Turing instability in Reaction-Diffusion models on complex networks

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Abstract
In this paper, the Turing instability in reaction-diffusion models defined on complex networks is studied. Here, we focus on three types of models which generate complex networks, i.e. the Erdős-Rényi, the Watts-Strogatz and the threshold network models. From analysis of the Laplacian matrices of graphs generated by these models, we reveal that the stable-unstable regions of a spatially homogeneous solution completely differ, depending on network structures. In particular, we approximately argue the existence of the stable-unstable regions in the cases of regular enhanced ring lattices which include regular circles, and networks generated by the threshold network model when the number of vertices is large enough.

Keywords: Turing instability; Reaction-diffusion models on networks; Complex network; Pattern formation

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1. Introduction

We can observe various types of pattern phenomena in nature. In order to understand the pattern formation mechanisms of such phenomena, mathematical models have been proposed and analyzed from the viewpoint of both numerical and theoretical studies. Among these models, reaction-diffusion systems have attracted many researchers ([1]). Though the system which describes a local interaction and a long-range dispersal between chemical substances or biological species is rather simple, Turing stated that spatially inhomogeneous structures can be formed in a self-organized way under certain conditions([2]). Since then, a lot of studies on the reaction-diffusion systems have been reported.

Turing considered the following system of partial differential equations:

\[
\begin{align*}
\frac{\partial u}{\partial t} &= d_u \Delta u + f(u, v), \\
\frac{\partial v}{\partial t} &= d_v \Delta v + g(u, v),
\end{align*}
\]  

(1.1)

where \( u = u(t, x) \) and \( v = v(t, x) \) indicate concentrations of chemical substances or population densities of biological species at time \( t \) and position \( x \), \( d_u \) and \( d_v \) mean respectively diffusion coefficients of \( u \) and \( v \), and the functions \( f(u, v) \) and \( g(u, v) \) express a local interaction between \( u \) and \( v \). In addition, Turing gave the following assumption on the reaction system without diffusion terms

\[
\begin{align*}
\frac{du}{dt} &= f(u, v), \\
\frac{dv}{dt} &= g(u, v).
\end{align*}
\]  

(1.2)

**Assumption.** The system (1.2) possesses an equilibrium point \((u, v) = (\bar{u}, \bar{v})\) and it is asymptotically stable.

In this framework, Turing derived a paradox that the equilibrium solution \((u, v) = (\bar{u}, \bar{v})\) in (1.1) with suitable boundary conditions can be destabilized in spite of adding the diffusion terms which possess a smoothing effect of spatial heterogeneity even though the equilibrium point \((u, v) = (\bar{u}, \bar{v})\) is stable in (1.2). This is well known as the diffusion-induced instability or the Turing instability. As a consequence of the Turing instability of \((u, v) = (\bar{u}, \bar{v})\), (1.1)
exhibits spatially inhomogeneous structures. Therefore, the Turing instability is regarded as important for the onset of the pattern formation on the reaction-diffusion systems. Let us review the Turing instability in a little more detail below.

1.1. Reaction-diffusion system on continuous media

In this subsection, we consider the following linear reaction-diffusion system in one space dimension:

\[
\frac{\partial u}{\partial t} = d_u \frac{\partial^2 u}{\partial x^2} + au + bv, \\
\frac{\partial v}{\partial t} = d_v \frac{\partial^2 v}{\partial x^2} + cu + dv,
\]

with periodic boundary conditions. We note that (1.3) is derived from the linearization of (1.1) around the equilibrium solution \((u, v) = (\bar{u}, \bar{v})\). Obviously, we know that (1.3) possesses the equilibrium solution \((u, v) = (0, 0)\). As conditions of the Turing instability, we assume that the equilibrium point \((u, v) = (0, 0)\) is asymptotically stable in the system of ordinary differential equations without diffusion terms

\[
\begin{align*}
\frac{du}{dt} &= au + bv, \\
\frac{dv}{dt} &= cu + dv,
\end{align*}
\]

so that, the parameters \(a, b, c\) and \(d\) satisfy the following conditions:

\(ad - bc > 0\) and \(a + d < 0\).

Under the condition (1.5), we consider the system including diffusion terms (1.3). Expressing a solution of (1.3) with periodic boundary conditions by the Fourier series, we obtain a sequence of systems of ordinary differential equations for each Fourier mode \(n\),

\[
\frac{d}{dt} \begin{pmatrix} u_n \\ v_n \end{pmatrix} = \begin{pmatrix} -d_u \left(\frac{2n\pi}{L}\right)^2 + a & b \\ c & -d_v \left(\frac{2n\pi}{L}\right)^2 + d \end{pmatrix} \begin{pmatrix} u_n \\ v_n \end{pmatrix},
\]

\(n = 0, 1, 2, \ldots\). From these linear systems, we find out the stability of the equilibrium solution \((u, v) = (0, 0)\) by investigating the sign of eigenvalues of the
matrix arising in (1.6). For each $n$, the bifurcation curve where an eigenvalue takes zero is given in $(d_u, d_v)$ plane as follows:

$$\Gamma_n = \{(d_u, d_v) \in \mathbb{R}^2 \mid (d_u k^2 - a)(d_v k^2 - d) - bc = 0\},$$

where $k = \frac{2n\pi}{L}$ and $k^2$ is an eigenvalue of $-\frac{\partial^2}{\partial x^2}$ with periodic boundary conditions. In addition, when we view $d_v$ as the function of $d_u$ for each $\Gamma_n$, the asymptote for each bifurcation curve is $d_u = \frac{a}{k^2}$. Therefore, when we define $D_n = \{(d_u, d_v) \in \mathbb{R}^2 \mid (d_u k^2 - a)(d_v k^2 - d) - bc < 0\}$, the unstable region of the spatially homogeneous state $(u, v) = (0, 0)$ is given by $\cup_{n=1}^{\infty} D_n$. When we use, for example, the following parameter values

$$\begin{pmatrix}
a & b \\
c & d
\end{pmatrix} = \begin{pmatrix}
1 & -2 \\
2 & -2
\end{pmatrix} \quad \text{and} \quad L = 1,$$

then the stable-unstable region on $(d_u, d_v)$ plane is shown in Figure 1. We know

![Figure 1: Stable-unstable region of the equilibrium solution $(u, v) = (0, 0)$ for the reaction-diffusion system (1.3) with periodic boundary conditions. The horizontal and vertical axes indicate $d_u$ and $d_v$, respectively.](image)

from this figure that the equilibrium solution $(u, v) = (0, 0)$ can be destabilized if the value of $d_v$ is larger than that of $d_u$ in addition to the condition (1.5). In other words, the difference between the two diffusion coefficients yields the instability of the spatially homogeneous state.

However, since (1.3) is a system describing an interaction and a dispersal between $u$ and $v$ on continuous media, it can not represent an interaction on
a spatially discrete environment such as dynamics of metapopulation, cellular networks of biological morphogenesis and networks of diffusively coupled chemical reactors. Therefore, in order to treat such situations, studies on reaction-diffusion models defined on networks have proceeded (3, 4, 5, 6).

1.2. Reaction-diffusion model on two vertices

In this subsection, we consider the Turing instability for a reaction-diffusion model defined on the simplest network with two vertices. Here, the vertices are regarded as a discrete environment such as patchy habitats, cells and chemical reactors. A local interaction on each vertex is given by

\[
\begin{align*}
  u_{it} &= f(u_i, v_i), \\
  v_{it} &= g(u_i, v_i),
\end{align*}
\]

for \( i = 1, 2 \), where \( u = u_i(t) \) and \( v = v_i(t) \) respectively represent densities or concentrations on the \( i \)th vertex at time \( t \) and the functions \( f(u_i, v_i) \) and \( g(u_i, v_i) \) indicate an interaction between \( u_i \) and \( v_i \) on the \( i \)th vertex. Here, when we take the flux of densities between two vertices based on Fick’s law into account, the following model is obtained:

\[
\begin{align*}
  u_{1t} &= d_u(u_2 - u_1) + f(u_1, v_1), \\
  u_{2t} &= d_u(u_1 - u_2) + f(u_2, v_2), \\
  v_{1t} &= d_v(v_2 - v_1) + g(u_1, v_1), \\
  v_{2t} &= d_v(v_1 - v_2) + g(u_2, v_2),
\end{align*}
\]

where \( d_u \) and \( d_v \) mean respectively the diffusivities of densities \( u \) and \( v \) between the vertices. We assume that (1.7) possesses an equilibrium state \((u_i, v_i) = (\overline{u}, \overline{v})\) \((i = 1, 2)\) and it is asymptotically stable. Then, the linearized system around the homogeneous state \((u_i, v_i) = (\overline{u}, \overline{v})\) of (1.8) is

\[
\begin{align*}
  U_{1t} &= d_u(U_2 - U_1) + aU_1 + bV_1, \\
  U_{2t} &= d_u(U_1 - U_2) + aU_2 + bV_2, \\
  V_{1t} &= d_v(V_2 - V_1) + cU_1 + dV_1, \\
  V_{2t} &= d_v(V_1 - V_2) + cU_2 + dV_2,
\end{align*}
\]
where \( a = f_u(u, v) \), \( b = f_v(u, v) \), \( c = g_u(u, v) \) and \( d = g_v(u, v) \). Therefore, the stability conditions on the parameter values are \( ad - bc > 0 \) and \( a + d < 0 \). (1.9) is rewritten as

\[
\begin{aligned}
\frac{d}{dt} \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} &= -d_u \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} + a \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} + b \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}, \\
\frac{d}{dt} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} &= -d_v \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} + c \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} + d \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}.
\end{aligned}
\]

The appearing matrix in the first term of the right hand side is called the Laplacian matrix of a graph. In this case, the graph consists of two vertices and an edge connecting them. Since this system of ordinary differential equations is linear, we can solve it and find that the bifurcation curve which divides the stable and unstable regions is obtained in \((d_u, d_v)\) plane as

\[
\Gamma = \{(d_u, d_v) \in \mathbb{R}^2 | (2d_u - a)(2d_v - d) - bc = 0\}.
\]

Thus, for a pair \((d_u, d_v)\) satisfying \((2d_u - a)(2d_v - d) - bc < 0\), we find that the homogeneous state \((U_i, V_i) = (0, 0) (i = 1, 2)\) is unstable. From this result, we know that the Turing instability occurs on the simplest network with two vertices. So far, the Turing instability arising in reaction-diffusion models defined on networks with a small number of vertices has been investigated ([3, 4, 5, 6]). Recently, studies on Turing patterns formed on complex networks with a large number of vertices have proceeded([7, 8]). However, a relation between the Turing instability and network structures with a large number of vertices is not clear. In this study, considering reaction-diffusion models on complex networks with a large number of vertices, we reveal the relation between network structures and the Turing instability. We emphasize that the Turing instability is an important concept as the onset of self-organized pattern formation. However, studies focusing on the Turing instability in reaction-diffusion models on complex networks with many vertices are very few.

This paper is organized as follow: in the next section, we formulate reaction-diffusion models defined on complex networks as an extension of (1.8), which we
discuss in this paper. Sections 3, 4 and 5 are devoted to computer-aided analysis of the Turing instability in reaction-diffusion models on networks generated by the Erdős-Rényi, the Watts-Strogatz and the threshold network models, respectively. We reveal that these analyses derive different results on the Turing instability of the equilibrium solution, depending on network structures. In section 6, we give theoretical results on the instability when the number of vertices is large enough. We complete this paper in section 7 where concluding remarks and future works are listed.

2. Formulation of reaction-diffusion model on a graph

As an analogy of the reaction-diffusion models on continuous media and on networks with a small number of vertices, we formulate a model on complex networks with a large number of vertices $N$.

When there is no connection between vertices, the dynamics on each vertex is described by a local interaction only as follows:

$$
\begin{align*}
    u_{it} &= f(u_i, v_i), \\
    v_{it} &= g(u_i, v_i),
\end{align*}
$$

(2.1)

where $i = 1, 2, \cdots, N$ and a pair $(u_i, v_i) = (u_i(t), v_i(t))$ denotes some quantities such as densities of biological species or concentrations of chemical substances on the $i$th vertex. As well as the Turing instability on continuous media, we assume the existence of an asymptotically stable equilibrium point $(u_i, v_i) = (\overline{u}, \overline{v})$ ($i = 1, \cdots, N$) for (2.1). This means that

$$
a + d < 0 \text{ and } ad - bc > 0
$$

(2.2)

from the information of the linearized system of (2.1) around $(\overline{u}, \overline{v})$, where $a = f_u(\overline{u}, \overline{v}), b = f_v(\overline{u}, \overline{v}), c = g_u(\overline{u}, \overline{v})$ and $d = g_v(\overline{u}, \overline{v})$. In addition to (2.2), we give an assumption on the interaction of quantities between vertices. In other words, taking connections between vertices into account, if the $i$th and the $j$th vertices are connected, we suppose that there are the fluxes between these vertices. On the other hand, there is no flux if two vertices are not connected.
We assume that the flux is given by Fick’s law of diffusion which means that the flux is proportional to the difference of quantities on the two vertices. Therefore, the dynamics of \( u_i \) and \( v_i \) on the \( i \)th vertex is described as

\[
\begin{align*}
  u_{it} &= d_u \sum_{j=1}^{N} A_{ij} (u_j - u_i) + f(u_i, v_i), \\
  v_{it} &= d_v \sum_{j=1}^{N} A_{ij} (v_j - v_i) + g(u_i, v_i),
\end{align*}
\]

(2.3)

where for \( i, j = 1, \ldots, N \),

\[
A_{ij} = \begin{cases} 
  1 & \text{if the } i \text{th and the } j \text{th vertices are connected,} \\
  0 & \text{if disconnected,}
\end{cases}
\]

and \( A_{ii} = 0 \) because we do not consider any self-loop in the present paper. Moreover, since we focus on undirected graphs, the matrix \( A \) with the elements \( A_{ij} \) is a symmetric matrix with \( A_{ji} = A_{ij} \). And, positive constants \( d_u \) and \( d_v \) mean respectively diffusivities of these quantities between vertices. Also, the degree of edges connecting to the \( i \)th vertex is expressed as \( k_i := \sum_{j=1}^{N} A_{ij} \).

Thus, for each \( i = 1, \ldots, N \), we can rewrite the flux term as

\[
\sum_{j=1}^{N} A_{ij} (u_j - u_i) = -\sum_{j=1}^{N} L_{ij} u_j,
\]

where \( L_{ij} = \delta_{ij} k_i - A_{ij} \) (\( \delta_{ij} \) is Kronecker’s delta). The matrix \( L \) with the elements \( L_{ij} \) is called the Laplacian matrix of a graph. We note that the Laplacian matrix \( L \) is varied according to network structures and eigenvalues of the Laplacian matrix \( L \) give an important information on the Turing instability of the equilibrium solution. We consider the following linear reaction-diffusion model with various types of network structures:

\[
\begin{align*}
  u_{it} &= -d_u \sum_{j=1}^{N} L_{ij} u_j + au_i + bv_i, \\
  v_{it} &= -d_v \sum_{j=1}^{N} L_{ij} v_j + cu_i + dv_i,
\end{align*}
\]

(2.4)
\[ i = 1, 2, \cdots, N. \] We call (2.4) a reaction-diffusion model on a graph. Obviously, we know that (2.4) possesses the asymptotically stable equilibrium solution \((u_i, v_i) = (0, 0) \ (i = 1, \cdots, N)\) when we impose the condition (2.2) on the parameters and \(d_u = d_v = 0\). In this paper, we use the following parameter values
\[
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix} =
\begin{pmatrix}
 1 & -2 \\
 2 & -2
\end{pmatrix}
\]
in all numerics. In order to specify the Laplacian matrix \(L\), different types of models are proposed. Below, we investigate a relation between the Turing instability and network structures which are generated by the various models. By using eigenvalues of the Laplacian matrix of a graph, we obtain bifurcation curves on \((d_u, d_v)\) plane
\[
\Gamma_i = \{(d_u, d_v) \in \mathbb{R}^2 | (d_u \lambda_i - a)(d_v \lambda_i - d) - bc = 0\},
\]
where \(\lambda_i\) is the \(i\)th eigenvalue of the Laplacian matrix of the graph. When we view \(d_v\) as the function of \(d_u\) for each \(\Gamma_i\), the asymptote of each curve is \(d_u = \frac{a}{\lambda_i}\). Therefore, we can indicate the unstable region of the equilibrium solution \((u_i, v_i) = (0, 0) \ (i = 1, \cdots, N)\) as \(\bigcup_{i=1}^{N} D_i\), where \(D_i = \{(d_u, d_v) \in \mathbb{R}^2 | (d_u \lambda_i - a)(d_v \lambda_i - d) - bc < 0\}\) for each eigenvalue \(\lambda_i\).

3. Networks generated by the Erdős-Rényi model

In this section, we deal with a reaction-diffusion model on a graph which is stochastically generated by the Erdős-Rényi model and reveal the stable-unstable region of the equilibrium solution \((u_i, v_i) = (0, 0) \ (i = 1, 2, \cdots, N)\) of (2.4) from the linear stability analysis with the aid of a computer. The stochastic graph with \(N\) vertices is produced as follows:

*Erdős-Rényi model.*

- An edge is set between each pair of distinct vertices with probability \(p\), independently of the other vertices.
Here, we consider a connected graph only. Obviously, the number of edges is zero when the probability \( p = 0 \), so this graph is disconnected. On the other hand, when \( p = 1 \), it becomes the complete graph with \( N \) vertices since every pair of distinct vertices is connected by an edge. Since the average degree of an Erdős-Rényi random graph is \( p(N - 1) \), the closer the value \( p \) approaches one or the larger the number of vertices is, the more edges a graph gets. In other words, this means that the mobility of substances \( u_i \) and \( v_i \) \((i = 1, \cdots, N)\) between vertices is fluxive due to a lot of connections, therefore we can expect that the equilibrium solution \((u_i, v_i) = (0, 0)\) \((i = 1, \cdots, N)\) tends to be stabilized. We are interested in the transition of the stable-unstable region of the equilibrium solution \((u_i, v_i) = (0, 0)\) \((i = 1, \cdots, N)\) for (2.4), depending on the link probability \( p \) and the number of vertices \( N \). In order to do that, we use the information on eigenvalues of the Laplacian matrix \( L \) of a graph. Figure 2 shows examples of an eigenvalue set of the Laplacian matrix \( L \) for each \( N \) when the link probability \( p \) is varied. We note that a graph is generated stochastically by the Erdős-Rényi model, so it appears that eigenvalues are rather fluctuated according to each graph structure, while those of the complete graph when \( p = 1 \) is deterministic. Figure 2(a) illustrates eigenvalues for \( N = 500 \) when the value of \( p \) is varied. The correspondence between the values of \( p \) and the colors of curves is indicated at the upper right of the figure. We can see from Figure 2(a) that all eigenvalues possess relatively small values when the link probability \( p \) is low. However, each eigenvalue increasingly approaches the corresponding one of the complete graph as the probability \( p \) tends to one. When \( p = 1 \), all the eigenvalues except one zero eigenvalue become \( N \) which is the same as the number of vertices. Moreover, since we consider connected graphs only, the smallest eigenvalue \( \lambda_1 \) is zero, which is simple, the second smallest eigenvalue \( \lambda_2 \) is greater than zero, and the largest eigenvalue \( \lambda_N \) is equal to or less than \( 2M \) for any \( p \), where \( M \) is the total number of edges. When the number of vertices is increased to \( N = 2000 \), \( N = 5000 \) and \( N = 10000 \), the number of eigenvalues is naturally varied, but qualitatively similar configurations of the eigenvalue distribution are obtained for each \( p \), as shown in Figures 2(b), (c)
Figure 2: Examples of an eigenvalue set of the Laplacian matrix for a graph generated by the Erdős-Rényi model. The numbers of vertices are respectively (a) $N = 500$, (b) $N = 2000$, (c) $N = 5000$ and (d) $N = 10000$, and the link probability between vertices $p$ is indicated in each figure. The horizontal and vertical axes mean eigenvalue numbers $i$ and these values $\lambda_i$, respectively. Eigenvalues are sorted in ascending order, that is $0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_N$.

and (d). We also observe that the similar tendency mentioned above will hold independently of the number of vertices $N$, that is, each eigenvalue converges to the corresponding one of the complete graph as the value of $p$ increasingly tends to one. However, we note that non-zero eigenvalues when $p = 1$ grow according to an increase of the number of vertices $N$.

Next, we illustrate the stable-unstable region of the equilibrium solution $(u_i, v_i) = (0, 0)$ ($i = 1, \cdots, N$) of (2.4) based on the eigenvalues of the Laplacian matrix $L$. Because of stochastic nature of network construction, we can easily speculate the fluctuation of eigenvalues, depending on network structures.
Therefore, we represent the stable-unstable region of the equilibrium solution with probability by repeating the following trial for 500 samples of the graph and taking the average:

- calculate eigenvalues of the Laplacian matrix for the network,
- compute the stable-unstable region of \((u_i, v_i) = (0, 0) (i = 1, \cdots, N)\) in \((d_u, d_v)\) plane, based on the bifurcation curves,
- represent the unstable region as one and the stable region as zero in \((d_u, d_v)\) plane.

Consequently, we present the probability that the equilibrium solution \((u_i, v_i) = (0, 0) (i = 1, \cdots, N)\) is destabilized, as shown in Figure 3 when \(N = 500\) and the value of \(p\) is varied. These figures mean that the black region denotes that the equilibrium solution \((u_i, v_i) = (0, 0) (i = 1, \cdots, N)\) is unstable with probability one and the white one denotes that it is unstable with probability zero. We express the probability between those with tones of gray. From this result, we know that the region where the equilibrium solution is unstable with high probability is large when \(p\) is small, while the unstable region gradually shrinks as the value of \(p\) approaches one. The probability \(p\) close to one easily sets an edge between each pair of distinct vertices, therefore, an increase of the total number of edges results in stabilizing the equilibrium solution \((u_i, v_i) = (0, 0) (i = 1, \cdots, N)\) of (2.4) through the active mobility of substances between vertices. On the other hand, if \((d_u, d_v)\) takes a pair of values in the vicinity of the origin, for instance \((d_u, d_v) = (0.001, 0.02)\), the equilibrium solution \((u_i, v_i) = (0, 0) (i = 1, \cdots, N)\) is stable with a high probability when \(p = 0.01\), but it gets to be included in the unstable region as \(p\) is increased. We find that there exists a certain parameter region such that the stability of \((u_i, v_i) = (0, 0) (i = 1, \cdots, N)\) changes from the stable state to the unstable one according to the values of \(p\). Figures 4 and 5 show the stable-unstable regions of the equilibrium solution for \(N = 2000\) and \(N = 5000\), respectively. From these figures, the similar situations hold even when the numbers of vertices are \(N = 12\).
Figure 3: The stable-unstable regions of the equilibrium solution \((u_i, v_i) = (0, 0)\) \((i = 1, \cdots, N)\) for (2.4) on the Erdős-Rényi random graphs. The number of vertices is \(N = 500\) and the link probability between vertices \(p\) is indicated at the bottom of figures. The black and white regions denote to be unstable with probability one and unstable with probability zero, respectively. The horizontal and vertical axes indicate \(d_u\) and \(d_v\), respectively.

2000 or \(N = 5000\), respectively. For the small value of \(p\), the region where the equilibrium solution \((u_i, v_i) = (0, 0)\) \((i = 1, \cdots, N)\) is destabilized with
Figure 4: The stable-unstable regions of the equilibrium solution \((u_i, v_i) = (0, 0)\) \((i = 1, \ldots, N)\) for (2.4) on the Erdős-Rényi random graphs. The number of vertices is \(N = 2000\) and the connection probability between vertices \(p\) is indicated at the bottom of figures. The black and white regions denote to be unstable with probability one and unstable with probability zero, respectively. The horizontal and vertical axes indicate \(d_u\) and \(d_v\), respectively.

high probability is wide. However, the unstable region shrinks gradually as to an increase of the value \(p\). The closer the probability \(p\) approaches one, the larger the second smallest eigenvalue \(\lambda_2\) (see Figure 2). This corresponds to that the region where the equilibrium solution \((u_i, v_i) = (0, 0)\) \((i = 1, \ldots, N)\) is destabilized with high probability is gradually narrow because we have the asymptote of the bifurcation curve \(d_u = \frac{\lambda_2}{\lambda_2}\) for the second smallest eigenvalue \(\lambda_2\). In the Erdős-Rényi model with a fixed \(p\), when the number of vertices increases, the average degree also increases. This means that the equilibrium solution \((u_i, v_i) = (0, 0)\) \((i = 1, \ldots, N)\) becomes difficult to be destabilized because the active mobility of substances promotes the homogenization. This expectation is fit for the numerical results in Figures 3, 4 and 5. For example,
for $p = 0.1$, we find that the unstable region becomes narrow, depending on the increase of the number of vertices. However, when we change $N$ and $p$ while keeping the average degree, the configurations of the eigenvalue distribution are quite similar, therefore, the stable-unstable regions of the equilibrium solution hardly change, as shown in Figure 6. Thus, in the Erdős-Rényi model, we can suggest that the stable-unstable region of the equilibrium solution changes, depending strongly on the average degree.
Figure 6: (Left) Examples of an eigenvalue set of the Laplacian matrix for (2.4) on an Erdős-Rényi random graph with the almost equal average degree. The horizontal and vertical axes mean eigenvalue numbers $i$ and these values $\lambda_i$, respectively. Eigenvalues are sorted in ascending order, that is $0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_N$. (Right) The stable-unstable regions of the equilibrium solution $(u_i, v_i) = (0, 0)$ ($i = 1, \cdots, N$) for (2.4) on the Erdős-Rényi random graphs. The horizontal and vertical axes indicate $d_u$ and $d_v$, respectively. The number of vertices $N$ and the link probability $p$ are indicated at the bottom of each figure.
4. Networks generated by the Watts-Strogatz model

We deal with reaction-diffusion models on complex networks generated by the Watts-Strogatz model in this section. Each graph is built by rewiring some edges of an enhanced ring in our study.

\textit{Watts-Strogatz model.}

1. We prepare an enhanced ring \( A_{ij} = 1 \) (0 < \( j - i + N \leq \frac{B}{2} \) or 0 < \( i - j + N \leq \frac{B}{2} \) mod \( N \)), or = 0 (otherwise).

2. For every pair \((i, j)\) such that \( A_{ij} = A_{ji} = 1 \), we change the values of \( A_{ij}, A_{ji} \) to zero with probability \( p \).

3. After finishing the prior procedure for all of the pairs \((i, j)\), we add new edges into the graph as follows: If we decided to change the values of \( A_{ij} \) and \( A_{ji} \) in the prior procedure, we equally choose either the number \( i \) or \( j \). When the number \( i \) (resp. \( j \)) has been chosen, we take a number \( k \) satisfying \( A_{ik} = A_{ki} = 0 \) (resp. \( A_{jk} = A_{kj} = 0 \)) and \( k - i + N, i - k + N > \frac{B}{2} \) mod \( N \) (resp. \( k - j + N, j - k + N > \frac{B}{2} \) mod \( N \)). Then we reset \( A_{ik} = A_{ki} = 1 \) (resp. \( A_{jk} = A_{kj} = 1 \)) for the chosen number \( k \). The reset means adding a new edge into the graph. This operation is repeated all over the pairs \((i, j)\) for which we changed the values of \( A_{ij} \) and \( A_{ji} \) in the prior procedure. The iteration should be performed in a suitable order of the pairs.

Note that the procedures 2 and 3 describe the rewiring of edges on the given graph (i.e. enhanced ring). Here, the network structures completely differ, depending on the rewiring probability \( p \). For example, when \( p = 0 \), it remains to be the regular ring lattice with the degree \( B \), and when \( p = 1 \), it possesses similar properties to an Erdős-Rényi random graph because all edges are randomly reconnected to other vertices though there is a restriction that an edge is never rewired within a vertex of \( B \) neighbors. Figure 7 shows examples of an eigenvalue set of the Laplacian matrix \( L \) of a network constructed by the Watts-Strogatz model, where \( B = 20 \), and the number of vertices \( N \) and probability \( p \) are
Figure 7: Examples of an eigenvalue set of the Laplacian matrix for a graph generated by the Watts-Strogatz model. The numbers of vertices are $N = 500$, $N = 2000$ and $N = 5000$, and $B = 20$. The reconnected probability $p$ is indicated in each figure. The horizontal and vertical axes mean the eigenvalue numbers and these values, respectively. Eigenvalues are sorted in ascending order, that is, $0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_N$.

varied. From this result, we know that for each $N$, the eigenvalues do not change drastically when the value of $p$ is varied. But, strictly speaking, for each fixed $N$, both small and large eigenvalues grow as the value of $p$ approaches one. In addition, the eigenvalue distributions for $p = 1$ are quite similar to those of graphs generated by the Erdös-Rényi model in Figure 6 where the average degree of the graphs is approximately 20. On the other hand, even though the number of vertices $N$ is varied, we find that the eigenvalue distributions are almost unchanged for each $p$ in the case of the Watts-Strogatz model. Based on these results, we illustrate the stable-unstable regions of the equilibrium
solution \((u_i, v_i) = (0, 0) \ (i = 1, \cdots , N)\) of \((2.4)\), as shown in Figures 8, 9 and 10. In this case also, since networks are constructed stochastically, we indicate the stable-unstable regions with probability by performing 500 trials and taking the average as well as the case of the Erdős-Rényi model. Figure 8 shows the stable-unstable regions with probability for \(N = 500\) when the value of \(p\) is varied. Interestingly, one can see from the figures that the unstable region of the equilibrium solution becomes narrow as the value of \(p\) increases. This means that the equilibrium solution \((u_i, v_i) = (0, 0) \ (i = 1, \cdots , N)\) of \((2.4)\) becomes difficult to be destabilized as a regular ring lattice changes into a random graph.

It seems that the small worldness of the networks is related to this phenomenon.

Moreover, the stable-unstable region for \(p = 1\) in Figure 8 is quite similar to that for \(p = 0.04\) and \(N = 500\) in Figure 6. When we increase the number of vertices, the similar tendency on the stable-unstable regions holds, that is, for each \(N\), the region where the equilibrium solution is destabilized with high probability becomes small as the value of \(p\) approaches one, as shown in Figure 9 and 10.

Besides, we note that the unstable region for \(p = 0\) grows when the number of vertices \(N\) increases. This comes from the behavior of small eigenvalues when \(N\) is varied. The fact that the small eigenvalues become smaller as \(N\) is larger implies the growth of the unstable region in the case of a regular ring lattice.

In other words, since the asymptote of the bifurcation curve is \(d_u = \frac{a}{\lambda_i}\) for each \(i\), small eigenvalues provide a large unstable region. When the number of vertices \(N\) is large enough, we will prove the existence of the stable-unstable region of the equilibrium solution \((u_i, v_i) = (0, 0) \ (i = 1, \cdots , N)\) for the regular ring lattice with \(p = 0\) in section 6.

5. Networks generated by the threshold network model

As the third model which produces complex networks, we introduce the threshold network model and perform the linear stability analysis of a reaction-diffusion model on a network generated by it. It is well known that this model can produce scale free networks \([9, 10, 11]\). The construction method of a graph...
Figure 8: The stable-unstable regions of the equilibrium solution \((u_i, v_i) = (0, 0)\) \((i = 1, \cdots, N)\) in a reaction-diffusion model on networks generated by the Watts-Strogatz model. The number of vertices is \(N = 500\), \(B = 20\) and the reconnection probability \(p\) is indicated at the bottom of each figure. The black and white regions denote to be unstable with probability one and unstable with probability zero, respectively. The horizontal and vertical axes indicate \(d_u\) and \(d_v\), respectively.
Figure 9: The stable-unstable regions of the equilibrium solution \((u_i, v_i) = (0, 0)\) \((i = 1, \ldots, N)\) in a reaction-diffusion model on networks generated by the Watts-Strogatz model. The number of vertices is \(N = 2000\), \(B = 20\) and the reconnection probability \(p\) is indicated at the bottom of each figure. The black and white regions denote to be unstable with probability one and unstable with probability zero, respectively. The horizontal and vertical axes indicate \(d_u\) and \(d_v\), respectively.
Figure 10: The stable-unstable regions of the equilibrium solution \((u_i, v_i) = (0, 0)\) \((i = 1, \cdots, N)\) in a reaction-diffusion model on networks generated by the Watts-Strogatz model. The number of vertices is \(N = 5000\), \(B = 20\) and the reconnection probability \(p\) is indicated at the bottom of each figure. The black and white regions denote to be unstable with probability one and unstable with probability zero, respectively. The horizontal and vertical axes indicate \(d_u\) and \(d_v\), respectively.
is as follows:

*threshold network model.*

- For \( i = 1, 2, \ldots, N \), a weight \( w_i \) is randomly chosen by some fixed distribution.

- Given the value \( \theta \) which is called threshold value, for every pair of distinct vertices \( i \) and \( j \), if the sum of weights \( w_i + w_j \) is greater than the threshold value \( \theta \), the two vertices are connected by an edge.

In this paper, we assume that the weight \( w_i \) follows the exponential distribution with a parameter \( \lambda \). From the construction by this model, we know that the generated graph is a complete graph when \( \theta = 0 \). On the other hand, it is said that this model can generate a scale-free network when the value of \( \theta \) is appropriate, that is, networks with a small number of hubs are generated. Then, the information on eigenvalues of the Laplacian matrix of a graph is important for the linear stability analysis of the equilibrium solution \((u_i, v_i) = (0, 0) (i = 1, \cdots, N)\) for (2.4). Figure 11 shows examples of an eigenvalue set of the Laplacian matrix of a graph generated by the threshold network model.

Here, we choose \( \theta \) as a parameter. For \( \theta = 0 \), all the eigenvalues except one zero eigenvalue take \( N \) which is the same as the number of vertices. When the value of \( \theta \) is increased, for example \( \theta = 1 \), the eigenvalues are divided into two cases, \( N \) or less than \( N \). And when \( \theta \) is relatively large (\( \theta = 7 \) and 8), we find that there exit a large number of small eigenvalues and a small number of large eigenvalues. We note that the condition on connectedness of networks for the threshold network model is \( \min_{i \in \{1, 2, \ldots, N\}} \{w_i\} + \max_{i \in \{1, 2, \ldots, N\}} \{w_i\} \geq \theta \). Therefore, the vertex with the largest weight is connected to all of the other vertices. When we change the number of vertices \( N \), the number of eigenvalues is naturally changed, but the configurations of eigenvalue distribution are almost unchanged, and the similar tendencies on eigenvalues hold. Based on the information on eigenvalues of the Laplacian matrix of the graphs, we illustrate the stable-unstable region of the equilibrium solution \((u_i, v_i) = (0, 0) (i = 1, \cdots, N)\) of (2.4).
Figure 11: Examples of an eigenvalue set of the Laplacian matrix for a graph generated by the threshold network model. The numbers of vertices are $N = 500$, $N = 2000$ and $N = 5000$, $\lambda = 1$ and $\theta$ is indicated in each figure. The horizontal and vertical axes mean respectively eigenvalue numbers and these values. Eigenvalues are sorted in ascending order, that is, $0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_N$.

as sections 3 and 4 we show it with probability by taking the average of 500 trials. Figures 12, 13 and 14 demonstrate the stable-unstable regions of the equilibrium solution with probability for $N = 500$, 2000 and 5000, respectively. When $\theta = 0$, these figures coincide with those of Figures 3, 4 and 5 for $p = 1$ because the graphs are complete in both cases. As the value of $\theta$ is increased, we know that the region where the equilibrium solution is destabilized with high probability becomes larger for a fixed $N$. Interestingly, when we focus on the case of $\theta = 8$, one can see from Figures 12, 13 and 14 that the probability of destabilization of the equilibrium solution becomes decreased as the number of
Figure 12: The stable-unstable regions of the equilibrium solution $(u_i, v_i) = (0, 0)$ $(i = 1, \cdots, N)$ in a reaction-diffusion model on networks generated by the threshold network model. The number of vertices is $N = 500$, $\lambda = 1$ and $\theta$ is indicated at the bottom of each figure. The black and white regions denote to be unstable with probability one and unstable with probability zero, respectively. The horizontal and vertical axes indicate $d_u$ and $d_v$, respectively.

vertices increases. For example, look at $(d_u, d_v) = (0.5, 8)$. When the value of $\theta$ is fixed and the number of vertices $N$ increases, the number of hubs also
Figure 13: The stable-unstable regions of the equilibrium solution \((u_i, v_i) = (0, 0)\) \((i = 1, \cdots, N)\) in a reaction-diffusion model on networks generated by the threshold network model. The number of vertices is \(N = 2000\), \(\lambda = 1\) and \(\theta\) is indicated at the bottom of each figure. The black and white regions denote to be unstable with probability one and unstable with probability zero, respectively. The horizontal and vertical axes indicate \(d_u\) and \(d_v\), respectively.

Increases. It seems that this is related to the reason why such a phenomenon occurs. That is, the increase of the number of hubs yields the active mobility.
Figure 14: The stable-unstable regions of the equilibrium solution \((u_i, v_i) = (0, 0)\) \((i = 1, \cdots, N)\) in a reaction-diffusion model on networks generated by the threshold network model. The number of vertices is \(N = 5000\), \(\lambda = 1\) and \(\theta\) is indicated at the bottom of each figure.

The black and white regions denote to be unstable with probability one and unstable with probability zero, respectively. The horizontal and vertical axes indicate \(d_u\) and \(d_v\), respectively.

The black and white regions denote to be unstable with probability one and unstable with probability zero, respectively. The horizontal and vertical axes indicate \(d_u\) and \(d_v\), respectively.

The black and white regions denote to be unstable with probability one and unstable with probability zero, respectively. The horizontal and vertical axes indicate \(d_u\) and \(d_v\), respectively.
the viewpoint of eigenvalue distribution of the Laplacian matrix of a graph

generated by the threshold network model when the number of vertices is large

enough.

6. Approximate analysis

In this section, we introduce some theoretical results on the Turing instability

in a reaction-diffusion model on a graph. The first result is on the Turing

instability on an enhanced graph. The second one is on eigenvalues of the

Laplacian matrix of a graph generated by the threshold network model.

6.1. Turing instability on an enhanced cycle

In this subsection, we deal with the stability of the equilibrium solution

\((u_i, v_i) = (0, 0) \ (i = 1, 2, \ldots, N)\) on a diffusion process on an enhanced cycle

when the number of its vertices is large enough. We carry out the analysis

under the condition \(bc < 0\). Ahead of the analysis of the enhanced cycle, we

make a brief discussion for a general graph \(G\) with \(N\) vertices. Let \(\mathcal{L}_+\) be the set

\[\{l \in \{0, 1, \ldots, N - 1\} \mid g(\sigma_G(l)) \geq 0\}\],

where \(\sigma_G(l) (l = 0, 1, \ldots, N - 1)\) are the eigenvalues of the Laplacian matrix \(L\) of the graph \(G\) and

\[g(s) = (d_u - d_v)^2 s^2 + 2(a - d)(d_u - d_v)s + (a - d)^2 + 4bc.\]

The function \(g(s)\) comes from a discriminant of a characteristic equation of a

2 \times 2 matrix which is a counterpart to the 2 \times 2 matrix in (1.6). Actually,

we get the discriminant of a characteristic equation of the matrix in (1.6) as

g\left(-\left(2n\pi/L\right)^2\right).\] After straightforwardly analyzing the characteristic polynomial

of the \(2N \times 2N\) matrix

\[
\left[\begin{array}{cc}
L_{ij} & d_u \\
0 & d_v
\end{array}\right] + \delta_{ij} \left[\begin{array}{cc}
a & b \\
c & d
\end{array}\right],
\]

we realize that the equilibrium solution is stable if and only if

\[
\begin{align*}
\min_{l \in \mathcal{L}_+} \tau(\sigma_G(l)) & > 0, \\
(a + d) & < 0.
\end{align*}
\]
where
\[ \tau(s) = d_ud_v s^2 + (dd_{uv} + ad_{uv}) s + ad - bc. \]
We suppose \( \min_{l \in L} \tau(\sigma_G(l)) > 0 \) for the null set \( L^+ = \phi \). Note that (2.4) is rewritten as
\[
\frac{d}{dt} \begin{pmatrix} u_i \\ v_i \end{pmatrix} = \sum_{j=1}^{N} \left\{ \begin{pmatrix} 0 & \delta_{ij} \\ d_u & 0 \end{pmatrix} + L_{ij} \begin{pmatrix} f_u & f_v \\ g_u & g_v \end{pmatrix} \right\} \begin{pmatrix} u_j \\ v_j \end{pmatrix}.
\]
From now on, we focus on an enhanced cycle with \( N \) vertices as the graph \( G \).
We express the enhanced cycle whose average degree is \( 2k \) \( (k \in \{1, 2, \ldots, [(N - 1)/2]\}) \), as a symbol \( C_{N,k} \). To be exact, the adjacency matrix of the enhanced cycle \( C_{N,k} \) is given by \( A_{ij} = 1 \) \((0 < j - i + N \leq k \text{ or } 0 < i - j + N \leq k \text{ mod } N), \text{ or } = 0 \) (otherwise). Figure 15 shows three examples of the enhanced cycle \( C_{N,k} \) with 10 vertices.

![Figure 15: Enhanced cycles with 10 vertices](a) \( k = 1 \) (b) \( k = 2 \) (c) \( k = 3 \)

We should note to have
\[
\sigma_{C_{N,k}}(l) = \begin{cases} 
0 & (l = 0), \\
-(2k + 1) + \frac{\sin(2k+1)}{\sin \frac{\pi}{N}} & (l = 1, 2, \ldots, N - 1).
\end{cases}
\]
By using (6.2), we can approximately analyze the stability of equilibrium solution for a large enough number \( N \) under the conditions
\[
\begin{align*}
\max \{|a|, 1\} + d &< 0, \\
ad - bc &> 0, \\
(a - d)^2 + 4bc &\geq 0, \\
a &\leq 1.
\end{align*}
\]
\[
(6.3)
\]
Assuming the parameters $a, b, c, d$ satisfies (6.3), we have a condition that the equilibrium solution gets stable for the number $N$ enough larger than the number $k$ (i.e. $k \ll N$) as follows. For $a \leq 0$, the solution becomes stable when $d_u, d_v > 0$. On the other hand, the condition for $a > 0$ is a little complicated. To describe it, we consider a function $S_k(x) (x \in [0,1])$ such that

$$S_k(x) = \begin{cases} 0 & (x = 0), \\ -(2k + 1) + \frac{\sin(2k+1)\pi x}{\sin \pi x} & (0 < x < 1), \end{cases}$$

which originates from (6.1). Then the equilibrium solution is stable under the condition

$$\begin{align*}
0 < d_v &< -\frac{dm(k)}{m(k)}(d_u + a) \\
0 < d_v &< -\frac{2bc}{a^2} \left( 1 + \sqrt{1 - \frac{ad}{bc}} \right) d_u + \frac{d}{a} \quad \text{(otherwise),}
\end{align*}$$

where $m(k) = \min_{x \in [0,1]} S_k(x)$. So, when we set $a > 0$, the bifurcation line of the equilibrium solution consists of a continuous line which is produced by both a hyperbolic curve and a linear line, as shown in Figure 16.

![Figure 16](image-url)

Figure 16: The bifurcation line of the equilibrium solution is given by both a hyperbolic curve and a linear line when the parameters $a, b, c, d$ ($a > 0$) satisfy (6.3).

While we have treated the enhanced cycle in the case of $k \ll N$, we can also approximately compute the stability of the equilibrium solution for the complete
graph $K_N$ with $N$ vertices, supposing that it does not have any self-loop. If the number $N$ is an odd integer and $k = (N - 1)/2$, the enhanced cycle $C_{N,k}$ is equivalent to the complete graph $K_N$. In the following result, we do not have to consider the assumption (6.3). For the complete graph $K_N$, the eigenvalues $\lambda_{\pm}(l) (l = 0, 1, \ldots, N - 1)$ of the matrix (6.1) are computed as

$$\lambda_{\pm}(l) = \begin{cases} 
\frac{a + d \pm \sqrt{(a - d)^2 + 4bc}}{2} & (l = 0), \\
\frac{h_{N,1}(d_u, d_v) \pm \sqrt{h_{N,2}(d_u, d_v)}}{2} & (l = 1, 2, \ldots, N - 1), 
\end{cases}$$

where

$$h_{N,1}(x, y) = -N(x + y) + a + d,$$

$$h_{N,2}(x, y) = N^2(x - y)^2 - 2N(a - d)(x - y) + (a - d)^2 + 4bc.$$

By using the similar method as the approximate analysis for the enhanced cycle, we see that the stability does not depend on $d_u, d_v$ as $N \gg 1$ and it is determined by the condition $\Re(\sqrt{(a - d)^2 + 4bc}) < -(a + d)$, where $\Re(z)$ means the real part of the complex number $z$.

6.2. Turing instability on networks generated by the threshold network model

Let $X_1, \ldots, X_N$ be a sequence of independent and identically distributed random variables with a common distribution function $F$. If we consider the threshold network model with the vertex weights $X_1, \ldots, X_N$ and the threshold value $\theta \in \mathbb{R}$, then

$$D_N(i) = \sum_{1 \leq j \leq N \atop j \neq i} I_{\{X_i + X_j > \theta\}}$$

is the degree of a vertex $i$, where $I_{\{X_i + X_j > \theta\}} = 1$ if $X_i + X_j > \theta$ and $= 0$ otherwise. Let

$$\nu_N(dx) = \frac{1}{N} \sum_{1 \leq i \leq N} \delta \left( x - \frac{D_N(i)}{N} \right) dx$$

be the empirical distribution of the normalized degree sequence $D_N(1)/N, \ldots, D_N(N)/N$ where $\delta(x)$ is the delta function. By the definition, the $m$-th moment of the
empirical distribution $\nu_N(dx)$ is $(1/N) \sum_{1 \leq i \leq N} (D_N(i)/N)^m$. Using the same arguments of the proof of Theorems 2 and 3 in [12] and the induction, we can easily show that

$$P \left( \lim_{n \to \infty} \frac{1}{n} \sum_{1 \leq i \leq n} \left[ \frac{D_n(i)}{n} \right]^m = E \left[ (1 - F(X_1))^m \right] \right) = 1.$$ 

This implies that the empirical distribution $\nu_n(dx)$ converges weakly to the distribution of the random variable $1 - F(X_1)$ with probability one.

On the other hand, the eigenvalues of the Laplacian matrix of the threshold network model are expressed as follows (see Theorems 2 and 3 in [13]):

$$\lambda_N(N - i) = \sharp \{ j : D_N(j) \geq N - i \},$$

for $1 \leq j \leq N$. Combining these two observations, we obtain the empirical distribution

$$\mu_N(d\lambda) = \frac{1}{N} \sum_{1 \leq i \leq N} \delta \left( \lambda - \frac{\lambda_N(N - i)}{N} \right) d\lambda$$

converges weakly to the distribution of the random variable $1 - F(X_1)$ with probability one. When $X_1$ follows the exponential distribution with a parameter $\lambda$, that is, the density function

$$f(x) = \begin{cases} 
\lambda e^{-\lambda x}, & x \geq 0, \\
0, & x < 0,
\end{cases}$$

we have the density function on eigenvalues of the Laplacian matrix of a graph generated by the threshold network model

$$1 - F(\theta - X_1) = \begin{cases} 
\delta_1(dk), & \theta \leq 0, \\
I_{(e^{-\lambda \theta}, 1)}(k) : e^{-\lambda \theta} + e^{-\lambda \theta} \cdot \delta_1(dk), & \theta > 0.
\end{cases}$$

Roughly speaking, this result implies that the number of small eigenvalues is quite small, compared to the number of large ones when we consider the case $\theta > 0$ to which the numerical results in section 5 correspond. Therefore, we can expect that the ratio of the number of small eigenvalues to the total number
of eigenvalues is decreased as $N$ tends to infinity. This insight explains our numerical results in Figures 12, 13 and 14. In other words, this suggests the reason why for each $(d_u, d_v)$, the probability that the equilibrium solution is destabilized is reduced as $N$ is increased.

7. Concluding remarks

In the present paper, we dealt with the Turing instability in a reaction-diffusion model defined on complex networks, and revealed that the stable-unstable regions strongly depend on network structures which are generated by models. However, we do not mention on network-organized patterns resulting from the Turing instability in this paper. Since the Turing instability is generally the onset of a self-organized pattern formation, we can expect that inhomogeneous patterns emerge in a self-organized way in a reaction-diffusion model defined on a network. Actually, the existence of such network-organized patterns is numerically discussed in [7] and the authors refer to several features of network-organized pattern formation. Moreover, recently, studies on the stable inhomogeneous patterns with a single differentiated node have proceeded from the viewpoint of bifurcation analysis in [8]. As an another typical solution in reaction-diffusion equations on continuous media, a traveling wave solution is well known, which moves with constant speed without changing the profile. In [14], wave-like phenomena are also observed in reaction-diffusion models on networks. However, a lot of things on the network-reaction-diffusion models are not clear. It is not easy to understand such phenomena arising in a reaction-diffusion system on networks due to stochastic nature of network structures and the large number of vertices (the large number of equations).

Though we dealt with three types of models i.e. the Erdős-Rényi, the Watts-Strogatz and the threshold network models, many models have been proposed besides the three models, for instance the Barabási-Albert model. Investigation of a relation between such models and the stable-unstable regions of the equilibrium solution is a future work. A challenging problem is to treat a nonlinear
system with the Laplacian matrix of a graph such as (2.3). In this paper, since we focus on the Turing instability of reaction-diffusion models on networks, the associated linear systems with network structures were considered. However, in order to understand the mechanism of network-organized pattern formation, we need to analyze the nonlinear system instead of the linear system. Potential applications of a reaction-diffusion model on networks are introduced in [7, 15]. It is also interesting to apply the theory to specific phenomena. Moreover, from the viewpoint of mathematical ecology, a two patches model is proposed in [16], where the migration rate of each species is influenced by its own and the other one’s density, that is the model on two patches possesses the cross-diffusion effect which is one of the nonlinear diffusion effects. The extension of the model on complex networks may contribute to the understanding of the spatially distribution of biological species. This problem is also left as a future work.

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