On the spectral theory of groups of automorphisms of S-adic nilmanifolds

BACHIR BEKKA and YVES GUIVARC’H

IRMAR, UMR-CNRS 6625 Université de Rennes 1, Campus Beaulieu,
F-35042 Rennes Cedex, France
(e-mail: bachir.bekka@univ-rennes1.fr, yves.guivarch@univ-rennes1.fr)

(Received 15 November 2021 and accepted in revised form 10 March 2023)

Abstract. Let $S = \{p_1, \ldots, p_r, \infty\}$ for prime integers $p_1, \ldots, p_r$. Let $X$ be an $S$-adic compact nilmanifold, equipped with the unique translation-invariant probability measure $\mu$. We characterize the countable groups $\Gamma$ of automorphisms of $X$ for which the Koopman representation $\kappa$ on $L^2(X, \mu)$ has a spectral gap. More specifically, let $Y$ be the maximal quotient solenoid of $X$ (thus, $Y$ is a finite-dimensional, connected, compact abelian group). We show that $\kappa$ does not have a spectral gap if and only if there exists a $\Gamma$-invariant proper subsolenoid of $Y$ on which $\Gamma$ acts as a virtually abelian group.

Key words: nilmanifolds, Koopman representation, spectral theory
2020 Mathematics Subject Classification: 37A05 (Primary); 22F30, 60B15, 60G50 (Secondary)

1. Introduction

Let $\Gamma$ be a countable group acting measurably on a probability space $(X, \mu)$ by measure-preserving transformations. Let $\kappa = \kappa_X$ denote the corresponding Koopman representation of $\Gamma$, that is, the unitary representation of $\Gamma$ on $L^2(X, \mu)$ given by

$$\kappa(\gamma)\xi(x) = \xi(\gamma^{-1}x) \quad \text{for all} \quad \xi \in L^2(X, \mu), \quad x \in X, \quad \gamma \in \Gamma.$$ 

We say that the action $\Gamma \curvearrowright (X, \mu)$ of $\Gamma$ on $(X, \mu)$ has a spectral gap if the restriction $\kappa_0$ of $\kappa$ to the $\Gamma$-invariant subspace

$$L^2_0(X, \mu) = \left\{ \xi \in L^2(X, \mu) : \int_X \xi(x) \, d\mu(x) = 0 \right\}$$

does not weakly contain the trivial representation $1_\Gamma$; equivalently, if $\kappa_0$ does not have almost invariant vectors, that is, there is no sequence $(\xi_n)_n$ of unit vectors in $L^2_0(X, \mu)$ such that

$$\lim_n \| \kappa_0(\gamma)\xi_n - \xi_n \| = 0 \quad \text{for all} \quad \gamma \in \Gamma.$$
The existence of a spectral gap admits the following useful quantitative version. Let \( \nu \) be a probability measure on \( \Gamma \) and \( \kappa_0(\nu) \) the convolution operator defined on \( L^2_0(X, \mu) \) by

\[
\kappa_0(\nu)\xi = \sum_{\gamma \in \Gamma} \nu(\gamma) \kappa_0(\gamma)\xi \quad \text{for all} \ \xi \in L^2_0(X, \mu).
\]

Observe that we have \( \|\kappa_0(\nu)\| \leq 1 \) and hence \( r(\kappa_0(\nu)) \leq 1 \) for the spectral radius \( r(\kappa_0(\nu)) \) of \( \kappa_0(\mu) \). Assume that \( \nu \) is aperiodic, that is, the support of \( \nu \) is not contained in the coset of a proper subgroup of \( \Gamma \). Then the action of \( \Gamma \) on \( X \) has a spectral gap if and only if \( r(\kappa_0(\nu)) < 1 \) and this is equivalent to \( \|\kappa_0(\nu)\| < 1 \); for more details, see the survey [Bekk16].

In this paper we will be concerned with the case where \( X \) is an \( S \)-adic nilmanifold, to be introduced below, and \( \Gamma \) is a subgroup of automorphisms of \( X \).

Fix a finite set \( \{p_1, \ldots, p_r\} \) of integer primes and set \( S = \{p_1, \ldots, p_r, \infty\} \). The product

\[
Q_s := \prod_{p \in S} Q_p = Q_\infty \times Q_{p_1} \times \cdots \times Q_{p_r}
\]

is a locally compact ring, where \( Q_\infty = \mathbb{R} \) and \( Q_p \) is the field of \( p \)-adic numbers for a prime \( p \). Let \( \mathbb{Z}[1/S] = \mathbb{Z}[1/p_1, \cdots, 1/p_r] \) denote the subring of \( \mathbb{Q} \) generated by \( 1 \) and \( \{1/p_1, \ldots, 1/p_r\} \). Through the diagonal embedding

\[
\mathbb{Z}[1/S] \rightarrow Q_s, \quad b \mapsto (b, \cdots, b),
\]

we may identify \( \mathbb{Z}[1/S] \) with a discrete and cocompact subring of \( Q_s \).

If \( G \) is a linear algebraic group defined over \( \mathbb{Q} \), we denote by \( G(R) \) the group of elements of \( G \) with coefficients in \( R \) and determinant invertible in \( R \), for every subring \( R \) of an overfield of \( \mathbb{Q} \).

Let \( U \) be a linear algebraic unipotent group defined over \( \mathbb{Q} \), that is, \( U \) is an algebraic subgroup of the group of \( n \times n \) upper triangular unipotent matrices for some \( n \geq 1 \). The group \( U(\mathbb{Q}_S) \) is a locally compact group and \( \Lambda := U(\mathbb{Z}[1/S]) \) is a cocompact lattice in \( U(\mathbb{Q}_S) \). The corresponding \( S \)-adic compact nilmanifold

\[
\text{Nil}_S = U(\mathbb{Q}_S)/U(\mathbb{Z}[1/S])
\]

will be equipped with the unique translation-invariant probability measure \( \mu \) on its Borel subsets.

For \( p \in S \), let \( \text{Aut}(U(\mathbb{Q}_p)) \) be the group of continuous automorphisms of \( U(\mathbb{Q}_p) \). Set

\[
\text{Aut}(U(\mathbb{Q}_S)) := \prod_{p \in S} \text{Aut}(U(\mathbb{Q}_p))
\]

and denote by \( \text{Aut}(\text{Nil}_S) \) the subgroup

\[
\{g \in \text{Aut}(U(\mathbb{Q}_S)) \mid g(\Lambda) = \Lambda\}.
\]

Every \( g \in \text{Aut}(\text{Nil}_S) \) acts on \( \text{Nil}_S \) preserving the probability measure \( \mu \).
The abelian quotient group

\[ \mathbf{U}(\mathbb{Q}_S) := \mathbf{U}(\mathbb{Q}_S)/[\mathbf{U}(\mathbb{Q}_S), \mathbf{U}(\mathbb{Q}_S)] \]

can be identified with \( \mathbb{Q}^d_S \) for some \( d \geq 1 \) and the image \( \Delta \) of \( \mathbf{U}(\mathbb{Z}[1/S]) \) in \( \mathbf{U}(\mathbb{Q}_S) \) is a cocompact and discrete subgroup of \( \mathbf{U}(\mathbb{Q}_S) \); so,

\[ \text{Sol}_S := \mathbf{U}(\mathbb{Q}_S)/\Delta \]

is a solenoid (that is, is a finite-dimensional, connected, compact abelian group; see [HeRo63, §25]). We refer to \( \text{Sol}_S \) as the \( S \)-adic solenoid attached to the \( S \)-adic nilmanifold \( \text{Nil}_S \). We equip \( \text{Sol}_S \) with the probability measure \( \nu \) which is the image of \( \mu \) under the canonical projection \( \text{Nil}_S \to \text{Sol}_S \).

Observe that \( \text{Aut}(\mathbb{Q}^d_S) \) is canonically isomorphic to \( \prod_{s \in S} \text{GL}_d(\mathbb{Q}_s) \) and that \( \text{Aut}(\text{Sol}_S) \) can be identified with the subgroup \( \text{GL}_d(\mathbb{Z}[1/S]) \). The group \( \text{Aut}(\text{Nil}_S) \) acts naturally by automorphisms of \( \text{Sol}_S \); we denote by

\[ p_S : \text{Aut}(\text{Nil}_S) \to \text{GL}_d(\mathbb{Z}[1/S]) \subset \text{GL}_d(\mathbb{Q}) \]

the corresponding representation.

**Theorem 1.** Let \( \mathbf{U} \) be an algebraic unipotent group defined over \( \mathbb{Q} \) and \( S = \{p_1, \ldots, p_r, \infty\} \), where \( p_1, \ldots, p_r \) are integer primes. Let \( \text{Nil}_S = \mathbf{U}(\mathbb{Q}_S)/\mathbf{U}(\mathbb{Z}[1/S]) \) be the associated \( S \)-adic nilmanifold and let \( \text{Sol}_S \) be the corresponding \( S \)-adic solenoid, respectively equipped with the probability measures \( \mu \) and \( \nu \) as above. Let \( \Gamma \) be a countable subgroup of \( \text{Aut}(\text{Nil}_S) \). The following properties are equivalent.

(i) The action \( \Gamma \curvearrowright (\text{Nil}_S, \mu) \) has a spectral gap.

(ii) The action \( p_S(\Gamma) \curvearrowright (\text{Sol}_S, \nu) \) has a spectral gap, where \( p_S : \text{Aut}(\text{Nil}_S) \to \text{GL}_d(\mathbb{Z}[1/S]) \) is the canonical homomorphism.

Actions with spectral gap of groups of automorphisms (or more generally groups of affine transformations) of the \( S \)-adic solenoid \( \text{Sol}_S \) have been completely characterized in [BeFr20, Theorem 5]. The following result is an immediate consequence of this characterization and of Theorem 1. For a subset \( T \) of \( \text{GL}_d(\mathbb{K}) \) for a field \( \mathbb{K} \), we denote by \( T^t = \{g^t \mid g \in T\} \) the set of transposed matrices from \( T \).

**Corollary 2.** With the notation as in Theorem 1, the following properties are equivalent.

(i) The action of \( \Gamma \) on the \( S \)-adic nilmanifold \( \text{Nil}_S \) does not have a spectral gap.

(ii) There exists a non-zero linear subspace \( W \) of \( \mathbb{Q}^d \) which is invariant under \( p_S(\Gamma)^t \) and such that the image of \( p_S(\Gamma)^t \) in \( \text{GL}(W) \) is a virtually abelian group.

Here is an immediate consequence of Corollary 2.

**Corollary 3.** With the notation as in Theorem 1, assume that the linear representation of \( p_S(\Gamma)^t \) in \( \mathbb{Q}^d \) is irreducible and that \( p_S(\Gamma)^t \) is not virtually abelian. Then the action \( \Gamma \curvearrowright (\text{Nil}_S, \mu) \) has a spectral gap.
Recall that the action of a countable group \( \Gamma \) by measure-preserving transformations on a probability space \((X, \mu)\) is strongly ergodic (see [Schm81]) if every sequence \((B_n)_n\) of measurable subsets of \(X\) which is asymptotically invariant (that is, which is such that \(\lim_n \mu(\gamma B_n \triangle B_n) = 0\) for all \(\gamma \in \Gamma\)) is trivial (that is, \(\lim_n \mu(B_n)(1 - \mu(B_n)) = 0\)). It is straightforward to check that the spectral gap property implies strong ergodicity and it is known that the converse does not hold in general.

The following corollary is a direct consequence of Theorem 1 (compare with [BeGu15, Corollary 2]).

**Corollary 4.** With the notation as in Theorem 1, the following properties are equivalent.

1. The action \( \Gamma \acts (\text{Nil}_S, \mu) \) has the spectral gap property.
2. The action \( \Gamma \acts (\text{Nil}_S, \mu) \) is strongly ergodic.

Theorem 1 generalizes our previous work [BeGu15], where we treated the real case (that is, the case \(S = \infty\)). This requires an extension of our methods to the \(S\)-adic setting, which is a non-straightforward task; more specifically, we had to establish the following four main tools for our proof:

- A canonical decomposition of the Koopman representation of \( \Gamma \) in \( L^2(\text{Nil}_S) \) as a direct sum of certain representations of \( \Gamma \) induced from stabilizers of representations of \( U(Q_S) \)—this fact is valid more generally for compact homogeneous spaces (see Proposition 9);
- A result of Howe and Moore [HoMo79] about the decay of matrix coefficients of algebraic groups (see Proposition 11);
- The fact that the irreducible representations of \( U(Q_S) \) appearing in the decomposition of \( L^2(\text{Nil}_S) \) are rational, in the sense that the Kirillov data associated to each one of them are defined over \( \mathbb{Q} \) (see Proposition 13);
- A characterization (see Lemma 12) of the projective kernel of the extension of an irreducible representation of \( U(Q_p) \) to its stabilizer in \( \text{Aut}(U(Q_p)) \).

Another tool we constantly use is a generalized version of Herz’s majoration principle (see Lemma 7).

Given a probability measure \( \nu \) on \( \Gamma \), our approach does not seem to provide quantitative estimates for the operator norm of the convolution operator \( \kappa_0(\nu) \) acting on \( L^2_0(\text{Nil}_S, \mu) \) for a general unipotent group \( U \). However, using known bounds for the so-called metaplectic representation of the symplectic group \( Sp_{2n}(Q_p) \), we give such estimates in the case of \( S\)-adic Heisenberg nilmanifolds (see §11).

**Corollary 5.** For an integer \( n \geq 1 \), let \( U = H_{2n+1} \) be the \((2n+1)\)-dimensional Heisenberg group and \( \text{Nil}_S = H_{2n+1}(Q_S)/H_{2n+1}(\mathbb{Z}[1/S]) \). Let \( \nu \) be a probability measure on the subgroup \( Sp_{2n}(\mathbb{Z}[1/S]) \) of \( \text{Aut}(\text{Nil}_S) \). Then

\[
\|\kappa_0(\nu)\| \leq \max\{\|\lambda_\Gamma(\nu)\|^{1/2n+2}, \|\kappa_1(\nu)\|\},
\]

where \( \kappa_1 \) is the restriction of \( \kappa_0 \) to \( L^2_0(\text{Sol}_S) \) and \( \lambda_\Gamma \) is the regular representation of the group \( \Gamma \) generated by the support of \( \nu \). In particular, in the case where \( n = 1 \) and \( \nu \) is aperiodic, the action of \( \Gamma \) on \( \text{Nil}_S \) has a spectral gap if and only if \( \Gamma \) is non-amenable.
2. Extension of representations

Let $G$ be a locally compact group which we assume to be second countable. We will need the notion of a projective representation. Recall that a mapping $\pi : G \to U(H)$ from $G$ to the unitary group of the Hilbert space $H$ is a projective representation of $G$ if the following assertions hold.

- $\pi(e) = I$.
- For all $g_1, g_2 \in G$, there exists $c(g_1, g_2) \in \mathbb{C}$ such that
  \[ \pi(g_1 g_2) = c(g_1, g_2)\pi(g_1)\pi(g_2). \]
- The function $g \mapsto \langle \pi(g)\xi, \eta \rangle$ is measurable for all $\xi, \eta \in H$.

The mapping $c : G \times G \to S^1$ is a 2-cocycle with values in the unit circle $S^1$. Every projective unitary representation of $G$ can be lifted to an ordinary unitary representation of a central extension of $G$ (for all this, see [Mack76] or [Mack58]).

Let $N$ be a closed normal subgroup of $G$. Let $\pi$ be an irreducible unitary representation of $N$ on a Hilbert space $H$. Consider the stabilizer $G_\pi = \{ g \in G \mid \pi^g \text{ is equivalent to } \pi \}$ of $\pi$ in $G$ for the natural action of $G$ on the unitary dual $\widehat{N}$ given by $\pi^g(n) = \pi(n^g)$. Then $G_\pi$ is a closed subgroup of $G$ containing $N$. The following lemma is a well-known part of Mackey’s theory of unitary representations of group extensions.

**Lemma 6.** Let $\pi$ be an irreducible unitary representation of $N$ on the Hilbert space $H$. There exists a projective unitary representation $\tilde{\pi}$ of $G_\pi$ on $H$ which extends $\pi$. Moreover, $\tilde{\pi}$ is unique, up to scalars: any other projective unitary representation $\tilde{\pi}'$ of $G_\pi$ extending $\pi$ is of the form $\tilde{\pi}' = \lambda \tilde{\pi}$ for a measurable function $\lambda : G_\pi \to S^1$.

**Proof.** For every $g \in G_\pi$, there exists a unitary operator $\tilde{\pi}(g)$ on $H$ such that

\[ \pi(g(n)) = \tilde{\pi}(g)\pi(n)\tilde{\pi}(g)^{-1} \quad \text{for all } n \in N. \]

One can choose $\tilde{\pi}(g)$ such that $g \mapsto \tilde{\pi}(g)$ is a projective unitary representation of $G_\pi$ which extends $\pi$ (see [Mack58, Theorem 8.2]). The uniqueness of $\pi$ follows from the irreducibility of $\pi$ and Schur’s lemma.

\[ \square \]

3. A weak containment result for induced representations

Let $G$ be a locally compact group with Haar measure $\mu_G$. Recall that a unitary representation $(\rho, K)$ of $G$ is weakly contained in another unitary representation $(\pi, H)$ of $G$, if every matrix coefficient

\[ g \mapsto \langle \rho(g)\eta \mid \eta \rangle \quad \text{for } \eta \in K \]

of $\rho$ is the limit, uniformly over compact subsets of $G$, of a finite sum of matrix coefficients of $\pi$; equivalently, if $\|\rho(f)\| \leq \|\pi(f)\|$ for every $f \in C_c(G)$, where $C_c(G)$ is the space of continuous functions with compact support on $G$ and where the operator $\pi(f) \in B(H)$ is defined by the integral
\[ \pi(f)\xi = \int_G f(g)\pi(g)\xi \, d\mu_G(g) \quad \text{for all } \xi \in \mathcal{H}. \]

The trivial representation \(1_G\) is weakly contained in \(\pi\) if and only if there exists, for every compact subset \(Q\) of \(G\) and every \(\varepsilon > 0\), a unit vector \(\xi \in \mathcal{H}\) which is \((Q, \varepsilon)\)-invariant, that is, such that

\[ \sup_{g \in Q} \|\pi(g)\xi - \xi\| \leq \varepsilon. \]

Let \(H\) be a closed subgroup of \(G\). We will always assume that the coset space \(H \backslash G\) admits a non-zero \(G\)-invariant (possibly infinite) measure on its Borel subsets. Let \((\sigma, \mathcal{K})\) be a unitary representation of \(H\). We will use the following model for the induced representation \(\pi := \text{Ind}^G_H \sigma\). Choose a Borel fundamental domain \(X \subset G\) for the action of \(G\) on \(H \backslash G\). For \(x \in X\) and \(g \in G\), let \(x \cdot g \in X\) and \(c(x, g) \in H\) be defined by

\[ xg = c(x, g)(x \cdot g). \]

There exists a non-zero \(G\)-invariant measure on \(X\) for the action \((x, g) \mapsto x \cdot g\) of \(G\) on \(X\). The Hilbert space of \(\pi\) is the space \(L^2(X, \mathcal{K}, \mu)\) of all square-integrable measurable mappings \(\xi : X \to \mathcal{K}\) and the action of \(G\) on \(L^2(X, \mathcal{K}, \mu)\) is given by

\[ (\pi(g)\xi)(x) = \sigma(c(x, g))(\xi(x \cdot g)), \quad g \in G, \xi \in L^2(X, \mathcal{K}, \mu), \quad x \in X. \]

Observe that, in the case where \(\sigma\) is the trivial representation \(1_H\), the induced representation \(\text{Ind}^G_H 1_H\) is equivalent to quasi-regular representation \(\lambda_{H \backslash G}\), that is, the natural representation of \(G\) on \(L^2(H \backslash G, \mu)\) given by right translations.

We will use several times the following elementary but crucial lemma, which can be viewed as a generalization of Herz’s majoration principle (see [BeGu15, Proposition 17]).

**Lemma 7.** Let \((H_i)_{i \in I}\) be a family of closed subgroups of \(G\) such that \(H_i \backslash G\) admits a non-zero \(G\)-invariant measure. Let \((\sigma_i, \mathcal{K}_i)\) be a unitary representation of \(H_i\). Assume that \(1_G\) is weakly contained in the direct sum \(\bigoplus_{i \in I} \text{Ind}^G_{H_i} \sigma_i\). Then \(1_G\) is weakly contained in \(\bigoplus_{i \in I} \lambda_{H_i \backslash G}\).

**Proof.** Let \(Q\) be a compact subset of \(G\) and \(\varepsilon > 0\). For every \(i \in I\), let \(X_i \subset G\) be a Borel fundamental domain for the action of \(G\) on \(H_i \backslash G\) and \(\mu_i\) a non-zero \(G\)-invariant measure on \(X_i\). There exists a family of vectors \(\xi_i \in L^2(X_i, \mathcal{K}_i, \mu_i)\) such that \(\sum_i \|\xi_i\|^2 = 1\) and

\[ \sup_{g \in Q} \sum_i \|\text{Ind}^G_{H_i} \sigma_i(g)\xi_i - \xi_i\|^2 \leq \varepsilon. \]

Define \(\varphi_i \in L^2(X_i, \mu_i)\) by \(\varphi_i(x) = \|\xi_i(x)\|\). Then \(\sum_i \|\varphi_i\|^2 = 1\) and, denoting by \((x, g) \mapsto x \cdot g\) the action of \(G\) on \(X_i\) and by \(c_i : X_i \times G \to H_i\) the associated map as above, we have

\begin{align*}
\|\text{Ind}^G_{H_i} \sigma_i(g)\xi_i - \xi_i\|^2 &= \int_{X_i} \|\sigma_i(c_i(x, g))(\xi_i(x \cdot g)) - \xi_i(x)\|^2 \, d\mu_i(x) \\
&\geq \int_{X_i} \|\sigma_i(c_i(x, g))(\xi_i(x \cdot g))\|^2 - \|\xi_i(x)\|^2 \, d\mu_i(x)
\end{align*}
\[= \int_{X_i} \|\xi_i(x \cdot g)\| - \|\xi_i(x)\|^2 \, d\mu_i(x)\]
\[= \int_{X_i} |\varphi_i(x \cdot g) - \varphi(x)|^2 \, d\mu_i(x)\]
\[= \|\lambda_{H_i \setminus G}(g) \varphi_i - \varphi_i\|^2,\]
for every \(g \in G\), and the claim follows. \(\square\)

4. Decay of matrix coefficients of unitary representations
We recall a few general facts about the decay of matrix coefficients of unitary representations. Recall that the projective kernel of a (genuine or projective) representation \(\pi\) of the locally compact group \(G\) is the closed normal subgroup \(P_\pi\) of \(G\) consisting of the elements \(g \in G\) such that \(\pi(g)\) is a scalar multiple of the identity operator, that is, such that \(\pi(g) = \lambda_\pi(g)I\) for some \(\lambda_\pi(g) \in S^1\).

Observe also that, for \(\xi, \eta \in H\), the absolute value of the matrix coefficient
\[C^\pi_{\xi, \eta} : g \mapsto \langle \pi(g)\xi, \eta \rangle\]
is constant on cosets modulo \(P_\pi\). For a real number \(p\) with \(1 \leq p < +\infty\), the representation \(\pi\) is said to be strongly \(L^p\) modulo \(P_\pi\), if there is a dense subspace \(D \subset H\) such that \(|C^\pi_{\xi, \eta}| \in L^p(G/P_\pi)\) for all \(\xi, \eta \in D\).

**Proposition 8.** Assume that the unitary representation \(\pi\) of the locally compact group \(G\) is strongly \(L^p\) modulo \(P_\pi\) for \(1 \leq p < +\infty\). Let \(k\) be an integer \(k \geq p/2\). Then the tensor power \(\pi^{\otimes k}\) is contained in an infinite multiple of \(\text{Ind}_{P_\pi}^G \lambda_k\pi\), where \(\lambda_\pi\) is the unitary character of \(P_\pi\) associated to \(\pi\).

**Proof.** Observe that \(\sigma := \pi^{\otimes k}\) is square-integrable modulo \(P_\pi\) for every integer \(k \geq p/2\). It follows (see [HoMo79, Proposition 4.2] or [HoTa92, Ch. V, Proposition 1.2.3]) that \(\sigma\) is contained in an infinite multiple of \(\text{Ind}_{P_\pi}^G \lambda_k\pi\). \(\square\)

5. The Koopman representation of the automorphism group of a homogeneous space
We establish a decomposition result for the Koopman representation of a group of automorphisms of an S-adic compact nilmanifold. We will state the result in the general context of a compact homogeneous space.

Let \(G\) be a locally compact group and \(\Lambda\) a lattice in \(G\). We assume that \(\Lambda\) is cocompact in \(G\). The homogeneous space \(X := G/\Lambda\) carries a probability measure \(\mu\) on the Borel subsets of \(X\) which is invariant by translations with elements from \(G\). Every element from
\[\text{Aut}(X) := \{\gamma \in \text{Aut}(G) \mid \gamma(\Lambda) \subset \Lambda\}\]
induces a Borel isomorphism of \(X\), which leaves \(\mu\) invariant, as follows from the uniqueness of \(\mu\).

Given a subgroup \(\Gamma\) of \(\text{Aut}(X)\), the following crucial proposition gives a decomposition of the associated Koopman \(\Gamma\) on \(L^2(X, \mu)\) as direct sum of certain induced representations of \(\Gamma\).
PROPOSITION 9. Let $G$ be a locally compact group and $\Lambda$ a cocompact lattice in $G$, and let $\Gamma$ be a countable subgroup of $\text{Aut}(X)$ for $X := G/\Lambda$. Let $\kappa$ denote the Koopman representation of $\Gamma$ associated to the action $\Gamma \curvearrowright X$. There exists a family $(\pi_i)_{i \in I}$ of irreducible unitary representations of $G$ such that $\kappa$ is equivalent to a direct sum

$$\bigoplus_{i \in I} \text{Ind}^\Gamma_{\Gamma_i}(\bar{\pi}_i|_{\Gamma_i} \otimes W_i),$$

where $\bar{\pi}_i$ is an irreducible projective representation of the stabilizer $G_i$ of $\pi_i$ in $\text{Aut}(G) \ltimes G$ extending $\pi_i$, and where $W_i$ is a finite-dimensional projective unitary representation of $\Gamma_i := \Gamma \cap G_i$.

Proof. We extend $\kappa$ to a unitary representation, again denoted by $\kappa$, of $\Gamma \ltimes G$ on $L^2(X, \mu)$ given by

$$\kappa(\gamma, g)\xi(x) = \xi(\gamma^{-1}(gx)) \quad \text{for all } \gamma \in \Gamma, g \in G, \xi \in L^2(X, \mu), x \in X.$$

Identifying $\Gamma$ and $G$ with subgroups of $\Gamma \ltimes G$, we have

$$\kappa(\gamma^{-1})\kappa(g)\kappa(\gamma) = \kappa(\gamma^{-1}(g)) \quad \text{for all } \gamma \in \Gamma, g \in G. \quad (*)$$

Since $\Lambda$ is cocompact in $G$, we can consider the decomposition of $L^2(X, \mu)$ into $G$-isotypical components: we have (see [GGPS69, Ch. I, §3, Theorem])

$$L^2(X, \mu) = \bigoplus_{\pi \in \Sigma} \mathcal{H}_\pi,$$

where $\Sigma$ is a certain set of pairwise non-equivalent irreducible unitary representations of $G$; for every $\pi \in \Sigma$, the space $\mathcal{H}_\pi$ is the union of the closed $\kappa(G)$-invariant subspaces $K$ of $\mathcal{H} := L^2(X, \mu)$ for which the corresponding representation of $G$ in $K$ is equivalent to $\pi$; moreover, the multiplicity of every $\pi$ is finite, that is, every $\mathcal{H}_\pi$ is a direct sum of finitely many irreducible unitary representations of $G$.

Let $\gamma$ be a fixed automorphism in $\Gamma$. Let $\kappa^\gamma$ be the conjugate representation of $\kappa$ by $\gamma$, that is, $\kappa^\gamma(g) = \kappa(\gamma^{-1}(g)) \gamma(\gamma^{-1}(g))^{-1}$ for all $g \in \Gamma \ltimes G$. On the one hand, for every $\pi \in \Sigma$, the isotypical component of $\kappa^\gamma|_G$ corresponding to $\pi$ is $\mathcal{H}_{\pi^\gamma^{-1}}$. On the other hand, relation $(*)$ shows that $\kappa(\gamma)$ is a unitary equivalence between $\kappa|_G$ and $\kappa^\gamma|_G$. It follows that

$$\kappa(\gamma)(\mathcal{H}_\pi) = \mathcal{H}_{\pi^\gamma} \quad \text{for all } \gamma \in \Gamma;$$

so, $\Gamma$ permutes the $\mathcal{H}_\pi$s among themselves according to its action on $\hat{G}$.

Write $\Sigma = \bigcup_{i \in I} \Sigma_i$, where the $\Sigma_i$ are the $\Gamma$-orbits in $\Sigma$, and set

$$\mathcal{H}_{\Sigma_i} = \bigoplus_{\pi \in \Sigma_i} \mathcal{H}_\pi.$$

Every $\mathcal{H}_{\Sigma_i}$ is invariant under $\Gamma \ltimes G$ and we have an orthogonal decomposition

$$\mathcal{H} = \bigoplus_i \mathcal{H}_{\Sigma_i}.$$

Fix $i \in I$. Choose a representation $\pi_i$ in $\Sigma_i$ and set $\mathcal{H}_i = \mathcal{H}_{\pi_i}$. Let $\Gamma_i$ denote the stabilizer of $\pi_i$ in $\Gamma$. The space $\mathcal{H}_i$ is invariant under $\Gamma_i$. Let $V_i$ be the corresponding representation of $\Gamma_i \ltimes G$ on $\mathcal{H}_i$. 
Choose a set $S_i$ of representatives for the cosets in
\[ \Gamma_i / \Gamma_i = (\Gamma \ltimes G) / (\Gamma_i \ltimes G) \]
with $e \in S_i$. Then $\Sigma_i = \{ \pi^+_i : s \in S_i \}$ and the Hilbert space $\mathcal{H}_{\Sigma_i}$ is the sum of mutually orthogonal spaces:
\[ \mathcal{H}_{\Sigma_i} = \bigoplus_{s \in S_i} \mathcal{H}^s_i. \]
Moreover, $\mathcal{H}^s_i$ is the image under $\kappa(s)$ of $\mathcal{H}_i$ for every $s \in S_i$. This means that the restriction $\kappa_i$ of $\kappa$ to $\mathcal{H}_{\Sigma_i}$ of the Koopman representation $\kappa$ of $\Gamma$ is equivalent to the induced representation $\text{Ind}_{\Gamma_i}^\Gamma V_i$.

Since every $\mathcal{H}_i$ is a direct sum of finitely many irreducible unitary representations of $G$, we can assume that $\mathcal{H}_i$ is the tensor product
\[ \mathcal{H}_i = \mathcal{K}_i \otimes \mathcal{L}_i \]
of the Hilbert space $\mathcal{K}_i$ of $\pi_i$ with a finite-dimensional Hilbert space $\mathcal{L}_i$, in such a way that
\[ V_i(g) = \pi_i(g) \otimes I_{\mathcal{L}_i} \quad \text{for all } g \in G. \] (**) Let $\gamma \in \Gamma_i$. By (**) above, we have
\[ V_i(\gamma)(\pi_i(g) \otimes I_{\mathcal{L}_i})V_i(\gamma)^{-1} = \pi_i(\gamma g \gamma^{-1}) \otimes I_{\mathcal{L}_i} \] (***) for all $g \in G$. On the other hand, let $G_i$ be the stabilizer of $\pi_i$ in $\text{Aut}(\Gamma) \ltimes G$; then $\pi_i$ extends to an irreducible projective representation $\tilde{\pi}_i$ of $G_i$ (see §2). Since
\[ \tilde{\pi}_i(\gamma)\pi_i(g) \tilde{\pi}_i(\gamma^{-1}) = \pi_i(\gamma g \gamma^{-1}) \quad \text{for all } g \in G, \]
it follows from (*** ) that $(\tilde{\pi}_i(\gamma^{-1}) \otimes I_{\mathcal{L}_i})V_i(\gamma)$ commutes with $\pi_i(g) \otimes I_{\mathcal{L}_i}$ for all $g \in G$. As $\pi_i$ is irreducible, there exists a unitary operator $W_i(\gamma)$ on $\mathcal{L}_i$ such that
\[ V_i(\gamma) = \tilde{\pi}_i(\gamma) \otimes W_i(\gamma). \]
It is clear that $W_i$ is a projective unitary representation of $\Gamma_i \ltimes G$, since $V_i$ is a unitary representation of $\Gamma_i \ltimes G$. \hfill \Box

6. **Unitary dual of solenoids**

Let $p$ be either a prime integer or $p = \infty$. Define an element $e_p$ in the unitary dual group $\widehat{\mathbb{Q}_p}$ of the additive group of $\mathbb{Q}_p$ (recall that $\mathbb{Q}_\infty = \mathbb{R}$) by $e_p(x) = e^{2\pi i x}$ if $p = \infty$ and $e_p(x) = \exp(2\pi i x)$ otherwise, where $[x] = \sum_{j=m}^{\infty} a_j p^j$ denotes the ‘fractional part’ of a $p$-adic number $x = \sum_{j=m}^{\infty} a_j p^j$ for integers $m \in \mathbb{Z}$ and $a_j \in \{0, \ldots, p-1\}$. Observe that $\text{Ker}(e_p) = \mathbb{Z}$ if $p = \infty$ and that $\text{Ker}(e_p) = \mathbb{Z}_p$ if $p$ is a prime integer, where $\mathbb{Z}_p$ is the ring of $p$-adic integers. The map
\[ \mathbb{Q}_p \to \widehat{\mathbb{Q}_p}, \quad y \mapsto (x \mapsto e_p(xy)) \]
is an isomorphism of topological groups (see [BeHV08, §D.4]).
7. Unitary representations of unipotent groups

Fix an integer \( d \geq 1 \). Then \( \widehat{Q}_p^d \) will be identified with \( Q_p^d \) by means of the map

\[
Q_p^d \to \widehat{Q}_p^d, \quad y \mapsto x \mapsto e_p(x \cdot y),
\]

where \( x \cdot y = \sum_{i=1}^d x_i y_i \) for \( x = (x_1, \ldots, x_d), y = (y_1, \ldots, y_d) \in Q_p^d \).

Let \( S = \{ p_1, \ldots, p_r, \infty \} \), where \( p_1, \ldots, p_r \) are integer primes. For an integer \( d \geq 1 \), consider the \( S \)-adic solenoid

\[
\text{Sol}_S = Q_p^d / \mathbb{Z}[1/S]^d,
\]

where \( \mathbb{Z}[1/S]^d = \mathbb{Z}[1/p_1, \ldots, 1/p_r]^d \) is embedded diagonally in \( Q_S = \prod_{p \in S} Q_p \). Then \( \text{Sol}_S \) is identified with the annihilator of \( \mathbb{Z}[1/S]^d \) in \( Q_S^d \), that is, with \( \mathbb{Z}[1/S]^d \) embedded in \( Q_S^d \) via the map

\[
\mathbb{Z}[1/S]^d \to Q_S^d, \quad b \mapsto (b, -b, \ldots, -b).
\]

Under this identification, the dual action of the automorphism group

\[
\text{Aut}(Q^d_S) \cong GL_d(\mathbb{R}) \times GL_d(Q_{p_1}) \times \cdots \times GL_d(Q_{p_r}).
\]

on \( \widehat{Q}_S^d \) corresponds to the right action on \( \mathbb{R}^d \times Q_{p_1}^d \times \cdots \times Q_{p_r}^d \) given by

\[
((g_\infty, g_1, \ldots, g_r), (a_\infty, a_1, \ldots, a_r)) \mapsto (g_\infty a_\infty, g_1 a_1, \ldots, g_r a_r),
\]

where \( (g, a) \mapsto ga \) is the usual (left) linear action of \( GL_d(\mathbb{k}) \) on \( \mathbb{k}^d \) for a field \( \mathbb{k} \).

7. Unitary representations of unipotent groups

Let \( U \) be a linear algebraic unipotent group defined over \( \mathbb{Q} \). The Lie algebra \( u \) is defined over \( \mathbb{Q} \) and the exponential map \( \exp : u \to U \) is a bijective morphism of algebraic varieties.

Let \( p \) be either a prime integer or \( p = \infty \). The irreducible unitary representations of \( U_p := U(\mathbb{Q}_p) \) are parametrized by Kirillov’s theory as follows.

The Lie algebra of \( U_p \) is \( u_p = u(\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}_p \), where \( u(\mathbb{Q}) \) is the Lie algebra over \( \mathbb{Q} \) consisting of the \( \mathbb{Q} \)-points in \( u \).

Fix an element \( f \) in the dual space \( u^*_p = \mathcal{H}om_{\mathbb{Q}_p}(u_p, \mathbb{Q}_p) \) of \( u_p \). There exists a polarization \( m \) for \( f \), that is, a Lie subalgebra \( m \) of \( u_p \) such that \( f([m, m]) = 0 \) and which is of maximal dimension. The induced representation \( \text{Ind}^{U_p}_M \chi_f \) is irreducible, where \( M = \exp(m) \) and \( \chi_f \) is the unitary character of \( M \) defined by

\[
\chi_f(\exp X) = e_p(f(X)) \quad \text{for all } X \in m,
\]

where \( e_p \in \widehat{Q}_p \) is as in §6. The unitary equivalence class of \( \text{Ind}^{U_p}_M \chi_f \) only depends on the co-adjoint orbit \( \text{Ad}^*(U_p) f \) of \( f \). The resulting map

\[
u^*_p / \text{Ad}^*(U_p) \to \widehat{U}_p, \quad \mathcal{O} \mapsto \pi_{\mathcal{O}},
\]

called the Kirillov map, from the orbit space \( u^*_p / \text{Ad}^*(U_p) \) of the co-adjoint representation to the unitary dual \( \widehat{U}_p \) of \( U_p \), is a bijection. In particular, \( U_p \) is a so-called type I locally compact group. For all of this, see [Kiri62] or [CoGr89] in the case of \( p = \infty \) and [Moor65] in the case of a prime integer \( p \).
The group \( \text{Aut}(U_p) \) of continuous automorphisms of \( U_p \) can be identified with the group of \( \mathbb{Q}_p \)-points of the algebraic group \( \text{Aut}(u) \) of automorphisms of the Lie algebra \( u \) of \( U \). Notice also that the natural action of \( \text{Aut}(U_p) \) on \( u_p \) as well as its dual action on \( u_p^* \) are algebraic.

Let \( \pi \in \hat{U}_p \) with corresponding Kirillov orbit \( O_\pi \) and \( g \in \text{Aut}(U_p) \). Then \( g(O_\pi) \) is the Kirillov orbit associated to the conjugate representation \( \pi g \).

**Lemma 10.** Let \( \pi \) be an irreducible unitary representation of \( U_p \). The stabilizer \( G_\pi \) of \( \pi \) in \( \text{Aut}(U_p) \) is an algebraic subgroup of \( \text{Aut}(U_p) \).

**Proof.** Let \( O_\pi \subset u_p^* \) be the Kirillov orbit corresponding to \( \pi \). Then \( G_\pi \) is the set of \( g \in \text{Aut}(U_p) \) such that \( g(O_\pi) = O_\pi \). As \( O_\pi \) is an algebraic subvariety of \( u_p^* \), the claim follows.

8. **Decay of matrix coefficients of unitary representations of \( S \)-adic groups**

Let \( p \) be an integer prime or \( p = \infty \) and let \( U \) be a linear algebraic unipotent group defined over \( \mathbb{Q}_p \). Set \( U_p := U(\mathbb{Q}_p) \).

Let \( \pi \) be an irreducible unitary representation of \( U_p \). Recall (see Lemma 10) that the stabilizer \( G_\pi \) of \( \pi \) in \( \text{Aut}(U_p) \) is an algebraic subgroup of \( \text{Aut}(U_p) \). Recall also (see Lemma 6) that \( \pi \) extends to a projective representation of \( G_\pi \). The following result was proved in [BeGu15, Proposition 22] in the case where \( p = \infty \), using arguments from [HoMo79]. The proof in the case where \( p \) is a prime integer is along similar lines and will be omitted.

**Proposition 11.** Let \( \pi \) be an irreducible unitary representation of \( U_p \) and let \( \tilde{\pi} \) be a projective unitary representation of \( G_\pi \) which extends \( \pi \). There exists a real number \( r \geq 1 \), only depending on the dimension of \( G_\pi \), such that \( \tilde{\pi} \) is strongly \( L^r \) modulo its projective kernel.

We will need later a precise description of the projective kernel of a representation \( \tilde{\pi} \) as above.

**Lemma 12.** Let \( \pi \) be an irreducible unitary representation of \( U_p \) and \( \tilde{\pi} \) a projective unitary representation of \( G_\pi \) which extends \( \pi \). Let \( O_\pi \subset u_p^* \) be the corresponding Kirillov orbit of \( \pi \). For \( g \in \text{Aut}(U_p) \), the following properties are equivalent.

(i) \( g \) belongs to the projective kernel \( P_{\tilde{\pi}} \) of \( \tilde{\pi} \).

(ii) For every \( u \in U_p \), we have

\[
g(u)u^{-1} \in \bigcap_{f \in O_\pi} \exp(\text{Ker}(f)).
\]

**Proof.** We can assume that \( \pi = \text{Ind}_M \chi_{f_0} \) for \( f_0 \in O_\pi \), and \( M = \exp \text{m} \) for a polarization \( \text{m} \) of \( f_0 \).

Let \( g \in \text{Aut}(U_p) \). If \( g \) is in the stabilizer \( G_\pi \) of \( \pi \) in \( \text{Aut}(U_p) \), recall (see Proof of Lemma 6) that

\[
\pi(g(u)) = \tilde{\pi}(g)\pi(u)\tilde{\pi}(g^{-1}) \quad \text{for all } u \in U_p.
\]
Since $\pi$ is irreducible, it follows from Schur’s lemma that $g \in P_{\tilde{\pi}}$ if and only if
\[ \pi(g(u)) = \pi(u) \quad \text{for all } u \in U_p \]
that is,
\[ g(u)u^{-1} \in \text{Ker}(\pi) \quad \text{for all } u \in U_p. \]

Now we have (see [BeGu15, Lemma 18])
\[ \text{Ker}(\pi) = \bigcap_{f \in \mathcal{O}_\pi} \text{Ker}(\chi_f), \]
and so $g \in P_{\tilde{\pi}}$ if and only if
\[ g(u)u^{-1} \in \bigcap_{f \in \mathcal{O}_\pi} \text{Ker}(\chi_f) \quad \text{for all } u \in U_p. \]

Let $g \in P_{\tilde{\pi}}$. Denote by $X \mapsto g(X)$ the automorphism of $u_p$ corresponding to $g$. Let $u = \exp(X)$ for $X \in u_p$ and $f \in \mathcal{O}_{\pi}$. Set $u_t = \exp(tX)$. By the Campbell Hausdorff formula, there exist $Y_1, \ldots, Y_r \in u_p$ such that
\[ g(u_t)(u_t)^{-1} = \exp(tY_1 + t^2Y_2 + \cdots + t^rY_r), \]
for every $t \in \mathbb{Q}_p$. Since
\[ 1 = \chi_f(g(u_t)(u_t)^{-1}) = e_p\left(f(tY_1 + t^2Y_2 + \cdots + t^rY_r)\right), \quad (\ast) \]
it follows that the polynomial
\[ t \mapsto Q(t) = tf(Y_1) + t^2f(Y_2) + \cdots + t^rf(Y_r) \]
takes its values in $\mathbb{Z}$ if $p = \infty$, and in $\mathbb{Z}_p$ (and so $Q$ has bounded image) otherwise. This clearly implies that $Q(t) = 0$ for all $t \in \mathbb{Q}_p$; in particular, we have
\[ \log(g(u)u^{-1}) = Y_1 + Y_2 + \cdots + Y_r \in \text{Ker}(f). \]

This shows that (i) implies (ii).

Conversely, assume that (ii) holds. Then clearly
\[ g(u)u^{-1} \in \bigcap_{f \in \mathcal{O}_\pi} \text{Ker}(\chi_f) \quad \text{for all } u \in U_p \]
and so $g \in P_{\tilde{\pi}}$. \qed

9. Decomposition of the Koopman representation for a nilmanifold

Let $U$ be a linear algebraic unipotent group defined over $\mathbb{Q}$. Let $S = \{p_1, \ldots, p_r, \infty\}$, where $p_1, \ldots, p_r$ are integer primes. Set
\[ U := U(\mathbb{Q}_S) = \prod_{p \in S} U_p. \]

Since $U$ is a type I group, the unitary dual $\widehat{U}$ of $U$ can be identified with the cartesian product $\prod_{p \in S} \widehat{U}_p$ via the map
\[
\prod_{p \in S} \hat{U}_p \to \hat{U}, \quad (\pi_p)_{p \in S} \mapsto \bigotimes_{p \in S} \pi_p,
\]
where \( \bigotimes_{p \in S} \pi_p = \pi_{i,\infty} \otimes \pi_{p_1} \otimes \cdots \otimes \pi_{p_p} \) is the tensor product of the \( \pi_p \).

Let \( \Lambda := U(\mathbb{Z}[1/S]) \) and consider the corresponding \emph{S-adic compact nilmanifold}
\[
\text{Nil}_S := U/\Lambda.
\]
equipped with the unique \( U \)-invariant probability measure \( \mu \) on its Borel subsets.

The associated \emph{S-adic solenoid} is
\[
\text{Sol}_S = \overline{U}/\overline{\Lambda},
\]
where \( \overline{U} := U/[U, U] \) is the quotient of \( U \) by its closed commutator subgroup \([U, U]\) and where \( \overline{\Lambda} \) is the image of \( U(\mathbb{Z}[1/S]) \) in \( \overline{U} \).

Set
\[
\text{Aut}(U) := \prod_{p \in S} \text{Aut}(U(\mathbb{Q}_p))
\]
and denote by \( \text{Aut}(\text{Nil}_S) \) the subgroup of all \( g \in \text{Aut}(U) \) with \( g(\Lambda) = \Lambda \). Observe that \( \text{Aut}(\text{Nil}_S) \) is a discrete subgroup of \( \text{Aut}(U) \), where every \( \text{Aut}(U_p) \) is endowed with its natural (locally compact) topology and \( \text{Aut}(U) \) with the product topology.

Let \( \Gamma \) be a subgroup of \( \text{Aut}(\text{Nil}_S) \). Let \( \kappa \) be the Koopman representation of \( \Gamma \ltimes U \) on \( L^2(\text{Nil}_S) \) associated to the action \( \Gamma \ltimes U \curvearrowright \text{Nil}_S \). By Proposition 9, there exists a family \((\pi_i)_{i \in I}\) of irreducible representations of \( U \), such that \( \kappa \) is equivalent to
\[
\bigoplus_{i \in I} \text{Ind}_{\Gamma_i \ltimes U}^\Gamma (\tilde{\pi}_i \otimes W_i),
\]
where \( \tilde{\pi}_i \) is an irreducible projective representation \( \tilde{\pi}_i \) of the stabilizer \( G_i \) of \( \pi_i \) in \( \text{Aut}(U) \ltimes U \) extending \( \pi_i \), and where \( W_i \) is a projective unitary representation of \( G_i \cap (\Gamma \ltimes U) \).

Fix \( i \in I \). We have \( \pi_i = \bigotimes_{p \in S} \pi_{i,p} \) for irreducible representations \( \pi_{i,p} \) of \( U_p \).

We will need the following more precise description of \( \pi_i \). Recall that \( u \) is the Lie algebra of \( U \) and that \( u(\mathbb{Q}) \) denotes the Lie algebra over \( \mathbb{Q} \) consisting of the \( \mathbb{Q} \)-points in \( u \).

Let \( u^*(\mathbb{Q}) \) be the set of \( \mathbb{Q} \)-rational points in the dual space \( u^* \); so, \( u^*(\mathbb{Q}) \) is the subspace of \( f \in u^* \) with \( f(X) \in \mathbb{Q} \) for all \( X \in u(\mathbb{Q}) \). Observe that, for \( f \in u^*(\mathbb{Q}) \), we have \( f(X) \in \mathbb{Q}_p \) for all \( X \in u_p = u(\mathbb{Q}_p) \).

A polarization for \( f \in u^*(\mathbb{Q}) \) is a Lie subalgebra \( m \) of \( u(\mathbb{Q}) \) such that \( f([m, m]) = 0 \) and which is of maximal dimension with this property.

**Proposition 13.** Let \( \pi_i = \bigotimes_{p \in S} \pi_{i,p} \) be one of the irreducible representations of \( U = U(\mathbb{Q}_S) \) appearing in the decomposition \( L^2(\text{Nil}_S) \) as above. There exist \( f_i \in u^*(\mathbb{Q}) \) and a polarization \( m_i \subset u(\mathbb{Q}) \) for \( f_i \) with the following property: for every \( p \in S \), the representation \( \pi_{i,p} \) is equivalent to \( \text{Ind}_{M_{i,p}}(\chi_{f_i}) \), where:

- \( M_{i,p} = \exp(m_{i,p}) \) for \( m_{i,p} = m_i \otimes \mathbb{Q} \mathbb{Q}_p \);
- \( \chi_{f_i} \) is the unitary character of \( M_{i,p} \) given by \( \chi_{f_i}(X) = e_p(f_i(X)) \), for all \( X \in m_{i,p} \), with \( e_p \in \mathbb{Q}_p \) as in \S 6.
Proof. The same result is proved in [Moor65, Theorem 11] (see also [Fox89, Theorem 1.2]) for the Koopman representation of $U(A)$ in $L^2(U(A)/U(Q))$, where $A$ is the ring of adeles of $Q$. We could check that the proof, which proceeds by induction of the dimension of $U$, carries over to the Koopman representation on $L^2(U(Q_S)/U(Z[1/S]))$, with the appropriate changes. We prefer to deduce our claim from the result for $U(A)$, as follows.

It is well known (see [Weil74]) that

$$A = \left( Q_S \times \prod_{p \notin S} Z_p \right) + Q$$

and that

$$\left( Q_S \times \prod_{p \notin S} Z_p \right) \cap Q = Z[1/S].$$

This gives rise to a well-defined projection $\varphi : A/Q \to Q_S/Z[1/S]$ given by

$$\varphi((a_S, (a_p)_{p \notin S}) + Q) = a_S + Z[1/S] \quad \text{for all } a_S \in Q_S, (a_p)_{p \notin S} \in \prod_{p \notin S} Z_p;$$

so the fiber over a point $a_S + Z[1/S] \in Q_S/Z[1/S]$ is

$$\varphi^{-1}(a_S + Z[1/S]) = \{(a_S, (a_p)_{p \notin S}) + Q \mid a_p \in Z_p \text{ for all } p\}.$$

This induces an identification of $U(Q_S)/U(Z[1/S]) = \text{Nil}_S$ with the double coset space $K_S \backslash U(A)/U(Q)$, where $K_S$ is the compact subgroup

$$K_S = \prod_{p \notin S} U(Z_p)$$

of $U(A)$. Observe that this identification is equivariant under translation by elements from $U(Q_S)$. In this way, we can view $L^2(\text{Nil}_S)$ as the $U(Q_S)$-invariant subspace $L^2(K_S \backslash U(A)/U(Q))$ of $L^2(U(A)/U(Q))$.

Choose a system $T$ of representatives for the $\text{Ad}^*(U(Q))$-orbits in $u^*(Q)$. By [Moor65, Theorem 11], for every $f \in T$, we can find a polarization $m_f \subset u(Q)$ for $f$ with the following property: setting

$$m_f(A) = m_f \otimes_Q A,$$

we have a decomposition

$$L^2(U(A)/U(Q)) = \bigoplus_{f \in T} \mathcal{H}_f$$

into irreducible $U(A)$-invariant subspaces $\mathcal{H}_f$ such that the representation $\pi_f$ of $U(A)$ in $\mathcal{H}_f$ is equivalent to $\text{Ind}_{M_f(A) \chi_f}^{U(A)} \chi_f$, where

$$M_f(A) = \exp(m_f(A))$$

and $\chi_f, A$ is the unitary character of $M_f(A)$ given by

$$\chi_f, A(\exp X) = e(f(X)) \quad \text{for all } X \in m_f(A);$$
here, $e$ is the unitary character of $A$ defined by
\[ e((a_p)_p) = \prod_{p \in \mathcal{P} \cup \{\infty\}} e_p(a_p) \quad \text{for all } (a_p)_p \in A, \]
where $\mathcal{P}$ is the set of integer primes.

We have
\[ L^2(K_S \backslash U(A)/U(Q)) = \bigoplus_{f \in T} \mathcal{H}_f^{K_S}, \]
where $\mathcal{H}_f^{K_S}$ is the space of $K_S$-fixed vectors in $\mathcal{H}_f$. It is clear that the representation of $U(Q_S)$ in $\mathcal{H}_f^{K_S}$ is equivalent to
\[ \text{Ind}_{U(Q_p)}^{U(Q_S)} \left( \bigotimes_{p \in S} \chi_{f,p} \right) = \bigotimes_{p \in S} (\text{Ind}_{U(Q_p)}^{U(Q_S)} \chi_{f,p}), \]
where $\chi_{f,p}$ is the unitary character of $M_f(Q_p)$ given by
\[ \chi_{f,p}(\exp X) = e_p(f(X)) \quad \text{for all } X \in m_f(Q_p). \]
Since $M_f(Q_p)$ is a polarization for $f$, each of the $U(Q_p)$-representations $\text{Ind}_{U(Q_p)}^{U(Q_S)} \chi_{f,p}$ and, hence, each of the $U(Q_S)$-representations
\[ \text{Ind}_{U(Q_S)}^{U(Q_p)} \left( \bigotimes_{p \in S} \chi_{f,p} \right) \]
is irreducible. This proves the claim. \hfill $\square$

We establish another crucial fact about the representations $\pi_i$ in the following proposition.

**Proposition 14.** With the notation of Proposition 13, let $O_Q(f_i)$ be the co-adjoint orbit of $f_i$ under $U(Q)$ and set
\[ \xi_{i,p} = \bigcap_{f \in O_Q(f_i)} \xi_p(f), \]
where $\xi_p(f)$ is the kernel of $f$ in $u_p$. Let $K_{i,p} = \exp(\xi_{i,p})$ and $K_i = \prod_{p \in S} K_{i,p}$.

(i) $K_i$ is a closed normal subgroup of $U$ and $K_i \cap \Lambda = K_i \cap U(Z[1/S])$ is a lattice in $K_i$.

(ii) Let $P_{\pi_i}$ be the projective kernel of the extension $\tilde{\pi}_i$ of $\pi_i$ to the stabilizer $G_i$ of $\pi_i$ in $\text{Aut}(U) \ltimes U$. For $g \in G_i$, we have $g \in P_{\pi_i}$ if and only if $g(u) \in uK_i$ for every $u \in U$.

**Proof.** (i) Let
\[ \xi_{i,Q} = \bigcap_{f \in O_Q(f_i)} \xi_Q(f), \]
where \( \xi_Q(f) \) is the kernel of \( f \) in \( u(Q) \). Observe that \( \xi_i, Q \) is an ideal in \( u(Q) \), since it is \( \text{Ad}(U(Q)) \)-invariant. So, we have

\[
\xi_i, Q = \xi_i(Q)
\]

for an ideal \( \xi_i \) in \( u \). Since \( f \in u^*(Q) \) for \( f \in \mathcal{O}_Q(f_i) \), we have

\[
\xi_i, p(f) = \xi_i, Q(f) \otimes Q Q_p
\]

and hence

\[
\xi_i, p = \xi_i(Q_p).
\]

Let \( K_i = \exp(\xi_i) \). Then \( K_i \) is a normal algebraic \( Q \)-subgroup of \( U \) and we have \( K_i, p = K_i(Q_p) \) for every \( p \); so,

\[
K_i = \prod_{s \in S} K_i(Q_p) = K_i(Q_S)
\]

and \( K_i \cap \Lambda = K_i(Z[1/S]) \) is a lattice in \( K_i \). This proves (i).

To prove (ii), observe that

\[
P_{\pi_i} = \prod_{p \in S} P_{i, p},
\]

where \( P_{i, p} \) is the projective kernel of \( \pi_i, p \).

Fix \( p \in S \) and let \( g \in G_i \). By Lemma 12, \( g \in P_{i, p} \) if and only if \( g(u) \in u K_i, p \) for every \( u \in U_p = U(Q_p) \). This finishes the proof. \( \square \)

10. Proof of Theorem 1

Let \( U \) be a linear algebraic unipotent group defined over \( Q \) and \( S = \{ p_1, \ldots, p_r, \infty \} \), where \( p_1, \ldots, p_r \) are integer primes. Set \( U := U(Q_S) \) and \( \Lambda := U(Z[1/S]) \). Let \( \text{Nil}_S = U/\Lambda \) and \( \text{Sol}_S \) be the \( S \)-adic nilmanifold and the associated \( S \)-adic solenoid as in §9. Denote by \( \mu \) the translation-invariant probability measure on \( \text{Nil}_S \) and let \( \nu \) be the image of \( \mu \) under the canonical projection \( \varphi : \text{Nil}_S \to \text{Sol}_S \). We identify \( L^2(\text{Sol}_S) = L^2(\text{Sol}_S, \nu) \) with the closed \( \text{Aut}(\text{Nil}_S) \)-invariant subspace

\[
\{ f \circ \varphi \mid f \in L^2(\text{Sol}_S) \}
\]

of \( L^2(\text{Nil}_S) = L^2(\text{Nil}_S, \mu) \). We have an orthogonal decomposition into \( \text{Aut}(\text{Nil}_S) \)-invariant subspaces

\[
L^2(\text{Nil}_S) = C_1\text{Nil}_S \oplus L^2_0(\text{Sol}_S) \oplus \mathcal{H},
\]

where

\[
L^2_0(\text{Sol}_S) = \left\{ f \in L^2(\text{Sol}_S) \mid \int_{\text{Nil}_S} f \, d\mu = 0 \right\}
\]

and where \( \mathcal{H} \) is the orthogonal complement of \( L^2(\text{Sol}_S) \) in \( L^2(\text{Nil}_S) \).

Let \( \Gamma \) be a subgroup of \( \text{Aut}(\text{Nil}_S) \). Let \( \kappa \) be the Koopman representation of \( \Gamma \) on \( L^2(\text{Nil}_S) \) and denote by \( \kappa_1 \) and \( \kappa_2 \) the restrictions of \( \kappa \) to \( L^2_0(\text{Sol}_S) \) and \( \mathcal{H} \), respectively.
Groups of automorphisms of $S$-adic nilmanifolds

Let $\Sigma_1$ be a set of representatives for the $\Gamma$-orbits in $\widehat{\text{Sol}}_S \setminus \{ \mathbf{1}_{\text{Sol}} \}$. We have

$$\kappa_1 \cong \bigoplus_{\chi \in \Sigma_1} \lambda_{\Gamma/\Gamma_\chi},$$

where $\Gamma_\chi$ is the stabilizer of $\chi$ in $\Gamma$ and $\lambda_{\Gamma/\Gamma_\chi}$ is the quasi-regular representation of $\Gamma$ on $\ell^2(\Gamma/\Gamma_\chi)$.

By Proposition 9, there exists a family $(\pi_i)_{i \in I}$ of irreducible representations of $U$, such that $\kappa_2$ is equivalent to a direct sum

$$\bigoplus_{i \in I} \text{Ind}_{\Gamma_i}^{\Gamma} (\tilde{\pi}_i|_{\Gamma_i} \otimes W_i),$$

where $\tilde{\pi}_i$ is an irreducible projective representation of the stabilizer $G_i$ of $\pi_i$ in $\text{Aut}(U)$ and where $W_i$ is a projective unitary representation of $\Gamma_i := \Gamma \cap G_i$.

**Proposition 15.** For $i \in I$, let $\tilde{\pi}_i$ be the (projective) representation of $G_i$ of $\pi_i$ in $\text{Aut}(U)$ and let $\Gamma_i$ be as above. There exists a real number $r \geq 1$ such that $\tilde{\pi}_i|_{\Gamma_i}$ is strongly $L^r$ modulo $P_{\tilde{\pi}_i} \cap \Gamma_i$, where $P_{\tilde{\pi}_i}$ is the projective kernel of $\tilde{\pi}_i$.

**Proof.** By Proposition 11, there exists a real number $r \geq 1$ such that the representation $\tilde{\pi}_i$ of the algebraic group $G_i$ is strongly $L^r$ modulo $P_{\tilde{\pi}_i}$. In order to show that $\tilde{\pi}_i|_{\Gamma_i}$ is strongly $L^r$ modulo $P_{\tilde{\pi}_i} \cap \Gamma_i$, it suffices to show that $\Gamma_i P_{\tilde{\pi}_i}$ is closed in $G_i$ (compare with the proof of [HoMo79, Proposition 6.2]).

Let $K_i$ be the closed $G_i$-invariant normal subgroup $K_i$ of $U$ as described in Proposition 14. Then $\overline{\Lambda} = K_i / \Lambda / K_i$ is a lattice in the unipotent group $\overline{U} = U / K_i$. By Proposition 14(ii), $P_{\tilde{\pi}_i}$ coincides with the kernel of the natural homomorphism $\varphi : \text{Aut}(U) \to \text{Aut}(\overline{U})$. Hence, we have

$$\Gamma_i P_{\tilde{\pi}_i} = \varphi^{-1}(\varphi(\Gamma_i)).$$

Now, $\varphi(\Gamma_i)$ is a discrete (and hence closed) subgroup of $\text{Aut}(\overline{U})$, since $\varphi(\Gamma_i)$ preserves $\overline{\Lambda}$ (and so $\varphi(\Gamma_i) \subset \text{Aut}(\overline{U} / \overline{\Lambda})$). It follows from the continuity of $\varphi$ that $\varphi^{-1}(\varphi(\Gamma_i))$ is closed in $\text{Aut}(U)$. $\square$

**Proof of Theorem 1.** We have to show that, if $1_{\Gamma}$ is weakly contained in $\kappa_2$, then $1_{\Gamma}$ is weakly contained in $\kappa_1$. It suffices to show that, if $1_{\Gamma}$ is weakly contained in $\kappa_2$, then there exists a finite-index subgroup $H$ of $\Gamma$ such that $1_H$ is weakly contained in $\kappa_1|_H$ (see [BeFr20, Theorem 2]).

We proceed by induction on the integer

$$n(\Gamma) := \sum_{p \in S} \dim Zc_p(\Gamma),$$

where $Zc_p(\Gamma)$ is the Zariski closure of the projection of $\Gamma$ in $GL_n(\mathbb{Q}_p)$.

If $n(\Gamma) = 0$, then $\Gamma$ is finite and there is nothing to prove.

Assume that $n(\Gamma) \geq 1$ and that the claim above is proved for every countable subgroup $H$ of $\text{Aut}(\text{Nil}_S)$ with $n(H) < n(\Gamma)$. 

Let $I_{\text{fin}} \subset I$ be the set of all $i \in I$ such that $\Gamma_i = G_i \cap \Gamma$ has finite index in $\Gamma$ and set $I_\infty = I \setminus I_{\text{fin}}$. With $V_i = \tilde{\pi}_i|_{\Gamma_i} \otimes W_i$, set

$$
\kappa_{2}^{\text{fin}} = \bigoplus_{i \in I_{\text{fin}}} \text{Ind}_{\Gamma_i}^\Gamma V_i \quad \text{and} \quad \kappa_{2}^\infty = \bigoplus_{i \in I_\infty} \text{Ind}_{\Gamma_i}^\Gamma V_i.
$$

Two cases can occur.

**First case:** $1_{\Gamma}$ is weakly contained in $\kappa_{2}^\infty$. Observe that $n(\Gamma_i) < n(\Gamma)$ for $i \in I_\infty$. Indeed, otherwise $Zc_p(\Gamma_i)$ and $Zc_p(\Gamma)$ would have the same connected component $C^0_p$ for every $p \in S$, since $\Gamma_i \subset \Gamma$. Then

$$
C^0 := \prod_{p \in S} C^0_p
$$

would stabilize $\pi_i$ and $\Gamma \cap C^0$ would therefore be contained in $\Gamma_i$. Since $\Gamma \cap C^0$ has finite index in $\Gamma$, this would contradict the fact that $\Gamma_i$ has infinite index in $\Gamma$.

By restriction, $1_{\Gamma_i}$ is weakly contained in $\kappa_{2}^\infty|_{\Gamma_i}$ for every $i \in I$. Hence, by the induction hypothesis, $1_{\Gamma_i}$ is weakly contained in $\kappa_{1}^\infty|_{\Gamma_i}$ for every $i \in I_\infty$. Now, on the one hand, we have

$$
\kappa_{1}^\infty|_{\Gamma_i} \cong \bigoplus_{\chi \in T_i} \lambda_{\Gamma_i/\Gamma \cap \Gamma_i},
$$

for a subset $T_i$ of $\widehat{\text{Sol}}_S \setminus \{1_{\text{Sol}}\}$. It follows that $\text{Ind}_{\Gamma_i}^\Gamma 1_{\Gamma_i} = \lambda_{\Gamma/\Gamma_i}$ is weakly contained in

$$
\bigoplus_{\chi \in T_i} \text{Ind}_{\Gamma_i}^\Gamma (\lambda_{\Gamma_i/\Gamma \cap \Gamma_i}) = \bigoplus_{\chi \in T_i} \lambda_{\Gamma/\Gamma \cap \Gamma_i},
$$

for every $i \in I_\infty$. On the other hand, since $1_{\Gamma}$ is weakly contained in

$$
\kappa_{2}^\infty \cong \bigoplus_{i \in I_\infty} \text{Ind}_{\Gamma_i}^\Gamma (\tilde{\pi}_i|_{\Gamma_i} \otimes W_i),
$$

Lemma 7 shows that $1_{\Gamma}$ is weakly contained in $\bigoplus_{i \in I_\infty} \lambda_{\Gamma/\Gamma_i}$. It follows that $1_{\Gamma}$ is weakly contained in

$$
\bigoplus_{i \in I_\infty} \bigoplus_{\chi \in T_i} \lambda_{\Gamma/\Gamma \cap \Gamma_i}.
$$

Hence, by Lemma 7 again, $1_{\Gamma}$ is weakly contained in

$$
\bigoplus_{i \in I_\infty} \bigoplus_{\chi \in T_i} \lambda_{\Gamma/\Gamma \cap \Gamma_i}.
$$

This shows that $1_{\Gamma}$ is weakly contained in $\kappa_{1}^\infty$.

**Second case:** $1_{\Gamma}$ is weakly contained in $\kappa_{2}^{\text{fin}}$. By the Noetherian property of the Zariski topology, we can find finitely many indices $i_1, \ldots, i_r$ in $I_{\text{fin}}$ such that, for every $p \in S$, we have

$$
Zc_p(\Gamma_{i_1}) \cap \cdots \cap Zc_p(\Gamma_{i_r}) = \bigcap_{i \in I_{\text{fin}}} Zc_p(\Gamma_i),
$$
Set \( H := \Gamma_{i_1} \cap \cdots \cap \Gamma_{i_r} \). Observe that \( H \) has finite index in \( \Gamma \). Moreover, it follows from Lemma 10 that \( \mathbb{Z}_p(\Gamma_{i_1}) \cap \cdots \cap \mathbb{Z}_p(\Gamma_{i_r}) \) stabilizes \( \pi_{i,p} \) for every \( i \in I_{\text{fin}} \) and \( p \in S \). Hence, \( H \) is contained in \( \Gamma_i \) for every \( i \in I_{\text{fin}} \).

By Proposition 9, we have a decomposition of \( \kappa_{i,\text{fin}}^{|H} \) into the direct sum

\[
\bigoplus_{i \in I_{\text{fin}}} (\tilde{\pi}_i \otimes W_i)|_H.
\]

By Propositions 11 and 15, there exists a real number \( r \geq 1 \), which is independent of \( i \), such that \( (\tilde{\pi}_i \otimes W_i)|_H \) is a strongly \( L_r \) representation of \( H \) modulo its projective kernel \( P_i \). Observe that \( P_i \) is contained in the projective kernel \( P_{\tilde{\pi}_i} \) of \( \tilde{\pi}_i \), since \( P_i = P_{\tilde{\pi}_i} \cap H \). Hence (see Proposition 8), there exists an integer \( k \geq 1 \) such that \((\kappa_{i,\text{fin}}^{|H})^\otimes k \) is contained in a multiple of the direct sum

\[
\bigoplus_{i \in I_{\text{fin}}} \text{Ind}_{P_i}^H \rho_i,
\]

for representations \( \rho_i \) of \( P_i \). Since \( 1_H \) is weakly contained in \( \kappa_{i,\text{fin}}^{|H} \) and hence in \((\kappa_{i,\text{fin}}^{|H})^\otimes k \), using Lemma 7, it follows that \( 1_H \) is weakly contained in

\[
\bigoplus_{i \in I_{\text{fin}}} \lambda_{\cdot H/P_i}.
\]

Let \( i \in I \). We claim that \( P_i \) is contained in \( \Gamma_\chi \) for some character \( \chi \) from \( \hat{\text{Sol}}_{S} \setminus \{1_{\text{Sol}_{S}}\} \). Once proved, this will imply, again by Lemma 7, \( 1_H \) is weakly contained in \( \kappa_1^{|H} \). Since \( H \) has finite index in \( \Gamma \), this will show that \( 1_\Gamma \) is weakly contained in \( \kappa_1 \) and conclude the proof.

To prove the claim, recall from Proposition 14 that there exists a closed normal subgroup \( K_i \) of \( U \) with the following properties: \( K_i \Lambda/K_i \) is a lattice in the unipotent algebraic group \( U/K_i \), \( K_i \) is invariant under \( P_{\tilde{\pi}_i} \) and \( P_{\tilde{\pi}_i} \) acts as the identity on \( U/K_i \). Observe that \( K_i \neq U \), since \( \pi_i \) is not trivial on \( U \). We can find a non-trivial unitary character \( \chi \) of \( U/K_i \) which is trivial on \( K_i \Lambda/K_i \). Then \( \chi \) lifts to a non-trivial unitary character of \( U \) which is fixed by \( P_{\tilde{\pi}_i} \) and hence by \( P_i \). Observe that \( \chi \in \hat{\text{Sol}}_S \), since \( \chi \) is trivial on \( \Lambda \).

### 11. An example: the \( S \)-adic Heisenberg nilmanifold

As an example, we study the spectral gap property for groups of automorphisms of the \( S \)-adic Heisenberg nilmanifold, proving Corollary 5. We will give a quantitative estimate for the norm of associated convolution operators, as we did in [BeHe11] in the case of real Heisenberg nilmanifolds (that is, in the case \( S = \{\infty\} \)).

Let \( K \) be an algebraically closed field containing \( \mathbb{Q}_p \) for \( p = \infty \) and for all prime integers \( p \). For an integer \( n \geq 1 \), consider the symplectic form \( \beta \) on \( K^{2n} \) given by

\[
\beta((x, y), (x', y')) = (x, y)^t J (x', y') \quad \text{for all} \ (x, y), (x', y') \in K^{2n},
\]

where \( J \) is the \((2n \times 2n)\)-matrix

\[
J = \begin{pmatrix}
0 & I \\
-I_n & 0
\end{pmatrix}.
\]
The symplectic group
\[ Sp_{2n} = \{ g \in GL_{2n}(K) \mid ^t g J g = J \} \]
is an algebraic group defined over \( \mathbb{Q} \).

The \((2n + 1)\)-dimensional Heisenberg group is the unipotent algebraic group \( H \) defined over \( \mathbb{Q} \), with underlying set \( K^{2n} \times K \) and product
\[(x, y, s)((x', y', t) = ((x + x', y + y'), s + t + \beta((x, y), (x', y'))),\]
for \((x, y), (x', y') \in K^{2n}, s, t \in K \).

The group \( Sp_{2n} \) acts by rational automorphisms of \( H \), given by
\[ g((x, y), t) = (g(x, y), t) \quad \text{for all } g \in Sp_{2n}, (x, y) \in K^{2n}, t \in K. \]

Let \( p \) be either an integer prime or \( p = \infty \). Set \( H_p = H(\mathbb{Q}_p) \). The center \( Z \) of \( H_p \) is \( \{(0, 0, t) \mid t \in \mathbb{Q}_p \} \). The unitary dual \( \hat{H}_p \) of \( H_p \) consists of the equivalence classes of the following representations:

- the unitary characters of the abelianized group \( H_p/Z \);
- for every \( t \in \mathbb{Q}_p \setminus \{0\} \), the infinite-dimensional representation \( \pi_t \) defined on \( L^2(\mathbb{Q}_p^n) \) by the formula
\[ \pi_t((a, b), s)\xi(x) = e_p(ts)e_p((a, x - b))\xi(x - b) \]
for \( ((a, b), s) \in H_p, \xi \in L^2(\mathbb{Q}_p^n), \) and \( x \in \mathbb{Q}_p^n \), where \( e_p \in \mathbb{Q}_p^{\ast} \) is as in §6.

For \( t \neq 0 \), the representation \( \pi_t \) is, up to unitary equivalence, the unique irreducible unitary representation of \( H_p \) whose restriction to the center \( Z \) is a multiple of the unitary character \( s \mapsto e_p(ts) \).

For \( g \in Sp_{2n}(\mathbb{Q}_p) \) and \( t \in \mathbb{Q}_p \setminus \{0\} \), the representation \( \pi_t^g \) is unitary equivalent to \( \pi_t \), since both representations have the same restriction to \( Z \). This shows that \( Sp_{2n}(\mathbb{Q}_p) \) stabilizes \( \pi_t \). We denote the corresponding projective representation of \( Sp_{2n}(\mathbb{Q}_p) \) by \( \omega_t^{(p)} \).

The representation \( \omega_t^{(p)} \) has different names: it is called the metaplectic representation, Weil’s representation or the oscillator representation. The projective kernel of \( \omega_t^{(p)} \) coincides with the (finite) center of \( Sp_{2n}(\mathbb{Q}_p) \) and \( \omega_t^{(p)} \) is strongly \( L^{4n+2+\epsilon} \) on \( Sp_{2n}(\mathbb{Q}_p) \) for every \( \epsilon > 0 \) (see [HoMo79, Proposition 6.4] or [Howe82, Proposition 8.1]).

Let \( S = \{ p_1, \ldots, p_r, \infty \} \), where \( p_1, \ldots, p_r \) are integer primes. Set \( H := H(\mathbb{Q}_S) \) and
\[ \Lambda := H(\mathbb{Z}[1/S]) = \{ ((x, y), s) : x, y \in \mathbb{Z}[1/S], s \in \mathbb{Z}[1/S] \}. \]

Let \( \text{Nil}_S = H/\Lambda \); the associated \( S \)-adic solenoid is \( \text{Sol}_S = \mathbb{Q}_S^{2n}/\mathbb{Z}[1/S]^{2n} \). The group \( Sp_{2n}(\mathbb{Z}[1/S]) \) is a subgroup of \( \text{Aut}(\text{Nil}_S) \). The action of \( Sp_{2n}(\mathbb{Z}[1/S]) \) on \( \text{Sol}_S \) is induced by its representation linearly by bijections on \( \mathbb{Q}_S^{2n} \).

Let \( \Gamma \) be a subgroup of \( Sp_{2n}(\mathbb{Z}[1/S]) \). The Koopman representation \( \kappa \) of \( \Gamma \) on \( L^2(\text{Nil}_S) \) decomposes as
\[ \kappa = 1_{\text{Nil}_S} \oplus \kappa_1 \oplus \kappa_2, \]
where \( \kappa_1 \) is the restriction of \( \kappa \) to \( L^2(\text{Sol}_S) \) and \( \kappa_2 \) the restriction of \( \kappa \) to the orthogonal complement of \( L^2(\text{Sol}_S) \) in \( L^2(\text{Nil}_S) \). Since \( Sp_{2n}(\mathbb{Q}_p) \) stabilizes every
infinite-dimensional representation of $H_p$, it follows from Proposition 13 that there exists a subset $I \subset \mathbb{Q}$ such that $\kappa_2$ is equivalent to a direct sum
\[ \bigoplus_{t \in I} \left( \bigotimes_{p \in S} (\omega_t(p) |_{\Gamma} \otimes W_t) \right), \]
where $W_t$ is a projective representation of $\Gamma$.

Let $\nu$ be a probability measure on $\Gamma$. We can give an estimate of the norm of $\kappa_2(\nu)$ as in [BeHe11] in the case of $S = \{\infty\}$. Indeed, by a general inequality (see [BeGu15, Proposition 30]), we have
\[ \|\kappa_2(\nu)\| \leq \|(\kappa_2 \otimes \overline{\kappa_2}) \otimes_k(\nu)\|^{1/2k}, \]
for every integer $k \geq 1$, where $\overline{\kappa_2}$ denotes the representation conjugate to $\kappa_2$. Since $\omega_t(p)$ is strongly $L^{2n+2+\epsilon}$ on $Sp_{2n}(\mathbb{Q}_p)$ for any $t \in I$ and $p \in S$, Proposition 8 implies that $(\kappa_2 \otimes \overline{\kappa_2}) \otimes^{(n+1)}$ is contained in an infinite multiple of the regular representation $\lambda_{\Gamma}$ of $\Gamma$. Hence,
\[ \|\kappa_2(\nu)\| \leq \|\lambda_{\Gamma}(\nu)\|^{1/2n+2} \]
and so,
\[ \|\kappa_0(\nu)\| \leq \max\{\|\lambda_{\Gamma}(\nu)\|^{1/2n+2}, \|\kappa_1(\nu)\|\}, \]
where $\kappa_0$ is the restriction of $\kappa$ to $L^2_{\mathbb{Z}}(\text{Nil}_S)$.

Assume that $\nu$ is aperiodic. If $\Gamma$ is not amenable then $\|\lambda_{\Gamma}(\nu)\| < 1$ by Kesten’s theorem (see [BeHV08, Appendix G]); so, in this case, the action of $\Gamma$ on $\text{Nil}_S$ has a spectral gap if and only if $\|\kappa_1(\nu)\| < 1$, as stated in Theorem 1.

Observe that, if $\Gamma$ is amenable, then the action of $\Gamma$ on $\text{Nil}_S$ or $\text{Sol}_S$ does not have a spectral gap; indeed, by a general result (see [JuRo79, Theorem 2.4]), no action of a countable amenable group by measure-preserving transformations on a non-atomic probability space has a spectral gap.

Let us look more closely to the case $n = 1$. We have $Sp_2(\mathbb{Z}[1/S]) = SL_2(\mathbb{Z}[1/S])$ and the stabilizer of every element in $\text{Sol}_S \setminus \{1_{\text{Sol}_S}\}$ is conjugate to the group of unipotent matrices in $SL_2(\mathbb{Z}[1/S])$ and hence amenable. This implies that $\kappa_1$ is weakly contained in $\lambda_{\Gamma}$ (see the decomposition of $\kappa_1$ appearing before Proposition 15); so, we have
\[ \|\kappa_1(\nu)\| < 1 \iff \Gamma \text{ is not amenable.} \]

As a consequence, we see that the action of $\Gamma$ on $\text{Nil}_S$ has a spectral gap if and only if $\Gamma$ is not amenable.

\textbf{References}

[BeFr20] B. Bekka and C. Francini. Spectral gap property and strong ergodicity for groups of affine transformations of solenoids. \textit{Ergod. Th. & Dynam. Sys.} \textbf{40}(5) (2020), 1180–1193.

[BeGu15] B. Bekka and Y. Guivarc’h. On the spectral theory of groups of affine transformations of compact nilmanifolds. \textit{Ann. Sci. Éc. Norm. Supér. (4)} \textbf{48} (2015), 607–645.

[BeHe11] B. Bekka and J.-R. Heu. Random products of automorphisms of Heisenberg nilmanifolds and Weil’s representation. \textit{Ergod. Th. & Dynam. Sys.} \textbf{31} (2011), 1277–1286.
[BeHV08] B. Bekka, P. de la Harpe and A. Valette. Kazhdan’s Property (T). Cambridge University Press, Cambridge, 2008.

[Bekk16] B. Bekka. Spectral rigidity of group actions on homogeneous spaces. Handbook of Group Actions. Volume IV (Advanced Lectures in Mathematics (ALM), 4). Eds. L. Ji and S.-T. Yau. International Press, Somerville, MA, 2018, pp. 563–622.

[Fox89] J. Fox. Adeles and the spectrum of compact nilmanifolds. Pacific J. Math. 140, 233–250 (1989).

[GGPS69] I. M. Gelfand, M. I. Graev and I. I. Pyatetskii-Shapiro, Representation Theory and Automorphic Functions. W.B. Saunders Company, Philadelphia, PA, 1969

[CoGr89] L. Corwin and F. Greenleaf. Representations of Nilpotent Lie Groups and their Applications, Cambridge University Press, Cambridge, 1989.

[HeRo63] E. Hewitt and K. Ross. Abstract Harmonic Analysis. Volume I (Die Grundlehren der mathematischen Wissenschaften, 115). Springer-Verlag, New York, 1963.

[HoMo79] R. Howe and C. C. Moore. Asymptotic properties of unitary representations. J. Funct. Anal. 32 (1979), 72–96.

[HoTa92] R. Howe and E. C. Tan. Non-abelian Harmonic Analysis. Springer, New York, 1992.

[Howe82] R. Howe. On a notion of rank for unitary representations of the classical groups. Harmonic Analysis and Group Representations. Ed. F. Talamanca. Liguori, Naples, 1982, pp. 223–331.

[JuRo79] A. del Junco and J. Rosenblatt. Counterexamples in ergodic theory. Math. Ann. 245 (1979), 185–197.

[Kiri62] A. A. Kirillov. Unitary representations of nilpotent Lie groups. Russian Math. Surveys 17 (1962), 53–104.

[Mack58] G. W. Mackey. Unitary representations of group extensions I. Acta Math. 99 (1958), 265–311.

[Mack76] G. W. Mackey. The Theory of Unitary Group Representations (Chicago Lectures in Mathematics). University of Chicago Press, Chicago, 1976.

[Moor65] C. C. Moore. Decomposition of unitary representations defined by discrete subgroups of nilpotent Lie groups. Ann. of Math. (2) 82 (1965), 146–182.

[Schm81] K. Schmidt. Amenability, Kazhdan’s property T, strong ergodicity and invariant means for ergodic group-actions. Ergod. Th. & Dynam. Sys. 1 (1981), 223–236.

[Weil74] A. Weil. Basic Number Theory. Springer-Verlag, Berlin, 1974.