ORDINARY SUBVARIETIES OF CODIMENSION ONE

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Abstract. In this paper we extend the properties of ordinary points of curves [10] to ordinary closed points of one-dimensional affine reduced schemes and then to ordinary subvarieties of codimension one.

Introduction.

Let $A$ be the local ring, at a point $x$, of an algebraic reduced variety $X$ over an algebraically closed field $k$. Denote with $e(A) = e$ the multiplicity of $X$ at $x$. Let $\mathfrak{m}$ be the maximal ideal of $A$. $\text{Spec}(G(A))$ is the tangent cone and $\text{Proj}(G(A))$ the projectivized tangent cone to $X$ at $x$. Furthermore, if $G(A)_{\text{red}} = \text{Spec}(G(A)/\text{nil}(G(A))$, then $\text{Spec}(G(A)_{\text{red}})$ is the tangent cone to $X$ at $x$, viewed as a set. If $\overline{A}$ is the normalization of $A$, the branches of $X$ at $x$ are the points of $\text{Spec}(A/\mathfrak{m}A)$.

If $A$ is the local ring, at a point $x$, of a curve $C$ and $\mathfrak{p}_i$, $i = 1, \ldots, n$, are the minimal primes of $G(A)$ then $\text{Spec}(G(A)/\mathfrak{p}_i)$ are the tangents of $C$ at $x$ (that is the tangents are the lines of $\text{Spec}(G(A)_{\text{red}})$ and correspond to the points of $\text{Proj}(G(A))$). It is well known [10, Lemma-Definition 2.1] that each branch has a tangent and that the following conditions are equivalent:

(a) The scheme $\text{Proj}(G(A))$ is reduced;

(b) $\text{Proj}(G(A))$ consists of $e$ points;

(c) there are $e = e(A)$ tangents at $x$;

(d) the curve $C$ has, at $x$, $e$ linear branches with distinct tangents.

A point $x$ which satisfies these conditions is said to be an ordinary point. Any nonsingular point of the curve ($e = 1$) is a trivial example of ordinary point. Generally, but not always [9, Section 4], an ordinary point has reduced tangent cone, in the sense that, if its tangents are in generic position, then the tangent cone to $C$ at $x$ (i.e. $G(A)$) is reduced [10, Theorem 3.3].

In this paper we extend these results to subvarieties of codimension one.

For this we need first to extend the notion of ordinary point of a curve to the notion of ordinary closed point of a one-dimensional reduced local affine scheme $\text{Spec}(A)$. In fact let $\mathfrak{m}$ be the maximal ideal of $A$ and $K$ be the algebraic closure of the residue field $k(\mathfrak{m})$ of $A$. Then the closed point of $\text{Spec}(A)$ is said to be ordinary if $\text{Proj}(G(A) \otimes_{k(\mathfrak{m})} K)$ is reduced [Definition 2.4].
We prove that the previous equivalences \((a) \iff (b) \iff (c) \iff (d)\) continue to hold in this more general setting if we suitably define (geometric) tangents and (geometric) branches and we prove that if the ordinary closed point of \(\text{Spec}(A)\) has (geometric) tangents in generic position then \(G(A) \otimes_{k(m)} K\) is reduced.

Then, basing on this, using results of [2] and the notion of normal flatness, we extend the properties of ordinary points to subvarieties of codimension one. Let \(X = \text{Spec}(R)\) be an equidimensional algebraic variety and \(Y = \text{Spec}(R/q)\) be an irreducible codimension one subvariety of \(X\) of multiplicity \(e = e(R_q)\). If \(A\) is the local ring of \(X\) at a closed point \(x\) of \(Y\) then \(x\) is said to be an ordinary point if \(\text{Spec}(G(A)_{\text{red}})\) is the union of \(e\) distinct linear spaces. \(Y\) is said to be an ordinary subvariety of \(X\) if the closed point of the one-dimensional scheme \(\text{Spec}(R_q)\) is ordinary. Assume that \(Y\) is nonsingular at \(x\) and \(X\) is normally flat along \(Y\) at \(x\), then \(x\) is an ordinary point of \(X\) if and only if \(X\) has, at \(x\), \(e\) linear branches with distinct tangent spaces. If these tangent spaces are in generic position then \(G(A)\) is reduced [Theorem 3.4]. Moreover \(Y\) is an ordinary subvariety of \(X\) if and only if there exists an open non-empty subset \(U\) of \(Y\) such that every closed point \(x\) of \(U\) is ordinary [Theorem 3.5].

This paper has been motivated by [12] in which the conductor of a variety with ordinary (multiple) subvarieties of codimension one is computed. We would like to thank G. Castaldo for pointing out the paper [2].

Throughout the paper all rings are supposed to be commutative, with identity and noetherian.

Let \(A\) be a local ring with maximal ideal \(m\). With \(H^0(A, n) = \dim_k(m^n/m^{n+1})\), \(n \in \mathbb{N}\), we denote the Hilbert function of \(A\) and with \(e(A)\) the multiplicity of \(A\) at \(m\). The embedding dimension \(\text{emd}(A)\) of \(A\) is given by \(H^0(A, 1)\). With \(A^h\) we denote the henselization of \(A\) and with \(A^h_S\) a fixed strict henselization of \(A\) (see [6, Number 32, Section 18] or [13]). If \(p\) is an ideal of \(A\), 
\[ G_p(A) = \bigoplus_{n \geq 0}(p^n/p^{n+1}) \]

is the associated graded ring with respect to \(p\). By \(G(A)\) we denote the associated graded ring with respect to \(m\). \(\text{Spec}(G(A))\) is the tangent cone to \(\text{Spec}(A)\) at its closed point. If \(S = \bigoplus_{n \geq 0} S_n\) is a standard graded finitely generated algebra over a field \(k\), with maximal homogeneous ideal \(n\), \(H(S, n) = \dim_k S_n = H(S_n, n)\) denotes the Hilbert function of \(S\) and \(\text{emd}(S) = H(S, 1) = \text{emd}(S_n)\) denotes the embedding dimension of \(S\). The multiplicity of \(S\) is \(e(S) = e(S_n)\). On has \(e(A) = e(G(A))\) and \(\text{emd}(A) = \text{emd}(G(A))\). If \(k(m)\) is the residue field of \(A\), \(\text{Hom}_{k(m)}(m/m^2, k(m))\) is the tangent space of \(\text{Spec}(A)\) at its closed point. If \(A\) is regular then the tangent cone and the tangent space of \(A\) coincide.

If \(R\) is any ring \(\dim(R)\) denotes the dimension of \(R\).

## 1. Generalities on branches and normal flatness

**Definition 1.1.** Let \(A\) be a local reduced ring with maximal ideal \(m\). Let \(K\) be the algebraic closure of the residue field \(k(m) = A/m\) and \(\overline{A}\) be the normalization of \(A\). A branch of \(\text{Spec}(A)\) at its closed point is a point of \(\text{Spec}(\overline{A}/m\overline{A})\). A geometric branch of \(\text{Spec}(A)\) at its closed point is a point of \(\text{Spec}(\overline{A}/m\overline{A} \otimes_{k(m)} K)\). If \(X\) is a scheme a branch (respectively a geometric branch) of \(X\) at a closed point \(x\) is a branch (respectively a geometric branch) at the closed point of \(\text{Spec}(A)\), where \(A\) is the local ring of \(X\) at \(x\).
Remark. Since $\mathcal{A}$ is integral over $A$ the ring $\mathcal{A}/m\mathcal{A}$ is zero-dimensional and then the ring $\mathcal{A}/m\mathcal{A} \otimes_{k(m)} K$ is zero-dimensional [6, N° 24, Corollaire 4.1.4]. Then the points of the schemes $\text{Spec}(\mathcal{A}/m\mathcal{A})$ and $\text{Spec}(\mathcal{A}/m\mathcal{A} \otimes_{k(m)} K)$ are closed.

**Lemma 1.2.** If $A$ is a local reduced ring, there is a canonical bijection between the set of branches of $\text{Spec}(A)$ and the set of minimal primes of $A^h$ and there is a canonical bijection between the set of geometric branches of $A$ and the set of minimal primes of $A^{sh}$.

**Proof.** [5, Theorem 2.1]

**Definition 1.3.** Let $A$ be a local reduced ring and $\gamma$ be a branch (respectively a geometric branch) of $\text{Spec}(A)$ at its closed point. If $p$ is the minimal prime of $A^h$ (respectively $A^{hs}$) corresponding to $\gamma$, the order of the branch (respectively the order of the geometric branch) $e(\gamma)$ is the multiplicity $e(A^h/p)$ (respectively $e(A^{sh}/p)$). The branch (respectively the geometric branch) $\gamma$ is linear if $A^h/p$ (respectively $A^{hs}/p$) is a regular ring.

**Remark.** The order of a geometric branch is independent from the strict henselization of $A$ chosen [1, Corollary 1.9]

**Proposition 1.4.** Let $A$ be a reduced local ring of dimension one or the local ring at a closed point of an algebraic equidimensional reduced variety (over an algebraically closed field $k$). Let $\gamma_i$, $i = 1, ..., n$ be the branches (respectively the geometric branches) of $\text{Spec}(A)$ at the closed point. Then:

(a) $e(A) = \sum_{i=1}^{n} e(\gamma_i)$;

(b) $\gamma_i$ is linear if and only if its order is one;

(c) $n = e(A)$ if and only if all the branches are linear.

**Proof.** (a), (b) (see [1, Propositions 2.7 and 2.10]). (c) is obvious consequence of (a) and (b). □

**Definition 1.5.** Let $p$ be a prime ideal of a local ring $A$. $A$ is normally flat along $p$ if $p^n/p^{n+1}$ is flat over $A/p$, for all $n \geq 0$. Note that $A$ is normally flat along $p$ if and only if $G_p(A)$ is free over $A/p$.

**Theorem 1.6.** Let $A$ be a local ring and $p$ a prime ideal of $A$ such that $A/p$ is regular of dimension $d$ and $A$ is normally flat along $p$. Then:

(a) there is an isomorphism of graded $k$-algebras

$$G(A) \cong (G_p(A) \otimes_{A/p} k)[T_1, ..., T_d]$$

(b) the following rings have the same Hilbert functions:

$$G_p(A) \otimes_{A/p} k \text{ (over } k) \text{ and } G(A_p) \text{ (over } k(p))$$

**Proof.** (a) [7, Corollary (21.11)].

(b) [8, Chapter II, Corollary 2]. □

Let $X = \text{Spec}(R)$ be a reduced variety over an algebraically closed field $k$ and $Y = \text{Spec}(R/p)$ be an irreducible subvariety $Y$ of $X$. 
Definition 1.7. If $x$ is any closed point of $Y$, $A$ is the local ring of $X$ at $x$ and $p = qA$ is the prime ideal in $A$ defining the subvariety $Y$, then:

(i) $X$ is normally flat along $Y$ at $x$ if $A$ is normally flat along $p$;
(ii) $Y$ is nonsingular at $x$ if $A/p$ is regular.

Theorem 1.8. There exists an open nonempty subset $U$ of $Y$ such that, for every closed point $x$ of $U$, $Y$ is nonsingular at $x$ and $X$ is normally flat along $Y$ at $x$.

Proof. It is well known that the nonsingular points of $Y$ form an open nonempty set and that $X$ is normally flat along $Y$ at the points of an open nonempty subset of $Y$ [7, Corollary (24.5)]. $\square$

2. Ordinary closed points of one-dimensional affine reduced schemes.

In this section $A$ is a reduced local one-dimensional ring of multiplicity $e = e(A)$ and maximal ideal $m$. With $K$ we denote the algebraic closure of the residue field $k(m)$ of $A$ and with $\bar{A}$ we denote the normalization of $A$.

Lemma 2.1. (a) There is a natural immersion $G(A) \subset G(A) \otimes_{k(m)} K$;
(b) $G(A) \otimes_{k(m)} K$ is one-dimensional;
(c) the Hilbert functions of $A$, $G(A)$ and $G(A) \otimes_{k(m)} K$ are the same, moreover $e(A) = e(G(A)) = e(G(A) \otimes_{k(m)} K)$ and $\text{emdim}(A) = \text{emdim}(G(A)) = \text{emdim}(G(A) \otimes_{k(m)} K)$.

Proof. (a) The immersion follows by the flatness of $G(A)$ over $k(m)$

(b) [6, N$^0$ 24, Corollaire 4.1.4]

(c) $G(A) \otimes K$ is the $K$-vector space obtained extending to $K$ the field of scalars of the $k(m)$-vector space $G(A)$, then the Hilbert functions of $G(A)$ (that is of $A$) and of $G(A) \otimes K$ are the same. This implies the equalities of the multiplicities and of the embedding dimensions. $\square$

Since, by Lemma 2.1, $G(A) \otimes_{k(m)} K$ is a graded one-dimensional algebra over the algebraically closed field $K$, $\text{Spec}((G(A) \otimes_{k(m)} K)_{\text{red}})$ is a union of lines each corresponding to a point of $\text{Proj}((G(A) \otimes_{k(m)} K))$.

Definition 2.2. the lines of $\text{Spec}((G(A) \otimes_{k(m)} K)_{\text{red}})$ are the geometric tangents of $\text{Spec}(A)$ at its closed point.

Let $n$ be a positive integer and $m^n : m^n = \{b \in A \mid bm^n \subset m^n\}$. We recall that the subring of $\bar{A}$, $B = \cup_{n>0}(m^n : m^n)$, is the ring obtained by blowing up the maximal ideal $m$ of $A$.

The natural isomorphism of schemes $\text{Spec}(B/mB) \cong \text{Proj}(G(A))$ [6, N$^0$ 32, Lemma 19.4.2] induces an isomorphism of schemes

$$\phi : \text{Spec}(B/mB \otimes_{k(m)} K) \longrightarrow \text{Proj}(G(A) \otimes_{k(m)} K)$$

By Definition 1.1 a geometric branch $\gamma$ of $\text{Spec}(A)$ is a point of $\text{Spec}(\bar{A}/m\bar{A} \otimes_{k(m)} K)$. The inclusion $B \subset \bar{A}$ induces a natural homomorphism of zero-dimensional rings

$$B/mB \otimes_{k(m)} K \longrightarrow \bar{A}/m\bar{A} \otimes_{k(m)} K$$
and then a morphism

$$\psi : \text{Spec}(\overline{A}/m\overline{A} \otimes_{k(m)} K) \longrightarrow \text{Spec}(B/mB \otimes_{k(m)} K)$$

Hence $\phi(\psi(\gamma))$ is a point of $\text{Proj}(G(A) \otimes_{k(m)} K)$.

**Definition 2.3.** The line of $\text{Spec}((G(A) \otimes_{k(m)} K)_{\text{red}})$ corresponding to $\phi(\psi(\gamma))$ is a geometric tangent of $\text{Spec}(A)$ which we call the geometric tangent to $\gamma$.

**Definition 2.4.** The closed point of the scheme $\text{Spec}(A)$ is said to be ordinary if $\text{Proj}(G(A) \otimes_{k(m)} K)$ is reduced.

**Theorem 2.5.** The following conditions are equivalent:

(a) the closed point of $\text{Spec}(A)$ is ordinary

(b) $\text{Proj}(G(A) \otimes_{k(m)} K)$ consists of $e$ points

(c) there are $e$ geometric tangents to $\text{Spec}(A)$

(d) $\text{Spec}(A)$ has $e$ linear geometric branches with distinct geometric tangents.

**Proof.** (a) $\Leftrightarrow$ (b). Set $R = G(A) \otimes_{k(m)} K$ and let $q_i, i = 1, \ldots, n$, be the minimal primary ideals of $R$ and $p_i = \sqrt{q_i}$ be the corresponding minimal primes. We have to prove that $\text{Proj}(R)$ is reduced if and only if $n = e = e(A)$. But $\text{Proj}(R)$ is reduced if and only if $q_i = p_i$, for any $i$. By standard facts on multiplicity we have $e(R) = \sum_{i=1}^{n} \lambda(q_i)e(R/p_i) = \sum_{i=1}^{n} \lambda(q_i)$, where $\lambda(q_i)$ denotes the length of the ideal $q_i$ (note that $e(R/p_i) = 1$ because $\text{Spec}(R/p_i)$ is a line). Moreover, by Lemma 2.1, (c), $e(A) = e(R)$ and then the claim follows from the fact that $\lambda(q_i) = 1$ if and only if $q_i$ is prime.

(b) $\Leftrightarrow$ (c). Follows immediately from Definition 2.2.

(c) $\Leftrightarrow$ (d). By definition each geometric branch has a geometric tangent and then the claim is clear if we consider Proposition 1.4. (c) \qed

**Example 2.6.** Let the normalization $\overline{A}$ be finite over $A$. If $k(m)$ has characteristic zero and $A$ is a seminormal ring the closed point of $\text{Spec}(A)$ is an ordinary point with $e(A) = \text{emdim}(A)$ tangents. In fact $A$ is seminormal if and only if $\text{emdim}(A) = e(A)$ and $G(A)$ is reduced [3, Theorem 1]. But, By Lemma 2.1, (a) we have the inclusion $G(A) \subset G(A) \otimes_{k(m)} K$. Hence $G(A) \otimes K$ is the extension to $K$ of the $k(m)$ algebra $G(A)$ and is reduced if $k(m)$ has characteristic zero. In general, if $A$ is seminormal, the closed point of $\text{Spec}(A)$ needs not to be ordinary. In fact let $k$ be a field of characteristic two with algebraic closure $\overline{k}$. Let $b^2 = a = -a \in k$, with $b \in \overline{k} - k$. Set $A = (k[X,Y,Z]/(Y^2 - aX^2 - X^3))(X,Y,Z)$. Clearly $e(A) = \text{emdim}(A) = 2$ and $G(A) = k[X,Y]/(Y^2 - aX^2)$ is reduced. Hence $A$ is seminormal. But $Y^2 - aX^2 = (Y + bX)^2$. Moreover $k(m) = k$ and $K = \overline{k}$. Then $G(A) \otimes_{k(m)} K = \overline{k}[X,Y]/(Y + bX)^2$ is not reduced.

Let $R = k[x_0, \ldots, x_r]$ be a reduced standard graded finitely generated $k$-algebra over an algebraically closed field $k$. Let $\text{dim}(R) = 1$ and set $e(R) = e$. Then $\text{Proj}(R) = \{ P_1, \ldots, P_e \} \subset \mathbb{P}^r$ is a finite set of points. Vice versa any set of projective points $\{ P_1, \ldots, P_e \} \subset \mathbb{P}^r$ is equal to $\text{Proj}(R)$ where $R$ is its homogeneous coordinate ring. It is well known that, for any $p \in \mathbb{N}$, $H^0(R, p) \leq \text{Min} \{ (e + 1)^p \}$.
Definition 2.7. The set \( \{ P_1, \ldots, P_e \} \subset \mathbb{P}^r \) is in generic position in \( \mathbb{P}^r \) (or the points \( P_1, \ldots, P_e \) are in generic position in \( \mathbb{P}^r \)) if the Hilbert function of \( R \) is maximal that is \( H^0(R, n) = \text{Min}\{e, \binom{n+r}{r}\} \) [11, Definition 3.1].

Remarks. 1. It is proved in [4, Theorem 4] that, for any \( e \) and \( r \), “generic position” is an open nonempty condition.

2. For details on the notion of points in generic position, see [11]

Example 2.8. It is easily seen that any set of points of \( \mathbb{P}^1 \) is in generic position.

Example 2.9. A set of \( \binom{n+r}{r} \) points in \( \mathbb{P}^r \) \((n > 0, r > 0)\) is in generic position if and only if they do not lie on a hypersurface of degree \( n \) [11, Corollary 3.4], in particular six points in \( \mathbb{P}^2 \) are in generic position if and only if they do not lie on a conic.

Let \( S = R[T_1, \ldots, T_n] \), \( n \geq 0 \), be the polynomial ring over a reduced standard graded finitely generated \( k \)-algebra \( R = k[x_0, \ldots, x_r] \) (\( k \) algebraically closed). Let \( \dim(R) = 1 \). Then \( \text{Spec}(S) = \{ L_1, \ldots, L_e \} \subset \mathbb{A}^{n+r+1} \), is a set of linear varieties of dimension \( n+1 \) all containing a linear subvariety (through the origin) \( L \) of dimension \( n \). Vice versa any set of such linear varieties is equal to \( \text{Spec}(S) \) where \( S \) is its affine coordinate ring.

Definition 2.10. The set of linear varieties \( \{ L_1, \ldots, L_e \} \) is in generic position in \( \mathbb{A}^{n+r+1} \) if the set of points \( \text{Proj}(R) \) is in generic position in \( \mathbb{P}^r \).

Example 2.11. A union of lines of \( \mathbb{A}^{r+1} \) through the origin is in generic position if the corresponding set of projective points of \( \mathbb{P}^r \) is in generic position. In particular, if \( \text{emdim}(A) = r+1 \), the set of geometric tangents \( \text{Spec}(\text{Proj}(G(A) \otimes_{k(m)} K)_{\text{red}}) \subset \mathbb{A}^{r+1} \) of \( \text{Spec}(A) \) at its maximal ideal is in generic position in \( \mathbb{A}^{r+1} \) if the Hilbert function of \( (G(A) \otimes_{k(m)} K)_{\text{red}} \) is maximal.

Theorem 2.12. Let \( \text{emdim}(A) = r+1 \). If the closed point of \( \text{Spec}(A) \) is ordinary with geometric tangents in generic position in \( \mathbb{A}^{r+1} \) then the rings \( G(B) \otimes_{k(m)} K \) and \( G(B) \) are reduced.

Proof. Set \( D = G(A) \otimes K \) and \( D_{\text{red}} = D/\text{nil}(D) \). By the surjective graded homomorphism \( \phi : D \rightarrow D_{\text{red}} \) we deduce that \( H(D_{\text{red}}, n) \leq H(D, n) \), for any \( n \). If we prove that \( H(D_{\text{red}}, n) = H(D, n) \) then \( \phi \) is an isomorphism and \( D \) is reduced (and \( G(A) \subset D \) [Lemma 2.1,(a)] is reduced). But \( H(D_{\text{red}}, n) \leq H(D, n) \leq \text{Min}\{e, \binom{n+r}{r}\} \), and \( H(D_{\text{red}}, n) = \text{Min}\{e, \binom{n+r}{r}\} \), by assumption, whence the result. \( \square \)

Remark. If \( A \) is the local ring at a point of a curve over an algebraically closed field \( k \) then \( k(m) = k \) and \( G(A) \otimes_{k(m)} K = G(A) \). Hence, in this case Theorems 2.5 and 2.12 give the results of [10, Lemma-Definition 2.1 and Theorem 3.3]. Although, by Theorem 2.12, an ordinary point of a curve has, in general, reduced tangent cone, there are classes of examples of ordinary points whose tangent cone is not reduced [9, Section 4].

3. Ordinary codimension one subvarieties.

In this section \( X = \text{Spec}(R) \) is a reduced variety over an algebraically closed field \( k \) and \( Y = \text{Spec}(R/q) \) is an irreducible codimension one subvariety \( Y \) of \( X \) (i.e., \( R_q \) is one-dimensional). By \( K \) we denote the algebraic closure of the residue field \( k \).
field \( k(q) \) of \( R \) at \( q \). We say that \( e(R_q) = e \) is the multiplicity of \( Y \) (on \( X \)) and we set \( \text{endim}(R_q) = r + 1 \). Note that, if \( A \) is the local ring of \( X \) at a closed point \( x \) of \( Y \) and \( p = qA \) is the prime ideal in \( A \) defining \( Y \), then \( A_p = R_q \). Hence \( k(q) \) is equal to the residue field \( k(p) \) of \( A \) in \( p \).

**Definition 3.1.** Let \( A \) be the local ring of \( X \) at a closed point \( x \) of \( Y \). Let \( \gamma \) be a branch of \( X \) at \( x \) and \( p \) the corresponding minimal prime of \( A^h \) [Lemma 1.2]. \( \text{Spec}(G(A^h/p)) \) is the tangent cone to the branch \( \gamma \) and \( \text{Spec}(G(A^h/p)_\text{red}) \) is the tangent cone, as a set, to \( \gamma \). If the branch \( \gamma \) is linear the ring \( A^h/p \) is regular [Definition 1.3] and then its tangent cone coincides with the tangent space and we call it the tangent space to the branch \( \gamma \).

**Theorem 3.2.** Let \( x \) be a point of \( Y \) and \( A \) be the local ring of \( X \) at \( x \). Suppose \( Y \) is nonsingular at \( x \) and \( X \) is normally flat along \( Y \) at \( x \). Let \( \gamma \) be a branch of \( X \) at \( x \). Then \( \text{Spec}(G(A)_\text{red}) \) (that is the tangent cone to \( X \) at \( x \), as a set) is union of linear varieties. Moreover the tangent cone to the branch \( \gamma \) is, as a set, a linear variety of \( \text{Spec}(G(A)_\text{red}) \) and vice versa each linear variety of \( \text{Spec}(G(A)_\text{red}) \) is the tangent cone, as a set, of at least one branch.

**Proof.** \( \text{Spec}(G(A)_\text{red}) \) is union of linear varieties if and only if the minimal primes of \( G(A)_\text{red} \) are generated by linear forms. Moreover, if \( p \) is the prime of \( Y \) in \( A \), \( (G_p(A) \otimes_{A/p} k)_\text{red} \) is a one-dimensional reduced \( k \)-algebra over the algebraically closed field \( k \) and then its minimal primes are generated by linear forms. But the isomorphism \( G(A) \cong (G_p(A) \otimes_{A/p} k)[T_1,...,T_d] \) of Theorem 1.6,(a) induces an isomorphism of graded \( k \)-algebras \( G(A)_\text{red} \cong (G_p(A) \otimes_{A/p} k)_\text{red}[T_1,...,T_d] \) and then also the minimal primes of \( G(A)_\text{red} \) are generated by linear forms.

The second claim is proved in [2,Theorem 4.5]. □

**Definition 3.3.** Let \( Y \) have multiplicity \( e \) on \( X \).

(a) A closed point \( x \) of \( Y \) is an ordinary point if, as a set, the tangent cone to \( X \) at \( x \) is the union of \( e \) linear varieties;

(b) \( Y \) is an ordinary subvariety of \( X \) if the closed point of the one-dimensional scheme \( \text{Spec}(R_q) \) is ordinary.

**Theorem 3.4.** Let \( Y \) have multiplicity \( e \) on \( X \) and \( x \) be a closed point of \( Y \). If \( Y \) is nonsingular at \( x \) and \( X \) is normally flat along \( Y \) at \( x \), then \( x \) is an ordinary point of \( X \) if and only if \( X \) has, at \( x \), \( e \) linear branches with distinct tangent spaces. Moreover if these tangent spaces are in generic position then the tangent cone of \( X \) at \( x \) is reduced.

**Proof.** Let \( A \) be the local ring of \( X \) at \( x \). If \( x \) is an ordinary point of \( X \) then \( \text{Spec}(G(A)_\text{red}) \) is the union of \( e \) linear varieties [Definition 3.3] which by Theorem 3.2 are the tangent spaces of \( e \) branches and these are linear by Proposition 1.4,(c). If \( x \) has \( e \) branches with distinct tangent spaces then, by Theorem 3.2 these are components of \( G(A)_\text{red} \) and then \( x \) is ordinary. Now we prove the second claim. Let \( p \) be the prime of \( Y \) in \( A \) and consider the isomorphism \( G(A)_\text{red} \cong (G_p(A) \otimes_{A/p} k)_\text{red}[T_1,...,T_d] \) induced by the isomorphism of Theorem 1.6,(a). By Definition 2.10 \( \text{Spec}(G(A)_\text{red}) \) is the union of \( e \) linear varieties in generic position if and only if \( \text{Proj}((G_p(A) \otimes_{A/p} k)_\text{red}) \) consists of points in generic position that is, if we set \( D = G_p(A) \otimes_{A/p} k \), \( H(D_\text{red},n) = \text{Min}\{e, (n+r) \} \) But the Hilbert function of \( G_p(A) \otimes_{A/p} k \) is equal to the Hilbert function of \( G(A) \otimes_{A/p} k \).
then $e = e(A) = e(G(A_p))$. Hence $H(D, n) \leq \text{Min}\{e, {n+r\choose r}\}$. Then $H(D_{\text{red}}, n) = H(D, n)$, and $D = D_{\text{red}}$ is reduced. Thus also $G(A)$, which is isomorphic to a polynomial ring over $D$, is reduced. \[\square\]

**Theorem 3.5.** The following conditions are equivalent:

(a) $Y$ is an ordinary subvariety of $X$

(b) there exists an open nonempty subset $U$ of $Y$ such that every closed point $x$ of $U$ is ordinary.

(c) there exists an open nonempty subset $U$ of $Y$ such that $X$ has $e$ linear branches with distinct tangent spaces, at any point of $U$.

**Proof.** Let $Y$ have multiplicity $e$ on $X$. Then $e(G(R_q) \otimes_{k(q)} K) = e$ [Lemma 2.1,(c)].

(a) $\Rightarrow$ (b) If $Y$ is ordinary $\text{Proj}(G(R_q) \otimes_{k(q)} K)$ has $e$ points (that is $e$ irreducible components) [Definition 3.3,(b) and Theorem 2.5,(a) $\Rightarrow$ (b)]. Then by [6, N° 28, Proposition 9.7.8] there exists an open nonempty subset $U_1$ of $Y$ such that, for every closed point $x$ of $U_1$, $G_p(A) \otimes_{A/p} k$ (the local ring of $X$ at $x$) has $e$ irreducible components and is $e$ points [see the proof of Theorem 2.5,(b) $\Leftrightarrow$ (a)]. Then the $e$ minimal primes of $G_p(A) \otimes_{A/p} k$ are generated by linear forms. Furthermore, by Theorems 1.6 and 1.8, there exists an open nonempty subset $U_2$ of $Y$ such that, for every closed point $x$ of $U_2$, $G(A)$ is isomorphic to a polynomial ring over $G_p(A) \otimes_{A/p} k$. Then if $x$ is a point of $U_1 \cap U_2$ the minimal primes of $G(A)$ are extensions of the minimal primes of the one dimensional finitely generated graded $k$-algebra $G_p(A) \otimes_{A/p} k$, hence they are generated by linear forms.

(b) $\Rightarrow$ (a) Suppose that, at every point $x$ of an open nonempty subset of $Y$, $\text{Spec}(G(A)_{\text{red}})$ has $e$ irreducible components, that is $G(A)_{\text{red}}$ has $e$ minimal primes. Then, by the graded isomorphism $G(A)_{\text{red}} \cong (G_p(A) \otimes_{A/p} k)_{\text{red}}[T_1, \ldots, T_d]$, on the points of an open set of $Y$, $(G_p(A) \otimes_{A/p} k)_{\text{red}}$ has $e$ minimal primes [6, N° 28, Proposition 9.7.8]. Hence the zero-dimensional scheme $\text{Proj}(G(A_p) \otimes_{k(p)} K) = \text{Proj}(G(R_q) \otimes_{k(q)} K)$ has $e$ points, and then, since by Lemma 2.1,(c) $e(G(A_p) \otimes_{k(p)} K) = e(G(A_p)) = e(A_p) = e$, is reduced [Theorem 2.5, (b) $\Rightarrow$ (a)].

(b) $\Leftrightarrow$ (c) It is an easy consequence of Theorem 1.8 and Theorem 3.4. \[\square\]

Let $Y$ be an ordinary subvariety of codimension one and of multiplicity $e$ on a reduced variety $X$. Let $x$ be a closed point of $Y$ such that $Y$ is nonsingular at $x$ and $X$ is normally flat along $Y$ at $x$. Then by Theorem 3.2 the tangent cone of $X$ at $x$ consists, as a set, of linear varieties, but the number of these can be less than $e$ and then $x$ is not an ordinary point of $X$ as the following example shows.

**Example 3.6.** Let $R = \mathbb{C}[X_1, X_2, X_3]/(X_1X_2^n - X_3^n) = \mathbb{C}[x_1, x_2, x_3]$ ($n \geq 2$) and $A$ be the local ring of $X = \text{Spec}(R)$ at the maximal ideal $(x_1, x_2, x_3)$. The non-normal locus of the hypersurface $X$ is the line $Y : x_2 = 0, x_3 = 0$ of multiplicity $n$ on $X$ and if $a \in \mathbb{C}$, $a \neq 0$, the tangent cone at the point $(a, 0, 0)$ of $L$ consists, as a set, of the $n$ distinct planes $x_3 = bx_2$, where $b^n = a$. Then $Y$ is an ordinary subvariety of $X$. Moreover $e(A) = e(R_q) = n$, where $q = (x_2, x_3)$. Then, by [7, Corollary (23.22)], $X$ is normally flat along $Y$ at $x$ and clearly $Y$ is non singular at $x$. But the tangent cone of $X$ at $(0, 0, 0)$ is $\text{Spec}(\mathbb{C}[X_1, X_2, X_3]/(X_3^n))$ and then, as a set, consists of the plane $x_3 = 0$. But $e = e(A) = n \geq 2$ and $(0, 0, 0)$ is not ordinary.
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References

[1] C. Cumino, *On the order of branches*, in "Commutative Algebra" (Marcel Dekker, ed.), Lecture Notes in Pure and Applied Mathematics, vol. 84, 1981, pp. 49-64.

[2] C. Cumino, *Tangent cones and analytic branches*, Rev. Roumaine Math. Pures Appl. 31 (1986), 843-854.

[3] E.D. Davis, *On the geometric interpretation of seminormality*, Proc. Am. Math. Soc. 68 (1978), 1-5.

[4] A.V. Geramita, F. Orecchia, *On the Cohen-Macaulay type of $s$-lines in $\mathbb{A}^{n+1}$*, J. Algebra 70 (1981), 116-140.

[5] S. Greco, *On the theory of branches*, Int. Symp. of Algebraic Geometry, Kioto (1977), 477-493.

[6] A. Grothendieck, J. Dieudonné, *Éléments de Géométrie Algébrique*, Publ. Math. I.H.E.S. Ch IV (1964-1967).

[7] M. Herrmann, S. Ikeda, U. Orbanz, *Equimultiplicity and blowing up*, Springer-Verlag, 1988.

[8] H. Hironaka, *Resolution of singularities of an algebraic variety over a field of characteristic zero*, Ann. Math. 79 (1964), 109-326.

[9] F. Orecchia, *One-dimensional local rings with reduced associated graded ring and their Hilbert function*, Manuscripta Math. 32 (1980), 391-405.

[10] F. Orecchia, *Ordinary singularities of algebraic curves*, Can. Math. Bull. 24 (1981), 423-431.

[11] F. Orecchia, *Points in generic position and conductors of curves with ordinary singularities*, J. London Math. Soc. 24 (1981), 85-96.

[12] F. Orecchia, *Points in generic position and conductor of varieties with ordinary multiple subvarieties of codimension one*, J. of Pure and Appl. Algebra, to appear.

[13] M. Raynaud, *Anneaux locaux henséliens*, Lecture Notes Math. 169 (1969), Springer Verlag.

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