Self-induced flavor instabilities of a dense neutrino stream in a two-dimensional model

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Introduction—Neutrino-neutrino interactions in dense neutrino gases provide a refractive term leading to a non-linear feedback in the flavor evolution, when the neutrino interaction potential \( \mu \sim \sqrt{2} G_F \nu \nu_e \) is comparable or larger than the neutrino vacuum oscillation frequency \( \omega = \Delta m^2 / 2E \). This effect can lead to a collective behavior of the dense neutrino ensemble (see [1] for a recent review). Remarkably, almost a decade ago it was realized that neutrino-neutrino interactions dominate the flavor evolution in the deepest core-collapse supernova (SN) regions producing self-induced flavor conversions [2–4]. The most important observable consequence of this type of flavor transitions would be the swap of the SN \( \nu_e \) and \( \bar{\nu}_e \) spectra with that of non-electron flavor \( \nu_x \) and \( \bar{\nu}_x \) in certain energy intervals [5].

Characterizing the SN neutrino flavor dynamics amounts to follow the spatial evolution of the neutrino fluxes. For a stationary neutrino emission, the kinetic equations of the \( \nu \) space-dependent occupation numbers \( \rho_{\nu_x,p} \) with momentum \( p \) at position \( x \) are [6, 7]

\[
\mathbf{v}_p \cdot \nabla_x \rho_{\nu_x,p} = -i [\Omega_{\nu_x,p}, \rho_{\nu_x,p}] ,
\]

with the Liouville operator in the left-hand side. Neglecting external forces and an explicit time dependence of the occupation numbers, this operator represents the drift term proportional to the neutrino velocity \( \mathbf{v}_p \), due to particle free streaming. On the right-hand-side the matrix \( \Omega_{\nu_x,p} \) is the full Hamiltonian containing the vacuum, matter and self-interaction terms. In particular, in non-isotropic neutrino gases, like the case of neutrinos streaming-off a SN core, the neutrino-neutrino interaction term contains multi-angle effects since the current-current nature of the low-energy weak interactions introduces an angle dependent term \( (1 - \mathbf{v}_p \cdot \mathbf{v}_q) \) between two interacting neutrino modes [2,8]. This term produces a net current so that test neutrinos moving in different directions would experience different refractive index. This in some cases challenges the collective behavior of the flavor evolution observed in an isotropic case, leading to flavor decoherence [9–11]. Multi-angle effects can also lead to a trajectory-dependent matter term, which if strong enough suppresses the self-induced conversions [12, 13].

The multi-angle flavor evolution described by the partial differential equations [1] has never been solved till now in its full complexity. Instead, numerical approaches have been typically based on the so-called “bulb model” [2, 4, 11], where it is assumed a spherical symmetry about the center of the SN and azimuthal symmetry about any radial direction. In this limit Eq. (1) reduces to an ordinary differential equation problem, projecting the evolution along the radial direction, i.e.

\[
\mathbf{v}_p \cdot \nabla_x \rightarrow v_r d/dr .
\]

Attempts to go beyond the bulb model have been proposed. For example, in [14] it was shown that assuming that the neutrino ensemble displays self-maintained coherence, the problem for generic geometries can be reduced to a one-dimensional case along the streamlines of the overall neutrino flux. However, the existence of a self-consistent coherent solution does not imply its stability. Indeed, it has been recently realized that instabilities may grow once one relaxes some symmetries of the bulb model, since neutrino-neutrino interactions can lead to spontaneous symmetry breaking effects. Notably, removing the assumption of axial symmetry in the \( \nu \) multi-angle term, a multi-azimuthal-angle instability has been discovered, even assuming a perfect spherically symmetric \( \nu \) emission [15–19]. Furthermore, also space and time homogeneity can be broken in a dense neutrino gas, so that it is not guaranteed that a quasi-stationary neutrino emission would lead to a stationary solution [20]. Finally, with a simple toy model it has been recently shown, by means of a stability analysis of the linearized equations of
motion, that self-induced oscillations can spontaneously break spatial symmetries \[21\]. All these findings suggest that the validity of the bulb model should be critically reconsidered and that a self-consistent solution of the SN neutrino flavor evolution can only be achieved by solving the complete multi-dimensional problem of Eq. \[1\].

As a further step in clarifying this issue we consider here the two-dimensional toy model discussed in \[21\], i.e. monochromatic neutrino streams emitted in a stationary way in two directions from an infinite boundary plane at \(z = 0\) with periodic conditions on \(x\) and translation invariance along the \(y\) direction. We assume a small perturbation for the initial symmetries of the flavor content in both the two emission modes and along the boundary in the \(x\) direction. Despite of its simplicity, this model with perturbed symmetries exhibits a rich phenomenology. Indeed, solving numerically the partial differential operator at left-hand-side reads

\[
\partial_t L_0 + \nabla_x \cdot L = \nabla_x \cdot L = 0 ,
\]

where first equality follows from the assumption of a stationary solution. Eq. \[3\] generalizes the lepton-number conservation law of the one dimensional case \[3\].

As first exploited in \[20\] (see also \[21\]) in the context of multi-dimensional neutrino oscillations, the partial differential equation problem like of Eq. \[3\] can be reduced to a tower of ordinary differential equations for the Fourier modes defined as

\[
P_{L(R),k}(z) = \int_{-\infty}^{+\infty} dx \ P_{L(R)}(x,z)e^{-ikx} ,
\]

and similarly for the antineutrino polarization vectors \(\overline{P}_{L,R}\). In the following, for simplicity we will consider a monochromatic perturbation in the \(x\) direction for the neutrino polarization vectors at the boundary at \(z = 0\). Since we start with a pure flavor state, we assume \(P_{L,R}(x,0) = 0\) and

\[
P_{L,R}(x,0) = (P_{L,R}^1(x,0) + \epsilon \cos(k_0 x) ,
\]

where this latter component of the polarization vector is proportional to the flavor content of the ensemble. The function \(P_{L,R}^1(x,0)\) indicates the unperturbed value of the polarization vectors, while \(k_0\) is the wave-number of the perturbation and \(\epsilon \ll \mu, \omega\) its small amplitude. It is easy to see that in this case, only higher harmonics of the fundamental mode with \(k_n = nk_0\) are excited. Defining \(P_{L,n} = k_0 P_{L,k_0}/(2\pi)\), from Eq. \[3\] one obtains

\[
\frac{d}{dz}P_{L,n}(z) = -i\sigma_{L,R} P_{L,n} + \omega B \times P_{L,n} + \mu \sum_{j=-\infty}^{+\infty} D_{R,n-j} \times P_{L,j} .
\]
An analogous set of coupled ordinary differential equations can be written for the $R$ mode and for the antineutrino polarization vectors. It is enough to follow the evolution for positive modes, $n \geq 0$, since the $P_{L,R}(x,z)$ and $\overline{P}_{L,R}(x,z)$ are real functions and therefore\
\begin{equation}
P^*_{(L,R),n} = P_{(L,R),-n}.
\end{equation}

Once the evolution of the harmonic modes is obtained from Eq. (11), the polarization vector in configuration space can be obtained by inverse Fourier transform.

**Numerical examples.**—To illustrate the behavior of the self-induced flavor conversions in our two-dimensional toy model, we consider a gas initially composed by only $\bar{\nu}_e$ and $\nu_e$. We normalize the polarization vectors to the $\nu_e$ number density and we assume an excess of $\nu_e$ over $\bar{\nu}_e$, i.e. $P_{\nu_e}^0/P_{\bar{\nu}_e}^0 = 1 + \alpha$. If the translational symmetry is assumed, i.e. $\epsilon = 0$ in Eq. (10), and the $L$ and $R$ modes are prepared identically, it is well known that the system is stable in normal mass hierarchy ($\Delta m^2 > 0$), while in inverted mass hierarchy ($\Delta m^2 < 0$) it exhibits a bimodal instability and behaves as a *flavor pendulum*, leading to periodic pair conversions $\nu_e \leftrightarrow \bar{\nu}_e$, that conserve the lepton number $L_0 = \alpha$ of Eq. (13). More recently, it has been shown that if the $L \leftrightarrow R$ symmetry is perturbed the system becomes unstable also in normal hierarchy, with a similar pendulum behavior [10]. Our further step is to perturb also the translational symmetry at the boundary.

We fix in Eq. (8) $\mu = 10$, $\omega = 1$, $\theta = 10^{-3}$, and we consider the normal mass hierarchy case (the result would be similar for the inverted mass hierarchy). We take as asymmetry parameter $\alpha = 0.3$. We assume $v_L = v_z = \sqrt{2}/2$. Perturbation in the $L \leftrightarrow R$ symmetry are introduced by a 1% difference in the initial conditions between these modes. Furthermore, to perturb the translational symmetry along the $x$ direction, we assume in Eq. (10) $\epsilon = 0.01$, and we take as perturbation frequency $k_0 = 0.2\sqrt{2}\omega\mu$, where the square-root expression is the proper frequency of the unperturbed flavor pendulum [3]. Correspondingly, $P_{L,1}^0(0) = P_{R,1}^0(0) = \epsilon$ (and analogously for antineutrinos), while the higher order harmonics are initially vanishing. We follow the evolution of the first $N = 600$ Fourier modes.

In Fig. 1 we show the flavor evolution of $\nu_e$ flavor content $\rho_{ee}$ in the $x-z$ plane, and its map on the bottom plane.

**FIG. 1:** Three dimensional evolution of the $\nu_e$ flavor content $\rho_{ee}$ in the $x-z$ plane, and its map on the bottom plane.

In Fig. 2 we represent the lepton density $L_0$ of Eq. (10) in the $x-z$ plane. Notice that as soon as the translational instability develops, lepton number shows a non trivial domain structure and that self-induced conversions lead to large space variations of the initial asymmetry $\alpha$.

In order to understand the origin of this flavor dynamics, in Fig. 3 we show a contour plot representing the evolution of the different Fourier modes $|P_{R,n}(z)|$ (in logarithmic scale) in the plane of $n-z$. The behavior of $|P_{L,n}(z)|$ would be similar (not shown). We realize that the breaking of the translational symmetry corresponds to the growth of the $n > 0$ modes occurring at $z > 2$. This dynamics can be seen as a diffusion process in the Fourier space, where a flavor wave caused by the flavor pendulum diffuses to higher harmonics (i.e. to smaller scales) as soon as the Fourier modes are ex-
cited by the non-linear interactions between the different modes. Note the analogy of this process with the multi-angle decoherence associated with a diffusion of excitations in the multipole space \cite{23}. Correspondingly, in the flavor evolution one observes the developments of spatial variations in the $x$ directions at smaller and smaller scales. In the example we are studying, at $z = 12$ about the first 300 harmonics are significantly excited. Indeed, the number of harmonics that one follows determine the range of validity of the numerical simulation. We checked that the number of excited harmonics is sensitive to the neutrino-neutrino interaction potential $\mu$, since this factor determines the strength of the terms responsible for the growth of the modes in the second line of Eq. (11).

The growth of the harmonics is also enhanced with the initial flavor asymmetry $\alpha$. Indeed, in the sum at right-hand-side of Eq. (11) the term $D_{(L,R),0}$ increases with the initial flavor asymmetry, pumping the higher order harmonics.

The behavior of the neutrino gas in our model has a nice analogy with the transition between laminar and turbulent behavior of a streaming fluid (see, e.g., \cite{24}). In this respect, it is useful to define an average neutrino velocity for the neutrino fluid at a point $\mathbf{x} = (x, z)$. Considering for example the $\nu_e$ flux, one has

$$
\langle \hat{v}_e \rangle_x = \frac{\varrho_{ee,L} \hat{v}_L + \varrho_{ee,R} \hat{v}_R}{\varrho_{ee,L} + \varrho_{ee,R}}
$$

(13)

Till the $L \leftrightarrow R$ symmetry is unbroken, we have $\langle \hat{v}_e \rangle_x \simeq v_z$. Then, when a $L-R$ asymmetry with variations in the $x$ direction is produced, the average velocity starts to acquire a transverse component in the $x$ direction. The “streamlines” of the neutrino flux are the solutions of

$$
\frac{dx}{ds} = \frac{\langle \hat{v}_e \rangle_x}{\langle \hat{v}_e \rangle_x} = \hat{F}_{e,x}
$$

(14)

where $s$ is a parameter along the line (see \cite{14}).

In Fig. 4 we show the streamlines defined in Eq. (14) (in vertical direction). One clearly sees that the transition between the coherent to incoherent flavor behavior observed in Fig. 1 corresponds to the change from a laminar to a turbulent regime. As soon as the translational symmetry is broken, the streamlines become irregular and are no longer parallel to the $z$ directions. Moreover, they exhibit large variations, as in the turbulent motion in a fluid, and tend to converge in preferred directions.

The behavior we found has a fascinating similarity with the non-linear instabilities of fluid flows, described by the Navier-Stokes equation, see e.g. \cite{24}. It remains to be explored if by developing this analogy, one can gain a deeper description of the dense neutrino gas.

**Conclusions.**— We presented the results of a study of self-induced flavor conversions in a simple two-dimensional model. As predicted from the stability analysis performed in \cite{21}, we find that the self-interacting neutrino gas can break the spatial symmetries of the initial conditions and the neutrino flavor content achieves large space variations. This implies that the coherent behavior of the neutrino gas is unstable under small spatial inhomogeneities. This process also leads to the formation of domains of different net lepton number $L_0$.

Our toy model is much simpler than any realistic SN neutrino case, since we are neglecting multi-angle effects, continuous energy spectra, ordinary matter effects and declining neutrino densities. All these effects would add additional complications and numerical challenges. However, if our results would apply also to the SN case this would radically change the current description of the self-induced flavor conversions, and would have interesting phenomenological consequences, like e.g., the generation of a self-induced direction-dependent asymmetry in the lepton number flux. These intriguing possibilities call for further efforts to go beyond the bulb model. This challenge would be quite demanding in terms of new computing time and/or novel approaches.

A.M. acknowledges useful discussions with Eligio Lisi, Antonio Marrone and Georg Raffelt. The work of A.M. was supported by the German Science Foundation (DFG) within the Collaborative Research Center 676 “Particles, Strings and the Early Universe”. G.M. is supported by INFN I.S. TASP. N.S. acknowledges support from the European Union FP7 ITN INVISIBLES (Marie Curie Actions, PITN- GA-2011- 289442).
[1] H. Duan, G. M. Fuller and Y. Z. Qian, “Collective Neutrino Oscillations,” Ann. Rev. Nucl. Part. Sci. 60, 569 (2010) [arXiv:1001.2799 [hep-ph]].
[2] H. Duan, G. M. Fuller, J. Carlson and Y. -Z. Qian, “Simulation of Coherent Non-Linear Neutrino Flavor Transformation in the Supernova Environment. 1. Correlated Neutrino Trajectories,” Phys. Rev. D 74, 105014 (2006) [astro-ph/0606616].
[3] S. Hannestad, G. G. Raffelt, G. Sigl and Y. Y. Y. Wong, “Self-induced conversion in dense neutrino gases: Pendulum in flavour space,” Phys. Rev. D 74, 105014 (2006) [Erratum-ibid. D 76, 029901 (2007)] [astro-ph/0606616].
[4] G. L. Fogli, E. Lisi, A. Marrone and A. Mirizzi, “Collective neutrino flavor transitions in supernovae and the role of trajectory averaging,” JCAP 0712, 010 (2007) [arXiv:0707.1998 [hep-ph]].
[5] B. Dasgupta, A. Dighe, G. G. Raffelt and A. Y. Smirnov, “Multiple Spectral Splits of Supernova Neutrinos,” Phys. Rev. Lett. 103, 051105 (2009) [arXiv:0904.3542 [hep-ph]].
[6] G. Sigl and G. Raffelt, “General kinetic description of relativistic mixed neutrinos,” Nucl. Phys. B 406, 423 (1993).
[7] P. Strack and A. Burrows, “Generalized Boltzmann formalism for oscillating neutrinos,” Phys. Rev. D 71, 093004 (2005) [hep-ph/0504035].
[8] Y. Z. Qian and G. M. Fuller, “Neutrino-neutrino scattering and matter enhanced neutrino flavor transformation in Supernovae,” Phys. Rev. D 51, 1479 (1995) [astro-ph/9406073].
[9] G. R. Raffelt and G. Sigl, “Self-induced decoherence in dense neutrino gases,” Phys. Rev. D 75, 083002 (2007) [hep-ph/0701182].
[10] R. F. Sawyer, “The multi-angle instability in dense neutrino systems,” Phys. Rev. D 79, 105003 (2009) [arXiv:0803.3319 [astro-ph]].
[11] A. Esteban-Pretel, S. Pastor, R. Tomàs, G. G. Raffelt and G. Sigl, “Decoherence in supernova neutrino transformations suppressed by deleptonization,” Phys. Rev. D 76, 125018 (2007) [arXiv:0706.4998 [astro-ph]].
[12] S. Chakraborty, T. Fischer, A. Mirizzi, N. Saviano and R. Tomas, “No collective neutrino flavor conversions during the supernova accretion phase,” Phys. Rev. Lett. 107, 151101 (2011) [arXiv:1104.4031 [hep-ph]].
[13] S. Sarikas, G. G. Raffelt, L. Hudepohl and H. T. Janka, “Suppression of Self-Induced Flavor Conversion in the Supernova Accretion Phase,” Phys. Rev. Lett. 108, 061101 (2012) [arXiv:1109.3601 [astro-ph.SR]].
[14] B. Dasgupta, A. Dighe, A. Mirizzi and G. G. Raffelt, “Collective neutrino oscillations in non-spherical geometry,” Phys. Rev. D 78, 033014 (2008) [arXiv:0805.3300 [hep-ph]].
[15] G. Raffelt, S. Sarikas and D. de Sousa Seixas, “Axial Symmetry Breaking in Self-Induced Flavor Conversion of Supernova Neutrino Fluxes,” Phys. Rev. Lett. 111, no. 9, 091101 (2013) [Erratum-ibid. 113, no. 23, 239903 (2014)] [arXiv:1305.7140 [hep-ph]].
[16] G. Raffelt and D. d. S. Seixas, “Neutrino flavor pendulum in both mass hierarchies,” Phys. Rev. D 88, 045031 (2013) [arXiv:1307.7625 [hep-ph]].
[17] H. Duan, “Flavor Oscillation Modes In Dense Neutrino Media,” Phys. Rev. D 88, 125008 (2013) [arXiv:1309.7377 [hep-ph]].
[18] A. Mirizzi, “Multi-azimuthal-angle effects in self-induced supernova neutrino flavor conversions without axial symmetry,” Phys. Rev. D 88, no. 7, 073004 (2013) [arXiv:1308.402 [hep-ph]].
[19] S. Chakraborty and A. Mirizzi, “Multi-azimuthal-angle instability for different supernova neutrino fluxes,” Phys. Rev. D 90, no. 3, 033004 (2014) [arXiv:1308.5255 [hep-ph]].
[20] G. Mangano, A. Mirizzi and N. Saviano, “Damping the neutrino flavor pendulum by breaking homogeneity,” Phys. Rev. D 89, no. 7, 073017 (2014) [arXiv:1403.1892 [hep-ph]].
[21] H. Duan and S. Shalgar, “Spontaneous breaking of spatial symmetries in collective neutrino oscillations,” arXiv:1412.7097 [hep-ph].
[22] H. Duan, G. M. Fuller and Y. Z. Qian, “Symmetries in collective neutrino oscillations,” J. Phys. G 36, 105003 (2009) [arXiv:0808.2046 [astro-ph]].
[23] G. Emanuel, “Analytical Fluid Dynamics,” CRC Press.
[24] Lun-Shin Yao, “Non-linear instabilities,” arXiv:1104.1207 [math-ph].