UNIQUENESS OF REDUCED ALTERNATING RATIONAL 3-TANGLES

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Abstract. Tangles were introduced by J. Conway. In 1970, he proved that every rational 2-tangle defines a rational number and two rational 2-tangles are isotopic if and only if they have the same rational number. So, from Conway’s result we have a perfect classification for rational 2-tangles. However, there is no similar theorem to classify rational 3-tangles.

In this paper, We introduce an invariant of rational \( n \)-tangles which is obtained from the Kauffman bracket. It forms a vector with Laurent polynomial entries. We prove that the invariant classifies the rational 2-tangles and the reduced alternating rational 3-tangles. We conjecture that it classifies the rational 3-tangles as well.

1. Introduction

A \( n \)-tangle is the disjoint union of \( n \) properly embedded arcs in the unit 3-ball. A rational \( n \)-tangle is a \( n \)-tangle \( \alpha_1 \cup \alpha_2 \cup \cdots \cup \alpha_n \) in a 3-ball \( B^3 \) such that there exists a homeomorphism of pairs \( \Phi : (B^3, \alpha_1 \cup \alpha_2 \cup \cdots \cup \alpha_3) \rightarrow (D^2 \times I, \{p_1, p_2, ..., p_n\} \times I) \) (a trivial tangle).

Two rational \( n \)-tangles, \( T, T' \), in \( B^3 \) are isotopic, denoted by \( T \approx T' \), if there is an orientation-preserving self-homeomorphism \( h : (B^3, T) \rightarrow (B^3, T') \) that is the identity map on the boundary.

Let \( S^2 \) be a sphere smoothly embedded in \( S^3 \) and let \( K \) be a link transverse to \( S^2 \). The complement in \( S^3 \) of \( S^2 \) consists of two open balls, \( B_1 \) and \( B_2 \). We assume that \( S^2 \) is the \( xz \)-plane \( \cup \{\infty\} \). Now, consider the projection of \( K \) onto the flat \( xy \)-plane. Then, the projection onto the \( xy \)-plane of \( S^2 \) is the \( x \)-axis and \( B_1 \) projects to the upper half plane and \( B_2 \) projects to the lower half plane. The projection gives us a link diagram, where we make note of over and undercrossings. The diagram of the link \( K \) is called a plat on \( 2n \)-strings, denoted by \( p_{2n}(w) \), if it is the union of a \( 2n \)-braid \( w \) and \( 2n \) unlinked and unknotted arcs which connect pairs of consecutive strings of the braid at the top and at the bottom endpoints and \( S^2 \) meets the top of the \( 2n \)-braid. (See the first and second diagrams of Figure 1.) Any link \( K \) in \( S^3 \) admits a plat presentation. i.e., \( K \) is ambient isotopic to a plat ([2], Theorem 5.1). The bridge (plat) number \( b(K) \) of \( K \) is the smallest possible number \( n \) such that there exists a plat presentation of \( K \) on \( 2n \) strings. We say that \( K \) is \( n \)-bridge link if the bridge number of \( K \) is \( n \). We remark that the braid group \( \mathbb{B}_{2n} \) is generated by \( \sigma_1, \sigma_2, \ldots, \sigma_{2n-1} \) which are twisting of two adjacent strings. For example, \( w = \sigma_2^{-1}\sigma_4^{-1}\sigma_3\sigma_1\sigma_5\sigma_2^{-1}\sigma_4^{-1}\sigma_2^{-1} \) is the word for the 6 braid of the first diagram of Figure 1.

Then we say that a plat presentation is standard if the \( 2n \)-braid \( w \) of \( p_{2n}(w) \) involves only \( \sigma_2, \sigma_3, \ldots, \sigma_{2n-1} \).
We remark that $K \cap B_2$ is a tangle.

Now, we define a **plat presentation** for rational $n$-tangles $p_{2n}(w) \cap B_2$ in $B_2$, denoted by $q_{2n}(w)$, if it is the union of a $2n$-braid $w$ and $n$ unlinked and unknotted arcs which connect pairs of strings of the braid at the bottom endpoints with the same pattern as in a plat presentation for a link and $\partial B_2$ meets the top of the $2n$-braid.

We note that $q_{2n}(w)$ is a rational $n$-tangle in $B_2$ since we can obtain a trivial rational $n$-tangle from the rational $n$-tangle by a sequence of half Dehn twists which are automorphisms of $B^3$ that preserve the six punctures.

We also say that $q_{2n}(w)$ is the **plat closure** of $q_{2n}(w)$ if it is the union of $q_{2n}(w)$ and $n$ unlinked and unknotted arcs in $B_1$ which connect pairs of consecutive strings of the braid at the top endpoints.

The tangle diagrams with the circles in Figure 2 give the diagrams of trivial rational 2, 3-tangles as in [1], [4], [5], [6]. The right sides of each pair of diagrams show the trivial rational 2, 3-tangles in $B_2$.

A tangle $T$ is **reduced alternating** if $T$ is alternating and $T$ does not have a self-crossing which can be removed by a Type I Reidemeister move.

We note that $q_{4}(w)$ is alternating if and only if $q_{4}(w)$ is alternating, possibly not reduced alternating.

In section 2, we introduce the Kauffman bracket of a rational tangle and discuss how to calculate it.
Then, we will prove that a vector from the Kauffman bracket of rational 2-tangle is an invariant which can classify the rational 2-tangles in section 3.

Finally, we will show that a vector from the Kauffman bracket of rational 3-tangle is an invariant which can classify the reduced alternating rational 3-tangles in section 4.

2. The Kauffman bracket and its calculation

Let $\Lambda = \mathbb{Z}[a, a^{-1}]$ and $L$ be a link.

We recall that the Kauffman bracket $\langle L \rangle \in \Lambda$ of a link $L$ is obtained from the three axioms

\[
\langle \bigcirc \rangle = 1 \\
\langle \bigtriangledown \rangle = a \langle \bigtriangleup \rangle + a^{-1} \langle \bigtriangleup \rangle \\
\langle L \cup \bigcirc \rangle = k \langle L \rangle ,
\]

where $k = -a^{-2} - a^2$.

The symbol $\langle \rangle$ indicates that the changes are made to the diagram locally, while the rest of the diagram is fixed.

The Kauffman polynomial $X_L(a) \in \Lambda$ is defined by

\[
X_L(a) = (-a^{-3})^{w(L)} \langle L \rangle,
\]

where the writhe $w(L) \in \mathbb{Z}$ is obtained by assigning an orientation to $L$, and taking a sum over all crossings of $L$ of their indices $e$, which are given by the following rule

\[
e (\bigtriangledown) = 1, \quad e (\bigtriangleup) = -1.
\]

Then we have the following theorem.

**Theorem 2.1.** ([7]) If $K$ is a 2-bridge link, then there exists a word $w$ in $B_4$ so that the plat presentation $p_4(w)$ is reduced alternating and standard and represents a link isotopic to $K$.

Since $p_4(w)$ is standard, we consider $B_3$ instead of $B_4$. Then, let $\sigma_1$ and $\sigma_2$ be the two generators of $B_3$. I want to emphasize here that we are changing from $\sigma_2$ and $\sigma_3$ to $\sigma_1$ and $\sigma_2$.

![Figure 3](image.png)

Goldman and Kauffman [5] define the **bracket polynomial** of the two tangle diagram $T = K \cap B_2$ as $\langle T \rangle = f(a) \langle T_0 \rangle + g(a) \langle T_\infty \rangle$, where the coefficients $f(a)$ and $g(a)$ are...
Laurent polynomials in \( a \) and \( a^{-1} \) that are obtained by starting with \( T \) and using the three axioms repeatedly until only the two trivial tangles \( T_0 \) and \( T_\infty \) in the expression given for \( T \) are left. We note that \( f(a) \) and \( g(a) \) are invariant under regular isotopy of \( T \), where regular isotopy is the equivalence relation of link diagrams that is generated by using the 2nd and 3rd Reidemeister moves only. So, we define the coefficients vector \( (f(a), g(a)) \) which is an invariant of the tangles.

Let \( \mathcal{A} = < T_0 > \) and \( \mathcal{B} = < T_\infty > \). So, \( < T > = f(a)\mathcal{A} + g(a)\mathcal{B} \).

We assume that \( T \) is a reduced alternating rational 2-tangle. Then we have \( w = \sigma_1^{\epsilon_1} \sigma_2^{-\epsilon_2} \cdot \cdots \cdot \sigma_1^{\epsilon_{2k-1}} \) for some positive (negative) integers \( \epsilon_i \) for \( 2 \leq i \leq 2k - 1 \) and non-negative (non-positive) integer \( \epsilon_1 \). We note that \( w \) needs to end at \( \sigma_1^{\pm 1} \). If \( w = \sigma_1^{\epsilon_1} \sigma_2^{-\epsilon_2} \cdot \cdots \cdot \sigma_1^{\epsilon_{2k-1}} \sigma_2^{\epsilon_{2k}} \) for some positive (negative) integer \( \epsilon_{2k} \) then it is not a reduced alternating form. i.e., we can twist the innermost unlinked and unknotted bottom arc to reduce some crossings to have fewer crossings for \( K \).

Suppose that \( q_4(w) \) is the plat presentation for the tangle \( T_1 \) so that \( w = \sigma_1^{\epsilon_1} \sigma_2^{-\epsilon_2} \cdot \cdots \cdot \sigma_1^{\epsilon_{2n-1}} \) for some positive (negative) integers \( \epsilon_i \) \( (2 \leq i \leq 2n - 1) \) and non-negative (non-positive) integer \( \epsilon_1 \).

Let \( A_1^{\pm 1} = \begin{pmatrix} -a^{\mp 1} & a^{\mp 1} \\ 0 & a^{\pm 1} \end{pmatrix} \) and \( A_2^{\pm 1} = \begin{pmatrix} a^{\pm 1} & 0 \\ a^{\mp 1} & -a^{\mp 3} \end{pmatrix} \).

Also, let \( A = A_1^{\epsilon_1} A_2^{-\epsilon_2} \cdots A_1^{\epsilon_{2n-1}} \).

Then we can calculate the two coefficients \( f(a) \) and \( g(a) \) of \( \mathcal{A} \) and \( \mathcal{B} \) of \( T_1 \) as follows.

**Theorem 2.2.** (Eliahou-Kauffman-Thistlethwaite [4]) Suppose that \( q_4(u) \) is the plat presentation of a rational two tangle \( T_1 \) which is alternating and standard so that \( u = \sigma_1^{\epsilon_1} \sigma_2^{-\epsilon_2} \cdot \cdots \cdot \sigma_1^{\epsilon_{2n-1}} \) for some positive (negative) integers \( \epsilon_i \) \( (1 \leq i \leq 2n - 1) \) and non-negative (non-positive) integer \( \epsilon_1 \). Then,

\[
<T_1> = f(a)\mathcal{A} + g(a)\mathcal{B}, \text{ where } f(a) \text{ and } g(a) \text{ are given by}
\]

\[
(f(a), g(a))^t = A \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \text{ i.e., the second column of } A, \text{ where } A = A_1^{\epsilon_1} A_2^{-\epsilon_2} \cdots A_1^{\epsilon_{2n-1}}.
\]

Then we can show the following lemma which helps us to show Theorem 3.2.

**Lemma 2.3.** Let \( A = A_2^m \) for some non-zero integer \( m \).

Then \( A = \begin{pmatrix} a^{m+2} + (-1)^{m+1} a^{-3m} \\ 1 + a^4 \end{pmatrix} \begin{pmatrix} 0 \\ (-1)^m a^{-3m} \end{pmatrix} \).

**Proof.** First, assume that \( m \) is a positive integer.
Let \( A = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \).

I will use induction on \( m \) to show this lemma.

We can easily check that if \( m = 1 \) then \( b_{11} = a, b_{12} = 0, b_{21} = a^{-1} \) and \( b_{22} = -a^{-3} \) from \( A_{2} \).

Suppose that the claim is true when \( m = k \), i.e., \( b_{11} = a^{k}, b_{12} = 0, b_{22} = (-1)^{k}a^{-3k} \), and

\[
b_{21} = \frac{a^{k+2} + (-1)^{k+1}a^{2-3k}}{1 + a^{4}}.
\]

Now, consider \( A_{2}^{k+1} = A_{2}A_{2}^{k} = \begin{pmatrix} b'_{11} & b'_{12} \\ b'_{21} & b'_{22} \end{pmatrix} \).

Then we have \( b'_{11} = aa^{k} = a^{k+1}, b'_{12} = 0, b'_{22} = -a^{-3}(-1)^{k}a^{-3k} = (-1)^{k+1}a^{-3(k+1)} \) and

\[
b'_{21} = a^{-1}(a^{k}) - a^{-3}\left(\frac{a^{k+2} + (-1)^{k+1}a^{2-3k}}{1 + a^{4}}\right) = a^{k-1} - \frac{a^{k-1} + (-1)^{k+1}a^{1-3k}}{1 + a^{4}} = a^{k+1} + (-1)^{k+1}a^{-3(k+1)}
\]

This completes the proof of the case that \( m \) is a positive integer.

Similarly, we can show the case that \( m \) is a negative integer.

\[\square\]

3. A NEW INVARIANT OF RATIONAL 2-TANGLES

For a Laurent polynomial \( p(a) \), let \( M(p(a)) \) be the maximal power of \( a \) in \( p(a) \) and \( m(p(a)) \) be the minimal power of \( a \) in \( p(a) \).

K. Murasugi [8] proved the theorem that for a link \( K \) with a reduced alternating projection onto the \( xy \)-plane, \( (M(X_{K}) - m(X_{K}))/4 \) is the same as the number of crossings of the reduced alternating projection of \( K \). Especially, we need the following theorem to prove Theorem 3.2.

**Theorem 3.1.** Suppose that \( K \) is a two-bridge link. Then \( (M(X_{K}) - m(X_{K}))/4 \) is the number of crossings of a reduced alternating projection of \( K' \) which is isotopic to \( K \).

Now, we show the following theorem which tells us the coefficients vector of the tangles classifies the rational 2-tangles.

**Theorem 3.2.** Suppose that \( T \) is the projection onto the \( xy \)-plane of a rational 2-tangle in \( B^{3} \) so that \(< T > = f(a) < T_{0} > + g(a) < T_{\infty} > \).

Then, \( T \approx T_{\infty} \) if and only if \( (f(a), g(a)) = (-a^{-3})^{k}(0, 1) \) for some integer \( k \).

**Proof.** First, we suppose that \( T \approx T_{\infty} \).
Since $T \approx T_\infty$, we get $T_\infty$ from $T$ after applying a sequence of a finite number of the three Reidemeister moves. We note that $f(a)$ and $g(a)$ are invariant under regular isotopy of $T$.

Now, consider the first Reidemeister moves as in Figure 4.

This implies that $(f(a), g(a)) = (-a^{-3})^k(0, 1)$ for some integer $k$ since $< T_\infty >= 0 < T_0 > +1 < T_\infty >$.

In order to show the opposite direction, assume that there is a non-trivial reduced alternating projection $T$ so that $< T >= (-a^{-3})^k < T_\infty >$.

Let $q_4(w)$ be the plat presentation of $T$ which is standard and reduced alternating.

Then, either $w = \sigma_1^{\epsilon_1}$ for a non-zero integer $\epsilon_1$ or $w = \sigma_2^{-\epsilon_0}\sigma_1^{\epsilon_1}\sigma_2^{-\epsilon_2}\sigma_1^{\epsilon_3} \cdots \sigma_1^{\epsilon_{2n-1}}$ for some positive (negative) integers $\epsilon_i$ (1 ≤ $i$ ≤ 2n − 1) and non-negative (non-positive) integer $\epsilon_0$.

If $w = \sigma_1^{\pm 1}$, then we see that $(f(a), g(a)) = (a^{\mp 1}, a^{\pm 1}) \neq (-a^{-3})^k(0, 1)$ for any $k$.

If $w = \sigma_1^{\epsilon_1}$ for $|\epsilon_1| ≥ 2$, then we have $p_4(w)$ which represents a reduced alternating link $K = \mathcal{T}$. So, we note that the Kauffman polynomial of $K$ is not trivial. However, we should have trivial Jones polynomial for $K$ since $< T >= (-a^{-3})^k(0, 1)$. This contradicts the assumption.

If $w = \sigma_2^{-\epsilon_0}\sigma_1$ for non-negative integer $\epsilon_0$, then $A = A_2^{-\epsilon_0}A_1$. By using Lemma 3.2, we have

$$(f(a), g(a)) = \left( a^{-\epsilon_0-1}, a^{-\epsilon_0+1} + \frac{(-1)^{-\epsilon_0}a^{3\epsilon_0+5}}{1+a^4} \right) \neq (-a^{-3})^k(0, 1) \text{ for any } k.$$  

Similarly, if $w = \sigma_2^{-\epsilon_0}\sigma_1^{-1}$ for non-positive integer $\epsilon_0$ then we have

$$(f(a), g(a)) = \left( a^{-\epsilon_0+1}, a^{-\epsilon_0+3} + \frac{(-1)^{-\epsilon_0}a^{3\epsilon_0-1}}{1+a^4} \right) \neq (-a^{-3})^k(0, 1) \text{ for any } k.$$
If $w = \sigma_2^{-\epsilon_0} \sigma_1^{\epsilon_1}$ for $\epsilon_1 \geq 2$, then we have $p_4(w)$ which represents a non-trivial alternating links. So, this violates the assumption too.

If $w = \sigma_2^{-\epsilon_0} \sigma_1^{\epsilon_1} \sigma_2^{-\epsilon_2} \sigma_1^{\epsilon_3} \cdots \sigma_1^{\epsilon_{2n-1}}$ for some positive (negative) integers $\epsilon_i$ ($1 \leq i \leq 2n-1$, $n \geq 2$) and non-negative (non-positive) integer $\epsilon_0$, then we have the reduced alternating presentation $v = \sigma_1^{\epsilon_1} \sigma_2^{-\epsilon_2} \cdots \sigma_1^{\epsilon_{2n-1}}$ for $p_4(v)$ for some positive (negative) integers $\epsilon_i$ ($1 \leq i \leq 2n-1$, $n \geq 2$).

This implies that $M(X_k) - m(X_k) \geq 4$ by Theorem 3.1.

However, if $(f(a), g(a)) = (0, (-a^{-3})^k)$ then $M(X_k) - m(X_k) = 0$ by Theorem 3.1.

This also contradicts the assumption.

Therefore, if $< T > = (-a^{-3})^k < T_\infty >$ then $T \approx T_\infty$.

This completes the proof.

\[\square\]

**Corollary 3.3.** Suppose that $T$ and $T'$ are the projections onto the $xy$-plane of two rational 2-tangles in $B^3$ so that $< T > = f(a) < T_0 > + g(a) < T_\infty >$ and $< T' > = f'(a) < T_0 > + g'(a) < T_\infty >$ respectively.

Then, $T \approx T'$ if and only if $(f'(a), g'(a)) = (-a^{-3})^k (f(a), g(a))$ for some integer $k$.

*Proof.* Let $q_4(w)$ be the plat presentation of the rational 2-tangle $T$ and $q_4(v)$ be the plat presentation of the rational 2-tangle $T'$.

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**Figure 5.**

Now, consider $e = w^{-1}w$ and $w^{-1}v$.

Let $T_1$ and $T'_1$ be the rational 2-tangles of the plat presentations $q_4(e)$ and $q_4(w^{-1}v)$ respectively.

We note that $T \approx T'$ if and only if $T_1 \approx T'_1$.

Then by Theorem 3.4, $T \approx T'$ if and only if $< T'_1 > = (-a^{-3})^k < T_\infty >$ for some $k$. 
Now, consider the following diagram to see that $T \approx T'$ if and only if $< T' > = (-a^{-3})^k < T >$ for some $k$.

This completes the proof.

4. A NEW INVARIANT OF RATIONAL 3-TANGLES

Now, suppose that a link $L$ has a 6-plat presentation $q_6(w)$. Then, let $w = \sigma_{k_1} \epsilon_1 \sigma_{k_2} \epsilon_2 \cdots \sigma_{k_{n-1}} \epsilon_n \sigma_{k_n}$ for some non-zero integers $\epsilon_i$ $(1 \leq i \leq n)$, where $k_i \in \{1, 2, 3, 4, 5\}$.

H. Cabrera-Ibarra [3] defined the bracket polynomial of the rational 3-tangle $T$ as $< T > = f_1^T(a) < 0_1 > + f_2^T(a) < 0_2 > + f_3^T(a) < 0_3 > + f_4^T(a) < 0_4 > + f_5^T < 0_5 >$, where $f_i^T(a)$ are Laurent polynomials in $a$ and $a^{-1}$ that are obtained by starting with $T$ and using the three axioms repeatedly until only the five trivial tangles $< 0_j >$ in the expression given for $T$ are left. (See Figure 6.)

Then, we define the vector $v_T = (f_1^T(a), f_2^T(a), ..., f_5^T(a))$ for a rational 3-tangle $T$. Then we note that the vector $v_T$ is an invariant under regular isotopy of $T$. Especially we have the following theorem.

**Theorem 4.1.** If $T \approx T'$ then $v_T = (-a^{-3})^k v_{T'}$ for some $k$.

**Proof.** It is the generalization of the proof for the one direction of Theorem 3.2. □

The link $L$ is $0_i$-closure of a rational 3-tangle $T$, denoted by $0_i(T)$, if $L$ is obtained from $T$ by connecting the six endpoints of $T$ as the pattern shown below which is depends on $0_i$. (See Figure 7.)
We say that $q_6(w)_i$ is a plat presentation of a rational 3-tangle $T$ under the connectivity pattern of bottom endpoints induced by $0$, as in Figure 8. For example, $q_6(w)_3 = q_6(w)$. Then we say that a rational 3-tangle $T$ is a reduced alternating 3-bridge tangle if $T$ is isotopic to one of the reduced alternating 6-plat presentations as in Figure 8, where $w$ is an element of $B_6$, so that it satisfies the condition that $T$ is a reduced alternating rational 3-tangle. Then we note that for $T = q_6(w)_i$ for some $i$, $w = \sigma_{k_1}^{\epsilon_1} \sigma_{k_2}^{\epsilon_2} \cdots \sigma_{k_n}^{\epsilon_n}$, where $\epsilon_i$ are positive (negative) if $k_i \in \{1, 3, 5\}$ and $\epsilon_j$ are negative (positive) if $k_j \in \{2, 4\}$ to have a reduced alternating 3-bridge tangle.

![Figure 8.](image-url)

Let $|T| = |\epsilon_1| + |\epsilon_2| + \cdots + |\epsilon_n|$. Then we have the following lemma.

**Lemma 4.2.** Suppose that $T$ is a reduced alternating 3-bridge tangle. If $|T| \geq 2$ then $0_i(T)$ is a reduced alternating link for some $i$.

**Proof.** Let $w$ be the element of $B_6$ so that $q_6(w)_i$ represents the reduced 3-bridge tangle $T$ for some $i$.

![Figure 9.](image-url)

First, assume that the first element of $w$ is $\sigma_{1}^{\pm 1}$. Then we can consider the two cases $(a)$ and $(b)$ as in Figure 9. The dotted line in $(a)$ means that it possibly has the crossings.

Consider $0_1$-closure of $T$ for the case $(a)$ and $0_2$-closure of $T$ for the case $(b)$ respectively as in Figure 9. We note that a crossing which is not a closest crossing to $S^2$ cannot be resolved by the first Reidemeister move since $q_6(w)_i$ represents a reduced 3-bridge tangle $T$ for some $i$.

The case $(c)$ and $(d)$ is the only two possible diagrams that allow the first Reidemeister move which is deleting the crossing obtained by $\sigma_{1}^{\pm 1}$.

Then we consider $0_4(T)$ instead of $0_1(T)$ or $0_2(T)$ to have the diagram $(c')$ and $(d')$ as in Figure 9.
We can check that \((d')\) leads a reduced alternating link since there is no more possible crossing which is closest to \(S^2\).

![Diagram](image)

**Figure 10.**

Now, assume that in the diagram \((c')\) there is a closest crossing by \(\sigma_{1}^{\pm 1}\) as in Figure 10.

In order to have the first Reidemeister move to delete the crossing by \(\sigma_{1}^{\pm 1}\), we need to have the diagram \((c' - 1)\) as in Figure 10. However, it contradicts the assumption that \(T\) is a reduced alternating rational 3-tangle.

The diagram \((c' - 2)\) shows us connecting patterns when \(|T| = 2\).

The cases that the first element of \(w\) is \(\sigma_{5}^{\pm 1}\) is analogous to this argument.

Now, we assume that the first element of \(w\) is \(\sigma_{2}^{\pm 1}\) or \(\sigma_{4}^{\pm 1}\) but not both \(\sigma_{1}^{\pm 1}\) and \(\sigma_{5}^{\pm 1}\).

Then by taking \(0_i\)-closure of \(T\), we can have a reduced alternating link \(0_i(T)\).

Now, we assume that the first element of \(w\) is \(\sigma_{3}^{\pm 1}\) but not any of \(\sigma_{1}^{\pm 1}, \sigma_{5}^{\pm 1}, \sigma_{2}^{\pm 1}\) and \(\sigma_{4}^{\pm 1}\).

Then by taking \(0_1\)-closure of \(T\), we can have a reduced alternating link \(0_4(T)\).

Therefore, if \(|T| \geq 2\) there exist \(0_i\)-closure of \(T\) so that the crossing which is closest crossing to \(S^2\) is not able to delete by the first Reidemeister move. This completes the proof.

\[ \square \]

**Lemma 4.3.** If \(|T| = 1\) then \(X_{0_i(T)} = (a^{-2})(a^{-2} - a^{-2})^t \) for \(i \in \{1, 2, 3, 4, 5\}\), where \(t\) is the number of components of \(X_{0_i(T)}\).

**Proof.** Consider the diagram in Figure 11. Since \(T\) has only one crossing, the cases \((a)\) and \((b)\) in Figure 11 are possible. Otherwise, \(0_i(T)\) is disconnected. Also, the only crossing of \(T\) should be deleted by the first Reidemeister move. This completes the proof of Lemma 4.3.

\[ \square \]

**Lemma 4.4.** Suppose that \(T\) and \(T'\) are reduced alternating 3-bridge tangles. If \(|T| = 1\) and \(|T'| \geq 2\), then \(\nu_T \neq (a^{-3})^k \nu_T\) for any \(k\).

**Proof.** Consider the diagram in Figure 11. Since \(T\) has only one crossing, the cases \((a)\) and \((b)\) in Figure 11 are possible. Otherwise, \(0_i(T)\) is disconnected. Also, the only crossing of \(T\) should be deleted by the first Reidemeister move. This completes the proof of Lemma 4.3.

\[ \square \]
Proof. By Lemma 4.2, there exists \(0_j\)-closure of \(T'\) so that \(0_j(T)\) is a reduced alternating link. Therefore, the minimal crossing number of \(0_i(T)\) is greater than or equal to 2.

However, the minimal crossing number of \(0_i(T)\) is zero.

So, we know that \(X_{0_j(T)} \neq (-a^{-3})^k X_{0_j(T)}\) for any \(k\) by the Tait conjecture which is proved by Murasugi [7].

If \(v_T = (-a^{-3})^k v_{T'}\) for some \(k\), then \(X_{0_j(T)} \neq (-a^{-3})^k X_{0_j(T)}\) for some \(k\). It contradicts the previous statement.

Therefore, \(v_T \neq (-a^{-3})^k v_{T'}\) for any \(k\).

\[\Box\]

**Lemma 4.5.** Suppose that \(T\) and \(T'\) are reduced alternating 3-bridge tangles and \(|T| \leq |T'| \leq 1\). Then \(T \approx T'\) if and only if \(v_T = (-a^{-3})^k v_{T'}\) for some \(k\).

Proof. It is enough to check that \(T\) with \(|T| = 1\) has a vector \(v_T\) which is different with the vector \(v_{T'} \in \{(1, 0, 0, 0, 0), (0, 1, 0, 0, 0), \ldots, (0, 0, 0, 0, 1)\}\).

It is easy to see that \(v_T\) has two entries that are non-zero.

\[\Box\]

**Lemma 4.6.** Suppose that \(T\) and \(T'\) are reduced alternating 3-bridge tangles, \(v_T = (-a^{-3})^k v_{T'}\) for some \(k\) and \(|T| \geq |T'| \geq 2\). Let \(T = q_0(w)_s\) and \(T' = q_0(u)_t\) for some \(s, t \in \{1, 2, 3, 4, 5\}\). Let \(u = \sigma_{k_1}' \sigma_{k_2}' \cdots \sigma_{k_m}'\) and \(v = \sigma_{k_1} \sigma_{k_2} \cdots \sigma_{k_m}\). Then \(k_i, k_i'\) are positive (negative) if \(\epsilon_i, \epsilon_i' \in \{1, 3, 5\}\) and \(k_i, k_i'\) are negative (positive) if \(\epsilon_i, \epsilon_i' \in \{2, 4\}\).

Proof. Suppose that \(k_i\) are positive if \(\epsilon_i \in \{1, 3, 5\}\) and \(k_i'\) are positive if \(\epsilon_j \in \{2, 4\}\).

Now, consider \(w' = u^{-1}w\) and \(u' = u^{-1}u = e\). Then \(q_0(w')_s\) and \(q_0(u')_t\) represent two rational 3-tangles, say, \(T_1\) and \(T'_1\).

Clearly, \(|T_1| = 0\) and \(|T| \geq |T| \geq 2\) since \(T_1\) is an alternating tangle and \(T\) is a reduced alternating tangle.

By Lemma 4.2, there exists a \(0_k\)-closure of \(T_1\) so that \(0_k(T_1)\) is a reduced alternating link with \(|0_k(T_1)| \geq 2\).

This implies that \(X_{0_k(T_1)} \neq (-a^{-3})^l X_{0_k(T_1)}\) for any integer \(l\). Therefore, \(v_{T_1} \neq (-a^{-3})^l v_{T'_1}\) for any integer \(l\).
It contradicts the assumption that $v_T = (-a^{-3})^l v_{T'}$ for some $l$ and this completes the proof. □

Now, we want to prove the following main theorem to classify reduced alternating rational 3-tangles.

**Theorem 4.7.** Suppose that $T$ and $T'$ are reduced alternating 3-bridge tangles. If $v_T = (-a^{-3})^k v_{T'}$ for some $k$, then $T \approx T'$.

**Proof.** Suppose that there exists two reduced alternating 3-bridge tangles $T$ and $T'$ so that $v_T = (-a^{-3})^k v_{T'}$ for some $k$, but $T \not\approx T'$.

Let $w$ and $w'$ be the elements of braid group $\mathbb{B}_6$ which represent $T$ and $T'$. i.e., $T = q_6(w)_i$ and $T = q_6(w')_j$ for some $i, j \in \{1, 2, 3, 4, 5, 6\}$.

Then choose $T$ and $T'$ so that $|T| + |T'|$ is minimal.

Then we note that $w$ and $w'$ cannot have the common $\sigma_i^\pm 1$ in the first element of the words for some $i \in \{1, 2, 3, 4, 5\}$. If they have, then consider $\sigma_i^\pm 1 w'$ and $\sigma_i^\pm 1 w$. Let $T_1$ and $T'_1$ be the rational 3-tangles which is presented by $\sigma_i^\pm 1 w$ and $\sigma_i^\pm 1 w'$ respectively. We note that $v_{T_1} = (-a^{-3})^k v_{T'_1}$. So, it contradicts the assumption that $|T| + |T'|$ is minimal.

By Lemma 5.4 and Lemma 5.5, if there exists a such example then $|T| \geq |T'| \geq 2$.

![Diagram](image)

**Figure 12.**

We claim that there exists $i$ ($1 \leq i \leq 5$) so that $0_i(T)$ is a reduced alternating link but $0_i(T')$ is not.

First, assume that $\sigma_i^\pm 1$ is the first element of the word $w$ and no other element of $w$ can be the first element if $w$ keeps reduced alternating property for $T$. i.e., there is no closest crossing to $S^2$ except the crossing by $\sigma_i^\pm 1$ with respect to the plat presentation.

Then, $0_1$, $0_2$- and $0_4$-closure of $T$ make reduced alternating links. (See (1)-(3).)
We note that 0₁-, 0₂- and 0₄-closure of \(T'\) also need to be reduced alternating links. Otherwise, \(X_{0_i(T)} \neq (-a^{-3})^k X_{0_i(T')}\) for any \(k\) by the Tait conjecture.

However, any possible cases for \(T'\) have 0₁-closure of \(T'\) which is not reduced alternating links for some \(i \in 1, 2, 4\). (See (a) – (d) of Figure 12.)

Now, assume that two crossings which are obtained by \(\sigma_1^{\pm 1}\) and \(\sigma_3^{\pm 1}\) are closest crossings to \(S^2\) as the diagrams (4), (5) in Figure 12.

Then 0₁- and 0₂-closure of \(T\) are reduced alternating links.

However, any possible cases for \(T'\) have 0₁- or 0₂-closure of \(T'\) which is not reduced alternating links. (See (e) and (f) of Figure 12.)

Now, assume that three crossings which are obtained by \(\sigma_1^{\pm 1}, \sigma_3^{\pm 1}\) and \(\sigma_5^{\pm 1}\) are closest crossings to \(S^2\) as the diagram (6) in Figure 12.

Then 0₁-closure of \(T\) is reduced alternating link. However, for the possible case for \(T', \, 0_1(T)\) is not reduced alternating link. (See (e) of Figure 12.)

The case that \(T\) has a crossing by the first element \(\sigma_5^{\pm}\) of \(w\) is analogous to the previous case.

![Figure 13.](image)

Now, consider the case that \(T\) has a crossing by the first element \(\sigma_2^{\pm}\) of \(w\).

First, assume that \(T\) has a crossing by the first element \(\sigma_2^{\pm}\) of \(w\) and no other element of \(w\) can be the first element if \(w\) keeps reduced alternating property of \(T\).

Then 0₃-, 0₄- and 0₅-closure of \(T\) make reduced alternating links. (See (1) – (3) of Figure 13.)

We note that 0₃-, 0₄- and 0₅-closure of \(T'\) also need to be reduced alternating links. Otherwise, \(X_{0_i(T)} \neq (-a^{-3})^k X_{0_i(T')}\) for any \(k\) by the Tait conjecture.
However, any possible cases for $T'$ have $0_i$-closure of $T'$ which is not reduced alternating links for some $i \in 3, 4, 5$. (See (a) – (d) of Figure 13.)

Assume that two crossings which are obtained by $\sigma_2^{\pm 1}$ and $\sigma_4^{\pm 1}$ are closest crossings to $S^2$ as the diagrams (4), (5) in Figure 13.

Then $0_3$- and $0_4$-closure of $T$ are reduced alternating links.

However, any possible cases for $T'$ have $0_3$- or $0_4$-closure of $T'$ which is not reduced alternating links. (See (e) and (g) of Figure 13.)

Now, assume that two crossings which are obtained by $\sigma_2^{\pm 1}$ and $\sigma_5^{\pm 1}$ are closest crossings to $S^2$ as the diagrams (6), (7) in Figure 13.

Then $0_4$- and $0_5$-closure of $T$ are reduced alternating links.

However, any possible cases for $T'$ have $0_4$- or $0_5$-closure of $T'$ which is not reduced alternating links. (See (e) – (g) of Figure 13.)

The case that $T$ has a crossing by the first element $\sigma_4^{\pm}$ of $w$ is analogous to the previous case.

Finally, we consider the case that $T$ has a crossing by the first element $\sigma_3^{\pm}$ of $w$.

First, assume that $T$ has a crossing by the first element $\sigma_3^{\pm}$ of $w$ and no other element of $w$ can be the first element if $w$ keeps reduced alternating property of $T$.

Then $0_1$-, $0_2$- and $0_5$-closure of $T$ make reduced alternating links. (See (1) – (3) of Figure 14.)

We note that $0_1$-, $0_2$- and $0_5$-closure of $T'$ also need to be reduced alternating links. Otherwise, $X_{0,(\overline{\sigma})} \neq (-a^{-3})^kX_{0,(\overline{\sigma})}$ for any $k$ by the Tait conjecture.
However, any possible cases for $T'$ have $0_i$-closure of $T'$ which is not reduced alternating links for some $i \in 1, 2, 5$. (See (a) – (d) of Figure 14.)

Assume that two crossings which are obtained by $\sigma^+_3$ and $\sigma^+_5$ are closest crossings to $S^2$ as the diagrams (4), (5) in Figure 14.

Then $0_1$- and $0_5$-closure of $T$ are reduced alternating links.

However, any possible cases for $T'$ have $0_1$- or $0_5$-closure of $T'$ which is not reduced alternating links. (See (e) and (g) of Figure 14.)

Therefore, there exists $i$ so that $0_i(T)$ is a reduced alternating link but $0_i(T')$ is not.

This implies that $X_{0_i(T)} \neq (-a^{-3})^k X_{0_i(T')}$ for any $k$ since the number of minimal crossings of $0_i(T)$ is strictly greater than the number of minimal crossings of $0_i(T')$. This implies that $v_T \neq (-a^{-3})^k v_{T'}$ for any $k$.

This contradicts the assume that $v_T = (-a^{-3})^k v_{T'}$ for some $k$.

This complete the proof of this theorem.

□

**Conjecture 4.8.** $T \approx T'$ if and only if $v_T = (-a^{-3})^k v_{T'}$ for some integer $k$.

![Figure 15](image)

**Figure 15.**

**Examples**: Figure 15 gives us examples of two rational 3-tangles that is distinguished by the invariant. We note that $0_3(T)$ is the Borromean rings. So, if you take any two of the three strings in $T$ then we get a trivial rational 2-tangle.

First of all, we see that $v_{0_3} = (0, 0, 1, 0, 0)$.

I found a method to calculate the Kauffman bracket of 6-plat presentations of links by using a presentation of braid group $\mathbb{B}_6$ into a group of $5 \times 5$ matrices.

By the method, we have $v_T = (-a^{-6} + 3a^{-2} - 3a^2 + a^6, -2a^4 + a^8 + 1, -2a^6 + a^{10}, -a^4, -2a^4 + a^8 + 1)$.

Therefore, $v_{0_3} \neq (-a^{-3})^k v_T$ for any $k$. 
This implies that $0_3 \ncong T$.

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