Small Support Approximate Equilibria in Large Games

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May 22, 2013

Abstract

In this note we provide a new proof for the results of Lipton et al. [3] on the existence of an approximate Nash equilibrium with logarithmic support size. Besides its simplicity, the new proof leads to the following contributions:

1. For $n$-player games, we improve the bound on the size of the support of an approximate Nash equilibrium.

2. We generalize the result of Daskalakis and Papadimitriou [4] on small probability games from the two-player case to the general $n$-player case.

3. We provide a logarithmic bound on the size of the support of an approximate Nash equilibrium in the case of graphical games.

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†The author wishes to thank Constantinos Daskalakis and Siddarth Barman for useful discussions and comments. The author gratefully acknowledges support from a Walter S. Baer and Jeri Weiss fellowship.
1 Introduction

The problem of the existence of a small-support approximate equilibrium (i.e., every player randomizes among small set of his actions) has been studied in the literature for the past two decades. Althofer [1] considered two-player zero-sum games and showed existence of approximately optimal strategies with support of size $O(\log m)$, where $m$ is the number of actions. Lipton, Markakis, and Mehta [3] later generalized this result to all two-player games; i.e., they showed existence of an approximate equilibrium with support of size $O(\log m)$. This result yields an exhaustive search algorithm for computing an approximate Nash equilibrium with a quasi-polynomial running time $(m^{\log m})$. This is the best-known bound today for computing an approximate Nash equilibrium. Daskalakis and Papadimitriou [4] generalized the technique of Lipton et al. [3] to prove that in two-player games an approximate Nash equilibrium can be computed in polynomial time in games that possess a small-probabilities Nash equilibrium (see definition in Section 4).

The related problem of the existence of a pure Nash equilibrium (an equilibrium with the minimal support) in subclasses of games has been studied in the literature for much longer; see, e.g., Rosenthal [7] and Shmeidler [8]. A recent paper by Azrieli and Shmaya [2] analyzes the relation between the influence that a player has on the payoffs of other players and the existence of an approximate Nash equilibrium. They show that if the influence is small enough, then such a game has an approximate pure Nash equilibrium.

In this note we provide a new proof for the results of Lipton et al. [3] and Daskalakis and Papadimitriou [4] using similar techniques to those developed by Azrieli and Shmaya [2]. Besides its simplicity, the new proof leads to the following contributions:

1. For $n$-player games we improve the bound on the size of the support
of an approximate Nash equilibrium from $O(n^2 \log m)$ (see Lipton et. al. \[3\]) to $O(n \log m)$ (see Corollary \[1\]).

2. We generalize the result of Daskalakis and Papadimitriou \[4\] from two-player games case to all $n$-player game cases (see Corollary \[3\]).

3. We provide a logarithmic bound $(O(\log n + \log m))$ on the size of the support of approximate Nash equilibrium in the case of graphical games. This bound is novel (see Theorem \[1\]).

The note is organized as follows. In Section \[2\] we present the notations and preliminaries that will be useful in our new proof. In Section \[3\] we state and prove the a result on graphical games; this result generalizes Lipton et al. \[3\]. In Section \[4\] we state and prove the result that generalizes the result of Daskalakis and Papadimitriou. Section \[5\] is a discussion.

2 Preliminaries

We consider $n$-player games where every player $i$ has a large number of actions. For simplicity, we will consider the case where all players have the same number of actions $m$. We will use the following standard notations. We denote by $A_i = \{1, 2, ..., m\}$ the actions set of player $i$, and by $A = \times_i A_i$ the actions profile set. The simplex $\Delta(A_i)$ is the set of mixed strategies of player $i$. We will assume that the payoffs of all players are in $[0, 1]$, and $u_i : A \to [0, 1]$ will denote the payoff function of player $i$. The payoff function $u_i$ can be multilinearly extended to $u_i : \Delta(A) \to [0, 1]$. The payoff functions profile is $u = (u_i)_{i=1}^n$, which is also called the game. A mixed action profile $x = (x_1, x_2, ..., x_n)$ is an Nash $\varepsilon$-equilibrium if for every action $a_i \in A_i$, it

\[Given a game where player $i$ has $m_i$ actions, we can consider an equivalent game where every player has $m = \max_i m_i$ actions. This can be done by adding $m - m_i$ strictly dominated actions to every player $i$.\]
holds that \( u_i(x) \geq u_i(a_i, x_{-i}) - \varepsilon \).

A mixed strategy \( x_i = (x_i(1), x_i(2), \ldots, x_i(m)) \) of player \( i \) will be called \( k \)-uniform if \( x_i(j) = c_j/k \), where \( c_j \in \mathbb{N} \) for every \( j = 1, 2, \ldots, m \). Note that the support of \( k \)-uniform strategy is of size at most \( k \). A mixed strategy profile \( x = (x_i)_{i=1}^n \) will be called \( k \)-uniform if every \( x_i \) is \( k \)-uniform.

We say that the payoff of player \( i \) depends on player \( j \) if there exists an action profile \( a_{-j} \) and a pair of actions \( a_j, a_j' \) of player \( j \) such that \( u_i(a_j, a_{-j}) \neq u_i(a_j', a_{-j}) \). A game where the payoff of every player depends on at most \( d \) other players will be called a graphical game of degree \( d \). Graphical games, introduced by Kearns et al. [5], express the situation where players are located on vertices of an underlying graph and their payoffs are influenced only by their neighbors’ actions. Note that every \( n \)-player game is a graphical game of degree \( n - 1 \).

### 2.1 Lipschitz games

Player \( i \) has a \( \lambda \)-Lipschitz payoff function if \( |u_i(a_j, a_{-j}) - u_i(a_j', a_{-j})| \leq \lambda \) for every \( i \neq j \) and every \( a_j, a_j' \in A_j \). The Lipschitz property means that a change of strategy of a single player \( j \neq i \) has little effect on the payoff of player \( i \). Note that player \( i \) can have a big effect on his own payoff. A game will be called \( \lambda \)-Lipschitz if the payoff functions of all players are \( \lambda \)-Lipschitz.

The following proposition is an important property of \( \lambda \)-Lipschitz games.

**Proposition 1.** If in an \( n \)-player game the payoff of player \( i \) depends on at most \( d \) players, and his payoff function is \( \lambda \)-Lipschitz, then for every pure action \( a_i \in A_i \) and for every mixed action profile of the opponents \( x_{-i} \), it
holds that

\[
x_{-i}(B) \geq 1 - 2\exp\left(-\frac{\delta^2}{d\lambda^2}\right)
\]

where \(B \subset A_{-i}\) is defined by

\[
B = \{a_{-i} : |u_i(a_i, a_{-i}) - u_i(a_i, x_{-i})| \leq \delta\}.
\]

In simple words, Proposition 1 claims that if we randomize an action profile \(a_{-i}\) according to \(x_{-i}\), then probably player \(i\) will have approximately the same outcome if he plays against \(a_{-i}\) or against \(x_{-i}\).

Proposition 1 is based on the concentration of measure phenomena for Lipschitz functions (see Ledoux [6]) and it is derived explicitly in Azrieli and Shmaya [2].

### 2.2 From general games to Lipschitz games

We present a very natural procedure that constructs for every game a corresponding game with the Lipschitz property.

Fix \(k \in \mathbb{N}\). Given a game \(u\) we construct a new game \(v = v(u, k)\) with \(kn\) players as follows. We “split” every player \(i\) into a population of \(k\) players \(i(1), i(2), ..., i(k)\). Each player \(i(j)\) plays the original game \(u\) against the aggregate behavior of the \(n - 1\) other populations of size \(k\).

Formally, it will be convenient to present \(A_i\) as the set of vectors \(\{e_1, e_2, ..., e_m\} \subset \mathbb{R}^m\), where \(e_j\) is the \(j - th\) unit vector in \(\mathbb{R}^m\). In such a representation the unit simplex \(\Delta^m := \{(x_j)_{j=1}^m : \sum_j = 1, x_j \geq 0\}\) is the set of mixed strategies \(\Delta(A_i)\). All players \(i(j)\) have the same actions set \(A_i\). The

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2By the notation \(x_{-i}(B)\), we refer to \(x_{-i}\) as a probability measure on \(A_{-i}\), and so \(x_{-i}(B)\) is the probability of the event \(B\).
payoff of player \(i_0(j_0)\) is defined by
\[
v_{i_0(j_0)}((a_{i(j)})_{1 \leq i \leq n, 1 \leq j \leq k}) = u_i \left( a_{i_0(j_0)}, \left( \frac{\sum_{j=1}^{k} a_{i(j)}}{k} \right)_{i \neq i_0} \right).
\]
Note that \(\sum_{j=1}^{k} a_{i(j)}/k \in \Delta^m\); therefore, this vector represents the mixed strategy of population \(i\).

**Remark 1.** The game \(v\) has the following two properties:

(P1) \(v\) is \(1/k\) Lipschitz, because a deviation of a single player \(i(j)\) changes the mixed strategy that is played by population \(i\) only by \(1/k\).

(P2) Every pure Nash \(\varepsilon\)-equilibrium of the game \(v\) corresponds to a \(k\)-uniform mixed Nash \(\varepsilon\)-equilibrium of the game \(u\). The corresponding mixed equilibrium will be the one where player \(i\) plays the aggregated strategy of population \(i\) in the game \(v^3\).

### 3 General Games and Graphical Games

**Theorem 1.** Every \(n\)-player graphical game of degree \(d\) with \(m\) actions for every player has a \(k\)-uniform Nash \(\varepsilon\)-equilibrium for \(k = \frac{8}{\varepsilon^2}d(\log n + \log m)\).

Usually graphical game models consider games with a large number of players \(n\) of constant degree \(d\). Theorem proves the existence of a relatively simple approximate Nash equilibrium where every player uses a strategy with a support that is logarithmically small on \(n\) and \(m\).

Lipton et al. \(3\) show that in every \(n\)-player game with \(m\) actions for every player there exists a \(k\)-uniform Nash \(\varepsilon\)-equilibrium for \(k = O(n^2 \log m)\).

\(3\)Moreover, the opposite direction is also true. Every \(k\)-uniform \(\varepsilon\)-equilibrium of \(u\) corresponds to a pure Nash \(\varepsilon\)-equilibrium of \(v\). The corresponding pure equilibrium will be the one where population \(i\) plays a pure profile with aggregated behavior \(x_i\), where \(x_i\) is the \(k\)-uniform strategy of player \(i\) in the \(\varepsilon\)-equilibrium in the game \(u\).
Theorem 1 applied to general games shows that in such games there exists a $k$-uniform Nash $\varepsilon$-equilibrium for $k = O(n \log m)$.

**Corollary 1.** Every $n$-player game of with $m$ actions for every player has a $k$-uniform Nash $\varepsilon$-equilibrium for $k = \frac{8}{\varepsilon^2} (n - 1)(\log n + \log m)$.

As a straightforward corollary of this result, we derive the following improvement to the oblivious algorithm for computing Nash approximate equilibrium in games with $n$ players.

**Corollary 2.** Let $k = \frac{8}{\varepsilon^2} (n - 1)(\log n + \log m)$. Then the oblivious algorithm\footnote{The term “oblivious algorithm” is from \cite{4}.} that exhaustively searches over the $k$-uniform strategies finds an $\varepsilon$-equilibrium in $O(m^2 \log m)$ steps.\footnote{Lipton et al. \cite{3} prove a bound of $O(m^{n^2 \log m})$ on the number of steps.}

**Proof of Theorem 1.** Let $k = \frac{8}{\varepsilon^2} (n - 1)(\log n + \log m)$. We construct the game $v = v(u, k)$ as presented in Section 2.2. We prove that the game $v$ possesses a pure Nash equilibrium, then, by Remark 1 (P2) this concludes the proof.

Moreover, we will prove that every $nk$-player $1/k$-Lipschitz graphical game of degree $dk$ has a pure Nash $\varepsilon$-equilibrium.

Consider a mixed action profile $x$ that is a (possibly mixed) Nash equilibrium of $v$. For every player $i$ and every action $b \in A_i$ of player $i$, we define the set of action profiles

$$E_{i,b} := A_i \times \{ a_{-i} : |v_i(b, a_{-i}) - v_i(b, x_{-i})| \leq \varepsilon/2 \} \subset A.$$ 

Every action $a^* \in \cap_{i,b} E_{i,b} \cap \text{support}(x)$ is a pure Nash $\varepsilon$-equilibrium according to the following inequality:

$$v_i(d, a^*_{-i}) \leq v_i(d, x_{-i}) + \frac{\varepsilon}{2} \leq v_i(a^*_i, x_{-i}) + \frac{\varepsilon}{2} \leq v_i(a^*_i, a^*_{-i}) + \varepsilon,$$
where the first inequality follows from \( a^* \in E_{i,d} \), the second from \( a^*_i \in \text{support}(x^i) \), and the third from \( a^* \in E_{i,a^*_i} \). Therefore it is enough to prove that the above intersection is not empty.

By proposition 1 we have
\[
x(E_{i,b}^c) \leq 2 \exp\left(\frac{-\varepsilon^2 k}{4d}\right).
\]
Putting \( k = \frac{8}{\varepsilon^2} d(\log n + \log m) \) we get \( x(E_{i,b}^c) \leq 1/(2nkm) \). There are \( nk \) players in \( v \), and \( m \) actions for every player. Therefore there are \( nkm \) events \( E_{i,b} \). Therefore, \( x(\cap E_{i,b}) \geq 1/2 > 0 \), which concludes the proof.

4 Small Probability Games

Following the terminology of [4], a profile of mixed actions \( x \) will be called a \( c \)-small probabilities profile if \( x_i(j) \leq c/m \) for every player \( i \) and every \( j \in A^i \). A game \( u \) will be called a \( c \)-small probability game if there exists a Nash equilibrium \( x \) that is a \( c \)-small probability profile.

Daskalakis and Papadimitriou [4] prove that in small probability two-player games the oblivious random algorithm that samples \( k \)-uniform strategies for \( k = \Theta(log m) \) finds an approximate Nash equilibrium in \( O(c^2 m \log c) \) steps, i.e., in polynomial time in \( m \). Here we generalize this result to general \( n \)-player games.

It will be convenient to think of the \( k \)-uniform strategies as a multiset that contains \( k \) ordered actions. In such a case the set of \( k \)-uniform strategy profiles is of size \( m^{kn} \).

**Theorem 2.** Let \( u \) be an \( n \)-player \( c \)-small probability games with \( m \) actions for every player, and let \( k = \frac{8}{\varepsilon^2} (n - 1)(\log n + \log m) \). Then, among the \( m^{kn} \) \( k \)-uniform strategy profiles in \( u \), the number of strategy profiles that forms
an Nash $\varepsilon$-equilibrium is at least

$$\frac{m^k n}{2(nm)^{\frac{1}{c^2}((n-1)n \ln c)}}.$$

**Corollary 3.** Fix $n$ and let $k = \frac{8}{\varepsilon^2}(n-1)(\log n + \log m)$. Then the oblivious algorithm that samples at random $k$-uniform strategies and checks whether it forms an $\varepsilon$-equilibrium finds such an $\varepsilon$-equilibrium in $c$-small probability games after $(nm)^{\frac{4}{c^2}((n-1)n \ln c)}$ samples in expectation, i.e., after polynomial time in $m$.

**Proof of Theorem 2.** Fix $k = \frac{8}{\varepsilon^2}(n-1)(\log n + \log m)$, and let $x$ be a $c$-small probability equilibrium of $u$. Consider the game $v = v(u, k)$ that is defined in Section 2.2. Note that the action profile where every player $i(j)$ plays the mixed action $x_i$ is a Nash equilibrium of the game $v$. Denote this equilibrium by $x_v$.

Following the same analysis that was done in the proof of Theorem 1, we define the sets $E_{i,b}$ and we know that $x_v(\cap_i b E_{i,b}) \geq 1/2$. Two different pure action profiles in $\cap_i b E_{i,b} \cap support(x_v)$ correspond to two different $k$-uniform Nash $\varepsilon$-equilibria in $u$. Let us show that there are many different action profiles in $\cap_i b E_{i,b} \cap support(x_v)$.

Note that $x_v(a) \leq (c/m)^nk$ because $x$ is a $c$-small probabilities profile. On the other hand, $x_v(\cap_i b E_{i,b}) \geq 1/2$. Therefore, there must be at least $m^{nk}/2c^{nk}$ different profiles in $\cap_i b E_{i,b} \cap support(x_v)$, which yield that there are at least $m^{nk}/2c^{nk}$ different $k$-uniform Nash $\varepsilon$-equilibria in $u$.

It only remains to evaluate the expression $c^{nk}$:

$$c^{nk} = (c^{n + n m})^{\frac{4}{c^2}((n-1)n)} = (n^{\ln c m \ln c})^{\frac{4}{c^2}((n-1)n)} = (n^m)^{\frac{12}{c^2}((n-1)n \ln c)}.$$
5 Discussion

This note contains a new approach to the problem of an approximate small support Nash equilibrium. Instead of considering the game itself, we can consider a population game where every player is replaced by a population of players and analyze the existence of an approximate pure Nash equilibrium in the population game. I believe that this approach might be useful for analyzing other interesting questions. For example, the question of characterizing the class of two-player games where an approximate Nash equilibrium with constant support exists might have the following interpretation: which two-population games with constant population size has a pure Nash equilibrium? Clearly, characterization of the above class is an important question because for those games there exists a polynomial-time exhaustive search algorithm for computing an approximate Nash equilibrium.

This paper provides an upper bound of $O(n \log m)$ on the size of the support of an approximate Nash equilibrium. It is known that the bound $\log m$ is tight even in two-player games (see Althofer [1]); i.e., there exists a two-player game where no Nash approximate equilibrium with a support smaller than $c \log m$ exists. The question whether the linear dependence on $n$ is also tight remains an open question.

**Open problem:** Does there exist an $n$-player $n$-action game where in every Nash approximate equilibrium at least one of the players plays a mixed action with support of size $f(n)$?

By Althofer [1] the answer to this question for $f(n) = c \log n$ is positive. What about $f(n) = cn^\alpha$ for $\alpha < 1$? What about $f(n) = cn$?
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