Exact correlations in the nonequilibrium stationary state of the noisy Kuramoto model

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Abstract

We obtain exact results on autocorrelation of the order parameter in the nonequilibrium stationary state of a paradigmatic model of spontaneous collective synchronization, the Kuramoto model of coupled oscillators, evolving in presence of Gaussian, white noise. The method relies on an exact mapping of the stationary-state dynamics of the model in the thermodynamic limit to the noisy dynamics of a single, non-uniform oscillator, and allows to obtain besides the Kuramoto model the autocorrelation in the equilibrium stationary state of a related model of long-range interactions, the Brownian mean-field model. Both the models show a phase transition between a synchronized and an incoherent phase at a critical value of the noise strength. Our results indicate that in the two phases as well as at the critical point, the autocorrelation for both the model decays as an exponential with a rate that increases continuously with the noise strength.

Keywords: stationary dynamics, nonequilibrium stationary state, synchronization, nonlinear dynamics

Some figures may appear in colour only in the online journal

1. Introduction

Characterizing the stationary state of a many-body interacting system evolving according to a given dynamics constitutes one of the primary objectives of statistical mechanics [1]. Complexity in the computation often stems from the many-body nature of the dynamics, and is further enhanced if the stationary state is out of equilibrium [2]. Indeed, the phase-space distribution in an equilibrium stationary state is given unequivocally by the Gibbs–Boltzmann weight independent of the underlying dynamics leading to its attainment, while that in a
nonequilibrium stationary state does not have a universal form but has to be obtained from an explicit consideration of the dynamics. While the phase-space distribution is a characterization of a one-time snapshot of the possible values of the dynamical variables in the stationary state, it is obviously of interest to consider how do the values at one time relate to those at another time. A measure of similarity of the values of dynamical variables at two different times as a function of the time lag between them is given by the autocorrelation function, which thereby provides valuable insights into the underlying dynamics. Autocorrelations in equilibrium may be deduced from the response of the system to small external perturbations by invoking the framework of the linear response theory [3]. By contrast, there is as yet no general procedure that allows for evaluation of autocorrelation in generic nonequilibrium stationary states, thus warranting the need to study model systems for which explicit formulas may be derived for autocorrelation.

In this work, we address the issue of obtaining exact results on the autocorrelation of a paradigmatic model showing spontaneous order in a nonequilibrium stationary state, the Kuramoto model. The model serves as a minimal framework to study the phenomenon of spontaneous synchronization among a population of coupled oscillating units of diverse natural frequencies [4]. Spontaneous synchrony is commonly observed in nature, e.g. in yeast cell suspensions [5], flashing fireflies [6], arrays of Josephson junctions [7], laser arrays [8], power-grid networks [9], and others. The Kuramoto model involves a set of limit-cycle oscillators of distributed frequencies that are coupled all-to-all through an interaction that depends sinusoidally on the difference of the phases between the oscillators [10–14]. The noisy Kuramoto model considers in addition the fact that the frequencies of the oscillators need not be constant in time but may have stochastic fluctuations in time. Denoting by \( \theta_i \in [0, 2\pi) \); \( i = 1, 2, \ldots, N \) the phase of the \( i \)th oscillator in a group of \( N \) oscillators, the dynamics of the model is given by a set of \( N \) coupled Langevin equations of the form [15]

\[
\frac{d\theta_i}{dt} = \omega_i + \frac{K}{N} \sum_{j=1}^{N} \sin(\theta_j - \theta_i) + \eta_i(t). \tag{1}
\]

Here, \( K \geq 0 \) is the coupling constant, \( \omega_i \in [-\infty, \infty] \) is the natural frequency of the \( i \)th oscillator, while the noise \( \eta_i(t) \) satisfies

\[
\langle \eta_i(t) \rangle = 0, \quad \langle \eta_i(t)\eta_j(t') \rangle = 2D\delta_{ij}\delta(t-t'), \tag{2}
\]

with \( D > 0 \) characterizing the strength of the noise and angular brackets denoting averaging over noise realizations. The frequencies \( \{\omega_i\}_{1 \leq i \leq N} \) denote a set of quenched disordered random variables distributed according to a common distribution \( g(\omega) \), with the latter obeying the normalization \( \int_{-\infty}^{\infty} d\omega \; g(\omega) = 1 \). As is often the case with most studies of the Kuramoto model, we will consider \( g(\omega) \) to be a unimodal distribution with a non-compact support, that is, one which is symmetric about its mean \( \langle \omega \rangle \equiv \int_{-\infty}^{\infty} d\omega \; \omega g(\omega) \), and which decreases monotonically and continuously to zero with increasing \( |\omega - \langle \omega \rangle| \).

Now, it is evident from equation (1) that the dynamics is invariant under the Galilean transformation \( \theta_i \rightarrow \theta_i + \langle \omega \rangle t \; \forall \; i \). In the particular case when the frequency term on the right hand side of the dynamics (1) is absent (i.e. \( \omega_i = 0 \; \forall \; i \)), or, when all the oscillators have the same frequency (equal to \( \omega_0 \), say, so that \( g(\omega) = \delta(\omega - \omega_0) \)) and one observes the dynamics (1) in a frame rotating uniformly with frequency \( \omega_0 \) with respect to an inertial frame, the corresponding equations of motion are given by

\[
\frac{d\theta_i}{dt} = \frac{K}{N} \sum_{j=1}^{N} \sin(\theta_j - \theta_i) + \eta_i(t). \tag{3}
\]
The above equations of motion constitute the so-called Brownian mean-field (BMF) model [16], and mimic the canonical ensemble dynamics, namely, the overdamped dynamics in contact with a heat bath at temperature \( T = D/k_B \), of a paradigmatic model of long-range interactions, the Hamiltonian mean-field model [17]. Here, \( k_B \) is the Boltzmann constant. Note that here we have set the damping coefficient to unity.

In terms of the so-called complex order parameter \( r \exp(i\psi) \) (with real \( r \) and \( \psi \) satisfying \( 0 \leq r \leq 1 \) and \( \psi \in [0,2\pi) \)), defined as

\[
    r(t) \exp(i\psi(t)) = \frac{1}{N} \sum_{j=1}^{N} \exp(i\theta_j(t)),
\]

the equations of motion (1) read

\[
    \frac{d\theta_j}{dt} = \omega_j + Kr(t) \sin(\psi(t) - \theta_j) + \eta_j(t),
\]

which makes it evident the mean-field nature of the dynamics: every oscillator evolves in a mean field of magnitude \( r \) produced by all the oscillators. The quantities \( r(t) \) and \( \psi(t) \) are obtained as

\[
    r(t) = \sqrt{r_s^2(t) + r_r^2(t)}, \quad \psi(t) = \tan^{-1}(r_s(t)/r_r(t)),
\]

where we have

\[
    (r_s(t), r_r(t)) = \frac{1}{N} \sum_{j=1}^{N} (\cos \theta_j(t), \sin \theta_j(t)).
\]

In the thermodynamic limit \( N \to \infty \), one may characterize the dynamics (1) in terms of a single-oscillator probability density \( f(\theta, \omega, t) \) defined such that \( f(\theta, \omega, t) d\theta \) gives out of oscillators that have natural frequency equal to \( \omega \) the fraction that have their phase values in \([\theta, \theta + d\theta] \) at time \( t \). The function \( f(\theta, \omega, t) \) satisfies \( f(\theta + 2\pi, \omega, t) = f(\theta, \omega, t) \forall \omega, t \) and the normalization

\[
    \int_{0}^{2\pi} d\theta f(\theta, \omega, t) = 1 \forall \omega, t.
\]

The time evolution of \( f \) obeys the Fokker–Planck equation [15]

\[
    \frac{\partial f}{\partial t} = -\frac{\partial}{\partial \theta} [(\omega + Kr \sin(\psi - \theta)) f] + D \frac{\partial^2 f}{\partial \theta^2},
\]

with

\[
    r(t) \exp(i\psi(t)) = \int_{-\infty}^{\infty} d\omega \, g(\omega) \int_{0}^{2\pi} d\theta \, \exp(i\theta)f(\theta, \omega, t).
\]

In the limit \( t \to \infty \), the dynamics (1) settles into a stationary state. Correspondingly, the single-oscillator density \( f(\theta, \omega, t) \) assumes a time-independent form \( f_{\text{st}}(\theta, \omega) \). Concomitantly, the quantities \( r \) and \( \psi \) assume the time-independent values \( r_{\text{st}} \) and \( \psi_{\text{st}} \), respectively. It may be shown that only under conditions mentioned above that reduce the set of equations (1)–(3) does the dynamics satisfy detailed balance. In this case, the dynamics settles into a stationary state that is in equilibrium [18]. Otherwise, the dynamics (1) relaxes into a generic nonequilibrium stationary state (NESS) [2, 18]. In the stationary state, measuring \( \theta_i^\text{st} \)'s with respect to \( \psi_{\text{st}} \) (thus setting \( r_{\text{st},y} = 0, r_{\text{st}} = r_{\text{st},x} \)), we obtain the stationary-state dynamics as
\[
\frac{d\theta}{dt} = \omega_i - K r_d \sin \theta_i + \eta(t). \tag{11}
\]

In the stationary state, it is known that for given \(g(\omega)\) and \(K\) and on tuning the noise strength \(D\), one observes a continuous phase transition from a low-\(D\) synchronized \((r_d \neq 0)\) to a high-\(D\) incoherent \((r_d = 0)\) phase at the critical value \(D_c\) that solves the equation [15]

\[
K = 2 \left[ \int_{-\infty}^{\infty} d\omega \frac{g(\omega)D_c}{D_c^2 + \omega^2} \right]^{-1}. \tag{12}
\]

In particular, for the BMF model, one obtains by substituting \(g(\omega) = \delta(\omega)\) in the above equation the critical noise strength as \(D_c = K/2\) [19]. In both the noisy Kuramoto and the BMF model, the quantity \(r_d\) decreases continuously from the value of unity at \(D = 0\) to zero at \(D = D_c\), and remains zero at higher values of \(D\).

It is our aim in this paper to characterize in the thermodynamic limit the stationary-state dynamics (11) in terms of autocorrelations of the dynamical variable \(\cos \theta\). The reason behind choosing such a dynamical variable is that the stationary order parameter is indeed a non-uniform oscillator characterized by its phase \(\cos \theta\), and so through such a choice, we will be monitoring the correlation between the phase coherence at two times in the stationary state. To this end, we define the autocorrelation function as

\[
C(\tau) \equiv \lim_{t \to \infty, \tau = \text{finite}} \langle \cos \theta(t) \cos \theta(t + \tau) \rangle, \tag{13}
\]

which is a measure of similarity of \(\cos \theta\)-values at two different times. Namely, \(C(\tau)\) is a measure of the possibility of observing a given \(\cos \theta\)-value (which could be contributed by any of the \(N\) oscillators) at one time and another given value at another time. Consequently, \(C(\tau)\) will be given by an appropriate joint probability, which according to standard notions of probability theory [20] may be expressed in terms of a conditional probability.

In order to proceed, we show the similarity of the stationary dynamics (11) with that of a single non-uniform oscillator of frequency \(\omega\) [21]. Contrary to an uniform oscillator for which the phase changes uniformly in time, a non-uniform oscillator is one in which the phase has a non-uniform variation in time: sometimes it speeds up and sometimes it slows down. Let us then first describe a single non-uniform oscillator characterized by its phase \(\theta \in [0, 2\pi]\), whose time evolution in presence of a Gaussian, white noise \(\eta(t)\) is given by the following Langevin dynamics:

\[
\frac{d\theta}{dt} = \omega - K \sin \theta + \eta(t). \tag{14}
\]

Here, \(\omega\) and \(K \geq 0\) are real constants, while \(\eta(t)\) satisfies

\[
\langle \eta(t) \rangle = 0, \quad \langle \eta(t) \eta(t') \rangle = 2D \delta(t - t'). \tag{15}
\]

For \(\omega = 0\), the equation of motion (14) corresponds to overdamped dynamics of \(\theta\) in a potential \(V(\theta) \equiv -K \cos \theta\) (with the damping coefficient set to unity) and in contact with a heat bath at temperature \(T = D/k_B\).

In the absence of noise (i.e. with \(D = 0\)), and provided we have \(K > \omega\), the dynamics (14) has two fixed points given by [21]

\[
\bar{\theta} = \sin^{-1}\left(\omega/K\right); \quad \cos \bar{\theta} = \pm \sqrt{1 - \omega^2/K^2}. \tag{16}
\]

In order to determine which of the two fixed points is linearly stable, we may linearize equation (14) about \(\bar{\theta}\), by writing \(\theta = \bar{\theta} + \delta \theta\), with \(|\delta \theta| \ll 1\). The linearized equation reads
\[ \frac{d\theta}{dt} = -K\delta \theta \cos \theta, \]
from which it is evident that the fixed point that satisfies \( \cos \theta > 0 \)

is linearly stable, while the other one is linearly unstable. Denoting the stable fixed point by \( \theta_{\text{stable}} \), we have \( \theta_{\text{stable}} = \sin^{-1}(\omega/K); \quad \cos \theta_{\text{stable}} = \sqrt{1-\omega^2/K^2}. \)

In the long-time limit, the dynamics (14) in absence of noise results in \( \theta \) settling into the fixed-point value \( \theta_{\text{stable}} \). In presence of weak noise \( (D \to 0) \), we expect \( \theta \) in the long-time limit to have a (narrow) distribution of values around \( \theta_{\text{stable}} \).

To characterize the behavior of the dynamics (14) in the stationary state, attained as \( t \to \infty \), let us introduce the quantity \( P(\theta, t) \) as a one-time probability density defined such that \( P(\theta, t) d\theta \) gives the probability to observe a value of the phase in the interval \( [\theta, \theta + d\theta] \) at time \( t \). One has the normalization \( \int_0^{2\pi} d\theta P(\theta, t) = 1 \forall t \); moreover, \( P(\theta, t) \) is \( 2\pi \)-periodic in \( \theta \): \( P(\theta + 2\pi, t) = P(\theta, t) \). For a given initial condition \( P(\theta, t = 0) \), the quantity \( P(\theta, t) \) evolves in time according to a Fokker–Planck equation that one may derive from equation (14) using standard procedure [22]. The equation reads

\[ \frac{\partial P(\theta, t)}{\partial t} = -\frac{\partial}{\partial \theta} [(\omega - K \sin \theta)P(\theta, t)] + D\frac{\partial^2 P(\theta, t)}{\partial \theta^2}. \] (17)

Comparing the dynamics (11) and (14), we arrive at the following useful analogy between the system of Kuramoto oscillators and a single non-uniform oscillator. First, let us club the Kuramoto oscillators into groups that have the same natural frequency \( \omega \). Then, the stationary-state dynamics of oscillators within each group is that of a non-uniform oscillator, equation (14), with the constant \( K \) equal to \( Kr_{0\alpha} \), where \( r_{0\alpha} \) is the global order parameter obtained from the stationary-state dynamics of oscillators across all groups. This analogy will be used in this paper to obtain results for the noisy Kuramoto model based on those for the single oscillator. The various steps of analysis would be (i) derive the conditional probability in the stationary state of the single non-uniform oscillator to observe given values of the phase at two different times, and (ii) use the derived results and the mapping between the single non-uniform oscillator and the Kuramoto oscillators mentioned above to obtain the stationary state conditional probability for the latter that will be required to evaluate (13). The method we employ to derive our results for the single non-uniform oscillator is based on a study of a Fokker–Planck equation of the form of equation (17) satisfied by the conditional probability. A general reference that summarizes techniques required to study such equations with the help of a Fourier expansion is the book by Risken [22].

The paper is structured as follows. In section 2, we obtain for the single non-uniform oscillator exact analytical results for the autocorrelation. We then use these results in section 3 to derive the core results of the paper, namely, the autocorrelation \( C(\tau) \) in the stationary state of the noisy Kuramoto model and the BMF model. We also compare our analytical results with those obtained from direct numerical integration of the dynamical equations of motion, demonstrating a very good agreement. The paper ends with conclusions.

### 2. The nonequilibrium stationary state of the single non-uniform oscillator

In this section, we study in detail the stationary state of the single non-uniform oscillator. We start with obtaining the form of the probability density \( P(\theta, t) \) in the stationary state. As \( t \to \infty \), one expects \( P(\theta, t) \) to relax to a stationary distribution \( P_{\text{st}}(\theta) \) that from equation (17) is seen to satisfy

\[ 0 = -\frac{\partial}{\partial \theta} [(\omega - K \sin \theta)P_{\text{st}}(\theta)] + D\frac{\partial^2 P_{\text{st}}(\theta)}{\partial \theta^2}. \] (18)
thereby implying that
\[ -D \frac{\partial \mathcal{P}_a(\theta)}{\partial \theta} + (\omega - K \sin \theta) \mathcal{P}_a(\theta) = J_a. \]  
(19)

Here, \( J_a \), a constant independent of \( \theta \), is the current in the stationary state, with the first and the second term on the left hand side accounting for the contribution due to diffusion and drift, respectively. This last equation is solved easily, with the value of \( J_a \) fixed by accounting for the 2\( \pi \)-periodicity of \( \mathcal{P}_a(\theta) \). One gets
\[
\mathcal{P}_a(\theta) = C(\omega) e^{(-K + K \cos \theta + i \omega \theta)/D} \left[ 1 + \left( \frac{e^{-2\pi \omega/D} - 1}{\int_0^{2\pi} d\theta' e^{(-\omega \theta' - K \cos \theta'/D)} / D} \right) \right],
\]  
(20)

where \( C(\omega) \) is a constant whose value may be fixed by employing the normalization condition: \( \int_0^{2\pi} d\theta \mathcal{P}_a(\theta) = 1 \). From equation (20), it may be checked that for \( \omega = 0 \), one has an equilibrium stationary state:
\[
\mathcal{P}_a(\theta) \propto \exp \left[ (-K + K \cos \theta)/D \right] \sim \exp [-V(\theta)/D];
\]  
(21)

substituting in the equation for \( J_a(\theta) \) given above, one finds that the current is identically zero for all \( \theta \), as it should be in equilibrium.

### 2.1. Stationary correlations

For the dynamics (14), let \( P(\theta, t; \theta', t') \) denote a conditional probability density. Namely, the quantity \( P(\theta, t; \theta', t')d\theta \) gives the probability that the phase has a value in the interval \( \theta, \theta + d\theta \) at time \( t \), given that it had the value \( \theta' \) at an earlier time \( t' \leq t \). The function \( P(\theta, t; \theta', t') \) is \( 2\pi \)-periodic in both \( \theta \) and \( \theta' \):
\[
P(\theta + 2\pi, t; \theta', t') = P(\theta, t; \theta', t'),
\]  
(22)

and satisfies the normalization condition
\[
\int_0^{2\pi} d\theta \ P(\theta, t; \theta', t') = 1 \ \forall \ \theta', t' \text{ and } \forall \ t \geq t'.
\]  
(23)

The time evolution of \( P(\theta, t; \theta', t') \) follows a Fokker–Planck equation that may be written down by using the Langevin equation (14). The equation is given by
\[
\frac{\partial P(\theta, t; \theta', t')}{\partial t} = -\frac{\partial}{\partial \theta} \left[ \left( \omega - K \sin \theta \right) P(\theta, t; \theta', t') \right] + D \frac{\partial^2 P(\theta, t; \theta', t')}{\partial \theta^2}.
\]  
(24)

Since \( P(\theta, t; \theta', t') \) is \( 2\pi \)-periodic in \( \theta \) and \( \theta' \), we may expand \( P \) in a Fourier series, as
\[
P(\theta, t; \theta', t') = \sum_{n,m=-\infty}^{\infty} \tilde{P}_{n,m}(t|t') e^{i(n\theta + m\theta')},
\]  
(25)

where the Fourier coefficients are given by
\[
\tilde{P}_{n,m}(t|t') = \frac{1}{(4\pi)^2} \int_0^{2\pi} d\theta \int_0^{2\pi} d\theta' \ P(\theta, t; \theta', t') e^{-i(n\theta + m\theta')}.
\]  
(26)

Since \( P(\theta, t; \theta', t) \) is real, we have \( \tilde{P}_{-n,-m}(t|t') = \tilde{P}_{n,m}^*(t|t) \), where \( * \) denotes complex conjugation; also, \( P(\theta, t; \theta', t') = \delta(\theta - \theta') \) implies that
\[ \tilde{P}_{n,m}(t'|t') = \frac{\delta_{n,-m}}{2\pi}. \] (27)

Using equation (20) and the Fourier expansion for \( P(\theta, t|\theta', t') \), we obtain the stationary correlation \( C(\tau, \omega) \equiv \lim_{t' \to \infty, \tau \to \infty} \langle \cos \theta(t') \cos \theta(t' + \tau) \rangle \) as

\[
C(\tau, \omega) = \lim_{t' \to \infty, \tau \to \infty} \int_0^{2\pi} d\theta \int_0^{2\pi} d\theta' \cos \theta \cos \theta' P(\theta, t \equiv t' + \tau|\theta', t')P_\omega(\theta') \\
= \lim_{t' \to \infty, \tau \to \infty} \pi \sum_{m=-\infty}^{\infty} \left( (\tilde{P}_{1,n}(t|t') + \tilde{P}_{-1,n}(t|t') \right) \int_0^{2\pi} d\theta e^{im\theta} \cos \theta \ P_\omega(\theta)). \] (28)

Now, equation (24) gives the time evolution of \( \tilde{P}_{n,m}(t|t') \) as

\[
\frac{\partial \tilde{P}_{n,m}(t|t')}{\partial t} = -(i\omega + Dn^2)\tilde{P}_{n,m}(t|t') + \frac{n\mathcal{K}}{2} \left( \tilde{P}_{n-1,m}(t|t') - \tilde{P}_{n+1,m}(t|t') \right). \] (29)

It then follows that \( \tilde{P}_{0,m}(t|t') \) is independent of \( t \), and thus, we have

\[
\tilde{P}_{0,m}(t|t') = \tilde{P}_{0,m}(t'|t) = \frac{\delta_{0,-m}}{2\pi}. \] (30)

Let us consider separately the cases \( \mathcal{K} \neq 0 \) and \( \mathcal{K} = 0 \).

2.1.1 The case \( \mathcal{K} \neq 0 \). For any pair of values \((n,m)\), the system of equation (29) is not closed and in fact involves an infinite hierarchy; to solve for \( \tilde{P}_{n,m} \) requires knowing \( \tilde{P}_{n+1,m} \) whose solution in turn requires knowing \( \tilde{P}_{n+2,m} \) and so on. Nevertheless, noting that for a given \( m \), only one of the \( \tilde{P}_{n,m} \)'s is non-zero at the initial time, that is, \( \tilde{P}_{n,m}(t'|t') = \delta_{n,-m}/(2\pi) \), the system of equations is solved quite easily by truncating it at a given value \( n = n_{\text{max}} \), that is, by stipulating that \( \tilde{P}_{n,m}(t|t') = 0 \) for \( n > n_{\text{max}} \), where \( n_{\text{max}} \) may be chosen to be as large as possible. In practice, in evaluating the correlation (28), we choose the same \( n_{\text{max}} \) for different \( m \)'s and restrict the values of \( m \) to the range \([-n_{\text{max}} : n_{\text{max}}] \), checking that a larger value of \( n_{\text{max}} \) does not lead to any significant change in the results obtained.

From the system of equation (29), one may obtain closed form expressions for \( \tilde{P}_{n,m} \) for the particular case of small \( \mathcal{K} \), when the equations can be solved perturbatively. To this end, we expand \( \tilde{P}_{n,m}(t|t') \) as a power series in \( \mathcal{K} \), as

\[
\tilde{P}_{n,m}(t|t') = \tilde{P}_{n,m}(0|t') + \mathcal{K}\tilde{P}_{n,m}^{(1)}(t|t') + \mathcal{K}^2\tilde{P}_{n,m}^{(2)}(t|t') + \ldots, \] (31)

where we have \( \tilde{P}_{n,m}^{(\alpha)}(t|t') = O(\mathcal{K}^\alpha) \) for \( \alpha \geq 1 \). Substituting in equation (29), and comparing terms of the same order in \( \mathcal{K} \) from both sides, we get

\[
\mathcal{K}^0 : \frac{\partial \tilde{P}_{n,m}^{(0)}(t|t')}{\partial t} = -(i\omega + Dn^2)\tilde{P}_{n,m}^{(0)}(t|t'), \] (32)

\[
\mathcal{K}^1 : \frac{\partial \tilde{P}_{n,m}^{(1)}(t|t')}{\partial t} = -(i\omega + Dn^2)\tilde{P}_{n,m}^{(1)}(t|t') + n\frac{\mathcal{K}}{2} \left( \tilde{P}_{n-1,m}^{(0)}(t|t') - \tilde{P}_{n+1,m}^{(0)}(t|t') \right), \] (33)

and so on. Since \( \tilde{P}_{n,m}(t'|t') = \delta_{n,-m}/(2\pi) \), we may take

\[
\tilde{P}_{n,m}^{(0)}(t|t') = \frac{\delta_{n,-m}}{2\pi}, \quad \tilde{P}_{n,m}^{(\alpha)}(t|t') = 0 \quad \forall \ \alpha \geq 1. \] (34)
Solving equation (32), we get
\[ \tilde{P}_{n,m}(t|t') = \frac{\delta_{n,-m}}{2\pi} \exp \left[ -\left( i\omega + Dn^2 \right)(t - t') \right], \]
which when used in equation (33) yields the solution
\[ \tilde{P}_{n,m}(t|t') = \frac{ne^{-i(\omega+D\tau)(t-t')}}{4\pi} \left[ \frac{\delta_{n,-1,-m}}{2nD - D + i\omega} \left( e^{-(2nD+D-i\omega)(t-t')} - 1 \right) \right. \\
+ \frac{\delta_{n+1,-m}}{2nD + D + i\omega} \left. \left( e^{-(2nD+D+i\omega)(t-t')} - 1 \right) \right]. \tag{36} \]

To order \( K \), substituting
\[ \tilde{P}_{n,m}(t|t') = \tilde{P}_{n,m}^{(0)}(t|t') + K\tilde{P}_{n,m}^{(1)}(t|t') \tag{37} \]
in equation (28) allows to obtain \( C(\tau, \omega) \) for the model (14) in the limit of small \( K \), as
\[ C(\tau, \omega) = \frac{1}{2} e^{-\left(D+i\omega\right)\tau} \int_0^{2\pi} d\theta \cos \theta e^{-i\theta} P_{m}(\theta) + \frac{K}{4} \left( \frac{1 - e^{-\left(D+i\omega\right)\tau}}{D + i\omega} \right) \int_0^{2\pi} d\theta \cos \theta P_{m}(\theta) \\
+ \frac{\left(e^{-\left(4D+2i\omega\right)\tau} - e^{-\left(D+i\omega\right)\tau}\right)}{3D + i\omega} \int_0^{2\pi} d\theta \cos \theta e^{-i2\theta} P_{m}(\theta) \right] + \text{c.c.}, \tag{38} \]
where c.c. denotes complex conjugate of the bracketed term.

For \( \omega = 0 \), when the dynamics (14) has an equilibrium stationary state
\[ P_{m}(\theta) = \exp \left[ -V(\theta)/D \right]/\int_0^{2\pi} d\theta \exp \left[ -V(\theta)/D \right], \tag{39} \]
thereby implying that
\[ \int_0^{2\pi} d\theta \cos \theta \sin(n\theta) P_{m}(\theta) = 0 \tag{40} \]
for non-zero integer \( m \), we obtain from equation (38) the equilibrium correlation for small \( K \) as
\[ C(\tau, \omega) \bigg|_{\omega=0} = e^{-D\tau} \int_0^{2\pi} d\theta \cos^2 \theta P_{m}(\theta) + \frac{K}{2} \left( \frac{1 - e^{-D\tau}}{D} \right) \int_0^{2\pi} d\theta \cos \theta P_{m}(\theta) \\
+ \frac{\left(e^{-4D\tau} - e^{-D\tau}\right)}{3D} \int_0^{2\pi} d\theta \cos \theta \cos(2\theta) P_{m}(\theta). \tag{41} \]

2.1.2. The case \( K = 0 \). For \( K = 0 \), equation (20) gives \( P_{m}(\theta) = 1/(2\pi) \), while equation (29) with the initial condition \( \tilde{P}_{n,m}(t'|t') = \delta_{n,-m}/(2\pi) \) has the solution
\[ \tilde{P}_{n,m}(t|t') = \frac{\delta_{n,-m}}{2\pi} \exp \left[ -\left( i\omega + Dn^2 \right)(t - t') \right]. \tag{42} \]
Equation (28) then yields the exact result
\[ C(\tau, \omega) = \frac{1}{2} \cos(\omega\tau)e^{-D\tau}. \tag{43} \]
We thus obtain \( C(0) = \langle \cos^2 \theta \rangle_{st} = \frac{1}{2} \). Putting \( \omega = 0 \), equation (43) gives the equilibrium correlation as

\[
C(\tau, \omega)|_{\omega=0} = \frac{1}{2} \exp(-D\tau). \tag{44}
\]

3. Results for the noisy Kuramoto model

In order to derive our results for the stationary state of the noisy Kuramoto model, equation (11), we employ the aforementioned exact analogy that exists between it and the dynamics (14) of the single non-uniform oscillator on setting the constant \( K \) in the latter to the value \( Kr_d \), where \( K \) is as usual the coupling constant of the Kuramoto model and \( r_{st} \) is the stationary Kuramoto order parameter. Consequently, we may write down for the noisy Kuramoto model the single-oscillator probability density in the stationary state by using equation (20) as \([15]\)

\[
f_{st}(\theta, \omega) = C(\omega) e^{-Kr_d Kr_a \cos \theta / D}\left[ 1 + \frac{(e^{-2\pi \omega/2D} - 1) \int_0^{2\pi} d\theta' e^{(-\omega \theta' - Kr_d \cos \theta') / D}}{\int_0^{2\pi} d\theta' e^{(-\omega \theta' - Kr_d \cos \theta') / D}} \right]. \tag{45}
\]

with the constant \( C(\omega) \) fixed by the normalization condition \( \int_0^{2\pi} d\theta \ f_{st}(\theta, \omega) = 1 \ \forall \ \omega \), and \( r_{st} \) determined from the self-consistent equation

\[
r_{st} = \int_{-\infty}^{\infty} d\omega \ g(\omega) \int_0^{2\pi} d\theta \ \cos \theta \ f_{st}(\theta, \omega). \tag{46}
\]

The single-oscillator \( \theta \)-distribution \( \mathcal{P}_{st}(\theta) \), defined as the probability density to observe a phase value equal to \( \theta \) in the stationary state, is obtained from \( f_{st}(\theta, \omega) \) as

\[
\mathcal{P}_{st}(\theta) = \int_{-\infty}^{\infty} d\omega \ g(\omega) f_{st}(\theta, \omega), \tag{47}
\]

while the stationary correlation \( C(\tau) \) for the Kuramoto oscillators is obtained as

\[
C(\tau) = \int_{-\infty}^{\infty} d\omega \ g(\omega) C(\tau, \omega). \tag{48}
\]

Results for the BMF model, which corresponds to the case \( g(\omega) = \delta(\omega) \), may be obtained by using equations (45)-(47), as

\[
\mathcal{P}_{st}(\theta) = \frac{e^{(Kr_d \cos \theta)/D}}{\frac{2\pi}{\int_0^{2\pi} d\theta \ e^{(Kr_d \cos \theta)/D}}}; \quad r_{st} = \frac{I_1(r_{st}/D)}{I_0(r_{st}/D)}, \tag{49}
\]

where \( I_0(x) \) is the modified Bessel function of the first kind.

For a given choice of \( g(\omega) \) and a given value of \( K \), the explicit steps involved in obtaining the correlation \( C(\tau) \) in the synchronized and the incoherent phase of the Kuramoto model are as follows. For \( D \geq D_c \), when \( K = Kr_d = 0 \), equation (43) gives

\[
C(\tau) = \frac{e^{-D\tau}}{2} \int_{-\infty}^{\infty} d\omega \ g(\omega) \cos(\omega\tau); \quad D \geq D_c. \tag{50}
\]

For \( D < D_c \), when \( K = Kr_d \neq 0 \), we first obtain the value of \( r_{st} \) by solving the self-consistent equation (46). We then solve for every value of \( \omega \) in the support of \( g(\omega) \) the system of equation (29) with the substitution \( K = Kr_d \), and use the solution \( \mathcal{P}_{n,m} \) in equation (28), with \( \mathcal{P}_{st}(\theta) \)
Figure 1. Stationary autocorrelation $C(\tau)$, equation (13), normalized by its value at $\tau = 0$, in the noisy Kuramoto (panel (a)) and the BMF model (panel (b)). For the former, the frequency distribution is taken to be a Gaussian centered at zero and with width given by $\sigma = 0.2$, thus yielding the critical noise strength $D_c \approx 0.43$, while that for the BMF model is given by $D_c = 0.5$. In both cases, we have taken the coupling constant to be $K = 1$. Here, the points refer to results obtained from numerical integration of the dynamics (system size: $N = 10^5$ for the noisy Kuramoto model and $N = 10^6$ for the BMF model), while continuous lines refer to analytical results obtained in the text. In panel (c), we show for $D \lesssim D_c$ and for both the models a comparison between numerical results and theory developed in this special case in the text; we have taken $D = 0.427$ for the Kuramoto model and $D = 0.49$ for the BMF model.
given by equation (47), to obtain $C(\tau, \omega)$. Finally, equation (48) yields the desired correlation. For the particular case of $D \lesssim D_c$, so that $K = Kr_{st}$ is small, we may use equation (38) to get

$$C(\tau) = \int_{-\infty}^{\infty} d\omega \ g(\omega) \left( \frac{1}{2} e^{-(D+i\omega)\tau} \int_0^{2\pi} d\theta \ \cos \theta \ e^{-i\phi} \mathcal{P}_{st}(\theta) \right)$$

$$+ \frac{Kr_{st}}{4} \left\{ (1 - e^{-(D+i\omega)\tau}) \int_0^{2\pi} d\theta \ \cos \theta \mathcal{P}_{st}(\theta) \right\}$$

$$+ \left( \frac{e^{-(4D+2i\omega)\tau} - e^{-(D+i\omega)\tau}}{3D + i\omega} \right) \int_0^{2\pi} d\theta \ \cos \theta \ e^{-2i\phi} \mathcal{P}_{st}(\theta) + c.c.; \ D \lesssim D_c.$$

(51)

The correlations in the equilibrium stationary state of the BMF model are obtained from equation (43) as

$$C(\tau) = \exp(-D\tau)/2; \ D \geq D_c,$$

while that for $D < D_c$ are obtained by first solving the self-consistent equation for $r_{st}$ given in equation (49) and then using this value to solve the system of equation (29) with $\omega = 0$, and finally using (28) with $\omega = 0$ and with $\mathcal{P}_{st}(\theta)$ given in equation (49). In particular, the result for $D \lesssim D_c$ is obtained from equation (41) with the substitution $K = Kr_{st}$.

Following the above procedure, and considering $g(\omega)$ to be a Gaussian distribution centered at zero and with width equal to $\sigma$, we show in figure 1(a) our analytical results for the noisy Kuramoto model compared against those obtained from direct numerical integration of the dynamics (1) for very large $N$ (note that all our analytical computations were done in the limit $N \to \infty$). The corresponding results for the BMF model are in figure 1(b), while the results for $D \lesssim D_c$ are shown in figure 1(c) for both the models. We have taken the coupling constant to be $K = 1$. The function $C(\tau)$ indeed decays to the value $(r_{st})^2$ for large $\tau$, as it should. In all cases, a very good agreement between theory and numerical results is evident from the plots, for values of $D$ both below and above the critical value $D_c$. The latter equals 0.5 for the BMF model and has the value $\approx 0.43$ for the noisy Kuramoto model (obtained by numerically solving equation (12)). The autocorrelation $C(\tau)$ in all cases decays as an exponential with a rate that increases continuously with the noise strength. An implication of this result is that the more noisy the dynamics is, the faster it takes to generate uncorrelated configurations.

4. Conclusions

In this work, we obtained exact analytical results on correlations in the nonequilibrium stationary state of a paradigmatic many-body interacting system showing spontaneous order, the Kuramoto model of coupled oscillators, evolving in presence of Gaussian, white noise. The method relies on an exact mapping of the stationary-state dynamics of the model to the noisy dynamics of a single, non-uniform oscillator, which could be possible due to the mean-field nature of the dynamics of the Kuramoto model. Namely, the dynamics may be thought as that of a single oscillator evolving in a mean-field due to its interaction with all the other oscillators. Although we considered the Kuramoto model that involves a sinusoidal interaction between the oscillators, the method is designed to work for any form of interaction between the oscillators so long as it is of the mean-field type. Investigations are underway to extract further implications of the mapping that allow to obtain useful and physically relevant results on both static and dynamic properties of both Kuramoto and related models. It would be interesting to obtain correlations in the stationary state of the Kuramoto model with inertia [14]. As is well known, inertia significantly changes the nature of the synchronization transition of
the model, and is thus expected to affect also the stationary-state correlation. On the analytical side, inertia brings in significant complexity in obtaining even the stationary state of the model [14]. Addressing inertial effects on autocorrelation is left for future work.

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