Quantum Analogue of the Neumann Function of Integer Order

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Abstract: q-Neumann function of integer order $N_n(x; q)$ is obtained and some of its properties are given. q-psi function which is used in deriving $N_n(x; q)$ is also introduced and some of its properties are presented.

March 1999

1. Introduction

The Hahn-Exton q-Bessel functions $J_\nu(x; q)$ which are closely connected to the quantum group of the plane motions are well studied \[1, 2, 3, 4\]. Note that for $\nu = n$ is integer the functions $J_n(x; q)$ and $J_{-n}(x; q)$ are not independent of each other. To our knowledge the quantum analogues of the Bessel functions of integer order which are independent of $J_n(x; q)$ have not been addressed.

The main purpose of this note is to introduce the second independent solution $N_n(x; q)$ of the q-Bessel difference equation. This function possess the same recurrence relations as the Hahn-Exton q-Bessel function and in $q \rightarrow 1^-$ becomes Neumann function of the order $n$. We call it the q-Neumann function of order $n$. This solution is non regular at $x = 0$. It is well known that non-regular solutions are important, since the Green functions are given

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in terms of them. For example the q-Legendre function of the second kind shows up as the Green function on the quantum sphere [3]. The recently obtained Green function on the quantum plane is in fact the superposition of the Hahn-Exton q-Bessel and q-Neumann functions of the order 0 [6].

Since the classical Neumann function of integer order $N_n(x)$ is obtained by taking the derivatives of the Bessel functions $J_\nu(x)$ and $J_{-\nu}(x)$ with respect to the order $\nu$, it involves the psi functions. Therefore to derive the q-Neumann function of integer order we first have to have q-psi function in hand.

The Section 2 is devoted to the introduction of the q-psi function and some of its properties which are employed to derive the q-Neumann function of integer order.

In Section 3 we obtained the q-Neumann function of integer order and presented some relations involving it.

2. q-Psi Function

We define the q-psi function as

$$\psi_q(\nu) = \frac{d}{d\nu} \log \Gamma_q(\nu),$$

(1)

where the q-gamma function $\Gamma_q(\nu)$ is defined by $(0 < q < 1)$

$$\Gamma_q(\nu) = (1 - q)^{1-\nu} \prod_{l=1}^{\infty} \frac{1 - q^l}{1 - q^{\nu+l}}.$$  

(2)

Many properties of the gamma function were derived by Askey [4]. It is obvious from (2) that $\Gamma_q(\nu)$ has poles at $\nu = 0, 1, 2, \ldots$. The residue at $\nu = -n$ is

$$\lim_{\nu \to -n} (\nu + n)\Gamma_q(\nu) = (-1)^n \frac{(q - 1)q^{-n(n+1)/2}}{\log q \Gamma_q(n + 1)}.$$  

(3)

The explicit form of $\psi_q(\nu)$ is

$$\psi_q(\nu) = -\log(1 - q) + \log q \sum_{l=0}^{\infty} \frac{q^{\nu+l}}{1 - q^{\nu+l}}.$$  

(4)
The recurrence relations and asymptotic conditions satisfied by this function are

\[ \psi_q(\nu + n) = \psi_q(\nu) - \log q \sum_{l=0}^{n-1} \frac{q^{\nu + l}}{1 - q^{\nu + l}}, \quad (5) \]

\[ \psi_q(\nu - n) = \psi_q(\nu) + \log q \sum_{l=0}^{n} \frac{q^{\nu - l}}{1 - q^{\nu - l}}, \quad (6) \]

and

\[ \lim_{\nu \to \infty} \psi_q(\nu) = - \log(1 - q); \quad \lim_{\nu \to -\infty} \psi_q(\nu) = \infty. \quad (7) \]

\( \psi_q(\nu) \) has poles at \( \nu = 0, 1, 2, \ldots \) with the residue

\[ \lim_{\nu \to -n} (\nu + n) \psi_q(\nu) = \log q \lim_{\nu \to -n} (\nu + n) \sum_{l=0}^{\infty} \frac{q^{\nu + l}}{1 - q^{\nu + l}} \]

\[ = \log q \lim_{\nu \to -n} \frac{(\nu + n)}{1 - q^{\nu + n}} = -1 \quad (8) \]

Equations (5) and (8) imply that

\[ \lim_{\nu \to \infty} \frac{\psi_q(\nu)}{\Gamma_q(\nu)} = (-1)^n q^{-n(n+1)/2} \frac{\log q}{1 - q} \Gamma_q(n + 1). \quad (9) \]

Before closing this section we like to present the \( q \to 1^- \) limit of \( \psi_q(\nu) \).

We first rewrite it as

\[ \psi_q(\nu) = \lim_{n \to \infty} \sum_{l=1}^{n} \left( \log \frac{1 - q^{l+1}}{1 - q^l} + \frac{q^{\nu + l - 1} \log q}{1 - q^{\nu + l - 1}} \right). \quad (10) \]

Taking the \( q \to 1^- \) limit in the finite sum in the above formula we have

\[ \lim_{q \to 1^-} \psi_q(\nu) = \lim_{n \to \infty} \left( \log(n + 1) - \sum_{l=1}^{n} \frac{1}{\nu + l - 1} \right) \]

\[ = \lim_{n \to \infty} \left( \log(n + 1) - \sum_{l=1}^{n+1} \frac{1}{l} + \sum_{l=1}^{n} \left( \frac{1}{l} - \frac{1}{\nu + l - 1} \right) + \frac{1}{n + 1} \right) \]

\[ = -C + \sum_{l=0}^{\infty} \left( \frac{1}{l + 1} - \frac{1}{\nu + l} \right) = \psi(\nu), \quad (11) \]
where
\[ C = \lim_{n \to \infty} \left( \sum_{l=1}^{n} \frac{1}{l} - \log n \right) = -\psi(1) \] (12)
is the Euler number \[8\]. It is then natural to define the q-Euler number
\[ C_q \equiv -\psi_q(1) \] (13)

3. q-Neumann Function of Integer Order

The Hahn-Exton q-Bessel function of order \( \nu \) is defined as
\[ J_{\nu}(x; q) = \sum_{k=0}^{\infty} \frac{(-1)^k q^{k(k+1)/2}}{\Gamma_q(k+1)\Gamma_q(k+\nu+1)} x^{2k+\nu} \] (14)
satisfies the q-difference equation
\[ J_{\nu}(q^{1/2}x; q) + J_{\nu}(q^{-1/2}x; q) + q^{-\nu/2}((1-q)^2x^2 - q^\nu - 1)J_{\nu}(x; q) = 0. \] (15)

For non-integer \( \nu \) let us define the function
\[ N_{\nu}(x; q) = \frac{\cos(\pi\nu)J_{\nu}(x; q) - q^{-\nu/2}J_{-\nu}(q^{-\nu/2}x; q)}{\sin(\pi\nu)} \] (16)
which is the second independent solution of the q-difference equation (15). It satisfies the same recurrence relations as the Hahn-Exton q-Bessel functions. For example we have
\[ q^{(\nu+1)/2}N_{\nu+1}(q^{1/2}x; q) - N_{\nu+1}(x; q) = (q - 1)xN_{\nu}(x; q), \] (17)
\[ q^{\nu/2}N_{\nu}(q^{-1/2}x; q) - N_{\nu}(x; q) = (q - 1)xN_{\nu+1}(x; q), \] (18)

Note that for integer \( \nu = n \) one has the property
\[ J_{-n}(x; q) = (-1)^n q^{n/2}J_n(q^{n/2}x; q). \] (19)

Therefore to derive the form of (16) for integer order we use the L’Hospital rule
\[ \pi N_n(x; q) = \frac{d}{d\nu} J_{\nu}(x; q) \bigg|_{\nu=n} - (-1)^n \frac{d}{d\nu} \left( q^{-\nu/2}J_{-\nu}(q^{-\nu/2}x; q) \right) \bigg|_{\nu=n}. \] (20)
Making use of (9) and (19) for $n \in \mathbb{Z}_+$ we get

$$\pi N_n(x; q) = 2 J_n(x; q) \log(q^{1/4} x) + \frac{\log q}{1 - q} \sum_{k=0}^{n-1} \frac{\Gamma_q(n-k)}{\Gamma_q(k+1)} x^{2k-n}$$

$$- \sum_{k=0}^{\infty} \frac{(-1)^k q^{k(k+1)/2}}{\Gamma_q(k+1)\Gamma_q(k+n+1)} \left( \psi_q(k+n+1) + \psi_q(k+1) + k \log q \right).$$

(21)

Using the recurrence relation (5) and the definition (13) we arrive at

$$\pi N_n(x; q) = 2 J_n(x; q)(\log(q^{1/4} x) + C_q) + \frac{\log q}{1 - q} \sum_{k=0}^{n-1} \frac{\Gamma_q(n-k)}{\Gamma_q(k+1)} x^{2k-n}$$

$$+ \log q \sum_{k=1}^{\infty} \frac{(-1)^k q^{k(k+1)/2}}{\Gamma_q(k+1)\Gamma_q(k+n+1)} \left( \sum_{l=1}^{k+n} \frac{q^l}{1 - q^l} + \sum_{l=1}^{k} \frac{1}{1 - q^l} \right)$$

$$+ \log q \frac{x^n}{\Gamma_q(n+1)} \sum_{k=1}^{n} \frac{q^k}{1 - q}.$$  

(22)

For $n = 0$ we follow the similar steps and obtain

$$\pi N_0(x; q) = 2 J_0(x; q)(\log(q^{1/4} x) + C_q)$$

$$+ \log q \sum_{k=1}^{\infty} \frac{(-1)^k q^{k(k+1)/2}}{\Gamma_q(k+1)\Gamma_q(k+1)} \sum_{l=1}^{k} \frac{1 + q^l}{1 - q^l}. $$

(23)

It is obvious that (22) and (23) become the usual Neumann functions in $q \to 1^-$ limit. From the construction it is clear that the q-Neumann functions of integer order satisfy the difference equation (13) and possess all recurrence relations satisfied by the Harh-Exton q-Bessel functions of integer order. We also have

$$N_{-n}(x; q) = (-1)^n q^{n/2} N_n(q^{n/2} x; q).$$

(24)

Before closing the section we note that several relations involving the q-Neumann functions can be obtained from those of the Hahn-Exton q-Bessel functions. For example using the product formula

$$\sum_{s=\infty}^{\infty} q^s J_x(q^{s/2}; q) J_{x-\nu}(q^{s/2}; q) J_\nu(rq^{(y+\nu+z)/2}; q) =$$

$$= J_0(rq^{(x+y)/2}; q) J_\nu(rq^{(\nu+y)/2}; q)$$

(25)
which is valid for \( r, x, y, \nu \in \mathbb{C}; \text{Re}(x) > -1, |r|^2 q^{1+\text{Re}(x)+\text{Re}(y)} < 1 \) and \( r \neq 0 \) we obtain the product formula for \( x = -y = \nu/2 \)

\[
\sum_{s=-\infty}^{\infty} q^s J_{\nu/2}(q^{s/2}; q) J_{-\nu/2}(q^{s/2}; q) N_\nu(r q^{\nu/4+z/2}; q) = J_0(r; q) N_\nu(r q^{\nu/4}; q).
\]

(26)

References

[1] Koelink, H. T., Duke Math. J., 76, 483 (1994).

[2] Vaksman, L. L., Korogodski, L. I., Soviet Math. Dokl., 39, 173 (1989).

[3] Woronowicz, S. L., Lett. Math. Phys., 23, 251 (1991); Commun. Math. Phys., 144, 417 (1992); Commun. Math. Phys., 136, 399 (1991).

[4] Koornwinder, T. H. and Swarttouw, R. F., Trans. Amer. Math. Soc., 333, 445 (1992).

[5] Ahmedov, H. and Duru, I. H., J. Phys. A: Math. Gen, 31, 5741 (1998).

[6] Ahmedov, H. and Duru, I. H.; Green function on the quantum plane, math.QA/9812108 (1998).

[7] Askey, R., Applicable Analysis 8, 125, (1978)

[8] Gradshtein, I. S. and Ryzhik, I. M.; Tables of Integrals, Series and Products. Academic Press, New York (1980).

[9] Koelink, H. T. and Swarttouw, R. F., J. Approx. Theory 81, 260 (1995).