COLOR OR COVER

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Abstract. Some properties of the triangulations of the 2-sphere and of the torus are proved. The proofs use the “coloring monodromy” and, more generally, the developing map of a spherical cone-surface onto the sphere. A relation to the Belyi surfaces is discussed.

1. A theorem about two odd vertices

Theorem 1. If a triangulation of the 2-sphere has exactly two vertices of odd degree, then these vertices are not adjacent.

There are triangulations with two non-adjacent vertices of odd degree, or with four pairwise adjacent odd vertices, see Figure 1. But nothing can bring together two single vertices of odd degree on the sphere.

Figure 1. Some possible configurations of vertices of odd degree.

By contrary, the torus and the projective plane allow triangulations of this sort, see Figure 2.

Figure 2. Triangulations of the projective plane and of the torus with two odd adjacent vertices.

More generally, degrees modulo \( k \) can be considered, see Theorem \( 

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2. Even triangulations of polygons

Assume we succeeded to find a triangulation of the sphere with exactly two odd degree vertices, which are neighbors. Remove the edge joining the odd vertices, and remove the two triangles on both sides of it. We obtain a triangulation of a sphere with a quadrangular hole, where all vertices, interior as well as those on the boundary, are of even degree. By stretching it to the plane, we obtain a triangulation of the square.

Thus, in order to prove Theorem 1, it suffices to prove that there is no triangulation of the square with all vertices of even degree. Why not to study a more general problem: for which \( n \) can an \( n \)-gon be triangulated with all vertices of even degrees? One tries in vain to do this with the quadrilateral and the pentagon, finds an even triangulation for a hexagon, and comes up with a construction that produces an even triangulation of an \((n + 3)\)-gon from one for an \( n \)-gon, see Figure 3.

\[
\begin{array}{c}
\text{no} \quad \text{no} \\
\text{\hspace{0.5cm}} \quad \text{\hspace{0.5cm}} \\
\text{n} \rightarrow n + 3
\end{array}
\]

Figure 3. Even triangulations of \( n \)-gons.

**Theorem 2.** There is a triangulation of an \( n \)-gon (vertices in the interior allowed, vertices on the sides forbidden) with all vertices of even degree if and only if \( n \) is divisible by 3.

**Proof.** The vertices of an even triangulation of an \( n \)-gon can be colored in three colors so that no two vertices of the same color are joined by an edge; equivalently, in every triangle all three colors must be present. Indeed, a coloring of a triangle induces colorings of all adjacent triangles. Thus we have little choice but to color one of the triangles and extend this coloring along all paths. The extensions along different paths ending at the same triangle agree because of the assumption that the degrees of all interior vertices are even: moving a path over a vertex of even degree does not change the colors at its terminal triangle.

Now look at the colors assigned to the corners of the \( n \)-gon. Since, by assumption, boundary vertices also have even degrees, the colors on the boundary repeat cyclically \( 1 \rightarrow 2 \rightarrow 3 \rightarrow 1 \rightarrow \cdots \). It follows that \( n \) must be divisible by 3.

To triangulate \( n \)-gons for \( n \) divisible by 3, use the inductive construction shown on Figure 3.
3. The coloring monodromy

Let’s try to color in three colors the vertices of a triangulated surface $M$ so that neighbors have different colors. As in the proof of Theorem 2, start by coloring the vertices of an arbitrary triangle and extend the coloring along every sequence of triangles with consecutive triangles sharing an edge. In general, the coloring of the last triangle depends on the choice of a sequence. Two homotopic sequences induce the same coloring if the homotopy doesn’t pass through vertices of odd degree.

**Definition 3.** Let $\Sigma$ be a triangulation of a surface $M$, possibly with boundary, and let $a_1, \ldots, a_n$ be all interior vertices of odd degree. Pick a triangle $\Delta_0$ and fix a coloring of its vertices, that is a bijection between its vertex set and the set $\{1, 2, 3\}$. The group homomorphism

$$\pi_1(M \setminus \{a_1, \ldots, a_n\}, \Delta_0) \to \text{Sym}_3$$

that maps every path to the associated re-coloring of the vertices of $\Delta_0$ is called the coloring monodromy.

More generally, the coloring monodromy of a triangulated $d$-dimensional manifold is a homomorphism

$$\pi_1(M \setminus M_{odd}, \Delta_0) \to \text{Sym}_{d+1}$$

where $M_{odd}$ is the odd locus of the triangulation, see Section 5. The notion was introduced in [3] under the name group of projectivities. Clearly, a triangulation is vertex-colorable (or balanced) if and only if its coloring monodromy is trivial.

**Second proof of Theorem 1.** Assume we have a triangulation of the sphere with two adjacent odd degree vertices $a$ and $b$. Let $\Delta_0$ be a triangle containing $a$ and $b$; color the vertices $a$ and $b$ with the colors 1 and 2, respectively. Then the path going around $a$ exchanges the colors 2 and 3, and the path around $b$ exchanges 1 and 3.

It follows that the image of the coloring monodromy

$$\pi_1(S^2 \setminus \{a, b\}) \to \text{Sym}_3$$

is the whole group $\text{Sym}_3$. On the other hand, $\pi_1(S^2 \setminus \{a, b\}) \cong \mathbb{Z}$, and there is no epimorphism from the group $\mathbb{Z}$ onto $\text{Sym}_3$. This contradiction shows that a triangulation with the assumed properties doesn’t exist. □
4. The minimal vertex-colored branched cover

The coloring monodromy not only detects whether the triangulation is colorable, but also allows to associate with every non-colorable triangulation a colorable branched cover. Namely, assume that some closed path forces us to recolor the vertices of a triangle. Then, instead of changing the colors, start a new layer of triangles. Doing this for all paths produces a branched cover over the triangulated surface, ramified over the vertices of odd degree.

Definition 4. Let \( \Sigma \) be a triangulation of a surface \( M \). Consider the set of colored triangles of \( \Sigma \):

\[
\{(\Delta, \phi) \mid \Delta \text{ a triangle of } \Sigma, \, \phi: \text{Vert}(\Delta) \to \{1, 2, 3\} \text{ a bijection}\}
\]

Glue two adjacent triangles if their colorings agree on their common side. All connected components of the resulting simplicial complex are isomorphic to some complex \( \tilde{\Sigma} \), which is called the unfolding of \( \Sigma \).

The unfolding comes together with a canonical projection \( \tilde{\Sigma} \to \Sigma \). If all vertices of \( \Sigma \) are even, then \( \tilde{\Sigma} \to \Sigma \) is a finite-sheeted covering, non-trivial if and only if some homotopically non-trivial loop recolors the vertices.

Example 5. In the 7-vertex triangulation of the torus all vertices have degree 6. Generators of the fundamental group recolor the vertices of a triangle in a cyclic way, see Figure 5. Therefore the unfolding is a vertex-colorable triangulation on 21 vertices triply covering the 7-vertex triangulation.

Theorem 6. Every even triangulation of the torus is either vertex-colorable or can be cut along a simple closed curve so that to become vertex-colorable.

Proof. It suffices to find a primitive element of \( \mathbb{Z}^2 \cong \pi_1(T) \) with the trivial coloring monodromy. There is at least one among the elements \( e_1, e_2, e_1 \pm e_2 \), where \((e_1, e_2)\) is a basis of \( \mathbb{Z}^2 \).  \( \square \)
5. Higher dimensions

The unfolding map $\tilde{\Sigma} \to \Sigma$ is ramified around the odd subcomplex $\Sigma_{odd}$ of $\Sigma$. This is the union of all $(d-2)$-simplices of $\Sigma$ incident to an odd number of $d$-simplices.

While the unfolding of every triangulated surface is again a surface, this is no more true for manifolds of dimension $d \geq 3$. For example, the unfolding of the double tetrahedron is the suspension over the torus. For $\tilde{\Sigma}$ to be a manifold, $\Sigma_{odd}$ must be a submanifold of $\Sigma$. In particular, if $\dim \Sigma = 3$, then $\Sigma_{odd}$ must be a knot or a link. This case, with $\Sigma$ a triangulated 3-sphere, was studied in [1].

The following are some restrictions on the odd subcomplex.

**Theorem 7.**

1. For every pure simplicial complex $\Sigma$, the fundamental cycle of its odd subcomplex $\Sigma_{odd}$ represents a zero element in $H_{d-2}(\Sigma; \mathbb{Z}_2)$.
2. If $d \equiv 0$ or $3 \pmod{4}$, then $|\Sigma_{odd}|$ is even; if $d \equiv 1$ or $2 \pmod{4}$, then $|\Sigma_{odd}|$ has the same parity as $|\Sigma|$. Here $|\Sigma_{odd}|$ and $|\Sigma|$ denote the number of $(d-2)$- and $d$-faces in $\Sigma_{odd}$ and $\Sigma$, respectively.
3. Let $\tau \subset \Sigma_{odd}$, $\dim \tau = d-3$. Assume that there are only two odd faces $\sigma_1, \sigma_2 \supset \tau$. Then $\sigma_1$ and $\sigma_2$ are not contained in the same $(d-1)$-simplex.

**Proof.** The complex $\Sigma_{odd}$ is the $\mathbb{Z}_2$-boundary of the $(d-1)$-skeleton of $\Sigma$.

For the second part, count the incidences between $d$- and $(d-2)$-faces of $\Sigma$. Every $d$-simplex has $\frac{d(d+1)}{2}$ faces of dimension $d-2$. This number is even if and only if $d \equiv 0$ or $3 \pmod{4}$. Therefore

$$|\Sigma_{odd}| \equiv \sum_{\dim \sigma = d-2} i_\sigma \frac{d(d+1)}{2} |\Sigma| \equiv \begin{cases} 0 \pmod{2}, & \text{if } d \equiv 0 \text{ or } 3 \pmod{4} \\ |\Sigma| \pmod{2}, & \text{if } d \equiv 1 \text{ or } 2 \pmod{4} \end{cases}$$

The third part of the theorem follows by considering the link of $\tau$. It is a triangulated 2-sphere whose odd subcomplex consists of two points $\sigma_1 \setminus \tau$ and $\sigma_2 \setminus \tau$. By Theorem 1 these points are not adjacent, which means that $\sigma_1$ and $\sigma_2$ don’t lie in the same $(d-1)$-simplex. □

Every codimension 2 submanifold $N \subset M$ which is a boundary mod 2 can be realized as the odd subcomplex of some triangulation of $M$, see [1, Proposition 5.1.3].

6. Belyi surfaces

A *Belyi function* is a holomorphic map from a compact Riemann surface $M$ to $\mathbb{C}P^1$ ramified over three points $0, 1, \infty$. Subdivide $\mathbb{C}P^1$ in two triangles with vertices $0, 1, \infty$ and color one of the triangles white and the other black. This induces a triangulation of $M$ with the vertices colored in three colors $0, 1, \infty$, and the triangles colored white and black, so that adjacent triangles have different colors. See [3] for more.
Note that the existence of a vertex-coloring does not imply the existence of a face-coloring, and vice versa, see Figure 6.

![Figure 6. Vertex-colored but not face-colored projective plane; face-colored but not vertex-colored torus.](image)

**Theorem 8.**
1. A vertex-colorable triangulation is face-colorable if and only if the underlying surface is oriented.
2. An even triangulation of an orientable surface is face-colorable if and only if the image of the coloring monodromy is either trivial or is generated by a 3-cycle.

**Proof.** A vertex- and face-colored triangulation can be oriented by choosing the (123)-orientation for every white and the (132)-orientation for every black triangle. Vice versa, if a vertex-colored triangulation is oriented, then color white every triangle with orientation (123) and color black every triangle with orientation (132).

For the second part, color the faces of the unfolding \(\tilde{\Sigma}\). The assumption on the monodromy implies that faces of \(\tilde{\Sigma}\) that project to the same face of \(\Sigma\) have the same color. This produces a face-coloring of \(\Sigma\). \(\square\)

Similarly to Definition 4, one defines a minimal face- and vertex-colored branched cover of a given triangulation. For this, take the set of vertex- and face-colored triangles \(\Delta, \phi, \epsilon\), where \(\epsilon \in \{w, b\}\), and glue a pair of adjacent colored triangles along their common edge if their vertex colors coincide and the face colors are opposite.

7. **Tetrahedron, octahedron, icosahedron, and a generalization of the main theorem**

The branched covers point of view leads to a third proof of Theorem 1.

**Third proof of Theorem 1.** Assume we have a triangulation of the sphere with two adjacent odd degree vertices \(a\) and \(b\). Make every triangle spherical with side lengths \(\frac{2\pi}{3}\) (such a triangle covers a hemisphere, with vertices equally spaced on the equator). The total angles around the even degree vertices are multiples of \(2\pi\), those around \(a\) and \(b\) are odd multiples of \(\pi\). Cut the triangulation along the edge \(ab\) and develop it onto the sphere. All edges
will be mapped to the equator, all vertices to three equally spaced points on the equator, and every triangle to the north or the south hemisphere (according to whether it is white or black, see the previous section). The images of $a$ and $b$ are at the distance $\frac{2\pi}{3}$ from each other, therefore both sides of the cut are mapped to the unique geodesic arc joining them. It follows that the angles around $a$ and $b$ are multiples of $2\pi$, which is a contradiction. \[\square\]

![Figure 7. The doubly covered triangle as a triangulation of the sphere.](image)

Other fairly regular triangulations of the sphere lead to the following.

**Theorem 9.** Let $k$ be a positive integer. If a triangulation of the 2-sphere has exactly two vertices whose degrees are not divisible by $k$, then these vertices are not adjacent.

**Proof.** The case $k = 2$ is already proved. For $k \geq 6$ there are no triangulations with such vertex degrees at all, since the sum of all vertex degrees is $6n - 12$. For $k = 3$, 4, or 5 cut along the edge joining the exceptional vertices and develop onto the surface of the tetrahedron, octahedron, or the icosahedron. As in the third proof of Theorem 1, both copies of the edge $ab$ must be mapped to the same edge of the polyhedron, which contradicts to the degrees of $a$ and $b$ not being multiples of $k$. \[\square\]

In the above proof, instead of a combinatorial developing map onto the surface of a regular polyhedron one can speak of the geometric developing map onto the sphere, where every triangle is realized as a spherical one with the angles $\frac{2\pi}{k}$.

The next theorem generalizes the well-known fact that there is no convex polyhedron with 12 pentagonal faces and one hexagonal face.

**Theorem 10.** There is no triangulation of the sphere with the degrees of all vertices except one divisible by 5.

Again, instead of 5 one can take any number $k$; but for $k = 2, 3, 4$ the theorem is trivial because the sum of all vertex degrees is divisible by 6.

**Proof.** If all vertex degrees except one are divisible by 5, then ignoring this vertex at first we can construct a branched cover of the sphere over the icosahedron. But going around the exceptional vertex we see that its degree must also be divisible by 5. \[\square\]
8. Final remarks

In [2] it was shown that there is no triangulation of the torus with all vertices of degree 6 except two of degrees 5 and 7. The proof uses the geometric structure arising when all triangles are viewed as euclidean equilateral ones.

The developing map argument from the previous section can be replaced by a monodromy argument. If all vertices except for \(a\) and \(b\) have degree divisible by \(k\), then we obtain a group homomorphism

\[\pi_1(\mathbb{S}^2 \setminus \{a, b\}) \to \text{Sym}(P)\]

where \(\text{Sym}(P)\) is the symmetry group of the polyhedron \(P\), which is the tetrahedron, the octahedron, or the icosahedron for \(k = 3, 4, 5\) respectively. The homomorphism arises from “rolling” the polyhedron \(P\) over the triangles of the triangulation. Going around a vertex of degree divisible by \(k\) produces an identity transformation. Going around \(a\) or \(b\) corresponds to rotations of \(P\) around two different axes. These generate a non-cyclic (and non-commutative) group, which cannot be the image of \(\mathbb{Z} \cong \pi_1(\mathbb{S}^2 \setminus \{a, b\})\).

The case \(k = 3\) can be described as coloring vertices of a triangulation in 4 colors so that the neighbors and the vertices lying across an edge from each other have different colors. For \(k = 4\) one colors the vertices in 6 colors imitating the numbers on the dice: the colors at the vertices lying across an edge must add up to 6. For \(k = 5\) a special arrangement of 12 colors is used, which makes the argument too intricate.

Theorem 1 implies the non-planarity of every graph on \(n\) vertices with \(3n - 6\) edges that has only two vertices of odd degree, which are adjacent. However, the simplest such graphs are vertex-splittings of \(K_5\).

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