NONREPETITIVE GAMES

JAROSŁAW GRZYTCZUK, JAKUB KOZIK, AND PIOTR MICEK

Abstract. Note. The results of this manuscript has been merged and published with another paper of the same authors: A new approach to nonrepetitive sequences.

A repetition of size \( h (h \geq 1) \) in a given sequence is a subsequence of consecutive terms of the form: \( xx = x_1 \ldots x_h x_1 \ldots x_h \). A sequence is nonrepetitive if it does not contain a repetition of any size. The remarkable construction of Thue asserts that 3 different symbols are enough to build an arbitrarily long nonrepetitive sequence. We consider game-theoretic versions of results on nonrepetitive sequences. A nonrepetitive game is played by two players who pick, one by one, consecutive terms of a sequence over a given set of symbols. The first player tries to avoid repetitions, while the second player, in contrast, wants to create them. Of course, by simple imitation, the second player can force lots of repetitions of size 1. However, as proved by Pegden [5], there is a strategy for the first player to build an arbitrarily long sequence over 37 symbols with no repetitions of size \( h > 1 \). Our techniques allow to reduce 37 to 6. Another game we consider is an erase-repetition game. Here, whenever a repetition occurs, the repeated block is immediately erased and the next player to move continues the play. We prove that there is a strategy for the first player to build an arbitrarily long nonrepetitive sequence over 8 symbols. Our approach is inspired by a new algorithmic proof of the Lovász Local Lemma due to Moser and Tardos [4] and previous work of Moser (his so called entropy compression argument).

1. Introduction

A repetition of size \( h (h \geq 1) \) in a sequence is a subsequence of consecutive terms of the form: \( xx = x_1 \ldots x_h x_1 \ldots x_h \). A sequence is nonrepetitive if it does not contain a repetition of any size.

In 1906 Thue [6] (see also [1]) proved that there exist arbitrarily long nonrepetitive sequences over only 3 different symbols. The method discovered by Thue is constructive and uses substitutions over a given set of symbols. Recently [3] a completely different approach to creating long nonrepetitive sequences emerged. Consider the following naive procedure: generate consecutive terms of a sequence by choosing symbols at random (uniformly and independently) and every time a repetition occurs, erase the repeated block and continue. For instance, if the generated sequence is \( abc \), we must cancel the last two symbols, which brings us back to \( abc \). By a simple counting one can prove that with positive probability the length of a constructed sequence exceeds any finite bound, provided the number of symbols is at least 4. This is slightly weaker than Thue’s result, but the argument seems to be more flexible for adaptations to other settings. It is proved in [3] that for every \( n \geq 1 \) and every sequence of sets \( L_1, \ldots, L_n \), each of size 4, there is a nonrepetitive sequence \( s_1, \ldots, s_n \) where \( s_i \in L_i \). The analogous statement for...
lists of size 3 remains an exciting open problem. In this paper we make use of the above-mentioned approach to games involving nonrepetitive sequences. The nonrepetitive game over a symbol set $S$ is played by two players in the following way. The players collectively build a sequence choosing from $S$, one by one, consecutive terms of the sequence. The first player, Ann, is trying to avoid repetitions, while the second player, Ben, not necessarily cooperates. Of course, just by mimicking Ann’s moves Ben can force a lot of repetitions of size 1. It turns out however that for large enough $S$ he cannot force any larger repetition at all! Pegden [5], using his extension of the Lovász Local Lemma, proved that Ann has a strategy in the nonrepetitive game to build an arbitrarily long sequence with no repetition of size greater than 1 over symbol set of size at least 37 (no matter how perfidiously Ben is playing). In this paper we prove (Theorem 2) that Ann can do the same on every set of symbols of size at least 6. On the other hand, Ben can easily force nontrivial repetitions in a game on just 3 symbols. Thus, the minimum size of a set of symbols required to ensure Ann’s strategy is 4, 5 or 6.

The erase-repetition game over a set of symbols $S$ is also a two-player game between Ann and Ben, say. As before they build a sequence picking symbols alternately from $S$ and appending them to the end of the sequence built so far. But this time whenever a repetition occurs the repeated block is immediately erased and the next player continues extending the remaining prefix of the sequence. We prove (Theorem 1) that there is a strategy for Ann in this game to build an arbitrarily long nonrepetitive sequence over at least 8 symbols.

The proof of the bound for the erase-repetition game is simpler, therefore it is presented first.

2. Preliminaries

We make some use of generating functions theory. We consider only algebraic functions. A generating functions $t(z) = \sum T_n z^n$ with positive radius of convergence is algebraic if there exists a nonconstant polynomial $P(z, t) \in \mathbb{C}[z, t]$ (defining polynomial) such that $P(z, t(z))$ is constantly zero within the disc of convergence of $t(z)$. It is a well known fact that, if the radius of convergence of $\sum T_n z^n$ is strictly greater than $\alpha$, then $T_n = o(\alpha^{-n})$. The following observation is fundamental in analysis of algebraic generating functions, the thorough study of which can be found in [2] (chapter VII.7).

Observation. Let $t(z) = \sum T_n z^n$ be a nonpolynomial algebraic generating function with defining polynomial $P(z, t)$. Then the radius of convergence of $t(z)$ is one of the roots of discriminant of $P(z, t)$ with respect to variable $t$ (i.e. the resultant of $P(z, t)$ and $\partial_t P(z, t)$ with respect to $t$).

The coefficients of the functions we use are nonnegative integers. In such cases it is known that the radius of convergence is not greater than 1, whenever a function has infinite number of nonzero coefficients. In order to bound the growth of the sequence of coefficients of such a function, we calculate discriminant of its defining polynomial $P(z, t)$ with respect to variable $t$, and look for its positive real root in the interval $(0, 1)$. If there is only one such root, it must be the radius of convergence of the function.

3. The erase-repetition game

Theorem 1. In the erase-repetition game over a symbol set of size 8, there exists a strategy for Ann to build an arbitrarily long nonrepetitive sequence.

Proof. We fix $n$ and prove that Ann has a strategy to build a nonrepetitive sequence of size $n$. In fact, the strategy for Ann will be randomized and we will show that
for every strategy of Ben there is an evaluation of random experiments leading to the sequence of size \( n \) against that strategy. The fact that for every strategy of Ben there is a strategy for Ann to build a sequence of size \( n \) implies that Ann has a strategy to build such a sequence in general. Then, by a routine application of König’s Infinity Lemma, we get the thesis.

Let \( C \) be the size of a symbol set. The argument to be presented turns out to work for \( C \geq 8 \). The strategy for Ann is the following: choose a random element distinct from the last three symbols in the sequence constructed so far. In this setting, Ann does not generate repetitions of size 1, 2 and 3. Obviously, Ben can cause many repetitions of size 1 but repetitions of size 2 and 3 are not possible. Indeed, in order to get a repetition of the form ’abcabc’ the last three symbols must be generated by Ben. Consider Ann’s move just before Ben puts ’b’ in the repeated block. As she could not play preceding symbol ’a’ she must have invoked a repetition. But all her repetitions are of size at least 4 and therefore the repeated block must have ended up with ’abca’. This would mean that she played ’a’ which is not possible as this symbol is not distinct from the last three in the current sequence.

Consider a repetition of size 4. Suppose, for a contradiction, that the size of a sequence after \( 2M \) moves is always (for any evaluation of Ann’s choices) less than \( n \). Ann generates exactly \( M \) elements. Let \( r_j (1 \leq j \leq M) \) be the \( j \)th symbol generated by Ann. Clearly, \( r_1, \ldots, r_M \) is a sequence of random variables with at least \((C - 3)^M\) possible evaluations. When we fix an evaluation of \( r_1, \ldots, r_M \) the course of the whole game is determined.

Let \( h_j (1 \leq j \leq 2M) \) be the length of the sequence generated after \( j \) moves (including possible erasure invoked by the \( j \)th move) and let \( d_1, \ldots, d_{2M} \) be the sequence of differences: \( d_1 = 1, d_j = h_j - h_{j-1} \) for \( 2 \leq j \leq 2M \). Note that \( d_j = 1 \) means that there is no erasure after \( j \)th move and \( d_j < 1 \) indicates that repeated block of size \( |d_j| + 1 \) was removed. A pair \((D, S)\) is a game log if there is an evaluation of \( r_1, \ldots, r_M \) such that \( D \) is the sequence of differences and \( S \) is the final sequence produced after \( 2M \) moves. A pair \((D, S)\) is a reduced game log if it is a log but with all zeros in \( D \) erased. Note that any sequence of differences \( D = (d_1, \ldots, d_m) \) in a reduced log satisfies:

\[
\begin{align*}
(i) & \quad m \leq 2M, \\
(ii) & \quad d_j \in \{1, -3, -4, -5, \ldots\}, \text{ for all } 1 \leq j \leq m, \\
(iii) & \quad \sum_{j=1}^{k} d_j \geq 1, \text{ for all } 1 \leq k \leq m.
\end{align*}
\]

Claim. Every reduced log corresponds to a unique evaluation of \( r_1, \ldots, r_M \).

Proof. Given a reduced log \(((d_1, \ldots, d_m), S_m)\) with \( S_m = (s_1, \ldots, s_k) \), we decode all random choices taken by Ann in two steps. First we reconstruct the sequence \( x_1, \ldots, x_m \) of all symbols introduced in the game except those (of Ben) generating repetitions of size 1. The introduced symbols generating repetitions of size 1 are called bad, other symbols are good. The move is good (bad) if a good (bad) symbol is introduced. The number of good moves played is exactly the size of the difference sequence in the reduced log, namely \( m \). Note that \( S_m \) is the sequence formed after the \( m \)th good move (even if a bad move is played afterwards, it does not change the sequence).

We reconstruct the sequence of good symbols backwards, i.e., we first decode \( x_m \), which is the last good symbol introduced, and the sequence \( S_{m-1} \) constructed after \( m - 1 \) good moves. Then, by simple iteration, we extract all the remaining good symbols \( x_{m-1}, \ldots, x_1 \).
If \( d_m = 1 \), then the \( m \)th good symbol introduced did not invoke a repetition. Thus, the last good symbol introduced is the last symbol of the final sequence, i.e. \( x_m = s_l \) and \( S_{m-1} = (s_1, \ldots, s_{l-1}) \).

If \( d_m \leq 0 \), then some symbols were erased after the \( m \)th good move. But since we know the size of the repetition, namely \( h = |d_m| + 1 \), and only one half of it was erased, we can read and copy the first part of the repeated block to restore \( S_{m-1} = (s_1, \ldots, s_l, s_{l-h+1}, \ldots, s_{l-1}) \) and \( x_m = s_l \).

Once we get all \( x_1, \ldots, x_m \), we read the sequence from the beginning and check whether the symbols agree with the strategy of Ben we fixed. The difference appears only where Ben introduces a bad symbol. There we extend the sequence with this symbol and continue. This way we reconstruct the sequence of all symbols introduced in the game and clearly every second symbol is chosen by Ann. \( \square \)

By a game walk we mean a sequence \( d_1, \ldots, d_m \) satisfying (ii), (iii) and additionally \( \sum_{j=1}^{M} d_j = 1 \). Let \( T_m \) be the number of gamewalks of length \( m \). By our assumption that Ann never wins, every feasible sequence of differences in a reduced log sums up to a number smaller than \( n \). The number of sequences satisfying (ii), (iii) but with a total sum \( k \) (for fixed \( k \geq 1 \)) is \( O(T_m) \). Finally, all feasible sequences of differences are of size at most \( 2M \). All this yields that the number of feasible difference sequences in a reduced log is at most \( 2M \cdot n \cdot O(T_{2M}) \). For a given feasible sequence of differences \( D \), the number of final sequences which can occur with \( D \) in a reduced log is bounded by \( C^n \). Thus, the number of reduced logs is bounded by

\[
2M \cdot n \cdot O(T_{2M}) \cdot C^n.
\]

We turn to the approximation of \( T_{2M} \). Every game walk \( d_1, \ldots, d_m \) is either a single step up (i.e., \( m = 1, d_1 = 1 \)), or it can be uniquely decomposed into \( |d_m| + 1 \) subsequent game walks of total length \( m - 1 \). The \( j \)th component of the decomposition is the substring between the last visit of height \( j - 1 \) and the last visit of height \( j \) (i.e. between the last \( k \) such that \( \sum_{i=1}^{k} d_i = j-1 \) and last \( l \) such that \( \sum_{i=1}^{l} d_i = j \)). This description together with the fact that if \( m < 1 \), then \( |d_m| + 1 \geq 4 \), certify that the generating function \( t(z) = \sum_{n \in N} T_n z^n \) satisfies the following functional equation:

\[
t(z) = z + z(t(z)^4 + t(z)^5 + \ldots),
\]

where the right hand side is \( z + \frac{t(z)^4}{1-t(z)} \). From this equation we extract a polynomial

\[
P(z, t) = zt^4 + t^2 - (1 + z)t + z
\]

that defines \( t(z) \). In the standard way we calculate the discriminant polynomial obtaining:

\[
-4 - 19z + 32z^2 - 2z^3 + 36z^4 + 229z^5.
\]

This polynomial has only one positive real root equal to 0.457..., which is greater than \( \frac{5}{2} \). Therefore \( T_{2n} = o(5^n) \).

By the claim the number of realizations is exactly the number of reduced logs. That gives

\[
(C - 3)^M \leq 2M \cdot n \cdot O(T_{2M}) \cdot C^n = o(5^M).
\]

Thus for \( C \geq 8 \) and sufficiently large \( M \) we obtain a contradiction. \( \square \)

4. The nonrepetitive game

**Theorem 2.** In the nonrepetitive game over a symbol set of size 6, there is a strategy for Ann to build an arbitrarily long sequence with no repetitions of size greater than 1.
Proof. We fix $n$ and prove that Ann has a strategy to build a sequence of size $n$ without repetitions of size greater than 1. As before we consider randomized Ann’s strategy and we show that for every strategy of Ben there is an evaluation of random experiments leading to the generation of a nonrepetitive sequence of size $n$. This means that Ben cannot have winning strategy. Therefore, there exists a winning strategy for Ann. Then, again, by a routine application of König’s lemma we get the thesis.

In this proof, by a repetition we mean a repetition of size greater than 1.

Let $s_1, \ldots, s_{m-1}$ be the sequence already generated in the game and suppose that it is Ann’s turn ($m$ is odd). The strategy for Ann goes as follows: choose any symbol at random, but

(i) exclude $s_{m-2}$,
(ii) if $s_{m-1} = s_{m-4}$, then exclude $s_{m-3}$,
(iii) if only one symbol has been excluded in (i) and (ii), then exclude $s_{m-4}$.

This strategy explicitly ensures that no repetitions of size 2 and 3 occur in the game. It turns out that also repetitions of size 4 are avoided. Suppose for a contradiction that at some point in the game a sequence with a suffix of the form $x_1x_2x_3x_4x_1x_2x_3x_4$ is produced. Suppose also that Ann introduces the last symbol, namely $x_4$. As she did not prevent a repetition of size 4, the rule (iii) of the strategy did not exclude a symbol and therefore rule (ii) must have been invoked. In particular, $x_3 = x_4$. But this means that in the previous move of Ann (when she introduced $x_2$ in the repeated block) the symbols excluded by (i) and (ii) were the same, so, rule (iii) must have been applied. But that rule excludes $x_2$, a contradiction. Analogous reasoning works for the case when Ben finishes a repetition of size 4.

Fix a strategy for Ben. We simulate the play between randomized Ann and this fixed strategy, and whenever a repetition of size $h$ occurs in the $m$th move (of the real game), we backtrack to the move $m - h + 1$. This means that we remove the whole repeated segment and continue the simulation starting from the move $m - h + 1$ again (with independent random experiments).

A search sequence is the sequence of consecutive symbols chosen by players in the simulation. Note that it is not possible for Ben to introduce three symbols in a row in the search sequence. Indeed, if he introduces two symbols in a row, then there must have been a repetition (of even size) after the first symbol. Thus, the second one is the same as the symbol just erased at this position (as Ben’s strategy is fixed in the simulation). This means that the second symbol could not generate repetition and therefore Ann is next to play in the simulation.

The weight of a search sequence is the number of symbols chosen by Ann in the sequence. Fix $M$ large enough. We are going to show that there is a scenario of the first $M$ random experiments (first $M$ moves of Ann) leading the simulation to an outcome sequence of size $n$. This will prove that Ann has a strategy to build a sequence of size $n$ against the fixed strategy of Ben. For a contradiction we suppose that all outcome sequences generated after $M$ moves of Ann in the simulation are of length less than $n$ for all possible evaluations of random experiments.

Clearly, a search sequence of weight $M$ is uniquely determined by the sequence of $M$ Ann’s choices. Let $r_1, \ldots, r_M$ be the symbols chosen by Ann. As she always chooses one symbol out of at least $C - 2$ symbols, the sequence $r_1, \ldots, r_M$ has at least $(C - 2)^M$ possible evaluations. A search sequence induced by an evaluation of $r_1, \ldots, r_M$ is called a realization of this evaluation.

Let $h_j$ be the length of the current sequence just before the $j$th step (move) of the simulation. The sequence ($h_j$) is called a height sequence. If Ann introduces a
symbol in the $j$th step, then her next move is in step $k \in \{j + 1, j + 2, j + 3\}$. There are only few possible extensions of the height sequence from $h_j$ up to $h_k$:

0. Ann makes no repetition in the $j$th step and Ben makes no repetition in the $(j + 1)$th step. In this case $k = j + 2$ and $h_{j+1} = h_j + 1$, $h_{j+2} = h_j + 2$.

1. Ann makes a repetition of odd size, at least 5, in the $j$th step and therefore she plays again in the $(j + 1)$th step. Here $k = j + 1$ and $h_k \leq h_j - 4$.

2. Ann makes a repetition of even size, at least 6, in the $j$th step and Ben plays no repetition in the $(j + 1)$th step. Here $k = j + 2$ and $h_k \leq h_j - 4$.

3. Ann makes no repetition in the $j$th step and Ben produces a repetition of even size, at least 6, in the $(j + 1)$th step. Here again $k = j + 2$ and $h_k \leq h_j - 4$.

4. Ann makes no repetition in the $j$th step. Ben makes a repetition of odd size, at least 5, in the $(j + 1)$th step. Then he plays no repetition in the $(j + 2)$th step. Here $k = j + 3$ and $h_k \leq h_j - 2$.

We want to get rid of some redundancy in the height sequence. More precisely, we encode the sequence of heights into its subsequence consisting of $h_j$’s corresponding to Ann’s moves with a little extra information. Let $h_j, h_k$ be again the heights of the current sequence right before any two consecutive moves of Ann. Note that

* If $h_k > h_j$, then the sequence of heights between $h_j$ and $h_k$ is of type (0).
* If $h_k = h_j - 2$, then the sequence of heights between $h_j$ and $h_k$ is of type (4).
* If $h_k \leq h_j - 4$, then the sequence of heights between $h_j$ and $h_k$ is of type (1), (2), (3) or (4).

Therefore, in order to record the whole height sequence it is enough to remember the subsequence $h_j', \ldots, h_M'$ of heights corresponding to Ann’s moves and additionally, if $h_{j+1}' \leq h_j' - 4$, to record type$(h_j', h_{j+1}') \in \{1, 2, 3, 4\}$, which is the type of the original height sequence between symbols corresponding to $h_j'$ and $h_{j+1}'$.

Finally, note that all the $h_j'$’s are even (as the current sequence before Ann’s move contains an even number of symbols). The reduced sequence of differences is:

\[ d_1 = 1, \quad d_{j+1} = (h_{j+1}' - h_j')/2 \text{ for } 1 \leq j < M, \] and the type function type$((j+1)) = \text{type}(h_j, h_{j+1})$, provided the latter is defined. Note that

(i) $d_j \leq 1$,

(ii) $\sum_{j=1}^k d_j \geq 1$, for all $1 \leq k \leq M$,

(iii) type$(d_j)$ is defined if and only if $d_j \leq -2$.

A pair $((D, \text{type}), S)$ is a search log if there is an evaluation of $r_1, \ldots, r_M$ such that $D$ is the reduced sequence of differences in the realization of $r_1, \ldots, r_M$, type is the type function of $D$, and $S$ is the final sequence produced after $M$ steps of Ann in this realization of the search procedure.

**Claim.** Every search log corresponds to a unique evaluation of $r_1, \ldots, r_M$.

**Proof.** Given a search log $((D, \text{type}_D), S)$ where $S = (s_1, \ldots, s_t)$ we decode the evaluation of $r_1, \ldots, r_M$ in a few steps. First we extract the height sequence $h_1, \ldots, h_m$ from $(D, \text{type}_D)$ and put additionally $h_{m+1} = |S|$. Now, we are going to describe how to reconstruct the sequence $x_1, \ldots, x_m$ of all symbols introduced in the simulation. This is done in backward direction, i.e., we decode first $x_m$ and the sequence $S_{m+1}$ constructed after $m - 1$ steps of the simulation. Then by simple iteration we extract all the remaining symbols $x_{m-1}, \ldots, x_1$.

If $h_{m+1} - h_m = 1$, then the introduction of $x_m$ did not invoke a repetition. Thus, $x_m$ is the last symbol in the final sequence $S$, i.e., $x_m = s_t$ and $S_{m+1} = (s_1, \ldots, s_t-1)$.

If $h_{m+1} - h_m \leq 0$, then some symbols were erased after the introduction of $x_m$. But we know the size of the repetition, namely $h = |h_{m+1} - h_m| + 1$, and
since only one half of it was erased, we can copy the appropriate block to restore $s_{m-1} = (s_1, \ldots, s_i, s_{i-1}, \ldots, s_{i-1})$ and $x_m = s_i$.

Once we get all $x_1, \ldots, x_m$ we read the sequence from the beginning and track the current sequence in the simulation. Every time the current sequence is of even length the next symbol is introduced by Ann.

By a typed search walk we mean a pair $((d_1, \ldots, d_M), \text{type})$ satisfying (i), (ii), (iii) and additionally $\sum_{j=1}^{M} d_j = 1$. Let $T_M$ be the number of search walks of length $M$ (i.e., $D$ is of length $M$). By our assumption that Ann never wins, every feasible sequence of differences in a typed search walk sums up to less than $n$. The number of typed search walks of length $M$ satisfying (i), (ii), (iii) with total sum $k$ (fixed $k \geq 1$) is $O(T_M)$. All this implies that the number of feasible typed search walks is $n \cdot O(T_M)$. For a given feasible typed search walk $(D, \text{type})$ the number of final sequences which can occur with $(D, \text{type})$ in a search log is bounded by $C^n$. Thus, the number of reduced logs is bounded by

$$n \cdot O(T_M) \cdot C^n.$$  

We turn to the approximation of $T_m$. Every search walk $((d_1, \ldots, d_m), \text{type})$ is either a single step up (i.e., $m = 1, d_1 = 1$), or it can be uniquely decomposed into $|d_m| + 1$ subsequent search walks of total length $m - 1$ and additionally into the type of $d_m$ if it is defined (i.e., if $d_m \leq -2$). This decomposition gives the following functional equation for the generating function $t(z)$:

$$t(z) = z + zt^2(z) + 4zt^3(z) + t^4(z) + t^5(z) + \ldots,$$

where $z$ stands for a trivial one-step-up walk, $zt^2(z)$ stands for the case $d_m = -1$ in which $d_m$ has no type, and the last term stands for the case $d_m \leq -2$. The right hand side of the equation is in fact equal to $z + zt^2(z) + 4zt^3(z) + \frac{t^4(z)}{4-t(z)}$. From that form we derive the defining polynomial for $t(z)$:

$$P(z, t) = -t + t^2 + z - tz + t^2z + 3t^3z.$$  

In the standard way we calculate the discriminant polynomial obtaining:

$$-1 - 12z + 24z^2 + 80z^3 + 288z^4.$$  

The radius of convergence of $t(z)$ is one of the roots of the above polynomial. This polynomial has only one positive real root in 0.2537... The root is greater than 1/4, therefore $T_M = O(4^M)$.

By the claim, the number of realizations is exactly the number of search logs. That gives

$$(C - 2)^M \leq n \cdot O(T_M) \cdot C^n = o(4^M).$$

Therefore for $C \geq 6$ and sufficiently large $M$ we obtain a contradiction.

References

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