Circular non-collision orbits for a large class of \( n \)-body problems.

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Abstract

We prove for a large class of \( n \)-body problems including a subclass of quasihomogeneous \( n \)-body problems, the classical \( n \)-body problem, the \( n \)-body problem in spaces of negative constant Gaussian curvature and a restricted case of the \( n \)-body problem in spaces of positive constant curvature for the case that all masses are equal and not necessarily constant that any solution for which the point masses move on a circle of not necessarily constant size has to be either a regular polygonal homographic orbit in flat space, or a regular polygonal rotopulsator in curved space, under the constraint that the minimal distance between point masses attains its minimum in finite time. Additionally, we prove that the same holds true if we add an extra mass at the center of that circle and find an explicit formula for the mass of each point particle in terms of the radius of the circle. Finally, we prove that for each order of the masses there is at most one polygonal homographic orbit for the case that the masses need not be constant.

1 Introduction

By \( n \)-body problems we mean problems where we are to determine the dynamics of a number of \( n \) point masses as dictated by a system of ordinary differential equations. The main \( n \)-body problem we will study in this paper is the problem of finding the orbits of point masses \( q_1,\ldots,q_n \in \mathbb{R}^2 \) and respective (not necessarily constant) masses \( m_1 > 0,\ldots, m_n > 0 \) determined
by the system of differential equations

\[ \ddot{q}_i = \sum_{j=1, j \neq i}^{n} m_j(q_j - q_i) f \left( \|q_j - q_i\|^2 \right), \quad (1.1) \]

where \( \| \cdot \| \) is the Euclidean norm, \( f : \mathbb{R}_{>0} \to \mathbb{R}_{>0} \) is a positive-valued scalar function and \( \sqrt{x} f(x) \) is a decreasing function. The study of \( n \)-body problems of this type has applications to, for example, atomic physics, celestial mechanics, chemistry, crystallography, differential equations and dynamical systems (see for example [1]–[8], [19], [22]–[29] and the references therein and [27] and the references therein for the case that the masses are not constant). A second \( n \)-body problem we will investigate is the \( n \)-body problem in spaces of constant Gaussian curvature, or curved \( n \)-body problem for short and is defined as follows: Let \( \sigma = \pm 1 \). The \( n \)-body problem in spaces of constant Gaussian curvature is the problem of finding the dynamics of point masses

\[ q_1, \ldots, q_n \in M^2_\sigma = \{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_1^2 + x_2^2 + \sigma x_3^2 = \sigma \}, \]

with respective masses \( m_1 > 0, \ldots, m_n > 0 \), determined by the system of differential equations

\[ \ddot{q}_i = \sum_{j=1, j \neq i}^{n} \frac{m_j(q_j - q_i) \cdot q_j}{(\sigma - \sigma(q_i \circ q_j)^2)^{\frac{3}{2}}} \cdot q_i, \quad i \in \{1, \ldots, n\}, \quad (1.2) \]

where for \( x, y \in M^3_\sigma \) the product \( \cdot \circ \cdot \) is defined as

\[ x \circ y = x_1 y_1 + x_2 y_2 + \sigma x_3 y_3. \]

The curved \( n \)-body problem for \( n = 2 \) goes back as far as the 1830s, but a working model for the \( n \geq 2 \) case was not found until 2008 by Diacu, Pérez-Chavela and Santoprete (see [16], [17] and [18]). This breakthrough then gave rise to further results for the \( n \geq 2 \) case in [9]–[15] and [20], [21] and [30]–[38]. See [15], [16], [17] and [18] for a historical overview. The study of the curved \( n \)-body problem has applications to for example geometric mechanics, Lie groups and algebras, non-Euclidean and differential geometry and stability theory, the theory of polytopes and topology (see for example [14]) and may give information about the geometry of the universe: For example: Diacu, Pérez-Chavela and Santoprete (see [16], [17]) showed that the configuration of the Sun, Jupiter and the Trojan asteroids cannot exist in curved space.
In this paper, we will consider solutions to (1.1) and (1.2) where either all point masses lie on a circle of nonconstant size, or all but one point masses lie on a circle of nonconstant size and the remaining point mass lies at the center of that circle. Specifically, we will look at solutions

\[ q_i(t) = r(t) \begin{pmatrix} \cos \theta_i(t) \\ \sin \theta_i(t) \end{pmatrix}, \ i \in \{1, \ldots, n\} \quad (1.3) \]

of (1.1), where \( r \) is a twice continuously differentiable, positive function and \( \theta_1 < \ldots < \theta_n \) are twice continuously differentiable functions, solutions

\[ q_i(t) = r(t) \begin{pmatrix} \cos \theta_i(t) \\ \sin \theta_i(t) \end{pmatrix}, \ i \in \{1, \ldots, n-1\}, \ q_n(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (1.4) \]

of (1.1), where \( r \) is a twice continuously differentiable, positive function and \( \theta_1 < \ldots < \theta_{n-1} \) are twice continuously differentiable functions, solutions

\[ q_i(t) = \begin{pmatrix} r(t) \cos \theta_i(t) \\ r(t) \sin \theta_i(t) \\ z(t) \end{pmatrix}, \ i \in \{1, \ldots, n\} \quad (1.5) \]

of (1.2), where \( r \) is a twice continuously differentiable, positive function and \( \theta_1 < \ldots < \theta_n, z(t) \) are twice continuously differentiable functions and solutions

\[ q_i(t) = \begin{pmatrix} r(t) \cos \theta_i(t) \\ r(t) \sin \theta_i(t) \\ z(t) \end{pmatrix}, \ i \in \{1, \ldots, n-1\}, \ q_n(t) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad (1.6) \]

of (1.2), where \( r \) is a twice continuously differentiable, positive function and \( \theta_1 < \ldots < \theta_{n-1}, z(t) \) are twice continuously differentiable functions. Additionally, if in (1.3), or (1.4) we have that \( \theta_i = \phi + \alpha_i \) for all \( i \in \{1, \ldots, N\} \), where \( \phi \) is a twice, continuously differentiable function, \( \alpha_i \in [0, 2\pi) \) is constant, \( N = n \) for a solution as described in (1.3) \( \) and \( N = n - 1 \) for a solution as described in (1.4), then we call the configuration \( q_1, \ldots, q_n \) a polygonal homographic orbit. If in (1.5), or (1.6) we have that \( \theta_i = \phi + \alpha_i \) for all \( i \in \{1, \ldots, N\} \), where \( \phi \) is a twice, continuously differentiable function, \( \alpha_i \in [0, 2\pi) \) is constant, \( N = n \) for a solution as described in (1.5) \( \) and \( N = n - 1 \) for a solution as described in (1.6), then we call the configuration \( q_1, \ldots, q_n \) a polygonal rotopulsator (see [15]). If in addition in (1.3), (1.4), (1.5), or (1.6) we have that \( r \) is constant, then we call respective solutions to (1.1), or (1.2) polygonal relative equilibria. It is well-known that polygonal
homographic orbits, polygonal rotopulsators and polygonal relative equilibria exist, but whether other solutions for which the point masses move on a circle exist seems to be an unexplored and mathematically nontrivial problem. To shed light onto at least a partial answer to that question, introducing functions

\[ \mu_1(t) = \min_{1 \leq j \leq n} \{ \theta_{j+1}(t) - \theta_j(t) \}, \]

where \( \theta_n = 2\pi + \theta_1 \), for solutions (1.3) and (1.5) and

\[ \mu_2(t) = \min_{1 \leq j \leq n-1} \{ \theta_{j+1}(t) - \theta_j(t) \}, \]

where \( \theta_n = 2\pi + \theta_1 \), for solutions (1.4) and (1.6), we will prove the following results:

**Theorem 1.1.** Let \( q_1, \ldots, q_n \) be a solution of (1.1) as described in (1.3), for which all masses are equal. If \( \mu_1 \) has a local minimum, then that solution is a polygonal homographic orbit with a regular polygon configuration.

**Corollary 1.2.** Let \( q_1, \ldots, q_n \) be a solution of (1.1) as described in (1.4), for which all masses are equal. If \( \mu_2 \) has a local minimum, then that solution is a polygonal homographic orbit with a regular polygon configuration.

**Corollary 1.3.** Let \( q_1, \ldots, q_n \) be a solution of (1.2) as described in (1.5), for which all masses are equal, \( \sigma = -1 \), or \( \sigma = 1 \) and \( r < \frac{1}{5}\sqrt{10} \). If \( \mu_1 \) has a local minimum, then that solution is a polygonal rotopulsator with a regular polygon configuration.

**Corollary 1.4.** Let \( q_1, \ldots, q_n \) be a solution of (1.2) as described in (1.5), for which all masses are equal, \( \sigma = -1 \), or \( \sigma = 1 \) and \( r < \frac{1}{5}\sqrt{10} \). If \( \mu_2 \) has a local minimum, then that solution is a polygonal rotopulsator with a regular polygon configuration.

**Corollary 1.5.** Let \( q_1, \ldots, q_n \) be a polygonal homographic orbit solution of (1.1) for the case that all masses are equal. Then the configuration of the point masses is a regular polygon.

**Corollary 1.6.** Let \( q_1, \ldots, q_n \) be a polygonal homographic orbit solution of (1.1) for the case that all masses of the \( q_i, i \in \{1, \ldots, n\} \), that lie on a circle are equal to a function \( m \). Then there exist constants \( a, b \in \mathbb{R} \) such that

\[
m = \frac{a^2 - r^n r^3}{r^4 \sum_{j=1}^{n-1} (1 - \cos \frac{2\pi j}{n}) f(2r^2(1 - \cos \frac{2\pi j}{n}))}
\]

and

\[
\phi' = \frac{a}{r^2}
\]
if the solution is as described in (1.3) and
\[ m = \frac{b^2 - (r'' + rMf(r))r^3}{r^4 \sum_{j=1}^{n-2} (1 - \cos \frac{2\pi j}{n-1})f(2r^2(1 - \cos \frac{2\pi j}{n-1}))} \]
and
\[ \phi' = \frac{b}{r^2} \]
if the solution is as described in (1.4), where \( M \) is the mass of the point mass that does not lie on the circle.

Corollary 1.7. If \( f \) is continuously differentiable, then for each order of the masses there exists at most one polygonal homographic or bit solution of (1.1).

Remark 1.8. The condition that \( \mu_1 \), or \( \mu_2 \) has a local minimum is not necessarily true for any solution \( q_1, \ldots, q_n \) of (1.1), or (1.2) where \( n \), or \( n - 1 \) point masses lie on the same circle. Proving the existence, or nonexistence, of solutions that are neither homographic orbits, nor rotopulsators where the point masses lie on a circle and \( \mu_1 \), or \( \mu_2 \) does not have a local minimum is a nontrivial matter that should be explored in future research.

Remark 1.9. Corollary 1.5 was implicitly already proven in [35] for the case that \( f \) in (1.1) is continuously differentiable. However, Corollary 1.5 shows that the same result holds true if the only conditions on \( f \) are that \( f \) is a positive function and \( x^2 f(x) \) is a decreasing function.

Remark 1.10. Because of the general nature of the \( n \)-body problem used in Theorem 1.1, Corollary 1.2 and Corollary 1.5, these results may help solve Problem 12 of the list by Albouy, Cabral and Santos (see [2]). Not only that: For the equal masses case, if the homographic orbit is a relative equilibrium, the proof of Corollary 1.5 (or rather the proof of Theorem 1.1) essentially becomes a one-line proof, as \( r' \) and \( \theta'' \), \( i \in \{1, \ldots, n\} \) are zero for polygonal relative equilibrium solutions and (2.5) reduces to \( 0 \geq 0 \), with the inequality strict if and only if the configuration is not a regular polygon.

Remark 1.11. Note that by Corollary 1.6, if \( r \) is the radius of the circle on which the point masses lie, it is possible to create polygonal homographic solutions of (1.1) with almost no conditions on \( r \).
Remark 1.12. Corollary 1.7 was proven in [35] for the case that the homo-
graphic orbit is a relative equilibrium and all masses are constant. Corol-
lary 1.7 shows that the assumptions that the homographic orbit is a relative
equilibrium and all masses are constant are not needed.

We will now prove Theorem 1.1 in section 2, Corollary 1.2 in section 3,
Corollary 1.3 in section 4, Corollary 1.4 in section 5, Corollary 1.5 in sec-
tion 6, Corollary 1.6 in section 7 and Corollary 1.7 is section 8.

2 Proof of Theorem 1.1

Let \( q_1, \ldots, q_n \) be a solution of (1.1) for which all point masses lie on a spher-
e of radius \( r \). Then we may write

\[
q_i = r T(\theta_i) \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ i \in \{1, \ldots, n\},
\]

\( (2.1) \)

where \( r \) and the \( \theta_i \) are scalar, differentiable functions and

\[
T(x) = \begin{pmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{pmatrix}.
\]

Inserting (2.1) into (1.1) and multiplying the result from the left by \( T(\theta_i)^{-1} \)
gives

\[
\left( \frac{r''}{2r'\theta_i'' + r\theta_i''} \right) = r \sum_{j=1, j \neq i}^{n} m_j \left( \frac{-(1 - \cos (\theta_j - \theta_i))}{\sin (\theta_j - \theta_i)} \right) f(2r^2(1 - \cos (\theta_j - \theta_i)))
\]

and consequently

\[
2r'\theta_i'' + r\theta_i'' = r \sum_{j=1, j \neq i}^{n} m_j \sin (\theta_j - \theta_i)f(2r^2(1 - \cos (\theta_j - \theta_i))). \quad (2.3)
\]

If we write \( \mathbb{R}_{\geq 0} = \bigcup_{k=1}^{\infty} I_k \), where the \( I_k \) are disjoint intervals for which
\( I_k = [a_k, b_k] \) and \( a_{k+1} = b_k \), then we can choose the \( a_k \) and \( b_k \) in such a way
that we can write \( \mu_1(t) = \theta_2(t) - \theta_1(t) \) for any \( t \in I_k, k \in \mathbb{N} \), relabeling
the point masses per interval \( I_k \) if necessary. Note that \( \mu_1 \) is continuous on \( \mathbb{R}_{\geq 0} \)
and differentiable on the interior of every \( I_k \). Note that

\[
\sin x f(r^2(1 - \cos x)) = \begin{cases} \sqrt{1 + \cos x} (\sqrt{1 - \cos x} f(2r^2(1 - \cos x))) & \text{for } x \in (0, \pi] \\ -\sqrt{1 + \cos x} (\sqrt{1 - \cos x} f(2r^2(1 - \cos x))) & \text{for } x \in (\pi, 2\pi), \end{cases}
\]

\[ 6 \]
so \( g(x) := \sin xf(2r^2(1-\cos x)) \) is a decreasing function on \((0, 2\pi)\), as \( \sqrt{yf(y)} \)
for \( y > 0 \) is a decreasing function, \( 1 - \cos x \) is an increasing function and
\( 1 + \cos x \) is a decreasing function. Additionally, for \( t \in I_k, k \in \mathbb{N}, \) we have
by construction that
\[
\theta_{j+1} - \theta_j \geq \theta_2 - \theta_1 \text{ for all } j \in \{1, \ldots, n\},
\]
or equivalently
\[
\theta_{j+1} - \theta_2 \geq \theta_j - \theta_1 \text{ for all } j \in \{1, \ldots, n\}. \tag{2.4}
\]
Also note that there are \( j \in \{1, \ldots, n\} \) for which (2.4) is a strict inequality
if and only if the configuration of the \( q_j \) is an irregular polygon. These two
observations essentially prove our theorem: As all masses are equal, write
\( m_j = m. \) Then by (2.3) in combination with (2.4) and the fact that \( g(x) \)
is a decreasing function on \((0, 2\pi), \) we get that for \( t \) in the interior of any
interval \( I_k \) that
\[
2r'\theta_1 + r\theta'' = r \sum_{j=2}^{n} m_j g(\theta_j - \theta_1) \geq \sum_{j=2}^{n} m_j g(\theta_{j+1} - \theta_2) = r \sum_{j=1, j \neq 2}^{n} m g(\theta_j - \theta_2)
\]
\[
= r \sum_{j=1, j \neq 2}^{n} m_j g(\theta_j - \theta_2) = 2r'\theta_2' + r\theta''_2. \tag{2.5}
\]
So by (2.5) we now have that \( 2r'\theta_2' + r\theta''_2 - (2r'\theta_1' + r\theta''_1) \leq 0 \) for all \( t \in I_k, \)
\( k \in \mathbb{N}. \) Multiplying both sides of this inequality with \( r \) and integrating this
inequality from any \( s \) to a value \( t, \) with \( s, t \in I_k, s < t, k \in \mathbb{N}, \) we get that
\( r^2(t)(\theta_2'(t) - \theta_1'(t)) \leq r^2(s)(\theta_2'(s) - \theta_1'(s)), \) or equivalently
\[
r^2(t)\mu_1'(t) \leq r^2(s)\mu_1'(s). \tag{2.6}
\]
Additionally, note that by construction \( \lim_{u \uparrow a_k} \mu_1'(u) \leq \lim_{u \downarrow a_k} \mu_1'(u), \) which means
that
\[
\lim_{u \uparrow a_k} r^2(u)\mu_1'(u) = r^2(a_k)(\lim_{u \uparrow a_k} \mu_1'(u)) \leq r^2(a_k)(\lim_{u \downarrow a_k} \mu_1'(u)) = \lim_{u \downarrow a_k} r^2(u)\mu_1'(u). \tag{2.7}
\]
With (2.6) and (2.7) in place, we can now continue our proof: It is given
that \( \mu_1 \) has a local minimum. If \( \mu_1(t) \) has a local minimum for a \( t = t_0, \) \( t_0 \)
in the interior of an interval \( I_k, \) then if we choose \( s \) and \( t \) in such a way that
\( s < t_0 < t, s, t \) close enough to \( t_0, \) we get by (2.6) that
\[
0 \leq r^2(t)\mu_1'(t) \leq r^2(t_0)\mu_1'(t_0) \leq r^2(s)\mu_1'(s) \leq 0,
\]
which means that the configuration of \(q_1, ..., q_n\) is a regular polygon. If \(\mu_1(t)\) has a local minimum for \(t = a_k\) for a certain \(k \in \mathbb{N}\), then by (2.6) and (2.7) and choosing \(s < a_k < t, s \in I_{k-1}, t \in I_k, \) if \(k \neq 1, t\) and \(s\) close enough to \(a_k\), then gives that

\[
0 \leq r^2(t)\mu_1'(t) \leq \lim_{u \uparrow a_k} r^2(u)\mu_1'(u) \leq \lim_{u \downarrow a_k} r^2(u)\mu_1'(u) \leq r^2(s)\mu_1'(s) \leq 0,
\]

meaning that the configuration of \(q_1, ..., q_n\) is again a regular polygon. This completes the proof.

3 Proof of Corollary 1.2

Let \(q_1, ..., q_n\) be a solution of (1.1) of the type described in (1.4), with \(m_n = M\). Then inserting the respective \(q_j, j \in \{1, ..., n\}\) into (1.1), subtracting \(-Mq_i f(\|q_i\|^2)\) from both sides and then multiplying the result on both sides from the left by \(T(\theta_i)^{-1}\) gives for \(i \neq n\)

\[
\left(\frac{r'' - r(\theta_i')^2}{2r\theta_i' + r\theta_i''}\right) + Mr \begin{pmatrix} 1 \\ 0 \end{pmatrix} f(r^2) = \sum_{j=1, j \neq i}^{n-1} m_j \begin{pmatrix} -(1 - \cos(\theta_j - \theta_i)) \\ \sin(\theta_j - \theta_i) \end{pmatrix} f(2r^2(1 - \cos(\theta_j - \theta_i))).
\]

(3.1)

The identity for the second coordinate in (3.1) is now the same as (2.3), except for the fact that now \(n\) has been replaced by \(n - 1\). Replacing every \(n\) in the proof of Theorem 1.1 after (2.3) with \(n - 1\) and \(\mu_1\) with \(\mu_2\) thus completes the proof for this theorem.

4 Proof of Corollary 1.3

Let \(q_1, ..., q_n\) be a solution of (1.2) for which all point masses lie on a sphere of radius \(r\). Then we may write

\[
q_i = rT(\theta_i) \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad i \in \{1, ..., n\},
\]

(4.1)

where the \(\theta_i\) are scalar, differentiable functions and

\[
T(x) = \begin{pmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{pmatrix}.
\]
Inserting (4.1) into (1.2) and multiplying the result for the first two coordinates from the left by $T(\theta_i)^{-1}$ gives

$$\left( r'' - r(\theta'_i)^2 \right) = r \sum_{j=1, j \neq i}^n m_j \left( -\left( \sigma(q_i \odot q_j - \cos(\theta_j - \theta_i)) \right) \sin(\theta_j - \theta_i) \right) \left( \sigma - \sigma(q_i \odot q_j)^2 \right)^{-\frac{3}{2}}$$

$$- r \sigma(q_i \odot q_i) \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \text{if } i \in \{1, \ldots, n\}$$

and by the identity for the second coordinate of (4.2), using that

$$q_i \odot q_j = r^2 \cos(\theta_j - \theta_i) + \sigma z^2 = -r^2 (1 - \cos(\theta_j - \theta_i)) + \sigma,$$

we have that

$$2r\theta'_i + r\theta''_i = r \sum_{j=1, j \neq i}^n m_j \sin(\theta_j - \theta_i) \left( 2r^2 (1 - \cos(\theta_j - \theta_i)) - \sigma r^2 (1 - \cos(\theta_j - \theta_i))^2 \right)^{-\frac{3}{2}}.$$  

(4.2)

Let

$$f(x^2) = (2r^2 x^2 - \sigma r^4 x^4)^{-\frac{3}{2}}.$$  

Then

$$\frac{d}{dx} \left( x f(x^2) \right) = \frac{d}{dx} \left( x^{-1} \left( 2r^2 - \sigma r^4 x^2 \right)^{-\frac{3}{2}} \right)$$

$$= -x^{-2} \left( 2r^2 - \sigma r^4 x^2 \right)^{-\frac{5}{2}} + 3r^4 \sigma \left( 2r^2 - \sigma r^4 x^2 \right)^{-\frac{5}{2}},$$

so $x^{\frac{1}{2}} f(x)$ is a decreasing function if $\sigma = -1$. Additionally, if $\sigma = 1$ and $r < \frac{1}{\sqrt{10}}$, then $x^{\frac{1}{2}} f(x)$ is a decreasing function as well. So if $\sigma = -1$, or if $\sigma = 1$, $r < \frac{1}{\sqrt{10}}$, then (4.3) has the required properties of (2.3) to continue as we did in the proof of Theorem 1.1. This completes the proof.

5 Proof of Corollary 1.4

Let $q_1, \ldots, q_n$ be a solution of (1.2) as described in (1.6). Then we may write for $i \neq n$ that

$$q_i = \begin{pmatrix} rT(\theta_i) \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \end{pmatrix},$$

(5.1)
where the $\theta_i$ are again scalar, differentiable functions and

$$T(x) = \begin{pmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{pmatrix}.$$ 

Inserting (5.1) into (1.2) and multiplying the result for the first two coordinates from the left by $T(\theta_i)^{-1}$ gives for $i \neq n$ that

$$\left( r'' - r(\theta_i')^2 \right) = r \sum_{j=1, j \neq i}^{n-1} m_j \left( \frac{-\sigma (q_i \odot q_j) - \cos (\theta_j - \theta_i)}{\sin (\theta_j - \theta_i)} \right) (\sigma - \sigma (q_i \odot q_j)^2)^{-\frac{3}{2}}$$

$$+ m_n \left( \begin{array}{c} 0 \\ 0 \end{array} \right) (\sigma - \sigma^2)^{-\frac{3}{2}} - r \sigma (\dot{q}_i \odot \dot{q}_i) \left( \begin{array}{c} 1 \\ 0 \end{array} \right)$$

(5.2)

and by the identity for the second coordinate of (5.2), using again that

$$q_i \odot q_j = r^2 \cos (\theta_j - \theta_i) + \sigma z^2 = -r^2 (1 - \cos (\theta_j - \theta_i)) + \sigma,$$

we have that

$$2r \theta_i' + r \theta_i'' = r \sum_{j=1, j \neq i}^{n-1} m_j \sin (\theta_j - \theta_i) \left( 2r^2 (1 - \cos (\theta_j - \theta_i)) - \sigma (r^2 (1 - \cos (\theta_j - \theta_i)) )^2 \right)^{-\frac{3}{2}},$$

(5.3)

which is the exact same formula as (4.3), except for the case that $n$ has been replaced with $n - 1$. The proof now holds by the same argument as in the proof of Theorem 1.3 after (4.3).

### 6 Proof of Corollary 1.5

If $q_1, \ldots, q_n$ is a polygonal homographic orbit of the type described in (1.3), then the $q_i$, $i \in \{1, \ldots, n\}$ have exactly the properties of the $q_i$ in the proof of Theorem 1.1 with the restriction that there exist a function $\phi$ and constants $\alpha_1, \ldots, \alpha_n \in [0, 2\pi)$ such that $\theta_i = \phi + \alpha_i$, $i \in \{1, \ldots, n\}$, so in that case the result holds by exactly the same argument as in the proof of Theorem 1.1.

If $q_1, \ldots, q_n$ is a polygonal homographic orbit of the type described in (1.4), then the $q_i$, $i \in \{1, \ldots, n - 1\}$ have exactly the properties of the $q_i$ in the proof of Theorem 1.3 with the restriction that there exist a function $\phi$ and constants $\alpha_1, \ldots, \alpha_{n-1} \in [0, 2\pi)$ such that $\theta_i = \phi + \alpha_i$, $i \in \{1, \ldots, n - 1\}$, so in that case the result holds by exactly the same argument as in the proof of Corollary 1.2. This proves that any polygonal homographic orbit solution of (1.1) has to have the configuration of a regular polygon.
7 Proof of Corollary 1.6

If \( q_1, \ldots, q_n \) is a polygonal homographic orbit of the type described in (1.3), then the \( q_i, i \in \{1, \ldots, n\} \) have exactly the properties of the \( q_i \) in the proof of Theorem 1.1, with the restriction that there exist a function \( \phi \) and constants \( \alpha_1, \ldots, \alpha_n \in [0, 2\pi) \) such that \( \theta_i = \phi + \alpha_i, i \in \{1, \ldots, n\} \). By Corollary 1.5, we have that the configuration of the point masses is a regular polygon, meaning that we may choose \( \alpha_i = \frac{2\pi i}{n} \), which means that by (2.2) we have that

\[
(r'' - r(\phi')^2) = r \sum_{j=1, j \neq i}^n m \left( - \left( 1 - \cos \left( \frac{2\pi(j-i)}{n} \right) \right) \right) f \left( 2r^2 \left( 1 - \cos \left( \frac{2\pi(j-i)}{n} \right) \right) \right)
\]

\[
= rm \sum_{j=1}^{n-1} \left( - (1 - \cos \left( \frac{2\pi j}{n} \right)) \right) f \left( 2r^2 \left( 1 - \cos \left( \frac{2\pi j}{n} \right) \right) \right).
\]

(7.1)

Let \( k = n - j \). As

\[
\sum_{j=1}^{n-1} \sin \left( \frac{2\pi j}{n} \right) f \left( 2r^2 \left( 1 - \cos \left( \frac{2\pi j}{n} \right) \right) \right)
\]

\[
= \sum_{k=1}^{n-1} \sin \left( \frac{2\pi(n-k)}{n} \right) f \left( 2r^2 \left( 1 - \cos \left( \frac{2\pi(n-k)}{n} \right) \right) \right)
\]

\[
= - \sum_{k=1}^{n-1} \sin \left( \frac{2\pi k}{n} \right) f \left( 2r^2 \left( 1 - \cos \left( \frac{2\pi k}{n} \right) \right) \right),
\]

we have that

\[
\sum_{j=1}^{n-1} \sin \left( \frac{2\pi j}{n} \right) f \left( 2r^2 \left( 1 - \cos \left( \frac{2\pi j}{n} \right) \right) \right) = 0,
\]

which means that by (7.1) we have that

\[
\left( r'' - r(\phi')^2 \right) = rm \sum_{j=1}^{n-1} \left( - \left( 1 - \cos \left( \frac{2\pi j}{n} \right) \right) \right) f \left( 2r^2 \left( 1 - \cos \left( \frac{2\pi j}{n} \right) \right) \right),
\]

(7.2)

which by the second vector components on both sides of (7.2) means that \( 2r'\phi' + r\phi'' = 0 \), or \( 2rr'\phi' + r^2\phi'' = 0 \), or equivalently that there exists a
constant $a \in \mathbb{R}$ such that $r^2 \phi' = a$, which by the first vector components on both sides of (7.2) means that
\[ r'' - \frac{a^2}{r^3} = r m \sum_{j=1}^{n-1} - \left(1 - \cos \left(\frac{2\pi j}{n}\right)\right) f \left(2r^2 \left(1 - \cos \left(\frac{2\pi j}{n}\right)\right)\right), \]
which shows that indeed
\[ m = \frac{a^2 - r''r^3}{r^4 \sum_{j=1}^{n-1} (1 - \cos \frac{2\pi j}{n}) f(2r^2(1 - \cos \frac{2\pi j}{n}))} \tag{7.3} \]
if the solution is as described in (1.3).

If $q_1, \ldots, q_n$ is a polygonal homographic orbit of the type described in (1.4), then the $q_i$, $i \in \{1, \ldots, n-1\}$ have exactly the properties of the $q_i$ in the proof of Theorem 1.2 with the restriction that there exist a function $\phi$ and constants $\alpha_1, \ldots, \alpha_{n-1} \in [0, 2\pi)$ such that $\theta_i = \phi + \alpha_i$, $i \in \{1, \ldots, n-1\}$. By Corollary 1.5, we have that the configuration of the point masses is a regular polygon, meaning that we may choose $\alpha_i = \frac{2\pi i}{n-1}$, which means that by (3.1) we have that
\[ \left(\frac{r'' - r(\theta'_i)^2}{2r \theta'_i + r \theta''_i}\right) + r M \begin{pmatrix} 1 \\ 0 \end{pmatrix} f(r^2) = \sum_{j=1}^{n-1} m \left(-\frac{1 - \cos \left(\frac{2\pi j}{n-1}\right)}{\sin \left(\frac{2\pi j}{n-1}\right)}\right) f \left(2r^2 \left(1 - \cos \left(\frac{2\pi j}{n-1}\right)\right)\right). \tag{7.4} \]

Repeating the calculation that leads from (7.1) to (7.3) with (7.1) replaced with (7.4) then gives that there exists a constant $b \in \mathbb{R}$ such that
\[ m = \frac{b^2 - (r'' + r M f(r))r^3}{r^4 \sum_{j=1}^{n-2} (1 - \cos \frac{2\pi j}{n-1}) f(2r^2(1 - \cos \frac{2\pi j}{n-1}))} \]
if the solution is as described in (1.4), where $M$ is the mass of the point mass that does not lie on the circle. This completes the proof.

8 Proof of Corollary 1.7

If $q_1, \ldots, q_n$ is a polygonal homographic solution as in (1.3), then we may write that $\theta_i = \phi + \alpha_i$, $i \in \{1, \ldots, n\}$, where $0 \leq \alpha_1 < \ldots < \alpha_n < 2\pi$ are
constants and $\phi$ is a twice continuously differentiable function. By (2.2) this means that
\[
\left( \frac{r'' - r(\phi')^2}{2r'\phi' + r\phi''} \right) = r \sum_{j=1, j \neq i}^{n} m_j \left( -\frac{(1 - \cos(\alpha_j - \alpha_i))}{\sin(\alpha_j - \alpha_i)} \right) f(2r^2(1 - \cos(\alpha_j - \alpha_i))), \ i \in \{1, \ldots, n\}. \tag{8.1}
\]

Additionally, by (1.1), using the cross product, where we interpret the $q_i$ as three dimensional vectors with their third components zero, we have that
\[
\sum_{i=1}^{n} m_i q_i \times \ddot{q}_i = \sum_{i=1}^{n} m_i q_i \times \sum_{j=1, j \neq i}^{n} m_j (q_j - q_i) f \left( \|q_j - q_i\|^2 \right)
\]
\[
= \sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} m_i m_j q_i \times (q_j - q_i) f \left( \|q_j - q_i\|^2 \right)
\]
\[
= \sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} m_i m_j q_i \times q_j f \left( \|q_j - q_i\|^2 \right),
\]
and as
\[
\sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} m_i m_j q_i \times q_j f \left( \|q_j - q_i\|^2 \right) = \sum_{j=1}^{n} \sum_{i=1, i \neq j}^{n} m_j m_i q_j \times q_i f \left( \|q_i - q_j\|^2 \right)
\]
\[
= -\sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} m_i m_j q_i \times q_j f \left( \|q_j - q_i\|^2 \right),
\]
this means that
\[
\sum_{i=1}^{n} m_i q_i \times \ddot{q}_i = 0. \tag{8.2}
\]

Let
\[
R(x) = \begin{pmatrix}
\cos x & -\sin x & 0 \\
\sin x & \cos x & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]
Then
\[ q_i \times \ddot{q}_i = rR(\phi + \alpha_i) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \times \left( (r'' - r(\phi')^2)R(\phi + \alpha_i) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + (2r\phi' + r\phi'')R(\phi + \alpha_i) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) \]

\[ = 0 + r(2r\phi' + r\phi'')R(\phi + \alpha_i) \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad i \in \{1, \ldots, n\}, \]

so by (8.2), we have that
\[ r(2r\phi' + r\phi'') = 0 \]
and consequently that
\[ \phi' = \frac{c}{r} \]
for a constant \( c \in \mathbb{R} \), which means by (8.1) that
\[
\left( r'' - \frac{c^2}{r} \right) = r \sum_{j=1, j \neq i}^{n} m_j \left( -\frac{(1 - \cos (\alpha_j - \alpha_i))}{\sin (\alpha_j - \alpha_i)} \right) f(2r^2(1 - \cos (\alpha_j - \alpha_i))), \quad i \in \{1, \ldots, n\}. \]

(8.3)

If we fix \( t \), then \( r(t), r''(t) \) and the right-hand side of (8.3) are constant. Writing \( A^2 = -\frac{1}{r} \left( r'' - \frac{c^2}{r} \right) \) (note that the right hand side of the first component of (8.3) is positive) then allows (8.3) to be written as
\[
r \left( A^2 \right) = r \sum_{j=1, j \neq i}^{n} m_j \left( \frac{1 - \cos (\alpha_j - \alpha_i))}{\sin (\alpha_j - \alpha_i)} \right) f(2r^2(1 - \cos (\alpha_j - \alpha_i))), \quad i \in \{1, \ldots, n\}. \]

(8.4)

In [35] it was proven that any polygonal relative equilibrium solution of (1.1) \( q_1, \ldots, q_n \), where
\[ q_i(t) = r \begin{pmatrix} \cos (At + \alpha_i) \\ \sin (At + \alpha_i) \end{pmatrix}, \quad i \in \{1, \ldots, n\}, \]
r > 0, \( A, \alpha_1, \ldots, \alpha_n \) constants, has to solve a system like (8.4) (see [35], (2.1)) and this was then used to prove that for each order of the masses at most one polygonal relative equilibrium solution exists. So for any fixed \( t \), we have that (8.4) has at most one solution. So by extension, we have that for any order of the masses at most one polygonal homographic solution of (1.1) exists.

If our polygonal homographic solution is of the form (1.4), then we repeat our argument that leads from (8.1) to (8.4) and start with (3.1) instead. This completes our proof.
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