On Tzitzeica Curves According to Q-Frame in Euclidean 3-Space

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Abstract: Tzitzeica curve is a special spatial curve, which introduced Gheorghe Tzitzeica in 1911. Gheorghe Tzitzeica defined this curve as follows; for a spatial curve \( \phi \), "if the ratio of its torsion \( \tau \) and the square of the distance \( d \) from the origin to the osculating plane at an arbitrary point of the curve is constant, then this spatial curve is a Tzitzeica curve in Euclidean space." Moreover Gheorghe Tzitzeica defined a special surface which is named Tzitzeica surface in 1907. In this surface, the asymptotic lines of Tzitzeica surfaces with negative Gaussian curvature are Tzitzeica curves. Also, the ratio of its Gaussian curvature \( K \) and the distance \( d \) from the origin to the tangent plane at any arbitrary point of the surface is constant. In this paper, we study the Tzitzeica curve via \( q \)-frame in \( \mathbb{E}^3 \). Firstly, we redefine the Tzitzeica curve \( q \)-frame in Euclidean 3-space. Then, it is obtained some conditions for the Tzitzeica curve as a spherical curve. Finally, we investigate harmonic vector cases of the Tzitzeica curve according to \( q \)-frame in Euclidean space.

Keywords: Tzitzeica curve, \( q \)-frame, Spherical curve

Introduction

Tzitzeica curve satisfies the following condition

\[
\frac{\tau}{d_{osc}^2} = a,
\]

where \( a \neq 0 \in \mathbb{R} \), and \( d_{osc} = \langle B, \phi \rangle \) and Tzitzeica surface satisfies the following condition

\[
\frac{K}{d_{osc}} = a,
\]

where \( a \neq 0 \in \mathbb{R} \), (Bayram et al., 2018, Bila, 2012 and Crâsmareanu, 2002). Then, these curves and surfaces are often used in scientific research from the past to the present with geometric properties. For example, Crâsmareanu studied the cylindrical form of Tzitzeica curves in (Crâsmareanu, 2002), Karacan gave the elliptic cylindrical form of Tzitzeica curves in (Karacan & Bukcu, 2009), Agnev computed the effect of a general centro-affine transformation on Tzitzeica surface in (Agnew et al., 2010) and finally, Bayram characterized Tzitzeica curves in \( \mathbb{E}^3 \) in terms of their curvatures, (Bayram et al., 2018). See more details (Bobe et al., 2012, Yazici et al., 2022)

Although there are many studies on these curves and surfaces in Euclidean space and Minkowski space, it is almost never studied according to \( q \)-frame in Euclidean space. We redefine the Tzitzeica curve \( q \)-frame in Euclidean 3-space. Then, it is obtained some conditions for the Tzitzeica curve as a spherical curve. Finally, we give harmonic vector cases of the Tzitzeica curve according to \( q \)-frame in Euclidean space.
Preliminaries

Let $\phi : I \subseteq \mathbb{R} \to \mathbb{R}^3$ be an unit speed curve. The $q$-frame of the curve $\phi$ is obtained as follows

$$t = \phi', \quad n_q = \frac{\times z}{\|x \times z\|}, \quad b_q = t \times n_q, \tag{1}$$

where $n_q$ is the quasi-normal vector, $b_q$ is the quasi-binormal vector, and $z$ is the projection vector. The $q$ formulas are expressed as

$$\begin{bmatrix}
\nabla \phi\cdot t \\
\nabla \phi\cdot n_q \\
\nabla \phi\cdot b_q 
\end{bmatrix} = \begin{bmatrix} 0 & k_1 & k_2 \\
-k_1 & 0 & k_3 \\
-k_2 & -k_3 & 0 
\end{bmatrix} \begin{bmatrix} t \\
n_q \\
b_q 
\end{bmatrix}, \tag{2}$$

where $k_i$ are called $q$-curvatures $(1 \leq i \leq 3)$,

$$k_1 = k \cos \gamma = \langle t', n_q \rangle,$$
$$k_2 = -k \sin \gamma = \langle t', b_q \rangle,$$
$$k_3 = dy + \tau = -\langle n_q, b_q' \rangle, \tag{3}$$

and $\gamma$ is the angle between the principal normal vector and the quasi normal vector. In this paper, it will be assumed that the projection vector $z = (0,0,1)$. If $\phi$ and $z$ are parallel, the vector $z$ can be chosen as $z = (0,1,0)$, (Dede & Ekici, 2018).

On the other hand, let $\phi$ be a space curve with the arclength parameter $s$, and $\nabla \phi' = \frac{d}{ds}$. The Laplacian operator $\Delta$ of $\phi$ can write as, (Kocayigit et al., 2016),

$$\Delta = -\nabla \phi' \cdot \nabla \phi'.$$

Definition 2.1 Let $\phi : I \subseteq \mathbb{R} \to \mathbb{R}^3$ be a unit speed curve. The curve $\phi$ is called a harmonic tangent vector if

$$\Delta t = 0.$$

Also, The curve $\phi$ is called a harmonic 1-type tangent vector if

$$\Delta t = \lambda t,$$

where $\lambda \in \mathbb{R}$, (Kocayigit et al., 2016).

Definition 2.2 Let $\phi : I \subseteq \mathbb{R} \to \mathbb{R}^3$ be a unit speed curve. The curve $\phi$ is called a harmonic principal vector if

$$\Delta n = 0.$$

Also, The curve $\phi$ is called a harmonic 1-type principal normal vector if

$$\Delta n = \lambda n,$$

where $\lambda \in \mathbb{R}$, (Kocayigit et al., 2016).

Definition 2.3 Let $\phi : I \subseteq \mathbb{R} \to \mathbb{R}^3$ be a unit speed curve. The curve $\phi$ is called a harmonic binormal vector if

$$\Delta b = 0.$$

Also, The curve $\phi$ is called a harmonic 1-type binormal vector if

$$\Delta b = \lambda b,$$

where $\lambda \in \mathbb{R}$, (Kocayigit et al., 2016).
Some Characterizations of Tzitzeica Curves

In this chapter, we will define Tzitzeica curves according to $q$-frame in $\mathbb{E}^3$. Then, we will do some characterizations of Tzitzeica curves with $q$-curvatures.

Definition 3.1 Let $\phi : I \subset \mathbb{R} \rightarrow \mathbb{E}^3$ be a unit speed curve, $\{t, n_q, b_q\}$ be its $q$-frame, and $k_1, k_2, k_3$ be $q$-curvatures. If the inner product of the two vectors $\alpha$ and $\beta$ in Euclidean space is denoted by $g(\alpha, \beta)$, the curve $\phi$ that provides the following condition is called the Tzitzeica curve:

$$k_3 = \ell g\left(b_q, \phi\right),$$

where $k_3 \neq 0$, $k_1 > 0$, and $k_2 > 0$.

Proposition 3.2 Let $\phi : I \subset \mathbb{R} \rightarrow \mathbb{E}^3$ be a unit speed Tzitzeica curve, $\{t, n_q, b_q\}$ be its $q$-frame. Then, the Tzitzeica curve $\phi$ provides the following equality

$$k_2k_3g(\phi, t) + k_3^2g(\phi, n_q) + k_3'g(\phi, b_q) = 0.$$  (5)

Proof. Because the Tzitzeica curve $\phi$ provides eq. (4), $\ell \neq 0$. In this case, the proof is clear.

Theorem 3.3 Let $\phi : I \subset \mathbb{R} \rightarrow \mathbb{E}^3$ be a unit speed Tzitzeica curve, $\{t, n_q, b_q\}$ be its $q$-frame, and $k_1, k_2, k_3$ be $q$-curvatures. The Tzitzeica curve $\phi$ is a spherical curve if and only if

$$(-k_1^2 - k_2^2)g(\phi, t) + (k_1' - k_2k_3)g(\phi, n_q) + (k_1k_3 + k_2')g(\phi, b_q) = 0.$$  (6)

Proof. If the Tzitzeica curve $\phi$ is a spherical curve, we can write

$$\|\phi\| = r.$$

Take the derivative last equation, it is obtained as

$$g(\phi, t) = 0.$$

If the last equation is taken again derivative, we can write as

$$(-k_1^2 - k_2^2)g(\phi, t) + (k_1' - k_2k_3)g(\phi, n_q) + (k_1k_3 + k_2')g(\phi, b_q) = 0.$$

In this case, the proof is completed.

Proposition 3.4 Let $\phi : I \subset \mathbb{R} \rightarrow \mathbb{E}^3$ be a unit speed spherical Tzitzeica curve, $\{t, n_q, b_q\}$ be its $q$-frame, and $k_1, k_2, k_3$ be $q$-curvatures. The following equations can be written for the curvatures of the spherical Tzitzeica curve;

$$k_1 = \frac{k_3' - k_2^2}{k_3},$$

$$k_2 = \frac{k_1' - k_2^2}{k_3},$$

$$k_3 = \frac{k_1^2 + k_2^2}{k_2}.$$  (7)

Proof. The proof is completed considering the Proposition 3.2 and the Theorem 3.3 together.

Theorem 3.5 Let $\phi : I \subset \mathbb{R} \rightarrow \mathbb{E}^3$ be a unit speed Tzitzeica curve, $\{t, n_q, b_q\}$ be its $q$-frame, and $k_1, k_2, k_3$ be $q$-curvatures. If the position vector of the Tzitzeica curve is

$$\phi(s) = \pi_1(s)t(s) + \pi_2(s)n_q(s) + \pi_3(s)b_q(s),$$

$$\pi_1(s) = \pi_2(s) = \pi_3(s) = \frac{1}{\sqrt{\ell^2 + k_1^2 + k_2^2}},$$

$$\pi_1'(s) = \frac{k_1}{\ell},$$

$$\pi_2'(s) = \frac{k_2}{\ell},$$

$$\pi_3'(s) = \frac{k_3}{\ell}.$$  (8)

Proof. The proof is completed considering the Proposition 3.2 and the Theorem 3.3 together.
the following equations can be written

\[ \pi_1' - \pi_2k_1 - \pi_3k_2 = 1, \]
\[ \pi_1k_1 + \pi_2' - \pi_3k_3 = 0, \]
\[ \pi_1k_2 + \pi_2k_3 + \pi_3' = 0. \]

Theorem 3.6 Let \( \phi: I \subset \mathbb{R} \rightarrow \mathbb{R}^3 \) be a unit speed Tzitzeica curve, \( \{t, n_q, b_q\} \) be its \( q \)-frame, and \( k_1, k_2, k_3 \) be \( q \)-curvatures. The differential equation characterizing Tzitzeica curve according to its tangent vector is obtained

\[ \mu_1 \nabla_{\phi}' t + (k_1 \eta_2 - k_2 \eta_1) \nabla_{\phi}' \eta_2 + (k_3 \eta_1 + \lambda_2 \eta_2) \nabla_{\phi}' \lambda_1 + \eta_1 k_2 \lambda_1 - \eta_2 k_1 \lambda_1 t = 0, \]  

where

\[ \lambda_1 = -k_1^2 - k_2^2, \]
\[ \lambda_2 = k_1' - k_2k_3, \]
\[ \lambda_3 = k_1k_3 + k_2', \]
\[ \mu_1 = k_2 \lambda_2 - k_1 \lambda_3, \]
\[ \eta_1 = \lambda_1 k_1 + \lambda_2^2 - \lambda_2 k_3, \]
\[ \eta_2 = \lambda_1 k_2 + \lambda_3^2 + \lambda_2 k_3. \]

Proof. Assume that \( \phi \) is the Tzitzeica curve and \( \{t, n_q, b_q\} \) is its \( q \)-frame. Then, the following equations can be written

\[ \nabla_{\phi}' t = k_1 n_q + k_2 b_q, \]  

\[ \nabla_{\phi}' \eta_2 = (-k_2 - k_2^2) t + (k_1' - k_2k_3) n_q + (k_1k_3 + k_2') b_q, \]

And

\[ \nabla_{\phi}' \lambda_1 = (-k_1^2 - k_1 k_2 + k_2' - 2k_2k_3 - k_3') n_q + (-k_1 k_2 - k_2^2 + 2k_1' k_3 - k_2k_3' + k_1 k_3 + k_2') b_q. \]

Considering eq. (9) and eq. (10) together, \( q \)-normal vector and \( q \)-binormal vector of the Tzitzeica curve obtained as

\[ n_q = \frac{1}{k_2k_3 - k_1k_2} \left( k_2 \nabla_{\phi}' t - \lambda_3 \nabla_{\phi}' \lambda_1 t - k_2 \lambda_3 t \right), \]

and

\[ b_q = \frac{1}{k_2k_3 - k_1k_2} \left( -k_1 \nabla_{\phi}' t + \lambda_2 \nabla_{\phi}' \lambda_1 t + k_1 \lambda_3 t \right). \]

Then, we can write

\[ L_1 \nabla_{\phi}' t + L_2 \nabla_{\phi}' \eta_2 + L_3 \nabla_{\phi}' \lambda_1 + L_4 t = 0, \]

where

\[ L_1 = 2k_1 k_2 \lambda_2 \lambda_3 - k_2^2 \lambda_2^2 - k_2^2 \lambda_3^2, \]
\[ L_2 = -(k_2 \eta_1 + k_1 \eta_2), \]
\[ L_3 = \lambda_3 \eta_1 + \lambda_2 \eta_2, \]
\[ L_4 = (2k_1 k_2 \lambda_2 \lambda_3 - k_2^2 \lambda_2^2 - k_2^2 \lambda_3^2)(\lambda_3 k_2 + \lambda_2 k_1 - \lambda_1') + \eta_1 k_2 \lambda_1 + \eta_2 k_1 \lambda_1. \]
Corollary 3.7 Let $\phi: I \subset \mathbb{R} \to \mathbb{R}^3$ be a unit speed Tzitzeica curve, $\{t, n_q, b_q\}$ be its $q$-frame, and $k_1, k_2, k_3$ be $q$-curvatures. The Tzitzeica curve $\phi$ is not both harmonic tangent vector and harmonic 1-type tangent vector.

Proof. Assume that $\phi$ is a Tzitzeica curve. Considering the Definition 2.1 and the Corollary 3.7 together, if the curve $\phi$ is a harmonic 1-type tangent vector, the following equations can be written

$$k_1^2 + k_2^2 = 0,$$

$$k_1' - k_2k_3 = 0,$$

$$k_1k_2 + k_3' = 0.$$

In this case, we can write $k_1 = k_2 = 0$, but the last equations conflict the Definition 3.1. That is, the Tzitzeica curve $\phi$ is not harmonic 1-type tangent vector according to $q$-frame in Euclidean space.

The other case can be shown similarly. That is, the Tzitzeica curve $\phi$ is not harmonic tangent vector according to $q$-frame in Euclidean space.

Theorem 3.8 Let $\phi: I \subset \mathbb{R} \to \mathbb{R}^3$ be a unit speed Tzitzeica curve, $\{t, n_q, b_q\}$ be its $q$-frame, and $k_1, k_2, k_3$ be $q$-curvatures. The differential equation characterizing Tzitzeica curve according to its $q$-normal vector is obtained

$$\mu_3\nabla^2_{\phi}n_q - (\eta_3k_3 + \eta_2k_1)\nabla^2_{\phi}n_q + (\eta_3\lambda_3 - \eta_2\lambda_1)\nabla_{\phi}n_q = 0,$$

where

$$\lambda_4 = -k_1' - k_2k_3,$$

$$\lambda_5 = -k_1^2 - k_3^2,$$

$$\lambda_6 = -k_1k_2 + k_3',$$

$$\mu_2 = k_1\lambda_3 + \lambda_1k_3,$$

$$\eta_3 = \lambda_1' + \lambda_2k_1 - \lambda_3k_2,$$

$$\eta_4 = \lambda_1k_2 + \lambda_2k_3 + \lambda_3'.$$

Proof. Assume that $\phi$ is the Tzitzeica curve and $\{t, n_q, b_q\}$ is its $q$-frame. Then, the following equations can be written

$$\nabla_{\phi}n_q = -k_1t + k_3b_q,$$

(13)

$$\nabla^2_{\phi}n_q = (-k_1' - k_2k_3)t + (-k_1^2 - k_3^2)n_q + (-k_1k_2 + k_3')b_q,$$

(14)

and

$$\nabla^2_{\phi}n_q = (-k_1'k_2k_3 - 2k_2k_3 - k_1k_2^2 - k_1k_2k_3 + k_1k_2^2)t$$

$$+ (-3k_1k_2k_3 + k_1k_2k_3 - 3k_1^2k_3')n_q$$

$$+ (-2k_1k_2 - k_2k_3 - k_2^2k_3 - k_2^3 - k_1k_2' + k_2')b_q.$$

The proof is completed.

Corollary 3.9 Let $\phi: I \subset \mathbb{R} \to \mathbb{R}^3$ be a unit speed Tzitzeica curve, $\{t, n_q, b_q\}$ be its $q$-frame, and $k_1, k_2, k_3$ be $q$-curvatures. The Tzitzeica curve $\phi$ is not both harmonic $q$-normal vector and harmonic 1-type $q$-normal vector.

Proof. The proof is clear from the proof of the Corollary 3.7.

Theorem 3.10 Let $\phi: I \subset \mathbb{R} \to \mathbb{R}^3$ be a unit speed Tzitzeica curve, $\{t, n_q, b_q\}$ be its $q$-frame, and $k_1, k_2, k_3$ be $q$-curvatures. The differential equation characterizing Tzitzeica curve according to its $q$-binormal vector is obtained

$$\mu_3\nabla^2_{\phi}b_q - (\eta_3k_3 + \eta_2k_1)\nabla^2_{\phi}b_q - (\eta_3\lambda_3 - \eta_2\lambda_1)\nabla_{\phi}b_q$$

$$+ (\eta_3k_3\lambda_3 - \eta_2k_2\lambda_3 - \mu_3(\lambda_3k_2 - \lambda_3k_3 + \lambda_3'))b_q = 0,$$

(16)
where
\[ \lambda_7 = -k_2' + k_1k_3, \]
\[ \lambda_8 = -k_1k_2 - k_3, \]
\[ \lambda_9 = -k_2^2 - k_3^2, \]
\[ \mu_3 = \lambda_8k_2 - \lambda_7k_3, \]
\[ \eta_5 = \lambda_7' - \lambda_8k_1 - \lambda_9k_2, \]
\[ \eta_6 = \lambda_7k_1 + \lambda_8 - \lambda_9k_3. \]

Proof. Assume that \( \phi \) is the Tzitzeica curve and \( \{t, n_q, b_q\} \) is its q-frame. Then, the following equations can be written
\[ \nabla_{\phi} \mathbf{b}_q = -k_2t - k_3n_q, \quad (17) \]
\[ \nabla_{\phi}^2 \mathbf{b}_q = (-k_2' + k_1k_3)t + (-k_1k_2 - k_3')n_q + (-k_2^2 - k_3^2)b_q, \quad (18) \]
and
\[ \nabla_{\phi}^3 \mathbf{b}_q = (\lambda_7' - \lambda_8k_1 - \lambda_9k_2)t + (\lambda_7k_1 + \lambda_8' - \lambda_9k_3)n_q + (\lambda_7k_2 - \lambda_8k_3 + \lambda_9'b_q. \]

Corollary 3.11 Let \( \phi: I \subset \mathbb{R} \rightarrow \mathbb{E}^3 \) be a unit speed Tzitzeica curve, \( \{t, n_q, b_q\} \) be its q-frame, and \( k_1, k_2, k_3 \) be q-curvatures. The Tzitzeica curve \( \phi \) is not both harmonic q-binormal vector and harmonic 1-type q-binormal vector.

Proof. The proof is clear from the proof of the Corollary 3.7.

Conclusion

Tzitzeica curve is a special spatial curve which is introduced by Gheorghe Tzitzeica in 1911. We redefine the Tzitzeica curve according to q-frame in Euclidean 3-space. We obtain the necessary and sufficient condition for a Tzitzeica curve to be a spherical curve. Then, we give some curvature properties satisfied by these curves. Finally, we obtain differential equations that characterize Tzitzeica curves according to q-frame and we determine if the Tzitzeica curve can both be harmonic and harmonic 1-type vector for three cases according to q-frame in Euclidean 3-space.

Scientific Ethics Declaration

The authors declare that the scientific ethical and legal responsibility of this article published in EPSTEM journal belongs to the authors.

Acknowledgements or Notes

This article was presented as an oral presentation at the International Conference on Research in Engineering, Technology and Science (www.icrets.net) conference held in Baku/Azerbaijan on July 01-04, 2022.

This work was supported by the Scientific Research Projects Commission of Selcuk University under the contract number 22701016.

References

Agnew, A., Bobe, A., Boskoff, W., & Suceava, B. (2010). Tzitzeica curves and surfaces. The Mathematica Journal, 12, 1-18.
Bayram, B., Tunc, E., Arslan, K., & Ozturk, G. (2018). On Tzitzeica curves in Euclidean 3-space, *Facta Universitatis, Series: Mathematics and Informatics*, 409-416.

Bila, N. (2012). *Symmetry reductions for the Tzitzeica curve equation*, Math and Computer Science Faculty Working Papers.

Bobe, A., Boskoff, W. G., & Ciucă, M. G. (2012). Tzitzeica-Type centro-affine invariants in Minkowski spaces. *Analele Universității Ovidius Constanța-Seria Matematică*, 20(2), 27-34.

Crâsmareanu, M. (2002). Cylindrical Tzitzeica curves implies forced harmonic oscillators. *Surfaces*, 1, 32.

Dede, M., & Ekici, C. (2018). Directional Bertrand curves. *Gazi University Journal of Science*, 31(1), 202-211.

Karacan, M. K., & Bukcu, B. (2009). On the elliptic cylindrical Tzitzeica curves in Minkowski 3-space. *Sci. Magna*, 5, 44-48.

Kocayigit, H., Kazaz, M., & Ari, Z. (2016). *Some Characterizations of space curves according to Bishop frame in Euclidean 3-space E3*. Ankara Matematik Gunleri, TOBB Ekonomi ve Teknoloji Universitesi Matematik Bolumu, Ankara, 3-4.

Yazici, B. D., Karakuş, S. O., & Tosun, M. (2022). On framed Tzitzeica curves in Euclidean space. *Facta Universitatis, Series: Mathematics and Informatics*, 313-325.

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To cite this article:
Sariaydin, M.T & Yazla, A. (2022). On Tzitzeica curves according to Q-Frame in Euclidean 3-Space. *The Eurasia Proceedings of Science, Technology, Engineering & Mathematics (EPSTEM)*, 17, 19-25.