Weighted inequalities for quasilinear integral operators on the semi-axis and applications to Lorentz spaces

D. V. Prokhorov and V. D. Stepanov

Abstract. A precise characterization of inequalities in weighted Lebesgue spaces with positive quasilinear integral operators of iterative type on the half-axis is given. All cases of positive integration parameters are treated, including the case of supremum. Applications to the solution of the well-known problem of the boundedness of the Hardy-Littlewood maximal operator in weighted Lorentz \( \Gamma \)-spaces are given.

Bibliography: 41 titles.

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§ 1. Introduction

Let \( \mathbb{R}_+ := [0, \infty) \). Let \( \mathcal{M} \) denote the set of measurable functions on \( \mathbb{R}_+ \) and \( \mathcal{M}^+ \subset \mathcal{M} \) the subset of nonnegative functions. If \( 0 < p \leq \infty \) and \( v \in \mathcal{M}^+ \), we define

\[
L_v^p := \left\{ f \in \mathcal{M} : \|f\|_{L_v^p} := \left( \int_0^\infty |f(x)|^p v(x) \, dx \right)^{1/p} < \infty \right\},
\]

\[
L_v^\infty := \left\{ f \in \mathcal{M} : \|f\|_{L_v^\infty} := \text{ess sup}_{x \geq 0} v(x) |f(x)| < \infty \right\}.
\]

Let \( 0 < q \leq \infty \) and \( w \in \mathcal{M}^+ \). We consider quasilinear operators on \( \mathcal{M}^+ \) of the form

\[
(Tf)(x) = \left( \int_x^\infty w(y) \left( \int_y^\infty k(y, z) f(z) \, dz \right)^q \, dy \right)^{1/q},
\]

\[
(\mathcal{T}f)(x) = \left( \int_0^x w(y) \left( \int_y^\infty k(z, y) f(z) \, dz \right)^q \, dy \right)^{1/q},
\]

\[
(Sf)(x) = \left( \int_x^\infty w(y) \left( \int_y^\infty k(z, y) f(z) \, dz \right)^q \, dy \right)^{1/q},
\]

\[
(\mathcal{S}f)(x) = \left( \int_0^x w(y) \left( \int_y^\infty k(y, z) f(z) \, dz \right)^q \, dy \right)^{1/q}.
\]

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where \( k(x, y) \geq 0 \) is a measurable function and for \( q = \infty \) the right-hand sides are to be replaced by

\[
(T f)(x) = \text{ess sup}_{y \geq x} w(y) \int_0^y k(y, z) f(z) \, dz,
\]

\[
(T f)(x) = \text{ess sup}_{y \geq x} k(y, x) w(y) \int_0^y f(z) \, dz,
\]

and so on for the other operators.

Let \( u, v, w \in \mathcal{M}_+ \) be weights, let \( 1 \leq p \leq \infty \) and \( 0 < r \leq \infty \). Our aim is to characterize the weighted inequalities

\[
\| T f \|_{L^u} \leq C_T \| f \|_{L^p_u}, \quad f \in \mathcal{M}, \quad (1.1)
\]

\[
\| \mathcal{T} f \|_{L^u} \leq C_{\mathcal{T}} \| f \|_{L^p_u}, \quad f \in \mathcal{M}, \quad (1.2)
\]

\[
\| S f \|_{L^u} \leq C_S \| f \|_{L^p_u}, \quad f \in \mathcal{M}, \quad (1.3)
\]

\[
\| \mathcal{S} f \|_{L^u} \leq C_{\mathcal{S}} \| f \|_{L^p_u}, \quad f \in \mathcal{M}, \quad (1.4)
\]

and

\[
\| T f \|_{L^u} \leq C_T \| f \|_{L^p_u}, \quad f \in \mathcal{M}, \quad (1.5)
\]

\[
\| \mathcal{T} f \|_{L^u} \leq C_{\mathcal{T}} \| f \|_{L^p_u}, \quad f \in \mathcal{M}, \quad (1.6)
\]

\[
\| S f \|_{L^u} \leq C_S \| f \|_{L^p_u}, \quad f \in \mathcal{M}, \quad (1.7)
\]

\[
\| \mathcal{S} f \|_{L^u} \leq C_{\mathcal{S}} \| f \|_{L^p_u}, \quad f \in \mathcal{M}, \quad (1.8)
\]

where the function \( k(x, y) \geq 0 \) on \([0, \infty)^2\) satisfies Oinarov’s condition: \( k(x, y) = 0 \) if \( x < y \) and there exists a constant \( D \geq 1 \) independent of \( x \geq z \geq y \geq 0 \) such that

\[
\frac{1}{D} (k(x, z) + k(z, y)) \leq k(x, y) \leq D(k(x, z) + k(z, y)), \quad (1.9)
\]

where the constants \( C_T \) and so on are taken to be the smallest possible. If \( q = r < \infty \), these inequalities reduce to generalized Hardy-type inequalities, which have been well studied (see, for instance, [1]–[3] with further extensions and improvements in [4]–[10] and other papers). The case \( q = \infty \) is closely related to recently initiated
studies of supremum operators [11]–[17]. If \( k(x, y) \equiv 1 \), inequality (1.4) plays an important role in analysis on Morrey-type spaces (see [18]–[22]). In particular, for certain values of the parameters \( p, q, r \) for (1.2) this was solved in [23] and [24], and for (1.4) in [22]. A complete solution of this case was given in [25] and [26].

Using a new method we characterize inequalities (1.1)–(1.8) with kernel \( k(x, y) \) satisfying (1.9) for all parameters \( 1 \leq p \leq \infty, 0 < r \leq \infty \) and \( 0 < q \leq \infty \). The cases \( p = \infty \) and \( r = \infty \) are trivial and the interval \( 0 < p < 1 \) is excluded because in this case it can be shown that if, say, \( C_T < \infty \), then \( C_T = 0 \) (see [27], Theorem 2, for details).

Sections 2 and 3 are devoted to the study of (1.1)–(1.4) and §§4 and 5 to (1.5)–(1.8). It is interesting to note that the second part is based in part on the first. Finally, in §6 we illustrate our results by a solution of the well-known problem of a sharp characterization of the \( \Gamma^p(v) \to \Gamma^q(w) \) boundedness for the Hardy-Littlewood maximal operator for all \( 0 < p, q < \infty \), including the most difficult cases missed in [28] and [29].

We use the symbols := and =: to define new quantities and we use \( \mathbb{Z} \) for the set of integers. For positive functionals \( F \) and \( G \) we write \( F \preceq cG \) for some positive constant \( c \) which depends only on irrelevant parameters. \( F \approx G \) means that \( F \approx cG \) or \( F = cG \). \( \chi_E \) denotes the characteristic function (indicator) of the set \( E \). Indeterminates of the form \( 0 \cdot \infty, \infty / \infty \) and \( 0 / 0 \) are set to be zero.

**§ 2. The operators \( T \) and \( S \)**

Suppose for simplicity that

\[
\int_0^t u < \infty \quad \forall t > 0,
\]

and define the functions \( \sigma: [0, \infty) \to [0, \infty] \), \( \sigma^{-1}: [0, \infty) \to [0, \infty) \) by the formulae (here \( \inf \emptyset = \infty \))

\[
\sigma(x) := \inf \left\{ y > 0 : \int_0^y u \geq 2 \int_0^x u \right\}, \quad \sigma^{-1}(x) := \inf \left\{ y > 0 : \int_0^y u \geq \frac{1}{2} \int_0^x u \right\}.
\]

The functions \( \sigma \) and \( \sigma^{-1} \) are increasing; furthermore,

\[
\int_0^{\sigma^{-1}(x)} u = \frac{1}{2} \int_0^x u
\]

for each \( x \in [0, \infty) \), and if \( \sigma(x) < \infty \), then

\[
\int_0^{\sigma(x)} u = 2 \int_0^x u.
\]

Let \( \sigma^m, m \in \mathbb{N} \), be the composition of \( m \) functions \( \sigma \) and similarly for \( \sigma^{-m} \). For \( 0 < c < d \leq \infty \) and \( f \in \mathcal{M}^+ \) we put

\[
(H_{c,d}f)(x) := \chi_{[c,d]}(x) \int_{\sigma^{-1}(c)}^x k(x, z)f(z) \, dz,
\]

\[
(H_{c}f)(x) := \chi_{[c,\infty]}(x) \int_0^x k(x, z)f(z) \, dz.
\]
Theorem 1. Let
\[ 1 \leq p < \infty, \quad 0 < r < \infty, \quad 0 < q \leq \infty, \quad \frac{1}{s} := \left( \frac{1}{r} - \frac{1}{p} \right)_+. \]

For (1.1) to hold it is necessary and sufficient that, for all \( f \in \mathfrak{M}^+ \), the inequalities
\begin{align*}
\left( \int_0^\infty u(x) \left( \int_0^\infty w \right) \right) & \frac{r}{q} \left( \int_0^x k(x, z) f(z) dz \right) dx \right) \right)^{1/r} \leq A_0 \| f \|_{L_p^v}, \\
\left( \int_0^\infty u(x) \left( \int_0^\infty [k(x, z)]^q w(z) dz \right) \right) & \frac{r}{q} \left( \int_0^x f \right) dx \right) \right)^{1/r} \leq A_1 \| f \|_{L_p^v}
\end{align*}
hold when \( q < \infty \), the inequalities
\begin{align*}
\left( \int_0^\infty u(x) \left( \operatorname{ess sup}_{y \geq x} w(y) \right) \right) & \frac{r}{q} \left( \int_0^x k(x, z) f(z) dz \right) dx \right) \right)^{1/r} \leq A_0 \| f \|_{L_p^v}, \\
\left( \int_0^\infty u(x) \left( \operatorname{ess sup}_{y \geq x} [w(y)k(y, x)] \right) \right) & \frac{r}{q} \left( \int_0^x f \right) dx \right) \right)^{1/r} \leq A_1 \| f \|_{L_p^v}
\end{align*}
hold when \( q = \infty \), and that the constant
\[ A_2 := \begin{cases} 
\sup_{t \in (0, \infty)} \left( \int_0^t u \right) \frac{1}{r} \| H_t \|_{L_p^v \rightarrow L_q^w}, & p \leq r, \\
\left( \int_0^\infty u(x) \left( \int_0^x u \right) \right) \frac{s}{p} \| H_{s^{-1}(x), s(x)} \|_{L_p^v \rightarrow L_q^w} dx \right) \right)^{1/s}, & r < p,
\end{cases} \]
is finite. Moreover, \( C_T \approx A_0 + A_1 + A_2 \).

Proof. Let \( n_0 \in \mathbb{Z} \) be an integer such that \( 2^{n_0} < \int_0^\infty u \). Put
\[ a_{n_0} := \inf \left\{ y > 0 : \int_0^y u \geq 2^{n_0} \right\}, \]
\[ a_{n+1} := \sigma(a_n) \quad \text{for } n \geq n_0, \quad a_{n-1} := \sigma^{-1}(a_n) \quad \text{for } n \leq n_0. \]
Set \( N := \sup \{ n \in \mathbb{Z} : a_n < \infty \} \). If \( N < \infty \), we set \( a_{N+1} := \infty \). Note that \( a_{n-1} = \sigma^{-1}(a_n) \) and \( \sigma(a_n) = a_{n+1} \) for all \( n \leq N \).

We suppose first that \( q < \infty \).

Sufficiency. Let \( \Delta_n := [a_n, a_{n+1}) \). Using (1.9) and the relation
\[ \sum_{n \in \mathbb{Z}} 2^n \left( \sum_{i \geq n} \lambda_i \right)^s \approx \sum_{n \in \mathbb{Z}} 2^n \lambda_n^s \]
Since \( k \) estimated as follows:

\[
\int_0^\infty [Tf]^r u = \sum_{n \leq N} \int_{\Delta_n} [Tf]^r u \approx \sum_{n \leq N} 2^n \left( \int_{a_n}^\infty w(y) \left( \int_0^y k(y, z) f(z) \, dz \right)^q \, dy \right)^{r/q}
\]

\[
\approx \sum_{n \leq N} 2^n \left( \int_{\Delta_n} w(y) \left( \int_0^y k(y, z) f(z) \, dz \right)^q \, dy \right)^{r/q}
\]

\[
\approx \sum_{n \leq N} 2^n \left( \int_{a_n}^\infty w(y) \left( \int_{a_{n-1}}^y k(y, z) f(z) \, dz \right)^q \, dy \right)^{r/q}
\]

\[
+ \sum_{n \leq N} 2^n \left( \int_{\Delta_n} w(y) \left( \int_0^{a_{n-1}} k(y, z) f(z) \, dz \right)^q \, dy \right)^{r/q} =: J_1^r + J_2^r.
\]

Since \( k(y, z) \approx k(y, x) + k(x, z) \) for \( y \in \Delta_n, x \in \Delta_{n-1} \) and \( z \in (0, a_{n-1}) \), \( J_2^r \) can be estimated as follows:

\[
J_2^r \approx \sum_{n \leq N} \int_{a_{n-1}}^{a_n} u(x) \, dx \left( \int_{\Delta_n} w(y) \left( \int_{a_{n-1}}^y k(y, z) f(z) \, dz \right)^q \, dy \right)^{r/q}
\]

\[
\approx \sum_{n \leq N} \int_{a_{n-1}}^{a_n} u(x) \left( \int_{a_n}^{a_{n-1}} w(y)[k(y, x)]^q \, dy \right)^{r/q} \, dx \left( \int_{0}^{a_{n-1}} f \right)^{r/q}
\]

\[
+ \sum_{n \leq N} \int_{a_{n-1}}^{a_n} u(x) \left( \int_{0}^{a_{n-1}} k(x, z) f(z) \, dz \right) \, dx \left( \int_{\Delta_n} w \right)^{r/q}
\]

\[
\lesssim \sum_{n \leq N} \int_{a_{n-1}}^{a_n} u(x) \left( \int_{x}^{\infty} w(y)[k(y, x)]^q \, dy \right)^{r/q} \left( \int_{0}^{x} f \right)^{r/q} \, dx
\]

\[
+ \sum_{n \leq N} \int_{a_{n-1}}^{a_n} u(x) \left( \int_{x}^{\infty} w \right)^{r/q} \left( \int_{0}^{x} k(x, z) f(z) \, dz \right)^r \, dx
\]

\[
\lesssim (A_1^r + A_0^r) \|f\|_{L_v^p}.
\]

To bound \( J_1^r \) above we write

\[
J_1^r \approx \sum_{n \leq N} 2^n \|H_{a_n, a_{n+1}} f\|_{L_w^q}^r \lesssim \sum_{n \leq N} \left( \int_{a_{n-1}}^{a_n} u \right) \|H_{a_n, a_{n+1}}\|_{L_v^p \to L_w^q}^r \left( \int_{a_{n-1}}^{a_n} f \right)^{r/p}
\]

If \( p \leq r \), we use Jensen’s inequality and obtain

\[
J_1 \lesssim \sup_{n \leq N} \left( \int_{a_{n-1}}^{a_n} u \right)^{1/r} \|H_{a_n, a_{n+1}}\|_{L_v^p \to L_w^q} \|f\|_{L_v^p} \leq A_2 \|f\|_{L_v^p}.
\]
If $r < p$, we use Hölder’s inequality with exponents $s/r$ and $p/r$ and obtain

\[
J_1^s \lesssim \sum_{n \leq N} \left( \int_{a_n}^{a_{n+1}} u \right)^{s/r} \|H_{\alpha_n, \alpha_{n+1}}\|^s_{L^p_v \to L^p_v} \|f\|^s_{L^p_v}
\]

\[
\lesssim \sum_{n \leq N} \left( \int_{a_n}^{a_{n+1}} u \right)^{s/p} \left( \int_0^{a_n} u \right)^{p/r} \|H_{\sigma^{-1}(\alpha_{n+1}), \sigma(\alpha_n)}\|^s_{L^p_v \to L^p_w} \|f\|^s_{L^p_w} \leq A_2 \|f\|^s_{L^p_w}
\]

Thus

\[
\|Tf\|_{L^p_u} \lesssim (A_0 + A_1 + A_2) \|f\|_{L^p_v}
\]

and the upper bound $C_T \lesssim A_0 + A_1 + A_2$ is proved.

Necessity. Since

\[
(Tf)(x) \geq \left( \int_x^\infty w(y) \left( \int_0^x k(y, z)f(z) \, dz \right)^q \, dy \right)^{1/q}
\]

\[
\gtrsim \left( \int_x^\infty w \right)^{1/q} \int_0^x k(x, z)f(z) \, dz,
\]

inequality (1.1) implies (2.1) and $C_T \gtrsim A_0$. Moreover,

\[
(Tf)(x) \geq \left( \int_x^\infty w(y) \left( \int_0^x k(y, z)f(z) \, dz \right)^q \, dy \right)^{1/q}
\]

\[
\gtrsim \left( \int_x^\infty [k(y, x)]^q w(y) \, dy \right)^{1/q} \int_0^x f.
\]

Then (1.1) implies (2.2) and $C_T \gtrsim A_1$. It follows from (1.1) that

\[
C_T \|f\|_{L^p_v} \geq \left( \int_0^t u \right)^{1/r} \|H_t f\|_{L^p_w}, \quad f \in \mathcal{M}^+,
\]

for any $t \in (0, \infty)$. Hence

\[
C_T \geq \sup_{t \in (0, \infty)} \left( \int_0^t u \right)^{1/r} \|H_t\|_{L^p_w \to L^q_w}
\]
and the lower bound $C_T \gtrsim A_2$ is proved for $p \leq r$. Now let $r < p$. Then we have

$$A_2^s = \int_0^\infty u(x) \left( \int_0^x u \right)^{s/p} \|H_{\sigma^{-1}(x),\sigma(x)}\|_{L^p \to L^q}^s \, dx$$

$$= \sum_{n \leq N} \int_{a_n}^{a_{n+1}} u(x) \left( \int_0^x u \right)^{s/p} \|H_{\sigma^{-1}(x),\sigma(x)}\|_{L^p \to L^q}^s \, dx$$

$$\leq \sum_{n \leq N} \left( \int_{a_n}^{a_{n+1}} u \right)^{s/p} \|H_{\sigma^{-1}(a_n),\sigma(a_{n+1})}\|_{L^p \to L^q}^s \, dx$$

$$\approx \sum_{n \leq N} (2^{s})^{s/r} \|H_{a_{n-1},a_{n+2}}\|_{L^p \to L^q}^s =: \mathcal{A}^s_2.$$  

Let $\theta \in (0,1)$ be arbitrary. For each $n \leq N$ there exists $f_n \in \mathcal{M}^+$ such that $\text{supp} \ f_n \subset (a_{n-2},a_{n+2})$, $\|f_n\|_{L^p} = 1$ and

$$\|H_{a_{n-1},a_{n+2}} f_n\|_{L^p} \geq \theta \|H_{a_{n-1},a_{n+2}}\|_{L^p \to L^q}.$$  

Set

$$g_n := (2^s)^{s/(pr)} \|H_{a_{n-1},a_{n+2}}\|_{L^p \to L^q}^s f_n, \quad g := \sum_{n \leq N} g_n.$$  

We find that

$$\|g\|_{L^p}^p = \sum_{j \leq N} \int_{a_j}^{a_{j+1}} \left( \sum_{n \leq N} g_n(x) \right)^p \, dx = \sum_{j \leq N} \int_{a_j}^{a_{j+1}} \left( \sum_{n = j-1}^{j+1} g_n(x) \right)^p \, dx$$

$$\leq \sum_{j \leq N} \int_{a_j}^{a_{j+1}} g_j(x) \, dx = \sum_{j \leq N} (2^j)^{s/r} \|H_{a_{j-1},a_{j+2}}\|_{L^p \to L^q}^s = \mathcal{A}^s_2.$$  

Finally, applying (1.1)

$$C_T^p \mathcal{A}^s_2 \geq \|g\|_{L^p} \geq \int_0^\infty [Tg]^r \, u \geq \sum_{n \leq N} \int_{a_{n-1}}^{a_{n+1}} [Tg]^r \, u$$

$$\geq \sum_{n \leq N} \left( \int_{a_{n-2}}^{a_{n+1}} u \right) \|H_{a_{n-1},a_{n+2}} g\|_{L^q}^r \geq \sum_{n \leq N} (2^n)^{s/r} \|H_{a_{n-1},a_{n+2}} g\|_{L^q}^r \geq \sum_{n \leq N} (2^n)^{s/r} \|H_{a_{n-1},a_{n+2}} f_n\|_{L^q}^r \geq \theta^r \mathcal{A}^s_2.$$  

Thus $C_T \gtrsim \theta \mathcal{A}_2$. Hence $C_T \gtrsim \theta A_2$ and the required lower bound $C_T \gtrsim A_0 + A_1 + A_2$ follows.

The case $q = \infty$ is treated similarly, replacing (2.6) by a trivial modification

$$\sum_{n \in \mathbb{Z}} 2^n \left( \sup_{i \geq n} \lambda_i \right)^s \approx \sum_{n \in \mathbb{Z}} 2^n \lambda_n^s.$$  

(2.7)
Remark 1. For $p = \infty$ we have

$$C_T = \left\| T\left( \frac{1}{v} \right) \right\|_{L_0^p}$$

and for $r = \infty$

$$C_T = \sup_{t \geq 0} U(t) \| H_t \|_{L_p^\infty \to L_w^\infty},$$

where $U(t) := \text{ess sup}_{0 \leq x \leq t} u(x)$.

Now for $0 < c < d < \infty$ and $f \in \mathcal{M}$ we put

$$(H^*_c,d f)(x) := \chi_{[c,d)}(x) \int_x^d k(z,x) f(z) \, dz,$$

$$(H^*_c f)(x) := \chi_{[c,\infty)}(x) \int_x^\infty k(z,x) f(z) \, dz.$$

**Theorem 2.** Let

$$1 \leq p < \infty, \quad 0 < r < \infty, \quad 0 < q \leq \infty, \quad \frac{1}{s} := \left( \frac{1}{r} - \frac{1}{p} \right)_+.$$

Then (1.3) is satisfied if and only if, for all $f \in \mathcal{M}$, the inequalities

$$\left( \int_0^\infty u(x) \left( \int_x^{\sigma^2(x)} w \right)^{r/q} \left( \int_{\sigma^2(x)}^{\infty} k(z,\sigma^2(x)) f(z) \, dz \right)^r \, dx \right)^{1/r} \leq A_0 \| f \|_{L_p^r},$$

$$\left( \int_0^\infty u(x) \left( \int_x^{\sigma^2(x)} [k(\sigma^2(x),z)]^q w(z) \, dz \right)^{r/q} \left( \int_{\sigma^2(x)}^{\infty} f \right)^r \, dx \right)^{1/r} \leq A_1 \| f \|_{L_p^r}$$

hold for $q < \infty$ or the inequalities

$$\left( \int_0^\infty u(x) \left[ \text{ess sup}_{y \in (x,\sigma^2(x))] w(y) \right]^r \left( \int_{\sigma^2(x)}^{\infty} k(z,\sigma^2(x)) f(z) \, dz \right)^r \, dx \right)^{1/r} \leq A_0 \| f \|_{L_p^r},$$

$$\left( \int_0^\infty u(x) \left[ \text{ess sup}_{y \in (x,\sigma^2(x))] [w(y)k(\sigma^2(x),y)] \right]^r \left( \int_{\sigma^2(x)}^{\infty} f \right)^r \, dx \right)^{1/r} \leq A_1 \| f \|_{L_p^r}$$

hold for $q = \infty$, and the constant

$$A_2 := \left\{ \begin{array}{ll}
\sup_{t \in (0,\infty)} \left( \int_0^t u \right)^{1/r} \| H^*_t \|_{L_p^r \to L_w^r}, & p \leq r, \\
\left( \int_0^\infty u(x) \left( \int_x^\infty u \right)^{s/p} \| H^*_\sigma^{-1}(x),\sigma(x) \|_{L_p^r \to L_w^r} \, dx \right)^{1/s}, & r < p,
\end{array} \right.$$
Proof. Let the sequence \( \{a_n\} \) be the same as in the proof of Theorem 1 and let \( q < \infty \).

Sufficiency. We have

\[
\int_0^\infty [Sf]^r u = \sum_{n \leq N} \int_{\Delta_n} [Sf]^r u \approx \sum_{n \leq N} 2^n \left( \int_{a_n}^{\infty} w(y) \left( \int_y^{\infty} k(z, y) f(z) \, dz \right)^q dy \right)^{r/q}
\]

\[
\approx \sum_{n \leq N} 2^n \left( \int_{\Delta_n} w(y) \left( \int_y^{a_{n+2}} k(z, y) f(z) \, dz \right)^q dy \right)^{r/q}
\]

\[
\approx \sum_{n \leq N} 2^n \left( \int_{\Delta_n} w(y) \left( \int_{a_{n+2}}^\infty k(z, y) f(z) \, dz \right)^q dy \right)^{r/q}
\]

\[
+ \sum_{n \leq N} 2^n \left( \int_{\Delta_n} w(y) \left( \int_{a_{n+2}}^\infty k(z, y) f(z) \, dz \right)^q dy \right)^{r/q} =: I_1^r + I_2^r.
\]

Since \( k(z, y) \approx k(z, \sigma^2(x)) + k(\sigma^2(x), y) \) for \( y \in \Delta_n, x \in \Delta_{n-1} \) and \( z \in (a_{n+2}, \infty) \), the term \( I_2^r \) can be estimated as follows:

\[
I_2^r \lesssim \sum_{n \leq N} \int_{a_{n-1}}^{a_n} u(x) \left( \int_{\Delta_n} w(y) \left( \int_{a_{n+2}}^\infty k(z, y) f(z) \, dz \right)^q dy \right)^{r/q} \, dx
\]

\[
\lesssim \sum_{n \leq N} \int_{a_{n-1}}^{a_n} u(x) \left( \int_{\Delta_n} w(y) \left( \int_{\sigma^2(x)}^\infty k(z, y) f(z) \, dz \right)^r dx \right)^{r/q}
\]

\[
+ \sum_{n \leq N} \int_{a_{n-1}}^{a_n} u(x) \left( \int_{\Delta_n} [k(\sigma^2(x), y)]^q w(y) \, dy \right)^{r/q} \left( \int_{\sigma^2(x)}^\infty f \, dx \right)^r
\]

\[
\lesssim \left( A_0^r + A_1^r \right) \left( \int_0^{\infty} f^p v \right)^{r/p}.
\]

To estimate \( I_1^r \) we write

\[
I_1^r \lesssim \sum_{n \leq N} \left( \int_{a_{n-1}}^{a_n} u \right) \left\| H_{a_n, a_{n+1}}^* \right\|_{L^p_v \to L^q_w} \left( \int_{a_n}^{a_{n+2}} f^p v \right)^{r/p}.
\]

If \( p \leq r \) then by Jensen’s inequality

\[
I_1 \lesssim \sup_{n \leq N} \left( \int_{a_{n-1}}^{a_n} u \right)^{1/r} \left\| H_{a_n, a_{n+1}}^* \right\|_{L^p_v \to L^q_w} \left\| f \right\|_{L^p_v} \lesssim A_2 \left\| f \right\|_{L^p_v}.
\]
If $r < p$, then using Hölder’s inequality with exponents $s/r$ and $p/r$, similarly to the proof of Theorem 1 we find that

$$I_1^s \lesssim \sum_{n \leq N} \left( \int_{a_n}^{a_{n+1}} u \right)^{s/r} \| H_{a_{n+1},a_n}^{s} \|_{L^p_{L^q_w}} \| f \|_{L^s_w}^s$$

$$\lesssim \sum_{n \leq N} \left( \int_{a_n}^{a_{n+1}} u(x) \left( \int_0^x u \right)^{s/p} \| H^{s-1}_{\sigma^{-1}(x),\sigma(x)} \|_{L^p_{L^q_w}} \| f \|_{L^p_w}^{s/p} \right) \| f \|_{L^p_w} \lesssim A_2^s \| f \|_{L^p_w}^s.$$

Thus, $C_S \lesssim A_0 + A_1 + A_2$.

Necessity. Since

$$(Sf)(x) \gtrsim \| \chi_{[x,\sigma^2(x)]} \|_{L^q_w} \int_{\sigma^2(x)}^\infty k(z,\sigma^2(x)) f(z) \, dz,$$

(1.3) implies (2.10) and $C_S \gtrsim A_0$. Also,

$$(Sf)(x) \geq \left( \int_x^{\sigma^2(x)} w(y) \left( \int_{\sigma^2(x)}^\infty k(z,y) f(z) \, dz \right)^q \, dy \right)^{1/q}$$

$$\gtrsim \left( \int_x^{\sigma^2(x)} w(y) k(\sigma^2(x),y) \, dy \right)^{1/q} \int_{\sigma^2(x)}^\infty f.$$ 

Therefore, (1.3) implies (2.11) and $C_S \gtrsim A_1$.

Now let $t \in (0,\infty)$. Then it follows from (1.3) that

$$C_S \| f \|_{L^p_w} \geq \left( \int_0^t u \right)^{1/r} \| H^{s}_{t} f \|_{L^q_w}, \quad f \in \mathcal{M}^+.$$ 

Hence

$$C_S \geq \sup_{t \in (0,\infty)} \left( \int_0^t u \right)^{1/r} \| H^{s}_{t} \|_{L^p_w \rightarrow L^q_w},$$

and therefore $C_S \gtrsim A_2$ for $p \leq r$.

Now let $r < p$. As in the proof of Theorem 1, we have

$$A_2^s = \int_0^\infty u(x) \left( \int_0^x u \right)^{s/p} \| H_{\sigma^{-1}(x),\sigma(x)}^{s} \|_{L^p_{L^q_w}} \, dx$$

$$\lesssim \sum_{n \leq N} (2^n)^{s/r} \| H_{a_{n+1},a_n+2}^{s} \|_{L^p_{L^q_w}}.$$

Let $\theta \in (0,1)$ be arbitrary. For each $n \leq N$ there exists $f_n \in \mathcal{M}^+$ such that $\text{supp} \, f_n \subset [a_{n-1},a_{n+3}]$, $\| f_n \|_{L^p_w} = 1$ and

$$\| H_{a_{n-1},a_n+2}^{s} f_n \|_{L^q_w} \geq \theta \| H_{a_{n-1},a_n+2}^{s} \|_{L^p_{L^q_w}}.$$ 

Put

$$g_n := (2^n)^{s/(pr)} \| H_{a_{n-1},a_n+2}^{s} \|_{L^p_{L^q_w}} f_n, \quad g := \sum_{n \leq N} g_n.$$
Then
\[ \|g\|_{L^p}^p = \sum_{j \leq N} \int_{a_j}^{a_{j+1}} \left( \sum_{n \leq N} g_n(x) \right)^p v(x) \, dx = \sum_{j \leq N} \int_{a_{j-2}}^{a_{j+1}} \left( \sum_{n=j-2}^{j+1} g_n(x) \right)^p v(x) \, dx \]
\[ \lesssim \sum_{j \leq N} \int_{a_{j-1}}^{a_{j+1}} g_j(x)^p v(x) \, dx = \sum_{j \leq N} (2^j)^{s/r} \|H^*_{a_{j-1},a_{j+1}}\|_{L^p \to L^q}^s. \]

Now,
\[ \int_0^\infty [Sg]^r u \lesssim \theta^r \sum_{n \leq N} (2^n)^{s/r} \|H^*_{a_{n-1},a_{n+1}}\|_{L^p \to L^q}^s \]
and we obtain \( C_S \gtrsim \Lambda_2 \) for \( r < p \), and \( C_S \gtrsim \Lambda_0 + \Lambda_1 + \Lambda_2 \) as in the proof of Theorem 1. The case \( q = \infty \) is treated analogously.

**Remark 2.** A precise characterization of inequalities (2.1)–(2.4) and (2.10)–(2.13) and sharp estimates of the norms \( \|H_t\|_{L^p \to L^q}, \|H^*_t\|_{L^p \to L^q} \), \( \|H^{(1)}_{\sigma^{-1}(x),\sigma(x)}\|_{L^p \to L^q} \), and \( \|H^*_{\sigma^{-1}(x),\sigma(x)}\|_{L^p \to L^q} \) are known and can be found (in various, but equivalent forms) in [9], [10] and [8], where an integral form of the criterion for \( 0 < q < 1 \) was found.

**Remark 3.** For \( p = \infty \) we have
\[ C_S = \left\| S \left( \frac{1}{\nu} \right) \right\|_{L^\infty} \]
and for \( r = \infty \)
\[ C_S = \sup_{t \geq 0} U(t) \|H^*_t\|_{L^p \to L^q}, \]
where \( U(t) := \text{ess sup}_{0 \leq x \leq t} u(x) \).

**§ 3. The operators \( \mathcal{J} \) and \( \mathcal{J}^* \)**

To find criteria for (1.2) and (1.4) assume for simplicity that \( 0 < \int_t^\infty u < \infty \) for all \( t > 0 \) and define the functions \( \zeta : [0, \infty) \to [0, \infty) \) and \( \zeta^{-1} : [0, \infty) \to [0, \infty) \) by
\[ \zeta(x) := \sup \left\{ y > 0 : \int_y^\infty u \geq \frac{1}{2} \int_x^\infty u \right\}, \]
\[ \zeta^{-1}(x) := \sup \left\{ y > 0 : \int_y^\infty u \geq 2 \int_x^\infty u \right\}. \]
where \( \sup \varnothing = 0 \). Let \( \zeta^m, m \in \mathbb{N} \), be the composition of \( m \) copies of \( \zeta \) and similarly for \( \zeta^{-m} \). For \( 0 \leq c < d < \infty \) and \( f \in \mathcal{M}^+ \) put

\[
(H_{c,d} f)(x) := \chi_{(c,d]}(x) \int_{c}^{d} k(z, x) f(z) \, dz,
\]

\[
(H_{d} f)(x) := \chi_{(0,d]}(x) \int_{0}^{d} k(z, x) f(z) \, dz,
\]

\[
(H_{c,d}^* f)(x) := \chi_{(c,d]}(x) \int_{c^{-1}(x)}^{d} k(z, x) f(z) \, dz,
\]

\[
(H_{d}^* f)(x) := \chi_{(0,d]}(x) \int_{0}^{d} k(z, x) f(z) \, dz.
\]

Similarly to the previous section we prove the following theorems.

**Theorem 3.** Let \( \frac{1}{6} < p \leq \infty \), \( 0 < r < \infty \), \( 0 < q \leq \infty \), \( \frac{1}{s} := \left( \frac{1}{r} - \frac{1}{p} \right)^+ \).

For (1.2) to hold it is necessary and sufficient that, for all \( f \in \mathcal{M}^+ \), the inequalities

\[
\left( \int_{0}^{\infty} u(x) \left( \int_{0}^{x} w \right)^{r/q} \left( \int_{x}^{\infty} k(z, x) f(z) \, dz \right)^{r} \, dx \right)^{1/r} \leq A_0 \| f \|_{L_p^v},
\]

\[
\left( \int_{0}^{\infty} u(x) \left( \int_{0}^{x} k(x, y)^{q} w(y) \, dy \right)^{r/q} \left( \int_{x}^{\infty} f \right)^{r} \, dx \right)^{1/r} \leq A_1 \| f \|_{L_p^v}
\]

hold for \( q < \infty \) or the inequalities

\[
\left( \int_{0}^{\infty} u(x) \left[ \text{ess sup}_{y \in (0,x)} w(y) \right]^{r} \left( \int_{x}^{\infty} k(z, x) f(z) \, dz \right)^{r} \, dx \right)^{1/r} \leq A_0 \| f \|_{L_p^v},
\]

\[
\left( \int_{0}^{\infty} u(x) \left[ \text{ess sup}_{y \in (0,x)} [w(y)k(x, y)] \right]^{r} \left( \int_{x}^{\infty} f \right)^{r} \, dx \right)^{1/r} \leq A_1 \| f \|_{L_p^v}
\]

hold for \( q = \infty \), and that the constant

\[
\mathcal{A}_2 := \begin{cases} \sup_{t \in (0,\infty)} \left( \int_{t}^{\infty} u \right)^{1/r} \| H_t \|_{L_p^v \rightarrow L_p^w}, & p \leq r, \\ \left( \int_{0}^{\infty} u(x) \left( \int_{x}^{\infty} u \right)^{s/p} \| H_{\zeta^{-1}(x), \zeta(x)} \|_{L_p^v \rightarrow L_p^w} \, dx \right)^{1/s}, & r < p, \end{cases}
\]

is finite. Moreover, \( C_\varnothing \approx \mathcal{A}_0 + \mathcal{A}_1 + \mathcal{A}_2 \).

**Theorem 4.** Let

\[
1 \leq p < \infty, \quad 0 < r < \infty, \quad 0 < q \leq \infty, \quad \frac{1}{s} := \left( \frac{1}{r} - \frac{1}{p} \right)^+.
\]
For (1.4) to hold it is necessary and sufficient that, for all $f \in \mathcal{M}^+$, the inequalities
\[
\left( \int_0^\infty u(x) \left( \int_0^x w(y) k(y, \zeta^{-2}(x)) \, dy \right)^{r/q} \left( \int_0^\infty k(\zeta^{-2}(x), z) f(z) \, dz \right)^r \, dx \right)^{1/r} \leq A_0 \| f \|_{L_0^r},
\]
\[
\left( \int_0^\infty u(x) \left( \int_0^x w(y) k(y, \zeta^{-2}(x)) \, dy \right)^{r/q} \left( \int_0^\infty k(\zeta^{-2}(x), z) f(z) \, dz \right)^r \, dx \right)^{1/r} \leq A_1 \| f \|_{L_0^r},
\]
hold for $q < \infty$ or the inequalities
\[
\left( \int_0^\infty u(x) \left( \sup_{y \in (\zeta^{-2}(x), x]} w(y) \right)^r \left( \int_0^\infty k(\zeta^{-2}(x), z) f(z) \, dz \right)^r \, dx \right)^{1/r} \leq A_0 \| f \|_{L_0^r},
\]
\[
\left( \int_0^\infty u(x) \left( \sup_{y \in (\zeta^{-2}(x), x]} w(y) k(y, \zeta^{-2}(x)) \right)^r \left( \int_0^\infty k(\zeta^{-2}(x), z) f(z) \, dz \right)^r \, dx \right)^{1/r} \leq A_1 \| f \|_{L_0^r},
\]
hold for $q = \infty$, and that the constant
\[
A_2 := \left\{ \begin{array}{ll}
\sup_{t \in (0, \infty)} \left( \int_0^\infty u(x) \right)^{1/r} & p \leq r, \\
\left( \int_0^\infty u(x) \right)^{s/p} & r < p,
\end{array} \right.
\]
is finite. Moreover, $C_{\gamma} \approx A_0 + A_1 + A_2$.

§ 4. The operators T and S

Let the functions $\sigma$ and $\sigma^{-1}$ be the same as in § 2. For $0 < c < d \leq \infty$ and $f \in \mathcal{M}^+$ we put
\[
(H_{c,d}f)(x) := \chi_{[c,d]}(x) \int_x^\infty f(z) \, dz, \quad (H_{c}f)(x) := \chi_{[c,\infty]}(x) \int_0^x f(z) \, dz,
\]
\[
(H_{c,d}^*f)(x) := \chi_{[c,d]}(x) \int_x^\infty f(z) \, dz, \quad (H_{c}^*f)(x) := \chi_{[c,\infty]}(x) \int_x^\infty f(z) \, dz.
\]

Theorem 5. Let
\[
1 \leq p < \infty, \quad 0 < r < \infty, \quad 0 < q \leq \infty, \quad \frac{1}{s} := \left( \frac{1}{r} - \frac{1}{p} \right) +.
\]
For (1.5) to hold it is necessary and sufficient that
\[
B := B_0 + B_1 + B_2 < \infty,
\]
where $B_0$ and $B_1$ are the smallest possible constants in the inequalities
\[
\left( \int_0^\infty u(x) \left( \int_x^\infty k(y, x) w(y) \, dy \right)^{r/q} \left( \int_0^x f(z) \, dz \right)^r \, dx \right)^{1/r} \leq B_0 \| f \|_{L_0^r},
\]
\[
\left( \int_0^\infty u(x) \left( \int_x^\infty k(\sigma^2(x), x) \right)^{r/q} \left( \int_0^\infty w(y) \left( \int_0^y f(z) \, dz \right)^{r/q} \, dy \right)^r \, dx \right)^{1/r} \leq B_1 \| f \|_{L_0^r},
\]
for $q < \infty$ or
\[
\left( \int_0^\infty u(x) \left[ \text{ess sup}_{y \geq x} k(y, x) w(y) \right]^r \left( \int_0^x f \right)^r \, dx \right)^{1/r} \leq B_0 \|f\|_{L^p_v}, \tag{4.4}
\]
\[
\left( \int_0^\infty u(x) |k(\sigma^2(x), x)|^r \left( \text{ess sup}_{y \geq \sigma^2(x)} w(y) \right) \left( \int_0^y f \right)^r \, dx \right)^{1/r} \leq B_1 \|f\|_{L^p_v}, \tag{4.5}
\]
for $q = \infty$ and $B_2$ is defined by
\[
B_2 := \begin{cases}
\sup_{t > 0} \left( \int_0^t u(x) \right)^{1/r} \left\|H_{\sigma(t)}\right\|_{L^p_v \to L^q_{w(\cdot)k(\cdot, t)}}', & p \leq r, \\
\left( \int_0^\infty u(x) \left( \int_0^x u \right)^{s/p} \|H_{\sigma^{-1}(x), \sigma^2(x)}\|_{L^p_v \to L^q_{w(\cdot)k(\cdot, \sigma^{-1}(x))}} \, dx \right)^{1/s}, & r < p.
\end{cases}
\tag{4.6}
\]
Moreover, $C_T \approx B$.

**Proof.** Let the sequence $\{a_n\}$ be the same as in the proof of Theorem 1 and let $q < \infty$.

**Sufficiency.** We write
\[
J := \int_0^\infty [Tf]^r u = \sum_{n \leq N} \int_{a_n}^{a_n+1} [Tf]^r u
\]
\[
\approx \sum_{n \leq N} 2^n \left( \int_{a_n}^{a_n+2} k(y, a_n) w(y) \left( \int_0^y f \right)^q \, dy \right)^{r/q} \approx J_1 + J_2,
\]
where
\[
J_1 := \sum_{n \leq N} 2^n \left( \int_{a_n}^{a_n+2} k(y, a_n) w(y) \left( \int_0^y f \right)^q \, dy \right)^{r/q},
\]
\[
J_2 := \sum_{n \leq N} 2^n \left( \int_{a_n}^{a_n+2} k(y, a_n) w(y) \left( \int_0^y f \right)^q \, dy \right)^{r/q}.
\]

**Estimate of $J_1$.** We have
\[
J_1 \approx \sum_{n \leq N} 2^n \left( \int_{a_n}^{a_n+2} k(y, a_n) w(y) \left( \int_{a_n-1}^y f \right)^q \, dy \right)^{r/q}
\]
\[
+ \sum_{n \leq N} 2^n \left( \int_{a_n}^{a_n+2} k(y, a_n) w(y) \, dy \right)^{r/q} \left( \int_{a_n-1}^y f \right)^r = J_{1,1} + J_{1,2}.
\]

For $J_{1,2}$ we write
\[
J_{1,2} \approx \sum_{n \leq N} \int_{a_n}^{a_n+1} u(x) \, dx \left( \int_{a_n}^{a_n+2} k(y, a_n) w(y) \, dy \right)^{r/q} \left( \int_0^{a_n-1} f \right)^r
\]
\[
\leq \int_0^\infty u(x) \left( \int_x^\infty k(y, x) w(y) \, dy \right)^{r/q} \left( \int_0^x f \right)^r \, dx \leq B_0 \left( \int_0^\infty f^p v \right)^{r/p}.
\]
For $J_{1,1}$ we write

$$J_{1,1} \approx \sum_{n \leq N} 2^n \left( \int_{a_n}^{a_{n+2}} k(y, a_n) w(y) (H_{a_n, a_{n+2}} f(y))^q dy \right)^{r/q}$$

$$\lesssim \sum_{n \leq N} 2^n \left\| H_{a_n, a_{n+2}} \right\|_{L^p_v \rightarrow L^q_v (k(\cdot, a_n))} \left( \int_{a_{n-1}}^{a_{n+2}} f^p dy \right)^{r/p}.$$  

If $p \leq r$, by Jensen's inequality we obtain

$$J_{1,1} \lesssim B_2^r \| f \|_{L^p_v}.$$  

If $r < p$, by Hölder's inequality

$$J_{1,1} \lesssim \left( \sum_{n \leq N} (2^n)^{s/r} \left\| H_{a_n, a_{n+2}} \right\|_{L^p_v \rightarrow L^q_v (k(\cdot, a_n))} \right)^{r/s} \| f \|_{L^p_v}$$

$$\lesssim \left( \int_{a_n}^{a_{n+1}} \left( \int_{a_n}^{a_{n+2}} u^s \left\| H_{a_{n+1-1}, a_{n+1}} \right\|_{L^p_v \rightarrow L^q_v (k(\cdot, a_{n+1}))} \right)^{r/s} \| f \|_{L^p_v}$$

$$\lesssim B_2^r \| f \|_{L^p_v}.$$  

Thus,

$$J_1 \lesssim (B_0 + B_2)^r \| f \|_{L^p_v}.$$  \hfill (4.7)

Estimate of $J_2$. Set

$$h(y) := w(y) \left( \int_0^y f \right)^q.$$  

Using (1.9) we obtain

$$\int_{a_{n+2}}^{a_{n+3}} k(y, a_n) h(y) dy = \sum_{i \geq n} \int_{a_{i+2}}^{a_{i+3}} k(y, a_n) h(y) dy$$

$$\approx \sum_{i \geq n} \int_{a_{i+2}}^{a_{i+3}} k(y, a_{i+1}) h(y) dy + \sum_{i \geq n} \int_{a_{i+2}}^{a_{i+3}} k(a_{i+1}, a_n) h(y) dy$$

$$\approx \sum_{i \geq n} \int_{a_{i+1}}^{a_{i+3}} k(y, a_{i+1}) h(y) dy + \sum_{i \geq n} \int_{a_{i+2}}^{a_{i+3}} k(a_{i+1}, a_n) h(y) dy =: I_{1,n} + I_{2,n}.$$  

Similarly to the proof of (4.7) we obtain

$$\sum_{n \leq N} 2^n I_{1,n}^{r/q} \approx J_1 \lesssim (B_0 + B_2)^r \| f \|_{L^p_v}.$$  \hfill (4.8)

By [30], Lemma 3.1, there exists $\alpha \in (0, 1)$ such that

$$k(a_{i+1}, a_n) \lesssim \left( \sum_{j=n}^i [k(a_{j+1}, a_j)]^{\alpha} \right)^{1/\alpha}, \quad i \geq n.$$  \hfill (4.9)
By Minkowski’s inequality

\[ I_{2,n} \lesssim \sum_{i \geq n} \left( \sum_{j=n}^{i} |k(a_{j+1}, a_j)|^\alpha \right)^{1/\alpha} \int_{a_{j+2}}^{a_{i+3}} h(y) \, dy \]

\[ \lesssim \left( \sum_{j \geq n} |k(a_{j+1}, a_j)|^\alpha \left( \int_{a_{j+2}}^{\infty} h \right)^{\alpha} \right)^{1/\alpha}. \]

Hence

\[ \sum_{n \leq N} 2^n I_{2,n}^{r/q} \lesssim \sum_{n \leq N} 2^n \left( \sum_{j \geq n} |k(a_{j+1}, a_j)|^\alpha \left( \int_{a_{j+2}}^{\infty} h \right)^{\alpha} \right)^{r/(q\alpha)} \]

\[ \approx \sum_{n \leq N} 2^n k(a_{n+1}, a_n)^{r/q} \left( \int_{a_{n+2}}^{\infty} h \right)^{r/q} \]

\[ \approx \sum_{n \leq N} \left[ \int_{a_{n-1}}^{a_n} u \right] k(\sigma^2(a_{n-1}), a_n)^{r/q} \left( \int_{\sigma^2(a_n)}^{\infty} h \right)^{r/q} \]

\[ \lesssim \int_{0}^{\infty} u(x) k(\sigma^2(x), x)^{r/q} \left( \int_{\sigma^2(x)}^{\infty} w(y) \left( \int_{0}^{y} f \, dy \right)^{q} \right)^{r/q} \, dx \leq B_1^r \| f \|_{L_p^r}. \]

In the case when \( q = \infty \) we write

\[ \text{ess sup}_{y \in [a_{n+2}, \infty)} k(y, a_n) h(y) = \sup_{i \geq n} \text{ess sup}_{y \in [a_{i+2}, a_{i+3})} k(y, a_n) h(y) \]

\[ \lesssim \sup_{i \geq n} \text{ess sup}_{y \in [a_{i+1}, a_{i+3})} k(y, a_{i+1}) h(y) + \sup_{i \geq n} k(a_{i+1}, a_n) \text{ess sup}_{y \in [a_{i+2}, a_{i+3})} h(y) \]

\[ =: I_{1,n} + I_{2,n}. \]

The estimate (4.8) follows in the same way. We also have

\[ I_{2,n} \lesssim \left( \sup_{i \geq n} \sum_{j=n}^{i} |k(a_{j+1}, a_j)|^\alpha \left( \text{ess sup}_{y \in [a_{i+2}, a_{i+3})} h(y) \right)^{\alpha} \right)^{1/\alpha} \]

\[ \leq \left( \sum_{j \geq n} |k(a_{j+1}, a_j)|^\alpha \left( \text{ess sup}_{y \in [a_{j+2}, \infty)} h(y) \right)^{\alpha} \chi_{[n, i]}(j) \right)^{1/\alpha} \]

\[ = \left( \sum_{j \geq n} |k(a_{j+1}, a_j)|^\alpha \left( \text{ess sup}_{y \in [a_{j+2}, \infty)} h(y) \right)^{\alpha} \right)^{1/\alpha}, \]

and the inequality \( \sum_{n \leq N} 2^n I_{2,n}^{r/q} \lesssim B_1^r \| f \|_{L_p^r} \) follows in this case too. Thus,

\[ J_2 \lesssim (B_0 + B_1 + B_2)^r \| f \|_{L_p^r} \]

and the upper bound \( C_T \lesssim B_0 + B_1 + B_2 \) is proved.

Necessity. Suppose (1.5) holds, that is,

\[ \left( \int_{0}^{\infty} (\int_{x}^{\infty} k(y, x) w(y) \left( \int_{0}^{y} f \, dy \right)^{r/q} u(x) \, dx \right)^{1/r} \right) \leq C_T \left( \int_{0}^{\infty} f^p v \right)^{1/p} \quad \text{(4.10)} \]
for all \(f \in \mathcal{M}^+\). Restricting the integration on the left-hand side, \((0, y) \to (0, x)\), we see that \(C_T \geq B_0\). Analogously, if \((x, \infty) \to (\sigma^2(x), \infty)\), \(k(y, x) \geq k(\sigma^2(x), x)\), then \(C_T \geq B_1\). If \((0, \infty) \to (0, t)\), \((x, \infty) \to (t, \infty)\) and \(k(y, x) \geq k(y, t)\), then
\[
C_T \gtrsim \left( \int_0^t u \right)^{1/r} \| H_t \|_{L^p_v \to L^q_{L^w(k \cdot t)}} \tag{4.11}\]
for all \(t > 0\). Consequently, \(C_T \gtrsim B_2\) for \(p \leq r\).

In the case when \(r < p\) we write
\[
B_2^s = \frac{1}{2} \int_0^\infty \max \left( \frac{1}{2}, \left( \frac{\| H_{\sigma^{-1}(x), \sigma^2(x)} \|_{L^p_v \to L^q_{L^w(k \cdot \sigma^{-1}(x))}}}{\| H_{\sigma^{-1}(x), \sigma^2(x)} \|_{L^p_v \to L^q_{L^w(k \cdot \sigma^{-1}(x))}}} \right) \right) \ dx.
\]

Let \(\theta \in (0, 1)\) be arbitrary. Then for each \(n \leq N\) there exists \(f_n \in \mathcal{M}^+\) such that \(\supp f_n \subseteq [a_{n-2}, a_{n+3}]\), \(\| f_n \|_{L^p_v} = 1\) and
\[
\| H_{a_{n-1}, a_{n+3}} f_n \|_{L^q_{L^w(k \cdot a_{n-1})}} \geq \theta \| H_{a_{n-1}, a_{n+3}} \|_{L^p_v \to L^q_{L^w(k \cdot a_{n-1})}}.
\]

Put
\[
g_n := (2^n)^{s/p} \| H_{a_{n-1}, a_{n+3}} \|^s_{L^p_v \to L^q_{L^w(k \cdot a_{n-1})}} f_n, \quad g := \sum_{n \leq N} g_n.
\]

We have
\[
\| g \|_{L^p_v} \geq \sum_{j \leq N} \int_{a_j}^{a_{j+1}} \left( \sum_{n \leq N} g_n(x) \right)^p v(x) \ dx = \sum_{j \leq N} \int_{a_{j-2}}^{a_{j+1}} \left( \sum_{n=j-2}^{j+2} g_n(x) \right)^p v(x) \ dx
\]
\[
\geq \sum_{j \leq N} \int_{a_{j-2}}^{a_{j+3}} g_j(x)^p v(x) \ dx = \sum_{j \leq N} (2^j)^{s/r} \| H_{a_{j-1}, a_{j+3}} \|^s_{L^p_v \to L^q_{L^w(k \cdot a_{j-1})}} = B_2^s.
\]

Finally, applying (1.1)
\[
C_T B_2^r \gtrsim C_T \| g \|_{L^p_v} \geq \int_0^\infty \| T g \| \cdot u \geq \sum_{n \leq N} \int_{a_{n-2}}^{a_{n-1}} \| T g \| \cdot u
\]
\[
\geq \sum_{n \leq N} \int_{a_{n-2}}^{a_{n-1}} \| H_{a_{n-1}, a_{n+3}} g \|^s_{L^q_{L^w(k \cdot a_{n-1})}} \geq \sum_{n \leq N} 2^n \| H_{a_{n-1}, a_{n+3}} g_n \|^s_{L^q_{L^w(k \cdot a_{n-1})}}
\]
\[
= \sum_{n \leq N} (2^n)^{s/r} \| H_{a_{n-1}, a_{n+3}} \|^s_{L^p_v \to L^q_{L^w(k \cdot a_{n-1})}} \| H_{a_{n-1}, a_{n+3}} f_n \|^s_{L^q_{L^w(k \cdot a_{n-1})}} \geq \theta^r B_2^s.
\]

Thus, \(C_T \geq \theta B_2\). Hence \(C_T \geq \theta B_2\) and the lower bound \(C_T \geq B_0 + B_1 + B_2\) follows, as required.
Remark 4. Similarly to (2.8) and (2.9) we have

\[ C_T = \left\| T \left( \frac{1}{v} \right) \right\|_{L^p_v}, \quad p = \infty, \quad (4.12) \]

\[ C_T \approx \sup_{t \geq 0} U(t) \| H_t \|_{L^p_v \rightarrow L^q_w(\cdot, t)}, \quad r = \infty. \quad (4.13) \]

Theorem 6. Let

\[ 1 \leq p < \infty, \quad 0 < r < \infty, \quad 0 < q \leq \infty, \quad \frac{1}{s} := \left( \frac{1}{r} - \frac{1}{p} \right)_+. \]

For (1.7) to hold it is necessary and sufficient that

\[ \mathbb{B} := \mathbb{B}_0 + \mathbb{B}_1 + \mathbb{B}_2 < \infty, \quad (4.14) \]

where \( \mathbb{B}_0 \) and \( \mathbb{B}_1 \) are the smallest possible constants in the inequalities

\[ \left( \int_0^\infty u(x) \left( \int_x^{\sigma(x)} k(y, x) w(y) dy \right)^{r/q} \left( \int_x^{\sigma(x)} f \right)^r dx \right)^{1/r} \leq \mathbb{B}_0 \| f \|_{L^p_v}, \quad (4.15) \]

\[ \left( \int_0^\infty u(x) k(\sigma^2(x), x)^{r/q} \left( \int_x^{\sigma^2(x)} w(y) \left( \int_y^{\sigma(x)} f \right)^q dy \right)^{r/q} dx \right)^{1/r} \leq \mathbb{B}_1 \| f \|_{L^p_v}, \quad (4.16) \]

when \( q < \infty \) and in

\[ \left( \int_0^\infty u(x) \left[ \text{ess sup}_{x \leq y \leq \sigma^2(x)} k(y, x) w(y) \right]^r \left( \int_x^{\sigma(x)} f \right)^r dx \right)^{1/r} \leq \mathbb{B}_0 \| f \|_{L^p_v}, \quad (4.17) \]

\[ \left( \int_0^\infty u(x) \left[ k(\sigma^2(x), x)^r \left( \text{ess sup}_{y \geq \sigma^2(x)} w(y) \int_y^{\sigma(x)} f \right)^r dx \right]^{1/r} \leq \mathbb{B}_1 \| f \|_{L^p_v}, \quad (4.18) \]

when \( q = \infty \). The constant \( \mathbb{B}_2 \) is given by

\[ \mathbb{B}_2 := \begin{cases} \sup_{t > 0} \left( \int_0^t u \right)^{1/r} \left\| H_t \right\|_{L^p_v \rightarrow L^q_w(\cdot, t)}, & p \leq r, \\ \left( \int_0^\infty u(x) \left( \int_0^x u \right)^{s/p} \left\| H^{*-1}_{\sigma^2(x), \sigma^2(x)} \right\|_{L^p_v \rightarrow L^q_w(\cdot, \sigma^{-1}(x))} dx \right)^{1/s}, & r < p. \end{cases} \quad (4.19) \]

Moreover, \( C_S \approx \mathbb{B} \).

Proof. Let the sequence \( \{ a_n \} \) be the same as in the proof of Theorem 5, and let \( q < \infty \).

Sufficiency. We write

\[ J := \int_0^\infty [Sf]^r u = \sum_{n \leq N} \int_{a_n}^{a_{n+1}} [Sf]^r u \]

\[ \approx \sum_{n \leq N} 2^n \left( \int_{a_n}^{\infty} k(y, a_n) w(y) \left( \int_y^{\infty} f \right)^q dy \right)^{r/q} \approx J_1 + J_2, \]
where

\[
J_1 := \sum_{n \leq N} 2^n \left( \int_{a_n}^{a_{n+2}} k(y, a_n) w(y) \left( \int_y^\infty f \, dy \right)^{r/q} \right), \\
J_2 := \sum_{n \leq N} 2^n \left( \int_{a_{n+2}}^{a_{n+4}} k(y, a_n) w(y) \left( \int_y^\infty f \, dy \right)^{r/q} \right).
\]

**Estimate of** \(J_1\). **We have**

\[
J_1 \approx \sum_{n \leq N} 2^n \left( \int_{a_n}^{a_{n+2}} k(y, a_n) w(y) \left( \int_y^{a_{n+3}} f \, dy \right)^{r/q} \right) \\
+ \sum_{n \leq N} 2^n \left( \int_{a_n}^{a_{n+2}} k(y, a_n) w(y) \, dy \right)^{r/q} \left( \int_{a_{n+3}}^{\infty} f \, dy \right)^r = J_{1,1} + J_{1,2}.
\]

For \(J_{1,2}\) we write

\[
J_{1,2} \approx \sum_{n \leq N} \int_{x=a_{n-1}}^{x=a_n} u(x) \, dx \left( \int_{a_n}^{a_{n+2}} k(y, a_n) w(y) \, dy \right)^{r/q} \left( \int_{a_{n+3}}^{\infty} f \, dy \right)^r \\
\lesssim \int_0^{\infty} u(x) \left( \int_x^{\sigma^3(x)} k(y, x) w(y) \, dy \right)^{r/q} \left( \int_{\sigma^3(x)}^{\infty} f \, dx \right)^r \leq B_0^r \left( \int_0^{\infty} f^p w \right)^{r/p}.
\]

For \(J_{1,1}\) we write

\[
J_{1,1} \approx \sum_{n \leq N} 2^n \left( \int_{a_n}^{a_{n+2}} k(y, a_n) w(y) \left( H^*_{a_n, a_n+2} f(y) \right)^{q} \, dy \right)^{r/q} \\
\lesssim \sum_{n \leq N} 2^n \left\| H^*_{a_n, a_n+2} \right\|_{L^p_v \rightarrow L^q_w(k(\cdot, a_n))} \left( \int_{a_n}^{a_{n+3}} f^p \, dy \right)^{r/p}.
\]

If \(p \leq r\), then by Jensen’s inequality

\[
J_{1,1} \lesssim B_2^r \left\| f \right\|_{L^p_v}.
\]

If \(r < p\), then by Hölder’s inequality

\[
J_{1,1} \lesssim \left( \sum_{n \leq N} \left( 2^n \right)^{s/r} \left\| H^*_{a_n, a_n+2} \right\|_{L^s_v \rightarrow L^q_w(k(\cdot, a_n))} \right)^{r/s} \left\| f \right\|_{L^p_v} \\
\lesssim \left( \sum_{n \leq N} \int_{a_n}^{a_{n+1}} u \left( \int_{a_n}^{a_{n+1}} u \right)^{s/p} \left\| H^*_{\sigma^{-1}(a_n+1), \sigma^2(a_n)} \right\|_{L^s_v \rightarrow L^q_w(k(\cdot, \sigma^{-1}(a_n+1)))} \right)^{r/s} \left\| f \right\|_{L^p_v} \\
\lesssim B_2^r \left\| f \right\|_{L^p_v}.
\]

Thus

\[
J_1 \lesssim (B_0 + B_2)^r \left\| f \right\|_{L^p_v}.
\]  (4.20)
Estimate of $J_2$. Set $h(y) := w(y)\left(\int_{y}^{\infty} f\right)^{q}$. Arguing similarly to the proof of Theorem 5 we obtain

$$J_2 \lesssim (\mathcal{B}_0 + \mathcal{B}_1 + \mathcal{B}_2)^r \|f\|_{L^p_r}.$$  

Necessity. Suppose that (1.3) holds, that is,

$$\left(\int_{x}^{\infty} k(y, x)w(y)\left(\int_{y}^{\infty} f\right)^{q} dy\right)^{r/q} u(x) dx \quad \leq \quad C_{\mathcal{S}} \left(\int_{0}^{\infty} f^p v \right)^{1/p}$$  

for all $f \in \mathfrak{M}^+$. Restricting the integration on the left hand side, $(x, \infty) \to (x, \sigma^3(x))$ and $(y, \infty) \to (\sigma^3(x), \infty)$, we see that $C_{\mathcal{S}} \geq \mathcal{B}_0$. Analogously, if $(x, \infty) \to (\sigma^2(x), \infty)$ and $k(y, x) \gtrsim k(\sigma^2(x), x)$, then $C_{\mathcal{S}} \gtrsim \mathcal{B}_1$. The proof that $C_{\mathcal{T}} \gtrsim \mathcal{B}_2$ is similar to the proof that $C_{\mathcal{T}} \gtrsim \mathcal{B}_2$.

Remark 5. For the extreme values we have

$$C_{\mathcal{S}} = \left\| S \left(\frac{1}{v}\right) \right\|_{L^p_0}, \quad p = \infty,$$

$$C_{\mathcal{S}} \approx \sup_{t \geq 0} U(t)\|H^*_t\|_{L^p_0} \to L^q_{\omega(\cdot)k(\cdot, t)}, \quad r = \infty.$$  

§ 5. The operators $\mathcal{T}$ and $\mathcal{S}$

Suppose $\zeta, \zeta^{-1} : [0, \infty) \to [0, \infty)$ are the same as in § 3. For $0 \leq c < d < \infty$ and $f \in \mathfrak{M}^+$ we define the operators

$$(\mathcal{T}_{c,d}f)(x) := \chi_{(c,d)}(x) \int_{x}^{c(d)} f(z) dz, \quad (\mathcal{T}_{d}f)(x) := \chi_{(0,d]}(x) \int_{x}^{\infty} f(z) dz,$$

$$(\mathcal{T}^*_{c,d}f)(x) := \chi_{(c,d)}(x) \int_{\zeta^{-1}(c)}^{x} f(z) dz, \quad (\mathcal{T}^*_{d}f)(x) := \chi_{(0,d]}(x) \int_{x}^{\infty} f(z) dz.$$  

The following theorems hold.

Theorem 7. Let

$$1 \leq p < \infty, \quad 0 < r < \infty, \quad 0 < q \leq \infty, \quad \frac{1}{s} := \left(\frac{1}{r} - \frac{1}{p}\right)_+.$$  

For (1.6) to hold it is necessary and sufficient that, for all $f \in \mathfrak{M}^+$, the inequalities

$$\left(\int_{0}^{\infty} u(x)\left(\int_{0}^{x} k(x, y)w(y) dy\right)^{r/q} \left(\int_{x}^{\infty} f dx\right)^{r} \right)^{1/r} \leq \mathcal{B}_0 \|f\|_{L^p_0},$$

$$\left(\int_{0}^{\infty} u(x)[k(x, \zeta^{-2}(x))]^{r/q} \left(\int_{0}^{\zeta^{-2}(x)} w(y)\left(\int_{y}^{\infty} f dy\right)^{q} dx\right)^{r/q} \right)^{1/r} \leq \mathcal{B}_1 \|f\|_{L^p_0}$$

hold for $q < \infty$ or the inequalities

$$\left(\int_{0}^{\infty} u(x)\left[\overset{\text{ess sup}}{y \in (0,x)} k(x, y)w(y)\right]^{r} \left(\int_{x}^{\infty} f dx\right)^{r} \right)^{1/r} \leq \mathcal{B}_0 \|f\|_{L^p_0},$$

$$\left(\int_{0}^{\infty} u(x)[k(x, \zeta^{-2}(x))]^{r} \left(\overset{\text{ess sup}}{y \in (0,\zeta^{-2}(x))} w(y)\left(\int_{y}^{\infty} f dx\right)^{r} \right)^{1/r} \leq \mathcal{B}_1 \|f\|_{L^p_0}.$$  

hold for $q = \infty$, and that the constant

$$
B_2 := \begin{cases}
\sup_{t \in (0, \infty)} \left( \int_{t}^{\infty} u \right)^{1/r} \| \tilde{f}_t \|_{L^q_{w(\cdot)k(t, \cdot)}}^r, & p \leq r, \\
\left( \int_{0}^{\infty} u(x) \left( \int_{x}^{\infty} u \right)^{s/p} \| \tilde{f}_{1}(x) \|_{L^q_{w(\cdot)k(x^2, \cdot)}}^s \right)^{1/s}, & r < p,
\end{cases}
$$

is finite. Moreover, $C_{\infty} \approx B_0 + B_1 + B_2$.

**Theorem 8.** Let

$$1 \leq p < \infty, \quad 0 < r < \infty, \quad 0 < q \leq \infty, \quad \frac{1}{s} := \left( \frac{1}{r} - \frac{1}{p} \right) +$$

For (1.8) to hold it is necessary and sufficient that, for all $f \in M^+$, the inequalities

$$
\left( \int_{0}^{\infty} u(x) \left( \int_{x}^{\infty} k(x, y)w(y) \right)^{r/q} \left( \int_{0}^{\zeta^{-3}(x)} f \right) \right)^{1/r} \leq B_0 \| f \|_{L^p_v},
$$

$$
\left( \int_{0}^{\infty} u(x)[k(x, \zeta^{-2}(x))]^{r/q} \left( \int_{0}^{\zeta^{-2}(x)} w(y) \left( \int_{0}^{y} f \right) \right) \right)^{1/r} \leq B_1 \| f \|_{L^p_v},
$$

hold for $q < \infty$ or the inequalities

$$
\left( \int_{0}^{\infty} u(x) \left[ \text{ess sup}_{y \in (0, \zeta^{-2}(x))} k(x, y)w(y) \right]^{r} \left( \int_{0}^{\zeta^{-3}(x)} f \right) \right)^{1/r} \leq B_0 \| f \|_{L^p_v},
$$

$$
\left( \int_{0}^{\infty} u(x)[k(x, \zeta^{-2}(x))]^{r} \left( \text{ess sup}_{y \in (0, \zeta^{-2}(x))} w(y) \int_{0}^{y} f \right) \right)^{1/r} \leq B_1 \| f \|_{L^p_v},
$$

hold for $q = \infty$, and that the constant

$$
B_2 := \begin{cases}
\sup_{t \in (0, \infty)} \left( \int_{t}^{\infty} u \right)^{1/r} \| \tilde{f}_t \|_{L^q_{w(\cdot)k(t, \cdot)}}^r, & p \leq r, \\
\left( \int_{0}^{\infty} u(x) \left( \int_{x}^{\infty} u \right)^{s/p} \| \tilde{f}_{1}(x) \|_{L^q_{w(\cdot)k(x^2, \cdot)}}^s \right)^{1/s}, & r < p,
\end{cases}
$$

is finite. Moreover, $C_{\infty} \approx B_0 + B_1 + B_2$.

**§ 6.** $\Gamma^p(v) \to \Gamma^q(w)$ boundedness of the maximal operator

The maximal Hardy-Littlewood operator is defined by

$$
Mf(x) := \sup_B \frac{1}{\text{mes } B} \int_B |f(y)| dy,
$$

where the supremum is taken over all balls centred at $x \in \mathbb{R}^n$. The Lorentz $\Gamma$-spaces were introduced by Sawyer [31] while working on a characterization of the boundedness of the maximal operator in weighted Lorentz spaces (see also [30] and [32]–[36]). More precisely, if $v \in M^+$ and $0 < p < \infty$, then

$$
\Gamma^p(v) = \left\{ f \text{ measurable on } \mathbb{R}^n : \left( \int_{0}^{\infty} [f^{**}(x)]^p v(x) dx \right)^{1/p} < \infty \right\},
$$
where \( f^{**}(x) := \frac{1}{x} \int_0^x f^*(t) \, dt \) and
\[
f^*(t) := \inf \left\{ s > 0 : \text{mes} \{ x : |f(x)| > s \} \leq t \right\}.
\]

It is known ([37], Theorem 3.8) that
\[
[Mf]^*(x) \approx \frac{1}{x} \int_0^x f^*.
\]

Therefore, \( M : \Gamma_1^p(v) \to \Gamma_1^q(u) \) boundedness is equivalent to the weighted inequality
\[
\left( \int_0^\infty \left[ \frac{1}{x} \int_0^x \left( \frac{1}{y} \int_0^y f \right) \, dy \right]^q u(x) \, dx \right)^{1/q} \leq C \left( \int_0^\infty \left( \frac{1}{t} \int_0^t f \right)^p v(t) \, dt \right)^{1/p},
\]
\[
(6.1)
\]

holding; this is restricted to the cone \( \mathcal{M}^1 \subset \mathcal{M}^+ \) of all nonincreasing functions. Moreover, the smallest possible constant \( C \) is equivalent to the norm of \( M : \Gamma_1^p(v) \to \Gamma_1^q(u) \):
\[
C \approx \|M\|_{\Gamma_1^p(v) \to \Gamma_1^q(u)} := \sup_{0 \neq f \in \Gamma_1^p(v)} \frac{\|Mf\|_{\Gamma_1^q(u)}}{\|f\|_{\Gamma_1^p(v)}}.
\]

Inequality (6.1) was first characterized in the case \( 1 < p = q < \infty, u = v \) (see [38], Theorem 5.1) and for \( 1 < p, q < \infty \) and \( u \neq v \) in [28], Theorem 3.3 and [29], Theorem 5.1 (see also [39]).

Using Theorems 3 and 4 we solve the problem for all \( p, q, 0 < p, q < \infty \), and our criteria have an explicit integral form.

Let \( \Omega_{1,0} := \{ g \in \mathcal{M}^1, t g(t) \in \mathcal{M}^1 \} \). Then \( F(t) = \frac{1}{t} \int_0^t f \in \Omega_{1,0} \) for any \( f \in \mathcal{M}^1 \) and \( F^p \in \Omega_{p,0} := \{ g \in \mathcal{M}^1, t^p g(t) \in \mathcal{M}^1 \} \). Making the substitution \( G = F^p \), (6.1) becomes equivalent to
\[
\left( \int_0^\infty \left( \frac{1}{x} \int_0^x G^{1/p} \right)^q u(x) \, dx \right)^{p/q} \leq C^p \int_0^\infty G v, \quad G \in \Omega_{p,0},
\]
\[
(6.2)
\]

and using Lemma 2.3 in [29] we reduce (6.2) to the inequality
\[
\left( \int_0^\infty \left( \frac{1}{x} \int_0^x \left( \int_0^\infty h(z) \, dz \right)^{1/p} \right)^q u(x) \, dx \right)^{p/q} \leq C^p \int_0^\infty h V, \quad h \in \mathcal{M}^+, \quad (6.3)
\]

where
\[
V(z) = \int_0^\infty \frac{v(y) \, dy}{yp + z^p}.
\]

Since
\[
\int_0^\infty h(z) \, dz \approx \int_y^\infty \frac{h(z) \, dz}{zp} + \frac{1}{yp} \int_0^y h(z) \, dz,
\]

\[
(6.3)
\]
(6.3) is characterized by the following pair of inequalities:
\[
\left( \int_0^\infty \left( \int_0^x \left( \int_y^\infty h(z) \, dz \right)^{1/p} \, dy \right)^q \, u(x) \, dx \right)^{p/q} \leq C_1^p \int_0^\infty h(t) t^p V(t) \, dt, \quad h \in M^+,
\]
\[
\left( \int_0^x \left( \int_0^y \left( \int_0^\infty h(z) \, dz \right)^{1/p} \, dy \right)^q \, u(x) \, dx \right)^{p/q} \leq C_2^p \int_0^\infty hV, \quad h \in M^+,
\]
which are of the form (1.2) and (1.4), respectively. Moreover,
\[
C \approx C_1 + C_2.
\]
Hence applying Theorems 3.1 and 3.2 we see that
\[
C_1 \approx \mathcal{A}_0 + \mathcal{A}_2, \quad (6.4)
\]
\[
C_2 \approx \mathcal{A}_0 + \mathcal{A}_2, \quad (6.5)
\]
where the constants of type A are defined in (3.1) and (3.2) for (6.4) and in (3.3) and (3.4) for (6.5) under an appropriate choice of the weights, the function \(\zeta\) and the auxiliary operators.

For simplicity suppose that
\[
0 < \int_0^\infty \frac{u(s)}{s^q} \, ds < \infty
\]
for all \(t > 0\). Now the functions \(\zeta\) and \(\zeta^{-1}\) are defined by
\[
\zeta(x) := \sup \left\{ y > 0 : \int_y^\infty \frac{u(s)}{s^q} \, ds \geq \frac{1}{2} \int_x^\infty \frac{u(s)}{s^q} \, ds \right\},
\]
\[
\zeta^{-1}(x) := \sup \left\{ y > 0 : \int_y^\infty \frac{u(s)}{s^q} \, ds \geq 2 \int_x^\infty \frac{u(s)}{s^q} \, ds \right\}.
\]

For \(0 \leq c < d < \infty\) and \(h \in M^+\) we put
\[
(\mathcal{H}_{c,d} h)(x) := \chi(c,d)(x) \int_x^\infty h, \quad (\mathcal{H}_d h)(x) := \chi(0,d)(x) \int_x^\infty h,
\]
\[
(\mathcal{H}^*_{c,d} h)(x) := \chi(c,d)(x) \int_x^{x_{c-1}} h, \quad (\mathcal{H}_d^* h)(x) := \chi(0,d)(x) \int_0^{x_{c-1}} h.
\]

By Theorem 3, \(\mathcal{A}_0\) is the smallest possible constant in the inequality
\[
\left( \int_0^\infty u(x) \left( \int_x^\infty h \right)^{q/p} \, dx \right)^{p/q} \leq \mathcal{A}_0^p \int_0^\infty h(z) z^p V(z) \, dz, \quad h \in M^+,
\]
and \(\mathcal{A}_2\) is defined by
\[
\mathcal{A}_2^p := \left\{ \begin{array}{ll}
\sup_{t \in (0,\infty)} \left( \int_t^\infty \frac{u(s)}{s^q} \, ds \right)^{p/q} \| \mathcal{H} \|_{L^1_{z^p V(z)} \to L^{1/p}} & \text{if } p \leq q,
\end{array} \right.
\]
\[
\left( \int_0^\infty \frac{u(x)}{x^q} \left( \int_x^\infty \frac{u(s)}{s^q} \, ds \right)^{q/(p-q)} \| \mathcal{H}_{\zeta^{-1}} \|_{L^1_{z^p V(z)} \to L^{1/p}} \, dx \right)^{(p-q)/q} & \text{if } q < p.
\]
Also by Theorem 4, $A_0$ is the best possible constant in the inequality
\[
\left( \int_0^\infty \frac{u(x)}{x} \left( \log \frac{x}{\zeta^{-2}(x)} \right)^q \left( \int_0^{\zeta^{-2}(x)} h \right)^{q/p} \frac{dt}{x} \right)^{p/q} \leq A_0^p \int_0^\infty h V, \quad h \in \mathcal{M}^+,
\]
and $A_2$ is determined from
\[
A_2^p := \begin{cases}
\sup_{t \in (0, \infty)} \left( \int_t^\infty \frac{u(x)}{x} \left( \log \frac{x}{\zeta^{-2}(x)} \right)^q \left( \int_x^{\zeta^{-2}(x)} h \right)^{q/p} \frac{dt}{x} \right)^{p/q} \| \mathcal{H}_t \|_{L^{1/p}_V \to L^{1/p}_V}, & p \leq q, \\
\left( \int_0^t \frac{u(x)}{x} \left( \log \frac{x}{\zeta^{-2}(x)} \right)^q \left( \int_x^{\zeta^{-2}(x)} h \right)^{q/p} \frac{dt}{x} \right)^{p/q} \| \mathcal{H}_{\zeta^{-2}(x)} \|_{L^{1/p}_V \to L^{1/p}_V}, & q < p.
\end{cases}
\]
(6.7)

By well-known results ([40], Ch. XI, §1.5, Theorem 4, see also [30], Theorem 1.1 and [41], Theorem 3.3) we have
\[
A_0^p = \sup_{t > 0} \left( \int_0^t \frac{u}{1/t} \right)^{p/q}, \quad p \leq q,
\]
(6.8)
and
\[
A_0^p \approx \left( \int_0^\infty \left[ t^p V(t) \right]^{1/(q-p)} \left( \int_0^t \frac{u}{1/t} \right)^{q/(p-q)} \frac{u(t) dt}{1/t} \right)^{(p-q)/q}, \quad q < p.
\]
(6.9)

Analogously, we obtain
\[
A_0^p = \sup_{t > 0} \left( \int_0^\infty \frac{u(x)}{x} \left( \log \frac{x}{\zeta^{-2}(x)} \right)^q \frac{dt}{x} \right)^{p/q}, \quad p \leq q,
\]
(6.10)
and for $q < p$,
\[
A_0^p \approx \left( \int_0^\infty \left( \int_x^\infty \frac{u(s)}{s^q} \left( \log \frac{s}{\zeta^{-2}(s)} \right)^q \frac{ds}{s} \right)^{q/(p-q)} \frac{u(x)}{x} \left( \log \frac{x}{\zeta^{-2}(x)} \right)^q \frac{dx}{x} \right)^{(p-q)/q}.
\]
(6.11)

Again applying Theorem 4 in [40], Ch. XI, §1.5 and Theorem 3.3 in [41] we obtain
\[
\| \mathcal{H}_t \|_{L^{1/p}_V \to L^{1/p}_V} = [V(t)]^{-1}, \quad 0 < p \leq 1,
\]
\[
\| \mathcal{H}_t \|_{L^{1/p}_V \to L^{1/p}_V} \approx \left( \int_0^t [V(x)]^{1/(1-p)} \frac{dx}{x} \right)^{p-1}, \quad p > 1,
\]
and so it follows from (6.6) for $p \leq q$ that
\[
A_2 = \sup_{t \in (0, \infty)} \left( \int_t^\infty \frac{u(s)}{s^q} \frac{ds}{s} \right)^{1/q} [V(t)]^{-1/p}, \quad 0 < p \leq 1,
\]
(6.12)
\[
A_2 \approx \sup_{t \in (0, \infty)} \left( \int_t^\infty \frac{u(s)}{s^q} \frac{ds}{s} \right)^{1/q} \left( \int_0^t [V(x)]^{1/(1-p)} \frac{dx}{x} \right)^{1/p'}, \quad p > 1,
\]
(6.13)
where $p' := p/(p - 1)$. In the same way

$$
\| \mathcal{H}_{\zeta^{-1}(x), \zeta(x)} \|_{L^1_{zV(z)}} \rightarrow L^1_{1/p} = \left[ \frac{\zeta(x) - \zeta^{-1}(x)}{\zeta(x)} \right]^p \frac{1}{V(\zeta(x))}, \quad 0 < p \leq 1,
$$

$$
\| \mathcal{H}_{\zeta^{-1}(x), \zeta(x)} \|_{L^1_{zV(z)}} \rightarrow L^1_{1/p} \approx \left( \int_{\zeta^{-1}(x)}^{\zeta(x)} \left[ t^p V(t) \right]^{1/(1-p)} (t - \zeta^{-1}(x))^{1/(p-1)} dt \right)^{p-1},
$$

$p > 1$.

Hence from (6.6) we see that for $q < p$

$$
\mathcal{A}_2 \approx \left( \int_0^\infty \frac{u(x)}{x^q} \left( \int_x^\infty \frac{u(s)}{s^q} ds \right)^{q/(p-q)} \frac{\zeta(x) - \zeta^{-1}(x)}{\zeta(x) [V(\zeta(x))]^{1/p}} dx \right)^{(p-q)/(pq)},
$$

if $0 < p \leq 1$ and

$$
\mathcal{A}_2 \approx \left( \int_0^\infty \frac{u(x)}{x^q} \left( \int_x^\infty \frac{u(s)}{s^q} ds \right)^{q/(p-q)} \frac{1}{t^p V(t)} \right)^{1/(p-1)} \frac{x}{p-1} \left( \int_0^t [V(x)]^{1/(1-p)} \left( \log \frac{t}{s} \right)^{1/(p-1)} dx \right)^{p-1},
$$

if $p > 1$.

In a similar way,

$$
\| \mathcal{H}_t^* \|_{L^1_{1/p} \rightarrow L^1_{1/y}} = \sup_{s \in (0, t)} [V(s)]^{-1} \left( \log \frac{t}{s} \right)^p, \quad 0 < p \leq 1,
$$

$$
\| \mathcal{H}_t^* \|_{L^1_{1/p} \rightarrow L^1_{1/y}} \approx \left( \int_0^t [V(x)]^{1/(1-p)} \left( \log \frac{t}{s} \right)^{1/(p-1)} dx \right)^{p-1}, \quad p > 1.
$$

Now it follows from (6.7) for $p \leq q$ that

$$
\mathbf{A}_2 = \sup_{t \in (0, \infty)} \left( \int_t^\infty \frac{u(s)}{s^q} ds \right)^{1/q} \sup_{s \in (0, t)} [V(s)]^{-1/p} \log \frac{t}{s}, \quad 0 < p \leq 1,
$$

$$
\mathbf{A}_2 \approx \sup_{t \in (0, \infty)} \left( \int_t^\infty \frac{u(s)}{s^q} ds \right)^{1/q} \left( \int_0^t [V(x)]^{1/(1-p)} \left( \log \frac{t}{s} \right)^{1/(p-1)} dx \right)^{1/p'}, \quad p > 1.
$$

We have

$$
\| \mathcal{H}_{\zeta^{-1}(x), \zeta(x)}^* \|_{L^1_{1/p} \rightarrow L^1_{1/y}} = \sup_{s \in (\zeta^{-1}(x), \zeta(x))} \frac{\left( \log \frac{\zeta(x)}{s} \right)^p}{V(s)}, \quad 0 < p \leq 1,
$$

$$
\| \mathcal{H}_{\zeta^{-1}(x), \zeta(x)}^* \|_{L^1_{1/p} \rightarrow L^1_{1/y}} \approx \left( \int_{\zeta^{-1}(x)}^{\zeta(x)} [V(t)]^{1/(1-p)} \left( \log \frac{\zeta(x)}{t} \right)^{1/(p-1)} dt \right)^{p-1}, \quad p > 1.
$$
Thus, from (6.7) for \( q < p \), we find that
\[
A_2 \approx \left[ \int_0^\infty \frac{u(x)}{x^q} \left( \int_x^\infty \frac{u(s)}{s^q} \, ds \right)^{q/(p-q)} \right. \\
\times \left. \sup_{s \in (\zeta^{-1}(x), \zeta(x))} \frac{\left( \log \frac{\zeta(x)}{s} \right)^p}{V(s)} \, dx \right]^{(p-q)/(pq)}
\] (6.18)
if \( 0 < p \leq 1 \) and
\[
A_2 \approx \left( \int_0^\infty \frac{u(x)}{x^q} \left( \int_x^\infty \frac{u(s)}{s^q} \, ds \right)^{q/(p-q)} \right. \\
\times \left. \left( \int_{\zeta^{-1}(x)}^{\zeta(x)} \frac{\left( \log \frac{\zeta(x)}{t} \right)^{1/(p-1)}}{V(t)} \, dt \right)^{q(p-1)/(p-q)} \, dx \right)^{(p-q)/(pq)}
\] (6.19)
if \( p > 1 \).

Finally, we obtain the following result.

**Theorem 9.** Let \( 0 < p, q < \infty \). Then for the maximal Hardy-Littlewood operator
\[
\| M \|_{\Gamma_p(v) \to \Gamma_q(u)} \approx A_0 + A_2 + A_0 + A_2,
\] (6.20)
where the constants on their right-hand side are determined by (6.8)–(6.11) for \( A_0 \) and \( A_2 \) and by (6.12)–(6.19) for \( A_2 \) and \( A_2 \).

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**Dmitrii V. Prokhorov**
Steklov Mathematical Institute
of the Russian Academy of Sciences, Moscow;
Computing Center of the Far Eastern Branch
of the Russian Academy of Sciences,
Kim Yu Chena 65, 680000 Khabarovsk, Russia
E-mail: prohorov@as.khb.ru

**Vladimir D. Stepanov**
Steklov Mathematical Institute
of the Russian Academy of Sciences, Moscow;
RUDN University, Moscow
E-mail: stepanov@mi.ras.ru

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