Normal extensions escape from the class of weighted shifts on directed trees

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Dedicated to Professor Franciszek H. Szafraniec on the occasion of his 70th birthday

Abstract. A formally normal weighted shift on a directed tree is shown to be a bounded normal operator. The question of whether a normal extension of a subnormal weighted shift on a directed tree can be modeled as a weighted shift on some, possible different, directed tree is answered.

1. Introduction

The notion of a weighted shift on a directed tree has been introduced and studied extensively in [8]. As shown therein, the class of weighted shifts on directed trees is wide enough to contain operators with some subtle properties including hyponormal operators whose squares are not hyponormal and non-hyponormal paranormal operators. It is well-known that there are no (classical) unilateral or bilateral weighted shifts with the aforesaid properties. In two recent papers [9, 10] new examples of unbounded operators with pathological properties have been constructed, each of them being implemented as a weighted shift on a special directed tree. The former contains an example of a hyponormal operator whose square has trivial domain, the latter an example of a non-hyponormal operator the $C^\infty$-vectors of which generate Stieltjes moment sequences.

In [8, 2] the question of subnormality of bounded and unbounded weighted shifts on directed trees has been studied. Criteria for subnormality of such operators written in terms of consistent systems of probability measures have been established. When analyzing subnormality, a question arises as to whether a normal extension of a nonzero subnormal weighted shift on a directed tree $T$ with nonzero weights can be modeled (up to unitary equivalence) as a weighted shift on some, possibly different, directed tree. As shown in Section 4 in most instances this is not the case. The only exceptional cases are those in which the directed tree

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\( \mathcal{F} \) is isomorphic to either \( \mathbb{Z} \) or \( \mathbb{Z}_+ \) (cf. Theorem 4.2). In the latter case the normal extension can be modeled as a weighted shift on a directed tree which comes from \( \mathbb{Z} \) by gluing a leaf to the directed tree \( \mathbb{Z} \) at the vertex 0 (cf. Remark 4.3).

2. Preliminaries

In what follows, \( \mathbb{Z} \) stands for the set of all integers and \( \mathbb{C} \) for the set of all complex numbers. We also use the following notation

\[
Z_+ = \{n \in \mathbb{Z}: n \geq 0\} \quad \text{and} \quad \mathbb{N} = \{n \in \mathbb{Z}: n \geq 1\}.
\]

Let \( A \) be an operator in a complex Hilbert space \( \mathcal{H} \) (all operators considered in this paper are linear). Denote by \( \mathcal{D}(A) \), \( \mathcal{R}(A) \) and \( A^* \) the domain, the range and the adjoint of \( A \) (in case it exists). A closed densely defined operator \( N \) in \( \mathcal{H} \) is said to be \textit{normal} if \( N^*N = NN^* \) (equivalently: \( \mathcal{D}(N) = \mathcal{D}(N^*) \) and \( \|N^*h\| = \|Nh\| \) for all \( h \in \mathcal{D}(N) \)). For this and other facts concerning unbounded operators we refer the reader to \[7, 11, 12, 13\]. A densely defined operator \( S \) in \( \mathcal{H} \) is said to be \textit{subnormal} if there exists a complex Hilbert space \( \mathcal{K} \) and a normal operator \( N \) in \( \mathcal{K} \) such that \( \mathcal{H} \subseteq \mathcal{K} \) (isometric embedding) and \( Sh = Nh \) for all \( h \in \mathcal{D}(S) \); such an \( N \) is called a \textit{normal extension} of \( S \). We refer the reader to \[10\] for the theory of bounded subnormal operators, \[16, 17, 18, 19\] for the foundations of the theory of unbounded subnormal operators and \[2, 11, 12, 13\] for research related to special classes of subnormal operators. From now on, \( B(\mathcal{H}) \) stands for the \( C^* \)-algebra of all bounded operators \( A \) in \( \mathcal{H} \) such that \( \mathcal{D}(A) = \mathcal{H} \). We write \( \text{lin} \mathcal{F} \) for the linear span of a subset \( \mathcal{F} \) of \( \mathcal{H} \).

Let \( \mathcal{F} = (V, E) \) be a directed tree (\( V \) stands for the set of all vertices of \( \mathcal{F} \) and \( E \) for the set of all edges of \( \mathcal{F} \)). If \( \mathcal{F} \) has a root, which will always be denoted by \textit{root}, then we write \( V^0 := V \setminus \{\text{root}\} \); otherwise, we put \( V^0 = V \). Set \( \text{Chi}(u) = \{v \in V: (u, v) \in E\} \) for \( u \in V \). If for a given vertex \( u \in V \) there exists a unique vertex \( v \in V \) such that \( (v, u) \in E \), then we denote it by \( \text{par}(u) \). The correspondence \( u \mapsto \text{par}(u) \) is a partial function from \( V \) to \( V \). For \( n \in \mathbb{N} \), the \( n \)-fold composition of the partial function \( \text{par} \) with itself will be denoted by \( \text{par}^n \). Let \( \text{par}^0 \) stand for the identity map on \( V \). We say that \( \mathcal{F} \) is \textit{leafless} if \( V = \{u \in V: \text{Chi}(u) \neq \emptyset\} \). A vertex \( u \in V \) is called a \textit{branching vertex} of \( \mathcal{F} \) if \( \text{Chi}(u) \) consists of at least two vertices. For a subset \( W \) of \( V \), we define \( \text{Chi}(W) = \bigcup_{v \in W} \text{Chi}(v) \) and \( \text{Des}(W) = \bigcup_{n=0}^{\infty} \text{Chi}^n(W) \), where

\[
\text{Chi}^0(W) = W, \quad \text{Chi}^{n+1}(W) = \text{Chi}(\text{Chi}^n(W)), \quad n \in \mathbb{Z}_+.
\]

For \( u \in V \), we put \( \text{Chi}^n(u) = \text{Chi}^n(\{u\}) \) and \( \text{Des}(u) = \text{Des}(\{u\}) \). The functions \( \text{Chi}^n(\cdot) \) and \( \text{Des}(\cdot) \) have the following properties (see e.g., \[2\] Proposition 2.2.1):

\[
\text{Chi}^n(u) = \{w \in V: \text{par}^n(w) = u\}, \quad n \in \mathbb{Z}_+, u \in V, \quad (2.1)
\]

\[
\text{Des}(u) = \bigcup_{n=0}^{\infty} \text{Chi}^n(u), \quad u \in V, \quad (2.2)
\]

\[
\text{Des}(u_1) \cap \text{Des}(u_2) = \emptyset, \quad u_1, u_2 \in \text{Chi}(u), \quad u_1 \neq u_2, \quad u \in V, \quad (2.3)
\]

where the symbol \( \bigcup \) is reserved to denote pairwise disjoint union of sets.

Let \( \ell^2(V) \) be the Hilbert space of all square summable complex functions on \( V \) equipped with the standard inner product. For \( u \in V \), we define \( e_u \in \ell^2(V) \) to
be the characteristic function of the one point set \{u\}. The family \{\varepsilon_u\}_{u \in V} is an orthonormal basis of \ell^2(V); we call it the canonical orthonormal basis of \ell^2(V). Set

\[ \varepsilon_V = \text{LIN}\{\varepsilon_u: u \in V\}. \]

Given \( \Lambda = \{\lambda_v\}_{v \in V^*} \subseteq \mathbb{C} \), we define the operator \( S_\Lambda \) on \( \ell^2(V) \) by

\[ D(S_\Lambda) = \{ f \in \ell^2(V): A_\mathcal{F} f \in \ell^2(V) \} , \]

\[ S_\Lambda f = A_\mathcal{F} f, \quad f \in D(S_\Lambda), \]

where \( A_\mathcal{F} \) is the map defined on functions \( f: V \to \mathbb{C} \) via

\[ (A_\mathcal{F}f)(v) = \begin{cases} \lambda_v \cdot f(\text{par}(v)) & \text{if } v \in V^o, \\ 0 & \text{if } v = \text{root}. \end{cases} \]

\( S_\Lambda \) is called a weighted shift on the directed tree \( \mathcal{F} \) with weights \( \{\lambda_v\}_{v \in V^*} \). Note that any weighted shift \( S_\Lambda \) on \( \mathcal{F} \) is a closed operator (cf. [8 Proposition 3.1.2]). Combining Propositions 3.1.3, 3.1.8, 3.4.1 and 3.1.7 of [8], we get the following properties of \( S_\Lambda \) (from now on, we adopt the convention that \( \sum_{v \in \mathcal{G}} x_v = 0 \)).

**Proposition 2.1.** Let \( S_\Lambda \) be a weighted shift on a directed tree \( \mathcal{F} \) with weights \( \Lambda = \{\lambda_v\}_{v \in V^*} \). Then the following assertions hold:

(i) \( e_u \) is in \( D(S_\Lambda) \) if and only if \( \sum_{v \in \text{Chi}(u)} |\lambda_v|^2 < \infty \); if \( e_u \in D(S_\Lambda) \), then

\[ S_\Lambda e_u = \sum_{v \in \text{Chi}(u)} \lambda_v e_v \quad \text{and} \quad \|S_\Lambda e_u\|^2 = \sum_{v \in \text{Chi}(u)} |\lambda_v|^2, \quad (2.4) \]

(ii) \( S_\Lambda \) is densely defined if and only if \( \varepsilon_V \subseteq D(S_\Lambda) \),

(iii) \( S_\Lambda \in B(\ell^2(V)) \) if and only if \( \alpha_\Lambda := \sup_{u \in V} \sum_{v \in \text{Chi}(u)} |\lambda_v|^2 < \infty \); moreover, if \( S_\Lambda \in B(\ell^2(V)) \), then \( \|S_\Lambda\|^2 = \alpha_\Lambda \),

(iv) if \( S_\Lambda \) is densely defined, then \( \varepsilon_V \subseteq D(S^*_\Lambda) \) and

\[ S^*_\Lambda e_u = \begin{cases} \overline{\lambda_u} e_{\text{par}(u)} & \text{if } u \in V^o, \\ 0 & \text{if } u = \text{root}, \end{cases} \quad u \in V, \quad (2.5) \]

(v) \( S_\Lambda \) is injective if and only if \( \mathcal{F} \) is leafless and \( \sum_{v \in \text{Chi}(u)} |\lambda_v|^2 > 0 \) for every \( u \in V \).

3. Formal normality - a general structure

Recall that a densely defined operator \( N \) in a complex Hilbert space \( \mathcal{H} \) is said to be formally normal if \( D(N) \subseteq D(N^*) \) and \( \|N^* h\| = \|N h\| \) for all \( h \in D(N) \) (cf. [4] [5]). In this section we show that formally normal weighted shifts on directed trees are always bounded and normal.

**Proposition 3.1.** If \( S_\Lambda \) is a nonzero weighted shift on a directed tree \( \mathcal{F} \) with weights \( \Lambda = \{\lambda_v\}_{v \in V^*} \), then the following three conditions are equivalent:

(i) \( S_\Lambda \) is formally normal,

(ii) there exists a sequence \( \{u_n\}_{n=-\infty}^\infty \subseteq V \) such that

\[ u_{n-1} = \text{par}(u_n) \quad \text{and} \quad |\lambda_{u_{n-1}}| = |\lambda_{u_n}| \quad \text{for all } n \in \mathbb{Z}, \]

and \( \lambda_v = 0 \) for all \( v \in V \setminus \{u_n: n \in \mathbb{Z}\} \),

(iii) \( S_\Lambda \in B(\ell^2(V)) \) and \( S_\Lambda \) is normal.
Since, by (3.3), the vector space $X$ view of the polarization formula, we have
$$
\langle S, S \rangle_{\mathcal{H}} = \langle S, S \rangle_{\mathcal{H}} e_u = \|S e_u\|^2 e_u \quad \text{for all } u \in V. 
$$
(3.1)

Assertions (ii) and (iv) of Proposition 2.1 imply that
$$
\text{applying (2.4) and (2.5) separately to } S \text{ yields } \langle S, S \rangle_{\mathcal{H}} e_u = S_S S \lambda e_u \quad \text{for all } u \in V^0. 
$$
(3.2)

In view of (3.1) and (3.2), we have
$$
\delta V \subseteq \mathcal{D}(S^* S \lambda) \cap \mathcal{D}(S_S S \lambda). 
$$
(3.3)

The formal normality of $S \lambda$ yields $\langle S^* S \lambda f, f \rangle = \langle S \lambda S^* S \lambda f, f \rangle$ for all $f \in X$. Hence, in view of the polarization formula, we have $\langle S^* S \lambda f, g \rangle = \langle S \lambda S^* S \lambda g, f \rangle$ for all $f, g \in X$. Since, by (3.3), the vector space $X$ is dense in $\ell^2(V)$, we obtain $S^* S \lambda e_u = S \lambda S^* S \lambda e_u$ for all $u \in V$. This, combined with (3.1) and (3.2), shows that
$$
\|S \lambda e_u\|^2 e_u = |\lambda_u|^2 e_u + \sum_{v \in \chi(\text{par}(u)) \setminus \{u\}} \lambda_v \lambda_u e_v, \quad u \in V^0. 
$$
(3.4)

Note that if $u \in V^0$ is such that $\|S \lambda e_u\| > 0$, then by (3.3) we have $\|S \lambda e_u\| = |\lambda_u|$, $\lambda_u \neq 0$ and $\lambda_v = 0$ for all $v \in \chi(\text{par}(u)) \setminus \{u\}$, which, in view of (2.1), implies that
$$
\|S \lambda e_{\text{par}(u)}\| = \|S \lambda e_u\| > 0. 
$$

Using an induction argument, we show that the following implication holds for all $u \in V$ and $m \in \mathbb{Z}_+$ such that $\text{par}^m(u) \in V$:
$$
\text{if } \|S \lambda e_u\| > 0 \Rightarrow \|S \lambda e_{\text{par}^k(u)}\| = \|S \lambda e_u\| \quad \text{for all } k = 0, \ldots, m. 
$$
(3.5)

Now we prove that if $u_1, u_2 \in V$ are such that $\|S \lambda e_{u_1}\| > 0$ and $\|S \lambda e_{u_2}\| > 0$, then $\|S \lambda e_{u_1}\| = \|S \lambda e_{u_2}\|$. Indeed, by [8] Proposition 2.1.4, there exists $u \in V$ such that $u_1, u_2 \in \text{Des}(u)$. It follows from (2.1) and (2.2) that there are $n_1, n_2 \in \mathbb{Z}_+$ such that $\text{par}^m(u_1) = u = \text{par}^m(u_2)$. This fact combined with (3.5) leads to $\|S \lambda e_{u_1}\| = \|S \lambda e_{u_2}\| = \|S \lambda e_{u}\|$. This implies that $\sup_{u \in V} \|S \lambda e_u\| < \infty$, which together with assertions (i) and (iii) of Proposition 2.1 yields $S \lambda \in B(\ell^2(V))$. Hence $S \lambda$ is a bounded normal operator.

(iii)⇒(i) Evident.

(ii)⇒(iii) It follows from (ii) that for any $u \in V$ the set $\{v \in \chi(u) : \lambda_v \neq 0\}$ has at most one element. Consequently, again by (ii), $\sup_{u \in V} \sum_{v \in \chi(u)} |\lambda_v|^2 = |\lambda_u|^2 < \infty$, which, combined with Proposition 2.1 (iii), implies that $S \lambda \in B(\ell^2(V))$. Now applying (2.4) and (2.5) separately to $u \in \{u_n : n \in \mathbb{Z}\}$ and $u \in V \setminus \{u_n : n \in \mathbb{Z}\}$, we verify that $S^* S \lambda e_u = S \lambda S^* S \lambda e_u$ for all $u \in V$, which yields the normality of $S \lambda$.

(iii)⇒(ii) Apply [8] Lemma 8.1.5 (this is the only case in which we use the assumption that $S \lambda$ is a nonzero operator).

Combining Proposition 3.1 with [8] Proposition 8.1.6, we see that the only directed tree admitting formally normal weighted shifts with nonzero weights is isomorphic to $\langle \mathbb{Z}, \{(n, n + 1) : n \in \mathbb{Z}\}\rangle$. 

\[\Box\]
4. Modeling normal extensions on weighted shifts

In this final section we will discuss the following question: under what circumstances can a normal extension of a subnormal weighted shift on a directed tree $\mathcal{F}$ be modeled as a weighted shift on a directed tree $\mathcal{F}$ (no relationship between $\mathcal{F}$ and $\mathcal{F}$ is required). Here and in what follows, by the unilateral shift we mean the weighted shift on the directed tree $(\mathbb{N}, \{(n, n+1) : n \in \mathbb{N}\})$ (respectively: the bilateral shift on $\ell^2(\mathbb{N})$) we mean the weighted shift on the directed tree $(\mathbb{Z}, \{(n, n+1) : n \in \mathbb{Z}\})$ (respectively: $(\mathbb{Z}, \{(n, n+1) : n \in \mathbb{Z}\})$) with all weights equal to 1. These two particular directed trees are denoted simply by $\Lambda$ and $\Gamma$, respectively.

Lemma 4.1. Let $S_\lambda$ be a nonzero subnormal weighted shift on a directed tree $\mathcal{F}$. Suppose $S_\lambda$ has a normal extension $N$ which is a weighted shift on a directed tree $\hat{\mathcal{F}} = (\hat{\mathcal{V}}, \hat{\mathcal{E}})$ (we do not assume that $\mathcal{F}$ is a directed subtree of $\hat{\mathcal{F}}$). Then $N \in \mathcal{B}^0(\ell^2(\hat{\mathcal{V}}))$ and $N = \alpha U \oplus 0$, where $\alpha$ is a positive real number, $U$ is a unitary operator which is unitarily equivalent to the bilateral shift on $\ell^2(\mathbb{Z})$ and 0 is the zero operator on $\ell^2(\hat{\mathcal{V}}) \oplus \mathcal{D}(U)$. Moreover, $\mathcal{R}(S_\lambda) \subseteq \mathcal{D}(U)$.

Proof. Denote by $\{\lambda_v\}_{v \in \hat{\mathcal{V}}}$ the weights of $N$. It follows from Proposition 3.1 that $N$ is a bounded operator on $\ell^2(\hat{\mathcal{V}})$ and that there exist a positive real number $\alpha$ and a sequence $\{u_n\}_{n=-\infty}^{\infty} \subseteq \hat{\mathcal{V}}$ such that

\begin{align*}
u_{n-1} &= \text{par}_\mathcal{F}(u_n) \quad \text{for all } n \in \mathbb{Z}, \quad (4.1) \\
\hat{\lambda}_v &= 0 \quad \text{for all } v \in \hat{\mathcal{V}} \setminus \{u_n : n \in \mathbb{Z}\}, \quad (4.2) \\
|\hat{\lambda}_{u_n}| &= \alpha \quad \text{for all } n \in \mathbb{Z}, \quad (4.3)
\end{align*}

where $\text{par}_\mathcal{F}(\cdot)$ refers to the directed tree $\mathcal{F}$. Set $X = \{u_n : n \in \mathbb{Z}\}$ and $Y = \hat{\mathcal{V}} \setminus X$. We deduce from $(2.4)$, $(4.1)$ and $(4.2)$ that the spaces $\ell^2(X)$ and $\ell^2(Y)$ are invariant for $N$ (and thus $N = N|_{\ell^2(X)} \oplus N|_{\ell^2(Y)}$, $N|_{\ell^2(Y)} = 0$ and $N\hat{e}_{u_n} = \hat{\lambda}_{u_{n+1}} \hat{e}_{u_{n+1}}$ for all $n \in \mathbb{Z}$, where $\hat{e}_v$ is the canonical orthonormal basis of $\ell^2(\hat{\mathcal{V}})$. Applying $(4.3)$ and $[15]$ Corollary 1, p. 52 we get the required decomposition $N = \alpha U \oplus 0$. Hence, we have $\mathcal{R}(S_\lambda) \subseteq \mathcal{R}(N) = \mathcal{R}(U) = \mathcal{D}(U)$, which completes the proof. □

Regarding Lemma 4.1 we note that bounded subnormal operators with normal extensions of the form $U \oplus 0$, where $U$ is a unitary operator, have been characterized in $[6]$.

Now we show that the only nonzero subnormal weighted shifts on directed trees with nonzero weights whose normal extensions can be modeled as weighted shifts on directed trees are those that are unitarily equivalent to either positive scalar multiples of the bilateral shift on $\ell^2(\mathbb{Z})$ or positive scalar multiples of “small” perturbations of the unilateral shift on $\ell^2(\mathbb{Z}_+)$. 

Theorem 4.2. Let $S_\lambda$ be a nonzero subnormal weighted shift on a directed tree $\mathcal{F}$ with nonzero weights $\lambda = \{\lambda_v\}_{v \in \hat{\mathcal{V}}}$, Suppose $S_\lambda$ has a normal extension $N$ which is a weighted shift on a directed tree $\hat{\mathcal{F}} = (\hat{\mathcal{V}}, \hat{\mathcal{E}})$ (we do not assume that $\mathcal{F}$ is a directed subtree of $\hat{\mathcal{F}}$). Then the directed tree $\mathcal{F}$ is isomorphic to either $\mathbb{Z}$ or $\mathbb{Z}_+$. In the former case, $S_\lambda$ is unitarily equivalent to a positive scalar multiple of the bilateral shift on $\ell^2(\mathbb{Z})$. In the latter case, $S_\lambda$ is unitarily equivalent to a positive scalar multiple of a unilateral weighted shift on $\ell^2(\mathbb{Z}_+)$ with weights $\{\vartheta, 1, 1, 1, \ldots\}$, where $\vartheta \in (0, 1]$. 

PROOF. In view of Lemma 4.1 the operator $N$ (and consequently $S_{\lambda}$) is bounded and $N = \alpha U \oplus 0$, where $\alpha$ and $U$ are as in Lemma 4.1. Without loss of generality, we can assume that $\alpha = 1$. It follows from Lemma 4.1 that there exists a unitary isomorphism $W: \mathcal{D}(U) \to L^2(\mathbb{T})$ such that $WU = MW$, where $L^2(\mathbb{T})$ is the Hilbert space of all square summable Borel functions on $\mathbb{T} := \{ z \in \mathbb{C} : |z| = 1 \}$ with respect to the normalized Lebesgue measure $m$ on $\mathbb{T}$ that is given by

$$m(\sigma) = \frac{1}{2\pi} \int_{0}^{2\pi} \chi_{\sigma}(e^{it}) \, dt, \quad \sigma \text{ - Borel subset of } \mathbb{T},$$

($\chi_{\sigma}$ stands for the characteristic function of $\sigma$), and $M \in B(L^2(\mathbb{T}))$ is defined by $(MF)(z) = zf(z)$ a.e. $[m]$ for $f \in L^2(\mathbb{T})$.

By [8] Proposition 5.1.1, the directed tree $\mathcal{T}$ is leafless. To prove that the directed tree $\mathcal{T}$ is isomorphic to either $\mathbb{Z}$ or $\mathbb{Z}_+$, it is enough to show that $\mathcal{T}$ has no branching vertex. Suppose that, contrary to our claim, $\mathcal{T}$ has a branching vertex $u \in V$. Then, by (2.3), there exist $u_1, u_2 \in V$ such that

$$\text{Des}(u_1) \cap \text{Des}(u_2) = \emptyset. \quad (4.4)$$

It follows from [8] Lemma 6.1.1 that $S_{\lambda}^k e_{u_j} \in \ell^2(\text{Chi}(n)(u_j))$ for all $n \in \mathbb{Z}_+$ and $j = 1, 2$. This fact, combined with (2.2) and (4.4), implies that

the vectors $\{ S_{\lambda}^k e_{u_1} : k \in \mathbb{Z}_+ \} \cup \{ S_{\lambda}^l e_{u_2} : l \in \mathbb{Z}_+ \}$ are pairwise orthogonal. \quad (4.5)

Since $\mathcal{R}(S_{\lambda}) \subseteq \mathcal{D}(U)$, we see that $S_{\lambda}e_{u_j} \in \mathcal{D}(U)$ for $j = 1, 2$. Set $f_j = W S_{\lambda} e_{u_j} \in L^2(\mathbb{T})$ for $j = 1, 2$. In view of Proposition 2.1(v), the operator $S_{\lambda}$ is injective. Hence $\| f_j \| > 0$ for $j = 1, 2$. It follows from the equality $N = U \oplus 0$ that

$$M^n f_j = W U^n (S_{\lambda} e_{u_j}) = W S_{\lambda}^{n+1} e_{u_j}, \quad n \in \mathbb{Z}_+, \ j = 1, 2. \quad (4.6)$$

Combining conditions (4.5) and (4.6), we deduce that the vectors $\{ M^k f_1 : k \in \mathbb{Z}_+ \} \cup \{ M^l f_2 : l \in \mathbb{Z}_+ \}$ are pairwise orthogonal. Thus the following equalities hold for all $k \in \mathbb{N}$, $l \in \mathbb{Z}_+$ and $j = 1, 2$:

$$\int_{\mathbb{T}} z^k |f_j(z)|^2 \, d m(z) = \langle M^k f_j, f_j \rangle = 0 = \langle f_j, M^k f_j \rangle = \int_{\mathbb{T}} z^k |f_j(z)|^2 \, d m(z),$$

$$\int_{\mathbb{T}} z^l |f_1(z) f_2(z)| \, d m(z) = \langle M^l f_1, f_2 \rangle = 0 = \langle f_1, M^l f_2 \rangle = \int_{\mathbb{T}} z^l |f_1(z) f_2(z)| \, d m(z).$$

This yields

$$\int_{\mathbb{T}} z^k |f_j(z)|^2 \, d m(z) = 0, \quad k \in \mathbb{Z} \setminus \{0\}, \ j = 1, 2, \quad (4.7)$$

$$\int_{\mathbb{T}} z^l |f_1(z) f_2(z)| \, d m(z) = 0, \quad l \in \mathbb{Z}. \quad (4.8)$$

It follows from (4.7) that for every complex trigonometric polynomial $p$ on $\mathbb{T}$,

$$\int_{\mathbb{T}} |p| f_j|^2 \, d m = \| f_j \|^2 \int_{\mathbb{T}} p \, d m, \quad j = 1, 2.$$

Since complex trigonometric polynomials are uniformly dense in the Banach space of continuous functions on $\mathbb{T}$, we infer from the Riesz representation theorem (cf. [14] Theorem 6.19) that $\int_{\sigma} |f_j|^2 \, d m = \| f_j \|^2 m(\sigma)$ for all Borel subsets $\sigma$ of $\mathbb{T}$. This implies that

$$|f_j| = \| f_j \| \text{ a.e. } [m]. \quad (4.9)$$
A similar argument applied to \((4.8)\) yields
\[
f_1 \overline{f_2} = 0 \text{ a.e. } [m].
\] (4.10)
Combining \((4.9)\) and \((4.10)\) with the fact that \(\|f_j\| > 0\) for \(j = 1, 2\), we conclude that \(m(T) = 0\), which contradicts the fact that \(m(T) = 1\). This shows that the directed tree \(T\) is isomorphic to either \(Z\) or \(Z^+\).

First we consider the case in which \(T\) is isomorphic to \(Z\). Without loss of generality, we can assume that \(T\) coincides with the directed tree \(Z\). Since the weights of \(S_\lambda\) are nonzero, we get \(\delta_Z \subseteq R(S_\lambda) \subseteq D(U)\), which yields \(\ell^2(Z) \subseteq D(U)\). Hence \(U\) is a unitary extension of \(S_\lambda\). This implies that \(S_\lambda\) is an isometric bilateral weighted shift on \(\ell^2(Z)\). As a consequence of \([15\) Corollary 1, p. 52], the operator \(S_\lambda\) is unitarily equivalent to the bilateral shift on \(\ell^2(Z)\).

Consider now the case in which \(T\) is isomorphic to \(Z^+\). Again without loss of generality, we can assume that \(T\) coincides with the directed tree \(Z^+\). Since the weights of \(S_\lambda\) are nonzero, we obtain \(\text{LIN}\{\epsilon_n: n \in \mathbb{N}\} \subseteq R(S_\lambda) \subseteq \ell^2(\mathbb{N})\), and so \(\ell^2(\mathbb{N}) = R(S_\lambda) \subseteq D(U)\). As \(U\) is a unitary operator and \(U \oplus 0\) extends \(S_\lambda\), we deduce that \(S_\lambda|_{R(S_\lambda)}\) is an isometry. Hence, we have
\[
\|S_\lambda^\dagger \epsilon_1\|_2^2 = \frac{1}{|\lambda_1|^2} \|S_\lambda \epsilon_0\|_2^2 = \frac{1}{|\lambda_1|^2} \|S_\lambda \epsilon_0\|_2^2 = 1, \quad n \in \mathbb{Z}^+.
\]
This implies that \(\{\|S_\lambda^\dagger \epsilon_1\|_2^2\}_{n=0}^\infty\) is a Stieltjes moment sequence with a representing measure \(\delta_1\) (\(\delta_1\) is the Borel probability measure on \([0, \infty)\) concentrated at the point 1). Since \(S_\lambda\) is subnormal, we deduce that \(\{\|S_\lambda^\dagger \epsilon_0\|_2^2\}_{n=0}^\infty\) is a Stieltjes moment sequence (see e.g., \([8\) Theorem 6.1.3]). Hence, by applying \([8\) Lemma 6.1.10] to \(u = 0\), we get \(|\lambda_1| \leq 1\). The fact that \(S_\lambda|_{R(S_\lambda)}\) is an isometry yields \(|\lambda_n| = 1\) for all \(n \geq 2\). Using \([15\) Corollary 1, p. 52], we conclude that \(S_\lambda\) is unitarily equivalent to a unilateral weighted shift on \(\ell^2(\mathbb{Z}^+)\) with weights \(\{\vartheta, 1, 1, 1, \ldots\}\), where \(\vartheta \in (0, 1]\). This completes the proof. \(\square\)

**Remark 4.3.** We show how to model a normal extension of a nonzero subnormal weighted shift \(S_\lambda\) on a directed tree \(T\) with nonzero weights by means of a weighted shift on some directed tree. As in the proof of Theorem \([12\] we consider only the case of \(\alpha = 1\). By this theorem we have only two possibilities: either the directed tree \(T\) is isomorphic to \(Z\) and \(S_\lambda\) is normal (and so \(S_\lambda\) is the required model), or the directed tree \(T\) is isomorphic to \(Z^+\) and \(S_\lambda\) is a unilateral weighted shift on \(\ell^2(Z^+)\) with weights \(\{\vartheta, 1, 1, 1, \ldots\}\), where \(\vartheta \in (0, 1]\). In the latter case, we fix \(\omega \notin Z\) and define the directed tree \(\hat{T} = (\hat{V}, \hat{E})\) by
\[
\hat{V} = \{\omega\} \cup Z \quad \text{and} \quad \hat{E} = \{(0, \omega)\} \cup \{(n, n+1): n \in Z\}.
\]
Then \(\hat{T}\) is rootless and 0 is a unique branching vertex of \(\hat{T}\). Let \(N\) be the weighted shift on \(\hat{T}\) with weights \(\{\lambda_v\}_{v \in \hat{V}}\) given by \(\lambda_v = 0\) for \(v = \omega\) and \(\lambda_v = 1\) for \(v \in Z\).

By Proposition \([5\) Proposition 5.1] \(N\) is a bounded normal operator on \(\ell^2(\hat{V})\). Define the sequence \(\{\hat{e}_n\}_{n=0}^\infty\) in \(\ell^2(\hat{V})\) by
\[
\hat{e}_n = \begin{cases} 
\sqrt{(1-\vartheta^2)} \hat{e}_\omega + \vartheta \hat{e}_0 & \text{for } n = 0, \\
\hat{e}_n & \text{for } n \in \mathbb{N}.
\end{cases}
\]
Denote by \(\mathcal{H}\) the closure of \(\text{LIN}\{\hat{e}_n\}_{n=0}^\infty\) in \(\ell^2(\hat{V})\). Then \(\{\hat{e}_n\}_{n=0}^\infty\) is an orthonormal basis of \(\mathcal{H}\). It is clear that \(N\hat{e}_0 = \vartheta \hat{e}_1\) and \(N\hat{e}_n = \hat{e}_{n+1}\) for all \(n \in \mathbb{N}\). As a
consequence, \( N(\mathcal{H}) \subseteq \mathcal{H} \) and the operator \( N|_{\mathcal{H}} \) is unitarily equivalent to a unilateral weighted shift on \( \ell^2(\mathbb{Z}_+) \) with weights \( \{\vartheta, 1, 1, 1, \ldots\} \). Hence, \( N \) is the required model of a normal extension of \( S_{\lambda} \). It is worth noting that the directed tree \( \mathcal{T} \) is isomorphic to many directed subtrees of \( \hat{\mathcal{T}} \). However, if \( \vartheta \in (0, 1) \), then \( \mathcal{T} \) could not be regarded as a directed subtree of \( \hat{\mathcal{T}} \). Indeed, otherwise \( V \subseteq \mathbb{Z} \subseteq \hat{V} \) and so \( S_{\lambda} \) is isometric as the restriction to \( \ell^2(V) \) of the unitary operator \( N|_{\ell^2(\mathbb{Z})} \). This contradicts the fact that for \( \vartheta \in (0, 1) \), \( S_{\lambda} \) is not an isometry.

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