Generalized cut method for computing the edge-Wiener index

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Abstract

The edge-Wiener index of a connected graph $G$ is defined as the Wiener index of the line graph of $G$. In this paper it is shown that the edge-Wiener index of an edge-weighted graph can be computed in terms of the Wiener index, the edge-Wiener index, and the vertex-edge-Wiener index of weighted quotient graphs which are defined by a partition of the edge set that is coarser than $\Theta^*$-partition. Thus, already known analogous methods for computing the edge-Wiener index of benzenoid systems and phenylenes are greatly generalized. Moreover, reduction theorems are developed for the edge-Wiener index and the vertex-edge-Wiener index since they can be applied in order to compute a corresponding index of a (quotient) graph from the so-called reduced graph. Finally, the obtained results are used to find the closed formula for the edge-Wiener index of an infinite family of graphs.

1 Introduction

The Wiener index of a connected graph $G$, denoted as $W(G)$, is defined as the sum of distances between all (unordered) pairs of vertices. More precisely,

$$W(G) = \sum_{\{u,v\} \subseteq V(G)} d_G(u,v) = \frac{1}{2} \sum_{u \in V(G)} \sum_{v \in V(G)} d_G(u,v),$$
where $d_G(u, v)$ is the standard shortest path distance between vertices $u$ and $v$ of graph $G$. The Wiener index was for the first time introduced by H. Wiener in 1947 [28]. Since it has good correlations with a large number of physico-chemical properties of organic molecules and also possesses various interesting mathematical properties, it has been studied in many papers.

On the other hand, the edge-Wiener index of a graph was independently introduced in [14, 16]. In [14] it was suggested that the distance between two edges $e, f$ of a graph $G$, here denoted by $d^0_G(e, f)$, should be defined as the distance between the corresponding vertices in the line graph $L(G)$ of $G$, i.e.

$$d^0_G(e, f) = d_{L(G)}(e, f),$$

since in this way the pair $(E(G), d^0_G)$ forms a metric space. Therefore, the edge-Wiener index of a connected graph $G$ is defined as

$$W_e(G) = \sum\{e, f\} \subseteq E(G) d^0_G(e, f) = \frac{1}{2} \sum_{e \in E(G)} \sum_{f \in E(G)} d^0_G(e, f).$$

In other words, $W_e(G)$ is just the Wiener index of the line graph of $G$, i.e. $W_e(G) = W(L(G))$ and hence, the edge-Wiener index was investigated before it was formally introduced (for example, see [9, 10, 11]).

However, for edges $e = xy, f = ab$ of a graph $G$ it is also possible to set

$$d^1_G(e, f) = \min\{d_G(x, a), d_G(x, b), d_G(y, a), d_G(y, b)\},$$

which was denoted by $\hat{d}_G(e, f)$ in [15]. Such definition gives us another version of the edge-Wiener index, denoted by $\hat{W}_e(G)$. More precisely,

$$\hat{W}_e(G) = \frac{1}{2} \sum_{e \in E(G)} \sum_{f \in E(G)} d^1_G(e, f).$$

Note that in [14] the numbers $W_e(G)$ and $\hat{W}_e(G)$ were denoted by $W_{e0}(G)$ and $W_{e1}(G)$, respectively. It is easy to observe that for any two distinct edges $e, f$ of a graph $G$ it holds $d^0_G(e, f) = d^1_G(e, f) + 1$ and therefore, $W_e(G)$ and $\hat{W}_e(G)$ are connected in the following way (cf. [14, 16]):

$$W_e(G) = \hat{W}_e(G) + \left(\frac{|E(G)|}{2}\right). \tag{1}$$

For our purposes, $d^1_G$ turns out to be more convenient than $d^0_G$. Hence, from technical reasons we also write $d_G$ instead of $d^1_G$, i.e. for any two edges $e, f \in E(G)$ we set

$$d_G(e, f) = d^1_G(e, f).$$
Some recent results on the edge-Wiener index can be found in [5, 8, 20, 21, 23, 27]. Moreover, it is worth mentioning that the edge-Wiener index is closely related to the edge-Hosoya polynomial of a graph [3]. In addition, for a recent survey on edge-Wiener descriptors see [13].

The cut method is a very useful tool for calculating distance-based topological indices and usually reduces the problem of calculating a topological index to the problem of calculating some indices of smaller graphs obtained by the edge cuts, see [18] for a recent survey. This method is often applied on benzenoid systems [6] or on partial cubes [25, 26]. In particular, some methods for computing the edge-Wiener index were developed, for instance, in [1, 2, 7, 29].

In [15] it has been proved that the edge-Wiener index of a benzenoid system can be computed by using the Wiener index, the edge-Wiener index, and the vertex-edge-Wiener index of the three weighted quotient trees obtained from elementary cuts. Later, a similar result has been established for phenylenes [30] (note that phenylenes and benzenoid systems represent important chemical graphs). In this paper, we generalize greatly the results from [15, 30] and prove that the edge-Wiener index of an edge-weighted graph can be calculated in terms of the Wiener index, the edge-Wiener index, and the vertex-edge-Wiener index of weighted quotient graphs defined by a partition of the edge set that is coarser than $\Theta^*$-partition. Therefore, our method is not restricted to some specific family of graphs and neither to partial cubes, but can be applied on any graph with at least two $\Theta^*$-classes. Consequently, the mentioned result can be used to develop very efficient algorithms for calculating the edge-Wiener index of important chemical graphs or networks and also to easily find closed formulas for some families of graphs. Such methods were recently developed also for some other distance-based topological indices: the Wiener index [19], the revised (edge-)Szeged index [22], the degree distance [4], the Graovac-Pisanski index [24].

As already mentioned, our method converts the problem of calculating the edge-Wiener index of a graph to the problem of calculating the three indices of weighted quotient graphs. However, it turns out that in some cases such a quotient graph can be further shrunk into the so-called reduced graph, so that the indices of the original graph can be computed from the reduced graph. In [17] it was shown that the Wiener index of a vertex-weighted graph can be computed from the Wiener index of a reduced graph. Therefore, we prove such results also for the edge-Wiener index and the vertex-edge-Wiener index.

The paper is organized in five sections. In the next section, some basic definitions and preliminary results are stated. In Section 3, it is firstly shown how the distance between two
edges can be computed by using the corresponding distances in quotient graphs. Moreover, we use this result to obtain the cut method for computing the edge-Wiener index. Furthermore, the reduction theorems are proved in Section 4. Finally, in Section 5 the obtained results are applied to an infinite family of graphs in order to calculate the closed formula for the edge-Wiener index.

2 Preliminaries

Unless stated otherwise, the graphs considered in this paper are simple, finite, and connected.

For a vertex \( v \) of a graph \( G \), we denote by \( N_G(v) \), or shortly by \( N(v) \), the open neighbourhood of \( v \), i.e. the set of vertices that are adjacent to \( v \). The distance between two vertices and the two distances between two edges have already been defined in the previous section. In addition, the distance between a vertex \( v \in V(G) \) and an edge \( e = xy \in E(G) \) is

\[
d_G(v, e) = \min\{d_G(v, x), d_G(v, y)\}.
\]

The vertex-edge-Wiener index of a graph \( G \) was defined in [15] as

\[
W_{ve}(G) = \sum_{v \in V(G)} \sum_{e \in E(G)} d_G(v, e).
\]

However, in [16] the vertex-edge-Wiener index was defined with the additional factor \( 1/2 \).

Next, we introduce the Wiener index, both versions of the edge-Wiener index, and the vertex-edge-Wiener index of weighted graphs. Let \( \mathbb{R}_0^+ = [0, \infty) \). If \( G \) is a graph and \( w : V(G) \to \mathbb{R}_0^+ \), \( w' : E(G) \to \mathbb{R}_0^+ \) are given weights, then \( (G, w) \), \( (G, w') \), and \( (G, w, w') \) are the vertex-weighted graph, the edge-weighted graph, and the vertex-edge-weighted graph, respectively. The corresponding Wiener indices of these graphs are defined as follows [15]:

\[
W(G, w) = \frac{1}{2} \sum_{u \in V(G)} \sum_{v \in V(G)} w(u)w(v)d_G(u, v),
\]

\[
W_e(G, w') = \frac{1}{2} \sum_{e \in E(G)} \sum_{f \in E(G)} w'(e)w'(f)d_G^0(e, f),
\]

\[
\tilde{W}_e(G, w') = \frac{1}{2} \sum_{e \in E(G)} \sum_{f \in E(G)} w'(e)w'(f)d_G(e, f),
\]

\[
W_{ve}(G, w, w') = \sum_{v \in V(G)} \sum_{e \in E(G)} w(v)w'(e)d_G(v, e).
\]

Two edges \( e = xy, f = ab \) of a graph \( G \) are in relation \( \Theta, e\Theta f \), if

\[
d_G(x, a) + d_G(y, b) \neq d_G(x, b) + d_G(y, a).
\]
Sometimes, this relation is also referred to as Djoković-Winkler relation. We can easily see that relation $\Theta$ is reflexive and symmetric, but not necessarily transitive. Therefore, its transitive closure (i.e. the smallest transitive relation containing $\Theta$) is denoted by $\Theta^*$. A sugraph $H$ of $G$ is an isometric subgraph if $d_G(u, v) = d_H(u, v)$ for all $u, v \in V(H)$ and an isometric subgraph of a hypercube is called a partial cube. It is known that in a partial cube relation $\Theta$ is always transitive, so $\Theta = \Theta^*$. Moreover, the class of partial cubes contains many interesting chemical graphs (for example benzenoid systems and phenylenes). For more information, see [12].

Let $E = \{E_1, \ldots, E_t\}$ be the $\Theta^*$-partition of the set $E(G)$. A partition $\{F_1, \ldots, F_r\}$ of $E(G)$ is said to be coarser than $E$ if each set $F_i$ is the union of one or more $\Theta^*$-classes of $G$.

Suppose $G$ is a graph and $E' \subseteq E(G)$ is some subset of its edges. The quotient graph $G/E'$ is the graph whose vertices are connected components of the graph $G \setminus E'$ and two such components $X$ and $Y$ are adjacent in $G/E'$ if and only if some vertex from $X$ is adjacent to a vertex from $Y$ in graph $G$. If $F = XY \in E(G/E')$ is an edge in graph $G/E'$, then we denote by $F$ also the set of edges of $G$ that have one end vertex in $X$ and the other end vertex in $Y$, i.e. $F = \{xy \in E(G) | x \in V(X), y \in V(Y)\}$.

Let $G$ be a connected graph and $\{F_1, \ldots, F_r\}$ a partition coarser than $\Theta^*$-partition. For any $i \in \{1, \ldots, r\}$, we define the function $\ell_i : V(G) \rightarrow V(G/F_i)$ as follows: for any $v \in V(G)$ let $\ell_i(v)$ be the connected component of the graph $G \setminus F_i$ that contains $v$. The result of the following lemma was obtained in [19]. For the complete proof see also [24].

**Lemma 2.1** [19, 24] Let $G$ be a connected graph. If $\{F_1, \ldots, F_r\}$ is a partition coarser than $\Theta^*$-partition, then for any $u, v \in V(G)$ it holds

$$d_G(u, v) = \sum_{i=1}^{r} d_{G/F_i}(\ell_i(u), \ell_i(v)).$$

When $\{F_1, \ldots, F_r\} = \{E_1, \ldots, E_t\}$ is the $\Theta^*$-partition, the function $\ell : V(G) \rightarrow V(G/E_1) \sqcup \cdots \sqcup V(G/E_t)$, defined with $\ell(v) = (\ell_1(v), \ldots, \ell_t(v))$ for any $v \in V(G)$, is called the canonical isometric embedding (see [12] for more details).

### 3 The main result

In this section we prove that the edge-Wiener index of a graph can be computed from the corresponding quotient graphs. Firstly, we define a function which maps any edge of $G$ either to
a vertex or to an edge of a quotient graph. Note that our definition generalizes the corresponding
definition from [15].

**Definition 3.1** Let $G$ be a connected graph and $\{F_1, \ldots, F_r\}$ a partition coarser than $\Theta^*-partition$. For any $i \in \{1, \ldots, r\}$, we introduce the function $\alpha_i : E(G) \to V(G/F_i) \cup E(G/F_i)$
by

$$\alpha_i(xy) = \begin{cases} \ell_i(x) \in V(G/F_i); & \ell_i(x) = \ell_i(y), \\
\ell_i(x)\ell_i(y) \in E(G/F_i); & \ell_i(x) \neq \ell_i(y), \end{cases}$$

where $xy$ is an arbitrary edge of $G$.

We can now show how the distance between two edges can be computed by using the distances in
the quotient graphs. In the proof, we use some ideas from [15] [22] but many additional insights
are also needed.

**Theorem 3.2** If $G$ is a connected graph and $\{F_1, \ldots, F_r\}$ a partition coarser than $\Theta^*-partition$, then for every $e, f \in E(G)$,

$$d_G(e, f) = \sum_{i=1}^{r} d_{G/F_i}(\alpha_i(e), \alpha_i(f)).$$

**Proof.** Let $e = xy$ and $f = ab$. Moreover, without loss of generality we can assume that
$d_G(e, f) = d_G(x, a)$. By Lemma 2.1 it follows

$$d_G(e, f) = \sum_{i=1}^{r} d_{G/F_i}(\ell_i(x), \ell_i(a)).$$

To finish the proof, we show that $d_{G/F_i}(\alpha_i(e), \alpha_i(f)) = d_{G/F_i}(\ell_i(x), \ell_i(a))$ for any $i \in \{1, \ldots, r\}$. Therefore, choose an arbitrary $i \in \{1, \ldots, r\}$ and consider the following cases.

**Case 1.** $\ell_i(x) = \ell_i(y)$ and $\ell_i(a) = \ell_i(b)$.
We obtain $\alpha_i(e) = \ell_i(x)$ and $\alpha_i(f) = \ell_i(a)$. Therefore, the result follows.

**Case 2.** $\ell_i(x) \neq \ell_i(y)$ and $\ell_i(a) = \ell_i(b)$.
Obviously, $e \in F_i$ and for any $j \in \{1, \ldots, r\}, j \neq i$, it holds $\ell_j(x) = \ell_j(y)$. Thus, we obtain

$$d_{G/F_j}(\ell_j(x), \ell_j(a)) = d_{G/F_j}(\ell_j(y), \ell_j(a))$$

for all $j \neq i$. Moreover, by Lemma 2.1 it follows

$$d_G(x, a) - d_G(y, a) = \sum_{j=1}^{r} d_{G/F_j}(\ell_j(x), \ell_j(a)) - \sum_{j=1}^{r} d_{G/F_j}(\ell_j(y), \ell_j(a))$$

$$= \sum_{j=1}^{r} (d_{G/F_j}(\ell_j(x), \ell_j(a)) - d_{G/F_j}(\ell_j(y), \ell_j(a)))$$

$$= d_{G/F_i}(\ell_i(x), \ell_i(a)) - d_{G/F_i}(\ell_i(y), \ell_i(a)).$$
Since \( d_G(x,a) \leq d_G(y,a) \), we now deduce \( d_{G/F_i}(\ell_i(x), \ell_i(a)) \leq d_{G/F_i}(\ell_i(y), \ell_i(a)) \) and therefore,
\[
d_{G/F_i}(\alpha_i(e), \alpha_i(f)) = d_{G/F_i}(\ell_i(x) \ell_i(y), \ell_i(a)) = d_{G/F_i}(\ell_i(x), \ell_i(a)).
\]

**Case 3.** \( \ell_i(x) = \ell_i(y) \) and \( \ell_i(a) \neq \ell_i(b) \).

This case is similar to Case 2.

**Case 4.** \( \ell_i(x) \neq \ell_i(y) \) and \( \ell_i(a) \neq \ell_i(b) \).

It is clear that \( e, f \in F_i \) and for any \( j \in \{1, \ldots, r\}, j \neq i \), it holds \( \ell_j(x) = \ell_j(y) \) and \( \ell_j(a) = \ell_j(b) \). Suppose that \( d_G(e,b) = d_G(z,b) \), where \( z \in \{x, y\} \). We can calculate
\[
d_G(e,a) - d_G(e,b) = d_G(x,a) - d_G(z,b) = \sum_{j=1}^{r} d_{G/F_j}(\ell_j(x), \ell_j(a)) - \sum_{j=1}^{r} d_{G/F_j}(\ell_j(z), \ell_j(b)) = \sum_{j=1}^{r} (d_{G/F_j}(\ell_j(x), \ell_j(a)) - d_{G/F_j}(\ell_j(z), \ell_j(b))) = d_{G/F_i}(\ell_i(x), \ell_i(a)) - d_{G/F_i}(\ell_i(z), \ell_i(b)).
\]

Since \( d_G(e,a) = d_G(x,a) \leq d_G(e,b) \), we now get
\[
d_{G/F_i}(\ell_i(x), \ell_i(a)) \leq d_{G/F_i}(\ell_i(z), \ell_i(b)) \tag{2}
\]

Obviously, \( d_G(x,a) \leq d_G(y,a) \), \( d_G(z,b) \leq d_G(x,b) \), and \( d_G(z,b) \leq d_G(y,b) \). By using similar reasoning as in Case 2, we can show
\[
d_{G/F_i}(\ell_i(x), \ell_i(a)) \leq d_{G/F_i}(\ell_i(y), \ell_i(a)),
\]
\[
d_{G/F_i}(\ell_i(z), \ell_i(b)) \leq d_{G/F_i}(\ell_i(x), \ell_i(b)),
\]
\[
d_{G/F_i}(\ell_i(z), \ell_i(b)) \leq d_{G/F_i}(\ell_i(y), \ell_i(b)).
\]

From these inequalities and from inequality (2) we conclude
\[
d_{G/F_i}(\alpha_i(e), \alpha_i(f)) = d_{G/F_i}(\ell_i(x) \ell_i(y), \ell_i(a) \ell_i(b)) = d_{G/F_i}(\ell_i(x), \ell_i(a)).
\]

In each case it holds \( d_{G/F_i}(\alpha_i(e), \alpha_i(f)) = d_{G/F_i}(\ell_i(x), \ell_i(a)) \), which completes the proof. \( \square \)

Let \( (G, w') \) be a connected edge-weighted graph and \( \{F_1, \ldots, F_r\} \) a partition coarser than \( \Theta^* \)-partition. The quotient graphs \( G/F_i, i \in \{1, \ldots, r\} \), are extended to weighted graphs \( (G/F_i, \lambda_i), (G/F_i, \lambda'_i), (T_i, \lambda_i, \lambda'_i) \) in the following way:
Theorem 3.3

The following theorem is the main result of the paper.

**Theorem 3.3** If \((G, w')\) is an edge-weighted connected graph and \(\{F_1, \ldots, F_r\}\) a partition coarser than \(\Theta^*-\)partition, then

\[
\tilde{W}_e(G, w') = \sum_{i=1}^{r} \left( W(G/F_i, \lambda_i) + \tilde{W}_e(G/F_i, \lambda'_i) + W_{ve}(G/F_i, \lambda_i, \lambda'_i) \right).
\]

**Proof.** By using Theorem 3.2 we get

\[
\tilde{W}_e(G, w') = \frac{1}{2} \sum_{e \in E(G)} \sum_{f \in E(G)} w'(e)w'(f)d_G(e, f)
\]

\[
= \frac{1}{2} \sum_{e \in E(G)} \sum_{f \in E(G)} w'(e)w'(f) \left( \sum_{i=1}^{r} d_{G/F_i}(\alpha_i(e), \alpha_i(f)) \right)
\]

\[
= \sum_{i=1}^{r} \left( \frac{1}{2} \sum_{e \in E(G)} \sum_{f \in E(G)} w'(e)w'(f)d_{G/F_i}(\alpha_i(e), \alpha_i(f)) \right).
\]

For any \(i \in \{1, \ldots, r\}\), we denote by \(E_1^i\) and \(E_2^i\) the set of edges of \(G\) that are mapped by function \(\alpha_i\) to a vertex or to an edge, respectively. More precisely,

\[
E_1^i(G) = \{ e \in E(G) | \alpha_i(e) \in V(G/F_i) \}, \quad E_2^i(G) = \{ e \in E(G) | \alpha_i(e) \in E(G/F_i) \}.
\]

Obviously, it holds \(E_1^i \cup E_2^i = E(G)\) and \(E_1^i \cap E_2^i = \emptyset\) for any \(i \in \{1, \ldots, r\}\). Therefore, for two distinct edges of \(G\) we have three possibilities: both edges belong to \(E_1^i\), both edges belong to \(E_2^i\), or one edge belongs to \(E_1^i\) and the other belongs to \(E_2^i\). Consequently, the inner sums can be partitioned into three parts:

\[
\tilde{W}_e(G, w') = \sum_{i=1}^{r} \left( \frac{1}{2} \sum_{e \in E_1^i(G)} \sum_{f \in E_1^i(G)} w'(e)w'(f)d_{G/F_i}(\alpha_i(e), \alpha_i(f)) \right)
\]

\[
+ \frac{1}{2} \sum_{e \in E_2^i(G)} \sum_{f \in E_2^i(G)} w'(e)w'(f)d_{G/F_i}(\alpha_i(e), \alpha_i(f))
\]

\[
+ \sum_{e \in E_1^i(G)} \sum_{f \in E_2^i(G)} w'(e)w'(f)d_{G/F_i}(\alpha_i(e), \alpha_i(f)).
\]
Let $X, Y$ be two arbitrary distinct connected components of $G \setminus F_i$. Obviously, for any $e, e' \in E(X)$ and $f, f' \in E(Y)$ it holds $d_{G/F_i}(X, Y) = d_{G/F_i}(\alpha_i(e), \alpha_i(f)) = d_{G/F_i}(\alpha_i(e'), \alpha_i(f'))$. Moreover,

$$\sum_{e \in E(X)} \sum_{f \in E(Y)} w'(e)w'(f)d_{G/F_i}(\alpha_i(e), \alpha_i(f)) = d_{G/F_i}(X, Y) \sum_{e \in E(X)} \sum_{f \in E(Y)} w'(e)w'(f) = \lambda_i(X)\lambda_i(Y)d_{G/F_i}(X, Y).$$

Let $E, F$ be two arbitrary distinct edges of the graph $G/F_i$. Obviously, for any $e, e' \in E$ and $f, f' \in F$ it holds $d_{G/F_i}(E, F) = d_{G/F_i}(\alpha_i(e), \alpha_i(f)) = d_{G/F_i}(\alpha_i(e'), \alpha_i(f'))$. Moreover,

$$\sum_{e \in E} \sum_{f \in F} w'(e)w'(f)d_{G/F_i}(\alpha_i(e), \alpha_i(f)) = d_{G/F_i}(E, F) \sum_{e \in E} \sum_{f \in F} w'(e)w'(f) = \lambda'_i(E)\lambda'_i(F)d_{G/F_i}(E, F).$$

Finally, let $X$ be a connected component of $G \setminus F_i$ and $F$ an edge of the graph $G/F_i$. Obviously, for any $e, e' \in E(X)$ and $f, f' \in F$ it holds $d_{G/F_i}(X, F) = d_{G/F_i}(\alpha_i(e), \alpha_i(f)) = d_{G/F_i}(\alpha_i(e'), \alpha_i(f'))$. Moreover,

$$\sum_{e \in E(X)} \sum_{f \in F} w'(e)w'(f)d_{G/F_i}(\alpha_i(e), \alpha_i(f)) = d_{G/F_i}(X, F) \sum_{e \in E(X)} \sum_{f \in F} w'(e)w'(f) = \lambda_i(X)\lambda'_i(F)d_{G/F_i}(X, F).$$

From the obtained calculations we finally conclude

$$\widehat{W}_e(G, w') = \sum_{i=1}^{r} \left( \frac{1}{2} \sum_{x \in V(G/F_i)} \sum_{y \in V(G/F_i)} \lambda_i(x)\lambda_i(y)d_{G/F_i}(x, y) \right) + \frac{1}{2} \sum_{E \in E(G/F_i)} \sum_{F \in E(G/F_i)} \lambda'_i(E)\lambda'_i(F)d_{G/F_i}(E, F)$$

$$+ \sum_{x \in V(G/F_i)} \sum_{F \in E(G/F_i)} \lambda_i(x)\lambda'_i(F)d_{G/F_i}(X, F)$$

$$= \sum_{i=1}^{r} \left( W(G/F_i, \lambda_i) + \widehat{W}_e(G/F_i, \lambda'_i) + W_{ve}(G/F_i, \lambda_i, \lambda'_i) \right)$$

and the proof is complete. \qed

If we set $w'(e) = 1$ for any $e \in E(G)$, the following corollary follows by equality (1).

**Corollary 3.4** If $G$ is a connected graph and $\{F_1, \ldots, F_r\}$ a partition coarser than $\Theta^*$-partition, then

$$W_e(G) = \sum_{i=1}^{r} \left( W(G/F_i, \lambda_i) + \widehat{W}_e(G/F_i, \lambda'_i) + W_{ve}(G/F_i, \lambda_i, \lambda'_i) \right) + \left( \frac{|E(G)|}{2} \right),$$
where \( \lambda_i : V(G/F_i) \to \mathbb{R}_0^+ \), \( \lambda'_i : E(G/F_i) \to \mathbb{R}_0^+ \) are defined as follows: \( \lambda_i(X) \) is the number of edges in the connected component \( X \) and \( \lambda'_i(XY) \) is the number of edges in \( G \) that have one end vertex in \( X \) and the other end vertex in \( Y \).

4 Reduction theorems

It was shown in [17] that in some cases the problem of computing the Wiener index of a vertex-weighted graph can be reduced to the problem of computing the Wiener index of a smaller graph obtained by a special reduction. Such a reduction can be defined, for example, by the so-called relation \( R \), which is important also elsewhere since the vertices that are in this relation are often called twins. In this section, we develop analogous results for the edge-Wiener index and for the vertex-edge-Wiener index, since these indices are needed to efficiently compute the edge-Wiener index in terms of Corollary 3.4. An example showing how these reductions can be used is provided in the next section.

Let \( G \) be a graph. Two vertices \( u \) and \( v \) are in relation \( R \) if \( N(u) = N(v) \). It is easy to see that \( R \) is an equivalence relation on the set of vertices \( V(G) \). The \( R \)-equivalence class containing \( v \) will be denoted with \( [v]_R \). The following theorem and corollary were obtained in [17]

**Theorem 4.1** [17] Let \((G, w)\) be a connected vertex-weighted graph, \( c \in V(G) \), and \( C = [c]_R = \{c_1, \ldots, c_k\} \). If \((G', w')\) is a graph defined by \( G' = G \setminus (C \setminus \{c\}) \), \( w'(c) = \sum_{x \in C} w(x) \), and \( w'(v) = w(v) \) for any \( v \in V(G) \setminus C \), then

\[
W(G, w) = W(G', w') + \sum_{\{c_i, c_j\} \subseteq C} 2w(c_i)w(c_j).
\]

**Corollary 4.2** [17] Let \((G,w)\) be a connected vertex-weighted graph, \( c \in V(G) \), \( C = [c]_R = \{c_1, \ldots, c_k\} \), and \( w(c_i) = a \) for any \( i \in \{1, \ldots, k\}, a \in \mathbb{R}_0^+ \). If \((G', w')\) is defined as in Theorem 4.1, then

\[
W(G, w) = W(G', w') + a^2k(k-1).
\]

However, in [17] it was assumed that a weight \( w \) is a function to the set \( \mathbb{R}^+ = (0, \infty) \), but the same proof works also when \( w : V(G) \to \mathbb{R}_0^+ \).

In the rest of the section, we define the reduced graph \( G' \) as in Theorem 4.1. For any \( c \in V(G) \), let \( C = [c]_R = \{c_1, \ldots, c_k\} \) and \( G' \) the graph defined by \( G' = G \setminus (C \setminus \{c\}) \). Let us denote \( N(c) = N(c_i) = \{n_1, \ldots, n_s\} \) for any \( i \in \{1, \ldots, k\} \). Moreover, we denote

\[
I(c_i) = \{c_in_j \in E(G) \mid j \in \{1, \ldots, s\}\},
\]
Using these facts we can proceed as follows:

$I(n_j) = \{c_i n_j \in E(G) | i \in \{1, \ldots, k\}\}$

for any $i \in \{1, \ldots, k\}$ or $j \in \{1, \ldots, s\}$. In other words, $I(c_i)$ is the set of edges that are incident with vertex $c_i$ and $I(n_j)$ is the set of edges that are incident to $n_j$ and a vertex from $C$. We also set $I(C) = \bigcup_{i \in \{1, \ldots, k\}} I(c_i) = \bigcup_{j \in \{1, \ldots, s\}} I(n_j)$, which represents the set of edges that are incident to a vertex of $C$. In addition, for any $i \in \{1, \ldots, k\}$, $j \in \{1, \ldots, s\}$, let $I(C)_{ij} = I(C) \setminus (I(c_i) \cup I(n_j))$.

Furthermore, for any weight $w : V(G) \to \mathbb{R}_0^+$ we define the weight $w' : V(G') \to \mathbb{R}_0^+$ by $w'(c) = \sum_{i=1}^{k} w(c_i)$ and $w'(v) = w(v)$ for any $v \in V(G) \setminus C$. Finally, for any weight $w_e : E(G) \to \mathbb{R}_0^+$ we define the weight $w'_e : E(G') \to \mathbb{R}_0^+$ in the following way: $w'_e(cn_j) = \sum_{i=1}^{k} w_e(c_i n_j)$, $j \in \{1, \ldots, s\}$, and $w'_e(e) = w_e(e)$ for any $e \in E(G) \setminus I(C)$. Now we can state our results.

**Theorem 4.3** If $(G, w_e)$ is a connected edge-weighted graph, $c \in V(G)$, $C = [c]_R = \{c_1, \ldots, c_k\}$, and $N(c) = \{n_1, \ldots, n_s\}$, then

$$
\hat{W}_e(G, w_e) = \hat{W}_e(G', w'_e) + \frac{1}{2} \sum_{c_i n_j \in I(C)} \sum_{e \in I(C)_{ij}} w_e(c_i n_j) w_e(e).
$$

**Proof.** If $|C| = 1$, $(G', w'_e) = (G, w_e)$ and the result obviously follows. Therefore, let $c_1 = c$ and $k \geq 2$. We can easily see

(i) $d_G(c_i n_j, e) = d_G(c_r n_j, e)$ holds for any $c_i, c_r \in C$, $n_j \in N(c)$, and $e \in E(G) \setminus I(C)$,

(ii) $d_G(e, f) = d_G(e, f)$ holds for any two edges $e, f \in E(G) \setminus I(C)$,

(iii) $d_G(c_i n_j, c_r n_t) = 1$ holds for any $c_i n_j, c_r n_t \in I(C)$, $i \neq r$, $j \neq t$.

Using these facts we can proceed as follows:

$$
\hat{W}_e(G, w_e) = \sum_{\{e, f\} \subseteq E(G)} w_e(e) w_e(f) d_G(e, f)
$$

$$
= \sum_{e \in E(G) \setminus I(C)} \sum_{j=1}^{s} \sum_{i=1}^{k} w_e(e) w_e(c_i n_j) d_G(c_i n_j, e)
$$

$$
+ \sum_{\{e, f\} \subseteq E(G) \setminus I(C)} w_e(e) w_e(f) d_G(e, f)
$$

$$
+ \sum_{\{e, f\} \subseteq I(C)} w_e(e) w_e(f) d_G(e, f).
$$
Corollary 4.4 follows.

\[ \sum_{e \in E(G) \setminus I(C)} \sum_{j=1}^{s} w_e(e)d_G(cn_j,e) \sum_{i=1}^{k} w_e(c_in_j) \]

\[ + \sum_{\{e,f\} \subseteq E(G) \setminus I(C)} w_e(e)w_f(f)d_G(e,f) \]

\[ + \frac{1}{2} \sum_{c_in_j \in I(C)} \sum_{e \in I(C)} w_e(c_in_j)w_e(e)d_G(c_in_j,e). \]

Therefore, we finally deduce

\[ \widehat{W}_e(G, w_e) = \sum_{e \in E(G')} \sum_{j=1}^{s} w'_e(e)w'_e(cn_j,e)d_G(cn_j,e) \]

\[ + \sum_{\{e,f\} \subseteq E(G')} w'_e(e)w'_f(f)d_G(e,f) \]

\[ + \frac{1}{2} \sum_{c_in_j \in I(C)} \sum_{e \in I(C)} w_e(c_in_j)w_e(e)d_G(c_in_j,e). \]

which finishes the proof. \( \square \)

When all the edges from \( I(C) \) have the same weight, the above theorem can be simplified as follows.

**Corollary 4.4** Let \((G, w_e)\) be a connected edge-weighted graph, \( c \in V(G), \) \( C = [c]_R = \{c_1, \ldots, c_k\}, \) and \( N(c) = \{n_1, \ldots, n_s\}. \) If \( w_e(e) = a \) for all \( e \in I(C), \) \( a \in \mathbb{R}_0^+, \) then

\[ \widehat{W}_e(G, w_e) = \widehat{W}_e(G', w'_e) + \frac{a^2ks}{2}(k-1)(s-1). \]

Next, we prove similar results also for the vertex-edge-Wiener index.

**Theorem 4.5** If \((G, w, w_e)\) is a connected vertex-edge-weighted graph, \( c \in V(G), \) \( C = [c]_R = \{c_1, \ldots, c_k\}, \) and \( N(c) = \{n_1, \ldots, n_s\}, \) then

\[ W_{ve}(G, w, w_e) = W_{ve}(G', w', w'_e) + \sum_{i=1}^{k} \sum_{c_i \in C \setminus \{c_i\}} \sum_{e \in I(C_i)} w(c_i)w_e(e). \]

**Proof.** If \(|C| = 1, (G', w', w'_e) = (G, w, w_e)\) and the result obviously follows. Therefore, let \( c_1 = c \) and \( k \geq 2. \) We can easily obtain

(i) \( d_G(c_i, e) = d_G(c_r, e) \) holds for any \( c_i, c_r \in C \) and \( e \in E(G) \setminus I(C), \)

(ii) \( d_G(v, c_in_j) = d_G(v, c_rn_j) \) holds for any \( v \in V(G) \setminus C, \) \( c_i, c_r \in C, \) and \( n_j \in N(c), \)
(iii) \( d_G(v, e) = d_{G'}(v, e) \) holds for any \( v \in V(G) \setminus C \) and \( e \in E(G) \setminus I(C) \),

(iv) \( d_G(c, c_rn_j) = 1 \) holds for any \( c, c_r \in C, i \neq r \), and \( n_j \in I(c_r) \).

Using these facts we can proceed as follows:

\[
W_{ve}(G, w, w_e) = \sum_{v \in V(G)} \sum_{e \in E(G)} w(v)w_e(e)d_G(v, e)
\]

\[
= \sum_{e \in E(G) \setminus I(C)} \sum_{i=1}^{k} w(c_i)w_e(e)d_G(c_i, e)
\]

\[
+ \sum_{e \in V(G) \setminus C} \sum_{j=1}^{s} \sum_{i=1}^{k} w(v)w_e(c_i n_j)d_G(v, c_i n_j)
\]

\[
+ \sum_{e \in V(G) \setminus C} \sum_{e \in E(G) \setminus I(C)} w(v)w_e(e)d_G(v, e)
\]

\[
+ \sum_{i=1}^{k} \sum_{e \in I(C)} w(c_i)w_e(e)d_G(c_i, e)
\]

\[
= \sum_{e \in E(G) \setminus I(C)} w_e(e)d_G(c, e) \sum_{i=1}^{k} w(c_i)
\]

\[
+ \sum_{e \in V(G) \setminus C} \sum_{j=1}^{s} \sum_{i=1}^{k} w(v)d_G(v, c_i n_j) \sum_{i=1}^{k} w(c_i n_j)
\]

\[
+ \sum_{e \in V(G) \setminus C} \sum_{e \in E(G) \setminus I(C)} w(v)w_e(e)d_G(v, e)
\]

\[
+ \sum_{i=1}^{k} \sum_{c_r \in C \setminus \{c_i\}} \sum_{n_j \in N(c_r)} w(c_i)w_e(c_i n_j).
\]

Hence, we finally deduce

\[
W_{ve}(G, w, w_e) = \sum_{e \in E(G') \setminus I(e)} w'(c)w'_e(e)d_{G'}(c, e)
\]

\[
+ \sum_{e \in V(G') \setminus \{c\}} \sum_{j=1}^{s} w'(v)w'_e(c, c_n_j)d_{G'}(v, c_n_j)
\]

\[
+ \sum_{e \in V(G') \setminus \{c\}} \sum_{e \in E(G') \setminus I(e)} w'(v)w'_e(e)d_{G'}(v, e)
\]

\[
+ \sum_{i=1}^{k} \sum_{c_r \in C \setminus \{c_i\}} \sum_{e \in I(c_r)} w(c_i)w_e(e)
\]

\[
= W(G', w', w'_e) + \sum_{i=1}^{k} \sum_{c_r \in C \setminus \{c_i\}} \sum_{e \in I(c_r)} w(c_i)w_e(e)
\]

and the proof is complete. \(\square\)
Corollary 4.6 Let \((G, w, w_e)\) be a connected vertex-edge-weighted graph, \(c \in V(G)\), \(C = [c]_R = \{c_1, \ldots, c_k\}\), and \(N(c) = \{n_1, \ldots, n_s\}\). If \(w(c_i) = a\) for any \(i \in \{1, \ldots, k\}\) and \(w_e(e) = b\) for all \(e \in I(C)\), \(a, b \in \mathbb{R}_0^+\), then
\[
W_{ve}(G, w, w_e) = W_{ve}(G', w', w'_e) + ab(k - 1)s.
\]

5 An example

In this final section we apply the obtained results on an infinite family of graphs to shown how these results can be used to efficiently compute the edge-Wiener index. For \(m, n \geq 1\), let \(G_{m,n}\) be the graph shown in Figure 1 which has \(m\) horizontal layers of hexagons and \(n\) vertical layers of hexagons. We obtain \(|V(G_{m,n})| = (m + 1)(2n + 1)\) and \(|E(G_{m,n})| = 3mn + m + 2n\).

![Figure 1: Graph \(G_{m,n}\).](image)

Firstly, we have to determine the \(\Theta^*\)-classes of \(G_{m,n}\). It is easy to see that any two edges on a shortest path can not be in relation \(\Theta\). Moreover, any two diametrically opposite edges in an isometric even cycle of a graph are in relation \(\Theta\). By using these facts, it is not difficult to check that graph \(G_{m,n}\) has \(\Theta^*\)-classes \(D_1, \ldots, D_m\) and \(E_1, \ldots, E_n\) as shown in Figure 2. Also, it is easy to see that for \(m \geq 2\) graph \(G_{m,n}\) is not a partial cube since relation \(\Theta\) is not transitive.

Next, we define \(F_1 = \bigcup_{i=1}^m D_i\) and \(F_2 = \bigcup_{i=1}^n E_i\). Obviously, \(\{F_1, F_2\}\) is a partition of the set \(E(G_{m,n})\) that is coarser than \(\Theta^*\)-partition. The quotient graph \(G_{m,n}/F_1\) is isomorphic to the path on \(m + 1\) vertices and the weights \(\lambda_1, \lambda'_1\) are calculated as in Corollary 3.4, see Figure 3. Therefore, we compute:
Figure 2: $\Theta^*$-classes of $G_{m,n}$.

Figure 3: Weighted quotient graph $(G_{m,n}/F_1, \lambda_1, \lambda'_1)$, $G_{m,n}/F_1 \cong P_{m+1}$.

$$W(G_{m,n}/F_1, \lambda_1) = 4n^2 \sum_{i=1}^{m} \sum_{j=i+1}^{m+1} (j - i) = \frac{2n^2}{3} (m^3 + 3m^2 + 2m),$$

$$\tilde{W}_e(G_{m,n}/F_1, \lambda'_1) = (n+1)^2 \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} (j - i - 1) = \frac{(n+1)^2}{6} (m^3 - 3m^2 + 2m),$$

$$W_{ee}(G_{m,n}/F_1, \lambda_1, \lambda'_1) = 2n(n+1) \left( \sum_{i=2}^{m+1} \sum_{j=1}^{i-1} (i - j - 1) + \sum_{i=1}^{m} \sum_{j=i}^{m} (j - i) \right)$$

$$= \frac{2n(n+1)}{3} (m^3 - m).$$

On the other hand, the quotient graph $G_{m,n}/F_2$ is depicted in Figure 4. Weights $\lambda_2, \lambda'_2$ are again calculated in terms of Corollary 3.3. The sets of vertices $C_1, \ldots, C_n$ are shown in the same figure and each of them contains $m + 1$ vertices.

Note that any two vertices from $C^i$, $i \in \{1, \ldots, n\}$, are in relation $R$. Therefore, we perform the reduction from Section 4 exactly $n$ times on $G_{m,n}/F_2$ and obtain the vertex-edge-weighted graph which will be denoted as $(H, \lambda_3, \lambda'_3)$. Obviously, $H$ is isomorphic to the path on $2n + 1$ vertices, see Figure 5.
Figure 4: Weighted quotient graph \((G_{m,n}/F_2, \lambda_2, \lambda'_2)\) with sets of vertices \(C^1, \ldots, C^n\).

Figure 5: Weighted graph \((H, \lambda_3, \lambda'_3), H \cong P_{2n+1}\).

Hence, we can calculate:

\[
W(H, \lambda_3) = m^2 \sum_{i=0}^{n-1} \sum_{j=i+1}^{n} ((2j+1) - (2i+1)) = \frac{m^2}{3} (n^3 + 3n^2 + 2n),
\]

\[
\tilde{W}_e(H, \lambda'_3) = (m+1)^2 \sum_{i=1}^{2n-1} \sum_{j=i+1}^{2n} (j - i - 1) = \frac{2(m+1)^2}{3} (2n^3 - 3n^2 + n),
\]

\[
W_{ve}(H, \lambda_3, \lambda'_3) = m(m+1) \left( \sum_{i=1}^{n} \sum_{j=1}^{2i} (2i+1 - j - 1) + \sum_{i=0}^{n-1} \sum_{j=2i+1}^{2n} (j - 2i - 1) \right)
\]

\[
= \frac{m(m+1)}{3} (4n^3 + 3n^2 - n).
\]

By Corollaries 4.2, 4.4, and 4.6 we obtain

\[
W(G_{m,n}/F_2, \lambda_2) = W(H, \lambda_3) + 0 \cdot n,
\]

\[
\tilde{W}_e(G_{m,n}/F_2, \lambda'_2) = \tilde{W}_e(H, \lambda_3) + m(m+1)n,
\]

\[
W_{ve}(G_{m,n}/F_2, \lambda_2, \lambda'_2) = W_{ve}(H, \lambda_3, \lambda'_3) + 0 \cdot n.
\]

Finally, by Corollary 3.4 it follows

\[
W_e(G_{m,n}) = \frac{1}{6} \left( 9m^3n^2 + 18m^2n^3 + 6m^3n + 36m^2n^2 + 24mn^3 + m^3 \\
+ 24m^2n + 24mn^2 + 8n^3 + 15mn - m - 2n \right).
\]

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