CANONICAL DECOMPOSITION OF MANIFOLDS WITH FLAT REAL PROJECTIVE STRUCTURE INTO \((n-1)\)-CONVEX MANIFOLDS AND CONCAVE AFFINE MANIFOLDS.

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Abstract. We try to understand the geometric properties of \(n\)-manifolds \((n \geq 2)\) with geometric structures modeled on \((\mathbb{R}P^n, \text{PGL}(n+1, \mathbb{R}))\), i.e., \(n\)-manifolds with projectively flat torsion free affine connections. We define the notion of \(i\)-convexity of such manifolds due to Carriére for integers \(i\), \(1 \leq i \leq n - 1\), which are generalization of convexity. Given a real projective \(n\)-manifold \(M\), we show that the failure of an \((n-1)\)-convexity of \(M\) implies an existence of a certain geometric object, \(n\)-crescent, in the completion \(\hat{M}\) of the universal cover \(\tilde{M}\) of \(M\). We show that this further implies the existence of a particular type of affine submanifold in \(M\) and give a natural decomposition of \(M\) into simpler real projective manifolds, some of which are \((n-1)\)-convex and others are affine, more specifically concave affine. We feel that it is useful to have such decomposition particularly in dimension three. Our result will later aid us to study the geometric and topological properties of radiant affine 3-manifolds leading to their classification. We get a consequence for affine Lie groups.

1. Introduction

From Ehresmann’s definition of geometric structures on manifolds, a real projective structure on a manifold is given by a maximal atlas of charts to \(\mathbb{R}P^n\) with transition functions extending to projective transformations. This device lifts the real projective geometry locally and consistently on a manifold. A differentio-geometric definition of a real projective structure is a projectively flat torsion-free connection. An equivalent way to define a real projective structure on a manifold \(M\) is to give an immersion \(\tilde{M} \to \mathbb{R}P^n\), a so-called a developing map, equivariant with respect to a so-called holonomy homomorphism.
$h : \pi_1(M) \to \text{PGL}(n + 1, \mathbb{R})$ where $\pi_1(M)$ is the group of deck transformations of the universal cover $M$ and $\text{PGL}(n + 1, \mathbb{R})$ is the group of deck transformations of $\mathbb{R}P^n$. Each of these description of a real projective structure gives rise to the unique descriptions of the other two kinds. For convenience, we will assume that the dimension $n$ of manifolds is greater than or equal to 2 throughout this paper unless stated otherwise.

The global geometric and topological properties of real projective manifolds are completely unknown, and are thought to be very complicated. The study of real projective structure is a fairly obscure area with only handful of global results, as it is a very young field with many open questions, however seemingly unsolvable by traditional methods. The complication comes from the fact that many compact manifolds are geodesically incomplete, and often the holonomy groups are far from being discrete lattices and thought to be far from being small such as solvable. There are some early indication that this field however offers many challenges for applying linear representations of discrete groups (which are not lattices), group cohomology, classical convex and projective geometry, affine and projective differential geometry, real algebraic geometry, and analysis. (Since we cannot hope to mention them here appropriately, we offer as a reference the Proceedings of Geometry and Topology Conference at Seoul National University in 1997 [19].)

This area is also an area closely related to the study of affine structures, which are more extensively studied with regard to affine Lie groups.

Riemannian hyperbolic manifolds admits a canonical real projective structure, via the Klein model of hyperbolic geometry as the hyperbolic space embeds as the interior of a standard ball in $\mathbb{R}P^n$ and the isometry group $\text{PSO}(1, n)$ as a subgroup of the group $\text{PGL}(n + 1, \mathbb{R})$ of projective automorphisms of $\mathbb{R}P^n$ (see [22] and [9]).

They belong to the class of particularly understandable real projective manifolds which are convex ones. A **convex real projective manifold** is a quotient of a convex domain in an affine patch of $\mathbb{R}P^n$, i.e., the complement of a codimension one subspace with the natural affine structure of a complete affine space $\mathbb{R}^n$, by a properly discontinuous and free action of a group of real projective transformations. It admits a Finsler metric, which has many nice geometric properties of a negatively curved Riemannian manifold though the curvature is not bounded in the sense of Alexandrov.

Affine manifolds naturally admit a canonical real projective structure since an affine space is canonically identified with the complement of codimension one subspace in the real
projective space $\mathbb{R}P^n$ and affine automorphisms are projective. In particular, euclidean manifolds are projective.

Not all real projective manifolds are convex (see [26] and [22]). However, in dimension two, we showed that closed real projective manifolds are built from convex surfaces. That is, a compact real projective surface of negative Euler characteristic with geodesic boundary or empty boundary decomposes along simple closed geodesics into convex surfaces (see [9], [10], [12], and [18]).

Also, recently, Benoist [5] classified all real projective structures, homogeneous or not, on nilmanifolds some of which are not convex. Again, decompositions into parts admitting homogeneous structures were the central results. His student Dupont [20] classifies real projective structures on 3-manifolds modelled on Sol.

The real projective structures on 3-manifolds are unexplored area, which may give us some insights into the topology of 3-manifolds along with hyperbolic or contact structures on 3-manifolds.

Let us state an interesting fact: All eight types of 3-dimensional homogeneous Riemannian manifolds, i.e., manifolds with hyperbolic, spherical, euclidean, $S^2 \times S^1$, $H^2 \times S^1$, Sol, Nil, or $\text{SL}(2,\mathbb{R})$-structures, admit canonical real projective structures since the models of each of the eight 3-dimensional geometry can be realized as pairs of open domains in an affine or real projective space or their cover, and subgroups of groups of projective automorphisms of such domains. Thus, all Seifert spaces and atoroidal Haken manifolds admit real projective structures.

We might ask whether (i) real projective 3-manifolds decomposes into pieces which admit one of the above geometries. or (ii) conversely pieces with such geometric structures can be made into a real projective structures by perturbations. (These are questions by Thurston.)

A question by Goldman (see [1, p. 336]) is that does all irreducible (Haken) 3-manifolds admit real projective structure? A very exciting development will come from discovering ways to put real projective structures on 3-manifolds other than from homogeneous Riemannian structures perhaps starting from triangulations of 3-manifolds. (A related question asked by John Nash after his showing that all smooth manifolds admit real algebraic structure is that when does a manifold admit a rational structure, i.e., an atlas of charts with transition functions which are real rational functions. Real projective manifolds are rational manifolds with more conditions on the transition functions.)
These questions are at the moment very mysterious. This paper initiates some methods to study these with regard to the question (i). We will decompose real projective $n$-manifolds into concave affine real projective $n$-manifolds and $(n-1)$-convex real projective $n$-manifolds.

To do this, we will extend and refine the techniques involved in proving the result in dimension two. We work on $n \geq 2$ case although $n = 2$ case was already done in [9] and [10]. The point where this paper improves the papers [9] and [10] even in $n = 2$ case is that we will be introducing the notion of two-faced submanifolds which makes decomposition easier to understand.

In three-dimensional case, our resulting decompositions into 2-convex 3-manifolds and concave affine 3-manifolds seem to be along totally geodesic surfaces, which hopefully will be essential in 3-manifold topology terminology. Thus, our remaining task is to see if 2-convex real projective 3-manifolds admit nice decompositions or at least nice descriptions.

Our result will be used in the decomposition of radiant affine 3-manifolds, which are 3-manifolds with flat affine structure whose affine holonomy group fixes a point of an affine space (see [14]). In particular, we will be proving there the Carrière conjecture (see [8]) that every radiant affine 3-manifolds admit a total section to the radial flow which exists naturally on radiant affine manifolds with the help from Barbot’s work [4], [3] (also see his survey article [2]). This will result in the classification of radiant affine 3-manifolds.

Let us begin to state our theorems. Let $T$ be an $(i+1)$-simplex in an affine space $\mathbb{R}^n$, $i + 1 < n$, with sides $F_1, F_2, \ldots, F_{i+2}$. A real projective manifold is said to be $i$-convex if every real projective immersion

$$T^o \cup F_2 \cup \cdots \cup F_{i+2} \to M$$

extends to one from $T$ itself.

**Theorem 1.1** (Main). *Suppose that $M$ is a compact real projective $n$-manifold with totally geodesic or empty boundary. If $M$ is not $(n-1)$-convex, then $M$ includes a compact concave affine $n$-submanifold $N$ of type I or II or $M^o$ includes the two-faced $(n-1)$-submanifold.*

We will define the term “two-faced $(n-1)$-submanifolds of type I and II” in Definitions 6.2 and 7.1 which arise in separate constructions. But they are totally geodesic and are quotients of an open domain in an affine space by a group of projective transformations. They are canonically defined. We will define the term concave affine $n$-submanifold in
Definition 9.1: A concave affine \( n \)-manifold \( M \) is a compact real projective manifold with concave boundary such that its universal cover is a union of overlapping \( n \)-crescents. An \( n \)-crescent is a convex \( n \)-ball whose bounding sides except one is in the “infinity” in the completion of the universal or holonomy cover (see Section 3). Their interiors are projectively diffeomorphic to either a half-space or an open hemisphere. They are really generalization of half-spaces as one of the side is at “infinity” or “missing”. The manifold-interior \( M^o \) of a concave affine manifold admits a projectively equivalent affine structure of very special nature. We expect them to be very limited.

Let \( A \) be a properly imbedded \((n-1)\)-manifold in \( M^o \), which may or may not be two-sided and not necessarily connected or totally geodesic. The so-called splitting \( S \) of \( M \) along \( A \) is obtained by completing \( M - N \) by adding boundary which consists of either the union of two disjoint copies of components of \( A \) or a double cover of components of \( A \) (see the beginning of Section 10).

A manifold \( N \) decomposes into manifolds \( N_1, N_2, \ldots \) if there exists a properly imbedded \((n-1)\)-submanifold \( \Sigma \) so that \( N_i \) are components of the manifold obtained from splitting \( M \) along \( \Sigma \); \( N_1, N_2, \ldots \) are said to be the resulting manifolds of the decomposition.

**Corollary 1.1.** Suppose that \( M \) is compact but not \((n-1)\)-convex. Then

1. after splitting \( M \) along the two-faced \((n-1)\)-manifold \( A_1 \) arising from hemispheric \( n \)-crescents, the resulting manifold \( M^s \) decomposes properly into concave affine manifolds of type I and real projective \( n \)-manifolds with totally geodesic boundary which does not include any concave affine manifolds of type I.

2. We let \( N \) be the disjoint union of the resulting manifolds of the above decomposition other than concave affine ones. After cutting \( N \) along the two-faced \((n-1)\)-manifold \( A_2 \) arising from bihedral \( n \)-crescents, the resulting manifold \( N^s \) decomposes into concave affine manifolds of type II and real projective \( n \)-manifolds with convex boundary which is \((n-1)\)-convex and includes no concave affine manifold of type II.

Furthermore, \( A_1 \) and \( A_2 \) are canonically defined and the decompositions are also canonical in the following sense: If \( M^s \) equals \( N \cup K \) for \( K \) the union of concave affine manifolds of type I in \( M^s \) and \( N \) the closure of the complement of \( K \) includes no concave affine manifolds of type I, then the above decomposition agree with the decomposition into components of submanifolds in (1). If \( N^s \) equals \( S \cup T \) for \( T \) the union of concave affine manifold of type II in \( N^s \) and \( S \) the closure of the complement of \( T \) that is \((n-1)\)-convex
and includes no concave affine manifold of type II, then the decomposition agree with the decomposition into components of submanifolds in (2).

If \( A_1 = \emptyset \), then we define \( M^s = M \) and if \( A_2 = \emptyset \), then we define \( N^s = N \). In Section 7, we will give a 2-dimensional example with a nontrivial splitting (see Example 7.2).

We note that \( M, M^s, N, N^s \) have totally geodesic or empty boundary, as we will see in the proof. The final decomposed pieces of \( N^s \) are not so. Concave affine manifolds of type II have in general boundary concave seen from its inside and the \((n-1)\)-convex real projective manifolds have convex boundary seen from inside (see Section 3).

Compare this corollary with what we have proved in [9] and [10] in the language of this paper, as the term “decomposition” is used somewhat differently there.

**Theorem 1.2.** Let \( \Sigma \) be a compact real projective surface with totally geodesic or empty boundary. Suppose \( \chi(\Sigma) < 0 \). Then \( \Sigma \) decomposes along the union of disjoint simple closed curves into convex real projective surfaces.

Our Corollary 1.1 is strong enough to imply Theorem 1.2, but we need to work out the classification of concave affine 2-manifolds to do so. As a corollary to Corollary 1.1, we get one further result.

**Corollary 1.2.** Suppose \( \chi(\Sigma) = 0 \). Then \( \Sigma \) decomposes into convex annuli, Möbius bands, and concave affine 2-manifolds of Euler characteristic 0.

This paper will be written as self-contained as possible on projective geometry and will use no highly developed machinery but will use perhaps many aspects of discrete group actions and geometric convergence in the Hausdorff sense joined in a rather complicated manner. Objects in this papers are all very concrete ones. To grasp these ideas, one only need to have some graduate student in geometry understanding and visualization of higher-dimensional projective and spherical geometry.

A holonomy cover of \( M \) is given as the cover of \( M \) corresponding to the kernel of the developing map. We often need not look at the universal cover but the holonomy cover as it carries all information and we can define the developing map and holonomy homomorphism from it. The so-called Kuiper completion or projective completion of the universal or holonomy cover is the completion with respect to a metric pulled from \( S^n \) by a developing map. This notion was introduced by Kuiper for conformally flat manifolds (see Kuiper [24]).
In Section 1, we will give preliminary definitions and define and classify convex sets in $S^n$. In Section 2, we discuss the Kuiper or projective completions $\tilde{M}$ or $\tilde{M}_h$ of the universal cover $\tilde{M}$ or the holonomy cover $M_h$ respectively and convex subsets of them, and discuss how two convex subsets may intersect, showing that in the generic case they can be read from their images in $S^n$.

The main ideas in this paper is to get good geometric objects in the universal cover of $M$. Loosely speaking, we illustrate our plan as follows:

(i) For a compact manifold $M$ which is not $(n-1)$-convex, obtain an $n$-crescent in $\tilde{M}_h$.
(ii) Divide the case into two cases where $\tilde{M}_h$ includes hemispheric $n$-crescents and where there are only $n$-crescents which are bihedral.
(iii) We derive a certain equivariance properties of hemispheric $n$-crescents or the unions of a collection of bihedral $n$-crescents equivalent to each other under the equivalence relation generated by the overlapping relation. That is, we show that any two of such sets either agree, are disjoint, or meet only in the boundary.
(iv) We show that the boundary where the two collections meet covers a closed codimension-one submanifold called the two-faced submanifolds. If we split along these, then the collection is now truly equivariant. From the equivariance, we obtain a submanifold covered by them called the concave affine manifold. This completes the proof of the Main Theorem.
(v) Apply the Main Theorem in sequence to prove Corollary 1.1; that is, we split along the two-faced manifolds and obtain concave affine manifolds for hemispheric $n$-crescent case and then bihedral $n$-crescent case.

In Section 4, we prove a central theorem that given a real projective manifold which is not $(n-1)$-convex, we can find an $n$-crescent in the projective completion. The argument is the blowing up or pulling back argument as we saw in 4.

In Section 5, we generalize the transversal intersection of crescents to that of $n$-crescents (see 9). This shows that they intersect in a manageable manner as their sides in the frontier extend each other and the remaining sides intersecting transversally.

In Section 6, when $\tilde{M}_h$ includes a hemispheric $n$-crescent, we show how to obtain a two-faced $(n-1)$-submanifold. This is accomplished by the fact that two hemispheric crescents are either disjoint, equal, or meet only in the boundary, i.e., at a totally geodesic $(n-1)$-manifold. In Section 7, we assume that $\tilde{M}_h$ includes no hemispheric $n$-crescent but includes bihedral $n$-crescents. We define equivalence classes of bihedral $n$-crescents.
Two bihedral $n$-crescents are equivalent if there exists a chain of bihedral $n$-crescents overlapping with the next ones in the chain. This enables us to define $\Lambda(R)$ the union of $n$-crescents equivalent to a given $n$-crescent $R$. Given $\Lambda(R)$ and $\Lambda(S)$ for two $n$-crescents $R$ and $S$, they are either disjoint, equal, or meet at a totally geodesic $(n-1)$-submanifold. We obtain a two-faced $(n-1)$-submanifold from the totally geodesic $(n-1)$-submanifolds, which covers closed totally geodesic $(n-1)$-submanifolds in $M$.

In Section 8, we show what happens to $n$-crescents if we take submanifolds or splits manifolds in the corresponding completions of the holonomy cover. They are all preserved.

In Section 9, we prove the Main Theorem: If there is no two-faced submanifold of type I, then two hemispheric $n$-crescents are either disjoint or equal. The union of all hemispheric $n$-crescents is invariant under deck transformations and hence covers a submanifold in $M$, a concave affine manifold of type I. If there is no two-faced submanifold of type II, then $\Lambda(R)$ and $\Lambda(S)$ for two $n$-crescents $R$ and $S$ are either disjoint or equal. Again since the deck transformation group acts on the union of $\Lambda(R)$ for all $n$-crescents $R$, the union covers a manifold in $M$, which is a concave affine manifold of type II.

In Section 10, we prove Corollary 1.1. That is, we decompose real projective manifolds. We show that when we have a two-faced submanifold, we can cut $M$ along these. The result does not have a two-faced submanifold and hence can be decomposed into $(n-1)$-convex ones and properly concave affine manifolds as in Section 8. In Section 11, we will show some consequence or modification of our result for affine Lie groups. In Appendix A, we show that a real projective manifold is convex if and only if it is a quotient of a convex domain in $\mathbb{S}^n$. In Appendix B, we study some questions on shrinking sequences of convex balls in $\mathbb{S}^n$ that are needed in Section 4.

A real projective structure on a Lie group is left-invariant if left-multiplications preserve the real projective structure. The following theorem is also applicable to real projective structures on homogeneous manifolds invariant with respect to a proper group actions (see Theorem 11.2).

**Theorem 1.3.** Let $G$ be a Lie group with left-invariant real projective structure. Then either $G$ is $(n-1)$-convex or $\tilde{G}$ is projectively diffeomorphic to the universal cover of the complement of a closed convex set in $\mathbb{R}^n$ with induced real projective structure.

The $(n-1)$-convexity of affine structures are defined similarly. This theorem easily translates to one on affine Lie groups.
Corollary 1.3. Suppose that $G$ has a left-invariant affine structure. Then either $G$ is $(n - 1)$-convex or $\tilde{G}$ is affinely diffeomorphic to the universal cover of a complement of a closed convex set in $\mathbb{R}^n$ with induced affine structure.

We benefited greatly from conversations with Thierry Barbot, Yves Benoist, Yves Carrière, William Goldman, Craig Hodgson, Michael Kapovich, Steven Kerckhoff, Hyuk Kim, François Labourie, John Millson, and William Thurston.
In this foundational section, we will discuss somewhat slowly the real projective geometry of \( \mathbb{R}P^n \) and the sphere \( S^n \), and discuss convex subsets of \( S^n \). We will give classification of convex subsets and give topological properties of them. We end with the geometric convergence of convex subsets. (We assume that the reader is familiar with convex sets in affine spaces, which are explain in Berger [6] and Eggleston [21] in detailed and complete manner.)

The real projective space \( \mathbb{R}P^n \) is the quotient space of \( \mathbb{R}^{n+1} - \{O\} \) by the equivalence relation \( x \sim y \) if \( x = sy \) for two nonzero vectors \( x \) and \( y \) and a nonzero real number \( s \). The group \( \text{GL}(n+1, \mathbb{R}) \) acts on \( \mathbb{R}^{n+1} - \{O\} \) linearly and hence on \( \mathbb{R}P^n \), but not effectively. However, the group \( \text{PGL}(n+1, \mathbb{R}) \) acts on \( \mathbb{R}P^n \) effectively. The action is analytic, and hence any element acting trivially in an open set has to be the identity transformation. (We will assume that \( n \geq 2 \) for convenience.)

Real projective geometry is a study of the invariant properties of the real projective space \( \mathbb{R}P^n \) under the action of \( \text{PGL}(n+1, \mathbb{R}) \). Given an element of \( \text{PGL}(n+1, \mathbb{R}) \) we identify it with the corresponding projective automorphism of \( \mathbb{R}P^n \).

Here by a real projective manifold, we mean an \( n \)-manifold with a maximal atlas of charts to \( \mathbb{R}P^n \) where the transition functions are projective. This lifts all local properties of real projective geometry to the manifold. A real projective map is an immersion from a real projective \( n \)-manifold to another one which is projective under local charts. More precisely, a function \( f : M \to N \) for two real projective \( n \)-manifolds \( M \) and \( N \) is real projective if it is continuous and for each pair of charts \( \phi : U \to \mathbb{R}P^n \) for \( M \) and \( \psi : V \to \mathbb{R}P^n \) for \( N \) such that \( U \) and \( f^{-1}(V) \) overlap, the function

\[
\psi \circ f \circ \phi^{-1} : \phi(U \cap f^{-1}(V)) \to \psi(f(U) \cap V)
\]

is a restriction of an element of \( \text{PGL}(n+1, \mathbb{R}) \) (see Ratcliff [25]).

It will be very convenient to work on the simply connected, spheres \( S^n \) the double cover of \( \mathbb{R}P^n \) as \( S^n \) is orientable and it is easier to study convex sets. We may identify the standard unit sphere \( S^n \) in \( \mathbb{R}^{n+1} \) with the quotient space of \( \mathbb{R}^{n+1} - \{O\} \) by the equivalence relation \( x \sim y \) if \( x = sy \) for nonzero vectors \( x \) and \( y \) and \( s > 0 \). As above \( \text{GL}(n+1, \mathbb{R}) \) acts on \( S^n \). The subgroup \( \text{SL}_{\pm}(n+1, \mathbb{R}) \) of linear maps of determinant \( \pm 1 \) acts on \( S^n \) effectively. We see easily that \( \text{SL}_{\pm}(n+1, \mathbb{R}) \) is a double cover of \( \text{PGL}(n+1, \mathbb{R}) \).
We denote by \( \text{Aut}(S^n) \) the isomorphic group of automorphisms of \( S^n \) which is induced by an element of \( \text{SL}_{\pm}(n + 1, \mathbb{R}) \).

Since \( \mathbb{R}P^n \) has an obvious chart to itself, namely the identity map, it has a maximal atlas containing this chart. Hence, \( \mathbb{R}P^n \) has a real projective structure. Since \( S^n \) is a double cover of \( \mathbb{R}P^n \), and the covering map \( p \) is a local diffeomorphism, it follows that \( S^n \) has a real projective structure. We see easily that each element of \( \text{Aut}(S^n) \) are real projective maps. Conversely, each real projective automorphism of \( S^n \) is an element of \( \text{Aut}(S^n) \) as the actions are locally identical with those of elements of \( \text{Aut}(S^n) \). There is a following convenient commutative diagram:

\[
\begin{array}{ccc}
S^n & \xrightarrow{g} & S^n \\
\downarrow p & & \downarrow p \\
\mathbb{R}P^n & \xrightarrow{g'} & \mathbb{R}P^n
\end{array}
\] (1)

where given a real projective automorphism \( g \), a real projective map \( g' \) always exists and given \( g' \), we may obtain \( g \) unique up to the antipodal map \( A_{S^n} \) which sends \( x \) to \(-x\) for each unit vector \( x \) in \( S^n \). (Note that \( S^n \) with this canonical real projective structure is said to be a real projective sphere.)

The standard sphere has a standard Riemannian metric \( \mu \) of curvature 1. We denote by \( d \) the distance metric on \( S^n \) induced from \( \mu \). The geodesics of this metric are arcs on a great circles parameterized by \( d \)-length. This metric is projectively flat, and hence geodesics of the metric agrees with projective geodesics up to choices of parametrization.

A convex line is an embedded geodesic in \( S^n \) of \( d \)-length less than or equal to \( \pi \). A convex set is a subset of \( S^n \) such that any two points of \( A \) is connected by a convex segment in \( A \). A simply convex subset of \( S^n \) is a convex subset such that every pair of point is connected by a convex segment of \( d \)-length \( < \pi - \epsilon \) for a positive number \( \epsilon \). (Note that all these are projectively invariant properties.) A singleton, i.e., the set consisting of a point, is convex and simply convex.

A great 0-dimensional sphere is the set of points antipodal to each other. This is not convex. A great \( i \)-dimensional sphere in \( S^n \) for \( i \geq 1 \) is convex but not simply convex. A great \( i \)-dimensional hemisphere, \( i \geq 1 \), is the closure of a component of a great \( i \)-sphere \( S^i \) removed with a great \((i - 1)\)-sphere \( S^{i-1} \) in \( S^i \). It is a convex but not simply convex subset. A 0-dimensional hemisphere is simply a singleton.

Given a codimension one subspace \( \mathbb{R}P^{n-1} \) of \( \mathbb{R}P^n \), the complement of \( \mathbb{R}P^n \) can be identified with an affine space \( \mathbb{R}^n \) so that geodesic structures agree, i.e., the projective
geodesics are affine ones and vice versa up to parameterization. Given an affine space \( \mathbb{R}^n \), we can compactify it to a real projective space \( \mathbb{R}P^n \) by adding points (see Berger [6]). Hence the complement \( \mathbb{R}P^n - \mathbb{R}P^{n-1} \) is called an affine patch. An open \( n \)-hemisphere in \( S^n \) maps homeomorphic onto \( \mathbb{R}P^n - \mathbb{R}P^{n-1} \) for a subspace \( \mathbb{R}P^{n-1} \). Hence, the open \( n \)-hemisphere has a natural affine structure of \( \mathbb{R}^n \) whose geodesic structure is same as that of the projective structure. An open \( n \)-hemisphere is sometimes called an affine patch. A bounded set in \( \mathbb{R}^n \) convex in the affine sense is convex in \( S^n \) by our definition when \( \mathbb{R}^n \) is identified with the open \( n \)-hemisphere in this manner.

We give a definition given in [25]: A pair of points \( x \) and \( y \) is proper if they are not antipodal. A minor geodesic connecting a proper pair \( x \) and \( y \) is the shorter arc in the great circle passing through \( x \) and \( y \) with boundary \( x \) and \( y \).

The following theorem shows the equivalence of our definition to the definition given in [25] except for pairs of antipodal points.

**Theorem 2.1.** A set \( A \) is a convex set or a pair of antipodal points if and only if for each proper pair of points \( x, y \) in \( A \), \( A \) includes a minor geodesic \( xy \) in \( A \) connecting \( x \) and \( y \).

**Proof.** If \( A \) is convex, then given two proper pair of points the convex segment in \( A \) connecting them is clearly a minor geodesic. A pair of antipodal points has no proper pair.

Conversely, let \( x \) and \( y \) be two points of \( A \). If \( x \) and \( y \) are proper then since a minor geodesic is convex, we are done. If \( x \) and \( y \) are antipodal, and \( A \) equals \( \{ x, y \} \), then we are done. If \( x \) and \( y \) are antipodal, and there exists a point \( z \) in \( A \) distinct from \( x \) and \( y \), then \( A \) includes the minor segment \( zx \) and \( yz \) and hence \( zx \cup yz \) is a convex segment connecting \( x \) and \( y \); \( A \) is convex.

By above theorem, we see that our convex sets satisfy the properties in Section 6.2 of [25]. Let \( A \) be a nonempty convex subset of \( S^n \). The *dimension* of \( A \) is defined to be the least integer \( m \) such that \( A \) is included in a great \( m \)-sphere in \( S^n \). If \( \dim(A) = m \), then \( A \) is included in a unique great \( m \)-sphere which we denote by \( < A > \). The interior of \( A \), denoted by \( A^\circ \), is the topological interior of \( A \) in \( < A > \), and the boundary of \( A \), denoted by \( \partial A \), is the topological boundary of \( A \) in \( < A > \). The closure of \( A \) is denoted by \( \text{Cl}(A) \) and is a subset of \( < A > \). \( \text{Cl}(A) \) is convex and so is \( A^\circ \). (These are from Ratcliff [25].) Moreover, the intersection of two convex sets is either convex or is a pair of antipodal points by the above theorem. Hence, the intersection of two convex sets is convex if it
contains at least three points, and it contains a pair of nonantipodal points, or one of the sets does not contain a pair of antipodal points.

A convex hull of a set is the minimal convex containing the set.

Lemma 2.1. Let $A$ be a convex set. $A^\circ$ is not empty unless $A$ is empty.

Proof. Let $<A>$ have dimension $k$. Then $A$ has to have at least $k+1$ points $p_1, \ldots, p_{k+1}$ in general position as unit vectors in $\mathbb{R}^{n+1}$ since otherwise every $(k+1)$-tuple of vectors are dependent and $A$ is a subset of a great sphere of lower dimension. The convex hull of the points $p_1, \ldots, p_{k+1}$ is easily shown to be a spherical simplex with vertices $p_1, \ldots, p_{k+1}$. The simplex is obviously a subset of $A$, and the interior of the simplex is included in $A^\circ$. $\square$

We give the following classification of convex sets in the following two theorems.

Theorem 2.2. Let $A$ be a convex subset of $S^n$. Then $A$ is one of the following sets:

- a great sphere $S^i$, $1 \leq i \leq n$,
- an $i$-dimensional hemisphere $H^i$, $0 \leq i \leq n$,
- a proper convex subset of an $i$-hemisphere $H^i$.

Proof. We will prove by induction on dimension $m$ of $<A>$. The theorem is obvious for $m = 0, 1$. Suppose that the theorem holds for $m = k - 1, k \geq 2$.

Suppose now that the dimension of $A$ equals $m$ for $m = k$. Let us choose a hypersphere $S^{m-1}$ in $<A>$ intersecting with $A^\circ$. Then $A_1 = A \cap S^{m-1}$ is as one of the above (1), (2), (3). The dimension of $A_1$ is at least one, i.e., $m - 1 \geq 1$. Suppose $A_1 = S^{m-1}$. As $A^\circ$ has two points $x, y$ respectively in components of $<A> - S^{m-1}$, taking the union of segments from $x$ to points of $S^{m-1}$, and segments from $y$ to points of $S^{m-1}$, we obtain that $A = <A>$.

If $A_1$ is as in (2) or (3), then choose an $(m-1)$-hemisphere $H$ containing $A_1$ with boundary a great $(m-2)$-sphere $\partial H$. Consider the collection $\mathcal{P}$ of all $(m-1)$-hemispheres including $\partial H$. Then $\mathcal{P}$ has a natural real projective structure of a great circle, and let $A'$ be the set of the $(m-1)$-hemispheres in $\mathcal{P}$ whose interior meets $A$. Then since a convex segment in $<A> - \partial H$ project to a convex segment in the circle $\mathcal{P}$, it follows that $A'$ has the property that any two proper pair of points of $A'$ is connected by a minor geodesic, and by Theorem 2.1 $A$ is either a pair of antipodal points or a convex subset.

Let $-H$ denote the closure of the complement of $<A> - H$. Then the interior of $-H$ do not meet $A$ as it does not meet $A_1$. Hence $A'$ is a subset of $\mathcal{P} - \{-H\}$. 

If \( A' \) is a pair of antipodal points, then \( A' \) must be \( \{H, -H\} \), and this is a contradiction. Since \( A' \) is a proper convex subset of \( \mathcal{P} \), \( A' \) must be a convex subset of a 1-hemisphere \( I \) in \( \mathcal{P} \). This means that only the interior of \( (m-1) \)-hemispheres in \( I \) meets \( A \), and there exists an \( m \)-hemisphere in \( <A> \) including \( A \). Thus \( A \) either equals this \( m \)-hemisphere or a proper convex subset of it.

\[ \square \]

**Theorem 2.3.** Let \( A \) be a proper convex subset of an \( i \)-dimensional hemisphere \( H^i \) for \( i \geq 1 \). Then exactly one of the following holds:

- \( \partial A \) contains a unique maximal great \( j \)-sphere \( S^j \) for some \( 0 \leq j \leq i-1 \), which must be in \( \partial H^i \) and its closure is the union of \( (j+1) \)-hemispheres with common boundary \( S^j \), or
- \( A \) is a simply convex subset of \( H^i \), in which case \( A \) can be realized as a bounded convex subset of perhaps another open \( i \)-hemisphere \( K^i \) identified with an affine space \( \mathbb{R}^i \).

**Proof.** We assume without loss of generality that \( A \) is closed by taking a closure of \( A \) if necessary. If \( A \) includes a pair of antipodal points, then \( A \) do not satisfy the Kobayashi’s criterion [17]. Then by Section 1.4 of [17], we have the first item.

If \( A \) includes no pair of antipodal points, then let \( m \) be the dimension of \( <A> \) and we do the induction over \( m \). If \( m = 0, 1 \), then the second item is obvious. Suppose we have the second item holding for \( m = k-1 \), where \( k \geq 2 \). Now let \( m = k \), and choose a great sphere \( S^{m-1} \) meeting \( A^o \), and let \( A_1 = A \cap S^{m-1} \). Then \( A_1 \) is another simply convex set. Hence, \( A_1 \) is a bounded convex subset of an open \( (m-1) \)-hemisphere \( K \) identified as an affine space \( \mathbb{R}^{m-1} \). Hence \( A_1 \) does not meet \( \partial K \). As in the proof of Theorem 2.2, we let \( \mathcal{P} \) be the set of all \( (m-1) \)-hemispheres with boundary in \( \partial K \), which has a natural real projective structure of a great circle. As in the proof, we see that the subset \( A' \) of \( \mathcal{P} \) consisting of hemispheres whose interior meets \( A \) is a convex subset of a 1-hemisphere in \( \mathcal{P} \). The boundary of \( A' \) consists of two hemispheres \( H_1 \) and \( H_2 \). Since \( A' \) is connected, \( H_1 \) and \( H_2 \) bound a convex subset \( L \) in \( <A> \) and \( H_1 \) and \( H_2 \) meet in a \( \mu \)-angle less than or equal to \( \pi \).

If the angle between \( H_1 \) and \( H_2 \) equals \( \pi \), then \( H_1 \cup H_2 \) is a great \( (m-1) \)-sphere, and \( H^o_1 \) and \( H^o_2 \) includes two points \( p, q \) of \( A \) respectively which are not antipodal. Since \( A \) is convex, \( \overline{pq} \) is a subset of \( A \); since \( p \) and \( q \) is not antipodal, \( \overline{pq} \) meets \( \partial K \) by geometry, a contradiction.

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Since the angle between $H_1$ and $H_2$ is less than $\pi$, it is now obvious that there exists an $m$-hemisphere $H$ containing $A$ and meeting $L$ only at $\partial K$. Hence $A$ is a convex subset of $H^o$. Since $A$ is compact, $A$ is a bounded convex subset of $H^o$. \hfill \box

An $m$-bihedron in $S^n$ is the closure of a component of a great sphere $S^m$ removed with two great spheres of dimension $m - 1$ in $S^m \ (m \geq 1)$. A 1-bihedron is a simply convex segment.

**Lemma 2.2.** A compact convex subset $K$ of $S^n$ including an $(n-1)$-hemisphere is either the sphere $S^n$ itself, a great $(n-1)$-sphere, an $n$-hemisphere, or an $n$-bihedron.

**Proof.** Let $H$ be the $(n-1)$-hemisphere in $K$ and $s$ the great circle perpendicular to $H$ at the center of $H$. Then since $K$ is convex, $s \cap K$ is a convex subset of $s$ or a pair of antipodal point as in the proof of Theorem 2.2. If $s \cap K = s$, then every segment from a point of $s$ to a point of $H$ belongs to $K$ by convexity. Thus, $K = S^n$. Depending on whether $s \cap K$ is a pair of antipodal points, a great segment or a simply convex segment (see [9]), $K$ is a great $(n-1)$-sphere, an $n$-hemisphere or an $n$-bihedron. \hfill \box

We will now discuss the topological properties of convex sets.

**Theorem 2.4.** Let $A$ be a convex $m$-dimensional subset of $S^n$ other than a great sphere. Then $A^o$ is homeomorphic to an open $m$-ball, $\text{Cl}(A)$ the compact $m$-ball, and $\partial A$ to the sphere of dimension $m - 1$.

**Proof.** If $A$ is zero or one dimensional, then the theorem is obvious. Assume that $m \geq 2$ from now on. $A$ is a subset of a closed $m$-hemisphere $H$. Choose a point $p$ of $H$ in $A^o$, which must be a point of $A^o \cap H^o$. Then for each point $q$ of $A$, $[p, q), [p, q) = \overline{pq} - \{q\}$ is a subset of $A^o$ (see Theorem 6.2.2 of [25]) where $\overline{pq}$ denotes the unique convex segment connecting $p$ and $q$. Hence, for each ray $r$ from $p$ to a point of $\partial H$, $r \cap A$ is a ray with endpoints $r$ and a point $q_r$ in $r \cap \partial A$ so that $[p, q_r) \subset A^o$. We define a function $f : \partial H \to \mathbb{R}$ by letting $f(x)$ equal the $d$-length of $\overline{pq}$ where $r$ is the ray from $p$ to $x$. Then $f$ is obviously bounded.

We claim that $f$ is a continuous function. Suppose not. Then there exists a sequence $y_i$ in $\partial H$ converging to a point $y$ of $\partial H$ so that $|f(y_i) - f(y)| > \delta$ for a small positive number $\delta$. Then we see that the corresponding $q_{r_i}$ for $r_i$ a ray connecting $p$ and $y_i$ converges the limit $y^*$ on the ray $r$ distinct from $y$. Since $\partial A$ is closed, we see that $y^*$ is a point of $\partial A$. Whether $y$ lies within $[p, q_r)$ or in $r - [p, q_r]$, we get contradiction by Theorem 6.2.2 of [25].
Now, we can follow Section 11.3.1 of Berger\(^{[3]}\) to see that \(A^o\) is homeomorphic to \(H^o\), \(A\) to \(H\), and \(\partial A\) to the sphere of dimension \(m - 1\).

Let \(A\) be an arbitrary subset of \(S^n\). A **supporting** hypersphere \(L\) for \(A\) is a great \((n - 1)\)-sphere containing \(x\) in \(A\) such that the two closed hemispheres determined by \(L\) includes \(A\) and \(x\) respectively. We say that \(L\) is the **supporting** hypersphere for \(A\) at \(x\).

**Theorem 2.5.** Let \(A\) be a convex subset of \(S^n\), other than \(S^n\) itself. Then for each point of \(\partial A\), there exists a supporting hypersphere for \(A\) at \(x\).

**Proof.** If the dimension \(i\) of \(A\) is 0, this is trivial. Assume \(i \geq 1\). If \(A\) is a great \(i\)-sphere or an \(i\)-hemisphere \(i \geq 1\), it is obvious. If not, then \(A\) is contained an \(i\)-hemisphere, say \(H\). Then \(A^o\) is a convex subset of the affine space \(H^o\). Hence, there exists a supporting hyperplane \(K\) for \(A^o\) at \(x\) by Proposition 11.5.2 of \([1]\). The hyperplane \(K\) equals \(L \cap H^o\) for a great \((i - 1)\)-sphere \(L\) in \(< A >\). Thus any great \((n - 1)\) sphere \(P\) meeting \(< A >\) at \(\partial H\) is the supporting hypersphere for \(A\) at \(x\).

We define a Hausdorff distance between all compact subsets of \(S^n\). We say that two compact subsets \(X, Y\) have distance less than \(\epsilon\), if \(X\) is in a \(d\)-\(\epsilon\)-neighborhood of \(Y\) and \(Y\) is in one of \(X\). This defines a metric on the space of all compact subsets of \(S^n\).

Suppose that a sequence of compact sets \(K_i\) converges to \(K_\infty\). If \(x \in K_\infty\), then by definition for any positive number \(\epsilon\), there exists an \(N\) so that for \(i > N\), there exists a point \(x_i \in K_i\) so that \(d(x, x_i) < \epsilon\). Also, given a point \(x\) of \(S^n\), so that a sequence \(x_i \in K_i\) converges to \(x\), then \(x\) lies in \(K_\infty\). If otherwise, \(x\) is at least \(\delta\) away from \(K_\infty\) for \(\delta > 0\), and so the \(\delta/2\)-\(d\)-neighborhood of \(K_\infty\) is disjoint from an open neighborhood \(J\) of \(x\). But since \(x_i \in J\) for \(i\) sufficiently large, this contradicts \(K_i \to K_\infty\).

**Theorem 2.6.** Given a sequence of compact convex subsets \(K_i\) of \(S^n\), we can always choose a subsequence converging to a subset \(K_\infty\). \(K_\infty\) is compact and convex. Also if \(K_i\) are \(i\)-balls, then \(K_\infty\) is a convex ball of dimension less than or equal to \(i\). If \(\dim K_\infty = n\), then we have \(\bigcup_{i=1}^\infty K_i^o \supset K_\infty^o\). In this case \(\partial K_i \to \partial K_\infty\).

**Proof.** The first statement follows from the well-known compactness of the spaces of compact subsets of compact Hausdorff spaces under Hausdorff metrics.

For each point \(x\) of \(K_\infty\), there exists a sequence \(x_i \in K_i\) converging to \(x\). Choose arbitrary two points \(x\) and \(y\) of \(K_\infty\), and sequences \(x_i \in K_i\) and \(y_i \in K_i\) converging to \(x\) and \(y\) respectively. Then there exists a segment \(\overline{x_iy_i}\) of \(d\)-length \(\leq \pi\) in \(K_i\) connecting \(x_i\) and \(y_i\). Since the sequence of \(\overline{x_iy_i}\) is a sequence of compact subsets of \(S^n\), we may assume
that a subsequence converges to a compact subset $L$ of $S^n$. By the above paragraph $L \subset K_\infty$. By elementary geometry, it is easy to see that $L$ is a segment of $d$-length $\leq \pi$. Thus $K_\infty$ is convex.

If $K_i$ are $i$-balls, then $K_i \subset H_i$ for $i$-hemispheres $H_i$. We choose a subsequence $i_j$ of $i$ so that $H_{i_j}$ converges to an $i$-hemisphere $H$. If follows that $K_\infty$ is a subset of $H$ by the paragraph above our lemma since $K_{i_j}$ converges to $K_\infty$. Thus, $K_\infty$ is an compact convex subset of $H_\infty$, which shows that $K_\infty$ is a convex ball of dimension $\leq i$.

The third and final statements follow as in Section 2 of Appendix of [9]. The dimension does not play a role.
3. Convex subsets in the Kuiper completion of the universal and holonomy covers.

In this second foundational section, we begin by lifting the development pair to the real projective sphere $S^n$. To make our discussion much more simpler, we will define a completion, called a Kuiper completion or projective completion, by inducing the Riemannian metric of the sphere to the universal cover $\tilde{M}$ or the holonomy cover $M_h$ of $M$ and then completing them in the Cauchy sense. Then we define the ideal set to be the completion removed with $\tilde{M}$ or $M_h$, i.e., points infinitely far away from points of $\tilde{M}$ or $M_h$.

We will define convex sets in these completions, which are always isomorphic to ones in $S^n$. Then we will introduce $n$-crescents, which are convex $n$-balls in the completions where a side or an $(n-1)$-hemisphere in the boundary lies in the ideal sets. We show how two convex subsets of the completion may intersect; their intersection properties are described by their images in $S^n$ under the developing map. Finally, we describe the dipping intersection, the type of intersection which will be useful in this paper, and on which our theory of $n$-crescents depend heavily as we will see in Section 5.

We will assume that our manifolds in this paper have dimension $\geq 2$ unless stated otherwise. Let $M$ be a real projective $n$-manifold. Then $M$ has a development pair $(\text{dev}, h)$ of an immersion $\text{dev} : \tilde{M} \to \mathbb{R}P^n$, called a developing map, and a holonomy homomorphism $h : \pi_1(M) \to \text{PGL}(n+1, \mathbb{R})$ satisfying $\text{dev} \circ \gamma = h(\gamma) \circ \text{dev}$ for every $\gamma \in \pi_1(M)$. Such a pair is determined up to an action of an element $\vartheta$ of $\text{PGL}(n+1, \mathbb{R})$ as follows:

$$(\text{dev}, h(\cdot)) \mapsto (\vartheta \circ \text{dev}, \vartheta \circ h(\cdot) \circ \vartheta^{-1}).$$

Developing maps are obtained by analytically extending coordinate charts in the atlas. Holonomy homomorphisms are obtained from the chosen developing map. See Ratcliff [25] for more details. The development pair characterizes the real projective structure, and hence another way to give a real projective structure to a manifold is to find a pair $(f, k)$ where $f$ is an immersion $\tilde{M} \to \mathbb{R}P^n$ which is equivariant with respect to the homomorphism $k$ from the group of deck transformations to $\text{PGL}(n+1, \mathbb{R})$.

We assume that the manifold-boundary $\delta M$ of a real projective manifold $M$ is totally geodesic unless stated otherwise. This means that for each point of $\delta M$, there exist an open neighborhood $U$ and a lift $\phi : U \to S^n$ of a chart $U \to \mathbb{R}P^n$ so that $\phi(U)$ is a
nonempty intersection of a closed \( n \)-hemisphere with a simply convex open set. (By an \( n \)-hemisphere, we mean a closed hemisphere unless we mention otherwise.) \( \delta M \) is said to be \textit{convex} if there exists an open neighborhood \( U \) and a chart \( \phi \) for each point of \( \delta M \) so that \( \phi(U) \) is a convex domain in \( S^n \). \( \delta M \) is said to be \textit{concave} if there exists a chart \( (U, \phi) \) for each point of \( \delta M \) so that \( \phi(U) \) is the complement of a convex open set in an open simply convex subset of \( S^n \). Note that if \( M \) has totally geodesic boundary, then so do all of its covers. The same is true for convexity and concavity of boundary.

**Lemma 3.1.** Suppose that a connected totally geodesic \((n-1)\)-submanifold \( S \) of \( M \) of codimension \( \geq 1 \) intersects \( \delta M \) in its interior point. Then \( S \subset \delta M \).

**Proof.** The intersection point must be a tangential intersection point. Since \( \delta \tilde{M} \) is a closed subset of \( \tilde{M} \), the set of intersection of \( S \) and \( \delta \tilde{M} \) is an open and closed subset of \( S \). Hence it must be \( S \).

**Remark 3.1.** If \( \delta M \) is assume to be convex, the conclusion holds also. This was done in \([3]\) in dimension 2. The proof for the convex boundary case is the same as the dimension 2.

**Remark 3.2.** Given any two real projective maps \( f_1, f_2 : N \to \mathbb{RP}^n \) on a real projective manifold \( N \), they differ by an element of \( \text{PGL}(n+1, \mathbb{R}) \), i.e., \( f_2 = \zeta \circ f_1 \) for a projective automorphism \( \zeta \) as they are charts restricted to an open set, and they must satisfy the equation there, and by analyticity everywhere. Given two real projective automorphisms \( f_1, f_2 : N \to S^n \), we have that \( p \circ f_1 = \zeta \circ p \circ f_2 \) for \( \zeta \) in \( \text{PGL}(n+1, \mathbb{R}) \). By equation \([1]\), there exists an element \( \zeta' \) of \( \text{Aut}(S^n) \) so that \( p \circ \zeta' = \zeta \circ p \) where \( \zeta' \) and \( A_{S^n} \circ \zeta' \) are the only automorphisms satisfying the equation. This means that \( p \circ f_1 = p \circ \zeta' \circ f_2 \), and hence it follows easily that \( f_1 = \zeta' \circ f_2 \) or \( f_1 = A_{S^n} \circ \zeta' \circ f_2 \) by analyticity of developing maps. Hence, any two real projective maps \( f_1, f_2 : N \to S^n \) differ by an element of \( \text{Aut}(S^n) \).

We agree to lift our developing map \( \text{dev} \) to the standard sphere \( S^n \), the double cover of \( \mathbb{RP}^n \), where we denote the lift by \( \text{dev}' \). Then for any deck transformation \( \vartheta \) of \( \tilde{M} \), we have \( \text{dev}' \circ \vartheta = h'(\vartheta) \circ \text{dev}' \) by the above remark. Hence \( \vartheta \mapsto h'(\vartheta) \) is a homomorphism, and we see easily that \( h' \) is a lift of \( h \) for the covering homomorphism \( \text{Aut}(S^n) \to \text{PGL}(n+1, \mathbb{R}) \).

The pair \((\text{dev}', h')\) will from now on be denoted by \((\text{dev}, h)\), and they satisfy \( \text{dev} \circ \gamma = h(\gamma) \circ \text{dev} \) for every \( \gamma \in \pi_1(M) \), and moreover, given a real projective structure, \((\text{dev}, h)\) is determined up to an action of \( \vartheta \) of \( \text{Aut}(S^n) \) as in equation \([2]\) by the above remark.
The sphere $S^n$ has a standard metric $\mu$ so that its projective structure is projectively equivalent to it; i.e., the geodesics agree. Let us denote by $d$ the distance metric induced from $\mu$. From the immersion $\text{dev}$, we induce a Riemannian metric $\mu$ of $\tilde{M}$, and let $d$ denote the induced distance metric on $\tilde{M}$. The Cauchy completion of $(\tilde{M}, d)$ is denoted by $(\tilde{M}, \tilde{d})$, which we say is the Kuiper completion or projective completion of $\tilde{M}$. We define the frontier $\tilde{M}_\infty = \tilde{M} - \tilde{M}$.

Note that $\text{dev}$ extends to a distance decreasing map, which we denote by $\text{dev}$ again. Since for each $\vartheta \in \text{Aut}(S^n)$, $\vartheta$ is quasi-isometric with respect to $d$, and each deck transformations $\varphi$ of $\tilde{M}$ locally mirror the metrical property of $h(\varphi)$, it follows that the deck transformations are quasi-isometric (see [3]). Thus, each deck transformation of $\tilde{M}$ extends to a self-homeomorphism of $\tilde{M}$. The extended map will be still called a deck transformation and will be denoted by the same symbol $\varphi$ if so was the original deck transformation denoted. Finally, the equation $\text{dev} \circ \vartheta = h(\vartheta) \circ \text{dev}$ still holds for each deck transformation $\vartheta$.

The kernel $K$ of $h : \pi_1(M) \to \text{Aut}(S^n)$ is well-defined since $h$ is well-defined up to conjugation. Since $\text{dev} \circ \vartheta = \text{dev}$ for $\vartheta \in K$, we see that $\text{dev}$ induces a well-defined immersion $\text{dev}' : \tilde{M}/K \to S^n$. We say that $\tilde{M}/K$ the holonomy cover of $M$, and denote it by $M_h$. We identify $K$ with $\pi_1(M_h)$. Since any real projective map $f : M_h \to S^n$ equals $\vartheta \circ \text{dev}'$ for $\vartheta$ in $\text{Aut}(S^n)$ by Remark 3.2, it follows that $\text{dev} \circ \varphi = h'(\varphi) \circ \text{dev}$ for each deck transformation $\varphi \in \pi_1(M)/\pi_1(M_h)$. Thus, $\varphi \mapsto h'(\varphi)$ is a homomorphism $h' : \pi_1(M)/\pi_1(M_h) \to \text{Aut}(S^n)$, which is easily seen to equal $h' = h \circ \Pi$ for the quotient homomorphism $\Pi : \pi_1(M) \to \pi_1(M)/\pi_1(M_h)$.

Moreover, by Remark 3.2, $(\text{dev}', h')$ is determined up to an action of $\vartheta$ in $\text{Aut}(S^n)$ as in equation 4. Conversely, such a pair $(f, k)$ where $f : M_h \to S^n$ equivariant with respect to the homomorphism $k : \pi_1(M)/\pi_1(M_h) \to \text{Aut}(S^n)$ determines a real projective structure on $M$. From now on, we will denote $(\text{dev}', h')$ by $(\text{dev}, h)$, and call them a development pair.

Given $\text{dev}$, we may pull-back $\mu$, and complete the distance metric $d$ to obtain $\tilde{M}_h$, the completion of $M_h$, which is again called a Kuiper completion. We define the frontier $M_{h,\infty}$ to be $\tilde{M}_h - M_h$. As before the developing map $\text{dev}$ extends to a distance decreasing map, again denoted by $\text{dev}$, and each deck transformation extends to a self-homeomorphism $M_h \to \tilde{M}_h$, which we call a deck transformation still. Finally, the equation $\text{dev} \circ \vartheta = h(\vartheta) \circ \text{dev}$ still holds for each deck transformation $\vartheta$.
Figure 1. A figure of $\tilde{M}_h$. The thick dark lines indicate $\partial M_h$ and the dotted lines the ideal boundary $M_{h,\infty}$, and 2-crescents in them in the right. They can have as many “pods” and what looks like “overlapping”. Such pictures happen if we graft annuli into convex surfaces.

A subset $A$ of $\tilde{M}$ is a convex segment if $\text{dev}| A$ is an imbedding onto a convex segment in $S^n$. $M$ is convex if given two points of the universal cover $\tilde{M}$, there exists a convex segment in $\tilde{M}$ connecting these two points. A subset $A$ of $\tilde{M}$ is convex if given points $x$ and $y$ of $A$, $A$ includes a convex segment containing $x$ and $y$. We say that $A$ is a tame subset if it is a convex subset of $\tilde{M}$ or a convex subset of a compact convex subset of $\tilde{M}$. If $A$ is tame, then $\text{dev}| A$ is an imbedding onto $\text{dev}(A)$ and $\text{dev}| \text{Cl}(A)$ for the closure $\text{Cl}(A)$ of $A$ onto a compact convex set $\text{Cl}(\text{dev}(A))$. The interior $A^o$ of $A$ is defined to be the set corresponding to $\text{Cl}(\text{dev}(A))^o$ and the boundary $\partial A$ the subset of $\text{Cl}(A)$ corresponding to $\partial \text{Cl}(\text{dev}(A))$. Note that $\partial A$ may not equal the manifold boundary $\delta A$ if $A$ has a (topological) manifold structure. But if $A$ is a compact convex set, then $\text{dev}(A)$ is a manifold by Theorem 2.4, i.e., a sphere or a ball, and $\partial A$ has to equal $\delta A$. In this case, we shall use $\delta A$ over $\partial A$.

Definition 3.1. An $i$-ball $A$ in $\tilde{M}$ is a compact subset of $\tilde{M}$ such that $\text{dev}| A$ is a homeomorphism to an $i$-ball (not necessarily convex) in a great $i$-sphere and its manifold interior $A^o$ is a subset of $\tilde{M}$. A convex $i$-ball is an $i$-ball that is convex.
Note that a tame set in $\tilde{M}$ which is homeomorphic to an $i$-ball is not necessarily an $i$-ball in this sense; that is, its interior may not be a subset of $\tilde{M}$. We will say it is a *tame topological* $i$-ball but not $i$-ball or convex $i$-ball.

We define the terms convex segments, convex subset, tame subset, $i$-ball and convex $i$-ball in $\tilde{M}$ in the same manner as for $\tilde{M}$.

We will from now on will be working on $\tilde{M}_h$ only; however, all of the materials in this section will work for $\tilde{M}$ as well, and much of the materials in the remaining section will work also; however, we will not say explicitly as the readers can easily figure out these details.

**Definition 3.2.** An $n$-ball $A$ of $\tilde{M}_h$ is said to be an $n$-*bihedron* if $\text{dev}|A$ is a homeomorphism onto an $n$-bihedron. The $n$-bihedron is bounded by two $(n - 1)$-dimensional hemispheres; the corresponding subsets of $A$ are said to be the *sides* of $A$ (see [11]). An $n$-ball $A$ of $\tilde{M}_h$ is said to be an $n$-*hemisphere* if $\text{dev}|A$ is a homeomorphism onto an $n$-hemisphere in $S^n$.

A bihedron is said to be an $n$-*crescent* if one of its side is a subset of $M_{h,\infty}$ and the other side is not. An $n$-hemisphere is said to be an $n$-*crescent* if a subset in the boundary corresponding to an $(n - 1)$-hemisphere under $\text{dev}$ is a subset of $M_{h,\infty}$ and the boundary itself is not a subset of $M_{h,\infty}$.

In this paper, we will often omit ‘$n$’ from $n$-bihedron. To distinguish, a *bihedral* $n$-*crescent* is an $n$-crescent that is a bihedron, and a *hemispheric* $n$-*crescent* is an $n$-crescent that is otherwise.

Contrast to Definition [5.1], we define an $m$-*bihedron* for $1 \leq m < n$, to be only a tame topological $m$-ball whose image under $\text{dev}$ is an $m$-bihedron in a great $m$-sphere in $S^n$, and an $m$-*hemisphere*, $0 \leq m \leq n - 1$, to be one whose image under $\text{dev}$ is an $m$-hemisphere. So, we do not necessarily have $A^o \subset M_h$ when $A$ is one of these.

**Example 3.1.** Let us give two trivial examples of real projective $n$-manifolds to demonstrate $n$-crescents (see [5] for 2-dimensional examples).

Let $\mathbb{R}^n$ be an affine patch of $S^n$ with standard affine coordinates $x_1, x_2, \ldots, x_n$ and $O$ the origin. Consider $\mathbb{R}^n-\{O\}$ quotient out by the group $\langle g \rangle$ where $g : x \to 2x$ for $x \in \mathbb{R}^n-\{0\}$. Then the quotient is a real projective manifold diffeomorphic to $S^{n-1} \times S^1$. Denote the manifold by $N$, and we see that $N_h$ can be identified with $\mathbb{R}^n-\{O\}$. Thus, $\tilde{N}_h$ equals the closure of $\mathbb{R}^n$ in $S^n$; that is, $\tilde{N}_h$ equals an $n$-hemisphere $H$, and $N_{h,\infty}$ is the union of $\{O\}$ and the boundary great sphere $S^{n-1}$ of $H$. Moreover, the closure of the set
given by $x_1 + x_2 + \cdots + x_n > 0$ in $H$ is an $n$-bihedron and one of its face is included in $S^{n-1}$. Hence, it is an $n$-crescent.

Let $H_1$ be the open half-space given by $x_1 > 0$, and $l$ the line $x_2 = \cdots = x_n = 0$. Let $g_1$ be the real projective transformation given by $(x_1, x_2, \ldots, x_n) \mapsto (2x_1, x_2, \ldots, x_n)$ and $g_2$ that given by

$$(x_1, x_2, \ldots, x_n) \mapsto (x_1, 2x_2, \ldots, 2x_n).$$

Then the quotient manifold $L$ of $H_1 - l$ by the commutative group generated by $g_1$ and $g_2$ is diffeomorphic to $S^{n-2} \times S^1 \times S^1$, and we may identify its holonomy cover $\tilde{L}_h$ with $H_1 - l$ and $\tilde{L}_h$ with the closure $\text{Cl}(H_1)$ of $H_1$ in $S^n$. Clearly, $\text{Cl}(H_1)$ is an $n$-bihedron bounded by an $(n - 1)$-hemisphere that is the closure of the hyperplane given by $x_1 = 0$ and an $(n - 1)$-hemisphere in the boundary of the affine patch $\mathbb{R}^n$. Therefore, $L_{h,\infty}$ is the union of $H_1 \cap l$ and two $(n - 1)$-hemispheres that form the boundary of $\text{Cl}(H_1)$. $\text{Cl}(H_1)$ is not an $n$-crescent since $\text{Cl}(H_1) \cap L_{h,\infty} \supset l \cap H_1^o \neq \emptyset$. In fact, $\text{Cl}(H_1)$ includes no $n$-crescents.

Let $R$ be an $n$-bihedron. If $R$ is an $n$-bihedron, then we define $\alpha_R$ to be the interior of the side of $R$ in $M_{h,\infty}$ and $\nu_R$ the other side. If $R$ is an $n$-hemisphere, then we define $\alpha_R$ to be the union of the interiors of all $(n - 1)$-hemispheres in $\partial R \cap M_{h,\infty}$ and define $\nu_R$ the complement of $\alpha_R$ in $\partial R$. Clearly, $\nu_R$ is a tame topological $(n - 1)$-ball.

Let us now discuss about how two convex sets may meet. Let $F_1$ and $F_2$ be two convex $i$-, $j$-balls in $\tilde{M}_h$ respectively. We say that $F_1$ and $F_2$ overlap if $F_1^o \cap F_2^o \neq \emptyset$, which is equivalent to $F_1 \cap F_2^o \neq \emptyset$ or $F_1^o \cap F_2 \neq \emptyset$.

**Theorem 3.1.** If $F_1$ and $F_2$ overlap, then $\text{dev}|F_1 \cup F_2$ is an imbedding onto $\text{dev}(F_1) \cup \text{dev}(F_2)$ and $\text{dev}|F_1 \cap F_2$ onto $\text{dev}(F_1) \cap \text{dev}(F_2)$. Moreover, if $F_1$ and $F_2$ are $n$-balls, then $F_1 \cup F_2$ is an $n$-ball, and $F_1 \cap F_2$ is a convex $n$-ball.

**Proof.** The proof is a direct generalization of that of Theorem 1.7 of [1].

The above theorem follows from

**Proposition 3.1.** Let $A$ be a $k$-ball in $\tilde{M}_h$ and $B$ an $l$-ball. Suppose that $A^o \cap B^o \neq \emptyset$, $\text{dev}(A) \cap \text{dev}(B)$ is a compact manifold in $S^n$ with interior equal to $\text{dev}(A^o) \cap \text{dev}(B^o)$ and $\text{dev}(A^o) \cap \text{dev}(B^o)$ is arcwise-connected. Then $\text{dev}|A \cup B$ is a homeomorphism onto $\text{dev}(A) \cup \text{dev}(B)$.

**Proof.** This follows as in its affine version Lemma 6 in [10].
In the following, we describe a useful geometric situation modelled on “dipping a bread into a bowl of milk”. Let $D$ be a convex $n$-ball in $\mathcal{M}_h$ such that $\partial D$ includes a tame subset $\alpha$ homeomorphic to an $(n-1)$-ball. We say that a convex $n$-ball $F$ is dipped into $(D, \alpha)$ if the following statements hold:

- $D$ and $F$ overlap.
- $F \cap \alpha$ is a convex $(n-1)$-ball $\beta$ with $\delta \beta \subset \delta F$ and $\beta^o \subset F^o$.
- $F - \beta$ has two convex components $O_1$ and $O_2$ such that $\text{Cl}(O_1) = O_1 \cup \beta = F - O_2$ and $\text{Cl}(O_2) = O_2 \cup \beta = F - O_1$.
- $F \cap D$ is equal to $\text{Cl}(O_1)$ or $\text{Cl}(O_2)$.

(The second item sometimes is crucial in this paper.) We say that $F$ is dipped into $(D, \alpha)$ nicely if the following statements hold:

- $F$ is dipped into $(D, \alpha)$.
- $F \cap D^o$ is identical with $O_1$ and $O_2$.
- $\delta(F \cap D) = \beta \cup \xi$ for a topological $(n-1)$-ball $\xi$, not necessarily convex or tame, in the topological boundary $\text{bd}F$ of $F$ in $\mathcal{M}_h$ where $\beta \cap \xi = \delta \beta$.

As a consequence, we have $\delta \beta \subset \text{bd}F$. (As above this is a crucial point.) (The nice dipping occurs when the bread does not touch the bowl.)

![Figure 2](image)

**Figure 2.** Various examples of dipping intersections. Loosely speaking $\alpha$ plays the role of the milk surface, $F, F'$, and $F''$ the breads, and $D^o$ the milk. The left one indicates nice dippings, and the right one not a nice one.

The direct generalization of Corollary 1.9 of [9] gives us:

**Corollary 3.1.** Suppose that $F$ and $D$ overlap, and $F^o \cap (\delta D - \alpha^o) = \emptyset$. Assume the following two equivalent conditions:

- $F^o \cap \alpha \neq \emptyset$.
- $F \not\subset D$.
Then $F$ is dipped into $(D, \alpha)$. If $F \cap (\delta D - \alpha^o) = \emptyset$ furthermore, then $F$ is dipped into $(D, \alpha)$ nicely. \hfill \qed

**Example 3.2.** In Example 3.1, choose a compact convex ball $B$ in $\mathbb{R}^n - \{O\} = N_h$ intersecting $R$ in its interior but not included in $R$. Then $B$ dips into $(R, P)$ nicely where $P$ is the closure of the plane given by $x_1 + \cdots + x_n = 1$. Also let $S$ be the closure of the half plane given by $x_1 > 0$. Then $S$ dips into $(R, P)$ but not nicely.

Consider the closure of the set in $\tilde{N}_h$ given by $0 < x_1 < 1$ and that of the set $0 < x_2 < 1$. Then these two sets do not dip into each other for any choice of $(n-1)$-balls in their respective boundaries to play the role of $\alpha$.

Since $\text{dev}$ restricted to a small open sets are charts, and the boundary of $M_h$ is convex, each point $x$ of $M_h$ has a compact ball-neighborhood $B(x)$ so that $\text{dev}|B(x)$ is an imbedding onto a compact convex ball in $S^n$ (see Section 1.11 of [3]). $\text{dev}(B(x))$ can be assumed to be a $d$-ball with center $\text{dev}(x)$ and radius $\varepsilon > 0$ intersected with an $n$-hemisphere $H$ so that $\delta M_h \cap B(x)$ corresponds to $\partial H \cap \text{dev}(B(x))$. Of course, $\delta M_h \cap B(x)$ or $\partial H \cap \text{dev}(B(x))$ may be empty. We say that such $B(x)$ is an $\epsilon$-tiny ball of $x$ and $\epsilon$ the $d$-radius of $B(x)$. Thus, for an $\epsilon$-tiny ball $B(x)$, $\delta M_h \cap B(x)$ is a compact convex $(n-1)$-ball, and the topological boundary $\text{bd}B(x)$ equals the closure of $\delta B(x)$ removed with this $(n-1)$-ball.

**Lemma 3.2.** If $B(x)$ and an $n$-crescent $R$ overlap, then either $B(x)$ is a subset of $R$ or $B(x)$ is dipped into $(R, \nu_R)$ nicely.

**Proof.** Since $\text{Cl}(\alpha_R) \subset M_{h,\infty}$ and $B(x) \subset M_h$, Corollary 3.1 implies the conclusion. \hfill \qed
4. \((n - 1)\)-Convexity

In this section, we introduce \(m\)-convexity. Then we state Theorem 4.4 central to this section, which relates the failure of \((n - 1)\)-convexity with an existence of \(n\)-crescents, or half-spaces. The proof of theorem is similar to what is in Section 5 in [9]. We first choose a sequence of points \(q_i\) converging to a point \(x\) in \(F_1 \cap M_{h,\infty}\). Then we pull back \(q_i\) to points \(p_i\) in the closure of a fundamental domain by a deck transformation \(\vartheta_i^{-1}\). Then analogously to [9], we show that \(T_i = \vartheta_i^{-1}(T)\) “converges to” a nondegenerate convex \(n\)-ball. Showing that \(\text{dev}(T_i)\) converges to an \(n\)-bihedron or an \(n\)-hemisphere is much more complicated than in [9]. The idea of the proof is to show that the sequence of the images under \(\vartheta_i\) of the \(\epsilon-(n - 1)\)-d-balls in \(\vartheta_i^{-1}(F_1)\) with center \(p_i\) have to degenerate to a point when \(x\) is chosen specially. So when pulled back by \(\vartheta_i^{-1}\), the balls become standard ones again, and \(F_1\) must blow up to be an \((n - 1)\)-hemisphere under \(\vartheta_i^{-1}\).

An \(m\)-simplex \(T\) in \(\check{M}\) is a tame subset of \(\check{M}_h\) such that \(\text{dev}\lvert_T\) is an imbedding onto an affine \(m\)-simplex in an affine patch in \(S^n\).

**Definition 4.1.** We say that \(M\) is \(m\)-convex, \(0 < m < n\), if the following holds. If \(T \subset \check{M}_h\) be an \((m + 1)\)-simplex with sides \(F_1, F_2, \ldots, F_{m+2}\) such that \(T^0 \cup F_2 \cup \cdots \cup F_{m+2}\) does not meet \(M_{h,\infty}\), then \(T\) is a subset of \(M_h\).

**Theorem 4.1.** Let \(T\) be an affine \((m+1)\)-simplex in an affine space with faces \(F_1, F_2, \ldots, F_{m+2}\). The following are equivalent:

(a) \(M\) is \(m\)-convex.

(b) Any (nonsingular) real projective map \(f\) from \(T^0 \cup F_1 \cup F_2 \cup \cdots \cup F_{m+2}\) to \(M\) extends to one from \(T\).

(c) a cover of \(M\) is \(m\)-convex.

**Proof.** The proof of the equivalence of (a) and (b) is the same as the affine version Lemma 1 in [16]. The equivalence of (b) and (c) follows from the fact that a real projective map to \(M\) always lifts to its cover. \(\square\)

**Theorem 4.2.** \(M\) is not convex if and only if there exists an \((m + 1)\)-simplex with a side \(F_1\) such that \(T \cap M_{h,\infty} = F_1^0 \cap M_{h,\infty} \neq \emptyset\).

**Proof.** The proof is same as Lemma 3 in [16]. \(\square\)
Figure 3. The tetrahedron in the left fails to detect non-2-convexity but the right one is detecting non-2-convexity.

Remark 4.1. It is easy to see that $i$-convexity implies $j$-convexity whenever $i \leq j < n$. (See Remark 2 in [16]. The proof is the same.)

Theorem 4.3. The following are equivalent:

- $M$ is 1-convex.
- $M$ is convex.
- $M$ is real projectively isomorphic to a quotient of a convex domain in $S^n$.

The proof is similar to Lemma 8 in [16]. Since there are minor differences between affine and real projective manifolds, we will prove this theorem in Appendix A.

Let us give examples of real projective $n$-manifolds one of which is not $(n-1)$-convex and the other $(n-1)$-convex.

As in Figure 3, $\mathbb{R}^3$ removed with a complete affine line or a closed wedge, i.e. a set defined by the intersection of two half-spaces with non-parallel boundary planes is obviously 2-convex. But $\mathbb{R}^3$ removed with a discrete set of points or a convex cone defined as the intersection of three half-spaces with boundary planes in general position is not 2-convex.

We recall Example 3.1. Let $\mathbb{R}^n$ be an affine patch of $S^n$ with standard affine coordinates $x_1, x_2, \ldots, x_n$ and $O$ the origin. Consider $\mathbb{R}^n - \{O\}$ quotient out by the group $< g >$ where $g : x \to 2x$ for $x \in \mathbb{R}^n - \{0\}$. Then the quotient is a real projective manifold diffeomorphic to $S^{n-1} \times S^1$. Denote the manifold by $M$, and we see that $M_h$ can be identified with $\mathbb{R}^n - \{0\}$. Thus, $M_h$ equals the closure of $\mathbb{R}^n$ in $S^n$; that is, $M_h$ equals an $n$-hemisphere $H$, and $M_{h,\infty}$ is the union of $\{O\}$ and the boundary great sphere $S^{n-1}$ of $H$. Consider an $n$-simplex $T$ in $\mathbb{R}^n$ given by $x_i \leq 1$ for every $i$ and $x_1 + x_2 + \cdots + x_n \geq 0$. Then the face of $T$ corresponding to $x_1 + x_2 + \cdots + x_n = 0$ contains the ideal point $O$ in its interior. Therefore, $M$ is not $(n-1)$-convex. Moreover, the closure of the set given by $x_1 + x_2 + \cdots + x_n > 0$ in $M_h = H$ is an $n$-bihedron and one of its face is included in $S^{n-1}$. Hence, it is an $n$-crescent.
Let $H_1$ be the open half-space given by $x_1 > 0$, and $l$ the line $x_2 = \cdots = x_n = 0$. Let $g_1$ be the real projective transformation given by $(x_1, x_2, \ldots, x_n) \mapsto (2x_1, x_2, \ldots, x_n)$ and $g_2$ that given by $(x_1, x_2, \ldots, x_n) \mapsto (x_1, 2x_2, \ldots, 2x_n)$. Then the quotient manifold $M$ of $H_1 - l$ by the commutative group generated by $g_1$ and $g_2$ is diffeomorphic to $S^{n-2} \times S^1 \times S^1$, and we may identify $M_h$ with $H_1 - l$ and $\tilde{M}_h$ with the closure $\text{Cl}(H_1)$ of $H_1$ in $S^n$. Clearly, $\text{Cl}(H_1)$ is an $n$-bihedron bounded by an $(n-1)$-hemisphere that is the closure of the hyperplane given by $x_1 = 0$ and an $(n-1)$-hemisphere in the boundary of the affine patch $\mathbb{R}^n$. Therefore, $M_{h,\infty}$ is the union of $H_1 \cap l$ and two $(n-1)$-hemispheres that form the boundary of $\text{Cl}(H_1)$. The intersection of an $n$-simplex $T$ in $S^n$ with the boundary $(n-1)$-hemispheres or $l$ is not a subset of the interior of a face of $T$. It follows from this that $M$ is $(n-1)$-convex.

The main purpose of this section is to prove the following principal theorem:

**Theorem 4.4.** Suppose that a compact real projective manifold $M$ with empty or totally geodesic boundary is not $(n-1)$-convex. Then the completion $\tilde{M}_h$ of the holonomy cover $M_h$ includes an $n$-crescent.

We may actually replace the word “total geodesic” with “convex” and the proof is same step by step. However, we need this result at only one point of the paper so we do not state it. We can also show that the completion $\tilde{M}$ of the universal cover $\tilde{M}$ also includes an $n$-crescent. The proof is identical with $\tilde{M}$ replacing $\tilde{M}_h$. Another way to do this is of course as follows: once we obtain an $n$-crescent in $\tilde{M}_h$ we may lift it to one in $\tilde{M}$ but we omit showing how this can be done.

**Remark 4.2.** As $M$ is $(n-1)$-convex, we may assume that $\tilde{M}$ or $M_h$ is not projectively diffeomorphic to an open $n$-bihedron or an open $n$-hemisphere. This follows since if otherwise, $\tilde{M}$ is convex and hence $(n-1)$-convex. (We will need this weaker statement later.)

A point $x$ of a convex subset $A$ of $S^n$ is said to be exposed if there exists a supporting great $(n-1)$-sphere $H$ at $x$ such that $H \cap A = \{x\}$ (see Section 2 and Berger [6, p. 361]).

To prove Theorem 4.4, we follow Section 5 of Choi [1]: Since $M$ is not $(n-1)$-convex, $\tilde{M}_h$ includes an $n$-simplex $T$ with a face $F_1$ such that $T \cap M_{h,\infty} = F_1^o \cap M_{h,\infty} \neq \emptyset$ by Theorem 4.2, where $\text{dev}|T : T \to \text{dev}(T)$ is an imbedding onto the $n$-simplex $\text{dev}(T)$. Let $K$ be the convex hull of $\text{dev}(F_1 \cap M_{h,\infty})$ in $\text{dev}(F_1)^o$, which is simply convex as $\text{dev}(F_1)$ is simply convex.
As $K$ is simply convex, we see that $K$ can be considered as a bounded convex subset of an affine patch, i.e., an open $n$-hemisphere. We see easily that $K$ has an exposed point in the affine sense in the open hemisphere, which is easily seen to be an exposed point in our sense as a hyperplane in the affine patch is the intersection of a hypersphere with the affine patch.

Let $x'$ be an exposed point of $K$. Then $x' \in \text{dev}(F_1 \cap M_{h,\infty})$, and there exists a line $s'$ in the complement of $K$ in $\text{dev}(F_1)^o$ ending at $x'$. Let $x$ and $s$ be the inverse images of $x'$ and $s'$ in $F_i^o$ respectively.

Let $F_i$ for $i = 2, \ldots, n+1$ denote the faces of $T$ other than $F_1$. Let $v_i$ for each $i$, $i = 1, \ldots, n+1$, denote the vertex of $T$ opposite $F_i$. Let us choose a monotone sequence of points $q_i$ on $s$ converging to $x$ with respect to $d$.

Choose a fundamental domain $F$ in $M_h$ such that for every point $t$ of $F$, there exists a $2\epsilon$-tiny ball of $t$ in $M_h$ for a positive constant $\epsilon$ independent of $t$. We assume $\epsilon \leq \pi/8$ for convenience. Let us denote by $F_{2\epsilon}$ the closure of the $2\epsilon$-neighborhood of $F$, and $F_\epsilon$ that of the $\epsilon$-neighborhood of $F$.

For each natural number $i$, we choose a deck transformation $\vartheta_i$ and a point $p_i$ of $F$ so that $\vartheta_i(p_i) = q_i$. We let $v_{j,i}$, $F_{j,i}$, and $T_i$, $i = 1, 2, \ldots, j = 1, \ldots, n+1$, denote the images under $\vartheta_i^{-1}$ of $v_j$, $F_j$, and $T$ respectively. Let $n_i$ denote the outer-normal vector to $F_{1,i}$ at $p_i$ with respect to the spherical Riemannian metric $\mu$ of $M_h$.

We choose subsequences so that each sequence consisting of $\text{dev}(v_{j,i}), \text{dev}(F_{j,i}), \text{dev}(T_i), n_i$, and $p_i$ converge geometrically with respect to $d$ for each $j, j = 1, \ldots, n+1$ respectively. Since $p_i \in F$ for each $i$, the limit $p$ of the sequence of $p_i$ belongs to $\text{Cl}(F)$. We choose an $\epsilon$-tiny ball $B(p)$ of $p$. We may assume without loss of generality that $p_i$ belongs to the interior $\text{int}B(p)$ of $B(p)$. Since the action of the deck transformation group is properly discontinuous and $F_{j,i} = \vartheta_i^{-1}(F_j)$ for a compact set $F_j$, there exists a natural number $N$ such that

$$F_{2\epsilon} \cap F_{j,i} = \emptyset \text{ for each } j, i, j > 1, i > N; \quad (3)$$

so $B(p) \cap F_{j,i} = \emptyset$ for $j > 1$. (This corresponds to Lemma 5.4 in [9] or $B(p)$ dips into $(T_i, F_{1,i})$ for each $i$, $i > N$, by Corollary 3.1.

Lemma 3 of the appendix of [9] holds for manifolds of higher dimensions as the dimension of the sphere $S^2$ do not matter in the proof. Thus, there exists an integer $N_1,$
$N_1 > N$, such that $T_i$ includes a common open ball for $i > N_1$. Let $T_\infty$ be the limit of $\text{dev}(T_i)$. Since $\text{dev}(T_i)$ includes a common ball for $i > N_1$, it follows that $T_\infty$ is a closed convex $n$-ball in $S^n$ (see Section 2 of the appendix of [9]).

Let $F_{j,\infty}$ denote the limit of $\text{dev}(F_{j,i})$. Then $\bigcup_{j=1}^{n+1} F_{j,\infty}$ is the boundary $\partial T_\infty$ by Theorem 2.6.

Theorem 4 of the appendix of [9] also holds for higher dimensional real projective manifolds as there are no dimensional assumptions that matter. Thus, $\tilde{M}_h$ includes a convex $n$-ball $T^u$ and convex sets $F^u_j$ such that $\text{dev}$ restricted to them are imbeddings onto $T_\infty$ and $F_{j,\infty}$ respectively. We have $F^u_j \subset M_{h,\infty}$ for $j \geq 2$ from the same theorem since $F_{j,i}$ is ideal.

We will prove below that $F_{1,\infty}$ is an $(n-1)$-hemisphere. It follows that $T_\infty$ is a compact convex $n$-ball in $S^n$ including the $(n-1)$-hemisphere $F_{1,\infty}$ in its boundary $\delta T_\infty$. By Lemma 2.7, $T_\infty$ is an $n$-bihedron or an $n$-hemisphere. As $\bigcup_{j\geq 2} F^u_j$ is a subset of $M_{h,\infty}$, if $F^u_1 \subset M_{h,\infty}$, then $\tilde{M}_h = T^u$ and $M_h$ equals the interior of $T^u$ by the following lemma 4.1. Since $M_h$ is not projectively diffeomorphic to an open $n$-bihedron or an open $n$-hemisphere by premise, $F^u_1$ is not a subset of $M_{h,\infty}$. Since $T^u$ is bounded by $F^u_1$ and $F^u_2 \cup \cdots \cup F^u_{n+1} \subset M_{h,\infty}$, it follows that $T$ is an $n$-crescent. This completes the proof of Theorem 4.4.

Lemma 4.1. Suppose that $\tilde{M}_h$ includes an $n$-ball $B$ with $\delta B \subset M_{h,\infty}$. Then $M_h$ equals $B^o$. 

\[\Box\]
Proof. Since \( \partial B \cap M_h \subset \delta B \) and \( \delta B \subset M_{h,\infty} \), it follows that \( \partial B \cap M_h \) is empty. Hence, \( M_h \subset B \).

We will now show that \( F_1^u \) is an \((n-1)\)-dimensional hemisphere. This corresponds to Lemma 5.5 of [9] showing that one of the side is a segment of \( d \)-length \( \pi \). (Note that this process may require us to choose further subsequences of \( T_i \). However, since \( \text{dev}(F_{1,i}) \) is assumed to converge to \( F_{1,\infty} \), we see that we need to only show that a subsequence of \( \text{dev}(F_{1,i}) \) converges to an \((n-1)\)-hemisphere.)

The sequence \( \text{dev}(q_i) = h(\vartheta_i)\text{dev}(p_i) \) converges to \( x' \). Since \( p_i \in F_1 \), \( M_h \) includes an \( \epsilon \)-tiny ball \( B(p_i) \) and a \( 2\epsilon \)-tiny ball \( B'(p_i) \) of \( p_i \). Let \( W(p_i) = F_{1,i} \cap B(p_i) \) and \( W'(p_i) = F_{1,i} \cap B'(p_i) \). We assume that \( i > N_1 \) from now on.

We now show that \( W(p_i) \) and \( W(p'_i) \) are “whole” \((n-1)\)-balls of \( d \)-radius \( \epsilon \) and \( 2\epsilon \), i.e., they map to such balls in \( S^n \) under \( \text{dev} \) respectively, or they are not “cut off” by the boundary \( \delta F_{1,i} \):

If \( p_i \in \delta M_h \), then the component \( L \) of \( F_{1,i} \cap M_h \) containing \( p_i \) is a subset of \( \delta M_h \) by Lemma 3.1. This component is a submanifold of \( \delta M_h \) with boundary \( \delta F_{1,i} \). Since \( \delta F_{1,i} \) is a subset of \( \bigcup_{j \geq 2} F_{j,i} \), and \( B(p_i) \) is disjoint from it by equation 3. \( \delta M_h \cap B(p_i) \) is a subset of \( L^o \). Thus, \( W(p_i) \) equals the convex \((n-1)\)-ball \( \delta M_h \cap B(p_i) \) with boundary in \( \partial B(p_i) \) and is a \( d \)-ball in \( F_{1,i}^o \) of dimension \((n-1)\) of \( d \)-radius \( \epsilon \) and center \( p_i \). It certainly maps to an \((n-1)\)-ball of \( d \)-radius \( \epsilon \) with center \( \text{dev}(p_i) \).

If \( p_i \in M_{h}^o \), then since \( F_{1,i} \) passes through \( p_i \), and \( F_{j,i} \cap B(p_i) = \emptyset \) for \( j \geq 2 \), it follows that \( B(p_i) \) dips into \( (T_i, F_{1,i}) \) nicely by Corollary 3.1. Thus \( W(p_i) \) is an \((n-1)\)-ball with boundary in \( \partial B(p_i) \), and an \( \epsilon \)-\( d \)-ball in \( F_{1,i}^o \) of dimension \((n-1)\) with center \( p_i \).

Similar reasoning shows that \( W'(p_i) \) is a \( 2\epsilon \)-\( d \)-ball in \( F_{1,i}^o \) of dimension \((n-1)\) with center \( p_i \) for each \( i \).

Since \( \vartheta_i(W(p_i)) \subset F_1 \), and \( \text{dev}(F_1) \) is a compact set, we may assume without loss of generality by choosing subsequences of \( \vartheta_i \) that the sequence of the subsets \( \text{dev}(\vartheta_i(W(p_i))) \) of \( \text{dev}(F_1) \), equal to \( h(\vartheta_i)\text{dev}(W(p_i)) \), converges to a set \( W_\infty \) containing \( x' \) in \( \text{dev}(F_1) \).

Since \( \text{dev}T^u \) is an imbedding onto \( T_\infty \), there exists a compact tame subset \( W^u \) in \( F_1 \) such that \( \text{dev} \) restricted to \( W^u \) is an imbedding onto \( W_\infty \). \( \vartheta_i(W(p_i)) \) is a subjugated sequence of the sequence whose elements equal \( T \) always. Since \( W(p_i) \) is a subset of a compact set \( F_t \), it follows that \( \vartheta_i(W(p_i)) \) is ideal (see Lemma 5.4 of [9]), and \( W^u \subset M_{h,\infty} \) by Lemma 4 of appendix of [9]. We obtain \( W^u \subset F_1 \cap M_{h,\infty} \).

For the proof, the fact that \( x' \) is exposed will play a role.
Figure 5. The pull-back process with $W(p_i)$s.

**Proposition 4.1.** $W_\infty$ consists of the single point $x'$.

Suppose not. Then as $\text{dev}(\vartheta_i(W(p_i)))$ does not converge to a point, there has to be a sequence $\{\text{dev}(\vartheta_i(z_i))\}$, $z_i \in W(p_i)$, converging to a point $z'$ distinct from the limit $x'$ of $\{\text{dev}(q_i)\}$. Since we have $\text{dev}(q_i) = \text{dev}(\vartheta_i(p_i))$, we choose $s_i$ to be the $d$-diameter of $W(p_i)$ containing $z_i$ and $p_i$, as a center. We obtained a sequence of segments $s_i \in W(p_i)$ passing through $p_i$ of $d$-length $2\epsilon$ so that the sequence of segments $\text{dev}(\vartheta_i(s_i))$ in $\text{dev}(F_1)$ converges to a nontrivial segment $s$ containing $x'$, and $z'$, satisfying $s \subset W_\infty \subset \text{dev}(M_{h,\infty} \cap F_1)$.

Since $s$ is a nontrivial segment, the $d$-length of $h(\vartheta_i)(\text{dev}(s_i))$ is bounded below by a positive constant $\delta$ independent of $i$. Since $h(\vartheta_i)(\text{dev}(s_i))$ is a subset of the $(n-1)$-simplex $\text{dev}(F_1)$, which is a simply convex compact set, the $d$-length of $h(\vartheta_i)(\text{dev}(s_i))$ is bounded above by $\pi - \delta'$ for some small positive constant $\delta'$. Let $s'_i$ be the maximal segment in $W'(p_i)$ including $s_i$. Then the $d$-length of $h(\vartheta_i)(\text{dev}(s'_i))$ also belongs to the interval $[\delta, \pi - \delta']$.

**Lemma 4.2.** Let $S^1$ be a great circle and $o, s, p, q$ distinct points on a segment $I$ in $S^1$ of $d$-length $< \pi$ with endpoints $o$ and $q$. Let $f_i$ be a sequence of projective maps $I \to S^n$ so that $d(f_i(o), f_i(s))$ and $d(f_i(p), f_i(q))$ lie in the interval $[\eta, \pi - \eta]$ for some positive constant $\eta$ independent of $i$. Then all of the $d$-distances between $f_i(o)$, $f_i(s)$, $f_i(p)$, and $f_i(q)$ are bounded below by a positive constant independent of $i$. 

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The lemma follows from these two statements.

Appendix of [9]. Since by a projective map \( f \) each subsequence of \( s \) contains \( (\vartheta) \) consisting of points \( \vartheta \) corresponding to the unit vector having an oriented angle of \( 1 \) with \( (1,0) \) in \( \mathbb{R}^2 \). Since \( s_i \) and \( s'_i \) are the diameters of balls of \( d \)-radius \( \epsilon \) and \( 2\epsilon \) with center \( p_i \) respectively, for the segment \([-2\epsilon,2\epsilon]\) consisting of points \( \vartheta \) satisfying \(-2\epsilon \leq \vartheta \leq 2\epsilon \) in \( S^1 \), we parameterize \( s'_i \) by a projective map \( f_i : [-2\epsilon,2\epsilon] \to s'_i \), isometric with respect to \( d \), so that the endpoints of \( s_i' \) correspond to \(-2\epsilon \) and \( 2\epsilon \), the endpoints of \( s_i \) to \(-\epsilon \) and \( \epsilon \), and \( p_i \) to 0.

Lemma 4.2 applied to \( k_i = h(\vartheta_i) \circ \text{dev} \circ f_i \) shows that \( d(k_i(2\epsilon),k_i(\epsilon)) \) and \( d(k_i(-2\epsilon),k_i(-\epsilon)) \) are bounded below by a positive constant since \( d(k_i(\epsilon),k_i(-\epsilon)) \) and \( d(k_i(2\epsilon),k_i(-2\epsilon)) \) are bounded below by a positive number \( \delta \) and above by \( \pi - \delta \). Since \( k_i(2\epsilon) \) and \( k_i(-2\epsilon) \) are endpoints of \( h(\vartheta_i)(\text{dev}(s'_i)) \) and \( k_i(\epsilon) \) and \( k_i(-\epsilon) \) those of \( h(\vartheta_i)(\text{dev}(s_i)) \), a subsequence of \( h(\vartheta_i)(\text{dev}(s'_i)) \) converges to a segment \( s' \) in \( \text{dev}(F_1) \) including \( s \) in its interior. Hence, \( s' \) contains \( x' \) in its interior.

Since \( s'_i \) is a subset of \( F_{2\epsilon} \), a compact subset of \( M_h \), it follows that the corresponding subsequence of \( \vartheta_i(s'_i) \) is ideal in \( F_1 \). Hence \( s' \subset \text{dev}(F_1 \cap M_{h,\infty}) \subset K \) by Theorem 4 of Appendix of [9]. Since \( x' \) is not an endpoint of \( s' \) but an interior point, this contradicts our earlier choice of \( x' \) as an exposed point of \( K \). \( \square \)

Since \( W_{\infty} \) consists of a point, it follows that the sequence of the \( d \)-diameter of \( h(\vartheta_i)(\text{dev}(W(p_i))) \) converges to zero, and the sequence converges to the singleton \( \{x'\} \).

Let us introduce a \( d \)-isometry \( g_i \), which is a real projective automorphism of \( S^n \), for each \( i \) so that each \( g_i(\text{dev}(W(p_i))) \) is a subset of the great sphere \( S^{n-1} \) including \( \text{dev}(F_1) \).
and hence $h(\vartheta_i) \circ g_i^{-1}$ acts on $S^{n-1}$. We may assume without loss of generality that the $d$-isometries $g_i$ converges to an isometry $g$ of $S^n$. Thus, $h(\vartheta_i) \circ g_i^{-1}(g_i(\text{dev}(W(p_i))))$ converges to $x'$, and $g_i \circ h(\vartheta_i)^{-1}(\text{dev}(F_1))$ converges to $g(F_{1,\infty})$ as we assumed in the beginning of the pull-back process. By Proposition 4.2, we see that $g(F_{1,\infty})$ is an $(n-1)$-hemisphere, and we are done.

The proof of the following proposition is left to Appendix B as the proof may distract us too much.

**Proposition 4.2.** Suppose we have a sequence of $\epsilon$-$d$-balls $B_i$ in a real projective sphere $S^m$ for some $m \geq 1$ and a sequence of projective maps $\varphi_i$. Assume the following:

- The sequence of $d$-diameters of $\varphi_i(B_i)$ goes to zero.
- $\varphi_i(B_i)$ converges to a point, say $p$.
- For a compact $m$-ball neighborhood $L$ of $p$, $\varphi_i^{-1}(L)$ converges to a compact set $L_\infty$.

Then $L_\infty$ is an $m$-hemisphere.
5. The transversal intersection of $n$-crescents

From now on, we will assume that $M$ is compact and with totally geodesic or empty boundary. We will discuss about the transversal intersection of $n$-crescents, generalizing that of crescents in two-dimensions [9].

First, we will show that if two hemispheric $n$-crescents overlap, then they are equal. For transversal intersection of two bihedral $n$-crescents, we will follow Section 2.6 of [9].

Our principal assumption is that $M_h$ is not projectively diffeomorphic to an open $n$-hemisphere or $n$-bihedron, which will be sufficient for the results of this section to hold. This is equivalent to assuming that $\tilde{M}$ is not projectively diffeomorphic to these. This will be our assumption in Sections 5 to 8. In applying the results of Sections 5 to 8 in Sections 9, 10, we need this assumption.

For the following theorem, we may even relax this condition even further:

**Theorem 5.1.** Suppose that $M_h$ is not projectively diffeomorphic to an open hemisphere. Suppose that $R_1$ and $R_2$ are two overlapping $n$-crescents that are hemispheres. Then $R_1 = R_2$, and hence $\nu_{R_1} = \nu_{R_2}$ and $\alpha_{R_1} = \alpha_{R_2}$.

**Proof.** We use Lemma 5.1 as in [9]: By Theorem 3.1, $\text{dev}|R_1 \cup R_2$ is an imbedding onto the union of two $n$-hemispheres $\text{dev}(R_1)$ and $\text{dev}(R_2)$ in $S^n$. If $R_1$ is not equal to $R_2$, then $\text{dev}(R_1)$ differs from $\text{dev}(R_2)$, $\text{dev}(R_1)$ and $\text{dev}(R_2)$ meet each other in a convex $n$-bihedron, $\text{dev}(R_1) \cup \text{dev}(R_2)$ is homeomorphic to an $n$-ball, and the boundary $\delta(\text{dev}(R_1) \cup \text{dev}(R_2))$ is the union of two $(n - 1)$-hemispheres meeting each other in a great $(n - 2)$-sphere $S^{n-2}$.

Since $\alpha_{R_1}$ and $\alpha_{R_2}$ are disjoint from any of $R_1^o$ and $R_2^o$ respectively, the images of $\alpha_{R_1}$ and $\alpha_{R_2}$ do not intersect any of $\text{dev}(R_1^o)$ and $\text{dev}(R_2^o)$ respectively by Theorem 3.1. Therefore, $\text{dev}(\alpha_{R_1})$ and $\text{dev}(\alpha_{R_2})$ are subsets of $\delta(\text{dev}(R_1) \cup \text{dev}(R_2))$. Since they are open $(n - 1)$-hemispheres, the complement of $\text{dev}(\alpha_{R_1}) \cup \text{dev}(\alpha_{R_2})$ in $\delta(\text{dev}(R_1) \cup \text{dev}(R_2))$ equals $S^{n-1}$, and $\text{dev}(\alpha_{R_1}) \cup \text{dev}(\alpha_{R_2})$ is dense in $\delta(\text{dev}(R_1) \cup \text{dev}(R_2))$. Since $\text{dev}|R_1 \cup R_2$ is an imbedding, it follows that $R_1 \cup R_2$ is an $n$-ball, and the closure of $\alpha_{R_1} \cup \alpha_{R_2}$ equals $\delta(R_1 \cup R_2)$. Hence, $\delta(R_1 \cup R_2) \subset M_{h,\infty}$. By Lemma 4.1 it follows that $M_h = R_1^o \cup R_2^o$, and $M_h$ is boundaryless. By the following lemma, this is a contradiction. Hence, $R_1 = R_2$. 

**Lemma 5.1.** Let $N$ be a closed real projective $n$-manifold. Suppose that $\text{dev}: \tilde{N}_h \to S^n$ is an imbedding onto the union of $n$-hemispheres $H_1$ and $H_2$ meeting each other in an
$n$-bihedron or an $n$-hemisphere. Then $H_1 = H_2$, and $N_h$ is projectively diffeomorphic to an open $n$-hemisphere.

Proof. Let $(\text{dev}, h)$ denote the development pair of $N$, and $\Gamma$ the deck transformation group. As $\text{dev}|N_h$ is a diffeomorphism onto $H_1^o \cup H_2^o$, a simply connected set, we have $N_h = \tilde{N}$.

Suppose that $H_1 \neq H_2$. Then $H_1 \cup H_2$ is bounded by two $(n-1)$-hemispheres $D_1$ and $D_2$ meeting each other on a great sphere $S^{n-2}$, their common boundary. Since the interior angle of intersection of $D_1$ and $D_2$ is greater than $\pi$, $\delta H_i - D_i$ is an open hemisphere included in $\text{dev}(\tilde{N})$ for $i = 1, 2$. Defining $O_i = \delta H_i - D_i$ for $i = 1, 2$, we see that $O_1 \cup O_2$ is $h(\Gamma)$-invariant since $\delta(H_1 \cup H_2)$ is $h(\Gamma)$-invariant. This means that the inverse image $\text{dev}^{-1}(O_1 \cup O_2)$ is $\Gamma$-invariant.

Let $O'_i = \text{dev}^{-1}(O_i)$. Then elements of $\Gamma$ either act on each of $O'_1$ and $O'_2$ or interchange them. Thus, $\Gamma$ includes a subgroup $\Gamma'$ of index one or two acting on each of $O'_1$ and $O'_2$. Since $N_h$ is a simply connected open ball, and so is $O'_1$, it follows that the $n$-manifold $\tilde{N}/\Gamma'$ and an $(n-1)$-manifold $O'_1/\Gamma'$ are homotopy equivalent. Since $\tilde{N}/\Gamma'$ is a finite cover of a closed manifold $N$, $\tilde{N}/\Gamma'$ is a closed manifold. Since the dimensions of $\tilde{N}/\Gamma'$ and $O'_1/\Gamma$ are not the same, this is shown to be absurd by computing $\mathbb{Z}_2$-homologies. Hence we obtain that $H_1 = H_2$, and since $\text{dev}(\tilde{N})$ equals the interior of $H_1$, $\tilde{N}$ is diffeomorphic to an open $n$-hemisphere.

\[\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure6.png}
\caption{Transversal intersections in dimension two.}
\end{figure}\]
Suppose that $R_1$ is an $n$-crescent that is an $n$-bihedron. Let $R_2$ be another bihedral $n$-crescents with sets $\alpha_{R_2}$ and $\nu_{R_2}$. We say that $R_1$ and $R_2$ intersect transversally if $R_1$ and $R_2$ overlap and the following conditions hold ($i = 1, j = 2$; or $i = 2, j = 1$):

1. $\nu_{R_1} \cap \nu_{R_2}$ is an $(n - 2)$-dimensional hemisphere.
2. For the intersection $\nu_{R_1} \cap \nu_{R_2}$ denoted by $H$, $H$ is an $(n - 2)$-hemisphere, $H^o$ is a subset of the interior $\nu_{R_1}^o$, and $\text{dev}(\nu_{R_1})$ and $\text{dev}(\nu_{R_2})$ intersect transversally at points of $\text{dev}(H)$.
3. $\nu_{R_i} \cap R_j$ is a tame $(n - 1)$-bihedron with boundary the union of $H$ and an $(n - 2)$-hemisphere $H'$ in the closure of $\alpha_{R_j}$ with its interior $H'^o$ in $\alpha_{R_j}$.
4. $\nu_{R_i} \cap R_j$ is the closure of a component of $\nu_{R_i} - H$ in $\bar{M}_h$.
5. $R_i \cap R_j$ is the closure of a component of $R_j - \nu_{R_i}$.
6. Both $\alpha_{R_i} \cap \alpha_{R_j}$ and $\alpha_{R_i} \cup \alpha_{R_j}$ are homeomorphic to open $(n - 1)$-dimensional balls, which are locally totally geodesic under $\text{dev}$.

Note that since $\alpha_{R_i}$ is tame, $\alpha_{R_i} \cap \alpha_{R_j}$ is tame. (See Figures 6 and 7.)

By Corollary 5.1, the above condition mirrors the property of intersection of $\text{dev}(R_1)$ and $\text{dev}(R_2)$ where $\text{dev}(\alpha_{R_i})$ and $\text{dev}(\alpha_{R_2})$ are included in a common great sphere $S^{n-1}$ of dimension $(n - 1)$, $\text{dev}(R_1)$ and $\text{dev}(R_2)$ included in a common $n$-hemisphere bounded by $S^{n-1}$ and $\text{dev}(\nu_{R_1})^o$ and $\text{dev}(\nu_{R_2})^o$ meets transversally (see Theorem 3.1).

**Example 5.1.** In the example 3.2, $R$ is an $n$-crescent with the closure of the plane $P$ given by the equation $x_1 + \cdots + x_n = 0$ equal to $\nu_R$. $\alpha_R$ equals the interior of the intersection of $R$ with $\delta H$. $\nu_S$ is the closure of the plane given by $x_1 = 0$ and $\alpha_S$ the interior of the intersection of $S$ with $\delta H$. Clearly, $R$ and $S$ intersect transversally.

Using the reasoning similar to Section 2.6. of [9], we obtain:

**Theorem 5.2.** Suppose that $R_1$ and $R_2$ are overlapping. Then either $R_1$ and $R_2$ intersect transversally or $R_1 \subset R_2$ and $R_2 \subset R_1$.

**Remark 5.1.** In case $R_1 \subset R_2$, we see easily that $\alpha_{R_1} = \alpha_{R_2}$ since the sides of $R_1$ in $M_{h, \infty}$ must be in that of $R_2$. Hence, we also see that $\nu_{R_1}^o \subset R_2^o$ as the topological boundary of $R_1$ in $R_2$ must lie in $\nu_{R_1}$.

The proof is entirely similar to that in [9]. A good thing to have in mind is the configurations of the images of two $n$-crescents in $S^n$ meeting in many ways. We will see that only the configuration as indicated above will happen.
Figure 7. A three-dimensional transversal intersection seen in two viewpoints

Assume that we have \( i = 1 \) and \( j = 2 \) or have \( i = 2 \) and \( j = 1 \), and \( R_1 \not\subset R_2 \) and \( R_2 \not\subset R_1 \). Since \( \text{Cl}(\alpha_{R_i}) \subset M_{h,\infty} \), Corollary 3.1 and Theorem 3.1 imply that \( R_j \) dips into \((R_i, \nu_{R_i})\) and \( \text{dev}|R_i \cup R_j \) is an imbedding onto \( \text{dev}(R_i) \cup \text{dev}(R_j) \). Hence, the following statements hold:

- \( \nu_{R_i} \cap R_j \) is a convex \((n - 1)\)-ball \( \alpha_i \) such that
  \[
  \delta \alpha_i \subset \delta R_j, \alpha_i^{\circ} \subset R_j^{\circ}.
  \]
- \( R_i \cap R_j \) is the convex \( n \)-ball that is the closure of a component of \( R_j - \alpha_i \).

Since \( \alpha_i^{\circ} \) is disjoint from \( \nu_{R_j}, \alpha_i^{\circ} \) is a subset of a component \( C \) of \( \nu_{R_i} - \nu_{R_j} \).

Lemma 5.2. If \( \nu_{R_i} \) and \( \nu_{R_j} \) meet, then they do so transversally; i.e, their images under \( \text{dev} \) meet transversally. If \( \nu_{R_i} \) and \( \alpha_{R_j} \) meet, then they do so transversally.

Proof. Suppose that \( \nu_{R_i} \) and \( \nu_{R_j} \) meet and they are tangential. Then \( \text{dev}(\nu_{R_i}) \) and \( \text{dev}(\nu_{R_j}) \) both lie on a common great \((n - 1)\)-sphere in \( S^n \) by the geometry of \( S^n \) and hence \( \nu_{R_i} \cap \nu_{R_j} = \nu_{R_i} \cap R_j \) by Theorem 3.1. Since \( \nu_{R_i} \cap R_j \) includes an open \((n - 1)\)-ball \( \alpha_i^{\circ} \), this contradicts \( \alpha_i^{\circ} \subset R_j^{\circ} \).

Suppose that \( \nu_{R_i} \) and \( \alpha_{R_j} \) meet and they are tangential. Then \( \nu_{R_i} \cap \text{Cl}(\alpha_{R_j}) = \nu_{R_i} \cap R_j \) as before. This leads to contradiction similarly. \( \square \)
We now determine a preliminary property of $\alpha_i$. Since $\alpha_i$ is a convex $(n-1)$-ball in $\nu_{R_i}$ with topological boundary in $\delta R_i \cup \delta R_j$, we obtain
\[
\delta \alpha_i \subset \delta \nu_{R_i} \cup (\delta R_j \cap \nu_{R_i}^o) \subset \delta \nu_{R_i} \cup (\nu_{R_j} \cap \nu_{R_i}^o) \cup (\alpha_{R_i} \cap \nu_{R_i}^o).
\]
Hence, we have
\[
\delta \alpha_i = (\delta \alpha_i \cap \delta \nu_{R_i}) \cup (\delta \alpha_i \cap \nu_{R_j} \cap \nu_{R_i}^o) \cup (\delta \alpha_i \cap \alpha_{R_i} \cap \nu_{R_i}^o).
\]

If $\alpha_{R_j}$ meets $\nu_{R_i}^o$, then since $\alpha_{R_j}$ is transversal to $\nu_{R_i}^o$ by Lemma 5.2, $\alpha_{R_j}$ must intersect $R_i^o$ by Theorem 3.1. Since $\alpha_{R_j} \subset M_{h,\infty}$, this is a contradiction. Thus, $\alpha_{R_j} \cap \nu_{R_i}^o = \emptyset$. We conclude
\[
\delta \alpha_i = (\delta \alpha_i \cap \delta \nu_{R_i}) \cup (\delta \alpha_i \cap \nu_{R_j} \cap \nu_{R_i}^o). \tag{4}
\]

Let us denote by $H$ the set $\nu_{R_i} \cap \nu_{R_j}$. Consider for the moment the case where $\nu_{R_j} \cap \nu_{R_i}^o \neq \emptyset$. Then $\nu_{R_j} \cap \nu_{R_i}^o$ is a tame topological $(n-2)$-ball from the transversality in Lemma 5.2 and Theorem 3.1. If $H$ has boundary points, i.e., points of $\delta H$, in $\nu_{R_i}^o$, then $\nu_{R_i}^o - H$ would have only one component. Since the boundary of $\alpha_i$ in $\nu_{R_i}^o$ is included in $H$, the component must equal $\alpha_i^o$. Since $\alpha_i$ is a convex $(n-1)$-ball, $\alpha_i^o$ is convex; this is a contradiction. It follows that $H$ is an $(n-2)$-ball with boundary in $\delta \nu_{R_i}$ and the interior $H^o$ in $\nu_{R_i}^o$, i.e., $H$ separates $\nu_{R_i}$ into two convex components, and the closures of each of them are ($n-1$)-bihedrons. Since $\nu_{R_i}$ is an $(n-1)$-hemisphere, $H$ is an $(n-2)$-hemisphere. Moreover, $\alpha_i$ is the closure of a component of $\nu_{R_i} - H$. (Use Theorem 3.1 to see the possible configurations of the images of these sets in $S^n$.)

We need to consider only the following two cases by interchanging $i$ and $j$ if necessary:

(i) $\nu_{R_j}^o \cap \nu_{R_i}^o \neq \emptyset$.
(ii) $\nu_{R_j} \cap \nu_{R_i}^o = \emptyset$ or $\nu_{R_i} \cap \nu_{R_j}^o = \emptyset$.

(i) Since $\alpha_i$ is the closure of a component of $\nu_{R_i} - H$, $\alpha_i$ is an $(n-1)$-bihedron bounded by an $(n-2)$-hemisphere $H$ and another $(n-2)$-hemisphere $H'$ in $\delta \nu_{R_i}$. Since $H'$ is a subset of the closure of $\alpha_{R_i}$, it belongs to $M_{h,\infty}$ and hence disjoint from $R_i^o$ while $R_i^o \subset M_h$.

Since $H'$ is a subset of $R_j$, we have $H' \subset \delta R_j$. Since $\nu_{R_i}$ is transversal to $\nu_{R_j}$, $H'$ is not a subset of $\nu_{R_j}$; thus, $H'^o$ is a subset of $\alpha_{R_j}$, and $H'$ that of $\text{Cl}(\alpha_{R_j})$ by Theorem 3.1. This completes the proof of the transversality properties (1)–(4) in case (i).
(5) By dipping intersection properties, \( R_i \cap R_j \) is the closure of a component of \( R_j - \alpha_i \) and hence that of \( R_j - \nu_{R_i} \).

(6) Since \( H' \) is a subset of \( \alpha_{R_i} \), \( \alpha_{R_j} - H' \) has two components \( \beta_1 \) and \( \beta_2 \), homeomorphic to open \((n - 1)\)-balls. By (5), we may assume without loss of generality that \( \beta_1 \subset R_i \), and \( \beta_2 \) is disjoint from \( R_i \). Since \( \beta_1 \subset M_{h, \infty} \), we have \( \beta_1 \subset \delta R_i \). As \( \alpha_1 \) is a component of \( \alpha_{R_j} \) removed with \( H' \), we see that \( \beta_1 \) is an open \((n - 1)\)-bihedron bounded by \( H' \) in \( \delta \nu_{R_i} \) and an \((n - 2)\)-hemisphere \( H'' \) in \( \delta \nu_{R_j} \).

Since the closure of \( \beta_1 \) belongs to \( R_i \), we obtain that \( H'' \subset R_i \) and \( H'' \) is a subset of \( \alpha_j \), where \( \alpha_j = \nu_{R_j} \cap R_i \). As \( H'' \) is a subset of \( \delta \nu_{R_j} \), and \( \alpha_j \) is the closure of a component of \( \nu_{R_j} \) removed with \( H \), we obtain \( H'' \subset \delta \alpha_j \).

Since (i) holds for \( i \) and \( j \) exchanged, we obtain, by a paragraph above the condition (i), \( H^o \) belongs to \( \nu_{R_i}^o \cap \nu_{R_j}^o \). By (1)–(4) with values of \( i \) and \( j \) exchanged, \( \alpha_j \) is an \((n - 1)\)-bihedron bounded by \( H \) and an \((n - 2)\)-hemisphere \( H'' \) with interior in \( \alpha_{R_i} \) and is the closure of a component of \( \nu_{R_j} - H \). Since \( H'' \) is an \((n - 1)\)-hemisphere in \( \delta \alpha_j \), and so is \( H'' \), it follows that \( H'' = H'' \).

Since \( \beta_1 \) has the boundary the union of \( H' \) in \( \delta \nu_{R_i} \) and \( H'' \), \( H'' = H'' \), with interior in \( \alpha_{R_i} \), and \( \beta_2 \) is a convex subset of \( R_i \), looking at the bihedron \( \text{dev}(R_i) \) and the geometry of \( S^n \) show that \( \beta_1 \subset \alpha_{R_i} \). Thus, we obtain \( \beta_1 \subset \alpha_1 \cap \alpha_{R_j} \). We see that \( \text{dev}(\alpha_{R_i}) \) and \( \text{dev}(\alpha_{R_j}) \) are subsets of a common great \((n - 1)\) -sphere; it follows easily by Theorem 3.1 that \( \beta_1 = \alpha_{R_i} \cap \alpha_{R_j} \). Hence, \( \alpha_{R_i} \cap \alpha_{R_j} \) and \( \alpha_{R_i} \cup \alpha_{R_j} \) are homeomorphic to open \((n - 1)\)-balls, and under \( \text{dev} \) they map to totally geodesic \((n - 1)\)-balls in \( S^n \).

(ii) Assume \( \nu_{R_j}^o \cap \nu_{R_i} = \emptyset \) without loss of generality. Then \( H \) is a subset of \( \delta \nu_{R_j} \). Since \( R_i \) dips into \( (R_j, \nu_{R_j}) \), we have that \( \alpha_j \neq \emptyset \). Since the boundary of \( \alpha_j \) in \( \nu_{R_j}^o \) is included in \( H \) (see equation [3]), we have \( \alpha_j = \nu_{R_j} \). It follows that \( \nu_{R_j}^o \) is a subset of \( R_i^o \), and \( R_i \cap R_j \) is the closure of a component of \( R_i - \nu_{R_j} \). Since \( \nu_{R_j} \) is an \((n - 1)\)-hemisphere, and \( R_i \) is an \( n \)-bihedron, the uniqueness of \((n - 1)\)-spheres in an \( n \)-bihedron (Theorem 2.3) shows that \( \delta \nu_{R_i} = \delta \nu_{R_j} \). Thus, the closures of components of \( R_i - \nu_{R_j} \) are \( n \)-bihedrons with respective boundaries \( \alpha_{R_i} \cup \nu_{R_j} \) and \( \nu_{R_i} \cup \nu_{R_j} \). By Corollary 3.1, \( R_i \cap R_j \) is the closure of either the first component or the second one.

In the first case, \( R_i^o \cap R_j^o \) is an open subset of \( R_j^o \) since \( R_i^o \) is open in \( M_{h}^o \). The closure of \( R_i^o \) in \( M_{h} \) equals \( R_i^o \cup (\nu_{R_j} \cap M_{h}) = R_i \cap M_{h} \). Since \( \nu_{R_j} \), which contains \( \nu_{R_i} \cap M_{h} \), do not meet \( R_j \) in the first case being in the other component of \( R_i - \nu_{R_j} \), we see that the
intersection of the closure of $R^o_i$ in $M_h$ with $R^o_j$ is same as $R^o_i \cap R^o_j$. Thus, $R^o_i \cap R^o_j$ is open and closed subset of $R^o_j$. Hence $R^o_i \subset R^o_j$ and $R_i \subset R_j$. This contradicts our hypothesis.

In the second case, $\text{dev}|R_i \cup R_j$ is a homeomorphism to $\text{dev}(R_i) \cup \text{dev}(R_j)$. As $\alpha_{R_i}$ and $\alpha_{R_j}$ are subsets of $M_{h,\infty}$, their images under $\text{dev}$ may not meet that of $R^o_i \cup R^o_j$. Hence, $\text{dev}(R_i) \cup \text{dev}(R_j)$ is an $n$-ball bounded by $\text{dev}(\text{Cl}(\alpha_{R_i}))$ and $\text{dev}(\text{Cl}(\alpha_{R_j}))$. We obtain that $R_i \cup R_j$ is the $n$-ball bounded by $(n-1)$-dimensional hemispheres $\text{Cl}(\alpha_{R_i})$ and $\text{Cl}(\alpha_{R_j})$.

Since $\text{Cl}(\alpha_{R_i})$ and $\text{Cl}(\alpha_{R_j})$ are subsets of $M_{h,\infty}$, Lemma 4.1 shows that $M_h = R_i \cup R_j$ and $M_h = R^o_i \cup R^o_j$; thus, $M_h = \tilde{M}$ and $M$ is a closed manifold. The image $\text{dev}(R_1) \cup \text{dev}(R_2)$ is bounded by two $(n-1)$-hemispheres meeting each other on a great sphere $S^{n-2}$, their common boundary. Since $M_h$ is not projectively diffeomorphic to an open $n$-hemisphere or an open $n$-bihedron, the interior angle of intersection of the two boundary $(n-1)$-hemisphere should be greater than $\pi$. However, Lemma 5.1 contradicts this.

**Remark 5.2.** Using the same proof as above, we may drop the condition on the Euler characteristic from Theorem 2.6 of [9] if we assume that $\tilde{M}$ is not diffeomorphic to an open 2-hemisphere or an open lune. This is weaker than requiring that the Euler characteristic of $M$ is less than zero. So, our theorem is an improved version of Theorem 2.6 of [9].

**Corollary 5.1.** Let $R_1$ and $R_2$ be bihedral $n$-crescents and they overlap. Then $\text{dev}(\alpha_{R_1})$ and $\text{dev}(\alpha_{R_2})$ are included in a common great $(n-1)$-sphere $S^{n-1}$, and $\text{dev}(R_1)$ and $\text{dev}(R_2)$ are subsets of a common great $n$-hemisphere bounded by $S^{n-1}$. Moreover, $\text{dev}(R_i - \text{Cl}(\alpha_{R_i}))$ is a subset of the interior of this $n$-hemisphere for $i = 1, 2$. \qed
6. n-CRESCEUTS THAT ARE n-HEMISPHERES AND THE TWO-FACED SUBMANIFOLDS

In this section, we introduce the two-faced submanifolds arising from hemispheric n-crescents. We showed above that if two hemispheric n-crescents overlap, then they are equal. We show that if two hemispheric n-crescents meet but do not overlap, then they meet at the union of common components of their \( \nu \)-boundaries, which we call copied components. The union of all copied components becomes a properly imbedded submanifold in \( M_h \) and covers a properly imbedded submanifold in \( M \). This is the two-faced submanifold.

Lemma 6.1. Let \( R \) be an n-crescent. A component of \( \delta M_h \) is either disjoint from \( R \) or is a component of \( \nu_R \cap M_h \). Moreover, a tiny ball \( B(x) \) of a point \( x \) of \( \delta M_h \) is a subset of \( R \) if \( x \) belongs to \( \nu_R \cap M_h \), and, consequently, \( x \) belongs to the topological interior \( \text{int} R \).

Proof. If \( x \in \delta M_h \), then a component \( F \) of the open \((n-1)\)-manifold \( \nu_R \cap M_h \) intersects \( \delta M_h \) tangentially, and by Lemma 3.1, it follows that \( F \) is a subset of \( \delta M_h \). Since \( F \) is a closed subset of \( \nu_R \cap M_h \), \( F \) is a closed subset of \( \delta M_h \). Since \( F \) is an open manifold, \( F \) is open in \( \delta M_h \). Thus, \( F \) is a component of \( \delta M_h \).

Since \( x \in \text{int} B(x) \), \( B(x) \) and \( R \) overlap. As \( \text{Cl}(\alpha_R) \) is a subset of \( M_{h,\infty} \), we have \( \text{bd} R \cap B(x) \subset \nu_R \) and \( \nu_R \cap B(x) = F \cap B(x) \) for a component \( F \) of \( \nu_R \cap M_h \) containing \( x \). Since \( F \) is a component of \( \delta M_h \), we obtain \( F \cap B(x) \subset \delta B(x) \); since we have \( \text{bd} R \cap B(x) \subset \delta B(x) \), it follows that \( B(x) \) is a subset of \( R \).

Suppose that \( \tilde{M}_h \) includes an n-crescent \( R \) that is an n-hemisphere. Then \( M_h \cap R \) is a submanifold of \( M_h \) with boundary \( \delta R \cap M_h \). Since \( R \) is an n-crescent, \( \delta R \cap M_h \) equals \( \nu_R \cap M_h \). Let \( B_R \) denote \( \nu_R \cap M_h \).

Let \( S \) be another hemispheric n-crescent, and \( B_S \) the set \( \nu_S \cap M_h \). By Theorem 5.1, we see that either \( S \cap R^o = \emptyset \) or \( S = R \). Suppose that \( S \cap R \neq \emptyset \) and \( S \) does not equal \( R \). Then \( B_S \cap B_R \neq \emptyset \). Let \( x \) be a point of \( B_S \cap B_R \) and \( B(x) \) the tiny ball of \( x \). Since \( \text{int} B(x) \cap R \neq \emptyset \), it follows that \( B(x) \) dips into \( (R, \nu_R) \) or \( B(x) \) is a subset of \( R \) by Lemma 3.2. Similarly, \( B(x) \) dips into \( (S, \nu_S) \) or \( B(x) \) is a subset of \( S \). If \( B(x) \) is a subset of \( R \), then \( S \) intersects the interior of \( R \). Theorem 5.1 shows \( S = R \), a contradiction. Therefore, \( B(x) \) dips into \( (R, \nu_R) \) and similarly into \( (S, \nu_S) \). If \( \nu_R \) and \( \nu_S \) intersect transversally, then \( R \) and \( S \) overlap. This means a contradiction \( S = R \). Therefore, \( B_S \) and \( B_R \) intersect tangentially at \( x \).
If \( x \in \delta M_h \), Lemma 6.1 shows that \( B(x) \) is a subset of \( R \). This contradicts a result of the above paragraph. Thus, \( x \in M_h^o \). Hence, we conclude that \( B_R \cap B_S \cap M_h \subset M_h^o \).

Since \( B_R \) and \( B_S \) are closed subsets of \( M_h \), and \( B_R \) and \( B_S \) are totally geodesic and intersect tangentially at \( x \), it follows that \( B_R \cap B_S \) is an open and closed subset of \( B_R \) and \( B_S \) respectively. Thus, for components \( A \) of \( B_R \) and \( B \) of \( B_S \), either we have \( A = B \) or \( A \) and \( B \) are disjoint. Therefore, we have proved:

**Theorem 6.1.** Given two hemispheric \( n \)-crescents \( R \) and \( S \), we have either \( R \) and \( S \) disjoint, or \( R \) equals \( S \), or \( R \cap S \) equals the union of common components of \( \nu_R \cap M_h \) and \( \nu_S \cap M_h \).

Readers may easily find examples where \( \nu_R \cap M_h \) and \( \nu_S \cap M_h \) are not equal in the above situations especially if \( M \) is an open manifold.

**Definition 6.1.** Given a hemispheric \( n \)-crescent \( T \), we say that a component of \( \nu_T \cap M_h \) is *copied* if it equals a component of \( \nu_U \cap M_h \) for some hemispheric \( n \)-crescent \( U \) not equal to \( T \).

Let \( c_R \) be the union of all copied components of \( \nu_R \cap M_h \) for a hemispheric \( n \)-crescent \( R \). Let \( A \) denote \( \bigcup_{R \in \mathcal{H}} c_R \) where \( \mathcal{H} \) is the set of all hemispheric \( n \)-crescents in \( M_h \). \( A \) is said to be the *pre-two-faced submanifold arising from hemispheric \( n \)-crescents*.

**Proposition 6.1.** Suppose that \( A \) is not empty. Then \( A \) is a properly imbedded totally geodesic \((n-1)\)-submanifold of \( M_h^o \) and \( p \mid A \) is a covering map onto a closed totally geodesic imbedded \((n-1)\)-manifold in \( M^o \).

First, given two \( n \)-crescents \( R \) and \( S \), \( c_R \) and \( c_S \) meet either in the union of common components or in an empty set: Let \( a \) and \( b \) be respective components of \( c_R \) and \( c_S \) meeting each other. Then \( a \) is a component of \( \nu_R \cap M_h \) and \( b \) that of \( \nu_S \cap M_h \). Since \( R \cap S \neq \emptyset \), either \( R \) and \( S \) overlap or \( a = b \) by the above argument. If \( R \) and \( S \) overlap, \( R = S \) and hence \( a \) and \( b \) must be the identical component of \( \nu_R \cap M_h \) and hence \( a = b \). Hence, \( A \) is a union of mutually disjoint closed path-components that are components of \( c_R \) for some \( n \)-crescent \( R \). In other words, \( A \) is a union of path-components which are components of \( c_R \) for some \( R \).

Second, given a tiny ball \( B(x) \) of a point \( x \) of \( M_h \), we claim that no more than one path-component of \( A \) may intersect \( \text{int} B(x) \): Let \( a \) be a component of \( c_R \) intersecting \( \text{int} B(x) \). Since copied components are subsets of \( M_h^o \), \( a \) intersects \( B(x)^o \) and hence \( B(x) \)
is not a subset of \( R \). By Lemma 3.2, \( \nu_R \cap B(x) \) is a compact convex \((n-1)\)-ball with boundary in \( \text{bd} B(x) \). Since it is connected, \( a \cap B(x) = \nu_R \cap B(x) \), and \( B(x) \cap R \) is the closure of a component \( C_1 \) of \( B(x) - (a \cap B(x)) \) by Corollary 3.1. Since \( a \) is copied, \( a \) is a component of \( \nu_S \cap M_h \) for an \( n \)-crescent \( S \) not equal to \( R \), and \( B(x) \cap S \) is the closure of a component \( C_2 \) of \( B(x) - (a \cap B(x)) \). Since \( R \) and \( S \) do not overlap, it follows that \( C_1 \) and \( C_2 \) are the two disjoint components of \( B(x) - (a \cap B(x)) \).

Suppose that \( b \) is a component of \( c_T \) for an \( n \)-crescent \( T \) and \( b \) intersects \( \text{int} B(x) \) also. If the \((n-1)\)-ball \( b \cap B(x) \) intersects \( C_1 \) or \( C_2 \), then \( T \) overlaps \( R \) or \( S \) respectively and hence \( T = R \) or \( T = S \) respectively by Theorem 5.1; therefore, we have \( a = b \). This is absurd. Hence \( b \cap B(x) \subset a \cap B(x) \) and \( T \) overlaps with either \( R \) or \( S \). Since these are hemispheric crescents, we have either \( T = R \) or \( T = S \) respectively; therefore, \( a = b \).

Since given a tiny ball \( B(x) \) for a point \( x \) in \( M_h \), no more than one distinct path-component of \( A \) may intersect \( \text{int} B(x) \), each path-component of \( A \) is an open subset of \( A \). This shows that \( A \) is a totally geodesic \((n-1)\)-submanifold of \( M^o_h \), closed and properly imbedded in \( M^o_h \).

Let \( p : M_h \to M \) be the covering map. Since \( A \) is the deck transformation group invariant, we have \( A = p^{-1}(p(A)) \) and \( p|A \) covers \( p(A) \). The above results show that \( p(A) \) is a closed totally geodesic manifold in \( M^o \).

**Definition 6.2.** \( p(A) \) for the union \( A \) of all copied components of hemispheric \( n \)-crescents in \( \tilde{M}_h \) is said to be the two-faced \((n-1)\)-manifold of \( M \) arising from hemispheric \( n \)-crescents (or type I).

Each component of \( p(A) \) is covered by a component of \( A \), i.e., a copied component of \( \nu_R \cap M_h \) for some crescent \( R \). Since \( \alpha_R \) is the union of the open \((n-1)\)-hemispheres in \( \delta R \), \( \nu_R \cap M_h \) lies in an open \((n-1)\)-hemisphere, i.e., an affine patch in the great \((n-1)\)-sphere \( \delta R \). Hence, each component of \( p(A) \) is covered by an open domain in \( \mathbb{R}^n \).

We end with the following observation:

**Proposition 6.2.** Suppose that \( A = \bigcup_{R \in H} c_R \). Then \( A \) is disjoint from \( S^o \) for each hemispheric \( n \)-crescent \( S \) in \( \tilde{M}_h \).

**Proof.** If not, then a point \( x \) of \( c_R \) meets \( S^o \) for some hemispheric \( n \)-crescent \( S \). But if so, then \( R \) and \( S \) overlap, and \( R = S \). This is a contradiction. \( \square \)
7. \textit{n-crescents that are n-bihedrons and the two-faced submanifolds}

In this section, we will define an equivariant set \( \Lambda(R) \) for a bihedral \( n \)-crescent \( R \). We discuss its properties which are exactly same as those of its two-dimensional version in \([3]\). Then we discuss the two-faced submanifolds that arises from \( \Lambda(R) \)'s: We show that \( \Lambda(R) \) and \( \Lambda(S) \) for two \( n \)-crescents are either equal or disjoint or meet at their common boundary components in \( M_h \). The union of all such boundary components for \( \Lambda(R) \) for every bihedral \( n \)-crescents \( R \) is shown to be a properly imbedded submanifold in \( M_h \) and cover a compact submanifold in \( M \). The proof of this fact is similar to those in the previous section.

We will suppose in this section that \( \tilde{M}_h \) includes no hemispheric crescent; we assume that all \( n \)-crescents in \( \tilde{M}_h \) are bihedrons. Two bihedral \( n \)-crescents in \( \tilde{M}_h \) are equivalent if they overlap. This generates an equivalence relation on the collection of all bihedral \( n \)-crescents in \( \tilde{M}_h \); that is, \( R \sim S \) if and only if there exists a sequence of bihedral \( n \)-crescents \( R_i, i = 1, \ldots, n \), such that \( R_1 = R, R_n = S \) and \( R_{i-1} \cap R_i^\circ \neq \emptyset \) for \( i = 2, \ldots, n \).

We define

\[
\Lambda(R) := \bigcup_{S \sim R} S, \quad \delta_\infty \Lambda(R) := \bigcup_{S \sim R} \alpha_S, \quad \Lambda_1(R) := \bigcup_{S \sim R} (S - \nu_R).
\]

\textbf{Example 7.1.} Consider the universal cover \( L \) of \( H^o - \{O\} \) where \( H \) is a 2-hemisphere in \( S^2 \). Then it has an induced real projective structure with developing map equal to the covering map \( c \). There is a nice parameterization \((r, \theta)\) of \( L \) where \( r \) denotes the \( d \)-distance of \( c(x) \) from \( O \) and \( \theta(x) \) the oriented total angle from the lift of the positive \( x \)-axis for \( x \in L \), i.e., one obtained by integrating the 1-form \( d\theta \). Here \( r \) is from \([0, \pi/2]\) and \( \theta \) in \((-\infty, \infty)\). \( L \) is hence a holonomy cover of itself as it is simply connected. \( \tilde{L} \) may be identified with the universal cover of \( H - \{O\} \) with a point \( O' \) added to make it a complete space where \( O' \) maps to \( O \) under the extended developing map \( c \). (We use the universal covering space since the holonomy cover gives us trivial examples. Of course, the holonomy cover of a universal cover is itself.)

A crescent in \( \tilde{L} \) is the closure of a lift of an affine half space in \( H^o - \{O\} \). (Recall that \( H^o \) has an affine structure.) A special type of a crescent is the closure of the set given by \( \theta_0 \leq \theta \leq \theta_0 + \pi \). Given a crescent \( R \) in \( \tilde{L} \), we see that \( \Lambda(R) \) equals \( \tilde{L} \).

We may also define another real projective manifold \( N \) by an equation \( f(\theta) < r < \pi/2 \) for a function \( f \) with values in \((0, \pi/2)\). Then \( \tilde{N} \) equals the closure of \( N \) in \( \tilde{L} \). Given a
crescent $R$ in $\hat{N}$, we see that $\Lambda(R)$ may not equal to $\hat{N}$ in especially in case $f$ is not a convex function (as seen in polar coordinates). See figure 8.

For a higher dimensional example, let $H$ be a 3-hemisphere in $S^3$, and $l$ a segment of $d$-length $\pi$ passing through the origin. Let $L$ be the universal cover of $H^o - l$. Then $L$ becomes a real projective manifold with developing map the covering map $c : L \rightarrow H^o - l$. The holonomy cover of $L$ is $L$ itself. The completion $\hat{L}$ of $L$ equals the completion of the universal cover of $H - l$ with $l$ attached to make it a complete space. A 3-crescent is the closure of a lift of an open half space in $H - l$. Given a 3-crescent $R$, $\Lambda(R)$ equals $\hat{L}$.

We introduce coordinates on $H^o$ so that $l$ is now the $z$-axis. Note that $L$ is parameterized by $(r, \theta, \phi)$ where $r(x)$ equals the $d$-distance from $O$ to $c(x)$ and $\phi$ the angle that $Oc(x)$ makes with the ray in $L$ from the origin in a given direction, and $\theta(x)$ the angle from the lift of the half-$xz$-plane given by $x > 0$. We may also define other real projective manifolds by equation $f(\theta, \phi) < r < \pi/2$ for $f : \mathbb{R} \times (0, \pi) \rightarrow (0, \pi/2)$. The readers may work out how the completions might look and what $\Lambda(R)$ may look when $R$ is a 3-crescent. We remark that for certain $f$ which converges to $\pi/2$ as $\phi \rightarrow 0$ or $\pi$, we may have no 3-crescents in the completion of the real projective manifold given by $f$.

Even higher-dimensional examples are given in a similar spirit by removing sets from such covers. We will see that what we gave are really typical examples of $\Lambda(R)$.

![Diagram](image_url)

**Figure 8.** Figures of $\Lambda(R)$. 46
Let us state the properties that hold for these sets: The proofs are exactly as in [9].

\[
\int \Lambda(R) \cap M_h = \int (\Lambda(R) \cap M_h)
\]
\[
\text{bd}\Lambda(R) \cap M_h = \text{bd}(\Lambda(R) \cap M_h) \cap M_h
\]
(see Lemma 6.4 [9]). For \( \vartheta \) a deck transformation, from definitions we easily obtain

\[
\vartheta(\Lambda(R)) = \Lambda(\vartheta(R)),
\]
\[
\vartheta(\delta_\infty \Lambda(R)) = \delta_\infty \Lambda(\vartheta(R))
\]
\[
\vartheta(\Lambda_1(R)) = \Lambda_1(\vartheta(R))
\]
(5)
\[
\int \vartheta(\Lambda(R)) \cap M_h = \vartheta(\int \Lambda(R) \cap M_h) \cap M_h = \vartheta(\int \Lambda(R) \cap M_h)
\]
\[
\text{bd}\vartheta(\Lambda(R)) \cap M_h = \vartheta(\text{bd}\Lambda(R) \cap M_h) \cap M_h = \vartheta(\text{bd}\Lambda(R) \cap M_h).
\]

The sets \( \Lambda(R) \) and \( \Lambda_1(R) \) are path connected. \( \delta_\infty \Lambda(R) \) is an open \((n-1)\)-manifold. Since Theorem 5.2 shows that for two overlapping \( n \)-crescents \( R_1 \) and \( R_2 \), \( \alpha_{R_1} \) and \( \alpha_{R_2} \) extend each other into a larger \((n-1)\)-manifold, there exists a unique great sphere \( S^{n-1} \) including \( \text{dev}(\delta_\infty \Lambda(R)) \) and by Corollary 5.1, a unique component \( A_R \) of \( S^n - S^{n-1} \) such that \( \text{dev}(\Lambda(R)) \subset \text{Cl}(A_R) \) and \( \text{dev}(\Lambda(R) - \text{Cl}(\delta_\infty \Lambda(R))) \subset A_R \). For a deck transformation \( \vartheta \) acting on \( \Lambda(R) \), \( A_R \) is \( h(\vartheta) \)-invariant. \( \Lambda_1(R) \) admits a real projective structure as a manifold with totally geodesic boundary \( \delta_\infty \Lambda(R) \).

**Proposition 7.1.** \( \Lambda(R) \cap M_h \) is a closed subset of \( M_h \).

*Proof.* This follows as in Section 6.1 of [9]. We show that each point of \( \text{bd}\Lambda(R) \cap M_h \) belongs to \( \Lambda(R) \) by using a sequence of points converging to it and a sequence of \( n \)-crescents containing it using sequences of crescents (see Lemma 9.1). \( \square \)

**Lemma 7.1.** \( \text{bd}\Lambda(R) \cap M_h \) is a properly imbedded topological submanifold of \( M^h \), and \( \Lambda(R) \cap M_h \) is a topological submanifold of \( M_h \) with concave boundary \( \text{bd}\Lambda(R) \cap M_h \).

*Proof.* Let \( p \) be a point of \( \text{bd}\Lambda(R) \cap M_h \). Since \( \Lambda(R) \) is closed, \( p \) is a point of a crescent \( R' \) equivalent to \( R \).

Let \( B(p) \) be an open tiny ball of \( p \). Since by Lemma 6.1, \( \text{bd}\Lambda(R) \cap M_h \) is a subset of \( M^h \), \( B(p)^o \) is an open neighborhood of \( p \). Since \( B(p)^o \cap \Lambda(R) \) is a closed subset of \( B(p)^o \), \( O = B(p)^o - \Lambda(R) \) is an open subset.

We claim that \( O \) is a convex subset of \( B(p)^o \). Let \( x, y \in O \). Then let \( s \) be the segment in \( B(p) \) of \( d \)-length \( \leq \pi \) connecting \( x \) and \( y \). If \( s^o \cap \Lambda(R) \neq \emptyset \), then a point \( z \) of \( s^o \)
belongs to an $n$-crescent $S$, $S \sim R$. If $z$ belongs to $S^o$, since $s$ must leave $S$, $s$ meets $\nu_S$ and is transversal to $\nu_S$ at the intersection point. Since a maximal line in the bihedron $S$ transversal to $\nu_S$ have an endpoint in $\alpha_S$, at least one endpoint of $s$ belongs to $S^o$, which is a contradiction. If $z$ belongs to $\nu_S$ and $s$ is transversal to $\nu_S$ at $z$, the same argument gives us a contradiction. If $z$ belongs to $\nu_S$ and $s$ is tangential to $\nu_S$ at $z$, then $s$ is included in the component of $\nu_S \cap M_h$ containing $z$ since $s \subset M_h$ is connected. Since $x$ and $y$ belong to $O$, this is a contradiction. Hence $s \subset O$, and $O$ is convex.

Since $O$ is convex and open, $bdO$ in $M_h$ is homeomorphic to an $(n-1)$-sphere by Lemma 2.4. The boundary $bd_{B(p)^o}O$ of $O$ relative to $B(p)^o$ equals $bdO \cap B(p)^o$. Hence, $bd_{B(p)^o}O$ is an imbedded open $(n-1)$-submanifold of $B(p)^o$. Since $bd\Lambda(R) \cap B(p)^o = bd_{B(p)^o}O$, $bd\Lambda(R) \cap M_h$ is an imbedded $(n-1)$-submanifold. 

Figure 9. A pre-two-faced submanifold.

Using the same argument as in Section 6.2 of [9] (see Lemma 6.4 of [9]), we obtain the following lemma:
Lemma 7.2. If \( \text{int}\Lambda(R) \cap M_h \cap \Lambda(S) \neq \emptyset \) for an \( n \)-crescent \( S \), then \( \Lambda(R) = \Lambda(S) \). Moreover, if for a crescent \( S \), \( \Lambda(R) \cap M_h \) and \( \Lambda(S) \cap M_h \) meet and they are distinct, then \( \Lambda(R) \cap \Lambda(S) \cap M_h \) is a subset of \( \text{bd}\Lambda(R) \cap M_h \) and \( \text{bd}\Lambda(S) \cap M_h \).

Assume now that \( \Lambda(R) \) and \( \Lambda(S) \) are distinct and meet each other. Consequently \( R \) and \( S \) are not equivalent. Let \( x \) be a common point of \( \text{bd}\Lambda(R) \) and \( \text{bd}\Lambda(S) \), and \( B(x) \) a tiny-ball neighborhood of \( x \). By Lemma 7.1, \( x \in M_h^o \) and so \( x \in B(x)^o \). Let \( T \) be a crescent equivalent to \( R \) containing \( x \), and \( T' \) that equivalent to \( S \) containing \( x \). Then \( T \cap B(x) \) is the closure of a component \( A \) of \( B(x) - P \) for a totally geodesic \( (n - 1) \)-ball \( P \) in \( B(p) \) with boundary in \( \text{bd}B(x) \) by Lemma 5.2. Moreover, \( \nu_T \cap B(x) = P \) and \( T^o \cap B(x) = A \), and \( A \) is a subset of \( \text{int}\Lambda(R) \). Let \( B \) denote \( B(x) \) removed with \( A \) and \( P \). Similarly, \( T' \cap B(x) \) is the closure of a component \( A' \) of \( B(x) - P' \) for a totally geodesic \( (n - 1) \)-ball \( P' \), \( P' = \nu_{T'} \cap B(x) \) with boundary in \( \text{bd}B(x) \), and \( A' \) is a subset of \( T'^o \) in \( \text{int}\Lambda(S) \). Since \( T^o \subset \text{int}\Lambda(R) \), \( T' \subset \Lambda(S) \), \( T' \cap B(x) \) and \( T^o \cap B(x) \) are disjoint, and \( P, P' \ni x \), it follows that \( P = P' \) and \( B = A' \); that is, \( P \) and \( P' \) are tangential. (We have that \( P = P' = \nu_S \cap B(x) = \nu_{S'} \cap B(x) \).

Since \( B \) is a subset of \( \text{int}\Lambda(S) \), \( B \) contains no point of \( \Lambda(R) \) by Lemma 7.2, and similarly \( A \) contains no point of \( \Lambda(S) \). Thus, \( \Lambda(R) \cap B(x) \) is a subset of the closure of \( A \), and \( \Lambda(S) \cap B(x) \) is that of \( B \). Since \( A \subset \text{int}\Lambda(R) \) and \( B \subset \text{int}\Lambda(S) \), it follows that

\[
\begin{align*}
A &= \text{int}\Lambda(R) \cap B(x), & B &= \text{int}\Lambda(S) \cap B(x), \\
P \cup A &= \Lambda(R) \cap B(x), & P \cup B &= \Lambda(S) \cap B(x), \\
P &= \text{bd}\Lambda(R) \cap B(x) = \text{bd}\Lambda(S) \cap B(x).
\end{align*}
\]

Hence, we have \( P = \text{bd}\Lambda(R) \cap \text{bd}\Lambda(S) \cap B(x) \) and \( P \) is a totally geodesic \( (n - 1) \)-ball with boundary in \( \text{bd}B(x) \) and our point \( x \) belongs to \( P^o \), to begin with. Since this holds for an arbitrary choice of a common point \( x \) of \( \text{bd}\Lambda(R) \) and \( \text{bd}\Lambda(S) \), a tiny ball \( B(x) \) of \( x \), it follows that \( \text{bd}\Lambda(R) \cap \text{bd}\Lambda(S) \cap M_h \) is an imbedded totally geodesic open \( (n - 1) \)-submanifold in \( M_h^o \). It is properly imbedded since \( B(x) \cap \text{bd}\Lambda(R) \cap \text{bd}\Lambda(S) \) is compact for every choice of \( B(x) \).

The above paragraph also shows that \( \text{bd}\Lambda(R) \cap \text{bd}\Lambda(S) \cap M_h \) is an open and closed subset of \( \text{bd}\Lambda(R) \cap M_h \). Therefore, for components \( B \) of \( \text{bd}\Lambda(R) \cap M_h \) and \( B' \) of \( \text{bd}\Lambda(S) \cap M_h \) where \( R \not\sim S \), either we have \( B = B' \) or \( B \) and \( B' \) are disjoint. If \( B = B' \), the above paragraph shows that \( B \) is a properly imbedded totally geodesic open \( (n - 1) \)-submanifold in \( M_h \).
We say that a component of \( \text{bd} \Lambda(R) \cap M_h \) is copied if it equals a component of \( \text{bd} \Lambda(S) \cap M_h \) for some \( n \)-crescent \( S \) not equivalent to \( R \). Let \( c_R \) be the union of all copied components of \( \text{bd} \Lambda(R) \cap M_h \).

**Lemma 7.3.** Each component of \( c_R \) is a properly imbedded totally geodesic \((n-1)\)-manifold, and equals a component of \( \nu_T \cap M_h \) for fixed \( T, T \sim R \) and that of \( \nu_{T'} \cap M_h \) for fixed \( T', T' \sim S \), where \( S \) is not equivalent to \( R \).

**Proof.** From above arguments, we see that given \( x \) in a component \( C \) of \( c_R \), and a tiny ball \( B(x) \) of \( x \), there exists a totally geodesic \((n-1)\)-ball \( P \) with \( \delta P \subset \text{bd}B(x) \) so that a component of \( B(x) - P \) is included in \( T, T \sim R \) and the other component in \( T' \) for \( T' \) equivalent to \( S \) but not equivalent to \( R \).

Since \( P \) is connected, \( P \subset C \). Let \( y \) be another point of \( C \) connected to \( x \) by a path \( \gamma \) in \( C \), a subset of \( M_h \). Then we can cover \( \gamma \) by a finitely many tiny balls. By induction on the number of tiny balls, we see that \( y \) belongs to \( \nu_T \cap M_h \) and \( \nu_{T'} \cap M_h \) for fixed \( T \) and \( T' \).

From this induction, we obtain for each point \( y \) of \( C \) and a tiny ball \( B(y) \) of \( y \), an existence of an \((n-1)\)-ball \( P \) satisfying

\[
\text{bd}P \subset \text{bd}B(y), P \subset \nu_T \cap \nu_{T'}, \text{ and } P \subset \text{bd} \Lambda(R) \cap \text{bd} \Lambda(S) \cap M_h.
\]

Since \( y \) belongs to the interior of \( P \) and \( P \subset C \) for any choice of \( y \), \( C \) is open in \( \nu_T \cap M_h \).

Since \( C \) is a closed subset of \( M_h \), \( C \) is a component of \( \nu_T \cap M_h \). Similarly, \( C \) is a component of \( \nu_{T'} \cap M_h \). \( \square \)

Let \( A \) denote \( \bigcup_{R \in \mathcal{B}} c_R \) where \( \mathcal{B} \) denotes the set of representatives of the equivalence classes of bihedral \( n \)-crescents in \( \tilde{M}_h \). \( A \) is said to be the pre-two-faced submanifold arising from bihedral \( n \)-crescents. \( A \) is a union of path-components that are totally geodesic \((n-1)\)-manifolds closed in \( M^o_h \).

**Proposition 7.2.** Suppose that \( A \) is not empty. Then \( A \) is a properly imbedded submanifold of \( M_h \) and \( p|A \) is a covering map onto a closed totally geodesic imbedded \((n-1)\)-manifold in \( M^o \).

**Proof.** We follow the argument in Section \( \Box \) somewhat repetitively. Every pair of two components \( a \) of \( c_R \) and \( b \) of \( c_S \) for \( n \)-crescents \( R \) and \( S \) where \( R, S \in \mathcal{B}, \) are either disjoint or identical. Hence, \( A \) is a union of disjoint closed path-components that are some components of \( c_R \) for \( R \in \mathcal{B} \). This is proved exactly as in Section \( \Box \).
Second, given a tiny ball \( B(x) \) of a point \( x \) of \( M_h \), no more than one path-component of \( A \) may intersect \( \text{int} B(x) \). Let \( a \) be a component of \( c_R \) intersecting \( \text{int} B(x) \). By Lemma 6.1, \( a \) is a component of \( \nu_S \cap M_h \) for \( S \sim R \) and that of \( \nu_T \cap M_h \) for \( T \) not equivalent to \( S \). Furthermore, \( \nu_S \cap B(x) \) is a compact convex \((n - 1)\)-ball with boundary in \( \text{bd} B(x) \). Since it is connected, \( a \cap B(x) = \nu_S \cap B(x) \), and \( B(x) \cap S \) is the closure of a component \( C_1 \) of \( B(x) - (a \cap B(x)) \). Similarly, \( a \cap B(x) = \nu_T \cap B(x) \), and \( B(x) \cap T \) is the closure of a component \( C_2 \) of \( B(x) - (a \cap B(x)) \). Since \( S \) and \( T \) do not overlap, it follows that \( C_1 \) and \( C_2 \) are the two disjoint components of \( B(x) - (a \cap B(x)) \).

Suppose that \( b \) is a component of \( c_U \) for \( U \in \mathcal{B} \) intersecting \( \text{int} B(x) \) also. By Lemma 6.1, \( b \) is a component of \( \nu_{T'} \cap M_h \) for \( T' \sim U \). If the \((n - 1)\)-ball \( b \cap B(x) \) intersects \( C_1 \) or \( C_2 \), then \( U \sim S \) or \( U \sim T \) and \( \Lambda(U) = \Lambda(R) \) or \( \Lambda(U) = \Lambda(T) \) by Lemma 7.2. The characterization of \( B(x) \cap \Lambda(R) \) and \( B(x) \cap \Lambda(T) \) in Lemma 7.3 implies that \( a = b \). This is absurd. Hence, \( b \cap B(x) \subset a \cap B(x) \), and \( T' \) overlaps with at least one of \( S \) or \( T \), and we have \( a = b \) as above.

Since given a tiny ball \( B(x) \) no more than one distinct path-component of \( A \) may intersect \( \text{int} B(x) \), \( A \) is a properly imbedded closed submanifold of \( M^o_h \). The rest of the proof of proposition is the same as that of Proposition 6.1.

Let \( p : M_h \rightarrow M \) be the covering map. Since \( A \) is the deck transformation group invariant, we have \( A = p^{-1}(p(A)) \) and \( p|A \) covers \( p(A) \). The above results show that \( p(A) \) is a closed totally geodesic manifold in \( M^o \).

**Definition 7.1.** For the union \( A \) of all copied components of \( \Lambda(R) \) for bihedral \( n \)-crescents \( R \) in \( \hat{M}_h \), we say \( A \) is the two-faced \((n - 1)\)-manifold of \( M \) arising from bihedral \( n \)-crescents (or type II).

Each component of \( p(A) \) is covered by a component of \( A \), i.e., a component of \( \nu_R \cap M_h \) for some bihedral \( n \)-crescent \( R \). Hence, each component of \( p(A) \) is covered by open domains in \( \mathbb{R}^n \) as in Section 4.

We end with the following observation:

**Proposition 7.3.** Suppose \( \hat{M}_h \) includes no hemispheric \( n \)-crescents and \( A = \bigcup_{R \in \mathcal{B}} c_R \). Then \( A \) is disjoint from \( \mathbb{R}^n \) for each \( n \)-crescent \( R \).

**Proof.** The proof is same as that of Proposition 6.2.

**Example 7.2.** Finally, we give an example in dimension 2. Let \( \vartheta \) be the projective automorphism of \( S^2 \) induced by the diagonal matrix with entries 2, 1, and 1/2. Then \( \vartheta \)
has fixed points $[\pm 1, 0, 0]$, $[0, \pm 1, 0]$, and $[0, 0, \pm 1]$ corresponding to eigenvalues $2, 1, 1/2$. Given three points $x, y, z$ of $S^2$, we let $xyz$ denote the segment with endpoints $x$ and $z$ passing through $y$ if there exists such a segment. If $x$ and $y$ are not antipodal, then let $xy$ denote the unique minor segment with endpoints $x$ and $y$. We look at the closed lune $B_1$ bounded by $[0, 0, 1][1, 0, 0][0, 0, -1]$ and $[0, 0, 1][0, 1, 0][0, 0, -1]$, which are to be denoted by $l_1$ and $l_2$, and the closed lune $B_2$ bounded by $[1, 0, 0][0, -1, 0][-1, 0, 0]$ and $[1, 0, 0][0, 0, 1][-1, 0, 0]$, which are denoted by $l_3$ and $l_4$. We consider the domain $U$ given by $U = B_1^o \cup B_2^o \cup l_1^o \cup l_4^o - \{[1, 0, 0], [0, 0, 1]\}$. Since there exists a compact fundamental domain of the action of $< \vartheta >$, $U/ < \vartheta >$ is a compact annulus $A$ with totally geodesic boundary. $U$ is the holonomy cover of $A$. The projective completion of $U$ can be identified with $B_1 \cup B_2$. It is easy to see that $B_1$ is a 2-crescent with $\alpha_{B_1} = l_2^o$ and $\nu_{B_1} = l_1$ and $B_2$ one with $\alpha_{B_2} = l_3^o$ and $\nu_{B_2} = l_4$. Also, any other crescent is a subset of $B_1$ or $B_2$. Hence $\Lambda(B_1) = B_1$ and $\Lambda(B_2) = B_2$ and the pre-two-faced submanifold $L$ equals $[1, 0, 0][0, 0, 1]^o$. $L$ covers a simple closed curve in $A$ give by $[1, 0, 0][0, 0, 1]^o/ < \vartheta >.$
8. Preservation of crescents after splitting

In this section, we consider somewhat technical questions: How does the \( n \)-crescents in the completions of the holonomy cover of a submanifold become in those of the holonomy cover of an ambient manifolds? What happens to \( n \)-crescents in the completion of a manifold when we split the manifold along the two-faced manifolds. The answers will be that there are one-to-one correspondence: Propositions 8.1, 8.2, and 8.3. In the process, we will define splitting manifolds precisely and show how to contruct holonomy covers of split manifolds.

Let \( M_h \) be the holonomy cover of \( M \) with development pair \((\text{dev}, h)\) and the group of deck transformations \( G_M \). Let \( N \) be a submanifold (not necessarily simply connected) in \( M \) of codimension 0 with an induced real projective structure. Then \( p^{-1}(N) \) is a codimension 0 submanifold of \( M_h \). Choose a component \( A \) of \( p^{-1}(N) \). Then \( A \) is a submanifold in \( M_h \) and \( p|A \) covers \( N \) with the deck transformation group \( G_A \) equal to the group of deck transformations of \( M_h \) preserving \( A \).

We claim that \( A \) is a holonomy cover of \( N \) with development pair \((\text{dev}|A, h')\) where \( h' \) is a composition of the inclusion homomorphism and \( h: G_M \to \text{Aut}(S^n) \). First, for each closed path in \( A \) which lifts to one in \( N \) obviously has a trivial holonomy (see Section 8.4 in [25]). Given a closed path with a trivial holonomy, it lifts to a closed path in \( M_h \) with a base point in \( A \). Since \( A \) is a component of \( p^{-1}(N) \), it follows that the closed path is in \( A \). Therefore, \( A \) is the holonomy cover of \( N \).

Let us discuss about the Kuiper completion of \( A \). The distance metric on \( A \) is induced from the Riemannian metric on \( A \) induced from \( S^n \) by \( \text{dev}|A \). The completion of \( A \) with respect to the metric is denoted by \( \hat{A} \) and the set of ideal points \( A_\infty \); that is, \( A_\infty = \hat{A} - A \). Note that \( \hat{A} \) may not necessarily equal the closure of \( A \) in \( M_h \). A good example may consists of the complement of the closure of the positive axis in \( \mathbb{R}^2 \) as \( A \) and \( M_h \) as \( \mathbb{R}^2 \).

Let \( i: A \to M_h \) be an inclusion map. Then \( i \) extends to a distance nonincreasing map \( \hat{i}: \hat{A} \to \text{Cl}(A) \subset M_h \).

**Lemma 8.1.**  
(i) \( \hat{i}^{-1}(M_{h,\infty}) \) is a subset of \( A_\infty \).  
(ii) If \( A \) is closed as a subset of \( M_h \), then \( i(A_\infty) \subset M_{h,\infty} \). Thus, in this case, \( i(A_\infty) = M_{h,\infty} \).
(iii) Let $P$ be a submanifold in $A$ with convex interior $P^o$. Then the closure $P'$ of $P$ in $\tilde{A}$ maps isometrically to the closure $P''$ of $P$ in $\tilde{M}_h$ under $\tilde{i}$. Here $P'$ and $P''$ are convex.

(iv) $\tilde{i}$ maps $P' \cap A_\infty$ homeomorphic onto $P'' \cap M_{h,\infty}$.

Proof. (i) If $x$ is a point of $i^{-1}(M_{h,\infty})$, then $x$ does not belong to $A$ since otherwise $i(x) = i(x) \in M_h$.

(ii) Suppose not. Then there exists a point $x$ in $M_h$ such that $x = i(y)$ for $y \in A_\infty$. There exists a sequence of points $y_i \in A$ converging to $y$. The sequence of points $i(y_i)$ converges to a point $x$ since $i$ is distance decreasing. Since $A$ is closed, this means $x \in A$ and $y = x$. This is a contradiction.

(iii) Since $i|P^o$ is an isometry with respect to $d|A$ and $d$ on $M_h$, the third part follows.

(iv) Let $K$ be the inverse image of $P'' \cap M_{h,\infty}$ under $i|P'$. By (i), we obtain $K \subset A_\infty$. By (ii), we see $i(P' \cap A_\infty) \subset P'' \cap M_{h,\infty}$. 

Let $R$ be an $n$-crescent in $\tilde{M}_h$, and consider a submanifold $R' = R \cap M_h$ with convex interior $R^o$. If $R'$ is a subset of a submanifold $A$ of $M_h$, then the above lemma shows that the closure $R''$ of $R'$ in $\tilde{A}$ is isometric to $R$ under $\tilde{i}$. By the above lemma, we obtain that $R''$ is also a crescent with $\alpha_{R''} = \tilde{i}^{-1}(\alpha_R)$, and $\nu_{R''} = \tilde{i}^{-1}(\nu_R)$. Moreover, if $R$ is bihedral (resp. hemispheric), then $R''$ is bihedral (resp. hemispheric).

Conversely, let $R$ be an $n$-crescent in $\tilde{A}$. By Lemma 8.1, $i|R : R \to \tilde{i}(R)$ is an imbedding, and the closure of $i(R \cap A)$ equals $\tilde{i}(R)$ and is a convex $n$-ball. By Lemma 8.1, $\tilde{i}(R)$ is an $n$-crescent, which is bihedral (resp. hemispheric) if $R$ is bihedral (resp. hemispheric).

(Note that $\alpha_{\tilde{i}(R)} = \tilde{i}(\alpha_R)$ and $\nu_{\tilde{i}(R)} = \tilde{i}(\nu_R)$.)

Thus, we have proved.

Proposition 8.1. There exists a one to one correspondence of all bihedral $n$-crescents in $\tilde{A}$ and those in $\text{Cl}(A)$ in $\tilde{M}_h$ given by $R \leftrightarrow R'$ for a bihedral $n$-crescent $R$ in $\tilde{A}$ and $R'$ one in $\text{Cl}(A)$ if and only if $R^o = R'^o$. The same statement holds for hemispherical $n$-crescents.

We give a precise definition of splitting. Let $N$ be a real projective $n$-manifold with a properly imbedded $(n-1)$-submanifold $A$. We take an open regular neighborhood $N$ of $A$. Then $N$ is an $I$-bundle over $A$. Let us enumerate components of $A$ by $A_1, \ldots, A_n, \ldots$ and corresponding components of $N$ by $N_1, \ldots, N_n, \ldots$ which are regular neighborhoods of $A_1, \ldots, A_n, \ldots$ respectively. (We do not require the number of components to be finite.)

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For an \( i \), \( N_i \) is an \( I \)-bundle over \( A_i \). By parameterizing each fiber by a real line, \( N_i \) becomes a vector bundle over \( A \) with a flat linear connection. We see that there is a subgroup \( G_i \) of index at most two in \( \pi_1(A_i) \) with trivial holonomy. The single or double cover \( \tilde{N}_i \) of \( N_i \) corresponding to \( G_i \) is a product \( I \)-bundle over \( \tilde{A}_i \) the cover of \( A_i \) corresponding to \( G_i \), considered as a submanifold of \( \tilde{N}_i \).

Since \( N_i \) is a product or twisted \( I \)-bundle over \( A_i \), \( N_i - A_i \) has one or two components. If \( N_i - A_i \) has two components, then we take the closure of each components in \( N_i \) and take their disjoint union \( \tilde{N}_i \) which has a natural inclusion map \( l_i : N_i - A_i \rightarrow \tilde{N}_i \). If \( N_i - A_i \) has one component, then \( \pi_1(A_i) \) has an index two subgroup. Take the double cover \( \tilde{N}_i \) of \( N_i \) so that it is now a product \( I \)-bundle over the corresponding double cover \( \tilde{A}_i \) in \( \tilde{N}_i \). Then \( N_i - A_i \) lifts and imbeds onto a component of \( \tilde{N}_i - \tilde{A}_i \). We denote by \( \tilde{N}_i \) the closure of this component in \( \tilde{N}_i \). There is a natural lift \( l_i : N_i - A_i \rightarrow \tilde{N}_i \), which is an imbedding. After we do this for each \( i, i = 1, \ldots, n, \ldots \), we identify \( N - A \) and the disjoint union \( \bigsqcup_{i=1}^n \tilde{N}_i \) of all \( N_i \) by the maps \( l_i \). Then the resulting manifold \( M \) is said to be obtained from \( N \) by splitting along \( A \). Obviously, \( M \) is compact if \( N \) is compact and \( M \) has totally geodesic boundary if \( A \) is totally geodesic. When \( A \) is not empty, \( M \) is said to be the split manifold obtained from \( N \) along \( A \) (this is just for terminological convenience). Also, we see that for each component of \( A \), we get either two copies or a double cover of the component of \( A \) in the boundary of the split manifold \( M \). They are newly created by splitting. There is a natural quotient map \( q : M \rightarrow N \) by identifying these new faces to \( A \), i.e., \( q|q^{-1}(A) : q^{-1}(A) \rightarrow A \) is a two-to-one covering map.

Now, suppose that \( N \) is compact and \( N_h \) its holonomy cover with development pair \((\text{dev}, h)\). We let \( G_N \) the group of deck transformations of the covering map \( p : N_h \rightarrow N \). Suppose that \( A \) has finitely many components. Then \( p^{-1}(A) \) is a properly imbedded \((n - 1)\)-manifold in \( N_h \). If one splits \( N_h \) along \( p^{-1}(A) \), then it is easy to see that the split manifold \( M' \) covers the manifold \( M \) of \( N \) split along \( A \) with covering map \( p' \) obtained from extending \( p \). However, \( M' \) may not be connected; for each component \( M_i \) of \( M \), we choose a component \( M'_i \) of \( M' \) covering that component. \( M_i \) contains exactly one component \( P_i \) of \( N - A \), and \( M'_i \) includes exactly one component \( P'_i \) of \( N_h - p^{-1}(A) \) as a dense open subset. Thus, \( \bigsqcup_{i=1}^n P'_i \) covers \( \bigsqcup_{i=1}^n P_i \) and \( \bigsqcup_{i=1}^n M'_i \) covers \( M \). (Note that covers in this section are not necessarily connected ones.)
Remark 8.1. The submanifold $p^{-1}(A)$ is orientable since great $(n-1)$-spheres in $S^n$ are orientable and $\text{dev}$ maps $p^{-1}(A)$ into a great $(n-1)$-sphere as an immersion. Thus, there are no twisted $I$-bundle neighborhoods of components of $p^{-1}(A)$ as $N_h$ is orientable also.

The developing map $\text{dev}|P_i$ uniquely extends to a map from $M_i'$ as an immersion for each $i$; we denote by $\text{dev}' : \coprod_{i=1}^n M_i' \to S^n$ the map obtained this way. The deck transformation group $G_i$ of the covering map $p|P_i' : P_i' \to P_i$ equals the subgroup of $G_N$ consisting of deck transformations of $N_h$ preserving $P_i'$. It is easy to see that the action of $G_i$ naturally extends to $M_i'$ and becomes the group of deck transformations of the covering map $p|M_i' : M_i' \to M_i$. For each $i$, we define $h_i : G_i \to \text{Aut}(S^n)$ by $h \circ l_i$ where $l_i : G_i \to G_N$ is the homomorphism induced from the inclusion map. We choose base points of $P_i$ and $P_i'$ respectively, and this gives us a map from the fundamental group $\pi_1(P_i)$ to $G_i$. We define the holonomy homomorphism $h_i' : \pi_1(P_i) \to \text{Aut}(S^n)$ by first mapping to $G_i$, and then followed by $h_i$.

We claim that $P_i'$ is the holonomy cover of $P_i$ with development pair $(\text{dev}'|P_i', h_i)$; that is, for each $P_i$ and $P_i'$ that $\ker h_i'$ equals the fundamental group of $P_i'$. Since $N_h$ is the holonomy cover of $N$, $h$ is injective; for each element of $G_i$ other than the identity maps to a nontrivial element in $\text{Aut}(S^n)$. Hence $\ker h_i'$ equals the kernel of the map $\pi_1(P_i) \to G_i$ given by path lifting to $P_i'$, and $\ker h_i'$ equals $\pi_1(P_i')$.

Moreover, since $M_i'$ is obtained from $P_i'$ by attaching boundary, it follows that $M_i'$ is the holonomy cover of $M_i$ with development pair $(\text{dev}'|M_i', h_i)$. We will sometimes say that the disjoint union $\coprod_{i=1}^n M_i'$ is a holonomy cover of $M = \coprod_{i=1}^n M_i$.

Suppose that there exists a nonempty pre-two-faced submanifold $A$ of $N_h$ arising from hemispheric $n$-crescents. Then we can split $N$ by $p(A)$ to obtain $M$ and $N_h$ by $A$ to obtain $M'$, and $M'$ covers $M$ under the extension $p'$ of the covering map $p : N_h \to N$.

We claim that the collection of hemispheric $n$-crescents in $\check{N}_h$ and the completion $\check{M}'$ of $M'$ are in one to one correspondence. Let $q : M' \to N_h$ denote the natural quotient map identifying the newly created boundary components which restricts to the inclusion map $N_h - A \to N_h$. We denote by $A'$ the set $q^{-1}(A)$, which are newly created boundary components of $M'$. Let $M'$ denote the projective completion of $M'$ with the metric $d$ extended from $N_h - A$. Then as $q$ is distance decreasing, $q$ extends to a map $\check{q} : M' \to \check{N}_h$ which is one-to-one and onto on $M' - A' \to N_h - A$.

Lemma 8.2. $\check{q}$ maps $A'$ to $A$, $M'$ to $N_h$, and $M'_\infty$ to $N_{h,\infty}$. (Which implies that $\check{q}^{-1}(A) = A'$, $\check{q}^{-1}(N_h) = M'$, and $\check{q}^{-1}(N_{h,\infty}) = M'_\infty$.)
Proof. We obviously have $\tilde{q}(A') = q(A') = A$ and $\tilde{q}(M') = q(M') = N_h$.

If a point $x$ of $\tilde{M}'$ is mapped to that of $A$, then let $\gamma$ be a path in $N_h - A$ ending at $\tilde{q}(x)$ in $A$. Then we may lift $\gamma$ to a path $\gamma'$ in $M' - A'$. $\tilde{q}(x)$ has a small compact neighborhood $B$ in $N_h$ where $\gamma$ eventually lies in, and as the closure $B'$ of a component of $B - A$ is compact, there exists a compact neighborhood $B''$ in $M'$ mapping homeomorphic to $B'$ under $q$ and $\gamma'$ eventually lies in $B''$. This means that $x$ lies in $B''$ and hence in $M'$. As $q^{-1}(A) = A'$, $x$ lies in $A'$. Thus, $\tilde{q}^{-1}(A) = A'$ and points of $M'_\infty$ cannot map to a point of $A$.

Using path lifting argument, we may show that $\tilde{q}(M'_\infty)$ is a subset of $A \cup N_{h,\infty}$ as $\tilde{q}|M' - A' \to N_h - A$ is a local $d$-isometry. Hence, it follows that $\tilde{q}(M'_\infty) \subset N_{h,\infty}$. \qed

First, consider the case when $\tilde{N}_h$ includes a hemispheric $n$-crescent $R$. As by Proposition 6.2, $R^o$ is a subset of $N_h - A$, $R^c$ is a subset of $M'$. The closure $R'$ of $R^o$ in $\tilde{M}'$ is naturally an $n$-hemisphere as $\text{dev}|R^o$ is an imbedding onto an open $n$-hemisphere in $S^n$. As $\tilde{q}$ is a $d$-isometry restricted to $R^o$, it follows that $\tilde{q}|R' : R' \to R$ is an imbedding.

Lemma 8.2 shows that $(\tilde{q}|R')^{-1}(\alpha_R)$ is a subset of $M'_\infty$. Thus, $R'$ includes an open $(n - 1)$-hemisphere in $\delta R' \cap M'_\infty$, which shows that $R'$ is a hemispheric $n$-crescent. ($\delta R'$ cannot belong to $M'_\infty$ by Lemma 4.1.)

Now if an $n$-crescent $R$ is given in $\tilde{M}'$, then we have $R^o \subset M' - A'$, and $\tilde{q}(R)$ is obviously an $n$-hemisphere as the closure $R'$ of $R^o$ in $\tilde{N}_h$ is an $n$-hemisphere and equals $\tilde{q}(R)$. Since $\tilde{q}(\alpha_R)$ is a subset of $N_{h,\infty}$ by Lemma 8.2, $\tilde{q}(R)$ is a hemispheric $n$-crescent.

**Proposition 8.2.** There exists a one-to-one correspondence between all hemispheric $n$-crescents in $\tilde{N}_h$ and those of $M'$ by the correspondence $R \leftrightarrow R'$ if and only if we have $R^o = R'^o$.

**Corollary 8.1.** If $\tilde{N}_h$ includes a hemispheric $n$-crescent, then the projective completion of the holonomy cover of at least one component of the split manifold $M$ along the two-faced submanifold, also includes a hemispheric $n$-crescent.

Proof. Let $M_i$ be the components of $M$ and $M'_i$ its holonomy cover as obtained earlier in this section; let $P_i$ be the component of $N - p(A)$ in $M_i$ and $P'_i$ that of $N_h - A$ in $M'_i$ so that $P'_i$ covers $P_i$. We regard two components of $N_h - A$ to be equivalent if there exists a deck transformation of $N_h$ mapping one to the other. Then $P_i$ is a representative of an equivalence class $A_i$. As the deck transformation group is transitive in an equivalence class $A_i$, we see that given two elements $P^a_i$ and $P^b_i$ in $A_i$, the components $M^a_i$ and $M^b_i$
of \(M\) including them respectively are projectively isomorphic as the deck transformation
sending \(P^a_i\) to \(P^b_i\) extends to a projective map \(M^a_i \to M^b_i\), and hence to a quasi-isometry \(\tilde{M}^a_i \to \tilde{M}^b_i\). Thus, if no \(\tilde{M}'_i\) includes a hemispheric \(n\)-crescent, then it follows that \(\tilde{M}'\) do not also. This contradicts Proposition 8.2.

Now, we suppose that \(\tilde{N}_h\) includes no hemispheric \(n\)-crescents \(R\) but includes some bihedral \(n\)-crescents. Let \(A\) be the pre-two-faced submanifold of \(N_h\) arising from bihedral \(n\)-crescents. We split \(N\) by \(p(A)\) to obtain \(M\) and \(N_h\) by \(A\) to obtain \(M'\), and \(M'\) covers \(M\) under the extension \(p'\) of the covering map \(p : N_h - A \to N - p(A)\).

By same reasonings as above, we obtain

**Proposition 8.3.** There exists a one-to-one correspondence between all bihedral \(n\)-crescents in \(\tilde{N}_h\) and those of \(\tilde{M}'\) by correspondence \(R \leftrightarrow R'\) if and only if \(R^o = R'^o\).

**Corollary 8.2.** If \(\tilde{N}_h\) includes a bihedral \(n\)-crescent, then the projective completion of the holonomy cover of at least one component of the split manifold \(M\) along the two-faced submanifold, also includes a bihedral \(n\)-crescent.
9. The proof of the Main theorem

In this section we prove Theorem 1.1 using the previous three-sections, in a more or less straightforward manner. We will start with hemispheric crescent case and then the bihedral case.

**Definition 9.1.** A concave affine manifold $M$ of type I is a compact real projective manifold such that its completion $\hat{M}_h$ of the holonomy cover $M_h$ of $M$ is an $n$-crescent that is an $n$-hemisphere, or a finite disjoint union of such real projective manifolds. A concave affine manifold $M$ of type II is a compact real projective manifold with concave or totally geodesic boundary so that its completion $\hat{M}_h$ of the holonomy cover $M_h$ equals $\Lambda(R)$ for an $n$-crescent $R$ in $\hat{M}_h$ and no $n$-crescent is an $n$-hemisphere, or is a finite disjoint union of such real projective manifolds. We allow $M$ to have nonsmooth boundary that is concave, i.e., not necessarily totally geodesic.

Concave affine manifolds admit natural affine structures: If $M$ is a concave affine manifold of type I, from the properties proved in the above Section 8, $\text{dev}(\hat{M}_h)$ equals an $n$-hemisphere $H$. Since the holonomy group acts on $H$, it restricts to an affine transformation group in $H^o$. Since a projective transformation acting on an affine patch is affine, the interior $M^o$ has a compatible affine structure. If $M$ is one of the second type, then for each bihedral $n$-crescent $R$, $\text{dev}$ maps $R \cap M_h$ into the interior of an $n$-hemisphere $H$ (see Section 7). Hence, it follows that $\text{dev}$ maps $M_h$ into $H^o$. Since given a deck transformation $\vartheta$, we have $\vartheta(\Lambda(R)) = \Lambda(\vartheta(R)) = \hat{M}_h$, we obtain $\text{int}\Lambda(\vartheta(R)) \cap \text{int}\Lambda(R) \cap M_h \neq \emptyset$ and $R \sim \vartheta(R)$ by Lemma 7.2. This shows that $\Lambda(R) = \Lambda(\vartheta(R)) = \vartheta(\Lambda(R))$ and $\delta_\infty \Lambda(R) = \delta_\infty \Lambda(\vartheta(R)) = \vartheta(\delta_\infty \Lambda(R))$ for each deck transformation $\vartheta$ by equation 5. Since $\text{dev}(\delta_\infty \Lambda(R))$ is a subset of a unique great sphere $S^{n-1}$, it follows that $h(\vartheta)$ acts on $S^{n-1}$ and since $\text{dev}(M_h)$ lies in $H^o$, the holonomy group acts on $H^o$. Therefore, $M$ has a compatible affine structure.

If $M$ is a concave affine manifold of type I, then $\delta M$ is totally geodesic since $M_h = R \cap M_h$ for an hemispheric $n$-crescent $R$ and $\delta M_h = \nu_R \cap M_h$. If $M$ is one of type II, then $\delta M$ is concave, as we said in the definition above.

For the purpose of the following lemma, we also define two $n$-crescents $S$ and $T$, hemispheric or bihedral, to be equivalent if there exists a chain of $n$-crescents $T_1, T_2, \ldots, T_n$ so that $S = T_1$ and $T = T_i$ and $T_i$ and $T_{i+1}$ overlaps for each $i = 1, \ldots, n - 1$. We will use this definition in this section only.
Lemma 9.1. Let $x_i$ be a sequence of points of $M_h$ converging to a point $x$ of $M_h$, and $x_i \in R_i$ for $n$-crescents $R_i$ for each $i$. Then for any choice of an integer $N$, we have $R_i \sim R_j$ for infinitely many $i, j \geq N$. Furthermore, if each $R_i$ is an $n$-hemisphere, then $R_i = R_j$ for infinitely many $i, j \geq N$. Finally $x$ belongs to an $n$-crescent $R$ for $R \sim R_i$ for each $i \geq N$.

Proof. Let $B(x)$ be a tiny ball of $x$. Assume $x_i \in \text{int}B(x)$ for each $i$. We can choose a smaller $n$-crescent $S_i$ in $R_i$ so that $x_i$ now belongs to $\nu_{S_i}$ with $\alpha_{S_i}$ subset of $R_i$ as $R_i$ are geometrically “simple”, i.e., a convex $n$-bihedron or an $n$-hemisphere.

Since $B(x)$ cannot be a subset of $S_i$, $S_i \cap B(x)$ is the component of the closure of $B(x) - a_i$ for $a_i = \nu_{S_i} \cap B(x)$ an $(n - 1)$-ball with boundary in $\text{bd}B(x)$. Let $v_i$ be the outer-normal vector at $x_i$ to $\nu_{S_i}$ for each $i$. Choose a subsequence $i_j$, with $i_1 = N$, of $i$ so that the sequence $v_{i_j}$ converges to a vector $v$ at $x$. A generalization of Lemma 3 of the Appendix of \[8\] shows that there exists a common open ball $\mathcal{P}$ in $R_{i_j}$ for $j \geq K$ for some integer $K$. (The proofs are identical.) Thus $S_{i_j} \sim S_{i_k}$ for $j, k \geq K$. Since $R_{i_j} \sim S_{i_j}$ as $S_{i_j}$ is a subset of $R_{i_j}$, we have that $R_{i_j} \sim R_{i_k}$ for $j, k \geq K$.

We assume that each sequence of $\text{dev}(S_{i_j})$ and $\text{dev}(\text{Cl}(\alpha_{S_{i_j}}))$ respectively converge to compact sets in $S^n$ under the Hausdorff topology by choosing a subsequence if necessary.

As $S_{i_j}$ forms a cored sequence, Lemma 4 of the Appendix of \[8\] shows that there exists a convex $n$-ball $S$ containing $\mathcal{P}$, such that $\text{dev}(S)$ equals the limit of $S_{i_j}$. There exists a set $\alpha$ in $S$ which maps to the limit $\alpha_\infty$ of $\text{dev}(\text{Cl}(\alpha_{S_{i_j}}))$, and as $\text{Cl}(\alpha_{S_{i_j}})$ forms a subjugated ideal sequence, it follows that $\alpha$ lies in $M_{h, \infty}$. As $\text{Cl}(\alpha_{S_{i_j}})$ includes an $(n - 1)$-hemisphere, so does $\alpha_\infty$. Thus $\text{dev}(S)$ is either an $n$-bihedron or an $n$-hemisphere. Therefore, $S$ is an $n$-crescent. Obviously, $S \sim S_{i_j}$ for $j \geq K$.

Since the sequence consisting of $x_i$ is also a subjugated sequence of $S_i$ with $\text{dev}(x_i) \to \text{dev}(x)$, we see that there exists a point $y$ in $S$ mapping to $\text{dev}(x)$ under $\text{dev}$. As $B(x)$ and $S$ overlap, $\text{dev}|B(x) \cup S$ is an imbedding. This shows that $x = y$. \[\square\]

We begin the proof of the Main Theorem \[11\]. Actually, what we will be proving the following theorem, which together with Theorem \[14\] implies Theorem \[11\].

Theorem 9.1. Suppose that $M$ is a compact real projective $n$-manifold with totally geodesic or empty boundary, and that $M_h$ is not real projectively diffeomorphic to an open $n$-hemisphere or $n$-bihedron. Then the following statements hold:
• If $\tilde{M}$ includes a hemispheric $n$-crescent, then $M$ includes a compact concave affine $n$-submanifold $N$ of type I or $M^o$ includes the two-faced $(n-1)$-submanifold arising from hemispheric $n$-crescent.

• If $\tilde{M}$ includes a bihedral $n$-crescent, then $M$ includes a compact concave affine $n$-submanifold $N$ of type II or $M^o$ includes the two-faced $(n-1)$-submanifold arising from bihedral $n$-crescent.

First, we consider the case when $\tilde{M}_h$ includes an $n$-crescent $R$ that is an $n$-hemisphere.

Suppose that there is no copied component of $\nu_T \cap M_h$ for every hemispheric $n$-crescent $T$. Recall from Section 6 that either $R = S$ or $R$ and $S$ are disjoint for every pair of hemispheric $n$-crescents $R$ and $S$.

Let $x \in M_h$ and $B(x)$ the tiny ball of $x$. Then only finitely many distinct hemispheric $n$-crescents intersect a compact neighborhood of $x$ in $\text{int}B(x)$. Otherwise, there exists a sequence of points $x_i$ converging to a point $y$ of $\text{int}B(x)$, where $x_i \in \text{int}B(x)$ and $x_i \in R_i$ for hemispheric $n$-crescents $R_i$ for mutually distinct $R_i$, but Lemma 9.1 contradicts this.

Consider $R \cap M_h$ for a hemispheric $n$-crescent $R$. Since $R$ is a closed subset of $\tilde{M}_h$, $R \cap M_h$ is a closed subset of $M_h$. Let $A$ be the set $\bigcup_{R \in H} R \cap M_h$. Then $A$ is a closed subset of $M_h$ by above.

Since $R \cap M_h$ is a submanifold for each $n$-crescent $R$, $A$ is a submanifold of $M_h$, a closed subset. Since the union of all hemispheric $n$-crescents $A$ is deck transformation group invariant, we have $p^{-1}(p(A)) = A$. The above results show that $p|A$ is a covering map onto a compact submanifold $N$ in $M$, and $p|R \cap M_h$ is a covering map onto a component of $N$ for each hemispheric $n$-crescent $R$.

Since the components of $A$ are locally finite in $M_h$, it follows that $N$ has only finitely many components. Let $K$ be a component of $N$. In the beginning of Section 8, we showed that $R \cap M_h$ is a holonomy cover of $K$. Let $\bar{K}$ be the projective completion of $R \cap M_h$. The closure of $R \cap M_h$ in $\bar{K}$ is a hemispheric $n$-crescent identical with $\bar{K}$ as a set by Proposition 8.1. Hence, $K$ is a concave affine manifold of type I.

If there is a copied component of $\nu_T \cap M_h$ for some hemispheric $n$-crescent $T$, Proposition 6.1 implies the Main theorem.

Now, we assume that $\tilde{M}_h$ includes only $n$-crescents that are $n$-bihedrons. Suppose that there is no copied component of $\text{bd} \Lambda(T) \cap M_h$ for every bihedral $n$-crescents $T$, $T \in \mathcal{B}$. Then either $\Lambda(R) = \Lambda(S)$ or $\Lambda(R)$ and $\Lambda(S)$ are disjoint for $n$-crescents $R$ and $S$, $R, S \in \mathcal{B}$, by Lemma 7.2.
Using this fact and Lemma 9.1, we can show similarly to the proof for the hemispheric $n$-crescents that $A = \bigcup_{R \in \mathcal{B}} \Lambda(R) \cap M_h$ is closed: We showed that $\Lambda(R) \cap M_h$ is a closed subset of $M_h$ in Section 7. For each point $x$ of $M_h$ and a tiny ball $B(x)$ of $x$, there are only finitely many mutually distinct $\Lambda(R_i)$ intersecting a compact neighborhood of $x$ in $\text{int} B(x)$ for $n$-crescents $R_i$: Otherwise, we get a sequence $x_i, x_i \in \text{int} B(x)$, converging to $y, y \in \text{int} B(x)$, so that $x_i \in \Lambda(R_i)$ for $n$-crescents $R_i$ with mutually distinct $\Lambda(R_i)$, i.e., $R_i$ is not equivalent to $R_j$ whenever $i \neq j$. Then $x_i \in S_i$ for an $n$-crescent $S_i$ equivalent to $R_i$. Lemma 9.1 implies $S_i \sim S_j$ for infinitely many $i, j \geq N$. Since $S_i \sim R_i$, this contradicts the fact that $\Lambda(R_i)$ are mutually distinct.

The subset $A$ is a submanifold since each $\Lambda(R) \cap M_h$ is one for each $R, R \in \mathcal{B}$. Similarly to the hemisphere case, since $p^{-1}(p(A)) = A$, we obtain that $p|A$ is a covering map onto a compact submanifold $N$ in $M$, and $p|\Lambda(R) \cap M_h, R \in \mathcal{B}$, is a covering map onto a component of $N$. $N$ has finitely many components since the components of $A$ are locally finite in $M_h$ by the above paragraph.

Let $K$ be the component of $N$ that is the image of $\Lambda(R) \cap M_h$ for $R, R \in \mathcal{B}$. As in the beginning of Section 8, $\Lambda(R) \cap M_h$ is a holonomy cover of $K$. Let $\tilde{K}_h$ denote the projective completion of $\Lambda(R) \cap M_h$. For each crescent $S, S \sim R$, the closure $S'$ of $S \cap M_h$ in $\tilde{K}_h$ is an $n$-crescent (see Section 8). It follows that each point $x$ of $\Lambda(R) \cap M_h$ is a point of a crescent $S'$ in $\tilde{K}$ equivalent to the crescent $R'$, the closure of $R \cap M_h$ in $\tilde{K}$. Since $\tilde{K}$ thus is the union of equivalent bihedral $n$-crescents, $N$ is a concave affine manifold of type II.

When there are copied components of $\text{bd}\Lambda(R) \cap M_h$, then Proposition 7.2 completes the proof of the Main theorem. \qed
10. The proof of the Corollary 1.1

In this section, we will prove Corollary 1.1. The basic tools are already covered in previous three sections. As before, we study hemispheric case first, and the proof of the bihedral case is entirely similar but will be spelled out. We will also give a proof of Corollary 1.2 here.

We give a cautionary note before we begin: We will assume that $M$ is not $(n-1)$-convex, and so $M_h$ is not projectively diffeomorphic to an open $n$-bihedron or an open $n$-hemisphere, so that we can apply various results in Sections 5 – 8, such as the intersection properties of hemispheric and bihedral $n$-crescents. We will carry out various decompositions of $M$ in this section. Since in each of the following step, the result are real projective manifolds with nonempty boundary if a nontrivial decomposition had occurred, it follows that their holonomy covers are not projectively diffeomorphic to open $n$-bihedrons and open $n$-hemispheres. So our theory in Sections 5 – 8 continues to be applicable.

Also, in this section, covering spaces need not be connected. This only complicates the matter of identifying the fundamental groups with the deck transformation groups, where we will be a little cautious. Note that even for disconnected spaces we can define projective completions as long as immersions to $S^n$, i.e., developing maps, are defined since we can always pull-back the metrics in that case.

We show a diagram of manifolds that we will be obtaining in the construction. The ladder in the first row is continued to the next one. Consider it as a one continuous ladder.

\[
\begin{align*}
M & \Rightarrow_{p(A_1)} M^s \Rightarrow N \amalg K \\
\uparrow p & \uparrow \\
M_h & \Rightarrow_{A_1} M_h^s \Rightarrow N_h \amalg \bigcup_{R \in H} R \cap N_h
\end{align*}
\]

\[
\Rightarrow_{p(A_2)} N^s \amalg K \Rightarrow S \amalg T \amalg K
\]

\[
\Rightarrow_{A_2} N_h^s \amalg \bigcup_{R \in H} R \cap N_h \Rightarrow S_h \amalg \bigcup_{R \in B} \Lambda(R) \cap N_h \amalg \bigcup_{R \in H} R \cap N_h
\]

where the notation $\Rightarrow_A$ means to split along a submanifold $A$ if $A$ is compact and means to split and take appropriate components to obtain a holonomy cover if $A$ is noncompact, $\Rightarrow$ means to decompose and to take appropriate components, $\amalg$ means a disjoint union and other symbols will be explained as we go along. Note that when any of $A_1, K, A_2, T$ is empty, then the operation of splitting or decomposition does not take place and the
next manifolds are identical with the previous ones. For convenience, we will assume that all of them are not empty in the proof.

Let $M$ be a compact real projective $n$-manifold with totally geodesic or empty boundary. Suppose that $M$ is not $(n - 1)$-convex, and we will now be decomposing $M$ into various canonical pieces.

Since $M$ is not $(n - 1)$-convex, $\tilde{M}_h$ includes an $n$-crescent (see Theorem 4.4). By Theorem 9.1, $M$ has a two-faced $(n - 1)$-manifold $S$, or $M$ includes a concave affine manifold.

Suppose that $\tilde{M}_h$ has a hemispheric $n$-crescent, and that $A_1$ is a pre-two-faced submanifold arising from hemispheric $n$-crescents. (As before $A_1$ is two-sided.) Let $M^s$ denote the result of the splitting of $M$ along $p(A_1)$, and $M'$ that of $M_h$ along $A_1$, and $A'_1$ the boundary of $M'$ corresponding to $A_1$, “newly created from splitting.” We know from Section 9 that there exists a holonomy cover of $M^s$ that is a union of suitable components of $M'$. This completes the construction of the first column of arrows in equation 7.

We now show that $M^s$ now has no two-faced submanifold. Let $\tilde{M}'$ denote the projective completion of $M'$. Suppose that two hemispheric $n$-crescents $R$ and $S$ in $\tilde{M}'$ meet at a common component $C$ of $\nu_s \cap S$ and $\nu_s \cap M'$ and that $R$ and $S$ are not equivalent. Proposition 3.1 applied to $M'$ shows that $C \subset M'^s$; in particular, $C$ is disjoint from $A'_1$.

There exists a map $\hat{q} : \tilde{M}' \to \tilde{M}_h$ extending the quotient map $q : M' \to M_h$ identifying newly created split faces in $M'$. There exist hemispheric $n$-crescents $\hat{q}(R)$ and $\hat{q}(S)$ in $\tilde{M}_h$ with same interior as $R^o$ and $S^o$ included in $M_h - A_1$ by Proposition 5.2.

Since $\nu_R \cap \nu_S \cap M'$ belongs to $M' - A'_1 = M_h - A_1$, it follows that $\hat{q}(\nu_R)$ and $\hat{q}(\nu_S)$ meet in $M_h - A_1$. Since obviously $q(\nu_R) \subset R'$ and $q(\nu_S) \subset S'$, we have that $R'$ and $S'$ meet in $M_h - A_1$. However, since $R'$ and $S'$ are hemispheric $n$-crescents in $\tilde{M}_h$, $q(C)$ is a subset of the pre-two-faced submanifold $A_1$. This is a contradiction. Therefore, we have either $R = S$ or $R \cap S = \emptyset$ for $n$-crescents $R$ and $S$ in $\tilde{M}'$. Finally, since $\tilde{M}_h^s$ is a subset of $\tilde{M}'$ obviously, we also have $R = S$ or $R \cap S = \emptyset$ for $n$-crescents $R$ and $S$ in $\tilde{M}_h^s$.

The above shows that $M^s$ has no two-faced submanifold arising from hemispheric $n$-crescents. Let $H$ denote the set of all hemispheric $n$-crescents in $\tilde{M}_h^s$. As in Section 3, $p|_{\bigcup_{R \in H} R} : \bigcup_{R \in H} R \cap M_h^s$ is a covering map to the concave affine manifold $K$ of type I. Since any two hemispheric $n$-crescents are equal or disjoint, it is easy to see that $p^{-1}(K^o) = \bigcup_{R \in H} R^o$. Then $N, N = M^s - K^o$, is a real projective $n$-manifold with totally geodesic boundary; in
fact, $M^s$ decomposes into $N$ and $K$ along totally geodesic $(n-1)$-dimensional submanifold

$$p(\bigcup_{R \in H} \nu_R \cap M^s_h).$$

We see that $p^{-1}(N)$ equals $M^s_h - p^{-1}(K^\circ)$, and so $M^s_h - p^{-1}(K^\circ)$ covers $N$. As we saw in Section 8, we may choose a component $L_i'$ of $M^s_h - p^{-1}(K^\circ)$ for each component $L_i$ of $N$ covering $L_i$ as a holonomy cover with developing map $\text{dev}|L_i'$ and holonomy homomorphism as described there; $\bigcup_{i=1}^n L_i'$ becomes a holonomy cover $N_h$ of $N$.

We will show that $\tilde{N_h}$ includes no hemispheric $n$-crescent, which implies that $N$ contains no concave affine manifold of type I. This completes the construction of the second column of arrows of equation 7.

The projective completion of $L_i'$ is denoted by $\tilde{L_i'}$. The projective completion $\tilde{N_h}$ of $N_h$ equals the disjoint union of $\tilde{L_i'}$. Since the inclusion map $i : L_i' \to M^s_h$ is distance decreasing, it extends to $\tilde{i} : \tilde{N_h} \to \tilde{M^s_h}$. If $\tilde{N_h}$ includes any hemispheric $n$-crescent $R$, then $\tilde{i}(R)$ is a hemispheric $n$-crescent by Proposition 8.1. First, since $\tilde{i}(R) \cap N_h$ is a subset of $p^{-1}(K)$ by the construction of $K$, and $\tilde{i}(R) \cap M^s_h \supset R \cap N_h$, it follows that $R \cap N_h \subset p^{-1}(K)$. Second, since $N_h \cap p^{-1}(K)$ equals $p^{-1}(\text{bd}K)$, it follows that $N_h \cap p^{-1}(K)$ includes no open subset of $M^s_h$. Finally, this is a contradiction while $R \cap N_h \supset R^o$. Therefore, $\tilde{N_h}$ includes no hemispheric $n$-crescent. (We may see after this stage, the completions of the covers of the spaces never includes any hemispheric $n$-crescents as the splitting and taking submanifolds do not affect this fact which we showed in Section 8.)

Now we go to the second stage of the construction. Suppose that $\tilde{N_h}$ includes bihedral $n$-crescents and $A_2$ is the two-faced $(n-1)$-submanifold arising from bihedral $n$-crescents. Then we obtain the splitting $N^s$ of $N$ along $p(A_2)$.

We split $N_h$ along $A_2$ to obtain $N^s$. Then the holonomy cover $N^s_h$ of $N^s$ is a disjoint union of components of $N^s$ chosen for each component of $N^s$. Let $\tilde{N^s_h}$ denote the completion.

The reasoning using Proposition 8.1 as in the seventh paragraph above shows that $\Lambda(R) = \Lambda(S)$ or $\Lambda(R) \cap \Lambda(S) = \emptyset$ for every pair of bihedral $n$-crescents $R$ and $S$ in $\tilde{N^s_h}$. The proof of Theorem 9.1 shows that $N^s$ contains a concave affine manifold $T$ of type II with the covering map

$$p|\bigcup_{R \in B} \Lambda(R) \cap N^s_h : \bigcup_{R \in B} \Lambda(R) \cap N^s_h \to K.$$
where $\mathcal{B}$ denotes the set of representatives of the equivalence classes of bihedral $n$-crescents in $\tilde{N}_h^s$. And we see that $N^s - T^o$ is a real projective manifold with convex boundary. We let $S = N^s - T^o$. Thus $N^s$ decomposes into $S$ and $T$.

For each component $S_i$ of $S$, we choose a component $S_i'$ of $N_h^s - p^{-1}(T^o)$. Then $\bigsqcup S_i'$ is a holonomy cover $S_h$ of $S$. The projective completion $\tilde{S}_h$ equals the disjoint union $\bigsqcup \tilde{S}_i'$. As in the fourth paragraph above, we can show that $\tilde{S}_h$ includes no hemispheric or bihedral $n$-crescent using Proposition 8.1. If $S$ is not $(n-1)$-convex, then $\tilde{S}_h$ includes an $n$-crescent since Theorem 4.4 easily generalizes to the case when the real projective manifold $M$ has convex boundary instead of totally geodesic one or empty one. Since $\tilde{S}_h$ does not include a bihedral $n$-crescent, it follows that $S$ is $(n-1)$-convex.

Now, we will show that the decomposition of Corollary 4.1 is canonical. First, the two-faced submanifolds $A_1$ and $A_2$ are canonically defined. Now let $M = M^s$ and $N = N^s$ for convenience.

First, suppose that $M$ decomposes into $N'$ and $K'$ where $K'$ is a submanifold whose components are concave affine manifolds of type I and $N'$ is the closure of $M - K'$ and $N'$ includes no concave affine manifold of type I. This means that $\tilde{N}_h'$ includes no hemispheric $n$-crescent where $\tilde{N}_h'$ is the completion of the holonomy cover $N_h'$ of $N'$. We will show that $N' = N$ and $K' = K$.

Let $K'_i$, $i = 1, \ldots, n$, be the components of $K'$, and let $K'_{i,h}$ their respective holonomy cover, $\tilde{K}'_{i,h}$ the projective completions. Each $\tilde{K}'_{i,h}$ equals a hemispheric $n$-crescent $R_i$. We claim that $p^{-1}(K')$ equals $\bigcup_{R \in \mathcal{H}} R \cap M_h$ where $\mathcal{H}$ is the set of all $n$-crescents in $\tilde{M}_h$. This will prove our claim in the above paragraph.

Each component $K'_i$ of $p^{-1}(K'_i)$ is a holonomy cover of $K'_i$ (see Section 8). Let $l'_i$ denote the lift of the covering map $K'_{h,i} \to K'_i$ to $K'_i$ which is a homeomorphism. $\text{dev} \circ l'_i$ is a developing map for $K'_{h,i}$ as it is a real projective map (see Ratcliff 23). We may put a metric $d$ on $K'_{h,i}$ induced from $d$ on $S^n$, a quasi-isometric to any such choice of metric, using developing maps. Thus, we may identify $K'_{i,j}$ with $K'_{h,i}$ and their completions for a moment. From the definition of concave affine manifolds of type I, the completion of $K'_{h,i}$ equals a hemispheric $n$-crescent $R_i$, and $K'_{h,i} = R_i \cap K'_{h,i}$. Proposition 8.1 shows that there exists a hemispheric $n$-crescent $R'_i$ in $\tilde{M}_h$ with identical interior as that of $R_i$, and clearly $R'_i$ includes $K'_i$ in $\tilde{M}_h$ so that $K'_i = R'_i \cap M_h$. Since this is true for any component $K'_i$, we have that $p^{-1}(K')$ is a union of hemispheric $n$-crescents intersected with $M_h$ and a subset of $\bigcup_{R \in \mathcal{H}} R \cap M_h$. 

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Suppose that there exists a hemispheric $n$-crescent $R$ in $\hat{M}_h$ so that $R \cap M_h$ is not a subset of $p^{-1}(K')$. Suppose $R$ meet $p^{-1}(K')$. Then $R$ meets a hemispheric $n$-crescent $S$ where $S \cap M_h \subset p^{-1}(K')$. If $R$ and $S$ overlap, then $R = S$, which is absurd. Thus, $R$ and $S$ may meet only at $\nu_R \cap M_h$ and $\nu_S \cap M_h$. This gives us a component of a pre-two-faced submanifold. By our assumption on $M$, this does not happen. Thus we see that $R$ and $S$ are disjoint, and $R^o$ is a subset of $p^{-1}(M - K'^o)$. Since each component of $p^{-1}(M - K'^o)$ is a holonomy cover of a component of $M - K'^o$, it follows that the completion of a holonomy cover of a component of $M - K'^o$ includes a hemispheric $n$-crescent by Proposition 8.1. Obviously this hemispheric $n$-crescent intersected with the holonomy cover covers a concave affine manifold of type I. As this contradicts our assumption about $M - K'^o$, we have that $p^{-1}(K')$ equals $\bigcup_{R \in H} R \cap M_h$.

Second, if $N$ decomposes into $S'$ and $T'$ where $T'$ is a concave affine manifold of type II, and $S'$ do not contain any concave affine manifold of type II, then we claim that $S' = S$ and $T' = T$. As above, this follows if the completion of the holonomy cover of each component of $S'$ does not contain any bihedral $n$-crescents. The proof of this claim is exactly analogous to the above three paragraphs. This completes the proof of Corollary 1.1.

To end this section, we supply the proof of Corollary 1.2. Corollary 1.1 shows that the real projective surface $\Sigma$ decomposes into convex real projective surfaces and concave affine surfaces along simple closed geodesics. By Euler characteristic considerations, their Euler numbers are all zero.

Remark 10.1. Finally, we remark that Corollary 1.1 also holds if we simply assume that $M_h$ is not real projectively diffeomorphic to an open $n$-hemisphere or $n$-bihedron. In this case the decomposition could be a trivial one. Given the assumption in Corollary 1.1, the decomposition has to take place at least once.
Finally, we end with an application to affine Lie groups. Let $G$ be a Lie group with a left-invariant real projective structure, which means that $G$ has a real projective structure and the group of left translations are projective automorphisms. Consider the holonomy cover $G_h$ of $G$. Then $G_h$ is also a Lie group with the induced real projective structure, which is clearly left-invariant.

As before if $G$ is not $(n-1)$-convex, then $G_h$ is not projectively diffeomorphic to an open $n$-bihedron or an open $n$-hemisphere.

**Theorem 11.1.** If $G$ is not $(n-1)$-convex as a real projective manifold, then the projective completion $\hat{G}_h$ of $G_h$ includes an $n$-crescent $B$.

**Proof.** This is proved similarly to Theorem 4.4 by a pull-back argument. The reason is that the left-action of $G_h$ on $G_h$ is proper and hence given two compact sets $K$ and $K'$ of $G_h$, the set $\{g \in G_h | g(K) \cap K' \neq \emptyset\}$ is a compact subset of $G_h$. That is, all arguments of Section 4 go through by choosing an appropriate sequence $\{g_i\}$ of elements of $G_h$ instead of deck transformations. Obviously, if $G_h$ contains a cocompact discrete subgroup, then this is a corollary of Theorem 4.4. But if not, we have to do this parallel argument. 

Suppose that from now on $\hat{G}_h$ includes an $n$-crescent $B$. Since the action of $G_h$ on $G_h$ is transitive, $\hat{G}_h$ equals the union of $g(B)$ for $g \in G_h$. We claim that $g(B) \sim g'(B)$ for every pair of $g$ and $g'$ in $G_h$: That is, there exists a chain of $n$-crescents $B_i$, $i = 1, \ldots, k$, of same type as $B$ so that $B_1 = g(B)$, $B_i$ overlaps with $B_{i+1}$ for each $i = 1, \ldots, n-1$, and $B_k = g'(B)$: Let $p$ be a point of $B^o$ and $B(p)$ a tiny ball of $p$ in $B^o$. We can choose a sequence $g_0, \ldots, g_n$ with $g_0 = g$ and $g_n = g'$ where $g_{i+1}^{-1}g_i(p)$ belongs to $B(p)^o$. In other words, we require $g_{i+1}^{-1}g_i$ to be sufficiently close to the identity element. Then $g_i(B^o) \cap g_{i+1}(B^o) \neq \emptyset$ for each $i$. Hence $g(B) \sim g'(B)$ for any pair $g, g' \in G_h$.

If $B$ is an $n$-hemisphere, then we claim that $g(B) = B$ for all $g$ and $G_h = B \cap G_h$. The proof of this fact is identical to that of Theorem 5.1 but we have use the following lemma instead of Lemma 5.1.

**Lemma 11.1.** Suppose that $\text{dev} : \hat{G}_h \to S^n$ is an imbedding onto the union of two $n$-hemispheres $H_1$ and $H_2$ meeting each other on an $n$-bihedron or an $n$-hemisphere. Then $H_1 = H_2$, and $G_h$ is projectively diffeomorphic to an open $n$-hemisphere.
Proof. If $H_1$ and $H_2$ are different, as in the proof of Lemma 5.1 we obtain two $(n-1)$-dimensional hemispheres $O_1$ and $O_2$ in $G_h$ where a subgroup of index one or two in $G_h$ acts on. Since the action of $G_h$ is transitive, this is clearly absurd.

So if $B$ is an $n$-hemisphere, then we obtain $G_h = B \cap G_h$. Since $G_h$ is boundaryless, $\delta B$ must consist of ideal points. This contradicts the definition of $n$-crecents. Therefore, every $n$-crecent in $\tilde{G}_h$ is a bihedral $n$-crecent.

The above shows that $\tilde{G}_h = \Lambda(B)$ for a bihedral $n$-crecent $B$, $G_h$ is a concave affine manifold of type II and hence so is $G$: To prove this, we need to show that two overlapping $n$-crecents intersect transversally as the proof for the Lie group case is slightly different. This is proved as in the proof of Theorem 5.2 using the above Lemma 11.1 instead of Lemma 5.1.

Let $H$ be a Lie group acting transitively on a space $X$. It is well-known that for a Lie group $L$ with left-invariant $(H, X)$-structure, the developing map is a covering map onto its image, an open subset (see Proposition 2.2 in Kim [23]). Thus $\text{dev}: G_h \to S^n$ is a covering map onto its image.

Recall that $\text{dev}$ maps $\Lambda(B)^o$ into an open subset of an open hemisphere $H$, and $\delta\Lambda(B)$ is mapped into the boundary $S^{n-1}$ of $H$. Each point of $G_h$ belongs to $S^o$ for an $n$-crecent $S$ equivalent to $R$ since the action of $G_h$ on $G_h$ is transitive (see above). Since each point of $\text{dev}(G_h)$ belongs to the interior of an $n$-bihedron $S$ with a side in $S^{n-1}$, the complement of $\text{dev}(G_h)$ is a closed convex subset of $H^o$. Thus, $\text{dev}|G_h$ is a covering map onto the complement of a convex closed subset of $\mathbb{R}^n$. As $\tilde{G}$ covers $G_h$, we see that this completes the proof of Theorem 1.3.

An affine $m$-convexity for $1 \leq m < n$ is defined as follows. Let $M$ be an affine $n$-manifold, and let $T$ be an affine $(m+1)$-simplex in $\mathbb{R}^n$ with sides $F_1, F_2, \ldots, F_{m+2}$. Then every nondegenerate affine map $f: T^o \cup F_2 \cup \cdots \cup F_{m+2} \to M$ extends to one $T \to M$ (see [10] for more details).

If $G$ has a left-invariant affine structure, then $G$ has a compatible left-invariant real projective structure. By Theorem 1.3, $G$ is either $(n-1)$-convex or $G_h$ is a concave affine manifold of type II. It is easy to see that the $(n-1)$-convexity of $G$ in the real projective sense implies the $(n-1)$-convexity of $G$ in the affine sense.

As before, if $G_h$ is a concave affine manifold of type II, the argument above shows that $G_h$ is mapped by $\text{dev}$ to the complement of a closed convex set in $\mathbb{R}^n$. This completes the proof of Corollary 1.3.
Finally, we easily see that the following theorem holds with the same proof as the Lie group case:

**Theorem 11.2.** Let $M$ be a homogenous space on which a Lie group $G$ acts transitively and properly. Suppose $M$ have a $G$-invariant real projective structure. Then $M$ is either $(n - 1)$-convex, or $M$ is concave affine. The same holds for $G$-invariant affine structures.
Appendix A. Proof of Theorem 4.3

The proofs are a little sketchy here; however, they are elementary.

Theorem A.1. The following are equivalent:

1. $M$ is 1-convex.
2. $M$ is convex.
3. $M$ is real projectively isomorphic to a quotient of a convex domain in $S^n$.

(1)$\rightarrow$(2): Since $M$ is 1-convex, $\tilde{M}$ is 1-convex. Any two points $x$ and $y$ in $\tilde{M}$ are connected by a chain of segments $s_i$, $i = 1, \ldots, n$, of $d$-length $< \pi$ with endpoints $p_i$ and $p_{i+1}$ so that $s_i \cap s_{i+1} = \{p_{i+1}\}$ exactly. This follows since any path may be covered by tiny balls which are convex. We will show that $x$ and $y$ is connected by a segment of $d$-length $\leq \pi$.

Assume that $x$ and $y$ are connected by such a chain with $n$ being a minimum. We can assume further that $s_i$ are in general position, i.e., $s_i$ and $s_{i+1}$ do not extend each other as an imbedded geodesic for each $i = 1, \ldots, n - 1$, which may be achieved by perturbing the points $p_2, \ldots, p_n$, unless $n = 2$ and $s_1 \cup s_2$ form a segment of $d$-length $\pi$; in which case, there is nothing to prove since $s_1 \cup s_2$ is the segment we need. To show we can achieve this, we take a maximal sequence of segments which extend each other as geodesics. Suppose that $s_i, s_{i+1}, \ldots, s_j$ form such a sequence for $j > i$. Then the total length of the segment will be less than $\pi|j - i|$. We divide the sequence into new segments of equal $d$-length $s_i', s_{i+1}', \ldots, s_j'$ where $s_k'$ has new endpoints $p_k'$, $p_{k+1}'$ for $i \leq k \leq j$ where $p_i' = p_i$ and $p_{j+1}' = p_{j+1}$. Then we may change $p_k'$ for $k = i + 1, \ldots, j$ toward one-side of the segments by a small amount generically. Then we see that new segments $s_i'', s_{i+1}'', \ldots, s_j''$ are in general position together with $s_{i-1}$ and $s_{j+1}$. This would work unless $j - i = 2$ and the total $d$-length equals $\pi$ since changing $p_{i+1}'$ still preserves $s_i'' \cup s_{i+1}''$ to be a segment of $d$-length $\pi$. However, since $n \geq 3$, we may move $p_i'$ or $p_{j+1}'$ in some direction to put the segments into the general position.

Let us choose a chain $s_i$, $i = 1, \ldots, n$, with minimal number of segments in general position. We assume that we are not in case when $n = 2$ and $s_1 \cup s_2$ forming a segment of $d$-length $\pi$.

We show that the number of the segments equals one, which shows that $\tilde{M}$ is convex. If the number of the segments is not one, then we take $s_1$ and $s_2$ and parameterize each of them by projective maps $f_i : [0, 1] \rightarrow s_i$, $i = 1, 2$, so that $f_i(0) = p_2$. Then since $p_2$ is
a point of its tiny ball $B(p_2)$, it follows that there exists a nondegenerate real projective map $f_t : \triangle \to B(p_2)$ where $\triangle$ is a triangle in $\mathbb{R}^2$ with vertices $(0,0), (t,0)$, and $(0,t)$ and $f_t(0,0) = p_2$, $f_t(s,0) = f_1(s)$ and $f_t(0,s) = f_2(s)$ for $0 \leq s \leq t$. $f_t : \triangle \to \tilde{M}$ is always an imbedding since $\text{dev} \circ f_t$ is a nondegenerate projective map $\triangle \subset S^n \to S^n$.

We consider the subset $A$ of $[0,1]$ so that $f_t : \triangle \to \tilde{M}$ is defined. Then $A$ is open in $[0,1]$ since as $f(\triangle)$ is compact, there exists a convex neighborhood of it in $\tilde{M}$ where $\text{dev}$ restricts to an imbedding. $A$ is closed by 1-convexity: we consider the union $K = \bigcup_{t \in A} f_t(A)$. Then the closure of $K$ in $\tilde{M}$ is a compact triangle in $\tilde{M}$ with two sides in $s_1$ and $s_2$. Thus, two sides of $K$ and $K^o$ are in $\tilde{M}$. By 1-convexity, $K$ itself is in $\tilde{M}$. Thus, $\sup A$ also belongs to $A$ and $A$ is closed. Hence $A$ must equal $[0,1]$ and there exists a segment $s'_1$ of $d$-length $< \pi$, namely $f_1((1,0)(0,1))$, connecting $p_1$ and $p_3$. This contradicts the minimality, and $x$ and $y$ are connected by a segment of $d$-length $\leq \pi$.

(2)$\rightarrow$(3) We choose a point $x$ in $\tilde{M}$. For each point $y$ of $\tilde{M}$, there exists a segment of $d$-length $\leq \pi$ connecting $x$ and $y$. This implies that $\text{dev}(\tilde{M})$ is a convex subset of $S^n$. As $\tilde{M}$ is convex, the closure of $\tilde{M}$ in $\tilde{M}$ is convex as we explain in Section 3. Thus $\tilde{M}$ is a tame set, and $\text{dev} : \tilde{M} \to S^n$ is an imbedding onto a convex subset of $S^n$. Hence $\text{dev}|\tilde{M}$ is an imbedding onto a convex subset of $S^n$. Now, the equation $\text{dev} \circ \vartheta = h(\vartheta) \circ \text{dev}$ holds for each deck transformation $\vartheta$ of $\tilde{M}$. Therefore, it follows that $\text{dev}$ induces a real projective diffeomorphism $\tilde{M}/\pi_1(M) \to \text{dev}(\tilde{M})/h(\pi_1(M))$.

(3)$\rightarrow$(1) This part is straightforward using the classification of convex sets in Theorem 2.2 since $\tilde{M}$ can be identified with a convex domain in $S^n$. 

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Proposition B.1. Suppose we have a sequence of $\epsilon$-d-balls $B_i$ in a real projective sphere $S^n$ for some $n \geq 1$ and a fixed positive number $\epsilon$ and a sequence of projective maps $\varphi_i$. Assume the following:

- The sequence of d-diameters of $\varphi_i(B_i)$ goes to zero.
- $\varphi_i(B_i)$ converges to a point, say $p$.
- For a compact n-ball neighborhood $L$ of $p$, $\varphi_i^{-1}(L)$ converges to a compact set $L_\infty$.

Then $L_\infty$ is an $n$-hemisphere.

Recall that $R^{n+1}$ has a standard euclidean metric and $d$ on $S^n$ is obtained from it by considering $S^n$ as the standard unit sphere in $R^{n+1}$.

The Cartan decomposition of Lie groups states that a real reductive Lie group $G$ can be written as $KTK$ where $K$ is a compact Lie group and $T$ is a maximal real tori. Since $\text{Aut}(S^n)$ is isomorphic to $\text{SL}_\pm(n+1, R)$, we see that $\text{Aut}(S^n)$ can be written as $O(n+1)D(n+1)O(n+1)$ where $O(n+1)$ is the orthogonal group acting on $S^n$ as the group of isometries and $D(n+1)$ is the group of determinant 1 diagonal matrices with positive entries listed in nonincreasing order where $D(n+1)$ acts in $S^n$ as a subgroup of $\text{GL}(n+1, R)$ acting in the standard manner on $S^n$. In other words, each element $g$ of $\text{Aut}(S^n)$ can be written as $i(g)d(g)i'(g)$ where $i(g), i'(g)$ are isometries and $d(g) \in D(n+1)$ (see Carrière [7] and Choi [16], and also [15]).

We may write $\varphi_i$ as $K_{1,i} \circ D_i \circ K_{2,i}$ where $K_{1,i}$ and $K_{2,i}$ are d-isometries of $S^n$ and $D_i$ is a projective map in $\text{Aut}(S^n)$ represented by a diagonal matrix of determinant 1 with positive entries. More precisely, $D_i$ has $2n+2$ fixed points $[\pm e_0], \ldots, [\pm e_n]$, the equivalence classes of standard basis vectors $\pm e_0, \ldots, \pm e_n$ of $R^{n+1}$, and $D_i$ has a matrix diagonal with respect to this basis; the diagonal entries $\lambda_i$, $i = 0, 1, \ldots, n$, are positive and in nonincreasing order. Let $O_{[e_0]}$ denote the open hemisphere containing $[e_0]$ whose boundary is the great sphere $S^{n-1}$ containing $[\pm e_j]$ for all $j$, $j \geq 1$, and $O_{[-e_0]}$ that containing $[-e_0]$ with the same boundary set.

Recall that $\varphi_i = K_{1,i} \circ D_i \circ K_{2,i}$. Let us denote by $q_i = \varphi_i(p_i)$ for the d-center $p_i$ of the ball $B_i$. Since $\varphi_i(B_i)$ converges to $p$, and $K_{1,i}$ is an isometry, the sequence of the d-diameter of $D_i \circ K_{2,i}(B_i)$ goes to zero as $i \to \infty$. We may assume without loss of generality that $D_i(K_{2,i}(B_i))$ converges to a set consisting of a point by choosing a subsequence if necessary. By the following lemma [B.1], $D_i(K_{2,i}(B_i))$ converges to one of the attractors
We may assume without loss of generality that $D_i(K_{2,i}(B_i))$ converges to $[e_0]$. Since $L$ is an $n$-dimensional ball neighborhood of $p$, $L$ includes a $d$-ball $B_δ(p)$ in $S^n$ with center $p$ with radius $δ$ for some positive constant $δ$. There exists a positive integer $N$ so that for $i > N$, we have

$$\varphi_i(p_i) \subset B_δ/2(p)$$

for the $d$-ball $B_δ/2(p)$ of radius $δ/2$ in $S^n$. Hence $B_δ/2(q_i)$ is a subset of $L$ for $i > N$.

Since $K_{1,i}^{-1}(q_i) = D_i \circ K_{2,i}(p_i)$, the sequence $K_{1,i}^{-1}(q_i)$ converges to $[e_0]$ by the second paragraph above. There exists an integer $N_1 > N$ such that $K_{1,i}^{-1}(q_i)$ is of $d$-distance less than $δ/4$ from $[e_0]$ for $i > N_1$. Since $K_{1,i}^{-1}$ is a $d$-isometry, $K_{1,i}^{-1}(B_δ/4(q_i))$ includes the ball $B_δ/4([e_0])$ for $i > N_1$. Hence $K_{1,i}^{-1}(L)$ includes $B_δ/4([e_0])$ for $i > N_1$.

Since $[e_0]$ is an attractor under the action of the sequence $\{D_i\}$ by Lemma 3.1, the images of $B_δ/4([e_0])$ under $D_i^{-1}$ eventually include any compact subset of $O([e_0])$. Thus, $D_i^{-1}(B_δ/4([e_0]))$ converges to $\mathrm{Cl}(O_1)$ geometrically, and up to a choice of subsequence $K_{2,i}^{-1} \circ D_i^{-1}(B_δ/4([e_0]))$ converges to an $n$-hemisphere. The equation

$$\varphi_i^{-1}(L) = K_{2,i}^{-1} \circ D_i^{-1} \circ K_{1,i}^{-1}(L)$$

$$\supset K_{2,i}^{-1} \circ D_i^{-1}(B_δ/4([e_0])).$$  \hspace{1cm} (8)

shows that $\varphi_i^{-1}(L)$ converges to an $n$-hemisphere. \hfill \Box

The straightforward proof of the following lemma is left to the reader.

**Lemma B.1.** Let $K_i$ be a sequence of $\varepsilon$-$d$-balls in $S^n$ and $d_i$ a sequence of automorphisms of $S^n$ that are represented by diagonal matrices of determinant 1 with positive entries for the standard basis with the first entry $λ_i$ the maximum. Suppose $d_i(K_i)$ converges to the set consisting of a point $y$. Then there exists an integer $N$ so that for $i > N$, the following statements hold:

1. $[e_0]$ and $[-e_0]$ are attracting fixed points of $d_i$.
2. $y$ equals $[e_0]$ or $[-e_0]$.
3. The eigenvalue $λ_i$ of $d_i$ corresponding to $e_0$ and $-e_0$ is strictly larger than the eigenvalues corresponding to $±e_j$, $j = 1, \ldots, n$.
4. $λ_i/λ_i' \rightarrow +\infty$ for the maximum eigenvalue $λ_i'$ of $d_i$ corresponding to $±e_j$, $j = 1 \ldots n$. 

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