On Baxterized Solutions of Reflection Equation and Integrable Chain Models

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Abstract. Non-polynomial baxterized solutions of reflection equations associated with affine Hecke and affine Birman-Murakami-Wenzl algebras are found. Relations to integrable spin chain models with nontrivial boundary conditions are discussed.

Introduction

A reflection equation was introduced by Cherednik in [1] as an additional factorization condition for a boundary S-matrix which describes the motion of relativistic particles on a half line. In the context of 2-dimensional integrable field theories with boundaries the reflection equations and their solutions were investigated in [2]. These equations and their solutions (boundary $K$-matrices) were also used in [3] for a formulation of integrable spin chains with nonperiodic boundary conditions and studied in many subsequent papers (see, e.g., [4], [5], [6] and references therein).

To investigate reflection equations and their solutions, it is convenient to use their universal formulation in terms of generators of a group algebra of a braid group and its quotients like Hecke $H_M$ or Birman-Murakami-Wenzl

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$BMW_M$ algebras. In this approach, solutions of the Yang-Baxter equation are represented as elements of the above algebras (baxterized elements), while solutions of the reflection equations are expressed in terms of generators of affine extensions of $H_M$ or $BMW_M$. Many papers are devoted to investigations of baxterized solutions of the reflection equation in the Hecke algebra case (see, e.g., [7], [6] and references therein). Conversely, not so much is known about baxterized solutions of the reflection equation in the Birman-Murakami-Wenzl algebra case (see, however, [8]).

In this paper, new baxterized solutions of the reflection equation of the Hecke and Birman-Murakami-Wenzl types are found. These solutions are rational functions of affine generators which automatically satisfy a unitarity condition. For the cyclotomic quotients of the affine Hecke and BMW algebras, when affine generators satisfy a polynomial equation of a finite degree, these solutions can be reduced to polynomial ones. In particular, for the cyclotomic affine Hecke algebra, we reproduce solutions obtained recently in [7].

In the last Section we discuss applications of the reflection equation solutions to the formulation of the integrable chain systems with nontrivial boundary conditions.

1 Solutions of reflection equations for the Hecke algebra

A braid group $B_{M+1}$ is generated by elements $\sigma_i \ (i = 1, \ldots, M)$ subject to relations:

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \quad \sigma_i \sigma_j = \sigma_j \sigma_i \text{ for } |i - j| > 1.$$  \hspace{1cm} (1.1)

An A-Type Hecke algebra $H_{M+1} \equiv H_{M+1}(q)$ (see, e.g., [10] and references therein) is a quotient of the group algebra of the braid group $B_{M+1}$ by a Hecke relation

$$\sigma_i^2 - 1 = \lambda \sigma_i, \quad (i = 1, \ldots, M),$$  \hspace{1cm} (1.2)

where $\lambda := (q - q^{-1})$ and $q \in \mathbb{C}\backslash\{0\}$ is a parameter.

Let $x \in \mathbb{C}$ be a spectral parameter. The baxterized elements [9]

$$\sigma_n(x) := \sigma_n - x\sigma_n^{-1} \in H_{M+1},$$  \hspace{1cm} (1.3)

solve the Yang-Baxter equation:

$$\sigma_n(x) \sigma_{n-1}(xy) \sigma_n(y) = \sigma_{n-1}(y) \sigma_n(xy) \sigma_n(x).$$  \hspace{1cm} (1.4)

Let $a$ be any solution of the equation $a - \frac{1}{a} = \lambda$; that is, $a$ equals $q$ or $(-q^{-1})$. Then, for $x \neq aq, -aq^{-1}$, the elements (1.3) can be represented in the form

$$\sigma_n(x) = \left(a x^{-1} - a^{-1}\right) \frac{\sigma_n + a x}{\sigma_n + a x^{-1}}.$$  \hspace{1cm} (1.5)
The element (1.5) does not depend on the choice of $a = \pm q^{\pm 1}$. The normalized element

$$\tilde{\sigma}_n(x; a) := \frac{1}{a x - a^{-1}} \sigma_n(x),$$

satisfies the unitarity condition $\tilde{\sigma}_n(x; a) \tilde{\sigma}_n(x^{-1}; a) = 1$.

An affine Hecke algebra $\hat{H}_{M+1}$ (see, e.g., [11], Chapter 12.3) is an extension of the Hecke algebra $H_{M+1}$. The algebra $\hat{H}_{M+1}$ is generated by elements $\sigma_i$ ($i = 1, \ldots, M$) of $H_{M+1}$ and affine generators $y_k$ ($k = 1, \ldots, M + 1$) which satisfy:

$$y_{k+1} = \sigma_k y_k \sigma_k, \quad y_k y_j = y_j y_k, \quad y_j \sigma_i = \sigma_i y_j \quad (j \neq i, i + 1).$$

(1.7)

The elements $\{y_k\}$ generate a commutative subalgebra in $\hat{H}_{M+1}$ while symmetric functions of $y_k$ form a center in $\hat{H}_{M+1}$.

The aim of this section is to find solutions $y_n(x)$ of the reflection equation:

$$\sigma_n \left( x z^{-1} \right) y_n(x) \sigma_n(x z) y_n(z) = y_n(z) \sigma_n(x z) y_n(x) \sigma_n \left( x z^{-1} \right),$$

(1.8)

where $y_n(x)$ is an element of the affine Hecke algebra and is a function of the spectral parameter $x$. We call $y_n(x)$ a local solution of (1.8) if $[y_n(x), \sigma_i] = 0 \quad (\forall i \neq n - 1, n)$. An additional requirement on the solution $y_n(x)$ is the unitarity condition: $y_n(x)y_n(x^{-1}) = 1$.

Consider a subalgebra $\hat{H}_2^{(n)} \subset \hat{H}_{M+1}$ which is generated by two elements $\sigma_n, y_n$ (in fact we consider an extension of $\hat{H}_2^{(n)}$ by formal series in $y_n$). Symmetric functions of two variables $y_n$ and $y_{n+1} := \sigma_n y_n \sigma_n$ are central in the algebra $\hat{H}_2^{(n)}$.

We make a following Ansatz

$$y_n(x) = Z_1(x) + Z_2(x) y_n,$$

(1.9)

where $Z_i(x)$ are central elements in $\hat{H}_2^{(n)}$.

A direct calculation shows that eq. (1.8) is satisfied iff

$$\frac{Z_1(x)}{Z_2(x)} = \frac{\gamma - z^{(2)} x}{x - x^{-1}},$$

(1.10)

where $\gamma$ is a central element in $\hat{H}_2^{(n)}$ independent of the spectral parameter $x$ and $z^{(2)} = y_n + y_{n+1}$.

Thus a general solution of the reflection equation (1.8) within the Ansatz (1.9) is:

$$y_n(x) = Z_2(x) \left( y_n + \frac{\gamma - z^{(2)} x}{x - x^{-1}} \right),$$

(1.11)

where $Z_2(x)$ is an arbitrary central element in $\hat{H}_2^{(n)}$. In this form the solution is not quite appropriate since it is not obviously local (e.g., it explicitly
depends on \( y_{n+1} \). However, one can achieve the locality (and unitarity) of the solution \( y_n(x) \) by specifying the central element \( Z_2(x) \) in (1.11). We now formulate the main result of this section.

**Proposition 1.** In the case of the affine Hecke algebra, a local unitary solution of the reflection equation (1.8) is

\[
y_n(x) = \frac{y_n - \xi x}{y_n - \xi x^{-1}},
\]

where \( \xi \in \mathbb{C} \) is an arbitrary parameter.

**Proof.** The unitarity property: \( y_n(x)y_n(x^{-1}) = 1 \) of the element (1.12) is obvious. One can rewrite (1.12) in the form

\[
y_n(x) \equiv Z_2(x) \left( y_n + \frac{y'_n - x z^{(2)}}{x - x^{-1}} \right)
\]

(1.13)

where \( \gamma' = \xi + (y_n y_{n+1})\xi^{-1} \) and

\[
Z_2(x) := \frac{\xi(x - x^{-1})}{(y_n - \xi x^{-1})(y_n + \xi x^{-1})},
\]

are central elements in \( \hat{H}_2^{(n)} \). The function (1.13) has the form (1.11) which means that (1.12) is the solution of eq.(1.8).

**Remark 1.** The solution (1.12) is regular: \( y_n(\pm 1) = 1 \) (see [3]). A general (within the Ansatz (1.9)) solution is obtained by a multiplication of (1.12) by an arbitrary scalar function \( f(x) \) such that \( f(x)f(x^{-1}) = 1 \).

**Remark 2.** If the parameter \( \xi \) in (1.12) is not equal to 0, \( \xi \neq 0 \), it can be set to 1 by an automorphism \( \sigma_k \rightarrow \sigma_k, y_n \rightarrow \xi y_n \) of the affine Hecke algebra \( \hat{H}_{M+1} \). In the case \( \xi = 0 \), the solution (1.12) becomes trivial, \( y_n(x) = 1 \).

The element (1.12) stays a solution of (1.8) if we substitute instead of \( \xi \) any element of \( \hat{H}_{M+1} \) central in \( \hat{H}_2^{(n)} \).

**Remark 3.** In a cyclotomic affine Hecke algebra, the generator \( y_1 \) satisfies an additional characteristic equation

\[
y_1^{m+1} + \sum_{k=0}^{m} \alpha_k y_1^k = 0,
\]

(1.14)

where \( \alpha_k \) are constants and \( m \) is a positive integer. The automorphism mentioned in Remark 2 is obviously broken. In this case the solution (1.12) is equivalent to a polynomial one. Indeed, the characteristic equation (1.14) can be written in the form

\[
\frac{1}{y_1 - \xi x^{-1}} = \sum_{k=0}^{m} b_k(x) y_1^k,
\]

(1.15)
where \((\alpha_{m+1} := 1)\)

\[ b_{m-k}(x) = b_m(x) \sum_{r=0}^{k} \alpha_{m-r+1} (\xi/x)^{k-r}, \quad b_m(x) = - \left( \sum_{r=0}^{m+1} \alpha_r (\xi/x)^r \right)^{-1}. \]

A substitution of (1.15) in (1.12) for \(n = 1\) gives a polynomial in \(y_1\) solution

\[ y_1(x) = (y_1 - \xi x) \sum_{k=0}^{m} b_k(x) y_1^k = b_m(x) \sum_{k=0}^{m} \left( \frac{b_{k-1}(x)}{b_m(x)} - \xi x \frac{b_k(x)}{b_m(x)} - \alpha_k \right) y_1^k, \quad (b_{-1}(x) := 0), \quad (1.16) \]

of the reflection equation (1.8). The solution (1.16) was obtained in [7].

We stress that the polynomial solution (1.16), which easily follows from the general simple formula (1.12), is valid only in the cyclotomic case and for \(n = 1\).

For \(\xi \neq 0\) the solution (1.16) is called principal in [7]. Note that in (1.16) one can perform a special limit: \(\xi \to 0, \alpha_0 \to 0, \alpha_0/\xi \to \zeta\), where \(\zeta\) is a constant (in this case, due to eq.(1.14), the element \(y_1\) is not invertible). As a result we obtain, in this particular case, following solutions of (1.8)

\[ y_1(x) = 1 + \frac{x - x^{-1}}{\alpha_1 x^{-1} + \zeta} \tilde{y}_1 \left( \tilde{y}_1 := y_1^n + \sum_{k=1}^{m} \alpha_k y_1^{k-1}, \quad y_1 \tilde{y}_1 = 0 \right); \]

these solutions are called small in [7].

Below we explicitly consider few special examples for the polynomial solutions (1.16) in the cases \(m = 1, 2, 3\) and \(\xi \neq 0\).

**Case \(m = 1\).** In this case the characteristic identity (1.14) for the affine element \(y_1\) is of order 2: \(y_1^2 + \alpha_1 y_1 + \alpha_0 = 0\) and the affine Hecke algebra is usually called the \(B\)-type Hecke algebra. We rewrite the characteristic identity in a form (to simplify notation we set \(y := y_1\))

\[ \frac{1}{y - \xi x^{-1}} = - \frac{(y + \alpha_1 + \xi x^{-1})}{(\xi^2 x^{-2} + \alpha_1 \xi x^{-1} + \alpha_0)}. \]

A substitution of this equation in (1.12) gives the simplest polynomial solution [6]:

\[ y(x) = \frac{(x - x^{-1})}{(\xi x^{-2} + \alpha_1 x^{-1} + \alpha_0/\xi)} \left( y + \frac{x \alpha_1 + \xi + \alpha_0/\xi}{x - x^{-1}} \right). \quad (1.17) \]

**Case \(m = 2\).** In this case the identity (1.14) is of order 3 and the formula (1.16) gives a solution of the reflection equation [7]

\[ y(x) = \frac{\xi (x - x^{-1})}{(\xi^2 + \alpha_2 \xi + \alpha_1 \xi + \alpha_0)} \left( y^2 + \left( \frac{\xi}{x} + \alpha_2 \right) y + \frac{\xi^2 x + \alpha_2 \xi + \alpha_1 x + \alpha_0 \xi}{x - x^{-1}} \right) \]
Hecke type 

and references therein) that the quantum matrix (with the skew invertible 

\eta \in \mathbb{F}

\exists \text{ invertible, that is, } \begin{cases} \ \eta \in \mathbb{F}, \\ \exists \text{ invertible, that is, } \eta \in \mathbb{F}. \end{cases}

where 

\begin{align*}
\sum_{r=0}^3 \alpha_r (r^3) y = x^2 + \frac{\lambda x^3 + \alpha_2 x^2 + \alpha_3 x + \alpha_4}{y}, \quad (\alpha_3 := 1).
\end{align*}

Case \( m = 3 \). The characteristic identity is of order 4 and the solution (1.16) takes the form

\begin{align*}
y(x) &= \frac{\xi(x-x^{-1})}{\sum_{r=0}^3 \alpha_r (r^3)} \left( y^3 + \left( \frac{\xi^2 + \alpha_1}{x} + \alpha_2 \xi + \frac{\alpha_3}{x} \right) + \frac{\alpha_4}{y} \right), \quad (\alpha_4 := 1).
\end{align*}

Remark 4. Let \( V \) be an \( N \)-dimensional vector space. In an \( R \)-matrix representation \( \rho_R \): \( \hat{H}_{M+1} \to \text{End}(V \otimes (M+1)) \) of the affine Hecke algebra \( \hat{H}_{M+1} \),

\begin{align*}
\rho_R(\sigma_n) = \hat{R}_n, \quad \rho_R(y_n) = L_n,
\end{align*}

one has \( (n = 1, \ldots, M) \)

\begin{align*}
\hat{R}_n \hat{R}_{n+1} \hat{R}_n &= \hat{R}_{n+1} \hat{R}_n \hat{R}_{n+1}, \quad (1.20) \\
\hat{R}_1 L_1 \hat{R}_1 L_1 &= L_1 \hat{R}_1 L_1 \hat{R}_1, \quad (1.21) \\
L_{n+1} &= \hat{R}_n L_n \hat{R}_n, \quad (1.22)
\end{align*}

where \( \hat{R} \in \text{End}(V \otimes V) \) is the Hecke type \( R \)-matrix, \( \hat{R}^2 = \lambda \hat{R} + I_{\otimes 2} \), and \( I \) is an identity matrix in \( \text{End}(V) \). The operator \( L_1 = \rho_R(y_1) \) is taken in a form \( \hat{L}_1 = L \otimes I_{\otimes M} \) where \( L \) is a \( N \times N \) matrix whose entries are generators of a unital associative algebra \( \mathcal{A} \) which is called reflection equation algebra.

We recall that for the Hecke algebra one can define inductively a set of antisymmetrizers [9]:

\begin{align*}
A_{1 \rightarrow 1} &= 1, \quad A_{1 \rightarrow k+1} = A_{1 \rightarrow k} \tilde{\sigma}_k(a^{2k}; a) A_{1 \rightarrow k}, \quad (k = 1, 2, \ldots), \quad (1.24)
\end{align*}

where \( \tilde{\sigma}_k(x; a) \) is the unitary baxterized element (1.6) and \( a = q \) (for \( a = -q^{-1} \) eqs.(1.24) define the set of symmetrizers). The Hecke type \( R \)-matrix is called the \( R \)-matrix of height \( (m + 1) \) if \( \rho_R(A_{1 \rightarrow m+2}) = 0 \) and the operator \( \rho_R(A_{1 \rightarrow m+1}) \) has rank 1. We also require that the \( R \)-matrix is skew invertible, that is, \( \exists F \in \text{End}(V \otimes V) \) such that \( Tr_3(F_{13}R_{32}) = P_{12} \), where \( P_{12} \in \text{End}(V \otimes V) \) is a permutation matrix. For such \( R \)-matrices (e.g., the \( U_q(gl(m+1)) \) Drinfeld-Jimbo \( R \)-matrix in the vector representation is the Hecke type \( R \)-matrix of height \( (m + 1) \), it was shown in [13] (see also [12] and references therein) that the quantum matrix \( L \) of the reflection equation algebra \( \mathcal{A} \) satisfies the characteristic identity of order \( (m + 1) \) whose coefficients are central in \( \mathcal{A} \). It means that in the \( R \)-matrix representation (1.19) (with the skew invertible \( R \)-matrix of finite height) the affine Hecke algebra
is effectively the cyclotomic one (over the center) and hence the general solutions (1.12) of the reflection equation (1.8) are reduced to the polynomial solutions (1.16).

We note also that a Temperley-Lieb algebra is a quotient of the Hecke algebra by a relation $A_{1-3} = 0$. Hence the $R$-matrix representation of the Temperley-Lieb algebra is associated to the Hecke type $R$-matrix of height 2. In this case (e.g. in the formulation of integrable XXZ spin chain models with nontrivial boundary conditions) it is enough to use the simplest polynomial solution (1.17).

**Remark 5.** Consider the $R$-matrix representation $\rho_R$ of the affine Hecke algebra $\hat{H}_{M+1}$ (1.19), where $\hat{R} \in \text{End}(V^\otimes 2)$ is the standard Drinfeld-Jimbo $R$-matrix for $U_q(\mathfrak{gl}(N))$:

$$\hat{R} = \sum_i q e_{ii} \otimes e_{ii} + \sum_{i \neq j} e_{ij} \otimes e_{ji} + \lambda \sum_{j > i} e_{ii} \otimes e_{jj}, \quad (1.25)$$

$$\hat{R}^2 = \lambda \hat{R} + I^\otimes 2, \quad \lim_{q \to 1} (\hat{R}_{12}) = P_{12}. \quad (1.26)$$

Here $e_{ij}$ are matrix units and $P_{12}$ is the permutation matrix. The $R$-matrix (1.25) is skew invertible Hecke type $R$-matrix of height $N$.

Let $L^+$ (respectively, $L^-$) $\in \text{Mat}(N)$ be upper (respectively, lower) triangular operator-valued invertible matrices which satisfy:

$$\hat{R}_{12} L^+_2 L^+_1 = L^+_2 L^+_1 \hat{R}_{12},$$
$$\hat{R}_{12} L^-_2 L^-_1 = L^-_2 L^-_1 \hat{R}_{12}, \quad (1.26)$$
$$\hat{R}_{12} L^+_2 L^-_1 = L^+_2 L^-_1 \hat{R}_{12}.$$

According to the approach of [14], $L^\pm$ are matrices of Cartan type generators of $U_q(\mathfrak{gl}(N))$. It is known that the formula

$$\rho_R(y_1) = L_1 := (1/L^-) L^+ \otimes I^\otimes M \quad (1.27)$$

defines a solution of the reflection equation (1.21).

For the operator (1.27), the solution (1.12) for $n = 1$ takes the form

$$\rho_R(y_1(x)) = \frac{1}{(L_1^+ - \xi x^{-1} L_1^-)} (L_1^+ - \xi x L_1^-) =: K_1(x). \quad (1.28)$$

The operator $K(x)$ in the formula (1.28) is represented in the factorized form $L^{-1}(\xi/x)L(\xi x)$, where the operator-valued matrix

$$L(x) = (L^+ - x L^-), \quad (1.29)$$

defines the evaluation representation (see, e.g., [11]) of the affine algebra $U_q(\hat{gl}(N))$ and satisfies the intertwining relations (see, e.g., [17])

$$\hat{R}_{12}(x) L_2(xy) L_1(y) = L_2(y) L_1(xy) \hat{R}_{12}(x). \quad (1.30)$$
Using (1.30) one can immediately check that (1.28) solves the reflection equation (1.8) for \( n = 1 \) written in the \( R \)-matrix form

\[
\hat{R}_{12}(x/z) K_1(x) \hat{R}_{12}(x z) K_1(z) = K_1(z) \hat{R}_{12}(x z) K_1(x) \hat{R}_{12}(x/z) ,
\]

where

\[
\hat{R}_{12}(x) := \hat{R}_{12} - x \hat{R}_{12}^{-1} = \rho_R(\sigma_1(x)) ,
\]
is the \( R \)-matrix image of the baxterized element (1.3).

## 2 Solutions of reflection equations for the BMW algebra

In this section, we search for solutions of the reflection equations (1.8) for the Birman-Murakami-Wenzl (BMW) algebra. The BMW algebra \( BMW_{M+1} \) is generated by elements \( \sigma_n \) (1.1) and elements \( \kappa_n \) (\( n = 1, \ldots, M \)) which satisfy following relations [15]:

\[
\kappa_n \sigma_n = \sigma_n \kappa_n = \nu \kappa_n , \tag{2.1}
\]

\[
\sigma_n - \sigma_n^{-1} = \lambda (1 - \kappa_n) , \tag{2.2}
\]

\[
\kappa_n \sigma_n^{\pm 1} \kappa_n = \nu^{\mp 1} \kappa_n , \tag{2.3}
\]

where \( \lambda = q - q^{-1} \) and \( \nu \in \mathbb{C} \setminus \{0, \pm q^{\pm 1}\} \) is an additional parameter of the algebra. The BMW algebra is a finite dimensional quotient of the group algebra of the braid group \( B_{M+1} \).

For the BMW algebra the baxterized elements (which solve the Yang-Baxter equation (1.4)) have the form [16], [17]:

\[
\sigma_n(x) = \frac{1}{\nu + ax^{-1}} \left( a (x^{-1} - 1) \sigma_n + \nu (1 - x) \sigma_n^{-1} + \lambda (a + \nu) \right)
= (\sigma_n - x \sigma_n^{-1}) + \lambda \frac{(\nu + a)}{(\nu + ax^{-1})} \kappa_n = (a x^{-1} - a^{-1}) \frac{\sigma_n + ax}{\sigma_n + ax^{-1}} , \tag{2.4}
\]

where \( x \) is a spectral parameter and \( a \) is a solution of the equation \( \lambda = a - a^{-1} \), i.e. \( a = \pm q^{\pm 1} \). The last expression in (2.4) coincides with the form (1.5) of the baxterized element for the Hecke algebra. However, in the case of the BMW algebra, two different possible values of \( a = \pm q^{\pm 1} \) lead to two different baxterized solutions.

We consider the affine algebra \( BMW_{M+1} \) with generators \( \sigma_n, \kappa_n, (n = 1, \ldots, M), y_n (n = 1, \ldots, M + 1) \), \( c \) and \( Q^{(k)} \) (\( k \) is a positive integer) subject to the relations (1.1), (1.7), (2.1) – (2.3) together with

\[
\kappa_1 y_1 \sigma_1 y_1 \sigma_1 = c \kappa_1 = y_1 \sigma_1 y_1 \sigma_1 \kappa_1 , \tag{2.5}
\]

\[
\kappa_1 y_1^k \kappa_1 = Q^{(k)} \kappa_1 . \tag{2.6}
\]
The generators $Q^{(k)}$ and $c$ are assumed to be central. We additionally suppose that the elements $y_1$ and $c$ are invertible and moreover the element $c$ has a square root in the algebra $BMW_{M+1}$. It is natural to introduce elements $Q^{(k)}$ with $k \leq 0$, which satisfy the relation (2.6) as well; we have $Q^{(0)} = (\nu^{-1} + \lambda - \nu)/\lambda$; the elements $Q^{(k)}$ with $k < 0$ are expressed in terms of $Q^{(k)}$ with $k > 0$ by

$$Q^{(-n)} = \nu^2 c^{-n} Q^{(n)} + \lambda \nu \sum_{j=1}^{n-1} c^{-j} \left( Q^{(2j-n)} - Q^{(j)} Q^{(j-n)} \right).$$  \tag{2.7}$$

A subalgebra generated by the elements $\sigma_n$ and $\kappa_n$ is the usual BMW algebra $BMW_{M+1}$. The abelian subalgebras which are generated by sets $\{c, Q^{(k)}, y_1\}$ and $\{c, Q^{(k)}\}$ can be denoted $BMW_1$ and $BMW_0$, respectively. Note that our definition of the affine BMW algebra is slightly different from the definition of the affine BMW algebra accepted in [18] and [8], where the central elements $Q^{(k)}$ and $c$ are taken to be constants (see, however, the definition of a "degenerate affine Wenzl algebra" in [19]).

The affine Hecke algebra $\hat{H}_{M+1}$ is isomorphic to a subalgebra, generated by the elements $\sigma_n$ and $y_n$, in the quotient of the affine BMW algebra $BMW_{M+1}$ by an ideal generated by $\{\kappa_n\}$. Having this in mind, it is natural to look for a solution of the reflection equation (1.8) (for the BMW algebra case) in the form (1.12). However, contrary to the case of the affine Hecke algebra, the algebra $BMW_{M+1}$ does not possess the automorphism $y_n \to \zeta y_n$ (see Remark 2 of the previous Section) because of the relations (2.5). So we expect that the parameter $\xi$ cannot be a free parameter of the solution (1.12) and should be fixed in the BMW algebra case. Indeed, we have the following result.

**Proposition 2.** For the affine BMW algebra a local unitary solution of the reflection equation (1.8) (with $\sigma_n(x)$ defined in (2.4)) is

$$y_n(x) = f(x) \frac{y_n - \xi x}{y_n - \xi x^{-1}},$$  \tag{2.8}$$

where $\xi$ is a central element in $BMW_{M+1}$ fixed by $\xi^2 = -a c/\nu$ and $f(x)$ is a scalar function such that $f(x)f(x^{-1}) = 1$.

**Proof.** Formula (2.8) can be checked by quite lengthy brute force calculations.

Without loss of generality we will consider the equation (1.8) and its solutions (2.8) for $n = 1$. In this case, it is sufficient to consider an affine subalgebra $BMW_2$ with generators $\sigma_1, \kappa_1, y_1, c$ and $Q^{(k)}$ ($k > 0$). In addition to $\{c, Q^{(k)}\}$, the center of the algebra $BMW_2$ contains elements $(y_1 y_2)$ and $f(y_1, y_2)(c - y_1 y_2)$, where $f(y_1, y_2)$ is any symmetric function of $y_1$ and $y_2 := \sigma_1 y_1 \sigma_1$.

One can use (2.8) to construct solutions for a cyclotomic affine BMW algebra when the generator $y_1$ satisfies a polynomial characteristic identity.
It can be done by the method which was explained in Remark 3 of the previous Section. Solutions are given by the same formula (1.16), where the parameter $\xi$ should be fixed as in Proposition 2. It is worth noting that in the BMW case the parameters $\alpha_k$ in (1.14) are not all independent; they are related by conditions

$$Q^{(m+1-r)} + \sum_{k=0}^{m} \alpha_k Q^{(k-r)} = 0 \quad (r = 0, 1, \ldots, m + 1),$$

(2.9)

which follow from (1.14).

The solution (2.8) is not general. Below we present two exceptional solutions of (1.8) (for $n = 1$) for cyclotomic affine BMW algebras of orders 2 and 4. In general these solutions cannot be produced from (2.8). To simplify formulas, we omit all scalar factors which are needed for the unitarity of $y_1(x)$ and use a concise notation $y := y_1, y(x) := y_1(x)$.

(1) For the characteristic identity of degree 2,

$$y^2 + \alpha_1 y + \alpha_0 = 0,$$

(2.10)

the compatibility conditions (2.9) imply relations between the elements $Q^{(k)}$, $\alpha_0$ and $\alpha_1$. First,

$$\alpha_1 = -\frac{\lambda(c + \nu^2\alpha_0)}{c(\nu^{-1} - \nu + \lambda)} Q^{(1)}.$$  

(2.11)

Then we have two possibilities.

(1a) The element $Q^{(1)}$ is free while the values of the parameters $\alpha_0$ and $\alpha_1$ become fixed:

$$\alpha_0 = -\frac{c}{a\nu}, \quad \alpha_1 = -\frac{Q^{(1)} \nu \lambda}{a\nu + 1}.$$  

(2.12)

Due to the characteristic identity, the elements $Q^{(k)}$ with $k > 1$ can be expressed in terms of $Q^{(1)}$.

We have in this case the same solution as in (1.17):

$$y(x) = y + \frac{\alpha_1 x + A}{x^2 - x^ {-1}},$$  

(2.13)

where $A$ is an arbitrary constant.

Here the value of the parameter $a$ ($a = q$ or $a = -q^{-1}$) should be the same as in the definition of the Baxterized element (2.4) which enters the reflection equation (1.8). The special case of (2.13) with $A = -\frac{c}{\xi \nu}(a + 1/a)$, $(\xi^2 = -a c/\nu)$, can be produced from the rational solution (2.8). Another special example of (2.13) was presented in [8].

(1b) The values of the elements $Q^{(k)}$ are fixed,

$$Q^{(k)} = \frac{\nu^{k/2}}{\nu^k} Q^{(0)}.$$  

(2.14)
In this case the parameter $\alpha_0$ is free, the characteristic polynomial for $y$ factorizes, \((y - c^{1/2}/\nu)(y - \alpha_0\nu/c^{1/2}) = 0\), and the solution (2.8) produces the solution (2.13) with $A = \xi + \alpha_0/\xi$.

(2) For the characteristic identity of degree 4,
\[ y^4 + \alpha_3 y^3 + \alpha_2 y^2 + \alpha_1 y + \alpha_0 = 0 , \tag{2.15} \]
the formula
\[ y(x) = \bar{\alpha}_0 y + \frac{\alpha_3 \bar{\alpha}_0 x + \alpha_1}{(x - x^{-1})} - x \frac{\alpha_0}{y} \tag{2.16} \]
where $\bar{\alpha}_0^2 = \alpha_0$, defines a solution of the reflection equation if
\[
\begin{align*}
\alpha_0 &= -\frac{c^2}{\nu a}, \quad \alpha_1 = -\frac{c}{\nu} \left( \frac{\alpha_3}{\nu} + \lambda Q^{(1)} \right), \\
\alpha_2 &= -\frac{\nu \lambda}{a \nu + 1} \left( Q^{(1)} \alpha_3 + Q^{(2)} \right) + \frac{\lambda c}{a}, \\
\alpha_3 &= -\frac{Q^{(3)} - \frac{\nu \lambda Q^{(1)} Q^{(2)}}{a \nu + 1} - \frac{c}{\nu a} Q^{(1)}}{Q^{(2)} - \frac{\nu \lambda Q^{(1)} Q^{(3)}}{a \nu + 1} - \frac{c}{\nu a} Q^{(0)}},
\end{align*}
\tag{2.17}
\]
(here $Q^{(0)} = (1/\nu + \lambda - \nu)/\lambda$). The choice of the parameters $\alpha_i$ in (2.17) is consistent with the constraints (2.9).

The solution (2.16) has the form of the special small solution of the reflection equation in the Hecke algebra case (see [7]).

**Remark 6.** The $R$-matrix representation of the affine BMW algebra is defined by eqs. (1.19)–(1.23) and the corresponding matrix versions of relations (2.1)–(2.6). As for the Hecke algebra case, one can define a set of antisymmetrizers for the BMW algebra [20], [17], [21]
\[
A_{1\to1} = 1, \quad A_{1\to n+1} = A_{1\to n} \bar{\sigma}_n(a^{2n};a) A_{1\to n} \quad (n = 1, 2, \ldots), \tag{2.18}
\]
where the unitary element $\bar{\sigma}_n(x;a)$ is given by (1.6) with the BMW Baxterized element (2.4) and $a = q$ (for $a = -q^{-1}$ eqs.(2.18) define the set of symmetrizers). If the BMW type $R$-matrix (additional information about the BMW type $R$-matrices can be found in [14], [17], [23]) is skew invertible, $\rho_R(A_{1\to n+1}) = 0$ and $\rho_R(A_{1\to n}) \neq 0$, then the corresponding quantum matrix $L$ of the reflection equation algebra $\mathcal{A}$ (1.21) satisfies the characteristic identity of order $m$ whose coefficients are central in $\mathcal{A}$ [20] (see also [22]). In this case, for the solutions of the reflection equation (1.8), one can use (2.13), (2.16) and the polynomial solutions of the type (1.16), where $\xi^2 = -ac/\nu$ and $y_1$ is the generator of the cyclotomic affine BMW algebra.
3 Relations to integrable spin chain models

Consider the reflection equation (1.8) in the generic case when $\sigma_n(x)$ is a baxterized element of the group algebra of $B_{M+1}$ (or its quotients). In this case, we have the following statement:

**Proposition 3.** Let $y_1(x)$ be a local (i.e., $[y_1(x), \sigma_m] = 0 \forall m > 1$) solution of (1.8) for $n = 1$. Then

$$\bar{y}_n(x) = \sigma_{n-1}(x) \cdots \sigma_1(x) y_1(x) \sigma_1(x) \cdots \sigma_{n-1}(x), \quad (3.1)$$

is a (non-local) solution of the reflection equation (1.8) for $n > 1$.

**Proof.** We prove (3.1) by induction. The element in the right hand side of eq.(3.1) commutes with $\sigma_m$ for $m > n$. Let $\bar{y}_{k-1}(x)$ be a solution of (1.8) for $n = k - 1$ and $[\bar{y}_{k-1}(x), \sigma_m] = 0$ ($m > k - 1$). Then, using (1.4) and (1.8) with $n = k - 1$, we obtain that

$$\bar{y}_k(x) = \sigma_{k-1}(x) \bar{y}_{k-1}(x) \sigma_{k-1}(x), \quad (3.2)$$

takes (1.8) for $n = k$.

Consider a direct product: $\hat{H}_{M+1} = \hat{H}_{M+1} \otimes \hat{H}_0$ of the affine Hecke algebra $\hat{H}_{M+1}$ and an abelian algebra $\hat{H}_0$ generated by commutative elements $Q^{(k)}_D$ ($k \in \mathbb{Z}$). The algebras $\hat{H}_{M+1}$ admit a chain of inclusions

$$\hat{H}_0 \subset \hat{H}_1 \subset \hat{H}_2 \subset \ldots \hat{H}_M \subset \hat{H}_{M+1}$$

defined on the generators as

$$\hat{H}_n \ni (Q^{(k)}_D, y_1, \sigma_i) \longrightarrow (Q^{(k)}_D, y_1, \sigma_i) \in \hat{H}_{n+1} \quad (i = 1, \ldots, n - 1);$$

the algebra $\hat{H}_1$ is generated by elements $(Q^{(k)}_D, y_1)$. We equip the algebra $\hat{H}_{M+1}$ with linear mappings $Tr_{D(n+1)}: \hat{H}_{n+1} \rightarrow \hat{H}_n$ (from the algebra $\hat{H}_{n+1}$ to its subalgebra $\hat{H}_n$) such that $\forall X, Y \in \hat{H}_n$ and $\forall Z \in \hat{H}_{n+1}$ we have

$$Tr_{D(n+1)}(XZY) = XTr_{D(n+1)}(YZ),$$

$$Tr_{D(n+1)}(\sigma^\pm_1 X \sigma^\mp_1) = Tr_{D(n)}(X),$$

$$Tr_{D(n+1)}(1) = D^{(0)}_n, \quad Tr_{D(n+1)}(\sigma^\pm_n) = D^{(\pm)}_n, \quad (3.3)$$

$$Tr_{D(1)}(y^k_1) = Q^{(k)}_D, \quad Tr_{D(n)}Tr_{D(n+1)}(\sigma_n Z) = Tr_{D(n)}Tr_{D(n+1)}(Z\sigma_n),$$

where $D^{(m)}_n \in \mathbb{C} \setminus \{0\}$ are constants subject to the relation $D^{(+)}_n = D^{(-)}_n + \lambda D^{(0)}_n$ which follows from the Hecke condition (1.2). In the $R$-matrix representation of $\hat{H}_{M+1}$ (1.19) – (1.23) the maps $Tr_{D(n+1)}$ (when $D^{(m)}_n$ do not depend on $n$) are nothing but a quantum trace.
Proposition 4. The following identity (cross-unitarity) holds for the baxterized element $\sigma_n(x)$ (1.3):

$$Tr_{D(n+1)}(\sigma_n(x)Y_n \sigma_n(b_n/x)) = \eta(x) \eta(b_n/x) Tr_{D(n)}(Y_n) \quad (\forall Y_n \in \hat{H}_n), \quad (3.4)$$

where $\eta(x) := (1 - x)$, $b_n := D_n^{(+1)}/D_n^{(-1)}$.

Proof. Using the definition (3.3) of the map $Tr_{D(n+1)}$ and the explicit form of the baxterized element (1.3) we obtain for the left-hand side of (3.4)

$$Tr_{D(n+1)} \left( ((1 - x)\sigma_n + \lambda x) \ Y_n \ ((1 - b_n x^{-1})\sigma_n^{-1} + \lambda) \right)$$

$$= (1 - x)(1 - b_n x^{-1}) Tr_{D(n+1)} \left( \sigma_n Y_n \sigma_n^{-1} \right) + \lambda \left( D_n^{(+1)} - b_n D_n^{(-1)} \right) Y_n,$$

which is equivalent to (3.4). \hfill \bullet

Proposition 5. Let an element $\tilde{y}_n(x) \in \hat{H}_n$ be a solution of (1.8) (e.g., the solution defined in (3.1)). Then elements $\tau(x) \in \hat{H}_{n-1}$

$$\tau(x) = Tr_{D(n)}(\tilde{y}_n(x)) \quad (3.5)$$

form a commutative family, $[\tau(x), \tau(z)] = 0 \ (\forall x, z)$.

Proof. Using properties (3.3), the identity (3.4) and the reflection equation (1.8) we deduce

$$\tau(x) \tau(z) = Tr_{D(n)}(\tilde{y}_n(x) \tau(z))$$

$$= \frac{1}{\eta_n(xz)} Tr_{D(n,n+1)}(\tilde{y}_n(x) \sigma_n(xz) \tilde{y}_n(z) \sigma_n(b_n/(xz)))$$

$$= \frac{1}{\eta_n(xz)} Tr_{D(n,n+1)} \left( \sigma_n^{-1}(x/z) \tilde{y}_n(z) \sigma_n(xz) \tilde{y}_n(x) \sigma_n(b_n/(xz)) \sigma_n(x/z) \right)$$

$$= \frac{1}{\eta_n(xz)} Tr_{D(n,n+1)} \left( \tilde{y}_n(z) \sigma_n(xz) \tilde{y}_n(x) \sigma_n(b_n/(xz)) \right) = \tau(z) \tau(x),$$

where $Tr_{D(n,n+1)} := Tr_{D(n)} Tr_{D(n+1)}$ and $\eta_n(x) = \eta(x) \eta(b_n/x)$. This proves the commutativity of the family $\tau(x)$. \hfill \bullet

In the $R$-matrix representation (1.19) – (1.23), the element $\tau(x)$ (with $\tilde{y}_n(x)$ taken in the form (3.1)) is nothing but an example of a Sklyanin monodromy matrix [3] in the case when the algebra $T_+$ from [3] is realized trivially, $T_+ = 1$. Thus, the element $\tau(x)$ can be used for a formulation of integrable chain models with nontrivial boundary conditions on one end of the chain. We consider two possibilities for the solution $y_1(x)$ in (3.1): (1) $y_1^{(1)}(x) = 1$ and (2) $y_1^{(2)}(x) = (y_1 - \xi x)(y_1 - \xi/x)^{-1}$ (see (1.12)). In these cases the corresponding monodromy elements and local Hamiltonians belong to the affine Hecke algebra $\hat{H}_{n-1}$:

$$\tau^{(1)}(x) = Tr_{D(n)} \left( \sigma_{n-1}(x) \cdots \sigma_2(x) \sigma_1^2(x) \sigma_2(x) \cdots \sigma_{n-1}(x) \right) \quad (3.6)$$
\[
\tau^{(2)}(x) = Tr_{D(n)} \left( \sigma_{n-1}(x) \cdots \sigma_1(x) \frac{y_1 - \xi x}{y_1 - \xi} \sigma_1(x) \cdots \sigma_{n-1}(x) \right), \quad (3.7)
\]

\[
H^{(1)} = \sum_{m=1}^{n-2} \sigma_m + \text{const}, \quad H^{(2)} = \sum_{m=1}^{n-2} \frac{\lambda \xi}{y_1 - \xi} + \text{const}. \quad (3.8)
\]

The Hamiltonians \( H^{(1)} \) (resp., \( H^{(2)} \)) are obtained by differentiating \( \tau^{(1)}(x) \) (resp., \( \tau^{(2)}(x) \)) with respect to \( x \) at the point \( x = 1 \). The Hamiltonian \( H^{(1)} \), obtained for the choice \( y_1(x) = 1 \) (which corresponds to free boundary conditions at both ends of the chain), is known. The polynomial analogues of the Hamiltonian \( H^{(2)} \) in the cyclotomic case (which corresponds to a non-trivial boundary condition on one end of the chain) can be deduced in a straightforward manner, as in Remark 3 in Section 1.

To formulate integrable chain systems with nontrivial boundary conditions on both ends of the chain, we need to introduce a "conjugated" reflection equation

\[
\sigma_n(x/z) \tilde{y}_n(z) \sigma_n(b_n/(xz)) \tilde{y}_n(x) = \tilde{y}_n(x) \sigma_n(b_n/(xz)) \tilde{y}_n(z) \sigma_n(x/z). \quad (3.9)
\]

We note that solutions of (3.9) are related to the solutions of the reflection equation (1.8) by means of identities

\[
\tilde{y}_n(x) = y_n \left( \frac{b_n^{1/2}}{x} \right) \quad \text{or} \quad \tilde{y}_n(x) = y_n^{-1} \left( \frac{x}{b_n^{1/2}} \right). \quad (3.10)
\]

Now we consider an \( R \)-matrix representation \( \rho_R \) of the algebra \( \hat{H}_{M+1} \). The homomorphism \( \rho_R \) is described in (1.19) – (1.23) and we also have \( \rho_R(Q_D^{(k)}) = Tr_{D(1)}(L_k) \), where \( Tr_D \) is a standard quantum trace defined for any skew-invertible \( R \)-matrix [17], [24]. More precisely, let \( F \) be the skew inverse of \( \hat{R} \),

\[
Tr_2(F_{12} \hat{R}_{23}) = P_{13} = Tr_2(\hat{R}_{12} F_{23}).
\]

Define the operator \( D \) by

\[
D_1 := Tr_2(F_{12}).
\]

Then

\[
Tr_{D(n)}(E) := Tr_n(D_n \cdot E), \quad \forall E \in \text{End}(V^\otimes(M+1)).
\]

Here \( Tr_n = Tr_{V_n} \) is a usual trace in the copy number \( n \) of the space \( V \) in the product \( V^\otimes(M+1) \). For the Hecke type \( R \)-matrices, this definition of the quantum trace implies that the parameters \( D_n^{(\pm)} \) from (3.3) and \( b_n = D_n^{(\pm)}/D_n^{(-)} \) do not depend on \( n \) and are equal to

\[
D_n^{(\pm)} = 1, \quad D_n^{(-)} = 1 - \lambda Tr(D), \quad b_n = b := (1 - \lambda Tr(D))^{-1}.
\]

In a particular example of the standard \( R \)-matrix of the \( GL_q(n|m) \)-type we have \( b = q^{2(n-m)} \).
The $R$-matrix version of the reflection equation (1.8) is
\[
\hat{R}_n(x/z) K_n(x) \hat{R}_n(xz) K_n(z) = K_n(z) \hat{R}_n(xz) K_n(x) \hat{R}_n(x/z),
\]
where $\hat{R}_n(x) := \rho_R(\sigma_n(x))$ and $K_n(x) := \rho_R(y_n(x))$.

Let $\tilde{K}(x)$ be a $(N \times N)$ matrix whose entries are scalar functions of the spectral parameter $x$:
\[
[K_{j_1 i_1}(x), K_{j_2 i_2}(z)] = 0 \quad (\forall i_1, i_2, j_1, j_2 = 1, \ldots, N).
\]

Assume that the matrix $\tilde{K}(x)$ is a solution of the conjugated reflection equation (3.9) written in the $R$-matrix representation
\[
\hat{R}_n(x/z) \tilde{K}_n(z) \hat{R}_n(b/(xz)) \tilde{K}_n(x) = \tilde{K}_n(x) \hat{R}_n(b/(xz)) \tilde{K}_n(z) \hat{R}_n(x/z),
\]
where $\tilde{K}_n(z)$ is taken in the local form
\[
\tilde{K}_n(z) = I^\otimes(n-1) \otimes \tilde{K}(z) \otimes I^\otimes(M+1-n),
\]
we call $\tilde{K}_n(z)$ local since it acts nontrivially only in the factor $V_n$ in $V^\otimes(M+1)$.

We have (cf. Theorem 1 in [3]):

**Proposition 6.** Let $K_n(x)$ be a solution of (3.11) which acts nontrivially only in the first $n$ factors of $V^\otimes(M+1)$. Assume that the matrix $\tilde{K}_n(x)$ is a local scalar solution (3.12), (3.14) of the conjugated reflection equation (3.13) and all entries of $\tilde{K}(x)$ commute with all entries of $K_n(z)$. Then the elements $t(x)$ defined by
\[
t(x) = Tr_{D(n)} \left( K_n(x) \tilde{K}_n(x) \right)
\]
form a commutative family, $[t(x), t(z)] = 0$ ($\forall x, z$).

**Proof.** Using properties of the quantum trace which follow from the definition of the mapping (3.3) and the $R$-matrix version of the identity (3.4) we obtain
\[
t(x) t(z) = Tr_{D(n)} \left( K_n(x) \left( Tr_{D(n)}(K_n(z) \tilde{K}_n(z)) \right) \tilde{K}_n(x) \right)
\]
\[
= \frac{1}{\eta'(x z)} Tr_{D(n)} Tr_{D(n+1)} \left( K_n(x) \hat{R}_n(xz) K_n(z) \tilde{K}_n(z) \hat{R}_n(b/(xz)) \tilde{K}_n(x) \right),
\]
where $\eta'(x z) = \eta(x z) \eta(b/(xz))$. Now we apply the reflection equations (3.11) and (3.13) and again use the properties (3.3) and the identity (3.4)) to rewrite the last expression in the form
\[
= \frac{1}{\eta'(x z)} Tr_{D(n)} Tr_{D(n+1)} \left( K_n(z) \hat{R}_n(xz) K_n(x) \tilde{K}_n(x) \hat{R}_n(b/(xz)) \tilde{K}_n(z) \right)
\]
\[
= Tr_{D(n)} \left( K_n(z) \left( Tr_{D(n)}(K_n(x) \tilde{K}_n(x)) \right) \tilde{K}_n(z) \right) = t(z) t(x).
\]
The proof is finished.
Let \( K_n(x) \) in (3.15) be the image of the element \( \tilde{y}_n(x) \) (3.1) in the \( R \)-matrix representation, \( K_n(x) = \rho_R(\tilde{y}_n(x)) \). With this choice we obtain the commutative family \( t(x) \in \text{End}(V^{\otimes(n-1)}) \):

\[
t(x) = Tr_{D(n)} \left( \hat{R}_{n-1}(x) \cdots \hat{R}_1(x) K_1(x) \hat{R}_1(x) \cdots \hat{R}_{n-1}(x) \tilde{K}_n(x) \right), \quad (3.16)
\]

where \( K_1(x) \) is any local solution of (3.11) for \( n = 1 \). The element \( t(x) \) (3.16) is an example of the Sklyanin monodromy matrix [3] in the case of a commutative representation \( \tilde{K}(x) \) of the algebra \( \mathcal{T}_n \) [3]. We see that in this case (which is rather general from the point of view of applications) the proof of the commutativity of \( t(x) \) simplifies.

As usual (see [3]) one can consider the element \( t(x) \) (3.16) as a generating function for integrals of motion of an integrable open chain model of length \( n-1 \). If we take in (3.16) the regular solution \( K_1(x) \) (i.e. \( K_1(1) = 1 \)) of (3.11) and take into account the relation \( \hat{R}(1) = \lambda \), we obtain a local Hamiltonian of the chain model with nontrivial boundary conditions

\[
\mathcal{H}^{(0)} = \sum_{m=1}^{n-2} \hat{R}_m - \frac{\lambda}{2} K_1'(1) + \frac{Tr_{D(n)}(\hat{R}_{n-1} \tilde{K}_n(1))}{Tr_{D(n)}(\tilde{K}_n(1))} \cdot (3.17)
\]

This Hamiltonian is obtained by differentiating \( t(x) \) with respect to \( x \) at the point \( x = 1 \). Consider the regular solution \( K_1(x) \) of the type (1.12),

\[
K_1(x) = \rho_R(y_1(x)) = \frac{L_1 - \xi_1 x}{L_1 - \xi_1 x^{-1}}, \quad (\hat{R}_1 L_1 \hat{R}_1 L_1 = L_1 \hat{R}_1 L_1 \hat{R}_1),
\]

where \( \xi_1 \) is a constant. There are two local solutions \( \tilde{K}_n(x) \) of (3.13) which can be used in (3.16) and (3.17)

\[
1) \quad \tilde{K}_n(x) = 1, \quad 2) \quad \tilde{K}_n(x) = \frac{\tilde{L}_n - \xi_2 b^{1/2} x^{-1}}{\tilde{L}_n - \xi_2 b^{-1/2} x}. \quad (3.18)
\]

Here in the case 2) the matrix \( \tilde{K}_n(x) \) is chosen according to the equations (3.10) and (1.12), where \( \tilde{L}_n = \rho_R(y_n) \) is a scalar and local (see (3.12) and (3.14)) solution of the constant reflection equation \( \hat{R}_n \tilde{L}_n \hat{R}_n \tilde{L}_n = \tilde{L}_n \hat{R}_n \tilde{L}_n \hat{R}_n \).

In the first case of (3.18) the monodromy matrix (3.16) and a local Hamiltonian of the corresponding integrable chain model are given by ”\( R \)-matrix images” of the elements (3.6), (3.7), (3.8). In the second case of (3.18) we obtain an integrable chain model (with nontrivial boundary conditions for both ends of the chain) with the Hamiltonian

\[
\mathcal{H}^{(3)} = \sum_{m=1}^{n-2} \hat{R}_m + \frac{\lambda \xi_1}{L_1 - \xi_1} + \frac{1}{\xi} Tr_{D(n)} \left( \hat{R}_{n-1} \frac{\tilde{L}_n - \xi_2 b^{1/2}}{\tilde{L}_n - \xi_2 b^{-1/2}} \right), \quad (3.19)
\]

where \( \xi' := Tr_{D} \left( \frac{\tilde{L} - \xi_2 b^{1/2}}{\tilde{L} - \xi_2 b^{-1/2}} \right) \) is a parameter depending on the choice of the matrix \( \tilde{L} \). The polynomial analogues of the Hamiltonian (3.19) in the cyclotomic case, when the matrices \( L_1 \) and \( \tilde{L}_n \) satisfy the polynomial characteristic
identities of the type (1.14), can be obtained straightforwardly, as in Remark 3 in Section 1.

One can generalize the above construction to the case of the BMW algebra. We do not present here the detailed description of this construction. The analogue of Proposition 4 reads

**Proposition 7.** The following identity (cross-unitarity) holds for the BMW baxterized element $\sigma_n(x)$ (2.4)

$$Tr_{D(n+1)}(\sigma_n(x) Y_n \sigma_n(z)) = \eta(x) \eta(z) Tr_{D(n)}(Y_n) \quad (\forall Y_n \in \overline{BMW}_n), \quad (3.20)$$

where

$$\eta(x) := (1 - x) \frac{(a\nu x + 1)}{(\nu x + a)}, \quad x z = \frac{a^2}{\nu^2},$$

and the map $Tr_{D(n)}: \overline{BMW}_n \to \overline{BMW}_{n-1}$ is defined as follows:

$$\kappa_n Y_n \kappa_n = \frac{1}{\nu} Tr_{D(n)}(Y_n) \kappa_n .$$

The proof of (3.20) consists in a direct but rather lengthy calculation which we omit. The conjugated reflection equation for the BMW algebra is given by (3.9) with $b_n = a^2/\nu^2$. The elements $\sigma_n(x)$ are the baxterized elements (2.4). The commutative BMW monodromy elements are given by the same formulas (3.5) and (3.16), where $\bar{y}_n(x)$ is defined in (3.1) with the baxterized elements $\sigma_n(x)$ (2.4); $\bar{R}_n(x)$ are the $R$-matrix images of (2.4); $\bar{K}(x)$ and $\tilde{K}(x)$ are the corresponding solutions of (3.11) and (3.13). The formulas for the commutative monodromy elements (3.6) and (3.7) (with $\xi^2 = -a c/\nu$) are valid in the BMW case as well.

We present the analogues of the Hamiltonians (3.8), (3.17) and (3.19) in the case of the BMW algebra. The analogue of $H^{(2)}$ (3.8) is

$$H^{(4)} = \sum_{m=1}^{n-2} \left( \sigma_m + \frac{\lambda \nu}{\nu + a} \kappa_m \right) + \frac{\lambda \xi}{y_1 - \xi} + \text{const} , \quad (3.21)$$

where $\sigma_m, \kappa_m, y_1$ are generators of the affine BMW algebra, $\xi^2 = -a c/\nu$ as in Proposition 2 and the parameter $a$ takes two values $\pm q^{\pm 1}$. In the case $y_1 = 1$ we obtain the Hamiltonian (the BMW analogue of $H^{(1)}$)

$$H^{(5)} = \sum_{m=1}^{n-2} \left( \sigma_m + \frac{\lambda \nu}{\nu + a} \kappa_m \right) + \text{const} \quad (3.22)$$

for the integrable chain models with trivial boundary conditions. The same Hamiltonian (3.22) defines an integrable system on a periodic chain if we identify $\sigma_k = \sigma_{n-2+k}$ [17].
The analogue of (3.17) is

\[ H^{(6)} = \sum_{m=1}^{n-2} \left( \hat{R}_m + \frac{\lambda \nu}{\nu + a} \hat{K}_m \right) - \frac{\lambda}{2} K'_1(1) + \frac{\lambda}{T_{D(n)}} \left( (\tilde{R}_{n-1} + \frac{\lambda \nu}{\nu + a} \tilde{K}_{n-1}) \tilde{K}(1) \right), \]

where \( \hat{K}_m = \rho_R(\kappa_m) \) and we have two different integrable systems for \( a = \pm q^{\pm 1} \). One can use the solutions (2.8), (2.13) and (2.16) in (3.23). In particular, one can substitute:

\[ K_1(x) = \frac{L_1 - \xi x}{L_1 - \xi x^{-1}}, \quad \tilde{K}_n(x) = \frac{\tilde{L}_n - \xi b^{1/2} x^{-1}}{\tilde{L}_n - \xi b^{-1/2} x}, \]

where \( \xi^2 = -a c/\nu, \ b = \pm a/\nu \). As a result we deduce the BMW analogue of (3.19)

\[ H_7 = \sum_{m=1}^{n-2} \left( \hat{R}_m + \frac{\lambda \nu}{\nu + a} \hat{K}_m \right) + \frac{\lambda \xi}{L_1 - \xi} + \frac{1}{\xi''} T_{D(n)} \left( (\tilde{R}_{n-1} + \frac{\lambda \nu}{\nu + a} \tilde{K}_{n-1}) \tilde{K}(1) \right), \]

where \( \xi'' = T_{D(n)}(\tilde{K}(1)) \) and \( \tilde{K}(1) = (\tilde{L}_n - \xi b^{1/2})(\tilde{L}_n - \xi b^{-1/2})^{-1} \).

**Remark 7.** We expect that any representation of the affine Hecke or BMW algebras will give integrable chain models with the Hamiltonians (3.8) or (3.21) and (3.22), respectively. In particular, one can use in (3.21), (3.22) the \( SO_q(N) \), \( Sp_q(2m) \) and \( OSP_q(N|2m) \) \( R \)-matrix representations of the algebra \( BMW_{M+1} \) (see, e.g., [17]) or \( BMW_M \) to formulate integrable spin chain models of \( SO, Sp \) or \( OSP \) types. In these cases, the parameter \( \nu \) is fixed as follows: \( \nu = q^{1-N} \) for \( SO(N) \)-type, \( \nu = -q^{-1-2m} \) for \( Sp(2m) \)-type [14], [20] and \( \nu = q^{1+2m-N} \) for \( OSP(N|2m) \)-type [17] (there is also a special case \( \nu = -q^{1+2m-2n} \) which corresponds to \( OSP_q(2m|2n) \) [17]). The Yangian limits of the corresponding \( R \)-matrices lead to the consideration of \( SO, Sp \) [25] or \( OSP \) [5] invariant spin chain models. These Yangian models are generalizations of the XXX Heisenberg models of magnets.

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