The existence of intrinsic rotating wave solutions of a flame/smoldering-front evolution equation

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Abstract

The Kuramoto-Sivashinsky equation for Jordan curves is used to model the smoldering combustion of a sheet of paper. Here, the behavior of a rotating wave bifurcating from an expanding circle solution to this equation is mathematically analyzed. We also present some numerical examples in which the rotating waves are visualized.

Keywords Kuramoto-Sivashinsky equation, rotating wave, smoldering combustion

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1. Introduction

The interfacial dynamics of flame fronts have long been studied in the field of combustion fields especially within the past forty years. Starting from the pioneering works by Kuramoto and Tsuzuki [1] and Sivashinsky [2], the so-called Kuramoto-Sivashinsky equation has been studied both in terms of combustion theory and dynamical systems theory of mathematics:

\[
f_t + \frac{1}{2} f_s^2 + (\alpha - 1) f_{ss} + 4 f_{ssss} = 0. \quad (1)
\]

Here \(f(x,t) \in \mathbb{R}\) describes a graph of curved propagating gaseous-flame front, \(\alpha > 0\) is a scaled Lewis number, and \(f_t = \partial f/\partial t, f_s = \partial f/\partial x, f_{ss} = (f_x)_x, \) and so forth.

Frankel and Sivashinsky [3] derived the flame-evolution equation for a family of smooth Jordan curves (\(\Gamma(t)\)) in the plane \(\mathbb{R}^2\) (which is equivalent to (1) on a certain space and time scale) as follows:

\[
X_t = VN + WT, \quad V = V_0 + (\alpha - 1) \kappa + \delta \kappa_{ss}. \quad (2)
\]

Here, \(\delta\) is a positive parameter corresponding to the value of 4 in (1), and the solution curve \(\Gamma(t)\) is parametrized by \(X : [0, 2\pi] \times [0, T] \rightarrow \mathbb{R}^2\) such that \(\Gamma(t) = \{X(u,t); u \in [0,2\pi]\}\). The velocity of the front is \(X_t\), which can be decomposed in the normal direction \(N\) and the tangential direction \(T\), where \(T = X_s\) is the unit tangential vector and \(N = -T^\perp\) is the unit normal vector \((a,b)^\perp = (-b,a)\). Here and hereafter, we use \(F_u = \partial F/\partial u\) and \(F_s = g^{-1}F_u\) to denote the formal differential with respect to the arc-length parameter, \(s = s(u,t) = \int_u^u g(u)du\) to denote the integral of the local length \(g(u,t) = |X_u|^2\), and \(F_{uu}\) to denote \((F_u)_{uu}\). We now consider the normal component of the velocity \(X_u\) given by a linear combination of the constant speed \(V_0\), the curvature \(\kappa = \text{det}(X_u, X_{uu})g^{-3/2}\), and the normal second derivative of \(\kappa\) with respect to \(s\).

Epstein and Gage [4] showed that the shape of the flame front \(\Gamma(t)\) is determined by the normal velocity \(V\) only, and that the tangential velocity \(W\) does not affect the shape of the curve \(\Gamma(t)\). From an interfacial-dynamics point of view, if the parameter is \(\alpha > 1\), the second term \((\alpha - 1)\kappa\) induces an instability, and the third term \(\delta \kappa_{ss}\) smooths the shape of the unstable fronts.

Goto et al. [5] proposed a fast and simple numerical method for front tracking of (2) and reported that (2) is valid not only for propagating gaseous-flame fronts, but also for expanding smoldering fronts over thin solids.

This paper aims to analyze the dynamics of the solution to (2), employing local bifurcation theory to elucidate the behavior of the smoldering wavefront. In Section 2, we derive a perturbation equation from the trivial solution, which is an expanding circle. This perturbation equation is then reduced to a system of ODEs around the first bifurcation point of the trivial solution in Section 3. In Section 4, we discuss the existence of a rotating wavefront while the smoldering front spreads outward. Such rotating waves are visualized in Section 5, and concluding remarks and some future works are presented in the final section.

2. The perturbation equation

Recall that the tangential velocity \(W\) does not affect the shape of evolving curves, as mentioned in Section 1. By taking the inner product of (2) and \(N\), we have

\[
X_t \cdot N = V. \quad (3)
\]

Eq. (3) has a circular solution given as follows:

\[
X(u,t) = X_R = R(t)g(u), \quad g = \left(\cos u, \sin u\right), \quad (4)
\]

where \(R(t)\) must satisfy \(\dot{R}(t) = V_0 + (\alpha - 1)/R(t)\), i.e.,

\[
\frac{1}{V_0} \left(\frac{R(t) - R(0)}{V_0 R(t) + \alpha - 1} - \frac{\alpha - 1}{V_0} \log \frac{V_0 R(t) + \alpha - 1}{V_0 R(0) + \alpha - 1}\right) = t.
\]
Definition 1  We call solution (4) the trivial solution or the circle solution.

The following property is easy to show.

Proposition 2 Let $\alpha - 1 > 0$. Then for any $R(0) > 0$, the solution $R(t)$ is strictly monotone decreasing, and $R(t) \to +\infty$ holds as $t \to +\infty$.

We now consider a perturbation, say $h(u, t)$, of the trivial solution such that $X = X_R = h(u, t)y$. By denoting $\partial^\nu X/\partial u^3 = F_j y + G_j y^+$, (3) yields

$$
\dot{h} + \frac{V_0}{G_1} h + \left[ (\alpha - 1) + 3g - 3g^{-1} - g_{uu} \right] h + \frac{G_1}{G_2} F_j G_1 - F_4 F_j G_1 - F_4 + F_3 G_1 - F_2 G_1 - \frac{3}{2} g^2 - 1 \frac{1}{2} g^2 = 0.
$$

We set $\varepsilon = |\alpha - 1| > 0$ as a small parameter and rescale (5) from $h$ to $f$ via $\varepsilon$ such that $h(u, t) = \varepsilon f(u, \tau)$, and $\tau = \varepsilon^2 t$, and $r = \varepsilon^{1/2} R$. Thus (5) can be rewritten as

$$
\begin{align*}
&\left[ f + \frac{\delta}{R} f_{uuu} + \frac{1}{2} \frac{\delta}{R^2} \left( \text{sgn}(\alpha - 1) + \frac{\delta}{R} \right) f_{uu} + \frac{\text{sgn}(\alpha - 1)}{R^2} f - \frac{V_0}{2R^2} + \frac{\varepsilon^2}{2} \right] = 0.
\end{align*}
$$

Since $\alpha - 1 = \text{sgn}(\alpha - 1)$, omitting the term $o(\varepsilon^2)$ and replacing $f$ with $h$, one can extract the following equation in the original scale:

$$
\dot{h} = \mathcal{L} h + \mathcal{N}(h),
$$

where

$$
\mathcal{L} = -\frac{\delta}{R^2} \frac{\partial^4}{\partial u^4} - \frac{1}{R^2} \left( \alpha - 1 + \frac{\delta}{R} \right) \frac{\partial^2}{\partial u^2} - \frac{\alpha - 1}{R^2},
$$

$$
\mathcal{N}(h) = \frac{V_0}{2R^2} \left( \frac{\partial h}{\partial u} \right)^2.
$$

3. Bifurcation analysis

Since $R(t)$ is a known function of $t$, the evolution equation (6) is a non-autonomous system with respect to $t$. To apply the standard bifurcation theory around the circle solution, $R(t)$ is regarded as the bifurcation parameter $R$ in $\mathcal{H}^4_{\text{loc}} = \{ h \in \mathcal{H}^4_{\text{loc}} ; h(u) = h(u + 2\pi) \}$. Substituting the Fourier expansion $h(u, t) = \sum_{m \in \mathbb{Z}} h_m(t) e^{-i\nu m u}$, $h_m(\cdot) \in \mathbb{C}$ into (6), we obtain an infinite-dimensional dynamical system:

$$
h_m = \lambda_m h_m - \frac{V_0}{2R^2} \sum_{m_1 m_2 \leq m} m_1 m_2 h_{m_1} h_{m_2},
$$

where $\lambda_{\pm 1} = 0$ and

$$
\lambda_{|m| \geq 2} = -\frac{\delta m^2}{R^4} + \frac{m^2}{R^2} \left( \alpha - 1 + \frac{\delta}{R} \right) - \frac{\alpha - 1}{R^2}.
$$

We consider (7) under the phase space $\mathcal{F} = \{ (h_m)_{m \in \mathbb{Z}} ; h_m = h_{-m}, \| (h_m)_{m \in \mathbb{Z}} \|_2^2 < \infty \}$ with the norm $\| (h_m)_{m \in \mathbb{Z}} \|_2^2 = \sum_{m \in \mathbb{Z}} (1 + m^2)^{1/2} |h_m|^2$. Note that $h_{-m} = h_{m}$ follows from $h(\cdot, \cdot) \in \mathbb{R}$, and that it is known that the linearized operator becomes a generator for the analytic semi-group (see Section 2 in Brauner et al. [6]).
such that the local invariant manifold $M_{loc}^c$ is expressed as

$$M_{loc}^c = \{ (\mu, h_1, h_2) \in \mathbb{R}^2 | h_m = g_m(\mu, h_1, h_2, h_3) \}$$

Note that $g_m = \mathcal{O}(\varepsilon^2)$ holds when $\| (h_j) \| \in (\pm 1, \pm 2)$. Finally, substituting

$$g_3 = \frac{2V_0}{\lambda_3 R_2} h_1 b_2 + \mathcal{O}((\mu, h_1, h_2, h_3)^3),$$

$$g_4 = \frac{2V_0}{\lambda_4 R_2^2} h_2^2 + \mathcal{O}((\mu, h_1, h_2, h_3)^3)$$

into (7), we obtain (8).

(QED)

4. Equilibria

4.1 Rotating wave solution

Set $h_j(t) = r_j(t)e^{i \theta_j(t)}$ for $j = 1, 2$ and $\phi = 2\theta_1 - \theta_2$. We analyze the following system, truncating the fourth-order or higher terms in (8):

$$\begin{cases}
\dot{r}_1 = r_1 r_2(a_1 \cos \phi + a_2 r_2), \\
\dot{r}_2 = \lambda_3 r_2 + \frac{b_1 r_1^2}{r_2} \cos \phi + r_2(b_2 r_1^2 + b_3 r_2^2), \\
\dot{\phi} = -\left(\frac{2a_1 r_2}{r_1^2} + \frac{b_1 r_1^2}{r_2}\right) \sin \phi.
\end{cases}
$$

In this paper, we call the rotating wave solution a non-trivial stationary solution of (9) under $\sin \phi \neq 0$. Hence, in the phase space $\{ (r_1, r_2, \phi) \}$, the equilibrium corresponding to the rotating-wave solution is

$$r_1 = \sqrt{\frac{2a_1}{b_1} r_2}, \quad r_2 = \sqrt{\frac{\lambda_3 b_1}{2a_1 b_2 - b_1(2a_2 + b_3)}},$$

$$\cos \phi = -\frac{a_2}{a_1} r_2.$$

Therefore, the necessary condition for the existence of a rotating wave is $a_1 b_1 = -V_0^2/R_2^4 < 0$, which holds for any $V_0 > 0$, $\alpha > 1$, and $\delta > 0$. We note that $|\cos \phi| \neq 0$, and then $\phi \neq \pm \pi/2$ holds. It should also be remarked that the individual phase equations are written as

$$\dot{\theta}_1 = -a_1 r_2 \sin \phi, \quad \dot{\theta}_2 = \frac{b_1 r_1^2}{r_2} \sin \phi.
$$

The rotating wave in (8) means that the phase difference $\phi$ remains constant, but $\theta_1$ and $\theta_2$ both increase or both decrease linearly with respect to time; this is the reason for the rotation.

The rotating-wave solution is approximately inherent in the solution structure of the system (6), and the leading term of the solution for (6) is given by

$$h(u, t) \sim 2(r_1 \cos \{\theta_1(t) + u\} + r_2 \cos \{\theta_2(t) + 2u\}).$$

Then, the linearized matrix at (10) is given by

$$\begin{pmatrix}
0 & p a_1 r_2^2 & -p a_1 r_2^2 \sin \phi \\
2pr_2^2 \left(\frac{b_2 - a_2 b_1}{a_1}\right) & 2(b_3 - a_2 r_2^2) - 2a_1 r_2^2 \sin \phi & -2b_1 p \sin \phi \\
-2b_1 p \sin \phi & -4a_1 \sin \phi & 0
\end{pmatrix},$$

where $p = \sqrt{-2a_1/b_1}$. Since the characteristic polynomial of (12) is $\lambda^3 + c_1 \lambda^2 + c_2 \lambda + c_3 = 0$, where

$$c_1 = 2r_2^2(a_2 - b_1),$$

$$c_2 = 2r_2^2 \left[\frac{a_2 b_1 - a_1 b_2}{a_1} + a_1(4a_1 - b_1 p^2 \sin^2 \phi)\right],$$

$$c_3 = 4a_1 p^2 r_2^2 [2(a_2 b_1 - a_1 b_2) + b_1 b_3] \sin^2 \phi,$$

from the Routh–Hurwitz stability criterion, the rotating wave solution is unstable for all $V_0 > 0$, $\delta > 0$, and $\alpha > 1$.

4.2 2-mode stationary solution

If we restrict the dynamical system (9) to the phase space $\{ (r_1, r_2, \phi) | r_1 = 0 \}$, then we can find that the 2-mode equilibria $(r_1, r_2, \phi) = (0, r_2, \phi)$ bifurcated from the trivial equilibrium, where $r_2 = \sqrt{-\lambda_2/b_3}$ and $\phi = 0$ or $\pi$. The linearized eigenvalues of $(0, r_2, 0)$ and $(0, r_2, \pi)$ are $(r_2(a_1 + a_2 r_2), \lambda_2 + 2b_3 r_2^2, -2a_1 r_2)$ and $(r_2(-a_1 + a_2 r_2), \lambda_2 + 3b_3 r_2^2, 2a_1 r_2)$, respectively.

5. Numerical experiments

5.1 Numerical solution of (6)

In this section, we show the numerical experiments to (6) and (9). Let

$$\delta = 4.0, \quad \alpha = 1.2, \quad \text{and} \quad V_0 = 1.0. \quad (13)$$

If we set $R = R_* + 0.01$, then we have $\lambda_2 = 1.670 \cdots \times 10^{-1}$; therefore, the rotating-wave equilibrium is given by

$$(r_1, r_2, \phi) = (1.686 \cdots \times 10^{-2}, 5.961 \cdots \times 10^{-3}, 1.552 \cdots).$$

The linearized eigenvalues are

$$7.433 \cdots \times 10^{-5} \pm 5.150 \cdots \times 10^{-4}i,$$

$$-1.113 \cdots \times 10^{-5}.$$

Moreover, the individual phase equations are

$$\dot{\theta}_1 = -1.486 \cdots \times 10^{-4}, \quad \dot{\theta}_2 = -2.973 \cdots \times 10^{-4}.$$

By applying the explicit method to (6), we can reproduce the traveling-wave solution numerically, as shown in Fig. 2, which corresponds to the rotating-wave solution. Since the rotating wave is unstable, the wave pattern can be observed to fluctuate little by little even when the initial value is set around the equilibrium.

5.2 Numerical solution of (9)

Fig. 3 indicates the numerical solution of the dynamical system (9) utilizing the standard fourth-order Runge–Kutta method. Here, the parameters are $\lambda_2 = 0.00001, a_1 = 1/40, a_2 = -3/40, b_1 = -1/160, b_2 = -3/80$ and $b_3 = -1/45$, which can be calculated from (13). The initial value is set around the rotating wave equilibrium (10). Since our analysis is constrained in small-amplitude solutions, the rotating effect may be invisible. However, numerical experiments suggest that rotation phenomena of the smoldering combustion front may be observed, even when the rotating wave is unstable.
E. Armbruster et al. [9] showed that a sufficiently small perturbation can be generated by an appropriate scaling parameter \( \delta \); it connects the two equilibria \((0,0,0)\) and \((s^2,0,0)\). Therefore, the amplitude of this oscillates periodically in time. They also showed that the heteroclinic cycle, which is visualized in Fig. 3, was suggested. From the viewpoint of combustion science, rotating-wave phenomena can be observed in the interfacial dynamics of the front in paper-smoldering combustion. We emphasize that our method offers a new approach for analyzing the instability of the front dynamics given by the interfacial equation.

We intend to address an analytical proof of the existence and stability of the heteroclinic cycle suggested by the numerical experiment (Fig. 3) in a future work. The essential difficulty is that dynamical system (8) has co-dimension one bifurcation, while that treated by Armbruster et al. [9] has co-dimension two.

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