On visualization of the linearity problem for mapping class groups of surfaces

Yasushi Kasahara

Abstract

We derive two types of linearity conditions for mapping class groups of orientable surfaces; one for once-punctured surface, and the other for closed surface, respectively. For the once-punctured case, the condition is described in terms of the action of the mapping class group on the deformation space of linear representations of the fundamental group of the corresponding closed surface. For the closed case, the condition is described in terms of the vector space generated by the isotopy classes of essential simple closed curves on the corresponding surface. The latter condition also describes the linearity for the mapping class group of compact orientable surface with boundary, up to center.

1 Introduction

Let $\Sigma_g$ be an orientable closed surface of genus $g$, and $M_g$ its mapping class group. By definition, $M_g$ is the group of the isotopy classes of the orientation preserving homeomorphisms of $\Sigma_g$. We also consider the once-punctured surface $\Sigma_{g,*}$, by which we mean the pair of the surface $\Sigma_g$ and a distinguished marked point $* \in \Sigma_g$. The mapping class group of $\Sigma_{g,*}$, denoted by $M_{g,*}$, is defined as the group of the isotopy classes of the orientation preserving homeomorphisms which preserve the marked point where the isotopy is assumed to always fix the marked point.

One of the fundamental problems on mapping class groups of surfaces is the linearity problem. It asks whether or not the mapping class group in question admits a faithful finite dimensional linear representation over some field. If a group admits a faithful finite dimensional linear representation over a particular field $K$, we will say that the group is $K$-linear. As for the case of genus $g = 1$, it is classically known that both $M_1$ and $M_{1,*}$ are isomorphic to $\text{SL}(2, \mathbb{Z})$, and hence are $\mathbb{Q}$-linear. The linearity of $M_2$ was established rather recently by Korkmaz [10] and Bigelow–Budney [2], using the celebrated works by Bigelow [1] and Krammer [14] that Artin’s braid groups are linear. However, the linearity of $M_g$ for $g \geq 3$, and also the linearity of $M_{g,*}$ for $g \geq 2$, both seem to remain open.

The purpose of this paper is to derive two types of conditions each of which is equivalent to the linearity of $M_{g,*}$, and that of $M_g$, respectively. For the former case, we show, for $g \geq 2$, that the $K$-linearity of $M_{g,*}$ is equivalent to the existence of a faithful $K$-linear representation of $\pi_1(\Sigma_{g,*})$ which represents a global fixed point of the action of $M_g$ on the corresponding deformation space (Theorem 3.2). We also combine this condition with the recent results by Franks–Handel [7] and Korkmaz [12] to observe that such a global fixed point does not exist for low degrees with $K = \mathbb{C}$ (Remark 3.3). For the latter case, we obtain a linearity condition for $M_g$ with $g \geq 3$, in terms of the vector space generated by the isotopy classes of essential simple
closed curves on $\Sigma_g$ (Corollary 4.9). The argument for the latter case also obtains a linearity condition \textit{up to center} for the case of a connected orientable compact surface of genus $g \geq 1$ with boundary (Theorem 4.8).

In the following, we first prepare useful generalities on the linearity of group in Section 2 which does not depend on any mapping class group material. We then apply these generalities in Section 3 and Section 4 respectively, to the case of $M_g, \ast$, and to the case of the mapping class group of connected compact orientable surface possibly with boundary. While we need in Section 3 the whole results of Section 2, we need in Section 4 only the results of Section 2.1.

Acknowledgements. The author is grateful to Makoto Sakuma for valuable discussions and a comment related to Remark 3.4. He is grateful to Masatoshi Sato for an enlightening conversation which was helpful in arranging the organization of this paper.

The author is partially supported by the Grant-in-Aid for Scientific Research (C) (No.23540102) from the Japan Society for Promotion of Sciences.

2 Generalities

Let $M$ be an arbitrary group, and $S$ a nonempty subset of $M$ which is invariant under conjugation. In later sections, $M$ will denote the mapping class group of a surface, and $S$ will be a certain geometric object on which the group $M$ acts naturally, and $S$ will be embedded in $M$ in such a way that the conjugation action coincides with the natural action.

Let $\varphi : M \to G$ be an arbitrary group homomorphism into another group $G$. In this section, we first observe that the restriction of $\varphi$ to $S$ implies an estimate for the size of the kernel of $\varphi$, which detects the injectivity of $\varphi$ when the centralizer of $S$ in $M$ is trivial. We then consider further the case that $S$ is actually a normal subgroup $\Gamma$ of $M$, which leads, under a certain condition, to a $K$-linearity condition for $M$ in terms of the deformation space of representations of $\Gamma$, by taking as $G$ the general linear groups over the field $K$.

Notation

We denote by $B(S)$ the group of all the bijections of the set $S$. Given a group $\Gamma$, we denote the group of all the automorphisms of $\Gamma$ by $\text{Aut} (\Gamma)$. Furthermore, we denote the group consisting of the inner automorphisms of $\Gamma$ by $\text{Inn} (\Gamma)$. We have the obvious inclusion $\text{Inn} (\Gamma) \triangleleft \text{Aut} (\Gamma) \subset B(\Gamma)$. The centralizer of $S$ in $M$ is the subgroup of $M$ consisting of those elements which commute with each element of $S$, and is denoted by $C_M(S)$. The center of $M$ is the centralizer of $M$ itself in $M$ and is denoted by $Z(M)$.

2.1 The size of kernel

To an arbitrary group homomorphism $\varphi : M \to G$, we associate the homomorphism $A_\varphi : M \to \text{Inn} (G) \subset \text{Aut} (G)$ defined by

$$A_\varphi (f)(g) = \varphi(f) \cdot g \cdot \varphi(f)^{-1} \quad (f \in M, g \in G).$$

For each $f \in M$, consider to restrict the action of $A_\varphi (f)$ on $G$ to the subset $\varphi(S)$. Since $S$ is invariant under conjugation in $M$, $\varphi(S)$ is invariant under the conjugation by $\varphi(f)$. Therefore, the restriction of the action of $A_\varphi (f)$ to $\varphi(S)$ induces a bijection of the set $\varphi(S)$. Hence the
composition of this restriction with $A\varphi$ induces a homomorphism, which we denote by $A_S\varphi : \mathcal{M} \to B(\varphi(S))$. It is explicitly described by the formula
\[
A_S\varphi(f)(\varphi(s)) = \varphi(f \cdot s \cdot f^{-1}) \quad (f \in \mathcal{M}, s \in S).
\] (2.2)

The next lemma estimates the size of $\text{Ker } \varphi$ from its behaviour on $S$.

**Lemma 2.1.** For any homomorphism $\varphi : \mathcal{M} \to G$ and the conjugacy invariant subset $S$, it holds that $\text{Ker } \varphi \subseteq \text{Ker } A_S\varphi$. In particular, if $\varphi$ is injective on $S$, then $\text{Ker } \varphi \subseteq C_S(\mathcal{M})$.

**Proof.** The former statement is obvious from the definition of $A_S\varphi$. Suppose next that $\varphi$ is injective on $S$. Then the formula (2.2) says that the homomorphism $A_S\varphi$ is nothing but the conjugation action of $\mathcal{M}$ on $S$, under the identification of $\varphi(S)$ with $S$. In particular, we have $\text{Ker } A_S\varphi = C_S(\mathcal{M})$. Now the former statement implies $\text{Ker } \varphi \subseteq C_S(\mathcal{M})$. \qed

**Remark 2.2.** If $S$ is a normal subgroup $\Gamma$ of $\mathcal{M}$, then the homomorphism $A_S\varphi$ obviously takes its value in $\text{Aut } (\varphi(\Gamma))$, and we will denote it by $A_{\Gamma}\varphi : \mathcal{M} \to \text{Aut } (\varphi(\Gamma))$.

### 2.2 Deformation space for the case of $S = \Gamma \triangleleft \mathcal{M}$

From now through this section, we assume that $S$ is a normal subgroup $\Gamma$ of $\mathcal{M}$. Then the restriction to $\Gamma$ of the given homomorphism $\varphi : \mathcal{M} \to G$ is obviously a homomorphism of group. In this subsection, we consider conversely the problem when a given homomorphism $\Gamma \to G$ extends to a homomorphism of $\mathcal{M}$ and record a useful necessary condition in terms of the deformation space of homomorphisms $\Gamma \to G$. To do so, we first recall the definition of deformation space, following Goldman [9].

Let $R_G(\Gamma) = \text{Hom } (\Gamma, G)$ denote the set of all homomorphisms $\Gamma \to G$. For any $\alpha \in \text{Aut } (\Gamma), h \in \text{Aut } (G)$, the correspondence
\[
(\alpha, h) \cdot \phi = h \circ \phi \circ \alpha^{-1} \quad (\phi \in R_G(\Gamma))
\]
defines an action of the product group $\text{Aut } (\Gamma) \times \text{Aut } (G)$ on $R_G(\Gamma)$. Next, we define the homomorphism $\iota_\Gamma : \mathcal{M} \to \text{Aut } (\Gamma)$ as $A_{\Gamma}(\text{id}_G)$, which is explicitly described by
\[
\iota_\Gamma(f)(\gamma) = f \cdot \gamma \cdot f^{-1} \quad f \in \mathcal{M} \text{ and } \gamma \in \Gamma. \quad (2.3)
\]

Via this $\iota_\Gamma$, the action of $\text{Aut } (\Gamma)$ on $R_G(\Gamma)$ induces an action of $\mathcal{M}$ on $R_G(\Gamma)$:
\[
f \cdot \phi = \phi \circ \iota_\Gamma(f)^{-1} \quad (2.4)
\]

Now, let $\iota : G \to \text{Inn } (G) (\subseteq \text{Aut } (G))$ denote the homomorphism defined by
\[
\iota(g)(x) = gxg^{-1} \quad (g, x \in G) \quad (2.5)
\]

The deformation space $X_G(\Gamma)$ is defined as the quotient $R_G(\Gamma)/\iota(G)$. Since the actions of $\text{Aut } (\Gamma)$ and $\text{Aut } (G)$ on $R_G(\Gamma)$ are commutative, the action of $\text{Aut } (\Gamma)$, and hence that of $\mathcal{M}$, on $R_G(\Gamma)$ descend to the action on $X_G(\Gamma)$. On the other hand, for $\gamma \in \Gamma$ the action of $\iota_\Gamma(\gamma)$ on $\phi$ coincides with that of the inner automorphism induced by $\phi(\gamma)^{-1} \in G$. Hence the normal subgroup $\Gamma$ of $\mathcal{M}$ acts on $X_G(\Gamma)$ trivially, and therefore the action of $\mathcal{M}$ on $X_G(\Gamma)$ descends to that of the quotient group $\mathcal{M} = \mathcal{M}/\Gamma$. In this terminology, the necessary condition mentioned above is as follows.
Lemma 2.3. If $\phi \in R_G(\Gamma)$ extends to a homomorphism $\mathcal{M} \to G$, then its representing class $[\phi] \in X_G(\Gamma)$ is a global fixed point of the $\overline{\mathcal{M}}$-action on $X_G(\Gamma)$.

Proof. Suppose that $\varphi : \mathcal{M} \to G$ is a homomorphism extending $\phi \in R_G(\Gamma)$. We then have, for every $f \in \mathcal{M}$, $\gamma \in \Gamma$,

$$
(f \cdot \phi)(\gamma) = \phi(\iota_\Gamma(f)^{-1}(\gamma)) = \phi(f^{-1} \cdot \gamma \cdot f)
= \varphi(f^{-1} \cdot \gamma \cdot f) = \varphi(f^{-1})\varphi(\gamma)\varphi(f^{-1})^{-1}
= \varphi(f^{-1})\phi(\gamma)\varphi(f^{-1})^{-1}
$$

Namely, the action of $f$ on $\phi$ coincides with that of the inner automorphism of $G$ defined by $\varphi(f)^{-1} \in G$, and hence $f \cdot [\phi] = [\phi]$ in $X_G(\Gamma)$. \hfill \Box

2.3 A global fixed point and a homomorphism of $\mathcal{M}$

Even if provided a homomorphism $\phi \in R_G(\Gamma)$ representing a global fixed point of the $\overline{\mathcal{M}}$-action on $X_G(\Gamma)$, it does not necessarily extend to a homomorphism $\mathcal{M} \to G$ because the centralizer $C_G(\phi(\Gamma))$ is possibly non trivial. In this subsection, instead of considering a direct extension of $\phi$, we associate to it another homomorphism of $\Gamma$, denoted by $a_\Gamma \phi$, and extend $a_\Gamma \phi$ to a homomorphism of $\mathcal{M}$. This is always possible because the extension is to be defined as the unique natural $\mathcal{M}$-action on $\phi(\Gamma)$ induced by the conjugation action of $\mathcal{M}$ on $\Gamma$, and the assumption that $[\phi]$ is a global fixed point assures the well-definedness of this action. If $\phi$ is further a linear representation, we will show that the extended homomorphism naturally gives rise to a linear representation of $\mathcal{M}$ with the same kernel.

Now, for an arbitrary homomorphism $\phi \in R_G(\Gamma)$ we define the homomorphism $a_\Gamma \phi : \Gamma \to \text{Aut}(\phi(\Gamma))$ by

$$
a_\Gamma \phi(\gamma)(\phi(x)) = \phi(\gamma \cdot x \cdot \gamma^{-1}) \quad (\gamma, x \in \Gamma). 
$$

Note this is same as the formula (2.2) for $A_\Gamma \varphi$ with $\Gamma = S$, if restricted to $\Gamma$.

Lemma 2.4. If $\phi \in R_G(\Gamma)$ represents a global fixed point of the $\overline{\mathcal{M}}$-action on $X_G(\Gamma)$, then $a_\Gamma \phi$ extends to a unique homomorphism $\Psi : \mathcal{M} \to \text{Aut}(\phi(\Gamma))$ satisfying

$$
\Psi(f) \circ \phi = \phi \circ \iota_\Gamma(f) \quad \text{for all } f \in \mathcal{M}
$$

where $\iota_\Gamma$ denotes the conjugation action of $\mathcal{M}$ on $\Gamma$.

Furthermore, if $G$ is the general linear group $\text{GL}(n, K)$ of degree $n$ over a field $K$, then $\Psi$ extends naturally to a linear representation $\bar{\Psi}$ of $\mathcal{M}$ which is defined over $K$, with the same kernel as $\text{Ker} \Psi$, and of degree at most $n^2$.

Remark 2.5. If $\phi$ extends to a homomorphism $\varphi : \mathcal{M} \to G$, then the uniqueness of $\Psi$ in the lemma implies that $\Psi$ coincides with $A_\Gamma \varphi$ described in Remark 2.2.

Proof of Lemma 2.4. Suppose that $\phi \in R_G(\Gamma)$ represents a global fixed point $[\phi] \in X_G(\Gamma)$ for the action of $\overline{\mathcal{M}}$. Then for each $f \in \mathcal{M}$, we have

$$
f^{-1} \cdot [\phi] = [\phi] \quad \text{in } X_G(\Gamma),
$$
and hence there exists an \( x_f \in G \) such that \( f^{-1} \cdot \phi = \iota(\phi \cdot (x_f \cdot \phi) \cdot (x_f)^{-1}) \), where \( \iota \) denotes the inner automorphism of \( G \) defined by \( x_f \) as (2.5). By formula (2.4), this means
\[
\phi \circ \iota(f) = \iota(x_f) \circ \phi
\] (2.8)
where \( \iota(f) \) denotes the conjugation action by \( f \) as (2.3). In other words, the following diagram is commutative:

\[
\begin{array}{ccc}
\Gamma & \xrightarrow{\phi} & G \\
\iota(f) \downarrow & & \iota(x_f) \downarrow \\
\Gamma & \xrightarrow{\phi} & G
\end{array}
\]

In view of this commutative diagram, the automorphism \( \iota(x_f) \) preserves \( \phi(\Gamma) \). In general, the element \( x_f \) satisfying (2.8) is not unique because of possibly non-trivial \( C_G(\phi(\Gamma)) \). However, since the left-hand side of (2.8) does not contain \( x_f \), the action of \( \iota(x_f) \) on \( \phi(\Gamma) \) does not depend on the choice of \( x_f \). Therefore, we obtain a well-defined set map
\[
\Psi : \mathcal{M} \rightarrow \text{Aut}(\phi(\Gamma))
\]
by sending each \( f \in \mathcal{M} \) to the restriction of \( \iota(x_f) \) to \( \phi(\Gamma) \). This is what we need. In fact, the property (2.8) is exactly the same as (2.7) to be satisfied. Also, the uniqueness of \( \Psi \) is a direct consequence of (2.7). We now need only to check that \( \Psi \) is a homomorphism of group, which is straightforward in view of the following commutative diagram for \( g, h \in \mathcal{M} \), together with (2.7):

\[
\begin{array}{ccc}
\Gamma & \xrightarrow{\phi} & \phi(\Gamma) \\
\iota(gf) \downarrow & & \iota(x_{gf}) \downarrow \\
\Gamma & \xrightarrow{\phi} & \phi(\Gamma) \\
\iota(g) \downarrow & & \iota(x_g) \downarrow \\
\Gamma & \xrightarrow{\phi} & \phi(\Gamma)
\end{array}
\]

This completes the proof of the former part of the lemma.

Next, suppose further that \( G = \text{GL}(n, K) \). Then our global fixed point \([\phi]\) is represented by a linear representation \( \phi : \Gamma \rightarrow \text{GL}(n, K) \). Therefore, for each \( f \in \mathcal{M} \), an element \( x_f \in G = \text{GL}(n, K) \) satisfying (2.8) defines a linear isomorphism of \( \text{End}(n, K) \), the space of all square matrices of degree \( n \) over \( K \), by the correspondence
\[
M \mapsto x_f \cdot M \cdot x_f^{-1} \quad (M \in \text{End}(n, K)),
\] (2.9)
whose restriction to \( \phi(\Gamma) \) coincides with \( \Psi(f) \).

Now let \( W \) denote the \( K \)-subspace of \( \text{End}(n, K) \) spanned by the set \( \phi(\Gamma) \). This is clearly a finite dimensional vector space over \( K \), with dimension at most \( n^2 \). Since the linear isomorphism (2.9) preserves the generating set \( \phi(\Gamma) \), it restricts to a linear isomorphism of \( W \) which is independent of the choice of \( x_f \) and is an extension of \( \Psi(f) \). We denote this linear isomorphism of \( W \) by \( \tilde{\Psi}(f) \), which defines a homomorphism
\[
\tilde{\Psi} : \mathcal{M} \rightarrow \text{GL}(W).
\]
Here \( \text{GL}(W) \) denotes the group of the \( K \)-linear isomorphisms of \( W \). It is easy to see \( \text{Ker} \tilde{\Psi} = \text{Ker} \Psi \). This completes the proof of Lemma 2.4. \( \square \)
2.4 A linearity condition for $\mathcal{M}$

As a consequence of the above Lemmas 2.1, 2.3, and 2.4 we have the following.

**Proposition 2.6.** Let $\Gamma$ be a normal subgroup of $\mathcal{M}$ with trivial centralizer $C_{\mathcal{M}}(\Gamma)$. Then the group $\mathcal{M}$ is $K$-linear if and only if there exists a faithful linear representation $\phi \in R_{GL(n,K)}(\Gamma)$ for some $n$ which represents a global fixed point of the action of $\mathcal{M} = \mathcal{M}/\Gamma$ on $X_{GL(n,K)}(\Gamma)$. Furthermore, if such a global fixed point exits, then $\mathcal{M}$ admits a faithful linear representation over $K$ of degree at most $n^2$.

**Proof.** If $\mathcal{M}$ admits a faithful linear representation $\varphi : \mathcal{M} \to GL(n, K)$, then its restriction to $\Gamma$ is a faithful linear representation, which represents, by Lemma 2.3, a global fixed point of the $\mathcal{M}$-action on $X_{GL(n,K)}(\Gamma)$. Suppose next, for some $n$, that a faithful representation $\phi \in R_{GL(n,K)}(\Gamma)$ represents a global fixed point of the $\mathcal{M}$-action on $X_{GL(n,K)}(\Gamma)$. Then the latter part of Lemma 2.4 gives the linear representation $\tilde{\Psi} : \mathcal{M} \to GL(W)$, of degree at most $n^2$. Since $\text{Ker} \tilde{\Psi} = \text{Ker} \Psi$, we have only to show $\text{Ker} \Psi$ is trivial. To do this, we consider the restriction of $\Psi$ to $\Gamma$, which coincides with $\alpha_1 \phi$ defined by (2.6). By the assumption that $\phi$ is faithful, the kernel of $\alpha_1 \phi$ coincides with the center of $\Gamma$, which is trivial by the assumption $C_{\mathcal{M}}(\Gamma) = \{1\}$. Therefore, the homomorphism $\Psi$ is injective on $\Gamma$. Now Lemma 2.1 for $\varphi = \Psi$ and $S = \Gamma$ implies that $\text{Ker} \Psi$ is trivial.

3 Once-punctured surface

Recall that $\mathcal{M}_{g,*}$ denotes the mapping class group of the once-punctured surface $\Sigma_{g,*}$. In this section, we apply the results in the previous section to the case $\mathcal{M} = \mathcal{M}_{g,*}$ with $g \geq 2$, and $\Gamma = \pi_1(\Sigma_{g,*})$, the fundamental group of $\Sigma_g$ with base point $* \in \Sigma_g$, to derive a condition equivalent to the linearity of $\mathcal{M}_{g,*}$.

The mapping class group $\mathcal{M}_{g,*}$, for $g \geq 1$, naturally acts on $\pi_1(\Sigma_{g,*})$, which gives rise to a homomorphism

$$\cdot_* : \mathcal{M}_{g,*} \to \text{Aut}(\pi_1(\Sigma_{g,*})),$$

(3.1)

By the classical Dehn–Nielsen theorem, this homomorphism is injective (c.f. Farb–Margalit [5]).

We henceforth assume that $g \geq 2$. Then, forgetting the puncture induces a homomorphism of $\mathcal{M}_{g,*}$ onto $\mathcal{M}_g$, the mapping class group of the closed surface $\Sigma_g$. The kernel of this homomorphism is canonically isomorphic to $\pi_1(\Sigma_{g,*})$, and hence we obtain the Birman exact sequence [3] (see also [5]):

$$1 \to \pi_1(\Sigma_{g,*}) \xrightarrow{i} \mathcal{M}_{g,*} \xrightarrow{j} \mathcal{M}_g \to 1$$

(3.2)

The following is a fundamental property of the Birman exact sequence:

$$f \cdot i(\gamma) \cdot f^{-1} = i(f_*(\gamma))$$

(3.3)

for $f \in \mathcal{M}_{g,*}$, and $\gamma \in \pi_1(\Sigma_{g,*})$, where $f_*$ denotes the natural action of $f$ on $\pi_1(\Sigma_{g,*})$.

In other words, under the identification of $\pi_1(\Sigma_{g,*})$ with its image via $i$, the natural action of $\mathcal{M}_{g,*}$ on $\pi_1(\Sigma_{g,*})$ coincides with the conjugation action in $\mathcal{M}_{g,*}$. Since the former is faithful by the Dehn–Nielsen theorem, it follows that the centralizer $C_{\mathcal{M}_{g,*}}(i(\pi_1(\Sigma_{g,*})))$ is trivial. Hereafter, we identify $\pi_1(\Sigma_{g,*})$ with its image in $\mathcal{M}_{g,*}$ via $i$ and omit the symbol $i$ if no confusion may occur.

Now, let $G$ be an arbitrary group. We first record a direct consequence of Lemma 2.1.
Proposition 3.1. Let \( \varphi : \mathcal{M}_{g,*} \to G \) denote an arbitrary homomorphism. Then \( \varphi \) is injective if and only if its restriction to \( \pi_1(\Sigma_g,*) \) is injective.

Before we proceed further, we remark that in view of (3.3), the action of \( \mathcal{M}_{g,*} \) on \( R_G(\pi_1(\Sigma_g,*)) \) defined by (2.4) can be written as

\[
f \circ \varphi = \varphi \circ f^{-1} \quad (f \in \mathcal{M}_{g,*}, \varphi \in R_G(\pi_1(\Sigma_g,*)))
\]

This action descends to an action of \( \mathcal{M}_g \) on \( X_G(\pi_1(\Sigma_g,*)) \), where \( \mathcal{M}_g \) is identified with \( \overline{\mathcal{M}} = \mathcal{M}/T \).

Now the desired condition for the linearity of \( \mathcal{M}_{g,*} \) is given by Proposition 2.6 as follows.

Theorem 3.2. Let \( K \) be a field. Then \( \mathcal{M}_{g,*} \) is \( K \)-linear if and only if there exists a \textit{faithful} linear representation \( \varphi \in R_{GL(n,K)}(\pi_1(\Sigma_g,*)) \) for some \( n \) which represents a global fixed point of the natural action of \( \mathcal{M}_g \) on \( X_{GL(n,K)}(\pi_1(\Sigma_g,*)) \). Furthermore, if such a global fixed point exists, then \( \mathcal{M}_{g,*} \) admits a faithful linear representation over \( K \) of degree at most \( n^2 \).

Remark 3.3 (Complex linear representations of low degrees). Assume \( K = \mathbb{C} \) and \( g \geq 3 \). Then a recent result by Franks–Handel [7] implies that any linear representation of \( \mathcal{M}_{g,*} \) of degree less than \( 2g \) is trivial (see also Korkmaz [11]). In addition, Korkmaz [12] proved that the kernel of any non-trivial linear representation of \( \mathcal{M}_{g,*} \) of degree \( 2g \) coincides with the Torelli group, and in particular, \( \pi_1(\Sigma_g,* \) is contained in the kernel. Therefore, when \( n \leq \sqrt{2g} \), one can apply the construction of Lemma 2.4 for \( K = \mathbb{C} \) to these results to observe that any global fixed point of the \( \mathcal{M}_g \)–action on \( X_{GL(n,C)}(\pi_1(\Sigma_g,*)) \) is represented by a linear representation of \( \pi_1(\Sigma_g,* \) with abelian image.

Remark 3.4. As described in Goldman [9], the recent dynamical study of the action of mapping class group on deformation space of the fundamental group of the corresponding surface has revealed the phenomenon that this action is sometimes properly discontinuous or ergodic, on some of its connected components. For the former case, there are no global fixed points at all, and also for the latter case, there cannot exist so many global fixed points. On the other hand, we see that Lemma 2.3 for \( (\mathcal{M},\Gamma) = (\mathcal{M}_{g,*},\pi_1(\Sigma_g,*)) \) implies that every finite dimensional linear representation of \( \mathcal{M}_{g,*} \) represents a global fixed point of the action of \( \mathcal{M}_g \) on the corresponding deformation space. Therefore, this phenomenon seems to explain at least partially why so few linear representations of \( \mathcal{M}_{g,*} \) are known so far.

Remark 3.5. Let \( \Sigma_{g,1} \) be an orientable connected surface of genus \( g \) with one boundary component. We denote by \( \mathcal{M}_{g,1} \) the mapping class group of \( \Sigma_{g,1} \) whose homeomorphisms and isotopies are assumed to fix the boundary pointwisely. By Paris–Rolfsen [13, Corollary 4.2], \( \mathcal{M}_g \) contains \( \mathcal{M}_{h,1} \) as a subgroup if \( 0 \leq h < g \). Hence the linearity of \( \mathcal{M}_g \) for \( g \geq 2 \) implies the linearity of \( \mathcal{M}_{h,1} \) for every \( h \) with \( 0 \leq h < g \). On the other hand, since \( \mathcal{M}_{g,*} \) is isomorphic to the quotient of \( \mathcal{M}_{g,1} \) by its center, the linearity of \( \mathcal{M}_{g,1} \) implies that of \( \mathcal{M}_{g,*} \) (c.f. Brendle–Hamidi-Tehrani [4, Lemma 3.4]). Therefore, we observe that if \( \mathcal{M}_{h,1} \) is \textit{not linear} for some \( h \geq 2 \), it implies that \( \mathcal{M}_g \) is \textit{not linear} for all \( g > h \). As mentioned in Introduction, the first \( h \) for which the linearity of \( \mathcal{M}_{h,1} \) is unknown is 2.

In view of these the linearity of \( \mathcal{M}_{2,*} \) seems an interesting problem, at present, to resolve the linearity of \( \mathcal{M}_g \) for general \( g \).
A comparison with the automorphism group of a free group

Let \( F_r \) be a free group of rank \( r \geq 2 \). We denote by \( \text{Aut}(F_r) \) the group of all automorphisms of \( F_r \). Since \( F_r \) is center free, the inner automorphism group of \( F_r \) can be naturally identified with \( F_r \) so that we have an analogue of the Birman exact sequence:

\[
1 \rightarrow F_r \rightarrow \text{Aut}(F_r) \rightarrow \text{Out}(F_r) \rightarrow 1
\]

where \( \text{Out}(F_r) \) denotes the outer automorphism group of \( F_r \). It can be easily checked that the action of \( \text{Aut}(F_r) \) on \( F_r \) by conjugation coincides with the natural action. In particular, the centralizer \( C_{\text{Aut}(F_r)}(F_r) \) is trivial. Hence Proposition 2.6 gives a necessary and sufficient condition for the linearity of \( \text{Aut}(F_r) \) provided \( (M, \Gamma) = (\text{Aut}(F_r), F_r) \).

On the other hand, Formanek–Procesi [6] has shown that \( \text{Aut}(F_r) \) is not linear if \( r \geq 3 \). For the case of \( r = 2 \), the \( \mathbb{C} \)-linearity of \( \text{Aut}(F_2) \) was established by Krammer [13]. Therefore, we see:

**Proposition 3.6.** (1) For \( r \geq 3 \), and any field \( K \), there does not exist a faithful linear representation in \( R_{GL(n,K)}(F_r) \) which represents a global fixed point of the \( \text{Out}(F_r) \)-action on \( X_{GL(n,K)}(F_r) \).

(2) For \( r = 2 \) and \( K = \mathbb{C} \), there exists such a faithful linear representation in \( R_{GL(n,\mathbb{C})}(F_2) \) for certain \( n \).

It might be an interesting problem to find out a direct proof of this proposition.

4 Compact orientable surface

Let \( \Sigma_{g,n} \) be a connected compact orientable surface of genus \( g \geq 1 \) with \( n \geq 0 \) boundary components. We denote by \( \mathcal{M}_{g,n} \) the mapping class group of \( \Sigma_{g,n} \), the group of isotopy classes of orientation preserving homeomorphisms of \( \Sigma_{g,n} \) where both the homeomorphisms and the isotopies are assumed to always fix the boundary of \( \Sigma_{g,n} \) pointwisely.

4.1 Preliminary

We first collect necessary results on \( \mathcal{M}_{g,n} \) to apply the results in Section 2.1. Let \( S = S(\Sigma_{g,n}) \) be the set of the isotopy classes of the essential simple closed curves on \( \Sigma_{g,n} \). Here a simple closed curve is said to be essential if it is not homotopic to a point nor parallel to any boundary component of \( \Sigma_{g,n} \).

We define a mapping \( \iota : S \rightarrow \mathcal{M}_{g,n} \) by sending each \( C \in S \) to the right-handed Dehn twist along \( C \). The natural action of \( \mathcal{M}_{g,n} \) on \( S \) has the effect on Dehn twists as follows:

\[
f \cdot \iota(C) \cdot f^{-1} = \iota(f(C)) \quad (f \in \mathcal{M}_{g,n}, C \in S)
\]

**Fact 4.1.** The mapping \( \iota \) is injective.

The proof of this fact, which can be found in [3], is done by showing that for two distinct elements \( C_1 \) and \( C_2 \in S \), the natural actions of the Dehn twists along them on \( S \) are different. On the other hand, every element of the center of \( \mathcal{M}_{g,n} \), denoted by \( Z(\mathcal{M}_{g,n}) \), acts trivially on \( S \) (c.f. Fact 4.1 and the formula (4.1)). Therefore, we can slightly generalize this fact as follows:

**Lemma 4.2.** For \( C_1, C_2 \in S \), if \( C_1 \neq C_2 \), then \( \iota(C_1) \iota(C_2)^{-1} \notin Z(\mathcal{M}_{g,n}) \).
We also need the following.

**Lemma 4.3.** The centralizer $C_{M_{g,n}}(\iota(S))$ coincides with the center $Z(M_{g,n})$.

We give a direct proof of this lemma while it is essentially a restatement of the well-known fact that the natural action of any element of $M_{g,n}$ on $S$ is trivial if and only if it lies in $Z(M_{g,n})$.

**Proof of Lemma 4.3.** Due to Gervais [8], $M_{g,n}$ is generated, if $g \geq 2$, by $\iota(S)$, and if $g = 1$, by $\iota(S)$ together with the Dehn twists along the simple closed curves parallel to the boundary components of $\Sigma_{g,n}$. Since the latter type of Dehn twists are contained in $Z(M_{g,n})$, each element of the centralizer always commutes with a generating set of $M_{g,n}$. Hence we have $C_{M_{g,n}}(\iota(S)) = Z(M_{g,n})$.

**Remark 4.4.** Some literature includes in $S$ the isotopy classes of simple closed curves parallel to boundary components of $\Sigma_{g,n}$. In that case, since the Dehn twists along such simple closed curves lie in the center, Lemma 4.2 does not hold in general, while Fact 4.1 remains true with a slight modification in the proof mentioned above (c.f. [15, Proposition 3.6]).

### 4.2 Linearity condition

We now apply the results in Section 2.1 to derive a linearity condition for $M_g$ with $g \geq 3$. We first observe that the injectivity of an arbitrary homomorphism of $M_{g,n}$ can be detected, up to center, by its restriction to $\iota(S)$.

**Lemma 4.5.** Let $\varphi : M_{g,n} \to G$ be an arbitrary group homomorphism. Then the composition $\varphi \circ \iota$ is injective if and only if $\ker \varphi \subset Z(M_{g,n})$.

**Proof.** Suppose first that $\varphi \circ \iota$ is injective. Then, since $\iota$ is injective by Fact 4.1, Lemma 2.1 for $(M, S) = (M_{g,n}, \iota(S))$ implies that the kernel of $\varphi$ is contained in the centralizer $C_{M_{g,n}}(\iota(S))$. On the other hand, we have $C_{M_{g,n}}(\iota(S)) = Z(M_{g,n})$ by Lemma 4.3. Therefore, we have $\ker \varphi \subset Z(M_{g,n})$.

Conversely, suppose $\ker \varphi \subset Z(M_{g,n})$. Then the injectivity of $\varphi \circ \iota$ follows immediately from Lemma 4.2.

**Remark 4.6.** Even if $\varphi \circ \iota$ is not injective, Lemma 2.1 still implies $\ker \varphi \subset \ker A_{\iota(S)}\varphi$ where the homomorphism $A_{\iota(S)}\varphi : M_{g,n} \to B(\varphi \circ \iota(S))$ is defined by (2.2) and is described by

$$f \in M_{g,n} \mapsto (\varphi \circ \iota(C) \mapsto \varphi \circ \iota(f(C))) \quad \text{for} \ C \in S.$$

Now we recall from Paris–Rolfsen [15, Theorem 5.6] that the center of $M_{g,n}$ is completely known as follows:

1. For the case of $n = 0$, it holds $M_{g,n} = M_g$, and if $g \geq 3$, $Z(M_g)$ is trivial; if $g = 1$, or 2, then $Z(M_g) \cong \mathbb{Z}/2\mathbb{Z}$ and is generated by the hyperelliptic involution.

2. For the case of $n = 1$ and $g = 1$, $Z(M_{1,1}) \cong \mathbb{Z}$ and is generated by the “half-twist” along the simple closed curve parallel to the boundary component.

3. Otherwise, $Z(M_{g,n})$ is an free abelian group of rank $n$, and is generated by the Dehn twists along the simple closed curves parallel to the boundary components.
In particular, Lemma 4.5 implies the injectivity condition for an arbitrary homomorphism of $\mathcal{M}_g = \mathcal{M}_{g,0}$ for $g \geq 3$:

**Corollary 4.7.** Suppose $g \geq 3$. Then an arbitrary homomorphism of $\mathcal{M}_g$ is injective if and only if it is injective on $\iota(S)$.

Now, for a field $K$, we present a $K$-linearity condition for $\mathcal{M}_{g,n}$, “up to center,” in terms of $S$. We denote by $K[S]$ the $K$-vector space freely generated by the set $S$. The natural action of $\mathcal{M}_{g,n}$ on $S$ defines the structure of an $\mathcal{M}_{g,n}$-module on $K[S]$ by linearity.

**Theorem 4.8.** The mapping class group $\mathcal{M}_{g,n}$ admits a finite dimensional linear representation over $K$ with kernel contained in the center $Z(\mathcal{M}_{g,n})$ if and only if $K[S]$ has an $\mathcal{M}_{g,n}$-invariant subspace of finite codimension such that the projection $p : K[S] \to K[S]/V$ is injective on the subset $S$.

As just mentioned above, since the center of $\mathcal{M}_g = \mathcal{M}_{g,0}$ is trivial for $g \geq 3$, this theorem gives a $K$-linearity condition for $\mathcal{M}_g$ for $g \geq 3$.

**Corollary 4.9.** Suppose $g \geq 3$. Then $\mathcal{M}_g$ is $K$-linear if and only if $K[S]$ has an $\mathcal{M}_g$-invariant subspace $V$ of finite codimension such that the natural projection $K[S] \to K[S]/V$ is injective on the set $S$.

Now we prove Theorem 4.8. Suppose first that $\varphi : \mathcal{M}_{g,n} \to \text{GL}(m,K)$ is a linear representation with $\text{Ker} \varphi \subset Z(\mathcal{M}_{g,n})$. We define the Adjoint representation of $\varphi$ as

$$\text{Ad}\varphi : \mathcal{M} \to \text{GL}(\text{End} (m,K)), \quad \text{Ad}\varphi(f)(X) = \varphi(f) \cdot X \cdot \varphi(f)^{-1}$$

for $f \in \mathcal{M}_{g,n}$, and $X \in \text{End} (m,K)$. Here, $\text{GL}(\text{End} (m,K))$ denotes the group of the $K$-linear isomorphisms of $\text{End} (m,K)$. This defines the structure of an $\mathcal{M}_{g,n}$-module on $\text{End} (m,K)$. The mapping $\varphi \circ \iota$ extends by linearity to a $K$-linear homomorphism $\phi : K[S] \to \text{End} (m,K)$. By the equality (4.1), $\phi$ is also $\mathcal{M}_{g,n}$-equivariant. Now, take $V = \text{Ker} \phi$. Then, $V$ is an $\mathcal{M}_{g,n}$-invariant subspace of $K[S]$, and the quotient $K[S]/V$ is isomorphic to the image of $\phi$, which is of finite dimension. Since the projection $K[S] \to K[S]/V$ restricted to $S$ is essentially the composition $\varphi \circ \iota$, it is injective by Lemma 4.5. This proves the first implication of the theorem.

Next, suppose conversely that $V$ is an $\mathcal{M}_{g,n}$-invariant subspace of $K[S]$ such that the quotient $K[S]/V$ is of finite dimension and the projection $p : K[S] \to K[S]/V$ is injective on $S$. Then, the natural action of $\mathcal{M}_{g,n}$ on $S$ induces a finite dimensional linear representation $\varphi_V : \mathcal{M}_{g,n} \to \text{GL}(K[S]/V)$ where $\text{GL}(K[S]/V)$ denotes the group of the $K$-linear isomorphisms of $K[S]/V$. Assume $f \in \text{Ker} \varphi_V$. Then, for every $C \in S$, we have

$$p(f(C)) = \varphi_V(f)(p(C)) = p(C).$$

Hence, by the injectivity of $p$ on $S$, we have $f(C) = C$ for every $C \in S$. Then the formula (4.1) implies $f \in C_{\mathcal{M}_{g,n}}(\iota(S))$. Therefore, by Lemma 4.3 we have $f \in Z(\mathcal{M}_{g,n})$. This completes the proof of Theorem 4.8. \qed
References

[1] S. J. Bigelow, Braid groups are linear, J. Amer. Math. Soc. 14 (2001), no. 2, 471–486 (electronic).

[2] S. J. Bigelow and R. D. Budney, The mapping class group of a genus two surface is linear, Algebr. Geom. Topol. 1 (2001), 699–708 (electronic).

[3] J. S. Birman, Braids, links, and mapping class groups, Princeton University Press, 1975.

[4] T. E. Brendle and H. Hamidi-Tehrani, On the linearity problem for mapping class groups, Algebr. Geom. Topol. 1 (2001), 445–468.

[5] B. Farb and D. Margalit, A primer on mapping class groups, Princeton Mathematical Series, vol. 49, Princeton University Press, Princeton, NJ, 2012.

[6] E. Formanek and C. Procesi, The automorphism group of a free group is not linear, J. Algebra 149 (1992), no. 2, 494–499.

[7] J. Franks and M. Handel, Triviality of some representations of $\text{MCG}(S_g)$ in $\text{GL}(n, C)$, $\text{Diff}(S^2)$ and $\text{Homeo}(T^2)$, preprint, arXiv:1102.4584 (2011).

[8] S. Gervais, A finite presentation of the mapping class group of a punctured surface, Topology 40 (2001), no. 4, 703–725.

[9] W. M. Goldman, Mapping class group dynamics on surface group representations, Problems on mapping class groups and related topics, Proc. Sympos. Pure Math., vol. 74, Amer. Math. Soc., 2006, pp. 189–214.

[10] M. Korkmaz, On the linearity of certain mapping class groups, Turkish J. Math. 24 (2000), no. 4, 367–371.

[11] _____, Low-dimensional linear representations of mapping class groups, preprint, arXiv:1104.4816v2 (2011).

[12] _____, The symplectic representation of the mapping class group is unique, preprint, arXiv:1108.3241 (2011).

[13] D. Krammer, The braid group $B_4$ is linear, Invent. Math. 142 (2000), no. 3, 451–486.

[14] _____, Braid groups are linear, Ann. of Math. (2) 155 (2002), no. 1, 131–156.

[15] L. Paris and D. Rolfsen, Geometric subgroups of mapping class groups, J. Reine Angew. Math. 521 (2000), 47–83.

YASUSHI KASAHARA
DEPARTMENT OF MATHEMATICS
KOCHI UNIVERSITY OF TECHNOLOGY
TOSAYAMADA, KAMI CITY, KOCHI
782-8502 JAPAN
E-mail: kasahara.yasushi@kochi-tech.ac.jp