ON SOME DIMENSIONAL PROPERTIES OF 4-MANIFOLDS

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Abstract. It is shown, under the assumption of Jensen’s principle ♦, that if for a complex $L$ with $[L] \geq [S^4]$ there exists a metrizable compactum whose extension dimension is $L$, then there exists a differentiable, countably compact, perfectly normal and hereditarily separable 4-manifold whose extension dimension is also $[L]$.

1. Introduction

It was shown in [12] that under the assumption of Jensen’s principle ♦ there exists a differentiable $n$-manifold $M^n_m$, $n \geq 4$, of any given Lebesgue dimension $m$ where $m > n$. This manifold is countably compact, perfectly normal and hereditarily separable. Under the same set-theoretical assumption ♦ for any countable ordinal number $\alpha > 4$ there exists [13] a 4-manifold $M^4_\alpha$ with $\text{Ind} M^4_\alpha = \alpha$. [13] also contains examples of: (a) weakly infinite-dimensional 4-manifolds without the large inductive dimension and (b) strongly infinite-dimensional 4-manifolds. Recently it was shown [5] that for a given countable complex $L$, with $[L] \geq [S^4]$ and which serves as the extension dimension of a metrizable compactum, there exists a differentiable 4-manifold $M = M^{4,L}$ with e-dim $M = [L]$. It should be emphasized that it is still unknown whether the extension dimension of a metrizable compactum is realized by a countable complex. Below we construct a differentiable 4-manifold with similar properties for any, not necessarily countable, complex.

2. Preliminaries

We recall that a subset $U \subseteq X$ of a space $X$ is functionally open in $X$ if there is a continuous map $\varphi : X \to \mathbb{R}$ such that $U = \varphi^{-1}(\mathbb{R} - \{0\})$. Also, we say that $X$ is at most $n$-dimensional (and write $\text{dim} X \leq n$)

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if every finite functionally open cover \( U \) of \( X \) has a finite functionally open refinement \( V \) of order \( \leq n + 1 \). The latter means that \( \cap V' = \emptyset \) for any family \( V' \subseteq U \) consisting of at least \( n + 2 \) elements.

For normal spaces this definition is equivalent to the usual definition of Lebesgue dimension. The next statement is well known (see, [19], for example).

**Proposition 2.1.** For every space \( X \) we have \( \dim X = \dim \beta X \).

We assume that reader is familiar with notions of a CW-complex, a simplicial complex with the metric topology and an absolute neighborhood retract in the category \( \mathcal{M} \) of metrizable spaces (\( ANR(\mathcal{M})\)-space) (see, for instance, [14]). In what follows, by a simplicial complex we mean any simplicial complex with the metric topology. Let us note here that all simplices are assumed to be closed which implies that every finite simplicial complex is compact. By an ANR we mean an \( ANR(\mathcal{M})\)-space.

**Theorem 2.2.** [14, Theorem 3.3.10] Every simplicial complex is an ANR.

The next statement, which is a corollary of [14, Theorem 5.2.1], allows us to consider only simplicial complexes.

**Theorem 2.3.** Every CW-complex has a homotopy type of a simplicial complex.

**Definition 2.4.** Following [3] we say that a space \( Z \) is an absolute extensor of a normal space \( X \) and write \( Z \in AE(X) \) if for each closed subspace \( Y \) of \( X \) any map \( f : Y \to Z \) has an extension \( \bar{f} : X \to Z \).

The next statement is an immediate corollary of the above definition.

**Proposition 2.5.** If \( Z \in AE(X) \), \( X \) is a normal space and \( Y \) is a closed in \( X \), then \( Z \in AE(Y) \).

**Definition 2.6.** Let \( X \) and \( Z \) be normal spaces. Recall that \( Z \) is an absolute neighborhood extensor of a space \( X \) (notation: \( Z \in ANE(X) \)) if for every closed subspace \( Y \subseteq X \) any map \( f : Y \to Z \) has an extension \( \bar{f} : U \to Z \) where \( U \) is a neighborhood of \( Y \) in \( X \).

**Proposition 2.7.** Let \( X \) be a normal countably compact space and let \( L \) be a simplicial complex. Then \( L \in ANE(X) \).

**Proof.** Let \( f : Y \to L \) be a map of a closed subset \( Y \subseteq X \). Since \( L \) is metrizable, \( f(Y) \) is compact. Hence \( f(Y) \) is contained in some finite subcomplex \( K \subseteq L \). But every finite complex is an ANE for any normal space. Thus, there is an extension \( \bar{f} : U \to K \) of \( f \) defined on an open neighborhood \( U \) of \( Y \) in \( X \). \( \square \)
Proposition 2.8. The following conditions are equivalent for every countably compact normal space \( X \) and every simplicial complex \( L \):

(1) \( L \in AE(X) \).

(2) \( L \in AE(\beta X) \).

Proof. (1) \( \implies \) (2). By Definition 2.4, we need to check that for every closed set \( Y \subseteq \beta X \) any map \( f : Y \to L \) has an extension \( \bar{f} : \beta X \to L \). By Proposition 2.7, there is an extension \( f_1 : U \to L \), where \( U \) is a neighborhood of \( Y \) in \( \beta X \). Let \( U_1 \) be a smaller neighborhood of \( Y \) in \( \beta X \) such that \( \text{cl}_{\beta X} U_1 \subseteq U \). Set \( F = X \cap \text{cl}_{\beta X} U_1 \) and let \( f_2 = f_1|F \).

By condition (1), there is an extension \( \bar{f}_2 : X \to L \). As in the proof of Proposition 2.7, \( \bar{f}_2(X) \) is contained in some finite complex \( K \subseteq L \). But as was noted above \( K \) is compact. Hence the map \( f_2 \) can be extended to a map \( \bar{f} : \beta X \to K \subseteq L \). It remains to show that \( \bar{f}|Y = f \). But \( \bar{f}|F = f_1 \). Hence, since \( F \) is dense in \( \text{cl}_{\beta X} U_1 \), we have \( \bar{f}|\text{cl}_{\beta X} U_1 = f_1 \).

On the other hand, \( f_1|Y = f \).

(2) \( \implies \) (1). Let \( Y \) be a closed subset of \( X \) and let \( f : Y \to L \) be a map. Set \( F = \text{cl}_{\beta X} Y \). Since \( Y \) is closed in a normal space \( X \), \( F = \beta Y \). Then \( f \) can be extended to a map \( f_1 : F \to L \) because \( f(Y) \) lies in some finite complex \( K \subseteq L \). Now, by condition (2), the map \( f_1 : F \to L \) can be extended to a map \( \bar{f}_1 : \beta X \to L \). It only remains to note that the map \( f = \bar{f}_1|X \) extends \( f \). Proposition 2.8 is proved. \( \square \)

Proposition 2.9. Let \( X \) be a countably compact normal space, \( F \) be its closed subset and \( U = X - F \). Suppose \( L \in AE(F) \) and \( L \in AE(Y) \) for every closed in \( X \) set \( Y \subseteq U \). Then \( L \in AE(X) \).

Proof. By Definition 2.4, we need to verify that for every closed set \( A \subseteq X \) any map \( f : A \to L \) has an extension \( \bar{f} : X \to L \). Let \( f_0 = f|(A \cap F) \). Since \( L \in AE(F) \), the map \( f_0 \) can be extended to a map \( \bar{f}_0 : F \to L \). Define the map \( f_1 : A \cup F \to L \) by letting \( f_1|A = f \) and \( f_1|F = \bar{f}_0 \). Clearly, \( f_1 \) is continuous. By Proposition 2.7, the map \( f_1 \) has an extension \( \bar{f}_1 : V \to L \), where \( V \) is a neighborhood of \( A \cup F \) in \( X \). Take a neighborhood \( V_1 \) of \( A \cup F \) such that \( \text{cl}(V_1) \subseteq V \) and let \( Y = X - V_1 \). \( Y_1 = \text{Bd}(V_1) \), \( g = \bar{f}_1|Y_1 \). Then \( Y \) is closed in \( X \) and \( Y_1 \) is closed in \( Y \). By condition \( L \in AE(Y) \). Hence, the map \( g \) has an extension \( \bar{g} : Y \to L \). Finally, define a map \( \bar{f} : X \to L \) by letting \( \bar{f}|Y = g \) and \( \bar{f}|\text{cl}(V_1) = \bar{f}_1 \).

Evidently, \( \bar{f} \) is well defined and continuous. It is also clear that \( \bar{f}|A = f \). Proposition 2.9 is proved. \( \square \)

Next we define a relation \( \leq \) on the class of all simplicial complexes. Following [8] we say that \( K \leq L \) if for every normal countably compact
space $X$ the condition $K \in A\!E(X)$ implies the condition $L \in A\!E(X)$. The relation $\leq$ is reflexive and transitive and, consequently, it is a relation of preorder. This preorder induces the following equivalence relation:

$$K \sim L \iff K \leq L \text{ and } L \leq K.$$ 

For a simplicial complex $L$ by $[L]$ we denote the class of all complexes which are equivalent to $L$. These classes $[L]$ are called extension types.

**Remark 2.10.** Relation $L \in A\!E(X)$, preorder $\leq$ and extension types $[L]$ can be defined for different classes of spaces $X$. A. N. Dranishnikov [9] defined relation $L \in A\!E(X)$ for the class $\mathcal{MLC}$ of all metrizable locally compact spaces. In [8] he defined this relation for the class $\mathcal{C}$ of all compact Hausdorff spaces. One can define relation $\leq_\sigma$ and associated concepts for arbitrary class $\sigma$ of topological spaces. Let $\mathcal{MC}$ be the class of all compact metrizable spaces and $\mathcal{CC}$ be the class of all normal countably compact spaces.

**Proposition 2.11.** For any simplicial complexes $K$ and $L$ the following conditions are equivalent:

1. $K \leq_{\mathcal{MC}} L$;
2. $K \leq_{\mathcal{C}} L$;
3. $K \leq_{\mathcal{CC}} L$;
4. $K \leq_{\mathcal{MLC}} L$;

The equivalence (1)$\iff$(2) was proved in [10, Theorem 11]. For the equivalence (2)$\iff$(3) consult with Proposition 2.8. As for the equivalence (1)$\iff$(4) it follows from Theorem 2.18 and the next trivial statement.

**Proposition 2.12.** Let $X_\alpha$, $\alpha \in \mathcal{A}$ be a discrete family of normal spaces. Then for any simplicial complex $L$

$$L \in A\!E(\otimes\{X_\alpha : \alpha \in \mathcal{A}\}) \iff L \in A\!E(X_\alpha) \text{ for each } \alpha \in \mathcal{A}.$$ 

If $\sigma$ is a class of topological spaces, then by $E_\sigma$ we denote the class of all extension types of all simplicial complexes generated by the relation $\leq_\sigma$. In view of Proposition 2.11 we shall use a simpler notation: $E$ and $\leq$.

**Definition 2.13.** (see [3][8]) Let $X$ be a countably compact normal space. Its extension dimension $\text{e-dim} \ X$ is defined as the smallest extension type $[L]$ of simplicial complexes, satisfying condition $L \in A\!E(X)$.

**Proposition 2.14.** [8, §3, Proposition 1] For any compactum $X$ there exists unique extension type $[L]$ such that $\text{e-dim} \ X = [L]$. 

Proposition 2.15. [8, §3, Proposition 2] The correspondence e-dim maps the class $C$ epimorphically onto the class $E$.

Propositions 2.8 and 2.14 yield

**Proposition 2.16.** For any normal countably compact space $X$ there exists unique extension type $[L]$ such that $e$-dim $X = e$-dim $\beta X = [L]$.

Propositions 2.8 and 2.15 yield

**Proposition 2.17.** The correspondence e-dim maps the class $CC$ epimorphically onto the class $E$.

**Theorem 2.18.** Suppose that a normal countably compact space $X$ is the union of its closed subsets $X_i$, $i \in \omega$. If $e$-dim $X_i \leq [L]$ for each $i \in \omega$, then $e$-dim $X \leq [L]$.

The proof of the above statement repeats the proof (see, for instance, [1]) of classical countable sum theorem for Lebesgue dimension dim for normal spaces by means of extension of mappings into $S^n$. The main feature of the sphere $S^n$ exploited in that proof is $S^n \in ANE(X)$. The corresponding property $L \in ANE(X)$ in our case is guaranteed by Proposition 2.7.

For further references we formulate just mentioned description of the Lebesgue dimension as a separate statement. Obviously it provides the main link between the theory of Lebesgue dimension and the theory of extension dimension.

**Theorem 2.19.** For any normal space $X$,

$$\dim X \leq n \iff e\text{-dim} X \leq [S^n].$$

3. **On a realization of dimensional types by manifolds**

We recall one result from [13] in a more convenient for us form.

**Theorem 3.1.** For an arbitrary metrizable compactum $C$, assuming $\Diamond$, there exists a differentiable, countably compact, perfectly normal, hereditarily separable 4-manifold $M_C^4$ such that $\beta M_C^4 - M_C^4$ is a metrizable compactum homeomorphic to the disjoint sum of $C$ and some open subset $U$ of the 3-dimensional sphere $S^3$.

The manifold $M_C^4$ is a manifold of type $M_\varphi^4$ from [13], where $\varphi = \varphi_C : B^4 \to B_\varphi^4$ is a quotient mapping, defined on the closed ball $B^4$, with the following properties.

Let the sphere $S^3$ be the boundary of $B^4$. There exists a closed set $A \subseteq S^3$ such that

(i) $A = \varphi^{-1}\varphi(A)$;
(ii) \( \varphi(A) = C \);

(iii) each fiber \( \varphi^{-1}(y) \), \( y \in C \), is a non-degenerate continuum nowhere dense in \( S^3 \).

(iv) \( \varphi^{-1}(y) = \{x\} \) for every \( x \in B^4 - A \).

Thus, \( \varphi(S^3) \equiv S^3_\varphi \) is homeomorphic to the disjoint sum of \( C \) and \( S^3 - A \). By [13, Proposition 2.3], \( \beta M^4_C - M^4_C = S^3_\varphi \).

Let \( \Lambda \) be the class of all complexes and let

\[ \Lambda^0 = \{ L \in \Lambda : [L] = \text{e-dim } X \text{ for some metrizable compactum } X \} \]

By Proposition 2.14, for every metrizable compactum \( X \) there is a complex \( L \in \Lambda^0 \) such that \( \text{e-dim } X = [L] \). Set

\[ \Lambda_1^0 = \{ L \in \Lambda^0 : [L] \geq [S^4] \} \]

The next theorem is the main result of this section.

**Theorem 3.2.** For an arbitrary complex \( L \in \Lambda_1^0 \), assuming \( \diamondsuit \), there exists a differentiable, countably compact, perfectly normal, hereditarily separable 4-manifold \( M = M^{4,L} \) such that \( \text{e-dim } M = [L] \).

**Proof.** We use the scheme of the proof of [5, Theorem 3.1], where a similar result was obtained for countable complexes. The only difference is that in our situation we can not apply auxiliary results for countable complexes which were used in [5].

Consider a complex \( L \in \Lambda^0_1 \). By definition of \( \Lambda^0_1 \), there is a metrizable compactum \( C \) such that \( \text{e-dim } C = [L] \).

Set \( M = M^4_C \), where \( M^4_C \) is a manifold from Theorem 3.1. We claim that this is a required manifold. First of all, \( M \) is countably compact. Hence, in view of Proposition 2.8,

\[ \text{e-dim } M = \text{e-dim } \beta M \]  

Further, by Corollary 2.5 and Theorem 3.1, we have

\[ \text{e-dim } \beta M \geq \text{e-dim } (\beta M - M) \geq \text{e-dim } C = [L] \]

Now we apply Proposition 2.9 to the pair \( (S^3_\varphi, C) \). Since \( S^3_\varphi - C \) is open in \( S^3 \), Theorem 2.19 yields

\[ \text{e-dim } S^3_\varphi \leq \max \{ [S^3], [L] \} = [L] \]

Finally, let us apply Proposition 2.9 to the pair \( (\beta M, S^3_\varphi) \). Since \( \dim Y \leq 4 \) for any compactum \( Y \subseteq M \), by Theorem 2.19 and inequality (3.4), we obtain

\[ \text{e-dim } \beta M \leq \max \{ [S^n], [L] \} = [L] \]
Inequalities (3.3) and (3.5) yield
\[ e\dim \beta M = [L]. \]
Thus, equality (3.2) finishes the proof of Theorem 3.2. \(\square\)

As corollaries of Theorem 3.2 we discuss several examples of complexes \(L \in \Lambda^i_4\) with certain curious properties. First of all we recall two results needed for our discussion.

Proposition 3.3. [4, Proposition 2.6] Let \(K\) be an \(n\)-dimensional locally compact polyhedron. Then \(e\dim K = [S^n]\).

Proposition 3.4. [4, Corollary 2.3] Let \(L\) be a simplicial complex homotopy dominated by a finite complex. Then there exists a metrizable compactum \(X\) such that \(e\dim X = [L]\).

Remark 3.5. It follows from the proof of Theorem 3.2 that for every complex \(L \in \Lambda^i_4\) there exists a metrizable compactum \(C\) such that
\[ (3.6) \quad e\dim M^{4, L} = e\dim M^{4, C} = e\dim C = [L]. \]

Example 3.6. Let \(\mathcal{L} = \{S^n : n \geq 4\}\) and \(C_n = I^n\). Then from (3.6) and Proposition 3.3 we obtain that \(\dim M^{4, S^n} = n\) – the fact proved earlier in [12].

Definition 3.7. Let \(L_n = M(\mathbb{Z}_2, n + 1) \vee S^{n+1}\), where \(M(\mathbb{Z}_2, n + 1)\) is the Moore complex, i.e. the complex obtained from \((n + 1)\)-dimensional disk \(B^{n+1}\) by attaching to its boundary \(S^n\) the disk \(B^{n+1}\) via the map \(S^n \to S^n\) of degree 2. It is clear that \(L_n\) is a finite complex with 
\[ [S^n] < [L_n] < [S^{n+1}]. \]

Example 3.8. Let \(\mathcal{L} = \{L_n : n \geq 4\}\) and let \(C_n\) be a metrizable compactum with \(e\dim C_n = [L_n]\) (see Proposition 3.4). Then \(e\dim M^{4, L_n} = [L_n]\).

Corollary 3.9. Assuming \(\Diamond\), there exists a differentiable, countably compact, perfectly normal, hereditarily separable 4-manifold \(M^4\) such that \([S^3] < e\dim M^4 < [S^5]\).

4. On the Dimension of Products of Manifolds

The Stone-Čech remainder \(\beta X - X\) of a space \(X\) is denoted by \(X^*\).

Lemma 4.1. Let \(M_i\) be a countably compact \(n_i\)-manifold of dimension \(\dim M_i = m_i, \ i = 1, 2\). Then
\[ \dim(M_1 \times M_2) = \max\{n_1 + m_2, n_2 + m_1, \dim(M_1^* \times M_2^*)\}. \]
Proof. Because each manifold is a $k$-space (being first countable) it follows from [11, Theorem 3.10.13] that $M_1 \times M_2$ is countably compact. Hence, by Gliksberg’s theorem [15], $M_1 \times M_2$ is pseudocompact and $\beta(M_1 \times M_2) = \beta M_1 \times \beta M_2$. By Proposition 2.1, $\dim(M_1 \times M_2) = \dim(\beta(M_1 \times M_2))$. So we have to find out the exact value of $\dim(\beta M_1 \times \beta M_2)$.

Let $X = \beta M_1 \times \beta M_2$, $F = M_1^* \times M_2^*$ and $U = X - F$. By Dowker’s theorem [6],

$$\dim X = \max\{\dim F, k\},$$

where

$$k = \sup\{\dim Y : Y \subseteq U, Y \text{ is closed in } X\}.$$  

(4.1)

It is clear, that each $Y$ from (4.2) is contained in some $Y' = (K_1 \times \beta M_2) \cup (\beta M_1 \times K_2)$, where $K_i \subseteq M_i$ is a finite sum of $n_i$-dimensional cubes, $i = 1, 2$. By Morita’s theorem [18],

$$\dim(K \times Z) = \dim K + \dim Z,$$

whenever $Z$ is a paracompact space and $K$ is a compact polyhedron. By the finite sum theorem for $\dim$, (4.3) yields

$$\dim(K_i \times \beta M_j) = \dim K_i + \dim \beta M_j.$$  

(4.3)

Consequently,

$$\dim Y' = \max\{\dim(K_1 \times \beta M_2), \dim(\beta M_1 \times K_2)\} \overset{(4.4)}{=} \max\{n_1 + \dim \beta M_2, \dim \beta M_1 + n_2\} \overset{\text{Proposition 2.1}}{=} \max\{n_1 + \dim M_2, \dim M_1 + n_2\} = \max\{n_1 + m_2, m_1 + n_2\}.$$  

Equality (4.1) finishes the proof of Lemma 4.1.

Corollary 4.2. Let $M_i$ be a countably compact $n_i$-manifold of dimension $\dim M_i = m_i$, $i = 1, 2$. If $\max\{n_1 + m_2, n_2 + m_1\} \leq \dim(M_1^* \times M_2^*)$, then $\dim(M_1 \times M_2) = \dim(M_1^* \times M_2^*)$.

Proposition 4.3. Let $M_1$ and $M_2$ be countably compact manifolds. Then

$$\dim(M_1 \times M_2) \leq \dim M_1 + \dim M_2.$$  

Proof. According to Lemma 4.1, we only need to check that $\dim(M_1^* \times M_2^*) \leq m_1 + m_2$. But for any compact spaces $X_1$ and $X_2$ we have (see [16])

$$\dim(X_1 \times X_2) \leq \dim X_1 + \dim X_2.$$
Hence,
\[
\dim(M_1^* \times M_2^*) \leq \dim M_1^* + \dim M_2^* \leq \dim \beta M_1 + \dim \beta M_2 = \\
\dim M_1 + \dim M_2 = m_1 + m_2.
\]

Proposition 4.3 is proved. ∎

The next statement is an immediate corollary of the countable sum theorem for Lebesgue dimension.

**Lemma 4.4.** Let \( X_i \) be metrizable compacta, \( i = 1, 2 \). Let \( F_i \) be a closed subset of \( X_i \), and let \( U_i = X_i - F_i \). Then
\[
\dim(X_1 \times X_2) = \\
\max\{\dim(U_1 \times U_2), \dim(U_1 \times F_2), \dim(F_1 \times U_2), \dim(F_1 \times F_2)\}.
\]

**Theorem 4.5.** Let \( m \) be a natural number such that \( m \geq 5 \). Then, assuming \( \diamond \), there exists a differentiable, countably compact, perfectly normal, hereditarily separable 4-manifold \( M = M_m \) such that \( \dim M = m \) and \( \dim(M \times M) = 2m - 1 < 2 \dim M \).

**Proof.** Let \( B \) be a two-dimensional metrizable compactum such that \( \dim(B \times B) = 3 \). Such a compactum was constructed by V. G. Boltyanski [2]. Let \( C = B \times I^{m-2} \). Then in accordance with (4.3),
\[
\dim C = m,
\]
(4.5)
\[
\dim(C \times C) = 2m - 1.
\]
(4.6)

Let \( M = M_C^4 \) be a manifold from Theorem 3.1. We claim that \( M \) is a required manifold. Indeed, by the properties of \( M_C^4 \), the set \( M^* - C = U \) is homeomorphic to an open subset of \( S^3 \). Consequently, Lemma 4.1 and (4.6) yield \( \dim(M^* \times M^*) = 2m - 1 \). In this situation Corollary 4.2 finishes the proof of theorem 4.5. ∎

**Question 4.6.** Does there exist a 4-manifold \( M \) such that
\[
2 \dim M - \dim(M \times M) \geq 2 ?
\]

A similar question about two different manifolds has a positive solution.

**Theorem 4.7.** Let \( m_1, m_2 \) and \( r \) be natural numbers such that \( 5 \leq m_1 \leq m_2 \) and \( 4 + m_2 \leq r < m_1 + m_2 \). Then assuming \( \diamond \) there exist differentiable, countably compact, perfectly normal, hereditarily separable 4-manifolds \( M_1 \) and \( M_2 \) of dimension \( \dim M_i = m_i \) such that
\[
\dim(M_1 \times M_2) = r < m_1 + m_2 = \dim M_1 + \dim M_2.
\]
Proof. We follow the proof of Theorem 4.5. First let us recall the following result [7, §2, Corollary 2].

- For all natural numbers $m_1, m_2$ and $r$ such that $m_1 \leq m_2$ and $m_2 < r \leq m_1 + m_2$, there exist metrizable compacta $X_1$ and $X_2$ such that $\dim X_i = m_i$ and $\dim(X_1 \times X_2) = r$.

Set $M_i = M_{X_i}^4$, where $X_1$ and $X_2$ are the above mentioned compacta with $m_1, m_2$ and $r'$ satisfying inequalities $m_2 < r' \leq m_1 + m_2$. From Lemma 4.4 we get $\dim(M_1^* \times M_2^*) = \max\{3 + m_2, r'\}$. On the other hand, for $k$ from (4.2) we have $k = \dim Y' = 4 + m_2$. In view of Lemma 4.1 we have $\dim(M_1 \times M_2) = \max\{4 + m_2, r'\}$. Theorem 4.7 is proved. \qed

Remark 4.8. As we have seen the dimension of the product of manifolds can be much less than the sum of their dimensions. But, since our manifolds $M_i$ are countably compact, by Proposition 4.3 we have $\dim(M_1 \times M_2) \leq \dim M_1 + \dim M_2$.

Question 4.9. Are there manifolds $M_1$ and $M_2$ such that $\dim(M_1 \times M_2) > \dim M_1 + \dim M_2$?

5. On the dimension of subsets of 4-manifolds

Theorem 5.1. Assuming $\diamond$, there exists an infinite-dimensional, differentiable, countably compact, perfectly normal, hereditarily separable 4-manifold $M^4$ such that for every closed set $F \subseteq M^4$ we have

either $\dim F \leq 4$ or $\dim F = \infty$.

Proof. Let $M^4 = M_C^4$, where $M_C^4$ is a manifold from Theorem 3.1 and $C$ is well known Henderson’s infinite-dimensional compactum with no positive-dimensional compact subsets [17]. By Proposition 2.1 and Theorem 3.1, $\dim M^4 = \infty$. Now let $F$ be a closed subset of $M^4$ such that $\dim F \geq 5$. Then in view of Proposition 2.1,

$$\dim \beta F \geq 5.$$ 

But, since $M^4$ is normal, $\beta F = \overline{\text{cl}_{M^4} F}$. Set

$$A = (\overline{\text{cl}_{M^4} F}) \cap (\beta M^4 - M^4).$$

By Proposition 2.9 for the pair $(\beta F, A)$, we have $\dim A \geq 5$. But in accordance with Theorem 3.1, $\beta M^4 - M^4$ is a disjoint sum of $C$ and some open subset of $S^3$. Hence,

$$\dim(A \cap C) \geq 5.$$
Therefore, by the property of Henderson’s compactum, \( \dim(A \cap C) = \infty \). Consequently,
\[
\dim F = \dim \beta F \geq \dim C \geq \dim(A \cap C) = \infty.
\]
Theorem 5.1 is proved.

The next statement is a generalization of Theorem 3.2.

**Theorem 5.2.** Let \( \mathcal{L} \) be a countable subset of \( \Lambda^0_4 \) (see Theorem 3.2). Then, assuming \( \Diamond \), there exists a differentiable, countably compact, perfectly normal, hereditarily separable 4-manifold \( M^4 \) which admits a family \( U_L, L \in \mathcal{L}, \) of open subsets of extension dimension \( e \)-dim \( U_L = [L] \). Moreover, one can choose sets \( U_L \) in such a way that

1. either \( \text{cl}(U_L) \cap \text{cl}(U_{L'}) = \emptyset \) if \( L \neq L' \),
2. or \( \cap \{ U_L : L \in \mathcal{L} \} \neq \emptyset \).

**Proof.** By Theorem 3.2, there is a metrizable compactum \( X_L \) of extension dimension \( e \)-dim \( X_L = [L] \). Let \( C \) be the Alexandroff compactification of the discrete sum \( \oplus \{ X_L : l \in \mathcal{L} \} \) of these compacta. Now we define \( M^4 \) as a manifold \( M^4_C \) from Theorem 3.1. Since \( \{ X_L : L \in \mathcal{L} \} \) is a discrete family in a compact space \( \beta M^4 \), there is a disjoint family of neighborhoods \( V_L \) of \( X_L \) in \( \beta M^4 \). We may assume also, that
\[
\text{cl}_\beta M^4(V_L) \cap \text{cl}_\beta M^4(V_{L'}) = \emptyset \text{ if } L \neq L'.
\]

Now, in case (1), we set
\[
(5.1) \quad U_L = V_L \cap M^4.
\]
To realize case (2) we take an open metrizable subset \( U \subseteq M^4 \) and set
\[
(5.2) \quad U_L = (U \cup V_L) \cap M^4.
\]

**Claim.** \( e \)-dim \( (\text{cl}_M^4 U_L) = [L] \).

**Proof.** By Proposition 2.8, it suffices to verify that
\[
(5.3) \quad e \text{-dim } \beta (\text{cl}_M^4 U_L) = [L].
\]

But
\[
(5.4) \quad \beta (\text{cl}_M^4 U_L) = \text{cl}_\beta M^4 (\text{cl}_M^4 U_L) = \text{cl}_\beta M^4 (U_L).
\]

On the other hand, according to (5.1), \( U_L \) is dense in \( V_L \). Hence,
\[
(5.5) \quad \text{cl}_\beta M^4(U_L) = \text{cl}_\beta M^4(V_L).
\]

Let \( \Phi_L = M^4 \cap \text{cl}_\beta M^4(V_L) \) and let \( F_L = \text{cl}_\beta M^4(V_L) - M^4 \). Since \( \Phi_L \) is dense in \( \text{cl}_\beta M^4(V_L) \), we have
\[
(5.6) \quad \text{cl}_\beta M^4(\Phi_L) = \Phi_L \cup F_L = \text{cl}_\beta M^4(V_L).
\]
Hence,
\[(5.7) \quad \beta \Phi_L = \Phi_L \cup F_L.\]
For every compactum \(Y \subseteq \Phi_L\) we have \(\dim Y \leq 4\). On the other hand, \(F_L \supseteq X_L\) and \(F_L \cap X_{L'} = \emptyset\) whenever \(L \neq L'\). Hence, \(\text{e-dim} F_L = [L]\).

Now we apply proposition 2.9 to the pair \((\beta \Phi_L, F_L)\). We have
\[(5.8) \quad \text{e-dim} \beta \Phi_L = \text{e-dim} F_L = [L].\]

Finally, conditions (5.7), (5.6), (5.5) and (5.4) give us the required equality (5.3). This finishes proof of our Claim.

In order to prove the equality \(\text{e-dim} U_L = [L]\) we need more general version of Theorem 2.18. Its proof is based on the fact that a countable simplicial complex is an \(ANE\) for the class of all normal spaces.

- Suppose that a normal space \(X\) is the union of its closed countably compact subsets \(X_i, i \in \omega\). If \(\text{e-dim} X_i \leq [L]\) for each \(i \in \omega\), then \(\text{e-dim} X \leq [L]\).

In order to finish the proof of Theorem 5.2 represent \(U_L\) as the union of an increasing sequence
\[U_L^0 \subseteq \text{cl} M^4 U_L^0 \subseteq U_L^1 \subseteq \cdots\]
and apply our claim. Theorem 5.2 is proved. \(\square\)

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