Action of Virasoro operators on Hall-Littlewood polynomials

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Abstract

In this paper, we prove formulas for the action of Virasoro operators on Hall-Littlewood polynomials at roots of unity.

1 Introduction

Hall-Littlewood functions played important roles in representation theory of symmetric groups and finite general linear groups (c.f. survey article [Mor76]). This class of symmetric functions includes ordinary Schur functions and Schur Q-functions as specializations. It is well known that these two special classes of functions give polynomial tau-functions for KP and BKP hierarchies (c.f. [DJM] and [Y]). Schur Q-functions were also used to study Kontsevich-Witten and Brezin-Gross-Witten tau functions which are generating functions of certain intersection numbers on moduli spaces of stable curves (c.f. [MM20], [A20], [A21], [LY1], [LY2]). Recently Mironov and Morozov proposed to use Hall-Littlewood functions specialized at roots of unity to study generalized Kontsevich matrix models (c.f. [MM21]). An interesting problem is to investigate whether Hall-Littlewood functions can also be used to study Gromov-Witten invariants since they are natural generalizations of intersection numbers on moduli spaces of stable curves. Virasoro constraints are powerful tools in the study of matrix models and Gromov-Witten invariants (c.f. [EHX] and [CK]). In fact Kontsevich-Witten and Brezin-Gross-Witten tau functions are determined by the Virasoro constraints up to a scalar. This was the starting point for the proof of Q-polynomial expansion formulas for these two tau functions in the approach given in [LY1] and [LY2]. To adapt this approach to more general models in Gromov-Witten theory and matrix models, it is important to know how Virasoro operators acting on Hall-Littlewood functions. The main purpose of this paper is to give formulas for the action of Virasoro operators on Hall-Littlewood functions specialized at roots of unity.

In this paper, we will consider Hall-Littlewood functions $Q_\lambda(t; \rho)$ as polynomials of variables $t = (t_1, t_2, ...)$, where $t_i$ is the $r_i$-th power sum function in the theory of symmetric functions (c.f. MacDonald’s book [Mac]). These polynomials are indexed by partitions $\lambda$ and also depend on a parameter $\rho \in \mathbb{C}$ (Here $\rho$ corresponds to the parameter $t$ in [Mac]).

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They are Schur polynomials when $\rho = 0$ and Schur Q-polynomials when $\rho = -1$. If $\rho$ is
the $n$-th root of unity $\xi_n$, we will denote the corresponding Hall-Littlewood polynomials
by $Q^{(n)}_\lambda$, i.e.

$$Q^{(n)}_\lambda(t) = Q_\lambda(t; \xi_n).$$

When computing derivatives of $Q_\lambda$, it is convenient to extend the definition of $Q_\lambda$ to the
case where $\lambda \in \mathbb{Z}^l$. This was already considered in the original article \[L\]. It is also
rather natural from the view point of Jing’s vertex operator approach to Hall-Littlewood
polynomials in \[J\].

For any $n \geq 2$ and $m \in \mathbb{Z}$, define

$$L_m^{(n)} := \sum_{k \geq 1, n/k} k t_k \frac{\partial}{\partial t_{knm}} + \frac{1}{2} \sum_{k \geq 1, n/k} n^{-1} \frac{\partial^2}{\partial t_k \partial t_{mn-k}} - \frac{1}{2} \sum_{k \geq 1, n/k} n^{-1} k(mn + k)t_{kn} + \delta_m,0 \frac{n^2 - 1}{24}. \quad (1)$$

Here we set $t_k = 0$ and $\frac{\partial}{\partial t_k} = 0$ if $k \leq 0$. Hence $L_m^{(n)}$ does not contain quadratic terms
t$^{kn-mn-k}$ if $m \geq 0$, and it does not contain second order derivative terms if $m \leq 0$. These
operators form a Virasoro algebra since they satisfy the following bracket relation

$$[L_i^{(n)}, L_j^{(n)}] = n(i - j)L_{i+j}^{(n)} + \delta_{i+j,0} \frac{n^2(n - 1)(i^3 - i)}{12} \quad (2)$$

for all $i, j \in \mathbb{Z}$. Note that operators $\frac{1}{n}L_m^{(n)}, m \in \mathbb{Z}$, satisfy the standard Virasoro bracket
relation.

The first main result of this paper is the following:

**Theorem 1.1** For any $n \geq 2$, $m \geq 0$, and $\lambda \in \mathbb{Z}^l$, we have

$$L_m^{(n)}Q^{(n)}_\lambda = \sum_{i=1}^l \lambda_i Q^{(n)}_{\lambda - mn \epsilon_i} + \sum_{k=1}^{mn-1} \sum_{i,j=1}^l (1 - \xi_n^{-k})Q^{(n)}_{\lambda - k \epsilon_i - (mn-k) \epsilon_j} + \delta_{m,0} \frac{n^2 - 1}{24} Q^{(n)}_\lambda, \quad (3)$$

where

$$\lambda - a \epsilon_i := (\lambda_1, \cdots, \lambda_i - a, \cdots, \lambda_l) \quad (4)$$

for $\lambda = (\lambda_1, \cdots, \lambda_l)$ and $a \in \mathbb{Z}$.

Since Virasoro constraints in Gromov-Witten theory and in matrix models usually
start with the $L_{-1}$-constraint which corresponds to the so called string equation, it is also
important to know the formula for $L_m^{(n)}Q^{(n)}_\lambda$ with $m < 0$. Let $\mathcal{P}$ be the set of partitions,
i.e. $(\lambda_1, \cdots, \lambda_l) \in \mathcal{P}$ if $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l > 0$. The second main result of this paper is the following:
Theorem 1.2 For any \( n \geq 2, \ m \geq 1, \) and \( \lambda \in \mathbb{Z}^l \), we have
\[
L_{-m}^{(n)} Q_{(n)}^{(n)} = \sum_{i=1}^{l} \left( \lambda_i + \frac{m(n-1)}{2} \right) Q_{\lambda+mn\epsilon_i}^{(n)} + \sum_{k=1}^{l} \sum_{\substack{i,j=1 \ \text{or} \ \text{even}, \ \text{odd} \ \text{or} \ \text{even}, \ \text{odd}}} \sum_{n|k} \xi_n^k Q_{\lambda+k\epsilon_i+(mn-k)\epsilon_j}^{(n)} + \sum_{i=1}^{l} \sum_{\mu \in \mathcal{P}} \sum_{|\mu|=k} c_{\mu}(\xi_n) Q_{(\lambda+(mn-k)\epsilon_i,\mu)}^{(n)}
\]
\[
+ \frac{1}{2} \sum_{k=1}^{mn-1} \left( \sum_{\mu \in \mathcal{P}} \sum_{|\mu|=k} c_{\mu}(\xi_n) Q_{(\lambda,\mu+(mn-k)\epsilon_j)}^{(n)} + \sum_{|\mu|=k} c_{\mu}(\xi_n) c_{\nu}(\xi_n) Q_{(\lambda,\mu,\nu)}^{(n)} \right),
\]
where \( |\mu| := \sum_i \mu_i \) if \( \mu = (\mu_1, \mu_2, \cdots) \), and coefficients \( c_{\mu}(\xi_n) \) will be given in Proposition 2.4 (In fact, \( c_{\mu}(\xi_n) \) are related to special cases of Green’s polynomials).

Note that there are many ways to construct Virasoro operators. Our choice of Virasoro operators \( L_{m}^{(n)} \) agrees with those used in [MM21], where only \( m > 0 \) cases were considered. These are essential parts of the operators used in the Virasoro constraints for generalized Kontsevich matrix models. Other types of Virasoro operators might be obtained from \( L_{m}^{(n)} \) by adding finitely many pure derivative terms of order 1 or 2 for \( m > 0 \). The action of such extra terms on Hall-Littlewood polynomials can be easily computed from the well-known derivative formula (see equation (17) below). For \( m < 0 \), one may need to add finitely many quadratic terms like \( t_it_m \) to obtain new Virasoro operators. The action of such quadratic terms on Hall-Littlewood polynomials \( Q_{\lambda}(t;\rho) \) can be easily computed from the following formula
\[
p_{r}Q_{\lambda} = \sum_{i=1}^{l(\lambda)} Q_{\lambda+r\epsilon_i} + \sum_{\substack{\mu \in \mathcal{P} \ \text{such that} \ |\mu|=r}} c_{\mu}(\rho) Q_{(\lambda,\mu)}
\]
for all \( \lambda \in \mathbb{Z}^l \), where \( p_r = rt_r \) corresponds to the \( r \)-th power sum function in the theory of symmetric functions, and an explicit formula for coefficients \( c_{\mu}(\rho) \) will be given in Proposition 2.4. We will give a proof of this formula in section 2 since we could not find it in the literature. For \( \rho = 0 \), i.e. for the Schur polynomials, this formula is well known and is related to Murnaghan-Nakayama rule (c.f. [Mac]). For \( \rho = -1 \), i.e. for Schur Q polynomials, this formula should also be well known (see, for example, [B] and [LY1]). In Section 5 of [B], it was attempted but without success to find such a formula for general \( Q_{\lambda}(t;\rho) \).

Remark 1.3 We notice that some formulas for \( L_{m}^{(n)} Q_{(n)}^{(n)} \) with \( m \geq 1 \) were given in [MM21] without proof under the assumption that \( \lambda \) and \( \lambda-km\epsilon_i-(n-k)m\epsilon_j \) for all \( 1 \leq k \leq n-1 \) and \( 1 \leq i, j \leq l \) are strict partitions, i.e. components of these vectors are strictly decreasing and non-negative. Equation (50) in [MM21] coincide with the corresponding formula in Theorem 1.1 for \( m = 1 \) with the understanding that the order of
components of \( \lambda \) should be reversed in [MM21]. For \( m \geq 2 \), it appears that some terms are missing in equations (51) and (52) in [MM21].

For \( n = 2 \), \( Q^{(n)}_{\lambda} \) is the Schur Q polynomial. Formulas for the action of a different set of Virasoro operators on \( Q^{(2)}_{\lambda} \) have been given in [ASY1] and [ASY2]. Note that variable \( t_k \) in those papers corresponds to \( 2t_k \) in the current paper. After this adjustment, the difference between \( L_m^{(2)} \) and the corresponding operator in [ASY1] is \( \frac{1}{4} \sum k \partial_k \partial_{2m-k} \) for \( m > 0 \). Using equation (17) below, it is easy to check that the main result in [ASY1] is equivalent to equation (3) for \( n = 2 \). One can also use equation (6) to check the equivalence of the main result in [ASY2] and equation (5) for \( n = 2 \). The proofs in [ASY1] and [ASY2] used Pfaffian expressions for Schur Q-polynomials, which could not be generalized to cases of \( n \neq 2 \). Our proofs of equations (3) and (5) work for all \( n \geq 2 \), and we will not use Pfaffian expressions for Schur Q-polynomials when \( n = 2 \).

Note that the class of functions \( Q^{(n)}_{\lambda} \) does not include Schur polynomials \( s_{\lambda}(t) = Q_{\lambda}(t; 0) \). The action of Virasoro operators on \( s_{\lambda} \) has the following simple form: For all \( \lambda \in \mathbb{Z}^l \) and \( m \geq 1 \),

\[
L_m^S \cdot s_\lambda = \sum_{i=1}^l \left( \lambda_i - \frac{2i + m - 1}{2} \right) s_{\lambda-m\varepsilon_i},
\]

where

\[
L_m^S := \sum_{k \geq 1} kt_k \partial_t_{k+m} + \frac{1}{2} \sum_{k=1}^{m-1} \partial_t_k \partial_t_{m-k}.
\]

Equation (7), or some equivalent forms of it, might have been implicitly used in the study of super-integrability of Gaussian Hermitian matrix models in [MMMR]. However, we could not find a literature where such formulas were explicitly written down. So we will give a proof of equation (7) in Appendix A where a formula of \( L_m^S \cdot s_\lambda \) with \( m < 0 \) will also be given.

This paper is organized as follows. In section 2, we review the definition of Hall-Littlewood polynomials and prove multiplication formula (6). In section 3, we prove Theorems 1.1 and 1.2. The proof of equation (7) will be given in Appendix A.

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2 Preliminaries

2.1 Hall-Littlewood polynomials

In this paper, Hall-Littlewood functions will be considered as polynomials of variables \( t = (t_1, t_2, ...) \), where \( rt_r = p_r \) corresponds to the \( r \)-th power sum function in the theory.
of symmetric functions. Hence it is natural to assign \( \deg t_r = r \). The ring \( \Lambda := \mathbb{C}[t] \) is isomorphic to the ring of symmetric functions. Set \( \Lambda(\rho) := \Lambda \otimes \mathbb{C}(\rho) \) where \( \rho \) is a parameter. A \( \mathbb{C}(\rho) \)-basis of \( \Lambda(\rho) \) is given by \( \{ t_{\lambda} \mid \lambda \in \mathcal{P} \} \), where \( t_{\lambda} := \prod_{i=1}^{l} t_{\lambda_i} \) for \( \lambda = (\lambda_1, \cdots, \lambda_l) \). Let \( l(\lambda) := l \) if \( \lambda = (\lambda_1, \cdots, \lambda_l) \). There is a natural scalar product on \( \Lambda(\rho) \) with values in \( \mathbb{C}(\rho) \) given by

\[
\langle t_\lambda, t_\mu \rangle = \delta_{\lambda,\mu} \frac{z_\lambda(\rho)}{\left( \prod_{i=1}^{l(\lambda)} \lambda_i \right)^2}
\]

for \( \lambda, \mu \in \mathcal{P} \), where \( \delta_{\lambda,\mu} \) is the Kronecker symbol,

\[
z_\lambda(\rho) := \frac{\prod_{k \geq 1} k^{m_k(\lambda)} m_k(\lambda)!}{\prod_{i=1}^{l(\lambda)} (1 - \rho^{\lambda_i})}
\]

with \( m_k(\lambda) := \#\{ j \mid \lambda_j = k \} \). Using this inner product, we can define the adjoint operator \( f^\perp \) of an operator \( f \) on \( \Lambda(\rho) \) by

\[
\langle fv, w \rangle = \langle v, f^\perp w \rangle
\]

for all \( u, v \in \Lambda(\rho) \). It is well known that

\[
t_r^\perp = \frac{1}{r(1 - \rho^r)} \frac{\partial}{\partial t_r}, \quad (9)
\]

where \( t_r \) is understood as the operator multiplying \( t_r \) (see, for example, [JL]).

We will follow the vertex operator realization of Hall-Littlewood polynomials introduced by Jing in [J]. Let \( B_n, n \in \mathbb{Z} \), be the operators on \( \Lambda(\rho) \) whose generating function \( B(u) := \sum_{n \in \mathbb{Z}} B_n u^n \) is given by

\[
B(u) = \exp \left( \sum_{n \geq 1} (1 - \rho^n) t_n u^n \right) \exp \left( - \sum_{n \geq 1} (1 - \rho^n) t_n^\perp u^{-n} \right),
\]

where \( u \) is a parameter. Then \( B_n \) is an operator of degree \( n \), i.e. \( \deg B_n w = \deg w + n \) if \( w \) is a homogeneous element in \( \Lambda(\rho) \). Hall-Littlewood polynomial associated with \( \lambda = (\lambda_1, \cdots, \lambda_l) \in \mathbb{Z}^l \) is defined by

\[
Q_\lambda(t; \rho) := B_{\lambda_1} \cdots B_{\lambda_l} \cdot 1. \quad (10)
\]

It follows that \( Q_\lambda(t; \rho) \) is a homogeneous polynomial of degree \( |\lambda| := \sum_{i=1}^{l} \lambda_i \) and \( Q_\lambda(t; \rho) = 0 \) if there exists \( 1 \leq j \leq l \) such that \( \sum_{i=j}^{l} \lambda_i < 0 \). Moreover \( Q_{(0)}(t; \rho) = B_0 \cdot 1 = 1 \), and the generating function for \( Q_{(n)}(t; \rho) = B_n \cdot 1 \) with \( n \geq 0 \) is given by

\[
\sum_{n \geq 0} Q_{(n)}(t; \rho) u^n = \exp \left\{ \sum_{n \geq 1} (1 - \rho^n) t_n u^n \right\}.
\]
Operators $B_n$ satisfy the following relation
\begin{equation}
B_{m-1}B_n - \rho B_m B_{n-1} = \rho B_n B_{m-1} - B_{n-1}B_m. \tag{11}
\end{equation}
If $r = n - m$ is a positive odd integer, repeatedly applying the above formula, we have
\begin{equation}
B_mB_n = \rho B_n B_m + (\rho^2 - 1) \sum_{i=1}^{\frac{r-1}{2}} \rho^{i-1}B_{n-i}B_{m+i}. \tag{12}
\end{equation}
Similarly if $r = n - m$ is a positive even integer, we have
\begin{equation}
B_mB_n = \rho B_n B_m + (\rho^2 - 1) \sum_{i=1}^{\frac{r-2}{2}} \rho^{i-1}B_{n-i}B_{m+i} + \rho^{r/2-1}(\rho - 1)B_{n-r/2}B_{m+r/2}. \tag{13}
\end{equation}
Equations (12) and (13) can be used to write equations for permuting adjacent components of $\lambda = (\lambda_1, \ldots, \lambda_l) \in \mathbb{Z}^l$ in $Q_\lambda$. For example, if $r = \lambda_{i+1} - \lambda_i$ is a positive odd integer, then by equation (12), we have
\begin{equation}
Q(\ldots, \lambda_i, \lambda_{i+1}, \ldots) = \rho Q(\ldots, \lambda_{i+1}, \lambda_i, \ldots) + (\rho^2 - 1) \sum_{k=1}^{(r-1)/2} \rho^{k-1}Q(\ldots, \lambda_{i+k}, \lambda_i, \lambda_{i+k}, \ldots) \tag{14}
\end{equation}
(see, for example, Lemma 1 in [Mor] and Example 2 in [Mac] III.2). Similar equations can be easily written down using equation (13) if $r = \lambda_{i+1} - \lambda_i$ is a positive even integer. Together with the fact that $Q_{(\lambda, 0)} = Q_\lambda$ and $Q_\lambda = 0$ if $\lambda_i < 0$, we can use these equations to represent all $Q_\lambda$ as linear combinations of $Q_\mu$ with $\mu \in \mathcal{P}$.

**Example 2.1** If $\rho = 0$, then $Q_\lambda = s_\lambda$ is the Schur polynomial. In this case
\begin{equation}
s(\ldots, \lambda_i, \lambda_{i+1}, \ldots) = \begin{cases} 
-s(\ldots, \lambda_{i+1}-1, \lambda_i+1, \ldots), & \text{if } \lambda_{i+1} - \lambda_i \geq 2, \\
0, & \text{if } \lambda_{i+1} - \lambda_i = 1.
\end{cases} \tag{15}
\end{equation}

**Example 2.2** If $\rho = -1$, then $Q_\lambda$ is the Schur $Q$ polynomial. In this case
\begin{equation}
Q(\ldots, \lambda_i, \lambda_{i+1}, \ldots) = -Q(\ldots, \lambda_{i+1}, \lambda_i, \ldots) + \delta_{\lambda_i, -\lambda_{i+1}}2(-1)^{\lambda_i}Q(\ldots, \lambda_i, \lambda_{i+1}, \ldots), \tag{16}
\end{equation}
which follows from equations (12), (13), and the fact that $B_n^2 = \delta_{n,0}$ for $n \in \mathbb{Z}$ when $\rho = -1$.

Classical definition of $Q_\lambda(t; \rho)$ usually requires $\lambda$ to be a partition (see, for example, [Mac]). However this restriction is not convenient when dealing with derivatives of $Q_\lambda(t; \rho)$. In fact, derivatives of Hall-Littlewood polynomials are given by the following simple formula (see, for example, p144 in [Mor76] and Theorem 2.4 in [JL]):
\begin{equation}
\frac{\partial}{\partial t_r}Q_\lambda(t; \rho) = (1 - \rho^r) \sum_{i=1}^{l} Q_{\lambda - r\epsilon_i}(t; \rho) \tag{17}
\end{equation}
for all $\lambda \in \mathbb{Z}^l$ and $r \geq 1$, where $\lambda - r\epsilon_i$ is defined by equation (4). Note that $\lambda - r\epsilon_i$ may no be a partition even if $\lambda$ is. This is one of the reasons we need to extend the definition of $Q_\lambda(t; \rho)$ to allow $\lambda \in \mathbb{Z}^l$. Equation (17) can be obtained by repeatedly applying the following relation (see, for example, equation (2.26) in [JL]):

$$t_r^p B_m = \frac{1}{r} B_{m-r} + B_m t_r^p.$$  

(18)

In the case that $\rho = \xi_n$ is the $n$-th root of unity, it follows from equation (17) that $Q_\lambda(t; \xi_n)$ does not depend on variables $t_{mn}$ for all positive integers $m$.

## 2.2 Multiplication formula

In this subsection, we give a proof for equation (6) which calculates the product of $p_r := rt_r$ and Hall-Littlewood polynomials. We need the following

**Lemma 2.3** For all $r \geq 1$ and $m \in \mathbb{Z}$,

$$p_r \circ B_m = B_m \circ p_r + B_{m+r},$$

(19)

where "$\circ$" is the composition of operators.

**Proof:** Note that

$$B(u) = \exp \left( \sum_{n \geq 1} \frac{1 - \rho^n}{n} p_n u^n \right) \exp \left( - \sum_{n \geq 1} u^{-n} \frac{\partial}{\partial p_n} \right).$$

The operator $p_r$ commutes with all operators on the right hand side of this equation except $\frac{\partial}{\partial p_r}$. It is straightforward to check that

$$\exp \left( a \frac{\partial}{\partial p_r} \right) \circ p_r = p_r \circ \exp \left( a \frac{\partial}{\partial p_r} \right) + a \exp \left( a \frac{\partial}{\partial p_r} \right)$$

(20)

for all $a$ which does not depend on $p_r$. One can prove this formula by applying both sides of the equation to an arbitrary function and using Leibniz rule. Hence we have

$$p_r \circ B(u)$$

$$= \exp \left( \sum_{n \geq 1} \frac{1 - \rho^n}{n} p_n u^n \right) \left\{ \exp \left( - \sum_{n \geq 1} u^{-n} \frac{\partial}{\partial p_n} \right) \circ p_r + u^{-r} \exp \left( - \sum_{n \geq 1} u^{-n} \frac{\partial}{\partial p_n} \right) \right\}$$

$$= B(u) \circ p_r + u^{-r} B(u).$$

Since $B(u) = \sum_{m \in \mathbb{Z}} B_m u^m$, the coefficient of $u^m$ in the above equation gives the desired formula. □

Now we give a more precise statement of equation (6):
Proposition 2.4 For all $r \geq 1$ and $\lambda \in \mathbb{Z}^l$, we have

$$p_r Q_\lambda = \sum_{i=1}^l Q_{\lambda + r e_i} + \sum_{\mu \in \mathcal{P}} c_\mu(\rho) Q_{(\lambda, \mu)},$$

where

$$c_\mu(\rho) = \frac{\rho^{n(\mu)} \phi(\mu) - 1 (\rho^{-1})}{b_\mu(\rho)}$$

with

$$n(\mu) := \sum_{i=1}^{l(\mu)} (i - 1) \mu_i, \quad \phi_k(\rho) := \prod_{i=1}^k (1 - \rho^i),$$

$$m_i(\mu) := \# \{ j | \mu_j = i \}, \quad b_\mu(\rho) := \prod_{i=1}^{\infty} \phi_{m_i(\mu)}(\rho).$$

Proof: For $\lambda = (\lambda_1, \cdots, \lambda_l) \in \mathbb{Z}^l$, repeatedly applying equation (19), we have

$$p_r \circ B_{\lambda_1} \cdots B_{\lambda_l} = \sum_{i=1}^l B_{\lambda_1} \cdots B_{\lambda_i + r} \cdots B_{\lambda_l} + B_{\lambda_1} \cdots B_{\lambda_l} \circ p_r.$$ 

(22)

The proposition is then obtained by applying both sides of this equation to 1 and using the formula

$$p_r = \sum_{\mu \in \mathcal{P}, |\mu| = r} c_\mu(\rho) Q_{\mu}.$$ 

(23)

Note that equation (23) follows from equation (7.1) and Example 2 in Section III.7 in [Mac]. □

Example 2.5 For the Schur polynomial case, $\rho = 0$. The coefficient $c_\lambda(0)$ should be understood as

$$c_\lambda(0) = \lim_{\rho \to 0} \frac{\rho^{n(\lambda)} \phi(\lambda) - 1 (\rho^{-1})}{b_\lambda(\rho)} = \begin{cases} (-1)^l(\lambda) - 1, & \text{if } \lambda = (k, 1, \ldots, 1) \text{ with } k \geq 1, \\ 0, & \text{otherwise}. \end{cases}$$

Together with equation (15), one can use Proposition 2.4 to recover the following well-known formula

$$p_r s_\lambda = \sum_{\mu \downarrow \lambda \text{ is a border strip of size } r} (-1)^{ht(\mu \setminus \lambda)} s_\mu$$

(24)

for $\lambda \in \mathcal{P}$ (c.f. Example 11 in Section I.3 in [Mac]). As pointed out in Example 5 in Section I.7 in [Mac], this formula implies the well-known Murnaghan-Nakayama rule for calculating characters of irreducible representations of the symmetric group.
Example 2.6 For the Schur Q-polynomial case, \( \rho = -1 \). In this case \( Q_\lambda \) does not depend on \( p_{2k} \) for all \( k \geq 1 \) and \( Q_\lambda = 0 \) if \( \lambda \in P \) is not strict. Moreover \( \phi_k(-1) = 0 \) if \( k \geq 2 \), and

\[
c_{\lambda}(-1) = \begin{cases} \frac{(-1)^m}{2}, & \text{if } \lambda = (k, m) \text{ with } k \geq m \geq 0, \\ 0, & \text{if } l(\lambda) \geq 3. \end{cases}
\]

Thus formula (6) in this case agrees with previous results in [B] and [LY1], which were proved using different methods.

Example 2.7 For \( \rho = \xi_n \) with \( n \geq 2 \), \( \phi_k(\rho) = \phi_k(\rho^{-1}) = 0 \) if \( k \geq n \). Hence \( c_{\lambda}(\xi_n) = 0 \) if \( l(\lambda) \geq n + 1 \).

3 Action of Virasoro operators on Hall-Littlewood polynomials

In this section, we prove Theorems 1.1 and 1.2 by induction on the length of \( \lambda \). Fix an integer \( n \geq 2 \). To carry out the induction process, we need to compute the commutator \( [\hat{L}_m^{(n)}, B_k] \), where \( \hat{L}_m^{(n)} \) is defined by equation (1). We first compute \( [\hat{L}_m^{(n)}, B_k] \) where

\[
\hat{L}_m^{(n)} := \sum_{k \geq 1} kt_k \frac{\partial}{\partial t_{k+mn}}.
\]

(25)

Recall \( p_k = kt_k \) for all \( k \). Operators \( \hat{L}_m^{(n)} \) can also be written as

\[
\hat{L}_m^{(n)} := \sum_{k \geq \iota_m} (1 - \rho^{k+mn}) p_k p_{k+mn}^{\perp},
\]

(26)

where

\[
\iota_m := \begin{cases} 1, & \text{if } m \geq 0, \\ 1 - mn, & \text{if } m < 0. \end{cases}
\]

(27)

For any integer \( N > \iota_m \), define

\[
[\hat{L}_m^{(n)}]_N := \sum_{k = \iota_m}^{N} (1 - \rho^{k+mn}) p_k p_{k+mn}^{\perp}.
\]

(28)

Recall that we have used the symbol ”\( \circ \)” for composition of operators. This symbol may be dropped if there is no ambiguity about the product of two operators.

We will need the following two lemmas to calculate \( [\hat{L}_m^{(n)}, B_r] \)

Lemma 3.1 For any \( r \in \mathbb{Z} \),

\[
[\hat{L}_m^{(n)}]_N \circ B_r = B_r \circ [\hat{L}_m^{(n)}]_N + \sum_{k = \iota_m}^{N} (1 - \rho^{k+mn}) \left( B_{r-mn} + B_{r+k} \circ p_{k+mn}^{\perp} + B_{r-k-mn} \circ p_k \right).
\]

(29)
Proof: Using equations (18) and (19), we obtain

\[ p_k p_j^+ \circ B_r = B_r \circ p_k p_j^+ + B_{r-j+k} + B_{r+k} \circ p_j^+ + B_{r-j} \circ p_k, \tag{30} \]

for all \( r \in \mathbb{Z} \) and \( j, k \geq 1 \). This implies the lemma. \( \square \)

Lemma 3.2 If \( N > \max \{ a, a+r \} \), then for any homogeneous polynomial \( f(t) \) of degree \( a \), we have

\[
\sum_{k=1}^{N} (1 - \rho^k) B_{r-k} \circ p_k \cdot f = \left\{ \begin{array}{ll}
N & - \sum_{k=1}^{N} (1 - \rho^k) \\
B_r \cdot f & - \sum_{k=1}^{N} (1 - \rho^k) B_{r+k} \circ p_k^+ \cdot f
\end{array} \right. \]  \tag{31}

for all \( r \in \mathbb{Z} \) and \( a \geq 0 \).

Proof: Note that

\[
\hat{L}_0^{(n)} = \sum_{k \geq 1} k t_k \frac{\partial}{\partial t_k} \quad \text{and} \quad [\hat{L}_0^{(n)}]_N = \sum_{k=1}^{N} k t_k \frac{\partial}{\partial t_k}.
\]

For any homogeneous polynomial \( g(t) \) of degree less than \( N \), \( \frac{\partial g}{\partial t_k} = 0 \) for all \( k > N \). Hence

\[
[\hat{L}_0^{(n)}]_N \cdot g = \hat{L}_0^{(n)} \cdot g = \deg(g) \cdot g.
\]

In particular, since \( f \) and \( B_r \cdot f \) are homogeneous of degree \( a \) and \( a + r \) respectively, we have

\[
[\hat{L}_0^{(n)}]_N \cdot f = a f \quad \text{and} \quad [\hat{L}_0^{(n)}]_N \circ B_r \cdot f = (a + r) B_r \cdot f.
\]

Applying both sides of equation (29) with \( m = 0 \) to function \( f \), we obtain the desired formula. \( \square \)

Now we are ready to compute \( [\hat{L}_m^{(n)}, B_r] \).

Proposition 3.3 Assume \( \rho = \xi_n \). Then for all \( r \in \mathbb{Z} \),

\[
[\hat{L}_m^{(n)}, B_r] = (r - mn) B_{r-mn} - \sum_{k=1}^{mn} (1 - \xi_n^k) B_{r-mn+k} \circ p_k^+ \tag{32}
\]

if \( m \geq 1 \), and

\[
[\hat{L}_m^{(n)}, B_r] = r B_{r-mn} - \sum_{k=1}^{-mn} (1 - \xi_n^k) B_{r-mn-k} \circ p_k \tag{33}
\]

if \( m \leq -1 \).

Proof: We only need to show that equalities hold when both sides of equations (32) and (33) acting on an arbitrary homogeneous polynomial \( f(t) \) of degree \( a \geq 0 \). Choose an integer \( N > \max \{ a, a+r \} + \iota_m \). Then

\[
[\hat{L}_m^{(n)}, B_r] \cdot f = ([\hat{L}_m^{(n)}]_N \circ B_r - B_r \circ [\hat{L}_m^{(n)}]_N) \cdot f.
\]
So by Lemma 3.1, we have

\[
\hat{L}_m^{(n)}(B_r) \cdot f = \sum_{k=\iota_m}^N (1 - \xi_n^k) \left\{ B_{r-mn} + B_{r+k} \circ p_{k+mn} + B_{r-k-mn} \circ p_k \right\} \cdot f
\] (34)

Here we have used the fact \( \xi_n^{k\pm mn} = \xi_n^k \) since \( \xi_n \) is the \( n \)-th root of unity. Applying Lemma 3.2 to the last term on the right hand side of equation (34), we have

\[
\sum_{k=\iota_m}^N (1 - \xi_n^k) B_{r-k-mn} \circ p_k \cdot f + \sum_{k=1}^{\iota_m-1} (1 - \xi_n^k) B_{r-k-mn} \circ p_k \cdot f
\]

\[
= \left\{ r - mn - \sum_{k=1}^N (1 - \xi_n^k) \right\} B_{r-mn} \cdot f - \sum_{k=1}^N (1 - \xi_n^k) B_{r-mn+k} \circ p_{k}^{\perp} \cdot f.
\] (35)

If \( m \geq 1 \), then \( \iota_m = 1 \). Combination of equations (34) and (35) gives the following

\[
\hat{L}_m^{(n)}(B_r) \cdot f = (r - mn) B_{r-mn} \cdot f - \sum_{k=1}^{mn} (1 - \xi_n^k) B_{r-mn+k} \circ p_{k}^{\perp} \cdot f
\]

\[
+ \sum_{k=N+1}^{N+mn} (1 - \xi_n^k) B_{r-mn+k} \circ p_{k}^{\perp} \cdot f.
\]

The last term on the right hand side of this equation vanishes since \( p_{k}^{\perp} \cdot f = 0 \) for \( k > N + mn > \deg(f) \). This proves equation (32) since \( f \) is an arbitrary homogenous polynomial.

If \( m \leq -1 \), then \( \iota_m = 1 - mn \). Combination of equations (34) and (35) gives the following

\[
\hat{L}_m^{(n)}(B_r) \cdot f = \left\{ r - mn - \sum_{k=1}^{-mn} (1 - \xi_n^k) \right\} B_{r-mn} \cdot f - \sum_{k=1}^{-mn} (1 - \xi_n^k) B_{r-k-mn} \circ p_k \cdot f
\]

\[
- \sum_{k=N+mn+1}^{-mn} \left( 1 - \xi_n^k \right) B_{r-mn+k} \circ p_k^{\perp} \cdot f.
\]

The last term on the right hand side of this equation vanishes since \( p_k^{\perp} \cdot f = 0 \) for \( k > N + mn > \deg(f) \). Moreover

\[
\sum_{k=1}^{-mn} (1 - \xi_n^k) = -mn - \xi_n - \xi_n^{-mn+1} \over 1 - \xi_n = -mn
\]

since \( \xi_n \) is the \( n \)-th root of unity. This proves equation (33) since \( f \) is an arbitrary homogenous polynomial. The proposition is thus proved. \( \square \)

**Corollary 3.4** For \( \rho = \xi_n, m \geq 1, \) and \( r \in \mathbb{Z}, \)

\[
\hat{L}_m^{(n)}(B_r) = r B_{r-mn} + \sum_{k=1}^{mn} \left( 1 - \xi_n^{-k} \right) B_{r-mn+k} \circ p_k^{\perp}
\] (36)
where
\[
\tilde{L}_m^{(n)} := \sum_{k \geq 1} kt_k \frac{\partial}{\partial t_k} + \frac{1}{2} \sum_{k=1}^{mn-1} \frac{\partial^2}{\partial t_k \partial t_{mn-k}}.
\]  

**Proof:** For \( m \geq 0 \), let
\[
W_m^{(n)} := mn - 1 \sum_{k=1}^{mn-1} (1 - \rho^k)(1 - \rho^{mn-k}) p_k^i p_{mn-k}^i.
\]

Then
\[
\tilde{L}_m^{(n)} = \tilde{L}_m^{(n)} + \frac{1}{2} W_m^{(n)}.
\]

Using equation (18) twice, we obtain
\[
[W_m^{(n)}, B_r] = B_{r-j-k} + B_{r-j} p_k^i + B_{r-k} p_j^i
\]
for all \( r \in \mathbb{Z} \) and \( j, k \geq 1 \). Hence
\[
[W_m^{(n)}, B_r] = \sum_{k=1}^{mn-1} (1 - \rho^k)(1 - \rho^{mn-k}) \left\{ B_{r-mn} + 2B_{r-mn+k} p_k^i \right\}.
\]

If \( \rho = \xi_n \), then \( \rho^{mn-k} = \xi_n^{mn-k} \), \( \sum_{k=1}^{mn} \xi_n^k = \sum_{k=1}^{mn} \xi_n^{mn-k} = 0 \), and
\[
\sum_{k=1}^{mn-1} (1 - \rho^k)(1 - \rho^{mn-k}) = \sum_{k=1}^{mn} (2 - \xi_n^k - \xi_n^{mn-k}) = 2mn.
\]

Hence
\[
[W_m^{(n)}, B_r] = 2mn B_{r-mn} + 2 \sum_{k=1}^{mn} (1 - \xi_n^k)(1 - \xi_n^{mn-k}) B_{r-mn+k} p_k^i.
\]

The corollary then follows from this equation and equation (32). \( \square \)

We are now ready to prove the first main result of this paper.

**Proof of Theorem 1.1** If \( \rho = \xi_n \), \( Q_\lambda(t; \rho) = Q_\lambda^{(n)} \) does not depend on variables \( t_{mn} \) for all \( m \geq 1 \). Hence
\[
L_m^{(n)} Q_\lambda^{(n)} = \tilde{L}_m^{(n)} Q_\lambda^{(n)},
\]
where \( \tilde{L}_m^{(n)} \) is defined by equation (37). For \( m = 0 \), equation (3) follows from the fact that \( Q_\lambda^{(n)} \) is homogeneous of degree \( |\lambda| \).

For \( m > 0 \), we prove this theorem by induction on \( l(\lambda) \). If \( l(\lambda) = 1 \) and \( \lambda = (\lambda_1) \), then
\[
L_m^{(n)} Q_\lambda^{(n)} = \tilde{L}_m^{(n)} B_{\lambda_1} \cdot 1 = \lambda_1 B_{\lambda_1-mn} \cdot 1 = \lambda_1 Q_\lambda^{(n)}^{(n)}_{\lambda-mn_1},
\]
where the second equality follows from equation (36) since \( \tilde{L}_m^{(n)} \cdot 1 = p_k^i \cdot 1 = 0 \). Hence equation (3) is true when \( l(\lambda) = 1 \).
If \( l(\lambda) = l > 1 \), let \( \lambda = (\lambda_1, \tilde{\lambda}) \) where \( \tilde{\lambda} = (\lambda_2, \cdots, \lambda_l) \). By equations (9) and (17),

\[
p_k^\perp \cdot Q_\lambda = \sum_{i=1}^{l} Q_{\lambda - k\epsilon_i}
\]

(41)

for all \( \lambda \in Z^l \) and \( k \geq 1 \). By equation (36),

\[
L_m^{(n)} Q_\lambda^{(n)} = \tilde{L}_m^{(n)} B_{\lambda_1} \cdot Q_\lambda^{(n)} = \left\{ B_{\lambda_1} \tilde{L}_m^{(n)} + \lambda_1 B_{\lambda_1 - mn} + \sum_{k=1}^{mn} (1 - \xi_n^{-k}) B_{\lambda_1 - mn + k} \circ p_k^\perp \right\} \cdot Q_\lambda^{(n)}
\]

\[
= B_{\lambda_1} L_m^{(n)} Q_\lambda^{(n)} + \lambda_1 Q_{\lambda - mn\epsilon_i}^{(n)} + \sum_{k=1}^{mn-1} (1 - \xi_n^{-k}) \sum_{i=2}^{l} Q_{\lambda - (mn - k)\epsilon_1 - \epsilon_i}^{(n)}.
\]

By induction hypothesis,

\[
B_{\lambda_1} L_m^{(n)} Q_\lambda^{(n)} = \sum_{i=2}^{l} \lambda_i Q_{\lambda - mn\epsilon_i}^{(n)} + \sum_{k=1}^{mn-1} \sum_{i,j=2}^{l} (1 - \xi_n^{-k}) Q_{\lambda - (mn - k)\epsilon_1 - \epsilon_i - \epsilon_j}^{(n)}.
\]

Combination of the above two equations gives equation (3). This finishes the proof of Theorem 1.1. \( \square \)

Before proving Theorem 1.2, we first prove the following simpler formula

**Theorem 3.5** For any \( n \geq 2, m \geq 1, \lambda \in Z^l \), we have

\[
\hat{L}_{-m}^{(n)} \cdot Q_\lambda^{(n)} = \sum_{i=1}^{l} \lambda_i Q_{\lambda + m\epsilon_i}^{(n)} + \sum_{k=1}^{mn} (\xi_n^k - 1) Q_{\lambda + k\epsilon_i + (mn - k)\epsilon_j}^{(n)}
\]

\[
+ \sum_{i=1}^{l} \sum_{k=1}^{mn} (\xi_n^k - 1) \sum_{\mu \in P} c_\mu(\xi_n) Q_{\lambda + (mn - k)\epsilon_i, \mu}^{(n)},
\]

(42)

where \( \hat{L}_{-m}^{(n)} \) is defined by equation (25).
Proof: If \( l(\lambda) = l \geq 1 \), let \( \lambda = (\lambda_1, \hat{\lambda}) \in \mathbb{Z}^l \) where \( \hat{\lambda} = (\lambda_2, \ldots, \lambda_l) \). Then we have

\[
\hat{L}_m \cdot Q^{(n)}_\lambda = \hat{L}_m B_{\lambda_1} \cdot Q^{(n)}_{\hat{\lambda}}
\]

\[
= \left\{ B_{\lambda_1} \hat{L}_m + \lambda_1 B_{\lambda_1 + mn} + \sum_{k=1}^{mn} (\xi_n^k - 1) B_{\lambda_1 - k + mn \circ p_k} \right\} \cdot Q^{(n)}_{\hat{\lambda}}
\]

\[
= B_{\lambda_1} \hat{L}_m Q^{(n)}_{\hat{\lambda}} + \lambda_1 Q^{(n)}_{\lambda + mn \epsilon_1}
\]

\[
+ \sum_{k=1}^{mn} \left( \xi_n^k - 1 \right) B_{\lambda_1 - k + mn} \left\{ \sum_{i=1}^{l-1} Q^{(n)}_{\lambda + k \epsilon_i} + \sum_{\mu \in P \atop |\mu|=k} c_\mu(\xi_n) Q^{(n)}_{(\lambda, \mu)} \right\}
\]

\[
= B_{\lambda_1} \hat{L}_m Q^{(n)}_{\hat{\lambda}} + \lambda_1 Q^{(n)}_{\lambda + mn \epsilon_1}
\]

\[
+ \sum_{k=1}^{mn} \left( \xi_n^k - 1 \right) \left\{ \sum_{i=2}^{l} Q^{(n)}_{\lambda + (mn - k) \epsilon_1 + k \epsilon_i} + \sum_{\mu \in P \atop |\mu|=k} c_\mu(\xi_n) Q^{(n)}_{(\lambda + (mn - k) \epsilon_1, \mu)} \right\}
\]

(43)

where the second equality follows from equation (33), the third equality follows from equation (6).

We prove the theorem by induction on \( l(\lambda) \). If \( l(\lambda) = 1 \), then \( \hat{\lambda} \) is the empty partition. In this case \( Q^{(n)}_{\hat{\lambda}} = 1 \) and \( \hat{L}_m Q^{(n)}_{\hat{\lambda}} = 0 \). Equation (43) coincides with equation (42). Hence the theorem holds for \( l(\lambda) = 1 \).

If \( l(\lambda) > 1 \), by induction hypothesis,

\[
B_{\lambda_1} \hat{L}_m Q^{(n)}_{\hat{\lambda}} = \sum_{i=2}^{l} \lambda_i Q^{(n)}_{\lambda + mn \epsilon_i} + \sum_{k=1}^{mn} \sum_{i,j=2 \atop i \neq j}^{l} (\xi_n^k - 1) Q^{(n)}_{\lambda + k \epsilon_i + (mn - k) \epsilon_j}
\]

\[
+ \sum_{k=1}^{mn} \sum_{i=2}^{l} (\xi_n^k - 1) c_\mu(\xi_n) Q^{(n)}_{(\lambda + (mn - k) \epsilon_1, \mu)}
\]

(44)

Combining equations (43) and (44), we obtain equation (42). Hence the theorem is proved. □

Now we are ready to prove the second main theorem of this paper.

Proof of Theorem 1.2: Let

\[
V^{(n)}_m := \sum_{k=1 \atop n \mid k}^{mn-1} p_k p_{mn-k}
\]

for \( m \geq 1 \). Since \( Q^{(n)}_{\lambda} \) does not depend on \( t_{kn} \) for all \( k \geq 1 \), we have

\[
L^{(n)}_m \cdot Q^{(n)}_{\lambda} = \hat{L}^{(n)}_m \cdot Q^{(n)}_{\lambda} + \frac{1}{2} V^{(n)}_m Q^{(n)}_{\lambda}
\]

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for all \( \lambda \). Applying equation (6) twice, we have

\[
V_m^{(n)} Q^{(n)}_{\lambda} = \sum_{k=1}^{mn-1} \left( \sum_{\mu \in \mathcal{P}, |\mu| = k} l(\mu) \sum_{j=1}^{l(\mu)} c_{\mu}(\xi_n) Q^{(n)}_{(\lambda, \mu+\lambda+\lambda+mn-k)\epsilon_j} \right) + \sum_{\mu, \nu \in \mathcal{P}, |\mu| = mn-k} c_{\mu}(\xi_n) c_{\nu}(\xi_n) Q^{(n)}_{(\lambda, \mu, \nu)}
\]

\[
+ m(n - 1) \sum_{i=1}^{mn} Q^{(n)}_{\lambda+\lambda+mn\epsilon_i} + 2 \sum_{k=1}^{mn-1} \sum_{i, j=1}^{l(\lambda)} Q^{(n)}_{\lambda+k\epsilon_i+(mn-k)\epsilon_j}
\]

\[
+ 2 \sum_{k=1}^{mn-1} \sum_{i=1}^{mn} \sum_{\mu \in \mathcal{P}, |\mu| = k} c_{\mu}(\xi_n) Q^{(n)}_{(\lambda+(mn-k)\epsilon_i+\lambda, \mu)}.
\]

The theorem then follows from equations (42) and (45). □

**Appendix**

### A Action of Virasoro operators on Schur polynomials

In this appendix, we study the action of Virasoro operators on Schur polynomials. These Virasoro operators are given by

\[
L^S_m := \sum_{k \geq 1} t_k \frac{\partial}{\partial t_{k+m}} + \frac{1}{2} \sum_{k=1}^{m-1} \frac{\partial^2}{\partial t_k \partial t_{m-k}} + \frac{1}{2} \sum_{k=1}^{m-1} k(-m-k)t_k t_{m-k}.
\]

The first order derivative part of this operator is

\[
\hat{L}^S_m = \sum_{k \geq 1} t_k \frac{\partial}{\partial t_{k+m}}.
\]

\( \hat{L}^S_m \) is just the operator \( \hat{L}_m^{(n)} \) defined in equation (25) with \( n = 1 \). Hence the same proof for Proposition 3.3 with \( \rho = 0 \) and \( n = 1 \) shows the following

**Lemma A.1** Assume \( \rho = 0 \). Then for all \( r \in \mathbb{Z} \),

\[
[\hat{L}_m^S, B_r] = (r - m)B_{r-m} - \sum_{k=1}^{m} B_{r-m+k} \circ p_k^\perp
\]

if \( m \geq 1 \), and

\[
[\hat{L}_m^S, B_r] = rB_{r-m} - \sum_{k=1}^{-m} B_{r-m-k} \circ p_k
\]

if \( m \leq -1 \).
Consequently, we have

**Corollary A.2** Assume \( \rho = 0 \). Then for all \( r \in \mathbb{Z} \),

\[
[L_m^S, B_r] = \left( r - \frac{m + 1}{2} \right) B_{r-m} - B_r \circ p_m^+ \tag{50}
\]

if \( m \geq 1 \), and

\[
[L_m^S, B_r] = \left( r - \frac{m + 1}{2} \right) B_{r-m} - B_r \circ p_{-m} \tag{51}
\]

if \( m \leq -1 \).

**Proof:** Define

\[
W_m^S = \sum_{k=1}^{m-1} \frac{\partial^2}{\partial t_k \partial t_{m-k}} \tag{52}
\]

for \( m \geq 0 \), and

\[
W_m^S = \sum_{k=1}^{-m-1} p_k p_{-m-k} \tag{53}
\]

for \( m < 0 \). Then

\[
L_m^S = \hat{L}_m^S + \frac{1}{2} W_m^S.
\]

By equation (39) with \( \rho = 0 \) and \( n = 1 \), we have

\[
[W_m^S, B_r] = (m - 1) B_{r-m} + 2 \sum_{k=1}^{m-1} B_{r-m+k} \circ p_k^+ \tag{54}
\]

for \( m > 0 \). This equation and equation (48) imply equation (50).

Applying equation (19) twice, we have

\[
p_k p_j \circ B_r = B_r \circ p_k p_j + B_{r+k} \circ p_j + B_{m+j} \circ p_k + B_{r+j+k} \tag{55}
\]

for all \( r \in \mathbb{Z} \) and \( j, k \geq 1 \). Consequently we have

\[
[W_m^S, B_r] = (-m - 1) B_{r-m} + 2 \sum_{k=1}^{-m-1} B_{r-m-k} \circ p_k \tag{56}
\]

for \( m < 0 \). This equation and equation (49) imply equation (51). \( \square \)

Now we are ready to prove equation (7), which we restate as the following theorem:

**Theorem A.3** For \( m \geq 1 \) and \( \lambda \in \mathbb{Z}^l \),

\[
L_m^S \cdot s_\lambda = \sum_{i=1}^{l} \left( \lambda_i - \frac{2i + m - 1}{2} \right) s_{\lambda - m\epsilon_i}.
\]

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Proof: We will prove this formula by induction on $l$. Write $\lambda = (\lambda_1, \ldots, \lambda_l)$ where $\lambda = (\lambda_2, \cdots \lambda_l)$. Then by equation (50), we have

$$L^S_m \cdot s_\lambda = L^S_m B_{\lambda_1} \cdot s_\lambda = \left\{ B_{\lambda_1} L^S_m + \left( \lambda_1 - \frac{m + 1}{2} \right) B_{\lambda_1 - m} - B_{\lambda_1} \circ p_m \right\} \cdot s_\lambda.$$

If $l = 1$, $\lambda$ is the empty partition and $s_\lambda = 1$. So $L^S_m \cdot s_\lambda = p_m \cdot s_\lambda = 0$ and $L^S_m \cdot s_\lambda = (\lambda_1 - \frac{m + 1}{2}) s_\lambda$. Hence the theorem holds for $l = 1$.

If $l > 1$, by induction hypothesis and equation (41), we have

$$L^S_m \cdot s_\lambda = \sum_{i=2}^l \left( \lambda_i - \frac{2(i - 1) + m - 1}{2} \right) s_{\lambda - \mu_i} + \left( \lambda_1 - \frac{m + 1}{2} \right) s_{\lambda - \mu_1} - \sum_{i=2}^l s_{\lambda - \mu_i}.$$

The theorem is thus proved. □

For the action of the negative branch of Virasoro operators on Schur polynomials, we have the following

**Theorem A.4** For $m \geq 1$ and $\lambda \in \mathbb{Z}^l$,

$$L^S_{-m} \cdot s_\lambda = \sum_{i=1}^l \left( \lambda_i - \frac{2i - 1 + m - 1}{2} \right) s_{\lambda + \mu_i} - \sum_{k=1}^m (-1)^{m-k} \left( l - k + \frac{m + 1}{2} \right) s_{(\lambda, k, 1^{m-k})},$$

(57)

where $1^j := (1, \cdots, 1) \in \mathbb{Z}^j$ for $j \geq 0$.

**Proof:** We first note that if $l = 0$, then $\lambda$ is an empty partition, $s_\lambda = 1$ and equation (57) becomes

$$L^S_{-m} \cdot 1 = \sum_{k=1}^m (-1)^{m-k+1} \left( -k + \frac{m + 1}{2} \right) s_{(1^{m-k})}. \quad (58)$$

The proof of this theorem is divided into proofs of statements in the following three steps.

**Step 1:** For each $m \geq 1$, equation (57) follows from equation (58).

We prove this statement by induction on $l$. Equation (58) is the base case with $l = 0$ for the induction. For $l \geq 1$, we write $\lambda = (\lambda_1, \ldots, \lambda_l)$ where $\lambda = (\lambda_2, \cdots, \lambda_l)$. Then by equation (51), we have

$$L^S_{-m} \cdot s_\lambda = L^S_{-m} B_{\lambda_1} \cdot s_\lambda = \left\{ B_{\lambda_1} L^S_{-m} + \left( \lambda_1 - \frac{m + 1}{2} \right) B_{\lambda_1 + m} - B_{\lambda_1} \circ p_m \right\} \cdot s_\lambda.$$

By Example 2.5, equation (6) becomes

$$p_r s_\mu = \sum_{i=1}^{l(\mu)} s_{\mu + r_\mu} + \sum_{k=1}^r (-1)^{r-k} s_{(\mu, k, 1^{r-k})} \quad (59)$$

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for all $r \geq 1$ and $\mu \in \mathbb{Z}^{l(\mu)}$. Using this equation and the induction hypothesis, we obtain

$$L_{-m}^S \cdot s_{\lambda} = \sum_{i=2}^{l} \left\{ \lambda_i - (i-1) + \frac{m+1}{2} \right\} s_{\lambda+m \epsilon_i} - \sum_{k=1}^{m} (-1)^{m-k} \left( L - k + \frac{m+1}{2} \right) s_{(\lambda,k,1^{m-k})}$$

$$+ \left( \lambda_1 - \frac{-m+1}{2} \right) s_{\lambda+m \epsilon_1} - \left\{ \sum_{i=2}^{l} s_{\lambda+m \epsilon_i} + \sum_{k=1}^{m} (-1)^{m-k} s_{(\lambda,k,1^{m-k})} \right\}$$

$$= \sum_{i=1}^{l} \left\{ \lambda_i - i + \frac{m+1}{2} \right\} s_{\lambda+m \epsilon_i} - \sum_{k=1}^{m} (-1)^{m-k} \left( L - k + \frac{m+1}{2} \right) s_{(\lambda,k,1^{m-k})}.$$ 

This completes the proof for the statement in step 1.

**Step 2:** Equation (57) holds for $m = 1$.

In fact, this statement follows from step 1 since equation (58) is a trivial equation with both sides equal to 0 when $m = 1$.

**Step 3:** Equation (58) holds for all $m \geq 1$.

We prove this statement by induction on $m$. For $m = 1$, (58) holds trivially since $L_{-1}^S \cdot 1 = 0$. By equation (59),

$$2L_{-2}^S \cdot 1 = p_1^2 \cdot 1 = p_1 s_{(1)} = s_{(2)} + s_{(1,1)}.$$ 

So equation (58) also holds for $m = 2$.

Now we consider the case for $m \geq 3$. By the Virasoro bracket relation (see, for example, Section 2.3 in [KR]), we have

$$(-m+2)L_{-m}^S = L_{-(m-1)}^S L_{-1}^S - L_{-1}^S L_{-(m-1)}^S.$$ 

Since $L_{-1}^S \cdot 1 = 0$, by the induction hypothesis, we have

$$(-m+2)L_{-m}^S \cdot 1 = -L_{-1}^S L_{-(m-1)}^S \cdot 1 = -L_{-1}^S \sum_{k=1}^{m-1} (-1)^{m-k} \left( -k + \frac{m}{2} \right) s_{(k,1^{m-1-k})}.$$ 

Note that $s_{(\ldots,1,2,\ldots)} = 0$ by Example 2.1. By equation (57) for $m = 1$, which was proved in step 2, we have

$$L_{-1}^S \cdot s_{(k,1^r)} = k s_{(k+1,1^r)} - (r+1) s_{(k,1^{r+1})}$$

for all $r \geq 0$. Hence, by equation (60), we have

$$(-m+2)L_{-m}^S \cdot 1 = \sum_{k=1}^{m-1} (-1)^{m-k+1} \left( -k + \frac{m}{2} \right) \left\{ k s_{(k+1,1^{m-1-k})} - (m-k) s_{(k,1^{m-k})} \right\}$$

$$= \sum_{k=1}^{m} (-1)^{m-k}(-m+2) \left( k - \frac{m+1}{2} \right) s_{(k,1^{m-k})}.$$
Divided both sides by \((-m+2)\), we obtain the desired formula. This proves the statement in step 3.

The theorem then follows from combination of step 1 and step 3. □

**Remark A.5** Equation (58) is equivalent to the following identity

\[
\sum_{k=1}^{m-1} p_k p_{m-k} = \sum_{k=1}^{m} (-1)^{m-k} (2k - m - 1) s_{(k,1^{m-k})}
\]

for all \(m \geq 1\).

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