On the Chow Ring of the Stack of truncated Barsotti-Tate Groups

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Abstract

We determine the Chow ring of the stack of truncated displays and more generally the Chow ring of the stack of $G$-zips. We also investigate the pull-back morphism of the truncated display functor. From this we can determine the Chow ring of the stack of truncated Barsotti-Tate groups over a field of characteristic $p$ up to $p$-torsion.

Introduction

Edidin and Graham ([EG2]) develop an equivariant intersection theory for actions of linear algebraic groups $G$ on algebraic spaces $X$. For such $G$-spaces they define $G$-equivariant Chow groups $A^*_G(X)$ generalizing Totaro’s definition of the $G$-equivariant Chow ring of a point in $[X]$. They are an invariant of the corresponding quotient stack $[X/G]$, i.e. they are independent of the choice of a presentation. Hence they can be used to define the integral Chow group of a quotient stack. If $X$ is smooth these groups carry a ring structure making them into commutative graded rings. Edidin and Graham used their theory to compute the Chow ring of the stacks $\mathcal{M}_{1,1}$ and $\mathcal{M}_{1,1}$ of elliptic curves. In an Appendix to that paper Vistoli computed the Chow ring of $\mathcal{M}_2$. Edidin and Fulghesu ([EF]) computed the integral Chow ring of the stack of hyperelliptic curves of even genus. In this article we investigate the Chow ring of the stack of truncated Barsotti-Tate groups over a field of characteristic $p > 0$.

Let us denote the stack of level-$n$ Barsotti-Tate groups by $BT_n$. A level-$n$ BT group has a height and a dimension, which are locally constant functions on the base. If $BT_n^{h,d}$ denotes the stack of level-$n$ BT groups of constant height $h$ and dimension $d$ we obtain a decomposition $BT_n = \coprod_{0 \leq d \leq h} BT_n^{h,d}$ into open and closed substacks. For example, if $A$ is an abelian scheme of relative dimension $g$ then its $p^g$-torsion subscheme $A[p^g]$ is a level-$n$ BT group of height $2g$ and dimension $g$.

Although $BT_n^{h,d}$ has a natural presentation $[X/GL_{p^{nh}}]$ as a quotient stack with quasi-affine and smooth $X$ (cf. [We]), it seems unlikely that this
presentation can be used directly to compute the Chow ring. Instead we relate the stack of truncated Barsotti-Tate groups to a stack whose Chow ring is easier to compute, but still closely related to the Chow ring of $BT_n$.

Our choice for this stack is the stack $\mathcal{D}_{isp}^n$ of truncated displays introduced in [La]. Displays were first introduced in [Zi] to provide a Dieudonne theory that is valid not only over perfect fields but more generally over $\mathbb{F}_p$-algebras or $p$-adic rings. While display are given by an invertible matrix with entries in the ring of Witt vectors $W(R)$, if a basis of the underlying modules is fixed, a truncated display is given by an invertible matrix over the truncated Witt ring $W_n(R)$.

Using crystalline Dieudonne theory one can associate to every $p$-divisible group a display. This induces a morphism $\phi: BT \to \mathcal{D}_{isp}$ from the stack of Barsotti-Tate groups to the stack of displays, which in turn induces a morphism

$$\phi_n: BT_n \to \mathcal{D}_{isp}^n,$$

compatible with the truncations on both sides. By [La] this morphism is a smooth morphism of smooth algebraic stacks over $k$ and an equivalence on geometric points.

**Theorem A.** The pull-back $\phi_n^*: A^*(\mathcal{D}_{isp}^n) \to A^*(BT_n)$ is injective and an isomorphism after inverting $p$.

Let us sketch the proof. Consider a field $L$ and a morphism $\text{Spec } L \to BT_n$. After base change to a finite field extension of $p$-power degree the fiber $\phi_n^{-1}(\text{Spec } L)$ is equal to the classifying space of an infinitesimal group scheme necessarily of $p$-power degree. It follows that the pull-back map of Bloch’s higher Chow groups $A_*(\text{Spec } L, m) \to A_*(\phi_n^{-1}(\text{Spec } L), m)$ becomes an isomorphism after inverting $p$. Using the long localization exact sequence the theorem follows from a limit argument and noetherian induction similar to that in [Qu, Proposition 4.1]. The injectivity assertion follows since $A^*(\mathcal{D}_{isp}^n)$ is $p$-torsion free.

Thus to compute the Chow ring of $BT_n$ at least up to $p$-torsion it suffices to compute the Chow ring of $\mathcal{D}_{isp}^n$, which is much easier due to the simpler presentation as a quotient stack. More precisely, if $\mathcal{D}_{isp}^{h,d}$ denotes the open and closed substack in $\mathcal{D}_{isp}^n$ of truncated displays with constant dimension $d$ and height $h$ we have

$$\mathcal{D}_{isp}^{h,d} = [\text{GL}_h(W_n(\cdot))/G_n^{h,d}],$$

where $W_n$ refers to the ring of truncated Witt vectors and $G_n^{h,d}$ is an extension of $\text{GL}_d \times \text{GL}_{h-d}$ by a unipotent group. The following result reduces the calculation of $A^*(\mathcal{D}_{isp}^n)$ to the case $n = 1$.

**Theorem B.** The pull-back $\tau_n^*: A^*(\mathcal{D}_{isp}) \to A^*(\mathcal{D}_{isp}^n)$ of the truncation map $\tau_n: \mathcal{D}_{isp} \to \mathcal{D}_{isp}^n$ is an isomorphism.
This is proved using the factorization
\[
[\text{GL}_h(W_n(\cdot))/G_{n,d}^h] \to [\text{GL}_h/G_n^h] \to [\text{GL}_h/G_1^h]
\]
of \(\tau_n\) and the fact that the first map is an affine bundle and that \(G_{n,d}^h\) is an extension of \(G_1^h\) by a unipotent group.

In a similar way one shows that the Chow ring of \(\text{Disp}^{h,d}_1\) coincides with that of the quotient stack
\[
[\text{GL}_h/(\text{GL}_d \times \text{GL}_{h-d})],
\]
where the action is given by conjugation with the Frobenius. This situation is a special case of Proposition 2.3.2.

**Theorem C.** The following equation holds
\[
A^*(\text{Disp}^{h,d}_1) = A^*_{\text{GL}_d \times \text{GL}_{h-d}}(\text{GL}_h)
\]
\[
= \mathbb{Z}[t_1, \ldots, t_h]^{S_d \times S_{h-d}}/((p-1)c_1, \ldots, (p^h-1)c_h),
\]
where \(c_1, \ldots, c_h\) are the elementary symmetric polynomials in the variables \(t_1, \ldots, t_h\).

Moreover, \(t_1, \ldots, t_d\) resp. \(t_{d+1}, \ldots, t_h\) are the Chern roots of the vector bundle \(\text{Lie}\) resp. \(\text{Lie}^\vee\) over \(\text{Disp}^{h,d}_1\). Here \(\text{Lie}\) is a vector bundle of rank \(d\) assigning to a display its Lie algebra and \(\text{Lie}^\vee\) is of rank \(h-d\) assigning to a display the dual Lie algebra of its dual display.

It follows that the \(\mathbb{Q}\)-vector space \(A^*(\text{Disp}^{h,d}_1)^\mathbb{Q}\) is finite dimensional of dimension \(\binom{h}{d}\), which also equals the number of isomorphism classes of truncated displays of level 1 with height \(h\) and dimension \(d\) over an algebraically closed field. We show that a basis is given by the cycles of the closures of the respective EO-Strata. We prove this fact in greater generality for the stack of \(G\)-zips (\cite{PWZ}) in Section 4.4. In this section we will also compute the Chow ring of the stack of \(G\)-zips for a connected algebraic zip datum. As in the case of displays the computation can be reduced to the situation of Proposition 2.3.2. In fact, truncated displays of level 1 are a special case of \(G\)-zips.

Now by the above results we gain the following information on the Chow ring of the stack of truncated Barsotti-Tate groups.

**Theorem D.** (i) We have
\[
A^*(BT_n^{h,d})_p = \mathbb{Z}[p^{-1}][t_1, \ldots, t_h]^{S_d \times S_{h-d}}/((p-1)c_1, \ldots, (p^h-1)c_h),
\]
where \(c_i\) denotes the \(i\)-th elementary symmetric polynomial in the variables \(t_1, \ldots, t_h\) and \(t_1, \ldots, t_d\) resp. \(t_{d+1}, \ldots, t_h\) are the Chern roots of \(\text{Lie}\) resp. \(\text{Lie}^\vee\).

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(ii) \( \dim \mathbb{Q} \ A^*(BT_n^{h,d})_\mathbb{Q} = \binom{h}{d} \) and a basis is given by the cycles of the closures of the EO-Strata.

(iii) 
\[
\text{Pic } BT_n^{h,d} = \begin{cases} 
\mathbb{Z}[p^{-1}]/(p-1) & \text{if } d = 0, h \\
\mathbb{Z}[p^{-1}] \times \mathbb{Z}[p^{-1}]/(p-1) & \text{else},
\end{cases}
\]
where the generator for the free resp. torsion part is \( \det(\text{Lie}) \) resp. \( \det(\text{Lie} \otimes \text{Lie}^\vee) \).

It would be interesting to know if the Chow ring of \( BT_n \) has \( p \)-torsion, and more specifically if the Picard group of \( BT_n \) has \( p \)-torsion. However, since \( \phi_n^* \) is injective and the Chow ring of \( \text{Disp}_n \) is \( p \)-torsion free, \( p \)-torsion in the Chow ring of \( BT_n \) cannot be constructed using displays.

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Terminology and Notation. Every scheme is assumed to be of finite type and separated over the base field \( k \). In Section 2 we assume \( k \) to be of characteristic \( p > 0 \). Algebraic groups are affine smooth group schemes over \( k \). We call an algebraic group \( G \) unipotent if \( G \) admits a filtration \( G = G_0 \supset G_1 \supset \ldots \supset G_e = \{1\} \) by subgroups such that \( G_i \) is normal in \( G_{i-1} \) with quotient isomorphic to \( \mathbb{G}_a \). The character group of an algebraic group \( G \) will be denoted by \( \hat{G} \). If \( X \) is a scheme \( A^*(X) \) will always denote the operational Chow ring of \( X \) (Fu, Chapter 17). \( A_*(X) \) resp. \( CH^*(X) \) will be the Chow group of \( X \) graded by dimension resp. codimension. If \( X \) is an algebraic space over \( k \) with a left action of an algebraic group \( G \) we will refer to \( X \) as a \( G \)-space. We write \([X/G]\) for the corresponding quotient stack. If \( G \) acts freely on \( X \), i.e. the stabilizer of every point is trivial, then \([X/G]\) is an algebraic space. In this case we will write \( X/G \) instead of \([X/G]\) and call \( X \to X/G \) the principal bundle quotient of \( X \) with structure group \( G \).

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1 Equivariant Intersection Theory

1.1 Equivariant Chow Groups

Consider an algebraic group $G$ over $k$. By [EG2, Lemma 9] we can find a representation $V$ of $G$, and an open subset $U$ in $V$ such that the complement of $U$ has arbitrary high codimension, and such that the principal bundle quotient $U/G$ exists in the category of schemes. If $X$ is an algebraic space on which $G$ acts then $G$ acts diagonally on $X \times U$ and we will denote the principal bundle quotient $(X \times U)/G$ by $X_G$.

Con**vention 1.1.1.** We call a pair $(V,U)$ consisting of a $G$-representation $V$ and an open subset $U$ a good pair for $G$ if $G$ acts freely on $U$, i.e. the stabilizer of every point is trivial. Sometimes we will call the quotient $X_G = (X \times U)/G$ a mixed space for the $G$-space $X$. If $(V,U)$ is a good pair for $G$ with $\text{codim}(U^c,V) > i$ we will also call $(X \times U)/G$ an approximation of $[X/G]$ up to codimension $i$.

If $X$ has dimension $n$ the $i$-th equivariant Chow group $A^G_i(X)$ is defined in the following way. Choose a good pair $(V,U)$ for $G$ such that the complement of $U$ has codimension greater than $n - i$. Then one defines

$$A^G_i(X) = A_{i+l-g}(X_G),$$

where $l$ denotes the dimension of $V$ and $g$ is the dimension of $G$. The definition is independent of the choice of the pair $(V,U)$ as long as $\text{codim}(U^c,V) > n - i$ holds ([EG2, Definition-Proposition 1]).

The equivariant Chow groups have the same functorial properties as ordinary Chow groups ([EG2, Section 2]). In particular, we have an operational equivariant Chow ring $A^G_*(X)$ ([EG2, Section 2.6]), i.e. an element $c \in A^G_*(X)$ consists of operations $c(Y \to X) : A^G_*(Y) \to A^G_{*-i}(Y)$ for each $G$-equivariant map $Y \to X$ that are compatible with flat pull-forward, proper push-forward and Gysin homomorphisms.

We will denote by $CH^*_G(X)$ the $G$-equivariant Chow group of $X$ graded by codimension. If $X$ is a pure dimensional $G$-scheme and $(V,U)$ a good
pair for $G$ with $\text{codim}(U^c, V) > i$ then

$$CH_G^i(X) = CH^i((X \times U)/G)$$

for all $j \leq i$. This motivates the term “approximation of $[X/G]$ up to codimension $i$” in Convention [1.1.1].

If $X$ is smooth then $CH^*_G(X)$ carries a ring structure which makes it into a commutative graded ring with unit element. Moreover, there is a natural isomorphism $A^*_G(X) \cong CH^*_G(X)$ of graded rings ([EG2, Proposition 4]).

By [EG2, Proposition 16] the equivariant Chow groups do not depend on the presentation as a quotient, meaning if $X$ is a $G$-space and $Y$ is an $H$-space such that $[X/G] \cong [Y/H]$, then $A^*_{i+g}(X) = A^*_{i+h}(Y)$, where $g = \dim G$ and $h = \dim H$. Hence one can define the Chow group of a quotient stack $[X/G]$ to be

$$A_i([X/G]) = A^*_{i+g}(X)$$

with $g = \dim G$. By [EG2, Proposition 19] one has $A^*([X/G]) \cong A_*([X/G])$, whenever $X$ is smooth.

### 1.2 Higher Equivariant Chow Groups

The reason why we shall need higher Chow groups is that they extend the localization exact sequence to the left. Higher Chow groups were introduced by Bloch in [B1]. For a scheme $X$ higher Chow groups $A_i(X, m)$ are defined as the homology of the complex $z_i(X, \ast)$, where $z_i(X, m)$ is the group of cycles of dimension $m + i$ in $X \times \Delta^m$ meeting all faces properly. For $m = 0$ one gets back the usual Chow group $A_*(X)$ and $A_i(X, m)$ may be non-trivial for $-m \leq i \leq \dim X$. The definition of these higher Chow groups also works for algebraic spaces.

In order to define $G$-equivariant versions $A^*_G(X, m)$ of higher Chow groups we need the homotopy property for the mixed spaces $X_G$, i.e. the pull-back map

$$A_*(X_G, m) \to A_*(E, m)$$

for a vector bundle $E$ over $X_G$ is an isomorphism. This is true for any scheme if $E$ is trivial by [B1, Theorem 2.1]. To prove the assertion for arbitrary vector bundles one needs the localization exact sequence of higher Chow groups proved by Bloch in the case of quasi-projective schemes: If $X$ is an equidimensional, quasi-projective scheme over $k$ and $Y \subset X$ a closed subscheme with complement $U = X - Y$, then there is a long exact sequence of higher Chow groups

$$\ldots \to A_*(Y, m) \to A_*(X, m) \to A_*(U, m) \to A_*(Y, m - 1)$$

$$\to \ldots \to A_*(Y) \to A_*(X) \to A_*(U) \to 0.$$ 

For a proof see [EG2, Lemma 4] and [B1, Theorem 3.1].
Remark 1.2.1. Levine extended Bloch’s proof of the existence of the long localization exact sequence to all separated schemes of finite type over $k$ ([Le, Theorem 1.7]). Hence for the equivariant higher Chow groups to be well defined it suffices that we can choose the mixed spaces to be separated schemes over $k$. However, in all applications we have in mind the conditions of Lemma 1.2.2 will be satisfied.

Lemma 1.2.2. Let $G$ be an algebraic group and $X$ a normal, quasi-projective $G$-scheme. Then for any $i > 0$ there is a representation $V$ of $G$ and an invariant open subset $U \subset V$ whose complement has codimension greater than $i$ such that $G$ acts freely on $U$ and the principal bundle quotient $(X \times U)/G$ is a quasi-projective scheme. In other words, the quotient stack $[X/G]$ can be approximated by quasi-projective schemes.

Proof. Embed $G$ into $\text{GL}_n$ for some $n$. Then there is a representation $V$ of $\text{GL}_n$ and an open subset $U \subset V$, whose complement has codimension greater than $i$ such that $U/\text{GL}_n$ is a Grassmannian (See [EG2, Lemma 9]). Since $\text{GL}_n$ is special the $\text{GL}_n/G$-bundle $\pi: U/G \to U/\text{GL}_n$ is locally trivial for the Zariski topology, and we will first show that $\pi$ is quasi-projective.

Since $\text{GL}_n/G$ is quasi-projective and normal there is an ample $\text{GL}_n$-linearizable line bundle $L \to \text{GL}_n/G$ ([Th, Section 5.7]). Then

$$(U \times L)/\text{GL}_n \to (U \times (\text{GL}_n/G))/\text{GL}_n = U/G$$

is a line bundle relatively ample for $\pi$. This shows that $\pi$ is quasi-projective. The same holds then for $U/G$. Again by [Th] Section 5.7 there is an ample $G$-linearizable line bundle on $X$. The pull-back to $X \times U$ is then relatively ample for the projection $X \times U \to U$. Applying [GIT, Proposition 7.1] to this situation yields the claim.  

Definition 1.2.3. (i) A pair $(V, U)$ will be called an admissible pair for a $G$-scheme $X$ if $(V, U)$ is a good pair for $G$ and if the mixed space $X_G$ is quasi-projective and (locally) equidimensional over $k$. $X$ will be called an admissible $G$-scheme if for any $i$ there is an admissible pair $(V, U)$ for $X$ with $\text{codim}(U^c, V) > i$.

(ii) If $X$ is an admissible $G$-scheme we define its higher equivariant Chow groups to be

$$A_i^G(X, m) = A_{i+l-g}(X_G, m),$$

where $g = \dim G$ and $X_G$ is formed from an $l$-dimensional admissible pair $(V, U)$ such that $\text{codim}(U^c, V) > \dim X + m - i$.

(iii) We will say that a stack $\mathcal{X}$ admits an admissible presentation if there exists an admissible $G$-scheme $X$ such that $\mathcal{X} = [X/G]$.

(iv) Let $\mathcal{X}$ be a quotient stack that admits a presentation $\mathcal{X} = [X/G]$ by an admissible $G$-scheme $X$. We define the higher equivariant Chow groups of $\mathcal{X}$ as

$$A_*^{\mathcal{X}}(\mathcal{X}, m) = A_*^{G}(X, m)$$

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where \( g = \dim G \).

**Remark 1.2.4.** The proof that Definition 1.2.3 (ii) resp. (iv) is independent of the choice of the admissible pair \((V, U)\) resp. the presentation \([X/G]\) is the same as for ordinary equivariant Chow groups (See Definition-Proposition 1 resp. Proposition 16 in [EG2]) by using the homotopy property for the mixed spaces.

**Remark 1.2.5.** We will frequently encounter the situation of a morphism \( T \to X \) of \( G \)-schemes such that \( T \) is open in a \( G \)-equivariant vector bundle over \( X \). We remark that, if \( X \) is an admissible \( G \)-scheme, so is \( T \). This follows since a vector bundle over a quasi-projective scheme is again quasi-projective.

**Lemma 1.2.6.** Let \( f : \mathcal{X} \to \mathcal{Y} \) be a flat map of quotient stacks of relative dimension \( r \). Then there is a flat pull-back map \( f^* : A_*(\mathcal{Y}) \to A_{*+r}(\mathcal{X}) \) between the Chow groups. If \( \mathcal{X} \) and \( \mathcal{Y} \) admit admissible presentations the same assertion holds for the higher Chow groups.

Furthermore, if \( \mathcal{X} \) and \( \mathcal{Y} \) are smooth then under the identification \( A_i(\mathcal{X}) = A^G_{\dim X - i}(\mathcal{X}) \) the above morphism is just the natural pull-back map between the operational Chow rings.

**Proof.** Consider presentations \( \mathcal{X} = [X/G] \) and \( \mathcal{Y} = [Y/H] \). By definition \( A_i(\mathcal{X}) = A^G_{i+g}(X) \) with \( g = \dim G \) and similar for \( A_i(\mathcal{Y}) \). Choose a good pair \((V_1, U_1)\) for \( G \) and a good pair \((V_2, U_2)\) for \( H \). Let \( l_i = \dim V_i \). As usual we will write \( X_G \) resp. \( Y_H \) for the mixed space \((X \times U_1)/G\) resp. \((Y \times U_2)/H\). Consider the fibresquare

\[
\begin{array}{ccc}
Z' & \to & Z \\
\downarrow & & \downarrow \\
X_G & \to & \mathcal{X}
\end{array}
\]
\[
\begin{array}{ccc}
& & Y_H \\
\downarrow & & \downarrow \\
& & \mathcal{Y}
\end{array}
\]

Then \( Z' \) is a bundle over \( X_G \) resp. \( Z \) with fiber \( U_2 \) resp. \( U_1 \) and \( Z' \to Y_H \) is a flat map of algebraic spaces of relative dimension \( l_1 + r \). Hence

\[
A_{i + l_1 + l_2 + r}(Z') = A_{i + l_1 + r}(X_G) = A_{i+r}(\mathcal{X})
\]

and we define \( f^* \) to be the ordinary pull-back of the flat map \( Z' \to Y_H \). The exact same construction works for the higher equivariant Chow groups if \( \mathcal{X} \) and \( \mathcal{Y} \) admit admissible presentations.

For the last part we recall that the isomorphism \( A^i(\mathcal{X}) \cong A^G_{\dim X - i}(X) \) maps \( c \in A^i(\mathcal{X}) \) to \( c(X_G \to \mathcal{X}) \cap [X_G] \in A^G_{\dim X - i}(X) \). Thus we need to check the equality

\[
f^*(d(Y_H \to \mathcal{Y}) \cap [Y_H]) = d(X_G \to \mathcal{X} \to \mathcal{Y}) \cap [X_G]
\]

for \( d \in A^i(\mathcal{Y}) \). This follows from the compatibility of \( d \) with flat pull-backs. \( \square \)
1.3 Auxiliary Results

Lemma 1.3.1. Let $X \rightarrow Y$ be a flat morphism of schemes and $Y' \rightarrow Y$ be a finite, flat and surjective map of degree $d$. Let $X' \rightarrow Y'$ be the base change of $X \rightarrow Y$ along $Y' \rightarrow Y$. Assume the pull-back $A_*(Y', m') \rightarrow A_*(X', m)$ becomes an isomorphism after inverting some integer $d'$. Then the pull-back $A_*(Y, m) \rightarrow A_*(X, m)$ is an isomorphism after inverting $dd'$. 

Proof. The injectivity of the pull-back $A_*(Y, m)_{dd'} \rightarrow A_*(X, m)_{dd'}$ follows from the exact diagram

$$
\begin{array}{c}
0 \rightarrow A_*(Y, m)_{dd'} \rightarrow A_*(Y', m)_{dd'} \\
\downarrow \quad \cong \quad \downarrow \\
0 \rightarrow A_*(X, m)_{dd'} \rightarrow A_*(X', m)_{dd'}
\end{array}
$$

and the surjectivity from the exact diagram

$$
\begin{array}{c}
A_*(Y', m)_{dd'} \rightarrow A_*(Y, m)_{dd'} \rightarrow 0 \\
\cong \\
A_*(X', m)_{dd'} \rightarrow A_*(X, m)_{dd'} \rightarrow 0
\end{array}
$$

where the horizontal maps in the first diagram are induced by pull-back and in the second diagram by push-forward. The commutativity of the second diagram is [Fu, Proposition 1.7].

Lemma 1.3.2. Let $T \rightarrow X$ be a morphism of quasi-projective schemes over $k$. We assume that $X$ is equidimensional and that $T \rightarrow X$ is flat of relative dimension $a$. Let $d, i \in \mathbb{Z}$ and for $x \in X$ let $h(x)$ denote the dimension of the closure of $\{x\}$ in $X$. If the pull-back $A_{i-h(x)}(\text{Spec} k(x), m)_{d} \rightarrow A_{i-h(x)+a}(T_x, m)_{d}$ is an isomorphism for every $x \in X$ and for any $m$, then $A_{i}(X, m)_{d} \rightarrow A_{i+a}(T, m)_{d}$ is an isomorphism.

Proof. We follow Quillen’s proof of the analogous result in higher K-theory ([Qu, Proposition 4.1]). First we may assume that $X$ is irreducible for if $X = W_1 \cup \ldots \cup W_r$ is a decomposition into irreducible components we may consider the long localization exact sequence of the pair $(W_1, X - W_1)$. By induction we are thus reduced to the irreducible case. Since the Chow groups only depend on the reduced structure, we may also assume that $X$ is reduced. Let $K$ denote the function field of $X$. We have

$$
A_{i-n}(\text{Spec} K, m) = \lim_{U} A_i(U, m),
$$

$$
A_{i-n+a}(T_K, m) = \lim_{U} A_{i+a}(T_U, m),
$$

where $U$ runs through open subsets of $T_K$. Since $A_{i}(X, m)_{d} \rightarrow A_{i+a}(T, m)_{d}$ is an isomorphism for every $x \in X$ and for any $m$,

$$
A_{i}(X, m)_{dd'} \rightarrow A_{i+a}(T, m)_{dd'}
$$

is an isomorphism.

[Qu]: Quillen, Algebraic K-Theory. Lecture Notes in Math., vol. 341, Springer, 1973.
where the limit goes over all non-empty open subsets of $X$ and $n$ denotes the dimension of $X$. In fact, it suffices to go over all non-empty open subsets with equidimensional complement, since for all non-empty open $U$ in $X$ there exists a non-empty open subset $U'$ contained in $U$ with equidimensional complement. We obtain a commutative diagram

$$
\begin{array}{ccc}
A_{i-n}(\text{Spec } K, m + 1) & \longrightarrow & \lim_{Y} A_{i}(Y, m) \\
\downarrow & & \downarrow \\
A_{i-n+a}(T_K, m + 1) & \longrightarrow & \lim_{Y} A_{i+a}(T_Y, m) \\
\downarrow & & \downarrow \\
A_{i-n}(\text{Spec } K, m) & \longrightarrow & \lim_{Y} A_{i}(Y, m - 1) \\
\downarrow & & \downarrow \\
A_{i-n+a}(T_K, m) & \longrightarrow & \lim_{Y} A_{i+a}(T_Y, m - 1)
\end{array}
$$

with exact rows, where the limit goes over all proper closed equidimensional subsets of $X$. After inverting $d$ the first and fourth vertical map become isomorphisms and we conclude by noetherian induction.

**Corollary 1.3.3.** Let $T \to X$ be a flat morphism of quasi-projective schemes over $k$ with fibers being affine spaces of some dimension $n$. Then the pullback $A_{*}(X, m) \to A_{*+n}(T, m)$ is an isomorphism.

**Proof.** This is an immediate consequence of Lemma 1.3.2. □

**Remark 1.3.4.** The assertion of the above corollary in the case $m = 0$ also holds without the quasi-projective assumption. One can use the same proof but using Gillet’s higher Chow groups. For his higher Chow groups a long localization exact sequence exists for arbitrary schemes. For details see Chapter 8 in [Gi].

**Lemma 1.3.5.** Let $K$ be a unipotent subgroup of an algebraic group $G$ such that the quotient $G/K$ is finite of degree $d$. Then the pull-back $A_{G}^{*}(m) \to A_{K}^{*}(m)$ is an isomorphism after inverting $d$.

**Proof.** Let $(V, U)$ be an admissible pair for $G$. Then $U/K \to U/G$ is a $G/K$-bundle locally trivial for the flat topology. By assumption on $G/K$ the morphism $U/K \to U/G$ is therefore finite, flat and surjective of degree $d$. It follows that the pull-back $A_{*}(U/G, m) \to A_{*}(U/K, m) \equiv A_{*}(U, m)$ is injective after inverting $d$. Also for sufficiently high degree we know that $A_{*}(\text{Spec } k, m) \to A_{*}(U, m)$ is surjective. Since we can assume the codimension of $U'$ in $V$ to be arbitrary high, we obtain the surjectivity of $A_{G}^{*}(m) \to A_{K}^{*}(m)$. □
Lemma 1.3.6. Let \( K/k \) be a Galois extension with Galois group \( G \) and let \( X \) be a scheme over \( k \). Then pulling back along \( X_K \to X \) induces an isomorphism \( A_*(X, m)_Q \cong A_*(X_K, m)_Q^G \). If \( K/k \) is a finite Galois extension of degree \( d \) it suffices to invert \( d \).

Proof. We first assume that \( K/k \) is finite of degree \( d \). Then on the level of cycles we have an injection \( z_*(X, \cdot)_d \hookrightarrow z_*(X_K, \cdot)_d^G \) since \( X_K \to X \) is finite, flat of degree \( d \). We claim that this map is also surjective. Let \( W \subset X_K \times_K \Delta_K \) be a subvariety meeting all faces properly. Let \( S \subset G \) be the isotropy group of \( W \). It suffices to see that \( \sum_{g \in G/S} [gW] \) lies in \( z_*(X, \cdot)_d \). For this consider the closed subscheme \( V = \bigcup_{g \in G/S} gw \) (equipped with the reduced structure). Then \( V \) is a \( G \)-invariant equidimensional subscheme of \( X_K \times_K \Delta_K \) that meets all faces properly. Thus it has a model \( \tilde{V} \) over \( k \) also meeting all faces properly. Finally all components \( gW \) have the same multiplicity 1 in the cycle \( [V] \) and therefore \( \sum_{g \in G/S} [gW] = [\tilde{V}_K] \). To complete the proof in the finite case it suffices now to note that taking \( G \)-invariants is an exact functor on the category of \( \mathbb{Z}[^d] \)-modules with \( G \)-action, hence \( H_*(z_*(X_K, \cdot)_d^G) = H_*(z_*(X, \cdot)_d)_d^G \). The general case follows from the finite case and the fact that \( A_*(X_K, m)^G = \lim_{\to} A_*(X_L, m)^{G(L/k)} \), where the limit goes over all finite Galois subextensions \( L/k \) of \( K \).

1.4 A Pull-Back Lemma

Throughout we consider the situation of an exact sequence

\[
0 \longrightarrow A \longrightarrow G \longrightarrow H \longrightarrow 0
\]

of algebraic groups and an admissible \( H \)-scheme \( X \) such that the induced \( G \)-action on \( X \) makes \( X \) also into an admissible \( G \)-scheme. These conditions are always satisfied if \( X \) is quasi-projective and normal by Lemma 1.2.2. We are then interested in properties of the pull-back homomorphism (Lemma 1.2.6)

\[
A_*(\mathbb{[X/H]}, m) \to A_*(\mathbb{[X/G]}, m).
\]

**Proposition 1.4.1.** Let

\[
0 \longrightarrow A \longrightarrow G \longrightarrow H \longrightarrow 0
\]

be an exact sequence of algebraic groups and \( X \) an admissible \( H \)-scheme such that the induced \( G \)-action makes \( X \) also into an admissible \( G \)-scheme. We also assume \( H \) to be special.

Let \( d \in \mathbb{Z} \) such that \( A_*(m) \to A_*(m)^{G(0)} \) becomes an isomorphism after inverting \( d \) for every field extension \( L \) of \( k \) and every \( m \). Then the pull-back \( A_*(\mathbb{[X/H]}, m) \to A_*(\mathbb{[X/G]}, m) \) becomes an isomorphism after inverting \( d \).
Proof. First note that the natural map $[X/G] \to [X/H]$ is flat of relative dimension $-a$ with $a = \dim A$. We can choose for any $i \in \mathbb{Z}$ an admissible pair $(V, U)$ for the $H$-action such that $A_{j+i}([(X \times U)/G], m) = A_j([X/G], m)$ and $A_{j+i}([(X \times U)/H, m] = A_j([X/H], m)$ for all $j > i$. Here $l$ denotes the dimension of $V$. Note that $X \times U$ is again an admissible $G$-scheme (cf. Remark 1.2.5). Replacing $X$ by $X \times U$ we may thus assume that $[X/H]$ is a quasi-projective scheme.

Let now $(X \times U)/G$ be a quasi-projective mixed space for $G$. Let $U$ be the quotient $U/A$. Then we can identify $(X \times U)/G$ with the quotient $(X \times U)/H$ and under this identification the map $(X \times U)/G \to X/H$ corresponds to the $U$-bundle $(X \times U)/H \to X/H$. It is Zariski locally trivial since $H$ is special. We are left to show that the pull-back of this map is an isomorphism after inverting $d$. This will follow from Lemma 1.3.2 once we have seen that the pull-back $A_{j-h(x)}(\text{Spec } k(x), m)_d \to A_{j-h(x)+l-a}(\bar{U}_{k(x)}, m)_d$ is an isomorphism for every $x \in X/H$. Here $h(x)$ is the dimension of the closure of $\{x\}$ in $X/H$. Let us write $L = k(x)$. Assuming the codimension of $U^c$ in $V$ to be sufficiently large we obtain by assumption

$$A_{j-h(x)}(\text{Spec } L, m)_d = A_{j-h(x)+l}(U_L, m)_d = A_{j-h(x)+l-a}(\bar{U}_L, m)_d.$$ 

For this recall $A_{j+l-a}(\bar{U}_L, m) = A_j^{U_L}(m)$ and $A_{j+l}(U_L, m) = A_j^{U}(m)$. This proves the claim. \qed

The above proposition applies to the following cases.

Corollary 1.4.2. In the situation of Proposition 1.4.1 the following assertions hold.

(i) If $A$ is unipotent then $A_*([X/H], m) \to A_*([X/G], m)$ is an isomorphism.

(ii) If $A$ is finite of degree $d$ then $A_*([X/H], m) \to A_*([X/G], m)$ becomes an isomorphism after inverting $d$.

Proof. The first part follows from Corollary 1.3.3 and the second part follows from Lemma 1.3.5 applied to the case $K = \{0\}$. \qed

The assumption on $H$ to be special is crucial for the proof of Proposition 1.4.1 since we need to know that the fibers of the $U$-bundle $(X \times U)/H \to X/H$ appearing in the proof are given by $U$ in order to apply Lemma 1.3.2. However, we have the following version when $H$ is finite.

Proposition 1.4.3. Let

$$0 \longrightarrow A \longrightarrow G \longrightarrow H \longrightarrow 0$$

be an exact sequence of algebraic groups and $X$ an admissible $H$-scheme such that the induced $G$-action makes $X$ also into an admissible $G$-scheme. We assume that $H$ is finite of degree $d$. 

\[12\]
Let $d' \in \mathbb{Z}$ such that $A^*_{A_L}(m) \to A^*_{[0]}(m)$ becomes an isomorphism after inverting $d'$ for every field extension $L$ of $k$ and any $m$. Then the pull-back $A_*([X/H], m) \to A_*([X/G], m)$ becomes an isomorphism after inverting $dd'$. 

Proof. We argue the same way as in Proposition 1.4.1 and then have to see that the pull-back of $(X \times \bar{U})/H \to X/H$ becomes an isomorphism after inverting $dd'$. As mentioned earlier we cannot apply Lemma 1.3.2 since the above $\bar{U}$-bundle is not locally trivial for the Zariski topology. Instead it becomes trivial after the finite, flat and surjective base change $X \to X/H$ of degree $d$, i.e. there is a cartesian diagram

\[
\begin{array}{ccc}
X \times \bar{U} & \rightarrow & X \\
| & & | \\
(X \times \bar{U})/H & \rightarrow & X/H.
\end{array}
\]

The claim thus follows from Lemma 1.3.1. 

Corollary 1.4.4. In the situation of Proposition 1.4.3 the following assertions hold.

(i) If $A$ is unipotent then $A_*([X/H], m)_d \to A_*([X/G], m)_d$ is an isomorphism.

(ii) If $A$ is finite of degree $d'$ then $A_*([X/H], m)_{dd'} \to A_*([X/G], m)_{dd'}$ is an isomorphism.

In the next proposition we show that the assertion of Proposition 1.4.1 is valid over $\mathbb{Q}$ for arbitrary $H$.

Proposition 1.4.5. Let

\[
\begin{array}{ccc}
0 & \rightarrow & A \\
& \rightarrow & G \\
& \rightarrow & H \\
& \rightarrow & 0
\end{array}
\]

be an exact sequence of algebraic groups and $X$ an admissible $H$-scheme such that the induced $G$-action makes $X$ also into an admissible $G$-scheme.

Assume $A^*_{A_L}(m)_\mathbb{Q} \to A^*_{[0]}(m)_\mathbb{Q}$ is an isomorphism for every field extension $L$ of $k$ and any $m$. Then the pull-back $A_*([X/H], m)_\mathbb{Q} \to A_*([X/G], m)_\mathbb{Q}$ is an isomorphism.

Proof. Using the notation of the proof of Proposition 1.4.1 we need to see that the pull-back of the $\bar{U}$-bundle $T := (X \times \bar{U})/H \to X/H$ is an isomorphism over $\mathbb{Q}$. It suffices to see that $A_*($Spec$k(x), m)_\mathbb{Q} \to A_*($Spec$k(x)^{sep}, m)_\mathbb{Q}$ is an isomorphism for $x \in X/H$. The above $U$-bundle may not be trivial for the Zariski topology, but we still have $T_{\bar{x}} = \bar{U}_{\bar{x}}$ and thus $A_*($Spec$k(x)^{sep}, m)_\mathbb{Q} \to A_*($Spec$k(x), m)_\mathbb{Q}$ is an isomorphism by assumption. The claim then follows from Lemma 1.3.6 and the fact that the Galois action is compatible with pull-back. 

\[
\end{array}
\]
Corollary 1.4.6. In the situation of Proposition 1.4.5 the following assertions hold.

(i) If $A$ is unipotent then $A_*(\lceil X/H \rceil, m) \to A_*(\lceil X/G \rceil, m)$ is an isomorphism.

(ii) If $A$ is finite then $A_*(\lceil X/H \rceil, m) \to A_*(\lceil X/G \rceil, m)$ is an isomorphism.

Lemma 1.4.7. Let $G$ be a split extension

$$0 \longrightarrow K \longrightarrow G \longrightarrow H \longrightarrow 0$$

of an algebraic group $H$ by a unipotent group $K$. Choose a splitting $H \hookrightarrow G$ and let $X$ be a normal, quasi-projective $G$-scheme. Then the pull-back map

$$A^G_*(X, m)_Q \to A^H_*(X, m)_Q$$

is an isomorphism. If $G$ is special, this above map is an isomorphism over $\mathbb{Z}$.

Proof. Let $(V, U)$ be an admissible pair for the $G$-action on $X$. It follows from the proof of Lemma 1.2.2 that $(V, U)$ is then also admissible for the induced $H$-action. The morphism $(X 	imes U)/H \to (X 	imes U)/G$ is a $G/H$-bundle. If $G$ is special this bundle is locally trivial for the Zariski topology. Hence the lemma follows from Corollary 1.3.3 in the special case and Lemma 1.3.6 and 1.3.2 in the general case.

1.5 The Restriction Map

We want to describe properties of the restriction map $res^T_G: A^G_*(X) \to A^T_*(X)$, where $T$ is a split torus in $G$. This map is defined via flat pull-back of the natural map $X_T \to X_G$ between the mixed spaces. Note that more generally one has a restriction map $res^T_H: A^G_*(X) \to A^H_*(X)$ for every subgroup $H$ of $G$. We will need the following result.

Theorem 1.5.1. Let $G$ be a connected reductive group with split maximal torus $T$ and Weyl group $W = W(G,T)$. Let $X$ be a $G$-scheme.

(i) $W$ acts on $A^T_*(X)$. Furthermore, the restriction morphism $A^G_*(X) \to A^T_*(X)$ induces a map $r: A^G_*(X) \to A^T_*(X)^W$.

(ii) Assume $X$ is smooth. Then $r$ is an isomorphism after tensoring with $\mathbb{Q}$.

Part (iii) is basically proved in [EG], where Edidin and Graham consider the case $X = \text{Spec} \ k$. However, there seems to be no complete proof of part (ii) in the literature. We therefore give a proof.

In the following $A^*(X; \mathbb{Q})$ will denote the operational Chow ring of $X$ consisting of characteristic classes with values in rational Chow groups, i.e. an element $c \in A^*(X; \mathbb{Q})$ assigns to each $T \to X$ a morphism

$$c(T \to X): A_*(T)_Q \to A_*(T)_Q$$
satisfying the usual compatibility conditions ([Fu, Section 17.1]). A proper map \( \pi : \tilde{X} \to X \) is called an envelope if for each irreducible subspace \( V \subset X \) there exists an irreducible subspace \( \tilde{V} \subset \tilde{X} \) such \( \pi \) maps \( \tilde{V} \) birationally onto \( V \).

**Remark 1.5.2.** There is a natural map \( A^*(X)_\mathbb{Q} \to A^*(X; \mathbb{Q}) \) and this map is an isomorphism if \( X \) is smooth. This follows from

\[
A^*(X)_\mathbb{Q} \cong A^*(X) \mathbb{Q} \cong A^*(X)^W \mathbb{Q}.
\]

We recall the following easy lemma.

**Lemma 1.5.3.** (i) Let \( \pi : \tilde{X} \to X \) be a proper surjective map. Then \( \pi_* : A_*(\tilde{X})_\mathbb{Q} \to A_*(X)_\mathbb{Q} \) is surjective and \( \pi^* : A^*(X; \mathbb{Q}) \to A^*(\tilde{X}; \mathbb{Q}) \) is injective.

(ii) Let \( \pi : \tilde{X} \to X \) be a birational envelope. Then \( \pi_* : A_*(\tilde{X}) \to A_*(X) \) is surjective and \( \pi^* : A^*(X) \to A^*(\tilde{X}) \) is injective.

**Proof.** The first part of (i) is [Ki, Proposition 1.3]. The first part of (ii) follows immediately from the definition of an envelope. The second part of (i) and (ii) are formal consequences of their first parts.

**Lemma 1.5.4.** Let \( G \) be a connected reductive group with split maximal torus \( T \) and Weyl group \( W = W(G, T) \). Let \( M \) be smooth and \( E \to M \) be a principal \( G \)-bundle. Consider a Borel subgroup \( B \supset T \). Then \( W \) acts on \( A^*(E/B) \) and pull-back induces an isomorphism \( A^*(M) \mathbb{Q} \cong A^*(E/B)_W \mathbb{Q} \).

**Remark 1.5.5.** This lemma is also mentioned (without proof) in [Vi, Section 2.5].

**Proof.** We identify \( W = N_G(T)/T \) and choose \( w \in N_G(T) \). Then \( w \) induces an automorphism \( w : E/T \to E/T \). This defines an action of \( W \) on \( A^*(E/T) = A^*(E/B) \). Since \( w \) lies in \( G \) the diagram

\[
\begin{array}{ccc}
E/T & \longrightarrow & E/G = M \\
\downarrow \quad w & \quad & \quad \\
E/T & \longrightarrow & E/T 
\end{array}
\]

commutes and this implies that the image of the pull-back \( A^*(M) \to A^*(E/B) \). We are left to show that

\[
A^*(M)_\mathbb{Q} \to A^*(E/B)^W_\mathbb{Q}
\]
is an isomorphism. Let us first show that $A_*(M)_Q \to A_*(E/B)_Q^W$ is surjective. For this the smoothness assumption on $M$ is not needed. We recall that every $G$-torsor is locally isotrivial by [Ra, XIV Lemma 1.4]. This means that there exists a covering of $M$ by open subsets $U$ with the property that for each $U$ there is a finite, etale and surjective map $U' \to U$ such that $E_{U'} = E \times_M U' \to U'$ becomes a trivial $G$-torsor. Let $V$ denote the complement of such an $U$ in $M$ and consider the commutative diagram

$$
\begin{array}{cccccc}
A_*(V)_Q & \longrightarrow & A_*(M)_Q & \longrightarrow & A_*(U)_Q & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
A_*(E_V/B)_Q^W & \longrightarrow & A_*(E/B)_Q^W & \longrightarrow & A_*(E_U/B)_Q^W & \longrightarrow & 0
\end{array}
$$

with exact rows. An easy diagram chase shows that if the first and last vertical map are surjective so is $A_*(M)_Q \to A_*(E/B)_Q^W$. Using noetherian induction we are thus reduced to the case that there exists a proper surjective map $M' \to M$ such that $E_{M'} \to M'$ is trivial. Since the diagram

$$
\begin{array}{cccccc}
A_*(M')_Q & \longrightarrow & A_*(E_{M'}/B)_Q^W & \\
\downarrow & & \downarrow & & \\
A_*(M)_Q & \longrightarrow & A_*(E/B)_Q^W
\end{array}
$$

commutes ([Fu, Proposition 1.7]) and since $A_*(E_{M'}/B)_Q^W \to A_*(E/B)_Q^W$ is surjective by part (i) of the previous lemma we are further reduced to the case of a trivial $G$-torsor $E = G \times M \to M$. Since $G/B$ has a decomposition into affine cells we obtain in this case $A_*(E/B)_Q = A_*(G/B)_Q^W \otimes A_*(M)_Q$. From [De, Section 8] we get $A_*(G/B)_Q = S_Q/(S_Q^W)$, where $S = \text{Sym}(T)$ and $S_Q^W$ denotes the submodule generated by homogeneous $W$-invariant elements of positive degree. Since $(S_Q/(S_Q^W))^W = Q$ we obtain $A_*(E/B)_Q^W = A_*(M)_Q$ as wanted.

By the previous lemma we know that $A^*(M; Q) \to A^*(E/B; Q)$ is injective but since $M$ (and therefore $E$) is smooth we obtain the injectivity of $A^*(M)_Q \to A^*(E/B)_Q$.

**Proof.** (of Theorem 1.5.1) The assertion (i) and (ii) are immediate consequences of Lemma 1.5.4. Under the assumption that $A^*_T(X)$ is $\mathbb{Z}$-torsion free the surjectivity of $r$ follows from part (ii) by using the argumentation of the proof of Lemma 5 in [EG].
2 The Chow Ring of the Stack of level-$n$ Barsotti-Tate Groups

2.1 The Stack of truncated Displays

Let $R$ be an $\mathbb{F}_p$-algebra. We denote by $W_n(R)$ the ring of truncated Witt vectors of length $n$. Let $I_{n,R} \subset W_n(R)$ be the image of the Verschiebung $W_{n-1}(R) \to W_n(R)$ and $J_{n,R} \subset W_n(R)$ be the kernel of the projection $W_n(R) \to W_{n-1}(R)$. The Frobenius on $R$ induces a ring homomorphism $\sigma: W_n(R) \to W_n(R)$ and the inverse of the Verschiebung induces a bijective $\sigma$-linear map $\sigma_1: I_{n+1,R} \to W_n(R)$. Note that $pR = 0$ implies $I_{n,R} J_{n,R} = 0$, hence we may view $I_{n+1,R}$ as a $W_n(R)$-module.

Truncated displays were introduced in [La]. Let us recall the necessary notations. We are only going to need the following description of truncated displays.

Definition 2.1.1. A truncated display of level $n$ over an $\mathbb{F}_p$-algebra $R$ is a triple $(L, T, \Psi)$ consisting of projective $W_n(R)$-modules $L$ and $T$ of finite rank and a $\sigma$-linear automorphism $\Psi: L \oplus T \to L \oplus T$.

A morphism between truncated displays is defined as follows. First we can use $\Psi$ to define $\sigma$-linear maps $F: L \oplus T \to L \oplus T, \quad l + t \mapsto p \Psi(l) + \Psi(t)$, $F_1: L \oplus (T \otimes_{W_n(R)} I_{n+1,R}) \to L \oplus T, \quad l + (t \otimes \omega) \mapsto \Psi(l) + \sigma_1(\omega) \Psi(t)$.

Then a morphism between two truncated displays $(L, T, \Psi)$ and $(L', T', \Psi')$ of level $n$ is given by a matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where $A \in \Hom(L, L')$, $B \in \Hom(T, T')$, $C \in \Hom(L, T' \otimes_{W_n(R)} I_{n+1,R})$ and $D \in \Hom(T, T')$ such that

\[
\begin{array}{cccc}
L \oplus T & \xrightarrow{F} & L \oplus T & \\
\downarrow & & \downarrow & \\
L' \oplus T' & \xrightarrow{F'} & L' \oplus T' & \\
\end{array}
\]

\[
\begin{array}{cccc}
L \oplus (T \otimes_{W_n(R)} I_{n+1,R}) & \xrightarrow{F_1} & L \oplus T & \\
\downarrow & & \downarrow & \\
L' \oplus (T' \otimes_{W_n(R)} I_{n+1,R}) & \xrightarrow{F'_1} & L' \oplus T' & \\
\end{array}
\]

commute.

The height of a truncated display is defined as the rank of $L \oplus T$ and the dimension as the rank of $T$. Both are locally constant functions on $\Spec R$. Let $\text{Disp}_n \to \Spec \mathbb{F}_p$ denote the stack of truncated displays of level $n$. That is for $R$ an $\mathbb{F}_p$-algebra $\text{Disp}_n(\Spec R)$ is the groupoid of truncated displays of level $n$. It is proved in [La, Proposition 3.15] that $\text{Disp}_n$ is a smooth Artin algebraic stack of dimension zero over $\mathbb{F}_p$ with affine diagonal.
For $h \in \mathbb{N}$ and $0 \leq d \leq h$ we denote by $\text{Disp}^{h,d}_n$ the open and closed substack of truncated displays of level $n$ with constant height $h$ and constant dimension $d$. Then

$$\text{Disp}_n = \bigsqcup_{h,d} \text{Disp}^{h,d}_n.$$ 

A Presentation of $\text{Disp}^{h,d}_n$. We will adopt the notation of the proof of Proposition 3.15 in [La]. Let $X_n^{h,d}$ be the functor on affine $\mathbb{F}_p$-schemes with $X_n^{h,d}(R) = \text{GL}_h(W_n(R))$. This is an affine open subscheme of $A^{nh^2}$. Furthermore, let $G_n^{h,d}$ be the functor such that $G_n^{h,d}(R)$ is the group of invertible matrices $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with $A \in \text{GL}_{h-d}(W_n(R))$, $B \in \text{Hom}(W_n(R)^d, W_n(R)^{h-d})$, $C \in \text{Hom}(W_n(R)^{h-d}, I_{n+1,R}^d)$ and $T \in \text{GL}_d(W_n(R))$. Then $G_n^{h,d}$ is a connected algebraic group of dimension $nh^2$.

**Remark 2.1.2.** Since $I_{2,R}$ is in bijection to $R$ via $\sigma_1$ we may view $G_1^{h,d}(R)$ as the group of invertible matrices with entries in $R$ with respect to the multiplication given by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} = \begin{pmatrix} AA' & AB' + BD' \\ C\sigma(A') + \sigma(D)C' & DD' \end{pmatrix},$$

where in the four blocks we have the usual matrix multiplication.

Let $\pi_n^{h,d}: X_n^{h,d} \to \text{Disp}_n^{h,d}$ be the functor that assigns to an invertible matrix $\Psi \in \text{GL}_h(W_n(R))$ the truncated display $(W_n(R)^{h-d}, W_n(R)^d, \Psi)$, where we view $\Psi$ as a $\sigma$-linear map $W_n(R)^h \to W_n(R)^h$ via $x \mapsto \Psi \cdot \sigma x$. Now if we let $G_n^{h,d}$ act on $X_n^{h,d}$ via

$$G \cdot \Psi = G\Psi \sigma_1(G)^{-1}$$

where $\sigma_1(G) = \begin{pmatrix} \sigma(A) & \sigma(B) \\ \sigma(C) & \sigma(D) \end{pmatrix}$, then every $G \in G_n^{h,d}$ defines an isomorphism $\pi_n^{h,d}(\Psi) \to \pi_n^{h,d}(G \cdot \Psi)$ of truncated displays. On the contrary if $G$ defines an isomorphism $\pi_n^{h,d}(\Psi) \to \pi_n^{h,d}(\Psi')$ then necessarily $\Psi' = G\Psi \sigma_1(G)^{-1}$. We thus obtain

**Theorem 2.1.3.** The functor $\pi_n^{h,d}$ induces an isomorphism of stacks

$$[X_n^{h,d}/G_n^{h,d}] \cong \text{Disp}^{h,d}_n.$$ 

There are the following two obvious vector bundles on $\text{Disp}^{h,d}_n$.

**Definition 2.1.4.** Let $\text{Spec} R \to \text{Disp}^{h,d}_n$ be a map corresponding to a truncated display $\mathcal{P} = (L,T,\Psi)$.

(i) We denote by $\text{Lie}$ the vector bundle of rank $d$ over $\text{Disp}^{h,d}_n$ that assigns to $\text{Spec} R \to \text{Disp}^{h,d}_n$ the vector bundle $\text{Lie}(\mathcal{P}) = T/I_{n,R}T$ of rank $d$ over $R$. 

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(ii) By \(\mathcal{L}ie^{\vee}\) we denote the vector bundle of rank \(h - d\) that assigns to 
\(\text{Spec } R \to \text{Disp}_{n,R}^{h,d}\) the vector bundle \(L/I_{n,R}L\) of rank \(h - d\) over \(R\).

**Remark 2.1.5.** The notation \(\mathcal{L}ie^{\vee}\) in the above definition stems from the fact that the dual of \(L/I_{n,R}L\) gives the Lie algebra of the dual display \(\mathcal{P}^{\dagger}\). For the definition of the dual display see [Zi, Definition 19].

**The Truncated Display Functor.** As already mentioned in the introduction the strategy for computing the Chow ring of the stack of truncated Barsotti-Tate groups is to relate it to the stack of truncated displays. This happens via the truncated display functor 
\[\phi_n: BT_n \to \text{Disp}_n\]
constructed in [La]. Let us briefly sketch the construction.

Let \(G\) be a \(p\)-divisible group over an \(\mathbb{F}_p\)-algebra \(R\). The ring of Witt vectors \(W(R)\) is \(p\)-adically complete and the ideal \(I_R\) in \(W(R)\) carries natural divided powers compatible with the canonical divided powers of \(p\). Let \(\mathbb{D}(G)\) denote the covariant Dieudonné crystal of \(G\). We can evaluate \(\mathbb{D}(G)\) at \(W(R) \to R\) and set \(P = \mathbb{D}(G)_{W(R)\to R}\) and \(Q = \text{Ker}(P \to \text{Lie}(G))\). Furthermore, let \(F^\sharp: P^\sigma \to P\) and \(V^\sharp: P \to P^\sigma\) be the maps induced by Frobenius and Verschiebung of \(G\). One can show that there are \(\sigma\)-linear maps \(F: P \to P\) resp. \(\tilde{F}: Q \to P\) compatible with base change in \(R\) such that \((P, Q, F, \tilde{F})\) is a display which induces the maps \(F^\sharp\) and \(V^\sharp\). See [La, Proposition 2.4] for the precise statement. This construction yields a 1-morphism
\[\phi: BT \to \text{Disp}\]
from the stack of Barsotti-Tate groups to the stack of displays. It is clear from the construction that the Lie algebra of \(G\) is equal to the Lie algebra of \(\phi(G)\) defined by \(P/Q\).

Moreover, one can prove that for all \(n\) there are maps \(\phi_n: BT_n \to \text{Disp}_n\) compatible with the truncation maps on both sides such that \(\phi\) is the projective limit of the system \((\phi_n)_{n \geq 1}\). The central result in [La] is that \(\phi_n\) is a smooth morphisms of smooth algebraic stacks over \(\mathbb{F}_p\) which is an equivalence on geometric points.

### 2.2 Group Theoretic Properties of \(G_n^{h,d}\)

We denote by \(R_{(n,m)}^{h,d}\) the kernel of the projection \(G_n^{h,d} \to G_m^{h,d}\) for \(m < n\) and by \(\tilde{R}_n^{h,d}\) the kernel of the projection \(G_n^{h,d} \to \text{GL}_{h-d} \times \text{GL}_d\). Note that \(G_n^{h,0} = \text{GL}_h(W_n(\cdot))\). We recall the following well known facts about the Witt ring. For an \(\mathbb{F}_p\)-algebra \(R\) we denote by \([\cdot]: R \to W_n(R)\) the map \(r \mapsto (r, 0, \ldots, 0)\) and \(\overline{\cdot}: W(R) \to W(R)\) is the Verschiebung.
Lemma 2.2.1. Let \( R \) be an \( \mathbb{F}_p \)-algebra and \( x, y \in R \). Then \( [x + y] - [x] - [y] \) lies in \( V^r W(R) \). Furthermore, \( V^r W(R) \cdot V^s W(R) \subset V^{r+s} W(R) \).

Proof. The first part follows immediately from the fact that \( V W(R) \) is the kernel of the ring homomorphism \( \mathbb{W}_0 : W(R) \to R \) and the fact \( \mathbb{W}_0([x]) = x \) for all \( x \in R \).

For the second part we may assume \( r \geq s \). We then write \( V^r x V^s y = V^r(x F^r V^s y) = p^s \cdot V^r(x F^{r-s} y) \). Since \( p R = 0 \) we have \( p(x_0, x_1, \ldots) = (0, x_0^p, x_1^p, \ldots) \) in \( W(R) \) and the lemma follows. \( \square \)

Lemma 2.2.2. (i) \( K_{(n,n-1)}^{h,d} \) is unipotent.

(ii) \( \tilde{K}_{n}^{h,d} \) is unipotent.

Proof. (i) First note that \( K_{(n,n-1)}^{h,0} = \ker(\GL_h(W_n(\cdot)) \to \GL_h(W_{n-1}(\cdot))) \) is unipotent. To see this we consider the Verschiebung \( V(\cdot) \) as a map \( W_n(R) \to W_n(R) \). Then by the above lemma the map

\[
\mathbb{G}_a^{h^2} \to K_{(n,n-1)}^{h,0}, \quad A \mapsto I_h + V^{n-1}[A]
\]

is an isomorphism of algebraic groups.

Next we show that \( K_{h,d}^{(n,n-1)} \) is unipotent. This is the group of matrices

\[
\begin{pmatrix} A & B \\ C & D \end{pmatrix}
\]

with \( A \in K_{(n,n-1)}^{h-d,0} \), \( B \in J_n^{(h-d) \times d} \), \( C \in J_{n+1}^{d \times (h-d)} \) and \( D \in K_{(n,n-1)}^{d,0} \). The multiplication in this group is given by

\[
\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} = \begin{pmatrix} AA' + AB'D' & BD' \\ CA' + DC' & DD' \end{pmatrix}
\]

Starting with the normal subgroup

\[
\begin{pmatrix} I_{n-d} & J_n^{(h-d) \times d} \\ J_{n+1}^{d \times (h-d)} & I_d \end{pmatrix},
\]

which is isomorphic to \( \mathbb{G}_a^{2d(h-d)} \), and then using the fact that \( K_{(n,n-1)}^{h-d,0} \) resp. \( K_{(n,n-1)}^{d,0} \) are isomorphic to \( \mathbb{G}_a^{(h-d)^2} \) resp. \( \mathbb{G}_a^{2d} \) one obtains a filtration of \( K_{(n,n-1)}^{h,d} \) by normal subgroups, whose successive quotients are isomorphic to a product of copies of \( \mathbb{G}_a \). Now we have an exact sequence

\[
0 \longrightarrow K_{(n,n-1)}^{h,d} \longrightarrow K_{(n,m)}^{h,d} \longrightarrow K_{(n-1,m)}^{h,d} \longrightarrow 0
\]

and by induction we may assume that \( K_{(n-1,m)}^{h,d} \) is unipotent. It follows that \( K_{(n,m)}^{h,d} \) is unipotent.

(ii) For \( n = 1 \) the assertion is obvious in view of Remark 2.1.2. For \( n > 1 \) we use the exact sequence

\[
0 \longrightarrow K_{(n,n-1)}^{h,d} \longrightarrow \tilde{K}_n^{h,d} \longrightarrow \tilde{K}_{n-1}^{h,d} \longrightarrow 0.
\]

By induction and part (i) it follows that \( \tilde{K}_n^{h,d} \) is unipotent. \( \square \)
Corollary 2.2.3. (i) $G_{n}^{h,d}$ is special.
(ii) $\tilde{K}_{n}^{h,d}$ is the unipotent radical of $G_{n}^{h,d}$.
(iii) The projection $X_{n}^{h,d} \to X_{1}^{h,d}$ is a trivial $K_{(n,1)}^{h,0}$-torsor.

Proof. We have the exact sequence

$$0 \longrightarrow \tilde{K}_{n}^{h,d} \longrightarrow G_{n}^{h,d} \longrightarrow \text{GL}_{h-d} \times \text{GL}_{d} \longrightarrow 0.$$ 

Now $\tilde{K}_{n}^{h,d}$ is unipotent, thus special. Since $\text{GL}_{h-d} \times \text{GL}_{d}$ is also special part (i) follows.

Clearly the projection $X_{n}^{h,d} \to X_{1}^{h,d}$ is a $K_{(n,1)}^{h,0}$-torsor by definition of $K_{(n,1)}^{h,0}$. It is trivial since $K_{(n,1)}^{h,0}$ is unipotent and $X_{1}^{h,d}$ is affine. $\Box$

2.3 The Chow Ring of $\text{Disp}_{n}$

We start with the following result which reduces the calculation of $A^{*}(\text{Disp}_{n})$ to the case $n = 1$.

Theorem 2.3.1. The pull-back

$$\tau_{n}^{*}: A^{*}(\text{Disp}_{1}^{h,d}) \to A^{*}(\text{Disp}_{n}^{h,d})$$

of the truncation $\tau_{n}: \text{Disp}_{n}^{h,d} \to \text{Disp}_{1}^{h,d}$ is an isomorphism.

Proof. Under the presentation $\text{Disp}_{n}^{h,d} = [X_{n}^{h,d}/G_{n}^{h,d}]$ the truncation $\tau_{n}$ is induced by the natural projections $X_{n}^{h,d} \to X_{1}^{h,d}$ and $G_{n}^{h,d} \to G_{1}^{h,d}$. Thus $\tau_{n}$ factors as

$$[X_{n}^{h,d}/G_{n}^{h,d}] \to [X_{1}^{h,d}/G_{1}^{h,d}] \to [X_{1}^{h,d}/G_{1}^{h,d}].$$

The pull-back of the second map is an isomorphism by Lemma 2.2.2 and Corollary 1.4.2. To show that the pull-back of the first map is also an isomorphism let us abbreviate $X = X_{1}^{h,d}$ and $G = G_{1}^{h,d}$. By part (iii) of Corollary 2.2.3 we know that $X_{1}^{h,d} = X \times K$ with $K = K_{(n,1)}^{h,0}$, and the projection $X \times K \to X$ is $G$-equivariant. Moreover, $K$ is an affine space by Lemma 2.2.2. After replacing $[X/G]$ by an appropriate mixed space (cf. Convention 1.1.1), i.e. replacing $X$ by $X \times U$ where $(V,U)$ is an admissible pair with high codimension, we may assume that $[X/G]$ is a quasi-projective scheme. We claim that $(X \times K)/G \to X/G$ is a Zariski locally-trivial affine bundle. Since $G$ is special by part (i) of Corollary 2.2.3 the principal $G$-bundle $X \to X/G$ is locally trivial for the Zariski topology and after replacing $X/G$ by an appropriate open subset we may assume $X = G \times (X/G)$. We then have an isomorphism $(G \times (X/G) \times K)/G \cong (X/G) \times K$ given by the assignment $(g, x, k) \mapsto (x, k')$, where $k'$ is defined by $g^{-1}(g, x, k) = (1, x, k')$. This proves the claim and hence the pull-back of the first map is also an isomorphism by Corollary 1.3.3. $\Box$
The main ingredient of the computation of $A^*\text{Disp}_1^{h,d}$ is the following Proposition

**Proposition 2.3.2.** Let $G$ be a connected split reductive group over a field $k$ with split maximal torus $T$. Consider an isogeny $\varphi: L \to M$, where $L$ and $M$ are Levi components of parabolic subgroups $P$ and $Q$ of $G$. Assume $T \subset L$ and let $g_0 \in G(k)$ such that $\varphi(T) = g_0 T$. Let $\tilde{\varphi}: T \to T$ denote the isogeny $\varphi$ followed by conjugation with $g_0^{-1}$. We write $S = \text{Sym}(\tilde{T}) = A^*_T$ and $S_+ = \hat{A}^1_T$. We have a natural action of $\tilde{\varphi}$ on $S$, that we will also denote by $\tilde{\varphi}$.

Consider the action of $L$ on $G$ by $\varphi$-conjugation. If $W_G = W(G,T)$ and $W_L = W(L,T)$ denote the respective Weyl groups we have

$$A^*_L(G)_Q = S^W_Q/(f - \tilde{\varphi} f | \ f \in S^W_{+L})_Q.$$  

If $G$ is special we have

$$A^*_L(G) = S^W_L/(f - \tilde{\varphi} f | \ f \in S^W_{+L}).$$

(Note that the action of $\tilde{\varphi}$ on $S^W_L$ is independent of the choice of $g_0$ since two choices differ by an element of $N_G(T)$.)

**Proof.** The case of special $G$ is proven in [Br. Proposition 1.1]. It remains to show $A^*_L(G)_Q = S^W_Q/(f - \varphi f | \ f \in S^W_{+L})_Q$ in the non-special case. Using the same argumentation as in the special case we arrive at

$$A^*_L(G)_Q = S^W_Q/(f - \varphi f | \ f \in S^W_{+L})_Q.$$  

Now by Theorem 1.5.1 we know $A^*_L(G)_Q = A^*_T(G)_Q^W$. Since $S^W \hookrightarrow S_Q$ is finite free ([Dc. Theorem 2 (d)]) it is also faithfully flat. Hence we obtain $S^W_Q \cap IS_Q = IS^W_Q$ and the assertion follows.

In the following we will write $c_i$ for the $i$-th elementary symmetric polynomial in the variables $t_1, \ldots, t_h$ and $c_i^{(j,k)}$ will denote the $i$-th elementary symmetric polynomial in the variables $t_j, \ldots, t_k$, where $1 \leq j < k \leq h$ and $1 \leq i \leq k - j + 1$. We then have $\mathbb{Z}[t_1, \ldots, t_n]^{S_h-d \times S_d} = \mathbb{Z}[c_1^{(1,h-d)}, \ldots, c^{(1,h-d)}_i, c^{(h-d+1,h)}_1, \ldots, c^{(h-d+1,h)}_d].$

**Theorem 2.3.3.**

$$A^*(\text{Disp}_1^{h,d}) = A^*_{\text{GL}_{h-d} \times \text{GL}_d}(\text{GL}_h)$$

$$= \mathbb{Z}[t_1, \ldots, t_n]^{S_h-d \times S_d}/((p-1)c_1, \ldots, (p^h-1)c_h),$$

where the $c_i^{(1,h-d)}$ resp. $c_i^{(h-d+1,h)}$ are the Chern classes of $\text{Lie}^\vee$ resp. $\text{Lie}$.
Proof. We have that $G_1^{h,d}$ is a split extension of the group $GL_{h-d} \times GL_d$ by the unipotent group \( \{ (E_{h-d} \ast E_d) \} \), where \( \ast \) denotes an arbitrary matrix (cf. Remark 2.1.2). The splitting is given by the canonical inclusion $GL_{h-d} \times GL_d \hookrightarrow G_1^{h,d}$. Hence by Lemma 1.4.7 we know

\[
A^\ast(\text{Disp}_{1}^{h,d}) = A^\ast_{GL_{h-d} \times GL_d}(GL_h),
\]

where the action of $GL_{h-d} \times GL_d$ on $GL_h$ is given by \( \sigma \)-conjugation. Since $GL_{h-d} \times GL_d$ is special with Weyl group $S_{h-d} \times S_d$ we obtain from Proposition 2.3.2

\[
A^\ast_{GL_{h-d} \times GL_d}(GL_h) = Z[t_1, \ldots, t_n]^{S_{h-d} \times S_d}/((p-1)c_1, \ldots, (p^h-1)c_h).
\]

The assertion that the $c_i^{(1,h-d)}$ resp. $c_i^{(h-d+1,h)}$ are the Chern classes of $\text{Lie}$ resp. $\text{Lie}^\vee$ follows from the following simple fact. Let us write $\mathcal{E}_d$ resp. $\mathcal{E}_{h-d}$ for the vector bundle over $[G_{1}^{h,d}/GL_h]$ resp. $[G_{1}^{h,d}/GL_d]$ that corresponds to the canonical representation of $GL_d$ resp. $GL_{h-d}$. Then $\text{Lie}$ is the pull-back of $\mathcal{E}_d$ under the natural map

\[
\text{Disp}_{1}^{h,d} = [GL_h/G_1^{h,d}] \longrightarrow [*/(GL_d \times GL_h)] \longrightarrow [*/GL_d]
\]

and similarly for $\text{Lie}^\vee$. 

\[
\begin{aligned}
\text{Corollary 2.3.4.} & \quad \text{Pic}^\ast(\text{Disp}_{1}^{h,d}) = \begin{cases} 
Z/(p-1)Z & \text{if } d = 0, h \\
Z \times Z/(p-1)Z & \text{else.}
\end{cases} \\
\end{aligned}
\]

A generator for the free resp. torsion part is $\text{det}(\text{Lie})$ resp. $\text{det}(\text{Lie} \otimes \text{Lie}^\vee)$. 

Proof. Note $Pic(\text{Disp}_{1}^{h,d}) = A^1\text{Disp}_{1}^{h,d}$ by [EG2 Corollary 1].

\[
\begin{aligned}
\text{Remark 2.3.5.} & \quad \text{There is also a more direct approach to compute the above Picard groups. By using a theorem of Rosenlicht, namely that for irreducible varieties } X \text{ and } Y \text{ the natural map } \mathcal{O}(X)^* \times \mathcal{O}(Y)^* \rightarrow \mathcal{O}(X \times Y)^* \text{ is surjective, it is not difficult to establish the following exact sequence} \\
& \quad \begin{align*}
\mathcal{O}(X)^*/k^* & \longrightarrow \hat{G} \longrightarrow \text{Pic}^G(X) \longrightarrow \text{Pic}(X)
\end{align*}
\end{aligned}
\]

for $G$ connected and $X$ an irreducible $G$-scheme. The first map assigns to a non-vanishing regular function on $X$ its eigenvalue. In our case we have $G = GL_{h-d} \times GL_d$ and $X = GL_h$. Then $\mathcal{O}(GL_h)^*/k^* = Z$ with generator given by the determinant and eigenvalue given by the character $(p-1)(\text{det}_{GL_{h-d}} + \text{det}_{GL_d}) \in \hat{G}$. Since $\text{Pic}(GL_h) = 0$ we again obtain $\text{Pic}^{GL_{h-d} \times GL_d}(GL_h) = Z \times Z/(p-1)Z$. 23
Remark 2.3.6. The fact that \((\det \mathcal{L}^e \otimes \det \mathcal{L}^e)^{p-1}\) is trivial can also be seen directly as follows: \((\det \mathcal{L}^e \otimes \det \mathcal{L}^e)^{p-1}\) being trivial means that \(\det \mathcal{L}^e \otimes \det \mathcal{L}^e\) is fixed under the pull-back of the Frobenius map \(\text{Frob} : \text{Disp}^{2,1}_1 \to \text{Disp}^{2,1}_1\) assigning to a display \(\mathcal{P}\) over an \(\mathbb{F}_p\)-algebra \(R\) the display \(\mathcal{P}^\sigma\) obtained by base change via the Frobenius \(\sigma : R \to R\). But by definition of a truncated display we have an isomorphism \(\Psi : L \oplus T \cong L^\sigma \oplus T^\sigma\) of \(R\)-modules. Taking the determinant of \(\Psi\) yields the desired isomorphism \(\det L \otimes \det T \cong \det L^\sigma \otimes \det T^\sigma\).

Remark 2.3.7. Let us put this result into context by relating it to the corresponding result for elliptic curves. Let \(\mathcal{M}_{1,1} \to \text{Spec } k\) denote the moduli stack of elliptic curves. A morphism \(\text{Spec } R \to \mathcal{M}_{1,1}\) corresponds to a pair \((C \to \text{Spec } R, \sigma)\) where \(C \to \text{Spec } R\) is a smooth projective curve of genus 1 and \(\sigma : \text{Spec } R \to C\) is a smooth section. We now have the following diagram

\[
\begin{array}{ccc}
\mathcal{M}_{1,1} & \xrightarrow{\phi} & \text{Disp}^{h=2,d=1}_n \downarrow \\
BT^{h=2,d=1} & \xrightarrow{\phi_1} & \text{Disp}^{h=2,d=1}_n
\end{array}
\]

where \(\mathcal{M}_{1,1} \to BT^{h=2,d=1}\) sends an elliptic curve \(C\) to its associated Barsotti-Tate group \(C[p^\infty]\). Let us consider the pull-back map \(A^*(\text{Disp}^{2,1}_1) \to A^*(\mathcal{M}_{1,1})\). In characteristic \(p\) different from 2 and 3 Edidin and Graham computed \(A^*(\mathcal{M}_{1,1}) = \mathbb{Z}[t]/(12t)\), where \(t\) is given by the first Chern class of the Hodge bundle on \(\mathcal{M}_{1,1}\) ([EG2 Proposition 21]).

By construction of the truncated display functor the pull-back of \(\mathcal{L}^e\) to \(\mathcal{M}_{1,1}\) is the dual of the Hodge bundle on \(\mathcal{M}_{1,1}\). Since the dual of an elliptic curve is the elliptic curve it follows from Remark 2.3.5 that the pull-back of \(\mathcal{L}^e\) is given by the Hodge bundle. Hence \(A^*(\text{Disp}^{2,1}_1) \to A^*(\mathcal{M}_{1,1})\) is the map

\[\mathbb{Z}[t_1,t_2]/((p-1)c_1,(p^2-1)c_2) \to \mathbb{Z}[t]/(12t)\]

that sends \(t_1\) to \(-t\) and \(t_2\) to \(t\). Note that \(p^2-1\) is divisible by 12 if and only if \(p \geq 5\). In particular, there can be no such map for \(p = 2,3\), and we deduce that the description \(A^*(\mathcal{M}_{1,1}) = \mathbb{Z}[t]/(12t)\) does not hold in characteristic 2 and 3.

2.4 The Chow Ring of the Stack of G-Zips

Let us first consider the case of F-zips introduced in [MW]. We denote by F-zip the stack of F-zips over a field \(k\) of characteristic \(p > 0\). For \(S\) a \(k\)-scheme \(F\text{-zip}(S)\) is the groupoid of F-zips over \(S\). If \(\tau : \mathbb{Z} \to \mathbb{Z}_{\geq 0}\) is a function with finite support we denote by F-zip\(^\tau\) the open and closed
substack of F-zips of type \( \tau \). Note that
\[
\text{F-zip} = \prod_{\tau} \text{F-zip}^{\tau}.
\]

The stacks \( \text{F-zip}^{\tau} \) are smooth Artin algebraic stacks over \( k \) which follows for example from the following representation as a quotient stack. Let \( X_{\tau} \) denote the \( k \)-scheme whose \( S \)-valued points are given by
\[
X_{\tau}(S) = \{ M = (M, C^\bullet, D^\bullet, \varphi_0) \mid M \text{ F-zip of type } \tau, M = O_S^h \}.
\]
This is a smooth scheme of dimension \( h^2 \). Here \( h = \sum_{i \in \mathbb{Z}} \tau(i) \) is also called the height of \( M \). The group \( \text{GL}_h \) acts on \( X_{\tau} \) by
\[
G \cdot M = (O_S^h, G(C^\bullet), G(D^\bullet), G\varphi_\bullet(G^{-1})^\sigma).
\]
It is easy to see that two F-zips over \( S \) of the above form are isomorphic if and only if they lie in the same \( \text{GL}_h(S) \)-orbit. Thus
\[
\text{F-zip}^{\tau} = [X_{\tau}/\text{GL}_h].
\]

An F-zip \( M \) of type \( \tau \) with support in \( \{0, 1\} \) over an \( \mathbb{F}_p \)-algebra \( R \) is just a tuple
\[
M = (M, C, D, \varphi_0, \varphi_1),
\]
where \( M \) is a projective \( R \)-module with submodules \( C \) and \( D \), which are direct summands of \( M \) and isomorphisms
\[
\varphi_0 : C^\sigma \to M/D, \quad \varphi_1 : (M/C)^\sigma \to D.
\]

**Lemma 2.4.1.** Let \( R \) be an \( \mathbb{F}_p \)-algebra. Then we have an equivalence of categories
\[
\text{Disp}_1(R) \to \prod_{\tau, \text{Supp}(\tau) \in \{0, 1\}} \text{F-zip}^{\tau}(R)
\]
given in the following way
\[
(L, T, \Psi) \mapsto (L \oplus T, T, \Psi^\sigma(L^\sigma), \Psi^\sigma |_{T^\sigma}, \Psi^\sigma |_{L^\sigma}).
\]
The above assignment commutes with pulling back. In particular, we get an isomorphism of stacks
\[
\text{F-zip}^{\tau} \cong \text{Disp}_1^{\tau(0)+\tau(1), \tau(1)}
\]
for every type \( \tau \) with support lying in \( \{0, 1\} \).

**Proof.** An inverse functor is given by the assignment
\[
(M, C, D, \varphi_0, \varphi_1) \mapsto (C, M/C, \varphi_0 \oplus \varphi_1).
\]
\( \square \)
There is more generally the stack of G-zips introduced in [PWZ]. Here G refers to an arbitrary reductive group. It is defined as follows. Let $Z$ be an algebraic zip datum, i.e. a 4-tuple $(G, P, Q, \varphi)$ consisting of a split reductive group $G$, parabolic subgroups $P$ and $Q$ and an isogeny $\varphi: P/R_u(P) \to Q/R_u(Q)$. To $Z$ one associates the group

$$E_Z = \{(p, q) \in P \times Q \mid \varphi(\pi_P(p)) = \pi_Q(q)\}.$$ 

Now $E_Z$ acts on $G$ by the rule

$$(p, q, g) \mapsto pgq^{-1}$$

and the quotient stack $[G/E_Z]$ is called the stack of $G$-zips. If $G$ is connected $Z$ is called a connected zip datum ([PWZ, Definition 3.1]).

Let us recall how the stack of F-zips is just a special case of this construction. For this let $\tau: Z \to \mathbb{Z}_{\geq 0}$ be a function with finite support, say $i_1 \leq \ldots \leq i_r$. If we denote $n_k = \tau(i_k)$, then $(n_1, \ldots, n_r)$ defines a partition of $h = \sum_k n_k$. We denote the standard parabolic of type $(n_1, \ldots, n_r)$ in $GL_h$ by $P^\tau$.

**Lemma 2.4.2.** Let $\tau: Z \to \mathbb{Z}_{\geq 0}$ be a function with finite support and $Z = (GL_h, P^\tau, P^{-\tau}, \sigma)$ be the algebraic zip datum with $P^{-\tau}$ the opposite parabolic of $P^\tau$ and $\sigma$ the Frobenius isogeny. Then there is an isomorphism of stacks

$$[GL_h/E_Z] \xrightarrow{\sim} F\text{-}zip^\tau.$$ 

**Proof.** Let $S$ be a $k$-scheme. We denote by $C^*_\tau$ the descending filtration

$$C^*_\tau = \mathcal{O}^h_S \supset \mathcal{O}^{n_1 + \ldots + n_{r-1}}_S \supset \ldots \supset \mathcal{O}^{n_1}_S \supset 0$$

in $\mathcal{O}^h_S$ given by the standard flag of type $(n_1, \ldots, n_r)$ and by $D^{\tau^{-}}_\tau$ the ascending filtration

$$D^{\tau^{-}}_\tau = 0 \subset \mathcal{O}^{n_r}_S \subset \ldots \subset \mathcal{O}^{n_r + \ldots + n_2}_S \subset \mathcal{O}^h_S,$$

given by the flag of type opposite to $(n_1, \ldots, n_r)$. To $g \in GL_h(S)$ we assign the F-zip

$$M_g = (\mathcal{O}^h_S, C^*_\tau, g(D^{\tau^{-}}_\tau), \varphi_*),$$

where $\varphi$ is given by the restriction of $g$ to the successive quotients of $C^*_\tau$. Note that we can consider $g$ as a $\sigma$-linear map.

If $(p, q)$ is an element of $E_Z$ we get an isomorphism $M_g \to M_{pq^{-1}}$ of F-zips induced by $p$. The fact that $p$ commutes with the $\varphi_i$ is exactly the condition $\sigma(\pi(p)) = \pi(q)$. On the other hand if an isomorphism $p: M_g \to M_{g'}$ of F-zips is given, we see that $g'^{-1}pg$ preserves the flag of type opposite to $(n_1, \ldots, n_r)$. Thus $q = g'^{-1}pg \in P^{-\tau}$ and again the compatibility of $p$ with the $\varphi_i$ implies the condition $\sigma(\pi(p)) = \pi(q)$. \qed
We can also use Proposition 2.3.2 to say something about the Chow ring of the stack of $G$-zips for an arbitrary connected algebraic zip datum.

**Definition 2.4.3.** We call an algebraic zip datum $Z = (G, P, Q, \varphi)$ special, if $G$ is special.

**Theorem 2.4.4.** Let $Z = (G, P, Q, \varphi)$ be a connected algebraic zip datum. Let $W_G = W(G, T)$ be the Weyl group of $G$ and $W_L = W(L, T)$ be the Weyl group of a Levi component $L$ of $P$ w.r.t. a split maximal torus $T \subset L$ of $G$. Let $g_0 \in G(k)$ such that $\varphi(T) = g_0 T$ and let $\tilde{\varphi} : T \to T$ denote the composition of $\varphi$ followed by conjugation with $g_0^{-1}$. Then $\tilde{\varphi}$ induces an action on $S = \text{Sym}(\hat{\epsilon})$ that we will also denote by $\tilde{\varphi}$. We then have
\[
A^*(\{E_Z/G]\}_Q) = S^W_L/(f - \tilde{\varphi} f \mid f \in S^W_G)_Q.
\]
If $Z$ is special we have
\[
A^*(\{E_Z/G]\)) = S^W_L/(f - \tilde{\varphi} f \mid f \in S^W_G).
\]
(Note that the action of $\tilde{\varphi}$ on $S^W_G$ is independent of the choice of $g_0$ since two choices differ by an element of $N_G(T)$.)

**Proof.** By definition of the group $E_Z$ we have a split exact sequence
\[
0 \longrightarrow R_u(P) \times R_u(Q) \longrightarrow E_Z \longrightarrow L \longrightarrow 0,
\]
where the splitting is given by $L \hookrightarrow E_Z$, $l \mapsto (l, \varphi(l))$. From Lemma 1.4.7 we deduce
\[
A^*(\{E_Z/G]\}_Q) = A^*_L(G)_Q,
\]
where the action of $L$ on $G$ is given by $\varphi$-conjugation. If $G$ is special the above equality holds over $\mathbb{Z}$. We conclude by Proposition 2.3.2.

**Example 2.4.5.** We consider the case $Z = (\text{Sp}(2n), P, P^-, \sigma)$, where $\sigma$ denotes the $q$-th power Frobenius. Recall that $\text{Sp}(2n)$ is special and the Weyl group of $\text{Sp}(2n)$ is the wreath product $S_n \wr (\mathbb{Z}/2\mathbb{Z}) = S_n \rtimes (\mathbb{Z}/2\mathbb{Z})^n$. It acts on $\text{Sym}(T) = \mathbb{Z}[t_1, \ldots, t_n]$ in the following way. $S_n$ acts by permuting the variables $t_1, \ldots, t_n$ and after identifying $\mathbb{Z}/2\mathbb{Z} = \{\pm 1\}$ an element $(\varepsilon_1, \ldots, \varepsilon_n) \in \mathbb{Z}/2\mathbb{Z}^n$ acts by $(\varepsilon_1, \ldots, \varepsilon_n) \cdot t_i = \varepsilon_i t_i$.

If $P$ is a Borel we obtain from the above theorem
\[
A^*(\{E_Z/\text{Sp}(2n)\})) = \mathbb{Z}[t_1, \ldots, t_n]/((q^2 - 1)c_1(t_2^2), \ldots, (q^{2n} - 1)c_n(t_2^2)).
\]
If $P$ is the maximal parabolic subgroup fixing a maximal isotropic subspace then $L = \text{GL}_n$ and $W_L = S_n$ and therefore
\[
A^*(\{E_Z/\text{Sp}(2n)\}) = \mathbb{Z}[c_1, \ldots, c_n]/((q^2 - 1)c_1(t_2^2), \ldots, (q^{2n} - 1)c_n(t_2^2)).
\]
It turns out that a \( \mathbb{Q} \)-basis of the Chow ring of the stack of \( G \)-zips is given by the closures of the orbits of the action of \( E_Z \) on \( G \). To prove this let us introduce the naive Chow group of a quotient stack.

**Definition 2.4.6.** Let \( G \) be an algebraic group and \( X \) be a \( G \)-scheme. Let \( Z_*(\mathcal{X}/G) \) be the free abelian group generated by the set of \( G \)-invariant subvarieties of \( X \) graded by dimension. Let \( W_i(\mathcal{X}/G) \) be the group \( \bigoplus_Y k(Y)^G \), where the sum goes over all \( G \)-invariant subvarieties of \( X \) of dimension \( i+1 \). There is the usual divisor map \( div : W_i(\mathcal{X}/G) \to Z_i(\mathcal{X}/G) \) and we define the \( i \)-th naive Chow group of \( \mathcal{X}/G \) to be

\[
A^0_i[\mathcal{X}/G] = Z_i(\mathcal{X}/G)/div(W_i(\mathcal{X}/G)).
\]

**Remark 2.4.7.** There is more generally a definition of naive Chow groups for arbitrary algebraic stacks ([Kr, Definition 2.1.4]) which in the case of a quotient stack agrees with the one given above. Thus the above definition is independent of the presentation as a quotient stack.

**Remark 2.4.8.** There is a natural map \( A^o_i[\mathcal{X}/G] \to A_*[\mathcal{X}/G] \). When \( X \) is Deligne-Mumford, i.e. the stabilizer of every point is finite and geometrically reduced, the induced map \( A^o_i[\mathcal{X}/G]_\mathbb{Q} \to A_*[\mathcal{X}/G]_\mathbb{Q} \) is an isomorphism of groups and an isomorphism of rings if \( \mathcal{X}/G \) is smooth ([Kr, Theorem 2.1.12 (ii)])

The stack of \( G \)-zips is not Deligne-Mumford. However, we still have the following proposition.

**Proposition 2.4.9.** Let \( G \) be a connected algebraic group and \( X \) be an admissible \( G \)-scheme (cf. Definition 1.2.3) with finitely many orbits such that the stabilizer of every point is an extension of a finite group by a unipotent group. Then \( A^o_*[\mathcal{X}/G]_\mathbb{Q} \to A_*[\mathcal{X}/G]_\mathbb{Q} \) is an isomorphism.

**Proof.** We prove this by induction on the number of orbits. Let \( U \) denote the open \( G \)-orbit and \( W \) its complement. We have a commutative diagram

\[
\begin{array}{cccccc}
0 & \longrightarrow & A^o_*[W/G]_\mathbb{Q} & \longrightarrow & A^o_*[X/G]_\mathbb{Q} & \longrightarrow & A^o_*[U/G]_\mathbb{Q} & \longrightarrow & 0 \\
0 & \longrightarrow & A_*[W/G]_\mathbb{Q} & \longrightarrow & A_*[X/G]_\mathbb{Q} & \longrightarrow & A_*[U/G]_\mathbb{Q} & \longrightarrow & 0 \\
\end{array}
\]

and we claim that the rows of this diagram are exact. Since there are only finitely many orbits every \( G \)-invariant subvariety \( Y \) of \( X \) is the closure of a \( G \)-orbit. Since \( Y \) admits a dense \( G \)-invariant subset every \( G \)-invariant rational function on \( Y \) is constant. It follows \( A^o_*[X/G] = \bigoplus_Z \mathbb{Z}[Z] \) where the sum goes over all \( G \)-orbits \( Z \) of \( X \). From this we obtain the exactness of the top row. For the exactness of the lower row we need to see that the
pull-back map $\mathbb{A}_*(\mathbb{A}([X/G], 1) \to \mathbb{A}_*(\mathbb{A}([U/G], 1))$ is surjective. But $\mathbb{A}_*(\mathbb{A}([U/G], m)$ is isomorphic to the classifying space of the stabilizer group scheme of $U$. By assumption and Corollary 1.4.4 we get that $\mathbb{A}_*(\mathbb{A}([U/G], m \to \mathbb{A}_*(\mathbb{A}([B\{0\}], m)$ is an isomorphism. Equivalently the pull-back of the structure morphism $\mathbb{A}([U/G] \to \text{Spec} k$ is an isomorphism for the higher Chow groups with rational coefficients and hence the claim follows.

Now the right vertical arrow is an isomorphism since both groups are isomorphic to $\mathbb{Q}$. By induction we may assume that the first vertical arrow is also an isomorphism.

Recall that an algebraic zip datum $Z$ is called orbitally finite if $G$ has finitely many $E_Z$-orbits ([PWZ, Definition 7.2]).

**Theorem 2.4.10.** Let $Z$ be an orbitally finite connected algebraic zip datum and $\mathbb{A}([G/E_Z]$ be the corresponding stack of G-Zips. Then the following assertions hold.

(i) $\mathbb{A}_*[G/E_Z] \mathbb{Q} \to \mathbb{A}_*[G/E_Z] \mathbb{Q}$ is an isomorphism.

(ii) $\mathbb{A}_*[G/E_Z] = \bigoplus Z \mathbb{Q}$ where the sum goes over all orbits $Z$.

In particular, the dimension of $\mathbb{A}_*[G/E_Z] \mathbb{Q}$ as a $\mathbb{Q}$-vector space is equal to the number of orbits.

**Proof.** The assumption of the previous proposition on the stabilizer group schemes hold by [PWZ, Theorem 8.1].

**Corollary 2.4.11.** Let $Z = (G, P, Q, \varphi)$ be a connected algebraic zip datum and $T$ be a split maximal torus of $G$ in a Levi component $L$ of $P$. If $Z$ is orbitally finite the $\mathbb{Q}$-vectorspace $A^*(\mathbb{A}([E_Z/G]) \mathbb{Q}$ is finite dimensional of dimension $|W_G/W_L|$, where as usual $W_G = W(G, T)$ is the Weyl group of $G$ and $W_L = W(L, T)$ is the Weyl group of $L$.

**Proof.** By the above theorem $\dim \mathbb{Q} A^*(\mathbb{A}([E_Z/G]) \mathbb{Q}$ equals the number of $E_Z$-orbits in $G$. This number equals $|W_G/W_L|$ by [PWZ, Theorem 7.5].

In the case of F-zips the above results read as follows.

**Corollary 2.4.12.** Let $\tau: Z \to \mathbb{Z}_{>0}$ be a function with finite support $i_1 \leq \ldots \leq i_r$ and $n_k = \tau(i_k)$. Let $h = \sum_n n_i$ be its height. Then the following holds

(i) $A^* F\text{zip}^\tau = \mathbb{Z}[t_1, \ldots, t_h]^{S_{n_1} \times \ldots \times S_{n_r}} / ((p-1)c_1, \ldots, (p^h - 1)c_h)$

with $c_i$ the $i$-th elementary symmetric polynomial in the variables $t_1, \ldots, t_h$.

(ii) $\text{Pic}(F\text{zip}^\tau) = \mathbb{Z}^{r-1} \times \mathbb{Z}/(p-1)\mathbb{Z}$

(iii) $\dim \mathbb{Q} A^*(F\text{zip}^\tau) \mathbb{Q} = \frac{h!}{n_1! \cdot \ldots \cdot n_r!}$
2.5 The Chow Ring of $BT_n$

The goal of this section is to prove the following result.

**Theorem 2.5.1.** The pull-back $\phi_n^* : A^*(\text{Disp}_n) \to A^*(BT_n)$ is injective and an isomorphism after inverting $p$.

We know that $\text{Disp}_n = \coprod_{d \leq h} \text{Disp}^{h,d}_n$ is a decomposition into open and closed substacks. The same holds for $BT_n$ and the morphism $\phi_n$ maps $BT^{h,d}_n$ to $\text{Disp}^{h,d}_n$. It suffices to prove the theorem for the restriction of $\phi_n$ to $BT^{h,d}_n$.

The following proposition is the crucial point in the proof of Theorem 2.5.1.

**Proposition 2.5.2.** Let $L$ be a field extension of $k$ and $\text{Spec} L \to \text{Disp}_n$ be a morphism. Then there is a finite field extension $L'$ of $L$ of $p$ power degree and an infinitesimal commutative group scheme $A$ over $L'$ such that the fiber $\phi_n^{-1}(\text{Spec} L')$ is the classifying space of $A$.

**Proof.** The diagonal $\Delta : BT_n \to BT_n \times \text{Disp}_n$ is flat and surjective by [La, Theorem 4.7]. This means that two Barsotti-Tate groups of level $n$ having the same associated display become isomorphic when pulled back to a suitable fppf-covering. It follows that the fiber $(BT_n)_L$ of a display $P$ over some field $L$ is a gerbe over $L$. If $L$ is perfect there is a truncated Barsotti-Tate group $G$ over $L$ with $\phi_n(G) = P$, i.e. $(BT_n)_L$ is a neutral gerbe. In this case $(BT_n)_L = \text{BAut}^o(G)$ where $\text{Aut}^o(G) = \text{Ker}(\text{Aut} G \to \text{Aut} P)$ is commutative and infinitesimal again by [La, Theorem 4.7]. If $L$ is not perfect we may consider the perfect hull $L_{p^{-\infty}}$ in an algebraic closure of $L$. Then $L \subset L_{p^{-\infty}}$ is purely inseparable and $(BT_n)_L(L_{p^{-\infty}})$ is non-empty. Since $(BT_n)_L(L_{p^{-\infty}}) = \lim_{L' \supset L}(BT_n)_L(L')$, where the limit goes over all finite subextensions $L \subset L' \subset L_{p^{-\infty}}$, we find some $L'$ such that $(BT_n)_L$ has a section corresponding to a truncated Barsotti-Tate group $G$ over $L'$. Thus $A = \text{Aut}^o(G)$ and $L'$ have the desired properties. $\square$

**Remark 2.5.3.** Over the open and closed substack of $BT_n$ consisting of level-$n$ BT-groups of constant dimension $d$ and codimension $c$ the degree of $\text{Aut}^o(G\text{univ})$ is $p^{ncd}$. See Remark 4.8 in [La].

Note that $\text{Disp}^{h,d}_n$ and $BT^{h,d}_n$ both admit admissible presentations in the sense of Definition 1.2.3. In the case of $\text{Disp}^{h,d}_n$ this follows from Theorem 2.1.3 and Lemma 1.2.2. To obtain the assertion for $BT^{h,d}_n$ we use [We, Proposition 1.8] which yields a presentation $BT^{h}_n = [Y^n_h/\text{GL}_{p^{nh}}]$ with $Y^n_h$ quasi-affine and of finite type over $k$. Now $BT^{h}_n$ is smooth over Spec $k$ ([La]). Hence $Y^n_h$ is also smooth and in particular normal and equidimensional.

We now consider the flat pull-back map

$$\phi_n^* : A_*(\text{Disp}^{h,d}_n, m) \to A_*(BT^{h,d}_n, m)$$

from Lemma 1.2.6.
Proposition 2.5.4. $\phi_n^*: A_*(\text{Disp}_n^{h,d}, m) \to A_*(BT_n^{h,d}, m)$ is an isomorphism after inverting $p$.

Proof. Let us write $\mathcal{X} = BT_n^{h,d}$ and $\mathcal{Y} = \text{Disp}_n^{h,d}$. We fix some $i_o \in \mathbb{Z}$ and show that $\phi_n: A_{i_o}(\text{Disp}_n^{h,d}, m)_p \to A_{i_o}(BT_n^{h,d}, m)_p$ is an isomorphism.

Consider an approximation of $\mathcal{Y}$ (cf. Convention 1.1.1) by a quasi-projective scheme $Y \to \mathcal{Y}$ so that $A_{i_o}(\mathcal{Y}, m) = A_{i_o}(Y, m)$ and similarly an approximation $X \to \mathcal{X}$ of $\mathcal{X}$. Let $r$ denote the relative dimension of $X \to \mathcal{X}$. Let $Z$ be the fibre product $X \times_{\mathcal{Y}} Y$. The morphism $Z \to Y$ is then smooth of relative dimension $r$ and we need to see that the pull-back $A_{i_o}(Y, m)_p \to A_{i_o+r}(Z, m)_p$ is an isomorphism. Note that $Z$ is again quasi-projective since it is open in a vector bundle over the quasi-projective scheme $X$ (cf. Remark 1.2.5). We have the following cartesian diagram

$$
\begin{array}{ccc}
Z_y & \longrightarrow & \mathcal{X}_k(y) \longrightarrow \text{Spec } k(y) \\
\downarrow \quad & \downarrow & \downarrow \\
Z & \longrightarrow & \mathcal{X}_Y \longrightarrow Y \\
\downarrow \quad & \downarrow & \downarrow \\
X & \longrightarrow & \mathcal{X} \longrightarrow \mathcal{Y}
\end{array}
$$

By Lemma 1.3.2 it suffices to see that $A_i(\text{Spec } k(y), m)_p \to A_{i+r}(Z_y, m)_p$ with $i = i_o - \dim \{y\}$ is an isomorphism. According to the previous proposition there is a finite field extension $K$ of $k(y)$ of $p$-power degree such that $K^n = BA$ holds for an infinitesimal group scheme $A$ over $K$.

Since $Z_K$ is open in a vector bundle over $\mathcal{X}_K$ of rank $r$ we have $Z_K = U/A$, where $U$ is open in a representation $V$ of $A$. Note that $V$ is of dimension $r$. Hence by choosing codim $X^c$ to be big enough, we may assume $A_i(\text{Spec } K, m) \to A_{i+r}(U, m)$ is an isomorphism. Since $A$ is of $p$-power degree it follows that the map $A_i(\text{Spec } K, m)_p \to A_{i+r}(Z_K, m)_p$ is an isomorphism. Now since the field extension $K \supset k(y)$ is of $p$-power degree it follows from Lemma 1.3.1 that $A_i(\text{Spec } k(y), m)_p \to A_{i+r}(Z_y, m)_p$ is also an isomorphism. We are done.

Proof. (of Theorem 2.5.1) Since $BT_n$ and $\text{Disp}_n$ are smooth the pull-back $(\phi_n)_p^*: A^*(\text{Disp}_n)_p \to A^*(BT_n)_p$ is an isomorphism by Lemma 1.2.0 and the proposition above. We already know $A^*(\text{Disp}_n)$ is $p$-torsion free by Theorem 2.3.3 and Theorem 2.3.1. Thus $\phi_n^*$ is injective.

Gathering the results of Chapter 4 we obtain

**Theorem 2.5.5.** (i) We have

$$A^*(BT_n^{h,d})_p = \mathbb{Z}[p^{-1}][t_1, \ldots, t_h]^{S_d \times S_n}/((p-1)c_1, \ldots, (p^h-1)c_h),$$

31
where $c_i$ denotes the $i$-th elementary symmetric polynomial in the variables $t_1, \ldots, t_h$ and $t_1, \ldots, t_d$ resp. $t_{d+1}, \ldots, t_h$ are the Chern roots of $\text{Lie}$ resp. $\text{Lie}^\vee$.

(ii) $\dim_{\mathbb{Q}} A^*(BT_{n}^{h,d})_{\mathbb{Q}} = \binom{h}{d}$ and a basis is given by the cycles of the closures of the $\text{EO}$-Strata.

(iii) 
\[
(Pic \ BT_{n}^{h,d})_p = \begin{cases} 
\mathbb{Z}[p^{-1}]/(p-1) & \text{if } d = 0, h \\
\mathbb{Z}[p^{-1}] \times \mathbb{Z}[p^{-1}]/(p-1) & \text{else},
\end{cases}
\]
where the generator for the free resp. torsion part is $\det(Lie)$ resp. $\det(Lie \otimes \text{Lie}^\vee)$.

Proof. By Theorem 2.5.1 we know $A^*(\text{Disp}_{n}^{h,d})_p \cong A^*(BT_{n}^{h,d})_p$. Moreover, we have $A^*(\text{Disp}_{n}^{h,d}) \cong A^*(\text{Disp}_{1}^{h,d})$ by Theorem 2.3.1 and $A^*(\text{Disp}_{1}^{h,d})$ was computed in Theorem 2.3.3. This proves part (i). By Lemma 2.4.1 and Lemma 2.4.2 we know that $\text{Disp}_{1}^{h,d}$ is isomorphic to the stack $[\text{GL}_h/\mathbb{Z}]$ corresponding to the Frobenius zip datum $Z = (\text{GL}_h, P, P^-, \sigma)$, where $P$ is the standard parabolic of type $(d, h)$, $P^-$ is the opposite parabolic and $\sigma$ is the Frobenius isogeny. Now the dimension of $A^*(\text{Disp}_{1}^{h,d})_{\mathbb{Q}}$ as a $\mathbb{Q}$-vectorspace follows from Corollary 2.4.12 and a basis is given by Theorem 2.4.10. This proves (ii). Finally (iii) follows from (i) together with the fact that $\text{Pic} \ BT_{n}^{h,d} = A^1(BT_{n}^{h,d})$. \qed

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