When is Existential Quantification Conservative?

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Abstract

We describe a sufficient condition for the process of left Kan extension to be a conservative functor. This is useful in the study of graphic Fourier transforms and quantum categories and groupoids.

1 Introduction

Let \( \mathcal{V} \) be a complete and cocomplete symmetric monoidal closed category. For various questions in “quantum” algebra (see [3]), we want to know when the functor

\[
\exists_N : [A, \mathcal{V}] \to [C, \mathcal{V}]
\]

(or \( \text{Lan}_N : \mathcal{P}(A) \to \mathcal{P}(C) \))

given by the coend formula

\[
\exists_N(f) = \int^a f(a) \otimes \mathcal{C}(Na, -)
\]

(see [2] for notation), is conservative (that is, reflects isomorphisms) for a given \( \mathcal{V} \)-functor

\[N : A \to C\]

with \( A \) a small \( \mathcal{V} \)-category. In this note we establish a “simple” sufficient condition for this to hold; namely, that \( A \) should also have a natural \( \mathcal{V} \)-opcategory structure (with mild assumptions on \( \mathcal{V} \)).

2 The main result

If a small \( \mathcal{V} \)-category \( A \) is equipped with \( \mathcal{V} \)-natural transformations

\[
\delta : A(a, b) \to A(c, b) \otimes A(a, c)
\]

\[
\epsilon : A(a, a) \to I
\]

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such that

$$\begin{array}{ccc}
A(a,b) & \xrightarrow{\delta} & A(b,b) \otimes A(a,b) \\
\downarrow 1 & & \downarrow \epsilon \otimes 1 \\
A(a,b) & \xleftarrow{=I \otimes A(a,b)} &
\end{array}$$

commutes, then the $\mathcal{V}$-functor

$$\exists_N = \text{Lan}_N : [A, \mathcal{V}] \to [C, \mathcal{V}]$$

is conservative for a given functor $N : A \to \mathcal{V}$ if we suppose that each component

$$N : A(a,b) \to C(Na, Nb)$$

is a regular mono (that is, the kernel of some pair of maps) and that the composite of regular monos in $\mathcal{V}$ is again a regular mono, and the functor

$$- \otimes \mathcal{V} : \mathcal{V} \to \mathcal{V}$$

preserves regular monos for each $X$ in $\mathcal{V}$.

The rest of this section is concerned with the proof of this statement.

Note: We shall use the term “opcat” to refer to any application using a $\mathcal{V}$-natural transformation of the form

$$fb \to \int_x A(x,b) \otimes f x$$

derived from the $\mathcal{V}$-natural transformation $\delta$ by use of the Yoneda expansion of a given $\mathcal{V}$-functor $f$. In fact, we can suppose that $A$ is merely a Frobenius category in the sense that the given family of maps

$$\delta : A(a,b) \to A(c,b) \otimes A(a,c)$$

is only $\mathcal{V}$-natural in $a$ and $b$ (and not necessarily in $c$) and that the family

$$\epsilon : A(a,a) \to I$$

is not necessarily $\mathcal{V}$-natural in $a$. We then use the $\mathcal{V}$-natural transformation

$$fb \to \prod_x A(x,b) \otimes f x$$

in the following calculations, where $\prod_x$ replaces $\int_x$, etc.

To show that $\exists_N$ is conservative, it suffices to show that the unit of the $\mathcal{V}$-adjunction $\exists_N \dashv [N, 1]$ is a regular mono (see [1] or [2], for example, and the
references therein to W. Tholen). To do this, we now consider the following diagram in \( \mathcal{V} \).

\[
\begin{array}{cccccc}
\int^a \mathcal{A}(a, b) \otimes f_a & \cong & \int^a \mathcal{C}(Na, Nb) \otimes f_a \\
\downarrow \text{opcat} & & \downarrow \text{opcat} & & \\
\int^a \int_x \mathcal{A}(a, x) \otimes \mathcal{A}(x, b) \otimes f_a & \rightarrow & \int^a \int_x \mathcal{A}(a, x) \otimes \mathcal{C}(Nx, Nb) \otimes f_a \\
\downarrow \text{can} & & \downarrow \text{can} & & \\
\int_x \mathcal{A}(a, x) \otimes \mathcal{A}(x, b) \otimes f_a & \rightarrow & \int_x \mathcal{A}(a, x) \otimes \mathcal{C}(Nx, Nb) \otimes f_a \\
\downarrow \text{can} & & \downarrow \text{can} & & \\
\int_x \mathcal{A}(x, b) \otimes f_x & \rightarrow & \int_x \mathcal{C}(Nx, Nb) \otimes f_x \\
\end{array}
\]

where the isomorphisms are by the Yoneda expansion of \( f \), and “can” denotes the canonical “interchange” maps. Since \( \int_x N \otimes 1 \) is a regular mono (by the hypotheses on \( \mathcal{V} \) and \( N \)), so also is its pullback along the right-hand composite map. Hence, since

\[
\text{opcat}: f_b \rightarrow \int_x \mathcal{A}(x, b) \otimes f_x
\]

is a coretraction, with left inverse the composite

\[
f_b \cong \int_x \mathcal{A}(x, b) \otimes f_x \cong \int_x \mathcal{A}(x, b) \otimes f_x \cong \int_x \mathcal{A}(a, x) \otimes f_a
\]

we have that

\[
\int^a N \otimes 1
\]

is the composite of a coretraction (which is always a regular mono) and a regular mono (the pullback of \( \int_x N \otimes 1 \)), so is a regular mono (by hypothesis on \( \mathcal{V} \)). Thus, the adjunction unit we are looking at, which is the composite

\[
f_b \rightarrow \int^a \mathcal{A}(a, b) \otimes f_a \rightarrow \int^a \mathcal{A}(a, b) \otimes f_a \rightarrow \int^a \mathcal{C}(Na, Nb) \otimes f_a
\]

is a regular monomorphism, as required.
3 A frequent generalisation

Similarly, given a $\mathcal{V}$-functor $N : \mathcal{A} \to \mathcal{E}$ with $\mathcal{A}$ small and $\mathcal{E}$ a $\mathcal{V}$-cocomplete $\mathcal{V}$-category, we have the standard $\mathcal{V}$-adjunction

$$(\epsilon, \eta) : L \dashv Y : \mathcal{E} \to [\mathcal{A}^{\text{op}}, \mathcal{V}]$$

where $Y(e)(a) = \mathcal{E}(Na, e)$ describes the $N$-Yoneda functor $Y$, and

$$L(f) = \int^a f_a \otimes Na$$

describes its left adjoint. Hence, with corresponding hypotheses on $\mathcal{A}$, $N$, and $\mathcal{V}$ as before, we can replace the right-hand side of the diagram displayed in §2 by the following composite (*):

$$\mathcal{E}\left(Nb, \int^a f_a \otimes Na\right) \xrightarrow{\text{opcat}} \mathcal{E}\left(Nb, \int^a \left(\int_x (A(a, x) \otimes f_x) \otimes Na\right)\right)$$

$$\xrightarrow{\text{can}} \mathcal{E}\left(Nb, \int^a \int_x (A(a, x) \otimes f_x) \otimes Na\right)$$

where the isomorphism comes from the Yoneda-lemma expansion

$$N_x \xrightarrow{\cong} \int^a A(a, x) \otimes Na$$

for $N$. Then, by the same argument as before, we obtain a regular mono in $\mathcal{V}$ from the commuting diagram

$$\mathcal{E}\left(Nb, \int^a f_a \otimes Na\right) \xrightarrow{\eta_{f,b}} \mathcal{E}\left(Nb, \int^a f_a \otimes Na\right) = YL(f)(b)$$

$$\xrightarrow{\text{opcat}} \int_a f_a \otimes A(b, a) \xrightarrow{(\ast)} \mathcal{E}\left(Nb, \int^a f_a \otimes Na\right)$$

for each unit-component $\eta_{f,b}$ of the $\mathcal{V}$-functor $f$ in $[\mathcal{A}^{\text{op}}, \mathcal{V}]$, provided the canonical natural transformation

$$X \otimes \mathcal{E}(Nb, Na) \to \mathcal{E}(Nb, X \otimes Na)$$

(†)
is a regular mono in \( V \) for each \( X \) in \( V \). Consequently, if this additional condition (†) holds on \( N \) and \( V \), then the left-adjoint \( V \)-functor

\[
L : [A^{\text{op}}, V] \rightarrow E
\]

is conservative.

The result of §2 can then be recovered from this latter result by putting \( E = [C, V] \), and taking the new \( N : A \rightarrow E \) to be the composite of the \( V \)-functor \( N^{\text{op}} : A^{\text{op}} \rightarrow C^{\text{op}} \) (this \( N \) from §2) with the Yoneda embedding

\[
C^{\text{op}} \subset [C, V],
\]

noting that \( A \) is a \( V \)-opcategory iff \( A^{\text{op}} \) is also a \( V \)-opcategory. The condition (†) holds for the new \( N \) since we have the isomorphisms

\[
X \otimes [C, V](C(Nb, -), C(Na, -)) \cong [C, V](C(Nb, -), X \otimes C(Na, -))
\]

by the Yoneda lemma applied twice.

### 4 Related Conditions

The following result is related to that of the earlier §2, but is much simpler.

Suppose that regular monomorphisms (that is, kernels in \( V \)) are closed under composition in \( V \) and also that \( n \) is a regular mono in \( V \) if \( mn \) and \( m \) are regular monos in \( V \). Then, provided both coproduct \( \Sigma \) and tensoring \( X \otimes - \) preserve regular monos in \( V \), we have that

\[
\exists N : [A, V] \rightarrow [C, V]
\]

is conservative if regular epimorphisms (that is, cokernels) split in the functor category \( [A^{\text{op}} \otimes A, V] \) and each component

\[
N : A(a, b) \rightarrow C(Na, Nb)
\]

of \( N : A \rightarrow C \) is a regular monomorphism in \( V \).

To establish this, one simply notes that the canonical regular epimorphism

\[
\sum_a S(a, a) \rightarrow \int^a S(a, a)
\]

splits naturally in \( S \in [A^{\text{op}} \otimes A, V] \), so that \( f^a N \otimes 1 \) is a regular monomorphism by the hypotheses on \( N \) and \( V \). This implies that the unit components

\[
\eta_{f, b} = fb \cong \int^a A(a, b) \otimes f a \xrightarrow{\int^a N \otimes 1} \int^a C(Na, Nb) \otimes f a
\]
are regular monomorphisms, as required for $\exists_N$ to be conservative.

Note: In practice, it is often the case that, under the given hypotheses on $[A^{op} \otimes A, V]$, each monomorphism of the form

$$N : A(-, b) \rightarrow C(N-, Nb)$$

splits naturally in $[A^{op}, V]$, in which case the result is obvious and generalises the familiar fact that $\exists_N$ is fully faithful if $N$ is.

Neither the condition in §2 and §3 that $A$ should have an opcategory structure, nor the condition in this section that regular epimorphisms should split in $[A^{op} \otimes A, V]$, is in any way necessary for the process of left Kan extension along the Yoneda embedding of $A^{op}$ into $[A, V]$, to be conservative. One notable example is the (left) Cayley functor

$$\exists_P : [A, V] \rightarrow [A^{op} \otimes A, V],$$

which is given by the coend formula

$$\exists_P(f) = \int^a f(a) \otimes P(a, -, -),$$

and is not fully faithful in general. This functor is conservative for each (small) $V$-promonoidal category $(A, P, J)$ defined over any complete and cocomplete base category $V$.

Finally we note that both the sufficient conditions mentioned immediately above are closely related to the splitting properties of regular epimorphisms in the functor category $[A, V]$. (by Maschke’s result).

Note: Any enquires regarding this article can be forwarded to the author through Micah McCurdy (Macquarie University), who kindly typed the manuscript.

References

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