Charged Fermions Tunnelling from Kerr-Newman Black Holes

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Abstract

We consider the tunnelling of charged spin-\(\frac{1}{2}\) fermions from a Kerr-Newman black hole and demonstrate that the expected Hawking temperature is recovered. We discuss certain technical subtleties related to the obtention of this result.

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1 Introduction

Semi-classical methods of modeling Hawking radiation as a tunnelling effect were developed over the past decade and have garnered a lot of interest [1]-[28]. The earliest work with black hole tunnelling was done by Kraus and Wilczek [1], and was refined by various researchers [2, 3, 4]. From this approach an alternative way of understanding black hole radiation emerged. In particular one can calculate the Hawking temperature in a manner independent of traditional Wick Rotation methods or the original method of modelling gravitational collapse [29]. Tunnelling provides not only a useful verification of thermodynamic properties of black holes but also an alternate conceptual means for understanding the underlying physical process of black hole radiation. For scalar field emission it has been shown to be very robust, having been successfully applied to a wide variety of interesting and exotic spacetimes, including the Kerr and Kerr-Newman cases [10, 11, 14], black rings [12], the 3-dimensional BTZ black hole [7, 13], the Vaidya spacetime [18], other dynamical black holes [19], Taub-NUT spacetimes [14], Gödel spacetimes [22], and dynamical horizons [19]. Tunnelling methods have even been applied to horizons that are not black hole horizons including those with cosmological horizons [5, 6, 25], and Rindler Spacetimes [4, 11, 24] for which it has been shown the Unruh temperature [30] is in fact recovered.

In general tunnelling methods involve calculating the imaginary part of the action for the (classically forbidden) process of s-wave emission across the horizon, which in turn is related to the Boltzmann factor for emission at the Hawking temperature. Different approaches exist for calculating the imaginary part of the action for the emitted particle. The first black hole tunnelling method developed was the Null Geodesic Method used by Parikh and Wilczek [3], which followed from the work of Kraus and Wilczek [1]. The other

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approach to black hole tunnelling is the Hamilton-Jacobi Ansatz used by Agheben et al [7], which is an extension of the complex path analysis of Padmanabhan et al [4]. Recently we have extended black hole tunnelling to include the emission of spin 1/2 fermions [24], demonstrating that fermion tunnelling from both a Rindler horizon and a generic non-rotating black hole recovers the expected results for temperature. While perhaps not surprising, the result is non-trivial insofar as fermionic vacua are distinct from bosonic vacua, and can lead to distinct physical results [31]. This work has been extended to describing fermionic tunnelling across horizons in other spacetimes such as the BTZ black hole [26], and dynamical horizons [27].

All tunnelling approaches use the fact that the WKB approximation of the tunnelling probability for the classically forbidden trajectory from inside to outside the horizon is given by:

\[ \Gamma \propto \exp(-2 \text{Im} I) \]  

(1)

where \( I \) is the classical action of the trajectory to leading order in \( \hbar \) (here set equal to unity). Where these methods differ is in how the action is calculated. For the Null Geodesic method the only part of the action that contributes an imaginary term is \( \int_{r_{\text{in}}}^{r_{\text{out}}} p_r dr \), where \( p_r \) is the momentum of the emitted null s-wave. Then by using Hamilton’s equation and knowledge of the null geodesics it is possible to calculate the imaginary part of the action. For the Hamilton-Jacobi ansatz it is assumed that the action of the emitted (scalar) particle satisfies the relativistic Hamilton-Jacobi equation. From the symmetries of the metric one picks an appropriate ansatz for the form of the action and plugs it into the Relativistic Hamilton-Jacobi Equation to solve. (For a detailed comparison of the Hamilton-Jacobi Ansatz and Null-Geodesic methods see [14]). The Hamilton-Jacobi Ansatz came from applying the WKB approximation to the Klein-Gordon equation. To lowest order in WKB this results in the Hamilton-Jacobi equation. For tunnelling of spin 1/2 particles it can be shown that applying the WKB approximation to the Dirac equation instead of the Klein-Gordon equation yields the tunnelling probability for fermions [24]. This was the first time that the tunnelling approach had been used to model spin 1/2 particles.

In this paper we extend the tunnelling method to model charged spin 1/2 particle emission from rotating black holes. To this end we apply the fermion tunnelling method to the Kerr-Newman black hole for both massless and massive charged particle emission. This extension introduces some non-trivial technical features associated with the choice of \( \gamma \) matrices. We confirm that spin 1/2 fermions are emitted at the expected Hawking Temperature from rotating black holes, providing further evidence for the universality of black hole radiation.
2 Charged Spin 1/2 Particle Emission From Kerr-Newman Black Holes

We will consider particle emission from the Kerr-Newman solution. The Kerr-Newman metric and vector potential are given by

\[ ds^2 = -f(r, \theta)dt^2 + \frac{dr^2}{g(r, \theta)} - 2H(r, \theta)dtd\phi + K(r, \theta)d\phi^2 + \Sigma(r, \theta)d\theta^2 \]

\[ A_a = \frac{er\Sigma(r)}{(dt)_a - a \sin^2 \theta (d\phi)_a} \] (2)

\[ f(r, \theta) = \frac{\Delta(r) - a^2 \sin^2 \theta}{\Sigma(r, \theta)} \]

\[ g(r, \theta) = \frac{\Delta(r)}{\Sigma(r, \theta)} \]

\[ H(r, \theta) = \frac{a \sin^2 \theta (r^2 + a^2 - \Delta(r))}{\Sigma(r, \theta)} \]

\[ K(r, \theta) = \frac{(r^2 + a^2)^2 - \Delta(r)a^2 \sin^2 \theta}{\Sigma(r, \theta)} \sin^2(\theta) \]

\[ \Sigma(r, \theta) = r^2 + a^2 \cos^2 \theta \]

\[ \Delta(r) = r^2 + a^2 + e^2 - 2Mr \]

Since the tunnelling method is not applicable to extremal black holes \[14\], we will assume a non-extremal black hole so that \( M^2 > a^2 + e^2 \). Consequently there are two horizons located at \( r_{\pm} = M \pm \sqrt{M^2 - a^2 - e^2} \).

It is convenient for our calculations to work with the function \( F(r, \theta) = -(g^{tt})^{-1} \) where

\[ F(r, \theta) = f(r, \theta) + \frac{H^2(r, \theta)}{K(r, \theta)} = \frac{\Delta(r)\Sigma(r, \theta)}{(r^2 + a^2)^2 - \Delta(r)a^2 \sin^2 \theta} \] (3)

and where the angular velocity at the black hole horizon is

\[ \Omega_H = \frac{H(r_+, \theta)}{K(r_+, \theta)} = \frac{a}{r_+^2 + a^2} \] (4)

We will only show the calculation explicitly for the spin up case; the final result is also the same for the spin down case as can be easily shown using the methods described below. In the non-rotating case a statistical argument was used to justify the assumption that overall a zero angular momentum state is maintained for fermion emission, because as many particles with spin pointing radially outward (spin up) would be emitted as particles with spin pointed radially inward (spin down). This argument is still valid in the rotating case: the statistical distribution of spins in the fermion emission spectrum should not alter the angular momentum of the black hole.

The Dirac equation with electric charge is:

\[ i\gamma^\mu(D_\mu - \frac{iq}{\hbar}A_\mu)\psi + \frac{m}{\hbar}\psi = 0 \] (5)

3
where:

\[
D_\mu = \partial_\mu + \Omega_\mu \tag{6}
\]

\[
\Omega_\mu = \frac{1}{2} i \Gamma^\alpha_\mu \Sigma_{\alpha\beta} \tag{7}
\]

\[
\Sigma_{\alpha\beta} = \frac{1}{4} i [\gamma^\alpha, \gamma^\beta] \tag{8}
\]

The \(\gamma^\mu\) matrices satisfy \(\{\gamma^\mu, \gamma^\nu\} = 2 g^{\mu\nu} \times 1\). We choose a representation for them in the form:

\[
\gamma^t = \frac{1}{\sqrt{F(r, \theta)}} \gamma^0 \quad \gamma^r = \sqrt{g(r, \theta)} \gamma^3 \quad \gamma^\theta = \frac{1}{\sqrt{\Sigma(r, \theta)}} \gamma^1 \tag{9}
\]

where the \(\gamma^a\)'s are simply the following chiral \(\gamma\)'s for Minkowski space

\[
\gamma^0 = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \quad \gamma^1 = \begin{pmatrix} 0 & \sigma^1 \\ \sigma^1 & 0 \end{pmatrix} \\
\gamma^2 = \begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix} \quad \gamma^3 = \begin{pmatrix} 0 & \sigma^3 \\ \sigma^3 & 0 \end{pmatrix} \tag{10}
\]

and the \(\sigma\)'s are the Pauli Matrices

\[
\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{11}
\]

and we denote \(\xi_{1/1}\) for the eigenvectors of \(\sigma^3\). Note that

\[
\gamma^5 = i \gamma^t \gamma^r \gamma^\theta \gamma^\phi = \frac{g}{\sqrt{FK\Sigma}} \left( -I + \frac{H}{\sqrt{FK}} \sigma^2 \right) \tag{12}
\]

is the resulting \(\gamma^5\) matrix.

The spin up (i.e. +ve r-direction) ansatz for the Dirac field, has the form:

\[
\psi_1(t, r, \theta, \phi) = \begin{bmatrix} A(t, r, \theta, \phi) \xi_\uparrow \\ B(t, r, \theta, \phi) \xi_\uparrow \end{bmatrix} \exp \left[ \frac{i}{\hbar} \int_I(t, r, \theta, \phi) \right] \tag{13}
\]

In order to apply the WKB approximation we insert the ansatz (13) for spin up particles into the Dirac Equation. Dividing by the exponential term and multiplying by \(\hbar\) the resulting equations to leading order
in $\hbar$ are

\[ 0 = -B \left[ \frac{1}{\sqrt{F(r, \theta)}} \partial_t I_1 + \sqrt{g(r, \theta)} \partial_r I_1 + \frac{H(r, \theta)}{K(r, \theta) \sqrt{F(r, \theta)}} \partial_\theta I_1 \right. \]

\[ + \frac{q_{er}}{\Sigma(r, \theta) \sqrt{F(r, \theta)}} \left( 1 - \frac{H(r, \theta)}{K(r, \theta) a \sin^2(\theta)} \right) \right] + Am \] (14)

\[ 0 = -B \left[ \frac{i}{\sqrt{K(r, \theta)}} (\partial_\phi I_1 - \frac{q_{er}}{\Sigma(r, \theta) a \sin^2 \theta} \right) \right] \] (15)

\[ 0 = A \left[ \frac{1}{\sqrt{F(r, \theta)}} \partial_t I_1 - \sqrt{g(r, \theta)} \partial_r I_1 + \frac{H(r, \theta)}{K(r, \theta) \sqrt{F(r, \theta)}} \partial_\theta I_1 \right. \]

\[ + \frac{q_{er}}{\Sigma(r, \theta) \sqrt{F(r, \theta)}} \left( 1 - \frac{H(r, \theta)}{K(r, \theta) a \sin^2(\theta)} \right) \right] + Bm \] (16)

\[ 0 = -A \left[ \frac{i}{\sqrt{K(r, \theta)}} (\partial_\phi I_1 - \frac{q_{er}}{\Sigma(r, \theta) a \sin^2 \theta} \right) \right] \] (17)

Note that although $A, B$ are not constant, their derivatives – and the components $\Omega_\mu$ – are all of order $O(\hbar)$ and so can be neglected to lowest order in WKB.

When $m \neq 0$ equations (14) and (16) couple whereas when $m = 0$ they decouple. We employ the ansatz

\[ I_1 = -Et + J \phi + W(r, \theta) \] (18)

and insert it into equations (14,17) (where we consider only the positive frequency contributions without loss of generality). To simplify the expressions we expand the equations near the horizon and find

\[ 0 = -B \left( \frac{-E + \Omega_H J + \frac{q_{er}}{r_+^2 + a^2}}{\sqrt{F_+(r_+, \theta)(r-r_+)}} + \sqrt{g_+(r_+, \theta)(r-r_+)} W_+(r, \theta) \right) + Am \] (19)

\[ 0 = -B \left( \frac{i}{\sqrt{K_+(r_+, \theta)}} (J - \frac{q_{er}}{\Sigma(r_+, \theta) a \sin^2 \theta} \right) \right] \] (20)

\[ 0 = A \left( \frac{-E + \Omega_H J + \frac{q_{er}}{r_+^2 + a^2}}{\sqrt{F_+(r_+, \theta)(r-r_+)}} - \sqrt{g_+(r_+, \theta)(r-r_+)} W_+(r, \theta) \right) + Bm \] (21)

\[ 0 = -A \left( \frac{i}{\sqrt{K_+(r_+, \theta)}} (J - \frac{q_{er}}{\Sigma(r_+, \theta) a \sin^2 \theta} \right) \right] \] (22)

where

\[ g_+(r_+, \theta) = \frac{\Delta_+(r_+)}{\Sigma(r_+, \theta)} = \frac{2r_+ - 2M}{r_+^2 + a^2 \cos^2(\theta)} \]

\[ F_+(r_+, \theta) = \frac{\Delta_+(r_+) \Sigma(r_+, \theta)}{(r_+^2 + a^2)^2} = \frac{(2r_+ - 2M)(r_+^2 + a^2 \cos^2(\theta))}{(r_+^2 + a^2)^2} \]
In the massless case it is possible to pull $\frac{1}{\sqrt{\Sigma(r, \theta)}}$ out of equations (19) and (21), making these equations independent of $\theta$. Furthermore, equations (20) and (22) have no explicit $r$ dependence. From this we can conclude that near the black horizon it is possible to further separate the function $W$

$$W(r, \theta) = W(r) + \Theta(\theta)$$

and we see that equations (20) and (22) both yield the same equation for $\Theta$ regardless of $A$ or $B$.

Equations (19) and (21) then have two possible solutions

$$A = 0 \quad \text{and} \quad W'(r) = W_+(r) = \frac{(E - \Omega_H J - \frac{qer}{r_+^2 + a^2})(r_+^2 + a^2)}{\Delta_r(r_+)(r - r_+)}$$

$$B = 0 \quad \text{and} \quad W'(r) = W_-(r) = \frac{-(E - \Omega_H J - \frac{qer}{r_+^2 + a^2})(r_+^2 + a^2)}{\Delta_r(r_+)(r - r_+)}$$

where the prime denotes a derivative with respect to $r$ and $W_{+/-}$ corresponds to outgoing/incoming solutions.

Since we are only concerned with calculating the semi-classical tunnelling probability, we will need to multiply the resulting wave equation by its complex conjugate. So the portion of the trajectory that starts outside the black hole and continues to the observer will not contribute to the final tunnelling probability and can be safely ignored (since it will be entirely real). Therefore, the only part of the wave equation that contributes to the tunnelling probability is the contour around the black hole horizon. For a visual representation of the deformation of the contour see Figure 1. This contour differs somewhat relative to conventions Padmanabhan defines [4], in which contours in the upper half plane are selected for both ingoing and outgoing particles. For the emission integral he multiplies his equivalent of $W_+(\partial S_0/\partial r)$ by a minus sign “where the minus sign in front of the integral corresponds to the initial condition that $\partial S_0/\partial r > 0$ at $r = r_1 < r_0$” [4]. Instead we choose the mathematically equivalent convention that the outgoing contour is in the lower half plane and so do not multiply by a minus sign.

The probabilities of crossing the horizon in each direction are proportional to

$$\text{Prob}[\text{out}] \propto \exp[-2 \text{Im } I] = \exp[-2(\text{Im } W_+ + \text{Im } \Theta)] \quad (23)$$

$$\text{Prob}[\text{in}] \propto \exp[-2 \text{Im } I] = \exp[-2(\text{Im } W_- + \text{Im } \Theta)] \quad (24)$$

To ensure that the probabilities are correctly normalized so that any incoming particles crossing the horizon have a 100% chance of entering the black hole we need to divide each equation by (24). From this the probability of going from outside to inside the horizon will be equal to 1 and this implies that the probability of a particle tunnelling from inside to outside the horizon is:

$$\Gamma \propto \frac{\text{Prob}[\text{out}]}{\text{Prob}[\text{in}]} = \frac{\exp[-2(\text{Im } W_+ + \text{Im } \Theta)]}{\exp[-2(\text{Im } W_- + \text{Im } \Theta)]} = \exp[-4 \text{Im } W_+] \quad (25)$$

Solving for $W_+$ yields

$$W_+(r) = \int \frac{(E - \Omega_H J - \frac{qer}{r_+^2 + a^2})(r_+^2 + a^2)}{\Delta_r(r_+)(r - r_+)}$$
Figure 1: Diagram of contours between black hole and observer for both outgoing and incoming trajectories
and after integrating around the pole (and dropping the + subscript) we obtain

\[ W = \frac{\pi i (E - \Omega HJ - \frac{qer_+}{r_+^2 + a^2})}{2r_+ - 2M} (r_+^2 + a^2) \]

\[ \text{Im} W = (E - \Omega HJ - \frac{qer_+}{r_+^2 + a^2}) \frac{\pi}{2} [\frac{r_+}{r_+^2 + a^2}]^2 (r_+ - M) \]

(26)

The resulting tunnelling probability is

\[ \Gamma = \exp[-2\pi \frac{r_+^2 + a^2}{(r_+ - M)} (E - \Omega HJ - \frac{qer_+}{r_+^2 + a^2})] \]

giving the expected Hawking temperature

\[ T_H = \frac{1}{2\pi r_+^2 + a^2} = \frac{1}{2\pi 2M(M + (M^2 - a^2 - e^2)^{\frac{1}{2}}) - e^2} \]

(27)

for a charged rotating black hole.

In the massive case equations (19) and (21) no longer decouple and analysis of the tunnelling is more subtle. We begin by eliminating the function \( W, (r, \theta) \) from these two equations so we can find an equation relating \( A \) and \( B \) in terms of known quantities. Subtracting \( B \times (21) \) from \( A \times (19) \) gives

\[ 0 = \frac{2AB(E - \Omega HJ - \frac{qer_+}{r_+^2 + a^2})}{\sqrt{F_r(r_+, \theta)}(r - r_+)} + mA^2 - mB^2 = 0 \]

(28)

\[ 0 = m\sqrt{F_r(r_+, \theta)}(r - r_+)(A) + 2(E - \Omega HJ - \frac{qer_+}{r_+^2 + a^2})(\frac{A}{B}) \]

\[ -m\sqrt{F_r(r_+, \theta)}(r - r_+) \]

(29)

and so

\[ \frac{A}{B} = \frac{(E - \Omega HJ - \frac{qer_+}{r_+^2 + a^2}) \pm \sqrt{(E - \Omega HJ - \frac{qer_+}{r_+^2 + a^2})^2 + m^2 F_r(r_+, \theta)(r - r_+)} m\sqrt{F_r(r_+, \theta)(r - r_+)} \]

where

\[ \lim_{r \to r_+} \left( \frac{(E - \Omega HJ - \frac{qer_+}{r_+^2 + a^2}) \pm \sqrt{(E - \Omega HJ - \frac{qer_+}{r_+^2 + a^2})^2 + m^2 F_r(r_+, \theta)(r - r_+)} m\sqrt{F_r(r_+, \theta)(r - r_+)} \right) = \left\{ \begin{array}{c} 0 \\ -\infty \end{array} \right\} \]

(30)

for the upper/lower sign respectively.

Consequently at the horizon either \( \frac{A}{B} \to 0 \) or \( \frac{A}{B} \to -\infty \), i.e. either \( A \to 0 \) or \( B \to 0 \). For \( A \to 0 \) at the horizon, we solve (21) in terms of \( m \) and insert into (19), obtaining

\[ W_r(r, \theta) = \frac{(E - \Omega HJ - \frac{qer_+}{r_+^2 + a^2}) (1 + \frac{A^2}{B^2})}{\sqrt{F_r(r_+, \theta)g_+ (r_+, \theta)(r - r_+)} (1 - \frac{A^2}{B^2})} \]

(31)
Note that the $\theta$-dependence drops out of this expression, i.e.

$$W_r(r, \theta) \equiv W'_r(r) = \frac{(E - \Omega_H J - \frac{qer}{r^2+a^2})}{\sqrt{F_r(r_+, \theta)g_r(r_+, \theta)(r-r_+)}} \left( 1 + \frac{\lambda^2}{4\pi^2} \right)$$

since $\frac{\lambda}{4\pi}$ is zero at the horizon the result of integrating around the pole is the same as in the massless case. For $B \to 0$ we can simply rewrite the expression (31) in terms of $\frac{B}{A}$ to get

$$W_r(r, \theta) \equiv W'_r(r) = \frac{-(E - \Omega_H J - \frac{qer}{r^2+a^2})}{\sqrt{F_r(r_+, \theta)g_r(r_+, \theta)(r-r_+)}} \left( 1 + \frac{B^2}{4\pi^2} \right)$$

Again, since the extra contributions vanish at the horizon, the result of integrating around the pole for $W$ in the massive case is the same as the massless case and we recover the Hawking temperature (27) for the Kerr-Newman black hole.

The spin-down case proceeds in a manner fully analogous to the spin-up case discussed above. Other than some changes of sign the equations are of the same form as the spin up case. For both the massive and massless cases the temperature (27) is obtained, implying that both spin up and spin down particles are emitted at the same rate.

### 3 Technical Issues

With rotating spacetimes the choice $\gamma$ matrices is quite relevant, not only for ease of calculation but also for tractability. In order to demonstrate this we will repeat the calculation for a different (and less convenient) choice of $\gamma$ matrices. We will also set the charge $q$ of the emitted particles to zero for simplicity.

Consider the choice

$$\tilde{\gamma}^t = \frac{1}{\sqrt{f(r, \theta)}} \left( \gamma^0 - \frac{H(r, \theta)}{\sqrt{F(r, \theta)K(r, \theta)}} \gamma^2 \right)$$

$$\tilde{\gamma}^r = \sqrt{g(r, \theta)} \gamma^3$$

$$\tilde{\gamma}^\theta = \frac{1}{\sqrt{\Sigma(r, \theta)}} \gamma^1$$

$$\tilde{\gamma}^\phi = \frac{1}{\sqrt{F(r, \theta)K(r, \theta)}} \gamma^2$$

where we use the same chiral $\gamma^a$ matrices (10) as before. This choice satisfies the correct anti-commutation relations $\{\tilde{\gamma}^\mu, \tilde{\gamma}^\nu\} = 2g^{\mu\nu}$, and corresponds to a different choice of tetrad basis for the metric.

Naively choosing the same ansatz as before

$$\psi_1(t, r, \theta, \phi) = \begin{bmatrix} A \\ 0 \\ B \\ 0 \end{bmatrix} \exp \left[ \frac{i}{\hbar} \int \right]$$

we find upon insertion into the (chargeless) Dirac equation (5) we obtain
\[-B \left( \frac{1}{\sqrt{f}} \partial_t I + \sqrt{g} \partial_r I \right) + Am = 0 \tag{33} \]
\[-B \left( i \sqrt{\frac{f}{FK}} \partial_\phi I - i \frac{H}{\sqrt{FK}} \partial_t I + \frac{1}{\sqrt{\Sigma}} \partial_{\theta} I \right) = 0 \tag{34} \]
\[A \left( \frac{1}{\sqrt{f}} \partial_t I - \sqrt{g} \partial_r I \right) + Bm = 0 \tag{35} \]
\[-A \left( i \sqrt{\frac{f}{FK}} \partial_\phi I - i \frac{H}{\sqrt{FK}} \partial_t I + \frac{1}{\sqrt{\Sigma}} \partial_{\theta} I \right) = 0 \tag{36} \]

Repeating the same kind of analysis as before (using the ansatz \( I = -Et + J\phi + W \)) we find from (33) and (35) that
\[W_{r \pm} = \pm \frac{E}{\sqrt{fg}} \]

Since \( f \) does not vanish at the horizon (except when \( \sin \theta = 0 \)), this expression does not have a simple pole at the horizon. It is not possible to solve the expression for arbitrary \( \theta \), and the calculation becomes intractable. This situation is analogous to what happens in the scalar field case if one naively applies the null geodesic method to a rotating black hole by trying to force \( \phi \) to be constant (id \( d\phi = 0 \)), as previously demonstrated [14].

In order to understand this issue in more detail it is useful to examine the similarity transformation between \( \gamma^\mu \) and \( \tilde{\gamma}^\mu \). We find that:
\[\tilde{\gamma}^\mu = S \gamma^\mu S^{-1}, \text{ for all } \mu \]
when:
\[S = \begin{pmatrix} aI - b\sigma^2 & 0 \\ 0 & aI + b\sigma^2 \end{pmatrix} \]
where
\[a = \frac{1}{2} \left( \sqrt{\frac{F}{f}} + 1 \right) \quad b = \frac{1}{2} \left( \sqrt{\frac{F}{f}} - 1 \right) \tag{37} \]

The transformation \( S \) is similar to a Lorentz boost in the \( \phi \) direction. Applying it to the spin up ansatz used previously we find
\[\tilde{\psi}_{\gamma}(t, r, \theta, \phi) = \begin{bmatrix} Aa \\ -Aib \\ Ba \\ Bib \end{bmatrix} \exp \left[ \frac{iI}{\hbar} \right] \tag{38} \]

As \( r \to \infty \) we see that \( a \to 1 \) and \( b \to 0 \), yielding the same spin up ansatz in this limit. Inserting (38) into the (chargeless) Dirac equation (5) and following the same procedure as before results in the same expression (27) for the temperature. This is not surprising since all we have done is applied a similarity transformation to the Dirac equation, and we shall not repeat the (somewhat more tedious) calculations here. Our point is to emphasize the importance of choosing an appropriate ansatz for a given choice of \( \gamma \) matrices.
4 Conclusions

We have successfully extended our approach of fermion tunnelling to model the emission of charged fermions from a rotating charged black hole. The analysis yields the expected temperature (27), consistent with black hole universality. However there are subtle technical issues involved with choosing an appropriate ansatz for the Dirac field consistent with the choice of $\gamma$ matrices, and failure to make such a choice leads to a breakdown in the method.

Computing corrections to the tunnelling probability by fully taking into account conservation of energy will yield corrections to the fermion emission temperature. In various scalar field cases this is inherent in the Parikh/Wilczek tunnelling method [3], [9]-[20] and can be incorporated into the Hamilton-Jacobi tunnelling approach [8]. Another avenue of research is to perform tunnelling calculations to higher order in WKB (in both the scalar field and fermionic cases) in order to calculate grey body effects. It is also worth investigating the possibility of calculating a density matrix for the emitted particles from a tunnelling approach in order to calculate correlations between particles. Work is in progress in these areas.

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