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Rings of $\hbar$-deformed differential operators

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In memory of Petr Kulish

Abstract

We describe the center of the ring $\text{Diff}_\hbar(n)$ of $\hbar$-deformed differential operators of type A. We establish an isomorphism between certain localizations of $\text{Diff}_\hbar(n)$ and the Weyl algebra $W_n$ extended by $n$ indeterminates.

1 Introduction

The ring $\text{Diff}_\hbar(n)$ of $\hbar$-deformed differential operators of type A appears in the theory of reduction algebras. A reduction algebra $R^A_g$ provides a tool to study decompositions of representations of an associative algebra $A$ with respect to its subalgebra in the situation when this subalgebra is the universal enveloping algebra of a reductive Lie algebra $g$ [M, AST]. We refer to [T, Zh] for the general theory and uses of reduction algebras.

Decompositions of tensor products of representations of a reductive Lie algebra $g$ is a particular case of a restriction problem, associated to the diagonal embedding of $U(g)$ into $U(g) \otimes U(g)$. The corresponding reduction algebra, denoted $D(g)$, is called “diagonal reduction algebra” [KO2]. A description of the diagonal reduction algebra $D(gl_n)$ in terms of generators and (ordering) defining relations was given in [KO2, KO3].

The diagonal reduction algebra $D(gl_n)$ admits an analogue of the “oscillator realization”, in the rings $\text{Diff}_\hbar(n, N)$, $N = 1, 2, 3, \ldots$, of $\hbar$-deformed differential operators, see [KO5]. The ring $\text{Diff}_\hbar(n, N)$ can be obtained by the reduction of the ring of differential operators in $nN$ variables (that is, of the Weyl algebra $W_{n, N} = W_n^\otimes N$) with respect to the natural action of $gl_n$. Similarly to the ring of $q$-differential operators [WZ], the algebra $\text{Diff}_\hbar(n, N)$ can be described in the R-matrix formalism. The R-matrix, needed here, is a solution of the so-called dynamical Yang–Baxter equation (we refer to [F, GV, ES] for different aspects of the dynamical Yang–Baxter equation and its solutions).

The ring $\text{Diff}_\hbar(n, N)$ is formed by $N$ copies of the ring $\text{Diff}_\hbar(n) = \text{Diff}_\hbar(n, 1)$. The aim of the present article is to investigate the structure of the ring $\text{Diff}_\hbar(n)$. Our first result is the description of the center of $\text{Diff}_\hbar(n)$: it is a ring of polynomials in $n$ generators.

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As follows from the results of [KO4], the ring $\text{Diff}_h(n)$ is a noetherian Ore domain. It is therefore natural to investigate its field of fractions and test the validity of the Gelfand–Kirillov-like conjecture [GK]. For the ring of $q$-differential operators, the isomorphism (up to a certain localization and completion) with the Weyl algebra was given in [O]. The second result of the present article consists in a construction of an isomorphism between certain localizations of $\text{Diff}_h(n)$ and the Weyl algebra $W_n$ extended by $n$ indeterminates. In particular, our isomorphism implies the isomorphism of the corresponding fields of fractions.

According to the general theory of reduction algebras, the ring $\text{Diff}_h(n)$ admits the action of Zhelobenko operators [Zh, KO1] by automorphisms. The Zhelobenko operators generate the action of the braid group $B_n$. However the Weyl algebra $W_n$ admits the action of the symmetric group $S_n$ by automorphisms. As a by-product of our construction we define the action of the symmetric group by automorphisms on the ring $\text{Diff}_h(n)$. Moreover these formulas can be generalized to produce the action of the symmetric group by automorphisms on the rings $\text{Diff}_h(n, N)$ for any $N$, and on the diagonal reduction algebra $D(\mathfrak{gl}_n)$. We conjecture that the general reduction algebra $R^A$ admits an action, by automorphisms, of the Weyl group of $\mathfrak{g}$.

Section 2 contains the definition of the ring of $h$-deformed differential operators and some of their properties used in the sequel. In Section 3 we present a family of $n$ quadratic central elements. We then describe an $n$-parametric family of “highest weight” representations of $\text{Diff}_h(n)$ and calculate values of the quadratic central elements in these representations. In Section 4 we introduce the necessary localizations of $\text{Diff}_h(n)$ and of the Weyl algebra, check the Ore conditions and establish the above mentioned isomorphism of the localized rings. In Section 5 we prove the completeness of the family of central elements constructed in Section 3. Then we describe the action of the symmetric group on $\text{Diff}_h(n, N)$ and $D(\mathfrak{gl}_n)$ as well as the action of the braid group, generated by Zhelobenko operators, on a localization of the Weyl algebra. Also, we present a $2n$-parametric family of representations of the algebra $\text{Diff}_h(n)$ implied by our construction.

**Notation**

Throughout the paper, $\mathfrak{t}$ denotes the ground ring of characteristic zero.

The symbol $s_i$ stands for the transposition $(i, i + 1)$.

We denote by $U(\mathfrak{h})$ the free commutative $\mathfrak{t}$-algebra in generators $\hat{h}_i$, $i = 1, \ldots, n$. Set $\hat{h}_{ij} = \hat{h}_i - \hat{h}_j \in \mathfrak{h}$. We define $\hat{U}(\mathfrak{h})$ to be the ring of fractions of $U(\mathfrak{h})$ with respect to the multiplicative set of denominators, generated by the elements $(h_{ij} + k)^{-1}$, $k \in \mathbb{Z}$. Let

$$\psi_i := \prod_{k:k>i} \hat{h}_{ik}, \psi'_i := \prod_{k:k<i} \hat{h}_{ik} \text{ and } \chi_i := \psi_i \psi'_i, \ i = 1, \ldots, n . \quad (1)$$

Let $\varepsilon_j$, $j = 1, \ldots, n$, be the elementary translations of the generators of $U(\mathfrak{h})$, $\varepsilon_j : \hat{h}_i \mapsto \hat{h}_i + \delta_i^j$. For an element $p \in \hat{U}(\mathfrak{h})$ we denote $\varepsilon_j(p)$ by $p[\varepsilon_j]$. 

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2 Definition and properties of rings of h-deformed differential operators

Let \( \hat{R} = \{ \hat{R}_{ij} \}_{i,j,k,l=1} \) be a matrix of elements of \( \hat{U}(\hbar) \), with nonzero entries

\[
\hat{R}_{ij} = \frac{1}{\hbar_{ij}}, \quad i \neq j, \quad \text{and} \quad \hat{R}_{ji} = \begin{cases} \frac{\hbar_{ij}^2 - 1}{\hbar_{ij}} , & i < j, \\ 1 , & i \geq j. \end{cases}
\]  

The matrix \( \hat{R} \) is the standard solution of the dynamical Yang–Baxter equation

\[
\sum_{a,b,u} \hat{R}_{ij}^{ab} \hat{R}_{bk}^{ur} [\varepsilon_a] \hat{R}_{au}^{mn} = \sum_{a,b,u} \hat{R}_{jk}^{ab} [\varepsilon_i] \hat{R}_{ia}^{mu} \hat{R}_{ub}^{nr} [\varepsilon_m]
\]

of type A.

The ring \( \text{Diff}_\hbar(n) \) of \( \hbar \)-deformed differential operators of type A is a \( \hat{U}(\hbar) \)-bimodule with the generators \( x^j \) and \( \bar{\partial}_j \), \( j = 1, \ldots, n \). The ring \( \text{Diff}_\hbar(n) \) is free as a one-sided \( \hat{U}(\hbar) \)-module; the left and right \( \hat{U}(\hbar) \)-module structures are related by

\[
\hat{h}_i x^j = x^j (\hat{h}_i + \delta_i^j) , \quad \hat{h}_i \bar{\partial}_j = \bar{\partial}_j (\hat{h}_i - \delta_i^j) .
\]

The defining relations for the generators \( x^j \) and \( \bar{\partial}_j \), \( j = 1, \ldots, n \), read (see [KO5])

\[
x^i x^j = \sum_{k,l} \hat{R}_{ij}^{kl} x^k x^l , \quad \bar{\partial}_i \bar{\partial}_j = \sum_{k,l} \hat{R}_{ji}^{lk} \bar{\partial}_k \bar{\partial}_l , \quad x^i \bar{\partial}_j = \sum_{k,l} \hat{R}_{ij}^{kl} \bar{\partial}_k x^l - \delta_i^j ,
\]

or, in components,

\[
x^i x^j = \frac{\hat{h}_{ij}}{\hbar_{ij}} x^j x^i , \quad i < j ,
\]

\[
\bar{\partial}_i \bar{\partial}_j = \frac{\hat{h}_{ij} - 1}{\hbar_{ij}} \bar{\partial}_j \bar{\partial}_i , \quad i < j ,
\]

\[
x^i \bar{\partial}_j = \begin{cases} \bar{\partial}_j x^i , & i < j , \\ \frac{\hat{h}_{ij}(\hat{h}_{ij} - 2)}{(\hat{h}_{ij} - 1)^2} \bar{\partial}_j x^i , & i > j , \\ \sum_j \frac{1}{1 - \hbar_{ij}} \bar{\partial}_j x^j - 1 . \end{cases}
\]
The ring $\text{Diff}_h(n)$ admits Zhelobenko automorphisms $\hat{q}_i$, $i = 1, \ldots, n - 1$, given by (see [KO5])

\[
\hat{q}_i(x^i) = -x^{i+1} \frac{\hat{h}_{i,i+1}}{h_{i,i+1} - 1}, \quad \hat{q}_i(x^{i+1}) = x^i, \quad \hat{q}_i(x^j) = x^j, \quad j \neq i, i + 1, \\
\hat{q}_i(\partial_i) = -\frac{\hat{h}_{i,i+1}}{h_{i,i+1}} \partial_{i+1}, \quad \hat{q}_i(\partial_{i+1}) = \partial_i, \quad \hat{q}_i(\partial_j) = \partial_j, \quad j \neq i, i + 1, \\
\hat{q}_i(\hat{h}_{i}) = \hat{h}_{s_i(i)}.
\]

The operators $\hat{q}_i$, $i = 1, \ldots, n - 1$, generate the action of the braid group, see [Zh, KO1].

The ring $\text{Diff}_h(n)$ admits an involutive anti-automorphism $\epsilon$, defined by

\[
\epsilon(\hat{h}_i) = \hat{h}_i, \epsilon(\partial_i) = \varphi_i x^i, \epsilon(x^i) = \bar{\partial}_i \varphi_i^{-1}, \text{ where } \varphi_i := \frac{\psi_i}{\psi_i[-\varepsilon_i]} = \prod_{k:k>i} \frac{\hat{h}_{ik}}{h_{ik} - 1}, i = 1, \ldots, n.
\]

The proof reduces to the formula

\[
\frac{\varphi_i[-\varepsilon_j]}{\varphi_i} = \frac{\hat{h}_{ij}^2 - 1}{\hat{h}_{ij}^2} \text{ for } 1 \leq i < j \leq n.
\]

The construction of central elements in the next Section uses the elements

\[
\Gamma_i := \bar{\partial}_i x^i \text{ for } i = 1, \ldots, n.
\]

We collect some properties of these elements.

**Lemma 1.** We have

(i) $\Gamma_i x^j = \frac{\hat{h}_{ij} + 1}{\hat{h}_{ij}} x^j \Gamma_i$ and $\Gamma_i \bar{\partial}_j = \frac{\hat{h}_{ij} - 1}{\hat{h}_{ij}} \bar{\partial}_j \Gamma_i$ for $i \neq j$, $i, j = 1, \ldots, n$.

(ii) $\hat{q}_i(\Gamma_j) = \Gamma_{s_i(j)}$ for $i = 1, \ldots, n - 1$ and $j = 1, \ldots, n$.

(iii) $\Gamma_i \Gamma_j = \Gamma_j \Gamma_i$ for $i, j = 1, \ldots, n$.

**Proof.** Formulas (i) and (ii) are obtained by a direct calculation; (iii) follows from (i). \qed

We will use the following technical Lemma.

**Lemma 2.** Let $\mathfrak{A}$ be an associative algebra. Assume that elements $\hat{h}_i$, $\hat{Z}_i$, $\bar{Z}_i \in \mathfrak{A}$, $i = 1, \ldots, n$, satisfy

\[
\hat{h}_i \hat{h}_j = \hat{h}_j \hat{h}_i, \quad \hat{h}_i \hat{Z}_j = \hat{Z}_j (\hat{h}_i + \delta^i_j), \quad \hat{h}_i \bar{Z}_j = \bar{Z}_j (\hat{h}_i - \delta^i_j), \text{ for } i, j = 1, \ldots, n.
\]
Let $\tilde{h}_{ij} := \tilde{h}_i - \tilde{h}_j$ and
\[ \tilde{\psi}_i := \prod_{k:k>i} \tilde{h}_{ik} , \quad \tilde{\psi}'_i := \prod_{k:k<i} \tilde{h}_{ik} , \quad i = 1, \ldots, n . \]

Assume that the elements $\tilde{h}_{ij}$ are invertible. Then
(i) the elements $\tilde{Z}_i$ satisfy
\[ \tilde{Z}_i \tilde{Z}_j = \frac{\tilde{h}_{ij} + 1}{\tilde{h}_{ij}} \tilde{Z}_j \tilde{Z}_i \quad \text{for} \quad i < j , \quad i, j = 1, \ldots, n \]
if and only if any of the two families $\{ \tilde{Z}^o_i \}_{i=1}^n$ or $\{ \tilde{Z}^{to}_i \}_{i=1}^n$ where
\[ \tilde{Z}^o_i := \tilde{\psi}_i \tilde{Z}_i , \quad \tilde{Z}^{to}_i := \tilde{Z}_i \tilde{\psi}'_i \] (13)
is commutative;
(ii) the elements $\tilde{Z}_i$ satisfy
\[ \tilde{Z}_i \tilde{Z}_j = \frac{\tilde{h}_{ij} - 1}{\tilde{h}_{ij}} \tilde{Z}_j \tilde{Z}_i \quad \text{for} \quad i < j , \quad i, j = 1, \ldots, n \]
if and only if any of the two families $\{ \tilde{Z}^o_i \}_{i=1}^n$ or $\{ \tilde{Z}^{to}_i \}_{i=1}^n$ where
\[ \tilde{Z}^o_i := \psi_i \tilde{Z}_i , \quad \tilde{Z}^{to}_i := \tilde{Z}_i \psi'_i \] (14)
is commutative.

**Proof.** A direct calculation. \[\square\]

## 3 Quadratic central elements

Let $e_k := \sum_{i_1 < \ldots < i_k} \tilde{h}_{i_1} \ldots \tilde{h}_{i_k}$, $k = 0, \ldots, n$, be the elementary symmetric functions in the variables $\tilde{h}_1, \ldots, \tilde{h}_n$. Set
\[ c_k := \sum_j \frac{\partial e_k}{\partial \tilde{h}_j} \Gamma_j - e_k , \]
where $\Gamma_j$, $j = 1, \ldots, n$, are the elements defined in (12).

It follows from Lemma 1 that $\tilde{q}_j(c_k) = c_k$ for all $j = 1, \ldots, n - 1$ and $k = 1, \ldots, n$.

**Proposition 3.** The elements $c_k$, $k = 1, \ldots, n$, belong to the center of the ring Diff$_h(n)$. 5
Proof. We shall use the generating functions

\[ e(t) := \sum_{k=0}^{n} e_k t^k = \prod_{i} (1 + \tilde{h}_i t) \]

and

\[ c(t) := \sum_{k=1}^{n} c_k t^k = u(t)e(t) + 1 \text{ with } u(t) := t \sum_{i} \frac{1}{1 + h_i t} \Gamma_i - 1 . \]

The expression \( u(t) \) is introduced for convenience; the denominator \( 1 + \tilde{h}_i t \), which is not defined in the ring \( \text{Diff}_{\tilde{h}}(n) \), vanishes in the combination \( u(t)e(t) \).

We shall check that the polynomial \( c(t) \) is central. We have

\[ x^j e(t) = \frac{1 + (\tilde{h}_j - 1) t}{1 + h_j t} e(t) x^j . \quad (15) \]

Next, it follows from Lemma 1 that

\[ x^j u(t) = \left( \sum_{k} \frac{t}{1 + \tilde{h}_k t} \frac{\tilde{h}_k}{h_k j} \Gamma_k \right) + \frac{t}{1 + (\tilde{h}_j - 1) t} \left( \sum_{k} \frac{1}{1 - \frac{1}{h_j k}} \Gamma_k - 1 \right) - 1 \right) x^j . \]

The coefficient of \( \Gamma_k \) in this expression is equal to \( \frac{1 + h_j t}{1 + (\tilde{h}_j - 1) t} \) for both \( k \neq j \) and \( k = j \). Therefore,

\[ x^j u(t) = \frac{1 + \tilde{h}_j t}{1 + (\tilde{h}_j - 1) t} u(t) x^j . \quad (16) \]

Combining (15) and (16) we find that \( c(t) \) commutes with \( x^j, j = 1, \ldots, n \). For \( \partial_j \) one can either make a parallel calculation or use the anti-automorphism (11).

Lemma 4. (i) The matrix \( V \), defined by \( V^k_j := \frac{\partial \psi_j}{\partial h_k} \), is invertible. Its inverse is

\[ (V^{-1})^j_i = \frac{(-1)^{j-1} \tilde{h}_j^{-n-j}}{\chi_i} , \]

where the elements \( \chi_i \) are defined in (1).

(ii) We have

\[ \chi_j \Gamma_j = \tilde{h}_j^n - \tilde{h}_j^n c(-\tilde{h}_j^{-1}) . \quad (17) \]

Proof. (i) See, e.g. [OP], Proposition 4.

(ii) Rewrite the equality \( c_k = \sum_j V^j_k \Gamma_j - e_k \) in the form

\[ \Gamma_j = \sum_k (V^{-1})^j_k (c_k + e_k) = \frac{1}{\chi_j} \sum_k (-1)^{k-1} \tilde{h}_j^{n-k} (c_k + e_k) = \frac{\tilde{h}_j^n}{\chi_j} (c(-\tilde{h}_j^{-1}) + e(-\tilde{h}_j^{-1}) - 1) . \]

Since \( e(-\tilde{h}_j^{-1}) = 0 \), we obtain (17). \( \square \)
**Highest weight representations.** The ring $\text{Diff}_h(n)$ admits an $n$-parametric family of “highest weight” representations. To define them, let $\mathfrak{D}_n$ be an $\bar{\text{U}}(\mathfrak{h})$-subring of $\text{Diff}_h(n)$ generated by $\{\bar{\partial}_i\}_{i=1}^n$. Let $\lambda := \{\lambda_1, \ldots, \lambda_n\}$ be a sequence of length $n$ of complex numbers such that $\lambda_i - \lambda_j \notin \mathbb{Z}$ for all $i, j = 1, \ldots, n, i \neq j$. Denote by $M_\lambda$ the one-dimensional $\mathfrak{g}$-vector space with the basis vector $|\rangle$. Under the specified conditions on $\lambda$ the formulas

$$\tilde{h}_i: |\rangle \mapsto \lambda_i |\rangle , \quad \tilde{\partial}_i: |\rangle \mapsto 0 , \quad i = 1, \ldots, n ,$$

define the $\mathfrak{D}_n$-module structure on $M_\lambda$. We shall call the induced representation $\text{Ind}_{\mathfrak{D}_n}^{\text{Diff}_h(n)} M_\lambda$ the “highest weight representation” of highest weight $\lambda$.

**Lemma 5.** The central operator $c_k, k = 1, \ldots, n$, acts on the module $\text{Ind}_{\mathfrak{D}_n}^{\text{Diff}_h(n)} M_\lambda$ by scalar multiplication on $-e_k|\rangle_{\tilde{h}_i, \lambda_i - 1}$, the evaluation of the symmetric function $-e_k$ on the shifted vector $\{\lambda_1 - 1, \ldots, \lambda_n - 1\}$.

**Proof.** It is sufficient to calculate the value of $c_k$ on the highest weight vector $|\rangle$. In terms of generating functions we have to check that

$$e(t)u(t): |\rangle \mapsto -\prod_i (1 + (\lambda_i - 1)t) |\rangle .$$

It follows from [KO5], section 3.3, that

$$\Gamma_j |\rangle = \frac{\chi_j[\varepsilon_j]}{\chi_j} |\rangle , \quad j = 1, \ldots, n .$$

Therefore we have to check that

$$e(t) \left( t \sum_i \frac{1}{1 + \tilde{h}_i t} \frac{\chi_j[\varepsilon_j]}{\chi_j} - 1 \right) : |\rangle \mapsto -\prod_i (1 + (\lambda_i - 1)t) |\rangle . \quad (18)$$

We use another formula from [KO5] (Note 3 after the proof of Proposition 4.3 in Section 4.2)

$$\prod_i \frac{\tilde{h}_0 - \tilde{h}_i - 1}{\tilde{h}_0 - \tilde{h}_i} + \sum_j \frac{1}{\tilde{h}_0 - \tilde{h}_j} \frac{\chi_j[-\varepsilon_j]}{\chi_j} = 1 ,$$

where $\tilde{h}_0$ is an indeterminate. After the replacements $\tilde{h}_0 \to t^{-1}$ and $\tilde{h}_j \to -\tilde{h}_j, j = 1, \ldots, n$, this formula becomes

$$\frac{e(t)[-\varepsilon]}{e(t)} + t \sum_j \frac{1}{1 + \tilde{h}_j t} \frac{\chi_j[\varepsilon_j]}{\chi_j} = 1 ,$$

where $\varepsilon = \varepsilon_1 + \cdots + \varepsilon_n$, which implies (18). □
4 Isomorphism between rings of fractions

It follows from the results of [KO4] that the ring Diff\(_h(n)\) has no zero divisors. Let \(S_x\) be the multiplicative set generated by \(x^j, j = 1, \ldots, n\). The set \(S_x\) satisfies both left and right Ore conditions (see, e.g., [A] for definitions): say, for the left Ore conditions we have to check only that for any \(x^k\) and a monomial \(m = \partial_{h_1} \ldots \partial_{h_n} x^{j_1} \ldots x^{j_n}\) there exist \(s \in S_x\) and \(m' \in \text{Diff}_h(n)\) such that \(sm = \tilde{m}x^k\). The structure of the commutation relations (6-9) shows that one can choose \(\tilde{s} = (x^k)\nu\) with sufficiently large \(\nu\). Denote by \(S^{-1}_x\text{Diff}_h(n)\) the localization of the ring \(\text{Diff}_h(n)\) with respect to the set \(S_x\).

Let \(W_n\) be the Weyl algebra, the algebra with the generators \(X^j, D_j, j = 1, \ldots, n\), and the defining relations

\[
X^iX^j = X^jX^i, \quad D_iD_j = D_jD_i, \quad D_iX^j = \delta_i^j + X^jD_i, \quad i, j = 1, \ldots, n.
\]

Let \(T\) be the multiplicative set generated by \(X^jD_j - X^kD_k + \ell, 1 \leq j < k \leq n, \ell \in \mathbb{Z}\), and \(X^j, j = 1, \ldots, n\). The set \(T\) satisfies left and right Ore conditions (see [KO4], Appendix). Denote by \(T^{-1}W_n\) the localization of \(W_n\) relative to the set \(T\).

Let \(a_1, \ldots, a_n\) be a family of commuting variables. We shall use the following notation:

\[
\mathcal{H}_j := D_jX^j, \quad \mathcal{H}_{jk} := \mathcal{H}_j - \mathcal{H}_k, \\
\Psi'_j := \prod_{k:k<j} \mathcal{H}_{jk}, \quad \Psi_j := \prod_{k:k>j} \mathcal{H}_{jk}, \\
C(t) := \sum_{k=1}^{n} a_k t^k, \quad \Upsilon_i := \mathcal{H}_{ij}^n (1 - C(-\mathcal{H}_{ij}^{-1})).
\]

The polynomial \(C\) has degree \(n\) so the element \(\Upsilon_i\) is a polynomial in \(\mathcal{H}_i, i = 1, \ldots, n\).

**Theorem 6.** The ring \(S^{-1}_x\text{Diff}_h(n)\) is isomorphic to the ring \(\mathfrak{f}[a_1, \ldots, a_n] \otimes T^{-1}W_n\).

**Proof.** The knowledge of the central elements (Proposition 3) allows to exhibit a generating set of the ring \(S^{-1}_x\text{Diff}_h(n)\) in which the required isomorphism is quite transparent.

In the localized ring \(S^{-1}_x\text{Diff}_h(n)\) we can use the set of generators \(\{\tilde{h}_i, x^i, \Gamma_i\}_{i=1}^{n}\) instead of \(\{h_i, x^i, \tilde{\partial}_i\}_{i=1}^{n}\). By Lemma 4 (ii), \(\{\tilde{h}_i, x^i, c_i\}_{i=1}^{n}\) is also a generating set. Finally, \(\mathfrak{B}_D := \{\tilde{h}_i, x^{\gamma_i}, c_i\}_{i=1}^{n}\), where \(x^{\gamma_i} := x^i\psi'_i, i = 1, \ldots, n\), is a generating set of the localized ring \(S^{-1}_x\text{Diff}_h(n)\) as well. It follows from Lemma 2 that the family \(\{x^{\gamma_i}\}_{i=1}^{n}\) is commutative. The complete set of the defining relations for the generators from the set \(\mathfrak{B}_D\) reads

\[
\tilde{h}_i \tilde{h}_j = \tilde{h}_j \tilde{h}_i, \quad \tilde{h}_i x^{\gamma_j} = x^{\gamma_j} (\tilde{h}_i + \delta_i^j), \quad x^{\gamma_i} x^{\gamma_j} = x^{\gamma_j} x^{\gamma_i}, \quad i, j = 1, \ldots, n, \quad c_i \text{ are central}, i = 1, \ldots, n.
\]
In the localized ring $\mathfrak{f}[a_1, \ldots, a_n] \otimes T^{-1}W_n$ we can pass to the set of generators $\mathfrak{B}_W := \{H_i, X^i, a_i\}_{i=1}^n$ with the defining relations
\begin{align*}
H_i H_j = H_j H_i, & \quad H_i X^j = X^j (H_i + \delta_i^j), & \quad X^i X^j = X^j X^i, & \quad i, j = 1, \ldots, n, \\
a_i \text{ are central,} & \quad i = 1, \ldots, n.
\end{align*}
(20)

The comparison of (19) and (20) shows that we have the isomorphism
\[
\mu : \mathfrak{f}[a_1, \ldots, a_n] \otimes T^{-1}W_n \rightarrow S_x^{-1}\text{Diff}_\hbar(n)
\]
given on our generating sets $\mathfrak{B}_D$ and $\mathfrak{B}_W$ by
\[
\mu : X^i \mapsto x^i \psi_i, \quad H_i \mapsto \tilde{h}_i, \quad a_i \mapsto c_i, \quad i = 1, \ldots, n.
\]
(21)

The proof is completed.

We shall now rewrite the formulas for the isomorphism $\mu$ in terms of the original generators of the rings $S_x^{-1}\text{Diff}_\hbar(n)$ and $\mathfrak{f}[a_1, \ldots, a_n] \otimes T^{-1}W_n$.

**Lemma 7.** We have
\[
\mu : X^i \mapsto x^i \psi_i, \quad D_i \mapsto (\psi_i)^{-1} \tilde{h}_i (x^i)\psi_i, \quad a_i \mapsto c_i, \quad i = 1, \ldots, n.
\]
(22)

and
\[
\mu^{-1} : \tilde{h}_i \mapsto H_i, \quad x^i \mapsto X^i \frac{1}{\psi_i}, \quad \partial_i \mapsto \Upsilon_i (X^i)\psi_i, \quad i = 1, \ldots, n.
\]
(23)

**Proof.** We shall comment only on the last formula in (23). Lemma 4 part (ii) implies that $\mu^{-1}(\chi_i \Gamma_i) = \Upsilon_i$ and the formula for $\mu^{-1}(\partial_i)$ follows since $\partial_i = \Gamma_i (x^i)^{-1}$.

\section{Comments}

We shall now establish several corollaries of our construction.

1. We can now give the description of the center of the ring $\text{Diff}_\hbar(n)$.

**Lemma 8.** The center of the ring $\text{Diff}_\hbar(n)$ is formed by polynomials in the elements $\{c_i\}_{i=1}^n$.

**Proof.** This is a direct consequence of the defining relations (19) for the generating set $\mathfrak{B}_D$. Indeed, any central element $\zeta$ must have $\hbar$-weight zero, so it belongs to a subring generated by $c_i$ and $\tilde{h}_i$, $i = 1, \ldots, n$. Interpret $\zeta$ as a rational function in $\tilde{h}_i$, $i = 1, \ldots, n$. Since $\zeta$ commutes with $x^{\alpha_i}$, $i = 1, \ldots, n$, this rational function is periodic, with period 1, with respect to any $\tilde{h}_i$, $i = 1, \ldots, n$. Therefore $\zeta$ belongs to the subring generated by $c_i$, $i = 1, \ldots, n$, as stated.

Another proof consists in using the isomorphism $\mu$ and the triviality of the center of the Weyl algebra.
2. The symmetric group $S_n$ acts by automorphisms on the algebra $W_n$,\[
\pi(X^j) = X^{\pi(j)}, \quad \pi(D_j) = D_{\pi(j)} \quad \text{for} \quad \pi \in S_n.
\]The isomorphism $\mu$ translates this action to the action of $S_n$ on the ring $S_{x}^{-1}\text{Diff}_h(n)$. It turns out that the subring $\text{Diff}_h(n)$ is preserved by this action. We present the formulas for the action of the generators $s_i$ of $S_n$.

\[
s_i(x^i) = -x^{i+1}h_{i,i+1}, \quad s_i(x^{i+1}) = x^i \frac{1}{h_{i,i+1}}, \quad s_i(x^j) = x^j \quad \text{for} \quad j \neq i, i + 1,
\]
\[
s_i(\delta_i) = -\frac{1}{h_{i,i+1}}\delta_{i+1}, \quad s_i(\delta_{i+1}) = h_{i,i+1}\delta_i, \quad s_i(\delta_j) = \delta_j \quad \text{for} \quad j \neq i, i + 1, \quad (24)
\]
\[
s_i(h_j) = \tilde{h}_{s_{i}(j)}.
\]

3a. For the R-matrix description of the diagonal reduction algebra $\mathcal{D}(\mathfrak{gl}_n)$ in [KO5] we used the ring $\text{Diff}_h(n, N)$ formed by $N$ copies of the ring $\text{Diff}_h(n)$. We do not know an analogue of the isomorphism $\mu$ for the ring $\text{Diff}_h(n, N)$. However a straightforward analogue of the formulas (24) provides an action of $S_n$ by automorphisms on the ring $\text{Diff}_h(n, N)$.

We recall that the ring $\text{Diff}_h(n, N)$ is a $\tilde{\mathbb{U}}(\mathfrak{h})$-bimodule with the generators $x^{j,\alpha}$ and $\bar{\partial}_{j,\alpha}$, $j = 1, \ldots, n, \alpha = 1, \ldots, N$. The ring $\text{Diff}_h(n, N)$ is free as a one-sided $\tilde{\mathbb{U}}(\mathfrak{h})$-module; the left and right $\tilde{\mathbb{U}}(\mathfrak{h})$-module structures are related by\[
\tilde{h}_i x^{j,\alpha} = x^{j,\alpha}(\tilde{h}_i + \delta_i^j), \quad \tilde{h}_i \partial_{j,\alpha} = \partial_{j,\alpha}(\tilde{h}_i - \delta_i^j). \quad (25)
\]
The defining relations for the generators $x^{j,\alpha}$ and $\bar{\partial}_{j,\alpha}$, $j = 1, \ldots, n, \alpha = 1, \ldots, N$, read
\[
x^{i,\alpha} x^{j,\beta} = \sum_{k,l} \hat{R}^{ij}_{kl} x^{k,\beta} x^{l,\alpha}, \quad \partial_{i,\alpha} \partial_{j,\beta} = \sum_{k,l} \hat{R}^{ik}_{jl} \partial_{k,\beta} \partial_{l,\alpha}, \quad x^{i,\alpha} \partial_{j,\beta} = \sum_{k,l} \hat{R}^{ik}_{jl} \epsilon \partial_{k,\beta} x^{l,\alpha} - \delta_{\beta} \delta_{i}^j, \quad (26)
\]
or, in components,
\[
x^{i,\alpha} x^{j,\beta} = \frac{1}{h_{ij}} x^{i,\beta} x^{j,\alpha} + \frac{\hat{h}_{ij}}{h_{ij}^2} x^{i,\beta} x^{j,\alpha}, \quad x^{i,\alpha} x^{j,\beta} = -\frac{1}{h_{ij}} x^{j,\beta} x^{i,\alpha} + x^{i,\beta} x^{j,\alpha}, \quad 1 \leq i < j \leq n, \quad (27)
\]
\[
\partial_{i,\alpha} \partial_{j,\beta} = -\frac{1}{h_{ij}} \partial_{i,\beta} \partial_{j,\alpha} + \frac{\hat{h}_{ij}}{h_{ij}^2} \partial_{i,\beta} \partial_{i,\alpha}, \quad \partial_{j,\alpha} \partial_{i,\beta} = \frac{1}{h_{ij}} \partial_{j,\beta} \partial_{i,\alpha} + \partial_{i,\beta} \partial_{j,\alpha}, \quad 1 \leq i < j \leq n, \quad (28)
\]
\[
x^{i,\alpha} \partial_{j,\beta} = \bar{\partial}_{j,\beta} x^{i,\alpha}, \quad x^{i,\alpha} \partial_{i,\beta} = \bar{\partial}_{i,\beta} x^{i,\alpha}, \quad 1 \leq i < j \leq n, \quad (29)
\]
\[
x^{i,\alpha} \partial_{i,\beta} = \sum_{k=1}^{n} \frac{1}{1 - h_{ik}} \bar{\partial}_{k,\beta} x^{k,\alpha} - \delta_{\beta} \delta_{i}^\alpha, \quad 1 \leq i \leq n. \quad (30)
\]
Lemma 9. The maps $s_i, i = 1, \ldots, n - 1$, defined on the generators of $\text{Diff}_h(n, N)$ by

$$
\begin{align*}
    s_i(x^i,\alpha) &= -x^{i+1,\alpha} \tilde{h}_{i,i+1}^{-1} + x^{i,\alpha} \frac{1}{\tilde{h}_{i,i+1}}, \\
    s_i(\bar{\partial}_i,\alpha) &= -\tilde{\partial}_{i+1,\alpha}, \\
    s_i(\tilde{h}_j) &= \tilde{h}_{s_i(j)},
\end{align*}
$$

extend to automorphisms of the ring $\text{Diff}_h(n, N)$. Moreover, these automorphisms satisfy the Artin relations and therefore give the action of the symmetric group $S_n$ by automorphisms.

Proof. After the formulas (31) are written down, the verification is a direct calculation. \qed

3b. The operators $s'_i := \epsilon s_i \epsilon$, where $\epsilon$ is the anti-automorphism (11), generate the action of the symmetric group $S_n$ by automorphisms as well. The action of the automorphism $s'_i$, $i = 1, \ldots, n - 1$, involves only the element $\tilde{h}_{i,i+1}$ (as the action of the automorphism $s_i$) and is given by

$$
\begin{align*}
    s'_i(x^i,\alpha) &= -x^{i+1,\alpha} \tilde{h}_{i,i+1}^{-1} + x^{i,\alpha} \frac{1}{\tilde{h}_{i,i+1}}, \\
    s'_i(\bar{\partial}_i,\alpha) &= -\tilde{\partial}_{i+1,\alpha}, \\
    s'_i(\tilde{h}_j) &= \tilde{h}_{s'_i(j)},
\end{align*}
$$

4. The diagonal reduction algebra $\mathcal{D}(\mathfrak{gl}_n)$ is a $\tilde{U}(h)$-bimodule with the generators $L^j_i$, $i, j = 1, \ldots, n$. The defining relations of $\mathcal{D}(\mathfrak{gl}_n)$ are given by the reflection equation, see [KO5]

$$
\hat{R}_{12} L_1 \hat{R}_{12} L_1 - L_1 \hat{R}_{12} L_1 \hat{R}_{12} = \hat{R}_{12} L_1 - L_1 \hat{R}_{12},
$$

where $L = \{L^j_i\}_{i,j=1}^n$ is the matrix of generators (we refer to [C, S, RS, KS, IO, IOP, IMO1, IMO2] for various aspects and applications of the reflection equation).

For each $N$ there is a homomorphism ([KO5], Section 4.1)

$$
\tau_N: \mathcal{D}(\mathfrak{gl}_n) \rightarrow \text{Diff}_h(n, N) \text{ defined by } \tau_N(L^j_i) = \sum_{\alpha} x^{i,\alpha} \bar{\partial}_{i,\alpha}.
$$

Moreover $\tau_N$ is an embedding for $N \geq n$.

The formulas (31) show that the image of $\tau_N$ is preserved by the automorphisms $s_i$.

The element $s_i(\tau_N(L^j_k))$ can be written by the same formula for all $N$. Since $\tau_N$ is injective for $N \geq n$ we conclude that the formulas (31) induce the action of the symmetric group $S_n$ on the diagonal reduction algebra $\mathcal{D}(\mathfrak{gl}_n)$ by automorphisms.
The resulting formulas for the action of the automorphisms \( s_i, i = 1, \ldots, n - 1 \), on the generators \( L_j^i, j, k = 1, \ldots, n \), read

\[
\begin{align*}
s_i(L_j^i) &= -L_j^{i+1} h_{i,i+1}, \\
s_i(L_j^j) &= \frac{1}{h_{i,i+1}} L_j^i, \\
s_i(L_j^k) &= L_j^k, \\
s_i(L_{i+1}^i) &= -L_j^i, \\
s_i(L_{i+1}^i) &= -L_{i+1}^i, \\
s_i(L_{i+1}^i) &= L_{i+1}^i.
\end{align*}
\]

5. The isomorphism \( \mu \) can be also used to translate the action (10) of the braid group by Zhelobenko operators to the action of the braid group by automorphisms on the ring \( U(a_1, \ldots, a_n) \otimes T^{-1} W_n \). It turns out that this action preserves the subring \( T^{-1} W_n \). Moreover, let \( T_0 \) be the multiplicative set of \( T \) generated by \( X^j D_j - X^k D_k + \ell, 1 \leq j < k \leq n, \ell \in \mathbb{Z} \). Then the action of the operators \( \tilde{q}_i, i = 1, \ldots, n - 1 \), preserves the subring \( T_0^{-1} W_n \). We present the formulas for the action of the operators \( \tilde{q}_i, i = 1, \ldots, n - 1 \):

\[
\begin{align*}
\tilde{q}_i(X^i) &= \frac{1}{H_{i,i+1}} X^{i+1}, \\
\tilde{q}_i(X^{i+1}) &= X^i H_{i,i+1}, \\
\tilde{q}_i(D_i) &= D_{i+1} H_{i,i+1}, \\
\tilde{q}_i(D_{i+1}) &= \frac{1}{H_{i,i+1}} D_i, \\
\tilde{q}_i(D_j) &= D_j \quad \text{for } j \neq i, i + 1.
\end{align*}
\]

6. The isomorphism (23) allows to construct a \( 2n \)-parametric family of \( \text{Diff}_n(n) \)-modules different from the highest weight representations. Let \( \vec{\gamma} := \{\gamma_1, \ldots, \gamma_n\} \) be a sequence of length \( n \) of complex numbers such that \( \gamma_i - \gamma_j \notin \mathbb{Z} \) for all \( i, j = 1, \ldots, n, i \neq j \). Let \( V_{\vec{\gamma}} \) be the vector space with the basis

\[
v_j := (X^1)^{j_1+\gamma_1} (X^2)^{j_2+\gamma_2} \ldots (X^n)^{j_n+\gamma_n}, \quad \text{where } \vec{j} := \{j_1, \ldots, j_n\}, \quad j_1, \ldots, j_n \in \mathbb{Z}.
\]

Under the conditions on \( \vec{\gamma}, V_{\vec{\gamma}} \) is naturally a \( T^{-1} W_n \)-module. Define the action of the elements \( a_k \) on the space \( V_{\vec{\gamma}} \) by \( a_k : v_j \mapsto A_k v_j \) where \( \vec{A} := \{A_1, \ldots, A_n\} \) is another sequence of length \( n \) of complex numbers. Then \( V_{\vec{\gamma}} \) becomes an \( U[a_1, \ldots, a_n] \otimes T^{-1} W_n \)-module and therefore \( \text{Diff}_n(n) \)-module which we denote by \( V_{\vec{\gamma},\vec{A}} \). The central operator \( c_k \) acts on \( V_{\vec{\gamma},\vec{A}} \) by scalar multiplication on \( A_k \).

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