AN EXPLICIT POSITIVITY PRESERVING NUMERICAL SCHEME FOR CIR/CEV TYPE DELAY MODELS WITH JUMP

I. S. STAMATIOU

Abstract. We consider mean-reverting CIR/CEV processes with delay and jumps used as models on the financial markets. These processes are solutions of stochastic differential equations with jumps, which have no explicit solutions. We prove the non-negativity property of the solution of the above models and propose an explicit positivity preserving numerical scheme, using the semi-discrete method, that converges in the strong sense to the exact solution. We also make some minimal numerical experiments to illustrate the proposed method.

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1. Introduction

Throughout, let $T > 0$ and $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0 \leq t \leq T}, \mathbb{P})$ be a complete probability space, meaning that the filtration $\{\mathcal{F}_t\}_{0 \leq t \leq T}$ satisfies the usual conditions, i.e. is right continuous and $\mathcal{F}_0$ includes all $\mathbb{P}$–null sets. Let $W_{t,\omega} : [0, T] \times \Omega \to \mathbb{R}$ be a one-dimensional Wiener process adapted to the filtration $\{\mathcal{F}_t\}_{0 \leq t \leq T}$ and $\tilde{N}(t) = N(t) - \lambda t$ a compensated Poisson process with intensity $\lambda > 0$ independent of $W_t$. Consider the following stochastic delay differential equation (SDDE) with jump,

$$
(1.1) \quad x_t = \begin{cases} 
\xi_0 + \int_0^t (k_1 - k_2 x_{s^-}) ds + \int_0^t k_3 b(x_{s-}) x_s^\alpha dW_s + \int_0^t g(x_{s^-}) d\tilde{N}_s, & t \in [0, T], \\
\xi(t), & t \in [-\tau, 0],
\end{cases}
$$

where $x_{s^-} = \lim_{r \uparrow s} x_r$, the coefficient $b \in \mathcal{C}(\mathbb{R}_+, \mathbb{R}_+),$ $g : \mathbb{R} \to \mathbb{R}$ is the jump coefficient assumed deterministic for simplicity, the function $\xi \in \mathcal{C}([-\tau, 0], (0, \infty))$ and $\tau > 0$ is a positive constant which represents the delay. The quantities $k_i$ are positive constants and the real number $\alpha \in [1/2, 1)$. The diffusion coefficient does not satisfy standard assumptions, i.e. linear growth conditions, therefore classical results on the existence and uniqueness of solution and boundness of the moments as in [1] and [2] are not applicable here. Model (1.1) includes many models used in mathematical finance describing quantities such as interest rates and volatilities, see [3] and Table 1.

### Table 1. Models included in general model (1.1).

| Jump Coefficient $g(x)$ | Delay Coefficient $b(x)$ | Diffusion exponent $\alpha$ | Model/Ref |
|--------------------------|--------------------------|----------------------------|-----------|
| 0                        | 1                        | 1/2                        | CIR, [4]  |
| 0                        | $x^\gamma$               | 1/2                        | delay CIR, [5],[6] |
| $\delta x$              | 1                        | 1/2                        | CIR with jump, [7], [8] |
| $\delta x$              | $x^\gamma$               | 1/2                        | delay CIR with jump, [9], [10] |
| $\delta x, \delta \sin x, \delta x/(1 + x)$ | 1 | (1/2,1) | CEV, [11] |
| 0                        | 1                        | (1/2,1)                    | CEV, [11] |
| 0                        | $x^\gamma$               | (1/2,1)                    | delay CEV, |
| $\delta x$              | 1                        | (1/2,1)                    | CEV with jump, [12] |
| $\delta x, \delta \sin x, \delta x/(1 + x)$ | 1 | [1/2,1) | CIR/CEV with jump, [13] |

Numerical approximations of SDDEs with jumps like (1.1) are necessary for simulations of the paths $x_t(\omega)$, or for approximation of functions of $x(T)$ or functionals of the form $\mathbb{E} F(x)$, where $F : \mathcal{C}([0, T], \mathbb{R}) \to \mathbb{R}$, so as to obtain the expected payoff of an option. The numerical analysis of jump models become more computationally complex proportional to the jump intensity [14].

We assume the following conditions for the delay coefficient $b$ and the jump coefficient $g$.

**Assumption A** The delay diffusion coefficient $b(\cdot)$ is $\gamma$-Hölder continuous where $\gamma > 0$ i.e.

$$
\sup_{x < z} \frac{|b(x) - b(z)|}{(x - z)^\gamma} := C_\gamma < \infty,
$$

and the jump coefficient $g(\cdot)$ satisfies for a constant $L > 0$ the following relation $\forall x \in \mathbb{R}$

$$
\left( |g'(x)| < L \leq 1, \text{ and } g(0) = 0 \text{ OR } g(x) > 0 \text{ and } |g'(x)| < L \right)
$$

---

$^1\mathcal{C}(A, B)$ the space of continuous functions $\phi : A \to B$ with norm $\|\phi\| = \sup_{u \in A} \phi(u)$.
Moreover assume $\|\xi\| < A_\xi$ for a constant $A_\xi$.

Let $\mathcal{T}$ be a jump-adapted time partition of the interval $[-\tau, T]$, see Section 2 for details. We introduce the following numerical method for the approximation of the solution of (1.1), for the jump-extended CIR/CEV model with delay, which we call Jump Adapted Semi-Discrete method (JASDM) with $y_t = \xi(t)$ when $t \in [-\tau, 0]$ and for $k = 0, 1, \ldots, n_T - 1$,

\[
\begin{align*}
y_{k+1}^- &= (z_{k+1})^2, \\
y_{k+1} &= y_{k+1}^- + g(y_{k+1}^-)\Delta \tilde{N}_k,
\end{align*}
\]

where

\[
z_t = \sqrt{y_{k}} \left(1 - \frac{k_2 \Delta_k}{1 + k_2 \theta \Delta_k} + \frac{k_1 \Delta_k}{1 + k_2 \theta \Delta_k} - \frac{(k_3)^2}{4(1 + k_2 \theta \Delta_k)^2 (1 + b(y_{k-\tau})\Delta_k^m)} (y_{k})^{2\alpha - 1} \Delta_k \right) \\
+ \frac{k_3}{2(1 + k_2 \theta \Delta_k)} \frac{b(y_{k-\tau})}{1 + b(y_{k-\tau})\Delta_k^m} (y_{k})^{\alpha - \frac{1}{2}} (W_t - W_{t_k})
\]

and $\Delta_k = t_{k+1} - t_k$, $\Delta \tilde{N}_k := \tilde{N}(t_{k+1}) - \tilde{N}(t_k) = \Delta N_k - \lambda \Delta_k$ and $\theta \in [0, 1]$ represents the level of implicitness. The positive constant $m$ is chosen to be equal to $1/4$, see proof of Proposition 4.12. The increment $\Delta N_k = 1$ when $t_{k+1}$ is a jump time and zero otherwise. For the derivation of (1.2) briefly saying we apply an analogue of the semi-discrete method [15] (and [16] for the case $\alpha = 1/2$) for the numerical approximation of the SDDE (1.1) without jump, that is in between jump times with the appropriate modification when the jump times occur, see Section 2 for details.

The numerical scheme (1.2) is positivity preserving by construction and converges to the true solution $x_t$ of model (1.1) and this is stated in our main result, Theorem 2.8.

A basic feature of the semi-discrete method, originally proposed in [17], is that it is explicit, converges strongly in the mean square sense to the exact solution of the original SDE, preserves positivity (see application in Wright-Fisher model [18]), and it does not explode in some super-linear problems [19], [20]. The purpose of this paper is on the one hand to propose the general model (1.1) and to generalize the semi-discrete method to include SDDEs with jumps that arise in the area of financial mathematics in valuating options.

There has been a lot of effort in numerical approximations of models of type (1.1) mainly in cases where no jumps are included, and we simply refer to Table 1 and references therein. The main difficulty in producing strongly converging numerical schemes, as already stated, is the non-Lipschitz nature of the coefficients. The recent focus on positivity preserving methods for these problems can be stated in [13], [6].

In Section 2 we collect all the main results. First, we investigate the nonnegative solutions of (1.1) and the mean-reversion property. Then we examine properties of the proposed numerical scheme (1.2), such as strong convergence, boundness of moments and the mean-reversion property. Section 3 gives some numerical examples confirming theoretical results. Section 4 is devoted to the proofs of all the results and in the last subsection we provide some concluding remarks.

2. Setting and main results

Our first goal, is to provide the well-posedness of (1.1), that is the existence and uniqueness of solution of the CIR/CEV model with delay, as well as the positivity of the solution process. Instead of proving existence, uniqueness and non-negativity of solution of (1.1) directly, we
treat the following SDDE with jump

\begin{equation}
    x_t = \begin{cases} 
        \xi_0 + \int_0^t (k_1 - k_2 x_s) ds + \int_0^t k_3 b(|x_{s-}|)|x_{s-}|^\alpha dW_s + \int_0^t g(x_s) d\tilde{N}_s, & t \in [0, T], \\
        \xi(t), & t \in [-\tau, 0]. 
    \end{cases}
\end{equation}

We first show that (2.3) has finite \( p \)-moments.

**Lemma 2.1** [Moment bounds for (2.3)] Let \( x_t \) be the solution of (2.3). Then for any \( p > 0 \) there exists a positive constant \( A \) such that

\begin{equation}
    \mathbb{E} \sup_{-\tau \leq t \leq T} |x_t|^p \leq A.
\end{equation}

We then examine the uniqueness and positivity of (2.3).

**Proposition 2.2** [Uniqueness and positivity of (2.3)] There exists a unique and nonnegative solution for equation (2.3), i.e. \( x_t > 0 \) almost surely (a.s.).

Using the previous results we easily reach the following theorem which is our first main result. We also provide the proof since it is short.

**Theorem 2.3** [Existence, Uniqueness and positivity of (1.1)] There exist a unique positive solution for the jump-extended CIR/CEV model (1.1) and the \( p \)-th moment of the solution is bounded.

**Proof of Theorem 2.3.** It is an immediate consequence of Lemma 2.1 and Proposition 2.2 since when \( x \) positive \( |x| = x \).

We also give inverse moment bounds of \( (x_t) \) which will be used in the proof of our main convergence results, Theorems 2.7 and 2.8.

**Lemma 2.4** [Inverse Moment bounds for (1.1)] Let \( x_t \) be the solution of (1.1), then for any \( p < 0 \) there exists a positive constant \( \hat{A} \) such that

\begin{equation}
    \sup_{-\tau \leq t \leq T} \mathbb{E}(x_t)^p \leq \hat{A}.
\end{equation}

We now examine the mean reversion property which holds for model (1.1) with no delay and jump, see [7, Theorem 2.2].

**Theorem 2.5** [Mean-reversion] The jump-extended CIR/CEV model (1.1) preserves the mean reversion property, i.e.

\begin{equation}
    \lim_{t \to \infty} \mathbb{E}x_t = \frac{k_1}{k_2}.
\end{equation}

**Proof of Theorem 2.5.** We take expectations on both sides of (1.1) to get

\[ \mathbb{E}x_t = \mathbb{E}x_0 - \int_0^t (k_2 \mathbb{E}x_s - k_1) ds, \]
where we also used the boundedness of the moments of \((x_t)\) and the fact that \((N_s)\) is a martingale. Now, let \(\psi_t := \mathbb{E}x_t\). Then \(\psi_t\) satisfies the following integral equation

\[
\psi_t = \psi_0 - \int_0^t (k_2\psi_s - k_1)ds,
\]

or \((\psi_t)' + k_2\psi_t = k_1\) and multiplying by \(e^{k_2t}\) each side of the equality implies \(\psi_t = \frac{k_1}{k_2} + (\mathbb{E}\xi_0 - \frac{k_1}{k_2}) e^{-k_2t}\). We therefore verify (2.6).

We rewrite the stochastic integral equation (1.1), denoting by a superscript \(g\) the dependence on the jump coefficient as

\[
x_t^g = \begin{cases} 
\xi_0 + \int_0^t (k_1 - k_2x_s)ds + \int_0^t k_3b(x_{s-\tau})x_s^\alpha dW_s + \int_0^t g(x_s-)d\tilde{N}_s, & t \in [0,T], \\
\xi(t), & t \in [-\tau,0].
\end{cases}
\]

and in a similar way we denote the semi-discrete scheme \(y_t^g\). Let the horizon \(T\) be a multiple of \(\tau\), i.e. \(T = N_0\tau\), where \(N_0 \in \mathbb{N}\) and discretize the interval \([-\tau,T]\) with equidistant steps of size \(\Delta = \tau/l\) for some \(l \in \{2,3,\ldots\}\) and \(t_n = n\Delta = n\tau/l\), where \(n = -l,-l+1,\ldots,N\). Now, construct a jump-adapted time partition

\[
\mathcal{T} = \{-\tau = t_{-l} < t_{-l+1} < \ldots < 0 = t_0 < t_1 < \ldots < t_{N_{\tau}} = T\},
\]

produced by a superposition of the jump times to the deterministic equidistant grid. Here, \(N_t = \max\{n : t_n \leq t\}\) for \(t \leq T\) and \(\mathbb{P}(t_{n+1} - t_n \leq \Delta) = 1\). The discretization \(\mathcal{T}\) is path-dependent with maximum step-size \(\Delta\). For \(t \in [t_k,t_{k+1})\) and \(k \geq 0\) it holds

\[
\begin{cases}
    x_{t_{k+1}} = x_{t_k} + \int_{t_k}^{t_{k+1}} (k_1 - k_2x_s)ds + \int_{t_k}^{t_{k+1}} k_3b(x_{s-\tau})x_s^\alpha dW_s, \\
    x_{t_{k+1}} = x_{t_{k+1}^-} + g(x_{t_{k+1}^-})\Delta \tilde{N}_k,
\end{cases}
\]

where \(\Delta \tilde{N}_k := N((k+1)\Delta) - N(k\Delta) - \lambda \Delta\) are the compensated Poisson increments. Between jump times (1.1) evolves like a delay CIR/CEV process without jumps. Theorem 2.3 implies that also the delay CIR/CEV model with no jumps has a unique nonnegative solution, i.e. \(x_t^0 \geq 0\) a.s. Therefore, the idea is to approximate the process between jump times with a positivity preserving numerical method and then take into account the jumps. In the following we will use the ideas in [13] where the mean-reverting CIR/CEV jump process was considered with no delay.

Let

\[
f_0(x,y) = k_1 - k_2(1-\theta)x - \frac{(k_3)^2}{4(1 + k_2\theta\Delta)} \left( \frac{b^2(z)}{1 + b(z)\Delta m} \right) x^{2\alpha-1} - k_2\theta y 
\]

\[
+ \frac{(k_3)^2}{4(1 + k_2\theta\Delta)} \left( \frac{b^2(z)}{1 + b(z)\Delta m} \right) x^{2\alpha-1}
\]

\[
(2.7)
\]

and

\[
g(x,y,z) = k_3 \frac{b(z)}{1 + b(z)\Delta m} x^{\alpha - \frac{1}{2}} \sqrt{y},
\]

\[
(2.8)
\]
where \( f_\theta(x, x, x) = k_1 - k_2 x \) and \( g(x, x, z) = k_3 \frac{b(z)}{1 + k(z)\Delta m} x^\alpha \).

Consider the following process

\[
y_t = y_{t_n} + f_1(y_{t_n}, y_{t}, y_{t_n}) \cdot \Delta + \int_{t_n}^t f_2(y_{t_n}, y_{t_n-r})ds + \int_{t_n}^t \text{sgn}(z_s)g(y_{t_n}, y_{s}, y_{t_n-r})dW_s,
\]

with \( y_0 = \xi_0 \) a.s. or more explicitly

\[
y_t = y_{t_n} + \left( k_1 - k_2 (1 - \theta) y_{t_n} - \frac{(k_3)^2}{4(1 + k_2 \theta \Delta)} \frac{b^2(y_{t_n-r})}{(1 + b(y_{t_n-r})\Delta m)^2} (y_{t_n})^{2\alpha - 1} - k_2 y_{t_n} \right) \cdot \Delta + \int_{t_n}^t \frac{(k_3)^2}{4(1 + k_2 \theta \Delta)} \frac{b^2(y_{t_n-r})}{(1 + b(y_{t_n-r})\Delta m)^2} (y_{t_n})^{2\alpha - 1} ds + \int_{t_n}^t \text{sgn}(z_s)\sqrt{y_s}dW_s,
\]

(2.9)

for \( t \in (t_n, t_{n+1}] \), where \( \theta \in [0, 1] \) represents the level of implicitness and

\[
z_t = \sqrt{y_n} + \frac{k_3}{2(1 + k_2 \theta \Delta)} \frac{b(y_{t_n-r})}{1 + b(y_{t_n-r})\Delta m} (y_{t_n})^{\alpha - 1/2} (W_t - W_{t_n}),
\]

with

(2.10)

\[
y_n := y_{t_n} \left( 1 - \frac{k_2 \Delta}{1 + k_2 \theta \Delta} \right) + \frac{k_1 \Delta}{1 + k_2 \theta \Delta} - \frac{(k_3)^2}{4(1 + k_2 \theta \Delta)^2} \frac{b^2(y_{t_n-r})}{(1 + b(y_{t_n-r})\Delta m)^2} (y_{t_n})^{2\alpha - 1} \Delta.
\]

Note that \( y_n = J_n/(1 + k_2 \theta \Delta) \) where

\[
J_n = \left( y_{t_n} (1 - k_2 (1 - \theta) \Delta) + k_1 \Delta - \frac{(k_3)^2}{4(1 + k_2 \theta \Delta)} \frac{b^2(y_{t_n-r})}{(1 + b(y_{t_n-r})\Delta m)^2} (y_{t_n})^{2\alpha - 1} \Delta \right) \mathbb{I}_{y_n \geq 1}
\]

\[
+ \left( y_{t_n} (1 - k_2 (1 - \theta) \Delta) + k_1 \Delta - \frac{(k_3)^2}{4(1 + k_2 \theta \Delta)} \frac{b^2(y_{t_n-r})}{(1 + b(y_{t_n-r})\Delta m)^2} (y_{t_n})^{2\alpha - 1} \Delta \right) \mathbb{I}_{y_n < 1}
\]

\[
\geq y_n^{2\alpha - 1} \left( 1 - k_2 (1 - \theta) \Delta - \frac{(k_3)^2}{4(1 + k_2 \theta \Delta)} \sqrt{\Delta} \right) \mathbb{I}_{y_n \geq 1}
\]

\[
+ \left( y_{t_n} (1 - k_2 (1 - \theta) \Delta) - \frac{(k_3)^2}{4(1 + k_2 \theta \Delta)} \sqrt{\Delta} (y_{t_n})^{2\alpha - 1} \right) \mathbb{I}_{\Delta^{1/4} y_n < 1}
\]

\[
+ \left( y_{t_n} (1 - k_2 (1 - \theta) \Delta) + k_1 \Delta - \frac{(k_3)^2}{4(1 + k_2 \theta \Delta)} \sqrt{\Delta} (y_{t_n})^{2\alpha - 1} \right) \mathbb{I}_{y_n \leq \Delta^{1/4}}
\]

\[
\geq 1 - k_2 (1 - \theta) \Delta - \frac{(k_3)^2}{4(1 + k_2 \theta \Delta)} \sqrt{\Delta}
\]

\[
y_{t_n} \left( 1 - k_2 (1 - \theta) \Delta - \frac{(k_3)^2}{4(1 + k_2 \theta \Delta)} \Delta^{1/4} \right) \mathbb{I}_{\Delta^{1/4} y_n < 1}
\]

\[
+ \left( y_{t_n} (1 - k_2 (1 - \theta) \Delta) + k_1 \Delta - \frac{(k_3)^2}{4(1 + k_2 \theta \Delta)} \Delta^{(2\alpha + 1)/4} \right) \mathbb{I}_{y_n \leq \Delta^{1/4}}.
\]

and \( m \leq 1/4 \) to be specified later. In the above inequality, the first term is nonnegative when \( \Delta < (k_2 (1 - \theta) + (k_3)^2/4)^{-2} \wedge (k_2 (1 - \theta))^{-1} \) the second term is nonnegative when
\[ \Delta < (k_2(1 - \theta) + (k_3)^2/4)^{-1} \] and for the last term we have

\[ \left( y_{t_n} (1 - k_2(1 - \theta) \Delta) + k_1 \Delta - \frac{(k_3)^2}{4(1 + k_2 \theta \Delta)} \Delta^{(2\alpha + 1)/4} \right) \mathbb{I}_{y_{t_n} \leq \Delta^{1/4}} \]

\[ \geq \left( y_{t_n} \frac{(k_3)^2}{4} + k_1 \Delta - \frac{(k_3)^2}{4(1 + k_2 \theta \Delta)} \sqrt{\Delta} \right) \mathbb{I}_{y_{t_n} \leq \Delta^{1/4}} \geq 0, \]

when \( \Delta < \frac{1 - (k_3)^2/4}{k_2(1 - \theta)}. \)

Furthermore, (2.9) has jumps at nodes \( t_n \). Solving for \( y_t \), we end up with the following explicit scheme

\[
y_t = y_n + \int_{t_n}^t \frac{(k_3)^2}{4(1 + k_2 \theta \Delta)^2} \frac{b^2(y_{t_n} - \tau)}{(1 + b(y_{t_n} - \tau) \Delta m)(y_{t_n})^{2\alpha - 1}} ds \]

\[ + \frac{k_3}{1 + k_2 \theta \Delta} \frac{b(y_{t_n} - \tau)}{1 + b(y_{t_n} - \tau) \Delta m(y_{t_n})^{\alpha - \frac{1}{2}}} \int_{t_n}^t \text{sgn}(z_s) \sqrt{\gamma_s} dW_s, \]

with solution in each step given by [21, (4.39), p.123]

\[ y_t = (z_t)^2. \]

We remove the term \( \text{sgn}(z_s) \) from (2.9) by considering the process

\[ \tilde{W}_t = \int_0^t \text{sgn}(z_s) dW_s, \]

which is a martingale with quadratic variation \( <\tilde{W}_t, \tilde{W}_t> = t \) and thus a standard Brownian motion w.r.t. its own filtration, justified by Lévy’s theorem [22, Th. 3.16, p.157]. Therefore, the compact form of (2.9) becomes

\[
y_t = \xi_0 + \int_0^t (k_1 - k_2(1 - \theta) y_s - k_2 \theta y_s) ds \]

\[ + \int_{t_n+1}^t (k_1 - k_2(1 - \theta) y_n - \frac{(k_3)^2}{4(1 + k_2 \theta \Delta)^2} \frac{b^2(y_{t_n} - \tau)}{(1 + b(y_{t_n} - \tau) \Delta m)(y_{t_n})^{2\alpha - 1}} - k_2 \theta y_t) ds \]

\[ + k_3 \int_0^t \frac{b(y_{t_n} - \tau)}{1 + b(y_{t_n} - \tau) \Delta m(y_{t_n})^{\alpha - \frac{1}{2}}} \sqrt{\gamma_s} d\tilde{W}_s, \]

for \( t \in (t_n, t_{n+1}] \) where

\[ s = t_j, s \in (t_j, t_{j+1}], j = 0, \ldots, n, \quad \tilde{s} = \begin{cases} t_{j+1}, & \text{for } s \in [t_j, t_{j+1}], j = 0, \ldots, n - 1. \\ t, & \text{for } s \in [t_n, t]. \end{cases} \]

Consider also the process

\[
\tilde{x}_t = \xi_0 + \int_0^t (k_1 - k_2 \tilde{x}_s) ds + \int_0^t k_3 b(\tilde{x}_{s-\tau})(\tilde{x}_s)^{\alpha} d\tilde{W}_s, \quad t \in [0, T],
\]

with \( \tilde{x}_t = \xi(t), t \in [-\tau, 0]. \) The process \( (x'_t) \) and the process \( (\tilde{x}_t) \) of (2.13) have the same distribution. We show in the following that \( \mathbb{E} \sup_{0 \leq t \leq T} |y_t - \tilde{x}_t|^2 \to 0 \) as \( \Delta \downarrow 0 \) thus \( \mathbb{E} \sup_{0 \leq t \leq T} |y_t - x_t|^2 \to 0 \) as \( \Delta \downarrow 0. \) To simplify notation we write \( \tilde{W}, (\tilde{x}_t) \) as \( W, (x_t) \). We end
up with the following explicit scheme
\[ y_{t_0} = y_n + \int_{t_n}^t \left( \frac{(k_3)^2}{4(1 + k_2 \theta \Delta)} \frac{b^2(y_{t_n} - \tau)}{(1 + b(y_{t_n} - \tau) \Delta \Delta m)^2} (y_{t_n})^{2 \alpha - 1} ds \right) + \frac{k_3}{1 + k_2 \theta \Delta} \frac{b(y_{t_n} - \tau)}{(1 + b(y_{t_n} - \tau) \Delta \Delta m)^{\alpha - \frac{1}{2}}} \int_{t_n}^t \sqrt{y_s} dW_s, \]
(2.14)

where \( y_n \) is as in (2.11).

The following result shows that the (JASDM) method preserves the mean-reversion property for appropriate step size \( \Delta \) and implicitness parameter \( \theta \).

**Theorem 2.6** [Mean-reversion Property] The (JASDM) method (1.2) preserves the mean-reversion property in the following sense,
\[ \text{lim}_{n \to \infty} \mathbb{E}[y_{t_n}] \leq \frac{k_1}{k_2 \theta}, \]
(2.15)
when \( \Delta(1 - \theta) < 1/k_2 \) and \( 0 < \theta \leq 1 \).

**Assumption B** Let \( \Delta > 0 \) be such that \( \Delta < \left( \frac{1}{k_2(1 - \theta)(k_3)^2/4} \right) \wedge \left( \frac{1}{k_2(1 - \theta)(k_3)^2/4} \right) \wedge \frac{4 - (k_3)^2}{4k_2(1 - \theta)} \) for \( \theta \in [0, 1] \).

In the following we show the strong convergence of the semi-discrete method to the true solution of model (1.1) without jumps.

**Theorem 2.7** [Strong convergence when no jumps are included] Suppose Assumptions A and B hold. Then the delay semi-discrete method \( y_{t_n}^0 \), with \( m = 1/4 \), converges in the mean square sense to the true solution of the delay CIR/CEV model, i.e. (1.1) with \( g \equiv 0 \), with order of convergence given by
\[ \mathbb{E} \sup_{0 \leq t \leq T} |y_{t_n}^0 - x_t^0|^2 \leq \begin{cases} C \Delta^{(\alpha - 1/2)\wedge \gamma}, & \alpha \in (1/2, 1), \\ C \Delta^{\gamma(1/2)}, & \alpha = 1/2, \end{cases} \]
(2.16)
where \( C \) is a positive constant independent of \( \Delta \).

The (JASDM) method reads
\[ y_{t_{k+1}} = y_{t_{k+1}} - g(y_{t_{k+1}}) \Delta \tilde{N}_k \]
\[ = y_{t_{k+1}} - g(y_{t_{k+1}})(\mathbb{I}_{\{t_{k+1} \text{ is a jump time}\}} - \lambda \Delta_k), \]
where \( y_{t_{k+1}} \) approximates \( x_{t_{k+1}} \). Assumption A implies for any \( r > 0 \),
\[ r > g(r)(1 - \lambda \Delta_k) \geq r - (1 - \lambda \Delta_k)Lr > r(1 - L) > 0, \]
when \( L < 1 \) and \( r - g(r)\lambda \Delta_k \geq r(1 - \lambda L \Delta_k) > 0 \), for any \( \Delta < (\lambda L)^{-1} \). In case \( g(\cdot) > 0 \)
\[ r + g(r)(1 - \lambda \Delta_k) > g(r)(1 - \lambda \Delta_k) > 0, \]
for any \( \Delta < \lambda^{-1} \). Therefore, \( y_{t_{k+1}} > 0 \) a.s. when \( \Delta < \lambda^{-1}(L^{-1} \wedge 1) \).

The results of Theorem 2.7 are valid in the case jumps are included in the model and this is stated in the following result.

**Theorem 2.8** [Strong convergence of (JASDM) to (1.1)] Suppose Assumption A and B hold and \( \Delta < \lambda^{-1}(L^{-1} \wedge 1) \). Then the jump adapted semi-discrete method (1.2) converges in the
mean square sense to the true solution of the jump-extended CIR/CEV model (1.1) with order of convergence given by

\begin{equation}
\mathbb{E} \sup_{0 \leq t \leq T} |y_t^\theta - x_t^\theta|^2 \leq \begin{cases} 
C \Delta^{(\alpha-1/2)\wedge \gamma}, & \alpha \in (1/2, 1), \\
C \Delta^{\gamma \wedge (1/2)}, & \alpha = 1/2,
\end{cases}
\end{equation}

where $C$ is a positive constant independent of $\Delta$. $\square$

3. Numerical illustration

In this section, we present some numerical results that illustrate the positivity preserving and the order of strong convergence of (JASDM). We examine models with coefficients as in [13, Sec. 4] where CIR/CEV jump models are considered with no delay. We consider the following SDEs with jumps,

\begin{equation}
\begin{aligned}
&dx_t = (k_1 - k_2x_t^-)dt + k_3b(x_{t-})^dW_t + g(x_{t-})d\tilde{N}_t, \quad t \in [0, 1], \\
x_t = \xi(t), \quad t \in [-1, 0],
\end{aligned}
\end{equation}

that is model (1.1) with parameters from Table 2. Let $\mathcal{T}$ be a jump-adapted time partition of the interval $[-1, 1]$. The (JASDM) method for $\theta = 1/2$ reads $y_t = \xi(t)$ when $t \in [-1, 0]$ and for $k = 0, 1, \ldots, n_{\mathcal{T}} - 1$, see (1.2)

\begin{equation}
\begin{cases}
y_{t+k}^- = (z_{t+k})^2, \\
y_{t+k} = y_{t+k}^- + g(y_{t+k}^-)(\Delta N_k - \Delta k),
\end{cases}
\end{equation}

where

\begin{align*}
z_t &= \sqrt{y_t \left(1 - \frac{2k_2\Delta_k}{2 + k_2\Delta_k}\right) + \frac{2k_1\Delta_k}{2 + k_2\Delta_k} - \frac{(k_3^2}{(2 + k_2\Delta_k)^2}}(1 + b(y_{t,k-1})\Delta_k^{1/4})^2(y_{t,k})^{2\alpha - 1}\Delta_k \\
&\quad + \frac{k_3}{1 + k_2\Delta_k} \frac{b(y_{t,k-1}) + b(y_{t,k-1})\Delta_k^{1/4}}{(y_{t,k})^{\alpha - 1/2}}(W_t - W_{t,k})
\end{align*}

and $\Delta_k = t_{k+1} - t_k$. First, we plot two one-path simulations of (3.18) for different parameters of Table 2. We use (JASDM) with $\Delta = 2^{-14}$ as the exact solution, see Figures 1, 2, 3 and 4.

We estimate the endpoint $L^2$-norm $\epsilon = \sqrt{\mathbb{E}|y(T) - x_T|^2}$, of the difference between the (JASDM) method at step size $\Delta$ and the exact solution of (3.18). We compute $M$ batches of $L$ simulation paths, where each batch is estimated by $\hat{\epsilon}_j = \frac{1}{L} \sum_{i=1}^L |y_{i,j}(T) - y_{i,j}^{(\text{ref})}(T)|^2$ and the Monte Carlo estimator of the error is

\begin{equation}
\hat{\epsilon} = \sqrt{\frac{1}{ML} \sum_{j=1}^M \sum_{i=1}^L |y_{i,j}(T) - y_{i,j}^{(\text{ref})}(T)|^2}
\end{equation}
and requires \( M \cdot L \) Monte Carlo sample paths. We also use \( M \cdot L \) Poisson paths. The reference solution is calculated using the method at a very fine time grid, \( \Delta = 2^{-14} \). We have shown in Theorem 2.8 that the (JASDM) converges strongly to the exact solution, so we use the (JASDM) method as a reference solution. We simulate \( 50 \cdot 100 = 5000 \) paths, where the choice of the number of Monte Carlo paths is adequately large, so as not to significantly hinder the mean-square errors. We compute the approximation error (3.20) and present the results in Table 3. We also give a graphical illustration for the delay CIR jump models in a \( \log_2 \)-\( \log_2 \) scale in Figure 5 and for the delay CEV jump models in Figure 6.

Assumption B requires \( \Delta < 0.42 \land 0.18 \land 0.64 = 0.18 \) therefore step \( \Delta = 2^{-5} \) is sufficient. Moreover, \( \lambda = L = 1 \) therefore the other requirement for the step size reads \( \Delta < 1 \), see Theorem 2.8. For the delay CIR jump model with \( \gamma = 1 \), the convergence rate should be at least 1/4 and for the CEV type models with \( (\alpha, \gamma) = (0.7, 1/2) \) and \( (\alpha, \gamma) = (0.9, 1/2) \) the convergence rate should be at least 0.1 and 0.2 respectively.

4. Proofs

In this section we provide the proofs of all the results of Section 2.
Proof of Lemma 2.1. We define for any \( k > 0 \) the stopping time
\[
\tau_k := \inf \{ t \in [0, T] : |x_t| > k \}
\]
for the stochastic process \( x_t \) of (2.3), with the convection that \( \inf \emptyset = \infty \). The Hölder inequality and the Burkholder-Davis-Gundy inequality, see [3], imply for any \( t_1 \in [0, T] \) and
\[ k_1 = 0.24, k_2 = 3, k_3 = 0.4, \alpha = 0.5, \gamma = 1, \text{Ref. SD at N} = 16384 \]

(A) SET I

\[ k_1 = 2, k_2 = 2, k_3 = 1.5, \alpha = 0.5, \gamma = 1, \text{Ref. SD at N} = 16384 \]

(B) SET II

Figure 5. Error and step size for (JASDM) for CIR model (3.18) with \( \gamma = 1 \).

(A) \( \text{Case } (\alpha, \gamma) = (0.7, 1/2) \).

(B) \( \text{Case } (\alpha, \gamma) = (0.9, 1/2) \).

Figure 6. Error and step size for (JASDM) for CEV model (3.18) for different \( \alpha \).

\[ p > 2, \]

\[ \mathbb{E} \sup_{0 \leq t \leq t_1} |x_{t \wedge T_k}|^p \leq 4^p \mathbb{E} |\xi_0|^p + 4^{p-1} \mathbb{E} \left( \int_0^{t_1 \wedge T_k} \left| k_1 - k_2 x_s \right|^p ds \right) \]

\[ + 4^{p-1} (k_3)^p \mathbb{E} \left( \sup_{0 \leq t \leq t_1} \left| \int_0^{t \wedge T_k} b(|x_{s-}|) |x_s|^\alpha dW_s \right|^p \right) \]

\[ + 4^{p-1} \mathbb{E} \left( \sup_{0 \leq t \leq t_1} \left| \int_0^{t \wedge T_k} g(x_s) d\tilde{N}_s \right|^p \right) \]

\[ \leq 4^p \mathbb{E} |\xi_0|^p + 4^{p-1} T^{p-1} \mathbb{E} \int_0^{t_1 \wedge T_k} \left| k_1 - k_2 x_s \right|^p ds \]

\[ + 4^{p-1} (k_3)^p C_p \mathbb{E} \left( \int_0^{t_1 \wedge T_k} b(|x_{s-}|)^2 |x_s|^{2\alpha} ds \right)^{p/2} \]

\[ + 4^{p-1} C_{p, \lambda} \mathbb{E} \left( \int_0^{t_1 \wedge T_k} |g(x_s)|^p ds \right), \]  

(4.21)
where \( C_p, C_{p, \lambda} \) are constants depending on \( p \) and \( p, \lambda \) respectively. The first integral in (4.21) is bounded in the following way

\[
(4.22) \quad \mathbb{E} \int_0^{t_{1} \wedge \tau_k} |k_1 - k_2 x_{s-}|^p ds \leq 2^{p-1} (k_1)^p T^p + 2^{p-1} (k_2)^p \int_0^{t_{1}} \mathbb{E} |x_{s- \wedge \tau_k}|^p ds.
\]

The third integral in (4.21) can be estimated using Assumption A, as

\[
(4.23) \quad \mathbb{E} \left( \int_0^{t_{1} \wedge \tau_k} |g(x_{s-})|^p ds \right) \leq 2^{p-1} L^p \mathbb{E} \int_0^{t_{1}} \mathbb{E} |(x_{s- \wedge \tau_k})|^p ds + 2^{p-1} |g(0)|^p T.
\]

Moreover, by the elementary inequality \( u^{1-\beta} v^\beta \leq (1-\beta) u + \beta v \), for \( u, v \geq 0 \) and \( 0 \leq \beta < 1 \) and another application of Hölder’s inequality we get

\[
\mathbb{E} \left( \int_0^{t_{1} \wedge \tau_k} b(|x_{s-}|^2) |x_{s-}|^{2\alpha} ds \right)^{p/2} \leq \mathbb{E} \left( \int_0^{t_{1} \wedge \tau_k} (1-\alpha) b(|x_{s-}|)^{2/(1-\alpha)} + \alpha |x_{s-}|^2 ds \right)^{p/2} \leq 2^{\frac{p}{2}} \mathbb{E} \left( \int_0^{t_{1} \wedge \tau_k} (1-\alpha) b(|x_{s-}|)^{2/(1-\alpha)} ds \right)^{p/2} + \mathbb{E} \left( \int_0^{t_{1} \wedge \tau_k} \alpha |x_{s-}|^2 ds \right)^{p/2} \leq 2^{\frac{p}{2}} \mathbb{E} \left( \int_0^{t_{1} \wedge \tau_k} b(|x_{s- \wedge \tau_k}|)^{p/(1-\alpha)} ds \right) + 2^{\frac{p}{2}} \alpha T^{\frac{p}{2}} \int_0^{t_{1}} \mathbb{E} |x_{s- \wedge \tau_k}|^p ds \leq C + (2T(1-\alpha))^{\frac{p}{2}} \int_0^{t_{1}} \mathbb{E} |x_{s- \wedge \tau_k}| \mathbb{E} |x_{s- \wedge \tau_k}|^{p/(1-\alpha)} ds + 2^{\frac{p}{2}} \alpha T^{\frac{p}{2}} \int_0^{t_{1}} \mathbb{E} |x_{s- \wedge \tau_k}|^p ds,
\]

where the constant \( C \) depends on \( \mathbb{E} \left( |\xi| \right)^{\frac{p}{\alpha}} \) and also in the last step we used Assumption A. Plugging the last estimate and (4.22), (4.23) into (4.21) yields

\[
(4.24) \quad \mathbb{E} \sup_{0 \leq t \leq t_{1}} |x_{t \wedge \tau_k}|^p \leq A_p + B_p \int_0^{t_{1}} \mathbb{E} |x_{s- \wedge \tau_k}| \mathbb{E} |x_{s- \wedge \tau_k}|^{p/(1-\alpha)} ds + C_p \int_0^{t_{1}} \mathbb{E} |x_{s- \wedge \tau_k}|^p ds,
\]

where the constants \( A_p, B_p \) and \( C_p \) depend on \( p, \mathbb{E} \left( |\xi| \right), k_1, k_2, k_3, T, \lambda \) and \( \alpha \).

We consider a finite sequence \( \{p_1, p_2, \ldots , p_{[T/\tau] + 1}\} \) such that \( p_i > 2 \) and \( \gamma p_{i+1} < (1-\alpha) p_i \) for \( i = 1, 2, \ldots , [T/\tau] \) where \([z]\) denotes the integer part of \( z \). We work successively in the regions \([0, \tau], [0, 2\tau], \ldots \) e.t.c. until we cover the interval \([0, T] \) proving in each case the moment bounds. This idea can be found in the proofs of [9, Lemma 2.1] or [5, Theorem 2.1]. Take a \( t_1 \in [0, \tau] \). Then \( \mathbb{E} |x_{s- \tau_k}|^{p_1 \mathbb{E} |x_{s- \tau_k}|} \leq \mathbb{E} \left( |\xi| \right)^{\frac{p_1 \mathbb{E} |x_{s- \tau_k}|}{\alpha}} \) for \( s \in [0, t_{1}] \). Equation (4.24) yields

\[
\mathbb{E} \sup_{0 \leq t \leq t_{1}} |x_{t \wedge \tau_k}|^{p_1} \leq A_{p_1} + B_{p_1} T \mathbb{E} \left( |\xi| \right)^{\frac{p_1 \mathbb{E} |x_{s- \tau_k}|}{\alpha}} + C_{p_1} \int_0^{t_{1}} \mathbb{E} \sup_{0 \leq \tau \leq s} |x_{\tau \wedge \tau_k}|^{p_1} ds \leq (A_{p_1} + B_{p_1} T \mathbb{E} \left( |\xi| \right)^{\frac{p_1 \mathbb{E} |x_{s- \tau_k}|}{\alpha}})^e, \]

by application of the Gronwall inequality, which in turn implies

\[
(4.25) \quad \mathbb{E} \sup_{0 \leq t \leq t_{1}} |x_{t}|^{p_1} \leq (A_{p_1} + B_{p_1} T \mathbb{E} \left( |\xi| \right)^{\frac{p_1 \mathbb{E} |x_{s- \tau_k}|}{\alpha}})^e \quad \text{for} \quad t \leq t_{1},
\]

by application of Fatou’s theorem as we let \( k \to \infty \). We have

\[
\mathbb{E} \sup_{-\tau \leq t \leq \tau} |x_{t}|^{p_1} \leq \mathbb{E} \sup_{-\tau \leq t \leq 0} |x_{t}|^{p_1} + \mathbb{E} \sup_{0 \leq t \leq \tau} |x_{t}|^{p_1} \leq \mathbb{E} \left( |\xi| \right)^{p_1} + (A_{p_1} + B_{p_1} T \mathbb{E} \left( |\xi| \right)^{\frac{p_1 \mathbb{E} |x_{s- \tau_k}|}{\alpha}})^e := A_1.
\]
When $t_1 \in [0, 2\tau]$ we get by (4.24)

\[
\mathbb{E} \sup_{0 \leq t \leq t_1} |x_{t \wedge T_k}|^{p_2} \leq A_{p_2} + B_{p_2} \int_0^{2\tau} (\mathbb{E}|x_{s \wedge T_k} - x_t|^{p_1})^{\gamma^{p_2}_{p_1}} ds + C_{p_2} \int_0^{t_1} \mathbb{E} \sup_{0 \leq r \leq s} |x_{r \wedge T_k}|^{p_2} ds \\
\leq A_{p_2} + B_{p_2} \int_{-\tau}^{\tau} (\mathbb{E}|x_{s \wedge T_k}|^{p_1})^{\gamma^{p_2}_{p_1}} ds + C_{p_2} \int_0^{t_1} \mathbb{E} \sup_{0 \leq r \leq s} |x_{r \wedge T_k}|^{p_2} ds \\
\leq A_{p_2} + 2B_{p_2} \tau A_1^{\frac{\gamma^{p_2}_{p_1}}{1-(\alpha/n)p_1}} + \mathcal{C}_{p_2} \int_0^{t_1} \mathbb{E} \sup_{0 \leq r \leq s} |x_{r \wedge T_k}|^{p_2} ds \\
\leq (A_{p_2} + 2B_{p_2} \tau A_1^{\frac{\gamma^{p_2}_{p_1}}{1-(\alpha/n)p_1}}) e^{2C_{p_2} \tau}
\]

and once again by the Fatou theorem we have

\[
\mathbb{E} \sup_{0 \leq t \leq 2\tau} |x_t|^{p_2} \leq (A_{p_2} + 2B_{p_2} \tau A_1^{\frac{\gamma^{p_2}_{p_1}}{1-(\alpha/n)p_1}}) e^{2C_{p_2} \tau}.
\]

and analogously

\[
\mathbb{E} \sup_{-\tau \leq t \leq 2\tau} |x_t|^{p_2} \leq \mathbb{E} \sup_{-\tau \leq t \leq 0} |x_t|^{p_2} + \mathbb{E} \sup_{0 \leq t \leq 2\tau} |x_t|^{p_2} \\
\leq \mathbb{E} \|\xi\|^{p_2} + (A_{p_2} + 2B_{p_2} \tau A_1^{\frac{\gamma^{p_2}_{p_1}}{1-(\alpha/n)p_1}}) e^{2C_{p_2} \tau} := A_2.
\]

Repeating the previous steps we finally obtain that for any $p = p[T/\tau] + 1 > 2$ there exists a positive constant $A_{[T/\tau]+1}$ such that

\[
\mathbb{E} \sup_{-\tau \leq t \leq T} |x_t|^p \leq A_{[T/\tau]+1} := A.
\]

The case $0 < p \leq 2$ follows from the Hölder inequality since then $\mathbb{E}|x_t|^p \leq (\mathbb{E}|x_t|^3)^{p/3} \leq A^{p/3}$. Therefore (2.4) is true. \hfill \Box

**Proof of Proposition 2.2.** The uniqueness of the solution of (2.3) can be found in [23]. The non negativity property can be shown by adopting the Yamada-Watanabe approach. Let the non-increasing sequence $\{e_m\}_{m \in \mathbb{N}}$ with $e_m = e^{-m(m+1)/2}$ and $e_0 = 1$. We introduce the following sequence $\phi_m(x) = 0$ for $x \geq 0$ and

\[
\phi_m(x) = \int_0^{-x} dy \int_0^y \psi_m(u) du, \quad \text{for } x < 0,
\]

where the existence of the continuous function $\psi_m(u)$ with $0 \leq \psi_m(u) \leq 2/(mu)$ and support in $(e_m, e_{m-1})$ is justified by $\int_{e_{m-1}}^{e_m} (du/u) = m$. The following relations hold for $\phi_m \in C^2(\mathbb{R}, \mathbb{R})$ with $\phi_m(0) = 0$,

\[
x^+ - e_{m+1} \leq \phi_m(x) \leq x^+, \quad \text{for } x \in \mathbb{R},
\]

where $x^+ = -x \vee 0$,

\[
-1 \leq \phi'_m(x) \leq 0, \quad \text{if } x < -e_m \quad \text{and} \quad \phi'_m(x) = 0 \text{ otherwise};
\]

\[
|\phi''_m(x)| \leq \frac{2}{m|x|}, \quad \text{when } -e_{m-1} < x < -e_m \quad \text{and} \quad \phi''_m(x) = 0 \text{ otherwise}.
\]
Application of the Itô formula for the jump-extended CIR/CEV model reads
\[
\phi_m(x_t) = \phi_m(x_0) + \int_0^t \dot{\phi}_m(x_s)(k_1 - k_2 x_s - \lambda g(x_s))ds + \frac{(k_3)^2}{2} \int_0^t \phi_m''(x_s)b(|x_s-\tau|)^2|x_s|^2\alpha ds
\]
\[+ k_3 \int_0^t \phi_m'(x_s)b(|x_s-\tau|)|x_s|\alpha dW_s + \int_0^t (\phi_m(x_s) - g(x_s)) dN_s.
\]

Now, we take expectations in the above inequality, using the moment bounds of Lemma 2.1 and the properties of \(\phi_m\) to get in the case \(g(0) = 0\)
\[
E\phi_m(x_t) = \int_0^t E\phi_m'(x_s)(k_1 - k_2 x_s - \lambda g(x_s))ds + \frac{(k_3)^2}{2} \int_0^t E\phi_m''(x_s)b(|x_s-\tau|)^2|x_s|^2\alpha ds
\]
\[+ \lambda E \int_0^t (\phi_m(x_s) - g(x_s)) ds
\]
\[\leq \frac{(k_3C_\gamma)^2}{m} \int_0^t E(|x_s-\tau|^{2\gamma}|x_s|^{2\alpha-1}) ds + \frac{(k_3)^2}{m} \int_0^t E(b(|\xi_s-\tau|^2)|x_s|^{2\alpha-1}) ds
\]
\[+ k_2 \sup_{x \in \mathbb{R}} |\phi_m'(x)| \int_0^t E(x_s) ds + \lambda \sup_{x \in \mathbb{R}} |\phi_m'(x)| \int_0^t E(g(x_s))ds
\]
\[\leq \frac{C(k_3)^2}{m} A + (\lambda L + k_2) \int_0^t E(x_s) ds
\]
\[\leq \frac{C(k_3)^2}{m} A + (\lambda L + k_2) E\phi_m(x_0) + e_{m-1} ds,
\]
where \(C\) is a constant or
\[
E\phi_m(x_t) \leq \left(\frac{C(k_3)^2}{m} A + (\lambda L + k_2) Te_{m-1}\right) + (\lambda L + k_2) \int_0^t E\phi_m(x_s) ds
\]
\[\leq \left(\frac{C(k_3)^2}{m} A + (\lambda L + k_2) Te_{m-1}\right) e^{(\lambda L+k_2)t}.
\]
Note that the same upper bound may be used in the case \(g(x) > 0\) since then \(\phi_m(x_s) + g(x_s)) \leq \phi_m(x_s)\). Thus, using once more a property of \(\phi\) and the above bound we have
\[
E|x_t^- - e_{m-1}| \leq \left(\frac{C(k_3)^2}{m} A + (\lambda L + k_2) Te_{m-1}\right) e^{(\lambda L+k_2)t}
\]
and taking the limit \(m \to \infty\) implies \(E|x_t^-| \leq 0\) or \(P(x_t \leq 0) = 0\) for all \(t \geq 0\).

**Proof of Lemma 2.4.** We apply the Itô formula for the jump-extended CIR/CEV model and then take expectations to get
\[
E(x_t)^p = E(x_0)^p + pE \left(\int_0^t (x_s)^{p-1}(k_1 - k_2 x_s - \lambda g(x_s))ds\right)
\[+ \frac{(k_3)^2 p(p-1)}{2} E \left(\int_0^t (x_s)^{p-2+2\alpha} b(x_s-\tau)^2 ds\right) + \lambda E \left(\int_0^t (x_s + g(x_s))^p - (x_s)^p ds\right),
\]
where \(p < 0\). Now Assumption A yields \((x_s + g(x_s))^p \leq (x_s)^p\) and we can find a constant \(C > 0\) such that
\[
k_1 p x^{p-1} + \frac{(k_3)^2 p(p-1)}{2} x^{p-2+2\alpha} \leq C,
\]
thus
\[
\mathbb{E}(x_t)^p \leq \mathbb{E}(\xi_0)^p + C + CC_7^2 \int_0^t \mathbb{E}(x_{s-\tau})^2 \, ds + C \int_0^t \mathbb{E}(x_{s-})^p \, ds
\]
\[
\leq \mathbb{E}(\xi_0)^p + C + CC_7^2 At + C \int_0^t \mathbb{E}(x_{s-})^p \, ds
\]
\[
\leq (\mathbb{E}(\xi_0)^p + C + CC_7^2 At)e^{Ct},
\]
by an application of the Gronwall inequality and Lemma 2.1, where the constant \(C\) depends on \(\mathbb{E}(|b(\xi_{-\tau})|^2)\). Therefore,
\[
\sup_{-\tau \leq t \leq T} \mathbb{E}(x_t)^p \leq \sup_{\tau \leq 0} \mathbb{E}(x_t)^p + \sup_{0 \leq t \leq T} \mathbb{E}(x_t)^p
\]
\[
\leq \mathbb{E}\|\xi\|^p + (\mathbb{E}(\xi_0)^p + C + CC_7^2 At)e^{Ct} := \hat{A}.
\]

Proof of Theorem 2.6. The proof of the mean reversion property of (JASDM) method requires moment bounds which are shown shortly after in Lemma 4.9. We take expectations in (1.2), with a new constant \(C\), to get,
\[
\mathbb{E}y_{t_{n+1}} = \mathbb{E}y_{t_{n+1}} + \mathbb{E}g(y_{t_{n+1}})\Delta \tilde{N}_n
\]
\[
\leq \mathbb{E}y_{t_{n+1}} + \sqrt{\mathbb{E}g^2(y_{t_{n+1}})} \sqrt{\mathbb{E}(\Delta \tilde{N}_n)^2}
\]
\[
\leq \mathbb{E}(z_{t_{n+1}})^2 + L\sqrt{\mathbb{E}(y_{t_{n+1}})^2} \sqrt{\lambda \Delta} \leq \mathbb{E}(z_{t_{n+1}})^2 + C \sqrt{\Delta}.
\]

Moreover
\[
\mathbb{E}(z_{t_{n+1}})^2 \leq \mathbb{E}\left(y_n \left(1 - \frac{k_2 \Delta}{1 + k_2 \theta \Delta}\right) + \frac{k_1 \Delta}{1 + k_2 \theta \Delta} - \frac{(k_3)^2}{4(1 + k_2 \theta \Delta)^2 (1 + b(y_{t_{n-\tau}})\Delta^m)^2(y_{t_{n-\tau}})^{2\alpha - 1} \Delta}\right)
\]
\[
+ \frac{(k_3)^2}{4(1 + k_2 \theta \Delta)^2} \mathbb{E}\left(\frac{b^2(y_{t_{n-\tau}})}{(1 + b(y_{t_{n-\tau}})\Delta^m)^2(y_{t_{n-\tau}})^{2\alpha - 1} \Delta W_n)^2\right)
\]
\[
\leq \left(1 - \frac{k_2 \Delta}{1 + k_2 \theta \Delta}\right) \mathbb{E}y_n + \frac{k_1 \Delta}{1 + k_2 \theta \Delta},
\]
where we used that
\[
\mathbb{E}(\psi_{t_n}(\Delta W_n)^j) = \mathbb{E}(\mathbb{E}(\psi_{t_n}(\Delta W_n)^j|\mathcal{F}_{t_n})) = \mathbb{E}(\psi_{t_n} \mathbb{E}((\Delta W_n)^j|\mathcal{F}_{t_n})),
\]
for \(j = 1, 2\) and \(\mathbb{E}(\Delta W_n) = 0, \mathbb{E}(\Delta W_n)^2 = \Delta\), which implies
\[
\mathbb{E}y_{t_{n+1}} \leq \left(1 - \frac{k_2 \Delta}{1 + k_2 \theta \Delta}\right)^n \mathbb{E} \xi_0 + \left(\frac{k_1 \Delta}{1 + k_2 \theta \Delta} + C \sqrt{\Delta}\right) \left(1 - \frac{k_2 \Delta}{1 + k_2 \theta \Delta}\right)^{n-1} + \frac{k_1 \Delta}{1 + k_2 \theta \Delta} + C \sqrt{\Delta}.
\]
Now, we take the limit \(n \to \infty\) when \(\Delta(1 - \theta) < \frac{1}{k_2}\) to get \(\lim_{n \to \infty} \mathbb{E}y_n \leq \frac{k_1}{k_2} \theta\). \(\square\)

Proof of Theorem 2.7. We divide the proof is several steps. First we give moment bounds for the proposed numerical scheme. Then we estimate the error bound of the semi-discrete method. Using an auxiliary process \((h_t)\) we show strong convergence of \((h_t)\) to \((x_t^0)\) which carries on to the strong convergence of \((y_t^0)\) to \((x_t^0)\).

Lemma 4.9 [Moment bounds for numerical approximation] It holds that
\[
\mathbb{E}\sup_{\tau \leq t \leq T} (y_t)^p \leq A_p \mathbb{E}(\|\xi\| + k_1 T)^p,
\]
for any $p > 0$, where $A_p$ is a constant.

Proof of Lemma 4.9. We first observe that $(y_t)$ is bounded in the following way

$$0 \leq y_t \leq \xi_0 + \int_0^t k_1 ds + k_3 \int_0^t \frac{b(y_{S^-})}{1 + b(y_{S^-})} \Delta (y_s)^{\alpha - \frac{1}{2}} \sqrt{y_s} d\tilde{W}_s$$

$$\leq \xi_0 + k_1 T + k_3 \int_0^t \frac{b(y_{S^-})}{1 + b(y_{S^-})} \Delta (y_s)^{\alpha - \frac{1}{2}} \sqrt{y_s} d\tilde{W}_s =: u_t,$$

a.s., where the lower bound comes from the construction of $(y_t)$ and the upper bound follows from a comparison theorem. We will bound $(u_t)$ and therefore $(y_t)$, since $0 \leq y_t \leq u_t$ a.s. Set the stopping time $\tau_R := \inf \{ t \in [0, T] : u_t > R \}$, for $R > 0$ with the convention $\inf \emptyset = \infty$. Application of Itô’s formula on $(u_{t\wedge \tau_R})^p$ implies

$$(u_{t\wedge \tau_R})^p = (\xi_0 + k_1 T)^p + \frac{p(p-1)}{2} (k_3)^2 \int_0^{t\wedge \tau_R} (u_s)^{p-2} \frac{b^2(y_{S^-})}{(1 + b(y_{S^-})} \Delta (y_s)^{\alpha - \frac{1}{2}} \sqrt{y_s} d\tilde{W}_s$$

$$+ pk_3 \int_0^{t\wedge \tau_R} (u_s)^{p-1} \frac{b(y_{S^-})}{1 + b(y_{S^-})} \Delta (y_s)^{\alpha - \frac{1}{2}} \sqrt{y_s} d\tilde{W}_s$$

$$\leq (\xi_0 + k_1 T)^p + \frac{p(p-1)}{2} (k_3)^2 \int_0^{t\wedge \tau_R} (u_s)^{p-1} b^2 (y_{S^-})(y_s)^{2\alpha - 1} ds + M_t$$

$$\leq (\xi_0 + k_1 T)^p + \frac{p(p-1)}{2} (k_3)^2 \int_0^{t\wedge \tau_R} (u_s)^{p} ds + 2^{3p-3}(p-1)(k_3)^2 \int_0^{t\wedge \tau_R} b^2(p) (y_{S^-})^{2\alpha - 1} ds + M_t$$

$$\leq (\| \xi \| + k_1 T)^p + \left( \frac{(p-1)^2}{4} + 2^{3p-3}(p-1)(k_3 C\gamma)^2 \int_0^{t\wedge \tau_R} (2\alpha - 1)|y_s|^p + 2(1 - \alpha)|y_{S^-}|^{p\gamma/(1 - \alpha)} ds + M_t \right)$$

$$+ 2^{p-2}(p-1)2(1 - \alpha)(k_3 C\gamma)^2 \int_0^{t\wedge \tau_R} (u_s)^{p} ds$$

where in the second step we have used that $0 \leq y_t \leq u_t$ and

$$M_t := pk_3 \int_0^{t\wedge \tau_R} (u_s)^{p-1} \frac{b(y_{S^-})}{1 + b(y_{S^-})} \Delta (y_s)^{\alpha - \frac{1}{2}} \sqrt{y_s} d\tilde{W}_s,$$

in the third step the inequality $x^{p-1}y \leq \epsilon \frac{p-1}{p} x^p + \frac{1}{p} p \gamma, p > 1$ with $\epsilon = \frac{1}{2}$ and in the final step the elementary inequality used in the proof of Lemma 2.1 since $\frac{1}{2} < \alpha < 1$. Taking expectations in the above inequality and using that $M_t$ is a local martingale vanishing at 0, we get

$$\mathbb{E}(u_{t\wedge \tau_R})^p \leq A_p + B_p \int_0^t \mathbb{E}(u_{s\wedge \tau_R})^{\frac{\gamma}{\gamma - p}} ds + C_p \int_0^t \mathbb{E}(u_{s\wedge \tau_R})^p ds,$$

where the constants $A_p, B_p$ and $C_p$ depend on $p, \mathbb{E}\| \xi \|, k_1, k_2, k_3, T, C\gamma$ and $\alpha$. Now proceeding as in (4.24) we get

$$\mathbb{E} \sup_{-\tau \leq t \leq T} (u_t)^p \leq A_p,$$

and as a consequence

$$\mathbb{E} \sup_{-\tau \leq t \leq T} (y_t)^p \leq A_p,$$

for any $p > 0$. □
Lemma 4.10 \textit{[Error bound]} Let \( n_s \) be an integer such that \( s \in [t_{n_s}, t_{n_s+1}] \). Then
\[
\mathbb{E}|y_s - y_{\bar{s}}|^p \leq \hat{A}_p \Delta^{p/2}, \quad \mathbb{E}|y_s - y_{\bar{s}}|^p < \tilde{A}_p \Delta^{p/2},
\]
for any \( p > 0 \), where the positive constants \( \hat{A}_p, \tilde{A}_p \) do not depend on \( \Delta \). \hfill \Box

\textbf{Proof of Lemma 4.10.} First we take a \( p \geq 2 \). We get that
\[
|y_s - y_{\bar{s}}|^p = \left| \int_{t_{n_s}}^s (k_1 - k_2(1 - \theta)y_{\bar{s}} - k_2y_{\bar{s}}) du + \int_{t_{n_s}}^{t_{n_s+1}} k_2\theta y_s du - \int_{s}^{t_{n_s+1}} k_2\theta y_{\bar{s}} du \right|^p
\]
\[
+ \left| \int_{t_{n_s}}^s \left( k_1 - k_2(1 - \theta)y_{\bar{s}} - \frac{(k_3)^2}{4(1 + k_2\theta \Delta)} \frac{b^2(y_{t_{n_s} - \tau})}{(1 + b(y_{t_{n_s} - \tau})\Delta m)^2}(y_{t_{n_s} - \tau})^{2\alpha - 1} \right) du \right|^p
\]
\[
+ k_3 \int_{t_{n_s}}^s \frac{b(y_{t_{n_s} - \tau})}{1 + b(y_{t_{n_s} - \tau})\Delta m}(y_{\bar{s}})^{\alpha - \frac{1}{2}} \sqrt{y_{\bar{s}}u} d\tilde{W}_u \right|^p
\]
\[
\leq 5^{p-1} \left| \int_{t_{n_s}}^s (k_1 - k_2(1 - \theta)y_{\bar{s}} - k_2y_{\bar{s}}) du \right|^p + (k_2)^p \theta^p (y_{\bar{s}})^p (t_{n_s+1} - t_{n_s})^p + (k_2)^p \theta^p (y_{\bar{s}})^p (t_{n_s+1} - s)^p
\]
\[
+ \left| \int_{t_{n_s}}^s \left( k_1 - k_2(1 - \theta)y_{\bar{s}} - \frac{(k_3)^2}{4(1 + k_2\theta \Delta)} \frac{b^2(y_{t_{n_s} - \tau})}{(1 + b(y_{t_{n_s} - \tau})\Delta m)^2}(y_{t_{n_s} - \tau})^{2\alpha - 1} \right) du \right|^p
\]
\[
+ (k_3)^p \int_{t_{n_s}}^s \frac{b(y_{t_{n_s} - \tau})}{1 + b(y_{t_{n_s} - \tau})\Delta m}(y_{\bar{s}})^{\alpha - \frac{1}{2}} \sqrt{y_{\bar{s}}u} d\tilde{W}_u \right|^p)
\]
\[
\leq 5^{p-1} \left| \int_{t_{n_s}}^s |k_1 - k_2(1 - \theta)y_{\bar{s}} - k_2y_{\bar{s}}|^p du + (k_2)^p \theta^p ((y_{\bar{s}})^p + (y_{\bar{s}})^p) \Delta^p
\]
\[
+ \left| k_1 - k_2(1 - \theta)y_{\bar{s}} - \frac{(k_3)^2}{4(1 + k_2\theta \Delta)} \frac{b^2(y_{t_{n_s} - \tau})}{(1 + b(y_{t_{n_s} - \tau})\Delta m)^2}(y_{t_{n_s} - \tau})^{2\alpha - 1} \right|^p \Delta^p
\]
\[
+ (k_3)^p \int_{t_{n_s}}^s \frac{b(y_{t_{n_s} - \tau})}{1 + b(y_{t_{n_s} - \tau})\Delta m}(y_{\bar{s}})^{\alpha - \frac{1}{2}} \sqrt{y_{\bar{s}}u} d\tilde{W}_u \right|^p)
\]
where we have used the Cauchy-Schwarz inequality. Taking expectations in the above inequality and using Lemma 4.9 and Doob’s martingale inequality on the diffusion term we conclude
(4.26)
\[
\mathbb{E}|y_s - y_{\bar{s}}|^p \leq \hat{A}_p \Delta^{p/2},
\]
where the positive quantity \( \hat{A}_p \) except on \( p \), depends also on the parameters \( k_1, k_2, k_3, \theta, \alpha, \gamma \) but not on \( \Delta \). Now, for \( 0 < p < 2 \) we get
\[
\mathbb{E}|y_s - y_{\bar{s}}|^p \leq (\mathbb{E}|y_s - y_{\bar{s}}|^2)^{p/2} \leq \hat{A}_p \Delta^{p/2},
\]
where we have used Jensen’s inequality for the concave function \( \phi(x) = x^{p/2} \). Following the same lines, we can show that
\[
\mathbb{E}|y_s - y_{\bar{s}}|^p \leq \tilde{A}_p \Delta^{p/2},
\]
for any \( 0 < p \), where the positive quantity \( \tilde{A}_p \) except on \( p \), depends also on the parameters \( k_1, k_2, k_3, \theta, \alpha, \gamma \) but not on \( \Delta \). \hfill \Box

For the rest of the proof we rewrite the compact form of (2.12) in the following way
(4.27)
\[
y_t = \xi_0 + \int_0^t f_\theta(y_s, y_{\bar{s}}) ds + \int_0^t g(y_s, y_s, y_{t_{n_s} - \tau}) d\tilde{W}_s + \int_{t_{n_s+1}}^t f_1(y_{t_{n_s}}, y_t, y_{t_{n_s} - \tau}) ds,
\]
where $f_θ(·,·)$ is given by (2.7). The auxiliary process $(h_t)$, with $h_t = ξ(t), t \in [-τ, 0]$ is close to $(y_t)$ as shown in the next result.

**Lemma 4.11** [Moment bounds involving the auxiliary process] For any $s \in [0, T]$ it holds that

\[
|E[h_s - y_s]|^p \leq C_p Δ^p, \quad |E[h_s]|^p \leq C_h,
\]

and for $s \in [t_n, t_{n+1}]$ we have that

\[
|E[h_s - y_s]|^p \leq \hat{C}_p Δ^{p/2}, \quad |E[h_s - y_s]|^p \leq \hat{C}_p Δ^{p/2},
\]

for any $p > 0$, where the positive quantities $C_p, \hat{C}_p, \bar{C}_p, C_h$ do not depend on $Δ$.

**Proof of Lemma 4.11.** We have that

\[
|h_s - y_s|^p = \left| \int_s^{t_{n+1}} f_1(y_{t_n}, y_s, y_{t_{n-τ}}) du \right|^p \leq |t_{n+1} - s|^p |f_1(y_{t_n}, y_s, y_{t_{n-τ}})|^p,
\]

for any $p > 0$, where we have used (4.27). Using Lemma 4.9 we get the left part of (4.28). Now for $p > 2$ and noting that

\[
|E[h_s]|^p \leq 2^{p-1}|E[h_s - y_s]|^p + 2^{p-1}|E[y_s]|^p \leq 2^{p-1}C_p Δ^p + 2^{p-1}A_p E(∥ξ∥ + k_1T)^p \leq C_h,
\]

we get the right part of (4.28), where we have used Lemma 4.9. The case $0 < p < 2$ follows by Jensen’s inequality as in Lemma 4.10.

Furthermore, for $s \in [t_n, t_{n+1}]$ and $p > 2$ we derive that

\[
|E[h_s - y_s]|^p \leq 2^{p-1}|E[h_s - y_s]|^p + 2^{p-1}|E[y_s - y_s]|^p \leq 2^{p-1}C_p Δ^p + 2^{p-1}\bar{A}_p Δ^{p/2} \leq \hat{C}_p Δ^{p/2},
\]

where we have used (4.26) and in the same manner

\[
|E[h_s - y_s]|^p \leq 2^{p-1}C_p Δ^p + 2^{p-1}\bar{A}_p Δ^{p/2} \leq \bar{C}_p Δ^{p/2}.
\]

The case $0 < p < 2$ follows by Jensen’s inequality.

The following results examine the convergence of the auxiliary process $(h_t)$ to $(x_t)$ in $C^2$.

**Proposition 4.12** Let Assumptions A and B hold and take $m = 1/4$ in (2.12). Then we have

\[
E \sup_{-τ ≤ t ≤ T} |h_t - x_t|^2 \leq CΔ^{(α - \frac{1}{2})∧γ},
\]

where $C$ is independent of $Δ$. In case $α = 1/2$

\[
E \sup_{-τ ≤ t ≤ T} |h_t - x_t|^2 \leq CΔ^γ.
\]

**Proof of Proposition 4.12.** We estimate the difference $|E_t|^2 := |h_t - x_t|^2$.

It holds that

\[
f_θ(y_s, y_s) - f_θ(x_s, x_s) = (k_1 - k_2(1 - θ)y_s - k_2θy_s) - (k_1 - k_2x_s)
\]

\[
= -k_2(1 - θ)(y_s - x_s) - k_2θ(y_s - x_s)
\]

\[
= k_2(1 - θ)(h_s - y_s) + k_2θ(h_s - y_s) - k_2(h_s - x_s),
\]
for \( n, l \geq 20 \) I. S. STAMATIOU

where we have used the inequality (4.32)

Taking the supremum over all \( t \in [0, t_1] \) and then expectations we have

where in the second step we have used the Cauchy-Schwarz inequality and (4.31) and

Moreover, \( M_t := \int_0^t (g(y_\bar{u}, y_\bar{u}, y_\bar{u} - r) - k_3 b(x_{u - r}) x_{u}^\alpha) dW_u \).

Taking the supremum over all \( t \in [0, t_1] \) and then expectations we have

where in the second step we have used Lemma 4.10 and Doob's martingale inequality with \( p = 2 \), since \( M_t \) is an \( \mathbb{R} \)-valued martingale that belongs to \( \mathcal{L}^2 \). Moreover,

or

where we have used the inequality \((n^\alpha - l^n)(n^{1-\alpha} - l^{1-\alpha}) \leq 2\alpha |n - l|\), that is true for all \( n, l \geq 0 \) and \( 1/2 \leq \alpha \leq 1 \), and the property of Hölder continuous functions \(|x^\alpha - y^\alpha| \leq |x - y|^\alpha \) for \( \alpha \leq 1 \). Furthermore,
\[ |g(x_s, x_s, x_{s-}) - k_3b(x_{s-})x_s^n| \leq (k_3)^2(x_s)^{2\alpha}b^2(x_{s-}) \left( \frac{b(x_{s-})\Delta^m}{1+b(x_{s-})\Delta^m} \right)^2 \]
\[ \leq (k_3)^2(x_s)^{2\alpha}b^4(x_{s-})\Delta^{2m} \]
\[ \leq 8(k_3)^2(x_s)^{2\alpha}(C_\gamma^4|x_{s-}|^{4\gamma} + b^4(\xi_{s-}))\sqrt{\Delta}, \]
(4.34)
where we chose \( m = 1/4 \).

At this point, we use a stochastic time change. We define the process
\[
\zeta(t) := \int_0^t \frac{1536(k_3)^22^\alpha(C_\gamma^2|\bar{y}_{s-}|^{2\gamma} + b^2(\xi_{s-}))ds}{[(y_s)^{1-\alpha} + (x_s)^{1-\alpha}]^2}
\]
and the stopping time
\[
\tau_l := \inf\{s \in [0, T] : 6T(k_2)^2s + \zeta(s) \geq l \}.
\]
The process \( \zeta(t) \) is well defined since \( x_t > 0 \) a.s. and \( y_t \geq 0 \). We find that
\[
E[M_{t_1}]^2 = E \left| \int_0^{t_1} |g(y_s, y_s, y_{s-})\text{sgn}(z_s) - k_3b(x_{s-})x_s^n|dW_s \right|^2 \leq E \left( \int_0^{t_1} |g(y_s, y_s, y_{s-})\text{sgn}(z_s) - k_3b(x_{s-})x_s^n|^2ds \right)
\]
\[ \leq 24(k_3)^2E \left( \int_0^{t_1} (C_\gamma^2|y_{s-}|^{2\gamma} + b^2(\xi_{s-}))y_s y_s - y_s^{2\alpha} + (C_\gamma^2|y_{s-}|^{2\gamma} + b^2(\xi_{s-}))8\alpha^2 \frac{|y_s - y_s|^2 + |E_s|^2}{[(y_s)^{1-\alpha} + (x_s)^{1-\alpha}]^2} \right)ds
\]
\[ + 12(k_3C_\gamma)^22^\alpha E \left( \int_0^{t_1} (x_s)^{2\alpha} |h_s - y_s|^{2\gamma}ds \right) + 12(k_3C_\gamma)^22^\alpha E \left( \int_0^{t_1} (x_s)^{2\alpha} |E_s|^{2\gamma}ds \right)
\]
\[ + 32(k_3)^2\sqrt{\Delta} E \left( \int_0^{t_1} (x_s)^{2\alpha} (C_\gamma^4|x_{s-}|^{4\gamma} + b^4(\xi_{s-}))ds \right) + 2E \left( \int_0^{t_1} g^2(y_s, y_s, y_{s-}) \text{sgn}(z_s) - 1)^2ds \right)
\]
\[ \leq 24(k_3)^2E \left( \int_0^{t_1} \sqrt{E((C_\gamma^2|y_{s-}|^{2\gamma} + b^2(\xi_{s-}))y_s^2)} \sqrt{E(|y_s - y_s|^4\alpha - 2)}ds \right.
\]
\[ + \left. 12(k_3C_\gamma)^22^\alpha E \left( \int_0^{t_1} (x_s)^4 \text{sgn}(h_s - y_s|^{2\gamma}ds \right) + 12(k_3C_\gamma)^22^\alpha E \left( \int_0^{t_1} (x_s)^2 |E_s|^{2\gamma}ds \right)
\]
\[ + 32(k_3)^2\sqrt{\Delta} E \left( \int_0^{t_1} \text{sgn}(z_s) - 1)^2ds \right) \leq C \sqrt{\Delta} \quad \text{as} \quad \Delta \downarrow 0.
\]
Turning back to (4.32) we get
Relation (4.36) for \( t_1 = \tau \wedge \tau_1 \) implies
\[
\mathbb{E} \sup_{0 \leq t \leq t_1} (\mathcal{E}_t)^2 \leq C \Delta^{(\alpha - \frac{1}{2})^\wedge \gamma} + \int_0^{t_1} \mathbb{E} \sup_{0 \leq t \leq s} |\mathcal{E}_t|^2 (6 t (k_2)^2 s + \zeta_s)' ds + C \int_0^{t_1} \sqrt{\mathbb{E}(\mathcal{E}_s - \tau)^2} ds,
\]
where in the last step we have used Gronwall's inequality. Using again relation (4.36) for \( t_1 = \tau \wedge \tau_1 \) implies
\[
\begin{align*}
\mathbb{E} \sup_{0 \leq t \leq t_1} (\mathcal{E}_t)^2 & \leq C \Delta^{(\alpha - \frac{1}{2})^\wedge \gamma} + \int_0^{t_1} \mathbb{E} \sup_{0 \leq t \leq s} (\mathcal{E}_t)^2 (6 t (k_2)^2 s + \zeta_s)' ds + C \int_0^{t_1} \sqrt{\mathbb{E}(\mathcal{E}_s - \tau)^2} ds \\
& \leq C \Delta^{(\alpha - \frac{1}{2})^\wedge \gamma} + \int_0^{t_1} \mathbb{E} \sup_{0 \leq t \leq s} (\mathcal{E}_t)^2 du \\
& \leq C \Delta^{(\alpha - \frac{1}{2})^\wedge \gamma} \epsilon^1,
\end{align*}
\]
where in the last step we have used Gronwall's inequality. Using again relation (4.36) for \( t_1 \in [0, \tau] \) and under the change of variables \( u = 6 t (k_2)^2 s + \zeta_s \) we get
\[
\begin{align*}
\mathbb{E} \sup_{0 \leq t \leq t_1} (\mathcal{E}_t)^2 & \leq C \Delta^{(\alpha - \frac{1}{2})^\wedge \gamma} + \int_0^{6 (k_2)^2 T + \zeta_s} \mathbb{E} \sup_{0 \leq \tau \leq \tau_1} (I_6 (k_2)^2 r + \zeta_s)^2 du \\
& \leq \int_0^{6 (k_2)^2 T + \zeta_s} \mathbb{E} \sup_{0 \leq \tau \leq \tau_1} (\mathcal{E}_t)^2 du + \int_0^{6 (k_2)^2 T + \zeta_s} \mathbb{P}(6 t (k_2)^2 s + \zeta_s u) \mathbb{E} \left( \sup_{0 \leq t \leq \tau} (\mathcal{E}_t)^2 (6 t (k_2)^2 s + \zeta_s u) \right) du \\
& + C \Delta^{(\alpha - \frac{1}{2})^\wedge \gamma} \\
& \leq C \Delta^{(\alpha - \frac{1}{2})^\wedge \gamma} (e^{6 (k_2)^2 T + 1} + C \Delta^{(\alpha - \frac{1}{2})^\wedge \gamma}) \int_0^{\infty} \mathbb{P}(\zeta_s u) e^u du,
\end{align*}
\]
where in the last steps we have used (4.37). We proceed by showing that \( u \to \mathbb{P}(\zeta_s u) e^u \in L^1(\mathbb{R}_+). \) Markov's inequality implies
\[
\mathbb{P}(\zeta_s u) \leq e^{-\epsilon u} \mathbb{E}(e^{\epsilon \zeta_s})
\]
for any \( \epsilon > 0. \) Using (4.35) and the fact that \( x_s > 0 \) and \( y_s \geq 0 \) we can provide the following bound
\[
\zeta_s = \int_0^\tau \frac{1536 (k_3)^2 \alpha^2 (C_2^2 |y_{s-} - \tau|^2 + b^2(\xi_{s-}))}{[(y_s)^{-\alpha} + (x_s)^{-\alpha}]^2} ds \leq 768 (k_3 C_2)^2 \alpha^2 \int_0^\tau (y_{s-} - \tau)^2 (x_s)^{2-2\alpha} ds
\]
\[
\leq 1536 (k_3)^2 \alpha^2 (C_2^2 \sup_{\tau \leq t \leq s} (y_t)^2 + b^2(\xi_{s-})) \int_0^\tau (x_s)^{2-2\alpha} ds,
\]
thus
\[
(4.38) \quad \mathbb{E}(e^{\epsilon \zeta_s}) \leq \mathbb{E} \left( e^{\epsilon 1536 (k_3)^2 \alpha^2 (C_2^2 \sup_{\tau \leq t \leq s} (\xi_t)^2 + b^2(\xi_{s-}) \int_0^\tau (x_s)^{2-2\alpha} ds) \right).
\]

It remains to bound the exponential moments of \( (x_t). \) We work as in [15, Section 4]. First we find the dynamics of the transformation \( v = x^{2-2\alpha} \) by application of Itô's formula
\[
v_t = v_0 + \int_0^t \left( (1 - 2\alpha)(1 - \alpha)k_3^2 b^2(x_{s-}) + 2(1 - \alpha)k_1(v_s) \frac{dx_s}{K_0} - 2(1 - \alpha)k_2 v_s \right) ds + \int_0^t 2k_3(1 - \alpha)b(x_{s-}) \sqrt{v_s} dW_s,
\]
for \( t \in [0, t_1] \), where \( v_0 = (\xi_0)^{2-2\alpha} > 0 \). Then, by a comparison theorem [22, Prop. 5.2.18] we obtain that \( v_t \geq \zeta_t(q) > 0 \) a.s. or \( (v_t)^{-1} \leq (\zeta_t(q))^{-1} \) a.s. where the process \((\zeta_t(q))\) reads

\[
(4.39) \quad \zeta_t(q) = \zeta_0 + \int_0^t (q - (K_2 + \eta(q))) \zeta_s ds + \int_0^t K_3 \sqrt{\zeta_s} dW_s,
\]

for \( t \in [0, t_1] \) with \( \zeta_0(q) = v_0 \) and

\[
\eta(q) = \frac{(2\alpha - 1)(q - K_0) \frac{1}{m-1}}{(k_1)^{\frac{2\alpha}{m-1}}}.
\]

Process (4.39) is a square root diffusion process and when \( \frac{2q}{(K_3)^2} - 1 \geq 0 \) or

\[
(4.40) \quad q \geq 2(1 - \alpha)^2(k_3)^2(\xi_{s-\tau}),
\]

remains positive if \( \zeta_0(q) > 0 \). Therefore, \((x_t)^{2\alpha - 2} \leq (\zeta_t(q))^{-1} \) a.s. so it suffices to bound exponential inverse moments of \((\zeta_t)\). By [24, Th. 3.1] we have

\[
(4.41) \quad \mathbb{E} e^{\delta f_0^{\nu}(\zeta_t(q))^{-1} ds} \leq C_{HK}(\zeta_0)\frac{1}{\nu(q)} \left( \frac{\nu(q)^2 - 8}{(K_3)^2} \right),
\]

for \( 0 \leq \delta \leq \left( \frac{2q}{(K_3)^2} - 1 \right)^{2}(K_3)^8 \) where the positive constant \( C_{HK} \) is explicitly given in [24, (10)] depends on the parameters \( k_2, k_3, T, \alpha \), but is independent of \( \zeta_0 \). Thus the other condition that we require for parameter \( q \) is

\[
(4.42) \quad q \geq 2(1 - \alpha)^2 \sqrt{2} k_3 b(\xi_{s-\tau}) + 2(1 - \alpha)^2 k_3^2 b^2(\xi_{s-\tau}).
\]

When (4.42) is satisfied then (4.40) is satisfied too, thus there is actually no restriction on the coefficient \( \delta \) in (4.41) since we can always choose appropriately a \( q \) such that (4.42) holds satisfying

\[
q \geq 2(1 - \alpha)^2 \sqrt{2} k_3 (C_{\gamma}\|\xi\|^\gamma + b(0)) + 2(1 - \alpha)^2 k_3^2 (C_{\gamma}\|\xi\|^\gamma + b(0))^2.
\]

Relation (4.38) becomes

\[
(4.43) \quad \mathbb{E}(e^{e^{C_{\gamma}t}}) \leq \mathbb{E} \left( e^{1536(k_3)^2\alpha^2(2\alpha\|\xi\|^2 + b^2(\xi_{s-\tau})) f_0^{\nu}(\zeta_t(q))^{-1} ds} \right) \leq \mathbb{E} \left( e^{1536(k_3)^2\alpha^2(2\alpha\|\xi\|^2 + b^2(\xi_{s-\tau})) f_0^{\nu}(\zeta_t(q))^{-1} ds} \right).
\]

We therefore require that

\[
(4.44) \quad 1536(k_3)^2\alpha^2(2\alpha\|\xi\|^2 + b^2(\xi_{s-\tau}))e \leq (\nu(q))^2 \frac{(K_3)^2}{8}
\]

and can always find a \( \epsilon > 1 \), such the above relation holds by choosing appropriately \( q \) as discussed before. Relation (4.38) becomes \( \mathbb{E}(e^{\zeta_t}) \leq A \) and therefore

\[
\mathbb{P} (\zeta_t \geq u) \leq Ae^{-\epsilon u},
\]

implying

\[
\mathbb{E} \sup_{0 \leq t \leq t_1} (\xi_t)^2 \leq \mathbb{E} \left( e^{C\Delta(a^{-\frac{1}{2}})^{\gamma}}(e^{6(k_2)^2T^2} + 1) + C\Delta(a^{-\frac{1}{2}})^{\gamma} \int_0^\infty e^{(1-\epsilon)u} du \right)
\]

\[
\leq C\Delta(a^{-\frac{1}{2}})^{\gamma},
\]

by choosing \( \epsilon > 1 \). We apply again successively (4.36) for \( t_1 = kT \land \tau_k \) for \( k = 2, \ldots, N_0 \) to finally get

\[
(4.45) \quad \mathbb{E} \sup_{0 \leq t \leq T} (\xi_t)^2 \leq C\Delta(a^{-\frac{1}{2}})^{\gamma}.
\]
For the CIR delay model, case \( \alpha = 1/2 \), we just sketch the main differences since we may follow the same steps. We now use the estimates, (see (4.33) and (4.34))

\[
|g(y_s, y_{s-\tau}) - g(x_s, x_{s-\tau})|^2 \leq 8(k_3)^2(C_\gamma^2|y_{s-\tau}|^{2\gamma} + b^2(\xi_{s-\tau})) \left(\frac{|h_s - y_s|^2 + |\mathcal{E}_t|^2}{\sqrt{|y_s| + \sqrt{x_s}}^2}\right)
\]

(4.46) \quad + 2(k_3C_\gamma)^22^{(2\gamma-1)+}x_s|h_{s-\tau} - y_{s-\tau}|^{2\gamma} + 2(k_3C_\gamma)^22^{(2\gamma-1)+}x_s(\mathcal{E}_{s-\tau})^{2\gamma}

and

\[
|g(x_s, x_{s-\tau}) - k_3b(x_{s-\tau})\sqrt{x_s}|^2 \leq 8(k_3)^2x_s(C_\gamma^2|x_{s-\tau}|^{4\gamma} + b^2(\xi_{s-\tau}))\sqrt{\Delta},
\]

where we chose again \( m = 1/4 \) and, (see (2.8))

\[
g(y, z) = k_3 \frac{b(z)}{1 + b(z)\Delta_m \sqrt{y}}.
\]

The process \( \zeta \) reads

\[
\zeta(t) := \int_0^t \frac{256(k_3)^2(C_\gamma^2|y_{s-\tau}|^{2\gamma} + b^2(\xi_{s-\tau}))}{(\sqrt{|y_s| + \sqrt{x_s}})^2} ds:
\]

(4.48) and the stopping time now is

\[
\tau_t := \inf\{s \in [0, T]: 4T(k_2)^2s + \zeta(s) \geq t\}.
\]

We get the following bound

\[
\mathbb{E}\sup_{0 \leq t \leq t_1} (\mathcal{E}_t)^2 \leq C \Delta^{\gamma \wedge (1/2)}(e^{4(k_2)^2T^2} + 1) + C \Delta^{\gamma \wedge (1/2)} \int_0^\infty \mathbb{P}(\zeta_t \geq u)e^u du
\]

and it remains to bound the following exponential inverse moment, (see 4.38)

\[
\mathbb{E}(e^{\zeta_T}) \leq \mathbb{E} \left( e^{8(k_3)^2(C_\gamma^2||\xi||^{2\gamma} + b^2(\xi_{s-\tau}))\int_0^T \mathcal{E}_s^{-1}ds} \right).
\]

(4.49) The exponential inverse moment of the delay CIR model is finite, \( \mathbb{E}e^{\delta \int_0^T (\mathcal{E}_s)^{-1}ds} < A \), thus we can find an \( \epsilon > 1 \) such that (4.49) holds. We conclude as before,

\[
\mathbb{E}\sup_{0 \leq t \leq T} (\mathcal{E}_t)^2 \leq C \Delta^{\gamma \wedge (1/2)}.
\]

(4.50)

\[
\square
\]

In order to finish the proof of Theorem 2.7 we just use the triangle inequality, Lemma 4.11, Proposition 4.12 and (4.45) to get

\[
\mathbb{E}\sup_{0 \leq t \leq T} |y_t^0 - x_t^0|^2 \leq 2\mathbb{E}\sup_{0 \leq t \leq T} |h_t - y_t|^2 + 2\mathbb{E}\sup_{0 \leq t \leq T} |\mathcal{E}_t|^2 \leq 2C\Delta^2 + 2C\Delta^{(\alpha - \frac{1}{2})\wedge \gamma} \leq C\Delta^{(\alpha - \frac{1}{2})\wedge \gamma},
\]

for the delay CEV jump model and (4.50) for the CIR jump model

\[
\mathbb{E}\sup_{0 \leq t \leq T} |y_t^0 - x_t^0|^2 \leq 2C\Delta^2 + 2C\Delta^{\gamma \wedge (1/2)} \leq C\Delta^{\gamma \wedge (1/2)},
\]

where \( C \) is independent of \( \Delta \).

\[
\square
\]
Proof of Theorem 2.8. First we have to ensure that Denote $\mathcal{E}_k := y_{t_k} - x_{t_k}$. It holds that

$$|\mathcal{E}_k| = |\mathcal{E}_k^- + (g(y_{t_k}) - g(x_{t_k})) \Delta \tilde{N}_{k-1}|$$

$$\leq |\mathcal{E}_k^-| + |g(y_{t_k}) - g(x_{t_k})| \Delta \tilde{N}_{k-1}|$$

$$\leq |\mathcal{E}_k^-| + L|\mathcal{E}_k^-|(1 + \lambda \Delta)$$

$$\leq (1 + L(1 + \lambda \Delta))|\mathcal{E}_k^-|$$

where we used Assumption A. Therefore

$$\mathbb{E} \sup_{1 \leq k \leq N_T} |\mathcal{E}_k|^2 \leq (1 + L(1 + \lambda \Delta))^2 |\mathcal{E}_k^-|^2$$

$$\leq (1 + L(1 + \lambda \Delta))^2 C \Delta^{(\alpha - \frac{1}{2}) \wedge \gamma} \leq C \Delta^{(\alpha - \frac{1}{2}) \wedge \gamma},$$

by an application of Theorem 2.7, where $C$ is a positive constant independent of $\Delta$. We conclude that the jump adapted semi-discrete method (1.2) converges in the mean square sense to the true solution of the jump-extended CEV model (1.1) with order of convergence at least $((\alpha - \frac{1}{2}) \wedge \gamma)/2$. Moreover, the jump adapted semi-discrete method (1.2) converges in the mean square sense to the true solution of the jump-extended CIR model (1.1) with order of convergence at least $((1/2) \wedge \gamma)/2$. □

Conclusion. In this work we have considered models with applications in finance, described by SDDEs with jumps. We have somehow unified existing models in the general model (1.1) for which we prove uniqueness, positivity, moment boundness and the mean reversion property of the solution process. We then proposed the JASDM method to numerically approximate the solution process of the delay CIR/CEV model with jumps. The proposed scheme is nonnegative, strongly convergent in the mean square sense to the exact solution of (1.1), with bounded moments and an analogue of the mean reversion property. We intend to make more numerical experiments to illustrate the impact of the diffusion exponent $\alpha$ and the delay coefficient $\gamma$ on the convergence of the (JASDM).

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E-mail address: joniou@gmail.com, ioannis.stamatiou@ouc.ac.cy, istamatiou@uniwa.gr