An Improved Lower Bound to the Number of Neighbors Required for the Asymptotic Connectivity of Ad Hoc Networks

Sanquan Song, Dennis L. Goeckel, Don Towsley

Abstract

Xue and Kumar [3] have established that the number of neighbors required for connectivity of wireless networks must grow as $\Theta(\log N)$, and [3] also established that the actual number required lies between 0.074 $\log N$ and 5.177 $\log N$. In this short paper, by recognizing that connectivity results for networks where the nodes are distributed according to a Poisson point process can often be applied to the problem of [3], we are able to improve the lower bound. In particular, we show that a network with nodes distributed in a unit square according to a 2D Poisson point process of parameter $N$ will be asymptotically disconnected with probability one if the number of neighbors is less than 0.129 $\log N$. Moreover, $0.129 \log \left( N + \frac{\pi}{4} - \sqrt{\frac{2N}{2} + \frac{\pi^2}{16}} \right)$ is not enough for an

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asymptotically connected network with $N$ nodes uniformly in a unit square, hence improving the lower bound from [3].

**Index Terms**

Wireless networks, ad hoc networks, connectivity, power control.

**I. INTRODUCTION**

Due to their widespread applicability, wireless ad hoc networks have attracted significant research interest in recent years. In an ad hoc network, each node is connected with several nearby nodes directly and thus to others by relay via these neighbors; thus, the neighbors of each node eventually determine the network connectivity. Based on this observation, researchers have defined the $k$-neighbor network model by assuming that each node adjusts its power to maintain a link with its $k$ closest neighbors [3], and then studied network connectivity performance as a function of $k$.

As $k$ increases, network connectivity improves. For a network with $N$ nodes, if $k = N - 1$, any pair of nodes can communicate directly, which is the best achievable connectivity. However, node power must increase to achieve such connectivity, which leads to more signal interference and lower network capacity [2]. Thus, given the requirement that the network be connected, the minimal $k$ that provides such is desired.

Researchers used to believe that there exists a “magic number” such as $k = 6$, $k = 8$ or $k = 3$, that leads to good network connectivity [13][14][15][16]. While such a number might be sufficient for connectivity of small-scale networks, Xue and Kumar [3] find that a large-scale network is disconnected with probability one when a fixed $k$ is employed. They study the analogous problem in the dense network case [3] and show that, if there are $N$ nodes uniformly located in a unit square, each node should be connected with $\Theta(\log N)$ nearest neighbors so that the network is connected with probability one asymptotically as $N$ goes to infinity. The exact value of $k$ that guarantees the connectivity should be more than $0.074 \log N$ and less than $5.1774 \log N$. 
In addition to the $k$-neighbor model, there exists the $r$-radius model for a wireless ad hoc network, where all nodes employ the same radio power, and thus each node can establish a direct link with any other node within some fixed distance $r$. Gupta and Kumar [1] find that in a network with $N$ nodes uniformly distributed in a unit area disk, the network is connected with probability one as $N \to \infty$ if and only if $\pi r^2 = \frac{\log N + c(N)}{N}$ and $\lim_{N \to \infty} c(N) = \infty$. Naturally, in this case, the expected number of neighbors of one node is $N \pi r^2 = \log N + c(N)$. Comparing this result with that for the $k$-neighbor model, it is reasonable to conjecture that the true value of $k$ should take the form of $\log N + c(N)$. However, this has not been established.

The argument in [3] leading to the lower bound of $0.074 \log N$ is very complicated. The most important reason for this complexity is the dependence of the nodes placed in two non-overlapping areas for a network consisting of $N$ nodes uniformly distributed in a unit square (denoted by $G(N)$ here). However, such a dependence does not exist in a network $G^{\text{Poisson}}(N)$, where the total number of nodes is a Poisson random variable with parameter $N$. Inspired by Lemma 4 in [17], it is possible to study the connectivity performance of the $k$-neighbor network $G^{\text{Poisson}}(N)$ and then establish a link from $G^{\text{Poisson}}(N)$ to $G(N)$. In this paper, we use this approach to improve the lower bound on the number of neighbors required for the asymptotic connectivity of ad hoc networks.

II. $0.129 \log N$ NEIGHBORS ARE NECESSARY FOR CONNECTIVITY

We focus on $G^{\text{Poisson}}(N)$, where nodes are distributed according to a two-dimensional Poisson point process. First, a disconnection pattern for the network is defined. Then the probability of there existing at least one such pattern is studied and a lower bound to this probability is obtained, which is a function of $k$. Thus, for any $k$ that makes this lower bound go to one, a lower bound of $k$ below which the network will be disconnected asymptotically with probability one is obtained. Finally, a link is made from $G^{\text{Poisson}}(N)$ to $G(N)$ that yields the desired result.
A. A Scenario for Disconnection

Definition 1: \( B_r(X) \): a disk centered at \( X \) with radius \( r \).

Definition 2: Trap of type \( d(r, a, L) \): a structure with three disks centered at the same point \( X_0 \), namely \( B_r(X_0) \), \( B_{(1+a)r}(X_0) \) and \( B_{(1+2a)r}(X_0) \) (see Figure 1). Furthermore, \( L \) non-overlapping disks of radius \( ar/2 \) (denoted by \( B_{ar/2}(Y_i), i \in \{1, \ldots, L\} \)) are evenly spaced in the annulus of inner radius \((1 + a)r\) and outer radius \((1 + 2a)r\). We call this structure a trap of type \( d(r, a, L) \).

Such a structure will cause a disconnection when the nodes are distributed according to some rules and the parameter \( L \) is large enough, thereby motivating the name ‘trap’.

\[ \text{Fig. 1: Trap of } d(r, a, L). \]

Definition 3: \( k \)-filling event: a trap of type \( d(r, a, L) \), where there exist \( k \) nodes in the disk \( B_r(X_0) \) and in each of the disks \( B_{ar/2}(Y_i), i \in \{1, \ldots, L\} \), and no additional nodes elsewhere in the disk \( B_{(1+2a)r}(X_0) \) (see Figure 2).

Consider the number of non-overlapping disks of diameter \( ar \) that can be placed in the annulus of inner radius \((1 + a)r\) and outer radius \((1 + 2a)r\).

Lemma 1: Let \( L_{\text{max}}(a) \) denote the maximum value that \( L \) can take for a trap of type \( d(r, a, L) \). Then
Fig. 2: A $k$-filling event for a trap of type $d(r, a, L)$, where $k = 4$ and $L = 5$.

$L_{\text{max}}(a)$ is a function of $a$ given by:

$$L_{\text{max}}(a) = \left\lfloor \frac{\pi}{\arcsin\left(\frac{a}{2 + 3a}\right)} \right\rfloor$$  \hspace{1cm} (1)

**Proof:** See details in Figure 3.

$$\alpha = 2 \arcsin \left( \frac{ar/2}{(1 + 3a/2)r} \right) = 2 \arcsin \left( \frac{a}{2 + 3a} \right)$$ \hspace{1cm} (2)

$$\Rightarrow L \leq \left\lfloor \frac{\pi}{\arcsin\left(\frac{a}{2 + 3a}\right)} \right\rfloor$$ \hspace{1cm} (3)

**Lemma 2:** For a given $a$, $\exists L_{\text{min}}(a) \leq L_{\text{max}}(a)$ such that $\forall L \in [L_{\text{min}}(a), L_{\text{max}}(a)]$, the $k$-filling event
that occurs in a trap of type $d(r, a, L)$ implies that the nodes in the center disk $B_r(X_0)$ of this trap are disconnected with the nodes outside $B_r(X_0)$ and, hence, the network is disconnected. (See Figure 4)

**Proof:**

![Diagram](image)

Fig. 4: Estimation of $L_{\text{min}}(a)$ ($L_0 = 5$ in this case).

See Figure 4 Consider a node $X_2$ which lies outside of the disk $B_{(1+2a)r}(X_0)$. $X_2$ will choose $k$ nearest nodes as its neighbors. If it selects one of the nodes in $B_r(X_0)$, a link from $B_r(X_0)$ to the outside of $B_{(1+2a)r}(X_0)$ exists, and the $k$-filling event for this trap of type $d(r, a, L)$ does not imply a disconnection scenario. Therefore, in order to guarantee that the $k$-filling event for this specified structure leads to a disconnection, we need to increase $L$, the number of the sub-disks in the annulus, so that a disk centered outside of $B_{(1+2a)r}(X_0)$ and tangent with $B_r(X_0)$ must contain at least one of the sub-disks, say $B_{ar/2}(Y_i)$ entirely. This guarantees that each node not in $B_{(1+2a)r}(X_0)$ contains a sufficient number of nodes ($\geq k$) closer to it than any node in $B_r(X_0)$. Furthermore, we just need to find the value of $L$ that is large enough so that any disk $B_{2ar}(X_1)$, which is centered on the boundary of $B_{(1+2a)r}(X_0)$, must contain at least one $B_{ar/2}(Y_i)$.

From (2), $\alpha$ is fixed given $a$. Let $\beta$ denote the angle between two neighbors $B_{ar/2}(Y_i)$ and $B_{ar/2}(Y_{i+1})$. Let $\beta_0$ denote the corresponding $\beta$ satisfying the condition that $B_{2ar}(X_1)$ is tangent with $B_{ar/2}(Y_i)$ and...
$B_{ar/2}(Y_5)$. Furthermore, for $\beta = \beta_0$, $B_{ar}(X_1)$ is tangent with $B_{ar/2}(Y_1)$, $B_{ar/2}(Y_5)$ and $B_{(1+a)r}(X_0)$ (see Figure 2). Therefore, we can put a disk of radius $ar/2$ in $B_{ar}(X_1)$, denoted as $B_{ar/2}(Y_1)$ in Figure 2 that is tangent with $B_{(1+a)r}(X_0)$ and $B_{(1+2a)r}(X_0)$ but does not contact $B_{ar/2}(Y_1)$ and $B_{ar/2}(Y_5)$. From above analysis, we know that $\beta_0 > \alpha$ for any $a$. Obviously, given the condition that $\beta \leq \beta_0$, the $k$-filling event that occurs in the corresponding trap of type $d(r, a, L(a))$ implies that the nodes in the center disk $B_r(X_0)$ of this trap are disconnected with the nodes outside $B_r(X_0)$ and, hence, the network is disconnected. Thus, for any $L(a)$ such that

$$L(a) \geq \left\lceil \frac{\pi}{2 \arcsin\left(\frac{a}{2+3a}\right)} \right\rceil = \left\lceil \frac{2\pi}{\alpha + \beta_0} \right\rceil \quad (4)$$

the corresponding $\beta$ is no greater than $\beta_0$. Therefore, the $k$-filling event that occurs in the trap of type $d(r, a, L(a)), L(a) \geq \left\lceil \frac{\pi}{2 \arcsin\left(\frac{a}{2+3a}\right)} \right\rceil$, implies that the network is disconnected. Thus, it yields:

$$L_{\min}(a) \leq \left\lceil \frac{\pi}{2 \arcsin\left(\frac{a}{2+3a}\right)} \right\rceil \quad (5)$$

which is less than $L_{\max}(a) = \left\lfloor \frac{\pi}{\arcsin(a/(2+3a))} \right\rfloor$.

**Lemma 3:** The maximum number $S$ of non-overlapping traps of type $d(r, a, L)$ that can be placed in a unit square is given by:

$$S = \left\lfloor \frac{1}{2(1+2a)r} \right\rfloor^2 \quad (6)$$

**Proof:** Clearly, we can divide the square into sub-squares of edge length $2(1+2a)r$ and put one trap into one sub-square. So we have:

$$S = \left\lfloor \frac{1}{2(1+2a)r} \right\rfloor^2 \quad (7)$$

**Lemma 4:** In $G^{\text{Poisson}}(N)$, the probability of a $k$-filling event for a trap of type $d(r, a, L)$ is:

$$P_{k-\text{filling}} = \frac{(N\pi r^2)^k}{k!} \left\lceil \frac{(N\pi a^2 r^2/4)^k}{k!} \right\rceil^L e^{-N\pi(1+2a)^2 r^2} \quad (8)$$

**Proof:** A $k$-filling event for a trap of type $d(r, a, L)$ means that there are $k$ nodes in each of the disks $B_r(X_0)$ and $B_{ar/2}(Y_i), i \in \{1, \ldots, L\}$ and no additional node elsewhere in the disk $B_{(1+2a)r}(X_0)$. 
Therefore:

\[
P_{k-filling} = \frac{(N \pi r^2)^k}{k!} e^{-N \pi r^2} e^{-N \pi (1+2a)^2 r^2 - \pi r^2 - L \pi a^2 r^2 / 4} \left[ \frac{(N \pi a^2 r^2 / 4)^k}{k!} e^{-N \pi a^2 r^2 / 4} \right]^L
\]  

(9)

\[
P_{k-filling} = \frac{(N \pi r^2)^k}{k!} \left[ \frac{(N \pi a^2 r^2 / 4)^k}{k!} \right]^L e^{-N \pi (1+2a)^2 r^2}
\]  

(10)

B. Probability for the Disconnection

1) Disconnection in \( G^{Poisson}(N) \): From the analysis in the last section, we know that the existence of a \( k \)-filling event for a trap of type \( d(r, a, L) \), \( L > L_{min}(a) \), in the network means that the network is disconnected. Furthermore, if the probability of existing at least one such \( k \)-filling event in the network is one, the network is disconnected with probability one.

Divide the unit square into non-overlapping sub-squares of edge length \( 2(1 + 2a)r \) and put one trap of type \( d(r, a, L) \) into each sub-square. As \( r, a \) and \( L \) are free parameters, which do not influence the real network connectivity, we can choose any \( r, a \) and \( L \) to facilitate the proof. In the following derivations, \( r \downarrow 0 \) as \( N \uparrow \infty \); \( a \) will be a constant as \( N \uparrow \infty \); \( L \) is a function of \( a \) and therefore it will vary between \( L_{min} \) and \( L_{max} \). Naturally, the feasible number of traps in the square \( S \uparrow \infty \) as \( N \uparrow \infty \) since \( r \downarrow 0 \). Furthermore, we assume that the expected number of nodes in \( B_r(X_0) \) of a trap \( N \pi r^2 \) goes to infinity as \( N \uparrow \infty \). Since the number of nodes in \( B_r(X_0) \) is a Poisson random variable with a parameter increasing to infinity, the probability for it to be some number \( k \) shrinks to zero regardless of the value of \( k \). Therefore, the probability of a \( k \)-filling event of a trap, denoted by \( P_{k-filling} \), shrinks to zero as \( N \uparrow \infty \). If a network \( G^{Poisson}(N) \) is connected, there does not exist a \( k \)-filling event when we place \( S \) non-overlapping traps of type \( d(r, a, L) \) into it. Since the distributions of nodes in separate traps are independent, the probability that the network \( G^{Poisson}(N) \) is connected is bounded by:

\[
P_{Connected}^{Poisson} \leq (1 - P_{k-filling})^S \rightarrow e^{-SP_{k-filling}}
\]  

(11)
From the above derivations, we can see that if \( SP_{k-filling} \to \infty \), the network will be disconnected with probability one asymptotically. Hence consider:

\[
SP_{k-filling} = \left[ \frac{1}{2(1+2a)r} \right]^2 \left( \frac{N\pi r^2}{k!} \right)^L e^{-N\pi(1+2a)^2r^2} \tag{12}
\]

Considering the same expression with the floor function replaced yields:

\[
f(a, r) = \left[ \frac{1}{2(1+2a)r} \right]^2 \left( \frac{N\pi r^2}{k!} \right)^L e^{-N\pi(1+2a)^2r^2}
= \left[ \frac{1}{2(1+2a)} \right]^2 \left( \frac{N\pi}{k!} \right)^L \left( \frac{(N\pi a^2/4)^k}{k!} \right)^L e^{-N\pi(1+2a)^2r^2}, \tag{14}
\]

and, as \( r \downarrow 0 \):

\[
SP_{k-filling} = f(a, r) - o(f(a, r)) \tag{15}
\]

Hence, it is sufficient to study \( f(a, r) \). Recalling (11)(15), we can maximize \( f(a, r) \) by selecting appropriate values of \( a, L, c, k, \) and \( r \) to obtain a tighter upper bound of \( P_{\text{Poisson Connected}} \). Therefore, letting \( \frac{\partial f(a, r)}{\partial (r^2)} = 0 \) yields \( r^2 = \frac{k(L+1)-1}{N\pi(1+2a)^2} \). Substituting \( r^2 = \frac{k(L+1)}{N\pi(1+2a)^2} \) into \( f(a, r) \) and assuming \( k \uparrow \infty \) as \( N \uparrow \infty \) yields:

\[
f \left( a, \left[ \frac{k(L+1)}{N\pi(1+2a)^2} \right]^{1/2} \right) = \frac{1}{4(1+2a)^2} \frac{N^{kL+k_L+\pi(k+1)2kL}}{k!(L+1)^{4kL+1}} \frac{k^{(L+1)+1+k_{L+1}}}{L^{k_{L+1}-1}} e^{-kL-k}
= \frac{1}{4kL+k} \frac{N\pi k^{kL+1}(L+1)^{kL+k+1}a^{2kL}}{(k!)^{L+1}} e^{-kL-k}
\geq \frac{1}{4kL+k} \frac{N\pi k^{kL+1}(L+1)^{kL+k+1}a^{2kL}}{(k!)^{L+1}} \frac{e^{-kL-k}}{\sqrt{2\pi k^{1/2}e^{-k+1/12k}}} \tag{18}
= \frac{1}{4(2\pi)^{(L+1)^{2/(L+3)}}(L+1)^{(L+3)^{2/(L+3)}}} e^{-\frac{L+1}{12k}} \tag{19}
\]

Assuming that \( k = c \log N \) and that \( a, L, c \) are constants yields:

\[
f \left( a, \left[ \frac{k(L+1)}{N\pi(1+2a)^2} \right]^{1/2} \right) \geq \frac{1}{4(2\pi)^{(L+1)^{2/(L+3)}}(L+1)} \frac{c \log N}{(c \log N)^{(L+3)^{2/(L+3)}}} \frac{(L+1)^{2kL}}{4^{L+1}(1+2a)^{2(L+1)}} e^{c \log N} \tag{20}
= \Theta \left( \frac{N}{(c \log N)^{(L+3)^{2/(L+3)}}} \right) e^{-\frac{L+1}{12k}} \tag{21}
\]
Let $y = c \log \frac{(L+1)^{L+1}2^L}{4^L(1+2a)^{2(L+1)}}$. We claim that if $y > -1$, $f(a, r)$ goes to infinity, because:

$$f \geq \Theta\left(\frac{N \ast N^y}{(c \log N)^{(L+3)/2}}\right)$$

$$= \Theta\left(\frac{N^{1+y}}{(c \log N)^{(L+3)/2}}\right)$$

$$\to \infty \text{ for } y > -1$$

From $y > -1$, we can find easily:

$$c < \left(-\log \frac{(L + 1)^{L+1}2^L}{4^L(1+2a)^{2(L+1)}}\right)^{-1} \quad \text{where } a \in (0, \infty), L \in [L_{\text{min}}, L_{\text{max}}]$$

Since the probability for a disconnected network is constructive, for any $a, r$ and $L$ that fit the assumptions, the $c$ obtained in (25) guarantees disconnectivity with probability one asymptotically if $k = c \log N$. Thus, by exhaustive numerical search, we can select $a$ and $L$ to maximize $c$. The result is given by: $a = 3.6, L_{\text{max}} = 11, L_{\text{min}} \leq 6, L = 11, c < 0.129$. Thus if $k < 0.129 \log N$, $f(a, r)$ will go to infinity, and, hence, $SP_{k\text{-filling}}$ will go to infinity. Since $P_{\text{Poisson Connected}}^{\text{Poisson}} \approx e^{-SP_{k\text{-filling}}}$, $P_{\text{Poisson Connected}}^{\text{Poisson}}$ will go to zero as $N \uparrow \infty$.

In this way, we obtain:

**Theorem 1:** For a $G_{\text{Poisson}}(N)$, if each node connects with $k$ nearest neighbors, where $k < 0.129 \log N$, the network will be disconnected with probability one as $N$ goes to infinity.

In this part, $a$ and $L$ are two parameters used to bound $P_{\text{Poisson Connected}}^{\text{Poisson}}$. It is natural to conjecture that any kind of square divisions and any kind of traps can be applied to bound the probability for network connection. In order to minimize $f(a, r)$ in (13), we set $N \pi (1 + 2a)^2 r^2 = k(L + 1)$, which is just the expected number of nodes in a $B_{(1+2a)r}(X_0)$. We achieve the maximal value of $c$ when $L = L_{\text{max}}$ because the expected number of nodes in a trap increases as $N \uparrow \infty$. So a trap of type $d(a, r, L)$ is likely to have less empty area and therefore as many $L$ sub-disks as possible. The model of a $k$-filling event can be extended by allowing more than $k$ nodes in the disk $B_r(X_0)$ and $B_{ar/2}(Y_i)$ (see Figure 4). This extension represents a more general case of disconnection than the original $k$-filling event and might be useful for improving the bound further.
2) Disconnection in $G(N)$: One of the most important differences between $G(N)$ and $G^{\text{Poisson}}(N)$ is that, due to the fixed total number of nodes, the probability of the $k$-filling events of non-overlapping traps are not independent in $G(N)$, which introduces technical difficulties if we directly apply the method above developed for $G^{\text{Poisson}}(N)$. It is the dependence of these events that is the reason for the complication of [3]. Now, we want to find a connection between $G^{\text{Poisson}}(N)$ and $G(N)$ so that Theorem I can be applied.

Let $h(N)$ be the actual number of nodes in the unit square for $G^{\text{Poisson}}(N)$. Obviously, $h(N)$ is a Poisson distributed random variable with parameter $N$. Recall Lemma 4 of [17]:

$$\lim_{N \to \infty} P \left( h(N) : h(N) \in \left[ N - \sqrt{\frac{\pi N}{2}}, N + \sqrt{\frac{\pi N}{2}} \right] \right) = 1$$

(26)

Thus as $N$ increases to infinity, although the true number of nodes in the square is a random variable, the ratio between the fluctuation $\sqrt{\frac{\pi N}{2}}$ and $N$ goes to zero with probability one. Thus, it is natural to infer that Theorem I is also correct for $G(N)$.

Here is the brief idea of the following proof: we just assume that a network $G(N + \sqrt{\frac{\pi N}{2}})$ with $k < 0.129 \log N$ will be connected with strictly positive probability as $N \uparrow \infty$. Then we find that for any $h(N) \in [N - \sqrt{\frac{\pi N}{2}}, N + \sqrt{\frac{\pi N}{2}}]$, the network $G(h(N))$ with $k < 0.129 \log N$ will be connected with strictly positive probability, too. Therefore, we get a conclusion that a network $G^{\text{Poisson}}(N)$ with $k < 0.129 \log N$ will be connected with strictly positive probability, which contradicts Theorem I. So the assumption is incorrect, and the following theorem arises:

**Theorem 2:** For a network $G(N)$ in a unit square, if each node connects with $k_0(N)$ nearest neighbors, where $k_0(N) < 0.129 \log \left( N + \frac{\pi}{4} - \sqrt{\frac{\pi N}{2}} + \frac{\pi^2}{16} \right)$, there is:

$$\liminf_{N \to \infty} P_{\text{Con}}(G(N), k_0(N)) = 0$$

(27)

where $P_{\text{Con}}(G(N), k_0(N))$ denotes the probability for the event that the network $G(N)$ with parameter $k_0(N)$ is connected.

**Proof:** See Appendix I. ■
Since $\log\left(\frac{N + \pi/4 - \sqrt{\pi N/2 + \pi^2/16}}{\log N}\right) \to 1$ as $N \to \infty$, this theorem improves the lower bound for the network connectivity from $0.074\log N$ to $0.129\log N$.

III. Conclusion

In this paper, we improve the lower bound on the number of neighbors required for the asymptotic connectivity of a dense ad hoc network. Critical to the proof is the use of the $G^{\text{Poisson}}(N)$ model, for which the distributions of nodes in non-overlapping areas are not dependent. The result is then extended from $G^{\text{Poisson}}(N)$ to the $G(N)$ model of interest, resulting in an improvement in the lower bound for the latter model to $0.129\log N$.

APPENDIX I

PROOF OF THEOREM [2]

Proof: Since the convergence of $\{P_{\text{Con}}(G(h(N)), k < 0.129\log N), N = 1, 2, \ldots\}$, is unknown, “lim inf” is used instead of “lim” throughout the proof. Naturally, we have:

\[
\begin{align*}
\lim_{N \to \infty} \inf P \left( G \left( N + \sqrt{\pi N/2} \right) \text{ is connected with } k < 0.129 \log N \right) \\
= \lim_{M \to \infty} \inf P \left( G(M) \text{ is connected with } k < 0.129 \log \left( M + \frac{\pi}{4} - \sqrt{\frac{\pi M}{2} + \frac{\pi^2}{16}} \right) \right) \quad (28) \\
\leq \lim_{M \to \infty} \inf P \left( G(M) \text{ is connected with } k < 0.129 \log \left( (M + 1) + \frac{\pi}{4} - \sqrt{\frac{\pi(M + 1)}{2} + \frac{\pi^2}{16}} \right) \right) \quad (29) \\
= \lim_{N \to \infty} \inf P \left( G \left( (N + \sqrt{\pi(N)/2} - 1 \right) \text{ is connected with } k < 0.129 \log N \right) \quad (30) \\
\leq \lim_{M \to \infty} \inf P \left( G(M) \text{ is connected with } k < 0.129 \log \left( M + \frac{\pi}{4} + \sqrt{\frac{\pi M}{2} + \frac{\pi^2}{16}} \right) \right) \quad (31) \\
= \lim_{N \to \infty} \inf P \left( G \left( N - \sqrt{\pi N/2} \right) \text{ is connected with } k < 0.129 \log N \right) \quad (32)
\end{align*}
\]

Similarly, $\forall h(N) \in \left[ N - \sqrt{\pi N/2}, N + \sqrt{\pi N/2} \right]$:

\[
\begin{align*}
\lim_{N \to \infty} \inf P \left( G \left( N + \sqrt{\pi N/2} \right) \text{ is connected with } k < 0.129 \log N \right) \\
\leq \lim_{N \to \infty} P \left( G(h(N)) \text{ is connected with } k < 0.129 \log N \right)
\end{align*}
\]
Assume that:

$$\liminf_{N \to \infty} P \left( G \left( N + \sqrt{\pi N/2} \right) \right) \text{ is connected with } k < 0.129 \log N > 0 \quad (34)$$

and seek a contradiction. Recalling (33):

$$\liminf_{N \to \infty} \frac{1}{2} P \left( G \left( N + \sqrt{\pi N/2} \right) \right) \text{ is connected with } k < 0.129 \log N$$

$$< \liminf_{N \to \infty} P \left( G \left( h(N) \right) \right) \text{ is connected with } k < 0.129 \log N \quad (35)$$

Combined with (28), (29) and (30), this yields:

$$\liminf_{M \to \infty} \frac{1}{2} P \left( G \left( M \right) \right) \text{ is connected with } k < 0.129 \log \left( M + \frac{\pi}{4} - \sqrt{\frac{\pi M}{2} + \frac{\pi^2}{16}} \right)$$

$$= \liminf_{N \to \infty} \frac{1}{2} P \left( G \left( N + \sqrt{\pi N/2} \right) \right) \text{ is connected with } k < 0.129 \log N \quad (36)$$

$$< \liminf_{N \to \infty} P \left( G \left( N + \sqrt{\pi N/2} - 1 \right) \right) \text{ is connected with } k < 0.129 \log N$$

$$= \liminf_{M \to \infty} P \left( G \left( M \right) \right) \text{ is connected with } k < 0.129 \log \left( (M + 1) + \frac{\pi}{4} - \sqrt{\frac{\pi (M + 1)}{2} + \frac{\pi^2}{16}} \right) \quad (38)$$

Then, it is obvious that: \( \exists N_0 \), such that \( \forall N_1 > N_0, \forall N_2 > N_0 \),

$$\inf_{n_1 \geq N_1} \frac{1}{2} P \left( G \left( n_1 \right) \right) \text{ is connected with } k < 0.129 \log \left( n_1 + \frac{\pi}{4} - \sqrt{\frac{\pi n_1}{2} + \frac{\pi^2}{16}} \right)$$

$$< \inf_{n_2 \geq N_2} P \left( G \left( n_2 \right) \right) \text{ is connected with } k < 0.129 \log \left( (n_2 + 1) + \frac{\pi}{4} - \sqrt{\frac{\pi (n_2 + 1)}{2} + \frac{\pi^2}{16}} \right) \quad (39)$$

$$\leq \inf_{n_2 \geq N_2} P \left( G \left( n_2 \right) \right) \text{ is connected with } k < 0.129 \log \left( n_2 + \frac{\pi}{4} + \sqrt{\frac{\pi n_2}{2} + \frac{\pi^2}{16}} \right) \quad (40)$$

Thus, we have shown that there exists \( N_0, \forall N > 0 \) such that \( N - \sqrt{\pi N/2} > N_0 \), for any \( h(N) \) such that \( h(N) \in \left[ N - \sqrt{\pi N/2}, N + \sqrt{\pi N/2} \right] \), there is:

$$\inf_{n \geq N} \frac{1}{2} P \left( G \left( n + \sqrt{\pi n/2} \right) \right) \text{ is connected with } k < 0.129 \log n$$

$$< \inf_{n \geq N} P \left( G \left( g(n) \right) \right) \text{ is connected with } k < 0.129 \log n \quad (41)$$
From Theorem 1, let $k_0 < 0.129 \log N$ and let $T$ be the number of nodes in the square. Then,

$$\lim_{N \to \infty} P_{\text{Con}}(G_{\text{Poisson}}(N), k_0) = \lim_{N \to \infty} \sum_{j=0}^{\infty} P_{\text{Con}}(G(j), k_0) P(T = j)$$

(42)

$$= \lim_{N \to \infty} \sum_{j=N-\sqrt{\pi N/2}}^{N+\sqrt{\pi N/2}} P_{\text{Con}}(G(j), k_0) P(T = j)$$

(43)

$$\geq \lim_{N \to \infty} \sum_{j=N-\sqrt{\pi N/2}}^{N+\sqrt{\pi N/2}} \inf_{n \geq j} P_{\text{Con}}(G(n), k_0) P(T = j)$$

(44)

$$> \lim_{N \to \infty} \sum_{j=N-\sqrt{\pi N/2}}^{N+\sqrt{\pi N/2}} \inf_{n \geq N} \frac{1}{2} P_{\text{Con}} \left( G \left( n + \sqrt{\frac{\pi}{2}} \right), k_0 \right) P(T = j)$$

(45)

$$= \lim_{N \to \infty} \frac{1}{2} P_{\text{Con}} \left( G \left( N + \sqrt{\frac{\pi}{2}} \right), k_0 \right)$$

(46)

Combined with (28), this yields:

$$0 < \lim_{N \to \infty} \inf P \left( G(N) \text{ is connected with } k < 0.129 \log \left( N + \frac{\pi}{4} - \sqrt{\frac{\pi N}{2} + \frac{\pi^2}{16}} \right) \right)$$

$$= \lim_{N \to \infty} \inf P \left( G \left( N + \sqrt{\frac{\pi N}{2}} \right) \text{ is connected with } k < 0.129 \log N \right)$$

(47)

$$< \lim_{N \to \infty} 2P(G_{\text{Poisson}}(N) \text{ is connected with } k < 0.129 \log N)$$

(48)

which contradicts Theorem 1. Thus, the assumption (34) is not correct.

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