The generalized good cut equation

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Abstract

The properties of null geodesic congruences (NGCs) in Lorentzian manifolds are a topic of considerable importance. More specifically NGCs with the special property of being shear-free or asymptotically shear-free (as either infinity or a horizon is approached) have received a great deal of recent attention for a variety of reasons. Such congruences are most easily studied via solutions to what has been referred to as the ‘good cut equation’ or the ‘generalization good cut equation’. It is the purpose of this paper to study these equations and show their relationship to each other. In particular we show how they all have a four-complex-dimensional manifold (known as \(\mathcal{H}\)-space, or in a special case as complex Minkowski space) as a solution space.

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1. Introduction

Shear-free (and its generalization to asymptotically shear-free) null geodesic congruences (NGCs) in Lorentzian spacetimes have played a variety of important roles in spacetime geometry. They first appeared in the search for algebraically special solutions of Maxwell’s equations in curved spacetimes [1]. This was followed by the discovery of the vacuum twist-free algebraically special metrics of Robinson and Trautman [2] and the extension, via the remarkable Goldberg–Sachs theorem [3], to all algebraically special vacuum metrics. Penrose then showed the close connection of shear-free congruences, via the Kerr theorem, with twistor theory [4]. More recently the asymptotically shear-free congruences (as null infinity is approached) have been used to give physical interpretations to the asymptotic fields in asymptotically flat spacetimes [5]. This latter case has been generalized to congruences that become shear-free as a non-expanding horizon in a spacetime is approached [6].

All the cases are governed by solutions to different versions of the same type of partial differential equation, referred to as ‘good-cut equations’ (GCEq). The GCEq or its generalization, the generalized GCEq (G\(^2\)CEq), are second-order PDEs that live on 3-manifolds. Although it is very likely that the discussion could be generalized to arbitrary...
3-surfaces in a Lorentzian manifold (or more specifically to arbitrary null surfaces), we confine ourselves to the cases where the 3-surfaces are specifically Penrose’s future null infinity, \( \mathcal{I}^+ \), or a vacuum non-expanding horizon, \( \mathcal{H} \) (cf [7, 8]). For the general discussion we will refer to both \( \mathcal{I} \) and/or \( \mathcal{H} \) as \( \mathcal{N} \). In each of these cases \( \mathcal{N} \) has topology \( S^2 \times \mathbb{R} \), and is foliated by null geodesics congruences whose shear and divergence both vanish on \( \mathcal{N} \). Though these manifolds are real 3-surfaces in the real spacetime, we must consider their complexification, \( \mathcal{N}_C \) (i.e. their analytic continuation, at least a small way, into the complexification of the spacetime). The coordinatization of \( \mathcal{N} \) is given by Bondi-like coordinates: \( (\zeta, \bar{\zeta}) \), which label the null generators of \( \mathcal{N} \), and are the stereographic coordinates on the \( S^2 \) portion of \( \mathcal{N} \) (\( S^2 \) need not be a metric sphere), while the coordinate \( u \) parametrizes the cross-sections of \( \mathcal{N} \). For \( \mathcal{N}_C \), \( u \) is allowed to take complex values close to the real, while \( \bar{\zeta} \) goes over to an independent variable \( \tilde{\zeta} \) close to the complex conjugate of \( \zeta \). To avoid the notational nuisance of repeatedly saying that \( \zeta \approx \tilde{\zeta} \), we will simply use \( \bar{\zeta} \) to mean an independent complex variable taking values at, or close to, the complex conjugate of \( \zeta \). The context should make its usage clear.

The distinction between the GCEq and the \( G^2 \) CEq is that the former lives on a 3-surface \( \mathcal{N} \) whose \( u \)-constant cross-sections are metric spheres, while for the latter equation the metric is arbitrary.

In section 2, we first discuss the geometric meaning of the GCEq and its connection to (asymptotically) shear-free NGCs. This is followed by a review of the known properties of solutions to the GCEq: its four-complex-dimensional solution space and its relationship to twistor theory. In section 3 we show that the \( G^2 \) CEq can be transformed directly to the GCEq so that the solution spaces of both equations and properties are equivalent. We demonstrate the utility of this result by applying it to the twistor space associated with horizon shear-free NGCs at a non-expanding horizon. Section 4 concludes with a discussion of possible applications of our findings.

2. The good cut equations

Before describing the GCEq we first discuss a notational choice. As mentioned earlier, the 3-surface \( \mathcal{N} \) is described by an \( S^2 \) worth of null geodesics with the cross-sections given by \( u = \) constant. The metric of the 2-surface cross-sections is expressed in stereographic coordinates \( (\zeta, \bar{\zeta}) \) so that the metric takes the conformally flat form:

\[
ds^2 = \frac{4d\zeta \, d\bar{\zeta}}{P^2(\zeta, \bar{\zeta})},
\]

with \( P(\zeta, \bar{\zeta}) \) an arbitrary smooth non-vanishing function on \( (\zeta, \bar{\zeta}) \), the extended complex plane (Riemann sphere). In the special case of a metric sphere we take \( P = P_0 = 1 + \zeta \bar{\zeta} \),

while in general we write

\[
P = V(\zeta, \bar{\zeta})P_0.
\]

The \( G^2 \) CEq contains the general \( P \), while the special case using \( P_0 \) yields the GCEq.

For the most general situation, the \( G^2 \) CEq can be written as a differential equation for the function \( u = G(\zeta, \bar{\zeta}) \):

\[
\bar{\partial}_\zeta \left( V^2 P_0^2 \partial_\zeta G \right) = \sigma(G, \zeta, \bar{\zeta}),
\]

or

\[
P_0^2 \partial_\zeta^2 G + 2 \left( P_0^2 V^{-1} \partial_\zeta V + P_0 \zeta \right) \partial_\zeta G = V^{-2} \sigma(G, \zeta, \bar{\zeta}).
\]
When $V = 1$ we have the GCEq:

$$\delta_0^2 G \equiv \partial_\xi (P_0^2 \partial_\xi G) = \sigma (G, \zeta, \bar{\zeta}).$$

(5)

Further, when the arbitrary spin-weight 2-function $\sigma (G, \zeta, \bar{\zeta})$ vanishes, we have the homogeneous GCEq:

$$\partial_\xi (P_0^2 \partial_\xi G) = 0.$$ 

(6)

In the following section it will be shown that the $G^2$CEq can, by a (non-obvious) coordinate transformation, be transformed into equation (5). We can simply describe the solutions (and its properties) of equation (4). Chief among these properties is the fact the solution space to (5) is a complex 4-manifold with a natural Einstein metric; hence, taking any curve in this solution space produces a 1-parameter family of good cut functions which in turn describe a foliation of 3-surface $\mathcal{N}$.

Remark 1. Solutions to the GCEq or $G^2$CEq, $u = G(\zeta, \bar{\zeta})$, known as ‘good cut functions’ describe cross-sections of $\mathcal{N}$ that are referred to as ‘good cuts’. From the tangents to these good cuts, $L = \partial_\xi \sigma (G, \eta, \bar{\eta})$, one can construct null directions (out of $\mathcal{N}$) into the spacetime itself that determine a NGC whose shear vanishes at $\mathcal{N}$. (See the following section.) When $\mathcal{N} = \mathbb{S}^3$, these are asymptotically shear-free NGCs; when $\mathcal{N}$ is a non-expanding horizon, these are vacuum-horizon-shear-free NGCs.

2.1. Solution space of the GCEq

In this and the following subsection we will be concerned only with the GCEq (5) and its solutions.

The key fact about the solution space to the GCEq is that it forms a complex 4-manifold, known as $\mathcal{H}$-space, for sufficiently regular $\sigma (G, \zeta, \bar{\zeta})$ (which is assumed here). This manifold of solutions can be shown to possess a vacuum Einstein metric with anti-self-dual Weyl tensor. Although the rigorous proof of the existence and properties of $\mathcal{H}$-space requires the use of Kodaira deformation theory and Penrose’s nonlinear graviton construction [9], we can give a simple intuitive argument here. The solutions will be written as

$$u = G(z^a, \zeta, \bar{\zeta}).$$

(7)

with $z^a$ (appearing as four constants of integration) the $\mathcal{H}$ -space coordinates.

From the properties of the (sphere) $\delta_0$-operator, one immediately has that the general regular solution of the homogeneous equation (6) is given by the four-parameter function

$$G_0(z^a, \zeta, \bar{\zeta}) = z^a l_0(z, \bar{z}) = \frac{\sqrt{2}}{2} z^0 + \frac{1}{2} z^i Y^0_i (\zeta, \bar{\zeta}),$$

$$l_0(z, \bar{z}) = \frac{\sqrt{2}}{2 P_0} (1 + \zeta \bar{z}, \zeta + \bar{z}, i \zeta - i \bar{z}, -1 + \zeta \bar{z}).$$

(8)

The inhomogeneous equation can then be rewritten as the integral equation

$$G = G_0(z^a, \zeta, \bar{\zeta}) + \int_{\mathcal{S}} \sigma (G, \eta, \bar{\eta}) K^+_{0, -2}(\eta, \bar{\eta}; \zeta, \bar{\zeta}) dS_\eta,$$

(9)

with

$$K^+_{0, -2}(\zeta, \bar{\zeta}, \eta, \bar{\eta}) \equiv \frac{1}{4 \pi} \frac{(1 + \zeta \eta)^2(\eta - \zeta)}{(1 + \eta \bar{\eta})(\bar{\eta} - \zeta)},$$

(10)

$$dS_\eta = 4i \frac{d\eta \wedge d\bar{\eta}}{(1 + \eta \bar{\eta})^2}.$$
where $K_{0,\ldots,2}^+(\zeta, \bar{\zeta}, \eta, \bar{\eta})$ is the Green’s function for the $\delta^0_0$-operator [10]. By iterating this equation, with $G_0(z^a, \zeta, \bar{\zeta}) = z^a l_0(\zeta, \bar{\zeta})$ being the zeroth iterate,

$$G_n(\zeta, \bar{\zeta}) = z^a l_n(\zeta, \bar{\zeta}) + \int_{S^2} K_{0,\ldots,2}^+(\zeta, \bar{\zeta}, \eta, \bar{\eta}) \sigma(G_{n-1}, \eta, \bar{\eta}) \, dS_\eta,$$

one easily sees how the four $z^a$ enter the solution: the four constants originate from the solution to the homogeneous equation.

We thus have the result that the solutions to the GCEq,

$$u = G(z^a, \zeta, \bar{\zeta}),$$

define a four-parameter family of cuts of $\mathcal{M}$, each cut labeled by the $H$-space points, $z^a$. By choosing an arbitrary analytic curve in the $H$-space, $z^a = \xi^a(\tau)$, with $\tau$ an arbitrary complex parameter, we have a one-parameter family of cuts of $\mathcal{M}$:

$$u = Z(\tau, \zeta, \bar{\zeta}) = G(\xi^a(\tau), \zeta, \bar{\zeta}).$$

Each choice of the curve $z^a = \xi^a(\tau)$ yields an asymptotically shear-free NGC by the following construction.

Considering $\mathcal{M}$ as embedded in a Lorentzian spacetime, at each point of $\mathcal{M}$ we can construct its past (or future) light-cone. The sphere of null directions can be coordinitized by complex stereographic coordinates, $(L, L)$. A field of null directions pointing backward (or forward) from $\mathcal{M}$ can be written as

$$L = L(\mathcal{M}) = L(u, \zeta, \bar{\zeta}).$$

It is known that the field of null directions that is given parametrically on $\mathcal{M}$ by

$$L(u, \zeta, \bar{\zeta}) = \delta_0 Z(\tau, \zeta, \bar{\zeta}),$$

$$u = Z(\tau, \zeta, \bar{\zeta}),$$

describes the null direction field of an asymptotically shear-free NGC [5, 11–13]. When we are dealing with the homogeneous equation on $\mathcal{B}$ in Minkowski space, the NGC turns out to be shear-free everywhere.

We point out that the solution space comes naturally with the $(H$-space) complex metric [12]

$$ds^2_{(H)} = g_{(H)ab} \, dz^a \, dz^b \equiv \left( \frac{1}{8\pi} \int_{S^2} \frac{dS}{(dG)^2} \right)^{-1},$$

$$dG = \nabla_a G \, dz^a,$$

$$dS = 4i \frac{d\zeta \wedge d\bar{\zeta}}{(1 + \zeta \bar{\zeta})^2},$$

which is Ricci flat and has anti-self-dual conformal (Weyl) curvature. For the special case of solutions to the homogeneous GCEq, this metric reduces to the complex Minkowski metric.

2.2. The good cut equation and twistors

The study of the GCEq is intimately related to Penrose’s twistor theory. Although we will not provide an extensive review of twistor theory here, we will include the briefest of overviews to set the stage for the following discussion. The interested reader need only consult [14, 15] for a more in-depth introduction and discussion. For our purposes, twistor space is the complex projective 3-space $\mathbb{P}^3 \simeq \mathbb{C}P^3$, charted with homogeneous coordinates $Z^a = (\omega^a, \pi_A)$, where $\omega$ and $\pi$ are un-primed and primed Weyl spinors respectively. A (projective) twistor is any
point $Z^a \in \mathbb{P}T$. Twistor space is related to points $x$ in complex Minkowski spacetime by the incidence relation:

$$\omega^A = i x^{AA'} \pi_{A'}.$$  \hspace{1cm} (16)

where $x^{AA'}$ is the usual spinor representation (where a vector index is replaced by a pair of primed and un-primed spinor indicies):

$$x^{AA'} = \frac{1}{\sqrt{2}} \begin{pmatrix} t + x & y - iz \\ y + iz & t - x \end{pmatrix}.$$  

Equation (16) can be used to determine that a point in $\mathbb{P}T$ corresponds to a null geodesic in complex Minkowski space, while a point $x$ in complex Minkowski space corresponds to a line $L_x \simeq \mathbb{C}P^1 \subset \mathbb{P}T$. The geometry of real Minkowski spacetime is recovered on null twistor space, defined as

$$\mathbb{P}N = \{ Z^a \in \mathbb{P}T : \omega^A \bar{\pi}_A + \pi^{A'} \bar{\omega}_{A'} = 0 \}.$$  

In other words, $Z^a$ corresponds to a real null geodesic in Minkowski spacetime if and only if $Z^a \in \mathbb{P}N$; and $L_x$ corresponds to a real point in Minkowski spacetime if and only if $L_x \subset \mathbb{P}N$.

We can also chart $\mathbb{P}T$ with non-homogeneous coordinates, which are most useful when studying the GCEq. Assuming that $\pi_A \neq 0$ (which corresponds to excluding the point at infinity by (16)) and working on a patch where $\pi_{0'} \neq 0$, we can write [16]

$$(\omega^0, \pi_A) = (r \mu^0, -ir \mu^1, r, -r \xi),$$

where $r \in \mathbb{C}$ is a common scaling factor and $\xi$ can be shown to be equivalent to the complex stereographic angle on $S^2$ introduced earlier. This means that $(\xi, \mu^0, \mu^1)$ can be interpreted as coordinates on $\mathbb{P}T \simeq \mathbb{C}P^3$ via

$$\begin{align*}
\xi &= -\frac{\pi_1}{\pi_0}, \\
\mu^0 &= -i \frac{\omega^0}{\pi_0}, \\
\mu^1 &= i \frac{\omega^1}{\pi_0}.
\end{align*}$$

A curved twistor space $\mathbb{P}T$ can also be constructed for any complex spacetime that is Ricci flat with anti-self-dual conformal curvature by the nonlinear graviton construction (cf [14]). Such curved twistor spaces have a similar correspondence with null geodesics and points in the complex spacetime, although the curves $L_x$ will no longer be lines, as in (16).

We now discuss the relationship between the above discussion of twistor space and the GCEq.

Starting with the homogeneous GCEq, twistor space can be constructed in the following manner. Treating the variable $\xi = \xi_0$ for the moment as a fixed constant, the homogeneous GCEq (6) becomes a second-order ODE for $u = G(\xi)$. Its solution is determined by two initial conditions: the value of $G$ and its first derivative at $\xi$ equal to the complex conjugate of $\xi$ (denoted by $\bar{\xi}_0$), i.e. at $u_0 = G(\xi_0)$ and $L_0 = (1 + \bar{\xi}_0 \xi_0) \partial_{\xi_0} G(\xi_0)$. The curves so determined are defined as projective twistors, with $\mathbb{P}T$ being the collection of all such curves. We then adopt $(\xi_0, u_0, L_0)$ as local coordinates on $\mathbb{P}T$, with the relationship to the standard twistor coordinates given earlier by

$$(\xi_0, \mu_0, \mu_1) = (\xi_0, u_0 - \bar{\xi}_0 L_0, \xi_0 u_0 + L_0).$$
These relations come directly from the integration of the ODE:
\[
\partial \bar{\zeta} (1 + \zeta_0 \bar{\zeta})^2 \partial_{\bar{\zeta}} G = 0,
\]
\[
\partial \zeta G = (1 + \zeta_0 \bar{\zeta})^{-2} \alpha_0,
\]
\[
\bar{L} = \delta G \equiv (1 + \zeta_0 \bar{\zeta}) \partial_{\bar{\zeta}} G = (1 + \zeta_0 \bar{\zeta})^{-1} \alpha_0,
\]
\[
\implies u = G = \alpha_1 - \zeta_0^{-1} (1 + \zeta_0 \bar{\zeta})^{-1} \alpha_0.
\]

The pair of integration constants \((\alpha_0, \alpha_1)\) are determined directly in terms of the initial conditions \((u_0, \bar{L}_0)\). Defining \((\mu_0, \mu_1)\) by
\[
\alpha_0 = \mu_1 - \zeta \mu_0, \quad \alpha_1 = \mu_1 \zeta^{-1},
\]
we have
\[
u = (\mu_1 \bar{\zeta} + \mu_0)(1 + \zeta_0 \bar{\zeta})^{-1}
\]
\[
\bar{L} = (\mu_1 - \zeta_0 \mu_0)(1 + \zeta_0 \bar{\zeta})^{-1}
\]
\[
u_0 = (\mu_1 \bar{\zeta}_0 + \mu_0)(1 + \zeta_0 \bar{\zeta}_0)^{-1}
\]
\[L_0 = (\mu_1 - \zeta_0 \mu_0)(1 + \zeta_0 \bar{\zeta}_0)^{-1}.
\]

Suppose two different twistors were chosen with the local coordinates \((\zeta_0, u_0, L_0)\) and \((\zeta_1, u_1, L_1)\), and their respective solution curves \(G_{1,2}\) equated with an arbitrary four-parameter regular solution of the form \((8)\)
\[
G_0(e^a, \zeta, \bar{\zeta}) = e^a l_a(\zeta, \bar{\zeta}) = \frac{\sqrt{2}}{2} e^0 + \frac{1}{2} e^1 Y^0(\zeta, \bar{\zeta}),
\]
at \(\zeta = \zeta_0\) and \(\zeta = \zeta_1\). This yields four linear algebraic equations to determine the four coordinates \(e^a\) in terms of the four \((u_0, L_0, u_1, L_1)\). This construction is totally equivalent to the use of the twistor incidence relationship \((16)\) to determine the spacetime points \(x_A^A\) from a pair of projective twistors, since the two twistors \((\zeta_0, u_0, L_0)\) and \((\zeta_1, u_1, L_1)\) uniquely determine a line \(L_x \subset \mathbb{P}^T\). This is a linear relationship, in that the choice of any pair of twistors determines both the spacetime point \(x_A^A\) and a line in projective twistor space, where any pair of points on the line determine the same point \(x_A^A\).

The attempt to apply this construction to the inhomogeneous equation, equation \((5)\), fails; the relationship is no longer linear. One obtains instead a differential equation describing a (nonlinear) curve in a curved twistor space such that any point and its tangent vector on the curve determines a point in \(\mathcal{H}\)-space. The argument we give for this is very heuristic in the sense we are assuming (without proof) that several implicit algebraic equations can be inverted. When the situation is sufficiently close to that of the homogeneous equation (i.e. for suitably small \(\sigma(u, \zeta, \bar{\zeta})\)), this should not be a problem.

By rewriting equation \((5)\) as
\[
P_0^2 \partial_{\zeta} \partial_{\bar{\zeta}} G + 2P_0 \zeta \partial_{\bar{\zeta}} G = \sigma(G, \zeta, \bar{\zeta}),
\]
or in the compressed form
\[
\partial_{\bar{\zeta}}^2 G = S(G, \partial_{\bar{\zeta}} G, \zeta, \bar{\zeta}),
\]
\[S \equiv P_0^{-2} \sigma(G, \zeta, \bar{\zeta}) - 2P_0^{-1} \zeta \partial_{\bar{\zeta}} G,
\]
we see that there are two different types of solutions.

The first comes from the condition that \(\zeta = \zeta_0\) is taken as a constant and equation \((17)\) can be treated as a second-order ODE for \(G\) as a function of \(\bar{\zeta}\) whose solution depends on two constants of integration, \((\alpha, \beta)\). They can be taken as the initial value and the first derivative
of G at some arbitrary point \( \tilde{\xi} = \tilde{\zeta}_0 \). On a fiducial (or special) curve we take \( \tilde{\zeta}_0 \) to be the complex conjugate of \( \zeta_0 : \tilde{\zeta}_0 = \overline{\zeta}_0 \). The solution will then be written as

\[
u = G^{(1)}(\alpha(\overline{\zeta}_0), \beta(\overline{\zeta}_0), \zeta_0, \tilde{\zeta}). \tag{18}\]

This curve in \((u, \tilde{\zeta})\) space, labeled by \((\alpha, \beta, \zeta_0)\), is identified as a twistor with local coordinates \((\alpha(\overline{\zeta}_0), \beta(\overline{\zeta}_0), \zeta_0)\). (It should be noted that if the initial value point, \(\tilde{\zeta}_0\), on the twistor curve is changed, the new values of \(\alpha\) and \(\beta\) are easily found.) By freeing up \(\zeta_0\), i.e. allowing \(\zeta_0 \to \zeta\), to vary and letting \(\alpha, \beta\) be functions of both \(\zeta\) and the fixed initial value point \(\tilde{\zeta}_0\), we obtain a one-parameter family of twistors (or a twistor-space curve):

\[
u = G^{(1)}(\alpha(\zeta, \overline{\zeta}_0), \beta(\zeta, \overline{\zeta}_0), \zeta, \tilde{\zeta}). \tag{19}\]

The second type of solutions are the regular ones, equation (13), that depend on four constants, the \(H\)-space coordinates, \(z^a\):

\[
u = G^{(2)}(z^a, \zeta, \tilde{\zeta}). \tag{20}\]

Holding the \(z^a\) fixed and varying \(\zeta\), we obtain a one-parameter family of twistors (i.e. a curve in \(P^T\)).

The question is: How can the two functions \(\alpha(\zeta, \overline{\zeta}_0), \beta(\zeta, \overline{\zeta}_0)\) be chosen so that two sets of solutions coincide?

This is accomplished by first equating the two types of solution and their first derivatives at \(\tilde{\zeta}_0 = \bar{\zeta}_0\):

\[
G^{(1)}(\alpha(\zeta, \overline{\zeta}_0), \beta(\zeta, \overline{\zeta}_0), \zeta, \tilde{\zeta}_0) = G^{(2)}(z^a, \zeta, \tilde{\zeta}_0),
\]

\[
\partial_{\overline{\zeta}_0} G^{(1)}(\alpha(\zeta, \overline{\zeta}_0), \beta(\zeta, \overline{\zeta}_0), \zeta, \tilde{\zeta}_0) = \partial_{\overline{\zeta}_0} G^{(2)}(z^a, \zeta, \tilde{\zeta}_0).
\]

We thus have a pair of implicit equations whose algebraic solution for \(\alpha\) and \(\beta\) has the form

\[
\alpha(\zeta, \overline{\zeta}_0) = A(z^a, \zeta, \overline{\zeta}_0), \tag{23}\]

\[
\beta(\zeta, \overline{\zeta}_0) = B(z^a, \zeta, \overline{\zeta}_0). \tag{24}\]

It is easy to derive, by the following argument, a (parametric) pair of second-order ODEs for \(\alpha\) and \(\beta\) where (23) and (24) are the solutions and \(z^a\) are the constants of integration. First take the \(\zeta\) derivative of (23) and (24):

\[
\alpha'(\zeta, \overline{\zeta}_0) = \partial_\zeta A(z^a, \zeta, \overline{\zeta}_0), \tag{25}\]

\[
\beta'(\zeta, \overline{\zeta}_0) = \partial_\zeta B(z^a, \zeta, \overline{\zeta}_0). \tag{26}\]

The two pairs, equations (23), (24) and (25), (26), determine implicitly

\[
z^a = 2^a(\alpha, \beta, \alpha', \beta', \zeta, \overline{\zeta}_0). \tag{27}\]

Finally taking the \(\zeta\) derivative of (25) and (26):

\[
\alpha''(\zeta, \overline{\zeta}_0) = \partial_\zeta \partial_\zeta A(z^a, \zeta, \overline{\zeta}_0), \tag{28}\]

\[
\beta''(\zeta, \overline{\zeta}_0) = \partial_\zeta \partial_\zeta B(z^a, \zeta, \overline{\zeta}_0). \tag{29}\]

and eliminating \(z^a\), via (27), leaves the pair

\[
\alpha''(\zeta, \overline{\zeta}_0) = A(\alpha, \beta, \alpha', \beta', \zeta, \overline{\zeta}_0), \tag{30}\]

\[
\beta''(\zeta, \overline{\zeta}_0) = B(\alpha, \beta, \alpha', \beta', \zeta, \overline{\zeta}_0). \tag{31}\]
The solutions determine, for each set of constants, $\zeta^a$, a curve in the asymptotic twistor space.

Though it might be (and probably should be) possible to construct $A$ and $B$ directly from the GCEq (i.e. from $\sigma(u, \zeta, \bar{\zeta})$), at the present we do not know how to do this. Nevertheless it is nice to see that such curves do exist and determine points in the $H$-space. Hence, we have confirmed precisely what is known from twistor theory: a point in $H$-space (a vacuum, anti-self-dual complex spacetime) corresponds to a curve in a curved twistor space. Indeed, since $H$-space reduces to complex Minkowski space in the case of the homogeneous GCEq, we saw (as expected) that points in this trivial $H$-space corresponded to lines in $\mathbb{P}T$.

3. Equivalence of good cut equations: from the $G^2$CEq to the GCEq

In this section we show how, by a coordinate transformation of the (independent) complex stereographic coordinates $(\zeta, \bar{\zeta})$, the generalized GCEq ($G^2$CEq) can be transformed into the GCEq. It must be remembered from our notation that $\bar{\zeta}^*$ (or $\bar{\zeta}$) is close to, but is not necessarily, the complex conjugate of $\zeta^*$ (or $\zeta$).

We first rewrite the GCEq with stereographic coordinates $(\zeta^*, \bar{\zeta}^*)$ as

$$\bar{\delta}^2_{0, G} = \partial_{\zeta^*} \left( P^*_0 \partial_{\zeta^*} G \right) = \sigma^*(G, \zeta^*, \bar{\zeta}^*),$$

(32)

and the $G^2$CEq as

$$\bar{\delta}^2 G = \partial_{\zeta^*} \left( V^2 P^2_0 \partial_{\zeta^*} G \right) = \sigma(G, \zeta, \bar{\zeta}).$$

(34)

We now apply the coordinate transformation

$$\bar{\xi^*} = \frac{\bar{\xi} + W}{1 - W\xi} \equiv N(\xi, \bar{\xi}),$$

(35)

$$\xi^* = \xi,$$  

(36)

with $W$ (a spin-weight 1-function) defined from

$$V^{-2} = 1 + \bar{\delta}_0 W = 1 + P_0 \partial_{\xi} W - W\xi,$$

$$P_0 = 1 + \xi\bar{\xi},$$

to equation (34). Substituting the derived relations,

$$P^*_0 = 1 + \xi\bar{\xi}^* = \frac{1 + \xi\bar{\xi}}{1 - W\xi} = \frac{P_0}{1 - W\xi},$$

$$\partial_{\xi} G = \partial_{\zeta^*} G \cdot \partial_{\xi} N,$$

$$\partial_{\bar{\xi}} G = \partial_{\zeta^*} G \cdot (\partial_{\bar{\xi}} N)^2 + \partial_{\zeta^*} G \cdot \partial_{\bar{\xi}}^2 N,$$

$$\partial_{\xi} N = V^{-2} - W\xi$$

$$= \frac{2V^{-3}\partial_{\bar{\xi}} V}{(1 - W\xi)^3} - \frac{2V^{-3}\partial_{\xi} V}{(1 - W\xi)^2},$$

into equation (34), we have, after a bit of algebra,

$$\bar{\delta}^2_{0, G} = \partial_{\zeta^*} \left( P^*_0 \partial_{\zeta^*} G \right) = F(\zeta^*, \bar{\zeta}^*) \sigma(G, \zeta(\zeta^*, \bar{\zeta}^*), \bar{\zeta}^*(\zeta^*, \bar{\zeta}^*)) \equiv \sigma^*(G, \zeta^*, \bar{\zeta}^*),$$

namely equation (32), the GCEq.
Hence, we see that the $G^2$CEq is really equivalent to the GCEq via the coordinate transformation (35). This means that the study of the $G^2$CEq on a general 3-surface $\mathcal{N}$ can be reduced to the study of the properties of the GCEq on a 3-surface whose cross-sections are metric spheres. Although the form of the coordinate transformation is far from obvious, this equivalence is not totally un-expected, since the GCEq is known to be conformally invariant.

3.1. Application: vacuum non-expanding horizons

In [6], it was shown that the condition for a neighborhood of a vacuum non-expanding horizon $\mathcal{H}$ to be foliated by a NGC whose shear vanishes at the horizon takes the form of a time-independent $G^2$CEq:

$$\delta^2 G = \sigma(\zeta, \bar{\zeta}),$$

where the right-hand side does not depend on $u$. It is not hard to prove that the solution space to this equation is a complex 4-manifold, but it is unclear if this solution space possesses the ordinary $\mathcal{H}$-space metric. Equivalently, we want to know: Is there a twistor space associated with the generalized GCEq on the horizon $\mathcal{H}$?

Answering this question is now almost trivial in light of our previous discussion. Since we continue to work with the complexified surface, a complex supertranslation

$$u \rightarrow u + f(\zeta, \bar{\zeta})$$

shifts the function $\sigma$ by Sachs’ theorem:

$$\sigma(\zeta, \bar{\zeta}) \rightarrow \sigma(\zeta, \bar{\zeta}) + \delta^2 f(\zeta, \bar{\zeta}).$$

Hence, we can choose $f$ to set $\sigma = 0$ on the entire horizon. This leaves us with the homogeneous $G^2$CEq:

$$\delta^2 G = 0.$$

But by the coordinate transformation (35), we know that this is equivalent to the homogeneous GCEq, which we saw had complex Minkowski space as its solution space. From section 2.2, we know that complex Minkowski space corresponds to (flat) twistor space $\mathcal{P}^T$, so it follows that there is a flat twistor space associated with horizon-shear-free NGCs intersecting any vacuum non-expanding horizon.

4. Conclusion

In this note we have studied an old topic: the GCEq and its solution space. In section 2, we saw how both flat and curved twistor spaces could be used to explicitly understand the relationship between points in $\mathcal{H}$-space and solutions to the GCEq on a 3-surface $\mathcal{N}$. In the setting where $\mathcal{N}$ has non-metric sphere cross-sections (such as non-expanding horizons embedded in spacetime), generalized GCEq arise, which differ from the ordinary GCEq in the definition of the $\delta$-operator on the $S^2$ portion of the topology. Section 3 demonstrated that these generalized GCEq are related to the GCEq by a simple coordinate transformation on the complex stereographic coordinates $(\zeta, \bar{\zeta})$. This means that the study of the $G^2$CEq on $\mathcal{N}$ with an arbitrary conformal factor on the sphere topology reduces to the study of the normal GCEq on metric 2-spheres. As an example, we saw that this observation immediately implies that

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3 A vacuum non-expanding horizon is a null 3-submanifold in a Lorentzian spacetime which has vanishing divergence and shear, is topologically $S^2 \times \mathbb{R}$ and is fibered over $S^2$ by null curves with the vacuum Einstein equations holding in a neighborhood of the horizon.
the solution space associated with the $G^2$CEq on a vacuum non-expanding horizon is complex Minkowski space, and that the corresponding twistor space is the flat $\mathbb{P}^1_T$.

In particular, this indicates that recently developed physical identification theories based on the study of solutions to the GCEq on $\mathbb{I}^+$ could be adapted to non-conformal local 3-surfaces embedded in spacetime (cf [5]). In the future, we hope to apply these results to prior studies of non-expanding horizons (e.g. [6]) in the hope of developing a local physical identification theory which could identify physical quantities such as mass, linear momentum and angular momentum flux at null 3-surfaces in spacetime.

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