Schauder bases in Dirac modules over quaternions.

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Abstract

Dirac modules over the quaternion skew field are investigated on a compact domain relative to the supremum norm and Hardy’s norm with the parameter $1 < p < \infty$ as well. An existence of Schauder bases in them is proved. Procedures for construction of such bases are outlined.

1 Introduction.

The theory of Schauder bases in Banach spaces is the important part of Functional Analysis [15, 19, 24, 48]. As it is known now it exists not in all Banach spaces. But an existence or finding of it in concrete classes of Banach spaces is frequently a serious problem related with their particular structure [15, 19, 22, 23, 24, 38]-[42, 48].

On the other hand, hypercomplex analysis over Clifford algebras and quaternions in particular is developing fast (see [4], [10] - [16], [25, 26, 27, 29], [47] and references therein). It has many applications not only in mathematics, but also in natural sciences [13, 14, 17]. For example it permits to integrate new types of partial differential equations (see [16, 30] and references therein).

One of the classical examples is the solution of Klein-Gordon’s hyperbolic partial differential equation by Dirac with the help of quaternions [7, 13, 14]. Since that time the partial differential operator of the first order over...
quaternions used by Dirac for the decomposition of the hyperbolic partial
differential Klein-Gordon operator is widely used and is frequently known
under his name.

The quaternion skew field $\mathbf{H}$ is associative and non-commutative. It is
the algebra over the real field $\mathbf{R}$, but it is not the algebra over the complex
field $\mathbf{C}$, because the center of $\mathbf{H}$ is the real field.

Each complex holomorphic or harmonic function has locally a power series
expansion, but for their concrete Banach spaces it is frequently a serious
problem whether they have Schauder bases $[3, 22, 23, 38]-[41, 48]$.

In the paper $[3]$ an existence of a Schauder basis in the Banach algebra
$A(K)$ of all holomorphic functions on the open unit circle $K$ in the complex
field $\mathbf{C}$ having continuous bounded extensions on its closure $\bar{K}$ was proved
and a procedure for its construction was described.

Characterizing complex holomorphic functions by the condition $\bar{\partial}f(x) =
0$ one can consider their quaternion analog $\sigma f = 0$, where $\bar{\partial}f(x) = \partial f(x)/\partial \bar{x}$
for the complex variable $x$, whilst $\sigma$ is the Dirac operator over the quaternion
skew field. Nevertheless, the technique presented in $[3]$ does not work over
the quaternion skew field $\mathbf{H}$ because of specific features of $\mathbf{H}$ and $\ker(\sigma)$.
Moreover, in that work very particular properties of complex holomorphic
functions on $K$, functions on the circumference $S^1$ and the commutativity
of the complex field were used for the proof, which are not valid over the
quaternion skew field. For example, converging power series over quaternions
are more complicated and contain additives like $a_{n,1} z^{k_1} \ldots a_{n,m} z^{k_m}$, where $z$
is a variable in a domain $\Omega$ in $\mathbf{H}$, $k_1, \ldots, k_m$ are non-negative integers, $m$ is a
natural number, while $a_{n,1}, \ldots, a_{n,m}$ are quaternion coefficients. Moreover, in
$[3]$ it was used that each complex number $x$ can be written as $x = |x| e^{i\phi}$,
where $\phi$ is a real parameter (argument of $x$). This is not the case in $\mathbf{H}$,
because $z = |z| e^M$ with $M$ in the non-commutative purely imaginary domain
$\mathcal{I} = \mathbf{R}i_1 \oplus \mathbf{R}i_2 \oplus \mathbf{R}i_3 \subset \mathbf{H}$ such that $Re(M) = 0$, but generally $e^M$
and $e^N$ do not commute, when purely imaginary quaternions $M, N \in \mathcal{I}$ do not
commute.

Using the commuting bounded approximation property it was proved anew in $[38]$ that the disk algebra $A(K)$ has a basis and that the Hardy
space $H^p(K)$ has an unconditional basis. On the multi-dimensional torus $K^N$ these results were extended in [22, 23].

In this work the kernel of the Dirac operator $\sigma$ is investigated on a domain $\tilde{\Omega}$ in $H$ quasi-conformal with the closed unit ball $\tilde{B}$ (see Section 3.1). The Dirac operator is considered from the space of continuously differentiable functions $f$ on $\tilde{\Omega}$ into the space of continuous functions on $\tilde{\Omega}$. That is $\sigma f$ is taken on an open domain $\Omega$ such that the function $f$ has a continuous bounded extension on $\tilde{\Omega}$, where $\tilde{\Omega}$ is the closure of $\Omega$. It appears to be a left module over the quaternion skew field. But apart from the complex case it is not an algebra even over the real field. In our paper we elaborate a new technique different from previous works.

In Section 3 of this article Dirac modules over the quaternion skew field $H$ are investigated. For this purpose an analog of the Stone-Weierstrass theorem over quaternions is proved in Section 2. An existence of a Schauder basis in a Dirac module supplied with the supremum norm is proved in Theorem 3.15 and Corollary 3.16. Procedures for construction of such bases are outlined. Moreover, relative to Hardy’s norm with the parameter $1 < p < \infty$ Theorem 3.18 about unconditional bases in Dirac modules is proved.

Main results of this paper are obtained for the first time.

2 Banach spaces over the quaternion skew field.

1. Definitions and Notes. An $R$ linear space $X$ which is also left and right $H$ module will be called an $H$ vector space. We present $X$ as the direct sum

$$ (DS) \quad X = X_0i_0 \oplus \ldots \oplus X_3i_3, $$

where $X_0, \ldots, X_3$ are pairwise isomorphic real linear spaces, where $i_0, \ldots, i_3$ are generators of the quaternion skew field $H$ such that $i_0 = 1$, $i_k^2 = -1$ and $i_ki_j = -i_ji_k$ for each $k \geq 1$ and $j \geq 1$ so that $k \neq j$.

Let $X$ be an $R$ linear normed space which is also left and right $H$ module such that
(1) \(0 \leq \|ax\|_X = |a|\|x\|_X\) and 
(2) \(\|xa\|_X = |a|\|x\|_X\) and 
(3) \(\|x + y\|_X \leq \|x\|_X + \|y\|_X\)

for all \(x, y \in X\) and \(a \in H\). Such space \(X\) will be called an \(H\) normed space.

Suppose that \(X\) and \(Y\) are two normed spaces over \(H\). A continuous \(R\) linear mapping \(\theta : X \to Y\) is called an \(R\) linear homomorphism. If in addition \(\theta(bx) = b\theta(x)\) and \(\theta(xb) = \theta(x)b\) for each \(b \in H\) and \(x \in X\), then \(\theta\) is called a homomorphism of \(H\) (two sided) modules \(X\) and \(Y\).

If a homomorphism is injective, then it is called an embedding (\(R\) linear or for \(H\) modules correspondingly).

If a homomorphism \(h\) is bijective and from \(X\) onto \(Y\) so that its inverse mapping \(h^{-1}\) is also continuous, then it is called an isomorphism (\(R\) linear or of \(H\) modules respectively).

2. Definitions. One says that a real vector space \(Z\) is supplied with a scalar product if a bi-\(R\)-linear bi-additive mapping \(<,> : Z^2 \to R\) is given satisfying the conditions:

(1) \(<x, x> \geq 0, \quad <x, x> = 0\) if and only if \(x = 0\);
(2) \(<x, y> = <y, x>\);
(3) \(<ax + by, z> = a<x, z> + b<y, z>\) for each real numbers \(a, b \in R\) and vectors \(x, y, z \in Z\).

Then an \(H\) vector space \(X\) is supplied with an \(H\) valued scalar product, if a bi-\(R\)-linear bi-\(H\)-additive mapping \(<*,*> : X^2 \to H\) is given such that

(4) \(<f, g> = \sum_{j,k} <f_j, g_k > i^j_k,\)

where \(f = f_0i_0 + ... + f_3i_3, \quad f, g \in X, \quad f_j, g_j \in X_j, \) each \(X_j\) is a real linear space with a real valued scalar product, \((X_j, <*,*>)\) is real linear isomorphic with \((X_k, <*,*>)\) and \(<f_j, g_k> \in R\) for each \(j, k\). The scalar product induces the norm:

(5) \(\|f\| := \sqrt{<f, f>}\).

An \(H\) normed space or an \(H\) vector space with an \(H\) valued scalar product complete relative to its norm will be called an \(H\) Banach space or an \(H\) Hilbert space respectively.

3. Banach spaces of continuous functions over the quaternion skew field. As usually the quaternion skew field \(H\) is supplied with its
standard norm topology: $|z| = \sqrt{zz^*}$ for each $z \in H$. Considered as the real normed space the quaternion skew field $H$ has the real shadow which is the Euclidean space $\mathbb{R}^4$.

Let $C(U, H)$ denote the set of all continuous functions on a canonical closed subset $U$ in $H$. It is an $\mathbb{R}$-linear space and a left- and right- $H$-module. Moreover, $C(U, H)$ is the algebra over the real field with the pointwise addition and multiplication of functions. Supply it with the norm

\[(1) \quad \|f\|_{C(U, H)} := \sup_{z \in U} |f(z)|.\]

Relative to this norm the space $C(U, H)$ is Banach. There are the following quaternion analogs of the Stone-Weierstrass theorem.

4. **Theorem.** Suppose that $A$ is a separating points subalgebra in $C(T, H)$ such that $Z(A) = \emptyset$ (that is, functions from $A$ have no any common zero in $T$), where $T$ is a Hausdorff compact topological space, then this set $A$ is (everywhere) dense in $C(T, H)$.

**Proof.** For proving this theorem we use the classical Stone-Weierstrass theorem. It states: let $A$ be a separating points subalgebra in $C(T, \mathbb{R})$ such that $Z(A) = \emptyset$, where $T$ is a Hausdorff compact topological space, then this set $A$ is (everywhere) dense in $C(T, \mathbb{R})$ (see, for example, §4.10 Theorem A in [8]). The algebra $C(U, H)$ is isomorphic with $[C(U, \mathbb{R})i_0] \oplus \ldots \oplus [C(U, \mathbb{R})i_3]$. This decomposition induces the algebras $A_0, \ldots, A_3$ over the real field so that $A = A_0i_0 \oplus \ldots \oplus A_3i_3$. The algebra $A$ is the left and right module over $H$, consequently, $i_k A = A$ for each $k$, since $|i_k z| = |z|$ for each $z \in H$. Therefore, the algebras $A_0, \ldots, A_3$ over the real field are pairwise isomorphic. Moreover, $Z(A) = \emptyset$ implies that $Z(A_0) = \emptyset$, since $f(z) \neq 0$ means that one of the components $f_k(z)$ is non-zero, while $A_0$ and $A_k$ are isomorphic, where $f(z) = \sum_k f_k(z) i_k$ with real-valued components $f_k$, $f \in A$, $f_k \in A_k$ for each $k$. Since $A_0$ is dense in $C(U, \mathbb{R})$, then $A$ is dense in $C(U, H)$.

5. **Corollary.** If $U$ is a canonical closed bounded subset in $H^k$, then the family $\mathcal{H}(U, H^m)$ of all $H$-differentiable functions $f : U \to H^m$ is dense in $C(U, H^m)$, where $k, m \in \mathbb{N}$.

**Proof.** The set $U$ is closed and bounded in $H^k$, as the $\mathbb{R}$-vector topological space $H^k$ is locally compact, since $k \in \mathbb{N}$, consequently, $U$ is compact. Therefore, the statement of this corollary follows from Theorem 4, since
\( \mathcal{H}(U, H^m) \) is the subalgebra in \( C(U, H^m) \) and \( Z(\mathcal{H}(U, H^m)) = \emptyset \).

6. **Theorem.** Suppose that a canonical closed domain \( U \) in the quaternion skew field \( H \) is compact. Then the set \( \mathcal{P}(U, H) \) of all polynomials \( P_n : U \to H \) is dense in \( C(U, H) \).

**Proof.** To prove this theorem we use the preceding theorem. Particularly, we take \( T = U \).

To rewrite a function from the real variables \( z_j \) in the \( z \)-representation or vice versa the following identities are used:

\[
(1) \quad z_j = (-zi_j + i_j 2^{-1} \{-z + \sum_{k=1}^{3} i_k (zi_k^*)\})/2 \\
(2) \quad z_0 = (z + 2^{-1} \{-z + \sum_{k=1}^{3} i_k (zi_k^*)\})/2,
\]

where \( z \) is a quaternion number decomposed as

\[
(3) \quad z = z_0i_0 + \ldots + z_3i_3 \in H
\]

with \( z_j \in \mathbb{R} \) for each \( j \), \( i_k^* = \bar{i}_k = -i_k \) for each \( k > 0 \), \( i_0 = 1 \), since \( i_j(i_ji_k) = -i_k \) and \( (i_ki_j)i_j = -i_k \) for each \( j > 0 \), also \( i_ji_k = -i_ki_j \) for each \( j \neq k \) with \( j > 0 \) and \( k > 0 \), while \( i_k(i_0i_k^*) = 1 \) for each \( k \). Formulas \((1-3)\) define the real-linear projection operators \( \pi_j : H \to \mathbb{R} \) so that

\[
(4) \quad \pi_j(z) = z_j
\]

for each quaternion number \( z \in H \) and every \( j = 0, 1, 2, 3 \).

Let \( f \in C(U, H) \). The canonical closed domain \( U \) in the quaternion skew field \( H \) is compact. It has the real shadow \( V \) in \( \mathbb{R}^4 \). Each function \( f : U \to H \) can be written in the form

\[
(5) \quad f(z) = \sum_{j=0}^{3} f_j(z)i_j,
\]

where \( f_j : U \to \mathbb{R} \), \( i_j \) is the standard generator of the quaternion skew field \( H \) for each \( j = 0, 1, 2, 3 \). On the real shadow \( V \) to each \( f_j \) a function \( g_j \) of real variables \( z_0, z_1, \ldots, z_3 \) corresponds due to equalities \((1-3)\) above. Thus, if \( f : U \to H \) is continuous on \( U \), then each \( g_j : V \to \mathbb{R} \) is continuous on \( V \). Vice versa if \( g_j : V \to \mathbb{R} \) is continuous on \( V \) for each \( j = 0, \ldots, 3 \), then \( f : U \to H \) is continuous on \( U \) due to formulas \((1-4)\).

The set of all real-valued polynomials in the variable \( z = z_0i_0 + \ldots + z_3i_3 \in U \) forms the algebra \( A_0 \) over \( \mathbb{R} \), since the sum of polynomials and the product
of polynomials from $A_0$ is again a polynomial. Each polynomial of the form

(6) $q(z) = \pi_j((z - u)^n)$

belongs to $A_0$, where $z, u \in U, \ n \in \mathbb{N}, \ u$ is a marked quaternion parameter
for $q$. Since

(7) $(z-u)^n = z^n - z^{n-1}u - (z^{n-2}u)z + \ldots + (-1)^{n-1}((zu)u)u + (-u)^n$,

then Formulas (1 - 3) imply that $q(z)$ is the polynomial with $H$ coefficients
in the variable $z$, since $H$ is associative. If $x$ and $y$ are two distinct points in $U$, then there exists $k \in \{0, 1, 2, 3\}$ such that $x_k \neq y_k$. Then the function

(8) $g(z) := (z_k - x_k)^n$,

where $n \geq 1$ is a natural number, separates points $x$ and $y$. In view of Formulas (1 - 3) this function $g$ expresses as the real-valued polynomial $Q_{n,k}(z)$ in the variable $z \in U$ so that $Q_{n,k}(x) = 0$ and $Q_{n,k}(y) \neq 0$. Thus the algebra $A_0$ separates points in $U$ such that $Z(A_0) = \emptyset$. Applying Theorem 4 one gets the statement of this theorem. In more details the end of the proof is the following.

Then in view of the classical Stone-Weierstrass theorem (see above) $A_0$ is (everywhere) dense in $C(U, \mathbb{R})$. On the other hand, Formula (5) means that $C(U, H)$ is isomorphic with $[C(U, \mathbb{R})]i_0 \oplus \ldots \oplus [C(U, \mathbb{R})]i_3$. The same Formula (5) implies that the algebra $A = \mathcal{P}(U, H)$ of all quaternion valued polynomials on $U$ in the quaternion variable $z \in U$ is isomorphic with $(A_0 i_0) \oplus \ldots \oplus (A_0 i_3)$, since each $P_n(z) \in \mathcal{P}(U, H)$ has the form:

(9) $P_n(z) = \sum_{j=0}^3 P_{n,j}(z)i_j$

with $P_{n,j}(z) = \pi_j(P_n(z))$ being real-valued polynomials on $U$ and with $H$ expansion coefficients, because $\mathbb{R}i_j \subset H$ for each $j = 0, \ldots, 3$. Therefore, $\mathcal{P}(U, H)$ is (everywhere) dense in $C(U, H)$. That is, for each $f \in C(U, H)$ and for each $\epsilon > 0$ a polynomial $P_n(z) \in \mathcal{P}(U, H)$ exists such that

$$\|f - P_n\|_{C(U, H)} < \epsilon.$$ 

3 Dirac modules over quaternions.

1. Dirac module. The quaternion skew field $H$ is associative and non-commutative. It has the standard basis $\{i_0, i_1, i_2, i_3\}$ over the real field $\mathbb{R}$
such that \( i_0 = 1, i_1i_2 = i_3, i_1^2 = i_2^2 = i_3^2 = -1, i_0i_j = i_j \) for each \( j \), while \( i_ji_k = -i_ki_j \) for each \( j \neq k \geq 1 \). We consider the Dirac operator \( \sigma : C^1(\Omega, \mathcal{H}) \to C(\Omega, \mathcal{H}) \), where \( \sigma f(z) = \sum_{j=0}^3 (\partial f(z)/\partial z_j)i_j \) for a differentiable function \( f(z) \), \( z = \sum_{j=0}^3 z_ji_j, z \in \mathcal{H}, z_j \in \mathbb{R}, \ z^* = z_0i_0 - z_1i_1 - z_2i_2 - z_3i_3 \) denotes the conjugated quaternion \( z \).

More generally we consider an associative Clifford algebra \( \mathcal{X} \) of dimension \( l \) over \( \mathcal{H} \) and the kernel of the Dirac operator \( ker(\sigma) \) in \( C^1(\Omega, \mathcal{Y}) \) or its submodule, when it is indicated. Where \( \Omega \) is a domain in \( \mathcal{X} \), while \( \mathcal{Y} \) denotes a finite dimensional (two sided) module over \( \mathcal{X} \) with basis \( q_1, ..., q_m \),

\[
(1) \quad \sigma f(z) = \sum_{k=1}^l \sum_{j=0}^3 (\partial f(z)/\partial z_{j,k})e_ki_j,
\]

\( e_k \) are basic elements in \( \mathcal{X} \) over \( \mathcal{H} \), such that \( e_me_k^* = \delta_{m,k}e_1, e_m^* = (-1)^{p(m)}e_m \) with \( p(1) = 2, p(m) \in \{1, 2\} \) for each \( 2 \leq m \leq n, e_ki_j = i_je_k \) and \( e_1i_j = i_j \) for each \( j \) and \( k \),

\[
(2) \quad z = \sum_{k=1}^l \sum_{j=0}^3 z_{j,k}e_ki_j
\]

with \( z_{j,k} \in \mathbb{R} \) for each \( j, k \), whilst \( z \in \mathcal{X} \), \( l \) is a natural number,

\[
(3) \quad z^* = \sum_{k=1}^l \sum_{j=0}^3 z_{j,k}e_k^*i_j^*.
\]

As usually \( C^m(\Omega, \mathcal{Y}) \) and \( C(\Omega, \mathcal{Y}) \) stand for spaces of \( m \) times continuously differentiable (by all real variables \( z_{j,k} \)) and continuous respectively functions on a domain \( \Omega \) in \( \mathcal{X} \) with values in \( \mathcal{Y} \). If \( \Omega \) is a real \( C^m \)-manifold embedded into \( \mathcal{X} \), then in the standard way using charts of an atlas \( At(\Omega) \) spaces \( C^m(\Omega, \mathcal{Y}) \) and partial derivatives \( D^af \) in local coordinates are defined. For the unit sphere \( S^{n-1} \) the traditional atlas consisting of two charts is considered (see, for example, [18]).

We denote by \( K(B, \mathcal{Y}) \) the space of all \( C^1 \) functions \( f \) on \( B \) with values in \( \mathcal{Y} \) such that \( \sigma f(z) = 0 \) for each \( z \in B \) and a function \( f(z) \) has a bounded continuous extension on the closure \( \overline{B} = \{ z : z \in \mathcal{X}, |z| \leq 1 \} \) of the open unit ball \( B \) in \( \mathcal{X} \). The family \( K(B, \mathcal{Y}) \) is considered relative to the \( C \) norm

\[
(4) \quad \| f \|_{C(B, \mathcal{Y})} := \sup\{|f(z)| : z \in \overline{B}\},
\]

where \( |z|^2 = \sum_{k=1}^m |kz|^2 \) for each \( z = z_1q_1 + ... + mz_{m}, z \in \mathcal{Y} \), where \( kz \in \mathcal{X} \)
for each \( k \), whilst \( |y|^2 = \sum_{k=1}^l |y_k|^2 \) for any \( y = y_1e_1 + \ldots + y_le_l \in \mathcal{X} \) with \( y_k \in A_r \) for each \( k \).

2. **Proposition.** The family \( K(B, \mathcal{Y}) \) has the structure of the left \( \mathbb{H} \) module. If \( f \in K(B, \mathcal{Y}) \), then \( f \) is harmonic on \( B \), that is \( \Delta f(z) = 0 \) for each \( z \in B \), where \( \Delta \) denotes the Laplace operator.

**Proof.** From Formula 1(1) it follows that

\[
\sigma^* \sigma g(z) = \sigma \sigma^* g(z) = \Delta g(z) \text{ for each } g \in C^2(B, \mathcal{Y}),
\]

\[
\Delta f(z) = \sum_{k=1}^l \sum_{j=0}^3 \frac{\partial^2 f(z)}{\partial z_{j,k}^2} e_k^* i_j^*.
\]

Then from \( \sigma f(z) = 0 \) for each \( z \in B \) it follows that \( \sigma^* \sigma f(z) = 0 \) on \( B \), consequently, each \( f \in K(B, \mathcal{Y}) \) is harmonic, i.e. \( \Delta f(z) = 0 \) on \( B \). On the other hand, for each \( f \in K(B, \mathcal{Y}) \) and \( b \in \mathbb{H} \) we get that

\[
\sigma bf(z) = b \sum_{k=1}^l \sum_{j=0}^3 \frac{\partial f(z)}{\partial z_{j,k}} e_k^* i_j = b[\sigma f(z)],
\]

since the quaternion skew field is associative and \( \mathcal{Y} \) is the left \( \mathbb{H} \) module. Thus the condition \( f \in K(B, \mathcal{Y}) \) implies \( bf \in K(B, \mathcal{Y}) \) for each quaternion \( b \). Then for every \( f, g \in K(B, \mathcal{Y}) \) and \( b, c \in \mathbb{H} \) we infer that \( \sigma(bf + cg) = b[\sigma f] + c[\sigma g] = 0 \) and \( \sigma(bcf) = \sigma(b(cf)) = b\sigma(cf) = (bc)\sigma f = 0 \) on \( B \). Thus \( K(B, \mathcal{Y}) \) is the left module over the quaternion skew field.

3. **Corollary.** The left modules \( K(B, \mathcal{Y}) \) and \( K(B, \mathcal{X}) q_1 \oplus \ldots \oplus K(B, \mathcal{X}) q_m \) are isomorphic, where \( q_1, \ldots, q_m \) is a basis of \( \mathcal{Y} \) over \( \mathcal{X} \).

4. **Remark.** The quaternion skew field is also the particular case of the Clifford algebra. Functions with values in the Clifford algebra \( \mathcal{X} \) which satisfy the equation \( \sigma f = 0 \) are called Clifford analytic, where \( \sigma \) is the Dirac operator over \( \mathcal{X} \) (see also Chapter 2 §3 in [13]). The non trivial Clifford algebras different from \( \mathbb{R}, \mathbb{C} \) and \( \mathbb{H} \) have divisors of zero. Therefore, in Section 1 the case was considered, when \( K(B, \mathcal{Y}) \) is the left module over the quaternion skew field (see Proposition 2). But due to the analog of the Riemann mapping theorem (see §2.47 in [31] and §2.1.5.7 in [33]) over quaternions for a wide class of domains \( \Omega \) it is sufficient to consider the open unit ball \( B = \{z : z \in \mathbb{H}^n, |z| < 1\} \). Generally one can consider a domain \( \Omega \)
which is $C^\omega$ diffeomorphic with the unit ball $B = \{ z : z \in \mathbb{H}^n, |z| < 1 \}$ in $\mathbb{H}^n$, 1 $n \in \mathbb{N}$, where $C^\omega$ denotes the class of all locally analytic functions from $\Omega$ into $B$.

It can be lightly seen that there are $f, g \in K(B, \mathbb{H})$ the product of which is not in $K(B, \mathbb{H})$, because the quaternion skew field is non commutative and $[(\partial f(z)/\partial z_{j,k})i_j]g$ generally can not be equal to $[(\partial f(z)/\partial z_{j,k})g]i_j$. Thus apart from the complex case $K(B, \mathcal{Y})$ is not an algebra even for $\mathcal{Y} = \mathbb{H}$.

Due to Proposition 2 and Corollary 3 it is sufficient to prove that $K(B, \mathcal{X})$ has a Schauder basis, then the left module $K(B, \mathcal{Y})$ would have it as well.

5. Theorem. Let $\Omega$ be a connected domain in $\mathbb{H}^n$. Let also $u : \Omega \to \mathbb{H}$ be a harmonic function on $\Omega$. If $|u|$ has a maximum in $\Omega$, then $u$ is constant.

Proof. Suppose that $|u|$ has a maximum value $q$ at a point $x \in \Omega$. Then $u(x)\ast u(x) = |u(x)|^2$ and hence there exists $b \in \mathbb{H}$ such that $bu(x) = q$ and $|b| = 1$. Evidently the real part $Re(bu)$ is the harmonic function on $\Omega$ and attains its maximum value at $x$ according to Theorem 1.4 [1]. Then the equalities $|bu(z)| = |b||u(z)| = |u(z)|$ and the inequality $|u(z)| \leq q$ imply that $Im(bu) := bu - Re(bu) = 0$ on $\Omega$. Thus $bu$ and hence $u$ is constant on $\Omega$, since the quaternion skew field $\mathbb{H}$ has not divisors of zero.

6. Note. For Clifford analytic functions the analog of the latter theorem is contained in §§3.28 and 3.30 [13].

Let $P(x, y) = (1 - |x|^2)/|x-y|^n$ be the Poisson kernel on the $n$-dimensional over the real field $\mathbb{R}$ unit ball $\mathcal{B}$ in the Euclidean space $\mathbb{R}^n$ with $n > 2$, where $x \neq y \in \mathcal{B}$. Then for the unit ball $B$ in $\mathbb{H}^l$, where $l \in \mathbb{N}$, using the real shadow $\mathcal{B}$ of it one gets the Poisson kernel on $B$ with $n = 4l$:

1. $P(z, \xi) = (1 - |z|^2)/|z - \xi|^n$ for each $z \neq \xi \in \bar{B}$. We put

$$S_{n-1}^+ := \{ z \in S^{n-1} : z_{0,1} > 0 \} \text{ and } S_{n-1}^- := \{ z \in S^{n-1} : z_{0,1} < 0 \},$$

where $S^{n-1} = \partial B$ denotes the unite sphere. We consider the function $U(z) := \hat{P}(\chi_{S_{n-1}^+} - \chi_{S_{n-1}^-})(z)$, where $\chi_A$ denotes the characteristic function of subset $A$, $\chi_A(x) = 1$ for each $x \in A$, while $\chi_A(x) = 0$ for any $x \notin A$, $\hat{P}$ is the integral operator:

$$2. \quad \hat{P}(g)(z) := \int_{S^{n-1}} g(\xi) P(z, \xi) \psi(d\xi),$$
where $\psi$ denotes the normalized Riemann volume element (Borel measure)
on $S^{n-1}$, $\psi(S^{n-1}) = 1$.

7. **Theorem.** Let $f$ be a harmonic function $f : B \to \mathbf{H}$ with $f(0) = 0$,where $B$ is an open unit ball in $\mathbf{H}^l$, $l \in \mathbf{N}$. Then

$$\begin{align*}
(1) & \quad |f(z)| \leq U(|z|i_0e_1) \\
(2) & \quad f = cU \circ T \text{ on } B.
\end{align*}$$

**Proof.** A function $f$ can be written in the form $f(z) = \sum_{j=0}^3 f_j(z)i_j$,where $f_j$ are real-valued functions. Therefore the function $f$ is harmonic, $\Delta f = 0$, if and only if each function $f_j$ is harmonic. Then we take a constant $b \in \mathbf{H}$ of modulus 1 for a given $z \in B$ such that $|f(z)| = bf(z)$, since $\mathbf{H}$ is the associative skew field. The Laplace operator $\Delta$ is invariant under each transformation $Q$ such that the corresponding operator $[Q]$ on the real shadow $\mathbf{R}^n$ is orthogonal, consequently, the function $f \circ Q^{-1}$ is harmonic as well. Moreover, $\Delta(bf) = b\Delta f$, since the Laplace operator is the partial differential operator in real variables with real coefficients, while the real field is the center of the quaternion skew field $\mathbf{H}$, hence $bf$ is harmonic, when $f$ is harmonic. One can take in particular $Q$ such that $[Q]$ is orthogonal and $Q|z|i_0e_1 = z$. In view of Theorem 6.16 [1] applied to the real part $Re(bf \circ Q^{-1})$ one gets inequality (1) on $B$.

On the other hand, the equality in Formula (1) implies that $f(z) = b^*U(Q|z|i_0e_1)$. But $|bf(x)| \leq U(x)$ for each $x \in B$. For the real valued function $Re(bf)$ the equality at $z \neq 0$ in $B$ implies that

$$\begin{align*}
(3) & \quad Re(bf)(x) = U(x) \text{ for each } x \in B
\end{align*}$$
due to Theorem 6.16 [1]. Therefore, $Im(bf(x)) = 0$ for each $x \in B$ by already proved Inequality (1) and Property (3). Thus Formula (2) is fulfilled with $c = b^*$.

8. **Theorem.** For each function $f \in K(B, \mathcal{X})$ there exists a continuous
function \( h : S^{n-1} \to X \) such that

\[
(1) \quad f(z) = \int_{S^{n-1}} h(\xi)\sigma^*_z P(z, \xi)\psi(d\xi)
\]

for each \( z \in B \), where \( B \) is an open unit ball with center at zero in \( X \) (see §1), \( S^{n-1} = \partial B \).

**Proof.** According to Proposition 2 from the inclusion \( f \in K(B, X) \) it follows that a function \( f \) is harmonic. Moreover, Formulas 2(1 – 3) imply that

\[
(2) \quad f = \sigma^*_z g
\]

for some harmonic function \( g \) on \( B \), since \( \Delta = \sigma \sigma^* \) and the operators \( \sigma \) and \( \sigma^* \) commute on \( C^2 \), while each harmonic function on \( B \) is infinite differentiable by Theorem 1.18 [1]. Indeed, one equation (2) can be written as the system of \( 4l \) linear partial differential equations with constant real coefficients in \( 4l \) real variables and real functions \( f_{j,p}, g_{j,p} \):

\[
(3) \quad f_{j,p}(z) = \sum_{t,q; m,k; i=i_j; e_p e_t = e_p} \partial g_{m,q}(z) / \partial z_{k,t} + (-1)^{\phi(m,k)} \partial g_{k,q}(z) / \partial z_{m,t},
\]

where \( j = 0, 1, 2, 3; \ p = 1, ..., l; \) natural numbers \( \phi(m,k) \in \{1, 2\} \) are such that

\[
(4) \ i_m i_k = (-1)^{\phi(m,k)} i_k i_m.
\]

Since \( f \) is continuous on \( \bar{B} \), then the function \( g \) is also continuous on \( \bar{B} \). In view of Theorem 1.17 [1] the function \( g \) can be written in the form

\[
(5) \quad g(z) = \hat{P}(g|_{S^{n-1}})(z),
\]

where the integral operator is given by Formula 6(2), \( g|_{S^{n-1}} \) denotes the restriction of \( g \) on \( S^{n-1} \), since the Poisson kernel is real, \( P(z, \xi) \in \mathbb{R} \) for each \( z \neq \xi \in \bar{B} \), and hence the operator \( \hat{P} \) is \( \mathbf{H} \) linear. Evidently the integral in Formula (1) converges uniformly on each smaller closed ball \( \bar{B}_R \) of radius \( 0 < R < 1 \) with center at zero in \( B \). By the theorem of analysis about differentiation of integrals depending on parameters (see §XVII.2.3 in [19]) the identity is valid

\[
(6) \quad \sigma^*_z \hat{P}(g|_{S^{n-1}})(z) = \int_{S^{n-1}} g(\xi)\sigma^*_z P(z, \xi)\psi(d\xi)
\]

for each \( z \in B \). Thus Formulas (2, 5, 6), 2(1) and 6(2) lead to the representation (1).
9. Remark. Theorem 8 shows that the left module $K(B, \mathcal{X})$ is infinite dimensional.

For each $y, z \in \mathcal{X}$ (see §1) we put

$$(1) \quad <y, z> := \sum_{p=1}^{l} y_p^* z_p$$

to be the quaternion valued scalar product. Let $\mu$ be a Lebesgue measure on the real shadow $\mathbb{R}^n$, then $L^2(B, \mathcal{X})$ denotes the space of all $\mu$-measurable functions $f : B \to \mathcal{X}$ such that $\|f\|_2 < \infty$, where $\|f\|_2 := \sqrt{(f, f)}$, the $\mathcal{H}$ valued scalar product is given by the integral:

$$(2) \quad (f, g) := \frac{\int_B <f(z), g(z)> \mu(dz)}{\mu(B)}.$$  

10. Theorem. Let $f \in C^1(\bar{B}, \mathcal{X})$ be a function harmonic on $B$, and let $g \in C^1(\bar{B}, \mathbb{R})$ be a real valued function, then

$$(1) \quad (\sigma^* f, \sigma^* g) = \int_{S^{n-1}} <\sigma^* f(y), y^*> g(y) \psi(dy).$$

Proof. Definition 9(1) of the scalar product implies that $<y, z> = y^* z$. We consider the spherical coordinates in $\mathbb{R}^n$ related with the Cartesian coordinates by the formulas:

$$\begin{align*}
(1) & \quad x_1 = r \cos(\theta_1), \\
& \quad x_2 = r \sin(\theta_1) \cos(\theta_2), \ldots, \\
& \quad x_{n-1} = r \sin(\theta_1) \sin(\theta_2) \cdots \sin(\theta_{n-2}) \cos(\theta_{n-1}), \\
& \quad x_n = r \sin(\theta_1) \sin(\theta_2) \cdots \sin(\theta_{n-2}) \sin(\theta_{n-1}), \\
& \text{where } r = |x| \geq 0, 0 \leq \theta_j \leq \pi \\
& \text{for each } j = 1, \ldots, n-2, \text{while } 0 \leq \theta_{n-1} \leq 2\pi \text{ (see §12.1 in [49])}, x = \phi(z), \phi: \mathcal{X} \to \mathbb{R}^n \text{ is the real linear isometry such that, } x_1 = z_{0,1}, \ldots, x_4 = z_{3,1}, \ldots, x_{n-3} = z_{0,3}, \ldots, x_n = z_{3,l}, \ n = 4l, \ z \in \mathcal{X}, \ x \in \mathbb{R}^n. \text{ The Jacobian} \\
& \quad J = J(r, \theta) = r^{n-1} \sin^{n-2}(\theta_1) \sin^{n-3}(\theta_2) \cdots \sin(\theta_{n-2}) \\
& \text{is positive for } r > 0 \text{ and } 0 < \theta_j < \pi \text{ for each } j = 1, \ldots, n-2, \text{ where } \theta = (\theta_1, \ldots, \theta_{n-1}). \text{ Then} \\
& \quad \mu(dz) = Jdrd\theta_1 \cdots d\theta_{n-1} = \psi(dy)dr, \text{ where } y = z/r \text{ for } r > 0. \text{ This transformation from Cartesian to spherical coordinates can be presented as product of the dilation } x \to rx \text{ and of } n-1 \text{ orthogonal transformations with matrices } A_1, \ldots, A_{n-1} \text{ depending on one angle parameter } \theta_1, \ldots, \theta_{n-1} \text{ so that} \end{align*}
\[ x = (1, 0, \ldots, 0)rA_1, \ldots, A_{n-1}, \] where \( A_k \) is the \( n \times n \) matrix with 1 as diagonal elements \((j, j)\) for \( j \neq k \) and \( j \neq k + 1\), \((A_k)_{k,k} = (A_k)_{k+1,k+1} = \cos(\theta_k), \)

\((A_k)_{k,k+1} = -(A_k)_{k+1,k} = \sin(\theta_k),\) others elements of \( A_k \) are zero, where a vector \( x \) is written as the one row matrix. That is one can consider a sequence of \( n \) transformations. Using the chain rule one can express the Dirac operator in spherical coordinates as:

\[
(2) \quad \sigma_t f(z(t)) = \sum_{k=1}^{n} \left[ \frac{\partial f(z(t))}{\partial t} \right] \alpha_k(t),
\]

where \((t_1, \ldots, t_n) = (r, \theta_1, \ldots, \theta_{n-1})\), functions \( \alpha_k(t) \) have values in \( \mathcal{X} \),

\[
(3) \quad \alpha_1(t) = \cos(\theta_1)e_1i_0 + \sin(\theta_1)\cos(\theta_2)e_1i_1 + \ldots + \sin(\theta_1)\sin(\theta_{n-2})\sin(\theta_{n-1})e_1i_3,
\]

\[
(4) \quad \alpha_k(t) = r^{n-2}J^{-1} \sum_{j=0}^{3} \sum_{p=1}^{l} \beta_{k,j,p}(\theta) e_{p}i_{j},
\]

where \( \beta_{k,j,p}(\theta) \) are definite products \( \beta_{k,j,p}(\theta) = \prod_{m=1}^{n-1} \sin^{a(m)}(\theta_m) \cos^{b(m)}(\theta_m) \),

where \( a(m) = a_{k,j,p}(m) \) and \( b(m) = b_{k,j,p}(m) \) are nonnegative integers for each \( m = 1, \ldots, n-1 \). Thus \( J(t)\alpha_k(t) \) are infinite differentiable functions for each \( k \). In particular, for \( l = 1 \) one has:

\[
(5) \quad \alpha_2 = r^2J^{-1}[ -\sin^3(\theta_1)\sin(\theta_2)i_0 + \sin^2(\theta_1)\cos(\theta_1)\sin(\theta_2)\cos(\theta_2)i_1 \\
+ \sin^2(\theta_1)\cos(\theta_1)\sin^2(\theta_2)\cos(\theta_3)i_2 + \sin^2(\theta_1)\cos(\theta_1)\sin^2(\theta_2)\sin(\theta_3)i_3 ]
\]

\[
(6) \quad \alpha_3 = r^2J^{-1}[ -\sin(\theta_1)\sin^2(\theta_2)i_1 + \sin(\theta_1)\sin(\theta_2)\cos(\theta_2)\cos(\theta_3)i_2 \\
+ \sin(\theta_1)\sin(\theta_2)\cos(\theta_2)\sin(\theta_3)i_3 ]
\]

\[
(7) \quad \alpha_4 = r^2J^{-1}[ -\sin(\theta_1)\sin(\theta_3)i_2 + \sin(\theta_1)\cos(\theta_3)i_3 ]
\]

In spherical coordinates the adjoint Dirac operator is:

\[
(6) \quad \sigma_t^* f(z(t)) = \sum_{k=1}^{n} \left[ \frac{\partial f(z(t))}{\partial t} \right] \alpha_k^*(t).
\]

The operator \( \sigma^* \) in Cartesian coordinates or correspondingly \( J\sigma^* \) in spherical coordinates defines a vector field \( Y^* \) with coefficients in \( \mathcal{X} \). For the Dirac operator \( \sigma \) in Cartesian coordinates the corresponding vector field

\[
Y = \sum_{k=1}^{l} \sum_{j=0}^{3} e_{k}i_{j}(\partial/\partial z_{j,k})
\]

has constant Clifford coefficients \( e_{k}i_{j} \) of unit norm. Then

\[
\sigma^*[\sigma^* f]^* g = \langle \sigma^* f, \sigma^* g \rangle,
\]

since \( \Delta f = 0 \) and the function \( g \) is real valued, while \( \mathbf{R} \) is contained in the

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center of the Clifford algebra $\mathcal{X}$. Under the composition of the mappings $\phi$ and $x \mapsto t = (r, \theta)$ the images of $B$ and $S^{n-1}$ are $Q$ and $\partial Q$ respectively, where $Q := [0, 1] \times [0, \pi]^{n-2} \times [0, 2\pi]$, consequently,

$$(\sigma^* f, \sigma^* g) = \int_B \sigma^*[(\sigma^* f)^*(z)g(z)]\mu(dz) = \int_Q \sigma^*[(\sigma^* f)^*(z((r, \theta)))g(z(r, \theta))]Jdrd\theta_1...d\theta_{n-1}.$$  

For an even dimensional unit ball $B = B_n$ its volume $\mu(B) = V_n(B) = \pi^{n/2}/(n/2)!$ and the unnormalized surface area of the unit sphere $S^{n-1}$ in $\mathbb{R}^{n-1}$ is $nV_n(B)$, where $n$ is the dimension of $B$ over $\mathbb{R}$, $\mu([0,1]^n) = 1$, (see Appendix A in [1] or [49]). In view of Stokes’ theorem XV.3.5 [49] applied to the integrals in the latter formula one gets

$$(\sigma^* f, \sigma^* g) = \sum_{p=1}^{n} \int_{\partial Q_p} <\sigma^* f(z(t)), \gamma^*(t) > g(z(t))dt_1\wedge...\wedge dt_{p-1}\wedge dt_{p+1}\wedge dt_{n}\bigg|_{\partial \mathcal{Q}} = n \int_{\partial Q} <\sigma^* f(y), y^* > g(y)\psi(dy),$$

since $y \in S^{n-1}$ is a vector orthogonal to $S^{n-1}$ at $y$ and directed outwards $B$ and of unit norm, $|y| = 1$, whilst $(ab)^* = b^*a^*$ for each $a,b \in \mathcal{X}$, where

$$d\gamma|_{\partial Q} = J \sum_{p=1}^{n} (-1)^{p+1} \alpha_p \chi_{\partial Q_p} dt_p, \quad \partial Q_p := \{ t \in \partial Q : t_p = 0 \text{ or } t_p = b_p \},$$

where the orientation of $Q$ is consistent with that of $\partial Q$ and the orientation of $B$ is consistent with that of $\partial B = S^{n-1}$ (see also §XV.3 [49]), $b_1 = 1$, $b_p = \pi$ for $p = 2,...,n-1$, $b_n = 2\pi$.

11. **Note.** Using the identity 10(1) we define the scalar product on $L^2(S^{n-1}, \mathbb{R}) \cap C^1(S^{n-1}, \mathbb{R})$:

$$(1) \quad [f,g] := n \int_{S^{n-1}} <\sigma^* \dot{P}(f)(y), y^* > \dot{P}(g)(y)\psi(dy).$$

Let the operator $T$ on $C^1(S^{n-1}, \mathcal{X})$ be defined by the formula

$$(2) \quad Tf := \sigma^* \dot{P}[f].$$

12. **Corollary.** The restriction of the operator $T$ (see 11(2)) to $C^1(S^{n-1}, \mathbb{R})$ induces the real linear isometry from $V := (L^2(S^{n-1}, \mathbb{R}) \cap C^1(S^{n-1}, \mathbb{R}))/Y$ into $L^2(\mathcal{B}, \mathcal{X})$ relative to the corresponding scalar products $[*,*]$ and $(*,*)$, where $Y = T^{-1}(0) = ker(T)$.

**Proof.** The calculation of $\sigma^*_zP(z,w)$ gives

$$(1) \quad \sigma^*_zP(z,w) = -\frac{2|z-w|^2z^* + n(1-|z|^2)(z-w)^*}{|z-w|^{n+2}}$$
for each \( z \neq w \in \mathcal{B} \). Therefore, for each \( f \in L^2(\mathcal{B}, \mathcal{X}) \) we have \( \sigma^* \hat{P}(f)(z) \in \mathcal{X} \) for any \( z \in \mathcal{B} \) and Formula 9(2) leads to the inequality \( < f(z), f(z) > \geq 0 \), since \( < f(z), f(z) > \geq 0 \) for each \( z \in \mathcal{B} \), whilst \( \mu(dz) \) is the nonnegative Lebesgue measure. There is the inclusion \( C^1(S^{n-1}, \mathcal{X}) \subset L^2(S^{n-1}, \mathcal{X}) \). If \( f \in C^1(S^{n-1}, \mathbb{R}) \), then the function \( \hat{P}(f) \) is harmonic on \( \mathcal{B} \) and continuously differentiable on \( \mathcal{B} \) due to Theorems 1.14 and 1.17 [1].

At the same time, the equality \( (f, f) = 0 \) is equivalent to \( < f(z), f(z) > = 0 \) for almost all \( z \in \mathcal{B} \), that in its turn is equivalent to \( f(z) = 0 \) for almost all \( z \in \mathcal{B} \) according to 9(1), consequently, \( \sqrt{(f, f)} \) is the norm on \( L^2(\mathcal{B}, \mathcal{X}) \). Therefore, \( \sqrt{|g, g|} = \sqrt{(Tg, Tg)} \) is the norm on the real quotient space \( \mathcal{V} \).

If \( g \in C^1(S^{n-1}, \mathbb{R}) \), then \( < \sigma^* \hat{P}(g)(y), y^* > \in \mathcal{X} \) for each \( y \in S^{n-1} \) as follows from Formulas (1), 1(1, 3), 2(3) and 9(1). In view of Equalities (1), 9(2) and 10(1) the inclusion \( [f, g] \in \mathbb{R} \) is valid for each \( f, g \in C^1(S^{n-1}, \mathbb{R}) \) due to the polarization identity

\[
[f, g] = \frac{[f + g, f + g] - [f - g, f - g]}{4},
\]

since

\[
(\sigma^*_x P(z, w))(\sigma_x P(z, \xi)) + (\sigma^*_x P(z, \xi))(\sigma_x P(z, w)) = \Delta_z [P(z, w)P(z, \xi)] \in \mathbb{R}
\]

and \( (\sigma^*_x P(z, w))^* = \sigma_x P(z, w) \) for every \( w, \xi, z \in \mathcal{B} \) with \( w \neq z \) and \( \xi \neq z \). On the other hand, \( \hat{P}(g) \) is the harmonic function on \( \mathcal{B} \). This implies that \( [f, f] = 0 \) if and only it \( f \in \ker(T) \). Thus \( T : \mathcal{V} \to L^2(\mathcal{B}, \mathcal{X}) \) is the isometry, which is linear over the real field, where \( \mathcal{V} \) is supplied with the scalar product \( [f, g] = (Tf, Tg) \) and the norm \( \|g\| = \sqrt{|g, g|} \).

13. **Theorem.** On the Banach space \( C^1(S^{n-1}, \mathbb{R}) \) the integral

\[
(1) \quad (\sigma^*_x P(z, \xi), \sigma^*_x P(z, w)) := \int_{\mathcal{B}} < \sigma^*_x P(z, \xi), \sigma^*_x P(z, w) > \mu(dz)
\]

defines the generalized function \( (\sigma^*_z \hat{P}(\delta_{z, \xi})g(\xi), \sigma^*_z \hat{P}(\delta_{z, w})f(w)) \) (continuous \( \mathbb{R} \)-bilinear functional with values in \( \mathcal{X} \)) by \( f \) and \( g \), where \( f, g \in C^1(S^{n-1}, \mathbb{R}) \).

**Proof.** The delta function \( \delta_{x, y} \) on the sphere \( S^{n-1} \) is characterized by the formula

\[
(2) \quad \int_{S^{n-1}} \delta_{x, y} g(y) \psi(dy) = \int_{S^{n-1}} \delta_{y, x} g(y) \psi(dy) = g(\xi)
\]

for each \( g \in C(S^{n-1}, \mathbb{R}) \) and \( \xi \in S^{n-1} \). Take any delta sequence of continuous nonnegative functions \( g_m : S^{n-1} \to \mathbb{R} \) such that

\[
\int_{S^{n-1}} g_m(y) \psi(dy) = 1
\]
and there exists $0 < \epsilon < 1$ for which
\[
\int_{y \in S^{n-1} : |y - w| < \epsilon/m} g_m(y) \psi(dy) \geq 1 - 1/m
\]
for each $m \in \mathbb{N}$, that is the limit exists $\lim_m g_m(y) = \delta_w(y)$ relative to the weak* topology in the topological dual space $C^*(S^{n-1},\mathbb{R})$ (see §6.4 [13]), where
\[
\int_{S^{n-1}} \delta_w(y)g(y)\psi(dy) = g(w)
\]
for each $g \in C(S^{n-1},\mathbb{R})$, $(C, C^*)$ is the dual pair. In particular this is for $g(y) = \sigma_z^*P(z,y)$, when $z \in B$ and $y \in S^{n-1}$. Since $\hat{P}[f](y) = f(y)$ for each $f \in C(S^{n-1},\mathbb{R})$ and $y \in S^{n-1}$ (see Theorems 1.14 and 1.17 in [1]) taking the limit one arrives to the equality
\[
(3) \quad \hat{P}[\delta_w]|_{S^{n-1}} = \lim_m \hat{P}[g_m]|_{S^{n-1}} = \lim_m g_m|_{S^{n-1}} = \delta_w|_{S^{n-1}}
\]
in $C^*(S^{n-1},\mathbb{R})$. On the other hand, $\int_{S^{n-1}} \sigma_z^*P(z,y)\delta_w(y)\psi(dy) = \sigma_z^*P(z,w)$ for each $z \in B$ and $w \in S^{n-1}$.

For the direct product of delta functions and every $f, g \in C(S^{n-1},\mathbb{R})$ the equality $(\delta_{\xi,y} \times \delta_{w,y})(f(\xi) \times g(w)) = f(y)g(y)$ is valid. Therefore, by Fubini’s theorem
\[
\int_{S^{n-1}} f(\xi) \int_{S^{n-1}} (\delta_{\xi,y} \times \delta_{w,y})\psi(dy)\psi(d\xi) = \int_{S^{n-1} \times S^{n-1}} f(\xi)(\delta_{\xi,y} \times \delta_{w,y})\psi(dy)\psi(d\xi)
\]
\[
= f(w) = \int_{S^{n-1}} \delta_{\xi,w}f(\xi)\psi(d\xi),
\]
correspondingly, $\int_{S^{n-1}} (\delta_{\xi,y} \times \delta_{w,y})\psi(dy) = \delta_{\xi,w}$. Therefore, from Fubini’s theorem, Theorem 10 and Equality (3) we infer for each $f \in C^1(S^{n-1},\mathbb{R})$ that
\[
\int_{S^{n-1}} (\sigma_z^*P(z,\xi), \sigma_z^*P(z,w))f(w)\psi(dw) :=
\]
\[
\int_{S^{n-1}} \int_B < \int_{S^{n-1}} (\sigma_z^*P(z,y))\delta_{\xi,y}\psi(dy), \int_{S^{n-1}} \sigma_z^*P(z,q)\delta_{w,q}\psi(dq) > f(w)\mu(dz)\psi(dw)
\]
\[
= n \int_{S^{n-1}} \int_{S^{n-1}} \int_{S^{n-1}} \int_{S^{n-1}} < (\sigma_z^*P(z,y), z^* > \delta_{\xi,y} \times \delta_{w,q}f(w)P(z,q)\psi(dy)\psi(dq)\psi(dw)\psi(dz)
\]
\[
= n \int_{S^{n-1}} \int_{S^{n-1}} < (\sigma_z^*P(z,y), z^* > \delta_{\xi,y} \times \delta_{w,z}f(w)\psi(dy)\psi(dw)\psi(dz)
\]
\[
= n \int_{S^{n-1}} < (\sigma_z^*P(w,y), w^* > f(w)\delta_{\xi,y}\psi(dy)\psi(dw)
\]
\[
= n \int_{S^{n-1}} < (\sigma_z^*P(w,\xi), w^* > \hat{P}(f)(w)\psi(dw)
\]
\[
= \int_B < (\sigma_z^*P(w,\xi), (\sigma_z^*\hat{P}(f))(w)) > \mu(dw) = (\sigma_z^*P(\delta_{z,\xi}), \sigma_z^*\hat{P}(f)(z)).
\]
Finally we get \((\sigma_\ast^z \hat{P}(\delta_z, \xi), \sigma_\ast^z \hat{P}(f)(z))g(\xi) = (\sigma_\ast^z \hat{P}(\delta_z, \xi)g(\xi), \sigma_\ast^z \hat{P}(\delta_z, w)f(w))\).

The continuity of this real bilinear functional follows from the equalities

\[(\sigma_\ast^z \hat{P}(\delta_z, \xi)g(\xi), \sigma_\ast^z \hat{P}(\delta_z, w)f(w)) = [f, g] := n \int_{S^{n-1}} <\sigma \ast \hat{P}(f)(y), y^* > \hat{P}(g)(y)\psi(dy)\]

and the estimate

\[\| [f, g] \| \leq n \| f \|_{C^1(S^{n-1}, R)} \| g \|_{C^1(S^{n-1}, R)}, \]

since \(\sigma \ast \hat{P}(f)(y)|_{S^{n-1}} = \sigma \ast f(y)|_{S^{n-1}} = \hat{P}(\sigma \ast f(y)|_{S^{n-1}})|_{S^{n-1}}.\)

14. Proposition. The operators \(\hat{P} : C(S^{n-1}, \mathcal{X}) \to K(B, \mathcal{X})\) and \(\sigma \ast \hat{P} : C^1(S^{n-1}, \mathcal{X}) \to K(B, \mathcal{X})\) are continuous, \(\hat{P}\) is left and right \(H\) linear, while \(\sigma \ast \hat{P}\) is left \(H\) linear.

Proof. The integral operator \(\hat{P}\) has the real integral kernel \(P\) and the Borel measure \(\psi\) on \(S^{n-1}\) in the integral is nonnegative, hence \(\hat{P}\) is left and right \(H\) linear. In §2 it was proved, that the operator \(\sigma \ast\) is left \(H\) linear, consequently, the composite operator \(\sigma \ast \hat{P}\) is left \(H\) linear. Since \(f(z) \mapsto f(z) - f(0)\) is the continuous mapping of \(C(S^{n-1}, H)\) into itself, \(\hat{P}(f)\) is harmonic on \(B\) and \(\hat{P}(f)|_{S^{n-1}} = f|_{S^{n-1}}\) for each \(f \in C(S^{n-1}, H)\), then from Theorem 7 it follows that the mapping \(\hat{P} : C(S^{n-1}, H) \to K(B, H)\) is continuous.

On the other hand, \(K(B, \mathcal{X}) = \bigoplus_{p=1}^l K(B, \mathcal{H})e_p\), hence the operator \(\hat{P}\) from \(C(S^{n-1}, \mathcal{X})\) into \(K(B, \mathcal{X})\) is also continuous. Analogously the mapping \(\hat{P} : C^1(S^{n-1}, \mathcal{X}) \to C^1(\bar{B}, \mathcal{X})\) is continuous, since each harmonic function is infinite differentiable. The operator \(\sigma \ast : C^1(\bar{B}, \mathcal{X}) \to C(\bar{B}, \mathcal{X})\) is continuous, consequently, the operator \(\sigma \ast \hat{P}\) from \(C^1(S^{n-1}, \mathcal{X})\) into \(K(B, \mathcal{X})\) is continuous as well.

15. Theorem. The left module \(K(B, \mathcal{X})\) is Banach and has a Schauder basis over the quaternion skew field \(H\).

Proof. By the left \(H\) span \(l - \text{span}_H\Psi\) of a set \(\Psi\) in a left module over \(H\) we mean all finite sums \(a_1f_1 + ... + a_lf_l\) with \(a_1, ..., a_l \in H\) and \(f_1, ..., f_l \in \Psi\), where \(l \in N\). At the same time the operators \(\sigma\) and \(\sigma \ast\) map \(C^1(\mathbb{R}^n, \mathcal{X})\) into \(C(\mathbb{R}^n, \mathcal{X})\) and hence \(C^1(S^{n-1}, \mathcal{X})\) into \(C(S^{n-1}, \mathcal{X})\). According to Tietze-Urysohn’s theorem (see §2.1.8 in [9]) each continuous function \(u : S^{n-1} \to \mathbb{R}\) has a continuous extension \(u : \bar{B} \to \mathbb{R}\), consequently, each continuous
function \( f : S^{n-1} \to \mathcal{X} \) has a continuous extension \( f : \overline{B} \to \mathcal{X} \), since \( \overline{B} \) is the normal topological space and \( S^{n-1} \) is closed in it.

In view of Theorem 8 and Formulas 10(2–4) we have, that \( \hat{P}(f|_{S^{n-1}})|_{S^{n-1}} = f|_{S^{n-1}} \) and \( \hat{P}(f|_{S^{n-1}}) \) is harmonic on \( B \) so that if \( g \in C^1(S^{n-1}, \mathcal{X}) \), then

\[
(1) \quad \sigma^* \hat{P}(g|_{S^{n-1}})|_{S^{n-1}} = \sigma^* g|_{S^{n-1}}.
\]

If \( f \in C(\overline{B}, \mathcal{X}) \), then the linear system of partial differential equations with real coefficients \( 8(3) \) has a solution \( g \in C^1(\overline{B}, \mathcal{X}) \) (see Theorem 2.4.1 [36] and references therein), hence \( \sigma^*(C^1(S^{n-1}, \mathcal{X})) = C(S^{n-1}, \mathcal{X}) \). This implies that \([l-\text{span}_H \sigma^*(C^1(S^{n-1}, R))] \cap C(S^{n-1}, H) \) is dense in \( C(S^{n-1}, H) \) and hence \( \bigoplus_{p=1}^l l-\text{span}_H [\sigma^*(C^1(S^{n-1}, R)] \) is dense in \( C(S^{n-1}, \mathcal{X}) \). At the same time we have from Formula 2(3), that the kernel of the restriction to \( C^1(\overline{B}, R) \) of the operator \( \sigma^* \) consists of all constant functions, that is \( \text{ker}(\sigma^*|_{C^1(\overline{B}, R)}) = R \).

Then we take a Schauder basis \( \{f_m : m \in \mathbb{N}\} \) in \( C(S^{n-1}, H) \) (see Theorem 2.4 above), consequently, its left \( H \) span is dense in \( C(S^{n-1}, H) \). Therefore, \( \{f_m e_k : m \in \mathbb{N}, k = 1, ..., l\} \) is the Schauder basis in the left \( H \) module \( C(S^{n-1}, \mathcal{X}) \). For each \( f_m \) then we choose a particular solution \( g_m \in C^1(S^{n-1}, \mathcal{X}) \) of the equation \( \sigma^* g_m = f_m \) restricted to \( S^{n-1} \) as above (see §8 also). We modify this basis \( \{f_m e_k : m, k\} \) in such manner that each \( g_m \) is real valued, which is possible, since \( C^1(S^{n-1}, \mathcal{X}) = \bigoplus_{p=1}^l (1\text{-span}_H g_p) \) is harmonic on \( B \), moreover, each \( \hat{P}(g_m) \) is real valued.

From Formulas 1(2), 9(1, 2), 10(1), 11(1) and Corollary 12 it follows that \( [g_m, g_p] \in H \) for any \( m, p \in \mathbb{N} \). Each quaternion equation \( ax = b \) or \( xa = b \) has a solution \( x \in H \), when \( a \neq 0 \), where \( a, b \in H \). Then using Schmidt’s orthogonalization and normalization procedures relative to the scalar product \([*, *] \) applied to \( \{g_m : m\} \) we get functions \( u_m = \sum_{k=1}^{m} a_m k f_k \) and \( v_m = \sum_{k=1}^{m} a_m k g_k \) such that \([v_m, v_p] = \delta_{m,p} \) for each \( m, p \in \mathbb{N} \), where \( a_m k \in H \) are quaternion constants, since each \( g_k \) is real valued, while \( \delta_{m,p} \) is Kroneker’s delta-symbol, \( \delta_{m,p} = 0 \) when \( m \neq p \) whilst \( \delta_{m,m} = 0 \) for every \( m, p \in \mathbb{N} \). Certainly, these functions are related by the equation \( u_m = \sigma^* v_m \) for each natural number \( m \). Thus each function \( v_m \) is \( H \) valued.

The operator \( \hat{P} \) is continuous from \( C(S^{n-1}, \mathcal{X}) \) into \( K(B, \mathcal{X}) \). At the same time a solution of the equation \( \sigma^* g = h \) depends continuously on \( h \in \mathbb{R} \).
$C(\tilde{B}, \mathcal{X})$ as it is known from the theory of systems of linear partial differential equations with constant coefficients. This means that the anti-derivation operator $\Upsilon_{\sigma^*}$ is continuous from $C(\tilde{B}, \mathcal{X})$ into $C^1(\tilde{B}, \mathcal{X})$ (see Theorem 2.4.1 in [36] and references therein). Therefore, there exists a continuous anti-derivation operator denoted by $\Upsilon_{\sigma^*}|_{S^{n-1}}$ from $C(S^{n-1}, \mathcal{X})$ into $C^1(S^{n-1}, \mathcal{X})$.

Let $Q$ be the left Banach module over $\mathbf{H}$ which let be the closure in $C^1(S^{n-1}, \mathcal{X})$ of $l - \text{span}_\mathbf{H}\{v_m e_k : m \in \mathbb{N}, k = 1, \ldots, l\}$. Therefore, this Banach module $Q$ is contained in $L^2(S^{n-1}, \mathcal{X})$. By the construction above we infer that $\sigma^* Q = C(S^{n-1}, \mathcal{X})$. The continuity of the anti-derivation operator $\Upsilon_{\sigma^*}|_{S^{n-1}}$ implies that $\{v_m e_k : m \in \mathbb{N}, k = 1, \ldots, l\}$ is the Franklin system in $Q$ relative to the scalar product $[*, *]$, since

$$ (2) \quad \sigma^* \hat{P}(\Upsilon_{\sigma^*}|_{S^{n-1}} f)|_{S^{n-1}} = f = \hat{P}(f)|_{S^{n-1}} $$

for each continuous function $f : S^{n-1} \rightarrow \mathcal{X}$, also $Q \subset (L^2(S^{n-1}, \mathcal{X}) \cap C^1(S^{n-1}, \mathcal{X}))$ and $[v_m, v_p] = \delta_{m,p}$ for each $m, p \in \mathbb{N}$.

Suppose that $h_n$ is a fundamental sequence in $K(B, \mathcal{X})$, then $h_n|_{S^{n-1}}$ is a fundamental sequence in $C(S^{n-1}, \mathcal{X})$, where $n \in \mathbb{N}$. Therefore, the limit $\lim_{n \rightarrow \infty} h_n|_{S^{n-1}} = y$ exists in $C(S^{n-1}, \mathcal{X})$. Then from the continuity of the operator $\sigma^* \hat{P} \Upsilon_{\sigma^*}|_{S^{n-1}}$, Identities (2) and Theorem 8 we infer that the limit

$$ \lim_{n \rightarrow \infty} h_n = h = \sigma^* \hat{P} (\Upsilon_{\sigma^*}|_{S^{n-1}} y) $$

exists in $K(B, \mathcal{X})$ relative to the $C(\tilde{B}, \mathcal{X})$ norm. Hence $K(B, \mathcal{X})$ is the Banach left module over the quaternion skew field.

If $h \in K(B, \mathcal{X})$, then from the embedding $K(B, \mathcal{X}) \hookrightarrow L^2(\tilde{B}, \mu, \mathcal{X})$ and from Theorems 10,13 and Corollary 12 it follows that

$$ h = \sum_{m=1}^{\infty} \sum_{k=1}^{l} \beta_{m,k} w_{m,k}, $$

where $w_{m,k} = Tv_m e_k$,

$$ h = \sum_{k=1}^{l} h_k e_k $$

with $h_k \in K(B, \mathbf{H})$ for each $k = 1, \ldots, l$, $\beta_{m,k} = (h_k^*, w_{m,k}^*)$. That is, each expansion coefficient $\beta_{m,k}$ for a function $h$ is unique. Since

$$ (h_k^*, w_{m,k}^*) = \int_\mathcal{B} h_k(z) w_{m,k}^*(z) \mu(dz), $$

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each functional $\beta_{m,k} : K(B, H) \to H$ is left $H$ linear, where $\beta_{m,k} = \beta_{m,k}(h_k)$, since the quaternion skew field is associative. From Cauchy-Bunyakovsky-Schwarz’s inequality we infer that
\[
|(h_k^*, w_{m,k}^*)| \leq \|h_k\|_{L^2(B, H)} \|w_{m,k}\|_{L^2(B, H)} \leq \|h_k\|_{C(B, H)} \|w_{m,k}\|_{L^2(B, H)},
\]
since $\|w_{m,k}\|_{L^2(B, H)} = 1$ and $\mu(B) < \infty$. Therefore, each functional $\beta_{m,k}$ is continuous from $(K(B, H), \| * \|_{C(B, H)})$ into $H$ and $\|\beta_{m,k}\| \leq 1$ for every $m$ and $k$. The operator $\sigma^* \hat{P}_\sigma|_{S^{n-1}}$ is continuous from $C(S^{n-1}, \mathcal{X})$ onto $K(B, \mathcal{X})$ due to Formulas (1, 2), Theorem 8 and the proof above. Thus \{\text{\small{$w_{m,k}$}} : m \in \mathbb{N}, k = 1, ..., l\} is the Schauder basis in $K(B, \mathcal{X})$. Moreover, it is the Franklin system due to Corollary 12.

16. Corollary. The left module $K(B, \mathcal{Y})$ has Schauder bases.

Proof. From the construction of the Schauder basis in $K(B, \mathcal{X})$ and from Remark 4 we get that there exists a Schauder basis in $K(B, \mathcal{Y})$ as well.

17. Notation. Let $H^p(\bar{B}, \mathcal{Y})$ denote the (Hardy) left $H$-module of all measurable functions $f : \bar{B} \to \mathcal{Y}$ satisfying the condition:
\[
(1) \quad \|f\|_{H^p(\bar{B}, \mathcal{Y})} := \left[ \sup_{0<r\leq 1} r^{1-n} \int_{S^{n-1}} \|f(ry)\|_\mathcal{Y} \psi(dy) \right]^{1/p} < \infty,
\]
where $1 < p < \infty$. As usually $W^m_p(\Omega, \mathcal{Y})$ stands for the Sobolev space of all functions $f : \Omega \to \mathcal{Y}$ such that their partial derivatives $D^\alpha f$ are measurable for each $|\alpha| \leq m$ and
\[
(2) \quad \|f\|_{W^m_p(\Omega, \mathcal{Y})} := \left[ \sum_{|\alpha| \leq m} \int_\Omega \|D^\alpha f(y)\|_\mathcal{Y} \psi(dy) \right]^{1/p} < \infty,
\]
where $1 < p < \infty$, $\Omega$ is a Riemann $C^m$ manifold, $\psi$ is a Borel measure (that is a volume element) on $\Omega$. Particularly, one gets $W^0_p(\Omega, \mathcal{Y}) = L^p(\Omega, \mathcal{Y})$ for $m = 0$. Then we denote by $K_p(B, \mathcal{Y})$ the space of all functions $f : \bar{B} \to \mathcal{Y}$ satisfying the conditions:
\[
(3) \quad f|_B \in C^1(B, \mathcal{Y}) \text{ and }
(4) \quad \sigma f(z) = 0 \text{ for each } z \in B \text{ and }
(5) \quad f \in H^p(\bar{B}, \mathcal{Y}).
\]
This left $H$-module $K_p(B, \mathcal{Y})$ is supplied with the norm inherited from $H^p(\bar{B}, \mathcal{Y})$.

18. Theorem. If $1 < p < \infty$, then the left module $K_p(B, \mathcal{X})$ is Banach and has an unconditional basis over the quaternion skew field $H$. 

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The Proof of this theorem is analogous to that of Theorem 15 and Corollary 16 with the following modifications.

In view of Theorem 6.12 [1] the mapping \( f \mapsto \hat{P}(f) \) is the surjective isometry from \( L^p(S^{n-1}, \mathbb{R}) \) onto \( H^p(\bar{B}, \mathbb{R}) \), where \( 1 < p < \infty \). On the other hand, the restriction of the Dirac operator \( \sigma^* \) to the unit sphere \( S^{n-1} \) maps the Sobolev space \( W_1^p(S^{n-1}, \mathcal{X}) \) into \( L^p(S^{n-1}, \mathcal{X}) \), whilst the anti-derivation operator \( \Upsilon_{\sigma^*}|_{S^{n-1}} \) restricted to \( S^{n-1} \) maps the Lebesgue space \( L^p(S^{n-1}, \mathcal{X}) \) into \( W_1^p(S^{n-1}, \mathcal{X}) \). Then the linear system of partial differential equations with real coefficients 8(3) has a solution \( g \in W_1^p(\bar{B}, \mathcal{X}) \) for each \( f \in L^p(\bar{B}, \mathcal{X}) \), hence \( \sigma^*(W_1^p(S^{n-1}, \mathcal{X})) = L^p(S^{n-1}, \mathcal{X}) \). Then from Formula 2(3) we infer, that the kernel of the restriction to \( W_1^p(\bar{B}, \mathbb{R}) \) of the operator \( \sigma^* \) consists of all functions \( f \) each partial derivative \( \partial f(z)/\partial z_{m,k} \) of which is zero almost everywhere on \( \bar{B} \). From Lebesgue’s theorems (see Theorems 2 and 3 in §VI.4 [21]) applied by each real variable \( z_{m,k} \) it follows that such \( f \in \ker(\sigma^*) \) is almost everywhere constant on \( \bar{B} \), since

\[
\int_0^{z_{m,k}} \left( \partial f(z)/\partial z_{m,k} \right) dz_{m,k} = f(z + (x_{m,k} - z_{m,k})e_ki_m) - f(0)
\]

for each \( z + (x_{m,k} - z_{m,k})e_ki_m \in \bar{B} \), where \( z \in \bar{B} \). Thus we get that

\[
\ker(\sigma^*|_{W_1^p(\bar{B}, \mathbb{R})}) = \mathbb{R}.
\]

Hence

1. \( \sigma^* \hat{P}(g|_{S^{n-1}})|_{S^{n-1}} = \sigma^*g|_{S^{n-1}} \) almost everywhere on \( S^{n-1} \) for each \( g \in W_1^p(S^{n-1}, \mathcal{X}) \). Moreover, we deduce that

2. \( \sigma^* \hat{P}(\Upsilon_{\sigma^*}|_{S^{n-1}}f)|_{S^{n-1}} = f = \hat{P}(f)|_{S^{n-1}} \)

almost everywhere on \( S^{n-1} \) for each \( f \in L^p(S^{n-1}, \mathcal{X}) \). Therefore, \([l - span_{\mathcal{H}}\sigma^*(W_1^p(S^{n-1}, \mathbb{R})) \cap L^p(S^{n-1}, \mathcal{H})] \) is dense in \( L^p(S^{n-1}, \mathcal{H}) \) and hence \( \bigoplus_{p=1}^l l - span_{\mathcal{H}}[\sigma^*(W_1^p(S^{n-1}, \mathbb{R})]e_p \) is dense in \( L^p(S^{n-1}, \mathcal{X}) \). Then we infer that the composite operator \( \sigma^* \hat{P}\Upsilon_{\sigma^*}|_{S^{n-1}} \) is continuous from \( L^p(S^{n-1}, \mathcal{X}) \) onto \( K_p(B, \mathcal{X}) \) due to Theorem 6.12 [1], Formulas 15(1, 2) and Theorem 8.

Evidently, there is the continuous embedding of \( K_p(B, \mathcal{X}) \) into \( L^p(\bar{B}, \mathcal{X}) \).

Let \( h_n \) be a fundamental sequence in \( K_p(B, \mathcal{X}) \), then \( h_n|_{S^{n-1}} \) is a fundamental sequence in \( L^p(S^{n-1}, \mathcal{X}) \) according to Formula 17(1). Hence the limit \( \lim_{n \to \infty} h_n|_{S^{n-1}} = y \) exists in \( L^p(S^{n-1}, \mathcal{X}) \). Using the continuity of the operator \( \sigma^* \hat{P}\Upsilon_{\sigma^*}|_{S^{n-1}} \), Identities (2), Theorem 8 and Theorem 6.12 [1] we
deduce that the limit
\[ \lim_{n \to \infty} h_n = h = \sigma^* \hat{P}(\Upsilon_{\sigma^*}|_{S^{n-1}y}) \]
exists in \( K_p(B, \mathcal{X}) \) relative to the norm 17(1), consequently, \( K_p(B, \mathcal{X}) \) is the left Banach module over the quaternion skew field.

According to Proposition 1.c.8 and Theorem 1.c.9 in volume 2 of the book [24] the Banach space \( L^p(S^{n-1}, \mathbb{R}) \) has an unconditional basis. Then we choose an unconditional basis \( \{ f_m : m \in \mathbb{N} \} \) in \( L^p(S^{n-1}, \mathcal{H}) \). Therefore, \( \{ f_m e_k : m \in \mathbb{N}, k = 1, ..., l \} \) is the unconditional basis in the left \( \mathcal{H} \) module \( L^p(S^{n-1}, \mathcal{X}) \). For each \( f_m \) then we choose a particular solution \( g_m \in W^p_1(S^{n-1}, \mathcal{X}) \) of the equation \( \sigma^* g_m = f_m \) restricted to \( S^{n-1} \).

One can put \( Q \) to be the left Banach module over \( \mathcal{H} \) which is the closure in the Sobolev space \( W^p_1(S^{n-1}, \mathcal{X}) \) of \( l - \text{span}_\mathcal{H} \{ g_m e_k : m \in \mathbb{N}, k = 1, ..., l \} \). Therefore, we deduce that \( \sigma^* Q = L^p(S^{n-1}, \mathcal{X}) \). From Theorem 6.12 [1] and Formulas (1, 2) above it follows that \( \{ \sigma^* \hat{P}(g_m e_k) : m, k \} \) is the unconditional basis in \( K_p(B, \mathcal{X}) \).

19. Remark. More concrete bases can be constructed with the help of Theorem 15 and Corollary 16 as it is outlined below. In the real Hilbert space \( L^2(S^{n-1}, \mathbb{R}) \) supplied with the standard scalar product
\[ \{ f, g \} := \int_{S^{n-1}} f(y) g(y) \psi(dy) \]
the subspace of harmonic polynomials restricted on \( S^{n-1} \) is dense, where \( f \) and \( g \) are functions from \( S^{n-1} \) into \( \mathbb{R} \), while \( f \) and \( g \in L^2(S^{n-1}, \mathbb{R}) \). Moreover, the decomposition
\[ L^2(S^{n-1}, \mathbb{R}) = \bigoplus_{m=0}^{\infty} \mathcal{H}_m(S^{n-1}) \]
is valid (see Theorem 5.8 [1]), where \( \mathcal{H}_m(\mathbb{R}^n) \) denotes the space of all harmonic real homogeneous polynomials \( P_m(x) \) of degree \( m \) on \( \mathbb{R} \), that is \( P_m(tx) = t^m P_m(x) \) for each \( x \in \mathbb{R}^n \) and \( t \in \mathbb{R} \), \( \Delta P_m(x) \equiv 0 \), whilst the vector space \( \mathcal{H}_m(S^{n-1}) \) is the restriction of \( \mathcal{H}_m(\mathbb{R}^n) \) to \( S^{n-1} \). According to Theorem 5.34 [1] the set \( \{ D^\alpha |x|^{2-n} : |\alpha| = m, \alpha_1 = 0 \text{ or } 1 \} \) is a basis for the real vector space \( \mathcal{H}_m(S^{n-1}) \). Therefore, \( l - \text{span}_\mathcal{H} \bigcup_{m=0}^{\infty} \{ D^\alpha |x|^{2-n} : |\alpha| = m, \alpha_1 = n)
0 or 1} |_{S^{n-1}} \text{ is dense in } C^1(S^{n-1}, \mathbb{R}), \text{ since } C^1(S^{n-1}, \mathbb{R}) \subset L^2(S^{n-1}, \mathbb{R}) \text{ and } \mathcal{H}_m(S^{n-1}) \subset C^1(S^{n-1}, \mathbb{R}) \text{ for each } m.

From the decomposition of the Banach space \( C(S^{n-1}, \mathbb{H}) = \bigoplus_{j=0}^{3} C(S^{n-1}, \mathbb{R})_{ij} \) it follows that the left \( \mathbb{H} \)-span of the set \( \{ D^\alpha \sigma^*_x | x |^{2-n} \ : \ |\alpha| = m, \alpha_1 = 0 \text{ or } 1; m = 1, 2, ... \} |_{S^{n-1}} \) is dense in \( C(S^{n-1}, \mathbb{H}) \), since \( D^\alpha \) and \( \sigma^*_x \) commute, where \( x = (x_1, ..., x_n) \in \mathbb{R}^n \), \( D^\alpha = \partial^{\alpha_1} / \partial x_1^{\alpha_1} ... \partial x_n^{\alpha_n} \), \( \alpha = (\alpha_1, ..., \alpha_n) \), \( \alpha_k \in \{ 0, 1, 2, ... \} \) for each \( k = 1, ..., n \), \( |\alpha| := \alpha_1 + ... + \alpha_n \). Using the selection procedure of §15 and Corollary 16 one gets from this system a Schauder basis in \( K(B, \mathcal{Y}) \).

It is possible also to take as the starting point Schauder bases in \( C^1(S^{n-1}, \mathbb{R}) \). In several works (see [5, 45, 46] and references therein) Schauder bases were constructed in the Banach spaces \( C^m([0,1]^k, \mathbb{R}) \).

Using two chart atlas of the \( C^\infty \) Riemann manifold \( S^{n-1} \) and a Schauder basis in \( C^m(D, \mathbb{R}) \) one can construct a Schauder basis in \( C^m(S^{n-1}, \mathbb{R}) \), where \( D := \{ x : x \in \mathbb{R}^{n-1} ; |x| \leq 1 \} \), \( m \geq 1 \). In \( C(S^{n-1}, \mathbb{R}) \) a Schauder basis exists due to Weierstrass theorem and in \( C(S^{n-1}, \mathbb{H}) \) according to Theorem 2.4.

Above Clifford algebras and modules were considered over the quaternion skew field \( \mathbb{H} \). One can also consider Clifford algebras \( \mathcal{X} \) and modules \( \mathcal{Y} \) both over \( \mathbb{R} \) or \( \mathbb{C} \) and construct Schauder bases in \( K(B, \mathcal{Y}) \) and in \( K_p(B, \mathcal{Y}) \) with \( 1 < p < \infty \) over the field either \( \mathbb{R} \) or \( \mathbb{C} \) correspondingly using results of this paper.

Apart from the left \( \mathbb{H} \)-module \( K_p(B, \mathcal{Y}) \) with \( 1 < p < \infty \), the left \( \mathbb{H} \)-module \( K(B, \mathcal{Y}) \) is not expected to have an unconditional basis, since \( C(0,1) \) does not have an unconditional basis and does not even embed in a space with an unconditional basis (see page 2 in volume 2 of the book [24]).

The results of this paper can be used for a subsequent investigation of kernels \( ker(\sigma) \) of the Dirac operator \( \sigma \), integration and solution of partial differential equations.

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