Testing of Metric-Field Equations of Gravitation by Binary Pulsar

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Abstract

Testing of the gravitation equations, proposed by one of the authors earlier, by a binary pulsar is considered. It has been shown that the formulas for the gravitation radiation of the system resulting from the equations do not contradict the available observations data.

1 Introduction

In paper [1] gravitation equations which do not lead to a physical singularity in the center of the spherically symmetric field were proposed. These equations also predict that there can exist stable supermassive compact configurations of the degeneration Fermi-gas without an events horizon [2]. The equations do not contradict classical tests at the distances from the center which are much larger than the Schwarzschild radius. In the present paper we find the power of the gravitational-wave radiation from a close binary system and use the result to find the deceleration of the orbital period of the pulsar PSR1913+16 conditioned by the gravitational radiation. In this case we deal with a moderately strong gravitation field and use the definition of gravitational energy that follows from the gravitation equations under consideration.

2 Equations of Field

Thirring [3] proposed that gravitation can be described as a tensor field \( \kappa \) of spin two in Pseudo-Euclidean space-time \( E_4 \) where the Lagrangian action describing the motion of test particles in a given field is of the form

\[
L = m_p c \ g \left( \frac{\kappa}{\kappa} \right)_{x}^{1=2} \quad : \quad (1)
\]

In this equation \( g \) is a tensor function of \( \kappa \), \( m_p \) is the mass of the particle, \( c \) is the speed of light and \( \kappa = \frac{dx}{dt} \).

A theory based on that action must be invariant under the gauge transformations that are a consequence of the existence of "extra" components of the tensor \( \kappa \). Transformations give rise to some transformations \( g \) ! Therefore, the field equations for \( g \) (\( \kappa \)) and equations of the motion of the test...
particle must be invariant under these transformations of the tensor $g_i$. Equations of gravitation that are invariant with respect to arbitrary gauge transformations were proposed in [1]. These equations are of the form

$$B; B B = 0;$$

(2)

The equations are vacuum bimetric equations for the tensor

$$B = ;$$

(3)

(Greek indices run from 0 to 3), where

$$= (n + 1)^1 h^1 ;$$

(4)

$$= (n + 1)^1 ;$$

(5)

are the Christoffel symbols of the pseudo-Euclidean space-time $E_4$ whose fundamental tensor is $h_i$, are the Christoffel symbols of the Riemannian space-time $V_4$ with dimension $n = 4$, whose fundamental tensor is $g_i$. The semi-colon in eqs. (2) denotes the covariant differentiation in $E_4$.

A peculiarity of eqs. (2) is that they are invariant under arbitrary transformations of the tensor $g$ retaining invariant the equations of motion of a test particle, i.e. geodesics in $V_4$. In other words, the equations are geodesic-invariant. Thus, the tensor field $g$ is defined up to geodesic mappings of space-time $V_4$ (in the analogous way as the potential $A$ in electrodynamics is determined up to gauge transformations). A physical sense have only geodesic invariant values. The simplest object of that kind is the object $B$ which can be named the strength tensor of the gravitation field. The coordinate system is defined by the used measurement instruments and is given.

The Christoffel symbols are transformed under the geodetic (i.e. projective) mappings as follows:

$$= + ' + ' ;$$

(6)

where $'$ is a vector-function of $x$. This equation has a simple interpretation in an $(n+1)$-dimensional manifold $M_5$ where the admissible coordinates transformations are of the form

$$\tilde{x} = \tilde{x} (x^0; x^1; x^2; x^3);$$

(7)

$$\tilde{x}^4 = x^4 \int^Z \; dx :$$

(8)

Namely, eq. (6) can be interpreted as the transformation of 4-components of the connection coefficient $A^A_{B C} (A; B; C = 0:4) \in M_5$ under the transformation (8) if the condition $\tilde{\epsilon}_4 = 1$ is satisfied.

For this reason we will consider the tensor $g$ as 4-components of 5-dimensional tensor

$$g_{A B} = g^4 g_4;$$

(9)

The components $g_4$ are transformed under (8) as follows

$$\tilde{g}_4 = g + g_4 ' + g_4 ;$$

(10)
\[ g_{44} = g_{44} + \alpha g \] (11)

Eqs. (11) coincides with the transformation of the object (or \( Q = \)) under (8), if \( g_{44} = n + 1 \). For this reason, we will assume that

\[ g_{AB} = g_{Q}^{Q} (n + 1) \] (13)

We will assume also that (10) is the transformation of the tensor \( g \) under the geodesic mappings of \( V_{4} \). Then there exists the geodesic-invariant tensor

\[ G = g_{Q}^{Q} (n + 1) \] (14)

and the geodesic-invariant generalization of the Einstein equations with matter source are of the form

\[ B = k (T + 2G T) \] (15)

where \( k = 8 \) \( c^2 \), \( T \) is the matter energy-momentum tensor, \( T = G T \). At the gauge conditions \( Q = 0 \) they coincide with the Einstein equations.

Consider now the question about the definition of the energy of gravitational field in the used theory. Set in eqs. (15) \( T = c^2 u u \); where \( u \) is the 4-velocity of matter points. At the small macroscopic velocities of the matter we can set \( u_0 = 1 \) and \( u_i = 0 \). Then, the 00-component of eq (15) can be written in the form

\[ B_{00} = (c^2 + t_{00}) \] (16)

where \( k = 2 \), and \( t_{00} \) is the 00-component of the tensor

\[ t = 1 B B \] (17)

Setting

\[ B_{00} = c^2 \partial U \partial x \] (18)

we obtain the equation

\[ U = (c^2 + t_{00}) \] (19)

In the absence of the term \( t_{00} \) and at the distances from the central masses much larger than the Schwarzschild radius this equation coincides with the Poisson equation for the Newtonian gravity potential. For this reason it is natural to expect that in general case eq. (19) is the differential equation for a generalization of the gravitational potential too and the term \( t_{00} \), which is in the equation additively with \( c^2 \); is the energy of the gravitational field.

To verify this assumption let us find the energy of gravitational field of the point mass \( M \) as the following integral in the Pseudo-Euclidean space-time:

\[ E = t_{00} dV \] (20)

In the Newtonian theory this integral is divergent. In our case we have

\[ t_{00} = 2 \partial \partial x \] (21)
and, therefore, using spherical coordinates, we find

\[ E = \frac{1}{8} \frac{r^2 c^4}{G} J \]

where

\[ J = \frac{\partial V}{\partial r} = \frac{4}{3\pi} B (1,1=3) \]

and

\[ B (z;w) = \frac{z^{1/2}}{(1 + t)^{z+w}} \]

is B-function. Using the equality

\[ B (z;w) = \frac{(z)}{(z + w)} \]

we obtain

\[ E = \frac{r_0 c^4}{2G} = M c^2 \]

We arrive at the conclusion that the energy of the point mass is finite and the rest energy of the point particle is caused by its gravitational field. The spatial components of the vector \( P = t_0 \) are equal to zero. Due to these facts we may consider (at least in a weak-field approximation) the tensor \( t \) as the energy-momentum tensor of gravitational field.

### 3 Gravitation radiation of a binary system

In no gravitation case \( g = 0 \), where \( g \) is the metric tensor of Pseudo-Euclidean space-time \( E_4 \). For this reason it is natural to suppose that in a weak-field approximation

\[ g = + h \]

To simplify eq. (15) let us choose the following gauge condition

\[ D + \frac{n}{n+1} D = 0 \]

where \( D \) is the Christoffel symbols of space-time \( V_4 \), in which the derivatives are replaced by the covariant ones in \( E_4 \). Equality (28) is covariant and, therefore, does not imply restrictions for choosing the coordinate system. Using (27) we obtain to the first order in \( h \)

\[ B = \frac{1}{2} h, + h, + \frac{1}{2} (n+1)^{1/n} h, + h, \]

and the gauge conditions (28):

\[ h, = (n+1)^{1/n} h, \]

Consider a system of the slowly moving bodies of a finite volume where \( T \neq 0 \) in the vicinity of the coordinates origin, and suppose that \( f_{00}, f_{01}, f_{12} \) (Latin
indexes run from 1 to 3). Using an orthogonal system of coordinates with the metric \( \text{diag}(1;1;1;1) \) let us find the gravitation radiation in the wave zone.

Equations (15) now are of the form

\[
2 = 2kT ;
\]

where

\[
\text{diag}(1;1;1;1) = h ;\quad 1=2 \quad h ;
\]

The general solution of eq. (31) for departing waves is of the form

\[
Z = \frac{k}{2} T x \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial x^0} \right) \delta^{(4)} x^0 ;
\]

where \( x^0 \) is the radius-vector of an infinitely small element of the source and \( x \) is the radius vector of the point in which the field is calculated.

If the origin of the coordinate system coincides with the masses center of the radiating system, the solution of eq. (31) is given by

\[
0_0 = \text{const} + o(\epsilon^2) ;\quad 0_1 = o(\epsilon^2) ;
\]

\[
i_{ij} = \frac{k^4}{c^2} T \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial x^0} \right) x^0 ;
\]

where

\[
I_{ij} = \frac{1}{c^2} T_{00} x^0 \delta^{(4)} 0^0 ;
\]

the points denote the differentiation with respect to time \( t \) and \( \text{const} \) denotes an additional term to the newtonian potential. Therefore, the solution of eq. (33) for weak gravitational waves can be written in the form

\[
0 = 0 ;
\]

\[
i_k = \frac{k^4}{c^2} T I_{ik} \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial x^0} \right) x^0 ;
\]

Therefore, the lowest mode of the gravitational radiation is the quadrupole one.

Let us find the number of the independent states of the field in vacuum. According to (31) the functions in vacuum obey the equation

\[
2 = 0
\]

and gauge conditions (30), which give four constraint equations. Equations (37) also gives four constraints. Taking also into account that for weak waves \( 0_1 = 0 \), we have only two independent components of \( \text{const} \). It follows also from (30) and (37) that the trace of the tensor is equal to zero. Thus,

\[
0 = 0 ;\quad i_{ij} = 0 ;\quad k_k = 0 ;
\]

where the coma denotes the partial derivative. Therefore, the plane gravitational wave is a transversal with the zero trace.

Now let us find the flow of energy of the gravitational radiation. Starting from eq. (17), we find that the flow of energy carried by the wave in the radial direction is given by

\[
cb_{ik} = \frac{n_k}{4k} h_{ij} \rightarrow_{i \rightarrow j} \epsilon^0
\]
where \( n^k = x^k = r \) and the symbols \( \mathcal{h}_i \) denote the period \( T \) averaging of the function \( f(t) \) by the formula

\[
\mathcal{h}_f(t) = \frac{1}{T} \int_0^T f(t) \, dt.
\]

We can also satisfy condition (40) by projecting the tensor to the plane perpendicular to the vector \( n^i \) with the help of the operator

\[
P_{ij} = \eta_{ij} \, n_i \, n_j.
\]

(43)

As a result, the transversal-traceless part of the tensor is

\[
P^{\text{T}T}_{ij} = \frac{1}{2} P_{ij} \left( P_{rs} \, r_s \right) + \frac{1}{2} \mathcal{h}_{ij} \, J_{ij} \, n_i n_j
\]

(44)

It should also be noted that the transversal-traceless part of the quadrupole momentum \( I_{ij} \), in (38), and the magnitude \( J_{ij} = I_{ij} \, 1 = 3 \, I_{kk} \) are the same, which can be verified immediately. For this reason

\[
i_j = \frac{k}{4} \, J_{ij}.
\]

(45)

Finally, we obtain

\[
c_t\tau = \frac{n_r \, k}{64 \, c^2 \, \mathcal{h}_{ij} \, J_{ij}} \, \mathcal{h}_{ij} \, J_{ij} + \frac{1}{2} \mathcal{h}_{ij} \, J_{ij} \, n_i n_j
\]

(46)

The averaging time rate of the energy flow change can be found by integration of eq. (46) over the sphere of the radius \( r \):

\[
\frac{dE}{dt} = \int_{\mathcal{C} r} \mathcal{h}_{ij} \, J_{ij} \, n_i n_j \, \sin d\tau
\]

\[
= \frac{3}{5 c^2} \mathcal{h}_{ij} \, J_{ij} \, n_i n_j
\]

(47)

were \( E \) is the total mechanical energy of the radiating system. This formula completely coincides with the one in general relativity.

Consider a system of two gravitationally bound point masses \( m_1 \) and \( m_2 \). Assuming that the distance between the bodies is much larger than the Schwarzschild radius we can use the Newtonian mechanics. Since \( E = G M = (2a) \), where \( M = m_1 + m_2 \), \( a \) is the large semiaxes of the orbit and the orbital period is \( P_b = 2 \, a^3 \, (M \, G) \, 1^{2} \), the relative velocity of increasing of the orbital period is given by

\[
\frac{dP_b}{P_b} = \frac{3 \, d\eta}{2 \, a} = \frac{3 \, dE}{2 \, E}.
\]

(48)

It gives

\[
\frac{dP_b}{P_b} = \frac{192}{5 c^2} \, \mathcal{G} \, G \, f \, \eta \, m_1 \, m_2 \, M \, 1 = 3 \,
\]

(49)

where \( f(\eta) = 1 \, \mathcal{G} \, 7^{2} \, 1 + 73 = 24 \, \mathcal{G} \, 2 + 37 = 96 \, \mathcal{G} \). It should be noted that in the above calculations the shift of the periastron is not taken into account since at the used one period-averaged the shift is very small.
4 Application to PSR1913+16

Let us use the above results for an analysis of the binary pulsar PSR 1913+16 in the framework of the Blandford-Teukolsky model \([4]\). Namely, we take into account the dependence of the phase of the detected radiation on the following observable parameters that are the functions of the unknown masses \(m_1\) and \(m_2\).

1. Increasing the orbital period because of the gravitational radiation \((49)\).
2. The rate of the periastron shift of the pulsar orbit which according to \([2]\) is given by
   \[
   \frac{d \varpi}{d t} = \frac{6 GM}{a^2 (1 - e^2)^2 c^2} + \frac{G^2 M^2}{2a^2 (1 - e^2)^2 c^4} f_1 (\theta); \]
   where \(f_1 (\theta) = 54 + 16e^2 - e^4\). The second term for the considered system is about \(0.01\%\) of the total value of \(\frac{d \varpi}{d t}\) and, therefore must be taken into account.
3. The magnitude
   \[
   \delta \equiv \frac{t_1 - t_2}{\sin (\Phi)}; \]
   where \(t_1\) is the moment of the radiation of the pulse, measured in the inertial frame of reference of the distant observer, \(t_2\) is the same moment, measured in the proper frame of reference of the pulsar, \(\Phi\) is the eccentric anomaly.

Supposing that 4-vector \(k^0 = t^! = c \delta = c \gamma g\) of a photon, the motion of which is described by the Lagrangian \((1)\), satisfies the equality \(g \cdot k = 0\), we obtain approximately \(k^0 = t^! = c \gamma g_{00}\), where \(!\) is the frequency of the electromagnetic wave and \(g_{00}\) is given by \((5)\).

\[
\gamma_{00} = 1 - \frac{r_g}{r} (r_g + r_g^3)^{1/3}; \]

Let \(v_p\) be the pulsar velocity relative to the masses center of the system. Then we obtain the following relation between the measured frequencies of the signal \(!_0\) and \(!\) in the frames of the observer and pulsar

\[
\frac{!}{!_0} = \frac{1}{1 - \frac{r_g}{r} (r_g + r_g^3)^{1/3}} \frac{v_p}{v_c}; \]

where \(\#\) is the angle between the wave direction and the direction of the motion of the source radiation, \(v_c = (\mathbf{r}^3 + r_g^3)^{1/3}\); \(r_g\) is the gravitational radius of the pulsar companion, \(r\) is the distance between the pulsar and its companion. From \((5)\) we obtain, setting \(\# = \#_0 = 2\), the relation between time intervals \(d\) and \(dt\) in the reference frames of the pulsar and observer

\[
dt = \frac{1}{1 - \frac{r_g}{r} (r_g + r_g^3)^{1/3}} \frac{v_p}{v_c} \frac{r_g}{2r} \frac{v_p^2}{2c^2} dt; \]

which with the used accuracy coincides with the same relation in general relativity.

The Shapiro effect is too small (the same as in general relativity) and is not taken into account in the model under consideration.

Proceeding from these results, we obtain in an analogy with \([4]\)

\[
\frac{P_b}{P_b} = \frac{2}{\gamma_g} \left( \frac{m_1 + 2m_2}{2ac^2 (m_1 + m_2)^2} \right); \]

Fig. 1 shows that in the plane \(m_1, m_2\) the curves \(P_b = P_b (m_1, m_2), ! = ! (m_1, m_2), \)

\(\Rightarrow (m_1, m_2)\) intersect in one point that means that the theory does not contradict the
observation data. The value of the full system mass $M$, resulting from the measured value of $P_b$ and $e$ by eq. (50) is equal to $(2.82845 \pm 0.00004)M$ and the masses of the pulsar and its companion, found by (55), are equal to $(1.441 \pm 0.003)M$ and $(1.387 \pm 0.003)M$, respectively. These results differ very little from the ones resulting from general relativity [5] $(1.442 \pm 0.003)M$ and $(1.386 \pm 0.003)M$. Due to a kinematic effect in our Galaxy [6] the small correction $(0.017 \pm 0.005)M^2$ must be added to the found theoretical value of $P_b = (2.40249 \pm 0.00029)M$. Taking into account this correction we obtain that the ratio of the observational value of $P_b$ to the found theoretically is equal to $1.0023 \pm 0.0047$.

Fig. 1 The plots of the functions $= \left( m_1, m_2 \right)$ (curve 1), $ = ! \left( m_1, m_2 \right)$ (curve 2), $P_b = B^b \left( m_1, m_2 \right)$ (curve 3).

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