ECONOMICAL ADJUNCTION OF SQUARE ROOTS TO GROUPS

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How large must an overgroup of a given group be in order to contain a square root of any element of the initial group? We give an almost exact answer to this question (the obtained estimate is at most twice worse than the best possible) and state several related open questions.

0. Introduction

The solvability of equations over groups has been extensively studied (see, e.g., [GeRo62], [Levin62], [Lynd80], [Brod84], [EdHo91], [Howie91], [Kly99], [Kar95], [FeRo96], [Kly97], [ClGo00], [EdJu00], [JuHa03], [Kly06] and the references therein). In these papers, it was proven that, under some conditions, an equation \( w(x) = 1 \) with coefficients from a group \( G \) is solvable over \( G \), i.e., there exists a group \( H \) containing \( G \) as a subgroup and an element \( h \in H \) such that \( w(h) = 1 \). In this paper, we study a quantitative question: how large must such a group \( H \) be? Even for simple equations whose solvability is well known, this question turns out to be very difficult and we restrict ourselves to the simplest nontrivial equation \( x^2 = g \).

Certainly, the answer strongly depends on the initial group \( G \). For example, if the order of \( G \) is odd, then the role of \( H \) can be played by the group \( G \) itself; if \( G \) is cyclic, then \( H \) may be taken twice larger than \( G \), etc. The most interesting is to estimate the order of \( H \) “in the worst case”. We obtain an estimate at most twice worse than the best possible. Namely, we prove the following theorem.

Main theorem. Each finite group \( G \) embeds into a group of order \( 2|G|^2 \) in which all elements of \( G \) are squares. There exist infinitely many pairwise nonisomorphic finite groups \( G_i \) and elements \( g_i \in G_i \) such that no group of order smaller than \( |G_i|^2 \) can contain \( G_i \) together with a quadratic root of \( g_i \).

Actually, we consider two problems: economical adjunction of a solution to an equation \( x^2 = g \) and economical adjunction of solutions to all equations of such form. The main theorem shows that, in both cases, a group of order \( 2|G|^2 \) is enough, but a group of order less than \( |G|^2 \) is not enough sometimes.

The behaviour of solution sets of equations in finite groups has been thoroughly studied (see, e.g., [Frob93], [Hall96], [Solo06], [Stru96], and the references therein). Unfortunately, we are unable to use these nontrivial results.

The first assertion of the theorem is not new and can be proven easily (see Section 1). In Section 2, we prove the second assertion. In the last section, we state several open questions about economical adjunction of solutions of equations to groups.

Notation which we use is mainly standard. Note only that if \( k \in \mathbb{Z} \) and \( x \) and \( y \) are elements of a group, then \( x^y \), \( x^{y^k} \), and \( x^{-y} \) denote \( y^{-1}xy \), \( y^{-1}x^ky \) and \( y^{-1}x^{-1}y \), respectively. If \( X \) is a subset of a group, then \( |X| \), \( \langle X \rangle \), and \( \langle \langle X \rangle \rangle \) denote the cardinality of \( X \), the subgroup generated by \( X \), and the normal subgroup generated by \( X \), respectively. The letter \( \mathbb{Z} \) denotes the set of integers. The symbol \( \mathbb{Z}_n \) denotes the group or the ring \( \mathbb{Z}/n\mathbb{Z} \). The multiplicative group of the ring \( \mathbb{Z}_n \) is denoted by \( \mathbb{Z}_n^* \). The automorphism group of \( G \) is denoted by \( \text{Aut} \, G \). The symbol \( D_p \) denotes the dihedral group of order \( 2p \). The stabiliser of a point \( a \) under an action of a group \( G \) is denoted by \( \text{St}_G(a) \). A reflection is an element of \( D_p \) not belonging to its subgroup \( \mathbb{Z}_p \).

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1. Wreath products and the proof of the first assertion of the theorem

The first assertion of the theorem is well known [Levin62]: the wreath product

\[
G \wr \mathbb{Z}_2 = \left\{ \left( \begin{array}{cc} g_1 & 0 \\ 0 & g_2 \end{array} \right) \mid g_1, g_2 \in G \right\} \cup \left\{ \left( \begin{array}{cc} 0 & g_1 \\ g_2 & 0 \end{array} \right) \mid g_1, g_2 \in G \right\}
\]

of \( G \) and the cyclic group of order 2 is a group of order \( 2|G|^2 \) containing square roots of all elements of \( G \), assuming that group \( G \) embeds into the wreath product \( G \wr \mathbb{Z}_2 \) as the diagonal:

\[
g \mapsto \left( \begin{array}{cc} g & 0 \\ 0 & g \end{array} \right)
\]

Indeed, \( \left( \begin{array}{cc} g & 0 \\ 0 & g \end{array} \right)^2 = \left( \begin{array}{cc} g^2 & 0 \\ 0 & g^2 \end{array} \right) \). This is the simplest special case of the Levin theorem. The full statement can be found in the last section of this paper.

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2. Dihedral groups and the proof of the second assertion of the theorem

The second assertion of the main theorem is implied by the following fact.

**Theorem 1.** If \( p \in 4\mathbb{Z} + 3 \) is prime, \( \tilde{G} \) is a group containing the dihedral subgroup \( G = D_p \), and the reflection \( g \in G \) is the square of some element \( x \in G \), then \( |\tilde{G}| \geq |G|^2 \).

To prove this theorem, we need several simple lemmata.

**Lemma 1.** If \( H_1 \) and \( H_2 \) are subgroup of a group \( H \), then \( |H| \geq \frac{|H_1||H_2|}{|H_1 \cap H_2|} = |H_1H_2| \).

We leave the proof of this lemma to the readers.

**Lemma 2.** If \( D_p = G \subseteq \langle G, x \rangle = \tilde{G} \) and \( x^2 = g \), where \( g \in G \) is a reflection, then either \( G \triangleleft \tilde{G} \) or \( G \cap G^x = \langle g \rangle \).

**Proof.** Clearly, \( g \in G \cap G^x \). The group \( D_p \) has exactly two subgroups containing \( g \). If \( G \cap G^x = \langle g \rangle \), then we have nothing to prove. If \( G \cap G^x = G \), then \( G = G^x \) and hence \( G \triangleleft \tilde{G} \), because \( \langle G, x \rangle = \tilde{G} \).

**Lemma 3.** If \( D_p = G \triangleleft \tilde{G} \), where \( p \in 3 + 4\mathbb{Z} \) is prime, then no reflection \( g \in G \) is a square in \( \tilde{G} \).

**Proof.** The subgroup \( Z_p \subset D_p = G \triangleleft \tilde{G} \) is the commutator subgroup of \( G \) and, therefore, it is characteristic in \( G \) and normal in \( \tilde{G} \). The group \( G \) acts on \( Z_p \) by conjugations. The reflection \( g \) acts as \(-1 \in Z_p = \text{Aut } Z_p \), and it is well known that \(-1 \) is not a square in \( Z_p^* \) if \( p \in 3 + 4\mathbb{Z} \). This completes the proof.

Now, we proceed to prove the main theorem. We may assume that \( \tilde{G} = \langle G, x \rangle \). Let \( K \) be the set of all subgroups of \( \tilde{G} \) conjugate with \( G \). The group \( \tilde{G} \) acts transitively on \( K \) by conjugations.

**Lemma 4.** \( |\tilde{G}| \geq |K| \cdot |G| \).

**Proof.** \( |\tilde{G}| = |K| \cdot |	ext{St}_{\tilde{G}}(G)| \geq |K| \cdot |G| \), because \( G \subseteq \text{St}_{\tilde{G}}(G) \).

Consider the complete undirected graph \( \Gamma \) with vertex set \( K \). We call an edge \((G^{h_1}, G^{h_2})\)

- green if \( |G^{h_1} \cap G^{h_2}| = 2 \);
- yellow if \( |G^{h_1} \cap G^{h_2}| = p \);
- red if \( |G^{h_1} \cap G^{h_2}| = 1 \).

Clearly, all edges are coloured. If there is at least one red edge, then the assertion follows readily from Lemma 1. So, we assume that there are no red edges.

**Lemma 5.** All vertices \( K \) and yellow edges form a graph \( Y \) whose connected components are complete graphs. All these components contain the same number of vertices.

**Proof.** The first assertion follows immediately from that \( D_p \) has a unique subgroup of order \( p \). The second assertion follows from that the action of \( \tilde{G} \) on \( K \) is transitive and preserves colours of edges.

**Lemma 6.** The number of green edges incident to the vertex \( G \) is a positive multiple of \( p \).

**Proof.** Each green edge incident to \( G \) corresponds to one of the \( p \) reflections \( g \in G \). Thus, the edges incident to \( G \) are divided into \( p \) classes. These classes have the same number of edges, because all reflections in \( G \) are conjugate and, therefore, any class is mapped onto any other class by an automorphism of the graph. This means that the number of green edges incident to \( G \) is a multiple of \( p \).

The graph has a green edge, namely, the edge \((G, G^x)\), where \( x^2 = g \in G \) is a reflection. Indeed, \( G^x \cap G \supseteq g \), but \( G^x \neq G \) (because otherwise \( G \) would be a normal subgroup in \( \tilde{G} = \langle G, x \rangle \), which contradicts Lemma 3). Therefore, \( G^x \cap G = \langle g \rangle_2 \) and the lemma is proven.

We continue the proof of the theorem. Suppose that there are at least \( 2p \) green edges incident to \( G \). Then the graph has at least \( 2p + 1 \) vertices, i.e., \( |K| > 2p \), and, by Lemma 4, \( |\tilde{G}| \geq |K| \cdot |G| > 2p|G| = |G|^2 \), as required.

According to Lemma 6, it remains to consider the case, where there are exactly \( p \) green edges incident to each vertex.

Suppose that \( u \) is the number of vertices in each connected component of \( Y \) (Lemma 5), and \( v \) is the number of these components. Then

\[ p = (\text{the number of green edges incident to } G) = (v - 1)u \]

(because each vertex not joined with \( G \) by a yellow edge is joined with \( G \) by a green edge).

The equality \( p = (v - 1)u \) means (by virtue of the primeness of \( p \)) that either \( v = 2 \) and \( u = p \) or \( v = p + 1 \) and \( u = 1 \).

In the first case, \( |K| = 2p \), and the assertion follows from Lemma 4: \( |\tilde{G}| \geq 2p|G| = |G|^2 \).
In the second case, \( |K| = p + 1 \), and the graph \( \Gamma \) is a complete graph all of whose edges are green. The group \( \tilde{G} \) acts on this graph, and the action of \( G \) on set vertices different from \( G \) is isomorphic to the action of \( G \) by conjugations on the set of its subgroups of order two (this isomorphism maps a group \( G^h \) to the subgroup \( G^h \cap G \)). In particular, the conjugation by a reflection \( g \) is a permutation of vertices of the graph which fixes exactly two points \((G \) and \( G^h\), where \( G \cap G^h = \{g\} \)) and, therefore, this permutation decomposes into a product of \( \frac{|G|}{2} \) independent transpositions. This permutation is odd, because \( p \in 3 + 4Z \), which contradicts the assumption that \( g \) is a square in \( \tilde{G} \). The main theorem is proven.

3. Higher degree roots and other open questions

The question arises: what is the best possible estimate?

**Question 1.** Do there exist infinitely many finite groups \( G \) such that, for some \( g \in G \), each overgroup of \( G \) in which \( g \) is a square has order at least \( 2|G|^2 \)?

The following proposition shows that, for dihedral groups, our theorem can not be strengthened, and to answer Question 1 one must study groups close to simple.

**Proposition 1.** If a finite group \( G \) and its element \( g \) satisfy at least one of the conditions

a) \( G \) does not coincide with its commutator subgroup;

b) \( G \) does not coincide with the normal closure of \( g \);

c) \( G \) has a nontrivial normal subgroup of odd order,

then \( G \) embeds into a group \( H \) of order at most \( |G|^2 \) in which the element \( g \) is a square.

**Proof.** The following lemma shows that under condition a) or b) a proper subgroup of the wreath product \( G \wr Z_2 \) (see Section 1) can be taken for \( H \). If condition c) holds, then we can take a proper quotient of this wreath product for \( H \); this follows from Lemma 8 (see below).

**Lemma 7.** In the wreath product \( G \wr Z_2 \), the subgroup \( H \) generated by \( G \) embedded diagonally and the quadratic root \( \left( \begin{array}{cc} 0 & g \\ 1 & 0 \end{array} \right) \) of an element \( g \in G \) has the form

\[
H = \left\{ \left( \begin{array}{cc} g_1 & 0 \\ 0 & g_2 \end{array} \right) ; \ g_1g_2^{-1} \in [\langle g \rangle , G] \right\} \cup \left\{ \left( \begin{array}{cc} 0 & g_1 \\ g_2 & 0 \end{array} \right) ; \ g_1g_2^{-1} \in g \langle \langle g \rangle , G \rangle \right\},
\]

where \([\langle g \rangle , G]\) is the mutual commutator subgroup of \( G \) and the normal closure of \( g \) in \( G \).

**Proof.** The natural epimorphism \( \varphi : G \to G/ [\langle g \rangle , G] \) induces a homomorphism \( \Phi : G \wr Z_2 \to (G/ [\langle g \rangle , G]) \wr Z_2 \). The right-hand side of the required equality is \( \Phi^{-1}(\Phi(H)) \). Therefore, it is sufficient to prove that \( H \) contains the kernel of \( \Phi \). But \( \ker \Phi \) is generated (as a subgroup) by the elements of the form

\[
\left( \begin{array}{cc} [g^x, y] & 0 \\ 0 & 1 \end{array} \right) \quad \text{and} \quad \left( \begin{array}{cc} 1 & 0 \\ 0 & [g^x, y] \end{array} \right), \quad \text{where} \ x, y \in G,
\]

which lie in \( H \), as the following equalities shows:

\[
\left( \begin{array}{cc} x^{-1} & 0 \\ 0 & x^{-1} \end{array} \right) \left( \begin{array}{cc} 0 & g \\ 1 & 0 \end{array} \right) \left( \begin{array}{cc} x & 0 \\ 0 & x \end{array} \right) = \left( \begin{array}{cc} 0 & g^x \\ 1 & 0 \end{array} \right), \quad \left( \begin{array}{cc} 0 & g^x \\ 1 & 0 \end{array} \right) \left( \begin{array}{cc} y & 0 \\ 0 & y \end{array} \right) = \left( \begin{array}{cc} 0 & g^xy \\ 1 & 0 \end{array} \right), \quad \left( \begin{array}{cc} 1 & 0 \\ 0 & [g^x, y] \end{array} \right) = \left( \begin{array}{cc} 0 & 0 \\ 1 & [g^x, y] \end{array} \right).
\]

**Lemma 8.** If \( N \) is a normal abelian subgroup of a group \( G \), then the set

\[
K = \left\{ \left( \begin{array}{cc} x & 0 \\ 0 & x^{-1} \end{array} \right) ; \ x \in N \right\}
\]

is a normal subgroup of the wreath product \( G \wr Z_2 \). If the order of \( N \) is odd, then the intersection of \( K \) and \( G \) (embedded into the wreath product diagonally) is trivial.

Conversely: each nontrivial normal subgroup of this wreath product trivially intersecting \( G \) contains a nontrivial abelian normal subgroup of the specified form.

**Proof.** The set \( K \) is a normal subgroup, obviously. Clearly, \( K \cap G = \{ x \in N ; \ x^2 = 1 \} \); therefore, \( K \) intersects \( G \) trivially, if the order of \( N \) is odd.

*) The Feit–Thompson theorem about the solvability of odd-order groups [FeTh63] implies that property c) is equivalent to the existence of a nontrivial abelian odd order subgroup of \( G \).
It is well known that an arbitrary nontrivial normal subgroup $X$ of a wreath product nontrivially intersects the base (see, e.g., [KaMe82]). If

$$1 \neq u = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \in X,$$

then

$$\left[ u, \begin{pmatrix} y & 0 \\ 0 & y \end{pmatrix} \right] = \begin{pmatrix} [x,y] & 0 \\ 0 & 1 \end{pmatrix} = v \in X \ni \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{-1} v \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & [x,y] \end{pmatrix} = w$$

and, therefore,

$$vw = \begin{pmatrix} [x,y] & 0 \\ 0 & [x,y] \end{pmatrix} \in X \cap G = \{1\}, \text{ i.e., } [x,y] = 1.$$

But then

$$X \ni \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{-1} u \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} y & 0 \\ 0 & x \end{pmatrix} = t$$

and, therefore,

$$ut = \begin{pmatrix} xy & 0 \\ 0 & xy \end{pmatrix} \in X \cap G = \{1\}, \text{ i.e., } xy = 1.$$

Thus, the intersection of $X$ and the base of the wreath product has the form

$$K = \left\{ \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} \mid x \in N \right\},$$

where $N$ is a subset of $G$. This implies that the set $N$ must be an abelian normal subgroup. Lemma 8 is proven.

These lemmata prove Proposition 1 and, in addition, show that, if $G$ satisfies neither condition a), b), nor c) (e.g., if the group $G$ is nonabelian simple), then the wreath product $G \wr \mathbb{Z}_2$ has neither proper subgroups nor proper quotient containing $G$ and the square root $\left( \begin{smallmatrix} a & g \\ 1 & 0 \end{smallmatrix} \right)$ of $g$.

Now, consider higher degree roots and solutions to other equations. The point of departure in our investigation is the Levin theorem, whose complete statement looks as follows.

**Levin theorem** ([Levin62]). The wreath product $G \wr \mathbb{Z}_n$ (whose order is $n|G|^n$) of a group $G$ and the cyclic group of order $n$ contains solutions to all positive equations of degree $n$ over the group $G$.

A **positive equation of degree $n$ over a group $G$** is an equation of the form

$$g_1 x g_2 x \ldots g_n x = 1 \text{ where } g_1, \ldots, g_n \in G.$$

So, the question arises: does the Levin theorem give the best possible estimate?

**Question 2.** Do there exist infinitely many finite groups $G$ such that each overgroup of $G$ containing solutions to all positive equations of degree $n$ over $G$ has order at least $n|G|^n$?

One can formulate a more daring conjecture.

**Question 3.** Do there exist infinitely many finite groups $G$ such that each overgroup $H$ of $G$ in which all elements of $G$ are $n$th powers (of elements of $H$) has order at least $n|G|^n$?

What can be said about economical adjoining of solutions to other (i.e., nonpositive) equations? For example, the Gerstenhaber–Rothaus theorem [GeRo62] in combination with Mal’tsev’s theorem about residual finiteness of finitely generated linear groups [Mal40] gives the following fact.

**Proposition 2.** ([GeRo62] + [Mal40]). Each finite group $G$ embeds into a finite group $H$ containing solutions to all nonsingular equations of length $n$ over $G$.

A **nonsingular equation of length $n$ over a group $G$** is an equation of the form

$$g_1 x^{e_1} g_2 x^{e_2} \ldots g_n x^{e_n} = 1, \text{ where } g_i \in G, \ varepsilon_i \in \{\pm 1\}, \text{ and } \sum \varepsilon_i \neq 0.$$

The proof of the Gerstenhaber–Rothaus theorem is nice but nonconstructive. So, it is difficult to write out any estimate of the order of $H$.

**Question 4.** How to estimate $|H|$ via $|G|$ and $n$ in Proposition 2?

For $n = 1$, the answers to Questions 2, 3, and 4 is, obviously, positive. A nonsingular equation of length two has the form $g_1 x^2 g_2 x^2 = 1,$ where $\varepsilon \in \{\pm 1\}$, and a linear change of variables reduces it to the form $x^2 = g$; therefore, the main result of this paper gives an answer to the “twice weakened” versions of these questions for $n = 2$. What occurs for other $n$ we do not know.
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