ON CERTAIN QUASICONFORMAL AND ELLIPTIC MAPPINGS

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Abstract. Let $\overline{\mathbb{D}}$ be the closure of the unit disk $\mathbb{D}$ in the complex plane $\mathbb{C}$ and $g$ be a continuous function in $\overline{\mathbb{D}}$. In this paper, we discuss some characterizations of elliptic mappings $f$ satisfying the Poisson’s equation $\Delta f = g$ in $\mathbb{D}$, and then establish some sharp distortion theorems on elliptic mappings with the finite perimeter and the finite radial length, respectively. The obtained results are the extension of the corresponding classical results.

1. Preliminaries and main results

Let $\mathbb{D} = \{ z : |z| < 1 \}$ denote the open unit disk in the complex plane $\mathbb{C}$ and let $\mathbb{T} = \partial \mathbb{D}$ be the unit circle. Furthermore, we denote by $\mathcal{C}^{m}(\Omega)$ the set of all complex-valued $m$-times continuously differentiable functions from $\Omega$ into $\mathbb{C}$, where $\Omega$ is a subset of $\mathbb{C}$ and $m \in \{0, 1, 2, \ldots \}$. In particular, $\mathcal{C}(\Omega) := \mathcal{C}^{0}(\Omega)$ denotes the set of all continuous functions in $\Omega$. Let $G$ be a domain of $\mathbb{C}$ with $\overline{G}$ be its closure.

We use $d_G(z)$ to denote the Euclidean distance from $z$ to the boundary $\partial G$ of $G$. Especially, we always use $d(z)$ for $d_D(z)$.

For a real $2 \times 2$ matrix $A$, the matrix norm and the matrix function are defined by 

$$\|A\| = \sup\{|Az| : |z| = 1\}, \quad \text{and} \quad l(A) = \inf\{|Az| : |z| = 1\},$$

respectively. For $z = x + iy \in \mathbb{C}$, the formal derivative of a complex-valued function $f = u + iv$ is given by 

$$D_f := \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix},$$

so that 

$$\|D_f\| = |f_z| + |f_\bar{z}| \quad \text{and} \quad l(D_f) = ||f_z| - |f_\bar{z}||,$$

where 

$$f_z = (f_x - if_y)/2, \quad \text{and} \quad f_\bar{z} = (f_x + if_y)/2.$$

We use $J_f := \det D_f = |f_z|^2 - |f_\bar{z}|^2$ to denote the Jacobian of $f$.

A sense-preserving homeomorphism $f$ from a domain $\Omega$ onto $\Omega'$, contained in the Sobolev class $W^{1,2}_{loc}(\Omega)$, is said to be a $K$-quasiconformal mapping if, for $z \in \Omega$, 

$$\|D_f(z)\|^2 \leq K \det D_f(z), \quad \text{i.e.,} \quad \|D_f(z)\| \leq Kl(D_f(z)),$$

where $K \geq 1$.

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A mapping $f \in C^1(\Omega)$ is said to be an \textit{elliptic mapping} (or $(K,K')$-\textit{elliptic mapping}) if there are constants $K \geq 1$ and $K' \geq 0$ such that $f$ satisfies the following partial differential inequality
\[ \|Df(z)\|^2 \leq K J_f(z) + K', \]
for all $z \in \Omega$. Let $\Omega_1$ and $\Omega_2$ be two subdomains of $\mathbb{C}$. A sense-preserving homeomorphism $f : \Omega_1 \to \Omega_2$ is said to be a $(K,K')$-\textit{quasiconformal mapping} if $f$ is absolutely continuous on lines in $\Omega_1$, and there are constants $K \geq 1$ and $K' \geq 0$ such that
\[ \|Df(z)\|^2 \leq K J_f(z) + K', \quad z \in \Omega_1. \]
Obviously, a $(K,K')$-quasiconformal mapping $f \in C^1(\Omega)$ is an elliptic mapping in $\Omega$. In particular, if $K' = 0$, then the $(K,K')$-quasiconformal mappings are $K$-quasiconformal (cf. [2, 21]). Moreover, if a $(K,K')$-quasiconformal mapping is harmonic, then it is said to be harmonic $(K,K')$-quasiconformal.

In 1968, Martio [25] has discussed the conditions for the $K$-quasiconformality of harmonic mappings from the closed unit disk onto itself. The harmonic $K$-quasiconformal mappings of Riemannian manifolds have been considered by Goldberg and Ishihara (see [13, 12]). Tam and Wan [32] have investigated some properties of harmonic quasiconformal diffeomorphisms and the universal Teichmuller space. In 2002, Pavlović [31] has generalized the corresponding results of Martio [25]. Kalaj [16], Partyla and Sakan [28] have investigated the $K$-quasiconformality of harmonic mappings from the unit disk onto bounded convex domains. See [22, 18, 24, 29, 33] and the references therein for detailed discussions on this topic. Recent papers [2, 21] and [20] bring much attention on the topic of $(K,K')$-quasiconformal mappings in the plane. This paper continues the study of previous work of [4, 5] and is mainly motivated by the articles of Finn and Serrin [11], and Kalaj and Mateljević [21]. In order to state our main results, we need to recall some basic definitions and some results which motivate the present work.

For $\theta \in [0,2\pi]$ and $z,w \in \mathbb{D}$ with $z \neq w$, let
\[ G(z,w) = \log \frac{1 - z\overline{w}}{z - w} \quad \text{and} \quad P(z,e^{i\theta}) = \frac{1 - |z|^2}{|1 - ze^{-i\theta}|^2}, \]
denote the Green function and the (harmonic) Poisson kernel, respectively.

Let $\psi : \mathbb{T} \to \mathbb{C}$ be a bounded integrable function, and $g \in C(\mathbb{D})$. The solution to the Poisson equation
\[ \begin{cases} \Delta f = g & \text{in } \mathbb{D}, \\ f = \psi & \text{in } \mathbb{T}, \end{cases} \]
is given by
\[ f(z) = P[\psi](z) - G[g](z), \]
where
\[ G[g](z) = \frac{1}{2\pi} \int_{\mathbb{D}} G(z,w)g(w)dA(w), \quad P[\psi](z) = \frac{1}{2\pi} \int_0^{2\pi} P(z,e^{it})\psi(e^{it})dt, \]
\[ (1.2) \]
and \(dA(w)\) denotes the Lebesgue measure on \(\mathbb{D}\). It is well known that if \(\psi\) and \(g\) are continuous in \(\mathbb{T}\) and in \(\overline{\mathbb{D}}\), respectively, then \(f = P[\psi] - G[g]\) has a continuous extension \(\tilde{f}\) to the boundary, and \(\tilde{f} = \psi\) in \(\mathbb{T}\) (see [15, pp. 118-120] and [22, 17]).

A continuous increasing function \(\omega : [0, \infty) \to [0, \infty)\) with \(\omega(0) = 0\) is called a majorant if \(\omega(t)/t\) is non-increasing for \(t > 0\) (see [9, 10, 30]). For \(\alpha > 0\) and a majorant \(\omega\), we use \(B^\alpha_\omega(\mathbb{D})\) to denote the generalized Bloch-type space of all functions \(f \in C^1(\mathbb{D})\) with \(\|f\|_{B^\alpha_\omega(\mathbb{D})} < \infty\), where

\[
\|f\|_{B^\alpha_\omega(\mathbb{D})} = |f(0)| + \sup_{z \in \mathbb{D}} \{\|D_f(z)\|\omega(d^\alpha(z))\}.
\]

For a given bounded integrable function \(\psi \in L^1(\mathbb{T})\) and a given \(g \in C(\overline{\mathbb{D}})\), let

\[
\mathcal{F}_{g,\psi}(\mathbb{D}) = \{f \in C^2(\mathbb{D}) : \Delta f = g \text{ in } \mathbb{D} \text{ and } f = \psi \text{ in } \mathbb{T}\}.
\]

Clearly, functions in \(\mathcal{F}_{0,\psi}(\mathbb{D})\) are harmonic in \(\mathbb{D}\). Furthermore, if \(f \in \mathcal{F}_{g,\psi}(\mathbb{D})\), then \(f + G[g]\) is harmonic in \(\mathbb{D}\), and thus, has the representation

\[
(1.3) \quad f + G[g] = h_1 + h_2,
\]

where \(h_1\) and \(h_2\) are analytic in \(\mathbb{D}\) (cf. [8]), and \(G[g]\) is defined in (1.2).

In [21], Kalaj and Mateljević proved that a harmonic diffeomorphism between two bounded Jordan domains with \(C^2\) boundaries is a harmonic \((K, K')\)-quasiconformal mapping for some constants \(K \geq 1\) and \(K' \geq 0\) if and only if it is Lipschitz continuous (see [21, Theorem 1.1 and Corollary 1.3]). This result can be considered as an extension of the corresponding results of Martio [25], Pavlović [31], and Partyka and Sakan [29]. For related investigations on this topic, we refer to [28, 33]. In the following, we will give some characterizations of elliptic mappings without the \(C^2\) boundary hypothesis of the image domains.

**Theorem 1.1.** Suppose that \(\omega\) is a given majorant. For a given bounded integrable function \(\psi \in L^1(\mathbb{T})\) and a given \(g \in C(\overline{\mathbb{D}})\), let \(f \in \mathcal{F}_{g,\psi}(\mathbb{D})\) be a univalent and sense-preserving mapping, and \(f(\mathbb{D})\) be a convex domain. If there are two positive constants \(C_1, C_2\) and \(\alpha \in [0, 1]\) such that for any \(z_1, z_2 \in \mathbb{D}\) with \(z_1 \neq z_2\),

\[
(1.4) \quad \frac{\omega \left(\frac{1}{(1 + |z_1|)(1 + |z_2|)}^{\frac{1-\alpha}{\alpha}}\right)}{C_1} \leq \frac{|f(z_1) - f(z_2)|}{|z_1 - z_2|} \leq \frac{C_1}{\omega \left(\frac{1}{(d(z_1)d(z_2))^{\frac{1-\alpha}{\alpha}}}\right)}
\]

and

\[
\int_0^1 \frac{dt}{\omega \left(\frac{1}{(d(\Phi(t)))^{\frac{1-\alpha}{\alpha}}}\right)} \leq C_2,
\]

then \(f\) is an elliptic mapping, where \(\Phi(t) := f^{-1}(f(z_1) + t(f(z_2) - f(z_1)))\).

**Remark 1.2.** In particular, if \(\alpha = 1\) in (1.4), then \(f\) is bi-Lipschitz. In this situation, Theorem 1.1 is trivial because all univalent and sense-preserving bi-Lipschitz mappings are quasiconformal mappings (see Chapter 14.78 in [14]). However, quasiconformal mappings are not necessarily bi-Lipschitz, not even Lipschitz (see Example 1.3).
Example 1.3. Let
\[
f(z) = \begin{cases} 
z \log^\alpha \left( \frac{e}{|z|^2} \right) & \text{for } z \in \mathbb{D} \setminus \{0\}, \\
0 & \text{for } z = 0,
\end{cases}
\]
where \( \alpha \in (0,1/2) \) is a constant. Then \( f \) is a quasiconformal self-homeomorphism of \( \mathbb{D} \). However, \( f \) is not Lipschitz at the origin (cf. [23]).

Proposition 1.4. For a given bounded integrable function \( \psi \in L^1(\mathbb{T}) \) and a given \( g \in C(\mathbb{D}) \), let \( f = h_1 + \overline{h_2} - G[g] \in F_{g,\psi}(\mathbb{D}) \), where \( h_1 \) and \( h_2 \) are analytic in \( \mathbb{D} \). If there is a constant \( C_3 \in [1,2) \) such that for any \( z_1, z_2 \in \mathbb{D} \),
\[
(1.5) \quad |f(z_1) - f(z_2)| \leq C_3|h_1(z_1) - h_1(z_2)|,
\]
then \( f \) is an elliptic mapping.

We remark that the inverse of Proposition 1.4 does not necessarily hold (see Example 1.5).

Example 1.5. For \( z \in \mathbb{D} \), let \( f(z) = 3z|z|^2 - z|z|^8 \). Then

1. \( f \) is a \((1,729/(2^{16/3}))\)-quasiconformal mapping in \( \mathbb{D} \);
2. \( f \) is not a \( K \)-quasiconformal mapping for any \( K \geq 1 \);
3. \( f \) does not satisfy the inequality (1.5).

Proof. We first prove the univalence of \( f \). Suppose on the contrary that \( f \) is not univalent. Then there are two distinct points \( z_1, z_2 \in \mathbb{D} \) such that \( f(z_1) = f(z_2) \), which implies that
\[
(1.6) \quad z_1|z_1|^2(3 - |z_1|^6) = z_2|z_2|^2(3 - |z_2|^6).
\]

**Case 1.** If \( |z_1| = |z_2| \), then, by (1.6), we have \( z_1 = z_2 \). This is a contradiction with the assumption.

**Case 2.** If \( |z_1| \neq |z_2| \), then (1.6) reduces to \( |z_1|^3(3 - |z_1|^6) = |z_2|^3(3 - |z_2|^6) \), and consequently
\[
(|z_1|^3 - |z_2|^3)(3 - |z_1|^6 - |z_2|^6 - |z_1|^3|z_2|^3) = 0.
\]
This implies \( |z_1| = |z_2| = 1 \), which violates the hypothesis. Hence \( f \) is univalent.

Also, for \( z \in \mathbb{D} \), elementary calculations lead to
\[
f_z(z) = |z|^2(6 - 5|z|^6) \quad \text{and} \quad f_\overline{z}(z) = z^2(3 - 4|z|^6),
\]
which give that
\[
(1.7) \quad \|D_f(z)\|^2 - J_f(z) \leq \|D_f(z)\|^2 = \max_{|z|\in[0,1]} \left\{ 81|z|^4(1 - |z|^6)^2 \right\} = 729/(2^{16/3})
\]
and
\[
(1.8) \quad \lim_{|z|\to1-} \frac{|f_\overline{z}(z)|}{|f_z(z)|} = 1.
\]
The first assertion and the second assertion easily follows from (1.7) and (1.8), respectively. The last assertion is obvious. \( \square \)
For \( r \in (0, 1) \) and \( f \in C^1(D) \), the perimeter of the curve \( C(r) = \{ w = f(re^{i\theta}) : \theta \in [0, 2\pi] \} \), with counting multiplicity, is defined by (cf. \([3, 5, 7]\))

\[
\ell_f(r) = \int_0^{2\pi} |df(re^{i\theta})| = r \int_0^{2\pi} \left| f_z(re^{i\theta}) - e^{-2i\theta} f_z(re^{i\theta}) \right| d\theta.
\]

In particular, let \( \ell_f(1) = \sup_{0 < r < 1} \ell_f(r) \). Let us recall the following distortion theorem for \( K \)-quasiconformal harmonic mappings with finite perimeter.

**Theorem A.** ([5, Theorem 2]) Let \( f(z) = \sum_{n=0}^\infty a_n z^n + \sum_{n=1}^\infty b_n \overline{z}^n \) be a \( K \)-quasiconformal harmonic mapping. If \( \ell_f(1) < \infty \), then for \( n \geq 1 \),

\[
|a_n| + |b_n| \leq \frac{K \ell_f(1)}{2n\pi},
\]

\[
\|D_f(z)\| \leq \frac{\ell_f(1) \sqrt{R}}{2\pi(1 - |z|)}
\]

and \( f \in B^1_r(D) \), where \( \omega(t) = t \). In particular, if \( K = 1 \), then the above two estimates are sharp, and the extreme function is \( f(z) = z \).

Concerning the generalized form of inequality (1.9), Mateljević \([26]\) proved the following result: Let \( f(z) = \sum_{n=0}^\infty a_n z^n + \sum_{n=1}^\infty b_n \overline{z}^n \) be a harmonic mapping with \( \ell_f(1) < \infty \). Then, for \( n \geq 1 \), the inequality

\[
|a_n| + |b_n| \leq \frac{\ell_f(1)}{n\pi}
\]

holds (see \([26, \text{Theorem 10}]\)). Moreover, Kalaj \([19]\) improved the inequality (1.10) and obtained a sharp inequality for harmonic diffeomorphisms of \( D \). It reads as follows.

**Theorem B.** If \( f \) is a harmonic sense-preserving diffeomorphism of \( D \) onto a Jordan domain \( \Omega \) with rectifiable boundary of length \( 2\pi R \), then the sharp inequality

\[
|f_z(z)| \leq \frac{R}{1 - |z|^2}, \quad z \in D
\]

holds, where \( R \) is a positive constant. If the equality in (1.12) is attained for some \( a \), then \( \Omega \) is convex, and there is a holomorphic function \( \mu : \mathbb{D} \to \mathbb{D} \) and a constant \( \theta \in [0, 2\pi] \), such that

\[
F(z) := e^{-i\theta} f \left( \frac{z + a}{1 + z\overline{a}} \right) = R \left( \int_0^2 \frac{dt}{1 + t^2 \mu(t)} + \int_0^2 \frac{\mu(t) dt}{1 + t^2 \mu(t)} \right).
\]

Moreover, every function \( f \) defined by (1.13) is a harmonic diffeomorphism and maps \( \mathbb{D} \) to a Jordan domain bounded by a convex curve of length \( 2\pi R \) and the inequality (1.12) is attained for \( z = a \).

The following result is also a generalization of Theorem A.
Theorem 1.6. Let $K \geq 1$, $K' \geq 0$ and $R > 0$ be constants. If $f(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n z^n$ is a harmonic $(K, K')$-quasiconformal mapping of $\mathbb{D}$ onto a Jordan domain $\Omega$ with rectifiable boundary of length $2\pi R$, then for $n \geq 1$,

$$
|a_n| + |b_n| \leq \frac{\sqrt{K' + KR}}{n},
$$

(1.14)

$$
\|D_f(z)\| \leq \left( R + \frac{-R + \sqrt{K' + KK' + K^2 R^2}}{1 + K} \right) \frac{1}{1 - |z|^2}, \quad \text{for } z \in \mathbb{D},
$$

(1.15)

and $f \in \mathcal{B}_1^1(\mathbb{D})$.

In particular, if $K' = K - 1 = 0$ and $\Omega = \mathbb{D}$, then the estimates of (1.14) is sharp, and the extreme function is $f(z) = z$ for $z \in \mathbb{D}$. Moreover, if $K' = K - 1 = 0$ and $\Omega = \mathbb{D}$, then the equal sign occurs in (1.15) for some fixed $z = a$ if and only if $f(z) = e^{it} \frac{i-z}{1-az}$, where $t \in [0, 2\pi]$.

We remark that if $K' = 0$ and $K \in [1, 2]$, then the inequality (1.14) is better than (1.11).

Let $f \in \mathcal{C}^1(\mathbb{D})$. Then, for $\theta \in [0, 2\pi]$, the radial length of the curve $C_\theta(r) = \{w = f(\rho e^{i\theta}) : 0 \leq \rho \leq r < 1\}$, with counting multiplicity, is defined by

$$
\ell_r^\ast(r, \theta) = \int_0^r |df(\rho e^{i\theta})| = \int_0^r \left|f_z(\rho e^{i\theta}) + e^{-2i\theta} f(\rho e^{i\theta})\right| d\rho.
$$

In particular, let

$$
\ell_1^\ast(1, \theta) = \sup_{0 \leq r < 1} \ell_r^\ast(r, \theta).
$$

We refer the reader to [5, 6] for some discussion of the radial length. In particular, the following result establishes the Fourier coefficient estimates of $K$-quasiconformal harmonic mappings with the finite radial length.

Theorem C. ([3, Theorem 4]) Let $f(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n z^n$ be a harmonic $K$-quasiconformal mapping in $\mathbb{D}$. If $\ell_1^\ast(1) = \sup_{\theta \in [0, 2\pi]} \ell_r^\ast(1, \theta) < \infty$, then

$$
|a_n| + |b_n| \leq K \ell_1^\ast(1) \quad \text{for } n \geq 1.
$$

Moreover, if $K = 1$, then the estimate (1.16) is sharp and the extreme function is $f(z) = \ell_1^\ast(1) z$ for $z \in \mathbb{D}$.

We improve Theorem C into the following form.

Theorem 1.7. For $K \geq 1$ and $K' \geq 0$, let $f(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n z^n$ be a harmonic $(K, K')$-quasiconformal mapping in $\mathbb{D}$. If $\ell_1^\ast(1) = \sup_{\theta \in [0, 2\pi]} \ell_r^\ast(1, \theta) < \infty$, then for $n \geq 1$,

$$
|a_n| + |b_n| \leq \sqrt{K'} + K \ell_1^\ast(1).
$$

(1.17)

In particular, if $K' = K - 1 = 0$, then the estimate (1.17) is sharp, and the extreme function is $f(z) = \ell_1^\ast(1) z$ for $z \in \mathbb{D}$.
The proof of Theorems 1.1, 1.6 and 1.7, and Proposition 1.4 will be presented in Section 2.

2. The proofs of the main results

In this section, we shall prove Theorems 1.1 and 1.7, and Propositions 1.4 and 1.6. We start with some useful Lemmas.

Lemma D. ([4, Lemma 6]) Let \( \omega \) be a majorant and \( \nu \in [0, 1] \). Then for \( t \in [0, \infty) \),

\[
\omega(\nu t) \geq \nu \omega(t).
\]

Lemma 2.1. Let \( \omega \) be a majorant and \( \alpha \in [0, 1] \) be a constant. Suppose that \( f \in C^1(D) \) is univalent, and \( f(D) \) is a convex domain. Then the following two statements are equivalent:

(a) For any \( z_1, z_2 \) with \( z_1 \neq z_2 \), there exists a constant \( C_4 \) such that

\[
\frac{1}{C_4} \omega \left( (1 + |z_1|)(1 + |z_2|) \right)^{(1-\alpha)/2} \leq \frac{|f(z_1) - f(z_2)|}{|z_1 - z_2|} \leq \frac{C_4}{\omega \left( (d(z_1)d(z_2))^{1-\alpha}/2 \right)}.
\]

(b) For any \( z \in D \), there is a constant \( C_5 > 0 \) such that

\[
\frac{1}{C_5} \omega \left( (1 + |z|)^{1-\alpha} \right) \leq l(D_f(z)) \leq \|D_f(z)\| \leq \frac{C_5}{\omega \left( (d(z))^{1-\alpha} \right)}.
\]

Proof. We first prove (a) \( \Rightarrow \) (b). For \( \theta \in [0, 2\pi] \) and \( z = x + iy \in D \), elementary computations lead to (see (1.1))

\[
f_z(z) + f_{\bar{z}}(z) = f_x(z) \quad \text{and} \quad i(f_z(z) - f_{\bar{z}}(z)) = f_y(z)
\]

and therefore,

\[
(2.1) \quad f_z(z) \cos \theta + f_y(z) \sin \theta = f_z(z) e^{i\theta} + f_{\bar{z}}(z) e^{-i\theta}.
\]

For \( r \in [0, 1 - |z|] \), let \( \xi = z + re^{i\theta} \). Then, by (a) and (2.1), we obtain

\[
\|D_f(z)\| = \max_{\theta \in [0, 2\pi]} |f_z(z) \cos \theta + f_y(z) \sin \theta| = \max_{\theta \in [0, 2\pi]} \left\{ \lim_{r \to 0^+} \frac{|f(z) - f(\xi)|}{|z - \xi|} \right\}
\]

\[
\leq \max_{\theta \in [0, 2\pi]} \left\{ \lim_{r \to 0^+} \frac{C_4}{\omega \left( (d(z)d(z + re^{i\theta}))^{1-\alpha}/2 \right)} \right\} = \frac{C_4}{\omega \left( (d(z))^{1-\alpha} \right)}.
\]
and
\[ \|D_f(z)\| \geq I(D_f(z)) = \min_{\theta \in [0, 2\pi]} |f_x(z) \cos \theta + f_y(z) \sin \theta| \]
\[ = \min_{\theta \in [0, 2\pi]} \left\{ \lim_{r \to 0^+} \frac{|f(z) - f(\xi)|}{|z - \xi|} \right\} \]
\[ \geq \min_{\theta \in [0, 2\pi]} \left\{ \lim_{r \to 0^+} \frac{\omega \left( \left(1 + |z| \right) \left(1 + |z + re^{i\theta}|\right)^{\frac{1-\alpha}{2}} \right)}{C_4} \right\} \]
\[ = \frac{\omega \left( (1 + |z|)^{1-\alpha} \right)}{C_4}. \]

Now we prove (b) \(\Rightarrow\) (a). For \(t \in [0, 1]\) and \(z_1, z_2 \in \mathbb{D}\) with \(z_1 \neq z_2\), let
\[ \chi(t) = z_1t + (1 - t)z_2. \]

Since
\[ d(\chi(t)) \geq 1 - t|z_1| - (1 - t)|z_2| \geq 1 - t - (1 - t)|z_2| = (1 - t)d(z_2) \]
and
\[ d(\chi(t)) \geq 1 - t|z_1| - (1 - t)|z_2| \geq 1 - t|z_1| - (1 - t) = td(z_1), \]
we see that
\[ (d(\chi(t)))^{1-\alpha} \geq (t(1-t)d(z_1)d(z_2))^{\frac{1-\alpha}{2}}. \]

By calculations, we have
\[ |f(z_1) - f(z_2)| = \left| \int_0^1 \left( f_w(\chi(t))\chi'(t) + f_{\bar{w}}(\chi(t))\overline{\chi'(t)} \right) dt \right| \]
\[ \leq |z_1 - z_2| \int_0^1 \|D_f(\chi(t))\| dt, \]

where \(w = \chi(t)\).

Now, we estimate the integral on the right. By (b) and (2.2), we have
\[ \int_0^1 \|D_f(\chi(t))\| dt \leq \int_0^1 \frac{C_5 dt}{\omega \left( (d(\chi(t)))^{1-\alpha} \right)} \leq \int_0^1 \frac{C_5 dt}{\omega \left( (t(1-t)d(z_1)d(z_2))^{\frac{1-\alpha}{2}} \right)} , \]

which, together with Lemma D, implies that
\[ \int_0^1 \|D_f(\chi(t))\| dt \leq \frac{C_5}{\omega \left( (d(z_1)d(z_2))^{\frac{1-\alpha}{2}} \right)} \int_0^1 \frac{dt}{t^{\frac{1-\alpha}{2}} (1 - t)^{\frac{1-\alpha}{2}}}, \]
\[ = \frac{\Gamma^2 \left( \frac{1+\alpha}{2} \right) C_5}{\Gamma(1 + \alpha) \omega \left( (d(z_1)d(z_2))^{\frac{1-\alpha}{2}} \right)}, \]

where \(\Gamma\) denotes the usual Gamma function.
It follows from (2.3) and (2.4) that
\[
\frac{|f(z_1) - f(z_2)|}{|z_1 - z_2|} \leq \frac{\Gamma^2 \left( \frac{1 + \alpha}{2} \right)}{\Gamma(1 + \alpha)} \frac{C_5}{\omega \left( (d(z_1)d(z_2))^{\frac{1-\alpha}{2}} \right)}.
\]

On the other hand, for any \(z_1, z_2 \in \mathbb{D}\) with \(z_1 \neq z_2\), let \(w_1 = f(z_1)\) and \(w_2 = f(z_2)\). For \(t \in [0, 1]\), let
\[
(2.5) \quad \gamma(t) = tw_1 + (1 - t)w_2
\]
be the straight line segment connecting \(w_1\) and \(w_2\). Since \(f(\mathbb{D})\) is a convex domain, we see that \(\gamma(t) \subset f(\mathbb{D})\) and \(\eta(t) = f^{-1}(\gamma(t)) \subset \mathbb{D}\) for \(t \in [0, 1]\). It is not difficult to know that
\[
1 + |\eta(t)| \geq \frac{1 + |z_1|}{2} \text{ and } 1 + |\eta(t)| \geq \frac{1 + |z_2|}{2},
\]
which, together with Lemma D, implies that
\[
(2.6) \quad \omega \left( (1 + |\eta(t)|)^{1 - \alpha} \right) \geq \frac{\omega \left( ((1 + |z_1|)(1 + |z_2|))^{\frac{1-\alpha}{2}} \right)}{2^{1 - \alpha}}.
\]

By (b) and (2.6), we obtain
\[
(2.7) \quad \int_0^1 l(D_f(\eta(t)))|\eta'(t)|dt \geq \frac{1}{C_5} \int_0^1 \omega \left( (1 + |\eta(t)|)^{1 - \alpha} \right) |\eta'(t)|dt \\
\geq \frac{\omega \left( ((1 + |z_1|)(1 + |z_2|))^{\frac{1-\alpha}{2}} \right)}{2^{1 - \alpha}C_5} \int_0^1 |\eta'(t)|dt \\
\geq \omega \left( ((1 + |z_1|)(1 + |z_2|))^{\frac{1-\alpha}{2}} \right) \frac{1}{2^{1 - \alpha}C_5} |z_1 - z_2|,
\]

It follows from (2.5) that
\[
|w_1 - w_2| = |\gamma'(t)| = \int_0^1 |\gamma'(t)|dt \\
= \int_0^1 \left| f_\zeta(\eta(t))\eta'(t) + f_\zeta(\eta(t))\eta'(t) \right| dt \\
\geq \int_0^1 l(D_f(\eta(t)))|\eta'(t)|dt,
\]
which, together with (2.7), yields that
\[
\frac{|w_1 - w_2|}{|z_1 - z_2|} \geq \omega \left( ((1 + |z_1|)(1 + |z_2|))^{\frac{1-\alpha}{2}} \right) \frac{1}{2^{1 - \alpha}C_5}.
\]

The proof of this lemma is finished. \(\square\)
Lemma E. ([17, Lemma 2.7]) If \( g \in C(\mathbb{D}) \), then
\[
\max \left\{ \frac{\partial}{\partial z} G[g](z), \frac{\partial}{\partial \bar{z}} G[g](z) \right\} \leq \frac{1}{3} \| g \|_\infty \text{ for } z \in \mathbb{D}.
\]

Lemma 2.2. Let \( f \in C^1(\mathbb{D}) \) be a sense-preserving mapping. Then \( f \) is an elliptic mapping if and only if there exist constants \( k_1 \in [0, 1) \) and \( k_2 \in [0, \infty) \) such that
\[
|f_z(z)| \leq k_1 |f_{\bar{z}}(z)| + k_2 \text{ for } z \in \mathbb{D}.
\]

Proof. We first prove the sufficiency. By (2.8), for \( z \in \mathbb{D} \), we have
\[
\|Df(z)\| \leq \left(1 + \frac{k_1}{1 - k_1}\right) l(Df(z)) + \frac{2k_2}{1 - k_1}
\]
\[
\leq \left(1 + \frac{k_1}{1 - k_1}\right) l(Df(z)) + \sqrt{\left(1 + \frac{k_1}{1 - k_1}\right)^2 l^2(Df(z)) + \frac{4k_2^2}{(1 - k_1)^2}}.
\]

**Case 1.** For all \( z \in \mathbb{D} \), \( \|Df(z)\| \leq \left(1 + \frac{k_1}{1 - k_1}\right) l(Df(z)) \).

In this case, it is easy to know that
\[
\|Df(z)\|^2 \leq \left(1 + \frac{k_1}{1 - k_1}\right) J_f(z),
\]
which implies that \( f \) is an elliptic mapping.

**Case 2.** There is a subset \( E \) of \( \mathbb{D} \) such that
\[
\|Df(z)\| > \left(1 + \frac{k_1}{1 - k_1}\right) l(Df(z)) \text{ for } z \in E.
\]

In this case, it follows from (2.9) that
\[
\left[\|Df(z)\| - \left(1 + \frac{k_1}{1 - k_1}\right) l(Df(z))\right]^2 \leq \left(1 + \frac{k_1}{1 - k_1}\right)^2 l^2(Df(z)) + \frac{4k_2^2}{(1 - k_1)^2} \quad \text{for } z \in E,
\]
which implies that
\[
\|Df(z)\|^2 \leq 2 \left(1 + \frac{k_1}{1 - k_1}\right) J_f(z) + \frac{4k_2^2}{(1 - k_1)^2}.
\]

On the other hand, for \( z \in \mathbb{D} \setminus E \), we have
\[
\|Df(z)\|^2 \leq \left(1 + \frac{k_1}{1 - k_1}\right) J_f(z),
\]
which, together with (2.10), implies that \( f \) is also an elliptic mapping in \( \mathbb{D} \).

Next, we show the necessity. If \( f \) is an elliptic mapping, then there exist constants \( K \geq 1 \) and \( K' \geq 0 \) such that
\[
\|Df(z)\|^2 \leq K J_f(z) + K' \text{ for } z \in \mathbb{D}.
\]
This gives that
\[ \|D_f(z)\| \leq \frac{Kl(D_f(z)) + \sqrt{(Kl(D_f(z)))^2 + 4K^2}}{2} \leq Kl(D_f(z)) + \sqrt{K^2}, \]
and consequently
\[ |f_\pi(z)| \leq \frac{K - 1}{K + 1}|f_z(z)| + \frac{\sqrt{K^2}}{1 + K}. \]
The proof of this lemma is complete. \( \square \)

**The proof of Theorem 1.1.** Differentiating both sides of the equation \( f^{-1}(f(z)) = z \) and then simplifying the resulting relations lead to the formulae

\[ (f^{-1})_w = \frac{T_z}{J_f} \quad \text{and} \quad (f^{-1})_{\bar{w}} = -\frac{f_\pi}{J_f}, \]

where \( w = f(z) \). Next, for \( z_1, z_2 \in \mathbb{D} \) with \( z_1 \neq z_2 \), we let
\[ \varphi(t) = t(f(z_1) - f(z_2)) + f(z_2), \]
where \( t \in [0, 1] \). Since \( f(\mathbb{D}) \) is a convex domain, we see that \( \varphi(t) \subset f(\mathbb{D}) \) and \( \Phi(t) := f^{-1}(\varphi(t)) \subset \mathbb{D} \) for \( t \in [0, 1] \). For \( z \in \mathbb{D} \), let \( F(z) = f(z) + G[g](z) \). Then \( F \) is harmonic in \( \mathbb{D} \), and \( F \) can be represented by \( F = h_1 + h_2 \) in \( \mathbb{D} \), where \( h_j \) (\( j = 1, 2 \)) are analytic in \( \mathbb{D} \).

With \( z = \Phi(t) \), we have \( \Phi(0) = z_2, \Phi(1) = z_1 \) and thus, by (2.11), we obtain
\[
\begin{align*}
  h_1(z_1) - h_1(z_2) &= \int_{z_2}^{z_1} h'_1(z)dz = \int_0^1 h'_1(\Phi(t))\Phi'(t)dt \\
  &= \int_0^1 h'_1(\Phi(t))\left( f_w^{-1}(\varphi(t))\varphi'(t) + f_{\bar{w}}^{-1}(\varphi(t))\overline{\varphi'(t)} \right)dt \\
  &= \int_0^1 h'_1(\Phi(t))\left( \frac{f_z(\Phi(t))}{J_f(\Phi(t))}\varphi'(t) - \frac{f_\pi(\Phi(t))}{J_f(\Phi(t))}\overline{\varphi'(t)} \right)dt,
\end{align*}
\]
which implies that
\[ |h_1(z_1) - h_1(z_2)| = \frac{1}{|J_f(\Phi(t))|} \left| \int_0^1 h'_1(\Phi(t))\left( \frac{f_z(\Phi(t))}{J_f(\Phi(t))} - \frac{f_\pi(\Phi(t))}{\varphi'(t)} \right)dt \right| \leq \int_0^1 \frac{h'_1(\Phi(t))||D_f(\Phi(t))||}{|J_f(\Phi(t))|}dt \leq \int_0^1 \frac{l(D_f(\Phi(t)) + |G[g]|_z(\Phi(t)) + |f_\pi(\Phi(t))|}{l(D_f(\Phi(t)))}dt. \]

It follows from Lemma 2.1 that there exists a positive constant \( C_6 \) such that
\[
\frac{|G[g]z(\Phi(t))| + |f(\Phi(t))|}{l(D_f(\Phi(t)))} \leq \frac{|G[g]z(\Phi(t))| + \|D_f(\Phi(t))\|}{l(D_f(\Phi(t)))} \\
\leq \frac{C_6|G[g]z(\Phi(t))| + \frac{C_6}{\omega((d(\Phi(t)))^{1-\alpha})}}{\omega((1 + |\Phi(t)|)^{1-\alpha})} \\
\leq \frac{C_6}{\omega(1)} \left( |G[g]z(\Phi(t))| + \frac{1}{\omega((d(\Phi(t)))^{1-\alpha})} \right),
\]
which, together with (2.12) and Lemma E, gives that

\[
(2.13) \quad \frac{|h_1(z_1) - h_1(z_2)|}{f(z_1) - f(z_2)} \leq 1 + \frac{C_6}{\omega(1)} \left( \int_0^1 |G[g]z(\Phi(t))|dt \\
+ \int_0^1 \frac{dt}{\omega((d(\Phi(t)))^{1-\alpha})} \right) \leq \mu,
\]
where

\[
\mu := 1 + \frac{C_6\|g\|_{\infty}}{3\omega(1)} + \frac{C_6C_2}{\omega(1)} \geq 1.
\]

For \( \theta \in [0, 2\pi] \) and any fixed point \( z \in \mathbb{D} \), let \( \varsigma = z + re^{i\theta} \), where \( r \in [0, 1 - |z|] \). By (2.13), we have

\[
\frac{1}{\mu} \lim_{r \to 0^+} \left| \frac{h_1(\varsigma) - h_1(z)}{\varsigma - z} \right| \leq \lim_{r \to 0^+} \left| \frac{h_1(\varsigma) - h_1(z)}{\varsigma - z} \right| + \frac{h_2(\varsigma) - h_2(z)}{\varsigma - z} + \frac{G[g](z) - G[g](\varsigma)}{\varsigma - z} \\
= |h_1'(z)e^{i\theta} + h_2'(z)e^{-i\theta} - (G[g]z(z)e^{i\theta} + G[g]z(z)e^{-i\theta})|
\]
which yields that

\[
(2.14) \quad \frac{1}{\mu} |h_1'(z)| \leq \min_{\theta \in [0,2\pi]} |h_1'(z) - G[g]z(z) + (h_2'(z) - G[g]z(z))e^{-2i\theta}| \\
= \min_{\theta \in [0,2\pi]} |f_z(z) + f_{\overline{z}}(z)e^{-2i\theta}| = l(D_f(z)).
\]

Since

\[
|h_1'| \geq |h_1' - G[g]z| - |G[g]z| = |f_z| - |G[g]z|,
\]
by (2.14) and Lemma E, we see that

\[
(2.15) \quad |f_z| \leq \left( \frac{\mu - 1}{\mu} \right) |f_z| + \frac{1}{3\mu} \|g\|_{\infty}.
\]
It follows from (2.15) and Lemma 2.2 that \( f \) is an elliptic mapping. The proof of this theorem is complete. \( \square \)
The proof of Proposition 1.4. For $\theta \in [0, 2\pi]$ and any fixed point $z \in \mathbb{D}$, let $\varsigma = z + re^{i\theta}$, where $r \in [0, 1 - |z|]$. Then by (2.1) and (1.5), we have

$$\|Df(z)\| = \max_{\theta \in [0, 2\pi]} |f_x(z)\cos \theta + f_y(z)\sin \theta| = \max_{\theta \in [0, 2\pi]} \left\{ \lim_{r \to 0^+} \frac{|f(\varsigma) - f(z)|}{\varsigma - z} \right\}$$

$$\leq C_3 \lim_{r \to 0^+} \left| \frac{h_1(s) - h_1(z)}{s - z} \right| = C_3|h_1'(z)|$$

$$\leq C_3|f_x(z)| + C_3|G[g]_2(z)|,$$

which, together with Lemma E, yields that

(2.16) $|f_x(z)| \leq (C_3 - 1)|f_x(z)| + \frac{C_3}{3}\|g\|_{\infty}$.

It follows from (2.16) and Lemma 2.2 that $f$ is an elliptic mapping. The proof of this proposition is complete.

The proof of Theorem 1.6. We first prove (1.14). Since $\|Df(z)\|^2 \leq KJ_f(z) + K'$, we see that

(2.17) $\|Df(z)\| \leq \frac{Kl(Df(z)) + \sqrt{(Kl(Df(z)))^2 + 4K'}}{2} \leq Kl(Df(z)) + \sqrt{K'}$, where $z \in \mathbb{D}$. It follows from (2.17) that

$$\ell_f(r) = r \int_0^{2\pi} |f_x(re^{i\theta}) - e^{-2i\theta}f_x(re^{i\theta})| \, d\theta \geq r \int_0^{2\pi} l(Df(re^{i\theta})) \, d\theta$$

$$\geq r \int_0^{2\pi} \left( \frac{\|Df(re^{i\theta})\| - \sqrt{K'}}{K} \right) \, d\theta,$$

which gives that

(2.18) $r \int_0^{2\pi} \|Df(re^{i\theta})\| \, d\theta \leq 2\pi r \sqrt{K'} + K\ell_f(r)$.

For $n \geq 1$ and $r \in (0, 1)$, it follows from Cauchy’s integral formula that,

$$na_n = \frac{1}{2\pi i} \int_{|z|=r} \frac{\partial f(z)}{\partial z} \frac{dz}{z^n} \quad \text{and} \quad nb_n = \frac{1}{2\pi i} \int_{|z|=r} \left( \frac{\partial f(z)}{\partial z} \right) \frac{dz}{z^n},$$

which, together with the inequality (2.18), implies that

$$n(|a_n| + |b_n|) \leq \frac{1}{2\pi r^n} \int_0^{2\pi} r \|Df(re^{i\theta})\| \, d\theta$$

$$\leq \frac{1}{2\pi r^n} \left( 2\pi r \sqrt{K'} + K\ell_f(1) \right)$$
Consequently,
\[(2.19) \quad |a_n| + |b_n| \leq \frac{\sqrt{K'}}{n} + \frac{K\ell_f(1)}{2n\pi} = \frac{\sqrt{K'}}{n} + \frac{K\pi}{n}.\]

This proves (1.14)

Next, we prove (1.15). For \(z \in \mathbb{D}\), it follows from the inequality
\[\|D_f(z)\| \leq KJ_f(z) + K',\]
that
\[|f_{\pi}(z)| \leq -|f_{\pi}(z)| + \sqrt{K' + KK' + K^2|f_{\pi}(z)|^2},\]
which, together with (1.12), yields that
\[\|D_f(z)\| \leq \left( R + \frac{-R + \sqrt{K' + KK' + K^2R^2}}{1 + K} \right) \frac{1}{1 - |z|^2}.\]

The proof of the theorem is complete. \[\square\]

**Theorem F**. ([1, Theorem 2]) Let \(\phi\) be subharmonic in \(\mathbb{D}\). If for all \(r \in [0, 1)\),
\[A(r) = \sup_{\theta \in [0, 2\pi]} \int_0^r \phi(\rho e^{i\theta}) d\rho \leq 1,\]
then \(A(r) \leq r\).

**The proof of Theorem 1.7.** Since \(f\) is a univalent \((K, K')\)-elliptic mapping, we see that
\[\|D_f(z)\|^2 \leq K\|D_f(z)\|l(D_f(z)) + K'\] for \(z \in \mathbb{D}\).

This gives that
\[(2.20) \quad \|D_f(z)\| \leq \frac{Kl(D_f(z)) + \sqrt{(Kl(D_f(z)))^2 + 4K'}}{2} \leq Kl(D_f(z)) + \sqrt{K'}.\]

It follows from (2.20) that, for \(\theta \in [0, 2\pi]\) and \(r \in (0, 1)\),
\[\ell_f^*(r, \theta) = \int_0^r |f_{\pi}(\rho e^{i\theta}) + e^{-2i\theta}f_{\pi}(\rho e^{i\theta})| d\rho \geq \int_0^r l(D_f(\rho e^{i\theta})) d\rho \geq \frac{1}{K} \int_0^r \left( \|D_f(\rho e^{i\theta})\| - \sqrt{K'} \right) d\rho\]
and consequently
\[(2.21) \quad \int_0^r \|D_f(\rho e^{i\theta})\| d\rho \leq \sqrt{K'}r + K\ell_f^*(r, \theta) \leq \sqrt{K'} + K\ell_f^*(1).\]

Inequality (2.21) and Theorem F lead to
\[(2.22) \quad \int_0^r \|D_f(\rho e^{i\theta})\| d\rho \leq \left( \sqrt{K'} + K\ell_f^*(1) \right) r.\]
The Cauchy integral formula shows that, for $\rho \in (0,1)$,
\[
na_n = \frac{1}{2\pi i} \int_{|z|=\rho} \frac{\partial f(z)}{\partial z} \frac{dz}{z^n} \quad \text{and} \quad nb_n = \frac{1}{2\pi i} \int_{|z|=\rho} \left( \frac{\partial f(z)}{\partial z} \right) \frac{dz}{z^n},
\]
which yields that
\[
(2.23) \quad n(|a_n| + |b_n|) \leq \frac{1}{2\pi \rho^n} \int_0^{2\pi} \rho \|D_f(\rho e^{i\theta})\| d\theta.
\]
Then combining (2.22) and (2.23) gives the final estimate for $|a_n| + |b_n|$, namely,
\[
2\pi n(|a_n| + |b_n|) \int_0^r \rho^{n-1} d\rho \leq \int_0^{2\pi} \left( \int_0^r \|D_f(\rho e^{i\theta})\| d\rho \right) d\theta \leq \int_0^{2\pi} \left( \sqrt{K'} + K\ell_f^*(1) \right) r d\theta
\]
and consequently
\[
|a_n| + |b_n| \leq \inf_{r \in (0,1)} \left( \frac{\sqrt{K'} + K\ell_f^*(1)}{r^{n-1}} \right) = \sqrt{K'} + K\ell_f^*(1).
\]
The proof of this theorem is complete.

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