Separating Rank Logic from Polynomial Time

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Abstract—In the search for a logic capturing polynomial time the most promising candidates are Choiceless Polynomial Time (CPT) and rank logic. Rank logic extends fixed-point logic with counting by a rank operator over prime fields. We show that the isomorphism problem for CFI graphs over \( \mathbb{Z}_2^i \) cannot be defined in rank logic, even if the base graph is totally ordered. However, CPT can define this isomorphism problem. We thereby separate rank logic from CPT and in particular from polynomial time.

Index Terms—rank logic; polynomial time; choiceless polynomial time; CFI graphs

I. INTRODUCTION

The question of a logic capturing polynomial time (PTIME) is one of the central open questions in the field of descriptive complexity theory [1]. This question [2] asks whether there is a logic within which we can define exactly the polynomial-time computable properties of finite relational structures. The two most promising candidates for such a logic are Choiceless Polynomial Time and rank logic [3]. In this paper we rule out rank logic as a candidate. We show that rank logic neither captures PTIME nor Choiceless Polynomial Time.

Rank logic was introduced in [4] and extends fixed-point logic with counting (IFP+C) by a rank operator. Using this rank operator, the rank of definable matrices can be accessed in the logic. Multiple variants of rank logic were proposed. In its first version [4], rank logic comes with a rank operator \( \text{rk}_p \) for each prime \( p \). If the vertex set of a finite structure is \( A \), then an \( A^k \times A^k \) matrix is defined by a term \( \Phi(\bar{x}, \bar{y}) \) by setting the entry indexed by \( \bar{u}, \bar{v} \) to the value \( \Phi(\bar{u}, \bar{v}) \), to which \( \Phi \) evaluates in the structure. The rank operator \( \text{rk}_p \) evaluates to the rank of said matrix over \( \mathbb{F}_p \). When working with \( A^k \times A^k \) matrices, we call the rank operator \( k \)-ary.

Crucially, rank logic can define the isomorphism problem of the so-called CFI graphs or — for short — rank logic defines the CFI query. These graphs were given by Cai, Füredi, and Immmerman [5] to separate IFP+C from PTIME. From a base graph, one obtains a CFI graph by replacing every vertex with a particular gadget and by connecting the gadgets of adjacent vertices. A connection between two gadgets can either be straight or twisted. CFI graphs with the same parity of twisted connections are isomorphic. So for each base graph there is a pair of non-isomorphic CFI graphs. CFI graphs implicitly define a linear equation system over \( \mathbb{F}_2 \). These systems can be used to distinguish the non-isomorphic graphs. One graph satisfies the system and the other does not. This approach is expressible in rank logic by defining the matrix corresponding to the linear equation system and checking its rank.

CFI graphs can be generalized to work not only over \( \mathbb{F}_2 \) but over other fields \( \mathbb{F}_p \) or even groups (see e.g. [6], [7], [8]). Grädel and Pakusa [9] used these graphs to show that extending IFP+C by the rank operators \( \text{rk}_p \) is not sufficient to capture PTIME. They show that if the CFI graphs are defined over \( \mathbb{F}_p \) then rank operators \( \text{rk}_q \) for another prime \( q \neq p \) cannot define the CFI query over \( \mathbb{F}_p \). In particular, there is no formula defining the CFI query for all CFI graphs over an arbitrary prime field. An alternative variant of rank logic was proposed in [9], [10], [11], [12]. It replaces the rank operators \( \text{rk}_p \) for fixed fields by a unified rank operator \( \text{rk} \), where the formula defines the prime in terms of the structure. This second variant of rank logic can define the CFI query over all prime fields.

Another example demonstrating the expressiveness of rank logic are multipedes. These graphs come also with an isomorphism problem, which cannot be defined in IFP+C but in rank logic [13], [10]. Moreover, rank logic captures PTIME on the class of structures with color class size two [14]. An open question in [15] is whether rank logic can express the solvability of linear equation systems over finite rings.

In this paper, we show that rank logic fails to define the CFI query over the rings \( \mathbb{Z}_{2^i} \) for every \( i \in \mathbb{N} \). As in the case for fields, we consider the class of CFI graphs over all rings \( \mathbb{Z}_{2^i} \) and not a fixed one. This eliminates rank logic as a candidate for a logic capturing PTIME. As for \( \mathbb{F}_2 \), the isomorphism problem for CFI graphs over \( \mathbb{Z}_{2^i} \) can be translated to a linear equation system over \( \mathbb{Z}_{2^i} \). Hence, we also answer the question for solvability of linear equation systems over finite rings in the negative. Even more, we separate rank logic from Choiceless Polynomial Time and not only from PTIME.

Choiceless Polynomial Time (CPT) was introduced in [16]. It is a logic manipulating hereditarily finite sets and expresses all usual operations of finite sets. The key point is, that by definition of CPT it is impossible to pick an arbitrary element out of a set. If one wants to process an element in a set, one has to process all of them. This makes CPT choiceless and thereby isomorphism-invariant. CPT can define the isomorphism problem of CFI graphs over \( \mathbb{F}_2 \) if the base graph is totally ordered [17] (which is sufficient to separate IFP+C from PTIME). More generally, CPT captures PTIME on the class of structures with bounded color class size, where the automorphism group of each color class is abelian [14].

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This result is established by showing that CPT defines the solvability problem of a certain class of linear equation systems. Grädel and Grohe suggested that this class of equation systems might be a candidate for separating CPT from rank logic [3]. The result on bounded abelian color classes in [14] can be strengthened to not necessarily bounded color classes, as long as a total order of the automorphism group of each color class is given [12]. Using this result we show that CPT indeed defines the CFI query over \( \mathbb{Z}_2^i \) for every \( i \in \mathbb{N} \) for totally ordered base graphs. Hence, we separate rank logic not only from PTIME but also from CPT.

a) Our Techniques: We consider CFI graphs over \( \mathbb{Z}_2^i \). The automorphism groups of these graphs are 2-groups. We show that in this case formulas containing the uniform rank operator \( r_k \) without a fixed prime can be translated to formulas only using the \( r_k \) operator. This strengthens a result from [9] stating that on CFI graphs, whose automorphism group are \( p \)-groups, rank operators \( r_{k_q} \) for \( q \neq p \) can be simulated by IFP+C. Hence, if there is a formula defining the CFI query over \( \mathbb{Z}_2^i \), we can assume that it only uses the rank operator \( r_k \).

To prove that rank logic cannot define the CFI query over \( \mathbb{Z}_2^i \), we use game-based methods. For IFP+C there is the well-known Ehrenfeucht-Fraïssé-like pebble game, called the bijective pebble game [18]. It can be used to show that a property is not IFP+C definable. Such a game also exists for the extension by rank operators [19]. In the process of the game, ranks of matrices are computed, which are defined over the two graphs, on which the game is played. To show that a property is not definable in rank logic, it must always be possible to play in a way that the ranks of corresponding matrices for the two graph are equal.

During the rank-pebble game the entries of two matrices (one for each graph) are partitioned. Then one has to consider all labellings of the corresponding parts with values (so 0 and 1 in the case of \( \mathbb{F}_2 \)). This makes it in particular hard to prove that for every such labelling the two matrices have the same rank. To overcome this problem, we actually prove a stronger result. Two matrices have the same rank if and only if they are equivalent. This leads Dawar and Holm [19] to the invertible-map game, where instead of matrix equivalence matrix similarity is required. In fact, simultaneous similarity of two sequences of matrices is required. As matrix similarity implies matrix equivalence (and so equality of ranks), this game is potentially more expressive in the sense that it can distinguish more structures.

We use the invertible-map game to prove that there are non-isomorphic CFI graphs which cannot be distinguished by rank logic. The benefit of the invertible-map game is that we do not have to deal with all possible labellings of the values in a definable matrix. Instead, we can partition the matrix and then show that the corresponding parts of the two matrices are simultaneously similar. Indeed, we show that for every \( k \) there is an \( i \) such that Duplicator has a winning strategy in the \( k \)-ary invertible-map game played on CFI graphs over \( \mathbb{Z}_2^i \), whenever the base graph is sufficiently connected and its girth is sufficiently large. Requiring large connectivity is common for these arguments [5], [9], but the girth condition is specific for our construction.

The challenge for Duplicator in the invertible-map game is to come up with an invertible matrix proving simultaneous similarity of two sequences matrices. To construct such matrices, we use two ingredients. The first is a class of combinatorial objects, we call blurers. In some sense, blurers are symmetric objects, which contain a well-defined asymmetry. We use this asymmetry to “blur” the twist between two non-isomorphic CFI graphs over multiple edges in the graphs. Because of the symmetry of the blurers, the blurred twist cannot be detected in most cases and the graphs seem to be isomorphic. In particular, if we only consider 1-ary rank operators, blurers suffice to show that the CFI query is not definable. Considering \( k \)-ary rank operators for \( k > 1 \) becomes inherently more difficult. While the argument for the 1-ary case is a local argument, we have to consider \( k \)-tuples, where vertices of different parts of the graph are combined. We design and use blurers such that there is exactly one gadget at which the edges become apparent, between which the twist is blurred. Only in the case that both the column and the row index of the matrix fix a vertex of that gadget, blurers are not sufficient to prove similarity. In this situation, one can spot the edges, between which we blur the twist. Here we use the second idea. The critical indices all fix a vertex of the same gadget. If we fix one vertex of a gadget, then all vertices of that gadget can be distinguished, i.e., the gadget is separated into singleton orbits. Hence, we can make an arbitrary vertex of the critical gadget a parameter, recurse on the arity of the rank operator (as we now only have to consider \( (k-1) \)-tuples as indices), and obtain similarity matrices for \( (k-1) \)-ary rank operators. Each matrix blurs the “recursive” twist at one of the edges, to which the blurer moves the twist. Because the girth of the base graph is large enough, these similarity matrices “act” on “independent” parts of the graph and can be combined with the blurer to obtain a similarity matrix for the \( k \)-ary case.

b) Related Work: In [18] Hella shows a hierarchy for generalized Lindström quantifiers, showing that the expressiveness strictly increases with the arity. A similar result can also be given for rank logic [10], [11]. In that light the increased complexity of our approach for the \( k \)-ary case compared to the 1-ary case is not surprising.

Closely related to computing ranks is checking linear equation systems for solvability. Atserias, Bulatov, and Dawar proved that IFP+C does not define solvability of linear equation systems over finite rings [20]. Solvability of linear equation systems of prime-power fields is definable in rank logic [10]. So, as the CFI query corresponds to solving certain equation systems, a variant of rank operators using prime-power fields probably does not define the CFI query over \( \mathbb{Z}_2^i \).

While IFP+C fails to capture PTIME for CFI graphs, there are many other graph classes on which IFP+C captures PTIME. These include e.g. graphs with excluded minors [21] and graphs with bounded rank width [22]. While showing that rank logic defines the CFI query for prime fields is rather simple, for CPT this is a non-trivial result. The already mentioned result
by Dawar, Richerby, and Rossman [17] uses deeply nested sets and is restricted to totally ordered base graphs. This result was strengthened by Pakusa, Schalthöfer, and Selman [23] to base graphs with logarithmic color class size. Recently, the result for bounded abelian color classes by Abu Zaid, Grädel, Grohe, and Pakusa [14] was extended by Lichter and Schwichtenberg [24] to graphs with bounded color classes with dihedral colors and also for certain structures of arity three.

c) Structure of this Paper: We first introduce the variants of rank logic and the invertible-map game in Sections III and IV. Then we give a CFI construction suitable for our arguments in Section V. Next, we discuss matrices defined over CFI structures in Section VI and develop a criterion for invertibility of such matrices over $\mathbb{F}_2$. This will be used in Section VII, where we treat the case of 1-ary rank operators and introduce blurers. The following Section VIII defines the notion of the active region of a matrix, which is the part of a graph, on which it has a “non-trivial effect”. This is crucial for the case of general $k$-ary rank operators in Section IX to successfully combine the matrices obtained from recursion. There we also generalize blurers to the $k$-ary case. In Section X we separate rank logic from CPT. The full version of this paper containing all proofs is available at [25].

II. PRELIMINARIES

We denote with $[k]$ the set $\{1, \ldots, k\}$. Let $N$ and $I$ be finite sets. Then $N^I$ is the set of $I$-indexed tuples over $N$. For a tuple $t \in N^I$ the entry for index $i \in I$ is $t(i)$. For $k \in \mathbb{N}$, $t \in N^k = N^{[k]}$, and $i \leq k$ we write $t_i$ for the $i$-th entry. Let $I' \subseteq I$ and $\tilde{t} \in N^{I'}$. The restriction of $\tilde{t}$ to $I'$ is $\tilde{t}|_{I'} \in N^{I'}$. If $K$ is a set of subsets of $I$, we use $\bigcup_{K}$ for $\bigcup_{i \in K}$. For finite index sets $I$ and $J$, a $I \times J$ matrix $\mathcal{M}$ over $N$ is a mapping $\mathcal{M} : I \times J \to N$. We write $\mathcal{M}(i, j)$ for the entry at index $(i, j)$.

The identity matrix is $1$. The ring of integers modulo $j$ is $\mathbb{Z}_j$. Its elements are $\{0, \ldots, j-1\}$. For a tuple $\bar{a} \in \mathbb{Z}_j^I$ we set $\sum_{i \in I} a(i)$ and likewise for a function $g : I \to \mathbb{Z}_j$.

A (relational) signature $\tau = \{R_1, \ldots, R_k\}$ is a set of relation symbols associated with arities $\tau_i \in \mathbb{N}$ for all $i \in [k]$. A $\tau$-structure $\mathfrak{A}$ is a tuple $\mathfrak{A} = (A, R_1^{\mathfrak{A}}, \ldots, R_k^{\mathfrak{A}})$ where $R_i^{\mathfrak{A}} \subseteq A^{\tau_i}$ for all $i \in [k]$. The universe of $\mathfrak{A}$ is always denoted by $A$. We only consider finite structures. A pebbled structure is a pair $(\mathfrak{A}, \bar{u})$ of a relational structure and a tuple $\bar{u} \in A^k$. Two pebbled structures $(\mathfrak{A}, \bar{u})$ and $(\mathfrak{B}, \bar{v})$ are isomorphic, if there is an isomorphism $\varphi : A \to B$ such that $\varphi((\mathfrak{A}, \bar{u})) = (\mathfrak{B}, \bar{v})$.

Let $G = (V, E)$ be a simple graph. The distance of $x, y \in V$ in $G$ is $d_G(x, y)$. For sets $X, Y \subseteq V$ we set $d_G(X, Y) := \min_{x \in X, y \in Y} d_G(x, y)$ and likewise $d_G(x, Y)$. The set of neighbors of a vertex $x \in V$ is $N_G(x)$. The $k$-neighborhood of $x$ is $N_G^k(x) := \{y \in V \mid d_G(x, y) \leq k\}$. The graph $G$ is $k$-connected, if $|V| > k$ and if for every $V' \subseteq V$ of size at most $k-1$, $G - V'$ is connected. The girth of $G$ is the length of the shortest cycle in $G$. If $G$ has girth at least $2k + 1$, then for every $x \in V$ the induced subgraph $G[N_G^k(x)]$ is a tree.

Let $\Gamma$ be a finite permutation group with domain $N$. Let $p$ be a prime. If for every $\sigma \in \Gamma$ there is an $\ell$ such that $\sigma$ is of order $p^\ell$, then $\Gamma$ is called a $p$-group. The orbit of $n \in N$ is the set $\{m \in N \mid \sigma(n) = m \text{ for some } \sigma \in \Gamma\}$. A $k$-orbit is a maximal set $P \subseteq N^k$, such that for every $\bar{n}, \bar{m} \in P$, there is a $\sigma \in \Gamma$ such that $\sigma(\bar{n}) = \bar{m}$. We write $\operatorname{orb}_k(\Gamma)$ for the set of $k$-orbits of $\Gamma$. The group $\Gamma$ is transitive if $|\operatorname{orb}_1(\Gamma)| = 1$.

If additionally $|\Gamma| = |N|$, then $\Gamma$ is called regular.

III. RANK LOGIC

In this section we consider rank logic, an extension of inflationary fixed-point logic with counting by a rank operator. Let $\mathfrak{A} = (A, R_1, \ldots, R_k)$ be a $\tau$-structure. We set $\tau^\# := \tau \cup \{+, 0, 1\}$ and $\mathfrak{A}^\# := (A, R_1, \ldots, R_k, \mathbb{N}, +, -0, 1)$ to be the two-sorted $\tau^\#$-structure obtained from the disjoint union of $\mathfrak{A}$ and $\mathbb{N}$.

a) Fixed-Point Logic with Counting: We define IFP+C, the fixed-point logic with counting (proposed in [26]), also see [27]). IFP+C is a two-sorted logic using the signature $\tau^\#$ with $element$ variables ranging over the vertices of the input structure and $number$ variables ranging over the natural numbers. We use the letters $x$ and $y$ for element variables, the greek letter $\nu$ for a numeric variable, and letters $t$ and $r$ for numeric terms. For a tuple we write $\bar{x}$, $\bar{\nu}$, and $t$ respectively.

IFP+C formulas are built from first-order formulas, a fixed-point operator, and counting terms. To ensure polynomial-time evaluation, quantification over numeric variables needs to be bounded: Whenever $\varphi$ is an IFP+C formula, $\nu$ is a numeric, possibly free variable in $\varphi$ and $t$ is a closed numeric term, then $\exists \nu \leq t. \varphi$ is an IFP+C formula, where $Q \in \{\forall, \exists\}$. We now define (inflationary) fixed-points. Let $R$ be a relation symbol. We allow relations relating vertices with numbers. Let $\varphi$ be an IFP+C formula with free variables $\bar{x}$ and $\bar{\nu}$. Then $\exists \bar{x} \bar{\nu} R \bar{x} (\bar{\nu})$ is an IFP+C formula. Here, $t$ is a tuple of $|\bar{\nu}|$ closed numeric terms which bound the values of $\bar{\nu}$.

To relate element and numeric variables, IFP+C possesses counting terms that count the number of different values for some variables satisfying a formula. Let $\varphi$ be an IFP+C formula as above with free variables $\bar{x}$ and $\bar{\nu}$. Then $\# \bar{x} \bar{\nu} t. \varphi$ is a numeric IFP+C term. Again, $t$ is a tuple of closed numeric terms bounding the range of $\bar{\nu}$.

An IFP+C formula (or term) is then evaluated over $\mathfrak{A}^\#$. For a numeric term $\Phi(\bar{x}, \bar{\nu})$, we denote with $\Phi^\# : A^{[\bar{x}]} \times \mathbb{N}^{[\bar{\nu}]} \to \mathbb{N}$ the function, that maps the possible values of the free variables of $\Phi$ to the value that $\Phi$ takes in $\mathfrak{A}^\#$. Similarly, for a formula $\varphi(\bar{x}, \bar{\nu})$ we write $\varphi^\# : A^{[\bar{x}]} \times \mathbb{N}^{[\bar{\nu}]}$ for the set of values for the free variables satisfying $\varphi$. Then e.g. the evaluation $\Phi(\bar{x}, \bar{\nu})$ of a counting term for a formula $\varphi(\bar{x}, \bar{\nu})$ is defined as the number of $\bar{w} \in A^{[\bar{\nu}]} \times \mathbb{N}^{[\bar{\nu}]}$ such that $n_i \leq t_i$ for all $i \in |\bar{\nu}|$ and $\Phi(\bar{w})\bar{\nu} \in \varphi^\#$.

b) Rank Logic: We now consider the extension of IFP+C by a rank operator. We follow the definition in [10]. Let $\Phi(\bar{x}, \bar{y})$ be a numeric term such that $k := |\bar{x}| = |\bar{y}|$ and let $r$ be a closed numeric term. Then $\operatorname{rk}(\bar{x}, \bar{y}) := (\Phi, r)$ is a numeric term. The term $\Phi$ defines a $A^k \times A^k$ matrix $M^\Phi_{\bar{X}, \bar{Y}}$ over $\mathbb{N}$: $M^\Phi_{\bar{X}, \bar{Y}}(\bar{x}, \bar{y}) := \Phi(\bar{x}, \bar{y})$. Let $p := r^\Phi$ be prime. Then $\operatorname{rk}(\bar{x}, \bar{y}) := (\Phi, r)^\Phi$ is the rank of $(M^\Phi_{\bar{X}, \bar{Y}} \mod p)$ over $F_p$. For non-prime values of $p$, the value is 0. We omitted parameters for readability. We say for convenience that $k$ is the $arity$ of...
the operator, although it is actually $2k$. The logic IFP+R is the extension of IFP+C by the rank operator $rk$. The restriction to square matrices does not limit the expressive power.

We set IFP+$R\Omega$ for a set of prime numbers $\Omega$ to be the variant of IFP+$R$, in which we have instead of the rank operator $rk$ a different rank operator $rk_p$ working over $\mathbb{F}_p$ for every prime $p \in \Omega$. That is, we have to fix the field in the formula independently of the structure. This is not the case for the operator $rk$, where we can determine the value for $p$ by another term that evaluates differently for different structures.

c) Choiceless Polynomial Time: Choiceless Polynomial Time (CPT) is a logic different from IFP+C. CPT formulas manipulate hereditarily finite sets. They are choiceless in the sense that they either process all elements of such sets or none. It is not possible to pick an arbitrary element from a set. By these conditions, all sets constructed by a CPT term are closed under automorphisms of the input structure. Polynomial evaluation is guaranteed by explicit polynomial bounds and the number of steps and sizes of the constructed sets. We omit a formal definition of CPT here and refer to [3], [12].

A relational structure $\mathfrak{A}$ has $q$-bounded colors, if one relation is a total preorder partitioning the universe into equivalence classes, called color classes, of size at most $q$. It has abelian colors, if the automorphism group of the induced substructure of every color class is abelian.

**Theorem 1 ([14]).** CPT captures $\text{PTIME}$ on $q$-bounded relational structures with abelian colors.

This result can be slightly strengthened from bounded color class size to ordered colors. A structure $\mathfrak{A}$ has ordered colors, if for every color class $C$ there is a total order on the automorphism group of the induced substructure $\mathfrak{A}[C]$ of $C$.

**Theorem 2 ([12]).** CPT captures $\text{PTIME}$ on structures with ordered abelian colors.

### IV. The Invertible-Map Game

For IFP+C there is an equivalent Ehrenfeucht-Fraïssé-like game called the bijective pebble game [18]. It is a game between two players called Spoiler and Duplicator played on two pebbled structures. The aim of Spoiler is to prove that the two structures can be distinguished in IFP+C, where Duplicator tries to show the converse. Such a game also exists for rank logic (called matrix-equivalence game in [19]). It extends the bijective pebble game with ranks. Instead of looking at this pebble game, we consider the invertible-map game [18]. It is more expressive than the rank-pebble game in the sense that if Duplicator has a winning strategy in the invertible-map game, he has a winning strategy in the rank-pebble game, too. To show that rank logic cannot distinguish two structures, it suffices to show that Duplicator has a winning strategy in the invertible-map game. The game is defined as follows:

Let $k$ and $m$ be two positive integers with $2k \leq m$ and let $\Omega$ be a finite and nonempty set of primes. The invertible-map game $\mathcal{M}^{m,k,\Omega}$ is played on two pebbled structures $(\mathfrak{A}, \bar{a})$ and $(\mathfrak{B}, \bar{b})$ with $|\bar{a}| = |\bar{b}| \leq m$ of the same vocabulary. For each structure there are $m$ pebbles labeled with $1, \ldots, m$. On $a_i$ and $b_i$, there are pebbles with the same label for all $i \in [|\bar{a}|]$. If $|\bar{u}| < m$, not all pebbles are used. There are two players called Spoiler and Duplicator. If $|A| \neq |B|$, then Spoiler wins the game. Otherwise, a round of the game is played as follows:

First, Spoiler chooses a prime $p \in \Omega$ and picks up $2k$ pebbles from $\mathfrak{A}$ and the corresponding pebbles (with the same labels) from $\mathfrak{B}$. Second, Duplicator picks a partition $P$ of $A_k \times B_k$ and another one $Q$ of $B_k \times B_k$ such that $|P| = |Q|$. Furthermore, he picks an invertible matrix $S$ over $\mathbb{F}_p$, such that the matrix induces a total and bijective map $f: P \rightarrow Q$ and an invertible matrix $S$ satisfying $\chi^P = S \cdot \chi^Q \cdot S^{-1}$. Here $\chi^P$ (respectively $\chi^Q$) is the characteristic $A_k \times A_k$ matrix of $P$ (respectively the $B_k \times B_k$ matrix of $Q$), which we treat as a 0/1 matrix over $\mathbb{F}_p$. To say it differently, Duplicator can pick a bijection $f: P \rightarrow Q$ and an invertible matrix $S$ satisfying $\chi^P = S \cdot \chi^f(P) \cdot S^{-1}$ for all $P \in \mathcal{P}$, i.e., the characteristic matrices of $P$ and $Q$ are simultaneously similar. Lastly, Spoiler chooses a block $P \in \mathcal{P}$, a tuple $\bar{u} \in P$, and a tuple $\bar{v} \in f(P)$. Then for each $i \in [2k]$ she places a pebble on $u_i$ and the corresponding pebble on $v_i$.

After a round, Spoiler wins the game if the pebbles do not define a partial isomorphism or if Duplicator was not able to respond with a matrix satisfying the condition above. Duplicator wins the game if Spoiler fails to win forever. Duplicator has a winning strategy if he can win the game starting at $(\mathfrak{A}, \bar{u})$ and $(\mathfrak{B}, \bar{v})$ in any case independently of the actions of Duplicator. Likewise, Duplicator has a winning strategy, if he can always win the game. In that case, we write $(\mathfrak{A}, \bar{u}) \equiv_{m,k,\Omega} (\mathfrak{B}, \bar{v})$.

Finally, we consider the game with a bounded number of rounds: The $\ell$-round invertible-map game $\mathcal{M}^{m,k,\Omega}_\ell$ proceeds exactly as $\mathcal{M}^{m,k,\Omega}$ but stops after $\ell$ rounds. Duplicator wins, if Spoiler did not win in $\ell$ rounds. In the following, we use the invertible-map game instead of the rank-pebble game, because we prove a stronger result and also can simplify proofs.

**Lemma 3 ([19]).** Let $K$ be a class of finite $\tau$-structures and $P$ be a property of $K$-structures. If for every $k, m \in \mathbb{N}$ with $2k \leq m$ and every finite set of primes $\Omega$ there is a pair of structures $(\mathfrak{A}, \mathfrak{B})$, such that $\mathfrak{A}$ satisfies $P$, $\mathfrak{B}$ does not satisfy $P$, and $\mathfrak{A} \equiv_{m,k,\Omega} \mathfrak{B}$, then $P$ is not IFP+$R\Omega$ definable.

Here $\mathcal{P}$ is the set of all primes. If we fix a set of primes $\Omega$ in Lemma 3, then the property $P$ is not IFP+$R\Omega$ definable. Lemma 3 is proved in [19] for the $(m,k,\Omega)$-rank-pebble game, which induces the equivalence $\equiv_{m,k,\Omega}$. Then the authors show that $\equiv_{m,k,\Omega}$ refines $\equiv_{R}$. Whether the relation $\equiv_{m,k,\Omega}$ strictly refines $\equiv_{R}$ is an open problem.

### V. CFI Structures

We define a variant of the well-known CFI graphs. Starting from a base graph, for every vertex in the base graph a gadget is constructed. Then gadgets arising from adjacent vertices in the base graph are connected. In the seminal paper of Cai, Fürer, and Immerman [5] these gadgets consist of inner...
and outer vertices. A color class of outer vertices has automorphism group $\mathbb{Z}_2$ and a color class of inner vertices realizes the automorphism group $\{\bar{a} \in \mathbb{Z}_2^d | \sum \bar{a} = 0\}$ of its $d$ adjacent color classes of outer vertices. Two gadgets are connected by connecting the corresponding outer vertices. A connection can either by “straight” or “twisted”. This construction generalizes to other groups than $\mathbb{Z}_2$. In [8] a construction for the gadgets for abelian groups can be found. We are interested in cyclic groups $\mathbb{Z}_2^d$. The following construction only uses the inner vertices and directly connects the inner vertices of two gadgets. For $\mathbb{Z}_2$ this can be found in [28].

A base graph is a simple, connected, and totally ordered graph. Let $G = (V,E,\leq)$ be a base graph. Consider the additive group of $\mathbb{Z}_2^d$. For each vertex $x \in V$ we define a gadget consisting of vertices $A_x$ and two families of relations:

$$A_x := \{\bar{a} \in \mathbb{Z}_2^{dG(x)} | \sum \bar{a} = 0\}, \quad x \in V$$

$$I_{x,y} := \{(\bar{a}, \bar{b}) \in A_x^2 | \bar{a}(y) = \bar{b}(y)\}, \quad x,y \in V, y \in N_G(x)$$

$$C_{x,y} := \{(\bar{a}, \bar{b}) \in A_x^2 | \bar{a}(y) + 1 = \bar{b}(y)\}, \quad x,y \in V, y \in N_G(x)$$

Consider the sets $A_{x,y,c} := \{\bar{a} \in A_x | \bar{a}(y) = c\}$ for every $y \in N_G(x)$ and $c \in \mathbb{Z}_2^d$. The relation $I_{x,y}$ realizes these sets by disjoint cliques, one for each $A_{x,y,c}$. The relation $C_{x,y}$ induces a directed cycle $A_{x,y,c}, A_{x,y,c+1}, \ldots, A_{x,y,c+1}$ on these sets for a fixed $y$ by adding directed complete bipartite graphs between subsequent cliques. Thereby it realizes the group $\mathbb{Z}_2^d$ on the sets $A_{x,y,c}$. By the condition $\sum \bar{a} = 0$ on the vertices, the automorphism group of a gadget is isomorphic to $\{\bar{a} \in \mathbb{Z}_2^d | \sum \bar{a} = 0\}$ where $d$ is the degree of $x$.

Now we connect gadgets. Let $g : E \to \mathbb{Z}_2^d$ be a function defining the values by which the edges are twisted. For every edge $\{x,y\} \in E$ we connect the gadgets of the incident vertices. We obtain the CFI structure $\text{CFI}_{\mathbb{Z}_2^d}(G,g)$ as follows:

$$E_{(x,y),c} := \{(\bar{a}, \bar{b}) | \bar{a} \in A_x, \bar{b} \in A_y, \bar{a}(y) + \bar{b}(x) = c\},$$

$$\preceq := \{(\bar{a}, \bar{b}) | \bar{a} \in A_x, \bar{b} \in A_y, x \leq y\},$$

$$R_I := \{(\bar{a}, \bar{b}, \bar{a}', \bar{b}') | (\bar{a}, \bar{b}) \in I_{x,y}, (\bar{a}', \bar{b}') \in I_{x',y'},$$

$$(x,y) \leq (x',y')\},$$

$$A := \bigcup_{x \in V} A_x, \quad R_{E,c} := \bigcup_{e \in E} E_{e,c+g(e)}$$

where $\{x,y\} \in E$ and $c \in \mathbb{Z}_2^d$. The unions above are meant to be disjoint. The relation $R_C$ is defined similarly to $R_I$ for the $C_{x,y}$. The $I_{x,y}$ and $C_{x,y}$ are obtained as the equivalence classes of $R_I$ and $R_C$. We say that vertices $\bar{a} \in A_x$ originate from $x$ or that their origin is $x$ and write $\text{orig}(\bar{a}) := x$. We extend this to tuples and define the origin of $(\bar{a}_1, \ldots, \bar{a}_j) \in A^j$ as $\text{orig}((\bar{a}_1, \ldots, \bar{a}_j)) := (\text{orig}(\bar{a}_1), \ldots, \text{orig}(\bar{a}_j))$. We will often use this as the set $\{\text{orig}(\bar{a}_1), \ldots, \text{orig}(\bar{a}_j)\}$ and write $x \in \text{orig}(\bar{a})$. If $A$ is a set of tuples with the same origin, we set $\text{orig}(A) = \text{orig}(\bar{a})$ for some (and thus all) $\bar{a} \in A$.

For CFI structures it is well-known that $\text{CFI}_{\mathbb{Z}_2^d}(G,g) \cong \text{CFI}_{\mathbb{Z}_2^d}(G,f)$ if and only if $\sum e \in E g(e) = \sum e \in E f(e)$. So, up to isomorphism, there are $2^\theta$ many CFI structures of $G$.

Other CFI Constructions: We relate our CFI construction to other ones. The sets $A_{x,y,c}$ in our construction correspond to the outer vertices in the classical construction. We only use one type of vertices to avoid case distinctions. One can alternatively use outer vertices and replace the inner vertices by relations of higher arity (see e.g. [18]). The arity depends on the degree of the base graph. For our purpose, the signature has to be independent of the degree. Our construction always has arity 4. But the number of relations varies with the group $\mathbb{Z}_2^d$. We could use a single ternary relation encoding the $R_{E,c}$. It suffices to use $R_{E,0}$ only to obtain a structure with the same automorphism group. The other $R_{E,c}$ are definable in IFP+C. But with all the $R_{E,c}$ included, local isomorphism types will be more informative (cf. Section V-B). Most properties of the structures transfer between the different constructions. All lemmas in this section are based on known ideas.

Lemma 4. The automorphism group of $\text{CFI}_{\mathbb{Z}_2^d}(G,g)$ is an abelian 2-group.

A. Isomorphisms

Let $x \in V$ and $\bar{a} \in \mathbb{Z}_2^{dG(x)}$ satisfy $\sum \bar{a} = 0$. We define an action of $\bar{a}$ on vertices of $\mathbb{A}$ with origin $x$. Let $A_x \subseteq A$ be the vertices with origin $x$ and $u \in A_x$. Then we define $\bar{a}(u) := u$ such that $v(y) = u(y) + \bar{a}(y)$ for all $y \in N_G(x)$. Because $\sum \bar{a} = 0$, we have $\bar{a}(u) \in A_x$, too. Let $f, g : E \to \mathbb{Z}_2^d$. An edge $e \in E$ is twisted if $f(e) \neq g(e)$.

Definition 5 (Path Isomorphism). Let $\bar{s} = (x_1, \ldots, x_n)$ be a simple path in $G$. Let for $1 < i < n$ the tuple $\bar{a}_i \in \mathbb{Z}_2^{dG(x_i)}$ satisfy $\bar{a}_i(x_{i-1}) = c_i$, $\bar{a}_i(x_{i+1}) = -c_i$, and $\bar{a}_i(y) = 0$ for all other $y \in N_G(x_i)$. The path isomorphism $\bar{\pi}[\bar{s}]$ is defined by

$$\bar{\pi}[\bar{s}][u] := \begin{cases} \bar{a}_i(u) & \text{if orig}(u) = x_i \text{ and } 1 < i < n, \\ u & \text{otherwise}. \end{cases}$$

Let $e_1 = \{x_1, x_2\}$ and $e_2 = \{x_{n-1}, x_n\}$. If exactly $e_1$ and $e_2$ are twisted and $g(e_1) = f(e_1) + c$ and $g(e_2) = f(e_2) - c$, then $\bar{\pi}[\bar{s}]$ is an isomorphism $\text{CFI}_{\mathbb{Z}_2^d}(G,f) \to \text{CFI}_{\mathbb{Z}_2^d}(G,g)$. Isomorphisms between CFI structures satisfying $\sum f = \sum g$, which are twisted at most two edges, can be composed by multiple path isomorphisms. A special case of such isomorphisms will play an important role later.

Definition 6 (Star Isomorphism). Let $z \in V$, $\bar{s}_1, \ldots, \bar{s}_\ell$ be simple paths, $\bar{s}_i = (x_1^{i}, \ldots, x_{\ell}^{i})$, $\ell \leq d$, $x_i^{\ell} = z$ for all $i \in [\ell]$, and the $\bar{s}_i$ be disjoint apart from $z$. We call $\bar{s}_1, \ldots, \bar{s}_\ell$ a star.

For $c \in \mathbb{Z}_2^d$ such that $\sum c = 0$ we define the star-isomorphism

$$\bar{\pi}^*[\bar{s}_i, \ldots, \bar{s}_\ell][u] := \begin{cases} \bar{c}^*[u] & \text{if orig}(u) = z, \\ \bar{\pi}[\bar{s}_i][u] & \text{if orig}(u) \in \bar{s}_i \setminus \{z\}, \\ u & \text{otherwise}, \end{cases}$$

where $\bar{c}^* \in \mathbb{Z}_2^{dG(z)}$ such that $\bar{c}^*(x_{i-1}^{\ell}) = c_i$ for all $i \in [\ell]$ and $\bar{c}^*(y) = 0$ for all other $y \in N_G(z)$.

Let $e_1 = \{x_1^{i}, x_2^{i}\}$ for all $i \in [\ell]$. If exactly the $e_i$ are twisted and $g(e_i) = f(e_i) + c_i$ for all $i \in [\ell]$, then $\bar{\pi}^*[\bar{s}_1, \ldots, \bar{s}_\ell]$ is an isomorphism $\text{CFI}_{\mathbb{Z}_2^d}(G,f) \to \text{CFI}_{\mathbb{Z}_2^d}(G,g)$. 
B. Orbits of CFI Structures

Let $k,m \in \mathbb{N}$, $G = (V,E,\leq)$ be a $(k+m+1)$-connected base graph, $f: E \to \mathbb{Z}_{2n}$, $\mathcal{A} = \text{CFI}_{2s}(G,f)$, and $\bar{p} \in A^m$. We analyze the structure of $k$-orbits of the pebbled structure $(\mathcal{A},\bar{p})$, i.e., orbits of $k$-tuples. Let $\text{Aut}(\mathcal{A},\bar{p})$ be the automorphism group of $(\mathcal{A},\bar{p})$ and $\text{orbs}_k((\mathcal{A},\bar{p}))$ be the set of all $k$-orbits. This is a partition of $A^k$, such that $\bar{u},\bar{v} \in A^k$ are in the same orbit if and only if there is a $\varphi \in \text{Aut}(\mathcal{A},\bar{p})$ such that $\varphi(\bar{u}) = \bar{v}$.

**Definition 7** (Type of a Tuple). The isomorphism type of a pebbled structure is the class of all isomorphic structures. The type of a tuple $\bar{u} \in A^k$ in $(\mathcal{A},\bar{p})$ is the pair $(\text{orig}(\bar{u}), T)$, where $T$ is the isomorphism type of $(\mathcal{A}[\bar{p}\bar{u}],\bar{p}\bar{u})$. We only say type of $\bar{u}$ if the corresponding structure is clear.

Using known arguments [7] we show that for every $\bar{u},\bar{v} \in A^k$ there is an automorphism $\varphi \in \text{Aut}(\mathcal{A},\bar{p})$ such that $\varphi(\bar{u}) = \bar{v}$ if and only if $\bar{u}$ and $\bar{v}$ have the same type.

**Corollary 8.** For every $P \in \text{orbs}_k((\mathcal{A},\bar{p}))$ there is a type such that $P$ contains exactly the tuples of that type.

**Definition 9** (Type of an Orbit). The type of a $k$-orbit in $(\mathcal{A},\bar{p})$ is the type of its contained tuples.

**Corollary 10.** For every pair $g,h: E \to \mathbb{Z}_{2s}$, which does not twist an edge contained in $\text{orig}(\bar{p})$, it holds that $\text{orbs}_k((\text{CFI}_{2s}(G,g),\bar{p})) = \text{orbs}_k((\text{CFI}_{2s}(G,h),\bar{p}))$ and $\text{Aut}(\text{CFI}_{2s}(G,g),\bar{p})) = \text{Aut}(\text{CFI}_{2s}(G,h),\bar{p}))$.

While the orbit partitions of $(\text{CFI}_{2s}(G,g),\bar{p})$ and $(\text{CFI}_{2s}(G,h),\bar{p})$ are equal, it is not true that an orbit $P \in \text{orbs}_k((\text{CFI}_{2s}(G,g),\bar{p}))$ has the same type in $(\text{CFI}_{2s}(G,h),\bar{p})$ and in $(\text{CFI}_{2s}(G,h),\bar{p})$.

**Lemma 11.** Suppose $g,h: E \to \mathbb{Z}_{2s}$ do not twist an edge contained in $\text{orig}(\bar{p})$. Then for every $P \in \text{orbs}_k((\text{CFI}_{2s}(G,g),\bar{p}))$ there is a $Q \in \text{orbs}_k((\text{CFI}_{2s}(G,h),\bar{p}))$ of the same type.

**Lemma 12.** Let $P \in \text{orbs}_k((\mathcal{A},\bar{p}))$ and $\Gamma$ be the permutation group on $P$ induced by $\text{Aut}(\mathcal{A},\bar{p})$. Then $\Gamma$ is a regular abelian 2-group.

C. Composition of Orbits

For our arguments, $k$-orbits, whose origins do not induce a connected subgraph in the base graph $G$, play a special role. We now analyze their structure. Let $\bar{u} \in A^k$ and $N,M$ be a partition of $\text{orig}(\bar{u})$ into two parts (as a set). We introduce notation for splitting $\bar{u}$ into its parts of $N$ and $M$ and for recovering $\bar{u}$ again. We write $\bar{u}_N$ for the tuple obtained from $\bar{u}$ by deleting all entries, whose origin is not in $N$ and likewise for $M$. We define a concatenation operation $\bar{u}_N \cdot \sigma \bar{u}_M := \sigma(\bar{u}_N \cdot \bar{u}_M)$ for a permutation $\sigma$ of $[k]$. For a suitable $\sigma$ we have $\bar{u} = \bar{u}_N \cdot \sigma \bar{u}_M$. We are interested in permutations satisfying the former equation. Then $\sigma$ is almost always fixed by the context and we use juxtaposition $\bar{u}_N \bar{u}_M$. We define a similar operation $\bar{u}_N \cdot \bar{u}_M$ for orbits: Let $P \in \text{orbs}_k((\mathcal{A},\bar{p}))$. Then $P|_N$ denotes the set $\{\bar{u}_N \mid \bar{u} \in P\}$. We define $P|_N \cdot \sigma P|_M := \{\bar{u}_N \cdot \sigma \bar{u}_M \mid \bar{u}_N \in P|_N, \bar{u}_M \in P|_M\}$ and leave out the $\sigma$ if clear from the context. We also use this notation if $N$ and $M$ are sets of sets, such that $\bigcup N$ and $\bigcup M$ form a partition of $\text{orig}(\bar{u})$.

**Definition 13** (Components of Tuples and Orbits). Let $\bar{u} \in A^k$ and $N \subseteq \text{orig}(\bar{u})$. We call $\bar{u}$ a component of $\bar{u}$ if $N$ is a connected component of $G[\text{orig}(\bar{u})]$. We call $\bar{u}$ disconnected if it has more than one component. Likewise, a $k$-orbit $P \in \text{orbs}_k((\mathcal{A},\bar{p}))$ is disconnected if $P$ contains some (and thus only) disconnected tuples. A set $N \subseteq \text{orig}(P)$ is a component of $P$ if $N$ is a connected component of $G[\text{orig}(P)]$.

If a $k$-orbit $P$ is disconnected, then $P$ can be split into multiple $k'$-orbits for $k' < k$.

**Lemma 14.** Let $P \in \text{orbs}_k((\mathcal{A},\bar{p}))$ and $M$ and $N$ be a partition of the components of $P$. Then $P = P|_M \cdot \sigma P|_N$ and $P|_M \in \text{orbs}_k((\mathcal{A},\bar{p}))$ and $P|_N \in \text{orbs}_k((\mathcal{A},\bar{p}))$ for suitable $k_m$ and $k_n$ such that $k_m + k_n = k$.

Another important case for our arguments are $k$-orbits, where we pick a vertex $z$ in their origin and fix some entries with origin $z$. Then we obtain orbits of $(\mathcal{A},\bar{p}w)$ with an additional parameter $w$ with origin $z$.

**Lemma 15.** Let $P \in \text{orbs}_k((\mathcal{A},\bar{p}))$, $K \subseteq [k]$, and $\text{orig}(P|_K) = \{z\}$. For every $\bar{u} \in A^{|K|}$ and $w \in A$ with $\text{orig}(\bar{w}) = \{z\}$ and $\text{orig}(w) = \bar{w}$ the set $Q := \{\bar{u}|_{[k]-K} \mid \bar{u} \in P, \bar{u}|_K = \bar{w}\}$ satisfies $Q \in \text{orbs}_k((\mathcal{A},\bar{p}w)) \cup \{\emptyset\}$.

D. Rank Logic on CFI Structures

**Definition 16.** Let $\mathcal{K}$ be class of base graphs. Then $\text{CFI}_{2s}(\mathcal{K}) := \{\text{CFI}_{2s}(G,g) \mid G = (V,E,\leq) \in \mathcal{K}, q \in \mathbb{N}, g: E \to \mathbb{Z}_{2s}\}$ is the class of all CFI structures over $\mathcal{K}$.

The following lemma refines a result of [9], [7].

**Lemma 17.** Let $\mathcal{K}$ be a class of base graphs and $\varphi$ an IFP+$\forall\exists$ formula. Then there is a IFP+$\forall\exists\{\forall\}$ formula $\psi$, that is equivalent to $\varphi$ on $\text{CFI}_{2s}(\mathcal{K})$.

VI. Matrices over CFI Structures

Let $G = (V,E,\leq)$ be a base graph, $\mathcal{A} = \text{CFI}_{2s}(G,f)$, and $\mathcal{B} = \text{CFI}_{2s}(G,g)$ with universes $A$ and $B$ for some functions $f,g: E \to \mathbb{Z}_{2s}$. Furthermore, let $k, m \in \mathbb{N}$, $\bar{p} \in A^m$, and $S$ be a $A^k \times B^m$ matrix over $\mathbb{F}_2$. In the invertible-map game, Duplicator has to come up with an invertible matrix in each round for some partitioning of the $2k$-tuples. We want that Duplicator plays with the orbit partitions. In this section we develop a criterion for invertibility of such matrices. Let $P_j = \text{orbs}_j((\mathcal{A},\bar{p}))$ and $Q_j = \text{orbs}_j((\mathcal{B},\bar{p}))$ for $j \in [k,2k]$.

**Definition 18** (Blurring the Twist). The matrix $S$ $k$-blurs the twist (between $(\mathcal{A},\bar{p})$ and $(\mathcal{B},\bar{p})$) if $S$ is invertible and $\chi^P \cdot S = S \cdot \chi^Q$ for every $P \in P_{2k}$ and $Q \in Q_{2k}$ which are of the same type.
Note that by Corollary 8 two different orbits have different types and that by Lemma 11 for each $P \in \mathcal{P}_{2k}$ there is a $Q \in \mathcal{Q}_{2k}$ of the same type. So we indeed get a bijection between the orbits and Duplicator can use the matrix $S$ in the invertible-map game. Because $S$ is invertible, $\chi^P, S = S, \chi^Q$ is equivalent to $\chi^P = S, \chi^Q, S^{-1}$. By showing the former we do not need the inverse $S^{-1}$ explicitly. We introduce combinatorial conditions on $S$, which guarantee that $S$ is invertible. The set of orbits $\mathcal{P}_k$ and $\mathcal{Q}_k$ partition $S$ into a block matrix. Each $P \in \mathcal{P}_k$ corresponds to a subset of the rows of $S$ and each $Q \in \mathcal{Q}_k$ corresponds to a subset of the columns of $S$. We denote with $S_{P \times Q}$ the corresponding submatrix of $S$.

**Definition 19 (Orbit-Diagonal Matrix).** We call $S$ orbit-diagonal (over $(\mathfrak{A}, \bar{p})$ and $(\mathfrak{B}, \bar{p})$), if $S_{P \times Q} \neq 0$ if and only if $P$ and $Q$ are orbits of the same type.

As already seen, for $P \in \mathcal{P}_k$ there is exactly one $Q \in \mathcal{Q}_k$ of the same type. So orbit-diagonal matrices are block-diagonal matrices, where orbits of the same type build the blocks.

**Definition 20 (Orbit-Invariant Matrix).** The matrix $S$ is called orbit-invariant, if for every $P \in \mathcal{P}_k$, $Q \in \mathcal{Q}_k$, and every $\varphi \in \text{Aut}(\mathfrak{A}, \bar{p}) = \text{Aut}(\mathfrak{B}, \bar{p})$ (cf. Corollary 10) the matrix $S$ satisfies $\varphi(S_{P \times Q}) = S_{P \times Q}$.

Note the difference between an orbit-invariant matrix and e.g. an automorphism invariant matrix. An automorphism invariant matrix has to be invariant under applying the same automorphism to the matrix. For $P, P' \in \mathcal{P}_k$ and $Q, Q' \in \mathcal{Q}_k$ such that $P$ and $Q$ (respectively $P'$ and $Q'$) have the same type, an orbit-invariant matrix must be invariant under applying an automorphism $\varphi$ to the $P \times Q$ block and another automorphism $\psi$ to the $P' \times Q'$ block, even if there is no single automorphism inducing the action of $\varphi$ and $\psi$ on the corresponding blocks.

**Definition 21 (Odd-Filled Matrix).** A matrix over $\mathbb{F}_2$ is odd-filled if for every row the number of ones it contains is odd.

**Lemma 22.** If $S$ is orbit-diagonal, orbit-invariant, and odd-filled, then $S$ is invertible.

## VII. The Arity 1 Case

In this section we show how Duplicator can "survive" one round in the $\mathcal{M}_{m,1}^{(2)}$ game played on CFI structures over $\mathbb{Z}_{2^q}$ for $q \geq 2$. This corresponds to IFP+$\mathcal{R}_{(2)}$ only with rank operators of arity 1. Constructing a winning strategy will then be straightforward using existing techniques. The arity 1 case introduces basic techniques and serves as a base case for higher arities later.

Let $q \geq 2$, $m \in \mathbb{N}$, $G = (V, E, \leq)$ be a $(m + 3)$-connected base graph, $t \in V$ be of degree $d \geq 3$, and $\{t, t'\} \in E$. The connectivity is needed to apply the lemmas of Section V-B. Let $f, g : E \rightarrow \mathbb{Z}_{2^q}$ such that $g(\{t, t'\}) = f(\{t, t'\}) + 2^{q-1}$ and $g(e) = f(e)$ for all $e \in E \setminus \{t, t'\}$. We use $m$ as the number of parameters and play with $m + 2$ pebbles. We set $\mathfrak{A} = \text{CFI}_{2^q}(G, f)$ and $\mathfrak{B} = \text{CFI}_{2^q}(G, g)$. Let $\bar{p} \in A^m = B^m$ such that $\text{dist}_G(t, \text{orig}(\bar{p})) \geq 3$. For $x \in V$ let $A_x = B_x$ be the vertices of the two structures originating from $x$, i.e. the vertices of the gadget for $x$.

The key idea is to "distribute" the twist over multiple edges, such that it cannot be detected by Spoiler. For this, we introduce blurers, the key ingredient to define the desired similarity matrix.

**Definition 23.** Let $\Xi \subseteq \mathbb{Z}_{2^q}^d$. For $b \in \mathbb{Z}_{2^q}$ and $j \in [d]$ we define $\#_{j,b}(\Xi) := |\{a \in \Xi | a_j = b\}|$ mod 2. The set $\Xi$ is called a $(q, d)$-blurer if it satisfies

1. $\sum \xi = 0$ for all $\xi \in \Xi$,
2. $\#_{1,2^q-1}(\Xi) = 1$,
3. $\#_{j,0}(\Xi) = 1$ for all $1 \leq j \leq d$, and
4. $\#_{j,b}(\Xi) = 0$ for all other pairs of $b \in \mathbb{Z}_{2^q}$ and $j \in [d]$.

A blurer $\Xi$ consists solely of tuples satisfying $\xi = 0$, i.e., we can later turn every $\xi \in \Xi$ into an automorphism. But when looking at a single index and summing over all $\xi \in \Xi$, it looks like that there is a twist at index 1 and no twist at all other indices. This is why we can use a blurer to define a matrix blurring the twist by summing over (local) automorphisms.

**Lemma 24.** The size $|\Xi|$ of every $(q, d)$-blurer is odd. For every $d \geq 3$, there is a $(q, d)$-blurer.

**Proof:** We have $|\Xi|$ mod 2 $= \sum_{b \in \mathbb{Z}_{2^q}} \#_{1,b}(\Xi)$ $= \#_{1,2^q-1}(\Xi) + \sum_{b \in \mathbb{Z}_{2^q} \setminus \{2^q-1\}} \#_{1,b}(\Xi)$ $= 1$ by the blurer Conditions 2 and 4. For $d \geq 3$ set $\Xi := 2^{d-2} \cdot \{(3,0,1,0,\ldots,0), (3,1,0,0,\ldots,0), (2,1,1,0,\ldots,0)\}$. Let $P_j := \text{orb}_j((\mathfrak{A}, \bar{p}))$ and $Q_j := \text{orb}_j((\mathfrak{B}, \bar{p}))$ for every $j \in [2]$. For $P \in \mathcal{P}_2$ we set $P_i := P_i$ for every $i \in [2]$ and likewise for a $Q \in \mathcal{Q}_2$. By Corollary 8 we have that $P_i = A_x = B_x$ if $x = \text{orig}(P_i)$ and $\text{dist}_G(x, \text{orig}(\bar{p})) > 1$. Moreover, every $P \in \mathcal{P}_1$ is also in $\mathcal{Q}_1$ and has the same type in $(\mathfrak{A}, \bar{p})$ as in $(\mathfrak{B}, \bar{p})$.

Let $\Xi$ be a $(q, d)$-blurer and $N_G(t) = \{e_1', \ldots, e_d'\}$ such that $e_i' = t_i'$. Then we can see $\xi \in \Xi$ also as a tuple $\xi \in \mathbb{Z}_{2^{q-1}}^{N_G(t)}$. Thus, $\xi$ acts on vertices $u$ originating from $t$ denoted by $\xi(u)$ (cf. Section V-A for a definition of the action). Note that every $\xi \in \Xi$ extends to an automorphism of $(\mathfrak{A}, \bar{p})$ (and so of $(\mathfrak{B}, \bar{p})$): by Corollary 8 the gadget of $t$ consists of a single orbit because $\text{dist}_G(t, \text{orig}(\bar{p})) \geq 3$, i.e., $A_t \in \mathcal{P}_1$. We define an $A \times B$ orbit-diagonal matrix $S$ over $\mathbb{F}_2$, where $S_p := S_{P \times P}$:

$$S_p(u, v) := \begin{cases} 1 & \text{if orig}(P) \neq u \text{ and } v = v, \\ 1 & \text{if orig}(P) = t \text{ and } \xi(u) = v, \\ 0 & \text{for some } \xi \in \Xi, \\ 0 & \text{otherwise.} \end{cases}$$

Of particular interest is the 1-orbit $P_t$ with origin $t$. As already seen, we have $P_t = A_t \in \mathcal{P}_1$, because $\text{dist}_G(t, \text{orig}(\bar{p})) \geq 3$ by assumption. So $P_t$ is in particular the unique 1-orbit with origin $t$. For all other orbits $P \in \mathcal{P}_1$, we have $S_p = 1$.

**Lemma 25.** The matrix $S$ is orbit-invariant.

**Proof:** Let $P \in \mathcal{P}_1$, $\varphi \in \text{Aut}((\mathfrak{A}, \bar{p}))$, $u \in P$, and $v \in Q = P \in \mathcal{Q}_1$. If $P \neq P_t$, clearly $\varphi(S_p) = \varphi(1) = 1 = S_p$. Otherwise, $P = P_t$. Then $S_p(\varphi(u), \varphi(v)) = 1$ if and only if
\[\xi(\varphi(u)) = \varphi(v)\] for some \(\xi \in \Xi\). Because the automorphism group of \(\mathfrak{A}\) is abelian and \(\xi\) extends to an automorphism, we have \(\varphi(v) = \xi(\varphi(u)) = \varphi(\xi(u))\), which is the case if and only if \(v = \xi(u)\) if and only if \(SP(A, B, C, D)\) is symmetric.

**Lemma 26.** The matrix \(S\) is odd-filled.

Proof: Let \(P \subseteq P_1\). For \(P \neq P_1\), the number of ones in a row of \(SP(A, B, C, D)\) is one and thus odd. In \(SP(A, B, C, D)\), the number of ones in a row is \(|\Xi|\) because \(\xi(u) \neq \xi'(u)\) if \(\xi \neq \xi'\) (Lemma 12) and if \(u \in P_1\), then \(\xi(u) \in P_1\) for every \(\xi \in \Xi\). With Lemma 24 we have that \(|\Xi|\) is odd.

**Corollary 27.** The matrix \(S\) is invertible.

We want to define \(f: P_2 \to Q_2\) such that it maps an orbit to another orbit of the same type. By Corollary 10 we know that \(P_2 = Q_2\) by Lemma 11 that a type-preserving bijection exists. Let \(P \subseteq P_2\) with origin \((x, y)\). If \(\{t', t\} \neq \{x, y\}\), we set \(f(P) = P\). Otherwise if \((t', t) = \{x, y\}\) then \(P\) has a different type in \((\mathfrak{A}, \mathfrak{B})\) than in \((\mathfrak{B}, \mathfrak{B})\). Every vertex in \(P_1\) is related with every vertex in \(P_2\) via some \(R_{E,C}\). By Corollary 8 we have that \(P = E_{(t', t), a}\) for some \(a \in Z_{2^\infty}\) (recall our assumption \(dist_G(t, \mathfrak{B}) \geq 3\) and so only the relation between \((u, v) \in P\) determines the type of \(P\)). We set \(f(P) = E_{(t', t), a+2^{n-1}}\), which has the same type in \(B\) because of the twist (cf. Figure 1). The case of \((t', t)\) is symmetric.

**Lemma 28.** Let \(P \subseteq P_2\). Then \(\chi_{SS} = S = \chi_{f(P)}\).

Proof: Let \(\text{orig}(P) = (x, y)\) and set \(Q := f(P)\). Clearly \(P \subseteq P_1 \times P_2\). We also have \(P_1 = Q_1\) and \(P_2 = Q_2\) (as seen earlier by Corollary 8). Then \(P_1 \times P_2\) block is the only nonzero block of \(\chi_{SS} = S\). Because \(S\) is orbit diagonal, \(\chi_{SS}\) has only one nonzero block, namely the \(P_1 \times Q_2\) block, which satisfies \((\chi_{SS})_{P_1 \times Q_2} = \chi_{P_1 \times P_2} \cdot SP(A, B, C, D)\). Likewise we have \((\chi_{SS})_{P_1 \times Q_2} = \chi_{P_1 \times Q_2} \cdot SP(A, B, C, D)\). Recall that we have set \(SP(A, B, C, D) = SP(A, B, C, D)\). We identify \(\chi_{SS}\) with \(SP(A, B, C, D)\) and likewise for \(\chi_{f(P)}\). So we are left to show that \(\chi_{SS} = SP(A, B, C, D)\).

a) Case \(t \notin \{x, y\}\): Then \(Q = f(P) = P\) and \(\chi_{SS} = \chi_{f(P)} = \chi(P) \cdot 1 = \chi_{SS} = \chi_{f(P)}\).

b) Case \(x = y = t\): Then \(Q = f(P) = P\). As seen already, we have \(P_1 = P_2 = Q_1 = Q_2 = P_1\). So if \(u \in P_1\), then \(\xi^{-1}(u) \in P_1\). We obtain \((\chi_{SS} \cdot SP(A, B, C, D))(u, v) = \sum_{w \in P_1} \chi_{SS}(u, w) \cdot SP(A, B, C, D)(w, v) = \sum_{w \in P_1} \chi_{SS}(u, w, v) = \sum_{w \in P_1} \chi_{SS}(u, v) = (\sum_{w \in P_1} \chi_{SS}(u, v, w) = (\sum_{w \in P_1} \chi_{SS}(u, v))\).

c) Case \(y = t\) and \((x, t) \in E\): (The case \(x = t\) and \((t, y) \notin E\) is symmetric): Likewise to the previous case we have \(P_2 = P_1\) and

\[
(\chi_{SS} \cdot SP(A, B, C, D))(u, v) = \sum_{\xi \in \Xi} \chi_{SS}(u, \xi^{-1}(v))
\]

\[
= \begin{cases} 1 & \text{if } |\{\xi \in \Xi \mid (u, \xi^{-1}(v)) \in P\}| \text{ is odd,} \\ 0 & \text{otherwise.} \end{cases}
\]

Let \(P = E_{(t, x), a}\) for some \(a \in Z_{2^\infty}\) (cf. the definition of \(f\)) and \((u, v) \in E_{(t, y), b}\) for some \(b \in Z_{2^\infty}\). Then by definition of \(E_{(t, x), b}\) it holds that \(u(t) + v(x) = b\). Let \(i \in [d]\) such that \(x = e_i\). Then \((u, \xi^{-1}(v)) \in P = E_{(t, x), a}\) if and only if \(u(t) + \xi^{-1}(v)(x) = a\) if and only if \(\xi(i) = b - a\) because \(u(t) + \xi^{-1}(v)(x) = (u(t) + v(x)) - \xi(i) = b - \xi(i)\). We see that \(|\{\xi \in \Xi \mid (u, \xi^{-1}(v)) \in P\}| = |b - a(\Xi)|\).

Define \(c := 2^{n-1}\) if \(i = 1\) (and so \(x = t\)) and \(c := 0\) otherwise. Then \(b - a(\Xi) = 1\) if and only if \(b - a = c\) by the properties of a bluer. It follows that

\[
(\chi_{SS} \cdot SP(A, B, C, D))(u, v) = \begin{cases} 1 & \text{if } b - a = c, \\ 0 & \text{otherwise.} \end{cases}
\]

If \(i\) is \(\neq t\), then \(c = 0\) and \((\chi_{SS} \cdot SP(A, B, C, D))(u, v) = 1\) if and only if \(b = a\), that holds if and only if \((u, v) \in P\). So \(\chi_{SS} \cdot SP(A, B, C, D) = \chi_{SS} \cdot SP(A, B, C, D)\), because \(Q = f(P) = P\).

If \(i = 1\) (so \(x = t\)), we have that \((\chi_{SS} \cdot SP(A, B, C, D))(u, v) = 1\) if and only if \(b = a = 2^{n-1}\), i.e., \(a + 2^{n-1} = b\). But that holds by definition of \(f\) if and only if \((u, v) \in Q = f(P) = E_{(t, x)\cdot a+2^{n-1}}\) and so \(\chi_{SS} \cdot SP(A, B, C, D) = \chi_{SS} \cdot SP(A, B, C, D)\).

d) Case \(y = t\) and \((x, t) \notin E\): (The case \(x = t\) and \((t, y) \notin E\) is symmetric). By the assumption that \(dist_G(t, \text{orig}(P)) \geq 3\) the type of \((u, v)\) and \((u, v')\) for \(u \in A_x\) and \(v, v' \in A_t\) is equal. So by Corollary 8 we have \((u, v) \in P\) if and only if \((u, v') \in P\). We compute:}

\[
(\chi_{SS} \cdot SP(A, B, C, D))(u, v)
\]

\[
= \begin{cases} 1 & \text{if } |\{\xi \in \Xi \mid (u, \xi^{-1}(v)) \in P\}| \text{ is odd,} \\ 0 & \text{otherwise,} \end{cases}
\]

\[
= \chi_{SS}(u, v).
\]

This holds because \((u, v) \in P\) if and only if \((u, \xi^{-1}(v)) \in P\). Let \(D := \{\xi \in \Xi \mid (u, \xi^{-1}(v)) \in P\}\). So if \((u, v) \in P\), then \(D = \Xi\) and \(|D| = |\Xi|\) is odd (Lemma 24). If \((u, v) \notin P\), then \(D = \emptyset\) and \(|D| = 0\). As seen before, we have \(\chi_{SS} \cdot SP(A, B, C, D) = \chi_{SS} \cdot SP(A, B, C, D)\) because \(Q = f(P) = P\).

**Corollary 29.** The matrix \(S\) blurs the twist between \((\mathfrak{A}, \mathfrak{B})\) and \((\mathfrak{B}, \mathfrak{B})\).

It is not hard to see that Duplicator is able to win the 1-round game \(M^{m,1}(2)\) using the matrix \(S\). By using known
techniques he can also manage to win the $\mathcal{M}_{m,1,2}$ game. We will discuss this in more detail for the general $k$-ary case. Constructing such a similarity matrix $S$ in the $k$-ary case turns out to be more difficult.

VIII. The Active Region of a Matrix

To construct matrices for the $k$-ary case, we need to consider the part of $S$, where $S$ “has a non-trivial effect”. Intuitively, this means that $S$ is locally not the identity matrix. In the 1-ary case, this was the gadget of $t$. We will call these parts of $S$ the active region. Conversely, for parts where $S$ is not active, we will know that $S$ is locally the identity matrix.

Let $G = (V, E; \leq)$ be a base graph, $\mathfrak{A} = \text{CFI}_{2^k}(G, f)$, and $\mathfrak{B} = \text{CFI}_{2^k}(G, g)$ with universes $A$ and $B$ for some functions $f, g: E \to 2^{2^k}$. As in Section VI, let $k, m \in \mathbb{N}$, $\bar{p} \in A^m$, and $S$ be a $A^k \times B^k$ matrix over $\mathbb{F}_2$. For $k \in \mathbb{N}$, the $N$-components $C_N(P)$ of an orbit $P$ is the set of components of $P$ of satisfying $C \subseteq N$. For $N \subseteq V$, the $N$-components $C_N(P)$ of an orbit $P$ is the set of components of $P$ of satisfying $C \subseteq N$.

Definition 30 (Active Region). Let $P \in \mathbb{P}_k$ and $Q \in \mathbb{Q}_k$ have the same type and let $C$ be a component of $P$ (and so of $Q$). The matrix $S$ is active on $C$ if there are $\bar{u}, \bar{v} \in P, \bar{v} \in Q$ such that $\bar{u} \cap C \neq \emptyset$ and $S(\bar{u}, \bar{v}) = 1$. We write $A_S(P) = A_S(Q)$ for the set of components of $P$, on which $S$ is active, and $N_S(P)$ for the remaining components.

The active region $A(S)$ of $S$ is the smallest set satisfying the following:

- $C \subseteq A(S)$ for every $C \subseteq A(S)$ and every $P \in \mathbb{P}_k$.
- Let $P, P' \in \mathbb{P}_k$ and $Q, Q' \in \mathbb{Q}_k$ be arbitrary orbits such that $C_A(S)(P) = C_A(S)(P') = C_A(S)(Q) = C_A(S)(Q') = A$, both $P$ and $Q$ (respectively $P'$ and $Q'$) have the same type, and thus $N_S(P) = N_S(Q)$ (respectively $N_S(P) = N_S(Q')$).

The second condition states that $S(\bar{u}, \bar{v})$ only depends on the components of $\bar{u}$ and $\bar{v}$, on which $S$ is active, as long as the entries for the other components are equal (otherwise $S(\bar{u}, \bar{v}) = 0$ anyway). That is, we can replace the nonactive parts with any other $(\bar{u}_{\bar{N}_S(P)})$ with $\bar{u}_{\bar{N}_S(P)}$ and $\bar{v}_{\bar{N}_S(Q)}$ with $\bar{v}_{\bar{N}_S(Q')}$ and $S$ has the same entry. We first consider the matrix blurring the twist defined in Section VII:

Lemma 31. Assume we are in the situation of Section VII and let $S$ be the matrix defined there. Then $A_S = \{t\}$.

We continue in the general case. It easy to see that if $P$ and $Q$ have the same type, those origins contain no vertex of $A(S)$, then $S_{P \times Q} = 1$. Around a twist, $S$ has to be active (or zero), because if the origin of $P$ contains a twisted edge, then $P \cap Q = \emptyset$ and hence the nonzero entry has different row and column indices.

We consider products $S \cdot T$. Let $\mathfrak{H} = \text{CFI}_{2^k}(H, h)$ with universe $H$ for $h: E \to 2^{2^k}$, $T$ be a $B^k \times H^k$ matrix over $\mathbb{F}_2$, and $\mathbb{R}_k = \text{orb}_{\mathfrak{H}}(\bar{h}, \bar{p})$. If the active regions of $S$ and $T$ are disjoint, the product $S \cdot T$ is reminiscent to an entry-wise product. If we sum over the active region of $S$, the contribution of $S$ vanishes.

Lemma 32. Let $S$ and $T$ be orbit-diagonal, orbit-invariant, and odd-filled, $A(S) \cap A(T) = \emptyset$, $P \in \mathbb{P}_k$, $Q \in \mathbb{Q}_k$, and $R \in \mathbb{R}_k$ be of the same type, and $M$ and $N$ be a partition of the components of $P$, such that $C_A(S)(P) \subseteq M$ and $C_A(T)(P) \subseteq N$. Then for every $\bar{u} \in P$ and $\bar{v} \in R$ we have $S \cdot T(\bar{u}, \bar{v}) = S(\bar{u}_M \bar{u}_N, \bar{w}_M \bar{w}_N) \cdot T(\bar{w}_M \bar{w}_N, \bar{u}_M \bar{u}_N)$ and

\[
\sum_{\bar{u}_M \in P \cap M} (S \cdot T)(\bar{u}_M, \bar{u}_N) = T(\bar{w}_M \bar{w}_N, \bar{u}_M \bar{u}_N).
\]

IX. The Arity $k$ Case

We now construct a similarity matrix for $k$-ary rank operators. Constructing and verifying the suitability of this matrix will be quite technical and intricate. We discuss the difficulties we have to overcome and the idea behind the construction. All technical details are given in the full version [25].

a) Orbits of the Same Type: Let again $\mathfrak{A}$ and $\mathfrak{B}$ be two CFI structures, such that a single edge $\{t, t'\}$ of the base graph $G$ is twisted. Let $\bar{p}$ be parameters, whose origin has sufficient distance to the twisted edge. We have seen in Section VII that 1-orbits have the same type in $(\mathfrak{B}, \bar{p})$ as in $(\mathfrak{B}, \bar{p})$. For a $k$-orbit $P$ this is not the case whenever $\{t, t'\} \subseteq \text{orig}(P)$. Ultimately, our goal is to construct an orbit-invariant, orbit-diagonal, and odd-filled similarity matrix $S$ blurring the twist. Because the blocks on the diagonal of $S$ arise from orbits of the same type and because the characteristic matrices of orbits of the same type have to be simultaneously similar, we first want to define a bijection $\text{orb}_{\mathfrak{B}}(\mathfrak{A}, \bar{p}) \rightarrow \text{orb}_{\mathfrak{B}}(\mathfrak{B}, \bar{p})$ for every $k' \leq 2k$ that preserves the orbit types. For this, we want to construct a function $\tau: A^{\leq 2k} \rightarrow B^{\leq 2k}$ that preserves the type of tuples. Then $\tau$ preserves orbit types, too. To do so, we pick a path $(z, \ldots, t, t')$ such that $\text{dist}_{\mathfrak{A}}(t', z) > 2k$ and consider a path-isomorphism $\varphi_{\tau}$ which twists the edge $\{t, t'\}$ and the edge incident to $z$ in the chosen path. That is, between $\varphi(\mathfrak{A})$ and $\mathfrak{B}$ an edge incident to $z$ is twisted but the edge $\{t, t'\}$ is not. For the moment assume that we only consider connected tuples and thus only connected orbits. Let $\tau$ be the function that applies the path-isomorphism $\varphi_{\tau}$ to every tuple $\bar{u}$ with $\{t, t'\} \subseteq \text{orig}(\bar{u})$ and is trivial on all others. Let $\bar{u} \subseteq A^{\leq 2k}$ be such a tuple with $\{t, t'\} \subseteq \text{orig}(\bar{u})$. Because $\text{dist}_{\mathfrak{A}}(t', z) > 2k$ and because we are considering connected tuples, we have that $z \notin \text{orig}(\bar{u})$. Hence $\mathfrak{B}[\text{orig}(\bar{u})] = \varphi(\mathfrak{A})[\text{orig}(\bar{u})]$ (here $\mathfrak{B}[\text{orig}(\bar{u})]$ denotes the induced substructures of $\mathfrak{B}$ of all vertices $\bar{v}$ with $\text{orig}(\bar{v}) \in \text{orig}(\bar{u})$).

Corollary 33. If $k' \leq 2k$ and $P \in \text{orb}_{\mathfrak{B}}((\mathfrak{A}, \bar{p}))$ then $\tau(P) \in \text{orb}_{\mathfrak{B}}((\mathfrak{B}, \bar{p}))$ and $\tau(P)$ has the same type as $P$.

b) Generalized Blurers: Next we transfer the concept of a blurer to the $k$-ary case. In Definition 23, we required that the number of tuples in a blurer with a given value at a given index satisfies some properties. In the $k$-ary case, we require similar properties, but now we do not consider one index but a set of $k$ many indices.
Figure 2. The star $s_1, \ldots, s_d$. Because the girth of the base graph is large enough, each path $s_i$ is contained in its own tree rooted at $z$ (shown in gray).

**Definition 34** (Blurer). Let $d \geq k$, $\Xi \subseteq \mathbb{Z}_d^l$, and $a \in \mathbb{Z}_{2^k}$. For $N \subseteq |d|$ of size $k$ and $b \in \mathbb{Z}_{2^k}$, we define
\[ \#_{N,b}(\Xi) := |\{ e \in \Xi \mid e|N = b \}| \mod 2. \]

The set $\Xi$ is called a $(k, q, a, d)$-blurer if it satisfies the following for all $N \subseteq |d|$ with $|N| = k$:

1. $\sum_\Xi e = 0$ for all $e \in \Xi$.
2. If $1 \in N$, then $\#_{N,(a,0,\ldots,0)}(\Xi) = 1$.
3. If $1 \notin N$, then $\#_{N,0}(\Xi) = 1$. 
4. $\#_{N,b}(\Xi) = 0$ for all other pairs of $N$ and $b$.

Showing the existence of such blurers is more difficult, in particular we have to use, for a given $k$, the ring $\mathbb{Z}_{2^k}$ for a sufficiently large $q$ for the CFI structures $\mathfrak{A}$ and $\mathfrak{B}$. The additional parameter $a$ is needed for recursion and becomes apparent later on.

**Lemma 35.** There is a $(2^i - 1, 1, i, 2^i - 1)$-blurer for every $i \in \mathbb{N}$.

In the 1-ary case, we translated a tuple $\xi \in \Xi$ to an automorphism of the gadget of $t$. We now describe the approach in the $k$-ary case. Let $\Xi$ be a blurer for arity $k$ consisting of tuples of length $d$, that is $\Xi$ is a $(k, q, a, d)$-blurer for some suitable $q$ and $a \in \mathbb{Z}_{2^k}$. We now require that the base graph $G$ is $d$-regular. Recall that in Section VII we blurred the twist between the edges incident to $t$. One of these edges was the twisted edge $\{t, t'\}$. Because we now have to consider $2k$-tuples, it is not possible to blur the twist around a single vertex. Instead, we choose vertices $e_1, \ldots, e_d$ and $e_1', \ldots, e_d'$, such that $e_1 = t, e_1' = t'$, and there are simple paths $\bar{s}_i = (z, \ldots, e_i, e_i')$ of length at least $2k$ forming a star, i.e., the $\bar{s}_i$ are disjoint apart from $z$ (cf. Figure 2). Here we in particular choose $\bar{s}_1$ to be the path we used to define the tuple-type-preserving map $\tau$. We ensure that such paths exist by requiring that the girth of $G$ is large enough. We use the blurer to distribute the twist across the edges $\{e_i, e_i'\}$. In the 1-ary case, an element in a blurer corresponds to an automorphism of the gadget of $t$, or likewise to a star-isomorphism, where the paths of the star have length one. In the $k$-ary case we identify a $\xi \in \Xi$ with the star-isomorphism $\varphi_\xi := \pi^* [\xi, \bar{s}_1, \ldots, \bar{s}_d]$. Again in order to preserve the type of tuples, we only apply $\xi$ to tuples $\bar{u}$ satisfying $\{e_i, e_i'\} \not\subset \text{orig}(\bar{u})$ for all $i \in [d]$. That is on such a $\bar{u}$, the action of $\xi$ could also be defined by an automorphism. This turns $\xi$ into a “star-automorphism” (instead of an automorphism of a single gadget in the 1-ary case). If we only had connected tuples, this approach would be sufficient to construct a similarity matrix (and in particular we could even use easier blurers). However, disconnected tuples complicate matters.

**c) Disconnected Orbits:** Now we have to consider disconnected tuples and orbits. While with connected tuples the approach described above is local (we only consider the $2k$-neighborhood of $z$), we now have tuples containing vertices from very different places in the structure. But then these vertices belong to different components (cf. Definition 13). Lemma 14 tells us that the components of disconnected orbits are independent whenever the connectivity of $G$ is sufficiently large. In a first step we salvage the previous approach by applying the path- and the star-isomorphism not to whole tuples, but only to components of tuples. That is, if a component $C$ contains the twisted edge $\{t, t'\}$ we apply the type-preserving map $\tau$ to this component and if $\{e_i, e_i'\} \not\subset C$ for all $i \in [d]$ we apply the star-automorphisms $\xi$ this component. (Note that $\xi$ is only different from the identity map if $C$ intersects with a path $\bar{s}_i$). With this approach we can blur the twists for most of the orbits, i.e., if we were to define the matrix $S$ in that way, then $S$ would be indeed a similarity matrix for most of the orbits. So we discuss the problematic orbits, for which this approach fails. The blurer $\Xi$ can be successfully used, if we only consider $k$ of its $d$ indices. In the 1-ary case, we could blur the twist because the twisted edge only depended on $1$ of the at least $3$ indices. But in the $k$-ary case, there are orbits whose origin contains one of the edges $\{e_i, e_i'\}$ and also the center $z$ of the star. The action of a star-automorphism $\xi \in \Xi$ on vertices with origin $z$ depends on all $d$ entries of $\xi$, because all paths $\bar{s}_i$ come together at the center $z$. For connected $2k$-tuples, this cannot happen, because $\text{dist}_G(z, e_i') > 2k$ for all $i \in [d]$. But for disconnected tuples this is of course possible. This is why we have to distinguish two kinds of $k$-orbits.

**Definition 36.** Let $P$ be a $k$-orbit of $(\mathfrak{A}, \bar{p})$. If $z \not\in \text{orig}(P)$, we call $P$ blurable.

For non-blurable orbits, we need the following technique.

**d) Recursive Blurring:** Now consider a $2k$-orbit $P$, such that both $P_1 := P|_{\{1, \ldots, k\}}$ and $P_2 := P|_{\{k+1, \ldots, 2k\}}$ are not blurable. Let us quickly recall the 1-ary case. We could blur the twist in Lemma 28, because we summed over the tuples $\xi(\bar{u})\bar{v}$ for all $\xi \in \Xi$. For a 2-orbit $P$, whose origin was the twisted edge $\{t, t'\}$, w.l.o.g. the origin of $P_1$ is $t$ and thus depends on all entries of the blurer. But $P_2$ only depended on one index of the blurer, so we could apply the blurer property, and the twist vanishes. The 2-orbits for which both $P_1$ and $P_2$ have origin $\{t\}$ do not cover the twisted edge.

Now consider the $k$-ary case again. Here of course there are orbits $P$ such that both $P_1$ and $P_2$ are non-blurable and they contain the twisted edge in their origins. Let $\bar{u} \in P_1$
and \( \bar{v} \in P_2 \). Both \( \bar{u} \) and \( \bar{v} \) contain a vertex with origin \( z \) and the blurer properties do not apply, and the orbit of \( \xi(\bar{u}) \bar{v} \) is different for every \( \xi \in \Xi \) (fixing one vertex of origin \( z \) separates the gadget of \( z \) into singleton orbits). That is, we now can detect the twists introduced by each \( \xi \) at the edges \( \{e_i, e'_i\} \). But this only happens at orbits where both \( P_1 \) and \( P_2 \) are non-blurable. To blur these twists, we follow a recursive approach. Let \( P_2 \) be some vertex with origin \( z \) and let \( \mathfrak{B}_\xi := \varphi_\xi^{-1}(\mathfrak{B}) \). We use the inverse \( \varphi_\xi^{-1} \) of the star-isomorphism \( \varphi_\xi \) here because we do not only want to blur the twist between \( \mathfrak{A} \) and \( \mathfrak{B} \), but also to revert the twists introduced by \( \varphi_\xi \). For every \( \xi \), we get recursively a matrix \( S^\xi \) that \( (k - 1) \)-blurs the twist between \( (\mathfrak{A}, \bar{p}_\xi z) \) and \( (\mathfrak{B}_\xi, \bar{p}_\xi z) \). We can use these matrices for the \( k \)-orbits \( P_1 \) and \( P_2 \), because by Lemma 15 we obtain a \( (k - 1) \)-orbit \( P_1 \) of \( (\mathfrak{A}, \bar{p}_\xi z) \) when fixing a vertex with origin \( z \) (and likewise for \( P_2 \)). Here the need arises to blur a twist by \( a \in \mathbb{Z}_{2\nu} \) for CFI structures defined over \( \mathbb{Z}_{2\nu} \), where \( a \neq 2^{\nu - 1} \), because a \( \xi \in \Xi \) contains entries different from \( 2^{\nu - 1} \). Therefore, the parameter \( a \) is introduced in Definition 34.

Combining the blurers \( \Xi \) with the \( S^\xi \) becomes formally tedious. For a blurable \( k \)-orbit \( P \) we define

\[
S_{P \times \tau(P)}(\bar{u}, \bar{v}) := \sum_{\xi \in \Xi, \tau(\xi(\bar{u})) = \bar{v}} 1.
\]

For a non-blurable \( k \)-orbit \( P \), we intuitively want to define \( S \) so that we sum over \( S^\xi(\xi(\bar{u}^z), \bar{v}^z) \) for all \( \xi \in \Xi \), that is, if \( u \) is the entry with origin \( z \). Actually, we have to use an additional type-preserving map \( \tau_z \) here to map tuples of \( (\mathfrak{B}_{\xi}, \bar{p}_\xi z) \) to tuples of \( (\mathfrak{B}_z, \bar{p}_z) \) with the same type and sum over \( S^\xi(\xi(\bar{u}^z), \tau_z^{-1}(\bar{v}^z)) \). Here, we need to take the inverse \( \varphi_\xi^{-1} \) because \( \mathfrak{B}_\xi = \varphi_\xi^{-1}(\mathfrak{B}) \). We finally obtain for non-blurable \( k \)-orbits \( P \)

\[
S_{P \times \tau(P)}(\bar{u}, \bar{v}) := \sum_{\xi \in \Xi, \tau(\xi(\bar{u})) = \bar{v}} S^\xi(\xi(\bar{u}^z), \tau_z^{-1}(\bar{v}^z)),
\]

where \( \bar{u}_z \) denotes the vertex deleted in \( \bar{u}^z \).

e) Active Region and Blurers: Recall that \( S \) is defined for blocks of \( k \)-orbits. Blocks for blurable \( k \)-orbits are defined using the blurer \( \Xi \), blocks for non-blurable \( k \)-orbits are defined using \( \Xi \) and the matrices \( S^\xi \). With this approach we can show that \( S \) is a similarity matrix for orbits \( P \) if either both \( P_1 \) and \( P_2 \) are blurable or neither blurable. In the former case we use the blurer property, in the latter case we use induction. So now consider the case where \( P_1 \) is blurable and \( P_2 \) is not or vice versa. Similar to our argument in the 1-ary case for orbits whose origin is the twisted edge \( \{t, t'\} \), we can argue with the blurer properties here. But in the \( k \)-ary case there is the following problem: We have to show that \( \chi^P \cdot S = S \cdot \chi^Q \) (for \( Q = \tau(P) \), which is of the same type). The right multiplication uses the block of \( S \) of the blurable orbit \( \tau(P_1) = Q_1 \), but the left multiplication uses the block of the non-blurable orbit \( P_2 \).

So on the left side, the matrices \( S^\xi \) appear. Because we want to argue using the blurer properties, we want to ensure that the effect of the \( S^\xi \) in the left multiplication vanishes. Because \( P_1 \) is blurable, only the action of the \( S^\xi \) on at most \( k \) of the edges \( \{e_i, e'_i\} \) matters. To make this intuition formal, we need to ensure that the action of \( S^\xi \) on \( \{e_i, e'_i\} \) is independent of the action on \( \{e_j, e'_j\} \) for \( i \neq j \). So we obtain \( S^\xi \) as \( S^\xi := \sum_{i \neq j} S_i^\xi \), such that \( S_i^\xi \) blurs only the twist at the edge \( \{e_i, e'_i\} \) between \( \mathfrak{A} \) and \( \mathfrak{B}_i \). We then use the notion of the active region of Section VIII. We bound the active region of the \( S^\xi \) by the \( (r(k))-neighborhood of \( e_i \) (for a suitable \( r(k) \)).

If we now choose the paths \( S_i \) to not only have length greater than \( 2k \), but actually greater than \( \max(2k, r(k)) \), the active regions of \( S_i^\xi \) and \( S_j^\xi \) are disjoint for \( i \neq j \). By Lemma 32 the contribution of the \( S_i^\xi \) for all \( i \in [d] \), on which \( P_1 \) does not depend, vanishes. Then we can use the blurer properties to show that actually the effect of the at most \( k \) remaining \( S_i^\xi \) vanishes, too. But now, the matrices \( S^\xi \) do not appear in the left multiplication \( \chi^P \cdot S \) anymore and we can use the blurer properties once more to show that \( S \) is a similarity matrix for orbits where \( P_1 \) is blurable and \( P_2 \) is not.

f) The Final Lemma: We saw that the connectivity, the girth, and the degree of the base graph \( G \) must be large enough. Also, to ensure existence of the needed blurers, the ring \( \mathbb{Z}_{2\nu} \) must be chosen for a sufficiently large \( q \). Because we recurse on the arity \( k \), we define the required bounds recursively. In the following definitions let \( i \in \mathbb{N} \) such that \( 2^{i-1} - 1 < k \leq 2^i - 1 \).

\[
r(k) := \begin{cases} 1 & \text{if } k = 1, \\ \max\{4 \cdot r(k-1) + 2, 2k + 2\} & \text{otherwise,} \\
\end{cases}
\]

\[
\theta(k) := \begin{cases} 1 & \text{if } k = 1, \\ i + \theta(k-1) & \text{otherwise,} \\
\end{cases}
\]

\[
d(k, m) := \begin{cases} 3 + m & \text{if } k = 1, \\ \max\{2^{i+1} + m - 1, d(k-1, m+1)\} & \text{otherwise,} \\
\end{cases}
\]

\[
q(k) := 1 + \theta(k).
\]

All in all, we outlined how we prove the following lemma:

**Lemma 37.** For every \( k, m \in \mathbb{N} \), every regular and \( (m + 2k + 1) \)-connected base graph \( G = (V, E, \leq) \) of girth at least \( 2r(k+1) \) and degree \( d \geq d(k, m) \), every edge \( \{t, t'\} \in E \), every \( q \geq q(k) \), and every \( \theta = a \cdot 2^q(k) \in \mathbb{Z}_{2\nu} \) (for an arbitrary \( a \in \mathbb{Z}_{2\nu} \)), \( \mathfrak{B} = \mathcal{Codd}(\mathcal{G}, \mathfrak{B}) \) such that \( f, g: E \to \mathbb{Z}_{2\nu} \), \( f(e) = g(e) \) for all \( e \in E \setminus \{t, t'\} \) and \( g(\{t, t'\}) = f(\{t', t\}) + \theta \). For every \( m \)-tuple \( \bar{p} \in A_m = B^{m+ \mathbb{N}} \) satisfying \( \text{dist}_{\mathcal{G}}(\tau(P), \text{orig}(\mathfrak{p})) > r(k+1) \), there is an orbit-diagonal, orbit-invariant, and odd-filled matrix \( S \) that \( k \)-blurs the twist between \( (\mathfrak{A}, \bar{p}) \) and \( (\mathfrak{B}, \bar{p}) \) and satisfies \( A(S) \subseteq N^r_{\mathcal{G}}(k+1)(t) \).

The arity 1 case in Section VII serves as base case when using the blurers from Lemma 37. This is needed to blur twists different from \( 2^{i-1} - 1 \). We should mention that we checked our construction in the proof for \( k \leq 2 \) on the computer. For larger \( k \) it was computationally not tractable.
X. SEPARATING RANK LOGIC FROM CPT

We finally are ready to separate rank logic from CPT. To apply Lemma 37, we need to construct a sufficient class of base graphs. To prove the following lemma, we use that cages have high connectivity [29] and that cages exist because Ramanujan graphs exist [30].

Lemma 38. For every $n$ there is a graph, which is regular, has degree at least $n$, girth at least $n$, and is $n$-connected.

Lemma 39. Let $G = (V, E)$ be a $d$-regular graph of girth $2(\ell+2)+1$ for some $\ell \in \mathbb{N}$. Then for every set $V' \subseteq V$ of size $|V'| \leq d$, there is a vertex $x \in V$ such that $\text{dist}_G(V', x) > \ell$.

Theorem 40. There is a class of base graphs $K$, such that for every $m, k \in \mathbb{N}$ satisfying $2k \leq m$, there is a $G = (V, E, \leq) \in K$ and a $q \in \mathbb{N}$ such that $\text{CFI}_{2^q}(G, g) \equiv_{\text{M}, k}^{m, k} \text{CFI}_{2^q}(G, f)$, where $g, f : E \to \mathbb{Z}_{2^q}$ satisfy $\sum g > \sum f + 2^{q-1}$.

Proof: Let $K$ be the class of graphs given by Lemma 38 for every $n \in \mathbb{N}$ equipped with some total order. Let $2k \leq m$ and $G \in K$ such that $G$ has degree $d \geq d(k, m) \geq m$, girth at least $2(r(k+1)+2)+1$, and $G$ is at least $(m+2k+1)$-connected. Let $q := q(k), e = \{x, y\} \in E$, and $g, f : E \to \mathbb{Z}_{2^q}$ such that $g(e') = f(e')$ for all $e' \in E \setminus \{e\}$. Let $\mathfrak{A} = \text{CFI}_{2^q}(G, g)$ and $\mathfrak{B} = \text{CFI}_{2^q}(G, f)$. We consider the case where $m$ pebbles are placed on the structures. Starting with fewer pebbles does not provide more insights. Let $\bar{u} \in A^m$ and $\bar{v} \in B^m$ such that the type of $\bar{u}$ is the same as of $\bar{v}$. We assume $\mathfrak{A}, \mathfrak{B}$. Let $P \in \text{orbs}_m(\mathfrak{A}, \bar{u})$ contain $\bar{u}$ and $Q \in \text{orbs}_m(\mathfrak{B}, \bar{v})$ contain $\bar{v}$. Because $\bar{u}$ and $\bar{v}$ have the same type, $P$ and $Q$ have the same type. Because $e \not\subseteq \text{orig}(\bar{u}), P = Q$. That is, there is an automorphism $\varphi \in \text{Aut}(\mathfrak{B})$ such that $\varphi(\bar{u}) = \bar{v}$ (Corollary 8). So we can continue the game on $(\mathfrak{A}, \bar{u})$ and $\varphi((\mathfrak{B}, \bar{v}) = (\mathfrak{B}, \bar{u})$. Spoiler picks up the corresponding $2k$ pebbles on each graph leaving us with $(\mathfrak{A}, \bar{w})$ and $(\mathfrak{B}, \bar{w})$, where $\bar{w} \in A^{m-2k}$ and $\bar{w}$ has the same type in $\mathfrak{A}$ and $\mathfrak{B}$.

There is a vertex $z \in V$ such that $\text{dist}_G(\text{orig}(\bar{w}), z) > 2r(k+1)$ by Lemma 39. Because $e \not\subseteq \text{orig}(\bar{u})$ and $G$ is $(m+1)$-connected there is a path $\bar{s} = (x, y, \ldots, z)$ such that $\bar{s}$ and $\text{orig}(\bar{w})$ are disjoint apart from $x$ (we assume w.l.o.g. that $e \cap \text{orig}(\bar{u}) \subseteq \{x\}$ because by assumption $e \not\subseteq \text{orig}(\bar{v})$). Then we apply the path isomorphism $\psi := \pi[2^{q-1}, s]$ and continue the game on $(\mathfrak{A}, \bar{w})$ and $\psi((\mathfrak{B}, \bar{w}) = (\mathfrak{B}, \bar{w})$.

Duplicator chooses the partitions $P := \text{orbs}_{2k}((\mathfrak{A}, \bar{w}))$ and $Q := \text{orbs}_{2k}((B', \bar{w}))$ of $A^k \times A^k$ and $B^k \times B^k$. He chooses the invertible matrix $S$ that $k$-blurs the twist between $(\mathfrak{A}, \bar{w})$ and $(B', \bar{w})$ given by Lemma 37. By construction, the conditions of the lemma are satisfied. The matrix $S$ induces a map $P \to Q$ mapping $P \mapsto Q$. If only if if $P$ and $Q$ have the same type (Definition 18). Spoiler chooses orbits $P \in P$ and $Q \in Q$ of the same type and $\bar{u}' \in P$ and $\bar{w}' \in Q$. Then $\bar{u}'$ has the same type in $(\mathfrak{A}, \bar{w})$ than $\bar{v}'$ in $(B', \bar{w})$. So $\bar{w}w' \bar{u}'$ and $\bar{w}w'$ induce a partial isomorphism and the next round starts.

We are in the same situation as before: $\bar{w}w' \bar{u}'$ and $\bar{w}w'$ have the same type. Duplicator can apply his strategy once more.

So he has a winning strategy in the $H^{m, k}$ game.

Theorem 41. There is a class of $\tau$-structures $K$, such that IFP+$R$ $\not< \text{PTIME}$ on $K$ and CPT $< \text{PTIME}$ on $K$.

Proof: Let $K'$ be the class of base graphs from Theorem 40. Set $K := \text{CFI}_{2^q}(K')$. We show that the CFI query is not IFP+$R$-definable. So for $\mathfrak{A} = \text{CFI}_{2^q}(G, g) \in K$ we want to determine whether it holds that $\sum g = 0$. By Lemma 17 it suffices to show that the CFI query is not IFP+$R_{\{2\}}$-definable. The claim follows with Lemma 3 and Theorem 40.

We now argue that CPT captures PTIME on K. By Theorem 2 it suffices to show that $K'$ is a class of structures with abelian and ordered colors, i.e., the automorphism group of every color class is abelian and can be ordered. By Lemma 4 the colors are abelian. Consider the automorphism group $\Gamma$ of a single gadget (i.e. a color class). As seen in Section V, each automorphism in $\Gamma$ can be identified with a tuple $\bar{a} \in \mathbb{Z}_{2^q}^k$. This identification is easily CPT-definable using the relations $C_{x,y}$. Surely, tuples $\mathbb{Z}_{2^q}^k$ can be ordered lexicographically.

XI. DISCUSSION

We showed that rank logic does not capture CPT and in particular not PTIME on the class of CFI structures over $\mathbb{Z}_{2^q}$, even if the base graph is totally ordered. To do so, we used combinatorial objects called blurers and a recursive approach over the arity. The non-locality of $k$-tuples for $k > 1$ increased the difficulty of $k$-ary rank operators dramatically compared to the 1-ary case. It was suggested in [9] that CFI graphs over $\mathbb{Z}_4$ are a separating example for rank logic and PTIME. We require rings $\mathbb{Z}_{2^q}$ for $i > 2$ for higher arities to ensure that blurers exist. Our computer experiments for verifying Lemma 37 for $k \leq 2$ indicate that the CFI query over $\mathbb{Z}_4$ is possibly definable in rank logic using rank operators of higher arity.

There are various definitions of rank logic, which slightly differ in the way the matrices in the rank operator are defined. In particular, there is an extension, in which rank operators not only bind universe variables, but also numeric variables [9], [11], [12]. It is not clear whether this extension is more expressive or not. However, for a suitable adaptation of the invertible-map game, which also supports numeric variables in the rank operator, we strongly believe that our arguments work exactly the same. In fact, we think that at least in the invertible-map game numeric variables do not increase the expressiveness and thus our arguments directly apply.

A natural question is how rank logic can be extended so that it can define the CFI query. We have shown that it is not sufficient to compute ranks over finite fields only. However, it is not clear how rank logic can be extended to rings $\mathbb{Z}_4$. Over rings, there are several non-equivalent notions of the rank of a matrix. For a discussion see e.g. [15], [12]. As opposed to rank logic, solvability logic can easily be extended to rings and thus should be able to define the CFI query over all $\mathbb{Z}_4$. Notably, such an extension would also capture PTIME on structures with bounded and abelian colors [12].

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