D-Dimensional Gravity from \((D+1)\) Dimensions

Steve Rippl\(^1a\), Carlos Romero\(^1b\) & Reza Tavakol\(^1c\)*

\(^1\) School of Mathematical Sciences
Queen Mary & Westfield College
Mile End Road
London E1 4NS, UK

\(^2\) Departamento de Física
Universidade Federal da Paraíba
C. Postal 5008 - J. Pessoa -Pb
58059-970 - Brazil

January 5, 2022

Abstract

We generalise Wesson’s procedure, whereby vacuum \((4 + 1)\)–dimensional field equations give rise to \((3 + 1)\)–dimensional equations with sources, to arbitrary dimensions. We then employ this generalisation to relate the usual \((3 + 1)\)–dimensional vacuum field equations to \((2 + 1)\)–dimensional field equations with sources and derive the analogues of the classes of solutions obtained by Ponce de Leon. This way of viewing lower dimensional gravity theories can be of importance in establishing a relationship between such theories and the usual 4-dimensional general relativity, as well as giving a way of producing exact solutions in \((2 + 1)\) dimensions that are naturally related to the vacuum \((3 + 1)\)–dimensional solutions. An outcome of this correspondence, regarding the nature of lower dimensional gravity, is that the intuitions obtained in \((3 + 1)\) dimensions may not be automatically transportable to lower dimensions.

We also extend a number of physically motivated solutions studied by Wesson and Ponce de Leon to \((D + 1)\) dimensions and employ the equivalence between the \((D + 1)\) Kaluza-Klein theories with empty \(D\)–dimensional Brans-Dicke theories (with \(\omega = 0\)) to throw some light on the solutions derived by these authors.

*e-mail (a) sfr@maths.qmw.ac.uk, (b) car@maths.qmw.ac.uk, (c) reza@maths.qmw.ac.uk
1 Introduction

Since the advent of general relativity (GR), the aim of a purely geometrical description of all physical forces as well as that of a geometrical origin for the matter content of the Universe has been pursued \( [5, 7, 15] \). A simple and elegant idea in this connection, recently put forward by Wesson, Ponce de Leon and Coley \( [17, 18, 4] \), involves starting from the vacuum 5-dimensional Kaluza-Klein equations

\[
(5) R_{ab} = 0, \quad (1)
\]

where \( R_{ab} \) is the Ricci tensor in the 5-dimensional space. They show that equation (1) gives rise to 4-dimensional Einstein equations with sources in the form

\[
(4) G_{\alpha\beta} = (4) T_{\alpha\beta}, \quad (2)
\]

provided the extra terms related to the fifth dimension are appropriately used to define an energy-momentum tensor \( T_{\alpha\beta} \) \( [18] \). Furthermore, it has been shown that for certain cosmological solutions of the vacuum \( (4 + 1) \) field equations \( [10] \), the resulting sources in the \( (3 + 1) \) dimensions can be interpreted as matter with reasonable equations of state \( [17] \).

Parallel to these developments, there has recently been a great deal of interest in lower dimensional gravity theories \( [3, 9, 11] \). The usual motivation for this body of work is that by enormously simplifying the difficulties usually associated with the Einstein field equations in \( (3+1) \) dimensions, such theories can act as “laboratories” in which models can be more readily constructed to study a number of phenomena, ranging from quantum gravitational effects \( [12] \) and topological defects \( [6] \) to black holes \( [3, 11] \). The hope is that these lower dimensional models are of value in understanding their analogues in \( (3 + 1) \) dimensions.

The problem, however, is that such theories have radically different properties to the 4-dimensional GR, for example, the non-existence of a Newtonian limit \( [9] \). It is therefore not at all clear that intuitions obtained from the study of lower dimensional models would be of relevance in 4-dimensional GR. Similarly, it is not clear that \( (3 + 1) \)-dimensional intuitions are automatically transportable to \( (2 + 1) \) dimensions. We shall, for example, see that there exist vacuum Einstein solutions in \( (3 + 1) \) dimensions which give rise to solutions in \( (2 + 1) \) dimensions with corresponding sources, which on the basis of intuitions imported from ordinary gravity could be ruled out as “non-physical”. The real physics, however, takes place in (at least) \( (3 + 1) \) dimensions and the notion of physicality of solutions in lower dimensions can only make sense in relation to the usual \( (3 + 1) \) theory. An important issue is, therefore, to understand how lower dimensional theories are related to the usual 4-dimensional theory.

Our aims here are threefold:

(i) to generalise the scheme put forward by Wesson et al. for relating the vacuum \( (D + 1) \)-dimensional theories to \( D \)-dimensional theories with sources, and generalising some of the physically motivated solutions to \( D \) dimensions,

(ii) to employ the equivalence between the \( (D + 1) \)-dimensional Kaluza-Klein theory with the empty Brans-Dicke theory in \( D \) dimensions in order to elucidate some of the solutions derived by Wesson and others, and

(iii) to employ this scheme as a tool to study the relation between lower dimensional gravity theories and the usual 4-dimensional vacuum GR, as well as a way of producing new solutions in lower dimensional gravity theories.

2 \( D \)-dimensional gravity from \( (D + 1) \) dimensions

In this section we generalise the work of Wesson and Ponce de Leon \( [18] \) to \( (D + 1) \) dimensions. Our motivation for this extension is twofold. Firstly to justify, mathematically, the analogue of the procedure

\footnote{By \( D \)-dimensional we mean \( ((D - 1), 1) \)-dimensional.}
by Wesson et al in going from (3 + 1) to (2 + 1) dimensions, which will be discussed in section (4) below. Secondly, our aim is to ultimately consider a generalisation of the scheme of Wesson et al, which would involve a “cascade” of such steps. There are, however, important differences here after the first step and we shall return to these in a future publication.

To do this we start with the \((D + 1)\)-dimensional source-free Kaluza-Klein field equations \(^{[2]}\) and in line with the ansatz adopted in \([18]\), we take the metric to be in the form

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & g_{\alpha\beta} & 0 & 0 & 0 \\
0 & 0 & g_{\alpha\beta} & 0 & 0 \\
0 & 0 & 0 & 0 & g_{DD}
\end{pmatrix},
\]

where both the \((D+1)\) and \(D\)-dimensional metrics \(g_{ab}\) and \(g_{\alpha\beta}\) are in general dependent on the coordinate \(x^D\) and as in \([18]\) we write the \(g_{DD}\) as

\[
g_{DD} = \phi^2, \quad g^{DD} = \frac{1}{\phi^2}. \tag{3}
\]

Now proceeding as in \([18]\) and starting from the \((D + 1)\)-dimensional Ricci tensor expressed in terms of the \((D + 1)\) Christoffel symbols

\[
R_{ab} = (\Gamma^c_{ab})_c - (\Gamma^c_{ac})_b + \Gamma^c_{ab}\Gamma^d_{cd} - \Gamma^c_{ad}\Gamma^d_{bc}, \tag{4}
\]

and letting \(a \rightarrow \alpha, \ b \rightarrow \beta\), we obtain the \(D\)-dimensional part of the \((D + 1)\)-dimensional Ricci tensor indicated by \((D+1)R_{\alpha\beta}\) thus:

\[
(D+1)R_{\alpha\beta} = (D)R_{\alpha\beta} + (\Gamma^\alpha_{\beta\alpha})_D - (\Gamma^\alpha_{\alpha\beta})_D + \Gamma^\alpha_{\alpha\beta}\Gamma^d_{D\nu} + \Gamma^D_{\alpha\beta}\Gamma^d_{Dd} - \Gamma^d_{\alpha\beta}\Gamma^D_{Dd}. \tag{5}
\]

Substituting in \((5)\) from \((3)\), we obtain

\[
(D+1)R_{\alpha\beta} = (D)R_{\alpha\beta} - \frac{\phi\alpha\beta}{\phi} + \frac{1}{2\phi^2} \left( \frac{\phi_D}{\phi} g_{\alpha\beta,D} - g_{\alpha\beta,DD} + g^{\lambda\mu}g_{\alpha\lambda,D}g_{\beta\mu,D} - \frac{1}{2}g^{\mu\nu}g_{\mu\nu,D}g_{\alpha\beta,D} \right), \tag{6}
\]

and similarly we obtain an analogous expression for \((D+1)R_{DD}\),

\[
(D+1)R_{DD} = (\Gamma^D_{DD})_\alpha - (\Gamma^D_{D\alpha})_D + \Gamma^D_{DD}\Gamma^\beta_{D\beta} + \Gamma^D_{\alpha\beta}\Gamma^D_{D\beta} - \Gamma^D_{D\beta}\Gamma^\beta_{DD}. \tag{7}
\]

Now the source free field equations in \((D + 1)\) dimensions are given by

\[
(D+1)R_{ab} = 0 \implies (D+1)R_{\alpha\beta} = 0, \quad (D+1)R_{DD} = 0, \quad (D+1)R_{\alpha D} = 0. \tag{8}
\]

From the first of these \((D+1)R_{\alpha\beta} = 0\) we obtain (using \([3]\))

\[
(D)R_{\alpha\beta} = \frac{\phi\alpha\beta}{\phi} - \frac{1}{2\phi^2} \left( \frac{\phi_D}{\phi} g_{\alpha\beta,D} - g_{\alpha\beta,DD} + g^{\lambda\mu}g_{\alpha\lambda,D}g_{\beta\mu,D} - \frac{1}{2}g^{\mu\nu}g_{\mu\nu,D}g_{\alpha\beta,D} \right), \tag{9}
\]

The second equation \((D+1)R_{DD} = 0\) results in

\[
\phi \Box \phi = -\frac{1}{4}g^{\lambda\beta,D}g_{\lambda\beta,D} - \frac{1}{2}g^{\lambda\beta}g_{\lambda\beta,DD} + \frac{\phi_D}{2\phi} g^{\lambda\beta}g_{\lambda\beta,D}, \tag{10}
\]

\footnote{In this section the Latin and Greek indices run from 0 to \(D\) and 0 to \((D - 1)\) respectively.}
One can now define an effective energy-momentum tensor by
\[
^{(D)}T_{\alpha\beta} = \frac{\phi}{\phi} - \left(1 - \frac{1}{2g^2}\left[\frac{\phi}{\phi^2} g_{\alpha \beta, D} - g_{\alpha \beta, DD} + g^{\lambda \mu} g_{\alpha \lambda, D} g_{\beta \mu, D} - \frac{1}{2} g^{\mu \nu} g_{\mu \nu, D} g_{\alpha \beta, D} \right] + \frac{g_{\alpha \beta}}{4} \left[ g^{\mu \nu, D} g_{\mu \nu, D} + (g^{\mu \nu} g_{\mu \nu, D})^2 \right] \right),
\]
where
\[
^{(D)}T_{\alpha\beta} = ^{(D)}R_{\alpha\beta} - \frac{1}{2} ^{(D)}R g_{\alpha \beta}.
\]

In this way the \((D+1)\)-dimensional source free Einstein type equations are related to a \(D\)-dimensional theory with sources, just as in the 5-dimensional case treated in [18].

Finally, the last equation in (8) \((^{(D+1)}R_{\alpha D} = 0)\) allows us to define the \(D\)-dimensional tensor \(P_{\beta \alpha} \) such that this field equation can be expressed as
\[
P_{\beta \alpha} = 0 \text{ with } P_{\beta \alpha} = \frac{1}{2\sqrt{g^D}} \left( g^{\beta \alpha} g_{\mu \alpha, D} - \delta^\beta_\alpha g^{\rho \nu} g_{\rho \nu, D} \right).
\]

As has already been observed in [18], these equations have the appearance of conservation laws, the full meaning of which is not clear at the present.

3 \(D\)-dimensional cosmological solutions

Within the context of the Wesson’s scheme (from 5 to 4 dimensions) discussed in section (1), a number of attempts have been made to construct cosmological solutions in 4 dimensions. In particular, Wesson and Ponce de Leon [17, 18] have obtained Friedmann-Lemaître-Robertson-Walker (FLRW) type solutions with flat spatial sections. These cosmological solutions essentially fall into two classes, depending upon whether or not they possess a Killing vector in their fifth dimension.

In this section we present a generalisation of the more physically relevant solutions due to Wesson and Ponce de Leon and, to begin with, we consider the solution in 5 dimensions in the form
\[
ds^2 = dt^2 - t(dx^2 + dy^2 + dz^2) - t^{-1}d\psi^2,
\]
which possesses a Killing vector in its fifth dimension. This spacetime has two important features: firstly it has a shrinking fifth dimension with time \(t\), and secondly the 4-dimensional equations it gives rise to produce a FLRW model with a radiative equation of state \(p = \frac{1}{3}\rho\), where
\[
\rho = \frac{3}{4t^2}, \quad p = \frac{1}{4t^2}.
\]

Thus in this sense the fifth dimension “generates” effective sources (with non-zero \(\rho\) and \(p\)) and these in turn act to curve the 4-dimensional spacetime.

It is useful here to emphasise a different interpretation of this scheme by recalling that for 5-dimensional Kaluza-Klein spacetimes with a symmetry group generated by \(\frac{\partial}{\partial \psi}\) (implying that the metric is independent of the extra spatial dimension \(\psi\)) the field equations
\[
^{(5)}G_{ij} = 0,
\]
are formally identical to the vacuum Brans-Dicke field equations in four dimensions with the free parameter \(\omega = 0\) [3]. Therefore Wesson’s solution [4] should correspond to a vacuum FLRW solution of the Brans-Dicke theory with \(\omega = 0\). This turns out to be the O’Hanlon-Tupper solution [5] given by
\[
ds^2 = dt^2 - t^q(dx^2 + dy^2 + dz^2), \quad \phi(t) = \phi_0 t^r,
\]
where \(\phi\) is the Brans-Dicke scalar field, \(\phi_0\) is a constant and the parameters \(r\) and \(q\) are given by
\[
\frac{1}{r} = -\frac{1}{2} \left[1 \pm \sqrt{3(2\omega + 3)}\right], \quad q = \frac{1}{3}(1 - r).
\]

---

\(^3\)We are grateful to the referee for pointing out that this solution was first found by Belinsky and Khalatnikov [1].
Substituting $\omega = 0$ into (17) gives the Wesson’s solution for $r = -\frac{1}{2}$ (incidentally, $r = 1$ leads to Minkowski spacetime which by equation (13) give $T_{\alpha\beta} = 0$). We should note, however, that despite the mathematical equivalence of the two solutions, the fact that they come from different physical theories makes them conceptually distinct. Indeed the O’Hanlon-Tupper solution represents an empty universe, the curvature of which is generated by a non-static scalar field - usually related to the Newtonian constant of gravitation through $\phi \sim G^{-1/2}$.

Now in order to generalise solution (14) from (4+1) to ($D+1$) dimensions, we consider the generalised spatially flat FLRW line element in the form

$$ds^2 = dt^2 - R^2(t) \sum_{i=1}^{D-1} (dx^i)^2 - \phi^2(t) d\psi^2.$$  \hspace{1cm} (19)

The vacuum Einstein field equations in the ($D+1$) dimensions are given by (8) which for the metric (19) become

$$(D - 1)\ddot{R} \frac{\dot{R}}{R} + \ddot{\phi} \frac{\phi}{R} = 0$$ \hspace{1cm} (20)

$$(D - 1)\dot{\phi} \frac{\dot{R}}{R} + \ddot{\phi} \frac{\phi}{R} = 0$$ \hspace{1cm} (21)

$$\ddot{R} \frac{\dot{R}}{R} + (D - 2)\frac{\dot{R}^2}{R} + \dot{\phi} \frac{\dot{R}}{R} = 0.$$ \hspace{1cm} (22)

The solution of these equations is given by

$$R(t) = R_0 \left[ \frac{Dt}{2} + A \right]^{(2/D)} \hspace{1cm} \phi(t) = \frac{B}{R^{(2/D)-1}(t)}.$$ \hspace{1cm} (23)

where $R_0, A$ and $B$ are integration constants. An obvious coordinate transformation then allows the metric (19) to be put in the following form

$$ds^2 = dt^2 - t^{4/D} \sum_{i=1}^{D-1} (dx^i)^2 - t^{(4/D-2)} d\psi^2,$$ \hspace{1cm} (24)

which clearly shows that for $D = 4$ the Wesson-O’Hanlon-Tupper metric is recovered. This latter result may be viewed as a Kasner-like universe in ($D+1$) dimensions being compactified to a Friedmann-like universe in $D$ dimensions for an arbitrary $D > 2$. In this sense, the compactification property of this solution is robust with respect to changes in dimensions for all $D > 2$. Furthermore, it can be easily seen from the equation (23) that a comoving observer will measure a matter density $\rho$ given by

$$\rho = \ddot{\phi} \frac{\phi}{t^2} \left( \frac{1}{D} - 1 \right) \left( \frac{2}{D} - 1 \right)$$ \hspace{1cm} (25)

and a pressure given by

$$p = \frac{2}{D t^2} \left( 1 - \frac{2}{D} \right).$$ \hspace{1cm} (26)

which are related by $p = \rho/(D - 1)$ (the general $D$-dependent radiative equation of state), where $p \to 0$ as $D$ increases.

Now let us turn our attention to the solutions given by Ponce de Leon [14]. He considers solutions which depend on the fifth coordinate ($\psi$) and by using a method of separation of variables solves the field equations to give eight families of solutions. The analogues of these classes of solutions - in the case of going from four to three dimensions - will be discussed in the next section. Here we will consider only two

---

4 Again, it was brought to our attention after submitting this paper that an equivalent solution had been discovered in another context by Sato [16].
classes of his solutions which are potentially of more physical interest in the context of 4—dimensions. Up to coordinate transformations these solutions are given by

\[ ds^2 = \frac{\Lambda}{3} \psi^2 dt^2 - \psi^2 e^{2\sqrt{\Lambda/3t}} (dx^2 + dy^2 + dz^2) - d\psi^2 \]  

and

\[ ds^2 = \psi^2 dt^2 - t^{2/\alpha} \psi^{2/(1-\alpha)} (dx^2 + dy^2 + dz^2) - \alpha^2 (1 - \alpha)^{-2} t^2 d\psi^2, \]

where \( \Lambda \) and \( \alpha \) are arbitrary parameters. Now as shown by Ponce de Leon, the solutions (27) and (28) may be interpreted in 4 dimensions (with \( \psi = \text{constant} \)) as de Sitter and FLRW metrics respectively. However, we should point out that although they are claimed to be independent solutions [14], they in fact represent the same spacetime in five dimensions, as both have vanishing curvature. Furthermore, the natural generalisations of (27) and (28) are given by

\[ ds^2 = \frac{\Lambda}{3} \psi^2 dt^2 - \psi^2 e^{2\sqrt{\Lambda/3t}} \sum_{i=1}^{D-1} dx_i^2 - d\psi^2, \]  

and

\[ ds^2 = \psi^2 dt^2 - t^{2/\alpha} \psi^{2/(1-\alpha)} \sum_{i=1}^{D-1} dx_i^2 - \alpha^2 (1 - \alpha)^{-2} t^2 d\psi^2, \]

which are also flat in \((D + 1)\) dimensions. In \( D \) dimensions the metric (29) represents a generalised vacuum de Sitter solution with a cosmological constant \( \Lambda_D = \frac{\Lambda}{6}(D - 1)(D - 2) \). So in this way, one could say that the cosmological constant (as well as matter) can be viewed as the manifestation of the extra dimension of spacetime. Similarly, the solution (30) gives rise to a \( D \)—dimensional universe with energy density and pressure given respectively by

\[ \rho_D = \frac{(D - 1)(D - 2)}{2\alpha^2 \psi^2 t^2}, \quad p_D = \frac{2(D - 2)\alpha - (D - 1)(D - 2)}{2\alpha^2 \psi^2 t^2}, \]

which amounts to a perfect fluid equation of state with \( p_D = \lambda_D \rho_D \), where \( \lambda_D = \frac{2\alpha}{D-1} - 1 \) and \( \frac{D-1}{2} \leq \alpha \leq D - 1 \), resulting in a \( D \)—dimensional generalisation of Ponce de Leon’s result [14].

4 Source free GR and lower gravity theories

As was discussed in section (1), lower dimensional gravity theories have in general radically different properties from the usual 4-dimensional gravity. Furthermore, the notion of what is physical in lower dimensional gravity is not strictly speaking understood. It is in fact not clear whether such a question can be answered purely in reference to the lower dimensional theories themselves. Therefore, any mechanism that allows a bridge to be established between such theories and the ordinary 4-D gravity is of vital interest. One possible way to proceed is to make sure that the ansatz employed in \((2 + 1)\)-dimensional studies have at least some relation to the \((3 + 1)\) Einstein theory, and one way of doing this is to start by concentrating on those solutions in \((2 + 1)\) dimensions that are related to their vacuum \((3 + 1)\)—dimensional counterparts via the generalisation of Wesson’s procedure given in section (2). Apart from being mathematically possible (shown in section (2)), and having the desirable property of establishing a bridge, we think this is justified as a possible mechanism, especially in view of the absence (as far as we are aware) of other physically more suggestive bridges.

Here, as a first step, we employ this generalisation to derive the analogues of the classes of solutions obtained by Ponce de Leon [14] in \((2 + 1)\) dimensions. We therefore similarly start by letting the 4-dimensional vacuum spacetime to be Ricci-flat, homogenous and isotropic and for the sake of comparison take as our line element the four dimensional analogue of the line element proposed by Wesson [17] in the form

\[ ds^2 = e^\alpha dt^2 - e^b (dx^2 + dy^2) - e^c d\psi^2, \]  

where \( \Lambda \) and \( \alpha \) are arbitrary parameters. Now as shown by Ponce de Leon, the solutions (27) and (28) may be interpreted in 4 dimensions (with \( \psi = \text{constant} \)) as de Sitter and FLRW metrics respectively. However, we should point out that although they are claimed to be independent solutions [14], they in fact represent the same spacetime in five dimensions, as both have vanishing curvature. Furthermore, the natural generalisations of (27) and (28) are given by

\[ ds^2 = \frac{\Lambda}{3} \psi^2 dt^2 - \psi^2 e^{2\sqrt{\Lambda/3t}} \sum_{i=1}^{D-1} dx_i^2 - d\psi^2, \]  

and

\[ ds^2 = \psi^2 dt^2 - t^{2/\alpha} \psi^{2/(1-\alpha)} \sum_{i=1}^{D-1} dx_i^2 - \alpha^2 (1 - \alpha)^{-2} t^2 d\psi^2, \]

which are also flat in \((D + 1)\) dimensions. In \( D \) dimensions the metric (29) represents a generalised vacuum de Sitter solution with a cosmological constant \( \Lambda_D = \frac{\Lambda}{6}(D - 1)(D - 2) \). So in this way, one could say that the cosmological constant (as well as matter) can be viewed as the manifestation of the extra dimension of spacetime. Similarly, the solution (30) gives rise to a \( D \)—dimensional universe with energy density and pressure given respectively by

\[ \rho_D = \frac{(D - 1)(D - 2)}{2\alpha^2 \psi^2 t^2}, \quad p_D = \frac{2(D - 2)\alpha - (D - 1)(D - 2)}{2\alpha^2 \psi^2 t^2}, \]

which amounts to a perfect fluid equation of state with \( p_D = \lambda_D \rho_D \), where \( \lambda_D = \frac{2\alpha}{D-1} - 1 \) and \( \frac{D-1}{2} \leq \alpha \leq D - 1 \), resulting in a \( D \)—dimensional generalisation of Ponce de Leon’s result [14].
where \(a, b,\) and \(c\) are differentiable functions of \(t\) and \(\psi\) only. The vacuum 4-dimensional field equations are given by

\[
G_{00} = e^{-a}[(b_{,\psi})^2 + 2b_{,t}c_{,\psi}] + e^{-c}[-3(b_{,\psi})^2 + 2b_{,\psi}c_{,\psi} - 4b_{,\psi\psi}] = 0 \tag{33}
\]
\[
G_{03} = a_{,\psi}b_{,\psi} - b_{,\psi}b_{,t} + b_{,t}c_{,\psi} - 2b_{,t}c_{,\psi} = 0 \tag{34}
\]
\[
G_{11} = G_{22} = e^{-c}[(a_{,\psi})^2 + a_{,\psi}b_{,\psi} - a_{,\psi}c_{,\psi} + 2a_{,\psi} + (b_{,\psi})^2 + b_{,\psi}c_{,\psi} + 2b_{,\psi\psi}] \tag{35}
\]
\[
\quad + e^{-a}[a_{,t}b_{,t} + a_{,t}c_{,t} - (b_{,t})^2 + b_{,t}c_{,t} - 2b_{,t}c_{,t} - c_{,t}c_{,t}] = 0
\]
\[
G_{33} = e^{-c}[2a_{,\psi}b_{,\psi} + (b_{,\psi})^2] + e^{-a}[2a_{,\psi}b_{,\psi} - 3(b_{,\psi})^2 - 4b_{,\psi\psi}] = 0. \tag{36}
\]

To find the solutions of (33)-(36) we proceed similarly to Ponce de Leon [14], and assume that the metric coefficients are separable in their arguments, thus

\[
e^a = N(\psi)T(t) \tag{37}
\]
\[
e^b = P(\psi)S(t) \tag{38}
\]
\[
e^c = M(\psi)K(t), \tag{39}
\]

where \(N, P, M\) are undefined differentiable functions of \(\psi\) and \(T, S, K\) are undefined differentiable functions of \(t\). Substitution of equations (37)-(39) into equation (33) yields

\[
\frac{S_t}{S} \left( \frac{P_{,\psi}}{P} - \frac{N_{,\psi}}{N} \right) = \frac{P}{P} \frac{K_t}{K}. \tag{40}
\]

The following analogous classes of solutions to those given by Ponce de Leon [14] were obtained in \((3 + 1)\) dimensions. The related 3-dimensional metrics can then be obtained from the vacuum 4-dimensional solutions by assuming \(\psi = \text{constant} \).

(i) Letting \(S \) and \(K\) to be constants, we find the following relationships between \(P(\psi), M(\psi)\) and \(P(\psi), N(\psi)\) from field equations (33) and (34),

\[
\frac{(P_{,\psi})^4}{P} = C_1 M^2, \quad P = \frac{C_2}{N^2}. \tag{41}
\]

where \(C_1\) and \(C_2\) are arbitrary constants. Rescaling \(t\) and \(\psi\) according to \(T^* dt = \text{const.} d\tilde{t}\) and \(M^* d\psi = \text{const.} d\tilde{\psi}\), as well as removing unnecessary constants, the metric takes the form

\[
ds^2 = \tilde{\psi}^{-\frac{2}{3}} d\tilde{t}^2 - \tilde{\psi}^{-\frac{4}{3}} (d\tilde{x}^2 + d\tilde{y}^2) - d\tilde{\psi}^2, \tag{42}
\]

which for \(\tilde{\psi} = \text{constant}\) corresponds to the \((2 + 1)\)-dimensional Minkowski metric.

(ii) Assuming \(P\) and \(N\) to be constants we can again find relationships between \(S, K\) and \(T\) of exactly the same functional form as (41). This yields the following metric,

\[
ds^2 = dt^2 - \tilde{t}^2 (dx^2 + dy^2) - \tilde{t}^{-\frac{2}{3}} d\tilde{\psi}^2, \tag{43}
\]

which in \((2+1)\) dimensions is the O’Hanlon-Tupper solution [17]. The empty \((3+1)\)-dimensional solution has the interesting property that the fourth coordinate \(\tilde{\psi}\) will shrink relative to the two space dimensions \(x\) and \(y\) as \(\tilde{t}\) increases. The density and pressure in \((2+1)\) dimensions are given by \(\rho = \frac{1}{3\tilde{t}^2}, p = \frac{2}{3\tilde{t}^2}\), which correspond to the 3-dimensional radiative equation of state \(p = \frac{\rho}{3}\).

(iii) Assuming \(S\) and \(P\) to be constants and rescaling \(t\) and \(\psi\) such that \(T\) and \(M\) become constants, equation (55) yields the following relations

\[
2K\ddot{K} - \dot{K}^2 - FK = 0 \tag{44}
\]

and

\[
2N\ddot{N} - \dot{N}^2 - FN = 0 \tag{45}
\]
where $F$ is a constant. For the case $F = 0$ equations \((44)\) and \((45)\) can be integrated in terms of elementary functions to generate the flat metric
\[
ds^2 = \bar{\psi}^2 dt^2 - (dx^2 + dy^2) - \bar{r}^2 d\bar{\psi}^2,
\]
which for $\bar{\psi} = \text{constant}$ again corresponds to the $(2 + 1)$-dimensional Minkowski metric.

Now if $S_\alpha$ and $P_\psi$ are non-zero, then equation \((10)\) becomes
\[
P \left( \frac{P_\psi}{P} \right) - \frac{N_\psi}{N} = \frac{S}{S_\alpha} \frac{K_\alpha}{K},
\]
where $\alpha$ is an arbitrary constant. Solving for $N$ and $K$ in terms of $P$ and $S$ respectively we can write the metric coefficients as,
\[
e^a = C_3 [P(\psi)]^{(1-\alpha)} T(t) \tag{48}
\]
\[
e^b = P(\psi) S(t) \tag{49}
\]
\[
e^c = C_4 M(\psi)[S(t)]^{\alpha}, \tag{50}
\]
where $C_3$ and $C_4$ are arbitrary constants. Substituting \((48)-(50)\) back into the field equation \((33)\) yields,
\[
\frac{C_4 S_\alpha^2 S^{(\alpha-2)}}{C_3} (1 + 2\alpha) = \frac{P^{(1-\alpha)}}{M} \left( 4 \frac{P_{\psi\psi}}{P} - 2 \frac{P_{\psi}}{P} M_{\psi} - \left( \frac{P_{\psi}}{P} \right)^2 \right). \tag{51}
\]

Now proceeding analogously to \((14)\), the following cases arise:

**iv)** If $\alpha = -\frac{1}{2}$ and we let $P = \psi$ and $T = 1$, then the metric becomes
\[
ds^2 = C_3 \bar{\psi}^2 dt^2 - \bar{\psi} S(dx^2 + dy^2) - C_4 (C_4 \bar{\psi} S)^{-\frac{1}{2}} d\bar{\psi}^2, \tag{52}
\]
which upon using using \((56)\) gives the following equation for $S(t)$:
\[
4 \frac{S_{tt}}{S^2} - \frac{S_t^2}{S^2} = \frac{4C_3}{C_4 C_1 \bar{\psi}^2}. \tag{53}
\]
This equation has a solution of the form $S(t) \sim t^{-4}$, with the constants satisfying $C_3/(C_4 C_1^{1/2}) = 16$. Hence, with suitable reparametrisation, the metric \((52)\) takes the following flat form
\[
ds^2 = \bar{\psi}^4 dt^2 - \bar{\psi} t^{-4} (dx^2 + dy^2) - \frac{1}{16} \bar{\psi}^{-\frac{4}{3}} t^2 d\bar{\psi}^2. \tag{54}
\]
This solution has the interesting property that spatial coordinates $x$ and $y$ contract with increasing $t$. The density and pressure in this case are given by $\rho = \frac{4}{t^4}, p = \frac{6}{t^6}$, which results in an equation of state that violates both the strong and dominant energy conditions \([10]\). This is an example of how vacuum solutions in $(3 + 1)$ dimensions can induce sources in $(2 + 1)$ dimensions which according to intuitions derived from the usual $(3 + 1)$-dimensional theory would be termed as “non-physical”.

**v)** For the case $\alpha \neq -\frac{1}{2}$, equation \((51)\) can be split up into the following 2 equations,
\[
\frac{C_4 S_\alpha^2 S^{(\alpha-2)}}{C_3} (1 + 2\alpha) = \beta, \tag{55}
\]
\[
\frac{P^{(1-\alpha)}}{M} \left( 4 \frac{P_{\psi\psi}}{P} - 2 \frac{P_{\psi}}{P} M_{\psi} - \left( \frac{P_{\psi}}{P} \right)^2 \right) = \beta, \tag{56}
\]
where $\beta$ is a nonzero constant. From \((53)\) we find the following relationship between $S(t)$ and $T(t)$,
\[
T(t) = \gamma S^{(\alpha-2)} S_\alpha^2 \tag{57}
\]
This yields the Robertson-Walker-de Sitter metric, where by setting Minkowski spacetime, they give rise (via Wesson’s scheme) to non-equivalent embeddings which generate the solutions (46), (54), (64), (65) and (66) are all flat in (3 + 1) dimensions, i.e. they represent the same while in solution (v) the time coordinate actually shrinks. It is worthwhile mentioning here that although de Leon, it can be seen that solutions (i), (ii), and (iii), although different in details of the functional pressure are given by

\[
\rho = C_4 \left( 1 + 2\alpha \right) \frac{E}{\beta}.
\]

while (58) yields the relation between \( P(\psi) \) and \( M(\psi) \),

\[
P_\psi^2 = \left[ E + \frac{\beta P^{\left(\alpha + \frac{1}{2}\right)}}{\alpha + \frac{1}{2}} \right] \frac{MP^\frac{1}{2}}{2},
\]

where \( E \) is a constant of integration. Substituting (57) and (59) into (48)-(50) gives

\[
e^a = \gamma C_3 P^{\left(1 - \alpha\right)} S^{\alpha - 2} S_t^2
\]

\[
e^b = PS
\]

\[
e^c = 2C_4 P_\psi^2 \left[ E + \frac{\beta P^{\left(\alpha + \frac{1}{2}\right)}}{\alpha + \frac{1}{2}} \right]^{-1} S^\alpha,
\]

which upon substitution in field equation (36) yields the remaining cases as those satisfying the condition \((2\alpha - 3)(1 + 2\alpha)E = 0\). As we have already considered the case \( \alpha = -\frac{3}{2} \), we shall now consider the cases of \( \alpha = \frac{3}{2} \) and \( E = 0 \) in turn.

If \( \alpha = \frac{3}{2} \) and we choose the coordinate transformation \( \gamma C_3 S_t^2 dt^2 = d\bar{t}^2 \) and \( P_\psi^2 d\psi^2 = d\bar{\psi}^2 \), again removing unnecessary constants, then the corresponding solution becomes

\[
ds^2 = (\bar{\psi}\bar{t})^{-\frac{1}{\alpha}} d\bar{t}^2 - \bar{t}\bar{\psi}(dx^2 + dy^2) - \bar{\psi}^{-\frac{1}{\alpha}} \left[ E + \bar{\psi}^2 \right]^{-1} \bar{t}^2 d\bar{\psi}^2.
\]

This indicates that the coefficient of \( d\bar{t} \) decreases as \( \bar{t} \) increases. The density and pressure are given by \( \rho = \frac{1}{2\gamma \bar{t}^2} \), \( \rho = \frac{1}{2\gamma \bar{t}^2} \), which imply a radiative equation of state in (2 + 1) dimensions with \( p = \frac{\gamma}{2} \).

\textbf{(vi) - (viii)} The case where \( E = 0 \) follows through exactly as in [14], thus we are left with the three cases where (a) \( \alpha = 0 \), (b) \( \alpha = 1 \), and (c) \( \alpha \neq \frac{1}{2}, 0, 1, \frac{3}{2} \). The appropriate solutions (which turn out to be flat) will in each case become:

\[
(a) \quad ds^2 = C\bar{\psi}^2 d\bar{t}^2 - \bar{\psi}^2 \exp \left[ 2\sqrt{\frac{\alpha}{Ct}} \right] (dx^2 + dy^2) - d\bar{\psi}^2.
\]

This yields the Robertson-Walker-de Sitter metric, where by setting \( C = \Lambda/3 \) the scale factor can be written in its usual form of \( e^{2\sqrt{\frac{\Lambda}{3t}}} \). Hence, we recover the (2 + 1) dimensional equivalent of equation (20).

\[
(b) \quad ds^2 = d\bar{t}^2 - \bar{t}^2 \exp \left[ 2\bar{\psi} \right] (dx^2 + dy^2) - \bar{t}^2 d\bar{\psi}^2.
\]

This (3+1)-dimensional spacetime leads to a flat (2+1)-dimensional model with the density and pressure given by \( \rho = \frac{1}{2} \) and \( p = 0 \), corresponding to a dust filled universe.

\[
(c) \quad ds^2 = \bar{\psi}^2 d\bar{t}^2 - \bar{t}^2 \bar{\psi}^2 (dx^2 + dy^2) - \frac{\alpha^2 \bar{t}^2}{(1 - \alpha r)^2} d\bar{\psi}^2.
\]

which yields a 3-dimensional FLRW metric with scale factor proportional to \( t^{2/\alpha} \), where the density and pressure are given by \( \rho = \frac{1}{4\alpha \bar{t}} \) and \( p = \frac{1}{4\alpha (2 - \frac{1}{\alpha})} \). The resulting equation of state, \( p = (\alpha - 1)p \), is the special case of equation (24) in (2 + 1) dimensions. Comparing our results with those obtained by Ponce de Leon, it can be seen that solutions (i), (ii), and (iii), although different in details of the functional forms of their metric coefficients, are qualitatively the same. This is also true of the solutions (vi)-(viii) which are in fact identical to those obtained by Ponce de Leon. Solutions (iv) and (v) are, however, different. Solution (iv) has the property that the \( x \) and \( y \) spatial coordinates shrink with increasing \( t \), while in solution (v) the time coordinate actually shrinks. It is worthwhile mentioning here that although the solutions (46), (54), (64), (65) and (66) are all flat in (3 + 1) dimensions, i.e. they represent the same Minkowski spacetime, they give rise (via Wesson’s scheme) to non-equivalent embeddings which generate
(2 + 1) spacetimes with different extrinsic curvatures.

It is also worth pointing out that Wesson’s procedure as employed here provides a simple mechanism for generating 3-dimensional spaces which have their symmetries induced by the symmetries of the 4-dimensional space. For example, equation (32) represents a class of line elements with axial symmetry in four dimensions which generates a class of circularly symmetric spacetimes in 3 dimensions. Finally, the examples given above were meant to act as an illustration of the idea put forward in this paper. It would be of interest to consider other embeddings, including further non-flat embedding spacetimes.

5 Conclusions

We have generalised the procedure put forward by Wesson and co-workers for setting up a correspondence between vacuum 5-dimensional Kaluza-Klein theory with 4-dimensional Einstein theory with sources to arbitrary dimensions, and applied this generalisation to connect the ordinary vacuum 4-dimensional gravity with (2 + 1)-dimensional theory with sources. This correspondence between the vacuum Einstein theory and lower dimensional theories is of potential importance in view of the great deal of effort that is currently going into the study of lower dimensional theories, and particularly in view of the radical differences between such theories and 4-dimensional gravity. Furthermore, this correspondence gives a mechanism for generating solutions in (2 + 1) dimensions that are naturally related to vacuum Einstein solutions.

Finally an important question with regards to lower dimensional theories is how to choose “reasonable” forms of the field equations and the energy-momentum tensor. Our work shows that vacuum Einstein solutions may not necessarily correspond to solutions in lower dimensions with reasonable (in the sense of 4-dimensions) equations of state, which highlights the fact that intuitions regarding physical reasonableness may not automatically be transportable to lower dimensions.

Further study of this correspondence is in progress and will be reported in future.

Acknowledgements CR was partially supported by CNPq (Brazil) and would like to thank the School of Mathematical Sciences for hospitality. SR was supported by the award of a PPARC studentship and RT was supported by SERC UK Grant number H09454.
References

[1] Belinsky, V. A. and Khalatnikov, I. M. 1972, Zh. Eksp. Teor. Fiz., 63, 1121 (Engl. transl. 1973 Sov. Phys. JETP 36, 591).

[2] Brans, C. and Dicke, R. H. 1961, Phys. Rev., 124, 539.

[3] Chan, K. C. K. and Mann, R. B. 1993, Class. Quantum Grav., 10, 913.

[4] Coley, A.A. 1994, Ap. J., 47, 585.

[5] Davidson, A. and Owen, D. A. 1985, Phys. Lett. B., 155, 247.

[6] Deser, S., Jackiw, R. and 'tHooft, G. 1984, Ann. Phys., 152, 220.

[7] Einstein, A. 1956, “The Meaning of Relativity”, Princeton University Press, Princeton.

[8] Freund, P. G. O. 1982, Nucl. Phys., B209, 146.

[9] Giddings, S., Abbott, J. and Kuchar, K., 1984, Gen. Rel. Grav., 16, 751.

[10] Hawking, S. W. and Ellis, G. F. R. 1973, “The Large Scale Structure of Space-Time”, Cambridge University Press, Cambridge.

[11] Mann, R. B., “Lower Dimensional Black Holes: Inside and Out”, preprint 1995.

[12] Odintsov, S.D. and Shapiro, I.L., 1992, Mod. Phys. Lett. A7, 437; Eliadze, E and Odinstov, S.D., 1993, Nucl. Phys. B399, 581.

[13] O’Hanlon, J. and Tupper, B. J. 1972, Nuovo Cimento, B7, 305.

[14] Ponce de Leon, J. 1988, Gen. Rel. Grav., 20, 539.

[15] Salam, A. 1980, Rev. Mod. Phys., 52, 525.

[16] Sato, H. 1984, Prog. Theor. Phys 72 98.

[17] Wesson, P. S. 1992, Astrophys. J., 394, 19.

[18] Wesson, P. S. and Ponce de Leon, J. 1992, J. Math. Phys., 33 (11), 3883.