Quasilinear elliptic problem without Ambrosetti–Rabinowitz condition involving a potential in Musielak–Sobolev spaces setting

Soufiane Maatouk and Abderrahmane El Hachimi

Center of Mathematical Research and Applications of Rabat (CeReMAR), Laboratory of Mathematical Analysis and Applications (LAMA), Department of Mathematics, Faculty of Sciences, Mohammed V University, Rabat, Morocco

ABSTRACT

In this paper, we consider the quasilinear elliptic problem with potential

\[
(P) \begin{cases}
-\text{div}(\phi(x, |\nabla u|) \nabla u) + V(x)|u|^{q(x)-2}u = f(x, u) & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]

where \( \Omega \) is a smooth bounded domain in \( \mathbb{R}^N (N \geq 2) \), \( V \) is a given function in a generalized Lebesgue space \( L^{s(x)}(\Omega) \), and \( f(x, u) \) is a Carathéodory function satisfying suitable growth conditions. Using variational arguments, we study the existence of weak solutions for (P) in the framework of Musielak–Sobolev spaces. The main difficulty here is that the nonlinearity \( f(x, u) \) considered does not satisfy the well-known Ambrosetti–Rabinowitz condition.

1. Introduction

Let \( \Omega \subset \mathbb{R}^N (N \geq 2) \) be a bounded smooth domain. Assume that \( \phi : \Omega \times [0, +\infty) \rightarrow [0, +\infty) \) is a Carathéodory function such that for all \( x \in \Omega \), we have

\[
(\phi) \begin{cases}
\phi(x, 0) = 0, & \phi(x, t) \text{ is strictly increasing}, \\
\phi(x, t).t > 0, & \forall t > 0 \text{ and } \phi(x, t).t \rightarrow +\infty \text{ as } t \rightarrow +\infty.
\end{cases}
\]

In this paper, we study the quasilinear elliptic problem

\[
(P) \begin{cases}
-\text{div}(\phi(x, |\nabla u|) \nabla u) + V(x)|u|^{q(x)-2}u = f(x, u) & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]

where \( V \) is a potential belonging to \( L^{s(x)}(\Omega) \), \( q \) and \( s : \widehat{\Omega} \rightarrow (1, \infty) \) are continuous functions and \( f : \Omega \times \mathbb{R} \rightarrow \mathbb{R} \) is a Carathéodory function which satisfies some suitable growth conditions. Precise conditions concerning the functions \( q, s, f \) and \( V \) will be given hereafter.

CONTACT Abderrahmane El Hachimi aelhachi@yahoo.fr

© 2020 Informa UK Limited, trading as Taylor & Francis Group
Problem \((P)\) appears in many branches of mathematical physics and has been studied extensively in recent years. From an application point of view, this problem has its backgrounds in such hot topics as image processing, nonlinear electrorheological fluids and elastic mechanics. We refer the readers to [1,2] and the references therein for more background of applications. In particular, when \(\phi(x,t) = t^{p(x) - 2}\), where \(p\) is a continuous function on \(\overline{\Omega}\) with the condition \(\min_{x \in \overline{\Omega}} p(x) > 1\), the operator involved in \((P)\) is the \(p(x)\)-Laplacian operator, i.e. \(\Delta p(x) u := \text{div}(|\nabla u|^{p(x) - 2} \nabla u)\). This differential operator is a natural generalization of the \(p\)-Laplacian operator \(\Delta p u := \text{div}(|\nabla u|^{p - 2} \nabla u)\) where \(p > 1\) is a real constant. Note that the \(p(x)\)-Laplacian operator possesses more complicated nonlinearities than the \(p\)-Laplacian operator (for example, it is nonhomogeneous), so more complicated analysis has to be carefully carried out.

The interest in analyzing this kind of problems is also motivated by some recent advances in the study of problems involving nonhomogeneous operators in divergence form. We refer for instance to the results in [3–12]. The studies for \(p(x)\)-Laplacian problems have been extensively considered by many researchers in various ways (see e.g. [3,11,14,15]). It should be noted that our problem \((P)\) enables the presence of many other operators such as double-phase and variable exponent double-phase operators.

Before moving forward, we give a review of some results related to our work. We start by the case where the potential \(V \equiv 0\) on \(\Omega\). Fan and Zhang [14], proved the existence of a nontrivial solution and obtained infinitely many solutions for a Dirichlet problem involving the \(p(x)\)-Laplacian operator. In [16], Boreanu, Pucci and Radulescu studied the question of multiplicity of solutions for a class of anisotropic elliptic equations with variable exponent. Always, in the anisotropic setting, we also refer the reader to [17,18]. Clément et al. [19], established the existence of a nontrivial solution for more general quasilinear equation in the framework of Orlicz-Sobolev spaces, in the case where the function \(\phi\) considered in \((P)\) is independent of \(x\), i.e. \(\phi(x,t) = \phi(t)\). In the Orlicz-Sobolev space setting, Bonanno et al. [5–7], proved interesting results concerning the existence of infinitely many weak solutions for a nonhomogeneous eigenvalue Dirichlet problem (see also [8]). Liu and Zhao [20], obtained the existence of a nontrivial solution and infinitely many solutions for a quasilinear equation related to problem \((P)\) in the framework of Musielak–Sobolev spaces (see also [9]).

In the above mentioned papers, the authors assumed, among other conditions, that the nonlinearity \(f\) satisfy the well-known Ambrosetti–Rabinowitz condition ((A–R) condition for short); which, for the \(p\)-Laplacian operator, asserts that there exist two constants \(M > 0\) and \(\theta > p\), such that

\[
0 < \theta F(x,t) \leq f(x,t)t, \quad \forall \; |t| \geq M,
\]

where \(F(x,t) = \int_0^t f(x,s) \, ds\). Clearly, this condition implies the existence of two positive constants \(c_1, c_2\) such that

\[
F(x,t) \geq c_1 |t|^\theta - c_2, \quad \forall \; (x,t) \in \Omega \times \mathbb{R}.
\]  \(\text{(1)}\)

This means that \(f\) is \(p\)-superlinear at infinity in the sense that

\[
\lim_{|t| \to +\infty} \frac{F(x,t)}{|t|^p} = +\infty.
\]  \(\text{(2)}\)
This type of condition was introduced by Ambrosetti and Rabinowitz in their famous paper [21] and has since become one of the main tools for finding solutions to elliptic problems of variational type; especially in order to prove the boundedness of Palais–Smale sequence of the energy functional associated with such a problem. Unfortunately, there are several nonlinearities which are \( p \)-superlinear but do not satisfy the (A–R) condition. For instance, if we take \( f(x,t) = |t|^{p-2}t \ln(1 + |t|) \), then we can check that for any \( \theta > p \), \( F(x,t)/|t|^{\theta} \to 0 \) as \( |t| \to +\infty \). However, many recent types of research have been made to drop the (A–R) condition (see e.g. [4,10,11,22] and references therein).

In [4], the authors studied a similar problem as that in [19] and proved the existence of at least a nontrivial solution under the assumptions on the nonlinearity \( f \): there exist an \( N \)-function \( \Gamma \) (cf. [23]) and positive constants \( C, R \) such that

\[
\Gamma \left( \frac{F(x,t)}{|t|^{\phi_0}} \right) \leq CF(x,t), \quad \forall (x, |t|) \in \Omega \times [R, +\infty), \tag{3}
\]

and

\[
\lim_{|t| \to +\infty} \frac{f(x,t)}{|t|^{\phi_0}} = +\infty, \quad \lim_{|t| \to 0} \frac{f(x,t)}{|t|^{\phi_0}} = \lambda, \tag{4}
\]

where \( \bar{F}(x,t) := f(x,t)t - \phi_0^0 F(x,t) \), \( \lambda \), some nonnegative constant and \( \phi_0, \phi_0^0 \) are defined in relation (9) below (when \( \phi(x,t) = \phi(t) \) independent of \( x \)) with specific assumptions. It should be noted that the condition (3) is a type of ”nonquadraticity condition at infinity”, which was first introduced by Costa and Magalhães [24] for the Laplacian operator (with \( \phi_0 = \phi_0^0 = 2 \)) as follows.

\[
\liminf_{|t| \to +\infty} \frac{\bar{F}(x,t)}{|t|^{\sigma}} \geq a > 0,
\]

holds for some \( \sigma > 0 \). We would also like to mention that this condition plays an important role in proving the boundedness of Palais–Smale sequences.

In [22] also, the authors considered a similar problem as that in [19] and proved the existence of a nontrivial solution under the assumptions on the nonlinearity \( f \): there exist \( \mu_1, \mu_2 > 0 \) such that

\[
\lim_{|t| \to +\infty} \frac{F(x,t)}{|t|^{\phi_0^0}} = +\infty, \quad \lim_{|t| \to 0} \frac{f(x,t)}{|t|^{\phi_0^0 - 1}} = 0, \tag{5}
\]

\[
\bar{F}(x,t) \leq \bar{F}(x,s) + \mu_1, \quad \forall (x, s) \in \Omega \times (0,s) \quad \text{or} \quad \forall (x,t) \in \Omega \times (s,0), \tag{6}
\]

and

\[
\bar{H}(ts) \leq \bar{H}(t) + \mu_2, \quad \forall t \geq 0 \quad \text{and} \quad s \in [0,1], \tag{7}
\]

where \( \bar{H}(t) := \phi_0^0 \Phi(t) - \phi(t) t^2 \) with \( \Phi(t) = \int_0^t \phi(s) s \, ds \).

On the other hand, in the few last studies, studies on double phase problems have attracted more and more interest and many results have been obtained. Especially, in [10] the authors proved the existence of a nontrivial solution and obtained infinitely many solutions for a double phase problem without (A–R) condition. More precisely, they considered the problem \( (P) \) (with \( V \equiv 0 \)) with the function \( \phi(x,t) = t^{p-2} + a(x) t^{q-2} \), where \( a: \Omega \mapsto [0, +\infty) \) is Lipschitz continuous, \( 1 < p < q < N, q/p < 1 + 1/N \) and the nonlinearity \( f \) satisfies the assumptions (5) and (6) above with \( \phi_0 = p \) and \( \phi_0^0 = q \). In [25] however, the
authors considered the same previous problem and proved the existence of infinitely many
solutions; but instead of hypotheses (5) and (6) the nonlinearity $f$ is supposed to satisfy the
assumption (3) above where $\Gamma(t) = |t|^\sigma$ with $\sigma > \max\{1, N/p\}$, and $F(x, t) \geq 0$ for any
$(x, |t|) \in \Omega \times [R, +\infty)$ is such that $\lim_{|t| \to +\infty} F(x, t)/|t|^\sigma = +\infty$. In the same paper, the
authors obtained also similar existence result under the assumption instead of (3): there
exist $\mu > q$ and $\theta > 0$ such that

$$\mu F(x, t) \leq tf(x, t) + \theta |t|^p, \quad \forall (x, t) \in \Omega \times \mathbb{R}.$$ 

Now, we give some review results concerning the case where the potential $V \not\equiv 0$ on $\Omega$. In
[3], Abdou and Marcos, proved the existence of multiple solutions for a Dirichlet problem
involving the $p(x)$-Laplacian operator with a changing sign potential $V$ belonging to a
generalized Lebesgue space $L^{q(x)}(\Omega)$ when the nonlinearity $f$ satisfies some growth
condition under (A-R) condition. In that work, the main assumptions on the variable
exponents $q(\cdot), s(\cdot)$ and $p(\cdot)$ are such that: $q, s, p \in C_{+}(\bar{\Omega})$ (see notation below) and satisfy
$1 < q(x) < p(x) \leq N < s(x)$ for any $x \in \bar{\Omega}$.

Recently, in [12] the authors proved the existence of nontrivial nonnegative and non-positive
solutions, and obtained infinitely many solutions for the quasilinear equation
$-\text{div}A(x, \nabla u) + V(x)|u|^{q(x)-2}u = f(x, u)$ in $\mathbb{R}^N$, where the divergence type operator has
behaviors like $|\xi|^{q(x)-2}$ for small $|\xi|$ and like $|\xi|^{p(x)-2}$ for large $|\xi|$, where $1 < a(\cdot) \leq p(\cdot) < q(\cdot) < N$. In that paper, it is supposed that the potential $V \in L^1_{\text{loc}}(\mathbb{R}^N)$ verifies $V(\cdot) \geq V_0 > 0$, $V(x) \to +\infty$ as $|x| \to +\infty$ and that the nonlinearity $f$ satisfies some
growth condition with the assumption instead of (A-R) condition: there exist constants
$M, C_1, C_2 > 0$ and a function $a$ such that

$$C_1 |t|^{q(x)}[\ln(e + |t|)]^{a(x)-1} \leq C_2 \frac{f(x, t)t}{\ln(e + |t|)} \leq f(x, t)t - s(x)F(x, t), \quad \forall (x, |t|) \in \mathbb{R}^N \times [M, +\infty), \quad (8)$$

where $\text{ess}_{x \in \mathbb{R}^N} \inf(a(x) - q(x)) > 0$, $q(\cdot) \leq s(\cdot)$ and $\text{ess}_{x \in \mathbb{R}^N} \inf(p^*(x) - s(x)) > 0$ with
$p^*(x) := Np(x)/(N - p(x))$. Related to this subject, we refer the readers to some important
results concerning the study of the eigenvalue problems (see [5–8,15, 26–28] and the
references therein).

A main motivation of our current study is that, to the best of our knowledge, there is little
research considering both the potential $V \not\equiv 0$ and nonlinearity $f$ without (A-R) condition
for more general quasilinear equation in the framework of Musielak–Sobolev spaces. In
this paper, our main goal is to show the existence of weak solutions to the problem $(P)$.
Firstly, by using standard lower semicontinuity argument, we prove the existence of weak
solutions under the condition that $V \in L^1(\Omega)$ has changing sign, and the nonlinearity $f$
satisfies the condition $(f_0)$ below. Secondly, we establish the existence of at least a nontrivial
solution and the existence of infinitely many solutions by using Mountain Pass Theorem
and Fountain Theorem respectively, where $V \in L^1(\Omega)$ has constant sign and the nonlinearity $f$
does not satisfy the (A-R) condition. For these purposes, we propose a set of growth
conditions under which we are able to check the Palais–Smale condition. More precisely,
we prove the boundedness of Palais–Smale sequences by using a similar condition to that
in (3) above instead of (A-R) condition.
The paper is organized as follows: In Section 2, we recall some definitions and basic properties about Musielak–Orlicz–Sobolev spaces and variable exponent Lebesgue-Sobolev spaces. In Section 3, we state our main results and in Section 4 we give the proofs. Finally, in Section 5, we give an application of our main results.

2. Preliminary results

In the study of nonlinear partial differential equations, it is well known that more general functional space can handle differential equations with more nonlinearities. For example, the $p$-Laplacian equations correspond to the classical Sobolev space setting, the $p(x)$-Laplacian equations correspond to the variable exponent Sobolev space setting, etc. Concerning the problem $(P)$, Musielak–Sobolev spaces are the adequate functional spaces corresponding to the solutions. We shall therefore start by recalling some basic facts about these spaces. For more details we refer the readers to the papers [20,29–31].

Define

$$\Phi_1(x,t) = \int_0^t \phi(x,s)ds, \quad \forall t \geq 0.$$  

Since the function $\phi$ satisfies the condition $(\phi)$, then $\Phi$ is a generalized $N$-function; that is, for each $t \in [0, +\infty)$, $\Phi(\cdot, t)$ is measurable and for a.e. $x \in \Omega$, $\Phi(x, \cdot)$ is continuous, even, convex, with $\Phi(x, 0) = 0$, $\Phi(x, t) > 0$, for $t > 0$ and satisfies the conditions

$$\lim_{t \to 0^+} \frac{\Phi(x, t)}{t} = 0 \quad \text{and} \quad \lim_{t \to +\infty} \frac{\Phi(x, t)}{t} = +\infty.$$  

Denote $N(\Omega)$ the set of generalized $N$-functions. For $\Phi \in N(\Omega)$, the Musielak–Orlicz space $L^\Phi(\Omega)$ is defined by

$$L^\Phi(\Omega) := \left\{ u : u : \Omega \to \mathbb{R} \text{ is measurable, and } \exists \lambda > 0 \text{ such that } \int_\Omega \Phi\left(x, \frac{|u(x)|}{\lambda}\right)dx < \infty \right\},$$  

endowed with the Luxemburg norm

$$\|u\|_{L^\Phi(\Omega)} = \|u\|_\Phi := \inf \left\{ \lambda > 0 : \int_\Omega \Phi\left(x, \frac{|u(x)|}{\lambda}\right)dx \leq 1 \right\}.$$  

Next, define the Musielak–Sobolev space $W^{1, \Phi}(\Omega)$ by

$$W^{1, \Phi}(\Omega) := \left\{ u \in L^\Phi(\Omega) : |\nabla u| \in L^\Phi(\Omega) \right\},$$  

endowed with the norm

$$\|u\|_{W^{1, \Phi}(\Omega)} = \|u\|_{1, \Phi} := \|u\|_\Phi + \|\nabla u\|_\Phi,$$

where $\|\nabla u\|_\Phi = \|\nabla u\|_\Phi$.

Remark 2.1: In the particular case where $\Phi(x, t) = \Phi(t)$ is independent of $x$, $W^{1, \Phi}(\Omega)$ is actually an Orlicz-Sobolev space; while in the case where $\Phi(x, t) = |t|^{p(x)}$, this space becomes the variable exponent Sobolev space $W^{1, p(x)}(\Omega)$.
The function $\tilde{\Phi} : \Omega \times [0, +\infty) \to [0, +\infty)$ defined by
$$\tilde{\Phi}(x, t) = \sup_{s>0}(ts - \Phi(x, s)), \quad \text{for } x \in \Omega \quad \text{and} \quad t \geq 0,$$
is called the complementary function to $\Phi$ in the sense of Young. Observe that the function $\tilde{\Phi}$ belongs to $N(\Omega)$, and that $\Phi$ is also the complementary function to $\tilde{\Phi}$. Furthermore, $\Phi$ and $\tilde{\Phi}$ satisfy the Young inequality
$$st \leq \Phi(x, t) + \tilde{\Phi}(x, s), \quad \text{for } x \in \Omega \quad \text{and} \quad s, t \geq 0.$$

Throughout this paper, we assume that there exist two positive constants $\phi_0$ and $\phi^0$ such that
$$1 < \phi_0 \leq \frac{\phi(x, t)^2}{\Phi(x, t)} \leq \phi^0 < N, \quad \text{for } x \in \Omega \quad \text{and} \quad t > 0. \quad (9)$$

This relation gives the following result (see [32, Proposition 2.1]).

**Lemma 2.1:** Let $u \in L^\Phi(\Omega)$ and $\rho, t \geq 0$. Then, we have
$$\min \left\{ \rho^{\phi_0}, \rho^{\phi^0} \right\} \Phi(x, t) \leq \Phi(x, \rho t) \leq \max \left\{ \rho^{\phi_0}, \rho^{\phi^0} \right\} \Phi(x, t), \quad (10)$$
$$\min \left\{ \|u\|^{\phi_0}_\Phi, \|u\|^{\phi^0}_\Phi \right\} \leq \int_\Omega \Phi(x, |u(x)|) \, dx \leq \max \left\{ \|u\|^{\phi_0}_\Phi, \|u\|^{\phi^0}_\Phi \right\}. \quad (11)$$

Using the previous lemma we can easily show the following result.

**Proposition 2.1:** The function $\Phi$ satisfies the $(\Delta_2)$-condition, that is, there exist a positive constant $C > 0$ such that
$$\Phi(x, 2t) \leq C\Phi(x, t), \quad \text{for } x \in \Omega \quad \text{and} \quad t \geq 0.$$

Concerning the complementary function $\tilde{\Phi}$, we have the following analog lemma (see [33]).

**Lemma 2.2:** Let $u \in L^{\tilde{\Phi}}(\Omega)$ and $\rho, t \geq 0$. Then, we have
$$\min \left\{ \rho^{(\phi_0)\prime}, \rho^{(\phi^0)\prime} \right\} \tilde{\Phi}(x, t) \leq \tilde{\Phi}(x, \rho t) \leq \max \left\{ \rho^{(\phi_0)\prime}, \rho^{(\phi^0)\prime} \right\} \tilde{\Phi}(x, t), \quad (12)$$
$$\min \left\{ \|u\|^{(\phi_0)\prime}_\tilde{\Phi}, \|u\|^{(\phi^0)\prime}_\tilde{\Phi} \right\} \leq \int_\Omega \tilde{\Phi}(x, |u(x)|) \, dx \leq \max \left\{ \|u\|^{(\phi_0)\prime}_\tilde{\Phi}, \|u\|^{(\phi^0)\prime}_\tilde{\Phi} \right\}, \quad (13)$$
where $(\phi_0)\prime = \phi_0/(\phi_0 - 1)$ and $(\phi^0)\prime = \phi^0/(\phi^0 - 1)$.

**Remark 2.2:** From Lemma 2.2, the complementary function $\tilde{\Phi}$ also satisfies $(\Delta_2)$-condition.

Since both $\Phi$ and $\tilde{\Phi}$ satisfy the $(\Delta_2)$-condition, then we have the following result.
**Proposition 2.2 ([20]):** The following assertions hold true.

(1) \( L^\Phi(\Omega) = \{ u : u : \Omega \to \mathbb{R} \text{ is measurable, and } \int_\Omega \Phi(x, |u(x)|) \, dx < \infty \} \).

(2) For any sequence \( (u_n) \) in \( L^\Phi(\Omega) \), we have
   
   \( \int_\Omega \Phi(x, |u_n(x)|) \, dx \to 0 \text{ (resp. } 1; +\infty) \iff \| u_n \|_\Phi \to 0 \text{ (resp. } 1; +\infty) \),

(3) Let \( u \in L^\Phi(\Omega) \) and \( v \in L^\Phi(\Omega) \). Then, the Hölder type inequality holds true.

\[
\left| \int_\Omega u(x)v(x) \, dx \right| \leq 2\| u \|_\Phi \| v \|_{\tilde{\Phi}}.
\]

(4) \( \phi(x, |u(x)|)u(x) \in L^\Phi(\Omega) \) provided that \( u \in L^\Phi(\Omega) \).

Let \( \Phi, \Psi \in N(\Omega) \). We say that \( \Phi \) is weaker than \( \Psi \), and denote \( \Phi \preceq \Psi \), if there exist positive constants \( K_1, K_2 \) and a nonnegative function \( h \in L^1(\Omega) \) such that

\[
\Phi(x, t) \leq K_1 \Psi(x, K_2 t) + h(x), \quad \text{for } x \in \Omega \quad \text{and} \quad t \geq 0.
\]

By Theorem 8.5 in [31], the embeddings

\[
L^\Psi(\Omega) \hookrightarrow L^\Phi(\Omega) \quad \text{and} \quad L^\Phi(\Omega) \hookrightarrow L^\Psi(\Omega)
\]

are continuous provided that \( \Phi, \Psi \) are such that \( \Phi \preceq \Psi \).

We say that \( \Phi \in N(\Omega) \) is locally integrable if \( \Phi(\cdot, t_0) \in L^1(\Omega) \), for every \( t_0 > 0 \). Note that when \( \Phi \) is locally integrable, then \( (L^\Phi(\Omega), \| \cdot \|_\Phi) \) is a separable Banach space (see [30, 31]).

In this paper, we shall need the assumptions:

\( \phi_1 \) \( \inf_{x \in \Omega} \Phi(x, 1) = c_1 > 0 \).

\( \phi_2 \) For every \( t_0 > 0 \) there exists \( c = c(t_0) > 0 \) such that

\[
\frac{\Phi(x, t)}{t} \geq c, \quad \text{and} \quad \tilde{\Phi}(x, t) \geq c \quad \text{for } x \in \Omega \quad \text{and} \quad t \geq t_0.
\]

It is easy to see that \( \phi_2 \Rightarrow \phi_1 \). Moreover, in the case where \( \Phi \) is independent of \( x \), \( \phi_1 \) and \( \phi_2 \) hold automatically and \( \Phi \) is automatically locally integrable.

By assumption \( \phi_1 \), we have the embeddings

\[
L^\Phi(\Omega) \hookrightarrow L^1(\Omega) \quad \text{and} \quad W^{1,\Phi}(\Omega) \hookrightarrow W^{1,1}(\Omega).
\]

In addition, by assuming \( \Phi \) and \( \tilde{\Phi} \) both locally integrable and satisfying \( \phi_2 \), we conclude that \( L^\Phi(\Omega) \) is reflexive, and that the mapping \( J : L^\Phi(\Omega) \to (L^\Phi(\Omega))^* \) defined by

\[
\langle J(v), w \rangle = \int_\Omega v(x)w(x) \, dx, \quad \forall \, v \in L^\Phi(\Omega), \quad \forall \, w \in L^\Phi(\Omega),
\]

is a linear isomorphism and \( \| J(v) \|_{(L^\Phi(\Omega))^*} \leq 2\| v \|_{L^\Phi(\Omega)} \) (see [31, p.189]).
Denote $W^{1,\Phi}_0(\Omega)$ the closure of $\mathcal{C}_0^{\infty}(\Omega)$ in $W^{1,\Phi}(\Omega)$ and by $D^{1,\Phi}_0(\Omega)$ the completion of $\mathcal{C}_0^{\infty}(\Omega)$ in the norm $\|\nabla u\|_\Phi$. It is clear that $W^{1,\Phi}_0(\Omega) = D^{1,\Phi}_0(\Omega)$ in the case where $\|\nabla u\|_\Phi$ is an equivalent norm in $W^{1,\Phi}_0(\Omega)$.

By assuming that $\Phi$ is locally integrable and satisfies $(\phi_1)$, $W^{1,\Phi}(\Omega)$, $W^{1,\Phi}_0(\Omega)$ and $D^{1,\Phi}_0(\Omega)$ are clearly separable Banach spaces, and we have

$$W^{1,\Phi}_0(\Omega) \hookrightarrow W^{1,\Phi}(\Omega) \hookrightarrow W^{1,1}(\Omega),$$

$$D^{1,\Phi}_0(\Omega) \hookrightarrow D^{1,1}_0(\Omega) = W^{1,1}_0(\Omega).$$

In addition, these spaces are reflexive if $L^\Phi(\Omega)$ is reflexive.

In this work, we need to use some standard tools such as the Poincaré inequality and some compactness results for embeddings in Musielak–Sobolev spaces. For this reason, we shall suppose the supplementary assumptions on $\Phi$:



$(H_1)$ $\Omega \subset \mathbb{R}^N (N \geq 2)$ is a bounded domain with the cone property and $\Phi \in N(\Omega)$.

$(H_2)$ $\Phi : \overline{\Omega} \times [0, +\infty) \to [0, +\infty)$ is continuous and $\Phi(x, t) \in (0, +\infty)$ for $x \in \overline{\Omega}$ and $t \in (0, +\infty)$.

Now, let $\Phi$ satisfies $(H_1)$ and $(H_2)$. Then, for each $x \in \overline{\Omega}$, the function $\Phi(x, \cdot) : [0, +\infty) \to [0, +\infty)$ is a strictly increasing homeomorphism. Denote $\Phi^{-1}(x, \cdot)$ the inverse function of $\Phi(x, \cdot)$. We also assume the condition

$(H_3)$

$$\int_0^1 \frac{\Phi^{-1}(x, t)}{t^{(N+1)/N}} \, dt < +\infty, \quad \forall x \in \overline{\Omega}.$$ 

Define the function $\Phi_*^{-1} : \overline{\Omega} \times [0, +\infty) \to [0, +\infty)$ by

$$\Phi_*^{-1}(x, s) = \int_0^s \frac{\Phi^{-1}(x, \tau)}{\tau^{(N+1)/N}} \, d\tau, \quad \text{for } x \in \overline{\Omega} \text{ and } s \in [0, +\infty), \quad (14)$$

Then, by assumption $(H_3)$, $\Phi_*^{-1}$ is well defined, and for each $x \in \overline{\Omega}$, $\Phi_*^{-1}(x, \cdot)$ is strictly increasing, $\Phi_*^{-1}(x, \cdot) \in C^1((0, +\infty))$ and the function $\Phi_*^{-1}(x, \cdot)$ is concave.

Set $T(x) = \lim_{s \to +\infty} \Phi_*^{-1}(x, s)$, for all $x \in \overline{\Omega}$. Then, $T(x) \in (0, +\infty]$.

Define the function $\Phi_* : \overline{\Omega} \times [0, +\infty) \to [0, +\infty)$ by

$$\Phi_*(x, t) = \begin{cases} 
s, & \text{if } x \in \overline{\Omega}, \ t \in [0, T(x)) \text{ and } \Phi_*^{-1}(x, s) = t, \\
+\infty, & \text{if } x \in \overline{\Omega}, \ t \geq T(x). \end{cases}$$

Then, we have $\Phi_* \in N(\Omega)$, and for each $x \in \overline{\Omega}$, $\Phi_*(x, \cdot) \in C^1((0, T(x))).$ $\Phi_*$ is called the Sobolev conjugate function of $\Phi$.

Let $X$ be a metric space and $f : X \to (-\infty, +\infty]$ be an extended real-valued function. For $x \in X$ with $f(x) \in \mathbb{R}$, the continuity of $f$ at $x$ is well defined. Now, for $x \in X$ with $f(x) = +\infty$, we say that $f$ is continuous at $x$ if given any $M > 0$, there exists a neighborhood $U$ of $x$ such that $f(y) > M$ for all $y \in U$. We say that $f : X \to (-\infty, +\infty]$ is continuous on $X$ if
$f$ is continuous at every $x \in X$. Define $\text{Dom}(f) = \{x \in X : f(x) \in \mathbb{R}\}$ and denote $C^{1,0}(X)$ the set of all locally Lipschitz continuous real-valued functions defined on $X$.

**Remark 2.3:** Suppose that $\Phi$ satisfies $(H_2)$. Then, for each $t_0 \geq 0$, $\tilde{\Phi}(x, t_0)$ and $\Phi_*(x, t_0)$ are bounded.

Concerning the function $\Phi_*$ and the operator $T$, we suppose that

$(H_4) \quad T : \overline{\Omega} \to [0, +\infty]$ is continuous on $\overline{\Omega}$ and $T \in C^{1,0}(\text{Dom}(T))$.

$(H_5) \quad \Phi_* \in C^{1,0}(\text{Dom}(\Phi_*))$ and there exist positive constants $C_0, \delta_0 < 1/N$ and $t_0 \in (0, \min_{x \in \overline{\Omega}} T(x))$ such that

$$|\nabla_x \Phi_*(x, t)| \leq C_0 (\Phi_*(x, t))^{1+\delta_0},$$

for $x \in \Omega$ and $t \in [t_0, T(x))$ provided that $\nabla_x \Phi_*(x, t)$ exists.

**Remark 2.4:** Examples of generalized $N$-function $\Phi$ satisfying the above assumptions and covering the case of variable exponent space, double-phase space, and variable exponent double-phase space, are given in [13].

Let $\Phi, \Psi \in N(\Omega)$. We say that $\Phi$ essentially grows more slowly than $\Psi$ and we write $\Phi \ll \Psi$, if for any $k > 0$,

$$\lim_{t \to +\infty} \frac{\Phi(x, kt)}{\Psi(x, t)} = 0, \quad \text{uniformly for } x \in \Omega.$$

Obviously, if $\Phi \ll \Psi$ then $\Phi \preceq \Psi$.

Now, we recall the embedding theorems for Musielak–Sobolev spaces (see [20, 29]).

**Theorem 2.1:** Assume $(H_1)$–$(H_5)$ hold. Then, the assertions hold true:

1. There is a continuous embedding $W^{1,\Phi}(\Omega) \hookrightarrow L^{\Phi_*}(\Omega)$.
2. Suppose that $\Psi \in N(\Omega)$, $\Psi : \Omega \times [0, +\infty) \to [0, +\infty)$ is continuous, and $\Psi(x, t) \in (0, +\infty)$ for $x \in \Omega$ and $t \in (0, +\infty)$. If $\Psi \ll \Phi_*$. Then, the embedding $W^{1,\Phi}(\Omega) \hookrightarrow \hookrightarrow L^{\Psi}(\Omega)$ is compact.

In particular, as $\Phi \ll \Phi_*$ then we have

**Theorem 2.2:** Assume $(H_1)$–$(H_5)$ hold. Then, the assertions hold true:

1. The embedding $W^{1,\Phi}(\Omega) \hookrightarrow \hookrightarrow L^{\Phi}(\Omega)$ is compact.
2. The Poincaré type inequality

$$\|u\|_{\Phi} \leq C \|\nabla u\|_{\Phi} \quad \text{for } u \in W^{1,\Phi}_0(\Omega),$$

holds.

We finish the recall of Musielak–Sobolev spaces properties by stating a lemma (see [33]).
Lemma 2.3: Let \( u \in L^{\Phi^*}(\Omega) \) and \( \rho, t \geq 0 \). Then, we have

\[
\min \left\{ \rho^{\phi_0^*}, \rho^{(\phi^0)^*} \right\} \Phi_*(x, t) \leq \Phi_*(x, \rho t) \leq \max \left\{ \rho^{\phi_0^*}, \rho^{(\phi^0)^*} \right\} \Phi_*(x, t),
\]

where \( (\phi_0)^* = N\phi_0/(N - \phi_0) \) and \( (\phi^0)^* = N\phi^0/(N - \phi^0) \).

Now, we give some background facts concerning the variable exponent Lebesgue spaces. For more details on the basic properties of these spaces, we refer the reader to the papers [34, 35].

Set

\[
\mathcal{C}_+(\overline{\Omega}) = \{ h \in C(\overline{\Omega}) : h(x) > 1 \text{ for any } x \in \overline{\Omega} \}.
\]

For any \( h \in \mathcal{C}_+(\overline{\Omega}) \), denote

\[
h^- = \min_{x \in \overline{\Omega}} h(x), \quad h^+ = \max_{x \in \overline{\Omega}} h(x),
\]

and for any \( q(x) \in \mathcal{C}_+(\overline{\Omega}) \), define the variable exponent Lebesgue space

\[
L^{q(x)}(\Omega) := \left\{ u : u : \Omega \to \mathbb{R} \text{ is measurable with } \int_{\Omega} |u(x)|^{q(x)} \, dx < \infty \right\},
\]

endowed with the norm

\[
\|u\|_{L^{q(x)}(\Omega)} = \|u\|_{q(x)} := \inf \left\{ \lambda > 0 : \int_{\Omega} \frac{|u(x)|^{q(x)}}{\lambda} \, dx \leq 1 \right\}.
\]

We recall that the variable exponent Lebesgue spaces are separable and reflexive Banach spaces. Let \( L^{q'(x)}(\Omega) \) denote the conjugate space of \( L^{q(x)}(\Omega) \) with \( 1/q(x) + 1/q'(x) = 1 \).

For any \( u \in L^{q(x)}(\Omega) \) and \( v \in L^{q'(x)}(\Omega) \), the Hölder type inequality

\[
\int_{\Omega} |uv| \, dx \leq \left( \frac{1}{q^-} + \frac{1}{q^+} \right) \|u\|_{q(x)} \|v\|_{q'(x)},
\]

holds true. Moreover, if \( 0 < |\Omega| < \infty \) and \( q_1, q_2 \) are variable exponents such that \( q_1(x) \leq q_2(x) \) almost everywhere in \( \Omega \), then the embedding \( L^{q_2(x)}(\Omega) \hookrightarrow L^{q_1(x)}(\Omega) \) is continuous. Furthermore, if we define the mapping \( \rho_q : L^{q(x)}(\Omega) \to \mathbb{R}^+ \) by

\[
\rho_q(u) = \int_{\Omega} |u|^{q(x)} \, dx,
\]

then the relations hold true:

\[
\min\{\|u\|_{q(x)}, \|u\|_{q(x)}^{q'}\} \leq \rho_q(u) \leq \max\{\|u\|_{q(x)}, \|u\|_{q(x)}^{q'}\}, \quad (18)
\]

\[
\|u\|_{q(x)} < 1 (= 1, > 1) \iff \rho_q(u) < 1 (= 1, > 1), \quad (19)
\]

\[
\|u_n - u\|_{q(x)} \to 0 \iff \rho_q(u_n - u) \to 0, \quad \forall \ (u_n), u \in L^{q(x)}(\Omega), \quad (20)
\]

\[
\|u_n\|_{q(x)} \to +\infty \iff \rho_q(u_n) \to +\infty, \quad \forall \ (u_n) \in L^{q(x)}(\Omega). \quad (21)
\]

We recall also a proposition, which will be used later.
Proposition 2.3 ([36]): Let \( p \) and \( q \) be measurable functions such that \( p \in L^\infty(\Omega) \) and \( 1 < p(x)q(x) \leq \infty \) for a.e. \( x \in \Omega \). Let \( u \in L^{s(x)}(\Omega), u \neq 0 \). Then, it yields that

\[
\|u\|_{p(x)q(x)} \leq 1 \Rightarrow \|u\|^p_{p(x)q(x)} \leq \|u|^p_{p(x)q(x)},
\]

\[
\|u\|_{p(x)q(x)} \geq 1 \Rightarrow \|u\|^p_{p(x)q(x)} \leq \|u|^p_{p(x)q(x)}.
\]

In particular when \( p(x) = p \) is a constant, then

\[
\|u\|^p_{q(x)} = \|u\|^p_{pq(x)}.
\]

3. Main results

In this section we state the main results of this paper. We will study the problem \((P)\) when \( q \in C_+(\Omega) \) and the potential \( V : \Omega \to \mathbb{R} \) is nontrivial and belongs to \( L^{s(x)}(\Omega) \) with \( s \in C(\Omega) \). Before dealing with our main results in this section, we introduce the assumptions for \( f(x, u) \).

\( (f_0) \) There exist \( \Psi \in N(\Omega) \) satisfying the assumption (2) of Theorem 2.1, and two positive constants \( \psi_0 \) and \( \psi^0 \) such that

\[
1 < \psi_0 \leq \frac{\phi(x, t)}{\psi(x, t)} \leq \psi^0, \quad \text{for } x \in \Omega \quad \text{and} \quad t > 0,
\]

\[
|f(x, t)| \leq C_1 \psi(x, |t|) + h(x), \quad \text{for } (x, t) \in \Omega \times \mathbb{R},
\]

where \( C_1 \) is a positive constant, \( 0 \leq h \in L^\Psi(\Omega) \), and \( \psi : \Omega \times \mathbb{R}^+ \to \mathbb{R}^+ \) is a continuous function and \( \psi(x, t) = \int_0^t \psi(x, s)ds \), for all \( x \in \Omega \).

\( (f_1) \) There exist \( \Gamma \in N(\Omega) \) satisfying the assumptions of \((H_2)\), and two positive constants \( \gamma_0 \) and \( \gamma^0 \) such that

\[
1 < \frac{\Gamma}{\phi_0} < \gamma_0 \leq \frac{\gamma(x, t)}{\Gamma(x, t)} \leq \gamma^0, \quad \text{for } x \in \Omega \quad \text{and} \quad t > 0,
\]

\[
\Gamma \left( x, \frac{F(x, t)}{|t|\phi_0} \right) \leq C_2 H(x, t), \quad \text{for } x \in \Omega \quad \text{and} \quad |t| \geq M,
\]

where \( C_2, M \) are positive constants, \( H(x, t) = f(x, t)t - vF(x, t) \), for all \((x, t) \in \Omega \times \mathbb{R} \) with \( v = \phi^0 \) if \( V \leq 0 \) a.e. on \( \Omega \) and \( v = q^+ \) if \( V \geq 0 \) a.e. on \( \Omega \), and \( \gamma : \Omega \times \mathbb{R}^+ \to \mathbb{R}^+ \) is a continuous function and \( \Gamma(x, t) = \int_0^t \gamma(x, s)ds \), for all \( x \in \Omega \).

\( (f_2) \) \( \lim_{|t| \to +\infty} \frac{F(x, t)}{|t|\phi_0} = +\infty \), uniformly for \( x \in \Omega \).

\( (f_3) \) \( f(x, t) = o(|t|\phi(x, t)) \) as \( t \to 0 \), uniformly for \( x \in \Omega \).

\( (f_4) \) \( f(x, -t) = -f(x, t) \) for all \((x, t) \in \Omega \times \mathbb{R} \).

To summarize all assumptions concerning the function \( \Phi \), in what follows we shall say that the function \( \Phi \) satisfies the assumption \((\Phi)\) if: \( \Phi \) satisfies the assumption \((\phi)\), \( \Phi \) satisfies \((9) \) and \((H_1)-(H_3)\), both \( \Phi \) and \( \tilde{\Phi} \) are locally integrable and satisfy \((\phi_2)\). Hence, under the assumption \((\Phi)\), the spaces \( L^\Phi(\Omega), W^{1, \Phi}(\Omega), W^{1, \Phi}_0(\Omega) \) are separable reflexive Banach
spaces. Therefore, we can apply the embedding theorems for Musielak–Sobolev spaces in Theorems 2.1 and 2.2.

**Definition 3.1:** A function $u \in W_0^{1,\Phi}(\Omega)$ is said to be a weak solution of problem (P) if

$$
\int_{\Omega} \phi(x,|\nabla u|)\nabla u \nabla v \, dx + \int_{\Omega} V(x)|u|^{q(x)-2}uv \, dx = \int_{\Omega} f(x,u)v \, dx, \quad \forall v \in W_0^{1,\Phi}(\Omega),
$$

holds. Our main results in this paper are given by the following theorems.

**Theorem 3.1:** Assume that the assumptions ($\Phi$) and $(f_0)$ hold. Furthermore, assume that $\max\{\psi^0,q^+\} < \phi_0$ and $s(x) > q(x)(\phi_0)^* / ((\phi_0)^* - q(x))$ for every $x \in \Omega$. Then, problem (P) has a weak solution.

In order to obtain the second main result, we assume that $f$ satisfies the condition $(f_0')$ instead of $(f_0)$:

$(f_0')$ We assume that (22) of $(f_0)$ holds and that

$$
|f(x,t)| \leq C_1(\psi(x,|t|) + 1), \quad \text{for } (x,t) \in \Omega \times \mathbb{R},
$$

where $C_1$ is a positive constant.

**Theorem 3.2:** Assume that the assumptions ($\Phi$) and $(f_0')$-(f_3) hold. Furthermore, assume that $\phi^0 < \min\{\psi_0,q^-\}$, $\max\{\psi^0,q^+\} < (\phi_0)^*$, $q^+ - \frac{1}{2}\phi_0 < q^-$ and $s(x) > q(x)(\phi_0)^* / ((\phi_0)^* - q(x))$ for every $x \in \Omega$. If $V$ has a constant sign a.e. on $\Omega$, then the problem (P) has a nontrivial weak solution.

**Theorem 3.3:** Assume that the assumptions of Theorem 3.2 hold. If the function $f$ satisfies $(f_4)$, then the problem (P) has a sequence of weak solutions $(\pm u_n)_{n \in \mathbb{N}} \subseteq W_0^{1,\Phi}(\Omega)$ such that $I(\pm u_n) \to +\infty$ as $n \to +\infty$.

In order to prove Theorem 3.3, we will use the following Fountain theorem (see [37] for details). Let $(X,\|\cdot\|)$ be a real reflexive Banach space such that $X = \overline{\bigoplus_{j \in \mathbb{N}^*} X_j}$ with $\dim(X_j) < +\infty$ for any $j \in \mathbb{N}^*$. For each $k \in \mathbb{N}^*$, we set $Y_k = \bigoplus_{j=1}^k X_j$ and $Z_k = \bigoplus_{j=k}^\infty X_j$.

**Proposition 3.1 (Fountain theorem):** Let $(X,\|\cdot\|)$ be a real reflexive Banach space and $I \in C^1(X,\mathbb{R})$ an even functional. If for each sufficiently large $k \in \mathbb{N}^*$, there exist $\rho_k > r_k > 0$ such that the conditions hold:

1. $\inf_{\|u\| = \rho_k} I(u) \to +\infty$ as $k \to +\infty$,
2. $\max_{\|u\| = \rho_k} I(u) \leq 0$,
3. $I$ satisfies the Palais–Smale condition for every $c > 0$,

then $I$ has a sequence of critical values tending to $+\infty$. 
4. Proofs of the main results

In this section we give the proofs of our main results. We note that in these results we always have \( s(x) > q(x)(\Phi_0)^*/((\Phi_0)^* - q(x)) \) for every \( x \in \overline{\Omega} \) and \( \max \{ \psi, q^+ \} < (\Phi_0)^* \).

Define the functional \( I : W_{0,1}^1(\Omega) \to \mathbb{R} \) by the formula
\[
I(u) = \mathcal{H}(u) + J(u) - F(u),
\]
where,
\[
\mathcal{H}(u) = \int_{\Omega} \Phi(x, |\nabla u|) \, dx, \quad J(u) = \int_{\Omega} \frac{V(x)}{q(x)} |u|^{q(x)} \, dx, \quad \text{and} \quad F(u) = \int_{\Omega} F(x, u) \, dx,
\]
with \( F(x, t) = \int_0^t f(x, s) \, ds \).

**Proposition 4.1:** The functional \( I \) is well defined and \( I \in C^1(W_{0,1}^1(\Omega), \mathbb{R}) \) with the derivative given by
\[
\langle I'(u), v \rangle = \int_{\Omega} \phi(x, |\nabla u|) \nabla u \nabla v \, dx + \int_{\Omega} V(x) |u|^{q(x)-2} uv \, dx
\]
\[
- \int_{\Omega} f(x, u) v \, dx, \quad \forall u, v \in W_{0,1}^1(\Omega).
\]

**Proof:** Firstly, it is clear that \( \mathcal{H} \) is well defined on \( W_{0,1}^1(\Omega) \). Furthermore, by similar arguments as used in the proof of [32, Lemma 4.2], we have \( \mathcal{H} \in C^1(W_{0,1}^1(\Omega), \mathbb{R}) \) and its derivative is given by
\[
\langle \mathcal{H}'(u), v \rangle = \int_{\Omega} \phi(x, |\nabla u|) \nabla u \nabla v \, dx, \quad \forall u, v \in W_{0,1}^1(\Omega).
\]

Secondly, the functional \( J \) is well defined. Indeed, since \( s(x) > q(x)(\Phi_0)^*/((\Phi_0)^* - q(x)) \) for every \( x \in \overline{\Omega} \), then it is clear that \( s \in C_+(\overline{\Omega}) \) and \( s(x) > q(x) \) for every \( x \in \overline{\Omega} \). Furthermore, by a simple computation we have
\[
1 < s'(x)q(x) < (\Phi_0)^* \quad \text{and} \quad 1 < \alpha(x) := \frac{s(x)q(x)}{s(x) - q(x)} < (\Phi_0)^*, \quad \forall x \in \overline{\Omega}. \tag{28}
\]

Thus,
\[
\max_{x \in \overline{\Omega}} \frac{s'(x)q(x)}{s(x) - q(x)} : = \frac{s'(x_0)q(x_0)}{s(x) - q(x)} < (\Phi_0)^* \quad \text{and} \quad \max_{x \in \overline{\Omega}} \alpha(x) := \alpha(x_0) < (\Phi_0)^*.
\]

Using Lemma 2.3 and \((H_5)\), we obtain
\[
\lim_{t \to +\infty} \frac{|kt|^{s'(x)q(x)}}{\Phi_*(x, t)} \leq \frac{k^{s'(x_0)q(x_0)}}{\Phi_*(x, 1)} \lim_{t \to +\infty} \frac{1}{t(\Phi_0)^* - s'(x_0)q(x_0)} = 0 \text{ uniformly for } x \in \Omega. \tag{29}
\]

Using the same arguments as above we have that
\[
\lim_{t \to +\infty} \frac{|kt|^{\alpha(x)}}{\Phi_*(x, t)} = 0, \text{ uniformly for } x \in \Omega. \tag{30}
\]
Hence, (29) and (30) imply that $|t|^q(x) u(x) \ll \Phi_*$ and $|t|^\alpha(x) u \ll \Phi_*$, respectively. Thus, from Theorem 2.1 we have the compact embeddings

$$W_0^{1,\Phi}(\Omega) \hookrightarrow \hookrightarrow L^{q(x)}(\Omega),$$

and

$$W_0^{1,\Phi}(\Omega) \hookrightarrow \hookrightarrow L^{\alpha(x)}(\Omega).$$

Now, by using the Hölder inequality, Proposition 2.3, and (31), we have for all $u$ in $W_0^{1,\Phi}(\Omega)$

$$|\mathcal{J}(u)| \leq c_0 \|V\|_{s(x)} \|u\|_{q(x)} \leq c_1 \|V\|_{s(x)} \max\{\|u\|_{s(x)}^q \|u\|_{s(x)}^{q+}\} \leq c_2 \|V\|_{s(x)} \max\{\|u\|_{1,\Phi}^{q-},\|u\|_{1,\Phi}^{q+}\}$$

where $c_i, i = 0, 1, 2$, are positive constants. Hence, $\mathcal{J}$ is well defined. Moreover, since $q^+ < (\phi_0)^*$ then, as in the proof of relation (29), the space $W_0^{1,\Phi}(\Omega)$ is compactly embedded in $L^{q(x)}(\Omega)$. Therefore, using (32) and following the same arguments as in the proof of [15, Proposition 2], we obtain $\mathcal{J} \in C^1(W_0^{1,\Phi}(\Omega), \mathbb{R})$ and

$$\langle \mathcal{J}'(u), v \rangle = \int_\Omega V(x)|u|^{q(x)-2}u v \, dx, \quad \forall \, u, v \in W_0^{1,\Phi}(\Omega).$$

Finally, from the properties of $\Psi$, $\Psi(x, k)$ is bounded for any positive constant $k$. Using Lemma 2.1 and the fact that $\psi^0 < (\phi_0)^*$ we obtain for any $k > 0$

$$\lim_{t \to +\infty} \frac{\Psi(x, k)}{\Phi_*(x, t)} \leq \frac{\Psi(x, k)}{\Phi_*(x, 1)} \lim_{t \to +\infty} \frac{1}{t(\phi_0)^* - \psi^0} = 0, \text{ uniformly for } x \in \Omega. \tag{34}$$

Hence, $\Psi \ll \Phi_*$, which implies by Theorem 2.1 that

$$W_0^{1,\Phi}(\Omega) \hookrightarrow \hookrightarrow L^\Psi(\Omega). \tag{35}$$

Consequently, from (23), the functional $\mathcal{F}$ is well defined and $\mathcal{F} \in C^1(W_0^{1,\Phi}(\Omega), \mathbb{R})$ with its derivative given by

$$\langle \mathcal{F}'(u), v \rangle = \int_\Omega f(x, u) v \, dx, \quad \forall \, u, v \in W_0^{1,\Phi}(\Omega).$$

The proof of this proposition is now complete. \hfill \blacksquare

**Proposition 4.2:**

(i) The mapping $\mathcal{H}': W_0^{1,\Phi}(\Omega) \to (W_0^{1,\Phi}(\Omega))^*$ defined by

$$\langle \mathcal{H}'(u), v \rangle = \int_\Omega \phi(x, |\nabla u|) \nabla u \nabla v \, dx, \quad \forall \, u, v \in W_0^{1,\Phi}(\Omega), \tag{36}$$

is a bounded, coercive, strictly monotone homeomorphism, and is of type $(S_+)$, namely,

$$u_n \to u \text{ in } W_0^{1,\Phi}(\Omega) \quad \text{and} \quad \limsup_{n \to \infty} \langle \mathcal{H}'(u_n), u_n - u \rangle \leq 0$$

imply that $u_n \to u$ in $W_0^{1,\Phi}(\Omega)$.

where $\to$ and $\to$ denote the weak and strong convergence in $W_0^{1,\Phi}(\Omega)$, respectively.
(ii) The functional $\mathcal{F}$ is sequentially weakly continuous, namely, $u_n \rightharpoonup u$ in $W_0^{1,\Phi}(\Omega)$ implies $\mathcal{F}(u_n) \to \mathcal{F}(u)$. In addition, the mapping $\mathcal{F}^* : W_0^{1,\Phi}(\Omega) \to (W_0^{1,\Phi}(\Omega))^*$ defined by

$$
\langle \mathcal{F}^*(u), v \rangle = \int_{\Omega} f(x, u)v \, dx, \quad \forall \, u, \, v \in W_0^{1,\Phi}(\Omega),
$$

is a completely continuous linear operator.

**Proof:** We refer the reader to [9, Theorem 2.2] for the proof of the first item and to [20, Lemma 4.1] for that of the second one. 

Note that, by Proposition 4.1 and Definition 3.1, $u$ is a weak solution of problem $(P)$ if and only if $u$ is a critical point of the functional $I$. Hence, we shall use critical point theory tools to show our main results.

To establish Theorem 3.1, we shall prove that the functional $I$ has a global minimum.

**Proof of Theorem 3.1:** Firstly, we show that $I$ is coercive, namely, $I(u) \to +\infty$ as $\|u\|_{1,\Phi} \to +\infty$. From (23), we have

$$|F(x, t)| \leq C_0 \Psi(x, t) + h(x)|t|, \quad \forall \, (x, t) \in \Omega \times \mathbb{R}.$$  

Then, by applying Lemma 2.1, Poincaré’s and Hölder’s inequalities, and using similar arguments as in the proof of relation (33), we obtain

$$I(u) = \int_{\Omega} \Phi(x, |\nabla u|) \, dx + \int_{\Omega} \frac{V(x)}{q(x)}|u|^{q(x)} \, dx - \int_{\Omega} F(x, u) \, dx \\geq\|u\|_{1,\Phi}^{\phi_0} - c_1 \|V\|_{s(x)} \|u\|_{1,\Phi}^{q(x) +} - c_2 \|\psi^0\|_{1,\Phi} - c_3 \|h\|_{1,\Phi} \|u\|_{1,\Phi}. $$

Using the fact that $W_0^{1,\Phi}(\Omega)$ is compactly embedded in $L^\Psi(\Omega)$ (see the proof of relation (35)), the previous inequality becomes

$$I(u) \geq\|u\|_{1,\Phi}^{\phi_0} - c_1 \|V\|_{s(x)} \|u\|_{1,\Phi}^{q(x) +} - c_2 \|\psi^0\|_{1,\Phi} - c_3 \|h\|_{1,\Phi} \|u\|_{1,\Phi}. $$

Since $1 < q^+ < \phi_0$ and $\psi^0 < \phi_0$, we then have $I(u) \to +\infty$ as $\|u\|_{1,\Phi} \to +\infty$. To complete the proof, we only need to show that the functional $I$ is weakly lower semi-continuous, namely, $u_n \rightharpoonup u$ in $W_0^{1,\Phi}(\Omega)$ implies $I(u) \leq \liminf_{n \to \infty} I(u_n)$. Suppose that $u_n \rightharpoonup u$ in $W_0^{1,\Phi}(\Omega)$. Since the functional $\mathcal{H} \in C^1(W_0^{1,\Phi}(\Omega), \mathbb{R})$ is strictly convex ($\mathcal{H}'$ being strictly monotone), then we have $\mathcal{H}(u_n) > \mathcal{H}(u) + \langle \mathcal{H}'(u), u_n - u \rangle$ which implies that $\mathcal{H}$ is weakly lower semi-continuous on $W_0^{1,\Phi}(\Omega)$. Concerning the functional $\mathcal{J}$; since $W_0^{1,\Phi}(\Omega)$ is compactly embedded in $L^{s(x)}(\Omega)$ (see the proof of relation (31)), then $u_n \to u$ in $L^{s(x)}(\Omega)$. This fact combined with relation (33) yields that $\mathcal{J}(u_n) \to \mathcal{J}(u)$. Finally, from Proposition 4.2-ii), $\mathcal{F}$ is sequentially weakly continuous. Then, we have $\mathcal{F}(u_n) \to \mathcal{F}(u)$. 

We shall now prove Theorem 3.2 by using the Mountain Pass Theorem (see [21]). Since the proof of this theorem is quite long, we will divide it into several lemmas. Firstly, we
show that the functional \( I \) has a geometrical structure. Secondly, we show that \( I \) satisfies the Palais–Smale condition at level \( \bar{c} \) (see the Definition 4.2). To this end, we show that any Palais–Smale sequence at the level \( \bar{c} \) for \( I \) (see the Definition 4.1) is bounded in \( W^{1,\Phi}_0(\Omega) \), and then has a strongly convergent subsequence.

Let us start with a lemma.

**Lemma 4.1:** Assume that \( \phi^0 < q^- \leq q^+ < (\phi_0)^* \), \( q^+ - \frac{1}{2}\phi_0 < q^- \) and \( s(x) > q(x)(\phi_0)^*/((\phi_0)^* - q(x)) \) for every \( x \in \overline{\Omega} \). Then, for any function \( V \in L^{s(x)}(\Omega) \) we have

\[
\int_{\Omega} |V(x)||u|^{q(x)} \, dx \leq C\|V\|_{s(x)}^{\alpha/r^-} \left[ M_1 + M_2 \left( \|u\|_{1,\Phi}^{2(q^+ - \theta)} + \|u\|_{1,\Phi}^{2(\theta r^+) - r^-} \right) \right],
\]

where \( \alpha = r^+ \) if \( \|V\|_{s(x)} > 1 \) and \( \alpha = r^- \) if \( \|V\|_{s(x)} \leq 1 \), and \( 2(\theta r^+) - r^- < 2(q^+ - \theta) < 2(q^+ - \theta) < \phi_0 \), for some measurable function \( r \) and positive constants \( C \) and \( \theta \).

**Proof:** Since we have \( q^+ - \frac{1}{2}\phi_0 < q^- \), then there exists \( \theta > 0 \) such that \( q^+ - \frac{1}{2}\phi_0 < \theta < q^- \). This fact implies that \( 2(q^- - \theta) < 2(q^+ - \theta) < \phi_0 \) and \( 1 + \theta - q^+ > 0 \). Let \( r \) be any measurable function satisfying

\[
\max \left\{ \frac{s(x)}{1 + \theta s(x)}, \frac{(\phi_0)^*}{\phi_0)^* + \theta - q(x)} \right\} < r(x)
\]

\[
< \min \left\{ \frac{s(x)(\phi_0)^*}{(\phi_0)^* + \theta s(x)}, \frac{1}{1 + \theta - q(x)} \right\},
\]

\[
\theta \left( \frac{r^+}{r^-} + 1 \right) < q^-,
\]

\( \forall \, x \in \Omega \).

It is clear that \( r \in L^\infty(\Omega) \) and \( 1 < r(x) < s(x) \). Now, by using Hölder’s inequality, we get

\[
\int_{\Omega} |V(x)||u|^{q(x)} \, dx \leq C\|V\|_{s(x)}^{\theta} \|r(x)\|_{s(x)} \|u|^{q(x)-\theta}\|_{(r(x))'}.
\]

Without loss of generality, we may assume that \( \|V(x)||u|^\theta\|_{r(x)} > 1 \). Using Hölder’s inequality, relation (18), and Proposition 2.3, we obtain

\[
\|V\|^\theta_{r(x)} \leq \left[ \int_{\Omega} |V(x)|^{r(x)}|u|^{\theta r(x)} \, dx \right]^{1/r^-}
\]

\[
\leq C_1 \|V\|^{r(x)}_{s(x)/r(x)} \|u|^{\theta r(x)}_{s(x)/s(x)} \|u|^{1/r^-}_{(r(x))'}
\]

\[
\leq C_2 \|V\|_{s(x)}^{\alpha/r^-} \left( 1 + \|u|^{\theta r^+}/r^-_{(r(x))} \right),
\]

where \( \alpha = r^+ \) if \( \|V\|_{s(x)} > 1 \) and \( \alpha = r^- \) if \( \|V\|_{s(x)} \leq 1 \).
Now, by applying the same arguments as above, we obtain
\[
|||u||^q(x)\|_{(r(x))'} \leq 1 + ||u||^{q+}\|_{(q(x)-\theta)(r(x))'}.
\] (43)

Since \( r(x) \) is chosen such that (39) is fulfilled then
\[
1 < \theta r(x) \left( \frac{s(x)}{r(x)} \right)' < (\phi_0)^* \quad \text{and} \quad 1 < (q(x) - \theta)(r(x))' < (\phi_0)^*, \quad \forall \ x \in \Omega.
\]

Since \( \Phi_* \) satisfies (H5), then by using Lemma 2.3, we have \(|t|^\theta(r(x)(s(x)/r(x))' \leq \Phi_*^\theta \text{ and } |t|^{q(x)-\theta}(r(x))' \leq \Phi_*^{q^+} \) imply that \( L^\Phi(\Omega) \) is continuously embedded in \( L^{\theta (r(x)(s(x)/r(x))'}(\Omega) \) and in \( L^{(q(x)-\theta)(r(x))'}(\Omega) \). Therefore, from Theorem 2.1, \( W^1_{0,\Phi}(\Omega) \) is continuously embedded in \( L^{\theta (r(x)(s(x)/r(x))'}(\Omega) \) and in \( L^{(q(x)-\theta)(r(x))'}(\Omega) \). Consequently, the relations (42) and (43) become
\[
\|V||u||^\theta \|_{r(x)} \leq C'\|V\|^\theta_{s(x)} \left(1 + \|u||^\theta r^+ / r^-\right)
\] (44)

and
\[
\|u||^q(x)-\theta \|_{(r(x))'} \leq C'' \left(1 + \|u||^{q^+ - \theta}\right),
\] (45)

respectively. Substituting (44) and (45) into (41), and using Young's inequality we obtain
\[
\int_\Omega |V(x)||u||^q(x) dx \leq C\|V\|^\theta_{s(x)} \left[M_1 + M_2 \left(\|u||^{2(q^+ - \theta)}_{1,\Phi} + \|u||^{2(\theta r^+ / r^-)}_{1,\Phi}\right)\right],
\] (46)

where \( C, M_1, \) and \( M_2 \) are positive constants.

**Lemma 4.2:** Assume that the assumptions (\( \Phi \), \( f_0' \), \( f_2 \) and \( f_3 \)) hold. Furthermore, assume that \( \phi_0 < \min\{\psi_0, q^-\}, \max\{\psi_0, q^+\} < (\phi_0)^*, q^+ - \frac{1}{2}\phi_0 < q^- \), and \( s(x) > q(x)(\phi_0)^*/((\phi_0)^* - q(x)) \) for every \( x \in \Omega \). Then, the functional \( \mathcal{I} \) has a geometrical structure; that is, \( \mathcal{I} \) satisfies the properties:

(i) There exist \( \rho > 0 \) and \( \beta > 0 \) such that \( \mathcal{I}(u) \geq \beta \) for any \( u \in W^1_{0,\Phi}(\Omega) \) with \( \|u||_{1,\Phi} = \rho \).

(ii) There exists \( u_0 \in W^1_{0,\Phi}(\Omega) \) such that \( \|u_0||_{1,\Phi} > \rho \) and \( \mathcal{I}(u_0) \leq 0 \).

**Proof:** (i) Firstly, from \( f_0' \) and \( f_3 \) it follows that, for all given \( \epsilon > 0 \) there exists \( C(\epsilon) > 0 \), such that
\[
|F(x, t)| \leq \epsilon \Phi(x, t) + C(\epsilon) \Psi(x, t), \quad \forall \ (x, t) \in \Omega \times \mathbb{R}.
\] (47)

Using Lemma 2.1, the Poincaré inequality, and the fact that \( W^1_{0,\Phi}(\Omega) \) is compactly embedded in \( L^\Psi(\Omega) \), we obtain
\[
\int_\Omega |F(x, t)| dx \leq \epsilon \max\{\|u||^{\phi_0}_{1,\Phi}, \|u||^{\phi_0}_{1,\Phi}\} + C' (\epsilon) \max\{\|u||^{\psi_0}_{1,\Phi}, \|u||^{\psi_0}_{1,\Phi}\}.
\] (48)

By the same arguments as in the proof of relation (33), we obtain
\[
\int_\Omega \frac{V(x)}{q(x)} |u||^q(x) \leq C\|V||^\theta_{s(x)} \max\{\|u||^{q^-}_{1,\Phi}, \|u||^{q^+}_{1,\Phi}\}.
\] (49)
Now, by using the definition of $I$ in (27), Lemma 2.1, and the relations (48) – (49), we get

\[ I(u) = \int_{\Omega} \Phi(x, |\nabla u|) \, dx + \int_{\Omega} \frac{V(x)}{q(x)} |u|^{q(x)} \, dx - \int_{\Omega} F(x, u) \, dx \]

\[ \geq \min \{ ||u||_{1,\Phi}^{\phi^0}, ||u||_{1,\Phi}^{\phi^0} \} - C ||V||_{s(x)} \max \{ ||u||_{1,\Phi}^{q^+}, ||u||_{1,\Phi}^{q^+} \} \]

\[ - \epsilon \max \{ ||u||_{1,\Phi}^{\phi^0}, ||u||_{1,\Phi}^{\phi^0} \} - C'(\epsilon) \max \{ ||u||_{1,\Phi}^{\psi^0}, ||u||_{1,\Phi}^{\psi^0} \}, \]

which implies, for all $u \in W_{0}^{1,\Phi}(\Omega)$ with $||u||_{1,\Phi} < 1$, that

\[ I(u) \geq ||u||_{1,\Phi}^{\phi^0} - C ||V||_{s(x)} ||u||_{1,\Phi}^{q^+} - \epsilon ||u||_{1,\Phi}^{\phi^0} - C'(\epsilon) ||u||_{1,\Phi}^{\psi^0} \]

\[ \geq \frac{1}{2} ||u||_{1,\Phi}^{\phi^0} - C ||V||_{s(x)} ||u||_{1,\Phi}^{q^+} - C'(\epsilon) ||u||_{1,\Phi}^{\psi^0} \]

\[ = ||u||_{1,\Phi}^{\phi^0} \left( \frac{1}{2} - C ||V||_{s(x)} ||u||_{1,\Phi}^{(q^+)-\phi^0} - C'(\epsilon) ||u||_{1,\Phi}^{(\psi^0)-\phi^0} \right). \]  

(50)

Since $(q^+) - \phi^0 > 0$ and $\psi^0 - \phi^0 > 0$, then from (50) we can choose $\beta > 0$ and $\rho > 0$ such that $I(u) \geq \beta > 0$ for any $u \in W_{0}^{1,\Phi}(\Omega)$ with $||u||_{1,\Phi} = \rho$.

(ii) From (f2), it follows that for any $L > 0$ there exists a constant $C_L := C(L) > 0$ depending on $L$, such that

\[ F(x, t) \geq L|t|^\phi^0 - C_L, \quad \forall (x, t) \in \Omega \times \mathbb{R}. \]  

(51)

Let $w \in W_{0}^{1,\Phi}(\Omega)$ with $w > 0$. We take $t > 1$ large enough to ensure that $||tw||_{1,\Phi} > 1$. Then, from (51), Lemma 2.1 and Lemma 4.1, we have

\[ I(tw) \]

\[ = \int_{\Omega} \Phi(x, |t\nabla w|) \, dx + \int_{\Omega} \frac{V(x)}{q(x)} |tw|^{q(x)} \, dx - \int_{\Omega} F(x, tw) \, dx \]

\[ \leq t^{\phi^0} ||w||_{1,\Phi}^{\phi^0} + C ||V||_{s(x)}^{\phi^0} \left[ M_1 + M_2 \left( t^{2(q^+)+\phi^0} ||w||_{1,\Phi}^{2(q^+)+\phi^0} + t^{2(\theta r^+)+r^0} ||w||_{1,\Phi}^{2(\theta r^+)+r^0} \right) \right] \]

\[ - Lt^{\phi^0} \int_{\Omega} |w|^{\phi^0} \, dx + C_L |\Omega| \]

\[ = t^{\phi^0} \left( ||w||_{1,\Phi}^{\phi^0} - L \int_{\Omega} |w|^{\phi^0} \, dx \right) \]

\[ + C ||V||_{s(x)}^{\phi^0} \left[ M_1 + M_2 \left( t^{2(q^+)+\phi^0} ||w||_{1,\Phi}^{2(q^+)+\phi^0} + t^{2(\theta r^+)+r^0} ||w||_{1,\Phi}^{2(\theta r^+)+r^0} \right) \right] + C_L |\Omega|. \]

By choosing $L > 0$ such that $||w||_{1,\Phi}^{\phi^0} - L \int_{\Omega} |w|^{\phi^0} \, dx < 0$ and using the fact that $2(\theta r^+/r^0) < 2(q^+/\phi^0) < \phi^0$, we obtain $I(tw) \to -\infty$ as $t \to +\infty$. The proof of this lemma is complete.

\[ \square \]

**Remark 4.1:** Note that in the proof of the geometrical structure lemma we do not need any sign condition on the potential $V$. 
Now, we define the level at \( \tilde{c} \) as
\[
\tilde{c} = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)),
\]
where \( \Gamma = \{ \gamma \in C([0,1], W^{1,\Phi}_0(\Omega)) : \gamma(0) = 0, \gamma(1) = u_0 \} \) is the set of continuous paths joining 0 and \( u_0 \), where \( u_0 \in W^{1,\Phi}_0(\Omega) \) is defined in the previous lemma. Let us recall the standard definitions of Palais–Smale sequence at the level \( \tilde{c} \) and Palais–Smale condition at the level \( \tilde{c} \) for a functional \( I \in C^1(E, \mathbb{R}) \), where \( E \) is a Banach space.

**Definition 4.1:** Let \( E \) be a Banach space with dual space \( E^* \) and \( (u_n) \) a sequence in \( E \). We say that \( (u_n) \) is a Palais–Smale sequence at the level \( \tilde{c} \) for a functional \( I \in C^1(E, \mathbb{R}) \) if
\[
I(u_n) \to \tilde{c}, \quad \text{and} \quad \| I'(u_n) \|_{E^*} \to 0.
\]

**Definition 4.2:** We say that a functional \( I \) satisfies the Palais–Smale condition at the level \( \tilde{c} \) if any Palais–Smale sequence at the level \( \tilde{c} \) for \( I \) possesses a convergent subsequence.

Note that by Lemma 4.2, the existence of a Palais–Smale sequence at the level \( \tilde{c} \) for our functional \( I \) is ensured. This can be observed directly from the proof given in [21].

Now, in order to prove that the functional \( I \) satisfies the Palais–Smale condition, we shall first show that any Palais–Smale sequence for \( I \) is bounded. To this end, we have the following lemma.

**Lemma 4.3:** Assume that the assumptions (\( \Phi \)) and (\( f'_0 \)) – (\( f_3 \)) hold. Furthermore, assume that \( \phi^0 < \min(\psi_0, q^-) \), \( \max(\psi_0, q^+) < (\phi_0)^* \), \( q^+ - \frac{1}{2} \phi_0 < q^- \), and \( s(x) > q(x)(\phi_0)^* / ((\phi_0)^* - q(x)) \) for every \( x \in \Omega \). If \( V \) has a constant sign a.e. on \( \Omega \), then any Palais–Smale sequence at the level \( \tilde{c} \) for \( I \) is bounded in \( W^{1,\Phi}_0(\Omega) \).

**Proof:** Let \( (u_n) \) be a Palais–Smale sequence at the level \( \tilde{c} \) for \( I \) in \( W^{1,\Phi}_0(\Omega) \). We prove by contradiction that \( (u_n) \) is bounded in \( W^{1,\Phi}_0(\Omega) \). Let us assume that \( (u_n) \) is unbounded in \( W^{1,\Phi}_0(\Omega) \), that is, \( \| u_n \|_{1,\Phi} \to +\infty \).

Set \( v_n := u_n / \| u_n \|_{1,\Phi} \). It is clear that \( (v_n) \) is bounded in \( W^{1,\Phi}_0(\Omega) \). Hence, there exists a subsequence denoted again \( (v_n) \) such that \( v_n \) converges weakly to \( v \) in \( W^{1,\Phi}_0(\Omega) \). Since \( W^{1,\Phi}_0(\Omega) \) is compactly embedded in \( L^\Phi(\Omega) \) (see the proof of relation (35)), then \( v_n \) converges strongly to \( v \) in \( L^\Phi(\Omega) \), and then a.e. in \( \Omega \).

Define \( \Omega_{\neq} := \{ x \in \Omega : |v(x)| \neq 0 \} \). We consider two possible cases: \( |\Omega_{\neq}| = 0 \) or \( |\Omega_{\neq}| > 0 \). Firstly, we assume that \( |\Omega_{\neq}| = 0 \), that is, \( v = 0 \) a.e. in \( \Omega \). From the definition of \( I \) in (27), Lemma 2.1, and the fact that \( \| u_n \|_{1,\Phi} \to +\infty \), we get
\[
\| u_n \|^\phi^0_{1,\Phi} \leq I(u_n) - \int_\Omega \frac{V(x)}{q(x)} |u_n|^{q(x)} \, dx + \int_\Omega F(x, u_n) \, dx
\leq I(u_n) + \frac{1}{q^-} \int_\Omega |V(x)| \| u_n \|^{q(x)} \, dx + \int_\Omega F(x, u_n) \, dx,
\]
which implies that
\[
1 \leq \frac{I(u_n)}{\| u_n \|^\phi^0_{1,\Phi}} + \frac{1}{q^- \| u_n \|^\phi^0_{1,\Phi}} \int_\Omega |V(x)| \| u_n \|^{q(x)} \, dx + \int_\Omega \frac{F(x, u_n)}{\| u_n \|^\phi^0_{1,\Phi}} \, dx,
\]

(53)
Now, we show that all terms of the right-hand side of (53) tend to zero when \( n \) is large enough, which is the desired contradiction. Since \((u_n)\) is a Palais–Smale type sequence, then \((I(u_n))\) is bounded. Hence, the first term of the right-hand side of (53) tends to zero as \( n \) is large enough. For the second one, from Lemma 4.1, we get

\[
\frac{1}{q^-} \|u_n\|_{1,\Phi}^{\phi_0} \int_{\Omega} |V(x)| |u_n|^{q(x)} \, dx \leq C \|V\|_{s(x)}^{\alpha/r} M_1 + M_2 \left( \|u_n\|_{1,\Phi}^{2(q^- - \theta)} + \|u_n\|_{1,\Phi}^{2(q^+ - \theta - \gamma)} \right) \frac{\|u_n\|_{1,\Phi}^{\phi_0}}{q^-}.
\]

(54)

Since \(2(\theta r^+) - \gamma < 2(q^- - \theta) < 2(q^+ - \theta) < \phi_0\), then passing to the limit in (54), we obtain

\[
\frac{1}{q^-} \|u_n\|_{1,\Phi}^{\phi_0} \int_{\Omega} |V(x)| |u_n|^{q(x)} \, dx \to 0, \quad \text{as } n \to +\infty.
\]

Hence, the second term tends to zero as \( n \) is large enough. For the third term, on the one hand it follows from the definition of \( F \) that

\[
\int_{\{ |u_n| \leq M \}} \frac{|F(x, u_n)|}{\|u_n\|_{1,\Phi}^{\phi_0}} \, dx \leq \frac{C(M)}{\|u_n\|_{1,\Phi}^{\phi_0}},
\]

(56)

where \( C(M) \) is a positive constant depending on \( M \) defined in (25). On the other hand, by using Hölder’s inequality we get

\[
\int_{\{ |u_n| > M \}} \frac{F(x, u_n)}{\|u_n\|_{1,\Phi}^{\phi_0}} \, dx = \int_{\{ |u_n| > M \}} \frac{F(x, u_n)}{|u_n|^{\phi_0}} \|v_n|^{\phi_0} \, dx \leq 2 \left( \int_{\{ |u_n| > M \}} \frac{F(x, u_n)}{|u_n|^{\phi_0}} \chi_{\{ |u_n| > M \}} \right) \left( \left\| |v_n|^{\phi_0} \chi_{\{ |u_n| > M \}} \right\|_{\Gamma} \right).
\]

Without loss of generality, we may suppose that \( |F(x, u_n)|/|u_n|^{\phi_0} \chi_{\{ |u_n| > M \}} \|_{\Gamma} > 1 \). Then, from Lemma 2.1 we get

\[
\left\| \frac{F(x, u_n)}{|u_n|^{\phi_0}} \chi_{\{ |u_n| > M \}} \right\|_{\Gamma} \leq \left[ \int_{\{ |u_n| > M \}} \Gamma \left| \frac{F(x, u_n)}{|u_n|^{\phi_0}} \right| \, dx \right]^{1/\gamma_0}.
\]

Hence, it follows from (25) that

\[
\left\| \frac{F(x, u_n)}{|u_n|^{\phi_0}} \chi_{\{ |u_n| > M \}} \right\|_{\Gamma} \leq C \left[ \int_{\Omega} H(x, u_n) \, dx \right]^{1/\gamma_0} + C',
\]

(57)

where \( C \) and \( C' \) are positive constants independent of \( n \).

In the case where \( V \leq 0 \) a.e. on \( \Omega \), then from the definition of the functional \( I \) we get

\[
\phi^0 I(u_n) - \langle I'(u_n), u_n \rangle = \int_{\Omega} \left[ \phi^0 \Phi(x, |\nabla u_n|) - \phi(x, |\nabla u_n|)|\nabla u_n|^2 \right] \, dx
\]

\[
+ \int_{\Omega} V(x) \left( \frac{\phi^0}{q(x)} - 1 \right) |u_n|^{q(x)} \, dx + \int_{\Omega} (f(x, u_n) u_n - \phi^0 F(x, u_n)) \, dx.
\]

(58)
From (9) and the fact that $\phi^0 < q^- \leq q(x)$, the first and the second terms of the right-hand side of (58) are nonnegative. Hence, the relation (58) becomes

$$\phi^0 \mathcal{I}(u_n) - \langle \mathcal{I}'(u_n), u_n \rangle \geq \int_{\Omega} H(x, u_n) \, dx.$$  \hfill (59)

It follows from (59) that, $\int_{\Omega} H(x, u_n) \, dx \leq C$, for $n$ large enough.

Now, in the case where $V \geq 0$ a.e. on $\Omega$, then from the definition of the functional $\mathcal{I}$ we get

$$q^+ \mathcal{I}(u_n) - \langle \mathcal{I}'(u_n), u_n \rangle = \int_{\Omega} \left[ q^+ \Phi(x, |\nabla u_n|) - \phi(x, |\nabla u_n|)|\nabla u_n|^2 \right] \, dx$$

$$+ \int_{\Omega} V(x) \left( \frac{q^+}{q(x)} - 1 \right) |u_n|^q(x) \, dx + \int_{\Omega} H(x, u_n) \, dx.$$  \hfill (60)

Since $q(x) \leq q^+$, then following the same arguments as for (59), we have also $\int_{\Omega} H(x, u_n) \, dx \leq C$, for $n$ large enough. This fact combined with relation (57) yields

$$\left\| \frac{F(x, u_n)}{|u_n|^{\phi_0} \chi_{|u_n| > M}} \right\|_{\Gamma} \leq C, \text{ for } n \text{ large enough},$$  \hfill (61)

where $C$ is a positive constant independent of $n$.

Now, it remains to show that $||v_n||_{\phi_0 \chi_{|u_n| > M}} \|_{\tilde{\Gamma}} \to 0$ as $n \to +\infty$. Let $K(x, t) := \tilde{\Gamma}(x, t)^{\phi_0}$. Since $\phi_0 > 1$ and $\tilde{\Gamma} \in N(\Omega)$, then it is clear that $K \in N(\Omega)$. Moreover, since $\Gamma$ satisfies (H2) then $K$ verifies the assumption (2) of Theorem 2.1 and by Remark 2.3, $K(x, k)$ is bounded for each $k > 0$. Using Lemmas 2.1 and 2.3, we get

$$\lim_{t \to +\infty} \frac{K(x, kt)}{\Phi^*_\phi(x, t)} \leq \lim_{t \to +\infty} \frac{K(x, k)}{\Phi^*_\phi(x, 1)} \frac{1}{t^{\phi_0(y_0)' - (\phi_0)^*}},$$

where $(\gamma_0)' = \gamma_0/\gamma_0 - 1$ is defined as in (24). Since $N/\phi_0 < \gamma_0$, then $\phi_0(y_0)' < (\phi_0)^*$. From the last inequality it follows that

$$\lim_{t \to +\infty} \frac{K(x, kt)}{\Phi^*_\phi(x, t)} = 0, \text{ uniformly for } x \in \Omega.$$

Thus, form Theorem 2.1, $W_0^1,\phi(\Omega)$ is compactly embedded in $L^K(\Omega)$, which implies that

$$\int_{\Omega} \tilde{\Gamma}(x, |v_n|^{\phi_0}) \, dx \to 0, \text{ as } n \to +\infty.$$  

Consequently,

$$||v_n||_{\phi_0 \chi_{|u_n| > M}} \|_{\tilde{\Gamma}} \to 0, \text{ as } n \to +\infty.$$  \hfill (62)

Hence, passing to the limit in (53) and using (55), (61) and (62), we obtain a contradiction.

Secondly, we assume that $|\Omega| > 0$. Then obviously, $|u_n| = |v_n||u_n|_{1,\phi} \to +\infty$ in $\Omega_{\neq}$. Hence, for some positive real $M$ we have $\Omega_{\neq} \subset \{x \in \Omega : |u_n| \geq M\}$, for $n$ large enough.
Therefore, using Lemma 2.1 we get
\[
\frac{\mathcal{I}(u_n)}{\|u_n\|_{1,\Phi}^{\phi_0}} \leq 1 + \frac{1}{q^- \|u_n\|_{1,\Phi}^{\phi_0}} \int_{\Omega} |V(x)||u_n|^q(x) \, dx - \int_{\Omega} \frac{F(x, u_n)}{\|u_n\|_{1,\Phi}^{\phi_0}} \, dx
\]
\[
= 1 + \frac{1}{q^- \|u_n\|_{1,\Phi}^{\phi_0}} \int_{\Omega} |V(x)||u_n|^q(x) \, dx - \int_{\{\|u_n\| \leq M\}} \frac{F(x, u_n)}{\|u_n\|_{1,\Phi}^{\phi_0}} \, dx
\]
\[
- \int_{\{\|u_n\| > M\}} \frac{F(x, u_n)}{\|u_n\|_{1,\Phi}^{\phi_0}} |v_n|^{\phi_0} \, dx.
\]

Now, using relations (55), (56), assumption (f2) and Fatou’s Lemma, we obtain a contradiction. Hence, \((u_n)\) is bounded in \(W_0^{1,\Phi}(\Omega)\). The proof of this lemma is complete. □

Remark 4.2: The preceding lemma holds true under a slightly weaker assumption than the hypothesis ‘\(V\) has a constant sign’. Indeed, assume that there exists a constant \(\rho\) such that \(\phi_0 \leq \rho \leq q^+\) and \(V(x)(\rho/(q(x) - 1)) \geq 0\) a.e. on \(\Omega\). Then, by taking \(H(x, t) = f(x, t)t - \rho F(x, t)\) and following the same arguments as for (58) – (59), we obtain the previous conclusion.

To finish the proof of the Palais–Smale condition for \(\mathcal{I}\), we only need to show the following lemma.

Lemma 4.4: Assume that the assumptions of Lemma 4.3 hold. Then, the Palais–Smale sequence at the level \(\tilde{c}\) for \(\mathcal{I}\) possesses a convergent subsequence.

Proof: Let \((u_n)\) be a Palais–Smale sequence at the level \(\tilde{c}\) for \(\mathcal{I}\) in \(W_0^{1,\Phi}(\Omega)\). Then, \(\mathcal{I}'(u_n) \to 0\) in \((W_0^{1,\Phi}(\Omega))^*\) and from Lemma 4.3, \((u_n)\) is bounded in \(W_0^{1,\Phi}(\Omega)\). As \(W_0^{1,\Phi}(\Omega)\) is reflexive, then there exists a subsequence denoted again \((u_n)\) such that \(u_n\) converges weakly to \(u\) in \(W_0^{1,\Phi}(\Omega)\). From Proposition 4.2-(i), the mapping \(\mathcal{H}'\) is of type \((S_+)\). Thus, to conclude the result of this lemma it suffices to show that
\[
\lim \sup_{n \to \infty} \langle \mathcal{H}'(u_n), u_n - u \rangle \leq 0.
\]

Indeed, using the definition of \(\mathcal{I}'\) in Proposition 4.1, we have
\[
\langle \mathcal{H}'(u_n), u_n - u \rangle = \langle \mathcal{I}'(u_n), u_n - u \rangle + \langle \mathcal{F}'(u_n), u_n - u \rangle - \langle \mathcal{J}'(u_n), u_n - u \rangle.
\]

It is clear that,
\[
\langle \mathcal{I}'(u_n), u_n - u \rangle \to 0.
\]

From Proposition 4.2-(ii), \(\mathcal{F}'\) is a completely continuous linear operator. Hence,
\[
\langle \mathcal{F}'(u_n), u_n - u \rangle \to 0.
\]

Now, it remains to show that \(\langle \mathcal{J}'(u_n), u_n - u \rangle \to 0\); that is
\[
\int_{\Omega} V(x)|u_n|^{q(x)-2} u_n(u_n - u) \, dx \to 0.
\]

From the assumptions, we have \(1 < q(x) < (\phi_0)^*\) and \(1 < \alpha(x) < (\phi_0)^*\), for every \(x \in \overline{\Omega}\), where \(\alpha(x) := s(x)q(x)/(s(x) - q(x))\). Then, as in the proof of relation (32), the space...
$W_0^{1,\Phi}(\Omega)$ is compactly embedded in $L^{q(x)}(\Omega)$ and in $L^{\alpha(x)}(\Omega)$. Since $(u_n)$ is bounded in $W_0^{1,\Phi}(\Omega)$, then $u_n$ converges strongly to $u$ in $L^{\alpha(x)}(\Omega)$. Consequently, using Hölder’s inequality and Proposition 2.3, then relation (67) holds thanks to the inequality

$$\left| \int_{\Omega} V(x)|u_n|^{q(x)-2}u_n(u_n - u) \, dx \right| \leq C_0 \|V\|_{L^{q(x)}}\|u_n\|^{q(x)-1}_{q(x)}\|u_n - u\|_{\alpha(x)}$$

$$\leq C_1 \|V\|_{L^{q(x)}}\|u_n\|^\tau_{q(x)}\|u_n - u\|_{\alpha(x)},$$

where $C_1$ is a positive constant independent of $n$ and $\tau \in \{q^- - 1, q^+ - 1\}$. Finally, it follows from (65), (66) and (67) that (63) holds. Hence, since $\mathcal{H}'$ is of type $(S_+)$, then $u_n$ converges strongly to $u$ in $W_0^{1,\Phi}(\Omega)$. The proof of Theorem 3.2 is complete. ■

Next, as $W_0^{1,\Phi}(\Omega)$ is a reflexive and separable Banach space, there exist $(e_j)_{j \in \mathbb{N}^*} \subseteq W_0^{1,\Phi}(\Omega)$ and $(e^*_j)_{j \in \mathbb{N}^*} \subseteq (W_0^{1,\Phi}(\Omega))^*$ such that

$$W_0^{1,\Phi}(\Omega) = \text{span}\{e_j : j \in \mathbb{N}^*\}, \quad (W_0^{1,\Phi}(\Omega))^* = \text{span}\{e^*_j : j \in \mathbb{N}^*\}$$

and

$$\langle e_i, e^*_j \rangle = \begin{cases} 1, & i = j \\ 0, & i \neq j. \end{cases}$$

For $k \in \mathbb{N}^*$ denote

$$X_j = \text{span}\{e_j\}, \quad Y_k = \bigoplus_{j=1}^{k} X_j, \quad \text{and} \quad Z_k = \bigoplus_{j=k}^{\infty} X_j.$$

**Proof of Theorem 3.3:** Denote

$$\beta_k := \sup \left\{ \int_{\Omega} \Psi(x, |u|) \, dx : \|u\|_{1,\Phi} = 1, u \in Z_k \right\}.$$ 

Since $\Psi \ll \Phi$, then $\lim_{k \to +\infty} \beta_k = 0$ (see [20, Lemma 4.3]). Now, we verify the conditions of Fountain theorem. It follows from assumption (f4) that $\mathcal{F}$ is even, hence the functional $\mathcal{I}$ is even. From Lemmas 4.3 and 4.4, $\mathcal{I}$ satisfies the Palais–Smale condition; hence the condition (3) of Fountain theorem holds. It remains to prove that conditions (1) and (2) in Fountain theorem hold.

1. By (f$'_0$), it follows that

$$|F(x, t)| \leq C(\Psi(x, t) + |t|), \quad \forall (x, t) \in \Omega \times \mathbb{R}. \quad (68)$$

Let $u \in Z_k$ with $\|u\|_{1,\Phi} > 1$. From the definition of $\mathcal{I}$ in (27), Lemmas 2.1, 4.1 and Poincaré’s inequality, we obtain

$$\mathcal{I}(u) = \int_{\Omega} \Phi(x, |\nabla u|) \, dx + \int_{\Omega} \frac{V(x)}{q(x)}|u|^{q(x)} \, dx - \int_{\Omega} F(x, u) \, dx$$

$$\geq \|u\|^\beta_{1,\Phi} - C\|V\|_{L^{\alpha(x)}}^{q(x)} \left[ M_1 + M_2 \left( \|u\|^2_{1,\Phi} + \|u\|_{1,\Phi}^{2(\theta r^+)} \right) \right]$$

$$- C_1 \int_{\Omega} \Psi(x, |u|) \, dx - C_2 \|u\|_{1,\Phi}, \quad (69)$$
where we recall that $\alpha = r^+$ if $\|V\|_{s(x)} > 1$ and $\alpha = r^-$ if $\|V\|_{s(x)} \leq 1$. Furthermore, from Lemma 2.1, we have
\[
\int_{\Omega} \Psi(x, |u|) \, dx = \int_{\Omega} \Psi \left( x, \|u\|_{1, \Phi}, \frac{|u|}{\|u\|_{1, \Phi}} \right) \, dx \\
\leq \|u\|_{r, \Omega}^0 \int_{\Omega} \Psi \left( x, \frac{|u|}{\|u\|_{1, \Phi}} \right) \, dx.
\]

Using the definition of $\beta_k$, the relation (69) becomes
\[
\mathcal{I}(u) \geq \|u\|_{1, \Phi}^{\phi_0} - C \|V\|_{s(x)}^{\alpha/r^-} \left[ M_1 + M_2 \left( \|u\|_{1, \Phi}^{2(\gamma^+ - \theta)} + \|u\|_{1, \Phi}^{2(\gamma^+) r^-} \right) \right] \\
- C_1 \|u\|_{1, \Phi}^{\phi_0} \beta_k - C_2 \|u\|_{1, \Phi}.
\]

Now, let $u_k \in Z_k$ with $\|u\|_{1, \Phi} = r_k = (2C_1 \beta_k)^{1/(\phi_0 - \phi_0)}$. Since $\phi_0 < \psi_0^0$ and $\lim_{k \to +\infty} \beta_k = 0$, then $r_k \to +\infty$ as $k \to +\infty$. Thus, we have
\[
\mathcal{I}(u) \geq (2C_1 \beta_k)^{\phi_0/(\phi_0 - \psi_0^0)} - C \|V\|_{s(x)}^{\alpha/r^-} \left[ M_1 + M_2 \left( r_k^{2(\gamma^+ - \theta)} + r_k^{2(\gamma^+) r^-} \right) \right] \\
- C_1 (2C_1 \beta_k)^{\psi_0^0/(\phi_0 - \psi_0^0)} \beta_k - C_2 r_k.
\]

That is
\[
\mathcal{I}(u) \geq \frac{1}{2} r_k \phi_0 - C \|V\|_{s(x)}^{\alpha/r^-} \left[ M_1 + M_2 \left( r_k^{2(\gamma^+ - \theta)} + r_k^{2(\gamma^+) r^-} \right) \right] - C_2 r_k.
\]

Since $2(\gamma^+) r^- < 2(\gamma^+ - \theta) < 2(\gamma^+ - \theta) < \phi_0$ and $1 < \phi_0$, we then have
\[
\inf_{\{u \in Z_k, \|u\| = r_k\}} \mathcal{I}(u) \to +\infty \text{ as } k \to +\infty.
\]

(2) Let $w \in Y_k$ with $w > 0$, $\|w\|_{1, \Phi} = 1$ and $t > 1$. Then, applying Lemma 2.1, (51) gives
\[
\mathcal{I}(tw) \leq t^{\phi_0} \left( \|w\|_{1, \Phi}^{\phi_0} - L \int_{\Omega} |w|^{\phi_0} \, dx \right) \\
+ C \|V\|_{s(x)}^{\alpha/r^-} \left[ M_1 + M_2 \left( t^{2(\gamma^+ - \theta)} \|w\|_{1, \Phi}^{2(\gamma^+ - \theta)} + t^{2(\gamma^+) r^-} \|w\|_{1, \Phi}^{2(\gamma^+) r^-} \right) \right] \\
+ C_2 |\Omega|.
\]

Clearly, we can choose $L > 0$ so that $\|w\|_{1, \Phi}^{\phi_0} - L \int_{\Omega} |w|^{\phi_0} \, dx < 0$. So, since $2(\gamma^+) r^- < 2(\gamma^+ - \theta) < \phi_0$ then we have $\mathcal{I}(tw) \to -\infty$ as $t \to +\infty$. Thus, there exists $t > r_k > 1$ such that $\mathcal{I}(tw) < 0$. By setting $\rho_k = \tilde{t}$, we therefore obtain
\[
\max_{\{u \in Y_k, \|u\| = \rho_k\}} \mathcal{I}(u) \leq 0.
\]

The proof of this theorem is complete. 

\[\blacksquare\]
5. Application

Let us give an example of function \( f \) satisfying the assumptions \((f_0')-(f_3)\) and for which our main Theorems 3.2 and 3.3 hold.

Let us fix \( \Phi(x,t) = 1/p(x)|t|^p(x) \), with \( p \in C^{1-0}(\Omega) \). Then, the operator \( \text{div}(\phi(x,|\nabla u|)\nabla u) \) involved in (P) is the \( p(x) \)-Laplacian operator, i.e. \( \Delta_{p(x)}u := \text{div}(|\nabla u|^{p(x)-2}\nabla u) \). In this case, we have \( \phi_0 = p^- \) and \( \phi^0 = p^+ \) with \( 1 < p^- \leq p(x) \leq p^+ < N \).

In the case where \( V \geq 0 \) a.e. on \( \Omega \), we take \( F(x,t) = |t|^q\ln(1+|t|) \), with \( q^+ + 1 < Np^-/(N-p^-) \). The derivative with respect to \( t \) of \( F(x,t) \) is given by \( F'(x,t) := f(x,t) = q^+|t|^{q^-2t\ln(1+|t|) + t|t|^{q^+1} - 1/(1 + |t|)} \) and we have: \( f(x,t)t - q^+F(x,t) = |t|^{q^-}/(1 + |t|) \).

It is clear that \( f \) satisfies the assumptions \((f_0'),(f_2)-(f_4)\). Moreover, since \( F(x,t)/|t|^\theta \to 0 \) for all \( \theta > q^+ \) then from (1), \( f \) does not satisfy the (A–R) condition. Now, it remains to show that the assumption \((f_1)\) holds. To this end, let us consider the function \( \Gamma(x,t) = |t|^\beta \), where \( 1 < N/p^- < \beta < q^+/q^+ - p^- \). Then, \( \Gamma(x,F(x,t)/|t|^\theta) = |t|^{\theta(q^+ - p^-)\ln^\beta(1 + |t|)} \). Since \( \beta(q^+ - p^-) < q^+ \), then \( |t|^{\beta(q^+ - p^-)\ln^\beta(1 + |t|)} \to 0 \) as \( |t| \to +\infty \). Hence, the assumption \((f_1)\) holds.

Now, in the case where \( V \leq 0 \) a.e. on \( \Omega \) we can take \( F(x,t) = |t|^p\ln(1 + |t|) \). By the same arguments above, the choice of \( \Gamma(x,t) = |t|^\beta \), where \( 1 < N/p^- < \beta < p^+/p^+ - p^- \), ensures easily that \( f \) verifies the assumptions \((f_0')-(f_4)\). Consequently, the main Theorems 3.2 and 3.3 hold.

Remark 5.1:  
(1) In the case where \( V \leq 0 \) a.e. on \( \Omega \), we cannot take the same function \( F \) considered in the first case, i.e. \( F(x,t) = |t|^q\ln(1 + |t|) \). Indeed, in this case the nonlinearity \( f \) satisfies the (A–R) condition.

(2) As in the first remark, when \( V \geq 0 \) a.e. on \( \Omega \) we cannot consider the function \( F(x,t) = |t|^p\ln(1 + |t|) \) as an example of application. Indeed, in this case we have

\[
f(x,t)t - q^+F(x,t) = (p^+ - q^+)|t|^{p^+\ln(1 + |t|)} + \frac{|t|^{p^+1}1 + |t|}{1 + |t|} < 0,
\]

for \( |t| \) large enough,

and hence, assumption \((f_1)\) is not satisfied by the nonlinearity \( f \).

Disclosure statement

No potential conflict of interest was reported by the author(s).

ORCID

Soufiane Maatouk  http://orcid.org/0000-0001-6608-7098
Abderrahmane El Hachimi  http://orcid.org/0000-0002-3292-1977

References

[1] Chen Y, Levine S, Rao R. Variable exponent, linear growth functionals in image restoration. SIAM J Appl Math. 2006;66(4):1383–1406.
[2] Růžička M. Electrorheological fluids: modeling and mathematical theory. Berlin: Springer-Verlag; 2000. (Lecture Notes in Math.; 1748).
[3] Abdou A, Marcos A. Existence and multiplicity of solutions for a Dirichlet problem involving perturbed $p(x)$-Laplacian operator. Electron J Differ Equ. 2016;197:1–19.
[4] Carvalho MLM, Goncalves JVA, Da Silva ED. On quasilinear elliptic problems without the Ambrosetti–Rabinowitz condition. J Math Anal Appl. 2015;426(1):466–483.
[5] Bonanno G, Molica Bisci G, Rădulescu V. Quasilinear elliptic non-homogeneous Dirichlet problems through Orlicz–Sobolev spaces. Nonlinear Anal. 2012;75(12):4441–4456.
[6] Bonanno G, Molica Bisci G, Rădulescu V. Arbitrarily small weak solutions for a nonlinear eigenvalue problem in Orlicz–Sobolev spaces. Monatsh Math. 2011;165(3-4):305–318.
[7] Bonanno G, Molica Bisci G, Rădulescu V. Infinitely many solutions for a class of nonlinear eigenvalue problems in Orlicz–Sobolev spaces. C R Acad Sci Paris. 2011;349:263–268.
[8] Bonanno G, Molica Bisci G, Rădulescu V. Existence of three solutions for a non-homogeneous Neumann problem through Orlicz–Sobolev spaces. Nonlinear Anal. 2011;74(14):4785–4795.
[9] Fan X. Differential equations of divergence form in Musielak–Sobolev spaces and a sub-supersolution method. J Math Anal Appl. 2012;386(2):593–604.
[10] GueB, LuDJ, LuaJE. Multiple solutions for a class of double phase problem without the Ambrosetti–Rabinowitz conditions. Nonlinear Anal. 2019;188:294–315.
[11] Li G, Rădulescu V, Repovš DD, et al. Nonhomogeneous Dirichlet problems without the Ambrosetti–Rabinowitz condition. Topol Methods Nonlinear Anal. 2018;51(1):55–77.
[12] Rădulescu V, Zhang Q. Double phase anisotropic variational problems and combined effects of reaction and absorption terms. J Math Pures Appl. 2018;118:159–203.
[13] Wang B, Liu D, Zhao P. Hölder continuity for nonlinear elliptic problem in Musielak–Orlicz–Sobolev space. J Differ Equ. 2019;266(8):4835–4863.
[14] Fan X, Zhang QH. Existence of solutions for $p(x)$-Laplacian Dirichlet problem. Nonlinear Anal. 2003;52(8):1843–1852.
[15] Kefi K. $p(x)$-Laplacian with indefinite weight. Proc Amer Math Soc. 2011;139(12):435–4360.
[16] Boreanu MM, Pucci P, Rădulescu V. Multiplicity of solutions for a class of anisotropic elliptic equations with variable exponent. Complex Var Elliptic Equ. 2011;56:755–767.
[17] Mihăilescu M, Pucci P, Rădulescu V. Nonhomogeneous boundary value problems in anisotropic Sobolev spaces. C R Acad Sci Paris Ser I. 2007;345:561–566.
[18] Mihăilescu M, Pucci P, Rădulescu V. Eigenvalue problems for anisotropic quasilinear elliptic equations with variable exponent. J Math Anal Appl. 2008;340:687–698.
[19] Clément Ph, García-Huidobro M, Manásevich R, et al. Mountain pass type solutions for quasilinear elliptic equations. Calc Var Partial Differ Equ. 2000;11(1):33–62.
[20] Liu D, Zhao P. Solutions for a quasilinear elliptic equation in Musielak–Orlicz–Sobolev spaces. Nonlinear Anal. 2015;26:315–329.
[21] Ambrosetti A, Rabinowitz PH. Dual variational methods in critical point theory and applications. J Funct Anal. 1973;14(4):349–381.
[22] Chung NT, Toan HQ. On a nonlinear and non-homogeneous problem without (A-R) type condition in Orlicz–Sobolev spaces. Appl Math Comput. 2013;219(14):7820–7829.
[23] Rao MN, Ren ZD. Theory of Orlicz spaces. New York: Marcel Dekker; 1985.
[24] Costa DG, Magalhães CA. Variational elliptic problems which are nonquadratic at infinity. Nonlinear Anal. 1994;23(11):1401–1412.
[25] Ge B, Chen ZY. Existence of infinitely many solutions for double phase problem with sign-changing potential. RACSAM Rev R Acad Cienc Exactas Fis Nat Ser A Mat. 2019;113:1–12.
[26] Benouhiba N. On the eigenvalues of weighted $p(x)$-Laplacian on $\mathbb{R}^N$. Nonlinear Anal. 2011;74(1):235–243.
[27] Kim IH, Kim YH. Mountain pass type solutions and positivity of the infimum eigenvalue for quasilinear elliptic equations with variable exponents. Manuscripta Math. 2015;147:169–191.
[28] Mihăilescu M, Rădulescu V, Repovš D. On a non-homogeneous eigenvalue problem involving a potential: an Orlicz–Sobolev space setting. J Math Pures Appl. 2010;93(2):132–148.
[29] Fan X. An imbedding theorem for Musielak–Sobolev spaces. Nonlinear Anal. 2012;75(4):1959–1971.
[30] Hudzik H. On generalized Orlicz-Sobolev space. Funct Approx Comment Math. 1976; 4:37–51.
[31] Musielak J. Orlicz spaces and modular spaces. Berlin: Spring-Verlag; 1983. (Lecture Notes in Math; 1034).
[32] Mihăilescu M, Rădulescu V. Neumann problems associated to non-homogeneous differential operators in Orlicz–Sobolev spaces. Ann Inst Fourier. 2008;58(6):2087–2111.
[33] Fukagai N, Ito M, Narukawa K. Positive solutions of quasilinear elliptic equations with critical Orlicz–Sobolev nonlinearity on $\mathbb{R}^N$. Funkcial Ekvac. 2006;49(2):235–267.
[34] Fan X, Zhao D. On the spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$. J Math Anal Appl. 2001;263(2):424–446.
[35] Kovacik O, Rakosnik J. On spaces $L^{p(x)}(\Omega)$ and $W^{k,p(x)}(\Omega)$. Czechoslovak Math J. 1991;41(4):592–618.
[36] Edmunds D, Rákosník J. Sobolev embeddings with variable exponent. Studia Math. 2000;143(3):267–293.
[37] Willem M. Minimax theorems. Boston: Birkhäuser; 1996.