Gaussian Signal Detection With Product Arrays

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ABSTRACT The paper specifies the probability density function (PDF) for the detection statistic for a product processor of colinear arrays. The product processor’s detection PDF is a scaled product of the detection statistic with modified Bessel functions. Using the PDF, this research compares the product processor’s detection performance against a conventional beamforming (CBF) linear array with an equal number of sensors. For the basic detection case of a single known signal in Gaussian noise with a single Gaussian plane wave interferer, receiver operation characteristics curves and mean discriminating information over a range of interferer power illustrate that the product processor’s performance is inferior to the CBF detector with an equal number of sensors over all possible interferer locations. The detection performance of a product processor matches or outperforms the CBF detector only for high power interferers in specific locations.

INDEX TERMS Coprime arrays, detection, Gaussian, nested arrays, product arrays, sparse arrays.

I. INTRODUCTION

Conventional beamforming (CBF) for a linear array involves summing the weighted sensor signals to reinforce the signal of interest and reduce noise. Product processing for linear arrays comprises dividing the array into two subarrays, computing the subarray outputs for a shared direction and multiplying the subarray outputs. Many researchers in acoustics, radar, and radio astronomy have compared the CBF and product processing [1]–[10] in terms of direction of arrival and detection performances.

Berman and Clay demonstrated as early as 1957 that the directional characteristics of a CBF processing standard ULA can be achieved by taking the average of the products of sensor signals [1]. Several researchers have pointed out that while product processing can offer improved resolution over CBF processing for a given number of sensors, the product beampattern suffers from higher side lobes and reduction in array gain [2]–[6], [11]. Miller and Lee considered a special case of product arrays where a linear array is split into two subarrays that are joined end to end and derived the probability density functions (PDFs) of the product output for plane wave signals in presence of additive Gaussian noise [7], [8]. Miller and Lee compared detection performance of CBF and product processing for the specific product arrays and concluded that the CBF processing is slightly better than the product processing for equal number of sensors. However, their analysis does not address other product arrays where the subarrays are not joined end to end.

A product array can achieve higher resolution than a standard ULA with the equal number of sensors, but at the cost of higher side lobes and grating lobes [12]. Some researchers have designed prudent arrays that have undersampled subarrays but disambiguate grating lobes in the product beampattern [11], [13]–[19]. These arrays have the potential to offer the resolution of a standard ULA with many more sensors. Kefalas discusses multiplying the beampatterns of sparse receiver and transmitter arrays to suppress the grating lobe ambiguities in [13] by providing specific examples of product arrays. However, Kefalas fails to stipulate general guidelines on the array design. Davies and Ward proposed a judicious design that nests a short standard ULA with a longer undersampled ULA. Pal and Vaidyanathan proposed a generalized version of Davies and Ward’s design where multiple levels of nesting arrays are detailed [18], [19]. Mitra discussed designing sparse arrays based upon the polynomial factorization of product beampattern [15]. Vaidyanathan and Pal proposed coprime arrays in 2011 that interleave two undersampled ULAs with coprime undersampling factors and exhibited effective cancellation of grating lobes using product processing [16], [17]. These sparse product arrays such as nested and coprime arrays allow us to
estimate $O(MN)$ directions of arrival using only $O(M + N)$ sensors [16]–[23].

The pros and cons of product arrays (especially sparse product arrays) against CBF ULAs in signal estimation have been outlined in detail in [24]. However, the advantages and disadvantages of using product arrays in detection of signals have not been fully explored. Reference [25] describes the fundamental PDF results for the basic binary hypothesis case for signal detection for a special type of product arrays, namely coprime arrays. This paper focuses on the performance of a general product processor when the received signals include one or more interferers in addition to the desired signal and noise. Most real world problems of interest have competing sources whose power rivals or exceeds that of the source of interest. In such cases, comparing the basic signal plus noise against noise hypothesis test can be misleadingly optimistic. Also, when the underlying subarrays in a product array are sparse, the product beam pattern has higher side lobes at some directions. Although many product arrays have the same PDF for the basic detection problem, their varying side lobe structures result in substantial differences in their performances when there are plane wave interferers present as described in Section III. Moreover, the detection statistic PDF of a product array converges to the CBF ULA detection statistic PDF asymptotically for equal number of sensors [25], but that does not account for very different side lobe heights possible between product and CBF arrays.

Section II defines the signal model and specifies the product processor output PDF. Section III evaluates detection performance measures and compares them between CBF and product processing. Section IV summarizes the results.

Conventions: Boldface lowercase math symbols denote vectors and boldface uppercase math symbols denote matrices. $(\cdot)^H$ denotes Hermitian (conjugate transpose) operator. $a \sim \mathcal{CN}(\mu, \sigma^2)$ means $a$ is a complex random variable with proper normal distribution with mean $\mu$ and variance $\sigma^2$ [26]. $a \sim \mathcal{N}(\mu, \sigma^2)$ means $a$ is a real random variable with normal distribution with mean $\mu$ and variance $\sigma^2$, $a \sim \mathcal{E}(1/\alpha)$ means $a$ is a real random variable with exponential distribution with mean $\alpha$ and variance $\alpha^2$ [27]. $E_{p(x)}(X)$ means the expectation of random variable $X$ with respect to the PDF $p(x)$. The quantity $D_{KL}(p_1|p_0)$ is the Kullback-Leibler divergence [28] between the PDFs $p_1$ and $p_0$ given by $D_{KL}(p_1|p_0) = E_{p_1} \left\{ \log \left( \frac{p_1}{p_0} \right) \right\}$.

II. PRODUCT ARRAY

A product array consists of two linear subarrays, whose outputs are multiplied during the processing of the received signal. The two subarrays are henceforth called Subarray 1 and Subarray 2. Subarray 1 consists of $M_s$ sensors and $N_1 \lambda/2$ intersensor spacing, where $\lambda$ is the wavelength of the signal to be sampled. Subarray 2 consists of $N_v$ sensors and $M_2 \lambda/2$ intersensor spacing. This definition of a product array encompasses a broad category of multiplicative linear arrays where one or both of the individual subarrays could be sparse, the subarrays could be either interspersed or joined end to end, and the subarrays could share any number of sensors.

A coprime array ([16]), and a nested array ([18], [19]) are both examples of product arrays [24]. A coprime array is a product array with $M$ and $N$ coprime integers. The number of sensors in the subarrays of a coprime array can be chosen in various ways as described in [29]–[33]. Fig. 1 illustrates the formation of a coprime product array. Fig. 1(a) shows the individual subarrays and Fig. 1(b) shows a product array formed by combining the subarrays. Similarly, a nested array is a product array with $M_e = cM$ and $N = 1$, where $c$ is an integer.

Fig. 2 depicts a plane wave with wavelength $\lambda/2$ impinging on the product array making an angle $\theta_s$ with the array axis. Fig. 3 depicts the product array detector. The product array detector applies conventional beamforming (CBF) to the individual subarrays, multiplies one subarray’s CBF output with the complex conjugate of the other subarray’s CBF output, and generates the magnitude of the product as the detection statistic. Assume that both signal and noise are independent zero mean proper Gaussian random variables [26]. The inputs to Subarray 1 and Subarray 2 are $x_1 = a_1 v_1(\theta_s) + n_1$ and $x_2 = a_2 v_2(\theta_s) + n_2$, respectively, where $a_s \sim \mathcal{CN}(0, \sigma^2_s)$, $n_1 \sim \mathcal{CN}(0, \sigma_{n_1}^2 I_{M_e})$, and $n_2 \sim \mathcal{CN}(0, \sigma_{n_2}^2 I_{N_v})$. The signal components
are $a_1 y_{1s}(\theta_s)$ and $a_2 y_{2s}(\theta_s)$ and the noise components are $n_1$ and $n_2$. The $M_e$ and $N_e$ element vectors $v_{1s}(\theta_s)$ and $v_{2s}(\theta_s)$ are the array manifold vectors at direction $\theta_s$ for the subarrays which are given by
\[
v_{1s}(\theta_s) = \begin{bmatrix} 1 \\ e^{-j \pi a_1 M} \\ \vdots \\ e^{-j \pi a_1 (M_e-1) N} \end{bmatrix},
\]
and
\[
v_{2s}(\theta_s) = \begin{bmatrix} 1 \\ e^{-j \pi a_2 M} \\ \vdots \\ e^{-j \pi a_2 (N_e-1) M} \end{bmatrix},
\]
respectively. The CBF weights for the two subarrays are $w_1(\theta) = v_1(\theta)/M_e$ and $w_2(\theta) = v_2(\theta)/N_e$, where $v_1(\theta)$ and $v_2(\theta)$ are the array manifold vectors at direction $\theta$ for the subarrays. The dependence of the weight vectors and the array manifold vectors on direction $\theta$ will be suppressed henceforth for notational convenience.

The subarrays' CBF outputs are linear combination of the received signals at different sensors, i.e., $y_1 = w_1 H x_1$ and $y_2 = w_2 H x_2$. When the subarrays are steered to the signal direction, Subarray 1’s CBF output and Subarray 2’s conjugated CBF output are $y_1 = a_1 + y_{1s}^H n_1/M_e = a_1 + \eta_1$, and $y_2^* = (a_2 + y_{2s}^H n_2/N_e)^* = a_2^* + \eta_2$, where $\eta_1 = y_{1s}^H n_1/M_e \sim CN(0, \sigma_n^2/M_e)$, $\eta_2 = (y_{2s}^H n_2/N_e)^* \sim CN(0, \sigma_n^2/N_e)$. The CBF outputs are also zero mean random Gaussian: $y_1 \sim CN(0, \sigma_1^2 = \sigma_n^2 + \sigma^2_1/M_e)$, $y_2 \sim CN(0, \sigma_2^2 = \sigma_n^2 + \sigma^2_2/N_e)$. The product processor detection statistic is the magnitude of the product $y = |y_1 \cdot y_2^*| = |y_1| \cdot |y_2|$. The output $y$ is the product of two correlated Rayleigh distributed random variables and its PDF is given by [25],[34],[35]
\[
g(y) = \frac{4y}{\sigma_1 \sigma_2} I_0(2y|\rho|) \cdot K_0(2y) , \tag{1}
\]
where $r = \frac{1}{\sigma_1 \sigma_2 (1 - |\rho|^2)}$ and $\rho$ is the correlation coefficient between the Gaussian variables $y_1$ and $y_2$ which is given by (see Appendix IV-A for the derivation)
\[
\rho = \frac{\Psi + \frac{\alpha}{M_e N_e}}{\sqrt{\left(\Psi + \frac{1}{M_e}\right) \left(\Psi + \frac{1}{N_e}\right)}} \tag{2}
\]
where $\Psi = \frac{\sigma^2}{\sigma_n^2}$ is the input SNR and $\alpha$ is the number of sensors shared.

The mean and variance of the product processor's detection statistic are functions of the variances of the subarrays' CBF outputs, and the correlation coefficient between the subarrays' CBF outputs. The expression for the mean is
\[
\mu = \frac{\pi}{4} \sigma_1 \sigma_2 (1 - |\rho|^2)^2 F(1.5, 1.5; 1; |\rho|^2), \tag{3}
\]
where $F(v; \beta; y; z)$ is a hypergeometric series [36] of the form $F(v; \beta; y; z) = 1 + \frac{v \cdot \beta \cdot z}{\gamma(y+1) \cdot 1 \cdot 2} z^2 + \frac{v(v+1) \cdot (v+2) \cdot \beta(v+1)\beta+1}{\gamma(y+1)(y+2) \cdot 1 \cdot 3} z^3 + \cdots$. See Appendix IV-B for the derivation. The expression for the variance is
\[
\sigma^2 = (1 + |\rho|^2) \sigma_1^2 \sigma_2^2 - \mu^2, \tag{4}
\]
see Appendix IV-C for the derivation.

The detection statistic PDF in (1) depends only on the input signal variance $\sigma_1^2$, the input noise variance $\sigma_n^2$, the number of sensors in the subarrays $M_e$, $N_e$ and the number of sensors shared by the two subarrays $\alpha$. The PDF in (1) does not depend on the intersensor spacings of the subarrays. Consequently, for additive white noise, the detection performance of a product array is independent of its array geometry, but instead only depends on the number of sensors in each subarray and the number of sensors shared between the subarrays. However, when there are interferers present in the scenario, the array structure becomes pertinent in the detection performance because of the presence of the array manifold vector in the statistics of the product output. When there are $P$ plane wave interferers present, the mean and variance retain the same functional forms as (3) and (4) but incorporate the modified value of the correlation coefficient resulting from the presence of interferers. The modified value of the correlation coefficient is
\[
\rho = \left(\frac{\sum_{k=1}^{P} \Phi_k |y_{1s}^H v_{1,k}|^2}{M_e} + \frac{\alpha}{M_e N_e}\right)^{-0.5} \times \left(\frac{\sum_{k=1}^{P} \Phi_k |y_{2s}^H v_{2,k}|^2}{N_e} + \frac{1}{N_e}\right)^{-0.5}, \tag{5}
\]
where \( \mathbf{v}_{i,k} \) and \( \mathbf{v}_{i,2,k} \) are the array manifold vectors for the two subarrays at the \( k \)-th interferer location. The variable \( Q_k \) is the interferer to noise ratio (INR) for the \( k \)-th interferer. The subarrays’ output variances are \( \sigma_i^2 = \sigma_s^2 + \sum_{k=1}^{P} \frac{\sigma_{i,k}^2 |\mathbf{v}_{i,k}^H \mathbf{v}_{i,k}|^2}{M_e} + \frac{\sigma_n^2}{M_e} \) and \( \sigma_i^2 = \sigma_s^2 + \sum_{k=1}^{P} \frac{\sigma_{i,k}^2 |\mathbf{v}_{i,2,k}^H \mathbf{v}_{i,2,k}|^2}{N_e^2} + \frac{\sigma_n^2}{N_e} \). The presence of the array manifold vectors in equation (5) indicates the dependence of the mean and variance on the array structure. See Appendix IV-D for the derivation of the statistics.

A standard uniform linear array (ULA) [12] can be considered a degenerate case of the product array described in Section II because of the structural similarities between the standard ULA case and other product processors. The detection statistic for a standard ULA with input \( \mathbf{x} \) and output \( \mathbf{w}^H \mathbf{x} \) is the output power \( y = |\mathbf{w}^H \mathbf{x}|^2 \) which can be written as a product in the form of Fig. 3 by noting \( |\mathbf{w}^H \mathbf{x}|^2 = |\mathbf{w}^H \mathbf{x}|(|\mathbf{w}^H \mathbf{x}|)^* \). A standard ULA is product array with \( M_e = N_e, M = N = 1 \) and the number of shared sensors in the subarrays \( \alpha = M_0 \). The detection statistic PDF for a standard ULA with \( L \) sensors is an exponential function

\[
f(Y) = \frac{1}{\mu_L} \exp \left(-\frac{y}{\mu_L}\right), \tag{6}
\]

where \( \mu_L \) is the expected value of the output. When there are no interferers present, the expected value \( \mu_L \) is \( \sigma_s^2 + \frac{\sigma_n^2}{L} \). When there are \( P \) interferers present, the expected value \( \mu_L \) is \( \sigma_s^2 + \sum_{k=1}^{P} \frac{\sigma_{i,k}^2 |\mathbf{v}_{i,k}^H \mathbf{v}_{i,k}|^2}{L^2} + \frac{\sigma_n^2}{L} \), where the \( L \)-element vectors \( \mathbf{v}_i \) and \( \mathbf{v}_{i,k} \) are the array manifold vectors at the signal and \( k \)-th interferer directions respectively. The variance, \( \sigma_i^2 \), of the output for the standard ULA is the square of the expected value, \( \mu_L^2 \).

### III. PRODUCT PROCESSOR’S DETECTION PERFORMANCE

This section evaluates two detection performance measures (receiver operating characteristics and discriminating information) for a product array using the PDF in (1) and compares the detection performances of two special cases of product arrays — coprime array and nested array — with a standard ULA and a non-uniform linear array that use CBF processing. The detection performance for the arrays are assessed under two conditions. The first condition is when the interferers are uniformly randomly distributed in bearing. The second condition is a worst-case scenario when the interferer is located at the peak side lobe position for the power pattern for each beamformer. The first condition relates to the performance one would expect as interferers move within the environment over time. The second condition relates to the performance expected when an intelligent adversary is attempting to jam the detection performance of the system. The conditions were assessed both when the interferer and source had equal power, and also when the interferer was much stronger than the source.

#### A. RECEIVER OPERATION CHARACTERISTIC

Evaluating the PDFs found in the previous section produces the Receiver Operating Characteristics (ROCs) required to compare the detection performance of the different array processors. The ROC considers a modification to the classic binary hypothesis test for detection, in that the null hypothesis \( \mathcal{H}_0 \) is that the array receives both noise and plane wave interferers. The alternate hypothesis \( \mathcal{H}_1 \) is that the array receives a signal, noise, and plane wave interferers. The null and alternate hypotheses with \( P \) interferers for the \( j = 1, 2 \) subarrays are represented as

\[
\mathcal{H}_0 : \mathbf{x}_{j,\text{null}} = \sum_{k=1}^{P} a_{i,k} \mathbf{v}_{i,k} + \mathbf{n}_j,
\]

\[
\mathcal{H}_1 : \mathbf{x}_{j,\text{alt}} = a_s \mathbf{v}_s + \sum_{k=1}^{P} a_{i,k} \mathbf{v}_{i,k} + \mathbf{n}_j
\]

where \( a_s \) is the complex random amplitude of the signal and \( a_{i,k} \) is the complex random amplitude of the \( k \)-th interferer, with the \( i \) subscript denoting an interferer. For the special case of the standard ULA, the null and alternate hypotheses are

\[
\mathcal{H}_0 : \mathbf{x}_{\text{L,null}} = \sum_{k=1}^{P} a_{i,k} \mathbf{v}_{i,k} + \mathbf{n}_L
\]

\[
\mathcal{H}_1 : \mathbf{x}_{\text{L,alt}} = a_s \mathbf{v}_s + \sum_{k=1}^{P} a_{i,k} \mathbf{v}_{i,k} + \mathbf{n}_L
\]

The probabilities of false alarm and detection for the product array are

\[
P_{FA,p} = \Pr(y > \gamma | \mathcal{H}_0) = \int_{\gamma}^{\infty} g(y | \mathcal{H}_0) dy \tag{7}
\]

and

\[
P_{D,p} = \Pr(y > \gamma | \mathcal{H}_1) = \int_{\gamma}^{\infty} g(y | \mathcal{H}_1) dy, \tag{8}
\]

where the variable \( \gamma \) is the detection threshold and the subscript \( (p) \) represents the product processor. Similarly, the probabilities of false alarm and detection for the CBF ULA are

\[
P_{FA,c} = \Pr(y > \gamma | \mathcal{H}_0) = \exp \left( \frac{-\gamma}{\mu_{L0}} \right) \tag{9}
\]

and

\[
P_{D,c} = \Pr(y > \gamma | \mathcal{H}_1) = \exp \left( \frac{-\gamma}{\mu_{L1}} \right). \tag{10}
\]

where the variables \( \mu_{L0} \) and \( \mu_{L1} \) are the output expected values for null and alternate hypotheses respectively and the subscript \( (c) \) represents the CBF processor.

The \( P_{FA} \) and \( P_{D} \) expressions in (7) and (8) lend themselves to the evaluation of receiver operation characteristic (ROC) curves for product detectors such as coprime array and nested array while the \( P_{FA} \) and \( P_{D} \) expressions in (9) and (10) facilitate the computation of the ROC curves for CBF ULA and CBF non-uniform linear array (NULA). Fig. 4 depicts two
product arrays and two CBF arrays used for the evaluation and comparison of ROC curves. Fig. 4A. is a coprime array with $M_e = 14$, $N = 3$, $N_e = 21$, and $M = 2$. Fig. 4B. is a nested array with $M_e = 14$, $N = 1$, $N_e = 21$, and $M = 2$. Fig. 4C. is a standard ULA with $L = 28$ which can be considered a product array with $M_e = 28$, $N = 1$, $N_e = 28$, and $M = 1$. Fig. 4D. depicts a NULA that has the same sensor locations as the coprime array in Fig. 4 but it uses CBF processing. All four detectors have 28 sensors. The coprime array, nested array, and NULA also have equal aperture guaranteeing equal resolution. In addition, the coprime array has 7 periods of a basic coprime array with $M = 2$ and $N = 3$. Having 7 periods of the basic coprime array ensures that the peak side lobe height matches a full ULA’s peak side lobe height of $-13$ dB [29].

For the evaluation of ROC curves, consider a scenario with one plane wave Gaussian signal, one plane wave Gaussian interferer, and spatially white Gaussian noise where the signal, interferer, and noise are mutually uncorrelated. The SNR is 0 dB for the signal of interest. The top panel of Fig. 5 depicts the average ROC curves evaluated for the four detectors in Fig. 4 using log axes for both $P_{FA}$ and $P_D$. The figure illustrates the average ROC curves considering all possible interferer locations. The interferer location is assumed to be uniformly distributed in bearing, $\theta$, and the interferer power to noise power ratio (INR) is 0 dB. Hence, even though there is an interferer in the scenario, it is negligible because of its low power. Both coprime and nested arrays’ ROCs were evaluated using (7) and (8) with the global adaptive quadrature method of numerical integration [37]. The green dash-dot line represents the standard ULA ROC curve evaluated using the analytical expressions for $P_{FA,c}$ and $P_{D,c}$ in (9) and (10) while the purple dot line represents the ROC curve for the non-uniform linear array (NULA) that has the same sensor locations as the coprime array but uses CBF processing. The figure suggests that based on the ROC measure, the average detection performance of the CBF ULA is better than all other detectors being considered except when the probability of false alarm needs to be very low. The product coprime detector proves to be better than the other detectors when very low false alarm is required. The bottom panel of Fig. 5 is for INR of 20 dB. The bottom panel figure shows that the presence of a strong interferer in the scenario causes all four detectors’ performances to deteriorate in general. The CBF ULA is superior to the other detectors in terms of the ROC curves for the range of probability of detection shown in the figure. As in the 0 dB INR case, when the false alarm needs to be very low, the coprime product detector outperforms the CBF ULA (not seen in the figure) but the probability of detection is also too low to be useful at that point and hence not shown in the graph.

The top panel of Fig. 6 depicts the ROC curves evaluated for the same four detectors in Fig. 4 at 0 dB SNR and 0 dB INR. Instead of plotting the average ROC curves,
the figure considers the worst case scenario for each detector. The worst case scenario is when the interferer is at the same location as the peak side lobe of the detector and the locations are $\theta = 10.8^\circ$, $\theta = 86.2^\circ$, $\theta = 84.1^\circ$, and $\theta = 0^\circ$ respectively for the coprime array, nested array, CBF ULA, and the CBF NULA in Fig. 4. Comparison of the ROC curves in the top panel of Fig. 6 shows that in the presence of an interferer with very low power, the CBF ULA detector outperforms the product arrays and the CBF NULA with an equal number of sensors. However, the presence of high power interferer at the worst locations can render the product coprime array better in detection performance than their CBF counterparts as evidenced by the bottom panel of Fig. 6.

B. MEAN DISCRIMINATING INFORMATION

Mean discriminating information (MDI) is the Kullback-Leibler Divergence between the alternate hypothesis and the null hypothesis [28], [38]

$$MDI = D_{KL}(p(y|H_1)||p(y|H_0)).$$ (11)

The MDI serves as a detection performance measure since according to Chernoff-Stein lemma, when the probability of missed detection, $P_{md}$, is upper bounded by a small positive number $\epsilon$, then

$$\lim_{n \to \infty, \epsilon \to 0} P_{FA} = \exp(-n \times MDI),$$ (12)

where $n$ is the number of snapshots. Higher values of MDI indicate faster decay of the probability of false alarm and hence better asymptotic detection performance. For a CBF detector, the MDI evaluated using its null hypothesis distribution $E(\mu_{L0})$ and alternate hypothesis distribution $E(\mu_{L1})$, is

$$MDI = \log\left(\frac{\mu_{L0}}{\mu_{L1}}\right) + \frac{\mu_{L1}}{\mu_{L0}} - 1$$ [39]. For a product array, there is no closed form expression for the MDI but the PDF specified in (1) facilitates the numerical evaluation of the MDI.

Fig. 7 illustrate the MDI plots evaluated as a function of INR for the same four detectors in Fig. 4 at 0 dB SNR for the source of interest. The top panel of Fig. 7 depicts the average MDI evaluated for the coprime array (brown line), nested array (black line), CBF ULA (green line), and CBF NULA (purple line) in Fig. 4. The average is computed over all possible interferer locations and assuming the interferer location is equiprobable in bearing, $\theta$. The INR varies from 0 dB to 20 dB. Similarly, the bottom panel of Fig. 7 depicts the MDI evaluated for the same four detectors when the interferer is at the highest peak side lobe location for all detectors. It is evident in the bottom panel of Fig. 7 that in a scenario where an interferer is weak or non-existent, the CBF ULA detector performs better than the other detectors with an equal number of sensors while in a scenario with a strong enough interferer, the coprime product array surpasses the CBF detectors. Therefore, the bottom panel of Fig. 7 corroborates the inference made with Fig. 6 for the worst case scenarios. On the other hand, the average MDI plots in the top panel of Fig. 7 might seem to portray a
different story than the average ROC curves in Fig. 5 at a first glance. However, it should be noted that the ROC examined in Section III-A is a measure of detection performance for the case of one snapshot whereas the MDI in (11) provides a measure of detection for an asymptotic case when the number of snapshots is infinity. Averaging the processor output over various independent values improves the performance of all four detectors because of reduction in noise variance at the output. However, averaging the detection output over the equal number of snapshots seems to help the four detectors by various amounts, which explains why the orders of the four detectors in Fig. 5 and the top panel of Fig. 7 are different.

Overall, comparing the ROC and MDI values demonstrates that the coprime array exhibits better detection performance than the CBF ULA when there is a strong interferer at the peak side lobe locations, in both single snapshot and many snapshots cases. The ROC curve in the bottom panel of Fig. 6 showed the coprime array’s superiority over the CBF ULA for a single snapshot case whereas the MDI plot in the bottom panel of Fig. 7 showed the coprime array’s superiority over the CBF ULA for the multiple snapshots case. Fig. 8 verifies the MDI values for the case of one interferer with 20 dB INR present at worst side lobe locations using Chernoff-Stein lemma in (12). When the number of observations is sufficiently high and the probability of missed detection is upper bounded by a small positive number, the natural log of the probability of false alarm divided by the total number of observations equals the negative of MDI. Fig. 8 compares the negative of MDI values for the four detectors evaluated using the Chernoff-Stein lemma (dot lines in Fig. 8) and actual values obtained using the probability of false alarm (solid lines in Fig. 8). To evaluate the probability of false alarm for over 400 snapshots case in Fig. 8, invoking the central limit theorem, the Gaussian distribution function was assumed for the average processor output. The method proposed by Nuttall in [40], [41] was used in the evaluation of Gaussian tail PDFs encountered in the computation of probability of false alarm.

When the number of snapshots is high and the interferer is present at an arbitrary direction cosine rather than at the worst peak side lobe locations of the four detectors, product detectors can still outperform the CBF detector based on the interferer location. The bar graph in Fig. 9 depicts the pairwise comparison of the four detectors for the case of 100 snapshots and indicates the percentage of interferer locations one detector performs better than the other as the interferer location is varied. For each interferer location, one detector is considered better than the other if the total area under its ROC curve is more than the total area under the ROC curve of the other detector. The first block of the 6 bars compares coprime against CBF ULA, coprime against CBF NULA, coprime against nested, CBF NULA against CBF ULA, CBF NULA against nested, and nested against CBF ULA respectively for 0 dB INR. Similarly, the second and third blocks are for 10 dB INR and 20 dB INR. When the INR is 0 dB, the first bar representing coprime against CBF ULA shows that the coprime detector is better than the CBF ULA in about 3% (brown component in the first bar) of the interferer locations, the two detectors perform equally in 97% (gray component in the first bar) of the interferer locations, and the CBF ULA does not perform better than the coprime detector for any interferer location when the INR is 0 dB. Similarly, Fig. 9 allows comparison of any other pair of the four detectors at different INR levels.

The CBF ULA’s detection performance is superior to the other detectors in general even though the CBF ULA has less resolution than the other three detectors. The coprime product array matches both peak side lobe height and the number of sensors of the CBF ULA and has higher resolution than the CBF ULA. However, its detection performance lags behind the CBF ULA. Even the product nested array, which has higher peak side lobe height than the coprime array, achieves commensurate and in some cases, even better detection performance than the coprime array. The power beampatterns depicted in Fig. 10 reconcile these seemingly discordant points. The CBF ULA beampattern (green line) has wider
main lobe width than the other three detectors because of its shorter aperture. However, the CBF ULA beampattern has a consistent side lobe roll off factor of $-6$ dB/octave. The total side lobe area is less than all other detectors despite its wider main lobe which implies more white noise suppression than the other detectors. This explains why the CBF ULA’s overall detection performance surpasses all other detectors. The coprime array’s beampattern (brown line) and the nested array’s beampattern (black line) have equal main lobe width because of equal aperture and the coprime beampattern has lower peak side lobe height. However, the total side lobe area of the coprime beampattern is still higher than the nested beampattern which explains why the nested detector outperforms the coprime detector. The CBF NULA (purple line) has some very high and wide side lobes, but other narrow and low side lobes. There are several ranges of bearing where the CBF NULA has better side lobes pattern than the product beampatterns justifying better average MDI than the product detectors.

**IV. CONCLUSION**

This paper identified the conditional PDFs of the outputs of a broad category of sparse array geometries processed with a product processor. The paper compared the performance of the CBF ULA detector with both nested and coprime sparse array processors, as well as CBF processing for the CSA sensor geometry. The detection performance was assessed for both average performance for uniformly distributed interferer locations, and in worst-case scenarios when the interferer was located at the peak side lobe bearing for the beamformer. The CBF ULA achieved the best detection performance for the random interferer location scenarios for when the interferer was much louder than the source of interest, and also when the interferer power was comparable to the source of interest. The CBF ULA also achieved the best detection performance in the worst case scenarios when the interferer was comparable in power to the source of interest. The CSA achieves the best detection performance in one scenario: when a much louder interferer appears in the peak side lobe location.

These results suggest that although sparse arrays offer advantages in identifying the directions of arrival (DOAs) for more sources than sensors through co-array processing [16]–[19], these processors sacrifice detection performance for any given look direction in many common array processing scenarios. The ability to detect the sources in an array processing environment is an important prerequisite to the subspace DOA algorithms such as MUSIC commonly exploited by these sparse array processors. The subspace algorithms assume an accurate estimate of the number of signals present in the data, which must be obtained by direct detection as studied here, or a model order estimator. However, there is presently no model order estimator for the statistics of product processing of sparse arrays. Consequently, the results of this study suggest that sparse array algorithms employing product processing may be best suited for high SNR snapshot rich environments where the primary challenge is estimating the DOAs for many sources, and detecting the sources over the background noise is not the primary challenge faced by array processing algorithms.

**APPENDIXES**

**A. CORRELATION COEFFICIENT BETWEEN SUBARRAY OUTPUTS**

The subarray outputs are $y_1$ and $y_2$ with zero mean and variances $\sigma_1^2$, $\sigma_2^2$. The correlation coefficient between $y_1$ and $y_2$ is

$$\rho = \frac{E[y_1 y_2^*]}{\sigma_1 \sigma_2}. \quad (13)$$

Plugging in the expressions for $y_1$ and $y_2$, yields

$$\rho = \frac{E[|a_s|^2 + a_s \eta_1 + a_s^2 \eta_2^2 + \eta_1 \eta_2]}{\sigma_1 \sigma_2}. \quad (14)$$

Since signal is independent of the noise components $\eta_1$ and $\eta_2$ and the variables $\eta_1$ and $\eta_2$ are zero mean, $\rho$ simplifies to

$$\rho = \frac{E[|a_s|^2 + \eta_1 \eta_2]}{\sigma_1 \sigma_2}. \quad (15)$$

Plugging in the values of $\sigma_1$, $\sigma_2$ and input signal variance, we get

$$\rho = \frac{\sigma_s^2 + E[\eta_1 \eta_2]}{\sqrt{(\sigma_s^2 + \sigma_n^2 M_e) (\sigma_s^2 + \sigma_n^2 N_e)}}. \quad (16)$$

Since the subarrays share $\alpha$ sensors, $\rho$ simplifies to

$$\rho = \frac{\sigma_s^2 + \frac{\sigma_n^2 \cdot \alpha}{M_e N_e}}{\sqrt{(\sigma_s^2 + \sigma_n^2 M_e) (\sigma_s^2 + \sigma_n^2 N_e)}}. \quad (17)$$
Substituting $\Psi = \sigma_s^2/\sigma_n^2$, we get

$$\rho = \frac{\Psi + \frac{\alpha}{M_e N_e}}{\sqrt{\left(\Psi + \frac{1}{M_e}\right) \left(\Psi + \frac{1}{N_e}\right)}}. \quad (18)$$

**B. MEAN OF THE PRODUCT PROCESSOR DETECTION STATISTIC**

The mean of the product processor detection statistic is

$$\mu = \int_0^\infty y g(y) dy = \frac{4}{(1 - |\rho|^2)\sigma_1^2\sigma_2^2} \times \int_0^\infty y^2 I_0 \left(\frac{2y|\rho|}{\sigma_1\sigma_2(1 - |\rho|^2)}\right) K_0 \left(\frac{2y}{\sigma_1\sigma_2(1 - |\rho|^2)}\right) dy.$$  

The integral in the above equation simplifies to [36, page 684]

$$\frac{2(\Gamma(1.5))^2}{\left(\sigma_1\sigma_2(1 - |\rho|^2)\right)^3} F(1.5, 1.5; 1; |\rho|^2).$$

Hence, $\mu = (\pi/4)\sigma_1\sigma_2(1 - |\rho|^2)^2 F(1.5, 1.5; 1; |\rho|^2)$, since $(\Gamma(1.5))^2 = \pi/4$ [36, page 895].

**C. VARIANCE OF THE PRODUCT PROCESSOR DETECTION STATISTIC**

The second moment of the product processor detection statistic is

$$M_2 = \int_0^\infty y^2 g(y) dy = \frac{4}{(1 - |\rho|^2)\sigma_1^2\sigma_2^2} \times \int_0^\infty y^3 I_0 \left(\frac{2y|\rho|}{\sigma_1\sigma_2(1 - |\rho|^2)}\right) K_0 \left(\frac{2y}{\sigma_1\sigma_2(1 - |\rho|^2)}\right) dy.$$  

The integral in the above equation simplifies to [36, page 684]

$$\frac{4}{\left(\sigma_1\sigma_2(1 - |\rho|^2)\right)^4} F(2, 2; 1; |\rho|^2).$$

The hypergeometric series $F(2, 2; 1; |\rho|^2)$ is [36, page 1005]

$$F(2, 2; 1; |\rho|^2) = \frac{(2)^2 |\rho|^2 + (2 \times 3)^2 (|\rho|^2)^2 + (2 \times 3 \times 4)^2 (|\rho|^2)^3 + \cdots}{(1 \times 2)^2 (|\rho|^2)^2}.$$

$$= \sum_{n=0}^\infty \frac{(n + 1)!^2}{(n!)^2} (|\rho|^2)^n = \sum_{n=0}^\infty (n + 1)^2 (|\rho|^2)^n$$

$$= \sum_{n=0}^\infty n^2 (|\rho|^2)^n + 2 \sum_{n=0}^\infty n (|\rho|^2)^n + \sum_{n=0}^\infty (|\rho|^2)^n$$

$$= \frac{(|\rho|^2)^2 + |\rho|^2}{(1 - |\rho|^2)^2} + \frac{2|\rho|^2}{(1 - |\rho|^2)^2} + \frac{1}{1 - |\rho|^2}$$

$$= 1 + |\rho|^2 (1 - |\rho|^2)^3.$$  

Hence, the second moment is $M_2 = \frac{4}{(1 - |\rho|^2)\sigma_1^2\sigma_2^2} \times \frac{4}{\left(\sigma_1\sigma_2(1 - |\rho|^2)\right)^4} F(1.5, 1.5; 1; |\rho|^2) = (1 + |\rho|^2)^2\sigma_1^2\sigma_2^2$

and the variance of the product processor is

$$\sigma^2 = M_2 - \mu^2 = (1 + |\rho|^2)^2\sigma_1^2\sigma_2^2 \times \left(\frac{\pi}{4}\sigma_1\sigma_2(1 - |\rho|^2)^2 F(1.5, 1.5; 1; |\rho|^2)\right)^2.$$

**D. DETECTOR OUTPUT STATISTICS WHEN THERE ARE INTERFERERS PRESENT**

When there are $P$ interferers present, the inputs to Subarray 1 and Subarray 2 are $x_1 = a_s v_{1} + \sum_{k=1}^P a_{i,k} v_{1,k} + n_1$ and $x_2 = a_s v_{2} + \sum_{k=1}^P a_{i,k} v_{2,k} + n_2$ respectively, where $a_{i,k} \sim \mathcal{C} \mathcal{N}(0, \sigma_s^2)$ and other variables are as defined in Section II. The signal components are $a_s v_{1}$ and $a_s v_{2}$, the interferer components are $a_{i,k} v_{1,k}$ and $a_{i,k} v_{2,k}$, and the noise components are $n_1$ and $n_2$. The $M_e$ and $N_e$ element vectors $v_{1,k}$ and $v_{2,k}$ are the array manifold vectors at $k^{th}$ the interferer direction $\theta_{i,k}$.

The subarrays’ CBF outputs when steered to the signal direction are

$$y_1 = a_s + \sum_{k=1}^P a_{i,k} v_{1}^H v_{1,k} + \frac{v_{1}^H n_1}{M_e}$$

$$= a_s + \sum_{k=1}^P a_{i,k} v_{1}^H v_{1,k} + \eta_1,$$

and

$$y_2 = a_s + \sum_{k=1}^P a_{i,k} v_{2}^H v_{2,k} + \frac{v_{2}^H n_2}{N_e}$$

$$= a_s + \sum_{k=1}^P a_{i,k} v_{2}^H v_{2,k} + \eta_2,$$

where $\eta_1$ and $\eta_2$ are described in Section II. The CBF outputs are also zero-mean proper Gaussians with variances

$$\sigma_1^2 = \sigma_s^2 + \frac{\sum_{k=1}^P |a_{i,k}|^2 |v_{1,k}|^2}{M_e^2} + \frac{\sigma_n^2}{M_e}$$

and

$$\sigma_2^2 = \sigma_s^2 + \frac{\sum_{k=1}^P |a_{i,k}|^2 |v_{2,k}|^2}{N_e^2} + \frac{\sigma_n^2}{N_e}.$$

The correlation coefficient between $y_1$ and $y_2$ is $\rho = E[y_1 y_2^\ast]/(\sigma_1 \sigma_2)$. Since the signal, interferers, and noise are zero-mean independent variables, we get

$$\rho = \frac{E\left[|a_s|^2 + \sum_{k=1}^P |a_{i,k}|^2 (v_{1,k}^H v_{1,k}) \right]}{\sigma_1 \sigma_2}.$$
Plugging in the values of $\sigma_1$, $\sigma_2$, input signal variance, input interferers’ variances, and input noise variance and simplifying, we get

$$
\rho = \frac{\sigma_n^2}{\sigma_i^2} + \frac{\sum_{k=1}^{P} \sigma_{i,k}^2 |v_1, k|^2 |v_2, k|^2}{M_k N_k} + \frac{\sigma_n^2}{M_k N_k}
$$

Substituting $\Psi = \sigma_i^2/\sigma_n^2$ and $\Phi_k = \sigma_{i,k}^2/\sigma_n^2$, we get

$$
\rho = \left( \frac{\sum_{k=1}^{P} \Phi_k |v_1, k|^2 |v_2, k|^2}{M_k N_k} - \frac{\alpha}{M_k N_k} \right) \Psi + \frac{\sum_{k=1}^{P} \Phi_k |v_1, k|^2}{M_k^2} + \frac{1}{M_k} - 0.5 \Psi + \frac{\sum_{k=1}^{P} \Phi_k |v_1, k|^2}{N_k^2} + \frac{1}{N_k} - 0.5
$$

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REFERENCES

[1] A. Berman and C. S. Clay, “Theory of time averaged roduct arrays,” J. Acoust. Soc. Amer., vol. 29, no. 7, pp. 805–812, Jul. 1957.

[2] D. G. Tucker, “Signal/noise performance of multiplexer (or correlation) and addition (or integrating) types of detector,” Proc. IEE C, Monographs, vol. 102, no. 2, pp. 181–190, Sep. 1955.

[3] V. Welsby and D. Tucker, “Multiplicative receiving arrays,” J. Brit. Inst. Radio Eng., vol. 19, no. 369–382, Jun, 1959. [Online]. Available: http://digital-library.theiet.org/content/journals/10.1049/jbri.1959.0043

[4] D. G. Tucker, “Multiplicative arrays in radio-astronomy and sonar systems,” Radio Electron. Engineer, vol. 25, no. 2, pp. 113–118, 1965.

[5] V. G. Welsby, “Multiplicative receiving arrays: The angular resolution of targets in a sonar system with electronic scanning,” J. Brit. Inst. Radio Eng., vol. 22, no. 1, pp. 5–12, Jul. 1961. [Online]. Available: http://digital-library.theiet.org/content/journals/10.1049/jbri.1961.0077

[6] P. Cath, “Three methods of nonlinear processing for direction finding,” Res. Inst., Univ. Michigan, Ann Arbor, MI, USA, Tech. Rep. 85, Aug. 1958.

[7] H. Miller and J. Lee, “Signal detection and bearing estimation by squarerlaw and multiplicative array processors,” in Proc. Ocean IEEE Intl. Conf. Ocean Environ., Sep. 1973, pp. 475–480.

[8] L. Miller and J. Lee, “The probability density function for the output of an analog cross-correlator with correlated bandpass inputs,” IEEE Trans. Inf. Theory, vol. 20, no. 4, pp. 433–440, Jul. 1974.

[9] J. Faran and R. Hills, “Correlators for signal reception,” Harvard Acoust. Res. Lab., Tech. Rep MR-364-903, Sep. 1952.

[10] D. C. Fakley, “Comparison between the performances of a time-averaged product array and an inacross correlator,” J. Acoust. Soc. Amer., vol. 31, no. 10, pp. 1307–1314, Oct. 1959.

[11] B. Steinberg, Principles of Aperture and Array System Design; Including Random and Adaptive Arrays. Hoboken, NJ, USA: Wiley, 1976.

[12] H. V. Trees, Optimum Array Processing (Detection, Estimation Modulation Theory). Hoboken, NJ, USA: Wiley, 2002.

[13] G. Kefalas, “An aperture distribution technique for product-array antennas,” IEEE Trans. Antennas Propag., vol. 16, no. 1, p. 125, Jan. 1968.

[14] D. E. N. Davies and C. R. Ward, “Low sidelobe patterns from thinned arrays using multiplicative processing,” IEE Proc. F Commun., Radar Signal Process., vol. 127, no. 1, pp. 9–23, 1980.

[15] J. G. Mitra, M. Mondal, M. K. Tchobanou, and G. J. Dolecek, “General polynomial factorization-based design of sparse periodic linear arrays,” IEEE Trans. Ultrason., Ferroelectr., Freq. Control, vol. 57, no. 9, pp. 1952–1966, Sep. 2010.

[16] P. Vaidyanathan and P. Pal, “Sparse sensing with co-prime samplers and arrays,” IEEE Trans. Signal Process., vol. 59, no. 2, pp. 573–586, Feb. 2011.

[17] P. Vaidyanathan and P. Pal, “Theory of sparse coprime sensing in multiple dimensions,” IEEE Trans. Signal Process., vol. 59, no. 8, pp. 3592–3608, Aug. 2011.

[18] P. Pal and P. Vaidyanathan, “Nested arrays: A novel approach to array processing with enhanced degrees of freedom,” IEEE Trans. Signal Process., vol. 58, no. 8, pp. 4167–4181, Aug. 2010.

[19] P. Pal and P. Vaidyanathan, “Nested arrays in two dimensions, part I: Geometrical considerations,” IEEE Trans. Signal Process., vol. 60, no. 9, pp. 4694–4705, Sep. 2012.

[20] K. Wage, “When two wrongs make a right: Combining aliased arrays to find source sounds,” Acoust. Today, vol. 14, no. 3, pp. 48–56, 2018.

[21] K. Adhikari, “Beamforming with semi-coprime arrays,” J. Acoust. Soc. Amer., vol. 145, no. 5, pp. 2841–2850, May 2019. doi: 10.1121/1.5100281.

[22] V. Chavali, K. E. Wage, and J. R. Buck, “Multiplicative and min processing of experimental passive sonar data from thinned arrays,” J. Acoust. Soc. Amer., vol. 144, no. 6, pp. 3262–3274, Dec. 2018.

[23] K. Adhikari and B. Drozdenko, “Symmetry-imposed rectangular coprime and nested arrays for direction of arrival estimation with multiple signal classification,” IEEE Access, vol. 7, pp. 153217–153229, 2019.

[24] K. Adhikari and J. R. Buck, “Spatial spectral estimation with product processing of a pair of colinear arrays,” IEEE Trans. Signal Process., vol. 65, no. 9, pp. 2389–2401, May 2017.

[25] K. Adhikari and J. R. Buck, “Gaussian signal detection by coprime sensor arrays,” in Proc. IEEE Int. Conf. Acoust., Speech Signal Process. (ICASSP), Apr. 2015, pp. 2379–2383.

[26] P. Schreier and L. Scharf, Statistical Signal Processing of Complex-Valued Data: The Theory of Improper and Noncircular Signals. Cambridge, U.K.: Cambridge Univ. Press, 2010.

[27] A. Papoulis, Probability, Random Variables, and Stochastic Processes (Electrical Engineering). New York, NY, USA: McGraw-Hill, 1991. [Online]. Available: http://opac.inria.fr/record=b1077486

[28] T. Cover and J. Thomas, Elements of Information Theory. Hoboken, NJ, USA: Wiley, 2006. [Online]. Available: https://books.google.com/books?id=EuhBlaW31hc

[29] K. Adhikari, J. R. Buck, and K. E. Wage, “Extending coprime sensor arrays to achieve the peak side lobe height of a full uniform linear array,” EURASIP J. Adv. Signal Process., vol. 2014, no. 1, p. 148, Sep. 2014, doi: 10.1155/2014/1952–148.

[30] S. Qin, Y. D. Zhang, and M. G. Amin, “Generalized coprime array configurations for Direction-of-Arrival estimation,” IEEE Trans. Signal Process., vol. 63, no. 6, pp. 1377–1390, Mar. 2015.

[31] Y. Liu and J. R. Buck, “Detecting Gaussian signals in the presence of interferers using the coprime sensor arrays with the min processor,” in Proc. 49th Asilomar Conf. Signals, Syst. Comput., Nov. 2015, pp. 370–374.

[32] Y. Liu and J. R. Buck, “Super-resolution DOA estimation using a coprime sensor array with the min processor,” in Proc. 50th Asilomar Conf. Signals, Syst. Comput., Nov. 2016, pp. 944–948.

[33] Y. Liu and J. R. Buck, “Spatial spectral estimation using a coprime sensor array with the min processor,” in Proc. IEEE Sensor Array Multichannel Signal Process. Workshop (SAM), Jul. 2016, pp. 1–5.

[34] K. Miller, Multidimensional Gaussian Distributions (SIAM Series in Applied Mathematics). Hoboken, NJ, USA: Wiley, 1964. [Online]. Available: http://books.google.com/books?id=9dJpOClUAYC

[35] M. Simon, Probability Distributions Involving Gaussian Random Variables: A Handbook for Engineers and Scientists (International Series in Engineering and Computer Science). Springer, 2007. [Online]. Available: http://books.google.com/books?id=zjJdP0CJUAYC

[36] I. Gradshetyen and I. Ryzhik, Table of Integrals, Series, and Products. New York, NY, USA: Academic, 1900.

[37] Mathworks. Integral. Accessed: May 30, 2018. [Online]. Available: https://www.mathworks.com/help/matlab/ref/integral.html?s_tid=doc_ta
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