FINDING THE CLOSEST NORMAL STRUCTURED MATRIX

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Abstract. Given a structured matrix $A$ we study the problem of finding the closest normal matrix with the same structure. The structures of our interest are: Hamiltonian, skew-Hamiltonian, per-Hermitian, and perskew-Hermitian. We develop a structure-preserving Jacobi-type algorithm for finding the closest normal structured matrix and show that such algorithm converges to a stationary point of the objective function.

1. Introduction

The problem of finding the closest normal matrix $X$ to any unstructured matrix $A \in \mathbb{C}^{n \times n}$ in the Frobenius norm

$$\min_{X \in \mathcal{N}} \|X - A\|_F,$$

where $\mathcal{N}$ stands for the set of normal matrices, was an open question for a long time. It was solved independently by Gabriel \cite{2, 3} and Ruhe \cite{10}. A nice summary of important findings is given by Higham in \cite{4}. In this paper we are interested in the structure-preserving version of problem (1.1). That is, given a structure $S$ and matrix $A \in S$, we are looking for

$$\min_{X \in \mathcal{N} \cap S} \|X - A\|_F.$$

The following theorem from \cite{1} states the solution of (1.1) using a maximization problem formulation. See \cite{4, Theorem 5.2} for a full set of references. Notation $\mathcal{U}$ stands for the set of unitary matrices.

**Theorem 1.1.** Let $A \in \mathbb{C}^{n \times n}$ and let $X = UDU^H$, where $U \in \mathcal{U}$ and $D \in \mathbb{C}^{n \times n}$ is diagonal. Then $X$ is a nearest normal matrix to $A$ in the Frobenius norm if and only if

(a) $\|\text{diag}(U^H AU)\|_F = \max_{Q \in \mathcal{U}} \|\text{diag}(Q^H AQ)\|_F$, and

(b) $D = \text{diag}(U^H AU)$.

Thus, the problem of finding the closest normal matrix to $A \in \mathbb{C}^{n \times n}$ can be transformed into a problem of finding a unitary similarity transformation $Q$ which makes the sum of squares of the diagonal elements of $Q^H AQ$ as large as possible. Instead of solving the minimization problem (1.1), one can address the dual maximization problem

$$\max_{Q \in \mathcal{U}} \|\text{diag}(Q^H AQ)\|_F^2.$$

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It is well known that if $A$ is normal, then it can be unitarily diagonalizable. Since we focus on matrices that are not normal, the goal is to make the matrix $Q^HAQ$ “as diagonal as possible”. Then, the closest normal matrix to $A$ is obtained as $X = Q \text{diag}(Q^HAQ)Q^H$.

We consider four classes of matrices:

- Hamiltonian $\mathcal{H} = \{ A \in \mathbb{C}^{2n \times 2n} \mid (JA)^H = JA \}$,
- skew-Hamiltonian $\mathcal{W} = \{ A \in \mathbb{C}^{2n \times 2n} \mid (JA)^H = -JA \}$,
- per-Hermitian $\mathcal{M} = \{ A \in \mathbb{C}^{m \times m} \mid (FA)^H = FA \}$,
- perskew-Hermitian $\mathcal{K} = \{ A \in \mathbb{C}^{m \times m} \mid (FA)^H = -FA \}$,

where

$$J = J_{2n} = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} \in \mathbb{R}^{2n \times 2n}, \quad F = F_m = \begin{bmatrix} \ddots & 1 \\ & \ddots & \vdots \\ & 1 & \ddots \end{bmatrix} \in \mathbb{R}^{m \times m}. \quad (1.4)$$

A unitary similarity transformation $Z^HAZ$, $Z \in \mathcal{U}$ is, in general, not structure-preserving. Therefore, in order to get $Z^HAZ \in \mathcal{S}$ for $A \in \mathcal{S}$, matrix $Z$ needs to have an additional structure. Transformations that keep the structure of the sets $\mathcal{H}$ and $\mathcal{W}$ are symplectic transformations

$$\mathcal{S}_p = \{ Z \in \mathbb{C}^{2n \times 2n} \mid Z^HJZ = J \},$$

and transformations that keep the structure of the sets $\mathcal{M}$ and $\mathcal{K}$ are perplectic transformations

$$\mathcal{P}_p = \{ Z \in \mathbb{C}^{m \times m} \mid Z^HFZ = F \}.$$

Both groups of symplectic and perplectic matrices form manifolds. Hamiltonian matrices form the tangent subspace on the manifold of symplectic matrices at the identity. It is easy to check this. For symplectic matrices $M$ we have $h(M) := M^HJM - J = 0$. Tangent space at the identity is the set of matrices $\{ A \mid Dh(I)A = 0 \}$. Using linear approximation we get

$$h(I + A) = h(I) + Dh(I)A + O(||A||^2),$$

and since $h(I) = 0$,

$$Dh(I)A = (I + A)^HJ(I + A) - J = A^HJ + JA + O(||A||^2) = 0.$$ 

Matrices that satisfy the equation $A^HJ + JA = 0$ are indeed Hamiltonian matrices. Orthogonal space at the identity is orthogonal complement of the set of Hamiltonian matrices, which is the set of skew-Hamiltonain matrices. In the same way one can check that perskew-Hermitian matrices form the tangent subspace on the manifold of perplectic matrices at the identity, and per-Hermitian matrices form its orthogonal subspace. Transformations from a manifold preserve the structure of the matrices from the corresponding tangent or orthogonal subspace.

In the algebraic setting, one can look at the symplectic and perplectic groups as Lie groups. Hamiltonian and skew-Hamiltonian matrices are Lie algebra and Jordan algebra of the symplectic group, respectively, while per-Hermitian and perskew-Hermitian matrices are Jordan algebra and Lie algebra of the perplectic group, respectively. Transformations from a Lie group preserve the structure of the corresponding Jordan or Lie algebra.

Both geometric and algebraic interpretation of the studied matrix structures are given in Table [1]. One can also find more about these structures in the existing literature, e.g., [6, 11].

In Section [2] we give structured analogues of Theorem [1.1] and formulate the corresponding versions of minimization problem (1.2). Then in Section [3] we develop the Jacobi-type algorithm for solving the minimization problems defined in Section [2] and prove its convergence in Section [4]. Finally, in Section [5] we present some numerical results.
We study minimization problem (1.2). Theorem 1.1 suggests to find a unitary matrix $U$ that maximizes $\|\text{diag}(U^H AU)\|_F$. Let us explore how that approach can be used with the structure-preserving constrain.

2.1. Hamiltonian and skew-Hamiltonian matrices. A Hamiltonian matrix $H \in \mathbb{C}^{2n \times 2n}$ can be written as a $2 \times 2$ block matrix

$$H = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & -H_{11}^H \end{bmatrix}, \quad \text{where } H_{12}^H = H_{12}, \ H_{12}^{H} = H_{21}, \ H_{11}, H_{12}, H_{21} \in \mathbb{C}^{n \times n}. \quad (2.1)$$

Moreover, a skew-Hamiltonian matrix $W \in \mathbb{C}^{2n \times 2n}$ can be written as a $2 \times 2$ block matrix

$$W = \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{11}^H \end{bmatrix}, \quad \text{where } W_{12}^H = -W_{12}, \ W_{21}^H = -W_{21}, \ W_{11}, W_{12}, W_{21} \in \mathbb{C}^{n \times n}. \quad (2.2)$$

It follows from (2.1) and (2.2), respectively, that any diagonal Hamiltonian matrix has the form

$$D_H = \begin{bmatrix} D & 0 \\ 0 & -D^H \end{bmatrix},$$

while any skew-Hamiltonian diagonal matrix has the form

$$D_W = \begin{bmatrix} D & 0 \\ 0 & D^H \end{bmatrix},$$

where $D = \text{diag}(d_1, \ldots, d_n)$. Also, it is easy to check that for every skew-Hamiltonian matrix $W \in \mathcal{W}$ there is a Hamiltonian matrix $H \in \mathcal{H}$ (and for every $H \in \mathcal{H}$ there is $W \in \mathcal{W}$) such that $W = iH$.

Therefore, all results obtained for Hamiltonian matrices will imply analogue results for skew-Hamiltonian matrices.

In order to obtain a result analogue to that in Theorem 1.1, we use the Schur decomposition for Hamiltonian matrices given in [8].

**Theorem 2.1** ([8]). If $H \in \mathbb{C}^{2n \times 2n}$ is a Hamiltonian matrix whose eigenvalues have nonzero real parts, then there exists a unitary

$$U = \begin{bmatrix} U_{11} & U_{12} \\ -U_{12} & U_{11} \end{bmatrix}, \quad U_{11}, U_{12} \in \mathbb{C}^{n \times n},$$

such that

$$U^H HU = \begin{bmatrix} T & M \\ 0 & -T^H \end{bmatrix}, \quad T, M \in \mathbb{C}^{n \times n}, \quad (2.3)$$

where $T$ is upper triangular and $M^H = M$. 

| manifold | tangent subspace at $I$ | orthogonal subspace at $I$ |
|----------|-------------------------|-----------------------------|
| symplectic | Hamiltonian | skew-Hamiltonian |
| perplectic | perskew-Hermitian | per-Hermitian |
| Lie group | Lie algebra | Jordan algebra |

Table 1. Geometric and algebraic setting for the structured matrices
The following lemma is a special case of Theorem 2.1 for Hamiltonian and skew-Hamiltonian normal matrices.

Lemma 2.2. (i) If $H \in \mathbb{C}^{2n \times 2n}$ is a normal Hamiltonian matrix whose eigenvalues have nonzero real parts, then there exists a unitary symplectic $U \in \mathbb{C}^{2n \times 2n}$ and diagonal $D \in \mathbb{C}^{n \times n}$ such that

$$H = U \begin{bmatrix} D & 0 \\ 0 & -D^H \end{bmatrix} U^H. \tag{2.4}$$

(ii) If $W \in \mathbb{C}^{2n \times 2n}$ is a normal skew-Hamiltonian matrix whose eigenvalues have nonzero imaginary parts, then there exists a unitary symplectic $U \in \mathbb{C}^{2n \times 2n}$ and diagonal $D \in \mathbb{C}^{n \times n}$ such that

$$W = U \begin{bmatrix} D & 0 \\ 0 & D^H \end{bmatrix} U^H. \tag{2.5}$$

Proof. (i) The Schur decomposition of $H \in \mathcal{H}$ is as in relation (2.3). Matrices $H$ and $U^H U$ are normal. For permutation $P = \begin{bmatrix} I_n \\ F_n \end{bmatrix}$, where $F_n$ is as in (1.4), matrix

$$P^T (U^H U) P = P^T \begin{bmatrix} T & M \\ 0 & -T^H \end{bmatrix} P \tag{2.6}$$

is normal and triangular. Normal triangular matrix must be diagonal. Diagonal elements of $P^T (U^H U) P$ are the same as of $U^H U$. Therefore, in (2.6) we conclude that $M = 0$, $T = D$ is diagonal, and

$$U^H U = \begin{bmatrix} D & 0 \\ 0 & -D^H \end{bmatrix}.$$

This gives relation (2.4).

It is easy to check that matrix $U$ is indeed symplectic. We have

$$JU = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} \\ -U_{12} & U_{11} \end{bmatrix} = \begin{bmatrix} -U_{12} & U_{11} \\ -U_{11} & -U_{12} \end{bmatrix} = U J.$$

Now, since $U$ is unitary, it follows that $U^H J U = J$.

(ii) Let $W \in \mathcal{W}$. Then $W = iH$ for some $H \in \mathcal{H}$. If $W$ is normal, then $H$ is also normal. If the eigenvalues of $W$ have nonzero imaginary parts, then eigenvalues of $H$ have nonzero real parts. Hence, we can apply first assertion of this lemma on $H$. This gives

$$H = U \begin{bmatrix} \tilde{D} & 0 \\ 0 & -\tilde{D}^H \end{bmatrix} U^H.$$

For $D = i\tilde{D}$ it follows

$$W = iH = U \begin{bmatrix} i\tilde{D} & 0 \\ 0 & -i\tilde{D}^H \end{bmatrix} U^H = U \begin{bmatrix} D & 0 \\ 0 & D^H \end{bmatrix} U^H.$$

Using the decompositions from Lemma 2.2 we will prove Theorems 2.4 and 2.5 which are structured analogues to Theorem 1.1 for structures $\mathcal{H}$ and $\mathcal{W}$, respectively. Before that, we need one more auxiliary result.
Lemma 2.3. For a general matrix \( M \) we have
\[
\|M\|^2 = \|M - \text{diag}(M)\|^2 + \|\text{diag}(M)\|^2.  
\] (2.7)

Proof. Let \( M \) be an arbitrary matrix. Its orthogonal projection to the subspace of diagonal matrices is \( \text{diag}(M) \). On the other hand, null-matrix \( \mathbf{0} \) also belongs to the subspace of the diagonal matrices. Hence, matrices \( M, \text{diag}(M) \) and \( \mathbf{0} \) are vertices of a right-angled triangle with legs \( \|\text{diag}(M)\| \) and \( \|M - \text{diag}(M)\| \) and the hypotenuse \( \|M\| \).

Now, equation (2.3) follows from the Pythagoras’ theorem. \( \square \)

Theorem 2.4. Let \( A \in \mathbb{C}^{2n \times 2n} \) be a Hamiltonian matrix and let \( X = ZDZ^H \), where \( Z \) is symplectic unitary and \( D \) is Hamiltonian diagonal. Then \( X \) is a normal Hamiltonian matrix with no purely imaginary eigenvalues, closest to \( A \) in the Frobenius norm, if and only if

(a) \( \|\text{diag}(Z^H AZ)\|_F = \max_{Q \in \mathcal{U} \cap \mathcal{S}p} \|\text{diag}(Q^H AQ)\|_F \), and

(b) \( D = \text{diag}(Z^H AZ) \).

Proof. Let \( A \in \mathcal{H} \). By \( X_0 \) denote the closest normal Hamiltonian matrix to \( A \). If \( A \) is already normal, the distance between \( A \) and \( X_0 \) is zero. Otherwise,

\[
\min_{X \in N \cap \mathcal{H}} \|A - X\|_F = \|A - X_0\|_F.  
\] (2.8)

Let \( X_0 = ZD_0Z^H \) be the Schur decomposition of \( X_0 \in N \cap \mathcal{H} \), like in (2.4), \( D_0 \in \mathcal{H} \). Then

\[
\|A - X_0\|_F = \|A - ZD_0Z^H\|_F = \|Z^H AZ - D_0\|_F,
\]
and relation (2.8) is transformed into

\[
\min_{D \in \mathcal{H} \text{ diagonal}} \|Z^H AZ - D\|_F = \|Z^H AZ - D_0\|_F.
\]

The closest diagonal matrix to \( Z^H AZ \) is its orthogonal projection to the subspace of diagonal matrices, which is simply \( \text{diag}(Z^H AZ) \). This gives \( D_0 = \text{diag}(Z^H AZ) \) and implies assertion (b).

To obtain (a), take \( D = \text{diag}(Q^H AQ) \in \mathcal{H} \), \( Q \) unitary symplectic. Matrix \( Q(\text{diag}(Q^H AQ))Q^H \) is normal and its distance from \( A \) is at least \( X_0 \). Thus,

\[
\|A - Q(\text{diag}(Q^H AQ))Q^H\|_F^2 \geq \|A - X_0\|_F^2,
\]
\[
\|Q^H AQ - \text{diag}(Q^H AQ)\|_F^2 \geq \|Z^H AZ - Z^H X_0Z\|_F^2.  
\] (2.9)

On the left-hand side of (2.9) we use Lemma (2.3) for \( M = Q^H AQ \), while on the right-hand side we use the same lemma for \( M = Z^H AZ \). We get

\[
\|Q^H AQ\|_F^2 - \|\text{diag}(Q^H AQ)\|_F^2 \geq \|Z^H AZ\|_F^2 - \|Z^H X_0Z\|_F^2,
\]
\[
\|\text{diag}(Q^H AQ)\|_F^2 \leq \|Z^H X_0Z\|_F^2.
\]
\[
\max_{Q \in \mathcal{U} \cap \mathcal{S}p} \|\text{diag}(Q^H AQ)\|_F^2 = \|Z^H X_0Z\|_F^2.
\]
Conversely, let (a) and (b) hold for \( X_0 \in \mathcal{N} \cap \mathcal{H} \). There exists a closest normal Hamiltonian matrix because both set of normal and set of Hamiltonian matrices are closed. Assume that \( X_0 \) is not the closest, that is

\[
\min_{X \in \mathcal{N} \cap \mathcal{H}} \| A - X \|_F \neq \| A - X_0 \|_F.
\]

Then \( \| A - X \| < \| A - X_0 \| \), for some \( X \in \mathcal{N} \cap \mathcal{H} \). Take \( X_0 = ZD_0Z^H \), \( X = QDQ^H \) from the Schur decomposition (2.4). It follows from (b) that \( D_0 = \text{diag}(Z^H AZ) \) and \( D = \text{diag}(Q^H AQ) \). Using the argument (2.7) again, we get

\[
\| A - Q \text{diag}(Q^H AQ)Q^H \|_F < \| A - Z \text{diag}(Z^H AZ)Z^H \|_F,
\]

\[
\| Q^H AQ - \text{diag}(Q^H AQ) \|_F^2 < \| Z^H AZ - \text{diag}(Z^H AZ) \|_F^2,
\]

\[
\| Q^H AQ \|_F^2 - \| \text{diag}(Q^H AQ) \|_F^2 < \| Z^H AZ \|_F^2 - \| \text{diag}(Z^H AZ) \|_F^2,
\]

\[
\max_{Q \in U \cap S_p} \| \text{diag}(Q^H AQ) \|_F^2 > \| \text{diag}(Z^H AZ) \|_F^2,
\]

which is contradiction with (a).

\[ \square \]

**Theorem 2.5.** Let \( A \in \mathbb{C}^{2n \times 2n} \) be a skew-Hamiltonian matrix and let \( X = ZDZ^H \), where \( Z \) is symplectic unitary and \( D \) is skew-Hamiltonian diagonal. Then \( X \) is a normal skew-Hamiltonian matrix with no real eigenvalues, closest to \( A \) in the Frobenius norm, if and only if

(a) \( \| \text{diag}(Z^H AZ) \|_F = \max_{Q \in U \cap S_p} \| \text{diag}(Q^H AQ) \|_F \), and

(b) \( D = \text{diag}(Z^H AZ) \).

**Proof.** The proof is the same as for the Hamiltonian case from Theorem 2.4, but instead of the Schur decomposition (2.4) it uses (2.5).

\[ \square \]

2.2. **Per-Hermitian and perskew-Hermitian matrices.** A per-Hermitian matrix \( M \in \mathbb{C}^{2n \times 2n} \) can be written as a \( 2 \times 2 \) block matrix

\[
M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & F M_{11} H F \end{bmatrix},
\]

where \( (FM_{12})^H = FM_{12}, (FM_{21})^H = FM_{21} \).

\( M_{11}, M_{12}, M_{21} \in \mathbb{C}^{n \times n} \). The elements of the antidiagonal of \( M_{12} \) and \( M_{21} \) have to be real. A perskew-Hermitian matrix \( K \in \mathbb{C}^{2n \times 2n} \) can be written as a \( 2 \times 2 \) block matrix

\[
K = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & -FK_{11} H F \end{bmatrix},
\]

where \( (FK_{12})^H = FK_{12}, (FK_{21})^H = FK_{21} \).

\( K_{11}, K_{12}, K_{21} \in \mathbb{C}^{n \times n} \). The elements of the antidiagonal of \( K_{12} \) and \( K_{21} \) have to be imaginary (or zero). A diagonal per-Hermitian matrix \( D_M \) has to be of the form

\[
D_M = \begin{bmatrix} D & 0 \\ 0 & FD H F \end{bmatrix},
\]

while a diagonal perskew-Hermitian matrix \( D_K \) is given by

\[
D_K = \begin{bmatrix} D & 0 \\ 0 & -FD H F \end{bmatrix},
\]

where \( D = \text{diag}(d_1, \ldots, d_n) \). Also, for every perskew-Hermitian matrix \( K \in \mathcal{K} \) there is a per-hermitian matrix \( M \in \mathcal{M} \), and viceversa, such that

\[
K = \imath M.
\]
The next lemma gives Schur-like decomposition of per-Hermitian and perskew-Hermitian normal matrices.

**Lemma 2.6.**

(i) If $A \in \mathbb{C}^{2n \times 2n}$ is a normal per-Hermitian matrix whose eigenvalues have nonzero imaginary parts, then there exists a unitary perplectic $U \in \mathbb{C}^{2n \times 2n}$ and diagonal $D \in \mathbb{C}^{n \times n}$ such that

$$A = U \begin{bmatrix} D & 0 \\ 0 & RDH^H \end{bmatrix} U^H.$$  \hfill (2.10)

(ii) If $A \in \mathbb{C}^{2n \times 2n}$ is a normal perskew-Hermitian matrix whose eigenvalues have nonzero real parts, then there exists a unitary perplectic $U \in \mathbb{C}^{2n \times 2n}$ and diagonal $D \in \mathbb{C}^{n \times n}$ such that

$$A = U \begin{bmatrix} D & 0 \\ 0 & -RDH^H \end{bmatrix} U^H.$$  \hfill (2.11)

**Proof.**  

(i) First, notice that eigenvalues of per-Hermitian matrix $A$ come in complex conjugate pairs $(\lambda, \bar{\lambda})$ with $\lambda$ and $\bar{\lambda}$ having the same algebraic multiplicity. Let us verify this. If $\lambda \in \sigma(A)$, then $\bar{\lambda} \in \sigma(A^H)$. Since $\sigma(A^H) = \sigma(RA^HR)$ and $RA^HR = A$, we have $\bar{\lambda} \in \sigma(A)$.

Let $\{\lambda_1, \ldots, \lambda_p, \bar{\lambda}_1, \ldots, \bar{\lambda}_p\}$ be the eigenvalues of $A$ and let $v_1, \ldots, v_n$ be a complete set of orthogonal eigenvectors corresponding to $\lambda_1, \ldots, \lambda_p$. Set $V = [v_1 \ v_2 \ \cdots \ v_n] \in \mathbb{C}^{2n \times n}$. If $v_i, v_j \in \mathbb{C}^n$ are eigenvectors of $A$ for $\lambda_i$ and $\lambda_j$, respectively, then $v_i^H R_{2n} v_j \neq 0$ only if $\lambda_i = \lambda_j$. For $i \neq j$ we have $\lambda_i \neq \lambda_j$ and since all eigenvalues of $A$ have nonzero imaginary parts, we have $\lambda \neq \bar{\lambda}$ for all $\lambda \in \sigma(A)$. This implies that

$$V^H R_{2n} V = 0.$$  \hfill (2.12)

Define $U := [V \ R_{2n} V R_n] \in \mathbb{C}^{2n \times 2n}$. Using identity (2.12) along with $V^H V = I_n$ and $RR = I$ it is easy to check that $U$ is unitary

$$U^H U = \begin{bmatrix} V^H \\ R_n V^H R_{2n} \end{bmatrix} \begin{bmatrix} V & R_{2n} V R_n \end{bmatrix} = \begin{bmatrix} V^H V & V^H R_{2n} V R_n \\ R_n V^H R_{2n} V & R_n V^H R_{2n} R_{2n} V R_n \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ 0 & I_n \end{bmatrix} = I_{2n},$$

and perplectic

$$U^H R_{2n} U = \begin{bmatrix} V^H \\ R_n V^H R_{2n} \end{bmatrix} R_{2n} \begin{bmatrix} V & R_{2n} V R_n \end{bmatrix} = \begin{bmatrix} V^H R_{2n} V \\ R_n V^H R_{2n} R_{2n} V \\ R_n V^H R_{2n} R_{2n} R_{2n} V R_n \end{bmatrix} \begin{bmatrix} 0 & R_n \\ R_n & 0 \end{bmatrix} = R_{2n}.$$

Then, using the fact that $\text{span}(V)$ is invariant subspace for $A$, that $AV = VB$ for some $B$, and $RAR = A^H$, we have

$$U^H AU = \begin{bmatrix} V^H AV & V^H AR_{2n} V R_n \\ R_n V^H R_{2n} AV & R_n V^H R_{2n} AR_{2n} V R_n \end{bmatrix} = \begin{bmatrix} V^H AV & V^H AR_{2n} V R_n \\ R_n V^H R_{2n} VB & R_n V^H A^H V R_n \end{bmatrix} = \begin{bmatrix} D & 0 \\ 0 & R_n D^H R_n \end{bmatrix},$$

for $D = \text{diag}(\lambda_1, \ldots, \lambda_p)$. Since $A$ is assumed to be normal, $U^H AU$ is upper triangular and normal. Therefore, it is diagonal, that is $V^H AR_{2n} V R_n = 0$, which gives decomposition (2.10).
(ii) If $A$ is perskew-Hermitian, its eigenvalues come in pairs $(\lambda, -\bar{\lambda})$ and the assumption of nonzero real parts assures that $\lambda \neq -\bar{\lambda}$. Further on, the proof follows the same reasoning as above.

We use decompositions from Lemma 2.6 to get the results analogue to those in Theorems 2.4 and 2.5.

**Theorem 2.7.** Let $A \in \mathbb{C}^{2n \times 2n}$ be a per-Hermitian matrix and let $X = ZDZ^H$, where $Z$ is perplectic unitary and $D$ is per-Hermitian diagonal. Then $X$ is a normal per-Hermitian matrix with no real eigenvalues, closest to $A$ in the Frobenius norm, if and only if

- (a) $\|\text{diag}(Z^H AZ)\|_F = \max_{Q \in U \cap P} \|\text{diag}(Q^H AQ)\|_F$, and
- (b) $D = \text{diag}(Z^H AZ)$.

**Proof.** The proof follows the lines of the proof of Theorem 2.4. Instead of symplectic we have perplectic matrices and instead of decomposition (2.4) we use (2.10). □

**Theorem 2.8.** Let $A \in \mathbb{C}^{2n \times 2n}$ be a perskew-Hermitian matrix and let $X = ZDZ^H$, where $Z$ is perplectic unitary and $D$ is perskew-Hermitian diagonal. Then $X$ is a normal perskew-Hermitian matrix with no purely imaginary eigenvalues, closest to $A$ in the Frobenius norm, if and only if

- (a) $\|\text{diag}(Z^H AZ)\|_F = \max_{Q \in U \cap P} \|\text{diag}(Q^H AQ)\|_F$, and
- (b) $D = \text{diag}(Z^H AZ)$.

**Proof.** The proof follows the lines of the proof of Theorem 2.4. Instead of symplectic we have perplectic matrices and instead of decomposition (2.4) we use (2.11). □

3. **Jacobi-type algorithm for finding the closest normal matrix with a given structure**

Based on the results from Section 2 we can formulate the structured analogues of the maximization problem (1.3). Assuming that $A$ is Hamiltonian or skew-Hamiltonian, it follows from Theorems 2.4 and 2.5 respectively, that a dual maximization formulation of the minimization problem (1.2) is

$$\max_{Z \in U \cap S_P} \|\text{diag}(Z^H AZ)\|_F^2, \quad (3.1)$$

while in per-Hermitian or perskew-Hermitian case Theorems 2.7 and 2.8 imply the form

$$\max_{Z \in U \cap P} \|\text{diag}(Z^H AZ)\|_F^2. \quad (3.2)$$

We develop the Jacobi-type algorithm for solving (3.1) and (3.2). In both cases this is an iterative algorithm

$$A^{(k+1)} = R_k^H A^{(k)} R_k, \quad k \geq 0, \quad A^{(0)} = A, \quad (3.3)$$

where $R_k$ are structure-preserving rotations. The goal of the $k$th step of (3.3) is to make the Frobenius norm of the diagonal of $A^{(k+1)}$ as big as possible. To achieve that we take the pivot pair $(i_k, j_k)$ and choose the appropriate rotation form and the rotation angles. We obtain unitary structure-preserving matrix $Z$ that solves (3.1) (or (3.2)) as the product of these rotations. Then, we form the closest normal matrix as

$$X = Z(\text{diag}(Z^H AZ)Z^H).$$
3.1. **Structure-preserving rotations.** Let us say more about the structure-preserving rotations used in (3.3) Symplectic and perplectic rotations that we use in each iterative step (3.3) can be formed by embedding one or more Givens rotations

\[
G = \begin{bmatrix}
\cos \phi & -e^{i\alpha} \sin \phi \\
\bar{e}^{i\alpha} \sin \phi & \cos \phi
\end{bmatrix}
\]  

(3.4)

into an identity matrix \(I_{2n}\). To simplify the notation, set \(c = \cos \phi, s = e^{i\alpha} \sin \phi\). Then matrix \(G\) can be written as \(G = \begin{bmatrix} c & -s \\ \bar{s} & c \end{bmatrix}\). In our algorithm we use three kinds of symplectic and three kinds of perplectic embeddings to form rotations \(R = R(i,j,\phi,\alpha)\). Our rotations are similar to the structured rotations from [6], but note that the definitions of symplectic and perplectic matrices in [6] slightly differ.

We start with symplectic rotations. If we insert only one Givens rotation \(G\) from (3.4) into \(I_{2n}\), we get a symplectic matrix only if this is done in a very special way. Matrix \(G\) must be inserted on the intersection of \(i\)th and \((n+i)\)th column and row and \(\alpha\) must be zero. Therefore, we get

\[
R(i,j,\phi,\alpha) = R(i,n+i,\phi,0) = \begin{bmatrix}
\cos \phi & -\sin \phi \\
\sin \phi & \cos \phi
\end{bmatrix}
\]

\(i\)  

\(n+i\)  

(3.5)

All elements that are not explicitly written are as in \(I_{2n}\). Notice that in a Hamiltonian matrix entries on positions \((i,n+i)\) have only real and in skew-Hamiltonian matrix purely imaginary values. That is why real rotations are adequate here. The second type of symplectic rotations that we use is the symplectic direct sum of two Givens rotations, that is,

\[
R(i,j,\phi,\alpha) = \begin{bmatrix}
c & -s \\
\bar{s} & c
\end{bmatrix}
\]

\(i\)  

\(j\)  

(3.6)

Further on, we need concentric embedding of two Givens rotations given by

\[
R(i,j,\phi,\alpha) = \begin{bmatrix}
c & -s \\
\bar{s} & c
\end{bmatrix}
\]

\(i\)  

\(j\)  

(3.7)

Pair \((i,j)\) in matrices (3.5), (3.6) and (3.7) is called pivot pair. Usually in the Jacobi-type methods pivot pairs are taken from the upper triangle, \(P = \{(i,j) \mid 1 \leq i < j \leq 2n\}\). Here, because of the double embeddings in rotations (3.6) and (3.7), instead of \((i,j)\) one could equally
say that the pivot pair is \((n + i, n + j)\), or \((j - n, n + i)\), respectively. Therefore, we do not need to go through all pairs from \(\mathcal{P}\), but its subset of \(n^2\) positions.

Rotations (3.5): \((i, n + i)\), \(1 \leq i \leq n\) \(\leftarrow n\) pivot positions

Rotations (3.6): \((i, j)\), \(1 \leq i < j \leq n\) \(\leftarrow n(n - 1)/2\) pivot positions

Rotations (3.7): \((i, j)\), \(1 \leq i < n, n + i < j \leq n\) \(\leftarrow n(n - 1)/2\) pivot positions

For better understanding we depict the pivot positions on a \(10 \times 10\) matrix. Rotations (3.5), (3.6), and (3.7) are used on positions ◦, □, and ◄, respectively,

\[
\begin{bmatrix}
\text{●} & \text{●} & \text{●} & \text{●} & \text{●} & \text{●} & \text{●} & \text{●} & \text{●} & \text{●} \\
\text{●} & \text{⊙} & \text{★} & \text{★} & \text{★} & \text{★} & \text{★} & \text{★} & \text{★} & \text{★} \\
\text{●} & \text{●} & \text{●} & \text{●} & \text{●} & \text{●} & \text{●} & \text{●} & \text{●} & \text{●} \\
\text{●} & \text{●} & \text{●} & \text{●} & \text{●} & \text{●} & \text{●} & \text{●} & \text{●} & \text{●} \\
\text{●} & \text{●} & \text{●} & \text{●} & \text{●} & \text{●} & \text{●} & \text{●} & \text{●} & \text{●} \\
\text{●} & \text{●} & \text{●} & \text{●} & \text{●} & \text{●} & \text{●} & \text{●} & \text{●} & \text{●} \\
\text{●} & \text{●} & \text{●} & \text{●} & \text{●} & \text{●} & \text{●} & \text{●} & \text{●} & \text{●} \\
\text{●} & \text{●} & \text{●} & \text{●} & \text{●} & \text{●} & \text{●} & \text{●} & \text{●} & \text{●} \\
\text{●} & \text{●} & \text{●} & \text{●} & \text{●} & \text{●} & \text{●} & \text{●} & \text{●} & \text{●} \\
\text{●} & \text{●} & \text{●} & \text{●} & \text{●} & \text{●} & \text{●} & \text{●} & \text{●} & \text{●} \\
\end{bmatrix}
\]

(3.8)

Considering the double embeddings on positions ◦ and □, we see that the whole upper triangle is covered in the following way,

\[
\begin{bmatrix}
\text{●} & \text{●} & \text{●} & \text{●} & \text{●} & \text{●} & \text{●} & \text{●} & \text{●} & \text{●} \\
\text{●} & \text{⊙} & \text{★} & \text{★} & \text{★} & \text{★} & \text{★} & \text{★} & \text{★} & \text{★} \\
\text{●} & \text{●} & \text{●} & \text{●} & \text{●} & \text{●} & \text{●} & \text{●} & \text{●} & \text{●} \\
\text{●} & \text{●} & \text{●} & \text{●} & \text{●} & \text{●} & \text{●} & \text{●} & \text{●} & \text{●} \\
\text{●} & \text{●} & \text{●} & \text{●} & \text{●} & \text{●} & \text{●} & \text{●} & \text{●} & \text{●} \\
\text{●} & \text{●} & \text{●} & \text{●} & \text{●} & \text{●} & \text{●} & \text{●} & \text{●} & \text{●} \\
\text{●} & \text{●} & \text{●} & \text{●} & \text{●} & \text{●} & \text{●} & \text{●} & \text{●} & \text{●} \\
\text{●} & \text{●} & \text{●} & \text{●} & \text{●} & \text{●} & \text{●} & \text{●} & \text{●} & \text{●} \\
\text{●} & \text{●} & \text{●} & \text{●} & \text{●} & \text{●} & \text{●} & \text{●} & \text{●} & \text{●} \\
\text{●} & \text{●} & \text{●} & \text{●} & \text{●} & \text{●} & \text{●} & \text{●} & \text{●} & \text{●} \\
\end{bmatrix}
\]

On the other hand, perplectic rotation can be obtained by embedding only one Givens rotation \(G\) only when \(\alpha = -\frac{\pi}{2}\) and such \(G\) is inserted on the intersection of the \(i\)th and the \((2n - i + 1)\)th column and row. Then we have

\[
R(i, j, \phi, \alpha) = R(i, 2n - i + 1, \phi, -\frac{\pi}{2}) = \begin{bmatrix}
\cos \phi & i \sin \phi \\
i \sin \phi & \cos \phi
\end{bmatrix}
\begin{bmatrix}
i \\
2n - i + 1
\end{bmatrix}
\]

(3.9)
When embedding two Givens rotations, we have more freedom. Perplectic direct sum embedding is given by

\[
R(i, j, \phi, \alpha) = \begin{bmatrix}
c & -s \\
\bar{s} & c \\
c & \bar{s} \\
-s & c
\end{bmatrix}
\begin{bmatrix}
i \\
j \\
2n - j + 1 \\
2n - i + 1
\end{bmatrix}.
\] (3.10)

Finally, perplectic interleaved embedding of two Givens rotations is

\[
R(i, j, \phi, \alpha) = \begin{bmatrix}
c & -s \\
\bar{s} & c \\
2n - j + 1 \\
\bar{s} & c \\
2n - i + 1
\end{bmatrix}
\begin{bmatrix}
i \\
j \\
2n - j + 1 \\
2n - i + 1
\end{bmatrix}.
\] (3.11)

Like it was the case with double embeddings (3.6) and (3.7), for the pivot position in both rotations (3.10) and (3.11) one can also choose \((2n - j + 1, 2n - i + 1)\) instead \((i, j)\). Here we are considering the following positions of pivot pairs.

Rotations (3.9): \((i, j), 1 \leq i \leq n, j = 2n - i + 1 \leftrightarrow n \) pivot positions

Rotations (3.10): \((i, j), 1 \leq i < j \leq n \leftrightarrow n(n - 1)/2 \) pivot positions

Rotations (3.11): \((i, j), 1 \leq i < n \leq j \leq 2n - i \leftrightarrow n(n - 1)/2 \) pivot positions

Again, we depict this on a \(10 \times 10\) matrix denoting the pivot positions corresponding to (3.9), (3.10), and (3.11) by \(\circ\), \(\diamond\), and \(\Box\), respectively. We have

3.2. Algorithm. Knowing the shape of the structure-preserving rotations we write the Jacobi-type algorithm for solving the maximization problem (3.1). In each step we take a pivot pair \((i, j)\). The pivot position implies rotation form (3.5), (3.6) or (3.7), for which we compute the rotation angles. Then we perform one iterative step (3.3) and update the symplectic unitary transformation matrix \(Z_{k+1} = Z_k R_k, \ k \geq 0\). Note that there is no need to set up matrices \(R_k\).

Algorithm 1. Jacobi-type algorithm for solving maximization problem (3.1)
Input: $A \in \mathbb{C}^{2n \times 2n}$ Hamiltonian or skew-Hamiltonian.
Output: symplectic unitary $Z$

$k = 0$
$A^{(1)} = A$
$Z_1 = I$

repeat

for $i = 1, \ldots, n - 1$ do

for $j = i + 1, \ldots, n$ do

$k = k + 1$
Find $\phi_k, \alpha_k$ for $R_k = R(i, j, \phi_k, \alpha_k)$ as in \(3.6\)

$A^{(k+1)} = R^H_k A^{(k)} R_k$

$Z_{k+1} = Z_k R_k$

end for

for $j = n + i + 1, \ldots, 2n$ do

$k = k + 1$
Find $\phi_k, \alpha_k$ for $R_k = R(i, j - n, \phi_k, \alpha_k)$ as in \(3.7\)

$A^{(k+1)} = R^H_k A^{(k)} R_k$

$Z_{k+1} = Z_k R_k$

end for

end for

for $i = 1, \ldots, n$ do

$k = k + 1$
Find $\phi_k, \alpha_k$ for $R_k = R(i, n + i, \phi_k, \alpha_k)$ as in \(3.5\)

$A^{(k+1)} = R^H_k A^{(k)} R_k$

$Z_{k+1} = Z_k R_k$

end for

until convergence

The order in which pivot pairs are taken defines a pivot strategy. Algorithm \(1\) uses a cyclic pivot strategy. In general, cyclic strategies are periodic strategies with the period equal to the number of possible pivot positions. In our case that means that during the first $n^2$ steps in \(3.3\) (and later on during any consecutive $n^2$ steps) we take all pivot positions corresponding to those marked in \(3.8\), each of them exactly once. This process defines one cycle. We repeat such cycles until the convergence is obtained. Specifically, reading from Algorithm \(1\) we take pivot positions marked with $\blacklozenge$ in the row-wise order, then positions $\square$ in the row-wise order, and positions $\circ$ again row by row.

As we will see in Section \(4\) the convergence does not depend on the order inside one cycle and our proof holds for any cyclic pivot strategy. Therefore, for loops in Algorithm \(1\) can be altered depending on the pivot strategy. Nevertheless, in order to ensure the convergence one must check that each pivot pair satisfies the condition of Lemma \(4.4\)

$$|\langle \text{grad} f_H(Z), Z \dot{R}(i, j, 0, \alpha) \rangle| \geq \frac{2}{\sqrt{4n^2 - 2n}} \| \text{grad} f_H(Z) \|_F.$$ 

If this is not true for some pivot pair, that pair is skipped.

Algorithm \(1\) can easily be modified for solving minimization problem \(3.2\). Instead of rotations \(3.5\), \(3.6\) and \(3.7\) symplectic rotations \(3.9\), \(3.10\) and \(3.11\) are used. The inner for loop for the rotation \(3.10\) has to be modified to ‘for $j = n + 1 : 2n - i$ do’.
3.3. The choice of rotation angles $\phi$ and $\alpha$. In the $k$th step of Algorithm 1 one should chose rotation angles $\phi_k$ and $\alpha_k$ such that $R_k = R(i_k, j_k, \phi_k, \alpha_k)$ maximizes the Frobenius norm of the diagonal of $A^{(k+1)} = R_k^H A^{(k)} R_k$. Here we show how $\phi$ and $\alpha$ are obtained in the case of symplectic rotations. The same reasoning holds for perplectic rotations.

Denote the $k$th pivot position by $(i_k, i_k)$. If $R_k$ is of the form (3.5), after one iteration two diagonal elements on positions $(i_k, i_k)$ and $(j_k, j_k)$ are changed. If $R_k$ is double Givens rotation (3.6) or (3.7), then four diagonal elements are changed. However, for $A$ Hamiltonian or skew-Hamiltonian matrix we have

\[
\begin{align*}
|\text{Re}(a_{n+i,n+i})| &= |\text{Re}(a_{ii})|, \\
|\text{Re}(a_{n+j,n+j})| &= |\text{Re}(a_{jj})|, \\
|\text{Re}(a_{j-n-j-n})| &= |\text{Re}(a_{jj})|, \\
\end{align*}
\]

(3.12)

From (3.12a) and (3.12b) it follows that for rotations (3.6) it is enough to consider only the changes on positions $(i_k, i_k)$ and $(j_k, j_k)$, $1 \leq i_k, j_k \leq n$. The same conclusion follows from (3.12a) and (3.12c) for rotations (3.7), with $1 \leq i_k \leq n < j_k \leq 2n$. Thus it is always enough to consider only the changes induced by one Givens rotation and the following computation holds for all rotations (3.5), (3.6), and (3.7).

For the simplicity of notation, denote $A^{(k+1)} = A' = (a'_{ij}), A^{(k)} = A = (a_{ij}), \phi_k = \phi, \alpha_k = \alpha$. Consider the pivot submatrix

\[
\begin{bmatrix}
  a'_{ii} & a'_{ij} \\
  a'_{ji} & a'_{jj}
\end{bmatrix}
= \begin{bmatrix}
  \cos \phi & -e^{i\alpha} \sin \phi \\
  e^{-i\alpha} \sin \phi & \cos \phi
\end{bmatrix}^H \begin{bmatrix}
  a_{ii} & a_{ij} \\
  a_{ji} & a_{jj}
\end{bmatrix} \begin{bmatrix}
  \cos \phi & -e^{i\alpha} \sin \phi \\
  e^{-i\alpha} \sin \phi & \cos \phi
\end{bmatrix}
\]

We need

\[
|a'_{ii}|^2 + |a'_{jj}|^2 \rightarrow \max.
\]

(3.14)

Set $a_{rs} = x_{rs} + y_{rs}i$. Condition (3.14) becomes

\[
|x'_{ii} + y'_{ii}i|^2 + |x'_{jj} + y'_{jj}i|^2 = (x'_{ii})^2 + (y'_{ii})^2 + (x'_{jj})^2 + (y'_{jj})^2 \rightarrow \max.
\]

(3.13)

From (3.13) it follows

\[
\begin{align*}
x'_{ii} + y'_{ii}i &= (x_{ii} + y_{ii}i) \cos^2 \phi + (x_{jj} + y_{jj}i) \sin^2 \phi \\
&\quad + (x_{ij} \cos \alpha - x_{ij} \sin \alpha + y_{ij} \cos \alpha + y_{ij} \sin \alpha) \sin \phi \cos \phi \\
&\quad + (x_{ji} \cos \alpha + x_{ji} \sin \alpha + y_{ji} \cos \alpha - y_{ji} \sin \alpha) \sin \phi \cos \phi,
\end{align*}
\]

\[
\begin{align*}
x'_{jj} + y'_{jj}i &= (x_{ii} + y_{ii}i) \sin^2 \phi + (x_{jj} + y_{jj}i) \cos^2 \phi \\
&\quad - (x_{ij} \cos \alpha + x_{ij} \sin \alpha + y_{ij} \cos \alpha + y_{ij} \sin \alpha) \sin \phi \cos \phi \\
&\quad - (x_{ji} \cos \alpha - x_{ji} \sin \alpha + y_{ji} \cos \alpha - y_{ji} \sin \alpha) \sin \phi \cos \phi.
\end{align*}
\]

Splitting the real and imaginary part and using $e^{i\alpha} = \cos \alpha + i \sin \alpha$ gives

\[
\begin{align*}
x'_{ii} &= x_{ii} \cos^2 \phi + x_{jj} \sin^2 \phi + (x_{ij} \cos \alpha + y_{ij} \sin \alpha) \sin \phi \cos \phi + (x_{ji} \cos \alpha - y_{ji} \sin \alpha) \sin \phi \cos \phi, \\
y'_{ii} &= y_{ii} \cos^2 \phi + y_{jj} \sin^2 \phi + (-x_{ij} \sin \alpha + y_{ij} \cos \alpha) \sin \phi \cos \phi + (x_{ji} \sin \alpha + y_{ji} \cos \alpha) \sin \phi \cos \phi, \\
x'_{jj} &= x_{ii} \sin^2 \phi + x_{jj} \cos^2 \phi - (x_{ij} \cos \alpha + y_{ij} \sin \alpha) \sin \phi \cos \phi + (x_{ji} \cos \alpha - y_{ji} \sin \alpha) \sin \phi \cos \phi, \\
y'_{jj} &= y_{ii} \sin^2 \phi + y_{jj} \cos^2 \phi - (-x_{ij} \sin \alpha + y_{ij} \cos \alpha) \sin \phi \cos \phi + (x_{ji} \sin \alpha + y_{ji} \cos \alpha) \sin \phi \cos \phi.
\end{align*}
\]

Define the function $g : (-\frac{\pi}{2}, \frac{\pi}{2}) \times (-\frac{\pi}{2}, \frac{\pi}{2}) \rightarrow \mathbb{R}$,

\[
g(\phi, \alpha) = (x'_{ii})^2 + (y'_{ii})^2 + (x'_{jj})^2 + (y'_{jj})^2.
\]

(3.15)
Finding $\phi$ and $\alpha$ that maximize $g$ will give the solution of the maximization problem (3.14). Partial derivatives of $g$ are
\[ 0 = \frac{\partial}{\partial \phi} g(\phi, \alpha) = 2 \cos \alpha \cos 4\phi \left( (x_{ij} + x_{ji})(x_{ii} - x_{jj}) + (y_{ij} + y_{ji})(y_{ii} - y_{jj}) \right) + 2 \sin \alpha \cos 4\phi \left( (x_{ii} - x_{jj})(y_{ij} - y_{ji}) + (x_{ji} - x_{ij})(y_{ii} - y_{jj}) \right) + \sin 4\phi \left( x_{ij}^2 + x_{ji}^2 + y_{ij}^2 + y_{ji}^2 - (x_{ii} - x_{jj})^2 - (y_{ii} - y_{jj})^2 \right) + 2 \cos 2\alpha (x_{ij}x_{ji} + y_{ij}y_{ji}) + 2 \sin 2\alpha (x_{ji}y_{ij} - x_{ij}y_{ji}), \tag{3.16} \]
\[ 0 = \frac{\partial}{\partial \alpha} g(\phi, \alpha) = 2 \sin^2 2\phi \left( (x_{ij}y_{ji} - x_{ij}y_{ji}) \cos 2\alpha - (x_{ij}x_{ji} + y_{ij}y_{ji}) \sin 2\alpha \right) + \sin 2\phi \cos 2\phi \left( (x_{ii} - x_{jj})(y_{ij} - y_{ji}) - (x_{ij} - x_{ji})(y_{ii} - y_{jj}) \right) \cos \alpha + \left( (x_{ij} + x_{ji})(x_{jj} - x_{ii}) - (y_{ij} + y_{ji})(y_{ii} - y_{jj}) \right) \sin \alpha. \tag{3.17} \]

We take a closer look at (3.17) and distinguish between different cases.
- The trivial solution is $\phi = 0$. The transformation matrix $R$ will be the identity.
- If $\phi = \frac{\pi}{4}$, relation (3.17) simplifies to
  \[ 0 = (x_{ji}y_{ij} - x_{ij}y_{ji}) \cos 2\alpha - (x_{ij}x_{ji} + y_{ij}y_{ji}) \sin 2\alpha. \]

Then we either have $x_{ij}x_{ji} + y_{ij}y_{ji} = 0$ and $\alpha = \pm \frac{\pi}{4}$ or
\[ \tan 2\alpha = \frac{x_{ji}y_{ij} - x_{ij}y_{ji}}{x_{ij}x_{ji} + y_{ij}y_{ji}}. \tag{3.18} \]

Otherwise, we divide (3.17) by $\cos^2 2\phi$ and set $t = \tan 2\phi$. We obtain a quadratic equation in $t$,
\[ K_2(\alpha) t^2 + K_1(\alpha) t = 0, \]
where
\[ K_1(\alpha) = \left( (x_{ii} - x_{jj})(y_{ij} - y_{ji}) - (x_{ij} - x_{ji})(y_{ii} - y_{jj}) \right) \cos \alpha + \left( (x_{ij} + x_{ji})(x_{jj} - x_{ii}) - (y_{ij} + y_{ji})(y_{ii} - y_{jj}) \right) \sin \alpha, \]
\[ K_2(\alpha) = 2 \left( (x_{ji}y_{ij} - x_{ij}y_{ji}) \cos 2\alpha - (x_{ij}x_{ji} + y_{ij}y_{ji}) \sin 2\alpha \right). \]

Since $\phi \neq 0$, we have $t \neq 0$ and
\[ t = -\frac{K_1(\alpha)}{K_2(\alpha)}. \tag{3.19} \]

We have $K_2(\alpha) \neq 0$ because $\phi \neq \frac{\pi}{2}$.

Now we consider relation (3.16). Again, we distinguish between different cases.
- If $\alpha = \frac{\pi}{2}$, relation (3.16) simplifies to
  \[ 0 = 2 \left( (x_{ii} - x_{jj})(y_{ij} - y_{ji}) + (x_{ji} - x_{ij})(y_{ii} - y_{jj}) \right) \cos 4\phi + \left( (x_{ij} - x_{ji})^2 + (y_{ij} - y_{ji})^2 - (x_{ii} - x_{jj})^2 - (y_{ii} - y_{jj})^2 \right) \sin 4\phi. \]

Then we either have $\phi = \pm \frac{\pi}{8}$ or
\[ \tan 4\phi = \frac{-2 \left( (x_{ii} - x_{jj})(y_{ij} - y_{ji}) + (x_{ji} - x_{ij})(y_{ii} - y_{jj}) \right)}{(x_{ij} - x_{ji})^2 + (y_{ij} - y_{ji})^2 - (x_{ii} - x_{jj})^2 - (y_{ii} - y_{jj})^2}. \tag{3.20} \]
Otherwise, we substitute
\[
\cos 4\phi = \frac{1 - t^2}{1 + t^2}, \quad \sin 4\phi = \frac{2t}{1 + t^2},
\]
\[
\cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha, \quad \sin 2\alpha = 2\sin \alpha \cos \alpha
\]
in (3.16) and obtain
\[
2 \cos \alpha (1 - t^2)((x_i + x_j)(x_{ii} - x_{jj}) + (y_{ij} + y_{ji})(y_{ii} - y_{jj}))
+ 2 \sin \alpha (1 - t^2)((x_i - x_j)(y_{ij} - y_{ji}) + (x_{ij} - x_{ji})(y_{ii} - y_{jj}))
+ 2t(x_{ij}^2 + x_{ji}^2 + y_{ij}^2 + y_{ji}^2 - (x_{ii} - x_{jj})^2 - (y_{ii} - y_{jj})^2)
+ 2(\cos^2 \alpha - \sin^2 \alpha)(x_{ij}x_{ji} + y_{ij}y_{ji}) + 4 \cos \alpha \sin \alpha (x_{ij}y_{ij} - x_{ij}y_{ji})) = 0.
\]
Using (3.19) we multiply the obtained equation with $K_2(\alpha)^2$ and get
\[
\cos \alpha (K_2(\alpha)^2 - K_1(\alpha)^2)((x_i + x_j)(x_{ii} - x_{jj}) + (y_{ij} + y_{ji})(y_{ii} - y_{jj}))
+ \sin \alpha (K_2(\alpha)^2 - K_1(\alpha)^2)((x_i - x_j)(y_{ij} - y_{ji}) + (x_{ij} - x_{ji})(y_{ii} - y_{jj}))
- K_1(\alpha)K_2(\alpha)(x_{ij}^2 + x_{ji}^2 + y_{ij}^2 + y_{ji}^2 - (x_{ii} - x_{jj})^2 - (y_{ii} - y_{jj})^2)
+ 2(\cos^2 \alpha - \sin^2 \alpha)(x_{ij}x_{ji} + y_{ij}y_{ji}) + 4 \cos \alpha \sin \alpha (x_{ij}y_{ij} - x_{ij}y_{ji})) = 0. \tag{3.21}
\]
The left-hand side in (3.21) is a sum of expressions of the form $C \cos^k \alpha \sin^l \alpha$, for $k + l = 3$ or $k + l = 5$, and different $C \in \mathbb{R}$. If we take a closer look, we see that the summands where $k + l = 5$ can be reduced. Precisely, the sum of all expressions $C \cos^k \alpha \sin^l \alpha$ such that $k + l = 5$ equals
\[
-8 \left( x_{ij}^2y_{ji}y_{ii} - x_{ij}x_{ji}y_{ji} + x_{ii}x_{ji}y_{ji} - x_{ij}x_{jj}y_{ij} + x_{ij}y_{ii}y_{ii} - x_{ii}x_{ij}y_{ji} + x_{ij}x_{jj}y_{jji} - x_{ii}y_{ij}y_{ji} \right)
+ x_{ij}y_{ij}^2y_{ji} + x_{ii}y_{ij}y_{ji} - x_{ij}y_{ii}y_{ji} - x_{ij}y_{ij}y_{ji} - x_{ij}x_{jj}y_{ij} - x_{ii}y_{ij}y_{ji} + x_{ij}x_{jj}y_{jj} + x_{ij}y_{ij}y_{jj} \cos \alpha
+ (-x_{ij}x_{ji}x_{ji} - x_{ii}x_{ij}x_{ii} + x_{ij}x_{ji}x_{jj} + x_{ij}x_{jj}x_{jj} - x_{ij}y_{ii}y_{jj} - x_{ii}x_{ij}y_{jj} + x_{ij}x_{jj}y_{jj} - x_{ii}y_{ij}y_{jj} - x_{ij}x_{jj}y_{jj} - x_{ii}y_{ij}y_{jj}) \cdot \cos \alpha
- y_{ij}y_{ij}^2y_{jj} - x_{ii}x_{ij}y_{jj} + x_{ij}x_{jj}y_{jj} - x_{ij}y_{ij}y_{jj} + x_{ij}y_{ij}y_{jj} + x_{ij}y_{ij}y_{jj} + y_{ij}y_{ij}y_{jj} + x_{ii}y_{ij}y_{jj} + y_{ij}y_{ij}y_{jj} \cdot \sin \alpha \right).
\cdot (\cos^2 \alpha + \sin^2 \alpha).
\cdot (-x_{ij}y_{ij} \cos^2 \alpha + x_{ij}y_{ij} \cos^2 \alpha + 2x_{ij}x_{ji} \cos \alpha \sin \alpha + 2y_{ij}y_{ji} \cos \alpha \sin \alpha + x_{ij}y_{ij} \sin^2 \alpha - x_{ij}y_{ji} \sin^2 \alpha).
\]
Using only the fact that $\cos^2 \alpha + \sin^2 \alpha = 1$ we see that this is again a sum of expressions $C \cos^k \alpha \sin^l \alpha$ for $k + l = 3$. Therefore, the left-hand side in (3.21) is a sum of expressions of the form $C \cos^k \alpha \sin^l \alpha$ for $k + l = 3$.

Thus we can divide equation (3.21) by $\cos^3 \alpha$ and set $\tau = \tan \alpha$. We get a cubic equation in $\tau$,
\[
C_3\tau^3 + C_2\tau^2 + C_1 + C_0\tau = 0. \tag{3.22}
\]
Equation (3.22) has at least one real solution. For each real solution we substitute $\alpha = \arctan \tau$ into (3.19) to obtain $t$, and hence $\phi = \frac{1}{2} \arctan t$.

Finally, we take the pair $(\phi, \alpha)$ from among all possible solutions that gives the largest value of function $g$. Algorithm 2 summarizes the process of computing $\phi$ and $\alpha$.

**Algorithm 2.** Rotation angles in Algorithm 1

*Form $g(\phi, \alpha)$ as in (3.15).*

*Case 1: $(\phi_1, \alpha_1) = (0, 0).$*
Case 2: \((\varphi_2, \alpha_2) = (\frac{\pi}{4}, \pm\frac{\pi}{4})\), or \((\varphi_2, \alpha_2) = (\frac{\pi}{4}, \alpha_2)\) as in (3.18).

Case 3: \((\varphi_3, \alpha_3) = (\pm\frac{\pi}{8}, \frac{\pi}{2})\), or \((\varphi_3, \alpha_3) = (\varphi_3, \frac{\pi}{2})\) with \(\alpha_3\) as in (3.19).

Case 4: \((\varphi_4, \alpha_4)\) with \(\alpha_4 = \arctan \tau\) for all real solutions \(\tau\) of (3.22) and the corresponding \(\varphi_4\) from (3.19).

Choose that pair \((\varphi, \alpha)\) which gives the largest value of \(g(\varphi, \alpha)\).

4. Convergence of Algorithm 1

In this section we provide a convergence proof for Algorithm 1. We will discuss only the case of Hamiltonian matrices, the proof for the other three structures follows in the same way with only minor modifications. Related to the maximization problem (3.1) we define the objective function

\[
\tilde{f}_H : \mathcal{U} \cap \mathcal{S}p \to \mathbb{R}_{\geq 0}, \quad \tilde{f}_H(Z) = \|\text{diag}(Z^HHZ)\|_F^2 = \sum_{j=1}^{2n} |\langle A(Ze_j, Z_e_j)\rangle|^2.
\]

We show that Algorithm 1 converges to the stationary point of this function. In particular, we will prove the following theorem.

**Theorem 4.1.** Let \((Z_k, k \geq 0)\) be the sequence generated by Algorithm 1. Every accumulation point of \((Z_k, k \geq 0)\) is a stationary point of function \(f_H\) from (4.1).

The proof of Theorem 4.1 uses the technique from [5], and is based on Polak’s theorem on model algorithms [9, Section 1.3, Theorem 3]. We will need three auxiliary results from Lemmas 4.2, 4.4, and (4.5).

Before we move to Lemma 4.2 that gives the structure of \(\text{grad} f_H(Z)\), let us say a bit more about the function \(f_H\). As a real valued function of a complex variable is complex differentiable only if it is constant, the function \(f_H\) is not complex differentiable. That is, \(\frac{\partial f_H}{\partial z_{jk}}\) does not exist for \(Z = [z_{jk}] \in \mathbb{C}^{2n \times 2n}\). But with \(z_{jk} = \text{Re}(z_{jk}) + i\text{Im}(z_{jk})\), partial derivatives

\[
\frac{\partial f_H}{\partial \text{Re}(z_{jk})} \quad \text{and} \quad \frac{\partial f_H}{\partial \text{Im}(z_{jk})}, \quad j, k = 1, \ldots, 2n,
\]

do exist as this involves only real differentiation. We identify \(\mathbb{C}\) with \(\mathbb{R}^{1 \times 2}\), \(\mathbb{C}^{2n}\) with \(\mathbb{R}^{2n \times 2n}\) and \(\mathbb{C}^{2n \times 2n}\) with \(\mathbb{R}^{2n \times 2n \times 2}\). Consequently, \(f_H\) is viewed as a real-valued function on the Euclidian space \(\mathbb{R}^{2n \times 2n \times 2}\). As such, it is differentiable and its gradient is a matrix

\[
\text{grad} f_H(Z) = \left[ \frac{\partial f_H}{\partial \text{Re}(z_{jk})} + i \frac{\partial f_H}{\partial \text{Im}(z_{jk})} \right]_{j, k=1}^{2n}.
\]

**Lemma 4.2.** The gradient of \(f_H\) from (4.1) can be expressed as

\[
\text{grad} f_H(Z) = ZX,
\]

where \(\text{diag}(X) = 0\) and \(X\) is skew-Hermitian Hamiltonian.

**Proof.** We will not be able to determine \(\text{grad} f_H(Z)\) directly. Instead, we define function \(\tilde{f} : \mathbb{C}^{2n \times 2n} \to \mathbb{R}_{\geq 0},\)

\[
\tilde{f}(Z) = \sum_{j=1}^{2n} |\langle A(Ze_j, Z_e_j)\rangle|^2,
\]

on a larger domain. Then \(f_H\) is the restriction of \(\tilde{f}\) to \(\mathcal{U} \cap \mathcal{S}p\). We first determine \(\text{grad} \tilde{f}(Z)\).
To that end we define a new function $g : \mathbb{C}^{2n} \rightarrow \mathbb{R}$, $g(z) = |\langle A z, z \rangle|^2$. It allows us to rewrite $\tilde{f}$ as

$$\tilde{f}(Z) = \sum_{j=1}^{2n} g(Ze_j).$$

Then

$$\text{grad} \tilde{f}(Z) = \begin{bmatrix} \nabla g(Ze_1) & \cdots & \nabla g(Ze_{2n}) \end{bmatrix},$$

where

$$\nabla g(z) = \begin{bmatrix} \frac{\partial g}{\partial \text{Re}(z_j)} + i \frac{\partial g}{\partial \text{Im}(z_j)} \end{bmatrix}_{j=1}^{2n},$$

as $g$ is real differentiable.

In order to determine $\nabla g(z)$ we use Taylor expansion of $g$,

$$g(z + h) = g(z) + \langle \nabla g(z), h \rangle_{\mathbb{R}} + O(\|h\|^2),$$

for $h \in \mathbb{C}^{2n}$ and $\langle u, v \rangle_{\mathbb{R}} = \Re(\langle u, v \rangle)$. We have

$$g(z + h) - g(z) = |\langle A(z + h), z + h \rangle|^2 - |\langle A z, z \rangle|^2$$

$$= |\langle A z, z \rangle + \langle A z, h \rangle + \langle A h, z \rangle + \langle h, h \rangle|^2 - |\langle A z, z \rangle|^2$$

$$= 2\Re(\langle A z, z \rangle + \langle A z, h \rangle + \langle h, h \rangle) + O(\|h\|^2)$$

$$= \Re(\langle 2(A z, z)Az + 2(A z, z)A^H z, h \rangle) + O(\|h\|^2).$$

Then

$$g(z + h) - g(z) = 2\langle (A z, z)Az + (A z, z)A^H z, h \rangle_{\mathbb{R}} + O(\|h\|^2).$$

Relation (4.2) implies

$$\nabla g(z) = 2\langle (A z, z)Az + (A z, z)A^H z \rangle_{\mathbb{R}}.$$

With this, we have described $\text{grad} \tilde{f}(Z)$.

Further on, $\text{grad} f_H(Z)$ is obtained by projecting $\text{grad} \tilde{f}(Z)$ onto the tangent space of unitary symplectic matrices at $Z$. We have

$$\text{grad} f_H(Z) = \pi(\text{grad} \tilde{f}(Z)).$$

For any unitary (and symplectic) matrix $Z$, matrix

$$Y := Z^H \text{grad} \tilde{f}(Z)$$

does exist. Thus we can write $\text{grad} \tilde{f}(Z) = ZY$. Then

$$\text{grad} f_H(Z) = ZX,$$

where $X = \pi(Y)$. Since the tangent space of the group of unitary symplectic matrices at the identity are skew-Hermitian Hamiltonian matrices, it only remains to prove that $\text{diag}(X) = 0$.

The diagonal of $Y$ from (4.3) is given by

$$\text{diag}(Y) = \text{diag}(Z^H \text{grad} \tilde{f}(Z)) = (\langle \nabla g(Ze_j), Ze_j \rangle)_{j=1}^{2n}.$$

Further on,

$$\langle \nabla g(z), z \rangle = 2\langle (A z, z)Az + (A z, z)A^H z \rangle_{\mathbb{R}} = 4|\langle A z, z \rangle|^2 \in \mathbb{R}.$$

Therefore, $\text{diag}(Y)$ is real. Its projection onto the space of skew-Hermitian matrices will give zeros on the diagonal of $X$, that is $\text{diag}(X) = 0$. $\square$
Remark 4.3. The orthogonal projection of \( Y = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} \) onto the subspace of skew-Hermitian Hamiltonian matrices is given by

\[
\begin{bmatrix}
B & C \\
-C & B
\end{bmatrix}, \quad B = \frac{Y_{11} + Y_{22} - Y_{11}^H - Y_{22}^H}{4}, \quad C = \frac{Y_{12} - Y_{21} + Y_{12}^H - Y_{21}^H}{4}.
\]

Lemma 4.2 is used in Lemma 4.4.

Lemma 4.4. For every symplectic unitary \( Z \in \mathbb{C}^{2n \times 2n} \) there is symplectic rotation \( R(i, j, \phi, \alpha) \) such that

\[
\langle \langle \text{grad}_H f(Z), Z \hat{R}(i, j, 0, \alpha) \rangle \rangle \geq \eta \| \text{grad}_H f(Z) \|_F, \quad \eta = \frac{2}{\sqrt{4n^2 - 2n}},
\]

where \( \hat{R}(i, j, 0, \alpha) \) denotes \( \frac{\partial}{\partial \phi} R(i, j, \phi, \alpha) \bigg|_{\phi=0} \).

Proof. Obviously, if \( \| \text{grad}_H f(Z) \|_F = 0 \), the assertion holds for any rotation. Thus, assume that \( \| \text{grad}_H f(Z) \|_F \neq 0 \). From Lemma 4.2 we know that \( \text{grad}_H f(Z) = ZX \). Hence, for \( X = [x_{ij}]_{i,j=1}^{2n} \) and \( |x| = |x_{rs}| = \max_{i \neq j} |x_{ij}| > 0 \) we have

\[
\| \text{grad}_H f(Z) \|_F = \| ZX \|_F = \| X \|_F \leq \sqrt{4n^2 - 2n} |x|.
\]

On the other hand,

\[
\langle \langle \text{grad}_H f(Z), Z \hat{R}(i, j, 0, \alpha) \rangle \rangle = \text{Re}(\text{trace}((\text{grad}_H f(Z))^H Z \hat{R}(i, j, 0, \alpha)))
\]

\[
= \text{Re}(\text{trace}((ZX)^H Z \hat{R}(i, j, 0, \alpha)))
\]

\[
= \text{Re}(\text{trace}(X^H \hat{R}(i, j, 0, \alpha))).
\]

Let us first consider a unitary symplectic rotation \( R(i, j, \phi, \alpha) \) of the form (3.6). Then

\[
\hat{R}(i, j, \phi, \alpha) = \begin{bmatrix}
-\sin \phi & -e^{i\alpha} \cos \phi \\
-e^{-i\alpha} \cos \phi & -\sin \phi
\end{bmatrix}
\]

and

\[
\hat{R}(i, j, 0, \alpha) = \begin{bmatrix}
-e^{i\alpha} & e^{-i\alpha} \\
e^{-i\alpha} & -e^{i\alpha}
\end{bmatrix},
\]

where in the matrices on the right-hand side all elements that are not explicitly given are zero. It follows from Lemma 4.2 that matrix \( X \) is skew-Hermitian and Hamiltonian. For \( x = x_{ij} \) we
have $x_{ji} = -\bar{x}$, $x_{n+i,n+j} = x$ and $x_{n+j,n+i} = -\bar{x}$. This gives

$$X^H \hat{R}(i, j, 0, \alpha) = \begin{bmatrix} -x & -e^{i\alpha} \\ \bar{x} & e^{-i\alpha} \\ -x & -e^{i\alpha} \end{bmatrix}$$

where in $X^H$ only four relevant entries at the positions $(i, j)$, $(j, i)$, $(n+i, n+j)$ and $(n+j, n+i)$ are given. Now (4.5) implies

$$\langle \text{grad} f_H(Z), Z \dot{R}(i, j, 0, \alpha) \rangle |_R = \text{Re}(-2xe^{-i\alpha} - 2\bar{x}e^{i\alpha}) = -4\text{Re}(xe^{-i\alpha}).$$

Choose $\tilde{\alpha}$ such that

$$e^{-i\tilde{\alpha}} = \text{sgn}(\bar{x}) = \frac{\bar{x}}{|\bar{x}|}. \quad (4.6)$$

Then

$$|\langle \text{grad} f_H(Z), Z \dot{R}(i, j, 0, \tilde{\alpha}) \rangle |_R| = 4\text{Re}(x\frac{\bar{x}}{|\bar{x}|}) = 4|x|. \quad (4.7)$$

Using relation (4.4) we obtain

$$|\langle \text{grad} f_H(Z), Z \dot{R}(i, j, 0, \tilde{\alpha}) \rangle |_R| \geq \frac{4}{\sqrt{4n^2 - 2n}} \|\text{grad} f_H(Z)\|_F = 2\eta \|\text{grad} f_H(Z)\|_F.$$
Lemma 4.5. Let \((Z_k, k \geq 0)\) be the sequence generated by Algorithm 4.1. For every \(\hat{Z} \in U \cap S_p\) with \(\text{grad} f_H(\hat{Z}) \neq 0\), there exist \(\epsilon > 0\) and \(\delta > 0\) such that
\[
\|Z_k - \hat{Z}\|_F < \epsilon \quad \Rightarrow \quad f_H(Z_{k+1}) - f_H(Z_k) \geq \delta.
\]

Proof. As \(\text{grad} f_H(\hat{Z}) \neq 0\), there exists \(\epsilon > 0\) such that
\[
\eta_1 := \min_{\|Z - \hat{Z}\|_F < \epsilon} \|\text{grad} f_H(Z)\|_F > 0.
\]

For a fixed \(k\) we define the differentiable function \(h_k : \mathbb{R} \times \mathbb{R} \to \mathbb{R}\),
\[
h_k(\phi, \alpha) = f_H(Z_k R(i_k, j_k, \phi, \alpha)),
\]
where \(R(i_k, j_k, \phi, \alpha)\) is a unitary symplectic rotation as in Subsection 3.1. As a part of Algorithm 4.1, Algorithm 2 returns \(\phi_k\) and \(\alpha_k\) such that \(f_H\) is maximized. Since \(R(i_k, j_k, 0, \alpha) = I\) for any \(\alpha\), we have
\[
h_k(0, \alpha) = f_H(Z_k) \quad \text{and} \quad \max_{\phi, \alpha} h_k(\phi, \alpha) = h_k(\phi_k, \alpha_k) = f_H(Z_{k+1}).
\]

Take \(\tilde{\alpha}\) as in (4.6) and define another function \(H_k : \mathbb{R} \to \mathbb{R}\),
\[
H_k(\phi) = h_k(\phi, \tilde{\alpha}).
\]
The Taylor expansion of \(H_k\) around 0 yields
\[
H_k(\phi_k) = H_k(0) + H_k'(0)\phi_k + \frac{1}{2} H_k''(\xi)\phi_k^2, \quad 0 < \xi < \phi_k.
\]
Let \(M = \max |H_k''(\xi)| < \infty\). Then
\[
H_k(\phi_k) - H_k(0) \geq H_k'(0)\phi_k - \frac{1}{2} M\phi_k^2.
\]

The derivative of \(H_k\) is
\[
H_k'(\phi) = \frac{\partial}{\partial \phi} f_H(Z_k R(i_k, j_k, \phi, \tilde{\alpha})) = \langle \text{grad} f_H(Z_k R(i_k, j_k, \phi, \tilde{\alpha})), Z_k \dot{\phi}\rangle_{\mathbb{R}},
\]
and in particular,
\[
H_k'(0) = \langle \text{grad} f_H(Z_k), Z_k \dot{\phi}\rangle_{\mathbb{R}}.
\]
From Lemma 4.4 and relation (4.8) we obtain
\[
|H_k'(0)| \geq \eta \|\text{grad} f_H(Z_k)\|_F \geq \eta \min_{\|Z - \hat{Z}\|_F < \epsilon} \|\text{grad} f_H(Z)\|_F = \eta \eta_1.
\]

From (4.9), (4.10) and (4.11), for any \(\phi\), we have
\[
f_H(Z_{k+1}) - f_H(Z_k) = h_k(\phi_k, \alpha_k) - h_k(0, \tilde{\alpha}) - h_k(0, \tilde{\alpha})
\]
\[
= H_k(\phi) - H_k(0) \geq H_k'(0)\phi - \frac{1}{2} M\phi^2.
\]

Choose \(\phi = \frac{H_k'(0)}{M}\). Finally, from (4.13) and (4.12) we obtain
\[
f_H(Z_{k+1}) - f_H(Z_k) \geq \frac{H_k'(0)^2}{M} - \frac{H_k'(0)^2}{2M} = \frac{H_k'(0)^2}{2M} \geq \frac{\eta \eta_1^2}{2M} = \delta.
\]
Now we can prove Theorem 4.1 by contradiction.

Proof of Theorem 4.1. Suppose that \( \tilde{Z} \) is an accumulation point of Algorithm 1. Then there is a subsequence \( \{Z_j\} \), \( j \in K \subseteq \mathbb{N} \) such that \( Z_j \) converges to \( \tilde{Z} \).

Assume that \( \tilde{Z} \) is not a stationary point of \( f_H \), that is \( \text{grad} f_H(\tilde{Z}) \neq 0 \). Then, for any \( \epsilon > 0 \), there is \( k_0 \in K \) such that \( \|Z_k - \tilde{Z}\| < \epsilon \) for every \( k > k_0 \). Lemma 4.5 implies that \( f_H(Z_{k+1}) - f_H(Z_k) \geq \delta > 0 \). Therefore, \( f_H(Z_k) \to \infty \) when \( k \to \infty \). Though, if \( Z_k \) converges, \( f_H(Z_k) \) should converge, too. This gives a contradiction. \( \square \)

5. Numerical experiments

We present some numerical experiments for the Hamiltonian and skew-Hamiltonian case. We set up a random \( 2n \times 2n \) Hamiltonian matrix \( H \) as in (2.1) by generating a random \( n \times n \) matrix \( H_{11} \) and random \( n \times n \) Hermitian matrices \( H_{12} \) and \( H_{21} \). Also, we set up a random \( 2n \times 2n \) skew-Hamiltonian matrix \( W \) as in (2.2) using a random \( n \times n \) matrix \( W_{11} \) and random \( n \times n \) skew-Hermitian matrices \( W_{12} \) and \( W_{21} \). All tests were done in Matlab R2019b.

First, we see how the matrix norm moves to the diagonal during three iterations of Algorithm 1. In Figure 1 the change in absolute value of the matrix entries is given. We start with a random \( 50 \times 50 \) Hamiltonian matrix \( H \) and show \( H^{(k)} \), \( k = 1, 2, 3 \). We can observe that the underlying matrix becomes diagonally dominant already after the first iteration. During the next iterations norm on the diagonal increases.

In general, Hamiltonian matrix cannot be diagonalized using symplectic rotations. Then Algorithm 1 will diagonalize it as much as possible. In Figure 2 we show the convergence of \( \|\text{diag}(A^{(k)})\|_F \), \( k = 1, \ldots, 20 \), for two \( 100 \times 100 \) Hamiltonian matrices, one that can not be diagonalized using only symplectic rotations and the other one that can. We see how \( \|\text{diag}(A^{(k)})\|_F \) approaches \( \|A\|_F \). When \( A \) can be diagonalized using only symplectic rotations, then complete norm of \( A \) can be moved to its diagonal, so \( \|\text{diag}(A^{(k)})\|_F \) becomes equal to \( \|A\|_F \).

Theorems 2.4, 2.5, 2.7 and 2.8 all include condition on the matrix eigenvalues. In Hamiltonian and perskew-Hermitian case, eigenvalues should not be purely imaginary, while in skew-Hamiltonian and per-Hermitian case they should not be real. Hamiltonian matrices from Figures 1 and 2 have all eigenvalues with non-zero real part, that is no purely imaginary eigenvalues. Still, in practice, Algorithm 1 does not display any difference if the condition on the eigenvalues is not satisfied. Figure 3 gives the convergence of \( \|\text{diag}(A^{(k)})\|_F \) compared to \( \|A\|_F \) for two \( 50 \times 50 \) skew-Hamiltonian matrices, one with no and the other one with some real eigenvalues.

Recall that in Algorithm 1 any cyclic pivot ordering can be used since the convergence proof from Section 4 does not depend on the ordering inside one sweep. In all previous examples we used pivot ordering

\[
O_1 = (1, 2), (1, 3), \ldots, (1, n), (2, 3), \ldots, (2, n), \ldots, (n - 1, n),
(1, n + 1), (2, n + 2), \ldots, (n, 2n),
(1, n + 2), (1, n + 3), \ldots, (1, 2n), (2, n + 3), \ldots, (2, 2n), \ldots, (n - 1, 2n).
\]

Now we will compare the convergence using two different cyclic orderings. The first one is \( O_1 \). The second one is “bottom to top” ordering from [7]. Keep in mind that we take pivot positions from the upper triangle, while in [7] they are taken from the lower triangle. This transforms “bottom to top” ordering into “right to left”, meaning that instead of

\[
(2n, 1), (2n - 1, 1), \ldots, (2, 1), (2n, 2), \ldots, (3, 2), \ldots, (2n, 2n - 1)
\]
Figure 1. Change in the absolute value of matrix entries.

Figure 2. Convergence of $\|\text{diag}(A^{(k)})\|_F$, $A \in \mathcal{H}$. 
we have

\[(1, 2n), (1, 2n - 1), \ldots, (1, 2), (2, 2n), \ldots, (2, 3), \ldots, (2n - 1, 2n).\]  

(5.1)

Besides, because our algorithm uses double rotations, it does not take all pivot positions listed above, but its subset, as shown in (5.3). Thus, “bottom to top” ordering applied to our situation is a subset of (5.1) given by

\[O_2 = (1, 2n), (1, 2n - 1), \ldots, (1, n + 1), (1, n), (1, n - 1), \ldots, (1, 2),\]

\[(2, 2n), (2, 2n - 1), \ldots, (2, n + 2), (2, n), (2, n - 1), \ldots, (2, 3),\]

\[(3, 2n), \ldots, (3, n + 3), (3, n), \ldots, (3, 4), \ldots, (n - 1, 2n), (n - 1, n), (n, 2n).\]

In Figure 4 we present the convergence results for both orderings \(O_1\) and \(O_2\) on two random \(50 \times 50\) Hamiltonian matrices, one that can not and one that can be completely diagonalized by unitary symplectic transformations.
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