Eigenvalues and forbidden subgraphs I

Vladimir Nikiforov
Department of Mathematical Sciences, University of Memphis,
Memphis TN 38152, USA

March 30, 2022

Abstract

Suppose a graph $G$ have $n$ vertices, $m$ edges, and $t$ triangles. Letting $\lambda_n(G)$ be the largest eigenvalue of the Laplacian of $G$ and $\mu_n(G)$ be the smallest eigenvalue of its adjacency matrix, we prove that

$$\lambda_n(G) \geq \frac{2m^2 - 3mt}{m(n^2 - 2m)},$$
$$\mu_n(G) \leq \frac{3n^3t - 4m^3}{nm(n^2 - 2m)},$$

with equality if and only if $G$ is a regular complete multipartite graph.

Moreover, if $G$ is $K_{r+1}$-free, then

$$\lambda_n(G) \geq \frac{2mn}{(r-1)(n^2 - 2m)}$$

with equality if and only if $G$ is a regular complete $r$-partite graph.

Keywords: $K_r$-free graph, graph Laplacian, largest eigenvalue, smallest eigenvalue, forbidden subgraphs

1 Introduction

Our notation is standard (e.g., see [2] and [4]); in particular, $G(n)$ stands for a graph of order $n$, and $G(n, m)$ stands for a graph of order $n$ and size $m$. We write $t(G)$ for the number of triangles of a graph $G$, $A(G)$ for its adjacency matrix, and $D(G)$ for the diagonal matrix of its degree sequence. The Laplacian of $G$ is defined as $L(G) = D(G) - A(G)$. Given a graph $G = G(n)$, the eigenvalues of $A(G)$ are $\mu_1(G) \geq \ldots \geq \mu_n(G)$ and the eigenvalues of $L(G)$ are $0 = \lambda_1(G) \leq \ldots \leq \lambda_n(G)$.

In this note we study how $\lambda_n(G)$ and $\mu_n(G)$ depend on the number of certain subgraphs of $G$. In [8] we showed that if $r \geq 2$ and $G$ is a $K_{r+1}$-free graph with $n$ vertices and $m$ edges, then

$$\mu_n(G) < -\frac{2}{r} \left( \frac{2m}{n^2} \right)^r n.$$  \hfill (1)
Here we prove a similar inequality for $\lambda_n(G)$.

**Theorem 1** If $r \geq 2$ and $G = G(n, m)$ is a $K_{r+1}$-free graph, then

$$\lambda_n(G) \geq \frac{2mn}{(r-1)(n^2-2m)}$$
with equality if and only if $G$ is a regular complete $r$-partite graph.

We deduce Theorem 1 from more general results.

**Theorem 2** If $G = G(n, m)$, then

$$6nt(G) \geq (n + \lambda_n(G)) \sum_{u \in V(G)} d^2(u) - 2nm\lambda_n(G)$$
with equality if and only if $G$ is a complete multipartite graph, and

$$\lambda_n(G) \geq \frac{2m^2 - 3nt(G)}{m(n^2-2m)}n$$
with equality if and only if $G$ is a regular complete multipartite graph.

Inequality (4) suggests a similar inequality for $\mu_n(G)$.

**Theorem 3** If $G = G(n, m)$, then

$$\mu_n(G) \leq \frac{3n^3t(G) - 4m^3}{nm(n^2-2m)}$$
with equality if and only if $G$ is a regular complete multipartite graph.

From Theorem 3 we effortlessly deduce results complementary to results of Serre, Li, and Cioabă (e.g., see [6] and its references). Note that these authors study regular graphs of fixed degree and large order; in contrast, our results are meaningful for any graph $G = G(n)$ with average degree $\gg n^{1/2}$. Here is an immediate consequence of Theorem 3.

**Corollary 4** If $0 \leq \varepsilon \leq 1$ and $G = G(n, m)$ is a graph with $t(G) \leq \varepsilon (m/n)^3$, then

$$\mu_n(G) \leq -(1-\varepsilon) \frac{4m^2}{n^3}.$$

In other words, graphs with small $|\mu_n(G)|$ abound in triangles. Likewise, graphs with small $|\mu_n(G)|$ have cycles of all lengths up to $O(m^2/n^3)$.

**Corollary 5** If $3 \leq r \leq n/2$ and a graph $G = G(n, m)$ contains no cycle of length $r$, then

$$\mu_n(G) \leq -\frac{4m^2}{n^3} + 2(r-3).$$
2 Proofs

For any vertex \( u \), write \( \Gamma (u) \) for the set of its neighbors, \( t(u) \) for the number of triangles containing it, and \( t'(u) \) for \( e(V(G) \setminus \Gamma (u)) \).

Proof of Theorem 2 It is known (e.g., see [3]) that for any partition \( V(G) = V_1 \cup V_2 \),

\[
\lambda_n(G) \geq \frac{e(V_1, V_2) n}{|V_1||V_2|}.
\] (6)

Therefore, for every \( u \in V(G) \) and partition \( V_1 = \Gamma (u), V_2 = V(G) \setminus \Gamma (u) \),

\[
\lambda_n(G) d(u)(n - d(u)) \geq ne(\Gamma (u), V(G) \setminus \Gamma (u)).
\] (7)

In view of \[
\sum_{v \in \Gamma(u)} d(v) = e(\Gamma (u), V(G) \setminus \Gamma (u)) + 2t(u),
\]

summing (7) for all \( u \in V(G) \), we find that

\[
\lambda_n(G) \sum_{u \in V(G)} d(u)(n - d(u)) \geq n \sum_{u \in V(G)} \sum_{v \in \Gamma(u)} d(v) - 2n \sum_{u \in V(G)} 2t(u)
\]

\[
= n \sum_{u \in V(G)} d^2(u) - 6nt(G),
\]

and (3) follows. Now (4) follows from (3), in view of \( \sum_{u \in V(G)} d^2(u) \geq 4m^2/n \).

If equality holds in (3), then equality holds in (7) for every \( u \in V(G) \). Note that if equality holds in (3), then all vertices from \( V_2 \) are connected to the same number of vertices from \( V_1 \) (for a detailed proof of this result see, e.g., [4].) In our selection of \( V_1 \) and \( V_2 \) the vertex \( u \) is joined to all vertices from \( \Gamma (u) \), hence all vertices from \( V(G) \setminus \Gamma (u) \) are joined to all vertices from \( \Gamma (u) \). Consequently, \( G \) contains no induced subgraph of order 3 with exactly one edge; hence, \( G \) is complete multipartite.

If equality holds in (4), then \( G \) is a complete multipartite graph; as \( \sum_{u \in V(G)} d^2(u) = 4m^2/n \), \( G \) is regular. \( \square \)

Proof of Theorem 1 Since \( G \) is \( K_{r+1} \)-free, \( \Gamma (u) \) induces a \( K_r \)-free graph for every \( u \in V(G) \). Thus, Turán’s theorem implies that

\[
t(u) \leq \frac{r - 2}{2(r - 1)} d^2(u).
\]

Summing this inequality for all \( u \in V(G) \), we obtain

\[
6t(G) \leq \frac{r - 2}{r - 1} \sum_{u \in V(G)} d^2(u).
\]
This, in view of (3), implies that

\[
\frac{n - 2}{r - 1} \sum_{u \in V(G)} d^2(u) \geq (n + \lambda_n(G)) \sum_{u \in V(G)} d^2(u) - 2nm\lambda_n(G).
\]

Using \(\sum_{u \in V(G)} d^2(u) \geq 4m^2/n\), the result follows after simple algebra.

If equality holds in (2), then equality holds in (3), implying that \(G\) is a complete multipartite graph. The condition for equality in Turán’s theorem implies that the neighborhood of every vertex is a complete \((r - 1)\)-partite graph, thus, \(G\) is \(r\)-partite. Finally, we have \(\sum_{u \in V(G)} d^2(u) = 4m^2/n\), so \(G\) is regular, completing the proof. \(\square\)

### 2.1 Proof of Theorem 3

To prove Theorem 3, we need two propositions and a lemma.

**Proposition 6** For every graph \(G = G(n, m)\),

\[
2 \sum_{u \in V(G)} d(u) (t'(u) - t(u)) = 4m^2 - 4 \sum_{uv \in E(G)} d(u) d(v)
\]

**Proof** For every \(u \in V(G)\), we have

\[
2t(u) = \sum_{v \in \Gamma(u)} d(v) - e(V_1, V_2)
\]

\[
2t'(u) = \sum_{v \in V(G) \setminus \Gamma(u)} d(v) - e(V_1, V_2) = 2m - e(V_1, V_2) - \sum_{v \in \Gamma(u)} d(v).
\]

Hence,

\[
2(t'(u) - t(u)) d(u) = 2md(u) - 2d(u) \sum_{v \in \Gamma(u)} d(v);
\]

summing this equality for all \(u \in V(G)\), we obtain the required equality. \(\square\)

**Proposition 7** For every graph \(G = G(n, m)\)

\[
2 \sum_{uv \in E(G)} d(u) d(v) \geq 4m^2 + \sum_{u \in V(G)} d^3(u) - nd^2(u)
\]
Proof We have
\[
2 \sum_{uv \in E(G)} d(u) d(v) = \sum_{uv \in E(G)} d^2(u) + d^2(v) - (d(u) - d(v))^2 \\
\geq \sum_{u \in V(G)} d^3(u) - \frac{1}{2} \sum_{u \in V(G)} \sum_{v \in V(G)} (d(u) - d(v))^2 \\
= \sum_{u \in V(G)} d^3(u) - \frac{1}{2} \sum_{u \in V(G)} \sum_{v \in V(G)} d^2(u) + d^2(v) - 2d(u) d(v) \\
= \sum_{u \in V(G)} d^3(u) - n \sum_{u \in V(G)} d^2(u) + 4m^2,
\]
completing the proof.

Lemma 8 Let \(0 \leq x_1 \leq \ldots \leq x_n \leq 1\) be reals with \(x_1 + \ldots + x_n = ns\). Then,
\[
\sum_{i=1}^{n} 2x_i^3 - (2 + s)x_i^2 \geq ns^3 - 2ns^2. \tag{8}
\]

Proof Setting \(\varphi(x) = 2x^3 - (2 + s)x^2\), we routinely find that:
(i) \(\varphi(x)\) decreases for \(0 \leq x \leq (s + 2)/3\) and increases for \((s + 2)/3 \leq x \leq 1\);
(ii) \(\varphi(x)\) is concave for \(0 \leq x \leq (s + 2)/6\);
(iii) \(\varphi(x)\) is convex for \((s + 2)/6 \leq x \leq 1\).

Let \(F(x_1, \ldots, x_n) = \sum_{i=1}^{n} \varphi(x_i)\) and suppose \(x_1 \leq \ldots \leq x_n\) are such that \(F(x_1, \ldots, x_n)\) is minimal, subject to the conditions of the lemma; clearly, we may assume that \(x_1 > 0\).

If \(x_1 \geq (s + 2)/6\), \(\text{(iii)}\) implies that \(x_1 = \ldots = x_n\) and the proof is completed. Assume \(x_1 < (s + 2)/6\); we shall show that this assumption leads to a contradiction. Note first that \(x_2 \geq (s + 2)/6\); otherwise, for sufficiently small \(\varepsilon > 0\), \(\text{(ii)}\) implies that
\[
F(x_1 - \varepsilon, x_2 + \varepsilon, x_3, \ldots, x_n) < F(x_1, x_2, x_3, \ldots, x_n),
\]
contradicting the choice of \(x_1, \ldots, x_n\). Using \(\text{(iii)}\) again, we find that \(x_2 = \ldots = x_n\). Now, setting \(z = x_2\), we see that \(0 < x_1 < s\) and \(s < z < ns/(n - 1)\), and that the function
\[
f(z) = F(x_1, z, \ldots, z) \\
= 2(ns - (n - 1)z)^3 + 2(n - 1)z^3 - (2 + s)((n - 1)z - ns)^2 - (2 + s)(n - 1)z^2
\]
has a local minimum in the interval
\[
s < z < ns/(n - 1). \tag{9}
\]
We have

\[
f'(z) = -6(n - 1)((n - 1)z - ns)^2 + 6(n - 1)z^2 - 2(n - 1)(2 + s)((n - 1)z - ns) - 2(2 + s)(n - 1)z
\]
\[
= -6(n - 1)\left(n((n - 2)z - ns)(z - s) + \frac{(2 + s)}{3}n(z - s)\right)
\]
\[
= -6n(n - 1)(z - s)\left((n - 2)z - (n - 2)s - 2s + \frac{(2 + s)}{3}\right)
\]
\[
= -6n(n - 1)(n - 2)(z - s)\left(z - \left(s + \frac{5s - 2}{3(n - 2)}\right)\right).
\]

In view of (9), the local minimum of \(f(z)\) must be attained at

\[z_0 = s + \frac{5s - 2}{3(n - 2)},\]

implying, in particular, that \(z_0 > s\). But since \(f'(z) > 0\) for \(s < z < z_0\) and \(f'(z) < 0\) for \(z > z_0\), we see that \(f(z)\) has a local maximum at \(z_0\). This contradiction completes the proof. \(\square\)

**Proof of Theorem** In [3] it is proved that for any partition \(V(G) = V_1 \cup V_2\),

\[
\mu_n(G) \leq \frac{2e(V_1)}{|V_1|} + \frac{2e(V_2)}{|V_2|} - \frac{2m}{n}.
\] (10)

Hence, for every \(u \in V(G)\) and partition \(V_1 = \Gamma(u), V_2 = V(G) \setminus \Gamma(u)\),

\[
\mu_n(G) \leq \frac{2e(V_1)}{d(u)} + \frac{2e(V_2)}{n - d(u)} - \frac{2m}{n} = \frac{2t(u)}{n} + \frac{2t'(u)}{d(u) - n} - \frac{2m}{n},
\] (11)

and therefore,

\[
\mu_n(G)(n - d(u))d(u) \leq 2t(u)(n - d(u)) + 2t'(u)d(u) - \frac{2m}{n}d(u)(n - d(u))
\]
\[
= 2nt(u) + d(u)(2t'(u) - 2t(u)) - 2md(u) + \frac{2m}{n}d^2(u).
\]

Summing this inequality for all \(u \in V(G)\), in view of \(\mu_n(G) \leq 0\) and

\[
\sum_{u \in V(G)}(n - d(u))d(u) \leq \frac{2m}{n}(n^2 - 2m),
\] (12)

we obtain

\[
\mu_n(G)\frac{2m}{n}(n^2 - 2m) = 6nt(G) + \sum_{u \in V(G)}d(u)(2t'(u) - 2t(u)) - 4m^2 + \frac{2m}{n}\sum_{u \in V(G)}d^2(u).
\]
Propositions 6 and 7 imply that
\[ \mu_n(G) \frac{2m}{n} (n^2 - 2m) \leq 6nt(G) - 4 \sum_{uv \in E(G)} d(u) d(v) + \frac{2m}{n} \sum_{u \in V(G)} d^2(u) \]
\[ \leq 6nt(G) - 8m^2 - 2 \sum_{u \in V(G)} d^3(u) + \left( \frac{2m}{n} + 2n \right) \sum_{u \in V(G)} d^2(u). \]
Assume for convenience that \( V(G) = \{1, \ldots, n\} \) and \( d(1) \leq \ldots \leq d(n) \). Setting \( x_i = d(i)/n, 1 \leq i \leq n \), and \( s = 2m/n^2 \), Lemma 8 implies that
\[ -2 \sum_{u \in V(G)} d^3(u) + \left( \frac{2m}{n} + 2n \right) \sum_{u \in V(G)} d^2(u) \leq 8m^2 - \frac{8m^3}{n^2}; \]
therefore,
\[ \mu_n(G) \leq \frac{6nt(G) - 8m^3/n^2}{(2m/n) (n^2 - 2m)}, \]
and (5) follows.

If equality holds in (5), then equality holds in (12); thus, \( G \) is regular. Also, equality holds in (11) for every \( u \in V(G) \). Some algebra shows that for regular graphs inequality (10) is equivalent to (6); hence, as in the proof of Theorem 2, \( G \) is a complete multipartite graph. The proof is completed. \( \square \)

Proof of Corollary 5 Let \( G \) have \( m \) edges and \( t \) triangles. If \( G \) has no \( C_r \) for some \( r \leq n/2 \), then \( e(G) \leq n^2/4 \) (1, p. 150). Since the neighborhood of any vertex \( u \) has no path of order \( r - 1 \), by a theorem of Erdős and Gallai [7], the neighborhood of \( u \) induces at most \( (r - 3) d(u)/2 \) edges, i.e., \( 2t(u) \leq (r - 3) d(u) \). Summing over all vertices, we see that \( 3t \leq (r - 3) m \). Hence, Theorem 3 implies that
\[ \mu_n(G) \leq \frac{3n^3t - 4m^3}{nm(n^2 - 2m)} \leq \frac{n^3(r - 3) m - 4m^3}{nm(n^2 - 2m)} \leq - \frac{4m^2}{n^3} + 2(r - 3), \]
completing the proof. \( \square \)

Acknowledgment Part of this research was completed while the author was visiting the Institute for Mathematical Sciences, National University of Singapore in 2006. The author is also indebted to Béla Bollobás for his kind support.

References

[1] B. Bollobás, Extremal Graph Theory, Academic Press, 1978.
[2] B. Bollobás, Modern Graph Theory, Graduate Texts in Mathematics, 184, Springer-Verlag, New York (1998), xiv+394 pp.

[3] B. Bollobás, V. Nikiforov, Graphs and Hermitian matrices: eigenvalue interlacing, Discrete Math. 289 (2004), 119-127.

[4] B. Bollobás, V. Nikiforov, Graphs and Hermitian matrices: exact interlacing, submitted.

[5] D. Cvetković, M. Doob, H. Sachs, Spectra of Graphs, VEB Deutscher Verlag der Wissenschaften, Berlin, 1980, 368 pp.

[6] S. M. Cioabă, On the extreme eigenvalues of regular graphs. J. Combin. Theory Ser. B 96 (2006), 367–373.

[7] P. Erdős, T. Gallai On maximal paths and circuits of graphs. Acta Math. Acad. Sci. Hungar 10 (1959) 337–356.

[8] V. Nikiforov, The smallest eigenvalue of $K_r$-free graphs, Discrete Math. 306 (2006), 612-616.