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Algorithms for Convex Quadratic Programming

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1 Introduction

The main interest of this diploma thesis is to describe and compare different, practically successful solution methods for general convex quadratic problems with arbitrary linear constraints. Therefore we first define the convex quadratic program (QP) as

$$\min_x \frac{1}{2} x^T Q x + d^T x$$ \hspace{1cm} (1.1a)

subject to \hspace{1cm} \begin{align*}
B x &= c, \hspace{1cm} (1.1b) \\
b &\leq x &\leq a, \hspace{1cm} (1.1c)
\end{align*}

where \(Q\) is a positive definite \(n \times n\) matrix, \(B\) is a \(m \times n\) matrix, \(a, b, d \in \mathbb{R}^n\), and \(c \in \mathbb{R}^m\). This problem has received considerable interest in the literature. We recall some of the recent contributions.

Solution methods like the augmented Lagrangian method, active-set methods and interior point methods are among the most popular approaches to solve (1.1), and can be traced back to the 1960’s.

The so-called augmented Lagrangian method was first proposed by Hestenes [25] and Powell [42]. More recent contributions making use of the augmented Lagrangian idea are from Conn et al. [10], [9], who designed the nonlinear programming code LANCELOT, Dostál [13], who describes a semi-monotonic augmented Lagrangian algorithm for solving large convex quadratic programming problems, and Murtagh and Saunders [35], [36], who developed a software called MINOS that combines efficient sparse-matrix techniques as in the revised simplex method with stable quasi-Newton methods for handling the nonlinearities.

Active set methods for convex QP are the implementation of Gill and Murray [21] called QPOPT, the software of Gould and Toint [23] called QPA, and Fletcher’s code BQPD [14].

Another solution strategy consists in using (primal-dual) interior point methods. Here we want to mention the recent contributions of Vanderbei [43], [44], [45], who designed the software package LOQO, of Mészáros [33], who built up the
solves BPMPD, and of Gertz and Wright [18], [19], who developed the object-oriented software package OOQP. Further important interior-point solvers for QP are CPLEX [1], KNITRO [8], [46], and MOSEK [2].

For comparison of some of the above mentioned algorithms and methods, we refer to the benchmarking articles of Mittelmann [34], Dolan et al. [11], [12], and Gould and Toint [24].

Especially we want to mention that we developed our own contribution to solve (1.1) that we present in Chapter 5. It consists in combining the method of multipliers with an infeasible active-set method. Our approach is iterative. In each step we calculate an augmented Lagrange function. Then we minimize this function using an infeasible active-set method that was already successfully applied to similar problems, see the paper of Kunisch and Rendl [30]. After this we update the Lagrange multiplier for the equality constraints. Finally we try to solve (1.1) directly, again with the infeasible active-set method, starting from the optimal solution of the actual Lagrange function. Computational experience with our method indicates that typically only few (most of the time only one) outer iterations (multiplier-updates) and also only few (most of the time less than ten) inner iterations (minimization of the Lagrange function and trying to solve (1.1) directly) are required to reach the optimal solution.

The diploma thesis is organized as follows. We close this chapter with some notation used throughout. In Chapter 2 we show the equivalence of different QP problem formulations and present some important so-called direct methods for solving equality-constrained QPs. We cover the most important aspects for practically successful interior point methods for linear and convex quadratic programming in Chapter 3. Chapter 4 deals with ingredients for practically efficient feasible active set methods. Finally Chapter 5 provides a close description of our Lagrangian infeasible active set method and further gives a convergence analysis of the subalgorithms involved.

**Notation:** The following notation will be used throughout. $M := \{1, \ldots, m\}$ and $N := \{1, \ldots, n\}$ are two sets of integer numbers. For a subset $A \subseteq N$ and $x \in \mathbb{R}^n$ we write $x_A$ for the components of $x$ indexed by $A$, i.e. $x_A := (x_i)_{i \in A}$. The complement of $A$ will be denoted by $\bar{A}$. If $P$ is a matrix and $A$ and $E$ are subsets of $N$, then $P_{A,E}$ is the submatrix of $P$, with rows indexed by $A$ and columns indexed by $E$. If $A = E$ we write $P_A$ for $P_{A,A}$. By $P_{A,E}^T$ we identify the submatrix of $P^T$, with rows indexed by $A$ and columns indexed by $E$. For $a, b \in \mathbb{R}^n$ we write $a \circ b$ to denote the vector of element-wise products, $a \circ b := (a_i b_i)_{i \in N}$. 
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In this chapter we show the equivalence of different QP problem formulations in Section 2.1 and then in Section 2.2 we present some important so-called direct methods for solving equality-constrained quadratic programs.

2.1 Different Problem Formulations

The general quadratic program can be stated as

\[
\begin{align*}
\min_x \quad & \frac{1}{2} x^\top Q x + x^\top d \\
\text{subject to} \quad & a_i^\top x = c_i, \quad i \in \mathcal{E}, \quad (2.1a) \\
& a_i^\top x \leq c_i, \quad i \in \mathcal{I}, \quad (2.1b)
\end{align*}
\]

where \( Q \) is a symmetric \( n \times n \) matrix, \( \mathcal{E} \) and \( \mathcal{I} \) are finite sets of indices, and \( d, x \) and \( \{a_i\}, \ i \in \mathcal{E} \cup \mathcal{I} \), are vectors in \( \mathbb{R}^n \). If the Hessian matrix \( Q \) is positive definite, we say that (2.1) is a strictly convex QP, and in this case the problem is often similar in difficulty to a linear program. Nonconvex QPs, in which \( Q \) is an indefinite matrix, can be more challenging because they can have several stationary points and local minima.

We can convert the inequality constraints in the above formulation of a QP by introducing a vector of slack variables \( z \) and writing

\[
\begin{align*}
\min_x \quad & \frac{1}{2} x^\top Q x + x^\top d \\
\text{subject to} \quad & a_i^\top x = c_i, \quad i \in \mathcal{E}, \quad (2.2a) \\
& a_i^\top x + z = c_i, \quad i \in \mathcal{I}, \quad (2.2b) \\
& z_i \geq 0, \quad i \in \mathcal{I}. \quad (2.2c)
\end{align*}
\]
We can further transform this formulation by splitting $x$ into its nonnegative and nonpositive parts, $x = x^+ - x^-$, where $x^+ = \max(x, 0) \geq 0$ and $x^- = \max(-x, 0) \geq 0$. The problem (2.2) can now be written as

$$\min_{(x^+, x^-, z)} \frac{1}{2} \begin{pmatrix} x^+ \\ x^- \\ z \end{pmatrix}^T \begin{pmatrix} Q & 0 & 0 \\ 0 & Q & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x^+ \\ x^- \\ z \end{pmatrix} + \begin{pmatrix} x^+ \\ x^- \\ z \end{pmatrix}^T \begin{pmatrix} d \\ -d \\ 0 \end{pmatrix}$$

subject to

$$\begin{pmatrix} a_i \\ -a_i \\ 0 \end{pmatrix}^T \begin{pmatrix} x^+ \\ x^- \\ z \end{pmatrix} = c_i, \quad i \in \epsilon$$

$$\begin{pmatrix} a_i \\ -a_i \\ 1 \end{pmatrix}^T \begin{pmatrix} x^+ \\ x^- \\ z \end{pmatrix} = c_i, \quad i \in \iota$$

$$\begin{pmatrix} x^+ \\ x^- \\ z \end{pmatrix} \geq 0.$$

Now setting

$$\begin{pmatrix} x^+ \\ x^- \\ z \end{pmatrix} = \bar{\pi},$$

$$\begin{pmatrix} Q & 0 & 0 \\ 0 & Q & 0 \\ 0 & 0 & 0 \end{pmatrix} = \bar{Q},$$

$$\begin{pmatrix} d \\ -d \\ 0 \end{pmatrix} = \bar{d},$$

$$\begin{pmatrix} A_{\epsilon} \\ -A_{\epsilon} \\ 0_{n \times k} \end{pmatrix} = B,$$

where
we obtain

\[
\min \frac{1}{2} x^\top Q x + x^\top d \quad (2.3a)
\]
subject to \( B x = c, \quad (2.3b) \)
\( \bar{x} \geq 0. \quad (2.3c) \)

Hence, we showed that (2.3) is equivalent to (2.1) and it depends on the considered algorithm what representation of the quadratic problem is preferable. Furthermore we want to mention that we can also convert inequality constraints of the form \( x \leq a \) or \( Ax \geq c \) to equality constraints by adding or subtracting slack variables:

\[
x \leq a \iff x + w = a, \; w \geq 0,
\]

\[
Ax \geq c \iff Ax - w = c, \; w \geq 0.
\]

### 2.2 Solution Methods For Equality-Constrained QPs

In this section we consider direct solution methods for quadratic programs in which only equality constraints are present. We define them as follows:

\[
\min x \frac{1}{2} x^\top Q x + x^\top d \quad (2.4a)
\]
subject to \( Ax = c, \quad (2.4b) \)

where \( Q \) is a positive-definite \( n \times n \) matrix, \( A \) is a \( m \times n \) non-singular matrix, \( d \) is a vector in \( \mathbb{R}^n \) and \( c \) is a vector in \( \mathbb{R}^m \).

The KKT conditions for this problem are
where

\[ K = \begin{bmatrix} Q & A^\top \\ A & 0 \end{bmatrix} \] (2.6)

and

\[
\begin{align*}
g &= c + Qx, \\
h &= Ax - b, \\
p &= x^* - x.
\end{align*}
\]

These problems appear often as subproblems in algorithms that solve general QPs with inequality constraints (see, for example, the subproblems for feasible active-set methods described in Section 4.2) and therefore it is very important to find ways to solve them efficiently. In addition to the direct solution methods described in this section, there also exist iterative solution methods like the conjugate gradient method applied to the reduced system and the projected conjugate gradient method. For a further discussion of these iterative methods see, for example, Nocedal and Wright [40, Section 16.3], Conn, Gould, and Toint [9] and Burke and Moré [7].

2.2.1 Factoring the full KKT system

One option for solving (2.5) is the use of a triangular factorization of \( K \) and then make backward and forward substitution. To discuss this option we need some theoretical knowledge about the definiteness of \( K \). Therefore let us give a result that states that the KKT matrix \( K \) is always indefinite. We define

\[
\text{inertia}(S) \overset{\text{def}}{=} (n_+, n_-, n_0)
\]

where \( n_+ \) is the number of positive eigenvalues of \( S \), \( n_- \) denotes the number of negative eigenvalues of \( S \) and \( n_0 \) gives the number of zero eigenvalues of \( S \). Now we can state a result that characterizes the inertia of \( K \).
Theorem 1 Let \( K \) be given by (2.6), and suppose that \( A \) has rank \( m \). Then

\[
\text{inertia}(K) = \text{inertia}(Z^\top QZ) + (m, m, 0),
\]

where \( Z \) is an \( n \times (n - m) \) matrix whose columns are a basis of the null space of \( A \). That is, \( Z \) has full rank and satisfies \( AZ = 0 \). Therefore, if \( Z^\top QZ \) is positive definite, \( \text{inertia}(K) = (n, m, 0) \).

The proof of this result is given in Forsgren and Gill [15, Lemma 4.1] or Gould [22, Lemma 3.4], for example.

Because of indefiniteness of \( K \), we cannot use the Cholesky factorization to solve (2.5). The use of Gaussian elimination has the disadvantage that it ignores symmetry. Therefore the most effective approach is to use a symmetric indefinite factorization\(^1\) which has the form

\[
P^\top SP = LBL^\top,
\]

where \( S \) is a general symmetric matrix, \( P \) is a permutation matrix, \( L \) is a unit lower triangular matrix and \( B \) is a block-diagonal matrix with either \( 1 \times 1 \) or \( 2 \times 2 \) blocks. We use the symmetric permutations defined by \( P \) to improve the numerical stability of the computation and, if \( S \) is sparse, to maintain sparsity.

Now to solve (2.5), we first compute a factorization of KKT matrix \( K \):

\[
P^\top KP = LBL^\top,
\]

and then use the calculated factors in the following way to arrive the solution:

---

\(^1\) The computational cost of a symmetric factorization is typically about half the cost of Gaussian elimination
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solve $L_x = P^T \begin{bmatrix} g \\ h \end{bmatrix}$ to obtain $z$;
solve $B \hat{z} = z$ to obtain $\hat{z}$;
solve $L^T \bar{z} = \hat{z}$ to obtain $\bar{z}$;
set $\begin{bmatrix} -p \\ \lambda^* \end{bmatrix} = P \bar{z}$.

The by far most expensive operation in this approach is the performance of the factorization (2.7). This factoring of the KKT matrix $K$ is quite effective for many problems. It may be expensive, however, if $K$ is sparse and the heuristics for choosing $P$ are not able to maintain this sparsity in $L$ and therefore $L$ becomes dense.

2.2.2 The Schur-complement method

We assumed for the equality-constrained QP (2.4) that $Q$ is positive definite. Therefore we can multiply the first equation in (2.5) by $AQ^{-1}$ and then subtract the second equation to get the following equation in $\lambda^*$ alone:

$$(AQ^{-1}A^\top)\lambda^* = (AQ^{-1}g - h).$$

As $AQ^{-1}A^\top$, the so called Schur complement of $Q$, is also positive definite (because we assumed that $A$ has full rank), we can calculate $\lambda^*$ as:

$$\lambda^* = (AQ^{-1}A^\top)^{-1}(AQ^{-1}g - h),$$

and then obtain $p$ from the first equation of (2.5):

$$p = Q^{-1}(A^\top \lambda^* - g).$$

Using the Schur-Complement method we need to invert $Q$, as well as to compute a factorization of the $m \times m$ matrix $AQ^{-1}A^\top$. Therefore, the method is most effective if $Q$ is well conditioned and easy to invert or if $Q^{-1}$ is known explicitly through a quasi-Newton updating formula or if the number of constraints $m$ is small.
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2.2.3 The null-space method

The null-space method does not require nonsingularity of $Q$ but only full rank of $A$ and positive definiteness of $Z^\top QZ$, where $Z$ is the null-space basis matrix.

Let us partition the vector $p$ in (2.5) into two components, so that:

$$p = Yp_y + Zp_z, \quad (2.8)$$

where $Z$ is $n \times (n - m)$, $Y$ is $n \times m$, $p_y$ is a vector in $\mathbb{R}^m$ and $p_z$ is a vector in $\mathbb{R}^{n-m}$.

Thereby we choose $Y$ and $Z$ with the following properties:

$$[Y|Z] \in \mathbb{R}^{n \times n} \text{ is nonsingular, } AZ = 0.$$  

Since $A$ has full rank, so does $A[Y|Z] = [AY|0]$ and therefore $AY$ is nonsingular and has rank $m$.

Now we substitute $p$ with the help of (2.8) in $Ap = -h$, which gives

$$(AY)p_y = -h.$$  

We can make $p_y$ explicit, as $AY$ is nonsingular:

$$p_y = -(AY)^{-1}h. \quad (2.9)$$

To determine $p_z$ we use the first equation of (2.5) to obtain

$$-QYp_y - QZp_z + A^\top \lambda^* = g,$$

and then multiply it by $Z^\top$:

$$(Z^\top QZ)p_z = -Z^\top QYp_y - Z^\top g. \quad (2.10)$$
To calculate $p_z$ from this equation, we can use, for example, a Cholesky factorization of $Z^\top QZ$. After that we can compute the total step $p$ by using (2.8). Finally we can obtain $\lambda^*$ by multiplying the first equation of (2.5) by $Y^\top$

\[(AY)^\top \lambda^* = Y^\top (g + Qp),\]

and then solving this equation for $\lambda^*$.

The main computational effort of the null-space method lies in the determination of the not uniquely defined matrix $Z$. If we choose $Z$ to have orthonormal columns, then the conditioning of $Z^\top QZ$ is at least as good as that of $Q$ itself, but an orthonormal $Z$ is often expensive to compute (especially if $A$ is sparse). On the other hand if we choose $Z$ in a different, computationally cheaper way, the reduced system (2.10) may become ill conditioned. Therefore the null-space method is preferable compared with the Schur-complement method when it is more expensive to invert $Q$ and compute factors of $AQ^{-1}A^\top$ than to compute $Z$ and factors of $Z^\top QZ$ and $AY$. This is most of the time the case if the number of equality constraints $m$ is large and therefore the matrices $Z$ and $Z^\top QZ$ have low dimensions.

---

2 For an orthonormal $Z$ the corresponding $Y$ can be calculated by a QR factorization of $A^\top$, for details see Section 4.4.
3 Interior Point Methods

This chapter is devoted to the description of practically successful interior point methods for linear and convex quadratic programming. In Section 3.1 we mention some basis information about the exciting historical development of the interior point methods as first real competitor of the simplex method. Section 3.2 is used to present the central components of interior point methods on the basis of the simple linear programming framework. After that we show in Section 3.3 that the generalisation of interior point methods to QPs is a natural and easy one, especially if we compare it with the serious differences between the simplex method and active set methods for QPs.

3.1 A short historical review

Starting with the seminal paper of Karmarkar [27] in 1984, interior point methods in mathematical programming have been the most important research area in optimization since the development of the simplex method for linear programming. Interior point methods have strongly influenced mathematical programming theory, practice and computation. For example linear programming is no longer synonymous with the simplex method, and linear programming is shown as a special case of nonlinear programming due to these developments.

On the theoretical side, permanent research led to better computational complexity bounds for linear programming, quadratic programming, linear complementarity problems, semi-definite programming and some classes of convex programming problems. On the computational side, the performance of tools for linear and nonlinear programming improved greatly, as the sudden appearance of credible competition for the active set methods initiated significant improvements in implementations.

Interior-point methods arose from the search for algorithms with better theoretical properties than the simplex method. As Klee and Minty [29] showed, the simplex method can be inefficient on certain pathological problems. Roughly speaking, the time required to solve a linear program may be exponential in the size of the problem, as measured by the number of unknowns and the amount of storage needed for the problem data. For almost all practical problems, the simplex method is much more efficient than this bound would suggest, but its poor worst-case complexity
motivated the development of new algorithms with better guaranteed performance. The first such method was the ellipsoid method, proposed by Khachiyan [28], which finds solution in time that is at worst polynomial in the problem size. Unfortunately, this method approaches its worst-case bound on all problems and is not competitive with the simplex method in practice.

Karmarkar’s projective algorithm [27], announced in 1984, also has the polynomial complexity property, but it came with the added attraction of good practical behavior. The initial claims of excellent performance on large linear programs were never fully borne out, but the announcement prompted a great deal of research activity which gave rise to many new methods.

In the first years after Karmarkar’s initial paper, research in linear programming was concentrated on finding algorithms that worked with the primal problem, but had better complexity bounds or were easier to implement than the original method. A next crucial step was done by Megiddo [31] in 1987, when he described a framework for primal-dual algorithms. To take into account the primal and the dual problem proved to be extraordinarily productive. The primal-dual viewpoint led to new algorithms with best practical and also interesting theoretical properties. Furthermore it formed the basis for transparent extensions to convex programming and linear complementarity. The basis algorithm for most current practical linear programming software was described by Mehrotra in 1989 [32].

Some years later, Nesterov and Nemirovskii published their theory of self-concordant functions [37] which was the main tool to extend algorithms for linear programming based on the primal log-barrier function to more general classes of convex problems like semi-definite programming and second-order cone programming. Later on, Nesterov and Todd [38, 39] did further extending work along these lines. Interior point methods have also been frequently used in such areas as control theory, structural optimization, combinatorial and integer programming and linear algebra for different decomposition methods.

In the next sections, we will concentrate on central trajectory methods using the primal-dual framework, because these algorithms have the best practical features in the class of interior point methods. Furthermore we will concentrate on linear and convex quadratic programming. Readers interested also in affine scaling or potential reduction methods or in algorithms using only the primal or only the dual variables or in further topics like linear complementarity problems, semi-definite programming, self-duality or and theoretical run-time properties are referred to three survey articles of Forsgren et al. [16], Freund and Mizuno [17] and Potra and Wright [41] and two comprehensive books of Wright [47] and Ye [48] about interior point methods.
3.2 Linear Programming

In this section we present the central components of interior point methods on the basis of the simple linear programming framework. We consider the linear programming problem in standard form

\[
\begin{align*}
\min & \quad c^\top x \\
\text{subject to} & \quad Ax = b, \\
& \quad x \geq 0,
\end{align*}
\]

(3.1a) (3.1b) (3.1c)

where \(c\) and \(x\) are vectors in \(\mathbb{R}^n\), \(b\) is a vector in \(\mathbb{R}^m\), and \(A\) is an \(m \times n\) matrix with full row rank. The dual problem for (3.1) is

\[
\begin{align*}
\max & \quad b^\top \lambda \\
\text{subject to} & \quad A^\top \lambda + s = c, \\
& \quad s \geq 0,
\end{align*}
\]

(3.2a) (3.2b) (3.2c)

where \(\lambda\) is a vector in \(\mathbb{R}^m\) and \(s\) is a vector in \(\mathbb{R}^n\).

3.2.1 The KKT system and Newton’s method

Solutions of (3.1) together with (3.2) are characterized by the KKT conditions:

\[
\begin{align*}
A^\top \lambda + s &= c, \\
Ax &= b, \\
x_i s_i &= 0, \quad i = 1, 2, \ldots, n, \\
x \geq 0, \\
s \geq 0.
\end{align*}
\]

(3.3a) (3.3b) (3.3c) (3.3d) (3.3e)

Primal-dual methods find solutions \((x^*, \lambda^*, s^*)\) of this system by applying variants of Newton’s method to the three equalities (3.3a) - (3.3c) and modifying the search directions and step lengths so that the inequalities (3.3d) and (3.3e) are satisfied strictly in every iteration. The equations (3.3a) and (3.3b) are linear and (3.3c) is
only mildly nonlinear. So these three equations are not difficult to solve by themselves. However, the problem becomes much more difficult when we add the non-negativity requirements (3.3d) and (3.3e), which give rise to all the complications in the design and analysis of interior-point methods.

To derive primal-dual interior-point methods we restate the first three equations (3.3a) - (3.3c) of the above KKT-system in a slightly different form by means of a mapping $F$ from $\mathbb{R}^{2n+m}$ to $\mathbb{R}^{2n+m}$:

$$F(x, \lambda, s) = \begin{bmatrix} A^T \lambda + s - c \\ Ax - b \\ XSe \end{bmatrix} = 0 \quad (3.4)$$

where

$$X = \text{diag}(x_1, \ldots, x_n),$$

$$S = \text{diag}(s_1, \ldots, s_n),$$

and $e = (1, \ldots, 1)^\top$. Primal-dual methods generate iterates $(x^k, \lambda^k, s^k)$ that satisfy the bounds (3.3d) and (3.3e) strictly. This property is the origin of the term interior-point. By respecting these bounds, the method avoids solutions, that satisfy $F(x, \lambda, s) = 0$ but not (3.3d) or (3.3e). These so-called spurious solutions abound and do not provide useful information about solutions of (3.1) or (3.2), so it makes sense to exclude them altogether from the region of search.

### 3.2.2 The duality measure and the centering parameter

Like most iterative algorithms in optimization, primal-dual interior-point methods have two basic ingredients; a procedure for determining the step and a measure of the desirability of each point in the search space. An important component of the measure of desirability is the average value of the pairwise products $x_i s_i$, $i = 1, \ldots, n$, which are all positive when $x > 0$ and $s > 0$. This quantity is known as the duality measure and is defined as follows:

$$\mu = \frac{1}{n} \sum_{i=1}^{n} x_i s_i = \frac{x^\top s}{n}. \quad (3.5)$$
The procedure for determining the search direction has its origins in Newton’s method for the nonlinear equations (3.4). Newton’s method forms a linear model of (3.4) around the current point and obtains the search direction \((\Delta x, \Delta \lambda, \Delta s)\) by solving the following system of linear equations:

\[
\begin{bmatrix}
0 & A^\top & I \\
A & 0 & 0 \\
S & 0 & X
\end{bmatrix}
\begin{bmatrix}
\Delta x \\
\Delta \lambda \\
\Delta s
\end{bmatrix}
= 
\begin{bmatrix}
-r_c \\
-r_b \\
-XSe
\end{bmatrix}
\]  

(3.6)

where

\[
r_b = Ax - b,
\]

\[
r_c = A^\top \lambda + s - c.
\]

Usually, a full step along this direction would violate the bounds, so we perform a line search along the Newton direction and define the new iterate as

\[
(x^+, \lambda^+, s^+) = (x, \lambda, s) + \alpha (\Delta x, \Delta \lambda, \Delta s),
\]

for some line search parameter \(\alpha \in (0, 1]\). We often can take only a small step along this direction (\(\alpha << 1\)) before violating the conditions \(x > 0\) and \(s > 0\). Hence, the pure Newton direction (3.6), sometimes known as the affine scaling direction, often does not allow us to make much progress towards a solution.

Most primal-dual methods use a less aggressive Newton direction, one that does not aim directly for a solution of (3.4) but rather for a point whose pairwise products \(x_i s_i\), \(i = 1, \ldots, n\), are reduced to a lower average value - not all the way to zero. So we take a Newton step toward a point for which \(x_i s_i = \sigma \mu\), where \(\mu\) is defined by (3.5) and \(\sigma \in [0, 1]\) is the reduction factor that we wish to achieve in the duality measure on this step. The modified step equation is then

\[
\begin{bmatrix}
0 & A^\top & I \\
A & 0 & 0 \\
S & 0 & X
\end{bmatrix}
\begin{bmatrix}
\Delta x \\
\Delta \lambda \\
\Delta s
\end{bmatrix}
= 
\begin{bmatrix}
-r_c \\
-r_b \\
-XSe + \sigma \mu e
\end{bmatrix}
\]  

(3.7)
When $\sigma > 0$, it usually is possible to take a longer step $\alpha$ along the direction defined by (3.7) before violating the bounds. Therefore $\sigma$ is called the centering parameter.

The concrete choices of $\sigma$ and $\alpha$ are crucial to the performance of interior-point methods. Therefore techniques for controlling these parameters, directly and indirectly, give rise to a wide variety of methods with diverse properties.

Although software for implementing interior point methods does usually not start from a feasible point $(x^0, \lambda^0, s^0)$ that fulfills:

\begin{align*}
Ax^0 &= b, \\
A^T \lambda^0 + s^0 &= c,
\end{align*}

most of the historical development of theory and algorithms assumed that these conditions are satisfied. Assuming this, a comprehensive convergence analysis can be presented in just a few pages using only basic mathematical tools and concepts (see, for example, Wright [47, Chapter 5] or Nocedal and Wright [40, Section 14.1]). Analysis of the infeasible case follows the same principles, but is considerably more complicated in the details.

Practical implementations of interior-point algorithms work with an infeasible starting point and infeasible iterations, maintain strict positivity of $x$ and $s$ throughout and take at each iteration a Newton-like step involving a centering component. Several aspects of ’theoretical‘ algorithms are typically ignored, while several enhancements are added that have a significant effect on practical performance. Next we describe the algorithmic enhancements that are found in a typical implementation of an infeasible-interior-point method (for further details consult the paper of Mehrotra [32]).

### 3.2.3 Corrector and centering steps

A key feature of practical algorithms is their use of corrector steps that compensate for the linearization error made by the Newton affine-scaling step in modeling equation (3.3c). Consider the affine-scaling direction $(\Delta x^{aff}, \Delta \lambda^{aff}, \Delta s^{aff})$ defined by
If we take a full step in this direction, we obtain

\[(x_i + \Delta x^{aff}_i)(s_i + \Delta s^{aff}_i) = x_is_i + x_i\Delta x^{aff}_i + s_i\Delta s^{aff}_i + \Delta x^{aff}_i \Delta s^{aff}_i = \Delta x^{aff}_i \Delta s^{aff}_i.\]

That is, the updated value of \(x_is_i\) is \(\Delta x^{aff}_i \Delta s^{aff}_i\) rather than the ideal value 0. We can solve the following system to obtain a step \((\Delta x^{cor}, \Delta \lambda^{cor}, \Delta s^{cor})\) that attempts to correct for this deviation from the ideal:

\[
\begin{bmatrix}
0 & A^\top & I \\
A & 0 & 0 \\
S & 0 & X
\end{bmatrix}
\begin{bmatrix}
\Delta x^{cor} \\
\Delta \lambda^{cor} \\
\Delta s^{cor}
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
-XSe
\end{bmatrix}.\tag{3.9}
\]

In many cases, the combined step \((\Delta x^{aff}, \Delta \lambda^{aff}, \Delta s^{aff}) + (\Delta x^{cor}, \Delta \lambda^{cor}, \Delta s^{cor})\) does a better job of reducing the duality measure than does the affine-scaling step alone.

A further important ingredient for a good practical algorithm is the use of centering steps, with an adaptive choice of the centering parameter \(\sigma_k\). Thereby the affine-scaling step can be used as the basis of a successful heuristic for choosing \(\sigma_k\).

If the affine-scaling step reduces the duality measure significantly, there is not much need for centering, so a smaller value of \(\sigma_k\) is appropriate. Conversely, if not much progress can be made along this direction before reaching the boundary of the non-negative orthant, a larger value of \(\sigma_k\) will ensure that the next iterate is more centered, so a longer step will be possible form this next point. The following scheme calculates the maximum allowable step lengths along the affine-scaling direction:

\[
\alpha_{aff}^\text{pri} \overset{\text{def}}{=} \min(1, \min_{i: \Delta x^{aff}_i < 0} \frac{x_i}{\Delta x^{aff}_i}), \tag{3.10a}
\]

\[
\alpha_{aff}^\text{dual} \overset{\text{def}}{=} \min(1, \min_{i: \Delta s^{aff}_i < 0} \frac{s_i}{\Delta s^{aff}_i}). \tag{3.10b}
\]
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Now we can define $\mu_{aff}$ to be the value of $\mu$ that would be obtained by using these step lengths:

$$
\mu_{aff} = (x + \alpha_{aff}^{pri} \Delta e^a) ^\top (s + \alpha_{aff}^{dual} \Delta s^a) / n.
$$

(3.11)

The centering parameter $\sigma$ is chosen according to the following heuristic:

$$
\sigma = \left( \frac{\mu_{aff}}{\mu} \right)^3.
$$

(3.12)

To summarize, computation of the search direction requires the solution of two linear systems. First (3.8) is solved to obtain the affine-scaling direction, also known as the predictor step. This step is used to define the right-hand side for the corrector step and to calculate the centering parameter from (3.10) - (3.12). Second, the search direction is calculated solving

$$
\begin{bmatrix}
0 & A^\top & I \\
A & 0 & 0 \\
S & 0 & X
\end{bmatrix}
\begin{bmatrix}
\Delta x \\
\Delta \lambda \\
\Delta s
\end{bmatrix} =
\begin{bmatrix}
-r_c \\
-r_b \\
-X Se - \Delta X_{aff} \Delta s_{aff} e + \sigma \mu e
\end{bmatrix}.
$$

(3.13)

Note that the predictor, corrector and centering contributions have been aggregated on the right-hand side of this system. The coefficient matrix in both linear systems (3.8) and (3.13) is the same. Thus, the factorization of the matrix needs to be computed only once and the marginal cost of solving the second system is relatively small.

3.2.4 Finding an optimal step length

Practical implementations typically calculate the maximum step lengths that can be taken in the $x$ and $s$ variables without violating nonnegativity separately:

$$
\alpha_{k, max}^{pri} \overset{\text{def}}{=} \min (1, \min_{i: \Delta x_i < 0} \frac{x_i}{\Delta x_i}),
$$

$$
\alpha_{k, max}^{dual} \overset{\text{def}}{=} \min (1, \min_{i: \Delta s_i < 0} \frac{s_i}{\Delta s_i}).
$$

(3.14) has no solid analytical justification, but appears to work well in practice
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and then take a step length of slightly less than this maximum:

\[ \alpha_{pri}^k = \eta_k \alpha_{pri}^k_{\text{max}}, \]  
\[ \alpha_{dual}^k = \eta_k \alpha_{dual}^k_{\text{max}}, \]

(3.15a)  
(3.15b)

where \( \eta_k \in [0.9, 1.0] \) is chosen to accelerate the asymptotic convergence. Therefore we want that \( \eta_k \to 1 \) as the iterates approach the primal-dual solution. Then we obtain a new iterate by setting

\[ x^{k+1} = x^k + \alpha_{pri}^k \Delta x^k, \]

\[ (\lambda^{k+1}, s^{k+1}) = (\lambda^k, s^k) + \alpha_{dual}^k (\Delta \lambda^k, \Delta s^k). \]

As the step \( (\Delta x^k, \Delta \lambda^k, \Delta s^k) \) adjusts the infeasibility in the KKT conditions

\[ A \Delta x^k = -r_b^k, \]
\[ A^\top \Delta \lambda^k + \Delta s^k = -r_c^k \]

we have for the infeasibilities at the new iterate \( k + 1 \)

\[ r_b^{k+1} = (1 - \alpha_{pri}^k) r_b^k, \]
\[ r_c^{k+1} = (1 - \alpha_{dual}^k) r_c^k. \]

3.2.5 Choosing a starting point

Choice of the starting point is an important practical issue with a significant effect on the robustness of the algorithm. A poor choice \( (x^0, \lambda^0, s^0) \) satisfying only (3.3d) and (3.3e) often leads to failure in convergence. We describe here a heuristic that finds a starting point that satisfies (3.3a) and (3.3b) reasonably well, while maintaining (3.3d) and (3.3e) and additionally avoiding too large values of these components.

First we find a vector \( \tilde{x} \) of minimum norm satisfying (3.3b):

\[ A \Delta x^k = -r_b^k, \]
\[ A^\top \Delta \lambda^k + \Delta s^k = -r_c^k \]

we have for the infeasibilities at the new iterate \( k + 1 \)

\[ r_b^{k+1} = (1 - \alpha_{pri}^k) r_b^k, \]
\[ r_c^{k+1} = (1 - \alpha_{dual}^k) r_c^k. \]
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\[
\min_{x} \frac{1}{2} x^\top x \\
\text{subject to } Ax = b,
\]

and a vector \((\tilde{\lambda}, \tilde{s})\) satisfying (3.3a) such that \(\tilde{s}\) has minimum norm:

\[
\min_{(\lambda, s)} \frac{1}{2} s^\top s \\
\text{subject to } A^\top \lambda + s = c.
\]

The optimal values \((\tilde{x}, \tilde{\lambda}, \tilde{s})\) can be written explicitly as follows:

\[
\tilde{x} = A^\top (AA^\top)^{-1} b, \quad (3.16a) \\
\tilde{\lambda} = (AA^\top)^{-1} Ac, \quad (3.16b) \\
\tilde{s} = c - A^\top \tilde{\lambda}. \quad (3.16c)
\]

In general, \(\tilde{x}\) and \(\tilde{s}\) will have nonpositive components, so are not suitable for use as a starting point. Therefore define

\[
\delta_x = \max\left(-\frac{3}{2} \min_i \tilde{x}_i, 0\right), \\
\delta_s = \max\left(-\frac{3}{2} \min_i \tilde{s}_i, 0\right),
\]

and adjust the \(\tilde{x}\) and \(\tilde{s}\) vectors so that they fulfill (3.3d) and (3.3e):

\[
\hat{x} = \tilde{x} + \delta_x e, \\
\hat{s} = \tilde{s} + \delta_s e.
\]

To ensure that our starting points are not too close to zero and not too dissimilar, we define them finally as:

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\begin{align*}
x^0 &= \xi + \delta_x e, \quad (3.17a) \\
\lambda^0 &= \tilde{\lambda}, \quad (3.17b) \\
s^0 &= \tilde{s} + \delta_s e. \quad (3.17c)
\end{align*}

where

\begin{align*}
\delta_x &= \frac{1}{2} \xi^T \delta \xi, \\
\delta_s &= \frac{1}{2} \xi^T \delta s \xi.
\end{align*}

The computational cost of finding \((x^0, \lambda^0, s^0)\) by this scheme is about the same as one step of the primal-dual method.

### 3.2.6 A practical primal-dual method

Finally we put together the different, above mentioned ingredients for a practically successful algorithm in Table 3.1 below.

| Practical Predictor-Corrector Algorithm |
|------------------------------------------|
| Calculate \((x^0, \lambda^0, s^0)\) using \((3.16) - (3.17)\); |
| \(k = 0\) |
| \textbf{repeat} |
| \textbf{Set} \((x, \lambda, s) = (x^k, \lambda^k, s^k)\) and calculate \((\Delta x^{aff}, \Delta \lambda^{aff}, \Delta s^{aff})\) by solving \((3.8)\); |
| \textbf{Calculate} \(\mu\) by using \((3.5)\); |
| \textbf{Use additionally} \((3.10), (3.11)\) and \((3.12)\) to calculate \(\sigma\); |
| \textbf{Solve} \((3.13)\) for \((\Delta x, \Delta \lambda, \Delta s)\); |
| \textbf{Calculate} \(\alpha^{pri}\) and \(\alpha^{dual}\) as in \((3.15)\); |
| \textbf{Set} \(x^{k+1} = x^k + \alpha^{pri}_k \Delta x;\) |
| \textbf{Set} \((\lambda^{k+1}, s^{k+1}) = (\lambda^k, s^k) + \alpha^{dual}_k (\Delta \lambda, \Delta s);\) |
| \textbf{k} = k + 1; |
| \textbf{Set} \(r^k = (Ax^k - b, A^\top \lambda^k + s^k - c, XSe);\) |
| \textbf{until} norm\((r^k) < \varepsilon\) (for a given \(\varepsilon > 0\)). |

Table 3.1: Formal specification of a practical algorithm

As there are examples for that this algorithm diverges, no convergence theory is available for the algorithm. Simple safeguards could be incorporated into the
method to force it into the convergence framework of existing methods or to im-
prove its robustness, but many practical codes do not implement these safeguards
because failures are rare.

3.3 Extensions To Convex Quadratic Programming

The interior point approach, introduced for linear programming in the previous sec-
tion, can also be applied to convex quadratic programs through simple extensions
of the linear programming algorithm.

To keep the description of the interior point method simple, we consider a QP with
only inequality constraints:

\[
\begin{align*}
\min & \frac{1}{2} x^\top Q x + d^\top x \\
\text{subject to} & \ Ax \geq c,
\end{align*}
\]

where \(Q\) is symmetric and positive definite, \(d\) and \(x\) are vectors in \(\mathbb{R}^n\), \(c\) is a vector
in \(\mathbb{R}^m\), and \(A\) is an \(m \times n\) matrix.

3.3.1 The KKT system and Newton’s method

The KKT conditions for (3.18) are :

\[
\begin{align*}
A^\top \lambda - Q x &= d, \\
Ax - s &= c, \\
\lambda_i s_i &= 0, \quad i = 1, 2, \ldots, m, \\
\lambda &\geq 0, \\
s &\geq 0.
\end{align*}
\]

Since \(Q\) is positive definite, these KKT conditions are necessary and sufficient, and
therefore we can solve (3.18) by finding solutions of (3.19).

Given a current iterate \((x, s, \lambda)\) that satisfies (3.19d) and (3.19e), we can define the
complementary measure \(\mu\) as

\[\text{If equality constraints are also present, they can be incorporated by simple extensions to the}
\text{method described below} \]

---

\(5\) If equality constraints are also present, they can be incorporated by simple extensions to the
method described below.
As in the previous section, we derive a practical, path-following, primal-dual algorithm by considering the perturbed KKT conditions:

$$F(x, \lambda, s; \sigma \mu) = \begin{bmatrix} A^\top \lambda - Qx - d \\ Ax - c - s \\ \Lambda S e - \sigma \mu e \end{bmatrix} = 0$$ (3.21)

where

$$\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n),$$

$$S = \text{diag}(s_1, \ldots, s_n),$$

and $\sigma \in [0, 1]$. The solutions of (3.21) for all positive values $\sigma$ and $\mu$ define the central path. This is a trajectory that leads to the solution of the quadratic program as $\sigma \mu$ tends to zero.

By applying Newton’s method to (3.21), we obtain the linear system

$$\begin{bmatrix} Q & 0 & -A^\top \\ A & -I & 0 \\ 0 & \Lambda & S \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta s \\ \Delta \lambda \end{bmatrix} = \begin{bmatrix} -r_d \\ -r_c \\ -\Lambda S e + \sigma \mu e \end{bmatrix}$$ (3.22)

where

$$r_c = Ax - s - c,$$

$$r_d = Qx - A^\top \lambda + d.$$
3.3.2 Finding the optimal step length

We define the new iterate as

\[(x^{k+1}, s^{k+1}) = (x^k, s^k) + \alpha_k^{pri}(\Delta x^k, \Delta s^k),\]  
\[\lambda^{k+1} = \lambda^k + \alpha_k^{dual}\Delta \lambda^k,\]

(3.23a, 3.23b)

where \((\alpha_k^{pri}, \alpha_k^{dual})\) are selected so as to (approximately) minimize the optimality measure

\[
\|Q x^{k+1} - A^\top \lambda^{k+1} + d\|_2^2 + \|Ax^{k+1} - s^{k+1} - c\|_2^2 + (s^{k+1})^\top \lambda^{k+1},
\]

subject to

\[0 \leq \alpha_k^{pri} \leq \alpha_k^{pri},\]  
\[0 \leq \alpha_k^{dual} \leq \alpha_k^{dual},\]

(3.24a, 3.24b, 3.24c)

where \(x^{k+1}, s^{k+1}\) and \(\lambda^{k+1}\) are defined as functions of the step lengths through (3.23) and \(\alpha_k^{pri}\) and \(\alpha_k^{dual}\) are given by

\[\alpha_k^{pri} = \max\{\alpha \in (0, 1] : s^k + \alpha \Delta s^k \geq (1 - \tau_k)s^k\},\]
\[\alpha_k^{dual} = \max\{\alpha \in (0, 1] : \lambda^k + \alpha \Delta \lambda^k \geq (1 - \tau_k)\lambda^k\},\]

whereby the parameter \(\tau_k \in (0, 1)\) controls how far we back off from the maximum step for which the conditions \(s^k + \alpha \Delta s^k \geq 0\) and \(\lambda^k + \alpha \Delta \lambda^k \geq 0\) are satisfied.

3.3.3 Choosing a starting point

As for linear programming, the efficiency and robustness of our practical algorithm can be greatly enhanced by choosing a good starting point. Therefore we want to present at least a simple heuristic that improves the choice of the starting point by moving an initial starting point \((x, s, \lambda)\) form the user far enough away from the boundary of the region \((s, \lambda) \geq 0\) to permit the algorithm to take long steps on early iterations. Therefore, our heuristic computes the affine scaling step \((x^{aff}, s^{aff}, \lambda^{aff})\) from \((x, s, \lambda)\) and then sets
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\[ s_0 = \max(1, |\bar{s} + \Delta s^{aff}|), \]  
\[ \lambda_0 = \max(1, |\bar{\lambda} + \Delta \lambda^{aff}|), \]  
\[ x_0 = \bar{x}. \] (3.25a, 3.25b, 3.25c)

3.3.4 A practical primal-dual algorithm

The most popular practical algorithms for convex quadratic programming are, as practical interior point methods for linear programming, based on Mehrotra’s predictor-corrector idea (for details see [32]). Therefore we first compute an affine scaling step \((\Delta x^{aff}, \Delta \lambda^{aff}, \Delta s^{aff})\) by setting \(\sigma = 0\) in (3.22). The following scheme calculates the maximum allowable step lengths along the affine-scaling direction:

\[ \alpha_{pri}^{aff} \overset{\text{def}}{=} \min(1, \min_{i: \Delta s_i^{aff} < 0} \frac{s_i}{\Delta s_i^{aff}}), \] (3.26a)
\[ \alpha_{dual}^{aff} \overset{\text{def}}{=} \min(1, \min_{i: \Delta \lambda_i^{aff} < 0} \frac{\lambda_i}{\Delta \lambda_i^{aff}}). \] (3.26b)

Using the above definitions, we set \(\mu^{aff}\) in accordance with the definition of \(\mu\) in (3.20) to be:

\[ \mu^{aff} = (s + \alpha_{pri}^{aff} \Delta x^{aff})^\top (\lambda + \alpha_{dual}^{aff} \Delta \lambda^{aff})/n. \] (3.27)

The centering parameter \(\sigma\) is chosen according to the following heuristic\(^6\)

\[ \sigma = \left( \frac{\mu^{aff}}{\mu} \right)^3. \] (3.28)

Next we define the corrector step that aims to improve the affine scaling step as

\(^6\) (3.28) has no solid analytical justification, but appears to work well in practice
Finally, the total step is obtained by solving the following system:

\[
\begin{bmatrix}
Q & 0 & -A^\top \\
A & -I & 0 \\
0 & \Lambda & S
\end{bmatrix}
\begin{bmatrix}
\Delta x^{\text{cor}} \\
\Delta s^{\text{cor}} \\
\Delta \lambda^{\text{cor}}
\end{bmatrix} =
\begin{bmatrix}
0 \\
0 \\
-\Delta X^{\text{aff}} \Delta S^{\text{aff}} e
\end{bmatrix}.
\]

(3.29)

Finally, we put together the different, above mentioned ingredients for a practical successful algorithm in Table 3.2 below.

Table 3.2: Formal specification of a practical algorithm
4 Feasible Active-Set Methods

This chapter deals with the description of practically successful active set methods for convex quadratic programming. In the following sections we consider the most important aspects of feasible active-set methods like the working set, the subproblems at each iteration, the smart choice of a starting point and the usage of updating factorizations. We conclude the chapter with a comparison of active-set and interior point methods.

For the description of feasible active-set methods we use the following problem formulation:\footnote{This formulation is equivalent to the other formulations of general QPs presented in this thesis. For details see Section 2.1.}

\[
\begin{align*}
\min_x & \quad \frac{1}{2} x^\top Q x + x^\top d \\ 
\text{subject to} & \quad a_i^\top x = c_i, \quad i \in \varepsilon, \\ & \quad a_i^\top x \geq c_i, \quad i \in \iota,
\end{align*}
\]

where $Q$ is a symmetric, positive definite $n \times n$ matrix, $\varepsilon$ and $\iota$ are finite sets of indices, and $d, x$ and $\{a_i\}, \ i \in \varepsilon \cup \iota$, are vectors in $\mathbb{R}^n$.

4.1 Active Sets And Working Sets

We now describe active-set methods for solving the quadratic program, given by (4.1).

If the contents of the optimal active set $A(x^*)$, given by

\[
A(x^*) = \{i \in \varepsilon \cup \iota | a_i^\top x^* = c_i\}
\]
were known in advance, we could find the solution $x^*$ easily. Of course, we usually do not have prior knowledge of $A(x^*)$ and therefore determination of this set is the main challenge facing active-set algorithms for quadratic programs.

The simplex method starts by making a guess of the optimal active set, then repeatedly uses gradient and Lagrange multiplier information to drop one index from the current estimate of $A(x^*)$ and add a new index, until optimality is detected. Active-set methods for quadratic programs differ from the simplex method in that the iterates and the solution $x^*$ are not necessarily vertices of the feasible region.

There are primal, dual and primal-dual versions of active-set methods. We will explain now primal methods, which are steadily decreasing the objective function (4.1a) while remaining feasible with respect to the primal problem.

Primal active-set methods find a step from one iterate to the next by solving a quadratic subproblem in which some of the inequality constraints (4.1c), additionally to the equations (4.1b), are treated as equalities. This set of equations is called the working set and is denoted as $W_k$ at the $k$th iterate $x_k$. We further assume that the gradients $a_i$ of the constraints in $W_k$ are linearly independent.

### 4.2 The Subproblems

The first step in every iteration is to check whether the current iterate $x_k$ minimizes (4.1a) in the subspace defined by $W_k$. If this is not the case, we solve an equality-constrained quadratic subproblem, in which the constraints belonging to $W_k$ are included and the other inequality constraints are temporarily disregarded, to determine a step $p$, defined as

$$p = x - x_k, \quad (4.2)$$

Now, by substituting (4.2) in (4.1a), we get

$$\min_p \frac{1}{2} p^\top Qp + g_k p + \phi_k,$$

where

---

8 If we use a linearly independent subset of the gradients $a_i, i \in \{1, \ldots, m\}$ as initial working set $W_0$, the definition of the step length in (4.6) ensures that the linear independence is maintained for the subsequent working sets $W_k, k \geq 1$. 

---

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4 Feasible Active-Set Methods

\[ g_k = Qx_k + d, \]
\[ \phi_k = \frac{1}{2} x_k^\top Qx_k + d^\top x_k, \]

are independent of \( p \). Therefore the subproblem to be solved at the \( k \)th iteration can be written as

\[
\begin{align*}
\min_p & \quad \frac{1}{2} p^\top Q p + g_k p \\
\text{subject to} & \quad a_i^\top p = 0, \quad i \in W_k.
\end{align*}
\] (4.3a)

We can solve the subproblem for example by a symmetric indefinite factorization or by the Schur-complement method or by the Null-Space method (for details see Section 2.2). For the solution of this subproblem, denoted by \( p_k \), we have

\[ a_i^\top (x_k + \alpha p_k) = a_i^\top x_k = c_i, \quad \forall \alpha, \quad i \in W_k. \] (4.4)

If the direction \( p_k \) is nonzero, the objective function is strictly decreasing (as \( Q \) is positive definite) along this direction (for a proof see, for example, Nocedal and Wright [40, Theorem 16.6.]).

Now we have to decide how far to move along the direction \( p_k \). We set

\[ x_{k+1} = x_k + \alpha_k p_k, \] (4.5)

where we choose the step-length parameter \( \alpha_k \) in order to maximize the decrease in (4.1a) to be the largest value in the range \([0, 1]\) for which all constraints are satisfied:

\[ \alpha_k \overset{\text{def}}{=} \min\left(1, \min_{i \notin W_k, a_i^\top p_k < 0} \frac{c_i - a_i^\top x_k}{a_i^\top p_k}\right) \] (4.6)

The constraint \( i \), for which the minimum in (4.6) is achieved, is called blocking constraint. It is also possible for \( \alpha_k \) to be zero, because some constraint \( i \) could fulfill \( a_i^\top p_k < 0 \) and additionally be active at \( x_k \) without belonging to \( W_k \).
If $\alpha_k < 1$, a new working set $W_{k+1}$ is constructed by adding one of the blocking constraints to $W_k$.

We continue to do this until we reach a point $\hat{x}$ that minimizes (4.3) over its current working set $\hat{W}$. Such a point $\hat{x}$ satisfies the KKT conditions for the subproblem:

$$
\begin{bmatrix}
Q & A_k^T \\
A_k & 0
\end{bmatrix}
\begin{bmatrix}
-\hat{p} \\
\hat{\lambda}^+
\end{bmatrix} =
\begin{bmatrix}
g \\
h
\end{bmatrix},
$$

where

$$g = d + Q\hat{x},$$

$$h = A_k\hat{x} - c,$$

and $A_k$ is the Jacobian of the constraints in (4.3) and $c_k$ is the vector whose components are $c_i, i \in W_k$. Furthermore $p = 0$ at $\hat{x}$ and therefore we have that

$$\sum_{i \in \hat{W}} a_i \hat{\lambda}_i = g = Q\hat{x} + d,$$

for some Lagrange multipliers $\hat{\lambda}_i, i \in \hat{W}$. It follows that $\hat{x}$ and $\hat{\lambda}$ satisfy the first three KKT conditions for the original quadratic program (4.1)

$$Qx^* + c - \sum_{i \in A(x^*)} \lambda_i^* a_i = 0,$$  

$$a_i^T x^* = c_i, \ \forall i \in A(x^*),$$  

$$a_i^T x^* > c_i, \ \forall i \in \iota \setminus A(x^*),$$  

$$\lambda_i^* \geq 0, \ \forall i \in \iota \cap A(x^*),$$

if we define

$$\hat{\lambda}_i = 0 \ \forall i \in \iota \setminus \hat{W},$$

and consider the step length control defined in (4.6).
We now take a look at the fourth equation of the above KKT system (4.7d), which concerns the inequality constraints in \( \hat{W} \). If these multipliers are all nonnegative, our solution \((\hat{x}, \hat{\lambda})\) is the global optimum for (4.1).

If, on the other hand, one or more multipliers are negative, the objective function, given by (4.1a), can be decreased by dropping one of these constraints. Thus, the next step is to remove the most negative multiplier \( \lambda^9 \) from \( \hat{W} \) and solve the subproblem, given by (4.3), for the new working set.

It can be shown that the optimal value \( p \) of this new subproblem gives a direction that is feasible with respect to the dropped constraint (for a proof see, for example, Nocedal and Wright [40, Theorem 16.5.]).

Hence, we have at least at every second iteration a direction \( p_k \) that guarantees together with the assumption that the step length \( \alpha_k \) is nonzero for every \( p_k \neq 0 \) that we have a strict decrease in the objective function after two iterations. This fact finally guarantees finite termination of our algorithm (for details see Nocedal and Wright [40, Section 16.5]).

### 4.3 Choosing A Starting Point

Various techniques can be used to determine an initial feasible point. One such is to use a two-phase approach, where in Phase I an auxiliary linear program is designed so that an initial basic feasible point is trivial to find. This problem can be solved with the simplex method and its solution gives a basic feasible point for the original (4.1) (for details see, for example, Nocedal and Wright [40, Section 13.5]).

An alternative approach is a penalty (or 'big M') method that includes a measure of infeasibility in the objective function that is zero at the solution. We introduce a scalar artificial variable \( \eta \) into (4.1) to get a measure of the constraint violation. So we solve the modified problem

\[ \text{(4.1)} \]

\[ \text{(4.3)} \]

---

\(^9\) This choice is motivated by a sensitivity analysis concerning the removal of the Lagrange multipliers, which shows that the rate of decrease in the objective function is proportional to the negative magnitude of the multiplier. However the step length along the resulting direction may be small because of some blocking constraint. That’s why the amount of decrease in the objective function is not guaranteed to be greater than for other negative multipliers. Furthermore the magnitude of the multipliers is dependent on the scaling of the corresponding constraints. Therefore, as for the simplex method in linear programming, strategies that are less sensitive to scaling often give better practical results.
for some large value of $M$. It can be shown by using the theory of Lagrange multipliers (see Theorem 2 in Section 5.4) that if there exist feasible points for the original problem (4.1), then for $M$ sufficiently large, the solution of (4.8) will have $\eta = 0$ and the value of $x$ will be also optimal for (4.1).

To solve (4.1) we therefore use some heuristic to choose $M$, then solve (4.8) and increase $M$ if $\eta > 0$ until $\eta$ becomes zero. A feasible starting point for (4.8) can be obtained easily by just taking some guess $\tilde{x}$ and then choosing $\eta$ large enough so that all constraints are satisfied.

4.4 Updating Factorizations

In this subsection we explain an updating technique that is crucial to the efficiency of the above presented active-set method.

As the working set can change by at most one index at every iteration in the active-set method presented in this chapter, the KKT matrix of the current iteration differs in at most one row and one column from the KKT matrix of the previous iteration. Therefore we can compute the matrix factors needed to solve the current subproblem by updating the factors computed at the previous iteration. The total cost of the updating is in general cheaper than solving the new system from the scratch.

We limit our discussion here to the null-space method, described in (2.8) - (3.1a), but their are also ways to make an update for the other methods presented in Subsection 2.2. Suppose that the $m \times n$ matrix $A$ has $m$ linearly independent rows and assume that the orthogonal $n \times m$ matrix $Y$ and the orthogonal $n \times n - m$ matrix $Z$ are defined by means of a QR factorization of $A^\top$ in the following way:\[10\]

\[10\] As $Z$ is not uniquely defined there are also other possible definitions of $Z$ (for details see Subsection 2.2)
where $\Pi$ is a permutation matrix and $R$ is a square, nonsingular, upper triangular $m \times m$ matrix.

Now let us take a look at the case where one constraint $a$ is added to the working set. Our new constraint matrix $\overline{A}^\top$ has full column rank and is equal to $[A^\top a]$. As $Y$ and $Z$ are orthogonal, we have

$$
\overline{A}^\top = [Y \ Z \ \hat{Q}^\top a],
$$

(4.9)

where $\gamma$ is a scalar and $\hat{Q}$ is a orthogonal matrix that transforms $Z^\top a$ in the following way:

$$
\hat{Q}(Z^\top a) = \begin{bmatrix} \gamma \\ 0 \end{bmatrix}.
$$

From (4.9) we can see that the new factorization has the form

$$
\overline{A}^\top = [Y \ Z\hat{Q}^\top] \begin{bmatrix} R \\ 0 \end{bmatrix},
$$

where

$$
\Pi = \begin{bmatrix} \Pi & 0 \\ 0 & 1 \end{bmatrix},
$$

$$
\overline{R} = \begin{bmatrix} R & Y^\top a \\ 0 & \gamma \end{bmatrix}.
$$

Now we choose $Z$ to be the last $n - m - 1$ columns of $Z\hat{Q}^\top$ to finish the update.

To update $Z$, we need to account for the cost of obtaining $\hat{Q}$ and the cost for calculating $Z\hat{Q}$, which is of order $n(n - m)$. This is less expensive than computing the
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new factors from scratch, which causes cost of order $n^2 m$, especially when the null space is small.

In the case that we want to remove an index from the working set, we have to remove a row from $R$ and thus disturb its upper triangular property by introducing a number of nonzeros on the diagonal immediately below the main diagonal. We can restore the upper diagonal property by applying a sequence of plane rotations that introduce a number of inexpensive transformations into $Y$. The updated matrix $Z$ is then the current matrix $Z$ augmented by a single column $z$:

$$Z = \begin{bmatrix} z & Z \end{bmatrix}.$$

The total cost of the updating depend on the location of the removed column but is in general cheaper than computing the QR factors from the scratch (for details see Gill et al. [20, Section 5]).

Let us next consider the reduced Hessian $Z^T QZ$. For problem (4.3), $h = 0$ in (2.5) and therefore $p_y$, given by (2.9), is also zero. Thus the equation for null-space vector $p_z$ reduces from (2.10) to

$$(Z^T QZ) p_z = -Z^T g.$$  

To update the Cholesky factorization of the reduced Hessian

$$Z^T QZ = LL^T$$

a series of inexpensive, elementary operations can be used. Furthermore we can update the reduced gradient $Z^T g$ at the same time as $Z$ (for details see Nocedal and Wright [40, Section 16.7]).

4.5 Comparison Of Active-Set And Interior Point Methods

Interior point methods share common features that distinguish them from the active set methods. Each interior point iteration is expensive to compute and can make significant progress towards the solution, while the active set methods usually require a large number of inexpensive iterations. Geometrically, the active set methods for
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QP differ from the simplex method in that the iterates are not necessarily vertices of the feasible region. Interior point methods approach the boundary of the feasible set only in the limit. They may approach the solution either from the interior or exterior of the feasible region, but they never actually lie on the boundary of this region.

The numerical comparison of active-set and interior point methods for convex quadratic programming, executed by Gould and Toint [24], indicates that interior-point methods are generally much faster on large problems. If some warm start information is available, however, the active set methods are generally preferable. Although a lot of research has been focused on improving the warm-start ability of interior point methods, the full potential of interior point methods in this area is not yet known.
This chapter provides a description of a Lagrangian infeasible active-set method for convex quadratic programming. In the following sections we give detailed information about the different parts of the algorithm and then take a look at the algorithm’s convergence behaviour.

To describe the algorithm let \( a, b, d \in \mathbb{R}^n \), \( c \in \mathbb{R}^m \), \( A \in \mathbb{R}^{m \times n} \) and \( Q = Q^T \) be given, with \( Q \) a positive definite \( n \times n \) matrix. We consider a convex quadratic minimization problem with equality constraints and simple bound constraints:

\[
\begin{align*}
\text{min } J(x) & \quad \text{subject to } h(x) = 0 \text{ and } b \leq x \leq a, \\
\end{align*}
\]

where

\[
\begin{align*}
J(x) & := \frac{1}{2} x^T Q x + d^T x, \\
h(x) & := B x - c.
\end{align*}
\]

The KKT-system for (5.1) is given by

\[
\begin{align*}
B^T \lambda + Q x + d + s + t & = 0, \\
B x & = c, \\
s \circ (x - b) & = 0, \\
t \circ (x - a) & = 0, \\
x - b & \geq 0, \\
a - x & \geq 0, \\
s & \leq 0, \\
t & \geq 0.
\end{align*}
\]

\[\text{[11]}\] This formulation is equivalent to the other formulations of general QPs presented in this thesis. For details see Section 2.1.
It is well known that a vector $x$ together with vectors $\lambda \in \mathbb{R}^m$, $s \in \mathbb{R}^n$ and $t \in \mathbb{R}^n$ of Lagrange multipliers for the equality and bound constraints furnishes a global minimum of (5.1) if and only if $(x, \lambda, s, t)$ satisfies the KKT-system.

We now describe in some detail the approach sketched above. Therefore first we give a survey of the main components of our algorithm in Table 5.1 and then take a closer look at the important parts of our approach in the following sections.

### Prototype Algorithm

**Input:** $Q$ symmetric, positive definite $n \times n$ matrix, $A$ $n \times m$ matrix, $a, b, d \in \mathbb{R}^n$, $c \in \mathbb{R}^m$, $A_1 \subseteq N$ and $A_2 \subseteq N$, $A_1 \cap A_2 = \emptyset$ e.g. $A_1 = \emptyset$, $A_2 = \emptyset$

**Output:** $(x, \lambda, s, t)$ optimal solution

repeat until $(x, \lambda, s, t)$ is optimal

- Calculate the actual augmented Lagrange function.
- Minimize the augmented Lagrange function applying an infeasible active set method.
- As initial active set use the optimal active set of the last iteration.
- Update $\lambda$.
- If there has been a change in the active set, try to solve problem (P) directly again using an infeasible active set method.
- Compare the solution of the direct approach with the one you have got from minimizing the augmented Lagrange function and take the “better” one.

| Table 5.1: Description of the algorithm |

### 5.1 Outer Algorithm: The Augmented Lagrangian Method

We make use of the so-called augmented Lagrangian method, which was first proposed by Hestenes [25] and Powell [42], in our outer algorithm. Therefore we define the augmented Lagrangian function, which is a combination of the Lagrangian function and the quadratic penalty function, as:

$$\mathcal{L}_A(x, \lambda; \sigma) = J(x) + \lambda^\top (Bx - c) + \frac{\sigma}{2} \|Bx - c\|^2 \quad (5.3)$$

We now try to solve the problem:
Instead of the general quadratic program (5.1), rewriting (5.4) by using (5.3) gives

\[
\min_x \frac{1}{2} x^\top \tilde{Q} x + \tilde{d}^\top x + e
\]
subject to \( b \leq x \leq a \)

where

\[
\tilde{Q} = Q + \sigma B^\top B,
\]
\[
\tilde{d} = d + B^\top (\lambda + \sigma c),
\]
\[
e = \frac{\sigma}{2} c^\top c.
\]

Next we introduce an algorithm that fixes \( \lambda \) at the current estimate \( \lambda_k \) at its \( k \)th iteration, fixes the penalty parameter \( \sigma \) to some well-chosen value, and performs minimization with respect to \( x \), of course considering the simple bound constraints for \( x \). Using \( x_k \) to denote the approximate minimizer of \( \mathcal{L}_A(x, \lambda_k; \sigma) + (s_k + t_k)x \), we have by the first order optimality conditions that

\[
0 \approx Q x_k + d + B^\top (\lambda_k - \sigma (B^\top x_k - c)) + s_k + t_k
\]

Comparing this with the first order optimality condition for the general QP (5.1), given by (5.2a), we get

\[
\lambda^* \approx \lambda_k - \sigma (B x_k - c).
\]

Therefore we update \( \lambda \) by the rule
\[ \lambda_{k+1} = \lambda_k - \sigma (Bx_k - c) \]  \hspace{1cm} (5.7)

This first order updating rule is, for example, given by Nocedal and Wright [40] in formula (17.39) or by Bertsekas [5]. But they deduced it without considering bound constraints.

We will show later on (in Theorem (2)) that (under some conditions) we can solve problem (5.1) by iteratively solving problem (5.4) and updating \( \lambda \).

## 5.2 Inner Algorithm For Minimizing The Augmented Lagrange Function

To solve problem (5.4) we use an infeasible active set method. This method was already successfully applied to constrained optimal control problems (see Bergounioux et al. [3, 4]) and to convex quadratic problems with simple bound constraints (see [30]).

First we take a look at the KKT system for problem (5.4):

\[
\begin{align*}
\tilde{Q}x + \tilde{d} + s + t & = 0 \quad \text{(5.8a)} \\
\circ(x - b) & = 0 \quad \text{(5.8b)} \\
\circ(a - x) & = 0 \quad \text{(5.8c)} \\
x - b & \geq 0 \quad \text{(5.8d)} \\
a - x & \geq 0 \quad \text{(5.8e)} \\
s & \leq 0 \quad \text{(5.8f)} \\
t & \geq 0 \quad \text{(5.8g)}
\end{align*}
\]

The crucial step in solving (5.4) is to identify those inequalities which are active on the lower bound and those which are active on the upper bound, i.e. the sets \( A_1 \subseteq N \) and \( A_2 \subseteq N \ (A_1 \cap A_2 = \emptyset) \), where the solution to (5.4) satisfies \( x_{A_1} = b_{A_1} \) and \( x_{A_2} = a_{A_2} \). Then, with \( I := N \setminus (A_1 \cup A_2) \), we must have \( s_I = 0, t_I = 0, s_{A_2} = 0 \) and \( t_{A_1} = 0 \).

To compute the remaining elements \( x_I, s_{A_1} \) and \( t_{A_2} \) of \( x, s \) and \( t \), we use (5.8a) and partition the equations and variables according to \( A_1, A_2 \) and \( I \):
5 A Lagrangian Infeasible Active-Set Method

\[
\begin{pmatrix}
\tilde{Q}_{A_1} & \tilde{Q}_{A_1,A_2} & \tilde{Q}_{A_1,I} \\
\tilde{Q}_{A_2,A_1} & \tilde{Q}_{A_2} & \tilde{Q}_{A_2,I} \\
\tilde{Q}_{I,A_1} & \tilde{Q}_{I,A_2} & \tilde{Q}_I
\end{pmatrix}
\begin{pmatrix}
x_{A_1} \\
x_{A_2} \\
x_I
\end{pmatrix}
+ \begin{pmatrix}
\tilde{d}_{A_1} \\
\tilde{d}_{A_2} \\
\tilde{d}_I
\end{pmatrix}
+ \begin{pmatrix}
s_{A_1} \\
s_{A_2} \\
s_I
\end{pmatrix}
+ \begin{pmatrix}
t_{A_1} \\
t_{A_2} \\
t_I
\end{pmatrix}
= 0 \quad (5.9)
\]

The third set of equations can be solved for \(x_I\), because \(\tilde{Q}_I\) is by assumption positive definite:

\[
x_I = -\tilde{Q}_I^{-1}(\tilde{d}_I + \tilde{Q}_{I,A_1}b_{A_1} + \tilde{Q}_{I,A_2}a_{A_2}). \quad (5.10)
\]

Substituting this into the first and second set of equations implies

\[
s_{A_1} = -\tilde{d}_{A_1} - \tilde{Q}_{A_1,N}x \quad (5.11)
\]
\[
t_{A_2} = -\tilde{d}_{A_2} - \tilde{Q}_{A_2,N}x \quad (5.12)
\]

If our guesses for \(A_1\) and \(A_2\) would have been correct, then \(x_I \geq b_I, s_{A_1} \leq 0\) and \(t_{A_2} \geq 0\) would have to hold. Suppose this is not the case. Then we need to make a new 'guess' for \(A_1\) and \(A_2\), which we denote by \(A_1^+\) and \(A_2^+\). Let us first look at \(s_{A_1}\). If \(s_i < 0\), this confirms our previous guess \(i \in A_1\), so we include \(i\) also in \(A_1^+\). Consider now \(t_{A_2}\). If \(t_i > 0\), this confirms our previous guess \(i \in A_2\), so we include \(i\) also in \(A_2^+\). Let us finally look at \(x_I\). If \(x_i < b_i\) we set \(x_i = b_i\) in the next iteration and hence we include \(i\) in \(A_1^+\). On the other hand if \(x_i > a_i\) we set \(x_i = a_i\) in the next iteration and therefore we include \(i\) in \(A_2^+\). Formally we arrive at

\[
A_1^+ := \{i : x_i < b_i \text{ or } s_i < 0\} \quad (5.13a)
\]
\[
A_2^+ := \{i : x_i > a_i \text{ or } t_i > 0\}. \quad (5.13b)
\]

So in each step of this iterative approach, we maintain the first order optimality condition and the complementary constraints associated to problem \((5.4)\), given by \((5.8a), (5.8b)\) and \((5.8c)\). As initial active sets we take the empty sets in the first iteration and the 'best' (in terms of norm minimization of the equality constraints) active sets, we have found so far, for all consecutive iterations. The iterates of the algorithm are well defined, because in each step we get a unique solution for all \(A_1 \subseteq N\) and \(A_2 \subseteq N\), due to \(\tilde{Q} \succ 0\).
5.3 Inner Algorithm For Solving The Problem Directly

After solving the quadratic program (5.4) for the actual $\lambda_k$, we try to solve our general quadratic program with equality constraints and simple bound constraints, given by (5.1), directly by making use of the active sets that belong to the optimal value of (5.4) as initial active sets.

We again use the infeasible active set method described above. Solving the system consisting of the first order optimality condition, given by (5.2a), and the equality constraint, given by (5.2b), under the additional constraints that $x_{A_1} = b_{A_1}$, $x_{A_2} = a_{A_2}$, $s_I = 0$, $t_I = 0$, $s_{A_2} = 0$, $t_{A_1} = 0$ leads to

$$
\begin{pmatrix}
Q_I & B_{I,M}^T \\
Q_{A_1,I} & B_{A_1,M}^T \\
Q_{A_2,I} & B_{A_2,M}^T \\
B_{M,I} & 0
\end{pmatrix}
\begin{pmatrix}
x_I \\
\lambda
\end{pmatrix} =
\begin{pmatrix}
-d_I - Q_{I,A_1} b_{A_1} - Q_{I,A_2} a_{A_2} \\
-d_{A_1} - s_{A_1} - Q_{A_1,A_1} b_{A_1} - Q_{A_1,A_2} a_{A_2} \\
-d_{A_2} - t_{A_2} - Q_{A_2,A_1} b_{A_1} - Q_{A_2,A_2} a_{A_2} \\
c - B_{M,A_1} b_{A_1} - B_{M,A_2} a_{A_2}
\end{pmatrix}.
$$

(5.14)

Making $x_I$ explicit in the first set of equations of (5.14)

$$
x_I = Q_I^{-1}(-B_{I,M}^T \lambda - d_I - Q_{I,A_1} b_{A_1} - Q_{I,A_2} a_{A_2})
$$

(5.15)

and using this in the fourth set of equations of (5.14) gives

$$
B_{M,I} Q_I^{-1} B_{I,M} \lambda = B_{M,A_1} b_{A_1} + B_{M,A_2} a_{A_2} - 
\begin{pmatrix}
c - B_{M,I}(Q_I^{-1}(d_I + Q_{I,A_1} b_{A_1} + Q_{I,A_2} a_{A_2}))
\end{pmatrix}.
$$

(5.16)

If $B_{M,I} Q_I^{-1} B_{I,M}$ is invertible, (5.16) can be solved for $\lambda$:

$$
\lambda = (B_{M,I} Q_I^{-1} B_{I,M})^{-1}(B_{M,A_1} b_{A_1} + B_{M,A_2} a_{A_2} - 
\begin{pmatrix}
c - B_{M,I}(Q_I^{-1}(d_I + Q_{I,A_1} b_{A_1} + Q_{I,A_2} a_{A_2}))
\end{pmatrix}.
$$
By using $\lambda$ in (5.15), we can calculate $x_I$. Finally making use of $\lambda$ and $x_I$ in the second and third set of equations of (5.14) yields $s_{A_1}$ and $t_{A_2}$:

$$s_{A_1} = -d_{A_1} - Q_{A_1}b_{A_1} - Q_{A_1}A_2a_{A_2} - Q_{A_1}x_J - B_{A_1,M}^\top\lambda \quad (5.17)$$
$$t_{A_2} = -d_{A_2} - Q_{A_2}a_{A_2} - Q_{A_2}A_1b_{A_1} - Q_{A_2}x_J - B_{A_2,M}^\top\lambda \quad (5.18)$$

If our guess for $A_1$ and $A_2$ would have been correct, then $b_I \leq x_I \leq a_I$, $s_{A_1} \leq 0$ and $t_{A_2} \geq 0$ would have to hold. If this is not the case we arrive at a new active sets $A_1^+$ and $A_2^+$, formally defined by

$$A_1^+ := \{ i : x_i < b_i \text{ or } s_i < 0 \} \quad (5.19)$$
$$A_2^+ := \{ i : x_i > a_i \text{ or } t_i > 0 \}. \quad (5.20)$$

If we cannot go on with the direct approach, because $B_{M,I}Q_I^{-1}B_{I,M}^\top$ is not invertible for our current active sets $A_1$ and $A_2$ or because we have reached a maximum number of iterations $K < \infty$ we start a new outer iteration by calculating a new augmented Lagrange function.

Computational experience with our method indicates that typically only few (most of the time only one) outer iterations (multiplier-updates) and also only few (most of the time less than ten) inner iterations (minimization of the Lagrange function and trying to solve (5.1) directly) are required to reach the optimal solution.

To investigate the convergence behaviour of the algorithm we look at convergence results for the augmented Lagrangian method and we examine the convergence of our inner algorithms.

### 5.4 Convergence Analysis Of The Augmented Lagrangian Method

In this section we give a convergence result for the augmented Lagrangian method and then take a closer look on an assumption made in Theorem 2.

The following result, given by Bertsekas [5] [6], gives conditions under which there is a minimizer of $\mathcal{L}_\lambda(x, \lambda; \sigma)$ that lies close to $x^*$ and gives error bounds both for $x_k$ and the updated multiplier estimate $\lambda^{k+1}$ obtained from solving the subproblem at iteration $k$. 

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**Theorem 2** Let \( x^* \) be a strict local minimizer and a regular point of \((P)\). Furthermore let \( \sigma \) be a positive scalar such that \( \nabla^2_{xx} \mathcal{L}_A(x^*, \lambda^*; \sigma) \succ 0 \). Then there exist positive scalars \( \delta, \varepsilon, \) and \( \kappa \) such that:

(a) For all \((\lambda_k, \sigma)\) in the set \( D \subset \mathbb{R}^{m+1} \) defined by

\[
D = \{ (\lambda_k, \sigma) \mid \| \lambda_k - \lambda^* \| < \delta \sigma, \sigma \leq \sigma \},
\]

the problem

\[
\min_x \mathcal{L}_A(x, \lambda_k; \sigma) \text{ subject to } b \leq x \leq a, \| x - x^* \| \leq \varepsilon
\]

has a unique solution \( x_k \). Moreover, we have

\[
\| x_k - x^* \| \leq \kappa \| \lambda_k - \lambda^* \| / \sigma.
\]

(b) For all \((\lambda_k, \sigma) \in D\), we have

\[
\| \lambda_{k+1} - \lambda^* \| \leq \kappa \| \lambda_k - \lambda^* \| / \sigma
\]

where \( \lambda_{k+1} \) is given by (5.7).

(c) For all \((\lambda_k, \sigma) \in D\), the matrix \( \nabla^2_{xx} \mathcal{L}_A(x_k, \lambda_k; \sigma) \succ 0 \).

For a proof of the above Theorem see Bertsekas [5, 6].

Now we examine the positive definiteness assumption of the above theorem closer.

**Theorem 3** Let \( x^* \) be a strict local minimizer and a regular point of \((P)\). Then

\[
\nabla^2_{xx} \mathcal{L}_A(x^*, \lambda^*; \sigma) \succ 0 \iff \sigma > \max \{ -e_1, \ldots, -e_m \}
\]

where \( e_1, \ldots, e_m \) are the eigenvalues of \( \nabla h(x^*) \left[ \nabla^2_{xx} \mathcal{L}_A(x^*, \lambda^*; 0) \right]^{-1} h(x^*) \).
For a proof of the above Theorem see again Bertsekas [5].

Finally we adapt the above result to our problem structure. For problem (5.4)

\[
\{\nabla h(x^\star)\left[\nabla^2_{xx} \mathcal{L}_A(x^\star, \lambda^\star; 0)\right]^{-1} h(x^\star)\}^{-1} = (AQ^{-1}A^\prime)^{-1}.
\]

Therefore the assumption

\[
\nabla^2_{xx} \mathcal{L}_A(x^\star, \lambda^\star; \sigma) \succ 0
\]

in Theorem 2 is fulfilled for all \(\sigma > 0\).

5.5 Convergence Analysis Of The Kunisch-Rendl Method

In this section we generalize the proof idea used for the finite step convergence result for the Kunisch-Rendl method with only upper bounds (for a proof see Kunisch and Rendl [30]) to the case where we have lower and upper bounds. The main aim of this section is to argue why the proof idea does not work in this more general case any more.

5.5.1 Index partition

To investigate the behaviour of the algorithm, we look at two consecutive iterations. Suppose that some iteration is carried out with the active sets \(A^k_1 \subseteq N\) and \(A^k_2 \subseteq N\) \((A^k_1 \cap A^k_2 = \emptyset)\) (for \(k \geq 1\)), yielding \((x^k_s, s^k_t, t^k_i)\) as solution of the KKT system (5.8) for the current active sets. According to (5.13), the new active sets are

\[
A^{k+1}_1 := \{i : x^k_i < b_i \text{ or } s^k_i < 0\}
\]
\[
A^{k+1}_2 := \{i : x^k_i > a_i \text{ or } t^k_i > 0\}.
\]

Let \((x^{k+1}_s, s^{k+1}_t, t^{k+1}_i)\) denote the solution of the KKT system (5.8) for the active sets \(A^{k+1}_1\) and \(A^{k+1}_2\). To avoid too many superscripts, we write
Given $A$ and $B$, we have the set of inactive variables $I = N \setminus (A \cup B)$ and we find that $x, s, t, C, D, u, v$ are determined by

$$x_A = b_A, \quad x_B = a_B, \quad s_I = s_B = 0, \quad t_I = t_A = 0, \quad \tilde{Q}x + \tilde{d} + s + t = 0$$

$$C = \{i : x_i < b_i \text{ or } s_i < 0\} \quad \text{and} \quad D := \{i : x_i > a_i \text{ or } t_i > 0\}$$

$$J = N \setminus (C \cup D), \quad y_C = b_C, \quad y_D = a_D, \quad u_J = u_D = 0, \quad v_J = v_C = 0, \quad \tilde{Q}y + \tilde{d} + u + v = 0$$

The following partition of $N$ into mutually disjoint subsets will be useful in our analysis. We first partition $A$ into

$$S := \{i \in A : s_i \geq 0\} \quad (5.26)$$

and $A \setminus S$ and $B$ into

$$T := \{i \in B : t_i \leq 0\} \quad (5.27)$$

and $B \setminus T$. The set $I$ is partitioned into

$$U := \{i \in I : x_i < b_i\} \quad (5.28)$$

$$V := \{i \in I : x_i > a_i\} \quad (5.29)$$

and $I \setminus (U \cup V)$. In Table 5.2 we summarize the relevant information about $x, s, t, y, u, v$ for this partition. A nonspecified entry indicates that the domain of the associated variable cannot be constrained.

On the basis of the above table, we define the sets $K$ and $L$ that give the indices of lower and upper infeasibility of $y$: 

\[ (A, B, x, s, t) \text{ for } (A^k_A, x^k_A, s^k_A, t^k_A) \text{ and } (C, D, y, u, v) \text{ for } (A^k_{A_1}, A^k_{A_2}, x^k_A, s^k_A, t^k_A) \]
### 5.5.2 The merit function

Let us define our merit function as

\[ L_c, d(x, s, t) = \tilde{f}(x) + \frac{c}{2} \|g(x)\|^2 + \frac{d}{2} \|h(x)\|^2 \]  

(5.32)

where
\[ \tilde{J}(x) = x^\top \tilde{Q}x + \tilde{d}^\top x \] (5.33)
\[ g(x) = \max(b - x, 0) \] (5.34)
\[ h(x) = \max(x - a, 0) \] (5.35)

In the remainder of this section we shall investigate the value of (5.32) along the iterates of the algorithm:

\[ L_{c,d}(y,u,v) - L_{c,d}(x,s,t). \] (5.36)

First, we consider the changes of the objective function during consecutive iterations. One cannot expect a monotone decrease of \( \tilde{J}(x) \) as the iterates may be infeasible.

**Lemma 1** Let \((x,s,t), (y,u,v), U \text{ and } V\) be given as above and \(W := (U \cup V)\). Then, we have

\[ \tilde{J}(y) - \tilde{J}(x) = \frac{1}{2} (y - x)^\top \begin{pmatrix} \tilde{Q}_W & 0 \\ 0 & -\tilde{Q}_W \end{pmatrix} (y - x). \] (5.37)

**Proof.** We use the \(Q\)-inner product, \( \langle a, b \rangle_Q := a^\top Qb \), with the associated norm \( \|a\|_Q^2 := \langle a, a \rangle_Q \) and get

\[ \tilde{J}(y) - \tilde{J}(x) = \frac{1}{2} \|y\|_Q^2 - \frac{1}{2} \|x\|_Q^2 + z^\top \tilde{d}, \] (5.38)

where \( z = y - x \). Using the following identity

\[ \|a\|_Q^2 - \|b\|_Q^2 = 2 \langle a - b, a \rangle_Q - \|a - b\|_Q^2, \] (5.39)

on the right hand side of (5.38) we obtain

\[ \tilde{J}(y) - \tilde{J}(x) = -\frac{1}{2} z^\top \tilde{Q}z + z^\top (\tilde{Q}y + \tilde{d}) \] (5.40)
Considering that \( \tilde{Q}y + \tilde{d} = -u - v \), we get

\[
J(y) - J(x) = -\frac{1}{2} z^T \tilde{Q}z - z^T (u + v). \tag{5.41}
\]

Now \( u_i = v_i = 0 \) for \( i \in S \cup T \cup (I \setminus W) \) and \( z_i = 0 \) on \( (A \setminus S) \cup (B \setminus T) \). Therefore \( z^T (u + v) = \sum_{i \in W} z_i (u_i + v_i) \). Furthermore \( u + v - s - t = -\tilde{Q}z \) and \( u_i + v_i - s_i - t_i = u_i + v_i \) for \( i \in W \), and hence

\[
- \sum_{i \in W} z_i (u_i + v_i) = \sum_{i \in W} z_i (\tilde{Q}z)_i = z^T \left( \begin{array}{cc}
\tilde{Q}_{W,N} & 0 \\
0 & 0
\end{array} \right) z. \tag{5.42}
\]

Summarizing, we see that

\[
\bar{J}(y) - \bar{J}(x) = -\frac{1}{2} z^T \left( \begin{array}{cc}
\tilde{Q}_W & \tilde{Q}_{W,W} \\
\tilde{Q}_{W,W} & \tilde{Q}_W
\end{array} \right) z + z^T \left( \begin{array}{cc}
\tilde{Q}_W & \frac{1}{2} \tilde{Q}_{W,W} \\
\frac{1}{2} \tilde{Q}_{W,W} & 0
\end{array} \right) z. \tag{5.43}
\]

Lemma 2 Let \((x,s,t), (y,u,v), U, V, K \) and \( L \) be given as above. Then we have

\[
\|g(y)\|^2 - \|g(x)\|^2 = \sum_{i \in K} |y_i - b_i|^2 - \sum_{i \in U} |x_i - b_i|^2 \tag{5.43}
\]

as well as

\[
\|h(y)\|^2 - \|h(x)\|^2 = \sum_{i \in L} |y_i - b_i|^2 - \sum_{i \in W} |x_i - b_i|^2 \tag{5.44}
\]

Proof. The claim follows from the fact that \( x \) is infeasible on the lower bound precisely on \( U \) and on the upper bound precisely on \( V \) (see Table 5.2). Moreover, by the definition of the sets \( K \) and \( L \) (see (5.30) and (5.31)), the variable \( y \) is infeasible on the lower bound on \( K \) and on the upper bound on \( L \). \( \square \)

In summary we have proved the following result.
Proposition 1  For every two consecutive triples \((x,s,t)\) and \((y,u,v)\) we have

\[
L_{c,d}(y,u,v) - L_{c,d}(x,s,t) = \frac{1}{2}(y-x)^\top \begin{pmatrix} \tilde{Q}_W & 0 \\ 0 & -\tilde{Q}_W \end{pmatrix} (y-x) + \frac{c}{2} \sum_{i \in K} |y_i - b_i|^2 + \frac{d}{2} \sum_{i \in L} |y_i - b_i|^2 - \frac{c}{2} \sum_{i \in U} |x_i - b_i|^2 \right) - \frac{d}{2} \sum_{i \in V} |x_i - b_i|^2.
\]

(5.45)

Let us introduce \(\mu := \lambda_{\min}(\tilde{Q}) > 0\) as the smallest eigenvalue of \(\tilde{Q}\) and then formulate Proposition 2.

Proposition 2  For every two consecutive triples \((x,s,t)\) and \((y,u,v)\) we have

\[
2(L_{c,d}(y,u,v) - L_{c,d}(x,s,t)) = \|\tilde{Q}\| \|z_W\|^2 - \mu \|z_W\|^2 + c \|z_K\|^2 + d \|z_L\|^2 - c \|z_U\|^2 - d \|z_V\|^2
\]

(5.46)

Proof. We first note that for \(i \in K\) we have \(x_i \geq b_i\) and for \(i \in L\) we have \(x_i \leq a_i\). Hence \(0 < b_i - y_i \leq x_i - y_i\) for \(i \in K\) and \(0 < y_i - a_i \leq y_i - x_i\) for \(i \in L\), and therefore

\[
\sum_{i \in K} (y_i - b_i)^2 \leq \|z_K\|^2
\]

\[
\sum_{i \in K} (y_i - a_i)^2 \leq \|z_L\|^2.
\]

Furthermore we have \(y_U = b_U\) and \(y_V = a_V\), and hence

\[
\sum_{i \in U} (x_i - b_i)^2 = \|z_U\|^2
\]

\[
\sum_{i \in V} (x_i - a_i)^2 = \|z_V\|^2.
\]

Using Proposition 1 we get
\[ 2(L_{c,d}(y,u,v) - L_{c,d}(x,s,t)) = \|\tilde{Q}\|\|z_W\|^2 - \mu\|z_W\|^2 + \\
   c\|z_K\|^2 + d\|z_L\|^2 - c\|z_U\|^2 - d\|z_V\|^2 \]

5.5.3 The need to bound \(\|z_K\|\) and \(\|z_L\|\)

The next goal should be to bound \(\|z_K\|\) and \(\|z_L\|\) in terms of \(\|z\|\). On \(K_1\) and \(L_2\) we have

\[ s_{K_1} \geq 0, s_{L_2} \geq 0, t_{K_1} = t_{L_2} = 0, u_{K_1} = u_{L_2} = 0, v_{K_1} = v_{L_2} = 0, \]

and therefore

\[ (\tilde{Q}z)_{K_1} = s_{K_1} \geq 0 \quad \text{and} \quad (\tilde{Q}z)_{L_2} = s_{L_2} \geq 0. \]

On \(L_1\) and \(K_2\) we have

\[ s_{L_1} = s_{K_2} = 0, t_{L_1} \leq 0, t_{K_2} \leq 0, u_{L_1} = u_{K_2} = 0, v_{L_1} = v_{K_2} = 0, \]

and therefore

\[ (\tilde{Q}z)_{L_1} = t_{L_1} \leq 0 \quad \text{and} \quad (\tilde{Q}z)_{K_2} = t_{K_2} \leq 0. \]

On \(K_3\) and \(L_3\) we have

\[ s_{K_3} = s_{L_3} = 0, t_{K_3} = t_{L_3} = 0, u_{K_3} = u_{L_3} = 0, v_{K_3} = v_{L_3} = 0, \]

and thus
\[(\tilde{Q}z)_K = (\tilde{Q}z)_L = 0.\]

It follows that
\[
(\tilde{Q}z)_K = \tilde{Q}_K z_K + \tilde{Q}_{K,K}\tilde{z}_K = \begin{pmatrix} s_{K1} \\ t_{K2} \\ 0 \end{pmatrix}
\]
\[
(\tilde{Q}z)_L = \tilde{Q}_L z_L + \tilde{Q}_{L,L}\tilde{z}_L = \begin{pmatrix} t_{L1} \\ s_{L2} \\ 0 \end{pmatrix}.
\]

Taking the inner product with \(z_K\) and \(z_L\) respectively, we obtain
\[
z_K^T(\tilde{Q}z)_K = z_K^T\tilde{Q}_K z_K + z_K^T\tilde{Q}_{K,K}\tilde{z}_K = s_{K1}z_{K1} + t_{K2}z_{K2}
\]
\[
z_L^T(\tilde{Q}z)_L = z_L^T\tilde{Q}_L z_L + z_L^T\tilde{Q}_{L,L}\tilde{z}_L = t_{L1}z_{L1} + s_{L2}z_{L2}
\]

where
\[
s_{K1}z_{K1} \leq 0 \text{ but } t_{K2}z_{K2} \geq 0 \text{ and }
\]
\[
t_{L1}z_{L1} \leq 0 \text{ but } s_{L2}z_{L2} \geq 0.
\]

Thus we cannot derive the equations
\[
z_K^T\tilde{Q}_K z_K \leq z_K^T\tilde{Q}_{K,K}\tilde{z}_K \text{ or } z_K^T(\tilde{Q}z)_K \leq 0 \text{ and } \quad (5.47a)
\]
\[
z_L^T\tilde{Q}_L z_L \leq z_L^T\tilde{Q}_{L,L}\tilde{z}_L \text{ or } z_L^T(\tilde{Q}z)_L \leq 0 \quad (5.47b)
\]
that we would need to bound \(\|z_K\|\) and \(\|z_L\|\) in terms of \(\|z\|\). If we could derive the above equations \((5.47)\), the rest of the proof would be very similar to the one for only upper bounds in the paper of Kunisch and Rendl \([30]\). We would have to set \(c := d := \|\tilde{Q}\| + \mu\) and define the conditions \((C1)\) and \((C2)\) slightly differently as
5 A Lagrangian Infeasible Active-Set Method

condition (C1) \[ 2 \cdot \text{cond}(\tilde{Q}) < (\frac{\mu}{\nu})^2 - 2 \]

condition (C2) \[ 2 \cdot \text{cond}(\tilde{Q}) < (\frac{q}{r})^2 - 2, \]

where the diagonal matrix \( D := \text{diag}(q_{11}, \ldots, q_{nn}) \) is consisting of the main diagonal elements of \( \tilde{Q} \) and

\[
\begin{align*}
\nu &:= \max \{ \| \tilde{Q}_{A \pi} \| : A \subset N, A \neq 0, A \neq N \}, \\
q &:= \min \{ q_{ii} : i \in N \}, \\
r &:= \| Q - D \| , \\
\text{cond}(\tilde{Q}) &:= \frac{\lambda_{\max}(\tilde{Q})}{\lambda_{\min}(\tilde{Q})}.
\end{align*}
\]

Although we cannot prove the convergence of the Lagrangian infeasible active-set method presented in Chapter 5, the method converges very fast in practice, as we will see in the following section.

5.6 Computational Experience

In this we look at the practical behaviour of our algorithm by considering a variety of test problems.

The only nontrivial inputs to our algorithm are the initial active sets \( A_1 \) and \( A_2 \), the initial Lagrange multiplier \( \lambda_0 \) and the penalty parameter \( \sigma \). Our algorithm is quite insensitive to their choice.\(^{12}\)

5.6.1 Randomly Generated Dense Problems

At first we study in some detail randomly generated problems, where \( Q \) and \( B \) are dense matrices. We vary the number of variables \( n \) and the number of equality constraints \( m \). In order for the reader to be able to reproduce some of the following results, we provide the MATLAB commands that we used to generate the data \( Q \), \( B \), \( d \), \( a \), \( b \) and \( c \).

\(^{12}\) \( A_2 \) and \( A_2 \) are chosen as empty sets, \( \lambda_0 \) as zero vector and \( \sigma \) equal to 10000
n = 500; (or n = 1000 or ... or n = 15000)
m = 5; (in general: m = n/100; m = n/10; m = n/2)
rand('seed',n+m)
x = rand(n,1);
B = rand(m,n);
c = B*x;
d = rand(n,1);
Z = rand(n)-0.5;
Q = Z'*Z + eye(n);
b = zeros(n,1);
a = ones(n,1);

In Table 5.3 below we summarize the key performance features of our algorithm for different problem sizes:

- The number of $\lambda$-Updates, which is equal with the number of outer iterations,
- the number of iterations we run to solve the inner problem (5.4),
- the number of iterations we carry out to try to solve the underlying problem (5.1) directly,
- the time that is needed on a workstation with 3 GHz and 10 GB RAM get the optimal solution.

| n       | m       | $\lambda$-updates | Iter. inner problem | Iter. underlying problem | seconds |
|---------|---------|--------------------|---------------------|--------------------------|---------|
| 500     | 50      | 1(0)               | 7(0.67)             | 1(0)                     | 0.09(0.01) |
|         | 250     | 1(0)               | 7.9(0.74)           | 1(0)                     | 0.2(0.02)  |
| 1000    | 100     | 1(0)               | 7.9(0.57)           | 1.1(0.32)                | 0.5(0.06)  |
|         | 500     | 1(0)               | 8.7(0.82)           | 1(0)                     | 1.0(0.04)  |
| 3000    | 300     | 1(0)               | 9.0(0.47)           | 1.1(0.32)                | 9.7(1.15)  |
|         | 1500    | 1(0)               | 9.5(0.71)           | 1.6(0.52)                | 23.9(4.4)  |
| 5000    | 500     | 1                  | 9                   | 1                        | 37.2     |
|         | 2500    | 1                  | 11                  | 2                        | 118      |
| 10000   | 1000    | 1                  | 10                  | 1                        | 282      |
|         | 5000    | 1                  | 10                  | 2                        | 864      |
| 15000   | 1500    | 1                  | 10                  | 1                        | 963      |
|         | 7500    | 1                  | 10                  | 2                        | 2896     |

Table 5.3: Dense data: Key performance features of our algorithm for different problem sizes and structures
5.6.2 Randomly Generated Sparse Problems

Next we study randomly generated problems, where $Q$ and $B$ are sparse matrices with 10 nonzero entries per row in average. We vary the number of variables $n$ and the number of equality constraints $m$. In order for the reader to be able to reproduce some of the following results, we provide the MATLAB commands that we used to generate the data $Q$, $A$, $d$, $b$ and $c$, where by $nz$ we denote the average number of nonzero entries per row in $Q$ and $A$.

```matlab
» n = 500; (or n = 1000 or ... or n = 20000)
» m = 5; (in general: m = n/100; m = n/10; m = n/2)
» rand('seed',n+m)
» x = rand(n,1);
» B = rand(m,n);
» for i = 1:m;
    y = rand(n,1);
    for j = 1:n;
        if y(j) > nz/n;
            B(i,j) = 0;
        end;
    end;
end;
» c = B*x;
» d = rand(n,1);
» Z = sprand(n,n,0.1);
» Q = Z'*Z + eye(n);
» for i = 1:n;
    y = rand(n,1);
end;
```

The low standard deviations justify this action.

Until $n=3000$ we perform 10 runs for different random data and give the expectation value and in parenthesis the standard deviation. For larger problems we only make one run in order to save time.\(^{13}\)

We can see that the algorithm always needs only one outer iteration. It takes no more than 11 iterations to solve the inner problem (5.4) and no more than 2 further iterations to finally get the exact numerical solution for the underlying problem (5.1). If we compare these results with the ones we obtained using the same algorithm on the same problem data but without considering upper bounds, we recognize that we need more iterations to solve the inner problem (5.4), but less time, as the systems of equations we have to solve are smaller (for further details see \([26, Section 4.1]\)).
for j = i:n;
    if \( y(j) > \frac{nz}{n} \);
        Q(i,j) = 0;
        Q(j,i) = 0;
    end;
end;

» Q = Q + (\text{abs}(\min(\text{eig}(Q))) + 1) \ast \text{eye}(n);
» b = \text{zeros}(n,1);
» a = \text{ones}(n,1);

In Table 5.4 below we summarize the key performance features of our algorithm for different problem sizes.

| n    | m    | \( \lambda \)-updates | Iter. inner problem | Iter. underlying problem | seconds |
|------|------|-------------------------|----------------------|-------------------------|---------|
| 500  | 50   | 1(0)                    | 4.3(0.48)            | 1(0)                    | 0.06(0) |
|      | 250  | 1(0)                    | 5.6(0.70)            | 1.7(0.48)               | 0.2(0.04) |
| 1000 | 100  | 1(0)                    | 4.8(0.42)            | 1.3(0.48)               | 0.3(0.04) |
|      | 500  | 1(0)                    | 6.2(0.79)            | 2.1(0.57)               | 1.3(0.22) |
|      | 300  | 1(0)                    | 5.7(0.67)            | 1.9(0.32)               | 6.3(0.55) |
|      | 1500 | 3.1(6.64)               | 9.6(8.22)            | 2.9(0.99)               | 32.6(12.8) |
| 5000 | 500  | 1                       | 6                    | 2                       | 25.1    |
|      | 2500 | 1                       | 8                    | 3                       | 135     |
| 10000| 1000 | 1                       | 6                    | 2                       | 186     |
|      | 5000 | 1                       | 6                    | 4                       | 1235    |
| 15000| 1500 | 1                       | 6                    | 3                       | 797     |
|      | 7500 | 1                       | 8                    | 4                       | 4208    |

Table 5.4: Sparse data: Key performance features of our algorithm for different problem sizes and structures

Until \( n=3000 \) we perform 10 runs for different random data and give the expectation value and in parenthesis the standard deviation. For larger problems we only make one run in order to save time.\(^{14}\)

We can see that the algorithm most of the time needs only one outer iteration. It takes no more than 8 iterations to solve the inner problem (5.4) and no more than 4 further iterations to finally get the exact numerical solution for the underlying

\(^{14}\) The low standard deviations justify this action.
problem (5.1). If we compare these results with the ones we obtained using the same algorithm on the same problem data but without considering upper bounds, we recognize that we need more iterations to solve the inner problem (5.4), but less time, as the systems of equations we have to solve are smaller (for further details see [26, Section 4.2]).
6 Discussion

The main interest of this diploma thesis was to describe and compare different, practically successful solution methods for general convex quadratic problems with arbitrary linear constraints.

Therefore we showed the equivalence of different QP problem formulations and presented some important so-called direct methods for solving equality-constrained QPs in Chapter 2. After this, we covered the most important aspects for practically successful interior point and active-set methods for convex quadratic programming in Chapter 3 and Chapter 4 respectively.

Finally, as the core of the diploma thesis, we presented a combination of the augmented Lagrangian method with an infeasible active set method as a new algorithm for solving QPs efficiently in Chapter 5.

Among the special features of this algorithm are its ability to find the exact numerical solution of the problem and the fact that at each iteration level the size of the linear system that must be solved is determined by the currently inactive set that can be significantly smaller than the total set of variables. As a consequence the proposed algorithm differs significantly from the interior point methods that we described in Chapter 3. Because of its ability to ‘correct’ many active variables to inactive ones and vice versa in each iteration and its computationally cheap definition of the new active sets, the algorithm also seems preferable to the feasible active set methods presented in Chapter 4.

From the numerical experiments in Section 5.6 we observe that the algorithm can mostly find the optimal solution in the first try to solve the system directly. This is certainly one of its distinguishing practical features. Furthermore the total number of iterations is frequently quite insensitive with respect to data and initialization.

The next step of research will be to compare our algorithm with other fast software for solving QPs on different test problems.
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