Space-Time Curvature Signatures in Bose-Einstein Condensates

Tonatiuh Matos1,‡ and Eduardo Gomez2,‡

1Departamento de Física, Centro de Investigación y Estudios Avanzados del IPN
A. P. 14–740, 07000, México, D.F., México.
2Instituto de Física, Universidad Autónoma de San Luis Potosí, San Luis Potosí 78290, México

We derive a generalized Gross-Pitaevskii (GP) equation immersed on an electromagnetic and a weak gravitational field starting from the covariant Complex Klein-Gordon field in a curved space-time. We compare it with the GP equation where the gravitational field is added by hand as an external potential. We show that there is a small difference of order $g_0/c^2$ between them that could be measured in the future using Bose-Einstein Condensates (BEC). This represents the next order correction to the Newtonian gravity in a curved space-time.

PACS numbers: 67.85.Jk, 37.25.+k, 04.80.-y

Among the four forces of nature, the gravitational force is the hardest one to study. It is orders of magnitude weaker than the other three forces and requires astronomical masses to see corrections beyond the basic Newtonian formula. The lab scale masses that we can manipulate introduce gravitational forces that are very hard to measure accurately. This is reflected in the poor precision that we currently have on the determination of the Newtonian constant of gravitation ($G = 6.67384(80) \times 10^{-11} \text{kg}^{-1}\text{m}^3\text{s}^{-2}$) [1].

Most of the experiments that measure the gravitational force use basically the same method pioneered by Cavendish [2]. Atomic interferometers have been recently introduced as an alternative way to implement sensitive gravimeters [3]. Here, cold atoms are launched vertically and subjected to a sequence of laser pulses that split, recombine and interfere the atoms. Using a Bose-Einstein Condensate (BEC) reduces the ballistic expansion of the atomic cloud and allows for a larger fall time thus increasing the precision of the gravimeters.

Sensitive atomic gravimeters are useful in the detection of gravitational waves [4], tests to Einstein’s general relativity [5], study of short range gravitational forces [6] and coherent evolution of delocalized quantum objects [7]. Measurements of $g$ (the gravitational acceleration of the earth) typically reach 9 digits of precision. This has been recently improved by two order of magnitude by Planck. Improvements to this include using a 146 m tower [9], launching a BEC in free fall in a 10 m tower [8]. Planned measurements of $g$ at the Instituto de Física, Universidad Autónoma de San Luis Potosi typically reach 9 digits of precision. This has been recently improved by two order of magnitude by Planck. Improvements to this include using a 146 m tower [9], launching a BEC in free fall in a 10 m tower [8]. Planned measurements of $g$ at the Instituto de Física, Universidad Autónoma de San Luis Potosi typically reach 9 digits of precision. This has been recently improved by two order of magnitude by Planck. Improvements to this include using a 146 m tower [9], launching a BEC in free fall in a 10 m tower [8].

BECs are usually described by a Gross-Pitaevskii (GP) equation, where an external field is added in order to confine the system or to include the earth’s gravitational acceleration ($g_0$) [12]. Nevertheless, the GP equation is non-relativistic and non-covariant; it is not invariant under Lorentz transformations unlike the electromagnetic field. A GP equation obtained this way could only show Newtonian effects of gravity. A more complete description is obtained by treating the gravitational field not as a force, but as the curvature of space-time as in Einstein’s theory of relativity. In Ref. [13] it was shown that the Klein-Gordon equation in a curved space takes the GP form after a simple transformation and includes extra terms that are interpreted as the gravitational field and a finite temperature contribution. The gravitational components include terms beyond the Newtonian gravity. In this work we propose that these terms could become the focus of future experimental measurements.

I. KLEIN–GORDON IN A WEAK GRAVITATIONAL FIELD

References [14] and [15] derive the relativistic theory of a BEC in a quantum field theory description. Starting from the Klein-Gordon equation in a flat space-time, a generalized GP equation is obtained for relativistic and finite temperature fields. The generalization for an expanding universe is given in [13]. Here we apply the same approach on a curved space for a BEC living in a weak gravitational field, the Klein Gordon equation for a complex scalar field is given by

$$\Box \Phi - \frac{dV}{d\Phi^*} = (\nabla_\mu + i \frac{e}{\hbar c} A_\mu)(\nabla_\mu + i \frac{e}{\hbar c} A_\mu)\Phi - \frac{dV}{d\Phi^*} = \frac{2m^2}{\hbar^2} U_{\text{ext}} \Phi,$$

(1)

where $\Phi$ is the complex scalar field and $A_\mu$ is the corresponding electromagnetic four vector. In this work we use the Mexican hat scalar field potential given by

$$V = \frac{m^2 c^2}{\hbar^2} |\Phi|^2 + \frac{\lambda}{4\hbar^2} |\Phi|^4.$$

(2)

We add an external interaction $U_{\text{ext}}$ that can represent for example the potential that confines the condensate

---

* Part of the Instituto Avanzado de Cosmología (IAC) collaboration http://www.iac.edu.mx/
1 Electronic address: tmatos@fis.cinvestav.mx
2 Electronic address: egomez@fisica.uaslp.mx
like a laser, a box or the gravitational field seen as an
external one. The metric of a weak gravitational field is

\[ ds^2 = -(1 + 2\psi)dt^2 + (1 - 2\phi)g_{ij}dx^idx^j, \]

where \( \psi, \phi \) are the gravitational potentials and \( g_{ij} \) is
the 3-dimensional flat-space metric. The Klein-Gordon
equation with this metric reads

\[ \Box_{\text{NEW}} \Phi - \frac{dV}{d\Phi^*} + ic[1 + 2\phi] \nabla \cdot \nabla \Phi + 2i\hat{c}[1 + 2\phi] \nabla \Phi = 0. \]

The metric not only produces the gravitational potential,
but it also affects the kinetic energy term, something that
does not happen in the traditional treatment that adds
the gravitational potential by hand.

In a curved space-time in the weak field limit the
Maxwell equations read

\[ \nabla \cdot \mathbf{E} + \nabla (\psi - 3\phi) \cdot \mathbf{E} = \rho \]
\[ \mathbf{E} + (\psi - 3\phi) \mathbf{E} + \mathbf{B} \times \nabla (\psi - 3\phi) - \nabla \times \mathbf{B} = \mathbf{J} \]
\[ \nabla \cdot \mathbf{B} - 4\mathbf{B} \cdot \nabla \phi = 0 \]

(6)

where \( \rho \) and \( \mathbf{J} \) are the charge density and
electric current respectively and the electric \( \mathbf{E} \) and magnetic \( \mathbf{B} \) fields are defined by the relations

\[ \mathbf{B} = -\frac{\nabla \times \mathbf{A}}{(1 - 2\phi)^2} \]
\[ \mathbf{E} = \frac{1}{(1 - 2\phi)(1 + 2\psi)} \left( \frac{\partial \mathbf{A}}{\partial t} - \nabla \phi \right) \]

(7)

To write down the Klein-Gordon equation (Eq. [1]) in its
GP form we apply the transformation \( \Phi = \Psi e^{-imc^2t/i\hbar} \)

\[ i(1 - 2\phi)\hbar \dot{\Psi} = -\frac{\hbar^2}{2m} \Box_{\text{NEW}} \Psi + \frac{\lambda}{2m} |\Psi|^2 \Psi + mc^2(\phi + U_{\text{ext}})\Psi - \frac{e^2}{2m}((1 + 2\phi)A^2 - (1 - 2\psi)\dot{\phi} + \nabla (\psi - \phi) \cdot \mathbf{A} + \frac{1}{c} (\psi + 3\phi) \dot{\phi}) \]
\[ - \frac{e^2}{2m}((1 + 2\phi)A^2 - (1 - 2\psi)\phi) - \frac{e}{m} [(1 + 2\phi) \mathbf{A} \cdot \nabla \Psi - (1 - 2\psi)\phi(\frac{1}{c} \Psi - i\dot{m} \Psi)]. \]

(8)

where \( \dot{m} = mc^2/h \). This last equation is again the Klein-
Gordon equation (Eq. [1]), but written in terms of the
function \( \Psi \).

If we remove in Eq. [5] the gravitational fields \( \phi \) and
\( \psi \) and we set the charge \( e = 0 \), we recover the Gross Pitaveskii equation in the non-relativistic limit. Therefore
we interpret Eq. [5] as the generalization of the GP
equation for a relativistic charged Bose particle in elec-
 tromagnetic media in a weak gravitational field.

In Einstein’s theory with a weak gravitational field we
have \( \phi = \psi \). In some alternative theories of gravity this
identity does not follow [18]. For simplicity, we assume
a static gravitational field \( (\partial_t \phi = \dot{\phi} = 0) \) like that of the
Earth. The result after taking the Newtonian limit is

\[ i(1 - 2\phi)\hbar \dot{\Psi} = -\frac{\hbar^2}{2m} \Box_{\text{NEW}} \Psi + \frac{\lambda}{2m} |\Psi|^2 \Psi + mc^2 \phi \Psi + mc^2 U_{\text{ext}} \Psi - \frac{e^2}{2m}((1 + 2\phi)A^2 - (1 - 2\phi)\phi) \Psi - \frac{e}{m} [(1 + 2\phi) \mathbf{A} \cdot \nabla \Psi - (1 - 2\phi)\phi(\frac{1}{c} \Psi - i\dot{m} \Psi)]. \]

(9)

We interpret Eq. [9] as the generalization of the GP
equation in a gravitational field immersed in an electro-
 magnetic potential.

We first study the difference between the traditional
approach to Newtonian gravity to the curved space-time
derivation used in this work. For that we neglect the
electromagnetic field \( (e = 0) \). Taking \( \phi = 0 \), we recover
the flat space result that corresponds to the usual GP
equation

\[ i\hbar \dot{\Psi} = -\frac{\hbar^2}{2m} \nabla^2 \Psi + \frac{\lambda}{2m} |\Psi|^2 \Psi + me^2 U_{\text{ext}} \Psi \]

(10)

Taking \( U_{\text{ext}} = gz/c^2 \) we obtain the gravitational po-
tential of the Earth \( (mgz) \). Here the gravitational in-
teraction is added as an external potential. This case
represents the Newtoninan version of a BEC in a grav-
itational field.
In the version of Einstein we introduce the gravitational potential through the space-time curvature. Taking $U_{ext} = 0$ and $\phi = gz/c^2$ gives the gravitational potential as well, but there is a correction since $\phi$ appears also in the $\Psi$ term of Eq. 9. To determine the size of the correction lets consider an atomic interferometer. Here an atom (or the BEC wavefunction) is splitted in two, and the two components evolve in free fall for some time before recombining them. The separation is usually smaller than 1 cm [8], but it is conceivable to achieve separations up to 1 m in the near future by combining large free fall times with large momentum transfer. A $z = 1$ m separation gives a $\phi \simeq 10^{-16}$, that is small compared to the 1 in the $\Psi$ term of Eq. 9.

The combination $gz/c^2$ is the perturbative parameter, in analogy to the term $v^2/c^2$ of special relativity. The observation of the correction requires doing atomic interferometry with 16 digits of precision. The current record is at 11 digits [3], but the projected sensitivity of the space project is at 15 digits of precision [10]. Having the atoms in space goes in the wrong direction since that eliminates $g$. Instead, the present work suggests moving towards atomic interferometry in strong gravitational fields.

It is possible to reach higher values of $g$ by taking advantage of the equivalence principle. Instead of the gravitational field of the Earth one could accelerate the complete experimental setup. This is clearly complicated, but in the case of acceleration introduced by Bloch oscillations things might be simpler. Here it is only necessary to sweep the phase of the two counter propagating lasers beams to produce an accelerated lattice [13]. There is a price to pay in the precision of the measurement due the reduced measurement time since, for example, a linear acceleration of 100 $g$ gives already a displacement of 5 m in 0.1 s. An alternative is to have the BEC in a centrifugal force. Here the gravitational field is $\phi = v^2/c^2 \ln(R/R_0)$, with $v_c$ the tangential velocity of the BEC, $R$ the radius of the centrifuge and $R_0$ an arbitrary reference radius. For $R = 1$ m and $v_c > 3.2$ m/s the centrifugal force is bigger than the gravitational force of the Earth.

We apply a Madelung transformation $\Psi = \sqrt{n}e^{iS}$ to Eq.9 with $e = 0$, with $n$ representing the density number of particles and $S$ the velocity super potential

$$\mathbf{v} = \frac{\hbar}{m} \nabla S.$$  \hspace{1cm} (11)

The real and imaginary parts of Eq. 9 in terms of the $n$ and $S$ variables are

$$(1 - 2\phi)n + (1 + 2\phi)\nabla \cdot (nv) - (1 - 2\phi)\dot{j} = 0 \hspace{1cm} (12)$$

$$+ \frac{\lambda}{2m^2} \nabla^2 - \frac{\hbar^2}{2m^2c^2} \nabla \cdot \nabla \frac{\sqrt{n}}{\sqrt{n}} + \frac{1}{2}(1 - 2\phi)\frac{v^2}{c^2} = 0 \hspace{1cm} (13)$$

respectively, with the flux $j = nv$ and

$$j = n\frac{\hbar}{mc^2} \dot{S} = \frac{n}{c}$$ \hspace{1cm} (14)

We interpret Eq. 12 as the generalized continuity equation in gravitational and electromagnetic fields. This equation differs from the one derived using pure fluid mechanics in the factors in front of the density and velocity terms. Equations 12 and 13 are the Klein-Gordon equation in a weak gravitational field, in other words, they are the Einstein-Klein-Gordon equations written in the variables $n$ and $S$. We calculate the gradient of Eq. 13 to obtain the corresponding momentum equation

$$\nabla F_{ext} + n(1 + 2\phi) + 2\frac{\nabla^2}{c^2}F_{\phi} - \nabla p + nF_{\phi} = \sigma$$ \hspace{1cm} (15)

where $p$ is the pressure with a state equation $p = (\lambda/m^2)n^2$, $F_{ext} = -\nabla U_{ext}$ and the gravitational force by $F_{\phi} = -c^2\nabla \phi$. Equations 12 and 15 correspond to the Klein-Gordon equation (Eq. 1) written in terms of $n$ and $v$.

The difference between the two procedures to include the gravitational potential becomes evident from Eq. 15. If we consider the gravitational field as an external force (non-covariant case), then space-time is flat, $\phi = 0$ and thus $F_{\phi} = 0$. In this case the hydrodynamic equation corresponds to the traditional one. If instead we use the covariant form of the equations we must set $F_{ext} = 0$ and include the gravitational force in the $F_{\phi}$ term. The coefficient in front of both terms differs by $2\phi/c^2 = 2gz/c^2$ which is the same parameter we obtained before.

In summary, we present a field theoretical approach to describe a BEC in a curved space. The derivation results in a generalized GP equation in a gravitational and electromagnetic fields. We identify the expansion parameter $gz/c^2$ that gives corrections that could be verified with precise enough atomic interferometry. This makes BECs in strong gravitational (or accelerated) fields a very interesting candidate to study corrections beyond Newtonian gravity.

Acknowledgments

This work was partially supported by CONACyT México under grants CB-2009-01, no. 132400, CB-2011, no. 166212, Xiuhoool cluster at Cinvestav and 10101/131/07 C-234/07 of the Instituto Avanzado de Cosmologia (IAC) collaboration (http://www.iac.edu.mx/). EG aknowledges support from CONACyT and Fundación Marcos Moslinsky.
[1] 2010 CODATA recomended values.
[2] P.J. Mohr, B.N. Taylor and D.B. Newell, Rev. Mod. Phys. 84, 1527 (2012).
[3] M. Kasevich and S. Chu, Phys. Rev. Lett. 67, 181 (1991).
[4] S. Dimopoulos, P.W. Graham, J.M. Hogan, M.A. Kasevich and S. Rajendran, Phys. Rev. D 78, 122002 (2008).
[5] S. Dimopoulos, P.W. Graham, J.M. Hogan and M.A. Kasevich, Phys. Rev. Lett. 98, 111102 (2007).
[6] G. Ferrari, N. Poli, F. Sorrentino and G.M. Tino, Phys. Rev. Lett. 97, 060402 (2006).
[7] B. Lamine, R. Hervé, A. Lambrecht and S. Reynaud, Phys. Rev. Lett. 96, 050405 (2006).
[8] S.M. Dickerson, J.M. Hogan, A. Sugarbaker, D.M.S. Johnson and M. Kasevich, Phys. Rev. Lett. 111, 083001 (2013).
[9] T.V. Zoest et al., Science 328, 1540 (2010).
[10] R. Geiger et al., Nat. Commun. 2, 474 (2011).
[11] G.M. Tino et al., Nucl. Phys. B Proc. Suppl. 243-244, 203 (2013).
[12] Claus Lämmerzahl, Phys. Lett. A 203, 12 (1995).
[13] A. Suarez and T. Matos, Mon. Not. Roy. Astron. Soc. 416, 87 (2011). [arXiv:1101.4039 [gr-qc]].
[14] T. Matos and A. Suarez, Europhys. Lett. 96, 56005 (2011). [arXiv:1110.3114 [gr-qc]].
[15] T. Matos and A. Suarez, arXiv:1103.5731 [gr-qc].
[16] Chung-Pei Ma and Edmund Bertschinger, Astrophys. J. 455, 7 (1995).
[17] L.A. Ureña-Lopez, Phys. Rev. D 90, 027306 (2014). [arXiv:1310.8601 [astro-ph.CO]].
[18] A. Aviles and J. L. Cervantes-Cota, Phys. Rev. D 84, 083515 (2011). [Erratum-ibid. D 84, 089905 (2011)] [arXiv:1108.2457 [astro-ph.CO]].
[19] P. Cladé, E. de Mirandes, M. Cadoret, S. Guellati-Khélifa, C. Schwob, F. Nez, L. Julien and F. Biraben, Phys. Rev. A 74, 052109 (2006).
[20] C. J. Pethick and H. Smith, Bose-Einstein Condensates in Dilute Gases. Cambridge University Press, (2008)
[21] T. Matos, E. Castellanos and A. Suarez, arXiv:1207.4416 [physics.gen-ph].