Fundamental group schemes of generalized Kummer variety

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Abstract. Let $k$ be an algebraically closed field of characteristic $p > 3$. Let $A$ be an abelian surface over $k$. Fix an integer $n \geq 1$ such that $p \nmid n$ and let $K^{[n]}$ be the $n$-th generalized Kummer variety associated to $A$. In this article, we show that the $S$-fundamental group scheme and the Nori’s fundamental group scheme of $K^{[n]}$ are trivial.

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1. Introduction

Let $A$ be an abelian surface over an algebraically closed field $k$. For any positive integer $n$, let $S^n(A)$ be the $n$-th symmetric product of $A$ and $A^{[n]}$ be the Hilbert scheme of closed subschemes of length $n$ on $A$. Let $\varphi : A^{[n]} \to S^n(A)$ denote the Hilbert–Chow morphism and $\Sigma : S^n(A) \to A$ be the addition morphism. Let $\rho := \Sigma \circ \varphi : A^{[n]} \to A$ be the composition morphism. If $n \geq 2$, then the fiber $K^{[n]} := \rho^{-1}(e)$ over the identity element $e$ of $A$ is a projective scheme of dimension $2(n - 1)$, called the $n$-th generalized Kummer variety associated to $A$. For example, $K^{[2]}$ is just the Kummer $K3$ surface associated to $A$. For a complex abelian surface $A$, it is known that the $n$-th generalized Kummer variety $K^{[n]}$ associated to $A$ is simply connected as shown by Beauville in [1, Proposition 8].

Let $X$ be a connected, reduced and complete scheme over a perfect field $k$. Fix a $k$-rational point $x \in X$. Nori introduced a $k$-group scheme $\pi^N(X, x)$ associated to essentially finite locally free sheaves on $X$ in [11] and further extended the definition of $\pi^N(X, x)$ to connected and reduced $k$-schemes in [12]. In [3], Biswas et al. defined the notion of $S$-fundamental group scheme $\pi^S(X, x)$ for $X$, a smooth projective curve over any algebraically closed field $k$ as a group scheme associated to a certain Tannakian category of locally free sheaves on $X$. This is generalized to higher-dimensional connected smooth projective $k$-schemes and studied by Langer [9,10]. In general, $\pi^S(X, x)$ carries more information than $\pi^N(X, x)$ and $\pi^{\text{ét}}(X, x)$. A precise definition of the above objects are given in Sect. 2. The fundamental group scheme is an interesting object in algebraic geometry of positive characteristics. It is an interesting problem to determine $\pi^{\text{ét}}(X, x)$, $\pi^N(X, x)$ and $\pi^S(X, x)$ for well-known varieties. Some articles addressing these types
of questions are [2,13]. In [2], Biswas and Hogadi have studied the fundamental group of a variety with quotient singularities. In [13], Paul and Sebastian have computed the fundamental group schemes of Hilbert schemes of $n$ points on irreducible smooth projective surfaces over algebraically closed fields of positive characteristic.

In this paper, our goal is to compute the fundamental group schemes of the generalized Kummer varieties over algebraically closed field of characteristic $p > 3$. The following theorem is the main result of this article.

**Theorem (Theorem 5.1.1).** Let $k$ be an algebraically closed field of characteristic $p > 3$ such that $p \nmid n$. Then the $S$-fundamental group scheme $\pi^S(K[n], n[e])$ of the $n$-th generalized Kummer variety $K[n]$ over $k$ is trivial.

As a corollary, we get as follows.

**Theorem. (Corollary 5.2.2).** Let $k$ be an algebraically closed field of characteristic $p > 3$ such that $p \nmid n$. Then the Nori’s fundamental group scheme $\pi^N(K[n], n[e])$ of the $n$-th generalized Kummer variety $K[n]$ over $k$ is trivial.

Now we describe the organization of this paper. In Sect. 2, we recall the definitions of fundamental group schemes from [12] and [9]. We also recall some properties which will be needed later. In Sect. 3, we show that for an étale Galois cover $f : X \rightarrow Y$ with Galois group $G$, there is a short exact sequence $1 \rightarrow \pi^S(X, x_0) \rightarrow \pi^S(Y, y_0) \rightarrow G \rightarrow 1$. In Sect. 4, we show some results we need about the $n$-th generalized Kummer variety $K[n]$. In Sect. 5, we use some results from [13] about the fundamental group scheme of $A^n$ to prove that the fundamental group schemes of $K[n]$ are trivial.

## 2. Fundamental group scheme

In the rest of this article, unless mentioned otherwise, $k$ will denote an algebraically closed field of characteristic $p > 0$.

### 2.1 S-fundamental group scheme

Let $X$ be a complete connected reduced scheme defined over $k$.

**DEFINITION 2.1.1**

A locally free sheaf $E$ on $X$ is called nef if $O_{P(E)}(1)$ is nef on the projectivization $P(E)$ of $E$. A locally free sheaf $E$ on $X$ is called numerically flat if both $E$ and its dual $E^\vee$ are nef.

Let $f : X \rightarrow Y$ be a surjective morphism of complete $k$-varieties. Then the locally free sheaf $E$ on $Y$ is nef if and only if $f^*E$ is nef [8, p. 303, Proposition 1]. Similarly, since pull back commutes with dualization, we have the following.

**Lemma 2.1.2.** Let $f : X \rightarrow Y$ be a surjective morphism of complete $k$-varieties. Let $E$ be a locally free sheaf on $Y$. Then $E$ is numerically flat if and only if $f^*E$ is numerically flat.
Let $C^{nf}(X)$ denote the full subcategory of the category of coherent sheaves on $X$ whose objects are all numerically flat locally free sheaves. Let $\text{Vect}_k$ be the category of finite dimensional $k$-vector spaces. Let $x \in X$ be a $k$-rational point of $X$ and $T_x : C^{nf}(X) \to \text{Vect}_k$ be the fiber functor which sends an object $E$ of $C^{nf}(X)$ to its fiber $E_x$ over $x$. Then $(C^{nf}(X), \otimes, T_x, \mathcal{O}_X)$ is a neutral Tannaka category [10, Section 2.1]. The affine $k$-group scheme $\pi^S(X, x)$ Tannaka dual to this category is called the $S$-fundamental group scheme of $X$ with base point $x$ [10, Definition 2.1].

Let $X$ and $Y$ be two complete $k$-varieties and $x \in X, y \in Y$ be $k$-rational points. Let $f : X \to Y$ be a morphism between them with $f(x) = y$. Then pullback by $f$ induces a map $f^* : C^{nf}(Y) \to C^{nf}(X)$ which is clearly a morphism of Tannakian categories, i.e.,

$$f^* : (C^{nf}(Y), \otimes, T_y, \mathcal{O}_Y) \to (C^{nf}(X), \otimes, T_x, \mathcal{O}_X).$$

This induces a morphism of $S$-fundamental group schemes:

$$\tilde{f} : \pi^S(X, x) \to \pi^S(Y, y).$$

Let $X$ and $Y$ be two complete $k$-varieties and $x \in X, y \in Y$ be $k$-rational points. We consider the projection morphisms $\text{pr}_X : X \times_k Y \to X$ and $\text{pr}_Y : X \times_k Y \to Y$. Let $\hat{\text{pr}}_X$ and $\hat{\text{pr}}_Y$ be the morphisms induced on $S$-fundamental group schemes by the two projections. Then by [10, Theorem 4.1], we have the following theorem.

\textbf{Theorem 2.1.3.} \textit{The natural homomorphism}

$$\hat{\text{pr}}_X \times \hat{\text{pr}}_Y : \pi^S(X \times_k Y, (x, y)) \to \pi^S(X, x) \times_k \pi^S(Y, y)$$

\textit{is an isomorphism.}

\section*{2.2 Nori’s fundamental group scheme}

Let $X$ be a connected, proper and reduced $k$-scheme.

A locally free sheaf $E$ on $X$ is said to be finite if there are distinct non-zero polynomials $f, g \in \mathbb{Z}[t]$ with non-negative coefficients such that $f(E) \cong g(E)$.

\textbf{DEFINITION 2.2.1}

A locally free sheaf $E$ on $X$ is said to be essentially finite if there exist two numerically flat locally free sheaves $V_1, V_2$ and finitely many finite locally free sheaves $F_1, \ldots, F_n$ on $X$ with $V_2 \subseteq V_1 \subseteq \bigoplus_{i=1}^n F_i$ such that $E \cong V_1 / V_2$.

Let $EF(X)$ denote the full subcategory of the category of coherent sheaves on $X$ whose objects are essentially finite locally free sheaves on $X$. Let $x \in X$ be a $k$-rational point of $X$ and $T_x : EF(X) \to \text{Vect}_k$ be the functor defined by sending an object $E \in EF(X)$ to its fiber $E_x$ at $x$. Then $(EF(X), \otimes, T_x, \mathcal{O}_X)$ is a neutral Tannakian category. The affine $k$-group scheme $\pi^N(X, x)$ representing the functor of $k$-algebras $\text{Aut}^\otimes (T_x)$ is called Nori’s fundamental group scheme of $X$ based at $x$. (The definition of the functor $\text{Aut}^\otimes (T_x)$ can be found in [4, Section 1].)
3. A short exact sequence of fundamental group schemes

For any group scheme $G$ over $k$, let $\text{Rep}_k(G)$ denote the category of representations of $G$ into finite dimensional $k$-vector spaces. Let $\theta : G \rightarrow H$ be a homomorphism of affine group schemes over $k$ and let

$$\theta^* : \text{Rep}_k(H) \rightarrow \text{Rep}_k(G)$$

be the functor given by sending a representation $\alpha : H \rightarrow \text{GL}(V)$ to $\alpha \circ \theta : G \rightarrow \text{GL}(V)$.

Let $f : X \rightarrow Y$ be a morphism between two smooth projective $k$-varieties and $f(x_0) = y_0$. Then we have seen in Section 2 that $f$ induces a morphism of $S$-fundamental group schemes

$$\tilde{f} : \pi^S(X, x_0) \rightarrow \pi^S(Y, y_0).$$

Further, let $f$ be an étale Galois cover with Galois group $G$. Then the group $G$ has a right action on $X$. Each $g \in G$ defines an automorphism, say $\theta_g \in \text{Aut}(X)$. This induces a left action of $G$ on $\mathcal{O}_X$, which is defined by $g \cdot f = f \circ \theta_g$, where $g \in G$ and $f \in \Gamma(U, \mathcal{O}_X)$, $U$ being an open set in $X$. Let $\sigma : G \rightarrow \text{GL}(V)$ be any object of $\text{Rep}_k(G)$. Then $\mathcal{O}_X \otimes_k V$ is a locally free sheaf on $X$ with a left action of $G$ on it, given by

$$g \cdot (f \otimes v) = (f \circ \theta_g) \otimes \sigma(g)(v)$$

for any $g \in G$, $v \in V$ and $f \in \Gamma(U, \mathcal{O}_X)$, $U$ being an open set in $X$. Let $(\mathcal{O}_X \otimes_k V)^G$ be the subsheaf of $G$-invariants in $(\mathcal{O}_X \otimes_k V)$ with this action of $G$. Then $E_\sigma := (\mathcal{O}_X \otimes_k V)^G$ is a locally free sheaf on $Y$ (see [11, Section 2, p. 31]). The pullback of $E_\sigma$ on $X$ is trivial (by [11, Theorem 2.6]). So by Lemma 2.1.2, it follows that $E_\sigma$ is numerically flat on $Y$. Thus each $\sigma \in \text{Rep}_k(G)$ gives an associated locally free sheaf $E_\sigma \in \text{C}^{nf}(Y)$. This defines a functor $P^*_f : \text{Rep}_k(G) \rightarrow \text{C}^{nf}(Y)$. By the equivalence of categories $\text{C}^{nf}(Y)$ and $\text{Rep}_k(\pi^S(Y, y_0))$, we get a morphism $P_f : \pi^S(Y, y_0) \rightarrow G$ of group schemes. We will denote this map just by $P$ when no confusion can arise. So given an étale Galois cover $f : X \rightarrow Y$ with Galois group $G$, we get a sequence of morphisms of group schemes:

$$\pi^S(X, x_0) \xrightarrow{\tilde{f}} \pi^S(Y, y_0) \xrightarrow{P} G.$$

**Remark 3.0.1.** By [9, Lemma 6.2], there exists a natural faithfully flat homomorphism $\pi^S(Y, y_0) \rightarrow \pi^{\acute{e}t}(Y, y_0)$ of group schemes. Moreover, $P$ is a composition of the natural maps

$$\pi^S(Y, y_0) \rightarrow \pi^{\acute{e}t}(Y, y_0) \rightarrow G,$$

both of which are faithfully flat. So $P$ is faithfully flat.

We will use Theorem A.1 of [5, Appendix A] to prove our main result of this section.

**PROPOSITION 3.0.2** (Theorem A.1 [5])

Let $L \xrightarrow{q} G \xrightarrow{\pi} A$ be a sequence of homomorphisms of affine group schemes over a field $k$ with induced sequence of functors $\text{Rep}_k(A) \xrightarrow{\pi^*} \text{Rep}_k(G) \xrightarrow{q^*} \text{Rep}_k(L)$. Then

1. the map $q : L \rightarrow G$ is a closed immersion if and only if any object of $\text{Rep}_k(L)$ is a subquotient of object of the form $q^*(V)$ for some $V \in \text{Rep}_k(G)$. 

(2) Assume that \( q \) is a closed immersion and \( r \) is faithfully flat. Then the sequence \( L \xrightarrow{q} G \rightarrow A \) is exact if and only if the following conditions hold:

(a) For an object \( V \in \text{Rep}_k(G) \), \( q^*(V) \) in \( \text{Rep}_k(L) \) is trivial if and only if \( V \cong r^*U \) for some \( U \in \text{Rep}(A) \).

(b) Let \( W_0 \) be the maximal trivial subobject of \( q^*(V) \) in \( \text{Rep}_k(G) \), then there exists \( V_0 \subset V \) in \( \text{Rep}_k(G) \), such that \( q^*(V_0) \cong W_0 \).

(3) Any \( W \) in \( \text{Rep}_k(L) \) is a quotient of \( q^*(V) \) for some \( V \in \text{Rep}_k(G) \).

**Lemma 3.0.3.** Let \( f : X \rightarrow Y \) be an étale Galois cover of smooth projective \( k \)-varieties with a finite Galois group \( G \). Let \( x_0 \in X \) and \( y_0 \in Y \) be \( k \)-points such that \( f(x_0) = y_0 \). Then the induced morphism on S-fundamental group schemes \( \tilde{f} : \pi^S(X, x_0) \rightarrow \pi^S(Y, y_0) \).

**Proof.** By Proposition 3.0.2(1), we need only to show that any object of \( C^{nf}(X) \) is a subquotient of an object of the form \( f^*V \) for some \( V \in C^{nf}(Y) \). So let \( E \) be a numerically flat locally free sheaf on \( X \). Then \( f_*E \) is a locally free sheaf on \( Y \) as \( f \) is an étale Galois cover. Consider the Cartesian diagram

\[
\begin{array}{ccc}
X \times_Y X & \cong & G \times_k X \\
\downarrow^\mu & & \downarrow^f \\
X & \overset{f}{\rightarrow} & Y
\end{array}
\]

where \( \mu \) is the group action of \( G \) on \( X \) and \( pr \) denotes the natural projection \( G \times_k X \rightarrow X \).

Since \( f \) is flat, we have \( f^*f_*E \cong \mu_*pr^*E \) and \( \mu_*pr^*E \cong \bigoplus_{g \in G} g^*E \cong \bigoplus_{g \in G} E \) is numerically flat. So \( f^*f_*E \) is numerically flat. By Lemma 2.1.2, it follows that \( f_*E \) is numerically flat. We take \( V = f_*E \in C^{nf}(Y) \). As \( E \subset \bigoplus_{g \in G} g^*E \), \( E \) is a subquotient of \( f^*V \). This proves that \( \tilde{f} \) is a closed immersion. \( \square \)

The main result of this section is the following.

**Theorem 3.0.4.** Let \( f : X \rightarrow Y \) be an étale Galois cover of smooth projective \( k \)-varieties with a finite Galois group \( G \). Let \( x_0 \in X \) and \( y_0 \in Y \) be \( k \)-points such that \( f(x_0) = y_0 \). Then the sequence of morphisms of group schemes

\[
1 \rightarrow \pi^S(X, x_0) \xrightarrow{\tilde{f}} \pi^S(Y, y_0) \xrightarrow{P} G \rightarrow 1
\]

is exact.

**Proof.** We have already proved that \( \tilde{f} \) is a closed immersion in Lemma 3.0.3. Also by Remark 3.0.1, \( P \) is faithfully flat. We are left to show that the sequence is exact in the middle. By Proposition 3.0.2(2), we need to show the following:

(a) For any numerically flat locally free sheaf \( E \in C^{nf}(Y) \), \( f^*E \) is trivial if and only if \( E = P^*U \) for some \( U \in \text{Rep}_k(G) \).
(b) Let $V \in C^{nf}(Y)$ and $W_0$ be the maximal trivial subobject of $f^*V$. Then there exists $V_0 \subseteq V$ in $C^{nf}(Y)$ such that $f^*V_0 = W_0$.

(c) Any numerically flat locally free sheaf $W \in C^{nf}(X)$ is a quotient of $f^*V$ for some numerically flat locally free sheaf $V \in C^{nf}(Y)$.

To prove (a), let $E$ be a numerically flat locally free sheaf on $Y$. If $E = P^*U = (\mathcal{O}_X \otimes_k U)^G$ for some $U \in \text{Rep}_\mathbb{k}(G)$, then $f^*E$ is a trivial bundle on $X$. For the other direction, let $f^*E$ is trivial. Then we need to show that $E$ is associated to some representation of $G$. Consider the locally free sheaf

$$E' := f_\ast f^*E$$

on $Y$. As $f$ is faithfully flat, the natural map $E \to E'$ is injective. Let

$$\sigma : \pi^S(Y, y_0) \to \text{GL}(W) \quad \text{and} \quad \sigma' : \pi^S(Y, y_0) \to \text{GL}(W')$$

be the representations of $\pi^S(Y, y_0)$ which corresponds to $E$ and $E'$ respectively. We need to show that $\sigma$ factors through $P$ as $\sigma'$ is a subrepresentation of $\sigma$, it is enough to show that $\sigma'$ factors via $P$ or equivalently $E'$ is associated to some representation of $G$. Since $f^*E$ is trivial, so $E' = f_\ast f^*E$ is isomorphic to a finite direct sum of $f_\ast \mathcal{O}_X$. So again it is enough to show that $f_\ast \mathcal{O}_X$ is associated to a representation of $G$. Let $W$ be the group ring $k[G]$ (viewing as a $k$-vector space) with left regular representation of $G$ on it, i.e., $h \in G$ sends an element $\sum_{g \in G} d_g[h] \in k[G]$ to $\sum_{g \in G} d_g[hg]$. It is easy to see that $(\mathcal{O}_X \otimes_k W)^G$ is isomorphic to $f_\ast \mathcal{O}_X$ as $\mathcal{O}_Y$-modules which proves that $f_\ast \mathcal{O}_X$ is associated to the representation $W$ of $G$. This proves (a).

To prove (b), let $V \in C^{nf}(Y)$ and $W_0$ be a maximal trivial subbundle of $f^*V \in C^{nf}(X)$. Then $f^*V$ corresponds to a representation $\sigma : \pi^S(X, x_0) \to \text{GL}(T)$ and $W_0$ corresponds to the maximal trivial subrepresentation of $\sigma$, say $\sigma_0 : \pi^S(X, x_0) \to \text{GL}(T_0)$, $T_0 \subset T$ are the $k$-vector spaces. Now let $g \in G$. Then $g^*W_0$ is again a trivial subbundle of $f^*V$ and hence corresponds to a trivial subrepresentation of $\sigma$, say $\sigma'_0 : \pi^S(X, x_0) \to \text{GL}(T'_0)$. If $g^*W_0 \neq W_0$, then $T_0 \neq T'_0$. But then $\pi^S(X, x_0)$ acts trivially on $T_0 + T'_0$ which contradicts the maximality of $T_0$. So we must have $g^*W_0 = W_0$ for any $g \in G$. Thus $W_0$ is $G$-equivariant. By descent theory, there exists a subsheaf $V_0 \subset V$ such that $f^*V_0$ is isomorphic to $W_0$. In fact, if we take $V_0$ to be the locally free sheaf $(f_\ast W_0)^G$ on $Y$, then the natural map $f^*V_0 \to W$ is an isomorphism of sheaves on $X$. This proves (b).

Now to prove (c), let $W$ be a numerically flat locally free sheaf on $X$. Taking $V = f_\ast W$, we have seen in the proof of Lemma 3.0.3 that $V$ is a numerically flat locally free sheaf on $Y$ and $f^*V = \bigoplus_{g \in G} g^*W$. So clearly $W$ is a quotient of $f^*V$, which proves (c). This completes the proof of exactness of the sequence (3.0.1).

4. Smoothness of generalized Kummer variety

All results in this section are well known to experts. However, the author could not find a good reference and so we include these for the benefit of the reader and the author.

Let $A$ be an abelian surface over $k$ and $n$ be a positive integer. Let $e \in A$ denotes the identity element of $A$. The $n$-th symmetric product $S^n(A)$ of $A$ is the quotient $A^n/S_n$ of the $n$-fold product of $A$ by the symmetric group $S_n$. It is a normal projective variety of dimension $2n$ over $k$. The Hilbert scheme $A^{[n]}$ of closed subschemes of length $n$ on $A$ is a
smooth projective variety of dimension $2n$ over $k$. Note that $A^{[1]} \cong A$, so we also assume that $n \geq 2$. We have the Hilbert–Chow morphism

$$\varphi : A^{[n]} \to S^n(A)$$

which is given by sending $Z \in A^{[n]}$ to

$$\sum_{p \in \text{Supp}(Z)} l(O_{Z,p})[p] \in S^n(A),$$

where $\text{Supp}(Z) = \{ p \in A : O_{Z,p} \neq 0 \}$ denotes the support of the 0-cycle $Z$ in $A$, and $l(O_{Z,p})$ is the length of the local ring $O_{Z,p}$ as a module over itself. The Hilbert–Chow morphism is explained in detail in [6, Chapter 7]. Let $\psi : A^n \to S^n(A)$ be the quotient map. We have the addition morphism $+: A^n \to A$ which factors through $S^n(A)$ giving the morphism $\Sigma : S^n(A) \to A$. Let $\rho := \Sigma \circ \varphi : A^{[n]} \to A$ be the composition morphism. The fiber $K^{[n]} := \rho^{-1}(e)$ is the $n$-th generalized Kummer variety associated to $A$. For arbitrary $n$, $K^{[n]}$ is not necessarily smooth. An example is shown in [14] in the case $\text{char}(k) = 2$. However, we will see that if $\text{char}(k) \nmid n$, then $K^{[n]}$ is a smooth projective variety of dimension $2n - 1$ over $k$.

We need the following well known result for later use. Its proof is left to the reader.

**Lemma 4.0.1.** Let $X, Y, Z$ be the Noetherian projective schemes over an algebraically closed field $k$ and $Y$ be an integral. Let $f : X \to Y$ be a morphism of schemes over $Z$, that is, we have the commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow & & \downarrow \\
Z & \xrightarrow{} & \text{Z}
\end{array}
$$

Suppose $f$ is finite and for each closed point $z \in Z$ the induced morphism on fibers $f_z : X_z \to Y_z$ is an isomorphism. Then $f$ is an isomorphism.

4.1 A Galois cover of the Hilbert scheme

The group scheme $A$ acts on itself by translation, say $T_a : A \to A$ denotes the translation by an element $a \in A$. There is an induced action of $A$ on the Hilbert scheme $A^{[n]}$. Under this action, each element $a \in A$ sends a closed point $Z$ of the Hilbert scheme $A^{[n]}$ to its pullback $T^*_a Z$, viewing $Z$ as a closed subscheme of $A$ of length $n$. We have the commutative diagram

$$
\begin{array}{ccc}
A^{[n]} & \xrightarrow{T^+_a} & A^{[n]} \\
\downarrow{\rho} & & \downarrow{\rho} \\
A & \xrightarrow{T_{na}} & A
\end{array}
$$

The action of $A$ on $A^{[n]}$ gives the morphism of group schemes $\nu : A \times_k A^{[n]} \to A^{[n]}$ which sends a closed point $(a, Z) \in A \times_k A^{[n]}$ to $T^*_a Z \in A^{[n]}$. Let $n : A \to A$ be the multiplication by $n$ morphism.
Lemma 4.1.1. The following commutative diagram is Cartesian:

\[
\begin{array}{ccc}
A \times_k K^n & \xrightarrow{\nu} & A^n \\
\downarrow^{pr_A} & & \downarrow^{\rho} \\
A & \xrightarrow{n} & A
\end{array}
\] (4.1.1)

where \( pr_A : A \times_k K^n \rightarrow A \) is the natural projection onto \( A \).

Proof. Let \( B \) be the fiber product as in the Cartesian diagram:

\[
\begin{array}{ccc}
B & \xrightarrow{\beta} & A^n \\
\downarrow^{\alpha} & & \downarrow^{\rho} \\
A & \xrightarrow{n} & A
\end{array}
\]

The morphisms \( pr_A : A \times_k K^n \rightarrow A \) and \( \nu : A \times_k K^n \rightarrow A^n \) gives a unique morphism \( \Phi : A \times_k K^n \rightarrow B \) satisfying \( \beta \circ \Phi = \nu \) and \( \alpha \circ \Phi = pr_A \). Any fiber of \( \nu \) over a closed point of \( A^n \) is finite and so is true for \( \beta \). So the fiber of \( \Phi \) over any closed point of \( B \) is finite. Moreover, \( \Phi \) is projective, so \( \Phi \) is a finite morphism. Next we consider the commutative triangle

\[
\begin{array}{ccc}
A \times_k K^n & \xrightarrow{\Phi} & B \\
\downarrow^{pr_A} & & \downarrow^{\alpha} \\
A & \xrightarrow{\nu} & A
\end{array}
\] (4.1.2)

For any closed point \( a \in A \), pulling back the triangle (4.1.2) along the inclusion \( \iota_a : \{a\} \hookrightarrow A \), we get a map \( \Phi_a : \{a\} \times_k K^n \rightarrow B_a \) between the fibers over \( a \). By definition of \( K^n \), \( \Phi_e \) is an isomorphism. Again \( \Phi_a \) is just the pullback of \( \Phi_e \) along the translation \( T_{-a} : A \rightarrow A \). It follows that \( \Phi_a \) is also an isomorphism. So we are in the following situation: we have the commutative diagram (4.1.2) such that \( \Phi \) is finite and \( \Phi_a \) is isomorphism for all closed points \( a \in A \). Using Lemma 4.0.1, it follows that \( \Phi \) is an isomorphism. Consequently, (4.1.1) is a Cartesian diagram. \( \square \)

Now if \( p \nmid n \), then \( n : A \rightarrow A \) is a Galois cover with Galois group \( A[n] \), the group of \( n \)-torsion points. It follows from Lemma 4.1.1 that \( \nu \) is also a Galois cover with Galois group \( A[n] \). So \( A \times_k K^n \) is a Galois cover of the Hilbert scheme \( A^n \) and hence \( A \times_k K^n \) is smooth. As a consequence we get the following corollary.

**COROLLARY 4.1.2**

Let \( \text{char}(k) \nmid n \). Then \( K^n \) is smooth.

5. **Fundamental group scheme of generalized Kummer variety**

We consider the abelian surface \( A \) over the algebraically closed field \( k \) of characteristic \( p > 3 \) and \( n \) be a positive integer not divisible by \( p \). Let \( e \in A \) be the identity element of \( A \). We can easily check that (or by [10, Theorem 6.1]) we have the following remark.
Remark 5.0.1. The $S$-fundamental group scheme $\pi^S(A, e)$ of an abelian variety $A$ is commutative.

In [13, Section 4.3], Paul and Sebastian have defined a morphism of group schemes

$$\delta : (\pi^S(A, e))^n \to \pi^S(A^{[n]}, n[e]),$$

and this factors as

$$\xymatrix{ (\pi^S(A, e))^n \ar[r]^-\delta & \pi^S(A^{[n]}, n[e]) \ar[r]^-\overline{\delta} & \pi^S(A, e)_{ab} }$$

where $m$ is the composition of the natural homomorphism $(\pi^S(A, e))^n \to ((\pi^S(A, e))_{ab})^n$ followed by the product morphism $((\pi^S(A, e))_{ab})^n \to (\pi^S(A, e))_{ab}$. The map $\overline{\delta}$ is an isomorphism by [13, Theorem 5.3.11]. In view of Remark 5.0.1, we rewrite diagram (5.0.1) as

$$\xymatrix{ (\pi^S(A, e))^n \ar[r]^-\delta & \pi^S(A^{[n]}, n[e]) \ar[r]^-\overline{\delta} & \pi^S(A, e) \ar[r]^-\delta & \pi^S(A, e)_{ab} }$$

(5.0.2)

Recall that $\psi : A^n \to S^n(A)$ is the quotient map and $\varphi : A^{[n]} \to S^n(A)$ is the Hilbert–Chow morphism.

Lemma 5.0.2. We have the following commutative diagram:

$$\xymatrix{ (\pi^S(A, e))^n \ar[r]^-\delta & \pi^S(A^{[n]}, n[e]) \ar[r]^-\overline{\delta} & \pi^S(A^n, \psi(e, \ldots, e)) \ar[r]^-\overline{\psi} & \pi^S(A^{[n]}, n[e]) }$$

(5.0.3)

Proof. To prove this lemma, we first recall how the map $\delta$ is defined in [13]. A point $y \in S^n(A)$ can be written as

$$\sum_{j=1}^r n_j a_j,$$

where $a_1, \ldots, a_r \in A$ are distinct points with multiplicities $n_1 \geq n_2 \geq \cdots \geq n_r \in \mathbb{Z}_{\geq 0}$ respectively, such that $\sum_{j=1}^r n_j = n$. The $r$ tuple $(n_1, \ldots, n_r)$ is called the type of $y$. Let $W \subset S^n(A)$ denote the open subset consisting of points of type $(1, 1, \ldots, 1)$ and $(2, 1, 1, \ldots, 1)$. Let $V$ denote the open subset $\varphi^{-1}(W)$ in $A^{[n]}$. Then $A^{[n]} \setminus V$ has codimension 2 in $A^{[n]}$ and $A^n \setminus \psi^{-1}(W)$ has codimension $\geq 4$ in $A^n$. Let $j : \psi^{-1}(W) \hookrightarrow A^n$ denote the inclusion. The morphism $\mathcal{G}$ defined by

$$\mathcal{G}(E) := (j_*\psi^w \varphi_* (E|V)))^{\vee\vee}$$

for an object $E \in C^{nf}(A^{[n]})$

is a functor between Tannakian categories

$$\mathcal{G} : C^{nf}(A^{[n]}) \to C^{nf}(A^n).$$
The functor $G$ induces the map $\delta : (\pi^S(A, e))^n \to \pi^S(A[n], n[e])$ on $S$-fundamental group schemes (see [13, Section 4.3]). In [7, Chapter III, Corollary 11.4], it is proved that for a birational projective morphism $h : X \to Y$ of Noetherian integral schemes with $Y$ being normal, we have $h_*(\mathcal{O}_X) = \mathcal{O}_Y$. Using this result for the birational morphism $\varphi$ and the fact that $X \setminus \psi^{-1}(W)$ has codimension $\geq 2$ in $X$, we get that $G\varphi^* (E) = \psi^*(E)$ for any $E \in C^n(S^n(A))$. This shows that $\widehat{\varphi} \circ \delta = \widehat{\psi}$ which proves the lemma. □

5.1 $S$-fundamental group scheme of $K^{[n]}$

**Theorem 5.1.1.** Let $k$ be an algebraically closed field of characteristic $p > 3$ and $n$ be an integer $\geq 2$ such that $p \nmid n$. Then the $S$-fundamental group scheme $\pi^S(K^{[n]}, n[e])$ of the $n$-th generalized Kummer variety $K^{[n]}$ over $k$ is trivial.

**Proof.** We have the Cartesian diagram (4.1.1).

$$
\begin{array}{c}
A \times_k K^{[n]} \xrightarrow{\nu} A^n \\
\downarrow \varphi_A \downarrow \rho \\
A \xrightarrow{n} A
\end{array}
$$

As $p \nmid n$, $n$ is a Galois cover with Galois group $A[n]$, the group of $n$-torsion points and $\nu$ is also a Galois cover with Galois group $A[n]$. Moreover, $A$, $K^{[n]}$ and $A^n$ are all smooth projective variety as we have seen in Sect. 4. So using Theorem 3.0.4, we get the following commutative diagram whose rows are exact.

$$
\begin{array}{c}
1 \to \pi^S(A \times_k K^{[n]}, (e, n[e])) \xrightarrow{\overline{\varphi}} \pi^S(A^n, n[e]) \to A^n \to 1 \\
\downarrow \overline{\varphi}_A \downarrow \overline{\rho} \\
1 \to \pi^S(A, e) \xrightarrow{\overline{n}} \pi^S(A, e) \to A^n \to 1
\end{array}
$$

By Theorem 2.1.3, we have the isomorphism

$$
\overline{\varphi}_A \times \overline{\rho}_K^{[n]} : \pi^S(A \times_k K^{[n]}, (e, n[e])) \xrightarrow{\sim} \pi^S(A, e) \times_k \pi^S(K^{[n]}, n[e]).
$$

So to prove that $\pi^S(K^{[n]}, n[e])$ is trivial, it is enough to show that $\overline{\rho}$ is an isomorphism. Now the map $\overline{\rho}$ is the composition $\overline{\Sigma} \circ \overline{\varphi}$ and by Lemma 5.0.2, we have $\overline{\varphi} \circ \delta = \overline{\psi}$. It follows that

$$
\overline{\rho} \circ \delta = \overline{\Sigma} \circ \overline{\varphi} \circ \delta = \overline{\Sigma} \circ \overline{\psi} = \overline{\gamma}.
$$

Now the map on the $S$-fundamental group schemes induced by the addition morphism $+: A^n \to A$ is the product morphism $m : (\pi^S(A, e))^n \to \pi^S(A, e)$, identifying $\pi^S(A^n, (e, \ldots, e))$ as $(\pi^S(A, e))^n$. In other words, we have $\overline{\gamma} = m$. So we have the commutative diagram

$$
\begin{array}{c}
(\pi^S(A, e))^n \xrightarrow{\delta} \pi^S(A[n], n[e]) \\
\downarrow m \downarrow \varphi \\
\pi^S(A, e) \xrightarrow{\overline{n}} \pi^S(A, e)
\end{array}
$$
In (5.0.2), we observe that \( \delta = \tilde{\delta} \circ m \). Thus the above commutative diagram becomes as follows:

\[
\begin{array}{ccc}
(\pi^S(A, e))^n & \xrightarrow{m} & \pi^S(A, e) \\
\downarrow & & \downarrow \tilde{\rho} \\
\pi^S(A, e) & \xrightarrow{\tilde{\delta}} & \pi^S(A^{[n]}, n[e])
\end{array}
\]

i.e., \( \tilde{\rho} \circ \tilde{\delta} \circ m = m \). From surjectivity of \( m \), it follows that \( \tilde{\rho} \circ \tilde{\delta} \) is the identity morphism on \( \pi^S(A, e) \). Since \( \tilde{\delta} \) is an isomorphism (by [13, Theorem 5.3.11]), \( \tilde{\rho} \) is also an isomorphism. This proves that \( \pi^S(K^{[n]}, n[e]) \) is trivial. \( \square \)

5.2 Nori’s and étale fundamental group schemes of \( K^{[n]} \)

We will use [9, Lemma 6.2] for this section.

**Lemma 5.2.1 (Lemma 6.2, [9]).** Let \( X \) be a complete connected reduced \( k \)-scheme and let \( x \in X \) be \( k \)-rational points. Then there exist natural faithfully flat homomorphisms

\[
\pi^S(X, x) \to \pi^N(X, x) \to \pi^\text{ét}(X, x)
\]

of affine group schemes.

**COROLLARY 5.2.2**

Let \( k \) be an algebraically closed field of characteristic \( p > 3 \) and \( n \) be an integer \( \geq 2 \) such that \( p \nmid n \). Then the Nori’s fundamental group scheme \( \pi^N(K^{[n]}, n[e]) \) and étale fundamental group scheme \( \pi^\text{ét}(K^{[n]}, n[e]) \) of the \( n \)-th generalized Kummer variety \( K^{[n]} \) over \( k \) are trivial.

**Proof.** Using Theorem 5.1.1 and Lemma 5.2.1, the assertion is clear. \( \square \)

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