Theory of an electron asymmetric scattering on skyrmion textures in two-dimensional systems

K S Denisov

Ioffe Institute, 194021 St. Petersburg, Russia
E-mail: denisokonstantin@gmail.com

Received 5 March 2020, revised 19 May 2020
Accepted for publication 26 May 2020
Published 16 July 2020

Abstract
We discuss in detail the electron scattering pattern on skyrmion-like magnetic textures in two-dimensional geometry. The special attention is focused on analyzing the scattering asymmetry, which is a precursor of the topological Hall effect. We present analytical results valid in the limiting regimes of strong and weak coupling, we analyze analytically the conditions when the transverse response acquires a quantized character determined by the topological charge of a magnetic texture, we also derive the numerical scheme that gives access to the exact solution of the scattering problem. We describe how the electron scattering asymmetry is modified due to an additional short-range impurity located inside a magnetic skyrmion. Based on the numerical computations we investigate the properties of the asymmetric scattering for an arbitrary magnitude of the interaction strength and the topology of a magnetic texture, we also account for the presence or absence of a scalar impurity.

Keywords: topological Hall effect, magnetic skyrmion, skew scattering, Berry phase

(Some figures may appear in colour only in the online journal)

1. Introduction
Rapidly growing physics of the topological magnetic textures [1–4] pays a special attention to the emerging nontrivial electrodynamics [5–7] and to the topological Hall effect (THE) in particular. This phenomenon is essential for the understanding of transport properties of the skyrmion-lattices [8–12] and other systems hosting magnetic skyrmions both in the discretized [13–15] and non-regular geometries [16–21]. Moreover, the systems enabling the formation of individual magnetic skyrmions have recently gained a specific interest in view of applications in the skyrmion-based logic devices [3, 4, 22]. In particular, the significant progress has been achieved due to the recent advances in imaging techniques that have lead to the discovery of different material platforms hosting skyrmions with size ranging from sub-100 nm [16, 21, 23–25] down to sub-10 nm [26–28]. The subsequent implementation requires operating with skyrmions purely by electrical means, at that the topological Hall effect is widely discussed as a direct method for the skyrmion detection. The major argument in favor of this concept is based on the assumption that THE differentiates reliably between topologically unequal states of the magnetization, i.e. that the corresponding contribution to the Hall resistivity is proportional to the total number of magnetic skyrmion multiplied by the topological charge (the winding number) [1, 12, 29] and that it is independent of particular sample details. However, the recent transport studies of systems with individual nanoscale magnetic textures have reported a rich pattern of data contradicting this basic scheme [13, 15, 16]. In fact, one generally observes that the THE magnitude is considerably renormalized compared with the estimation based purely on topological arguments, which suggests that the THE contribution can be modified due to different physical processes.

Recently various theoretical studies have been conducted to address this issue [30–35]. Without accounting for the spin–orbit interaction driven effects [11, 33, 36], the diagrammatic analysis [30, 31], the tight-binding modeling
and the scattering theory approach simultaneously indicate that the contribution to the electric Hall current induced by a single magnetic skyrmion is generally renormalized either due to the nonadiabaticity of carrier spin dynamics or due to an additional scattering on impurities. For instance, the strong scalar disorder can lead to the substantial increase in the THE magnitude even for the relatively weak exchange coupling. Alternatively, the quantized contribution to the Hall current in case of the ballistic electron motion inside a single skyrmion appears to be realized only for the large skyrmion size. Moreover, the consideration of an electron scattering on a spin texture using the symmetry analysis and the numerical calculations revealed that THE additionally exhibits complex spin-dependent features driven by the nonadiabatic electron spin motion. Given that the ongoing experimental studies have begun to provide more information on transport properties of systems with individual skyrmions, a systematic research addressing the diversity of the topological Hall effect features is highly needed.

In this manuscript we present the comprehensive study of the electron asymmetric scattering on skyrmion-like textures covering various scenarios behind the topological Hall effect modification. We clarify the conditions when the asymmetric scattering rates are directly determined by the topology of a magnetic texture; this case typically corresponds to the conventional estimations of THE. We derive the analytical expressions for the scattering rates in the weak coupling regime and analyze the corresponding properties of the asymmetric scattering. Encouraged by the experimental indications that skyrmions tend to be captured by structural defects, we analyze how an additional impurity potential affects the asymmetric scattering. We describe in detail the numerical scheme that gives access to the investigation of the electron scattering on a spin texture for an arbitrary coupling strength and allowing for the presence of an additional scalar perturbation; we discuss some general features of the transverse response obtained numerically using the considered method. The presented numerical scheme can be straightforwardly generalized for more complex effective band models, e.g. for the recently considered spin-orbital systems. Finally, we discuss the modification of the electron asymmetric scattering on skyrmions for the single spin-subband regime.

The paper is organized as follows. In section 2 we introduce the general framework for the electron scattering on a skyrmion-like texture. In section 3 we present the analytic descriptions for the strong and weak coupling regimes. The numerical scheme is further developed in section 4 to address the exact solution of the generalized scattering problem. In section 5 we analyze various scattering scenarios based on the numerical calculations. In sections 3.1, 5.2 and 5.4 we examine the role of the skyrmion topology on the Hall current. The scattering on electrically charged skyrmions is discussed in sections 3.2 and 5.3. The scattering features driven by the nonadiabaticity of the electron spin motion are presented in sections 3.2 and 5.1.

2. Scattering framework

2.1. Skyrmion scattering potential

We consider a 2D system with the electron Hamiltonian \( \mathcal{H} \) given by:

\[
\mathcal{H} = \mathcal{H}_0 + V(r), \quad \mathcal{H}_0 = \frac{\mathbf{p}^2}{2m_0} - \frac{\Delta}{2} \sigma_z, \tag{1}
\]

here \( \mathcal{H}_0 \) describes the electron free motion, \( m_0 \) is an effective mass, \( \mathbf{p} \) is the momentum operator, \( \Delta > 0 \) is the spin splitting of the electron subbands, the \( r \)-dependent term \( V(r) \) is a scattering potential. The spectrum shown in figure 1 consists of two parabolas shifted by \( \Delta \), the energies are given by \( E_{\pm} = \hbar^2 k^2/2m_0 - s\Delta \), where \( s = \pm 1/2 \) is the electron spin projection onto the \( z \) axis. There are two regimes with respect to the position of the electron energy \( E \) (see figure 1), either \( E > \Delta/2 \) so both spin subbands are available for a free motion, or \( E < \Delta/2 \) and the propagation in the spin-down subband is suppressed. In what follows we denote the wavevectors at energy \( E \) according to the following notation:

\[
(E > \Delta/2) \quad \hbar k_x = \sqrt{2m_0(E + s\Delta)}, \quad \hbar k = \sqrt{2m_0E}. \tag{2}
\]

In this paper we consider the scattering potential \( V(r) \) of the following form:

\[
V(r) = -g \left( \frac{v_1(r)}{u(r)} e^{-i(\chi\phi + \gamma)} \frac{u(r)e^{i(\chi\phi + \gamma)}}{v_2(r)} \right), \tag{3}
\]

where \( r = (r, \phi) \), \( g \) is a coupling constant, \( v_{1,2}(r) \) and \( u(r) \) are dimensionless real functions of \( r \) only, the parameter \( \chi = \pm 1, \pm 2, \ldots \) takes integer values, \( \gamma \) is an arbitrary phase. We will assume that the potential has a localized character such that \( v_{1,2}, u \to 0 \) at \( r > r_0 \), where \( r_0 \) is a localization radius. Of key importance is the dependence of \( V(r) \) on the polar angle \( \phi \) entering in its off-diagonal components. The \( \phi \)-dependence leads to the non-commutativity of the Hamiltonian with the operator of angular momentum \( -i\partial_\phi \), as a result the electron scattering on \( V(r) \) gets an asymmetric character. A comprehensive description of this phenomenon is the main subject of the present paper.

Let us comment on physics underlying the chosen form of \( V(r) \). This type of potentials is relevant for magnetic materials when an electron interacts with a single chiral spin texture, such as magnetic skyrmion. Let \( \mathbf{n}(r) \) be a unit vector directed along the local magnetization. For an individual chiral spin texture \( \mathbf{n}(r) \) can be generally written as:

\[
\mathbf{n}(r) = \left( n_{||}(r) \cos(\chi\phi + \gamma), n_{\perp}(r) \sin(\chi\phi + \gamma), \eta_z(r) \right), \tag{4}
\]

where \( (\chi, \gamma) \) correspond to the vorticity and the helicity of the spin texture, respectively, and \( n_{\perp}(r) \) describe the radial profiles. In what follows we assume that at \( r \geq r_0 \) one has \( n_\perp \to 1, n_{||} \to 0 \), while the orientation of \( \mathbf{n} \) right in the center of a spin texture is arbitrary (e.g., in figure 1 we show the spin texture with \( n(r \to 0) = e_z \)). The scattering potential \( V(r) \) from
equation (3) appears due to an electron exchange interaction with a static magnetization field of this shape. If no other perturbation is present we can relate \( g \) to an exchange interaction constant and the functions \( v_{1,2}(r), u(r) \) to the spin profiles:

\[
v_1(r) = -v_2(r) = n_z(r) - 1, \quad u(r) = n_y(r) \tag{5}
\]

The uniform background component outside the texture core \( n(\mathbf{r} > r_0) = e_z \) gives rise to the spin subband splitting \( \Delta = 2g \) in this case.

We shall mention that when an electron scattering is induced entirely by the perturbation of the magnetization it is necessary to assume the additional coupling \( v_1 = -v_2 \) between the diagonal components of \( V(r) \). In what follows, however, we will develop the theory with no restrictions on \( v_{1,2}(r) \) functions. By making this generalization we can also take into account an additional scalar potential \( U_0 \) superimposed on a spin texture. This situation has a great practical interest, as there are numerous experimental observations that skyrms tend to be pinned by structural defects \([16, 28]\), i.e., by charged impurities. In particular, the approximation \( v_1 \approx v_2 \) instead of \( v_1 = -v_2 \) can be considered to describe the extreme regime with \( U_0 \gg g \). The theory presented in this work allows us to analyze both these cases.

2.2. Scattering rates

In this work we treat the scattering problem using the \( \hat{T}(z) \)-operator which satisfies the Lippmann–Schwinger equation:

\[
\hat{T}(z) = V + V\hat{G}_0(z)\hat{T}(z), \tag{6}
\]

where \( \hat{G}_0(z) = (z - \hat{H}_0)^{-1} \) is the Green operator corresponding to the free Hamiltonian, and \( V \) corresponds to the scattering potential \( V(r) \) defined in equation (3). Since \( V(r) \) is a \( 2 \times 2 \) matrix there are generally four scattering channels. To describe an elastic electron scattering with the energy \( E \) from \((\mathbf{k}, s)\) to \((\mathbf{k}', s')\) states one deals with the \( T \)-matrix on a mass shell:

\[
T_{kk'}^{ss'} \equiv \lim_{\delta \to 0} \langle k|T(E + i\delta)|k'\rangle_{s's}. \tag{8}
\]

The square modulus of so defined \( T \)-matrix elements \( |T_{kk'}^{ss'}|^2 \) determine the scattering rates. In particular, the differential scattering cross-section in 2D geometry is defined as [43]:

\[
\frac{d\sigma_{kk'}}{d\theta} = \frac{m_0^2}{2\pi \hbar^2 k_s} |T_{kk'}^{ss'}|^2, \tag{7}
\]

where \( \theta \) is the scattering angle, i.e., the angle between \( \mathbf{k} \) and \( \mathbf{k}' \).

In what follows, however, we will describe the scattering using the symmetric \( \mathcal{G}_{kk'}^{ss'} = \mathcal{G}_{kk'}^{ss} \) and asymmetric \( \mathcal{J}_{kk'}^{ss'} = -\mathcal{J}_{kk'}^{ss} \) dimensionless functions defined as:

\[
|\mathcal{T}_{kk'}^{ss'}|^2 = \frac{1}{\nu_0} \left( \mathcal{G}_{kk'}^{ss'} + \mathcal{J}_{kk'}^{ss'} \right), \quad \nu_0 = \frac{m_0}{2\pi \hbar^2}. \tag{8}
\]

Here the prefactor corresponds to the two-dimensional density of states \( \nu_0 \). The reason to extract \( \nu_0 \) explicitly from \( |\mathcal{T}_{kk'}^{ss'}|^2 \) becomes clear when using the classical description presented in section 3.1; it also allows for a compact and natural representation of the Hall resistivity, see the details in reference [32].

The asymmetric terms \( \mathcal{J}_{kk'}^{ss'} \) leading to the Hall response appear due to the intrinsic angular asymmetry of the considered chiral potentials. The integral quantities describing the transverse currents are given by:

\[
\mathcal{J}_{ss'} = \int_0^{2\pi} \mathcal{J}_{kk'}^{ss'} \sin \theta \, d\theta. \tag{9}
\]

In the following sections we calculate \( \mathcal{G}_{kk'}^{ss'}, \mathcal{J}_{kk'}^{ss'} \) and consider the properties of an electron scattering in various regimes and for different potential shapes.

3. Analytical results

Let us comment on some general features associated with the scattering on chiral potentials \( V(r) \). There are two types of dynamic processes that occur when the electron moves through the scattering region. Firstly, there is the evolution of its orbital trajectory due to the change of the momentum in a given potential. Secondly, there is the electron spin rotation driven by its coupling with the spatially inhomogeneous magnetization field. As an important consequence, these two processes affect each other leading to the appearance of the scattering asymmetry.

In order to develop an analytical description for some limiting regimes we should distinguish the role of parameters affecting both orbital and spin motions. For instance, the magnitude of \( kr_0 \) determines two orbitally different regimes if we assume \( g \) = const. Namely, there is the transition from quantum isotropic scattering (described perturbatively) at \( kr_0 \ll 1 \) to the quasiclassical low-angle motion at \( kr_0 \gg 1 \). The character of an electron spin motion in its turn changes essentially depending on the magnitude of the adiabatic parameter, which is determined as \( \lambda_0 = \frac{\omega_{\text{ex}}}{\tau_{\text{bf}}} \); here \( \omega_{\text{ex}} = 2g/\hbar \) corresponds to the energy difference between spin up and spin down states, and \( \tau_{\text{bf}} = (2\pi r_0)/v \) is the time of electron presence inside a texture core, \( v = \sqrt{2E/\hbar m_0} \) is the electron velocity. The magnitude of \( \lambda_0 \) shows if the electron has enough time for its spin to become adiabatically co-aligned with the local magnetization direction (\( \lambda_0 \ll 1 \)). Otherwise the perturbation is rather instantaneous (\( \lambda_0 \lesssim 1 \)) so that after flying out of the potential region the electron spin only experiences a small rotation with respect to its initial direction. Naturally, the latter scenario \( \lambda_0 \lesssim 1 \) is accompanied by the activation of the spin-flip scattering channels; the adiabatic regime is on the contrary featured by the suppression of the spin-flip processes.
The adiabatic parameter can be written in form \( \lambda_a = (2g/E) \cdot (kr_0) \), which indicates that the change of the potential radius \( r_0 \) will affect both the orbital and the spin motions simultaneously. As a result it is reasonable to treat the so-called weak coupling regime \( (\lambda_a \lesssim 1) \) on the basis of the perturbation theory along with the condition \( kr_0 \lesssim 1 \). To consider the opposite adiabatic regime it is natural to use the quasiclassical approximation and to assume \( kr_0 \gg 1 \). Below in this section we present the analytical results for these two limiting regimes. The numerical scheme is further developed in section 4 to address the exact solution of the scattering problem.

3.1. Classical scattering and the adiabatic spin motion

In this section we consider the referent regime of the electron adiabatic motion typically assumed when analyzing the topological Hall effect in ferromagnets [13, 15, 16]. Namely, we derive the expressions for the total asymmetric rates \( \mathcal{J}_{se} \) from equation (9) valid in the classical and adiabatic limits. In other words we assume that the scattering potential radius significantly exceeds the electron wavelength \( kr_0 \gg 1 \) and that the electron spin quantization axis is adiabatically co-aligned with local magnetization \( (\lambda_a \gg 1) \). In this case the scattering problem finds a rather elegant solution based on classical mechanics.

As a starting point we use equation (7) and express \( \mathcal{J}_{ss} \) for the spin conserving scattering channels (the spin-flip scattering is suppressed \( \mathcal{J}_{sf} \approx 0 \) in the adiabatic limit) via the spin-dependent differential scattering cross-sections \( d\sigma_s/d\theta \):

\[
\mathcal{J}_{ss} = \frac{p_x}{2\pi\hbar} \int_0^{2\pi} \frac{d\sigma_s}{d\theta} \sin\theta \, d\theta,
\]

(10)

here \( \theta \) is the scattering angle, \( p_x = \hbar k_x \) is the spin-dependent momenta from equation (2). In the assumed approximation we are allowed to calculate \( d\sigma_s/d\theta \) by accounting for the classical trajectories of the electrons moving initially as a uniform incident beam, see figure 2. An electron approaching from the left boundary and having the impact parameter \( y \) is deflected by the scattering angle \( \theta(y) \) being the function of \( y \), the electron momentum \( p_x \), after flying out of the scattering region has both projections \( p_x' = p_x \cos(\theta(y)) \), \( p_y' = p_y \sin(\theta(y)) \). We note that considering \( \theta = \theta(y) \) in equation (10) as a function of the impact parameter and taking into account that \( d\sigma_s/d\theta = d\gamma(\theta)/d\theta \) one can rewrite the formula for \( \mathcal{J}_{ss} \) in the following way:

\[
\mathcal{J}_{ss} = \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p_y'(y) \, dy,
\]

(11)

which is now expressed only in terms of the transverse momentum projection \( p_y'(y) \) resulting from the classical trajectory with the impact parameter \( y \).

The next step is to determine \( p_y'(y) \) which an electron gains moving inside a chiral potential. The condition of the adiabaticity indicates that the electron spin quantization axis rotates in space following the local direction of the magnetization \( \mathbf{m}(r) \). During this process the electron wavefunction acquires a geometrical phase, which is the Berry phase when considered along a closed loop [44]. Essentially, it also manifests itself as the appearance of an effective magnetic field acting on the electron orbital motion \( [5, 12, 29] \).

This effect finds a classical explanation, which is described in detail by Aharonov and Stern [45]. The equation of motion for an electron experiencing the adiabatic rotation of its spin axis is given:

\[
\frac{dp_y}{dr} = \frac{|e|}{c} [v \times \mathbf{B}'(r)],
\]

(12)

here \( \mathbf{B}' \) is an effective spin-dependent magnetic field [12] which is responsible for the appearance of the topological Hall effect. It is worth mentioning that the sign of the effective magnetic field \( \mathbf{B}' \) is opposite for two electron spin state, so the resulting asymmetry \( \mathcal{J}_{+} = -\mathcal{J}_{-} \) in the adiabatic regime is spin-dependent which naturally leads to the spin Hall effect. The effective field can be related to the geometrical characteristic of the magnetization, namely:

\[
\mathbf{B}_{z}^{\pm1/2}(r) = \pm \phi_0 \rho_{sk}(r), \quad \rho_{sk}(r) = \frac{1}{4\pi} \mathbf{n}(r) \cdot \left[ \partial_r \mathbf{n}(r) \times \partial_t \mathbf{n}(r) \right],
\]

(13)

where \( \phi_0 = \hbar c/|e| \) is the magnetic flux quantum, and \( \rho_{sk}(r) \) is the skyrmion density; we note that the unit vector in equation (13) obeys \( \mathbf{n}(r \to \infty) = \mathbf{e}_z \), so the sign of \( \rho_{sk} \) is merely determined by \( \chi \). The total integral over \( \rho_{sk} \) gives the topological charge of a spin texture:

\[
Q = \int \rho_{sk}(r) \, dr = 0, \pm1, \pm2, \ldots,
\]

(14)

the configurations having \( Q \neq 0 \) are classified as magnetic skyrmions [1]. Using equation (12) we express \( p_y'(y) \) in the following way:

\[
p_y'(y) = -\frac{|e|}{c} \int v_s(t) B'_z(x(t),y(t)) \, dt,
\]

(15)

where the integration goes over the time of the electron presence inside a scattering region, and \( (x(t),y(t)) \) is its classical trajectory. We further assume that the scattering has a small-angle character (it is typical for large scale potentials \( kr_0 \gg 1 \)), i.e. the obtained transverse momentum \( p_y' \ll p_y \) is small compared to its initial value. At that one can further simplify the integration in equation (15) by replacing the coordinate \( y(t) \) of the real trajectory by its initial position \( y(t) \approx y \) and by imposing the integration \( v_s(t) \, dt \approx dx \) over the straight line.
The resulting expression for \( p_0'(y) \) is given:
\[
p_0'(y) = -\frac{|e|}{c} \int B_0'(x(t), y) v_y(t) \, dt \approx -\frac{|e|}{c} \int_\infty^\infty B_0'(x, y) \, dx.
\]  
(16)

Finally, we substitute this formula into equation (11) and get for the total asymmetric rates:
\[
J_{\downarrow\uparrow} = -J_{\uparrow\downarrow} = \int \rho_{\downarrow\uparrow}(r) \, dr = Q.
\]  
(17)

Thus we obtained a remarkable finding, namely \( J_{\alpha\gamma} \) are entirely determined by the topological charge \( Q \) of a spin texture. The magnitude of the transverse current appears to be robust and independent of a particular distribution of the magnetization inside a skyrmion core. Moreover, the value of the topological charge determines the maximum magnitude of the transverse response that can be achieved due to an asymmetric scattering on a single skyrmion. Let us emphasize, however, that the topological quantization does not have a universal character. Indeed, the result from equation (17) remains valid only upon three additional assumptions, namely the classical character of an electron motion, the small-angle character of the scattering and the adiabaticity of the electron spin motion. Apart from these assumptions the topology of a spin texture ceases to be the unique requirement for the appearance of the scattering asymmetry. In particular, equation (25) for \( \mathcal{J}_{k^0}^{\alpha\gamma} \) obtained via the perturbation theory does not reflect any topological features of a scattering potential; the asymmetric scattering will take place independently of \( Q \) in that case.

The present consideration is equally applicable to the single subband case (\( \Delta > 2E \)). At sufficiently large potential radius the condition of adiabaticity becomes fulfilled as well, at that the Berry phase approach is also valid. Therefore we are able to determine the magnitude of the total asymmetric rate for the single subband regime will be also determined by equation (17).

It is worth mentioning that the topological quantization of the Hall current can be also obtained by different arguments, e.g. via the semiclassical approach [46] or by the direct introduction of the Berry’s phase [12].

### 3.2. Perturbation theory

In this section we will get analytical expressions for the scattering rates \( G_{k^0}^{\alpha\gamma}, \mathcal{J}_{k^0}^{\alpha\gamma} \) using the perturbation theory. We assume \( \lambda_\alpha \lesssim 1 \) and that the scattering potential has a short-range character, which means that \( V(r) \) is nonzero only at \( kr_0 \lesssim 1 \). The starting point is equation (6) written for the T-matrix on a mass shell:
\[
T_{k^0}^{\alpha\gamma} = V_{k^0}^{\alpha\gamma} + \sum_{g, \gamma'} \frac{V_{kg}^{\alpha\gamma'} T_{k^0}^{\gamma'g}}{E - \varepsilon_{k^0}^g + i0},
\]  
(18)

where \( E \) is the electron energy, \( \varepsilon_{k^0}^g \) is the band spectrum given by equation (1), and \( V_{k^0}^{\alpha\gamma} \) is the matrix element of the scattering potential:
\[
V_{k^0}^{\alpha\gamma} = -g \left( -ie^i\chi_{\alpha\gamma} v_1(q) \right) \right) u(q) \),
\[
\delta T_{k^0}^{\alpha\gamma} = \sum_{g, \gamma'} \frac{V_{kg}^{\alpha\gamma'} V_{gk}^{\gamma'\alpha}}{E - \varepsilon_{kg}^\gamma + i0} = \mathcal{P} \sum_{g, \gamma'} \frac{V_{kg}^{\alpha\gamma'} V_{gk}^{\gamma'\alpha}}{E - \varepsilon_{kg}^\gamma}
\]
\[
\quad - i\pi \sum_{g, \gamma} \delta \left( E - \varepsilon_{kg}^\gamma \right) V_{kg}^{\alpha\gamma'} V_{gk}^{\gamma'\alpha}.
\]  
(23)

The first term in equation (23) is the correction to \( G_{k^0}^{\alpha\gamma} \) and we will neglect it (here \( \mathcal{P} \) stands for the principal value). The second term gives rise to \( \mathcal{J}_{k^0}^{\alpha\gamma} \) via the interference with the first Born approximation terms \( V_{k^0}^{\alpha\gamma} \) in the square modulus of
tering channel. Taking into account the explicit forms of $\zeta$ and $\epsilon$, we have that the coupling between imaginary part of the product ($V_{kk}^{\uparrow\uparrow}$) depends on whether the scattering potential is purely magnetic texture ($\chi_{\parallel}$) or it also contains an nonmagnetic impurity is positively or negatively charged. In this regard a magnetic spintron electrostatic environment can dramatically affect the symmetry properties of the topological Hall effect such as its dependence on a carrier spin polarization.

The spin-dependent structure of the asymmetric scattering can be understood based on the spin chirality arguments. In case of a pure magnetic texture the Hall response is driven by the scalar spin chirality $M_1 \cdot [M_2 \times M_3]$ [47–49] which is irrelevant to the electron spin state; naturally it leads to the spin-independent skew scattering [40]. On the contrary, when the scalar potential is present the scattering asymmetry can be induced due to the mixed product $S \cdot [M_2 \times M_3]$ [33] composing of both the magnetization vector spin chirality $[M_2 \times M_3]$ and the electron spin $S$, this mechanism thus gives rise to the spin Hall effect.

Let us further consider the case when only one (spin-up) subband is activated ($\Delta > 2E$). The second-order correction to the $T$-matrix relevant for the scattering asymmetry is written:

$$\delta T_{kk}^{\uparrow\uparrow} = -i\nu \int_0^{2\pi} d\varphi \cdot \nu_{\parallel}(E)V_{gg}^{\uparrow\downarrow} V_{kk}^{\downarrow\uparrow},$$

(28)

where $\nu_{\parallel}(E)$ is the density of states in the corresponding spin subband. Since the energy $E$ lies below the bottom of $\epsilon_{\parallel}(k)$ spectrum we have $\nu_{\parallel}(E) = 0$, which leads to the disappearance of $\delta T_{kk}^{\uparrow\uparrow}$ and of the asymmetric scattering $T_{kk}^{\uparrow\downarrow} = 0$ correspondingly. Therefore the topological Hall effect is strongly suppressed in the regime $2E < \Delta, kr_0 \lesssim 1$.

4. Numerical solution (methods)

In this section we present the scheme for the numerical calculations of $T$-matrix describing the electron scattering on potentials from equation (3). The exact solution of the scattering problem is especially important for the investigation of the spin motion crossover which occurs when one passes from a perturbative scattering to the classical motion. This section has mostly a methodological character as it provides an alternative platform for the numerical studies of THE. The readers mainly interested in physical properties of the scattering can go straight to section 5, where the results obtained by the numerical calculations are discussed in detail.

The considered potentials $V(r)$ have an important feature, namely the following commutator turns out to be zero:

$$[V(r), -i\partial_0 + \chi \hat{\sigma}_z/2] = 0.$$

(29)
The operator $-i\partial_\phi + \chi \sigma_z/2$ has the meaning of $z$-component of the total angular momentum. The existence of such an integral of motion allows us to separate the polar coordinates $r = (r, \phi)$ in the Schrödinger equation, which opens up a way towards the application of the phase theory of scattering. However, a specific angular structure of the associated eigenfunctions modifies the decomposition of $T$-matrix on its partial scattering parameters. The further consideration goes as follows. Firstly in section 4.2 we derive the expansion of $T$-matrix in terms of the eigenstates associated with $-i\partial_\phi + \chi \sigma_z/2$, and secondly in section 4.3 we adjust the phase-function method for the numerical calculations of the scattering parameter $s$. The analysis given in sections 4 and 5 is applicable for the case $E > \Delta/2$ when two spin subbands are activated. The single subband regime is considered separately in section 5.4.

### 4.1. Angular harmonics

The angular harmonics $\psi_m(r)$ corresponding to the operator $-i\partial_\phi + \chi \sigma_z/2$ are given by:

$$\psi_m(r, \phi) = e^{i m \phi} \psi_m(r),$$

where $m = 0, \pm 1, \pm 2, \ldots$ takes integer values, the functions $\psi_m$ depend only on $r$. One can naturally see that $\psi_m$ are the eigenfunctions of $-i\partial_\phi + \chi \sigma_z/2$ with the eigenvalue $m + \chi/2$. The functions $a_m(r), b_m(r)$ are determined by the explicit form of $\psi_{1,2}(r)$. The two-component function $g_m \equiv (a_m, b_m)^T$ satisfies the following matrix equation:

$$\mathcal{H}_m g_m(r) = -\omega_0 \hat{W}(r) g_m(r),$$

$$\mathcal{H}_m = \begin{pmatrix} \frac{1}{r} \partial_r r \partial_r - \frac{m^2}{r^2} + k_1^2 & 0 \\ 0 & \frac{1}{r} \partial_r r \partial_r - \frac{(m + \chi)^2}{r^2} + k_1^2, \end{pmatrix},$$

$$\hat{W}(r) = \begin{pmatrix} v_1(r) & u(r) \\ u(r) & v_2(r) \end{pmatrix},$$

where $\omega_0 = 2m_0 \omega/\hbar^2$ and we assume $E > \Delta/2$. Solving equation (31) gives us the relevant scattering parameters needed for the computation of $T$-matrix.

### 4.2. Decomposition of $T$-matrix

We note that the left part of equation (31) describes an electron free motion, so that away from the scattering potential $r > r_0$ the right side of equation (31) is absent and there are two independent cylindrical waves $g^{(1)}_m(r), g^{(2)}_m(r)$ given by:

$$g^{(1)}_m = \begin{pmatrix} J_m(k_1 r) - k_{11m} Y_m(k_1 r) \\ -k_{12m} Y_{m+\chi}(k_1 r) \end{pmatrix},$$

$$g^{(2)}_m = \begin{pmatrix} -k_{12m} Y_m(k_1 r) \\ J_{m+\chi}(k_1 r) - k_{22m} Y_{m+\chi}(k_1 r) \end{pmatrix},$$

where $J_m, Y_m$ are the Bessel functions of the first and second kind respectively, the matrices $K_m$ of constant coefficients $K^{ij}_m (i, j = 1, 2)$ are determined by $\hat{W}(r)$ profile at $r < r_0$. Our goal is to express $T$-matrix through $K_m$ coefficients.

Let us consider a wave function $\Psi(r)$ which satisfies the full equation (1) with energy $E$ and which has the following asymptotic form away from the scattering potential $r \gg r_0$:

$$\Psi(r, \phi) = \psi_{in} + \psi_{sc}, \quad \psi_{in} = \begin{pmatrix} e^{ik_1 r} u_1 \\ e^{ik_1 r} u_2 \end{pmatrix},$$

$$\psi_{sc}(r, \phi) = \frac{1}{\sqrt{r}} \begin{pmatrix} e^{i k'_{11} r} \left( f_{111} u_1 + f_{112} u_2 \right) \\ e^{i k'_{12} r} \left( f_{121} u_1 + f_{122} u_2 \right) \end{pmatrix},$$

where $r = (r, \phi)$ is the radius vector in the polar coordinates, the function $\psi_{in}$ describes the incident plane wave with energy $E$ and momentum direction $\mathbf{n}' = (\cos \varphi', \sin \varphi')$, the magnitude of the wavevector differs for two spin subbands $k'_i = (k'_i, \varphi')$, where $k'_i$ are given in equation (2), the polar angle $\varphi'$ corresponds to the direction on the incident flux; the coefficients $u_{1,2}$ determine the incident spin polarization of the electron $(|u_1|^2 + |u_2|^2) = 1$. The second term $\psi_{sc}$ corresponds to the outgoing cylindrical scattered wave, $f_{ij}(\varphi, \varphi')$ is the scattering amplitude; here $\varphi$ is regarded as the polar angle of the scattered plane wave described by the wavevectors $k_i = (k_i, \varphi)$ so that the scattering angle is defined as $\theta = \varphi - \varphi'$. The scattering amplitude $f_{ij}(\varphi, \varphi')$ is connected with $T$-matrix at the mass shell as [43]:

$$T_{kk'}^{ij} = \frac{\hbar^2}{m_0^2} \frac{2\pi k_k}{l} f_{ij}(\varphi, \varphi').$$

We further decompose $\Psi(r)$ over the set of the angular harmonics $\psi_m$ from equation (30):

$$\Psi(r, \phi) = \sum_m e^{-im\phi} \left( \mathcal{A}^{i1}_m \psi_i(r) + \mathcal{A}^{i2}_m \psi_{sc}(r) \right),$$

where $\mathcal{A}^{i1,2}_m$ are some coefficients, and the functions $\psi^{1,2}_i$ taken at $r > r_0$ correspond to two linearly independent solutions $g^{(1,2)}_m(r)$ given by equation (32). The divergent and convergent parts of the full $\Psi = \Psi^* + \Psi^-$ and the incident $\psi_{in} = \psi_{in}^* + \psi_{in}^-$ wave functions at $r > r_0$ are given by:

$$\psi_{in}^* = \frac{1}{2} \sum_m e^{im\phi} \left( u_1 H^*_m(k_1 r) + u_2 H^*_m(k_2 r) \right),$$

$$\psi_{in}^- = \sum_m e^{im\phi} \left[ \mathcal{A}^{1}_m \left( \begin{pmatrix} 1 \pm i K_{m1} \\ \mp i K_{m2} \end{pmatrix} H^*_m(k_{1r}) \right) \right. \left. + \mathcal{A}^{2}_m \left( \begin{pmatrix} \mp i K_{m1} \\ 1 \pm i K_{m2} \end{pmatrix} H^*_m(k_{2r}) \right) \right],$$

where $\bar{\gamma} = \gamma + \chi \varphi'$, and $H^*_m$ are the Hankel functions of the first and second kind respectively. The scattered wave $\psi_{sc} = \Psi - \psi_{in}^*$ does not contain a convergent part, which
brings us to the following system of equations on $A_m^{1,2}$:

$$
\begin{pmatrix}
1-ik_m^{11} & -ik_m^{12} \\
-ik_m^{21} & 1-ik_m^{22}
\end{pmatrix}
\begin{pmatrix}
\hat{A}_m^1 \\
\hat{A}_m^2
\end{pmatrix}
=
\begin{pmatrix}
u_1 \\
u_2 e^{i\gamma}
\end{pmatrix}, \quad \delta = \pi\chi/2 - \gamma.
$$

(37)

The solutions of these equations are given by: $(\hat{A}_m^1, \hat{A}_m^2)^T = (I - i\hat{K}_m)^{-1}(u_1, u_2 e^{i\gamma})^T$, where $I$ is the unit matrix $2 \times 2$. Let us mention the appearance of an additional phase factor $\delta$. The comparison of the scattered wave $\psi_{sc} = \Psi^+ - \psi_{in}^+$ in the asymptotic region for the Hankel function $H_m^\pm(x) \to (-i)^m e^{\mp i\pi/2}x^m$ with the expression for $\psi_{sc}$ containing $f_{\omega}(\varphi, \varphi')$ leads us to the following expression for the scattering amplitude $f_{\omega}(\varphi, \varphi')$:

$$
f_{\omega}(\varphi, \varphi') = \frac{1}{\sqrt{2\pi}} \sum_{m} \sum_{n} S_{mn}^{11} \left( e^{i\chi(n+\gamma')/2} - 1 \right) \left( e^{i\chi(n+\gamma')/2} + 1 \right) e^{i\chi(n+\gamma')/2},
$$

(38)

where $\gamma' = \gamma - \pi\chi/2$ and we introduced the partial $S_m$-matrices according to:

$$
\hat{S}_m = \left(I + ik_m^0\right) \cdot \left(I - ik_m^0\right)^{-1}.
$$

(39)

This type of coupling between $\hat{S}_m$ and $\hat{K}_m$ is common for multichannel scattering problems. Using the relation (34) we finally get the decomposition of $T$-matrix:

$$
T_{kk'}^{\omega} = \frac{1}{2\pi r_0} \sum_{m} \sum_{n} S_{mn}^{11} \left( e^{i\chi(n+\gamma')} - 1 \right) \left( e^{i\chi(n+\gamma')} + 1 \right) e^{i\chi(n+\gamma')},
$$

(40)

where the coefficients $S_{mn}$ are determined by a particular spatial profile $\hat{W}(r)$.

4.3. Phase-function method

In this section we describe the numerical method for the calculation of $\hat{S}_m$, $\hat{K}_m$ parameters entering in equations (40) and (39). The Schrödinger equation is of the second order thus it requires two boundary conditions. In order to eliminate the necessity to address the wave function asymptotics at $r \gg r_0$ one uses the so-called phase function method [50, 51], which replaces the second order equation (31) by the first order nonlinear Cauchy problem for a set of scattering parameters. The Cauchy problem can be further solved using the standard computational software. Here we provide step by step derivation of this method purposely, so that one could straightforwardly adjust the similar calculations for more complex band structures.

Let us write equation (31) in the following form:

$$
\left( \hat{H}_m^0 + i\hat{r}^{-1}\partial_r \hat{r} \partial_r \right) g_m(r) = -\omega_\chi\hat{W}(r)g_m(r),
$$

(41)

where $\hat{H}_m^0 = \text{diag} \left( k_m^1 - m^2/r^2, k_m^2 - (m + \chi)^2/r^2 \right)$ and $\hat{r}$ is the unit matrix $2 \times 2$. Following the textbook [50, 51] for a multichannel scattering we present the functions $g_m(r)$ as:

$$
g_m(r) = \left( J_m(r) - \hat{Y}_m(r)\hat{K}_m(r) \right) C_m(r),
$$

$$
J_m(r) = \begin{pmatrix} J_m(k_r) & 0 \\ 0 & J_m^{-1}(k_r) \end{pmatrix},
$$

(42)

$$
\hat{Y}_m(r) = \begin{pmatrix} Y_m(k_r) & 0 \\ 0 & Y_m+\chi(k_r) \end{pmatrix},
$$

where the $2 \times 2$ matrix $\hat{K}_m(r)$ and the two-component column $C_m(r)$ are some functions of the coordinate $r$. Outside the scattering region $r > r_0$ the functions $g_m(r)$ can be presented in form (32) with the matrix of $r$-dependent coefficients $\hat{K}_m$; at that the normalization column $C_m$ will describe the polarization structure of an electron state. In order to endow $\hat{K}_m(r)$ with the meaning of the real scattering parameters on the potential $\hat{W}(r)$ cut off at point $r < r_0$ one has to impose the additional condition for the derivative of $g_m$:

$$
\frac{dg_m}{dr} = \left( \frac{dJ_m}{dr} - \frac{dY_m}{dr} \hat{K}_m \right) C_m(r).
$$

(43)

The matrices $\hat{K}_m(r)$ satisfying both equations (41) and (43) and taken at the boundary point $\hat{K}_m(r_0)$ will correspond to the real scattering parameters of a potential $\hat{W}(r)$. The introduced functions $\hat{K}_m(r)$ are called the phase functions.

We further proceed with the derivation of the first order equation on $\hat{K}_m(r)$. Let us substitute $g_m(r)$ in equation (41) and take into account the condition (43). The term containing only the second derivatives of $g_m$ can be written as:

$$
\frac{1}{r} \frac{d}{dr} \left( J_m - Y_m \hat{K}_m \right) C_m = \left( J_m' + (\hat{r}^0 + \hat{r}^2) \hat{K}_m \right) C_m
$$

$$
+ \left( J_m - Y_m \hat{K}_m \right) \hat{C}_m = \left( J_m - Y_m \hat{K}_m \right) \hat{C}_m.
$$

(44)

The terms in the square brackets from above cancel out $\hat{H}_m^0 g_m$ term in equation (41), thus equation (41) does not contain the second derivatives:

$$
\left( J_m - Y_m \hat{K}_m \right) \frac{d\hat{C}_m}{dr} - \hat{Y}_m \frac{d\hat{K}_m \hat{C}_m}{dr} = -\omega_\chi \hat{W}(r) \left( J_m - Y_m \hat{K}_m \right) \hat{C}_m.
$$

(45)

The next step is to express $d\hat{C}_m/dr$ through $d\hat{K}_m/dr$. After multiplying this formula on $J_m - \hat{K}_m \hat{Y}_m$ we get for the left side of the equation:

$$
\left( J_m - \hat{K}_m \hat{Y}_m \right) \frac{d\hat{C}_m}{dr}
$$

$$
- \left( J_m - \hat{K}_m \hat{Y}_m \right) \hat{K}_m \frac{d\hat{K}_m \hat{C}_m}{dr} + \left[ \hat{K}_m, \hat{W}_m \right] \frac{d\hat{C}_m}{dr},
$$

(46)

where $\hat{W}_m = \hat{Y}_m J_m - J_m \hat{Y}_m$ is the Wronskian matrix of the Bessel functions and $\left[ \hat{K}_m, \hat{W}_m \right]$ is the commutator of two matrices. Using the condition (43) one can further express...
we demonstrate the computed scattering pattern starts narrowing into the forward direction. The asymmetric rates show a sin-like dependence on the scattering angle, the type of the scattering asymmetry is unique for all scattering channels, which is in full agreement with the results of equations (25) and (26). The magnitude of the asymmetric rates is significantly decreased compared to its value in the adiabatic limit; for the chosen parameters g/E = 0.2, r0 = 0.75 one has |J(kr0)| ∼ 10^{-6}, which indicates the strong influence of the (kr0)^8 factor. The middle panel demonstrates the crossover regime, here the scattering pattern starts narrowing into the forward direction and the asymmetric rates gradually lose a certain preferable direction. The right panel shows the scattering in the adiabatic regime. The spin-flip scattering channels are suppressed in this case. The small-angle scattering taking place for the spin-conserving channels is featured by the pronounced spin-dependent asymmetry, the latter indicates the presence of the spin-dependent magnetic fields due to the Berry phase, see section 3.1.

The symmetry crossover between the spin-dependent and the spin-independent Hull responses has been firstly discovered in [38]; the detailed discussion of its features can be found in [32]. Let us mention that to describe the behavior of the scattering rates in the crossover regime one necessarily has to address the exact solution of the scattering problem, at that the numerical scheme from section 4 is of special importance.

5.2. Adiabatic limit and the topology

In section 3.1 we demonstrated that when both the classical and the adiabatic conditions are fulfilled (kr0, λ ≫ 1) the total asymmetric rates J_{as} are quantized with the magnitude determined by the topological charge Q of a magnetization field. In this section we examine this effect based on the numerical calculations. We also discuss the deviations from the purely topological scenario and analyze the renormalization of the asymmetric scattering rates driven by the nonadiabaticity of the electron spin motion. We consider two purely magnetic potentials featured by the different topology of parental spin textures. We take χ = 1 and use the following spin texture profiles (the parameterization is introduced in equation (50), v_{1}(r) = -v_{2}(r) = n_{1}(r) - 1, \ u(r) = n_{1}(r), \ n_{1}^2(r) + n_{2}^2(r) = 1.

(50)

We parameterize the functions n_{z} = \cos(\Theta(r)), n_{\parallel} = \sin(\Theta(r)) using the azimuthal angle of the magnetization field \Theta(r). The spin splitting of the electron subbands \Delta = 2g. In the calculations shown below we take \chi = 1 and make use of the following skyrmion profile:

\Theta(r) = \pi \left(1 - \frac{r}{r_0}\right), \quad r < r_0.

(51)

In figure 3 we demonstrate the computed scattering pattern along with the \theta-dependence of the asymmetric rates J_{as}(\Theta) for different scattering regimes. The left panel corresponds to the weak-coupling regime. As we discussed in section 3.2, the scattering within the spin-conserving channels indeed has an isotropic-like character, while the spin-flip channels are characterized by the suppression of the forward scattering. The asymmetric rates show a sin-like dependence on the scattering angle, the type of the scattering asymmetry is unique for all scattering channels, which is in full agreement with the results of equations (25) and (26). The magnitude of the asymmetric rates is significantly decreased compared to its value in the adiabatic limit; for the chosen parameters g/E = 0.2, r0 = 0.75 one has |J(kr0)| ∼ 10^{-6}, which indicates the strong influence of the (kr0)^8 factor. The middle panel demonstrates the crossover regime, here the scattering pattern starts narrowing into the forward direction and the asymmetric rates gradually lose a certain preferable direction. The right panel shows the scattering in the adiabatic regime. The spin-flip scattering channels are suppressed in this case. The small-angle scattering taking place for the spin-conserving channels is featured by the pronounced spin-dependent asymmetry, the latter indicates the presence of the spin-dependent magnetic fields due to the Berry phase, see section 3.1.

The symmetry crossover between the spin-dependent and the spin-independent Hull responses has been firstly discovered in [38]; the detailed discussion of its features can be found in [32]. Let us mention that to describe the behavior of the scattering rates in the crossover regime one necessarily has to address the exact solution of the scattering problem, at that the numerical scheme from section 4 is of special importance.

5.2. Adiabatic limit and the topology

In section 3.1 we demonstrated that when both the classical and the adiabatic conditions are fulfilled (kr0, λ ≫ 1) the total asymmetric rates J_{as} are quantized with the magnitude determined by the topological charge Q of a magnetization field. In this section we examine this effect based on the numerical calculations. We also discuss the deviations from the purely topological scenario and analyze the renormalization of the asymmetric scattering rates driven by the nonadiabaticity of the electron spin motion. We consider two purely magnetic potentials featured by the different topology of parental spin textures. We take χ = 1 and use the following spin texture profiles (the parameterization is introduced in equation (50), v_{1}(r) = -v_{2}(r) = n_{1}(r) - 1, \ u(r) = n_{1}(r), \ n_{1}^2(r) + n_{2}^2(r) = 1.

(50)

We parameterize the functions n_{z} = \cos(\Theta(r)), n_{\parallel} = \sin(\Theta(r)) using the azimuthal angle of the magnetization field \Theta(r). The spin splitting of the electron subbands \Delta = 2g. In the calculations shown below we take \chi = 1 and make use of the following skyrmion profile:

\Theta(r) = \pi \left(1 - \frac{r}{r_0}\right), \quad r < r_0.

(51)

In figure 3 we demonstrate the computed scattering pattern along with the \theta-dependence of the asymmetric rates J_{as}(\Theta) for different scattering regimes. The left panel corresponds to the weak-coupling regime. As we discussed in section 3.2, the scattering within the spin-conserving channels indeed has an isotropic-like character, while the spin-flip channels are characterized by the suppression of the forward scattering. The asymmetric rates show a sin-like dependence on the scattering angle, the type of the scattering asymmetry is unique for all scattering channels, which is in full agreement with the results of equations (25) and (26). The magnitude of the asymmetric rates is significantly decreased compared to its value in the adiabatic limit; for the chosen parameters g/E = 0.2, r0 = 0.75 one has |J(kr0)| ∼ 10^{-6}, which indicates the strong influence of the (kr0)^8 factor. The middle panel demonstrates the crossover regime, here the scattering pattern starts narrowing into the forward direction and the asymmetric rates gradually lose a certain preferable direction. The right panel shows the scattering in the adiabatic regime. The spin-flip scattering channels are suppressed in this case. The small-angle scattering taking place for the spin-conserving channels is featured by the pronounced spin-dependent asymmetry, the latter indicates the presence of the spin-dependent magnetic fields due to the Berry phase, see section 3.1.

The symmetry crossover between the spin-dependent and the spin-independent Hull responses has been firstly discovered in [38]; the detailed discussion of its features can be found in [32]. Let us mention that to describe the behavior of the scattering rates in the crossover regime one necessarily has to address the exact solution of the scattering problem, at that the numerical scheme from section 4 is of special importance.

5.2. Adiabatic limit and the topology

In section 3.1 we demonstrated that when both the classical and the adiabatic conditions are fulfilled (kr0, λ ≫ 1) the total asymmetric rates J_{as} are quantized with the magnitude determined by the topological charge Q of a magnetization field. In this section we examine this effect based on the numerical calculations. We also discuss the deviations from the purely topological scenario and analyze the renormalization of the asymmetric scattering rates driven by the nonadiabaticity of the electron spin motion. We consider two purely magnetic potentials featured by the different topology of parental spin textures. We take χ = 1 and use the following spin texture profiles (the parameterization is introduced in equation (50), v_{1}(r) = -v_{2}(r) = n_{1}(r) - 1, \ u(r) = n_{1}(r), \ n_{1}^2(r) + n_{2}^2(r) = 1.

(50)

We parameterize the functions n_{z} = \cos(\Theta(r)), n_{\parallel} = \sin(\Theta(r)) using the azimuthal angle of the magnetization field \Theta(r). The spin splitting of the electron subbands \Delta = 2g. In the calculations shown below we take \chi = 1 and make use of the following skyrmion profile:

\Theta(r) = \pi \left(1 - \frac{r}{r_0}\right), \quad r < r_0.

(51)

In figure 3 we demonstrate the computed scattering pattern along with the \theta-dependence of the asymmetric rates J_{as}(\Theta) for different scattering regimes. The left panel corresponds to the weak-coupling regime. As we discussed in section 3.2, the scattering within the spin-conserving channels indeed has an isotropic-like character, while the spin-flip channels are characterized by the suppression of the forward scattering. The asymmetric rates show a sin-like dependence on the scattering angle, the type of the scattering asymmetry is unique for all scattering channels, which is in full agreement with the results of equations (25) and (26). The magnitude of the asymmetric rates is significantly decreased compared to its value in the adiabatic limit; for the chosen parameters g/E = 0.2, r0 = 0.75 one has |J(kr0)| ∼ 10^{-6}, which indicates the strong influence of the (kr0)^8 factor. The middle panel demonstrates the crossover regime, here the scattering pattern starts narrowing into the forward direction and the asymmetric rates gradually lose a certain preferable direction. The right panel shows the scattering in the adiabatic regime. The spin-flip scattering channels are suppressed in this case. The small-angle scattering taking place for the spin-conserving channels is featured by the pronounced spin-dependent asymmetry, the latter indicates the presence of the spin-dependent magnetic fields due to the Berry phase, see section 3.1.

The symmetry crossover between the spin-dependent and the spin-independent Hull responses has been firstly discovered in [38]; the detailed discussion of its features can be found in [32]. Let us mention that to describe the behavior of the scattering rates in the crossover regime one necessarily has to address the exact solution of the scattering problem, at that the numerical scheme from section 4 is of special importance.
Figure 3. The scattering pattern and the asymmetric scattering rates $J_{ss}(\theta) = J_{ss}^{\lambda \lambda}$. The dependence of the asymmetric $J_{ss}$ rates on the potential radius in units $kr_0$ for $Q = 1$ (solid lines) and $Q = 0$ (dashed lines) spin configurations, the ratio $g/E = 0.5$.

$\Delta = 2g$:

(Q = 1) $\Theta(r) = \pi \left( 1 - \frac{r}{r_0} \right)$, $r < r_0$;

(Q = 0) $\Theta(r) = 2\pi \frac{r}{r_0} \left( 1 - \frac{r}{r_0} \right)$, $r < r_0$.

In figure 4(b) we demonstrate the computed dependence of $J_{ss}$ for the spin-conserving channels on the spin texture radius when entering into the classical and adiabatic limits ($g/E = 0.5$ is fixed). This figure shows that there is a saturation of $J_{ss}$ when $kr_0$ increases, the limiting magnitude of $|J_{ss}|$ approaches $Q = 1$ for the skyrmion configuration (solid lines) and goes down to zero for the topologically uncharged texture $Q = 0$ (dashed lines). This is a clear manifestation of the topological features discussed in section 3.1, namely the asymmetric rates approach the limiting values $J_{\uparrow \downarrow} = -J_{\downarrow \uparrow} = Q$ determined by the topology; $J_{ss}$ thus become independent of a particular magnetization profile and its spatial size.

Let us further discuss the modification of the asymmetric scattering rates when going away from the adiabatic limit. In figure 4(a) we show the evaluated dependence of $J_{ss}$ on the spin texture radius for the crossover region, i.e. when $\lambda_a \gtrsim 1$. As the limiting value of so introduced dimensionless functions $J_{ss}$ (see equations (8) and (9)) is fixed $|J_{ss}| \to 1$, the computed magnitude of $J_{ss}$ shown in figure 4 explicitly demonstrates the renormalization of the transverse current magnitude compared with the topological estimations. In particular, based on figure 4 we conclude that the topological Hall effect experiences the suppression to one order of magnitude over a wide range of texture sizes, namely from $kr_0 = 10$ to $kr_0 = 3$. Additionally, it is well seen from figure 4(a) that the topological charge ceases to affect the magnitude of $J_{ss}$ as we enter into the crossover regime. Indeed, the asymmetric rates for the electron scattering on $Q = (0, 1)$ spin textures are not only close in
values but also show quite similar dependence on the texture radius.

5.3. Scattering on electrically charged skyrmions

In this section we study the electron asymmetric scattering in case when a scalar potential is present additionally to a magnetic skyrmion. We make use of the following parameterization for the diagonal elements $v_{1,2}$:

$$v_1(r) = n_e(r) - 1 + \delta U(r), \quad v_2(r) = 1 - n_e(r) + \delta U(r),$$  \hspace{1cm} (53)

the off-diagonal component $u(r) = n_e(r) = \sqrt{1 - n_e^2(r)}$ remains determined by $n_e(r)$, see equation (50). The subband splitting $\Delta = 2g$. For the numerical calculations we use the skyrmion profile from equation (51) and the following form of the potential $\delta U(r) = U_0 \, e^{-|r|/R^2}$ with $R$ being its localization radius.

In figure 5 we demonstrate the scattering pattern and the $\theta$-dependence of $J_{kk}^{\parallel}$ for different skyrmion radii. In this plot only $r_0$ is varied, other parameters such as the localization radius $R$, scattering energy $E$, $g$ and $U_0$ remain unchanged. The left panel of figure 5 corresponds to the perturbative scattering regime, here the scalar potential and the skyrmion are close in size. It is seen from the left panel that $J_{kk}^{\parallel\parallel}$, $J_{kk}^{\parallel\perp}$ have different signs, which is consistent with the results of the weak coupling theory section 3.2 and equation (27) predicting that the non-magnetic component of $V(r)$ restores the spin Hall effect and suppresses the spin-independent scattering caused by a pure magnetic texture. The right panel of figure 5 corresponds to the adiabatic regime with respect to the skyrmion size; here $\delta U(r)$ is kept localized $(r_0/R = 31)$ so its length $R$ is small compared to $r_0$. The scattering rates in this regime have a similar structure to that in case of a pure magnetic skyrmion (see figure 3). Therefore in case of a large skyrmion the short-range scalar perturbation does not affect significantly the asymmetric scattering, the latter is mainly produced during a lingering electron motion in the skyrmion texture surrounding $\delta U(r)$, at that the general arguments given in section 3.1 for the classical regime remain applicable. The middle panel in figure 5 corresponds to the intermediate case, here both the scattering pattern and $J_{kk}^{\parallel}$ are strongly influenced by the presence of $\delta U(r)$. We note that the scattering asymmetry in the weak coupling regime depends on the sign of $U_0$. Data shown in figure 5 is obtained for the positive value $U_0 = 30$, at that the scattering asymmetry for the spin up and the spin down states shown in the left panel differs from that in the classical limit. Increasing $r_0$ drives the scattering channels to switch their asymmetry, at that a complex scattering pattern at the intermediate region is indeed expected.

5.4. One spin subband regime

In this section we consider the case when $E < \Delta/2$ and only the spin up subband is available for a free motion (see figure 1). Firstly we adjust the phase function method for this regime and secondly we discuss some numerical results.

The following matrix equations on $g_m(r)$ functions should be used instead of equation (31):

$$\mathcal{H}_m^r g_m(r) = -\omega_i \mathcal{W}(r) g_m(r),$$

$$\mathcal{H}_m^r = \begin{pmatrix} 1/r \partial_r r \partial_r - \frac{m^2}{r^2} + \lambda_i^2 & 0 \\ 0 & \frac{1/r \partial_r r \partial_r - (m + \chi)^2}{r^2} - x^2 \end{pmatrix},$$

$$\mathcal{W}(r) = \begin{pmatrix} v_1(r) & u(r) \\ u(r) & v_2(r) \end{pmatrix},$$  \hspace{1cm} (54)

where we introduced the real parameter $h_{\mathcal{S}} = \sqrt{2m_0(\Delta/2 - E)}$. Two independent solutions $g_m^1, g_m^2$ of equation (54) away from the scattering potential $r > r_0$ are given by:

$$g_m^1 = \frac{I_m(kr) - K_m^{11} Y_m(kr)}{-K_m^{11} K_m^{\pm 11}(xr)},$$

$$g_m^2 = \frac{-K_m^{12} Y_m(kr)}{I_m + K_m^{11} K_m^{\pm 11}(xr)},$$  \hspace{1cm} (55)

where $I_m, K_m$ are modified Bessel functions of the first and second kind respectively. Among all $K_m^{ij}$ coefficients only $K_m^{11}$ remains relevant for the scattering properties.

Indeed, the asymptotic form of the propagating wavefunction at $r \gg r_0$ contains only spin-up state:

$$\Psi(r, \varphi) = \left( e^{ikr} + e^{ikr} \frac{f(\varphi, \varphi')}{\sqrt{r}} \right) |\uparrow\rangle. \hspace{1cm} (56)$$

Since $I_m$ entering in $|\downarrow\rangle$ state diverges at $r \gg r_0$ the expansion of $\Psi'$ over the partial harmonics $g_m^{12}$ cannot contain the admixture of $g_m^1$ functions. Therefore the scattering amplitude is given by a conventional single-channel decomposition:

$$f(\varphi, \varphi') = \frac{1}{\sqrt{2\pi i k_r}} \sum_m e^{im(\varphi - \varphi')}(S_m - 1), \quad S_m = \frac{1 + iK_m}{1 - iK_m}. \hspace{1cm} (57)$$

The phase-function method is adjusted for the one spin subband case using equations (54) and (55). We introduce the phase functions $\tilde{K}_m(r)$ according to the following notation:

$$g_m(r) = \left( \tilde{Q}_m - Z_m \tilde{K}_m \right) C_m(r),$$

$$\frac{dg_m}{dr} = \left( \frac{d\tilde{Q}_m}{dr} - \frac{dZ_m}{dr} \tilde{K}_m \right) C_m(r),$$

$$\tilde{Q}_m(r) = \begin{pmatrix} I_m(kr) & 0 \\ 0 & -\sqrt{2\pi} K_m^{\pm 11}(xr) \end{pmatrix},$$

$$\tilde{Z}_m(r) = \begin{pmatrix} Y_m(kr) & 0 \\ 0 & -\sqrt{2\pi} K_m^{11}(xr) \end{pmatrix},$$  \hspace{1cm} (58)

where the normalization constant $\sqrt{2/\pi}$ is introduced to make the Wronskian $\tilde{Q}_m \tilde{Z}_m - \tilde{Q}_m \tilde{Z}_m = I \times 2/\pi r$ proportional to the unity matrix. Using the functions $g_m$ from equation (58)
Figure 5. The scattering pattern and the asymmetric scattering rates $J_{ss}(\theta) \equiv J_{kk}^{ss}$ in case of an additional short-range scalar potential. The parameters: $U_0 = 30$, $kR = 0.45$, $g/E = 0.3$.

![Graphs showing scattering patterns](image)

and making the similar transformations of equation (54) as were described in section 4.3 we get the following equations for $K_m(r)$ and $S_m(r) = (I + iK_m) \cdot (I + i\tilde{K}_m)^{-1}$ matrix functions:

$$
\frac{dK_m}{dr} = \frac{\pi r}{2} \omega_0 \left( \hat{Q}_m - \tilde{K}_m \hat{Z}_m \right) W(r) \left( \hat{Q}_m - \hat{Z}_m \hat{K}_m \right),
$$

$$
\frac{dS_m}{dr} = \frac{i\pi r}{4} \omega_0 \left( \hat{Q}_m - i\hat{Z}_m + \hat{S}_m \cdot (\hat{Q}_m + i\hat{Z}_m) \right) \times W(r) \left( \hat{Q}_m - i\hat{Z}_m + (\hat{Q}_m + i\hat{Z}_m) \cdot \hat{S}_m \right).
$$

(59)

Let us mention that to calculate the scattering amplitude only $\tilde{K}_m \equiv K_m^T$ element of the whole matrix is needed.

We further apply this technique to study the scattering on a purely magnetic potential with the parameterization presented in equation (50). In figure 6 we demonstrate the dependence of the total asymmetric rate $J_{\uparrow\uparrow}$ on the magnetic texture radius $r_0$ for two topologically different configurations (in figure 6 the absolute value $|J_{\uparrow\uparrow}|$ is shown; $J_{\uparrow\uparrow}$ remains negative in accordance with the previous considerations). Here we take $\chi = 1$ and use the spin texture profiles from equation (52), the subband spin splitting $\Delta = 2g$. As follows from our calculations $J_{\uparrow\uparrow}$ is strongly suppressed at small $kr_0 \lesssim 1$. This fact has been already mentioned in section 3.2 when considering the perturbative region: it is due to the vanishing of the third-order correlator in equation (28). We also notice that the asymptotic behavior of $J_{\uparrow\uparrow}$ at $k/r_0 \gg 1$ is similar to that observed for two opened spin subbands (see section 5.2 and figure 4). Namely, the asymmetric rate $J_{\uparrow\uparrow} \rightarrow -1$ saturates for the topologically charged configuration $Q = 1$ while going to zero for the uncharged one $Q = 0$. This result stems from the Berry phase description section 3.1 valid for each subband independently. It is worth mentioning, however, that in the intermediate region when neither perturbative theory nor the Berry phase approach are valid the asymmetric scattering generally persists independently of a spin texture topology. As follows from figure 6 in the range of $5 \lessapprox 2k_r0 \lessapprox 10$ the scattering rate $J_{\uparrow\uparrow}$ has approximately the same magnitude for $Q = 0, 1$ configurations.

6. Conclusions

To summarize, we have considered different processes that have an impact on the asymmetric electron scattering on a skyrmion-like magnetic texture. We have analytically demonstrated that the magnitude of the transverse electron current
is fully determined by the topological charge of a magnetic texture only if the electron orbital and spin motions have a classical character. Beyond this condition the electron scattering pattern is significantly modified. In particular, we have shown that in the weak coupling regime the asymmetric scattering rates have the same angular structure independently of a scattering potential profile. Moreover, the obtained analytical expressions for the scattering rates in the weak coupling regimes demonstrate the fast suppression of the asymmetric rate magnitude with the decrease in a texture size. We have also analyzed the behavior of the scattering pattern in case of the electron scattering on an electrically charged skyrmion. We argue that the presence of a short-range impurity is mostly important in the nonadiabatic regime. At that the charge transverse response due to magnetic texture might be superseded by the spin Hall effect and the magnitude of the charge transverse response due to magnetic texture might be.

Acknowledgments

The author thanks I.V. Rozhansky and N.S. Averkiev for helpful and fruitful discussions. The analytical theory and the investigation of the asymmetric scattering in the presence of a scalar impurity were supported by the Russian Science Foundation (Project 18-72-10111). The derivation of numerical methods and the investigation of a single spin-subband regime were supported by the Russian Foundation of Basic Research (Grant 19-52-12066). KSD thanks the Foundation for the Advancement of Theoretical Physics and Mathematics “BASIS”.

ORCID iDs

K S Denisov https://orcid.org/0000-0002-1477-3667

References

[1] Nagaosa N and Tokura Y 2013 Nat. Nanotechnol. 8 899
[2] Yu X, Koshibae W, Tokunaga Y, Shibata K, Taguchi Y, Nagaosa N and Tokura Y 2018 Nature 564 95
[3] Fert A, Reyren N and Cros V 2017 Nat. Rev. Mater. 2 17031
[4] Wiesendanger R 2016 Nat. Rev. Mater. 1 16044
[5] Sundaram G and Niu Q 1999 Phys. Rev. B 59 14915
[6] Franz C et al 2014 Phys. Rev. Lett. 112 186601
[7] Buhl P M, Freimuth F, Blügel S and Mokrousov Y 2017 Phys. Status Solidi Rapid Res. Lett. 11 1700007
[8] Neubauer A, Pfleiderer C, Binz B, Rosch A, Ritz R, Niklowitz P G and Böni P 2009 Phys. Rev. Lett. 102 186602
[9] Leroux M, Stolt M, Jin S, Pete D, Reichhardt C and Maiorov B 2018 Sci. Rep. 8 15510
[10] Kazanazwa N, Onos Y, Arima T, Okuyama D, Ohoyama K, Wakimoto S, Kakurai K, Ishiwata S and Tokura Y 2011 Phys. Rev. Lett. 106 156603
[11] Spencer C et al 2018 Phys. Rev. B 97 214406
[12] Bruno P, Dugaev V K and Taillefumier M 2004 Phys. Rev. Lett. 93 096806
[13] Zeissler K et al 2018 Nat. Nanotechnol. 13 1161
[14] Kazanazwa N, Kubota M, Tsuzukza A, Kozuka Y, Takahashi K S, Kawasaki M, Ichikawa M, Kagawa F and Tokura Y 2015 Phys. Rev. B 91 041122
[15] Maccariello D, Legrand W, Reyren N, Garcia K, Bouzehouane K, Collins C, Sos V and Fert A 2018 Nat. Nanotechnol. 13 233–7
[16] Raju M, Yagil A, Soumyanarayanan A, Tan A, Almoalem A, Ma F, Auslaender O and Panagopoulos C 2019 Nat. Commun. 10 966
[17] Ohuchi Y, Kozuka Y, Uchida M, Ueno K, Tsuzukza A and Kawasaki M 2015 Phys. Rev. B 91 245115
[18] Yun Y, Ma Y, Su T, Xing W, Chen Y, Yao Y, Cai R, Yuan W and Han W 2018 Phys. Rev. Mater. 2 034201
[19] Liu C, Zang Y, Ruan W, Gong Y, He K, Ma X, Xue Q-K and Wang Y 2017 Phys. Rev. Lett. 119 176809
[20] Karube K et al 2018 Sci. Adv. 4 eaar7043
[21] Wang L et al 2018 Nat. Mater. 17 1087–94
[22] Zhang X, Zhou Y, Ezawa M, Zhao G and Zhao W 2015 Sci. Rep. 5 11369
[23] Moreau-Lhuishe C et al 2016 Nat. Nanotechnol. 11 444
[24] Legrand W et al 2017 Nano Lett. 17 2703
[25] Soumyanarayanan A et al 2017 Nat. Mater. 16 898
[26] Romming N, Hanneken C, Menzel M, Bickel J, Wolter B, von Bergmann K, Kubetzka A and Wiesendanger R 2013 Science 341 636
[27] Romming N, Kubetzka A, Hanneken C, von Bergmann K and Wiesendanger R 2015 Phys. Rev. Lett. 114 177203
[28] Meyer S, Perini M, von Malottki S, Kubetzka A, Wiesendanger R, von Bergmann K and Heinz S 2019 Nat. Commun. 10 3823
[29] Ye J, Kim Y, Millis A J, Shraiman B I, Majumdar P and Telnanovic Z 1999 Phys. Rev. Lett. 83 3737
[30] Nakazawa K, Bibes M and Kohno H 2018 J. Phys. Soc. Japan 87 033705
[31] Nakazawa K and Kohno H 2019 Phys. Rev. B 99 174425
[32] Denisov K S, Rozhansky I V, Averkiev N S and Lăhderanta E 2018 Phys. Rev. B 98 195439
[33] Ishizuka H and Nagaosa N 2018 New J. Phys. 20 123027
[34] Ishizuka H and Nagaosa N 2018 Sci. Adv. 4 eaap9962
[35] Ndiaye P B, Akosa C A and Mancho A 2017 Phys. Rev. B 95 064426
[36] Akosa C A, Li H, Tatar G and Tretiakov O A 2019 Appl. Phys. Rev. 12 054032
[37] Ohe J-L, Ohtsuki T and Kramer B 2007 Phys. Rev. B 75 245313
[38] Denisov K S, Rozhansky I V, Averkiev N S and Lăhderanta E 2017 Sci. Rep. 7 17204
[39] Rozhansky I V, Denisov K S, Lifshits M, Averkiev N S and Lăhderanta E 2019 Phys. Status Solidi b 20 1900033
[40] Denisov K S, Rozhansky I V, Averkiev N S and Lăhderanta E 2016 Phys. Rev. Lett. 117 027202
[41] Araki Y and Nomura K 2017 Phys. Rev. B 96 165303
[42] Wang C-Z, Xu H-Y and Lai Y-C 2020 Phys. Rev. B 103 013705
[43] Adhikari S K 1986 Am. J. Phys. 54 362
[44] Berry M V 1984 Proc. R. Soc. 392 45
[45] Aharonov Y and Stern A 1992 Phys. Rev. Lett. 69 3593
[46] Zhang S-S and Heinonen O 2018 Phys. Rev. B 97 134401
[47] Tatar G and Kawamura H 2002 J. Phys. Soc. Japan 71 2613
[48] Lazuta A, Maleyev S and Toperverg B 1978 Phys. Lett. A 65 348
[49] Udalov O G and Fracarman A 2014 Phys. Rev. B 90 064202
[50] Babikov V V 1976 Phase-Function Method in Quantum Mechanics (Moscow: Nauka)
[51] Babikov V V 1967 Usp. Fiz. Nauk 92 3