Distributed Optimization with Uncertain Communications

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Abstract

We consider the distributed optimization problem for the sum of convex functions where the underlying communications network connecting agents at each time is drawn at random from a collection of directed graphs. We propose a modified version of the subgradient-push algorithm that is provably almost surely convergent to an optimizer on any such sequence of random directed graphs. We also find an upper bound of the order of $\sim O(1/\sqrt{t})$ on the convergence rate of our proposed algorithm, establishing the first convergence bound in such random settings.

1 Introduction

Distributed optimization of a sum of convex functions is concerned with solving the following problem: a network of nodes $\mathcal{V} = \{v_1, v_2, ..., v_n\}$, each having access to a private local convex function $f_i : \mathbb{R}^d \rightarrow \mathbb{R}$, aim to solve the minimization problem

$$\minimize F(z) \triangleq \sum_{i=1}^{n} f_i(z), \quad z \in \mathbb{R}^d,$$

in a distributed manner, i.e., only by exchanging limited information on their estimate of the optimizer. To be more precise, assume that there is

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a communication layer between the nodes, which for now the reader can assume to be fixed over time, and at each time $t \geq 0$, nodes can exchange information about their state, which is their local estimate of a solution of (1). We are seeking for an algorithm where each node utilizes the private local convex function along with the estimates received from its neighbours to solve (1). The importance of the problem stems from its applications in a variety of contexts, ranging from sensor localization [2] and distributed electricity generation and smart grids [3, 4] to statistical learning [5].

Distributed optimization is by now a mature subject, with a large volume of literature devoted to it. The main focus of this work is on conditions imposed on the communication network, particularly in scenarios where the communication graph is drawn at random, in order to insure convergence to a solution of (1); clearly, practical implementation of distributed optimization algorithms relies on guaranteed performance in scenarios with deficiencies or uncertainties in communications. The importance of this issue has been pointed out in [6]. With this in mind, we use the focus on communication network as a guideline when navigating through the literature in Section 1.2, admittedly missing some important results related to other aspects.

In order to proceed in a precise manner, we start with some mathematical preliminaries; the reader can safely move to Section 1.2 and return to the preliminaries as needed.

1.1 Mathematical Preliminaries

Let $\mathbb{R}$ denote the set of real numbers, and let $\mathbb{R}_{\geq 0}$ and $\mathbb{Z}_{\geq 0}$ denote the set of non-negative real numbers and integers, respectively. For a set $A$, we write $S \subset A$ if $S$ is a proper subset of $A$, and we call the empty set and $A$ trivial subsets of $A$. The complement of $S$ is denoted by $S^c$. Let $|S|$ denote the cardinality of a finite set $S$. We view all vectors in $\mathbb{R}^n$ as column vectors, where $n$ is a positive integer. We denote by $\| \cdot \|$ and $\| \cdot \|_1$, the standard Euclidean norm and the 1-norm on $\mathbb{R}^n$, respectively. The notation $A'$ and $v'$ will refer to the transpose of the matrix $A$ and the vector $v$, respectively. We use $\mathbb{R}^{n \times n}_{\geq 0}$ to denote the set of $n \times n$ non-negative real-valued matrices. A matrix $A \in \mathbb{R}^{n \times n}_{\geq 0}$ is column-stochastic if each of its columns sums to 1; row-stochastic matrices are defined similarly, and when both conditions are satisfied, we refer to $A$ as doubly stochastic. For a given $A \in \mathbb{R}^{n \times n}_{\geq 0}$ and any nontrivial $S \subset [n] \triangleq \{1, \ldots, n\}$, we define $A_{S^c} \triangleq \sum_{i \in S, j \in S^c} A_{ij}$. 


1.1.1 Graph theory

A (weighted) directed graph \( G \triangleq (\mathcal{V}, \mathcal{E}, W) \) consists of a node set \( \mathcal{V} \triangleq \{v_1, v_2, \ldots, v_n\} \), an edge set \( \mathcal{E} \subseteq \mathcal{V} \times \mathcal{V} \), and a weighted adjacency matrix \( W \in \mathbb{R}_{\geq 0}^{n \times n} \), with \( W_{ji} > 0 \) if and only if \((v_i, v_j) \in \mathcal{E}\), in which case we say that \( v_i \) is connected to \( v_j \). Similarly, given a matrix \( W \in \mathbb{R}^{n \times n}_{\geq 0} \), one can associate to \( W \) a directed graph \( G = (\mathcal{V}, \mathcal{E}) \), where \((v_i, v_j) \in \mathcal{E}\) if and only if \( W_{ji} > 0 \), and hence \( W \) is the corresponding weighted adjacency matrix for \( G \). The in-neighbors and the out-neighbors of \( v_i \) are the set of nodes \( N_{i}^{\text{in}} = \{j \in [n] : W_{ij} > 0\} \) and \( N_{i}^{\text{out}} = \{j \in [n] : W_{ji} > 0\} \), respectively. The out-degree of \( v_i \) is \( d_i = |N_{i}^{\text{out}}| \); we simply drop the in and out indices when the graph is undirected. In the directed graph \( G = (\mathcal{V}, \mathcal{E}, W) \), a path is sequence of distinct nodes \( v_{i_1}, \ldots, v_{i_k} \) for some \( k \in [n] \) such that \((v_{i_j}, v_{i_{j+1}}) \in \mathcal{E}\) for all \( j \in [k-1] \). A directed graph is strongly connected if there is a path between any pair of nodes. If the directed graph \( G = (\mathcal{V}, \mathcal{E}, W) \) is strongly connected, we say that \( W \) is irreducible. For graphs \( G_1 = (\mathcal{V}, \mathcal{E}_1) \) and \( G_2 = (\mathcal{V}, \mathcal{E}_2) \) on the node set \( \mathcal{V} \), \( G = G_1 \cup G_2 \) is the graph on the node set \( \mathcal{V} \) with the edge set \( \mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2 \).

1.1.2 Sequences of random column-stochastic matrices

Let \( \mathcal{S}_n \) be the set of \( n \times n \) column-stochastic matrices, and let \( \mathcal{F}_{\mathcal{S}_n} \) denote the Borel \( \sigma \)-algebra on \( \mathcal{S}_n \). Given a probability space \((\Omega, \mathcal{B}, \mu)\), a measurable function \( W : (\Omega, \mathcal{B}, \mu) \rightarrow (\mathcal{S}_n, \mathcal{F}_{\mathcal{S}_n}) \) is called a random column-stochastic matrix, and a sequence \( \{W(t)\} \) of such measurable functions on \((\Omega, \mathcal{B}, \mu)\) is called a random column-stochastic matrix sequence; throughout, we assume that \( t \in \mathbb{Z}_{\geq 0} \). Note that for any \( \omega \in \Omega \), one can associate a sequence of directed graphs \( \{G(t)(\omega)\} \) to \( \{W(t)(\omega)\} \), where \((v_i, v_j) \in \mathcal{E}(t)(\omega)\) if and only if \( W_{ji}(t)(\omega) > 0 \). This in turn defines a sequence of random directed graphs on \( \mathcal{V} = \{v_1, \ldots, v_n\} \), which we denote by \( \{G(t)\} \).

1.2 Consensus-based optimization

The main idea behind most of the algorithms for solving (1), which goes back to [7] and is fully developed in [8, 9], is to use a combination of consensus algorithm and subgradient flow. In this sense, it is natural to assume that the agents step in the direction given by the subgradient of their own local cost functions, while taking into account the states communicated by their neighbours through some averaging process. As shown in [8, 9], for the
setting where the graph is undirected, connected, and fixed over time, by

tuning the learning rate, it is possible to guarantee that the states of all agents converge to a solution of (1). More precisely, consider a connected undirected graph $G$ on $n \geq 1$ nodes, with the adjacency matrix $A$ whose entries are $a_{ij}$, with $a_{ii} > 0$, for all $i, j \in \{1, \ldots, n\}$. Suppose that each node initially has a state denoted by $x_i(0)$, which is an estimate of a solution to (1). Assume that $A$ is fixed over iterations of the algorithm, and at each time $(t+1) \geq 1$, node $v_i$ updates its variables according to

$$x_i(t+1) = \sum_{j \in N_i(t)} a_{ij} \frac{x_j(t)}{|N_i(t)|} - \alpha(t)g_i(t),$$

where $g_i(t)$ is a subgradient of the convex function $f_i$ at $x_i(t)$. It has been noted in [8] that the algorithm above is still applicable when the underlying graph is directed, but with doubly stochastic weighting. In fact, the proof can be extended to accommodate deterministic time-varying sequences of such graphs. Particularly relevant to our work is [10], where it is shown that in fact the analysis above can be extended to sequences of random graphs, as long as the weighting is doubly stochastic. The continuous-time algorithms introduced in [11] provably works on sequences of strongly connected weight-balanced directed graphs, however, as shown in [11] such algorithms do not extend beyond this class. The major issue is that consensus dynamics, i.e., (2) with functions set to zero, do not necessarily converge to the average of the initial state when the underlying graph is not weight-balanced; this is an essential ingredient in guaranteeing convergence to an optimizer.

1.3 Subgradient push-sum algorithm

To keep this document fully self-contained, we plan to provide a brief introduction to the so-called push-sum algorithm, which is designed to achieve average consensus on sequences of graphs that are not necessarily weight-balanced. We first wish to note that without the requirement of consensus on average, the convergence properties of the consensus dynamics on random graphs under very mild ergodicity conditions, in particular the so-called infinite-flow property, is by now a mature subject and has been extensively studied, see [12] and references within. In spite of this, the extension of such results to "average" consensus is far from obvious, because of the difficulties that arise from the nonlinear nature of the push-sum dynamics. The main idea of this algorithm is that agents carry a second state at each time, with initial values set to one, which adjust for the imbalances in graph; by
dividing the states of the agents at each time by this second state, one can guarantee convergence to the average of initial states for a fixed strongly connected directed graph.

Let us introduce the push-sum dynamics, introduced in [13], formally: consider a network of nodes $\mathcal{V} = \{v_1, v_2, \ldots, v_n\}$, where node $v_i \in \mathcal{V}$ has an initial state $x_i(0) \in \mathbb{R}$. The push-sum algorithm, proposed in [13], is defined as follows. Each node $v_i$ maintains and updates, at each time $t \geq 0$, two state variables $x_i(t)$ and $y_i(t)$. The first state variable is initialized to $x_i(0)$ and the second one is initialized to $y_i(0) = 1$, for all $i \in [n]$. At time $t \geq 0$, node $v_i$ sends $\frac{x_i(t)}{d_i(t)}$ and $\frac{y_i(t)}{d_i(t)}$ to its out-neighbors in the random directed graph $G(t) = (\mathcal{V}, \mathcal{E}(t), W(t))$, which we assume to contain self-loops at all nodes for all $t \geq 0$. At time $(t + 1)$, node $v_i$ updates its state variables according to

$$x_i(t + 1) = \sum_{j \in N_{in}(i)} \frac{x_j(t)}{d_j(t)},$$

$$y_i(t + 1) = \sum_{j \in N_{in}(i)} \frac{y_j(t)}{d_j(t)},$$

$$z_i(t + 1) = \frac{x_i(t + 1)}{y_i(t + 1)}.$$

Here, $z_i(t + 1)$ is the estimate by node $v_i$ of the average $\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i(0)$. As a by-product, in [14], it is shown that as long as the sequence of graphs satisfy some uniform strong connectivity over time, the state $z_i(t)$ is guaranteed to converge to $\bar{x}$ for all $i$; in fact, one can obtain convergence rates. In a recent work [15], we established an ergodicity criteria for the sequence of column-stochastic matrices corresponding to the push-sum protocol, and demonstrated that a large class of time-varying sequences of random directed graphs satisfy such conditions. Applying this result to a random sequence of graphs generated using a time-varying $B$-irreducible probability matrix, we obtained the first known convergence rates for the push-sum algorithm in random settings. It turns out that pairing the push-sum protocol (3) with subgradient flow leads to a distributed algorithm with guaranteed convergence to a solution of (1), when the directed graph is time-varying but satisfies some uniform strong connectivity. It is shown in [16, Remark 5] that in the particular case where the graph is undirected, the results are actually extendable to random settings. This being said, extending such results to general random directed graphs, as we demonstrate, is highly non-trivial; we discuss this further in our statement of contributions below.
Finally, even though this is beyond the scopes of this paper, we wish to refer the reader to the recent sequence of results \[16\], where geometric convergence rates are established for distributed optimization, and the very recent works \[17\], \[18\] where the important issue of dependency of such bounds with respect to the size of the network is studied.

1.4 Contributions of this work

The main focus of our work is on establishing provably convergent dynamics that achieve distributed optimization on time-varying sequences of random directed graphs, without making doubly stochastic assumptions; the problem is of paramount importance, and to best of our knowledge for most parts completely unresolved. For reasons that are too technical to summarize here but will be clarified later, our prior work in \[15\] cannot be immediately extended to distributed optimization setting. We overcome this issue by introducing a modified version of the subgradient-push algorithm which, under suitable connectivity assumptions, provably converges to a solution of the distributed optimization problem over sequences of random directed graphs, establishing the first result of this kind. The main idea is to introduce a lower bound for the updates, hence excluding the complicated issues that can arise due to the presence of division of two correlated random variables. We also provide convergence rates for the proposed algorithm.

2 Statement of the Problem

Consider the distributed optimization problem \[1\]. Suppose now that the communication layer between nodes at discrete time instances \(t \geq 0\) is specified by a sequence of random directed graphs \(\{G(t)\}\). Starting with some initial estimate of an optimal solution, at each time \(t\), each node communicates with its neighbors in \(G(t)\): sends its values to its out-neighbors and updates its values according to those of its in-neighbors. We assume that the set of optimal solutions is nonempty. One standing assumption throughout this paper is that each node knows its out-degree at every time, which is shown to be necessary in \[19\]. Our main objective is to show that a modified version of the so-called subgradient-push algorithm successfully accomplishes the task of minimizing \(F(z)\) in a distributed fashion, and under the assumption that the communication network is directed and random.
This key point distinguishes our work from the existing results in the literature [8, 10, 14, 20, 21].

3 Modified Subgradient-Push Algorithm

In the subgradient-push (SP) algorithm, each node \(v_i\) maintains and updates, at each time \(t \geq 0\), two vector variables \(x_i(t), w_i(t) \in \mathbb{R}^d\) as well as a scalar variable \(y_i(t) \in \mathbb{R}\). For each \(i \in [n]\), the vector \(x_i(0)\) is initialized to the estimate of an optimal solution of node \(v_i\) and \(y_i(0) = 1\). At time \(t \geq 0\), node \(v_i\) sends \(x_i(t)\) and \(y_i(t)\) to its out-neighbors in the directed graph of the available communication channels \(\overline{G}(t) = (V, \overline{E}(t))\), which is assumed to contain self-loops at all nodes. At time \((t + 1)\), node \(v_i\) updates its variables according to

\[
\begin{align*}
\mathbf{w}_i(t + 1) &= \sum_{j \in N_{\text{in}}^i(t)} \frac{x_j(t)}{d_j(t)}, \\
y_i(t + 1) &= \sum_{j \in N_{\text{in}}^i(t)} \frac{y_j(t)}{d_j(t)}, \\
z_i(t + 1) &= \frac{\mathbf{w}_i(t + 1)}{y_i(t + 1)}, \\
x_i(t + 1) &= \mathbf{w}_i(t + 1) - \alpha(t + 1)\mathbf{g}_i(t + 1),
\end{align*}
\]

where \(\mathbf{g}_i(t + 1)\) is a subgradient of the convex function \(f_i\) at \(z_i(t + 1)\) and \(\alpha(t) = \frac{1}{t^\gamma}\) for some \(0.5 < \gamma < 1\). For this choice if \(\alpha(t)\) we have

\[
\sum_{t=1}^{\infty} \alpha(t) = \infty, \quad \text{and} \quad \sum_{t=1}^{\infty} \alpha^2(t) < \infty.
\]

At each time \(t\), \(z_i(t)\) is the estimate by node \(v_i\) of a minimizer of \(F(z)\). For each \(i \in [n]\), \(N_{\text{in}}^i(t)\) is the set of in-neighbors of \(v_i\) and \(d_i(t)\) is its out-degree in \(\overline{G}(t)\). We assume that all functions \(f_i\) are Lipschitz continuous, i.e., for all \(i \in [n]\) there exists \(L_i\) such that \(\|\mathbf{g}_i\| \leq L_i\). For our future analysis we define \(L = \sum_{i=1}^{n} L_i\). In this paper, as in [14], we use boldface only for the vectors in \(\mathbb{R}^d\).

We are now in a position to motivate the algorithm proposed in this work. In order to do this, we need to at the least make a hint on what happens when one attempts to extend the convergence greatness of [14] to setting where the graph is selected at random. As shown in for example [10],
in undirected settings, or directed by with doubly stochastic weighting, the same algorithm can be used along sample paths and that leads to a convergence result in expectation. Now, one hopes to prove a similar result for (4) in random settings, with an important caveat that this algorithm heavily relies on division of two correlated random variables. More importantly, the proof of the convergence in deterministic settings heavily relies on bounding the denominator away from zero. This issue makes it not possible to carry the analysis of [15] to random setting. The main novelty of this work is to modify this algorithm, while still using its useful features; this is what we describe next.

3.1 The Modified Subgradient-Push Algorithm

Here we introduce the modified subgradient-push algorithm (MSP). In the MSP algorithm, each node $v_i \in V$ sends its values to its out-neighbors in $\bar{G}(t)$ only if $y_i(t) \geq \frac{1}{n^2}$. In this sense, at time $t$ node $v_i$ only receives information from the subset $N_{in}^i(t) \subseteq \bar{N}_{in}^i(t)$ of its in-neighbors given by

$$N_{in}^i(t) = \bar{N}_{in}^i(t) \backslash \{v_j \mid y_j(t) < \frac{1}{n^2}\}.$$  

Indeed, this construction induces an effective communication network graph $G(t) = (V, E(t), W(t))$ at time $t$ with the same set of nodes as $\bar{G}(t)$ and the set of edges $E(t) \subseteq \bar{E}(t)$. Similar to the SP algorithm, the MSP algorithm is given as

$$w_i(t+1) = \sum_{j \in \bar{N}_{in}^i(t)} \frac{x_j(t)}{d_j(t)},$$

$$y_i(t+1) = \sum_{j \in \bar{N}_{in}^i(t)} \frac{y_j(t)}{d_j(t)},$$

$$z_i(t+1) = \frac{w_i(t+1)}{y_i(t+1)},$$

$$x_i(t+1) = w_i(t+1) - \alpha(t+1)g_i(t+1), \quad (5)$$

where $\bar{N}_{in}^i(t)$ is replaced with $\bar{N}_{in}^i(t)$ for all $i \in [n]$ and $t \geq 0$. Here the $d_i(t)$ are the out-degrees of nodes in the effective communication network graph $G(t)$. Throughout this article, whenever we write $G(t)$, or $\bar{N}_{in}^i(t)$, we refer to the subgraph of $G(t)$ resulting from this modification.
4 Random Setting and Main Result

We next state our assumptions on the sequence of random graphs; as we mentioned in Section 3.1, we denote by $\bar{G}(t) = (\mathcal{V}, \mathcal{E}(t))$ the graph of the available communication channels and by $G(t) = (\mathcal{V}, \mathcal{E}(t), W(t))$ the communication network graph utilized by the nodes at time $t$.

**Assumption 1** Let $\mathcal{G} = \{\bar{G}_1, \bar{G}_2, \ldots, \bar{G}_{2^{n^2-n}}\}$ be the set of all the possible graphs of available communication channels on the node set $\mathcal{V}$ with self-loops at all nodes. We assume that the sequence of realized available communication graphs $\{\bar{G}(t)\}$ satisfies:

(i) at each time $t \geq 0$, the corresponding graph for communications is $\bar{G}_b$ with probability $\mathbb{P}(\bar{G}(t) = \bar{G}_b) = p_b$, where $b \in [2^{n^2-n}]$;

(ii) $\bigcup_{b:p_b>0} \bar{G}_b$ is strongly connected, where $p_b$ is as in part (i);

(iii) the graphs $\bar{G}(t)$ are independent of each other.

This assumption states that the graphs of available communication channels are drawn independently from the set of all possible graphs on the node set $\mathcal{V}$ with self-loops at all nodes. Note that part (ii) imposes a mild connectivity assumption on this sequence of graphs, which is analogous to deterministic settings.

The main results of this paper is as follows.

**Theorem 1** Consider the MSP algorithm (5) and suppose that the sequence of available communication channels $\{\bar{G}(t)\}$ satisfies Assumption 1.

(i) Then we have

$$\lim_{t \to \infty} z_i(t) = z^*, \quad \text{for all } i \in [n],$$

almost surely, where $z^*$ is an optimal solution of (1).

(ii) Define $S(0) = 0$ and $S(t) = \sum_{s=0}^{t-1} \alpha(s + 1)$ for all $t \geq 1$. Assume that every node $i$ maintains the vector $\tilde{z}_i(t) \in \mathbb{R}^d$ initialized with any $\tilde{z}_i(0) \in \mathbb{R}^d$ with the following update rule:

$$\tilde{z}_i(t + 1) = \frac{\alpha(t + 1)z_i(t + 1) + S(t)\tilde{z}_i(t)}{S(t + 1)}, \quad \text{for } t \geq 0,$$
Then, we have for all $t \geq 1$, $i \in [n]$, and any $z^* \in Z^*$,

$$
\mathbb{E}[F(\tilde{z}_i(t+1)) - F(z^*)] \leq \Gamma \left[ \frac{n\|\bar{x}(0) - z^*\|_1}{2} + \left( 1 + \frac{1}{2\gamma - 1} \right) \frac{L^2}{2n} 
+ \frac{60L}{\delta(1-\lambda)} \sum_{j=1}^{n} \|x_j(0)\|_1
+ \frac{60dL^2}{\delta(1-\lambda)} \left( 1 + \frac{1}{2\gamma - 1} \right) \right],
$$

where $\Gamma = \frac{(1-\gamma)}{(t+2)^{1-\gamma}}.$

The rest of this paper is dedicated to the proof of this result.

5 Convergence Analysis

Similar to [14], rather than directly dealing with the subgradients, we start by defining a perturbed version of the push-sum algorithm subject to the modification that we have made. In this section, for most parts, we work with scalar variables. In the proof of Theorem 1, we apply the results of this section to solve (1) with vector variables.

5.1 Modified Perturbed-Push Algorithm

Here we describe the modified perturbed-push (MPP) algorithm. In this algorithm, each node $v_i$ maintains and updates scalar variables $x_i(t), w_i(t)$ and $y_i(t)$, where the $x_i(0)$ are arbitrary scalars and the $y_i(0)$ are initialized to 1, for all $i \in [n]$. The update rule at time $t + 1$ is

$$
w_i(t + 1) = \sum_{j \in N_i^v(t)} \frac{x_j(t)}{d_j(t)},
$$

$$
y_i(t + 1) = \sum_{j \in N_i^v(t)} \frac{y_j(t)}{d_j(t)},
$$

$$
z_i(t + 1) = \frac{w_i(t + 1)}{y_i(t + 1)},
$$

$$
x_i(t + 1) = w_i(t + 1) + \epsilon_i(t + 1),
$$

(6)

where the $\epsilon_i(t)$ are perturbations at time $t$, to be specified later. Similar to the MSP algorithm, node $v_i$ shares its values only if $y_i(t) \geq \frac{1}{\lambda \pi}$, and the
and the \( N_i^{\text{in}}(t) \) are the in-neighbors of the nodes in the effective communication network graph \( G(t) \), respectively.

It is useful to write the algorithm in a matrix form. Hence, for all \( t \geq 0 \), we let the column-stochastic matrix \( W(t) \) be the weighted adjacency matrix associated with the effective communication network graph \( G(t) \) with entries

\[
W_{ij}(t) = \begin{cases} \frac{1}{d_j(t)} & \text{if } j \in N_i^{\text{in}}(t), \\ 0 & \text{otherwise}. \end{cases}
\]  

(7)

Writing the MPP algorithm in matrix form, we have for all \( t \geq 0 \)

\[
w(t + 1) = W(t)x(t),
\]

\[
y(t + 1) = W(t)y(t),
\]

\[
z_i(t + 1) = \frac{w_i(t + 1)}{y_i(t + 1)}, \quad \text{for all } i \in [n]
\]

\[
x(t + 1) = w(t + 1) + \epsilon(t + 1),
\]  

(8)

where \( \epsilon(t) = (\epsilon_1(t), \ldots, \epsilon_n(t))' \) is the vector of perturbations at time \( t \). Here, \( w(t) = (w_1(t), \ldots, w_n(t)) \in \mathbb{R}^n \), \( z(t) = (z_1(t), \ldots, z_n(t)) \in \mathbb{R}^n \), and in this sense we have treated each component of the individual agent’s state variables separately. In the analysis of the MPP algorithm, we assume that \( \|\epsilon(t)\|_1 \leq \frac{U}{n} \). This assumption holds when we regard the subgradient term in the MSP algorithm as perturbation. Throughout this paper, the \( W(t) \) denote the adjacency matrices associated with the effective communication graphs defined in (7).

5.2 Convergence of the MPP algorithm

We study the convergence properties of MPP algorithm. We first tackle the issue of connectivity of the sequence of matrices that are generated through the MPP algorithm. One of the key properties that we need to ensure for the sequence of random matrices induced by the MPP algorithm is the directed infinite flow property, which we recall next.

**Definition 1** [15, Definition 3] We say that a sequence of matrices \( \{W(t)\} \) has the directed infinite flow property if for any non-trivial \( S \subset [n] \)

\[
\sum_{t=0}^{\infty} W_{SS^c}(t) = \infty, \quad \text{almost surely.}
\]
The following proposition presents an upper bound on how well the sequences $z_i(t+1)$ estimate the average $\bar{x}(t) := \frac{1}{n} x(t)$ for each sample path, when the sequence of matrices $\{W(t)\}$ has the directed infinite flow property. This will allow us to state our first connectivity result in a random setting.

**Proposition 1** Consider the MPP algorithm and suppose that the sequence $\{W(t)\}$ has the directed infinite flow property, almost surely. Then, we have

$$|z_i(t+1) - \bar{x}(t)| \leq \frac{2}{y_i(t+1)} \left( \Lambda_{t,0}\|x(0)\|_1 + \sum_{s=1}^{t} \Lambda_{t,s}\|\epsilon(s)\|_1 \right),$$

almost surely, where $\Lambda_{t,s} \in (0,1)$ for all $t \geq s \geq 0$.

**Proof** Let us start the proof by defining the shorthand notation, $W(t:s) = W(t)W(t-1)\cdots W(s)$ for $t \geq s \geq 0$. Since we assume that the sequence $\{W(t)\}$ has the directed infinite flow property, almost surely, by [15, Proposition 3], there exist $\phi(t) \in \mathbb{R}^n$ and $\Lambda_{t,s} \in (0,1)$ such that for all $i,j \in [n]$ and $t \geq s$,

$$\left|W(t:s)\right|_{ij} - \phi_i(t) \leq \Lambda_{t,s}.$$  \hspace{1cm} (10)

Now, for all $t \geq s \geq 0$ define $D(t:s) = W(t:s) - \phi_i(t)1_n'$. Following similar steps as in [14] proof of Lemma 1], we obtain

$$|z_i(t+1) - \bar{x}(t)| \leq \frac{2}{y_i(t+1)} \left( \max_j \|D(t:0)\|_{ij}\|x(0)\|_1 + \sum_{s=1}^{t} \max_j \|D(t:s)\|_{ij}\|\epsilon(s)\|_1 \right)$$

$$\leq \frac{2}{y_i(t+1)} \left( \Lambda_{t,0}\|x(0)\|_1 + \sum_{s=1}^{t} \Lambda_{t,s}\|\epsilon(s)\|_1 \right),$$

where the second inequality follows from (10).  \hspace{1cm} □

In the following, we state our first result on the connectivity of the sequence of matrices $\{W(t)\}$ for the given random setting. The main challenge, in contrast to the setting of [15], is that in addition to time dependency, this sequence also depends on the states, namely the $y_i(t)$'s.
Lemma 1 Consider the MPP algorithm (8) and let $W(t)$ given in (7) be the weighted adjacency matrix of the graph induced by the available communication channel at time $t$, and suppose that Assumption 1 holds for the sequence $W(t)$. Then, 

(i) for all $t \geq 0$, we have

$$\mathbb{P}(W(t + 2n - 3 : t) \text{ irreducible}) \geq p > 0,$$

where $p = (\min_{b:p_b > 0} p_b)^{2n-2}$.

(ii) $\{W(t)\}$ has the directed infinite flow property.

Proof We start by proving (i). Similar to the setting in the push-sum algorithm in [13, Proposition 2.2], it’s easy to check that \(\sum_{i \in [n]} y_i(t) = n\) for all $t$, which is known as mass conservation property. Hence, there exists a node $v_{i_0} \in V$ for which we have \(y_{i_0}(t) \geq 1\). (11)

Since we assume that the graphs of available communication channels contain self-loops at all nodes, each node $i$ sends $\frac{1}{d_i}$ share of its values to itself. In addition, $d_i \leq n$ for all $i$, therefore, in each iteration of the MSP algorithm $y_i(t') \geq \frac{1}{n} y_i(t' - 1)$. This along with (11), implies that

$$y_{i_0}(t') \geq \frac{1}{n^{n-1}}, \text{ for all } t \leq t' \leq t + n - 1.$$

Since $\bigcup_{b:p_b > 0} G_b$ is strongly connected, by assumption, there exists a graph $G_{b_1}$ with probability $p_{b_1} > 0$, in which, $v_{i_0}$ is connected to some other node $v_{i_1}$, i.e., $e_1 := (v_{i_0}, v_{i_1}) \in E_{b_1}$ for some $v_{i_1} \in V \{v_{i_0}\}$. When $G(t) = G_{b_1}$, $v_{i_1}$ receives $\frac{1}{d_{i_0}}$ share of the values of the node $v_{i_0}$. Hence, we have $y_{i_1}(t + 1) \geq \frac{1}{n}$ which again implies that

$$y_{i_1}(t') \geq \frac{1}{n^{n-1}}, \text{ for all } t + 1 \leq t' \leq t + n - 1.$$

Similarly, there exists $G_{b_2}$ with probability $p_{b_2} > 0$ such that either $(v_{i_0}, v_{i_2}) \in E_{b_2}$ or $(v_{i_1}, v_{i_2}) \in E_{b_2}$ for some $v_{i_2} \in V \{v_{i_0}, v_{i_1}\}$. When $G(t) = G_{b_1}$ and $G(t + 1) = G_{b_2}$, we have $y_{i_2}(t + 2) \geq \frac{1}{n^2}$ and hence,

$$y_{i_2}(t') \geq \frac{1}{n^{n-1}}, \text{ for all } t + 2 \leq t' \leq t + n - 1.$$
We set \( e_2 \) as one of these edges. Continuing this argument, we obtain the event 
\[
\mathcal{A}_1 = \{ \tilde{G}(t) = \bar{G}_{b_1}, \ldots, \tilde{G}(t + n - 2) = \bar{G}_{b_{n-1}} \},
\]
and the directed spanning tree \( S_1 = \{ e_1, \ldots, e_{n-1} \} \) of \( \bigcup_{t'=t}^{t'+n-2} G(t) \) rooted at \( v_{i_0} \) with \( v_{i_n} \) as a leaf, noting that \( e_i \in \mathcal{E}_{t+i-1} \) for all \( i \in [n - 1] \). Now, starting from the node \( v_{i_n} \) and following similar argument as above, we can find the event
\[
\mathcal{A}_2 = \{ \tilde{G}(t + n - 1) = \bar{G}_{b_n}, \ldots, \tilde{G}(t + 2n - 3) = \bar{G}_{b_{2n-2}} \}
\]
and the directed spanning tree \( S_2 = \{ e_n, \ldots, e_{2n-2} \} \) of \( \bigcup_{t'=t}^{t'+2n-3} G(t) \) rooted at \( v_{i_2n} \). By construction, the set \( S_1 \cup S_2 \) is a strongly connected subgraph of \( \bigcup_{t'=t}^{t'+2n-3} G(t') \), which occurs with probability
\[
\mathbb{P}(A_1 \cap A_2) = \prod_{i=1}^{2n-2} p_{b_i} \geq \left( \min_{b_{p_b}>0} p_b \right)^{2n-2} =: p > 0.
\]
By \([8, \text{Lemma 1}]\), for this event \( W(t + 2n - 3 : t) \) is irreducible and hence,
\[
\mathbb{P}(W(t + 2n - 3 : t) \text{ irreducible}) \geq p.
\]
This completes the proof of (i). The proof of Part (ii) follows similar steps as the proof of Lemma 1 in \([15]\) and is omitted here.

A useful consequence of this result and Proposition \([1]\) is stated next.

**Corollary 1** Consider the MPP algorithm \([8]\) and let \( W(t) \) given in \((7)\) be the weighted adjacency matrix of the graph induced by the available communication channel at time \( t \), and suppose that Assumption 1 holds. We have
\[
|z_i(t + 1) - \bar{x}(t)| \leq \frac{2}{\delta} \left( \Lambda_{t,0} \| x(0) \|_1 + \sum_{s=1}^{t} \Lambda_{t,s} \| \epsilon(s) \|_1 \right),
\]
almost surely, where \( \Lambda_{t,0} \in (0,1) \) for all \( t \geq s \geq 0 \) and \( \delta = \frac{1}{n^{2n}} \).

**Lemma 2** Consider the MPP algorithm \([8]\) and let \( W(t) \) given in \((7)\) be the weighted adjacency matrix of the graph induced by the available communication channel at time \( t \), and suppose that Assumption 1 holds. Then
(i) for all \( t \geq s \geq 0 \) we have

\[
P(\Lambda_{t,s} > 2\lambda^{t-s}) \leq 13e^{-c_1(t-s)} \quad \text{and} \quad E[\Lambda_{t,s}] \leq 15\lambda^{t-s},
\]

where \( \lambda = \left(1 - \frac{1}{n^2p}\right)^{\frac{p}{2nB}} \in (0, 1) \), \( c_1 = \frac{p^2}{4B} \) and \( B = 2n - 2 \);

(ii) we have that

\[
P(\lim_{t \to \infty} \Lambda_{t,s} = 0) = 1.
\]

**Proof** The proof of part (i) is similar to the lines of the proof of Lemma 3 in [15] and is omitted here. Part (ii) is a direct consequence of part (i) and the Borel-Cantelli lemma.

**Lemma 3** Consider the MPP algorithm (8) and let \( W(t) \) given in (7) be the weighted adjacency matrix of the graph induced by the available communication channel at time \( t \), and suppose that Assumption 1 holds. In addition, assume that the perturbations \( \epsilon_i(t) \) are bounded as follows:

\[
\|\epsilon(t)\|_1 \leq U, 
\]

for some scalar \( U > 0 \). Then,

(i) \( \lim_{t \to \infty} |z_i(t+1) - \bar{x}(t)| = 0 \), almost surely;

(ii) \( \sum_{t=0}^{\infty} \alpha(t+1) |z_i(t+1) - \bar{x}(t)| < \infty \), almost surely;

(iii) we have

\[
E \left[ \frac{1}{\sum_{k=0}^{t} \alpha(k+1)} \sum_{k=0}^{t} \alpha(k+1) |z_i(k+1) - \bar{x}(k)| \right] 
\leq \frac{2c_2 \Gamma}{\delta(1 - \lambda)} \cdot \left( \|x(0)\|_1 + U \left( 1 + \frac{1}{2\gamma - 1} \right) \right),
\]

where \( \Gamma \) is defined in Theorem 1.

**Proof** We start by proving (i). By Corollary 1, for all \( t \) we have

\[
|z_i(t+1) - \bar{x}(t)| \leq \frac{2}{\delta} \left( \Lambda_{t,0}\|x(0)\|_1 + \sum_{s=1}^{t} \Lambda_{t,s}\|\epsilon(s)\|_1 \right).
\]
Hence, it suffices to show that the right hand side converges to zero, almost surely, as $t \to \infty$. As shown in Lemma 2 (ii), $\lim_{t \to \infty} \Lambda_{t,0} = 0$, almost surely and therefore, it remains to show that

$$\lim_{t \to \infty} \sum_{s=1}^{t} \Lambda_{t,s} \|\epsilon(s)\|_1 = 0,$$

almost surely. For all $t \geq 1$ define $\tau_t := \left\lceil \frac{2}{c_1} \ln(t) \right\rceil$, where $c_1$ is a scalar constant given in Lemma 2 (i). In order to study this limit, we break the summation into the following two summations

$$\sum_{s=1}^{t} \Lambda_{t,s} \|\epsilon(s)\|_1 = \sum_{s=1}^{t-\tau_t} \Lambda_{t,s} \|\epsilon(s)\|_1 + \sum_{s=t-\tau_t+1}^{t} \Lambda_{t,s} \|\epsilon(s)\|_1. \quad (14)$$

For the first summation on the right hand side of (14) we have

$$\mathbb{P}\left( \sum_{s=1}^{t-\tau_t} \Lambda_{t,s} \|\epsilon(s)\|_1 > \sum_{s=1}^{t-\tau_t} 2\lambda^{t-s} \|\epsilon(s)\|_1 \right) \leq \mathbb{P}\left( \bigcup_{s=1}^{t-\tau_t} \{ \Lambda_{t,s} > 2\lambda^{t-s} \} \right) \leq \sum_{s=1}^{t-\tau_t} \mathbb{P}(\Lambda_{t,s} > 2\lambda^{t-s}) \leq \sum_{s=1}^{t-\tau_t} 13e^{-c_1(t-s)} \quad \text{(by (13))} \leq \frac{13e^{c_1}}{e^{c_1} - 1} \tau_t \leq \frac{13e^{c_1}}{e^{c_1} - 1} t^{-2}. \quad (15)$$

If we sum (15) over $t$ we have

$$\sum_{t=1}^{\infty} \mathbb{P}\left( \sum_{s=1}^{t-\tau_t} \Lambda_{t,s} \|\epsilon(s)\|_1 > \sum_{s=1}^{t-\tau_t} 2\lambda^{t-s} \|\epsilon(s)\|_1 \right) \leq \frac{13e^{c_1}}{e^{c_1} - 1} \sum_{t=1}^{\infty} t^{-2} < \infty.$$

Hence, using Borel-Cantelli lemma, there exists $t'$ such that for all $t \geq t'$

$$\sum_{s=1}^{t-\tau_t} \Lambda_{t,s} \|\epsilon(s)\|_1 \leq \sum_{s=1}^{t-\tau_t} 2\lambda^{t-s} \|\epsilon(s)\|_1, \quad \text{almost surely.}$$
It is shown in [14, Lemma 1 (a)] that for our choice of $\lambda$ and $\epsilon(s)$,

$$\lim_{t \to \infty} \sum_{s=1}^{t} \lambda^{t-s}\|\epsilon(s)\|_1 = 0,$$

therefore,

$$\lim_{t \to \infty} \sum_{s=1}^{t} \Lambda_{t,s}\|\epsilon(s)\|_1 = 0, \quad \text{almost surely.} \quad (16)$$

For the second summation on the right hand side of (14) recall that by assumption $\|\epsilon(s)\|_1 \leq \frac{U}{s^\gamma}$. This along with the fact that $\Lambda_{t,s} \in (0,1)$ implies that

$$\sum_{s=t-\tau_t+1}^{t} \Lambda_{t,s}\|\epsilon(s)\|_1 \leq \sum_{s=t-\tau_t+1}^{t} \|\epsilon(s)\|_1 \leq \sum_{s=t-\tau_t+1}^{t} \frac{U}{s^\gamma}$$

$$\leq \tau_t \cdot \max_{t-\tau_t < s \leq t} \frac{U}{s^\gamma}$$

$$\leq \frac{2U\ln(t) + 1}{\left(t - \frac{2}{\epsilon_t} \ln(t) - 1\right)^\gamma} =: \delta_t.$$  

It is easy to show that $\lim_{t \to \infty} \delta_t \to 0$, which along with (16) gives us our desired result.

Now we prove part (ii). By Corollary 1 we have

$$\sum_{t=0}^{\infty} \alpha(t) \cdot |z(t) - \bar{x}(t)|$$

$$\leq \sum_{t=0}^{\infty} \alpha(t) \left( \Lambda_{t,0}\|x(0)\|_1 + \sum_{s=1}^{t} \Lambda_{t,s}\|\epsilon(s)\|_1 \right)$$

$$\leq \frac{2}{\delta} \left( \sum_{t=0}^{\infty} \alpha(t) \Lambda_{t,0}\|x(0)\|_1 + \sum_{t=0}^{\infty} \sum_{s=1}^{t} \Lambda_{t,s}\alpha(s)\|\epsilon(s)\|_1 \right).$$

Similar to part (i), it can be seen that the first term on the right-hand side is finite, almost surely. For the second term on the right-hand side, we break
the summation into two summations as follows
\[
\sum_{t=1}^{\infty} \sum_{s=1}^{t} \Lambda_{t,s} \alpha(s) \|\epsilon(s)\|_1 = \sum_{t=1}^{\infty} \sum_{s=1}^{t-\tau_t} \Lambda_{t,s} \alpha(s) \|\epsilon(s)\|_1 + \sum_{t=1}^{\infty} \sum_{s=t-\tau_t+1}^{t} \Lambda_{t,s} \alpha(s) \|\epsilon(s)\|_1. \tag{17}
\]

For the first summation on the right-hand side of (17), similar to part (i) we have
\[
\mathbb{P}\left( \sum_{s=1}^{t-\tau_t} \Lambda_{t,s} \alpha(s) \|\epsilon(s)\|_1 > \sum_{s=1}^{t-\tau_t} 2\lambda^t s^{-\alpha} \alpha(s) \|\epsilon(s)\|_1 \right) \leq \mathbb{P}\left( \bigcup_{s=1}^{t-\tau_t} \{ \Lambda_{t,s} > 2\lambda^t s^{-\alpha} \} \right) \leq \frac{13e^{c_1}}{e^{c_1} - 1} t^{-2}. \tag{19}
\]

If we sum (19) over \( t \) we have
\[
\sum_{t=1}^{\infty} \mathbb{P}\left( \sum_{s=1}^{t-\tau_t} \Lambda_{t,s} \alpha(s) \|\epsilon(s)\|_1 > \sum_{s=1}^{t-\tau_t} 2\lambda^t s^{-\alpha} \alpha(s) \|\epsilon(s)\|_1 \right) \leq \frac{13e^{c_1}}{e^{c_1} - 1} \sum_{t=1}^{\infty} t^{-2} < \infty.
\]

Hence, using Borel-Cantelli lemma, there exists \( t'' \) such that for all \( t \geq t'' \)
\[
\sum_{s=1}^{t-\tau_t} \Lambda_{t,s} \alpha(s) \|\epsilon(s)\|_1 \leq \sum_{s=1}^{t-\tau_t} 2\lambda^t s^{-\alpha} \alpha(s) \|\epsilon(s)\|_1, \quad \text{almost surely.}
\]

It is shown in [14, Lemma 1 (b)] that for our choice of \( \lambda \) and \( \epsilon(s) \),
\[
\lim_{t \to \infty} \sum_{s=1}^{t-\tau_t} \lambda^t s^{-\alpha} \alpha(s) \|\epsilon(s)\|_1 < \infty,
\]
therefore,
\[
\lim_{t \to \infty} \sum_{s=1}^{t-\tau_t} \Lambda_{t,s} \alpha(s) \|\epsilon(s)\|_1 < \infty, \quad \text{almost surely.} \tag{20}
\]

For the second summation on the right-hand side of (17) we have
\[
\sum_{t=1}^{\infty} \sum_{s=t-\tau_t+1}^{t} \Lambda_{t,s} \alpha(s) \|\epsilon(s)\|_1 \leq \sum_{t=1}^{\infty} \sum_{s=t-\tau_t+1}^{t} \frac{U}{s^{2\gamma}} \leq \Delta,
\]
where
\[ \Delta = U \sum_{t=1}^{\infty} \frac{2}{c_1} \ln(t) + 1 \frac{1}{(t - \frac{2}{c_1} \ln(t) - 1)^{2\gamma}} \]

Since \( 0.5 < \gamma < 1 \) we have \( \Delta < \infty \), which along with (20) gives us our desired result.

Finally, we prove part (iii). Using Corollary [1] we have
\[
\sum_{k=1}^{t} \alpha(k + 1) |z_i(k + 1) - \overline{x}(k)| \leq \sum_{k=1}^{t} \frac{1}{(k + 1)^\gamma} \frac{2}{\delta} \left( \Lambda_{k,0} \|x(0)\|_1 + \sum_{s=1}^{k} \Lambda_{k,s} \|\epsilon(s)\|_1 \right)
\]
\[
= \frac{2\|x(0)\|_1}{\delta} \sum_{k=1}^{t} \frac{1}{(k + 1)^\gamma} \Lambda_{k,0}
\]
\[
+ \frac{2}{\delta} \sum_{k=1}^{t} \sum_{s=1}^{k} \frac{1}{(k + 1)^\gamma} \Lambda_{k,s} \|\epsilon(s)\|_1
\]
\[
\leq \frac{2\|x(0)\|_1}{\delta} \sum_{k=1}^{t} \Lambda_{k,0} + \frac{2U}{\delta} \sum_{k=1}^{t} \sum_{s=1}^{k} \frac{1}{s^{2\gamma}} \Lambda_{k,s}.
\]

Taking expectations from both sides, using (13), we have
\[
E \left[ \sum_{k=1}^{t} \alpha(k + 1) |z_i(k + 1) - \overline{x}(k)| \right] \leq E \left[ \frac{2\|x(0)\|_1}{\delta} \sum_{k=1}^{t} \Lambda_{k,0} + \frac{2U}{\delta} \sum_{k=1}^{t} \sum_{s=1}^{k} \frac{1}{s^{2\gamma}} \Lambda_{k,s} \right]
\]
\[
\leq 30\|x(0)\|_1 \sum_{k=1}^{t} \Lambda_{k,0} + \frac{30U}{\delta} \sum_{k=1}^{t} \sum_{s=1}^{k} \frac{1}{s^{2\gamma}} \Lambda_{k,s}
\]
\[
\leq 30\|x(0)\|_1 \left( \frac{\lambda}{1 - \lambda} + \frac{30U}{\delta} \sum_{k=1}^{t} \sum_{s=1}^{k} \frac{1}{s^{2\gamma}} \right)
\]

For the second term on the right-hand side we have
\[
\sum_{k=1}^{t} \sum_{s=1}^{k} \frac{1}{s^{2\gamma}} \lambda^{k-s} = \sum_{s=1}^{t} \frac{1}{s^{2\gamma}} \sum_{k=s}^{t} \lambda^{k-s} \leq \sum_{s=1}^{t} \frac{1}{s^{2\gamma}} \frac{1}{1 - \lambda} \leq \frac{1}{1 - \lambda} \left( 1 + \int_{1}^{\infty} \frac{du}{u^{2\gamma}} \right)
\]
\[
= \frac{1}{1 - \lambda} \left( 1 + \frac{1}{2\gamma - 1} \right).
\]

Following similar steps as in [13, Corollary 3]
\[
E \left[ \sum_{k=0}^{t} \alpha(k + 1) |z_i(k + 1) - \overline{x}(k)| \right] \leq \frac{30}{\delta(1 - \lambda)} \left( \|x(0)\|_1 + U \left( 1 + \frac{1}{2\gamma - 1} \right) \right).
\]
In addition, we have the following inequality
\[
\sum_{k=0}^{t} \alpha(k + 1) = \sum_{k=0}^{t} \frac{1}{(k+1)^\gamma} \geq \int_{0}^{t} \frac{dk}{(k+2)^\gamma} = \frac{(t+2)^{1-\gamma} - 1}{1-\gamma} \triangleq \Gamma^{-1} \quad (22)
\]
Therefore, by (21) and (22), we obtain our desired result. \(\square\)

Proof [Theorem 1] We start by proving part (i). Having established all the necessary results to accommodate the random nature of the underlying communication networks, the remaining steps of the proof follows similar steps as in \[14\] [Proof of Theorem 1]. In particular, by applying the result in Lemma 3 (i) to each coordinate, we obtain
\[
\lim_{t \to \infty} \|z_i(t+1) - \bar{x}(t)\| = 0 \text{ for all } i \in [n],
\]
where \(\bar{x}(t) = \frac{1}{n} \sum_{i=1}^{n} x_i(t)\) for all \(t \geq 0\). Moreover, by Lemma 3 (ii)
\[
\sum_{t=0}^{\infty} \alpha(t+1)\|z_i(t+1) - \bar{x}(t)\| < \infty \text{ for all } i \in [n].
\]
On the other hand, by Lemma 4 in the Appendix
\[
\|\bar{x}(t+1) - z^*\|^2 \leq \|\bar{x}(t) - z^*\|^2 - \frac{2\alpha(t+1)}{n} (F(\bar{x}(t)) - F(z^*))
+ \frac{4\alpha(t+1)}{n} \sum_{i=1}^{n} L_i \|z_i(t+1) - \bar{x}(t)\|
+ \alpha^2(t+1) \frac{L^2}{n^2},
\]
for all the optimal solutions \(z^*\). Therefore, with our choice of \(\alpha(t)\), all the conditions of Lemma 5 in the Appendix are satisfied and we obtain the desired result.

The proof of part (ii) is similar to the lines of the proof of Theorem 2 in \[14\] and is omitted. \(\square\)

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Appendix

Lemma 4 [14, Lemma 8] Suppose that $F(z) = \sum_{i=1}^{n} f_i(z)$, where the $f_i(z)$ are convex functions over $\mathbb{R}^d$. In addition, suppose that the $f_i(z)$ are Lipschitz continuous with Lipschitz constants $L_i < \infty$. Then for all $v \in \mathbb{R}^d$ and $t \geq 0$,

$$\
\|\bar{x}(t+1) - v\|^2 \leq \|\bar{x}(t) - v\|^2 \\
- \frac{2\alpha(t+1)}{n}(F(\bar{x}(t)) - F(v)) \\
+ \frac{4\alpha(t+1)}{n} \sum_{i=1}^{n} L_i \|z_i(t+1) - \bar{x}(t)\| \\
+ \alpha^2(t+1) \frac{L^2}{n^2}.
$$

Lemma 5 [14, Lemma 7] Consider a convex minimization problem $\min_{x \in \mathbb{R}^m} f(x)$, where $f : \mathbb{R}^m \to \mathbb{R}$ is a continuous function. Assume that the solution set $X^*$ of the problem is nonempty. Let $\{x_t\}$ be a sequence such that for all $x^* \in X^*$ and all $t \geq 0$,

$$\
\|x_{t+1} - x^*\|^2 \leq \|x_t - x^*\|^2 - \beta_t(f(x_t) - f(x)) + c_t,
$$

where $\beta_t \geq 0$ and $c_t \geq 0$ for all $t \geq 0$, with $\sum_{t=0}^{\infty} \beta_t = \infty$ and $\sum_{t=0}^{\infty} c_t < \infty$. Then the sequence $\{x_t\}$ converges to some solution $x^* \in X^*$. 