No-Substitution $k$-means Clustering with Optimal Center Complexity and Low Memory

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Abstract

We consider $k$-means clustering in the online no-substitution setting where one must decide whether to take each data point $x_t$ as a center immediately upon streaming it and cannot remove centers once taken. Our work is focused on the arbitrary-order assumption where there are no restrictions on how the points $X$ are ordered or generated. Algorithms in this setting are evaluated with respect to their approximation ratio compared to optimal clustering cost, the number of centers they select, and their memory usage.

Recently, Bhattacharjee and Moshkovitz (2020) defined a parameter, $\text{Lower}_{\alpha,k}(X)$ that governs the minimum number of centers any $\alpha$-approximation clustering algorithm, allowed any amount of memory, must take given input $X$. To complement their result, we give the first algorithm that takes $\tilde{O}(\text{Lower}_{\alpha,k}(X))$ centers (hiding factors of $k$, $\log n$) while simultaneously achieving a constant approximation and using $\tilde{O}(k)$ memory in addition to the memory required to save the centers. Our algorithm shows that in the no-substitution setting, it is possible to take an order-optimal number of centers while using little additional memory.

Keywords: $k$-means; No-substitution clustering; approximation algorithms; online algorithms.

1. Introduction

Clustering is an indispensable tool in unsupervised learning. The goal of $k$-means clustering is to find centers that partition a dataset into $k$ clusters in a way that minimizes the $k$-means cost. It is well-known that this problem is NP-hard (Aloise et al. (2009); Dasgupta (2008)), and there is no known algorithm that will find the minimum cost clustering. However, many approximation algorithms promising to find a clustering with cost at most $\alpha$ times the optimal cost, are known, e.g. Aggarwal (2007); Arthur and Vassilvitskii (2006); Kanungo et al. (2004). The best of these algorithms can find approximations for constant $\alpha$ regardless of the input dataset.

In this work, we consider the online, no-substitution setting, a term introduced in Hess and Sabato (2020). In this setting, data points are streamed one by one, and for each point we decide whether or not to include it as a center. The size of the data stream is unknown, and our decisions are final in that a point is gone forever if we do not select it and can’t be later removed if we do select it. The objective is a bi-criteria: we must (1) select enough centers so that the resulting $k$-means cost of clustering the entire dataset with the chosen centers is at most $\alpha$ times the optimal, where $\alpha$ is any fixed constant, and (2) minimize the number of centers chosen. This setting has applications where center selection is mapped to some irreversible action, such as patient selection for a clinical trial.
In this general setting, Bhattacharjee and Moshkovitz (2020) show a lower bound Lower_{α,k}(X) such that any α-approximation algorithm must select Ω(Lower_{α,k}(X)) when streamed on X. While the same paper presents an O(k^3)-approximation algorithm that selects ˜O(k^2Lower_{α,k}(X)), this algorithm uses O(n) memory and runs an offline clustering algorithm each time a new point is streamed. However, this result does show that it is possible to be tight with the lower bound up to factors of k, log n. In addition, the best true online α-approximation algorithm selects ˜O(k log γ^*) centers where γ^* is the maximum ratio between the furthest two points and closest two points in an optimal cluster (Bhaskara and Rwanpathirana, 2020; Liberty et al., 2016). However, there exist datasets where γ^* is arbitrarily large, yet Lower_{α,k}(X) is small for any α. Thus, this algorithm does not achieve optimal center complexity.

We provide a new online, no-substitution algorithm that achieves a 9 approximation and selects ˜O(kLower_{9,k}(X)) centers, which is order-optimal with the best known lower bound. Our algorithm uses O(k) memory in addition to the centers it selects, and its running time for each streamed point is no more than the number of centers it selects. While the aforementioned algorithms were able to achieve either order-optimal center complexity or low memory, our algorithm is the first to provide both guarantees.

1.1. Summary of Our Results

We begin with our main result which governs the performance of OnlineCluster (Algorithm 1). We use L to denote the cost function (formally defined in Section 2), and we let L_k(X) denote the optimal k-means cost of clustering X. We also let S denote the set of cluster centers outputted by OnlineCluster.

**Theorem 1** For any input X = \{x_1, x_2, \ldots, x_n\}, Algorithm 1 has the following properties (with all probabilities taken over its own randomness).

1. (Approximation Factor) With probability at least 0.95, L(X, S) ≤ 9L_k(X).

2. (Center Complexity) With probability at least 0.95, |S| ≤ O(k \ln k \log^2 nLower_{9,k}(X)).

3. (Auxiliary Memory) Algorithm 1 always uses at most ˜O(k) additional memory.

OnlineCluster combines ideas from both Liberty et al. (2016) and Bhattacharjee and Moshkovitz (2020) to achieve its guarantees. Now we describe the processing methods that the existing algorithms use to decide if an incoming point x_t should be a center. Since we will extensively refer to these processing methods later, we give them names.

In a **type 1 processing** (Liberty et al., 2016), x_t is selected with probability \( \frac{D_t^2}{R_t} \), where D_t is the distance between x_t and the nearest center, and R_t is a “threshold” at time t. If R_t is appropriately chosen, the expected center complexity will be O(k), and an O(1)-approximation factor is possible. However, the appropriate value of R_t depends on the entire stream, and cannot be computed. Instead, Liberty et al. use a doubling scheme that doubles R_t when too many centers are being selected and show that this converges to the appropriate value of R_t while incurring a center complexity of ˜O(k log γ^*).

In a **type 2 processing** (Bhattacharjee and Moshkovitz, 2020), x_t is selected with probability inversely proportional to the size of the offline cluster containing x_t. These offline clusters are recomputed before processing x_t, at each time t. This method is able to quickly adapt to changing
Table 1: Comparison of existing no-substitution algorithms with our algorithm. The memory column contains memory in addition to the selected centers. Thus, the total memory used by each algorithm is the sum of its memory and its center complexity.

data scales: if data suddenly becomes extremely spread out, the offline clustering will immediately detect this change, whereas the algorithm of Liberty et al. may take many points as it updates to an appropriate value of $R_t$.

The key new innovation of our algorithm is that OnlineCluster leverages a modified version of the classic $k$-centers clustering algorithm both to decides when to apply which type of processing, and to execute each type of processing based on of Charikar et al. (2003). Charikar et al. (2003)’s algorithm can easily maintain a low memory nearly optimal $k$-center clustering. While this clustering is relatively poor for $k$-means, it is nevertheless able to give a good indication on when the data is extremely spread out, meaning that a type 2 processing is necessary, or when the data is more mixed, meaning that a type 1 processing is necessary. Furthermore, our modified version of this algorithm allows us to provide rough estimates on the optimal value of $R_t$ (for type 1 processings) as well as the sizes of an offline clustering (for type 2 processings).

We now include a comparison of Algorithm 1’s performance with the other main algorithms in this setting. The key takeaway is that our algorithm is the first algorithm to achieve success simultaneously across all 3 criteria. While Bhaskara and Rwanpathirana (2020) achieves a good approximation ratio and (additional) memory requirement, their center complexity has a dependence on the aspect ratio which can be quite poor (as discussed in Bhattacharjee and Moshkovitz (2020). By contrast, Bhattacharjee and Moshkovitz (2020) achieves an order optimal center complexity (up to factors of $k^2 \log n$) but has a high approximation factor and a very high memory requirement.

As a final note, our algorithm actually improves on the center complexity given in Bhattacharjee and Moshkovitz (2020) by reducing the quadratic dependence on $k$ to a linear dependence. This improvement can be non-trivial if $k$ is set to be substantially larger than $\log n$.

1.2. Related Work

No-substitution setting. Perhaps the most well-known example of the no-substitution setting for an online algorithm is the secretary problem (Babaioff et al., 2009). The setting has also been used in online submodular maximization (Buchbinder et al., 2018). For the $k$-means problem, existing work makes assumptions about the dataset $X$. First, Hess and Sabato (2020) and Moshkovitz (2019) assume $X$ is generated by some random process rather than an arbitrary stream as we have here. Also, Hess and Sabato (2020) and Bhaskara and Rwanpathirana (2020) assume the aspect ratio of $X$ is bounded.
Lower bounds for online, no-substitution \( k \)-means are known as well. Moshkovitz (2019) show that there exist datasets for which any algorithm must take \( \Omega(n) \) centers. Bhattacharjee and Moshkovitz (2020) improve this by showing a dataset-specific lower bound of \( \Omega(Lower_{\alpha,k}(X)) \).

**Online facility location.** Online facility location is similar to no-substitution \( k \)-means and was first studied in a classic work by Meyerson (2001). In online facility location, facilities are created immediately when streamed and cannot be closed later. There are two significant differences between this and our setting. First, the cost incurred by the algorithm upon opening a set of facilities \( F \) is \( |F| + \sum_{t=1}^{n} d(x_t, F_t)^2 \), where \( F_t \) is the set of facilities at time \( t \). Our cost function does not have a factor of \( |F| \) in it which can be significant: if distances are of order \( n \), then a trivial solution with \( |F| = n \) becomes optimal. Furthermore, a point \( x_t \) is charged using its distance to the facilities that currently exist, \( F_t \), whereas in the no-substitution setting, \( x_t \) is charged using its closest distance to all the centers eventually chosen. Several works (Lang, 2018; Fotakis, 2011; Feldkord and Meyer auf der Heide, 2018) consider variants of online facility location.

**Streaming with limited memory.** Data streaming algorithms for a wide range of problems are known; for a full discussion, see Aggarwal (2007) and Guha et al. (2003). Algorithms for \( k \)-means in the streaming model are known such as those by Braverman et al. (2011), Shindler et al. (2011), and Ailon et al. (2009). However, this setting is much different from the online, no-substitution setting because they permit previous points to be added to or removed from the set of centers.

### 2. Preliminaries

For a dataset \( X \subset \mathbb{R}^d \), the \( k \)-means cost of \( X \) with respect to a set \( S \subset \mathbb{R}^d \) is defined as

\[
\mathcal{L}(X, S) = \sum_{x \in X} \min_{c \in C} d(x, c)^2,
\]

where \( d(x, y) \) denotes Euclidean distance. For a natural number \( k \), we denote \( \mathcal{L}_k(X) \) as the smallest achievable \( k \)-centers cost out of all sets of size \( k \). Formally, \( \mathcal{L}_k(X) = \min |S| = k \mathcal{L}(X, S) \).

The following is a folklore lemma that allows us to bound \( k \)-means costs in many situations.

**Lemma 2 (Center Shifting Lemma)** Let \( X \subset \mathbb{R}^d \) be a finite set of points, and let \( \mu = \frac{1}{|X|} \sum_{x \in X} x \) be its average. Then for any \( s \in \mathbb{R}^d \), \( \mathcal{L}(X, \{s\}) = \mathcal{L}(X, \{\mu\}) + |S| d(s, \mu)^2 \).

When \( k = 1 \), this lemma establishes that the best \( k \)-means cost of \( X \) is its centroid. We will use this lemma extensively in later analyses.

Next, we review the online, no-substitution framework. In this setting, one streams through a dataset \( X = \{x_1, \ldots, x_n\} \), one point at a time. At each time step, an algorithm must choose whether to accept the new point, which means that it will be used a center in the eventual output, or to reject the new point, which means that this point can no longer be used as a center. In this work, we assume that \( n \) is not known ahead of time.

Algorithms are evaluated with a bi-criteria: in terms of their **approximation factor** and in terms of their **center complexity**. More precisely, we say that an online algorithm \( A \) is an \( \alpha \)-approximation if for any input \( X \) and any ordering of that input, it returns a set \( S \subset X \) such that \( \mathcal{L}(X, S) \leq \alpha \mathcal{L}_k(X) \) with probability at least 0.95, where the probability is taken over the randomness in \( A \). Similarly, we say that \( A \) has center complexity \( \beta \) on input \( X \) if with probability
at least 0.95 it returns a set of clusters $S$ such that $|S| \leq \beta$. As before, the probability is taken over
the randomness in $A$. Note that the choice of 0.95 is arbitrary: it is not difficult to apply standard
amplifying techniques to increase this probability to $1 - \delta$ at the cost of increasing the number of
centers chosen by a factor of $\log \frac{1}{\delta}$.

**Additional Notation:** We let $X = \{x_1, \ldots, x_n\}$ be our input dataset, and let $X_t = \{x_1, \ldots, x_t\}$
denote the first $t$ elements of $X$. We also let $C_1^t, C_2^t, \ldots, C_k^t$ denote the optimal $k$-means clustering
of $X_t$. Namely, if $S_t = \{s_1^t, \ldots, s_k^t\}$ achieves $L(X_t, S_t) = L_k(X_t)$, then $C_i^t$ are those points which
are closest to $s_i^t$ out of all points in $S_t$.

In general, we refer to the value of any parameter in Algorithm 1 at time $t$ by adding a subscript $t$. For example, $R_t$ denotes the value of $R$ after the point $x_t$ has been processed.

**Lower Bounds:** We now briefly review the parameter $\text{Lower}_{\alpha,k}(X)$ introduced in Bhattacharjee and Moshkovitz (2020). We begin by defining the $l$-fold diameter $\text{diam}_l(X)$ of a set $X \subseteq \mathbb{R}^d$ as the smallest real
number $D$ such that $X$ can be partitioned into $l$ disjoint sets each of which have diameter at most $D$.

Using this notion, for any $\alpha > 1$, we define $\text{Lower}_{\alpha,k}(X)$ as the longest sequence of points
$y_1, y_2, \ldots, y_r$ where $y_i$ are distinct elements of $X$ that satisfy

$$d(y_i, \{y_1, \ldots, y_{i-1}\}) > \sqrt{i\alpha} \text{diam}_{k-1}(\{y_1, \ldots, y_{i-1}\})$$

for all $2 \leq i \leq r$.

Bhattacharjee and Moshkovitz (2020) showed that this parameter gives a lower bound on the
center complexity of any $\alpha$-approximation algorithm run on $X$ in worst case order. The intuition
here is that the points $y_1, y_2, \ldots, y_r$ become increasingly spread out at an exponential rate. As a
consequence, any online algorithm must select all of them (when presented in that order) in order
to maintain an approximation factor of $\alpha$. Formally, we have the following theorem:

**Theorem 3** Bhattacharjee and Moshkovitz (2020) Let $X$ be an arbitrary set of points. There exists
an ordering of $X$ such that any online $\alpha$-approximation algorithm has center complexity at least

$$\frac{18}{19} \text{Lower}_{\alpha,k}(X)$$

when run on that ordering.

Full details regarding this definition and lower bound can be found in Appendix A. This theorem
gives the best possible center complexity we can hope to achieve in the no-substitution setting.
While Bhattacharjee and Moshkovitz (2020) provide an algorithm which achieves center complexity
of $\text{Lower}_{k^3,k}(X)$ and a $k^3$-approximation, our algorithm is the first to use just $\tilde{O}(k)$ additional
memory.

### 3. Algorithm Description

We now describe OnlineCluster, our no-substitution $k$-means clustering algorithm. The pseudocode
for OnlineCluster appears in Algorithm 1. The main idea behind OnlineCluster is to perform a type
1 or type 2 processing, depending on a property of the online $k$-center clustering of the current set
of points. We now describe the variables and the three main components of OnlineCluster in detail.
Algorithm 1: OnlineCluster\((X)\)

**Input:** A stream of points \(X = \{x_1, x_2, \ldots, x_n\}\)

**Output:** A set of cluster centers \(S \subseteq X\)

1. \(S \leftarrow \{x_i : 1 \leq i \leq k\}\);
2. \(R, F \leftarrow 0\);
3. \(P, Q, Z \leftarrow \text{Online}_k\text{centers}(\{x_1, \ldots, x_k\})\);
4. for \(t = k + 1, \ldots, n\) do
   5. \(P, Q, Z \leftarrow \text{Online}_k\text{centers}(x_t)\);
   6. if \(Q \leq 4(t + 2)P\) then
      7. with probability \(\frac{P^2}{24k\ln(k+10)}\), \(R \leftarrow \frac{P^2}{24k\ln(k+10)}\);
      8. \(F \leftarrow 0\);
      9. \(S \leftarrow S \cup \{x_t\}\);
   10. else
      11. \((z, m) \leftarrow \arg\min_Z d(z, x_t)\);
      12. with probability \(\frac{12\ln(k+10)}{m}\), \(S \leftarrow S \cup \{x_t\}\);
   13. end
5. return \(S\);

**Algorithm Variables**

Lines 1-4 initialize the main variables of Algorithm 1. The variable \(S\) is the cluster centers that our algorithm selects for the its \(k\)-means clustering. Because we are in the no-substitution setting, points are never removed from \(S\). Line 1 initializes \(S\) by setting it to be the first \(k\) points. This forms a trivial clustering for the first \(k\) points.

The variables \(R, F\) are used for type 1 clusterings as a selection threshold and a doubling counter, respectively. The variable \(R\) inversely influences the probability of selecting a new point in a type 1 processing. In Line 2, \(R\) is initialized to 0, as there is no obvious way to set \(R\) without knowing anything about \(X\), and its value is updated later before doing any type 1 processings. The variable \(F\) counts how many points were selected since \(R\) was last doubled. This is used to help \(R\) converge to its “best” value, as if many points are being taken, then \(R\) is set too low.

The variables \(P, Q, Z\) contain information about an online \(k\)-centers clustering of \(X_t\). They are computed using a classic online \(k\)-centers clustering algorithm Charikar et al. (1997) which we denote by Online\(_k\)centers. The variable \(Z\) is the set of \(k\) centers returned by the algorithm, the variable \(P\) is the \(k\)-centers cost of these centers, and the variable \(Q\) is the minimum distance between two centers in \(Z\). These variables are used to decide which type of processing to perform and are used in type 2 processings. Using results from Charikar et al. (1997), the variables \(P, Z\) satisfy:

**Theorem 4** Charikar et al. (1997) Let \(Z_t\) and \(P_t\) denote the values of \(Z\) and \(P\) returned by Online\(_k\)centers when called at time \(t\). Let \(\text{OPT}^{k,\text{center}}_t\) denote the optimal set of \(k\) centers for \(k\) centers clustering at time \(t\). Then the following hold:
1. \(|Z_t| \leq k\).

2. \(\max_{1 \leq s \leq t} d(x_s, Z_t) \leq 4P_t\)

3. \(\max_{1 \leq s \leq t} d(x_s, OP_{t}^{k,\text{center}}) \geq \frac{P_t}{2}\)

4. There exists \(k + 1\) distinct points in \(X_t\) whose pairwise distances are all at least \(P_t\).

**Corollary 5** For all times \(t\), \(\frac{P_t^2}{2} \leq L_k(X_t) \leq 16tP_t^2\).

**Proof** By part 4 of Theorem 4 and the pigeonhole principle, there exists two points of distance least \(P_t\) that are clustered together in any \(k\)-means clustering. This gives a minimum cost of at least \(\frac{P_t^2}{2}\) with equality holding only when the cluster center is the midpoint of these points, implying the first inequality.

By parts 1 and 2 of Theorem 4, we see that the points in \(Z_t\) immediately give a \(k\) clustering where each point induces cost at most \(16P_t^2\). The second inequality follows.

**Deciding Processing Type** The logic for deciding the type of processing to perform appears in lines 5-6 of Algorithm 1. Line 5 simply updates the \(k\) centers, and line 6 does a type 1 processing if and only if \(Q \leq 4(t + 2)P\). This condition roughly means that \(X_t\) is not too spread out, and in this case, OnlineCluster processes the next point with a type 1 processing. When \(Q \geq 4(t + 2)P\), then intuitively the points easily spread out into \(k\) well-separated clusters. In this case, OnlineCluster processes the next point with a type 2 processing. Section 4 contains a full discussion of why type 1 and type 2 processings work in their respective scenarios.

**Type 1 Processings** Pseudocode for Type 1 processings appears in lines 7-14 of Algorithm 1. In a type 1 processing, \(x_t\) is selected with probability \(\frac{d(x_t, S_t)^2}{R}\). The rest of the code in type 1 processings is used to find a good value for \(R\), since if \(R\) is too small, then many centers are taken, and if \(R\) is too large, then points which could form good centers to new clusters may be missed. We will later show that when \(R\) reaches a sweet spot of \(R^* = O(\frac{L_k(X_t)}{12k \ln(k+10)})\), type 1 processings will not take too many centers and achieve a good approximation ratio. This is a critical step of our analysis of type 1 processings.

In order to quickly converge to \(R^*\), type 1 processings use two methods. First, in lines 11-14, OnlineCluster updates the counter \(F\) that grows whenever a point is taken with a type 1 processing. When this counter is too high, OnlineCluster doubles \(R\) (as too many centers are being taken) and resets the counter. Second, in lines 7-9, OnlineCluster increases \(R\) to be at least \(\frac{P_t^2}{24k \ln(k+10)}\).

By Corollary 5, \(\frac{P_t^2}{2} \leq L_k(X_t)\). Thus, \(R\) is never set above \(\frac{L_k(X_t)}{12k \ln(k+10)}\), meaning this increase does not set it above \(R^*\). Lines 7-9 ensure that \(R\) increases quickly when distant points are encountered.

**Type 2 Processings** Pseudocode for Type 2 processings appear in lines 16-17 of Algorithm 1. In a type 2 processing, OnlineCluster computes the size \(m\) of the \(k\)-center cluster closest to \(x\) and selects \(x\) with probability \(\frac{12 \ln(k+10)}{m}\). The key intuition behind type 2 processings is that if an optimal \(k\)-means clustering \(C_1^*, \ldots, C_k^*\) is known in advance, then selecting point \(x_t\) with probability \(\hat{O}(\frac{1}{|C_t^*|})\), where \(C_t^* = \min_{i=1}^{k} d(x_t, C_t^i)\), produces a set of centers with size \(\hat{O}(k)\) and good approximation factor. We do not know an optimal clustering in advance, but we are able to modify Online\_k\_centers
to keep estimates $m^1, \ldots, m^k$ of the sizes of its $k$-center clustering. This modified algorithm appears in Appendix B. The modified algorithm represents the set $Z$ as $Z = \{(z^1, m^1), \ldots, (z^k, m^k)\}$, where $m^i$ is the estimated size of the size of the cluster with center $z_i$. The modification satisfies the following:

**Proposition 6** Let $Q$ denote the distance between the nearest two centers in $Z$. Suppose that $Q > 4(t + 2)P$. Let $C^1_t, C^2_t, \ldots, C^k_t$ denote the optimal $k$-means clustering of $\{x_1, x_2, \ldots, x_t\}$. Then $Z$ consists of $k$ pairs, $(z^1, m^1), (z^2, m^2), \ldots, (z^k, m^k)$ such that the following hold for all $1 \leq i \leq k$: $z^i \in C^i_t$, $m^i = |C^i_t|$, and $d(z^i, x) \leq 4P_t$ for all $x \in C^i_t$.

**Proof** See Appendix B.

It may seem strange that this proposition applies to the optimal $k$-means clustering of $X_t$ given that Online $k$-centers is a $k$-center algorithm. However, when $Q > 4(t + 2)P$, the optimal $k$-center clustering and $k$-means clustering are the same because the points are so spread out. Using Proposition 6, Line 16 finds $m = |C^*|$, and thus line 17 selects $x_t$ with probability $\tilde{O}(\frac{1}{|C^*|})$. This produces our desired bounds on approximation ratio and center complexity.

4. **Proof of Theorem 1**

The auxiliary memory of Algorithm 1 is clearly $\tilde{O}(k)$: we only store $k$ additional points $(Z)$ and maintain integer counts assigned to each point. The variables $R, F$ constitute just $\tilde{O}(1)$ additional memory as well. Thus, to prove Theorem 1, it suffices to understand the approximation factor and the center complexity. We will consider both of these quantities separately in Sections 4.2 and 4.3. Now, we prove an important result about $R$ that will be used in both later sections.

4.1. Proving a Bound on $R$

Our later proofs about type 1 processings depend on $R$ neither being too large or too small, and lying in a sweet spot value of $O(\frac{\mathcal{L}_k(X_t)}{12k \ln(k+10)})$. The following lemma helps us establish that $R$ likely does not grow beyond this value.

**Proposition 7** Let $R_t$ denote the value of $R$ after the point $x_t$ is processed by Algorithm 1. With probability at least $1 - \frac{1}{120k}$ over the randomness of the algorithm, $R_t \leq \frac{\mathcal{L}_k(X_t)}{12k \ln(k+10)}$ for all $k \leq t \leq n$.

Observe that this proposition holds simultaneously over all times $t$. That is, there is a large chance $(1 - \frac{1}{120k})$ that all values of $R$ during the course of the algorithm will be appropriately bounded. Also, note that for our purposes, we optimize $R$ to be $\frac{\mathcal{L}_k(X_t)}{12k \ln(k+10)}$. The additional factor of $12 \ln k$ is helpful for bounding constants throughout both the approximation factor and the center complexity. The proof of this proposition follows the ideas from Liberty et al. (2016) in a relatively straightforward manner. The main difference is that some extra care is needed to bound all values of $R$ rather than the last one. A full proof can be found in Appendix C.
4.2. Bounding the Approximation Factor

We begin by proving that Algorithm 1 is an $O(1)$-approximation algorithm. Broadly speaking, our strategy will be to show that we are highly likely to select points that are “close” to the optimal cluster centers of $X$. We refer to such points as “good.”

**Good points:** For any finite set $C \subseteq \mathbb{R}^d$, we let $g \in C$ be a good point if $\mathcal{L}(C,g) \leq 3\mathcal{L}(C)$. By the Center Shifting Lemma, this is equivalent to $d(g,\mu)^2 \leq \frac{2\mathcal{L}(C)}{\mu}$, where $\mu$ denotes the optimal $k$-means center of $C$. Because $\mathcal{L}(S) = \sum_{s \in S} d(s, \mu)^2$, Markov’s inequality implies that at least half the points in $C$ are good (see Bhattacharjee and Moshkovitz (2020) for a complete proof).

Recall that Algorithm 1 deals with points through either type 1 or type 2 processings. Crucially, the decision of whether or not to apply a type 1 processing or a type 2 processing is uniquely determined by the values of $P, Q, Z$ and the point in question $x_t$. Since $P, Q, Z$ are all deterministic functions of $X$ (i.e. Charikar et al. (1997)’s algorithm is not randomized), it follows that the type of selection made is deterministic as well. We will make use of this fact by handling the cases of type 1 and type 2 processings separately. We now give a lemma that governs the behavior of type 1 processings.

**Lemma 8** Let $C \subseteq X$ be a subset, such that at least half of the good points in $C$ are type 1 processed. Then with probability at least $1 - \frac{1}{10k}$ over the randomness of Algorithm 1,

$$\mathcal{L}(C,S) \leq 5\mathcal{L}(C) + \frac{2\mathcal{L}_k(X)}{k}.$$  

**Proof** For any time $t$, define $\Gamma_t = d(x_t, S_{t-1})^2$ to be the squared distance used for type 1 selecting $x_t$. $\Gamma_t$ is a random variable because the set $S_{t-1}$ is stochastically determined by the randomness in Algorithm 1. Let $x_{l_1}, x_{l_2}, \ldots, x_{l_p}$ denote the good points in $C$ that are type 1 processed. Since at least half of all points in $C$ are good, it follows that $p \geq \frac{|C|}{4}$. Then we let $A$ denote the event that one of the following two things occur:

1. Some point among $x_{l_1}, \ldots, x_{l_p}$ is selected in $S$.
2. $\min(\Gamma_{l_1}, \Gamma_{l_2}, \ldots, \Gamma_{l_p}) \leq \frac{\mathcal{L}_k(X)}{k|C|}$.

$A$ can be thought of as a desirable event. The first condition would imply a good point is chosen, meaning that the cost of clustering $C$ is small, and the second condition would imply that some point we have already selected is close to a good point. We will show that if $A$ is likely to occur, and that if so our cost of clustering $C$ is small.

**Claim 1:** $\Pr[A] \geq 1 - \frac{1}{10k}$.

For any event $B$, observe that $\Pr[\overline{A}] \leq \Pr[\overline{A} \cap \overline{B}] + \Pr[\overline{A} \cap B]$ where $\overline{A}, \overline{B}$ denote the complements of $A, B$. Let $B$ denote the event that $R_n \leq \frac{\mathcal{L}_k(X_n)}{12k \ln(k+10)}$. By Proposition 7, $\Pr[\overline{B}] \leq \frac{1}{30k}$. Thus, it suffices to bound $\Pr[\overline{A} \cap B]$.

For $1 \leq j \leq p$, let $E_j$ denote the event that $\Gamma_{l_j} \leq \frac{\mathcal{L}_k(X_n)}{k|C|}$, $R_{l_j} \leq \frac{\mathcal{L}_k(X_n)}{12k \ln(k+10)}$, and let $F_j$ denote the event that $x_{l_j}$ is not selected as a center. Then $\overline{A} \cap B$ occurs only if $E_1, F_1, \ldots, E_p, F_p$ all occur.
Thus, it suffices to bound the probability of all these events simultaneously occurring. To that end, for any \( j \geq 1 \),

\[
\Pr[F_j \mid E_1, F_1, \ldots, E_{j-1}, F_{j-1}] \leq \max(0, 1 - \frac{\mathcal{L}_k(X_n)/(k|C|)}{\mathcal{L}_k(X_n)/(12k \ln(k + 10))}) \leq \exp(-\frac{12 \ln(k + 10)}{|C|}).
\]

This is because the distribution of the event \( F_j \) is entirely determined by the values of \( \Gamma_{t_j} \) and \( R_{l_j} \), and thus \( \frac{\Pr[\cdot]}{R_{l_j}} \) is bounded if we condition on \( E_j \). Multiplying this equation over all \( j \), we find that

\[
\Pr[\bigcap B] \leq \exp(-\frac{12p \ln(k+10)}{|C|}) \leq \exp(-3 \ln(k+10)) = \frac{1}{(k+10)^3} \text{ since } p \geq \frac{|C|}{4}.
\]

Substituting this, we find that

\[
\Pr[\bigcup B] = \Pr[\bigcap B] + \Pr[B] \leq \frac{1}{(k+10)^3} + \frac{1}{120k} \leq \frac{1}{60k}.
\]

**Claim 2:** If \( A \) occurs, then \( \mathcal{L}(C, S) \leq 5\mathcal{L}(C) + \frac{2\mathcal{L}_k(X)}{k} \).

Let \( c \) denote the optimal cluster center of \( C \). Then by the definition of a good point, we see that \( d(c, x_{t_j}) \leq \sqrt{\frac{2\mathcal{L}_k(C)}{|C|}} \). Since \( A \) occurs, either one of these good points is selected in \( S \), or something close to one of them is. It follows that there is some \( x \in S \) such that \( d(c, x) \leq d(c, x_{t_j}) + \sqrt{\frac{\mathcal{L}_k(X)}{k|C|}} \).

Substituting this and applying the Center Shifting Lemma, we have

\[
\mathcal{L}(C, S) = \mathcal{L}(C, x) \\
= \mathcal{L}(C) + |C|d(c, x)^2 \\
\leq \mathcal{L}(C) + |C| \left( \sqrt{\frac{2\mathcal{L}(C)}{|C|}} + \sqrt{\frac{\mathcal{L}_k(X)}{k|C|}} \right)^2 \\
\leq 5\mathcal{L}(C) + \frac{2\mathcal{L}_k(X)}{k}.
\]

Next, we handle type 2 processings. We begin with a useful lemma.

**Lemma 9** Suppose \( x_t \) and \( x_{t'} \) are type 2 processed where \( t < t' \). Let \( C_t^1, \ldots, C_t^k \) and \( C_t'^1, \ldots, C_t'^k \) denote the optimal \( k \)-means clusterings of \( X_t \) and \( X_{t'} \) respectively. Then for all \( 1 \leq i \leq k \), there exists \( 1 \leq j \leq k \) such that \( C_t^j \subseteq C_t'^i \).

**Proof** Assume towards a contradiction that there exists \( i \) such that \( C_t^j \nsubseteq C_t'^i \) for all \( 1 \leq j \leq k \). Then there exists \( x, y \in C_t^i \) with \( x \in C_t^a \) and \( y \in C_t^b \) for \( a \neq b \). Let \( z_t^1, \ldots, z_t^k \) and \( z_t'^1, \ldots, z_t'^k \) denote the points of \( Z \) at times \( t' \) and \( t \) respectively. By repeatedly using Proposition 6 and the triangle inequality, we have

\[
d(z_t^a, z_t'^b) \leq d(z_t^a, x) + d(z_t'^b, y) + d(x, y) \\
\leq 4P_t + 4P_t + d(x_t, z_t'^i) + d(y_t, z_t^i) \\
\leq 8P_t + 4P_t + 4P_t \\
\leq 16P_t.
\]
The last inequality holds since $P$ is monotonically non-decreasing. Because $x_{i'}$ was type 2 processed, $Q_{i'} > 4(t + 2)P_{i'} > 16P_{i'}$. This contradicts the definition of $Q_{i'}$ which implies our claim. ■

Let $n' \leq n$ denote the last time during which $x_{n'}$ is type 2 processed, and let $C_{n'}^1, C_{n'}^2, \ldots, C_{n'}^k$ denote the optimal $k$ means clustering of $X_{n'}$.

**Lemma 10** Fix $1 \leq i \leq k$. Suppose that at least half of the points in $C_{n'}^i$ are type 2 processed. Then with probability at least $1 - \frac{1}{60k}$ over the randomness of Algorithm 1, $\mathcal{L}(C_{n'}^i, S) \leq 3\mathcal{L}(C_{n'}^i)$.

**Proof** Let $x_{l_1}, x_{l_2}, \ldots, x_{l_p}$ denote all the good points in $C_{n'}^i$ that are type 2 processed. By the definition of a good point, if any of these points are chosen, then $\mathcal{L}(C_{n'}^i, S) \leq 3\mathcal{L}(C_{n'}^i)$. Thus, it suffices to bound the probability that at least one of these points is chosen.

For any $1 \leq j \leq p$, let $C_{l_j}^j, \ldots, C_{l_j}^k$ denote the optimal $k$-means clustering of $X_{l_j}$. Since $x_{l_j}$ is type 2 processed, it is selected with probability $\frac{12\ln(k + 10)}{|C_{l_j}^i|}$ where $x_{l_j} \in C_{l_j}^i$. Since $x_{l_j} \in C_{n'}^i$, by Lemma 9, $C_{l_j}^i \subseteq C_{n'}^i$. Thus, $x_{l_j}$ is selected with probability at least $\frac{12\ln(k + 10)}{|C_{n'}^i|}$. As a result, the probability that no $x_{l_j}$ is selected is at most

$$
\left(1 - \frac{12\ln(k + 10)}{|C_{n'}^i|}\right)^p \leq \exp\left(-\frac{12p\ln(k + 10)}{|C_{n'}^i|}\right) \leq \exp(-3\ln(k + 10)) \leq \frac{1}{60k}.
$$

We are now prepared to prove that Algorithm 1 is an $O(1)$-approximation algorithm.

**Proposition 11** With probability at least $\frac{19}{20}$ over the randomness of Algorithm 1, $\mathcal{L}(X, S) \leq 9\mathcal{L}(S)$.

**Proof** Let $D^1, D^2, \ldots, D^k$ denote the optimal $k$-means clustering of $\{x_{n'+1}, x_{n'+2}, \ldots, x_n\}$, where $x_{n'}$ is the last point to be type 2 processed. Then by definition, all points in $D^i$ are type 1 processed for $1 \leq i \leq k$. By Lemma 8, with probability at least $1 - \frac{1}{60k}$, $\mathcal{L}(D^i, S) \leq 5\mathcal{L}(D^i) + \frac{\mathcal{L}_k(X)}{k}$. Applying a union bound, we see that with probability at least $\frac{19}{20}$, this holds simultaneously over $1 \leq i \leq k$.

Next, for each $C_{n'}^i$, either at least half the good points in it are type 1 processed, or half the good points in it are type 2 processed. By applying Lemmas 8 and 10, we see that in either case, $\mathcal{L}(C_{n'}^i, S) \leq 5\mathcal{L}(C_{n'}^i, S) + \frac{2\mathcal{L}_k(X)}{k}$ with probability at least $1 - \frac{1}{60k}$. Applying a union bound, we see that with probability at least $\frac{19}{20}$, this holds simultaneously over $1 \leq i \leq k$.

Applying a union bound over both the $D^i$s and the $C_{n'}^i$s, we see that with probability at least $\frac{29}{30} \geq \frac{19}{20}$,

$$
\mathcal{L}(X, S) = \sum_{i=1}^k \mathcal{L}(D^i, S) + \sum_{i=1}^k \mathcal{L}(C_{n'}^i, S) \\
\leq \sum_{i=1}^k 5\mathcal{L}(D^i) + \frac{2\mathcal{L}_k(X)}{k} + \sum_{i=1}^k 5\mathcal{L}(C_{n'}^i) + \frac{2\mathcal{L}_k(X)}{k} \\
= 5\mathcal{L}_k(\{x_{n'+1}, \ldots, x_n\}) + 5\mathcal{L}_k(\{x_1, \ldots, x_{n'}\}) + 4\mathcal{L}_k(X) \\
\leq 9\mathcal{L}_k(X).
$$
4.3. Bounding the Center Complexity

We bound the number of points selected through type 1 and type 2 processings separately.

**Type 1 processings:** Let $1 \leq t_1, t_2, \ldots, t_r \leq n$ be defined so that $t_1$ is the first time in which we perform a type 1 processing, and for $i \geq 1$, $t_i + 1$ is the first time after $t_i$ for which we perform a type 1 processing and $P_{t_i + 1} > 8\sqrt{t_i + 1} (t_i + 5)P_{t_i}$. If no such $t_i$ exists, we terminate the sequence.

These times are instances in which the parameter $P$ increases by a significant amount, and consequently between any $t_{i-1}, t_i$, the parameter $P$ stays relatively bounded. This implies that the desired value of $R$ during this time period stays relatively bounded implying that relatively few centers are chosen through type 1 processings in this interval.

**Lemma 12** For $1 \leq i \leq r$, let $T_i = \{x_{t_i}, \ldots, x_{t_i+1-1}\}$ where $t_{i+1} - 1$ is defined as $n$. Then with probability at least $1 - \frac{1}{120k}$, there are at most $O(k \ln k \log^2 n)$ type 1 selections made in $T_i$ for all $1 \leq i \leq r$.

**Proof** By Proposition 7, with probability at least $1 - \frac{1}{120k}$, $R_t \leq \frac{L_k(X_t)}{12k \ln(k+10)}$ for all $k \leq t \leq n$. It suffices to show that if this occurs, the number of type 1 selections made in each $T_i$ is bounded as desired.

Fix any $i$ and let $t = t_i$ and $t' = t_i + 1 - 1$. Then by the definition of $\{t_j\}$, $P_{t'} \leq 8\sqrt{t'} (t' + 5)P_t$.

By step 8 in Algorithm 1, $R_t \geq \frac{P_t^2}{24k \ln(k+10)}$. Applying Corollary 5, we have

$$R_{t'} \leq \frac{L_k(X_{t'})}{12k \ln(k+10)} \leq \frac{16t'P_{t'}^2}{12k \ln(k+10)} \leq \frac{32 \cdot 64(9t')(t' + 5)^2P_{t'}^2}{24k \ln(k+10)} \leq 2048(9)(t'+ 5)^2R_t.$$

This implies that $R$ can be doubled at most $2 - 48(9t')(t' + 5)^2$ times in $T_i$. Since every $289k \ln(k+10) \log n$ type 1 selections corresponds to a doubling, by multiplying these we see that at most $O(k \ln k \log^2 n)$ type 1 selections can be made in $T_i$.

Lemma 12 bounds the number of type 1 selections made between any 2 times $t_i$ and $t_{i+1}$. To bound the total, we now bound the number of intervals, $r$, in terms of the parameter $\text{Lower}_{9,k}(X)$.

Recall that this parameter, defined in *Bhattacharjee and Moshkovitz* (2020), gives a lower bound on the number of centers any 9-approximation algorithm must take on input $X$. Full details about it can be found in Appendix A.

**Lemma 13** $r \leq \text{Lower}_{9,k}(X)$.

**Proof** By the definition of $\text{Lower}_{9,k}(X)$, it suffices to show that for all $i$, there exists $b \in \{t_i + 1, t_i + 2, \ldots, t_{i+1}\}$ such that for all $x_a \in X_{t_i}$, $d(x_a, x_b) > \sqrt{9} \text{diam}_{k-1}(X_{t_i})$, as this will allow us to construct a $(9, k)$-sequence with precisely $r$ elements (simply keep selecting the appropriate point $x_b$ from each interval $[t_i + 1, t_{i+1}]$).

Fix $i$ and for convenience let $t$ and $t'$ denote $t_i$ and $t_{i+1}$. Assume towards a contradiction that no such $x_b$ exists, that is, for all $x_b \in X_{t'}$, there exists $x_a \in X_t$ such that $d(x_a, x_b) \leq$
\[ \sqrt{9t} \text{diam}_{k-1}(X_t). \] Let \( z^1_t, z^2_t, \ldots, z^k_t \) denote the \( k \) centers that are in \( Z_t \). While it is possible that \( Z_t \) contains fewer than \( k \) centers, this case is easy to handle.

By Theorem 4, for all \( x_a \in X_t \), there exists \( j \) such that \( d(x_a, z^j_t) \leq 4P_t \). Since there exist \( z^j_t \) and \( z'^j_t \) such that \( d(z^j_t, z'^j_t) \leq Q_t \), it follows that \( X_t \) can be partitioned into \( k - 1 \) regions each of which have diameter at most \( Q_t + 8P_t \).

It follows that for all \( x_b \in X_{t'} \), there exists \( 1 \leq j \leq k \) such that

\[ d(x_b, z^j_t) \leq d(x_b, x_a) + d(x_a, z^j_t) \leq \sqrt{9t}(Q_t + 8P_t) + 4P_t \leq \sqrt{9t}(t + 5)P_t, \]

with the last inequality holding since \( t' \) is a type 1 selection meaning \( Q_t \leq 4(t + 2)P_t \). This implies that the \( k \)-centers cost of clustering \( X_{t'} \) is at most \( 4\sqrt{9t}(t + 5)P_t \) which implies \( P_{t'} \leq 4\sqrt{9t}(t + 5)P_t \) (Theorem 4). However, this is a contradiction, because \( t \) and \( t' \) are consecutive elements in the sequence \( \{t_i\} \) which means \( P_{t'} > 8\sqrt{9t}(t + 5)P_t \).

**Type 2 Processing:** Let \( k \leq s_1 < s_2 < \cdots < s_q \leq n \) be defined as follows. Let \( s_1 \) be first time that we do a type 2 processing. For \( i \geq 1 \), let \( s_{i+1} \) denote the first time for which \( x_{s_{i+1}} \) is type 2 processed and there exists \( 1 \leq a, b, c \leq k \) with \( a \neq b \) such that \( C^a_{s_i}, C^b_{s_i} \subseteq C^c_{s_{i+1}} \). If no such time exists, we terminate the sequence.

**Lemma 14** Fix any \( 1 \leq i \leq q \). Over the randomness of Algorithm 1, the expected number of points that are selected by type 2 processing from \( \{x_{s_1}, \ldots, x_{s_{i+1}-1}\} \) is at most \( 24 \ln(k + 10) \ln n \).

**Proof** Let \( T \subseteq \{x_{s_1}, \ldots, x_{s_{i+1}-1}\} \) be the set of points that are type 2 processed. For any \( x_i \in T \), we claim that the optimal \( k \)-means clusters of \( X_i \) can be ordered such that \( C^j_{s_i} \subseteq C^j_{t} \) for all \( 1 \leq j \leq k \).

Observe that by Lemma 9, every set \( C^j_{s_i} \) must be a subset of some \( C^j_{t} \). However, by the definition of the sequence \( s \), it is not possible for \( C^a_{s_i} \) and \( C^b_{s_i} \) to both be subsets of the same cluster \( C^c_{t} \) for otherwise we would have \( t = s_i \). Thus, each cluster \( C^j_{s_i} \) is contained in a unique cluster \( C^j_{t} \). By relabeling clusters as necessary, we can force \( j = j' \) giving the claim.

Applying this argument twice, we can also see that for \( x_i, x_{t'} \in T \) with \( t < t' \), \( C^j_{i} \subseteq C^j_{t'} \) for \( 1 \leq j \leq k \). Letting \( t_{\text{max}} \) denote the last time in \( T \), we see that for all \( x_t \in T \) and all \( 1 \leq j \leq k \) \( C^j_{t} = C^j_{t_{\text{max}}} \cap X_t \).

Finally, each point \( x_t \in T \) is selected with probability \( \frac{12\ln(k+10)}{|C^j_t|} \) where \( x_t \in C^j_t \). Thus, the expected number of points selected from \( T \) is \( \sum_{x_t \in T} \frac{12\ln(k+10)}{|C^j_t|} \). Let \( T^j = T \cap C^j_{t_{\text{max}}} \), then

\[ \sum_{x_t \in T} \frac{12\ln(k+10)}{|C^j_t|} = \sum_{j=1}^{k} \sum_{x_t \in T^j} \frac{12\ln(k+10)}{|C^j_t|} \leq \sum_{j=1}^{k} \sum_{f=1}^{n} \frac{12\ln(k+10)}{f} \leq 24k \ln(k + 10) \ln n, \]

where the inequality comes from the fact that each additional point must strictly increase the size of the cluster it is in (as it is literally the new point). We also add a factor of 2 to properly bound the harmonic sum with \( \ln n \). The lemma then follows by linearity of expectation.
Lemma 15  The length $q$ of the sequence $\{s_1, \ldots, s_q\}$ satisfies $q \leq \text{Lower}_{9,k}(X)$.

Proof  The idea of this proof follows that of Lemma 13. We claim that for all $i$, there exists $s_i < b \leq s_{i+1}$ such that for all $x_i \in X_{s_i}$, $d(x_i, x_{s_i}) > \sqrt{9s_i \text{diam}_{k-1}(X_{s_i})}$. Note that this claim implies the lemma from the definition of $\text{Lower}_{9,k}(X)$.

At time $s_{i+1}$, by Theorem 4, all of the points in $X_{s_{i+1}}$ can be partitioned into $k$ sets each with diameter at most $8P_{s_{i+1}}$. Furthermore, by Proposition 6, these $k$ sets are precisely $C_{s_{i+1}}^1, \ldots, C_{s_{i+1}}^k$.

By the definition of the sequence $s$ in conjunction with Lemma 9, all of the points in $X_{s_i}$ are contained within at most $k - 1$ of the clusters $C_{s_{i+1}}^1, \ldots, C_{s_{i+1}}^k$. This implies that $\text{diam}_{k-1}(X_{s_i}) \leq 8P_{s_{i+1}}$. Without loss of generality, it follows that $C_{s_{i+1}}^1$ and $X_{s_i}$ are disjoint. In this case, we claim that $x_b = z_{s_{i+1}}^1$ suffices.

Observe that for any $x \in X_{s_i}$, $d(x_b, x) \geq Q_{s_{i+1}} - 4P_{s_{i+1}}$, since $x$ has distance at most $4P_{s_{i+1}}$ to the cluster center $z_{s_{i+1}}^1$ corresponding to it. Since $s_{i+1}$ is a type 2 selection, we have that $Q_{s_{i+1}} > 4(s_{i+1} + 1)P_{s_{i+1}}$. It thus follows that $d(x_b, x) \geq 4(s_{i+1} + 1)P_{s_{i+1}}$. Substituting our previous bound on $\text{diam}_{k-1}(X_{s_i})$ we see that for $s_{i+1} > 36$, $d(x_b, x) \geq 4(s_{i+1} + 1)P_{s_{i+1}} > \sqrt{9s_{i+1}(8P_{s_{i+1}})} \geq \sqrt{9s_{i+1}\text{diam}_{k-1}(X_{s_i})}$.

Note that any issues caused for $s_{i+1} \leq 36$ can easily be circumvented by just selecting the first 36 points (which trivially changes our center complexity).

Putting it all together:  We now bound the overall center complexity of Algorithm 1.

Proposition 16  There exists a constant $\beta > 0$ such that with probability at least $10/19$ over the randomness of Algorithm 1, $|S| \leq \beta k \ln k \log^2 n\text{Lower}_{9,k}(X)$.

Proof  By Lemmas 12 and 13, we immediately see that with probability at least $1 - \frac{1}{100}$, at most $O(k \ln k \log^2 n\text{Lower}_{9,k}(X))$ selections are made from type 1 processings. By Lemmas 14 and 15, we see that the expected number of points chosen by type 2 processings is $O(k \ln k \ln n\text{Lower}_{9,k}(X))$. However, observe that every type 2 selection is in fact an independent event. Type 2 processings have no dependence on $S$. Because of this, we can easily apply a Chernoff bound to see that with probability at least $1 - \frac{1}{100}$, we do indeed select $O(k \ln k \ln n\text{Lower}_{9,k}(X))$ points via type 2 processings. A union bound implies the proposition.

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**Appendix A. Lower bounds from Bhattacharjee and Moshkovitz (2020)**

Finally, we review the center complexity lower bounds in Bhattacharjee and Moshkovitz (2020).

For any input dataset, $X$, Bhattacharjee and Moshkovitz defined a central parameter, $OC_k(X)$ which they used to express both their lower bounds and upper bounds on the center complexity of an online $O(1)$-approximation algorithm. In this paper, we will modify their definitions to construct a new parameter $Lower_{\alpha,k}(X)$. $Lower_{\alpha,k}(X)$ is closely related to $OC_k(X)$, but enables us to give sharper results than those in $OC_k(X)$. To this end, we will review the definitions and lower bound from Bhattacharjee and Moshkovitz (2020) by phrasing them in terms of this modified parameter, $Lower_{\alpha,k}(X)$.

First, we define the $l$-fold diameter of a set $S \subseteq \mathbb{R}^d$ as the smallest real number $D$ such that $S$ can be partitioned into $l$ disjoint sets each of which have diameter at most $D$. That is,

$$diam_l(S) = \min_{S^1 \cup S^2 \cup \ldots \cup S^l = S} \min_{1 \leq i \leq l} diam(S^i).$$

Using this, we define the $(\alpha, k)$-sequences, which represent the “worst case” input for online $k$-means clustering in the no-substitution setting.

**Definition 17** Fix $\alpha > 1$, and let $x_1, x_2, \ldots, x_m$ be a sequence of points in $\mathbb{R}^d$ such that for all $1 < i \leq m$,

$$\min_{1 \leq j < i} d(x_i, x_j) > \sqrt{i\alpha}diam_{k-1}(\{x_1, \ldots, x_{i-1}\}).$$
Using this, we define $\text{Lower}_{\alpha,k}(X)$ for any set of points $X$.

**Definition 18** Let $\alpha > 1$, and $X$ be any set of $n$ points in $\mathbb{R}^d$. Then $\text{Lower}_{\alpha,k}(X)$ is defined as the length of the largest sequence of points, $x_{t_1}, x_{t_2}, \ldots, x_{t_r}$, whose elements are distinct elements of $X$ that can be ordered to form an $\alpha, k$-sequence.

**Proof of Theorem 3.** Let $A$ be an $\alpha$-approximation algorithm. Order $X$ so that the first $\text{Lower}_{\alpha,k}(X)$ points that are streamed form an $(\alpha, k)$-sequence. First, $A$ must select the first $k$ points since otherwise it would incur a non-zero cost for a set whose optimal cost is 0.

Next, let $k + 1 \leq t \leq \text{Lower}_{\alpha,k}(X)$. Suppose that $A$ does not select $x_t$. Then, the cost incurred, even if all points before $x_t$ are chosen, is at least $\min_{1 \leq i \leq t-1} d(x_t, x_i)^2$. Since our points form an $\alpha, k$-sequence, this cost is strictly larger than $\alpha t D_t^2$, where $D_t = \text{diam}_{k-1}(\{x_1, \ldots, x_{t-1}\})$.

On the other hand, observe that a cost of $\frac{tD_t^2}{2}$ is obtainable. To see this, note that any set of $m$ elements in $\mathbb{R}^d$ with diameter $D_t$ has a 1-clustering with cost $mD_t^2$. Thus, by clustering $\{x_1, \ldots, x_t\}$ by setting $x_t$ as its own cluster and partitioning the rest of the points into sets of diameter $D_t$, we obtain a cost of $tD_t^2$. This implies that whenever we fail to select $x_t$, our cost is at least $\alpha L_k(\{x_1, \ldots, x_t\})$. Thus, in order to be an $\alpha$-approximation algorithm, $A$ must select $x_t$ with probability at least $\frac{19}{20}$, as $A$ is an $\alpha$-approximation.

Let $B$ denote the smallest real number such that $A$ selects fewer than $B$ points from the first $\text{Lower}_{\alpha,k}(X)$ points with probability at least $\frac{19}{20}$. Then it follows that the expected number of points selected by $A$ is at most $\frac{19B}{20} + \frac{\text{Lower}_{\alpha,k}(X)}{20}$. However, since we know that $A$ selects each point with probability at least $\frac{19}{20}$, it follows by linearity of expectation that $\frac{19B}{20} + \frac{\text{Lower}_{\alpha,k}(X)}{20} \geq \frac{19\text{Lower}_{\alpha,k}(X)}{20}$.

Rearranging this equation gives that $B \geq \frac{18\text{Lower}_{\alpha,k}(X)}{19}$, which implies the desired result. $lacksquare$

**Appendix B. A modified online $k$-centers algorithm**

Recall that the goal of $k$-center clustering is to choose $k$ centers so that the maximum distance over all points in the dataset to their nearest center is minimized. Charikar et al. (1997) give a classic online algorithm that maintains an 8-approximation to the optimal $k$ center clustering at all times.

For our purposes, we need to augment their algorithm to maintain, in addition to the $k$ centers at all times, an approximation to the number of points that are “clustered” with each center. Specifically, given centers at time $t$, $z_1^t, z_2^t, \ldots, z_k^t$, we wish to know $|\{x_s : i = \arg \min_j d(x, z_i) \leq t\}|$ for all $1 \leq i \leq k$.

While directly maintaining these counts is potentially challenging, for our purposes, an approximation suffices. We begin by first giving a modified version of the $k$-centers algorithm of Charikar et al. (1997) that maintains guesses for these counts at all times. The central idea is to augment each stored center with a count that represents how many points are “assigned” to it. When centers are changed, we “merge” counts by simply adding assignment numbers (line 16).

In Algorithm 2, we let $Z$ denote the set of $k$ augmented clusters, and $P$ denote an approximate estimate of the radius of each cluster. This algorithm is functionally identical to Charikar et al. (1997): the selection of cluster centers is identical in both cases. As a consequence, the approximation guarantee of the algorithm is the same.

We now show that our augmentations, namely the integer counts that are associated with each element of $Z_t$, are reasonable approximations of the number of points that are assigned to each cluster center.
Algorithm 2: Online $k$-centers ($X$)

1. $Z \leftarrow \{(x_1, 1), (x_2, 1), \ldots, (x_k, 1)\}$;
2. $P \leftarrow \min_{(z,m),(z',m') \in Z} d(z, z')$;
3. for $t = k + 1, \ldots$ do
   4. $(z, m) = \arg \min_{Z} d(z, x_t)$;
   5. if $d(z, x_t) \leq 2P$ then
      6. $m \leftarrow m + 1$;
   7. else
      8. $Z \leftarrow Z \cup \{(x_t, 1)\}$;
      9. while $|Z| > k$ do
         10. $Z' \leftarrow \{}$;
         11. for $(z, m) \in Z$ do
            12. $(z', m') = \arg \min_{Z'} d(z, z')$;
            13. if $d(z, z') > 2P$ then
               14. $Z' = Z' \cup \{(z, m)\}$;
            15. else
               16. $m' \leftarrow m' + m$;
            17. end
         18. $Z \leftarrow Z'$;
         19. $P \leftarrow 2P$;
      20. end
   8. end

Proposition 6 Let $Q$ denote the distance between the nearest two centers in $Z$. Suppose that $Q > 4(t + 2)P$. Let $C^1, C^2, \ldots, C^k$ denote the optimal $k$-means clustering of $\{x_1, x_2, \ldots, x_t\}$. Then $Z$ consists of $k$ pairs, $(z^1, m^1), (z^2, m^2), \ldots, (z^k, m^k)$ such that the following hold for all $1 \leq i \leq k$: $z^i \in C^i$, $m^i = |C^i|$, and $d(z^i, x) \leq 4P_t$ for all $x \in C^i$.

Proof Let $G$ be a graph with vertices $\{x_1, \ldots, x_t\}$. We construct an edge between $x_a, x_b$ in two situations.

1. If during lines 4-5 at time $b$ of Algorithm 2 $(z, m) = \arg \min_{Z} d(z, x_b)$ and satisfies $d(x_a, x_b) \leq 2P$, then we construct an edge from $x_a$ to $x_b$.
2. If during line 16 the augmented point $(z, m)$ has nearest neighbor $(z', m')$ such that $d(z, z') \leq 2P$, then we construct an edge from $z$ to $z'$ (recall that all $(z, m) \in Z$ satisfies $z \in \{x_1, \ldots, x_t\}$).

We claim that $G$ satisfies the following properties:

1. Adjacent vertices in $G$ have distance at most $2P$.
2. $G$ comprises of $k$ connected components, $G^1, G^2, \ldots, G^k$.
3. Each element of $Z$ corresponds to a distinct connected component with $z^i \in G^i$ and $m^i = |G^i|$.
4. For all $x \in G^i$, $d(x, z^i) \leq 4P$.

Property 1. is trivial: $P$ is monotonically increasing, and edges are explicitly only added if the distance between the vertices is at most $2P$. To verify properties 2. and 3., observe that during all times $1 \leq s \leq t$, the number of connected components in the graph cannot exceed $k$. We either just connect the new vertex to precisely 1 connected component (lines 5-6) and update the values of $m$ accordingly. Otherwise, we repeatedly combine components of our graph (lines 15-16) until we have at most $k$ connected components. In all of these cases, it is easy to see that the vertices of $Z$ are selected so that no two of them are path-connected. Note that at time $t$, by assumption $|Z| = k$ meaning that we have precisely $k$ path connected components.

To verify property 4, observe that for all $x \in X_t$, $d(x, Z) \leq 4P$ by Theorem 4. Suppose that $x \in G^i$ but that $x$ has nearest neighbor in $Z$ equal to $z^j$ for some $i \neq j$. Since $x$ is path connected to $z^i$ by definition, $d(x, z^i) \leq 2tP$ (since each edge has length at most $P$). However, $d(z^i, z^j) \geq Q$ by the definition of $Q$. It follows by the triangle inequality that

$$4P + 2tP = d(x, z^j) + d(x, z^i) \geq d(z^i, z^j) \geq Q,$$

which clearly contradicts $Q > 4(t + 2)P$. Thus $z^i$ is the nearest neighbor of $x$ and $d(x, z^i) \leq 4P$.

Having verified the properties above, it suffices to show that $G^1, G^2, \ldots, G^k$ is the optimal $k$-means clustering of $X_t$. In particular, this will imply that $G^i = C^i$ for all $i$. To do this, it suffices to show that it is impossible in the optimal clustering of $X_t$ for any two vertices in $G$ that are not path connected to be clustered together.

First, observe that $\mathcal{L}(X_t, OPT^k_t) \leq t(4P)^2$. This is because we can easily take $z^1, \ldots, z^k$ to be $k$-means centers, and then apply property 4. of $G$ to bound the $k$-means cost function. Second, suppose that $x \in G^i$ and $x' \in G^j$ are clustered together in some $k$-means clustering of $X_t$ where $i \neq j$. Then observe that the cost of clustering just $x$ and $x'$ can be bounded by $\frac{d(x, x')^2}{2}$ (as they must be assigned to the same cluster center). Using this, we observe that our induced cost is at least

$$\frac{d(x, x')^2}{2} \geq \frac{(d(z^i, z^j) - 8P)^2}{2} \geq \frac{(4tP)^2}{2} > t(4P)^2,$$

since $t \geq 2$. This implies that $x$ and $x'$ cannot be clustered together in the optimal $k$-means clustering, which finishes the proof.
Appendix C. Proving Proposition 7

We begin with the following key lemma.

Lemma 19  Let $k \leq t \leq n$. Then the probability that $R_t$ is illicitly doubled, that is set to twice its previous value and also set beyond its desired threshold is small. That is,

$$\Pr[R_t > \frac{L_k(X_t)}{k} \text{ and } R_t > 2R_{t-1}] \leq \frac{1}{500kt^2}.$$  

Proof  The value of $R$ only changes in two ways, it is either doubled, or it is set to $\frac{p^2}{12k\ln(k+10)}$. By Theorem 4, there exist $k + 1$ points at time $t$ in $X_t$ with pairwise distances at least $\bar{P}$. Thus, it follows that $L(X_t, OPT^k_t) \geq \frac{p^2}{12k\ln(k+10)}$. This means that the only way $R$ can exceed its desired value of $\frac{L(X_t, OPT^k_t)}{12k\ln(k+10)}$ is through egregiously doubling. To this end, we see that $R_t > \frac{L_k(X_t)}{k}$ if and only if there exists $t' < t$ such that $R_{t'} > \frac{L_k(X_t)}{24k\ln(k+10)}$ and a doubling occurs between times $t'$ and $t$.

Next, recall that we are trying to bound the probability that not only is $R_t$ large, but that it was doubled during time $t$. This can only happen if at least $289k \ln(k+5) \log t$ centers are selected, as $R_t \leftarrow R_{t-1}$ only if $F_{t-1} = 289k \ln k \log t$. Thus, to bound the desired probability, we will bound the probability that at least $289k \ln k \log(t+5)$ centers are selected conditioning on $t'$ existing with $R_{t'} > \frac{L_k(X_t)}{24k\ln(k+10)}$.

Let $C^1, C^2, \ldots, C^k$ denote the optimal $k$ clustering of $X_t$, and let $c^1, c^2 \ldots c^k$ denote their respective centers. Our main idea will be to partition each $C^i$ into subsets based on their distance from the center $c^i$, and then examine each of these smaller subsets individually. We will argue that we are likely to select only a small number of points from each of these smaller subsets, and then apply a Chernoff bound over all the points to obtain our result.

Let $\gamma_i = \frac{L(C^i)}{|C^i|}$, and define

$$C^i_j = \{x : x \in C^i, 2^{j-1} \gamma_i \leq d(x, c^i)^2 < 2^j \gamma_i\}, \text{ for all } j \geq 1.$$  

We also let $C^i_0 = \{x : x \in C^i, d(x, c^i)^2 < \gamma_i\}$. $C^i_j$ can be thought of as the $j$th “ring” of points centered around $c^i$ with each ring having double the radius of the previous ring. Because $|C^i| \gamma_i = L(C^i)$, $C^i_j$ is empty for all $j > \log |C^i| + 1$. Since $|C^i| \leq t$, and $C^i_1$ is non-empty, it follows that we have at most $k \log t$ non-empty subsets that partition $\{x_{t'+1}, \ldots, x_t\}$.

For $t'+1 \leq s \leq t$, let $C(s)$ be the unique $C^i_j$ with $x_s \in C^i_j$, and let

$$E_s = \begin{cases} 1 & x_s \text{ is selected, } |C(s) \cap S_{s-1}| \geq 1 \\ 0 & \text{otherwise} \end{cases}.$$  

$E_s$ is an indicator representing that $x_s$ has been selected but is not the first element of its “ring” to be selected. Since there are at most $k \log t$ non-empty rings, it follows that the total number of selections made between times $t'+1$ and $t$ is at most $k \log t + \sum_{s=t'+1}^{t} E_s$. We will now analyze the latter sum using a modified Chernoff bound.

To bound $\mathbb{E}[E_s]$, define $\gamma(s) = 2^j \gamma_i$ if $C(s) = C^i_j$. Suppose that $|C(s) \cap S_{s-1}| \geq 1$ and let $x \in C(s) \cap S_{s-1}$. By the triangle inequality, $d(x_s, x)^2 \leq 4 \gamma(s)$. Since $x_s$ is selected with probability at most $\frac{d(x_s, x)^2}{R_s}$, implying $\Pr[E_s = 1] \leq \frac{48k \ln(k+10) \gamma(s)}{R_s} \leq \frac{96k \ln(k+10) \gamma(s)}{L_k(X_t)}$, with the
Let \( \frac{L_s(X_t)}{24k \ln(k+10)} < R_t \leq R_s \). Furthermore, this inequality holds regardless of the values of \( E'_s \) are for all \( s' < s \): adding further selections can only reduce the \( \Pr[E_s = 1] \).

This allows us to apply a modified Chernoff bound to study the concentration of the sum \( \sum E_s \).

Bounding its expectation, we see

\[
\mathbb{E} \left[ \sum_{s=t'+1}^{t} E_s \right] \leq \sum_{s=t'+1}^{t} \frac{96k \ln(k+10) \gamma(s)}{L_k(X_t)} \\
\leq \frac{96k \ln(k+10)}{L_k(X_t)} \sum_{i=1}^{k} \sum_{j=0}^{\log t+1} |C^i_j| 2^j \gamma^i \\
\leq \frac{96k \ln(k+10)}{L_k(X_t)} \sum_{i=1}^{k} \left[ \sum_{x \in C^i_0} \gamma^i + \sum_{j=1}^{\log t+1} \sum_{x \in C^i_j} 2d(x, c^i)^2 \right] \\
\leq \frac{96k \ln(k+10)}{L_k(X_t)} \sum_{i=1}^{k} \left[ L(C^i) + 2L(C^i) \right] \\
= 288k \ln(k+10).
\]

The main manipulations we make in the sequence above come from the fact that \( \gamma^i < d(x, c^i)^2 \leq 2^i \gamma^i \) for all \( x \in C^i_j \), and from the definition of \( L(C^i) \).

We desire the probability that \( E_{total} \leq 288k \ln(k+10) \log t \). Since, \( E_{total} \leq k \log t + \sum_{s=t'+1}^{t} E_s \), it suffices to bound the probability that \( \sum E_s \leq 288k \log t \). Using a Chernoff bound,

\[
\Pr \left[ \sum_{s=t'+1}^{t} E_s > 288k \log t \right] = \Pr \left[ \sum_{s=t'+1}^{t} E_s > 288k \ln(k+10) (1 + (\log t - 1)) \right] \\
\leq \exp \left( -288k \ln(k+10) \left( \frac{(\log t - 1)^2}{\log t + 1} \right) \right) \\
\leq \frac{1}{500kt^2}.
\]

We now restate and prove Proposition 7.

**Proposition 7** Let \( R_t \) denote the value of \( R \) after the point \( x_t \) is processed by Algorithm 1. With probability at least \( 1 - \frac{1}{20k} \) over the randomness of the algorithm, \( R_t \leq \frac{L_s(X_t)OPT^+}{12k \ln(k+10)} \) for all \( k \leq t \leq n \).

**Proof** (Proposition 7) Suppose that after running Algorithm 1 that some \( t \) satisfies \( R_t > \frac{L_s(X_t)}{12k \ln(k+10)} \).

Take \( t \) to be minimal so that \( R_s \leq \frac{L_s(X_t)}{12k \ln(k+10)} \) for all \( s < t \). Observe that we must have that \( R_t = 2R_{t-1} \); if this weren’t the case, then \( R_t = R_{t-1} \) which would contradict the minimality of \( t \). By Lemma 19, the probability of this occurring for a given value of \( t \) is at most \( \frac{1}{500kt^2} \). Thus, the probability of this occurring for some \( t \) can be bounded with a union bound as at most \( \sum \frac{1}{500kt^2} \leq \frac{1}{120k} \), which implies the result.