Generalized Nash Equilibrium Problems with Mixed-Integer Variables

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We consider generalized Nash equilibrium problems (GNEPs) with non-convex strategy spaces and non-convex cost functions. This general class of games includes the important case of games with mixed-integer variables for which only a few results are known in the literature. We present a new approach to characterize equilibria via a convexification technique using the Nikaido-Isoda function. To any given instance of the GNEP, we construct a set of convexified instances and show that a feasible strategy profile is an equilibrium for the original instance if and only if it is an equilibrium for any convexified instance and the convexified cost functions coincide with the initial ones. We develop this convexification approach along three dimensions: We first show that for quasi-linear models, where a convexified instance exists in which for fixed strategies of the opponent players, the cost function of every player is linear and the respective strategy space is polyhedral, the convexification reduces the GNEP to a standard (non-linear) optimization problem. Secondly, we derive two complete characterizations of those GNEPs for which the convexification leads to a jointly constrained or a jointly convex GNEP, respectively. These characterizations require new concepts related to the interplay of the convex hull operator applied to restricted subsets of feasible strategies and may be interesting on their own. Note that this characterization is also computationally relevant as jointly convex GNEPs have been extensively studied in the literature. Finally, we demonstrate the applicability of our results by presenting a numerical study regarding the computation of equilibria for three classes of GNEPs related to integral network flows and discrete market equilibria.

1. Introduction

The generalized Nash equilibrium problem constitutes a fundamental class of noncooperative games with applications in economics [13], transport systems [3] and electricity markets [2]. The differentiating feature of GNEPs compared to classical games is the flexibility to model dependencies among the strategy spaces of players, that is, the individual strategy space of every player depends on the strategies chosen by the rival players. Examples in which this aspect is crucial appear for instance in market games where discrete goods are traded and

*A one-page abstract appeared in the Proceedings of the 17th Conference on Web and Internet Economics, 2021 [27]
the buyers have hard spending budgets: effectively, the strategy space of a buyer depends on
the market price (set by the seller) as only those bundles of goods remain affordable that fit
into the budget. Other examples appear in transportation systems, where joint capacities (e.g.
road-, production- or storage capacity) constrain the strategy space of a player. For further
applications of the GNEP and an overview of the general theory, we refer to the excellent
survey articles of Facchinei and Kanzow [22] and Fischer et al. [25].

While the GNEP is a research topic with constantly increasing interest, the majority of
work is concerned with the continuous and convex GNEP, i.e., instances of the GNEP where
the strategy sets of every player may depend
strategy space of player
i
except
i
ON
GNEPs.

Let us introduce the model formally and first recap the standard pure Nash equilibrium
problem (NEP). For an integer
k
∈
\mathbb{N}
, let
[k] := \{1, \ldots , k\}. Let
N = [n]
be a finite set of players. Each player
i
∈
N
controls
the variables
x_i \in X_i \subseteq \mathbb{R}^{k_i}
. We call
x = (x_1, \ldots , x_n)
with
x_i \in X_i
for all
i \in N
a strategy profile and
X = X_1 \times \cdots \times X_n \subseteq \mathbb{R}^k
the strategy space, where
k := (k_1, \ldots , k_n)
and
\mathbb{R}^{(l_1, \ldots , l_s)} := \mathbb{R}^{\sum_{i=1}^{l_i}}
for any vector
l \in \mathbb{N}^s
and
s \in \mathbb{N}
. We use
standard game theory notation: for a strategy profile
x \in X
, we write
x = (x_i, x_{-i})
meaning that
x_i
is the strategy that player
i
plays in
x
and
x_{-i}
is the partial strategy profile of all players except
i
. The private cost of player
i
∈ \mathbb{N}
in strategy profile
x \in X
is defined by a function
\pi_i : \mathbb{R}^k \rightarrow \mathbb{R}
, \pi_i(x) = \pi_i(x_i)
. A (pure) Nash equilibrium is a strategy profile
x^* \in X
with
\pi_i(x^*) \leq \pi_i(y_i, x_{-i}^*)
for all
y_i \in X_i
, \ i \in \mathbb{N}
.

The GNEP generalizes the model by allowing that the strategy sets of every player may depend
on the rival players’ strategies. More precisely, for any
x_{-i} \in \mathbb{R}^{k_{-i}}
(using the notation
k_{-i} := (k_j)_{j \neq i}
), there is a feasible strategy set
X_i(x_{-i}) \subseteq \mathbb{R}^{k_i}
. In this regard, one can think of the strategy space of player
i \in N
represented by a set-valued mapping
X_i : \mathbb{R}^{k_{-i}} \rightrightarrows \mathbb{R}^{k_i}
. This leads to the notation of the combined strategy space represented by a mapping
X : \mathbb{R}^k \rightrightarrows \mathbb{R}^{k_i}
, \ x \mapsto \prod_{i \in \mathbb{N}} X_i(x_{-i})
with
y_i \in X_i(x_{-i})
for all
i \in \mathbb{N}
. The private cost function is given by
\pi_i : \mathbb{R}^k \rightarrow \mathbb{R}
for every player
i \in \mathbb{N}
. The problem of player
i \in \mathbb{N}
– given the rivals’ strategies
x_{-i} – is to solve the following minimization problem:
\inf_{y_i} \pi_i(x_i, x_{-i}) \ \text{s.t.:} \ y_i \in X_i(x_{-i}). \ \ (1)

A generalized Nash equilibrium (GNE) is a feasible strategy profile
x^* \in X(x^*)
with
\pi_i(x^*) \leq \pi_i(y_i, x_{-i}^*)
for all
y_i \in X_i(x_{-i}^*), \ i \in \mathbb{N}
.

We can compactly represent a GNEP by the tuple
I = (N, (X_i(\cdot))_{i \in \mathbb{N}}, (\pi_i)_{i \in \mathbb{N}})
. In the sequel of this paper, we will heavily use the Nikaido-Isoda function (short: NI-function), see [33].

**Definition 1 (NI-function).** Let an instance
I = (N, (X_i(\cdot))_{i \in \mathbb{N}}, (\pi_i)_{i \in \mathbb{N}})
I
of the GNEP be given. For any two vectors
x, y \in \mathbb{R}^k
, the NI-function is defined as:
\Psi(x, y) := \sum_{i \in \mathbb{N}} [\pi_i(x) - \pi_i(y_i, x_{-i})].

By defining
\hat{V}(x) := \sup_{y \in X(x)} \Psi(x, y)
we can recap the following well-known characterization of a generalized Nash equilibrium, see for instance Facchinei and Kanzow [22].
Theorem 1. For an instance $I$ of the GNEP the following statements are equivalent.

1. $x$ is a generalized Nash equilibrium for $I$.
2. $x \in X(x)$ and $\hat{V}(x) = 0$.
3. $x$ is an optimal solution of $\inf_{x \in X(x)} \hat{V}(x)$ with value zero.

This characterization does not rely on any convexity assumptions on the strategy spaces nor on the private cost functions of the players. Yet, the characterization seems computationally of limited interest as neither the Nikaido-Isoda function itself nor the fixed-point condition $x \in X(x)$ seems computationally tractable.

1.1. Our Results and Organization of the Paper

Our approach relies on a convexification technique applied to the original non-convex game leading to a new characterization of the existence of Nash equilibria for GNEPs. In particular, we derive for any instance $I$ of the GNEP a set of convexified instances $I^{\text{conv}}$. Roughly speaking, the latter set consists of all those instances $I^{\text{conv}} = (N, (X_i^{\text{conv}}(\cdot))_{i \in N}, (\phi_i)_{i \in N})$, where for all players $i \in N$ and rivals strategies $x_{-i}$ contained in a certain subset of $\mathbb{R}^{k-i}$, the convexified strategy space $X_i^{\text{conv}}(x_{-i})$ is given by the convex hull $\text{conv}(X_i(x_{-i}))$ of the original strategy space and the convexified private cost function $x_i \mapsto \phi_i(x_i, x_{-i})$ is the convex envelope of $x_i \mapsto \pi_i(x_i, x_{-i})$. Our main result (Theorem 2) states that for any $I^{\text{conv}} \in I^{\text{conv}}$, a strategy profile $x \in X(x)$ is a GNE for $I$ if and only if it is a GNE for $I^{\text{conv}}$ and the convexified cost functions coincide with the original ones. The proof is based on using the Nikaido-Isoda functions for both games $I^{\text{conv}}$ and $I$. While the convexified instances may admit an equilibrium under certain circumstances, this equilibrium might still not be feasible for the original non-convex game. The advantage of our convex reformulation, however, lies in the possibility that for some problems, it is computationally tractable to solve a convexified instance while preserving feasibility with respect to the original game. In this regard, we study several subclasses of GNEPs for which this methodology applies.

In Section 3 we consider quasi-linear GNEPs in which the cost functions of players are quasi-linear and the players’ strategy spaces are quasi-polyhedral sets, that is, they admit a structure which allows the convexified private cost functions to be chosen linearly for fixed strategies of the other players. Similarly, the convexified strategy sets can be described by polyhedra whenever the rivals’ strategies are fixed. By reformulating the $\hat{V}$ function of an associated convexified instance, we show in Theorem 3 that a quasi-linear GNEP can be modeled as a standard (non-linear) optimization problem. The reformulation uses linear duality of the players’ optimization problems and we note that this approach has been used before by Stein and Sudermann-Merx for the special case of linear GNEPs in which the cost functions and strategy sets can be described by linear functions in $x$.

In Section 4 we study jointly constrained GNEPs which are also called GNEPs with shared constraints. These games have the differentiating feature that the players’ strategy sets are restricted via a shared feasible set $X \subseteq \mathbb{R}^k$. If $X$ is convex, one speaks of a jointly convex GNEP but we do not impose this on $X$ a priori. GNEPs with shared constraints have been extensively studied in the literature and in this regard we analyze the structure of original non-convex GNEPs $I$ (not even jointly constrained) for which the set of convexified instances $I^{\text{conv}}$ contains a jointly constrained or even a jointly convex instance. To this end, we introduce the new classes of $k$-restrictive-closed and restrictive-closed GNEPs for which we show that they completely characterize whether or not $I^{\text{conv}}$ contains a jointly constrained or a jointly convex instance, respectively. The property of $(k)$-restrictive-closedness is for example fulfilled for all $\{0,1\}^k$ jointly constrained instances $I$ and thus admits interesting applications.

In Section 5 we present numerical results on the computation of equilibria for three classes of GNEPs related to integral network flows and discrete market equilibria which are shown
to belong to both classes of restrictive-closed and quasi-linear GNEPs. To find equilibria of an instance $I$, we propose two different methods based on our convexification result. Firstly, we present an approach where our quasi-linear reformulation is plugged into a standard non-convex solver (BARON). Secondly, we try to compute an integral GNE of a specific convexified instance $I_{\text{conv}} \in \mathcal{I}$. by implementing different procedures from the literature for solving a convex GNEP, enhanced by a simple rounding procedure in order to obtain an integral equilibrium. Perhaps surprisingly, it turned out that our quasi-linear approach was not only faster (on average) in finding specifically integral GNE for the original non-convex GNEP but also for computing (not necessarily integral) GNE for the convexified instances.

1.2. Related Work

**Continuous and Convex GNEPs.** GNEPs have been studied intensively in terms of equilibrium existence and numerical algorithms. It is fair to say, that the majority of works focus on the continuous and convex case, that is, the cost functions of players are convex (or at least continuous) and the strategy spaces are convex. One major reason for these restrictive assumptions lies in the lack of tools to prove existence of equilibria. Indeed, most existence results rely on an application of Kakutani’s fixed point theorem which in turn requires those convexity assumptions (e.g. Rosen [38]). We refer to the survey articles of Facchinei and Kanzow [22] and Fischer et al. [25] for an overview of the general theory.

We discuss in the following various approaches for computing GNE for convex and continuous GNEPs. Based on reducing the GNEP to the standard NEP, Facchinei and Sagratella [23] described an algorithm to compute all solutions of a jointly convex GNEP, where the joint restrictions are given by linear equality constraints. However, this algorithm does not terminate in finite time whenever there are infinitely many equilibria. Dreves [15, 16] tackled this problem via an algorithm which computes in finite time the whole solution set for two different types of GNEPs. In [15], he investigated affine GNEPs with one-dimensional strategy sets in which the players’ optimization problems are convex quadratic problems with a common linear constraint in $x$. The other type of GNEPs considered by Dreves [16] are the linear (not necessarily jointly convex) GNEPs introduced by Stein and Sudermann-Merx [47]. While Dreves investigated the computation of all solutions, Stein and Sudermann-Merx studied the smoothness of a certain gap function that arises via a suitable extension of the $\hat{V}$ function. The latter extension is based on a dualization approach regarding the second part of the NI-function, allowing for a reformulation of $\hat{V}(x) = \sup_{y \in X(x)} \Psi(x, y)$ as a minimization problem. Note that this dualization step will also play a key role in our analysis of quasi-linear GNEPs in Section 3. The applicability of the findings of Stein and Sudermann-Merx was demonstrated in [19] by Dreves and Sudermann-Merx where they investigated various numerical methods to compute equilibria of linear GNEPs, cf. also [48]. Returning to the jointly convex GNEP, Heusinger and Kanzow [49] presented an optimization reformulation using the Nikaido-Isoda function, assuming that the cost functions $\pi_i(x_i, x_{-i})$ of the players are (at least) continuous in $x$ and convex in $x_i$. Under the same assumptions concerning the cost functions, Dreves, Kanzow and Stein [18] generalized this approach to player-convex GNEPs, where additionally to the assumptions on the cost functions, the strategy sets are assumed to be described by $X_i(x_{-i}) = \{x_i \mid g_i(x_i, x_{-i}) \leq 0\}$ for a restriction function $g_i$ which is (at least) continuous in $x$ and convex in $x_i$. In comparison to this optimization reformulation, Dreves et al. [17] took a different approach to finding equilibria via the KKT conditions of the GNEP. Under sufficient regularity, e.g. $C^2$ cost- and restriction functions, they discuss how the KKT system of the GNEP may be solved in order to find generalized Nash equilibria.

While the assumptions concerning the cost- and restriction functions in the above papers are mild in the context of continuous GNEPs, it is a priori not clear, whether or not there exists a convexified instance in $\mathcal{I}_{\text{conv}}$ which fulfills them, and then allows for the application of algorithms from the domain of convex and continuous GNEPs. In this regard, we are concerned
in Section \[1\] with identifying subclasses of GNEPs which guarantee the existence of such well-behaved convexified instances in \( T^{\text{conv}} \).

**Non-Convex and Discrete GNEPs.** For non-convex and discrete GNEPs, much less is known regarding the existence and computability of equilibria. In fact, the only computational approach for finding pure GNE we are aware of are that of Sagratella \([41, 43]\). In the former, two different techniques for the subclass of so-called generalized potential games with mixed-integer variables are presented. Similar to the jointly convex GNEPs, in these potential games, the players are restricted through a common convex set \( X \) with the further restriction that some strategy components need to be integral and there is a potential function over the set \( X \). On the one hand, Sagratella introduced certain optimization problems with mixed-integer variables based on the fact that minimizers of the potential function correspond to a subset of generalized Nash equilibria. On the other hand, he showed that a Gauss-Seidel best-response (BR) algorithm may approximate equilibria arbitrarily well within a finite amount of steps in this setting. We remark, however, that a BR-algorithm is not a correct approach for GNEPs not admitting a potential function, because there are trivial examples (even for standard NEPs) in which a BR-algorithm may cycle forever and not terminate although equilibria exist. In particular, for GNEPs that are not jointly constrained, the best response of a player may lead to an infeasible overall strategy profile resulting in an empty best-response correspondence for some player. In this regard, Sagratella’s results as well as several interesting models that have emerged based upon his results in the domain of automated driving \([21]\), traffic control \([10]\) or transportation problems \([44]\) are not directly extendable to the general mixed-integer GNEP setting that we consider in this paper. In \([43]\), Sagratella generalized his approach in \([42]\) for standard NEPs (c.f. below) and considered mixed-integer GNEPs in which each player’s strategy set has partial integer constraints and depends on the other players’ strategies via a linear constraint in her own strategy and the other players’ integrally constrained strategies. Under further convexity and continuity assumptions regarding the cost functions, he then proposed a branch and bound method based on merit functions as well as a branch and bound method exploiting dominance of strategies for suitable cuts.

Besides the work of Sagratella, we are only aware of the paper \([1]\) by Ananduta and Grammatico which deals with mixed-integer GNEPs. They considered a model in which the dependency of costs and constraints can be additively separated between the continuous and integer variables. However, the authors did not directly deal with the mixed-integer GNEP but rather with a mixed-strategy extension of the game. That is, the players are assumed to choose probability distributions over their integral strategies and the game is solved using mixed-strategy GNEP without integrality constraints. Based upon the specific structure of the latter, the authors then introduced a Bregman forward-reflected-backward splitting and design distributed algorithm to compute equilibria. In this paper, we focus on pure GNE, partly because mixed or correlated strategies have no meaningful physical interpretation in some games; see also the discussion in Osborne and Rubinstein \([34, § 3.2]\) about critics of mixed Nash equilibria. In particular, the definition of a meaningful randomization concept for general GNEPs is non-trivial. This is, for instance, illustrated by the special class of separable GNEPs considered in \([1]\), where the proposed concept of mixed equilibria may associate a non-zero probability to strategy profiles which are not even feasible.

While mixed-integer GNEPs are fairly unexplored, there is a growing research regarding general formulations of non-convex (standard) NEPs. For instance, the so-called Integer Programming Games (IPGs) have recently gained interest in which the optimization problem of each player consists of minimizing a continuous function over a (fixed) polyhedron with partial integrality constraints. IPGs in a less general setting were first introduced by Köppe et al. \([30]\), where they investigated the computational complexity of computing pure NE. Along these lines, Carvalho et al. \([8, 9]\) investigated the existence \([8]\) and computation \([9]\) of NE as well as recently
published a tutorial on the computation of NE in IPGs in [7]. Extending the Sample generation method presented in [9] for IPGs, Crönert and Minner [12] proposed an algorithm for the complete enumeration of all mixed equilibria in finite games, i.e. games belonging to the standard NEP with strategy sets being finite and succinctly described by a finite amount of inequalities. Regarding more specialized subclasses of IPGs, Del Pia et al. [30] introduced a strongly polynomial-time algorithm to compute NE and derived several related complexity results for IPGs in which each player has a totally unimodular strategy set. On the basis of aggregating the players strategy spaces, this approach has been extended by Kleer and Schäfer [29]. Another subclass of IPGs was explored in [30] where the authors focus on games in which each player solves a mixed-binary quadratic program where the cost function is quadratic in the whole strategy profile. Leveraging a completely positive program reformulation established by Burer [5], the authors tackled the computation of NE via the associated KKT conditions. For IPGs in which the strategy spaces are boxes, Kirst, Schwarz and Stein [28] recently proposed a branch-and-bound algorithm that computes the set of all approximate equilibria for a given approximation error based on discarding rules that determine sets which do not contain equilibria. Finally regarding IPGs, Dragotto and Scatamacchia [14] presented an algorithm based on cutting planes to tackle the computation, enumeration and selection of pure NE in IPGs with purely integral strategies.

Departing from IPGs, Carvalho et al. [6] introduced the class of Reciprocally-Bilinear Games (RBGs), where for each player, the cost function is bilinear in her own and her rivals’ strategies while her strategy set is only required to have a convex hull whose closure is polyhedral. A Cut-and-Play algorithm is then presented which computes a mixed Nash equilibrium of an RBG based upon a scheme consisting of mainly two components: the computation of mixed Nash equilibria for approximated games and the eventual refinement of the approximations.

Finally, another class of non-convex NEPs was investigated by Sagratella [40] where a branch-and-bound algorithm that computes the set of all approximate equilibria combined with integrality constraints for all strategy components. Recently, Schwarze and Stein [46] were able to drop the latter convexity assumption of players’ cost functions by extending Sagratella’s approach via a branch-and-prune algorithm.

2. Convexification

In this section we will introduce for any instance \( I = (N, (X_i(\cdot)), (\pi_i)), (\pi_i))) \) of the GNEP a corresponding set of convexified instances \( I^{\text{conv}} \) of the GNEP. In order to describe this convexification method, we introduce the following concept of a refined domain:

**Definition 2.** Let \( I = (N, (X_i(\cdot)), (\pi_i))) \) be an instance of the GNEP. For any \( i \in N \) we define the refined domain of the set-valued mapping \( X_i : \mathbb{R}^{k_i} \rightarrow \mathbb{R}^{k_i} \) by

\[
\text{rdom} X_i := \left\{ x_{-i} \in \mathbb{R}^{k_{-i}} \mid \exists \tilde{x}_i \in \mathbb{R}^{k_i} : (\tilde{x}_i, x_{-i}) \in X((\tilde{x}_i, x_{-i})) \right\}.
\]

This concept is a refinement of the standard domain \( \text{dom} X_i := \left\{ x_{-i} \in \mathbb{R}^{k_{-i}} \mid X_i(x_{-i}) \neq \emptyset \right\} \). Clearly, the standard domain of \( X_i \) contains the refined domain. However, the refined domain is in general a proper subset as it also takes into account whether the rivals’ strategies \( x_{-i} \) are feasible or not. That is, only the rivals’ strategies \( x_{-i} \) are in the refinement where at least one feasible strategy \( \tilde{x}_i \in X_i(x_{-i}) \) for player \( i \) exists such that for any of her rivals \( j \neq i \), the strategy \( x_j \in X_j((\tilde{x}_i, x_{-i})) \) is feasible. Here, \( (\tilde{x}_i, x_{-ij}) \in \mathbb{R}^{k_{-i}} \) is the partial strategy profile of all players except \( j \) in which player \( l \) plays strategy \( x_l \) for \( l \notin \{i, j\} \) and \( \tilde{x}_i \) for \( l = i \). Alternatively, the refined domain of player \( i \) can be seen as the projection of the set of feasible strategy profiles \( x \in X(x) \) to the rivals’ strategy space \( \mathbb{R}^{k_{-i}} \). This idea of relevant strategies leads to the following definition of what we call quasi-isomorphic instances of the GNEP.
Definition 3. Two instances $I = (N, (X_i(\cdot))_{i \in N}, (\pi_i)_{i \in N})$ and $I' = (N, (X'_i(\cdot))_{i \in N}, (\pi'_i)_{i \in N})$ are called quasi-isomorphic, if for all $i \in N$ the refined domains coincide $\text{rdom}X_i = \text{rdom}X'_i$ and for all $x_{-i} \in \text{rdom}X_i$ the strategy sets and cost functions coincide, i.e. $X_i(x_{-i}) = X'_i(x_{-i})$ and $\pi_i(x_i, x_{-i}) = \pi'_i(x_i, x_{-i})$ for all $x_i \in X_i(x_{-i})$.

The above concept of quasi-isomorphic GNEPs $I, I'$ requires that the set of feasible strategy profiles of both games coincide and for each feasible strategy profile $x \in X(x)$ of $I$ the conditions for it to be a GNE in $I$ are precisely the same conditions as in $I'$ and vice versa.

Our convexification approach relies heavily on the concept of convex envelopes which we introduce next, cf. Rockafellar [37].

Definition 4. Let $f : M \to \mathbb{R}$ with $M \subseteq \mathbb{R}^l$ for some $l \in \mathbb{N}$ be a function into the extended reals. We denote by $\text{epi}(f) := \{ (x, y) \in M \times \mathbb{R} \mid y \geq f(x) \}$ its epigraph and call $\phi : \text{conv}(M) \to \mathbb{R} \cup \{-\infty\}$ with $\phi(x) := \inf\{ y \in \mathbb{R} \mid (x, y) \in \text{conv}(\text{epi}(f)) \}$ in $\mathbb{R} \cup \{-\infty\}$ the convex envelope of $f$.

Note that Rockafellar [37] Theorem 5.3] showed that the convex envelope is a well-defined convex function (which may attain the value $-\infty$). Based on this definition, we can now describe our convexification method.

Definition 5. Let $I = (N, (X_i(\cdot))_{i \in N}, (\pi_i)_{i \in N})$ and $I^{\text{conv}} = (N, (X^{\text{conv}}_i(\cdot))_{i \in N}, (\phi_i)_{i \in N})$ be two instances of the GNEP. We call $I^{\text{conv}}$ a convexified instance for $I$, if it fulfills for all $i \in N$ and $x_{-i} \in \text{rdom}X_i$ the following two criteria:

1. $X^{\text{conv}}_i(x_{-i}) = \text{conv}(X_i(x_{-i}))$, i.e. the convexified strategy space is given by the convex hull of the original strategy space.

2. $\phi_i(x_i, x_{-i}) = \phi^{\text{conv}}_i(x_i)$ for all $x_i \in \text{conv}(X_i(x_{-i}))$ where $\phi^{\text{conv}}_i(\cdot) : \text{conv}(X_i(x_{-i})) \to \mathbb{R}$ denotes the convex envelope of $\pi_i(\cdot, x_{-i}) : X_i(x_{-i}) \to \mathbb{R}$, $x_i \mapsto \pi_i(x_i, x_{-i})$.

Note that [2] only required to hold for $x_{-i} \in \text{rdom}X_i$ and not the whole $\mathbb{R}^{k-i}$. Similarly, [3] restricts the cost function of a convexified instance only for $x$ with $x_i \in \text{conv}(X_i(x_{-i}))$, $x_{-i} \in \text{rdom}X_i$ and not the whole $\mathbb{R}^k$. This degree of freedom leads to a whole set of convexified instances for $I$ which we denote by $I^{\text{conv}}(I)$. If the instance $I$ is clear from context, we will just speak of convexified instances and $I^{\text{conv}}$.

By definition, the cost function of an instance of the GNEP need to be real-valued. In this regard, the set $I^{\text{conv}}(I) \neq \emptyset$ is nonempty if and only if for all $i \in N$ and $x_{-i} \in \text{rdom}X_i$, the function $\pi_i(\cdot, x_{-i}) : X_i(x_{-i}) \to \mathbb{R}$ admits a real-valued convex envelope $\phi^{\text{conv}}_i(\cdot) : \text{conv}(X_i(x_{-i})) \to \mathbb{R}$. It thus follows immediately that the boundedness of [11] is sufficient for $I^{\text{conv}}(I) \neq \emptyset$. However, we remark that it is not necessary. In Section 3 it is shown that all player-linear mixed-integer GNEPs (cf. Definition [7] fulfill $I^{\text{conv}}(I) \neq \emptyset$, even though there may exist players with unbounded [11].

With our convexification method at hand, we can now describe our first main theorem.

Theorem 2. Let $I = (N, (X_i(\cdot))_{i \in N}, (\pi_i)_{i \in N})$ be an instance of the GNEP and $I^{\text{conv}} \in I^{\text{conv}}$ any convexified instance. For any $x \in X(x)$, the following assertions are equivalent.

1. $x$ is a generalized Nash equilibrium for $I$.

2. $x$ is a generalized Nash equilibrium for $I^{\text{conv}}$ and $\phi_i(x) = \pi_i(x)$ for all $i \in N$.

Proof. Let $I^{\text{conv}} \in I^{\text{conv}}$ be an arbitrary convexified instance. We first show that for every $x \in X(x)$, the inequality $\hat{V}^{\text{conv}}(x) \leq \hat{V}(x)$ holds, where $\hat{V}^{\text{conv}}$ is the $\hat{V}$ function for $I^{\text{conv}}$. For an arbitrary $x \in X(x)$, we have:

$$
\hat{V}^{\text{conv}}(x) = \sup_{y \in X^{\text{conv}}(x)} \sum_{i \in N} [\phi_i(x) - \phi_i(y_i, x_{-i})] = \sum_{i \in N} [\phi_i(x)] - \inf_{y \in X^{\text{conv}}(x)} \sum_{i \in N} [\phi_i(y_i, x_{-i})].
$$

(2)
As \( x \in X(x) \) and subsequently \( x_{-i} \in \text{dom} X_i \) for all \( i \in N \), we have by Definition 5.1 that \( y \in X^\text{conv}(x) \) if and only if \( y_i \in \text{conv}(X_i(x_{-i})) \) for all \( i \in N \). Furthermore the objective function \( \sum_{i \in N} [\phi_i(y_i, x_{-i})] \) is obviously separable in \( y \). Therefore, the following is true:

\[
\inf_{y \in X^\text{conv}(x)} \sum_{i \in N} [\phi_i(y_i, x_{-i})] = \sum_{i \in N} \inf_{y_i \in \text{conv}(X_i(x_{-i}))} [\phi_i(y_i, x_{-i})].
\]

As for all \( i \in N \) the function \( \text{conv}(X_i(x_{-i})) \to \mathbb{R}, y_i \mapsto \phi_i(y_i, x_{-i}) \) is the convex envelope of the function \( X_i(x_{-i}) \to \mathbb{R}, y_i \mapsto \pi_i(y_i, x_{-i}) \), the following equality holds:

\[
\inf_{y_i \in \text{conv}(X_i(x_{-i}))} \phi_i(y_i, x_{-i}) = \inf_{y_i \in X_i(x_{-i})} \pi_i(y_i, x_{-i}).
\]

Thus, we arrive at:

\[
\hat{\nu}^\text{conv}(x) = \sum_{i \in N} \phi_i(x) - \inf_{y \in X^\text{conv}(x)} \sum_{i \in N} [\phi_i(y_i, x_{-i})] \quad \text{by (2)}
\]

\[
= \sum_{i \in N} \phi_i(x) - \sum_{i \in N} \inf_{y_i \in X_i(x_{-i})} [\pi_i(y_i, x_{-i})] \quad \text{by (3) and (4)}
\]

\[
\leq \sum_{i \in N} [\pi_i(x)] - \sum_{i \in N} \inf_{y_i \in X_i(x_{-i})} [\pi_i(y_i, x_{-i})] \quad \text{by } \phi_i(x) \leq \pi_i(x), i \in N \text{ as } x \in X(x) \quad (5)
\]

\[
= \sum_{i \in N} [\pi_i(x)] - \inf_{y \in X(x)} \sum_{i \in N} [\pi_i(y, x_{-i})] \quad \text{by the same argumentation as for (4)}
\]

\[
= \hat{V}(x)
\]

Therefore, we have the inequality

\[
\hat{\nu}^\text{conv}(x) \leq \hat{V}(x) \quad \text{for all } x \in X(x)
\]

which allows us to prove the equivalence 4 \iff 2.

4 \implies 2: Let \( x \in X(x) \) be a generalized Nash equilibrium of I. Theorem 1 and inequality 6 imply that \( \hat{V}(x) = 0 \geq \hat{\nu}^\text{conv}(x) \). Furthermore \( x \in X^\text{conv}(x) \) as \( x \in X(x) \) and by observing that \( \hat{\nu}^\text{conv}(x) \geq 0 \) for all \( x \in X^\text{conv}(x) \) we conclude that \( \hat{\nu}^\text{conv}(x) = 0 \). Summarizing we have \( x \in X^\text{conv}(x) \) with \( \hat{\nu}^\text{conv}(x) = 0 \) which is equivalent to \( x \) being a generalized Nash equilibrium for \( I^\text{conv} \) by Theorem 1. Furthermore \( \hat{\nu}^\text{conv}(x) = 0 = \hat{V}(x) \) implies that the inequality in (5) must be tight, i.e. \( \sum_{i \in N} \pi_i(x) = \sum_{i \in N} \phi_i(x) \) holds. Together with \( \phi_i(x) \leq \pi_i(x) \), \( i \in N \) we thus get \( \phi_i(x) = \pi_i(x), i \in N \).

2 \implies 4: Let \( x \in X(x) \) be a generalized Nash equilibrium of \( I^\text{conv} \) and \( \phi_i(x) = \pi_i(x), i \in N \). Theorem 1 implies that \( \hat{\nu}^\text{conv}(x) = 0 \) while the equality \( \phi_i(x) = \pi_i(x), i \in N \) implies that the inequality in (5) is tight and therefore \( \hat{V}(x) = \hat{\nu}^\text{conv}(x) = 0 \) holds. Again, Theorem 1 yields that \( x \) is a generalized Nash equilibrium for \( I \) which finishes our proof.

Theorem 2 allows us to formulate the following characterization of a generalized Nash equilibrium:

Corollary 1. For an instance \( I \) of the GNEP and any \( I^\text{conv} \in \mathcal{I}^\text{conv} \), the following statements are equivalent.

1. \( x \) is a generalized Nash equilibrium for \( I \).
2. \( x \in X(x), \hat{\nu}^\text{conv}(x) = 0 \) and \( \phi_i(x) = \pi_i(x), i \in N \).
3. \( x \) is an optimal solution of (7) with value zero and \( \phi_i(x) = \pi_i(x), i \in N \).

\[
\inf_{x \in X(x)} \hat{\nu}^\text{conv}(x)
\]
Before we illustrate the above concepts in the following Example 1, let us briefly discuss the relevance of Theorem 2 and Corollary 1 in particular in regard of practical applications. In the latter, one may not have the precise description of the refined domains $\text{rdom} X_i$ nor convex envelopes $\partial^c X_i$ at hand as they may for example be just too difficult/expensive to compute. However, there might be types of instances $I$ for which one can a priori identify computationally tractable convexified instances without ever having to compute the refined domains or the convex envelopes in practice, cf. the instance $I^{\text{conv}}$ in Example 1. Thus, in the remainder of the paper, we will identify several types of original instances for which we can describe computationally tractable convexified instances.

We remark that a similar observation can be made from a theoretical standpoint. Identifying suitable, well-understood convexified instances may yield structural insights for the original ones. For example, there exist various conditions for a convex GNEP in the literature guaranteeing the existence of an equilibrium which may be used together with Theorem 2 to guarantee the suitability of the instance $I$.

**Example 1.** Let $I = (N, (X_i(\cdot))_{i \in N}, (\pi_i)_{i \in N})$ be a 2-player GNEP, i.e. $N = [2]$, where the strategy sets are 1-dimensional sets given by

\[
X_1(x_2) := X_1^{\text{rel}}(x_2) \cap \mathbb{Z} := \{ x_1 \in \mathbb{R} \mid (x_1 - 2)^2 + (x_2 - 2)^2 \leq 1 \} \cap \mathbb{Z}
\]

\[
X_2(x_1) := X_2^{\text{rel}}(x_1) \cap \mathbb{Z} := \{ x_2 \in \mathbb{R}_{\geq 0} \mid x_2 + x_1 \geq 2, 2x_2 - x_1 \leq 5, x_2 + 3x_1 \leq 9 \} \cap \mathbb{Z}
\]

for all $x_2, x_1 \in \mathbb{R}$ respectively.

For the cost functions we set $\pi_1(x) := x_1 \cdot x_2$ and $\pi_2(x) := (1 + |x_2|)^{x_1 - 1}$ for all $x \in \mathbb{R}^2$. The refined domains are described by

\[
\text{rdom} X_1 = \{ x_2 \in \mathbb{R} \mid \exists \tilde{x}_1 \in \mathbb{R} : \tilde{x}_1 \in X_1(x_2), x_2 \in X_2(\tilde{x}_1) \}
\]

\[
= \{ x_2 \in \mathbb{R} \mid \exists \tilde{x}_1 \in \mathbb{R} : \tilde{x}_1 \in \mathbb{Z}, (\tilde{x}_1 - 2)^2 + (x_2 - 2)^2 \leq 1, \\
x_2 \in \mathbb{Z}_{\geq 0}, x_2 + \tilde{x}_1 \geq 2, 2x_2 - \tilde{x}_1 \leq 5, x_2 + 3\tilde{x}_1 \leq 9 \}
\]

\[
= \{ 1, 2, 3 \}
\]

and

\[
\text{rdom} X_2 = \{ x_1 \in \mathbb{R} \mid \exists \tilde{x}_2 \in \mathbb{R} : x_1 \in X_1(\tilde{x}_2), \tilde{x}_2 \in X_2(x_1) \}
\]

\[
= \{ x_1 \in \mathbb{R} \mid \exists \tilde{x}_2 \in \mathbb{R} : x_1 \in \mathbb{Z}, (x_1 - 2)^2 + (\tilde{x}_2 - 2)^2 \leq 1, \\
\tilde{x}_2 \in \mathbb{Z}_{\geq 0}, \tilde{x}_2 + x_1 \geq 2, 2\tilde{x}_2 - x_1 \leq 5, \tilde{x}_2 + 3x_1 \leq 9 \}
\]

\[
= \{ 1, 2 \}.
\]

Alternatively, the set of feasible strategies is given by $\{(1, 2), (2, 1), (2, 2), (2, 3)\}$ with the projection $\{1, 2, 3\}$ to $\mathbb{R}^{k-1}$ and $\{1, 2\}$ to $\mathbb{R}^{k-2}$. In comparison, the standard domains are given by $\text{dom} X_1 = [1, 3]$ and $\text{dom} X_2 = [0, 3]$ since $X_1(x_2) \neq \emptyset$ iff $x_2 \in [1, 3]$ (cf. the green points in the first picture in Figure 1, e.g. $X_1(2.5) = \{2\}$) and analogously $X_2(x_1) \neq \emptyset$ iff $x_1 \in [0, 3]$.

By Definition 3 an instance $I^{\text{conv}}$ is a convexified instance for $I$ if and only if it fulfills:

---

*Figure 1: Representation of the strategy sets and the resulting set of feasible strategy profiles $x \in X(x)$ marked via red circles.*
1. $X_1^{\text{conv}}(x_2) = \text{conv}(X_1(x_2)) = \{2\}$ for $x_2 \in \{1, 3\}$ and $X_1^{\text{conv}}(x_2) = [1, 3]$ for $x_2 = 2$.

$X_2^{\text{conv}}(x_1) = \text{conv}(X_2(x_1)) = [1, 3]$ for $x_1 = 1$ and $X_2^{\text{conv}}(x_1) = [0, 3]$ for $x_1 = 2$.

2. $\phi_1(x) = x_1 \cdot x_2$ for $x_1 \in \{2\}, x_2 \in \{1, 3\}$ and $\phi_1(x) = 2x_1$ for $x_1 \in \{1, 3\}, x_2 \in \{2\}$.

$\phi_2(x) = 1$ for $x_2 \in \{1, 3\}, x_1 \in \{1\}$ and $\phi_2(x) = 1 + x_2$ for $x_2 \in [0, 3], x_1 \in \{2\}$.

As described in Definition 4, the above setting allows for a whole set of convexified instances. Some of these convexifications may have a natural and computational efficient representation for which even the domains $\text{rdom} X_i, i \in N$ need not be known a priori. To illustrate this aspect, let us further specify two different convexifications for our example that both belong to $\mathcal{I}^{\text{conv}}$: $\mathcal{I}^{\text{conv}} = (N, (X_i^{\text{conv}}(\cdot))_{i \in N}, (\phi_i)_{i \in N})$ and $\mathcal{I}^{\text{conv}} = (N, (\bar{X}_i^{\text{rel}}(\cdot))_{i \in N}, (\bar{\phi}_i)_{i \in N})$.

In the first game, we use the canonical relaxation of the original strategy sets for both players, i.e. $X_i^{\text{conv}}(x_{-i}) := X_i^{\text{rel}}(x_{-i})$ for $i \in \{2\}$ and all $x_{-i} \in \mathbb{R}^{k-i}$. Regarding the cost functions, we define $\phi_1(x) := \pi_1(x)$ and $\phi_2(x) := (x_1 - 1) \cdot x_2 + 1$ for all $x \in \mathbb{R}^2$ which is possible as $\pi_1(x)$ is linear for fixed $x_2 \in \text{rdom} X_2$. The possibility to use the canonical relaxations of the original strategy sets and the setting of $\phi_1 = \pi_1$ is of course not always possible, yet, in the following Section 3 we will identify certain types of original instances for which this is always possible.

In the second game, we set

$$\bar{X}_1^{\text{conv}}(x_2) := \begin{cases} \{2\}, & \text{if } x_2 \in \{1, 3\} \\ [1, 3], & \text{if } x_2 = 2 \\ \emptyset, & \text{else} \end{cases} \quad \text{and} \quad \bar{X}_2^{\text{conv}}(x_1) := X_2^{\text{rel}}(x_1) \text{ for all } x_1 \in \mathbb{R}.$$ 

As cost functions we define $\bar{\phi}_1(x) := \pi_1(x)$ and $\bar{\phi}_2(x) := \pi_2(x)$ for all $x \in \mathbb{R}^2$.

Both games in the above example fulfill 7 and 8, but their cost functions and strategy sets differ significantly whenever $x_{-i} \notin \text{rdom} X_i$. This results in different smoothness properties of the games. For instance, the convexified cost function $\bar{\phi}_2$ of player 2 in the first game is convex whenever $x_1$ is fixed for all $x_1 \in \mathbb{R}$ whereas in the second game, $\bar{\phi}_2$ is only convex for fixed $x_1$ when $x_1 \in [2, \infty)$. Similarly, the strategy sets of player 1 in the first game are described by a $C^\infty$ convex restriction function which contrasts the second game. If we now apply Corollary 7 to the two different convexifications $\mathcal{I}^{\text{conv}}$ and $\mathcal{I}^{\text{conv}}$, we obtain quite different optimization problems 7:

$$\inf_{x} \max_{y} x_1 \cdot x_2 + (x_1 - 1) \cdot x_2 + 1$$

$$- (y_1 \cdot x_2 + (x_1 - 1) \cdot y_2 + 1)$$

s.t.: $$(y_1 - 1)^2 + (x_2 - 2)^2 - 1 \leq 1$$ $$y_2 + x_1 \geq 2, \ 2y_2 - x_1 \leq 5, \ y_2 + 2x_1 \leq 5$$ $$(x_1 - 2)^2 + (x_2 - 2)^2 - 1 \leq 1$$ $$x_2 + x_1 \geq 2, \ 2x_2 - x_1 \leq 5, \ x_2 + 2x_1 \leq 5$$ $$y_1 \in \mathbb{R}, \ y_2 \in \mathbb{R}_{\geq 0}, \ x_1 \in \mathbb{Z}, \ x_2 \in \mathbb{Z}_{\geq 0}$$

$$\inf_{x} \max_{y} x_1 \cdot x_2 + (1 + |x_2|)^{x_1-1}$$

$$- (y_1 \cdot x_2 + (1 + |y_2|)^{x_1-1})$$

s.t.: $y_1 \in \bar{X}_1^{\text{conv}}(x_2)$ $$y_2 + x_1 \geq 2, \ 2y_2 - x_1 \leq 5, \ y_2 + 2x_1 \leq 5$$ $$x_1 \in \bar{X}_1^{\text{conv}}(x_2)$$ $$x_2 + x_1 \geq 2, \ 2x_2 - x_1 \leq 5, \ x_2 + 2x_1 \leq 5$$ $$y_1 \in \mathbb{R}, \ y_2 \in \mathbb{R}_{\geq 0}, \ x_1 \in \mathbb{Z}, \ x_2 \in \mathbb{Z}_{\geq 0}$$
Note again the effect of the two different convexifications on the structure of the resulting feasible sets and their objective functions: in contrast to the second problem, the first one admits a quasi-linear objective and a feasible set defined via smooth algebraic constraints. In the remainder of the paper, we will further identify necessary and sufficient conditions of an instance $I$ so that it allows for “well-behaved” convexified instances and consequently leads to more tractable optimization problems \(\mathbb{U}\).

3. Quasi-Linear GNEPs

3.1. MINLP-Reformulation

In what follows we identify a subclass of the GNEP such that the optimization problem \(\mathbb{U}\) becomes more accessible. The main obstacle when solving \(\mathbb{U}\) in the general case is the need to separately evaluate the function $\hat{V}^{\text{conv}}$ for any strategy profile $x$. This is due to the fact that the evaluation of $\hat{V}^{\text{conv}}$ at any $x$ is itself a maximization problem which ultimately leads to a computationally intractable optimization problem \(\mathbb{U}\) as the latter constitutes of a maximization problem nested within a minimization problem. For linear GNEPs, Stein and Sudermann-Merx \([47]\) showed that one can resolve this problem by dualizing the corresponding $\hat{V}$ function.

We define in the following the class of quasi-linear GNEPs $\mathcal{I}$ for which there exists a convexified instance $I^{\text{conv}} \in \mathcal{I}^{\text{conv}}$ admitting the property that $\hat{V}^{\text{conv}}(x)$ is a linear maximization problem, i.e. for every $i \in N$ and fixed $x_{-i} \in \mathbb{R}^{k-i}$, player $i \in N$ has a linear cost function as well as a polyhedral strategy set. While $I^{\text{conv}}$ does not need to belong to the linear GNEPs considered in \([47]\), their dualization idea is still applicable resulting in a reformulation of \(\mathbb{U}\) as an optimization problem \(\mathbb{R}\) in standard form.

**Definition 6.** An instance $I$ of the GNEP is called quasi-linear, if there exists $I^{\text{conv}} \in \mathcal{I}^{\text{conv}}$ which fulfills for every $i \in N$ the following two statements:

1. There exists a matrix-valued function $M_i : \text{rdom}X_i \rightarrow \mathbb{R}^{l_i \times k_i}$, $x_{-i} \mapsto M_i(x_{-i})$ and a vector-valued function $e_i : \text{rdom}X_i \rightarrow \mathbb{R}^{l_i}$, $x_{-i} \mapsto e_i(x_{-i})$ for an integer $l_i \in \mathbb{N}$, such that

   \[
   X_i^{\text{conv}}(x_{-i}) = \left\{ x_i \in \mathbb{R}^{k_i} \mid M_i(x_{-i})x_i \geq e_i(x_{-i}) \right\} \quad \text{for all } x_{-i} \in \text{rdom}X_i.
   \]

2. There exists a vector-valued function $C_i : \text{rdom}X_i \rightarrow \mathbb{R}^{k_i}$, $x_{-i} \mapsto C_i(x_{-i})$ such that

   \[
   \phi_i(x_i, x_{-i}) = C_i(x_{-i})^\top x_i \quad \text{for all } (x_i, x_{-i}) \in \mathbb{R}^{k_i} \times \text{rdom}X_i.
   \]

**Theorem 3.** Let $I$ be a quasi-linear GNEP with an instance $I^{\text{conv}} \in \mathcal{I}^{\text{conv}}$ as described in Definition \(\mathbb{U}\). Every optimal solution $(x, \nu)$ of \(\mathbb{R}\) with value zero and $\phi_i(x) = \pi_i(x), i \in N$ corresponds to a GNE $x$ of $I$ and vice versa.

\[
\inf_{x, \nu} \sum_{i \in N} C_i(x_{-i})^\top x_i - \sum_{i \in N} e_i(x_{-i})^\top \nu_i \quad \text{(R)}
\]

s.t.: $\nu_i^\top M_i(x_{-i}) = C_i(x_{-i})^\top$ for all $i \in N$,

$\nu_i \in \mathbb{R}_{\geq 0}^{l_i}$, $x_i \in X_i(x_{-i})$ for all $i \in N$.

**Proof.** Consider for an arbitrary but fixed $x \in X(x)$ the function $\hat{V}^{\text{conv}}(x)$ where the latter is again the $\hat{V}$ function corresponding to $I^{\text{conv}}$. From the proof of Theorem \(\mathbb{U}\) we already know:

\[
\hat{V}^{\text{conv}}(x) = \sum_{i \in N} \phi_i(x) - \sum_{i \in N} \inf_{y_i \in X_i^{\text{conv}}(x_{-i})} \phi_i(y_i, x_{-i}) = \sum_{i \in N} \phi_i(x) - \sum_{i \in N} \inf_{y_i \in X_i^{\text{conv}}(x_{-i})} C_i(x_{-i})^\top y_i.
\]
By using Definition 6.1, we arrive at the following linear optimization problem \( \text{LP}_i(x_{-i}) \) with its corresponding dual \( \text{DP}_i(x_{-i}) \) for the optimization problem of player \( i \) in the convexified game for the rivals’ strategies \( x_{-i} \):

\[
\begin{align*}
\inf_{x_{-i}} C_i(x_{-i})^\top y_i \\
\text{s.t.: } M_i(x_{-i})y_i \geq e_i(x_{-i}), \quad (\text{LP}_i(x_{-i}))
\end{align*}
\]

\[
\begin{align*}
\sup_{\nu_i} e_i(x_{-i})^\top \nu_i \\
\text{s.t.: } \nu_i^\top M_i(x_{-i}) = C_i(x_{-i})^\top, \quad (\text{DP}_i(x_{-i}))
\end{align*}
\]

\[\nu_i \in \mathbb{R}^l_{\geq 0}.\]

Note that \( \text{LP}_i(x_{-i}) \) attains its minimum if and only if the problem is bounded from below. In this case, we get by linear programming duality that the dual attains its maximum and their optimal objective values coincide. In the following let us denote by \( \text{DP}_i(x_{-i}) \) and \( \text{LP}_i(x_{-i}) \) also the corresponding optimal objective values with the convention that \( \text{DP}_i(x_{-i}) = -\infty \) if \( \text{LP}_i(x_{-i}) \) has no feasible solution. By this convention and the above argument, we have \( \text{DP}_i(x_{-i}) = \text{LP}_i(x_{-i}) \) which allows us to reformulate \( \hat{V}^\text{conv}(x) \) as:

\[
\hat{V}^\text{conv}(x) = \sum_{i \in N} \phi_i(x) - \sum_{i \in N} \text{LP}_i(x_{-i}) = \sum_{i \in N} \phi_i(x) - \sum_{i \in N} \text{DP}_i(x_{-i}).
\]

Since the \( n \) maximization problems \( \text{DP}_i(x_{-i}) \) are completely separable we can combine them to one maximization problem. Hence, by applying the representation of the convex envelopes from Definition 6.2, we can describe \( \hat{V} \) via the following optimization problem

\[
\begin{align*}
\inf_{\nu} \sum_{i \in N} C_i(x_{-i})^\top x_i - \sum_{i \in N} e_i(x_{-i})^\top \nu_i \\
\text{s.t.: } \nu_i^\top M_i(x_{-i}) = C_i(x_{-i})^\top \\
\quad \nu_i \in \mathbb{R}^l_{\geq 0}
\end{align*}
\]

(8) together with the property that \( \hat{V}(x) < \infty \) if and only if (S) attains its minimum at some \( \nu \) with the objective value \( \hat{V}(x) \). This allows us to relate to each \( x \in X(x) \) with \( \hat{V}(x) < \infty \) a feasible solution \( (x, \nu) \) of (R) with \( \nu \) being optimal for (S) and objective value equal to \( \hat{V}(x) \) and vice versa. Hence, any optimal solution \( (x, \nu) \) of (R) corresponds to an optimal solution \( x \) of (7) with the same objective value. Conversely, an optimal solution \( x \) of (7) with \( \hat{V}(x) < \infty \) can be identified with an optimal solution \( (x, \nu) \) of (R). Thus, the claim of the theorem follows by Corollary 4.

Theorem 3 is in particular interesting for quasi-linear GNEPs \( I \) in which the conditions \( x \in X(x) \) and \( \phi_i(x) = \pi_i(x), i \in N \) are computationally tractable. Such a situation is present in various interesting applications where instances \( I \) are used which belong to the class of what we call player-linear mixed-integer GNEPs.

**Definition 7.** An instance \( I \) belongs to the player-linear mixed-integer GNEPs, if for every \( i \in N \) the strategy space and cost functions are described by

\[
X_i(x_{-i}) = \left\{ x_i \in \mathbb{Z}^{s_i} \times \mathbb{R}^{k_i-s_i} \mid \tilde{M}_i(x_{-i})x_i \geq \tilde{e}_i(x_{-i}) \right\} \quad \text{for all } x_{-i} \in \text{rdom} X_i
\]

\[
\pi_i(x_i, x_{-i}) = \tilde{C}_i(x_{-i})^\top x_i \quad \text{for all } (x_i, x_{-i}) \in \mathbb{R}^{k_i} \times \text{rdom} X_i
\]

for a matrix-valued function \( \tilde{M}_i : \text{rdom} X_i \to \mathbb{R}^{l_i \times k_i} \) and vector-valued functions \( \tilde{e}_i : \text{rdom} X_i \to \mathbb{R}^{l_i}, x_{-i} \mapsto \tilde{e}_i(x_{-i}) \) and \( \tilde{C}_i : \text{rdom} X_i \to \mathbb{R}^{k_i}, x_{-i} \mapsto \tilde{C}_i(x_{-i}) \) for some \( s_i \in [k_i], l_i \in \mathbb{N} \).

Note that for a player-linear mixed-integer GNEP \( I \), the convex hull of the set \( X_i(x_{-i}) \) is a polytope (cf. Conforti et al. 11) for any \( x_{-i} \in \text{rdom} X_i \). Thus, the existence of a convexified instance fulfilling Definition 6.1 is guaranteed. The same holds true for Definition 6.2 since
for all $x_{-i} \in \text{rdom}X_i$ the convex envelope $\phi_i^{x_{-i}}(\cdot)$ of $\pi_i(\cdot, x_{-i})$ : $\text{conv}(X_i(x_{-i})) \rightarrow \mathbb{R}, x_i \mapsto \pi_i(x_i, x_{-i}) = \tilde{C}_i(x_{-i})^\top x_i$ is just $\pi_i(\cdot, x_{-i})$ due to the linearity of the latter. Therefore any player-linear mixed-integer GNEP is a quasi-linear GNEP and thus automatically fulfills $\mathcal{I}^{\text{conv}} \neq \emptyset$. Theorem 3 yields the following corollary.

**Corollary 2.** Let $I$ be a player-linear mixed-integer GNEP with an instance $I^{\text{conv}} \in \mathcal{I}^{\text{conv}}$ as described in Definition 4. Then, every optimal solution $(x, \nu)$ with value zero of the mixed-integer optimization problem (R) corresponds to a GNE $x$ of $I$ and vice versa.

Proof. The argumentation previous to the corollary shows that $\phi_i(x) = \pi_i(x), i \in N$ holds for any $x \in X(x)$. Thus, the corollary follows immediately by Theorem 3 and the fact that

$$\inf \sum_{i \in N} C_i(x_{-i})^\top x_i - \sum_{i \in N} e_i(x_{-i})^\top \nu_i$$

s.t.: $M_i(x_{-i}) x_i \geq e_i(x_{-i})$ for all $i \in N$,

$$\nu_i^\top M_i(x_{-i}) = C_i(x_{-i})^\top$$

for all $i \in N$, $\nu_i \in \mathbb{R}^i_{\geq 0}$, $x_i \in \mathbb{Z}^s_i \times \mathbb{R}^{k_i-s_i}$ for all $i \in N$.

Even for player-linear mixed-integer GNEPs, the computation of the matrix- and vector-valued functions $M_i, e_i, i \in N$ for a convexified instance $I^{\text{conv}}$ as described in Definition 4 may in general be quite complex. An exception to that is present in what we call hole-free GNEPs, cf. [32] for a similar concept in the realm of discrete convexity. Here we can use for all $i \in N$ the $I$-defining matrix- and vector-valued functions $\tilde{M}_i$ and $\tilde{e}_i$ instead of $M_i$ and $e_i$ in the above optimization problem (R). The key point of these hole-free represented GNEPs is the property that the strategy set $X_i(x_{-i})$ of a player $i$ is perfectly described for relevant strategies $x_{-i} \in \text{rdom}X_i$ in the sense that their convex hull coincides with their relaxation. Hence, the continuous relaxation of a hole-free instance $I$ is a convexified instance in $\mathcal{I}^{\text{conv}}(I)$.

**Definition 8.** We call a player-linear mixed-integer GNEP hole-free-represented, if for all $i \in N$ and $x_{-i} \in \text{rdom}X_i$, the following equality holds:

$$\text{conv} \left( \left\{ x_i \in \mathbb{Z}^s_i \times \mathbb{R}^{k_i-s_i} \mid M_i(x_{-i}) x_i \geq \tilde{e}_i(x_{-i}) \right\} \right) = \left\{ x_i \in \mathbb{R}^{k_i} \mid \tilde{M}_i(x_{-i}) x_i \geq \tilde{e}_i(x_{-i}) \right\}.$$  (10)
Note that the equality (10) does only need to hold for \( x_{-i} \in \text{rdom} X_i \) but not necessarily for all \( x_{-i} \) in the (potentially substantial) bigger set \( \text{dom} X_i \); as the following example illustrates.

**Example 2** (Capacitated Discrete Flow Games (CDFG)). We consider a directed graph \( G = (V, E) \) with nodes \( V \) and edges \( E \). There is a set of players \( N = \{1, \ldots, n\} \) where each \( i \in N \) is associated with an end-to-end pair \((s_i, t_i)\) \( \in V \times V \) as well as an individual constraint-vector \( c_i \in Z^E_{\geq 0} \). The strategy \( x_i \) of a player \( i \in N \) represents an integral \( s_i, t_i \)-flow with a flow value equal to her demand \( d_i \in \mathbb{N} \). Hereby, a player is restricted in her strategy choice by her capacity constraints, i.e., for given rivals’ strategies \( x_{-i} \), her flow \( x_i \) has to satisfy the restriction \( x_i \leq c_i - \sum_{s \neq i} x_s \). Thus the strategy set of a player \( i \in N \) is described by

\[
X_i(x_{-i}) = X'_i \cap \{ x_i \in Z^E_{\geq 0} | x_i \leq c_i - \sum_{s \neq i} x_s \} \quad \text{for all} \quad x_{-i} \in \mathbb{R}^{k-i},
\]

where \( X'_i = \{ x_i \in Z^E_{\geq 0} | Ax_i = b_i \} \) is the flow polyhedron of player \( i \) with \( A \) the arc-incidence matrix of the graph \( G \) and \( b_i \) the vector with \((b_i)_{s_i} = d_i, (b_i)_{t_i} = -d_i \), and zero, otherwise. Remark that the (standard) domain \( X_i \) is given by all (not necessarily integral) \( x_{-i} \in \mathbb{R}^{k-i} \) such that the intersection in (11) is nonempty. In contrast, by \( X_i(x_{-i}) \subseteq Z^E \), we may deduce that \( \text{rdom} X_i \subseteq \text{dom} X_i \cap Z^{k-i} \).

We define the cost functions by

\[
\pi_i(x_i, x_{-i}) := (\sum_{j \neq i} x_j)^\top C_i^1 x_i + C_i^2 x_i = (\sum_{j \neq i} x_j)^\top C_i^1 + C_i^2 x_i
\]

with \( C_i^1 \in \mathbb{R}^{E \times E} \) and \( C_i^2 \in \mathbb{R}^E \). Here, the first term can be interpreted as costs that arise through congestion whereas the second term represents congestion independent costs for player \( i \).

Clearly, the CDFG belongs to the player-linear mixed-integer GNEPs. Furthermore, it is hole-free-represented. To see this, it is sufficient to verify that for all \( x_{-i} \in \text{rdom} X_i \) the inclusion

\[
\{ x_i \in Z^E_{\geq 0} | Ax_i = b_i, x_i \leq c_i - \sum_{s \neq i} x_s \} \subseteq \text{conv}\left( \{ x_i \in Z^E_{\geq 0} | Ax_i = b_i, x_i \leq c_i - \sum_{s \neq i} x_s \} \right)
\]

holds as \( \supseteq \) is trivially fulfilled. Since \( \text{rdom} X_i \subseteq Z^{k-i} \), the restriction \( 0 \leq x_i \leq c_i - \sum_{s \neq i} x_s \) is an integral box-constraint for any \( x_{-i} \in \text{rdom} X_i \). Thus the polytope on the left has integral vertices since the flow polyhedron is box-tdi, see Edmonds and Giles [20] and Schrijver [45] for a definition of box-tdi and the aforementioned property of the flow polyhedron. These integral vertices are clearly contained in the right set and therefore the inclusion follows. Notice that for non-integral \( x_{-i} \), the inclusion \( \subseteq \) is in general false. Hence, the inclusion \( \subseteq \) is in general not true for all \( x_{-i} \in \text{dom} X_i \).

Let us motivate the hole-free GNEPs with another example regarding discrete market equilibria.

**Example 3** (Equilibria in Transportation Markets). Using the same terminology as in the previous Example 2 consider the situation in which the edges \( E \) are up for sale and each player wants to buy a single \( s_i, t_i \)-path \( x_i \) with the goal to maximize her linear utility \( U_i(x_i) = C_i^1 x_i, C_i \in Z^E \). The market manager wants to determine an integral price vector \( p \in Z^E_{\geq 0} \) for selling the edges such that every player receives a \( s_i, t_i \)-path \( x_i^* \in X'_i \) maximizing her quasi-linear utility \( x_i^* \in \arg\max_{x_i \in X'_i} \{ U_i(x_i) - p^\top x_i \} \) and unsold edges have prices equal zero. The tuple \( (x_i^*|_{i \in N}, p) \) is known as a competitive equilibrium, cf. e.g. [4].

We can model this situation as a GNEP with \( n + 1 \) players in which the first \( n \) players correspond to the \( n \) buyers and the \( n + 1 \)-th player to the market manager. We denote by \((x, p)\) a strategy profile and set the costs to the negated utility \( \pi_i(x_i, x_{-i}, p) = -(U_i(x_i) - p^\top x_i) \) for \( i \in [n] \).
and the costs of the market manager to \( \pi_{n+1}(p, x) = (1 - \sum_{i \in [n]} x_i) p \) with \( 1 = (1, \ldots, 1)^\top \in \mathbb{R}^E \).
For the strategy spaces we set \( X_i(x_{-i}, p) \equiv X_i^L \) to the flow polyhedron and \( X_{n+1}(x) := \mathbb{Z}^E_0 \) if \( \sum_{i \in [n]} x_i \leq 1 \) and \( X_{n+1}(x) := \emptyset \) else. It is straightforward to verify that any GNE of this GNEP corresponds to a competitive equilibrium and vice versa. Furthermore, it follows from the observations in Example 2 that the GNEP is a hole-free linear mixed-integer GNEP.

We conclude this section with another consequence of Theorem 3. Namely, under certain assumptions, the restriction of \( x \in X(x) \) in (7) may be obsolete and can be relaxed to \( x \in X^{\text{conv}}(x) \) as the feasibility of an optimal solution for the original game is a priori ensured. Such a case is described in the following corollary where additionally, the existence of generalized Nash equilibria of the instance \( I \) can be determined by solving a convex optimization problem. For the promised corollary, we need the following definition:

**Definition 9.** Let \( l \in \mathbb{N} \) and \( M \subseteq \mathbb{R}^l \) be arbitrary. We denote by \( E(M) \) the set of all extreme points of \( M \):

\[
E(M) := \{ x \in M \mid x \notin \text{conv}(M \setminus \{ x \}) \}.
\]

**Corollary 3.** Let \( I \) be a quasi-linear GNEP with an instance \( I^{\text{conv}} \in \mathcal{I}^{\text{conv}} \) as described in Definition 6., where the functions \( C_i(x_{-i}) \equiv C_i \) and \( e_i(x_{-i}) \equiv e_i \) are both constant in \( x_{-i} \) for all \( i \in N \). Furthermore, assume that

\[
F := \{(x, \nu) \in \mathbb{R}^{\sum k_i + l_i} \mid M_i(x_{-i})x_i \geq e_i, \nu_i^\top M_i(x_{-i}) = C_i^\top, i = 1, \ldots, n \}
\]

is convex and any \( (x, \nu) \in E(F) \) satisfies \( x \in X(x) \). Then, \( I \) has a generalized Nash equilibrium if and only if the following convex optimization problem has the optimal value 0.

\[
\inf_{x \in X(x)} \sum_{i \in N} C_i^\top x_i - e_i^\top \nu_i \quad \text{s.t.:} \quad (x, \nu) \in F
\]

**Proof.** Since \( I^{\text{conv}} \in \mathcal{I}^{\text{conv}}(I^{\text{conv}}) \), the latter is itself a quasi-linear GNEP. Thus, the optimization problem in (11) for \( I^{\text{conv}} \) and \( I^{\text{conv}} \in \mathcal{I}^{\text{conv}}(I^{\text{conv}}) \) instead of \( I \) and \( I^{\text{conv}} \) equals (12). The result then follows by Theorem 2 and 3 the equality \( \phi_i(x) = \pi_i(x), i \in N \) for all \( x \in X(x) \) as well as the fact that the optimization problem in (12) attains its minimum (if it exists) at an extreme point of \( F \) as the set \( F \) is convex and the objective function is linear. \( \square \)

### 3.2. Hole-free Linear Mixed-Integer GNEPs

Besides the approach described in Corollary 2 to compute equilibria of a player-linear mixed-integer GNEP, let us mention in the following another possibility for the special case of hole-free linear mixed-integer GNEPs, that is, hole-free player-linear mixed-integer GNEPs where \( M_i, C_i, i \in N \) are constant and \( e_i \) is linear. For this special class, the linear relaxation is not only a convexified instance but also belongs to the class of linear (continuous and convex) GNEPs. As mentioned in the introduction, Dreves [16] introduced an algorithm for linear GNEPs which computes the whole solution set in a finite amount of time. Hence, our convexification result in Theorem 2 shows that one can determine the whole solution set of the original instance \( I \) by applying Dreves’ algorithm to the linear relaxation, computing the whole solution set of the latter and determining all originally feasible solutions by re-administering the mixed-integer conditions. Note that the CDFG described above for the case of \( C_i^1 = 0 \in \mathbb{R}^{m \times m} \) belongs for example to the class of hole-free linear mixed-integer GNEPs.

### 4. Jointly Constrained GNEPs

In several interesting applications, the players’ strategy sets are restricted by coupled constraints.
**Definition 10.** We call an instance $I$ jointly constrained w.r.t. $X \subseteq \mathbb{R}^k$ if for all $i \in N$ and $x_{-i} \in \mathbb{R}^{k-1}$, the strategy set $X_i(x_{-i})$ has the following description:

$$X_i(x_{-i}) = \left\{ x_i \in \mathbb{R}^{k_i} \mid (x_i, x_{-i}) \in X \right\}.$$  

Notice that the joint restriction set $X \subseteq \mathbb{R}^k$ doesn’t need to be convex and may be discrete. This type of GNEP occurs for example in the domain of automated driving [21], traffic control [10] or transportation problems [44]. Before we investigate the structure of any convexified game $I_{\text{conv}} \in \mathcal{I}_{\text{conv}}$ of a jointly constrained GNEP, let us motivate this special type of GNEP further by the following example.

**Example 4 (Jointly Constrained Atomic Congestion Games).** We first describe the atomic (resource-weighted) congestion game, which is a generalization of the model of Rosenthal [39], without joint restrictions. The set of strategies available to player $i \in N = \{1, \ldots, n\}$ is given by $X_i \subseteq \times_{j \in E}(0, d_{ij})$ for weights $d_{ij} > 0$ and resources $j \in E = \{1, \ldots, m\}$. Note that by assuming $x_i \in \{0, 1\}^m$ for all $i \in N$, that is, $d_{ij} = 1$, we obtain the standard congestion game model of Rosenthal. The cost functions on resources are given by player-specific functions $c_{ij}(\ell(x))$, $j \in E$, $i \in N$, where $\ell(x) := \sum_{i \in N} x_i$. The private cost of a player $i \in N$ for strategy profile $x \in \prod_{i \in N} X_i$ is defined by $\pi_i(x_i, x_{-i}) := \sum_{j \in E} c_{ij}(\ell(x))x_{ij}$. This model can be generalized by allowing joint restrictions in the players’ strategy sets, that is, extending the above model to a jointly constrained GNEP with respect to a set $X \subseteq \prod_{i \in N} X_i$, e.g., if the usage of resources is bounded by hard capacities.

GNEPs with joint constraints were first studied in detail by Rosen in 1965 [38]. Since then, these GNEPs have been the object of a fairly intense study in the literature and became one of the best understood subclasses of the GNEP, see [22] for more details. Our goal is to identify properties of an instance $I$ so that a convexified instance $I_{\text{conv}} \in \mathcal{I}_{\text{conv}}$ exists that belongs to these well-understood jointly constrained/convex GNEPs. It seems quite natural to hope for a given jointly constrained instance $I$ that its convexification $\mathcal{I}_{\text{conv}}$ contains a jointly constrained GNEP. However, this is in general not the case as the example in Figure 2 illustrates.

![Figure 2: Example for a 2-player jointly constrained GNEP I w.r.t. $X \subseteq \mathbb{R}^{(1,1)}$ represented by the four black dots in the first picture. The prescribed strategy sets $X_{\text{prev}}(x_{-i}) = \text{conv}(X_i(x_{-i})), x_{-i} \in \text{rdom} X_i$ which rule out the possibility for $I_{\text{conv}}$ to be jointly constrained are represented in picture 2 and 3.](image)

In the above example, the instance $I$ is jointly constrained w.r.t. $X$, where any $I_{\text{conv}} \in \mathcal{I}_{\text{conv}}$ can not be jointly constrained w.r.t. some set $X_{\text{conv}}$. This becomes evident when one assumes that $I_{\text{conv}}$ would be jointly convex w.r.t. some set $X_{\text{conv}}$. Then $2 \in \text{rdom} X_1, \text{rdom} X_2$ and thus $(2, 2) \in \text{conv}(X_1(2)) \times 2 = X_{\text{conv}}^1(2) \times 2$ would imply $(2, 2) \in X_{\text{conv}}$ which contradicts $(2, 2) \notin \text{conv}(X_2(2)) \times 2 = X_{\text{conv}}^2(2) \times 2$.

As $X_{\text{conv}}^i(x_{-i})$ for $x_{-i} \notin \text{rdom} X_i$ is not a priori determined, only the prescribed strategy sets of player $i \in N$, $\text{Pres}_i := \bigcup_{x_{-i} \in \text{rdom} X_i} \text{conv}(X_i(x_{-i}))) \times x_{-i}$, may prohibit the possibility for $I_{\text{conv}}$ to contain jointly constrained instances as the example in Figure 2 illustrates. This leads to the
It is not hard to see that the equilibria of $\pi$ that is an instance $I$ as in Theorem 2 with constrained convexification which still falls under our main Theorem 2. One naive approach question whether or not we can adapt our convexification method in order to obtain a jointly constrained w.r.t. some set $X$ constrained $\pi$ exterms. We remark here that the cost functions $\phi$ as possible while putting as little effort as possible in the computation of the cost-functions reasonably in a computational regard as one wants as much regularity of the cost functions $\infty$ setting them to $+\infty$ otherwise. It is not hard to see that the equilibria of $I$ can be characterized by $I^{\text{ext}}$ in the same fashion as in Theorem 2 with $I^{\text{conv}}$. Yet this approach of extending $I^{\text{conv}}$ seems computationally of limited interest as the extended cost functions do not have any regularity. One may try to extend the cost functions in a original-equilibria-preserving and smooth manner instead of just setting them to $+\infty$ outside of $X^{\text{conv}}(x_{-i})$. Yet, it is not clear how to extend these functions reasonably in a computational regard as one wants as much regularity of the cost functions as possible while putting as little effort as possible in the computation of the cost-functions themselves. We remark here that the cost functions $\phi_i(x_i, x_{-i})$ of any convexified instance are by Definition only a priori determined on $\text{conv}(X_i(x_{-i}))$ for $x_{-i} \in \text{rdom}X_i$ and thus a similar problem as described above occurs w.r.t. the convexified cost functions $\phi_i$. But it is substantially easier to find any arbitrary smooth extension compared to finding a smooth extension which preserves original GNE. On top of that, the functions $\phi_i(\cdot, x_{-i}) : \text{conv}(X_i(x_{-i})) \to \mathbb{R}, x_{-i} \in \text{rdom}X_i$ may have a natural and smooth extension to the whole domain, as it is the case for most quasi-linear GNEPs for example. This gives rise to the question whether or not one can modify $I^{\text{conv}}$ by only extending the strategy spaces and, thus, without specifically tailoring the cost-functions to conserve original equilibria. However, one can quickly verify that this will in general lead to a loss of original GNE. To see this, let’s take a look back at the example in Figure 2. Assume that the cost function of player 2 is represented by $\phi_2(x_1, x_2) = -x_2$ on the whole $\mathbb{R}^2$. Let $I^{\text{ext}}$ be a jointly constrained extension of $I^{\text{conv}}$ as described above, but without changing the cost functions. Then $\{1, 2\} \subseteq X^{\text{ext}}(2)$ as $(1, 2), (2, 2) \in X^{\text{conv}}(2) \times 2 \subseteq X^{\text{ext}}$ and therefore the generalized Nash equilibrium $(x_1^*, x_2^*) = (2, 1)$ for $I$ would not remain a GNE for the extension $I^{\text{ext}}$.

As the example in Figure 2 shows, for general jointly constrained GNEPs $I$, there may exist $x_{-i} \in \text{rdom}X_i$ with a subsequently prescribed convexified strategy set $X^{\text{conv}}(x_{-i}) = \text{conv}(X_i(x_{-i}))$ which prohibits the possibility for $I^{\text{conv}}$ to contain a jointly constrained instance, showing that jointly constrainedness of $I$ is not sufficient for $I^{\text{conv}}$ to contain a jointly constrained instance. Perhaps surprising, the example in Figure 3 illustrates that it is also not a necessary condition.

![Figure 3](image-url)  

**Figure 3:** Example for a 2-player GNEP $I$ which is not jointly constrained but admits a jointly constrained convexified instance $I^{\text{conv}} \in I^{\text{conv}}$. The prescribed strategy sets are represented in picture 1 and 2 where the dots correspond to the original strategies. In the third picture is an example for a possible joint restriction set $X^{\text{conv}}$. 

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4.1. \( k \)-restrictive-closed GNEPs

The insights of the previous subsection raise the question which instances \( I \) admit jointly constrained instances in \( \mathcal{T}^{\text{conv}} \) and which do not. In order to answer this question we define some necessary concepts in the following.

**Definition 11.** For a vector \( k = (k_1, \ldots, k_n) \in \mathbb{N}^n \) and a set \( X \subseteq \mathbb{R}^k \), we

- define for \( i \in [n] \), \( x_{-i} \in \mathbb{R}^{k-i} \) the restriction \( \text{res}(X, x_{-i}) := \{ \hat{x} \in X \mid \hat{x} = (\hat{x}_i, x_{-i}) \} \) of \( X \) w.r.t. \( x_{-i} \).
- say that \( X \) is \( k \)-convex, if for all \( i \in [n] \) and \( x_{-i} \in \mathbb{R}^{k-i} \), the restriction \( \text{res}(X, x_{-i}) \) is convex. Note that any convex set \( X \) is also \( k \)-convex as \( \text{res}(X, x_{-i}) = X \cap (\mathbb{R}^{k_i} \times x_{-i}) \) is the intersection of two convex sets in this case.
- define the \( k \)-convex hull of \( X \) as the smallest \( k \)-convex set that contains \( X \), that is, we define \( \text{conv}^k(X) := \bigcap \{ Z \subseteq \mathbb{R}^k \mid Z \text{ is } k\text{-convex}, X \subseteq Z \} \).

The concept of \( k \)-convex sets is a special case of so-called \( \mathcal{O} \)-convex sets (see e.g. [23]). A set in \( \mathbb{R}^d \) for some \( d \in \mathbb{N} \) is \( \mathcal{O} \)-convex, if its intersection with every \( \mathcal{O} \)-line is connected, where an \( \mathcal{O} \)-line is a 1-dimensional intersection of several \( \mathcal{O} \)-hyperplanes. The latter are in turn hyperplanes that are parallel to some hyperplane contained in the orientation set \( \mathcal{O} \subseteq \mathcal{P}(\mathbb{R}^d) \) which is a subset of the power set of \( \mathbb{R}^d \) containing hyperplanes.

The following lemma shows that \( k \)-convex sets are \( \mathcal{O} \)-convex sets for a certain orientation set \( \mathcal{O} \). In particular, for the special case of \( k = (1, 1) \in \mathbb{N}^2 \), \( k \)-convexity reduces to orthogonal convexity (in 2 dimensions), see e.g. [35].

**Lemma 1.** For a vector \( k = (k_1, \ldots, k_n) \in \mathbb{N}^n \), a set \( X \subseteq \mathbb{R}^k \) is \( k \)-convex if and only if it is \( \mathcal{O} \)-convex for the orientation set containing the hyperplanes of the form \( H_a = \{ x \in \mathbb{R}^k \mid a^\top x = 0 \} \subseteq \mathbb{R}^k \) with \( a = (a_{ij})_{i \in N, j \in [k_i]} \in \mathbb{R}^k \) having zero entries for all \( i \in N \) and corresponding \( j \in [k_i] \) except for one \( i^* \in N \), i.e.

\[
\mathcal{O} := \left\{ H_a \subseteq \mathbb{R}^k \mid \exists i^* \in N : a_{ij} = 0, i \neq i^*, j \in [k_i] \right\}.
\]

**Proof.** We first show that the set of \( \mathcal{O} \)-lines is given by

\[
\mathcal{O}\text{-lines} = \left\{ \{ \lambda \cdot (\hat{x}_i, 0_{-i}) + x \in \mathbb{R}^k \mid \lambda \in \mathbb{R} \} \mid i \in N, \hat{x}_i \in \mathbb{R}^{k_i}, x \in \mathbb{R}^k \right\},
\]

where \( 0_{-i} \in \mathbb{R}^{k_i} \) denotes the vector only consisting of zeros. For the inclusion \( \supseteq \), let \( i \in N \) and \( \hat{x}_i \in \mathbb{R}^{k_i} \) be arbitrary. Let \( A \in \mathbb{R}^{k_i-1 \times k_i} \) be a matrix with \( \ker(A) = \{ \lambda \hat{x}_i \mid \lambda \in \mathbb{R} \} \) and denote by \( A_j \) the \( j \)-th row (interpreted as a column vector). Then the hyperplanes of the form \( \{ x \in \mathbb{R}^k \mid (A_j, 0_{-i})^\top x = 0 \} \in \mathcal{O} \) for all \( j \in [k_i-1] \) are contained in \( \mathcal{O} \). Similarly, we have \( \{ x \in \mathbb{R}^k \mid e_{ij}^\top x = 0 \} \in \mathcal{O} \) for all \( l \in N, l \neq i \) and \( j \in [k_i] \) where \( e_{ij} \) denotes the standard basis vector with a 1 at the \( l \)-th position. Intersecting all these above mentioned hyperplanes results in

\[
\left( \bigcap_{j \in [k_i-1]} \{ x \in \mathbb{R}^k \mid (A_j, 0_{-i})^\top x = 0 \} \right) \cap \left( \bigcap_{l \in N, l \neq i} \left( \bigcap_{j \in [k_i]} \{ x \in \mathbb{R}^k \mid e_{ij}^\top x = 0 \} \right) = \{ \lambda (\hat{x}_i, 0_{-i}) \in \mathbb{R}^k \mid \lambda \in \mathbb{R} \}
\]

which shows \( \supseteq \) in (13) as \( \mathcal{O} \)-lines are lines that are parallel to some line constructible as the intersection of hyperplanes in \( \mathcal{O} \).

In order to show \( \subseteq \) in (13), let \( \bigcap_{s \in [L]} \mathcal{H}_s = \{ \lambda \hat{x} \in \mathbb{R}^k \mid \lambda \in \mathbb{R} \} \) for some hyperplanes \( \mathcal{H}_s \in \mathcal{O}, s \in [L] \) for a \( L \in \mathbb{N} \). By the definition of \( \mathcal{O} \), we can represent the intersection as the linear equation system \( \text{diag}(A_1, \ldots, A_n)x = 0 \) for a block diagonal matrix consisting of some
matrices $A_i \in \mathbb{R}^{L_i \times k_i}, i \in N$ with $\sum_{i \in N} L_i = L$. Thus, $\ker(A_i) = \{\lambda \tilde{x}_i \mid \lambda \in \mathbb{R}\}$ needs to hold for any $i \in N$ which shows that only one $i \in N$ may exist with $\tilde{x}_i \neq 0_{k_i}$ as otherwise the intersection of the hyperplanes $\bigcap_{\alpha \in L} \mathcal{H}_s$ would not be 1-dimensional. Thus, $\subseteq$ in (13) holds.

Now we are ready to prove the equivalence of $k$-convexity and $O$-convexity. We start with the only if direction. Let $X \subseteq \mathbb{R}^k$ be $k$-convex and let $\mathcal{L} := \{\lambda \cdot (\tilde{x}_i, 0_{-i}) + x \in \mathbb{R}^k \mid \lambda \in \mathbb{R}\}$ be an arbitrary $O$-line. Then $\mathcal{L} \cap X = \mathcal{L} \cap \text{res}(X, x_{-i})$ which shows by $X$ being $k$-convex that $\mathcal{L} \cap X$ is a convex set as it is the intersection of two convex sets and thus is in particular connected.

For the if direction, let $X$ be an $O$-convex set. Now assume for contradiction that there exists $x_{-i} \in \mathbb{R}^{k_{-i}}$ such that $\text{res}(X, x_{-i})$ is not convex, that is, there exist $x^1 := (x^1_i, x_{-i}), x^2 := (x^2_i, x_{-i}) \in \text{res}(X, x_{-i})$ and $\alpha \in (0, 1)$ with $x^\alpha := \alpha x^1 + (1 - \alpha) x^2 \notin \text{res}(X, x_{-i})$, i.e. $x^\alpha \notin X$. This contradicts that $X$ is $O$-convex as $\mathcal{L} := \{\lambda \cdot (x^1_i - x^2_i, 0_{-i}) + (x^2_i, x_{-i}) \in \mathbb{R}^k \mid \lambda \in \mathbb{R}\}$ is an $O$-line with $\mathcal{L} \cap X$ not being connected as $x^\alpha \in \mathcal{L}$ and $x^1, x^2 \in \mathcal{L} \cap X$ but $x^\alpha \notin X$. \hfill \qed

As already noted in the previous section, the jointly constrainedness of $I$ is not necessary in order for $\mathcal{I}^{\text{conv}}$ to contain a jointly constrained instance which leads to the following definition of what we call $k$-restrictive-closed $GNEPs$.

**Definition 12.** Let $I$ be an instance of the GNEP. We define the complete (relevant) strategy set of player $i \in N$ by $S_i := \bigcup_{x_{-i} \in \text{rdom}X_i} X_i(x_{-i}) \times x_{-i}$ and denote their union over all players by $S := \bigcup_{i \in N} S_i$. An instance $I$ is called $k$-restrictive-closed (w.r.t. the $\text{conv}^k$-operator), if for all $i \in N$ and $x_{-i} \in \text{rdom}X_i$, the following equality holds:

$$\text{conv}^k(\text{res}(S_i, x_{-i})) = \text{res}(\text{conv}^k(S), x_{-i}).$$

(14)

The above concept of $k$-restrictive-closedness requires that for fixed $x_{-i} \in \text{rdom}X_i$, the $k$-convex hull of the restriction of $S_i$ w.r.t. $x_{-i}$ is equal to the restriction of $\text{conv}^k(S)$ w.r.t. $x_{-i}$. Remark that in the special case of a jointly constrained instance $I$ w.r.t. a restriction set $X$, the complete strategy set of each player $i \in N$ and subsequently their union equals the restriction set $X$, i.e. $S_i = S = X$. Thus, in the case of jointly constrained instances, one can identify $k$-restrictive-closedness by only investigating the joint restriction set $X$ as (14) becomes a condition solely for $X$.

We note in the following proposition that the inclusion $\subseteq$ in (14) always holds.

**Proposition 1.** In Definition 12, the inclusion $\subseteq$ in (14) holds for all $i \in N, x_{-i} \in \text{rdom}X_i$.

**Proof.** For $i \in N$ and $x_{-i} \in \text{rdom}X_i$ arbitrary, we have by the definition of $\text{conv}^k(S)$ that the restriction $\text{res}(\text{conv}^k(S), x_{-i})$ is convex and therefore also $k$-convex. As $\text{res}(S_i, x_{-i}) \subseteq \text{res}(\text{conv}^k(S), x_{-i})$, the results follows by the definition of the $\text{conv}^k$-operator. \hfill \qed

The following theorem gives various equivalent characterizations of $k$-restrictive-closedness. The first two equivalences $[1] \iff [2]$ and $[1] \iff [3]$ are interesting as they give geometric interpretations of $k$-restrictive-closed $GNEPs$. For instance, by $[1] \iff [3]$ one can easily verify that the example in Figure 2 is not $k$-restrictive-closed, as $x_{-2} = x_1 := 2 \in \text{rdom}X_2$, yet the point $(x_1, x_2) := (2, 2)$ of the restriction $\text{res}([\text{Pre}_1, x_{-2}) = \{(2, 1), (2, 2)\}$ is not contained in $\text{res}([\text{Pre}_2, x_{-2}) = \text{res}([\text{Pre}_2, 2) = \{(2, 1)\}$. Similar, by $[1] \iff [2]$ one can immediately verify that the example in Figure 3 (see below) is not $k$-restrictive-closed as $E(\text{res}(\text{conv}^k(S), x_{-i})) = \{(3, 1), (3, 4)\} \nsubseteq S_i = X$ for $i = 2$ and $x_{-2} = x_1 := 3$.

The last equivalence $[1] \iff [4]$ will allow us to show in the subsequent Theorem 5 that $k$-restrictive-closed $GNEPs$ with nonempty $\mathcal{I}^{\text{conv}}$ are exactly the $GNEPs$ $I$ which admit a jointly constrained convexification $I^{\text{conv}} \in \mathcal{I}^{\text{conv}}$. 

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Theorem 4. Let \( I \) be an instance of the GNEP. Then, the following statements are equivalent:

1. \( I \) is \( k \)-restrictive-closed.
2. \( E \left( \text{res} (\text{conv}^k(S), x_{-i}) \right) \subseteq S_i \) for all \( i \in N \) and \( x_{-i} \in \text{rdom}X_i \).
3. \( \text{res} (\text{Pre}_i, x_{-j}) \subseteq \text{res} (\text{Pre}_j, x_{-j}) \) for all \( i, j \in N \) and \( x_{-j} \in \text{rdom}X_j \).
4. For all \( x \in \mathbb{R}^k \) and \( i \in N \) the following implication holds:

\[ x_i \in \text{conv}(X_i(x_{-i})) \land x_{-i} \in \text{rdom}X_i \Rightarrow \forall j \in N : x_j \in \text{conv}(X_j(x_{-j})) \lor x_{-j} \notin \text{rdom}X_j. \]

Proof. We will prove the theorem by showing the equivalences \( 1 \iff 2 \iff 3 \iff 4 \) in this order. Before doing so, we observe that the following inclusion is valid:

\[ \bigcup_{i \in N} \text{Pre}_i \subseteq \text{conv}^k(S). \] (15)

To see this, we remark that we have the following equations for the prescribed strategy sets \( \text{Pre}_i \) of player \( i \in N \):

\[ \text{Pre}_i := \bigcup_{x_{-i} \in \text{rdom}X_i} \text{conv}(X_i(x_{-i})) \times x_{-i} = \bigcup_{x_{-i} \in \text{rdom}X_i} \text{conv} (\text{res} (S_i, x_{-i})) \]

\[ = \bigcup_{x_{-i} \in \text{rdom}X_i} \text{conv}^k (\text{res} (S_i, x_{-i})) \]

where the last equality follows as for arbitrary \( i \in N \) and \( x_{-i} \in \mathbb{R}^{k-i} \), the equality

\[ \text{conv}^k (\text{res} (S_i, x_{-i})) = \text{conv} (\text{res} (S_i, x_{-i})) \] (16)

holds. Clearly, \( \subseteq \) holds in (16) since any convex set is also \( k \)-convex. To see that \( \supseteq \) holds, we argue as follows: For any \( k \)-convex set \( Z \supseteq \text{res} (S_i, x_{-i}) \), the set \( \text{res} (Z, x_{-i}) \) needs to be convex by definition. Since the latter also contains \( \text{res} (S_i, x_{-i}) \), the inclusion \( \text{conv} (\text{res} (S_i, x_{-i})) \subseteq \text{res} (Z, x_{-i}) \subseteq Z \) holds which shows \( \supseteq \) in (16).

Thus, the inclusion in (15) follows by Proposition 1 and the following representation of \( \text{conv}^k(S) \):

\[ \text{conv}^k(S) = \bigcup_{i \in N} \bigcup_{x_{-i} \in \mathbb{R}^k} \text{res} (\text{conv}^k(S), x_{-i}). \]

Now we are ready to prove the equivalences:

1. \( \iff 2 \): Let \( i \in N \) and \( x_{-i} \in \text{rdom}X_i \) be arbitrary.

2. \( \iff 2 \): By 1 together with the equality in (16), we get

\[ E \left( \text{res} (\text{conv}^k(S), x_{-i}) \right) = E \left( \text{conv} (\text{res} (S_i, x_{-i})) \right) \subseteq \text{res} (S_i, x_{-i}) \subseteq S_i. \]

3. \( \iff 2 \): The following implications hold:

\[ 2 \Rightarrow E \left( \text{res} (\text{conv}^k(S), x_{-i}) \right) \subseteq \text{res} (S_i, x_{-i}) \]

\[ \Rightarrow \text{conv} \left( E \left( \text{res} (\text{conv}^k(S), x_{-i}) \right) \right) \subseteq \text{conv} (\text{res} (S_i, x_{-i})) \]

\[ \Rightarrow \text{res} (\text{conv}^k(S), x_{-i}) \subseteq \text{conv}^k (\text{res} (S_i, x_{-i})) \]

where the last inclusion follows by (16) and the convexity of \( \text{res} (\text{conv}^k(S), x_{-i}) \). Since the inclusion \( \supseteq \) in the last line always holds by Proposition 1 the claim follows.
2 \Rightarrow 3: Assume for contradiction that there exists \(i,j \in N\), \(x_{-j} \in \text{rdom}X_j\) and a strategy profile \(x := (x_j, x_{-j}) \in \text{res(Pre}_i, x_{-j}) \setminus \text{res(Pre}_j, x_{-j})\). By (15), we know that \(\text{res(Pre}_i, x_{-j}) \subseteq \text{res(conv}^k(S), x_{-j})\). As 2 and \(S_j \subseteq \text{Pre}_j\) holds, we know \(x \notin E(\text{res(conv}^k(S), x_{-j}))\). Thus, there exists a convex combination \(x = \sum_{j=1}^L \lambda_j(x^s_j, x_{-j})\) with \((x^s_1, x_{-j}), \ldots, (x^s_L, x_{-j}) \neq x\) extreme points of \(\text{res(conv}^k(S), x_{-j})\) and \(\lambda \in \Lambda_L := \{\alpha \in \mathbb{R}^L_{\geq 0} \mid \sum_{k=1}^L \alpha_k = 1\}\) for some \(L \in \mathbb{N}\). By 2 we have for all \(s \in [L]\) that \((x^s_j, x_{-j}) \in S_j\), that is, \(x^s_j \in X_j(x_{-j})\). This in turn implies that \(x_j \in \text{conv}(X_j(x_{-j}))\) which together with \(x_{-j} \in \text{rdom}X_j\) shows that \(x \in \text{Pre}_j\) which contradicts our assumption.

2 \Leftarrow 3: We first show that \(\text{conv}^k(S) = \bigcup_{j \in N} \text{Pre}_j\) holds. By our observation (15), we know that \(\supseteq\) always holds. Thus it suffices to show that \(\bigcup_{j \in N} \text{Pre}_j\) is \(k\)-convex since \(S \subseteq \bigcup_{j \in N} \text{Pre}_j\). In order to do so we have to show that \(\text{res}(\bigcup_{j \in N} \text{Pre}_j, x_{-i})\) is convex for all \(i \in N\) and \(x_{-i} \in \mathbb{R}^{k-1}\). For the latter set we have:

\[
\text{res}(\bigcup_{j \in N} \text{Pre}_j, x_{-i}) = \bigcup_{j \in N} \text{res(Pre}_j, x_{-i}) = \bigcup_{j \in N} \text{res(Pre}_i, x_{-i}) = \text{res(Pre}_i, x_{-i})
\]

where the penultimate equality is valid by 3. Since \(\text{res(Pre}_i, x_{-i})\) is clearly convex by definition, the set \(\text{res}(\bigcup_{j \in N} \text{Pre}_j, x_{-i})\) is in fact convex. Therefore \(\text{conv}^k(S) = \bigcup_{j \in N} \text{Pre}_j\) holds which implies for all \(i \in N\) and \(x_{-i} \in \text{rdom}X_i\):

\[
E(\text{res}(\text{conv}^k(S), x_{-i})) = E(\text{res}(\bigcup_{j \in N} \text{Pre}_j, x_{-i})) = E(\text{res}(\text{Pre}_i, x_{-i})) \subseteq \text{res}(S_i, x_{-i})
\]

where the penultimate equality follows by (17) and the last equality by the definition of \(\text{Pre}_i, S_i\).

\(3 \Rightarrow 4\): Let \(x \in \mathbb{R}^k\) with \(x_i \in \text{conv}(X_i(x_{-i}))\) and \(x_{-i} \in \text{rdom}X_i\). In what follows, let \(j \in N\) be arbitrary and assume \(x_{-j} \in \text{rdom}X_j\). Since 3 and \(x \in \text{res(Pre}_i, x_{-j})\) hold, it follows that \(x \in \text{res(Pre}_j, x_{-j})\) which implies \(x_j \in \text{conv}(X_j(x_{-j}))\).

\(3 \Leftarrow 4\): Assume for contradiction that there exists \(i,j \in N\), \(x_{-j} \in \text{rdom}X_j\) and a strategy profile \((\bar{x}_j, x_{-j}) \in \text{res(Pre}_i, x_{-j}) \setminus \text{res(Pre}_j, x_{-j})\). By \((\bar{x}_j, x_{-j}) \in \text{Pre}_i\) we may infer that \(x_i \in \text{conv}(X_i(\bar{x}_j, x_{-j}))\) and \((\bar{x}_j, x_{-j}) \in \text{rdom}X_i\). Subsequently by 4, \(\bar{x}_j \in \text{conv}(X_j(x_{-j}))\) since \(x_{-j} \in \text{rdom}X_j\). This implies \((\bar{x}_j, x_{-j}) \in \text{Pre}_j\) which contradicts our assumption. 

From the proof follows directly the following necessary condition for \(I\) to be \(k\)-restrictive-closed.

**Lemma 2.** If \(I\) is \(k\)-restrictive-closed, then the union of the prescribed strategy sets over all players equals the \(\text{conv}^k\)-hull of \(S\), i.e. \(\text{conv}^k(S) = \bigcup_{i \in N} \text{Pre}_i\).

However, note that the property \(\text{conv}^k(S) = \bigcup_{i \in N} \text{Pre}_i\) is not sufficient for \(I\) to be \(k\)-restrictive-closed as the example in Figure 4 shows.

**Theorem 5.** Let \(I\) be an instance of the GNEP with \(\mathcal{I}^\text{conv}(I) \neq \emptyset\). Then \(I\) is \(k\)-restrictive-closed if and only if the convexification \(\mathcal{I}^\text{conv}\) contains a jointly constrained instance. In this case, \(\mathcal{I}^\text{conv}\) contains for any set \(X^\text{conv}\) that satisfies

\[
\text{res}(X^\text{conv}, x_{-i}) = \text{res}(\text{conv}^k(S), x_{-i})
\]

for all \(i \in N\) and \(x_{-i} \in \text{rdom}X_i\),

a jointly constrained instance with \(X^\text{conv}\) as its joint restriction set.

**Proof.** We start with the only if direction. Let \(I^\text{conv}\) be a jointly constrained instance w.r.t. any restriction set \(X^\text{conv}\) as described above. Furthermore, let the cost functions of \(I^\text{conv}\) fulfill the requirements (Definition 32) for \(I^\text{conv}\) to belong to \(\mathcal{I}^\text{conv}\). Such cost functions exist due to \(\mathcal{I}^\text{conv}(I) \neq \emptyset\). We want to prove that \(I^\text{conv} \in \mathcal{I}^\text{conv}\). Thus, we have to show that the condition of Definition 31 for arbitrary \(i \in N\) and \(x_{-i} \in \text{rdom}X_i\) is fulfilled, that is:

\[
X^\text{conv}_i(x_{-i}) := \left\{ x_i \in \mathbb{R}^k_i \mid (x_i, x_{-i}) \in X^\text{conv} \right\} = \text{conv}(X_i(x_{-i}))
\]
Figure 4: A 2-player jointly constrained GNEP \( I \) w.r.t. \( X \subseteq \mathbb{R}^k, k := (1,1) \) represented by the black set in the first picture. In picture 2 the union of the prescribed strategy sets is represented which equals the \( k \)-convex hull (and even the regular convex hull) of \( X \). Yet, for \( i = 2 \) and \( x_{-2} = x_1 := 3 \) we have \( \text{conv}^k(\text{res}(X, x_{-i})) = \{(3,1)\} \subseteq \{3\} \times [1,4] = \text{res}(\text{conv}^k(X), x_{-i}) \). Remember that \( X = S_i = S \) holds for any \( i \in N \) in a jointly constrained instance.

which is equivalent to \( \text{res}(X^\text{conv}, x_{-i}) = \text{conv}(\text{res}(S_i, x_i)) \). The latter equality is valid due to \( I \) being \( k \)-restrictive-closed and \( X^\text{conv} \) fulfilling \( \{18\} \).

For the if direction, let \( I^\text{conv} \in \mathcal{I}^\text{conv} \) be jointly constrained w.r.t. \( X^\text{conv} \). Then for any \( x \in \mathbb{R}^k \) with \( x_i \in \text{conv}(X_i(x_{-i})) \) and \( x_{-i} \in \text{rdmx}_i \) for some \( i \in N \) we have \( x_i \in X^\text{conv}_i(x_{-i}) \) and by the jointly constrainedness of \( I^\text{conv} \) that \( (x_i, x_{-i}) \in X^\text{conv} \). Again by the jointly constrainedness of \( I^\text{conv} \), we get for all \( j \in N \) that \( x_j \in X^\text{conv}_j(x_{-j}) \) which implies that \( x_j \in \text{conv}(X_j(x_{-j})) \) if \( x_{-j} \in \text{rdmx}_j \). Therefore Theorem 4(4) \( \Rightarrow \{1\} \) shows that \( I \) is \( k \)-restrictive-closed.

Figure 5: Example for a 2-player jointly constrained GNEP \( I \) w.r.t. a \((1,1)\)-restrictive-closed \( X \subseteq \mathbb{R}^{(1,1)} \) represented by the four black dots in the first picture. In picture 2 and 3 are two suitable choices of \( X^\text{conv} \) which fulfill \( \{18\} \).

4.2. Restrictive-closed GNEPs

Jointly constrained GNEPs w.r.t. a convex restriction set are often referred to as jointly convex in the literature and constitute one of the best understood subclasses of the GNEP. Thus, for \( k \)-restrictive-closed GNEPs, the question arises whether or not we can chose a convex restriction set \( X^\text{conv} \). However, for general \( k \)-restrictive-closed GNEPs \( I \), there may exist \( x_{-i} \in \text{rdmx}_i \) with a subsequently prescribed convexified strategy set \( X^\text{conv}_i(x_{-i}) = \text{conv}(X_i(x_{-i})) \) which prohibit the possibility for \( I^\text{conv} \) to be jointly convex w.r.t. some convex set \( X^\text{conv} \). An example for such a situation is given in Figure 4 by \( x_{-2} := 2 \in \text{rdmx}_2 \) and \( X^\text{conv}_2(2) \). To see this, assume \( I^\text{conv} \in \mathcal{I}^\text{conv} \) was jointly convex w.r.t. some convex set \( X^\text{conv} \). The union of the complete strategy sets of the players in the convexified instance \( S^\text{conv} \) is then given by \( S^\text{conv} = X^\text{conv} \). Since \( X^\text{conv} \) is convex and \( S \subseteq S^\text{conv} \) clearly holds, we even have \( \text{conv}(S) \subseteq X^\text{conv} \). Thus, we
get:

\[ X_2^{\text{conv}}(2) := \text{conv}(X_2(2)) = \{2\} \]
\[ \subseteq [2, 8/3] = \{x \in \mathbb{R} \mid (2, x) \in \text{conv}(S)\} \]
\[ \subseteq \{x \in \mathbb{R} \mid (2, x) \in X^{\text{conv}}\} \]

by \( I^{\text{conv}} \in \mathcal{I}^{\text{conv}} \)

by \( \text{conv}(S) \subseteq X^{\text{conv}} \)

which contradicts that \( I^{\text{conv}} \) is jointly convex w.r.t. \( X^{\text{conv}} \).

In the previous section, we identified the restrictive-closedness for \( I \) w.r.t. the \( k \)-convex hull operator as the characterizing property for \( I^{\text{conv}} \) to contain a jointly constrained instance. It turns out that the restrictive-closedness of \( I \) w.r.t. the standard convex hull operator is the characterizing property of \( I \) for \( I^{\text{conv}} \) to contain a jointly convex instance.

**Definition 13.** We call an instance \( I \) restrictive-closed (w.r.t. the convex hull operator), if for all \( i \in \mathbb{N} \) and \( x_{-i} \in \text{rdom}X_i \) the equality

\[ \text{conv}(\text{res}(S_i, x_{-i})) = \text{res}(\text{conv}(S), x_{-i}) \]

holds. Note that \( \subseteq \) always holds, cf. Proposition 7

The above concept of restrictive-closed sets requires that for fixed \( x_{-i} \in \text{rdom}X_i \), the convex hull of the restriction of \( S_i \) w.r.t. \( x_{-i} \) is equal to the restriction of \( \text{conv}(S) \) w.r.t. \( x_{-i} \).

By observing that we have for any \( i \in \mathbb{N} \) and \( x_{-i} \in \text{rdom}X_i \):

\[ \text{conv}(\text{res}(S_i, x_{-i})) = \text{conv}^k(\text{res}(S_i, x_{-i})) \subseteq \text{res}(\text{conv}^k(S), x_{-i}) \subseteq \text{res}(\text{conv}(S), x_{-i}) \]

due to (16), Proposition 4 and \( \text{conv}^k(S) \subseteq \text{conv}(S) \) respectively, we can immediately state the following necessary conditions for restrictive-closedness of \( I \).

**Lemma 3.** If \( I \) is restrictive-closed, then

1. \( I \) is \( k \)-restrictive-closed.

2. for all \( i \in \mathbb{N} \) and \( x_{-i} \in \text{rdom}X_i \) the equality \( \text{res}(\text{conv}^k(S), x_{-i}) = \text{res}(\text{conv}(S), x_{-i}) \) holds.

Note that the second necessary condition is not sufficient, not even for \( k \)-restrictive-closedness of \( I \) as the example in Figure 4 illustrates.

Similar to the previous section, we can give two equivalent characterizations of restrictive-closedness. The equivalence 1 \( \iff \) 2 is again a geometric interpretation of restriction-closed GNEPs. With the help of it, one can easily verify that the example in Figure 4 is not restrictive-closed, as \( x_{-2} = x_1 := 2 \in \text{rdom}X_2 \) but the restriction \( \text{res}(\text{conv}(X), x_{-2}) = \{2\} \times [2, 8/3] \)
The equivalence $\textbf{1} \Leftrightarrow \textbf{3}$ will allow us to show in the subsequent Theorem 4 that restrictive-closed GNEPs are exactly the GNEPs $I$ which admit a jointly convex convexification $I^{\text{conv}} \in I^{\text{conv}}$.

**Theorem 6.** Let $I$ be an instance of the GNEP. Then the following statements are equivalent:

1. $I$ is restrictive-closed.
2. $E \big( \text{res} \left( \text{conv}(S), x_{-i} \right) \big) \subseteq S_i$ for all $i \in N$ and $x_{-i} \in \text{rdom}X_i$.
3. For all $i \in N$ and $x_{-i} \in \mathbb{R}^{k-i}$ the following implication holds:

$$x_{-i} \in \text{rdom}X_i \Rightarrow \text{conv}(X_i(x_{-i})) = \{ x_i \in \mathbb{R}^{k_i} \mid (x_i, x_{-i}) \in \text{conv}(S) \}.$$ 

**Proof.** $\textbf{1} \Leftrightarrow \textbf{2}$ Let $i \in N$ and $x_i \in \text{rdom}X_i$ be arbitrary.

$\textbf{1} \Rightarrow \textbf{2}$ We get by the restrictive-closedness of $I$:

$$E \left( \text{res} \left( \text{conv}(S), x_{-i} \right) \right) = E \left( \text{conv} \left( \text{res} \left( S_i, x_{-i} \right) \right) \right) \subseteq \text{res} \left( S_i, x_{-i} \right) \subseteq S_i.$$ 

$\textbf{1} \Leftrightarrow \textbf{2}$ The following implications hold:

$$\textbf{2} \Rightarrow E \left( \text{res} \left( \text{conv}(S), x_{-i} \right) \right) \subseteq \text{res} \left( S_i, x_{-i} \right)$$

$$\Rightarrow \text{conv} \left( E \left( \text{res} \left( \text{conv}(S), x_{-i} \right) \right) \right) \subseteq \text{conv} \left( \text{res} \left( S_i, x_{-i} \right) \right)$$

$$\Rightarrow \text{res} \left( \text{conv}(S), x_{-i} \right) \subseteq \text{conv} \left( \text{res} \left( S_i, x_{-i} \right) \right)$$

where the last inclusion follows by the convexity of $\text{res} \left( \text{conv}(S), x_{-i} \right)$. Since the inclusion $\supseteq$ in the last line always holds, the claim follows.

$\textbf{1} \Leftrightarrow \textbf{3}$: Let $i \in N$ and $x_{-i} \in \text{rdom}X_i$ be arbitrary.

$\textbf{1} \Rightarrow \textbf{3}$: This follows immediately by the definition of restrictive-closedness:

$$X_i^{\text{conv}}(x_{-i}) \times x_{-i} = \text{conv} \left( X_i(x_{-i}) \right) \times x_{-i} = \text{conv} \left( \text{res} \left( S_i, x_{-i} \right) \right)$$

$\textbf{1}$ \hspace{1em}$\text{res} \left( \text{conv}(S), x_{-i} \right) = \{ x_i \in \mathbb{R}^{k_i} \mid (x_i, x_{-i}) \in \text{conv}(S) \} \times x_{-i}$.

$\textbf{1} \Leftrightarrow \textbf{3}$: We calculate:

$$\text{conv} \left( \text{res} \left( S_i, x_{-i} \right) \right) = \text{conv} \left( X_i(x_{-i}) \right) \times x_{-i}$$

$\textbf{3}$ \hspace{1em}$\{ x_i \in \mathbb{R}^{k_i} \mid (x_i, x_{-i}) \in \text{conv}(S) \} \times x_{-i} = \text{res} \left( \text{conv}(S), x_{-i} \right)$.

$\square$

**Theorem 7.** Let $I$ be an instance of the GNEP with $I^{\text{conv}}(I) \neq \emptyset$. Then $I$ is restrictive-closed if and only if the convexification $I^{\text{conv}}$ contains a jointly convex instance. In this case, $I^{\text{conv}}$ contains for any convex set $X^{\text{conv}}$ that satisfies

$$\text{res} \left( X^{\text{conv}}(x_{-i}) \right) = \text{res} \left( \text{conv}(S), x_{-i} \right) \quad \text{for all } i \in N \text{ and } x_{-i} \in \text{rdom}X_i,$$

a jointly convex instance with $X^{\text{conv}}$ as its restriction set.

**Proof.** We start with the only if direction. Let $I^{\text{conv}}$ be a jointly convex instance w.r.t. any restriction set $X^{\text{conv}}$ as described above. Furthermore, let the cost functions of $I^{\text{conv}}$ fulfill the requirements for $I^{\text{conv}}$ to belong to $I^{\text{conv}}$ which exist due to $I^{\text{conv}}(I) \neq \emptyset$. We want to prove
that $I^{\text{conv}} \in \mathcal{I}^{\text{conv}}$. Thus, we have to show that for arbitrary $i \in N$ and $x_{-i} \in \text{rdom}X_i$, the strategy set $X_i^{\text{conv}}(x_{-i})$ of $I^{\text{conv}}$ fulfills:

$$X_i^{\text{conv}}(x_{-i}) := \left\{ x_i \in \mathbb{R}^{k_i} \mid (x_i, x_{-i}) \in X^{\text{conv}} \right\} \supseteq \text{conv}(X_i(x_{-i}))$$

which is equivalent to $\text{res}(X^{\text{conv}}, x_{-i}) = \text{conv}(\text{res}(S_i, x_i))$. The latter equality is valid due to $I$ being restrictive-closed and $X^{\text{conv}}$ fulfilling (19).

For the if direction, let $I^{\text{conv}} \in \mathcal{I}^{\text{conv}}$ be jointly convex w.r.t. $X^{\text{conv}}$. Then for any $i \in N$ and $x_{-i} \in \text{rdom}X_i$, we have

$$\text{conv}(X_i(x_{-i})) =: X_i^{\text{conv}}(x_{-i}) = \left\{ x_i \in \mathbb{R}^{k_i} \mid (x_i, x_{-i}) \in X^{\text{conv}} \right\}$$

by $I^{\text{conv}} \in \mathcal{I}^{\text{conv}}$ and $I^{\text{conv}}$ being jointly convex w.r.t. $X^{\text{conv}}$. Furthermore the jointly constrainedness implies that $S^{\text{conv}} = X^{\text{conv}}$. As $S \subseteq S^{\text{conv}}$ we get $\text{conv}(S) \subseteq X^{\text{conv}}$ due to $X^{\text{conv}}$ being convex. Thus the equality in (20) implies

$$\text{conv}(X_i(x_{-i})) = \left\{ x_i \in \mathbb{R}^{k_i} \mid (x_i, x_{-i}) \in X^{\text{conv}} \right\} \supseteq \left\{ x_i \in \mathbb{R}^{k_i} \mid (x_i, x_{-i}) \in \text{conv}(S) \right\}.$$ 

The above inclusion is in fact an equality as the proof of Theorem [6] (11 = [3]) together with the fact that $\text{conv}(\text{res}(S_i, x_{-i})) \subseteq \text{res}(\text{conv}(S), x_{-i})$ always holds shows that the inclusion $\text{conv}(X_i(x_{-i})) \subseteq \left\{ x_i \in \mathbb{R}^{k_i} \mid (x_i, x_{-i}) \in \text{conv}(S) \right\}$ is also always fulfilled. Thus, Theorem [6] (11 = [3]) shows that $I$ is restrictive-closed.

![Figure 7: Example for a 2-player restrictive-closed, jointly constrained GNEP $I$ w.r.t. $X \subseteq \mathbb{R}^{(1,1)}$ represented by the four black dots in the first picture. In picture 2 and 3 are two suitable choices of $X^{\text{conv}}$. The sets that have to coincide after (19) are represented by the 4 red lines.](image)

### 4.3. Applications

In this subsection we show how several interesting game classes belong to the restrictive-closed GNEPs. In order to do so, we present in the following a sufficient condition for restrictive-closedness which requires the definition of pseudo jointly constrained GNEPs.

**Definition 14.** We call an instance $I$ pseudo jointly constrained, if for all $i \in N$ and $x_{-i} \in \text{rdom}X_i$ the strategy set $X_i(x_{-i})$ can be described as:

$$X_i(x_{-i}) = \left\{ x_i \in \mathbb{R}^{k_i} \mid (x_i, x_{-i}) \in S \right\}.$$

Remark that any jointly constrained instance w.r.t. a restriction set $X$ is obviously pseudo jointly constrained as $S_i = X = S$ holds for all $i \in N$. Subsequently, Figure [2] shows that not every pseudo jointly constrained GNEP is $k$-restrictive-closed. Similarly, not every $k$-restrictive-closed GNEP is pseudo jointly constrained as the example in Figure [3] illustrates.
Lemma 4. Let \( I \) be a pseudo jointly constrained GNEP. Then \( I \) is restrictive-closed, if the projection \( P_i(S):=\{x_i\in\mathbb{R}^k_i\mid \exists x_{-i}: (x_i, x_{-i})\in S\} \) of \( S \) to the strategy space \( \mathbb{R}^k_i \) of player \( i\in N \) only consist of extreme points, i.e. \( E(P_i(S)) = P_i(S) \).

Proof. Let \( i\in N \) and \( x_{-i}\in \text{rdom}X_i \) be arbitrary. We have to show that the equality in Definition 13 holds. As mentioned before, \( \subset \) always holds. For the other inclusion \( \supseteq \) we argue that the following steps are valid: res\( (\text{conv}(S), x_{-i}) \subseteq \text{conv}(\text{res}(S, x_{-i})) \subseteq \text{conv}(\text{res}(S, x_{-i})). \)

For the first inclusion, let \( (x_i, x_{-i})\in \text{res}(\text{conv}(S), x_{-i}) \). Then there exists a convex combination \( (x_i, x_{-i}) = \sum_{s=1}^L \lambda_s x_s \) with \( x_s \in S, s\in [L] \) for some \( L\in\mathbb{N} \) and \( \lambda \in \Lambda_L \). Since \( x_{-i}\in \text{rdom}X_i \) there exists a \( x_i^*\in\mathbb{R}^k_i \) with \( (x_i^*, x_{-i})\in X((x_i^*, x_{-i}))) \). Subsequently \( (x_i^*, x_{-i})\in S \) and \( x_j\in P_j(S) = E(P_j(S)) \) for all \( j\neq i \). Similarly, as \( x^*\in S \) we have \( x_j^*\in P_j(S) \) for all \( j\in N, s\in [L] \). Therefore \( x_j = \sum_{s=1}^L \lambda_s x_s \), \( j\in N \) implies \( x_j^* = x_j \) for all \( s\in [L], j\neq i \). Therefore \( x^* = (x_i^*, x_{-i})\in \text{res}(S, x_{-i}), s\in [L] \) which shows that \( x\in \text{conv}(\text{res}(S, x_{-i})). \)

The second equality is a direct consequence of \( I \) being pseudo jointly constrained as pseudo jointly constrainedness is equivalent to \( \text{res}(S, x_{-i}) = \text{res}(S_i, x_{-i}) \) for all \( i\in N, x_{-i}\in \text{rdom}X_i \).

We get as a direct consequence of the above lemma that all 0,1 pseudo jointly constrained games are restrictive-closed.

Corollary 4. If \( I \) is a pseudo jointly constrained GNEP with \( S\subseteq \{0,1\}^k \), then \( I \) is restrictive-closed.

Proof. The projection of \( S\subseteq \{0,1\}^k \) to the strategy space of any player \( i\in N \) is a subset of the hypercube \( \{0,1\}^k_i \) which only consists of extreme points. Thus \( E(P_i(S)) = P_i(S) \) holds.

Another interesting class of GNEPs which belongs to the restrictive-closed GNEPs are the jointly constrained discrete flow games described in the following lemma.

Lemma 5. Let \( I \) be an instance of the GNEP as described in Example 2 but instead of linear individual capacity constraints, we consider a joint restriction imposed by a convex function \( g: \mathbb{R}^k \to \mathbb{R}^s \) for some \( s\in\mathbb{N} \). More precisely, the strategy set of player \( i\in N \) is described by

\[
X_i(x_{-i}) = \left\{ x_i\in\mathbb{Z}^m_{\geq 0} \mid Ax_i = b_i, \ g(x_i, x_{-i}) \leq 0 \right\} \quad \text{for all } x_{-i}\in\mathbb{R}^{k_i-1}.
\]

Furthermore let

\[
X := \prod_{i\in N} \left\{ x_i\in\mathbb{Z}^m_{\geq 0} \mid Ax_i = b_i \right\} \cap \left\{ x\in\mathbb{R}_{\geq 0}^k \mid g(x) \leq 0 \right\}.
\] (21)

If for all \( i\in N \) and

\[
x_{-i}\in \{\tilde{x}_{-i}\in\mathbb{R}^{k_i-1} \mid \exists \tilde{x}_i\in\mathbb{R}^{k_i} : (\tilde{x}_i, \tilde{x}_{-i})\in X\},
\] (22)

the restriction \( g(x_i, x_{-i}) \leq 0 \) for \( x_i \) is equivalent to an integral box-constraint \( a^{x_{-i}}_i \leq x_i \leq b^{x_{-i}}_i \) with \( a^{x_{-i}}_i, b^{x_{-i}}_i \in \mathbb{Z}^{k_i} \), then \( I \) is restrictive-closed.

Proof. It is not hard to see that \( I \) is quasi-isomorphic to a GNEP \( I' = (N, (X'_i(\cdot))_{i\in N}, (\pi_i)_{i\in N}) \) which has the same cost functions as \( I \) and is jointly constrained w.r.t. \( X \). As for two quasi-isomorphic instances the respective complete (relevant) strategy sets for a player \( i\in N \) coincide \( S_i = S'_i \), we get \( S = X \). Furthermore we have that \( \text{rdom}X_i = \text{rdom}X'_i \) where the latter is given by the set in (22). To verify the restrictive-closedness, we show that \( S = S' = X \) fulfills the condition stated in Definition 13. Let \( i\in N \) and \( x_{-i}\in \text{rdom}X_i \). Since \( \subset \) always holds we just have to show the inclusion \( \supseteq \). To prove this, define the relaxation of \( X \) by

\[
\bar{X} := \prod_{i\in N} \left\{ x_i\in\mathbb{Z}^m_{\geq 0} \mid Ax_i = b_i \right\} \cap \left\{ x\in\mathbb{R}_{\geq 0}^k \mid g(x) \leq 0 \right\}.
\] (23)
We argue that the following two inclusions are valid. Clearly, they imply that $\supseteq$ in Definition 13 holds.

\[
\{ x_i \in \mathbb{R}^n \mid (x_i, x_{-i}) \in \text{conv}(X) \} \subseteq \left\{ x_i \in \mathbb{R}^n \mid (x_i, x_{-i}) \in \hat{X} \right\}
\]

(24)

\[
\subseteq \text{conv} \left( \left\{ x_i \in \mathbb{R}^n \mid (x_i, x_{-i}) \in X \right\} \right).
\]

(25)

By the definition of $X$ it follows immediately that $X \subseteq \hat{X}$. Since $\hat{X}$ is convex, the inclusion $\text{conv}(X) \subseteq \hat{X}$ and thus also the inclusion (24) holds. By rewriting the sets for the inclusion (25) via the definition of $X$ and $\hat{X}$, we get equivalently:

\[
\{ x_i \in \mathbb{R}^m \mid Ax_i = b_i, \ g(x_i, x_{-i}) \leq 0 \} \subseteq \text{conv} \left( \left\{ x_i \in \mathbb{Z}_{\geq 0}^m \mid Ax_i = b_i, \ g(x_i, x_{-i}) \leq 0 \right\} \right).
\]

As $x_{-i} \in \text{rdom}X_i = \text{rdom}X_i'$ and thus fulfills (22), the restriction $g(x_i, x_{-i}) \leq 0$ is an integral box-constraint. Thus the polytope on the left has integral vertices since the flow polyhedron is box-tdi, cf. Example 2. These integral vertices are clearly contained in the right set and therefore the inclusion follows. Hence $I$ is restrictive-closed.

The above proof shows that the relaxed set $\hat{X}$ fulfills the equality stated in (19) and thus a convexified instance from $I^{\text{conv}}$ which is jointly convex w.r.t. $\hat{X}$ can be solved in order to derive insights into the original instance $I$ via our main Theorem 2. This is extremely convenient in a computational regard as $\text{conv}(X) \neq \hat{X}$ in general as the following instance of the CDFG shows. Note that the CDFG described in Example 2 belongs to the flow games introduced in Lemma 5.

![Figure 8: Example for a capacitated discrete flow game where $\text{conv}(X) \subseteq \hat{X}$](image)

**Example 5** (conv($X$) $\neq \hat{X}$ in general). Let $I$ be an instance of the CDFG where $N = \{1, 2\}$ and $G$ is given by the graph displayed in Figure 8. Both players have the same capacity $c_1 = c_2 = 1 \in \mathbb{R}^E$ and want to send one unit of flow. Then $X$ consists of only two elements, namely $X = \{(x^*_1, x^*_2), (x^*_1, x^*_2')\}$ where we denote by $x^*_1$ the flow sending one flow unit over the edge $(s_1, t_1)$ and by $x^*_2$ resp. $x^*_2'$ the unique path from $s_2$ to $t_2$ starting with the upper edge $c^*_2$ resp. lower edge $c^*_2$. The set $\hat{X}$ contains for example the point $\frac{1}{2} \cdot (x^*_1 + x^*_1, x^*_2 + x^*_2) \notin \text{conv}(X)$ where we define $x^*_1$ and $x^*_1$ analogously to $x^*_2$ and $x^*_2$, thus showing that conv($X$) $\subseteq \hat{X}$.

As another example for restrictive-closed GNEPs we revisit Example 4.

**Example 6** (continued). Assume that the weights $d_{ij} = 1$ are equal to one for all $i \in N$, $j \in E$. Then $X \subseteq \{0, 1\}^{n \cdot m}$ and thus by Corollary 4, we’re dealing with a restrictive-closed GNEP. Subsequently Theorem 7 shows that we can define $I^{\text{conv}} \in I^{\text{conv}}$ as a jointly convex GNEP w.r.t. any set $X^{\text{conv}}$ fulfilling (13). Concerning the cost functions of $I^{\text{conv}}$, we observe that the cost functions $\pi_i, i \in N$ are quasi-linear, i.e. they allow for convexified cost functions.
as in Definition 2. This is due to the fact that $X_i(x_{-i}) \to \mathbb{R}, x_i \mapsto \pi_i(x_i, x_{-i})$ is linear for all $x_{-i} \in \text{rdom}X_i$ and $i \in N$ which can be verified by the following description:

$$
\pi_i(x_i, x_{-i}) := \sum_{j \in E} c_{ij}(\ell_j(x))x_{ij} = c_i(1 + \sum_{j \neq i} x_j)^\top x_i \text{ for all } x_i \in X_i(x_{-i}) \subseteq \{0, 1\}^m
$$

Thus defining $\phi_i(x) := c_i(1 + \sum_{j \neq i} x_j)^\top x_i$ for all $x \in \mathbb{R}^k$ fulfills the restrictions for the cost functions of a convexified instance, i.e. Definition 2. With this definition of $I^{\text{conv}}$ and assuming that $c_i : \mathbb{R}^m \to \mathbb{R}^m$ is a smooth function, $I^{\text{conv}}$ is a jointly convex GNEP w.r.t. $X^{\text{conv}}$ with smooth cost-functions for the players and thus various methods to solve $I^{\text{conv}}$ are known in the literature.

5. Computational Study

In this section, we present numerical results on the computation of generalized Nash equilibria for Examples 2 and 3, i.e., the capacitated discrete flow games and transportation markets.

We will consider three different types of methods for the computation of equilibria. The first type (see Section 5.2.1) exploits the fact that the instances are hole-free GNEPs allowing us to apply the reformulation of the GNEP via Corollary 2. Note that this approach is correct (assuming enough run-time) and has the striking advantage that any positive lower bound of the resulting global optimization problem serves as a certificate for the non-existence of GNE.

The second type (see Section 5.2.2) uses Theorem 2 in the sense that we try to compute a GNE for the convexified GNEP (e.g. by finding local minimizer of the $\hat{V}$ function) and then check feasibility for the original non-convex GNEP. This approach has the advantage that it can use well-known numerical methods from the area of convex GNEPs and in addition it is in principle applicable to all GNEPs and not only quasi-linear GNEPs. On the down-side, this approach is only correct, if we were able to compute all GNE for the convexified instance and check them for original feasibility.

The third type (see Section 5.2.3) is a best response algorithm (which we term BR), where the players – whenever they can strictly improve their costs – update their strategy using a best response. Remark, however, that a BR-algorithm is not correct in general as it may not terminate due to cycling or may stop at infeasible strategy profiles for which a player’s optimization problem is infeasible. As a consequence, the BR heuristic is not applicable to the instances of Example 3 (cf. Section 5.3).

Let us emphasize that prior to our paper, there were no existing methods available in the literature that can deal with general non-convex or even quasi-linear GNEPs. Hence, a comparison of our proposed methods to some benchmark methods from the literature is not possible.

In the following subsections, we first describe the set of test instances which we generated. Then, we examine the aforementioned methods in more detail and conclude by presenting their numerical results in terms of the number of GNE found, certificates of non-existence and computation times on average.

5.1. Test Instances

We generated 10 different graphs $G = (V, E)$ for each $|V| \in \{10, 15, 20\}$ and each of the three different player set sizes $N \in \{2, 4, 10\}$. The edges of the graph were assigned randomly with each pair of nodes $a \neq b \in V$ having a $\{20\%, 15\%, 10\%\}$ chance for $|V| \in \{10, 15, 20\}$ to be connected by the directed arc $(a, b)$. Concerning the source sink pair of each player, we generated two types. Namely on the one hand a single source single sink type in which every player gets the same randomly selected (connected) source sink pair. On the other hand a multi source multi sink type in which each player has an individual randomly selected (connected) source sink pair. Similar, the weight of each player, i.e. the integral amount of flow each player wants to send, is either chosen uniformly at random from the range of 1 to 10 or set to 1 for
each player. In conclusion, we generated 10 graphs for each combination of \(|V|, |N|\), the two types concerning the source/sink assignment and the weight assignment, leading to a total of 360 different graph-player setups.

**JCDFG.** For the above described graph-player setups, we considered two different types of the CDFG. The first one is a jointly capacitated version (JCDFG) in which every player has the same capacity vector \(c_i = c, i \in N\). To generate capacities that have an impact on the strategy sets, we first chose the capacities uniformly at random from a relative small range of 1 to \(\max(n, d_1, \ldots, d_n)\). If the resulting strategy space is empty, the capacities are reassigned. This random reassignment is executed until either the strategy space is not empty anymore or a limit for the amount of reassignments is exceeded. In the latter case, the range of values in which the capacities are chosen is incremented by one and the procedure is repeated. Regarding the cost functions, we use \(\pi_i(x_i, x_{-i}) := (\sum_{j \neq i} x_j)^\top C_1^i x_i + C_2^i x_i\). If the player’s weights are arbitrary, \(C_1^i \in \mathbb{Z}^{n \times m} \geq 0\) and \(C_2^i \in \mathbb{Z}^m \geq 0\) are randomly generated with values in the range of 0 to 20. Otherwise, we set \(C_1^i = \text{diag}(C_2^i)\) as the diagonal matrix having \(C_2^i\) as its diagonal as the JCDFG can then be interpreted as a jointly constrained atomic congestion game in this case, cf. Example 4.

**ICDFG.** We also considered an individually capacitated version of the CDFG (ICDFG) by changing the above described jointly constrained instances only w.r.t. the capacities of the players. In contrast to above, players now have individual capacities, i.e. \(c_i \neq c_j\) in general, which are analogously generated to the jointly constrained case.

**Transportation Markets.** For the transportation markets of Example 3, we set the weights in all of the above graph-player setups to 1 and then considered only those graph-player setups in which at least one edge-disjoint path-allocation exists. This resulted in 74 instances. The players’ costs were then set to \(\pi_i(x_i, x_{-i}, p) := (p - C_2^i)^\top x_i\) (with \(C_2^i\) from the JCDFG). Hereby, \(p\) is the strategy of the market manager which we added on top of the previously existing players. The latter’s optimization problem is defined as described in Example 3 with the addition of an upper price bound \(p_e \leq \text{PB}, e \in E\) for which we considered three different cases \(\text{PB} \in \{20, 35, 50\}\).

### 5.2. Computing Generalized Equilibria

In order to compute original GNE, we use the continuous relaxation of the original games as convexification \(I^{\text{conv}}\) which is possible since GNEPs corresponding to the JCDFG, ICDFG and transportation markets are hole-free represented GNEPs. Based on this convexified instance, we implemented the following methods in MATLAB® in order to find equilibria of \(I\).

#### 5.2.1. Quasi-linear Reformulation

The first method is relying on the fact that both types of the CDFG and the transportation markets are hole-free-represented player-linear mixed-integer GNEPs for which Corollary 2 is applicable and consequently the problem of finding a GNE reduces to finding a global optimum of a MINLP. A striking advantage of this reformulation is its computational tractability as it allows for the application of global (MINLP) solvers such as BARON, cf. [31]. Note that these solvers typically require that the objective and restriction functions have an algebraic description, i.e. only consist of solver-supported operations like +, −, · etc., which is not the case for the non-reformulated, original problem (7). Furthermore, the possibility to use BARON comes with the additional benefit that BARON generates lower bounds on the optimal objective value during the search for a global optimum. A lower bound larger than zero is a certificate
for non-existence of equilibria and the computation can be exited as soon as such a bound is found, cf. the computational results for the transportation markets in Section 5.3.

5.2.2. Minimizing Variants of the $\hat{V}$ Function

The second type of methods is based on minimizing different variants of the $\hat{V}$ function of $I^{\text{conv}}$, rounding the found minimum and then verifying whether the rounded solution is a GNE. In contrast to the first method, here we do not deal with an optimization problem with a complete algebraic description, but each objective function ($\hat{V}$) call requires to solve $|N|$ linear minimization problems. Solving optimization problems with an objective function of such inaccessible form is a challenging task and we’re not aware of any solvers that may produce lower bounds on the optimal objective value and hence guarantee global minimality. Instead, we utilize the MATLAB® Optimization Toolbox and implemented the $\hat{V}$ function using the LP solver linprog and calculated local minima via the fmincon solver.

For the case of the jointly convex GNEPs and hence the convexification of the JCDFG, we are able to make use of the following regularization of the $\hat{V}$ function:

$$\hat{V}_\alpha(x) := \max_{y \in \hat{X}} \sum_{i \in N} \left[ \pi_i(x) - \pi_i(y_i, x_i) - \frac{\alpha}{2} \| x_i - y_i \|^2 \right]$$

where $\hat{X}$ is the joint constraint set of $I^{\text{conv}}$ (cf. (23)), $\| \cdot \|$ is the Euclidean norm and $\alpha > 0$ denotes a regularization parameter. Heusinger and Kanzow [49] showed that $\hat{V}_\alpha$ is bounded from below by zero, every feasible solution with value zero corresponds to a (normalized) GNE of $I^{\text{conv}}$ and is continuously differentiable. The latter fact is beneficial in a computational regard as it allows one to provide an analytic gradient, significantly speeding up the computation of a local minimum. In this regard, we also computed local minima via the fmincon solver of $\hat{V}_\alpha$ with $\alpha = 0.02$ for $I^{\text{conv}}$ of the JCDFG.

Although the transportation markets do not correspond to a jointly convex GNEP, we are still able to define a similar regularization by

$$\hat{V}_\alpha(x, p) := \pi_{n+1}(x, p) + \max_{y \in \hat{X}'} \sum_{i \in [n]} \left[ \pi_i(x, p) - \pi_i(y_i, x_i, p) - \frac{\alpha}{2} \| x_i - y_i \|^2 \right]$$

where the $n+1$-th player resembles the market manager and $\hat{X}'$ the continuous relaxation of the product of the flow polytopes $\times_{i \in [n]} X_i'$. By a similar argumentation as in [49], it is easy to see that this function is bounded from below by zero, every feasible solution with value zero corresponds to a GNE of $I^{\text{conv}}$ and is continuously differentiable. Hence, we also tried to computed (local) minima via the fmincon solver of $\hat{V}_\alpha$ with $\alpha = 0.02$.

An interesting question is whether and how one can adjust these techniques to find GNE of the convexified game but at the same time preserve original feasibility. We performed a first step into this direction by also implementing a penalized version of the above two methods in which we augmented the $\hat{V}/\hat{V}_\alpha$ function by the additive term $\frac{1}{mn} \sum_{i \in N} \sum_{j \in E} \sin(\pi \cdot x_{ij})^2$ which penalizes non-integrality, resulting in the functions $\hat{V}^{\text{pen}}/\hat{V}^{\text{pen}}_\alpha$. The idea here is that local minima found by the solver should be more likely to be integral and hence originally feasible. Yet, this penalty term must be viewed with caution as the computation of a single local minimum is likely to be more time consuming and new local minima with an objective value bigger than zero may be generated through this penalty term.

The fmincon solver requests a starting point. Thus, we computed an ordered and common set of 2000 random starting points by projecting random vectors in $[0, \max(n, d_1, \ldots, d_n)]^k$ to the set of feasible strategy profiles of $I^{\text{conv}}$. This is done by solving for each random vector $r$ the quadratic program $\min_{x \in \hat{X}} \| r - x \|^2$ via the quadprog solver of MATLAB. Note that the corresponding computation time was negligible. Beginning with the first starting point, a
(local) minimum is then computed of the respective objective function. Each component of this local minimum is then rounded to the nearest integer. The resulting integral vector is then checked for feasibility and whether or not it is a GNE of $I^{\text{conv}}$ by evaluating the $\hat{V}$ function for $I^{\text{conv}}$ at that point. If the rounded solution is not a GNE, the next (local) minimum is computed with the usage of the next starting point. This procedure is executed until either a GNE has been found, all starting vectors were tried or a time limit of one minute is exceeded, in which case the current computation is exited and no further (local) minima are computed.

5.2.3. Best Response Algorithm

The BR-algorithm is applied to the previously mentioned starting vectors until an original GNE was found or the time limit of one minute is exceeded. Hereby, starting with the first player, in each iteration the current strategy profile is updated by the current player’s best response which is computed by solving her corresponding integral linear program via BARON. If the current player’s optimization problem is infeasible, the BR-algorithm stops and is applied to the next starting vector.

5.3. Results

All methods have been implemented in MATLAB® R2023a on Windows 10 Enterprise. The computations have been performed on a machine with Intel Core i5-12500 and 32 GB of memory. An overview of the results can be seen in Table 1. The “GNE” column of a method displays how often an equilibrium was found while the “Time” column shows how long it took (in seconds) to compute the equilibrium on average. The “Non-Existence” column shows how often BARON was able to give a lower bound larger zero on the objective, i.e. giving a certificate for non existence of equilibria and the corresponding “Time” column shows how long it took (in seconds) to compute the lower bound. In order to illustrate the behaviour of the methods with respect to different player sets, we also present the results of the JCDFG and ICDFG subdivided into the three possibilities $N \in \{2, 4, 10\}$. To demonstrate the behaviour of the various methods for one instance-type, we also present in Figure 9 and Figure 10 boxplots of the performance of all methods based on 100 randomly generated instances of the type $(2,20,m,10)$ for the JCDFG and ICDFG. The diagrams show the distribution of the computation time (in seconds) of an integral GNE. The mark inside each box denotes the median, boxes represent lower and upper quartiles, and the whisker ends show the minimum and maximum, respectively, apart from possible outliers marked by a cycle.

The results regarding the transportation markets demonstrate that a BR heuristic may fail completely for certain types of GNEPs as infeasible strategy profiles may lead to the non-existence of best responses for players. In contrast, the results of the quasi-linear approach for these transportation markets illustrate the advantage that comes with our quasi-linear reformulation. Namely the possibility to generate certificates for non-existence of equilibria. In this regard, BARON was able to find in all market instances either a GNE or such a certificate.

Finally, remark that the overall weak performance of the $\hat{V}$ and $\hat{V}^{\text{pen}}$ approaches can be mainly attributed to the numerical complexity of fmincon when using differentiation. This task is time-demanding, requiring numerous costly $\hat{V}/\hat{V}^{\text{pen}}$ evaluations.
Table 1: The performances of the various methods applied to the different examples. The “GNE” column of a method displays how often an equilibrium was found while the “Time” column shows how long it took (in seconds) to compute the equilibrium on average. The “Non-Existence” column shows how often BARON was able to give a lower bound bigger zero on the objective, i.e. giving a certificate for non existence of equilibria and the corresponding “Time” column shows how long it took (in seconds) to compute the lower bound.

Figure 9: Boxplots of the performance of all methods with respect to the instance type (2,20,m,10) in the JCDFG. The diagrams show the distribution of the computation time (in seconds) of a original GNE. We did not include the time when no equilibrium was found. In this regard, the methods (a)-(f) found (100,80,80,100,100,96) equilibria respectively.
Figure 10: Boxplots of the performance of all methods with respect to the instance type (2,20,m,10) in the ICDFG. The diagrams show the distribution of the computation time (in seconds) of an original GNE. We did not include the time when no equilibrium was found. In this regard, the quasi-linear approach found 56 equilibria and provided a non-existence certificate in 20 cases. The methods (c)-(e) found (45,50,57) equilibria respectively.

6. Conclusions

We derived a new characterization of generalized Nash equilibria by convexifying the original instance $I$, leading to a set of more structured convexified instances $I^{\text{conv}}$ of the GNEP. This convexification approach is very general and thus its relevance is relying on the identification of classes of original instances and corresponding well-behaved convexified instances. We illustrated this by deriving for the three problem classes of quasi-linear, $k$-restrictive-closed and restrictive-closed GNEPs, respectively, new characterizations of the existence and computability of generalized Nash equilibria. We demonstrated the applicability of the latter by presenting various methods and corresponding numerical results for the computation of equilibria in the CDFG and transportation markets. In this regard, our convexification offers an approach to systematically tackle the poorly understood class of non-convex and discrete GNEPs via identifying original and corresponding well-behaved convexified instances in order to then draw conclusions for the original instance from the convexified one via our main Theorem. Therefore we believe that there is still untapped potential in our convexification method in order to obtain structural insights into the problem as well as pave the way for a more tractable computational approach.

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A. Appendix

A.1. Detailed Numerical Results

In Table 2 and 3 we present the numerical results in a more detailed fashion. That is, we subdivided the numerical results w.r.t. each instance type. As described above, for each instance type there are 10 different instances.

| Instance-type | GNE | Time | Non-Existence | Time | GNE | Time | GNE | Time | GNE | Time | BR |
|---------------|-----|------|---------------|------|-----|------|-----|------|-----|------|----|
| (2,10,s,1)    | 10  | 0.14 | 0             | -    | 9   | 4.66 | 10  | 0.33 | 10  | 0.33 | 10  | 0.93|
| (2,10,s,10)   | 10  | 0.18 | 0             | -    | 10  | 0.70 | 10  | 0.08 | 10  | 0.08 | 9   | 0.26|
| (2,10,m,1)    | 10  | 0.14 | 0             | -    | 10  | 10.12| 10  | 0.15 | 10  | 0.14 | 10  | 0.28|
| (2,10,m,10)   | 10  | 0.24 | 0             | -    | 10  | 15.32| 10  | 0.12 | 10  | 0.10 | 9   | 0.42|
| (2,15,s,1)    | 10  | 0.18 | 0             | -    | 10  | 18.91| 10  | 0.71 | 10  | 1.13 | 9   | 0.31|
| (2,15,s,10)   | 10  | 0.33 | 0             | -    | 9   | 17.06| 9   | 0.53 | 10  | 0.24 | 8   | 0.33|
| (2,15,m,1)    | 10  | 0.17 | 0             | -    | 10  | 19.36| 10  | 0.24 | 10  | 0.20 | 10  | 0.90|
| (2,15,m,10)   | 10  | 0.37 | 0             | -    | 10  | 33.74| 10  | 0.55 | 10  | 0.32 | 10  | 0.32|
| (2,20,s,1)    | 10  | 0.20 | 0             | -    | 9   | 15.98| 9   | 13.02| 10  | 2.02 | 10  | 0.90|
| (2,20,s,10)   | 10  | 0.44 | 0             | -    | 9   | 26.75| 7   | 23.28| 10  | 0.63 | 10  | 1.44|
| (2,20,m,1)    | 10  | 0.20 | 0             | -    | 9   | 28.36| 9   | 26.75| 10  | 1.93 | 10  | 0.32|
| (2,20,m,10)   | 10  | 0.44 | 0             | -    | 8   | 27.58| 10  | 0.50 | 10  | 0.55 | 10  | 0.27|
| (4,10,s,1)    | 10  | 0.22 | 0             | -    | 9   | 13.74| 10  | 14.66| 10  | 0.32 | 10  | 2.00|
| (4,10,s,10)   | 10  | 0.68 | 0             | -    | 9   | 10.77| 10  | 14.96| 10  | 0.40 | 10  | 0.34|
| (4,10,m,1)    | 10  | 0.21 | 0             | -    | 9   | 14.08| 9   | 12.76| 10  | 1.44 | 10  | 2.55|
| (4,10,m,10)   | 10  | 0.62 | 0             | -    | 8   | 24.38| 8   | 21.75| 10  | 0.80 | 10  | 0.64|
| (4,15,s,1)    | 10  | 0.37 | 0             | -    | 5   | 36.69| 4   | 39.25| 10  | 6.24 | 10  | 4.67|
| (4,15,s,10)   | 10  | 2.08 | 0             | -    | 0   | 0    |   0 | 0    | 10  | 5.71 | 10  | 5.75|
| (4,15,m,1)    | 9   | 0.37 | 0             | -    | 4   | 32.73| 6   | 33.66| 9   | 7.03 | 9   | 2.95|
| (4,15,m,10)   | 10  | 1.64 | 0             | -    | 2   | 32.34| 2   | 29.52| 10  | 2.34 | 10  | 4.21|
| (4,20,s,1)    | 10  | 0.47 | 0             | -    | 4   | 34.85| 4   | 33.88| 10  | 4.96 | 10  | 1.84|
| (4,20,m,1)    | 10  | 2.35 | 0             | -    | 2   | 17.67| 2   | 16.03| 10  | 13.06| 10  | 1.34|
| (4,20,m,10)   | 10  | 1.42 | 0             | -    | 5   | 42.39| 6   | 39.76| 10  | 4.02 | 10  | 2.74|
| (4,20,m,10)   | 10  | 2.31 | 0             | -    | 1   | 25.39| 1   | 26.66| 10  | 5.64 | 10  | 6.30|
| (10,10,s,1)   | 8   | 0.73 | 0             | -    | 4   | 29.08| 5   | 37.21| 9   | 4.71 | 9   | 4.51|
| (10,10,s,10)  | 10  | 7.34 | 0             | -    | 2   | 10.92| 2   | 11.14| 8   | 17.21| 8   | 11.96|
| (10,10,m,1)   | 10  | 0.70 | 0             | -    | 7   | 36.48| 6   | 25.91| 10  | 5.40 | 10  | 7.92|
| (10,10,m,10)  | 10  | 7.07 | 0             | -    | 1   | 5.41 | 1   | 4.74 | 8   | 14.77| 7   | 6.55|
| (10,15,s,1)   | 10  | 1.69 | 0             | -    | 0   | 0    |   0 | 0    | 10  | 9.17 | 8   | 8.50|
| (10,15,s,10)  | 10  | 15.08| 0             | -    | 0   | 0    |   0 | 0    | 10  | 9.17 | 8   | 13.07|
| (10,15,m,1)   | 7   | 1.60 | 0             | -    | 0   | 0    |   0 | 0    | 6   | 14.20| 6   | 9.33|
| (10,15,m,10)  | 10  | 33.70| 0             | -    | 0   | 0    |   0 | 0    | 2   | 6.79 | 2   | 20.22|
| (10,20,s,1)   | 6   | 1.98 | 0             | -    | 1   | 39.26| 1   | 39.97| 6   | 13.15| 6   | 14.75|
| (10,20,s,10)  | 8   | 29.80| 0             | -    | 0   | 0    |   0 | 0    | 4   | 25.34| 4   | 28.61|
| (10,20,m,1)   | 4   | 1.81 | 0             | -    | 0   | 0    |   0 | 0    | 6   | 12.52| 7   | 9.05|
| (10,20,m,10)  | 10  | 26.98| 0             | -    | 0   | 0    |   0 | 0    | 5   | 20.31| 5   | 20.32|

Table 2: The performances of the various methods applied to the JCDFG. The “GNE” column of a method displays how often an equilibrium was found while the “Time” column shows how long it took (in seconds) to compute the equilibrium on average. The “Non-Existence” column shows how often BARON was able to give a lower bound bigger zero on the objective, i.e. giving a certificate for non existence of equilibria and the corresponding “Time” column shows how long it took (in seconds) to compute the lower bound.
| Instance-type | Quasi-Linear | $\hat{V}$ | $\hat{V}^{pen}$ | BR |
|---------------|-------------|-----------|---------------|----|
|               | GNE Time    | Non-Existence | Time | GNE Time | GNE Time | GNE Time | GNE Time |
| (2,10,s,1)    | 10 0.19 0   | -          | 10 6.91       | 10 7.91 | 10 4.08 |
| (2,10,s,10)   | 7 0.18 0   | -          | 7 9.34        | 7 5.61  | 7 0.24 |
| (2,10,m,1)    | 9 0.14 0   | -          | 9 11.39       | 9 10.09 | 9 0.35 |
| (2,10,m,10)   | 4 0.18 0.23 | 3          | 4 13.07       | 4 13.28 | 4 0.25 |
| (2,15,s,1)    | 10 0.18 0  | -          | 9 18.74       | 9 18.70 | 9 1.77 |
| (2,15,s,10)   | 9 0.30 0   | -          | 7 23.70       | 7 24.66 | 8 2.73 |
| (2,15,m,1)    | 10 0.16 0  | -          | 10 20.20      | 10 18.68 | 10 3.02 |
| (2,15,m,10)   | 7 0.31 0.38 | 1          | 6 16.95       | 6 16.32 | 7 0.31 |
| (2,20,s,1)    | 10 0.20 0  | -          | 9 16.40       | 9 14.82 | 10 5.09 |
| (2,20,s,10)   | 6 0.32 0.48 | 1          | 7 22.20       | 7 19.62 | 8 0.27 |
| (2,20,m,1)    | 10 0.20 0  | -          | 10 23.56      | 10 25.71 | 10 1.10 |
| (2,20,m,10)   | 5 0.40 0.44 | 1          | 5 17.45       | 5 18.39 | 5 0.28 |
| (4,10,s,1)    | 9 0.23 0   | -          | 9 21.20       | 9 26.56 | 8 0.66 |
| (4,10,s,10)   | 8 0.35 0   | -          | 7 9.08        | 7 8.63  | 9 0.55 |
| (4,10,m,1)    | 7 0.20 0.54 | 2          | 7 16.16       | 5 15.15 | 7 0.54 |
| (4,10,m,10)   | 5 0.49 0   | -          | 4 12.02       | 5 21.45 | 5 0.52 |
| (4,15,s,1)    | 6 0.35 0   | -          | 2 42.98       | 2 43.80 | 7 2.09 |
| (4,15,s,10)   | 2 1.35 0   | -          | 0 -           | 0 -     | 5 0.85 |
| (4,15,m,1)    | 8 0.42 0   | -          | 3 47.19       | 2 27.11 | 7 2.09 |
| (4,15,m,10)   | 1 1.02 0.67 | 1          | 1 5.40        | 1 5.71  | 1 0.67 |
| (4,20,s,1)    | 7 0.46 0   | -          | 3 36.39       | 3 37.67 | 8 1.11 |
| (4,20,s,10)   | 4 1.76 0   | -          | 0 -           | 0 -     | 4 0.61 |
| (4,20,m,1)    | 9 0.42 0   | -          | 3 45.70       | 2 35.60 | 8 0.70 |
| (4,20,m,10)   | 2 1.10 0   | -          | 0 -           | 1 55.62 | 2 0.59 |
| (10,10,s,1)   | 3 0.83 1   | 5.21       | 2 19.11       | 4 32.08 | 4 1.93 |
| (10,10,s,10)  | 4 8.24 0   | -          | 1 21.72       | 2 35.24 | 4 1.73 |
| (10,10,m,1)   | 3 0.73 3   | 0.61       | 1 12.12       | 2 23.98 | 3 1.93 |
| (10,10,m,10)  | 2 3.89 0   | -          | 0 -           | 1 43.77 | 2 2.04 |
| (10,15,s,1)   | 4 1.82 0   | -          | 0 -           | 0 -     | 4 4.17 |
| (10,15,s,10)  | 5 11.89 0  | -          | 0 -           | 0 -     | 5 1.80 |
| (10,15,m,1)   | 2 2.15 0   | -          | 0 -           | 0 -     | 2 14.36 |
| (10,15,m,10)  | 1 30.69 1.51 | 5.11       | 0 -           | 0 -     | 1 1.80 |
| (10,20,s,1)   | 5 2.08 0   | -          | 1 54.64       | 0 -     | 5 2.01 |
| (10,20,s,10)  | 5 39.86 0  | -          | 0 -           | 1 40.44 | 6 2.01 |
| (10,20,m,1)   | 1 1.73 0   | -          | 1 45.32       | 1 33.58 | 1 2.26 |
| (10,20,m,10)  | 4 14.06 0  | -          | 0 -           | 0 -     | 4 1.78 |

Table 3: The performances of the various methods applied to the ICDFG. The “GNE” column of a method displays how often an equilibrium was found while the “Time” column shows how long it took (in seconds) to compute the equilibrium on average. The “Non-Existence” column shows how often BARON was able to give a lower bound bigger than zero on the objective, i.e. giving a certificate for non existence of equilibria and the corresponding “Time” column shows how long it took (in seconds) to compute the lower bound.