Exotic smooth $\mathbb{R}^4$ and certain configurations of NS and D branes in string theory

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Received Day Month Year
Revised Day Month Year

In this paper we show that in some important cases 4-dimensional data can be extracted from superstring theory such that a) the data are 4 Euclidean geometries embedded in standard $\mathbb{R}^4$, b) these data depend on NS and D brane charges of some string backgrounds, c) it is of potential relevance to 4-dimensional physics, d) the compactification and stabilization techniques are not in use, but rather are replaced. We analyze certain configurations of NS and D-branes in the context of SU(2) WZW model and find the correlations with different exotic smoothings of $\mathbb{R}^4$. First, the dynamics of D-branes in SU(2) WZW model at finite $k$, i.e. the charges of the branes, refers to the exoticness of ambient $\mathbb{R}^4$. Next, the correspondence between exotic smoothness on 4-space, transversal to the world volume of NS5 branes in IIA type, and the number of these NS5 branes follows. Finally, the translation of 10 dimensional string backgrounds to 4 Euclidean spaces embedded as open subsets in the standard $\mathbb{R}^4$ is achieved.

Keywords: exotic $\mathbb{R}^4$; string background; D-branes; brane charge.

PACS numbers: 11.25.Uv, 02.40K, 04.20Gx

1. Introduction

The problem with successful inclusion of effects of exotic open 4-manifolds like exotic $\mathbb{R}^4$ into any physical theory, is the notorious lack of an explicit coordinate-like presentation of these smooth manifolds. In the series of our recent papers we addressed this issue and worked out some relative techniques allowing for analytical treatment of small exotic $\mathbb{R}^4$'s [121111330]. Based on these results we will show in this paper a rather unexpected relation between configurations of D-branes in some exact string backgrounds with the exotic smoothness structure of the $\mathbb{R}^4$. This

*Based on the talk „Small exotic smooth $\mathbb{R}^4$ and string theory” given at the International Congress of Mathematicians, ICM2010, 19-28.08.2010, Hyderabad, India
relation is not only a pure formal correspondence but instead we see it as a way to 4-dimensional physics. The proposed realization is different and independent on various compactifications or model building techniques worked out so far by string theorists and aiming also at the description of our 4-dimensional world. Why do we develop any alternative to the well established phenomenological approach in string theory? The answer is rather direct. First, in spite of the substantial effort results worked out in string theory are highly ambiguous: there exist about $10^{500}$ possible backgrounds as candidates for real physics. Second, the appearance of exotic smoothness of Euclidean $\mathbb{R}^4$ in the formalism of string theory is a direct indication that this formalism deals with dimension 4 at the fundamental level. Even from a very general point of view, if different string constructions refer indeed to exotic 4-smoothness on Euclidean 4-space, there is a chance to reduce the above-mentioned ambiguity. When additionally we would have a good understanding of how 4-exotics refer to real physics there is a big chance to solve the puzzle of a relation between string theory and 4-dimensional physics. Even though much remain to be done this paper serves as the first important step into this direction.

The basic technical ingredient of the analysis of small exotic $\mathbb{R}^4$'s enabling uncovering many applications also in string theory is the relation between exotic (small) $\mathbb{R}^4$'s and non-cobordant codimension-1 foliations of $S^3$ as well gropes and wild embeddings as shown in [12]. The foliations are classified by the Godbillon-Vey classes as elements of the cohomology group $H^3(S^3, \mathbb{R})$. By using $S^1$-gerbes it was possible to interpret the integral elements $H^3(S^3, \mathbb{Z})$ as characteristic classes of the $S^1$-gerbes over $S^3$ [11]. In the next section we will explain the whole complex of ideas more carefully. The following section deal with the relation between string backgrounds and exotic $\mathbb{R}^4$. A discussion of the results closes the paper.

2. Exotic $\mathbb{R}^4$ and codimension-one foliations of the 3-sphere

The main line of the topological argumentation can be briefly described as follows:

1. In Bizaca’s exotic $\mathbb{R}^4$ one starts with the neighborhood $N(A)$ of the Akbulut cork $A$ in the K3 surface $M$. The exotic $\mathbb{R}^4$ is the interior of $N(A)$.
2. This neighborhood $N(A)$ decomposes into $A$ and a Casson handle representing the non-trivial involution of the cork.
3. From the Casson handle we construct a grope containing Alexanders horned sphere.
4. Akbulut’s construction gives a non-trivial involution, i.e. the double of that construction is the identity map.
5. From the grope we get a polygon in the hyperbolic space $\mathbb{H}^2$.
6. This polygon defines a codimension-1 foliation of the 3-sphere inside of the exotic $\mathbb{R}^4$ with an wildly embedded 2-sphere, Alexanders horned sphere.
7. Finally we get a relation between codimension-1 foliations of the 3-sphere and exotic $\mathbb{R}^4$. 
Now we will explain the details in this construction.

An exotic $\mathbb{R}^4$ is a topological space with $\mathbb{R}^4$-topology but with a different (i.e. non-diffeomorphic) smoothness structure than the standard $\mathbb{R}^4_{std}$ getting its differential structure from the product $\mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$. The exotic $\mathbb{R}^4$ is the only Euclidean space $\mathbb{R}^n$ with an exotic smoothness structure. The exotic $\mathbb{R}^4$ can be constructed in two ways: by the failure to arbitrarily split a smooth 4-manifold into pieces (large exotic $\mathbb{R}^4$) and by the failure of the so-called smooth h-cobordism theorem (small exotic $\mathbb{R}^4$). Here we will use the second method.

Consider the following situation: one has two topologically equivalent (i.e. homeomorphic), simply-connected, smooth 4-manifolds $M, M'$, which are not diffeomorphic. There are two ways to compare them. First one calculates differential-topological invariants like Donaldson polynomials \(^27\) or Seiberg-Witten invariants \(^2\) But there is another possibility: It is known that one can change a manifold $M$ to $M'$ by using a series of operations called surgeries. This procedure can be visualized by a 5-manifold $W$, the cobordism. The cobordism $W$ is a 5-manifold having the boundary $\partial W = M \sqcup M'$. If the embedding of both manifolds $M, M'$ in to $W$ induces homotopy-equivalences then $W$ is called an h-cobordism. Furthermore we assume that both manifolds $M, M'$ are compact, closed (no boundary) and simply-connected. As Freedman \(^31\) showed a h cobordism implies a homeomorphism, i.e. h-cobordant and homeomorphic are equivalent relations in that case. Furthermore, for that case the mathematicians \(^26\) are able to prove a structure theorem for such h-cobordisms:

Let $W$ be a h-cobordism between $M, M'$. Then there are contractable submanifolds $A \subset M, A' \subset M'$ together with a sub-cobordism $V \subset W$ with $\partial V = A \sqcup A'$, so that the h-cobordism $W \setminus V$ induces a diffeomorphism between $M \setminus A$ and $M' \setminus A'$. Thus, the smoothness of $M$ is completely determined (see also \(^34\)) by the contractible submanifold $A$ and its embedding $A \hookrightarrow M$ determined by a map $\tau : \partial A \to \partial A$ with $\tau \circ \tau = id_{\partial A}$ and $\tau \neq \pm id_{\partial A}(\tau$ is an involution). One calls $A$, the Akbulut cork. According to Freedman \(^31\), the boundary of every contractible 4-manifold is a homology 3-sphere. This theorem was used to construct an exotic $\mathbb{R}^4$. Then one considers a tubular neighborhood of the sub-cobordism $V$ between $A$ and $A'$. The interior $int(V)$ (as open manifold) of $V$ is homeomorphic to $\mathbb{R}^4$. If (and only if) $M$ and $M'$ are homeomorphic, but non-diffeomorphic 4-manifolds then $int(V) \cap M$ is an exotic $\mathbb{R}^4$. As shown by Bizaca and Gompf \(^15\), \(^16\) one can use $int(V)$ to construct an explicit handle decomposition of the exotic $\mathbb{R}^4$. We refer for the details of the construction to the papers or to the book \(^34\). The idea is simply to use the cork $A$ and add some Casson handle $CH$ to it. The interior of this construction is an exotic $\mathbb{R}^4$. Therefore we have to consider the Casson handle and its construction in more detail. Briefly, a Casson handle $CH$ is the result of attempts to embed a disk $D^2$ into a 4-manifold. In most cases this attempt fails and Casson \(^24\) looked for a substitute, which is now called a Casson handle. Freedman \(^31\) showed that every Casson handle $CH$ is homeomorphic to the open 2-handle $D^2 \times \mathbb{R}^2$ but in nearly all cases it is not diffeomorphic to the standard handle \(^32\), \(^33\). The Casson
handle is built by iteration, starting from an immersed disk in some 4-manifold $M$, i.e. a map $D^2 \to M$ with injective differential. Every immersion $D^2 \to M$ is an embedding except on a countable set of points, the double points. One can kill one double point by immersing another disk into that point. These disks form the first stage of the Casson handle. By iteration one can produce the other stages.

Finally consider not the immersed disk but rather a tubular neighborhood $D^2 \times D^2$ of the immersed disk, called a kinky handle, including each stage. The union of all neighborhoods of all stages is the Casson handle $CH$. So, there are two input data involved with the construction of a $CH$: the number of double points in each stage and their orientation $\pm$. Thus we can visualize the Casson handle $CH$ by a tree: the root is the immersion $D^2 \to M$ with $k$ double points, the first stage forms the next level of the tree with $k$ vertices connected with the root by edges etc. The edges are evaluated using the orientation $\pm$. Every Casson handle can be represented by such an infinite tree.

The main idea is the construction of a grope, an infinite union of surfaces with non-vanishing genus, from the Casson handle. But the grope can be represented by a sequence of polygons in the two-dimensional hyperbolic space $\mathbb{H}^2$. This sequence of polygons is replaced by one polygon with the same area. From this polygon we can construct a codimension-one foliation on the 3-sphere as done by Thurston. This 3-sphere is part of the boundary $\partial A$ of the Akbulut cork $A$. Furthermore one can show that the codimension-one foliation of the 3-sphere induces a codimension-one foliation of $\partial A$ so that the area of the corresponding polygons agree.

Thus we are able to obtain a relation between an exotic $\mathbb{R}^4$ (of Bizaca as constructed from the failure of the smooth h-cobordism theorem) and codimension-one foliation of the $S^3$. Two non-diffeomorphic exotic $\mathbb{R}^4$ implying non-cobordant codimension-one foliations of the 3-sphere described by the Godbillon-Vey class in $H^3(S^3, \mathbb{R})$ (proportional to the area of the polygon). This relation is very strict, i.e. if we change the Casson handle then we must change the polygon. But that changes the foliation and vice versa. Finally we obtained the result:

**The exotic $\mathbb{R}^4$ (of Bizaca) is determined by the codimension-1 foliations with non-vanishing Godbillon-Vey class in $H^3(S^3, \mathbb{R})$ of a 3-sphere seen as submanifold $S^3 \subset \mathbb{R}^4$.** We say: the exoticness is localized at a 3-sphere inside the small exotic $\mathbb{R}^4$.

### 3. Geometry of string backgrounds and exotic $\mathbb{R}^4$

In this section we take the point of view that exotic smoothness of some small exotic $\mathbb{R}^4$’s when localized on $S^3 \subset \mathbb{R}^4$, correspond to some string geometry given by so-called $B$-fields on $S^3$. The localization is understood as the representation of the exotics by 3-rd integral or real cohomologies of $S^3$. This correspondence is restricted to the classical limit of the geometry of string backgrounds seen as a curved Riemannian manifold with $B$-field. One can say that the small exotic smooth $\mathbb{R}^4$ given by a localized $S^3$ is described by string geometry of $B$-fields on this $S^3$. 
The correspondence can be extended to the string regime of finite volume of SU(2) WZW model.

3.1. SU(2) WZW model, D-branes and exotic \( R^4 \)

In this subsection we want to focus on the change of smooth structure on \( R^4 \). As explained above, we realize the plan by considering the changes as localized on \( S^3 \).

Following 12, 11, this change gives rise to stringy effects, since the changes can be described by computations in some 2D CFT, namely WZW models on \( SU(2) \) at finite level.

3.1.1. The dynamics of branes in the bosonic \( SU(2) \) WZW model

First we start with a discussion of the bosonic \( SU(2) \) WZW model and dynamics of branes in it. We deal here with \( S^3 \), i.e. the metric of string background has non-zero curvature. In general, a non-vanishing curvature \( R(g) \) w.r.t. a non-constant metric \( g \) of the background manifold \( (M, g) \) on which bosonic string theory is formulated, enforces the \( H \)-field on \( M \) to have a non-zero value. This fact can be seen by using the string field equations (see e.g. 38)

\[
R_{\mu\nu}(g) - \frac{1}{4} H_{\mu\rho\sigma} H_{\nu}^{\rho\sigma} = O(\alpha') \tag{1}
\]

where \( H = dB \) is the NSNS 3-form, \( B = B(x) dx^\mu \wedge dx^\nu \) is the B-field, and dilaton field has a fixed, constant value. Furthermore in the case of superstring theory this equation still holds true provided all RR background fields vanish 38.

D-branes in group manifold \( SU(2) \) (at the semi-classical limit) are determined by wrapping the conjugacy classes of \( SU(2) \), i.e. (degenerated) 2-spheres described by a 2-sphere \( S^2 \) having two poles each localized at a point. Due to the quantization conditions there are \( k + 1 \) D-branes on the level \( k \) SU(2) WZW model 29, 38, 5. To grasp the dynamics of the branes one should deal with the gauge theory on the stack of \( N \) D-branes on \( S^3 \), quite similar to the flat space case where noncommutative gauge theory emerges 14. Following this idea, the action for the brane dynamics is given by the following argumentation.

Given \( N \) branes of type \( J \) (on top of each other), where \( J \) is the representation of \( SU(2) \) i.e. \( J = 0, \frac{1}{2}, 1, \ldots, \frac{3}{2} \). Then the dynamics of these branes (see 23) is described by the noncommutative action:

\[
S_{N,J} = S_{YM} + S_{CS} = \frac{\pi^2}{k^2 (2J + 1)N} \left( \frac{1}{4} \text{tr}(F_{\mu\nu} F^{\mu\nu}) - \frac{i}{2} \text{tr}(f_{\mu\nu\rho} CS_{\mu\nu\rho}) \right). \tag{2}
\]

Here the curvature form \( F_{\mu\nu}(A) = iL_\mu A_\nu - iL_\nu A_\mu + i[A_\mu, A_\nu] + f_{\mu\nu\rho} A^\rho \) and the noncommutative Chern-Simons action reads \( CS_{\mu\nu\rho}(A) = L_\mu A_\nu A_\rho + \frac{1}{2} A_\mu [A_\nu, A_\rho] \). The fields \( A_\mu, \mu = 1, 2, 3 \) are defined on a fuzzy 2-sphere \( S^2_j \) and should be considered as \( N \times N \) matrix-valued, i.e. \( A_\mu = \sum_{j,a} a_\mu^{j,a} Y^j_a \) where \( Y^j_a \) are fuzzy spherical...
harmonics and $a^{\mu}_{j,a}$ are Chan-Paton matrix-valued coefficients. $L_\mu$ are generators of the rotations on fuzzy 2-spheres and they act only on fuzzy spherical harmonics [35]. The noncommutative action $S_{YM}$ was derived from Connes spectral triples of the noncommutative geometry and was aimed to describe Maxwell theory on fuzzy spheres [23]. The equations of motion derived from the stationery points of (2), read

$$L_\mu F^{\mu\nu} + [A_\mu, F^{\mu\nu}] = 0. \tag{3}$$

The solutions of (3) describe the dynamics of the branes, i.e. the condensation processes on the brane configuration $(N, J)$ which results in another configuration $(N', J')$. A special class of solutions, in the semi-classical $k \to \infty$ limit, can be obtained from the $N(2J + 1)$ dimensional representations of the algebra $su(2)$. For $J = 0$ one has $N$ branes of type $J = 0$, i.e. $N$ point-like branes in $S^3$ at the identity of the group. Given another solution corresponding to $J_N = \frac{N-1}{2}$, then one can show: this solution can be firstly interpreted as the brane wrapping around the $S^3_{J_N}$ sphere but also as the condensed state of $N$ point-like branes at the identity of $SU(2)$ [38].

$$(N, J) = (N, 0) \to (1, \frac{N-1}{2}) = (N', J') \tag{4}$$

Turning to the finite $k$ string regime of the $SU(2)$ WZW model one can make use of the techniques of the boundary CFT when applied to the analysis of Kondo effect [35]. It follows that there exists at the level of partition functions a continuous shift between $N \chi_j(q)$ and the interfered sum of characters $\sum_j N_{jN}^l \chi_l(q)$ where $N = 2J_N + 1$ (in the vanishing value of the coupling constant) and $N_{jN}^l$ are Verlinde fusion rule coefficients. In the case of $N$ point-like branes one can determine the decay product of these by considering open strings ending on the branes. The result on the partition function is

$$Z_{(N,0)}(q) = N^2 \chi_0(q)$$

which is continuously shifted to $N \chi_{I_N}(q)$ and next to $\sum_j N_{jN}^l \chi_l(q)$. As the result we have the decay process [38]

$$Z_{(N,0)}(q) \to Z_{(1,J_N)}$$

$$(N, 0) \to (1, J_N) \tag{5}$$

which extends the similar process derived at the semi-classical $k \to \infty$ limit in the effective gauge theory [41], however the representations $2J_N$ are bounded now, from the above, by $k$.

3.1.2. Brane charges and exotic $\mathbb{R}^4$

Given the above dynamics of branes in the WZW $SU(2)$ model at string regime, one can address the question of brane charges in a direct way. This is based on the
decay rule in the supersymmetric WZW SU(2) model. In this case we have a shift of the level by \( k \rightarrow k + 2 \) which measures the units of the NSNS flux through \( SU(2) = S^3 \). One can see the supersymmetric model as strings moving on \( SU(2) = S^3 \) with \( k + 2 \) units of NSNS flux. From the CFT point of view there exist currents \( J^a \) which satisfy \( k + 2 \) level of the Kac-Moody algebra and free fermionic fields \( \psi^a \) in the adjoint representation of \( su(2) \). However it is possible to redefine the bosonic currents as

\[
J^a + \frac{i}{k} f^a_{bc} \psi^b \psi^c
\]

which fulfill the current algebra commutation relation at the level \( k \). Here \( f^a_{bc} \) are the structure constants of \( su(2) \). The fields \( \psi^a \) commute with these currents. Thus we have the splitting of the supersymmetric WZW SU(2) model at level \( k + 2 \) as WZW SU(2) model at level \( k \) times the theory of free fermionic fields.

Thus there are \( k + 1 \) stable branes wrapping the conjugacy classes numbered by \( J = 0, \frac{1}{2}, \ldots, \frac{k}{2} \). The decaying process says that placing \( N \) point-like branes (each charged by the unit 1) at the pole \( e \) they can decay to the spherical brane \( J_N \) wrapping the conjugacy class. Taking more point-like branes to the stack at \( e \) gives the more distant \( S^2 \) branes until reaching the opposite pole \(-e\) where we have single point-like brane with the opposite charge \(-1\). Having identify \( k + 1 \) units of the charge with \(-1\) we arrive at the conclusion that the group of charges is \( \mathbb{Z}_{k+2} \).

More generally the charges of branes on the background \( X \) with non-vanishing \( H \in H^3(X, \mathbb{Z}) \) are described by the twisted K group \( K_H^*(X) \) (see e.g. [17]). In the case of \( SU(2) \) we get the group of RR charges as above for \( K = k + 2 \)

\[
K_H^*(S^3) = \mathbb{Z}_K
\]

Turning to the exotic \( \mathbb{R}^4 \) case based on [12], we have for a given nonzero integral class \( H \in H^3(S^3, \mathbb{Z}) \) exotic \( \mathbb{R}_H^4 \). And conversely, given this exotic \( \mathbb{R}_H^4 \) we recover the class \( H \in H^3(S^3, \mathbb{Z}) \). Certain topological conditions have to be fulfilled: the 3-sphere, i.e. \( SU(2) \), is seen as a part of the boundary of the Akbulut cork. It is the attachment of the Casson handle to the Akbulut cork, which determine the exotic \( \mathbb{R}_H^4 \). Thus we can correlate ambient exotic smoothness of \( \mathbb{R}_H^4 \) with the classes \( H \in H^3(S^3, \mathbb{Z}) \) provided \( S^3 \) is the part of the boundary of the Akbulut cork. Moreover, in [11] was shown that exotic smooth \( \mathbb{R}_H^4 \) deforms K-theory \( K(S^3) \) toward equivariant one \( K_H(S^3) \). Thus, we obtain the following important observation: certain small exotic \( \mathbb{R}_H^4 \)’s generate the group of RR charges of D-branes in the curved background of \( S^3 \subset \mathbb{R}^4 \).

Then we arrive at the correspondence:

**Theorem 3.1.**

The classification of RR charges of the branes on the background given by the group manifold \( SU(2) \) at the level \( k \) (hence the dynamics of D-branes in \( S^3 \) in stringy
regime) is correlated with the exotic smoothness on $\mathbb{R}^4$ containing this $S^3 = SU(2)$ as the part of the boundary of the Akbulut cork.

We can give yet another interpretation of the 4-exoticness which appears on flat $\mathbb{R}^4$ in this context. Exotic smoothness of $\mathbb{R}^4$, $\mathbb{R}^4_H$, determines the collection of stable D-branes on $SU(2)$ at the level $k$ of the WZW model, where $k = [H] \in H^3(S^3, \mathbb{Z})$. Thus, the string-finite $k$-level of the WZW model characterizes exotic 4-smoothness. Recall that in the case of $H = 0$ (e.g. $B$ constant in a flat space, i.e. in $k \to \infty$ limit) the smooth structure on $\mathbb{R}^4$ is the standard one,$^{12}$ Thus the exotic smoothness on $\mathbb{R}^4$ translates the 4-curvature to the non-zero H-field on $S^3$ of finite volume in string units. This is similar to the effect of string field equations relating $R$ and $H$ as in (1), though it holds now between different spaces ($\mathbb{R}^4$ and $S^3$).

3.2. $SU(2)$ WZW model in the geometry of the stack of NS5-branes

The manifold $SU(2) = S^3$ is the only group manifold which became relevant so far for the description of small exotic $\mathbb{R}^4$. From the other side it is the only one which appears directly as part of a string background (namely one generated by NS5-branes). That is why the connection of 4-exotics and string theory can be naturally formulated in the geometry of the stack of NS5-branes. Let us briefly describe this string background.$^{29,38,21}$

We consider a configuration of $k$ coincident supersymmetric NS5-branes in type II theory. The full fivebrane background is (in string frame)

$$
\begin{align*}
    ds^2 &= dx^2 + f(r)dy^2 \\
    e^{2\phi} &= g_s^2 f(r) \\
    f(r) &= 1 + \frac{k\alpha'}{r} \\
    H_{IJK} &= k\alpha' \epsilon_{IJK}
\end{align*}
$$

where $x$ are the $5 + 1$ longitudinal coordinates along NS5-branes referred to by indices $\mu, \nu, \text{etc.}$, $y$ being 4 transverse coordinates referred to by indices $I, J, K \ldots$ and $r = |y|, 1/\alpha' \sim$ string tension. The fields of this background are given by

$$
\begin{align*}
    e^{2\Phi} &= 1 + \sum_{j=1}^{k} \frac{l^2}{|y_j|^2} \\
    g_{IJ} &= e^{2\phi} \delta_{IJ} \\
    g_{\mu\nu} &= \eta_{\mu\nu} \\
    H_{IJK} &= -\epsilon_{IJKL} \partial^L \Phi
\end{align*}
$$

where $y_j, j = 1, \ldots, k$ are the positions of the NS5-branes. When the branes coincide at 0, $y_j = 0$, the near horizon solutions $y \to 0$, are
In the near-horizon limit \( r = |y|^2 \to 0 \), the background factorizes into a radial component as well in a \( S^3 \) and flat 6-dimensional Minkowski spacetime. Strings propagating at this limiting background are described by the exact world-sheet CFT with the target \( \mathbb{R}^5,1 \times \mathbb{R}_\phi \times S^3_k \). Here \( \mathbb{R}_\phi \) is the real line with the parameter \( \phi \) which is a scalar corresponding to the „linear dilaton”

\[
\Phi = \frac{\sqrt{12}}{2k} \phi
\]

\[\phi = \sqrt{\frac{2}{k}} \log \frac{1}{r} \]

The flat Minkowski space \( \mathbb{R}^5,1 \) is longitudinal to the directions of NS5-branes, \( S^3_k \) is \( SU(2)_k \) and is a level \( k \) WZW supersymmetric CFT (SCFT) on \( SU(2) \) as discussed in the previous section. This \( S^3 \) corresponds to the angular coordinates of the transversal \( \mathbb{R}^4 \). We see that the infinite „throat” \( \mathbb{R}_\phi \times S^3_k \), emerges with the metric of the background (in the string frame)

\[
ds^2 = dx_6^2 + d\phi^2 + kl_s d\Omega_3^2, g^2_s(\phi) = e^{-2\phi/\sqrt{\eta_s}}.
\]

This background is obtained in the near horizon geometry (i.e. \( \phi \to -\infty \) \( r \to 0 \)) of the stack of \( k - 2 \) NS5-branes in type II string theory and is in fact a SCFT on the throat. The NS5-branes are placed at \( \phi \to -\infty \) and string theory is in a strong-coupling regime, i.e. \( g_s \sim \exp(2\Phi) \). In the opposite limit \( \phi \to +\infty \), or \( r \to +\infty \) gives asymptotically flat 10-manifold and string theory is weakly coupled in that limit. This is essentially the CHS (Callan, Harvey, Strominger) exact string theory background where \( SU(2) \) WZW model appears at suitable level \( k \).

Given the CHS limiting geometry of \( N \) NS5-branes we have the 4-dimensional tube \( \mathbb{R}_\phi \times S^3 \). The volume of \( S^3 \) in string units is finite and correlated with the number of NS5-branes by \( N = k - 2 \). We take an exotic \( \mathbb{R}^4_H \) for \( [H] = k[] \in H^3(S^3, \mathbb{Z}) \). This can be achieved more directly by considering the Akbulut cork \( A_H \) with the boundary, \( \partial A_H = \Sigma_H \), a homology 3-sphere. As was shown in \( H \) \( \Sigma_H \) contains \( S^3 \) such that the codimension-1 foliations of it generates the foliations of \( \Sigma_H \). Expressed in a different way, this foliation is generated by the Casson handles attached to \( A \). Thus the Akbulut cork attached to the Casson handle (or an open neighborhood \( N(A) \) of the cork \( A \)) determines the small exotic smoothness of \( \mathbb{R}^4_H \). Two different exotic smoothness structures on the \( \mathbb{R}^4 \) are given by two non-cobordant codimension-one foliations of the 3-sphere. Moreover, the cobordism
classes of codimension-1 foliations of $S^3$ are classified by the Godbillon-Vey invariants which are elements of $H^3(S^3, \mathbb{R})$. In our case we deal with integral 3-rd cohomologies $[H] \in H^3(S^3, \mathbb{Z})$. Thus, the embedding of the Akbulut cork (determined by the Casson handle) in the ambient $\mathbb{R}^4$ is determined by the integral classes $k[\ ] \in H^3(S^3, \mathbb{Z})$. By using the diffeomorphism $\Sigma_H = \Sigma_H \# S^3$ we obtain a $S^3$ as part of the boundary $\Sigma_H$ of the Akbulut cork. By using the identification $S^3 = SU(2)$ in the context of the string background of $N$ NS5-branes we have the following result:

**Theorem 3.2.**

*In the geometry of the stack of NS5-branes in type II superstring theories, adding or subtracting a NS5-brane is correlated with the change of the smoothness structure on the transversal $\mathbb{R}^4$.*

Now the geometry of the tube $\mathbb{R}_\Phi \times S^3$ is defined with respect to the ambient standard $\mathbb{R}^4$. Interpreting the $S^3$ as part of the boundary of the Akbulut cork for some exotic smooth $\mathbb{R}^4_H$, the factor $S^3_k$ in the background is correlated with $\mathbb{R}^4_H$, $[H] = k[\ ] \in H^3(S^3, \mathbb{Z})$. Changing $k$ causes a change of smoothness for the ambient $\mathbb{R}^4$. As explained above, this smoothness is determined by the embedding of the Akbulut cork. However, the change of the smoothness of the transversal $\mathbb{R}^4$ affects the geometry of the tube. Thus the background $\mathbb{R}^{5,1} \times \mathbb{R}_\Phi \times SU(2)_k$ is correlated by the above topological arguments with another smooth geometry, namely $\mathbb{R}^{5,1} \times \mathbb{R}^4_H$ where we now interpret $\mathbb{R}^4_H$ as an exotic $\mathbb{R}^4$ transversal to $\mathbb{R}^{5,1}$. This is precisely the geometry which is sensitive to the number of NS5-branes as in Theorem 3.2. Thus string theory with CHS limiting geometry deals with the geometry of $\mathbb{R}^{5,1} \times \mathbb{R}^4_H$. However, possible correction terms in the DBI action for branes in such backgrounds can appear.

The construction of any smooth metric on an exotic $\mathbb{R}^4_H$ in explicit form is a very complicated mathematical task at present. Nevertheless string theory touches these intractable geometries and reflects their effects some brane configurations in the CHS limit. Thus string theory gives information relating exotic $\mathbb{R}^4$ regions of the background. Especially one should keep the following fact in mind: by working with an exotic $\mathbb{R}^4_H$ one has to forget the factorization like in the smooth CHS geometry, i.e. $\mathbb{R}_\Phi \times S^3$. This fact can be understood by considering the end of $\mathbb{R}^4$, i.e. this part of $\mathbb{R}^4$ which extends to infinity. The end of the $\mathbb{R}^4$ is $S^3 \times \mathbb{R}$. Then an exotic $\mathbb{R}^4$ has an exotic end $(S^3 \times \mathbb{R})_\gamma$. But $(S^3 \times \mathbb{R})_\gamma$ do not factorize (or is globally foliated by the leaves $S^3 \times \{t\}$ for all $t \in \mathbb{R}$) otherwise it is the end of the standard smooth $\mathbb{R}^4$. Thus we have the following correspondence: The change of exotic smoothness of $\mathbb{R}^4_H$ in $\mathbb{R}^{5,1} \times \mathbb{R}^4_H$ to the standard one, gives the factorization of the CHS limiting geometry. String theory in this limiting CHS geometry remembers from which $\mathbb{R}^4_H$ it was projected by the level $k$ of $SU(2)_k$ factor. Thus the information about the string background $\mathbb{R}^{5,1} \times \mathbb{R}_\Phi \times SU(2)_k$ is originally encoded purely geometrically as

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$a$ An end of a space $X$ is the limit of a sequence $U_1 \subset U_2 \subset \ldots$ of complements $U_\alpha = X \setminus K_\alpha$ where $K_1 \subset K_2 \subset \ldots$ is an ascending sequence of compact sets $K_\alpha$ whose interiors cover $X$. 


$\mathbb{R}^{5.1} \times \mathbb{R}^4_H$. To understand precisely this relation in analytical terms is possible. But we will present it in a separate paper, since this requires some mathematical work on exotic $\mathbb{R}^4_H$.

In that way we could have a theory producing some exact results of string theory. This theory would have an (exotic) 4-geometry (given by a foliation) on the Euclidean space $\mathbb{R}^4$ as the fundamental structure. The product factorization of the end $S^3 \times \mathbb{R}$ gives the standard smoothness and also the background of the string theory. Moreover, these (small) exotic $\mathbb{R}^4_H$ are all embeddable in standard $\mathbb{R}^4$ and carry string theory information about the exact string backgrounds and brane charges. Then we will obtain a full 4-dimensional description: exotic $\mathbb{R}^4_H$ embedded in $\mathbb{R}^4$ and projecting to the CHS string background of $k$ NS5-branes where $H = k[\cdot] \in H^3(S^3, \mathbb{Z})$. One goal of our future work is the collecting of arguments that such a theory could be considered as more fundamental than string theory itself. Instead we will propose here a general heuristic rule in our geometrical setting:

R1. Lifted D-branes probing the exotic geometry $\mathbb{R}^{5.1} \times \mathbb{R}^4_H$ are projected on to D-branes of type II string theory probing the factorized geometry, $\mathbb{R}^{5.1} \times \mathbb{R}_\phi \times SU(2)_k$ based on standard smooth product $\mathbb{R}_\phi \times S^3$. Again, the dependence of the calculations on $k$ is a direct consequence of the exoticness of $\mathbb{R}^4_H$. The projection is driven by the factorization of smooth $\mathbb{R}^4$ such that the topological product of axes becomes smooth.

„Lifted D-branes probing the exotic geometry $\mathbb{R}^{5.1} \times \mathbb{R}^4_H$” serves here merely as a description of the correspondence following Theorem 3.2. Furthermore we do not formulate any result depending on the actual existence of these D-branes. What is their exact status (in the lifted theory) will become evident when modified DBI action and solutions are presented.

Rule R1 is based on the observation in [12] that various nonstandard smoothings of $\mathbb{R}^4$ can be grasped by the effects of $H^3(S^3, \mathbb{Z})$ and that the factorization $\mathbb{R}_\phi \times S^3$ of the end gives the standard smooth $\mathbb{R}^4$. Following this rule we can consider many examples of D-branes in the above background (see e.g. [25, 31, 28, 35]), as referring to 4-exoticness. Let us discuss briefly the case of little string theory (LST) in this context.

Type II string theory on $\mathbb{R}^{5.1} \times \mathbb{R}_\phi \times SU(2)_k$ is given by the SCFT on the infinite „throat” of the background, i.e. $\mathbb{R}_\phi \times S^3$. Then this theory was proposed to be approachable via holography by using duality. The holographically dual theory is the 6-dimensional little string theory [28, 11]. LST has possible experimental signatures at the TeV scales after the compactification on the torus [9].

LST’s are non-local theories without gravity and can be described in the limit $g_s \rightarrow 0$ in the theory of $k$ NS5-branes. In this limit the bulk degrees of freedom decouple, hence gravity does. This 6-dimensional LST without gravity is holographically dual to the type II string theory formulated on the background $\mathbb{R}^{5.1} \times \mathbb{R}_\phi \times SU(2)_k$.

From the rule R1 above, it follows that LST refers also to exotic $\mathbb{R}^4_H$'s. However
the perturbative calculations are hardly performed in LST since the string coupling $g_s$ diverges in the dual string background along the tube, and LST is sensitive on this background. One usually regulates the geometry via chopping the tube. But the decomposition of the SCFT $SU(2)_k$ on $S^1_Y \times SU(2)_k/U(1)$ can be performed. Here $SU(2)_k/U(1)$ is the minimal $N = 2$ model at the level $k$ and $S^1_Y$ is the Cartan subalgebra of $SU(2)$ with the parameter $Y$. The dependence on $k$ is crucial in this reformulation since it refers to 4-exotics by theorem 3.2 and the rule R1. Thus we have the SCFT $\mathbb{R}_\phi \times S^1_Y \times SU(2)_k/U(1)$ instead of the tube $\mathbb{R}_\phi \times SU(2)_k$. The chopping of the strong coupling region is now performed by taking the SCFT $\mathbb{R}_\phi \times S^1_Y$ instead of $\mathbb{R}_\phi \times S^1_Y \times SU(2)_k$, which means replacing the background $\mathbb{R}^{5,1} \times \mathbb{R}_\phi \times SU(2)_k$ by $\mathbb{R}^{5,1} \times \mathbb{S}^3_{(2)} \times SU(2)_k$. This means, on the level of $k$ NS5-branes, the separation of these 5-branes along the transverse circle of radius $L$. Now the double-scale limit of LST is the one when taking both $g_s$ and $L$ to zero while $L/g_s$ remains constant.

Following 28 we can take systems of D4, D6-branes between separated NS5-branes. The various expressions like correlation functions can now be calculated perturbatively in the holographically dual 6-dimensional LST theory. Besides, suitable compactifications may refer to the spectra with the TeV scale of the standard model of particles. The dependence on $k$ of some of these expressions can be seen again as the signature of that these expressions were obtained by standard factorization and projection from non-factorized exotic 4-geometries.

**Exoticness of the 4-space transversal to the world-volume of NS5-branes, is reflected in specific perturbative spectra of D-branes when calculated in dual 6-dimensional LST. When compactifying this LST on 2 directions longitudinal to the 5-brane one gets spectra which could be sensitive on transversal exoticness of $\mathbb{R}^4$.**

This is in fact the reformulation of the rule R1. The NS5-branes backgrounds show that string theory computations „feel” the 4-exoticness. On the other hand, these 4-exotic regions contain information about branes in certain string backgrounds.

### 4. Discussion and conclusions

In this paper we tried to give a partial answer to the important question: Is it possible that string theory deals with 4-dimensional structures directly neither by implementing compactifications nor by phenomenological models-building, and these structures would have a physical meaning?

We propose that the structures are nonstandard smoothings of the Euclidean 4-space. Here we present the scenario where exotic smoothness on Euclidean $\mathbb{R}^4$ appears in string theory and is correlated with the charges, hence dynamics, of NS and D branes in certain string backgrounds. We gave topological reason for the above correspondence: when a WZW model on $SU(2)$ at the level $k$ appears in string theory as a part of exact string backgrounds then we will get a correspondence by placing this $SU(2) = S^3$ as a part of the boundary of the Akbulut cork. Thus exotic smoothness of the Euclidean $\mathbb{R}^4_k$ is determined by the 3-sphere localized at
the boundary of the Akbulut cork. In the CHS limiting geometry of the stack of $k$ NS5-branes we are able to show that there exists a 4-region of this solution which is projected from the unique exotic $\mathbb{R}^4_k$ via factorization. In that way the dependence on $k$ appears and the string solution remembers the original exoticness via this dependence. We conjecture, that a theory without projection should be considered as fundamental for string theory. In the CHS background the lifted theory describes small exotic $\mathbb{R}^4$’s as embedded in the standard $\mathbb{R}^4$. These data are projected on the factorized string background $\mathbb{R}^3,1 \times \mathbb{R}_\phi \times SU(2)_k$. Moreover, the close relation with string theory opens the possibility that exotic metrics on $\mathbb{R}^4_k$ could be derived from string theory calculations.

At present the construction of any metric for any exotic $\mathbb{R}^4$ is untractable. To recognize 4-exotic geometry from the point of view of string theory one could try to formulate a modified DBI action for D-branes in the lifted theory which would explore exotic 4-spaces, similarly as various D-branes do for transversal $\mathbb{R}_\phi \times SU(2)_k$. One possibility is the usage of D3-branes in this string background. These D3-branes fill the 3-space $\mathbb{R}_\phi \times S^2$ where 2-spheres are the conjugacy classes of $SU(2)$. Again, the number of allowed conjugacy classes depend on the level $k$ WZW model on $SU(2)$ whereas the standard smooth structure on $\mathbb{R}_\phi \times SU(2)$ follows from this factorization. It was proposed in [12] to consider some correlation functions between states in the WZW model, and use these states to characterize exotic 4-smoothness. One could also deal with a kind of superposition referring to quantum correlation functions. Or, one should consider wild embeddings of $S^2$ in $\mathbb{R}_\phi \times S^2$ which directly refers to exotic $\mathbb{R}^4$’s [12]. Then one loses the factorization, and, on the other hand, approaches the branes in string theory as a quantum object. These ideas will be the topic of our next paper.

Further, one could wonder whether exotic smooth $\mathbb{R}^4$’s are suitable objects when dealing with 4-dimensional physics: Are these structures relevant to physics at all? This important question was already answered, though partially in some research papers [20,18,7,9,11] and see also the textbook [10]. In particular in [12] we showed that exotic smoothness of an open 4-region in spacetime have the same effect as the existence of magnetic monopoles, i.e. exotic smoothness induces the quantization condition for the electric charge. By using [12] one finds many further arguments to consider the exotic $\mathbb{R}^4$’s as quantum object, i.e. the spacetime induces the quantization processes. The work on uncovering 4-dimensional physics from exotic $\mathbb{R}^4$’s by using the quantum-physical point of view is currently developed, which should be seen as complimentary to the string theory thread. Completing this work should give more essential understanding of the formalism of string theory as referring to 4-dimensional physics which is not covered by compactification. On the other hand our understanding of the phenomenon of exotic 4-smoothness on open manifolds has a chance to be broaden.
Acknowledgment

T.A. wants to thank C.H. Brans and H. Rosé for numerous discussions over the years about the relation of exotic smoothness to physics. J.K. benefited much from the explanations given to him by Robert Gompf regarding 4-smoothness several years ago, and discussions with Jan Śladkowski.

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