Next-to-next-to-leading order post-Newtonian spin-orbit Hamiltonian for self-gravitating binaries

Johannes Hartung, Jan Steinhoff

Theoretisch–Physikalisches Institut,
Friedrich–Schiller–Universität,
Max–Wien–Platz 1, 07743 Jena, Germany, EU

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Abstract

We present the next-to-next-to-leading order post-Newtonian (PN) spin-orbit Hamiltonian for two self-gravitating spinning compact objects. If at least one of the objects is rapidly rotating, then the corresponding interaction is comparable in strength to a 3.5PN effect. The result in the present paper in fact completes the knowledge of the post-Newtonian Hamiltonian for binary spinning black holes up to and including 3.5PN. The Hamiltonian is checked via known results for the test-spin case and via the global Poincaré algebra with the center-of-mass vector uniquely determined by an ansatz.

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1 Introduction

In the present paper the next-to-next-to-leading order post-Newtonian (PN) spin-orbit Hamiltonian for two self-gravitating spinning compact objects is derived. This Hamiltonian is of the order 3.5PN if at least one of the objects is rapidly rotating. Indeed, the result in the present paper completes the knowledge of the post-Newtonian Hamiltonian (and thus of the equations of motion) for binary spinning black holes up to and including 3.5PN. Besides the well-known Newtonian and 1PN Hamiltonians, previous results to this order for point-masses are the 2PN [1–3], 2.5PN [4, 5], 3PN [6–10], and 3.5PN [11, 12] Hamiltonians. For the spin part the leading order can be found in [13–16] and the next-to-leading order in [17–19]. At the 3.5PN level one also needs all Hamiltonians cubic in the spins derived in [20, 21]. Notice that so far these cubic Hamiltonians are known for binary black holes only, whereas all other mentioned Hamiltonians (including the one derived in the present paper) are valid for general compact objects [or have been generalized to this case, see [22–25] for the spin(1)-spin(1) level]. Further, tidal effects become very important for general compact objects like neutron stars, see, e.g., [26] and also [27].

The calculation in the present paper was performed within the canonical formalism of Arnowitt, Deser, and Misner (ADM) [28, 29], which was recently generalized to rotating objects at linear order in spin [30], see also [31–33]. For various other (noncanonical) derivations of post-Newtonian results at the point-mass level see [34–40] and references therein. The next-to-leading order in spin was also treated in [41–47]. Also more than two compact objects have been treated at high post-Newtonian orders [48–52]. Notice that the calculation in the present paper is comparable in complexity to the one of the 3PN point-mass Hamiltonian. In particular one has to check for certain integral contributions that can only be handled correctly within dimensional regularization, as they may lead to ambiguities when integrated in three spatial dimensions.
The result of the present paper was verified using the test-spin Hamiltonian in Kerr-spacetime given in [54] and using the global Poincaré algebra. For the latter the center-of-mass vector had to be determined uniquely from an ansatz with 68 coefficients.

It should be noted that the Hamiltonian derived in the present paper is yet only useful within the Taylor-expanded post-Newtonian series if at least one of the objects is rapidly rotating (due to the missing 4PN point-mass Hamiltonian). However, if the 4PN point-mass Hamiltonian can be derived in the future, then the result of the present paper must be included in the post-Newtonian series also when the spins are small. But the effective one-body approach for nonspinning objects is able to cover such higher post-Newtonian orders by calibration to numerical relativity, see [55, 56] and references therein. The result given in the present paper is thus expected to be valuable for the effective one-body formalism, just like the next-to-leading order one [57–59].

The paper is organized as follows. In Sect. 2 a brief outline of the calculation is given. The next-to-next-to-leading order spin-orbit Hamiltonian is presented in Sect. 3. A comparison with known results for the test-spin case is made. The Hamiltonian is further checked via the global Poincaré algebra in Sect. 4, where the center-of-mass vector is uniquely determined from an ansatz. In forthcoming papers we will derive the next-to-next-to-leading order spin(1)-spin(2) Hamiltonian and provide much more details on the calculation of the spin-orbit Hamiltonian, shown in the present paper, as well.

Three-dimensional vectors are written in boldface and their components are denoted by Latin indices. The scalar product between two vectors $\mathbf{a}$ and $\mathbf{b}$ is denoted by $(\mathbf{a} \cdot \mathbf{b})$. Our units are such that $c = 1$. There is no special convention for Newton’s gravitational constant $G$. In the results $P_a$ denotes the canonical linear momentum of the $a$th object, $z_a$ the position of the object, $m_a$ the mass of the object, $\mathbf{S}_a$ the spin of the object, $r_{ab} = |z_a - z_b|$ the relative distance between two objects, and $n_{ab} = (z_a - z_b)/r_{ab}$ the direction vector pointing from object $b$ to object $a$. In the integrands $r_a = |x - z_a|$, $n_a = (x - z_a)/r_a$, and $s_{ab} = r_a + r_b + r_{ab}$. In the binary case the object labels $a, b$ take only the values 1 and 2.

## 2 Outline of the calculation

In the following we present a short outline of our calculation – which will be discussed more detailed in a forthcoming publication – and cite the main literature necessary to undertake it.

For all computations we used XTensor [60], a free package for MATHEMATICA [61], especially because of its fast index canonicalizer based on the package XPERM [62]. We also used the package XPERT [63], which is part of XTensor, for performing the perturbative part of our calculations. Furthermore we wrote several MATHEMATICA packages ourselves for evaluating integrals.

First we generalized the derivation of the canonical formalism given in [30] to arbitrary dimensions. The initial action is of the same form as in $d = 3$ and the $(d + 1)$-split is also straightforward. The $d$-dependence enters the calculation via the relation between extrinsic curvature and field momentum, and via the decomposition of the metric and field momentum in the ADM transverse-traceless gauge. The calculation was done in $d$ dimensions due to possible appearance of ambiguities in three-dimensional integrals. Ambiguity means that one will get different results when one does an integration by parts in a certain integral. These ambiguities can only be ruled out or corrected by doing the UV-analysis explained in [10, 53], which relies on the complete $d$-dependence of the integrands. In the following UV-analysis always refers to the analysis of certain integrals via a Taylor expansion of the field expressions in the $r_1$ variable, extracting the critical $r_1$ powers, and averaging over the $n_1$ vectors afterwards to get the pole part of the integrals in $\varepsilon \equiv d - 3$.

The integrations by parts necessary to get the Hamiltonian presented in this paper were done like suggested in [6, 10, 32] to get comparable intermediate results. After accomplishing the integration by parts we ended up with a Hamiltonian density which can be split up into three parts: A kinetic field part (containing the kinetic energy of the propagating field degrees of freedom) a matter part (containing only field-matter-interactions), and an interaction part (containing interactions between matter fields and the propagating fields), see, e.g., [4, 5]. We checked intermediate results against [6, 30, 32, 33].

From the Hamiltonian obtained by integrating the density from above one can go to a Routhian (a Hamiltonian in matter degrees of freedom and a Lagrangian in the propagating field degrees of freedom) as suggested in [6, 33] and can eliminate the propagating degrees of freedom by inserting their approximate
solutions in terms of the matter variables [6]. Subsequent elimination of time derivatives corresponds to a coordinate transformation [64, 65].

After obtaining a suitable Hamiltonian density via the simplifications mentioned in the last paragraphs, one has to integrate all appearing terms in the density to get a full Hamiltonian. The appearing integrals can be divided into three types: the delta-type \( \int d^d x f(x) \), the Riesz-type \( \int d^d x n^1_1 \ldots n^d_1 n^1_2 \ldots n^d_2 r^\beta_1 r^\gamma_2 \) and the generalized Riesz-type \( \int d^d x n^1_1 \ldots n^d_1 n^1_2 \ldots n^d_2 r^\beta_1 r^\gamma_2 s_{1,12} \).

The delta-type integrals can be solved by the Partie-Finie regularization procedure mentioned in the appendices of, e.g., [6, 11, 33]. Another possibility to solve some of them is via the Riesz kernel method, where one inserts a Riesz kernel for the delta functions [53, 66]. This was done as an alternative way to check whether our algorithms work correctly, although we did not calculate all delta-type integrals via the Riesz kernel method because not all appearing inverse Laplacians can be solved via the method mentioned in [11]. Of course, using the Riesz kernel instead of a delta as source of the fields (to eliminate the necessity of distributional derivatives) makes the integration much more complicated because all delta-type integrals will change into integrals of the Riesz-type or the generalized Riesz-type.

The Riesz-type integrals can be solved by eliminating the \( n_1 \) and \( n_2 \) vectors via rewriting them into derivatives as shown in [6, 33] and solving the remaining scalar integrals via the Riesz-formula [53, 66]

\[
\int d^d x r^\alpha_1 r^\beta_2 \text{reg} = \frac{\pi^{d/2} \Gamma\left(\frac{\alpha+\beta}{2}\right) \Gamma\left(\frac{\beta+\gamma}{2}\right) \Gamma\left(-\frac{\alpha+\beta+\gamma}{2}\right)}{\Gamma\left(-\frac{\alpha+\beta+\gamma+2d}{2}\right)} r^{\alpha+\beta+\gamma}. \tag{1}
\]

The \( n_1 \) and \( n_2 \) vectors in the integrands of generalized Riesz-type cannot be eliminated via rewriting the vectors into derivatives. Instead one has to use the averaging procedure in prolate spheroidal coordinates in [6] to get rid of the \( n \) vectors. Afterwards one can use the generalized Riesz formula which was found by P. Jaranowski during his 3PN point-mass calculations (also given in [6]),

\[
\int d^d x r^\alpha_1 r^\beta_2 s_{1,12} \text{reg} = 2\pi \Gamma(\alpha+2)\Gamma(\beta+2)\Gamma(-\alpha-\beta-\gamma-4) \frac{I_{1/2}(\alpha+2,-\alpha-\gamma-2)}{\Gamma(-\gamma)}
+ I_{1/2}(\beta+2,-\beta-\gamma-2)
- I_{1/2}(\alpha+\beta+4,-\alpha-\beta-\gamma-4) - 1] r^{\alpha+\beta+\gamma+3}, \tag{2}
\]

which reduces to the formula for the integrals of the Riesz-type for \( \gamma \to 0 \). The function \( I_{1/2}(x, y) \) is the regularized incomplete Euler beta function which is defined as

\[
I_{1/2}(x, y) = \frac{B_{1/2}(x, y)}{B(x, y)}, \tag{3}
\]

with

\[
B_{1/2}(x, y) = \frac{1}{2^x \Gamma(x+y)} \Gamma\left(x, 1 + \frac{1}{2}\right), \quad B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \tag{4}
\]

being the incomplete Euler beta function and the Euler beta function respectively. Notice that all integrals of the generalized Riesz-type and all inverse Laplacians of two variables can only be solved in \( d = 3 \). Only the UV-singular part of those integrals can be evaluated in \( d \) dimensions.

It turns out after using the integration procedures mentioned above that all integrands of the generalized Riesz-type appearing at spin-orbit level have such a structure that the incomplete Euler beta functions appearing there can be reduced to Gamma functions and Polygamma functions, which could be handled very well by MATHEMATICA.

3 Result

To check our code we recalculated parts of the 3PN point-mass Hamiltonian given in [6, 10]. The next-to-next-to-leading order spin-orbit Hamiltonian we obtained as a result of the procedures discussed in Sect. 2.
is given by

$$H_{SNLO}^{3d} = \frac{G}{r_{12}} \left[ \left( \frac{7m_2(P_2^I)^2}{16m_1^2} + \frac{9(n_{12}P_1)(n_{12}P_2)P_2^I}{16m_1^2} + \frac{3P_1^2(n_{12}P_2)^2}{4m_1^2m_2^2} \right) + \frac{45(n_{12}P_1)(n_{12}P_2)^3}{16m_1^2m_2^2} + \frac{9P_1^2(n_{12}P_2)^2}{16m_1^2} - \frac{3(n_{12}P_2)^2(P_1P_2)}{16m_1^2m_2^2} \right]$$

$$- \frac{3(P_1^2)(P_2^2)}{16m_1^2m_2^2} - \frac{3(n_{12}P_1)(n_{12}P_2)P_2^I}{4m_1^2m_2^2} - \frac{15(n_{12}P_2)^2(n_{12}P_2)^2}{4m_1^2m_2^2} + \frac{3P_1^2(n_{12}P_2)^2}{2m_1^2m_2^2} - \frac{P_1^2(n_{12}P_2)^2}{2m_1^2m_2^2} + \frac{(P_1P_2)^2}{2m_1^2m_2^2}$$

$$+ \frac{3(n_{12}P_2)^2P_2^I}{2m_1^2m_2^2} - \frac{(P_1P_2)^2}{4m_1^2m_2^2} - \frac{3(n_{12}P_1)(n_{12}P_2)P_2^I}{2m_1^2m_2^2}$$

$$+ \frac{27(n_{12}P_1)(n_{12}P_2)^2}{16m_1^2m_2^2} - \frac{(n_{12}P_2)(P_1P_2)}{8m_1^2m_2^2} \left[ \frac{5(n_{12}P_1)P_2^I}{16m_1^2m_2^2} \right]$$

$$+ \frac{(n_{12}P_2)^2}{m_1^2m_2^2} ((P_1P_2S_1) + \frac{G^2}{r_{12}^2} \left[ \left( \frac{-3m_2(n_{12}P_1)^2}{2m_1^2} + \left( \frac{3m_2}{2m_1^2} + \frac{27m_2^2}{8m_1^2} \right) P_1^2 + \left( \frac{177}{16m_1} + \frac{11}{m_2} \right) (n_{12}P_2)^2 \right) + \frac{11}{2m_1} + \frac{9m_2}{2m_1^2} \right) (n_{12}P_1) (n_{12}P_2) + \left( \frac{23}{4m_1} + \frac{9m_2}{2m_1^2} \right) (P_1P_2)$$

$$+ \left( \frac{159}{16m_1} + \frac{37}{8m_2} \right) P_2^I (n_{12}P_1S_1) + \left( \frac{4(n_{12}P_1)^2}{m_1} + \frac{13P_1^2}{2m_1} \right)$$

$$+ \frac{5(n_{12}P_2)^2}{m_2} + \frac{53P_2^I}{8m_2} \left( \frac{211}{8m_1} + \frac{22}{m_2} \right) (n_{12}P_1) (n_{12}P_2)$$

$$+ \left( \frac{47}{8m_1} + \frac{5}{m_2} \right) (P_1P_2) (n_{12}P_1S_1) + \left( \frac{8}{m_1} + \frac{9m_2}{2m_1^2} \right) (n_{12}P_1)$$

$$+ \frac{59}{4m_1} + \frac{27}{2m_2} \left( n_{12}P_2 \right) \left( \frac{P_1P_2S_1} {2m_1^2} \right)$$

$$+ \frac{G^3}{r_{12}^2} \left[ \left( \frac{181n_{12}m_2}{16} + \frac{95m_2^3}{4} + \frac{75m_2^3}{8m_1} \right) \left( n_{12} \times P_1 \right) \right]$$

$$+ \left( \frac{21m_2^2}{4} + \frac{473m_1m_2^2}{16} + \frac{39m_2^3}{4} \right) \left( n_{12} \times P_2 \right) S_1 \right) + (1 \leftrightarrow 2).$$

Obviously there are no logarithmic dependencies of $r_{12}$ appearing.\(^1\) Also notice that the canonical antisymmetric spin tensor was rewritten in terms of the canonical spin vector, which is possible in $d = 3$. The $d$-dimensional UV-analysis described in \cite{10,53} and Sect. 2 gave contributions to intermediate expressions, however they exactly canceled in the final result. In contrast, for point-masses only the poles in $\varepsilon = d - 3$ canceled but a finite part remained. Further from a combinatorial point of view there are 66 algebraically different possible contributions to the Hamiltonian for each object (written in terms of the canonical spin tensor), but 24 of them do not appear in the canonical representation used here. The Hamiltonian is valid for any compact objects like black holes or neutron stars. It completes the knowledge

\(^1\)The published version of the present article contains a typo in the framed term, the coefficient should read $-\frac{5}{16}$ instead of $-\frac{15}{16}$. We thank S. Marsat for pointing this out. Notice that the term in question does not contribute in the center-of-mass frame.
of the dynamics up to and including 3.5PN for maximally rotating black holes. For other objects the $S^3$
Hamiltonians and the inclusion of tidal effects are missing. One can find a discussion of leading order tidal
effects in [26]. Notice that the coupling structure in this Hamiltonian reduces in the center-of-mass frame
to a pure (LS) structure with complicated coefficients. So the Hamiltonian is indeed a spin-orbit Hamiltonian.
We compared our result in the test-spin limit with the PN expanded Hamiltonians of a test-spin near
a Kerr black hole in ADM coordinates in [54, Eq. (6.20)] and got full agreement.\footnote{Note that there is a typo in [54] in Eq. (6.20), the last term $\frac{105}{8} m^2 (S^* \cdot L)$ has to be $\frac{75}{8} M^2 (S^* \cdot L)$. See the arXiv version of [54] for the correct equation. We thank E. Barausse and A. Buonanno for clarifying this issue.}

The matter variables appearing in the fully reduced matter-only Hamiltonian fulfill the standard Poisson bracket relations, namely

$$\{z^i_a, P_{aj}\} = \delta_{ij}, \quad \{\hat{S}_{a(i)}, \hat{S}_{a(j)}\} = \varepsilon_{ijk} \hat{S}_{a(k)}, \quad (6)$$

all other zero and the Hamiltonian can be used to get the time evolution of an arbitrary phase space function $A$
via

$$\frac{dA}{dt} = \{A, H\} + \frac{\partial A}{\partial t}. \quad (7)$$

Although the algorithms to be used at formal 3PN level are given above, the whole calculation is very hard. It is in particular much harder than (and really different from) the formal 2PN calculation at next-to-leading order spin-orbit level.

4 Approximate Poincaré algebra

In this section we check that the global Poincaré algebra is fulfilled in a PN approximate way, see, e.g.,
[9, 17]. Besides the Hamiltonian, the quantities entering the Poincaré algebra are the center-of-mass vector $G$,
the total linear momentum $P$, and the total angular momentum tensor $J^{ij} = -J^{ji}$. As the latter two are
the infinitesimal generators of translations and rotations, they can be expressed in terms of canonical
variables as $[9, 17]$

$$P = \sum_a P_a, \quad J^{ij} = \sum_a \left[ z^i_a P_{aj} - z^j_a P_{ai} + \hat{S}_{a(i)(j)} \right], \quad (8)$$

where the canonical spin tensor $\hat{S}_{a(i)(j)}$ is related to the canonical spin vector $\hat{S}_a$ via $\hat{S}_{a(i)(j)} = \varepsilon_{ijk} \hat{S}_{a(k)}$. Notice that the total angular momentum is not only the sum of the orbital angular momenta but also contains
the spin angular momenta. For the contributions of the propagating field degrees of freedom see, e.g., [31, 32]. As in [9, 17] we used an ansatz for the center-of-mass vector $G$ at next-to-next-to-leading order spin-orbit level which contains 68 unknown coefficients here. For comparison we mention that the 2PN binary
point-mass $G$-vector requires 20 unknown coefficients to be fixed and the 3PN binary point-mass $G$-vector requires 78 unknown coefficients to be fixed [9]. In [17, Eq. (5.9)] one can see the ansatz for the next-to-leading order case (which contains only 8 unknown coefficients).
But at the order considered here there will be additional linear momentum powers which increase the number of necessary coefficients significantly. 16 of them can be fixed by taking into account the \{\(G^i, P^j\)\} Poisson bracket relation appearing in the Poincaré algebra. The remaining 52 coefficients were uniquely fixed by evaluating the \{\(G, H\)\} Poisson brackets. The consistency of the solution obtained by evaluating the Poisson bracket above was checked by evaluating the \{\(G^i, G^j\)\} Poisson bracket relation and all remaining relations of the Poincaré algebra.

The center-of-mass vector at next-to-next-to-leading order spin-orbit level is given by

$$G^{3 N L O}_{50} = \left( \frac{P_1}{r_{12}} \right)^2 \left( P_1 \times \hat{S}_1 \right)$$

$$+ \left( P_2 \times \hat{S}_1 \right) \left[ \frac{G}{r_{12}} \left( \frac{3 (n_{12} P_1) (n_{12} P_2)}{8 m_1 m_2} - \frac{(P_1 P_2)}{8 m_1 m_2} \right) + \frac{G^2}{r_{12}^2} \left( \frac{47 m_1}{16} - \frac{21 m_2}{8} \right) \right]$$

$$+ \left( P_1 \times \hat{S}_1 \right) \left[ \frac{G}{r_{12}} \left( \frac{9 m_2 P_1^2}{16 m_1^2} - \frac{5 P_2^2}{8 m_1 m_2} \right) + \frac{G^2}{r_{12}^2} \left( \frac{57 m_2}{16} + \frac{15 m_2}{8 m_1} \right) \right].$$

Note that there is a typo in [54] in Eq. (6.20), the last term $\frac{105}{8} m^2 (S^* \cdot L)$ has to be $\frac{75}{8} M^2 (S^* \cdot L)$. See the arXiv version of [54] for the correct equation. We thank E. Barausse and A. Buonanno for clarifying this issue.
\begin{equation}
\begin{aligned}
&+ (n_{12} \times \hat{S}_1) \left[ \frac{G}{r_{12}} \left( \frac{9 (n_{12} P_1) (n_{12} P_2)}{16 m_1 m_2} + \frac{3 (n_{12} P_2) (P_1 P_2)}{8 m_1 m_2} + \frac{(n_{12} P_1) P_2^2}{16 m_1 m_2} \right) \\
&+ \frac{G^2}{r_{12}^2} \left( - \frac{5 m_2}{8} (n_{12} P_1) + \left\{ \frac{13 m_1}{8} + \frac{11 m_2}{4} \right\} (n_{12} P_2) \right) \right] \\
&- \frac{G}{r_{12}^2} P_1 \left( \frac{(n_{12} P_2) ((n_{12} \times P_2) \hat{S}_1)}{2 m_1 m_2} \right) \\
&+ \frac{G}{r_{12}^2} P_2 \left( - \frac{(n_{12} P_2) ((n_{12} \times P_2) \hat{S}_1)}{8 m_1 m_2} + \frac{(n_{12} P_1) ((n_{12} \times P_2) \hat{S}_1)}{2 m_1 m_2} \\
&- \frac{(P_1 \times P_2) \hat{S}_1}{8 m_1 m_2} \right) \\
&+ n_{12} \left[ \frac{G}{r_{12}} \left( \left\{ \frac{m_2 P_1^2}{16 m_1^1} + \frac{15 (n_{12} P_2)^2}{16 m_1 m_2} - \frac{3 P_2^2}{16 m_1 m_2} \right\} (n_{12} \times P_1) \hat{S}_1 \\
+ \left\{ - \frac{3 (n_{12} P_1) (n_{12} P_2)}{2 m_1 m_2} - \frac{P_1 P_2}{2 m_1 m_2} \right\} ((n_{12} \times P_2) \hat{S}_1) \\
+ \frac{3 (n_{12} P_2)}{8 m_1 m_2} \left( (P_1 \times P_2) \hat{S}_1 \right) \right] \\
&+ \frac{G^2}{r_{12}^2} \left\{ \frac{m_2}{2} + \frac{5 m_2^2}{4 m_1} \right\} (n_{12} \times P_1) \hat{S}_1 + \left\{ - 2 m_1 - 5 m_2 \right\} ((n_{12} \times P_2) \hat{S}_1) \right] \\
&+ \frac{z_1}{r_{12}} \left[ \frac{G}{r_{12}} \left( \frac{3 (n_{12} P_1) (n_{12} P_2)}{m_1 m_2} + \frac{(P_1 P_2) m_2}{m_1 m_2} \right) ((n_{12} \times P_2) \hat{S}_1) \\
+ \left\{ \frac{3 (n_{12} P_1)}{4 m_1^2} - \frac{2 (n_{12} P_2)}{m_1 m_2} \right\} ((P_1 \times P_2) \hat{S}_1) \\
+ \left\{ - \frac{5 m_2 P_1^2}{8 m_1^2} - \frac{3 (n_{12} P_1) (n_{12} P_2)}{4 m_1^2} - \frac{3 (n_{12} P_2)^2}{2 m_1 m_2} \\
- \frac{3 (P_1 P_2)}{4 m_1^2} + \frac{P_2^2}{4 m_1^2} \right\} ((n_{12} \times P_1) \hat{S}_1) \right] \\
&+ \frac{G^2}{r_{12}^2} \left\{ - \frac{11 m_2}{2} - \frac{5 m_2^2}{m_1} \right\} ((n_{12} \times P_1) \hat{S}_1) + \left\{ 6 m_1 + \frac{15 m_2}{2} \right\} ((n_{12} \times P_2) \hat{S}_1) \right] \\
&+ (1 \leftrightarrow 2). 
\end{aligned}
\end{equation}

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