EIGENVALUE BOUND FOR SCHRÖDINGER OPERATORS WITH UNBOUNDED MAGNETIC FIELD

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Abstract. In this paper we consider magnetic Schrödinger operators on the two-dimensional unit disk with a radially symmetric magnetic field which explodes to infinity at the boundary. We prove a bound for the eigenvalue moments and a bound for the number of negative eigenvalues for such operators.

1. Introduction

1.1. The current paper deals with eigenvalue bounds for magnetic Schrödinger operators. However at first we recall some classical results for non-magnetic case. Let $V(x)$ be a bounded measurable real-valued function on an open set $\Omega \subset \mathbb{R}^d$, $d \geq 1$. We consider the Schrödinger operator

$$H_{\Omega}(0,V) = -\Delta_{\Omega} - V$$

acting in $L^2(\Omega)$ subject to the Dirichlet conditions on the boundary of $\Omega$; 0 is the notation $H_{\Omega}(0,V)$ reflects the fact that there are no magnetic potential. Denote by $\{\lambda_j(\Omega,0,V)\}_{j=1}^N$ the eigenvalues of $H_{\Omega}(0,V)$ located below the bottom of the essential spectrum of $H_{\Omega}(0,V)$. As usual we renumber the eigenvalues in the non-decreasing order and repeat them according to their multiplicity. If $\Omega$ is bounded, the spectrum of $H_{\Omega}(0,V)$ is purely discrete, $N = \infty$, and the eigenvalues $\lambda_j(\Omega,0,V)$ accumulates at infinity. The main object of our studies are the so-called Riesz means given by

$$\text{tr}(H_{\Omega}(0,V))^\sigma = \sum_{\lambda_j(\Omega,0,V) \leq 0} |\lambda_j(\Omega,V)|^\sigma, \quad \sigma \geq 0. \quad (2)$$

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Here and in what follows the notation $f^\pm = (|f| \pm f)/2$ stays for the positive and negative parts of a number, a function or an operator. Note that for $\sigma = 0$ the quantity in (2) is the number of non-positive eigenvalues of $H_0(V)$. The first classical result in this area concerns the behaviour of $\text{tr}(H_0(V))^\sigma_-$ in the strong coupling limit. Namely, introducing a scaling parameter $\lambda > 0$ and replacing the potential $V$ by $\lambda V$ one gets the asymptotic formula

$$
\lim_{\lambda \to \infty} \lambda^{-\sigma-d/2} \text{tr}(H_0(\lambda V))^\sigma_- = L_{\sigma,d}^\text{cl} \int_\Omega V_+(z)^{\sigma+d/2} dz, \quad \sigma \geq 0 \quad (3)
$$

with the semiclassical constant

$$
L_{\sigma,d}^\text{cl} = \frac{\Gamma(\sigma + 1)}{(4\pi)^{\frac{d}{2}} \Gamma(\sigma + 1 + d/2)}. \quad (4)
$$

We assumed above that $V \in L^{\sigma+d/2}(\Omega)$. For $\sigma = 0$, $V \equiv \text{const}$ this result goes back to H. Weyl [18], therefore (3) is usually referred to as Weyl’s law.

The second classical result – Lieb-Thirring inequality – was established by E.H. Lieb and W. Thirring in [11]. It states that the right-hand-side in (3) is not only the limit of the left-hand-side, but also an upper bound (up to a multiplicative constant). Namely, for $\sigma > \max\{0, 1-d/2\}$ and $V \in L^{\sigma+d/2}(\Omega)$ the estimate

$$
\text{tr}(H_0(V))^\sigma_- \leq L_{\sigma,d} \int_\Omega V_+(z)^{\sigma+d/2} dz \quad (5)
$$

holds with certain positive constant $L_{\sigma,d}$. In fact, the above result was established in [11] for $\Omega = \mathbb{R}^d$, and then for an arbitrary domain $\Omega$ it holds immediately due to the inequality

$$
\text{tr}(H_0(V))^\sigma_- \leq \text{tr}(H_{\mathbb{R}^d}(0, \hat{V}))^\sigma_-, \quad (6)
$$

where $\hat{V}$ is the extension of $V$ by zero to $\mathbb{R}^d \setminus \Omega$; (6) follows easily from the min-max principle (see, e.g., [13]).

Note that estimate (3) remains valid for $\sigma = 0$, $d \geq 3$. This result was established independently by M. Cwikel [6], E.H. Lieb [9], and G.V. Rozenblyum [14, 15]. T. Weidl [17] proved that (5) also holds for $d = 1$, $\sigma = 1/2$. However for $d = 2$ and $\sigma = 0$ (5) does not hold. In this case one has the following estimate established by K. Chadan, N.N. Khuri, A. Martin, and T. T. Wu in [4] under the assumption that the potential $V$ is radially symmetric:

$$
\text{tr}(H_0(V))^0_- \leq 1 + \int_{\mathbb{R}^2} V_+(z)(1 + |\ln |z||) dz. \quad (7)
$$

1.2. Despite the rigorous study of Schrödinger operators [11], there has been much less investigation of Schrödinger operators with magnetic fields, which are in focus of the present paper. Let $\Omega$ be a open set in $\mathbb{R}^2$; in what follows the points in $\Omega$ will be denoted by $z$, its Cartesian coordinates will be denoted by $(x, y)$. Let

$$
A = (A_1, A_2) : \Omega \to \mathbb{R}^2 \quad \text{(magnetic potential)}, \quad V : \Omega \to \mathbb{R} \quad \text{(electric potential)}. \nonumber
$$
As above $V$ is assumed to be bounded and measurable. The two-dimensional magnetic Schrödinger operator is (formally) defined by

$$H_{\Omega}(A, V) = (i\nabla + A)^2 - V.$$  \hfill (8)

On $\partial \Omega$ we again prescribe the Dirichlet boundary conditions. The magnetic field $B$ is given by

$$B = \text{rot } A = \frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}.$$  

Again we denote the eigenvalues $H_{\Omega}(A, V)$ lying below the bottom of the essential spectrum by $\{\lambda_j(\Omega, A, V)\}_{j=1}^N$ renumbering them in the increasing order and with account of their multiplicities. Note, that if $\Omega$ is bounded and the vector potential $A$ satisfies mild regularity conditions, the magnetic Sobolev norm

$$\|(i\nabla + A)u\|_{L^2(\Omega)}^2, \quad u \in H^1_0(\Omega)$$

is equivalent to the non-magnetic one, whence one can easily deduce the discreteness of the spectrum of $H_{\Omega}(A, V)$, i.e. in this case one has $N = \infty,$ and eigenvalues $\lambda_j(\Omega, A, V)$ accumulates to infinity.

A. Laptev and T. Weidl \cite{12} proved that

$$\text{tr}(H_{\mathbb{R}^d}(A, V))^\sigma \leq \mathcal{L}_{\sigma,d}^1 \int_{\mathbb{R}^d} V_+(z)^{\sigma+d/2}dz, \quad \sigma \geq 3/2$$ \hfill (9)

provided $A \in L^2_{\text{loc}}(\mathbb{R}^d)$ and $V \in L^{\sigma+d/2}(\mathbb{R}^d)$. By the minimax principle estimate (9) also holds with an arbitrary bounded domain $\Omega$ instead of $\mathbb{R}^d$, provided $A \in L^2(\Omega)$ and $V \in L^{\sigma+d/2}(\Omega)$.

1.3. One of the models attracting considerable attention in the last ten years concerns magnetic Schrödinger operators on bounded domains $\Omega$ with magnetic fields satisfying

$$B(z) \to \infty \text{ as } z \to \partial \Omega.$$  \hfill (10)

Apparently, for the first time such model was treated in \cite{5}, where the authors established the essential self-adjointness of $H_{\Omega}(A, V)$ (defined on $C_0^\infty(\Omega)$) under certain assumptions on the growth of $B$ near the boundary of $\Omega$. The obtained results are of some technical interest due to their connection with to a special kind of magnetic confinement devices – tokamaks.

In the current paper for the above model we derive Lieb-Thirring-type inequality (Theorem 2.1) and also the estimate for the number of negative eigenvalues (Theorem 2.2) under the following restrictions: $\Omega$ is a unit disc, the magnetic field is radially symmetric with respect to the center of this disc, and also some additional conditions of the growth of $B$ near the boundary of $\Omega$ take place (cf. (12)). It is important that magnetic potentials we deal with are not necessary in $L^2(\Omega)$ and therefore we are not able to apply the results of \cite{12}.

We formulate the main results in Section 2. Note, that some other eigenvalue bounds for operators (8) with magnetic fields satisfying (10) were also derived in \cite{2, 3, 16} under more restrictive assumptions on $A$ and $B$. At the end of Section 2 we compare the estimates obtained in these works with the estimates presented in the present paper. Their proof are given in Section 3.
2. Results

Let $\Omega$ be the two-dimensional unit disk centered at the origin. We denote the points in $\Omega$ by $z = (r, \theta)$, where $(r, \theta)$ are polar coordinates (with respect to the center of $\Omega$).

We are given the bounded measurable function $V : \Omega \to \mathbb{R}$ (electric potential) and the radially symmetric function $B : \Omega \to \mathbb{R}$ (magnetic field) satisfying

$$\inf_{z \in \Omega} B(z) > 0, \quad (11)$$

$$B(z) = \frac{M}{(1 - |z|)^\alpha} + g(|z|) \quad (12)$$

with some $\alpha \in (0, 2]$ and $M > 0$, and a bounded measurable function $g : [0, \infty) \to \mathbb{R}$. It is easy to see that, up to a gauge transformation, the corresponding magnetic potential $A = (A_1, A_2)$ is given by

$$A_1(z) = -\sin \theta \cdot \int_0^r sB(s)ds, \quad A_2 = \cos \theta \cdot \int_0^r sB(s)ds, \quad z = (r, \theta).$$

Note that $A$ does not belong to $L^2(\Omega)$ as $\alpha \geq 3/2$.

We define the operator $H_{\Omega}(A, V)$ acting in $L^2(\Omega)$ by differential operation $(8)$, first on the smooth and compactly supported in $\Omega$ functions. In view of $(11)$ and the well-known lower bound (see, e.g., [1])

$$(H_{\Omega}(A, V)(u), u)_{L^2(\Omega)} \geq \int_{\Omega} (B(z) - \|V\|_{L^\infty(\Omega)}) |u(z)|^2 \, dz$$

one can construct the Friedrichs extension of $H_{\Omega}(A, V)$. For simplicity, we will use for this Friedrichs extension the same symbol $H_{\Omega}(A, V)$.

One can show that $H_{\Omega}(A, V)$ has a purely discrete spectrum. In fact, this fact will be established within the proof of Theorem 2.1. We denote the increasingly ordered sequence of the eigenvalues of $H_{\Omega}(A, V)$ by $\lambda_k = \lambda_k(\Omega, A, V), \ k = 1, 2, 3 \ldots .$

Further we will also need the function $\tilde{V} : \mathbb{R}_+ \to \mathbb{R}$ given by

$$\tilde{V}(r) := \text{ess sup}_{\theta \in [0, 2\pi)} V_+(r, \theta). \quad (13)$$

Now we are in position to give the main results of this work.

**Theorem 2.1.** For any $\sigma > 0$ the inequality

$$\text{tr} (H_{\Omega}(A, V))_\sigma^\sigma \leq C \int_0^1 \tilde{V}^{\sigma+1}(r) r \, dr \quad (14)$$

holds with some positive constant $C = C(B, \sigma)$ depending on $B$ and on $\sigma$.

For the radially symmetric potential $V$ our estimate $(14)$ coincides with the standard Lieb-Thirring inequality up to a constant depending on the magnetic field.

**Theorem 2.2.** The estimate

$$\text{tr} (H_{\Omega}(A, V))_0^0 \leq 1 + C_1(B) \int_0^1 \tilde{V}_+(r)(1 + |\ln r|) r \, dr \quad (15)$$

holds with some positive constant $C_1 = C_1(B)$ depending on $B$. 
For the radially symmetric potential $V$ our estimate (15) coincides with (7) up to a constant depending on the magnetic field.

**Discussion.** Estimates for Riesz means $\text{tr} (H_{\Omega}(A,V))^\sigma$ as $\sigma > 0$ and the magnetic field satisfies (10) have been obtained in [2, 3] under stronger restriction $B$ and $A$ comparing with those we treat in the present paper. Namely, in [2] the authors assumed that the total magnetic flux $\int_\Omega B(z) \, dz$ is less than $\pi$ (for $B$ satisfying (12) this does not hold for $\alpha \geq 1$); in [3] the right-hand side of the estimate explodes to infinity if $A \not\in L^2(\Omega)$.

As regard to the number of negative eigenvalues, we refer at first to the paper [8], where the author treated magnetic Schrödinger operators on $\mathbb{R}^2$ and for a large class of magnetic potential obtained the estimates resembling (7) (as in our Theorem 2.2, with constants depending on the magnetic field), again under the assumption that magnetic potentials are in $L^2_{\text{loc}}(\mathbb{R}^d)$ – thus we cannot use these results for all magnetic fields we treat in the present paper (recall that in the present paper the magnetic potential does not belong to $L^2(\Omega)$ if in the assumption (12) $\alpha$ is larger or equal then $3/2$).

If $\Omega$ is a disc and the magnetic field is radially symmetric and satisfies (10), some estimates (rather different from (15)) for the number of negative eigenvalues were obtained in [16] under additional assumption $B(z) \leq M(1 - |z|)^\alpha$, $\alpha \in (0, 3/2)$, $M > 0$.

In this case the underlying magnetic potential is again square integrable on $\Omega$.

**3. Proof of Theorems 2.1, 2.2**

In what follows, we use the same notation $\tilde{V}$ for the function of $r \in [0, \infty)$ defined by (13) as well for the radially symmetric function of $z = (r, \theta) \in \Omega$, whose values at $z$ with $|z| = r$ are defined by (13). That is, $\tilde{V}(r) = \tilde{V}(z)$ as $z = (r, \theta) \in \Omega$.

At first we observe that $H_{\Omega}(A,V) \leq H_{\Omega}(A,\tilde{V})$. Then by the minimax principle

$$\text{tr} (H_{\Omega}(A,V))^\sigma \leq \text{tr} (H_{\Omega}(A,\tilde{V}))^\sigma, \quad \sigma \geq 0.$$  

Hence it is sufficient to prove estimate (14) with $H_{\Omega}(A,\tilde{V})$ being replaced by $H_{\Omega}(A,V)$ in its left-hand-side.

Recall, that our magnetic potential $A(r, \theta)$ is given by

$$A(r, \theta) = \left( \frac{-\Phi(r)}{r} \sin \theta, \frac{\Phi(r)}{r} \cos \theta \right), \text{ where } \Phi(r) := \int_0^r sB(s) \, ds.$$  

We denote by $h_m(B,\tilde{V})$ the Friedrichs extensions of the operator being associated with the quadratic form $Q(h_m(B,\tilde{V}))$ in $L^2((0,1), 2\pi r \, dr)$,

$$Q(h_m(B,\tilde{V}))[v] = 2\pi \int_0^1 \left( |v'(r)|^2 + \frac{(m - \Phi(r))^2}{r^2} |v(r)|^2 \right) \, dr, \quad v \in C_0^\infty(0,1).$$  

The action of this operator is given by

$$h_m(B,\tilde{V}) = -\frac{d^2}{dr^2} - \frac{1}{r} \frac{d}{dr} + \frac{(m - \Phi(r))^2}{r^2} - \tilde{V}(r).$$
The radial symmetry of our potentials implies (see [7]) the direct sum decomposition

\[ H_\Omega(A, \tilde{V}) = \bigoplus_{m=-\infty}^{\infty} h_m(B, \tilde{V}) \]

with respect to the space decomposition

\[ L^2(\Omega, dx) = \bigoplus_{m=-\infty}^{\infty} L^2((0,1), 2\pi r dr), \]

\[ f \rightarrow (\ldots, f_{-1}, f_0, f_1, \ldots) \quad \text{with} \quad f(r, \theta) = \sum_{m=-\infty}^{\infty} e^{im\theta} f_m(r). \]

Our strategy is to prove that

\[ h_m(B, \tilde{V}) \geq \gamma h_m(0, \tilde{V}/\gamma), \]

with some constant \( \gamma > 0 \), and then to employ the standard Lieb-Thirring bound for the non-magnetic Schrödinger operator

\[ H_\Omega(0, \tilde{V}/\gamma) = \bigoplus_{m=-\infty}^{\infty} h_m(0, \tilde{V}/\gamma). \] (16)

Due to (11) \( \Phi(r) \geq 0 \), whence we immediately conclude that

\[ h_m(B, \tilde{V}) \geq \gamma_0 \left( -\frac{d^2}{dr^2} - \frac{1}{r} \frac{d}{dr} + \frac{m^2}{r^2} - \tilde{V}/\gamma_0 \right) \quad \text{as} \quad m \leq 0, \] (17)

where \( \gamma_0 := 1 \).

Now, let \( m > 0 \). We introduce the numbers \( r_m, r'_m \in (0, 1] \) in the following way:

- If \( \Phi(1) \geq 2m \), then \( r_m, r'_m \) are such numbers that \( \Phi(r_m) = m/2 \) and \( \Phi(r'_m) = 2m \) holds; obviously \( r_m, r'_m \in (0, 1] \).
- If \( m/2 < \Phi(1) < 2m \), then \( r_m \) is defined as above, while \( r'_m := 1 \).
- If \( \Phi(1) \leq m/2 \), we set \( r_m = r'_m := 1 \).

It is easy to see that

\[ m - \Phi(r) \geq \frac{m}{2} \quad \text{as} \quad r \in (0, r_m), \] (18)

\[ \Phi(r) - m \geq m \quad \text{as} \quad r \in (r'_m, 1). \] (19)

In the following, \( v \) be an arbitrary function from \( C_0^\infty(0,1) \) normalized by

\[ \|v\|_{L^2((0,1), 2\pi r dr)}^2 = 2\pi \int_0^1 v(r)r dr = 1. \] (20)

Note, that (20) imply the following simple estimate:

\[ \int_{r_m}^{r'_m} \frac{1}{r} |v(r)|^2 dr \leq \frac{1}{2\pi r_m^2}. \] (21)
If $\Phi(1) \leq m/2$, then inequality (18) holds for all $r \in (0, 1)$. Consequently,

$$Q(h_m(B, \tilde{V}))[v] \geq 2\pi \int_0^1 r \left( |v'(r)|^2 + \frac{m^2}{4r^2} |v(r)|^2 - \tilde{V}(r) |v(r)|^2 \right) dr,$$

whence, denoting $\gamma_1 := 1/4$, we get

$$h_m(B, \tilde{V}) \geq \gamma_1 \left(- \frac{d^2}{dr^2} - \frac{1}{r} \frac{d}{dr} + \frac{m^2}{r^2} - \tilde{V}/\gamma_1 \right)$$

as $m > 0$, $\Phi(1) \leq m/2$. (22)

For $\Phi(1) > m/2$ (which implies, in particular, $r_m < r'_m$), we consider separately two cases:

**Case I:**

$$2\pi \int_{r_m}^{r_m'} r |v(r)|^2 dr \leq \frac{1}{2}$$

**Case II:**

$$2\pi \int_{r_m}^{r_m'} r |v(r)|^2 dr > \frac{1}{2}.$$ 

**Case I.** At first we note that, due to (20), (23) is equivalent to

$$2\pi \int_{(0, r_m) \cup (r'_m, 1)} r |v(r)|^2 dr > \frac{1}{2}.$$ 

Inequality (26) together with (18) yields

$$2\pi \int_{(0, r_m) \cup (r'_m, 1)} \frac{(m - \Phi(r))^2}{r} |v(r)|^2 dr \geq \frac{m^2}{2} \int_{(0, r_m) \cup (r'_m, 1)} r |v(r)|^2 dr \geq \frac{m^2}{8}. \quad (26)$$

Combining (26) with (18)–(19) we find

$$\frac{1}{2\pi} Q(h_m(B, \tilde{V}))[v] \geq \int_0^1 r \left( |v'(r)|^2 - \tilde{V}(r) |v(r)|^2 \right) dr + \int_{(0, r_m) \cup (r'_m, 1)} \frac{(m - \Phi(r))^2}{r} |v(r)|^2 dr$$

$$\geq \int_0^1 r \left( |v'(r)|^2 - \tilde{V}(r) |v(r)|^2 \right) dr + \frac{1}{2} \int_{(0, r_m) \cup (r'_m, 1)} \frac{m^2}{4r} |v(r)|^2 dr + \frac{m^2}{32\pi}.$$ 

The above bound together with (21) implies

$$\frac{1}{2\pi} Q(h_m(B, \tilde{V}))[v] \geq \int_0^1 r \left( |v'(r)|^2 - \tilde{V}(r) |v(r)|^2 \right) dr$$

$$+ \frac{1}{8} \int_{(0, r_m) \cup (r'_m, 1)} \frac{m^2}{r} |v(r)|^2 dr + \frac{r_m^2}{16} \int_{r_m}^{r'_m} \frac{m^2}{r} |v(r)|^2 dr$$

$$\geq \frac{r_1^2}{16} \int_0^1 r \left( |v'(r)|^2 + \frac{m^2}{r^2} |v(r)|^2 - \frac{16}{r_1^2} \tilde{V}(r) |v(r)|^2 \right) dr.$$ 

Then, denoting $\gamma_2 := r_1^2/16$, we arrive at

$$h_m(B, \tilde{V}) \geq \gamma_2 \left(- \frac{d^2}{dr^2} - \frac{1}{r} \frac{d}{dr} + \frac{m^2}{r^2} - \tilde{V}/\gamma_2 \right)$$

as $m > 0$, $\Phi(1) > m/2$, (23) holds. (27)
Case II. We set $\kappa := \min \{ r'_m - r_m, r_m/2 \}$ and fix an arbitrary $\mu \in (0, 1)$ such that

$$\mu < \frac{(r_m - \kappa)\kappa}{8(r_m - r_m + \kappa)} \tag{28}$$

(such a choice of constants will become clear later). Again we have two possibilities:

**Case IIa.**

$$2\pi \int_{r_m - \kappa}^{r_m} r |v(r)|^2 \, dr > \mu \tag{29}$$

**Case IIb.**

$$2\pi \int_{r_m - \kappa}^{r_m} r |v(r)|^2 \, dr \leq \mu. \tag{30}$$

**Case IIa.** It is easy to see that (29) implies

$$2\pi \int_{r_m - \kappa}^{r_m} \frac{|v(r)|^2}{r} \, dr \geq \frac{\mu}{r_m^2}. \tag{31}$$

Repeating the similar calculations as in Case I and taking into account (18)-(19) and (31) we obtain the following estimate:

$$\frac{1}{2\pi} Q(h_m(B, \tilde{V}))[v] \geq \int_0^1 r \left( |v'(r)|^2 - \tilde{V}(r) |v(r)|^2 \right) \, dr$$

$$+ \int_{(0, r_m - \kappa) \cup (r'_m, 1)} \frac{(m - \Phi(r))^2}{r} |v(r)|^2 \, dr + \int_{r_m - \kappa}^{r_m} \frac{(m - \Phi(r))^2}{r} |v(r)|^2 \, dr$$

$$\geq \int_0^1 r \left( |v'(r)|^2 - \tilde{V}(r) |v(r)|^2 \right) \, dr$$

$$+ \int_{(0, r_m - \kappa) \cup (r'_m, 1)} \frac{m^2}{4r} |v(r)|^2 \, dr + \int_{r_m - \kappa}^{r_m} \frac{m^2}{4r} |v(r)|^2 \, dr$$

$$\geq \int_0^1 r \left( |v'(r)|^2 - \tilde{V}(r) |v(r)|^2 \right) \, dr$$

$$+ \int_{(0, r_m - \kappa) \cup (r'_m, 1)} \frac{m^2}{4r} |v(r)|^2 \, dr + \frac{m^2 \mu}{16\pi r_m^2}. \tag{32}$$

This together with (21) gives

$$\frac{1}{2\pi} Q(h_m(B, \tilde{V}))[v] \geq$$

$$\int_0^1 r \left( |v'(r)|^2 - \tilde{V}(r) |v(r)|^2 \right) \, dr$$

$$+ \int_{(0, r_m - \kappa) \cup (r'_m, 1)} \frac{m^2}{4r} |v(r)|^2 \, dr$$

$$+ \frac{1}{2} \int_{r_m - \kappa}^{r_m} \frac{m^2}{4r} |v(r)|^2 \, dr + \frac{\mu}{8} \int_{r_m - \kappa}^{r_m} \frac{m^2}{r} |v(r)|^2 \, dr$$

$$\geq \frac{\mu}{8} \int_0^1 r \left( |v'(r)|^2 + \frac{m^2}{r^2} |v|^2 - \frac{8\tilde{V}(r)}{\mu} |v|^2 \right) \, dr.$$
The latter means
\[ h_m(B, \tilde{V}) \geq \gamma_3 \left( -\frac{d^2}{dr^2} - \frac{1}{r} \frac{d}{dr} + \frac{m^2}{r^2} - \tilde{V}_3/\gamma_3 \right) \]
as \( m > 0 \), \( \Phi(1) > m/2 \), (24), (29) hold.

\[ \gamma_3 = \frac{\mu}{8}. \]

**Case IIb.** We need the following auxiliary lemma.

**Lemma 3.1.** Under assumptions (11)-(12) there exists a constant \( \tilde{C} = \tilde{C}(B) \) such that the following inequality takes place
\[ \Phi'(r) \geq \tilde{C} \Phi^2(r). \]

**Proof.** Recall, that
\[ \Phi'(r) = rB(r), \]
where \( B(r) \) is given by (12). One has the asymptotic formulae
\[ \Phi(r) = \begin{cases} 
\frac{M}{(\alpha - 1) (1 - r)^{\alpha - 1}} (1 + o(1)), & \alpha > 1, \\
M \ln(1 - r) (1 + o(1)), & \alpha = 1, \text{ as } r \to 1, \\
\Phi(r) = \mathcal{O}(1), & \alpha < 1 
\end{cases} \]

(36)

Taking into account that \( \alpha \) is assumed to be smaller than or equal to 2, one easily obtains from (35) and (36) the estimate (34) for the values of \( r \) being close to 1 (more precisely, for \( r \in [r', 1] \) with some \( r' < 1 \)). Finally, for \( r \in [0, r'] \) one can estimate \( \Phi \) as follows:
\[ (\Phi(r))^2 \leq \frac{\|B\|_{L^\infty(0, r')}^2}{4} r^4 \leq rB(r)/\tilde{C}, \]

where \( \tilde{C} = 4 \inf_{z \in \Omega} B(z) \) (recall, that \( \inf_{z \in \Omega} B(z) > 0 \)). The lemma is proven.

Let us return to the proof of the theorem. Recall, that we investigate **Case IIa**, which means that conditions (24) and (30) holds.

In view of (30) one can choose a point \( z \in (r_m - \kappa, r_m) \) such that
\[ |v(z)| \leq \left( \frac{\mu}{2\pi(r_m - \kappa)\kappa} \right)^{1/2}. \]

This inequality together with the fundamental theorem of calculus gives
\[ \frac{1}{2} < 2\pi \int_{z}^{r_m} r|v(r)|^2 dr = 2\pi \int_{z}^{r_m} r \left| \int_{z}^{r} v'(t) dt + v(z) \right|^2 dr \]
\[ \leq 4\pi \int_{z}^{r_m} r(r - z) \int_{z}^{r} |v'(t)|^2 dt dr + 4\pi|v(z)|^2(r'_m - z) \]
\[ \leq \frac{4\pi(r'_m - r_m + \kappa)^2}{z} \int_{z}^{r_m} r |v'(r)|^2 dr + \frac{2\mu(r'_m - r_m + \kappa)}{(r_m - \kappa)\kappa}. \]
Hence in view of (28)
\[ \int_0^{z_m} r|v(r)|^2 \, dr \geq \frac{16\pi (r_m' - r_m + \kappa)^2}{64\pi (r_m' - r_m)^2}. \tag{37} \]
Using the mean value theorem $\Phi(r_m') - \Phi(r_m) = \Phi'(r_m')(r_m' - r_m)$, where $r_m'$ is some point in $(r_m, r_m')$, the monotonicity of $\Phi$ (it follows from (11)), and Lemma (3.1) we obtain
\[ r_m' - r_m = \frac{\Phi(r_m') - \Phi(r_m)}{\Phi'(r_m')} = \frac{3m}{2\Phi'(r_m')} \leq \frac{3\Phi(r_m)}{C\Phi^2(r_m)} \leq \frac{3\Phi(r_m)}{C\Phi^2(r_m)} \leq \frac{6}{Cm}. \tag{38} \]
Finally, due to the choice of $\kappa$, one gets
\[ z \geq r_m/2 \geq r_1/2. \tag{39} \]
Combining (37)–(39) we conclude the existence of a constant $C'' = C''(B) > 0$ such that
\[ \int_0^{z_m} r|v(r)|^2 \, dr \geq C'' m^2. \]
This estimate together with (18)–(19) and (21) implies
\[ \frac{1}{2\pi} Q(h_m(B, \tilde{V}))[v] \geq \int_{(0,z)\cup(r_m,1)} r|v'(r)|^2 \, dr - \int_0^1 r\tilde{V}(r)|v(r)|^2 \, dr \\
+ \frac{1}{2} \int_z^{r_m} r|v'(r)|^2 \, dr + \frac{1}{2} C'' m^2 \\
+ \int_{(0,r_m)\cup(r_m,1)} \frac{(m - \Phi(r))^2}{r^2} |v(r)|^2 \, dr + \int_{r_m}^{r_m'} \frac{(m - \Phi(r))^2}{r} |v(r)|^2 \, dr \\
\geq \int_{(0,z)\cup(r_m,1)} r|v'(r)|^2 \, dr - \int_0^1 r\tilde{V}(r)|v(r)|^2 \, dr + \frac{1}{2} \int_z^{r_m} r|v'(r)|^2 \, dr \\
+ \int_{(0,r_m)\cup(r_m,1)} \frac{m^2}{4r} |v(r)|^2 \, dr + C'' \pi r_m^2 \int_{r_m}^{r_m'} \frac{m^2}{r^2} |v(r)|^2 \, dr \\
\geq \gamma_4 \int_0^1 r \left( |v'(r)|^2 + \frac{m^2}{r^2} |v(r)|^2 - \tilde{V}(r)/\gamma_4 |v(r)|^2 \right) \, dr, \]
where $\gamma_4 := \min\{1/4, C'' r_1^2\}$. Thus
\[ h_m(B, \tilde{V}) \geq \gamma_4 \left( -\frac{d^2}{dr^2} - \frac{1}{r} \frac{d}{dr} + \frac{m^2}{r^2} - \tilde{V}/\gamma_4 \right) \tag{40} \]
as $m > 0$, $\Phi(1) > m/2$, (24), (30) hold.

Combining inequalities (17), (22), (27), (33) and (40) we obtain the desired estimate
\[ \forall m \in \mathbb{Z} : \ h_m(B, \tilde{V}) \geq \gamma h_m(0, \tilde{V}/\gamma), \text{ where } \gamma = \min\{\gamma_0, \gamma_1, \gamma_2, \gamma_3, \gamma_4\} \]
(note, that $\gamma$ depends only on $B$). Consequently
\[ H_\Omega(A, \tilde{V}) \geq \gamma H_\Omega(0, \tilde{V}/\gamma). \tag{41} \]
Using (41) and taking into account that the spectrum of \( H_{\Omega}(0, \tilde{V}/\gamma) \) is purely discrete, we conclude by the min-min principle that the spectrum of \( H_{\Omega}(A, \tilde{V}/\gamma) \) is also purely discrete, moreover

\[
\forall \sigma \geq 0 : \quad \text{tr} \left( H_{\Omega}(A, \tilde{V}) \right)_-^\sigma \leq \gamma^\sigma \text{tr} \left( H_{\Omega}(0, \tilde{V}/\gamma) \right)_-^\sigma.
\]

Finally, applying for \( \sigma > 0 \) the Lieb-Thirring bound (5) (recall, that in the two-dimensional case (5) holds only for positive \( \sigma \)) we obtain from (42) the estimate

\[
\text{tr} \left( H_{\Omega}(A, \tilde{V}) \right)_-^\sigma \leq \gamma^\sigma L_{\sigma,2} \int_{\Omega} \left( \frac{\tilde{V}(z)}{\gamma} \right)^{\sigma+1} dz = \frac{L_{\sigma,2}}{\gamma} \int_0^1 r^{\tilde{V}_{\sigma+1}}(r) dr,
\]

where \( L_{\sigma,2} \) is a constant from (5). Thus Theorem 2.1 is proven. Similarly, Theorem 2.2 follows from (42) (with \( \sigma = 0 \)) and the Chadan-Khuri-Martin-Wu estimate (7).

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**References**

[1] J. Avron, I. Herbst, B. Simon, Schrödinger operators with magnetic fields. I: General interactions, Duke Math. J. 45 (1978), 847–883.
[2] D. Barseghyan, P. Exner, H. Kovářík, T. Weidl, Semiclassical bounds in magnetic bottles, Reviews in Mathematical Physics, 28 (1) (2016).
[3] D. Barseghyan, F. Truc, Magnetic Dirichlet Laplacian with radially symmetric magnetic field, Operator Theory: Advances and Applications (2019), to appear.
[4] K. Chadan, N. N. Khuri, A. Martin, T. T. Wu, Bound states in one and two spatial dimensions, Journal of Mathematical Physics 44, 406-422 (2003).
[5] Y. Colin de Verdière, F. Truc, Confining quantum particles with a purely magnetic field, Ann. Inst. Fourier 60 (2010), 2333–2356.
[6] M. Cwikel, Weak type estimates for singular values and the number of bound states of Schrödinger operators, Ann.Math. 106 (1977),93–102.
[7] L. Erdös, Gaussian decay of the magnetic eigenfunctions, Geometric and Functional Analysis, 6 (1996), 231-248.
[8] H. Kovarik. Eigenvalue bounds for two-dimensional magnetic Schrödinger operators, Journal of Spectral Theory, 1(4) (2011), DOI: 10.4171/JST/16.
[9] E.H. Lieb, The number of bound states of one body Schrödinger operators and the Weyl problem, Proc. A.M.S. Symp. Pure Math. 36 (1980), 241–252.
[10] E. H. Lieb, R. Seiringer, The stability of matter in quantum mechanics, Cambridge University Press, Cambridge, 2010.
[11] E.H. Lieb, W. Thirring, Inequalities for the moments of the eigenvalues of the Schrödinger Hamiltonian and their relation to Sobolev inequalities, in Studies in Mathematical Physics, Essays in Honor of Valentine Bargmann (E. Lieb, B. Simon and A.S. Wightman, eds.); Princeton University Press, Princeton (1976), 269–330.
[12] A. Laptev, T. Weidl: Sharp Lieb-Thirring inequalities in high dimensions, Acta Mathematica, 184 (2000), 87–111.
[13] M. Reed and B. Simon, Methods of Modern Mathematical Physics. IV. Analysis of Operators, Academic Press, New York–London, 1978.
[14] G.V. Rozenblyum, The distribution of the discrete spectrum for singular differential operators, Sov. Math. Dokl. 13 (1972), 245–249.
[15] G.V. Rozenblyum, Distribution of the discrete spectrum of singular differential operators, Soviet Math. 20 (1976), 63–71.

[16] F. Truc, Eigenvalue bounds for radial magnetic bottles on the disk, Asymptotic Analysis, 76, 2012, 233-248.

[17] T. Weidl, On the Lieb-Thirring constants $L_{\gamma,1}$ for $\gamma \geq 1/2$, Comm. Math. Phys. 178 (1996), 135–146.

[18] H. Weyl, Das asymptotische Verteilungsgesetz der Eigenwerte linearer partieller Differentialgleichungen (mit einer Anwendung auf die Theorie der Hohlraumstrahlung), Mathematische Annalen 71 (4) (1912), 441–479.