research interests are functional analysis, nonlinear functional analysis and interval analysis.

Yılmaz Yılmaz received MSc and PhD degrees in İnönü University, Malatya, Turkey. Currently he is a professor at İnönü University, Malatya, Turkey. His research interests are Functional analysis, sequence spaces, nonlinear functional analysis, Bifurcation theory.

PUBLIC INTEREST STATEMENT
The theory of quasilinear spaces was introduced by Aseev (1986). Aseev used the partial order relation when he defined quasilinear spaces and so he can give consistent counterparts of results in linear spaces. As known, the theory of inner product space and Hilbert spaces play a fundamental role in functional analysis and its applications. We know that any inner product space is a normed space and any normed space is a particular class of normed quasilinear space. Hence, this relation and Aseev’s work motivated us to examine quasilinear counterpart of inner product space in classical analysis. Thus, we introduce the concept of inner product quasilinear space. In this paper, we give some results related to floors of inner product quasilinear spaces. Also, some examples have been given which provide an important contribution to understand the structure of inner product quasilinear spaces.
We see from the definition of quasilinear space which given in Aseev (1986), the inverse of some elements of in quasilinear spaces may not be available. Yılmaz, Çakan, and Aytekin (2012), these elements are called as singular elements of quasilinear space. At the same time the others which have an inverse are referred to as regular elements. Then, in Çakan (2016), she noticed that the base of each singular elements of a combination of regular elements of the quasilinear space. Therefore, she defined the concept of the floor of an element in quasilinear space in Çakan (2016) which is very convenient for some analysis of quasilinear spaces. This work has motivated us to introduce some results about the floors of inner product quasilinear spaces, briefly, IPQLS.

In this paper, motivated by the work of Assev (1986) and Çakan (2016), we research some properties of floors of inner product quasilinear spaces and prove some theorems related to floor of a subset of an inner product quasilinear space. Further, we try to extend the results in quasilinear functional analysis. Our consequences gives us some information about the properties of floor of an inner product quasilinear space.

Let us give some notation and preliminary results given by Aseev (1986).

**Definition 1.1** A set $X$ is called a quasilinear space (QLS, for short), if a partial order relation “$\leq$”, an algebraic sum operation, and an operation of multiplication by real numbers are defined in it in such way that the following conditions hold for any elements $x, y, z, v \in X$ and any real numbers $\alpha, \beta \in \mathbb{R}$:

1. $x \leq x$;
2. $x \leq z$ if $x \leq y$ and $y \leq z$,
3. $x = y$ if $x \leq y$ and $y \leq x$,
4. $x + y = y + x$,
5. $x + (y + z) = (x + y) + z$,
6. there exists an element $\theta \in X$ such that $x + \theta = x$,
7. $\alpha \cdot (\beta \cdot x) = (\alpha \cdot \beta) \cdot x$,
8. $\alpha \cdot (x + y) = \alpha \cdot x + \alpha \cdot y$,
9. $1 \cdot x = x$,
10. $0 \cdot x = \theta$,
11. $(\alpha + \beta) \cdot x \leq \alpha \cdot x + \beta \cdot x$,
12. $x + z \leq y + v$ if $x \leq y$ and $z \leq v$,
13. $\alpha \cdot x \leq \alpha \cdot y$ if $x \leq y$.

A linear space is a quasilinear space with the partial order relation “$=$”. The most popular example which is not a linear space is the set of all closed intervals of real numbers with the inclusion relation “$\subseteq$”, algebraic sum operation

$$A + B = \{a + b: a \in A, \ b \in B\}$$

and the real scalar multiplication
\( \lambda \cdot A = \{ \lambda \cdot a : a \in A \} \).

We denote this set by \( \Omega_\lambda(A) \). Another one is \( \Omega(A) \), the set of all compact subsets of real numbers. By a slight modification of algebraic sum operation (with closure) such as

\[
A + B = \{ a + b : a \in A, \quad b \in B \}
\]

and by the same real scalar multiplication defined above and by the inclusion relation we get the nonlinear QLS, \( \Omega_\lambda(E) \) and \( \Omega(E) \), the space of all nonempty closed bounded and convex closed bounded subsets of some normed linear space \( E \), respectively.

**Lemma 1.1** Suppose that any element \( x \) in a QLS \( X \) has an inverse element \( x' \in X \). Then the partial order in \( X \) is determined by equality, the distributivity conditions hold, and consequently, \( X \) is a linear space (Aseev, 1986).

Suppose that \( X \) is a QLS and \( Y \subseteq X \). Then \( Y \) is called a subspace of \( X \) whenever \( Y \) is a QLS with the same partial order and the restriction to \( Y \) of the operations on \( X \). One can easily prove the following theorem using the condition of to be a QLS. It is quite similar to its linear space analogue (Yılmaz et al., 2012).

**Theorem 1.1** \( Y \) is a subspace of a QLS \( X \) if and only if \( \alpha \cdot x + \beta \cdot y \in Y \) for every \( x, y \in Y \) and \( \alpha, \beta \in \mathbb{R} \) (Yılmaz et al., 2012).

Let \( X \) be a QLS. An \( x \in X \) is said to be symmetric if \((-1) \cdot x = -x = x \), and \( X_d \) denotes the set of all such elements. \( \theta \) denotes the zero’s, additive unit of \( X \) and it is minimal, i.e. \( x = \theta \) if \( x \preceq \theta \). An element \( x \) is called inverse of \( x \) if \( x + x' = \theta \). The inverse is unique whenever it exists and \( x' = -x \) in this case. Sometimes \( x' \) may not be exist but \(-x \) is always meaningful in QLSs. An element \( x \) possessing an inverse is called regular, otherwise is called singular. For a singular element \( x \) we should note that \( x + x' \neq 0 \). Now, \( X_r \) and \( X_s \) stand for the sets of all regular and singular elements in \( X \), respectively. Further, \( X_r \), \( X_d \) and \( X \cup \{ \theta \} \) are subspaces of \( X \) and they are called regular, symmetric and singular subspaces of \( X \), respectively (Yılmaz et al., 2012).

**Proposition 1.1** In a quasilinear space \( X \) every regular element is minimal (Yılmaz et al., 2012).

**Definition 1.2** Let \( X \) be a QLS. A function \( \| \cdot \|_X : X \longrightarrow \mathbb{R} \) is called a norm if the following conditions hold (Aseev, 1986):

\[
\begin{align*}
&\text{(14)} \quad \| x \|_X > 0 \text{ if } x \neq 0, \\
&\text{(15)} \quad \| x + y \|_X \leq \| x \|_X + \| y \|_X, \\
&\text{(16)} \quad \| \alpha \cdot x \|_X = |\alpha| \| x \|_X, \\
&\text{(17)} \quad \text{if } x \preceq y, \text{ then } \| x \|_X \leq \| y \|_X, \\
&\text{(18)} \quad \text{if for any } \varepsilon > 0 \text{ there exists an element } x_\varepsilon \in X \text{ such that } x_\varepsilon \leq y + x, \text{ and } \| x_\varepsilon \|_X \leq \varepsilon \text{ then } x \preceq y. 
\end{align*}
\]

A quasilinear space \( X \) with a norm defined on it is called normed quasilinear space (NQLS, for short). It follows from Lemma 1.1 that if any \( x \in X \) has an inverse element \( x' \in X \), then the concept of NQLS coincides with the concept of a real normed linear space.

Let \( X \) be a NQLS. Hausdorff or norm metric on \( X \) is defined by the equality

\[
h_\varepsilon(x, y) = \inf \{ r \geq 0 : x \preceq y + \varepsilon, y \preceq x + \varepsilon, \| \varepsilon \| \leq r \}.
\]
Since \( x \leq y + (x - y) \) and \( y \leq x + (y - x) \), the quantity \( h_X(x, y) \) is well-defined for any elements \( x, y \in X \), and
\[
h_X(x, y) \leq \|x - y\|_X. \tag{1}
\]

It is not hard to see that this function satisfies all of the metric axioms.

**Lemma 1.2** The operations of algebraic sum and multiplication by real numbers are continuous with respect to the Hausdorff metric. The norm is continuous function respect to the Hausdorff metric (Aseev, 1986).

**Example 1.1** Let \( E \) be a Banach space. A norm on \( \Omega(E) \) is defined by
\[
\|A\|_{\Omega(E)} = \sup_{a \in A} \|a\|_E.
\]
Then \( \Omega(E) \) and \( \Omega_c(E) \) are normed quasilinear spaces. In this case, the Hausdorff metric is defined as usual:
\[
h_{\Omega, E}(A, B) = \inf \{ r \geq 0 : A \subset B + S_r(\theta), B \subset A + S_r(\theta) \},
\]
where \( S_r(\theta) \) denotes a closed ball of radius \( r \) about \( \theta \in X \) (Aseev, 1986).

**Definition 1.3** Let \( X \) be a QLS, \( M \subseteq X \) and \( x \in M \). The set of
\[
F^M_x = \{ z \in M : z \leq x \}
\]
is called floor in \( M \) of \( x \). In the case of \( M = X \) it is called only floor of \( x \) and written briefly \( F_x \) instead of \( F^X_x \) (Çakan, 2016).

Floor of an element \( x \) in linear spaces is \( \{ x \} \). Therefore, it is nothing to discuss the notion of floor of an element in a linear space.

**Definition 1.4** Let \( X \) be a QLS and \( M \subseteq X \). Then the union set
\[
\bigcup_{x \in M} F^M_x
\]
is called floor of \( M \) and is denoted by \( F^M \). In the case of \( M = X \), \( F^X \) is called floor of the qls \( X \).

On the other hand, the set
\[
F^X_M = \bigcup_{x \in M} F^X_x
\]
is called floor in \( X \) of \( M \) and is denoted by \( F^X_M \) (Çakan, 2016).

**Definition 1.5** Let \( X \) be a quasilinear space. \( X \) is called solid-floored quasilinear space whenever
\[
y = \sup \{ x \in X : x \leq y \}
\]
for every \( y \in X \). Otherwise, \( X \) is called nonsolid-floored quasilinear space (Çakan, 2016).

**Example 1.2** \( \Omega(\mathbb{R}) \) and \( \Omega_c(\mathbb{R}) \) are solid-floored quasilinear space. But singular subspace of \( \Omega_c(\mathbb{R}) \) is a nonsolid-floored quasilinear space.
Definition 1.6 Let $X$ be a QLS. Consolidation of floor of $X$ is the smallest solid-floored QLS $\tilde{X}$ containing $X$, that is, if there exists another solid-floored QLS $Y$ containing $X$ then $\tilde{X} \subseteq Y$.

Clearly, $\tilde{X} = X$ for some solid-floored QLS $X$. Further, $\Omega_c(\mathbb{R}^n) = \Omega_c(\mathbb{R}^n)$. For a QLS $X$, the set

$$ F_y^\circ = \{ z \in (\tilde{X}), z \leq y \}. $$

is the floor of $X$ in $\tilde{X}$.

Let us give an extended definition of inner product. This definition and some prerequisites are given by Y. Yılmaz. We can see following inner product as (set-valued) inner product on QLSs.

Definition 1.7 Let $X$ be a quasilinear space. A mapping $\langle \cdot, \cdot \rangle : X \times X \to \Omega(\mathbb{R})$ is called an inner product on $X$ if for any $x, y, z \in X$ and $\lambda \in \mathbb{R}$ the following conditions are satisfied:

19) if $x, y \in X$, then $\langle x, y \rangle \in \Omega_c(\mathbb{R})$, $\equiv \mathbb{R}$,

20) $\langle x + y, z \rangle \subseteq \langle x, z \rangle + \langle y, z \rangle$,

21) $\langle \lambda \cdot x, y \rangle = \lambda \cdot \langle x, y \rangle$,

22) $\langle x, y \rangle = \langle y, x \rangle$,

23) $\langle x, x \rangle \geq 0$ for $x \in X$, and $\langle x, x \rangle = 0 \Leftrightarrow x = 0$,

24) $\|\langle x, y \rangle\|_{\Omega_\mathbb{R}} = \sup \{ \| \langle a, b \rangle \|_{\Omega_\mathbb{R}} : a \in F_y^\circ, b \in F_y^\circ \}$,

25) if $x \leq y$ and $u \leq v$ then $\langle x, u \rangle \subseteq \langle y, v \rangle$,

26) if for any $\varepsilon > 0$ there exists an element $x_\varepsilon \in X$ such that $x \leq y + x_\varepsilon$ and $\langle x_\varepsilon, x_\varepsilon \rangle \subseteq S_\varepsilon(\theta)$ then $x \leq y$.

A quasilinear space with an inner product is called an inner product quasilinear space, briefly, IPQLS.

Example 1.3 One can see easily $\Omega_c(\mathbb{R})$, the space of closed real intervals, is an IPQLS with inner product defined by

$\langle A, B \rangle = \{ ab : a \in A, b \in B \}$. 

Every IPQLS $X$ is a normed QLS with the norm defined by

$$ \|x\| = \sqrt{\|\langle x, x \rangle\|_{\Omega_\mathbb{R}}} $$

for every $x \in X$. This norm is called inner product norm. Classical norm of $\Omega_c(\mathbb{R})$ (see Aseev, 1986) is generated by the above inner product.

Proposition 1.2 $x_n \to x$ and $y_n \to y$ in an IPQLS then $\langle x_n, y_n \rangle \to \langle x, y \rangle$.

An IPQLS is called Hilbert QLS, if it is complete according to the Inner product (norm) metric. $\Omega_c(\mathbb{R})$ is a Hilbert QLS.
Definition 1.8 (Orthogonality) An element \(x\) of an IPQLS \(X\) is said to be orthogonal to an element \(y \in X\) if

\[\|\langle x, y \rangle\|_{\Omega(\mathbb{R})} = 0.\]

We also say that \(x\) and \(y\) are orthogonal and we write \(x \perp y\). Similarly, for subsets \(m, n \subseteq X\) we write \(x \perp m\) if \(x \perp z\) for all \(z \in m\) and \(m \perp n\) if \(a \perp b\) for all \(a \in m\) and \(b \in n\).

An orthonormal set \(M \subset X\) is an orthogonal set in \(X\) whose elements have norm 1, that is, for all \(x, y \in M\)

\[\|\langle x, y \rangle\|_{\Omega(\mathbb{R})} = \begin{cases} 0, & x \neq y \\ 1, & x = y \end{cases}\]

Definition 1.9 Let \(A\) be a nonempty subset of an inner product quasilinear space \(X\). An element \(x \in X\) is said to be orthogonal to \(A\), denoted by \(x \perp A\), if \(\|\langle x, y \rangle\|_{\Omega(\mathbb{R})} = 0\) for every \(y \in A\). The set of all elements of \(X\) orthogonal to \(A\), denoted by \(A^\perp\), is called the orthogonal complement of \(A\) and is indicated by

\[A^\perp = \{ x \in X : \|\langle x, y \rangle\|_{\Omega(\mathbb{R})} = 0, \ y \in A \}.\]

For any subset \(A\) of an IPQLS \(X\), \(A^\perp\) is a closed subspace of \(X\).

2. Main results

In this section, we try to explore some properties of floor of an element in an inner product quasilinear space. We note that the concept of floor is unneeded in linear spaces. Because, the floor of a linear space is equal to itself.

In general, \((\lambda + \mu) \cdot A = \lambda \cdot A + \mu \cdot A\) equality is not satisfy in a quasilinear space for every \(\lambda, \mu \in \mathbb{R}\). For example; Let \(A = [-2, 1] \subseteq \Omega_c(\mathbb{R})\) and \(\lambda = 1, \mu = -1\), we have

\[(1 + (-1)) \cdot [-2, 1] = 0 \cdot [-2, 1] = \{0\}\]

But

\[1 \cdot [-2, 1] + (-1) \cdot [-2, 1] = [-2, 1] + [-1, 2] = [-3, 3].\]

Here, we see that \(\{0\} \neq [-3, 3]\). But, we can say that \((\lambda + \mu) \cdot A \subset \lambda \cdot A + \mu \cdot A\) inequality is provided for any \(A \in \Omega_c(\mathbb{R})\).

Definition 2.1 Let \(X\) be a quasilinear space. \(X\) is called homogenized quasilinear space if for every \(x \in X\) and \(a, \beta \geq 0\) the following condition is satisfied:

\[(a + \beta) \cdot x = a \cdot x + \beta \cdot x.\]

Clearly, every linear space is a homogenized quasilinear space. But the reverse is not true.

Let \(X\) be a normed linear space. Then \(\Omega_c(X)\) is a homogenized quasilinear space but \(\Omega(X)\) is non-homogenized quasilinear space.

Proposition 2.1 Let \(X\) be a homogenized IPQLS and \(x \in X\). Then \(F_x\) is convex subset of \(X\).

Proof Let \(X\) be a homogenized IPQLS. From Definition 1.3, we get

\[F_x = \{ a \in X : a \leq x \}\]
for a $x \in X$. So we have

$$a \leq x \text{ and } b \leq x$$

for every $a, b \in F_x$. From the condition (13), we get

$$\lambda \cdot a \leq \lambda \cdot x \text{ and } (1 - \lambda) \cdot b \leq (1 - \lambda) \cdot x$$

for all $0 \leq \lambda \leq 1$. Hence,

$$\lambda \cdot a + (1 - \lambda) \cdot b \leq \lambda \cdot x + (1 - \lambda) \cdot x.$$

Since, $X$ is a homogenized IPQLS,

$$\lambda \cdot x + (1 - \lambda) \cdot x = (\lambda + 1 - \lambda) \cdot x = x$$

for every $0 \leq \lambda \leq 1$. So, we obtain

$$\lambda \cdot a + (1 - \lambda) \cdot b \leq x.$$

Hence $\lambda \cdot a + (1 - \lambda) \cdot b \in F_x$. This completes the proof. \hfill \Box

Remark 2.1  
Floor of an element of an IPQLS $X$ is convex if and only if this IPQLS $X$ is homogenized.

If $X$ is not homogenized inner product quasilinear space in the above proposition, then $F_x$ is not convex since $(a + \beta) \cdot x \neq a \cdot x + \beta \cdot x$.

Proposition 2.2  
Let $X$ be an IPQLS and $A, B \subseteq X$. Then, we have

(a) $\{0\} \in F_A^\perp$,

(b) $F_{\{0\}} = \{0\}$,

(c) if $A \subseteq B$, then we get $F_A \subseteq F_B$ and $F_A^\perp \subseteq F_B^\perp$.

The proof of proposition is similar to the classical linear counterpart.

Theorem 2.1  
If $M$ is a convex subspace of Hilbert QLS $X$, then $F_M$ is complete and convex subspace of Hilbert QLS $X$.

Proof  
Let $a, b \in F_M$. Then, in view of Definition 1.3, there exist $x \in M$ such that $a \leq x$ and there exist $y \in M$ such that $b \leq y$. From (12) and (13), we have

$$a \cdot a + (1 - a) \cdot b \leq a \cdot x + (1 - a) \cdot y.$$  

Since $M$ is convex, we find $z \in M$ such that

$$a \cdot a + (1 - a) \cdot b \leq z.$$  

This proves that $a \cdot a + (1 - a) \cdot b \in F_M$.

Let $(a_n) \in F_M$ and $a_n \rightarrow a$ for some $a \in X$. Then for any $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that the following condition holds for any $n > N$:

$$a_n \leq a + a_{1n} \varepsilon, \quad a \leq a_{n} + a_{2n} \varepsilon, \quad \|a_{n}\| \leq \frac{\varepsilon}{2}. \quad (2)$$
On the other hand, if \((a_n) \in F_M\) then there exist \(a(x_n) \in M\) such that \(a_n \leq x_n\) for every \(n \in \mathbb{N}\). From here and above inequality, we get
\[
a \leq x_n + a_n, \quad \|a_n^2\| \leq \epsilon
\]
for every \(n \in \mathbb{N}\). By the (18), we have \(a \leq x_n\) for every \(n \in \mathbb{N}\). Now, we show that \(a\) is a regular element of \(X\). By Lemma 1.2, we know \(-a_n \rightarrow -a\) when \(a_n \rightarrow a\). So, for any \(\epsilon > 0\) there exists an \(n' \in \mathbb{N}\) such that
\[
-a_n \leq -a + b_n^\ast, \quad -a \leq -a_n + b_n^\ast, \quad \|b_n^\ast\| \leq \frac{\epsilon}{2}.
\]
Because of \((a_n) \in F_M\), \(a_n - a_n = 0\). By Lemma 1.2, (2) and (3), we get
\[
a_n - a_n \leq a - a + a_n^\ast + b_n^\ast, \quad a - a \leq a_n - a_n + a_n^\ast + b_n^\ast, \quad \|a_n^\ast + b_n^\ast\| \leq \epsilon
\]
and
\[
0 \leq a - a + a_n^\ast + b_n^\ast, \quad a - a \leq 0 + a_n^\ast + b_n^\ast, \quad \|a_n^\ast + b_n^\ast\| \leq \epsilon.
\]
From here, we have \(0 = a - a\) since \(X\) is a Hilbert QLS. This shows that \(a\) is a regular element of \(X\). Thus, since \((x_n) \in M\) for all \(n \in \mathbb{N}\), we obtain \(a \in F_M\). This proves that the set \(F_M\) is complete.

**Corollary 2.1** Let \(X\) be a Hilbert QLS and \(M\) is a convex subspace of \(X\). Then \(F_M\) is a complete subspace of \(X\) even if \(M\) is not complete.

**Proposition 2.3** If \(X\) is an IPQLS and \(x \in X\), then \(F_x\) is a closed.

**Proof** Let \((b_n) \in F_{F_M}\) and \(b_n \rightarrow b\) for some \(b \in X\). Then for all \(\epsilon > 0\) there exists an \(n_0 \in \mathbb{N}\) such that the following condition holds for any \(n > n_0\):
\[
b_n \leq b + c_n^\ast, \quad b \leq b_n + c_n^\ast, \quad \|c_n\| \leq \epsilon.
\]
Since \((b_n) \in F_{F_M}\), \(b_n \leq x\) for every \(n \in \mathbb{N}\). So, we have
\[
b \leq x
\]
since \(b \leq b_n + c_n^\ast \|c_n^\ast\|^2 = \|c_n^\ast, c_n^\ast\|_{\langle M, X \rangle} \leq \epsilon\). Also, we can show that \(b\) is regular element of \(X\) similar to the above proof. By Lemma 1.2, we know \(-b_n \rightarrow -b\) when \(b_n \rightarrow b\) and \(-b_n \rightarrow -b\). So, for any \(\epsilon > 0\) there exists an \(n_0 \in \mathbb{N}\) such that the following condition holds for any \(n > n_0\):
\[
b_n - b_n \leq -b + c_n^\ast, \quad b - b \leq b_n - b_n + c_n^\ast, \quad \|c_n\| \leq \epsilon.
\]
From here, we have \(0 = b - b\) since \(X\) is an IPQLS. This shows that \(b\) is a regular element of \(X\).

**Lemma 2.1** Let \(X\) be an IPQLS. A floor of any element of IPQLS \(X\) may not subspace of \(X\). But, the orthogonal complement of floor of any element of IPQLS \(X\) is subspace of \(X\).

**Proof** Let \(a, b \in F_x^\ast\). Definition of floor of an element \(a \leq x\) and \(b \leq x\) for \(a, x, b \in X\). Since \(X\) is an IPQLS, we have
\[
a \cdot a + b \cdot b \leq a \cdot x + b \cdot x
\]
for every \(a, b \in \mathbb{R}\). From here, we obtain \(a \cdot a + b \cdot b \notin F_x\) since \(a \cdot x + b \cdot x\) may not equal to \(x\) for all \(a, b \in \mathbb{R}\). So, \(F_x\) is not a subspace of \(X\). Now, let \(z \in F_x\) and \(c, d \in F_x^\ast\) for \(a, x \in X\). From (15), (20) and (21)
we have
\[
\| (z, \alpha \cdot c + \beta \cdot d) \|_{\Omega_c(\mathbb{R})} \leq \| (z, \alpha \cdot c) \|_{\Omega_c(\mathbb{R})} + \| (z, \beta \cdot d) \|_{\Omega_c(\mathbb{R})} = \alpha \| (z, c) \|_{\Omega_c(\mathbb{R})} + \beta \| (z, d) \|_{\Omega_c(\mathbb{R})} = 0
\]
So, we get \( \alpha \cdot c + \beta \cdot d \in F_\ast \) for all \( \alpha, \beta \in \mathbb{R} \).

**Remark 2.2** The floor of an subset of \( (\Omega_c(\mathbb{R}))_d \) is equal to the largest element according to the order relation of the \( \Omega_c(\mathbb{R}) \).

**Example 2.1** Let \( RZ = \{ [n, 0] : n \in \mathbb{R}^- \} \), the right-zero subset of \( \Omega_c(\mathbb{R}) \). By the definition of floor, we get

\[
F_{\text{RZ}} = \bigcup_{n \in \mathbb{R}^-} F_{[n, 0]_\mathbb{R}} = \bigcup_{n \in \mathbb{R}^-} \{ a \in (\Omega_c(\mathbb{R})), a \subseteq RZ \} = \{ (a) : a \in \mathbb{R}^- \} \bigcup \{ 0 \}.
\]

Similarly, if we say \( LZ = \{ [0, n] : n \in \mathbb{R}^+ \} \), the left-zero subset of \( \Omega_c(\mathbb{R}) \), we find

\[
F_{\text{LZ}} = \bigcup_{n \in \mathbb{R}^+} F_{[0, n]_\mathbb{R}} = \bigcup_{n \in \mathbb{R}^+} \{ a \in (\Omega_c(\mathbb{R})), a \subseteq LZ \} = \{ (a) : a \in \mathbb{R}^+ \} \bigcup \{ 0 \}.
\]

From here, we have

**Theorem 2.2** Suppose that \( X \) is an IPQLS and \( A, B \subseteq X \). If \( A \cup B = X \), then \( F_A \cup F_B = X \).

\[
F_{\text{RZ}} \cup F_{\text{LZ}} = \{ (a) : a \in \mathbb{R}^- \} \bigcup \{ 0 \} \bigcup \{ (a) : a \in \mathbb{R}^+ \} = \{ (c) : c \in \mathbb{R} \}.
\]

**Proof** It is easy to see that \( x \in X \), for every \( x \in F_A \cup F_B \). Let us consider \( x \in X \). From here, we know that \( x \in X \). Since \( A \cup B = X \), \( x \) is an element either \( A \) or \( B \).

If \( x \) is an element of \( A \), \( x \in F_A \) since \( x \in X \).
If \( x \) is an element of \( B \), \( x \in F_B \) since \( x \in X \). This implies \( x \in F_A \cup F_B \).

**Remark 2.3** Although, in an IPQLS \( X, F_A \cup F_B = X \), for all \( A, B \subseteq X \), the combination of \( A \) and \( B \) may not be equal to \( X \).

**Example 2.2** Let us consider the IPQLS \( X = \Omega_c(\mathbb{R}) \) and the subspaces \( A = X_r \) and \( B = X_r \). Clearly, \( A \cup B = X \) and \( F_A \cup F_B = X_r \). If we take \( C = LZ \) and \( D = RZ \) (\( LZ \) are subset of \( \Omega_c(\mathbb{R}) \) which is given in Example 2.1), we get \( F_{\text{RZ}} \cup F_{\text{LZ}} = X_r \). But \( C \cup D = RZ \cup LZ \neq X \).

**Acknowledgements**
The authors would like to record their gratitude to the reviewer for his careful reading and making some useful corrections which improved the presentation of the paper.

**Funding**
The authors received no direct funding for this research.

**Author details**
Hacer Bozkurt\(^1\)
E-mail: hacero.bozkurt@batman.edu.tr

Yılmaz Yılmaz\(^2\)
E-mail: yyilmaz44@gmail.com
\(^1\)Department of Mathematics, Batman University, 72100 Batman, Turkey.
\(^2\)Department of Mathematics, İnönü University, 44280 Malatya, Turkey.

**Citation information**
Cite this article as: Some new results on inner product quasilinear spaces, Hacer Bozkurt & Yılmaz Yılmaz, Cogent Mathematics (2016), 3: 1194801.
References
Aseev, S. M. (1986). Quasilinear operators and their application in the theory of multivalued mappings. Proceedings Steklov Institute Mathematics, 2, 23–52.
Çakan, S. (2016). Some new results related to theory of normed quasilinear spaces. Malatya: University of İnönü.
Lakshmikantham, V., Gnana Bhaskar, T., & Vasundhara Devi, J. (2006). Theory of set differential equations in metric spaces. Cambridge: Cambridge Scientific Publishers.
Rojas-Medar, M. A., Jiménez-Granero, M. D., Chalco-Canoa, Y., & Viera-Brandao, A. J. (2005). Fuzzy quasilinear spaces and applications. Fuzzy Sets and Systems, 152, 173–190.
Yılmaz, Y., Çakan, S., & Aytekin, Ş. (2012). Topological quasilinear spaces. Abstract and Applied Analysis. doi:10.1155/2012/951374