Running Coupling and the $\Lambda$-Parameter from SU(3) Lattice Simulations

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ABSTRACT

We present new results on the static $q\bar{q}$ potential from high statistics simulations on $32^4$ and smaller lattices, using the standard Wilson action at $\beta = 6.0, 6.4, \text{ and } 6.8$ on the Connection Machine CM-2. Within our statistical errors ($\approx 1\%$) we do not observe any finite size effects affecting the potential values, on varying the spatial lattice extent from $0.9\, fm$ up to $3.3\, fm$. We are able to see and quantify the running of the coupling from the Coulomb behaviour of the interquark force. From this we extract the ratio $\sqrt{\sigma}/\Lambda_L$. We demonstrate that scaling violations on the string tension can be considerably reduced by introducing effective coupling schemes, which allow for a safe extrapolation of $\Lambda_L$ to its continuum value. Both methods yield consistent values for $\Lambda$: $\Lambda_{\overline{MS}} = 0.558^{+0.017}_{-0.007} \times \sqrt{\sigma} = 246^{+7}_{-5}\, MeV$. At the highest energy scale attainable to us we find $\alpha(5\, GeV) = 0.150(3)$.

PACS numbers: 11.15.Ha, 12.38.Gc, 12.38.Aw

1Work supported by EC project SC1*-CT91-0642, DFG grant Schi 257/1-4
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1 Introduction

The experimental determination of the running coupling constant of QCD has reached a reasonable degree of accuracy [1] after two decades of research effort. This has stimulated considerable attention to compute this quantity from first principles, by use of lattice methods [2, 3, 4]. The lattice approach to the problem of matching perturbative and nonperturbative aspects of QCD is notoriously difficult because of the requirement of a high energy resolution. Nevertheless, computer experiments in pure SU(2) and SU(3) gauge theory have reached a precision that allows to ask rather detailed questions about the static quark-antiquark potential. The size of the available lattices (48\(^3\times 56\), in SU(2) gauge theory [3]) enables one to decrease the lattice spacing \(a\) into a regime where one can make contact to predictions of continuum perturbation theory. This has been done for the case of SU(2) by a study of the Coulomb behaviour of the interquark force in ref. [3]. In the case of SU(3), a lattice spacing of \(a^{-1} = 3.6\, GeV\) was achieved so far [6] on a 32\(^4\) lattice at \(\beta = 6.4\). This resolution is about the threshold for running coupling effects to become visible.

In this paper we want to present a detailed investigation of the running coupling in SU(3) gauge theory, by further reducing the lattice spacing to \(a^{-1} = 6.0\, GeV\). Within our analysis of the small distance regime, we will use a parameterization incorporating lattice effects. Being limited to lattice sizes up to 32\(^4\), we have to make sure that our results are not spoiled by finite size effects. For this reason we have worked on a variety of lattices, at each value of \(\beta\).

Once the running coupling has been extracted, we will be able to compare to perturbative predictions and estimate a value for the corresponding \(\Lambda_L\) parameter. We will see that this value is consistent with \(\Lambda_L\), as obtained from the string tension (by the use of the two-loop \(\beta\)-function [7]), after an extrapolation to \(a = 0\). In order to substantiate this result, we will improve on scaling violations (as expressed in the strong \(\beta\)-dependence of \(\Lambda_L\)) by replacing the bare coupling with suitable “effective” couplings [8, 9, 10, 11], measured on the lattice from the average plaquette. In this case, we will find nearly asymptotic scaling for \(\beta > 6.0\). The extrapolation to the continuum yields an estimate for \(\Lambda_L\) which is consistent within smaller errors with the value obtained from the running coupling.
2 Methods

2.1 Sampling

In order to maintain an appropriate stochastic movement of the gauge system through phase space with increasing $\beta$, we have combined one Cabibbo-Marinari pseudo-heatbath-sweep [12] over the three diagonal $SU(2)$ subgroups with four(nine) successive overrelaxation sweeps [13] for $\beta = 6.4(6.8)$. We reach an acceptance rate of 99.5% for an overrelaxation link update. For the heatbath we use the algorithm proposed by Kennedy and Pendleton [14] which has a high acceptance rate and can thus be efficiently implemented on a SIMD machine. We can afford iterating the algorithm until all link variables are changed. On our local 8K CM-2 we need 9.2$\mu$sec for an overrelaxation link update and 11.5$\mu$sec for a single Cabibbo-Marinari link update. This performance was achieved after rewriting the $SU(3)$ matrix multiply routines in assembler language. Measurements were started after 2000 – 10000 thermalization sweeps.

2.2 Smoothing Operators

In lattice gauge theory physical quantities of interest like masses, potential values, and matrix elements are related to asymptotic properties of exponentially decreasing correlation functions in Euclidean time, and therefore prone to be drowned in noise. So one is forced to improve operators in order to reach the desired asymptotic behaviour for the small $T$ region. We will shortly describe our particular improvement technique [1].

We start from the relation between Wilson loops, $W(R,T)$, and the (ground state) potential $V(R)$

$$W(R,T) = C(R) \exp \{-TV(R)\} \quad + \quad \text{excited state contributions.} \quad (1)$$

Our aim is to enhance — for each value of $R$ — the corresponding ground state overlap $C(R)$. Since the ground state wave function is expected to be smooth on an ultraviolet scale we concentrate on reducing noise by applying a local smoothing procedure on the spatial links: consider a spatial link variable $U_i(n)$, and the sum of the four spatial staples $\Pi_i(n)$ connected to it:

$$\Pi_i(n) = \sum_{j=\pm1,...,3} U_j(n) U_i(n+j) U_j^\dagger(n+i). \quad (2)$$

We apply a gauge covariant, iterative smoothing algorithm which replaces (in the same even/odd ordering as the Metropolis update) $U_i(n)$ by $U'_i(n)$ minimizing the local spatial action $S_i(n) = -\text{Re Tr}\{U_i(n)\Pi_i^\dagger(n)\}$, which is qualitatively a measure for the roughness of the gauge field. This is very similar to lattice cooling techniques already invented by previous authors [15, 16] except that we are cooling only within time-slices and thus not affecting the transfer matrix. Alternatively, this algorithm may be interpreted as substituting $U_i(n)$ by $\mathcal{P}(\Pi_i(n))$ where $\mathcal{P}$ denotes the projection operator onto the nearest
SU(3) matrix. In this sense it is a variant of the APE recursive blocking scheme \cite{17} with the coefficient of the straight link set to zero, but with even/odd updating. The latter feature renders the algorithm less memory consuming and seems to improve convergence. Contributions from excited states become increasingly suppressed, as we repeat this procedure. After 30(45) such smoothing steps at $\beta = 6.4(6.8)$ we reach values for the overlap $C(R)$ of 95(80)% for small (large) spatial separations $R$.

2.3 Extraction of Potential Values

For the extraction of the potential from the Wilson loop data we proceed essentially as described in ref. \cite{6}, with a slight modification that helps to carry out a straightforward error analysis. Instead of fitting the Wilson loops to the dependence

$$W_{\vec{R}}(T; C(\vec{R}), V(\vec{R})) := C(\vec{R}) \exp \left\{ -V(\vec{R}) T \right\} \quad (3)$$

for $T \geq T_{\text{min}}$ with some reasonable cutoff $T_{\text{min}}$ we take the local mass

$$V_{T_{\text{min}}}^{\vec{R}} = \ln \left\{ \frac{W_{\vec{R}, T_{\text{min}}}}{W_{\vec{R}, T_{\text{min}} + 1}} \right\} \quad (4)$$

as an estimator for the potential $V(\vec{R})$. By using this explicit formula for the calculation of $V(\vec{R})$, we are able to propagate the covariance matrix between Wilson loops to a covariance matrix for the potential values. This allows one to separate the determination of potential parameters from the measurement of the potential itself, helping to decrease the degrees of freedom and promoting stability within the fitting procedure. Note that the value of $V(\vec{R}) = V_{T_{\text{min}}}^{\vec{R}}$ does not differ appreciably from the result of a fit to eq. (3) because the latter is anyhow dominated by the lowest two $T$ data due to their small relative errors.

The optimization of the overlap $C(\vec{R})$ proceeds as described in ref. \cite{6}: The parameters $C(\vec{R})$ and $V(\vec{R})$ are fitted for different $T_{\text{min}}$ to the Wilson loop data separately for each smoothing step (and $\vec{R}$) according to eq. (3) by minimizing

$$\chi^2_{\vec{R}}(C(\vec{R}), V(\vec{R})) = \sum_{T_1, T_2} \left( W_{\vec{R}, T_1} - W_{\vec{R}}(T_1; C(\vec{R}), V(\vec{R})) \right) \left( C_{\vec{R}\vec{R}} \right)_{T_1 T_2}^{-1} \times \left( W_{\vec{R}, T_2} - W_{\vec{R}}(T_2; C(\vec{R}), V(\vec{R})) \right). \quad (5)$$

$C_{\vec{R}\vec{R}}$ denotes the covariance matrix which is estimated to be

$$C_{\vec{R}\vec{R}} = \frac{1}{N(N-1)} \sum_{i=1}^{N} \left( W_i(\vec{R}_1, T_1) - W(\vec{R}_1, T_1) \right) \left( W_i(\vec{R}_2, T_2) - W(\vec{R}_2, T_2) \right). \quad (6)$$

We have divided the timeseries of Wilson loops into $N$ successive subsets of given length $n$. $W_i(\vec{R}, T)$ stands for the average of the respective Wilson loop over the $i$th subset. $n$ should be chosen such that $\tau \ll n \ll N$, in order to cope with the autocorrelation time
\[ \tau. \] Afterwards for each value of \( R \) the smoothing step with highest ground state overlap \( C(R) \) is selected from the fits with reasonable \( \chi^2 \).

In a second step stability of local masses \( V_T(R) \) (eq. (\ref{eq:V_T})) against variation of \( T \) is checked, and \( T_{min}(R) \) is determined as the \( T \) value (plus one) from which onwards stability within errors is observed. For large \( R \) values we find \( T_{min} = 4 \). For simplicity we chose the same value for small \( R \).

As promised, we are now able to propagate the covariance matrix between different Wilson loops \( C_{T_1 T_2}^{R_1 R_2} \) to a covariance matrix between the potential values \( C_v^{R_1 R_2} \), by using the quadratic approximation

\[ C_v^{R_1 R_2} = \sum_{T_1, T_2} \frac{\partial V(R_1)}{\partial W(R_1, T_1)} C_{T_1 T_2}^{R_1 R_2} \frac{\partial V(R_2)}{\partial W(R_2, T_2)} \]

\[ = \frac{C_{T_1 T_2}^{R_1 R_2}}{W(R_1, T(R_1)) W(R_2, T(R_2))} + \frac{C_{T_1 T_2}^{R_1 R_2}}{W(R_1, T(R_1) + 1) W(R_2, T(R_2) + 1)} \]

\[ - \frac{C_{T_1 T_2}^{R_1 R_2}}{W(R_1, T(R_1) + 1) W(R_2, T(R_2))} + \frac{C_{T_1 T_2}^{R_1 R_2}}{W(R_1, T(R_1)) W(R_2, T(R_2) + 1)} \]

where \( T(R) \) is used as an abbreviation for \( T_{min}(R) \). With this covariance matrix we are able to fit the potential data to various parameterizations, incorporating all possible correlations between different operators measured on individual configurations as well as correlation effects within the Monte Carlo time series of configurations.

### 2.4 Measurements

The lattice parameters used for the simulations are collected in table \( \text{I} \) which includes quotations of 32\(^4 \) lattices at \( \beta = 6.0 \), and \( \beta = 6.4 \), as well as a 24\(^3 \times 32 \) lattice at \( \beta = 6.4 \) that have been simulated recently \( \text{[3]} \), and are reanalysed in the present investigation. The spatial extent of the lattices at \( \beta = 6.4 \) ranges from \( aL_s = 0.87 \) fm to 1.74 fm. At \( \beta = 6.8 \) lattice volumes of \((0.52 \text{ fm})^3\) and \((1.05 \text{ fm})^3\) have been realised. The resolution \( a^{-1} \) is varied from 1.9 GeV to 6.0 GeV.

Smoothened on- and off-axis Wilson loops were measured every 100 sweeps (every 50 sweeps for \( \beta = 6.0 \)). Up to \( N_{\max} = 30(45) \) smoothing steps were performed at \( \beta = 6.0, 6.4(6.8) \). The following spatial separations were realized: \( \bar{R} = Me_i \) with \( e_i = (1,0,0), (1,1,0), (2,1,0), (1,1,1), (2,1,1), (2,2,1) \). \( M \) was increased up to \( L_s/2 \) for \( i = 1,2,4 \), and up to \( L_s/4 \) for the remaining directions. Altogether this yields 72 different

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\( ^3 \)In order to check the validity of this approximation, we have moreover carried out a bootstrap analysis of our data on the 32\(^4 \) lattices. (This method is also shortly described in the appendix of ref. \( \text{[19]} \).) The resulting errors (and biased values) are almost identical with the results of our approximation, but the bootstrap method alone does not deliver reliable \( \chi^2 \) values (incorporating the correlation effects).

\( ^4 \)Note that we have adapted the physical scales from \( \sqrt{\sigma} = 420 \text{ MeV} \) to \( \sqrt{\sigma} = 440 \text{ MeV} \).
Table 1: The simulated lattices. Physical units correspond to the choice $\sqrt{\sigma} = 440\text{MeV}$ for the string tension. Errors ignore the experimental uncertainty within the value of the string tension.

| $V = L_S^3 \times L_T$ | $\beta = 6.0$ | $\beta = 6.4$ | $\beta = 6.8$ |
|-------------------------|-------------|-------------|-------------|
| $a/\text{fm}$          | 0.101 (2)   | 0.0544 (5)  | 0.0327 (5)  |
| $a^{-1}/\text{GeV}$    | 1.94 (5)    | 3.62 (4)    | 6.02 (10)   |
| $aL_S/\text{fm}$       | 3.25 (8)    | 1.31 (1)    | 1.74 (2)    |
| $(aL_T)^{-1}/\text{MeV}$ | 61 (1)    | 113 (1)    | 226 (2)    |
| Total # of sweeps      | 6100        | 11900       | 22000       |
| Thermalization phase   | 1000        | 2000        | 2100        |
| # of measurements      | 102         | 100         | 200         |

Separations $\vec{R}$ on the $32^3 \times L_T$ lattices. The time separations $T = 1, 2, \ldots, 10$ were used. Thus the total number of operators measured on one configuration ($V = 32^3 \times L_T$) is $72 \times 10 \times N_{\text{max}}$.

The potential values at $\beta = 6.0$, and $\beta = 6.4$ have been listed in our previous publication [6]. For convenience of the reader we collect the corresponding values for $\beta = 6.8$ in the appendix.

3 Results

3.1 $q\bar{q}$-Potential

We connect our investigation to the recent $SU(2)$ analysis by Chris Michael [3], and start from his ansatz:

$$V(\vec{R}) = V_0 + KR - e\left(\frac{1 - l}{R} + l 4\pi G_L(\vec{R})\right) + \frac{f}{R^2}.$$  \hspace{1cm} (8)

The lattice propagator for the one gluon exchange [20]

$$G_L(\vec{R}) = \int_{-\pi}^{\pi} \frac{d^3k}{(2\pi)^3} \frac{\cos(k \vec{R})}{4 \sum_i \sin^2(k_i/2)}.$$  \hspace{1cm} (9)

has been calculated numerically. The parameter $l$ is expected to be in the range $0 \leq l \leq 1$ and controls the violation of rotational symmetry on the lattice (within this ansatz). The term $f/R^2$ mocks deviations from a pure Coulomb behaviour and is expected to be positive to the extent that asymptotic freedom becomes visible in the effective Coulomb term $-(e - f/R)/R$.

A test of the ansatz eq. (8) implies that the “corrected” data $V(R) = V(\vec{R}) + \delta V(\vec{R})$ with

$$\delta V(\vec{R}) = el \left(4\pi G_L(\vec{R}) - 1/R\right)$$ \hspace{1cm} (10)

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Table 2: Fit results. Since the parameter values on the largest lattices are most precise, we refrain from citing results gained on smaller volumes as long as they are compatible with the stated numbers. For the $16^3 \times 64$ lattice at $\beta = 6.8$ this is not the case. Therefore we have listed both the standard fit result, and the parameter values with the string tension constrained to its $32^4$ value.

| Vol.    | $\beta = 6.0$ | $\beta = 6.4$ | $\beta = 6.8$ |
|---------|---------------|---------------|---------------|
| $32^4$  | $32^4$        | $32^4$        | $16^3 \times 64$ |
| $K$     | 0.0513 (25)   | 0.01475 (29)  | 0.00533 (18)  |
| $e$     | 0.275 (28)    | 0.315 (15)    | 0.311 (10)    |
| $V_0$   | 0.636 (10)    | 0.6013 (37)   | 0.5485 (24)   |
| $l$     | 0.64 (12)     | 0.5634 (55)   | 0.725 (87)    |
| $f$     | 0.041 (58)    | 0.075 (18)    | 0.094 (13)    |
| $R_{\text{min}}$ | 2          | $\sqrt{3}$    | $\sqrt{3}$    |
| $\chi^2/N_{DF}$ | 0.816    | 0.953       | 0.937         |

The global situation is depicted for the $32^4$ lattice at $\beta = 6.4$ in figure 1 where the corrected data points are plotted together with the interpolating fit $V(R) = V_0 + KR - e/R + f/R^2$, with fit parameters $V_0, K, e,$ and $f$ as given in table 2. Our potential fits yield $\chi^2/N_{DF} < 1$ as long as the first two data points are excluded. The stability of the string tension result with respect to cuts in $R$ is displayed in figure 2 (for $\beta = 6.4$, and 6.8).

For $\beta \geq 6.4$ the Coulomb coefficients $e$ are definitely different from the value $\pi/12 \approx 0.262$ predicted by the string vibrating picture \[21] for large $q\bar{q}$ separations. The self energy contribution $V_0$ follows the leading order expectation $V_0 \propto 1/\beta$. We emphasize that for all $\beta$ values the parameter $f$ is established to be positive as expected. In fact, this parameter tends to increase with $\beta$, weakening the Coulomb coupling for small distances.

A more sensitive representation of the scatter of the data points around the interpolating fit curve (obtained on the $32^4$ lattice) is shown in figure 3 (for $\beta = 6.4$). Note that the deviations are within a 1% band for the largest volume, once the first two data points are excluded. Decreasing the lattice spatially or in the time direction by a factor of two leaves the data points compatible with the interpolating curve, i.e. the finite size effects (FSE) are below our statistical accuracy. Nevertheless it pays to work on a $32^4$ lattice since the larger possible $q\bar{q}$ separations increase the lever arm needed to fix the long distance part of the potential.

At $\beta = 6.8$ we find indications of FSE by comparing results from the small lattice and the $32^4$ lattice. As the string tension appears not to suffer from these effects, we have fixed its value to that measured on the larger lattice in order to study FSE on the remaining parameters more directly. The largest FSE occurs for the lattice correction $^5$Three for $\beta = 6.0$.  

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[1] Reference to external content.

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[21] Reference to external content.
parameter \( l \). This may be due to the low momentum cutoff that starts to become visible on the scale of a few lattice spacings. By choosing the form of the one gluon exchange (eq. (9)), we have neglected this cutoff in the integral bounds.

We concentrate our interest here on short distance physics where the linear term is not yet dominating the potential. In the case of \( \beta = 6.4 \) the latter happens at \( R \approx 5 \). From figure 3 we conclude that reliable results can be extracted for a lattice as small as \( 16^3 \) for this \( \beta \)-value. In physical units this corresponds to a \( 27^3 \) lattice at \( \beta = 6.8 \). So a volume of \( 32^3 \) (or even smaller) appears to be sufficiently large for our purpose.

A synopsis of data for \( \beta = 6.0, 6.4 \), and 6.8, in physical units, is displayed in figure 4 with logarithmic ordinate ranging from 0.03 fm up to 1.9 fm. The three data sets collapse to a universal potential. The two curves correspond to a linear plus Coulomb parameterization, with the string tension \( \sigma = Ka^{-2} = (440 MeV)^2 \), and the strength of the Coulomb term determined by our fit to the \( \beta = 6.4 \) data \((e = 0.315, \text{full curve})\), and fixed to the L"uscher-value \((e = \pi/12, \text{dashed curve})\), respectively. The plot demonstrates the incompatibility of the data points with a pure Coulomb behaviour for short distances, and the necessity of additional terms like \( f/R^2 \).

### 3.2 Running Coupling

Our lattice analysis for the running coupling \( \alpha_{q\bar{q}}(R) \) closely follows the procedure suggested in ref. [3]. We start from the symmetric discretization in terms of the force \( F \)

\[
\alpha_{q\bar{q}}(R) = -\frac{3}{4} R_1 R_2 F(R) = \frac{3}{4} R_1 R_2 \frac{V(R_1) - V(R_2)}{R_1 - R_2}.
\] (11)

with \( R = (R_1 + R_2)/2 \). We take the corrected potential \( V(R_i) = V(\vec{R}_i) + \delta V(\vec{R}_i) \) with \( \delta V(\vec{R}_i) \) as given in eq. (10). Unlike ref. [3], however, we use all possible combinations \( \vec{R}_1, \vec{R}_2 \) with \( |\vec{R}_1 - \vec{R}_2| < 1.5 \).

The resulting data points are contained in figure 5a. In order to exhibit both the global behaviour, and the perturbative region \((R \to 0)\) we decided to use a logarithmic ordinate (in units of \( \sigma^{-1/2} \)). The latter region is expanded in the inset. We omitted all values with errors \( \Delta \alpha_{q\bar{q}}(R) > \alpha_{q\bar{q}}(R)/3 \) in order not to clutter the graph. In addition to the statistical error of the force \( F(R) \) we allow for a systematic error

\[
\Delta F_{syst}(R) = \left( \left( \frac{\Delta l}{l} \right)^2 + \left( \frac{\Delta e}{e} \right)^2 \right)^{1/2} |\delta F(R)|
\] (12)

with \( \delta F(R) = \frac{\delta V(\vec{R}_1) - \delta V(\vec{R}_2)}{R_1 - R_2} \). \( \Delta F_{syst} \) is typically of the order of 10% of the lattice correction \( \delta F(R) \).

Now we can proceed to analyse our \( \alpha_{q\bar{q}} \)-data in terms of the continuum large momentum expectation for the running coupling:

\[
\alpha_{q\bar{q}}(R) = \frac{1}{4\pi} \left( b_0 \ln (Ra\Lambda R)^{-2} + b_1/b_0 \ln \ln (Ra\Lambda R)^{-2} \right)^{-1},
\] (13)
with

\[ b_0 = \frac{11}{3} \frac{N_C}{16\pi^2}, \quad b_1 = \frac{34}{3} \left( \frac{N_C}{16\pi^2} \right)^2 \]  

being the first two coefficients of the weak coupling expansion of the \( SU(N_C) \) Callan-Symanzik \( \beta \)-function (eq. (23) below). In order to extract \( \Lambda_R \) we base our fits exclusively on data points at \( \beta = 6.8 \) with \( R_1, R_2 \geq \sqrt{3} \) on the r.h.s. of eq. (11). This is done in order to avoid the danger of “pollution” from discretization errors.

We now ask the question, within which \( R \) region our data are compatible — if at all! — with the asymptotic behaviour of eq. (13). We find that as long as \( R \sqrt{K} < 0.173 \) our fits yield results with reasonable \( \chi^2/N_{DF} \). This upper limit in \( R \) corresponds to \( 2.5 GeV \). Fitting the \( \beta = 6.8 \) data over this region we obtain

\[ \Lambda_R = (0.562 \pm 0.020 \pm 0.010) \sqrt{\sigma} \approx (247 \pm 10) MeV. \]  

(15)

The first error stems from the fit just described, while the second relates to the statistical uncertainty of the string tension within our lattice analysis. The corresponding fit curve with error bands is plotted in figure 5. As the data appear to oscillate the asymptotic curve one finds a systematic dependence on the \( R \) cut: \( \Lambda_R \) tends to be larger if more (low energy) data points are included and \textit{vice versa}. In this sense one might consider our value as an upper limit.

Exploiting the relation \( \Lambda_R = 30.19 \Lambda_L \) \cite{22} we get:

\[ \Lambda_L^0 = (18.6 \pm 0.7 \pm 0.3) \times 10^{-3} \sqrt{\sigma} \approx (8.19 \pm 0.33) MeV. \]  

(16)

This corresponds to the ratio

\[ \frac{\sqrt{\sigma}}{\Lambda_L^0} = 53.7 \pm 2.1. \]  

(17)

In figure 5b we have plotted \( \alpha \) versus the energy. At the largest realized energy scale we find \( \alpha_{q\bar{q}}(5 GeV) \approx 0.150(3) \).

Returning to the global structure of the data displayed in figure 5 we make three observations: 1. The small \( R \) contributions (circles, and triangles) follow very neatly the asymptotic perturbative prediction eq. (13), indicating very little discretization effects. 2. Over the whole \( R \) range the data sets for \( \beta = 6.4, \) and \( \beta = 6.8 \) coincide very nicely, giving evidence for scaling. 3. The deviations of the data from the asymptotic behaviour remain fairly small up to \( q \approx 1 GeV \) or \( \alpha_{q\bar{q}} \approx 0.4 \).

We conclude that lattice simulations can indeed make contact to the perturbative regime. Moreover, it is very satisfying to observe that the 2-loop-formula describes the lattice data down to a scale as small as \( 1 GeV \) — at least in the quenched approximation of QCD. One would expect that the situation in full QCD is fairly similar, concerning this property. In the infrared regime \((q < \sqrt{\sigma})\) the differences between both theories will be considerable: Because of the linear confining potential our expectation for the pure gauge sector is \( \alpha_{q\bar{q}}(q) \propto 1/q^2 \). This has to be confronted with the expression \( \alpha_{q\bar{q}}(q) \propto e^{-\mu/q}/q^2 \) for QCD with fermionic degrees of freedom where \( \mu \) stands for the screening mass.
Table 3: The lattice spacing $a$, and cutoff parameters $\Lambda_L$ calculated from the 2-loop-expansion eq. (18) in units of the string tension $\sigma$. $\Lambda_L$ is obtained by inserting the bare lattice coupling. For $\Lambda_L^{(1,2)}$ the $\beta_E^{(1,2)}$ effective couplings were used. A naive linear extrapolation to $a = 0$ leads to the results displayed in the second last row. Logarithmic extrapolations yield the values in the last row.

| $\beta$ | $a\sqrt{\sigma}$ | $\sqrt{\sigma}/\Lambda_L$ | $\sqrt{\sigma}/\Lambda_L^{(1)}$ | $\sqrt{\sigma}/\Lambda_L^{(2)}$ |
|--------|-------------------|-----------------------------|-------------------------------|-------------------------------|
| 5.7    | 0.4099 (24)       | 124.7 (0.7)                 | 63.3 (0.4)                    | 55.7 (0.3)                    |
| 5.8    | 0.3302 (30)       | 112.4 (1.0)                 | 63.0 (0.6)                    | 55.6 (0.5)                    |
| 5.9    | 0.2702 (37)       | 102.9 (1.4)                 | 61.2 (0.8)                    | 54.3 (0.7)                    |
| 6.0    | 0.2265 (55)       | 96.5 (2.3)                  | 60.0 (1.5)                    | 53.4 (1.3)                    |
| 6.2    | 0.1619 (19)       | 86.4 (1.0)                  | 56.9 (0.7)                    | 50.8 (0.6)                    |
| 6.4    | 0.1215 (12)       | 81.3 (0.8)                  | 55.7 (0.5)                    | 50.0 (0.5)                    |
| 6.8    | 0.0730 (12)       | 76.9 (1.3)                  | 55.7 (0.9)                    | 50.4 (0.8)                    |
| $\infty$ | lin. | 0 | 63.6 (2.4) | 53.1 (1.6) | 48.3 (1.4) |
|         | log. | 0 | 54^{+18}_{-15} & 53.2^{+6}_{-8} & 49.1^{+3}_{-5.9} |

3.3 Scaling

Normally one speaks of asymptotic scaling when the ratio $\sqrt{\sigma}/\Lambda_L$ remains constant on varying $\beta$ where

$$\Lambda_L = \frac{1}{a} \exp \left( -\frac{1}{2b_0 g^2} \right) \left( b_0 g^2 \right)^{-\frac{b_1}{2b_0}}$$  \hspace{1cm} (18)

(with $g^2 = 2N_C/\beta$) denotes the integrated two-loop $\beta$-function (eq. (25) below). In table 3 we have compiled our new results on the string tension together with previous results from refs. [6, 23]. As can be seen, we are still far away from the asymptotic scaling region up to $\beta = 6.8$.

We attempt to extrapolate $\Lambda_L^{-1}$ to the continuum limit by the use of a parameterization that takes into account the leading order expectation for scaling violations $O(1/\ln a)$:

$$\Lambda_L^{-1}(a) = \Lambda_L^{-1}(0) + \frac{C}{\sqrt{\sigma} \ln(Da\sqrt{\sigma})}$$  \hspace{1cm} (19)

We find the data compatible with this logarithmic behaviour, with $D \approx 1-2$, and $C \approx 20-80$. The fit parameters are not particularly stable with respect to a variation of the number of data points. The bandwith of extrapolations to the continuum limit is illustrated in figure 6a where we have plotted the extreme cases of a fit to our four low $a$ data points, and all seven data points (open circles). If we average the values obtained from these fits, and take the upmost and the lowest possible numbers as error bandwidth, we estimate the asymptotic value to be $\sqrt{\sigma} \Lambda_L(0)^{-1} = 54^{+18}_{-15}$ (full circle). We would like to mention that a naive linear extrapolation to the continuum limit yields the
Table 4: The average plaquette action $\langle S_\square \rangle$, measured on large lattice volumes. The values for $\beta \leq 5.9$ are taken from the collection in ref. [11] while the other numbers are our new results, obtained on $32^4$ lattices, and one $24^3 \times 32$ lattice ($\beta = 6.2$).

| $\beta$ | $\langle S_\square \rangle$ |
|---------|--------------------------|
| 5.7     | 0.45100 (80)            |
| 5.8     | 0.43236 (5)             |
| 5.9     | 0.41825 (6)             |
| 6.0     | 0.406262 (17)           |
| 6.2     | 0.386353 (8)            |
| 6.4     | 0.369353 (5)            |
| 6.8     | 0.340782 (5)            |

value $\sqrt{\sigma} \Lambda_L(0)^{-1} = 63.6(2.4)$ with (obviously) underestimated error. We take this as a warning for purely phenomenological continuum extrapolations.

In view of the uncertainty of the above number it would be highly desirable to improve the situation by developing a scheme within which the $a$ dependence of $\Lambda_L(a)$ is reduced. Parisi suggested many years ago a more “natural” expansion parameter $g_E$ [8], based on a mean field argument. His scheme was elaborated in refs. [9, 10, 11]. It works as follows: Let $c_n$ be the coefficients of the weak coupling expansion of the average plaquette

$$
\langle S_\square \rangle = \frac{1}{6V} \sum_\square \left( 1 - \frac{1}{N_C} \text{Re} \text{ Tr} U_\square \right) = \sum_{n=1}^{\infty} c_n g^{2n}.
$$

The idea, now, is to introduce an effective coupling in terms of the Monte Carlo generated average plaquette

$$
g_E^2 = \frac{\langle S_\square \rangle}{c_1} = g^2 + \frac{c_2}{c_1} g^4 + \frac{c_3}{c_1} g^6 + O(g^8),
$$

for which the first order expansion is exact. The hope is that the nonperturbative (or higher order perturbative) contributions that are resummed in the effective coupling $g_E$ may compensate high order terms in the $\beta$-function which are responsible for the scaling violations. Support for this expectation comes from the observed scaling of ratios of physical quantities (figures 4, 5) within the same $\beta$ region.

The coefficients $c_1$, and $c_2$ have been calculated previously [24], and an unpublished value for $c_3$, obtained by H. Panagopoulos, has been cited in ref. [11]. The numerical values are:

$$
c_1 = \frac{(N_C^2 - 1)}{(8N_C)}
$$
\[c_2 = (N_C^2 - 1)(0.0204277 - 1/(32N_C^2))/4\]  \\(c_3 = (N_C^2 - 1)N_C(0.0066599 - 0.020411/N_C^2 + 0.0343399/N_C^4)/6.\]

The plaquette values needed for the conversion into the effective coupling schemes are collected in table 4. The numbers for \(\beta \leq 5.9\) were taken from the collection in ref. [11].

Starting from the expansion

\[
\beta(g) = -\frac{dg}{d\ln a} = -\sum_{n=0}^{\infty} b_ng^{2n+3}
\]

of the \(\beta\)-function, one rewrites

\[
\beta(g_E) = -\frac{dg_E}{d\ln a} = -\frac{dg}{d\ln a} \frac{g}{g_E} \frac{dg_E^2}{dg^2}
\]

\[
= -b_0g_E^2 - b_1g_E^5 - b_2g_E^7 + \left(3b_0\left(\frac{c_2}{c_1}\right)^2 - \frac{c_3}{c_1}\right) - 2b_1\frac{c_2}{c_1} g^7 + \mathcal{O}(g^9).
\]

The first two terms in this weak coupling expansion remain unchanged under the substitution. Therefore, an integration again leads to eq. (18), but with a redefined integration constant

\[
\Lambda_E = \Lambda_L \exp\left(\frac{c_2}{2c_1b_0}\right) \approx 2.0756\Lambda_L \quad \text{(for SU}(3)).
\]

This factor is due to a shift of the effective \(\beta\) by a constant in the continuum limit: \(g_E^{-2} = g^{-2} - c_2/c_1 + \mathcal{O}(g^2)\). In the following we will refer to this scheme as the \(\beta_E^{(1)}\) scheme. As one can see from figure 6a (open squares), and table 3 this kind of (numerical) resummation of the asymptotic series eq. (20) leads to considerably reduced logarithmic corrections \((C \approx 2.5)\).

As an additional check of this improvement technique we consider in the following an “alternative” effective coupling scheme \(\beta_E^{(2)}\). Our idea is to introduce a coupling \(g_2\) by inverting the relation

\[
\langle S_\square \rangle = c_1g_2^2 + c_2g_2^4.
\]

This amounts to truncating the weak coupling expansion eq. (20) after the second term. A short calculation yields:

\[
\beta(g_2) = -b_0g_2^3 - b_1g_2^5 - b_2g_2^7 - 3b_0\frac{c_3}{c_1}g_2^7 + \mathcal{O}(g_2^9).
\]

Because of \(g_2^{-2} = g^{-2} + \mathcal{O}(g^2)\) the integration constant \(\Lambda_L\) remains unchanged in respect to the original bare coupling scheme.

\(\footnote{One can generalise this scheme by truncating in higher orders \(n\). This is of little interest, however (unless one is interested in numerical studies of the impact of a particular higher loop contribution on the observed scaling violations), since the \(\beta\)-function has only been calculated up to \(\mathcal{O}(g^5)\). Moreover, one would retrieve the bare coupling scheme at \(n\) sufficiently large.}
If we compare the third order terms of the two effective schemes (eqs. (26,29)), we find explicitly:

\[ \beta(g) = \beta(g_E(g)) + 5.3 \times 10^{-4} g^7 + O(g^9) = \beta(g_2(g)) + 4.02 \times 10^{-3} g^7 + O(g^9). \] (30)

This means that the correction of the $\beta$-function through the 3-loop-contribution is much larger for the $\beta^{(2)}$ than for the $\beta^{(1)}$ scheme\footnote{Note that the difference between the $\beta$-functions for both effective schemes is independent of $c_3$ to this order.}. Nevertheless, at least within the investigated $\beta$ region, the qualitative behaviour of both schemes is the same as can be seen in figure 6a. For the $\beta^{(2)}$ scheme the correction coefficient ($C \approx 0.9$) of the continuum extrapolation eq. (19) is even smaller than for the $\beta_E^{(1)}$ scheme. In figure 6a we have included the estimates for the asymptotic $\Lambda^{-1}_L$ values (and the $\Lambda^{-1}_L$ from the running coupling) as full symbols.

The extrapolated values for both effective schemes are, respectively:

\[ \sqrt{\sigma} = 53.2^{+2.6}_{-7.3} \Lambda^{(1)}_L \] (31)
\[ = 49.1^{+2.3}_{-5.9} \Lambda^{(2)}_L. \] (32)

Averaging these numbers that carry asymmetric errors leads to $\sqrt{\sigma} = 50.8^{+4.6}_{-1.0} \Lambda^{E}_L$. This result is in nice agreement with the ratio extracted from the running coupling ($\sqrt{\sigma} = 53.7(2.1) \Lambda^{E}_L$, eq. [17]). Using this additional information, we obtain:

\[ \sqrt{\sigma} = 51.6^{+0.7}_{-1.0} \Lambda_L. \] (33)

This result may be converted into any continuum renormalization scheme like the minimal subtraction ($\hat{MS}$) scheme. By exploiting the relation $\Lambda_{\hat{MS}} = 28.81 \Lambda_L$ [25], we get:

\[ \frac{\Lambda_{\hat{MS}}}{\sqrt{\sigma}} = 0.558^{+0.017}_{-0.007}. \] (34)

Let us finally comment that the two approaches presented in this paper for the determination of the QCD scale parameter $\Lambda$, namely to analyze (a) $g^2(\Lambda aR)$, and (b) its inverse $\Lambda a(g^2)$ in terms of the two-loop predictions eqs. (13,18), are complementary and supportive to each other because higher order corrections to methods (a) and (b) are anticorrelated. In our running coupling (string tension) analysis we observe the “effective” $\Lambda_L$ to decrease (increase) with the energy scale. Since the central value of our “upper limit” $\Lambda^{E}_L$ is smaller than that of our “lower limit” $\Lambda^{E}_L$ we are in the position to state relatively small errors for $\Lambda_{\hat{MS}}$.

In figure 6b we have plotted the $\Lambda^{-1}_L$ data versus $\beta$ in order to visualize the slow approach of the bare coupling data towards the asymptotic value, and the improvement achieved by the use of effective couplings.
4 Discussion

We have demonstrated that medium size computer experiments are able to determine the \( \Lambda \)-parameter of \( SU(3) \) Yang-Mills theory within a reasonable accuracy (that can compete with QCD experiments). For this result, it has been important to study both infrared, and ultraviolet aspects in order to verify the reliability of the continuum extrapolation. We might say that we have been lucky to get hold of \textit{asymptotia} within our means. This is due to the discovery that the running coupling \textit{constant} is well described within this theory by the two-loop formula down to a scale of about 1 GeV.

If nature continues to be nice to us, and the inclusion of dynamical quarks results only in a \( \beta \)-shift of quenched predictions it is possible to predict experimental numbers like \( \alpha_s(M_Z) \), as explained in ref. [2]. Obviously, it is preferable to repeat this study in full QCD on the level of TERAFLOPS power. In the meantime, further improvements of lattice techniques are of great interest. A promising route has been proposed by M. Lüscher \textit{et. al.} [26], and tested on \( SU(2) \) Yang-Mills theory. These authors start from a volume dependent coupling \( g(L) \) which allows them to reach large energies on small lattices.

After completion of this work we received a preprint by S.P. Booth, C. Michael, and collaborators [27] that contains a running coupling study for \( SU(3) \) gauge theory up to \( \beta = 6.5 \). Their results are fully consistent with ours.

Acknowledgements. We are grateful to Deutsche Forschungsgemeinschaft for the support given to our CM-2 project. We thank Peer Ueberholz and Randy Flesch for their kind support. One of the authors (G.B.) would like to thank Chris Michael, Edwin Laermann, Rainer Sommer, and Jochen Fingberg for helpful discussions about data analysis, and the different effective coupling schemes.
Appendix

A Potential Values

In this appendix we are stating the potential values measured on a $32^4$ lattice at $\beta = 6.8$. The corresponding numbers for the other $\beta$-values can be found in ref. [6]. The on- and off-axis paths are numbered in the following way:

| Path # | 1 | 2 | 3 | 4 | 5 | 6 |
|--------|---|---|---|---|---|---|
| Path $(X, Y, Z)$ | $(1, 0, 0)$ | $(1, 1, 0)$ | $(2, 1, 0)$ | $(1, 1, 1)$ | $(2, 1, 1)$ | $(2, 2, 1)$ |
| Elementary distance $M$ | 1 | 1.41 | 2.24 | 1.73 | 2.45 | 3 |

The results for the potential $V(\vec{R})$ (in lattice units), as well as for the “corrected” $V(R)$, and the corresponding ground state overlaps $C(\vec{R})$ are collected in table 5. The data is plotted (among the other curves) in figure 4.

Table 5: The potential values $V(\vec{R})$ (in lattice units $a^{-1}$), “corrected” values $V(R)$, and ground state overlaps $C(\vec{R})$ for $\beta = 6.8$, $V = 32^4$.

| $R$ | Path | $V(\vec{R})$ | $V(R)$ | $C(\vec{R})$ |
|-----|------|--------------|---------|--------------|
| 1.00 | 1    | 0.3107 (6)   | 0.3210 (10) | 0.950 (3) |
| 1.14 | 2    | 0.3855 (11)  | 0.3794 (12) | 0.951 (4) |
| 1.73 | 4    | 0.4188 (19)  | 0.4098 (20) | 0.946 (8) |
| 2.00 | 1    | 0.4236 (14)  | 0.4266 (14) | 0.929 (5) |
| 2.24 | 3    | 0.4428 (13)  | 0.4397 (14) | 0.934 (5) |
| 2.45 | 5    | 0.4559 (15)  | 0.4509 (15) | 0.936 (6) |
| 2.83 | 2    | 0.4696 (20)  | 0.4656 (20) | 0.923 (8) |
| 3.00 | 1    | 0.4725 (14)  | 0.4709 (14) | 0.931 (6) |
| 3.00 | 6    | 0.4751 (18)  | 0.4705 (19) | 0.924 (7) |
| 3.46 | 4    | 0.4906 (31)  | 0.4861 (31) | 0.923 (12) |
| 4.00 | 1    | 0.5000 (18)  | 0.4970 (19) | 0.916 (7) |
| 4.24 | 2    | 0.5079 (23)  | 0.5039 (23) | 0.939 (9) |
| 4.47 | 3    | 0.5105 (22)  | 0.5068 (22) | 0.916 (9) |
| $R$ | Path | $V(R)$ | $V'(R)$ | $C(R)$ |
|-----|------|--------|---------|--------|
| 4.90 | 5    | 0.5178 (28) | 0.5139 (28) | 0.913 (11) |
| 5.00 | 1    | 0.5193 (19) | 0.5159 (20) | 0.924 (8)   |
| 5.20 | 4    | 0.5230 (32) | 0.5190 (32) | 0.929 (13)  |
| 5.66 | 2    | 0.5312 (29) | 0.5273 (30) | 0.920 (12)  |
| 6.00 | 1    | 0.5325 (25) | 0.5289 (25) | 0.907 (10)  |
| 6.00 | 6    | 0.5357 (29) | 0.5317 (30) | 0.918 (12)  |
| 6.71 | 3    | 0.5421 (27) | 0.5383 (27) | 0.917 (11)  |
| 6.93 | 4    | 0.5469 (42) | 0.5430 (42) | 0.916 (16)  |
| 7.00 | 1    | 0.5463 (27) | 0.5426 (27) | 0.921 (11)  |
| 7.07 | 2    | 0.5474 (36) | 0.5436 (37) | 0.928 (15)  |
| 7.35 | 5    | 0.5504 (32) | 0.5466 (32) | 0.923 (13)  |
| 8.00 | 1    | 0.5568 (34) | 0.5531 (34) | 0.910 (13)  |
| 8.49 | 2    | 0.5623 (44) | 0.5584 (44) | 0.911 (17)  |
| 8.66 | 4    | 0.5644 (47) | 0.5605 (47) | 0.930 (19)  |
| 8.94 | 3    | 0.5663 (37) | 0.5625 (37) | 0.911 (15)  |
| 9.00 | 1    | 0.5671 (36) | 0.5633 (36) | 0.920 (14)  |
| 9.00 | 6    | 0.5651 (37) | 0.5612 (37) | 0.911 (15)  |
| 9.80 | 5    | 0.5733 (41) | 0.5695 (41) | 0.908 (16)  |
| 9.90 | 2    | 0.5745 (48) | 0.5707 (48) | 0.925 (19)  |
| 10.00| 1    | 0.5743 (44) | 0.5705 (44) | 0.904 (17)  |
| 10.39| 4    | 0.5777 (53) | 0.5739 (53) | 0.905 (21)  |
| 11.00| 1    | 0.5830 (49) | 0.5792 (49) | 0.913 (19)  |
| 11.18| 3    | 0.5818 (49) | 0.5780 (49) | 0.903 (20)  |
| 11.31| 2    | 0.5841 (50) | 0.5803 (50) | 0.898 (20)  |
| 12.00| 1    | 0.5887 (55) | 0.5849 (55) | 0.895 (21)  |
| 12.00| 6    | 0.5900 (55) | 0.5862 (55) | 0.901 (22)  |
| 12.12| 4    | 0.5941 (60) | 0.5902 (60) | 0.928 (24)  |
| 12.25| 5    | 0.5918 (53) | 0.5879 (53) | 0.912 (21)  |
| 12.73| 2    | 0.5962 (64) | 0.5923 (64) | 0.918 (26)  |
| 13.00| 1    | 0.5987 (56) | 0.5949 (56) | 0.912 (22)  |
| $R$  | Path | $V(R)$  | $V(\bar{R})$ | $C(R)$  |
|-----|------|---------|-------------|---------|
| 13.42 | 3    | 0.5998 (61) | 0.5960 (61) | 0.894 (24) |
| 13.86 | 4    | 0.6031 (75) | 0.5993 (75) | 0.895 (29) |
| 14.00 | 1    | 0.6055 (62) | 0.6017 (62) | 0.899 (24) |
| 14.14 | 2    | 0.6052 (73) | 0.6014 (73) | 0.893 (29) |
| 14.70 | 5    | 0.6096 (70) | 0.6058 (70) | 0.895 (27) |
| 15.00 | 1    | 0.6097 (68) | 0.6059 (68) | 0.895 (27) |
| 15.00 | 6    | 0.6102 (69) | 0.6064 (69) | 0.895 (27) |
| 15.56 | 2    | 0.6139 (81) | 0.6101 (81) | 0.904 (32) |
| 15.59 | 4    | 0.6163 (81) | 0.6125 (81) | 0.910 (32) |
| 15.65 | 3    | 0.6144 (73) | 0.6106 (73) | 0.895 (29) |
| 16.00 | 1    | 0.6151 (74) | 0.6113 (74) | 0.878 (29) |
| 16.97 | 2    | 0.6246 (88) | 0.6209 (88) | 0.886 (34) |
| 17.15 | 5    | 0.6248 (78) | 0.6210 (78) | 0.895 (31) |
| 17.32 | 4    | 0.6258 (94) | 0.6220 (94) | 0.883 (36) |
| 17.89 | 3    | 0.6296 (90) | 0.6258 (90) | 0.880 (35) |
| 18.00 | 6    | 0.6312 (88) | 0.6274 (88) | 0.885 (34) |
| 18.39 | 2    | 0.6337 (99) | 0.6299 (99) | 0.899 (39) |
| 19.05 | 4    | 0.6394 (109) | 0.6357 (109) | 0.900 (43) |
| 19.60 | 5    | 0.6402 (95) | 0.6364 (95) | 0.874 (36) |
| 19.80 | 2    | 0.6440 (105) | 0.6402 (105) | 0.882 (41) |
| 20.79 | 4    | 0.6486 (117) | 0.6448 (117) | 0.875 (44) |
| 21.00 | 6    | 0.6496 (108) | 0.6458 (108) | 0.879 (42) |
| 21.21 | 2    | 0.6526 (118) | 0.6489 (118) | 0.891 (46) |
| 22.52 | 4    | 0.6610 (131) | 0.6573 (131) | 0.889 (51) |
| 22.63 | 2    | 0.6545 (128) | 0.6508 (128) | 0.847 (48) |
| 24.00 | 6    | 0.6688 (123) | 0.6650 (123) | 0.863 (47) |
| 24.25 | 4    | 0.6694 (142) | 0.6657 (142) | 0.862 (53) |
| 25.98 | 4    | 0.6791 (151) | 0.6753 (151) | 0.866 (57) |
| 27.71 | 4    | 0.6908 (162) | 0.6871 (162) | 0.848 (60) |
References

[1] See e.g. the review article: T. Hebbeker, “Tests of QCD...”, to appear in Phys. Reports.

[2] A.X. El-Khadra, G. Hockney, A.S. Kronfeld, and P.B. Mackenzie, Fermilab preprint 91/354-T; P.B. Mackenzie, Nucl. Phys. B[Proc. Suppl.]26 (1992) 369.

[3] C. Michael, Phys. Lett. B283 (1992) 103.

[4] M. Lüscher, R. Sommer, U. Wolff, and P. Weisz, CERN preprint CERN-TH 6566/92.

[5] The UKQCD Collaboration: S.P. Booth, K.C. Bowler, D.S. Henty, R.D. Kenway, B.J. Pendleton, D.G. Richards, A.D. Simpson, A.C. Irving, A. McKerrell, C. Michael, P.W. Stephenson, M. Teper, and K. Decker, Phys. Lett. B275 (1992) 424.

[6] G.S. Bali and K. Schilling, Wuppertal preprint WUB 92-02, to appear in Phys. Rev. D46 (1992).

[7] A. and P. Hasenfratz, Phys. Lett. 93B (1980) 165; Nucl. Phys. B193 (1981) 210.

[8] G. Parisi, Proceedings of the xxth International Conference on High Energy Physics 1980, Madison, Eds. L. Durand, and L.G. Pondrom, American Institute of Physics, New York (1981) 1531.

[9] Y.M. Makeenko and M.I. Polikarpov, Nucl. Phys. B205 (1982) 386.

[10] S. Samuel, O. Martin, and K. Moriarty, Phys. Lett. 152B (1984) 87.

[11] J. Fingberg, U. Heller, and F. Karsch, Bielefeld preprint BI-TP 92-26

[12] N. Cabibbo and E. Marinari, Phys. Lett. 119B (1982) 387.

[13] S.L. Adler, Phys. Rev. D23 (1981) 2901.

[14] A. Kennedy and B. Pendleton, Phys. Lett. B156 (1985) 393.

[15] J. Hoek, M. Teper, and J. Waterhouse, Nucl. Phys. B288 (1987) 589.

[16] M. Campostrini, A. Di Giacomo, M. Maggiore, H. Panagopoulos, and E. Vicari, Phys. Lett. B225 (1989) 403.

[17] The APE Collaboration: M. Albanese et al., Phys. Lett. B192 (1987) 163.

[18] B. Efron, Ann. Statist. 7 (1979) 1.

[19] R. Gupta et. al., Phys. Rev. D36 (1987) 2813.
[20] C.B. Lang and C. Rebbi, *Phys. Lett.* **115B** (1982) 137.

[21] M. Lüscher, K. Symanzik, and P. Weisz, *Nucl. Phys.* **B173** (1980) 365; M. Lüscher, *Nucl. Phys.* **B180** (1981) 317.

[22] A. Billoire, *Phys. Lett.* **104B** (1981) 472.

[23] The $MT_C$ Collaboration: K.D. Born, R. Altmeyer, W. Ibes, E. Laermann, R. Sommer, T.F. Walsh, and P. Zerwas, *Nucl. Phys.* **B**[Proc. Suppl.] **20** (1991) 394.

[24] A. DiGiacomo and G.C. Rossi, *Phys. Lett.* **100B** (1981) 481; A. DiGiacomo and G. Paffati, *Phys. Lett.* **108B** (1982) 327; U. Heller and F. Karsch, *Nucl. Phys.* **B251** (1985) 254.

[25] R. Dashen and D.J. Gross, *Phys. Rev.* **D23** (1981) 2340.

[26] M. Lüscher, R. Narayanan, P. Weisz, and U. Wolff, DESY preprint 92-025, to appear in *Nucl. Phys.* **B**.

[27] S.P. Booth, D.S. Henty, A. Hulsebos, A.C. Irving, C. Michael, and P.W. Stephenson, Liverpool preprint LTH 285 (1992).
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