REES ALGEBRA OF A SQUAREFREE MONOMIAL IDEAL

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Abstract. Let $S = k[X_1, \ldots, X_n]$ be a polynomial ring, where $k$ is a field. This article deals with the defining ideal of the Rees algebra of squarefree monomial ideal generated in degree $n - 2$. As a consequence, we prove that Betti numbers of powers of the cover ideal of the complement graph of a tree do not depend on the choice of tree. Further, we study the regularity and Betti numbers of powers of cover ideals associated to certain graphs.

1. Introduction

Let $G$ be a graph on the vertex set $V = \{X_1, X_2, \ldots, X_n\}$, and $S = k[X_1, X_2, \ldots, X_n]$ be a polynomial ring over a field $k$. Then the ideal $I(G) = \langle X_i X_j : X_i \text{ is adjacent to } X_j \rangle$ is called the edge ideal of $G$. The Alexander dual of $I(G)$, denoted as $J(G)$, is called the cover ideal of $G$. The main problem of interest in this article is to study algebraic invariants such as Hilbert series, regularity and Betti numbers of powers of cover ideals associated to certain graphs. The regularity of a monomial ideal is an important invariant in commutative algebra and algebraic geometry and it measures the complexity of its minimal free resolution. It is known that for a homogeneous ideal $I \subset S$, $\text{reg}(I^s)$ is a linear function of $s$ for $s \gg 0$, i.e., there exist non-negative integers $a, b$ and $s_0$ such that $\text{reg}(I^s) = as + b$ for all $s \geq s_0$. This result was proved by Cutkosky, Herzog and Trung [2] and independently by Kodiyalam [15]. While the constant $a$ is given by the maximum degree of minimal generators of $I$. On the other hand, $b$ and $s_0$ are not well understood and problem of finding their values is addressed by many authors, see [1, 3, 6, 12, 13, 20]. The problem of computing the regularity or finding bounds on the regularity is a difficult problem. Thus one would like to provide bounds and give an explicit formula for the regularity of ideals associated to certain graphs (edge ideals, cover ideals). In the case of edge ideals and cover ideals, the regularity has been studied by various authors, e.g., see [1, 12, 13, 14]. Thus it is interesting to study the regularity of powers of cover ideals associated to certain graphs.

Another object of interest is the Rees algebra of an ideal. The Rees algebra of a homogeneous ideal $I \subset S$ is a bigraded algebra defined as $\mathcal{R}(I) = \oplus_{s \geq 0} I^s t^s$. Rees algebra helps to study the asymptotic behaviour of an ideal and useful in computing the integral closure of powers of an ideal. Rees algebra of an ideal $I$ provides a condition such that $I^s$ has a linear resolution for all $s \geq 1$. Römer in [18] gives an upper bound for the regularity of powers of a homogeneous ideal in terms of $x$-regularity of corresponding Rees algebra $\mathcal{R}(I)$. In particular, if $x$-regularity of $\mathcal{R}(I)$ is zero, then each power of $I$ has a linear resolution. For a homogeneous ideal $I$, the defining ideal of $\mathcal{R}(I)$ is studied by many authors, see [10, 11, 16], and D. Taylor in [19] studied it for a monomial ideal. Further, Villarreal in [21] gives an

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explicit description of the defining ideal of the Rees algebra of any square free monomial ideal generated in degree 2.

We now give a brief overview of the paper. Section 2 covers some basics of graph theory and commutative algebra which are used throughout the paper.

In section 3, we study Rees algebras of cover ideals of certain graphs. For a squarefree monomial ideal \( J \) generated in degree \( n - 2 \), we associate a graph \( G_J \). Further, we study the Rees algebra of \( J \) using the combinatorial properties of \( G_J \). In particular, we prove that if \( G_J \) is a tree, then \( \mathcal{R}(J) \) is a quadratic complete intersection, and hence \( J^s \) has a linear resolution for all \( s \geq 1 \). As a consequence, if \( J \) is the cover ideal of the complement graph of a tree, then the Hilbert series and Betti numbers of \( J^s \) do not depend on the choice of tree.

In section 4, we give an combinatorial formula to compute the Betti numbers of powers of cover ideal of a graph \( G \), where \( G \) is either a complete graph or a complement graph of a tree. In section 5, we compute the regularity of powers of the cover ideals of complete multipartite graphs. Hence, we settle Conjecture 4.10 and 4.11 given by A. V. Jayanthan and N. Kumar in [12].

2. Preliminaries

Let \( S = k[X_1, \ldots, X_n] \). We use the following notation in this article.

**Notation 2.1.**

a) \([n] = \{1, \ldots, n\}, n \in \mathbb{N}\).

b) \( \text{Mon}(S) = \) set of all monomials in \( S \).

c) For a monomial ideal \( I \), we denote \( M(I) \) be the minimal generating set of monomials of \( I \).

d) \( w(X_i) = \) weight assigned on each variable \( X_i \) in \( S \). For \( u = X_1^{a_1} \cdots X_n^{a_n} \in \text{Mon}(S) \), the weighted degree of \( u \) is given by \( \deg_w(u) = \sum_{i=1}^{n} a_i w(X_i) \). Further, if \( w(X_i) = 1 \) for all \( i \), then we use \( \deg(u) \) for \( \deg_w(u) \).

e) For \( u = X_1^{a_1} \cdots X_n^{a_n} \in \text{Mon}(S) \), we denote \( m_i(u) = a_i \).

f) For \( v \in \text{Mon}(S) \), we denote \( \text{supp}(v) = \{i : X_i \text{ divides } v\} \).

g) For any non empty subset \( F \subset [n] \), we set \( X_F = \prod_{i \in F} X_i \).

**Definition 2.2.**

i) A finite simple graph is an ordered pair \((V, E)\), where \( V \) is a collection of vertices and \( E \) is a collection of subsets of \( V \) with cardinality 2. The elements of \( E \) are called the edges of a graph \( G \). We assume that \( V = [n] \).

ii) Let \( G = (V, E) \) be a graph on the vertex set \( V \). Then the complement graph of \( G \), denoted by \( G^c \), is a graph on vertices \( V \) such that \( \{i, j\} \) is an edge of \( G^c \) if and only if \( \{i, j\} \notin E \).

iii) A graph \( G = (V, E) \) is called a complete graph if for every \( i, j \in V \), we have \( \{i, j\} \in E \).

iv) A subset \( C \subset [n] \) is called a cover set of \( G \) if for every \( \{i, j\} \in E \), we have \( \{i, j\} \cap C \neq \emptyset \). This set is called a minimal cover set if for any \( l \in C, C \setminus \{l\} \) is not a cover set of \( G \).

v) An independent set of a finite graph \( G \) is a subset \( S \subset V \) such that \( \{i, j\} \notin E \) for all \( i, j \in S \). The independent complex of a finite graph \( G \) is a simplicial complex \( \Delta(G) \) on \( V \) whose facets are the maximal independent subsets of \( G \).

vi) A cycle of length \( \ell \) in a graph \( G \) is a subgraph \( L \) of \( G \) with edge set \( E(L) = \{\{i_1, i_2\}, \{i_2, i_3\}, \ldots, \{i_\ell, i_1\}\} \).
such that \( i_r \neq i_s \) for \( r \neq s \).

ii) A chord of a cycle \( L \) in a graph \( G \) is an edge \( \{i, j\} \) of \( G \) such that \( \{i, j\} \) is not an edge of \( L \) and \( \{i, j\} \subset V(L) \). A graph \( G \) is called a chordal graph, if any cycle in \( G \) of length greater than 3 has a chord.

In the following example, we illustrate the minimal cover sets of a graph \( G \).

**Example 2.3.**

a) Let \( G \) be a graph with \( V = \{1, 2, 3, 4\} \) and \( E = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 1\}\} \).

In this case, minimal cover sets of \( G \) are \( \{1, 3\} \), \( \{2, 4\} \).

b) Let \( G \) be the complete graph on the vertex set \( V = [n] \). Then \( C \) is a minimal cover set of \( G \) if and only if there exists \( i \in [n] \) such that \( C = [n] \setminus \{i\} \).

**Definition 2.4** (Edge and Cover Ideals). Let \( G = (V, E) \) be a graph on the vertex set \( V = [n] \) and \( S = k[X_1, \ldots, X_n] \).

i) The edge ideal of \( G \), denoted by \( I(G) \), is defined as

\[
I(G) = \langle X_iX_j : \{i, j\} \in E \rangle.
\]

ii) The cover ideal of \( G \), denoted by \( J(G) \), is defined as

\[
J(G) = \left\langle \prod_{i \in C} X_i : C \text{ is a minimal cover set of } G \right\rangle.
\]

**Example 2.5.**

a) Let \( G \) be as in Example 2.3(a). Then \( J(G) = \langle X_1X_3, X_2X_4 \rangle \) is the cover ideal of \( G \).

b) Let \( G \) be the complete graph on the vertex set \( V = [n] \). Then the cover ideal of \( G \) is

\[
J(G) = \left\langle \prod_{j \neq i} X_j : i \in [n] \right\rangle.
\]

**Definition 2.6.** Let \( M \) be a finitely generated graded \( S \)-module.

i) Then \( \beta^S_{i,j}(M) = (\dim_k(\text{Tor}^S_i(M, k)))_j \) is called the \( (i, j) \)-th graded Betti number of \( M \).

ii) The regularity of \( M \), denoted as \( \text{reg}(M) \), is defined as

\[
\text{reg}(M) = \max\{j - i : \beta^S_{i,j}(M) \neq 0\}.
\]

iii) The projective dimension of \( M \), denoted as \( \text{pdim}_S(M) \), is defined as

\[
\text{pdim}_S(M) = \max\{i : \beta^S_{i,j}(M) \neq 0 \text{ for some } j\}.
\]

iv) Let \( \text{pdim}_S(M) = p \). If for each \( i \in \{0, 1, \ldots, p\} \), there exists a number \( d_i \) such that \( \beta^S_{i,j}(M) = 0 \) for \( j \neq d_i \), then \( M \) is said to have a pure resolution of type \( (d_0, d_1, \ldots, d_p) \). Further, if \( d_0 = d \) and \( d_i = d + i \) for all \( i \in [p] \), then we say that \( M \) has a linear resolution.
3. REES ALGEBRA OF A MONOMIAL IDEAL

Let $J \subset S$ be a squarefree monomial ideal generated in degree $n - 2$. Then for every $u \in M(J)$, there exist $i, j$ such that $[n] \setminus \text{supp}(u) = \{i, j\}$. Now we associate a graph $G_J$ to the ideal $J$ on the vertex set $[n]$ with edge set $\{(i, j) : [n] \setminus \text{supp}(u) = \{i, j\}$ for some $u \in M(J)\}$. In this section, we discuss the Rees algebra $\mathcal{R}(J)$ and give an explicit description of the defining ideal of $\mathcal{R}(J)$ in terms of properties of $G_J$. In particular, we find the defining ideal of the Rees algebra of the cover ideal of a graph whose complement graph is triangle free.

**Definition 3.1.** Let $I$ be a homogeneous ideal of $S$. Then the Rees algebra of $I$ is defined as $S[It] = \bigoplus_{s \geq 0} I^s t^s$, and it is denoted by $\mathcal{R}(I)$.

Let $I = \langle u_1, \ldots, u_r \rangle$ and $R = S[T_1, \ldots, T_r]$. Then there is a surjective ring homomorphism $\phi : R \rightarrow \mathcal{R}(I)$ by setting $\phi(X_i) = X_i$ for $i = 1, \ldots, n$ and $\phi(T_j) = u_j t$ for $j = 1, \ldots, r$. Set $K = \ker(\phi)$. The ideal $K$ is called the defining ideal of the Rees algebra. Further, note that $K = \bigoplus K_i$, where $K_i$ is a homogeneous component of degree $i$ in $T_j$-variables. If $K = RK_1$, then $I$ is said to be of linear type.

**Notation 3.2.** Let $I$ be a monomial ideal generated by $u_1, \ldots, u_r$. Let $I_s$ be a set of all non-decreasing sequences $\alpha = (i_1, \ldots, i_s)$ in $[r]$ of length $s$. Then for any $\alpha \in I_s$ we denote $u_\alpha = u_{i_1} u_{i_2} \cdots u_{i_s}$, $T_\alpha = T_{i_1} T_{i_2} \cdots T_{i_s}$, and $\hat{u}_{\alpha i} = u_{i_1} \cdots \hat{u}_{i_i} \cdots u_{i_s}$. For any $\alpha, \beta \in I_s$, we define $T_{\alpha, \beta} = \frac{\text{lcm}[u_\alpha, u_\beta]}{u_\beta} T_\beta - \frac{\text{lcm}[u_\alpha, u_\beta]}{u_\alpha} T_\alpha$.

Defining ideal of the Rees algebra of a monomial ideal is studied by D. Taylor in [19]. In order to prove Proposition 3.6, we use the following result.

**Theorem 3.3** (Taylor, [19]). Let $S = k[X_1, \ldots, X_n]$ and $I = \langle u_1, \ldots, u_r \rangle$ be a monomial ideal in $S$. Then $\mathcal{R}(I) \simeq R/K$, where $R = S[T_1, \ldots, T_r]$, $K = RK_1 + R(\bigcup_{s=2} K_s)$ with $K_s = \{T_{\alpha, \beta} : \alpha, \beta \in I_s\}$.

**Notation 3.4.** Let $S = k[X_1, \ldots, X_n]$ and $u$ be a squarefree monomial in $S$. Set $X = X_1 \cdots X_n$. We denote the monomial $\frac{X}{u}$ by $u'$.

**Remark 3.5.** Let $u, v$ be monomials in $S$. Then we have $\frac{\text{lcm}[u', v']}{v'} = \frac{\text{lcm}[u, v]}{u}$.

In the following proposition, we extend the result [21, Theorem 3.1] of Villarreal for any squarefree monomial ideal generated in degree $n - 2$.

**Proposition 3.6.** Let $J = \langle u_1, \ldots, u_r \rangle$ be a squarefree monomial ideal of $S$ generated in degree $n - 2$ and $K$ the defining ideal of the Rees algebra of $J$. Then $K = RK_1 + R(\bigcup_{s=2} P_s)$, where $P_s = \{T_{\alpha} - T_\beta : u_\alpha = u_\beta \text{ for some } \alpha, \beta \in J_s\}$.

**Proof.** For $s \geq 2$, let $\alpha = (i_1, \ldots, i_s)$ and $\beta = (j_1, \ldots, j_s) \in J_s$ such that $\alpha \neq \beta$. From the proof of [21, Theorem 3.1], it follows that there exist integers $l$ and $m$ and a monomial $v$
such that \( \text{lcm}[u'_\alpha, u'_\beta] = u'_i \hat{u}'_{\beta_m} v \). This implies that \( \frac{\text{lcm}[u'_\alpha, u'_\beta]}{\text{lcm}[u'_i, u'_{j_m}] u'_{\beta_m}} \) and \( \frac{\text{lcm}[u'_\alpha, u'_\beta]}{\text{lcm}[u'_i, u'_{j_m}] u'_{\beta_m}} \) are monomials in \( R \). By Theorem 3.3, we know that \( K_s \) is generated by polynomials of type

\[
T_{\alpha, \beta} = \frac{\text{lcm}[u'_\alpha, u'_\beta]}{u_\beta} T_{\beta} - \frac{\text{lcm}[u'_\alpha, u'_\beta]}{u_\alpha} T_{\alpha}.
\]

By Remark 3.5, we know that \( T_{\alpha, \beta} = \frac{\text{lcm}[u'_\alpha, u'_\beta]}{u_\beta} T_{\beta} - \frac{\text{lcm}[u'_\alpha, u'_\beta]}{u_\alpha} T_{\alpha} \). Now, one can note that

\[ T_{\alpha, \beta} = AT_m + B \hat{T}_\alpha, \]

where

\[
A = \frac{\text{lcm}[u'_\alpha, u'_\beta]}{\text{lcm}[u'_i, u'_{j_m}] u'_{\beta_m}} \left( \frac{\text{lcm}[\hat{u}'_{\alpha}, \hat{u}'_{\beta_m}] \hat{T}_{\beta_m}}{\hat{u}'_{\alpha}} - \frac{\text{lcm}[\hat{u}'_{\alpha}, \hat{u}'_{\beta_m}] \hat{T}_{\alpha}}{\hat{u}'_{\beta_m}} \right),
\]

and

\[
B = \frac{\text{lcm}[u'_\alpha, u'_\beta]}{\text{lcm}[u'_i, u'_{j_m}] u'_{\beta_m}} \left( \frac{\text{lcm}[u'_i, u'_{j_m}] T_{j_m}}{u'_i} - \frac{\text{lcm}[u'_i, u'_{j_m}] T_{i}}{u'_{j_m}} \right).
\]

Note that \( A \in K_{s-1} \) and \( B \in K_1 \), and hence we get \( K_s \subset R_1 K_{s-1} + R_{s-1} K_1 \). The backward mathematical induction completes the proof.

**Corollary 3.7.** Let \( J \) be a squarefree monomial ideal generated in degree \( n - 2 \) and \( G_J \) be the associated graph. Then \( J \) is linear type if and only if \( G_J \) is a forest or it has a unicycle with an odd length cycle.

**Proof.** From Proposition 3.6, it can be seen that \( J \) is of linear type if and only if for any \( s \geq 2 \), we have \( u_\alpha \neq u_\beta \) for all \( \alpha, \beta \in J_s \) with \( \alpha \neq \beta \). Note that \( u_\alpha = u_\beta \) if and only if \( u'_\alpha = u'_\beta \). It follows from [21 Corollary 3.2] \( u'_\alpha \neq u'_\beta \) for all \( \alpha, \beta \in J_s \) if and only if \( G_J \) is a forest or it is a unicycle with an odd length cycle.

**Remark 3.8.** Let \( J \) be a squarefree monomial ideal of \( S \) generated in degree \( n - 2 \).

i) If \( G_J \) is a connected graph, then \( J \) has a linear resolution.

ii) If \( G_J \) is a triangle free connected graph, then \( J \) is the cover ideal of the complement graph of \( G_J \).

iii) Assume that \( G_J \) is a tree. Since tree is a chordal graph, by [8 Theorem 9.2.3], we get that \( I(G_J) \) has a linear resolution. Hence using part (ii), we get \( J \) is Cohen-Macaulay.

Further, part (i) and \( \text{ht}(J) = 2 \) forces that

\[
\beta_{1,j}^S(J) = \begin{cases} n - 2 & \text{if } j = n - 1 \\ 0 & \text{otherwise.} \end{cases}
\]

**Lemma 3.9.** Let \( J \) be a squarefree monomial ideal generated in degree \( n - 2 \). If \( G_J \) is a tree, then \( R(J) \) is a quadratic complete intersection. In particular, \( J^s \) has a linear resolution for \( s \geq 1 \).

**Proof.** Let \( R = S[T_1, \ldots, T_{n-1}] \) and \( R(J) \simeq R/K \). By Corollary 3.7, we know that \( J \) is of linear type, and hence \( K \) is generated by \( \beta_{1,j}^S(J) = n - 2 \) elements. Note that \( \text{dim}(R) = 2n - 1 \) and \( \text{dim}(R(J)) = n + 1 \). This implies that \( \text{ht}(K) = n - 2 = \mu(K) \) which forces that \( K \) is generated by a regular sequence, and hence \( R(J) \) is a complete intersection.

Further, \( J \) is linearly presented and of linear type implies that \( K \) is generated in degree \((1,1)\). Since \( K \) is generated by a regular sequence of degree \((1,1)\), we get that \( \text{reg}_x(R(J)) = 0 \). Now, [8 Proposition 10.1.16] completes the proof.
Lemma 3.10. Let $T_1$ and $T_2$ be trees on $n$ vertices and $J_1$ and $J_2$ be cover ideals of their complement graphs, respectively. Then $H_{R(I)}(z_1, z_2) = H_{R(J)}(z_1, z_2)$ for all $s \geq 1$.

**Proof.** Note that if $H_{R(I)}(z_1, z_2)$ is the bigraded Hilbert series of the Rees algebra of a monomial ideal $I$, then

$$H_{I^s}(z_1) = \frac{1}{s!} \frac{\partial^s}{\partial z_2^s} (H_{R(I)}(z_1, z_2)) \bigg|_{z_2=0}.$$  

Thus it is enough to prove that $H_{R(J)}(z_1, z_2) = H_{R(J)}(z_1, z_2)$. This follows from the proof of Lemma 3.9. □

Corollary 3.11. Let $J_1$ and $J_2$ be as in Lemma 3.10. Then $\beta_J(J_1^s) = \beta_J(J_2^s)$ for all $s \geq 1$.

**Proof.** The proof of the corollary follows from Remark 3.8(i) and Lemma 3.10. □

The above corollary says that if $G$ is the complement graph of a tree, then the Betti numbers of powers of cover ideal do not depend on tree. However, if $G$ is the complement of a unicyclic graph, then above result does not hold. For example, let $G$ be the complement graph of a cycle $C_7$ and $H$ be the complement of $\{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{5, 1\}, \{5, 6\}, \{6, 7\}\}$. Then using Macaulay2 [5], we see that

$$196 = \beta_{2,16}^J \left( \frac{S}{J(G)^s} \right) \neq \beta_{2,16}^J \left( \frac{S}{J(H)^s} \right) = 195.$$

4. Betti Numbers of Powers of Cover Ideals

Let $K_n$ be a complete graph on $n$ vertices and $J(K_n)$ be its cover ideal. In this section our goal is to compute the Betti numbers of $J(K_n)^s$ for $s \geq 1$ which proves [12, Conjecture 4.10]. As a consequence, we find the Betti numbers of powers of cover ideal of the complement of a tree. Now for $s \geq 1$, observe that

$$M(J(K_n)^s) = \left\{ \frac{X^s}{w} : X = \prod_{i=1}^n X_i, w \in \text{Mon}(S), \deg(w) = s \right\}.$$  

Lemma 4.1. For $s \geq 1$, the ideal $J(K_n)^s$ has linear quotients with respect to the reverse lexicographic order of the generators.

**Proof.** Let $u = \frac{X^s}{w_1}, v = \frac{X^s}{w_2} \in M(J(K_n)^s)$ with $w_1, w_2 \in \text{Mon}(S)$ of degree $s$. Since we have $\deg(w_1) = \deg(w_2)$, for any $i$ with $m_i(w_1) > m_i(w_2)$ there exists some $j$ such that $m_j(w_2) > m_j(w_1)$. Further, note that $\deg \left( \frac{X^s}{X_i} w_1 \right) = s$. The remaining proof follows from [7, Lemma 1.3]. □

Definition 4.2. Let $I$ be a monomial ideal having linear quotients with respect to some order $u_1, \ldots, u_r$ of elements of $M(I)$. Then, for $2 \leq j \leq r$, we define $I_j = \langle u_1, \ldots, u_{j-1} \rangle$ and set $u_j = \{X_k : X_k \in I_j : u_j \}$. We set $A_t(I^s) = \{u \in M(I^s) : |\text{set}(u)| = t\}$.

We use the following result of Herzog and Takayama [7, Lemma 1.5], to compute the Betti numbers of $J(K_n)^s$.

Lemma 4.3 (Herzog-Takayama, [7]). Suppose that $I$ has linear quotients with respect to order $u_1, \ldots, u_r$ of generators of $I$ and $\deg(u_1) \leq \deg(u_2) \leq \cdots \leq \deg(u_r)$. Then the iterated mapping cone $F_i$, derived from the sequence $u_1, \ldots, u_m$, is a minimal graded free resolution of $S/I$ and for all $i > 0$, the symbols $f(\sigma; u)$ with $u \in M(I), \sigma \subset \text{set}(u), |\sigma| = i - 1$, forms a basis for $F_i$. Moreover, $\deg(f(\sigma, u)) = \deg(u) + |\sigma|$.  

Remark 4.4. Note that the proof of above Lemma remains valid even if we assign weight \(w(X_i) = w_i\) for each \(i\). Also in this case, \(\deg_w(f(\sigma, u)) = \deg_w(u) + \sum_{X_i \in \sigma} w_i\).

Let \(M(J(K_n)^s) = \{u_1, \ldots, u_r\}\), where \(u_1 > u_2 > \cdots > u_r\) in the reverse lexicographical order with respect to \(X_1 > X_2 > \cdots > X_n\). Then for \(2 \leq j \leq r\), we compute the set(\(u_j\)), which will be useful in proving Theorem 4.6.

Lemma 4.5. For \(2 \leq j \leq r\), let \(u_j = \frac{X^s}{v}\). Then set(\(u_j\)) = \(\{X_i : i < n \text{ and } i \in \text{supp}(v)\}\).

Proof. Let \(X_i \in \text{set}(u_j)\), where \(1 \leq i \leq n\). Then \(X_i u_j \in \langle u_1, \ldots, u_{j-1} \rangle\). In particular, there exists \(v_1 \in \text{Mon}(S)\) such that \(X_i u_j = u_t v_1\), for some \(1 \leq l \leq j - 1\). Note that \(\deg(u_t) = \deg(u_j)\) implies that \(\deg(v_1) = 1\), hence \(v_1 = X_{j_i}\) for some \(j_i \in [n]\). Thus, we get \(\frac{X_i u_j}{X_j} = u_t \in M(J(K_n)^s)\). This implies that that \(m_i \left(\frac{X_i u_j}{X_j}\right) \leq s\), and hence \(m_i \left(\frac{u_j}{X_j}\right) \leq s - 1\). Since \(X_i \neq X_j\), we get \(i \in \text{supp}(v)\). Suppose \(i = n\). Then \(X_n u_j = u_t X_{j_i}\) implies that \(m_{n}(u_t) > m_{n}(u_j)\), which is a contradiction to the fact that \(u_t > \text{revlex} \ u_j\).

Conversely, let \(i \in \text{supp}(v), i \neq n\). Since \(j \neq 1\), we know that \(X_n | u_j\). Now, consider a monomial \(u' = \frac{X_i u_j}{X_n}\). Clearly, \(u' > \text{revlex} \ u_j\) and \(X_i u_j = u' X_n\). This completes the proof. \(\square\)

Now we are in position to compute the Betti numbers of \(J(K_n)^s\).

Theorem 4.6. The Betti numbers of \(J(K_n)^s\) are given by

\[
\beta_i(J(K_n)^s) = \binom{n-1}{i} \binom{n-1-i+s}{n-1}.
\]

Proof. Firstly, we show that \(|A_t(J(K_n)^s)| = \binom{n-1}{t} \binom{n-1}{t}\). In view of Lemma 4.5, we get

\[
A_t(J(K_n)^s) = \left\{ \frac{X^s}{v} : v \in \text{Mon}(S), \deg(v) = s \text{ and } |\text{supp}(v) \cap [n-1]| = t \right\}.
\]

Since each monomial \(v \in A_t(J(K_n)^s)\) corresponds to a unique monomial \(v'\) in \((n-1)\) variables of degree less than or equal to \(s\) with \(|\text{supp}(v')| = t\), we get \(|A_t(J(K_n)^s)| = \binom{n-1}{t} \binom{n-1}{t}\). Now using Lemma 4.3 we get

\[
\beta_i^S(J(K_n)^s) = \sum_{t=0}^{n-1} \binom{n-1}{t} \binom{s}{t} \binom{t}{i} \binom{n-1-i+s}{n-1},
\]

where the last equality follows from the Chu-Vandermonde identity. \(\square\)

As an immediate consequence, we get the following result.

Theorem 4.7. Let \(T\) be a tree on \(n\) vertices and \(G\) the complement graph of \(T\). Let \(J(G)\) be the cover ideal of \(G\). Then the Betti numbers of \(J(G)^s\) are given by

\[
\beta_i(J(G)^s) = \binom{n-2}{i} \binom{n-2-i+s}{n-2}.
\]

Proof. By Corollary 3.11, we may assume that \(T\) be a star graph, and hence its complement is a complete graph on \(n-1\) vertices. Thus the result follows from Theorem 4.6. \(\square\)
5. Regularity of Powers of Cover Ideals of Complete Multipartite Graphs

In this section, our goal is to prove [12] Conjecture 4.11. Let $J(G)$ be the cover ideal of a complete $n$-partite graph $G$ on the vertex set $V$ with partition $V_1 \sqcup \cdots \sqcup V_n$, where $V_i = \{X_{i,j}\}$, $1 \leq j \leq w_i$. Then by taking $X_i = \prod_{j=1}^{w_i} X_{i,j}$, one can identify $J(G)$ with the cover ideal of a complete graph on vertices $X_1, \ldots, X_n$. We set $deg(X_{i,j}) = 1$ and $w(X_i) = w_i$. Thus to compute the regularity of powers of the cover ideal of a complete multipartite graphs, we compute the regularity of powers of the cover ideal of a complete graph $K_n$ on vertices $X_1, \ldots, X_n$ with $w(X_i) = w_i$.

**Notation 5.1.** Let $\sigma \subset \{X_1, \ldots, X_n\}$ and $u$ be a monomial in $S$. Then we denote

$$m(u, \sigma) = u \prod_{X_k \in \sigma} X_k.$$

**Remark 5.2.** Let $J(K_n)$ be the cover ideal of a complete graph $K_n$ on vertices $X_1, \ldots, X_n$ and $M(J(K_n)^s) = \{u_1, \ldots, u_r\}$, where $u_1 > u_2 > \cdots > u_r$ in the reverse lexicographical order. Further, if we assume $w(X_i) = w_i$ with $w_i \leq w_j$ for all $i \leq j$, then observe the following.

i) If $u \in M(J(K_n)^s)$ and $\sigma \subset \text{set}(u)$, then by Lemma 4.5, we get that for any $i$, $X_i^{s+1}$ does not divide $m(u, \sigma)$. In other words, we have $m_i(m(u, \sigma)) \leq s$, for all $i$. Further, if $|\sigma| = i - 1$, then $\text{deg}_w(m(u, \sigma)) \leq w_1(i - 1) + s \sum_{l=2}^{w_i} w_l$.

ii) It follows from Lemma 4.3 and Lemma 4.5 that $\text{pdim}_S(S/J(K_n)^s) = \begin{cases} s + 1 & \text{if } s < n - 1, \\ n & \text{otherwise.} \end{cases}$

Now we proceed to calculate the regularity of $S/J(K_n)^s$ in the above set-up.

**Theorem 5.3.** Let $S = k[X_1, \ldots, X_n]$ with $w(X_i) = w_i$ and $K_n$ be a complete graph on vertices $X_1, \ldots, X_n$. Further, if we assume $w_1 \leq \cdots \leq w_n$, then

$$\text{reg}(S/J(K_n)^s) = \begin{cases} \frac{s \sum_{l=1}^{n} w_l - (s + 1)}{s \sum_{l=1}^{n} w_l - w_1(s - n + 1) - n} & \text{if } s < n - 1, \\ \text{otherwise.} \end{cases}$$

**Proof.** Firstly, using Lemma 4.4 we get $J(K_n)^s$ has linear quotients with reverse lexicographic order. Now, Lemma 4.3 implies that a basis element of $i$th component $F_i$ of a graded minimal free resolution $F_*$ of $S/J(K_n)^s$ is given as following:

$$f(\sigma; u) \text{ with } u \in M(J(K_n)^s), \sigma \subset \text{set}(u), |\sigma| = i - 1,$$

where $\text{deg}_w(f(\sigma; u)) = \text{deg}_w(m(\sigma; u))$. This implies that $\beta_i^{s,J}(S/J(K_n)^s) \neq 0$ if and only if there exists some $\sigma \subset \text{set}(u)$ with $|\sigma| = i - 1$ such that $j = \text{deg}_w(m(\sigma; u))$. Let $d_i = \max\{j : \beta_i^{s,J}(S/J(K_n)^s) \neq 0\}$. Then

$$d_i = \max\{\text{deg}_w(m(\sigma; u)) : \sigma \subset \text{set}(u), |\sigma| = i - 1 \text{ and } u \in M(J(K_n)^s)\}.$$

By Remark 5.2(ii), it is easy to see that $s \geq i - 1$. Now, for $i < n$ take $u = X_s^{n(1)}X_s^{n-2}X_s^{n-1}$ with $\sigma = \{X_2, \ldots, X_i\}$, and for $i = n$ take $u = X_s^{n(1)}X_s^{n-2}X_s^{n-1}$ with $\sigma = \{X_1, \ldots, X_{i-1}\}$.

Now note that in the both cases $m(\sigma, u) = X_{i-1}X_{i-2}^sX_{i-1}^s$, and hence $\text{deg}_w(m(\sigma, u)) = w_1(i - 1) + s \sum_{l=2}^{w_i} w_l$. Thus Remark 5.2(i) gives $d_i = w_1(i - 1) + s \sum_{l=2}^{w_i} w_l$. Note that $d_i - i \leq d_{i+1} - (i + 1)$. Hence from Remark 5.2(ii), the result follows. □
As an immediate consequence, we get the following corollary.

**Corollary 5.4.** Let $G$ be a complete multipartite graph with partition on the vertex set $V_1 \sqcup \cdots \sqcup V_n$. If $|V_i| = w_i$ with $w_i \leq w_{i+1}$, then

$$
\text{reg}(S/J(G)^s) = \begin{cases} 
    s \sum_{l=1}^n w_l - (s + 1) & \text{if } s < n - 1, \\
    s \sum_{l=1}^n w_l - w_1(s - n + 1) - n & \text{otherwise}.
\end{cases}
$$

**References**

[1] A. Banerjee, *The regularity of powers of edge ideals*, J. Algebraic Combin., 41 (2015), no. 2, 303 – 321.

[2] S. Cutkosky, J. Herzog, and N. V. Trung, *Asymptotic behaviour of the Castelnuovo-Mumford regularity*, Compositio Math., 118 (1999), 243 – 261.

[3] H. Dao, C. Huneke, and J. Schweig, *Bounds on the regularity and projective dimension of ideals associated to graphs*, J. Algebraic Combin., 38 (2013), 37 – 55.

[4] A. Eagon and V. Reiner, *Resolutions of Stanley-Reisner Rings and Alexander Duality*, J. pure & Applied Algebra 130 (1998), 265 – 275.

[5] D. Grayson, M. Stillman, *Macaulay2, a software system for research in algebraic geometry*, available at http://www.math.uiuc.edu/Macaulay2/

[6] H. T. Hà and A. V. Tuyl, *Monomial ideals, edge ideals of hypergraphs, and their graded Betti numbers*, J. Algebraic Combin., 27 (2008), 215 – 245.

[7] J. Herzog, Y. Takayama, *Resolutions by mapping cone*, Homology, Homotopy Appl. 4 (2002), 277 – 294.

[8] J. Herzog and T. Hibi, *Monomial Ideals*, Graduate Texts in Mathematics, 260, Springer-Verlag London Ltd., London, 2011.

[9] J. Herzog and T. Hibi, *The depth of powers of an ideal*, J. Algebra, 291 (2005), 534 – 550.

[10] S. Huckaba, *On complete d-sequences and the defining ideals of Rees algebras*, Math. Proc. Cambridge Philos. Soc. 106 (1989), 445 – 458.

[11] C. Huneke, *On the symmetric and Rees algebra of an ideal generated by a d-sequence*, J. Algebra 62 (1980), 268 – 275.

[12] A. V. Jayanthan and N. Kumar, *Resolution and regularity of cover ideals of certain multipartite graphs*, arXiv:1709.05055.

[13] A. V. Jayanthan, N. Narayanan, and S. Selvaraja, *Regularity of powers of bipartite graphs*, J. Algebraic Comb., 47 (2018), no. 1, 17 – 38.

[14] M. Katzman, *Characteristic-independence of Betti numbers of graph ideals*, J. Combin. Theory Ser. A 113 (2006), no.3, 435 – 454.

[15] V. Kodiyalam, *Asymptotic behaviour of Castelnuovo-Mumford regularity*, Proc. Amer. Math. Soc., 128 (2000), 407 – 411.

[16] S. Morey, B. Ulrich, *Rees algebras of ideals with low codimension*, Proc. Amer.Math. Soc. 124 (1996), 3653 – 3661.

[17] I. Peeva, *Graded syzygies*. Algebra and Applications 14, Springer-Verlag London, Ltd., London, 2011.

[18] T. Römer, *Homological properties of bigraded algebras*, Illinois J. Math., 45 (2001), 1361 – 1376.

[19] D. Taylor, *Ideals generated by monomials in an R-sequence*, Ph.D. thesis, University of Chicago, 1966.

[20] A. V. Tuyl, *Sequentially Cohen-Macaulay bipartite graphs: vertex decomposability and regularity*, Arch. Math. (Basel), 93 (2009), 451 – 459.

[21] R. H. Villarreal, *Rees algebras of edge ideals*, Comm. Algebra, 23 (1995), 3513 – 3524.

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