INTEGRABLE N=2 SUPERSYMMETRIC FIELD THEORIES

JONATHAN EVANS\(^1\) AND TIMOTHY HOLLOWOOD\(^2\)

Theoretical Physics, 1 Keble Road, Oxford, OX1 3NP, U.K.

ABSTRACT

A classification is given of Toda-like theories with N=2 supersymmetry which are integrable by virtue of some underlying Lie superalgebra. In addition to the N=2 superconformal theories based on \(sl(m, m-1)\), which generalize the Liouville model, a family of massive N=2 theories based on the algebras \(sl(m, m)\)\(^{(1)}\) is found, providing natural generalizations of the sine-Gordon theory. A third family of models based on \(sl(m, m)\) which have global supersymmetry, a version of conformal invariance, but no superconformal invariance is also briefly discussed. Unlike their N=0 and N=1 cousins, the N=2 massive theories apparently cannot be directly thought of as integrable deformations of the corresponding N=2 superconformal theories. It is shown that these massive theories admit supersymmetric soliton solutions and a form for their exact S-matrices is conjectured.

\(^1\) evansjm@dionysos.thphys@prg.oxford.ac.uk

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\(^2\) holl@dionysos.thphys@prg.oxford.ac.uk

Address after 1st October: Theory Division, CERN, 1211 Geneva 23, Switzerland
1. Introduction: N=1 supersymmetric Toda theories

Given a Cartan matrix or a set of simple roots for some Lie algebra, one can construct an associated integrable bosonic Toda field theory (see for instance [1]). If the algebra is finite-dimensional then the resulting theory is massless and exhibits an extended conformal symmetry [2,3] whilst if the algebra is of affine Kac-Moody type, then the resulting theory is massive. In seeking to generalize this construction, it is important to stress that there is no obvious way to supersymmetrize a given bosonic Toda theory whilst maintaining integrability. One can, however, write down integrable Toda theories based on Lie superalgebras (see [4,5,6] and references therein) which contain both bosons and fermions but which are not supersymmetric in general. For superalgebras, unlike conventional Lie algebras, there can exist inequivalent bases of simple roots (up to isomorphism under the Weyl group) and each of these inequivalent bases leads to a distinct Toda theory. Each root of a superalgebra carries a $\mathbb{Z}_2$-grading which makes it either of ‘bosonic’ or ‘fermionic’ type and it turns out that it is precisely those simple root systems which are purely fermionic which give rise to supersymmetric Toda theories.

In this paper we shall consider integrable field theories with $N=2$ supersymmetry. We shall extend the treatment of the conformal case given in [6,7] and construct a family of massive $N=2$ Toda models which are natural integrable generalizations of the $N=2$ super sine-Gordon theory. A number of novel features arise compared to the bosonic situation and in particular these massive theories apparently cannot be interpreted as integrable deformations of $N=2$ superconformal Toda models in any immediate way. We will show that the massive $N=2$ theories admit supersymmetric soliton solutions, which are in fact related to the solitons found recently [8,9] in complex bosonic $\text{sl}(m)_1$ Toda theories. In the presence of these solitons the $N=2$ supersymmetry algebra acquires a central term dependent on the topological charge – in line with general arguments first given by Witten and Olive [10] – and we describe briefly how this leads to an equation for the classical soliton masses. We conclude with some comments and conjectures concerning $S$-matrices related to these theories.

To begin, we summarize in this section the construction of integrable $N=1$ supersymmetric Toda theories. Most of the material is standard, although we shall treat a number of aspects of the construction more thoroughly than is usual. This will prove essential in allowing us to understand some key features of the superalgebra models which have no counterparts in the bosonic cases.

To write down an integrable theory we require a contragredient Lie superalgebra (CLSA) [11,12,13] which can be defined in terms of a basis of simple roots. The $\mathbb{Z}_2$-graded commutation relations of the corresponding step operators can be specified by a Cartan matrix $a_{ij}$ which is $n\times n$-dimensional, symmetric without loss of generality, and of rank $r$, say. Specializing immediately to the case of a CLSA with purely fermionic simple root system, the associated Toda equations are

$$iD_+D_-\Phi_i + \frac{1}{\beta} \sum_j \mu_j a_{ij} \exp \beta \Phi_j = 0. \quad (1.1)$$
The \( \Phi_j \)'s are a set of \( n \) scalar superfields appropriate to two-dimensional super-Minkowski space: they are functions of the bosonic light-cone coordinates \( x^\pm = \frac{1}{2}(t \pm x) \) and of the real fermionic coordinates \( \theta^\pm \), in terms of which each field can be expanded

\[
\Phi_j = \phi_j + i\theta^+\psi_j+ + i\theta^-\psi_j- + i\theta^+\theta^-F_j. \tag{1.2}
\]

The super-derivatives \( D_\pm \) are defined by

\[
D_\pm = \frac{\partial}{\partial \theta^\pm} - i\theta^\pm \partial_\pm, \quad D_\pm^2 = -i\partial_\pm, \tag{1.3}
\]

and the equations are clearly invariant under transformations generated by the super-charges

\[
Q_\pm = \frac{\partial}{\partial \theta^\pm} + i\theta^\pm \partial_\pm, \quad Q_\pm^2 = i\partial_\pm. \tag{1.4}
\]

The coupling constant \( \beta \) is dimensionless while the quantities \( \mu_i \) are non-zero parameters with the the dimensions of mass which can be re-scaled by shifting the fields \( \Phi_i \).

The integrability of the Toda equations (1.1) can be established by viewing them as the zero-curvature conditions for a certain gauge-field in superspace (for details see \[4,6\] which follows the bosonic treatment given in \[1\]). The superfields \( \Phi_i \) can take either real or complex values. It is also consistent, as explained in \[7\], to impose a twisted reality condition of the form

\[
\Phi^*_i = \Phi_{\sigma(i)}, \tag{1.5}
\]

provided that \( \sigma \) is a symmetry of order two of the Cartan matrix, obeying

\[
a_{\sigma(i)\sigma(j)} = a_{ij}, \quad \mu^*_j = \mu_{\sigma(j)}, \quad \sigma^2 = 1. \tag{1.6}
\]

These are the most general possibilities known to be compatible with integrability and we shall see below that this choice is important in constructing \( N=2 \) theories.

The number of non-trivial superfield degrees of freedom in the Toda equations (1.1) is always equal to \( r \), the rank of the Cartan matrix, rather than to \( n \), its dimension. This follows because if \( \xi_j \) are the components of any null eigenvector of the Cartan matrix, then (1.1) implies that the field \( \Phi' = \sum_j \xi_j \Phi_j \) satisfies the free super-wave equation \( D_+D_-\Phi' = 0 \). We may therefore consistently set \( \Phi' = 0 \) and so reduce by one the number of independent superfields for every independent null eigenvector \( \xi \). It is convenient to regard the remaining Toda fields as comprising an \( r \)-dimensional vector \( \Phi \). One can then introduce \( n \) constant \( r \)-dimensional vectors \( \alpha_i \) and an inner-product denoted by a ‘dot’ such that

\[
\Phi_i = \Phi \cdot \alpha_i, \quad a_{ij} = \alpha_i \cdot \alpha_j, \quad \sum_i \xi_i \alpha_i = 0 \quad \forall \xi. \tag{1.7}
\]

The \( \alpha_i \)'s are actually projections of the simple roots of the CLSA and the inner-product is similarly induced from the natural invariant inner-product on the algebra. Although the precise details will not concern us here, an important point is that this inner-product can
have indefinite signature even when the CLSA is finite-dimensional. The Toda equations for the independent superfields can now be written

\[ iD_+D_- \Phi + \frac{1}{\beta} \sum_j \mu_j \alpha_j \exp \beta \alpha_j \cdot \Phi = 0, \]  

(1.8)

and they can be derived from the superspace Lagrangian density

\[ L = \frac{i}{2} \Phi \cdot D_+ \Phi - \frac{1}{\beta^2} \sum_j \mu_j \exp \beta \alpha_j \cdot \Phi. \]  

(1.9)

The character of this theory depends crucially on the relationship between \( r \) and \( n \).

If \( r = n \), the theory is superconformally invariant. At the classical level this symmetry can be realized in superspace by transformations of the form (see [6] for details)

\[ \Phi \to \Phi + \frac{2}{\beta} \rho \log (D_+ \xi^+ D_- \xi^-) \]  

(1.10)

where \( \xi^\pm(x^\pm, \theta^\pm) \) and the vector \( \rho \) is defined by \( \rho \cdot \alpha_i = \frac{1}{2} \), and thus owes its existence to the fact that the \( \alpha_j \)'s are linearly independent. A consequence of this symmetry is that \( \Phi \) can always be shifted so as to set \( \mu_j = 1 \), thereby eliminating any mass parameter from the theory; we adopt this convention from now on. A list of all superalgebras of this type was given in [5,6] and we reproduce it here for completeness: \( sl(m, m-1) = A(m-1, m-2) \) \( m \geq 2 \) with \( r = 2m-2 \); \( osp(2m+1, 2m) = B(m, m) \) \( m \geq 1 \) with \( r = 2m \); \( osp(2m-1, 2m) = B(m-1, m) \) \( m \geq 1 \) with \( r = 2m-1 \); \( osp(2m, 2m) = D(m, m) \) \( m \geq 2 \) with \( r = 2m \); \( D(2, 1; \alpha) \alpha \in \mathbb{R} \neq 0, -1 \) with \( r = 3 \).

The alternative is that \( r \) is strictly less than \( n \). In this case there is at least one linear relation (1.7) obeyed by the \( \alpha_j \)'s which is incompatible with the existence of a superconformal symmetry (1.10). Furthermore, the best we can do by making constant shifts in the superfields \( \Phi \) is to fix \( r \) ratios of the \( \mu_j \)'s so that some mass parameter always remains. A list of the superalgebras of this type can be found from the work of [13,5]: \( sl(m, m) \) \( m \geq 2 \) with \( r = 2m-2 \); \( sl(m, m)^{(1)} \) \( m \geq 2 \) with \( r = 2m-2 \); \( sl(2m, 2m)^{(2)} \) \( m \geq 1 \) with \( r = 2m \); \( sl(2m+1, 2m+1)^{(4)} \) \( m \geq 1 \) with \( r = 2m \); \( sl(2m, 2m-1)^{(2)} = A(2m-1, 2m-2)^{(2)} \) \( m \geq 1 \) with \( r = 2m-1 \); \( osp(2m+1, 2m)^{(1)} = B(m, m)^{(1)} \) \( m \geq 1 \) with \( r = 2m \); \( osp(2m, 2m-2)^{(1)} = D(m, m-1)^{(1)} \) \( m \geq 2 \) with \( r = 2m-1 \); \( osp(2m, 2m)^{(2)} = D(m, m)^{(2)} \) \( m \geq 2 \) with \( r = 2m-1 \); \( D(2, 1; \alpha)^{(1)} \) with \( r = 3 \).

Except for the first two families, all entries in the last list are infinite-dimensional Kac-Moody superalgebras with \( n = r+1 \). Each is constructed from the given finite-dimensional superalgebra by means of an outer automorphism of the indicated order. In all the corresponding Toda models one can shift \( \Phi \) so as to make \( \mu_j = \mu \xi_j \) where \( \mu \) is a residual mass scale and \( \xi_j \) is the unique null eigenvector of the Cartan matrix. The classical potential of the model then has a minimum at \( \Phi = 0 \) and one can see directly that the theory is massive. These cases are therefore exactly analogous to the bosonic affine theories, but when we consider instead the first two families on the list some unfamiliar features emerge.
The superalgebra $sl(m,m)$ is finite-dimensional, but despite this its Cartan matrix has a single null eigenvector, implying $n = r+1$. The reason is that this algebra has a one-dimensional centre (which in its defining representation is generated by the identity matrix). As we shall see later, the resulting Toda theory has no classical minimum to its potential so that it is not a massive theory. Neither is it superconformally invariant, however, because we have already explained that this relies on the $\alpha_j$’s being linearly independent. We shall reconcile these apparently conflicting facts later. Precisely because the Cartan matrix of $sl(m,m)$ already has one null eigenvector, the Cartan matrix of its untwisted affine extension $sl(m,m)^{(1)}$ has two null eigenvectors. This algebra is therefore unique in having $n = r+2$ which will prove important when we search for $N=2$ supersymmetric models. We shall see below that the resulting Toda theory is massive and has none of the unusual features of the previous case.

2. $N=2$ supersymmetric Toda theories

We now derive conditions for the general $N=1$ theory (1.9) to possess $N=2$ supersymmetry, generalizing the treatment of the conformal case given in [6,7]. We look for a second supersymmetry transformation of the field $\Phi$ defined by

$$Q'_\pm \Phi = JD_\pm \Phi, \quad (2.1)$$

where $J$ is some matrix which must be compatible with the chosen reality properties of $\Phi$. This Ansatz ensures that the new supersymmetry and the original supersymmetry anti-commute. If $Q'_\pm$ are to obey the same algebra (1.4) as $Q_\pm$ we must have

$$J^2 = -1. \quad (2.2)$$

For there to be no change in the action, which is the superspace integral of (1.9), it is clear that the kinetic and potential terms must be separately invariant under this transformation. The variation of the kinetic terms is a total superspace derivative if and only if $J$ is antisymmetric with respect to the inner-product:

$$\Phi \cdot (J\Phi') = -(J\Phi) \cdot \Phi'. \quad (2.3)$$

Given this, the potential terms will also vary into superspace derivatives if and only if each $\alpha_j$ is an eigenvector of $J$:

$$J\alpha_j = \lambda_j \alpha_j. \quad (2.4)$$

The compatibility of $J$ with the reality properties of $\Phi$ now amounts to some set of compatibility conditions for the eigenvalues $\lambda_j$.

The equations (2.2), (2.3) and (2.4) are necessary and sufficient conditions for the existence of a second supersymmetry in any $N=1$ Toda theory, whether massless or massive. On combining them we obtain the equivalent conditions

$$(\lambda_i + \lambda_j)a_{ij} = 0, \quad \lambda_j^2 = -1. \quad (2.5)$$
These are very restrictive: since $\lambda_j \neq 0$, a second supersymmetry requires that all the diagonal entries of the Cartan matrix must vanish and from the lists of $N=1$ theories given above one finds that the only algebras which meet this criterion are as follows.

(1) $sl(m, m-1)$ ($m \geq 2$) with $n = r = 2m-2$ and Cartan matrix

$$a = \begin{pmatrix}
0 & 1 & 1 & 1 & \cdots & 1 \\
1 & 0 & -1 & 1 & \cdots & 1 \\
& -1 & 0 & -1 & \cdots & 1 \\
& & -1 & 0 & -1 & \cdots & 1 \\
& & & & & \ddots & \ddots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 \\
& & & & & & & & & & & & & & & & & & & & & & & & & & & 1 \end{pmatrix},$$

(2.6)

(2) $sl(m, m)$ ($m \geq 2$) with $n = 2m-1$. The Cartan matrix is

$$a = \begin{pmatrix}
0 & 1 & 1 & 1 & \cdots & 1 \\
1 & 0 & -1 & 1 & \cdots & 1 \\
& -1 & 0 & -1 & \cdots & 1 \\
& & -1 & 0 & -1 & \cdots & 1 \\
& & & & & \ddots & \ddots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 \\
& & & & & & & & & & & & & & & & & & & & & & & & & & & 1 \end{pmatrix},$$

(2.7)

and it has a unique null eigenvector $(1, 0, 1, \ldots, 0, 1)$ so $r = 2m-2$.

(3) $sl(m, m)^{(1)}$ ($m \geq 2$) with $n = 2m$. The Cartan matrix is

$$a = \begin{pmatrix}
0 & -1 & 1 & 1 & \cdots & 1 \\
-1 & 0 & 1 & 1 & \cdots & 1 \\
1 & 0 & -1 & 1 & \cdots & 1 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 \\
& & & & & & & & & & & & & & & & & & & & & & & & & & & 1 \end{pmatrix},$$

(2.8)

and it has two linearly independent null eigenvectors $(1, 0, 1, \ldots, 1, 0)$ and $(0, 1, 0, \ldots, 0, 1)$ so that $r = 2m-2$.

In each of these cases there is a unique solution to the conditions (2.5) given by

$$\lambda_j = \pm i (-1)^j.$$  

(2.9)

To complete the analysis, we must now determine to what extent this solution is compatible with the various reality properties we may wish to choose for $\Phi$.

If, for any of the above algebras, the superfield $\Phi$ is real, then the solution (2.9) is clearly inconsistent with (2.1) and (2.4) and so there is no second supersymmetry in these cases. It is true that with the choice $\lambda_j = (-1)^j$, which is consistent with $\Phi$ real, then (2.1) and (2.4) would define a new fermionic invariance of the theory. But this invariance would
not be a bona fide supersymmetry because it would not obey the characteristic algebra (1.4). (Perhaps such an invariance has interesting consequences, but we shall not pursue this idea here.) Another possibility is that we take $\Phi$ to be a complex-valued superfield, and then (2.9) is clearly consistent. Each of the complex Toda theories based on the above algebras is therefore $N=2$ supersymmetric. The last possibility, which in a number of respects turns out to be the most interesting, is that $\Phi$ obeys some condition of the form (1.5) for a non-trivial symmetry $\sigma$. To determine whether such a choice is possible and whether (2.9) is then compatible with it, we will have to examine the Toda models based on each of these algebras in more detail.

The case of $\text{sl}(m,m-1)$ has been dealt with previously in [7] but we summarize it here for completeness. There is a non-trivial reality condition

$$\alpha_j \cdot \Phi^* = \alpha_{\sigma(j)} \cdot \Phi, \quad \sigma(j) = 2m - 1 - j, \quad j = 1, \ldots, 2m-2$$

where $\sigma$ is clearly a symmetry of the Cartan matrix (2.6) and (1.6) is satisfied because we have chosen $\mu_j = 1$. It is easy to see that (2.9) is compatible with this reality condition so that there is indeed an additional supersymmetry. The resulting theory is actually $N=2$ superconformally invariant and one can deduce the value of the (quantum) central charge of the Virasoro algebra [6] to be $c = 3(m-1) \left(1 + \frac{m^2}{\beta^2}\right)$, where $\beta$ is the coupling constant. In particular the case $m=2$ is the $N=2$ super-Liouville theory and with the choice $\beta^2 = -(k+2)$ for $k = 0, 1, 2, \ldots$ one recovers the $N=2$ unitary discrete series of allowed central charges [14]. For $m>2$ we expect similar ranges of values of the central charge to correspond to the unitary discrete series for certain $N=2$ super $W$-algebras.

It is instructive to consider the purely bosonic sectors of these models, which can be calculated by reducing the superspace Lagrangian to components and eliminating the auxiliary fields. On general grounds one expects the result to be a number of decoupled bosonic Toda theories and this is precisely what appears. For $m=2$, the $N=2$ super-Liouville theory, the bosonic part consists of the usual bosonic Liouville theory together with an additional free scalar field. More generally the $\text{sl}(m,m-1)$ theory leads to a direct sum of bosonic conformal Toda theories based on $\text{sl}(m)$ and $\text{sl}(m-1)$ together with one free boson. The twisted reality conditions necessary for $N=2$ supersymmetry imply similar twisted reality conditions on the bosonic sub-theories.

Turning next to the algebra $\text{sl}(m,m)$, we find that no non-trivial condition of the form (1.5) is allowed, because there is no non-trivial symmetry of the Cartan matrix (2.7). For this algebra then, it is only the complex Toda theory which is $N=2$ supersymmetric. We have alluded previously to the fact that any Toda theory based on $\text{sl}(m,m)$ has some rather bizarre properties; now that we have written down the Cartan matrix explicitly it is appropriate to elaborate on these points. Many of them can be traced to the particular form of the unique null eigenvector $\xi_j$ following (2.7).

First, we see that we have a linear relation $\sum_{j=1}^{m} \alpha_{2j-1} = 0$ which is explicitly incompatible with the existence of a vector $\rho$ obeying $\rho \cdot \alpha_j = \frac{1}{2}$. Hence, as stated earlier, it is definitely not possible to define superconformal transformations (1.10) which are symmetries of such a theory. On the other hand, the theory is not massive, because the following argument shows that there is no classical minimum to the potential. If there were a minimum, we could certainly shift $\Phi$ so that it occurred at $\Phi = 0$. But since all the even
entries of the null eigenvector $\xi_j$ vanish, it is impossible to shift the fields so as to make $\mu_j = \mu \xi_j$ for some $\mu$, because such shifts can only rescale the $\mu_j$. The linear term in the expansion of the potential about $\Phi = 0$ can therefore never vanish and so there can never be a minimum at this point.

This puzzling situation can be clarified to some extent by examining the component content of the simplest example: the theory based on $sl(2,2)$ with two independent real superfields. On eliminating auxiliary fields the resulting Lagrangian can be written in terms of two real bosons $\phi_1, \phi_2$ and two real fermions $\psi_{1\pm}, \psi_{2\pm}$ in the form

$$L = L_1 - L_2 + i\psi_2+ \partial_- \psi_{1+} - i\psi_2- \partial_+ \psi_{1-} - i\psi_{1+} \psi_{1-} \exp \frac{\beta}{2} (\phi_1 + \phi_2) - 2i\psi_2+ \psi_{2-} \cosh \frac{\beta}{2} (\phi_1 - \phi_2).$$

(2.11)

Here $L_j$ denotes the Liouville Lagrangian for the bosonic field $\phi_j$ with coupling constant $\beta$ so that the bosonic sector of the model is conformally invariant in the standard way. By inspection, this conformal symmetry can be uniquely extended to the whole action, but only if the fermions transform with non-standard integer conformal weights:

$$\psi_{1\pm} \rightarrow (0,0), \quad \psi_{2+} \rightarrow (1,0), \quad \psi_{2-} \rightarrow (0,1).$$

(2.12)

Such an assignment clearly implies that the supercurrent for the $\mathcal{N}=1$ global supersymmetry of the model cannot have the conformal dimensions required to extend this non-standard conformal symmetry to a superconformal symmetry. Thus we are able to reconcile the co-existence of a global supersymmetry and a version of conformal invariance with the absence of superconformal invariance. This is also consistent with the fact that one can eliminate all mass parameters from the theory, as we have done in (2.11), by redefinitions of the various component fields, whereas this cannot be achieved by constant shifts in the original superfields, as we remarked earlier.

Although we have considered just the simplest case, similar remarks should also apply to more complicated models. Thus for the complex Toda theory based on $sl(m,m)$ we expect that the supercurrents for the $\mathcal{N}=2$ global supersymmetry will have exotic conformal weights with respect to a similar non-standard conformal symmetry. The precise rôle played by this non-standard conformal symmetry is rather obscure at present. Nevertheless, there are some intriguing features associated with it, among which is the fact that the spin assignments for the fermions are those of a ghost system such as might be encountered in a topological theory. Perhaps these unusual features would repay a more thorough study.

Finally we come to a detailed examination of the Toda theories based on $sl(m,m)^{(1)}$. Starting from the general action (1.9), the existence of two null eigenvectors of the Cartan matrix (2.8) means that there are two independent mass parameters in the theory which cannot be absorbed away by shifting $\Phi$. More precisely, the particular form of the null eigenvectors means that we can set $\mu_{2j-1} = \mu$ and $\mu_{2j} = \mu'$, say. The Lagrangian then reads

$$L = \frac{i}{2} D_+ \Phi \cdot D_- \Phi - \frac{\mu}{\beta^2} \sum_{j=0}^{m-1} \exp \beta \alpha_{2j+1} \cdot \Phi - \frac{\mu'}{\beta^2} \sum_{j=0}^{m-1} \exp \beta \alpha_{2j} \cdot \Phi,$$

(2.13)
and it is clear that the classical minimum occurs at $\Phi = 0$. The resulting theory is massive, and it displays none of the peculiarities exhibited by the algebraically related model based on $sl(m, m)$.

Despite the occurrence of two parameters $\mu$ and $\mu'$ with the dimensions of mass, the classical masses of the elementary excitations of the theory depend just on the product $\mu \mu'$ and the ratio $\mu / \mu'$ appears only in the higher order couplings between bosons and fermions. It is therefore more accurate to describe this model as having a single mass scale together with an extra dimensionless coupling constant $\mu / \mu'$ in addition to $\beta$. We should stress also that the masses mentioned above are always real and positive even though the kinetic energy contains terms of negative sign in general (because of the indefinite nature of the superalgebra inner-product).

We must now consider the allowed reality choices for $\Phi$ in this case. There is certainly a non-trivial symmetry of (2.8) which we can use to write down a reality condition

$$\alpha_j \cdot \Phi^* = \alpha_{\sigma(j)} \cdot \Phi, \quad \sigma(j) = 2m - 1 - j, \quad j = 0, \ldots, 2m-1 \quad (2.14)$$

(Note the range of labelling chosen here and in (2.13) which will turn out to be convenient later.) But this is consistent with the Toda equations only if we set $\mu' = \mu$ so as to fulfill (1.6), thereby fixing the value of the additional dimensionless parameter discussed above. Having done so, it is easy to see that (2.14) is also compatible with (2.9) so that we have found another class of $N=2$ supersymmetric theories.

For $m=2$ the above construction gives the $N=2$ super sine-Gordon model which contains as its bosonic limit decoupled copies of both the sine- and sinh-Gordon theories. The fact that this really is the natural generalization of the conventional bosonic sine-Gordon theory is best appreciated in the language of $N=2$ superspace which we shall introduce in the next section. Some previous work linking integrability of this model with the algebra $sl(2,2)^{(1)}$ can be found in [15] and a recent discussion of its $S$-matrix is given in [16].

For $m>2$ we have found the natural multi-component integrable generalizations of the $N=2$ sine-Gordon theory. In the general case the bosonic limit consists of a direct sum of two bosonic Toda theories, each based on $sl(m)^{(1)}$. The twisted reality conditions on the superfields descend to these bosonic sub-theories in a way which is not entirely trivial. Nevertheless, the mass spectrum of these bosonic theories is independent of the reality condition used [7] and this fact provides a simple way of seeing that the mass spectrum of the superalgebra model is real and positive despite the possibility of indefinite kinetic energy. Details of how the reality conditions descend will be given elsewhere.

3. $N=2$ Superspace

We shall now take the extended supersymmetric theories constructed from $sl(m, m-1)$ and $sl(m, m)^{(1)}$ by using the reality conditions (2.10) and (2.14) and re-formulate them in $N=2$ superspace. In standard fashion, we can introduce an $N=2$ superspace by taking complex fermionic coordinates $\theta^\pm$ and their conjugates $\bar{\theta}^\pm$; the $N=1$ superspace we have been considering thus far is contained as the subspace with fermionic coordinates $\theta^\pm = \bar{\theta}^\pm$. 8
The $N=2$ superspace derivatives are

\[
D_\pm = \frac{\partial}{\partial \theta^\pm} - \frac{i}{2} \bar{\theta}^\pm \partial_{\theta^\pm}, \quad \overline{D}_\pm = \frac{\partial}{\partial \bar{\theta}^\pm} - \frac{i}{2} \theta^\pm \partial_{\bar{\theta}^\pm} \Rightarrow \{D_\pm, \overline{D}_\pm\} = -i \partial_{\theta^\pm} \tag{3.1}
\]

with all other brackets vanishing. A chiral $N=2$ superfield $\Psi$ obeys $\overline{D}_\pm \Psi = 0$ and its complex conjugate $\Psi^*$ is antichiral obeying $D_\pm \Psi^* = 0$. The generic form of an $N=2$ superspace action is

\[
S = \int d^2 x d^2 \theta d^2 \bar{\theta} K(\Psi_j^*, \Psi_j) - \int d^2 x d^2 \theta W(\Psi_j) - \int d^2 x d^2 \bar{\theta} W^*(\Psi_j^*), \tag{3.2}
\]

where $K$ is a hermitian form and $W$ is a holomorphic superpotential. Notice in particular how the interaction terms occur in complex conjugate pairs to make the action as a whole real.

The theories of interest to us have been defined in terms of $2(m-1)$ independent, complex $N=1$ superfields obeying (2.10) and (2.14). But there is an obvious way to identify a complex $N=1$ superfield with a chiral $N=2$ superfield, component by component. In the cases at hand it is convenient to introduce $m-1$ chiral $N=2$ superfields $\Psi_j$ by means of the identifications

\[
\Psi_j \leftrightarrow \alpha_j \cdot \Phi \quad j \text{ odd}, \quad \Psi_j \leftrightarrow \alpha_j \cdot \Phi^* \quad j \text{ even}, \quad j = 1, \ldots, m-1 \tag{3.3}
\]

which is consistent with (2.10) and (2.14). In both the conformal and massive cases the kinetic terms can now be written in $N=2$ language by choosing

\[
K = \sum_{i, j=1}^{m-1} \Psi_i^* k_{ij}^{-1} \Psi_j, \quad k = \begin{pmatrix}
0 & 1 & 0 & -1 \\
1 & 0 & -1 & 0 \\
0 & 0 & 0 & (1)^{m-1} \\
0 & (1)^{m-1} & (1)^{m-1} & (1)^{m}
\end{pmatrix} \tag{3.4}
\]

For the conformal theory based on $sl(m, m-1)$ the appropriate superpotential is

\[
W = \frac{1}{\beta^2} \sum_{j=1}^{m-1} \exp \beta \Psi_j, \tag{3.5}
\]

while for the massive theory based on $sl(m, m)^{(1)}$ it is

\[
W = \frac{\mu}{\beta^2} \left\{ \sum_{j=1}^{m-1} \exp \beta \Psi_j + \exp \left( -\beta \sum_{j=1}^{m-1} \Psi_j \right) \right\}. \tag{3.6}
\]

Notice that the rank $r$ is even for both algebras, corresponding to the hermitian structure of the kinetic terms. It is important for the affine theory that $n$ is also even, because the
general form of an $N=2$ superspace action requires that the exponential interaction terms occur in complex conjugate pairs. The unique property of the family $sl(m,m)^{(1)}$ in having $n = r+2$ is therefore intimately related to the extended supersymmetry present in this Toda theory.

The extended superspace formalism makes transparent why the models based on $sl(2,1)$ and $sl(2,2)^{(1)}$ are the natural $N=2$ extensions of the Liouville and sinh-Gordon theories. They can be formulated using a single chiral $N=2$ superfield and superpotentials

$$ W = \frac{1}{\beta^2} \exp \beta \Psi \quad \text{and} \quad W = \frac{2\mu}{\beta^2} \cosh \beta \Psi, $$

respectively, which are obvious generalizations of the bosonic cases. Due to the algebraic complexities hidden in the superspace notation, however, the bosonic sectors of such theories are not immediately obvious. As we stated earlier, the $N=2$ Liouville theory reduces to the bosonic Liouville theory plus one free scalar, whilst the $N=2$ sine-Gordon reduces to decoupled copies of the sinh-Gordon and sine-Gordon theories, on setting the fermions to zero. We should also point out the $N=2$ theories corresponding to higher values of $m$ are not simple generalizations of bosonic Toda theories.

4. Integrable Deformations and $N=2$ Toda Theories

Toda theories provide a framework for discussing integrable deformations of certain conformal field theories by particular primary operators in the following way [3,17]. Let $L$ be the Lagrangian for a conformal Toda theory based on some CLSA $g$ with some choice of simple roots. Then the massive theory defined by

$$ L + \lambda \begin{cases} \exp(\beta \chi \cdot \Phi) & \chi \text{ fermionic} \\ i\theta^+ \theta^- \exp(\beta \chi \cdot \Phi) & \chi \text{ bosonic} \end{cases} \tag{4.1} $$

will still be integrable if the vector $\chi$ appearing in the perturbing operator extends the simple root system for $g$ to a simple root system for some affine CLSA. In the simplest case $\chi$ is the negative of the highest root for $g$ which corresponds to the additional root of the untwisted affine CLSA $g^{(1)}$, but more generally the larger root system may be that of some twisted affine CLSA. Notice that the deformation preserves or breaks $N=1$ supersymmetry according to whether $\chi$ is graded fermionic or bosonic respectively.

It is natural to ask whether one can find Toda theories describing integrable perturbations which preserve $N=2$ supersymmetry. For the $N=2$ superconformal Toda model based on $sl(m,m-1)$, the simplest deformation corresponding to $sl(m,m-1)^{(1)}$ was considered in [6]. Starting from the basis of purely fermionic simple roots, it was found that $\chi$ is always bosonic so that the deformation breaks even the $N=1$ supersymmetry of the original theory. (This analysis assumed standard reality conditions for $\Phi$ but it carries over immediately to the case of twisted reality conditions relevant to the genuine $N=2$ theories considered here.) Now that we have analyzed the construction of $N=2$ massive theories more thoroughly, however, we can address the issue of $N=2$ deformations more systematically.
Our work above immediately suggests that we might try to interpret the massive $N=2$ theories based on $sl(m, m)^{(1)}$ as $N=2$ supersymmetric integrable deformations of the conformal theories based on $sl(m, m-1)$. These models have the same numbers of fields (because the ranks of the algebras are equal for fixed $m$) and the latter theories can be ‘embedded’ in the former in a natural way, as is evident from the Cartan matrices (2.6) and (2.8). We are thus led to consider a deforming operator in the $sl(m, m-1)$ model consisting of a sum of two exponentials

$$\lambda(\exp \beta \alpha_0 \cdot \Phi + \exp \beta \alpha_{2m-1} \cdot \Phi), \quad \alpha_0 = -\sum_{j=1}^{m-1} \alpha_{2j}, \quad \alpha_{2m-1} = -\sum_{j=1}^{m-1} \alpha_{2j-1}. \quad (4.2)$$

(the ratio of the coefficients is fixed by the condition of $N=2$ supersymmetry which required $\mu = \mu'$ in (2.13).) Unfortunately, a calculation reveals that the holomorphic and antiholomorphic dimensions of this perturbing operator are $\frac{1}{2}(1-m)$, which is always negative (as was noted for the Sine-Gordon case in [16]). This strongly suggests that such a simple interpretation purely at the level of Lagrangians is too naive, and so we shall not advocate it here. This is to be contrasted with the situation in the Landau-Ginzburg formalism where deformations can be considered directly at the level of the Lagrangian [18]. A more satisfactory route to clarifying the relationships between the $N=2$ Toda models considered here would be to investigate the ultra-violet limit of the quantum $S$-matrix of the affine theories (for recent relevant work see [19]).

5. Solitons in massive N=2 Toda theories

Both the complex and twisted (2.14) families of massive $N=2$ theories based on $sl(m, m)^{(1)}$ admit supersymmetric soliton solutions. Recent work [8,9] on solitons in bosonic Toda models, however, suggests that the complex theories may be the most natural setting in which to study such solutions, so we will concentrate on these here. In the bosonic case, solitons of the complex $sl(m)^{(1)}$ Toda equations have been constructed in [8]. (These theories are to be thought of as natural generalizations of the sine-Gordon model, whereas the conventional real Toda theories generalize the sinh-Gordon model.) It turns out that the classical masses of these solitons are real, in spite of the fact that the Hamiltonian is not hermitian. A form for the soliton $S$-matrix has also been proposed. This is is generically non-unitary, as expected, although it seems that for some particular region for the coupling constant $\beta$, unitarity is restored [9].

Consider the complex $N=2$ theory based on the CLSA $sl(m, m)^{(1)}$ and define for $j = 1, 2, \ldots, m$

$$\alpha_{2j-1} \cdot \Phi = \Phi_j^{(1)} - i\Phi_j^{(2)}, \quad \alpha_{2j-2} \cdot \Phi = i\Phi_{j-1}^{(2)} - \Phi_j^{(1)}, \quad (5.1)$$

with $\Phi_j^{(1,2)} \equiv \Phi_{j+m}^{(1,2)}$. In looking for classical solutions to such a field theory it is customary to set all fermions to zero and to consider the resulting purely bosonic equations. With the definitions above, this implies that $\phi^{(1)}$ satisfies a bosonic $sl(m)^{(1)}$ Toda equation with
coupling constant $\beta$, whereas $\phi^{(2)}$ satisfies a complex $sl(m)^{(1)}$ Toda equation with coupling constant $i\beta$:
\[
\partial_+ \partial_- \phi^{(2)}_j + \frac{\mu^2}{i\beta} \left( e^{i\beta(\phi^{(2)}_j - \phi^{(2)}_{j+1})} - e^{i\beta(\phi^{(2)}_{j-1} - \phi^{(2)}_j)} \right) = 0. \tag{5.2}
\]

We can now use the results of [8] directly to construct multi-soliton solutions. For example, the one soliton solutions are
\[
\phi^{(2)}_j(x, t) = -\frac{1}{i\beta} \log \left( \frac{1 + e^{\rho(x-\nu t) + \frac{2\pi i a}{m} j + \xi}}{1 + e^{\rho(x-\nu t) + \frac{2\pi i a}{m} (j-1) + \xi}} \right), \quad \phi^{(1)}_j = 0. \tag{5.3}
\]

where $\rho$, $\xi$ and $\nu$ are constants satisfying $\rho^2(1 - \nu^2) = 4\mu^2 \sin^2(\pi a/m)$ and $a = 1, 2, \ldots, m-1$. The topological charge of the soliton (5.3) is a weight of the $a$th fundamental representation of $sl(m)$ and the classical mass of the soliton is equal to
\[
M_a = \frac{4\mu m}{\beta^2} \sin \left( \frac{\pi a}{m} \right). \tag{5.4}
\]

We now show that these one soliton solutions are ‘supersymmetric’ in the sense that they are invariant under a particular supersymmetry transformation. Working in $N=1$ superspace, with the second supercharge being $Q'_\pm = J\phi_\pm$, as before, we consider the variation of the soliton solution under the transformation generated by $\epsilon^+ Q_+ + \epsilon^- Q_- + \epsilon'^+ Q'_+ + \epsilon'^- Q'_-$. The change in a general bosonic field configuration $\phi$ is automatically zero because the fermion fields vanish, so the only non-trivial equations result from requiring that the variation of the fermions should also vanish. This gives
\[
\delta \psi_+ = -\epsilon^+ \partial_+ \phi + \epsilon'^+ J\partial_+ \phi + \epsilon^- F + \epsilon'^- JF, \\
\delta \psi_- = -\epsilon^- \partial_- \phi + \epsilon'^- \partial_- \phi - \epsilon^+ F - \epsilon'^+ JF, \tag{5.5}
\]

where the auxiliary field takes its on-shell value:
\[
F = -\frac{\mu}{\beta} \sum_j \alpha_j \exp \beta \alpha_j \cdot \phi. \tag{5.6}
\]

One finds that these variations vanish for the restricted transformation $\epsilon^- = \lambda \epsilon^+$, $\epsilon'^- = \sigma \epsilon^+$ and $\epsilon'^+ = \lambda \sigma \epsilon^+$, on taking $\phi$ as given in (5.3), because it then satisfies the first order equation
\[
-(1 - \sigma J) \partial_+ \phi + \lambda(1 + \sigma J) F = 0, \tag{5.7}
\]
with
\[
\lambda = \sqrt{\frac{1 - \nu}{1 + \nu}}, \quad \sigma = \tan \left( \frac{\pi a}{2m} + \frac{\pi}{4} \right). \tag{5.8}
\]

There exists a formula relating the mass of such a supersymmetric soliton to its topological charge which can be derived from a careful treatment of the non-trivial central
terms in the superalgebra induced by such topological configurations \([10,20]\). In the present context one finds that this formula for the mass is

\[
M = \frac{\mu}{\beta^2} \int dx \frac{\partial}{\partial x} \left( \frac{1 + i\sigma}{1 - i\sigma} \sum_{j \text{ even}} e^{\beta \alpha_j \phi} + \frac{1 - i\sigma}{1 + i\sigma} \sum_{j \text{ odd}} e^{\beta \alpha_j \phi} \right),
\]

where \(\sigma\) is the parameter appearing in the supersymmetry transformation (5.8). It is straightforward to verify explicitly that with \(\phi\) given by (5.3) the masses (5.4) are correctly reproduced by (5.9). We therefore obtain a new way of seeing that these soliton masses must be real, which follows because the two terms in (5.9) are complex conjugates. We intend to present the above calculations in greater detail elsewhere, together with a more complete discussion of some related issues.

Although it is is somewhat premature to speculate on the possibilities for defining a consistent \(N=2\) massive quantum theory associated to these classical equations, experience with the bosonic theories is encouraging in that an exact \(S\)-matrix for the soliton solutions has been postulated [9]. In the present case with \(m=2\), giving the \(N=2\) super sine-Gordon theory, the \(S\)-matrix is thought to be related to the \(S\)-matrix of the bosonic sine-Gordon theory by a very simple way [16,19]

\[
S_{SG}^{N=2}(\beta_{N=2}; \theta) = S_{SG}^{N=0}(\beta_{N=0}; \theta) \otimes S_{N=2}[m=2](\theta),
\]

where \(S_{SG}\) is the \(S\)-matrix of the sine-Gordon theories of the indicated supersymmetry, and \(S_{N=2}[m](\theta)\) is one of the ‘minimal’ \(N=2\) supersymmetric \(S\)-matrices constructed in [19]. The coupling constant \(\beta_{N=2}\) turns out to be the bare coupling of the bosonic sine-Gordon theory. The fact that the final form is a product of a bosonic \(S\)-matrix and a fermionic \(S\)-matrix seems to be a characteristic feature of these types of theory [20]. It seems plausible that the \(m > 2\) theories would yield a similar structure with the bosonic \(S\)-matrix factor being replaced by the \(sl(m)\) soliton \(S\)-matrix of [9] (but without the factor \(S_{\text{min}}\), since in this case \(S_{N=2}[m]\) would provide the necessary pole structure). It would clearly be interesting to investigate the spectrum of states that follows from such an Ansatz, and also to consider the question of unitarity along the lines of [9]. A proposal for the theory giving this \(S\)-matrix is made in [19], however, it is not the same as our proposal, and its integrability is questionable given the cautionary remarks we made in the introduction regarding the construction of integrable supersymmetric theories.

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Note added: after submitting this paper for publication we became aware of ref. [21] (we thank M. Grisaru for bringing it to our attention). These authors consider field theories based on the algebras \(sl(m,m)^{(1)}\) and compute one-loop corrections to particle masses, although they do not discuss \(N=2\) supersymmetry. They also consider theories related to \(sl(m,m)\) and claim to find a quantum superconformal symmetry, at variance with our results. Clearly these points deserve more detailed investigations. Some additional references on super Toda theories can be found in [22,23,24].
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