Thick vortices in SU(2) lattice gauge theory

Srinath Cheluvaraja

Center for Computational Biology and Bioinformatics, University of Pittsburgh, Pittsburgh, PA, 15261

ABSTRACT

Three dimensional SU(2) lattice gauge theory is studied after eliminating thin monopoles and the smallest thick monopoles. Kinematically this constraint allows the formation of thick vortex loops which produce Z(2) fluctuations at longer length scales. The thick vortex loops are identified in a three dimensional simulation. A condensate of thick vortices persists even after the thin vortices have all disappeared. The thick vortices decouple at a slightly lower temperature (higher $\beta$) than the thin vortices and drive a phase transition.

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The role of thick vortices as a mechanism for confinement in $SU(2)$ LGT has been extensively discussed in the last few years [1]. Thick vortices are configurations analogous to domain walls in ferromagnets with a continuous symmetry—they are like thick Peierls contours. These vortices can reduce their free energy by increasing the size of their core [2]. However, the competition between the energy and the entropy prevents them from spreading indefinitely and a physical size emerges. At distances beyond this physical size an effective theory of vortices can be used to explain the long distance properties of the gauge theory. In pure $SU(2)$ gauge theories vortices are defined with respect to the center of the gauge group and have a $Z(2)$ magnetic charge. On the lattice, thick vortices should be distinguished from thin vortices. Thin vortices are like the vortices in a $Z(2)$ gauge theory— their core has a thickness of only one lattice spacing and they are suppressed by the plaquette term at large $\beta$. A thin vortex consists of a set of plaquettes with negative trace. It is well known that a dilute gas of such thin vortices produces an area law for the Wilson loop. Because of flux conservation such configurations can form (in three dimensions) closed loops or they can end at isolated points which absorb the vortex flux. These isolated points are called $Z(2)$ monopoles. The thin $Z(2)$ monopole density on a cube $c$ is given by

$$\rho_1(c) = (1/2)(1 - \text{sign}(\prod_{p \in \partial c} \text{tr}U(p))) . \quad (1)$$

In the above definition the product is taken over all the plaquettes bordering a cube. By definition, the monopole density is defined modulo 2. Now we turn to thick vortices. Thick vortices can be identified as sets of $dXd$ loop variables ($d \geq 2$) having negative trace. Unlike thin vortices the plaquettes in a thick vortex configuration always have a positive sign. It is clear that such configurations are only possible if the gauge group is continuous. A gas of such thick vortices will also disorder Wilson loops like thin vortices. Analogously, a thick vortex can end in a thick $Z(2)$ monopole. Its density (in three space-time dimensions) is given by

$$\rho_d(c_d) = (1/2)(1 - \text{sign}(\prod_{d \in \partial c_d} \text{tr}U(d))) , \quad (2)$$

where the product is now taken over all $dXd$ loops bordering a cube $c_d$ of side $d$. This density is again
defined modulo 2. The subscript \( d \) indicates that the density can be defined on any 3 dimensional cube of side \( d \). Thus, in three space-time dimensions thick monopoles have a finite size. They are monopoles because they violate the Bianchi identity. Densities of thin and thick vortices can be defined by counting the number of plaquettes and dXd loops respectively with negative trace. The interplay between monopoles and vortices prevents an understanding of the gauge theory exclusively in terms of one or the other. It should be clear that a whole hierarchy of such monopoles and vortices can be defined with different thicknesses. Unlike the thin monopoles which are suppressed by the plaquette action, the thick monopoles cost lesser energy because their energy is spread over a finite region. It was proposed in [2] that at low temperatures (larger \( \beta \)) vortices and monopoles of larger cross-section will dominate in determining long distance properties. These ideas were encapsulated in the effective \( Z(2) \) theory of confinement [3] where it was argued that the long distance properties of \( SU(2) \) theory becomes analogous to those of a \( Z(2) \) theory with a running coupling \( \beta(d) \) \( (\beta(d) \to \infty \text{ as } d \to \infty) \). As \( \beta \) increases the effective lattice spacing decreases and a thicker vortex will have the same size in physical units. If these vortices are condensed in the large \( \beta \) limit, the continuum limit at distances greater than the vortex size can be described by expanding around a vortex condensate. The above ideas summarise the effective \( Z(2) \) theory of confinement and it is an open question whether this scenario is in fact realised.

We should mention that there have been different approaches and results in the study of vortices in non-abelian gauge theories. The free energy of a thick vortex was calculated in [4] using twisted boundary conditions and it was shown to be finite. Another approach to defining vortices is by gauge fixing [6] which leaves a \( Z(2) \) symmetry intact. Vortices are defined in the gauge fixed theory just as in a \( Z(2) \) LGT. Studies of vortex excitations in this gauge have led to the observation of center dominance— the ungauged degrees (the center degrees of freedom) are sufficient to reproduce the string tension and some other non-perturbative quantities. More discussion of this approach can be found in [7][8][9]. Another approach developed in [5] and pursued further in [10] is to study monopoles and vortices using the \( Z(2) \times SO(3) \) decomposition of the \( SU(2) \) LGT. In this approach the role of the Wilson loop as a vortex counter is quite transparent and it is capable of handling thin and thick monopoles and vortices.
The presence of thick monopoles prevents the thick vortices from forming closed loops and makes it difficult to study their effects separately on the lattice. If the thick monopoles are removed it might be possible for us to see if thick vortices are present at all and then study their effects. With this aim in mind we study the $SU(2)$ theory after eliminating the thin and the smallest thick monopoles. This system can be regarded as a generalisation of the Mack-Petkova (MP) model where only thin monopoles are suppressed. In the MP model the elimination of thin monopoles increases the entropy of the vortex degrees of freedom at strong coupling and leads to a $Z(2)$ phase transition driven by thin vortices. We would like to see if the elimination of the thick monopoles leads to a separate transition driven by thick vortices. Just as in the Mack-Petkova model the elimination of the thin and thick monopoles changes only the short distance properties of the gauge theory and long distance properties like confinement will remain unaffected. This means that the physical continuum limit will still be at $\beta \to \infty$ but we hope to get a hint of the role of thick vortices at a larger value of $\beta$.

The model was simulated by introducing two chemical potential terms for the thin and thick monopole densities in addition to the Wilson action. The action used was

$$S = \frac{\beta}{2} \sum_p tr U(p) - \lambda_1 \sum_{c_1} \rho_1(c_1) - \lambda_2 \sum_{c_2} \rho_2(c_2).$$  \hspace{1cm} (3)

This model was studied in 3 dimensions mainly for reasons of computational speed and also because vortex lines are easier to identify than vortex sheets. Since in three dimensions the coupling constant is dimensionful the lattice spacing decreases by two if we increase $\beta$ by a factor of two. $\lambda_1, \lambda_2$ were chosen large so that the monopole densities are very small (in practice $\lambda_1 = \lambda_2 \approx 10$ resulted in identically zero densities for almost all thermalised configurations on $8^3$ lattices).

The simulation of this model presented its own share of difficulties. The environment of a single link is complicated because each link touches four thin monopoles and eight thick monopoles. Different strategies were attempted to simulate this model. Metropolis updating and a combination of heat bath and metropolis were both tried but metropolis updating was found to be more efficient provided the table of $SU(2)$ elements is
tuned regularly to get a reasonable acceptance. Unlike the pure SU(2) theory the lattice does not thermalise very quickly. It was found that in the phase where thick vortices are in abundance an ordered start (which has zero density of monopoles and vortices) took a longer time to reach the equilibrium distribution. Likewise, in the phase where thick vortices are absent a random start (which has a large number of thick vortices) took a longer time to reach the equilibrium distribution. These metastabilities do not arise in the pure SU(2) theory or the Mack-Petkova model and they appear to be linked to the thick vortices present in this model. Several approaches were tried to deal with this metastability like starting from configurations "closer" to the equilibrium configuration but the problem always remained. Nevertheless, with increased computational effort consistent results can be and were obtained. The simulation was performed in three dimensions mainly for reasons of computational speed. On the lattices used (8^3) it was found that simulating this model is more time consuming than simulating the four dimensional pure SU(2) model and hence we have confined most of our studies to 8^3 lattices although some preliminary results for 12^3 lattices were also performed.

Several observables like the plaquette, the density of thin and thick vortices, and Creutz ratios were studied in the simulations. Around $\beta \approx 2.9$ very long metastabilities were observed for the plaquette and 2X2 Wilson loops. At this point ordered and random configurations failed to converge and settled down to two different values as shown in Fig. 1. As we go away from this point in either direction these metastabilities disappeared and the two starts resulted in identical Monte-Carlo trajectories. For values of $\beta < 2.9$ a hot start was more efficient in reaching the equilibrium configuration while the opposite was true for $\beta > 2.9$.

With these observations in mind we plotted the plaquette and the density of thick vortices for different values of $\beta$. The jump in the plaquette and the density of thick vortices also occurs at $\beta \approx 2.9$ which is also the point where we observe two state behaviour. The density of thick (2X2) vortices drops abruptly to a very small value at the same value of $\beta$ as can be seen by comparing Fig. 2 and Fig. 3. The results in the graphs were obtained by doing simulations from 100,000 to 500,000 MC steps.

It is well known that 3 dimensional SU(2) LGT does not exhibit the crossover found in the four-dimensional theory but instead has a very smooth behaviour for the plaquette and other quantities. Elimination of only thin monopoles leads to a slight jump in the plaquette (at $\beta \approx 1.5$). The density of thin vortices
drops to zero at this point. The interesting result of our simulations is that eliminating thick monopoles causes another jump at a lower temperature (larger $\beta$). In the vicinity of this jump very long relaxation times are observed before a starting configuration reaches equilibrium. The sign of the $2X2$ Wilson loop is a rapidly fluctuating quantity near this transition and displays a more dramatic change than the plaquette (compare Fig. 2 with Fig. 3). It should be stressed that the thin vortices have all disappeared near this transition and the sign of the plaquette is a positive quantity in almost all configurations. $Z(2)$ monopoles of size $d = 1$ and $d = 2$ are also absent. The dominant configurations are those for which a $2X2$ Wilson loop has a negative sign such that all the plaquettes inside the loop have a positive sign—in other words, configurations which arise because the gauge group is continuous and non-abelian. In the graphs we do not include the points $\beta = 2.8$ and $\beta = 2.9$ because we observe strong metastability near the phase transition.

In three dimensions a doubling of $\beta$ corresponds to halving the lattice spacing and a thick vortex of twice the lattice spacing would have the same physical size as a thin vortex at the smaller $\beta$.

If $Z(2)$ fluctuations at longer length scales are present it is natural to ask if these fluctuations are caused by thick vortices. Indeed they are! Since vortex densities only provide information about the number of vortex strings present but do not give any information of the extent and size of the vortex loops we should track down the vortex loops and get a measure of the different loop sizes present. The set of $2X2$ loops with negative sign must form closed loops by flux conservation with the loops being defined on a lattice with twice the original lattice spacing. A measurement of the vortices on either side of the transition shows that they form very long loops in one phase and smaller loops in the other phase. The number of vortex loops $N(L)$ as a function of loop size $L$ at $\beta = 2.7$ on a $8^3$ lattice is given as follows:

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| L  | 8 | 12 | 16 | 20 | 28 | 32 | 36 | 44 | 60 | 64 |
|----|---|----|----|----|----|----|----|----|----|----|
| N(L)| 25| 12 | 12 | 6  | 1  | 3  | 1  | 4  | 1  | 1  |
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The same on a $12^3$ lattice:
The size of the largest loop increases with the lattice size and we expect it to be of thermodynamic significance.

A comparison of the first few Creutz ratios in the phase where vortices dominate with a nearby point in the vortex less also shows a reduction by a factor of more than four.

Determining the order of the phase transition needs a systematic finite size scaling study of the features observed on $8^3$ lattices. Studies on $12^3$ lattices show the same features observed on $8^3$ lattices. In particular, the metastability observed at $\beta = 2.9$ persists on the larger lattice and even the size of the discontinuity of the plaquette variable remains approximately the same. A clear signature of a first order transition is a double peaked histogram at the transition point. Despite many improvements in our Monte-Carlo algorithm we have been unable to observe any tunneling events and this has prevented us from making a more detailed study of the transition. Since thin $Z(2)$ vortices drive a second order transition in the 3 dimensional $Z(2)$ gauge theory, a first order transition due to thick vortices would imply that fluctuations of the non-centre degrees of freedom change the order of the transition.

The scaling properties of the $SU(2)$ theory only emerge in the $\beta \to \infty$ limit and the thick vortices we have discussed are irrelevant in that limit as they decouple at $\beta > 2.9$. However, at larger $\beta$ monopoles and vortices of greater thickness $d(\beta)$ will still be present and can be in a condensed phase. This can also be seen by studying the density of the vortices of thickness 3 and greater. These degrees of freedom are still important for $\beta > 2.9$ and they will decouple at a higher $\beta$ where thicker vortices will become the dominant excitations. As we approach the continuum limit there will be a hierarchy of such effective $Z(2)$ theories operating at larger and larger $\beta$ [3]; the model studied here is just the first member of the hierarchy. Eliminating monopoles of greater thickness should in principle unravel the entire hierarchy of $Z(2)$ like theories. If vortices are relevant for confinement the thick vortices will scale with $\beta$ so that a physical size of the order of a fermi results. This is one way in which the vortices of the $SU(2)$ LGT can escape the phase

| L | 8  | 12 | 16 | 20 | 24 | 28 | 32 | 36 | 40 | 48 | 52 | 56 | 60 |
|---|----|----|----|----|----|----|----|----|----|----|----|----|----|
| N(L) | 52 | 14 | 3  | 9  | 7  | 2  | 2  | 1  | 2  | 2  | 4  | 3  |    |

| L | 72 | 80 | 88 | 96 | 100 | 104 | 108 | 116 | 128 |
|---|----|----|----|----|-----|-----|-----|-----|-----|
| N(L) | 1  | 2  | 1  | 1  | 2   | 1   | 1   | 1   | 1   |
transition seen in the $Z(2)$ LGT and always be in the condensed phase. It is not easy to write down the effective $Z(2)$ theories for the different members of this hierarchy but the effective $Z(2)$ theory will be in its strong coupling phase \[4\] so that vortices are always abundant. In \[12\] an effective $Z(2)$ theory is derived for the $d = 1$ case in the strong coupling limit. Some properties of the effective $Z(2)$ theory (in four dimensions) are discussed in \[3\].

The main point of this investigation was to show that thick vortices emerge as important excitations once the thick monopoles are suppressed. The thick monopoles prevent the formation of long vortex loops and removing them enhances the vortex degrees of freedom. This leads to a jump in the value of the 2X2 Wilson loop at large $\beta$. The phase transition is driven by thick vortices, which unlike the thin vortices, are fluctuations at longer distances. The vortices responsible for these fluctuations can be explicitly identified. The string tension decreases dramatically across this transition as signalled by the change in theCreutz ratios. We have chosen to work in three dimensions mainly for reasons of computational speed and the relative ease with which vortex lines can be identified. In four dimensions, monopoles and vortices will be loops and sheets respectively.
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Figure 1: Metastability at $\beta = 2.9$.

Figure 2: Plaquette as a function of $\beta$.

Figure 3: Thick vortex density as a function of $\beta$. 

Figure 4: Sign of the 2X2 Wilson loop near the phase transition.