Fault tolerant controller for a class of additive faults: a quasi-continuous high-order sliding mode approach

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Abstract. In this paper a fault tolerant control strategy that combines the backstepping procedure and the quasi-continuous high-order sliding mode controller is proposed. The fault tolerance principle is based on a hierarchical application of the backstepping methodology ensuring the finite time convergence of the desired system states, in spite of the considered fault situations. The additive effect of the faults and disturbances is canceled out by the hierarchical application of the quasi-continuous controller ensuring fault-tolerance. The effect of Lebesgue measurable noise over the precision of the proposed controller is studied. Simulation results based on a nonlinear model of the F16 jet fighter show the efficiency of the proposed techniques.

1. Introduction

Automatic control reconfiguration and Fault Tolerant Control (FTC) can help avoid a potentially hazardous, out-of-tolerance or dangerous behavior of the controlled system, for example, see [1, 2, 3] and the references therein.

Designing a FTC scheme associated with each fault is usually carried out through passive or active FTC schemes [4]. In the passive approach, a unique control algorithm based on robust control techniques is designed to achieve the given objectives in healthy as well as in faulty situations. This is a simple and computationally attractive way to achieve fault tolerance without a fault diagnosis scheme. However, increasing robustness to some faults deteriorates the performance level in the fault-free mode. In contrast to passive designs, active FTC scheme exchanges some of the simplicity for an increased performance level, since it designs control laws that are dedicated to each fault of interest. Under the assumption that the fault is recoverable [5], stability and performance are therefore guaranteed in the presence of each fault by means of a reconfiguration mechanism that is activated when a Fault Detection an Isolation (FDI) unit detects and diagnoses any relevant signal deviation or failure. The available design methodologies in active FTC area include those based on $H_\infty$ control [6], linear quadratic control [7], switching control [8] and fault hiding paradigm [9] to name a few. They have the major advantage that the performance level for the nominal (no fault) operating mode can be preserved. However, the majority of the available methods are built on the unrealistic assumption that the FDI and FTC outputs are instantaneously available to provide decisions and/or actions to other subsystems. The problem of guaranteeing stability and performance of the overall active FTC scheme taking
into account both the FDI performances (detection delay, ... ) and re-configuration mechanism, has not been considered due to its complexity. In order to address this shortcoming, an attractive technique based on the dwell-time concept has recently been introduced in [10] to show how FDI and FTC performance interact and can be managed to have a global optimal solution. 

To avoid such type of complex management [10], considerable effort has been made in the area of adaptive control methods [11]. Due to its prominent adapting ability to handle the structural and parametric uncertainties and variations, adaptive control design schemes can be applied with neither control restructuring nor FDI processing. Based on backstepping technique the results of [12], [13] deal with actuator fault compensation for linear systems. These studies have been next extended to nonlinear systems in previous works [14, 15, 16]. A new backstepping scheme for parametric strict feedback nonlinear systems has been introduced in [17]. A smooth backstepping control based on exact identification of the matched and unmatched uncertainties and disturbances has been proposed for the nonlinear systems [18]. This technique exploits a High-Order Sliding Mode (HOSM) differentiator to identify the perturbations and additional HOSM differentiators are involved to facilitate the control law computation. In all works, the boundedness of all closed-loop signals can still be ensured. However, to the best of our knowledge, there is still no rigorous analysis of measurement noise effects in the area of fault tolerant backstepping technique.

In such a setting, the main contribution of this work consists of development of a technologically viable solution in a bounded noise environment. The contribution of this work is twofold: i) Firstly, the quasi-continuous High-Order Sliding Mode (HOSM) control theory [19] is applied to FTC context. It is shown that finite-time convergence can be achieved in spite of the existence of a class of faults. ii) Secondly, the effect of bounded measurement noises is analyzed.

The paper is organized as follows. Section 2 gives the problem statement. Section 3 is devoted to the proposed FTC strategy where all design and analysis steps are presented. Finally, section 4 shows the efficiency of the proposed scheme on a nonlinear model of the F16 jet aircraft.

2. Fundamentals

The proposed fault tolerant algorithm is designed using the backstepping procedure, and combining two main discontinuous algorithms, the quasi-continuous high-order sliding mode controller and the high-order sliding-modes arbitrary order differentiator. These algorithms are described in this Section.

2.1. Quasi-continuous controller [20]

Let us consider the nonlinear system described by the equation:

\[ \dot{x} = a(t, x) + b(t, x)u \]

where \( x \in \mathbb{R}^n \), \( u \in \mathbb{R} \) and \( a(t, x), b(t, x) : \mathbb{R}_+ \times \mathbb{R}^n \rightarrow \mathbb{R} \) are smooth nonlinear functions.

The “quasi-continuous” arbitrary-order sliding mode controller was developed to provide finite time establishment of sliding surfaces with relative degree greater than one [20, 21].

Let us introduce the variable \( \sigma(t, x) : \mathbb{R}_+ \times \mathbb{R}^n \rightarrow \mathbb{R} \) that possesses relative degree \( n \) with respect to the control input \( u \), i.e. it satisfies the following equality:

\[ \sigma^{(n)} = h(t, x) + g(t, x)u \]

where the superscript \( (n) \) stands for the \( n \)-th derivative of \( \sigma \), \( h(t, x) = \sigma^{(n)}|_{u=0} \) and \( g(t, x) = \frac{\partial}{\partial u} \sigma^{(n)} \) and satisfy the following inequalities \( |h(t, x)| \leq C, \quad 0 < K_m \leq g(t, x) \leq K_M \). The quasi-continuous arbitrary-order sliding mode controller is applied to bring to zero, in finite
time, the generic scalar sliding variable \( \sigma \). The recursive representation of the quasi-continuous controller for a generic variable \( \sigma \) is given as follows:

\[
\begin{align*}
\varphi_{0,n} &= \sigma, \quad N_{0,n} = |\sigma| \\
\psi_{0,n} &= \varphi_{0,n}/N_{0,n} = \text{sign} \, \sigma, \\
\varphi_{i,n} &= \sigma^{(i)} + \beta_i N_{i-1,n}^{(n-i)/(n-i+1)} \psi_{i-1,n}, \\
N_{i,n} &= |\sigma^{(i)}| + \beta_i N_{i-1,n}^{(n-i)/(n-i+1)}, \\
\psi_{i,n} &= \varphi_{i,n}/N_{i,n}
\end{align*}
\]

where \( i = 1, \ldots, n-1 \), and \( \beta_1, \ldots, \beta_{n-1} \) are positive numbers. Notice that, the quasi-continuous \( n \)-sliding controller is obtained as

\[
\begin{align*}
\dot{\sigma} &= -\alpha \psi_{n-1,n}(\sigma, \dot{\sigma}, \ldots, \sigma^{(n-1)}).
\end{align*}
\]

It was shown in [20] that provided \( \alpha > C \), and the tuning parameters \( \beta_1, \ldots, \beta_{n-1} \) are chosen sufficiently large in the given order then the control law defined by (1)-(6) enforce the following equality \( \sigma = \dot{\sigma} = \ldots = \sigma^{(n)} = 0 \) in finite time.

2.2. Arbitrary-order exact robust differentiator [22]

The above \( n \)-th order quasi-continuous controller requires the availability of the successive derivatives, up to the order \( n-1 \), of the sliding variables \( \sigma \). In order to reconstruct such derivatives exactly and in finite time, the well known Arbitrary-Order Exact Robust Differentiator [22] can be used.

The \( n \)-th order differentiator can be expressed in the following non-recursive form

\[
\begin{align*}
\dot{z}_0 &= z_1 - \kappa_n M^{1/n} |z_0 - \sigma|^{\frac{n}{n+1}} \text{sign}(z_0 - \sigma), \\
\dot{z}_1 &= z_2 - \kappa_{n-1} M^{1/(n-1)} |z_1 - \dot{z}_0|^{\frac{n}{n+2}} \text{sign}(z_1 - \dot{z}_0), \\
&\vdots \\
\dot{z}_i &= z_{i+1} - \kappa_{n-i} M^{1/(n-i)} |z_i - \dot{z}_{i-1}|^{\frac{n}{n+i+1}} \text{sign}(z_i - \dot{z}_{i-1}), \\
&\vdots \\
\dot{z}_n &= -\kappa_0 M \text{sign}(z_n - \dot{z}_{n-1})
\end{align*}
\]

for suitable positive constant coefficients \( \kappa_i \) and \( M \) to be chosen recursively large in the given order [22]. Under the assumption that the \( n \)-th derivative of the sliding variable \( \sigma \) is bounded, the gains \( M \) must satisfy the following constraint: \( M \geq |\sigma^{(n)}| \). According to [22] \( \kappa_0 = 1.1, \kappa_1 = 1.5, \kappa_2 = 2, \kappa_3 = 3, \kappa_4 = 5, \kappa_5 = 8 \) is one of the possible selections for the gains \( \kappa_i \). Hence, the differentiator (7) can be tuned by a suitable choice of parameters \( M \). Notice that in the absence of measurement noise, the following equalities are true after a finite time transient process:

\[
|z_i - \sigma^{(i)}| = 0 \quad i = 0, \ldots, n
\]

The presence of non-idealities like measurement noise and finite frequency commutation causes a bounded error in the estimated derivatives [22] and, as a result, a bounded loss of accuracy for the controller that uses the “noisy” derivative estimates [20, 21, 22].

The separation and robustness results relevant to the combined use of the differentiator and any \( n \)-sliding homogenous controller has been previously discussed [22], in these works the separation property is proven as a consequence of the finite-time convergence of the high-order sliding-mode differentiator.
3. Problem statement

Let us consider a nonlinear system in the following lower triangular form:

\[
\begin{align*}
\dot{x}_1 &= g_1(x_1, t) + B_1(x_1, t)x_2 + f_{c_1}(t) + w_1(t) \\
\vdots & \\
\dot{x}_i &= g_i(x_1, \ldots, x_i, t) + B_i(x_1, \ldots, x_i, t)x_{i+1} + f_{c_i}(t) + w_i(t) \\
\vdots & \\
\dot{x}_n &= g_n(x_1, \ldots, x_n, t) + B_n(x_1, \ldots, x_n, t)u + f_{c_n}(t) + f_a(t) + w_n(t)
\end{align*}
\]

where \( x_i \in \mathbb{R}^p, i = 2, \ldots, n - 1; \ x^T = [x_1^T \ x_2^T \ \cdots \ x_n^T], \ x \in \mathcal{X} \subseteq \mathbb{R}^{np}, \ u \in \mathbb{R}^p \) is the control signal, \( g_i(\cdot) \) are known nonlinear smooth vector fields, \( B_i(x_1, \ldots, x_i, t) \in \mathbb{R}^{p \times p} \) are known nonsingular function matrices, \( w_i(t) \) are disturbances, \( f_{c_i}(t) \) and \( f_a(t) \) are faults. In particular, \( f_{c_1}(t), \ldots, f_{c_n}(t) \) denotes the component fault (in an additive representation) and \( f_a(t) \) can model a class of actuator fault.

The considered faults satisfy the following restrictions

\[
\left\| \frac{d^{(n-i)}}{dt^{(n-i)}} \left( f_{c_i}(x_1, \ldots, x_i, t) + w_i(t) \right) \right\|_\infty \leq \Gamma_i^+(10)
\]

\[
f_{c_i}(x_1, \ldots, x_i, t) + w_i(t) \in C^{n-i}, \quad i = 1, \ldots, n - 1 \tag{11}
\]

\[
||f_{c_i}(t) + f_a(t) + w_n(t)|| \leq \Gamma_n^+ \tag{12}
\]

where \( \Gamma_i^+ \) is a known positive scalar constant and, \( \mathcal{C}^k \) denotes the set of functions with at least \( k \) bounded derivatives.

Differential equations are understood in the Filippov sense [23] in order to provide for the possibility to use discontinuous signals in controls. Filippov solutions coincide with the usual solutions, when the right-hand sides are continuous. It is assumed also that all considered inputs allow the existence of solutions and their extension to the whole semi-axis \( t \geq 0 \).

Given the control output \( y = x_1 \), the aim of this work is to provide a design methodology able to guarantee, the exact finite-time tracking of a smooth desired reference \( y_c \), in spite of the occurrence of the faults \( f_{c_i}(t), f_a(t) \) and the disturbances \( w_i(t) \). The whole state vector \( x \) and the control signal \( u \) are assumed available.

4. Fault tolerant control design

In this section, the backstepping strategy based on the hierarchical application of quasi-continuous controllers [19] is applied to the fault tolerant control design. The design is applied to each one of the components of the control output

4.1. Control Design

The control objective is the tracking of a smooth command signal \( y_c \) using the control signal \( u \). The control signal \( u \) is designed through a series of virtual controls \( \phi_1 \), defined recursively, in such a way that at each step the virtual control signal provides the tracking of the preceding coordinate of the system.

Step 1. Define as a virtual control \( x_2 = \phi_1 \), with

\[
\phi_1 = B_1^{-1}(x_1, t) [-g_1(x_1, t) + v_1(t)]
\]

\[
v_1^{(n-1)} = \begin{bmatrix}
-\alpha_{11} \psi_{n-1,1} \sigma_{11}^{(n-1)} \\
\vdots \\
-\alpha_{1p} \psi_{n-1,1} \sigma_{1p}^{(n-1)}
\end{bmatrix}
\]

\[
\phi_1 = B_1^{-1}(x_1, t) [-g_1(x_1, t) + v_1(t)]
\]
where $\sigma_1 = y - y_c$, the superscript $(n - 1)$ stands for the $(n - 1)$th derivative of the signal $v_1$, and the nonlinear term $\psi_{n-1,n}(:,:,i)$ will be designed further using the quasi-continuous high-order sliding mode controller in order to compensate for the fault and disturbance $f_{c_1}(t) + w_1$.

Step i. Define recursively the virtual controls $x_i = \phi_i$ as

$$
\phi_i = \frac{B_i^{-1}}{B_i^{-1}} (x_1, ..., x_i, t) [-g_i(x_1, ..., x_i, t) + v_i]
$$

(15)

Step i. Define recursively the virtual controls $x_i = \phi_i$ as

$$
\phi_i = \frac{B_i^{-1}}{B_i^{-1}} (x_1, ..., x_i, t) [-g_i(x_1, ..., x_i, t) + v_i]
$$

(16)

where $\sigma_i = x_i - \phi_{i-1}$ and the nonlinear control $\psi_{n-1,n-i+1}(:,:,i)$ will be designed further in order to compensate for the fault and disturbance $f_{c_1}(t) + w_1$.

Step n. The control is designed as

$$
u_n = \frac{B_n^{-1}}{B_n^{-1}} (x_1, ..., x_n, t) [-g_n(x_1, ..., x_n, t) + v_n]
$$

(17)

$$
u_n = \frac{B_n^{-1}}{B_n^{-1}} (x_1, ..., x_n, t) [-g_n(x_1, ..., x_n, t) + v_n]
$$

(18)

where $\sigma_n = x_n - \phi_{n-1}$.

**Theorem 1** Let us consider that inequalities (10) to (12) are satisfied for system (9). The backstepping based design (13)-(18) guarantees, after a finite-time transient, the exact tracking of the command signal $y = y_c$, even in the presence of faults and disturbances.

**Proof.**

Before proceeding with the proof of convergence, it is interesting to notice that the algorithm is composed by several instances of the quasi-continuous sliding mode controller. All those instances use the variable $\sigma$ and its derivatives, these derivatives can be obtained by means of several instances of the High-Order Sliding Mode differentiator.

First, we prove the convergence of the estimations of the high-order derivatives for a generic variable $\sigma$, after that, the proof of convergence of the output $y$ to the command signal $y_c$ is given assuming that the high-order derivatives of the surfaces have been already obtained by means of the differentiator.

Let us define the estimation error as $e_0 = z_0 - \sigma$ and $e_i = z_i - \sigma(i)$. The estimation error dynamics is given by the following expression:

$$
\begin{align*}
\dot{e}_0 &= e_1 - \kappa_1 M^{1/n} |e_0|^{\frac{n-1}{n+1}} \text{sign}(e_0), \\
\dot{e}_1 &= e_2 - \kappa_2 M^{1/(n-1)} |e_1 - \dot{e}_0|^{\frac{n-2}{n+2}} \text{sign}(e_1 - \dot{e}_0), \\
&\vdots \\
\dot{e}_i &= e_{i+1} - \kappa_{n-1} M^{1/(n-i)} |e_i - \dot{e}_{i-1}|^{\frac{n-2}{n+2}} \text{sign}(e_i - \dot{e}_{i-1}), \\
&\vdots \\
\dot{e}_{n-1} &= -\kappa_0 M \text{sign}(e_{n-1} - \dot{e}_{n-2}) + \sigma^{(n)}
\end{align*}
$$

(19)

It is important to remark that, given the appearance of the unknown term $\sigma^{(n)}$ in the last equation of (19), this equation is uncertain, hence the estimation error can be written in a
Step 1. Define the tracking error as $e_i = y - y_c = x_1 - x_{i_c}$. By differentiating this term is obtained:

$$\dot{e}_1 = g_1(x_1, t) + B_1(x_1, t)x_2 + f_{c_1}(t) + w_1 - \dot{x}_{c_1}$$

Using the virtual control $x_2 = \phi_1$ and by differentiating the above given equation (n-1) times the following equation is obtained:

$$\sigma_i^{(n)} = \left[ \frac{d^{n-1}}{dt^{n-1}}(f_{c_1}(t) + w_1) \right] + \epsilon_i^{(n-1)}$$

Given that the inequality (10) is satisfied for all $i = 0, ..., n - 1$, the above given equality could be written in a differential inclusion form as:

$$\sigma_i^{(n)} \in -[\Gamma_i^+ \Gamma_i^+] + \epsilon_i^{(n-1)}$$

Then, by design, each $\alpha_i, i = 1, ..., p$ satisfies $\alpha_i > 1$. The rest of the proof is a direct consequence of [20].

Step i. Let us introduce the $i$th component of the virtual tracking error as $\sigma_i = x_i - \phi_{i-1}$. The first derivative of $\sigma_i$ is given by:

$$\dot{\sigma}_i = g_i(x_1, ..., x_i, t) + B_i(x_1, ..., x_i, t)x_{i+1} + f_{c_i}(t) + w_i - \dot{\phi}_{i-1}$$

Replacing $x_{i+1}$ by the virtual control (15) and differentiating the equation until the control appears (i.e. differentiating $n-i-1$ times), the following equality is obtained:

$$\sigma_i^{(n-i+1)} = \left[ \frac{d^{n-i}}{dt^{n-i}}(f_{c_i}(t) + w_i) \right] + \epsilon_i^{(n-i)}$$

Using again inequality (10), the above given equality can be written in a differential inclusion form as:

$$\sigma_i^{(n-i+1)} \in -[\Gamma_i^+ \Gamma_i^+] + \epsilon_i^{(n-i)}$$

Notice again that $\alpha_i$ is designed to compensate for the bounded fault and disturbance, the rest of the proof is also a direct consequence of [20].

Step n. Finally, the $n$th component of the virtual tracking error takes on the form:

$$\sigma_n = x_n - \phi_{n-1}$$
Its dynamics is given by the following expression:

\[ \dot{x}_n = g_n(x_1, ..., x_n, t) + B_n(x_1, ..., x_n, t)u + f_{cn}(t) + f_\alpha(t) + w_n - \phi_{n-1} \]

By substitution of the control (17) the following equality is obtained:

\[ \dot{x}_n = f_{cn}(t) + f_\alpha(t) + w_n - \phi_{n-1} + v_n \]

As in the above described \( n - 1 \) steps, the fault is bounded according to (12), therefore \( \alpha_n \) should be chosen such that \( \alpha_n > \Gamma^+ \), then the equality could be written as a differential inclusion:

\[ \dot{x}_n \in \left[ \Gamma_n^+ \quad \Gamma_n^+ \right] + v_n \]

Notice that in Step \( n \), the discontinuous control \( v_n \) compensates for the faults \( f_{cn}(t) + f_\alpha(t) \) and the disturbance \( w_n \) that are only need to be bounded. The rest of the proof is a consequence of (20).

4.2. Noise effect analysis

The following result is taken from [22] and [20]. Let \( g_0 \) be measured with a bounded noise \( \epsilon \), \( |\epsilon| < \epsilon^+ \) for a known positive scalar \( \epsilon^+ \). Then, following inequalities are satisfied by differentiator (7):

\[ |z_i - g_0^{(i)}| \leq \mu_i (\epsilon^+)^{(n-i+1)/(n+1)}, \quad i = 0, ..., n \]

where \( \mu_i \) depends on the differentiator parameters. Therefore, the accuracy of the derivatives estimation depends on the selection of \( M \) in (7).

The application of the algorithm (1)-(17) requires the estimation of \( \sigma_{j_1}, \sigma_{j_2}, ..., \sigma_{j_k}^{(n-1)} \), this problem could be solved by means of (7), however the existence of noise inhibits an exact estimation of the derivatives. Therefore, the noise effect is transferred through the differentiator to the controller, in this sense, it follows from [20] that the accuracy of the controller in the presence of a bounded deterministic noise is given as follows:

\[ |\sigma_{j_k}| \leq \mu_{j_k}(\epsilon^+)^{n-i/n}, \quad i = 1, ..., n - 1 \]

where \( \mu_{j_k} \) are positive scalars that depends on the parameters of the differentiator and the controller.

5. Simulation example

5.1. System and fault modeling

In order to illustrate the application of the proposed methodology, the algorithm is applied to the model presented by [24] and [25], that corresponds to the nominal nonlinear model of an F-16 jet fighter at Mach=0.7, \( h=10000 \) ft, \( \alpha_{trim} = \theta_{trim} = 0.106803 \) rad, \( \delta_{c_{trim}} = -0.0295 \) rad, \( \beta_{t_{trim}} = p_{t_{trim}} = q_{t_{trim}} = r_{t_{trim}} = \varphi_{t_{trim}} = \delta_{a_{t_{trim}}} = \delta_{\varphi_{t_{trim}}} = 0 \). The dynamics is given by

\[ \begin{align*}
\dot{y} &= g_1(y,t) + B_1(y,t)z + f_{c_1} + w_1 \\
\dot{z} &= g_2(y,z,t) + B_2(y,z,t)\delta + f_{c_2} \\
\dot{\delta} &= -A_\delta(\delta - u)
\end{align*} \]

where the vector \( y \in \mathbb{R}^3 \) is composed of the command angles \( y = [\bar{\varphi} \quad \bar{\alpha} \quad \bar{\beta}]^T \) in [rad] (roll angle, angle of attack, sideslip angle, respectively); \( z \in \mathbb{R}^3 \) is a vector of angular rates in [rad/s] \( (z = [p \quad q \quad r]^T) \), roll, pitch and yaw rates, respectively); \( \delta \in \mathbb{R}^3 \) is a vector of control deflections in
\[ \delta = [\delta_u, \delta_c, \delta_r]^T, \] aileron, elevator and rudder deflections, respectively); \( u \in \mathbb{R}^3 \) is a control vector applied to actuator inputs. \( f_{c1}, f_{c2} \in \mathbb{R}^3 \) are the component faults caused by damage, in this example. \( \omega_1 \in \mathbb{R}^3 \) is the vector of disturbances and \( f_a \in \mathbb{R}^3 \) is the vector of actuator faults. 

The nonlinear vector-functions \( g_1(y, t) \in \mathbb{R}^3 \) and \( g_2(y, z, t) \in \mathbb{R}^3 \) are known and differentiable and, \( B_1(y, t) \in \mathbb{R}^{3 \times 3}; B_2(y, z, t) \in \mathbb{R}^{3 \times 3} \) are known and nonsingular matrices. It is considered that all the variables \( y, z, \delta \) are measurable.

The nonlinear functions are given as follows:

\[
g_1(y, t) = \begin{bmatrix} 0 \\ 0.083589 - 1.15y_2 \\ -0.297y_3 \\
\end{bmatrix}, \quad B_1(y, t) = \begin{bmatrix} 1 & 0 & 0 \\ -y_3 & 0.9937 & 0 \\ y_2 + 0.00085 & 0 & -0.9973 \\
\end{bmatrix}
\]

\[
g_2(y, z, t) = \begin{bmatrix} g_{21} \\ g_{22} \\ g_{23} \\
\end{bmatrix}, \quad B_2(y, z, t) = \begin{bmatrix} -50.933 & 0 & 10.177 \\ 0 & -19.5 & 0 \\ 4.125 & 0 & -6.155 \\
\end{bmatrix}
\]

\[
g_{21} = -0.1345z_1z_2 - 0.8225z_2z_3 - 53.48y_3 - 4.324z_1 - 0.224z_3; \\
g_{22} = 0.9586z_1z_3 - 0.0833(z_3^2 - z_1^2) - 1.94166 + 3.724y_2 - 1.262z_2; \\
g_{23} = -0.7256z_1z_2 + 0.1345z_2z_3 + 17.67y_3 + 0.234z_1 - 0.649z_3; \\
A_3 = diag\{20, 20, 20\}
\]

According to technical reports [26], physical limitations of actuators are \( \delta_u \in [-0.52, 0.52] \) and \( \delta_c \in [-0.52, 0.52] \) radians. The operating range of the vector \( y \) is \( \bar{\varphi} \in [-1.15, 1.15], \bar{\alpha} \in [-0.34, 0.34] \) and \( \bar{\beta} \in [-0.34, 0.34] \) radians.

Let \( y_c(t) \) be a real-time command reference profile. Desired flying qualities are given in terms of second order linear systems as follows:

\[
\frac{y_{c1}}{y_1} = \frac{2}{s^2 + 2.22s + 2}, \quad \frac{y_{c2}}{y_2} = \frac{4}{s^2 + 3s + 4}, \quad \frac{y_{c3}}{y_3} = \frac{16.98}{s^2 + 6.18s + 16.38},
\]

where \( y_1, y_2 \) and \( y_3 \) correspond to \( \bar{\varphi}, \bar{\alpha} \) and \( \bar{\beta} \) respectively. The control goal is to ensure the convergence of the output tracking errors to zero, i.e. \( \lim_{t \to \infty} |y_{c_i} - y_i| = 0 \), \( \forall i = 1, 2, 3 \), using the control signal \( u \) designed as in (17) in spite of external disturbances and fault occurrence.

To assess the proposed scheme, the following simulation commands are considered:

(i) The angle-of-attack command (with respect to a trim value \( \alpha_{\text{trim}} = 0.106803 \) rad) is two 3 seconds of duration pulses of 0.1 rad amplitude and polarities + and − during time intervals \([0, 3]\) and \([6, 9]\) s, respectively.

(ii) The roll-angle command is one pulse of 1.0 rad amplitude during the time interval of \([1, 9]\) s.

(iii) The sideslip angle command is zero for all time.

In the flying quality filters in equation (25) are used to filter the commands.

In this flying, an aircraft battle damage is considered to illustrate the workability of the proposed fault tolerant controller. According to the previous work [25], the effect of such kind of damage can be seen like a component fault where \( f_{c1} \) and \( f_{c2} \) of (26)-(27) can be computed by using the following modelling equations:

\[
f_{c1} = (\Delta g_1(y, z, \delta, t) + \Delta B_1(y, z, \delta, t)z) \\
f_{c2} = (\Delta g_2(y, z, \delta, t) + \Delta B_2(y, z, \delta, t)\delta)
\]

(26)

(27)

It is considered that the matrices \( (B_1(y, t) + \Delta B_1(y, z, \delta, t)) \) and \( (B_2(y, z, \delta, t) + \Delta B_2(y, z, \delta, t)) \) are not singular.

\[
\Delta g_1(\cdot) = 1 \cdot (t - 1.5)\Delta g_{11}(\cdot) + 1 \cdot (t - 5)\Delta g_{12}(\cdot) \\
\Delta B_1(\cdot) = 1 \cdot (t - 1.5)\Delta B_{11}(\cdot) + 1 \cdot (t - 5)\Delta B_{12}(\cdot) \\
\Delta g_2(\cdot) = 1 \cdot (t - 1.5)\Delta g_{21}(\cdot) + 1 \cdot (t - 5)\Delta g_{22}(\cdot) \\
\Delta B_2(\cdot) = 1 \cdot (t - 1.5)\Delta B_{21}(\cdot) + 1 \cdot (t - 5)\Delta B_{22}(\cdot)
\]
where \( 1 \cdot (\tau) \) is a unit step function and

\[
\Delta g_1(\cdot) = \begin{bmatrix} 0 & 0 \\ 0.0534\beta + 0.002466\delta_u + 0.0186\delta_r & 0 \\ -0.00005 & 0 \end{bmatrix}, \quad \Delta g_2(\cdot) = \begin{bmatrix} 0 & 0 \\ 0.04\alpha - 0.0885\delta_e & 0 \\ 0 & 0 \end{bmatrix}
\]

\[
\Delta B_1(\cdot) = \begin{bmatrix} 0 \\ 0.04\alpha - 0.0885\delta_e \\ -0.00005 \end{bmatrix}, \quad \Delta B_2(\cdot) = \begin{bmatrix} 0 \\ 0.04\alpha - 0.0885\delta_e \\ -0.00005 \end{bmatrix}
\]

\[
\Delta B_{11}(\cdot) = \begin{bmatrix} 0 \\ 0.04\alpha - 0.0885\delta_e \\ -0.00005 \end{bmatrix}, \quad \Delta B_{12}(\cdot) = \begin{bmatrix} 0 \\ 0.04\alpha - 0.0885\delta_e \\ -0.00005 \end{bmatrix}
\]

\[
\Delta B_{21}(\cdot) = \begin{bmatrix} 0 \\ 0.04\alpha - 0.0885\delta_e \\ -0.00005 \end{bmatrix}, \quad \Delta B_{22}(\cdot) = \begin{bmatrix} 0 \\ 0.04\alpha - 0.0885\delta_e \\ -0.00005 \end{bmatrix}
\]

**Remark 1** The cases of total failure of ailerons, elevator and rudder are not considered. For the total failure case, the control surfaces are completely disabled and can not generate rotational torques. This faulty situation cannot be recoverable [5].

The command signals and the corresponding response of the aircraft for the three cases are shown in Fig. 1. Notice that, even in the presence of disturbances, actuator faults and component faults, the proposed controller provides an appropriate tracking of the command signals. Notice that the tracking of the command signals is not affected by the appearance of faults (see \( t=1.5 \) and \( t=5 \)), the proposed controller is insensitive with respect to the actuator faults. It is important to remark that for all the cases the set of parameters is the same, i.e. \( \alpha_5 = 15, \alpha_z = 60, \alpha_y = 15, M_z = 200, M_y = 200 \), that correspond to all the gains \( \alpha \), for the different instances of the quasi-continuous controller, and \( M \), for the different instances of the High-Order Sliding Modes differentiator, for each set of coordinates \( \delta, z \) and \( y \). Here, the sub-indexes have been included to avoid any confusion. The tracking errors are presented in Fig. 2, notice that all the tracking errors are of the order \( 10^{-3} \). In the Fig. 3 the ailerons, elevators and rudder deflections are shown, all the deflections are in the valid rank of the aircraft.

![Figure 1. Tracking of the angles (ϕ, α, β).](image-url)
### 5.2. Noise effect

Let us consider now that the variables are affected by a bounded noise. For simulation purposes the measured signals, including the noise effect, takes the following form:

\[
\bar{y} = y + 0.003 \sin(2.5 + 2 \sin(1.8t)) + 0.003 \cos(5 + 4 \sin(2.5t)) \\
\bar{z} = z + 0.002 \sin(4 + 3.9 \sin(2.3t)) + 0.004 \cos(2 + \sin(1.5t)) \\
\bar{\delta} = \delta + 0.006 \sin(6 + 4 \sin(0.5t))
\]

For the case with noisy measurements the corresponding response of the fault tolerant controller is shown in the Figure 4. Notice that almost perfect tracking is obtained in spite of the bounded noise. The tracking error is shown in Figure 5, it is possible to verify that the maximal tracking error is proportional to \( \epsilon^+ \), as it was explained in Subsection 4.2. The commanded deflections are presented in the Figure 6. It is clear that the noise effect is not amplified by the controller.

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Figure 4. Tracking of the variables \( \phi, \alpha \) and \( \beta \) in the presence of bounded noise.

Figure 5. Tracking errors for the angles \( \phi, \alpha \) and \( \beta \) in the presence of bounded noise.

Figure 6. Vector of deflections (\( \delta_a \) ailerons, \( \delta_e \) elevators, \( \delta_r \) rudder) in the presence of bounded noise.
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