QUANTUM GEOMETRY OF ALGEBRA
FACTORISATIONS AND COALGEBRA BUNDLES

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Abstract

We develop the noncommutative geometry (bundles, connections etc.) associated to algebras that factorise into two subalgebras. An example is the factorisation of matrices $M_2(\mathbb{C}) = \mathbb{C}Z_2 \cdot \mathbb{C}Z_2$. We also further extend the coalgebra version of theory introduced previously, to include frame resolutions and corresponding covariant derivatives and torsions. As an example, we construct $q$-monopoles on all the Podleś quantum spheres $S^2_{q,s}$.

1. Introduction

In [8] it was shown that one can generalise the notion of principal bundles in noncommutative geometry to a very general setting in which the role of ‘coordinate functions’ on the base is played by a general (possibly noncommutative) algebra and the role of the ‘structure group’ (fibre) of the principal bundle is a coalgebra. In particular, it need not be a quantum group, which would be too restrictive for many interesting examples. In [6] the theory of modules or ‘associated bundles’ is extended to this case along the lines of the quantum group case in [7]. We apply this now to extend the recently introduced notion of a frame resolution [8], thereby bringing the coalgebra version of the gauge theory in line with the more restrictive quantum group gauge theory case.

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The paper begins, however, in Section 2 with a useful reformulation of coalgebra bundles entirely in terms of algebras. This is a theory where the role of ‘gauge group’ or fibre in the principal bundle is played by any algebra $A$ subject to a certain non-degeneracy ‘Galois’ condition for its action on the algebra $P$ of the total space of the bundle. The algebra $A$ plays the role of a classical (or quantum) enveloping algebra of a Lie algebra in usual (or quantum group) gauge theory, but now without any kind of Hopf algebra structure. Without the latter one cannot make general tensor products of representations so that it is indeed remarkable that the formulation of geometric notions is possible. This is what we outline, namely a gauge theory that has connections, principal bundles, associated bundles etc. using only algebras and in particular not requiring anything from the theory of quantum groups.

As such, the material in Section 2 should be rather more widely accessible than the coalgebra bundle version of the theory. In particular, it can be viewed as a critical first step towards a $C^*$ algebra or von Neumann algebra treatment. While beyond our scope to actually consider operator theory and topological completions here (we work algebraically), it offers the possibility to link up with and extend other approaches to noncommutative geometry based on $C^*$-algebras etc. We recall that in the $C^*$ algebra approach to noncommutative geometry, see [11], one traditionally works directly with vector bundles (as projective modules) and not principal bundles – one would expect that the latter would require some kind of group-like object such as a Hopf algebra but we see that this need not be the case. Also, although we do not develop a precise connection with the theory of subfactors at the present time, we note that our final data in terms of algebras is not unlike a subfactor inclusion. In that context one considers inclusions of von Neumann algebras with the larger one being viewed as some kind of ‘cross product’ of the smaller one by some kind of ‘paragroup’[27]. Similarly we show that if $A$ is an algebra acting on another algebra $P$ subject to a certain nondegeneracy condition then one can form a generalised ‘cross product’ (which we call the ‘Galois product’) of $P$ by $A$. In the subfactor case it is known that a special case corresponds to some kind of (weak) Hopf algebra[1], while similarly a special case of an algebra bundle corresponds to $A$ a Hopf algebra. The development of such an analogy on the one hand could provide a gauge theory of subfactors (as well as a coalgebraic version of some aspects of their theory) and on the other hand suggest the existence of a whole ‘Jones tower’ of bundles. It would also connect with gauge theory from the point of view of algebraic quantum field theory as in [14] [15] and many other works.
Linking up with $C^*$ or operator algebra results is the long-term motivation for the section. From a mathematical point of view, however, it should be stressed that our present results are strictly equivalent to a subset of the coalgebra bundle case. Part of the reformulation was already hinted at in [8] where part of the data was expressed as an algebra factorising into subalgebras $A, P$. The crucial exactness or ‘Galois’ condition in this form is what we provide now. It turns out to involve traces over the vector space of $A$, which essentially forces us to finite-dimensional $A$. From this it is clear that the theory can be developed in two ways to cover the infinite-dimensional case: either one needs to introduce operator completions which is the $C^*$ algebra or von Neumann algebra direction mentioned above, or one needs to replace $A$ by its dual, a coalgebra, which then works for infinite-dimensional coalgebras – this is the approach taken in [8] and in the remaining sections of the present paper.

In Section 3 and Section 4 we continue with new results in the coalgebraic setting. We provide the necessary formulation of associated bundles by exploiting the recent work [5]. A small generalisation of coalgebra bundles has been made in [9] and we will use in fact this formulation. Also, the notion of a connection which we use here requires less structure than the one introduced in [8]. In Section 5 we study frame resolutions at this level.

Finally, in Section 6, we show that the coalgebra theory allows one to include the crucial example of the monopole on the full 2-parameter family of Podleś quantum spheres [28]. Recall that Podleś classified all reasonable ‘quantum spheres’ covariant under the quantum group $SU_q(2)$, and until now the q-monopole has been constructed in [7] only for a diagonal subfamily (the so-called standard quantum spheres). The general case requires the more general coalgebra bundle theory. The bundle itself for all the quantum 2-spheres is in [4] and we provide on this now the required connection. Similarly it is clear from their construction that all of the q-deformed symmetric spaces in the classification of [26] should be constructable as coalgebra bundles, which includes the coalgebra bundle from which one would expect to project out a q-instanton. This is a second direction for further work.

We work algebraically over a general field $k$. We use the usual notations $\Delta c = c(1) \otimes c(2)$ for a coproduct of a coalgebra $C$ (summation understood). We also write $C^+ = \ker \varepsilon$ where $\varepsilon$ is the counit. We write $\nu \Delta(v) = v(1) \otimes v(\infty)$ for a left coaction on a vector space $V$, and $\Delta_V(v) = v(0) \otimes v(1)$ for a right coaction. We also denote by $\text{Hom}_A(V,W)$ the linear maps commuting with a right action of an algebra $A$ and
by \( A \mathrm{Hom}(V, W) \) those commuting with a left action. Similarly, \( \mathrm{Hom}^C(V, W) \) for maps commuting with a right coaction of a coalgebra \( C \) and \( C \mathrm{Hom}(V, W) \) for a left coaction. In general when we need to refer to the components of other elements \( \chi^# \), \( \Psi(a \otimes u) \) etc. of tensor product spaces we will use the upper bracket notation \( \chi^# = \chi^#(1) \otimes \chi^#(2) \) etc. again with summation understood.

Also, we recall that for any algebra \( P \), the universal 1-forms on \( P \) are \( \Omega^1 P = \ker \cdot P \subseteq P \otimes P \). The exterior derivative \( d : P \rightarrow \Omega^1 P \) is \( du = 1 \otimes u - u \otimes 1 \) for all \( u \in P \). This extends to higher forms (see [18] e.g.) \( \Omega^n P \subseteq P \otimes \cdots \otimes P \) characterised by the requirement that the products of all adjacent factors vanish, and \( d : \Omega^{n-1} P \rightarrow \Omega^n P \),

\[
d(u_1 \otimes \cdots \otimes u_n) = \sum_{k=1}^{n+1} (-1)^{k-1} u_1 \otimes \cdots \otimes u_{k-1} \otimes 1 \otimes u_k \otimes \cdots \otimes u_n. \tag{1}\]

With these definitions \( \Omega P = \bigoplus_{n=0}^{\infty} \Omega^n P \) is a graded differential algebra with product given by juxtaposition and multiplication in \( P \).

2. Galois actions and algebra factorisations

Although we will continue mainly in an algebra-coalgebra setting in later sections, we start with a more accessible version which depends only on algebras and which should be useful for the operator-algebraic version. We consider unital algebras and unital algebra maps. An algebra factorisation means an algebra \( X \) and subalgebras \( P, A \) such that the linear map \( P \otimes A \rightarrow X \) given by the product in \( X \) is an isomorphism.

**Proposition 2.1** Cf. [22, 20, 10] algebra factorisations are in 1-1 correspondence with algebras \( P, A \) and \( \Psi : A \otimes P \rightarrow P \otimes A \) such that

\[
\Psi(\cdot A \otimes \cdot) = (\cdot \otimes \cdot A)\Psi_{12}\Psi_{23}, \quad \Psi(1 \otimes u) = u \otimes 1, \quad \forall u \in P
\]

\[
\Psi(\cdot \otimes 1) = (\cdot \otimes \cdot)\Psi_{23}\Psi_{12}, \quad \Psi(1 \otimes a) = 1 \otimes a, \quad \forall a \in A.
\]

In this case, given \( e : A \rightarrow k \) a character, there is a left action

\[
\triangleright : A \otimes P \rightarrow P, \quad a \triangleright u = (\cdot \otimes e)\Psi(a \otimes u), \quad \forall a \in A, \quad u \in P.
\]

The subspace \( M = P_e = \{ m \in P | a \triangleright m = e(a)m \ \forall a \in A \} \) forms a subalgebra.

**Proof.** Details of the stated equivalence are in [20, pp. 299-300]. Given \( \Psi \) we define \( X = P \otimes A \) with product \( (u \otimes a)(v \otimes b) = u\Psi(a \otimes v)b \) for \( u, v \in P \) and \( a, b \in A \). Given
we define \( \Psi \) by \( au = \cdot_X \Psi(a \otimes u) \). The action \( \triangleright \) is also part of the proof in \([20]\) (where \( e = \varepsilon \) the counit of a bialgebra). There is a similar right action of \( P \) on \( A \) when \( P \) is equipped with a character, which we do not use. From the point of view of \( X \), \( e \) on \( A \) extends to a left \( P \)-module map \( e : X \to P \) obeying \( e(au) = a \triangleright u \) for all \( a \in A \) and \( u \in P \). Hence \( M = \{ u \in P | e(au) = ue(a) \, \forall a \in A \} \), from which it is clear that \( M \) is a subalgebra. One may also see this from the equations for \( \Psi \). ⊓ ⊔

Such factorisations are quite common. For example, they come up naturally as part of Hopf algebra factorisations\([21]\)\([20]\). Another example is the braided tensor product \( A \hat{\otimes} B \) of two algebras, see \([20]\). In our geometrical picture, \( P \) plays the role of the algebra of functions of the ‘total space’ of a principal bundle, and \( A \) plays the role of the group algebra of the structure group. The subalgebra \( M \) plays the role of the functions on the base. The algebra \( X \) is not usually considered but plays the role of the cross product \( C^*\)-algebra of the functions on the total space by the action of the structure group.

**Proposition 2.2** In the setting above, the map \( \tilde{\chi} : A \otimes P \otimes P \to P \) defined by \( \tilde{\chi}(a \otimes u \otimes v) = (a \triangleright u)v \) descends to \( \chi : A \otimes P \otimes_M P \to P \). We say that the factorisation is Galois if there exists \( \chi^\# : P \to P \otimes_M P \otimes A \) such that

\[
\text{Tr}_A(\chi^\# \circ \chi) = \text{id}_{P \otimes_M P}, \quad (\chi \otimes \text{id}_A)(\text{id}_A \otimes \chi^\#) = \tau : A \otimes P \to P \otimes A
\]

where \( \tau \) is the usual flip or transposition map. We call \( P(M, A, \Psi, e) \) a copointed algebra bundle.

**Proof.** We have \( a \triangleright (um) = e(auq) = e(u_i a^i m) = u_i e(a^i m) = u_i me(a^i) = (a \triangleright u)m \) for all \( a \in A \), \( u \in P \) and \( m \in M \), as required. Here \( \Psi(a \otimes u) = u_i \otimes a^i \) is a notation (sum over \( i \)). The rest is a definition. This can also be obtained from the \( \Psi \) equations. ⊓ ⊔

This is the analogue of the Galois condition in \([8]\), which in turn is motivated from the theory of quantum principal bundles and, independently, the theory of Hopf-Galois extensions in the Hopf algebra case. In geometrical terms the map \( \chi \) plays the role of the action of the Lie algebra \( g \) of the structure group of a principal bundle on its algebra of functions: if \( \xi \in g \) one has a left-invariant vector field \( \tilde{\xi} \) given by differentiating the action corresponding to \( \triangleright \). The element \( \chi^\# = \chi^\#(1) \in P \otimes_M P \otimes A \) is particularly important and plays the role of the ‘translation map’ of the principal bundle.
Notice, however, that a factorisation can be Galois only if \( A \) is finite-dimensional. This should not unduly worry us since our formulation is mainly intended as a precursor to an operator-theoretic treatment where infinite-dimensional \( A \) would be allowed subject to topological completions and trace class conditions. To avoid all that in the infinite-dimensional case one should of course use the coalgebra formulation as in later sections. Meanwhile, let us note that even finite-dimensional \( A \) are not uninteresting – the algebra \( P \) and the factorising algebra can in principle both be infinite-dimensional. A similar situation pertains with the theory of subfactors where the two von Neumann algebras are typically infinite-dimensional but the case where their ‘ratio’ is in some sense finite is still very interesting.

There is an obvious notion of a \( \Psi \)-module associated to an algebra factorisation, namely a left \( P \) module and \( A \) module \( V \) such that

\[
 a \triangleright (u \triangleright v) = \triangleright (\Psi(a \otimes u) \triangleright v), \quad \forall a \in A, \ u \in P, \ v \in V. \tag{2}
\]

Explicitly we require \( a \triangleright (u \triangleright v) = u, \triangleright (a^i \triangleright v), \) where \( \Psi(a \otimes u) = u_i \otimes a^i. \) This is what corresponds to a left \( X \)-module. This point of view suggests a natural slight generalisation of the above, replacing \( e \) by the requirement of a map \( \tilde{e} : A \rightarrow P. \)

**Proposition 2.3** Let \((P, A, \Psi)\) be an algebra factorisation datum as in Proposition 2.1. Linear maps \( \tilde{e} : A \rightarrow P \) such that

\[
 \tilde{e}(ab) = \Psi(a \otimes \tilde{e}(b))^{(1)} \tilde{e}(\Psi(a \otimes u)^{(2)}), \quad \tilde{e}(1) = 1, \quad \forall a, b \in A,
\]

are in 1-1 correspondence with extensions of the left regular action of \( P \) on itself to a \( \Psi \)-module structure on \( P \). Given \( \tilde{e} \), we define

\[
 a \triangleright u = \Psi(a \otimes u)^{(1)} \tilde{e}(\Psi(a \otimes u)^{(2)}), \quad \forall a \in A, \ u \in P
\]

and conversely, given such an extension, we set \( \tilde{e}(a) = a \triangleright 1. \) In this situation the space

\[
 M = M_{\tilde{e}} = \{m \in P| a \triangleright m = \tilde{e}(a)m, \ \forall a \in A\},
\]

is a subalgebra of \( P \) and \( \tilde{X} \) as in Proposition 2.2 descends to a map \( \chi. \)

**Proof.** We define the linear map \( \triangleright : A \otimes P \rightarrow P \) as stated and verify first equation (2) as

\[
 \Psi(a \otimes uv)^{(1)} \tilde{e}(\Psi(a \otimes uv)^{(2)}) = u_i \Psi(a^i \otimes v)^{(1)} \tilde{e}(\Psi(a^i \otimes v)^{(2)}) = \Psi(a \otimes u)^{(1)} (\Psi(a \otimes u)^{(2)} \triangleright v),
\]
where we used the second of factorisation properties in Proposition 2.1. We also used
the shorthand Ψ(a ⊗ u) = u_i ⊗ a^i as before. Next, we check that ⊲ is indeed an action,

(1) \( (ab) \triangleright v = \Psi(ab \otimes u)(1) \tilde{e}(\Psi(ab \otimes u)(2)) = \Psi(a \otimes u_i)(1) \tilde{e}(\Psi(a \otimes u_i)(2) b^i) \)

= \( \Psi(a \otimes u_i)(1)(\Psi(a \otimes u_i)(2) \triangleright \tilde{e}(b^i)) = a \triangleright (u_i \tilde{e}(b^i)) = a \triangleright (b \triangleright u) \)

as required. We used the first of the factorisation properties of Ψ and the assumed
condition on ˜e, which can be written as ˜e(ab) = a ⊲ ˜e(b) in terms of ⊲. We then used (2)
already proven. Finally, 1 ⊲ u = u˜e(1) = 1 so ⊲ is indeed an action. Conversely, given an
action ⊲ making \( P \) a Ψ-module we define ˜e(a) = a ⊲ 1. Then ˜e(ab) = a ⊲ (b ⊲ 1) = a ⊲ ˜e(b) and
a ⊲ u = a ⊲ (u ⊲ 1) = u_i(a^i ⊲ 1) = u_i ˜e(a^i) (using (2)), as required. The remaining facts
follow easily from the definition of \( M \). It can also be characterised equivalently as

\( M = \{ m \in P | a \triangleright (um) = (a \triangleright u)m, \forall u \in P, a \in A \} \) (3)

in view of (2) and the definition of ⊲. \( \square \)

We note that

**Lemma 2.4** *In the setting of Proposition 2.3, for any Ψ-module \( V \) there is a natural
notion of ‘invariant’ subspace

\( V_0 = \{ v \in V | a \triangleright v = \tilde{e}(a) \triangleright v, \forall a \in A \} \subseteq V \) (4)

which is a left \( M \)-module by restriction of the action of \( P \).*

*Proof.* For all \( a \in A, m \in M \) and \( v \in V_0 \), we have \( a \triangleright (m \triangleright v) = \triangleright (\Psi(a \otimes m) \triangleright v) = m_i \triangleright (\tilde{e}(a^i) \triangleright v) = (m_i \tilde{e}(a^i)) \triangleright v = (a \triangleright m) \triangleright v = (\tilde{e}(a)m) \triangleright v = \tilde{e}(a) \triangleright (m \triangleright v), \) so \( m \triangleright v \in V_0 \) as well. \( \square \)

The subalgebra \( M \) itself is a case of such an invariant subspace. When there is
a corresponding \( \chi^\# \), we call \( P(M,A,\Psi,\tilde{e}) \) an algebra bundle. The copointed case is
\( \tilde{e}(a) = e(a)1 \). The construction has a natural converse.

**Lemma 2.5** *In an algebra bundle, \( \chi^\# = \chi^\#(1) \) obeys

(a) \( \chi^\#(a) = a \triangleright \chi^\# \) for all \( a \in A \), where the action on \( P \otimes M P \otimes A \) is on its first
factor.

(b) \( \chi^\#(uv) = \chi^\#(u)(1) v \otimes \chi^\#(u)(2) \) for all \( u, v \in P \)

(c) \( \chi^\#(1) \triangleright (\chi^\#(2) \triangleright u) = u \otimes_M 1 \).

Here \( \chi^\#(1) \in P \otimes_M P \) and \( \chi^\#(2) \in A \) for all \( u \in P \).
Proof. From its definition, it is evident that

$$\chi(ab \otimes u \otimes v) = (ab \triangleright u)v = \chi(a \otimes b \triangleright u \otimes v), \quad \chi(a \otimes u \otimes vw) = \chi(a \otimes u \otimes v)w$$

for all $u, v, w \in P$ and $a, b \in A$. Parts (a) and (b) are just the corresponding properties for $\chi^\#$. Thus,

$$a\triangleright \chi^\# = (\text{Tr}_A \chi^\# \circ \chi \otimes \text{id})(a\triangleright \chi^\#) = (\text{id} \otimes f^a)(\chi(e_a \otimes a\triangleright \chi^\#(1))) \otimes \chi^\#(2)$$

$$= (\text{id} \otimes f^a)\chi^\#(\chi(e_a \otimes \chi^\#(u)) \otimes \chi^\#(v)) = (\text{id} \otimes f^a)\chi^\#(1) \otimes e_a a = \chi^\# a$$

where $\{e_a\}$ is a basis of $A$ and $\{f^a\}$ a dual basis. Similarly,

$$\chi^\#(u)^{(1)} v \otimes \chi^\#(u)^{(2)} = (\text{Tr}_A \chi^\# \circ \chi(\chi^\#(u)^{(1)})) \otimes \chi^\#(v)^{(2)}$$

$$= (\text{id} \otimes f^a)\chi^\#(\chi(e_a \otimes \chi^\#(u))) \otimes \chi^\#(v) = \chi^\#(uv).$$

We then deduce part (c) from part (b) as

$$\chi^\#(1)^{(1)}(\chi^\#(1)^{(2)} \triangleright u) = \chi^\#(1)^{(1)}(e_a \triangleright u)(f^a, \chi^\#(1)^{(2)}$$

$$= (\text{id} \otimes f^a)\chi^\#(1)(e_a \triangleright u) = (\text{id} \otimes f^a)\chi^\# = \chi^\#(u \otimes 1).$$

These correspond to important properties of the translation map in differential geometry derived in the Hopf algebraic setting in [3][31].

**Theorem 2.6** Let $P, A$ be algebras and $P$ a left $A$-module under an action $\triangleright$. We define $M$ by (3) and say that the action is Galois if $\chi$ defined as in Proposition 2.2 has a corresponding $\chi^\#$. In this case there exists a unique algebra factorisation $X = P \otimes_q A$ such that $P, A$ form an algebra bundle and $P$ is a $\Psi$-module (cf. eq. (3)) via product in $P$ and the action $\triangleright$. Explicitly,

$$\Psi(a \otimes u) = \chi(a \otimes u \chi^\#(1)) \otimes \chi^\#(2), \quad \tilde{e}(a) = a \triangleright 1, \quad \forall a \in A, \ u \in P.$$ 

We call the corresponding algebra factorisation $X = P \otimes_q A$ the Galois product associated to a Galois action of an algebra $A$ on an algebra $P$.

**Proof.** Here we define $M$ and $\tilde{\chi}$ directly from the action $\triangleright$; it is easy to see that $M$ is a subalgebra and that $\tilde{\chi}$ descends to a map $\chi$. We assume the existence of a corresponding $\chi^\#$ obeying the conditions in Proposition 2.2. For the purposes of this proof, we now
write \( \chi^# = \chi^{(1)} \otimes_M \chi^{(2)} \otimes \chi^{(3)} \) (a more explicit notation than the one before) and we let \( \chi' \) be a second copy of \( \chi^# \). Then the map \( \Psi \) explicitly reads

\[
\Psi(a \otimes u) = (a \triangleright (u \chi^{(1)})) \chi^{(2)} \otimes \chi^{(3)}
\]

and we have,

\[
(id \otimes \cdot_A)\Psi_{12} \Psi_{23} (a \otimes b \otimes u) = (id \otimes \cdot_A)\Psi_{12}(a \otimes (b \triangleright (u \chi^{(1)}))) \chi^{(2)} \otimes \chi^{(3)}
\]

\[
= (a \triangleright ((b \triangleright (u \chi^{(1)}))) \chi^{(2)} \chi^{(1)}) \chi^{(2)} \otimes \chi^{(3)}
\]

\[
= (a \triangleright ((b \triangleright (u \chi^{(1)}))) \chi^{(2)} \chi^{(3)} \triangleright \chi^{(1)}) \chi^{(2)} \otimes \chi^{(3)}
\]

\[
= (a \triangleright (b \triangleright (u \chi^{(1)}))) \chi^{(2)} \chi^{(3)} = \Psi(ab \otimes u)
\]

using parts (a) and then (c) of the lemma and that \( \triangleright \) is an action. On the other side, we have

\[
(\cdot_P \otimes id)\Psi_{23} \Psi_{12} (a \otimes u \otimes v) = (\cdot_P \otimes id)\Psi_{23}((a \triangleright (u \chi^{(1)})) \chi^{(2)} \otimes \chi^{(3)} \otimes v)
\]

\[
= (a \triangleright (u \chi^{(1)})) \chi^{(2)} \chi^{(3)} \triangleright (v \chi^{(1)}) \chi^{(2)} \otimes \chi^{(3)}
\]

\[
= (a \triangleright (uv \chi^{(1)})) \chi^{(2)} \otimes \chi^{(3)} = \Psi(a \otimes u)
\]

using part (c) of the lemma. The computations for \( \Psi(a \otimes 1) \) and \( \Psi(1 \otimes u) \) are more trivial and left to the reader. We need

\[
\chi^{(1)} \chi^{(2)} \otimes \chi^{(3)} = \chi^{(1)} \otimes \chi^{(2)} \otimes \chi^{(3)} = 1 \otimes 1
\]

for the latter case.

Hence we have a factorisation datum and by Proposition 2.1 we have an algebra \( X \) built on \( P \otimes A \) with the cross relations \( (1 \otimes a)(u \otimes 1) = \Psi(a \otimes u) \). We now define \( \tilde{\epsilon}(a) = a \triangleright 1 \) and check easily that \( \tilde{\epsilon}(ab) = a \triangleright \tilde{\epsilon}(b) \) as required, and that \( \Psi(a \otimes u)^{(1)} \tilde{\epsilon}(\Psi(a \otimes u)^{(2)}) = (a \triangleright (u \chi^{(1)})) \chi^{(2)} \chi^{(3)} \triangleright 1 = a \triangleright u \) by part (c) of the lemma. Hence is the converse to the preceding proposition.

To prove that \( P \) is a \( \Psi \)-module, we take any \( a \in A, u, v \in P \) and use the explicit form of \( \Psi \) above and part (c) of the lemma to compute

\[
\cdot \circ (\Psi(a \otimes u) \triangleright v) = (a \triangleright (u \chi^{(1)})) \chi^{(2)} \chi^{(3)} \triangleright v = a \triangleright (uv).
\]

Finally, suppose there is another factorisation \( \Psi' \) such that \( P \) is a \( \Psi' \)-module, and let \( \Psi'(a \otimes u) = u_i \otimes a^i \) for all \( a \in A, u \in P \). Then

\[
\Psi(a \otimes u) = (a \triangleright (u \chi^{(1)})) \chi^{(2)} \otimes \chi^{(3)} = u_i(a^i \triangleright \chi^{(1)}) \chi^{(2)} \otimes \chi^{(3)} = u_i \otimes a^i = \Psi'(a \otimes u),
\]

where we used the definition of \( \chi^# \). This proves the uniqueness of \( \Psi \). □
Example 2.7 Let $q$ be a primitive $n$’th root of 1. The $n \times n$ matrices factorise as $M_n(\mathbb{C}) = \mathbb{C}Z_n \cdot \mathbb{C}Z_n$, where the two copies of $\mathbb{Z}_n$ are generated by

$$g = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & q & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & q^n & 0 \\ 0 & \cdots & 0 & \cdots & q^n-1 \end{pmatrix}, \quad h = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

obeying $hg = qgh$, so

$$\Psi(h \otimes g) = qg \otimes h, \quad \Psi(1 \otimes g) = g \otimes 1, \quad \Psi(h \otimes 1) = 1 \otimes h, \quad \Psi(1 \otimes 1) = 1 \otimes 1.$$

The nontrivial character $e(h) = q$ gives

$$h \triangleright g^m = q^{m+1}g^m$$

and hence $M = \mathbb{C}1$. The result is Galois, with

$$\chi(h^m \otimes g^k \otimes g^l) = q^{m(k+1)}g^{k+l}, \quad \chi^*(g^m) = \sum_{a,b} q^{-ab}g^{b-1} \otimes g^{m-b+1} \otimes h^a.$$

Proof. We identify $A = \mathbb{C}Z_n = \mathbb{C}[h]/h^n = 1$ and $P = \mathbb{C}Z_n = \mathbb{C}[g]/g^n = 1$ as the two subalgebras. The relations $hg = (1 \otimes h)(g \otimes 1) = \Psi(h \otimes g) = q(g \otimes 1)(1 \otimes h)$ give the form of $\Psi$. This extends uniquely to a solution of the factorisation equations in Proposition 2.1 as $\Psi(h^m \otimes g^k) = q^{mk}g^k \otimes h^m$. Actually, this is an example of a braided tensor product $M_n(\mathbb{C}) = \mathbb{C}Z_n \otimes \mathbb{C}Z_n$ in the braided category of anyonic or $\mathbb{Z}_n$-graded spaces. The character $e$ then gives the action shown as $h \triangleright g^m = q^m g^m e(h)$. Hence $\sum_m a_m g^m \in M$ iff $a_m(q^{m+1} - q) = 0$ for all $m$, which means $M = \mathbb{C}1$. We also obtain $\chi$ as shown and one may verify that $\chi^*$ as stated fulfills the requirements in Proposition 2.2. □

In this example $A$ is actually a Hopf algebra and $\tilde{e}(h) = q1$ as here yields a bundle with is equivalent (in the coalgebra bundle version) to a Hopf algebra bundle as in [7]. On the other hand, other choices of $\tilde{e}$ yield algebra bundles which are not equivalent to Hopf algebra bundles, i.e. strict examples of our more general theory. We examine the $n = 2$ case in detail:

Example 2.8 The factorisation $M_2(\mathbb{C}) = \mathbb{C}Z_2 \cdot \mathbb{C}Z_2$ as above (with $q = -1$) admits a family of algebra bundle structures parametrized by $\theta \in [0, 2\pi)$, with

$$\tilde{e}(h) = \cos(\theta) + i g \sin(\theta).$$

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The associated Galois action is

\[ h \triangleright g^k = (-1)^k g^k (\cos(\theta) + i g \sin(\theta)). \]

Proof. We have \( A = \mathbb{C}[h]/h^2 = 1 \) and \( P = \mathbb{C}[g]/g^2 = 1 \). We require \( \tilde{e} \) of the form

\[ \tilde{e}(1) = 1, \quad \tilde{e}(h) = \alpha + i \beta g \]

(say) obeying the condition in Proposition 2.3. The non-empty case is \( 1 = \tilde{e}(1) = \tilde{e}(h) = \Psi(h \otimes \tilde{e}(h))^{(1)} \tilde{e}(\Psi(h \otimes \tilde{e}(h))^{(2)}) = \alpha \tilde{e}(h) - i \beta g \tilde{e}(h) = (\alpha - i \beta g)(\alpha + i \beta g) = \alpha^2 + \beta^2 \). This admits many solutions over \( \mathbb{C} \), a natural family being those where \( \alpha, \beta \) are real, i.e. on a circle parametrized by \( \theta \). On the other hand, in the case equivalent to a Hopf algebra bundle, \( P \) would be an \( A \)-module algebra. This happens when \( (h \triangleright g)^2 = \cos^2(\theta) - \sin^2(\theta) + 2i \sin(\theta) \cos(\theta) g = 1 \), which is exactly when \( \theta = 0, \pi \). The first case is trivial and the second is the \( n = 2 \) case of the preceding Example 2.6.

Next, we consider \( m = a + bg \) such that \( h \triangleright m = \tilde{e}(h)m \). It is easy to see that this happens iff \( b = 0 \), provided \( \sin(\theta) \neq 0 \) or \( \cos(\theta) \neq 0 \) (one of which is always the case). Hence \( M = \mathbb{C}1 \). Finally, we have \( \chi(h^m \otimes g^k \otimes g^l) = g^{k+l}((-1)^k (\cos(\theta) + i \sin(\theta) g)) \)

which we can write as a map \( P \otimes P \rightarrow A^* \otimes P \). Identifying \( A^* = \mathbb{CZ}_2 \) with generator \( c \), say, the map is

\[ g^k \otimes g^l \mapsto c_+ + c_- (-1)^k \cos(\theta) \otimes g^{k+l} + c_- i (-1)^k \sin(\theta) \otimes g^{k+l+1} \]

where \( c_\pm = (1 \pm c)/2 \). (This is the map \( \chi \) in the corresponding coalgebra bundle version). Invertibility of this map is equivalent to the existence of \( \chi^# \) in the present setting; in fact the map has determinant 1 in the obvious basis \( \{g^k \otimes g^l\} \) and \( \{c^k \otimes g^l\} \) and is therefore invertible. \( \square \)

The corresponding factorisation over \( \mathbb{R} \) is the quaternion algebra and provides a counterexample to the existence of \( \tilde{e} \):

Example 2.9 Over \( \mathbb{R} \), the quaternion algebra \( \mathbb{H} = \text{span}\{1, i, j, k\} \) obeying \( i^2 = j^2 = k^2 = -1 \) and \( ij = k \) etc., is a factorisation \( \mathbb{H} = \mathbb{R}[i] \mathbb{R}[j] \) where \( \mathbb{R}[i] = \mathbb{C} \) as a 2-dimensional algebra over \( \mathbb{R} \). One has

\[ \Psi(j \otimes i) = -i \otimes j, \quad \Psi(1 \otimes i) = i \otimes 1, \quad \Psi(j \otimes 1) = 1 \otimes j, \quad \Psi(1 \otimes 1) = 1 \otimes 1. \]

This factorisation admits no map \( \tilde{e} \).
Proof. The factorisation is evident, with $P = \mathbb{R}[i]$ and $A = \mathbb{R}[j]$ (the quotient of polynomials by the relation $i^2 = -1$ and $j^2 = -1$ respectively). Now suppose a linear map $\tilde{e} : \mathbb{R}[j] \to \mathbb{R}[i]$ of the form

$$\tilde{e}(1) = 1, \quad \tilde{e}(j) = \alpha + i\beta.$$ 

Then a similar computation to the one above yields this time $-1 = \tilde{e}(-1) = \tilde{e}(j.j) = \alpha^2 + \beta^2$, which has no solutions over $\mathbb{R}$. \(\square\)

Returning to the general theory,

**Proposition 2.10** An algebra bundle is trivial or ‘cleft’ if there is an invertible element $\Phi = \Phi^{(1)} \otimes \Phi^{(2)} \in P \otimes A^{op}$ such that

$$\Phi a = a \triangleright \Phi, \quad \forall a \in A,$$

where the product from the right is in $A$. In this case, $P \cong \text{Hom}_k(A, M)$ as a left $A$-module and right $M$-module.

Moreover, $(P, A, \Psi, \tilde{e})$ in Proposition 2.3 is a trivial (cleft) algebra bundle if there exists an invertible $\Phi \in P \otimes A^{op}$ obeying the above condition, with

$$\chi^\#(u) = \Phi^{(1)} \otimes \Phi^{-1}(u) \otimes \Phi^{-2}(2), \quad \forall u \in P.$$

Proof. The isom $\Theta : \text{Hom}_k(A, M) \to P$ is

$$\Theta(f) = \Phi^{(1)} f(\Phi^{(2)}), \quad \Theta^{-1}(u)(a) = \Phi^{-1}(u(\Phi^{-2}(2)a \triangleright u)).$$

Here $\Theta$ is a left $A$-module map since the image of $f$ is in $M$ and $\Phi$ obeys the condition above. Next, the latter condition is equivalent to the condition

$$\Psi(\Phi^{(1)}a \Phi^{-1}(2)) \Phi^{-2} = \phi(a)\Phi^{-1}$$

for $\Phi^{-1}$ (just compute $\tilde{e}(a) \otimes 1 = a \triangleright 1 \otimes 1 = a \triangleright (\Phi^{-1}(1) \Phi^{(1)}(2) \Phi^{-2}(2) \Phi^{-2}(2)$ using (2)). Hence

$$a \triangleright (\Phi^{-1}(1))(\Phi^{-2}(2) \triangleright u)) = a \triangleright (\Phi^{-1}(1)) \Phi^{-2}(2) \triangleright u = \tilde{e}(a)\Phi^{-1}(1)(\Phi^{-2}(2) \triangleright u),$$

i.e. $\Phi^{-1}(1)(\Phi^{-2}(2) \triangleright u \in M$ for all $u \in P$. In particular, this implies that $\Theta^{-1}(u) : A \to M$ as required. This then provides the required inverse since $\Theta \circ \Theta^{-1}(u) = \Phi^{(1)}(\Theta^{-1}(u))(\Phi^{(2)}) = \Phi^{-1}(1) \Phi^{-2}(2) \triangleright u = u$ from the definitions, and $(\Theta^{-1}(\Theta(f))(a) = \Phi^{-1}(1)(\Phi^{-2}(2) \triangleright \Theta(f)) = \Phi^{-1}(1)(\Phi^{-2}(2) \triangleright f) = \Phi^{-1}(1) f(\Phi^{(2)} \Phi^{-2}(2) a) = f(a)$ by the left $A$-module property of $\Theta$. 

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For the second part, given a factorisation datum and \( \bar{e} \), we define \( \chi^\# \) as shown. Then
\[
\chi(a \otimes \chi^\#(u)^{(1)}) \otimes \chi^\#(u)^{(2)} = (a \triangleright \Phi^{(1)}) \Phi^{-2}(u \otimes \Phi^{-2}\Phi^{(2)}) = \Phi^{(1)} \Phi^{-1}(u \otimes \Phi^{-2}\Phi^{(2)})a = u \otimes a
\]
by the property of \( \Phi \). On the other side,
\[
\text{Tr}_A \chi^\# \circ \chi(u \otimes v) = \Phi^{(1)} \otimes \Phi^{-1}(\Phi^{-2}\Phi^{(2)} \triangleright u)v = \Phi^{(1)} \Phi^{-1}(\Phi^{-2}\Phi^{(2)} \triangleright u) \otimes v = u \otimes v
\]
since \( \Phi^{-1}(\Phi^{-2}\Phi^{(2)} \triangleright (\Phi^{(2)} \triangleright u)) \in M \) by the same proof as above. Hence the action of \( A \) on \( P \) is Galois in this case. \( \square \)

**Proposition 2.11** A bundle automorphism is an invertible linear map \( P \to P \) which is a right \( M \)-module map and a left \( A \)-module map. The group of bundle automorphisms is in correspondence with invertible \( f \in P \otimes A^{\text{op}} \) such that
\[
\Psi(a \otimes f^{(1)})f^{(2)} = f^{(1)} \otimes f^{(2)}a, \quad \forall a \in A.
\]
When the bundle is trivial, such \( f \) correspond to invertible elements \( \gamma \in M \otimes A^{\text{op}} \) by \( f = \Phi \gamma \Phi^{-1} \) multiplied in \( P \otimes A^{\text{op}} \).

**Proof.** Note first of all that the set of such elements in \( P \otimes A^{\text{op}} \) form a group. Thus,
\[
\Psi(a \otimes f^{(1)}g^{(1)})g^{(2)}f^{(2)} = f^{(1)} \Psi(a \otimes g^{(1)})g^{(2)}f^{(2)} = f^{(1)}g^{(1)} \otimes f^{(2)}a = f^{(1)}g^{(1)} \otimes g^{(2)}f^{(2)}a
\]
when \( f, g \) obey this condition. The relation between such \( f \) and automorphisms \( F \) is
\[
F(u) = f^{(1)}(f^{(2)} \triangleright u), \quad f = F(\chi^\#^{(1)})\chi^\#^{(2)} \otimes \chi^\#^{(3)}.
\]
Thus, given \( f \) it is evident that \( a \triangleright F(u) = a \triangleright (f^{(1)}(f^{(2)} \triangleright u)) = \cdot \circ \Psi(a \otimes f^{(1)})f^{(2)} \triangleright u = f^{(1)}(f^{(2)}a \triangleright u) = F(a \triangleright u) \) by (2) and the property of \( f \), so \( F \) is a left \( A \)-module map (it is clearly a right \( M \)-module map as well). Also from this, it is immediate that the product in \( P \otimes A^{\text{op}} \) maps over to the composition of bundle transformations. Finally, the inverse of the construction is as shown using the properties of \( \chi^\# \). Thus,
\[
F(\chi^\#^{(1)})\chi^\#^{(2)}(\chi^\#^{(3)} \triangleright u) = F(u) \text{ by Lemma 2.5(c), and when } F \text{ is defined by } f, \text{ the inversion formula yields } (f^{(1)} \triangleright (f^{(2)} \triangleright \chi^\#^{(1)}))\chi^\#^{(2)} \otimes \chi^\#^{(3)} = f^{(1)} \chi(f^{(2)} \otimes \chi^\#^{(1)} \otimes \chi^\#^{(2)}) \otimes \chi^\#^{(3)} = f \text{ from the definition of } \chi^\#.
\]

In the case of a trivial bundle, we define \( f \) as shown and verify
\[
\Psi(a \otimes f^{(1)})f^{(2)} = \Psi(a \otimes \Phi^{(1)}\gamma^{(1)} \Phi^{-1}) \Phi^{-2}\gamma^{(2)}\Phi^{(2)}
\]
\[
= (\Phi^{(1)}\gamma^{(1)}) \Psi(a \otimes \Phi^{-1}) \Phi^{-2}\gamma^{(2)}\Phi^{(2)}
\]
\[
= (a \triangleright (\Phi^{(1)}\gamma^{(1)})) \Phi^{-1} \otimes \Phi^{-2}\gamma^{(2)}\Phi^{(2)} = f^{(1)} \otimes f^{(2)}a
\]
using the properties of $\Psi$, (5) and that $\gamma^{(1)} \in M$. Conversely, given $f$ we define $\gamma = \Phi^{-1} f \Phi$ (product in $P \otimes A^{\text{op}}$ and verify using (5) that $a \triangleright \gamma = \tilde{e}(a) \gamma$ so that $\gamma \in M \otimes A^{\text{op}}$. \hfill \Box

Next, even though $A$ is only an algebra, its action on $P$ extends naturally to tensor powers and hence to the universal exterior differentials $\Omega^n P \subseteq P^{\otimes (n+1)}$.

**Proposition 2.12** In the setting of Proposition 2.3, $\Omega^n P$ is a $\Psi^\bullet$-module with $P$ acting by left multiplication and

$$a \triangleright (u_0 \otimes \cdots \otimes u_n) = \Psi^{n+1}(a \otimes u_0 \otimes \cdots \otimes u_n)^{(1)} \tilde{e}[\Psi^{n+1}(a \otimes u_0 \otimes \cdots \otimes u_n)^{(2)}]$$

where $\Psi^\bullet|_{\Omega^n P} = \Psi^{n+1} = \Psi_{n,n+1} \cdots \Psi_{12}$ defines another factorisation datum $(\Omega P, \Psi^\bullet, A)$. It is such that $\Psi^\bullet \circ (\text{id} \otimes d) = (d \otimes \text{id}) \circ \Psi^\bullet$.

**Proof.** That $(\Omega P, \Psi^\bullet, A)$ is another factorisation datum is an elementary proof by induction repeatedly using the factorisation properties in Proposition 2.1 and the product in $\Omega P$ (which is just inherited from the product in $P$); it is left to the reader. Applying Proposition 2.3 to this new factorisation, with $\tilde{e} : A \to P \subseteq \Omega P$ then gives a $\Psi^\bullet$-module. We then restrict the action to ones of $A, P$. \hfill \Box

Armed with this, we can define a connection as an equivariant splitting of $\Omega^1 P \supseteq P(\Omega^1 M) P$ as in [7][8]. More precisely, we require that $\Pi$ has kernel $P(\Omega^1 M) P$, is a right $P$-module map and $\Pi \circ d$ is left $A$-module map. Such projections turn out to be in 1-1 correspondence with $\omega \in \Omega^1 P \otimes A$ such that

i) $\omega^{(1)} \tilde{e}(\omega^{(2)}) = 0$

ii) $\tilde{\chi}(a \otimes \omega^{(1)}) \otimes \omega^{(2)} = 1 \otimes a - \tilde{e}(a) \otimes 1$

iii) $\omega a = \Psi^\bullet(a \otimes \omega^{(1)}) \omega^{(2)}$

Here the correspondence is

$$\Pi(u \otimes v) = \omega^{(1)} \tilde{\chi}(\omega^{(2)} \otimes u \otimes v)$$

(using $\chi^\#$ in the reverse direction). We will provide this in more detail in the next section in the coalgebra setting.

There is also a theory of associated bundles. In fact, one has and needs two kinds of associated bundles; given an algebra bundle and a right $A$-module $V_R$ we have

$$E = \{ \sum_k v_k \otimes u_k \in V_R \otimes P \mid \sum_k v_k \triangleright a \otimes u_k = \sum_k v_k \otimes a \triangleright u_k, \ \forall a \in A \} \subseteq V_R \otimes P$$
as a natural right $M$-module by right multiplication in $P$. And given a left $A$-module $V_L$ we have

$$\bar{E} = \{ \sum_k u_k \otimes v_k \in P \otimes V_L \mid \sum_k \Psi(a \otimes u_k) \triangleright v_k = \sum_k e(a) u_k \otimes v_k, \forall a \in A \} \subseteq P \otimes V_L$$

as a natural left $M$-module. Here the $\bar{E}$ is the natural ‘invariant’ subspace from Lemma 2.4 for the $\Psi$-module structure of $P \otimes V_L$ provided by the following lemma.

**Lemma 2.13** If $V$ is a left $A$-module then $P \otimes V$ is a $\Psi$-module where $P$ acts by multiplication from the left and $A$ acts by $a \triangleright (u \otimes v) = \Psi(a \otimes u) \triangleright v$.

**Proof.** We check first that $A$ acts as shown. Thus, $(ab) \triangleright (u \otimes v) = \Psi(ab \otimes u) \triangleright v = \Psi(a \otimes (b \triangleright (u \otimes v)))^{(1)} \triangleright (b \triangleright (u \otimes v))^{(2)} = a \triangleright (b \triangleright (u \otimes v))$ using the definitions and the factorisation property of $\Psi$. Here $b \triangleright (u \otimes v) = (b \triangleright (u \otimes v))^{(1)} \otimes (b \triangleright (u \otimes v))^{(2)}$ is a notation. This then forms a $\Psi$-module since $a \triangleright (uu' \otimes v) = \Psi(a \otimes uu') \triangleright v = u_i \Psi(a^i \otimes u') \triangleright v = u_i (a^i \triangleright (u' \otimes v)) = \triangleright (\Psi(a \otimes u) \triangleright (u' \otimes v))$ as required. $\square$

Sections of these bundles are $M$-valued $M$-module maps from $E$, $\bar{E}$ respectively. When $P$ is flat over $M$ and $\Psi$ has a certain adjoint $\Psi^\#$, one can show that

$$\hom_A(V_L, P) \cong_M \hom(\bar{E}, M), \quad \hom(V_R, P)_0 \cong_M \hom_M(E, M)$$

as right $M$-modules, left $M$-modules respectively. In the first case, if $\varphi \in \hom_A(V_L, P)$ then the corresponding section of $\bar{E}$ is $s_\varphi(u \otimes v) = u \varphi(v)$. In the second case, $\hom(V_R, P)$ is a left $\Psi$-module in a similar manner to Lemma 2.13 (coinciding with it in the finite-dimensional case, namely $(a \triangleright \varphi)(v) = \tr_A \Psi(a \otimes \varphi(v))$). If $\varphi \in \hom(V_R, P)_0$ then the corresponding section of $\bar{E}$ is $s_\varphi(v \otimes u) = \varphi(v)u$. The proof of these assertions will be given in Section 4 in the coalgebra setting with $\chi^{-1}$ and $\psi^{-1}$ in the roles of $\chi^\#$ and $\Psi^\#$.

When $V_L$ and $V_R$ are finite-dimensional then

$$E = \hom_A(V_R^*, P), \quad \bar{E} = \hom(V_L^*, P)_0,$$

so that each bundle can be viewed as the space of sections of the other. Moreover, the constructions generalise directly to form-valued sections by using $\Psi^*$ in place of $\Psi$. One may then proceed to frame bundles etc. Thus, one has a covariant derivative

$$\nabla : E \to E \otimes \Omega^1 M, \quad \nabla : \bar{E} \to \Omega^1 M \otimes \bar{E}$$
associated to a suitable (strong) connection in the pointed case. By definition a frame resolution is an associated bundle equipped with a canonical form such that \( E \cong \Omega^1 M \), and in this case \( \nabla \) plays the role of Levi-Civita connection etc, along the lines in [23]. This and the rest of the theory will be provided in Section 4, in our preferred coalgebra bundle setting.

Finally, we give the situation in the case of trivial (cleft) algebra bundles. In this case sections correspond to ‘matter fields’ on the base \( M \),

\[
\text{Hom}_A(V_L, P) \cong \text{Hom}_k(V_L, M), \quad \text{Hom}(V_R, P)_0 = \text{Hom}_k(V_R, M).
\]

The first isomorphism sends \( \bar{f} \in \text{Hom}_k(V_L, M) \) to the map \( \varphi_{\bar{f}}(v) = \Phi(1)\bar{f}(\Phi(2)\triangleright v) \). The second isomorphism sends \( f \in \text{Hom}_k(V_R, M) \) to \( \varphi_f(v) = f(v \triangleleft \Phi^{-2}) \Phi^{-1} \). Similarly, (strong) connections \( \omega \) are determined by ‘gauge fields’ \( \alpha \in \Omega^1 M \otimes A \) such that \( \alpha^{(1)} \tilde{e}(\alpha^{(2)}) = 0 \), according to

\[
\omega = \Phi(1)\alpha^{(1)}\Phi^{-1} \otimes \Phi^{-2}(2)\alpha^{(2)}\Phi^{-2}(2) + \Phi^{(1)}\Delta\Phi^{-1} \otimes \Phi^{-2}(2)\Phi^{-2}(2).
\]

Proofs will again be given in the following sections, in the coalgebra setting. The covariant derivative on these matter fields and their gauge transformation by \( \gamma \in M \otimes A^{\text{op}} \) then take on the familiar form for algebraic gauge theory on trivial bundles (see [22, Sec. 3]).

### 3. Coalgebra bundles and connections

We switch now to the coalgebra version of the theory, where \( A \) is replaced by a coalgebra \( C \). This is the original theory of coalgebra bundles [8], which we extend further. The coalgebra version involves less familiar notations but has advantages in a purely algebraic treatment.

**Definition 3.1** [8] A coalgebra \( C \) and algebra \( P \) are entwined by \( \psi : C \otimes P \to P \otimes C \) if

\[
\psi \circ (\text{id} \otimes \cdot) = (\cdot \otimes \text{id}) \circ \psi_{23} \circ \psi_{12}, \quad \psi \circ (u \otimes 1) = 1 \otimes u, \quad \forall u \in P,
\]

\[
(\text{id} \otimes \Delta) \circ \psi = \psi_{12} \circ \psi_{23} \circ (\Delta \otimes \text{id}), \quad (\text{id} \otimes \varepsilon) \circ \psi = \varepsilon \otimes \text{id}.
\]

The triple \((P, C, \psi)\) is called an entwining structure.

We will often use the notation \( \psi(c \otimes u) = u_\alpha \otimes c^\alpha \) (summation over \( \alpha \) is understood). In this notation conditions [6] and [7] take a very simple explicit form

\[
(uv)_\alpha \otimes c^\alpha = u_\alpha v_\beta \otimes c^{\alpha\beta}, \quad 1_\alpha \otimes c^\alpha = 1 \otimes c,
\]
\[ u_\alpha \otimes c_\alpha (1) \otimes c_\alpha (2) = u_\beta \alpha \otimes c_\alpha (1) \otimes c_\alpha (2), \quad u_\alpha \varepsilon(c) = \varepsilon(c)u, \]

for any \( u, v \in P \) and \( c \in C \).

The entwining structure corresponds to an algebra factorisation in the case \( C \) finite-dimensional, built on \( A = C^{\text{op}} \) and \( P \), as explained in [8]. Similarly, if \( e \in C \) is grouplike, there is a right coaction \( \Delta_P : P \to P \otimes C \) defined by \( \Delta_P(u) = \psi(e \otimes u) \) and \( M = M_e = \{ u \in P | \Delta_P(u) = u \otimes 1 \} \) is a subalgebra. The map \( \check{\chi} : P \otimes P \to P \otimes C \) defined by \( \check{\chi}(u \otimes v) = u \Delta_P(v) \) descends to \( \chi : P \otimes_M P \to P \otimes C \) and we have a copointed coalgebra bundle \( P(M, C, \psi, e) \) when \( \chi \) is invertible and \( P \). This is the setting studied in [8].

We also note that for any entwining structure we have a natural category \( \mathbf{M}^P_C(\psi) \) of entwined modules. The objects are right \( P \)-modules and right \( C \)-comodules \( V \) such that for all \( v \in V, u \in P \)

\[ \Delta_P(v \triangleleft u) = v(0) \triangleleft \psi(v(1) \otimes u) := v(0) \triangleleft u_\alpha \otimes v(1)^\alpha, \quad (8) \]

The morphisms are right \( P \)-module and right \( C \)-comodule maps. The category \( \mathbf{M}^P_C(\psi) \) generalises the category of unifying or Doi-Koppinen modules [12][19] which unifies various categories studied intensively in the Hopf algebra theory (e.g. Drinfeld-Radford-Yetter (or crossed) modules, Hopf modules, relative Hopf modules, Long modules etc.).

The algebra \( P \) is an object in \( \mathbf{M}^P_C(\psi) \), with the right regular action of \( P \) (by multiplication) if and only if there exists an element \( \check{e} \in P \otimes C \) such that

\[ \check{e}^{(1)} \psi(\check{e}^{(2)} \otimes \check{e}'^{(1)}) \otimes \check{e}'^{(2)} = (\text{id} \otimes \Delta)\check{e}, \quad (\text{id} \otimes \varepsilon)\check{e} = 1 \]

(where \( \check{e}' \) is another copy of \( \check{e} \) and we use the notation \( \check{e} = \check{e}^{(1)} \otimes \check{e}^{(2)} \), etc.). In this case the coaction is

\[ \Delta_P(u) = \check{e}^{(1)} \psi(\check{e}^{(2)} \otimes u), \quad \forall u \in P. \]

Notice that \( \check{e} = \Delta_P(1) \). We then define

\[ M = \{ m \in P | \Delta_P(mv) = m\Delta_Pv \ \forall v \in P \} = \{ m \in P | \Delta_P(m) = m\check{e} \} \]

which is a subalgebra of \( P \), and proceed as above, requiring \( \chi \) to be bijective. We will call this a general coalgebra bundle \( P(M, C, \psi) \). The copointed case corresponds to the choice \( \check{e} = 1 \otimes e \).

There is also a converse: if \( P \) is an algebra and a right \( C \)-comodule, we say that the coaction is Galois if \( M \) defined as above is such that \( \chi \) is bijective. In this case there is
an entwining structure \[\psi(c \otimes u) = \chi(c^{(1)} \otimes c^{(2)}u), \quad \chi^{-1}(1 \otimes c) = c^{(1)} \otimes c^{(2)}, \quad \tilde{e} = \Delta_P(1)\]

and we have a coalgebra bundle. Because of these natural properties, we will work now with these slightly more general coalgebra bundles (or \(C\)-Galois extensions). Our preliminary goal in the present section is to make the evident generalisations of the copointed theory in [8] to this case.

Next, a coalgebra bundle is *trivial* cf[8] (or one says that the \(C\)-Galois extension is cleft) if there is a convolution invertible map \(\Phi : C \to P\) (the trivialisation or cleaving map) such that

\[
\Delta_P \circ \Phi = (\Phi \otimes \text{id}) \circ \Delta.
\]

By considering the equality \(1_0 \varepsilon(c) \otimes 1_1 = 1_0 \psi(1_1 \otimes \Phi(c_1)\Phi^{-1}(c_2))\) one finds that

\[
\psi(c_1 \otimes \Phi^{-1}(c_2)) = \Phi^{-1}(c)\Delta_P(1)
\]

which allows one to use the argument of the proof of [8, Proposition 2.9] to show that \(P \cong M \otimes C\) as a left \(M\)-module and right \(C\)-comodule.

We turn now to the theory of connections, based on the theory for the copointed case in [8]. As shown in [8, Proposition 2.2], given an entwining structure \((P, C, \psi)\) there is an entwining structure \((\Omega P, C, \psi^*)\), where

\[
\psi^* |_{C \otimes \Omega^{n-1} P} = \psi^n \equiv \psi_{n,n+1}\psi_{n-1,n} \cdots \psi_{12} : C \otimes P^{\otimes n} \to P^{\otimes n} \otimes C
\]

is the iterated entwining. Moreover,

\[
\psi^* \circ (\text{id} \otimes d) = (d \otimes \text{id}) \circ \psi^*.
\]

Therefore, given \(\tilde{e} : P \otimes C\) we have \(\Omega^n P \in M_{\Omega P}^C(\psi)\) with the action right multiplication by \(P\) and the coaction

\[
\Delta_{\Omega^n P} = (\cdot P \otimes \text{id})(\text{id} \otimes \psi^{n+1})(\tilde{e} \otimes \text{id}).
\]

**Definition 3.2** A connection on \(P(M, C, \psi)\) is a left \(P\)-module projection \(\Pi : \Omega^1 P \to \Omega^1 P\) such that (i) \(\ker \Pi = P(\Omega^1 M)P\) (ii) the map \(\Pi \circ d : P \to \Omega^1 P\) commutes with the right coaction.

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Proposition 3.3 Connections $\Pi$ are in 1-1 correspondence with $\omega : C \to \Omega^1 P$ such that

(i) $\tilde{e}^{(1)} \omega(\tilde{e}^{(2)}) = 0$
(ii) $\tilde{x} \circ \omega(c) = 1 \otimes c - \varepsilon(c) \tilde{e}$
(iii) $\psi^2(c(1) \otimes \omega(c(2))) = \omega(c(1)) \otimes c(2)$.

The correspondence is via $\Pi(udv) = uv(0) \omega(v(1))$ for all $u, v \in P$.

Proof. Assume first that there is $\omega$ satisfying (i)-(iii). Then the map $\Pi$ is well-defined since for all $u \in P, \Pi(u1) = u \omega e^{(1)}(e^{(2)}) = 0$, by (i). Next for any $u, v \in P, x M$ we have

$$\Pi(u(dx)v) = \Pi(u dv) = u(xv)(0) \omega((xv)(1)) - uxv(0) \omega(v(1)) = 0,$$

since $\Delta_p$ is left $M$-linear. On the other hand, if $\sum_i u^i dv^i \in \ker \Pi$, then using (ii) we have

$$0 = \sum_i \tilde{x}(u^i v^i(0) \omega(v^i(1))) = \sum_i (u^i v^i(0) \otimes v^i(1) - u^i v^i \tilde{e}) = \sum_i \tilde{x}(u^i dv^i).$$

Since $\ker \tilde{x} = P(\Omega^1 M)P$, we have $\ker \Pi \subseteq P(\Omega^1 M)P$, i.e., $\ker \Pi = P(\Omega^1 M)P$. Finally notice that for all $u \in P, \Pi(du) = u(0) \omega(u(1))$. Therefore

$$\begin{align*}
\Delta_{\Omega^1 P}(\Pi(du)) &= u(0) \psi^2(u(1) \otimes \omega(u(2))) \\
 &= u(0) \omega(u(1)) \otimes u(2) \quad \text{(by (iii))} \\
 &= \Pi(du(0)) \otimes u(1).
\end{align*}$$

Conversely, assume there is a connection in $P(M, C, \psi)$. This is equivalent to the existence of a map $\sigma : P \otimes C^+ \to \Omega^1 P$, where $C^+ = \ker \varepsilon$, such that $\tilde{x} \circ \sigma = \text{id}$ and $\Pi = \sigma \circ \tilde{x}$. Define $\omega(c) = \sigma(1 \otimes c - \varepsilon(c) \tilde{e})$. Clearly, (ii) holds. An immediate calculation verifies (i). The definition of $\Pi(udv) = uv(0) \omega(v(1))$, for all $u, v \in P$. Since $\Pi \circ d$ commutes with the coaction we have for all $u \in P$

$$u(0) \psi^2(u(1) \otimes \omega(u(2))) = u(0) \omega(u(1)) \otimes u(2).$$

Since $\chi$ is bijective, for any $c \in C$ there is $c(1) \otimes c(2) \in P \otimes_M P$ such that $c(1)c(2)(0) \otimes c(2)(1) = 1 \otimes c$. Thus we have

$$\psi^2(c(1) \otimes \omega(c(2))) = c(1)c(2)(0) \psi^2(c(2)(1) \otimes \omega(c(2)(2))) = c(1)c(2)(0) \omega(c(2)(1)) \otimes c(2)(2) = \omega(c(1)) \otimes c(2).$$
Therefore \( \omega \) satisfies (iii) and the proof of the proposition is completed. \( \square \)

Every connection \( \Pi \), induces a covariant derivative, \( D = d - \Pi \circ d : P \to \Omega^1 P \). In the copointed case \( D \) commutes with the right coaction, since \( d \) itself commutes with the right coaction.

**Proposition 3.4** If \( P(M, C, \psi, e) \) is a copointed trivial coalgebra bundle with trivialisation \( \Phi \) such that \( \Phi(e) = 1 \), and \( \alpha : C \to \Omega^1 M \) obeys \( \alpha(e) = 0 \), then

\[
\omega(c) = \Phi^{-1}(c(1))\alpha(c(2))\Phi(c(3)) + \Phi^{-1}(c(1))\Phi(c(2))
\]

is a connection.

**Proof.** We verify directly that \( \omega \) satisfies conditions (i)-(iii) of Proposition 3.3 with \( \tilde{e} = 1 \otimes e \). We have

\[
\omega(e) = \Phi^{-1}(e)\alpha(e)\Phi(e) + \Phi^{-1}(e)d\Phi(e) = d1 = 0.
\]

Next, take any \( c \in C \) and compute

\[
\tilde{\chi} \circ \omega(c) = \tilde{\chi}(\Phi^{-1}(c(1)) \otimes \Phi(c(2)) - \varepsilon(c)1 \otimes 1) = \Phi^{-1}(c(1))\Phi(c(2)) \otimes c(3) - \varepsilon(c)1 \otimes e = 1 \otimes c - \varepsilon(c)\tilde{e},
\]

where we used that the first summand in \( \omega \) is in \( P(\Omega^1 M)P \). Finally we have

\[
\psi^2(c(1) \otimes \omega(c(2))) = \psi^2(c(1) \otimes \Phi^{-1}(c(2))\alpha(c(3))\Phi(c(4))) + \psi^2(c(1) \otimes \Phi^{-1}(c(2))d\Phi(c(3)))
\]

\[
= \Phi^{-1}(c(1))\psi^2(e \otimes \alpha(c(2))\Phi(c(3))) + \Phi^{-1}(c(1))\psi^2(e \otimes d\Phi(c(2)))
\]

\[
= \Phi^{-1}(c(1))\alpha(c(2))\Delta_P(\Phi(c(3))) + \Phi^{-1}(c(1))d\Phi(c(2)) \otimes c(3)
\]

where we used that \( \Omega^1 P \in M_{\Omega^1 P}(\psi^\bullet) \) and (10) to derive the second equality, and that \( \alpha(c) \in \Omega^1 M, \Phi \) is an intertwiner and (11) to derive the third one. \( \square \)

For another class of examples one has coalgebra homogeneous spaces associated to coalgebra surjections \( \pi : P \to C \). Thus, let \( P \) be a Hopf algebra and \( M \) a subalgebra of \( P \) such that \( \Delta(M) \subseteq P \otimes M \) (an embeddable \( P \)-homogeneous quantum space). Define the quotient coalgebra \( C = P/(M^+P) \), where \( M^+ = \ker \varepsilon \cap M \) is the augmentation ideal. There is a natural right coaction of \( C \) on \( P \) given as \( \Delta_P = (\text{id} \otimes \pi) \circ \Delta \), where \( \pi : P \to C \) is the canonical surjection. It is clear that \( M \subseteq \{ u \in P | \Delta_P u = u \otimes e \} \), with

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\[ e = \pi(1), \text{ and we assume that this is an equality (this is known to hold for example if } P \text{ is faithfully flat as a left } M\text{-module). Then } P(M, C, \pi(1)) \text{ is a coalgebra bundle. Since } \tilde{e} = 1 \otimes \pi(1) \text{ we have } e = \pi(1), \text{ i.e. a copointed coalgebra bundle as in } \theta. \] 

In this case we know that if \( i : C \to P \) is a linear splitting of \( \pi \) such that

\[ i(e) = 1, \quad \varepsilon \circ i = \varepsilon, \quad i(c(2)) \otimes c(1) \odot (Si(c(2))(1))i(c(2))(3) = i(c(1)) \otimes c(2) \]

then

\[ \omega(c) = Si(c)(1)d\partial(c) \]

is a left-invariant connection and every left-invariant connection on the bundle is of this form (cf. \[\delta\]). The left-invariance here means that \( \Delta_{\Omega^1 P} \omega(c) = 1 \otimes \omega(c) \) for all \( c \in C \).

We use here the right action of \( P \) on \( C \) given by \( c \triangleright u = \pi(vu) \) for any \( v \in \pi^{-1}(c) \).

The theory of connections can be developed also for nonuniversal calculi \( \Omega^1(P) = \Omega^1 P/\mathcal{N} \) where \( \mathcal{N} \subseteq \Omega^1 P \) is a sub-bimodule, although the situation is slightly more complicated. We say that \( \Omega^1(P) \) is a differential calculus on \( P(M, C, \psi) \) iff it is covariant in the sense

\[ \psi^2(C \otimes \mathcal{N}) \subseteq \mathcal{N} \otimes C \]

so that the coaction \( \Delta_{\Omega^1 P} \) descends to \( \Omega^1(P) \). This is obtained from \( \psi^2_{\mathcal{N}} \) defined by

\[ \psi^2_{\mathcal{N}} \circ (\text{id} \otimes \pi_{\mathcal{N}}) = (\pi_{\mathcal{N}} \otimes \text{id}) \circ \psi^2 \]

where \( \pi_{\mathcal{N}} : \Omega^1 P \to \Omega^1(P) \) is the canonical surjection. We have

\[ \Delta_{\Omega^1(P)} = \tilde{e}^{(1)}\psi^2_{\mathcal{N}}(\tilde{e}^{(2)} \otimes ( )) \]

Let \( \mathcal{M} = (P \otimes C^+)/\bar{\chi} \mathcal{N} \) (and denote by \( \pi_{\mathcal{M}} \) the canonical surjection). This is a left \( P \)-module (since \( \bar{\chi} \) is left \( P \)-module map) by \( uv \cdot m = \sum_i \pi_{\mathcal{M}}(uv_i \otimes c_i) \) for any \( \sum v_i \otimes c_i \in \pi^{-1}_{\mathcal{M}}(m) \). We can then define

\[ \Lambda = \{ \lambda \in \mathcal{M} | \exists c \in C, \text{ s.t. } \lambda = \pi_{\mathcal{M}}(1 \otimes c - \varepsilon(c)\bar{e}) \} \]

The action provides a surjection \( P \otimes \Lambda \to \mathcal{M} \).

**Definition 3.5** A connection with a nonuniversal calculus is a left \( P \)-module projection \( \Pi : \Omega^1(P) \to \Omega^1(P) \) such that \( \ker \Pi = \Omega^1(P)_{\text{hor}} \) and \( \Pi \circ d \) commutes with the right coaction.
Proposition 3.6 Suppose $P \otimes \Lambda \cong \mathcal{M}$ by the surjection above. Then connections $\Pi$ on $\Omega^1(P)$ are in 1-1 correspondence with $\omega : \Lambda \to \Omega^1(P)$ such that

(i) $\chi^2 \circ \omega = 1 \otimes \text{id}$

(ii) $\psi^2(\chi(c_{1(1)} \otimes \omega(\pi(c_{1(2)})))) = \omega(\pi(c_{1(1)})) \otimes c_{1(2)}$ where $\pi(c) = \pi_M(1 \otimes c - \varepsilon(c) \tilde{e})$.

The correspondence is via $\Pi(udv) = u \sum_i v_i \omega(\lambda_i)$ for all $u, v \in P$ and $\sum_i v_i \otimes \lambda_i \in P \otimes \Lambda$ such that $\sum_i v_i \otimes \lambda_i = \chi^2(\text{dv})$.

Proof. The proof is analogous to the proof of Proposition 3.3. \qed

In the case of a homogeneous bundle where $P$ is a Hopf algebra and $e = \pi(1)$, a natural type of calculus $\Omega^1(P)$ is a left-covariant one defined by an ideal $Q$ in $\ker \varepsilon \subseteq P$.

Example 3.7 For a homogeneous bundle with left-covariant calculus, $\Lambda = C^+/\pi(Q)$ and $P \otimes \Lambda \cong \mathcal{M}$. Moreover if for all $q \in Q$, $u \in P$, $q(2) \otimes \pi(u(Sq(1))q(3)) \in Q \otimes C$, then $\Omega^1(P)$ is a calculus on $P(M, C, \psi, \pi(1))$. In particular, if $\Omega^1(P)$ is a bicovariant calculus on $P$ then it is a calculus on $P(M, C, \psi, \pi(1))$.

Proof. Recall that any element $n \in \mathcal{N}$ is of the form $n = \sum_i u^i S q^i(1) \otimes q^i(2)$ for some $u^i \in P$, $q^i \in Q$. For any $u \in P$, $q \in Q$ we have $u \otimes \pi(q) = \tilde{\chi}(u Sq(1) \otimes q(2)) \in \tilde{\chi}(\mathcal{N})$. On the other hand $\tilde{\chi}(\sum_i u^i S q^i(1) \otimes q^i(2)) = \sum_i u^i \otimes \pi(q^i) \in P \otimes Q$. This proves that $\tilde{\chi}(\mathcal{N}) = P \otimes \pi(Q)$. Therefore $\mathcal{M} = P \otimes C^+/\tilde{\chi}(\mathcal{N}) = P \otimes (C^+/\pi(Q))$, and $\Lambda = C^+/\pi(Q)$.

Finally, take any $c \in C$ and let $v \in \pi^{-1}(c)$. We have:

$$\sum_i \psi^2(c \otimes u^i S q^i(1) \otimes q^i(2)) = \sum_i u^i(1) S q^i(2) \otimes \psi(\pi(v u^i(2) S q^i(1))) \otimes q^i(3))$$

$$= \sum_i u^i(1) S q^i(2) \otimes q^i(3) \otimes \pi(v u^i(2) S q^i(1)) q^i(4)) \psi^2(q^i(2)) \psi^2(q^i(3)) \psi^2(q^i(4)).$$

By the assumption on $Q$ the last expression is in $\mathcal{N} \otimes Q$, so that the resulting calculus $\Omega^1(P)$ is a calculus on $P(M, C, \psi, \pi(1))$. If $Q$ defines a bicovariant calculus then $Q$ is Ad-stable, so that the required condition is immediately satisfied. \qed
4. Bijectivity of $\psi$ and strong connections

In this section we return to some technical considerations. For simplicity here and in most of what follows, we will concentrate on the universal differential calculus. First of all, we consider the question of when $\psi$ is bijective. It plays the role in the Hopf algebra case of having a bijective antipode, and allows us to relate left and right handed versions of the theory.

**Lemma 4.1** If $\psi$ is bijective then $P$ is a left $C$-comodule by

$$ p\Delta(u) = \psi^{-1}(u\tilde{e}). $$

Moreover, $M = \{ u \in P | p\Delta u = \psi^{-1}(\tilde{e})u\}$. 

**Proof.** This lemma is part of [5, Lemma 6.5]. $\square$

In the copointed case, it is easy to see that if $\psi$ is bijective then $P^{\otimes(n+1)}$ is a left $C$-comodule by $p_{\otimes(n+1)}\Delta = \psi^{-(n+1)}((\ ) \otimes e)$. This coaction restricts to $P \otimes M^{\otimes n}$ and $\Omega^n P$.

**Proposition 4.2** In the copointed case, let $\omega$ be a connection on $\Omega^1 P$ with $\psi$ bijective. Then $\bar{\Pi} : \Omega^1 P \to \Omega^1 P$ defined by

$$ \bar{\Pi}((du)v) = \omega(u_{(1)})u_{(\infty)}v $$

is a right-connection in the sense

(i) $\bar{D} = (id - \bar{\Pi}) \circ d$ is a left $C$-comodule map.

(ii) $\bar{\Pi}$ is a right $P$-module projection and $\ker \bar{\Pi} = P(\Omega^1 M)P$.

**Proof.** (i) We introduce the notation $\psi^{-1}(u \otimes c) = c_\alpha \otimes u^\alpha$, for all $c \in C$, $u \in P$. One easily finds that

$$ c_\alpha(1) \otimes c_\alpha(2) \otimes u^\alpha = c_{(1)\alpha} \otimes c_{(2)\beta} \otimes u^{\alpha\beta} \tag{12} $$

and $p\Delta(u) = e_\alpha \otimes u^\alpha$. We have

$$ \psi^2(u_{(1)} \otimes \omega(u_{(2)})u_{(\infty)}) = \omega(u_{(1)})\psi(u_{(2)} \otimes u_{(\infty)}) $$

$$ = \omega(e_{\alpha(1)})\psi(e_{\alpha(2)} \otimes u^\alpha) $$

$$ = \omega(e_\alpha)\psi(e_\beta \otimes u^{\alpha\beta}) $$

$$ = \omega(e_\alpha)u^\alpha \otimes e = \omega(u_{(1)})u_{(\infty)} \otimes e, $$

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where we used that $e$ is group-like and (12) to derive the third equality. This implies that
\[
\psi^{-2}(\omega(u(1))u(\infty) \otimes e) = u(1) \otimes \omega(u(2))u(\infty),
\]
which is precisely the left $C$-covariance of $\bar{\Pi} \circ d$ and, consequently, implies the left-covariance of $\tilde{D}$.

(ii) It is clear that $\bar{\Pi}$ is a right $P$-module map. The following diagram commutes:

\[
\begin{array}{cccccc}
0 & \longrightarrow & P(\Omega^1 M)P & \longrightarrow & \Omega^1 P & \longrightarrow & P \otimes C^+ & \longrightarrow & 0 \\
& & \downarrow \cong & & \downarrow \cong & & \downarrow \psi & & \\
0 & \longrightarrow & P(\Omega^1 M)P & \longrightarrow & \Omega^1 P & \longrightarrow & C^+ \otimes P & \longrightarrow & 0
\end{array}
\]

where $\tilde{\chi}_L = \psi^{-1} \circ \tilde{\chi}$ (explicitly, $\tilde{\chi}_L(u \otimes v) = u(1) \otimes u(\infty)v$). Since $P$ is a coalgebra principal bundle the top sequence is exact. Furthermore $\psi$ is bijective and $\rho\Delta$ is right $M$-linear thus the bottom sequence is also exact. It is split by the map $\sigma : C^+ \otimes P \rightarrow \Omega^1 P$, $\sigma(c \otimes u) = \omega(c)u$. Indeed,
\[
\tilde{\chi}_L \circ \sigma(c \otimes u) = \tilde{\chi}_L(\omega(c))u = \psi^{-1}(1 \otimes c)u = c \otimes u,
\]
where we used that $\tilde{\chi}_L$ is a right $P$-module map and that $\omega$ is a connection one-form (Proposition 3.3(ii)). Now notice that $\bar{\Pi} = \sigma \circ \tilde{\chi}_L$, and the fact that $\sigma$ is a splitting (i.e. $\tilde{\chi}_L \circ \sigma = \text{id}$) of the above sequence implies both that $\bar{\Pi}$ is a projection and has the kernel as stated. \hfill \Box

Finally, a connection is strong if $(\text{id} - \Pi) \circ d$ has its image in $(\Omega^1 M)P$ [10, Definition 2.1]. These are the connections most closely associated to the base and used in the theory of associated bundles etc. Recently, a simple condition for strongness was given in the Hopf algebra case, in [23]. This can be generalised to the coalgebra case.

**Proposition 4.3** A connection on a copointed coalgebra bundle $P(M, C, \psi, e)$ is strong iff
\[
(id \otimes \Delta_P)\omega(c) = 1 \otimes 1 \otimes c - \varepsilon(c)1 \otimes 1 \otimes e + \omega(c(1)) \otimes c(2). \tag{13}
\]
Furthermore, if $\psi$ is bijective then a connection is strong iff
\[
(p\Delta \otimes \text{id})\omega(c) = c \otimes 1 \otimes 1 - e \otimes 1 \otimes 1 \varepsilon(c) + c(1) \otimes \omega(c(2)).
\]
Proof. Assume that $\omega$ is strong. This is equivalent to the statement that
\[
(id \otimes \Delta_P) \circ D(u) = \Delta_{\Omega^1 P} \circ D(u), \quad \forall u \in P.
\] (14)
Using the explicit definition of $d$ and $D$, Proposition 3.3(iii), as well as the fact that $\Omega^1 P \in M_{\Omega^0 P}(\psi^*)$ one finds that (14) implies that
\[
(id \otimes \Delta_P)(u(0) \omega(u(1))) = u(0) \otimes 1 \otimes u(1) - u \otimes 1 \otimes e + u(0) \omega(u(1)) \otimes u(2).
\]
Next for all $c$, let $c^{(1)} \otimes c^{(2)} \in P \otimes M P$ be the translation map, i.e. $c^{(1)} \otimes c^{(2)} = \chi^{-1}(c \otimes 1)$. It means that $c^{(1)}c^{(2)}(0) \otimes c^{(2)}(1) = 1 \otimes c$. Using the above equality and the fact that $c^{(1)}c^{(2)} = \varepsilon(c)$, we have
\[
(id \otimes \Delta_P) \circ \omega(c) = (id \otimes \Delta_P)(c^{(1)} \otimes c^{(2)}(0) \omega(c^{(2)}(1))) = c^{(1)}(id \otimes \Delta_P)(c^{(2)}(0) \omega(c^{(2)}(1))
\]
\[
= c^{(1)}c^{(2)}(0) \otimes 1 \otimes c^{(2)}(1) - c^{(1)}c^{(2)} \otimes 1 \otimes e
\]
\[
+ c^{(1)}c^{(2)}(0) \omega(c^{(2)}(1)) \otimes c^{(2)}(2)
\]
\[
= 1 \otimes 1 \otimes c - \varepsilon(c)1 \otimes 1 \otimes e + \omega(c^{(1)}) \otimes c^{(2)},
\]
i.e. (13) holds. Conversely, an easy calculation reveals that (13) implies (14), i.e., the connection is strong as required.

The second assertion is obtained by applying $\psi^{-2}$ to (13). □

As in [23], the significance of this is that this is manifestly a ‘strongness’ condition for the left-handed theory with $\bar{\Pi}$. In studying the coalgebra frame resolutions we will need both the left and the right handed theories simultaneously, and we see that if one holds so does the other for a given $\omega$.

A situation where $\psi$ is bijective is a homogeneous bundle $\pi : P \to C$ with $P$ having bijective antipode.

**Proposition 4.4** For a homogeneous coalgebra bundle with bijective antipode, strong left-invariant connections are in 1-1 correspondence with splittings $i : C \to P$ of $\pi$ which are covariant with respect to $(id \otimes \pi) \circ \Delta$ and $(\pi \otimes id) \circ \Delta$, and such that $i(\pi(1)) = 1$ and $\varepsilon \circ i = \varepsilon$. In this case
\[
\omega(c) = Si(c) \vert_{(1)} di(c) \vert_{(2)}.
\]

**Proof.** Given such a splitting $i : C \to P$ of $\pi$, consider $\omega(c) = Si(c) \vert_{(1)} di(c) \vert_{(2)}$ as stated. The normalisation conditions imply that $\omega(\pi(1)) = 0$ and $\bar{\chi} \circ \omega(c) = 1 \otimes c - \varepsilon(c)1 \otimes \pi(1)$. 25
Also
\[
\psi^2(c_{(1)} \otimes \omega(c_{(2)})) = S i(c_{(2)})(1) d i(c_{(2)})(3) \otimes \pi(i(c_{(1)}) S i(c_{(2)})(1) i(c_{(2)})(4)) \\
= S i(c_{(2)})(1) d i(c_{(2)})(4) \otimes \pi(i(c_{(1)}) S i(c_{(2)})(2) i(c_{(2)})(5)) \quad (i \text{ is left-covariant}) \\
= S i(c_{(2)})(1) d i(c_{(2)})(2) \otimes \pi(i(c_{(2)})) \quad (i \text{ is right-covariant}) \\
= \omega(c_{(1)}) \otimes c_{(2)} \quad (\pi \text{ is split by } i)
\]

Proposition 3.3 implies that \(\omega\) is a connection one-form. Finally, compute
\[
(id \otimes \Delta_P)(\omega(c)) = S i(c_{(1)})(1) \otimes i(c_{(2)})(2) \otimes \pi(i(c_{(2)})(3)) - \varepsilon(c)1 \otimes 1 \otimes \pi(1) \\
= S i(c_{(1)})(1) \otimes i(c_{(1)})(2) \otimes c_{(2)} - \varepsilon(c)1 \otimes 1 \otimes \pi(1) \\
= \omega(c_{(1)}) \otimes c_{(2)} + 1 \otimes 1 \otimes c - \varepsilon(c)1 \otimes 1 \otimes \pi(1),
\]

where the use of the fact that \(i\) is a right covariant splitting was made in the derivation of the second equality. Proposition 4.3 now implies that the connection corresponding to \(\omega\) is strong.

Conversely, assume that there is a strong connection with the left-invariant connection form \(\omega\). Then the left-invariance of \(\omega\) implies that there exists a splitting \(i : C \rightarrow P\) of \(\pi\) such that \(\varepsilon \circ i = \varepsilon\) and \(\omega(c) = S i(c_{(1)}) d i(c_{(2)})(2)\) (cf. [9, Proposition 3.5]). The fact that \(\omega(\pi(1)) = 0\) implies that \(i(\pi(1)) = 1\). Applying \((id \otimes \Delta_P)\) to this \(\omega\) and using Proposition 1.3 one deduces that \(i\) is right-covariant. Bijectivity of \(S\) implies that \(\psi\) is bijective (cf. [3]). The left coaction induced by \(\psi^{-1}\) is \(P\Delta(u) = \pi(S^{-1} u_{(2)}) \otimes u_{(1)}\). By Proposition 4.3
\[
(P\Delta \otimes P)\omega(c) = \pi(i(c_{(2)})(1) \otimes S i(c_{(2)})(2) \otimes i(c_{(2)})(3)) - \varepsilon(c)\pi(1) \otimes 1 \otimes 1
\]

must be equal to
\[
c_{(1)} \otimes S i(c_{(2)})(1) \otimes i(c_{(2)})(2) - \varepsilon(c)\pi(1) \otimes 1 \otimes 1.
\]
Applying \(id \otimes S^{-1} \otimes \varepsilon\) to this equality one deduces that \(i\) must be left-covariant. This completes the proof. \(\square\)

This is the analogue for coalgebra bundles of the bicovariant formulation of strong canonical connections in the Hopf algebra case in [17].
5. Frame resolutions, covariant derivatives and torsion

In this section we define frame resolutions in the coalgebra setting, following the theory introduced recently in [23] in the Hopf algebra case. The theory depends heavily on the notion of associated bundles, so we recall these briefly. In the coalgebra case there are two kinds of associated bundles (which are equivalent in the Hopf algebra case), as studied recently in [5].

**Definition 5.1** Let $P(M, C)$ be a coalgebra bundle.

(i) The left associated bundle (or module) to a left $C$-comodule $V$ is $E = P \square_C V$.

(ii) The right associated bundle (or module) to a right $C$-comodule $V$ is $\bar{E} = (V \otimes P)v(0) \otimes \psi(v(1) \otimes u)$.

The cotensor product $W \square_C V$ here, between a left comodule $V$ and right comodule $W$ is defined by the exact sequence [24]

$$0 \rightarrow W \square_C V \hookrightarrow W \otimes V \xrightarrow{\Delta \otimes \text{id} - \text{id} \otimes \Delta} W \otimes C \otimes V.$$  

This is just the arrow reversal of the usual tensor product. Less conventional is the fixed subobject

$$(V \otimes P)_0 = \{ \sum_i v_i \otimes u_i \in V \otimes P \mid v_i(0) \otimes \psi(v_i(1) \otimes u_i) = v_i \otimes u_i \bar{e}^{(1)} \otimes \bar{e}^{(2)} \}.$$  

This is the natural analogue for coalgebra bundles of the associated bundles in the quantum group gauge theory of [7].

**Lemma 5.2** For a copointed coalgebra bundle $P(M, C, \psi, e)$, let $(P \otimes M^{\otimes n})_0 = \{ w \in P \otimes M^{\otimes n} \mid \psi^{n+1}(e \otimes w) = w \otimes e \}$ be the invariant subset of $P \otimes M^{\otimes n}$. If $\psi$ is bijective then $(P \otimes M^{\otimes n})_0 = M^{\otimes n+1}$.

**Proof.** Clearly $M^{\otimes n+1} \subseteq (P \otimes M^{\otimes n})_0$. If $w \in (P \otimes M^{\otimes n})_0$ then $\psi^{n+1}(e \otimes w) = w \otimes e$. Applying $\psi^{-(n+1)}$ one deduces that $\psi^{-(n+1)}(w \otimes e) = e \otimes w$. Let $w = \sum_i u_i \otimes m_1^i \otimes \cdots \otimes m_n^i$. Since for all $m \in M$, $\psi^{-1}(m \otimes e) = e \otimes m$ one immediately finds that $e \otimes \sum_i u_i \otimes m_1^i \otimes \cdots \otimes m_n^i = \sum_i \psi^{-1}(u_i \otimes e) \otimes m_1^i \otimes \cdots \otimes m_n^i$. This in turn implies that for all $i$, $u_i \in M$. $\square$

Now we can extend the notion of a strongly horizontal form from [4]
**Definition 5.3** Let $E$ be a left bundle associated to a cointerpointed coalgebra bundle $P(M,C,\psi,e)$ and a left $C$-comodule $V$. A right strongly tensorial $n$-form on $E$ is a linear map $\phi : V \to P(\Omega^n M)$ such that

$$\psi^{n+1} \circ (id \otimes \phi) \circ \nabla = \phi \otimes e,$$

(15)

By the extension of the notation above, the space of right strongly tensorial $n$-forms will be denoted by $\text{Hom}_0(V,P(\Omega^n M))$ (in $\mathcal{E}$ right strongly 0-forms $\text{Hom}_0(V,P)$ are denoted by $\text{Hom}_\psi(V,P)$). $\text{Hom}_0(V,P(\Omega^n M))$ has a right $M$-module structure defined by $(\phi \cdot m)(v) = \phi(v)m$.

**Proposition 5.4** Let $P(M,C,\psi,e)$ be a copointed coalgebra bundle with $\psi$ bijective and $P$ flat as a right $M$-module (or $V$-coflat as a left $C$-comodule). Then right strongly tensorial forms $\text{Hom}_0(V,P(\Omega^n M))$ and $M\text{Hom}(E,\Omega^n M)$ are isomorphic as right $M$-modules.

**Proof.** The proof of this proposition is analogous to the proof of [3, Theorem 4.3]. We include it here for completeness. The flatness (coflatness) assumption implies that $(P \otimes M P)\square_C V \cong P \otimes_M (P\square_C V)$, canonically (cf. [29, p. 172]). Thus there is a left $P$-module isomorphism $\rho : P \otimes_M E \to P \otimes V$, obtained as a composition of $\chi \otimes id$ with the canonical isomorphism $P \otimes C \square_C V \cong P \otimes V$, i.e., $\rho = \cdot \otimes id$, $\rho^{-1} = (\chi^{-1} \otimes id) \circ (id \otimes \nabla)$. Following [13], apply $\text{Hom}_E(-,P(\Omega^n M))$ to $\rho$ to deduce the right $M$-module isomorphism $\text{Hom}(V,P(\Omega^n M)) \cong \text{Hom}_M(E,P(\Omega^n M))$, given by $\phi \mapsto s\phi$. $s\phi(\sum_i u^i \otimes v^i) = \sum_i u^i \phi(v^i)$. For any $\phi \in \text{Hom}(V,P(\Omega^n M))$, $x = \sum_i u^i \otimes v^i \in E$ we have

$$\Delta_{\Omega P}(s\phi(x)) = \sum_i \Delta_{\Omega P}(u^i \phi(v^i)) = \sum_i u^i(0) \psi^{n+1}(u^i(1) \otimes \phi(v^i))$$

$$= \sum_i u^i \psi^{n+1}(v^i(1) \otimes \phi(v^i(\infty))),$$

since $\sum_i u^i(0) \otimes u^i(1) \otimes v^i = \sum_i u^i \otimes v^i(1) \otimes v^i(\infty)$ by the definition of $E = P\square_C V$. By Lemma 5.2, $\Omega^n M = (P(\Omega^n M))_0$, therefore $s\phi(x) \in \Omega^n M$ iff

$$\sum_i u^i \psi^{n+1}(v^i(1) \otimes \phi(v^i(\infty))) = \sum_i u^i \phi(v^i) \otimes e.$$

(16)

Clearly, (15) implies (16). Applying (16) to $\rho^{-1}(1 \otimes v)$ one easily finds that (16) implies (15). Therefore the right $M$-module isomorphism $\phi \mapsto s\phi$ restricts to the isomorphism $\text{Hom}_0(V,P(\Omega^n M)) \cong \text{Hom}_M(E,\Omega^n M)$ as required. $\square$
Proposition 5.4 is the coalgebra bundle version of [23, Lemma 3.1], and allows us to define similarly,

**Definition 5.5** cf [23, Definition 3.2] A coalgebra frame resolution of an algebra $M$ is a left bundle $E$ associated to a copointed coalgebra bundle $P(M, C, \psi, e)$ with bijective $\psi$, and $V$, together with a right strongly tensorial one-form $\theta : V \to P(\Omega^1 M)$ such that $s_{\theta} : E \to \Omega^1 M$ corresponding under Proposition 4.4 is an isomorphism of left $M$-modules.

As in [23], we can now proceed to deduce the left $M$-module isomorphism

$$\text{id} \otimes s_{\theta} : (\Omega^1 M)P \square_C V \cong \Omega^1 M \otimes_M \Omega^1 M = \Omega^2 M. \quad (17)$$

Here, the cotensor product is defined with respect to the right coaction $\Delta_{(\Omega^1 M)P} : (\Omega^1 M)P \to (\Omega^1 M)P \otimes C$ given by $\Delta_{(\Omega^1 M)P}(w) = \psi^2(e \otimes w)$ (it is an easy exercise which uses (6) to verify that $(\Omega^1 M)P$ is closed under this coaction).

Furthermore, given a frame resolution, we can now define a covariant derivative $\nabla : \Omega^1 M \to \Omega^2 M$ corresponding to a strong connection $\Pi$ in $P(M, C, \psi, e)$ by [23, Proposition 3.3]

$$\nabla = (\text{id} \otimes s_{\theta}) \circ (D \square_C \text{id}) \circ s_{\theta}^{-1} : \Omega^1 M \to \Omega^2 M. \quad (18)$$

The map $\nabla$ is well-defined since $D$ is an intertwiner so that the expression $D \square_C \text{id}$ makes sense. Furthermore, by the strongness assumption $D(P) \subseteq (\Omega^1 M)P$ so the isomorphism (17) implies that the output of $\nabla$ is in $\Omega^2 M$. Finally, it can be easily verified (cf. [23, Proposition 3.3]) that $\nabla(m \cdot w) = m \cdot \nabla w + dm \otimes_M w$, for any $m \in M$ and $w \in \Omega^1 M$, so that $\nabla$ is a connection on $\Omega^1 M$ as a left $M$-module.

Next, cf [23, Proposition 3.5], we define the torsion of a connection $\nabla$ by

$$T = d - \nabla : \Omega^1 M \to \Omega^2 M.$$  

By Proposition 5.4 this $T$ can be also viewed as a map $T : V \to P(\Omega^2 M)$ provided $P$ is $M$-flat.

**Proposition 5.6** If $\omega$ is a strong connection on $P(M, C, \psi, e)$ and $\psi$ is bijective then there is a covariant derivative

$$\hat{D} : \text{Hom}_0(V, P\Omega^n M) \to \text{Hom}_0(V, P\Omega^{n+1} M)$$

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given by
\[ \bar{D} \varphi(v) = d\varphi(v) + \omega(v(1))\varphi(v(\infty)). \]

In particular, \( T = \bar{D} \theta. \)

**Proof.** We first show that the map \( \bar{D} \) is well-defined. We will use the following notation for the connection one-form \( \omega(c) = \omega(c)^{(1)} \otimes \omega(c)^{(2)} \) (summation understood), for all \( c \in C. \) Take any \( \varphi \in \text{Hom}_0(V, P\Omega^n M), v \in V \) and compute

\[
(id \otimes \Delta_{\Omega P})\bar{D}\varphi(v) = 1 \otimes \varphi(v)(0) \otimes \varphi(v)(1) + d\varphi(v) \otimes e - 1 \otimes \varphi(v) \otimes e \\
+\omega(v(1))\otimes \Delta_{\Omega P}(\omega(v(1))^{(2)}\varphi(v(\infty))) \\
= 1 \otimes \varphi(v)(0) \otimes \varphi(v)(1) + d\varphi(v) \otimes e - 1 \otimes \varphi(v) \otimes e \\
+\omega(v(1))\otimes \varphi(v(\infty))) - 1 \otimes \psi^{n+1}(e \otimes \varphi(v) \\
+\omega(v(1))\psi^{n+1}(v(2) \otimes \varphi(v(\infty))) \\
= 1 \otimes \varphi(v)(0) \otimes \varphi(v)(1) + d\varphi(v) \otimes e - 1 \otimes \varphi(v) \otimes e \\
+\omega(v(1))\varphi(v(\infty))) + \omega(v(1))\varphi(v(\infty)) \otimes e \\
= \bar{D}\varphi(v) \otimes e,
\]

where we used that \( \Omega P \in M_{\Omega P}^C(\psi^*) \) to derive the second equality, then Proposition 1.3 to derive the third one and the fact that \( \varphi \in \text{Hom}_0(V, P\Omega^n M) \) to obtain the fourth equality. This shows that \( \bar{D}\varphi(v) \in P(\Omega^{n+1} M). \)

Next we need to show that \( \bar{D}\varphi \) satisfies (13). We have

\[
\psi^{n+2}(v(1) \otimes \bar{D}\varphi(v(\infty))) = \psi^{n+2}(v(1) \otimes d\varphi(v(\infty))) + \psi^{n+2}(v(1) \otimes \omega(v(2))\varphi(v(\infty))) \\
= (d \otimes \text{id})(\psi^{n+1}(v(1) \otimes \varphi(v(\infty))) \\
+\omega(v(1))\psi^{n+1}(v(2) \otimes \varphi(v(\infty)))) \\
= d\varphi(v) \otimes e + \omega(v(1))d\varphi(v(\infty)) \otimes e = \bar{D}\varphi(v) \otimes e,
\]

where we used the covariance of \( d \) with respect to \( \psi^* \), the fact that \( \Omega P \in M_{\Omega P}^C(\psi^*) \), and the covariance property of the connection one-form to derive the second equality. It is an easy exercise to verify that \( T = \bar{D}\theta. \)

Here \( \bar{D} \) extends \( \bar{D} \) in Section 4 to higher forms. Next, again following [23], we introduce left strongly tensorial forms and a quantum metric. Thus, let \( V \) be a right
C-comodule. A left strongly tensorial n-form is a map \( \varphi : V \to (\Omega^n M)P \) commuting with the right coaction of \( C \), where \( C \) coacts on \((\Omega^n M)P\) by \( \psi^{n+1}(e \otimes w) \).

**Proposition 5.7** Left strongly tensorial forms \( \text{Hom}^C(V, (\Omega^n M)P) \) and \( \text{Hom}_M(\tilde{E}, \Omega^n M) \) are isomorphic as left \( M \)-modules if \( P \) is faithfully flat as a left \( M \)-module (cf. [3] for a comprehensive review of the concept of faithful flatness).

*Proof.* This can be shown as [3, Theorem 5.4]. Given \( \varphi \in \text{Hom}^C(V, (\Omega^n M)P) \) the corresponding \( s\varphi \in \text{Hom}_M(\tilde{E}, \Omega^n M) \) is given by \( s\varphi(\sum_i v^i \otimes u^i) = \sum_i \varphi(v^i)u^i, \sum_i v^i \otimes u^i \in \tilde{E} \). Conversely given \( s \in \text{Hom}_M(\tilde{E}, \Omega^n M) \), the corresponding tensorial form is given by \( \varphi_s(v) = s(v \otimes 1) \).

On the other hand, for \( V \) a right \( C \)-comodule we have the covariant derivative \( D \) extending the \( D \) in Section 3 to higher forms.

**Proposition 5.8** If \( \omega \) is a strong connection on \( P(M, C, e) \) and \( \psi \) is bijective then there is a covariant derivative

\[
D : \text{Hom}^C(V, (\Omega^n M)P) \to \text{Hom}^C(V, (\Omega^{n+1} M)P)
\]

given by

\[
D\varphi(v) = d\varphi(v) + (-1)^{n+1}\varphi(v(0))\omega(v(1)).
\]

*Proof.* This proposition is a coalgebra bundle version of a similar statement in [10] for quantum group principal bundles. The proof is similar to the proof of Proposition 5.6. Take any right \( C \)-covariant \( \varphi : V \to (\Omega^n M)P \) and \( v \in V \) and compute

\[
(\Omega_P \Delta \otimes \text{id})D\varphi(v) = (-1)^{n+1}\varphi(v(1)) \varphi(v(\infty)) \otimes 1 + e \otimes d\varphi(v) + (-1)^n e \otimes \varphi(v) \otimes 1
\]
\[
+ (-1)^n \varphi(v(0)) \omega(v(1)) \varphi(v(\infty)) \otimes 1 + e \otimes d\varphi(v) + (-1)^n e \otimes \varphi(v) \otimes 1
\]
\[
+ (-1)^n \varphi(v(0)) \omega(v(1)) \varphi(v(\infty)) \otimes 1 + e \otimes d\varphi(v) + (-1)^n e \otimes \varphi(v) \otimes 1
\]
\[
+ (-1)^n \varphi(v(0)) \omega(v(1)) \varphi(v(\infty)) \otimes 1 + e \otimes d\varphi(v) + (-1)^n e \otimes \varphi(v) \otimes 1
\]
\[
+ (-1)^n \varphi(v(0)) \omega(v(1)) \varphi(v(\infty)) \otimes 1 + e \otimes d\varphi(v) + (-1)^n e \otimes \varphi(v) \otimes 1
\]
\[
+ (-1)^n \varphi(v(0)) \omega(v(1)) \varphi(v(\infty)) \otimes 1 + e \otimes d\varphi(v) + (-1)^n e \otimes \varphi(v) \otimes 1
\]
\[
+ (-1)^n \varphi(v(0)) \omega(v(1)) \varphi(v(\infty)) \otimes 1 + e \otimes d\varphi(v) + (-1)^n e \otimes \varphi(v) \otimes 1
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+ (-1)^n \varphi(v(0)) \omega(v(1)) \varphi(v(\infty)) \otimes 1 + e \otimes d\varphi(v) + (-1)^n e \otimes \varphi(v) \otimes 1
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+ (-1)^n \varphi(v(0)) \omega(v(1)) \varphi(v(\infty)) \otimes 1 + e \otimes d\varphi(v) + (-1)^n e \otimes \varphi(v) \otimes 1
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+ (-1)^n \varphi(v(0)) \omega(v(1)) \varphi(v(\infty)) \otimes 1 + e \otimes d\varphi(v) + (-1)^n e \otimes \varphi(v) \otimes 1
\]
\[
+ (-1)^n \varphi(v(0)) \omega(v(1)) \varphi(v(\infty)) \otimes 1 + e \otimes d\varphi(v) + (-1)^n e \otimes \varphi(v) \otimes 1
\]
\[
+ (-1)^n \varphi(v(0)) \omega(v(1)) \varphi(v(\infty)) \otimes 1 + e \otimes d\varphi(v) + (-1)^n e \otimes \varphi(v) \otimes 1
\]
\[
+ (-1)^n \varphi(v(0)) \omega(v(1)) \varphi(v(\infty)) \otimes 1 + e \otimes d\varphi(v) + (-1)^n e \otimes \varphi(v) \otimes 1
\]
\[
+ (-1)^n \varphi(v(0)) \omega(v(1)) \varphi(v(\infty)) \otimes 1 + e \otimes d\varphi(v) + (-1)^n e \otimes \varphi(v) \otimes 1
\]
\[
+ (-1)^n \varphi(v(0)) \omega(v(1)) \varphi(v(\infty)) \otimes 1 + e \otimes d\varphi(v) + (-1)^n e \otimes \varphi(v) \otimes 1
\]
\[
+ (-1)^n \varphi(v(0)) \omega(v(1)) \varphi(v(\infty)) \otimes 1 + e \otimes d\varphi(v) + (-1)^n e \otimes \varphi(v) \otimes 1
\]
\(-(-1)^{n+1} \varphi(v)_{(1)} \otimes \varphi(v)_{(\infty)} \otimes 1 - (-1)^{n} e \otimes \varphi(v) \otimes 1 \\
+(-1)^{n+1} e \otimes \varphi(v_{(0)}) \omega(v_{(1)})
\) \\
\(= e \otimes D \varphi(v). \)

The third equality follows from Proposition 4.3. Thus we deduce that \(D \varphi(v) \in (\Omega^{n+1}M) P.\)

The proof of the covariance of \(D \varphi\) is analogous to the corresponding part of the proof of Proposition 5.6. \(\Box\)

Finally, when \(V\) is a finite-dimensional left \(C\)-comodule we can identify \(\bar{E}\) with \(\text{Hom}(V, P)_0\) and \(\text{Hom}^C(V^*, (\Omega^n M) P)\) with \((\Omega^n M) P \square_C V\) and hence obtain

\((\Omega^n M) P \square_C V \cong \text{Hom}_M(\text{Hom}_0(V, P), \Omega^n M)\).

We can then define, cf [23], a metric on \(M\) as an element

\(\gamma \in (\Omega^1 M) P \square_C V\)

such that the corresponding map \(\text{Hom}_0(V, P) \to \Omega^1 M\) is an isomorphism. In the infinite dimensional case we do not have a bijection between these spaces, but we still obtain a map \(\text{Hom}_0(V, P) \to \Omega^1 M\) from \(\gamma\) and can require it to be suitably nondegenerate. If \(P(M, C, \psi, e)\) and \(V\) is a frame resolution of \(M\) then we can identify \((\Omega^1 M) P \square_C V\) with \(\Omega^2 M\), so that \(\gamma\) is a 2-form on \(M\).

Following [23], we can also define the cotorsion \(\Gamma \in \Omega^3 M\) of the metric as

\(\Gamma = (\text{id} \otimes s_\theta)(D \square_C \text{id})(\gamma).\)

Here, since \(\gamma\) is left strongly tensorial (and if \(D\) corresponds to a strong connection) then \(D \gamma\) is also left-strongly tensorial when viewed as a map on \(V^*\). Hence \((D \square_C \text{id}) \gamma \in (\Omega^2 M) P \square_C V\) as required here. In this context one has the following version of \(D\) that does not go through \(V^*\).

**Proposition 5.9** If \(\omega\) is a strong connection on \(P(M, C, \psi, e)\) and \(\psi\) is bijective then there is a covariant derivative

\[D : (\Omega^n M) P \square_C V \to (\Omega^{n+1} M) P \square_C V\]

given by

\[D(w \otimes v) = dw \otimes v + (-1)^{n+1} w \omega(v_{(1)}) \otimes v_{(\infty)}.\]
Proof. Dual to the proof of Proposition 5.8. □

Also provided in [23] is a general construction for frame resolutions on quantum group homogeneous bundles $\pi : P \to H$. We extend this now in the coalgebra setting $\pi : P \to C$, to embeddable homogeneous spaces. This more general setting is definitely needed since it includes, for example, the full family of quantum 2-spheres [28] considered in the next section. The following proposition generalises [23, Proposition 4.3] to include this case.

**Proposition 5.10** A quantum embeddable homogeneous space $M$ of $P$ corresponding to $\pi : P \to C$ has a coalgebra frame resolution with $V = M^+$, $\nu \Delta = (\pi \otimes \text{id}) \circ \Delta$ and $\theta : V \to P(\Omega^1 M)$, $\theta : v \mapsto Su(1) \otimes v(2)$.

**Proof.** The canonical entwining structure is $\psi(c \otimes h) = h_{(1)} \otimes \pi(gh_{(2)})$, where $g \in \pi^{-1}(c)$ (cf. [8, Example 2.5]). Since $\theta(v) \in P \otimes M$, as $M$ is a left $P$-comodule algebra, we find

$$
\psi^2(\pi(v(1)) \otimes \theta(v(2))) = Su(3) \otimes \psi(\pi(v(1)Su(2)) \otimes v(4)) = Su(1) \otimes \psi(\pi(1) \otimes v(2))
$$

$$
= Su(1) \otimes v(2) \otimes \pi(1) = \theta(v) \otimes \pi(1).
$$

Since $\chi(\theta(v)) = (Su(1)v(2) \otimes \pi(v(3))) = 1 \otimes \pi(v) = 0$ it follows that $\theta(v) \in P(\Omega^1 M)$. From the above calculation we conclude that $\varphi \in \text{Hom}_0(V, P(\Omega^1 M))$. Now, consider the map $r : \Omega^1 M \to P \otimes M$, $r(\sum m^i \otimes \hat{m}^i) = \sum m^i\hat{m}^i(1) \otimes \hat{m}^i(2)$. Applying id $\otimes \varepsilon$ to $r$ one immediately finds that $\text{Im}r \subseteq P \otimes V$. Similarly, applying the coaction equalising map for the cotensor product to $r$ one finds that $\text{Im}r \subseteq P \square_C V$. Finally using the same argument as in [23, Proposition 4.3] one proves that $r$ is the inverse of $s_\theta : P \square_C V \to \Omega^1 M$, $s_\theta : \sum_i u^i \otimes v^i \mapsto \sum_i u^iSu(1) \otimes v(2)$ as required. □

6. Monopole on all Quantum 2-spheres

Let $SU_q(2)$ be the standard matrix quantum group over the field $k = \mathbb{C}$, with generators $
\begin{pmatrix}
\alpha & \beta \\
\gamma & \delta
\end{pmatrix}
$
and relations $\alpha \beta = q \beta \alpha$, $\alpha \gamma = q \gamma \alpha$, $\alpha \delta = \delta \alpha + (q - q^{-1})\beta \gamma$, $\beta \gamma = \gamma \beta$, $\gamma \delta = q \delta \gamma$, $\alpha \delta - q \beta \gamma = 1$. Let

$$
\xi = s(\alpha^2 - q^{-1} \beta^2) + (s^2 - 1)q^{-1} \alpha \beta, \quad \eta = s(q \gamma^2 - \delta^2) + (s^2 - 1)\gamma \delta,
$$

$$
\zeta = s(q \alpha \gamma - \beta \delta) + (s^2 - 1)q \beta \gamma,
$$

where $s \in [0, 1]$. We define $C = SU_q(2)/J$ where $J = \{\xi - s, \eta + s, \zeta\}SU_q(2)$ is a coideal. We denote by $\pi$ the canonical projection $SU_q(2) \to C$. As shown in [4], the fixed point
subalgebra under the coaction of $C$ on $SU_q(2)$ is generated by $\{1, \xi, \eta, \zeta\}$, and can be identified with $S^{2}_{q,s}$, the 2-parameter quantum sphere in [28]. The standard quantum sphere discussed in [4] corresponds to $s = 0$. It has been recently proved [25] that the coalgebra $C$ is spanned by group-like elements. We begin by finding such a basis of $C$ explicitly.

**Proposition 6.1** Let

$$g^+_n = \pi(\prod_{k=0}^{n-1}(\alpha + q^k s \beta)), \quad g^-_n = \pi(\prod_{k=0}^{n-1}(\delta - q^{-k} s \gamma)), \quad n = 1, 2, \ldots$$

(all products increase from left to right). Then $g^\pm_n$ are group-like elements of $C$, and $\{e = \pi(1), g^\pm_n | n \in \mathbb{N}\}$ is a basis of $C$.

To prove Proposition 6.1 we will need the following

**Lemma 6.2** Let $\triangleright$ denote the right action of $SU_q(2)$ on $C$, induced by $\pi$. Then:

$$sg^+_n + 1 = g^+_n \triangleright (s \delta + q^{-n} \gamma) = g^+_n \triangleright (s \delta + q^{-n} \beta),$$

and

$$sg^-_n + 1 = g^-_n \triangleright (s \alpha - q^n \gamma) = g^-_n \triangleright (s \alpha - q^n \beta).$$

**Proof.** Using the commutation rules in $SU_q(2)$ one easily verifies that for all $s \in \mathbb{C}$, and $n \in \mathbb{N}$

$$(\alpha + q^{n-1} s \beta)(s \delta + q^{-n} \gamma) = (s \delta + q^{-n+1} \gamma)(\alpha + q^n s \beta).$$

(21)

Note that the form of $J = \ker \pi$ implies that for all $x \in SU_q(2)$

$$\pi((s \delta + \gamma)x) = s \pi((\alpha + s \beta)x), \quad s \pi((\delta - s \gamma)x) = \pi((s \alpha - \beta)x).$$

(22)

This, together with the identity (21) immediately implies that (19) holds for $n = 1$. Now, assume that (19) is true for an $n > 1$. Then, using the definition of $g^+_n$ as well as (21) we have:

$$g^+_n \triangleright (s \delta + q^{-n} \gamma) = g^+_n \triangleright (\alpha + q^{n-1} s \beta)(s \delta + q^{-n} \gamma)$$

$$= g^+_n \triangleright (s \delta + q^{-n+1} \gamma)(\alpha + q^n s \beta)$$

$$= sg^+_n \triangleright (\alpha + q^n s \beta) = sg^+_n + 1.$$
Therefore the first of equalities (19) holds for any \( n \in \mathbb{N} \). Since
\[
\pi(\beta x) = \pi(\gamma x), \quad \forall x \in SU_q(2),
\]
also the second of equalities (19) holds.

Equalities (20) are proven in an analogous way, by using the following identity
\[
(s\alpha - q^{n-1}\beta)(\delta - sq^{-n}\gamma) = (\delta - sq^{-n+1}\gamma)(s\alpha - q^n\beta).
\]
\( \square \)

**Proof of Proposition 6.1.** An easy calculation which uses (22) verifies that \( g_1^+ \) is group-like. Assume that \( g_n^+ \) is group-like for an \( n > 1 \). Using the definition of \( g_{n+1}^+ \) and this inductive assumption we have
\[
\Delta g_{n+1}^+ = g_n^+\alpha \otimes g_n^+\alpha + g_n^+\beta \otimes g_n^+\gamma + q^n s g_n^+\alpha \otimes g_n^+\beta + q^n s g_n^+\beta \otimes g_n^+\delta
\]
\[
= g_n^+\alpha \otimes g_n^+\alpha (\alpha + q^n s \beta) + q^n g_n^+\beta \otimes g_n^+\alpha (s \delta + q^{-n} \gamma)
\]
\[
= g_n^+\alpha \otimes g_{n+1}^+ + q^n s g_n^+\beta \otimes g_{n+1}^+ \quad \text{(Lemma 6.2)}
\]
\[
= g_{n+1}^+ \otimes g_{n+1}^+.
\]
Thus we conclude that \( g_n^+ \) is group-like for any \( n \). Similarly one proves that all the \( g_n^- \) are group-like. The proof that \( \pi(1), g_n^\pm \) span \( C \) is analogous to the proof of [4, Proposition 6.1]. \( \square \)

Proposition 6.1 gives an explicit description of the coalgebra bundle. We now construct a bicovariant splitting of \( \pi \) and hence a strong connection on it.

**Proposition 6.3** The map \( i : C \to SU_q(2) \) given by
\[
i(g_n^+) = \prod_{k=0}^{n-1} \frac{\alpha + q^k s(\beta + \gamma) + q^{2k}s^2 s^2}{1 + q^{2k}s^2}, \quad i(g_n^-) = \prod_{k=0}^{n-1} \frac{\delta - q^{-k}s(\beta + \gamma) + q^{-2k}s^2 s^2}{1 + q^{-2k}s^2}
\]
is bicovariant and splits \( \pi \).

**Proof.** An easy direct calculation which uses (22), (23), verifies that \( \Delta_{SU_q(2)}(i(g_1^+)) = i(g_1^+) \otimes g_1^+ \) and \( SU_q(2)\Delta(i(g_1^+)) = g_1^+ \otimes i(g_1^+) \). Now assume that there is \( n > 1 \) such that \( i(g_n^+) \) is bicovariant. Then we have
\[
\Delta_{SU_q(2)}(i(g_n^+)) = \frac{1}{1 + q^{2n}s^2} \Delta_{SU_q(2)}(i(g_{n+1}^+)(\alpha + s q^n(\beta + \gamma) + s^2 q^{2n} \delta))
\]

For the left coaction we have

\[ \text{Lemma 6.2 and (22), (23).} \]

\[ \text{Proposition 5.3 and the coproduct and antipode } S \text{ and } g. \]

From

\[ \text{SU}_q(1) + q^n s \gamma \text{ splits.} \]

\[ \text{Lemma 6.2} \]

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\[ \text{Lemma 6.2} \]

\[ \text{Lemma 6.2} \]

Thus we conclude that \( i(g_n^+) \) is bicovariant for all \( n \in \mathbb{N} \). Similarly one shows that \( i(g_n^-) \) is bicovariant.

The fact that \( i \) splits \( \pi \) can be proven inductively too, and in the proof one uses Lemma 6.2 and (22), (23). \( \square \)

Consequently, we have a strong connection on \( S_{q,s}^2 \) defined via the elements \( \omega_n^\pm = Si(g_n^\pm)(1) \otimes i(g_n^\pm)(2) \).

**Lemma 6.4** The elements \( \omega_n^\pm \) may be computed iteratively from

\[ (1 + q^{2n} s^2) \omega_{n+1}^+ = (\delta - q^{n+1} s \gamma) \omega_n^+ (\alpha + q^n s \beta) + (\alpha q^n s - q^{-1} \beta) \omega_n^+ (q^n s \delta + \gamma) \]

\[ (1 + q^{-2n} s^2) \omega_{n+1}^- = (q \gamma + q^{-n} s \delta) \omega_n^- (-\beta + q^{-n} s \alpha) + (\alpha + q^{-n-1} \beta) \omega_n^- (\delta - q^{-n} s \gamma) \]

and \( \omega_0^\pm = 1 \otimes 1 \).

**Proof.** From \( i(g_{n+1}^+) = i(g_n^+)(\alpha + q^n (\beta + \gamma) + q^{2n} s^2 \delta)/(1 + q^{2n} s^2) \) as in the proof of Proposition 5.3 and the coproduct and antipode \( S \) of \( SU_q(2) \) one has

\[ (1 + q^{2n} s^2) \omega_{n+1}^+ = S \alpha \omega_n^+ \alpha + S \beta \omega_n^+ \gamma + q^n s (S \beta \omega_n^+ \delta + S \alpha \omega_n^+ \beta + S \gamma \omega_n^+ \alpha + S \delta \omega_n^+ \gamma) \]

\[ + q^{2n} s^2 (S \delta \omega_n^+ \delta + S \gamma \omega_n^+ \beta) \]

\[ = \delta \omega_n^+ \alpha - q^{-1} \beta \omega_n^+ \gamma - q^{-n-1} s \beta \omega_n^+ \delta + q^n s \delta \omega_n^+ \beta - q^{n+1} s \gamma \omega_n^+ \alpha + q^n s \alpha \omega_n^+ \gamma \]

\[ + q^{2n} s^2 \alpha \omega_n^+ \delta - q^{2n+1} s^2 \gamma \omega_n^+ \beta \]

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which we then factorise as shown.

The computation for $\omega_{n+1}$ is similar. Actually, when $s \neq 0$ we may collect the two cases together as

$$(1 + q^2 s^2)\omega_{n+1}^\pm = (\delta \mp q^{n+1} s^{\pm 1} \gamma)\omega_n^\pm (\alpha \pm q^n s^{\pm 1} \beta) + (\alpha q^n s^\pm 1 \mp q^{-1} \beta)\omega_n^\pm (q^n s^{\pm 1} \delta \pm \gamma).$$

$\Box$

For example,

$$\omega(g_1^+) = \frac{1}{1 + s^2}((\delta - q s \gamma)d(\alpha + s \beta) + (\alpha s - q^{-1} \beta)d(\gamma + s \delta)) = \omega_1^+ - 1 \otimes 1.$$

A closed expression for $\omega$ on all $g_n^\pm$ is possible for nonuniversal differential calculi where commutation relations exist between differential forms and elements of $S^2_{q,s}$, along the lines of [7] for the standard $q$-monopole.

Finally, as an example of an associated bundle, let $V = \mathbb{C}$ with the right $C$-comodule structure $\Delta_V(1) = 1 \otimes g_1^+$. Here and in what follows we identify linear maps from $\mathbb{C}$ with their values at $1 \in \mathbb{C}$. Then the space of strongly tensorial zero-forms in Proposition 4.8 can be computed as

$$\text{Hom}^C(V, P) = \{ u \in P | \Delta_R u = u \otimes g_1^+ \} = \{ x(\alpha + s \beta) + y(\gamma + s \delta) | x, y \in S^2_{q,s} \}.$$  

The covariant derivative $D : \text{Hom}^C(V, P) \to \text{Hom}^C(V, (\Omega^1 M) P)$ can be computed as

$$D u = du - u \omega(g_1^+) = 1 \otimes u - u \omega_1^+$$

$$= 1 \otimes u - \frac{u}{1 + s^2}(\delta - q s \gamma, \alpha s - q^{-1} \beta) \otimes \left( \begin{array}{c} \alpha + s \beta \\ \gamma + s \delta \end{array} \right)$$

from the form of $\omega_1^+$. Here a matrix product (or vector-covector contraction) notation is used.

These Hom-spaces correspond to sections of a bundle $E$. From another point of view, we may consider $V_L = \mathbb{C}$ with the left coaction $\gamma \Delta(1) = g_1^+ \otimes 1$ and identify the associated bundle $E = P \Box C V_L = \text{Hom}^C(V, P)$ as the same space as above. Similarly, we identify $\Omega^1 M \otimes_M E = \text{Hom}^C(V, (\Omega^1 M) P)$. From this point of view we can consider the above covariant derivative as a map $\nabla : E \to \Omega^1 M \otimes_M E$.

Finally, from the form of $E$ given above it is clear that $E$ is a rank 2 projective module over $S^2_{q,s}$ along the same lines as the recent result over the standard $q$-sphere in [7]. We use the relation

$$(\delta - q s \gamma)(\alpha + s \beta) + (s \alpha - q^{-1} \beta)(\gamma + s \delta) = 1 + s^2,$$
holding in $SU_q(2)$ to verify that
\[
\mathbf{p} = \frac{1}{1+s^2} \begin{pmatrix} 1 - \zeta & \xi \\ -\eta & s^2 + q^{-2}\zeta \end{pmatrix} = \frac{1}{1+s^2} \begin{pmatrix} \alpha + s\beta \\ \gamma + s\delta \end{pmatrix} (\delta - q\gamma, s\alpha - q^{-1}\beta)
\]
obeys $\mathbf{p}^2 = \mathbf{p}$ as an $S_q$-valued $2 \times 2$-matrix, and that $(S_q^2)^2 \mathbf{p} = E$ by the identification of $(x,y)\mathbf{p}$ with $u = x(\alpha + s\beta) + y(\gamma + s\delta)$. In terms of this, (24) becomes
\[
\nabla((x,y)\mathbf{p}) = 1 \otimes (x,y)\mathbf{p} - (x,y)\mathbf{p} \otimes \mathbf{p} = (d(x,y))\mathbf{p} + (x,y)(d\mathbf{p})\mathbf{p} = (d(x,y)\mathbf{p})\mathbf{p}
\]
so that $\nabla$ is the Grassmannian connection associated to the projective module. Further details of the projector computation will be presented elsewhere. A similar result holds for general $n$ along the lines for the standard $q$-monopole in [17].

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