Family of potentials with power law kink tails

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Abstract

We provide examples of a large class of one-dimensional higher-order field theories with kink solutions which asymptotically have a power law tail either at one end or at both ends. In particular, we provide examples of a family of potentials which admit a kink as well as a mirror kink solution where all four ends of the two kinks, or only two extreme ends of the two kinks have a power law tail (while the two ends facing each other have an exponential tail). The remaining case includes the situation when the ends facing each other have a power law tail while the two extreme ends of the two kinks have an exponential tail. Further, we show that for a kink with a power law tail at either one end or at both ends, there is no gap between the zero mode and the continuum of the corresponding stability equation. This is in contrast with the kinks with an exponential tail at both ends, in which case there is always a gap between the zero mode and the continuum.

Keywords: high-order field theories, solitons, kink stability, asymptotics, successive phase transitions

1. Introduction

Recently it was found that certain higher-order field theories admit kink solutions with a power law tail either at both ends or with a power law tail at one end and an exponential tail at the other [1, 2]. An example of the latter is the octic potential studied in the context of massless mesons [3]. This is in contrast with almost all the kink solutions that have been discussed in the last four decades where the kink solutions have an exponential tail at both ends [4, 5], the prototype being the celebrated $\phi^4$ kink. The discovery of these power law kinks [6–9] has raised several interesting questions such as the strength and range of the kink–kink (KK) and kink–antikink (K–AK) force [10, 11], the possibility of resonances [12] and scattering [13],
stability analysis of such kinks [6], etc. From this perspective, it is worth noting that Manton’s celebrated method [4, 5] provides the answer for both the strength and the range of the KK and K–AK interactions in cases where they have an exponential tail at both ends.

The study of higher-order field theories, their attendant kink excitations as well as the associated kink interactions and scattering are important in a variety of physical contexts ranging from successive phase transitions [1, 2, 14, 15] to isostructural phase transitions [16] to models involving long-range interaction between massless mesons [3], as well as from protein crystallization [17] to successive phase transitions presumably driving the late time expansion of the Universe [18]. Thus, understanding kink behaviour in these models provides a useful insight into the properties of domain walls in materials, condensed matter, high-energy physics, biology and cosmology, which also serves as one of our main motivations here.

Recently several attempts have been made [10–12] to understand the nature of the KK and K–AK interaction in a $\phi^8$ model [3], which admits a kink and a mirror kink (and corresponding antikinks) which have exponential tails at the two extreme ends and power law tails at the two adjoining ends. Very recently, following Manton’s suggestion [11], the KK and K–AK long-range force was calculated analytically for a one-parameter family of potentials and then compared with detailed numerical simulations [19].

These studies raise several issues. For example, what is the force between a kink and the corresponding mirror kink in cases where both fall off with a power law tail at the two extreme ends, while the tails of the two adjoining kinks have exponential tails? Secondly, what is the force between a kink and the corresponding mirror kink in cases where there is a power law tail at all four ends? Besides this, is there a conceptual difference in the stability analysis when there is an exponential tail at both ends compared to when there is a power law tail either at both ends or at one end?

To facilitate such studies, in this paper we obtain kink solutions in a one-parameter family of potentials for which there is a power law tail at both ends. The kink stability analysis for all these cases shows that the zero mode and the continuum start at the same energy (i.e. zero), that is, there is no gap between the zero mode and the continuum. This is in contrast to kinks with exponential tails at both ends where there is always a gap between the zero mode and where the continuum begins. As an illustration, in appendix A we consider a one-parameter family of potentials which admit a kink solution with an exponential tail at both ends. The stability analysis for these kink solutions reveals the existence of a gap between the zero mode and the beginning of the continuum. Besides this, we construct a one-parameter family of kink and corresponding mirror kink solutions in models such that (i) the kink (and hence the mirror kink) has a power law tail at both ends, (ii) the kink (and hence the mirror kink) has a power law tail at the two extreme ends and an exponential tail at the two adjacent ends, and (iii) the two kinks have an exponential tail at the two extreme ends while they have a power law tail at the two adjacent ends. Unfortunately, in all these cases we are only able to obtain the kink solution in an implicit form; however, we are able to obtain the relevant asymptotic tails.

The plan of the paper is as follows. In section 2 we set up the notations and illustrate when one can have a kink solution with a power law tail and when one can have a kink solution with an exponential tail. Besides this, we consider the case of two adjoining kinks and point out the various possible forms for the kink tails in the two adjoining kink solutions. In section 3 we present a one-parameter family of potentials of the form $(\alpha^2 - \phi^2)^{2n+2}, n = 1, 2, 3...,$ all of which admit a kink solution from $-\alpha$ to $+\alpha$ with a power law tail at both ends. Unfortunately, all these kink solutions are expressed in an implicit form. The stability analysis of these kink solutions is performed in section 3.5 where we show that there is no gap between the zero mode and the continuum. In rest of the paper we construct a one-parameter family of potentials
corresponding to the various possible forms for the two adjoining kinks. In section 4 we present a one-parameter family of potentials of the form \( \phi^{2n+2}(a^2 - \phi^2)^2, n = 1, 2, 3, ... \), all of which admit a kink solution from 0 to \( a \) and a mirror kink from \(-a\) to 0 (and the corresponding anti-kinks). For all these kink solutions, while around \( \phi = 0 \) one has a power law tail, around \( \phi = a \) one has only an exponential tail. In section 5 we present a one-parameter family of potentials of the form \( \phi^2(a^2 - \phi^2)^{2n+2}, n = 1, 2, 3, ... \), which admit a kink solution from 0 to \( a \) and a mirror kink from \(-a\) to 0 (and the corresponding anti-kinks), and for which while around \( \phi = 0 \) one has an exponential tail, around \( \phi = a \) one has a power law tail. Finally, in section 6 we present a one-parameter family of potentials of the form \( \phi^{2n+2}(a^2 - \phi^2)^4, n = 2, 3, ... \), which admit a kink solution from 0 to \( a \) and a mirror kink from \(-a\) to 0 (and the corresponding anti-kinks). For these kinks, there is a power law tail around both \( \phi = 0 \) as well as \( \phi = a \) such that the power law tail around \( \phi = 0 \) has a slower fall-off (i.e. it goes like \( x^{-1/n} \) as \( x \to \pm \infty \) where \( n = 2, 3, 4, ... \)) compared to the power law tail around \( \phi = a \) (which always goes like \( x^{-1} \) as \( x \to \infty \)).

In appendix A we present a one-parameter family of potentials of the form \( \phi^2(\alpha^2 - \phi^2)^2 \) with \( n = 1, 2, 3, ... \), which admits a kink solution from 0 to \( a \) and a mirror kink from \(-a\) to 0, both of which have an exponential tail at both ends. For the entire family, we obtain the kink solutions in an explicit form, perform the linear stability analysis and show that there is always a gap between the zero mode and the continuum. In appendix B, for completeness we present a one-parameter family of potentials of the form \( \phi^2(\alpha^2 - \phi^2)^{2n+2}, n = 1, 2, 3, ... \), all of which admit a kink solution from \(-a\) to \(+a\) with a power law tail at both ends. In the special case of \( n = 1 \) we obtain the kink solution analytically. Using this kink solution we then obtain the zero mode and also carry out linear stability analysis and show explicitly that there is no gap between the zero mode and the continuum. In appendices C to E we provide details of the calculation performed in sections 4–6, respectively. For completeness, in appendix F we present a one-parameter family of potentials of the form \( (a^2 - \phi^2)^2(b^2 - \phi^2)^{2n+2} \) which admit three kink solutions. Two of these, i.e. from \( a \) to \( b \) (and a mirror kink from \(-b\) to \(-a\)) have a power law tail at one end and an exponential tail at the other end while the kink from \(-a\) to \( a \) has an exponential tail at both ends. Finally, in appendix G we present a one-parameter family of potentials of the form \( (a^2 - \phi^2)^{2n+2}(b^2 - \phi^2)^2 \) which admit three kink solutions. Two of these, i.e. from \( a \) to \( b \) (and a mirror kink from \(-b\) to \(-a\)) have an exponential tail at one end and a power law tail at the other, while the kink from \(-a\) to \( a \) has a power law tail at both ends.

2. Formalism

Consider a relativistic neutral scalar field theory in 1 + 1 dimensions with the Lagrangian density

\[
\mathcal{L} = \frac{1}{2} \left( \frac{\partial \phi}{\partial t} \right)^2 - \frac{1}{2} \left( \frac{\partial \phi}{\partial x} \right)^2 - V(\phi),
\]

which leads to the equation of motion
\[
\left( \frac{\partial^2 \phi}{\partial t^2} \right) - \left( \frac{\partial^2 \phi}{\partial x^2} \right) = -\frac{dV}{d\phi}.
\]

We assume that the potential \( V(\phi) \) is smooth and non-negative. Thus \( V(\phi) \) attains its global minimum value of \( V = 0 \) for one or more values of \( \phi \), which are the global minima of the theory. We shall choose \( V(\phi) \) such that it has two or more global minima so that one has a static kink and anti-kink solutions interpolating between the two adjoining global minima as \( x \) increases from \(-\infty\) to \(+\infty\). While the field equation for a static kink is a second-order ODE, it can be reduced to a first-order ODE using the so-called Bogomolny technique. The first-order ODE is given by

\[
\frac{d\phi}{dx} = \pm \sqrt{2V(\phi)}.
\]

The corresponding static kink energy (which also equals the corresponding anti-kink energy), and which is also referred to as the kink mass, is given by

\[
M_k = \int_{-\infty}^{\infty} \left( \frac{1}{2} \left( \frac{d\phi}{dx} \right)^2 + V(\phi) \right) dx.
\]

In view of the first-order equation (3), the kink mass \( M_k \) takes a simpler form

\[
M_k = \int_{\phi_a}^{\phi_b} \sqrt{2V(\phi)} d\phi,
\]

where as \( x \) goes from \(-\infty\) to \(+\infty\), the kink solution goes from one minimum \( \phi_a \) to the adjacent minimum \( \phi_b \).

One can perform linear stability analysis by considering

\[
\phi(x, t) = \phi_k(x) + \eta(x)e^{i\omega t},
\]

where \( \phi_k \) is the kink solution. On substituting \( \phi(x, t) \) as given by equation (6) in the field equation (2) and retaining terms of order \( \eta \), it is easily shown that \( \eta(x) \) satisfies a Schrödinger-like equation

\[
-\frac{d^2\eta}{dx^2} + \frac{d^2V(\phi)}{d\phi^2} \bigg|_{\phi=\phi_k(x)} \eta(x) = \omega^2 \eta.
\]

Here \( \phi_k(x) \) denotes the corresponding kink (or anti-kink) solution. It is well-known that the stability equation (7) always admits a zero-mode solution, i.e.

\[
\omega_0 = 0, \quad \eta_0(x) = \frac{d\phi_k(x)}{dx},
\]

where \( \eta_0(x) \) is nodeless, thereby guaranteeing the linear stability of the kink solution of any theory.

In this paper we shall choose various forms of the potential \( V(\phi) \) and obtain the corresponding kink and anti-kink solutions. While the majority of the known kink solutions have an exponential tail, only a few have so far been constructed having a power law tail. The recipe for constructing kink solutions with a power law tail or an exponential tail is clear and well known. Since a kink solution has finite energy it implies that the solution must approach one of the minima (vacua) \( \phi_0 \) of the theory as \( x \) approaches either \(+\infty\) or \(-\infty\). If the lowest non-vanishing derivative of the potential at the minimum has order \( m \), then by Taylor expanding
the potential at the minimum and writing the field close to it as \( \phi = \phi_0 + \eta \), one finds that the self-dual first-order equation in \( \eta \) implies that (assuming that the potential vanishes at the minimum)

\[
\frac{d\eta}{dx} \propto \eta^{m/2}.
\] (9)

Thus if \( m = 2 \) then \( \eta \propto e^{-\alpha x} \) while if \( m > 2 \) then \( \eta \propto 1/x^{2/(m-2)} \). We shall use this recipe to construct several one-parameter families of potentials with various possible forms of power law and exponential tails.

Using the above recipe one can construct models which can give us kink solutions with either a power law tail or an exponential tail. Even more importantly, in most of the cases discussed in this paper, we construct models which give rise to a kink and a mirror kink with various possible options for the kink tails. Let us denote the two adjoining kink solutions as kink 1 and kink 2, and without loss of generality we will assume that kink 1 is to the left of kink 2. One has two kink tails corresponding to kink solution one (which we denote by \( K_{1L} \) and \( K_{1R} \)) and two kink tails \( K_{2L} \) and \( K_{2R} \) for solution two. In table 1 we give all eight possible forms of the kink tails. The previously well-studied case is when all four tails (i.e. two tails of the first kink and two of the second kink) have an exponential fall-off which we denote by \( K_{1Le}, K_{1Re}, K_{2Le}, K_{2Re} \) and for simplicity we will denote such tails simply as \( eeee \). On the other hand, a recent study by several groups [10–12, 19] has concentrated on the case when the kink tails have the form \( K_{1Le}, K_{1Re}, K_{2Le}, K_{2Re} \) and we shall denote this possibility as \( eppe \). We remark that for both of these cases, a kink and the corresponding mirror kink are possible. In fact, there are two other possibilities for which kinks and corresponding mirror kinks are also possible and these are of the form \( peep \) and \( pppp \). Correspondingly, for the other four possibilities shown in table 1, one necessarily has to consider non-mirror kinks, since for them the kink tails are of the form \( eepp \) (and \( peee \), and without loss of generality one can consider one of these two possibilities) and \( pppe \) (and \( eppp \), and again without loss of generality we can consider only one of these two possibilities).

We now consider the various possible forms of the kink tails mentioned above and construct a one-parameter family of potentials corresponding to each of these six possibilities. To initiate the discussion, in the next section we consider models admitting only one kink solution with a power law tail at both ends.

3. Power law tails at both ends

In this section we present a one-parameter family of potentials

| \( K_{1L} \) | \( K_{1R} \) | \( K_{2L} \) | \( K_{2R} \) |
|---|---|---|---|
| \( e \) | \( e \) | \( e \) | \( e \) |
| \( e \) | \( p \) | \( p \) | \( e \) |
| \( p \) | \( e \) | \( e \) | \( p \) |
| \( p \) | \( p \) | \( p \) | \( p \) |
| \( e \) | \( e \) | \( e \) | \( p \) |
| \( p \) | \( p \) | \( p \) | \( e \) |
| \( e \) | \( p \) | \( p \) | \( p \) |
\[ V(\phi) = \lambda^2 (\phi^2 - a^2)^2, \quad n = 1, 2, 3, ... \] (10)

which admits a kink solution with a power law tail at both ends. This potential has degenerate minima at \( \phi = \pm a \) and \( V(\phi = \pm a) = 0 \). We therefore look for a kink solution which oscillates from \(-a\) to \(+a\) as \( x \) goes from \(-\infty\) to \(+\infty\). Thus for the kink solutions discussed in this section, \( \phi^2 \leq a^2 \). In these cases explicit analytic solutions are unfortunately not possible and we can only find implicit kink solutions. From the latter we can obtain how a kink profile falls off as \( x \to \pm \infty \). We will first discuss explicitly the case \( n = 1 \) (i.e. the \( \phi^8 \) field theory) and \( n = 2 \) (i.e. the \( \phi^{12} \) field theory), which have already been discussed in [1], and then generalize to arbitrary \( n \).

3.1. Case I: \( n = 1 \)

For \( n = 1 \), equation (10) leads to the potential

\[ V(\phi) = \lambda^2 (\phi^2 - a^2)^4. \] (11)

On using equation (3), the self-dual first-order equation is

\[ \frac{d\phi}{dx} = \pm \sqrt{2} \lambda (a^2 - \phi^2)^2. \] (12)

This is easily integrated by using the identity

\[ \int \frac{d\phi}{(a^2 - \phi^2)^n} = \frac{(2n - 3)}{2(n - 1)a^2} \int \frac{d\phi}{(a^2 - \phi^2)^{n-1}} + \frac{\phi}{2(n - 1)(a^2 - \phi^2)^{n-1}a^2}. \] (13)

We find

\[ \mu x = \frac{2a\phi}{a^2 - \phi^2} + \ln \left( \frac{a + \phi}{a - \phi} \right), \quad \mu = 4\sqrt{2} \lambda a^3. \] (14)

From here it is straightforward to show that

\[ \lim_{x \to -\infty} \phi(x) = -a + \frac{a}{\mu x} + ..., \quad \lim_{x \to +\infty} \phi(x) = a - \frac{a}{\mu x} + .... \] (15)

Notice that the precise expressions for the kink tails around both \( \phi = +a \) and \(-a\) are entirely decided by the first term on the right hand side of equation (14), i.e. by the term \( \frac{2a\phi}{(a^2 - \phi^2)} \).

3.2. Case II: \( n = 2 \)

For \( n = 2 \), equation (10) leads to the potential

\[ V(\phi) = \lambda^2 (\phi^2 - a^2)^6. \] (16)

On using the fact that for the kink solution between \(-a\) and \(+a\), \( \phi^2 < a^2 \) and the identity (13), the kink solution turns out to be

\[ \mu x = \frac{4a^3\phi}{3(a^2 - \phi^2)^2} + \frac{2\phi a}{(a^2 - \phi^2)} + \ln \left( \frac{a + \phi}{a - \phi} \right), \quad \mu = \frac{16}{3} \sqrt{2} \lambda a^5. \] (17)
Asymptotically, we find
\[
\lim_{x \to -\infty} \phi(x) = -a + \frac{a}{\sqrt{-3\mu x}} + \ldots, \quad \lim_{x \to +\infty} \phi(x) = a - \frac{a}{\sqrt{3\mu x}} + \ldots
\]  
(18)

Again observe that the kink tails around both \( \phi = +a \) and \( -a \) are entirely given by the first term on the right hand side of equation (17), i.e. by the term \( \frac{4a^4\phi}{\pi(a^2-\phi^2)^2} \).

3.3. General case: arbitrary \( n \)

The generalization to arbitrary \( n \) is now straightforward. In this case the potential is
\[
V(\phi) = \lambda^2(b^2 - a^2)^{2n+2}.
\]  
(19)

On using the identity (13) repeatedly (also see the integral 2.149(3) of [22]), one can show that
\[
\mu_x = \frac{2^n(n-1)!}{(2n-1)!!} \left[ \frac{a^{2n-1}\phi}{(a^2-\phi^2)^n} \right] + \sum_{k=1}^{n} \frac{(2n-1)(2n-3)\ldots(2n-2k+1)}{2^n(n-1)(n-2)\ldots(n-k)} \frac{a^{2n-2k-1}\phi}{(a^2-\phi^2)^{n-k}} + \ln \left( \frac{a+\phi}{a-\phi} \right),
\]  
(20)

where
\[
\mu = \frac{2^{n+3/2}\lambda a^{2n+1}n!}{(2n-1)!!}.
\]  
(21)

Asymptotically, the leading contribution as \( x \to \pm \infty \) comes from the first term on the right hand side of equation (20). We find that
\[
\lim_{x \to -\infty} \phi(x) = -a + \frac{a}{\left( \frac{(2n-1)!!}{(a^2-\phi^2)^n} \right)^{1/n}} + \ldots,
\]
\[
\lim_{x \to +\infty} \phi(x) = a - \frac{a}{\left( \frac{(2n-1)!!}{(a^2-\phi^2)^n} \right)^{1/n}} + \ldots
\]  
(22)

As expected, for \( n = 1, 2 \) these results for the kink tails agree with those given by equations (15) and (18), respectively. Thus the kink tail around both \( \phi = +a \) and \( -a \) goes like \( x^{-1/n} \) and hence as \( n \) increases, the kink–antikink interaction becomes highly nonlinear.

For completeness, in appendix B we consider kink solutions for the one-parameter family of potentials
\[
V(\phi) = \lambda^2(a^2 - \phi^2)^2 + n, \quad n = 1, 2, \ldots
\]  
(23)

For the special case of \( n = 1 \) and \( n = 2 \), we analytically obtain the kink solution in an explicit form.

3.4. Kink mass

Using equation (5) one can immediately estimate the kink mass for the entire family of potentials as given by equation (10). We find that
\[
M_k(n) = \frac{2^{4n+3/2}\lambda a^{4n+1}[(2n)!]^2}{(4n+1)!}.
\]  
(24)
For example, while
\[ M_k(n = 1) = \frac{16\sqrt{2}}{15} \lambda a^4, \quad M_k(n = 2) = \frac{256\sqrt{2}}{315} \lambda a^9. \]  \tag{25}
We note that for \( a = 1 \), the kink mass decreases as \( n \) increases.

### 3.5. Linear stability analysis

Let us consider the linear stability of the kink solutions discussed in this section. For all these kink solutions we note that in the \( \lim_{x \to \pm \infty} \phi(x) \to \pm a \), so that for all these kink solutions
\[ \lim_{x \to \pm \infty} (a^2 - \phi^2) \to 0. \]  \tag{26}
Now the potential \( V(x) \) in the stability equation (7) is given by
\[ V(x) = d^2 V(\phi)/d\phi^2|_{\phi=\phi_k(x)}. \]  \tag{27}
But since all the potentials \( V(\phi) \) discussed in this section are of the form
\[ V(\phi) = (a^2 - \phi^2)^{2n+2}, \quad n = 1, 2, 3,... \]  \tag{28}
hence upon differentiating this \( V(\phi) \) twice with respect to \( \phi \) it follows that at least a factor of \( (a^2 - \phi^2)^2 \) will always remain. But since the kink solution goes to \(-a\) or \(+a\) as \( x = \to \pm \infty \), hence the potential \( V(x) \) as given by equation (27) vanishes as \( x \to \pm \infty \). Using the standard results of quantum mechanics in one dimension, it then follows that for \( \omega^2 \geq 0 \), the spectrum of the Schrödinger-like equation (7) for the above class of potentials is continuous, with \( \omega^2 = 0 \) being the beginning of the continuum. But for any kink solution one knows that there exists a nodeless zero mode at \( \omega^2 = 0 \) which guarantees the linear stability of the kink solution. Thus there is no gap between the zero mode and the continuum in the case of the kink solutions discussed in this section, all of which have a power law tail.

This argument is rather general and follows for any kink solution for which there is a power law tail at either both or one of the ends, because in all these cases the continuum always begins at \( \omega^2 = 0 \). We have checked this for all the kink solutions with a power law tail discussed in this paper, and in all these cases we find that there is no gap between the zero mode and the continuum. In contrast, for kinks with an exponential tail at both ends, there is always a gap between the zero mode and the continuum. As an illustration, in appendix A we demonstrate this for kink solutions of the one-parameter family of potentials of the form \( \phi^2(\phi^2 - a^{2n})^2 \) where \( n = 1, 2, 3,... \).

### 4. Potentials admitting kink and mirror kink solutions with tails of the form e p p e

In the previous section we considered kink solutions of the form \( p p \) (without any mirror kinks). Kink solutions of the form \( p p p p \) are considered in section 6. In this section we present a one-parameter family of potentials of the form
\[ V(\phi) = \lambda^2 \phi^{2n+2}(a^2 - \phi^2)^2, \quad n = 1, 2, 3,... \]  \tag{29}
This potential has degenerate minima at \( \phi = 0, \pm a \) and \( V(\phi = 0, \pm a) = 0 \) which admits a kink from 0 to \( a \) and a mirror kink from \(-a\) to 0, and both the kinks have a power law tail at one end and an exponential tail at the other. We look for a kink solution which oscillates from
0 to $+\infty$ as $x$ goes from $-\infty$ to $+\infty$. Thus for the kink solution $\phi^2 \leq a^2$. We consider the two cases of odd $n$ (i.e. potentials of the form $\phi^{4n}$) and even $n$ (i.e. potentials of the form $\phi^{4n+2}$) separately. Unfortunately, in all these cases, explicit analytic solutions are not possible and we can only find implicit kink solutions. From the latter we can obtain how a kink profile falls off as $x \to \pm \infty$.

4.1. $\phi^{4n}$ potentials

Consider the class of potentials

$$V(\phi) = \lambda^2 \phi^{4n}(a^2 - \phi^2)^2, \quad n = 1, 2, 3, ... \quad (30)$$

We will first discuss explicitly the cases $n = 1$ (i.e. the $\phi^8$ field theory) and $n = 2$ (i.e. the $\phi^{12}$ field theory), which have already been discussed in [1] and then generalize to arbitrary $n$.

4.1.1. Case I: $n = 1$. On using equation (3), the self-dual first-order equation is

$$\frac{d\phi}{dx} = \pm \sqrt{2} \lambda a \phi^2 (a^2 - \phi^2). \quad (31)$$

In this case

$$\sqrt{2} \lambda x = \int \frac{d\phi}{\phi^2 (a^2 - \phi^2)} \quad (32)$$

The integrand on the right hand side can be written as

$$\frac{1}{a^2 \phi^2} + \frac{1}{a^4 (a^2 - \phi^2)}. \quad (33)$$

This is easily integrated with the solution

$$\mu x = -\frac{2a}{\phi} + \ln \frac{a + \phi}{a - \phi}, \quad \mu = 2 \sqrt{2} \lambda a^3. \quad (34)$$

Asymptotically,

$$\lim_{x \to -\infty} \phi(x) = a \frac{2a}{\mu x} \quad \lim_{x \to \infty} \phi(x) = a - 2ae^{-\mu x^2 - 2} \quad (35)$$

Thus the kink tail around $\phi = 0$ is entirely determined by the first term on the right hand side of equation (34), i.e. the term $\frac{2a}{\phi}$.

4.1.2. Case II: $n = 2$. On using equation (3), the self-dual first-order equation is

$$\sqrt{2} \lambda x = \int \frac{d\phi}{\phi^4 (a^2 - \phi^2)} \quad (36)$$

The integrand on the right hand side can be evaluated using partial fractions and it can be shown to be

$$\frac{1}{a^2 \phi^4} + \frac{1}{a^4 \phi^2} + \frac{1}{a^4 (a^2 - \phi^2)}. \quad (37)$$

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This is easily integrated with the solution
\[ \mu x = -\frac{2a^3}{3\phi^3} - \frac{2a}{\phi} + \ln \frac{a + \phi}{a - \phi}, \quad \mu = 2\sqrt{2\lambda a^5}. \] (38)

Asymptotically,
\[ \lim_{x \to -\infty} \phi(x) = \frac{2^{1/3}a}{(-3\mu x)^{1/3}}, \quad \lim_{x \to \infty} \phi(x) = a - 2ae^{-\mu x-2-2/3}. \] (39)

Thus the kink tail around \( \phi = 0 \) is again entirely determined by the first term on the right hand side of equation (38), i.e. by the term \(-\frac{2a^3}{3\phi^3}\). Further, on comparing the kink tail around \( \phi = a \) (i.e. as \( x \to +\infty \)) in the case of \( n = 1 \) and \( n = 2 \) as given by equations (35) and (39) respectively, we find that in both cases the kink tail has an exponential form
\[ \lim_{x \to \infty} \phi(x) = a - 2ae^{-\mu x-K}, \] (40)

where the constant \( K \) equals 2 in the \( n = 1 \) case while it is \( 2 + 2/3 = 8/3 \) in the \( n = 2 \) case, whereas the rest of the behaviour is the same in both cases.

4.1.3. Case III: general \( n \). The generalization to arbitrary \( n \) is now straightforward. On using equation (3), the self-dual first-order equation is
\[ \sqrt{2}\lambda x = \int \frac{d\phi}{\phi^{2n}(a^2 - \phi^2)}. \] (41)

The integrand on the right hand side can be evaluated using partial fractions and it can be shown to be
\[ \sum_{k=1}^{n} \frac{1}{a^{2k}\phi^{2n-2-2k}} + \frac{1}{a^{2n}(a^2 - \phi^2)}. \] (42)

This is easily integrated with the solution
\[ \mu x = -\frac{2a^{2n-1}}{(2n-1)\phi^{2n-1}} - \sum_{k=2}^{n} \frac{2a^{2n-2k+1}}{(2n-2k+1)\phi^{2n-2k+1}} + \ln \frac{a + \phi}{a - \phi}, \] (43)

where
\[ \mu = 2\sqrt{2\lambda a^{2n+1}}. \] (44)

Asymptotically,
\[ \lim_{x \to -\infty} \phi(x) = \frac{2^{1/(2n-1)}a}{[-(2n-1)\mu x]^{1/(2n-1)}}, \quad \lim_{x \to \infty} \phi(x) = a - 2ae^{-\mu x-B_n}, \] (45)

where \( B_n = 2 + 2/3 + 2/5 + \ldots + 2/(2n-1) \).

As expected, for \( n = 1, 2 \) these results for the kink tail agree with those given by equations (35) and (39), respectively. Thus for the class of potentials as given by equation (30), the kink tail around \( \phi = +a \) goes like \( e^{-\mu x} \) while it goes like \( x^{-1/(2n-1)} \) around \( \phi = 0 \) and hence as \( n \) increases, the kink–antikink interaction becomes highly nonlinear.
4.2. $\phi^{4n+2}$ potentials

Let us now consider the class of potentials

$$V(\phi) = \lambda^2 \phi^{4n+2}(a^2 - \phi^2)^2, \quad n = 1, 2, 3, ....$$  \hfill (46)

First we will consider the $n = 1$ case (which is already discussed in [1]) and then mention the results for the kink tail for the general $n$ case. Details of the $n = 2$ as well as the general $n$ case are given in appendix C.

4.2.1. Case I: $n = 1$. On using equation (3) the self-dual first-order equation is

$$\sqrt{2} \lambda x = \int \frac{d\phi}{\phi(a^2 - \phi^2)}. \hfill (47)$$

The integrand on the right hand side can be evaluated using partial fractions and it can be shown to be

$$\frac{1}{a^2 \phi^3} + \frac{1}{a^4 \phi} + \frac{\phi}{a^4(a^2 - \phi^2)}. \hfill (48)$$

This is easily integrated with the solution

$$\mu x = -\frac{a^2}{\phi^2} + \ln \frac{\phi^2}{a^2 - \phi^2}, \quad \mu = 2\sqrt{2}\lambda a^4. \hfill (49)$$

Asymptotically,

$$\lim_{x \to -\infty} \phi(x) = \frac{a}{\sqrt{-\mu x}}, \quad \lim_{x \to +\infty} \phi(x) = a - \frac{a}{2} e^{-\mu x - 1}. \hfill (50)$$

Thus the kink tail around $\phi = 0$ is entirely determined by the first term on the right hand side of equation (49), i.e. by the term $-\frac{a^2}{\phi^2}$.

Similar to the $n = 1$ case, one can obtain results for the $n = 2$ case and then generalize to arbitrary $n$, which are discussed in appendix C. As shown in appendix C for arbitrary $n$, while the kink tail falls off like $e^{-\mu x}$ around $\phi = a$, it falls off like $x^{-1/2n}$ around $\phi = 0$.

On combining the results of sections 4.1 and 4.2, we then conclude that for the one-parameter family of potentials as given by equation (29), while the kink tail falls off like $e^{-\mu x}$ around $\phi = a$, it falls off like $x^{-1/n}$ around $\phi = 0$.

4.3. Kink mass

Using equation (5) one can immediately estimate the kink mass for the entire family of potentials as given by equation (29). We find that

$$M_k = \frac{2^{3/2} \lambda a^{n+4}}{(n+2)(n+4)}, \quad n = 1, 2, 3, .... \hfill (51)$$

For example, while

$$M_k(n = 1) = \frac{2\sqrt{2}}{15} \lambda a^5, \quad M_k(n = 2) = \frac{2\sqrt{2}}{24} \lambda a^6. \hfill (52)$$

We note that for $a = 1$, the kink mass decreases as $n$ increases.
5. Potentials admitting kink and mirror kink solutions with tails of the form \(peep\)

In this section we present a one-parameter family of potentials of the form

\[
V(\phi) = \lambda^2 \phi^2 (a^2 - \phi^2)^{2n+2}, \quad n = 1, 2, 3, \ldots \tag{53}
\]

These potentials have degenerate minima at \(\phi = 0, \pm a\) and \(V(\phi = 0, \pm a) = 0\), hence admit a kink from 0 to \(a\) and a mirror kink from \(-a\) to 0, and both the kinks have a power law tail at one end with an exponential tail at the other. We look for a kink solution which oscillates from 0 to \(+a\) as \(x\) goes from \(-\infty\) to \(+\infty\). Thus for the kink solution \(\phi^2 \leq a^2\). In these cases explicit analytic solutions are not possible and we can only find implicit kink solutions. From the latter we can obtain how a kink profile falls off as \(x \to \pm \infty\).

We will first discuss the case \(n = 1\) (i.e. the \(\phi^{10}\) field theory) which has already been discussed in [1] and then mention the results on the kink tail for arbitrary \(n\). Details of \(n = 2\) (i.e. the \(\phi^{14}\) field theory) and the generalization to arbitrary \(n\) are given in appendix D.

5.1. Case I: \(n = 1\)

On using equation (3), the self-dual first-order equation is

\[
\frac{d\phi}{dx} = \pm \sqrt{2} \lambda \phi (a^2 - \phi^2)^2. \tag{54}
\]

In this case

\[
\sqrt{2} \lambda x = \int \frac{d\phi}{\phi(a^2 - \phi^2)^2}. \tag{55}
\]

The integrand on the right hand side can be written as partial fractions

\[
\frac{\phi}{a^2(a^2 - \phi^2)^2} + \frac{\phi}{a^4(a^2 - \phi^2)} + \frac{1}{a^4\phi}. \tag{56}
\]

This is easily integrated with the solution

\[
\mu x = \frac{a^2}{a^2 - \phi^2} \ln \frac{a^2 - \phi^2}{a^2}, \quad \mu = 2\sqrt{2} \lambda a^2. \tag{57}
\]

Asymptotically,

\[
\lim_{x \to -\infty} \phi(x) = ae^{\mu x/2 - 1}, \quad \lim_{x \to \infty} \phi(x) = a - \frac{a}{2\mu x}. \tag{58}
\]

Thus the kink tail around \(\phi = a\) is entirely determined by the first term on the right hand side of equation (57), i.e. the term \(\frac{a^2}{a^2 - \phi^2}\).

Similar to the \(n = 1\) case, one can obtain results for the \(n = 2\) case and then generalize to arbitrary \(n\), which are discussed in appendix D. As shown in appendix D for arbitrary \(n\), while the kink tail falls off like \(e^{\mu x/2}\) (as \(x\) goes to \(-\infty\)) around \(\phi = 0\), it falls off like \(x^{-1/2}\) (as \(x\) goes to \(+\infty\)) around \(\phi = a\).

5.2. Kink mass

Using equation (5) one can immediately estimate the kink mass for the entire family of potentials as given by equation (53). We find that
\[ M_k = \frac{\lambda a^{2n+4}}{\sqrt{2(n+2)}}, \quad n = 1, 2, 3, \ldots \]  

(59)

For example, while
\[ M_k(n = 1) = \frac{1}{3\sqrt{2}} \lambda a^6, \quad M_k(n = 2) = \frac{1}{4\sqrt{2}} \lambda a^8. \]  

(60)

Thus for \( a = 1 \), the kink mass decreases as \( n \) increases.

6. Potentials admitting kink and mirror kink solutions with tails of the form

\[ V(\phi) = \lambda^2 \phi^{2m+2}(a^2 - \phi^2)^{2n+2}, \quad n, m = 1, 2, 3, \ldots \]  

(61)

These potentials have degenerate minima at \( \phi = 0, \pm a \) and \( V(\phi = 0, \pm a) = 0 \) and admit a kink from 0 to \( a \), a mirror kink from \( -a \) to 0, and both kinks have a power law tail at both ends. We look for a kink solution which oscillates from 0 to \( +a \) as \( x \) goes from \(-\infty\) to \( +\infty \), respectively. Thus for the kink solution \( \phi^2 \leq a^2 \). In these cases explicit analytic solutions are not possible and we can only find implicit kink solutions. From the latter we can obtain how a kink profile falls off as \( x \to \pm \infty \).

In these models depending on whether \( m < n \) (\( m > n \)), one can have kink solutions for which the power law tail around \( \phi = a \) has slower (faster) asymptotic fall off compared to the power law tail around \( \phi = 0 \). We shall now consider a one-parameter family of potentials of both types (i.e. where \( m < n \) and \( m > n \)).

6.1. Models where the kink tail around \( \phi = \pm a \) has a slower asymptotic fall-off compared to the tail around \( \phi = 0 \)

For simplicity, let us consider the potentials
\[ V(\phi) = \lambda^2 \phi^4(a^2 - \phi^2)^{2n+2}, \quad n = 2, 3, \ldots \]  

(62)

Similar arguments also hold good in cases where \( \phi^4 \) in equation (62) is replaced by the potential \( \phi^{2m+2} \) with \( m < n \). This potential has degenerate minima at \( \phi = 0, \pm a \) and \( V(\phi = 0, \pm a) = 0 \) which admit a kink from 0 to \( a \), a mirror kink from \( -a \) to 0, and both kinks have a power law tail at both ends. We look for a kink solution which oscillates from 0 to \( +a \) as \( x \) goes from \(-\infty\) to \( +\infty \). Thus for the kink solution \( \phi^2 \leq a^2 \). In these cases explicit analytic solutions are not possible and we can only find implicit kink solutions. From the latter we can obtain how a kink profile falls off as \( x \to \pm \infty \).

We will first discuss explicitly the case \( n = 2 \) (i.e. \( \phi^16 \) field theory) and then directly mention results on the kink tail for arbitrary \( n \). Details of \( n = 3 \) and the generalization to arbitrary \( n \) are discussed in appendix E.1. For completeness we will also discuss the case of \( n = 1 \) in appendix E.1. It might be noted that for \( n = 1 \) we will see that the tails around \( \phi = a \) and \( \phi = 0 \) have the same asymptotic fall-off, while for \( n \geq 2 \) the kink tail around \( \phi = a \) has a slower fall-off compared to the tail around \( \phi = 0 \).
6.1.1 Case I: \( n = 2 \). On using equation (3) the self-dual first-order equation is
\[
\sqrt{2} \lambda x = \int \frac{d \phi}{\phi^2 (a^2 - \phi^2)^3}.
\] (63)

The integrand on the right hand side can be evaluated using partial fractions and it can be shown to be equal to
\[
\frac{1}{a^2 (a^2 - \phi^2)^3} + \frac{1}{a^4 (a^2 - \phi^2)^3} + \frac{1}{a^6 (a^2 - \phi^2)^3} + \frac{1}{a^8 \phi^2}.
\] (64)

This is easily integrated by making use of the identity (13) leading to the solution
\[
\mu x = \frac{a^3 \phi}{4 (a^2 - \phi^2)^2} - \frac{a}{\phi} + \frac{7 a \phi}{8 (a^2 - \phi^2)} + \frac{15}{16} \ln \frac{a + \phi}{a - \phi}.
\] (65)

Here \( \mu = \sqrt{2} a^7 \). Asymptotically,
\[
\lim_{x \to -\infty} \phi(x) = a + \mu x, \quad \lim_{x \to \infty} \phi(x) = a - a \sqrt{\frac{1}{16 \mu x}}.
\] (66)

Thus the kink tail around \( \phi = a \) is entirely determined by the first term on the right hand side of equation (65), i.e. by the term \( a^3 \phi / 4 (a^2 - \phi^2)^2 \). On the other hand, the kink tail around \( \phi = 0 \) is entirely determined by the second term on the right hand side of equation (65), i.e. by the term \( \phi / (a^2 - \phi^2) \).

Similar to the \( n = 2 \) case, one can obtain results for the \( n = 3 \) case and then generalize to arbitrary \( n \), which are discussed in appendix E.1. As shown in appendix E.1, for arbitrary \( n \), while the kink tail around \( \phi = 0 \) always goes like \( x^{1/n} \), it falls off like \( x^{-1} \) around \( \phi = a \).

6.1.2 Kink mass. Using equation (5) one can easily estimate the kink mass for the entire family of potentials as given by equation (62). We find that
\[
M_k = \frac{\lambda a^{2n+3} \Gamma(\frac{3}{2}) \Gamma(n + \frac{2}{2})}{\sqrt{2} \Gamma(n + \frac{3}{2})}, \quad n = 1, 2, 3, \ldots
\] (67)

For example, while
\[
M_k(n = 1) = \frac{\lambda a^2 2^{7/2}}{105}, \quad M_k(n = 2) = \frac{\lambda a^2 2^{9/2}}{315}.
\] (68)

We note that for \( a = 1 \), the kink mass decreases as \( n \) increases.

6.2 Models where the kink tail around \( \phi = 0 \) has a slower asymptotic fall-off compared to the tail around \( \phi = a \)

For simplicity, let us consider the one-parameter family of potentials
\[
V(\phi) = \lambda^2 \phi^{2n+2} (a^2 - \phi^2)^4, \quad n = 2, 3, \ldots
\] (69)

Similar arguments also hold good in cases where \( (a^2 - \phi^2)^4 \) in equation (69) is replaced by the potential \( (a^2 - \phi^2)^{2m+2} \) with \( m < n \).

This potential has degenerate minima at \( \phi = 0, \pm a \) and \( V(\phi = 0, \pm a) = 0 \), which admits a kink from 0 to \( a \), a mirror kink from \( -a \) to 0, and both kinks have a power law tail at both
ends. We look for a kink solution which oscillates from 0 to $+a$ as $x$ goes from $-\infty$ to $+\infty$. Thus for the kink solution $\phi^2 \leq a^2$.

We consider the two cases of odd $n$ (i.e. potentials of the form $\phi^{4n}$) and even $n$ (i.e. potentials of the form $\phi^{4n+2}$) separately. In these cases explicit analytic solutions are not possible and we can only find implicit kink solutions. From the latter we can obtain how a kink profile falls off as $x \to \pm \infty$.

6.2.1. $\phi^{4n}$ potentials. Consider the class of potentials

$$V(\phi) = \lambda^2 \phi^{4n}(a^2 - \phi^2)^4, \quad n = 2, 3, \ldots$$

(70)

We consider the case of $n = 2$ and $n = 3$ and then generalize to arbitrary $n$. We will see that for $n \geq 2$ the kink tail around $\phi = 0$ has a slower fall-off compared to the tail around $\phi = a$.

6.2.2. Case I: $n = 2$. On using equation (3), the self-dual first-order equation is

$$\frac{d\phi}{dx} = \pm \sqrt{2} \lambda \phi^4 (a^2 - \phi^2)^2.$$  

(71)

In this case

$$\sqrt{2} \lambda x = \int \frac{d\phi}{\phi^4 (a^2 - \phi^2)^2}.$$  

(72)

The integrand on the right hand side can be written as

$$1/a^4 \phi^2 + 1/a^4 (a^2 - \phi^2)^2 + 2/a^\phi \phi^2 + 2/a^\phi (a^2 - \phi^2).$$

(73)

This is easily integrated using equation (13) with the solution

$$\mu x = \frac{2a^3}{3\phi^2} + \frac{a\phi}{a^2 - \phi^2} - \frac{4a}{\phi} + \frac{5}{2} \ln \frac{a + \phi}{a - \phi}, \quad \mu = 2\sqrt{2} \lambda a^7.$$  

(74)

Asymptotically,

$$\lim_{x \to -\infty} \phi(x) = \frac{2^{1/3}a}{(-3\mu x)^{1/3}}, \quad \lim_{x \to \infty} \phi(x) = a - \frac{a}{2\mu x}.$$  

(75)

Thus the kink tail around $\phi = 0$ is entirely determined by the first term on the right hand side of equation (74), i.e. the term $-\frac{2^{1/3}a}{(-3\mu x)^{1/3}}$. On the other hand, the kink tail around $\phi = a$ is entirely determined by the second term on the right hand side of equation (74), i.e. the term $\frac{a}{2\mu x}$.

Similar to the $n = 2$ case, one can obtain results for the $n = 3$ case and then generalize to arbitrary $n$, which are discussed in appendix E.2. As shown in appendix E.2, for arbitrary $n$, while the kink tail falls off like $x^{-1}$ around $\phi = a$, it falls off like $x^{-1/(2n-1)}$ around $\phi = 0$ with $n = 2, 3, 4, \ldots$.

6.2.3. $\phi^{4n+2}$ potentials. Consider the class of potentials

$$V(\phi) = \lambda^2 \phi^{4n+2}(a^2 - \phi^2)^4, \quad n = 1, 2, 3, \ldots$$

(76)

We first discuss the cases of $n = 1$ and $n = 2$ and then generalize to arbitrary $n$. We will see that the kink tail around $\phi = 0$ has a slower fall-off compared to the tail around $\phi = a$. 

\[\text{A Khare and A Saxena} \]
6.2.4. Case I: $n = 1$. On using equation (3) the self-dual first-order equation is
\[ \sqrt{2} \lambda x = \int \frac{d\phi}{\phi^3(a^2 - \phi^2)^2}. \] (77)
The integrand on the right hand side can be evaluated using partial fractions and it can be shown to be
\[ \frac{1}{a^4\phi^3} + \frac{\phi}{a^4(a^2 - \phi^2)^2} + \frac{2}{a^6\phi} + \frac{2\phi}{a^6(a^2 - \phi^2)}. \] (78)
This is easily integrated with the solution
\[ \mu x = -\frac{a^2}{\phi} + \ln \frac{\phi}{a - \phi^2}, \quad \mu = 2\sqrt{2}\lambda a^6. \] (79)
Asymptotically,
\[ \lim_{x \to -\infty} \phi(x) = \frac{a}{\sqrt{-\mu x}}, \quad \lim_{x \to \infty} \phi(x) = a - \frac{a}{2\mu x}. \] (80)
Thus the kink tail around $\phi = 0$ is entirely determined by the first term on the right hand side of equation (79), i.e. by the term $-\frac{a^2}{\phi}$. On the other hand, the kink tail around $\phi = a$ is entirely determined by the second term on the right hand side of equation (79), i.e. by the term $\frac{\phi}{a - \phi^2}$.

Similar to the $n = 1$ case, one can obtain results for the $n = 2$ case and then generalize to arbitrary $n$ which are discussed in appendix E.3. As shown in appendix E.3, for arbitrary $n$, while the kink tail falls off like $x^{-1}$ around $\phi = a$, it falls off like $x^{-1/n}$ around $\phi = 0$.

On combining the results of the two subsections of section 6.2, it is then clear that for the one-parameter family of potentials as given by equation (69), while the kink tail falls off like $x^{-1}$ around $\phi = a$, it falls off like $x^{-1/n}$ around $\phi = 0$ where $n = 2, 3, 4, ...$.

6.3. Kink mass

Using equation (5) one can immediately estimate the kink mass for the entire family of potentials as given by equation (69). We find that
\[ M_k = 2^{7/2} \lambda a^{n+6} \frac{(n+2)(n+4)(n+6)}{(n+2)(n+4)(n+6)}, \quad n = 1, 2, 3, .... \] (81)

For example, while
\[ M_k(n = 1) = 2^{7/2} \frac{105}{105} \lambda a^7, \quad M_k(n = 2) = 2^{7/2} \frac{24}{24} \lambda a^8. \] (82)

We note that for $a = 1$, the kink mass decreases as $n$ increases.

7. Conclusion

In this paper we have presented a large variety of potentials which admit kink solutions with a power law tail at either one end or at both ends. For example, we have presented a one-parameter family of potentials all of which admit a kink solution which has a power law tail at both ends. We have considered the linear stability of these kink solutions and have shown that in no case is there a gap between the zero mode and the continuum. This is in contrast
with the well-known kink solutions (say of $\lambda \phi^4$ field theory) which have an exponential tail at both ends, and in which there is always a gap between the zero mode and the continuum. As an illustration, in appendix A we have constructed kink solutions for a one-parameter family of such potentials, discussed their linear stability and shown the existence of a gap between the zero mode and the continuum.

We have presented several different classes of models which simultaneously admit two kink solutions with various different forms of kink tails. For example, in section 4 we have presented a one-parameter family of potentials which admit a kink and a corresponding mirror kink with kink tails of the form $eppe$. As mentioned in the introduction, very recently we [19] calculated the KK and the K–AK force for the one-parameter family of potentials discussed in section 4 and have shown that while the KK force is repulsive, the K–AK force is attractive and both these forces fall off as $R^{-2(n+1)/n}$ where $R$ is the KK or K–AK separation while $n = 1, 2, ...$. In particular it has been shown that

$$F_{KK} = \left[ \frac{\Gamma\left(\frac{n}{2(n+1)}\right)\Gamma\left(\frac{1}{2(n+1)}\right)}{2^{(n+2)/2(n+1)}n\sqrt{n}} \right]^{2(n+1)/n} R^{-2(n+1)/n},$$

and

$$F_{K–AK} = -\left[ \sqrt{n} \Gamma\left(\frac{n}{2(n+1)}\right) \Gamma\left(\frac{1}{2(n+1)}\right) \right]^{2(n+1)/n} R^{-2(n+1)/n},$$

so that

$$\frac{F_{K–AK}}{F_{KK}} = -\left( \sin \left( \frac{\pi}{2(n+1)} \right) \right)^{2(n+1)/n}. \tag{85}$$

Notice that as $n$ increases, this ratio goes on decreasing and as $n \to \infty$, the ratio of the K–AK to KK force goes to zero. In contrast, for kinks with tails of the form $eeee$, the magnitude of the K–AK and KK forces is always equal.

In section 5 we have presented a one-parameter family of potentials which gives rise to a kink and the corresponding mirror kink with kink tails of the form $peep$. Further, in section 6 we have presented two distinct one-parameter families of potentials which admit kink and mirror kink solutions with kink tails of the form $pppp$. In one family of potentials we have shown that the power law tail around $\phi = 0$ has a slower fall-off compared to the tails around $\phi = \pm a$, while in the second case it is the opposite.

For completeness, in appendices F and G, we have presented two different one-parameter families of potentials which admit non-mirror kink solutions with kink tails of the form $eeep$ and $pppe$, respectively.

Finally, we note that our results could provide insight into domain wall properties with regard to specific models in high-energy physics [3], condensed matter physics [14–16], biology [17] and cosmology [18].

However, there are several open questions which need to be carefully investigated. Some of these are as follows, and we hope to address them in the near future.

1. Why is the ratio of the magnitude of the K–AK force to KK force always less than one, decreasing monotonically (as $n$ increases) for kink tails of the form $eppe$ as discussed in section 4, while it is always one for kink tails of the form $eeee$?
2. What are the KK and the K–AK forces in the case where kinks and the corresponding mirror kinks have kink tails of the form $peep$ as in section 5? Naively one might think that both the KK and the K–AK forces would be exponentially small in this case. However,
in view of the power law tails at the two extreme ends, it is not obvious to us that this is
indeed true. We surmise that the KK and the K–AK forces may in fact turn out to have a
power law dependence. Another open question regards the ratio of the K–AK force and
the KK force in cases where the kink tails are of the form \( peep \). We believe that this is a nontrivial question for which one
may require an entirely new formulation. In fact, even for kink tails of the form \( eep \) as
discussed in appendix F, it is not obvious to us that either the KK or the K–AK force is
exponentially small. This is because of the power law tail at one end of one of the kinks.

3. What about the KK and the K–AK forces when all four tails are of the form \( pppp \)?
There are two possibilities here, both of which have been discussed in section 6. In one case the
power law tail around \( \phi = \pm a \) has a slower asymptotic fall-off compared to the asymptotic fall-off around \( \phi = 0 \), while in the other case, the power law tail around \( \phi = \pm a \) has a faster fall-off compared to the power law tail around \( \phi = 0 \).

In the one-parameter family of potentials \( V(\phi) = \lambda^2 \phi^4 (\phi^2 - a^2)^{2n+2} \) with \( n = 2, 3, ... \) discussed in section 6, while the kink tail around \( \phi = 0 \) falls off like \( x^{-1} \), while the kink tail around \( \phi = a \) falls off like \( x^{-1/n} \) around \( \phi = \pm a \) with \( n = 2, 3, ... \). In this case, one
would think naively that the KK and the K–AK force would go like \( R^{-4} \) where \( R \) is the
KK or K–AK separation, we believe that because of the kink tail around \( \phi = \pm a \), the KK
and K–AK force may actually fall off slower than \( R^{-4} \). As far as we are aware, until now no calculation exists in the literature for either the KK or the K–AK force in this case. We
suspect that a conceptually new formulation is required to compute KK and K–AK forces
in this case.

On the other hand, in the one-parameter family of potentials \( V(\phi) = \lambda^2 \phi^2 (\phi^2 - a^2)^{2n+2} \) with \( n = 2, 3, ... \) discussed in section 6, the kink tail around \( \phi = 0 \) falls off like \( x^{-1/n} \) while it falls off like \( x^{-1} \) around \( \phi = \pm a \) with \( n = 2, 3, ... \). In this case, we suspect that the KK
as well as the K–AK force will indeed go like \( R^{-2(n+1)/n} \) where \( R \) is the KK or K–AK separation where \( n = 1, 2, 3, ... \). However, we suspect that the strength of the KK and
K–AK forces would be modified because of the power law tail around \( \phi = \pm a \). So far
as we are aware, until now no calculation exists in the literature on the KK or the K–AK force in cases where the kink tails are of the form \( pppp \). We suspect that even in this case a conceptually new formulation would be required to compute the KK and K–AK forces.

Similarly, in the case of kink tails of the form \( pppe \) as discussed in appendix G, it is not
obvious how the KK or the K–AK force will vary with \( R \) and how the strength of the KK
and K–AK force would be modified by the power law tail at \( \phi = \pm a \).

4. Can one rigorously prove our assertion that for kinks with a power law tail at either one
end or both ends, there is no gap between the zero mode and the continuum?

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Appendix A. Potentials with an exponential tail at both ends

For completeness, we now discuss a one-parameter family of potentials which admits a kink solution as well as a mirror kink solution (and the corresponding antikink solutions) all of which have an exponential tail at both ends [5]. Let us consider the family of potentials

$$ V(\phi) = \frac{\lambda^2}{2} \phi^2 (\phi^{2n} - a^{2n})^2, \quad n = 1, 2, 3, \ldots \quad (A.1) $$

Note that for $n = 1$ the existence of such kink solutions is well known [20, 21]. The self-dual equation is

$$ \frac{d\phi}{dx} = \pm \lambda \phi (a^{2n} - \phi^{2n}). \quad (A.2) $$

It is easily shown that this equation admits a kink and a mirror kink solution, one from 0 to $a$ and a mirror kink from $-a$ to 0. The kink solution from 0 to $a$ is given by

$$ \phi(x) = a \left[ \frac{(1 + \tanh(\beta x))}{2} \right]^{1/2n}, \quad (A.3) $$

where $\beta = n\lambda a^{2n}$.

Hence, asymptotically

$$ \lim_{x \to -\infty} \phi(x) = ae^{\mu x/2n}, \quad \lim_{x \to \infty} \phi(x) = a - \frac{a e^{-\mu x}}{4n}, \quad \mu = 2n\lambda a^{2n}. \quad (A.4) $$

The stability analysis is easily performed. The corresponding zero mode is

$$ \psi_0(x) \propto (1 + \tanh(\beta x))^{(\frac{1}{2n} - 1)} \text{sech}^2\beta x, \quad (A.5) $$

which is clearly nodeless. The kink potential is also easily calculated

$$ V(x) = V''(\phi_k) = \frac{\lambda^2 a^{2n}}{4} [9(8n^2 + 6n + 1) \text{sech}^2\beta x + (8n^2 - 2) \tanh \beta x + (8n^2 + 2)]. \quad (A.6) $$

Note that

$$ V(\infty) = 4\lambda^2 n^2 a^{2n}, \quad V(-\infty) = \lambda^2 a^{2n}. \quad (A.7) $$

Thus, unlike the potentials with a power law tail, in the case with exponential tails of the form $e^{e^{e^{e}}}$, there is a gap between the zero mode and the continuum.

Appendix B. Kink solutions for the potentials given by equation (23)

Let us consider the class of potentials

$$ V(\phi) = \lambda^2 |\phi^2 - a^{2n}|^{2n+1}, \quad n = 1, 2, 3, \ldots \quad (B.1) $$

These potentials have degenerate minima at $\phi = \pm a$ and $V(\phi = \pm a) = 0$. We therefore look for a kink solution which oscillates from $-a$ to $+a$ as $x$ goes from $-\infty$ to $+\infty$. Thus for the kink solution $\phi^2 \leq a^2$. Notice that while the potential (B.1) is continuous its derivative is not. However, since the kink solution oscillates from $\phi = -a$ to $\phi = +a$ as $x$ goes from $-\infty$ to $\infty$ so that $\phi^2 \leq a^2$, hence as far as such a kink solution is concerned, our results would be valid.

In fact an explicit analytical solution for $n = 1$ (i.e. $\phi^6$ field theory) has already been obtained [6]. Using this explicit solution we will perform the stability analysis and show that
indeed there is no gap between the zero mode and the continuum. We also show that even for $n = 2$ (i.e. $\phi^4$ field theory) one can obtain an explicit kink solution analytically while for higher values of $n$ one can only get an implicit kink solution. But in all cases one can determine the asymptotic behaviour of the kink as $x \to \pm \infty$ and show that there is no gap between the zero mode and the continuum.

**Case I: $n = 1$.**

$$V(\phi) = \lambda^2|\phi^2 - a^2|^3.$$  

(B.2)

Note that the kink solution for this potential has already been obtained [6]. The self-dual first-order equation is

$$\frac{d\phi}{dx} = \pm \sqrt{2V(\phi)}.$$  

(B.3)

This is easily integrated, yielding

$$\mu x = \frac{\phi}{\sqrt{a^2 - \phi^2}}, \quad \mu = \sqrt{2}\lambda a^2.$$  

(B.4)

Thus the kink solution is [6]

$$\phi(x) = \frac{a\mu x}{\sqrt{1 + \mu^2x^2}}.$$  

(B.5)

Asymptotically it behaves as

$$\lim_{x \to -\infty} \phi(x) = -a + \frac{a}{2\mu^2x^2} + \ldots, \quad \lim_{x \to \infty} \phi(x) = a - \frac{a}{2\mu^2x^2} + \ldots.$$  

(B.6)

**Case II: $n = 2$.**

In this case the self-dual first-order equation is

$$\frac{d\phi}{dx} = \pm \sqrt{2\lambda(a^2 - \phi^2)^{3/2}}.$$  

(B.7)

This is easily integrated yielding

$$\mu x = \frac{\phi(3a^2 - \phi^2)}{(a^2 - \phi^2)^{3/2}}, \quad \mu = 3\sqrt{2}\lambda a^2.$$  

(B.8)

On squaring and using $y = \phi^2/a^2$ we get the cubic equation

$$y^3 - 3y^2 + 3 \left(\frac{\mu^2x^2 + 3}{\mu^2x^2 + 4}\right)y - \frac{\mu^2x^2}{\mu^2x^2 + 4} = 0.$$  

(B.9)

Its solution is given by

$$1 - y = \frac{a^2 - \phi^2}{a^2} = \left(\frac{\mu^2x^2 + 4 - \mu x}{2(\mu^2x^2 + 4)^{3/2}}\right)^{1/3} + \left(\frac{\mu x + \sqrt{\mu^2x^2 + 4}}{2(\mu^2x^2 + 4)^{3/2}}\right)^{1/3}.$$  

(B.10)

Thus asymptotically

$$\lim_{x \to -\infty} \phi(x) = -a + \frac{a}{2(\mu^2x^2)^{1/3}}, \quad \lim_{x \to \infty} \phi(x) = a - \frac{a}{2(\mu^2x^2)^{1/3}}.$$  

(B.11)
Case III: General $n$. Consider the potential
\[ V(\phi) = \lambda^2 |\phi|^2 - a^2 |\phi|^{2n+1}, \]
for which, on using the identity (13) the solution is given by
\[ \mu x = a^2 n - 2 \phi(\mu^2 x^2) + \sum_{k=1}^{n-1} \frac{2^k (n-1) ... (n-k)}{(2n-3)(2n-5) ... (2n-2k-1)} \left( \frac{\mu^2 x^2}{a^2 - \phi^2}_{\mu^2 x^2} \right)^{n-k-1/2}, \]
with
\[ \mu = (2n-1) \sqrt{2} 2^{n-2}. \]
Asymptotically the leading contribution as $x \to \pm \infty$ comes from the first term. We find
\[ \lim_{x \to -\infty} \phi(x) = -a + \frac{a}{2 \mu^2 x^2} \left( \frac{1}{(2n-1)} \right), \quad \lim_{x \to \infty} \phi(x) = a - \frac{a}{2 \mu^2 x^2} \left( \frac{1}{(2n-1)} \right). \]
It is clear that this result agrees with that for $n = 1, 2$.

Using the exact kink solution for the $\phi^6$ case as given by equation (B.5) and using equation (8) one can immediately show that in this case the zero mode is
\[ \psi_0 = \frac{d \phi_k}{dx} = \frac{a \mu}{(1 + \mu^2 x^2)^{3/2}}, \]
which is indeed nodeless. Consider the relevant Schrödinger-like stability equation
\[ -\psi''(x) + V(x) \psi(x) = \omega^2 \psi. \]
On using equation (7), the corresponding potential $V(x)$ is given by
\[ V(x) = \frac{d^2 V}{d \phi^2}_{\phi = \phi_k} = \frac{-3 \mu^2 (1 - 4 \mu^2 x^2)}{(1 + \mu^2 x^2)^2}. \]
This potential vanishes at $x = \pm \infty$, has a minimum at $x = 0$ with $V_{\text{min}} = -3 \mu^2$ and has maxima at $\mu^2 x^2 = 1$ with $V = 9/4$. Note that $V$ vanishes at $x = \pm \infty$, and hence the continuum begins at the same energy as that of the zero mode. This is quite different from the case of the kinks with an exponential tail at both ends in which there is always a gap between the zero mode and the continuum. Even for the kink solution of the $\phi^{10}$ field theory as given by equation (B.9), we have checked that the corresponding $V(x)$ of the stability equation (B.17) vanishes at $x = \pm \infty$ and hence the continuum begins at the same energy as that of the zero mode, i.e. there is no gap between the zero mode and the continuum. These two examples thus support the fact that unlike the kinks with exponential tails at both ends, for the kink solution with a power law tail at either or both ends, there is no gap between the zero mode and the continuum.

Appendix C. Details of the kink solutions of equation (46) of section 4

We now discuss the kink solutions for a class of potentials given by equation (46). We have already discussed the $n = 1$ case in section 4 and we now discuss the $n = 2$ case and then generalize to arbitrary $n$. 

Case II: \( n = 2 \) On using equation (3) the self-dual first-order equation is
\[
\sqrt{2} \lambda x = \int \frac{d\phi}{\phi^5 (a^2 - \phi^2)}.
\] (C.1)

The integrand on the right hand side can be evaluated using partial fractions and it can be shown to be
\[
\frac{1}{a^2 \phi^5} + \frac{1}{a^4 \phi^3} + \frac{1}{a^6 \phi} + \frac{\phi}{a^6 (a^2 - \phi^2)}.
\] (C.2)

This is easily integrated with the solution
\[
\mu x = -\frac{a^4}{2 \phi^4} - \frac{a^2}{\phi^2} + \ln \frac{\phi^2}{a^2 - \phi^2}, \quad \mu = 2 \sqrt{2} \lambda a^6.
\] (C.3)

Asymptotically,
\[
\lim_{x \to -\infty} \phi(x) = a - \frac{a}{2} e^{-\mu x - 1/2}, \quad \lim_{x \to +\infty} \phi(x) = a - \frac{a}{2} e^{-\mu x - K},
\] (C.4)

where the constant \( K \) equals 1 in the \( n = 1 \) case while it is \( 1 + 1/2 = 3/2 \) in the \( n = 2 \) case and the rest of the behaviour is the same in both cases.

Case III: General \( n \) The generalization to arbitrary \( n \) is now straightforward. On using equation (3), the self-dual first-order equation is
\[
\sqrt{3} \lambda x = \int \frac{d\phi}{\phi^{2n+1} (a^2 - \phi^2)}.
\] (C.6)

The integrand on the right hand side can be evaluated using partial fractions and it can be shown to be
\[
\sum_{k=1}^{n+1} \frac{1}{a^{2k} \phi^{2n+3-2k}} + \frac{\phi}{a^{2n+2} (a^2 - \phi^2)}.
\] (C.7)

This is easily integrated with the solution
\[
\mu x = -\frac{a^{2n}}{n \phi^{2n}} - \sum_{k=2}^{n} \frac{a^{2n+2-2k}}{(n+1-k) \phi^{2n+2-2k}} + \ln \frac{\phi^2}{a^2 - \phi^2}.
\] (C.8)

Asymptotically,
\[
\lim_{x \to -\infty} \phi(x) = a - \frac{a}{2} e^{-\mu x - D_n}, \quad \lim_{x \to +\infty} \phi(x) = a - \frac{a}{2} e^{-\mu x - D_n},
\] (C.9)

where \( D_n = 1 + 1/2 + 1/3 + \ldots + 1/n \).

As expected, for \( n = 1, 2 \) these results for the kink tail agree with those given by equations (50) and (C.4), respectively.
Appendix D. Details of the kink solutions of equation (53) in section 5

We now discuss the kink solution for the class of potentials given by equation (53). We have already discussed the $n = 1$ case in section 5 and we now discuss the $n = 2$ case and then generalize to arbitrary $n$.

Case II: $n = 2$ On using equation (3) the self-dual first-order equation is

$$\sqrt{2} \lambda x = \int \frac{d\phi}{\phi(a^2 - \phi^2)^2}.$$  \hspace{1cm} (D.1)

The integrand on the right hand side can be evaluated using partial fractions and it can be shown to be equal to

$$\frac{\phi}{a^2(a^2 - \phi^2)^2} + \frac{\phi}{a^4(a^2 - \phi^2)^2} + \frac{\phi}{a^6(a^2 - \phi^2)^2} + \frac{1}{a^6 \phi}.$$  \hspace{1cm} (D.2)

This is easily integrated with the solution

$$\mu x = \frac{a^4}{2(a^2 - \phi^2)^2} + \frac{a^2}{(a^2 - \phi^2)^2} + \ln \frac{\phi^2}{(a^2 - \phi^2)^2}, \quad \mu = 2\sqrt{2} \lambda a^6.$$  \hspace{1cm} (D.3)

Asymptotically,

$$\lim_{x \to -\infty} \phi(x) = ae^{\mu x/2 - 2 - 1/2}, \quad \lim_{x \to \infty} \phi(x) = a - \frac{a}{(8\mu x)^{1/2}}.$$  \hspace{1cm} (D.4)

Thus the kink tail around $\phi = a$ is again entirely determined by the first term on the right hand side of equation (D.3), i.e. by the term $\frac{a^4}{2(a^2 - \phi^2)^2}$. Further, on comparing the kink tail around $\phi = 0$ (i.e. as $x \to -\infty$) in the case of $n = 1$ and $n = 2$ as given by equations (58) and (D.4) respectively, we find that in both cases the kink tail has an exponential fall off

$$\lim_{x \to -\infty} \phi(x) = ae^{\mu x/2 - K},$$  \hspace{1cm} (D.5)

where the constant $K$ equals 1 in the $n = 1$ case while it is $1 + \frac{1}{2} = \frac{3}{2}$ in the $n = 2$ case and the rest of the behaviour is the same in both cases.

Case III: General $n$ The generalization to arbitrary $n$ is now straightforward. On using equation (3), the self-dual first-order equation is

$$\sqrt{2} \lambda x = \int \frac{d\phi}{\phi(a^2 - \phi^2)^2}.$$  \hspace{1cm} (D.6)

The integrand on the right hand side can be evaluated using partial fractions and it can be shown to be equal to

$$\sum_{k=1}^{n+1} \frac{\phi}{a^{2k}(a^2 - \phi^2)^{n+2-k}} + \frac{1}{a^{2n+2} \phi}.$$  \hspace{1cm} (D.7)

This is easily integrated with the solution

$$\mu x = \frac{a^{2n}}{n(a^2 - \phi^2)^2} + \sum_{k=2}^{n} \frac{a^{2n+2-2k}}{(n+1-k)(a^2 - \phi^2)^{n+1-k}} + \ln \frac{\phi^2}{(a^2 - \phi^2)^2}.$$  \hspace{1cm} (D.8)
Asymptotically,
\[
\lim_{x \to -\infty} \phi(x) = ae^{\mu x/2 - D_n}, \quad \lim_{x \to +\infty} \phi(x) = a - \frac{a}{(n^{2/n})^{1/n}}, \quad (D.9)
\]
where \( D_n = 1 + 1/2 + 1/3 + \ldots + 1/n \).

As expected, for \( n = 1, 2 \) these results for the kink tail agree with those given by equations (58) and (D.4), respectively.

Appendix E

E.1. Details of the kink solutions of equation (62) in section 6

We now discuss the kink solutions for the class of potentials given by equation (62). We have already discussed the \( n = 2 \) case in section 6 and we now discuss the \( n = 3 \) case and then generalize to arbitrary \( n \). For completeness we first discuss the \( n = 1 \) case in which the kink tail around \( \phi = a \) as well as around \( \phi = 0 \) both fall off like \( x^{-1} \).

Case II: \( n = 1 \) On using equation (3) in equation (62), for \( n = 1 \) the self-dual first-order equation is
\[
\frac{d\phi}{dx} = \pm \sqrt{2} \lambda \phi^2 (a^2 - \phi^2)^2. \quad (E.1)
\]
In this case
\[
\sqrt{2} \lambda x = \int \frac{d\phi}{\phi^2(a^2 - \phi^2)^2}. \quad (E.2)
\]
The integrand on the right hand side can be written as
\[
\frac{1}{a^2(a^2 - \phi^2)} + \frac{1}{a^4(a^2 - \phi^2)} + \frac{1}{a^3\phi^2}. \quad (E.3)
\]
This is easily integrated using the identity (13) with the solution
\[
\mu x = \frac{a\phi}{2(a^2 - \phi^2)} - \frac{a}{\phi} + \frac{3}{4} \ln \frac{a + \phi}{a - \phi}, \quad \mu = \sqrt{2} \lambda a^5. \quad (E.4)
\]
Asymptotically,
\[
\lim_{x \to -\infty} \phi(x) = \frac{a}{-\mu x}, \quad \lim_{x \to +\infty} \phi(x) = a - \frac{a}{4\mu x}. \quad (E.5)
\]
Thus while the kink tail around \( \phi = a \) is entirely determined by the first term on the right hand side of equation (E.4), i.e. by the term \( \frac{a\phi}{2(a^2 - \phi^2)} \), the kink tail around \( \phi = 0 \) is entirely decided by the second term on the right hand side of equation (E.4), i.e. by the term \( -\frac{a}{\phi} \). Observe that in this case the kink tail falls off like \( x^{-1} \) around both \( \phi = a \) and \( \phi = 0 \).
Case III: $n = 3$ On using equation (3) in equation (62), for $n = 3$ the self-dual first-order equation is

$$ \frac{d\phi}{dx} = \pm \sqrt{2\lambda} \phi^2 (a^2 - \phi^2)^4. \quad (E.6) $$

In this case

$$ \sqrt{2} \lambda x = \int \frac{d\phi}{\phi^2 (a^2 - \phi^2)^4}. \quad (E.7) $$

The integrand on the right hand side can be written as

$$ \frac{1}{a^2(a^2 - \phi^2)^4} + \frac{1}{a^4(a^2 - \phi^2)^4} + \frac{1}{a^6(a^2 - \phi^2)^4} + \frac{1}{a^6 \phi^2}. \quad (E.8) $$

This is easily integrated using the identity (13) with the solution

$$ \mu x = \frac{a^5 \phi}{6(a^2 - \phi^2)^3} - \frac{a}{\phi} + O\left(\frac{\phi}{(a^2 - \phi^2)^p}\right) + K \ln \frac{a + \phi}{a - \phi}, \quad \mu = \sqrt{2} \lambda a^9, \quad p = 1, 2. \quad (E.9) $$

It may be noted that we have only explicitly specified those terms which are relevant for knowing the kink tail around $\phi = a$ as well as around $\phi = 0$. Asymptotically,

$$ \lim_{x \to -\infty} \phi(x) = \frac{a}{-\mu x}, \quad \lim_{x \to \infty} \phi(x) = \frac{a}{(48 \mu x)^{1/3}}. \quad (E.10) $$

Thus while the kink tail around $\phi = a$ is entirely determined by the first term on the right hand side of equation (E.9), i.e. by the term $\frac{a^5 \phi}{6(a^2 - \phi^2)^3}$, the kink tail around $\phi = 0$ is entirely decided by the second term on the right hand side of equation (E.9), i.e. by the term $-\frac{a}{\phi}$. Observe that in this case the kink tail falls off like $x^{-1}$ around $\phi = 0$ while it falls off like $x^{-1/3}$ around $\phi = a$.

Case IV: General $n$ The generalization to arbitrary $n$ is now straightforward. On using equation (3), the self-dual first-order equation is

$$ \sqrt{2} \lambda x = \int \frac{d\phi}{\phi^2 (a^2 - \phi^2)^{n+1}}. \quad (E.11) $$

The integrand on the right hand side can be evaluated using partial fractions and it can be shown to be

$$ \sum_{k=1}^{n+1} \frac{1}{a^{2k}(a^2 - \phi^2)^{n+2-k}} + \frac{1}{a^{2n+2} \phi^2}. \quad (E.12) $$

This is easily integrated using the identity (13) with the solution

$$ \mu x = \frac{a^{2n-1} \phi}{2n(a^2 - \phi^2)^n} - \frac{a}{\phi} + O\left(\frac{\phi}{(a^2 - \phi^2)^p}\right) + K \ln \frac{a + \phi}{a - \phi}, \quad \mu = \sqrt{2} \lambda a^{2n+3}, \quad p = 1, 2, \ldots, (n - 1). \quad (E.13) $$

where $\mu = \sqrt{2} \lambda a^{2n+3}$ while $p = 1, 2, \ldots, (n - 1)$. It may be noted that we have only explicitly specified those terms which are relevant for knowing the kink tail around $\phi = a$ as well as around $\phi = 0$. 
Asymptotically,

\[
\lim_{x \to -\infty} \phi(x) = \frac{a}{-\mu x}, \quad \lim_{x \to \infty} \phi(x) = a - \frac{a}{(n2^{n+1}/\mu x)^{1/n}}.
\]  

(E.14)

As expected, for \( n = 2, 3 \) these results for the kink tail agree with those given by equations (65) and (E.10), respectively.

\subsection*{E.2. Details of the kink solutions of equation (70) in section 6}

We now discuss the kink solution for the class of potentials given by equation (70). We have already discussed the \( n = 2 \) case in section 6 and we now discuss the \( n = 3 \) case and then generalize to arbitrary \( n \).

\textbf{Case II:} \( n = 3 \)  

On using equation (3), the self-dual first-order equation is

\[
\sqrt{2} \lambda x = \int \frac{d\phi}{\phi^2(a^2 - \phi^2)^{3/2}}.
\]  

(E.15)

The integrand on the right hand side can be evaluated using partial fractions and it can be shown to be

\[
\frac{1}{a^2 \phi^6} + \frac{1}{a^6 (a^2 - \phi^2)^2} + \frac{2}{a^6 \phi^4} + \frac{3}{a^8 \phi^2} + \frac{3}{a^{10} (a^2 - \phi^2)^2}.
\]  

(E.16)

This is easily integrated with the solution

\[
\mu x = -\frac{2a^5}{5\phi^5} + \frac{a\phi}{a^2 - \phi^2} - \frac{4a^3}{3\phi^3} - \frac{6a}{\phi} + \frac{7}{2} \ln \left( \frac{a + \phi}{a - \phi} \right), \quad \mu = 2\sqrt{2} \lambda a^9.
\]  

(E.17)

Asymptotically,

\[
\lim_{x \to -\infty} \phi(x) = \frac{2a}{(-5\mu x)^{1/3}}, \quad \lim_{x \to \infty} \phi(x) = a - \frac{a}{2\mu x}.
\]  

(E.18)

Thus the kink tail around \( \phi = 0 \) is again entirely determined by the first term on the right hand side of equation (E.17), i.e. by the term \( -\frac{2a^5}{5\phi^5} \). On the other hand, the kink tail around \( \phi = a \) is entirely determined by the second term on the right hand side of equation (E.17), i.e. by the term \( \frac{a\phi}{a^2 - \phi^2} \). It may be noted that the behaviour of the kink tail around \( \phi = a \) (i.e. when \( x \to \infty \)) is the same for \( n = 1, 2, 3 \).

It is then clear that even for arbitrary \( n \), the behaviour of the kink tail around \( \phi = a \) will be the same as for \( n = 1, 2, 3 \). On the other hand the behaviour of the kink tail around \( \phi = 0 \) (i.e. when \( x \to -\infty \)) will simply depend on the most singular term in \( \phi \) as \( \phi \to 0 \).

\textbf{Case III: General} \( n \)  

The generalization to arbitrary \( n \) is now straightforward. On using equation (3), the self-dual first-order equation is

\[
\sqrt{2} \lambda x = \int \frac{d\phi}{\phi^{2n}(a^2 - \phi^2)^{3/2}}.
\]  

(E.19)

The integrand on the right hand side can be evaluated using partial fractions and it can be shown to be

\[
\sum_{k=1}^{n} \frac{n - k + 1}{a^{2(n-k+2)}(a^2 - \phi^2)^{2k}} + \frac{1}{a^{2n}(a^2 - \phi^2)^{3/2}} + \frac{n}{a^{2n+1}(a^2 - \phi^2)^{2}}.
\]  

(E.20)
This is easily integrated with the solution
\[ \mu x = -\frac{2a^{2n-1}}{(2n-1)\phi^{2n-1}} + \frac{a\phi}{a^2 - \phi^2} - \sum_{k=1}^{n-1} \frac{2(n-k+1)a^{2k-1}}{(2k-1)\phi^{2k-1}} + \frac{(2n+1)}{2} \ln \frac{a + \phi}{a - \phi}, \] \hspace{1cm} (E.21)

where
\[ \mu = 2\sqrt{2}\lambda a^{2n+3}. \] \hspace{1cm} (E.22)

Asymptotically,
\[ \lim_{x \to -\infty} \phi(x) = \frac{2^{1/(2n-1)}a}{\mu x^{1/(2n-1)}}, \quad \lim_{x \to \infty} \phi(x) = a - \frac{a}{2\mu x}. \] \hspace{1cm} (E.23)

As expected, for \( n = 1, 2, 3 \) these results for the kink tail agree with those given by equations (E.5), (75) and (E.18), respectively. Thus the kink tail around \( \phi = a \) goes like \( x^{-1} \) while it goes like \( x^{-1/(2n-1)} \) around \( \phi = 0 \).

### E.3. Details of the kink solutions of equation (76) in section 6

We now discuss the kink solutions for a class of potentials given by equation (76). We have already discussed the \( n = 1 \) case in section 6 and we now discuss the \( n = 2 \) case and then generalize to arbitrary \( n \).

**Case II: \( n = 2 \)** On using equation (3) the self-dual first-order equation is
\[ \sqrt{2}\lambda x = \int \frac{d\phi}{\phi^5(a^2 - \phi^2)^2}. \] \hspace{1cm} (E.24)

The integrand on the right hand side can be evaluated using partial fractions and it can be shown to be
\[ \frac{1}{a^4\phi^5} + \frac{\phi}{a^6(a^2 - \phi^2)^2} + \frac{2}{a^8\phi^3} + \frac{3}{a^{10}\phi} + \frac{3\phi}{a^{12}(a^2 - \phi^2)}. \] \hspace{1cm} (E.25)

This is easily integrated with the solution
\[ \mu x = -\frac{a^4}{2\phi^5} + \frac{a^2}{a^2 - \phi^2} - \frac{2a^2}{\phi^3} + 3 \ln \frac{\phi^2}{a^2 - \phi^2}, \quad \mu = 2\sqrt{2}\lambda a^8. \] \hspace{1cm} (E.26)

Asymptotically,
\[ \lim_{x \to -\infty} \phi(x) = \frac{a}{(-2\mu x)^{1/4}}, \quad \lim_{x \to \infty} \phi(x) = a - \frac{a}{2\mu x}. \] \hspace{1cm} (E.27)

Thus the kink tail around \( \phi = 0 \) is again entirely determined by the first term on the right hand side of equation (E.26), i.e. by the term \(-\frac{a^4}{2\phi^5}\). On the other hand, the kink tail around \( \phi = a \) is again determined by the second term on the right hand side of equation (E.26), i.e. by the term \( \frac{a^2}{a^2 - \phi^2} \). It is worth pointing out that the kink tail around \( \phi = a \) is the same in the case of \( n = 1 \) and \( n = 2 \), and we shall see below that it is the same for any \( n \).

**Case III: General \( n \)** The generalization to arbitrary \( n \) is now straightforward. On using equation (3), the self-dual first-order equation is
\[ \sqrt{2} \lambda x = \int \frac{d\phi}{\phi^{2n+1}(a^2 - \phi^2)^2}. \] (E.28)

The integrand on the right hand side can be evaluated using partial fractions and it can be shown to be

\[ \sum_{k=1}^{n+1} \frac{n - k + 2}{a^{2(n-k+1)} \phi^{2(k-1)}} + \frac{\phi}{a^{2n+2}(a^2 - \phi^2)^2} + \frac{(n+1)\phi}{a^{2n+4}(a^2 - \phi^2)}. \] (E.29)

This is easily integrated with the solution

\[ \mu x = -\frac{a^{2n}}{n \phi^{2n}} + \frac{a^2}{a^2 - \phi^2} - \sum_{k=2}^{n} \frac{(n + 2 - k)\phi^{2(k-1)}}{(k - 1)\phi^{2(k-1)}} + (n + 1) \ln \frac{\phi^2}{a^2 - \phi^2}, \] (E.30)

where \( \mu = 2\sqrt{2} \lambda a^{2n+4} \). Asymptotically,

\[ \lim_{x \to -\infty} \phi(x) = -\frac{a}{(-n \mu x)^{1/2n}}, \quad \lim_{x \to \infty} \phi(x) = a - \frac{a}{2\mu x}. \] (E.31)

As expected, for \( n = 1, 2 \) these results for the kink tail agree with those given by equations (80) and (E.27), respectively.

**Appendix F. Potentials admitting kink solutions with tails of the form e e e p**

We now briefly discuss a family of potentials of the form

\[ V(\phi) = \lambda^2 (a^2 - \phi^2)^2 (b^2 - \phi^2)^{2n+2}, \quad b > a, \quad n = 1, 2, 3, \ldots. \] (F.1)

These potentials have degenerate minima at \( \phi = \pm a, \pm b \) and \( V(\phi = \pm a, \pm b) = 0 \) and admit a kink from \(-a\) to \(a\) and a kink from \(a\) to \(b\) as well as a mirror kink from \(-b\) to \(-a\). While the kink from \(-a\) to \(a\) has an exponential tail around both \( \phi = -a \) as well as \( \phi = a \), the kink from \(a\) to \(b\) has an exponential tail around \( \phi = a \) and a power law tail around \( \phi = b \). In these cases too explicit analytic solutions are not possible and we can only find implicit kink solutions. From the latter we can obtain how a kink profile falls off as \( x \to \pm\infty \).

**- a to a kink**

On using equation (3), the self-dual first-order equation for the potential (F.1) is

\[ \sqrt{2} \lambda x = \int \frac{d\phi}{(a^2 - \phi^2)(b^2 - \phi^2)^{n+1}}. \] (F.2)

The integrand on the right hand side can be evaluated using partial fractions and it can be shown to be

\[ \sum_{k=1}^{n+1} \frac{1}{(b^2 - a^2)(n-k+2)(b^2 - \phi^2)^k} + \frac{1}{(b^2 - a^2)^{n+1}(a^2 - \phi^2)}. \] (F.3)

This is easily integrated with the solution

\[ \mu x = \ln\left(\frac{a + \phi}{a - \phi}\right) - \frac{a\phi(b^2 - a^2)^n}{nb^2(b^2 - \phi^2)^n} + \text{Lower Order Terms}, \] (F.4)

where \( \mu = 2\sqrt{2} \lambda a(b^2 - a^2)^{n+1} \). Note that in equation (F.4) we have only specified those terms which contribute to the dominant asymptotic behaviour as \( x \to \pm\infty \). Asymptotically we find that
\[
\lim_{x \to -\infty} \phi(x) = -a + 2af(a, b)e^{\mu x}, \quad \lim_{x \to +\infty} \phi(x) = a - 2af(a, b)e^{-\mu x}.
\]  
(F.5)

Thus for the kink solution from \(-a\) to \(a\), the kink tail around both \(\phi = -a\) and \(\phi = a\) has an exponential tail.

**a to b kink**

On using equation (3), the self-dual first-order equation is now given by

\[
\sqrt{2} \lambda x = \int \frac{d\phi}{(\phi^2 - a^2)(b^2 - \phi^2)^{n+1}}.
\]  
(G.2)

The integrand on the right hand side can be evaluated using partial fractions and it can be shown to be

\[
\sum_{k=1}^{n+1} \frac{1}{(b^2 - a^2)^{(n-k+2)}(b^2 - \phi^2)^k} + \frac{1}{(b^2 - a^2)^{n+1}(\phi^2 - a^2)}. \tag{F.7}
\]

This is easily integrated with the solution

\[
\mu x = \ln\left(\frac{\phi - a}{\phi + a}\right) + \frac{a\phi(b^2 - a^2)^n}{nb^2(b^2 - \phi^2)^n} + \text{Lower Orders}, \tag{F.8}
\]

where \(\mu = 2\sqrt{2}a \lambda (b^2 - a^2)^{n+1}\). Note that in equation (F.8) we have only specified those terms which contribute to the dominant asymptotic behaviour as \(x \to \pm \infty\). Asymptotically,

\[
\lim_{x \to -\infty} \phi(x) = a - 2af(a, b)e^{\mu x}, \quad \lim_{x \to +\infty} \phi(x) = b - b \left[ \frac{a(b^2 - a^2)^n}{n2^nb^{2n+1}\mu^x} \right]^{1/n}.
\]  
(F.9)

Thus for the kink solution from \(a\) to \(b\), while around \(\phi = a\) one has an exponential tail, the kink tail around \(\phi = b\) goes like \(x^{-1/n}\) as \(x \to \infty\).

**Appendix G. Potentials admitting kink solutions with tails of the form \(p p p\)**

We now briefly discuss a family of potentials of the form

\[
V(\phi) = \lambda^2(\phi^2 - \phi^2)^{2n+2}(b^2 - \phi^2)^2, \quad b > a, \quad n = 1, 2, 3, \ldots. \tag{G.1}
\]

These potentials have degenerate minima at \(\phi = \pm a, \pm b\) and \(V(\phi = \pm a, \pm b) = 0\) and admit a kink from \(-a\) to \(a\) and a kink from \(a\) to \(b\) as well as a mirror kink from \(-b\) to \(-a\). While the kink from \(-a\) to \(a\) has a power law tail around both \(\phi = -a\) as well as \(\phi = a\), the kink from \(a\) to \(b\) has a power law tail around \(\phi = a\) and an exponential tail around \(\phi = b\). In these cases too explicit analytic solutions are not possible and we can only find implicit kink solutions. From the latter we can obtain how a kink profile falls off as \(x \to \pm \infty\).

**a to b kink**

On using equation (3), the self-dual first-order equation for the potential (G.1) is

\[
\sqrt{2} \lambda x = \int \frac{d\phi}{(b^2 - \phi^2)(\phi^2 - b^2)^{n+1}}. \tag{G.2}
\]

The integrand on the right hand side can be evaluated using partial fractions and it can be shown to be
\[
\sum_{k=1}^{n+1} \frac{(-1)^{n-k+2}}{(b^{2} - a^{2})^{(n-k+2)}(a^{2} - \phi^{2})^{k}} + \frac{(-1)^{n+1}}{(b^{2} - a^{2})^{n+1}(b^{2} - \phi^{2})^{k}}. 
\] (G.3)

This is easily integrated with the solution

\[
\mu x = \frac{b\phi(b^{2} - a^{2})^{n}}{n a^{2}(a^{2} - \phi^{2})^{n}} + (-1)^{n+1} \ln b + \phi + \text{Lower Order Terms},
\] (G.4)

where \( \mu = 2\sqrt{2} b \lambda (b^{2} - a^{2})^{n+1} \). Note that in equation (G.4) we have only specified those terms which contribute to the dominant asymptotic behaviour as \( x \to \pm \infty \). Asymptotically,

\[
\lim_{x \to -\infty} \phi(x) = -a + a \left[ \frac{-b(b^{2} - a^{2})^{n}}{n^{2} a^{2n+1} \mu x} \right]^{1/n},
\]

\[
\lim_{x \to +\infty} \phi(x) = a - a \left[ \frac{b(b^{2} - a^{2})^{n}}{n^{2} a^{2n+1} \mu x} \right]^{1/n}.
\] (G.5)

Thus for the kink solution from \(-a\) to \(a\), the kink tail around both \( \phi = -a \) and \( \phi = a \) has a power law tail.

**a to b kink**

On using equation (3), the self-dual first-order equation is now given by

\[
\sqrt{2} \lambda x = \int \frac{d\phi}{(b^{2} - \phi^{2})(\phi^{2} - a^{2})^{n+1}}.
\] (G.6)

The integrand on the right hand side can be evaluated using partial fractions and it can be shown to be

\[
\sum_{k=1}^{n+1} \frac{1}{(b^{2} - a^{2})^{(n-k+2)}(\phi^{2} - a^{2})^{k}} + \frac{1}{(b^{2} - a^{2})^{n+1}(b^{2} - \phi^{2})^{k}}. 
\] (G.7)

This is easily integrated with the solution

\[
\mu x = \ln b + \phi - \frac{b\phi(b^{2} - a^{2})^{n}}{n a^{2}(\phi^{2} - a^{2})^{n}} + \text{Lower Order Terms},
\] (G.8)

where \( \mu = 2\sqrt{2} a \lambda (b^{2} - a^{2})^{n+1} \). Note that in equation (G.8) we have only specified those terms which contribute to the dominant asymptotic behaviour as \( x \to \pm \infty \). Asymptotically,

\[
\lim_{x \to -\infty} \phi(x) = a + a \left[ \frac{-b(b^{2} - a^{2})^{n}}{n^{2} a^{2n+1} \mu x} \right]^{1/n},
\]

\[
\lim_{x \to +\infty} \phi(x) = b + 2bh(a, b)e^{-\mu x}.
\] (G.9)

Thus for the kink solution from \(a\) to \(b\), while the kink tail around \( \phi = b \) is exponential, the kink tail around \( \phi = a \) goes like \((-x)^{-1/n}\) as \( x \to -\infty \).

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