SURJECTIVITY FOR HAMILTONIAN LOOP GROUP SPACES

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Abstract. Let $G$ be a compact Lie group, and let $LG$ denote the corresponding loop group. Let $(X, \omega)$ be a weakly symplectic Banach manifold. Consider a Hamiltonian action of $LG$ on $(X, \omega)$, and assume that the moment map $\mu : X \to Lg^*$ is proper. We consider the function $|\mu|^2 : X \to \mathbb{R}$, and use a version of Morse theory to show that the inclusion map $j : \mu^{-1}(0) \to X$ induces a surjection $j^* : H_G^*(X) \to H_G^*(\mu^{-1}(0))$, in analogy with Kirwan’s surjectivity theorem in the finite-dimensional case. We also prove a version of this surjectivity theorem for quasi-Hamiltonian $G$-spaces.

1. Introduction

Let $G$ be a compact Lie group. A Hamiltonian $G$ space is a triple $(M, \omega, \mu)$, where $(M, \omega)$ is a symplectic manifold, and $\mu : M \to g^*$ is a moment map for a Hamiltonian action of $G$ on $M$.

If 0 is a regular value of $\mu$, then the symplectic quotient $M//G = \mu^{-1}(0)/G$ is a symplectic orbifold. In this case, $G$ acts locally freely on $\mu^{-1}(0)$, so the equivariant cohomology ring $H_G^*(\mu^{-1}(0))$ coincides with the cohomology ring $H^*(M//G)$. Therefore, the restriction map from $H_G^*(M)$ to $H_G^*(\mu^{-1}(0))$ induces a map $\kappa : H_G^*(M) \to H^*(M//G)$, which we call the Kirwan map.

A version of Morse theory due to Kirwan is an important tool in the study of the topology of Hamiltonian $G$-spaces and their symplectic quotients. Kirwan shows that for a compact Hamiltonian $G$-space $(M, \omega, \mu)$, the function $|\mu|^2 : M \to \mathbb{R}$ is an equivariantly perfect Morse function on $M$. This fact has the following striking consequence.

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1In this paper, we consider only cohomology groups with rational coefficients.
The main purpose of this paper is to generalize Theorem 1 to the case of Hamiltonian loop group actions on symplectic Banach manifolds with proper moment map.

Let $G$ be a compact Lie group, and let $\langle \cdot, \cdot \rangle$ be an invariant inner product on the Lie algebra $\mathfrak{g}$. Let $L_sG$ be the space of maps from $S^1$ to $G$ of Sobolev class $s > 1/2$; $L_sG$ is a Banach Lie group with the group structure given by pointwise multiplication. The Lie algebra $L_s\mathfrak{g}$ is given by the space $\Omega^0_s(S^1) \otimes \mathfrak{g}$ of maps from $S^1$ to $\mathfrak{g}$ of Sobolev class $s$. We define $L_s\mathfrak{g}^*$ to be the space $\Omega^{1}_{s-1}(S^1) \otimes \mathfrak{g}$ of $\mathfrak{g}$-valued one forms on $S^1$ of Sobolev class $s - 1$. Integration gives a natural non-degenerate pairing $\langle \cdot, \cdot \rangle$ of $L_s\mathfrak{g}$ with $L_s\mathfrak{g}^*$. Using this inner product, the space $L_s\mathfrak{g}^*$ can be identified with the space of connections on the trivialized principal $G$-bundle $G \times S^1 \to S^1$. This identification induces an action of the group $L_sG$ on $L_s\mathfrak{g}^*$ given by

$$\lambda \cdot \gamma = \lambda^{-1} d\lambda + \lambda^{-1} \gamma \lambda$$

where $\gamma \in L_s\mathfrak{g}^*$ and $\lambda \in L_sG$.

A Banach manifold $X$ is weakly symplectic if is is equipped with a closed two-form $\omega \in \Omega^2(X)$ such that the induced map $\omega^\flat: T_pX \to T_p^*X$ is injective. An action of the group $L_sG$ on $(X, \omega)$ is Hamiltonian if there exists an $L_sG$-equivariant moment map $\mu: X \to L_s\mathfrak{g}^*$ so that $\langle d\mu, \xi \rangle = i_{\xi} \omega$ for all $\xi \in L_s\mathfrak{g}$. We call the triple $(X, \omega, \mu)$ a Hamiltonian $L_sG$ space.

Now consider a Hamiltonian $L_sG$ space $(X, \omega, \mu)$ with proper moment map. In this case, if $0$ is a regular value of $\mu$, the symplectic quotient $X/\!/L_sG = \mu^{-1}(0)/G$ is a finite dimensional symplectic orbifold. Since $G$ acts locally freely on $\mu^{-1}(0)$, the equivariant cohomology ring $H^*_G(\mu^{-1}(0))$ coincides with the cohomology ring $H^*(X/\!/L_sG)$. Therefore, the restriction map $H^*_G(X) \to H^*_G(\mu^{-1}(0))$ induces a map

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2 This action is not the coadjoint action of $L_sG$, but arises instead from the coadjoint action of a central extension of $L_sG$. As a consequence it is an affine action rather than a linear action on the vector space $L_s\mathfrak{g}^*$.

3 Let a Lie group $G$ act on a symplectic manifold $(M, \omega)$, and let $\mu: M \to \mathfrak{g}^*$ satisfy $\langle d\mu, \xi \rangle = i_{\xi} \omega$ for all $\xi \in \mathfrak{g}$. If $G$ is compact, then one can choose $\mu$ to be equivariant with respect to the coadjoint action of $G$ on $\mathfrak{g}^*$. In general, however, this is not possible, and it is natural to consider other actions of $G$ on $\mathfrak{g}^*$, such as the action arising from a central extension in $(1.1)$. 

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\(\kappa: H^*_G(X) \rightarrow H^*(X//L_sG)\) which is the infinite-dimensional analog of the Kirwan map.

In this paper we consider the function \(|\mu|^2: X \rightarrow \mathbb{R}\), and show that it can be treated as an equivariantly perfect Morse function. We thus prove the following analog of Kirwan’s surjectivity theorem:

**Theorem 2** (Surjectivity for Hamiltonian loop group actions). Let \(G\) be a compact Lie group and choose an invariant inner product on \(\mathfrak{g}\). Let \(L_sG\) be the corresponding loop group, where \(s > 1/2\). Let \((X, \omega, \mu)\) be a Hamiltonian \(L_sG\)-space with proper moment map. Then the restriction \(H^*_G(X) \rightarrow H^*_G(\mu^{-1}(0))\) is surjective. In particular, if \(0\) is a regular value of \(\mu\), then the Kirwan map \(\kappa: H^*_G(X) \rightarrow H^*(X//L_sG)\) is surjective.

### 1.1. Proof of Theorem 1

We begin by reviewing Kirwan’s proof of surjectivity in the finite-dimensional case. Let \(G\) be a compact Lie group, and let \((M, \omega, \mu)\) be a Hamiltonian \(G\)-space with proper moment map. Consider the function \(f = |\mu|^2: M \rightarrow \mathbb{R}\). Since \(\mu^{-1}(0) = f^{-1}(0)\), it is enough to prove that the restriction \(H^*_G(M) \rightarrow H^*_G(f^{-1}(0))\) is surjective.

The first step in Kirwan’s argument is to show that \(f\) is **Morse in the sense of Kirwan** (see Definition \[9.1\]). As a result, for every component \(C\) of the critical set of \(f\) there exists a vector bundle \(E_C\) over \(C\), called the **negative normal bundle at \(C\)** (see Definition \[9.2\]). For sufficiently small \(\epsilon > 0\), let \(M_\pm = f^{-1}(-\infty, f(C) \pm \epsilon)\). Consider the long exact sequence of the pair \((M_+, M_-)\) in equivariant cohomology. By Proposition \[9.4\], we obtain a commuting diagram

\[
\begin{array}{ccc}
\cdots & \rightarrow & H^*_G(M_+, M_-) & \rightarrow & H^*_G(M_+) & \rightarrow & H^*_G(M_-) & \rightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
H^*_C - \lambda_C (C) & \rightarrow & H^*_G(C) & & & & & & \\
\end{array}
\]

where \(\lambda_C\) denotes the index of \(C\) and \(e_C \in H^*_G(C)\) denotes the equivariant Euler class of the negative normal bundle at \(C\).

Kirwan now applies the completion principle of Atiyah and Bott \[AB\] to show that \(f\) is an equivariantly perfect Morse function. The key idea is to show that the Euler classes \(e_C \in H^*_G(C)\) are not zero divisors. By the diagram above, this implies that the long exact sequence in equivariant relative cohomology splits into short exact sequences

\[
\begin{align*}
0 & \rightarrow H^*_G(M_+, M_-) \rightarrow H^*_G(M_+) \rightarrow H^*_G(M_-) \rightarrow 0.
\end{align*}
\]
Since $f$ is non-negative, Theorem 1 follows by induction on the critical levels.

To show that $e_C$ is not a zero divisor, Kirwan shows that for each component $C$ of the critical set there exists a subtorus $T \subset G$ and a $Z(T)$-invariant subset $B \subset C^T$

$G \times_{Z(T)} B \to C$

is an equivariant homeomorphism. Hence, we have a natural isomorphism $H^*_G(C) \simeq H^*_{Z(T)}(B)$; this isomorphism takes the $G$-equivariant Euler class of $E_C$ to the $Z(T)$-equivariant Euler class of $E_C^B$. Moreover, $(E_C)^T$ is a subset of the zero section of $E_C^-$. The fact that $e_C$ is not a zero divisor now follows from the Lemma below.

**Lemma 1.4** (Atiyah-Bott). Let $V$ be a complex vector bundle over a connected space $Y$, and let a compact Lie group $K$ act on $V$. If there exists a subtorus $T \subset K$ so that the fixed point set $V^T$ is the zero section of $V$, then the Euler class $e(V) \in H^*_K(Y)$ is not a zero divisor.

1.2. **Proof of Theorem 2.** The basic idea of this proof is straightforward: Morally, we would like to consider the function $f = |\mu|^2$ as a Morse function on $X$, and follow the proof of Kirwan’s surjectivity theorem for compact Hamiltonian $G$-spaces which we recalled in Section 1.2 above. The main technical hurdle we must face is that Kirwan’s extension of Morse theory has only been developed in the finite dimensional case. However, in the case where $X = L_sG/G$ is the Hamiltonian $L_sG$-space given in Example 1.5 below, the function $f$ is precisely the classical energy function of Morse and Bott [B]. In the spirit of [B], we will replace $X$ in the general case by a sequence of finite dimensional approximations.

First, in Section 2, we construct an **infinite approximating space** $\hat{X}$, along with an **energy function** $\hat{f}: \hat{X} \to \mathbb{R}$, and prove:

**Proposition 1** (Proposition 3.1). Let $G$ be a compact Lie group, and $L_sG$ be the corresponding loop group, where $s > 1/2$. Let $(X, \omega, \mu)$ be a Hamiltonian $L_sG$ space with proper moment map. Let $\hat{X}$ be the associated infinite approximating space, and let $\hat{f}: \hat{X} \to \mathbb{R}$ be the energy function.

The restriction map $H^*_G(X) \to H^*_G(f^{-1}(0))$ is surjective if and only if the restriction map $H^*_G(\hat{X}) \to H^*_G(\hat{f}^{-1}(0))$ is surjective.

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4In this paper, given a subgroup $K \subset G$, $Z(K)$ denotes the centralizer of $K$ in $G$. If $G$ acts on a space $X$, then $X^K$ denotes the set of points in $X$ fixed by $K$. 
Next, in Section 3, we construct a series of finite approximating spaces $Y_n$, each of which has an energy function $f_n: Y_n \to \mathbb{R}$. We then prove:

**Proposition 2** (Proposition 4.1). Let $(X, \omega, \mu)$ be a Hamiltonian $L_s G$ space with proper moment map. Let $\tilde{X}$ be the associated infinite approximating space and let $\tilde{f}: \tilde{X} \to \mathbb{R}$ be the energy function. Let $Y_n$ be the finite approximating manifolds, and let $f_n: Y_n \to \mathbb{R}$ be the energy functions.

If the restriction maps $H^*_G(Y_n) \to H^*_G(f^{-1}_n(0))$ are all surjections, then the restriction map $H^*_G(\tilde{X}) \to H^*_G(\tilde{f}^{-1}(0))$ is a surjection.

Since $f^{-1}(0) = \mu^{-1}(0)$, in order to prove Theorem 2, it is enough to show that the restriction $H^*_G(Y_n) \to H^*_G(f^{-1}_n(0))$ is surjective. Unfortunately the manifolds $Y_n$ are not symplectic, so that we cannot directly apply Kirwan’s results. Instead, in Sections 4 through 8 of this paper we work through the local calculations needed to prove the two results below.

**Proposition 3** (Proposition 8.1). The functions $f_n: Y_n \to \mathbb{R}$ are Morse in the sense of Kirwan.

**Proposition 4** (Proposition 8.2). Let $C$ be a component of the critical set of $f_n$ and let $E_C^-$ be the negative normal bundle at $C$. Then there exists a subtorus $T \subset G$ and a $Z(T)$ invariant subset $B \subset C^T$ so that the natural map $G \times_{Z(T)} B \to C$ is an equivariant homeomorphism. Moreover, $(E_C^-)^{T}$ is a subset of the zero section of $E_C^-$. We can now follow Kirwan’s proof. Proposition 8.1 shows that for each component of the critical set of $f_n$ we have a commuting diagram analogous to (1.2). As in Kirwan’s argument, Proposition 8.2 now shows that the equivariant Euler class of the negative normal bundle at each component of the critical set is not a zero divisor. Hence the long exact sequence in relative equivariant cohomology breaks up into short exact sequences as in (1.3). Since each $f_n$ is non-negative, it follows by induction that the restriction maps $H^*_G(Y_n) \to H^*_G(f^{-1}_n(0))$ are surjections.

1.3. **Examples.** The following are examples of Hamiltonian $L_s G$ spaces \[AMM\].

**Example 1.5.** [The minimal coadjoint orbit] Consider $X = L_s G \cdot 0$, the $L_s G$-orbit of the point $0 \in L_s g^*$, equipped with the Kirillov-Kostant
symplectic structure \([PS]\). The inclusion \(X \to Lg^*\) is a proper moment map for the natural \(LG\) action. As \(\text{Stab}(0) = G\), this space may be identified with the quotient \(LG/G\). In terms of the identification of \(X\) with \(LG/G\), the symplectic form arises from the \(G\)-invariant two-form \(\tilde{\omega} \in \Omega^2(LG)\) given by

\[
\omega|_{\lambda}(\xi, \eta) = \frac{1}{2\pi} \int_0^{2\pi} \langle \xi(t), \eta'(t) \rangle dt,
\]

where \(\xi, \eta \in TLG\) (see \([PS]\), p. 147.). The moment map for the \(LG\) action is the map \(\mu: LG/G \to Lg^*\) arising from the \(G\)-invariant map \(\tilde{\mu}: LG \to Lg^*\) defined by \(\tilde{\mu}(\lambda) = \lambda^{-1}d\lambda\). The manifold \(LG/G\) is diffeomorphic to the subgroup \(\Omega G = \{\lambda \in LG \mid \lambda(0) = e\}\) of based loops in \(LG\). In terms of this identification, the square of the moment map \(E = |\mu|^2\) is the energy functional \(E(\lambda) = \frac{1}{2\pi} \int_0^{2\pi} |\lambda^{-1}d\lambda|^2dt\), studied by Morse and one of the present authors (R.B.). The perfection of \(E\) as a Morse function plays an important role in the proof of the Periodicity Theorem \([B]\).

**Example 1.6.** [The generic coadjoint orbit] Consider next a generic point \(\xi \in Lg^*\) and the corresponding coadjoint orbit \(X = LG \cdot \xi\). We may as well take \(\xi \in t^*\). Then \(\text{Stab}(\xi) = T\), so that this space may be identified with \(LG/T\). It is therefore diffeomorphic to the space of paths in \(G\) beginning at \(e\) and ending at a point of the conjugacy class of \(\exp \xi\). Again, this is a Hamiltonian \(LG\) space with the Kirillov-Kostant symplectic form and with moment map given by the inclusion \(X \to Lg^*\).

**Example 1.7.** [Spaces of connections on 2-manifolds]

Let \(\Sigma\) be a compact, connected 2-manifold of genus \(h\) with boundary \(\partial \Sigma = S^1\). Consider the space \(A(\Sigma) = \Omega^1(\Sigma, g)\) of connections on the trivialized principal \(G\)-bundle \(G \times \Sigma \to \Sigma\). The space \(A(\Sigma)\) is a symplectic manifold, equipped with a Hamiltonian action of the gauge group \(G(\Sigma) = Map(\Sigma, G)\). The moment map \(J: A(\Sigma) \to \text{Lie}(G(\Sigma))^* = \Omega^2(\Sigma, g)\) is given by \(J(A) = F_A\), where \(F_A\) is the curvature of the connection \(A\). The group \(G(\Sigma)\) has a normal subgroup \(\tilde{G}(\Sigma, \partial \Sigma)\) defined by \(\tilde{G}(\Sigma, \partial \Sigma) = \{\gamma \in G(\Sigma) \mid \gamma|_{\partial \Sigma} = e\}\), and \(G(\Sigma)/\tilde{G}(\Sigma, \partial \Sigma) = LG\). Consider the reduced space \(A(\Sigma)/\tilde{G}(\Sigma, \partial \Sigma)\). This is a Hamiltonian \(LG\)-space with proper moment map \(\mu\). The reduced space \(X//LG\) is the moduli space of flat connections on the trivial principal \(G\)-bundle \(G \times \Sigma \to \Sigma\).

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2. Surjectivity for quasi-Hamiltonian $G$-spaces

In this section we will review Alexe’ev, Malkin, and Meinrenken’s \[\text{AMM}\] definition of quasi-Hamiltonian $G$-spaces, together with their proof that these spaces are in one-to-one correspondence with Hamiltonian $L_sG$-spaces with proper moment map. We then restate our surjectivity theorem in terms of quasi-Hamiltonian $G$-spaces.

We begin by motivating the definition of quasi-Hamiltonian $G$-spaces. Let $(X,\omega,\mu)$ be a Hamiltonian $L_sG$-space with proper moment map. The normal subgroup $\Omega_sG \subset L_sG$ given by the based maps in $L_sG$

$$\Omega_sG = \{ \lambda \in L_sG \mid \lambda(0) = e \}$$

acts freely on $L_s\mathfrak{g}^*$. Since $\mu$ is $L_sG$-equivariant, $\Omega_sG$ acts freely on $X$; thus $X$ is a the total space of a principal $\Omega_sG$ bundle $\pi: X \rightarrow M$. Since $\mu$ is also proper the quotient $M = X/\Omega_sG$ is a finite-dimensional compact manifold \[\text{AMM}\]. The action of $L_sG$ on $X$ induces an action of $L_sG/\Omega_sG = G$ on $M$. Furthermore, if we identify the space $L_s\mathfrak{g}^*$ with a space of connections on the trivial principal $G$-bundle on the circle, the map $\text{Hol}: L_s\mathfrak{g}^* \rightarrow G$ given by the value at time 1 of the holonomy of the connection is a fibration with fibre $\Omega_sG$. Using this fibration, the $L_sG$–equivariant moment map $\mu: X \rightarrow L_s\mathfrak{g}^*$ induces a $G$–equivariant map $\Phi: M \rightarrow G$, which intertwines the $G$ action on $M$ with the adjoint $G$ action on $G$. We thus obtain a commuting square

\[
\begin{array}{ccc}
X & \xrightarrow{\mu} & L_s\mathfrak{g}^* \\
\downarrow{\pi} & & \downarrow{\text{Hol}} \\
M & \xrightarrow{\Phi} & G
\end{array}
\]

showing that the principal $\Omega_sG$-bundle $\pi: X \rightarrow M$ is the pullback under $\Phi$ of the contractible principal $\Omega_sG$-bundle $\text{Hol}: L_s\mathfrak{g}^* \rightarrow G$.

Alexe’ev, Malkin and Meinrenken \[\text{AMM}\] give a set of conditions a $G$-space $M$ must satisfy in order to arise from a Hamiltonian $L_sG$ space by this construction. Given a compact Lie group $G$, choose an invariant inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$. Let $\theta_L$ and $\theta_R$ denote the left and right invariant Maurer-Cartan forms, respectively. and let $\xi = \frac{1}{12}[\theta_L,[\theta_L,\theta_L]] = \frac{1}{12}[\theta_R,[\theta_R,\theta_R]] \in \Omega^3(G)$ denote the bi-invariant $3$-form; if $G$ is simple, $\xi$ generates $H^3(G)$. We consider $G$ as a $G$-space by equipping it with the adjoint action. A quasi-Hamiltonian $G$-space (or $q$-Hamiltonian $G$-space) $(M,\sigma,\Phi)$ is a compact $G$–manifold $M$, along with a $G$-equivariant map $\Phi: M \rightarrow G$, and a two-form $\sigma \in \Omega^2(M)$, satisfying
(1) $\Phi^*\xi = d\sigma$
(2) $i_M^*\sigma = \left(\frac{1}{2}\Phi^*(H_L + H_R), \eta\right)$ for all $\eta \in \mathfrak{g}$.
(3) $\ker\sigma|_p = \{e_p^M | e \in \ker(Ad_{\Phi(p)} + 1)\}$ for all $p \in M$.

According to [AMM], Theorem 8.3, there is a one-to-one correspondence between Hamiltonian $L_sG$-spaces $(X,\omega,\mu)$ with proper moment map and quasi-Hamiltonian $G$-spaces $(M,\sigma,\Phi)$: the quasi-Hamiltonian $G$-space $M$ associated to a Hamiltonian $G$-space $X$ is given by $X/\Omega_sG$, the moment map $\Phi$ is the map induced on $M$ by $\mu$, and the two-form $\sigma$ is given by formula (31) in [AMM]. Given a Hamiltonian $L_sG$-space $X$ with proper moment map, the symplectic quotient $X//L_sG$ is given in terms of the corresponding quasi-Hamiltonian $G$-space $M$ by $X//L_sG = M//G := \Phi^{-1}(e)/G$.

**Example 2.2.** The quasi-Hamiltonian $G$-space corresponding to $X = LG/G$ in Example 1.4 is a point. The quasi-Hamiltonian $G$-space corresponding to $X = LG/T$ in Example 1.3 is the conjugacy class $C_{\exp\xi}$. In both cases, the $G$-valued moment map is given by inclusion. The quasi-Hamiltonian $G$-space corresponding to Example 1.7 is $M = G^{2h}$, equipped with the moment map $\Phi: M \rightarrow G$ given by $\Phi(a_1, \ldots, a_h, b_1, \ldots, b_h) = \prod_{i=1}^{2h}[a_i, b_i]$ (see [AMM]).

We are now ready to state a version of our main theorem for quasi-Hamiltonian $G$-spaces. For any $G$-space $X$, let $C^*_G(X) = C^*(X_G) = C^*(X \times_G EG)$ denote the singular cochain complex. Consider the fibration $p: G \times G EG \rightarrow BG$, and let $j: BG \rightarrow G \times G EG$ denote the inclusion of $BG$ as $\{e\} \times G EG$. Because the action of $G$ on itself is equivariantly formal (see [GS2], p. 186), we can find closed co-chains $b_i \in C^*_G(G)$ whose restrictions to the fiber $G$ generate the cohomology of $G$ as a ring. By replacing each $b_i$, if necessary, by $b_i - p^*(j^*(b_i))$, we may assume that $j^*(b_i) = 0$. If there exist $a_i \in C^*_G(M)$ so that $\Phi^*(b_i) = da_i$ then $a_i|_{\Phi^{-1}(e)}$ is closed.

**Theorem 3** (Surjectivity for quasi-Hamiltonian $G$-spaces). Let $G$ be a compact simply connected Lie group. Let $b_i \in C^*_G(G)$ satisfy $j^*(b_i) = 0$ and generate the cohomology of the fiber $G$ of the fibration $p$ as a ring.

Let $(M,\sigma,\Phi)$ be a quasi-Hamiltonian $G$-space. Assume that there exist co-chains $a_i \in C^*_G(M)$ such that $da_i = \Phi^*(b_i)$.

Then $H^*_G(\Phi^{-1}(e))$ is generated as a ring by the image of the restriction $H^*_G(M) \rightarrow H^*_G(\Phi^{-1}(e))$ along with the classes $[a_i|_{\Phi^{-1}(e)}]$.

In particular, if $e$ is a regular value of $\Phi$, then $H^*(M//G)$ is generated as a ring by the image of the Kirwan map $\kappa$ along with the classes $[a_i|_{\Phi^{-1}(e)}]$. 
Remark 2.3. The additional assumption made in our theorem, that the pull-back $\Phi^*(b_i)$ is exact for all $i$, is satisfied for the quasi-Hamiltonian $G$-spaces given in Examples 1.3, 1.6, and 1.7. We do not know of a proof that this condition follows in general from the definition of a quasi-Hamiltonian $G$-space. However, in the case where $G = SU(2)$, this is the case, and we have the following corollary.

Corollary 2.4. Let $(M, \sigma, \Phi)$ be a quasi-Hamiltonian $SU(2)$-space.

Then $H_{SU(2)}^*(\Phi^{-1}(e))$ is generated as a ring by the image of the restriction $H_{SU(2)}^*(M) \rightarrow H_{SU(2)}^*(\Phi^{-1}(e))$ along with the class $[\sigma_{\Phi^{-1}(e)}]$. In particular, if $e$ is a regular value of $\Phi$, then $H^*(M//G)$ is generated as a ring by the image of $\kappa$ and the class of the symplectic form.

Proof. In the Cartan model, the equivariant cohomology of a $G$-manifold $M$ is given by the cohomology of the complex $((\Omega^*(M) \otimes S(\mathfrak{g}^*))^G, \delta = d + i_{\eta^M})$, where $S(\mathfrak{g}^*)$ is the algebra of symmetric polynomials on $\mathfrak{g}$, and $\eta^M$ denotes the vector field on $M$ corresponding to $\eta$ for each $\eta \in \mathfrak{g}$; see e.g. [GS2]. If $G$ is a simple group, the generator of $H^3_G(G)$ is given by the differential form $\xi + \frac{1}{2} \Phi^*(\theta_L + \theta_R)$. Conditions (1) and (2) in the definition of a quasi-Hamiltonian $G$-space above state that $\delta \sigma = \Phi^*(\xi + \frac{1}{2} \Phi^*(\theta_L + \theta_R))$. We now apply Theorem 3.

Proof of Theorem 3. Consider the fibration $q: L_s \mathfrak{g}^* \times_G EG \rightarrow BG$, and let $i: BG \rightarrow L_s \mathfrak{g}^* \times_G EG$ denote the inclusion of $BG$ as $\{0\} \times_G EG$. Since $\text{Hol} \circ i = j$, $i^*(\text{Hol}^*(b_i)) = j^*(b_i) = 0$. Since $i$ is a homotopy equivalence, there exist $c_i \in C_G^*(L_s \mathfrak{g}^*)$ such that $dc_i = \text{Hol}^*(b_i)$. By replacing each $c_i$, if necessary, by $c_i - q^*(i^*(c_i))$, we may assume that $i^*(c_i) = 0$.

Fix a point $x \in BG$, and let $\gamma_i \in C^*(L_s \mathfrak{g}^*)$ and $\beta_i \in C^*(G)$ denote the restrictions of $c_i$ and $b_i$ to a fiber of $q^{-1}(x)$ and $p^{-1}(x)$, respectively. We now consider the principal $\Omega_s G$ bundle $\text{Hol}: L_s \mathfrak{g}^* \rightarrow G$. Then $d\gamma_i = \text{Hol}^*(\beta_i)$. Since the space $L_s \mathfrak{g}^*$ is contractible and $H^*(G)$ is generated by the cohomology classes $\beta_i$, which are classes of odd degree, a spectral sequence argument shows that the restriction of $\gamma_i$ to each fiber generates the cohomology $H^*(\Omega_s G)$ as a ring. Hence, the restriction of $c_i$ to each fiber generates the cohomology $H^*(\Omega_s G)$ as a ring.

Now note that $e_i = \mu^*(c_i) - \pi^*(a_i)$ are closed cochains on $X \times_G EG$ because $d(e_i) = \mu^*(dc_i) - \pi^*(da_i) = \mu^*(\text{Hol}^*(b_i)) - \pi^*(\Phi^*(b_i)) = 0$. Moreover, the restrictions of $e_i$ to each fiber of the fibration $X \times G EG \rightarrow M \times_G EG$ generate the cohomology of the fibre $\Omega_s G$ as a ring. By the Leray-Hirsch theorem, the cohomology $H^*_G(X)$ is then generated.
as a ring by the $e_i$ and the pull-backs of classes on $H^*_G(M)$ under $\pi$. Therefore, the image of the restriction map $H^*_G(X) \to H^*_G(\mu^{-1}(0))$ is also generated as a ring by the images of these classes. This image is just the image of the restriction map $H^*_G(X) \to H^*_G(\Phi^{-1}(e))$, along with the restrictions $e_i|_{\Phi^{-1}(e)}$ (recall that $\Phi^{-1}(e) = \mu^{-1}(0)$). To identify those restrictions, note that $\pi^*(a_i)|_{\mu^{-1}(0)} = a_i|_{\Phi^{-1}(e)}$. On the other hand, $\mu^*(c_j)|_{\mu^{-1}(0)} = 0$ since the restriction $i^*c_j$ vanishes. Thus $e_i|_{\mu^{-1}(0)} = a_i|_{\Phi^{-1}(e)}$.

By Theorem 2, the restriction map $H^*_G(X) \to H^*_G(\mu^{-1}(0))$ is surjective. It follows that $H^*_G(\Phi^{-1}(e))$ is generated as a ring by the image of $H^*_G(M)$ under the inclusion map $k: \Phi^{-1}(e) \to M$, along with the classes $a_i|_{\Phi^{-1}(e)}$. □

3. Path Spaces

In this section we show that in order to prove surjectivity for a Hamiltonian $L_sG$-space $X$, it is enough to prove surjectivity for an associated infinite-dimensional space $\hat{X}$. The spaces $X$ and $\hat{X}$ are related in the same way as the space of paths of Sobolev class $s > \frac{1}{2}$ is related to the space of piecewise smooth paths. The rest of the paper will be devoted to proving surjectivity for the space $\hat{X}$.

We first construct this space $\hat{X}$.

Let $\hat{P}_eG$ be the space of piecewise smooth based paths on $G$; that is, piecewise smooth maps $\lambda: [0, 1] \to G$ so that $\lambda(0) = e$, where $e \in G$ is the identity. We define a metric $\delta$ on $\hat{P}_eG$, following [B]: given paths $\lambda, \lambda' \in \hat{P}_eG$,

$$\delta(\lambda, \lambda') = \max_{t \in [0, 1]} d(\lambda(t), \lambda'(t)) + |J(\lambda) - J(\lambda')|,$$

where $d$ is an invariant metric on $G$ and $J$ is the length function on $\hat{P}_eG$. The group $G$ acts on $\hat{P}_eG$ by $(g \cdot \lambda)(t) = g^{-1}\lambda(t)g$. Let $\hat{\rho}: \hat{P}_eG \to G$ be the endpoint map.

Let $(X, \omega, \mu)$ be a Hamiltonian $L_sG$-space with a proper moment map. Let $(M, \sigma, \Phi)$ be the associated quasi-Hamiltonian $G$-space. Note that $X$ can be reconstructed from $\Phi: M \to G$ since

$$X = \{ (\gamma, m) \in L_s\mathfrak{g}^* \times M \mid \Phi(m) = \text{Hol}(\gamma) \}.$$ 

We introduce a new space $\hat{X}$, which we call the **infinite approximating space** associated to the Hamiltonian-$L_sG$ space $(X, \omega, \mu)$, given by

$$\hat{X} := \{ (\lambda, m) \in \hat{P}_eG \times M \mid \Phi(m) = \hat{\rho}(\lambda) \}.$$
The space $\hat{X}$ comes equipped with the diagonal $G$ action. Define the energy function $\hat{f}: \hat{X} \to \mathbb{R}$ by

$$\hat{f}(\lambda, m) = \int_{[0,1]} \left| \lambda^{-1} \frac{d\lambda}{dt} \right|^2 dt.$$ 

The main result of this section is the following.

**Proposition 3.1.** Let $G$ be a compact Lie group, and $L_{s}G$ be the corresponding loop group, where $s > 1/2$. Let $(X, \omega, \mu)$ be a Hamiltonian $L_{s}G$-space with proper moment map. Let $\hat{X}$ be the associated infinite approximating space, and let $\hat{f}: \hat{X} \to \mathbb{R}$ be the energy function.

The restriction map $H^*_G(X) \to H^*_G(f^{-1}(0))$ is surjective if and only if the restriction map $H^*_G(\hat{X}) \to H^*_G(\hat{f}^{-1}(0))$ is surjective.

Let $P_{e}G^*$ be the space of continuous maps from $[0,1]$ to $G$, endowed with the uniform topology. The group $G$ acts on $P_{e}G^*$ by $(g \cdot \lambda)(t) = g^{-1}\lambda(t)g$. Let $\rho^*: P_{e}G^* \to G$ be the endpoint map. Define the space $X^*$

$$X^* = \{(\gamma, m) \in P_{e}G^* \times M \mid \Phi(m) = \rho^*(\gamma)\}.$$ 

Define the $G$ action on $X^*$ as in the case of $\hat{X}$.

Note that the spaces $\hat{P}_{e}G$ and $P_{e}G^*$ are both equivariantly contractible. The space $\hat{X}$ comes equipped with an equivariant map $\hat{\mu}: \hat{X} \to \hat{P}_{e}G$ defined by $\hat{\mu}(\lambda, m) = \lambda$. Similarly, $P_{e}G^*$ comes equipped with an equivariant map $\mu^*: X^* \to P_{e}G^*$. Let $e$ denote the constant path at the identity in both $\hat{P}_{e}G$ and $P_{e}G^*$.

There is a natural equivariant inclusion map $\hat{i}: \hat{X} \to X^*$. Similarly, where $s > 1/2$, the map $	ext{hol}: L_{s}g^* \to P_{e}G^*$, which sends every $g$-valued one form to its holonomy, induces a equivariant map $i: X \to X^*$.

To prove Proposition 3.1, note that $f^{-1}(0) = \mu^{-1}(0) = \mu^{-1}(\epsilon) = \hat{\mu}^{-1}(\epsilon) = \hat{f}^{-1}(0)$. Moreover, the maps $i: X \to X^*$ and $\hat{i}: \hat{X} \to X^*$ commute with the restrictions to these subspaces. Therefore, it is enough to prove the following result.

**Lemma 3.2.** The maps $i: X \to X^*$ and $\hat{i}: \hat{X} \to X^*$ induce isomorphisms in equivariant cohomology.

Let $\hat{\Omega}G$ be the group of piecewise smooth based loops on $G$, i.e.,

$$\hat{\Omega}G = \{\lambda \in \hat{P}_{e}G \mid \lambda(0) = \lambda(1) = e\}.$$ 

Let $\Omega G^*$ be the space of continuous based loops on $G$. The endpoint maps $\hat{\rho}: \hat{P}_{e}G \to G$ and $\rho^*: P_{e}G^* \to G$ are (Serre) fibrations, with fibers $\hat{\Omega}G$ and $\Omega G^*$, respectively. The natural inclusion
map \( j : \hat{P}eG \rightarrow PeG^* \) is a \( G \)-equivariant map which preserves these fibrations.

Considering now the exact homotopy sequence of the fibrations, we get a map of long exact sequences:

\[
\cdots \rightarrow \pi_n(\hat{\Omega}G) \rightarrow \pi_n(\hat{P}eG) \rightarrow \pi_n(G) \rightarrow \cdots
\]

The map on the right is the identity, and the map in the middle is an isomorphism because both \( PeG^* \) and \( \hat{PeG} \) are contractible. By the five lemma, the inclusion \( \hat{\Omega}G \rightarrow QG^* \) also induces an isomorphism in all homotopy groups.

There are natural fibrations \( \hat{X} \rightarrow M \) and \( X^* \rightarrow \hat{M} \) with fibers \( \hat{\Omega}G \) and \( \Omega G^* \), respectively. The inclusion \( \hat{PeG} \rightarrow PeG^* \) is \( G \)-equivariant, and preserves these fibrations. Hence, we obtain a map of the homotopy quotient fibrations from \( \hat{X} \times_G EG \rightarrow M \times_G EG \) to \( X^* \times_G EG \rightarrow M \times_G EG \).

We then obtain a map of long exact sequences

\[
\cdots \rightarrow \pi_n(\hat{\Omega}G) \rightarrow \pi_n(\hat{X} \times_G EG) \rightarrow \pi_n(M \times_G EG) \rightarrow \cdots
\]

The map on the right side of this sequence is the identity; the map on the left is an isomorphism for all \( n \) by our previous argument. Thus the map \( j \) induces an isomorphism in all homotopy groups. The Whitehead theorem shows that the inclusion \( \hat{X} \rightarrow X^* \) induces an isomorphism in equivariant cohomology.

A similar argument shows that the inclusion \( X \rightarrow X^* \) also induces an isomorphism in equivariant cohomology.

4. Reduction to finite dimensions

The previous section showed that surjectivity for a Hamiltonian \( L_sG \)-space \( X \) follows from surjectivity for the associated infinite approximating space \( \hat{X} \). In this section, we show that the infinite approximating space can itself be approximated by a sequence of finite-dimensional
manifolds $Y_n$. We show that surjectivity holds for $\hat{X}$ if it holds for each $Y_n$.

Let $G$ be a compact Lie group. Let $(X, \omega, \mu)$ be a Hamiltonian $L_sG$ space with proper moment map. Let $(M, \sigma, \Phi)$ be the associated quasi-Hamiltonian $G$-space. For each positive integer $n$, let

$$X_n := \{(g_1, \ldots, g_n, m) \in G^n \times M \mid g_n = \Phi(m)\}.$$

Note that $X_n$ is diffeomorphic to $G^{n-1} \times M$. The space $X_n$ comes equipped with the diagonal $G$ action, where $G$ acts on each copy of $G$ by the adjoint action. The energy function $f_n : X_n \to \mathbb{R}$ is given by

$$f_n(g_1, \ldots, g_n, m) = n\rho(e, g_1)^2 + n\rho(g_1, g_2)^2 + \cdots + n\rho(g_{n-1}, g_n)^2.$$

Note that the energy function $f_n$ is $G$-invariant. There exists a positive number $\overline{\rho}$ such that any two points $p, q \in G$ of distance $d(p, q) < \overline{\rho}$ may be connected by a unique shortest geodesic. We define the finite approximating manifold $Y_n$ associated to $(X, \omega, \mu)$ by

$$Y_n := f_n^{-1}(-\infty, \frac{1}{2}n\overline{\rho}^2).$$

We can now state the main result of this section:

**Proposition 4.1.** Let $(X, \omega, \mu)$ be a Hamiltonian $L_sG$ space with proper moment map. Let $\hat{X}$ be the associated infinite approximating space, and let $\hat{f} : \hat{X} \to \mathbb{R}$ be the energy function. Let $Y_n$ be the finite approximating manifolds, and let $f_n : Y_n \to \mathbb{R}$ be the energy functions.

If the restriction maps $H^*_G(Y_n) \to H^*_G(f_n^{-1}(0))$ are all surjections, then the restriction map $H^*_G(X) \to H^*_G(\hat{f}^{-1}(0))$ is a surjection.

Given $p = (g_1, \ldots, g_n, m) \in Y_n$, let $g_0 = e$. Then

$$f_n(g_1, \ldots, g_n, m) = n \left( \sum_{i=0}^{n-1} \rho(g_i, g_{i+1})^2 \right) < \frac{1}{2}n\overline{\rho}^2.$$
Proposition 4.1 now follows immediately from the lemma below, which is adapted from [B] and [M].

Lemma 4.2. For any natural number $n$, $\beta: Y_n \to \hat{f}^{-1}(\mathcal{B})$ is a $G$-equivariant homotopy equivalence.

Proof. First, we define $\alpha: \hat{f}^{-1}(\mathcal{B}) \to Y_n$. Given $(\lambda, m) \in \hat{X}$, define $\alpha(\lambda, m) = (\lambda(1/n), \lambda(2/n), \ldots, \lambda(1), m)$.

This function is well-defined, since, by assumption $\lambda(1) = \Phi(m)$. It is also continuous. Moreover, this map takes $\hat{f}^{-1}(\mathcal{B})$ to $Y_n$ since

$$\int_{\frac{i}{n}}^{\frac{i+1}{n}} \left| \frac{d\lambda}{dt} \right|^2 dt \geq n\rho \left( \lambda \left( \frac{i}{n} \right), \lambda \left( \frac{i+1}{n} \right) \right)^2$$

for all $0 \leq i < n$.

As in [B], one may construct a homotopy $D_t$ from $\hat{f}^{-1}(\mathcal{B}) \subset \hat{X}$ to itself so that $D_0$ is the identity, and $D_1 = \beta \circ \alpha$. To construct this, one simply deforms the segment of $\lambda$ between $\lambda(\frac{i}{n})$ and $\lambda(\frac{i+1}{n})$ into the shortest geodesic joining $\lambda(\frac{i}{n})$ and $\lambda(\frac{i+1}{n})$. The intermediate paths will be the shortest geodesic joining $\lambda(\frac{i}{n})$ and $\lambda(\frac{i}{n} + \epsilon)$, followed by the original curve between $\lambda(\frac{i}{n} + \epsilon)$ and $\lambda(\frac{i+1}{n})$.

Finally, since $\alpha \circ \beta$ is already the identity, we are done. \hfill $\square$

5. Properties of the finite-dimensional approximation

To prove that the energy function $f_n: Y_n \to \mathbb{R}$ on the finite approximating manifold is Morse in the sense of Kirwan, one needs to find the critical sets of $f_n$ and to compute the Hessian on each critical set. In this section, we begin these computations.

The results in this section do not depend on the details of the properties of quasi-Hamiltonian $G$-spaces: Given any compact, connected $G$-manifold $M$, along with an equivariant map $\Phi: M \to G$, define $X_n, f_n,$ and $Y_n$ as in Section 4 above. The results in this section will then hold.

Lemma 5.1. Consider $y = (g_1, \ldots, g_n, m) \in Y_n$. Let $\beta(y) = (\lambda, m) \in \hat{X}$. The function $f_n$ is critical at $y$ if and only if following conditions hold:

(1) The path $\lambda$ is a geodesic.

(2) The velocity of $\lambda$ at time 1 is perpendicular to the image of $\Phi_*: T_m M \to T_{\Phi(m)} G$. 
Proof. (Compare with [3], p. 319.) Let \( X = (X_1, \ldots, X_{n-1}, X_n, Y) \) \( \in T_yY_n \) be a tangent vector. Note that \( X_n = \Phi_*(Y) \), but otherwise the \( X_i \) are independent.

Let \( g_0 = e \) and let \( X_0 \in T_*G \) be the zero vector. Let \( s_i \) denote the unique shortest geodesic between \( g_i \) and \( g_{i+1} \) for all \( 0 \leq i < n \), parametrized so that \( s_i(0) = g_i \) and \( s_i(1) = g_{i+1} \). Let \( \dot{s}_i(t) \) denote the unit tangent vector of \( s_i \) at time \( t \). Consider a one-parameter family of paths \( \theta(s, t) \) so that \( \theta(0, t) = s(t) \) and \( \frac{\partial \theta}{\partial s}(0, t) = W(t) \), where \( W(t) \) is the unique Jacobi field which is continuous on \( s(t) \) and smooth except possibly at \( \frac{i}{n} \), such that \( W(i/n) = X_i \).

By the first variation formula

\[
X(\rho^2(g_i, g_{i+1})) = 2 \langle \dot{s}_i(1), X_{i+1} \rangle - 2 \langle \dot{s}_i(0), X_i \rangle.
\]

Summing the terms, we get

\[
\frac{1}{2n}X(f_n) = \sum_{i=0}^{n-2} \langle \dot{s}_i(1) - \dot{s}_{i+1}(0), X_{i+1} \rangle + \langle \dot{s}_{n-1}(1), \Phi_*(Y) \rangle.
\]

This is zero for all \( X \in T_yY \) exactly if \( \dot{s}_i(1) = \dot{s}_{i+1}(0) \) for all \( i \) and \( \Phi_*(T_yY_n) \) is perpendicular to \( \dot{s}_{n-1}(1) \) \( \square \).

**Definition 5.2.** Consider any \( m \in M \) and \( \xi \in T_{\Phi(m)}G \) such that \( \xi \) is perpendicular to the image \( \Phi_*(T_mM) \). There exists a symmetric bilinear form

\[
H^\xi \Phi : T_mM \times T_mM \rightarrow \mathbb{R}
\]

defined as follows: Given \( Y \) and \( Y' \) in \( T_mM \), choose a smooth map \( \alpha \) from a neighborhood of \( (0, 0) \) in \( \mathbb{R}^2 \) to \( M \) so that

\[
\alpha(0) = m, \quad \frac{\partial \alpha}{\partial s}(0, 0) = Y, \quad \text{and} \quad \frac{\partial \alpha}{\partial t}(0, 0) = Y',
\]

where \( s \) and \( t \) are the coordinates on \( \mathbb{R}^2 \). Then

\[
H^\xi \Phi(Y, Y') = \left\langle \frac{D}{Ds} \frac{\partial}{\partial t}(\Phi \circ \alpha)(0, 0), \xi \right\rangle.
\]

Here we have denoted by \( D/Ds \) the covariant derivative given by the Levi-Civita connection associated to the bi-invariant metric on \( G \).

The proof that this is well-defined is analogous to the proof that the Hessian of a function is well-defined at a critical point.

**Lemma 5.3.** Let \( y = (g_1, \ldots, g_n, m) \in Y_n \) be a critical point of \( f_n \) and let \( \beta(y) = (\lambda, m) \). Consider \( \eta = (X_1, \ldots, X_n, Y) \) and \( \eta' = (X'_1, \ldots, X'_n, Y') \in T_yY_n \). Let \( W \) be the unique Jacobi field on \( \lambda \) which is smooth except possibly at \( g_i \), and such that \( W(0) = 0 \) and \( W(i/n) = X_i \) for all \( 1 < i \leq n \). Let \( \Delta_i \frac{DW}{Dt} \) denote the difference between the left and right hand limit.
of $\frac{DW}{Dt}$ at $g_i$. The Hessian $Hf_n$ of $f_n$ at $y$ is given by the following formula:

\begin{equation}
Hf_n(\eta, \eta') = \sum_{i<n} \left\langle \Delta_i \frac{DW}{Dt}, X_i' \right\rangle + \left\langle \frac{DW}{Dt}(1), X_n' \right\rangle + H^\xi \Phi(Y, Y').
\end{equation}

**Proof.** The second variation formula for fixed endpoints is easily extended to the case where variations in the path are allowed at $t = 1$. Given a geodesic $\lambda(t) = \exp(\xi t)$ on the group $G$, and let $\theta(s, t)$ be a variation of $\lambda$; that is $\theta(0, t) = \lambda(t)$, and $\theta(s, 0) = e$ for all $s$. Suppose $\theta$ is continuous everywhere and smooth except possibly where $t = i/n, i = 1, \ldots, n - 1$. Write $V = \theta_s(\partial/\partial t)$, $Y = \theta_s(\partial/\partial s)$, and $W = Y'|_{s=0}$. The vector field $W(t)$ is a broken Jacobi field along $\lambda$; we write $X_i = W(i/n)$. Let $E(s) := 1/2 \int_0^1 |\partial \theta/\partial t|^2 dt$. The familiar second variation formula (e.g. [M], Theorem 13.1) becomes $d^2 E/ds^2|_{s=0} = \sum_{i<n} \left\langle \Delta_i \frac{DW}{Dt}, X_i \right\rangle + \left\langle D_Y Y, V \right\rangle|_{s=0, t=1} + \langle W(1), \frac{DW}{Dt}(1) \rangle|_{s=0, t=1}$. The result follows.

□

Since $f_n$ is a $G$-invariant function, equation (5.4) gives the following result.

**Lemma 5.5.** Let $y = (g_1, \ldots, g_n, m) \in Y_n$ be a critical point of $f_n$ and let $\beta(y) = (\lambda, m)$. Given $\gamma \in \mathfrak{g}$, let $\hat{\gamma} \in T_m M$ be the value at $m$ of the vector field $\gamma^M$ on $M$ induced by $\gamma$. Let $W$ be the Jacobi field on the path $\lambda$ induced by conjugation by $\exp(s\gamma)$. Then

$$\langle DW/Dt(1), \Phi_s(Y') \rangle + H^\xi \Phi(\gamma^M, Y') = 0$$

for all $Y' \in T_m(M)$.

6. **Local normal forms for quasi-Hamiltonian $G$-spaces**

The main goal of this section is to prove a local normal form theorem for quasi-Hamiltonian $G$-spaces (Proposition 6.1). The proof of this theorem involves combining the local normal form theorem for Hamiltonian $G$-spaces [GS] with results from [AMM] which show that a quasi-Hamiltonian $G$-space is locally modelled on a Hamiltonian $G$-space. The normal form theorem of Proposition 6.1 will allow us to perform explicit calculations of the Hessians of the energy functions on the finite approximating manifolds, which we will use to show that these functions are Morse in the sense of Kirwan.

Let $G$ be a compact Lie group, and let $\langle \cdot, \cdot \rangle$ be an invariant metric on $\mathfrak{g}$. Given a subspace $\mathfrak{h} \subset \mathfrak{g}$, let $\mathfrak{h}^\perp \subset \mathfrak{g}$ denote its metric orthogonal complement. We will also use the metric to identify $\mathfrak{h}^*$ with $\mathfrak{h}$. 
Let \((M, \sigma, \Phi)\) be a quasi-Hamiltonian \(G\)-space. Given a subspace \(W \subset T_x M\), let \(W^\sigma \subset T_x M\) denote the subspace of \(\sigma\) orthogonal vectors. The \textbf{symplectic slice} at \(p \in M\) is the vector space
\[ V = (T_p O)^\sigma / (T_p O \cap (T_p O)^\sigma), \]
where \(O = G \cdot p\) is the \(G\) orbit of \(p\). Since by the axioms for quasi-Hamiltonian \(G\)-spaces, the kernel of \(\sigma_p\) is contained entirely in \(T_p O\), \(V\) is a symplectic vector space. The isotropy representation of \(\text{Stab}(p)\) on \(T_p M\) induces a representation on the symplectic slice, called the \textbf{slice representation}. Our main proposition is the following.

**Proposition 6.1.** Let \((M, \sigma, \Phi)\) be a quasi-Hamiltonian \(G\)-space. For any \(p \in M\), let \(H = \text{Stab}(p)\), \(K = \text{Stab}(\Phi(p))\), and \(V\) be the symplectic slice at \(p\).

There exists a neighbourhood of the orbit \(G \cdot p\) which is equivariantly diffeomorphic to a neighborhood of the orbit \(G \cdot [e, 0, 0]\) in
\[ Y := G \times_H ((\mathfrak{g}^0 \cap \mathfrak{k}) \times V). \]
In terms of this diffeomorphism, the \(G\)-valued moment map \(\Phi: M \rightarrow G\) may be written as
\[ \Phi([g, \gamma, v]) = \text{Ad}_g(\Phi(p) \exp(\gamma + \phi(v))), \]
where \(\phi: V \rightarrow \mathfrak{h}^* \cong \mathfrak{h}\) is the moment map for the slice representation.

To prove Proposition 6.1, we will need the local normal form theorem for Hamiltonian \(G\)-spaces given, for example, in Guillemin-Sternberg \([GS]\), Section 41. Given a subalgebra \(\mathfrak{h} \subset \mathfrak{g}\), let \(\mathfrak{h}^0 \subset \mathfrak{g}^*\) denote its annihilator.

**Proposition 6.2.** Let \((M, \omega, \mu)\) be a Hamiltonian \(G\)-space. For any \(p \in M\), let \(H = \text{Stab}(p)\), let \(K = \text{Stab}(\mu(p))\), and let \(V\) be the symplectic slice at \(p\). There exists a neighbourhood of the orbit \(G \cdot p\) which is equivariantly diffeomorphic to a neighborhood of the orbit \(G \cdot [e, 0, 0]\) in
\[ Y := G \times_H ((\mathfrak{h}^0 \cap \mathfrak{k}) \times V). \]
In terms of this diffeomorphism, the moment map \(\mu: M \rightarrow \mathfrak{g}^*\) may be written as
\[ \mu([g, \gamma, v]) = \text{Ad}_g^*(\mu(p) + \gamma + \phi(v)), \]
where \(\phi: V \rightarrow \mathfrak{h}^*\) is the moment map for the slice representation.

Here, the \textbf{symplectic slice} at \(p \in M\) is the vector space \(V = (T_p O)^\omega / (T_p O \cap (T_p O)^\omega), \)
where \(O = G \cdot p\) is the \(G\) orbit of \(p\).

In order to derive the normal form theorem for quasi-Hamiltonian \(G\)-spaces from the normal form theorem for Hamiltonian \(G\)-spaces, we
need the following two theorems about quasi-Hamiltonian $G$-spaces, both taken from \cite{AMM}, which show that locally, a quasi-Hamiltonian $G$-space may be modelled on a Hamiltonian $G$-space.

**Proposition 6.3** \cite{AMM}, Remark 3.3. Let $(M, \sigma, \Phi)$ be a quasi-Hamiltonian $G$-space. Let $U \subset \mathfrak{g}$ be a connected neighborhood of $0$ so that the exponential map is a diffeomorphism on $U$, and let $V = \exp U$. Then there exists a Hamiltonian $G$-space $(N, \omega, \nu)$ and an equivariant diffeomorphism $\psi: N \rightarrow \Phi^{-1}(V)$, so that the following diagram commutes

$$
\begin{array}{ccc}
N & \xrightarrow{\nu} & \mathfrak{g}^* \simeq \mathfrak{g} \\
\downarrow \psi & & \downarrow \exp \\
\Phi^{-1}(V) & \xrightarrow{\Phi|_{\Phi^{-1}(V)}} & G
\end{array}
$$

Consider the adjoint action of the Lie group $G$ on itself. Recall that for any $\zeta \in G$, $U_\zeta \subset K$ is a slice for the action of $G$ at $\zeta$ if $U_\zeta$ is preserved by the action of $K = Z(\zeta)$, the centralizer of $\zeta$, and if the natural map $G \times K U_\zeta \rightarrow G$ is an equivariant diffeomorphism onto its image. Note that a slice exists for every $\zeta \in G$.

**Proposition 6.5** \cite{AMM}, Proposition 7.1. Let $(M, \sigma, \Phi)$ be a quasi-Hamiltonian $G$-space. Given $\zeta \in G$, let $U_\zeta$ be a slice for the action of $G$ on itself at $\zeta$, and let $Y_\zeta := \Phi^{-1}(U_\zeta)$. Let $K = Z(\zeta)$ be the centralizer of $\zeta$. The quasi-Hamiltonian cross-section $(Y_\zeta, \sigma|_{Y_\zeta}, \Phi|_{Y_\zeta})$ is a quasi-Hamiltonian $K$-space.

We can now begin our proof.

*Proof of Proposition 6.1.* Let $p \in M$ and let $\zeta := \Phi(p)$. Let $U_\zeta$ be a slice for the action of $G$ on itself at $\zeta$. Let $Y_\zeta = \Phi^{-1}(U_\zeta)$. By Proposition 6.5, $(Y_\zeta, \sigma|_{Y_\zeta}, \Phi|_{Y_\zeta})$ is a quasi-Hamiltonian $K$-space. Define $\Psi: Y_\zeta \rightarrow K$ by $\Psi(m) = \zeta^{-1}\Phi(m)$. Since $\zeta$ is in the center of $K$, the triple $(Y_\zeta, \sigma|_{Y_\zeta}, \Psi|_{Y_\zeta})$ is also a quasi-Hamiltonian $K$-space. Moreover $\Psi(p) = e$. Therefore, by Proposition 6.3, there exists a Hamiltonian $K$-space $(N, \omega, \nu)$ that is $K$-equivariantly diffeomorphic to a neighborhood of $p$ in $Y_\zeta$, with the diffeomorphism carrying $\exp(\nu)$ to $\Psi$. A calculation using the explicit formula given in \cite{AMM} (equation 31) for the relation between the form $\sigma|_{Y_\zeta}$ and the form $\omega$ shows that the symplectic slice $V$ at the point corresponding to $p$ in $N$ is identical with the symplectic slice at $p$ in $Y_\zeta$. Finally, we apply the local normal form theorem for Hamiltonian $K$-spaces (Proposition 6.2). This shows that $N$ is locally equivariantly diffeomorphic to $K \times_H ((\mathfrak{h}^0 \cap \mathfrak{k}^*) \times V)$, with the
diffeomorphism carrying the moment map \( \nu \) to the map \([k, \alpha, v] \mapsto k \cdot (\alpha + \phi(v))\), where \( \phi \) is the moment map for the slice representation. \( \square \)

7. Properties of quasi-Hamiltonian \( G \)-spaces

In this section, we first prove the collection of results immediately below, which describe the first and second derivatives of a \( G \)-valued moment map. We conclude by proving a result which will be useful in characterizing the critical set of the functions \( f_n \). All these results are proved using the local normal form theorem from the previous section.

We will use the following notation. Let \( G \) be a compact Lie group, and let \( M \) be a \( G \)-space. Given \( m \in M \), let \( \text{stab}(m) := \text{Lie}(\text{Stab}(m)) \). Also, for any \( \xi \in \mathfrak{g} \), let \( M^\xi := \{ m \in M \mid e^\xi \cdot m = m \ \forall \ t \in \mathbb{R} \} \).

**Lemma 7.1** ([AMM], Proposition 4.1, part (3)). Let \((M, \sigma, \Phi)\) be a quasi-Hamiltonian \( G \)-space. Fix \( m \in M \); let \( \mathfrak{h} = \text{stab}(m) \). Then

\[
\Phi(m)^{-1}\Phi_*(T_m M) = \mathfrak{h}^\perp.
\]

**Lemma 7.2.** Let \((M, \sigma, \Phi)\) be a quasi-Hamiltonian \( G \)-space. Fix \( \xi \in \mathfrak{g} \) and \( m \in M^\xi \) such that \( \Phi(m) = \exp(\xi) \); let \( \mathfrak{h} = \text{stab}(m) \). Then

\[
\Phi(m)^{-1}\Phi_*(T_m M^\xi) = \mathfrak{h}^\perp \cap \mathfrak{g}^\xi.
\]

**Proposition 7.3.** Let \((M, \sigma, \Phi)\) be a quasi-Hamiltonian \( G \)-space. Fix \( \xi \in \mathfrak{g} \) and \( m \in M^\xi \) such that \( \Phi(m) = \exp(\xi) \). If \( X \in T_m(M^\xi) \), then \( X \) is in the null-space of \( H^\xi \Phi_m \).

**Proposition 7.4.** Let \((M, \sigma, \Phi)\) be a quasi-Hamiltonian \( G \)-space. Fix \( \xi \in \mathfrak{g} \) and \( m \in M^\xi \) such that \( \Phi(m) = \exp(\xi) \). Given \( X \in T_m(M) \) such that \( \Phi(m)^{-1}\Phi_*(X) = 0 \), if \( X \) is in the null-space of the Hessian \( H^\xi \Phi_m \), then \( X \in T_m(M^\xi) \).

Let \( \zeta = \Phi(m) \), \( H = \text{Stab}(m) \), and \( K = \text{Stab}(\zeta) \). Let \( V \) be the symplectic slice at \( m \), and let \( \phi: V \to \mathfrak{h}^* \simeq \mathfrak{h} \) be the moment map for the slice representation. By Proposition 5.1, there exists a neighborhood of \( G \cdot m \) equivariantly diffeomorphic to a neighborhood of \( G \cdot [e, 0, 0] \) in \( Y = G \times_H (\mathfrak{k} \cap \mathfrak{h}^\perp \times V) \). Under this identification, \( \Phi \) is given by \( \Phi([g, \gamma, v]) = \text{Ad}_g(\zeta \exp(\gamma + \phi(v))) \).

A direct computation shows that for \( p = [e, 0, 0] \in G \times_H (\mathfrak{k} \cap \mathfrak{h}^\perp \times V) \) and for any \((a, x, v) \in \mathfrak{h}^\perp \times \mathfrak{k} \cap \mathfrak{h}^\perp \times V = T_p(G \times_H (\mathfrak{k} \cap \mathfrak{h}^\perp \times V)) \),

\[
\Phi(p)^{-1}\Phi_*(a, x, v) = (1 - \text{Ad}_\zeta)(a) + x.
\]

Consider the map from \( h: \mathfrak{g} \to \mathfrak{g} \) given by \( h(a) = (1 - \text{Ad}_\zeta)(a) \). The kernel of \( h \) is given by \( \ker(h) = \mathfrak{k} \). On the other hand, if \( b \in \mathfrak{k} \), then \( \langle (1 - \text{Ad}_\zeta)(a), b \rangle = \langle a, (1 - \text{Ad}_{\zeta^{-1}})(b) \rangle = 0 \) for all \( a \in \mathfrak{g} \), so \( h(\mathfrak{g}) \subset \mathfrak{k}^\perp \).
By a dimension count, this implies that \( h(\mathfrak{g}) = \mathfrak{k}^\perp \). Lemma 7.1 now follows immediately from equation (7.3).

We now prove Lemma 7.2. Note that \( \Phi(p)^{-1}\Phi_*(TY^\xi) \) is contained in \( \mathfrak{h}^\perp \cap \mathfrak{g}^\xi \). On the other hand, since \( \exp(\xi) = \zeta, \mathfrak{g}^\xi \subset \mathfrak{k} \). Thus, given \( x \in \mathfrak{h}^\perp \cap \mathfrak{g}^\xi \), we have \( (0, x, 0) \in T_p Y^\xi \) and \( \Phi(p)^{-1}\Phi_*(0, x, 0) = x \).

We now consider the proofs of Proposition 7.3 and Proposition 7.4. As these involve the Hessian \( H^\xi \Phi \), we begin with a direct computation of this Hessian in local coordinates. To do so, choose \( g, \) and paths \( e, g \) as needed.

It is easy to see that that \( H^\xi \Phi = H^\xi_R \Phi \), and since the Levi-Civita connection is replaced by its analog for the connection obtained from the parallelism arising from left-translation to the origin and right-invariant translation to the origin, respectively. It is easy to see that that \( H^\xi_L \Phi = H^\xi_R \Phi \), and since the Levi-Civita connection is the average of the left- and right-invariant connections, \( H^\xi \Phi = H^\xi_L \Phi = H^\xi_R \Phi \). A direct calculation of \( H^\xi_L \Phi \) shows that given \( (a, x, v) \) and \( (a', x', v') \) in \( T_p Y \), the Hessian of \( \Phi \) at \( p \) is given by

\[
2H^\xi \Phi_p((a, x, v), (a', x', v')) = \langle \phi(v, v'), \xi \rangle + \langle [a, \text{Ad}_z a'] + [a', \text{Ad}_z a] + [x', (1 + \text{Ad}_z)(a)] + [x, (1 + \text{Ad}_z)(a')], \xi \rangle.
\]

Here, \( \phi(v, v') \) denotes the \( \mathfrak{h} \)-valued function on \( V \times V \) which gives rise to the quadratic moment map \( \phi: V \to \mathfrak{h}^* \cong \mathfrak{h} \).

We now prove Proposition 7.3. Given any \( (a, x, v) \in T_p Y^\xi = (\mathfrak{h}^\perp)^\xi \times (\mathfrak{h}^\perp)^\xi \times V^\xi \), and any \( (a', x', v') \in T_p Y \), we may apply equation (7.4) to obtain

\[
2H^\xi \Phi_p((a, x, v), (a', x', v')) = \langle [a, a'] + [a', a] + 2[x', a] + 2[x, a'] + \phi(v, v'), \xi \rangle = 0,
\]
as needed.

We now consider Proposition 7.4. By equation (7.3), the kernel of \( \Phi(p)^{-1}\Phi_* \) is \( \mathfrak{k} \cap \mathfrak{h}^\perp \times 0 \times V \). Furthermore, if \( (a, 0, v) \in \mathfrak{k} \cap \mathfrak{h}^\perp \times 0 \times V \) is in the nullspace of \( H^\xi \Phi_p \), then

\[
2H^\xi \Phi_p((a, 0, v), (0, 0, v')) = \langle \phi(v, v'), \xi \rangle = 0
\]
for all \( v' \in V \). Thus \( v \in V^\xi \) and \( (a, 0, 0) \) is also in the nullspace of \( H^\xi \Phi_p \). Note that since \( a \in \mathfrak{k} \cap \mathfrak{h}^\perp \) and \( \xi \in \mathfrak{k}, [a, \xi] \in \mathfrak{k} \cap \mathfrak{h}^\perp \). By equation (7.4),
It follows that \([a, \xi] = 0\), so that \((a, 0, v) \in T_p M^\xi\).

Finally we have the following result.

**Proposition 7.7.** Let \((M, \sigma, \Phi)\) be a quasi-Hamiltonian \(G\)-space. Define \(\Delta \subset \mathfrak{g} \times M\) by

\[
\Delta := \{(\xi, m) \in \mathfrak{g} \times M \mid \xi \in \text{stab}(m) \text{ and } \exp(\xi) = \Phi(m)\}.
\]

Fix \(r > 0\). The set \(\Delta \cap ((G \cdot \xi) \times M)\) is nonempty for a finite number of \(G\)-orbits \(G \cdot \xi\) with \(|\xi| < r\). Furthermore, for each \(\xi \in \mathfrak{g}\), the set \(\Delta \cap ((G \cdot \xi) \times M)\) has a finite number of connected components.

**Proof.** Since \(\Delta\) is closed and \(M\) is compact, it will suffice to show that for all \((\eta, m) \in \Delta\), there exists a \(G\)-invariant neighborhood \(U\) of the orbit \(G \cdot m\) so that \(\Delta \cap ((G \cdot \xi) \times U)\) is nonempty for only finitely many orbits \(G \cdot \xi\) with \(|\xi| < r\), and so that \(\Delta \cap ((G \cdot \eta) \times U)\) has finitely many connected components. So fix \((\eta, m) \in \Delta\).

Let \(\zeta := \Phi(m)\), \(H := \text{Stab}(m)\), and \(K := Z(\zeta)\). Since \((\eta, m) \in \Delta\), \(\eta \in \mathfrak{h}\). We can choose a maximal torus \(T\) for \(G\) so that \(\eta \in \mathfrak{k}\) and \(T \cap H\) is a maximal torus for \(H\).

Let \(V\) be the symplectic slice, and let \(\phi : V \to \mathfrak{h}^* \cong \mathfrak{h}\) be the moment map for the slice representation. By Proposition 6.1, there exists a \(G\)-invariant neighborhood \(U\) of \(G \cdot m\), an \(H\)-invariant neighborhood \(A\) of \(0 \in \mathfrak{h}^+ \cap \mathfrak{k}\), and an \(H\)-invariant neighborhood \(B\) of \(0 \in V\) so that \(U \cong G \times_H (A \times B)\). Under this identification, the moment map \(\Phi|_U\) is given by \(\Phi([g, a, b]) = \text{Ad}_g \zeta \exp(a + \phi(b))\). There exists a \(K\) invariant neighborhood \(Z\) of \(\zeta\) in \(K\) so that the natural map

\[
G \times_K Z \to G
\]

is an equivariant diffeomorphism onto its image. By shrinking \(U\) further, if necessary, we may assume that the set \(A \times \phi(B)\) is contained in the injectivity radius of the exponential map \(\exp : \mathfrak{k} \to K\) and also that \(\zeta \exp(A + \phi(B)) \subset Z\).

We first show that \(\Delta \cap ((G \cdot \xi) \times U)\) is nonempty for only finitely many orbits \(G \cdot \xi\) with \(|\xi| < r\). Given a conjugacy class \(\Lambda\) of subgroups of \(G\), we write \(U_\Lambda := \{u \in U \mid \text{Stab}(u) \in \Lambda\}\). Because \(U_\Lambda \neq 0\) for only a finite number of conjugacy classes, it will suffice to show that for every conjugacy class \(\Lambda\), \(\Delta \cap ((G \cdot \xi) \times U_\Lambda)\) is nonempty for only finitely many orbits \(G \cdot \xi\).

Fix \(\Lambda\) so that \(U_\Lambda \neq 0\). Choose a representative \(\tilde{H}\) of \(\Lambda\) so that \(T \cap \tilde{H}\) is a maximal torus of \(\tilde{H}\). Let \(B_r(0)\) denote the ball of radius \(r\) in \(\mathfrak{g}\),
and define, for $\xi \in \tilde{h} \cap t \cap B_r(0)$,

$$V_\xi := \{(a,b) \in A \times B \mid \text{Stab}(a,b) = \tilde{H} \text{ and } \exp(\xi) = \zeta \exp(a + \phi(b))\}.$$ 

The set $\Delta$ is $G$-invariant. Moreover every $G$-orbit in $\Delta \cap (g \times U\Lambda)$ contains a point $(\xi, u)$ so that $u$ is of the form $[e, a, b]$, $\text{Stab}(u) = \tilde{H}$, and $\xi \in t \cap h$. Hence, we will be done if we prove that $V_\xi$ is nonempty for only finitely many $\xi \in \tilde{h} \cap t \cap B_r(0)$. Let $\ell \subset t$ denote the integral lattice in the torus $t$. Then, for $\xi \in \tilde{h} \cap t \cap B_r(0)$,

$$V_\xi = \{(a,b) \in A \times B \mid \text{Stab}(a,b) = \tilde{H} \text{ and } \eta + a + \phi(b) - \xi \in \ell\}.$$

Now fix $\lambda \in \ell$. Since $A \times \phi(B)$ is a bounded subset of $\tilde{h}^\perp$, the set $\tilde{h} \cap t \cap B_r(0) \cap (A \times \phi(B) + \lambda + \eta)$ contains at most one point, and is empty for sufficiently large $\lambda$. This proves that $V_\xi$ is non-empty for only finitely many such $\xi$.

We will now show that $\Delta \cap ((G \cdot \eta \times U)$ has finitely many connected components. Every orbit in $\Delta \cap ((G \cdot \eta \times U)$ contains a point $(\xi, u)$ so that $u$ is of the form $[e, a, b]$ and $\xi \in t \cap h$. Since $G \cdot \eta \cap t$ is a finite set, it is enough to check that for all $\xi \in t \cap h \cap G \cdot \eta$, the set \{(a,b) \in A \times B \mid \exp(\xi) = \zeta \exp(a + \phi(b))\} is connected. Now, $\exp(\xi) = g^{-1}\zeta g$. So $\text{Ad}_g(\xi) = \zeta \exp(a + \phi(b))$. Hence $\exp(a + \phi(b)) = 1$, which implies that $a + \phi(b) = 0$, whence $a = 0$ and $\phi(b) = 0$. Since $\phi$ is quadratic, the set \{ $b \in B \mid \xi \in \text{stab}(b)$ and $\phi(b) = 0$\} is connected.

\[\square\]

8. Properties of the energy functionals

In this section, we will put together the results of the previous sections to prove that the energy functions $f_n : Y_n \to \mathbb{R}$ are self-perfecting Morse-Kirwan functions. The main results of this chapter are summarized in the following two propositions.

**Proposition 8.1.** The functions $f_n : Y_n \to \mathbb{R}$ are Morse in the sense of Kirwan.

**Proposition 8.2.** Let $C$ be a component of the critical set of $f_n$ and let $E_C^-$ be the negative normal bundle at $C$. Then there exists a subtorus $T \subset G$ and a $Z(T)$ invariant subset $B \subset C^T$ so that the natural map $G \times_{Z(T)} B \to C$ is an equivariant homeomorphism. Moreover, $(E_C^-)^T$ is a subset of the zero section of $E_C^-$. Here, $Z(T)$ denotes the centralizer of $T$.

We now concentrate on proving these Propositions.
Lemma 8.3. The function $f_n : Y_n \to \mathbb{R}$ is critical at a point $y = (g_1, \ldots, g_n, m) \in Y_n$ if and only if there exists $\xi \in \mathfrak{g}$ with $|\xi| < n\rho$ such that:

1. $g_i = \exp(\frac{1}{n}\xi)$ for all $i$
2. $m \in M^\xi$.

Proof. This follows immediately from Lemma 5.1, Lemma 7.1, and the fact that a path $\gamma : [0,1] \to G$ with $\gamma(0) = e$ is a geodesic if and only there exists $\xi \in \mathfrak{g}$ so that $\gamma(t) = \exp(t\xi)$. □

Lemma 8.3 and Proposition 7.7 give the following explicit description of the critical points of $f_n$.

Corollary 8.4. The critical set of the function $f_n$ has a finite number of components. Each connected component $C$ of the critical set of $f_n$ is the $G$-orbit of a connected component of the set 

\[
C_\xi := \left\{ \left( \exp\left(\frac{\xi}{n}\right), \ldots, \exp\left(\frac{(n-1)\xi}{n}\right), \exp(\xi), m \right) \in Y_n \mid m \in M^\xi \right\}
\]

for some $\xi \in \mathfrak{g}$.

Note that the components of the critical set of $f_n$ need not be manifolds.

Definition 8.6. Let $C$ be a connected component of the set $G \cdot C_\xi$ for some $\xi$. For a sufficiently small neighborhood $U \subset Y_n$ of $C$, define $\Sigma_C = (G \cdot Y_\xi^C) \cap U$.

Lemma 8.7. Let $C$ be a component of the critical set of $f_n$. Then $\Sigma_C$ is a locally closed submanifold of $Y_n$.

Moreover, there exists $\xi \in G$ and a connected component $B$ of $C_\xi$ so that the natural map $G \times_{\text{Stab}(\xi)} B \to C$ is an equivariant homeomorphism. Let $N(\Sigma_C)$ denote the normal bundle to $\Sigma_C$. Then $(N(\Sigma_C))^\xi$ is a subset of the zero section of $N(\Sigma_C)$.

Proof. Let $c \in C$; then $c \in C_\xi$ for some $\xi$. Let $B$ the connected component of $C_\xi$ containing $c$. First, note that $Y_n^\xi$ is a manifold, since it is the fixed point set of a subgroup of $G$. The stabilizer of $\exp(\frac{\xi}{n})$ is $\text{Stab}(\xi)$. (Recall that $Y_n$ is constructed so that $\exp(\frac{\xi}{n})$ is not conjugate to $e$ along the geodesic $\exp(\xi)$.) Thus for every $x \in B$, and hence all nearby points in $Y_n^\xi$, the stabilizer of $x$ is contained in $\text{Stab}(\xi)$. Moreover, the action of $\text{Stab}(\xi)$ takes $Y_n^\xi$ to itself. Therefore, $G \cdot Y_n^\xi = G \times_{\text{Stab}(\xi)} Y_n^\xi$ is a manifold near $C$. Similarly, by Corollary 8.4, $C$ is homeomorphic to $G \times_{\text{Stab}(\xi)} B$. Finally, it is clear that $\exp(\xi)$ acts freely on the complement of the zero section of the normal bundle to
and hence also on the complement of the zero section of the normal bundle to \( \Sigma_C \).

To finish our proof, we need the following fact about Lie groups.

**Lemma 8.8.** Let \( G \) be a compact Lie group. Let \( W \) be a Jacobi field along the geodesic \( t \to \exp(t\xi) \) for some \( \xi \in \mathfrak{g} \). If \( W(0) = 0 \) and \( \left[ \frac{DW}{Dt}(0), \xi \right] = 0 \), then \( [W(t), \xi] = 0 \) for all \( t \). Moreover, \( W(1) \) is a non-zero multiple of \( \frac{DW}{Dt}(1) \).

**Proof.** Given vector fields \( X, Y, \) and \( Z \) on \( G \), let \( R(X,Y)Z \) denote the curvature tensor. Fix \( \xi \in \mathfrak{g} \). Let \( \gamma \) be the geodesic \( \gamma(t) = \exp(t\xi) \).

Define a linear transformation \( K : T_eG \to T_eG \) by \( K(W) = R(\xi,W)\xi \).

Since \( K \) is self-adjoint, we may choose an orthonormal basis \( U_1, \ldots, U_n \) for \( T_eG \) so that \( K(U_i) = e_i U_i \), where \( e_i \) are the eigenvalues of \( K \). We can now extend the \( U_i \) to vector fields along \( \gamma \) by parallel translation. Any Jacobi field along \( \gamma \) which vanishes at 0 can now be written uniquely as \( W = \sum c_i w_i U_i \), where \( w_i = t \) if \( e_i = 0 \), \( w_i = \sin(\sqrt{e_i} t) \) if \( e_i > 0 \), and \( w_i = \sinh(\sqrt{-e_i} t) \) if \( e_i < 0 \) (See \([3]\)). From this, it is clear that \( \frac{DW}{Dt}(0) = \sum c_i d_i U_i \), where \( d_i \neq 0 \) for all \( i \).

Now, since \( R(X,Y)V = \frac{1}{4}[X,Y],V] \) for left invariant vector field, it follows that if \( [U_i, \xi] = 0 \), then \( e_i = 0 \). So we have \( W(t) = \sum_i c_i t U_i \) and \( \frac{DW}{Dt}(0) = \sum c_i U_i \). So, in fact \( W(1) = \frac{DW}{Dt}(0) \).

**Lemma 8.9.** Let \( C \) be any connected component of the critical set of \( f_n \). Then \( C \) is the subset of \( \Sigma_C \) on which \( f_n \) takes its minimum value.

**Proof.** Recall that \( \Sigma_C = G \cdot (Y^c_n \cap U) \), for a sufficiently small neighborhood \( U \) of \( C_\xi \), for some \( \xi \in \mathfrak{g} \).

Suppose that \( f_n \) takes its minimal value at \( x = (g_1, \ldots, g_n, m) \in Y^c_n \cap U \subset (\text{Stab}(\xi))^n \times M^c \). We will show \( x \in C_\xi \).

Since \( x \) is near \( C_\xi \) and \( \Phi(m) \in \text{Stab}(\xi) \), there exists \( \eta \in \text{stab}(\xi) \) near \( \xi \) such that \( \exp(\eta) = \Phi(m) \). Since \( \xi \) is in the center of \( \text{Stab}(\xi) \), and \( \eta \) is nearby, there are no conjugate points to \( e \) in \( \text{Stab}(\xi) \) along the geodesic \( t \to \exp(t\eta) \). So the shortest path from \( e \) to \( \exp(\eta) \) near \( t \to \exp(t\xi) \) is the geodesic \( t \to \exp(t\eta) \). Hence, we may assume \( x = (\exp(\eta/\xi), \exp(2\eta/\xi), \ldots, \exp(\eta), m) \).

Now compute \( f_n(x) \) using the normal form theorem. Let \( K = \text{Stab}(\xi) \), and let \( H = \text{Stab}(m) \). By Proposition \([4]\), a neighbourhood of \( m \) in \( M \) is given by a neighbourhood of \( (e, 0, 0) \) in \( Y = G \times K ((\mathfrak{h}^\perp \cap \mathfrak{f}) \times V) \); under this identification the moment map \( \Phi \) is identified with the map \( \Phi : Y \to G \) given by

\[
\Phi([g, \gamma, v]) = Ad_g(\xi \exp(\gamma + \phi(v))),
\]
where $\phi: V \to \mathfrak{h}^* \simeq \mathfrak{h}$ is the moment map for the linear $H$ action on the symplectic vector space $V$.

In terms of these coordinates, $f_n(x)$ is given by the the square of the length of the geodesic $\exp(t\eta)$ from $e$ to $\exp(\eta) = \zeta \exp((\gamma + \phi(v)))$, that is, by $|\eta|^2$. But $\eta \in \text{Lie}(C(\xi))$, so that $\zeta^{-1} \exp(\eta) = \exp(\eta - \xi)$. Thus $f_n(x) = |\eta|^2 = |\xi + \gamma + \phi(v)|^2$. Since the image of $(\mathfrak{h}^+ \cap \mathfrak{f}) \times V^\xi$ under $\gamma + \phi(v)$ is a hyperplane in $\mathfrak{g}$ through the origin and perpendicular to $\xi$, the distance from that hyperplane to $-\xi$ is minimized at the origin. It follows that the function $f_n(x)$ attains its minimum where $\gamma + \phi(v) = 0$, that is, where $\Phi(m) = \zeta$.

$\Box$

**Lemma 8.10.** Let $C$ be any connected component of the critical set. For every point $x \in C$, the null-space of the Hessian $Hf_n$ is contained in the tangent space to $\Sigma_C$.

**Proof.** Let $y = (g_1, \ldots, g_n, m) \in Y_n$ be a critical point of $f_n$. Then there exists $\xi \in \mathfrak{g}$ such that $g_i = \exp(i\xi/n)$ and $m \in M^\xi$. Denote by $\lambda$ the geodesic $t \to \exp(t\xi)$. Let $\eta = (X_1, \ldots, X_n, Y)$ be in the null space of the Hessian $Hf_n$. We need to show that $\eta$ is in the tangent bundle to $\Sigma_C$.

Let $W$ be the unique Jacobi field on $\lambda$ which is smooth except (possibly) at $g_i$, and such that $W(0) = 0$ and $W(i\xi/n) = X_i$ for all $1 < i \leq n$. Let $\eta' = (\Delta_1 \frac{DW}{Dt}, \ldots, \Delta_n \frac{DW}{Dt}, 0, 0)$. Then $Hf_n(\eta, \eta') = \sum_{i<n} |\Delta_i \frac{DW}{Dt}|^2$. Hence, $W$ must be smooth.

Let $\alpha = \frac{DW}{Dt}(0) \in \mathfrak{g}$. The exact sequence $0 \to \text{ker } (\mathfrak{g}, \cdot) \to \mathfrak{g} \to [\xi, \mathfrak{g}] \to 0$ shows that $\mathfrak{g} = \text{ker } ([\xi, \cdot]) \oplus [\xi, \mathfrak{g}]$. Hence $\alpha$ can be written as a sum $\alpha = \beta + [\gamma, \xi]$, where $[\beta, \xi] = 0$.

Let $W_\gamma$ be the unique unbroken Jacobi field so that $W_\gamma(0) = 0$ and $\frac{DW_\gamma}{Dt}(0) = [\gamma, \xi]$. Then $W_\gamma$ is the Jacobi field on the geodesic path $t \to \exp(t\xi)$ induced by conjugation by $\exp(s\gamma)$. Let $\hat{\gamma} \in T_m M$ be the value at $m$ of the vector field $\gamma^M$ on $M$ induced by $\gamma$. Then, by Lemma 5.3.

$$\langle DW_\gamma(Dt(1), \Phi_\gamma(Y')) + H^\xi \Phi(\hat{\gamma}, Y'), Y' \rangle = 0$$

for all $Y' \in T_m M$. Let $\eta_\gamma = (W_\gamma(1/n), \cdots, W_\gamma(1), \hat{\gamma})$. Then $\eta_\gamma$ is in the null-space of the Hessian. Therefore, $\eta - \eta_\gamma$ is also in the null-space of the Hessian. Moreover, since $\eta_\gamma$ is the value at $y$ of the vector field on $G^m \times M$ induced by $\gamma$, $\eta_\gamma$ is in the tangent space of $\Sigma_C$. Thus, to show that $\eta$ is an element of $T_y \Sigma_C$, it suffices to show that $\eta - \eta_\gamma$ is in the tangent space of $\Sigma_C$. Equivalently, we can assume that $\eta$ is such that $\langle DW/Dt(0), \xi \rangle = 0$.

But by Lemma 5.3 if $DW/Dt(0)$ commutes with $\xi$, so does $W(1)$, and moreover, $cDW/Dt(1) = W(1)$ for some non-zero real number $c$. 


By assumption, $\Phi^*(Y) = W(1)$. So by Lemma 7.2 there also exists $Y' \in T_m M_\xi$ so that $\Phi^*(Y') = W(1)$. But then $H^\xi \Phi (Y, Y') = 0$ because by Proposition 7.3 $Y'$ is in the null-space of $H^\xi \Phi$. Therefore, letting $\eta' = (X_1, \cdots, X_n, Y')$, we have

$$Hf_n(\eta, \eta') = \langle DW/Dt(1), \Phi^*(Y') \rangle = \langle DW/Dt(1), W(1) \rangle = c|DW/Dt(1)|^2.$$  

But our assumption is that $\eta$ is in the null space of the Hessian $Hf_n$. So we have $DW/Dt(1) = 0$ and $\eta = (0, \cdots, 0, Y)$ where $Y \in \ker (\Phi^*_y)$.

This means that for any $\eta' = (X'_1, \cdots, X'_n, Y')$, $Hf_n(\eta, \eta') = H^\xi \Phi (Y, Y') = 0$. Thus $Y$ is in the nullspace of the Hessian $H^\xi \Phi_y$. By Proposition 7.4, $Y \in T_m (M^\xi)$, so that $\eta \in T_y \Sigma_C$, as needed.

\[\square\]

**Proof of Propositions 8.1 and 8.2.** The set of critical points of $f_n$ has a finite number of components by Corollary 8.4. By Lemma 8.7, $\Sigma_C$ is a locally closed submanifold of $Y_n$. Moreover, there exists $\xi \in \mathfrak{g}$ and a connected component $B$ of $C_\xi$ so that the natural map $G \times_{\text{Stab}(\xi)} B \rightarrow C$ is an equivariant homeomorphism. Let $T$ be the closure of the one-parameter subgroup $\exp(\xi t)$ generated by $\xi$. By Lemma 8.7, $(N(\Sigma_C))^T$ is a subset of the zero section of the bundle $N(\Sigma_C)$. The $T$-action also endows $N(\Sigma_C)|_B$ with an orientation; this orientation extends to all of $N(\Sigma_C)$. Finally the submanifold $\Sigma_C$ satisfies conditions 2(a) and 2(b) of Definition 9.1 by Lemma 8.9 and Lemma 8.10. This proves Proposition 8.1. Since $E_C$ is a subbundle of $N(\Sigma_C)|_C$, Lemma 8.7 also proves Proposition 8.2.

\[\square\]

9. **Kirwan’s extension of Morse theory**

In this section we give a brief description of Kirwan’s extension of Morse theory as given in [Kir].

**Definition 9.1.** A smooth function $f: X \rightarrow \mathbb{R}$ is **Morse in the sense of Kirwan** if

1. The set of critical points of $f$ has a finite number of components.
2. For each component $C$ of the critical set there exists a locally closed submanifold $\Sigma_C$ with an orientable normal bundle in $X$ such that:
   a. $C$ is the subset of $\Sigma_C$ on which $f$ takes its minimum value.
   b. At every point $x \in C$, every element $\eta \in T_x M$ that lies in the null space of the Hessian lies in the tangent space $T_x \Sigma_C$. 

Given a component $C$ of the critical set, we call $\Sigma_C$ a minimizing manifold for $f$ along $C$.

**Definition 9.2.** Let $f$ be Morse in the sense of Kirwan. Given any critical set $C$ of $f$, and any point $p \in C$, the Hessian $Hf_p$ splits the normal bundle to $\nu(\Sigma_C)$ into positive, negative, and null eigenspaces. We denote by $E^-_{C}$, the negative normal bundle at $C$, the vector bundle on $C$ given by the negative eigenspaces of $Hf$, and define the index $\lambda_C$ of $C$ as the dimension of $E^-_{C}$.

This definition of a Morse function in the sense of Kirwan is not identical to the definition given in [Kir]. According to Kirwan, a function $f : X \rightarrow \mathbb{R}$ is minimally degenerate if

1. The set of critical points of $f$ is a finite union of disjoint closed subsets $C$ of $X$ on each of which $f$ takes a constant value.
2. For each component $C$ of the critical set there exists a locally closed submanifold $\Upsilon_C$ with an orientable normal bundle in $X$ such that:
   
   a. $C$ is the subset of $\Upsilon_C$ on which $f$ takes its minimum value.
   b. At every point $x \in C$, the tangent space $T_x \Upsilon_C$ is maximal among those subspaces of $T_x X$ where the Hessian $Hf_x$ is positive semidefinite.

However, we have the following Lemma:

**Lemma 9.3.** If a smooth function $f : X \rightarrow \mathbb{R}$ is Morse in the sense of Kirwan, it is minimally degenerate.

**Proof.** Suppose $f$ is Morse in the sense of Kirwan. Let $C$ be a critical set of $f$ and let $\Sigma_C$ be the corresponding minimizing manifold. Let $U_C$ be a tubular neighbourhood of $\Sigma_C$ and identify $U_C$ with $N\Sigma_C$. Let $\Upsilon_C$ be the positive normal bundle to $\Sigma_C$, considered as a submanifold of $U_C$. It is clear that for each $x \in C$, $T_x \Upsilon_C$ is maximal among subspaces where $Hf_x$ is positive semidefinite. $\square$

This lemma allows us to apply the results of Chapter 10 of [Kir], which constructs a version of Morse theory for minimally degenerate functions. We may summarize the results of this construction in the following

**Proposition 9.4.** Let $f$ be a minimally degenerate function on a smooth manifold $X$, and let $c$ be a critical value of $f$, corresponding to a critical set $C$. For $\epsilon$ sufficiently small, define

$$M_\pm = f^{-1}(-\infty, c \pm \epsilon).$$
Denote by $D^-$ the negative disc bundle to $\Sigma_C$, restricted to $C$, and by $S^-$ the corresponding sphere bundle. Then there exists a long exact sequence

\[ \cdots \to H^*(D^-, S^-) \to H^*(M_+) \to H^*(M_-) \to \cdots \]

(9.5)

\[ H^*_{-\dim D^-}(C) \xrightarrow{\cup e(D^-)} H^*(C) \]

where $e(D^-)$ is the Euler class of the bundle $D^-$. If a compact group $G$ acts on $X$, and the function $f$ is invariant, the same result holds in equivariant cohomology.

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