PARTIAL PRESERVATION OF FREQUENCIES
AND FLOQUET EXPONENTS OF INVARIANT TORI
IN THE REVERSIBLE KAM CONTEXT 2

M. B. Sevryuk

ABSTRACT. We consider the persistence of smooth families of invariant tori in the reversible context 2
of KAM theory under various weak nondegeneracy conditions via Herman’s method. The reversible
KAM context 2 refers to the situation where the dimension of the fixed point manifold of the reversing
involution is less than half the codimension of the invariant torus in question. The nondegeneracy
conditions we employ ensure the preservation of any prescribed subsets of the frequencies of the unper-
turbed tori and of their Floquet exponents (the eigenvalues of the coefficient matrix of the variational
equation along the torus).

To the blessed memory of Helmut Rüssmann
whose contribution to KAM theory is so substantial and versatile

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1. Introduction

1.1. Reversible contexts 1 and 2. Equilibria, periodic orbits, invariant tori filled up with quasi-
periodic motions (conditionally periodic motions with rationally independent frequencies) and their
asymptotic manifolds (in particular, homoclinic and heteroclinic trajectories) are key elements of
finite-dimensional dynamics. The importance of equilibria (invariant 0-tori) and periodic orbits (in-
vARIANT 1-tori) of autonomous flows was realized already by Poincaré and further emphasized from the
bifurcational viewpoint by Andronov and Hopf [26]. Quasi-periodic motions with \( n \geq 2 \) basic frequencies
are the subject of the Kolmogorov–Arnold–Moser (KAM) theory founded in the fifties and sixties
of the last century [1, 8, 10, 15, 17, 18, 20, 31, 35, 59]. According to KAM theory, the occurrence
of invariant tori of various dimensions carrying quasi-periodic motions and organized into Cantor-like
families is a generic property of nonintegrable dynamical systems. The possible dimensions of the
tori and the number of parameters of their Cantor families (as a rule, these families themselves form
complicated hierarchical conglomerates) depend strongly on the phase space structures the system in
question is assumed to preserve.

For instance, a typical autonomous Hamiltonian system with \( N \) degrees of freedom is expected to
admit isolated equilibria, one-parameter smooth families of periodic orbits (the parameter being just

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the energy value), and $n$-parameter Cantor families of isotropic invariant $n$-tori carrying quasi-periodic motions for each $n = 2, \ldots, N$ [1, 8, 10, 35]. The existence of other types of families of quasi-periodic motions filling up isotropic invariant tori is evidence for the presence of additional symmetries of the system. By the way, in a generic one-parameter family of periodic orbits, the period is not a constant and can be used as an alternative parameter.

In KAM theory, one considers various classes of dynamical systems, and the invariant tori sought for can relate to the corresponding phase space structures in different ways; thus, one sometimes speaks of particular contexts of KAM theory. The most explored finite-dimensional contexts are the dissipative context (with no special structures on the phase space), the volume preserving context (where one looks for invariant tori of volume preserving systems), the Hamiltonian isotropic context (where one examines isotropic invariant tori in Hamiltonian systems), and the so-called reversible context 1, see [5–9, 33, 39, 40, 43–45, 55]. Less familiar contexts are exemplified by the Hamiltonian coisotropic context (with coisotropic invariant tori) and the Hamiltonian atropic context (where the invariant tori to be constructed are atropic, i.e., neither isotropic nor coisotropic); see [10] for references on the Hamiltonian coisotropic and atropic contexts, as well as by the so-called conformally Hamiltonian context; see [11] and references therein. One more example is the reversible context 2 that the present paper is devoted to. Let us recall the relevant definitions and principal facts.

Definition 1.1 ([23, 34, 37]). Given an arbitrary set $\mathcal{M}$, a mapping $G : \mathcal{M} \to \mathcal{M}$ is called an involution of $\mathcal{M}$ if $G^2 = G \circ G$ is the identity transformation. A dynamical system is said to be reversible with respect to a smooth involution $G$ of the phase space (or $G$-reversible) if this system is invariant under the transformation $(p, t) \mapsto (Gp, -t)$, where $p$ is a point of the phase space and $t$ is the time (i.e., if $G$ casts the system in question into a system with the reverse time direction).

In the reversible KAM theory, one always deals with only those tori that are invariant under both the system itself and the reversing involution.

Lemma 1.1 ([7, 8, 37, 48]). Let an $n$-torus $T \subset M$ be invariant under both a $G$-reversible flow on $M$ and the corresponding reversing involution $G$. If $T$ carries quasi-periodic motions, then one can introduce a coordinate frame $x \in \mathbb{T}^n = (\mathbb{R}/2\pi \mathbb{Z})^n$ in $T$ such that the dynamics on $T$ takes the form $\dot{x} = \omega$ and the restriction of $G$ to $T$ takes the form $G|_T : x \mapsto -x$. Consequently, the set of fixed points of $G|_T$ consists of $2^n$ isolated points $(x_1, \ldots, x_n)$ where each $x_i$, $1 \leq i \leq n$, is equal either to 0 or to $\pi$.

The set Fix$G$ of fixed points of an involution $G : \mathcal{M} \to \mathcal{M}$ of a manifold $\mathcal{M}$ is a submanifold of $\mathcal{M}$ of the same smoothness class as $\mathcal{M}$ itself [2, 28] (the books [2, 13] present extensive information on the fixed point sets of involutions of various manifolds). However, in the framework of Lemma 1.1, different points of Fix$(G|_T) = (\text{Fix} G) \cap T$ may belong to connected components of Fix$G$ of different dimensions (see [33, 39] for several examples in the case $n = 1$). None of these dimensions can exceed the codimension $\text{codim} T$ of $T$ in the phase space because $\text{dim}(T \cap \text{Fix} G) = 0$.

Definition 1.2 ([7, 8]). Let all the connected components of Fix$G$ that intersect $T$ in the framework of Lemma 1.1 be of the same dimension $d_G$. The situation where the inequalities $\frac{1}{2} c_T \leq d_G \leq c_T$ hold (here $c_T = \text{codim} T$) is called the reversible context 1. The opposite situation where the inequality $d_G < \frac{1}{2} c_T$ holds is called the reversible context 2.

Note that for most involutions $G$ encountered in practice, the fixed point manifold Fix$G$ is not empty and all the connected components of Fix$G$ are of the same dimension; thus, $\text{dim Fix} G$ is well defined [23, 34].

The drastic differences between the two reversible contexts and the peculiarities of the reversible context 2 are discussed in detail in our previous articles [47–51]. Here we just demonstrate these differences in the trivial case $n = 0$, where the invariant tori in question are equilibria and their codimension is the phase space dimension. These equilibria should be invariant under the reversing involution $G$, i.e., they should be fixed points of $G$. 

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Example 1.1. Consider the involution $G : (u, v) \mapsto (u, -v)$ of $\mathbb{R}^{a+b}$, where $u \in \mathbb{R}^a$ and $v \in \mathbb{R}^b$, so that $\text{Fix} \, G = \{v = 0\}$ and $\dim \text{Fix} \, G = a$. A system

\[ \dot{u} = U(u, v), \quad \dot{v} = V(u, v) \]

is reversible with respect to $G$ if and only if if $U$ is odd in $v$ and $V$ is even in $v$. We are looking for the equilibria of such a system on the plane $\text{Fix} \, G$, i.e., for the points $u \in \mathbb{R}^a$ such that $U(u, 0) = 0$ and $V(u, 0) = 0$. However, $U(u, 0) = 0$, and the desired equilibria $(u, 0)$ are determined by the equation $V(u, 0) = 0$.

The reversible context 1 here corresponds to the inequality $\frac{1}{2}(a + b) \leq a$, i.e., $a \geq b$. Within this context, the equation $V(u, 0) = 0$ generally describes a smooth $(a - b)$-dimensional surface in $\text{Fix} \, G$. On the other hand, in the reversible context 2 (where $a < b$) one generally has no equilibria lying in $\text{Fix} \, G$. To obtain such equilibria, one has to let the system depend on at least $b - a$ external parameters. For a $G$-reversible system $\dot{u} = U(u, v, w)$, $\dot{v} = V(u, v, w)$ depending on a $c$-dimensional external parameter $w$ with $c \geq b - a$, one generically gets a smooth $(c - b + a)$-dimensional surface of equilibria in the product of the plane $\text{Fix} \, G$ and the parameter space $\mathbb{R}^c \ni w$.

Let $R \in \text{GL}(a + b, \mathbb{R})$ be an involutive matrix with eigenvalue 1 of multiplicity $a$ and eigenvalue $-1$ of multiplicity $b$. One says that a matrix $M \in \mathfrak{gl}(a + b, \mathbb{R})$ anti-commutes with $R$, or is infinitesimally reversible with respect to $R$, if $MR = -RM$. If this is the case, then the eigenvalues of $M$ come in pairs $(\lambda, -\lambda)$, and if $b \neq a$, then 0 is an eigenvalue of $M$ of multiplicity $t \geq |b - a|$ [21, 37, 38, 52] (generically $t = |b - a|$).

Consider the linearization of a $G$-reversible system in the setup of Example 1.1 around any equilibrium lying in $\text{Fix} \, G$. If $b \neq a$ then this linearization possesses the zero eigenvalue of multiplicity $t \geq |b - a|$ (generically $t = |b - a|$). The nonzero eigenvalues come in pairs $(\lambda, -\lambda)$.

1.2. Unperturbed systems in the reversible context 2. It is an appropriate time now to introduce some notation. Let $\mathbb{N}$ be the set of positive integers and let $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. Throughout the paper, we will denote by $|\cdot|$ the $\ell_1$-norm of vectors in $\mathbb{C}^s$, by $\|\cdot\|$ the $\ell_2$-norm of vectors in $\mathbb{R}^s$, and by $\langle \cdot, \cdot \rangle$ the inner product of two vectors in $\mathbb{R}^s$. A closed $s$-dimensional unit ball centered at a point $\mu \in \mathbb{R}^s$ is the set $B = \{p \in \mathbb{R}^s \mid \|p - \mu\| \leq \rho\}$ for a certain $\rho > 0$. For $s = 0$, this definition gives $\mu = 0$ and $B = \{0\} = \mathbb{R}^0$. The expression $O_s(\mu)$ will denote an unspecified neighborhood of a point $\mu \in \mathbb{R}^s$. If $d \in \mathbb{N}$ and $x, y, z, \ldots$ are certain variables, we will write $O_d(x, y, z, \ldots)$ instead of $O(\|x\|^d + |y|^d + |z|^d + \cdots)$. Instead of $O_1(\cdot)$, we will write just $O(\cdot)$.

Given a matrix $M \in \mathfrak{gl}(N, \mathbb{R})$, the expression $0_m \oplus M$ will denote the $(m + N) \times (m + N)$ block diagonal matrix whose first block is the $m \times m$ zero matrix and the second block is $M$. The space of $n \times N$ real matrices will be denoted by $\mathbb{R}^{n \times N}$, so that $\mathfrak{gl}(n, \mathbb{R}) = \mathbb{R}^{n \times n}$.

Recall also that a $C^1$-smooth mapping $F : \mathcal{M} \rightarrow N$ of smooth manifolds is said to be submersive at a point $\mu \in \mathcal{M}$ if $\dim \mathcal{M} \geq \dim N$ and the rank of the differential of $F$ is equal to $\dim N$ at $\mu$. If this is the case, then $F$ is also submersive at any point $\mu' \in \mathcal{M}$ sufficiently close to $\mu$.

Definition 1.3. Let $\mathcal{T}$ be an invariant $n$-torus of some flow on an $(n+N)$-dimensional manifold. This torus is said to be reducible (or Floquet) if in a neighborhood of $\mathcal{T}$, there exists a coordinate frame $x \in \mathbb{T}^n$, $\mathcal{X} \in \mathcal{O}_N(0)$ in which the torus $\mathcal{T}$ itself is given by the equation $\mathcal{X} = 0$ and the dynamical system takes the Floquet form $\dot{x} = \omega + O(\mathcal{X})$, $\dot{\mathcal{X}} = \Lambda \mathcal{X} + O_2(\mathcal{X})$ with $x$-independent vector $\omega \in \mathbb{R}^n$ and matrix $\Lambda \in \mathfrak{gl}(N, \mathbb{R})$. The vector $\omega$ (not determined uniquely) is called the frequency vector of the torus $\mathcal{T}$, while the matrix $\Lambda$ (not determined uniquely) is called the Floquet matrix of $\mathcal{T}$, and its eigenvalues are called the Floquet exponents of $\mathcal{T}$. The coordinates $(x, \mathcal{X})$ are called the Floquet coordinates for $\mathcal{T}$.

Note that the Floquet exponents of an equilibrium (where $n = 0$) are just the eigenvalues of the linearization of the vector field around this equilibrium.
In the overwhelming majority of works on KAM theory, the invariant tori under study are reducible. In particular, this is the case for all the papers on the reversible context 2 [47–51]. The Cantor families of reducible invariant tori in KAM theory are in fact Whitney smooth. This means that although the Floquet coordinates for the tori within a given s-parameter family are defined a priori on a certain Cantor-like subset of \( \mathbb{R}^s \), these coordinates can be continued to smooth (say, \( \mathcal{C}^\infty \)) functions defined in an open domain in \( \mathbb{R}^s \). For basic references on Whitney smoothness in KAM theory, see [8, 10].

The results of the present paper imply that the situation with reducible invariant tori of an arbitrary dimension \( n \) within the reversible context 2 is more or less similar to the trivial case \( n = 0 \) of Example 1.1. Namely, if the phase space codimension of each torus is equal to \( a + b \) and \( \dim \text{Fix} G = a < b \) (\( G \) being the reversing involution), then for \( n \geq 2 \) one needs at least \( b - a + 1 \) external parameters, to be more precise, \( b - a \) parameters for the same reasons as in the case \( n = 0 \) (cf. Proposition 6.1 in Sec. 6 below) and one more parameter to control resonances involving the frequencies and the imaginary parts of the Floquet exponents. Each torus possesses the zero Floquet exponent of Proposition 6.1 in Sec. 6 below) and one more parameter to control resonances involving the frequencies and the imaginary parts of the Floquet exponents. It is worthwhile to emphasize that in the four “conventional” KAM contexts (the reversible context 1, Hamiltonian isotropic context, volume preserving context, and dissipative context), a one-dimensional external parameter is always enough [7, 8, 40, 43–45] (with the exception of very special situations).

In fact, to run KAM theory for the reversible context 2, one has first to choose the unperturbed systems where the invariant tori are organized into a \((\epsilon - b + a)\)-parameter smooth (rather than Cantor) family. Following [50, 51], we will consider unperturbed systems of the form

\[
\begin{align*}
\dot{x} &= \Omega(\mu) + \Delta(\sigma, \mu) + \xi(y, z, \sigma, \mu), \\
\dot{y} &= \sigma + \eta(y, z, \sigma, \mu), \\
\dot{z} &= M(\mu)z + \zeta(y, z, \sigma, \mu),
\end{align*}
\]

where \( x \in \mathbb{T}^n \), \( y \in \mathcal{O}_m(0) \), \( z \in \mathcal{O}_{2p}(0) \) are the phase space variables, \( \sigma \in \mathcal{O}_m(0) \) and \( \mu \in \mathcal{O}_s(0) \) are external parameters \((n \in \mathbb{Z}_+, m \in \mathbb{N}, p \in \mathbb{Z}_+, s \in \mathbb{N})\), \( M \) is a \( 2p \times 2p \) matrix-valued function, \( \Delta = O(\sigma) \) and \( \xi = O(y, z) \), \( \eta = O_2(y, z) \), \( \zeta = O_2(y, z, \sigma) \). These systems are assumed to be reversible with respect to the involution

\[
G : (x, y, z) \mapsto (-x, -y, Rz),
\]

where \( R \in \text{GL}(2p, \mathbb{R}) \) is an involutive matrix with eigenvalues 1 and \(-1\) of multiplicity \( p \) each and \( M(\mu)R \equiv -RM(\mu) \). The dimension of the space \( \{ (\sigma, \mu) \} \) of external parameters is equal to \( m + s \). The systems (1.1) are “integrable” in the sense that they are \( \mathbb{T}^n\)-equivariant, i.e., the right-hand side of (1.1) is independent of the angular variable \( x \).

For \( \sigma = 0 \) and any value of \( \mu \), system (1.1) and involution (1.2) admit a common reducible invariant \( n \)-torus \( \{ y = 0, z = 0 \} \) with frequency vector \( \Omega(\mu) \in \mathbb{R}^n \) and Floquet matrix \( 0_m \oplus M(\mu) \in \text{gl}(m + 2p, \mathbb{R}) \). The codimension of this torus in the phase space is equal to \( m + 2p \) while \( \dim \text{Fix} G = p \) (in the previous notation, \( a = p, b = m + p > a \), and \( c = m + s \), so that \( c - b + a = s \)). The parameter \( \sigma \) is a remedy for a drift along the variable \( y \) in \( G\)-reversible perturbations of systems (1.1).

**Remark 1.1.** One may wonder why the equation for \( \dot{z} \) in (1.1) does not contain a term like \( \Pi(\sigma, \mu)z \) with \( \Pi = O(\sigma) \) and \( \Pi(\sigma, \mu)R \equiv -R\Pi(\sigma, \mu) \). The reason is that such a term can be incorporated into \( \zeta(y, z, \sigma, \mu) = O_2(y, z, \sigma) \); cf. Eqs. (2.2) in [50].

**Remark 1.2.** Within the so-called \textit{a posteriori} format of KAM theorems, one considers invariant tori in dynamical systems that are not assumed to be nearly integrable in any sense; see Chapter 4 of the book [20] and references therein. To the best of the author’s knowledge, the a posteriori approach to the reversible contexts has not been implemented yet.
1.3. Aim of the present paper. The eigenvalues of the matrix $M(\mu)$ in (1.1) come in pairs $(\lambda, -\lambda)$ for each $\mu$, and generically $\det M(\mu) \neq 0$.

**Definition 1.4.** Let a matrix $M \in \text{GL}(2p, \mathbb{R})$ anti-commute with an involutive $2p \times 2p$ matrix with eigenvalues 1 and $-1$ of multiplicity $p$ each. We write that the spectrum of $M$ has the form $\mathfrak{M}(\nu_1, \nu_2, \nu_3; \alpha, \beta)$ where $\nu_1, \nu_2, \nu_3 \in \mathbb{Z}^+$, $\nu_1 + \nu_2 + 2\nu_3 = p$, and $\alpha \in \mathbb{R}^{\nu_1+\nu_3}$, $\beta \in \mathbb{R}^{\nu_2+\nu_3}$ are two vectors with positive components, if $\det M \neq 0$ and the eigenvalues of $M$ have the form

$$\pm \alpha_1, \ldots, \pm \alpha_{\nu_1}, \pm i\beta_1, \ldots, \pm i\beta_{\nu_2},$$

$$\pm \alpha_{\nu_1+1} \pm i\beta_{\nu_2+1}, \ldots, \pm \alpha_{\nu_1+\nu_2} \pm i\beta_{\nu_2+\nu_3}.$$

Assume that the spectrum of $M(\mu)$ is simple and has the form $\mathfrak{M}(\nu_1, \nu_2, \nu_3; (\alpha(\mu), \beta(\mu)))$ for each $\mu \in \mathcal{O}_\sigma(0)$ where $\nu_1 + \nu_2 + 2\nu_3 = p$. For $\sigma = 0$ and any $\mu$, the reducible invariant $n$-torus $T_\mu = \{y = 0, z = 0\}$ of system (1.1) has the zero Floquet exponent of multiplicity $m$ and $2p$ nonzero Floquet exponents

$$\pm \alpha_j(\mu), \ 1 \leq j \leq \nu_1; \quad \pm i\beta_j(\mu), \ 1 \leq j \leq \nu_2;$$

$$\pm \alpha_{\nu_1+j}(\mu) \pm i\beta_{\nu_2+j}(\mu), \ 1 \leq j \leq \nu_3.$$  (1.3)

For the unperturbed systems (1.1), various KAM theorems can be formulated. Let us mention four setups.

A) First, one can establish the so-called “source” (or Broer–Huitema–Takens-like) theorem where the frequencies $\Omega_i(\mu)$, $1 \leq i \leq n$, and the nonzero Floquet exponents (1.3) of the unperturbed tori are assumed to depend on $\mu$ in the “most nondegenerate” way, i.e., the mapping

$$\mu \mapsto (\Omega(\mu), \alpha(\mu), \beta(\mu)) \in \mathbb{R}^{n+p}$$  (1.4)

is submersive. This case was dealt with in our paper [50]. According to the source theorem, all the unperturbed tori $T_\mu$ with frequencies and nonzero Floquet exponents satisfying a suitable Diophantine condition persist under small $G$-reversible perturbations of systems (1.1). The corresponding perturbed $n$-tori possess the same frequency vectors and Floquet matrices and constitute a Whitney smooth family. The submersivity of mapping (1.4) is analogous to the classical Kolmogorov nondegeneracy condition for the unperturbed Lagrangian invariant tori in the Hamiltonian isotropic context without external parameters [1, 10, 17].

B) Second, one may consider weaker nondegeneracy conditions yielding just partial preservation of the frequencies and nonzero Floquet exponents of the unperturbed tori $T_\mu$ under small $G$-reversible perturbations of systems (1.1). This means that one can set up a correspondence between the unperturbed and perturbed $n$-tori in such a way that a prescribed subcollection of the frequencies $\Omega_i(\mu)$, the positive real parts $\alpha_j(\mu)$ of the Floquet exponents (1.3), and the positive imaginary parts $\beta_j(\mu)$ of the Floquet exponents (1.3) of the unperturbed tori $T_\mu$ coincide with the matching subcollection of the spectral characteristics of the corresponding perturbed tori. The partial preservation theorem is the subject of the present paper. In the extreme case of very weak (Rüssmann-like [35, 36]) nondegeneracy conditions, perturbed systems still admit a Whitney smooth family of reducible invariant $n$-tori but it is impossible to assign the unperturbed tori to the perturbed ones in any reasonable way.

C) Third, one may examine the situation where reversible perturbations of systems (1.1) are nonautonomous and depend on time quasi-periodically with $N$ basic frequencies. In this setting studied in our paper [51], the perturbed tori in the extended phase space are of dimension $n + N$.

D) Fourth, assuming that $\nu_2 > 0$, it probably makes sense to look for invariant $(n + d)$-tori $\mathfrak{S}^{n+d}$ “around” the $n$-tori $T_\mu$, $d = 1, \ldots, \nu_2$, in systems (1.1) themselves and in their small $G$-reversible perturbations. One may speak of the excitation of the elliptic normal modes (i.e., of the purely imaginary Floquet exponents $\pm i\beta_j(\mu), 1 \leq j \leq \nu_2$) of the unperturbed tori $T_\mu$. This is the subject of future publications.

In the reversible context 1, Hamiltonian isotropic context, volume preserving context, and dissipative context, the four analogous setups have been more or less thoroughly explored; see the works [6–10, 17].
40–45] and references therein. The relevant source theorems were proven by Broer, Huitema, and Takens in [6, 9] (some generalizations are contained in the papers [4, 5, 55] of Broer’s group). Rüssmann-like nondegeneracy conditions were used in [7, 8, 10, 40] (see also [35] for Rüssmann’s original formulation in the Hamiltonian isotropic context), the general partial preservation theorems were deduced in [43, 45], the perturbations that are quasi-periodic in time were handled in [44], and the excitation of elliptic normal modes was treated in [8, 40–42].

Moreover, in all our works [7, 8, 10, 40, 43–45] devoted to the four “conventional” KAM contexts, the results in the setups B) and C) were obtained as corollaries of the corresponding source theorems (whence the name “source”). The main reduction technique we employed is called the Herman method. This method is specifically adapted to construct invariant tori in perturbations of integrable or partially integrable systems with weak nondegeneracy conditions. It was proposed by Herman in 1990 in his talk at an international conference on dynamical systems in Lyon (cf. [60, § 4.6.2]). The results in the setup D) in [8, 40–42] were obtained mainly as corollaries of the results in the setup B) with the weakest nondegeneracy conditions (thus, in the long run, as corollaries of the source theorems as well). In fact, the excitation of the elliptic normal modes is only possible in the volume preserving context for tori of phase space codimension 2 [42], in the Hamiltonian isotropic context [8, 41], and in the reversible context 1 [8, 40].

The idea of the Herman approach is (roughly speaking) as follows. First, by adding new external parameters, one achieves full control of the frequencies and Floquet exponents of the unperturbed tori (the appropriate analogue of mapping (1.4) becomes submersive). The corresponding source theorem can now be applied to the new systems. Now, using the Whitney smoothness of the family of the perturbed invariant tori, the implicit function theorem, and a suitable number-theoretical lemma concerning Diophantine approximations on submanifolds of Euclidean spaces (or, as one sometimes says, Diophantine approximations of dependent quantities), one can “extract” information on invariant tori in the original systems (i.e., the systems without the additional external parameters). Within this procedure, all the cumbersome and laborious “KAM machinery” is required only to prove the source theorem and is not needed any longer as one reduces theorems with degeneracies to the source theorem.

In the reversible context 2, we also used the Herman method in the setup C) [51] and apply this technique again in the present paper in the setup B). Thus, the present paper contributes to the program [46, 47] of carrying over the results of [6–10, 40–45] to the more involved reversible context 2 without making the proofs more complicated.

Remark 1.3. Partial preservation of frequencies (or frequency ratios) of the unperturbed invariant tori in the Hamiltonian isotropic context was first considered in [12, 24, 25]. These papers do not employ any Herman-type reduction techniques; accordingly, the proofs in [12, 24, 25] are very difficult and involve the so-called quasi-linear infinite iterative scheme.

Remark 1.4. The codimension of the tori $\mathcal{W}^{n+d}$ in the setup D) above is equal to $m + 2p - d$. Consequently, for $\frac{1}{2}(m + 2p - d) \leq p = \dim \text{Fix} G$, i.e., for $d \geq m$, the tori $\mathcal{W}^{n+d}$ pertain to the reversible context 1. Thus, while examining the excitation of elliptic normal modes, one may pass from the reversible context 2 to the context 1 (cf. [49]). Similarly, while studying the destruction of unperturbed invariant tori with resonant frequencies, one may pass from the reversible context 1 to the context 2 [47, 49, 50]. Indeed, when a resonant invariant torus $\mathcal{T}$ of a $G$-reversible system breaks up into a finite collection of perturbed invariant tori $\mathcal{W}_1, \ldots, \mathcal{W}_l$ of smaller dimension, it is possible that $\frac{1}{2} \text{codim} \mathcal{T} \leq \dim \text{Fix} G$, but $\frac{1}{2} \text{codim} \mathcal{W}_i > \dim \text{Fix} G$. One may suspect that the excitation of elliptic normal modes in the reversible context 2 is a much more complicated phenomenon than that in the context 1 (but it would be naive to hope that the destruction of resonant unperturbed tori is easier to study in the reversible context 2 than in the context 1). One more technique of “moving” from the reversible context 1 to the context 2 was developed in the paper [50] where we proved the source theorem for the reversible context 2. This theorem was obtained in [50] (also by Herman-like
arguments) as a corollary of the main result of the article [4], which concerns systems within the reversible context 1 with a singular normal behavior of the invariant tori.

**Remark 1.5.** The main tool in our first three papers [47–49] on the reversible context 2 was the Moser modifying terms theory [31].

**Remark 1.6.** We would like to emphasize that throughout this paper, the word “dissipative” means “relating to no structure on the phase space.” For instance, conformally Hamiltonian vector fields $V$ and conformally symplectic diffeomorphisms $A$, which have been extensively studied lately in KAM theory (see [11] and references therein), are defined by the identities $d(i_V \omega^2) \equiv \eta \omega^2$ and $A^* \omega^2 \equiv \pm e^\eta \omega^2$ with constant $\eta \neq 0$ and are therefore not dissipative in this sense ($\omega^2$ being a symplectic structure on the phase space). However, conformally Hamiltonian systems are dissipative in another sense, namely, their dynamics exhibits no conservative patterns.

As in our previous works on the “conventional” KAM contexts and the reversible context 2, we confine ourselves to analytic systems but there is no doubt that our results (Theorems 3.1 and 3.2 below) can be carried over to Gevrey regular or merely infinitely differentiable systems and even to $C^r$-smooth systems for finite (but sufficiently large) $r$. Similarly, the families of analytic perturbed invariant tori in Theorems 3.1, 3.2, and 4.1 below are claimed to be $C^\infty$-smooth in the sense of Whitney, but these families are certainly Gevrey regular in the sense of Whitney (cf. [55]).

The paper is organized as follows. In Sec. 2, we formulate the Diophantine lemma (Lemma 2.1) to be used in the Herman procedure. The main result of the paper (Theorem 3.1) is stated in Sec. 3. In Sec. 4, we give a precise formulation of the source theorem for the reversible context 2 (Theorem 4.1) in the form we need. A proof of the main result is presented in Sec. 5. Finally, in Sec. 6, we give a rigorous proof of the fact that the occurrence of invariant tori in the reversible context 2 requires many external parameters.

## 2. Diophantine Lemma

**Definition 2.1** ([43–45, 51]). Let $n \in \mathbb{Z}_+$ and $\nu \in \mathbb{Z}_+$. Given $\tau \geq 0$, $\gamma > 0$, and $L \in \mathbb{N}$, a pair of vectors

$$\Omega \in \mathbb{R}^n, \quad \beta \in \mathbb{R}^\nu$$

is said to be *affinely $(\tau, \gamma, L)$-Diophantine* if the inequality

$$\left|(\Omega, k) + (\beta, \ell)\right| \geq \gamma |k|^{-\tau}$$

holds for any $k \in \mathbb{Z}^n \setminus \{0\}$ and $\ell \in \mathbb{Z}^\nu$ such that $|\ell| \leq L$.

Clearly, if $n \in \mathbb{N}$ and a pair of vectors (2.1) is affinely $(\tau, \gamma, L)$-Diophantine, then the vector $\Omega \in \mathbb{R}^n$ is $(\tau, \gamma, L)$-Diophantine in the usual sense, so that $\tau \geq n - 1$. If $n = 0$, then a pair of vectors (2.1) is affinely $(\tau, \gamma, L, \nu)$-Diophantine for any $\tau, \gamma, L, \nu$, and $\beta \in \mathbb{R}^\nu$ [45].

**Definition 2.2** ([43–45, 51]). Let $s \in \mathbb{N}, n \in \mathbb{Z}_+, \nu \in \mathbb{Z}_+$. We will adopt the standard multiindex notation

$$q! = q_1!q_2! \cdots q_s!, \quad \mu^q = \mu_1^{q_1}\mu_2^{q_2} \cdots \mu_s^{q_s}, \quad D^q_{\mu} \Omega = \frac{\partial^{q!} \Omega}{\partial \mu_1^{q_1}\partial \mu_2^{q_2} \cdots \partial \mu_s^{q_s}},$$

where $q \in \mathbb{Z}_+^s, \mu \in \mathbb{R}^s$, and $\Omega$ is a (vector-valued) function $C^{[q]}$-smooth in $\mu$. Let $\mathcal{A} \subset \mathbb{R}^s$ be an open domain and let $Q \in \mathbb{N}, L \in \mathbb{N}$. Consider a pair of $C^Q$-smooth mappings $\Omega : \mathcal{A} \rightarrow \mathbb{R}^n, \beta : \mathcal{A} \rightarrow \mathbb{R}^\nu$. If $n > 0$, introduce the notation

$$\rho^Q(\mu) = \min_{\|\epsilon\|=1} \max_{J=1}^Q \max_{\|u\|=1} \left| \sum_{|q|=J} \left( D^q_{\mu} \Omega(\mu), e \right) \frac{u^q}{q!} \right|$$
$$(q \in \mathbb{Z}_+^n, e \in \mathbb{R}^n, u \in \mathbb{R}^s)$$ for $\mu \in \mathfrak{A}$. If $\nu > 0$, introduce the notation

$$\kappa^Q_{\ell}(\mu) = \max_{J=1}^Q \max_{\|u\|=1} \sum_{|q|=J}  \left| \left< D^q_{\mu} \Omega(\mu), k \right> + \left< D^q_{\mu} \beta(\mu), \ell \right> \right|$$

$q \in \mathbb{Z}_+^n, u \in \mathbb{R}^s$ for $\mu \in \mathfrak{A}, \ell \in \mathbb{Z}_+^n$. The pair of mappings $\Omega, \beta$ is said to be affinely $(Q, L)$-nondegenerate at a point $\mu \in \mathfrak{A}$ if one of the following four conditions is satisfied.

1. $n > 0, \nu > 0, \rho^Q(\mu) > 0$, and

$$\max_{1 \leq |q| \leq Q} \left| \left< D^q_{\mu} \Omega(\mu), k \right> + \left< D^q_{\mu} \beta(\mu), \ell \right> \right| > 0$$

$q \in \mathbb{Z}_+^n$ for all $k \in \mathbb{Z}_{+}^n$ and $\ell \in \mathbb{Z}_+^n$ such that $1 \leq |\ell| \leq L$ and $\|k\| \leq \kappa^Q_{\ell}(\mu)/\rho^Q(\mu)$.

2. $n > 0, \nu = 0$, and $\rho^Q(\mu) > 0$.

3. $n = 0, \nu > 0$, and $\kappa^Q(\mu) > 0$ for all $\ell \in \mathbb{Z}_{+}^n$ such that $1 \leq |\ell| \leq L$.

4. $n = \nu = 0$.

Note that for any (vector-valued) $C^J$-smooth function $H$ defined in $\mathfrak{A} \subset \mathbb{R}^s (J \in \mathbb{Z}_+)$ and any $\mu \in \mathfrak{A}, u \in \mathbb{R}^s$ one has

$$J! \sum_{|q|=J} D^q_{\mu} H(\mu) \frac{u^q}{q!} = \frac{d^J}{dt^J} H(\mu + tu) \bigg|_{t=0}$$

$q \in \mathbb{Z}_+^n$. The inequality $\rho^Q(\mu) > 0$ (for $n > 0$) means that the collection of all $(s+Q)^n - 1$ partial derivatives of $\Omega$ at $\mu$ of all the orders from 1 to $Q$ spans $\mathbb{R}^n$, i.e., the linear hull of these derivatives is $\mathbb{R}^n$ (a Rüssmann-type property [35]). The inequality $\kappa^Q(\mu) > 0$ (for $\nu > 0$ and some $\ell \in \mathbb{Z}_+^n \setminus \{0\}$) means that at least one of the $(s+Q)^n - 1$ partial derivatives of $\beta$ at $\mu$ of all the orders from 1 to $Q$ is not orthogonal to $\ell$. It is not hard to verify that if a pair of mappings $\Omega, \beta$ is affinely $(Q, L)$-nondegenerate at a point $\mu \in \mathfrak{A}$, then it is affinely $(Q, L)$-nondegenerate at any point $\mu' \in \mathfrak{A}$ sufficiently close to $\mu$.

**Lemma 2.1 ([45]).** Let $s \in \mathbb{N}, n \in \mathbb{Z}_+, \nu \in \mathbb{Z}_+, Q \in \mathbb{N},$ and $L \in \mathbb{N}$. Let also $\mathfrak{A} \subset \mathbb{R}^s$ be an open domain, $A \subset \mathfrak{A}$ a subset of $\mathfrak{A}$ diffeomorphic to a closed $s$-dimensional ball, and $B$ an arbitrary compact metric space. Suppose that mappings

$$\Omega: \mathfrak{A} \times B \rightarrow \mathbb{R}^n \quad \text{and} \quad \beta: \mathfrak{A} \times B \rightarrow \mathbb{R}^\nu$$

are $C^Q$-smooth in $a \in \mathfrak{A}$ and, moreover, all the partial derivatives of the functions $\Omega$ and $\beta$ with respect to $a_1, \ldots, a_s$ of any order from 1 to $Q$ are continuous in $(a, b) \in \mathfrak{A} \times B$ (rather than only in $a \in \mathfrak{A}$). Let the pair of mappings

$$a \mapsto \Omega(a, b) \in \mathbb{R}^n \quad \text{and} \quad a \mapsto \beta(a, b) \in \mathbb{R}^\nu$$

be affinely $(Q, L)$-nondegenerate at each point $a \in A$ for any fixed value of $b \in B$. Then

1. There exists a number $\delta > 0$ and

2. For every $n^{\text{add}} \in \mathbb{Z}_+, \nu^{\text{add}} \in \mathbb{Z}_+$, $\tau_s \geq \max(0, n^{\text{add}} - 1), \gamma_s > 0, \varepsilon \in (0, 1)$ and every $\tau$ such that $\tau > (n + n^{\text{add}})Q$ and $\tau \geq \tau_s$, there exists a number $\gamma = \gamma(\varepsilon, \tau, \gamma_s) > 0$ such that the following holds. Let

$$\bar{\Omega}: \mathfrak{A} \times B \rightarrow \mathbb{R}^n \quad \text{and} \quad \bar{\beta}: \mathfrak{A} \times B \rightarrow \mathbb{R}^\nu$$

be any mappings that are $C^Q$-smooth in $a \in \mathfrak{A}$ and such that all the partial derivatives of each component of the differences $\bar{\Omega} - \Omega$ and $\bar{\beta} - \beta$ with respect to $a_1, \ldots, a_s$ of any order from 1 to $Q$ are smaller than $\delta$ in absolute value everywhere in $\mathfrak{A} \times B$. Let

$$\Omega^{\text{add}}: B \rightarrow \mathbb{R}^{n^{\text{add}}} \quad \text{and} \quad \beta^{\text{add}}: B \rightarrow \mathbb{R}^{\nu^{\text{add}}}$$

(2.3)
be arbitrary mappings. Then, for any \( b \in B \) such that the pair of vectors \( \Omega^{\text{add}}(b), \beta^{\text{add}}(b) \) is affinely \((\tau, \gamma, L)\)-Diophantine, the Lebesque measure of the set of those points \( a \in A \) for which the pair of vectors

\[
\left( \Omega(a,b), \Omega^{\text{add}}(b) \right) \in \mathbb{R}^{n+n^{\text{add}}} , \quad \left( \beta(a,b), \beta^{\text{add}}(b) \right) \in \mathbb{R}^{\nu+\nu^{\text{add}}}
\]

is affinely \((\tau, \gamma, L)\)-Diophantine, is greater than \((1 - \varepsilon)\) \(\text{meas}_s A\).

Here and henceforth, \(\text{meas}_s\) denotes the Lebesgue measure in \(\mathbb{R}^s\). Some particular cases of Lemma 2.1 are formulated in [43, 44, 51].

**Example 2.1.** The compactness of \( B \) in Lemma 2.1 is essential. For instance, suppose that \( n \in \mathbb{N} \) and a pair of \( C^Q \)-smooth mappings

\[
\Omega_0 : \mathfrak{A} \to \mathbb{R}^n \quad \text{and} \quad \beta_0 : \mathfrak{A} \to \mathbb{R}^\nu
\]

is affinely \((Q, L)\)-nondegenerate at each point \( a \in A \). Let \( B = [1, +\infty) \) and \( \Omega(a,b) = \Omega_0(a)/b, \beta(a,b) = \beta_0(a)/b \). The pair of mappings (2.2) is affinely \((Q, L)\)-nondegenerate at each point \( a \in A \) for any fixed value of \( b \in B \). Assume also that all the partial derivatives of each component of functions (2.4) of any order from 1 to \( Q \) are smaller than a certain number \( \mathcal{D} < +\infty \) in absolute value everywhere in \( \mathfrak{A} \).

Given \( \delta > 0 \), let \( c_1 = \max(\mathcal{D}/\delta, 1) \) and choose an arbitrary number \( c_2 > c_1 \). Consider an arbitrary function \( \vartheta : B \to \mathbb{R} \) such that \( \vartheta(b) = 1 \) for \( 1 \leq b \leq c_1 \), \( 0 < \vartheta(b) < 1 \) for \( c_1 < b < c_2 \), and \( \vartheta(b) = 0 \) for \( b \geq c_2 \) (such a function can be chosen to be \( C^\infty \)-smooth, but we will not use this fact). Set

\[
\tilde{\Omega}(a,b) = \vartheta(b)\Omega(a,b) = \vartheta(b)\Omega_0(a)/b \quad \text{and} \quad \tilde{\beta}(a,b) = \vartheta(b)\beta(a,b) = \vartheta(b)\beta_0(a)/b.
\]

Since

\[
\frac{\mathcal{D}(1 - \vartheta(b))}{b} < \delta
\]

for any \( b \in B \), all the partial derivatives of each component of the differences \( \tilde{\Omega} - \Omega \) and \( \tilde{\beta} - \beta \) with respect to \( a \) of any order from 1 to \( Q \) are smaller than \( \delta \) in absolute value everywhere in \( \mathfrak{A} \times B \). Now let \( n^{\text{add}} = \nu^{\text{add}} = 0 \), i.e., let mappings (2.3) be absent. Given arbitrary \( \varepsilon \in (0, 1) \), \( \tau > nQ, \gamma > 0 \), one cannot assert that for any \( b \in B \) the Lebesgue measure of the set \( \mathcal{A}_b \) of those points \( a \in A \) for which the pair of vectors (2.5) is affinely \((\tau, \gamma, L)\)-Diophantine, is greater than \((1 - \varepsilon)\) \(\text{meas}_s A\). Indeed, \( \mathcal{A}_b \) is empty for \( b \geq c_2 \) because both vectors (2.5) are zero for each \( a \) if \( b \geq c_2 \).

### 3. The Main Result

In Secs. 3 and 4, we will sometimes write \( \{0 \in \mathbb{R}^s\} \) instead of \( \{0\} \) with \( 0 \in \mathbb{R}^s \).

Let \( n \in \mathbb{Z}_+, m \in \mathbb{N}, p \in \mathbb{Z}_+, s \in \mathbb{N} \). Consider an analytic \((m + s)\)-parameter family of analytic differential equations

\[
\begin{align*}
\dot{x} &= \Omega(\mu) + \Delta(\sigma, \mu) + \xi(y, z, \sigma, \mu) + f(x, y, z, \sigma, \mu), \\
\dot{y} &= \sigma + \eta(y, z, \sigma, \mu) + g(x, y, z, \sigma, \mu), \\
\dot{z} &= M(\mu)z + \zeta(y, z, \sigma, \mu) + h(x, y, z, \sigma, \mu),
\end{align*}
\]

where \( x \in \mathbb{T}^n \), \( y \in \mathcal{O}_m(0), z \in \mathcal{O}_{2p}(0) \) are the phase space variables, \( \sigma \in \mathcal{O}_m(0) \) and \( \mu \in \mathcal{O}_s(0) \) are external parameters, \( M \) is a \( 2p \times 2p \) matrix-valued function, \( \Delta = O(\sigma) \) and \( \xi = O(y, z) \), \( \eta = O_2(y, z) \), \( \zeta = O_2(y, z, \sigma) \). The functions \( \Omega, M, \Delta, \xi, \eta, \zeta \) are supposed to be fixed, whereas the terms \( f, g, h \) are small perturbations; cf. (1.1). Let systems (3.1) be reversible with respect to the phase space involution (1.2), where \( R \in \text{GL}(2p, \mathbb{R}) \) is an involutive matrix with eigenvalues 1 and \(-1\) of multiplicity \( p \) each, \( M(\mu)R \equiv -RM(\mu) \), and the spectrum of \( M(0) \) is simple. One may assume that the spectrum of \( M(\mu) \) is simple for each \( \mu \) and has the form \( \mathfrak{M}(\nu_1, \nu_2, \nu_3; \alpha(\mu), \beta(\mu)) \) where \( \nu_1 + \nu_2 + 2\nu_3 = p \) (see Definition 1.4). Introduce the notation \( \nu = \nu_2 + 2\nu_3 \in \mathbb{Z}_+ \).
Choose arbitrary (possibly empty) subsets of indices
\[ \mathcal{G}_1 \subset \{1; 2; \ldots; n\}, \quad \mathcal{G}_2 \subset \{1; 2; \ldots; \nu_1 + \nu_3\}, \quad \mathcal{G}_3 \subset \{1; 2; \ldots; \nu\}, \quad \mathcal{T} \subset \{1; 2; \ldots; s\} \]
such that
\[ 0 \leq \# \mathcal{G}_1 + \# \mathcal{G}_2 + \# \mathcal{G}_3 = \# \mathcal{T} \leq \min(n + p, s - 1). \]
Here and henceforth, \# denotes the number of elements of a finite set. We are interested in the preservation of the frequencies \( \Omega \) of the unperturbed invariant tori \( T_\mu = \{y = 0, z = 0\} \) at \( \sigma = 0 \) with \( i \in \mathcal{G}_1 \), the real parts \( \alpha_j \) of the Floquet exponents with \( j \in \mathcal{G}_2 \), and the imaginary parts \( \beta_j \) of the Floquet exponents with \( j \in \mathcal{G}_3 \). We will write
\[ \Omega_+ = (\Omega_i \mid i \in \mathcal{G}_1), \quad \Omega_- = (\Omega_i \mid i \notin \mathcal{G}_1), \]
\[ \alpha_+ = (\alpha_j \mid j \in \mathcal{G}_2), \quad \alpha_- = (\alpha_j \mid j \notin \mathcal{G}_2), \]
\[ \beta_+ = (\beta_j \mid j \in \mathcal{G}_3), \quad \beta_- = (\beta_j \mid j \notin \mathcal{G}_3), \]
\[ \mu_+ = (\mu_l \mid l \in \mathcal{T}), \quad \mu_- = (\mu_l \mid l \notin \mathcal{T}); \]
(3.2)
similar notation will be used below without special mention for vector quantities denoted by the letters \( \Omega, \alpha, \beta, \mu \) with superscripts or diacritical marks. Set
\[ \# \mathcal{G}_1 = d_1, \quad \# \mathcal{G}_2 = d_2, \quad \# \mathcal{G}_3 = d_3, \quad d_1 + d_2 + d_3 = d = \# \mathcal{T}, \]
so that
\[ 0 \leq d_1 \leq n, \quad 0 \leq d_2 \leq \nu_1 + \nu_3, \quad 0 \leq d_3 \leq \nu, \]
\[ 0 \leq d \leq \min(n + p, s - 1). \] (3.3)
For any vector \( b \in \mathbb{R}^d \), we will write
\[ b^1 = (b_1, \ldots, b_{d_1}) \in \mathbb{R}^{d_1}, \quad b^2 = (b_{d_1+1}, \ldots, b_{d_1+d_2}) \in \mathbb{R}^{d_2}, \quad b^3 = (b_{d_1+d_2+1}, \ldots, b_d) \in \mathbb{R}^{d_3}. \]
We will also use the notation
\[ \mathcal{P}_0 = (\Omega_+(0), \alpha_+(0), \beta_+(0)) \in \mathbb{R}^d. \]

**Theorem 3.1.** Suppose that either \( d = 0 \) or \( d > 0 \) and the Jacobian
\[ \frac{\partial(\Omega_+, \alpha_+, \beta_+)}{\partial \mu_+} \] (3.4)
of order \( d \) does not vanish at \( \mu = 0 \). This implies, in particular, that \( (\Omega_+, \alpha_+, \beta_+) \) can be used as a part of a new coordinate frame near the origin of \( \mathbb{R}^\nu \). In other words, there exists an analytic change of coordinates \( \mu = \mu(a, b) \) in a neighborhood of \( \mu = 0 \) such that
\[ a \in \mathcal{O}_{s-d}(0), \quad b \in \mathcal{O}_d(\mathcal{P}_0), \quad \mu(0, \mathcal{P}_0) = 0, \]
and
\[ (\Omega_+, \alpha_+, \beta_+)\big|_{\mu=\mu(a,b)} \equiv b, \]
more precisely,
\[ \Omega_+(\mu(a, b)) \equiv b^1, \quad \alpha_+(\mu(a,b)) \equiv b^2, \quad \beta_+(\mu(a, b)) \equiv b^3. \] (3.5)
Assume also that a change of coordinates \( \mu = \mu(a,b) \) with this property can be chosen in such a way that the pair of mappings
\[ a \mapsto \Omega_-(\mu(a,0)) \in \mathbb{R}^{n-d_1} \quad \text{and} \quad a \mapsto \beta_-(\mu(a,0)) \in \mathbb{R}^{\nu-d_3} \] (3.6)
is affinely \((Q, 2)
\)-nondegenerate at \( a = 0 \) for some number \( Q \in \mathbb{N} \) (see Definition 2.2).

Then there exist a closed \((s - d)\)-dimensional ball \( A \subset \mathbb{R}^{s-d} \) centered at the origin and a closed \( d\)-dimensional ball \( B \subset \mathbb{R}^d \) centered at the point \( \mathcal{P}_0 \) such that the following holds. Set
\[ \Gamma = \{\mu(a, b) \mid a \in A, b \in B\} \subset \mathbb{R}^s \] (3.7)
(0 ∈ Γ). Then for every complex neighborhood
\[ \mathcal{C} \subset (\mathbb{C}/2\pi\mathbb{Z})^n \times \mathbb{C}^{2m+2p+s} \] (3.8)
of the set
\[ \mathbb{T}^n \times \{0 \in \mathbb{R}^m\} \times \{0 \in \mathbb{R}^{2p}\} \times \{0 \in \mathbb{R}^m\} \times \Gamma, \] (3.9)
every \( \mathcal{L} \in \mathbb{N}, \varepsilon_1 > 0, \varepsilon_2 \in (0,1), \varepsilon_3 \in (0,1), \tau_\ast \geq \max(0,d_1 - 1), \gamma_\ast > 0, \) and every \( \tau \) such that \( \tau > nQ \) and \( \tau \geq \tau_\ast \), there are numbers \( \delta > 0 \) and \( \gamma \in (0,\gamma_\ast] \) with the following properties.

Suppose that the perturbation terms \( f, g, h \) in (3.1) can be holomorphically continued to the neighborhood \( \mathcal{C} \) and \(|f| < \delta, |g| < \delta, |h| < \delta \) in \( \mathcal{C} \). Consider the closed \((s-d)\)-dimensional ball \( \bar{A} \subset A \) centered at the origin and the closed \(d\)-dimensional ball \( \bar{B} \subset B \) centered at the point \( \mathfrak{P}_0 \) such that
\[ \text{meas}_{s-d} \bar{A} = (1 - \varepsilon_3) \text{meas}_{s-d} A, \quad \text{meas}_d \bar{B} = (1 - \varepsilon_3) \text{meas}_d B \] (3.10)
and set
\[ \bar{\Gamma} = \left\{ \mu(a,b) \mid a \in \bar{A}, b \in \bar{B} \right\} \subset \Gamma \] (3.11)
\((0 \in \bar{\Gamma})
. Then there exist functions
\[ \Theta : \bar{\Gamma} \rightarrow \mathbb{R}^m, \quad \Xi : \bar{\Gamma} \rightarrow \mathbb{R}^s, \] (3.12)
and a change of variables
\[ x = \bar{x} + X(\bar{x}, \mu), \]
\[ y = \bar{y} + Y^0(\bar{x}, \mu) + Y^1(\bar{x}, \mu)\bar{y} + Y^2(\bar{x}, \mu)\bar{z}, \]
\[ z = \bar{z} + Z^0(\bar{x}, \mu) + Z^1(\bar{x}, \mu)\bar{y} + Z^2(\bar{x}, \mu)\bar{z} \] (3.13)
for each \( \mu \in \bar{\Gamma} \) with \( \bar{x} \in \mathbb{T}^n, \bar{y} \in \mathcal{O}_m(0), \bar{z} \in \mathcal{O}_{2p}(0) \) such that the following is valid.

(i) Functions (3.12) are \( C^\infty \)-smooth, and all the partial derivatives of each component of the functions \( \Theta, \Xi, \bar{\Theta} - \bar{\Omega}, \bar{M} - M \) of any order from 0 to \( \mathcal{L} \) are smaller than \( \varepsilon_1 \) in absolute value everywhere in \( \bar{\Gamma} \). If \( d = 0 \) then \( \Xi \equiv 0 \). The coefficients \( X, Y^0, Y^1, Y^2, Z^0, Z^1, Z^2 \) in (3.13) are mappings ranging in \( \mathbb{R}^n, \mathbb{R}^m, \mathfrak{gl}(m, \mathbb{R}), \mathbb{R}^{m \times 2p}, \mathbb{R}^{2p}, \mathbb{R}^{2p \times m}, \mathfrak{gl}(2p, \mathbb{R}) \), respectively. These mappings are analytic in \( \bar{x} \) and \( C^\infty \)-smooth in \( \mu \). All the partial derivatives of each component of these mappings of any order from 0 to \( \mathcal{L} \) are smaller than \( \varepsilon_1 \) in absolute value everywhere in \( \mathbb{T}^n \times \bar{\Gamma} \).

(ii) For each \( \mu \in \bar{\Gamma} \), the change of variables (3.13) commutes with involution (1.2) in the sense that in the new variables \((\bar{x}, \bar{y}, \bar{z})\), the involution \( G \) takes the form
\[ G : (\bar{x}, \bar{y}, \bar{z}) \mapsto (-\bar{x}, -\bar{y}, R\bar{z}). \]
The identity \( \bar{\Phi}(\mu)R \equiv -R\bar{\Phi}(\mu) \) holds.

(iii) The spectrum of \( \bar{M}(\mu) \) is simple and has the form \( \mathfrak{M}(\nu_1, \nu_2, \nu_3; \bar{\alpha}(\mu), \bar{\beta}(\mu)) \) for each \( \mu \in \bar{\Gamma} \) (see Definition 1.4), and the identities
\[ \bar{\Omega}_+ = \Omega_+, \quad \bar{\alpha}_+ = \alpha_+, \quad \bar{\beta}_+ = \beta_+ \] (3.14)
are valid in \( \bar{\Gamma} \).

(iv) For any point \( b \in \bar{B} \) such that the pair of vectors \( b^1 \in \mathbb{R}^{d_1}, b^3 \in \mathbb{R}^{d_3} \) is affinely \((\tau_\ast, \gamma_\ast, 2)\)-Diophantine (see Definition 2.1), there exists a set \( \mathcal{G}_0 \subset A \) that satisfies the following conditions.
(a) \( \text{meas}_{s-d} \mathcal{G}_0 > (1 - \varepsilon_2) \text{meas}_{s-d} \bar{A} \).
(b) For any point \( a \in \mathcal{G}_0 \), the pair of vectors \( \bar{\Omega}(\mu^0) \in \mathbb{R}^n, \bar{\beta}(\mu^0) \in \mathbb{R}^s \) is affinely \((\tau, \gamma, 2)\)-Diophantine, where \( \mu^0 = \mu(a, b) \).
For any point \( a \in \mathcal{G}_b \), the perturbed system (3.1) with \( \mu = \mu^0 + \Xi(\mu^0) \) and \( \sigma = \Theta(\mu^0) \) takes the form
\[
\mathcal{F} = \widetilde{\Omega}(\mu^0) + O(\overline{\mathcal{F}}), \quad \overline{\mathcal{F}} = O_2(\overline{\mathcal{F}}), \quad \mathcal{F} = \widetilde{M}(\mu^0)\overline{\mathcal{F}} + O_2(\overline{\mathcal{F}})
\]
(3.15)
after the coordinate change (3.13) with \( \mu = \mu^0 \).

**Remark 3.1.** One can easily show that for \( d_1 \in \mathbb{N} \), the Lebesgue measure \( \text{meas}_d \) of the set of those points \( b \in \widetilde{B} \) for which the pair of vectors \( b^1 \in \mathbb{R}^{d_1}, b^3 \in \mathbb{R}^{d_3} \) is not affinely \((\tau_*, \gamma_*, 2)\)-Diophantine, tends to 0 as \( \gamma_* \to 0 \) for any fixed \( \tau_*> d_1 - 1 \). For \( d_1 = 0 \), this set is empty for any \( \tau_* \geq 0 \) and \( \gamma_* > 0 \).

Thus, consider an arbitrary point \( \mu^* = \mu(a^*, b^*) \in \widetilde{\Gamma} \) such that the pair of vectors \( b^{*1} = \Omega_+((\mu^*)^0) \in \mathbb{R}^{d_1}, b^{*3} = \beta_+(\mu^*) \in \mathbb{R}^{d_3} \) is affinely \((\tau_*, \gamma_*, 2)\)-Diophantine; see (3.5). The unperturbed invariant \( n \)-tori \( T_\mu, \mu \in \Gamma \), such that
\[
\Omega_+(\mu) = \Omega_+(\mu^*), \quad \alpha_+(\mu) = \alpha_+(\mu^*), \quad \beta_+(\mu) = \beta_+(\mu^*)
\]
(3.16)
constitute an \((s-d)\)-parameter smooth family: equalities (3.16) are equivalent to that \( \mu = \mu(a, b^*) \) for some \( a \in A \). Now choose any \( a \in \mathcal{G}_b \) and denote \( \mu(a, b^*) \) by \( \mu^0 \). The perturbed system (3.1) with the shifted parameter values \( \mu = \mu^0 + \Xi(\mu^0), \sigma = \Theta(\mu^0) \) and involution (1.2) admit a common reducible invariant \( n \)-torus \( \{ \overline{\mathcal{F}} = 0, \overline{\mathcal{F}} = 0 \} \) with frequency vector \( \overline{\Omega}(\mu^0) \) and Floquet matrix \( \overline{\mathcal{M}}(\mu^0) \), see (3.15). For the frequencies \( \overline{\Omega},(\mu^0) \) of this torus, the positive real parts \( \overline{\alpha},(\mu^0) \) of the Floquet exponents, and the positive imaginary parts \( \overline{\beta},(\mu^0) \) of the Floquet exponents, one has
\[
\overline{\Omega}_+(\mu^0) = \Omega_+(\mu^0), \quad \overline{\alpha}_+(\mu^0) = \alpha_+(\mu^0), \quad \overline{\beta}_+(\mu^0) = \beta_+(\mu^0)
\]
according to (3.14) and (3.16). All these perturbed tori constitute an \((s-d)\)-parameter Cantor family (the parameter being \( a \in \mathcal{G}_b \)). The torus \( \{ \overline{\mathcal{F}} = 0, \overline{\mathcal{F}} = 0 \} \) is analytic and depends on \((a, b^*)\) in a \( C^\infty \)-way in the sense of Whitney.

Such partial preservation of the frequencies and the real and imaginary parts of the Floquet exponents of the unperturbed tori \( T_\mu \) is essentially provided by two nondegeneracy conditions: a Broer–Huijtema–Takens-type condition [6, 9] on the components \( \Omega_+, \alpha_+, \beta_+ \) to be preserved (the Jacobian (3.4) does not vanish for \( \mu \in \mathbb{R}^+ \) near 0) and a Rüssmann-type condition [35] on the components \( \Omega_-, \beta_- \) (the pair of mappings (3.6) is affinely \((Q, 2)\)-nondegenerate for \( a \in \mathbb{R}^{s-d} \) near 0). The second condition requires \( s-d \) to be positive. This is the reason why we assume that \( d \leq s-1 \) in Theorem 3.1, although the bound \( d \leq \min(n+p,s) \) may seem more “natural” in (3.3) than \( d \leq \min(n+p,s-1) \).

In the coordinate transformation (3.13), the terms \( X, Y^0, \) and \( Z^0 \) are responsible for the invariance of the torus \( \{ \overline{\mathcal{F}} = 0, \overline{\mathcal{F}} = 0 \} \), while the terms \( Y^1\overline{\mathcal{F}}, Y^2\overline{\mathcal{F}}, Z^1\overline{\mathcal{F}}, \) and \( Z^2\overline{\mathcal{F}} \) are responsible for its reducibility, i.e., for the variational equation along \( \{ \overline{\mathcal{F}} = 0, \overline{\mathcal{F}} = 0 \} \); see a detailed discussion in [51].

**Remark 3.2.** The frequency vectors of the unperturbed invariant tori \( T_\mu = \{ y = 0, z = 0 \} \) of systems (1.1) are \( \Omega(\mu) \), and generically some of these tori are resonant, some are not. Theorem 3.1 (even with \( d = 0 \)) shows that a small generic \( G \)-reversible perturbation of these systems preserves the family of tori \( T_\mu \) but makes it Cantor-like (for \( n \geq 2 \) under mild nondegeneracy conditions). Thus, the reversible context 2 obeys the heuristic principle formulated in the paper [7, Sec. 2] and in the book [8, § 1.4.1].

In the four “conventional” KAM contexts (the reversible context 1, Hamiltonian isotropic context, volume preserving context, and dissipative context), we had the following picture [7, 8, 40, 43, 45]: if the differential equations depend on \( c \) external parameters and the reducible invariant tori constitute an \( s \)-parameter Cantor-like family in the product of the phase space and the space of external parameters, then always \( s \geq c \) and each torus has \( s - c \) zero Floquet exponents (if \( s = c \), then \( c \) is required to be
at least 1). In the reversible context 2, on the contrary, the inequality $c > s$ holds, and each torus has $c - s$ zero Floquet exponents. Indeed, in the framework of Theorem 3.1, $c = m + s$, $s = s$, and each torus possesses a zero Floquet exponent of multiplicity $m$. In all the five contexts, each perturbed torus has $|s - c|$ zero Floquet exponents.

**Remark 3.3.** In the Hamiltonian isotropic context and the reversible context 1, there are results on the frequency preservation where the nondegeneracy condition is formulated in terms of the Brouwer topological degree rather than in terms of the rank of a certain Jacobi-type matrix. The relevant references for reversible systems are [22, 56–58]. As to Hamiltonian systems, we confine ourselves to the papers [59, 61] (see also references therein). Of all these works, the papers [22, 56–58] employ the Herman approach. Some sets of nondegeneracy conditions in the Hamiltonian isotropic context and the reversible context 1 are reviewed in [19].

For $d = 0$, none of the frequencies and Floquet exponents of the unperturbed tori in Theorem 3.1 is required to be preserved (a Rüssmann-like situation [35, 36]). The simplest case where $d = 0$ and $p = 0$ was examined in the paper [50, Sec. 5]. Since the case of zero $d$ is very important, we present it as a separate theorem. Consider again system of differential equations (3.1).

**Theorem 3.2.** Let the pair of mappings

$$
\mu \mapsto \Omega(\mu) \in \mathbb{R}^n \quad \text{and} \quad \mu \mapsto \beta(\mu) \in \mathbb{R}^\nu
$$

be affinely $(Q, 2)$-nondegenerate at $\mu = 0$ for some number $Q \in \mathbb{N}$ (see Definition 2.2). Then there exists a closed $s$-dimensional ball $\Gamma \subset \mathbb{R}^n$ centered at the origin and such that the following holds. For every complex neighborhood (3.8) of the set (3.9) and every $\mathcal{L} \in \mathbb{N}$, $\varepsilon_1 > 0$, $\varepsilon_2 \in (0, 1)$, $\varepsilon_3 \in (0, 1)$, $\tau > nQ$, there are numbers $\delta > 0$ and $\gamma > 0$ with the following properties.

Suppose that the perturbation terms $f, g, h$ in (3.1) can be holomorphically continued to the neighborhood $\mathcal{C}$ and $|f| < \delta$, $|g| < \delta$, $|h| < \delta$ in $\mathcal{C}$. Consider the closed $s$-dimensional ball $\breve{\Gamma} \subset \Gamma$ centered at the origin and such that

$$
\text{meas}_s \breve{\Gamma} = (1 - \varepsilon_3) \text{meas}_s \Gamma.
$$

Then there exist functions

$$
\Theta : \breve{\Gamma} \to \mathbb{R}^m, \quad \breve{\Omega} : \breve{\Gamma} \to \mathbb{R}^n, \quad \breve{M} : \breve{\Gamma} \to \mathfrak{gl}(2p, \mathbb{R})
$$

and a change of variables (3.13) for each $\mu \in \breve{\Gamma}$ with $\breve{\tau} \in \mathbb{T}^n$, $\breve{\gamma} \in \mathcal{O}_m(0)$, $\breve{\varepsilon} \in \mathcal{O}_{2p}(0)$ such that the following is valid.

(i) Functions (3.17) are $C^\infty$-smooth, and all the partial derivatives of each component of the functions $\Theta, \breve{\Omega} - \Omega, \breve{M} - M$ of any order from 0 to $\mathcal{L}$ are smaller than $\varepsilon_1$ in absolute value everywhere in $\breve{\Gamma}$. The coefficients $X, Y^0, Y^1, Y^2, Z^0, Z^1, Z^2$ in (3.13) are mappings ranging in $\mathbb{R}^n$, $\mathbb{R}^m$, $\mathfrak{gl}(m, \mathbb{R})$, $\mathbb{R}^{m \times 2p}$, $\mathbb{R}^{2p}$, $\mathbb{R}^{2p \times m}$, $\mathfrak{gl}(2p, \mathbb{R})$, respectively. These mappings are analytic in $\breve{\tau}$ and $C^\infty$-smooth in $\mu$. All the partial derivatives of each component of these mappings of any order from 0 to $\mathcal{L}$ are smaller than $\varepsilon_1$ in absolute value everywhere in $\mathbb{T}^n \times \breve{\Gamma}$.

(ii) For each $\mu \in \breve{\Gamma}$, the change of variables (3.13) commutes with involution (1.2). The identity $\breve{M}(\mu) \breve{R} \equiv -\breve{R} \breve{M}(\mu)$ holds.

(iii) The spectrum of $\breve{M}(\mu)$ is simple and has the form $\mathfrak{m}(\nu_1, \nu_2, \nu_3; \breve{\beta}(\mu))$ for each $\mu \in \breve{\Gamma}$ (see Definition 1.4).

(iv) There exists a set $\mathcal{G} \subset \breve{\Gamma}$ that satisfies the following conditions.

(a) $\text{meas}_s \mathcal{G} > (1 - \varepsilon_3) \text{meas}_s \breve{\Gamma}$.

(b) For any point $\mu \in \mathcal{G}$, the pair of vectors $\breve{\Omega}(\mu) \in \mathbb{R}^n$, $\breve{\beta}(\mu) \in \mathbb{R}^\nu$ is affinely $(\tau, \gamma, 2)$-Diophantine (see Definition 2.1).
(c) For any point $\mu \in G$, the perturbed system (3.1) with $\sigma = \Theta(\mu)$ takes the form
\[ \tilde{x} = \tilde{\Omega}(\mu) + O(\tilde{y}, \tilde{z}), \quad \tilde{y} = O_2(\tilde{y}, \tilde{z}), \quad \tilde{z} = \tilde{M}(\mu)\tilde{z} + O_2(\tilde{y}, \tilde{z}) \] (3.18)
after the coordinate change (3.13).

Thus, for each $\mu \in G$, the perturbed system (3.1) and involution (1.2) admit a common analytic reducible invariant $n$-torus $\{\tilde{y} = 0, \tilde{z} = 0\}$ with frequency vector $\tilde{\Omega}(\mu)$ and Floquet matrix $0_m \oplus \tilde{M}(\mu)$, see (3.18). All such tori constitute an $s$-parameter Whitney $C^\infty$-smooth family.

4. The Source Theorem in the Reversible Context 2

The material of this section almost coincides with that of [51, Sec. 4]; we have included this section in the paper to achieve a self-contained presentation. To “make” mapping (1.4) submersive, one replaces $\Omega(\mu) + \Delta(\sigma, \mu)$ with an independent external parameter $\omega \in \mathbb{R}^n$ and assumes the mapping
\[ \mu \mapsto (\alpha(\omega, \mu), \beta(\omega, \mu)) \in \mathbb{R}^p \]
(for $M$ dependent on $\omega$) to be submersive for fixed $\omega$.

Let $n \in \mathbb{Z}_+$, $m \in \mathbb{N}$, $p \in \mathbb{Z}_+$, $s \in \mathbb{Z}_+$, and $\omega_* \in \mathbb{R}^n$. Consider an analytic $(m + n + s)$-parameter family of analytic differential equations
\[ \begin{align*}
\dot{x} &= \omega + \xi(y, z, \sigma, \omega, \mu) + f(x, y, z, \sigma, \omega, \mu), \\
\dot{y} &= \sigma + \eta(y, z, \sigma, \omega, \mu) + g(x, y, z, \sigma, \omega, \mu), \\
\dot{z} &= M(\omega, \mu)z + \zeta(y, z, \sigma, \omega, \mu) + h(x, y, z, \sigma, \omega, \mu),
\end{align*} \] (4.1)
where $x \in \mathbb{T}^n$, $y \in \mathcal{O}_m(0)$, $z \in \mathcal{O}_2(0)$ are the phase space variables, $\sigma \in \mathcal{O}_m(0)$, $\omega \in \mathcal{O}_n(\omega_*)$, $\mu \in \mathcal{O}_s(0)$ are external parameters, $M$ is a $2p \times 2p$ matrix-valued function, and $\xi = O(y, z)$, $\eta = O_2(y, z)$, $\zeta = O_2(y, z)$. The functions $M, \xi, \eta, \zeta$ are assumed to be fixed whereas the terms $f, g, h$ are small perturbations. Let systems (4.1) be reversible with respect to the phase space involution (1.2) where $R \in \text{GL}(2p, \mathbb{R})$ is an involutive matrix with eigenvalues 1 and $-1$ of multiplicity $p$ each, $M(\omega, \mu)R \equiv -RM(\omega, \mu)$, and the spectrum of $M(\omega_*, 0)$ is simple. One may assume that the spectrum of $M(\omega, \mu)$ is simple for any $\omega$ and $\mu$ and has the form $\mathfrak{M}(\nu_1, \nu_2, \nu_3; \alpha(\omega, \mu), \beta(\omega, \mu))$ where $\nu_1 + \nu_2 + 2\nu_3 = p$ (see Definition 1.4). Retain the notation $\nu = \nu_2 + \nu_3 \in \mathbb{Z}_+$.

**Theorem 4.1 ([50]).** Suppose that the mapping
\[ \mu \mapsto (\alpha(\omega, \mu), \beta(\omega, \mu)) \in \mathbb{R}^p \]
is submersive at the origin $\mu = 0$ (so that $s \geq p$). Then there exists a neighborhood $\mathcal{D} \subset \mathbb{R}^{n+s}$ of the point $(\omega_*, 0)$ such that for any closed set $\Gamma \subset \mathcal{D}$ that is diffeomorphic to an $(n + s)$-dimensional ball and contains the point $(\omega_*, 0)$ in its interior, the following holds. For every complex neighborhood $\mathcal{C} \subset (\mathcal{C}/2\pi \mathbb{Z})^n \times C^{2m+2p+n+s}$ of the set
\[ \mathbb{T}^n \times \{0 \in \mathbb{R}^m\} \times \{0 \in \mathbb{R}^{2p}\} \times \{0 \in \mathbb{R}^m\} \times \Gamma \]
and every $\mathcal{L} \in \mathbb{N}$, $\varepsilon > 0$, $\tau > n - 1$ ($\tau \geq 0$ for $n = 0$), $\gamma > 0$, there is a number $\delta > 0$ with the following properties.

Suppose that the perturbation terms $f, g, h$ in (4.1) can be holomorphically continued to the neighborhood $\mathcal{C}$ and $|f| < \delta$, $|g| < \delta$, $|h| < \delta$ in $\mathcal{C}$. Then for each $(\omega_0, \mu_0) \in \Gamma$, there exist points
\[ v(\omega_0, \mu_0) \in \mathbb{R}^m, \quad u(\omega_0, \mu_0) \in \mathbb{R}^n, \quad w(\omega_0, \mu_0) \in \mathbb{R}^s \] (4.2)
and a change of variables
\[ \begin{align*}
x &= \tilde{x} + X(\tilde{x}, \omega_0, \mu_0), \\
y &= \tilde{y} + Y^0(\tilde{x}, \omega_0, \mu_0) + Y^1(\tilde{x}, \omega_0, \mu_0)\tilde{y} + Y^2(\tilde{x}, \omega_0, \mu_0)\tilde{z}, \\
z &= \tilde{z} + Z^0(\tilde{x}, \omega_0, \mu_0) + Z^1(\tilde{x}, \omega_0, \mu_0)\tilde{y} + Z^2(\tilde{x}, \omega_0, \mu_0)\tilde{z}
\end{align*} \] (4.3)
with \( \overline{z} \in \mathbb{T}^n \), \( \overline{y} \in \mathcal{O}_n(0) \), \( \overline{z} \in \mathcal{O}_{2p}(0) \) such that the following is valid.

1. The functions \( u, v, w \) in (4.2) are \( C^\infty \)-smooth as functions in \((\omega_0, \mu_0)\), and all the partial derivatives of each component of these functions of any order from 0 to \( L \) are smaller than \( \varepsilon \) in absolute value everywhere in \( \Gamma \). The coefficients \( X, Y^0, Y^1, Y^2, Z^0, Z^1, Z^2 \) in (4.3) are mappings ranging in \( \mathbb{R}^n \), \( \mathbb{R}^m \), \( \mathcal{g}(m, \mathbb{R}) \), \( \mathbb{R}^{n\times 2p} \), \( \mathbb{R}^{2p} \), \( \mathbb{R}^{2p \times m} \), \( \mathcal{g}(2p, \mathbb{R}) \), respectively. These mappings are analytic in \( \mathcal{T} \) and \( C^\infty \)-smooth in \((\omega_0, \mu_0)\). All the partial derivatives of each component of these mappings of any order from 0 to \( L \) are smaller than \( \varepsilon \) in absolute value everywhere in \( \mathbb{T}^n \times \Gamma \).

2. For each \((\omega_0, \mu_0) \in \Gamma \), the change of variables (4.3) commutes with involution (1.2).

3. For any point \((\omega_0, \mu_0) \in \Gamma \) such that the pair of vectors \( \omega_0 \in \mathbb{R}^n \), \( \beta(\omega_0, \mu_0) \in \mathbb{R}^\nu \) is affinely \((\tau, \gamma, 2)\)-Diophantine (see Definition 2.1), system (4.1) at the parameter values

\[ \sigma = v(\omega_0, \mu_0), \quad \omega = \omega_0 + u(\omega_0, \mu_0), \quad \mu = \mu_0 + w(\omega_0, \mu_0) \]  

(4.4)

takes the form

\[ \hat{x} = \omega_0 + O(\overline{y}, \overline{z}), \quad \hat{y} = O_2(\overline{y}, \overline{z}), \quad \hat{z} = M(\omega_0, \mu_0) \overline{z} + O_2(\overline{y}, \overline{z}) \]  

(4.5)

after the coordinate transformation (4.3).

In fact, Theorem 4.1 is just a particular case of the main result of [50]; see a discussion in [51]. Consider any point \((\omega_0, \mu_0) \in \Gamma \) such that the pair of vectors \( \omega_0 \in \mathbb{R}^n \), \( \beta(\omega_0, \mu_0) \in \mathbb{R}^\nu \) is affinely \((\tau, \gamma, 2)\)-Diophantine. The perturbed system (4.1) at the shifted parameter values (4.4) has the reducible invariant \( n \)-torus \( \{ \overline{y} = 0, \overline{z} = 0 \} \) with the same frequency vector \( \omega_0 \) and Floquet matrix \( \text{O}_m \oplus M(\omega_0, \mu_0) \) [see (4.5)], as those of the reducible invariant \( n \)-torus \( \{ y = 0, z = 0 \} \) of system (4.1) without the terms \( f, g, h \) (the unperturbed system) at the parameter values \( \sigma = 0, \omega = \omega_0, \mu = \mu_0 \). The torus \( \{ \overline{y} = 0, \overline{z} = 0 \} \) is analytic and invariant under involution (1.2) and depends on \((\omega_0, \mu_0)\) in a \( C^\infty \)-way in the sense of Whitney.

5. A Proof of Theorem 3.1

Our goal is to deduce Theorem 3.1 from Theorem 4.1 following the general Herman-like scheme (see [45] for a similar reduction technique in the “conventional” KAM contexts). Let systems (3.1) satisfy the hypotheses of Theorem 3.1. Since \( M(\mu) \) depends on \( \mu \) analytically and the spectrum of \( M(0) \) is simple, one can introduce an additional parameter \( \chi \in \mathcal{O}_S(0) \) for an appropriate \( S \in \mathbb{Z}^+ \) and construct an analytic family \( M^{\text{new}}(\mu, \chi) \) of \( 2p \times 2p \) real matrices such that the following holds.

1. \( M^{\text{new}}(\mu, 0) \equiv M(\mu) \) and \( M^{\text{new}}(\mu, \chi)R \equiv -RM^{\text{new}}(\mu, \chi) \). As a consequence, one may assume that for any \( \mu \) and \( \chi \), the spectrum of \( M^{\text{new}}(\mu, \chi) \) is simple and has the form

\[ \mathfrak{M}(\nu_1, \nu_2, \nu_3; \alpha^{\text{new}}(\mu, \chi), \beta^{\text{new}}(\mu, \chi)), \]

where \( \alpha^{\text{new}}(\mu, 0) \equiv \alpha(\mu) \) and \( \beta^{\text{new}}(\mu, 0) \equiv \beta(\mu) \).

2. The mapping

\[ (\mu, \chi) \mapsto (\alpha^{\text{new}}(\mu, \chi), \beta^{\text{new}}(\mu, \chi)) \in \mathbb{R}^p \]

is submersive at \( \mu = 0, \chi = 0 \) (so that \( s + S \geq p \)).

The existence of a \( 2p \times 2p \) matrix-valued function \( M^{\text{new}} \) satisfying these conditions follows immediately from the theory of normal forms and versal unfoldings of infinitesimally reversible matrices [21, 38, 52]. It always suffices to set \( S = p \).

Now introduce one more additional parameter \( \omega \in \mathcal{O}_n(\Omega(0)) \) and consider the analytic \((m + s + S + n)\)-parameter family of analytic differential equations

\[ \dot{x} = \omega + \xi(y, z, \sigma, \mu) + f(x, y, z, \sigma, \mu), \]
\[ \dot{y} = \sigma + \eta(y, z, \sigma, \mu) + g(x, y, z, \sigma, \mu), \]
\[ \dot{z} = M^{\text{new}}(\mu, \chi)z + \zeta(y, z, \sigma, \mu) + h(x, y, z, \sigma, \mu). \]  

(5.1)
Systems (5.1) are reversible with respect to involution (1.2) and satisfy all the hypotheses of Theorem 4.1, with $\Omega(0), s + S, (\mu, \chi), M_{\text{new}}$ playing the roles of $\omega_s, s, \mu, M$, respectively.

Consider a closed ball $A \subset \mathbb{R}^{s-d}$ centered at the origin, a closed ball $B \subset \mathbb{R}^d$ centered at the point $\mathfrak{P}_0$, a closed ball $\Gamma_1 \subset \mathbb{R}^s$ centered at the origin, and a closed ball $\Gamma_2 \subset \mathbb{R}^n$ centered at the point $\Omega(0)$. If the balls $A$ and $B$ are small enough, the set $\Gamma(3.7)$ is well defined (and diffeomorphic to a closed $s$-dimensional ball). According to Theorem 4.1, if all the four balls $A, B, \Gamma_1, \Gamma_2$ are sufficiently small, then for every complex neighborhood (3.8) of set (3.9) and every $L \in \mathbb{N}, \gamma > n - 1 (\gamma \geq 0$ for $n = 0), \gamma > 0$, the following holds.

Suppose that the perturbation terms $f, g, h$ in (3.1) and (5.1) can be holomorphically continued to neighborhood (3.8) and are sufficiently small in (3.8). Then for any $\mu_0 \in \Gamma, \chi_0 \in \Gamma_1$, and $\omega_0 \in \Gamma_2$, there exist points

\[
\begin{align*}
  v(\omega_0, \mu_0, \chi_0) &\in \mathbb{R}^m, \quad u(\omega_0, \mu_0, \chi_0) \in \mathbb{R}^n, \\
  w(\omega_0, \mu_0, \chi_0) &\in \mathbb{R}^s, \quad W(\omega_0, \mu_0, \chi_0) \in \mathbb{R}^S
\end{align*}
\]

and a change of variables

\[
\begin{align*}
  x &= \mathfrak{x}(\mathfrak{x}, \omega_0, \mu_0, \chi_0), \\
  y &= \mathfrak{y} + \mathfrak{y}^0(x, \omega_0, \mu_0, \chi_0) + \mathfrak{y}^1(x, \omega_0, \mu_0, \chi_0) + \mathfrak{y}^2(x, \omega_0, \mu_0, \chi_0)\mathfrak{z}, \\
  z &= \mathfrak{z} + \mathfrak{z}^0(x, \omega_0, \mu_0, \chi_0) + \mathfrak{z}^1(x, \omega_0, \mu_0, \chi_0)\mathfrak{y} + \mathfrak{z}^2(x, \omega_0, \mu_0, \chi_0)\mathfrak{z}
\end{align*}
\]

with $\mathfrak{x} \in \mathbb{T}^n, \mathfrak{y} \in O_m(0), \mathfrak{z} \in O_2(0)$ such that the following is valid.

First, the functions $u, v, w, W$ in (5.2) are $C^\infty$-smooth. The coefficients $\mathfrak{x}, \mathfrak{y}^0, \mathfrak{y}^1, \mathfrak{y}^2, \mathfrak{z}^0, \mathfrak{z}^1, \mathfrak{z}^2$ in (5.3) are analytic in $\mathfrak{x}$ and $C^\infty$-smooth in $(\omega_0, \mu_0, \chi_0)$. All the mappings $u, v, w, W, \mathfrak{x}, \mathfrak{y}^0, \mathfrak{y}^1, \mathfrak{y}^2, \mathfrak{z}^0, \mathfrak{z}^1, \mathfrak{z}^2$ are small in the $C^\infty$-topology.

Second, for any $\mu_0 \in \Gamma, \chi_0 \in \Gamma_1$, and $\omega_0 \in \Gamma_2$, the change of variables (5.3) commutes with involution (1.2).

Third, for any points $\mu_0 \in \Gamma, \chi_0 \in \Gamma_1$, and $\omega_0 \in \Gamma_2$ such that the pair of vectors $\omega_0 \in \mathbb{R}^n$, $\beta_{\text{new}}(\mu_0, \chi_0) \in \mathbb{R}^\nu$ is affinely ($\pi, \gamma, 2$)-Diophantine, system (5.1) at the parameter values

\[
\begin{align*}
  \sigma &= v(\omega_0, \mu_0, \chi_0), \quad \omega = \omega_0 + u(\omega_0, \mu_0, \chi_0), \\
  \mu &= \mu_0 + w(\omega_0, \mu_0, \chi_0), \quad \chi = \chi_0 + W(\omega_0, \mu_0, \chi_0)
\end{align*}
\]

takes the form

\[
\begin{align*}
  \mathfrak{x} &= \omega_0 + O(\mathfrak{y}, \mathfrak{z}), \quad \mathfrak{y} = O_2(\mathfrak{y}, \mathfrak{z}), \quad \mathfrak{z} = M_{\text{new}}(\mu_0, \chi_0)\mathfrak{z} + O_2(\mathfrak{y}, \mathfrak{z})
\end{align*}
\]

after the coordinate transformation (5.3).

One may assume the balls $A$ and $B$ to be so small that $\Omega(\Gamma)$ lies in the interior of $\Gamma_2$. If the functions $u, v, w, W$ are sufficiently small, then the system of equations

\[
\begin{align*}
  \omega + u(\omega, \mu, \chi) &= \Omega(\mu + w(\omega, \mu, \chi)) + \Delta(v(\omega, \mu, \chi), \mu + w(\omega, \mu, \chi)), \\
  \chi + W(\omega, \mu, \chi) &= 0
\end{align*}
\]

with $\mu \in \Gamma$ can be solved with respect to $\omega$ and $\chi$:

\[
\omega = \varphi(\mu), \quad \chi = \psi(\mu),
\]

where $\varphi : \Gamma \to \Gamma_2$ and $\psi : \Gamma \to \Gamma_1$ are $C^\infty$-functions close to $\Omega$ and 0, respectively, in the $C^\infty$-topology. The key observation is that for any $\mu_0 \in \Gamma$, system (5.1) at the parameter values (5.4) with $\omega_0 = \varphi(\mu_0)$ and $\chi_0 = \psi(\mu_0)$ coincides with the original system (3.1) at the parameter values

\[
\begin{align*}
  \sigma &= v(\omega_0, \mu_0, \chi_0), \quad \mu = \mu_0 + w(\omega_0, \mu_0, \chi_0).
\end{align*}
\]

Indeed, if $\omega_0 = \varphi(\mu_0)$ and $\chi_0 = \psi(\mu_0)$, then Eqs. (5.6) imply that the values of the parameters $\sigma, \omega, \mu, \chi$ given by (5.4) satisfy the relations

\[
\omega = \Omega(\mu) + \Delta(\sigma, \mu), \quad \chi = 0.
\]

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Let \( \varepsilon_3 \in (0, 1) \). Consider the closed \((s - d)\)-dimensional ball \( A' \subset A \) centered at the origin and the closed \(d\)-dimensional ball \( B' \subset B \) centered at the point \( \mathfrak{p}_0 \) such that

\[
\text{meas}_{s-d} A' = (1 - \varepsilon_3)^{1/2} \text{meas}_{s-d} A, \quad \text{meas}_d B' = (1 - \varepsilon_3)^{1/2} \text{meas}_d B
\]

and set

\[
\Gamma' = \{ \mu(a, b) \mid a \in A', b \in B' \} \subset \Gamma
\]

\((0 \in \Gamma')\). If the functions \( u, v, w, W \) are small enough, then the equation

\[
\mu = \mu_0 + w(\varphi(\mu_0), \mu_0, \psi(\mu_0))
\]

with \( \mu \in \Gamma' \) can be solved with respect to \( \mu_0 \):

\[
\mu_0 = \Upsilon(\mu),
\]

where \( \Upsilon : \Gamma' \to \Gamma \) is a \( C^\infty \)-function close to the identity mapping \( \mu \mapsto \mu \) in the \( C^\mathcal{L} \)-topology.

We have arrived at the following conclusion. For any point \( \mu \in \Gamma' \), set

\[
\mu_0 = \Upsilon(\mu), \quad \omega_0 = \varphi(\Upsilon(\mu)), \quad \chi_0 = \psi(\Upsilon(\mu)).
\]

If the pair of vectors \( \omega_0, \beta_{\text{new}}(\mu_0, \chi_0) \) is affinely \((\tau, \gamma, 2)\)-Diophantine, then the original system \((3.1)\) at the parameter values \( \mu \) and \( \sigma = v(\omega_0, \mu_0, \chi_0) \) takes form \((5.5)\) after the coordinate transformation \((5.3)\).

Introduce the functions

\[
\hat{\Omega}(\mu) = \varphi(\Upsilon(\mu)), \quad \Psi(\mu) = \psi(\Upsilon(\mu)), \quad \hat{M}(\mu) = M_{\text{new}}(\Upsilon(\mu), \Psi(\mu)), \quad \hat{\alpha}(\mu) = \alpha_{\text{new}}(\Upsilon(\mu), \Psi(\mu)), \quad \hat{\beta}(\mu) = \beta_{\text{new}}(\Upsilon(\mu), \Psi(\mu)), \quad \hat{\Theta}(\mu) = v\left(\hat{\Omega}(\mu), \Upsilon(\mu), \Psi(\mu)\right)
\]

for \( \mu \in \Gamma' \) and

\[
\hat{X}(\pi, \mu) = \mathcal{X}\left(\pi, \hat{\Omega}(\mu), \Upsilon(\mu), \Psi(\mu)\right), \quad \hat{Y}^r(\pi, \mu) = \mathcal{Y}^r\left(\pi, \hat{\Omega}(\mu), \Upsilon(\mu), \Psi(\mu)\right), \quad r = 0, 1, 2, \quad \hat{Z}^r(\pi, \mu) = \mathcal{Z}^r\left(\pi, \hat{\Omega}(\mu), \Upsilon(\mu), \Psi(\mu)\right), \quad r = 0, 1, 2
\]

for \( \pi \in \mathbb{T}^n \) and \( \mu \in \Gamma' \). The mappings \( \hat{\Omega}, \Psi, \hat{M}, \hat{\alpha}, \hat{\beta}, \hat{\Theta} \) are \( C^\infty \)-smooth and the functions \( \hat{\Omega} - \Omega, \Psi, \hat{M} - M, \hat{\alpha} - \alpha, \hat{\beta} - \beta, \hat{\Theta} \) are small in the \( C^\mathcal{L} \)-topology. For any \( \mu \in \Gamma' \), one has \( \hat{M}(\mu)R = -R\hat{M}(\mu) \), and the spectrum of the \( 2p \times 2p \) matrix \( \hat{M}(\mu) \) is simple and has the form \( \mathfrak{m}\left(\nu_1, \nu_2, \nu_3; \hat{\alpha}(\mu), \hat{\beta}(\mu)\right) \).

The coefficients \( \hat{X}, \hat{Y}^0, \hat{Y}^1, \hat{Z}^0, \hat{Z}^1, \hat{Z}^2 \) are analytic in \( \pi \in \mathbb{T}^n \), \( C^\infty \)-smooth in \( \mu \in \Gamma' \), and small in the \( C^\mathcal{L} \)-topology provided that the perturbation terms \( f, g, h \) in \((3.1)\) are sufficiently small.

The conclusion we have come to so far can be reformulated as follows. If the pair of vectors \( \hat{\Omega}(\mu) \in \mathbb{R}^n, \beta(\mu) \in \mathbb{R}^n \) is affinely \((\tau, \gamma, 2)\)-Diophantine for some \( \mu \in \Gamma' \), then system \((3.1)\) at the parameter values \( \mu \) and \( \sigma = \hat{\Theta}(\mu) \) takes the form

\[
\hat{x} = \hat{\Omega}(\mu) + O(\overline{y}, \overline{z}), \quad \hat{y} = O_2(\overline{y}, \overline{z}), \quad \hat{z} = \hat{M}(\mu)\overline{z} + O_2(\overline{y}, \overline{z})
\]

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after the $G$-commuting coordinate change
\begin{align*}
x &= \bar{x} + \bar{X}(\bar{x}, \mu), \\
y &= \bar{y} + \bar{Y}^0(\bar{x}, \mu) + \bar{Y}^1(\bar{x}, \mu)\bar{y} + \bar{Y}^2(\bar{x}, \mu)\bar{z}, \\
z &= \bar{z} + \bar{Z}^0(\bar{x}, \mu) + \bar{Z}^1(\bar{x}, \mu)\bar{y} + \bar{Z}^2(\bar{x}, \mu)\bar{z}
\end{align*}
with $\bar{x} \in \mathbb{T}^n$, $\bar{y} \in \mathcal{O}_m(0)$, $\bar{z} \in \mathcal{O}_{2p}(0)$.

Consider the closed $(s - d)$-dimensional balls $\bar{A} \subset A'' \subset A'$ centered at the origin and the closed $d$-dimensional ball $\bar{B} \subset B'$ centered at the point $\mathcal{P}_0$ such that
\[
\text{meas}_{s-d} A'' = (1 - \varepsilon_3)^3/4 \text{meas}_{s-d} A
\]
and relations (3.10) hold. Define the sets $\bar{\Gamma} \subset \Gamma'' \subset \Gamma'$ by the equation
\[
\Gamma'' = \left\{ (a, b) \mid a \in A'', b \in \bar{B} \right\}
\]
(0 \in \Gamma'') and Eq. (3.11). Let the balls $A$ and $B$ be so small that Jacobian (3.4) vanishes nowhere in $\Gamma$ for $d \geq 1$. Then the system of equations
\begin{align*}
\tilde{\Omega}_+(\mu^*_+, \mu_-) &= \Omega_+(\mu^*_+, \mu_-), \\
\tilde{\alpha}_+(\mu^*_+, \mu_-) &= \alpha_+(\mu^*_+, \mu_-), \\
\tilde{\beta}_+(\mu^*_+, \mu_-) &= \beta_+(\mu^*_+, \mu_-)
\end{align*}
can be solved with respect to $\mu^*_+$ for $\mu = (\mu^*_+, \mu_-) \in \Gamma''$ provided that the functions $u, v, w, W$ are sufficiently small:
\[
\mu^*_+ = \mu_+ + \Xi_+(\mu^*_+, \mu_-) = \mu_+ + \Xi_+(\mu),
\]
where $\Xi_+ : \Gamma'' \to \mathbb{R}^d$ is a $C^\infty$-function small in the $C^\infty$-topology. “Complement” the mapping $\Xi_+$ with the zero function $\Xi_- : \Gamma'' \to \mathbb{R}^{s-d}$ in such a way that
\[
\Xi_+ = (\Xi_l \mid l \in \mathcal{I}), \quad \Xi_- = (\Xi_l \mid l \notin \mathcal{I})
\]
for the mapping $\Xi = (\Xi_+, \Xi_-) : \Gamma'' \to \mathbb{R}^s$, cf. (3.2). If $d = 0$, then $\Xi \equiv 0$. For $\mu \in \Gamma''$, one has $\mu + \Xi(\mu) \in \Gamma'$ and
\begin{align*}
\tilde{\Omega}_+(\mu + \Xi(\mu)) &= \Omega_+(\mu), \\
\tilde{\alpha}_+(\mu + \Xi(\mu)) &= \alpha_+(\mu), \\
\tilde{\beta}_+(\mu + \Xi(\mu)) &= \beta_+(\mu).
\end{align*}

Now set
\begin{align*}
\bar{\Omega}(\mu) &= \tilde{\Omega}(\mu + \Xi(\mu)), \\
\bar{M}(\mu) &= \tilde{M}(\mu + \Xi(\mu)), \\
\bar{\alpha}(\mu) &= \tilde{\alpha}(\mu + \Xi(\mu)), \\
\bar{\beta}(\mu) &= \tilde{\beta}(\mu + \Xi(\mu)), \\
\bar{\Theta}(\mu) &= \tilde{\Theta}(\mu + \Xi(\mu))
\end{align*}
for $\mu \in \Gamma''$ and
\begin{align*}
X(\bar{x}, \mu) &= \hat{X}(\bar{x}, \mu + \Xi(\mu)), \\
Y^r(\bar{x}, \mu) &= \hat{Y}^r(\bar{x}, \mu + \Xi(\mu)), \quad r = 0, 1, 2, \\
Z^r(\bar{x}, \mu) &= \hat{Z}^r(\bar{x}, \mu + \Xi(\mu)), \quad r = 0, 1, 2
\end{align*}
for \( \varpi \in \mathbb{T}^n \) and \( \mu \in \Gamma'' \). The mappings \( \Xi, \tilde{\Omega}, \tilde{M}, \tilde{\alpha}, \tilde{\beta}, \Theta \) are \( C^\infty \)-smooth and the functions \( \Xi, \tilde{\Omega} - \Omega, \tilde{M} - M, \tilde{\alpha} - \alpha, \tilde{\beta} - \beta, \Theta \) are small in the \( C^L \)-topology. For any \( \mu \in \Gamma'' \), one has \( \tilde{M}(\mu)R = -R\tilde{M}(\mu) \), and the spectrum of the \( 2p \times 2p \) matrix \( \tilde{M}(\mu) \) is simple and has the form \( \mathfrak{m}\left( \nu_1, \nu_2, \nu_3; \tilde{\alpha}(\mu), \tilde{\beta}(\mu) \right) \).

Identities (3.14) are valid in \( \Gamma'' \). The coefficients \( X, Y^0, Y^1, Y^2, Z^0, Z^1, Z^2 \) are analytic in \( \varpi \in \mathbb{T}^n \), \( C^\infty \)-smooth in \( \mu \in \Gamma'' \), and small in the \( C^L \)-topology provided that the perturbation terms \( f, g, h \) in (3.1) are small enough. For each \( \mu \in \Gamma'' \), the change of variables (3.13) commutes with involution (1.2).

For any \( \mu^0 \in \Gamma'' \) such that the pair of vectors \( \tilde{\Omega}(\mu^0), \tilde{\beta}(\mu^0) \) is affinely \((\tau, \gamma, 2)\)-Diophantine, system (3.1) at the parameter values \( \mu = \mu^0 + \Xi(\mu^0) \) and \( \sigma = \Theta(\mu^0) \) takes form (3.15) after the coordinate change (3.13) with \( \mu = \mu^0 \).

The pair of mappings (3.6) is affinely \((Q, 2)\)-nondegenerate at \( a = 0 \). One may assume the balls \( A \) and \( B \) (and, consequently, the balls \( \tilde{A} \) and \( \tilde{B} \)) to be so small that the pair of mappings

\[ a \mapsto \Omega_-(\mu(a, b)) \in \mathbb{R}^{n-d_1} \quad \text{and} \quad a \mapsto \beta_-(\mu(a, b)) \in \mathbb{R}^{n-d_3} \tag{5.7} \]

is affinely \((Q, 2)\)-nondegenerate at each point \( a \in \tilde{A} \) for any fixed value of \( b \in \tilde{B} \). Now the Diophantine Lemma 2.1 can be applied with

- \( s-d \geq 1, n-d_1, \nu-d_3, d_1, d_3 \), and 2 playing the roles of \( s, n, \nu, n^{add}, \nu^{add} \), and \( L \), respectively;
- \( \tilde{B}, \tilde{A} \), and the interior of \( A'' \) playing the roles of \( B, A \), and \( \mathfrak{A} \), respectively;
- mappings (5.7) playing the roles of mappings (2.2);
- the mappings \( b \mapsto b^1 \in \mathbb{R}^{d_1} \) and \( b \mapsto b^3 \in \mathbb{R}^{d_3} \) playing the roles of the mappings \( \Omega^{add} \) and \( \beta^{add} \), respectively;
- \( \varepsilon_2 \) playing the role of \( \varepsilon \).

According to Lemma 2.1, if \( L \geq Q \) and the differences \( \tilde{\Omega} - \Omega \) and \( \tilde{\beta} - \beta \) are sufficiently small in \( \Gamma'' \) in the \( C^L \)-topology, then the following is valid. Let \( \varepsilon_2 \in (0, 1), \tau_* \geq \max(0, d_1-1) \), and \( \gamma_* > 0 \). Suppose that \( \tau > nQ, \tau \geq \tau_* \), and \( \gamma \) is sufficiently small: \( 0 < \gamma \leq \gamma_0(\varepsilon_2, \tau, \gamma_*) \). Then for any point \( b \in \tilde{B} \) such that the pair of vectors \( b^1 \in \mathbb{R}^{d_1}, b^3 \in \mathbb{R}^{d_3} \) is affinely \((\tau_*, \gamma_*, 2)\)-Diophantine, the Lebesgue measure \( \text{meas}_{s-d} \) of the set \( \mathcal{G}_b \) of those points \( a \in \tilde{A} \) for which the pair of vectors \( \tilde{\Omega}(\mu(a, b)) \in \mathbb{R}^n, \tilde{\beta}(\mu(a, b)) \in \mathbb{R}^n \) is affinely \((\tau, \gamma, 2)\)-Diophantine, is greater than \((1-\varepsilon_2)\) \( \text{meas}_{s-d} \tilde{A} \). Indeed, denoting \( \mu(a, b) \) by \( \mu^0 \) and taking into account identities (3.5) and (3.14), one has

\[
\tilde{\Omega}(\mu^0) = \left( \Omega_+(\mu^0), \Omega_-(\mu^0) \right) = \left( \Omega_+(\mu^0), \Omega_-(\mu^0) \right) = \left( b^1, \tilde{\Omega}_-(\mu^0) \right),
\]

\[
\tilde{\beta}(\mu^0) = \left( \beta_+(\mu^0), \beta_-(\mu^0) \right) = \left( \beta_+(\mu^0), \beta_-(\mu^0) \right) = \left( b^3, \tilde{\beta}_-(\mu^0) \right).
\]

The proof of Theorem 3.1 is completed.

### 6. Invariant Tori in General Systems

Let \( n \in \mathbb{Z}_+, m \in \mathbb{N}, p \in \mathbb{Z}_+ \). Consider a system of differential equations

\[
\dot{x} = \varphi(x, y, z), \quad \dot{y} = \eta(y), \quad \dot{z} = \psi(x, y, z), \tag{6.1}
\]

where \( x \in \mathbb{T}^n \), \( y \in \mathcal{O}_m(0) \), \( z \in \mathcal{O}_{2p}(0) \) are the phase space variables, cf. (1.1) and (3.1), and suppose that this system is reversible with respect to the phase space involution (1.2), where \( R \in \text{GL}(2p, \mathbb{R}) \) is an involutive matrix. The reversibility condition means that

\[
\varphi(-x, -y, Rz) = \varphi(x, y, z), \quad \eta(-y) = \eta(y), \quad \psi(-x, -y, Rz) = -R \psi(x, y, z).
\]

Note that we impose no restrictions on the equations for \( \dot{x} \) and \( \dot{z} \) (apart from the reversibility) but the right-hand side of the equation for \( \dot{y} \) is assumed to be independent of \( x \) and \( z \). In the present section, we give a rigorous proof of the following statement.

**Proposition 6.1.** Let system (6.1) and involution (1.2) admit a common invariant torus carrying quasi-periodic motions. Then \( \eta(0) = 0 \).
The torus in Proposition 6.1 is not assumed to be of dimension \( n \) (not to mention to be close to the torus \( \{ y = 0, z = 0 \} \)).

In particular, suppose that system (6.1) depends on a \( c \)-dimensional parameter \( w \) with \( c < m \):

\[
\dot{x} = f(x, y, z, w), \quad \dot{y} = g(y, w), \quad \dot{z} = h(x, y, z, w).
\]

Generically the points \( \eta(0, w) \) constitute a \( c \)-dimensional surface in \( \mathbb{R}^m \) that does not contain the origin. Consequently, if \( c < m \), then a generic \( c \)-parameter family of \( G \)-reversible systems (6.1) admits an invariant torus carrying quasi-periodic motions (and invariant under the involution \( G \) as well) at no value of the parameter.

Proposition 6.1 is a particular case of the following more general statement.

**Proposition 6.2.** Let the system

\[
\dot{u} = U(u, v), \quad \dot{v} = V(v)
\]

of differential equations on the direct product \( A \times B = \{ (u, v) \} \) of manifolds \( A \) and \( B \) be reversible with respect to an involution \( G : (u, v) \mapsto (G_A(u), G_B(v)) \) where \( G_A \) and \( G_B \) are involutions of \( A \) and \( B \), respectively. Suppose that \( \text{Fix} \ G_B \) consists of a single point \( v^0 \in B \). Let system (6.2) and the involution \( G \) admit a common invariant torus carrying quasi-periodic motions. Then \( V(v^0) = 0 \).

In turn, the proof of Proposition 6.2 is based on the following lemma.

**Lemma 6.1.** Let \( F : \mathbb{T}^n \to K \) be a surjective continuous mapping of \( \mathbb{T}^n \) onto a compact topological space. Let \( g^t \) be a quasi-periodic flow on \( \mathbb{T}^n \), i.e., \( g^t(\phi) = \phi + \omega t (\phi \in \mathbb{T}^n) \) where \( \omega \in \mathbb{R}^n \) is a fixed vector with rationally independent components. Let also \( \mathcal{G}^t \) be a continuous action of \( \mathbb{R} \) on \( K \). Suppose that \( F \circ g^t = \mathcal{G}^t \circ F \). Then \( K \) is a torus of dimension not greater than \( n \), and \( \mathcal{G}^t \) is quasi-periodic.

It is hardly possible that Lemma 6.1 is new, but I have failed to find it in the literature.

Let us first deduce Proposition 6.2 from Lemma 6.1. Suppose that system (6.2) and the involution \( G \) admit a common invariant \( n \)-torus \( F(\mathbb{T}^n) \), where \( F = (F_A, F_B) \) is an embedding of \( \mathbb{T}^n \) into \( A \times B \) (\( F_A \) and \( F_B \) take \( \mathbb{T}^n \) to \( A \) and \( B \), respectively). Assume the torus \( F(\mathbb{T}^n) \) to carry a quasi-periodic flow \( F \circ g^t \circ F^{-1} \) where \( g^t \) is a quasi-periodic flow on \( \mathbb{T}^n \). Since \( F(\mathbb{T}^n) \) is invariant under \( G \), for each \( \phi \in \mathbb{T}^n \) there exists \( \phi' \in \mathbb{T}^n \) such that \( F_A(\phi') = G_A(F_A(\phi)) \) and \( F_B(\phi') = G_B(F_B(\phi)) \). Consequently, the sets \( F_A(\mathbb{T}^n) \) and \( F_B(\mathbb{T}^n) \) are invariant under the involutions \( G_A \) and \( G_B \), respectively. It is also clear that \( F_B(\mathbb{T}^n) \) is an invariant set of the equation \( \dot{v} = V(v) \) and \( F_B \circ g^t = \mathcal{G}^t \circ F_B \) where \( \mathcal{G}^t \) is the restriction of the flow of the vector field \( V \) to \( F_B(\mathbb{T}^n) \). According to Lemma 6.1, \( F_B(\mathbb{T}^n) \) is a \( k \)-torus for some \( k \) (\( 0 \leq k \leq n \)), and \( \mathcal{G}^t \) is a quasi-periodic flow on \( F_B(\mathbb{T}^n) \). According to Lemma 1.1, the torus \( F_B(\mathbb{T}^n) \) contains \( 2^k \) fixed points of the involution \( G_B \). Since \( \text{Fix} G_B = \{ v^0 \} \), we arrive at the conclusion that \( k = 0 \) and \( F_B(\mathbb{T}^n) = \{ v^0 \} \). Now the invariance of the set \( F_B(\mathbb{T}^n) \) under the flow of \( V \) implies that \( V(v^0) = 0 \).

It remains to prove Lemma 6.1. Let \( \Lambda = F^{-1}(F(0)) \subset \mathbb{T}^n \). Then \( \Lambda \) is a closed subset of \( \mathbb{T}^n \). Our first goal is to verify that \( F(\phi^1) = F(\phi^2) \) if and only if \( \phi^1 - \phi^2 \in \Lambda \). Indeed, consider the sequence \( \{ \tau_j \}_{j \in \mathbb{N}} \) of real numbers such that \( \lim_{j \to \infty} \omega \tau_j = 0 \) on \( \mathbb{T}^n \). If \( F(\phi^1) = F(\phi^2) \), then

\[
F(\phi^1 - \phi^2) = \lim_{j \to \infty} F(\phi^1 - \omega \tau_j) = \lim_{j \to \infty} \mathcal{G}^{-\tau_j}(F(\phi^1)) = \lim_{j \to \infty} \mathcal{G}^{-\tau_j}(F(\phi^2)) = \lim_{j \to \infty} F(\phi^2 - \omega \tau_j) = F(0),
\]

so that \( \phi^1 - \phi^2 \in \Lambda \). On the other hand, if \( \phi^1 - \phi^2 \in \Lambda \), i.e., \( F(\phi^1) = F(\phi^2) \), then

\[
F(\phi^1) = \lim_{j \to \infty} F(\phi^1 - \phi^2 + \omega \tau_j) = \lim_{j \to \infty} \mathcal{G}^{\tau_j}(F(\phi^1 - \phi^2)) = \lim_{j \to \infty} \mathcal{G}^{\tau_j}(F(0)) = \lim_{j \to \infty} F(\omega \tau_j) = F(\phi^2).
\]

In particular, if \( \phi^1 \in \Lambda \) and \( \phi^2 \in \Lambda \), then \( F(\phi^1 + \phi^2) = F(\phi^1) = F(0) \), so that \( \phi^1 + \phi^2 \in \Lambda \), and \( F(\phi^1 - \phi^1) = F(0) \), so that \( -\phi^1 \in \Lambda \). Thus, \( \Lambda \) is a closed subgroup of \( \mathbb{T}^n \), and there exists a natural bijection \( \mathbb{T}^n/\Lambda \to F(\mathbb{T}^n) \).
Now one can apply the Pontryagin duality theorem for locally compact Abelian groups (also known as the Pontryagin–van Kampen duality theorem). For the consequences of this fundamental theorem that we need and for their particular case that concerns \( T^n \), the reader is referred to, e.g., [29, Proposition 38] and [54, Corollary 1.2.2, p. 706]. According to the Pontryagin duality theorem, closed subgroups of \( T^n \) are characterized by their annihilators in the character group \( \chi(T^n) \approx \mathbb{Z}^n \), i.e., in the group of all the continuous homomorphisms \( \chi : T^n \rightarrow \mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z} \):

\[
\chi(\phi_1, \ldots, \phi_n) = m_1 \phi_1 + \cdots + m_n \phi_n, \quad m_1, \ldots, m_n \in \mathbb{Z}
\]

(the groups \( T^n \) and \( \mathbb{Z}^n \) are dual to each other). In other words, there exists a subgroup \( L \subset \mathbb{Z}^n \) such that

\[
\Lambda = \{ \phi \in T^n \mid m_1 \phi_1 + \cdots + m_n \phi_n = 0 \ \forall (m_1, \ldots, m_n) \in L \}.
\]

On the other hand, there is a matrix \( Q \in \text{SL}(n, \mathbb{Z}) \) such that

\[
LQ = \{ (0, \ldots, 0, q_1 r_1, \ldots, q_k r_k) \mid r_1, \ldots, r_k \in \mathbb{Z} \},
\]

where \( q_1 \geq \cdots \geq q_k \geq 1 \) are certain natural numbers (here the elements of \( \mathbb{Z}^n \) are regarded as row vectors and \( 0 \leq k \leq n \)). The rank of the lattice \( L \) is equal to \( k \). Introduce the new coordinate frame \( \psi = Q^{-1} \phi \) on the torus \( T^n \) (here the points of \( T^n \) are regarded as column vectors). In the new coordinate frame

\[
\Lambda = \{ (\psi_1, \ldots, \psi_{n-k}, 2\pi p_1/q_1, \ldots, 2\pi p_k/q_k) \},
\]

where

\[
0 \leq p_1 \leq q_1 - 1, \ldots, 0 \leq p_k \leq q_k - 1; \quad \psi_1, \ldots, \psi_{n-k} \in \mathbb{S}^1, \quad p_1, \ldots, p_k \in \mathbb{Z}_+,
\]

and \( \dim \Lambda = n - k \). Moreover, the flow \( g^t \) on \( T^n \) in the new coordinate frame is determined by the equation

\[
\dot{\psi} = Q^{-1} \dot{\phi} = Q^{-1} \omega = \overline{\omega} = (\overline{\omega}_1, \ldots, \overline{\omega}_n).
\]

The set \( K = F(T^n) \) is homeomorphic to \( T^n/\Lambda \approx \mathbb{T}^k \). The natural coordinates on the factor \( T^n/\Lambda \) and, consequently, on the set \( F(T^n) \) are the coordinates

\[
(q_1 \psi_{n-k+1}, \ldots, q_k \psi_n) \in \mathbb{T}^k.
\]

The flow \( \mathcal{G}^t \) on the \( k \)-torus \( F(T^n) \) is quasi-periodic with the frequency vector

\[
(q_1 \overline{\omega}_{n-k+1}, \ldots, q_k \overline{\omega}_n) \in \mathbb{R}^k.
\]

This completes the proof of Lemma 6.1.

**Remark 6.1.** Lemma 6.1 is probably related to the theory of minimal isometric systems; cf. [53, Proposition 2.6.7].

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M. B. Sevryuk
V. L. Talroze Institute of Energy Problems of Chemical Physics
of the Russia Academy of Sciences, Moscow, Russia
E-mail: 2421584@mail.ru, sevryuk@mccme.ru