Handles, Hooks, and Scenarios: A fresh Look at the Collatz Conjecture

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Summary. An operational approach to the Collatz Conjecture is presented. Scenarios are defined as strings of characters "s" (for "spike") and "d" (for "down") which symbolize the Collatz operations \((3m + 1)/2\) and \(m/2\) in a Collatz Series connecting two odd integers, the startnumber and the endnumber. It is shown that a scenario determines uniquely four integers, called startperiod \(A_M\), startphase \(B_M\), endperiod \(A_N\), and endphase \(B_N\) such that startnumber \(M = A_M \cdot k - B_M\), and endnumber \(N = A_N \cdot k - B_N\), where \(k\) is any natural number. Therefore, any scenario can be realized infinitely many times, with well-defined startnumber and endnumber for each realization. It is shown how the periods and phases for a given scenario are calculated. The results are used to prove that any odd (even) number in a Collatz Series is less than 8 (7) ("up"- or "down"-) steps away from an odd integer which is divisible by 3. They are also used for the construction of Collatz Series which exhibit prescribed regular graphics patterns. Finally, the bearing of the present work on the question of non-trivial cycles is discussed.

1 Introduction.

The Collatz Conjecture has been widely discussed in the mathematical literature and numerous authors have contributed to the discussion. Interesting reviews of the problem have been given by Lagarias (1996) and by Wirsching (2000).

For this paper a three-fold goal has been set.

First, to present, within a newly developed concept, some new results which hopefully will help to deepen the understanding of the Collatz Conjecture and advance future research.

Second, to use key notions which appeal directly to the intuition of the non-mathematician, rather than to the specialist in number theory. Thus we will employ common language terms instead of scientific terms wherever there is no risk of loss of precision. E.g. instead of describing an integer n
by the condition $n \equiv 1 \pmod{2}$ we will write ‘$n$ is odd’. To a large extent, scientific thinking depends on intuition and not every brain has been trained to think in terms of abstract formalisms.

Third, to emphasize some facts about Collatz Series which have not been accorded sufficient attention in the previous literature.

The paper is organized as follows. Section 2 introduces the notions of scenarios, their periods, phases, and their realizations. Section 3 explains the notion of hooks and gives closed form expressions for their periods and phases. Section 4 lays the ground for the practical calculation of periods and phases for a given scenario. Sections 5.1 and 5.2 are concerned with an important aspect of Collatz Backward Series. Section 5.3 gives an example for the degree of control over Collatz Series which is reached with the results of this paper. With the exception of the outline of a proof for propositions 2.2 and 2.3 and of the proofs for propositions 3.1, 5.1, and 5.2, no proof for any of the other propositions is given. The proofs were omitted to keep the length of the paper within reasonable bounds. They may be published elsewhere.

To facilitate reading the text, the ends of definitions and propositions are marked by the symbol $\Box$.

Let us start by recalling or defining some of the basic terms and notions relevant for the discussion.

**Definition 1.1** The term Collatz Series designates the sequence of integers which is generated by recursive application of the Collatz Rules to a given positive integer $m$:

Determine $3m + 1$ if $m$ is odd, and determine $m/2$ if $m$ is even.

The operation $3m + 1$ will be denoted by ‘u’ (for ‘up’), and the operation $m/2$ will be denoted by ‘d’ (for ‘down’). $\Box$

The result of an ‘u’ operation, $3m + 1$, is always an even integer and will, therefore, undergo at least one operation ‘d’.

A few more remarks about Collatz-related jargon may be useful. We will not only have to deal with Collatz Series, but also with the series generated by the inverse of the Collatz operations, i.e. the operations $(m - 1)/3$ and $2m$. Such a series will be called a Collatz Backward Series or, shorter, a ‘CBS’. The ordinary Collatz Series will also be referred to as a ‘CFS’ (Collatz Forward Series).

Since the Collatz operations are involving the integers 2 and 3, divisibility by 2 or 3 will be an important property of the integers encountered. Common language does already describe the divisibility by 2 in assigning the properties even or odd to an integer. It will be useful as well to establish convenient expressions to describe divisibility by 3.
Definition 1.2  An integer $n$ will be called
RC0 if $n/3$ has remainder 0 ($\iff n \equiv 0 \pmod{3}$),
RC1 if $n/3$ has remainder 1 ($\iff n \equiv 1 \pmod{3}$),
RC2 if $n/3$ has remainder 2 ($\iff n \equiv 2 \pmod{3}$).
RC stands for the mathematical term Residue Class.  □

E.g. we will say ‘6 is RC0’, or ‘7 is RC1’, or ‘8 is RC2’.

Here are some simple algebraic properties of these classes which we will need.
(a) $2n$ is RC0 iff $n$ is RC0, $2n$ is RC1 iff $n$ is RC2, $2n$ is RC2 iff $n$ is RC1.
(b) RC0 + RC1 = RC1, RC0 - RC1 = RC2.

With this in mind we can now say that an ‘up’ operation will always result in an even integer which is RC1. The ensuing ‘down’ operation will then yield an integer which is RC2. Further division by 2, if it is possible, will result in an integer which is RC1, and so on.

For this reason an integer which is RC0 cannot occur in a Collatz Series once an ‘up’ operation has been made. An odd RC0 integer can occur at most once in a Collatz Series.

Odd RC0 integers are thus singled out among the other integers in a Collatz Series.

Also, since an odd RC0 integer can only be reached (from even RC0 integers) by ‘down’ operations, the backward extension of a Collatz Series from there is unique. This means that an odd RC0 integer determines a Collatz Series uniquely, both in forward and in backward direction.

We have to recall here that starting with an RC1 or RC2 integer, the Collatz Backward Series (CBS) is not unique, i.e. for each such integer there is a backward tree with an infinite number of branches. However, there is only one way to go backwards from a RC0 integer, namely by repeated doubling.

In view of the above it is appropriate to give the odd RC0 integers a proper name.

Definition 1.3  An integer which is odd and RC0 (i.e. divisible by 3 with remainder 0) will be called a handle.  □

The term handle was chosen since both the Collatz forward series (CFS) and the Collatz backward series (CBS) are determined uniquely by such an integer. They are essential for the notion of the Complete Collatz Series. It comes down from infinity through divisions by 2, with RC0 integers of the form $2^n H$ ($H = 3M$, $M$ an arbitrary odd integer), until it reaches the handle $H$. From thereon the series is made up of integers RC1 or RC2.

We shall show later (Section 5.1) that every odd integer can be reached in less than 8 steps by a Collatz Series starting from a handle.
2 Scenarios and their Realization.

Since an operation ‘u’ is always followed by an operation ‘d’, the combined operation ‘ud’ will be denoted by ‘s’ (for spike, the graphical appearance of the operation).

It is clear that any Collatz Series can be described in terms of ‘spikes’ and ‘downs’.

**Definition 2.1** A sequence of Collatz operations connecting 2 odd integers will be called a *scenario*.

A scenario is thus described by a word, beginning with ‘s’ and followed by any combination of the characters ‘s’ or ‘d’.

**Definition 2.2** A series of integers which is described by a specified scenario will be called a *realization of the scenario*.

As any scenario starts with an ‘s’, the realization of a scenario starts with an odd integer.

A scenario literally spells out what happens between a startnumber $M$ and an endnumber $N$.

For a scenario with $\sigma$ operations ‘s’ and $\delta$ operations ‘d’ the general form is

\[ sd^{\delta_1} sd^{\delta_2} ... sd^{\delta_\alpha} ... sd^{\delta_{\sigma}} \]

where $\delta_1 + \delta_2 + ... + \delta_{\sigma} = \delta$. The exponents $\delta_\alpha$ ($\alpha = 1, 2, ... \sigma$) are describing the multiplicity of the respective operation ‘d’.

We are now arriving at a crucial point in our discussion. The simplest, almost trivial question about Collatz Series is to ask, ‘Given a startnumber, what is the series?’ This question is, of course, the same as asking ‘Given a startnumber, what is the scenario attached to it?’ Many authors of articles about the Collatz Conjecture have posed and answered this question.

Here we will turn it around and ask the simple, but non-trivial question

‘Given a scenario, what are the startnumbers?’

Our task will be to find the startnumbers which are compatible with a given scenario. In fact, we want to determine all realizations of a scenario.

**Proposition 2.1 (Periodicity of a Realization)** A scenario $S$ uniquely determines

- a pair of integers $A_M$ and $B_M$, with $A_M$ even, $B_M$ odd,
- a pair of integers $A_N$ and $B_N$, with $A_N$ even, $B_N$ odd,

such that for any $k = 1, 2, 3, ...$

$M_k = A_M \cdot k - B_M$ is the startnumber, and

$N_k = A_N \cdot k - B_N$ is the endnumber.
of the Collatz sequence described by the scenario. □

**Corollary 2.1** The RC-property of the startnumber \( M_k \) and the endnumber \( N_k \) is not changed under the replacement \( k \to k + 3p \) where \( p \) is any (positive) integer. □

**Definition 2.3** The Collatz Series starting with \( M_k \) and ending with \( N_k \) will be referred to as the \( k \)th realization of the scenario. The integer \( A_M (A_N) \) will be called the startperiod (endperiod). The integer \( B_M (B_N) \) will be called the startphase (endphase). □

**Proposition 2.2 (Startperiod)** Let \( \sigma \) and \( \delta \) denote, respectively, the numbers of spikes ('s') and downs ('d') in a scenario. The startperiod depends on the total number of operations 's' and 'd' and is given by
\[
A_M = 2 \cdot 2^{\sigma + \delta}.
\]

**Proposition 2.3 (Endperiod)** The endperiod depends only on the number of spikes in the scenario and is given by
\[
A_N = 2 \cdot 3^{\sigma}.
\]

**Outline of a proof.** Consider the simplest and most basic scenario which is just 's'. Apply the operation \( (3M+1)/2 \) to an unspecified odd startnumber \( M = 2k−1 \), where \( k \) may be any positive integer. The result is \( 3k−1 \) which is even or odd depending on \( k \). To get an odd endnumber, re-scale \( k \) to \( 2k \). This yields \( N_k = 6k−1 \). To get the correct values of the startnumber, \( M \) has to be re-scaled accordingly and then becomes \( M_k = 4k−1 \). To verify that these expressions are realizations of the scenario 's' for all integers \( k \), just apply the Collatz rules to \( M_k \). □

By similar reasoning one can derive the realizations for the next simplest scenarios, with \( k = 1, 2, 3,..., \)
- 'sd' : Startnumbers are \( M_k = 8k−7 \), endnumbers are \( N_k = 6k−5 \),
- 'ss' : Startnumbers are \( M_k = 8k−1 \), endnumbers are \( N_k = 18k−1 \).

Again, these expressions may be verified by applying the Collatz rules to the startnumbers.

Unlike the periods, the phases are not given by simple expressions. The reason is that they depend on other properties of a scenario, such as the order in which the 's' and 'd' appear. However, the following can be said.

**Proposition 2.4** Both startphase and endphase are odd integers. The startphases satisfy the condition \( 0 < B_M < A_M \). The endphases satisfy the conditions \( 0 < B_N < A_N \) and, in addition, they are RC1 or RC2 (and not RC0). □

Further remarks about endphases will be found in section 4. These con-
ditions are sufficient to assure that, in all realizations of the scenario, start-
numbers and endnumbers are odd integers and, in addition, that the end-
numbers are not divisible by 3.

3 Hooks.

We will now take a look at the building blocks of any Collatz Series.

**Definition 3.1** A $\delta$-hook is a scenario of the type $sd^\delta$. The exponential
notation indicates that the ‘down’ operation is performed $\delta$ times. □

Hooks got their name from the way they appear in graphics. Since all
scenarios are sequences of hooks hooked up to other hooks, this notion is
basic for the discussion of the Collatz conjecture.

Period and phase for startnumber and endnumber of a hook are rela-
tively easy to calculate. It turns out that the periods don’t, but the phases
do depend on the parity of $\delta$.

**Proposition 3.1** For a $\delta$-hook the following holds.

*Periods:* Startperiod: $A_M = 2^\delta+2$, Endperiod: $A_N = 6$, in accordance with propositions 2.2 and 2.3.

*Phases:*  
$\delta$ even: Startphase: $B_M = (2^\delta+1 + 1)/3$, Endphase: $B_N = 1$,
$\delta$ odd: Startphase: $B_M = (5 \cdot 2^\delta+1 + 1)/3$, Endphase: $B_N = 5$. □

Proof. The startnumbers $M_k$ and endnumbers $N_k$ of a $\delta$-hook are given by
$\delta$ even: $M_k = 2^\delta+2 \cdot k - (2^\delta+1 + 1)/3$, $N_k = 6k - 1$
$\delta$ odd: $M_k = 2^\delta+2 \cdot k - (5 \cdot 2^\delta+1 + 1)/3$, $N_k = 6k - 5$

with $k = 1, 2, 3, ...$

Using these expressions we obtain by straightforward calculation
$\delta$ even: $(3M_k + 1)/2 = 2^\delta(6k - 1)$,
$\delta$ odd: $(3M_k + 1)/2 = 2^\delta(6k - 5)$. □

It should be noted that the endphase of a hook with $\delta$ even (odd) is RC1
(RC2). Therefore, since the endperiod of a hook is always 6, the endnumber
of a hook with $\delta$ even (odd) is RC2 (RC1).

In view of the relevance of the parity of $\delta$ we shall call a hook with $\delta$ even
(odd) an even (odd) hook. Since the endnumber of any scenario and the last
hook in it have the same RC-property, we have

**Proposition 3.2** The endnumber of a scenario is RC2 (RC1) if the last
hook is even (odd). □
This simple fact is also evident from basics since the last (in fact, any) ‘s’ operation in a scenario sets the RC-property of the integer reached to 2.

Values for the periods and phases of the first 16 hooks are given in the table below.

| δ  | Startperiod | Startphase | Endperiod | Endphase |
|----|-------------|------------|-----------|----------|
| 0  | 4           | 1          | 6         | 1        |
| 1  | 8           | 7          | 6         | 5        |
| 2  | 16          | 3          | 6         | 1        |
| 3  | 32          | 27         | 6         | 5        |
| 4  | 64          | 11         | 6         | 1        |
| 5  | 128         | 107        | 6         | 5        |
| 6  | 256         | 43         | 6         | 1        |
| 7  | 512         | 427        | 6         | 5        |
| 8  | 1.024       | 171        | 6         | 1        |
| 9  | 2.048       | 1.707      | 6         | 5        |
| 10 | 4.096       | 683        | 6         | 1        |
| 11 | 8.192       | 6.827      | 6         | 5        |
| 12 | 16.384      | 2.731      | 6         | 1        |
| 13 | 32.768      | 27.307     | 6         | 5        |
| 14 | 65.536      | 10.923     | 6         | 1        |
| 15 | 131.072     | 109.227    | 6         | 5        |

The first 16 hooks with their periods and phases.

Figure 1 shows the graphics for various combinations of δ-hooks with δ = 0, 1, 2. Each hook is pictured as the straight line connecting the start-number with the endnumber. Since the intermediate even integers are not shown, the ‘hook appearance’ gets lost. Each column shows a number of realizations for the type of hook indicated below the horizontal axis. You have to imagine that each of these columns reaches upwards towards infinity! The number of realizations shown in the Figure is only limited by the space available on paper.

The graphics shows how different hooks will connect to form new scenarios.
1. The scenario ‘ssd’ which is made up of hooks from the first two columns, is shown with its first 10 realizations. This scenario has startnumbers $M_k = 16k - 5$ and endnumbers $N_k = 18k - 5$.
2. The scenario ‘sdsdd’ which is made up of hooks from the second and third column, is shown with its first 4 realizations. This scenario has startnumbers $M_k = 64k - 47$ and endnumbers $N_k = 18k - 13$.
3. The scenario ‘sdds’ which is made up of hooks from the third and fourth
column, is shown with its first 5 realizations. This scenario has startnumbers $M_k = 32k - 3$ and endnumbers $N_k = 18k - 1$.

4. The scenario ‘ssdd’ which is made up of hooks from the fourth and fifth column, is shown with its first 2 realizations. This scenario has startnumbers $M_k = 32k - 13$ and endnumbers $N_k = 18k - 7$.

5. The scenario ‘ssdsdd’ which is made up of hooks from the first three columns, is shown with its first 2 realizations. This scenario has startnumbers $M_k = 128k - 117$ and endnumbers $N_k = 54k - 49$.

6. The scenario ‘dsdsds’ which is made up of hooks from columns 2 through 4, is shown with its first 2 realizations. This scenario has startnumbers $M_k = 128k - 47$ and endnumbers $N_k = 54k - 19$.

7. The scenario ‘ssdsdds’ which is made up of hooks from columns 1 through 4, is shown with its first 2 realizations. This scenario has startnumbers $M_k = 256k - 117$ and endnumbers $N_k = 162k - 73$.

The trivial cycle is represented by the hook ‘sd’ in first realization. It is shown by the horizontal line at the bottom of the second column. Now, if there would be a non-trivial cycle it would be shown in a similar diagram, with many more hooks of various kind arranged in various order, and it would have a startnumber to the left at the same height as the endnumber to the right.

4 Calculation of Periods and Phases by Iteration.

Here we will describe how the periods and phases are changed when one operation is appended to a scenario.

Appending an ‘s’ is easy because this operation is acting directly on the odd endnumber of the given scenario.

Adding a ‘d’ is a bit more tricky since this operation does not act on the endnumber of the given scenario.

But in any case we start from a scenario $S$ with realizations

$$M_k = A_M \cdot k - B_M$$
$$N_k = A_N \cdot k - B_N$$

where $k = 1, 2, 3, ...$

The enlarged scenario $S'$ has realizations with startnumbers $M'_k = A'_M \cdot k - B'_M$ and endnumbers $N'_k = A'_N \cdot k - B'_N$, where $k = 1, 2, 3, ...$

Proposition 4.1 (Appending an ‘s’) Let $S' = Ss$ be the scenario with one operation ‘s’ added at the end of $S$.

The periods and phases of $S'$ are expressed by those of $S$ as follows.

When

$$\frac{1}{3}(3B_N - 1) \text{ odd:}$$

$$A'_M = 2A_M, B'_M = B_M$$
\[ A'_N = 3A_N, \quad B'_N = \frac{1}{2}(3B_N - 1). \]

\( \frac{1}{2}(3B_N - 1) \) even:
\[ A'_M = 2A_M, \quad B'_M = B_M + A_M, \]
\[ A'_N = 3A_N, \quad B'_N = \frac{1}{2}(3B_N + 3A_N - 1). \]

It is worth noting that in the case where \( \frac{1}{2}(3B_N - 1) \) is odd, the startphase remains unchanged. Simple reasoning shows that this is true for any \( B_N \) which can be written as \( B_N = 4j - 3 \), with any \( j \) that is RC1 or RC2 (not RC0!).

Likewise, in the case where \( \frac{1}{2}(3B_N - 1) \) is even, the new startphase is obtained by simply adding the startperiod to the startphase of the given scenario. The condition is satisfied by any \( B_N \) which can be written as \( B_N = 4j - 1 \), with any \( j \) that is RC0 or RC2 (not RC1!).

The restrictions on \( j \) are necessary to ensure that \( B_N \), and by implication \( N_k = A_Nk - B_N \), is not a handle.

**Proposition 4.2 (Appending a ‘d’)** Let \( S' = Sd \) be the scenario with one operation ‘d’ added at the end of \( S \).

The periods and phases of \( S' \) are expressed by those of \( S \) as follows.

\( \frac{1}{2}(B_N + \frac{1}{2}A_N) \) odd:
\[ A'_M = 2A_M, \quad B'_M = B_M + \frac{1}{2}A_M, \]
\[ A'_N = A_N, \quad B'_N = \frac{1}{2}(B_N + \frac{1}{2}A_N). \]

\( \frac{1}{2}(B_N + \frac{1}{2}A_N) \) even:
\[ A'_M = 2A_M, \quad B'_M = [B_M + 3A_M/2] \mod A_M, \]
\[ A'_N = A_N, \quad B'_N = [\frac{1}{2}(B_N + 3A_N/2)] \mod A_N. \]

The above formulas may be used (and have been used by the author) to write a program which calculates all periods and phases as well as the startnumber and endnumber for the \( k \)th realization of any given scenario, see next section.

## 5 Applications and Examples.

Given that handles are playing such a unique role in Collatz Series it is interesting to ask how far they are away from ordinary integers, i.e. integers which are even or odd, RC1 or RC2. By ‘how far away’ we mean the number of steps (up or down) by which the integer can be reached via a Collatz Series.

For reasons which will become clear in the following it turns out to be ad-
vantageous to consider separately odd and even target integers.

5.1 Hooking up odd Integers to Handles.

Proposition 5.1 Any odd integer is located on a Collatz Series within less than 8 steps from a handle. (One step is either an ‘up’ or a ‘down’ operation.) □

Proof. If the integer in question is odd and RC0, nothing has to be shown because it is itself a handle. We will show that hooks with $\delta < 6$ can provide the link between handles and any odd (none RC0) integer. These hooks are listed in the first column of the table below, followed by the number of (up or down) steps in the second column. In the fifth column we have entered the smallest values of $k$ for which the corresponding startnumber $M_k$ is a handle. These startnumbers are listed in the sixth column (headed ‘$M$’). The last column shows the corresponding endnumbers.

| Hook | Steps | Startnumber $M_k$ | Endnumber $N_k$ |
|------|-------|------------------|-----------------|
| $s$  | 2     | $4k - 1$         | $6k - 1$        |
| $sd$ | 3     | $8k - 7$         | $6k - 5$        |
| $sd^2$ | 4     | $16k - 3$       | $6k - 1$        |
| $sd^3$ | 5     | $32k - 27$     | $6k - 5$        |
| $sd^4$ | 6     | $64k - 11$     | $6k - 1$        |
| $sd^5$ | 7     | $128k - 107$     | $6k - 5$        |

The endnumbers (last column) comprise all odd non-RC0 integers which are in the interval (0, 18). It can be verified by direct calculation that for each scenario the target is reached.

Now replace $k$ (in col. 5, table above) by $k + 3p$ where $p$ is any positive integer. The startnumbers will become larger, but they will remain handles. The endnumbers will just be increased by $18p$.

For $p = 0$ the $k$-values, startnumbers, and endnumbers are shown in the table above (col. 5, 6, 7).

For $p = 1$ the resulting $k$-values, startnumbers, and endnumbers are shown in the table below (columns 5, 6, 7) and for $p = 2$ they are shown in columns 8, 9, 10.

| Hook | $k + 3p$ | $M_p$ | $N_p$ | $k$ | $M$ | $N$ | $k$ | $M$ | $N$ |
|------|---------|-------|-------|-----|-----|-----|-----|-----|-----|
| $s$  | $1 + 3p$ | $3 + 12p$ | $5 + 18p$ | 4  | 15  | 23 | 7  | 27 | 41  |
| $sd$ | $2 + 3p$ | $9 + 24p$ | $7 + 18p$ | 5  | 33  | 25 | 8  | 57 | 43  |
| $sd^2$ | $3 + 3p$ | $45 + 48p$ | $17 + 18p$ | 6  | 93  | 35 | 9  | 141 | 53 |
| $sd^3$ | $3 + 3p$ | $69 + 96p$ | $13 + 18p$ | 6  | 165 | 31 | 9  | 261 | 49 |
| $sd^4$ | $2 + 3p$ | $117 + 192p$ | $11 + 18p$ | 5  | 309 | 29 | 8  | 501 | 47 |
| $sd^5$ | $1 + 3p$ | $21 + 384p$ | $1 + 18p$ | 4  | 405 | 19 | 7  | 789 | 37 |
If we let \( p \) run through all natural numbers 0, 1, 2, 3, ..., all odd non-RC0 integers will occur as endnumbers in one of the hooks which begins with a handle.

The result does not mean, that a hook always provides the shortest way to connect an odd integer to a handle. For instance, the integer 19 is reached, via a hook, from the handle 405 in 7 steps as shown in the table above. But it is reached in 6 steps from the handle 33 (via the scenario sdsd).

### 5.2 Hooking up even Integers to Handles.

Proposition 5.1 applies to all odd integers which are not handles, i.e. the integers beginning 1, 5, 7, 11, 13, 17, 19, 23, 25, 29, 31, 35, 37, 41, 43, 47, 49, 53, etc. Naturally, the question arises if a similar result holds for even integers.

**Proposition 5.2** Any even integer (not RC0) is located on a Collatz Series within less than 7 steps from a handle. (One step is either an ‘up’ or a ‘down’ operation.) □

**Proof.** First of all it is clear that the doubles of the odd integers above will be found within less than 7 steps from a handle. These are the integers 2, 10, 14, 22, 26, 34, 38, 46, 50, 58, 62, 70, 74, 82, 86, 94, 98, 106, etc.

But how about the even integers in between, beginning with 4, 8, 16, 20, 28, 32, 40, 44, 52, 56, 64, 68, 76, 80, 88, 92, 100, 104, etc.? Note that the above series is made up of the 2 sub-series \( 12j − 8 \) and \( 12j − 4 \) where \( j = 1, 2, 3, \ldots \).

It is easily verified that the integers in the first sub-series are hooked up to handles in the following way (with \( p \) taking the values 1, 2, 3, ...).

For \( j = 3p \) the endnumber \( 36p − 8 \) is obtained from the handle \( 12p − 3 \) in one step (‘up’).

For \( j = 3p − 1 \) the endnumber \( 36p − 20 \) is obtained from the handle \( 48p − 27 \) in three steps (by the 1-hook ‘sd’).

For \( j = 3p − 2 \) the endnumber \( 36p − 32 \) is obtained from the handle \( 192p − 171 \) in five steps (by the 3-hook ‘sddd’).

Similarly, the integers in the second sub-series are hooked up to handles as follows.

For \( j = 3p \) the endnumber \( 36p − 4 \) is obtained from the handle \( 24p − 3 \) in two steps (by the 0-hook ‘s’).

For \( j = 3p − 2 \) the endnumber \( 36p − 28 \) is obtained from the handle \( 96p − 75 \) in four steps (by the 2-hook ‘sdd’).

For \( j = 3p − 1 \) the endnumber \( 36p − 16 \) is obtained from the handle \( 384j − 171 \) in six steps (by the 4-hook ‘sdddd’). ■

What is the significance of the result?

It shows that handles are lurking around the corner near all integers of a Collatz Series. And, in the unlikely case that a non-trivial cycle should exist, it would be surrounded by a *halo* of close-by handles. People who have dealt
with Collatz Backward Series (CBS) may have noticed that non-trivial CBS (i.e. those which are not obtained by merely doubling RC1 or RC2 integers) will sooner or later hit a handle. And, recalling that one in 3 integers is a handle, one realizes how difficult it would be for the CBS on a cycle to miss all the handles that are in its way.

But, what really counts here is the way of treating the problem: The periodicity of scenarios is used to transport some property about Collatz Series from one finite interval (the period of a scenario) to all intervals, i.e. to infinity. The method is like a ‘mathematical telescope’ allowing to make statements about numbers which are so large that there would not be enough available paper on earth to write them down, not even in fine print!

5.3 Collatz Series Sections without Hailstones.

Some of the publications on the Collatz conjecture have invoked the notion of hailstones to describe the seemingly random order in which integers are appearing in a Collatz Series.

To exemplify the benefits which can be drawn from propositions 4.1 and 4.2 we want to produce a startnumber which leads to the section of a Collatz Series that has a perfectly regular pattern. For example, let us look at the scenario $\mathcal{S} = (s^7d^4)^9$. The exponents indicate the multiplicity of the respective operation. So, in this scenario we have in total $7 \cdot 9 = 63$ operations ‘s’ and $4 \cdot 9 = 36$ operations ‘d’. In terms of the traditional up- and down operations these figures amount to 162 steps.

Beginning with the scenario ‘s’, with startnumbers $M_k = 4k - 1$ and end-numbers $N_k = 6k - 1$, we apply the results of the previous section 4 and perform the 98 remaining iterations to get the periods and phases of our scenario. The result is the following.

$$
\begin{align*}
\text{Startperiod} & = 1267650600228229401496703205376 \\
\text{Startphase} & = 1039655887956965120651972413057 \\
\text{Endperiod} & = 228912254686167498971899392854 \\
\text{Endphase} & = 187740985577201070748176480485.
\end{align*}
$$

Each of these 4 integers has the order $10^{30}$.

The first 3 realizations of the scenario are as follows.

1$^{st}$ Realization:

$$
\begin{align*}
\text{Startnumber} & = 227994712271264280844730792319, \\
\text{Endnumber} & = 411712688284473919023722912369.
\end{align*}
$$

Each of these 2 integers is of the order $10^{29}$.

2$^{nd}$ Realization:

$$
\begin{align*}
\text{Startnumber} & = 1495645312499493682341433997695, \\
\text{Endnumber} & = 2700835235146148908795622305223.
\end{align*}
$$

3$^{rd}$ Realization:
Startnumber = 276329591272723083838137203071,
Endnumber = 4989957782007823898567521698077.
Each of these 4 integers is of the order $10^{30}$.

Using a suitable software for handling large integers and the iteration formulas of section 4, all of the above integers were calculated in just the fraction of a second.

**All-Number graphics and Odd-Number graphics.** When it comes to the graphical representation of Collatz Series, we have to make the choice between two options. We may plot either all integers which are generated by up- or down operations, creating a AN graphics (AN stands for ‘All Numbers’), or we may plot only the odd integers which occur in a CFS, creating a ON graphics (ON stands for ‘Odd Numbers’). The latter has the clear advantage of displaying much less jitter than we would see in the AN graphics.

Figure 2 pictures, as ON-graphics, the first 3 realizations of the scenario. The curves show a nearly horizontal and slightly increasing regular zigzag pattern with the endnumbers less than twice the corresponding startnumbers. Once the scenario is terminated, the habitual, chaotic looking shape of the series is back. Figure 3 shows, also as ON-graphics, the section of the Collatz Series presenting the first 3 realizations of the scenario. In Figure 4, the scenario sections of the Collatz Series are shown in full detail, i.e. with all numbers pictured (AN-graphics).

Please note also the difference in counting steps in ON-graphics and AN-graphics. In ON-graphics (Figures 2, 3) we just count the odd integers (whose number equals the number of ‘s’ operations in any section of a Collatz Series). In AN-graphics (Figure 4) we count all integers, i.e. for a scenario with $\sigma$ ‘s’-operations and $\delta$ ‘d’-operations, the number of steps counted is $2\sigma + \delta$.

The result shows that the integers appearing in a Collatz Series are not always falling like ‘hailstones’. Rather they are what we want them to be, within the limits of what is allowed by the Collatz rules. We could construct Collatz Series with much longer scenarios than the one considered here, expressing any regular pattern. Most Collatz Series are looking like random patterns because the startnumbers were chosen at random!

In the above example, the scenario was derived from the requirement that, in first realization, the endnumber $N$ of the subscenario $s^7d^4$ should be close to the startnumber $M$, i.e. satisfy the condition $\rho = |(M - N)/N| \ll 1$.

For our scenario, $(s^7d^4)^9$, we have $\rho \approx 0.45$, but for the subscenario $s^7d^4$ it holds $\rho \approx 0.064$. 

13
6 Epilogue and Outlook.

Epilogue. After the present work on scenarios had been done, G. J. Wirsching pointed out that it might be related to that of Riho Terras. The paper of Terras appeared 1976 in Acta Arithmetica and can be found via a link at the web site [http://matwbn.icm.edu.pl/]. Terras work (which is not quite easy to understand by somebody who is not specialized in mathematics) is indeed related to the present work. In the following, that relation will be described, employing the terminology developed in this paper. The encoding vector defined by Terras is closely related, but logically not the same as the scenario we defined here. Both notions have in common that they characterize a sequence of Collatz operations. If you consider a scenario and replace in it each operation ‘s’ by ‘1’ and each operation ‘d’ by ‘0’, you get the corresponding encoding vector. (We remark in passing that the encoding vector is just a sequence of Boolean numbers, but does not really qualify as a vector since this term has a specific meaning in geometry). The difference of the two notions lies in the situations they cover. The scenario is defined such that the corresponding Collatz sequence will start and end with an odd integer. In contrast, the encoding vector corresponds to a Collatz sequence which may end in an even or an odd integer. For this reason, Terras finds for the period of his encoding representation (which is his startnumber) half the value of that which we have determined for the scenario. In other words, there are twice as many (numerical) realizations for an encoding vector than for a scenario. Furthermore, since the endnumbers may be even, the Terras encoding vectors cannot always be connected with each other. In contrast, scenarios can always be connected since they always end with an odd integer. Nevertheless, the result of Terras (his Theorem 1.2) which he termed himself a remarkable periodicity phenomenon stands out as a singular achievement.

Outlook. Where should future work go from here?
The first important goal should be to disprove the existence of non-trivial cycles. The trivial cycle (the first realization of the 1-hook ‘sd’, see Figure 1) has only one odd integer in it. If a non-trivial cycle exists it would have a great number (at least about $10^7$) of odd integers in it. Every odd integer in a cycle could be taken as the start- and endnumber of a scenario. If the number of odd integers is $\sigma$, there would be $\sigma$ scenarios to take into consideration. All these scenarios would have the same startperiod and the same endperiod. Their startphases would be different from each other and so would be their endphases. There would be $\sigma$ equations to be satisfied, stating for each scenario the condition that startnumber and endnumber are the same. If these equations would be incompatible, the question of non-trivial cycles would be settled.
Another question of interest for the Collatz problem would be the composition (i.e. the concatenation) of scenarios. How to determine the phases of the scenario \( S = S_1S_2 \) in terms of properties of the scenarios \( S_1 \) and \( S_2 \)? As to the *periods* there is no problem, since from propositions 2.2 and 2.3 we simply find the startperiod \( A_M = \tfrac{1}{2}A_M^1 \cdot A_M^2 \) and the endperiod \( A_N = \tfrac{1}{2}A_N^1 \cdot A_N^2 \). However, a much harder nut to crack would be to express the *phases* of \( S \) by the periods and phases of \( S_1 \) and \( S_2 \).

As a first step one should try to have a look at hooks. Since hooks are the building blocks of scenarios it would be useful to have rules for their composition. First calculations on the composition of a \( j \)-hook and a \( k \)-hook indicate that the result will depend, among other factors, on the RC-property of \( k \).

Still another interesting question would be about flat CBS. These are obtained by performing, each time an odd RC1 or RC2 integer is reached, only the minimal number of doublings necessary to get an even RC1 integer. That minimal number of doublings would be equal to one if the odd integer is RC2 and it would be equal two if the odd integer is RC1. Consider one fixed integer and the set of CBS going back from there. Among all CBS emanating from that integer, the flat CBS and the hooks are the two opposite extremes. While hooks represent the steepest increase in a CBS, flat CBS are the other extreme. Can an upper limit be posed on the number of steps necessary to reach a handle via a flat CBS?

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