Research on High Precision solution of Fractional partial differential equations under Heat conduction Model

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Abstract. Aiming at the heat conduction equation, the heat conduction model is a very typical example of the partial differential equation. This paper focuses on the development and preliminary knowledge of the partial differential equation, the establishment of the heat conduction model and the solution of the heat conduction model. The partial differential equation with fractional Laplace operator is a kind of typical fractional partial differential equation, which has important applications in the fields of science and engineering. Fractional Laplace operator is a kind of nonlocal quasi-differential operator, which is the infinitesimal generator of Lévy steady-state process. It is essentially different from the classical Laplace operator, which leads to the disappearance of some classical properties, which generally brings difficulties to the study of this kind of problems. It is very difficult to solve the explicit solution of partial differential equation with fractional Laplace operator. In order to improve the accuracy and speed of solving partial differential equation, a high precision method for solving partial differential equation based on deep learning network is proposed. A multilayer radial basis function neural network is constructed and its structural layers are determined. By introducing subnetworks into each layer of the multilayer radial basis function neural network, a composite multilayer radial basis function neural network is constructed to improve the real function approximation performance and operation accuracy of the multilayer radial basis function neural network. The high-precision compound multi-layer radial basis function neural network is used to solve the partial differential equation. By giving a specific example of solving the partial differential equation, the solution effect of the method is tested. The results show that the method has very high solution accuracy under the four-layer network structure, which can improve about 1.5 orders of magnitude compared with the first layer. Under different number of training samples, it has higher solution accuracy and speed, and the comprehensive performance is superior. Therefore, it is a practical but challenging work to study the existence of solutions of partial differential equations with fractional Laplace operators. This paper mainly describes the research progress and trends of the existence of solutions of several classes of partial differential equations with fractional Laplace operators, including some of the work done by the author in this field in recent years.
1. Introduction
Partial differential equation originated in the 18th century, when Euler proposed the second order equation of string vibration, which is the earliest partial differential equation. By the 19th century, gradually developed the partial differential equation of the subject, the French mathematician Fourier described in the analytical theory of thermal heat flow problems [1], it gives the famous Fourier solution to partial differential equations, at the same time he was also one of the first to put forward the heat flow problem of three dimensional space [2], he in order to promote the solution to the partial differential equation of the subject development had great power. Unfortunately, he only gave some solutions, but he did not prove the convergence of the solutions. In the 20th century, with the development of functional analysis and other disciplines, the great development of partial differential equations was promoted [3].

Many applied disciplines, such as physics, chemistry, biology, mechanics, agriculture, environment, medicine, and even social sciences such as economics, a large number of mathematical models can usually be boiled down to partial differential equations. As a special branch of theory of partial differential equations, partial differential equations of fractional order theory in recent years is widely used in fluid mechanics, quantum mechanics, electricity in analytical chemistry, viscoelastic mechanics, electric conduction of biological systems, signal and image processing, robot control, the solution of polymer chain, the chaos phenomenon, molecular spectroscopy, and other fields, and gradually became a hot issue of scientific research, specific see literature [4]. Compared with the integer order partial differential equation, the fractional order partial differential equation is not only more accurate and detailed than the integer order partial differential equation in describing natural phenomena and describing the dynamic process in production and life, but also can avoid the disadvantages of the integer order partial differential equation model not consistent with the scientific experimental results [5]. Soviet mathematician Sobolev expanded the understanding of the concept of solution, put forward the concept of generalized function, then the famous French mathematician Laurent Schwarz in-depth study and research, so that the subject of partial differential equations has a further development [6].

2. Partial differential equations with fractional Laplace operators
In this section, we first review the basic knowledge of fractional-order Laplace operators and fractional-order Sobolev Spaces, and then introduce the physical background of several classes of partial differential equations with fractional-order Laplace operators related to this paper.

2.1. Fractional order Schrodinger equation
Fractional order Schrodinger equation is fractional the basic equations of quantum mechanics. By extending the Feyman path integral, Laskin extended the Brown-type quantum mechanics path to the Levy type quantum mechanics path, which was obtained in the initial form as follows:

$$\frac{1}{i\hbar} \frac{\partial \varphi}{\partial t} = (-\Delta)^s \varphi + (V(x) - \omega) \varphi - f(x, \varphi), x \in \mathbb{R}^N \quad (1)$$

Where, $\text{i}$ represents the imaginary unit, $V$ represents the potential function, $\varphi$ represents the wave function, $0 < s < 1$, and the nonlinear term $f$ represents the interaction between multiple particles. What we consider is to find a standing wave solution to the equation:

$$\varphi(x, t) = e^{-igt} u(x) \quad (2)$$

That is to study the following equation:

$$(-\Delta)^s u + V(x) u = f(x, u), x \in \mathbb{R}^N \quad (3)$$
The standing wave solution of the equation can describe the phenomena such as the soliton in a beam, the thermal pulse propagation of an object, the motion of a superconducting electron in a magnetic field and the Bose-Einstein condensation effect. When $s = 1$, the above equation becomes a classical Schrodinger equation, have abundant research results.

2.2. The definite solution problem of second order partial differential equations

So, let's take the evolution problem as an example, in order for a partial differential equation to have a unique solution, we need to add an initial condition and a boundary condition to the equation. There are three types of boundary value conditions:

- Dirichlet boundary condition (first side value condition)
  \[ u(x,t)|_{\partial\Omega} = g_1(x,t) \]  

- Neumann Boundary Condition (Second Boundary Value Condition)
  \[ \left( \frac{\partial u}{\partial n} \right)_{\partial\Omega} = g_2(x,t) \]  

- Robin boundary condition (third boundary condition)
  \[ \left( \frac{\partial u}{\partial n} + \alpha u \right)_{\partial\Omega} = g_2(x,t) \]  

The second order partial differential equation, initial value condition and boundary value condition constitute the definite solution problem.

3. Heat conduction model

Heat conduction refers to the heat transfer phenomenon when there is no macroscopic motion in the medium, more strictly speaking, it refers to the real heat conduction in the solid. Heat conduction means that there is a temperature difference in an object or system. Heat can flow from a place with a high temperature to a place with a low temperature, and the rate of heat conduction is determined by the distribution of temperature field in the object. The partial differential equation is established by the heat conduction model. The solution of the partial differential equation can describe the change of the function during the heat transfer. The heat conduction problem describes how the temperature changes over time in a particular area. Heat conduction models can be divided into one-dimensional and three-dimensional models. In this paper, we will take one-dimensional heat conduction model as an example to give the derivation of the equation. Let $x$ be position, $t$ be time, the temperature at position $x$ at time $t$, and $Q$ be heat, and let the cross-sectional area of a one-dimensional bar be $S$ and volume $V$, we can get the following formula according to the law of conservation of energy and Fourier’s law of heat transfer.

We know that the velocity of heat flow is proportional to the gradient of the temperature function. Let $k$ be the thermal conductivity coefficient and $J$ be the velocity of heat flow

\[ J = -k\nabla u \]  

According to conservation of energy, the change of heat from time $T_1$ to time $T_2$ is

\[ dQ = c\rho \left[ u(x,t_1) - u(x,t_2) \right] dV \]
According to Gauss formula, the one-dimensional heat conduction model formula is

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$  \hspace{1cm} (9)$$

Similarly, we can also reduce the heat conduction model to the three-dimensional space. In this paper, we will not discuss too much. The method is the same as the derivation process of the one-dimensional heat conduction equation, so the three-dimensional heat conduction model formula is:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$  \hspace{1cm} (10)$$

4. High precision solution of partial differential equations for deep learning networks

Multilayer structure of radial basis function (RBF) neural network includes a number of single radial function network multi-layer radial basis function (RBF) neural network, its residual function on level 1 network implementation of timeliness, is in the first layer on the basis of radial basis function network using radial basis function (RBF) network layer 2, again through the accumulative level 1 fitting results and gained by the total residual function to obtain the overall network output; Based on the two-layer network, the third layer network is used to achieve the fitting of the residual function of the second layer network, and so on, the high-precision multi-layer radial basis function neural network is obtained, Can be represented as.

$$f(x) = f_1(x) + f_2(x) + f_3(x) + \cdots + f_k(x)$$  \hspace{1cm} (11)$$

The number of layers is determined and a sufficiently small positive number $\varepsilon$ is given. First, the generalized crossover rate of layer $k$ is denoized as $GCV_k$. If

$$\left( \frac{GCV_k - GCV_{k-1}}{GCV_{k-1}} \right) > \varepsilon$$

then the fitting error $e^k(1 \leq i \leq N)$ is implemented to construct the network of layer $(GCV_k - GCV_{k-1})/GCV_{k-1}$, $\varepsilon$, the layer network is abandoned and the layer K-1 network is retained. The formula of generalized crossover rate GCV is:

$$GCV_k = \frac{1}{N} \sum_{i=1}^{N} \left[ e^k_i - f_k(x_i) \right]^2 / \left[ 1 - \frac{1}{N} \text{tr}(H) \right]$$  \hspace{1cm} (12)$$

5. Example result analysis

In order to test the accuracy and speed of solving the partial differential equation, a certain partial differential equation is given, and the method in this paper is used to solve the given partial differential equation. The solving effect of the method in this paper is verified by comparing the solving results and solving process time with other methods. The given partial differential equation is as follows:

$$\nabla^2 u = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} = \left( \lambda^2 + \mu^2 \right) \exp \left( \lambda x_1 + \mu x_2 \right)$$  \hspace{1cm} (13)$$

The boundary conditions are $x_1 \in [0, 1]$ and $x_2 \in [0, 1]$. At the same time are:
\[
\begin{aligned}
&\mu = \exp(\lambda x_1 + \mu x_2), x_1 = 0, x_2 = 1 \\
&\nu = \lambda \exp(\lambda x_1 + \mu x_2), x_1 = 0, x_2 = 1
\end{aligned}
\] (14)

In the case of 50 training samples, the standard deviation and root mean square error of the solution results of the method in this paper are compared with those of the exact solution at different levels. The standard deviation and root mean square error of the method presented a trend of gradual decrease. Especially, when the method used in this paper was 2-tier structure, the standard deviation and root mean square error decreased by an order of magnitude compared with that of the 1-tier structure, and the decrease was the most significant. However, starting from the five-tier structure, the decline of standard deviation and root mean square error becomes smaller. Therefore, the four-tier structure proposed in this paper is used for solving the problem and compared with other methods.

![Fig 1. Standard deviation comparison.](image)

![Fig 2. Root mean square error comparison.](image)

According to Fig. 1, with the same number of training samples, the standard deviation and root mean square error of the solution results of the three methods are in descending order as the reduced basis finite element method > orthogonal basis function method > presented method. In addition, with the increase of the number of training samples, the standard deviation and root mean square error of the results of each method show a trend of decreasing gradually. This indicates that the solving accuracy is directly proportional to the number of training samples, and the solving accuracy of the proposed method is improved compared with the other two methods.

The time taken for solving the process of each method was counted, and the solving speed of each method was compared according to the statistical results, so as to further test the solving effect of the
method in this paper. The statistical results are shown in Figure 3. According to the analysis of Figure 2, with the increase of the number of training samples, the time of obtaining high-precision solutions of each method also increases. However, under the same number of training samples, the time of high-precision solution of the proposed method is lower than that of the other two methods, indicating that the proposed method has a higher solving speed when solving partial differential equations with high precision.

![Figure 3](image)

**Fig 3.** Each method was used to obtain the duration statistical results.

### 6. Conclusions

Deep learning network, this paper researches on the high precision solution of partial differential equations, through to the multiple radial basis function neural network into the way of compound subnet in each layer of network, build a composite multilayer radial basis function (RBF) neural network, on the basis of ascension in the past simple multi-layer precision of radial basis function (RBF) neural network. Fractional Laplace operators are a class of nonlocal singular quasi differential operators. Although it maintains some important properties of Laplace operators, such as self-conjugation and bounded property, its nonlocality and singularity bring substantial difficulties to the study of related problems. In this paper, we summarize the research trends and progress on the existence of solutions of partial differential equations with fractional Laplace operators. According to the current research status, a series of excellent research results have been obtained, but there are still some problems worthy of further discussion and research.

The heat conduction model problem is a very typical partial differential equation problem. There are many ways to solve it, and the difference method is one of them that is easy to understand and easy to operate. In addition to the difference method, we can also use MATLAB PDEPE function, toolbox toolbox to solve or Monte Carlo method. In this paper, the method of solving the one-dimensional heat conduction model problem is given. We can compare the other algorithm and the difference method which is simpler and more practical

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