Nitsche’s method for a Robin boundary value problem in a smooth domain

Yuki Chiba | Norikazu Saito

Graduate School of Mathematical Sciences, The University of Tokyo, Meguro, Japan

Correspondence
Yuki Chiba, Graduate School of Mathematical Sciences, The University of Tokyo, Komaba 3-8-1, Meguro, Tokyo 153-8914, Japan.
Email: ychiba@ms.u-tokyo.ac.jp

Abstract
We prove several optimal-order error estimates for a $P^1$ finite-element method applied to an inhomogeneous Robin boundary value problem (BVP) for the Poisson equation defined in a smooth bounded domain in $\mathbb{R}^n$, $n = 2, 3$. The boundary condition is weakly imposed using Nitsche’s method. The Robin BVP is interpreted as the classical penalty method with the penalty parameter $\varepsilon$. The optimal choice of the mesh size $h$ relative to $\varepsilon$ is a non-trivial issue. This paper carefully examines the dependence of $\varepsilon$ on error estimates. Our error estimates require no unessential regularity assumptions on the solution. Numerical examples are also reported to confirm our results.

KEYWORDS
finite element method, Nitsche’s method

1 | INTRODUCTION

Nitsche’s method [17] is well-known as a powerful method for imposing the Dirichlet boundary condition (DBC) in the finite element method (FEM). DBC is usually imposed by specifying the function values themselves at boundary nodal points. In contrast, Nitsche’s method is based on the method of “weak imposition” of DBC using penalty parameter. Actually, this strategy is useful for resolving the issue of spurious oscillations for non-stationary Navier–Stokes and convection–diffusion equations as was mentioned in Bazilevs et al. [6, 7].

In recent years, demand for computing complex boundary conditions has been increasing. Boundary conditions involving the Laplace–Beltrami operator, such as a dynamic boundary condition and a generalized Robin boundary condition play important roles in application to the reduced
fluid–structure interaction model and Cahn–Hilliard equation (see, e.g., [11], [18] and [8]). Nitsche’s method may be an effective approach to address these boundary conditions, and therefore, is worthy of a thorough investigation.

When numerically solving PDEs in a smooth domain, we often utilize polyhedral approximations of the domain. Generally, a facile approximation of the problem may result in a wrong numerical solution; the so-called Babuška’s paradox in [2, Section 5] is a remarkable example. Therefore, investigating not only the error caused by discretizations but also that caused by domain approximations is important. For the standard FEM, approximating domains is a common problem, and analysis of the energy norm is well-developed thus far. Only recently, the optimal order $W_1^\infty$ and $L_\infty$ stability and error estimates were established; refer to [14] for detail.

Consequently, we evaluate Nitsche’s method for PDEs in a smooth bounded domain. We study a finite-element method (FEM) applied to an inhomogeneous Robin boundary value problem (BVP) for the Poisson equation defined in a smooth bounded domain $\Omega$ in $\mathbb{R}^n$, $n = 2, 3$. The boundary condition on the boundary $\Gamma = \partial \Omega$ is weakly imposed using Nitsche’s method. We then derive several optimal-order error estimates under reasonable regularity assumptions on the solution. Specifically, we consider

$$-\Delta u = f \quad \text{in } \Omega \quad (1.1a)$$

$$\frac{\partial u}{\partial \nu} + \frac{1}{\varepsilon} u = g \quad \text{on } \Gamma. \quad (1.1b)$$

Therein, we suppose that $f \in L^2(\Omega)$, and $g \in H^1(\Omega)$ are given. Moreover, $\partial / \partial \nu$ denotes the differentiation along the outward unit normal vector $\nu$ to $\Gamma$ and $\varepsilon > 0$ is a constant. If $\varepsilon = 0$, we consider (1.1a) with the Dirichlet boundary condition

$$u = u_0 \quad \text{on } \Gamma \quad (1.1c)$$

where $u_0 \in H^2(\Omega)$ instead of (1.1b).

The case of a polyhedral domain with $\varepsilon > 0$ has already been addressed in Juntunen and Stenberg [12]. We are motivated by [12] and this paper is a generalization of [12] to a smooth domain. We study simultaneously the case $\varepsilon = 0$, that is the case of DBC.

If we are concerned with the Dirichlet BVP (1.1a) and (1.1c), the Robin BVP (1.1a) and (1.1b) with $g = \varepsilon^{-1} u_0$ implies the classical penalty method with the penalty parameter $\varepsilon \to 0$. (The $\varepsilon$ is interpreted as the penalty parameter in the classical penalty method. On the other hand, Nitsche’s method is introduced using the penalty parameter, which we will write as $\gamma$. The readers have to take care not to confuse the two.) FEM for this method is well studied so far; we refer to [3-5, 16] for example. In particular, Barret and Elliott [4] presented the error estimate in a smooth domain as

$$\| \tilde{u} - u_\varepsilon \|_{H^1(\Omega_h)} \leq C \left[ (h + \varepsilon^{-\frac{1}{2}}h^2 + \varepsilon^{\frac{1}{2}}) \| u \|_{H^4(\Omega)} + \varepsilon^{-\frac{1}{2}}h^2 \| \tilde{u}_0 \|_{H^4(\Omega_h)} \right]. \quad (1.2)$$

Therein, $h$ denotes the granularity parameter of the triangulation and $\Omega_h$ denotes a polyhedral approximation of $\Omega$ satisfying (2.1). (See also Remark 2.3.) The continuous $P^1$ finite element solution is represented by $u_\varepsilon$. The definition of function spaces and their norms are described in the end of this section. Moreover, $\tilde{u}$ and $\tilde{u}_0$ are suitable extensions of $u$ and $u_0$, respectively. The precise definition will be mentioned in the next section. The estimate (1.2) gives the optimal-order estimate for the $H^1$ norm by setting $\varepsilon = Ch^2$. However, we need a surplus regularity $u \in H^4(\Omega)$. Barret and Elliott [5] later studied the iso-parametric FEM for a similar problem and obtained similar results as ours. However, regularity assumptions slightly vary from ours.
This paper examines the dependence of error estimates on \( \varepsilon \). As a matter of fact, how to choose \( h \) relative to \( \varepsilon \) is a non-trivial issue for smooth domain cases. A suitable regularity assumption is another non-trivial issue as we recalled for the standard FEM above. These points were not discussed in Junutunen and Stenberg [12]. In fact, if considering a polyhedral domain, we can apply the Galerkin orthogonality and prove the error estimates uniformly in \( \varepsilon \) and \( \gamma \). However, we must address the estimations for the residual term \( r_h(v, \chi) \) (see Section 4) very carefully in a smooth domain case, since the Galerkin orthogonality is not applicable. Consequently, our error estimates for \( \varepsilon > 0 \) include factors \( \sqrt{h + \varepsilon} \) and \( h/\varepsilon \), which are not appeared in estimates reported in [12]. For example, if \( \varepsilon > 0 \), we succeed in deriving (see Theorem 1)

\[
\| \bar{u} - u_h \|_{h, \star} \leq C h \left( \left( 1 + \sqrt{h + \varepsilon} + \frac{h}{\varepsilon} \right) \| u \|_{2, \Omega} + \sqrt{h + \varepsilon} \| f \|_{\Omega} + \| g \|_{1, \Omega} \right),
\]

where \( \| \cdot \|_{h, \star} \) denotes the DG \( H^1 \) norm defined as (2.4). It makes this possible by applying some estimates reported in [14]. From this estimate, we deduce the optimal-order estimate for the DG \( H^1 \) norm by setting \( h \leq C \varepsilon \) under no further assumptions the smoothness of the solution and data. Moreover, this estimate indicates the following. If \( \varepsilon \) is sufficiently small, that is the Dirichlet-like condition is considered, we must set a sufficiently small \( h \). This is not surprising because the Dirichlet-like condition on the smooth boundary is approximated the boundary integral terms on the polyhedral boundary. On the other hand, if considering the pure DBC (\( \varepsilon = 0 \)) and reformulating the Nitsche’s method to fit the pure DBC, we derive (see Theorem 1)

\[
\| \bar{u} - u_h \|_{h, \star} \leq C h \left( \| u \|_{2, \Omega} + \| f \|_{\Omega} + \| u_0 \|_{2, \Omega} \right).
\]

That is, we can drop factors \( \sqrt{h + \varepsilon} \) and \( h/\varepsilon \) in this case. However, we could not remove factors \( \sqrt{h + \varepsilon} \) and \( h/\varepsilon \) in the case \( \varepsilon > 0 \) by our method of analysis. The validity of the appearance of these factors will be discussed in Section 6.

We also derive \( L^2 \) error estimates (see Theorem 2). The lack of the Galerkin orthogonality makes the analysis much more complicated. In particular, we assume a surplus regularity \( u \in W^{3, q}(\Omega), q > n \), for deducing the optimal-order estimate for the \( L^2 \) norm (see Theorem 2). In our opinion, this is an essential requirement; see Remark 2.2 and Section 6.

This paper comprises six sections. In Section 2, the continuous \( P^1 \) finite element approximation using Nitsche’s method and the main error estimates (Theorems 1 and 2) are described. After having presented some preliminary results in Section 3, we prove Theorems 1 and 2 in Sections 4 and 5, respectively. Finally, numerical examples are also reported to confirm our results in Section 6.

**Notation.** We list the notations used in this paper. We follow the standard notation of, for example, [1] for function spaces and their norms. In particular, for \( 1 \leq p \leq \infty \) and a positive integer \( j \), we use the standard Lebesgue space \( L^p(\mathcal{O}) \) and Sobolev space \( W^{j, p}(\mathcal{O}) \). Hereinafter, \( \mathcal{O} \) denotes a bounded domain in \( \mathbb{R}^n \). The inner product and norm of \( L^2(\mathcal{O}) \) are denoted, respectively, by \( \langle \cdot, \cdot \rangle_\mathcal{O} \) and \( \| \cdot \|_\mathcal{O} \). The norm of \( W^{j, p}(\mathcal{O}) \) is denoted by \( \| \cdot \|_{W^{j, p}(\mathcal{O})} \). As usual, we set \( H^j(\mathcal{O}) = W^{j, 2}(\mathcal{O}) \), and the semi-norm and norm of \( H^j(\mathcal{O}) \) are denoted by, respectively,

\[
|v|_{j, \mathcal{O}} = \left( \sum_{|a|=j} \| \frac{\partial^a v}{\partial x^a} \|_{\mathcal{O}}^2 \right)^{1/2}, \quad \| v \|_{j, \mathcal{O}} = \left( \sum_{i=0}^{j} \| v_i \|_{i, \mathcal{O}}^2 \right)^{1/2}.
\]

For \( S \subset \partial \mathcal{O} \), we define \( L^p(S) \) using a surface measure \( dS \) in a common approach. The inner product and norm of \( L^2(S) \) is denoted by, respectively, \( \langle \cdot, \cdot \rangle_S \) and \( \| \cdot \|_S \). Moreover, \( P^r(\mathcal{O}) \) denotes the set of all polynomials in \( \mathcal{O} \) of degree \( \leq r \).
NITSCHÉ'S METHOD AND THE MAIN RESULTS

We recall that $\Omega$ is a bounded domain in $\mathbb{R}^n$, $n = 2, 3$. Throughout this paper, we assume that the boundary $\Gamma$ of $\Omega$ is a $C^l$ boundary, where $l$ is an integer $\geq 2$.

From the general theory of elliptic PDEs, we know that the unique solution $u$ of (1.1) belongs to $H^2(\Omega)$ and satisfies $\|u\|_{2,\Omega} \leq C(\|f\|_\Omega + \|u_0\|_{2,\Omega} + \|g\|_{1,\Omega})$, where $C$ denotes a positive constant depending only on $\Omega$ and $\varepsilon$.

Let $\{\mathcal{T}_h\}_h$ be a regular family of triangulations $\mathcal{T}_h$ of a polyhedral domain $\Omega_h \subset \mathbb{R}^n$ in the sense of [9]. That is,

1. $\mathcal{T}_h$ is a set of closed $n$-simplices (elements) $K$, and
   
   $$\Omega_h = \text{Int} \left( \bigcup_{K \in \mathcal{T}_h} K \right);$$

2. The granularity parameter $h$ is defined as $h = \max_{K \in \mathcal{T}_h} h_K$, where $h_K$ denotes the diameter of $K$;

3. Any two elements of $\mathcal{T}_h$ meet only in entire common faces or sides or in vertices;

4. There exists a positive constant $c_{\text{reg}}$ satisfying $h_K \leq c_{\text{reg}} h_K$ for all $K \in \mathcal{T}_h \in \{\mathcal{T}_h\}_h$, where $h_K$ denotes the diameter of the inscribed ball of $K$.

We then introduce the boundary mesh $\mathcal{E}_h$ inherited from $\mathcal{T}_h$ by

$$\mathcal{E}_h := \{E \subset \partial \Omega_h : E \text{ is an } (n - 1) - \text{face of some } K \in \mathcal{T}_h\},$$

and the boundary $\Gamma_h := \partial \Omega_h$ is expressed as $\Gamma_h = \bigcup_{E \in \mathcal{E}_h} E$. We assume that $\Gamma_h$ is an approximate surface/polygon of $\Gamma$ in the sense that

\[
\text{every vertex of } E \in \mathcal{E}_h \text{ lies on } \Gamma. \tag{2.1}
\]

We use the continuous $P^1$ finite element space

$$V_h := \{\chi \in C(\overline{\Omega}) : \chi|_K \in P^1(K) \ (\forall K \in \mathcal{T}_h)\}.$$

Below we fix a sufficiently smooth domain $\tilde{\Omega} \subset \mathbb{R}^n$ such that

$$\overline{\Omega} \cup \Omega_h \subset \tilde{\Omega}.$$

Since $\Gamma$ is of class $C^l$, $l \geq 2$, the domain $\Omega$ admits a strong $l$-extension operator $P$. That is, $P$ is a linear operator of $W^{l,r}(\Omega) \to W^{l,r}(\tilde{\Omega})$ for any $1 \leq r < \infty$, and it satisfies

$$\langle (Pv)|_{\Omega} = v \text{ in } \Omega, \quad \|Pv\|_{W^{l,r}(\tilde{\Omega})} \leq C_{l,r,\Omega}\|v\|_{W^{l,r}(\Omega)} \ (0 \leq k \leq l),$$

where $C_{l,r,\Omega}$ denotes a positive constant depending only on $l$, $r$ and $\Omega$; see [1, Theorem 5.22] for example. Using this, we write

$$\tilde{f} := Pf, \quad \tilde{u}_0 := Pu_0, \quad \tilde{g} := Pg.$$

Following [12], we set

$$a_h(w, v) := (\nabla w, \nabla v)_\Omega + b_h(w, v),$$

$$b_h(w, v) := \sum_{E \in \mathcal{E}_h} \left[ -\frac{\gamma h_E}{\varepsilon + \gamma h_E} \left( \langle \frac{\partial w}{\partial v}, v \rangle_E + \langle w, \frac{\partial v}{\partial h} \rangle_E \right) + \frac{1}{\varepsilon + \gamma h_E} \langle w, v \rangle_E - \frac{\varepsilon \gamma h_E}{\varepsilon + \gamma h_E} \langle \frac{\partial w}{\partial v}, \frac{\partial v}{\partial h} \rangle_E \right].$$
Moreover, we set
\[ l_h(v) := (\tilde{f}, v)_{\Omega_h} + \sum_{E \in \mathcal{E}_h} \left[ \frac{\varepsilon}{\varepsilon + \gamma h_E} \langle \tilde{g}, v \rangle_E - \frac{\varepsilon \gamma h_E}{\varepsilon + \gamma h_E} \langle \tilde{g}, \frac{\partial v}{\partial v_h} \rangle_E \right] \]
for \( \varepsilon > 0 \) and
\[ l_h(v) := (\tilde{f}, v)_{\Omega_h} + \sum_{E \in \mathcal{E}_h} \left[ \frac{1}{\gamma h_E} \langle \tilde{u}_0, v \rangle_E - \langle \tilde{u}_0, \frac{\partial v}{\partial v_h} \rangle_E \right] \]

Therein, \( \gamma > 0 \) is a penalty parameter, \( h_E \) the diameter of \( E \), and \( v_h \) the outer unit normal vector to \( \Gamma_h \). Nitsche’s method for (1.1) is stated as follows:
\[ u_h \in V_h, \quad a_h(u_h, \chi) = l_h(\chi) \quad (\forall \chi \in V_h). \tag{2.3} \]

We use the following norms that depend on \( \varepsilon \) and \( h_E \):
\[ \| v \|_h^2 := \| \nabla v \|_{\Omega_h}^2 + \sum_{E \in \mathcal{E}_h} \frac{1}{\varepsilon + \gamma h_E} \| v \|_E^2, \]
\[ \| v \|_{h,*}^2 := \| v \|_h^2 + \sum_{E \in \mathcal{E}_h} h_E \| \frac{\partial v}{\partial v_h} \|_E^2. \tag{2.4} \]

We recall that \( \| \cdot \|_h \) and \( \| \cdot \|_{h,*} \) are equivalent on \( V_h \) uniformly in \( h \) and \( \varepsilon \). That is, we have
\[ \| v \|_h \leq \| v \|_{h,*} \leq C \| v \|_h \quad (v \in V_h). \tag{2.5} \]

Here and hereinafter, \( C \) denotes a generic positive constant which is independent of \( h \) and \( \varepsilon \). The value of \( C \) may be different at each occurrence. The inequalities (2.5) follow from the well-known inequality
\[ \sum_{E \in \mathcal{E}_h, E \subset \partial K} h_E \| \frac{\partial v}{\partial v_h} \|_E^2 \leq C \| \nabla v \|_K^2 \quad (v \in V_h, K \in T_h). \tag{2.6} \]

In fact, (2.6) is a readily obtainable consequence of the standard trace inequality,
\[ \| v \|_E^2 \leq Ch_E^{-1} \left( \| v \|_K^2 + h_K^2 \| \nabla v \|_K^2 \right) \quad (E \in \mathcal{E}_h, E \subset \partial K, v \in H^1(K)). \tag{2.7} \]

Nitsche’s method (2.3) admits a unique solution in view of the following basic result; see [12, Theorem 3.2].

**Lemma 2.1.** We have
\[ a_h(w, v) \leq C \| w \|_{h,*} \| v \|_{h,*} \quad (\forall w, v \in H^2(\Omega_h) + V_h). \tag{2.8a} \]

Moreover, there exists a positive constant \( \gamma_0 \) which is independent of \( h \) and \( \varepsilon \) such that we have for \( 0 < \gamma \leq \gamma_0 \),
\[ a_h(\chi, \chi) \geq C \| \chi \|_h^2 \quad (\forall \chi \in V_h). \tag{2.8b} \]

Actually, \( \gamma_0 \) can be taken as any positive number strictly smaller than \( 1/C \), where \( C \) denotes the constant appearing in (2.6); see [12, Theorem 3.2]. Below we always assume that
\[ 0 < \gamma \leq \gamma_0. \tag{2.9} \]

This assumption on \( \gamma \) will be used for applying (2.8a) and (2.8b). To deduce convergence results, we need an inverse assumption as
\[ h \leq C \min_{E \in \mathcal{E}_h} h_E =: h_{\text{min}}. \tag{2.10} \]

This condition says that the boundary mesh \( \mathcal{E}_h \) is quasi-uniform.

We are now in a position to state our main result.
Theorem 1 \((H^1\) estimates). Suppose that \(\Gamma\) is a \(C^2\) boundary. Let \(u \in H^2(\Omega)\) and \(u_h \in V_h\) represent the solutions of (1.1) and (2.3), respectively. Assume that (2.1), (2.9) and (2.10) are satisfied. Then, if \(\varepsilon > 0\), we have for sufficiently small \(h\),

\[
\|\bar{u} - u_h\|_{h^*,\Omega} \leq Ch \left[ \left( 1 + \sqrt{h + \varepsilon} + \frac{h}{\varepsilon} \right) \|u\|_{2,\Omega} + \sqrt{h + \varepsilon} \|f\|_{1,\Omega} + \|g\|_{1,\Omega} \right],
\]

where \(\bar{u} = Pu\). If \(\varepsilon = 0\), we have

\[
\|\bar{u} - u_h\|_{h^*,\Omega} \leq Ch(\|u\|_{2,\Omega} + \|f\|_{1,\Omega} + \|u_0\|_{2,\Omega}).
\]

Theorem 2 \((L^2\) estimates). Suppose that \(\Gamma\) is a \(C^3\) boundary. Let \(u \in H^2(\Omega)\) and \(u_h \in V_h\) represent the solutions of (1.1) and (2.3), respectively. Assume that (2.1), (2.9) and (2.10) are satisfied. Then, if \(\varepsilon > 0\), \(u \in W^{3,q}(\Omega)\) for some \(q > n\) and \(g \in H^3(\Omega)\), we have for sufficiently small \(h\),

\[
\|\bar{u} - u_h\|_{\Omega_h} \leq Ch^2(1 + \varepsilon)
\]

\[
\times \left[ \|u\|^2_{W^{3,q}(\Omega)} + \left( \sqrt{h + \varepsilon} + \frac{h}{\varepsilon} \right) \left( 1 + \frac{h}{\varepsilon} \right) \|u\|_{2,\Omega} + \|g\|_{1,\Omega} \right],
\]

where \(\bar{u} = Pu\). On the other hand, if \(\varepsilon = 0\) and \(u \in H^3(\Omega)\), we have

\[
\|\bar{u} - u_h\|_{\Omega} \leq Ch^2(1 + \varepsilon)(\|u\|_{1,\Omega} + \|u_0\|_{2,\Omega}).
\]

Remark 2.2. Theorem 1 reports that the optimal rate of convergence for the \(H^1\) error is achieved under a reasonable (minimal) regularity assumption on \(u\). On the other hand, we pose a somewhat surplus regularity \(u \in W^{3,q}(\Omega)\), \(q > n\), for deducing the optimal rate of convergence for the \(L^2\) error. In our opinion, this is an essential requirement. Actually, a numerical example reported in Section 6 shows the second-order convergence may not take place if \(u \in W^{3,q}(\Omega)\), \(q > n\).

Remark 2.3. We are assuming (2.1) for \(\Omega_h\). This can be replaced by

\[
\text{dist}(\Omega, \Omega_h) \leq Ch^2
\]

with some obvious modification of proofs.

### 3 \ | \ BOUNDARY-SKIN ESTIMATES

We collect some auxiliary results that will be used in the proof of the main results.

Let \(d(x)\) be the signed distance function defined by

\[
d(x) := \begin{cases} 
-\text{dist}(x, \Gamma), & x \in \Omega \\
\text{dist}(x, \Gamma), & x \in \mathbb{R}^n \setminus \Omega.
\end{cases}
\]

We define \(\Gamma(\delta) := \{x \in \mathbb{R}^n : |d(x)| < \delta\}\), which we call the boundary-skin region. Then, for a sufficiently small \(\delta\), the orthogonal projection \(\pi\) onto \(\Gamma\) exists such that

\[
x = \pi(x) + d(x)\nu(\pi(x)) \quad (x \in \Gamma(\delta), \ \pi(x) \in \Gamma).
\]

For the detail, see [10, Section 7]. If \(h\) is sufficiently small, \(\pi\) is defined on \(\Gamma_h \subset \Gamma(\delta)\). Moreover,

\[
\pi(E_h) := \{ \pi(E) : E \in E_h \}
\]

is a partition of \(\Gamma\). We choose sufficiently small \(h_0\) such that all these properties hold for any \(h \leq h_0\).
Now we can state the boundary-skin estimates. For the proof, refer to [15, Theorems 8.1, 8.2, and 8.3 and Lemma 9.1] and [14, Lemma A.1].

**Lemma 3.1** (Boundary-skin estimates). Let $\delta = ch^2$ with a positive constant $c$. We have

$$\|v_h - v \circ \pi\|_{L^\infty(\Gamma_h)} \leq Ch.$$  \hfill (3.2a)

For $f \in L^1(\Gamma)$, we have

$$\left| \int_{\pi(E)} f \, d\gamma - \int_{E} f \circ \pi \, d\gamma_h \right| \leq Ch^2 \int_{\pi(E)} |f| \, d\gamma \quad (E \in \mathcal{E}_h),$$  \hfill (3.2b)

where $d\gamma$ and $d\gamma_h$ is the surface elements on $\pi(E)$ and $E$, respectively. For $f \in W^{1,p}(\Gamma(\delta))$, we have

$$\|f - f \circ \pi\|_{L^p(\Gamma_h)} \leq C\delta^{-1/p} \|f\|_{W^{1,p}(\Gamma(\delta))},$$  \hfill (3.2c)

$$\|f\|_{L^p(\Gamma(\delta))} \leq C \left( \delta \|\nabla f\|_{L^p(\Gamma(\delta))} + \delta^{1/p} \|f\|_{L^p(\Gamma)} \right).$$  \hfill (3.2d)

Moreover, for $f \in W^{1,p}(\Omega_h)$, we have

$$\|f\|_{L^p(\Omega_h \setminus \Omega)} \leq C \left( \delta \|\nabla f\|_{L^p(\Omega_h \setminus \Omega)} + \delta^{1/p} \|f\|_{L^p(\Gamma_h)} \right).$$  \hfill (3.2e)

**Lemma 3.2.** For any $\chi \in V_h$,

$$\|\chi\|_{\Omega_h \setminus \Omega} \leq Ch\sqrt{\epsilon + h} \|\chi\|_h.$$  \hfill (3.3)

**Proof.** Using (3.2e), we have

$$\|\chi\|_{\Omega_h \setminus \Omega}^2 \leq C \left( h^2 \|\nabla \chi\|_{\Omega_h \setminus \Omega}^2 + h^2 \sum_{E \in \mathcal{E}_h} \|\chi\|_{E}^2 \right)$$

$$\leq C \left( h^2 \|\nabla \chi\|_{\Omega_h \setminus \Omega}^2 + h^2 \sum_{E \in \mathcal{E}_h} \frac{\epsilon + \gamma h}{\epsilon + \gamma h_E} \|\chi\|_{E}^2 \right)$$

$$\leq Ch^2(\epsilon + h) \|\chi\|_{h}^2.$$  \hfill ■

**Lemma 3.3.** If $\epsilon = 0$ and if $u \in H^2(\Omega)$ satisfies (1.1c), we have

$$\|\tilde{u} - u_0\|_{\Gamma_h} \leq Ch^2 \left( \|u\|_{2,\Omega} + \|u_0\|_{2,\Omega} \right).$$  \hfill (3.4a)

Moreover, if $\epsilon > 0$ and if $u \in H^2(\Omega)$ satisfies (1.1b), we have

$$\left\| \frac{\partial \tilde{u}}{\partial v_h} + \frac{\tilde{u}}{\epsilon} - \tilde{g} \right\|_{\Gamma_h} \leq Ch \left( \|u\|_{2,\Omega} + \frac{h}{\epsilon} \|u\|_{2,\Omega} + \|g\|_{1,\Omega} \right).$$  \hfill (3.4b)

**Proof.** First, consider the case $\epsilon = 0$. Let $\delta = ch^2$ as in Lemma 3.1. Since $u \circ \pi = u_0 \circ \pi$ on $\Gamma_h$, we have by (3.2c)

$$\|\tilde{u} - u_0\|_{\Gamma_h} \leq \|\tilde{u} - u_0 \circ \pi\|_{\Gamma_h} + \|u_0 \circ \pi - \tilde{u}_0\|_{\Gamma_h} \leq Ch\|\tilde{u}\|_{1,\Gamma(\delta)} + Ch\|u_0\|_{1,\Gamma(\delta)}$$

Using (3.2d) and the trace theorem, we have

$$\|\tilde{u}\|_{1,\Gamma(\delta)} \leq Ch(\|\nabla \tilde{u}\|_{2,\Gamma(\delta)} + \|u\|_{1,\Gamma})$$

$$\|\tilde{u}\|_{1,\Gamma(\delta)} \leq Ch^2(\epsilon + h) \|\chi\|_{h}^2.$$  \hfill ■
Similarly, we derive \( \| \tilde{u}_0 \|_{1, \Gamma(h)} \leq C h \| u_0 \|_{2, \Omega} \). Combining these estimates, we obtain (3.4a).

We proceed to the case \( \varepsilon > 0 \). Since \( (\nabla \tilde{u} \circ \pi) \cdot (\nu \circ \pi) + \varepsilon^{-1} \tilde{u} \circ \pi = \tilde{g} \circ \pi \) on \( \Gamma_h \), we have

\[
\left\| \frac{\partial \tilde{u}}{\partial v_h} + \frac{\tilde{u}}{\varepsilon} - \tilde{g} \right\|_{\Gamma_h} \leq \| \nabla \tilde{u} \cdot v_h - (\nabla \tilde{u} \circ \pi) \cdot (\nu \circ \pi) \|_{\Gamma_h} + \| \tilde{g} - \tilde{g} \circ \pi \|_{\Gamma_h} + \frac{1}{\varepsilon} \| \tilde{u} - \tilde{u} \circ \pi \|_{\Gamma_h}
= J_1 + J_2 + J_3.
\]

As above, we estimate as

\[
J_2 + J_3 \leq \frac{h^2}{\varepsilon} \| u \|_{2, \Omega} + h \| g \|_{1, \Omega}.
\]

On the other hand, by (3.2c) and (3.2a)

\[
J_1 \leq \| \nabla \tilde{u} \cdot v_h - \nabla \tilde{u} \cdot (\nu \circ \pi) \|_{\Gamma_h} + \| \nabla \tilde{u} \cdot (\nu \circ \pi) - (\nabla \tilde{u} \circ \pi) \cdot (\nu \circ \pi) \|_{\Gamma_h}
\leq C h \| u \|_{2, \Omega}.
\]

Summing up, we deduce (3.4b). □

## 4 PROOF OF THEOREM 1

We start with a version of the Strang lemma. To state it, we set

\[
\rho_h(v, \chi) = a_h(v, \chi) - l_h(\chi)
\]

for \( v \in H^2(\hat{\Omega}) + V_h \) and \( \chi \in V_h \).

**Lemma 4.1.** Under the same assumption of Theorem 1, we have

\[
\| \tilde{u} - u_h \|_{h, \star} \leq C \left[ \inf_{\xi \in V_h} \| \tilde{u} - \xi \|_{h, \star} + \sup_{\chi \in V_h} \frac{| r_h(\tilde{u}, \chi) |}{\| \chi \|_h} \right]. \tag{4.1}
\]

**Proof.** Letting \( \xi \in V_h \) and \( \chi = u_h - \xi \), we have

\[
\| \chi \|_h^2 \leq C a_h(\chi, \chi)
= C [a_h(\tilde{u} - \xi, \chi) - a_h(\tilde{u}, \chi) + l_h(\chi)]
\leq C [\| \tilde{u} - \xi \|_{h, \star} \| \chi \|_{h, \star} + | r_h(\tilde{u}, \chi) |]
\leq C [\| \tilde{u} - \xi \|_{h, \star} \| \chi \|_h + | r_h(\tilde{u}, \chi) |],
\]

where (2.8b), (2.8a), and (2.5) are applied. Therefore, we have

\[
\| \tilde{u} - u_h \|_{h, \star} \leq \| \tilde{u} - \xi \|_{h, \star} + \| u_h - \xi \|_{h, \star}
\leq \| \tilde{u} - \xi \|_{h, \star} + C \| u_h - \xi \|_h
\leq C \| \tilde{u} - \xi \|_{h, \star} + \frac{| r_h(\tilde{u}, \chi) |}{\| \chi \|_h}
\leq C \| \tilde{u} - \xi \|_{h, \star} + \sup_{\chi \in V_h} \frac{| r_h(\tilde{u}, \chi) |}{\| \chi \|_h},
\]

which implies (4.1) □
The standard Lagrange interpolation operator of \( C(\Omega) \to V_h \) is denoted by \( \Pi_h \). It is well-known that
\[
|w - \Pi_h w|_{m,K} \leq C h^{\min(2-m,2)} |w|_2, K \in T_h, m = 0, 1, 2.
\] (4.2)

As a direct application of (4.2) and (2.7), we derive
\[
|||w - \Pi_h w|||_h, \ast \leq C h |w|_2, \Omega (w \in H^2(\Omega)).
\] (4.3)

We can state the following proof.

Proof of Theorem 1. First consider the case \( \epsilon > 0 \). In view of (4.1) and (4.3), it suffices to prove
\[
\sup_{\chi \in V_h} \frac{|r_\epsilon(\tilde{u}, \chi)|}{||\chi||_h} \leq C h \left[ \left( 1 + \sqrt{h + \epsilon} \right) ||u||_{2,\Omega} + \sqrt{h + \epsilon} ||f||_{\Omega} + ||g||_{1,\Omega} \right].
\] (4.4)

Let \( \chi \in V_h \) be arbitrary. Applying the integration by parts, we have
\[
a_h(\tilde{u}, \chi) - l_h(\chi) = (-\Delta \tilde{u} - \tilde{f}, \chi)_{\Omega_h}
+ \sum_{E \in \mathcal{T}_h} \frac{\epsilon}{\epsilon + \gamma h_E} \left\langle \frac{\partial \tilde{u}}{\partial \nu_h} + \frac{\tilde{u}}{\epsilon} - \tilde{g}, \chi - \gamma h_E \frac{\partial \chi}{\partial \nu_h} \right\rangle_E.
\] (4.5)

Since \(-\Delta \tilde{u} = \tilde{f}\) in \( \Omega \), we have by (3.3)
\[
|I_1| = |(-\Delta \tilde{u} - \tilde{f}, \chi)_{\Omega_h}|
\leq ||\Delta \tilde{u} + \tilde{f}||_{\Omega_h \setminus \Omega} ||\chi||_{\Omega_h \setminus \Omega}
\leq C h \sqrt{h + \epsilon} (||u||_{2,\Omega} + ||f||_{\Omega}) ||\chi||_h.
\]

Using (3.4b),
\[
|I_2| \leq C \left\| \frac{\partial \tilde{u}}{\partial \nu_h} + \frac{\tilde{u}}{\epsilon} - \tilde{g} \right\|_{L^2(\Gamma_h)} ||\chi||_h, \ast
\leq C h \left( ||u||_{2,\Omega} + \frac{h}{\epsilon} ||u||_{2,\Omega} + ||g||_{1,\Omega} \right) ||\chi||_h.
\]

Summing up, we deduce (4.4).

We proceed to the case \( \epsilon = 0 \). It suffices to prove
\[
\sup_{\chi \in V_h} \frac{|r_\epsilon(\tilde{u}, \chi)|}{||\chi||_h} \leq C h \left( ||u||_{2,\Omega} + ||f||_{\Omega} + ||u_0||_{2,\Omega} \right).
\] (4.6)

In this case, we have \( r_\epsilon(\tilde{u}, \chi) = I_1 + I_3 \), where
\[
I_3 = \sum_{E \in \mathcal{T}_h} \gamma h_E \left\langle \tilde{u} - \tilde{u}_0, \chi - \gamma h_E \frac{\partial \chi}{\partial \nu_h} \right\rangle_E.
\]

We apply (2.10) and (3.4a) to obtain
\[
|I_3| \leq C h^{-1} \min \{ ||\tilde{u} - \tilde{u}_0||_{\Gamma_h}, ||\chi||_h, \ast \}
\leq C h (||u||_{2,\Omega} + ||u_0||_{2,\Omega}) ||\chi||_h.
\]

Therefore, (4.6) is proved.
5 | PROOF OF THEOREM 2

We use the Green function \( G : L^2(\Omega_h) \rightarrow H^2(\Omega) \) defined as follows. For \( \varepsilon > 0 \), \( z = G\eta \in H^2(\Omega) \) is the unique solution of

\[
- \Delta z = \begin{cases} 
\eta & \text{in } \Omega \cap \Omega_h \\
0 & \text{in } \Omega \setminus \Omega_h,
\end{cases} \tag{5.1a}
\]

\[
\frac{\partial z}{\partial v} + \frac{1}{\varepsilon} z = 0 \quad \text{on } \Gamma, \tag{5.1a}
\]

where \( \eta \in L^2(\Omega_h) \). For \( \varepsilon = 0 \), (5.1a) is replaced by

\[
z = 0 \quad \text{on } \Gamma. \tag{5.1b}
\]

For any \( \eta \in L^2(\Omega_h) \), \( G\eta \) admits the a priori estimate

\[
\|G\eta\|_{L^2(\Omega)} \leq C(1 + \varepsilon)\|\eta\|_{L^2(\Omega_h)}. \tag{5.2}
\]

We omit the proof since it is outside the scope of this paper. As a matter of fact, this can be verified by a standard method of difference quotient. For example, using estimate

\[
\|v\|_{L^2(\Omega_h)}^2 \leq C(1 + \varepsilon)\left( (\nabla v, \nabla v)_{\Omega} + \frac{1}{\varepsilon} (v, v)_{\Gamma} \right),
\]

and tracing the proof of [13, Theorem 3.3] carefully (we can ignore the contributions of \( \beta \) and norm \( \| \cdot \|_{H^1(\mathbb{K}_h)} \)), we find that the estimate (5.2) holds true. Moreover, if \( \Gamma \) is a \( C^4 \) boundary, the same proof of [19, Lemma 4.1] is also applicable.

**Lemma 5.1.** Under the same assumption of Theorem 2, we have

\[
\|\bar{u} - u_h\|_{L^2(\Omega_h)} \leq C(1 + \varepsilon)\max\left\{ \|\bar{u} - u_h\|_{L^2(\Omega_h)} + h\varepsilon_c(h, \varepsilon) \|\bar{u} - u_h\|_{L^2(\Omega_h)}, \varepsilon \right\}
\]

\[
+ \sup_{\eta \in L^2(\Omega_h)} \left[ \frac{\|z - \Pi_h z\|_{L^2(\Omega)}}{\|z\|_{L^2(\Omega)}} \|\bar{u} - u_h\|_{L^2(\Omega)} + |r_h(\bar{u}, \Pi_h z)| \right], \tag{5.3}
\]

where \( \bar{z} = P_z \in H^2(\Omega) \), \( z = G\eta \in H^2(\Omega) \) and

\[
c_1(h, \varepsilon) = \begin{cases} 
1 + h/\varepsilon & (\varepsilon > 0), \\
1 & (\varepsilon = 0).
\end{cases}
\]

**Proof.** First, suppose that \( \varepsilon > 0 \). In the similar way as the derivation of (4.5), we deduce

\[
(\chi, -\Delta v)_{\Omega_h} - a_h(\chi, v) - \sum_{E \in E_h} \frac{\varepsilon}{\varepsilon + \gamma h_E} \left( \chi - \gamma h_E \frac{\partial \chi}{\partial v_h} + \frac{1}{\varepsilon} v \right)_E \tag{5.4}
\]

for \( \chi \in H^2(\Omega_h) + V_h \) and \( v \in H^2(\Omega_h) \).

Let \( \eta \in L^2(\Omega_h) \) and \( \bar{z} = P_z = P(G\eta) \). We use the same symbol \( \eta \) to express the zero extension of \( \eta \) into \( \Omega \setminus \Omega_h \). Substituting (5.4) for \( \chi = \bar{u} - u_h \) and \( v = \bar{z} \), we obtain

\[
(\bar{u} - u_h, \eta)_{\Omega_h} = (\bar{u} - u_h, \eta)_{\Omega \setminus \Omega_h} + (\bar{u} - u_h, \eta)_{\Omega_h \setminus \Omega_h}
\]

\[
= (\bar{u} - u_h, -\Delta \bar{z})_{\Omega_h} + (\bar{u} - u_h, \eta + \Delta \bar{z})_{\Omega_h \setminus \Omega_h}
\]

\[
= a_h(\bar{u} - u_h, \bar{z}) + (\bar{u} - u_h, \eta + \Delta \bar{z})_{\Omega_h \setminus \Omega_h}
\]

\[
- \sum_{E \in E_h} \frac{\varepsilon}{\varepsilon + \gamma h_E} \left( \bar{u} - u_h - \gamma h_E \frac{\partial (\bar{u} - u_h)}{\partial v_h} + \frac{1}{\varepsilon} \bar{z} \right)_E
\]
We finally state the following proof.

Therefore, (5.3) holds even for Summing up, we deduce Thanks to (5.2), an estimation for $J_3$ is readily;

We apply Lemma 3.3 with $g = 0$ and derive

Summing up, we deduce

by using (2.8a) and (5.2). This implies (5.3) for $\varepsilon > 0$.

If $\varepsilon = 0$, $J_4$ is replaced by

We apply Lemma 3.3 with $u_0 = 0$ and get

Therefore, (5.3) holds even for $\varepsilon = 0$.

We finally state the following proof.

Proof of Theorem 2. We define following bilinear and linear forms:

$$a(w, v) = (\nabla w, \nabla v)_\Omega + b(w, v)$$

$$b(w, v) = \sum_{E \in \mathcal{E}_h} \left\{ -\frac{\gamma h_E}{\varepsilon + \gamma h_E} \left( \frac{\partial w}{\partial v} \right)_{\partial E} + \left( w, \frac{\partial v}{\partial v} \right)_{\partial E} \right\} + \frac{1}{\varepsilon + \gamma h_E} \left( w, v \right)_{\partial E}$$

$$- \sum_{E \in \mathcal{E}_h} \frac{\varepsilon}{\varepsilon + \gamma h_E} \left( \bar{u} - u_h - \frac{\gamma h_E}{\varepsilon + \gamma h_E} \frac{\partial (\bar{u} - u_h)}{\partial v_h}, \frac{\partial \bar{z}}{\partial v_h} \right)_{E}$$

$$= J_1 + J_2 + J_3 + J_4.$$
Let $u$ be the solution of (1.1). Since $\pi(\mathcal{E}_h)$ defined in (3.1) is a partition of $\Gamma$, we obtain

$$a(u, v) = I(v) \quad (\forall v \in H^s(\Omega), \forall s > 3/2).$$

First consider the case $\varepsilon > 0$. Recall that $\pi(\mathcal{E}) \subset \Gamma$ for $\mathcal{E} \in \mathcal{E}_h$, we have

$$a_h(\bar{u}, \bar{z}) - l_h(\bar{z}) = a_h(\bar{u}, \bar{z}) - a(u, z) + l(z) - l_h(\bar{z})$$

$$= \left( \int_{\Omega_h} (\nabla \bar{u} \cdot \nabla \bar{z} - \bar{f} \bar{z}) \, dx - \int_{\Omega_h} (\nabla u \cdot \nabla z - fz) \, dx \right)$$

$$+ \sum_{\mathcal{E} \in \mathcal{E}_h} \frac{1}{\varepsilon + \gamma \mathcal{E}} \left( \begin{array}{c} \gamma h \frac{\partial u}{\partial v} \bar{u} + \bar{v} - \bar{\mathcal{E}}_g, \bar{z} \\ \varepsilon h \frac{\partial u}{\partial v} + u - \mathcal{E}_g, \bar{z} \end{array} \right)$$

$$- \left( \int_{\Omega_h} (\nabla \bar{u} \cdot \nabla \bar{z} - \bar{f} \bar{z}) \, dx - \int_{\Omega_h} (\nabla u \cdot \nabla z - fz) \, dx \right)$$

$$+ \sum_{\mathcal{E} \in \mathcal{E}_h} \frac{1}{\varepsilon + \gamma \mathcal{E}} \left( \begin{array}{c} \gamma h \frac{\partial u}{\partial v} + \bar{u} - \bar{\mathcal{E}}_g, \bar{z} \\ \varepsilon h \frac{\partial u}{\partial v} + u - \mathcal{E}_g, \bar{z} \end{array} \right)$$

$$= : I_1 + I_2 - I_3.$$
We rearrange $A_2$ as

\[
\sum_{E \in E_h} -\gamma h_E \langle \nabla \tilde{u} \cdot (\nu_h - \nu \sigma), \tilde{z} \rangle_E + \sum_{E \in E_h} -\gamma h_E \langle \nabla \tilde{u} \cdot \nu \sigma, \tilde{z} \rangle_E - \langle \nabla u \cdot v, \tilde{z} \rangle_{\pi(E)}
\]

\[
+ \sum_{E \in E_h} \frac{\epsilon}{\epsilon + \gamma h_E} \left[ \langle \tilde{u}, \tilde{z} \rangle_E - \langle u, z \rangle_{\pi(E)} \right] - \sum_{E \in E_h} \frac{\epsilon}{\epsilon + \gamma h_E} \langle \tilde{g}, \tilde{z} \rangle_E - \langle g, z \rangle_{\pi(E)}
\]

\[
=: A_1 + A_2 + A_3 - A_4.
\]

Applying (3.2a), we estimate as

\[
|A_1| \leq C \frac{h}{\epsilon} \sum_{E \in E_h} \| \nabla \tilde{u} \cdot (\nu_h - \nu \sigma) \tilde{z} \|_{L^1(E)}
\]

\[
\leq C \frac{h^2}{\epsilon} \| \tilde{u} \|_{H^1} \| \tilde{z} \|_{H^1} \leq C \frac{h^2}{\epsilon} \| u \|_{L^2} \| z \|_{L^2}.
\]

In view of (5.1a) and (3.2c), we derive

\[
\| \tilde{z} \|_{H^1} \leq \| \tilde{z} - z \varpi \|_{H^1} + \| z \varpi \|_{H^1}
\]

\[
\leq C h \| z \|_{1, C} + C \| z \|_{1, \Gamma}
\]

\[
\leq C h \| z \|_{1, C} + C \| \epsilon \nabla z \cdot v \|_{1, \Gamma}
\]

\[
\leq C (\epsilon + h) \| z \|_{2, \Omega}.
\]

Therefore, we obtain

\[
|A_1| \leq C h^2 \left( 1 + \frac{h}{\epsilon} \right) \| u \|_{L^2} \| z \|_{L^2}.
\]

We rearrange $A_2$ as

\[
A_2 = \sum_{E \in E_h} -\gamma h_E \langle \nabla \tilde{u} \cdot \nu \sigma, \tilde{z} - z \varpi \rangle_E
\]

\[
+ \sum_{E \in E_h} -\gamma h_E \langle (\nabla \tilde{u} - \nabla u \varpi) \cdot \nu \sigma, z \varpi \rangle_E
\]

\[
+ \sum_{E \in E_h} -\gamma h_E \langle (\nabla u \varpi \cdot \nu \sigma, z \varpi) - \langle \nabla u \cdot v, z \rangle_{\pi(E)} \rangle
\]

\[
=: A_{21} + A_{22} + A_{23}.
\]

Using boundary skin estimates (3.2c) and (3.2b), we get

\[
|A_{21}| \leq C \| \nabla \tilde{u} \cdot \nu \sigma \|_{L^1} \| \tilde{z} - z \varpi \|_{H^1}
\]

\[
\leq C h \| \nabla \tilde{u} \cdot \nu \sigma \|_{L^1} \| \tilde{z} \|_{H^1}
\]

\[
\leq C h^2 \| u \|_{L^2} \| z \|_{L^2},
\]

\[
|A_{22}| \leq C \| (\nabla \tilde{u} - \nabla u \varpi) \cdot \nu \sigma \|_{L^2} \| z \varpi \|_{L^2}
\]

\[
\leq C h^2 \| \nabla \tilde{u} \|_{W^{1, \infty}} \| z \varpi \|_{L^2}
\]

\[
\leq C h^2 \| u \|_{W^{1, \infty}} \| z \|_{L^2},
\]

\[
|A_{23}| \leq C h^2 \| \nabla u \cdot v \|_{L^2} \leq C h^2 \| u \|_{L^2} \| z \|_{L^2}.
\]

Therein, we used the Sobolev inequality and (2.2) to derive

\[
\| \tilde{u} \|_{W^{2, \infty}(\Omega)} \leq C \| \tilde{u} \|_{W^{3, \infty}(\Omega)} \leq C \| u \|_{W^{3, \infty}(\Omega)}.
\]

(To apply (2.2) we are assuming that $\Gamma$ is a $C^3$ boundary.)
Therefore, we deduce
\[ |A_2| \leq Ch^2 \|u\|_{W^{3,q}(\Omega)} \|z\|_{2,\Omega}. \]

Similarly, we have
\[ |A_4| \leq C \left| \langle \tilde{g} - g, \tilde{z} - z \sigma \pi \rangle \Gamma_h \right| + \left| \langle \tilde{g} - g \sigma \pi, z \sigma \pi \rangle \Gamma_h \right| + \left| \langle g \sigma \pi, z \sigma \pi \rangle \Gamma_h - \langle g, z \rangle \Gamma \right| \leq Ch^2 \|g\|_{3,\Omega} \|z\|_{2,\Omega}. \]

A3 is rewritten as
\[
A_3 = \sum_{E \in \mathcal{E}_h} \frac{\epsilon}{\epsilon + \gamma h_E} \left( \frac{\tilde{u}}{\epsilon} \cdot z - z \sigma \pi \right)_E + \sum_{E \in \mathcal{E}_h} \frac{\epsilon}{\epsilon + \gamma h_E} \left( \frac{\tilde{u}}{\epsilon} - \frac{u \sigma \pi}{\epsilon} \right)_E \\
+ \sum_{E \in \mathcal{E}_h} \frac{\epsilon}{\epsilon + \gamma h_E} \left( \frac{u \sigma \pi}{\epsilon}, z \sigma \pi \right)_E - \left( \frac{u}{\epsilon}, z \right)_\pi(E) =: A_{31} + A_{32} + A_{33}.
\]

Using boundary skin estimates and (5.1a), we can perform estimations as
\[
|A_{31}| \leq C \frac{h^2}{\epsilon} \|\tilde{u}\|_{\Gamma_h} \|z\|_{1,\Gamma(\delta)} \\
\leq C \frac{h^2}{\epsilon} \|\tilde{u}\|_{\Gamma_h}(h\|z\|_{2,\Gamma(\delta)} + \|z\|_{1,\Gamma}) \\
\leq Ch^2 \left( 1 + \frac{h}{\epsilon} \right) \|u\|_{1,\Omega} \|z\|_{2,\Omega},
\]

and
\[
|A_{32}| \leq C \frac{1}{\epsilon} \|\tilde{u} - u \sigma \pi\|_{L^\infty(\Gamma_h)} \|z \sigma \pi\|_{L^1(\Gamma)} \\
\leq C \frac{h^2}{\epsilon} \|\tilde{u}\|_{W^{1,\infty}(\Gamma(\delta))} \|z \sigma \pi\|_{L^1(\Gamma_h)} \\
\leq Ch^2 \|u\|_{3,\Omega} \|z\|_{2,\Omega}.
\]

Moreover,
\[
|A_{33}| \leq C \frac{1}{\epsilon} \left| \langle u \sigma \pi, z \sigma \pi \rangle \Gamma_h - \langle u, z \rangle \Gamma \right| \\
\leq C \frac{h^2}{\epsilon} \|\tilde{u}z\|_{L^1(\Gamma)} \\
\leq Ch^2 \|u\|_{1,\Omega} \|z\|_{2,\Omega}.
\]

So, we get
\[
|A_3| \leq Ch^2 \left( \|u\|_{3,\Omega} + \frac{h}{\epsilon} \|u\|_{1,\Omega} \right) \|z\|_{2,\Omega}.
\]

Summing up, we deduce
\[
|I_2| \leq Ch^2 \left( \|u\|_{W^{3,q}(\Omega)} + \frac{h}{\epsilon} \|u\|_{2,\Omega} + \|g\|_{3,\Omega} \right) \|z\|_{2,\Omega}.
\]

We apply (3.4b) to obtain
\[
|I_3| \leq Ch^2 \left( \|u\|_{2,\Omega} + \frac{h}{\epsilon} \|u\|_{2,\Omega} + \|g\|_{1,\Omega} \right) \|z\|_{2,\Omega}.
\]

Consequently,
\[
|a_h(\tilde{u}, \tilde{z}) - l_h(\tilde{z})| \leq Ch^2 \left( \|u\|_{W^{3,q}(\Omega)} + \frac{h}{\epsilon} \|u\|_{2,\Omega} + \|g\|_{3,\Omega} \right) \|z\|_{2,\Omega}.
\]
In the same way as the proof of (4.4), we derive
\[ |a_h(\bar{u}, w) - l_h(w)| \leq Ch \left( ||u||_{2,\Omega} + \frac{h}{\varepsilon} ||u||_{2,\Omega} + ||g||_{1,\Omega} \right) ||w||_{h,*} \]
for \( w \in H^2(\tilde{\Omega}) + V_h \). By substituting \( w = \bar{z} - \Pi_h \tilde{z} \), we have
\[ |r_h(\bar{u}, \Pi_h \tilde{z})| \leq |a_h(\bar{u}, \tilde{z} - \Pi_h \tilde{z}) - l_h(\tilde{z} - \Pi_h \tilde{z})| + |a_h(\bar{u}, \tilde{z}) - l_h(\tilde{z})| \leq Ch^2 \left( ||u||_{W^3(\Omega)} + \frac{h}{\varepsilon} ||u||_{2,\Omega} + ||g||_{3,\Omega} \right) ||z||_{2,\Omega}. \]
Finally, we have
\[ ||\bar{u} - u_h||_{\Omega_h \setminus \Omega} \leq Ch ||\bar{u} - u_h||_{h,*} \]
and obtain
\[ ||\bar{u} - u_h||_{\Omega_h} \leq Ch(1 + \varepsilon)c_1(h, \varepsilon) ||\bar{u} - u_h||_{h,*} + C(1 + \varepsilon) \sup_{\eta \in L^2(\Omega_h)} \frac{h||z||_{2,\Omega} ||\bar{u} - u_h||_{h,*} + |r_h(\bar{u}, \Pi_h \tilde{z})|}{||z||_{2,\Omega}}. \]
This, together with (2.11), implies the estimate (2.12).
We proceed to the case \( \varepsilon = 0 \). We replace \( I_2 \) by
\[ I_2' = - \sum_{E \in \mathcal{E}_h} \langle \nabla \bar{u} \cdot (v_h - v \circ \pi), \tilde{z} \rangle_E - \sum_{E \in \mathcal{E}_h} \left[ \langle \nabla \bar{u} \cdot v \circ \pi, \tilde{z} \rangle_E - \langle \nabla u \cdot v, \tilde{z} \rangle_{\pi(E)} \right] \]
\[ + \sum_{E \in \mathcal{E}_h} \frac{1}{\gamma h_E} \left[ \langle \bar{u} - \bar{u}_0, \tilde{z} \rangle_E - \langle u - u_0, \tilde{z} \rangle_{\pi(E)} \right] \]
\[ =: A'_1 + A'_2 + A'_3, \]
and \( I_3 \) by
\[ I'_3 = \sum_{E \in \mathcal{E}_h} \left\langle \bar{u} - \bar{u}_0, \frac{\partial \tilde{z}}{\partial v_h} \right\rangle_E. \]
Using (5.1b), (3.2a) and (3.2c), we have
\[ |A'_1| \leq Ch \|\nabla \bar{u}\|_{\Gamma_h} \|\tilde{z} - z \circ \pi\|_{\Gamma_h} \leq Ch^2 \|u\|_{2,\Omega} \|z\|_{2,\Omega}, \]
and
\[ |A'_2| \leq |\langle \nabla \bar{u} \cdot v \circ \pi, \tilde{z} - z \circ \pi \rangle_E| \leq Ch^2 (\|u\|_{2,\Omega} + \|u_0\|_{2,\Omega}) \|z\|_{2,\Omega}. \]
We apply (3.4a) and (3.2c) to obtain
\[ |A'_3| \leq Ch_{\min}^{-1} \|\bar{u} - \bar{u}_0\|_{\Gamma_h} \|\tilde{z} - z \circ \pi\|_{\Gamma_h} \leq Ch^2 (\|u\|_{2,\Omega} + \|u_0\|_{2,\Omega}) \|z\|_{2,\Omega}. \]
Finally, we have by (3.4a)
\[ |I'_3| \leq \|\bar{u} - \bar{u}_0\|_{\Gamma_h} \left\| \frac{\partial \tilde{z}}{\partial v_h} \right\|_{\Gamma_h} \leq Ch^2 (\|u\|_{2,\Omega} + \|u_0\|_{2,\Omega}) \|z\|_{2,\Omega}. \]
Summing up,
\[
|r_h(u, \Pi_h \tilde{z})| \leq |a_h(u, \tilde{z} - \Pi_h \tilde{z}) - l_h(\tilde{z} - \Pi_h \tilde{z})| + |a_h(u, \tilde{z}) - l_h(\tilde{z})| \leq C h^2 \left( \|u\|_{L, \Omega} + \|u_0\|_{L, \Omega} + \|g\|_{L, \Omega} \right) \|\tilde{z}\|_{L, \Omega}.
\]
Therefore, (2.13) is proved.

\[\]

6 | NUMERICAL EXAMPLES

In this section, we present some numerical results to verify the validity of our error estimates. We consider the Poisson problem (1.1) in a disk \(\Omega = \{x \in \mathbb{R}^2 : |x| < 1\}\). The penalty parameter is chosen as \(\gamma = 0.1\). We calculate all numerical results using FreeFEM++.

First, we confirm the validity of the estimates in Theorems 1 and 2. We set \(f\), \(g\) and \(u_0\) so that a function \(u(x_1, x_2) = \sin(x_1) \sin(x_2)\) solves (1.1). Let \(u_h \in V_h\) be the solution of (2.3). Table 1 shows the \(H^1\) error \(\|u - u_h\|_{H^1}\) and Table 2 shows the \(L^2\) error \(\|u - u_h\|_{L^2}\) for several \(\varepsilon\)’s. Figure 1 shows the \(H^1\) error and the \(L^2\) error \(\|u - u_h\|_{L^2}\) for \(\varepsilon = 0, 1, 10\).

We observe that the convergence rates are almost \(O(h)\) for the \(H^1\) error and \(O(h^2)\) for the \(L^2\) error for each \(\varepsilon\). Thus, the optimal convergence rates actually take place and the estimates of Theorems 1 and 2 are confirmed. But the error is almost same for each \(\varepsilon\). So, there is further room for studying the dependence of \(\varepsilon\).

Subsequently, we consider the exact solution \(u(x_1, x_2) = ((x_1 + 1)^2 + x_2^2)^{\frac{3}{2}}\) and the corresponding \(f\), \(g\), and \(u_0\). Let \(\varepsilon = 1\). In this case, \(u \in H^2(\Omega)\) and \(u \notin W^{3,4}(\Omega)\) for any \(q > n\). That is, the assumption

### Table 1. \(H^1\) errors for the exact solution \(u = \sin(x_1) \sin(x_2)\).

| \(h\)     | \(\varepsilon = 100\) | \(\varepsilon = 1\) | \(\varepsilon = 0.01\) | \(\varepsilon = 0\) |
|-----------|----------------------|---------------------|-----------------------|------------------|
| 5.83 \times 10^{-1} | 3.01 \times 10^{-1} | 2.99 \times 10^{-1} | 2.95 \times 10^{-1} | 2.96 \times 10^{-1} |
| 3.01 \times 10^{-1} | 1.39 \times 10^{-1} | 1.39 \times 10^{-1} | 1.38 \times 10^{-1} | 1.38 \times 10^{-1} |
| 1.61 \times 10^{-1} | 6.58 \times 10^{-2} | 6.57 \times 10^{-2} | 6.57 \times 10^{-2} | 6.57 \times 10^{-2} |
| 7.93 \times 10^{-2} | 3.20 \times 10^{-2} | 3.20 \times 10^{-2} | 3.20 \times 10^{-2} | 3.20 \times 10^{-2} |
| 4.39 \times 10^{-2} | 1.59 \times 10^{-2} | 1.59 \times 10^{-2} | 1.59 \times 10^{-2} | 1.59 \times 10^{-2} |
| 2.46 \times 10^{-2} | 7.72 \times 10^{-3} | 7.72 \times 10^{-3} | 7.72 \times 10^{-3} | 7.72 \times 10^{-3} |
| 1.24 \times 10^{-2} | 3.90 \times 10^{-3} | 3.90 \times 10^{-3} | 3.90 \times 10^{-3} | 3.90 \times 10^{-3} |
| 6.17 \times 10^{-3} | 1.97 \times 10^{-3} | 1.97 \times 10^{-3} | 1.97 \times 10^{-3} | 1.97 \times 10^{-3} |

### Table 2. \(L^2\) errors for the exact solution \(u = \sin(x_1) \sin(x_2)\).

| \(h\)     | \(\varepsilon = 100\) | \(\varepsilon = 1\) | \(\varepsilon = 0.01\) | \(\varepsilon = 0\) |
|-----------|----------------------|---------------------|-----------------------|------------------|
| 5.83 \times 10^{-1} | 2.24 \times 10^{-2} | 1.87 \times 10^{-2} | 1.72 \times 10^{-2} | 1.72 \times 10^{-2} |
| 3.01 \times 10^{-1} | 5.23 \times 10^{-3} | 4.29 \times 10^{-3} | 3.55 \times 10^{-3} | 3.57 \times 10^{-3} |
| 1.61 \times 10^{-1} | 1.37 \times 10^{-4} | 1.11 \times 10^{-4} | 9.15 \times 10^{-4} | 9.21 \times 10^{-4} |
| 7.93 \times 10^{-2} | 3.42 \times 10^{-4} | 2.77 \times 10^{-4} | 2.28 \times 10^{-4} | 2.29 \times 10^{-4} |
| 4.39 \times 10^{-2} | 8.71 \times 10^{-5} | 7.03 \times 10^{-5} | 5.78 \times 10^{-5} | 5.81 \times 10^{-5} |
| 2.46 \times 10^{-2} | 2.01 \times 10^{-5} | 1.62 \times 10^{-5} | 1.37 \times 10^{-5} | 1.38 \times 10^{-5} |
| 1.24 \times 10^{-2} | 5.08 \times 10^{-6} | 4.09 \times 10^{-6} | 3.54 \times 10^{-6} | 3.58 \times 10^{-6} |
| 6.17 \times 10^{-3} | 1.32 \times 10^{-7} | 1.07 \times 10^{-7} | 9.12 \times 10^{-7} | 9.18 \times 10^{-7} |
of Theorem 2 does not satisfied. Figure 2 reports the $H^1$ error and the $L^2$ error for $\varepsilon = 1$. We see from Figure 2 that the convergence rate for the $H^1$ error is $O(h)$. However, the second-order convergence does not achieve for the $L^2$ error. Actually, we observe that the convergence rate for the $L^2$ error is $O(h^{2-\sigma})$ with some small $\sigma > 0$. This result is consistent with Theorem 2. Therefore, this result is strong evidence that the regularity condition $u \in W^{3, q}(\Omega)$ with $q > n$ is an essential requirement for deducing the optimal order convergence.
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DATA AVAILABILITY STATEMENT
The data that support the findings of this study are available from the corresponding author upon reasonable request.

ORCID
Yuki Chiba https://orcid.org/0000-0002-7119-6851

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