Research Article

Optimal Homotopy Asymptotic Method-Least Square for Solving Nonlinear Fractional-Order Gradient-Based Dynamic System from an Optimization Problem

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In this paper, we consider an approximate analytical method of optimal homotopy asymptotic method-least square (OHAM-LS) to obtain a solution of nonlinear fractional-order gradient-based dynamic system (FOGBDS) generated from nonlinear programming (NLP) optimization problems. The problem is formulated in a class of nonlinear fractional differential equations, (FDEs) and the solutions of the equations, modelled with a conformable fractional derivative (CFD) of the steepest descent approach, are considered to find the minimizing point of the problem. The formulation extends the integer solution of optimization problems to an arbitrary-order solution. We exhibit that OHAM-LS enables us to determine the convergence domain of the series solution obtained by initiating convergence-control parameter $C_j$’s. Three illustrative examples were included to show the effectiveness and importance of the proposed techniques.

1. Introduction

Consider a nonlinear programming-constrained optimization problems (NLPCOPs) of the form

$$\min_{x \in \mathbb{R}^n} f(x) \text{ subject to } g_k(x) \leq 0 \text{ and } h_k(x) = 0 \forall k \in I = \{1, 2 \ldots m\},$$

where $f : \mathbb{R}^n \rightarrow \mathbb{R}$, $h_k : \mathbb{R}^n \rightarrow \mathbb{R}$, and $g_k : \mathbb{R}^n \rightarrow \mathbb{R}$, $k$, are $C^2$ functions. Let $X_0 = \{x \in \mathbb{R}^n \mid h_k = 0, g_k \leq 0, i \in I\}$ be the feasible set of Equation (1), and we assume that $X_0$ is not empty. The general idea of obtaining an approximate analytical solution to Equation (1) is to transform to an unconstrained nonlinear programming problem by any suitable technique such as augmented Lagrange method, barrier method, and penalty method [1, 2]; it can then be solved by any unconstrained optimization numerical method like the steepest descent method, conjugate gradient method, and Newton method. In optimization, the penalty method is the most efficient method to transform a constrained optimization problem into an unconstrained optimization problem [3–5]. An efficient penalty function for equality and inequality problem Equation (1) is given below

$$P_{\text{penalty}}(h_k(x)) = \mu \frac{1}{\sigma} \sum_{i=1}^{p} (h_k(x))^\sigma,$$  \hspace{1cm} (2)

$$P_{\text{penalty}}(g_k(x)) = \mu \frac{1}{\sigma} \sum_{i=1}^{p} (\max \{0, g_k(x)\})^\sigma,$$  \hspace{1cm} (3)

where $\sigma = 2$. It can be seen that under some conditions, the solutions to Equation (1) are solutions of the unconstrained below [6],

[Reference to Hindawi Advances in Mathematical Physics, Volume 2020, Article ID 8049397, https://doi.org/10.1155/2020/8049397]
\[
\begin{aligned}
\min & \quad F(x, \mu) = f(x) + \mu \left( \frac{1}{\sigma} \sum_{k=1}^{p} (h_k(x))^\sigma + \sum_{i=1}^{p} \left( \max \{0, g_i(x)\} \right)^\sigma \right), \\
\text{subject to} & \quad x \in \Re^n,
\end{aligned}
\]
constant. Recently, Khalil et al. [34] have characterized a new fractional derivative operator, which is an extension of the usual conformable fractional derivative, to overcome these deficiencies. Besides these advantages, the conformable fractional derivative does not show the memory effect, which is inherent for the other classical fractional derivatives.

**Definition 1.** Let $f : [0, \infty) \rightarrow \mathbb{R}$ be a given function. The $\alpha$th order CFD of $f$ given by

$$T^\alpha (f)(x) = \lim_{\varepsilon \to 0} \frac{f(x + \varepsilon x^{1-\alpha}) - f(x)}{\varepsilon}, \quad \forall x > 0 \text{ and } \alpha \in (0, 1]$$

This new definition preserves many properties of the classical derivatives refer to [34, 35]. Some features that we will adopt as are follows:

**Theorem 2.** Let $0 < \alpha \leq 1$ and $(f, g)$ be $\alpha$-differentiable at a point $x > 0$; if $f$ is a differentiable function, then $(d^\alpha f)/(dx^\alpha) = x^{\alpha-\alpha} (df/dx)$.

**Definition 3.** $P^\alpha_n (f)(x) = P^\alpha_n (x^{\alpha-1}f) = \int_a^x ((f(t))/(t^{1-\alpha})) dt$, where the integral is the regular Riemann improper integral, and $\alpha \in (0, 1]$.

**Theorem 4.** Let $f$ be any continuous function in the domain of $P^\alpha$, then $T^\alpha P^\alpha_n (f)(x) = f(x)$ as $\varepsilon \to x > a$.

**Theorem 5.** Let $f : (a, b) \to \mathbb{R}$ be differentiable and $0 < \alpha \leq 1$. Then, for all $a < x < a$, we have $P^\alpha_n T^\alpha_n (f)(x) = f(x) - f(a)$.

2.2. The Elementary Concepts of OHAM-LS. We start from the fundamental principle of OHAM as described in [36–38]. Consider the IVPs

$$L_i(z_i(t)) + N_i(z_i(t)) + g_i(t) = 0 \quad t \in \varphi i = 1, 2, \ldots, m,$$

with initial conditions

$$z_i(b) = a_i,$$

where $L_i$ is a linear operator, $N_i$ is a nonlinear operator, $t$ is an independent variable, $z_i(t)$ is an unknown function, $\varphi$ is the problem domain, and $g_i(t)$ is a known function. According to OHAM, one can construct an homotopy map $H_i(\phi_i(t, p)) : \varphi \times [0, 1] \to \varphi$ which satisfies

$$\begin{align*}
(1 - p)[L_i(\phi_i(t, p)) + g_i(t)] \\
= H_i(p)[L_i(\phi_i(t, p)) + N_i(\phi_i(t, p)) + g_i(t)],
\end{align*}$$

where $p \in [0, 1]$ is an embedding parameter, $H_i(p)$ is a nonzero auxiliary function for $p \neq 0$, $H(0) = 0$, and $\phi_i(t, p)$ is an unknown function. Obviously, when $p = 0$ and $p = 1$, it holds that $\phi_i(t, 0) = z_i(t)$ and $\phi_i(t, 1) = z_i(t)$, respectively. Thus, as $p$ varies from 0 to 1, the solution $\phi_i(t, p)$ approaches from $z_i(t)$ to $z_i(t)$ where $z_i(t)$ is the initial guess that satisfies the linear operator which is obtained from Equation (8) for $p = 0$ as

$$L_i(z_{i,0}(t)) + g_i(t) = 0, \quad z_{i,0}(b) = 0.$$  \hspace{1cm} (9)

$H_i(p)$ is chosen in the form

$$H_i(p) = pC_1 + p^2C_2 + p^3C_3 + \ldots \quad j = 1, 2, \ldots, n,$$

where $C_j$ would be determined in the last part of this work. We consider Equation (8) in the form

$$\phi_i(t, p, C_j) = z_{i,0}(t) + \sum_{k=1}^{\infty} z_{i,k}(t, C_j)p^k \quad j = 1, 2, \ldots, n.$$  \hspace{1cm} (10)

Now substituting Equation (11) in Equation (8) and equating the coefficient of like power of $p$, we obtain the governing equation of $z_{i,0}(t)$ in a linear form, given in Equation (9). The first- and second-order problems are given by

$$L_i(z_{i,1}(t)) + g_i(t) = C_0N_i(z_{i,0}(t)), \quad z_{i,1}(b) = 0,$$ \hspace{1cm} (12)

$$L_i(z_{i,2}(t)) - L_i(z_{i,1}(t)) = C_2N_i(z_{i,0}(t)) + C_1[L_i(z_{i,1}(t)) + N_i(z_{i,1}(t))], \quad z_{i,2}(b) = 0,$$ \hspace{1cm} (13)

and the general governing equations for $z_{i,k}(t)$ are given by

$$L_i(z_{i,k}(t)) - L_i(z_{i,k-1}(t)) = \frac{k-1}{(k-1)!}\sum_{m=1}^{k-1} C_{j,m}[L_i(z_{i,k-m}(t)) + N_i(z_{i,k-m}(t))],$$

$$z_{i,k}(b) = 0, \quad k = 2, 3, \ldots,$$

where $N_{i,m}(z_{i,0}(t), z_{i,1}(t), \ldots, z_{i,m}(t))$ is the coefficient of $p^m$, obtained by expanding $N_i(\phi_i(t, p, C_j))$ in series with respect to the embedding parameter $p$

$$N_i(\phi_i(t, p, C_j)) = z_{i,0}(t) + \sum_{m=1}^{\infty} N_{i,m}(z_{i,0}(t))p^m,$$

where $\phi_i(t, p, C_j)$ is obtained from Equation (11). It should noted that $z_{i,k}$ for $k \geq 0$ is governed by the linear equations (9), (12), and (14) with linear initial conditions that come from the original problem, which can be easily solved.

It has been shown that the convergence of the series Equation (16) depends upon the $C_j$. If it is convergent at $p = 1$, we have

$$z_i(t, C_j) = z_{i,0}(t) + \sum_{k=1}^{\infty} z_{i,k}(t, C_j),$$

The result of the $m$th-order approximation is given as

$$\tilde{z}_i(t, C_j) = z_{i,0}(t) + \sum_{k=1}^{m} z_{i,k}(t, C_j),$$

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where the value $C_j$ values of convergence-control parameters. Several methods [39, 40] can be used to apply such a case does not arise for nonlinear problems. Solution Equations (20) and (21).

The correctness of the method by was first proposed by Evirgen and Özdemir [24].

Using the penalty function Equation (2) and (3) for Equation (24) with $\rho = 2$, the conformable FOGBDS model can be constructed as

$$T^\alpha x(t) = -\nabla_x F(x, \mu),$$

subject to the initial conditions

$$x_k(0) = x_{k0}, \quad k = 1 \cdots m.$$  

where $\nabla_x F(x, \mu)$ is the gradient vector of Equation (25) with respect to $x_k \in \mathbb{R}^n$ and $T^\alpha$ is the CFD of $0 < \alpha \leq 1$.

Note that a point $x_\ast$ is called an equilibrium point of Equation (25) if it satisfies the RHS of Equation (25). We reformulate fractional dynamic system Equation (25) as

$$T^\alpha x_k(t) = g_k(t, \mu, x_1, x_2 \cdots x_n), \quad k = 1, 2 \cdots, m.$$  

We used OHAM-LS to obtain the solution of system Equation (27) by constructing the following homotopy

$$T^\alpha x_k(t) = p g_k(t, \mu, x_1, x_2 \cdots x_n),$$

where $k = 1, 2, \ldots, n$ and $p \in [0, 1]$. If $p = 0$, Equation (28) becomes

$$T^\alpha x_k(t) = 0,$$

and when $p = 1$, the homotopy Equation (28) becomes

$$T^\alpha x_k(t) = g_k(t, \mu, x_1, x_2 \cdots x_n), \quad k = 1, 2 \cdots, m, t \in [0, 1], 0 < \alpha \leq 1,$$
subject to the initial conditions,
\[ x_k(b) = a_k, \quad k = 1, 2, \ldots, m. \] (31)

The correction functional for the system of conformable fractional nonlinear differential equation Equation (30), according to OHAM-LS, can be constructed as
\[
(1 - p)[T^a(\phi_k(t, p))] = H_k(p)[T^a(\phi_k(t, q)) + N\phi_k(t, q) + g_k(t, \mu, \phi_1(t, q), \phi_2(t, q), \ldots)\phi_k(t, q)],
\] (32)

Thus as p varies from 0 to 1, the solution \( \phi_k(t, p) \) approaches from \( x_{k,0}(t) \) to \( x_k(t) \) where \( x_{k,0}(t) \) is the initial guess that satisfies the linear operator which is obtained from Equation (32) for \( p = 0 \) as
\[
T^a(x_{k,0}(t)) = 0, \quad x_{k,0}(b) = 0.
\] (33)

\( H_k(p) \) is chosen in the form
\[
H_k(p) = pC_1 + p^2C_2 + p^3C_3 \cdots,
\] (34)

where \( C_j \) can be determined later. We get an approximate solution by expanding \( \phi_k(t, p, C_j) \) in Taylor’s series with respect to \( p \); we have
\[
\phi_k(t, p, C_j) = x_{k,0}(t) + \sum_{i=1}^m x_{k,i}(t, C_j)p^i, \quad j = 1, 2, \ldots, n.
\] (35)

Now using Equation (35) in Equation (32) and equating the coefficient of like power of \( p \), we obtain the governing equation of \( x_{k,0}(t) \) in a linear form, given in Equation (33). The 1st- and 2nd-order problems are given by
\[
T^a(x_{k,1}(t)) + g_k(t) = C_1N_0(x_{k,0}(t)), \quad x_{k,1}(b) = 0,
\] (36)

and the general governing equations for \( x_{k,i}(t) \) are given by
\[
T^a(x_{k,i}(t)) - T^a(x_{k,i-1}(t)) = C_jN_{k,0}(x_{k,0}(t)) + \sum_{m=1}^{i-1} C_{j,m}[T^a(x_{k,i-m}(t)) + N_{k,i-m}(x_{k,i-m}(t))], \quad x_{k,i}(b) = 0, \quad i = 2, 3, \ldots, m.
\] (37)

It has been shown that the convergence of the series Equation (38) depends upon the \( C_j \). If it is convergent at \( p = 1 \), one has
\[
x_k(t, C_j) = x_{k,0}(t) + \sum_{i=1}^m x_{k,i}(t, C_j).
\] (39)

The solution of Equation (30) is determined approximately in the form,
\[
\tilde{x}_k(t, C_j) = x_{k,0}(t) + \sum_{i=1}^m x_{k,i}(t, C_j), \quad j = 1, 2, \ldots, n.
\] (40)

Substituting Equation (40) in Equation (30), we get the following expression for the residual error
\[
R_k(t, C_j) = T^a(\tilde{x}_k(t, C_j)) + N(\tilde{x}_k(t, C_j)) + g_k(\tilde{x}_k(t, C_j)).
\] (41)

If \( R_k(t, C_j) = 0 \), then \( \tilde{x}_k(t, C_j) \) is the exact solution. Usually, such a case does not arise for nonlinear problems. Using the least square method as below minimizes the functional
\[
J_k(C_1, C_2, C_3, \ldots C_m) = \int_a^b R_k^2(t, C_1, C_2, C_3, \ldots C_m)dt,
\] (42)

where the value of \( a \) and \( b \) depends on the given problem.
\[
\psi_k = \frac{\partial J_k(C_k)}{\partial C_k} = 0, \quad k = 1, 2, \ldots, m.
\] (43)

With these known \( C_k \), the analytical approximate solution (of \( m \)th-order) is well determined.

The steps for optimal homotopy asymptotic method-least square (OHAM-LS) are as follows:

Step 1. We transform the nonlinear constrained optimization problem to the unconstrained optimization problem by a penalty method.

Step 2. We find the gradient of the unconstrained optimization problem, with given initial conditions.

Step 3. We choose the linear and nonlinear operators for OHAM-LS.

Step 4. We construct homotopy for the conformable fractional nonlinear differential equation which includes embedding parameter, auxiliary function, and the unknown function.

Step 5. We substitute the series solution results into the governing equation and equate to zero for an exact solution. Usually, such case a does not arise in nonlinear problems.
Step 6. We find the optimal values for $C_j$ by using the optimization method called least square method, for good analytical approximate solution.

3.1. Convergence Analysis of OHAM-LS with FOGBDS

**Theorem 8.** As long as the series $\tilde{x}_k(t, C_j) = x_{k,0}(t) + \sum_{i=1}^{m} x_{k,i}(t, C_j)$, $j = 1, 2, \ldots, n$ converges where $\tilde{x}_k(t, C_j)$ is governed by Equation (40) under the definitions Equations (37) and (38), it must be the solution of Equations (25) and (26).

**Proof.** If we assume $\sum_{m=1}^{\infty} \tilde{x}_{k,m}(t, C_j)$, $k = 1, 2 \ldots, n$, converges to $\tilde{x}_k(t, C_j)$, then

$$\lim_{m \to \infty} \tilde{x}_{k,m}(t, C_j) = 0 \forall k = 1, 2 \ldots n. \quad (44)$$

From Equation (37), we can write

$$\sum_{i=1}^{\infty} \left[ C_i N_{k,0}(x_{k,0}(t)) + \sum_{m=1}^{i-1} C_{jm} T^a(x_{k,j-m}(t)) + N_{k,j-m}(x_{k,j-1}(t)) \right]$$

$$= \sum_{i=1}^{\infty} \left[ T^a(x_{k,i}(t)) - T^a(x_{k,i-1}(t)) \right]$$

$$= \lim_{n \to \infty} \sum_{i=1}^{n} T^a(x_{k,i}(t)) - T^a(x_{k,i-1}(t))$$

$$= T^a x_{11}(t) + T^a x_{22}(t) - T^a x_{21}(t) + \ldots + \sum_{i=1}^{n} T^a x_{m_i}(t)$$

$$= T^a \lim_{n \to \infty} \sum_{m=1}^{n} x_{m}(t) = T^a \lim_{n \to \infty} x_{m}(t) = 0,$$

$$\sum_{i=1}^{\infty} \left[ C_i N_{k,0}(x_{k,0}(t)) + \sum_{m=1}^{i-1} C_{jm} T^a(x_{k,j-m}(t)) \right] + N_{k,j-m}(x_{k,j-1}(t)) = 0. \quad (45)$$

So, using above gives

$$= \sum_{m=1}^{\infty} \left[ T^a x_{k(m-1)} + N(x_{k(m-1)}) \right]$$

$$+ g_k \left( t, \mu, x_{1(m-1)}, x_{2(m-1)} \ldots x_{n(m-1)} \right), \quad (46)$$

$$= \sum_{m=1}^{\infty} T^a x_{k(m-1)} + \sum_{m=1}^{\infty} N(x_{k(m-1)})$$

$$+ \sum_{m=1}^{\infty} g_k \left( t, \mu, x_{k(m-1)} \right), \quad (47)$$

$$= T^a \tilde{x}_k(t, C_j) + N(\tilde{x}_k(t, C_j)) + g_k(\tilde{x}_k(t, C_j)). \quad (48)$$

From Equation (48), we have

$$T^a \tilde{x}_k(t, C_j) + N(\tilde{x}_k(t, C_j)) + g_k(\tilde{x}_k(t, C_j)) = 0 \forall k = 1, 2 \ldots m, \quad (49)$$

4. Numerical Examples and Results

In this section, three examples are presented to illustrate the efficiency of the new method for solving NLPCOP. The calculations are performed using maple software 2018, HP ENVY laptop 13 corei7 8th Gen 16GB.

**Example 1.** Consider the NLPCOP test problem from Schittkowski [54] (No. 216).

Minimize $$f(x) = 100(x_1^2 - x_2^2) + (x_1 - 1)^2,$$

subject to $$h(x) = x_1(x_1 - 4) - 2x_2 + 12 = 0,$$

whose exact solution is not known, but expected optimal solution is $x_1^* = 1.9993$, $x_2^* = 3.9998$. First, we transform the constraint problem to an unconstrained problem by quadratic penalty function for $\sigma = 2$; then, we have

$$f(x, \mu) = 100(x_1^2 - x_2^2) + (x_1 - 1)^2,$$

$$+ \frac{1}{\sigma}(x_1(x_1 - 4) - 2x_2 + 12)^2,$$

where $\mu \in \mathbb{R}^+$, and so that the nonlinear FOGBDS can be given as

$$T^a x_1(t) = -400(x_1^2 - 2x_2 - 2(x_1 - 1)$$

$$- \mu(2x_1 - 4)(x_1^2 - 4x_1 - 2x_2 + 12),$$

$$T^a x_2(t) = 200(x_1^2 - x_2) + 2\mu(x_1^2 - 4x_1 - 2x_2 + 12),$$

$$x_1(0) = 0, x_2(0) = 0,$$

where $0 < \alpha \leq 1$. By using OHAM-LS with auxiliary penalty variable $\mu = 200$, the terms of the OHAM-LS solutions for fractional order are acquired by using the concept of homotopy. According to Equation (6)), we choose the linear and nonlinear operators in the following forms:

$$L_1[\varphi_1(t, p)] = T^a \varphi_1(t, p),$$

$$L_2[\varphi_2(t, p)] = T^a \varphi_2(t, p),$$

$$N_1[\varphi_1(t, p)] = T^a \varphi_1(t, p) + 400(\varphi_1(t, p)^2 - \varphi_2(t, p))\varphi_1(t, p)$$

$$+ 2(\varphi_1(t, p)^2 - 1) + 200(2\varphi_1(t, p) - 4)\varphi_1(t, p)$$

$$+ (\varphi_1(t, p)^2 - 4\varphi_1(t, p) - 2\varphi_2(t, p) + 12),$$

$$N_2[\varphi_2(t, p)] = T^a \varphi_2(t, p) - 200(\varphi_1(t, p)^2 - \varphi_2(t, p))$$

$$- 400(\varphi_1(t, p)^2 - 4\varphi_1(t, p) - 2\varphi_2(t, p) + 12).$$

(53)
We can construct the following homotopy

\[
(1 - p)T^n \varphi_1(t, p) = H(p)\left[T^n \varphi_1(t, p) + 400(\varphi_0(t, p))^2 - \varphi_2(t, p)\varphi_1(t, p) + 2(\varphi_1(t, p) - 1) + 200(2\varphi_1(t, p) - 4)(\varphi_1(t, p)^2 - 4\varphi_1(t, p) - 2\varphi_2(t, p) + 12)\right],
\]

where

\[
\varphi_1(t, p) = x_{1,0}(t) + \sum_{j=1}^N x_{1,j}(t)p^j,
\]

\[
\varphi_2(t, p) = x_{2,0}(t) + \sum_{j=1}^N x_{2,j}(t)p^j,
\]

\[
H_k(p) = pC_1 + p^2C_2 + p^3 + C_3 + \cdots, \quad k = 1, 2 \cdots m.
\]

Substituting Equations (56)-(58) into Equations (54) and (55) and equating the coefficient of the same power of \( p \) result to the following set of linear FDEs.

\[
p^0 : T^n x_{1,0}(t) = 0,
\]

\[
p^0 : T^n x_{2,0}(t) = 0,
\]

\[
p^1 : T^n x_{1,1}(t) = 2000x_{1,0}^3C_1 + T^n x_{1,1}C_1 - 3600x_{2,0}x_{1,0}C_1 - 9600x_{2,0}^2C_1 - T^n x_{1,0} + 6400x_{2,0}C_1 + 3200x_{1,0}C_1 - 3840C_1 = 0,
\]

\[
p^1 : T^n x_{2,1}(t) = T^n x_{2,0}C_1 - 1800x_{2,0}C_1 - T^n x_{2,0}C_1 + 6400x_{1,0}C_1 + 3400x_{2,0}C_1 - 19200C_1 = 0,
\]

\[
p^2 : T^n x_{1,2}(t) = 2000x_{1,0}^2C_2 + 6000x_{1,0}^2x_{1,1}C_1 + T^n x_{1,1}C_2 + T^n x_{1,1}C_2 - 3600x_{2,0}x_{1,0}C_2 - 6000x_{2,0}^2C_2 - 9600x_{2,0}^2C_2 - 19200x_{1,0}x_{1,1}C_1 - 3600x_{1,0}x_{2,0}C_2 - T^n x_{1,0}C_2 + 6400x_{2,0}C_2 + 3200x_{1,0}C_2 + 3200x_{1,0}C_1 + 6400x_{2,1}C_1 - 3840C_2 = 0,
\]

\[
p^2 : T^n x_{2,2}(t) = T^n x_{2,0}C_2 + T^n x_{2,1}C_2 - 1800x_{2,0}^2C_2 - 3600x_{1,0}x_{1,1}C_2 - T^n x_{2,1}C_2 + 6400x_{1,0}C_2 + 6400x_{1,0}C_1 + 3400x_{2,0}C_2 + 3400x_{2,1}C_1 - 19200C_2 = 0.
\]

Applying the operator \( I^\alpha \) to both sides of Equations (59)-(64) with initial conditions given in Equation (5.6), we obtain

\[
x_{1,0}(t) = 0,
\]

\[
x_{2,0}(t) = 0,
\]

\[
x_{1,1}(t, C_1) = 384020t^{\frac{1}{10}}C_1,
\]

\[
x_{2,1}(t, C_1) = 192000t^{\frac{1}{10}}C_1,
\]

\[
x_{1,2}(t, C_1, C_2) = -6.759104020 \times 10^{-10}t^{\frac{11}{10}}C_1^2
\]

\[
- 384020t^{\frac{1}{10}} + 384020t^{\frac{11}{10}}C_1
\]

\[
+ 384020C_2t^{\frac{1}{10}},
\]

\[
x_{2,2}(t, C_1, C_2) = -1.555264000 \times 10^{-10}t^{\frac{11}{10}}C_1^2
\]

\[
- 192000t^{\frac{1}{10}}C_1^2 + 192000t^{\frac{11}{10}}C_1
\]

\[
+ 192000C_2t^{\frac{1}{10}}.
\]

Adding up the solution components Equations (65)-(70), the 2nd-order approximate solution obtained by OHAM-LS at \( \alpha = 0.9 \), for \( p = 1 \), are

\[
x_1(t, C_1, C_2) = (768040C_1 - 384020C_1^2 + 384020C_2)t^{\frac{1}{10}}
\]

\[
- 6.759104020 \times 10^{-10}t^{\frac{11}{10}}C_1,
\]

\[
x_2(t, C_1, C_2) = (384000C_1 - 192000C_1^2 + 192000C_2)t^{\frac{1}{10}}
\]

\[
- 1.555264000 \times 10^{-10}t^{\frac{11}{10}}C_1.
\]

For the calculations of \( C_1 \) and \( C_2 \) in \( x_1(t) \) and \( x_2(t) \) given in Equations (71) and (72), we apply the procedure mentioned in Equations (19)-(21); we obtain, for \( x_1(t) \),

\[
c[1] = 1.800506683 \times 10^{-6},
\]

\[
c[2] = 6.594892833 \times 10^{-6},
\]

and for \( x_2(t) \),

\[
c[1] = 0.111906918 \times 10^{-4},
\]

\[
c[2] = 0.2190543167 \times 10^{-4}.
\]

Substituting these optimal values into Equations (71) and (72) becomes

\[
x_1(t) = 4.196444315t^{\frac{1}{10}} - 1.084631569t^{\frac{11}{10}},
\]

\[
x_2(t) = 7.546421106t^{\frac{1}{10}} - 0.7996784175t^{\frac{11}{10}}.
\]

Table 1 shows the \( C_k \) at different values of \( \alpha \) for Example 1. Table 2 shows the comparisons and the absolute error between OHAM-LS and RK4 at different values of \( \alpha = 1 \). Figure 1 shows the analytical approximate solutions obtained by OHAM-LS for \( \alpha = 1, 0.9, 0.8 \), and 0.7 with RK4 at \( \alpha = 1 \).
using the concept of homotopy. According to Equation (6),

\[ \text{Consider the NLPCOPs test problem from Schittkowski [54] [No 320].} \]

First, the quadratic penalty function is used to subject to \( h(x) = \frac{x_1^2}{100} + \frac{x_2^2}{4} - 1 = 0. \)

This is a practical problem, and the exact solution is not known, but the expected optimal solution is \( x_1^* = 9.395, x_2^* = -0.6846. \) First, the quadratic penalty function is used to get the unconstrained optimization problem as follows:

\[
F(x, \mu) = (x_1 - 20)^2 + (x_2 + 20)^2 + \frac{1}{2} \mu \left( \frac{x_1^2}{100} + \frac{x_2^2}{4} - 1 \right)^2,
\]

where \( \mu \in \mathbb{R}^+ \) and so that the nonlinear FOGBDS be given as

\[
T_\alpha x_1(t) = 2x_1 - 40 + \mu \left( \frac{1}{50} x_1^3 + \frac{1}{20} x_1 x_2^2 - \frac{1}{50} x_1 \right),
\]

\[
T_\alpha x_2(t) = 2x_2 + 40 + \mu \left( \frac{1}{200} x_1^2 x_2^2 + \frac{1}{8} x_2^3 - \frac{1}{2} x_2 \right),
\]

\[ 0 < \alpha \leq 1, x_1(0) = 0, x_2(0) = 0. \]

By using OHAM-LS with \( \mu = 10^6, \) the terms of the OHAM-LS solutions for fractional order are acquired by using the concept of homotopy. According to Equation (6), we choose the linear and nonlinear operators in the following forms:

\[ L_1[\varphi_1(t, p)] = T_\alpha \varphi_1(t, p), \]

\[ N_1[\varphi_1(t, p)] = T_\alpha \varphi_1(t, p) - 2 \left( \varphi_1(t, p) + 40 \right) - 10^6 \left( \frac{1}{5000} \varphi_1(t, p)^3 + \frac{1}{200} \varphi_1(t, p) \varphi_2(t, p)^2 - \frac{1}{50} \varphi_1(t, p) \right), \]

\[ N_2[\varphi_2(t, p)] = T_\alpha \varphi_2(t, p) - 2 \varphi_2(t, p) - 40 - 10^6 \left( \frac{1}{200} \varphi_2(t, p) \varphi_1(t, p)^2 - \frac{1}{8} \varphi_2(t, p)^3 + \frac{1}{2} \varphi_2(t, p) \right), \]

\[ (1 - p)T_\alpha \varphi_1(t, p) = H(p) \left[ T_\alpha \varphi_1(t, p) - 2 \left( \varphi_1(t, p) + 40 \right) - 10^6 \left( \frac{1}{5000} \varphi_1(t, p)^3 + \frac{1}{200} \varphi_1(t, p) \right) \times \varphi_2(t, p)^2 - \frac{1}{50} \varphi_1(t, p) \right], \]

\[ (1 - p)T_\alpha \varphi_2(t, p) = H(p) \left[ T_\alpha \varphi_2(t, p) - 2 \varphi_2(t, p) - 40 \right. \left. - 10^6 \left( \frac{1}{200} \varphi_2(t, p) \varphi_1(t, p)^2 \right. \left. - \frac{1}{8} \varphi_2(t, p)^3 + \frac{1}{2} \varphi_2(t, p) \right) \right], \]

\[ \text{Table 1: Control-convergence parameters } C_k \text{ at different values of } \alpha. \]

| Variable | \( x_1(t) \) | \( x_1(t) \) | \( x_2(t) \) | \( x_2(t) \) |
|----------|---------------|---------------|---------------|---------------|
| \( \alpha \) | \( C_1 \) | \( C_2 \) | \( C_1 \) | \( C_2 \) |
| 1        | 1.912514527 \times 10^{-6} | 7.294797236 \times 10^{-6} | 0.011369873 \times 10^{-4} | 0.229361274 \times 10^{-4} |
| 0.9      | 1.800506863 \times 10^{-6} | 6.594892833 \times 10^{-6} | 0.111906918 \times 10^{-4} | 0.2190543167 \times 10^{-4} |
| 0.8      | 1.714313871 \times 10^{-6} | 5.524430129 \times 10^{-6} | 0.10992623 \times 10^{-4} | 0.201017632 \times 10^{-4} |
| 0.7      | 1.593611093 \times 10^{-6} | 5.294592861 \times 10^{-6} | 0.107191284 \times 10^{-4} | 0.197911283 \times 10^{-4} |

\[ \text{Table 2: Comparisons and absolute error between OHAM-LS and RK4, } \alpha = 1. \]

| \( t_k \) | OHAM-LS \( x_1(t) \) | OHAM-LS \( x_1(t) \) | RK4 \( x_1(t) \) | RK4 \( x_2(t) \) | Error \( x_1(t) \) | Error \( x_2(t) \) |
|----------|----------------|----------------|---------------|---------------|---------------|---------------|
| 0.0000   | 0.000000 | 0.000000 | 0.000000 | 0.000000 | 0.000000 | 0.000000 |
| 0.0005   | 1.970211 | 3.871721 | 1.970899 | 3.871887 | 0.000688 | 0.000166 |
| 0.0010   | 1.977134 | 3.907823 | 1.978274 | 3.907993 | 0.001136 | 0.00017 |
| 0.0013   | 1.981102 | 3.922403 | 1.981384 | 3.922554 | 0.000282 | 0.000151 |
| 0.0015   | 1.982211 | 3.930534 | 1.983132 | 3.930578 | 0.000921 | 4.4E-05 |
| 0.0020   | 1.983214 | 3.935653 | 1.984252 | 3.935654 | 0.001038 | 1E-06 |

Example 2. Consider the NLPCOPs test problem from Schittkowski [54] [No 320].

Minimize \( f(x) = (x_1 - 20)^2 + (x_2 + 20)^2, \)

subject to \( h(x) = \frac{x_1^2}{100} + \frac{x_2^2}{4} - 1 = 0. \)

\[ (76) \]

\[ (77) \]
where
\[
\varphi_1(t, p) = x_{1,0}(t) + \sum_{j=1}^{1} x_{1,j}(t) p^j, 
\]
\[
\varphi_2(t, p) = x_{2,0}(t) + \sum_{j=1}^{1} x_{2,j}(t) p^j, 
\]
\[
H_k(p) = p C_1 + p^2 C_2 + p^3 + C_3 + \cdots, \quad k = 1, 2. 
\]

Substituting Equations (85)-(87) into Equations (83) and (84) and equating the coefficient of the same powers of \( p \) yields the following set of linear FDEs:

\[
\rho^0 : T^a x_{1,0}(t) = 0, \tag{88}
\]
\[
\rho^1 : T^a x_{1,1}(t) = -200x_{1,0}^1 C_1 - 5000x_{1,0}^2 C_1 + T^a x_{1,0} C_1 - T^a x_{1,0} + 19998 x_{1,0} C_1 + 40 C_1, \tag{90}
\]
\[
\rho^1 : T^a x_{2,1}(t) = -125000x_{2,0}^3 C_1 - 5000x_{2,0}^2 C_1 - T^a x_{2,0} C_1 - 2x_{2,0} C_1 - T^a x_{2,0} + 50000x_{1,0} C_1 - 40 C_1 = 0, \tag{91}
\]

Figure 1: (a) Different values of \( \alpha \) (OHAM-LS; \( \alpha = 1 \), dot; \( \alpha = 0.9 \), dash; \( \alpha = 0.8 \), dash dot; and \( \alpha = 0.7 \), long dash) and RK4 (\( \alpha = 1 \), solid) at \( x_1 \).

(b) Different values of \( \alpha \) (OHAM-LS; \( \alpha = 1 \), dot; \( \alpha = 0.9 \), dash; \( \alpha = 0.8 \), dash dot; and \( \beta = 0.7 \), long dash) and RK4 (\( \alpha = 1 \), solid) at \( x_2 \).
\( \text{Table 3: Control-convergence parameters } C_k \text{ at different values of } \alpha. \)

| Variable | \( x_1(t) \) | \( x_1(t) \) | \( x_1(t) \) | \( x_2(t) \) |
|----------|---------------|---------------|---------------|---------------|
| \( \alpha \) | \( C_1 \) | \( C_2 \) | \( C_1 \) | \( C_2 \) |
| 1        | \(-0.1198434251 \times 10^{-3}\) | \(-0.02645325610\) | \(-1.3256727843 \times 10^{-5}\) | \(-2.527402984 \times 10^{-3}\) |
| 0.9      | \(-0.1208162856 \times 10^{-3}\) | \(-0.02826592550\) | \(-1.343994006 \times 10^{-5}\) | \(-2.536649902 \times 10^{-3}\) |
| 0.8      | \(-0.148762674 \times 10^{-3}\) | \(-0.02983123651\) | \(-1.3619012564 \times 10^{-5}\) | \(-2.550122356 \times 10^{-3}\) |
| 0.7      | \(-0.1598723560 \times 10^{-3}\) | \(-0.03154109428\) | \(-1.3801234527 \times 10^{-5}\) | \(-2.573641295 \times 10^{-3}\) |

\( E \) is obtained with initial conditions given in Equation (4.38), we obtain

\[ x_1(t) = \frac{1}{\alpha} \left( -400 C_1 - 9.999000 \times 10^6 t^{1/5} C_1^2 + 200 C_1^2 - 200 C_2^2 \right), \tag{100} \]

\[ x_2(t) = t^{1/5} \left( 400 C_1 - 2.50001000 \times 10^8 t^{1/5} C_1^2 - 200 C_1^2 + 200 C_2^2 \right). \tag{101} \]

For the calculations of \( C_1 \) and \( C_2 \) in \( x_1(t) \) and \( x_2(t) \) given in Equations (100) and (101), we apply the procedure mentioned in Equations (19)-(21), we obtain for \( x_1(t) \),

\[ c[1] = -0.1208162856 \times 10^{-3}, \tag{102} \]

\[ c[2] = -0.02826592550. \tag{103} \]

And for \( x_2(t) \),

\[ c[1] = -1.343994006 \times 10^{-5}, \tag{103} \]

\[ c[2] = -2.536649902 \times 10^{-3}. \tag{103} \]

Adding up the solution components Equations (94)-(99),

\[ x_1(t) = \frac{1}{\alpha} \left( -400 C_1 - 9.999000 \times 10^6 t^{1/5} C_1^2 + 200 C_1^2 - 200 C_2^2 \right), \tag{100} \]

\[ x_2(t) = t^{1/5} \left( 400 C_1 - 2.50001000 \times 10^8 t^{1/5} C_1^2 - 200 C_1^2 + 200 C_2^2 \right). \tag{101} \]

For the calculations of \( C_1 \) and \( C_2 \) in \( x_1(t) \) and \( x_2(t) \) given in Equations (100) and (101), we apply the procedure mentioned in Equations (19)-(21), we obtain for \( x_1(t) \),

\[ c[1] = -0.1208162856 \times 10^{-3}, \tag{102} \]

\[ c[2] = -0.02826592550. \tag{103} \]

And for \( x_2(t) \),

\[ c[1] = -1.343994006 \times 10^{-5}, \tag{103} \]

\[ c[2] = -2.536649902 \times 10^{-3}. \tag{103} \]

Substituting these optimal values into Equations (100) and (101), we have

\[ x_1(t) = \left( 5.701514534 + 0.1459511521 t^{1/5} \right) t^{1/5}, \tag{104} \]

\[ x_2(t) = \left( -0.5167059926 + 0.04515817783 t^{1/5} \right) t^{1/5}. \tag{104} \]

Table 3 shows the \( C_k \) at different values of \( \alpha \) for example 2. Table 4 show the comparisons and the absolute error between OHAM-LS and RK4 at different values of \( \alpha = 1. \).
Figure 2 show the analytical approximate solutions obtained by OHAM-LS for \( \alpha = 1, 0.9, 0.8, \) and 0.7 with RK4 at \( \alpha = 1 \).

Example 3. Consider the NLPCOP test problem from Schittkowski [54] (No. 300).

Minimize \( f(x) = x_1^2 + x_2^2 + 2x_3^2 + x_4^2 - 5x_1 - 5x_2 - 21x_3 + 7x_4, \)
subject to \( 8 - x_1^2 - x_2^2 - x_3^2 - x_4^2 + x_2 - x_3 + x_4 \leq 0, \)
\( 10 - x_1^2 - 2x_2^2 - x_3^2 + x_1 + x_4 \leq 0, \)
\( 5 - 2x_1^2 - x_2^2 - x_3^2 - 2x_1 + x_2 + x_4 \leq 0. \)

This is a practical problem, and the exact solution is not known, but the expected optimal solution is \( x_1^* = 0, x_2^* = 1, x_3^* = 2, \) and \( x_4^* = -1. \) From the above procedure, the second-order approximate solution obtained by OHAM-LS at \( \alpha = 0.9, \) for \( p = 1, \) is

\[
x_1^2(t) = (16100C_1 - 8050C_2^2 + 8050C_2) t^{1/10} + 1.54569500 \times 10^{3} t^{1/5} C_1^2,
\]
(106)

\[
x_2^2(t) = (-25900C_1 + 12950C_2^2 - 12950C_2) t^{1/10} - 3.75020500 \times 10^{3} t^{1/5} C_1^2,
\]
(107)
Table 5: Control-convergence parameters $C_k$ at different values of $\alpha$.

| Variable $\alpha$ | $x_3(t)$ | $x_2(t)$ | $x_3(t)$ |
|-------------------|----------|----------|----------|
|                   | $C_1$    | $C_2$    | $C_1$    |
| 1                 | $-4.470470112 \times 10^{-13}$ | $-0.1542801253 \times 10^{-3}$ | $-1.0784243190 \times 10^{-4}$ |
| 0.9               | $-4.494712729 \times 10^{-13}$ | $-0.1618198317 \times 10^{-3}$ | $-1.096696787 \times 10^{-4}$ |
| 0.8               | $-4.5167327196 \times 10^{-13}$ | $-0.1832920121 \times 10^{-3}$ | $-1.1079094521 \times 10^{-4}$ |
| 0.7               | $-4.5371220162 \times 10^{-13}$ | $-0.2087212810 \times 10^{-3}$ | $-1.1261409123 \times 10^{-4}$ |

Table 6: Control-convergence parameters $C_k$ at different values of $\alpha$.

| Variable $\alpha$ | $x_3(t)$ | $x_4(t)$ | $x_3(t)$ |
|-------------------|----------|----------|----------|
|                   | $C_1$    | $C_2$    | $C_1$    |
| 1                 | $-0.5711237191 \times 10^{-3}$ | $-5.921274832 \times 10^{-12}$ | $0.8973526178 \times 10^{-4}$ |
| 0.9               | $0.599293243 \times 10^{-3}$ | $-5.935109529 \times 10^{-12}$ | $0.8773054262 \times 10^{-4}$ |
| 0.8               | $-0.6190253261 \times 10^{-3}$ | $-5.935158452 \times 10^{-12}$ | $0.8696526178 \times 10^{-4}$ |
| 0.7               | $-0.6213153411 \times 10^{-3}$ | $-5.975232801 \times 10^{-12}$ | $0.8572034710 \times 10^{-4}$ |

Table 7: Comparisons and absolute error between OHAM-LS and RK4, $\alpha = 1$.

| $t_k$ | OHAM-LS$x_3(t)$ | OHAM-LS$x_4(t)$ | OHAM-LS$x_5(t)$ | RK4$x_3(t)$ |
|-------|----------------|----------------|----------------|-------------|
| 0.000 | 0.000000 | 0.000000 | 0.000000 | 0.000000 |
| 0.001 | 0.834165 | -0.805509 | 0.834260 | 0.834165 |
| 0.002 | 0.894101 | -0.859211 | 0.894137 | 0.894101 |
| 0.003 | 0.931112 | -0.963765 | 0.931137 | 0.931112 |
| 0.004 | 0.958211 | -0.978625 | 0.958313 | 0.958211 |
| 0.005 | 0.979623 | -0.991899 | 0.979937 | 0.979623 |

For the calculations of $C_1$ and $C_2$ in $x_3(t)$, $x_2(t)$, $x_3(t)$, and $x_4(t)$ given in Equations (4.79)-(4.82), we apply the procedure mentioned in Equations (19)-(21); we obtain for $x_3(t)$,

$$c[1] = 0, c[2] = 0,$$

and for $x_2(t)$,

$$c[1] = -4.494712729 \times 10^{-13},$$
$$c[2] = -0.1618198317 \times 10^{-4},$$

and for $x_3(t)$,

$$c[1] = -1.096696787 \times 10^{-14},$$
$$c[2] = 0.599293243 \times 10^{-3},$$

and for $x_4(t)$,

$$c[1] = -5.935109529 \times 10^{-12},$$
$$c[2] = 0.8773054262 \times 10^{-4}.$$

Substituting these optimal values into Equations (106)-(109), we have

$$\bar{x}_3(t) = 0,$$
$$\bar{x}_4(t) = 0.095566833 \times t^{1/10} - 7.576330095 \times 10^{-17} t^{1/15},$$
$$\bar{x}_5(t) = 3.113922656 \times t^{1/10} + 1.954769053 \times t^{1/15},$$
$$\bar{x}_4(t) = -2.023943344 \times t^{1/10} - 4.389780283 \times 10^{-15} t^{1/15}.$$

Tables 5 and 6 show the $C_k$ at different values of $\alpha$ for Example 3. Tables 7 and 8 show the comparisons and the absolute error between OHAM-LS and RK4 at $\alpha = 1$. Also, Figure 3 shows the comparisons of OHAM-LS at $\alpha = 1, 0.9, 0.8,$ and 0.7 with RK4 at $\alpha = 1$, which verifies the performance of the present method as an excellent tool for NLPCOPs. For $\alpha = 1$, it can be seen that the approximate analytical solution agrees with the ideal solution. Thus, as $\alpha$ approaches 1, the classical solution for the system is recovered.
Figure 3: (a) Different values of $\alpha$ (OHAM-LS; $\alpha = 1$ dot, $\alpha = 0.9$ dash, $\alpha = 0.8$ dash dot, and $\alpha = 0.7$ long dash) and RK4 ($\alpha = 1$, solid) at $x_2$. (b) Different values of $\alpha$ (OHAM-LS; $\alpha = 1$ dot, $\alpha = 0.9$ dash, $\alpha = 0.8$ dash dot, and $\alpha = 0.7$ long dash), and RK4 ($\alpha = 1$, solid) at $x_3$. (c) Different values of $\alpha$ (OHAM-LS; $\alpha = 1$ dot, $\alpha = 0.9$ dash, $\alpha = 0.8$ dash dot, and $\alpha = 0.7$ long dash), and RK4 ($\alpha = 1$, solid) at $x_4$. 
5. Conclusions

In this paper, we implemented OHAM-LS for solving non-linear FOGBDS from the optimization problem. The fractional derivative is considered in a new conformable fractional derivative sense. The optimization minimization approach of the least square method helps to obtain optimal values of the $C_j$ for accurate approximate analytical solutions. The comparisons between the fourth-order Runge-Kutta ($\alpha = 1$) and OHAM-LS show that our present method performs rapid convergence to the expected optimal solutions of the optimization problem. The results obtained are in close agreement with the exact solution, and those from the RK4 and OHAM-LS are reliable, dependable, and efficient for finding an approximate analytical solution for non-linear FOGBDS optimization problem.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflict of interest.

Authors’ Contributions

All authors have equal contributions and they read and approved the final version of the paper.

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