Gradient Flow Analysis on MILC HISQ Ensembles

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Motivation

- Any dimensionful quantity with a finite continuum value can be used for scale setting.
- Eventually an experimentally determined quantity is needed to calculate in physical units, but relative scale setting can be done without physical units.
- The ideal scale setting routine:
  ▶ easy and fast to compute
  ▶ low statistical errors
  ▶ insensitive to systematic effects
    (valence/sea quark mass, finite volume, etc.)
- Gradient flow is particularly promising for its ease, speed, low statistical error, and small dependence on the lattice spacing and masses.
Gradient Flow and Scales

- Gradient flow is a smoothing of the original gauge fields $U$ towards stationary points of the action $S$. \cite{Luscher:2010iy}
- Successive links $V(t)$ are updated in flowtime according to the diffusion equation,

$$
\frac{d}{dt} V(t)_{i,\mu} = -V_{i,\mu} \frac{\partial S(V)}{\partial V_{i,\mu}}, \quad V(0)_{i,\mu} = U_{i,\mu} \quad \left[ \frac{dA_{\mu}}{dt} = D_{\nu} F_{\nu\mu} \right]
$$

- Dimensionless quantities can be defined through the energy density $\langle E(t) \rangle$ and flowtime $t[a^2]$. \cite{Luscher:2010iy} and \cite{Borsanyi:2012zs}

$$
T(t) = t^2 \langle E(t) \rangle \quad W(t) = t \frac{d}{dt} T(t)
$$

- From which a dimensionful quantity can be determined at the fiducial point

$$
T(t_0) = W(w_0^2) = 0.3
$$
### Measurements of $w_0/a$ and $\sqrt{t_0}/a$ ($m'_s = m_s$)

| $a$(fm) | $m'_f/m'_s$ | volume   | $N_{run}/N_{bins}$ | $\sqrt{t_0}/a$ | $w_0/a$[%]          |
|---------|--------------|----------|--------------------|----------------|---------------------|
| 0.15    | 1/5          | $16^3 \times 48$ | 1021/127          | 1.1004(05)    | 1.1221(08)[0.07%]  |
| 0.15    | 1/10         | $24^3 \times 48$ | 1000/125          | 1.1092(03)    | 1.1381(05)[0.04%]  |
| 0.15    | 1/27         | $32^3 \times 48$ | 999/124           | 1.1136(02)    | 1.1468(04)[0.03%]  |
| 0.12    | 1/5          | $24^3 \times 64$ | 1040/70           | 1.3124(06)    | 1.3835(10)[0.07%]  |
| 0.12    | 1/10         | $32^3 \times 64$ | 999/66            | 1.3228(04)    | 1.4047(09)[0.06%]  |
| 0.12    | 1/10         | $40^3 \times 64$ | 1001/66           | 1.3226(03)    | 1.4041(06)[0.04%]  |
| 0.12    | 1/27         | $48^3 \times 64$ | 34/34             | 1.3285(05)    | 1.4168(10)[0.07%]  |
| 0.09    | 1/5          | $32^3 \times 96$ | 102/34            | 1.7227(08)    | 1.8957(15)[0.08%]  |
| 0.09    | 1/10         | $48^3 \times 96$ | 119/29            | 1.7376(05)    | 1.9299(12)[0.06%]  |
| 0.09    | 1/27         | $64^3 \times 96$ | 67/16             | 1.7435(05)    | 1.9470(13)[0.07%]  |
| 0.06    | 1/5          | $48^3 \times 144$ | 127/42           | 2.5314(13)    | 2.896(03)[0.11%]   |
| 0.06    | 1/10         | $64^3 \times 144$ | 38/19             | 2.5510(14)    | 2.948(03)[0.11%]   |
| 0.06    | 1/27         | $96^3 \times 192$ | 49/16             | 2.5833(07)    | 3.0118(19)[0.06%]  |

- MILC HISQ ensembles with $N_f = 2 + 1 + 1$ dynamical quarks, at physical $m'_s$.
- Significantly more configurations can be run for ensembles with $a < 0.12$fm.
- Almost all statistical errors have been reduced below 0.1%.
- $\sqrt{t_0}$ has consistently lower statistical errors than $w_0$.  

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Gradient Flow on HISQ  
LAT 2014
Measurements of $w_0/a$ and $\sqrt{t_0}/a$ ($m'_s < m_s$)

- MILC HISQ ensembles at lighter than physical $m'_s$.
- $m'_s$ and $m'$ are the sea quark masses of the ensemble.
- $m_s$ is the physical strange quark mass.
- All ensembles are at $a = 0.12$ fm.

| $m'_s / m_s$ | $m'_s / m_s$ | volume   | $N_{run} / N_{bins}$ | $\sqrt{t_0}/a$ | $w_0/a\%$  |
|---------------|---------------|----------|----------------------|----------------|------------|
| 0.10          | 0.10          | $32^3 \times 64$ | 102/25               | 1.3596(06)     | 1.4833(13)[0.08%] |
| 0.10          | 0.25          | $32^3 \times 64$ | 204/51               | 1.3528(04)     | 1.4676(10)[0.07%] |
| 0.10          | 0.45          | $32^3 \times 64$ | 205/51               | 1.3438(05)     | 1.4470(10)[0.07%] |
| 0.10          | 0.60          | $32^3 \times 64$ | 107/26               | 1.3384(08)     | 1.4351(16)[0.11%] |
| 0.175         | 0.45          | $32^3 \times 64$ | 133/33               | 1.3385(05)     | 1.4349(13)[0.09%] |
| 0.20          | 0.60          | $24^3 \times 64$ | 255/63               | 1.3297(06)     | 1.4169(12)[0.08%] |
| 0.25          | 0.25          | $24^3 \times 64$ | 255/63               | 1.3374(07)     | 1.4336(14)[0.10%] |
Stream Analysis: RHMC vs RHMD

- There are 3 HISQ ensembles with streams generated on RHMD and RHMC. $w_0$ was measured for streams of both types on 2 of these ensembles.
- No significant difference was found between $w_0/a$ measured on ensembles generated with RHMD compared to RHMC.
  - For $a = 0.09$ fm, $m_l/m_s = 1/27$: $w_0(RHMC)/w_0(RHMD) = 1.0009(12)$
  - For $a = 0.06$ fm, $m_l/m_s = 1/10$: $w_0(RHMC)/w_0(RHMD) = 1.0002(26)$
Charm Quark Mistuning

- From decoupling analysis, variations in the charm quark mass can be accounted for through the leading dependence of the QCD scale with three flavors $\Lambda_{QCD}^{(3)}$. If $Q \propto \Lambda_{QCD}^{(3)}$, then

$$\frac{\partial Q}{\partial m_c} = \frac{2}{27} \frac{Q}{m_c}$$

- The meson masses $aM_\pi$, $aM_K$, scales $w_0/a$, $\sqrt{t_0}/a$, and decay constant $aF_{p4s}$ were adjusted.
  - $F_{p4s}$ is the pseudoscalar decay constant with degenerate valence masses $m_{val} = 0.4m_s$ and physical sea quark masses
- Because of small errors in the quantities studied and large charm mass mistunings ($\sim 10\%$ in some cases) adjustments were often significant.
  - Meson mass corrections ranged from 0.3 to $7\sigma_{stat}$
  - Gradient flow scale corrections ranged from 0.5 to $18\sigma_{stat}$
- Adjustments were largest for the unphysical quark mass, $a = 0.15$ and $a = 0.06$ fm ensembles.
Chiral Expansion

- Including quark mass dependence allows us to include ensembles with \( m'_s \neq m_s \) and correct for mistuning errors.
- For the continuum, \( N_f = 2 + 1 \) theory the mass dependence of \( w_0 \) to NNLO is: [0. Bär and M. Golterman, Phys. Rev. D 89, 034505 (2014)]

\[
\begin{align*}
    w_0 &= w_{0,\text{ch}} \left( 1 + k_1 \frac{2M_K^2 + M^2}{(4\pi f)^2} \right) \\
    &+ \frac{1}{(4\pi f)^2} \left( (3k_2 - k_1)M^2_{\pi\mu\pi} + 4k_2M_K^2\mu_K + \frac{k_1}{3} (M^2_{\pi} - 4M_K^2)\mu_\eta + k_2M_{\eta\mu\eta}^2 \right) \\
    &+ k_4 \frac{(2M_K^2 + M^2_{\pi})^2}{(4\pi f)^4} + k_5 \frac{(M^2_K - M^2_{\pi})^2}{(4\pi f)^4} \right), \\
    \mu_Q &= \frac{M_Q^2}{(4\pi f)^2} \log \frac{M_Q^2}{\mu^2}
\end{align*}
\]

- Expansion for \( \sqrt{t_0} \) is identical in form, because the meson masses are independent of flowtime.
Combined Continuum Extrapolation

- We simultaneously performed the continuum extrapolation and meson mass interpolation, using $F_{p4s}$ to make all masses and the gradient flow scale dimensionless.

- We considered many different versions of the extrapolation/interpolation:
  - To extrapolate to the continuum, we included $\alpha_s a^2$ and possible higher orders of $a^2$ (with or without $\alpha_s$), up to $a^6$.
  - Both NLO and NNLO chiral expansions were considered.
  - Products between terms in the chiral and continuum expansions were included up to the same order as the highest of other included terms. For this purpose, $(\Lambda_{QCD} a)^2 \sim (M/(4\pi f))^2$.
  - Due to the large range of $M_K$ covered by the full set of ensembles, some fits drop ensembles with low values of $M_K$.

- Overall, we consider $5_{\text{cont}} \times 2_{\text{chiral}} \times 7_{\text{kaon}} = 70$ versions of the fit.
The central fit form is up to \((\alpha_s a^2)^3\), chiral NNLO, and across all values of \(M_k\).

- Only \(m_s = m_s^{\text{physical}}\) ensembles are plotted, but fit includes all \(m_s \leq m_s^{\text{physical}}\) ensembles.
- Dotted lines are for actual masses run; solid lines are for re-tuned masses per legend.
- Curvature typical of highly improved actions: “leading” term reduced, so “higher” terms evident.
Histogram only includes fits with p-value > 0.01.
This only included fits with $a^6$ or $(\alpha_s a^2)^3$ terms.
Both NLO and NNLO chiral expansions are represented:
  - For NLO, ensembles with $M_K/F_{p4s} > 0.8$ are included
  - For NNLO, all ensembles can be included
The central fit has $\chi^2/dof = 10.6/10$, $p = 0.39$, and is $0.1\sigma$ from the physical $0.06$ fm ensemble.

Half the full width of the histogram is used to conservatively estimate a systematic fit error of $4 \times 10^{-4}$.

There is also residual finite volume error in $F_{p4s}$ (coming from residual FV error in $f_\pi$, which is estimated using $\chi PT$) that cannot be corrected for, adding another systematic error of $2 \times 10^{-4}$ fm.

Result: $w_0 = 0.1722(2)_{\text{stat}}(4)_{a^2}(2)_{\text{FV}}(3)_{F_{p4s}}$ fm
First is the statistical error, then systematic error from the continuum extrapolation, residual finite volume effects, and the value of $F_{p4s}$ in MeV, respectively.
Comparison of $w_0$ and $\sqrt{t_0}$

- Comparison of $w_0 F_{pA}$ and $\sqrt{t_0} F_{pA}$ plotted for physical mass ensembles
- The fits are over the full dataset and for the same central fit form shown before.
- Discretization dependence of $\sqrt{t_0} F_{pA}$ is larger.
Mass Dependence

- The central fit can be used to construct the continuum mass dependence of $w_0$ with masses in units of $w_0$.

- This is useful for scale setting: measure $w_0/a$, $aM_\pi$, and $aM_K$ to construct the independent variables $x = (w_0 M_\pi)^2$ and $y = (w_0 M_K)^2$, then read off $w_0(x, y)(\text{fm})$ from the plot (or a corresponding interpolation).

- Lines are for fixed values of $y = (w_0 M_K)^2$ from 0.1 to 1.3 times the physical value.
Our preliminary value of

\[ w_0 = 0.1722(2)_{\text{stat}}(4)_{a^2(2)FV(3)F_{p4s}} \text{ fm} \]

agrees with HPQCD within 1\(\sigma\), but deviates from BMW by 1.7\(\sigma\) (joint) compared to their HEX smeared, Wilson result \(w_0 = 0.1755(18)(04)\) fm.

[HPQCD (R. J. Dowdall et al.), arXiv:1303.1670] [BMW (S. Borsanyi et al.), JHEP 1209 (2012) 010]

Some of this deviation may be due to the difference in \(N_f\). We will be computing \(w_0\) on the asqtad \(N_f = 2 + 1\) ensembles.

Charm mass mistunings can have a significant effect for precise quantities, such as \(w_0\).

Compared to Lat’13, systematic error from the extrapolation/interpolation is cut in half; this is primarily due to charm mass adjustments and the \(\chi PT\) handle on mass dependence.

Evidence was provided that the discretization effects of \(w_0F_{p4s}\) are smaller than \(\sqrt{t_0F_{p4s}}\), in agreement with what BMW found.