GLOBAL ANALYSIS, ANALYSIS ON MANIFOLDS

Aronszajn and Sikorski subcartesian differential spaces

by

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Summary. We construct a natural transformation between the category of Aronszajn subcartesian spaces and the category of subcartesian differential spaces, which is a subcategory of Sikorski differential spaces.

Differential spaces were introduced in 1967 by Sikorski [2]. A comprehensive presentation of Sikorski’s theory of differential spaces is contained in his book [3]. For the current state of the theory consult [4, Chpt. 1]. In this theory, the differential structure of a space is given by its algebra of smooth functions. In 1967 Aronszajn [1] introduced the notion of a subcartesian space with its smooth structure described in terms of an atlas, analogous to the standard definition of a differentiable manifold.

We can characterize a smooth Hausdorff manifold of dimension $n$ as a Hausdorff differential space $S$ such that every point $x \in S$ has an open neighbourhood $U$ diffeomorphic to an open subset $V$ of $\mathbb{R}^n$. Here the differential structures on $U$ and $V$ are generated by restrictions of smooth functions on $S$ and $\mathbb{R}^n$, respectively. We can weaken this definition by not requiring that $V$ is open in $\mathbb{R}^n$ and allowing $n$ to be an arbitrary non-negative integer depending on $U$. This leads to the following notion of a subcartesian differential space. A differential space $S$ is subcartesian if it is Hausdorff and every point $x \in S$ has a neighbourhood $U_x$ diffeomorphic to a differential subspace $V_x$ of $\mathbb{R}^{nx}$.

Proposition 1. A differential subspace $S$ of $\mathbb{R}^n$ is a subcartesian differential space. Moreover, if $h \in C^\infty(S)$, then for every point $x \in S$ there is

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an open neighbourhood $U$ of $x$ in $S \subseteq \mathbb{R}^n$ and a function $f \in C^\infty(\mathbb{R}^n)$ such that $h_{|U} = f_{|U}$.

Proof. Since the set $S$ is a subset of $\mathbb{R}^n$, it is Hausdorff. Hence, the differential subspace $S$ of $\mathbb{R}^n$ is subcartesian. Let $(x_1, \ldots, x_n)$ be the natural coordinate functions on $\mathbb{R}^n$. By definition of a differential structure generated by a family of functions (see [1]) there exists $F \in C^\infty(\mathbb{R}^n)$ such that $h_{|U} = F(x_1, \ldots, x_n)_{|U}$.

But $f = F(x_1, \ldots, x_n)$ is in $C^\infty(S)$. $lacksquare$

A subcartesian space of Aronszajn is a Hausdorff topological space $S$ endowed with an atlas $\mathfrak{A} = \{\varphi : U_\varphi \rightarrow V_\varphi\}$, where $\varphi : U_\varphi \rightarrow V_\varphi$ is a homeomorphism of an open subset $U_\varphi$ of $S$ onto a subset $V_\varphi$ of $\mathbb{R}^{n_\varphi}$, which has the following properties:

1. The domains $\{U_\varphi | \varphi \in \mathfrak{A}\}$ form an open cover of $S$.
2. For every $\varphi, \psi \in \mathfrak{A}$ and every $x \in U_\varphi \cap U_\psi$, there exists a $C^\infty$-mapping $s$ extending $\psi \circ \varphi^{-1} : \varphi(U_\varphi \cap U_\psi) \rightarrow \psi(U_\varphi \cap U_\psi)$ in a neighbourhood of $\varphi(x) \in \mathbb{R}^{n_\varphi}$. Also, there exists a $C^\infty$-mapping $t$ extending $\varphi \circ \psi^{-1} : \psi(U_\varphi \cap U_\psi) \rightarrow \varphi(U_\varphi \cap U_\psi)$ in a neighbourhood of $\psi(x) \in \mathbb{R}^{n_\psi}$.

As in the theory of manifolds, there may be different atlases giving rise to the same subcartesian structure. A subcartesian structure determines a maximal atlas, which can be interpreted as the union of all the atlases giving this structure.

In [5] Walczak characterized the largest class of differential spaces which satisfy a condition analogous to condition 2 of the definition of Aronszajn subcartesian spaces. However, he did not give an explicit proof that subcartesian and Aronszajn subcartesian spaces are equivalent. The goal of this paper is to show that the definitions of subcartesian and Aronszajn subcartesian spaces are equivalent and to construct a natural transformation between the categories of Aronszajn subcartesian spaces and of subcartesian differential spaces.

First, we give Aronszajn’s definition of a smooth map between Aronszajn subcartesian spaces. If $(S_1, \mathfrak{A}_1)$ and $(S_2, \mathfrak{A}_2)$ are subcartesian spaces of Aronszajn, a map $\chi : S_1 \rightarrow S_2$ is smooth if, for every $x \in S_1$, there exist $\varphi_1 \in \mathfrak{A}_1$ and $\varphi_2 \in \mathfrak{A}_2$ such that $x \in U_{\varphi_1}$, $\chi(x) \in U_{\varphi_2}$ and $\varphi_2 \circ \chi \circ \varphi_1^{-1} : V_{\varphi_1} \rightarrow V_{\varphi_2}$ extends to a $C^\infty$-map of a neighbourhood of $\varphi_1(x) \in \mathbb{R}^{n_{\varphi_1}}$ to a neighbourhood of $\varphi_2(\chi(x)) \in \mathbb{R}^{n_{\varphi_2}}$. A map $\chi : S_1 \rightarrow S_2$ is a diffeomorphism of Aronszajn’s subcartesian spaces if it is smooth, invertible and its inverse is smooth.

Let $S$ be a subcartesian differential space with a differential structure $C^\infty(S)$. By definition, for every $x \in S$, there exists a diffeomorphism $\varphi_x : U_x \subseteq S \rightarrow V_x \subseteq \mathbb{R}^{n_x}$. Here, $U_x$ is an open neighbourhood of $x$ in $S$ endowed
with the differential structure $C^\infty(U_x)$. Similarly, $V_x$ is a subset of $\mathbb{R}^{n_x}$ generated by pulling back the standard differential structure of $\mathbb{R}^{m_x}$ by the inclusion mapping $V_x \hookrightarrow \mathbb{R}^{n_x}$.

**Proposition 2.** Let $S$ be a differential space with $\mathfrak{A} = \{\varphi_x : U_x \rightarrow V_x\}_{x \in S}$. The pair $(S, \mathfrak{A})$ is an Aronszajn subcartesian space. Moreover, the topology of $(S, \mathfrak{A})$ is the same as the topology of the subcartesian differential space $(S, C^\infty(S))$.

**Proof.** (a) Let $S''$ be a differential subspace of $\mathbb{R}^n$. Denote by $\mathfrak{A}''$ the atlas for $S''$ defined above. In this case the inclusion map $\iota'' : S'' \hookrightarrow \mathbb{R}^n$ is smooth by definition of a differential subspace. Also, $S''$ is a Hausdorff topological subspace of $\mathbb{R}^n$.

For $k = i, j$ let $\varphi''_k : U''_k \rightarrow \varphi''_i(U''_k) \subseteq \mathbb{R}^{m_k}$ be two charts in $\mathfrak{A}''$ such that $U''_{ij} = U''_i \cap U''_j \neq \emptyset$. Then $V''_i = \varphi''_i(U''_i)$ is a differential subspace of $\mathbb{R}^{m_i}$, $V''_j = \varphi''_j(U''_{ij})$ is a differential subspace of $\mathbb{R}^{m_j}$, and the map $\varphi''_ij = \varphi''_{ij} \circ (\varphi''_i)^{-1} : V''_j \rightarrow V''_i$ is a diffeomorphism. Let $(z_1, \ldots, z_{m_i})$ be coordinates in $\mathbb{R}^{m_i}$, considered as maps $z_k : \mathbb{R}^{m_i} \rightarrow \mathbb{R}$, for $k = 1, \ldots, m_i$. If $\iota'' : V''_i \rightarrow \mathbb{R}^{m_i}$ is the inclusion map, then

$$\iota''_i \circ \varphi''_ij = (z_1 \circ \varphi''_{ij}, \ldots, z_{m_i} \circ \varphi''_ij) : V''_j \rightarrow \mathbb{R}^{m_i}$$

is smooth. For $k = 1, \ldots, m_i$, the function $z_k \circ \varphi''_ij$ is in $C^\infty(V''_j)$. By Proposition 1, for $y \in V''_j$ there exists an open neighbourhood $W''_k$ of $y$ in $V''_j \subseteq \mathbb{R}^{m_i}$ and a function $f''_k \in C^\infty(\mathbb{R}^{m_i})$ such that $(z_k \circ \varphi''_ij)|_{W''_k} = f''_k|_{W''_k}$. Let $W'' = W''_1 \cap \cdots \cap W''_k \cap \cdots \cap W''_{m_i}$. Then $f''_1|_{W''} = (z_k \circ \varphi''_ij)|_{W''}$ for $k = 1, \ldots, m_i$. Therefore,

$$(z_1 \circ \varphi''_{ij}, \ldots, z_{m_i} \circ \varphi''_ij)|_{W''} = (f''_1, \ldots, f''_{m_i})|_{W''} : W'' \rightarrow \mathbb{R}^{m_j},$$

and $(f''_1, \ldots, f''_{m_i}) : \mathbb{R}^{m_i} \rightarrow \mathbb{R}^{m_j}$ is an extension of

$$\varphi''_ij|_W = (\varphi''_i \circ (\varphi''_j)^{-1})|_W : W'' \rightarrow V''_i.$$ 

Similarly, we can construct local extensions of $\varphi''_j \circ (\varphi''_i)^{-1} : V''_i \rightarrow V''_j$. Therefore, for every differential subspace $S''$ of $\mathbb{R}^n$, $\mathfrak{A}''$ is an atlas. Hence, $(S'', \mathfrak{A}'')$ is an Aronszajn subcartesian space.

(b) Let $S'$ be a differential space diffeomorphic to a differential subspace $S''$ of $\mathbb{R}^n$ and let $\chi : S' \rightarrow S''$ be a diffeomorphism between these spaces. Then $S'$ is Hausdorff because $\chi$ is a homeomorphism. Let $\mathfrak{A}' = \{\varphi'\}$ be the atlas on $S''$ introduced in (a). It consists of charts $\varphi'' : U'' \rightarrow \varphi''(U'') \subseteq \mathbb{R}^m$, where $U''$ is an open subset of $S''$ and $\varphi''$ is a diffeomorphism onto its image. Then

$$\mathfrak{A}' = \{\varphi' : \varphi' = \chi^{-1} \circ \varphi'' \circ \chi : U' = \chi^{-1}(U'') \rightarrow \varphi'(U'') \subseteq \mathbb{R}^m \text{ for } \varphi'' \in \mathfrak{A}''\}$$

is an atlas on $S'$. Since $\chi : S' \rightarrow S''$ is a diffeomorphism of subcartesian
spaces, the atlas \( \mathfrak{A} \) satisfies condition 1 of the definition of Aronszajn subcartesian space.

For \( k = i, j \) let \( \varphi_k'' : U_k'' \to \varphi_k''(U'') \) be two charts in \( \mathfrak{A}'' \) such that \( U_{ij}'' = U_i'' \cap U_j'' \neq \emptyset \), as above. We follow the arguments given in (a). Let \( U_i' = \chi^{-1}(U_i'') \), and

\[
\varphi_i' = \varphi_i'' \circ \chi : U_i' \to \varphi_i'(U_i') = \varphi_i'' \circ \chi \circ \chi^{-1}(U_i'') = \varphi_i''(U_i'')
\]

Then

\[
U_{ij}' = U_i' \cap U_j' = \chi^{-1}(U_i'') \cap \chi^{-1}(U_j'') = \chi^{-1}(U_i'' \cap U_j'') = \chi^{-1}(U_{ij}'') \neq \emptyset.
\]

Moreover,

\[
\varphi_i'(U_{ij}'') = \varphi_i' \circ \chi^{-1}(U_i'' \cap U_j'') = \varphi_i'' \circ \chi \circ \chi^{-1}(U_i'' \cap U_j'') = \varphi_i''(U_i'' \cap U_j'') = V_{ij}'
\]

and \( \varphi_i'(U_{ij}'') \) are differential subspaces of \( \mathbb{R}^{m_i} \) and \( \mathbb{R}^{m_j} \), respectively, and the map

\[
\varphi_{ij} = \varphi_i' \circ (\varphi_j')^{-1} = (\varphi_i'' \circ \chi) \circ (\varphi_j'' \circ \chi)^{-1} = \varphi_i'' \circ (\varphi_j'' \circ \chi)^{-1} = \varphi_{ij}'' : V''_i \to V''_j
\]

is a diffeomorphism. In (a) we showed that this map satisfies condition 2 of the definition of Aronszajn subcartesian space. Hence \( (S', \mathfrak{A}') \), where \( S' \) is a differential space diffeomorphic to a differential subspace of \( \mathbb{R}^n \), is a subcartesian space of Aronszajn.

(c) Suppose that \( S \) is a subcartesian differential space. By definition \( S \) is Hausdorff. Moreover, each point \( x \in S \) has an open neighbourhood \( U_x \subseteq S \) and a diffeomorphism \( \varphi_x : U_x \to V_x \), where \( V_x \) is a differential subspace of \( \mathbb{R}^{n_x} \), for some \( n_x \in \mathbb{Z}_{\geq 0} \). By the arguments in (b), each differential subspace \( U_x \) of \( S \) is an Aronszajn subcartesian space. Since \( S \) is Hausdorff, it suffices to show that the maps \( \varphi_x \) also satisfy the second condition of the definition of Aronszajn subcartesian space. However, this condition is local, and is satisfied in each open neighbourhood \( U_x \) of the covering \( \{ U_x \} \) of \( S \). Hence, the Hausdorff subcartesian differential space \( S \) with the atlas \( \mathfrak{A} = \{ \varphi_x : U_x \to V_x \}_{x \in S} \) is a subcartesian space of Aronszajn.

The topology of the subcartesian differential space \( S \) is encoded in its differential structure. By definition, \( S \) is Hausdorff. Moreover, the domains of charts in \( \mathfrak{A} \) form an open cover of \( S \). Hence the topology on the Aronszajn subcartesian space \((S, \mathfrak{A})\) coincides with the topology of the subcartesian differential space \((S, C^\infty(S))\). \( \blacksquare \)

For \( i = 1, 2 \), let \( S_i \) be a subcartesian differential space with differential structure \( C^\infty(S_i) \), and let \( \mathfrak{A}_i \) be the set of diffeomorphisms of open subsets of \( S_i \) onto subsets of Euclidean spaces. By Proposition 2, \((S_1, \mathfrak{A}_1)\) and \((S_2, \mathfrak{A}_2)\) are subcartesian spaces of Aronszajn.
PROPOSITION 3. If $\chi : (S_1, C^\infty(S_1)) \to (S_2, C^\infty(S_2))$ is a smooth map of differential spaces and the atlases $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are constructed in terms of the differential structures $C^\infty(S_1)$ and $C^\infty(S_1)$ as in Proposition 2, then $\chi : (S_1, \mathfrak{A}_1) \to (S_2, \mathfrak{A}_2)$ is a smooth map of Aronszajn subcartesian spaces.

Proof. Suppose that $\chi : S_1 \to S_2$ is a smooth map of differential spaces. This means that $\chi^* f \in C^\infty(S_1)$ for each $f \in C^\infty(S_2)$. We want to show that $\chi : S_1 \to S_2$ is smooth in the sense of Aronszajn subcartesian differential spaces. According to the definition of Aronszjan subcartesian space, we have to show that for every $x \in S_1$, there exist $\varphi_1 : U_{\varphi_1} \to V_{\varphi_1}$ in $\mathfrak{A}_1$ and $\varphi_2 : U_{\varphi_2} \to V_{\varphi_2}$ in $\mathfrak{A}_2$, such that $x \in U_{\varphi_1}$, $\chi(x) \in U_{\varphi_2}$ and $\varphi_2 \circ \chi \circ \varphi_1^{-1} : V_{\varphi_1} \to V_{\varphi_2}$ extends to a $C^\infty$-map of a neighbourhood of $\varphi_1(x) \in \mathbb{R}^{n_\varphi}$ to a neighbourhood of $\varphi_2(\chi(x)) \in \mathbb{R}^{n_{\varphi_2}}$. In order to simplify the notation, we write $n_1 = n_{\varphi_1}$ and $n_2 = n_{\varphi_2}$.

For each $i = 1, 2$, the domains of the charts in $\mathfrak{A}_i$ cover $S_i$. Hence, for $x \in S_1$, there exists $\varphi_1 \in \mathfrak{A}_1$ such that $x \in U_{\varphi_1}$, and there exists $\varphi_2 \in \mathfrak{A}_2$ such that $\chi(x) \in U_{\varphi_2}$. For $i = 1, 2$, $\varphi_i : U_{\varphi_i} \to V_{\varphi_i}$ is a diffeomorphism of a differential subspace $U_{\varphi_i}$ of $S_i$ onto a differential subspace $V_{\varphi_i}$ of $\mathbb{R}^{n_i}$. Since $\chi : S_1 \to S_2$ is smooth in the sense of differential spaces, it follows that $\varphi_2 \circ \chi \circ \varphi_1^{-1} : V_{\varphi_1} \to V_{\varphi_2}$ is a smooth map of a differential subspace $V_{\varphi_1}$ of $\mathbb{R}^{n_1}$ to a differential subspace $V_{\varphi_2}$ of $\mathbb{R}^{n_2}$, and $\varphi_2 \circ \chi \circ \varphi_1^{-1}(\varphi_1(x)) = \varphi_2(\chi(x))$.

As in the proof of Proposition 2, let $(z_1, \ldots, z_{n_2})$ be coordinates in $\mathbb{R}^{n_2}$, considered as maps $z_k : \mathbb{R}^{n_2} \to \mathbb{R}$ for $k = 1, \ldots, n_2$. If $\iota_2 : V_{\varphi_2} \to \mathbb{R}^{n_2}$ is the inclusion map, then

$$\iota_2 \circ \varphi_2 \circ \chi \circ \varphi_1^{-1} = (z_1 \circ \varphi_2 \circ \chi \circ \varphi_1^{-1}, \ldots, z_{n_2} \circ \varphi_2 \circ \chi \circ \varphi_1^{-1}) : V_{\varphi_1} \to \mathbb{R}^{n_2}$$

is smooth. For $k = 1, \ldots, n_2$, the function $z_k \circ \varphi_2 \circ \chi \circ \varphi_1^{-1}$ lies in $C^\infty(V_{\varphi_1})$. By Proposition 1, for $y \in V_{\varphi_1}$, there exists an open neighbourhood $W_k$ of $y$ in $V_{\varphi_1} \subseteq \mathbb{R}^{n_1}$ and a function $f_k \in C^{\infty}(\mathbb{R}^{n_1})$ such that $(z_k \circ \varphi_2 \circ \chi \circ \varphi_1^{-1})|_{W_k} = f_k|_{W_k}$. Let $W = W_1 \cap \cdots \cap W_k \cap \cdots \cap W_{n_2}$. Then $f_k|_{W} = (z_k \circ \varphi_2 \circ \chi \circ \varphi_1^{-1})|_{W}$ for $k = 1, \ldots, n_2$. Therefore,

$$(z_1 \circ \varphi_2 \circ \chi \circ \varphi_1^{-1}, \ldots, z_{n_2} \circ \varphi_2 \circ \chi \circ \varphi_1^{-1})|_{W} = (f_1, \ldots, f_{n_2})|_{W} = W \to \mathbb{R}^{n_2},$$

and $(f_1, \ldots, f_{n_2}) : \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ is an extension of $(\varphi_2 \circ \chi \circ \varphi_1^{-1})|_{W} : W \to V_{\varphi_2}$.

The preceding argument can be repeated for every $x \in S_1$. Therefore, the map $\chi : S_1 \to S_2$ is smooth as a map of Aronszajn subcartesian spaces.

To prove the converse to Proposition 3, we need to determine a differential structure for an Aronszajn subcartesian space. Let $(S, \mathfrak{A})$ be a subcartesian space of Aronszajn. The atlas $\mathfrak{A}$ on $S$ consists of charts $\varphi : U_\varphi \to V_\varphi$, where $U_\varphi$ is an open subset of $S$, $V_\varphi$ is a subset of $\mathbb{R}^{n_\varphi}$, for some $n_\varphi \in \mathbb{N}$, and $\varphi$ is a diffeomorphism of $U_\varphi$ onto $V_\varphi$. We assume that $\{U_\varphi \mid \varphi \in \mathfrak{A}\}$ is an open cover of $S$. Let $\mathcal{F}$ be the following family of functions on the
Aronszajn subcartesian space \((S, \mathcal{A})\). A function \(f : S \to \mathbb{R}\) is in \(\mathcal{F}\), if for every \(x \in S\) there exists \(\varphi \in \mathcal{A}\) such that if \(x \in U_\varphi\), then there exists an open neighbourhood \(W_{\varphi(x)}\) of \(\varphi(x)\) in \(\mathbb{R}^{n_\varphi}\) and a function \(F_{\varphi(x)} \in C^\infty(\mathbb{R}^{n_\varphi})\) satisfying the condition

\[
(1) \quad f_{|\varphi^{-1}(V_\varphi \cap W_{\varphi(x)})} = (F_{\varphi(x)} \circ \varphi)_{|\varphi^{-1}(V_\varphi \cap W_{\varphi(x)})}.
\]

**Proposition 4.** The family of sets

\[
s = \{f^{-1}(I) \mid f \in \mathcal{F} \text{ and } I \text{ is an open interval in } \mathbb{R}\}
\]

is a subbasis for the topology of the Aronszajn subcartesian space \((S, \mathcal{A})\).

**Proof.** Every \(U_\varphi \in \mathcal{A}\) is an open set in \(S\). It is homeomorphic to a subset \(V_\varphi \subseteq \mathbb{R}^{n_\varphi}\) with topology induced by the inclusion map \(V_\varphi \hookrightarrow \mathbb{R}^{n_\varphi}\). Let \(C^\infty(V_\varphi)\) be the differential structure of \(V_\varphi\) generated by its inclusion into \(\mathbb{R}^{n_\varphi}\). In other words, \(C^\infty(V_\varphi)\) is generated by the family \(\mathcal{F}_{V_\varphi}\) of functions \(h : V_\varphi \to \mathbb{R}\) such that for every \(\varphi(x) \in V_\varphi\), there exists an open neighbourhood \(W_{\varphi(x)}\) of \(\varphi(x)\) in \(\mathbb{R}^{n_\varphi}\) and a function \(F_{\varphi(x)} \in C^\infty(\mathbb{R}^{n_\varphi})\) which satisfies the condition

\[
h_{|V_\varphi \cap W_{\varphi(x)}} = F_{\varphi(x)}{\mid}_{V_\varphi \cap W_{\varphi(x)}}.
\]

The topology of \((V_\varphi, C^\infty(V_\varphi))\) has a subbasis

\[
s_{V_\varphi} = \{h^{-1}(I) \mid h \in \mathcal{F}_{V_\varphi} \text{ and } I \text{ is an open interval in } \mathbb{R}\}.
\]

Since \(\varphi : U_\varphi \to V_\varphi\) is a homeomorphism, it follows that \(\varphi^{-1}(s_{V_\varphi})\) is a subbasis for the topology of \(U_\varphi\). But

\[
\varphi^{-1}(s_{V_\varphi}) = \{\varphi^{-1}(h^{-1}(I)) \mid h \in \mathcal{F}_{V_\varphi} \text{ and } I \text{ is an open interval in } \mathbb{R}\}
\]

\[
= \{(h \circ \varphi)^{-1}(I) \mid h \in \mathcal{F}_{V_\varphi} \text{ and } I \text{ is an open interval in } \mathbb{R}\}
\]

\[
\subseteq \{f^{-1}(I) \mid f \in \mathcal{F} \text{ and } I \text{ is an open interval of } \mathbb{R}\} = s.
\]

Hence

\[
(2) \quad \bigcup_{\varphi \in \mathcal{A}} \varphi^{-1}(s_{V_\varphi}) \subseteq s.
\]

Since \(S = \bigcup_{\varphi \in \mathcal{A}} U_\varphi\) and \(\varphi^{-1}(s_{V_\varphi})\) is a subbasis for the topology of \(U_\varphi\), it follows that \(\bigcup_{\varphi \in \mathcal{A}} \varphi^{-1}(s_{V_\varphi})\) is a subbasis for the topology of \(S\). The inclusion (2) ensures that \(s\) is a subbasis for the topology of \(S\). 

There is a differential structure \(C^\infty(S)\) on a set \(S\), which is generated by a family \(\mathcal{F}\) of functions on \(S\) (see [4]). Applying this construction to a subcartesian space \((S, \mathcal{A})\) of Aronszajn, and taking the family \(\mathcal{F}\) defined by equation (1), we get the differential structure \(C^\infty(S)\) on \(S\) determined by the atlas \(\mathcal{A}\) on \(S\). Proposition 4 ensures that the subcartesian space \((S, \mathcal{A})\) and the corresponding differential space \((S, C^\infty(S))\) have the same topology.
Proposition 5. Let \((S_1, \mathcal{A}_1)\) and \((S_2, \mathcal{A}_2)\) be Aronszajn subcartesian spaces, and let \((S_1, C^\infty(S_1))\) and \((S_2, C^\infty(S_2))\) be the corresponding subcartesian differential spaces. If \(\chi : (S_1, \mathcal{A}_1) \to (S_2, \mathcal{A}_2)\) is a smooth map of subcartesian spaces of Aronszajn, then \(\chi : (S_1, C^\infty(S_1)) \to (S_2, C^\infty(S_2))\) is a smooth map of subcartesian differential spaces.

Proof. By definition of Aronszajn subcartesian space, the assumption that the map \(\chi : (S_1, C^\infty(S_1)) \to (S_2, C^\infty(S_2))\) is smooth means that for every \(x_1 \in S_1\), there exist \(\varphi_1 : U_{\varphi_1} \to V_{\varphi_1}\) in \(\mathcal{A}_1\) and \(\varphi_2 : U_{\varphi_2} \to V_{\varphi_2}\) in \(\mathcal{A}_2\) such that \(x_1 \in U_{\varphi_1}\), \(x_2 = \chi(x_1) \in U_{\varphi_2}\) and \(\varphi_2 \circ \chi \circ \varphi_1^{-1} : V_{\varphi_1} \to V_{\varphi_2}\) extends to a \(C^\infty\)-map of a neighbourhood of \(\varphi_1(x_1) \in \mathbb{R}^{n_1}\) to a neighbourhood of \(\varphi_2(x_2) \in \mathbb{R}^{n_2}\), where \(n_1 = n_{\varphi_1}\) and \(n_2 = n_{\varphi_2}\), as in the proof of Proposition 2.

For \(i = 1, 2\), let \(F_i\) denote the space of functions on \(S_i\) determined by equation (1). It suffices to show that, for every \(f_2 \in F_2\) the pull-back \(f_1 = \chi^* f_2 = f_2 \circ \chi\) is in \(F_1\). In other words, we have to show that, for every \(x_1 \in S_1\), there exists \(\varphi_1 \in \mathcal{A}_1\) such that if \(x_1 \in U_{\varphi_1}\) then there exists an open neighbourhood \(W_{\varphi_1(x_1)}\) of \(\varphi_1(x_1)\) in \(\mathbb{R}^{n_1}\) and a function \(F_{\varphi_1(x_1)} \in C^\infty(\mathbb{R}^{n_1})\) satisfying

\[
\left. f_1 \right|_{\varphi_1^{-1}(W_{\varphi_1(x_1)})} = \left. (F_{\varphi_1(x_1)} \circ \varphi_1) \right|_{\varphi_1^{-1}(W_{\varphi_1(x_1)})}.
\]

On the other hand, \(f_1 = f_2 \circ \chi\), where \(f_2 \in F_2\). Therefore, for \(x_2 = \chi(x_1) \in S_2\), there exists \(\varphi_2 \in \mathcal{A}_2\) such that if \(x_2 \in U_{\varphi_2}\), then there exists an open neighbourhood \(W_{\varphi_2(x_2)}\) of \(\varphi_2(x_2)\) in \(\mathbb{R}^{n_2}\) and a function \(F_{\varphi_2(x_2)} \in C^\infty(\mathbb{R}^{n_2})\) satisfying

\[
\left. f_2 \right|_{\varphi_2^{-1}(W_{\varphi_2(x_2)})} = \left. (F_{\varphi_2(x_2)} \circ \varphi_2) \right|_{\varphi_2^{-1}(W_{\varphi_2(x_2)})}.
\]

By hypothesis \(\chi\) is a smooth map of the Aronszajn subcartesian space \((S_1, \mathcal{A}_1)\) into the Aronszajn subcartesian space \((S_2, \mathcal{A}_2)\). So the map \(\psi = \varphi_2 \circ \chi \circ \varphi_1^{-1} : V_{\varphi_1} \to V_{\varphi_2}\) extends to a \(C^\infty\)-map \(\tilde{\psi} : \tilde{W}_1 \to \tilde{W}_2\), where \(\tilde{W}_1\) is a neighbourhood of \(\varphi_1(x_1) \in \mathbb{R}^{n_1}\), and \(\tilde{W}_2\) is a neighbourhood of \(\varphi_2(x_2) = \varphi_2(\chi(x_1)) \in \mathbb{R}^{n_2}\). Without loss of generality, we may shrink \(W_{\varphi_1(x_1)}\), \(W_{\varphi_2(x_2)}\), \(\tilde{W}_1\) and \(\tilde{W}_2\) so that \(W_{\varphi_1(x_1)} = \tilde{W}_1\) and \(W_{\varphi_2(x_2)} = \tilde{W}_2\). Note that the existence of appropriate \(W_{\varphi_2(x_2)}\), \(\tilde{W}_1\) and \(\tilde{W}_2\) is guaranteed by the hypotheses that \(f_2\) and \(\chi\) are smooth. The existence of an appropriate \(W_{\varphi_1(x_1)}\) has to be established. Initially, we choose \(W_{\varphi_1(x_1)} = \tilde{W}_1 = \tilde{\psi}^{-1} \tilde{W}_2 = \tilde{\psi}^{-1}(W_{x_2})\). Later, we shall have to shrink \(W_{\varphi_1(x_1)}\) some more.

Having made these choices, we compute

\[
\left. (F_{\varphi_2(x_2)} \circ \varphi_2) \right|_{\varphi_2^{-1}(W_{\varphi_2(x_2)})} = F_{\varphi_2(x_2)}|_{\varphi_2^{-1}(W_{\varphi_2(x_2)})} = F_{\varphi_2(x_2)}|_{\tilde{\psi}(V_{\varphi_1}) \cap W_{\varphi_2(x_2)}} = F_{\varphi_2(x_2)}|_{\tilde{\psi}(V_{\varphi_1}) \cap \tilde{\psi}^{-1}(W_{\varphi_2(x_2)})}
\]
$$= (F\varphi_2(x_2) \circ \tilde{\psi})|_{V_{\varphi_1} \cap \tilde{\psi}^{-1}(W_{\varphi_2}(x_2))} = (F\varphi_2(x_2) \circ \tilde{\psi})|_{V_{\varphi_1} \cap W_{\varphi_1}(x_1)}$$

$$= ((F\varphi_2(x_2) \circ \tilde{\psi}) \circ \varphi_1)|_{\varphi_1^{-1}(V_{\varphi_1} \cap W_{\varphi_1}(x_1))}.$$ 

Since $\psi = \varphi_2 \circ \chi \circ \varphi_1^{-1}$ and $f_1 = f_2 \circ \chi$,

$$f_1|_{\varphi_1^{-1}(V_{\varphi_1} \cap W_{\varphi_1}(x_1))} = (f_2 \circ \chi)|_{\varphi_1^{-1}(V_{\varphi_1} \cap W_{\varphi_1}(x_1))}$$

$$= (f_2 \circ \varphi_1^{-1} \circ \psi \circ \varphi_1)|_{\varphi_1^{-1}(V_{\varphi_1} \cap W_{\varphi_1}(x_1))}$$

$$= (f_2 \circ \varphi_1^{-1})|_{\psi(V_{\varphi_1} \cap W_{\varphi_1}(x_1))}$$

$$= (f_2 \circ \varphi_1^{-1})(V_{\varphi_2} \cap W_{\varphi_2}(x_2))$$

$$= f_2|_{\varphi_2^{-1}(V_{\varphi_2} \cap W_{\varphi_2}(x_2))}.$$ 

Taking into account equations (3) through (6), we get

$$f_1|_{\varphi_1^{-1}(V_{\varphi_1} \cap W_{\varphi_1}(x_1))} = f_2|_{\varphi_2^{-1}(V_{\varphi_2} \cap W_{\varphi_2}(x_2))}$$

$$= (F\varphi_2(x_2) \circ \varphi_2)|_{\varphi_2^{-1}(V_{\varphi_2} \cap W_{\varphi_2}(x_2))}$$

$$= ((F\varphi_2(x_2) \circ \tilde{\psi}) \circ \varphi_1)|_{\varphi_1^{-1}(V_{\varphi_1} \cap W_{\varphi_1}(x_1))}.$$ 

Note that $F\varphi_2(x_2) \circ \tilde{\psi}$ is a smooth function on the open subset $\tilde{\psi}^{-1}(W_{\varphi_2}(x_2))$ of $\mathbb{R}^{n_1}$. However, it need not extend to a smooth function on $\mathbb{R}^{n_1}$. But we can always find an open subset $W_{\varphi_1}(x_1)$ of $\psi^{-1}(W_{\varphi_2}(x_2))$ containing $x_1$ and $F\varphi_1(x_1) \in C^\infty(\mathbb{R}^{n_1})$ such that $F\varphi_1(x_1)|_{W_{\varphi_1}(x_1)} = (F\varphi_2(x_2) \circ \tilde{\psi})|_{W_{\varphi_1}(x_1)}$. Equation (7) ensures that equation (3) is satisfied with the choices made here. This argument works for every $f_2 \in F_2$. Hence $\chi : (S_1, C^\infty(S_1)) \to (S_2, C^\infty(S_2))$ is a smooth map of differential spaces.

In proving Proposition 2 we have shown that to a given a subcartesian differential space $S$ with differential structure $C^\infty(S)$, we can associate a subcartesian space of Aronszajn with an atlas $\mathbb{A}$ consisting of diffeomorphisms $\varphi : U_\varphi \to V_\varphi \subseteq \mathbb{R}^{n_\varphi}$, where $U_\varphi$ is an open subset of $S$ and $V_\varphi$ is an arbitrary differential subspace of $\mathbb{R}^{n_\varphi}$. This construction gives a morphism $(S, C^\infty(S)) \to (S, \mathbb{A})$ from the category of subcartesian differential spaces to the category of subcartesian spaces of Aronszajn. Conversely, given a subcartesian space of Aronszajn $(S, \mathbb{A})$, in proving Proposition 5 we have constructed a morphism $(S, \mathbb{A}) \to (S, C^\infty(S))$ from the category of subcartesian spaces of Aronszajn to the category of subcartesian differential spaces.

**Theorem 6.** The morphisms

$$(S, C^\infty(S)) \to (S, \mathbb{A}) \quad \text{and} \quad (S, \mathbb{A}) \to (S, C^\infty(S)),$$

defined in the preceding paragraph, are inverses of each other.
Proof. Let S be a subcartesian differential space with differential structure \(C^\infty(S)\), and let \(\mathfrak{A}\) be the atlas in S consisting of diffeomorphisms

\[ \varphi : (U_\varphi, C^\infty(U_\varphi)) \rightarrow (V_\varphi, C^\infty(V_\varphi)), \]

where \(U_\varphi\) is an open subset of S, \(C^\infty(U_\varphi)\) is generated by the inclusion map \(i : U_\varphi \hookrightarrow S\) and \((V_\varphi, C^\infty(V_\varphi))\) is an arbitrary differential subspace of \(\mathbb{R}^{n_\varphi}\). If \(f \in C^\infty(S)\), then \(f|_{U_\varphi} \in C^\infty(U_\varphi)\) and \(f|_{U_\varphi} \circ \varphi^{-1} \in C^\infty(V_\varphi)\). This means that for every \(x \in V_\varphi \subseteq \mathbb{R}^{n_\varphi}\) there exists an open neighbourhood \(W_x\) of \(x\) in \(\mathbb{R}^{n_\varphi}\) and a smooth function \(F_x \in C^\infty(\mathbb{R}^{n_\varphi})\) such that

\[ (f|_{U_\varphi} \circ \varphi^{-1})|_{V_\varphi \cap W_x} = F_x|_{V_\varphi \cap W_x}. \]

Conversely, if a function \(f : S \rightarrow \mathbb{R}\) is such that, for every \(\varphi \in \mathfrak{A}\), \(f|_{U_\varphi} \circ \varphi^{-1} \in C^\infty(V_\varphi)\), then \(f|_{U_\varphi} \in C^\infty(U_\varphi)\). Since the differential structure \(C^\infty(U_\varphi)\) is generated by restrictions to \(U_\varphi\) of functions in \(C^\infty(S)\), the definition of differential structure ensures that \(f \in C^\infty(S)\) (see [4]).

Let \(\mathcal{F}\) be family of functions \(f\) on \(S\) such that, for every \(x \in S\), there exists \(\varphi \in \mathfrak{A}\) with \(U_\varphi\) an open subset of \(S\) containing \(x\), and there exists an open neighbourhood \(W_\varphi(x)\) of \(\varphi(x)\) in \(\mathbb{R}^{n_\varphi}\) and a function \(F_\varphi(x) \in C^\infty(\mathbb{R}^{n_\varphi})\) which satisfies

\[ f|_{\varphi^{-1}(V_\varphi \cap W_\varphi(x))} = (F_\varphi(x) \circ \varphi)|_{\varphi^{-1}(V_\varphi \cap W_\varphi(x))}. \]

The argument in the preceding paragraph shows that \(\mathcal{F} \subseteq C^\infty(S)\). Proposition 4 ensures that the topology of the differential structure on \(S\) generated by \(\mathcal{F}\) coincides with the topology of the original differential structure \(C^\infty(S)\). We need only show that the differential structure generated by \(\mathcal{F}\) coincides with the original differential structure \(C^\infty(S)\). Since \(\mathcal{F} \subseteq C^\infty(S)\), it follows that the differential structure generated by \(\mathcal{F}\) is contained in \(C^\infty(S)\). Conversely, let \(f \in C^\infty(S)\). For every \(\varphi \in \mathfrak{A}\) and \(y \in V_\varphi\) it satisfies equation [8]. If \(x = \varphi^{-1}(y)\) and \(F_x = F_y \circ \varphi^{-1}\), then \(f\) satisfies equation [9]. So for every \(\varphi \in \mathfrak{A}\), \(f|_{U_\varphi}\) coincides with the restriction to \(U_\varphi\) of a function in the differential structure generated by \(\mathcal{F}\). This implies that \(f|_{U_\varphi}\) lies in that structure. Hence the differential structure on \(S\) generated by \(\mathcal{F}\) is equal to the original differential structure \(C^\infty(S)\). \(\blacksquare\)

In the construction of the morphisms \((S, C^\infty(S)) \rightarrow (S, \mathfrak{A})\) and \((S, \mathfrak{A}) \rightarrow (S, C^\infty(S))\) we used only intrinsic constructions. Therefore these morphisms are natural transformations. In particular, all geometric structures and relations obtained in the category of subcartesian spaces of Aronszajn may be translated to the category of subcartesian differential spaces.

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