MONOPOLES ON SASAKIAN THREE-FOLDS

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Abstract. We consider monopoles with singularities of Dirac type on quasiregular Sasakian three-folds fibering over a compact Riemann surface \( \Sigma \), for example the Hopf fibration \( S^3 \to S^2 \). We show that these correspond to holomorphic objects on \( \Sigma \), which we call twisted bundle triples. These are somewhat similar to Murray’s bundle gerbes. A spectral curve construction allows us to classify these structures, and, conjecturally, monopoles.

1. Introduction

Since being introduced in the 1930’s by Dirac, monopoles on a three-fold have occupied a place of privilege in the understanding of gauge theory. Dirac’s monopoles were singular, defined over \( \mathbb{R}^3 \), and attached to the gauge group \( U(1) \). In the 1970’s, it was realized that introducing a non-Abelian gauge group allowed one to consider, on \( \mathbb{R}^3 \) at least, non-singular solutions. Like their close cousins, namely instantons on \( \mathbb{R}^4 \), monopoles on \( \mathbb{R}^3 \) allow a holomorphic interpretation, and this was used to great effect by several authors, for example Ward [28] and most notably Hitchin [9] in constructing solutions. In parallel, work of Nahm [21], and then Hitchin [10], tied this complex interpretation to the Nahm transform, giving a very effective dictionary which allowed the classification of monopoles in 1983 by Donaldson [7]. The work, originally done for the gauge group \( SU(2) \), was extended to classical gauge groups by Murray and Hurtubise in a series of papers [19, 12, 11], and then by Jarvis to arbitrary reductive groups [14].

Of course, one is not tied to \( \mathbb{R}^3 \), and one of the early extensions was to hyperbolic space; this case was studied in a beautiful paper by Atiyah [1]. One can show, however, that non-singular and non-trivial monopoles cannot exist unless the space has a suitably large infinity. In particular one cannot have them on a compact manifold. Thus, in the latter case, one is led to admitting some singularities, and those which first appeared in the work of Dirac, and their analogues for general gauge groups, seem to be the most appropriate.

It was realized quite early on that the Dirac-type singularity leads to some most interesting geometry. Indeed, Kronheimer, in his Oxford MSc thesis, [16], showed that the geometry of these Dirac monopoles is tied intimately to that of the Hopf fibration, and that one can define a lift of the singular monopole to a nontrivial fibration which smooths out the singularity. Pauly expanded and developed this idea in [24]. Meanwhile, the

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singular monopoles turned up in a variety of contexts, linked for example to Nahm transforms of smooth configurations by Charbonneau [4], and to gravitational instantons by Cherkis-Kapustin [6]. Most spectacularly, they mediate Hecke transforms, reinterpreted as a scattering by the monopole, and are an important ingredient in Witten-Kapustin’s gauge theoretic interpretation of the geometric Langlands correspondence [15].

In this last interpretation, one is looking at singular monopoles on the product of a Riemann surface and an interval in $\mathbb{R}$; this was examined by Norbury [23]. Charbonneau and Hurtubise, in [5], then took up the case of self-Hecke transformations — monopoles on the product of a Riemann surface and a circle. They proved a Kobayashi-Hitchin type correspondence for the singular monopoles, showing that they correspond to holomorphic vector bundles on a Riemann surface equipped with a meromorphic automorphism, thought of as the self-Hecke correspondence. These Kobayashi-Hitchin correspondences are a recurrent theme in gauge theory over Kähler manifolds, linking gauge theoretic solutions to certain field equations (the Hermite-Einstein condition) to holomorphic objects, allowing us, for example, to classify them. This has been developed most notably by Donaldson [8], Uhlenbeck-Yau [27], and Simpson [26] (see also [17]).

Of course, there is no reason to restrict one’s attention to the trivial line bundle over a Riemann surface, and the subject of this paper is to see what happens over a more general circle bundle $X$, which we will take to be positive. It turns out that the relevant geometry for our circle bundle is Sasakian geometry. A Sasakian structure exists on an arbitrary line bundle of positive degree on a compact Riemann surface. The relevant structure on the four-fold $X \times S^1$, instead of a Kähler structure, will be a Gauduchon metric. We mention that a simple example is the round three-sphere, fibered over the two-sphere; the four-fold is the Hopf surface. The Kobayashi-Hitchin paradigm extends to this situation, thanks to work of Buchdahl [3]. Unfortunately, his results only apply to the non-singular case. In the case studied by Charbonneau and Hurtubise [5], the four-fold is Kähler, and the result of Simpson, which allows singularities, enables one to conclude that the Kobayashi-Hitchin type correspondence is bijective. The corresponding generalization with singularities of Buchdahl’s theorem remains unproven, though the full generalization of Simpson’s results to the non-singular case with a Gauduchon metric in any dimension has recently been given by Jacob [13].

Nevertheless, we can show that there are quite interesting holomorphic objects, of a fairly novel type, attached to the gauge fields, and we can show that the correspondence is injective. These objects can either be thought of as living on the three-fold $X$ (where one must give a suitable definition of holomorphic objects) or on the base Riemann surface $\Sigma$. In the latter context, they give objects which are rather reminiscent of Murray’s bundle gerbes [20].

Section 2 of this paper is devoted to recalling the necessary Sasakian and Gauduchon geometry on our circle bundle $X$. Section 3 considers monopoles on this three-fold, and defines the holomorphic objects on the three-fold which correspond to it. Section 4 is quite
brief and discusses the Kobayashi-Hitchin correspondence. Section 5 discusses the links between the holomorphic objects we have defined, and their reductions to the Riemann surface. Section 6 discusses the more general case of a circle bundle over an orbifold Riemann surface.

2. Sasakian geometry

2.1. Quasiregular Sasakian manifolds. Let $X$ be a compact quasiregular Sasakian three-fold, with metric $g$. These are manifolds with a contact structure and a metric, compatible in the sense that there is a unit (Reeb) vector field $\xi$ orthogonal to the contact planes, which acts on the manifold as a Killing field. (A useful reference is the book of Boyer and Galicki [2].) The orbits of $\xi$ are compact; under this hypothesis the manifold has a circle action by isometries, and the quotient

$$
\pi : X \longrightarrow \Sigma
$$

by the flow is a compact orbifold Riemann surface equipped with a Kähler form $\omega$. The Kähler structure $\omega$ is uniquely determined by the condition that $\pi$ is a Riemannian submersion with respect to $g$ and $\omega$. The quasiregular Sasakian three-fold $(X, g, \xi)$ is called regular if the circle action on $X$ is free. So a regular Sasakian three-fold is a principal $S^1$-bundle over a compact Riemann surface $\Sigma$ equipped with a Kähler form $\omega$. From now on, until the end of Section 5 we specialize to the regular Sasakian manifolds.

Let $\alpha$ be the normalized contact form on $X$, so

$$
\alpha(\xi) = 1, \quad \alpha|_{\xi^\bot} = 0,
$$

where $\xi$ is the above Killing field. If, in addition, we take, locally, one-forms $dz, d\overline{z}$ on $\Sigma$, and pull them back to $X$ using $\pi$ in (2.1), one has a local basis of complex 1-forms

$$
\alpha, dz, d\overline{z}
$$
on $X$. Let

$$
\xi, v_z, v_{\overline{z}}
$$

be the dual basis of vector fields. While $\xi$ is the Reeb vector field, both $v_z, v_{\overline{z}}$ are orthogonal to $\xi$, as well as being mutually orthogonal. The contact property tells us that the 2-form $d\alpha$ is non-degenerate. In addition, it is a lift from the Riemann surface. The form on $X$ which pulls back to $d\alpha$ will be denoted by $\omega$. One has a basis

$$
d\alpha = \pi^*\omega = \sqrt{-1} \mu(z, \overline{z}) dz \wedge d\overline{z}, \quad \alpha \wedge dz, \quad \alpha \wedge d\overline{z}
$$

for the complex two-forms $X$, where $\omega$ is the above Kähler form on $\Sigma$. The Lie brackets of the vector fields are

$$
[v_z, v_{\overline{z}}] = -\sqrt{-1} \mu(z, \overline{z}) \xi, \quad [\xi, v_z] = [\xi, v_{\overline{z}}] = 0.
$$
These follow using the equation $\alpha([v_z, v_{\overline{z}}]) = -d\alpha(v_z, v_{\overline{z}})$ and the fact that the vector field $\xi$ is Killing. In these bases, one can give the metric by
\[ g = \pi^* h + \alpha \otimes \alpha \]
with $h$ being the Hermitian structure on $\Sigma$ associated to $\omega$. Finally, one has a volume form $d\alpha \wedge \alpha$ on the three-fold $X$.

2.2. Sasakian geometry and Kähler geometry. One defining property of a Sasakian three-fold is that on the cone $M = \mathbb{R}^+ \times X$ over $X$, the metric $dr^2 + r^2 g$ is Kähler, where $g$ is the Riemannian metric on $X$. The Kähler form is
\[ \Omega = r^2 d\alpha - 2r\alpha \wedge dr = r^2 \pi^* \omega - 2r\alpha \wedge dr. \quad (2.2) \]

On the cone $M$, there is a basis of vector fields $\xi_r, \frac{\partial}{\partial r}, v_z r, v_{\overline{z}} r$. They have constant norm in $r$, and the first two are mutually orthogonal while being orthogonal to the others.

2.2.1. Complex structures. We have the complex structure on $X$ inherited from the Riemann surface $\Sigma$, to which one adds $J(\frac{\partial}{\partial r}) = \xi_r$, and so
\[ J(\xi_r + \sqrt{-1}\frac{\partial}{\partial r}) = \sqrt{-1}(\xi_r + \sqrt{-1}\frac{\partial}{\partial r}), \quad J(v_z) = \sqrt{-1}v_z, \]
spanning the $(1,0)$ part of the complexified tangent space, dually,
\[ J(r\alpha - \sqrt{-1}dr) = \sqrt{-1}(r\alpha - \sqrt{-1}dr), \quad J(rd\overline{z}) = \sqrt{-1}(rdz), \]
spanning the $(1,0)$ forms, and
\[ r\alpha + \sqrt{-1}dr, \quad rd\overline{z} \]
for the $(0,1)$ forms. The real subspace of the $(1,1)$ forms is spanned by
\[ r\alpha \wedge dr, \quad r^2 d\alpha = \sqrt{-1}mr^2dz \wedge d\overline{z}, \quad \sigma_3 = Re((r\alpha - \sqrt{-1}dr) \wedge dz), \]
\[ \sigma_4 = Re((r\alpha + \sqrt{-1}dr) \wedge dz). \]

We have the volume form on $M$
\[ \Omega \wedge \Omega = -4r^2 d\alpha \wedge \alpha \wedge dr, \]
where $\Omega$ is constructed in (2.2).

2.3. Sasakian geometry and Gauduchon geometry. A pointwise positive $(1,1)$–form $\zeta$ on a complex surface is called Gauduchon if $\partial \overline{\partial} \zeta = 0$.

Now let us consider instead of $\Omega$, the form on $M$
\[ \tilde{\Omega} = \frac{1}{r^2} \Omega = d\alpha - 2\alpha \wedge \frac{dr}{r} = d\alpha - 2\alpha \wedge dt, \quad (2.3) \]
setting $t = \log(r)$. 
Lemma 2.1. The following holds:

\[
\partial \bar{\partial} \left( \frac{1}{r^2} \right) = -\sqrt{-1} \left( d\alpha + 2\alpha \wedge dt \right).
\]

As a consequence,

\[
\partial \bar{\partial} \left( \frac{1}{r^2} \Omega \right) = \partial \bar{\partial} \left( \frac{1}{r^2} \right) \wedge \Omega = 0,
\]

so that the form \( \tilde{\Omega} \) in (2.3) is Gauduchon.

The form \( \tilde{\Omega} \) in (2.3) is invariant under the flow of the Reeb vector field, and in addition is also invariant in the additive time \((t)-\) direction. We note that Lemma 2.1 shows that there is a time invariant Gauduchon metric \( \tilde{\Omega} \) on the manifold

\[ N = X \times S^1. \quad (2.4) \]

It is this compact Gauduchon surface \((N, \tilde{\Omega})\) that will be used.

Before giving a few geometric properties, we first re-scale the bases given above on \( X \). We had one-forms \( dz, d\bar{z} \) on \( \Sigma \), lifted to \( X \), giving a local basis of forms \( \alpha, dz, d\bar{z} \), and dually, vectors \( \xi, v_z, v_{\bar{z}} \). On \( N \) (defined in (2.4)), we use a basis \( \xi, \frac{\partial}{\partial t}, v_z, v_{\bar{z}} \); they have constant norm in \( t \) (with respect to \( \tilde{\Omega} \)), with the first two being orthogonal and normal to the latter two which are isotropic.

We have the complex structure \( J(\frac{\partial}{\partial t}) = \xi \), and so

\[ \xi + \sqrt{-1} \frac{\partial}{\partial t}, \quad v_z \]

span the \((1,0)\) part of the complexified tangent space. Dually,

\[ \alpha - \sqrt{-1} dt, \quad dz \]

span the \((1,0)\) forms, while

\[ \alpha + \sqrt{-1} dt, \quad d\bar{z} \]

span the \((0,1)\) forms. We have a real basis of \((1,1)\) forms

\[ d\alpha, \alpha \wedge dt, v_3 = \frac{v_3}{r^2}, v_4 = \frac{v_4}{r^2}. \]

Let \( \tilde{L} \) denote the exterior product operation of forms by \( \tilde{\Omega} \), and let \( \tilde{\Lambda} \) denote the adjoint of \( \tilde{L} \). If \( \eta \) is a \((p,q)\) form, then \([\tilde{L}, \tilde{\Lambda}](\eta) = (p + q - 2)\eta\). Thus, for a 2-form \( \eta \) on \( M \), one should have the component \((\tilde{L})^2 \tilde{\Lambda} (\eta) = \tilde{L} \tilde{\Lambda} \tilde{L} (\eta) = 2\tilde{L} (\eta)\), and so

\[ \tilde{\Lambda} (\eta) \tilde{\Omega} \wedge \tilde{\Omega} = 2\eta \wedge \tilde{\Omega}. \]

Let us compute the Laplacian over \( X \). The coframe \( \alpha, dz, d\bar{z} \) satisfies the condition that \( \alpha \) is orthogonal to the other two, and \( dz, d\bar{z} \) are isotropic, with \( g(\alpha, \alpha) = \frac{1}{2} \) and \( g(dz, d\bar{z}) = \mu^{-1} \). The volume form is \( \sqrt{-1} \mu \alpha \wedge dz \wedge d\bar{z} = d\alpha \wedge \alpha \). From the relation \( g(a,b) d\text{vol} = a \wedge * b \), one has

\[ *\alpha = \frac{1}{2} d\alpha, \quad *dz = \sqrt{-1} \alpha \wedge dz, \quad *d\bar{z} = -\sqrt{-1} \alpha \wedge d\bar{z}. \]
Hence, for a function $f$ on $X$
\[ *df = *(\xi(f) \alpha + v_\xi f) dz + v_\xi(f) d\bar{z} = \xi(f) \frac{d\alpha}{2} + v_\xi(f) \sqrt{-1} \alpha \wedge dz - v_\xi(f) \sqrt{-1} \alpha \wedge d\bar{z}, \]
\[ d*df = \xi^2(f) \frac{\alpha \wedge d\alpha}{2} + v_\xi(v_\xi(f)) \sqrt{-1} \alpha \wedge dz \wedge d\bar{z} + v_\xi(v_\xi(f)) \sqrt{-1} \alpha \wedge dz \wedge d\bar{z}, \]
\[ \Delta(f) = *d*df = \frac{1}{2} \xi^2(f) + \mu^{-1}(v_\xi(v_\xi(f)) + v_\xi(v_\xi(f))). \]

Similarly, given a vector bundle $E$ on $X$ equipped with a connection $\nabla$, one can extend $\nabla$ to an operator
\[ \nabla : \Gamma(E \otimes \wedge^k(X)) \rightarrow \Gamma(E \otimes \wedge^{k+1}(X)), \]
and one has on sections of $E$:
\[ \Delta(s) = *\nabla * \nabla(f) = \frac{1}{2} (\nabla_\xi)^2(s) + \mu^{-1}(\nabla_{v_\xi}(\nabla_{v_\xi}(s)) + \nabla_{v_\xi}(\nabla_{v_\xi}(s))), \]
with, on $L^2$ norms,
\[ \langle s, \Delta(s) \rangle = -\langle \nabla(s), \nabla(s) \rangle. \]

3. Bundles, Hermite-Einstein monopoles and holomorphic structures

3.1. The Hermite-Einstein condition. Now assume that we have a Hermitian vector bundle $E$ over $N$, equipped with a Hermitian Chern connection, which we write as:
\[ \nabla_{v_\xi} dz + \nabla_{v_\xi} d\bar{z} + \nabla_\xi \alpha + \nabla_{\frac{\partial}{\partial t}} dt \]
\[ = (v_\xi + A_{v_\xi}) dz + (v_\xi + A_{v_\xi}) d\bar{z} + (\xi + A_\xi) \alpha + (\frac{\partial}{\partial t} + \phi) dt. \]

Recall that the Lie brackets of our vector fields on $X$ are of the form
\[ [v_\xi, v_\eta] = -\sqrt{-1} \mu(z, \bar{z}) \xi, \ [\xi, v_\xi] = [\xi, v_\eta] = 0, \]
while on $N$, we have $[\frac{\partial}{\partial t}, \xi] = 0$. The curvature tensor is:
\[ F = (\sqrt{-1} \mu(z, \bar{z}))^{-1} [\nabla_{v_\xi}, \nabla_{v_\xi}] + \nabla_\xi) d\alpha + ([\nabla_\xi, \nabla_{v_\xi}] \alpha \wedge dt
\[ + [\nabla_\xi, \nabla_{v_\xi}] \alpha \wedge dz + [\nabla_\xi, \nabla_{v_\xi}] \alpha \wedge d\bar{z}
\[ + ([\nabla_{\frac{\partial}{\partial t}}, \nabla_{v_\xi}] dt \wedge dz + ([\nabla_{\frac{\partial}{\partial t}}, \nabla_{v_\xi}] dt \wedge d\bar{z}
\[ = F^{1,1} + F^{2,0} + F^{0,2}
\[ \overset{\text{def}}{=} F_\Sigma d\alpha + F_\alpha \alpha \wedge dt + F_3 \tilde{\gamma}_3 + F_4 \tilde{\gamma}_4 + F^{2,0} + F^{0,2}. \]

When the connection is invariant in the $t$-direction, it can be put in a gauge for which the connection matrices are invariant in the $t$-direction. If $\phi$ is the $dt$-component of the connection, the commutators $[\nabla_\eta, \nabla_{\frac{\partial}{\partial t}}]$ become $\nabla_\eta (\phi)$, in particular, $F_\alpha$ becomes $\nabla_\xi \phi$.

We fix once and for all a collection of points $P = \{p_1, \ldots, p_\ell\} \subset X$. Define $q_i := \pi(p_i) \in \Sigma$. We also fix sequences $k_i = (k_{i,1}, \ldots, k_{i,n})$ of integers associated with $\{p_i\}_{i=1}^\ell$, and order the indices so that $k_{i,1} \geq \ldots \geq k_{i,n}$. 
Definition 3.1. A $U(n)$-Hermite-Einstein monopole with constant $C$, Dirac singularities of type $k_i$ at $p_i$ will be a rank $n$ Hermitian vector bundle $E$ on $X$, equipped with a Hermitian connection $\nabla$ and a skew Hermitian endomorphism ("Higgs field") $\phi$, defined away from the points $p_i$, such that

- When lifted to $N$ (with $\phi$ becoming the connection component along the extra circle direction), the result is not only compatible with the metric but also the complex structure, so that $F^{0,2} = F^{2,0} = 0$. The $F^{0,2} = 0$ condition is
  $$[\nabla_{v_z}, \nabla_{\xi} - \sqrt{-1}\phi] = 0.$$  
  Taking complex conjugates, the $F^{2,0} = 0$ condition is
  $$[\nabla_{v_z}, \nabla_{\xi} + \sqrt{-1}\phi] = 0.$$

- The lifted connection satisfy the Hermite-Einstein condition
  $$\tilde{\Lambda}(F) = -\sqrt{-1} C \cdot 1.$$

More explicitly:

$$F_{\Sigma} - F_{\alpha} = \frac{2}{2}((\sqrt{-1} \mu(z, \overline{z}))^{-1}[\nabla_{v_z}, \nabla_{v_{\overline{z}}} + \nabla_{\xi}] - (\nabla_{\xi}\phi) = -\sqrt{-1}(C \cdot 1). \quad (3.2)$$

- The singularities at $p_i$ are of Dirac type, as defined below.

Definition 3.3. Let $Y$ be a three-manifold, equipped with a metric, and let $p$ be a point of $Y$. Let $R$ denote the locally defined function on $Y$ given by the geodesic distance to $p$. Let $(t, x, y)$ be coordinates centered at $p$ with respect to which the metric is of the form $(I + O(R))$ as $R \to 0$. Let $\psi, \theta$ be, as above, angular coordinates on the sphere $R = c$, so that $R, \psi, \theta$ provide standard spherical coordinates on a neighborhood $B^3$ of $p$ defined by the inequality $R < c$. We say that a solution to the Hermite-Einstein monopole equations $(E, \nabla, \phi)$ on $Y \setminus \{p\}$ has a singularity of Dirac type, with weight $k = (k_1, \cdots, k_n)$ at $p$ if

- there is a unitary isomorphism $I$ of the restriction of the bundle $E$ to $B^3 \setminus \{p\}$ with a direct sum of line bundles $L_{k_1} \oplus \cdots \oplus L_{k_n}$, where $L_k$ is the pullback from $S^2$ of the standard line bundle of degree $k$, and
- under the isomorphism $I$, in the trivializations of $E$ over the two open subsets $\theta \neq 0$ and $\theta \neq \pi$ of $B^3$ induced by standard trivializations of the line bundles $L_k$, so that the $E$-trivializations have transition function $\text{diag}(e^{\sqrt{-1}k_1 \psi}, \cdots, e^{\sqrt{-1}k_n \psi})$, one has, in both trivializations,
  $$\phi = \frac{\sqrt{-1}}{2R} \text{diag}(k_1, \cdots, k_n) + O(1), \quad \nabla(R\phi) = O(1).$$

3.2. Holomorphic structures. The aim is to highlight a Kobayashi-Hitchin correspondence for our monopoles: they should yield some holomorphic objects which classify them. Obviously, as $X$ is three-dimensional, this holomorphic data must either be linked to the
complex curve $\Sigma$, or to the complex surface $N$. In the end, we will do both; we begin by saying what our holomorphic objects on $N$ become once one restricts them to $X$.

Let $E$ be a complex $C^\infty$ vector bundle on $X$. Let $T \subset TX$ be the orthogonal complement to the vector field $\xi$. We note that $T$ being isomorphic under projection $d\pi$ to $T\Sigma$, has a complexification which splits as $\tilde{T}^{1,0}\Sigma \oplus \tilde{T}^{0,1}\Sigma$. So $\tilde{T}^{1,0}\Sigma$ and $\tilde{T}^{0,1}\Sigma$ are identified with $\pi^*T^{1,0}\Sigma$ and $\pi^*T^{0,1}\Sigma$ respectively.

**Definition 3.4.** A holomorphic structure on $E$ over $X$ (or, locally, on an open set of $X$) will be given by specifying first order operators

$$\nabla^{0,1}_\Sigma : \Gamma(E) \rightarrow \Gamma(E \otimes (\tilde{T}^{0,1}\Sigma)^*), \quad \nabla^c_\xi : \Gamma(E) \rightarrow \Gamma(E),$$

which are locally of the form

$$s \mapsto (\omega(s) + A^{0,1}_\Sigma(s))d\bar{z}, \quad s \mapsto \xi(s) + \phi^c(s),$$

and which commute.

A holomorphic structure on $E$ over $X$ is a reduction of a holomorphic structure for a bundle on $N = S^1 \times X$, corresponding to an integrable $\mathbf{\bar{\partial}}$ operator over $N$, invariant in the circle direction on $N = S^1 \times X$.

We note that given any open subset $U \subset \Sigma$, and any section $\psi : U \rightarrow X$ of $\pi$, the two operators $\nabla^{0,1}_\Sigma$ and $\nabla^c_\xi$ can be combined to give a $\mathbf{\bar{\partial}}$ operator for $E$ on $\psi(U)$. We can think of the result as a holomorphic bundle $E_\psi$ over $U$. Given two such sections $\psi, \tau$ on $U$, we can choose paths along the circle orbits from $\psi(U)$ to $\tau(U)$, and integrate $\nabla^c_\xi$ along these paths, from $\psi(U)$ to $\tau(U)$ to obtain a map

$$\rho_{\psi\tau} : E_{\psi(U)} \rightarrow E_{\tau(U)}. \quad (3.5)$$

If these paths are chosen in a continuous fashion, this $\rho_{\psi\tau}$ will be a holomorphic isomorphism; again it can be thought of as a vector bundle isomorphism $\rho_{\phi,\tau} : E_\psi \rightarrow E_\tau$ over $U$. We note that there are choices involved in the definition of $\rho_{\phi,\tau}$, as to the direction along the circle orbits and more generally the winding number. If $\psi(U)$ and $\tau(U)$ do not intersect, we choose to go from $\psi(U)$ to $\tau(U)$, in the positive direction of the circle action, less than one full circle. If $\tau = \psi$, one can also choose one full positive turn around the circle, giving a monodromy $G_\psi : E_\psi \rightarrow E_\psi$ over $U$.

**Definition 3.6.** A meromorphic structure on $E$ with poles at $P = \{p_1, \cdots, p_\ell\}$ is first a holomorphic structure on $E$ over the complement $X \setminus P$. One asks in addition that the structure be meromorphic at each $p_i$ in the following sense. Let $U$ be an open subset of $\Sigma$ containing $q_i = \pi(p_i)$. For any pair of sections of $\pi$

$$\psi, \tau : U \rightarrow X$$

with disjoint images, one constructs $\rho_{\psi\tau}$ in (3.5). The result is a holomorphic isomorphism away from $q_i$, and also at $q_i$ if the paths of integration of $\nabla^c_\xi$ on the fibers above $q_i$ do not
contain a \( p_j \). We ask that the map, in more generality, be meromorphic at \( q_i \) even if the paths of integration contain a \( p_j \).

Let us choose sections \( \psi, \tau \) of the projection \( \pi : X \to \Sigma \) with disjoint image on an open set \( U \) containing \( q_i \) such that the path along the circle orbit over \( q_i \) from \( \psi(q_i) \) to \( \tau(q_i) \) passes through \( p_i \) once in the positive direction, and not any other \( p_j \). Let us also choose a coordinate \( z \) on \( \Sigma \) with \( z = 0 \) corresponding to \( q_i \). We say that the meromorphic structure has a pole of type \( \vec{k}_i = (k_{i,1}, \ldots, k_{i,n}) \) at \( p_i \) if the map \( \rho_{\phi,\tau} : E_\psi \to E_\phi \) is of the form

\[
\rho_{\phi,\tau} = F(z)\text{diag}(z^{k_{i,1}}, z^{k_{i,2}}, \ldots, z^{k_{i,n}})G(z),
\]

with \( F, G \) holomorphic and invertible. We note that the order \( k_{i,1} \) of the “pole” can be either positive or negative.

For a monopole, the operators for the meromorphic structure are simply \( \nabla_{\psi^*}, \nabla_{\tau^*} = \nabla_\xi - \sqrt{-1} \phi \). We have, as in [5]:

**Proposition 3.1.** A U(\( n \))-Hermite–Einstein monopole with constant \( C \), Dirac singularities of type \( \vec{k}_i \) at \( p_i \) determines a meromorphic structure on \( E \) with poles at \( P \), of type \( \vec{k}_i \) at \( p_i \).

The integrability of the holomorphic structure away from the singularity follows from the equation \( F^{0,2} = 0 \) satisfied by an Hermite-Einstein monopole. The meromorphic behavior near the singularities follows from the fact that the singularities are of Dirac type, and is proven in [4, Proposition 2.5]. A local version of this structure, on the three-sphere, was considered by Pauly in [25].

### 3.3. A degree

Surfaces equipped with a Gauduchon metric give a well defined numerical degree for a holomorphic bundle, by integrating against the trace of the curvature of a Chern connection. The Gauduchon condition ensures that the integral is independent of the Hermitian structure on the bundle. It should be mentioned that unlike the Kähler case, the degree can move continuously in a family of vector bundles. In particular, the degree is no longer a topological invariant.

In our case, as the data is invariant along the time \( t \) direction, we obtain an expression for the degree of an \( t \)-invariant bundle \( E \) with connection \( \nabla \) and curvature \( F \) by integrating along \( t = 0 \), i.e., on the manifold \( X \):

\[
\deg(E) = \sqrt{-1} \text{Vol}(X)^{-1} \int_X i(\frac{\partial}{\partial t})(\text{tr}(F) \wedge \tilde{\Omega})
\]

\[
= \frac{\sqrt{-1}}{2} \text{Vol}(X)^{-1} \int_X \tilde{\Lambda}(\text{tr}(F))i(\frac{\partial}{\partial t})(\tilde{\Omega} \wedge \tilde{\Omega})
\]

\[
= \frac{\sqrt{-1}}{2} \text{Vol}(X)^{-1} \int_X \tilde{\Lambda}(\text{tr}(F))(4d\alpha \wedge \alpha),
\]
where \( i(\frac{\partial}{\partial t}) \) denotes the constriction of forms using the vector field \( \frac{\partial}{\partial t} \). Pursuing further, if one has the decomposition

\[
tr(F)^{1,1} = tr(F)_{\Sigma}(d\alpha) + tr(F)_{\alpha}(\alpha \wedge dt) + tr(F)_{3}\tilde{\sigma}_3 + tr(F)_{4}\tilde{\sigma}_4,
\]

then in view of the equality \( F_{\alpha} = \nabla_{\xi}(\phi) \), the integral becomes

\[
\deg(E) = \sqrt{-1} \text{Vol}(X)^{-1} \int_X (2tr(F)_{\Sigma} - tr(\nabla_{\xi}\phi)(d\alpha \wedge \alpha)). \tag{3.7}
\]

We remark that \( tr(\nabla_{\xi}\phi) = \xi(tr(\phi)) \); using the fact that \( \int_S \xi(tr(\phi))\alpha = 0 \), the integral in (3.7) is then

\[
\deg(E) = 2\sqrt{-1} \text{Vol}(X)^{-1} \int_X (tr(F)_{\Sigma}d\alpha) \wedge \alpha \tag{3.8}
\]

\[
= 2\sqrt{-1} \text{Vol}(X)^{-1} \int_X tr(F) \wedge \alpha.
\]

If the Hermite-Einstein equation is satisfied, the degree is \( 2nC \), where \( n \) is the rank and \( C \) is the constant in (3.2).

**Definition 3.9.** A meromorphic section of a meromorphic structure on \( E \rightarrow X \) is a \( C^\infty \) section of \( E \) over \( X \setminus P \) lying in the kernel of the operators \( \nabla^0,1_{\Sigma} \) and \( \nabla^c_{\xi} \).

**Proposition 3.2.** Consider \( C \) in (3.2). If \( C < 0 \), an Hermite–Einstein monopole has no non-zero meromorphic sections. If \( C = 0 \), the only possibility for a section \( s \) is as a covariant constant section, lying in the kernel of \( \phi \). In particular, \( s \) then defines a rank one Hermite–Einstein monopole summand, so that the monopole splits as a direct sum of a rank one monopole and a rank \( n - 1 \) monopole.

**Proof.** One has the identity

\[
\mu^{-1}(\nabla_{v_z} \nabla_{v_T}) - \frac{1}{4}(\nabla_{\xi} + \sqrt{-1}\phi)(\nabla_{\xi} - \sqrt{-1}\phi)
\]

\[
= \frac{\mu^{-1}}{2}(\nabla_{v_z} \nabla_{v_T} + \nabla_{v_T} \nabla_{v_z}) + \frac{\sqrt{-1}}{2}F_{\Sigma} - \sqrt{-1} \nabla_{\xi} + \frac{1}{2}(\nabla_{\xi}^2 + \phi^2 - \sqrt{-1}F_{\alpha})
\]

\[
= \frac{1}{2} \Delta + \frac{1}{4} \phi^2 + \frac{\sqrt{-1}}{4}(2F_{\Sigma} - F_{\alpha}) - \sqrt{-1} \frac{\nabla_{\xi}}{2}.
\]
In particular, applying \( \frac{1}{2} \Delta + \frac{1}{4} \phi^2 + \sqrt{-1} \frac{1}{4} (2 F_\Sigma - F_\alpha) - \sqrt{-1} \frac{1}{2} \nabla_\xi \) to a holomorphic section gives zero. Now start with a holomorphic section \( s \). We have

\[
\frac{C}{2} |s|^2_{L^2} = \int_X \langle s, \frac{\sqrt{-1}}{4} (2 F_\Sigma - F_\alpha) s \rangle d\text{vol} \\
\geq \int_X \langle s, \frac{1}{2} \Delta + \frac{1}{4} \phi^2 + \sqrt{-1} \frac{1}{4} (2 F_\Sigma - F_\alpha) \rangle d\text{vol} \\
= \int_X \langle s, \frac{1}{2} \Delta + \frac{1}{4} \phi^2 + \sqrt{-1} \frac{1}{4} (2 F_\Sigma - F_\alpha) - \sqrt{-1} \frac{1}{2} \nabla_\xi \rangle d\text{vol} \\
= 0.
\]

The third step involves an integration by parts. One checks that this causes no difficulties at the singularities. For the fourth, one has the fact that the integrals along the circles in \( X \) of \( \xi \langle s, s \rangle = 2 \langle s, \nabla_\xi (s) \rangle \) is zero. Thus, unless \( C \) is positive or zero, one finds \( s = 0 \). If \( C \) is zero, then we have \( \nabla (s) = \phi (s) = 0 \). This then tells us that the orthogonal complement of \( s \) is also an invariant summand under the connection, and that it also is invariant under \( \phi \). \( \square \)

Proposition 3.2 tells us in effect that our notion of degree gives an appropriate definition of stability. Indeed, if we have a vector bundle \( E \) on \( N \) with a \( \bar{\partial} \) operator \( (\nabla_v, \nabla_\xi - \sqrt{-1} \phi) \) which is integrable, one can extend it as the Chern connection, by specifying a Hermitian metric. We saw above that one then has a well defined degree, independent of the choice. One can define meromorphic sub-bundles as bundles invariant under \( (\nabla_v, \nabla_\xi - \sqrt{-1} \phi) \). Define the degree of meromorphic sub-bundles in the same way. Define the slope \( \mu (F) \) of any nonzero sub-bundle \( F \) in the usual way as the quotient of the degree by the rank.

**Definition 3.10.** We say that the bundle \( E \) on \( N \) is stable (respectively, semistable) if for all holomorphic sub-bundles \( 0 \neq F \subset \subset E \) invariant under translation by \( t \),

\[
\mu (F) < \mu (E) \quad (\text{respectively, } \mu (F) \leq \mu (E)).
\]

A semistable vector bundle is called polystable if it is a direct sum of stable vector bundles.

**Theorem 3.3.** A Hermite-Einstein monopole on \( X \) defines a polystable meromorphic structure. If the Hermite-Einstein is irreducible then the meromorphic structure is stable.

**Proof.** To see this, we make a few remarks.

- A meromorphic subbundle \( F \) of \( E \) of rank \( k \) defines a meromorphic section \( s_F \) of \( \text{Hom} (\wedge^k F, \wedge^k E) \), invariant under translation by \( t \), and so a section of \( L \otimes \wedge^k E \), with \( L \) being the line bundle \( \wedge^k F^* \).
• If $E, F$ are of degrees $k, k'$ respectively, and of ranks $n, n'$ respectively, the degree of $L \otimes \wedge^k E$ is $-k'n + kn'$. It is positive or negative depending on whether the difference $\mu(E) - \mu(F)$ of slopes is positive or negative.

• A Hermite-Einstein monopole structure on $E$ induces a natural Hermite-Einstein monopole structure on $L \otimes \wedge^k E$.

• A covariant constant section $s_F$ of $L \otimes \wedge^k E$, coming from a subbundle $F$ of $E$, induces a sub-monopole of $E$.

The theorem then follows from the preceding proposition. 

4. From meromorphic structures to monopoles

Thus, a Hermite-Einstein monopole on $X$ defines a semistable meromorphic structure. By the general Kobayashi-Hitchin correspondence, this should yield a bijective map. The main difficulty is in showing that the map is surjective: given a semistable meromorphic structure, one would like to find a hermitian structure on the bundle such that the result satisfies the Hermite-Einstein condition of Definition 3.1. This amounts to solving a heat equation on the metric. We remark:

(1) The case when the bundle $\pi : X \longrightarrow \Sigma$ is trivial, meaning $\pi$ is the projection of $\Sigma \times S^1$ to $S$, is covered in [5]. In this case the corresponding manifold $N$ is Kähler, and one can appeal to the basic theorem of Simpson ([26]), and show the existence of a solution to the equation away from $P$, corresponding to our meromorphic structure. The singularities fall into the category covered by Simpson’s theorem. One can then appeal to an idea developed in the work of Pauly ([24]) to show that the result has the right Dirac type singularities at $P$.

(2) In the case which concerns us, one would have the required theorem if there were no singularities. Indeed, on a general closed Gauduchon surface, the Kobayashi-Hitchin correspondence has been established by Buchdahl ([3]). More generally, Jacob ([13]) has proven the more general theorem of Simpson, but again only in the case where there are no singularities.

It thus seems likely that the theorem extends to the case that concerns us here. The main technical issue seems to be that in the Gauduchon case one does not have the Donaldson functional that controls the heat flow near the singularities. We would like to thank Adam Jacob for explaining this to us.

In any case, one still has the injectivity, as in ([5]): If one has two Hermite-Einstein monopoles $E, E'$, such that the corresponding meromorphic structures $\mathcal{E}, \mathcal{E}'$ are isomorphic (i.e., through a bundle map on $X \setminus P$ which intertwines the holomorphic structures, and preserves the singularity structure – they therefore have the same degree), one has a holomorphic section $s$ of the bundle $\mathcal{E}^* \otimes \mathcal{E}'$. On the other hand, one has an Hermite-Einstein monopole structure on $E^* \otimes E'$, with constant 0, and so the section $s$ must be covariant constant and commute with $\phi$, and so must define a monopole isomorphism.
5. Holomorphic data on the curve $\Sigma$

We continue with our assumption that the Sasakian manifold $X$ is regular. So the projection $\pi$ in (2.1) makes $X$ a principal $S^1$–bundle over the Riemann surface $\Sigma$.

5.1. Reducing to the curve. Suppose that we are given a meromorphic structure on a vector bundle $E$ over the Sasakian three-fold $X$. Let us cover $\Sigma$ by open sets $U_\alpha$, and choose sections

$$\psi_\alpha : U_\alpha \rightarrow X;$$

we assume that the images of these sections do not intersect, and that the images do not contain any $p_i$. We will also assume that enough $\psi_\alpha : U_\alpha \rightarrow X$ are chosen so that if $p_i, p_j$ lie on the same orbit (so that $q_i = q_j$), there is a $\psi_\alpha(q_i)$ lying on the positive path from $p_i$ to $p_j$. Let

$$Q := \{q_1, \ldots, q_\ell\} \quad \text{and} \quad \Sigma_0 := \Sigma \setminus Q.$$

We have holomorphic bundles $E_\alpha = E_{\psi_\alpha}$ over $U_\alpha$, obtained by restricting the holomorphic structure on $E$ to $\psi_\alpha(U_\alpha)$, and meromorphic maps (monodromies of $\nabla^c_\xi$ in the $\xi$ direction)

$$G_\alpha : E_\alpha \rightarrow E_\alpha,$$

which are isomorphisms on $\Sigma_0 \cap U_\alpha$, and have singularities at $Q \cap U_\alpha$. If $p_i$ is alone on its $S^1$ orbit, meaning $p_i = \pi^{-1}(q_i) \cap P$, then the singularity type at $q_i$ is $\vec{k}_i$. We also have maps

$$\rho_{\beta\alpha} : E_\alpha \rightarrow E_\beta,$$

defined over $U_\alpha \cap U_\beta$, which are obtained by integrating our partial connection $\nabla^c_\xi$ in the positive direction, from $U_\alpha$ to $U_\beta$; these are again meromorphic with polar divisor supported over $Q$, and elsewhere are isomorphisms over their domains of definition. There is a twisted cocycle condition:

$$\rho_{\alpha\beta}\rho_{\beta\alpha} = G_\alpha.$$

The twist is due to the fact that one is doing one complete turn around the circle going from $\psi_\alpha(U_\alpha \cap U_\beta)$ to $\psi_\beta(U_\alpha \cap U_\beta)$ to $\psi_\alpha(U_\alpha \cap U_\beta)$, as one is always moving in the positive direction. In the same way, one has on triple overlaps:

$$\rho_{\alpha\beta}\rho_{\beta\gamma} = \rho_{\alpha\gamma} \quad \text{or} \quad \rho_{\alpha\gamma}G_\gamma,$$

on each component of $U_\alpha \cap U_\beta \cap U_\gamma$ depending on whether the images under $\psi_\alpha, \psi_\beta, \psi_\gamma$ of the component occur cyclically as one goes along the orbits of the circle action in $X$, or not. We would like to understand the set of solutions $E_\alpha, G_\alpha, \rho_{\alpha\beta}$ to these equations, modulo the obvious transformations given by gauge transformations on the $E_i$. We refer to solutions of these equations as twisted bundle triples over $\Sigma$.

We first choose some explicit open subsets $U_\alpha$. One can trivialize the circle bundle $X \rightarrow \Sigma$ over the complement of any point. This reduces us to a local geometry near the point which is essentially that of a power $k$ of the Hopf fibration. Let us choose a
closed disk \( D_1 \) inside an open disk \( D_2 \) around a base point \( p \). Thinking of these disks in the plane, centered at the origin, let \( D_i \) be of radius \( i \), centered on the origin. Set \( U_0 = X \setminus D_1 \), and put \( U_s = \epsilon\)-neighborhood of the angular sector 
\[ \theta \in (2\pi(s-1)/(k+1), 2\pi s/(k+1)), \quad s = 1, \ldots, k+1 \]
in \( D_2 \). The open sets \( U_0, U_1, \ldots, U_{k+1} \) cover \( X \).

One can choose trivializations of the fibration \( X \rightarrow \Sigma \) on \( D_2, U_0 \) such that the trivialization over \( U_0 \) is \( \exp(\sqrt{-1}k\theta) \) times that on \( D_2 \). If one trivializes the bundle over \( U_s \) by \( \exp(2\pi\sqrt{-1}(-s+1/2)/(k+1)) \) times the trivialization on \( D_2 \), one obtains trivializations of the bundle satisfying our requirements: the trivializations on the \( U_s, s = 1, \ldots, k+1 \) are arranged anti-cyclically in the circle over overlaps, whereas the cyclic order on \( U_0 \cap U_s \cap U_{s+1} \) is \( s+1, 0, s \). Our cocycle conditions then become:

\[
\rho_{s,1}\rho_{1,s} = \rho_{s,t}\rho_{t,s} = G_s
\]

\[
\rho_{0,s}\rho_{s,0} = G_0
\]

\[
\rho_{s+2,s+1}\rho_{s+1,s} = \rho_{s+2,s}
\]

\[
\rho_{s,0}\rho_{0,s+1} = \rho_{s,s+1}.
\]

We then have:

**Proposition 5.1.** The correspondence between meromorphic structures over \( X \setminus P \) and twisted bundle triples is bijective.

5.2. **Rank one.** We now look at these equations in rank one. In this case the \( G_s \) are functions because they are endomorphisms. Also, since the cocycle equations tell us \( G_s \) are conjugate, these functions patch together to give a single meromorphic function \( G \).

We would like to find one solution to these equations, for a given \( G \), using our explicit cover. On our open set \( D_2 \), we suppose that \( G \) has neither zero nor pole, and fix a \((k+1)\)-th root \( G^{1/k+1} \) of \( G \). Let us choose a determination \( \log_s(z^{k+1}) \) of \( \log(z^{k+1}) \) on each \( U_s, s = 1, \ldots, k+1 \) with imaginary part going from 0 to \( 2\pi \). On the overlap \( U_s \cap U_{s+1} \), the two determinations differ by \( 2\pi\sqrt{-1} \). Let us set

\[
\rho_{s+1,s} = G^{1/k+1}, \quad s = 1, \ldots, k
\]

\[
T_{0,s} = G^{k \log_s(z^{k+1})}.
\]

One checks that this can be completed to a solution to the equations (5.1).

Given one solution, we can find all the others by tensoring with a line bundle on \( \Sigma \). Explicitly, if \( T_{\beta\alpha} \) are transition functions for a line bundle over \( \Sigma \), one can get from one solution of our twisted line bundle equations to another by \( \rho_{\beta\alpha} \mapsto \rho_{\beta\alpha} T_{\beta\alpha} \).

**Proposition 5.2.** For a given meromorphic function \( G \), the family of solutions in rank one to the twisted line bundle equations forms a torsor over the Picard group of the Riemann surface \( \Sigma \).
We note that the singularities $p_i$ of a monopole are constrained. Indeed, their types $k_i$ and their projections $q_i$ to $\Sigma$ determine the divisor $\sum_{i=1}^{t} k_iq_i$ of the function $G$, which is constrained by Abel’s theorem, imposing $g = \text{genus}(\Sigma)$ complex constraints on the divisor. There are also constraints on the angular coordinates $\theta_i$ of the points $p_i$ along the orbits. Indeed, we will see, in the Abelian case, that these are linked to the Hermite-Einstein constant $C$, and so, fixing $C$ gives us one real constraint.

To see this, let us consider a fixed rank one triple $\mathcal{E} = (E_\alpha, G, \rho_{\alpha, \beta})$, with singularities at $p_i$ of type $k_i$. Now choose an angular coordinate $\theta$ on $X$ near $p_1$, with

$$d\theta(\xi) = 1, \quad \theta(p_1) = 0,$$

and define a family $\mathcal{E}_t$ of triples by keeping the same data as $\mathcal{E}$, but moving the singular point $p_1$ along its circle orbit in the positive direction to $\theta = t$; call the result $p_1(t)$. Let $z : U \to D$ be a coordinate on $U \subset \Sigma$ with $q_1 \in U$ corresponding to $z = 0$, and $D$ the unit disk. We want to consider the difference in the Hermite-Einstein constants (the constant $C$ in (3.2)) between $\mathcal{E}_t$ and $\mathcal{E}$. This amounts to computing the induced Hermite-Einstein degree for the triple corresponding to $\text{Hom}(\mathcal{E}, \mathcal{E}_t)$. The triple corresponding to $\text{Hom}(\mathcal{E}, \mathcal{E}_t)$ has the property that away from the orbit through $p_1$, it is canonically identified with the trivial triple $(\mathcal{O}, \mathcal{I}, \mathcal{I})$. Near $p_1$, one must take two sections

$$\psi_- : D \to X, \quad \psi_+ : D \to X,$$

defined in coordinates by $\psi_-(z) = (z, -t/2)$, $\psi_+(z) = (z, t/2)$. The corresponding maps are $\rho_- = z^{-k}$, $\rho_+ = z^k$. Let $S$ denote the slit $\{z = 0, \theta \in [0, t]\}$ on $X$, with interior $S_0$ denoting the slit $\{z = 0, \theta \in (0, t)\}$. Under the correspondence we have with meromorphic structures on $X$, the triple for $\text{Hom}(\mathcal{E}, \mathcal{E}_t)$ defines a meromorphic bundle on $X - \{(z, \theta) = (0, 0), (0, t)\}$, trivialized away from $S$, and the holomorphic transition function to a neighborhood $V \subset \pi^{-1}(D)$ of $S_0$ is $z^{-k}$. Following the usual recipe, let $\tau$ be a bump function on $X$, equal to one outside $V$ and equal to zero on a neighborhood of $S_0$ of radius $1/2$. One has a Hermitian metric on our bundle given in the trivialization on $X \setminus S$ by $h = \tau + (1 - \tau)z\overline{z}^k$, inducing a Chern connection with curvature component $F_\Sigma(t)$ which has the expression

$$F_\Sigma(t) = \frac{\sqrt{-1}}{2} \mu(z, \overline{z}) \partial_z \overline{\partial}_z (\log(\tau + (1 - \tau)z\overline{z}^k)).$$

If one now computes the invariant

$$-2\sqrt{-1}\text{Vol}(X)C(t) = \int_X F(t) \wedge \alpha = \int_V F_\Sigma(t)d\alpha \wedge \alpha = \int_V F_\Sigma(t)d\alpha \wedge d\theta,$$

and compares $F_\tau$ with $F_t$, one gets

$$-2\sqrt{-1}\text{Vol}(X)(C(t') - C(t)) = \int_t^{t'} \int_D \partial_z \overline{\partial}_z (\log(\tau + (1 - \tau)z\overline{z}^k))d\tau \wedge d\overline{z}$$

$$= \int_t^{t'} \int_{z\overline{z} = 1/4} \partial_z (\log z\overline{z}^k)dz = k(t' - t).$$
Thus, \( C(t) - C(0) = \frac{k\sqrt{-1}}{\sqrt{\text{Vol}(X)}} t \).

We have thus seen that there are \( g = \text{genus}(\Sigma) \) parameters worth of different meromorphic structures for a fixed choice of \( p_i, \) multiplicities \( k_i, \) and Hermite-Einstein degree \( C, \) if there exists one. On the other hand, for these to exist, there are \( g \) complex constraints on \( q_i = \pi(p_i) \) along the curve \( \Sigma \) and one real constraint for the location of \( p_i \) along the circle orbits (for fixed \( C \)).

5.3. Higher rank. More generally, if one is dealing with vector bundles of higher rank \( n, \) one can take the determinant bundle and compute as above. Thus, if \( \text{tr}(\vec{k}_i) = \sum_j \vec{k}_{ij}, \) we have the following:

**Proposition 5.3.** Let

\[
  t = (t_1, \cdots, t_\ell).
\]

Let \((E_\alpha, G_\alpha, \rho_{\alpha,\beta})_t\) be obtained from \((E_\alpha, G_\alpha, \rho_{\alpha,\beta})_0\) by shifting the corresponding singularities \( p_i \) along their circle orbits by \( t_i. \) Then the Hermite-Einstein degree \( C(t) \) of \((E_\alpha, G_\alpha, \rho_{\alpha,\beta})_t\) is obtained from the Hermite-Einstein degree \( C(0) \) of \((E_\alpha, G_\alpha, \rho_{\alpha,\beta})_0\) by

\[
  C(t) = C(0) + \frac{\sqrt{-1}}{n\text{Vol}(X)} \sum_{i=1}^\ell \text{tr}(\vec{k}_i)t_i.
\]

To understand our parameter space in higher rank, of course, things are not so simple: matrices do not always commute. One can Abelianize the problem, however, by what is now a classical construction: passing to a spectral curve [10, 18].

We have noted that the endomorphisms \( G_\alpha \) are all conjugate to each other. This means that there is an invariant spectral curve \( S, \) cut out in \( \Sigma \times \mathbb{P}^1 \) by the equations

\[
  \det(G_\alpha(z) - \eta I) = 0.
\]

Moreover, over each \( U_\alpha, \) we have quotient sheaves \( L_\alpha \) supported over the spectral curve in \( U_\alpha \times \mathbb{P}^1. \) Let \( \sigma : \Sigma \times \mathbb{P}^1 \longrightarrow \Sigma \) be the projection. Consider the short exact sequence

\[
  0 \longrightarrow \sigma^*E_\alpha \otimes \mathcal{O}(-1) \overset{G_\alpha(z) - \eta I}{\longrightarrow} \sigma^*E_\alpha \longrightarrow L_\alpha \longrightarrow 0.
\]

This encodes the pair \((E_\alpha, G_\alpha), \) where \( E_\alpha = \sigma_*L_\alpha \) and \( G_\alpha = \sigma_*G(z) \eta, \) in other words, \( G_\alpha \) is multiplication by the fiber coordinate. Now let us consider overlaps: on \( U_\alpha \cap U_\beta, \) we have a diagram

\[
  \begin{array}{ccc}
  0 & \longrightarrow & \sigma^*E_\alpha \otimes \mathcal{O}(-1) \\
  & & \overset{G_\alpha(z) - \eta I}{\longrightarrow} \\
  & & \sigma^*E_\alpha \\
  & & \downarrow \rho_{\alpha}\beta \\
  0 & \longrightarrow & \sigma^*E_\beta \otimes \mathcal{O}(-1) \\
  & & \overset{G_\beta(z) - \eta I}{\longrightarrow} \\
  & & \sigma^*E_\beta \\
  & & \downarrow \rho'_{\beta}\alpha \\
  0 & \longrightarrow & \sigma^*E_\alpha \otimes \mathcal{O}(-1) \\
  & & \overset{G_\alpha(z) - \eta I}{\longrightarrow} \\
  & & \sigma^*E_\alpha \\
  & & \downarrow \rho_{\beta}\alpha \\
  0 & \longrightarrow & \sigma^*E_\beta \otimes \mathcal{O}(-1) \\
  & & \overset{G_\beta(z) - \eta I}{\longrightarrow} \\
  & & \sigma^*E_\beta \\
  & & \downarrow \rho'_{\alpha}\beta \\
  0 & \longrightarrow & \sigma^*E_\alpha \otimes \mathcal{O}(-1) \\
  & & \overset{G_\alpha(z) - \eta I}{\longrightarrow} \\
  & & \sigma^*E_\alpha \\
  & & \downarrow \rho_{\alpha}\beta \\
  0 & \longrightarrow & \sigma^*E_\beta \otimes \mathcal{O}(-1) \\
  & & \overset{G_\beta(z) - \eta I}{\longrightarrow} \\
  & & \sigma^*E_\beta \\
  & & \downarrow \rho'_{\beta}\alpha
  \end{array}
\]

On triple overlaps, one gets \( \rho'_{\gamma}\beta\rho'_{\beta}\alpha = \rho'_{\gamma}\alpha \) or \( \rho'_{\alpha}\beta \eta, \) depending on whether the images of the open subsets \( U_\alpha, U_\beta, U_\gamma \) are arranged cyclically or not in \( X. \) Now, if we suppose that the curve \( S \) is smooth, reduced, then \( L_\alpha \) will be line bundles. We thus have obtained a twisted line bundle over the spectral curve, and so we have the following:
Proposition 5.4. Fixing the spectral curve, the family of twisted vector bundles is a torsor over the Picard variety of the spectral curve.

Of course, here, if we want Hermite-Einstein monopoles, one must worry about stability. One advantage of the spectral curve approach is that if the spectral curve is irreducible and reduced, there are no subobjects, as there are no sub-spectral curves.

5.4. Gerbe-like structure. We close this section with the comment that the data in the meromorphic bundle structure on $X$ induces a structure which rather resembles Murray’s bundle gerbes. Indeed, if $X^{[2]} \rightarrow \Sigma$ is the fiber product of $X$ with itself, we have a $\mathbb{Z}$-fold cover $\tilde{X}^{[2]}$ of $X^{[2]}$, given as pairs of points $x, y$ in the fiber over $X$ plus a homotopy class of paths from $x$ to $y$ along the fiber. (The inverse image of a point in $\Sigma$ would thus be $S^1 \times \mathbb{R}$.)

For our meromorphic bundles, parallel transport by $\nabla_{\xi}(s) = 0$ along the $S^1$ preserves the eigenspaces of the holonomy, and so the kernel of the difference of the monodromy and any multiple $\eta I$ of the identity map. If $\pi : S \rightarrow \Sigma$ is the spectral curve, taking the fiber product $Y = S \times_\Sigma X \subset \mathbb{P}^1 \times X$. There is a well defined line bundle $L$, and so another line bundle $Hom_L$ along $Y^{[2]}$, again equipped with a natural section when one lifts to the $\mathbb{Z}$-cover $\tilde{Y}^{[2]}$.

6. Equivariant bundles on regular Sasakians

In this section we will reduce the study of holomorphic vector bundles on quasiregular Sasakians to that of holomorphic vector bundles on regular Sasakians.

Let $X$ be a quasiregular Sasakian threefold. The map $\pi$ in (2.1) fails to be a submersion outside finitely many points of $\Sigma$. Let $x_1, \cdots, x_m$ be the points of $\Sigma$ such that the complement

$$\Sigma_0 := \Sigma \setminus \{x_1, \cdots, x_m\}$$

satisfies the condition that the restriction

$$\pi|_{\pi^{-1}(\Sigma_0)} : \pi^{-1}(\Sigma_0) \rightarrow \Sigma_0$$

is a submersion. For any $z \in \pi^{-1}(x_i)$, $1 \leq i \leq m$, the isotropy of $z$ for the action of $S^1$ on $X$ constructed using the Reeb vector field is a nontrivial finite cyclic group. Let $\nu_i$ be the order the isotropy subgroup of $z \in \pi^{-1}(x_i)$.

We assume the following:

At least one of the following four conditions hold:
(1) genus(\(\Sigma\)) \(\geq\) 1,
(2) \(m \geq 3\),
(3) if genus(\(\Sigma\)) = 0 and \(m = 2\), then \(\nu_1 = \nu_2\), and
(4) genus(\(\Sigma\)) = 0 and \(m = 0\).

We now recall a theorem of Bundgaard–Nielsen–Fox; see [22, p. 29, Theorem 1.2.15] and [22, p. 26, Proposition 1.2.12].

**Theorem 6.1** (Bundgaard–Nielsen–Fox). *There is a finite Galois covering

\[
\delta : \tilde{\Sigma} \longrightarrow \Sigma
\]

such that

- \(\delta\) is unramified over the complement \(\Sigma_0\),
- for any \(1 \leq i \leq m\), the isotropy of any \(y \in \delta^{-1}(x_m)\) is a cyclic group of order \(\nu_i\).

Let \(\Gamma := \text{Gal}(\delta)\) be the Galois group of the covering \(\delta\).

Consider the fiber product \(\tilde{\Sigma} \times_{\Sigma} X\). Although it is singular, it has a natural desingularization \(\tilde{X}\) such that the natural projection

\[
\tilde{\delta} : \tilde{X} \longrightarrow X
\]

is a ramified Galois covering with Galois group \(\Gamma\), and the natural projection

\[
\tilde{\pi} : \tilde{X} \longrightarrow \tilde{\Sigma}
\]

defines a principal \(S^1\)–bundle. Therefore, \(\tilde{X}\) is a regular Sasakian manifold.

Holomorphic vector bundles on \(X\) are identified with \(\Gamma\)–equivariant holomorphic vector bundles on \(X\). This correspondence preserves semistability and polystability, because equivariant polystability (respectively, equivariant semistability) coincides with usual polystability (respectively, semistability). Also, all our constructions of twisted bundle triples go over; one simply then has a condition of equivariance that gets added to the mix.

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