Approximate solutions for localized modes appearing in resonant circuit array with an external shorted coil

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Abstract: Localized modes in resonant circuit array consisting of circular coils and capacitors are analytically investigated. First, an appropriate approximation of magnetic couplings between the coils is introduced. Then, analytical solutions are successfully derived by solving simultaneous polynomials for two different positions of the external coil. These solutions are evaluated by comparing with numerically exact solutions in the original problem. It is confirmed that the obtained solutions fit well the exact solutions when the overlapping and the diameter of the external coil are not too large.

Key Words: wireless power transfer, resonant circuit array, localized mode

1. Introduction

Wireless power transfer is nowadays widely used in our life. Since a new technique that enables to transfer of several-ten watts between two coils separated about one meter was demonstrated by A. Kurs \textit{et al.} \cite{1}, wireless power transfer using resonant phenomena have attracted again many researchers. The efficiency of the wireless power transfer is very sensitive for the relative position between the transmitting coil and the receiving coil. Arraying transmitting coils is an approach to reduce the sensitivity \cite{2, 3}. The receiving coil is almost free for its position. However, in the wireless power transfer system using arrayed transmitting coils, detecting the position of the receiving coil and controlling the current that flows in the transmitting coils are necessary.

Recent research has reported that a localized mode appears when a receiving coil approaches the surface of an array of transmitting coils in which each element of the array consists of a circular coil and capacitors, namely, a resonant circuit \cite{4}. The localized mode is known as a spatially localized and temporally periodic solution in linearly coupled resonators. The amplitude distribution is localized around an impurity in the coupled resonators and it decays exponentially as far away from the center.
Fig. 1. Amplitude distribution of (a) a typical resonant mode without impurity, (b) localized modes that appeared around an impurity (blue circle). The typical resonant mode has a spatially extended distribution in amplitude because of the coupling between neighboring resonators. If an impurity is added to the array, the localized mode appears around the impurity.

of the localized mode as shown in Fig. 1. For the array of transmitting coils, it is numerically shown that the current flow localizes around the place where the receiving coil is placed [5]. It implies that the magnetic flux concentrates automatically around the receiving coil without any sensors or controllers. However, the frequency of the localized mode strongly depends on the position of the receiving coil. Excitation of the localized mode by using a single frequency power source will be difficult because of the fluctuation of the frequency. The fluctuation can be suppressed by adjusting the overlapping of the transmitting coils [5]. This paper aims to construct approximate solutions of localized modes to contribute to establishing a design rule of a wireless power transfer system utilizing localized modes. The simplest approximation has already been reported in Ref. [6]. This paper introduces more precise solutions and discusses their errors with respect to the design parameters which are the ratio of the overlapping and the diameter of the receiving coil.

In Sec. 2, a dynamical model of a resonant circuit array consisting of circular coils and capacitors is introduced and an effect of an external coil is considered. After a general form of the model equations is derived, it is reduced to the nearest-neighbor coupling model. In Sec. 3, localized modes are analytically derived under appropriate approximations. The errors of the solutions are evaluated in Sec. 4 with respect to the design parameters. Lastly in Sec. 5, we will give a brief discussion of an application of the analytical solutions to design a wireless power transfer system.

2. Resonant circuit array

A resonant circuit consists of a planar circular spiral coil and a capacitor as shown in Fig. 2(a). The outer diameter of the coil is denoted by $D_{TX}$. It is assumed that several tens of the resonant circuits are arranged straight with an equal interval of $d$. If $d < D_{TX}$ the adjacent coils have an overlap. The ratio of the overlap to the outer diameter is defined as $c = (D_{TX} - d)/D_{TX}$, where $c$ is assumed to take a positive value between zero and 1/2.

The circuit diagram of the resonant circuit array is shown in Fig. 3(a), where the planar coil is modeled as an ideal inductor. Each resonant circuit is not connected directly by wires but is coupled through mutual induction with each other. The following equations are obtained by Kirchhoff’s voltage law:

$$v_n = \sum_{m=1}^{N} M_{nm} \frac{di_m}{dt},$$

(1)

Fig. 2. Schematic illustration of a planar circular spiral coil. In this paper, we assumed $D_{TX} = 20\,\text{mm}$, $0 < d < 1/2$. The overlapping ratio is defined as $c = (D_{TX} - d)/D_{TX}$.
where \( N \) is the total number of coils, \( v_n \) is the voltage of the capacitor in the \( n \)th resonant circuit, and \( i_n \) is the current flowing \( n \)th coil. The mutual inductance between \( n \)th and \( m \)th coil is denoted by \( M_{nm} \). Note that \( M_{nn} \) corresponds to the self-inductance of \( n \)th coil \( L_n \) and \( M_{nm} = M_{mn} \). By using the relationship between the voltage and the current \( i_n = -C_n \frac{dv_n}{dt} \), the second order differential equation is derived:

\[
i_n = -C_n \sum_{m=1}^{N} M_{nm} \frac{d^2 i_m}{dt^2} = -LC \sum_{m=1}^{N} k_{nm} \frac{d^2 i_m}{dt^2},
\]

where all the inductance \( L_n \) and the capacitance \( C_n \) are assumed to have the same values \( L \) and \( C \), respectively. \( k_{nn} = 1 \) and \( k_{nm} = k_{mn} \). Hereafter, \( LC \) is set at unity without loss of generality. The above equation is linear and can be represented by the matrix form like,

\[
i = -\begin{pmatrix}
1 & k_{12} & k_{13} & \cdots & k_{1N} \\
k_{12} & 1 & k_{23} & \cdots & \\
k_{13} & k_{23} & 1 & \cdots & \\
\vdots & \vdots & \vdots & \ddots & \\
k_{1N} & \cdots & \cdots & 1
\end{pmatrix}
\frac{d^2 i}{dt^2} = -K_0 \frac{d^2 i}{dt^2},
\]

where \( i = (i_1 \ i_2 \ \ldots \ i_N)^T \) and \( K_0 \) is \((N \times N)\) real symmetric matrix.

Let us consider that an external shorted coil is placed on the resonant circuit array. This situation is mimicking that a receiving coil is placed on the transmitting coils array. Figure 3(b) shows how the external coil interacts with each coil in the resonant circuit array. Each voltage of the capacitor becomes

\[
v_n = \sum_{m=1}^{N} \left( M_{nm} \frac{di_m}{dt} + M_{nR} \frac{di_R}{dt} \right),
\]

where \( i_R \) is the current flowing in the external coil and \( M_{nR} \) is the mutual inductance between \( n \)th coil and the external coil. The current \( i_R \) in Eq. (4) can be eliminated by using the following equation obtained by Kirchhoff’s voltage law for the external coil

\[
0 = L_R \frac{di_R}{dt} + \sum_{m=1}^{N} M_{mR} \frac{di_m}{dt}.
\]

Therefore, the differential equation (2) is modified as

\[
i_n = -C_n \left\{ \sum_{m=1}^{N} M_{nm} \frac{d^2 i_m}{dt^2} - \frac{M_{nR}}{L_R} \sum_{m=1}^{N} M_{mR} \frac{d^2 i_m}{dt^2} \right\} = -LC \sum_{m=1}^{N} (k_{nm} - k_{nR} k_{mR}) \frac{d^2 i_m}{dt^2},
\]

where \( k_{nm} \) is the coupling factor between \( n \)th and \( m \)th coils. Then \( k_{nn} = 1 \) and \( k_{nm} = k_{mn} \). Hereafter, \( LC \) is set at unity without loss of generality. The above equation is linear and can be represented by the matrix form like,

\[
i = -\begin{pmatrix}
1 & k_{12} & k_{13} & \cdots & k_{1N} \\
k_{12} & 1 & k_{23} & \cdots & \\
k_{13} & k_{23} & 1 & \cdots & \\
\vdots & \vdots & \vdots & \ddots & \\
k_{1N} & \cdots & \cdots & 1
\end{pmatrix}
\frac{d^2 i}{dt^2} = -K_0 \frac{d^2 i}{dt^2},
\]

where \( i = (i_1 \ i_2 \ \ldots \ i_N)^T \) and \( K_0 \) is \((N \times N)\) real symmetric matrix.

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\]

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\[
0 = L_R \frac{di_R}{dt} + \sum_{m=1}^{N} M_{mR} \frac{di_m}{dt}.
\]

Therefore, the differential equation (2) is modified as

\[
i_n = -C_n \left\{ \sum_{m=1}^{N} M_{nm} \frac{d^2 i_m}{dt^2} - \frac{M_{nR}}{L_R} \sum_{m=1}^{N} M_{mR} \frac{d^2 i_m}{dt^2} \right\} = -LC \sum_{m=1}^{N} (k_{nm} - k_{nR} k_{mR}) \frac{d^2 i_m}{dt^2},
\]

\[
\text{Fig. 3. Circuit diagram of the resonant circuit array with an external shorted coil.}
\]
where $M_{nR} = k_{nR}\sqrt{L_nL_R}$, $M_{mR} = k_{mR}\sqrt{L_mL_R}$, and all the capacitance and the inductance are uniform as mentioned above. Finally, a matrix form

$$i = -\begin{pmatrix}
1 - k_{1R}^2 & k_{12} - k_{1R}k_{2R} & k_{13} - k_{1R}k_{3R} & \cdots & k_{1N} - k_{1R}k_{NR} \\
 k_{12} - k_{1R}k_{2R} & 1 - k_{2R}^2 & k_{23} - k_{2R}k_{3R} & \cdots & \vdots \\
k_{13} - k_{1R}k_{3R} & k_{23} - k_{2R}k_{3R} & 1 - k_{3R}^2 & \cdots & \vdots \\
k_{1N} - k_{1R}k_{NR} & \cdots & \cdots & 1 - k_{NR}^2
\end{pmatrix} \frac{d^2i}{dt^2},$$

is obtained. $K_0$ is the coupling factor matrix of Eq. (3). The $(n, m)$ element of $K_1$ is $k_{nR}k_{mR}$ which is caused by mutual induction between each coil in the resonant circuit array and the external coil.

The coupling factor between two planar circular spiral coils $k$ is a nonlinear function with respect to the distance $x$ between them which is plotted in Fig. 4. The maximum of the coupling factor is located at $x = 0$. It becomes negative around $x \simeq D_{TX}/2$ and goes to zero as $x$ increases. The negative coupling factor means that the direction of the interlinkage magnetic flux is reversed as $x$ increases. If $c = 0$, then the interval of coils in the resonant circuit array becomes $d = D_{TX}$. For this case, $k_{nm} = k((n - m)D_{TX})$, and $k_{nm}$ is negligible for $|n - m| \geq 2$ as shown in the Fig. 4. Namely, only the nearest neighbor interaction in the transmitting coils is necessarily considered. Although the second nearest neighbor interaction $k_{n,n+2}$ becomes somewhat large when $c$ becomes close to 1/2, we simplify the matrix $K_0$ as tridiagonal matrix:

$$K_0 = \begin{pmatrix}
1 & k_N & 0 \\
k_N & 1 & k_N \\
0 & k_N & 1
\end{pmatrix},$$

where $k_N = k((1 - c)D_{TX})$.

![Fig. 4. Coupling factor between two planar circular spiral coils $k(x)$ and $k_{TR}(x)$, which is symmetrized by $k(x) = (k_{org}(x) + k_{org}(-x))/2$ for simplicity. The curves are obtained by numerically integrating the Neumann's formula. The red solid curve is the coupling factor between two identical coils. The diameter is $D_{TX} = 20$ mm. The blue dotted curve shows the coupling factor between an external coil of $D_{RX} = 35$ mm and a coil of $D_{TX} = 20$ mm in the resonant circuit array.](image-url)
3. Analytical solutions of localized modes

Localized modes are known to appear around an impurity or several impurities such as a difference of mass or spring constant in homogeneous coupled oscillators array. The frequency and the shape of the localized modes can be analytically obtained by using the Green’s function method [7]. Another technique using the transfer matrix has been presented in Ref. [8, 9] which is much simpler than the method of Green’s function. In addition, Teramoto and Takeno have introduced the time-dependent equations of motion for coupled oscillators array having impurities in Ref. [10]. Localized modes have been obtained as the asymptotic solutions of the time-dependent equations of motion [10, 11]. In this paper, a more straightforward method which is based on the eigen equation is used to derive analytical forms of localized modes.

3.1 Plane wave solution in resonant circuit array

For the case that no external coil is placed on the resonant circuit array, the equation of motion for the \( n \)th current is obtained as follows:

\[
i_n = \frac{d^2 i_n}{dt^2} - k_N \frac{d^2 i_{n+1}}{dt^2} - k_N \frac{d^2 i_{n-1}}{dt^2}.
\]  

(9)

Let us assume a plane wave, \( i_n(t) = a_0 \cos(\omega t - \kappa n) \), be a solution for Eq. (9), where \( \omega \) represents the angular frequency and \( \kappa \) the wavenumber of the traveling wave. The relationship between \( \omega \) and \( \kappa \), namely, the dispersion relation becomes

\[
\omega^2 = \frac{1}{1 + 2k_N \cos \kappa}.
\]  

(10)

Note that the absolute value of the coupling factor \( |k_N| \) must be less than 1/2 since the \( n \)th coil couples two neighboring coils, namely \((n \pm 1)\)th coils. The plane wave of \( \kappa = 0 \) has the minimum frequency \( \omega_0^2 = 1/(1 + 2k_N) \) if \( k_N > 0 \). On the other hand, the plane wave of \( \kappa = \pi \) has the maximum frequency \( \omega_{\pi}^2 = 1/(1 - 2k_N) \) for \( k_N > 0 \). The frequencies of the other plane waves whose wave number is \( 0 < \kappa < \pi \) are located between \( \omega_0 \) and \( \omega_{\pi} \). This is called the band in which the plane wave can transmit through the resonant circuit array. In other words, a solution of frequency inside the band has a spatially extended profile in the amplitude distribution. Note that the relationship between \( \omega_{\pi} \) and \( \omega_0 \) changes to \( \omega_{\pi} < \omega_0 \) for \( k_N < 0 \).

3.2 Localized mode centered on a site

Let us consider the simplest case that the external coil is placed concentrically with a coil in the resonant circuit array. Only the mutual inductance between the external coil and the coil beneath it is considered. Eq. (7) becomes

\[
i = - \begin{pmatrix}
\cdots & \cdots & \cdots & 0 \\
\cdots & k_N & k_N & 0 \\
0 & k_N & 1 - k^2_{\text{trans}} & k_N \\
& k_N & 0 & \cdots
\end{pmatrix} \begin{pmatrix}
\frac{d^2 i_1}{dt^2} \\
\frac{d^2 i_2}{dt^2} \\
\frac{d^2 i_3}{dt^2} \\
\vdots
\end{pmatrix} = -K_1 \begin{pmatrix}
\frac{d^2 i_1}{dt^2} \\
\frac{d^2 i_2}{dt^2} \\
\frac{d^2 i_3}{dt^2} \\
\vdots
\end{pmatrix}.
\]  

(11)

The local modification of the tridiagonal matrix \( K_0 \) will cause a localized mode which has localized profile in the amplitude distribution and its frequency locates outside the band [7]. Let us assume a localized mode \( i_1 \cos \omega_{\text{loc}} t \) where \( i_1 = a_0(\cdots b^2 b 1 b b^2 \cdots)^T \). \( a_0 \) is an arbitrary amplitude and \( b \) relates to the tail of the localized mode. The tail has to be decreased as the distance from the center of the localized mode increases. Therefore \( |b| \) must be less than 1. If the localized mode is a solution of Eq. (11), the vector \( i_1 \) satisfies
\[
\begin{align*}
i_\uparrow &= \omega_{\text{loc}}^2 K_1 i_\uparrow \\
\begin{pmatrix}
\vdots \\
b^2 \\
b \\
b^2 \\
\vdots
\end{pmatrix} &= \omega_{\text{loc}}^2 \begin{pmatrix}
k_N & k_N & & & 0 \\
& k_N & k_N & & \\
& k_N & 1 - k_{\text{TR0}}^2 & k_N \\
& & k_N & 1 \\
0 & & & & \\
\end{pmatrix} \begin{pmatrix}
\vdots \\
b^2 \\
b \\
b^2 \\
\vdots
\end{pmatrix}.
\end{align*}
\]

From this equation, two independent algebraic equations are obtained as follows:
\[
\begin{align*}
1 &= \omega_{\text{loc}}^2 \left(2bk_N + 1 - k_{\text{TR0}}^2\right), \\
b &= \omega_{\text{loc}}^2 \left(b^2k_N + b + k_N\right).
\end{align*}
\]

By solving this equation, two solutions of \(b \neq k_{\text{TR0}}\) are obtained. The tail of localized mode \(b^n\) has to be zero as \(n \to \infty\). Therefore,
\[
\begin{align*}
b &= \frac{k_{\text{TR0}}^2 \pm \sqrt{k_{\text{TR0}}^4 + 4k_N^2}}{2k_N}, \\
\omega_{\text{loc}}^2 &= \frac{1}{1 - \sqrt{k_{\text{TR0}}^2 + 4k_N^2}}.
\end{align*}
\]

The sign of the nearest neighbor coupling \(k_N\) determines the shape of the amplitude distribution and the frequency of the localized mode. For \(k_N > 0\), each current oscillates anti-phase to neighboring sites, while all the currents oscillate in-phase when \(k_N < 0\). The frequency of the localized mode is always above the band for any \(k_N \neq 0\).

The simplest case will be valid until the overlapping ratio of \(c\) is small. However, the mutual inductions between the external coil and the second nearest coils in the resonant circuit array, let’s say \(k_{\text{TR1}}\), have to be considered if the overlapping becomes large. The localized mode \(i_\uparrow\) is modified as
\[
i'_\uparrow = a_0(\cdots ab^2 \ a \ b \ a \ b \ ab^2 \cdots)^T,
\]
where \(1 > |a|\) and \(1 > |b|\), because we only focused on localized modes with a monotonically decaying tail. The tridiagonal matrix of Eq. (11) becomes
\[
K'_1 = \begin{pmatrix}
\cdots & \cdots & \cdots & \cdots \\
& k_N & 1 - k_{\text{TR1}}^2 & k_N - k_{\text{TR0}k_{\text{TR1}}} \;& -k_{\text{TR1}}^2 \\
& k_N - k_{\text{TR0}k_{\text{TR1}}} & 1 - k_{\text{TR0}}^2 & k_N - k_{\text{TR0}k_{\text{TR1}}} & k_{\text{TR0}k_{\text{TR1}}} \\
& -k_{\text{TR1}}^2 & k_N - k_{\text{TR0}k_{\text{TR1}}} & 1 - k_{\text{TR1}}^2 & k_N \\
0 & & & & \\
\end{pmatrix}.
\]

Therefore, modified algebraic equations are obtained from \(i'_\uparrow = \omega_{\text{loc}}^2 K'_1 i'_\uparrow\). For this case, three independent equations are obtained.
\[
\begin{align*}
\omega_{\text{loc}}^2 &= 1 + k_N \left(\frac{b}{b + 1}\right), \\
a &= \frac{k_Nb^2 + k_{\text{TR0}b} + k_N}{2(k_N - k_{\text{TR0}k_{\text{TR1}}})b}, \\
0 &= w_3b^3 + w_2b^2 + w_1b + w_0,
\end{align*}
\]
where \( w_0 = k_N, w_1 = k_{TR0}^2 + 2k_{TR1}^2, w_2 = 4k_{TR0}k_{TR1} - k_N, w_3 = 2k_{TR1}^2 \). The angular frequency coincides with \( \omega_0^2 \) or \( \omega_2^2 \) if \(|b| = 1\). The slope \( b \) is a real root of the cubic equation. By using Cardano’s method, explicit formulae of the slope will be obtained but they are not useful because the formulae are very complicated. In this paper, the roots of the cubic equation are obtained by computing eigenvalues of the companion matrix of the cubic equation. If three roots are all real, a root of the smallest absolute value is chosen for the slope \( b \). The most localized solution is only focused on in this paper because the strongly localized mode can be applicable to concentrate the magnetic flux around the receiving coil for the wireless power transfer system.

3.3 Localized mode centered between sites

When the external coil is placed at the center between neighboring coils in the resonant circuit array, Eq. (7) becomes

\[
\begin{pmatrix}
\ddots & \cdots & \ddots \\
 k_N & 1 & k_N \\
 k_N & 1 - k_{TR1/2}^2 & k_N - k_{TR1/2}^2 \\
 k_N & 1 - k_{TR1/2}^2 & k_N - k_{TR1/2}^2 \\
 0 & k_N & 1 \\
 & \ddots & \ddots & \ddots
\end{pmatrix}
\begin{pmatrix}
\vdots \\
 i_d \\
\vdots \\
 i_d \\
0
\end{pmatrix}
= \begin{pmatrix}
\vdots \\
k_{TR1/2} \ddots \\
0 \\
\ddots & \ddots & \ddots \\
0 & \ddots & \ddots
\end{pmatrix}
\begin{pmatrix}
\vdots \\
d^2i_d \\
\vdots \\
d^2i_d \\
0
\end{pmatrix}.
\]

(20)

Two types of localized mode can be considered for this situation.

3.3.1 Symmetric case

One of the localized mode can be assumed to be \( \begin{pmatrix}
\vdots \\
i_d \\
\vdots
\end{pmatrix} = a_0 \left( b^2 \ b \ 1 \ 1 \ b \ b^2 \ \cdots \right)^T \), which is called symmetric bond-centered mode in this paper. From \( \begin{pmatrix}
\vdots \\
i_d \\
\vdots
\end{pmatrix} = \omega_{loc}^2 K_{2} \begin{pmatrix}
\vdots \\
i_d \\
\vdots
\end{pmatrix} \), two equations

\[
\begin{cases}
1 = \omega_{loc}^2 \left( bk_N + 1 + k_N - 2k_{TR1/2}^2 \right), \\
b = \omega_{loc}^2 \left( b^2k_N + b + k_N \right)
\end{cases}
\]

are obtained. The slope of localized mode \( b \) and the frequency \( \omega_{loc}^2 \) are

\[
b = \frac{k_N}{k_N - 2k_{TR1/2}^2},
\]

\[
\omega_{loc}^2 = \frac{k_{TR1/2}^2}{k_N - 2k_{TR1/2}^2 + 1 + k_N - 2k_{TR1/2}^2} = \frac{1}{1 + k_N \left( b + \frac{1}{b} \right)}.
\]

(22)

(23)

where

\[
\begin{cases}
0 < b < 1 \quad \text{for} \quad k_N < 0, \quad k_{TR1/2} \neq 0 \\
-1 < b < 0 \quad \text{for} \quad 0 < k_N < k_{TR1/2}^2, \quad k_{TR1/2} \neq 0.
\end{cases}
\]

(24)

3.3.2 Antisymmetric case

Let us assume the antisymmetric bond-centered mode \( \begin{pmatrix}
\vdots \\
i_d \\
\vdots
\end{pmatrix} = a_0 \left( b^2 \ b \ 1 \ -1 \ -b \ -b^2 \ \cdots \right)^T \). By the same procedure mentioned above, the following two equations are obtained,

\[
\begin{cases}
1 = \omega_{loc}^2 \left( bk_N + 1 - k_N \right), \\
b = \omega_{loc}^2 \left( b^2k_N + b + k_N \right).
\end{cases}
\]

(25)

The mutual induction between the external coil and two nearest coils, the terms including \( k_{TR1/2} \) are canceled out. This means that the antisymmetric bond-centered mode \( \begin{pmatrix}
\vdots \\
i_d \\
\vdots
\end{pmatrix} \) does not exist. In fact, by solving Eq. (25), we obtain
\[ b = -1 \]
\[ \omega^2_{\text{loc}} = \frac{1}{1 - 2k_N} = \omega^2_{\pi}. \]

Therefore \( \mathbf{i}_{\uparrow \downarrow} \) corresponds to the zone boundary mode of \( \kappa = \pi \) and is not localized spatially.

### 3.3.3 Including next nearest interaction for symmetric case

As in Sec. 3.2, the next-nearest interaction between the resonant circuit array and the external coil is considered. The shape of the symmetric bond-centered mode becomes,

\[ \mathbf{i}'_{\uparrow \downarrow} = a_0 (\cdots ab^2 ab a 1 1 a ab^2 \cdots)^\top, \]

where \( 1 > |a| > |b| \). The tridiagonal matrix of Eq. (20) becomes

\[
K'_2 = \begin{pmatrix}
    k_N & 1 & 1-k^2_{\text{TR}/2} & k_N-k^{2}_{\text{TR}/2} & -k^{2}_{\text{TR}/2} & -k^{2}_{\text{TR}/2} \\
    1-k^2_{\text{TR}/2} & k_N & k_N-k^{2}_{\text{TR}/2} & 1-k^{2}_{\text{TR}/2} & k_N-k^{2}_{\text{TR}/2} & -k^{2}_{\text{TR}/2} \\
    k_N-k^{2}_{\text{TR}/2} & 1-k^{2}_{\text{TR}/2} & k_N & k_N-k^{2}_{\text{TR}/2} & 1-k^{2}_{\text{TR}/2} & k_N-k^{2}_{\text{TR}/2} \\
    -k^{2}_{\text{TR}/2} & k_N-k^{2}_{\text{TR}/2} & 1-k^{2}_{\text{TR}/2} & k_N & k_N-k^{2}_{\text{TR}/2} & -k^{2}_{\text{TR}/2} \\
    -k^{2}_{\text{TR}/2} & -k^{2}_{\text{TR}/2} & k_N-k^{2}_{\text{TR}/2} & 1-k^{2}_{\text{TR}/2} & k_N & k^2 \text{TR}/2 \\
0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}
\]

From \( \mathbf{i}'_{\uparrow \downarrow} = \omega^2_{\text{loc}} K'_2 \mathbf{i}'_{\uparrow \downarrow}, \)

\[
\begin{align*}
\omega^2_{\text{loc}} &= \frac{1}{1 + b \left( \frac{b + 1}{b} \right)}, \\
a &= \frac{k_N b^2 + (2k^2_{\text{TR}/2} - k_N)b + k_N}{(k_N - 2k^{2}_{\text{TR}/2}k^{2}_{\text{TR}/2})b}, \\
0 &= u_3 b^3 + u_2 b^2 + u_1 b + u_0,
\end{align*}
\]

where \( u_0 = k_N, \ u_1 = 2k^2_{\text{TR}/2} + 2k^2_{\text{TR}/2} - k_N, \ u_2 = 2k_{\text{TR}/2}(2k_{\text{TR}/2} - k_{\text{TR}/2}), \ u_3 = 2k^2_{\text{TR}/2}. \) The cubic polynomial of the slope \( b \) must be solved as well as in the case of the site-centered mode.

### 4. Comparison with numerical exact solutions

Localized mode caused by the external coil is obtained by computing eigenvalues and eigenvectors of the matrix \( K \) in Eq. (7). The eigenvalues \( \rho_\ell \) corresponds to inverse square of the angular frequency of \( \ell \)th mode, namely, \( \rho_\ell = 1/\omega^2_\ell \ (\ell = 1, \ldots, N). \) As mentioned in the previous section, the angular frequency of localized mode is always higher than the top of the band. Therefore, the eigenvector corresponding to the smallest eigenvalue will have a localized profile. Examples of localized modes are shown in Fig. 5. The numerically obtained eigenvectors are obviously localized around the position where the external coil is placed. Analytical solutions \( \mathbf{i}_\uparrow \) and \( \mathbf{i}_{\uparrow \downarrow} \), which are obtained by considering only the nearest neighbor interaction between the planar coils in the resonant circuit array and the external coil, are well fit to the numerical result. An error of the analytical solution \( \mathbf{i}_\uparrow \) from the numerical solutions are defined as \( e_\uparrow = || \mathbf{i}_\uparrow - \mathbf{i}'_\uparrow || \), where \( \mathbf{i}'_\uparrow \) is numerically obtained localized mode and its norm \( || \mathbf{i}'_\uparrow || \) is unity. \( a_0 \) of \( \mathbf{i}_\uparrow \) is set at \( a_0 = 1/|| \mathbf{i}'_\uparrow || \). In this case, \( e_\uparrow \approx 0.1129 \) and \( e_{\uparrow \downarrow} \approx 0.06372. \) By considering the second nearest-neighbor interactions, the errors are greatly reduced for both site- and bond-centered localized modes. As for the case shown in Fig. 5, \( e_\uparrow \approx 0.01101 \) and \( e_{\uparrow \downarrow} \approx 0.02418. \)

The matrices in Eqs. (11), (18), (20), and (29) do not include the next-nearest-neighbor interaction in the resonant circuit array \( k_{NN} \) which becomes prominent when \( c \) increases, for instance, \( k_{NN} \) at \( c = 0.5 \) is the same as \( k_N \) at \( c = 0. \) As shown in Fig. 6, the errors of the analytical solutions strongly depend on \( c. \) In particular they become large for \( c > 0.4. \) The red solid curves indicate the error of \( \mathbf{i}_\uparrow \). The values of error vary relatively large and it increases dramatically when \( c > 0.3. \) On the
Fig. 5. Examples of localized modes induced by an external coil placed on \( n = 10 \) for (a), between \( n = 10 \) and 11 for (b). Black circles and dotted lines are numerically exact solutions. Red triangles and lines are analytical solutions obtained by considering the nearest neighbor coupling between the external coil and the resonant array. Blue squares and dashed lines are results by considering the second nearest couplings. The overlapping ratio is set at \( c = 0.1 \). The number of coils in the resonant circuit array are \( N = 20 \). The diameter of the external coil is assumed to be \( D_{RX} = 35 \) mm.

Fig. 6. Relative errors between analytical solutions and numerically rigorous localized modes with respect to the ratio of overlapping \( c \). Simulation settings are the same as in Fig. 5 except \( c \).

On the other hand, the error of the improved solution \( i_1' \), drawn by the blue dashed curves, keeps quite low value in \( c < 0.4 \). Note that the intersection point of the red and blue curves in Fig. 6(a) corresponds to a critical value of \( c \) which makes \( k_{TR1} \) zero, namely, \( K_1 = K_1' \). The errors for the bond-centered mode are shown by the green dotted curves for \( i_{1\uparrow\uparrow} \) and the cyan dashed-dotted curves for \( i_{1\uparrow\uparrow}' \). The solution \( i_{1\uparrow\uparrow} \) is very precise until \( c \) is less than 0.4. When \( c > 0.4 \), the absolute value of the relative error of \( i_{1\uparrow\uparrow} \) becomes smaller than that of \( i_{1\uparrow\uparrow}' \). The reason is that \( |k_{NN} - k_{TR1}/k_{TR3/2}| \) is smaller than \( |k_{NN} + k_{TR1}/k_{TR3/2}| \) in Eq. (29) since \( k_{NN} \) increases and becomes comparable with \( k_{TR1}/k_{TR3/2} \). The corresponding element of \( K_2 \) is always zero. Therefore, \( i_1 \) is coincidently become more precise than \( i_{1\uparrow\uparrow} \) at \( c > 0.4 \).
Fig. 7. Relative errors of the angular frequency of a site-centered mode with respect to the ratio of overlapping $c$ and the diameter of the external coil $D_{RX}$. Red (Blue) colors indicate the analytically obtained angular frequency $\omega_1$ is greater (less) than that of numerically rigorous localized mode $\omega^*_1$. The scale of color bars between the above figures is adjusted to be the same.

The dependency of the relative error $\omega_1$ and $\omega'_1$ are computed for two-dimensional parameter space $(c, D_{RX})$. Only a narrow region in Fig. 7(a) is whitened where the solution $\omega_1$ is well fit to the numerically rigorous site-centered mode $\omega^*_1$. On the other hand, as shown in Fig. 7(b), the white region greatly expanded for $\omega'_1$. The blue-colored region appear only for $c > 0.4$ and $D_{RX} > 60$ mm. In this paper, the diameter of the circular coil in the resonant circuit array is fixed at $D_{TX} = 20$ mm. Then the third-nearest-neighbor interactions $k_{TR2} = k_{TR}(2D_{TX}(1-c))$ or more distant coil’s effect to the external coil are no longer negligible. A similar tendency is observed for the analytical solutions for the bond-centered mode.

5. Conclusion

Localized modes induced by placing an external coil to a resonant circuit array have analytically been derived by simplifying the coupling factor matrices $K$ appropriately. The simplest approximation leads to explicit formulae of the angular frequency $\omega$ and the decay rate $b$ of the tail of localized mode. The explicit representation enables us to consider existence regions of localized modes and the frequency difference between a site-centered and a bond-centered localized mode. However, increasing the ratio of overlapping of circular coils in the resonant circuit array $c$ makes the next-nearest-neighbor interaction $k_{NN}$ prominent. The simple solutions are not precise if $c$ becomes large.

Then the interaction between the external coil and the second closest coils in the resonant circuit array was considered. As a result, the third-order polynomials of the slope $b$ are obtained. Although an explicit formula can be write down, it will be very complicated and it will be difficult to derive the existence regions and the frequency difference. In this paper, the companion matrix of the polynomials is numerically solved. The computation cost is rather small than extracting the eigenvalue and the eigenvector from $N \times N$ matrix where $N$ is the number of coils in the resonant circuit array. Therefore, improved solutions are also useful for investigating the existence condition and the frequency difference. For the large diameter of the external coil $D_{RX}$, not only one or two coils beneath the external coil is affected but also three or more coils are coupled to the external coil. This implies that a much broader shape of amplitude distribution should be assumed. The improved solutions are valid until the diameter of the external coil is smaller than the three times of that in the resonant circuit array.

Analytical solutions are obtained only for the specific position of the external coil. It is a future work to relax the constraint of the position of the external coil. Besides, this paper derives the solutions only for one-dimensionally arranged resonant circuit arrays. The same method used in this paper will be applicable to localized modes in a two-dimensionally arranged system. We will try to construct two-dimensional localized mode solutions in the future. These solutions will be worthwhile to design a wireless power transfer system using localized modes in practical engineering even though
there exist parasitic resistances in transmitting and receiving coils. For high-efficiency wireless power transfer, high Q-factor coils are usually used \cite{12}. Thus, the parasitic resistances tend to be small, and the system will have resonant modes that are close to those in the conservative system discussed in this paper. Also, the receiving circuit in the practical wireless power transfer system requires a load resistor and a resonant capacitor. We will discuss in the future the design of the receiving circuit utilizing the localized modes.

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