ESTIMATES FOR EIGENVALUES OF THE OPERATOR $L_r$

GUANGYUE HUANG AND XUERONG QI

Abstract. In this paper, we consider an eigenvalue problem of the elliptic operator

$$L_r = \text{div}(T^r \nabla \cdot)$$

on compact submanifolds in arbitrary codimension of space forms $\mathbb{R}^N(c)$ with $c \geq 0$. Our estimates on eigenvalues are sharp.

1. Introduction

Let $x : M \to \mathbb{R}^N(c)$ be an $n$-dimensional orientable closed connected submanifold of an $N$-dimensional space form $\mathbb{R}^N(c)$ of constant sectional curvature $c$, where $\mathbb{R}^N(c)$ is Euclidean space $\mathbb{R}^N$ when $c = 0$, $\mathbb{R}^N(c)$ is a unit sphere $\mathbb{S}^N$ when $c = 1$, and $\mathbb{R}^N(c)$ is a hyperbolic space $\mathbb{H}^N$ when $c = -1$. Let $\{e_A\}_{A=1}^N$ be an orthonormal basis along $M$ such that $\{e_i\}_{i=1}^n$ are tangent to $M$ and $\{e_\alpha\}_{\alpha=n+1}^N$ are normal to $M$. Denote by $\{\theta_i\}_{i=1}^n$ and $\{\theta_\alpha\}_{\alpha=n+1}^N$ the dual frame, respectively. Then we have the following structure equation (see [6]):

$$dx = \sum_i \theta_i e_i,$$

$$de_i = \sum_j \theta_{ij} e_j + \sum_{\alpha,j} h_{ij}^\alpha \theta_j e_\alpha - c \theta_i x,$$

$$de_\alpha = -\sum_{i,j} h_{ij}^\alpha \theta_j e_i + \sum_\beta \theta_\alpha \beta e_\beta,$$

where $h_{ij}^\alpha$ denote the components of the second fundamental form of $x$. Let $B_{ij} = \sum_{\alpha=n+1}^N h_{ij}^\alpha e_\alpha$. If $r \in \{0, 1, \cdots, n-1\}$ is even, the operator $L_r$ is defined by

$$L_r(f) = \sum_{i,j} T^r_{ij} f_{ij},$$

and

$$L_{r-1}(f) = \sum_{i,j} T^{r-1}_{\alpha i j} f_{ij} e_\alpha.$$
Here $T^r$ is given by

$$T^r_{ij} = \frac{1}{r!} \sum_{i_1, \ldots, i_r} \delta^{j_1 \cdots j_r}_{1 \cdots i_r} (B_{i_1 j_1}, B_{i_2 j_2}) \cdots (B_{i_r-1 j_{r-1}}, B_{i_r j_r});$$  \hfill (1.6)

$$T^{r-1}_{\alpha ij} = \frac{1}{(r - 1)!} \sum_{i_1, \ldots, i_{r-1} \atop j_1, \ldots, j_{r-1}} \delta^{j_1 \cdots j_{r-1} j}_1 (B_{i_1 j_1}, B_{i_2 j_2}) \cdots (B_{i_{r-3} j_{r-3}}, B_{i_{r-2} j_{r-2}}) h_\alpha^i.$$  \hfill (1.7)

$\delta^{j_1 \cdots j_r}_{i_1 \cdots i_r}$ is the generalized Kronecker symbols. It has been shown in [6] that $T^r$ is symmetric and divergence-free. When $r$ is even,

$$L_r(f) = \operatorname{div}(T_r \nabla f),$$

and the corresponding $r$th mean curvature function $S_r$ and $(r + 1)$th mean curvature vector field $S_{r+1}$ are given by

$$S_r = \frac{1}{r!} \sum_{i_1, \ldots, i_r \atop j_1, \ldots, j_r} \delta^{j_1 \cdots j_r}_{1 \cdots i_r} (B_{i_1 j_1}, B_{i_2 j_2}) \cdots (B_{i_r j_r-1}, B_{i_r j_r})$$

$$= \frac{1}{r} T^{r-1}_{\alpha ij} h_\alpha^i$$

$$= \binom{n}{r} H_r;$$  \hfill (1.8)

$$S_{r+1} = \frac{1}{(r + 1)!} \sum_{i_1, \ldots, i_{r+1} \atop j_1, \ldots, j_{r+1}} \delta^{j_1 \cdots j_{r+1}}_{1 \cdots i_{r+1}} (B_{i_1 j_1}, B_{i_2 j_2}) \cdots (B_{i_{r+1} j_{r+1}}, B_{i_{r+1} j_{r+1}})$$

$$= \frac{1}{r + 1} T_r^r h_\alpha^i e_\alpha$$

$$= \binom{n}{r + 1} H_{r+1}. $$  \hfill (1.9)

It has also been shown in [6] that for any even integer $r \in \{0, 1, \ldots, n - 1\}$, we have

$$\operatorname{trace}(T^r) = (n - r) S_r $$ \hfill (1.10)

$$L_r(x) = (r + 1) S_{r+1} - c(n - r) S_r x, $$ \hfill (1.11)

$$L_r(e_\alpha) = - \sum_{i,j,k} T^r_{ij} h^\alpha_{jk} e_k - \sum_{i,j,k,\beta} T^r_{ij} h^\alpha_{jk} h^\beta_{ij} e_\beta + c \sum_{i,j} T^r_{ij} h^\alpha_{ij} x.$$ \hfill (1.12)

When $M$ is a hypersurface of a space form, we have

$$L_0(f) = \Delta(f), \quad L_1(f) = \Box(f) = (n H \delta_{ij} - h_{ij}) f_{ij},$$  \hfill (1.13)

where the operator $\Box$ was introduced by Cheng-Yau in [5] and studied by many mathematicians. In [1], Alencar, do Carmo and Rosenberg generalized Reilly’s inequality to more general operators $L_r$ than the Laplacian. That
is, they proved that when $M$ is an orientable closed hypersurface of $\mathbb{R}^{n+1}$ with $H_{r+1} > 0$,
\[
\lambda_1^{L_r} \int_M H_r \, dv \leq c(r) \int_M H_{r+1}^2 \, dv
\]  
(1.14)
and equality holds precisely if $M$ is a sphere. Here $c(r) = (n - r) \binom{n}{r}$. In [9], Grosjean obtained the following similar optimal upper bound for $\lambda_1^{L_r}$ of closed hypersurfaces of any space form with $H_{r+1} > 0$ and convex isometric immersion $x$:
\[
\lambda_1^{L_r} \text{vol}(M) \leq c(r) \int_M \frac{H_{r+1}^2 + cH_r^2}{H_r} \, dv
\]  
(1.15)
and equality holds if and only if $x(M)$ is an umbilical sphere. For eigenvalues of $L_r$ and some important elliptic operators, see also [2–4, 7, 8, 10] and references therein.

In this paper, we assume that $L_r$ is elliptic on $M$, for some even integer $r \in \{0, 1, \ldots, n-1\}$. The purpose of this paper is to study the following closed eigenvalue problem of the elliptic operator $L_r$:
\[
L_r(u) = -\lambda u
\]  
(1.16)
on compact submanifolds in arbitrary codimension of space forms. We know that the set of eigenvalues consists of a sequence
\[
0 = \lambda_0^{L_r} < \lambda_1^{L_r} \leq \lambda_2^{L_r} \leq \cdots \leq \lambda_k^{L_r} \cdots \to +\infty.
\]
Denote by $u_i$ the normalized eigenfunction corresponding to $\lambda_i^{L_r}$ such that $
\{u_i\}_0^\infty$ becomes an orthonormal basis of $L^2(M)$, that is
\[
\begin{align*}
L_r(u_i) &= -\lambda_i^{L_r} u_i, \\
\int_M u_i u_j \, dv &= \delta_{ij}, \quad \text{for any } i, j = 0, 1, \cdots.
\end{align*}
\]

We will prove the following results:

**Theorem 1.1.** Let $(M, g)$ be an $n$-dimensional orientable closed connected submanifold of a space form $\mathbb{R}^N(c)$ with $c \geq 0$. Assume that $L_r$ is elliptic on $M$, for some even integer $r \in \{0, 1, \ldots, n-1\}$. Then we have
\[
\lambda_1^{L_r} \int_M H_r \, dv \leq c(r) \int_M (|H_{r+1}|^2 + cH_r^2) \, dv; \quad (1.17)
\]
\[
\sum_{i=1}^n \sqrt{\lambda_i^{L_r}} \leq \frac{n}{\text{vol}(M)} \sqrt{(n - r) \int_M S_r \, dv \int_M (|H|^2 + c) \, dv}, \quad (1.18)
\]
where $c(r) = (n - r) \binom{n}{r}$.

In particular, for $c = 0$, the equality in (1.17) holds if and only if $M$ is a sphere in $\mathbb{R}^{n+1}$; for $c = 1$, the equality in (1.17) holds if and only if $x$ is $r$-minimal. For $c = 0$, the equality in (1.18) holds if and only if $M$ is a sphere in $\mathbb{R}^{n+1}$; for $c = 1$, the equality in (1.18) holds if and only if $x$ is minimal.
Using the fact
\[ \lambda_1^{L_r} \leq \lambda_2^{L_r} \leq \cdots \leq \lambda_n^{L_r}, \]
we have
\[ \sum_{i=1}^n \sqrt{\lambda_i^{L_r}} \geq n \sqrt{\lambda_1^{L_r}}. \]
Therefore, we obtain the following upper bound of the first eigenvalue \( \lambda_1^{L_r} \) from (1.18):

**Corollary 1.2.** Under the assumption of Theorem 1.1, we have
\[ \lambda_1^{L_r} \leq \frac{n - r}{(\text{vol}(M))^2} \int_M S_r \text{dv} \int_M (|H|^2 + c) \text{dv}. \] (1.19)

In particular, for \( c = 0 \), the equality in (1.19) holds if and only if \( M \) is a sphere in \( \mathbb{R}^{n+1} \); for \( c = 1 \), the equality in (1.19) holds if and only if \( x \) is minimal.

In particular,
\[ \sqrt{\lambda_1^{L_r}} \leq \sqrt{\lambda_2^{L_r}} \leq \cdots \leq \sqrt{\lambda_n^{L_r}} < \sum_{i=1}^n \sqrt{\lambda_i^{L_r}}. \]

Hence, we also obtain

**Corollary 1.3.** Under the assumption of Theorem 1.1, we have
\[ \lambda_n^{L_r} < \frac{n^2(n - r)}{(\text{vol}(M))^2} \int_M S_r \text{dv} \int_M (|H|^2 + c) \text{dv}. \] (1.20)

**Remark 1.1.** When \( N = n + 1 \) and \( c = 0 \), our estimate (1.17) becomes the result (1.14) of Alencar, do Carmo and Rosenberg in [1]. For \( N = n + 1 \) and \( c \geq 0 \), our estimate (1.17) seems like the estimate (1.15) of Grosjean in [9]. But our estimate (1.17) is independent of the convex isometric immersion.

**Remark 1.2.** Clearly, our estimate (1.19) is new. Moreover, we obtain estimates on high order eigenvalues of the elliptic operator \( L_r \) on submanifolds of space forms with arbitrary codimension.

## 2. Proof of Results

In order to complete our proof, we need the following lemma:

**Lemma 2.1.** Under the assumption of Theorem 1.1, for any function \( h_A \in C^2(M) \) satisfying
\[ \int_M h_A u_B u_B = 0, \quad \text{for } B = 1, \cdots, A - 1, \] (2.1)
we have
\[ \lambda_A^{L_r} \int_M \langle T^r \nabla h_A, \nabla h_A \rangle \text{dv} \leq \int_M |\text{div}(T^r \nabla h_A)|^2 \text{dv}; \] (2.2)
and
\[
\sqrt{\lambda_A^{L_r}} \int_M |\nabla h_A|^2 \, dv \leq \delta \int_M \langle T^r \nabla h_A, \nabla h_A \rangle \, dv + \frac{1}{4\delta} \int_M (\Delta h_A)^2 \, dv, \quad (2.3)
\]
where \( \delta \) is any positive constant.

Proof. We let \( \varphi_A = h_A u_0 - u_0 \int_M h_A u_0^2 \, dv \). Then
\[
\int_M \varphi_A u_0 \, dv = 0. \quad (2.4)
\]
It has been shown from (2.1) that
\[
\int_M \varphi_A u_B \, dv = 0, \quad \text{for } B = 1, \cdots, A - 1. \quad (2.5)
\]
Hence, we have from the Rayleigh-Ritz inequality
\[
\lambda_A^{L_r} \int_M \varphi_A^2 \, dv \leq - \int_M \varphi_A L_r(\varphi_A) \, dv. \quad (2.6)
\]
Since \( u_0 \) is a nonzero constant satisfying \( u_0^2 \, \text{vol}(M) = 1 \), and \( T^r \) is symmetric and divergence-free, a direct calculation yields
\[
- \int_M \varphi_A L_r(\varphi_A) \, dv = - \int_M \varphi_A \text{div}(T^r \nabla (h_A u_0)) \, dv
= \int_M \langle T^r \nabla (h_A u_0), \nabla (h_A u_0) \rangle \, dv
= u_0^2 \int_M \langle T^r \nabla h_A, \nabla h_A \rangle \, dv. \quad (2.7)
\]
Putting (2.7) into the inequality (2.6) gives
\[
\lambda_A^{L_r} \int_M \varphi_A^2 \, dv \leq u_0^2 \int_M \langle T^r \nabla h_A, \nabla h_A \rangle \, dv. \quad (2.8)
\]
We define
\[
\omega_A := - \int_M \varphi_A \text{div}(T^r \nabla (h_A u_0)) \, dv = u_0^2 \int_M \langle T^r \nabla h_A, \nabla h_A \rangle \, dv. \quad (2.9)
\]
Then (2.8) gives
\[
\lambda_A^{L_r} \int_M \varphi_A^2 \, dv \leq \omega_A. \quad (2.10)
\]
From the Schwarz inequality and (2.10), we obtain
\[
\lambda_L^r \omega_A^2 = \lambda_L^r \left( \int_M \varphi_A \text{div}(T^r \nabla (h_A u_0)) \, dv \right)^2 \\
\leq \lambda_L^r \left( \int_M \varphi_A^2 \, dv \right) \left( \int_M |\text{div}(T^r \nabla (h_A u_0))|^2 \, dv \right) \\
\leq \omega_A \int_M |\text{div}(T^r \nabla (h_A u_0))|^2 \, dv,
\] (2.11)
which gives
\[
\lambda_L^r \omega_A \leq \int_M |\text{div}(T^r \nabla (h_A u_0))|^2 \, dv. \tag{2.12}
\]
Combining (2.9) with (2.12) yields the inequality (2.2).

On the other hand, from the the Stokes formula, one gets
\[
-u_0 \int_M \varphi_A \Delta h_A \, dv = - \int_M \varphi_A \Delta (h_A u_0) \, dv \\
= - \int_M \left( h_A u_0 - u_0 \int_M h_A u_0^2 \, dv \right) \Delta (h_A u_0) \, dv \\
= \int_M |\nabla (h_A u_0)|^2 \, dv \\
= u_0^2 \int_M |\nabla h_A|^2 \, dv.
\]
Therefore, for any positive constant \( \delta \), we derive from (2.8)
\[
\sqrt{\lambda_L^r u_0^2} \int_M |\nabla h_A|^2 \, dv = - \sqrt{\lambda_L^r} u_0 \int_M \varphi_A \Delta h_A \, dv \\
\leq \delta \lambda_L^r \int_M \varphi_A^2 \, dv + \frac{1}{4\delta} u_0^2 \int_M (\Delta h_A)^2 \, dv \\
\leq \delta u_0^2 \int_M \langle T^r \nabla h_A, \nabla h_A \rangle \, dv + \frac{1}{4\delta} u_0^2 \int_M (\Delta h_A)^2 \, dv. \tag{2.13}
\]
The desired inequality (2.3) is obtained.

Proof of the estimate (1.17) in Theorem 1.1. For \( c = 0 \), according to the orthogonalization of Gram and Schmidt, we get that there exists an orthogonal
matrix $O = (O^B_A)$ such that
\[
\sum_{C=1}^{N} \int_M O^C_A x_C u_B = \sum_{C=1}^{N} O^C_A \int_M x_C u_B = 0, \quad \text{for } B = 1, \ldots, A - 1. \tag{2.14}
\]
Taking $h_A = \sum_{C=1}^{N} O^C_A x_C$ in (2.2), and summing over $A$ from 1 to $N$, we obtain
\[
\sum_{A=1}^{N} \lambda^L_A \int_M \langle T^r \nabla h_A, \nabla h_A \rangle \, dv \leq \sum_{A=1}^{N} \int_M |\text{div}(T^r \nabla h_A)|^2 \, dv. \tag{2.15}
\]
Since $L_r$ is elliptic, namely $T^r$ is positive definite, we have
\[
\sum_{A=1}^{N} \lambda^L_A \int_M \langle T^r \nabla h_A, \nabla h_A \rangle \, dv \geq \lambda^L_1 \sum_{A=1}^{N} \int_M \langle T^r \nabla h_A, \nabla h_A \rangle \, dv. \tag{2.16}
\]
Therefore, from (2.15) and the orthogonal matrix $O$, we derive
\[
\lambda^L_1 \sum_{A=1}^{N} \int_M \langle T^r \nabla x_A, \nabla x_A \rangle \, dv \leq \sum_{A=1}^{N} \int_M |\text{div}(T^r \nabla x_A)|^2 \, dv. \tag{2.17}
\]
Let $E_1, \ldots, E_N$ be a canonical orthonormal basis of $\mathbb{R}^N$, then $x_A = \langle E_A, x \rangle$ and
\[
\nabla (x_A) = \langle E_A, e_i \rangle e_i = E^\top_A, \tag{2.18}
\]
where $\top$ denote the tangent projection to $M$. Therefore,
\[
\sum_{A=1}^{N} \langle T^r \nabla x_A, \nabla x_A \rangle = \sum_{A=1}^{N} T^r_{ij} \langle E_A, e_i \rangle \langle E_A, e_j \rangle = T^r_{ij} \langle e_i, e_j \rangle = \text{trace}(T^r) = (n - r) S_r,
\]
which shows that $S_r > 0$ since $T^r$ is positive definite. Using the definition of $L_r$, we have
\[
\sum_{A=1}^{N} |\text{div}(T^r \nabla x_A)|^2 = \sum_{A=1}^{N} \langle E_A, L_r(x) \rangle^2 = \langle L_r(x) \rangle^2 = (r + 1)^2 |S_{r+1}|^2.
\]
Thus, we derive from (2.17)
\[
(n - r) \lambda^L_1 \int_M S_r \, dv \leq (r + 1)^2 \int_M |S_{r+1}|^2 \, dv. \tag{2.19}
\]
By virtue of the relationships between $S_r$ and $H_r$, we deduce to
\[
\lambda^L_1 \int_M H_r \, dv \leq c(r) \int_M |H_{r+1}|^2 \, dv. \tag{2.20}
\]
When \( c = 1 \),
\[
\mathbb{S}^N = \{ x \in \mathbb{R}^{N+1}; \ |x|^2 = x_0^2 + x_1^2 + \cdots x_N^2 = \frac{1}{c}\}.
\]

Using the similar method, we can derive
\[
\lambda_1^{L_r} \sum_{A=0}^N \langle T^r \nabla x_A, \nabla x_A \rangle \, dv \leq \sum_{A=0}^N \int M |\text{div}(T^r \nabla x_A)|^2 \, dv.
\tag{2.21}
\]

Putting
\[
\sum_{A=0}^N \langle T^r \nabla x_A, \nabla x_A \rangle = \text{trace}(T^r) = (n - r)S_r
\]
and
\[
\sum_{A=0}^N |\text{div}(T^r \nabla x_A)|^2 = |L_r(x)|^2 = (r + 1)^2 |S_{r+1}|^2 + c^2(n - r)^2 S_r^2 |x|^2
\]
\[
= (r + 1)^2 |S_{r+1}|^2 + c(n - r)^2 S_r^2
\]
into (2.21) gives
\[
(n - r)\lambda_1^{L_r} \int_M S_r \, dv \leq \int_M [(r + 1)^2 |S_{r+1}|^2 + c(n - r)^2 S_r^2] \, dv.
\tag{2.22}
\]

Hence, the desired estimate (1.17) is derived.

Next, we consider the case that equalities occur. If \( c \geq 0 \) and the equality in (1.17) holds, then inequalities (2.6), (2.11) and (2.16) become equalities. Hence, we have
\[
\lambda_1^{L_r} = \lambda_2^{L_r} = \cdots = \lambda_N^{L_r} = \mu;
\tag{2.23}
\]
\[
L_r(\varphi_A) = -\mu \varphi_A,
\tag{2.24}
\]
where \( \mu \) is a constant. When \( c = 0 \), from (2.24) and (1.11), we can infer that the vector field \( \varphi = (\varphi_1, \cdots, \varphi_N) \) is parallel with \( S_{r+1} \). Thus, we obtain
\[
\frac{1}{2}(\langle \varphi^2 \rangle, i) = \langle e_i, \varphi \rangle = 0,
\tag{2.25}
\]
which shows that \( |\varphi|^2 \) is constant. Hence \( M \) is a sphere in \( \mathbb{R}^{n+1} \). When \( c = 1 \) and the equality in (1.17) holds, it is easy to see that \( S_{r+1} = 0 \) by combining (2.24) with (1.11). That is to say that \( x \) is \( r \)-minimal.

\[\square\]

Proof of the estimate (1.18) in Theorem 1.1. For \( c = 0 \), we taking \( h_A = \sum_{C=1}^N O^C_A x_C \) in (2.3), where the matric \( O \) is given by (2.14). From (2.18), we get
\[
|\nabla h_A|^2 = |\nabla x_A|^2 = |E_A^T|^2 \leq |E_A|^2 = 1, \quad \forall A,
\tag{2.26}
\]
and
\[
\sum_{A=1}^{N} |\nabla h_A|^2 = \sum_{A=1}^{N} |\nabla x_A|^2 = n. \tag{2.27}
\]
Thus, we infer
\[
\sum_{A=1}^{N} \sqrt{\lambda_A^L} |\nabla h_A|^2 \\
\geq \sum_{i=1}^{n} \sqrt{\lambda_i^L} |\nabla h_i|^2 + \sqrt{\lambda_{n+1}^L} \sum_{\alpha=n+1}^{N} |\nabla h_\alpha|^2 \\
= \sum_{i=1}^{n} \sqrt{\lambda_i^L} |\nabla h_i|^2 + \sqrt{\lambda_{n+1}^L} \left( n - \sum_{j=1}^{n} |\nabla h_j|^2 \right) \tag{2.28}
\]
\[
= \sum_{i=1}^{n} \sqrt{\lambda_i^L} |\nabla h_i|^2 + \sqrt{\lambda_{n+1}^L} \sum_{j=1}^{n} (1 - |\nabla h_j|^2) \\
\geq \sum_{i=1}^{n} \sqrt{\lambda_i^L} |\nabla h_i|^2 + \sum_{j=1}^{n} \sqrt{\lambda_j^L} (1 - |\nabla h_j|^2) \\
= \sum_{i=1}^{n} \sqrt{\lambda_i^L}.
\]
From \( \sum_{A=1}^{N} (\Delta h_A)^2 = |\Delta(x)|^2 = |S_1|^2 = n^2|\mathbf{H}|^2 \). Hence, taking sum on \( A \) from 1 to \( N \) for (2.23), we have
\[
\sum_{i=1}^{n} \sqrt{\lambda_i^L} \text{vol}(M) \leq \delta(n-r) \int_{M} S_r \, dv + \frac{n^2}{4\delta} \int_{M} |\mathbf{H}|^2 \, dv. \tag{2.29}
\]
Minimizing the right hand side of (2.29) by taking
\[
\delta = \frac{n}{2} \sqrt{\frac{\int_{M} |\mathbf{H}|^2 \, dv}{(n-r) \int_{M} S_r \, dv}}
\]
yields
\[
\sum_{i=1}^{n} \sqrt{\lambda_i^L} \text{vol}(M) \leq n \sqrt{(n-r) \int_{M} S_r \, dv \int_{M} |\mathbf{H}|^2 \, dv}. \tag{2.30}
\]
When \( c = 1 \), the proof is similar. We omit it here.

When \( c \geq 0 \) and the equality in (1.18) holds, then inequalities (2.6), (2.13) and (2.28) become equalities. Hence, we have
\[
\lambda_1^L = \lambda_2^L = \cdots = \lambda_N^L = \mu; \tag{2.31}
\]
\[
\Delta(\varphi_A) = -\mu \varphi_A. \tag{2.32}
\]
Similarly, we infer that, for $c = 0$, the equality in (1.18) holds if and only if $M$ is a sphere in $\mathbb{R}^{n+1}$; for $c = 1$, the equality in (1.18) holds if and only if $x$ is minimal.

\[ \square \]

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College of Mathematics and Information Science, Henan Normal University, Xinxiang, Henan 453007, People’s Republic of China
E-mail address: hgy@henannu.edu.cn

School of Mathematics and Statistics, Zhengzhou University, Zhengzhou, Henan 450001, People’s Republic of China
E-mail address: xrrqi@zzu.edu.cn