Decomposition of Hyperbolic Semigroups

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Abstract - In this article, the Banach space is decomposed into the direct sum of two closed subspaces such that the semigroup becomes forward exponentially stable on one subspace and backward exponentially stable on another subspace. Hyperbolic semigroup is characterised in terms of the spectrum of its cogenerator. Further, we study the rescaled hyperbolic semigroup to analyse its spectrum.

Keywords - C0-semigroups, rescaled semigroups, restricted semigroups, cogenerator of semigroups, discrete

1. Introduction

The theory of Hyperbolicity is one of the basic sources in the study of partial differential equations. Hyperbolic semigroups are studied in the models of beams and waves as well as the transport equation and networks of non-homogeneous transmission lines. If \( (T(\xi))_{\xi \geq 0} \) is a hyperbolic semigroup, then every operator \( T(\xi) \) in the semigroup must itself be hyperbolic [1]. The literature on hyperbolic semigroups is very rich [2].

The classical results are presented in the books (5; 8; 14). Over the years, the notion of hyperbolicity was broadened (non-uniform hyperbolicity) [3] and relaxed (partial hyperbolicity) [4] to encompass a much larger class of systems. In [5], Herbert and Daniel obtain a complete characterization of Fredholm spectrum of the semigroup generated by sharp energy estimates [6]. The existence and uniqueness of conformal measures are briefly explained by [7]. [8] proved that the rate of convergence of the semigroup to the point one is exponential. Quasi hyperbolic semigroups [9] and hyperbolic dynamical systems [10] gives a path way to study the variations of hyperbolic semigroups. In contrast to the known result on this topic, this paper is motivated to study the decomposition of hyperbolic semigroups [11-15]. The aim of the present study is to bridge the gap explored from the above literature. Our results enable us to efficiently analyse the hyperbolic semigroup in terms of its spectrum of its cogenerator.

2. Preliminaries

Definition 2.1

A \( C_0 \)-semigroup \( T(\xi) \) on a Banach space \( X \) is called bounded if \( \sup_{\xi \geq 0} kT(\xi)k < \infty \).

We remark that if \( A \) be the generator of a bounded semigroup of \( T(\xi) \) then \( \sigma(A) \subset \{ z: \Re z \leq 0 \} \) but not conversely.

Definition 2.2

A \( C_0 \)-semigroup \( T(\xi) \) is called uniformly exponentially stable, if \( \exists \ M \geq 1 \) and \( \forall \ z > 0 \) such that \( kT(\xi)k \leq M e^{-z} \) for all \( \xi \geq 0 \).

Theorem 2.3

For \( C_0 \)-semigroup \( T(\xi) \) on a Banach space \( X \) the following assertions are equivalent:

(i) \( T(\xi) \) is uniformly exponentially stable.
(ii) \( \lim_{t \to \infty} kT(\xi)t = 0 \)
(iii) \( kT(\xi)k < 1 \) for some \( \xi > 0 \)
(iv) \( r(T_0) < 1 \) for some \( \xi > 0 \)
(v) \( r(T(\xi)) < 1 \) for all \( \xi > 0 \)

Definition 3.1

For a \( C_0 \)-semigroup \( T(\xi) \) on a Banach space \( X \) with generator \( A \) if the spectrum of \( T(\xi) \) is hyperbolic iff \( \sigma(T(\xi)) \cap \{ 0 \} = \phi \) for all \( \xi > 0 \).

Proof

(1) and (2) imply that \( \sigma(T(\xi)) \cap \{ 0 \} = \phi \) for all \( \xi > 0 \).

Combining Equations (1) and (2), we obtain \( \sigma(T(\xi)) \cap \{ 0 \} = \phi \) for all \( \xi > 0 \).

\( \Rightarrow \ r(T(\xi))^{-1} \leq e^{-z} \)

Since \( \sigma(T(\xi)) = \{ \lambda^{-1} : \lambda \in \sigma(T(\xi))^{-1} \} \) then \( |\lambda| \geq e^z \) for each \( \lambda \in \sigma(T(\xi)) \). So \( (T(\xi)) \cap aT = \phi \) for \( a \leq e^z \) (2)

Conversely, assume \( s > 0 \) such that \( \sigma(T(s)) \cap \{ 0 \} = \phi \) for all \( \xi > 0 \).

\( \Rightarrow \ r(T(s))^{-1} \leq e^{-z} \)

\( \Rightarrow \ r(T(s)) \geq e^z \)

\( \Rightarrow \ |\lambda| \leq e^z \) for each \( \lambda \in \sigma(T(\xi)) \). So \( (T(\xi)) \cap aT = \phi \) for \( a \leq e^z \) (2)

\( \Rightarrow \ r(T(s)) \cap \{ 0 \} = \phi \) for all \( \xi > 0 \).

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The operators \( P_\alpha = \frac{1}{2\alpha} \int_{|x|<\alpha} \frac{dx}{|x|} \) and \( P_s = I - P_\alpha \) define projection maps onto the corresponding subspaces 
\( X_s, X_\alpha \) and \( \sigma(T(s)/X_\alpha) = \sigma(T(s)) \cap \{ \eta \le e^{-s} \sigma(T(s)/X_\alpha) = \sigma(T(s)) \cap \eta |x| \ge e^{\alpha^s} \). Then the spectral radius of \((T(s)/X_\alpha)\) is given by \( r(T(s)/X_\alpha) \le e^{\alpha^s} \).

Then by theorem 2.3, for \( x \in X_s, \xi \ge 0, R M > 0 \) such that \( k(T(\xi)xk \le Me^{-\xi kx}. \) Now, \( T_d(s) \) of \( T(s) \) in \( X_s \) has the spectrum
\[ \sigma(T_d(s)) = \{ \lambda \in \sigma(T(s)) : |\lambda| \ge e^{\alpha^s} \} \]
and is so invertible on \( X_s \). This shows that \( (T_d(s)) \) is invertible for \( 0 < \xi < s \) and \( \lambda > s \).

Let \( n \in \mathbb{N} \) so that \( \lambda_1 < \xi \).

Then \( (T_{d}(s))^n = T_d(ns) = T_d(ns - \xi) \) and \( T_{d}(s) \) is invertible for \( \lambda > s \). Again, by theorem 2.3, \( k(T(\xi)xk \ge e^{\alpha^s} \)
\[ x \in X_s, \xi \ge 0. \]
Thus \( (T_{d}(\xi))^{\le 0} \) is hyperbolic.

**Theorem 3.3**

Let \( A \) be the generator and \( V \) be the co-generator of a \( C \) -
semigroup \((T_{d}(\xi))^{\le 0} \). Assume further \( \sigma(T_{d}(\xi)) \subset \Gamma \) holds. Using the identity \( V = (A - I + 2I(A - I)) = I - 2R(A) \) we get \( \sigma(V) = \{ \lambda \in \sigma(A) : 1 \} \)
\[ \lambda > 0. \]
Now \( \sigma(V) \subset \sigma(A) \) equivalent to \( \frac{1}{\lambda - \alpha} \in \sigma(V) \). For every real \( r \). This means \( \sigma(V) \subset \sigma(A) \).

4. **Rescaled Hyperbolic Semigroup**

If \( (T_{d}(\xi))^{\le 0} \) is a hyperbolic semigroup, it can be observed that the rescaled semigroup operator \( e^{r T_{d}(\xi)} \) has no spectrum in the annulus \( e^{(r - \alpha^s) \xi} \le |\xi| \le e^{(r + \alpha^s) \xi} \)
for every \( \alpha \in \mathbb{R} \).

**Definition 4.1**

For \( 0 < \lambda < \mu \). We call \( (T_{d}(\xi))^{\le 0} \) is \( (\lambda, \mu) \) hyperbolic whenever the rescaled operator \( \lambda \beta T_{d}(\xi) \) is hyperbolic for every \( \alpha \in \lambda, \mu \).

If \( 0 < \lambda < 1 \) and \( \mu > 1 \) then \( (\lambda, \mu) \) hyperbolic becomes a hyperbolic semigroup.

**Theorem 4.2**

A co-semigroup \((T_{d}(\xi))^{\le 0} \) is \( (\lambda, \mu) \) hyperbolic iff
\[ \sigma(T_{d}(\xi)) \cap \{ \lambda \le |\xi| \le \mu \} = \emptyset \forall \xi > 0. \]

**Proof**

\( (T_{d}(\xi))^{\le 0} \) is \( (\lambda, \mu) \) hyperbolic iff \( \lambda \beta T_{d}(\xi) \) is hyperbolic for every \( \alpha \in \lambda, \mu \).
\[ \sigma(T_{d}(\xi)) \cap \{ \lambda \beta e^{-\lambda \xi} \le |\xi| \le \lambda \beta e^{\lambda \xi} \} = \emptyset \forall \xi > 0. \]
Since \( (T_{d}(\xi))^{\le 0} \) is hyperbolic if and only if \( T(1) \) is hyperbolic, we have \( \sigma(T_{d}(\xi)) \cap \alpha \Gamma = \emptyset \) for every \( \alpha \in \lambda, \mu \). So, \( \sigma(T_{d}(\xi)) \)
cannot have any points in common with \( \{ \lambda \le |\xi| \le \mu \} \) as desired.

**Theorem 4.3**

If \( \sigma(T_{d}(\xi)) \subset \alpha \Gamma \) holds. Using the identity \( V = (A - I + 2I(A - I)) = I - 2R(A) \) we get \( \sigma(V) = \{ \lambda \in \sigma(A) : 1 \} \)
\[ \lambda > 0. \]
Now \( \sigma(V) \subset \sigma(A) \) equivalent to \( \frac{1}{\lambda - \alpha} \in \sigma(V) \). For every real \( r \). This means \( \sigma(V) \subset \sigma(A) \).

5. **Conclusion**

The Banach space is decomposed into the direct sum of two closed subspaces. This leads to characterise the hyperbolic semigroup in terms of the spectrum of its cogenerator. Also rescaled hyperbolic semigroup is developed to analyse its spectrum. By analyzing the spectrum, one can study the equality of spectral and growth bound for strongly continuous semigroup which have extensive applications in quantum theory and stochastic processes.

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