GROUND STATE SOLUTIONS FOR THE FRACTIONAL PROBLEMS WITH DIPOLE-TYPE POTENTIAL AND CRITICAL EXponent

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(Dedicated to Professor Goong Chen on the occasion of his seventieth birthday)

Abstract. We are concerned with ground state solutions of the fractional problems with dipole-type potential and critical exponent. Under certain conditions on the dipole-type potential and the parameter, we show that the structure of the Palais-Smale sequence goes to zero weakly, and establish the existence of ground state solution to the above problems by using a new analytical method not involving the concentration-compactness principle.

1. Introduction. Consider the fractional problems:
\[(−Δ)^s u − \mu \frac{Φ(x/|x|)}{|x|^{2s}} u = |u|^{2^*_s−2} u, \quad x ∈ \mathbb{R}^3, \quad (P_S)\]
\[(−Δ)^s u − \mu \frac{Φ(x/|x|)}{|x|^{2s}} u = (I_α * |u|^{2^*_s,α})|u|^{2^*_s,α−2} u, \quad x ∈ \mathbb{R}^3, \quad (P_{HLS})\]
where \(s ∈ (0, 1), α ∈ (0, 3), I_α \) is the Riesz potential, \(2^*_s \) is the Sobolev critical exponent, and \(2^*_s,α = \frac{2s + α}{3−2s} \) is the Hardy-Littlewood-Sobolev upper critical exponent. The dipole-type potential \(Φ \) and the parameter \(μ \) satisfy the condition:
\((Φ_1) \quad 0 ≤ Φ ∈ L^{\frac{N}{α}}(S^2) \) and \(μ ∈ (0, Λ_Φ) \), where \(Λ_Φ = C_{s,2} |S^2|^{\frac{2s}{N}} \norm{Φ}_{L^{\frac{N}{α}}(S^2)}^{−1} \)

where \(C_{s,2} = 2π \frac{Γ(\frac{N+2s}{4}) Γ(\frac{N+2s}{2})}{Γ(\frac{N−2s}{4}) Γ(−s)} \) is the best constant for the fractional Hardy inequality.

For \(s ∈ (0, 1) \), the fractional Laplacian \((-Δ)^s \) of a function \(u : \mathbb{R}^N \rightarrow \mathbb{R} \) can be defined by
\[(-Δ)^s u = \mathcal{F}^{-1}(|ξ|^{2s} \mathcal{F}(u)(ξ)), \quad ξ ∈ \mathbb{R}^N, \quad u ∈ C^∞_0(\mathbb{R}^N),\]

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where $\mathcal{F}(u)$ denotes the Fourier transform of $u$. The operator $(-\Delta)^s$ in $\mathbb{R}^N$ is a nonlocal pseudo-differential operator of the form
\[
(-\Delta)^s u(x) = C_{N,s} \text{P.V.} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x-y|^{N+2s}} \, dy,
\]
where P.V. is the Cauchy principal value and $C_{N,s}$ is a normalization constant. The fractional power of Laplacian is the infinitesimal generator of Lévy stable diffusion process which arises in anomalous diffusion in plasma, population dynamics, geophysical fluid dynamics, flames propagation, minimal surfaces and game theory, see [2,16,18].

In the quantum mechanics, a spin zero relativistic particle of charge $e$ and mass $m$ in the Coulomb field of an infinitely heavy nucleus of charge $Z$ are described by the Hamiltonian
\[
H(p, x) = \left(p^2 + m^2\right)^{\frac{1}{2}} - \frac{Ze^2}{|x|},
\]
Fall-Felli extended $H(p, x)$ to
\[
\hat{H}(p, x) = \left(p^2 + m^2\right)^s - \frac{\Phi(x)}{|x|^{2s}},
\]
and considered an operator with the singular potential [6]:
\[
\hat{H}(i\nabla, x) = (-\Delta + m^2)^s - \frac{\Phi(x)}{|x|^{2s}}.
\]
For $\Phi \equiv 0$, the operator $\hat{H}(i\nabla, x)$ reduces to $(-\Delta + m^2)^s$. In this study, we restrict our attention to $\hat{H}(i\nabla, x)$ with $m = 0$. As we see, equation $(\mathcal{P}_S)$ is related to the fractional Schrödinger equation:
\[
\frac{\partial \psi}{\partial t} = (-\Delta)^s \psi + V(x)\psi - |\psi|^{2s-2}\psi, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^3. \tag{1.1}
\]
When $s = 1$, it is the well-known Schrödinger equation which arises in quite a few physical contexts such as quantum mechanics, nonlinear optics, quantum field theory and Hartree-Fock theory [1, 15, 16]. When $s \in (0, 1) \setminus \{\frac{1}{2}\}$, it is the fractional Schrödinger equation, which was introduced by Laskin (see [9]) owing to the extension of the Feynman path integral, from the Brownian-like to Lévy-like quantum mechanical paths. When $s = \frac{1}{2}$, it is the half-wave equation derived from a water waves model (see [8]). Solitary waves solutions of equation (1.1) have the ansatz form $\psi(t, x) = e^{i\omega t} u(x)$, where $\omega \in \mathbb{R}$ is a constant and $u$ satisfies
\[
(-\Delta)^s u + (\omega + V(x))u = |u|^{2s-2} u, \quad x \in \mathbb{R}^3. \tag{1.2}
\]

If $\omega = 0$ and $V = \mu \frac{\Phi(x)}{|x|^{2s}}$, equation (1.2) reduces to equation $(\mathcal{P}_S)$. The existence of solution of equation $(\mathcal{P}_S)$ has been extensively studied by the variational methods under various hypotheses on the function $\Phi$. Here let us briefly recall some existing results. Cotsiolis-Travoularis [3] considered the existence and properties of solutions for equation $(\mathcal{P}_S)$ with $\Phi \equiv 0$. Dipierro et al [5] studied the existence and properties of solutions for equation $(\mathcal{P}_S)$ with $\Phi \equiv 1$. Fall-Felli [6] established the existence and properties of solution for equation $(\mathcal{P}_S)$ with $0 \leq \Phi \in L^\infty(\mathbb{S}^2)$. Su et al [14] established the existence of nonnegative solution $u \in D^{s,2}_{rad}(\mathbb{R}^3)$ to equation $(\mathcal{P}_S)$ with $0 \leq \Phi \in L^\frac{4}{2s}(\mathbb{S}^2)$. Note that $u \in D^{s,2}_{rad}(\mathbb{R}^3) \subset D^{s,2}(\mathbb{R}^3)$. Then the solution $u \in D^{s,2}(\mathbb{R}^3)$ may not be a ground state solution in $D^{s,2}(\mathbb{R}^3)$. 


Equation \( \mathcal{P}_{HLS} \) is related to the fractional Choquard equation:

\[
\frac{\partial \psi}{\partial t} = (-\Delta)^s \psi + V(x) \psi - (I_\alpha * |\psi|^{2^*_s})|\psi|^{2^*_s-2}\psi, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^3.
\]

When \( s = 1 \), it is the Choquard equation arising in the quantum theory of a polaron at rest [13]. When \( s = \frac{1}{2} \), it is the semi–relativistic Hartree equation which arises in the quantum theory for large systems of self-interacting, relativistic bosons [10]. Solitary waves solutions of equation (1.3) has ansatz of

\[
(-\Delta)^s u + (\omega + V(x))u = (I_\alpha * |u|^{2^*_s})|u|^{2^*_s-2}u, \quad x \in \mathbb{R}^3. \tag{1.4}
\]

Recently, the existence of solutions to equation (1.4) has been studied in [12]. If \( \omega = 0 \) and \( V = \mu \frac{\Phi(x)}{|x|^s} \), equation (1.4) reduces to equation \( \mathcal{P}_{HLS} \). The existence and properties of solutions of equation \( \mathcal{P}_{HLS} \) with \( \Phi \equiv 0 \) or \( \Phi \equiv 1 \) were presented in [4].

It is natural to ask whether there exists a ground state solution to equations \( \mathcal{P}_S \) and \( \mathcal{P}_{HLS} \) with \( 0 \leq \Phi \in L^\infty(\mathbb{R}^3) \). To the best of our knowledge, it seems that there is no affirmative answer in the literature. Actually, this is the primary motivation for us to study the existence of ground state solution to equations \( \mathcal{P}_S \) and \( \mathcal{P}_{HLS} \) with condition \((\Phi_1)\).

We now summarize our main results on ground state solution of equations \( \mathcal{P}_S \) and \( \mathcal{P}_{HLS} \) with condition \((\Phi_1)\).

**Theorem 1.1.** Assume that \( s \in (0, 1) \) and condition \((\Phi_1)\) holds. Then equation \( \mathcal{P}_S \) has a nonnegative ground state solution \( u \in D^{s,2}(\mathbb{R}^3) \).

**Theorem 1.2.** Assume that \( s \in (0, 1), \alpha \in (0, 3) \) and condition \((\Phi_1)\) holds. Then equation \( \mathcal{P}_{HLS} \) has a nonnegative ground state solution \( u \in D^{s,2}(\mathbb{R}^3) \).

Let us introduce the energy functional corresponding to equation \( \mathcal{P}_S \) by

\[
J_0(u) = \frac{1}{2} \|u\|_\Phi^2 - \frac{1}{2} \int_{\mathbb{R}^3} |u|^{2^*_s} \, dx.
\]

It is well-known that the Sobolev embedding

\[
D^{s,2}(\mathbb{R}^3) \hookrightarrow L^{2^*_s}(\mathbb{R}^3)
\]

is not compact, and so it is difficult to find a convergent subsequence in the Palais-Smale sequence of \( J_0 \). The ideas and methods described in [14, 17] become invalid. For example, for \( \Phi \equiv 1 \), from [17] we know that the equation

\[
(-\Delta)^s u - \mu \frac{u}{|x|^{2s}} = |u|^{2^*_s-2}u, \quad x \in \mathbb{R}^3, \tag{1.5}
\]

has the ground state solution and there exists a Palais-Smale sequence \( \{u_n\} \) for the associated energy functional that does not go to zero weakly. Unfortunately, when \( \Phi \) satisfies condition \((\Phi_1)\), we cannot evaluate the limits:

\[
\lim_{n \to \infty} \Phi \left( \frac{x + \frac{x_n}{\lambda_n}}{|x + \frac{x_n}{\lambda_n}|} \right) \text{, as } \frac{x_n}{\lambda_n} \to \infty
\]

and

\[
\lim_{n \to \infty} \int_{\mathbb{R}^3} \Phi \left( \frac{x + \frac{x_n}{\lambda_n}}{|x + \frac{x_n}{\lambda_n}|} \right) \frac{v_n \varphi}{|x + \frac{x_n}{\lambda_n}|^{2s}} \, dx.
\]
To overcome this difficulty, we propose an analytical method without the usage of concentration-compactness principle as follows.

- **Step 1.** We study the non-vanishing of the \((PS)_{c_0}\) sequence \(\{u_n\}\) of \(J_0\) in \(L^{2^*_s}(\mathbb{R}^3)\), where the \((PS)_{c_0}\) sequence \(\{u_n\}\) of \(J_0\) satisfies
  \[
  \lim_{n \to \infty} J_0(u_n) = c_0 \quad \text{and} \quad \lim_{n \to \infty} \|J_0'(u_n)\|_{D^{-s,2}(\mathbb{R}^3)} = 0,
  \]
  as \(n \to \infty\),

  and \(c_0 = \inf_{\Upsilon \in \Gamma_0} \sup_{t \in [0,1]} J_0(\Upsilon(t)) \) and \(\Gamma_0 = \{ \Upsilon \in C([0,1], D^{s,2}(\mathbb{R}^3)) | \Upsilon(0) = 0, J_0(\Upsilon(1)) < 0 \}\).

- **Step 2.** We present the structure of the \((PS)_{c_0}\) sequence tends to zero weakly in \(D^{s,2}(\mathbb{R}^3)\): if the \((PS)_{c_0}\) sequence \(\{u_n\}\) goes to zero weakly in \(D^{s,2}(\mathbb{R}^3)\), then sequence \(\{u_n\}\) either vanishes in \(L^{2^*_s}(B(0,1))\) or concentrates in \(L^{2^*_s}(B(0,1))\).

This is the crucial step of our approach, see Lemma 3.3.

- **Step 3.** By the dilation transformation \(u(x) \to t^{\frac{3-2s}{s}} u(tx)\), we construct a new \((PS)_{c_0}\) sequence \(\{\tilde{u}_n\}\). Applying non-vanishing of the \((PS)_{c_0}\) sequence in \(D^{s,2}(\mathbb{R}^3)\), we show that the new \((PS)_{c_0}\) sequence \(\{\tilde{u}_n\}\) neither vanishes in \(L^{2^*_s}(B(0,1))\) nor concentrates in \(L^{2^*_s}(B(0,1))\). From Step 2, \(\{\tilde{u}_n\}\) tends to nonzero weakly in \(D^{s,2}(\mathbb{R}^3)\).

It is worth mentioning that this method can be applied to not only the case of \(0 \leq \Phi \in L^{\frac{3}{2s}}(\mathbb{S}^2)\) but also the case of \(0 \leq \Phi \in L^\infty(\mathbb{S}^2)\).

The rest of this paper is organized as follows. In Section 2, we present some preliminary results. In Sections 3 and 4, we present the proofs of Theorems 1.1 and 1.2, respectively.

2. **Preliminaries.** For \(s \in (0,1)\), the space \(D^{s,2}(\mathbb{R}^3)\) is the completion of \(C_0^\infty(\mathbb{R}^3)\) with respect to the semi-norm

\[
\|u\|_{D^{s,2}(\mathbb{R}^3)}^2 = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u(x) - u(y)|^2}{|x - y|^{3+2s}} \, dx \, dy.
\]

For \(u \in D^{s,2}(\mathbb{R}^3)\), we have the fractional Hardy inequality

\[
C_{s,2} \int_{\mathbb{R}^3} \frac{|u|^2}{|x|^{2s}} \, dx \leq \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u(x) - u(y)|^2}{|x - y|^{3+2s}} \, dx \, dy,
\]

where \(C_{s,2}\) is defined by condition \((\Phi_1)\).

We introduce the measure \(d\theta\) induced by the Lebesgue measure on the unit sphere \(\mathbb{S}^2 \subset \mathbb{R}^3\). We denote by \(\| \cdot \|_{L^\infty(\mathbb{S}^2)}\) the quantity

\[
\|\Phi\|_{L^\infty(\mathbb{S}^2)} = \int_{\mathbb{S}^2} \Phi(\theta)^d \, d\theta.
\]

**Lemma 2.1 ([7]).** Let \(s \in (0,1)\) and \(0 \leq \Phi \in L^{\frac{3}{2s}}(\mathbb{S}^2)\). Then we have

\[
\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u(x) - u(y)|^2}{|x - y|^{3+2s}} \, dx \, dy \geq \Lambda_\Phi \int_{\mathbb{R}^3} \frac{\Phi(|x|) |u|^2}{|x|^{2s}} \, dx,
\]

where \(u \in D^{s,2}(\mathbb{R}^3)\) and \(\Lambda_\Phi = C_{s,2} |\mathbb{S}^2|^{\frac{2s}{3}} \|\Phi\|^{-1}_{L^{\frac{3}{2s}}(\mathbb{S}^2)}\).

In view of Lemma 2.1 and \(\mu \in (0, \Lambda_\Phi)\),

\[
\|u\|_\Phi = \left( \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u(x) - u(y)|^2}{|x - y|^{3+2s}} \, dx \, dy - \mu \int_{\mathbb{R}^3} \frac{\Phi(|x|) |u|^2}{|x|^{2s}} \, dx \right)^{\frac{1}{2}}
\]

is an equivalent norm in \(D^{s,2}(\mathbb{R}^3)\).
Lemma 2.2 ([11]). Let \( t, r > 1, 0 < \alpha < 3 \) with \( \frac{1}{t} + \frac{1}{r} + \frac{3-\alpha}{3} = 2 \), \( f \in L^t(\mathbb{R}^3) \) and \( h \in L^r(\mathbb{R}^3) \). There exists a sharp constant \( C(\alpha, t, r) > 0 \) independent of \( f, h \) such that
\[
\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|f(x)||h(y)|}{|x-y|^{3-\alpha}} \, dx \, dy \leq C(\alpha, t, r) \| f \|_{L^t(\mathbb{R}^3)} \| h \|_{L^r(\mathbb{R}^3)}.
\]
If \( t = r = \frac{6}{3+\alpha} \), then \( C(\alpha, t, r) = C(\alpha) = \pi^{-\frac{3-\alpha}{2}} \frac{\Gamma(\frac{\alpha}{3+\alpha})}{\Gamma(\frac{\alpha}{3})} \frac{\Gamma(\frac{2}{3})}{\Gamma(3)} \). We define the best constant of the fractional critical Sobolev inequality by
\[
S_s := \inf_{u \in D^{s,2}(\mathbb{R}^3) \setminus \{0\}} \frac{\| u \|_{D^{s,2}(\mathbb{R}^3)}^2}{\int_{\mathbb{R}^3} |u(x)|^{2s} \, dx}.
\] (2.2)
and define the best constant of the fractional critical Hardy-Littlewood-Sobolev inequality by
\[
S_{s,\alpha} := \inf_{u \in D^{s,2}(\mathbb{R}^3) \setminus \{0\}} \frac{\| u \|_{D^{s,2}(\mathbb{R}^3)}^2}{\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u(x)|^2 |u(y)|^{2s-\alpha}}{|x-y|^{3s-\alpha}} \, dx \, dy}.
\] (2.3)
We know that \( S_s \) and \( S_{s,\alpha} \) can be attained in \( \mathbb{R}^3 \), see [4].

3. Proof of Theorem 1.1.

Lemma 3.1. Assume that all conditions described in Theorem 1.1 hold. Then the following statements are true.

(i) The functional \( J_0 \) possesses the mountain pass geometry. There exists \( \{u_n\} \subset D^{s,2}(\mathbb{R}^3) \) such that
\[
J_0(u_n) \to c_0 \quad \text{and} \quad \| J_0'(u_n) \|_{D^{-s,2}(\mathbb{R}^3)} \to 0, \quad n \to \infty,
\]
and \( \{u_n\} \) is uniformly bounded in \( D^{s,2}(\mathbb{R}^3) \), where
\[
c_0 = \inf_{\gamma \in \Gamma_0} \max_{t \in [0,1]} J_0(\gamma(t)),
\]
\[
\Gamma_0 = \{ \gamma \in C([0,1], D^{s,2}(\mathbb{R}^3)) \mid \gamma(0) = 0, J_0(\gamma(1)) < 0 \}.
\]

(ii) For each \( u \in D^{s,2}(\mathbb{R}^3) \setminus \{0\} \), there exists a unique \( t_u > 0 \) such that \( t_u u \in \mathcal{N}_0 \), where
\[
\mathcal{N}_0 = \{ u \in D^{s,2}(\mathbb{R}^3) \mid (J_0'(u), u) = 0, \ u \neq 0 \}.
\]

(iii) We have \( c_0 = \bar{c}_0 = \bar{c}_0, \) where
\[
\bar{c}_0 = \inf_{u \in \mathcal{N}_0} J_0(u) \quad \text{and} \quad \bar{c}_0 = \inf_{u \in D^{s,2}(\mathbb{R}^3)} \max_{t \geq 0} J_0(tu).
\]

Proof. We separate the proof into three parts.

(i). For any \( u \in D^{s,2}(\mathbb{R}^3) \setminus \{0\} \), we have
\[
J_0(u) \geq \frac{1}{2} \| u \|_{\Phi}^2 - \frac{1}{2s} \| u \|_{\Phi}^{2s}.
\]
Note that \( 2 < 2s^* \). Then there exists a sufficiently small positive number \( \rho \) such that
\[
\zeta := \inf_{\| u \|_{\Phi} = \rho} J_0(u) > 0 = J_0(0).
\]
Given \( u \in D^{s,2}(\mathbb{R}^3) \setminus \{0\} \), we have
\[
J_0(tu) = \frac{t^2}{2} \| u \|_{\Phi}^2 - \frac{t^{2s}}{2s} \int_{\mathbb{R}^3} |u|^{2s} \, dx < 0
\]
for the large \( t \). We choose \( \hat{t} > 0 \) corresponding to \( u \) such that \( J_0(tu) < 0 \) for all \( t > \hat{t} \) and \( \|tu\|_\Phi > \rho \).

By the Mountain pass theorem, there exists \( \{u_n\} \subset D^{s,2}(\mathbb{R}^3) \) such that
\[
J_0(u_n) \to c_0 \quad \text{and} \quad \|J'_0(u_n)\|_{D^{-s,2}(\mathbb{R}^3)} \to 0, \quad \text{as} \ n \to \infty.
\]

It is easy to see that \( \{u_n\} \) is uniformly bounded in \( D^{s,2}(\mathbb{R}^3) \).

(ii). For each \( u \in D^{s,2}(\mathbb{R}^3) \) with \( u \not\equiv 0 \) and \( t \in (0, \infty) \), we set
\[
f_1(t) = J_0(tu) = \frac{t^2}{2} \|u\|_\Phi^2 - \frac{t^{2\gamma}}{2\gamma} \int_{\mathbb{R}^3} |u|^2^\gamma \, dx
\]
and
\[
f_1'(t) = t^2 \|u\|_\Phi^2 - t^{2\gamma - 1} \int_{\mathbb{R}^3} |u|^2^\gamma \, dx.
\]

We know that \( f_1'(\cdot) = 0 \) if and only if
\[
\|u\|_\Phi^2 = t^{2\gamma - 2} \int_{\mathbb{R}^3} |u|^2^\gamma \, dx.
\]

Set
\[
f_2(t) = t^{2\gamma - 2} \int_{\mathbb{R}^3} |u|^2^\gamma \, dx.
\]

We know that \( \lim_{t \to 0} f_2(t) \to 0 \), \( \lim_{t \to \infty} f_2(t) \to \infty \) and \( f_2(\cdot) \) is strictly increasing on \( (0, \infty) \). Then there exists a unique \( 0 < t_u < \infty \) such that
\[
\begin{align*}
&f_2(t) > \|u\|_\Phi^2, \quad t_u < t < \infty, \\
&f_2(t) = \|u\|_\Phi^2, \quad t = t_u, \\
&f_2(t) < \|u\|_\Phi^2, \quad 0 < t < t_u.
\end{align*}
\]

This implies \( t_u u \in \mathcal{N}_0 \). Moreover, we have
\[
\begin{align*}
f_1'(t) < 0, \quad &t_u < t < \infty, \\
f_1'(t) = 0, \quad &t = t_u, \\
f_1'(t) > 0, \quad &0 < t < t_u.
\end{align*}
\]

Hence, \( f_1(\cdot) \) admits a unique critical point \( t_u \) on \( (0, \infty) \) such that \( f_1(\cdot) \) takes the maximum at \( t_u \).

It suffices to show the uniqueness of \( t_u \) by assuming that \( 0 < \hat{t} < \bar{t} \) satisfy \( f_1'(\hat{t}) = f_1'((\bar{t})) = 0 \). Then
\[
\|u\|_\Phi^2 = f_2(t) = f_2(\bar{t}).
\]

This obviously yields a contradiction. Hence, for each \( u \in D^{s,2}(\mathbb{R}^3) \) with \( u \not\equiv 0 \), there exists a unique \( t_u > 0 \) such that \( t_u u \in \mathcal{N}_0 \).

(iii). Note that \( J_0 \) is bounded from below on \( \mathcal{N}_0 \) and \( \bar{c}_0 > 0 \). Indeed, it follows from Lemma 3.1 (ii) that \( \bar{c}_0 = \bar{c}_0 \). Notice that for any \( u \in D^{s,2}(\mathbb{R}^3) \setminus \{0\} \), there exists a large \( \tilde{t} > 0 \) such that \( J_0(tu) < 0 \). Define a path \( \gamma : [0,1] \to D^{s,2}(\mathbb{R}^3) \) by \( \gamma(t) = t t_u \). Clearly, \( \gamma \in \Gamma \) and \( \bar{c}_0 \leq \bar{c}_0 \).

On the other hand, for every \( \gamma \in \Gamma \), let \( f_3(t) := \langle J_0'(\gamma(t)), \gamma(t) \rangle \). Then \( f_3(0) = 0 \) and \( f_3(t) > 0 \) for the small \( t > 0 \). A direct calculation gives
\[
J_0(\gamma(1)) - \frac{1}{2} \langle J_0'(\gamma(1)), \gamma(1) \rangle \geq \left( 1 \frac{1}{2} - \frac{1}{2s} \right) \int_{\mathbb{R}^3} |u|^2^\gamma \, dx \geq 0,
\]
which implies
\[
\langle J_0'(\gamma(1)), \gamma(1) \rangle \leq 2 J_0(\gamma(1)) = 2 J_0(tu) < 0.
\]
Thus, there exists $\tilde{t} \in (0, 1)$ such that $f_3(\tilde{t}) = 0$, i.e. $\gamma(\tilde{t}) \in \mathcal{N}$ and $c_0 \geq \tilde{c}_0$.

3.1. **Non-vanishing of** $(PS)_{c_0}$ **sequence.** The following lemma is regarding the non-vanishing of the $(PS)_{c_0}$ sequence.

**Lemma 3.2.** Assume that all conditions described in Theorem 1.1 hold. Let $\{u_n\}$ be a $(PS)_{c_0}$ sequence of $J_0$ with $c_0 > 0$. Then there exists $C_1 > 0$ such that

$$\limsup_{n \to \infty} \int_{\mathbb{R}^3} |u_n|^{2^*} \, dx = C_1.$$ 

**Proof.** It is easy to see that $\{u_n\}$ is uniformly bounded in $D^{s,2}({\mathbb{R}^3})$. There exists $C > 0$ independent of $n$ such that

$$0 \leq P_n = \int_{\mathbb{R}^3} |u_n|^{2^*} \, dx \leq C.$$ 

Namely, $\{P_n\}$ is a bounded sequence in $\mathbb{R}$. According to the Bolzano-Weierstrass theorem, there exists an accumulation point $P_0$ in $\mathbb{R}$.

Let $E \subset [0, C] \subset \mathbb{R}$ be the set of all accumulation points of $\{P_n\}$. So $P_0 \in E \neq \emptyset$. By the definition of the superior limit and the set $E$, we know

$$\limsup_{n \to \infty} P_n = \sup E.$$ 

Applying $E \subset [0, C]$ and the supremum principle, we obtain the existence of $\sup E$. Then there exists $C_1 \in [0, C]$ such that

$$\limsup_{n \to \infty} \int_{\mathbb{R}^3} |u_n|^{2^*} \, dx = C_1.$$ 

To show $C_1 > 0$, we suppose on the contrary that

$$\lim_{n \to \infty} \int_{\mathbb{R}^3} |u_n|^{2^*} \, dx = 0. \tag{3.1}$$

From (3.1) and the definition of the $(PS)_{c_0}$ sequence, it follows that

$$c_0 + o_n(1) = \frac{1}{2} \|u_n\|^2_{2^*} \quad \text{and} \quad o_n(1) = \|u_n\|^2_{2^*},$$

which implies $c_0 = 0$. This contradicts $0 < c_0$. Hence, we know

$$\limsup_{n \to \infty} \int_{\mathbb{R}^3} |u_n|^{2^*} \, dx = C_1 > 0. \tag{3.2}$$

3.2. **Structure of the** $(PS)_{c_0}$ **sequence.** In this subsection, we show that the structure of the $(PS)_{c_0}$ sequence goes to zero weakly.

**Lemma 3.3.** Assume that all conditions described in Theorem 1.1 hold. Let $\{u_n\} \subset D^{s,2}({\mathbb{R}^3})$ be a $(PS)_{c_0}$ sequence at $c_0 > 0$, and $u_n \rightharpoonup 0$ in $D^{s,2}({\mathbb{R}^3})$ as $n \to \infty$. Then there exists $\epsilon > 0$ such that

either $\lim_{n \to \infty} \int_{B(0,1)} |u_n|^{2^*} \, dx = 0$ or $\limsup_{n \to \infty} \int_{B(0,1)} |u_n|^{2^*} \, dx \geq \epsilon$.

**Proof.** Let $\{u_n\}$ be a $(PS)_{c_0}$ sequence at $c_0 > 0$. Then

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{(u_n(x) - u_n(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+2s}} \, dx \, dy - \mu \int_{\mathbb{R}^3} \frac{\Phi(x/|x|)}{|x|^{2s}} u_n \varphi \, dx$$

$$= \int_{\mathbb{R}^3} |u_n|^{2^* - 2} u_n \varphi \, dx \tag{3.2}$$
for any $\varphi \in D^{s,2}(\mathbb{R}^3)$. Let $\xi \in C_0^\infty(\mathbb{R}^3)$ be a cut off function such that $\text{Supp}(\xi) = \overline{B}(0,2)$ and $\xi \equiv 1$ in $B(0,1)$. The following Sobolev embedding is compact:

$$D^{s,2}(\mathbb{R}^3) \hookrightarrow L^q(B(0,2)) \text{ for } q \in [2,2^*_s). \quad (3.3)$$

We present the proof of this lemma by three claims.

**Claim 1.** There exists $C_2 > 0$ such that

$$o_n(1) + C_2\|\xi u_n\|_2^2 \leq C\|\xi u_n\|_B^2 \left(\int_{B(0,2)} |u_n|^2 \, dx\right)^{2^*_s - 2}.$$ 

To prove Claim 1, we observe

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{(u_n(x) - u_n(y))(u_n(x)|\xi(x)|^2 - u_n(y)|\xi(y)|^2)}{|x-y|^{N+2s}} \, dx \, dy$$ 

$$= \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u_n(x) - u_n(y)|^2|\xi(x)|^2}{|x-y|^{N+2s}} \, dx \, dy$$ 

$$+ \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{(u_n(x) - u_n(y))u_n(y)(|\xi(x)|^2 - |\xi(y)|^2)}{|x-y|^{N+2s}} \, dx \, dy. \quad (3.4)$$

Using the inequality $|a|^2 - |b|^2 \leq |a-b||(|a| + |b|)$ for $a, b \in \mathbb{R}$ and Young’s inequality ($1 < \varepsilon < 2$), we get

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{(u_n(x) - u_n(y))u_n(y)(|\xi(x)|^2 - |\xi(y)|^2)}{|x-y|^{N+2s}} \, dx \, dy$$ 

$$\leq \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u_n(x) - u_n(y)||(|\xi(x)| + |\xi(y)||)| |u_n(y)||\xi(x) - \xi(y)||}{|x-y|^{N+2s}} \, dx \, dy$$ 

$$\leq \frac{1}{2\varepsilon} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u_n(x) - u_n(y)|^2(|\xi(x)|^2 + |\xi(y)|^2)}{|x-y|^{N+2s}} \, dx \, dy$$ 

$$+ \frac{\varepsilon}{2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u(y)|^2|\xi(x) - \xi(y)|^2}{|x-y|^{N+2s}} \, dx \, dy$$ 

$$= \frac{1}{\varepsilon} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u(x) - u(y)|^2|\xi(x)|^2}{|x-y|^{N+2s}} \, dx \, dy + \frac{\varepsilon}{2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u(y)|^2|\xi(x) - \xi(y)|^2}{|x-y|^{N+2s}} \, dx \, dy.$$ 

Substituting this into (3.4) yields

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{(u_n(x) - u_n(y))(u_n(x)|\xi(x)|^2 - u_n(y)|\xi(y)|^2)}{|x-y|^{N+2s}} \, dx \, dy$$ 

$$\geq \left(1 - \frac{1}{\varepsilon}\right) \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u_n(x) - u_n(y)|^2|\xi(x)|^2}{|x-y|^{N+2s}} \, dx \, dy$$ 

$$- \frac{\varepsilon}{2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u_n(y)|^2|\xi(x) - \xi(y)|^2}{|x-y|^{N+2s}} \, dx \, dy. \quad (3.5)$$
It follows that

\[
\begin{align*}
&\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u_n(x)\xi(x) - u_n(y)\xi(y)|^2}{|x-y|^{N+2s}} \, dx \, dy \\
&= \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|(u_n(x) - u_n(y))\xi(x) - u_n(y)(\xi(x) - \xi(y))|^2}{|x-y|^{N+2s}} \, dx \, dy \\
&\leq C \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u_n(x) - u_n(y)|^2|\xi(x)|^2}{|x-y|^{N+2s}} \, dx \, dy \\
&+ C \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u_n(y)|^2|\xi(x) - \xi(y)|^2}{|x-y|^{N+2s}} \, dx \, dy.
\end{align*}
\]  

(3.6)

Substituting (3.6) into (3.5), we obtain

\[
\begin{align*}
&\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u_n(x)\xi(x) - u_n(y)\xi(y)|^2}{|x-y|^{N+2s}} \, dx \, dy \\
&\leq C \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{(u_n(x) - u_n(y))(u_n(x)|\xi(x)|^2 - u_n(y)|\xi(y)|^2}{|x-y|^{N+2s}} \, dx \, dy \\
&+ C \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u_n(y)|^2|\xi(x) - \xi(y)|^2}{|x-y|^{N+2s}} \, dx \, dy.
\end{align*}
\]  

(3.7)

In view of the definition of \(\xi\) and \(u_n \to 0\) in \(D^{s,2}(\mathbb{R}^3)\), it follows from (3.3) that

\[
\begin{align*}
&\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u_n(y)|^2|\xi(x) - \xi(y)|^2}{|x-y|^{N+2s}} \, dx \, dy \\
&= \int_{B(0,2)\setminus B(0,1)} \int_{B(0,2)\setminus B(0,1)} \frac{|u_n(y)|^2|\xi(x) - \xi(y)|^2}{|x-y|^{N+2s}} \, dx \, dy \\
&\leq C \int_{B(0,2)\setminus B(0,1)} |u_n(y)|^2 \int_{B(0,2)\setminus B(0,1) \cap \{|x-y|\geq 1\}} \frac{1}{|x-y|^{N+2s}} \, dx \, dy \\
&+ C \int_{B(0,2)\setminus B(0,1)} |u_n(y)|^2 \int_{B(0,2)\setminus B(0,1) \cap \{|x-y|\leq 1\}} \frac{1}{|x-y|^{N+2s-2}} \, dx \, dy \\
&\leq C \int_{B(0,2)\setminus B(0,1)} |u_n|^2 \, dx.
\end{align*}
\]  

(3.8)

From (3.7) and (3.8), there exists \(C_2 > 0\) such that

\[
C_2 \|\Phi u_n\|_\Phi^2 \leq \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{(u_n(x) - u_n(y))(u_n(x)|\xi(x)|^2 - u_n(y)|\xi(y)|^2)}{|x-y|^{N+2s}} \, dx \, dy \\
- \zeta \int_{\mathbb{R}^3} \frac{\Phi(x/|x|)}{|x|^2} |u_n\xi|^2 \, dx.
\]  

(3.9)
Applying Hölder’s inequality, we derive
\[
\int_{\mathbb{R}^3} |u_n|^{\frac{2^*}{2} - 2} |u_n \xi|^2 dx \leq \int_{B(0,2)} |u_n|^{\frac{2^*}{2} - 2} |u_n \xi|^2 dx
\]
\[
\leq \left( \int_{B(0,2)} |u_n|^{\frac{2^*}{2} - 2} \frac{2^*}{2} dx \right)^{1 - \frac{2}{2^*}} \left( \int_{\mathbb{R}^3} |u_n \xi|^2 dx \right)^{\frac{2}{2^*}}
\]
\[
= \left( \int_{B(0,2)} |u_n|^{\frac{2^*}{2}} dx \right)^{\frac{2^* - 2}{2^*}} \left( \int_{\mathbb{R}^3} |u_n \xi|^2 dx \right)^{\frac{2}{2^*}}
\]
\[
\leq C \|\xi u_n\|_{\Phi}^2 \left( \int_{B(0,2)} |u_n|^{\frac{2^*}{2}} dx \right)^{\frac{2^* - 2}{2^*}}.
\] (3.10)

In view of the uniform boundedness of \(u_n\) \((n \to \infty)\), using (3.10) and testing (3.2) with \(\xi^2 u_n\), we obtain
\[
o_n(1) + C_2 \|\xi u_n\|_{\Phi}^2 \leq C \|\xi u_n\|_{\Phi} \left( \int_{B(0,2)} |u_n|^{2^*} dx \right)^{\frac{2^* - 2}{2^*}}.
\]

**Claim 2.** If \(\limsup_{n \to \infty} \|\xi u_n\|_{\Phi} > 0\), then there exists \(\epsilon > 0\) such that

\[
either \lim \int_{B(0,1)} |u_n|^{2^*} dx = 0 or \limsup_{n \to \infty} \int_{B(0,1)} |u_n|^{2^*} dx \geq \epsilon.
\]

Similar to Lemma 3.2, there exists \(0 \leq C_3 < \infty\) such that
\[
\limsup_{n \to \infty} \int_{B(0,2)} |u_n|^{2^*} dx = C_3.
\] (3.11)

It follows from \(\limsup_{n \to \infty} \|\xi u_n\|_{\Phi} > 0\) and Claim 1 that
\[
\limsup_{n \to \infty} \left( \int_{B(0,2)} |u_n|^{2^*} dx \right)^{\frac{2^* - 2}{2^*}} \geq C_2 > 0.
\] (3.12)

From (3.11) and (3.12), we get
\[
C_3 = \limsup_{n \to \infty} \int_{B(0,2)} |u_n|^{2^*} dx > 0.
\] (3.13)

Set \(A := \limsup_{n \to \infty} \int_{B(0,2) \setminus B(0,1)} |u_n|^{2^*} dx\). Then
\[
\limsup_{n \to \infty} \int_{B(0,2)} |u_n|^{2^*} dx \leq A + \limsup_{n \to \infty} \int_{B(0,1)} |u_n|^{2^*} dx.
\] (3.14)

It’s easy to see \(A \in [0, C_3]\). We now consider three cases as follows:

**Case (1).** If \(A = 0\), by using (3.13) and (3.14), we find
\[
\limsup_{n \to \infty} \int_{B(0,1)} |u_n|^{2^*} dx = \limsup_{n \to \infty} \int_{B(0,2)} |u_n|^{2^*} dx = C_3 > 0.
\] (3.15)
Claim 3. If \( \lim_{n \to \infty} \int_{B(0,1)} |u_n|^2 \, dx < C_4 \), then
\[
C_3 = \lim_{n \to \infty} \int_{B(0,2)} |u_n|^2 \, dx \leq A + \lim_{n \to \infty} \int_{B(0,1)} |u_n|^2 \, dx < A + C_4 = C_3.
\]
This yields a contradiction. So we have
\[
\limsup_{n \to \infty} \int_{B(0,1)} |u_n|^2 \, dx \geq C_4 > 0. \tag{3.16}
\]

Case (3). For \( A = C_3 = \limsup_{n \to \infty} \int_{B(0,2)} |u_n|^2 \, dx \), we consider two subcases:

1. \( \lim_{n \to \infty} \int_{B(0,1)} |u_n|^2 \, dx \) exists; or
2. \( \lim_{n \to \infty} \int_{B(0,1)} |u_n|^2 \, dx \) does not exist.

Case (3.1). If \( \lim_{n \to \infty} \int_{B(0,1)} |u_n|^2 \, dx \) exists, from (3.14) we have
\[
\limsup_{n \to \infty} \int_{B(0,2)} |u_n|^2 \, dx = A + \limsup_{n \to \infty} \int_{B(0,1)} |u_n|^2 \, dx.
\]
Substituting \( A = \limsup_{n \to \infty} \int_{B(0,2)} |u_n|^2 \, dx \) into the above expression leads to
\[
\limsup_{n \to \infty} \int_{B(0,1)} |u_n|^2 \, dx = \lim_{n \to \infty} \int_{B(0,1)} |u_n|^2 \, dx = 0. \tag{3.17}
\]

Case (3.2). Similar to (3.11), there exists \( C_5 \geq 0 \) such that
\[
\limsup_{n \to \infty} \int_{B(0,1)} |u_n|^2 \, dx = C_5.
\]
Note that \( \lim_{n \to \infty} \int_{B(0,1)} |u_n|^2 \, dx \) does not exist. We know
\[
\limsup_{n \to \infty} \int_{B(0,1)} |u_n|^2 \, dx > \liminf_{n \to \infty} \int_{B(0,1)} |u_n|^2 \, dx \geq 0,
\]
which implies
\[
\limsup_{n \to \infty} \int_{B(0,1)} |u_n|^2 \, dx = C_5 > 0. \tag{3.18}
\]
Set \( \epsilon = \min\{C_3, C_4, C_5\} \). From (3.15)-(3.18), we deduce
\[
\text{either } \lim_{n \to \infty} \int_{B(0,1)} |u_n|^2 \, dx = 0 \text{ or } \limsup_{n \to \infty} \int_{B(0,1)} |u_n|^2 \, dx \geq \epsilon.
\]

Claim 3. If \( \lim_{n \to \infty} \|\xi u_n\|_\Phi = 0 \), then \( \lim_{n \to \infty} \int_{B(0,1)} |u_n|^2 \, dx = 0 \).

Let \( \lim_{n \to \infty} \|\xi u_n\|_\Phi = 0 \). It follows from Sobolev’s inequality that
\[
0 = \lim_{n \to \infty} \|\xi u_n\|_\Phi \geq C \lim_{n \to \infty} \left( \int_{\mathbb{R}^3} |\xi u_n|^2 \, dx \right)^{\frac{1}{2}} \geq C \lim_{n \to \infty} \left( \int_{B(0,1)} |u_n|^2 \, dx \right)^{\frac{1}{2}},
\]
which implies
\[
\lim_{n \to \infty} \int_{B(0,1)} |u_n|^2 \, dx = 0.
\]
Combining Claims 2 and 3, we complete the proof of Lemma 3.3. \( \square \)
Proof of Theorem 1.1. Let \( \{u_n\} \) be a \((PS)_{c_0}\) sequence. It follows from Lemma 3.2 that
\[
\limsup_{n \to \infty} \int_{\mathbb{R}^3} |u_n|^2 \, dx = C_1 > 0.
\] (3.19)

Let \( \delta = \min\{C_1, \frac{\epsilon}{2}\} \), where \( \epsilon > 0 \) is the same as given in Lemma 3.3. According to (3.19), for any \( \delta' \in (0, \delta) \), there exists a sequence \( r_n \in \mathbb{R}^+ \) such that up to a subsequence, there holds
\[
\int_{B(0, r_n)} |u_n|^2 \, dx = \delta'.
\]

Define \( \tilde{u}_n = r_n^{\frac{N-2}{2}} u_n(r_n x) \). Then \( \tilde{u}_n \in D^{s,2}(\mathbb{R}^3) \) and
\[
J_0(\tilde{u}_n) \to c_0 \quad \text{and} \quad \int_{B(0,1)} |\tilde{u}_n|^2 \, dx = \delta'.
\] (3.20)

For all \( \varphi \in D^{s,2}(\mathbb{R}^3) \), we have
\[
|\langle J_0'(\tilde{u}_n), \varphi \rangle| = |\langle J_0'(u_n), \tilde{\varphi} \rangle| \leqslant \|J_0'(u_n)\|_{D^{s,2}(\mathbb{R}^3)} \|\tilde{\varphi}\|_{D^{s,2}(\mathbb{R}^3)} = o_n(1) \|\tilde{\varphi}\|_{D^{s,2}(\mathbb{R}^3)},
\]
where \( \tilde{\varphi} = r_n^{\frac{N-2}{2}} \varphi(\frac{x}{r_n}) \). Note that \( \|\tilde{\varphi}\|_{D^{s,2}(\mathbb{R}^3)} = \|\varphi\|_{D^{s,2}(\mathbb{R}^3)} \). Then
\[
\|J_0'(\tilde{u}_n)\|_{D^{s,2}(\mathbb{R}^3)} \to 0, \quad \text{as} \quad n \to \infty.
\]

Hence, there exists \( \tilde{u} \in D^{s,2}(\mathbb{R}^3) \) such that up to a subsequence, we have \( \tilde{u}_n \rightharpoonup \tilde{u} \) in \( D^{s,2}(\mathbb{R}^3) \).

We now show \( \tilde{u} \not\equiv 0 \). Otherwise, we suppose \( \tilde{u} = 0 \). From Lemma 3.3, we have
\[
\limsup_{n \to \infty} \int_{B(0,1)} |\tilde{u}_n|^2 \, dx = 0 \quad \text{or} \quad \limsup_{n \to \infty} \int_{B(0,1)} |\tilde{u}_n|^2 \, dx \geqslant \epsilon.
\]

This contradicts (3.20) \((0 < \delta' < \delta = \min\{C_1, \frac{\epsilon}{2}\})\). So it implies \( \tilde{u} \not\equiv 0 \). In view of \( \langle J_0'(\tilde{u}), \varphi \rangle = 0 \), we have \( \tilde{u} \in N_0 \).

It suffices to show \( \tilde{u}_n \to \tilde{u} \) strongly in \( D^{s,2}(\mathbb{R}^3) \). Using the Brézis-Lieb lemma, we get
\[
c_0 = \lim_{n \to \infty} J_0(\tilde{u}_n) - \lim_{n \to \infty} \frac{1}{2s} \langle J_0'(\tilde{u}_n), \tilde{u}_n \rangle = \lim_{n \to \infty} \left( \frac{1}{2} - \frac{1}{2s} \right) \|\tilde{u}_n\|_\Phi^2 \geqslant \left( \frac{1}{2} - \frac{1}{2s} \right) \|\tilde{u}\|_\Phi^2 = J_0(\tilde{u}) \geqslant c_0 = c_0.
\] (3.21)

Thus, we have
\[
\lim_{n \to \infty} \|\tilde{u}_n\|_\Phi^2 = \|\tilde{u}\|_\Phi^2.
\]

Using the Brézis-Lieb lemma again yields
\[
\lim_{n \to \infty} \|\tilde{u}_n\|_\Phi^2 - \lim_{n \to \infty} \|\tilde{u}_n - \tilde{u}\|_\Phi^2 = \|\tilde{u}\|_\Phi^2,
\]
which implies
\[
\lim_{n \to \infty} \|\tilde{u}_n - \tilde{u}\|_\Phi^2 = 0.
\]

From (3.21), we have \( J_0(\tilde{u}) = c_0 \). Taking \( \tilde{u}^- = \min\{\tilde{u}, 0\} \) as test function in \((PS)\) and using the inequality
\[
|x - y|^{1-l} |x - y|^{l} = |x^+ - y^+|^l, \quad l \geqslant 1,
\]
we can see
\[
\|\tilde{u}^-\|_{D^{s,2}(\mathbb{R}^3)}^2 \leqslant 0,
\]
which implies \( \tilde{u}^- = 0 \). That is, \( \tilde{u} \geq 0 \) in \( \mathbb{R}^3 \). This indicates that \( \tilde{u} \) is a nonnegative ground state solution of equation \( (\mathcal{P}_3) \). \( \square \)

4. **Proof of Theorem 1.2.** We introduce the energy functional associated with equation \((\mathcal{P}_{HLS})\) by

\[
J_\alpha(u) = \frac{1}{2} \|u\|_\Phi^2 - \frac{1}{2^*_\alpha} \int_{\mathbb{R}^3} (I_\alpha * |u|^{2^*_\alpha}) |u|^{2^*_\alpha} \, dx.
\]

We define the Nehari manifold on \( D^{s,2}(\mathbb{R}^3) \) by

\[
\mathcal{N}_\alpha = \{ u \in D^{s,2}(\mathbb{R}^3) : \langle J'_\alpha(u), u \rangle = 0, u \neq 0 \},
\]

and denote

\[
\bar{c}_\alpha = \inf_{u \in \mathcal{N}_\alpha} J_\alpha(u) \quad \text{and} \quad \tilde{c}_\alpha = \inf_{u \in D^{s,2}(\mathbb{R}^3)} \max_{t \geq 0} J_\alpha(tu).
\]

Similar to Lemmas 3.1 and 3.2, we can obtain the following results immediately.

**Lemma 4.1.** Assume that all conditions described in Theorem 1.2 hold. Then the following statements are true.

(i) The functional \( J_\alpha \) possesses the mountain pass geometry and there exists \( \{v_n\} \subset D^{s,2}(\mathbb{R}^3) \) such that

\[
J_\alpha(v_n) \to c_\alpha \quad \text{and} \quad \|J'_\alpha(v_n)\|_{D^{-s,2}(\mathbb{R}^3)} \to 0, \quad n \to \infty,
\]

and \( \{v_n\} \) is uniformly bounded in \( D^{s,2}(\mathbb{R}^3) \), where

\[
c_\alpha = \inf_{t \in [0,1]} J_\alpha(\bar{Y}(t)),
\]

\[
\bar{Y}_\alpha = \{ \bar{Y} \in C([0,1], D^{s,2}(\mathbb{R}^3)) : \bar{Y}(0) = 0, J_\alpha(\bar{Y}(1)) < 0 \}.
\]

(ii) For each \( u \in D^{s,2}(\mathbb{R}^3) \) \( \neq \{0\} \), there exists a unique \( t_u > 0 \) such that \( t_u \bar{Y} \in \mathcal{N}_\alpha \).

(iii) \( c_\alpha = \bar{c}_\alpha = \tilde{c}_\alpha \).

(iv) There exists \( C_0 > 0 \) such that

\[
\limsup_{n \to \infty} \int_{\mathbb{R}^3} |v_n|^{2^*_\alpha} \, dx = C_0 > 0.
\]

We discuss the structure of the \((PS)_{c_\alpha}\) sequence by considering two cases:

\[
2^*_\alpha > 2, \text{ or } \quad 2^*_\alpha \leq 2.
\]

For \( 2^*_\alpha > 2 \), similar to Lemma 3.3, we have

**Lemma 4.2.** Assume that all conditions described in Theorem 1.2 and \( 2^*_\alpha > 2 \) hold. Let \( \{v_n\} \subset D^{s,2}(\mathbb{R}^3) \) be a \((PS)_{c_\alpha}\) sequence at \( c_\alpha > 0 \), and \( v_n \rightharpoonup 0 \) in \( D^{s,2}(\mathbb{R}^3) \). Then there exists \( \epsilon_1 > 0 \) such that

\[
\text{either } \lim_{n \to \infty} \int_{B(0,1)} |v_n|^{2^*_\alpha} \, dx = 0 \quad \text{or} \quad \limsup_{n \to \infty} \int_{B(0,1)} |v_n|^{2^*_\alpha} \, dx \geq \epsilon_1.
\]

**Proof.** Let the sequence \( \{v_n\} \) be a \((PS)_{c_\alpha}\) sequence at \( c_\alpha > 0 \). Then

\[
\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{(v_n(x) - v_n(y))(\varphi(x) - \varphi(y))}{|x-y|^{N+2s}} \, dx \, dy - \mu \int_{\mathbb{R}^3} \Phi(|x|) \frac{v_n \varphi}{|x|^2} \, dx
\]

\[
= \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |v_n(y)|^{2^*_\alpha} |v_n(x)|^{2^*_\alpha} - 2 v_n(x) \varphi(x) \, dx \, dy
\]

\[(4.1)\]

for any \( \varphi \in D^{s,2}(\mathbb{R}^3) \).
It follows from Lemma 2.2 and Hölder’s inequality that
\[
\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{(\xi(x)v_n(x))^2|v_n(x)|^{2s_{\alpha}}|v_n(y)|^{2s_{\alpha}}}{|x-y|^{3-\alpha}} \, dx \, dy
\]
\[
\leq C(N, \alpha)\|v_n\|^2_{L^{2s_{\alpha}}(\mathbb{R}^3)} \left( \int_{B(0,2)} |\xi v_n|^2 |v_n|^{2s_{\alpha}} \, dx \right)^{\frac{s_{\alpha}}{2s_{\alpha}}},
\]
(4.2)
\[
\leq C(N, \alpha)\|v_n\|^2_{L^{2s_{\alpha}}(\mathbb{R}^3)}\|\xi v_n\|^2_{L^2(\mathbb{R}^3)} \left( \int_{B(0,2)} |v_n|^{2s_{\alpha}} \, dx \right)^{\frac{s_{\alpha}}{2s_{\alpha}}},
\]
\[
\leq C\|v_n\|^2_{L^{2s_{\alpha}}(\mathbb{R}^3)}\|\xi v_n\|^2_{L^2(\mathbb{R}^3)}\|v_n\|^2_{L^{2s_{\alpha}}(B(0,2))}.
\]

In view of the uniform boundedness of \(u_n\), using (3.9) and (4.2), and testing (4.1) with \(\xi^2 v_n\), we get
\[
o_n(1) + C_7\|\xi v_n\|^2_{\Phi} \leq \|\xi v_n\|^2_{\Phi}\|v_n\|^2_{L^{2s_{\alpha}}(B(0,2))}.
\]
The rest argument is closely similar to the proof of Lemma 3.3, so we omit it. □

**Remark 4.1.** Note that when \(2s_{\alpha} - 2 \leq 0\), we can not use Hölder’s inequality in (4.2). This is the key difference between Theorems 1.1 and 1.2.

**Lemma 4.3.** Assume that all conditions described in Theorem 1.2 and \(2s_{\alpha} \leq 2\) hold. Let \(\{v_n\} \subset D^{s,2}(\mathbb{R}^3)\) be a (PS)\(c_\alpha\) sequence at \(c_\alpha > 0\), and \(v_n \rightharpoonup 0\) in \(D^{s,2}(\mathbb{R}^3)\) as \(n \to \infty\). Then there exists \(\epsilon_2 > 0\) such that

\[
\text{either } \lim_{n \to \infty} \int_{B(0,1)} |v_n|^{2s_{\alpha}} \, dx = 0 \text{ or } \limsup_{n \to \infty} \int_{B(0,1)} |v_n|^{2s_{\alpha}} \, dx \geq \epsilon_2.
\]

**Proof.** Applying Lemma 2.2 and Hölder’s inequality, we obtain
\[
\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{(\xi(x)v_n(x))^2|v_n(x)|^{2s_{\alpha}}|v_n(y)|^{2s_{\alpha}}}{|x-y|^{3-\alpha}} \, dx \, dy
\]
\[
= \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{(\xi(x)v_n(x))^{2s_{\alpha}}|v_n(x)|^{2s_{\alpha}}|v_n(y)|^{2s_{\alpha}}}{|x-y|^{3-\alpha}} \, dx \, dy
\]
\[
\leq C(N, \alpha)\|v_n\|^2_{L^{2s_{\alpha}}(\mathbb{R}^3)} \left( \int_{B(0,2)} \left|\xi v_n\right|^{2s_{\alpha}} \, dx \right)^{\frac{2s_{\alpha}}{2s_{\alpha}}},
\]
(4.3)
\[
\leq C(N, \alpha)\|v_n\|^2_{L^{2s_{\alpha}}(\mathbb{R}^3)} \left( \int_{B(0,2)} |v_n|^{2s_{\alpha}} \, dx \right)^{\frac{2s_{\alpha}}{2s_{\alpha}}}.
\]

Using (3.9) and (4.3), and testing (4.1) with \(\xi^2 v_n\) yields
\[
o_n(1) + C_8\|\xi v_n\|^2_{\Phi} \leq \|v_n\|^2_{L^{2s_{\alpha}}(B(0,2))}.
\]
(4.4)

There are only two cases: (I) \(\limsup_{n \to \infty} \|\xi v_n\|_{\Phi} > 0\) and (II) \(\lim_{n \to \infty} \|\xi v_n\|_{\Phi} = 0\). We just show Case (I), because Case (II) can be processed in an analogous manner.

**Case (I).** Note that \(\{v_n\}\) is uniformly bounded in \(D^{s,2}(\mathbb{R}^3)\). Similar to Lemma 3.2, there exists \(C_9 \geq 0\) such that
\[
\limsup_{n \to \infty} \|\xi v_n\|^2_{\Phi} = C_9.
\]
From \( \limsup_{n \to \infty} \|\xi v_n\|_{\Phi}^2 > 0 \), we get
\[
C_\theta = \limsup_{n \to \infty} \|v_n\|_{\Phi}^2 > 0. \tag{4.5}
\]
It follows from (4.4) and (4.5) that
\[
\limsup_{n \to \infty} \|v_n\|_{L^{2^*_\alpha}(B(0,2))}^{2^*_\alpha} \geq C_\theta C_\delta > 0,
\]
which implies
\[
\limsup_{n \to \infty} \|v_n\|_{L^{2^*_\alpha}(B(0,2))} > 0.
\]
The rest proof is similar to that of Lemma 3.3, so we omit it. \(\square\)

**Proof of Theorem 1.2.** Let \( \{v_n\} \) be a \((PS)_{c_\alpha}\) sequence. Using Lemma 4.1 (iv), we get
\[
\limsup_{n \to \infty} \int_{\mathbb{R}^3} |v_n|^{2^*} dx = C_\delta > 0. \tag{4.6}
\]
Let \( \bar{\delta} = \min\{C_\delta, \xi \frac{\xi}{2}\} \), where \( \xi := \min\{\epsilon_1, \epsilon_2\} \). According to (4.6), for any \( \delta'' \in (0, \bar{\delta}) \), there exists a sequence \( r_n \in \mathbb{R}^+ \) such that up to a subsequence, there holds
\[
\int_{B(0,r_n)} |v_n|^{2^*} dx = \delta''.
\]
Let \( \tilde{v}_n = r_n^{\frac{N-2s}{2}} v_n(r_n x) \). Then \( \tilde{v}_n \in D^{s,2}(\mathbb{R}^3) \) and
\[
J_\alpha'(\tilde{v}_n) \to c_\alpha \text{ as } n \to \infty \text{ and } \int_{B(0,1)} |\tilde{v}_n|^{2^*} dx = \delta''. \tag{4.7}
\]
For all \( \varphi \in D^{s,2}(\mathbb{R}^3) \), we obtain
\[
|\langle J_\alpha'(\tilde{v}_n), \varphi \rangle| = |\langle J_\alpha'(v_n), \varphi \rangle| \leq \|J_\alpha'(v_n)\|_{D^{-s,2}(\mathbb{R}^3)} \|\varphi\|_{D^{s,2}(\mathbb{R}^3)} = o_n(1)\|\tilde{\varphi}\|_{D^{s,2}(\mathbb{R}^3)},
\]
where \( \tilde{\varphi} = r_n^{-\frac{N-2s}{2}} \varphi(\frac{x}{r_n}) \).

Note that \( \|\tilde{\varphi}\|_{D^{s,2}(\mathbb{R}^3)} = \|\varphi\|_{D^{s,2}(\mathbb{R}^3)} \). Then
\[
\|J_\alpha'(\tilde{v}_n)\|_{D^{-s,2}(\mathbb{R}^3)} \to 0, \text{ as } n \to \infty.
\]
Hence, there exists \( \tilde{v} \in D^{s,2}(\mathbb{R}^3) \) such that up to a subsequence, we have \( \tilde{v}_n \rightharpoonup \tilde{v} \) in \( D^{s,2}(\mathbb{R}^3) \).

We show \( \tilde{v} \neq 0 \). Otherwise, we suppose that \( \tilde{v} = 0 \). From Lemmas 4.2-4.3, it follows that
\[
\text{either } \lim_{n \to \infty} \int_{B(0,1)} |\tilde{v}_n|^{2^*} dx = 0 \text{ or } \limsup_{n \to \infty} \int_{B(0,1)} |\tilde{v}_n|^{2^*} dx \geq \tilde{\epsilon}.
\]
This contradicts (4.7) \( 0 < \delta'' < \tilde{\delta} = \min\{C_\delta, \xi \frac{\xi}{2}\} \). So we have \( \tilde{v} \neq 0 \). In view of \( \langle J_\alpha'(\tilde{v}), \varphi \rangle = 0 \), we obtain \( \tilde{v} \in N_\alpha \).

Similar to the proof of Theorem 1.1, we can get \( J_\alpha(\tilde{v}) = c_\alpha \) and \( \tilde{v} \geq 0 \). This indicates that \( \tilde{v} \) is a nonnegative ground state solution of equation \((P_{HLS})\). \(\square\)
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