Feynman path-integral approach to the QED$_3$ theory of the pseudogap

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In this work the connection between vortex condensation in a d-wave superconductor and the QED$_3$ gauge theory of the pseudogap is elucidated. The approach taken circumvents the use of the standard Franz-Tesanovic gauge transformation, borrowing ideas from the path-integral analysis of the Aharonov-Bohm problem. An essential feature of this approach is that gauge-transformations which are prohibited on a particular multiply-connected manifold (e.g. a superconductor with vortices) can be successfully performed on the universal covering space associated with that manifold.

1. Introduction

Recently, much attention has been focused on understanding the pseudogap$^{1,2,3,4,5}$ phenomena of the high-temperature superconductors. One possible explanation$^{7,8,9}$ invokes the notion that the pseudogap is due to the presence of pairing correlations above the superconducting transition temperature $T_c$. Within such a scenario, the lack of long-range phase coherence in the pseudogap is presumed to be due to the proliferation of vortex excitations in it. Following the approach of many recent papers$^{8,9,10,11,12}$, we consider the problem of coupling vortices to the quasiparticles of a d-wave superconductor. As discussed in Refs. 10, 11, there is an important distinction between the condensation of $\hbar c/2e$ (i.e. singly quantized) and $\hbar c/e$ (i.e. doubly quantized) vortices in a d-wave superconductor. Experimentally,$^{13}$ magnetic field-induced vortices seem to be exclusively of the singly quantized variety. We shall take this as evidence that, in considering vortex excitations in the pseudogap regime, it is sufficient to consider only $\hbar c/2e$ vortices.

The technical issues associated with $\hbar c/2e$ vortices have been discussed in Refs. 10, 11, 14 and amount to the fact that certain singular gauge transformations (which arise naturally when considering the Bogoliubov–de Gennes equation in the presence of vortices) lead to the presence of quasiparticle branch cuts. Here we shall adopt the point of view that these difficulties are not unique to the problem of performing gauge transformations in the presence of vortices: They arise when one considers the general problem of making gauge-transformations on multiply-connected manifolds. For example, consider the Aharonov-Bohm problem$^{15}$, in which one imagines attaching current leads to a doubly-connected metallic ring.
through which a solenoid penetrates. Even in the case of a solenoid with a radius
sufficiently small that the magnetic field is negligible in the ring, the presence of
a nonzero vector potential leads to quantum-mechanical interference of electrons
taking different paths around the ring while propagating between the leads. Thus,
one cannot make a gauge-transformation which eliminates the effect of the vector
potential on topologically distinct Feynman paths.

In a similar sense, the winding of the pair-potential phase around a superconduct-
ing vortex may not be trivially removed by a gauge transformation. In particular,
as noted above, making certain singular gauge-transformations in the presence
of $\hbar c/2e$ vortices leads to quasiparticle branch cuts. Our aim here is to discuss a
new approach to handling such branch cuts using the topological properties of path
integrals. Recently, Franz and Tesanovic $^{14}$ have introduced a singular gauge trans-
formation which avoids the introduction of quasiparticle branch cuts via a clever
trick involving splitting the vortices into two distinct groups ($A$ and $B$) and trans-
forming the electrons relative to group $A$ and the holes relative to group $B$. These
authors find that the action governing the quasiparticle dynamics of a d-wave super-
conductor in the presence of fluctuating vortices is given by the well-known problem
of three-dimensional quantum electrodynamics (QED$_3$): Dirac fermions coupled to
a fluctuating “Berry” gauge field $a$. However, as we shall discuss, the propagator is
not given by the usual propagator for QED$_3$; it is given by the associated gauge-
invariant propagator. This occurs because $a$ is not a physical (i.e. electromagnetic)
gauge-field; thus the propagator must not transform under gauge transformations
on $a$. Gauge-invariant propagators have appeared many times in the context of
strongly-correlated electron systems $^{16,17,18}$; in particular the gauge-invariant prop-
agator for QED$_3$ exhibits intriguing non-Fermi liquid behavior $^{11,18}$.

To obtain further insight into the physics of fluctuating vortices in d-wave su-
perconductors, here we shall take a novel approach inspired by Schulman’s $^{19}$ topo-
logical approach to the Aharonov-Bohm problem. Our task is to shed light on the
connection between fluctuating vortices and fluctuating gauge-fields. Towards this
end, we shall view a vortex as a “hole” which divides a superconductor into a
doubly-connected space. As in the Aharonov-Bohm problem, the fact that the su-
perconductor is multiply connected means that Feynman paths contributing to the
quasiparticle propagator fall into topologically distinct sectors. By making different
gauge transformations for topologically distinct Feynman paths, we shall see how
the quasiparticle branch cuts may be expressed in terms of the Berry gauge field $a$.
The purpose of this Paper is to arrive at the QED gauge theory of vortices in the
pseudogap regime of the cuprates via a path integral technique.

This Paper is organized as follows: In Sec. 2 we consider the problem of solv-
ing the Bogoliubov-de Gennes (BdG) equation in the presence of a static array of
vortices. Although our main interest will be in the problem of vortex fluctuations
in a d-wave superconductor, the static case will be sufficient to motivate the tech-
nical difficulties associated with vortices in superconductors. In Sec. 3, we review
some results from the theory of the Aharonov-Bohm effect $^{15}$ from a point of view
due originally to Schulman \(^{19}\) in which one considers path integrals in multiply-connected spaces. In Sec. 4, we revisit the BdG eigenproblem from the point of view of gauge transformations in multiply-connected spaces and isolate the effect of the quasiparticle branch cuts on Feynman paths. In Sec. 5, these branch cuts are represented via a functional integral over an auxiliary field \(a\); the theory is constructed to be explicitly invariant under gauge-transformations associated with this field. In Sec. 6 we extend the results of Sec. 4 and Sec. 5 to the case of dynamic vortex excitations, finally arriving at an expression for the Bogoliubov–de Gennes propagator (in the presence of vortices) as a gauge-invariant Green function. In Sec. 7 we conclude with a brief discussion of our results.

2. Bogoliubov-De Gennes Equation

The QED\(^3\) scenario of the pseudogap regime, like the nodal liquid scenario\(^{10}\) which preceded it, focuses on the effect of vortex excitations on the quasiparticles of a d-wave superconductor. In the present section, we provide motivation by examining the problem of static vortex excitations in a superconductor. The quasiparticle excitations of a superconductor are described by the Bogoliubov–de Gennes (BdG) equation

\[
H \begin{pmatrix} u_n \\ v_n \end{pmatrix} = E_n \begin{pmatrix} u_n \\ v_n \end{pmatrix} \tag{1}
\]

\[
H \equiv \begin{pmatrix} -(\nabla - ieA)^2 - \mu & e^{i\varphi/2} \Delta e^{-i\varphi/2} \\ e^{-i\varphi/2} \Delta^* e^{i\varphi/2} & (\nabla + ieA)^2 + \mu \end{pmatrix}, \tag{2}
\]

where \(\mu\) is the chemical potential, \(\varphi\) is the local superconducting phase, \(e\) is the electronic charge, and \(\Delta\) is the usual d-wave pairing operator, which we take to be given by \(\Delta = \Delta_0 \hat{p}_x \hat{p}_y\). We have chosen units in which \(\hbar^2/2m = 1\), with \(m\) being the quasiparticle mass. Here, \(u_n\) and \(v_n\) are the electron and hole parts of the BdG wavefunction, respectively. The electromagnetic gauge field \(A\) (which we shall often suppress) and \(\varphi\) conspire to give this theory the following local \(U(1)\) symmetry:

\[
A_\mu \rightarrow A_\mu - \frac{1}{e} \partial_\mu \chi, \tag{3}
\]

\[
u_n \rightarrow e^{i\chi} u_n, \tag{4}
\]

\[
u_n \rightarrow e^{-i\chi} v_n, \tag{5}
\]

\[
\varphi \rightarrow \varphi + 2\chi. \tag{6}
\]

In the presence of vortices, \(\varphi\) has singularities at the locations \(r_i\) of each vortex (of vorticity \(q_i = \pm 1\)):

\[
\nabla \times \nabla \varphi(r) = \sum_i 2\pi q_i \delta^{(2)}(r - r_i). \tag{7}
\]

A natural way to proceed with solving Eq. (1) is to attempt to include the effects of vortex excitations perturbatively by expanding in small phase gradients. As \(\varphi\) appears only in the exponential of the off-diagonal terms, one is motivated to perform
the following naive gauge transformation

\[ u_n \rightarrow \tilde{u}_n \equiv e^{-i\varphi/2} u_n, \]
\[ v_n \rightarrow \tilde{v}_n \equiv e^{i\varphi/2} v_n. \]

This effectively moves the phase from the off-diagonal components and introduces it as a phase gradient in the diagonal components of the BdG equation:

\[ H'(\tilde{u}_n \tilde{v}_n) = E_n (\tilde{u}_n \tilde{v}_n), \]
\[ H' \equiv \left( -(\nabla + \frac{i}{2} \nabla \varphi)^2 - \mu \right) \Delta \left( \nabla - \frac{i}{2} \nabla \varphi \right)^2 + \mu. \]

However, as discussed by Balents et al \(^\text{10}\), such a gauge transformation produces branch cuts in the Bogoliubov–de Gennes eigenstates. This can be seen by noting that, if we assume that \( u_n \) and \( v_n \) are single-valued functions then Eq. (9) must be solved under the condition that \( \tilde{u}_n \) and \( \tilde{v}_n \) each gain a factor of \( e^{i\pi} \) upon encircling every vortex. The authors of Ref. 10 argue that such branch cuts lead to frustration effects that would strongly favor the pairing of vortices. Thus, by only allowing doubly quantized vortices, there are no branch cuts in the fields \( \tilde{u}_n \) and \( \tilde{v}_n \) to worry about.

Here, we follow Franz and Tesanovic \(^\text{11}\) in assuming that the correct approach is to consider the condensation of singly-quantized vortices in a d-wave superconductor. They avoid the branch-cut difficulty by making a gauge transformation which does not directly introduce branch cuts but still keeps track of their physical effects. Briefly, their technique \(^\text{14}\) involves splitting the vorticies into two groups labelled “A” and “B” and then performing a gauge transformation of the form

\[ u_n \rightarrow u_n e^{i\varphi_A} \]
\[ v_n \rightarrow v_n e^{-i\varphi_B}, \]

where \( \varphi_{A(B)} \) is the phase associated only with the vortices in group \( A(B) \). This gauge transformation has the advantage of not introducing any branch cuts in the quasiparticle wave function while at the same time treating the electrons and holes on an equal footing. The Berry gauge field \( a \) emerges, upon averaging over all vortex configurations, as the difference of phase gradients \( \nabla \varphi_A - \nabla \varphi_B \).

3. Path-integral treatment of the Aharonov-Bohm effect

In Sec. 2 we discussed one difficulty associated with solving the BdG equation in the presence of vortex excitations, i.e., that naive gauge transformations of the form of Eq. (8) introduce branch cuts in the quasiparticle wavefunctions. To motivate the path integral technique which we shall use to handle these branch cuts, in the present section we review previously known results \(^\text{19,20,21,22,23}\) in a related system in which gauge transformations must be made with care: The Aharonov-Bohm problem \(^\text{15}\). Our aim is to find an expression for the propagator \( G(x,y,t) \) for an
electron constrained to a ring; we represent this physical space by the symbol $M$. A thin solenoid penetrates the origin and is described by a vector potential $A_r = 0$ and $A_\theta = \phi/2\pi r$, with $\phi$ being the total flux. The Schrödinger equation for $G(x,y,t)$ takes the form

$$(-i\partial_t - (\nabla - ieA)^2)G(x,y,t) = \delta(x - y)\delta(t).$$

(13)

From Feynman’s path-integral representation of quantum mechanics, we know that $G(x,y,t)$ may be expressed in terms of a sum over all paths of an amplitude for each path. In the present situation, the doubly-connected nature of $M$ indicates that such paths may be divided into homotopically distinct classes depending on how many times the path in a particular homotopy class winds around the origin. Thus, following Schulman \textsuperscript{19}, we may express $G$ in the form

$$G(x,y,t) = \sum_n G_n(x_n,y,t),$$

(14)

where the function $G_n(x_n,y,t)$ contains only Feynman paths that wind around the origin $n$ times. The subscript $n$ on $x$ reminds us that $x_n$ is the same as $x$ after having wound $n$ times around the origin. Let us consider the physical meaning of $G_n(x_n,y,t)$. The space $M$ on which Eq. (13) is to be solved has the topology of a torus. However, by keeping only Feynman paths which wind $n$ times, it is as if we are solving the same equation on a helix-shaped space which winds $n$ times: A helix (topologically, a line) is the universal covering space of a torus. We shall denote the universal covering space by the symbol $M^*$; the important properties of $M^*$ are as follows: 1) $M^*$ is locally equivalent to $M$, and 2) $M^*$ is simply connected. It is further true that the Schrödinger equation [i.e. Eq. (13)] is satisfied for each of the terms $G_n(x_n,y,t)$ entering Eq. (14) \textsuperscript{24}.

The fact that $M^*$ is simply connected means that the quantity $\exp\left(i e \int_{x_n}^{x_n} A \cdot ds\right)$ is well-defined on it and can be used to simplify the equation for $G_n$. Thus, by writing

$$G_n(x_n,y,t) = \tilde{G}_n(x_n,y,t) \exp\left(i e \int_{y_n}^{x_n} A \cdot ds\right),$$

(15)

it can be seen that $\tilde{G}_n$ satisfies

$$(-i\partial_t - \nabla^2)\tilde{G}_n(x_n,y,t) = \delta(x_n - y)\delta(t),$$

(16)

i.e., it is the Green function for the free-particle (by this we mean $A = 0$) Schrödinger equation on $M^*$, which we denote by $\tilde{G}^{(0)}_n(x_n,y,t)$ and has the explicit form \textsuperscript{20,21,22,23}

$$\tilde{G}^{(0)}_n(x_n,y,t) = \frac{1}{4\pi it} e^{\frac{i}{4t}(x^2+y^2)} \int_{-\infty}^{\infty} d\lambda e^{i\lambda(\theta' - \theta + 2\pi n)} I_\lambda\left(\frac{x y}{2it}\right),$$

(17)

where $I_\lambda$ is the modified Bessel function and we have written the coordinates $x_n$ and $y$ in terms of the radial coordinates $(x,\theta' + 2\pi n)$ and $(y,\theta)$, respectively. We emphasize that this is not the usual two-dimensional free particle propagator, but
that part of the propagator due to Feynman paths with winding number \( n \) (or, equivalently, the free propagator on \( M^* \)). The usual free-particle propagator may be obtained from Eq. (17) by summing over all winding numbers \( n \).

The next step is to insert \( \bar{G}^{(0)}(x_n, y, t) \) into Eq. (14) after having directly calculated the factor \( \exp \left( ie \int_{x_n}^{x} A \cdot ds \right) \):

\[
G(x, y, t) = \sum_{n} e^{ie(\theta' - \theta + 2\pi n)/2\pi} \bar{G}^{(0)}(x_n, y, t) = \frac{1}{4\pi t} e^{\frac{i}{2}(x^2 + y^2)} \sum_{m} e^{im(\theta' - \theta)} I_{|m - \alpha|} \left( \frac{xy}{2it} \right),
\]

where \( \alpha \equiv e\phi/2\pi \) measures the flux through the hole. Equation (19) is the single-particle propagator for the Aharonov-Bohm problem, which is obtained from Eq. (18) using Eq. (17) along with the Poisson summation formula.

For our purposes, we are primarily interested in Eq. (18), which exhibits the structure proposed by Schulman: The propagator is expressed as a sum of terms, each of which is a free propagator associated with a particular winding number (i.e., on the covering space) multiplied by a gauge-field dependent factor. Before proceeding, however, let us briefly note one feature of Eq. (19), namely that for integral \( \alpha \), the summation over \( m \) may be shifted by \( \alpha \) and Eq. (19) becomes the two-dimensional free propagator, i.e., the solution to Eq. (16) in free space. This is merely the statement that for integral \( \alpha \) there are no interference effects associated with the flux and the introduction of the universal covering space was unnecessary. For non-integral \( \alpha \), however, the universal covering space provides a natural way to identify Feynman paths in different homotopy classes which have different phase factors arising from the gauge field. In a similar fashion, we shall see that our technique for handling vortex excitations in d-wave superconductors by invoking the covering space is useful for the case of \( hc/2e \) vortices [for which the gauge transformation in Eq. (8) produced branch cuts] but unnecessary for doubly quantized vortices [for which the gauge transformation in Eq. (8) produced no branch cuts]. In the next section, we shall exploit the universal covering space to gain insight into the problem of \( hc/2e \) vortices in d-wave superconductors.

### 4. Vortices in a d-wave superconductor: Static case

Having discussed the covering-space approach to the Aharonov-Bohm problem, in the present section we apply these ideas to the context of multiple vortices in a d-wave superconductor. In this section we shall assume a high-temperature approximation in which the vortices can be taken to be static. The extension to quantum-vortex fluctuations will be straightforward and discussed in Sec. 6. Our starting point is the following expression for the BdG propagator averaged over vortex positions:

\[
(G(r, r'; \tau))_\phi \equiv \int D\phi \, G(r, r'; \tau) e^{-S[\phi]},
\]

where \( S[\phi] \) is the action of the gauge field.
\[(\partial_\tau + H)G(r, r'; \tau) = \delta^{(2)}(r - r')\delta(\tau),\]  
(21)

where \(G(r, r'; \tau)\) is the BdG Green function in the presence of a fixed pattern of vortices and \(S[\varphi]\) is the Boltzmann weight associated with a particular, static, pattern of vortices. The Hamiltonian \(H\) is given by Eq. (2); however, henceforth we shall not display the physical gauge field \(A\). It may be reinstated at the end of the calculation.

Each vortex configuration entering into Eq. (20) corresponds to a particular set of locations for the \(r_i\) in Eq. (7). Thus, as in the Aharonov-Bohm problem, the manifold \(M\) on which we aim to solve Eq. (21) is multiply connected, i.e., it has “holes” at each \(r_i\). To ascertain the physical importance of this multiply-connectedness, we exchange the original multiply-connected space \(M\) (on which Eq. (21) is defined) for its simply-connected covering space \(M^*\) on which a gauge transformation of the form of Eq. (8) may be performed. As we saw in Sec. 2, for the Aharonov-Bohm effect the relevant universal covering space was a line. In the present case, the fact that we shall always consider the case of multiple vortices means that the associated universal covering space is difficult to envision physically. Fortunately, as in the case of the Aharonov-Bohm effect, the covering space will merely serve as a way to keep track of the phase picked up by various Feynman paths.

Following our procedure in the preceding section, we express \(G(r, r'; \tau)\) as a sum of propagators, the Feynman paths of which are grouped into topologically distinct sectors:

\[G(r, r', \tau) = \sum_n G_n(r_n, r'; \tau),\]  
(22)

where the vector index \(n = (n_1, n_2, \cdots)\) keeps track of the winding number \(n_i\) associated with the \(i\)th vortex. Technically speaking, this winding number does not fully label the elements of the non-abelian homotopy group associated with the multiply-connected space in question, but this does not matter for the calculation we are attempting since all that matters is that there exists a label for these elements. On \(M^*\) we have at our disposal the quantity

\[\chi_n(r_n) \equiv \int_0^{r_n} \nabla \varphi(\bar{r}) \cdot d\bar{r},\]  
(23)

which we emphasize is not uniquely defined on the space \(M\). For simplicity, we have set \(r' = 0\). Physically, \(\chi_n\) represents a line integral along a particular path through the vortices. We write \(G_n(r_n, 0, \tau)\) in the form

\[G_n(r_n, 0, \tau) = \exp \left[ \frac{i}{2} \chi_n(r_n) \hat{\sigma}_3 \right] \tilde{G}_n(r_n, 0, \tau),\]  
(24)

where \(\hat{\sigma}_3\) is the usual Pauli matrix. As in Sec. 3, this amounts to making different gauge transformations for Feynman paths having different winding numbers. The \(\tilde{G}_n\) satisfy

\[(\partial_\tau + H') \tilde{G}_n(r_n, 0; \tau) = \delta^{(2)}(r_n)\delta(\tau),\]  
(25)
where $H'$ is given by Eq. (10), i.e., the Hamiltonian obtained via the naive gauge transformation of Sec. 2.

Having made a gauge transformation on the covering space, we proceed by computing the path-dependent gauge-transformation factor $\exp \left[ \frac{i}{2} \chi_n(r_n) \sigma_3 \right]$. The path-dependence of $\chi_n(r_n)$ is exhibited schematically in Fig. 1. Consider for example the loop made by paths B and C: since there is one vortex inside the loop, we have

$$\int_{B-C} \nabla \varphi(\bar{r}) \cdot d\bar{r} = \pm 2\pi,$$

depending on the vorticity of the enclosed vortex. Thus, $\chi_n(r_n)$ differs by $2\pi$ for paths B and C. More generally, we may define

$$\chi_n(r_n) = \varphi(r) - \varphi(0) + 2\pi \sum_i q_i n_i,$$

$$\exp \left[ \frac{i}{2} \chi_n(r_n) \sigma_3 \right] = \exp \left[ \frac{i}{2} (\varphi(r) - \varphi(0)) \sigma_3 e^{i\pi \sum_i q_i n_i} \right],$$

where it is important to note that in Eq. (28) we have used the fact that $\exp(i\pi n \sigma_3) = \exp(i\pi n)$ for integer $n$. We thus have the following expression for the averaged BdG Green function:

$$[G(r, r'; \tau)]_\varphi = \int D\varphi \ e^{-S[\varphi]} e^{\frac{i}{2} \varphi(r) \sigma_3} \left[ \sum_n \tilde{G}_n(r_n, r'_n; \tau) e^{i\pi \sum_i q_i n_i} \right] e^{-\frac{i}{2} \varphi(r') \sigma_3}.$$

Since we are considering only singly quantized vortices, so that $q_i = \pm 1$, the branch-cut factor $\exp i\pi \sum_i q_i n_i$ is given by $\pm 1$ for different trajectories and is the manifestation, within the present approach, of the quasiparticle branch cuts encountered in
Sec. 2. In Fig. 1, since they are separated by two vortices, the branch-cut factor is the same for paths A and B. In general, if we considered the condensation of pairs of vortices \( q_i = \pm 2 \), then these factors could be all taken to be unity and there would be no need to invoke the covering space construction. However, although there is a \( \mathbb{Z}_2 \) character to the physics of fluctuating branch cuts as discussed here, it has an origin which is distinct from that of the \( \mathbb{Z}_2 \) gauge theory of Senthil and Fisher, which, like the theory of Ref. 10, only incorporates doubly quantized vortices.

Before proceeding to the next stage of the calculation, we pause to note that the argument of the functional integral in Eq. (29) is the analogue, within the present context, of Eq. (18) in the calculation of the Aharonov-Bohm propagator in Sec. 3. For that case it is possible to calculate the propagator on the covering space [cf. Eq. (17)] explicitly. In the present case, however, \( \tilde{G}_n(r_n, r'; \tau) \) is most likely not analytically solvable and, in the next section, we proceed by making some simplifying approximations.

5. Effect of branch cuts on quasiparticle dynamics

In the present section, we shall attempt to evaluate the functional integral over vortex and spin-wave like phase fluctuations in Eq. (29). As discussed in Sec. 4, the effect of the branch cuts in the quasiparticle wavefunctions has been traced back to the "branch-cut" factor \( \exp \left( i \pi \sum_i q_in_i \right) \) in Eq. (28) which differs for homotopically inequivalent trajectories. If this factor were absent (e.g. if we had only allowed the proliferation of doubly-quantized vortices so that, in effect, \( \sum_i q_in_i \) would be constrained to be an even integer), then there would be no need to divide the equation for the propagator into topologically inequivalent sectors, and the gauge transformation given in Eq. (8) could have been made on \( M \).

Let us turn to the evaluation of the functional integral over phase fluctuations. Formally, \( \varphi \) appears in two distinct places in Eq. (29): Firstly, it appears in the definition of \( H' \) in Eq. (10). As our aim is to focus on the effect of branch cuts, here we neglect \( \nabla \varphi \) in \( H' \). This simplifying approximation is not technically necessary at this point. It is motivated by the fact that, were we to keep the local phase gradient as another ("Doppler") gauge field \( \mathbf{v} \) then, as discussed in Refs. 11, 12, \( \mathbf{v} \) would end up being irrelevant in the renormalization group sense. Secondly, \( \varphi \) appears in the exponential factors \( \exp (\pm i \varphi(r)\sigma_3/2) \) in Eq. (29). Let us examine the matrix structure of the argument of the functional integral by writing it as a \( 2 \times 2 \) matrix:

\[
\exp \left( \frac{2}{i} \varphi(r)\sigma_3 \right) \sum_n \tilde{G}_n e^{-\frac{2}{i} \varphi(r')\sigma_3} = \begin{pmatrix} \tilde{G}_{n,11} e^{\frac{2}{i} \varphi(r) - \varphi(r')} & \tilde{G}_{n,12} e^{\frac{2}{i} (\varphi(r) + \varphi(r'))} \\ \tilde{G}_{n,21} e^{-\frac{2}{i} (\varphi(r) + \varphi(r'))} & \tilde{G}_{n,22} e^{-\frac{2}{i} (\varphi(r) - \varphi(r'))} \end{pmatrix}. \tag{30}
\]

By writing out the full matrix form for the product of these three factors appearing in Eq. (29) we see that whereas the diagonal components contain a phase difference, the off-diagonal components contain a sum of phases and thus must vanish upon performing the average over the \( \varphi \). This merely implies that the off-diagonal components of the single-particle Green function must vanish in the pseudogap regime,
i.e., that superconductivity has been destroyed. Being phase differences, the phase factors appearing in the diagonal components do not vanish upon averaging over $\varphi$. We shall take them to be unity, invoking the same argument which we used for the phase gradients in $H'$.

By making the preceding approximations, we are assuming that the branch cuts factors lead to quantum interference effects that have a dominant effect on the low-energy quasiparticles. We thus arrive at

$$\langle G(r, r'; \tau) \rangle_\varphi \simeq \int \mathcal{D}\varphi \, e^{-S[\varphi]} \sum_n e^{i\pi \sum_i q_in_i}$$

$$\times \langle r_n, \tau \rangle \left[ \partial_\tau + \left( -\nabla^2 - \mu \Delta \frac{\partial p_x p_y}{\nabla^2 + \mu} \right)^{-1} \right]_{\text{diag.}} |r', 0\rangle,$$

where we have explicitly written the expression for $\tilde{G}_n(r_n, r'; \tau)$ within this approximation. The subscript “diag.” indicates that our expression for $\langle G(r, r'; \tau) \rangle_\varphi$ only includes the diagonal components of the expression on the right side of Eq. (31), the off-diagonal components being zero as noted above. Henceforth, we shall suppress this subscript, although all subsequent equations share this property that only the diagonal components are of physical interest.

The next step is to represent the branch-cut factor in terms of the flux of a field $a$, allowing us to evaluate the functional integral over vortex configurations in Eq. (31). To do this, we define $a$ such that its local “magnetic field” has singularities at the locations of all the vortices (but curl-free on $M^*$):

$$\nabla \times a(r) = \pi \sum_i q_i \delta^{(2)}(r - r_i).$$

(32)

To account for the factor $e^{i\pi \sum_i q_in_i}$, we write $\sum_i q_in_i$ as the flux through a carefully chosen loop:

$$\sum_i q_in_i = -\frac{1}{\pi} \int_n a \cdot dr + \frac{1}{\pi} \int_\Gamma a \cdot dr,$$

(33)

where the subscript $n$ indicates that the first integral winds along the path falling in the homotopy class labelled by $n$ (i.e. from $y$ to $x_n$ in $M^*$), and the subscript $\Gamma$ indicates that the integral is to be taken along some arbitrary fixed reference path $x_\Gamma(s)$:

$$\int_\Gamma a \cdot dr \equiv \int_0^1 ds a_\mu(x_\Gamma(s)) \frac{dx_\Gamma^\mu(s)}{ds},$$

(34)

where $x_\Gamma(0) = r'$ and $x_\Gamma(1) = r$. Thus, we have

$$\langle G(r, r'; \tau) \rangle_\varphi \simeq \int \mathcal{D}\varphi \, e^{-S[\varphi]} \int \mathcal{D}a \, e^{i\int_a \frac{a \cdot dr}{\pi}} \sum_n e^{-i \int_n a \cdot dr}$$

$$\times \langle r_n, \tau \rangle \left[ \partial_\tau + \left( -\nabla^2 - \mu \Delta \frac{\partial p_x p_y}{\nabla^2 + \mu} \right)^{-1} \right]_{\text{diag.}} |r', 0\rangle.$$
\times \delta(\nabla \times \mathbf{a}(\mathbf{r}) - \pi \sum_i q_i \delta^{(2)}(\mathbf{r} - \mathbf{r}_i)), \quad (35)

where we must emphasize that the field \( \mathbf{a} \) has entered in a gauge-invariant way, in the sense that \( G(\mathbf{r}, \mathbf{r}'; \tau)|_\varphi \) is manifestly dependent only on \( \nabla \times \mathbf{a} \). Although the quantity \( \int_n \mathbf{a} \cdot d\mathbf{r} \) in Eq. (33) is trajectory-dependent, the integrals along the path \( \Gamma \) are by definition the same for each term in Eq. (35). Before proceeding, we remark that this propagator is reminiscent of the gauge-invariant Green function discussed in Ref. (16).

Next, we make another gauge transformation of the form of Eq. (24), absorbing the factors \( \exp -i \int_n \mathbf{a} \cdot d\mathbf{r} \) into the Green function associated with that particular trajectory (or, rather, class of trajectories associated with a particular homotopy class in the presence of the vortex excitations). After making such a gauge transformation, the division of propagators into topologically distinct sectors is redundant (the field \( \mathbf{a} \) keeps track of the branch cut factors), leaving us with the following expression for the Green function:

\[
G(\mathbf{r}, \mathbf{r}'; \tau)|_\varphi \simeq \int \mathcal{D} \varphi e^{-S[\varphi]} \int \mathcal{D} \mathbf{a} e^{i \int_{\Gamma} \mathbf{a} \cdot d\mathbf{r}} \frac{1}{\partial_\tau + H[\mathbf{a}]} |_{\mathbf{r}', 0} \\
\times \delta(\nabla \times \mathbf{a}(\mathbf{r}) - \pi \sum_i q_i \delta^{(2)}(\mathbf{r} - \mathbf{r}_i)), \quad (36)
\]

\[
H[\mathbf{a}] \equiv \left( \frac{(\mathbf{p} + \mathbf{a})^2 - \mu}{\hat{D}} - \frac{\hat{D}}{(\mathbf{p} + \mathbf{a})^2 + \mu} \right), \quad (37)
\]

\[
\hat{D} \equiv \frac{\Delta_0}{2} [p_x + a_x] p_y + a_y + (p_y + a_y) (p_x + a_x). \quad (38)
\]

The most straightforward way to verify the equality of Eq. (35) and Eq. (36) is to examine the gauge-transformation properties of the arguments of the respective functional integrals. The next step is to evaluate the functional integral over vortex and spin-wave like excitations, leaving an effective functional integral over the field \( \mathbf{a} \). Formally, we may follow Franz and collaborators\textsuperscript{11}, writing

\[
e^{-S_a} \equiv \int \mathcal{D} \varphi e^{-S[\varphi]} \delta(\nabla \times \mathbf{a}(\mathbf{r}) - 2\pi \sum_i q_i \delta^{(2)}(\mathbf{r} - \mathbf{r}_i)). \quad (39)
\]

Using this definition of the vortex action, we arrive at our final expression for the single-particle BdG propagator in the presence of static vortex excitations:

\[
G(\mathbf{r}, \mathbf{r}'; \tau)|_\varphi \simeq \int \mathcal{D} \mathbf{a} e^{-S_a} e^{i \int_{\Gamma} \mathbf{a} \cdot d\mathbf{r}} \frac{1}{\partial_\tau + H[\mathbf{a}]} |_{\mathbf{r}', 0}. \quad (40)
\]

Equation (40) is essentially our main result, i.e., we have obtained a gauge-theory model for vortex excitations in a d-wave superconductor via a path-integral technique. However, we recall that until now we have considered only static vortex excitations, so that the functional integral over vortex excitations in, e.g., Eq. (20), is truly a thermal average. As we shall see, however, the generalization to arbitrary dynamic vortex excitations is straightforward within the path-integral technique. In the next section, we discuss this issue in detail.
6. Vortices in a d-wave superconductor: Dynamic case

In arriving at Eq. (40), we have made use of the Feynman path integral description of the BdG propagator in the presence of a static pattern of vortices. In the present section, we generalize this procedure to the case of arbitrary fluctuating vortices. The quantity we are interested in calculating is, formally, the same as was discussed in Secs. 4 and 5 and is given by

\[ G(r, r'; \tau) \equiv \left( \frac{1}{\partial_\tau + H} \right) \int D\phi e^{-S[\phi]} \langle r, \tau | 1 \partial_\tau + H | r', 0 \rangle, \]  

where

\[ H \equiv \left( \begin{array}{cc} -\nabla^2 - \mu & e^{i\varphi/2} \Delta e^{-i\varphi/2} \\ e^{-i\varphi/2} \Delta e^{i\varphi/2} & \nabla^2 + \mu \end{array} \right), \]  

with the only distinction from the preceding discussion is that now \( \phi \) is time-dependent and \( S[\phi] \) is an action, as opposed to a Boltzmann.

In the picture discussed in the preceding sections, the various Feynman paths were parametrized by the temporal variable and fell into homotopically inequivalent classes defined by the locations of the (static) vortices. Now imagine that the vortices are time-dependent. One may of course still construct a conventional Feynman path integral even though \( H \) is time-dependent. However, since the vortex positions evolve in time, Feynman paths parametrized by time do not fall into homotopically inequivalent classes in the same way. Thus, it is convenient to use a path-integral method which treats the time variable \( \tau \) on an equal footing with the spatial coordinates. This may be realized by using a so-called “fifth-parameter” path integral scheme. Within such a scheme, one introduces a fictitious temporal coordinate \( \lambda \) and constructs a path integral representation of the corresponding Green function in which the temporal variable in the Feynman paths is \( \lambda \), not \( \tau \). For example, consider the following Schrödinger equation:

\[ (i \partial_\lambda + \mathcal{L}) \Gamma(x, y; \lambda) = -i \delta(x - y) \delta(\lambda), \]  

where \( x \equiv (r, \tau) \) and \( y \equiv (r', 0) \). It is clear that the propagator \( \Gamma(x, y; \lambda) \) may be expressed as a path integral in which the paths are labeled by \( \lambda \). Furthermore, it may be simply related to the argument of Eq. (41) via

\[ \langle r, \tau | \frac{1}{\partial_\tau + H} | r'0 \rangle = \int_0^\infty d\lambda \Gamma(x, y; \lambda) e^{-\epsilon \lambda}, \]  

where \( \epsilon = 0^+ \).

The utility of the preceding discussion is that now we have a path-integral representation (i.e. that of Eq. (43) for the argument of Eq. (41) in which the variable \( \tau \) is treated like any other coordinate, so that instead of thinking of vortices in a two-dimensional \( XY \) model we envision vortex loops in a three-dimensional \( XY \) model. The vortices are “static” relative to the coordinate \( \lambda \), allowing us to split the Feynman paths into homotopically distinct sectors depending on how they wind relative to them. This amounts to constructing an equation for \( \Gamma(x, y; \lambda) \) that is of the
Feynman path-integral approach to the QED theory of the pseudogap

form of Eq. (22). Following the same steps as in the static case (i.e., making gauge transformations on the covering space, etc.), generalized from 2 to 3 dimensions, we find that $[G(r, r'; \tau)]_\varphi$ is given by

$$G(r, r'; \tau) \simeq \int \mathcal{D}a e^{-S_a} e^{i \int \mathbf{a} \cdot d\mathbf{r} / [\partial_\tau + i a_\tau + H[a](r', 0)]}$$

(46)

$$S_a = -K (\partial_{\nu} a_\mu - \partial_{\mu} a_\nu)^2 / 4,$$

(47)

where now the vector field $a$ has three components given by $a = (a_\tau, a_x, a_y)$. The boldface quantity $a$ still refers to the spatial vector $a = (a_x, a_y)$, and $H[a]$ is still given by Eq. (36) (although we again emphasize that now $a$ fluctuates dynamically). The line integral with subscript label $\Gamma$ is similarly generalized from Eq. (34) to now indicate a line integral from $(r', 0)$ to $(r, \tau)$. The gauge-invariant action $S_a$ (with $K$ being an appropriate coupling) is formally defined by an equation exactly analogous to Eq. (39); in Eq. (47) we have used the results of Franz et al.\textsuperscript{11} who computed it using (see also Ref. 12) a model of free fluctuating vortex loops. It is important to note however that this form for $S_a$ could be expected on general symmetry grounds, it being the simplest quadratic action which respects gauge invariance.

To make the connection between $[G(r, r'; \tau)]_\varphi$ and the propagator for the QED\textsubscript{3} gauge theory (i.e., to make the gauge symmetry manifest) we represent $[G(r, r'; \tau)]_\varphi$ in terms of a fermionic path integral

$$G(r, r'; \tau) \simeq \int \mathcal{D}\psi \mathcal{D}\psi^\dagger \mathcal{D}a e^{-S_\psi - S_a} \psi(r, \tau) \psi^\dagger(r', 0) e^{i \int \mathbf{a} \cdot d\mathbf{r}},$$

(48)

$$S_\psi \equiv \int d^3 x \psi^\dagger [\partial_\tau + i a_\tau + H[a]] \psi,$$

(49)

where $\psi^\dagger$ and $\psi$ represent Nambu spinor fields. The action $S_\psi + S_a$ is invariant under the gauge transformation $\psi \rightarrow e^{-i \chi} \psi$, $a_\mu \rightarrow a_\mu + \partial_\mu \chi$. In addition, we see that the factor $e^{i \int \mathbf{a} \cdot d\mathbf{r}}$ in the argument of Eq. (48) ensures that

$$\psi(r, \tau) \psi^\dagger(r', 0) e^{i \int \mathbf{a} \cdot d\mathbf{r}}$$

(50)

and therefore $[G(r, r'; \tau)]_\varphi$, is also gauge-invariant. One may also expand the fermions near the nodes of the d-wave order parameter; by combining the Nambu fields $\psi$ into four component Dirac spinors\textsuperscript{11} one may complete the connection to QED\textsubscript{3}. However, for the present purposes it is sufficient to have the above gauge theory be our final expression.

7. Discussion

In this Paper, we have explored the effect of vortices on the quasiparticle excitations of d-wave superconductors in an attempt to understand the pseudogap phenomena. Our final expression for the single-particle Green function (i.e., Eq. (46)) makes direct contact with the QED\textsubscript{3}\textsuperscript{11} scenario of the pseudogap phenomena. As discussed in Sec. 1, the original formulation of the QED\textsubscript{3} of the pseudogap phenomena\textsuperscript{11} relied on splitting the vortices into two groups ($A$ and $B$) and making a
gauge transformation of the electrons with respect to group $A$ and the holes with respect to $B$. The aim of such an approach is to avoid directly introducing branch cuts in the quasiparticle wavefunctions.

The approach presented here deals with the quasiparticle branch cuts in a more direct fashion, by realizing that the problem of solving the BdG equation in the presence of vortices may be formulated as a problem of making gauge transformations on multiply connected manifolds. By going to the universal covering space of the associated manifold, the branch cuts emerge in a controllable fashion as factors of $\pm 1$ multiplying various Feynman paths. The field $a$ arises in an attempt to mimic the effect of these branch-cuts; in particular the line integral factor in Eq. (48) comes from expressing the branch cut factors $\pm 1$ associated with various Feynman paths in terms of the flux of $a$. Although we have clarified the origin of the Berry gauge field in the BdG action, we have not clarified issues associated with the calculation of the single particle Green function [i.e. Eq. (48)] As has been noted in several recent papers $^{18,26,27}$, there are many technical difficulties associated with the calculation of this quantity due to the presence of the line integral over the gauge field. As noted in a recent preprint $^{28}$, however, for certain gauge-invariant quantities these line integrals do not appear.

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