Embeddings of Müntz Spaces: Composition Operators

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Abstract. Given a strictly increasing sequence $\Lambda = (\lambda_n)$ of nonegative real numbers, with $\sum_{n=1}^{\infty} \frac{1}{\lambda_n} < \infty$, the Müntz spaces $M^p_\Lambda$ are defined as the closure in $L^p([0,1])$ of the monomials $x^{\lambda_n}$. We discuss how properties of the embedding $M^2_\Lambda \subset L^2(\mu)$, where $\mu$ is a finite positive Borel measure on the interval $[0,1]$, have immediate consequences for composition operators on $M^2_\Lambda$. We give criteria for composition operators to be bounded, compact, or to belong to the Schatten–von Neumann ideals.

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Introduction

The Müntz–Szasz Theorem states that, if $0 = \lambda_0 < \lambda_1 < \cdots < \lambda_n < \cdots$ is an increasing sequence of nonnegative real numbers, then the linear span of $x^{\lambda_n}$ is dense in $C([0,1])$ if and only if $\sum_{n=1}^{\infty} \frac{1}{\lambda_n} = \infty$. When $\sum_{n=1}^{\infty} \frac{1}{\lambda_n} < \infty$, the closed linear span of the monomials $(x^{\lambda_n})_{n=0}^{\infty}$ in $L^p([0,1])$ for $1 \leq p < +\infty$ is a proper subspace of $L^p([0,1])$. These spaces, called Müntz spaces and denoted $M^p_\Lambda$, exhibit interesting properties that have not been very much investigated. We refer principally to the monographies [3, 5]; recent results appear in [1, 2, 8, 4].

In the paper [6], of which this work is a sequel, we investigated various properties and necessary conditions that allowed us to embed the Hilbert Müntz space $M^2_\Lambda$ into the Lebesgue space $L^2(\mu)$ for some positive measure $\mu$ on $[0,1]$. The boundedness, compactness and Schatten ideal properties of this embedding were studied.

The purpose of this paper is to provide applications of the theory introduced in [6] to composition operators on $M^2_\Lambda$. The plan of the paper is the following. After a section of preliminaries, we show in Section 2 that $M^p_\Lambda$...
is not an invariant subspace of composition operators in general. It is then natural to study composition operators as mapping $M^p_\Lambda$ into $L^p([0,1])$. This is done in the sequel: sufficient conditions for composition operators to be bounded, compact or belong to Schatten ideals are obtained in Section 3, and necessary conditions in Section 4.

1. Preliminaries

We denote by $m$ the Lebesgue measure on $[0,1]$. $L^p(\mu)$ shall be used to denote the space of Lebesgue integrable functions of order $p \in [1, \infty]$ with respect to the measure $\mu$ on $[0,1]$. We will frequently use $L^p$ to mean $L^p(m)$, and denote by $\| \cdot \|_p$ and $\| \cdot \|_{L^p(\mu)}$ the norms in $L^p(m)$ and $L^p(\mu)$ respectively.

Let us denote, for a set $S$ of nonnegative real numbers, the subspace $L^p_S = \text{closed span}\{x^t : t \in S\} \subset L^p$. When clear from the context, we shall denote by $L_S$ the space $L^p_S$.

Definition 1.1. Let $\Lambda$ be an increasing sequence of nonnegative real numbers with $\sum_{\lambda \in \Lambda} \frac{1}{\lambda} < \infty$. The Müntz space $M^p_\Lambda$ is defined to be the space $L^p_\Lambda$.

In this paper, $\Lambda$ shall always denote an increasing sequence of nonnegative real numbers with $\sum_{\lambda \in \Lambda} \frac{1}{\lambda} < \infty$. The functions in $M^p_\Lambda$ are continuous on $[0,1)$ and real analytic in $(0,1)$. A feature of the Müntz monomials $(x^\lambda)_{\lambda \in \Lambda}$ is that they form a minimal system in $M^p_\Lambda$, which means that for any $\lambda' \in \Lambda$

$$\text{dist}(x^{\lambda'}, L_{\Lambda\setminus\{\lambda'\}}) = \inf_{g \in L_{\Lambda\setminus\{\lambda'\}}} \|x^{\lambda'} - g\|_{L^p} > 0.$$ 

This can easily be extended to show that if $\Lambda' \subset \Lambda$ is a finite subset, then

$$L_{\Lambda'} \cap L_{\Lambda\setminus\Lambda'} = \{0\}. \quad (1.1)$$

The monograph [5] may be consulted for a discussion on the minimality of Müntz monomials.

We shall need the Clarkson-Erdős Theorem from [5]:

Theorem 1.2. Assume that $\sum_k \frac{1}{\lambda_k} < \infty$ and $\inf_k (\lambda_{k+1} - \lambda_k) > 0$. If $f \in M^p_\Lambda$ then there exist $b_k \in \mathbb{R}$ such that

$$f(x) = \sum_{k=1}^{\infty} b_k x^{\lambda_k} \quad \text{for } x \in [0,1),$$

where the series converges uniformly on compact subsets of $[0,1]$. Also, for any $\varepsilon > 0$, there is a constant $M > 0$ such that

$$|b_k| \|x^{\lambda_k}\|_{L^p} \leq (1 + \varepsilon)^{\lambda_k} \|f\|_{L^p} \quad \text{if } k \geq M. \quad (1.2)$$

A sequence $\Lambda$ is called lacunary if for some $\gamma > 1$ we have $\lambda_{n+1}/\lambda_n \geq \gamma$ for $n \geq 1$. More generally, $\Lambda$ is called quasilacunary if for some increasing sequence $\{n_k\}$ of integers with $N := \sup_k (n_{k+1} - n_k) < \infty$ and some $\gamma > 1$ we have $\lambda_{n_{k+1}}/\lambda_{n_k} \geq \gamma$. The main feature of lacunarity is that the monomials
\[ \lambda_{n}^{1/p} x^{\lambda_{n}} \] form a basis in each of the spaces \( M_{\Lambda}^{p} \). In particular, the sequence \((\lambda_{n}^{1/2} x^{\lambda_{n}})_{n \geq 1}\) forms a Riesz basis in \( M_{\Lambda}^{2} \).

If \( T : \mathcal{E} \to \mathcal{F} \) is a bounded operator between Banach spaces, we define by \( \| T \| \leq \inf_{K} \| T + K \| \) the essential norm of an operator, where the infimum is taken over all compact operators \( K : \mathcal{E} \to \mathcal{F} \). This norm measures how far an operator is from being compact. In particular, \( T \) is compact if and only if \( \| T \|_e = 0 \).

The Schatten–Von Neumann class \( S_{q}(\mathcal{H}_{1}, \mathcal{H}_{2}) \) is formed by the compact Hilbert space operators \( T : \mathcal{H}_{1} \to \mathcal{H}_{2} \) such that \( \| T \| = \sqrt{T^{*}T} : \mathcal{H}_{1} \to \mathcal{H}_{1} \) has a family of eigenvalues \( \{ s_{n}(T) \}_{n=1}^{\infty} \in \ell_{q} \). If we define

\[
\| T \|_{q} = \left( \sum_{n=1}^{\infty} s_{n}(T)^{q} \right)^{1/q},
\]

then we obtain a quasinorm for \( 0 < q < 1 \) and a norm for \( q \geq 1 \), with respect to which \( S_{q}(\mathcal{H}_{1}, \mathcal{H}_{2}) \) is complete. It is immediate that \( \| T \|_{q} \geq \| T \|_{q'} \) for \( q \leq q' \), hence \( S_{q} \subset S_{q'} \).

We now define \( \Lambda \)-embedding measures which were previously studied in [4] and [6]:

**Definition 1.3.** A positive measure \( \mu \) on \([0, 1]\) is called \( \Lambda_{p} \)-embedding, if there is a constant \( C > 0 \) such that

\[
\| g \|_{L^{p}(\mu)} \leq C \| g \|_{p}
\]

for all polynomials \( g \in M_{\Lambda}^{p} \). Whenever \( p \) is clear from the context, we will remove subscript \( p \) and use the notation \( \Lambda \)-embedding.

It follows easily from the definition (see [4]) that a \( \Lambda_{p} \)-embedding measure \( \mu \) has to satisfy \( \mu(1) = 0 \). Therefore, as in Remark 2.5 of [4], we may extend the embedding to all \( f \in M_{\Lambda}^{p} \): if \( \mu \) is \( \Lambda_{p} \)-embedding, then \( M_{\Lambda}^{p} \subset L^{p}(\mu) \)

\[
\| f \|_{L^{p}(\mu)} \leq C \| f \|_{p}
\]

for all \( f \in M_{\Lambda}^{p} \). For a \( \Lambda_{p} \)-embedding \( \mu \) we denote by \( i_{\mu}^{p} \) the embedding operator \( i_{\mu_{\mu}}^{p} : M_{\Lambda}^{p} \hookrightarrow L^{p}(\mu) \), which is bounded. If \( 0 < \varepsilon < 1 \), then the interval \([1 - \varepsilon, 1]\) will be denoted by \( J_{\varepsilon} \).

The next result is proved in [4] for \( p = 1 \), but the extension to all \( p \geq 1 \) is straightforward.

**Proposition 1.4.** Let \( M_{\Lambda}^{p} \) be a Müntz space, and suppose there exists \( \delta > 0 \) such that \( d\mu|_{J_{\delta}} = h dm|_{J_{\delta}} \) for some bounded measurable function \( h \) with \( \lim_{t \to 0} h(t) = a \). Then \( i_{\mu}^{p} \) is bounded and \( \| i_{\mu}^{p} \|_{e} = a^{1/p} \).

A new class of measures called sublinear measures was introduced in [4]. There they were used to characterize embedding operators \( i_{\mu}^{p} : M_{\Lambda}^{1} \hookrightarrow L^{1}(\mu) \) for the class of quasilacunary sequences \( \Lambda \).

**Definition 1.5.** A measure \( \mu \) is called sublinear if there is a constant \( C > 0 \) such that for any \( 0 < \varepsilon < 1 \) we have \( \mu(J_{\varepsilon}) \leq C\varepsilon \). The smallest such \( C \) will be denoted by \( \| \mu \|_{S} \). The measure \( \mu \) is called vanishing sublinear if \( \lim_{\varepsilon \to 0} \frac{\mu(J_{\varepsilon})}{\varepsilon} = 0 \). Furthermore, a measure \( \mu \) is called \( \alpha \)-sublinear if \( \mu(J_{\varepsilon}) \leq C\varepsilon^{\alpha} \) for some \( \alpha > 1 \).
The main embedding results in [6] are contained in the next two theorems:

**Theorem 1.6.** Let $\Lambda$ be lacunary and $\mu$ a positive measure on $[0,1]$. Then
(i) $i_\mu^2$ is bounded if $\mu$ is sublinear.
(ii) $i_\mu^2$ is compact if $\mu$ is vanishing sublinear.

The above results are shown in [6] to be true, after an interpolation argument, for all embeddings $i_\mu^p$ for $1 \leq p \leq 2$.

In [6], we also investigated conditions for measures that enabled the embedding $i_\mu^2$ to belong to $S_q$. We shall need the main results therein:

**Theorem 1.7.** Let $\mu$ be a positive measure on $[0,1]$. Then $i_\mu^2 \in S_q(M_\Lambda^2, L^2(\mu))$ for all $q > 0$ if either of the following is true
(i) $\mu$ has compact support in $[0,1)$,
(ii) $\Lambda$ is quasilacunary and $\mu$ is $\alpha$-sublinear.

Our goal is to apply these embedding results to composition operators. Recall that the pullback of a measure $\nu$ by $\phi$ is the measure $\phi^*\nu$ on $[0,1]$ defined by
$$\phi^*\nu(E) = \nu(\phi^{-1}(E))$$
for any Borel set $E$. If $g$ is a positive measurable function, then the formula
$$\int_0^1 g(\phi(x))dx = \int_{[0,1]} g d(\phi^*\mu)$$
is easily checked on characteristic functions, hence the usual argument extends it to all positive Borel functions on $[0,1]$. In particular, if we define $\mu = \phi^*\mu$ and choose $g = |f|^p$ for some $f \in L^p(\mu)$, then the map $J : L^p(\mu) \rightarrow L^p$ defined by $J(f) = f \circ \phi$ is an isometry.

Let $\phi$ be a Borel function on $[0,1]$ such that $\phi([0,1]) \subset [0,1]$. The composition operator $C_\phi$ is defined as
$$C_\phi(g) = g \circ \phi$$
for all polynomials $g \in M_\Lambda^p$. Just as we did for $i_\mu^p$, we can extend $C_\phi = J \circ i_\mu^p$ to all $f \in M_\Lambda^p$. Since $J$ is an isometry, we obtain the following results for composition operators.

**Lemma 1.8.** Define the measure $\mu = \phi^*\mu$. Then
(i) $C_\phi$ is bounded from $M_\Lambda^p$ to $L^p$ if and only if $\mu$ is a $\Lambda_\rho$-embedding measure.
(ii) $C_\phi$ is compact from $M_\Lambda^p$ to $L^p$ if and only if $i_\mu^p$ is compact.
(iii) $C_\phi \in S_q(M_\Lambda^2, L^2)$ if and only if $i_\mu^2 \in S_q(M_\Lambda^2, L^2(\mu))$. 
2. M"untz Spaces are not Invariant to Most Composition Operators

It has already appeared above that we study composition operators defined on $M^\Lambda_p$, but whose range space is $L^p$. The reason is that M"untz spaces are usually not invariant with respect to composition. This has already been noticed by Al Alam [2], in the case of M"untz space $M^\infty_\Lambda$, i.e. the closure of the span of monomials $x^{\lambda_n}$ in $L^\infty$, and operators $C_\phi$ with continuous $\phi$. The following result was proved therein.

Proposition 2.1. Let $\Lambda = (\lambda_k)_k \subset \mathbb{N}$ and $\sum_k \frac{1}{\lambda_k} < \infty$. Then
(i) $C_\phi M^\infty_\Lambda \not\subset M^\infty_\Lambda$ if $\phi = \alpha x^m + \beta x^n$ with $\alpha, \beta \neq 0$ and $m, n \in \mathbb{N}$.
(ii) $C_\phi M^\infty_\Lambda \not\subset M^\infty_\Lambda$ if $\phi$ is a polynomial with positive coefficients and more than one term.

In this section we will significantly extend these results to other values of $p \geq 1$ and functions $\phi$. We prove in Theorem 2.5 that $C_\phi M^p_\Lambda \not\subset M^p_\Lambda$ whenever $\phi$ is a function of the form $c_1 x^{s_1} + \ldots + c_l x^{s_l}$ with $c_i \in \mathbb{R}$ and $s_i \in \mathbb{R}^+$. These functions will be called real-exponent polynomials. This generalizes Proposition 2.1 and $\Lambda$ may not even satisfy the gap condition $\inf_k (\lambda_{k+1} - \lambda_k) > 0$. If we assume the gap condition, then Theorem 2.8 generalizes Proposition 2.1(i) for arbitrary $\Lambda \subset \mathbb{R}^+$.

We start with a result of A. Schinzel [7]:

Lemma 2.2. If $\phi$ is a polynomial with at least two terms and $\lambda \in \mathbb{N}$, then $\phi^\lambda$ has at least $\lambda + 1$ terms.

The next result is an analog of Lemma 2.2 for real-exponent polynomials.

Lemma 2.3. If $\phi$ is a real-exponent polynomial with at least two terms and $\lambda \in \mathbb{N}$, then $\phi^\lambda$ has at least $\lambda + 1$ terms.

Proof. Let $\phi(x) = c_1 x^{s_1} + \ldots + c_l x^{s_l}$ with $c_i \in \mathbb{R}\{0\}$ and $s_i \in \mathbb{R}^+$. Considering $\mathbb{R}$ as a vector space over the rationals $\mathbb{Q}$, choose a basis $r_1, \ldots, r_\tau > 0$ for the space spanned by $s_1, \ldots, s_l$ where $\tau \leq l$. Therefore

$$s_i = \sum_{j=1}^\tau a_{ij} r_j \quad \text{for} \quad i = 1, \ldots, l$$

where $a_{ij} \in \mathbb{Q}$. We may assume that $a_{ij} \in \mathbb{Z}$ by adjusting the $r_j$ suitably. We note that for any positive real number $N$, $\phi^\lambda$ has the same number of terms as $(x^N \phi)^\lambda$. So by choosing $N = b_1 r_1 + \ldots + b_\tau r_\tau$ with integers $b_j > |a_{ij}|$ for $i = 1, \ldots, l$ and $j = 1, \ldots, \tau$, we may also assume that each $a_{ij} r_j > 0$ hence $a_{ij} \in \mathbb{N}$. We then obtain

$$\phi(x) = \sum_{i=1}^l c_i x^{s_i} = \sum_{i=1}^l c_i (x^{r_1})^{a_{i1}} \ldots (x^{r_\tau})^{a_{i\tau}}.$$
We define a polynomial $\psi$ in $\tau$ variables by

$$\psi(Y_1, \ldots, Y_\tau) = \sum_{i=1}^l c_i Y_1^{a_{i1}} \ldots Y_\tau^{a_{i\tau}}.$$  

Define $\Phi$ to be the collection of monomial terms in $\phi^\lambda$ after reduction and cancellation, and $\Psi$ similarly for $\psi^\lambda$. Hence our goal is to prove that card$\Phi \geq \lambda + 1$. Since both $\phi$ and $\psi$ each have $l$ distinct monomial terms, the total number of possible products while computing $\phi^\lambda$ or $\psi^\lambda$ is $l^\lambda$.

We claim that whenever two such products $p(x) = k.(x^{r_1})^{m_1} \ldots (x^{r_\tau})^{m_\tau}$ and $q(x) = k'.(x^{r_1})^{m'_1} \ldots (x^{r_\tau})^{m'_\tau}$ reduce (respectively cancel) in $\phi^\lambda$, the corresponding products $p_\psi(Y_1, \ldots, Y_\tau) = k.Y_1^{m_1} \ldots Y_\tau^{m_\tau}$ and $q_\psi(Y_1, \ldots, Y_\tau) = k'.Y_1^{m'_1} \ldots Y_\tau^{m'_\tau}$ also reduce (resp. cancel) in $\psi^\lambda$, where $m_j, m'_j \in \mathbb{N}$. Indeed, it is obvious that $p$ and $q$ combine (resp. cancel) if and only if $m_1 r_1 + \ldots + m_\tau r_\tau = m'_1 r_1 + \ldots + m'_\tau r_\tau$. Since $r_1, \ldots, r_\tau$ are linearly independent over $\mathbb{Q}$, this is possible if and only if $m_j = m'_j$ for $j = 1, \ldots, \tau$. And this is equivalent to the reducing (resp. cancelling) of $p_\psi$ and $q_\psi$. This proves that card$\Phi = $ card$\Psi$.

Note that $\psi$ has at least two terms because $\phi$ has at least two terms. This implies that for some $1 \leq j' \leq \tau$, $\psi$ as a polynomial in $Y_{j'}$ has at least two terms. Applying Lemma 2.2 to $\psi'(Y_{j'}) := \psi(1, \ldots, Y_{j'}, \ldots, 1) = \sum_{i=1}^l c_i Y_1^{a_{ij'}}$, we see that $(\psi')^\lambda$ has at least $\lambda + 1$ terms. Therefore $\psi^\lambda$ has at least $\lambda + 1$ terms and card$\Psi \geq \lambda + 1$. Therefore card$\Phi \geq \lambda + 1$. 

The next lemma is a consequence of formula (1.1).

**Lemma 2.4.** Let $\Lambda = (\lambda_k)_k$ and $\sum_k \frac{1}{\lambda_k} < \infty$. If a real-exponent polynomial $c_1 x^{s_1} + \ldots + c_l x^{s_l}$ belongs to $M^p_\Lambda$, then $s_1, \ldots, s_l \in \Lambda$.

**Proof.** Given $c_1 x^{s_1} + \ldots + c_l x^{s_l} \in M^p_\Lambda = L_\Lambda$, suppose on the contrary that some subset $\Lambda' = \{s_{k_1}, \ldots, s_{k_m}\} \subset \{s_1, \ldots, s_l\}$ does not belong to $\Lambda$ and $\{s_1, \ldots, s_l\} \setminus \Lambda' \subset \Lambda$. Then

$$p(x) = (c_1 x^{s_1} + \ldots + c_l x^{s_l}) - (c_{k_1} x^{s_{k_1}} + \ldots + c_{k_m} x^{s_{k_m}}) \in L_\Lambda.$$  

This implies that $c_{k_1} x^{s_{k_1}} + \ldots + c_{k_m} x^{s_{k_m}} = c_1 x^{s_1} + \ldots + c_l x^{s_l} - p(x) \in L_\Lambda \cap L_{\Lambda'}$. But $L_{\Lambda'} \cap L_\Lambda = \{0\}$ by (1.1), a contradiction. 

**Theorem 2.5.** Suppose $\Lambda = (\lambda_k)_k \subset \mathbb{N}$ with $\sum_k \frac{1}{\lambda_k} < \infty$. If $\phi$ is a real-exponent polynomial with more than one term, then $C_{\phi} M^p_\Lambda \not\subset M^p_\Lambda$.

**Proof.** Let $\phi(x) = c_1 x^{s_1} + \ldots + c_l x^{s_l}$ with $c_i \in \mathbb{R} \setminus \{0\}$ and $s_i \in \mathbb{R}^+$. Then for any $\lambda \in \Lambda$, we get $\phi(x^{\lambda}) = \lambda^\lambda$ which has at least $\lambda + 1$ terms by Lemma 2.3. We may assume that these $\lambda + 1$ terms are nonzero multiples of $x^{s_1 \lambda}, x^{s_1}, \ldots, x^{s_\lambda - 1}, x^{s_1 \lambda}$ where $s_1 \lambda < t_1 < \ldots < t_{\lambda - 1} < s_l \lambda$. Suppose that $C_{\phi} M^p_\Lambda \subset M^p_\Lambda$, then Theorem 2.4 gives us $s_1 \lambda, t_1, \ldots, t_{\lambda - 1}, s_l \lambda \in \Lambda$. We construct a subsequence $(\lambda_{k_j})_j$ of $\Lambda$ as follows: Let $\lambda_{k_1} = \lambda_1$ and
inductively choose $\lambda_{k_j}$ such that $s_1\lambda_{k_j} > s_l\lambda_{k_{j-1}}$ for $j \geq 2$. Then the sequence

$$\Lambda^* := \bigcup_{j=1}^{\infty} \{s_1\lambda_{k_j}, t_1, \ldots, t_{\lambda_{k_j}-1}, s_l\lambda_{k_j}\}$$

is increasing and has distinct elements; moreover, $\Lambda^* \subset \Lambda$. So

$$\sum_{k=1}^{\infty} \frac{1}{\lambda_k} \geq \sum_{s \in \Lambda^*} \frac{1}{s} \geq \sum_{j=1}^{\infty} \sum_{i=1}^{\lambda_{k_j}+1} \frac{1}{s_l\lambda_{k_j}} \geq \sum_{j=1}^{\infty} \frac{1}{s_l} = \infty$$

and hence the contradiction implies $C_\phi M^p_\Lambda \not\subset M^p_\Lambda$. \hfill \Box

**Corollary 2.6.** Let $\Lambda \subset \mathbb{N}$ and $\phi$ be a real-exponent polynomial. Then the following are equivalent:

1. $C_\phi M^p_\Lambda \subset M^p_\Lambda$
2. $\phi(x) = \alpha x^\eta$ and $\Lambda = \{1, \eta, \eta^2, \ldots\}$ for some $0 \leq \alpha \leq 1$ and $\eta \in \mathbb{R}^+$
3. $C_\phi : M^p_\Lambda \to M^p_\Lambda$ is a bounded operator.

**Proof.**

(i) $\Rightarrow$ (ii). Theorem 2.5 implies that $\phi(x) = \alpha x^\eta$ for $\eta \in \mathbb{R}^+$ and $0 \leq \alpha \leq 1$ because $\phi([0, 1]) \subset [0, 1]$. Then $C^m_\phi(x^\lambda) = C^{m-1}_\phi(\alpha^\lambda x^{\lambda \eta}) = \ldots = K x^{\lambda \eta^m} \in M^p_\Lambda$ for any $\lambda \in \Lambda$, $m \in \mathbb{N}$ and some constant $K$. Hence $\lambda \eta^m \in \Lambda$ for all $\lambda \in \Lambda$ and $m \in \mathbb{N}$ by Lemma 2.4. Therefore $\Lambda = \cup_{\lambda \in \Lambda} \lambda \{1, \eta, \eta^2, \ldots\} = \Lambda \{1, \eta, \eta^2, \ldots\}$.

(ii) $\Rightarrow$ (iii). Suppose $\phi(x) = \alpha x^\eta$ with $0 \leq \alpha \leq 1$ and $\eta \in \mathbb{R}^+$. If $\alpha < 1$, then $\mu = \phi^* m$ is supported on $[0, \alpha]$ and $d\mu|_{J_{1-\alpha}} = 0$. Hence $||\phi^*||_e = 0$ by Proposition 1.4 and $C_\phi = J \circ \phi^*_\mu$ is compact. For $\alpha = 1$, the measure $\mu = \phi^* m$ satisfies

$$\int_{J_\delta} f d\mu = \int_{\phi^{-1}(J_\delta)} f \circ \phi dm = \int_{J_\delta} f \cdot (\phi^{-1})' dm = \int_{1-\delta}^1 f(x) \eta^{-1} x^{\frac{1}{\eta} - 1} dx$$

for any continuous $f$ and $0 < \delta < 1$. Therefore $d\mu|_{J_\delta} = h dm|_{J_\delta}$ where $h(x) = \eta^{-1} x^{\frac{1}{\eta} - 1}$ is bounded on $J_\delta$, and hence $C_\phi$ is bounded by Proposition 1.4. Moreover, for any $\lambda \in \Lambda$ we see that $C_\phi x^\lambda = \alpha^\lambda x^{\lambda \eta} \in M^p_\Lambda$. Hence by the density of linear span of monomials $x^\lambda$ in $M^p_\Lambda$ and continuity of $C_\phi$, we get $C_\phi M^p_\Lambda \subset M^p_\Lambda$. The last part $(iii) \Rightarrow (i)$ is trivial. \hfill \Box

It is easy to see that Theorem 2.5 and Corollary 2.6 can be extended to the case when $\Lambda \not\subset \mathbb{N}$, but contains a subsequence of integers. To go beyond this case, we need some preparation about real-exponent power series.

**Lemma 2.7.** Suppose $f(x) = \sum_k a_k x^{s_k}$ is a series such that $(s_k)_k \subset \mathbb{R}^+$ is the finite union of sequences that satisfy the gap condition. Then $f$ is uniformly convergent on some interval $[0, \rho]$ if $L := \lim \sup_k |a_k|^{1/s_k} < \infty$. Furthermore, if $f \equiv 0$ on $[0, \rho_0]$ for $\rho_0 \leq \rho$ then $a_k = 0$ for all $k$.

**Proof.** It is sufficient to prove the first part for the case when $(s_k)_k$ itself satisfies the gap condition; in the general case, we can write $f$ as a finite sum of uniformly convergent series.
Since $|a_kx^{sk}|^{1/sk} = |a_k|^{1/sk}|x|$, we get $\limsup_k |a_kx^{sk}|^{1/sk} < 1$ if and only if $|x| < L^{-1}$ (taking $L^{-1} = \infty$ if $L = 0$). So, for $L|x| < 1$, we get $\limsup_k |a_kx^{sk}|^{1/sk} < r < 1$ for some $r$ and hence there exists a positive integer $N$ such that $|a_kx^{sk}|^{1/sk} < r$ for $k \geq N$. Therefore

$$\sum_{k \geq N} |a_kx^{sk}| \leq \sum_{k \geq N} r^k < \infty$$

where the convergence follows from the ratio test and the gap condition because

$$\lim_{k \to \infty} \frac{r^{k+1}}{r^k} = \lim_{k \to \infty} r^{s_k+1} \leq r^\inf(s_k+1) \leq 1.$$

So $f(x)$ converges absolutely for $L|x| < 1$, and in particular converges uniformly on $[0, \rho]$ for some $\rho > 0$.

For the second part, suppose on the contrary that $a_1$ is the first non-zero coefficient. We see that

$$f(x) = \sum_{k \geq 1} a_kx^{sk} = a_1x^{s_1}(1 + \sum_{k > 1} \frac{a_k}{a_1} x^{sk-s_1})$$

where $(s_k - s_1)_k$ is again a union of finitely many series satisfying the gap condition and

$$\limsup_k \left| \frac{a_k}{a_1} \right|^{s_1}_{s_k} \leq \limsup_k \left( \left| \frac{1}{a_1} \right|^{s_1}_{s_k} \right) = L < \infty$$

hence $g(x) = 1 + \sum_{k > 1} \frac{a_k}{a_1} x^{sk-s_1}$ converges uniformly on some interval $[0, \rho_1]$. So $f(x) = a_1x^{s_1} g(x) = 0$ on $[0, r]$, where $r = \min\{\rho_0, \rho_1\}$. Therefore $g = 0$ on $(0, r]$ and hence on $[0, r]$ by continuity. A contradiction, since $g(0) = 1$. \qed

**Theorem 2.8.** Suppose $\Lambda \subset \mathbb{R}^+$ with $\sum_k \frac{1}{\lambda_k} < \infty$ satisfies the gap condition $\inf_k (\lambda_{k+1} - \lambda_k) > 0$. If $\phi = \alpha x^{\xi_1} + \beta x^{\xi_2}$ with $\alpha, \beta \neq 0$ and $\xi_1 < \xi_2 \in \mathbb{R}^+$, then $C_\phi M^p_\Lambda \not\subseteq M_\Lambda^p$.

**Proof.** If $\Lambda \subset \mathbb{N}$, then Theorem 2.5 proves the result. So we assume $\Lambda \not\subseteq \mathbb{N}$, hence there exists $\lambda \in \Lambda$ that is not an integer. Suppose that $C_\phi M^p_\Lambda \subset M^p_\Lambda$; then

$$C_\phi(x^\lambda) = (\alpha x^{\xi_1} + \beta x^{\xi_2})^\lambda = \alpha^\lambda x^{\lambda\xi_1} (1 + \frac{\beta}{\alpha} x^{\xi_2-\xi_1})^\lambda \in M_\Lambda^p.$$ 

Hence by the binomial series we can represent $C_\phi(x^\lambda)$ as

$$C_\phi(x^\lambda)(t) = \alpha^\lambda t^{\lambda\xi_1} \sum_{k=0}^\infty a_k t^{k(\xi_2-\xi_1)} = \alpha^\lambda \sum_{k=0}^\infty a_k t^{\lambda\xi_1+k(\xi_2-\xi_1)}$$

where the series converges for $|t| < \frac{\alpha^\lambda}{\beta} |t|^{\frac{1}{\lambda\xi_1}}$, in particular on $[0, \eta]$ for some $\eta < 1$. The sequence of exponents $(\lambda\xi_1 + k(\xi_2 - \xi_1))_k$ clearly satisfies the gap condition, while the coefficients

$$a_k = \frac{\left( \frac{\beta}{\alpha} \right)^k \lambda(\lambda-1)(\lambda-2)\ldots(\lambda-k+1)}{k!}.$$
satisfy
\[ L_1 := \limsup_{k \to \infty} |a_k|^{1/\lambda_1 + k(\zeta_2 - \zeta_1)} < \infty. \]
Similarly, by Theorem 1.2 there exists a sequence of scalars \( b_k \in \mathbb{R} \) such that
\[ C_\phi(x^\lambda)(t) = \sum_{k=1}^\infty b_k t^{\lambda_k} \]
and the series converges uniformly on compact subsets of \([0, 1]\). By (1.2), the coefficients \((b_k)_k\) satisfy
\[ L_2 := \limsup_{k \to \infty} |b_k|^{1/\lambda_k} \leq \limsup_{k \to \infty} [(1 + \varepsilon)(2\lambda_k + 1)^{1/\lambda_k} |f|^{1/\lambda_k}] < \infty. \]
Since both series representations coincide on \([0, \eta]\), the series defined by
\[ f(t) = \sum_{k=1}^\infty b_k t^{\lambda_k} - \alpha^\lambda \sum_{k=0}^\infty a_k t^{\lambda_k + k(\zeta_2 - \zeta_1)} = \sum_{k} \gamma_k t^{s_k} \]
vanishes on \([0, \eta]\). Since \((s_k)_k\) is the union of two series satisfying the gap condition and \(\limsup_k |\gamma_k|^1/s_k \leq L_1 + L_2 < \infty\), by Lemma 2.7 we get \(\gamma_k = 0\) for all \(k\). Since \(\lambda\) is not an integer, all the \(a_k\) are non-zero; this implies that \(\lambda \zeta_1 + k(\zeta_2 - \zeta_1)\) \(\in \Lambda\) for all \(k\). This contradicts the fact that \(\sum_k 1/\lambda_k < \infty\) and hence \(C_\phi M^R_\Lambda \notin M^R_\Lambda\). \(\square\)

3. Composition Operators on \(M^2_\Lambda\): direct results

The next result is essentially contained in the work of Chalendar, Fricain and Timotin [4]:

**Proposition 3.1.** Suppose the Borel function \(\phi : [0, 1] \to [0, 1]\) satisfies the following:

(a) \(\phi^{-1}(1) = \{x_1, \ldots, x_k\}\) is finite.

(b) There exists \(\varepsilon > 0\) such that, for each \(i = 1, \ldots, k\), \(\phi\) is continuous on \((x_i - \varepsilon, x_i + \varepsilon)\), \(\phi \in C^1((x_i - \varepsilon, x_i))\) and \(\phi \in C^1((x_i, x_i + \varepsilon))\).

(c) \(\phi'_-(x_i) > 0\) and \(\phi'_+(x_i) < 0\) for all \(i = 1, \ldots, k\).

(\(\phi'_-(x)\) and \(\phi'_+(x)\) denote the left and right derivatives at \(x\) respectively, which may be infinite).

(d) There exists \(\alpha < 1\) such that, if \(x \notin \cup_{i=1}^k (x_i - \varepsilon, x_i + \varepsilon)\), then \(\phi(x) < \alpha\).

Then \(C_\phi : M^2_\Lambda \to L^2\) is bounded and \(\|C_\phi\|_e = \sum_{i=1}^k L(x_i)\), where
\[ L(x_i) = \begin{cases} \frac{1}{\phi'_-(x_i)} + \frac{1}{|\phi'_+(x_i)|} & \text{if } x_i \in (0, 1), \\ \frac{1}{\phi'_-(x_i)} & \text{if } x_i = 1, \\ \frac{1}{|\phi'_+(x_i)|} & \text{if } x_i = 0. \end{cases} \]
In particular, if \(\phi'_-(x_i) = \infty\) and \(\phi'_+(x_i) = -\infty\) for all \(i = 1, \ldots, k\), then \(C_\phi\) is compact.

We intend to go beyond the regularity assumptions in Proposition 3.1.
**Definition 3.2.** If \( \phi : [0, 1] \rightarrow [0, 1] \) is a Borel function and \( \alpha = \text{ess sup}_{[0,1]} \phi \), then a point \( x \in [0, 1] \) is an essential point of maximum for \( \phi \) if \( \text{ess sup}_E \phi = \alpha \) for every neighborhood \( E \) of \( x \). Denote by \( M_\phi \) the set of all essential points of maxima of \( \phi \), and by \( V_\varepsilon \) the neighborhood of \( M_\phi \) defined for each \( \varepsilon > 0 \) by

\[
V_\varepsilon = \{ x \in [0, 1] : \text{dist}(x, M_\phi) < \varepsilon \}.
\]

**Lemma 3.3.** The following statements are true:

(i) \( M_\phi \) is non-empty and closed,

(ii) \( \text{ess sup}_\varepsilon \phi |_{[0,1] \setminus V_\varepsilon} < \alpha \) for all \( \varepsilon > 0 \),

(iii) for every \( \varepsilon > 0 \) there exists a \( \delta_0 > 0 \) such that \( \phi^{-1}([\alpha - \delta, \alpha]) \subset V_\varepsilon \) almost everywhere whenever \( 0 < \delta < \delta_0 \).

**Proof.** (i). If \( M_\phi \) were empty, then every point \( x \in [0, 1] \) would have a neighborhood \( N_x \) such that \( \text{ess sup}_{N_x} \phi < \alpha \) and all such \( N_x \) would cover \([0, 1]\). Choosing a finite subcover so that \( \bigcup_{k=1}^m N_{x_k} = [0, 1] \), we see that

\[
\text{ess sup}_{[0,1]} \phi = \max_k \{ \text{ess sup}_{N_{x_k}} \phi \} < \alpha.
\]

The contradiction yields \( M_\phi \neq \emptyset \). To prove that \( M_\phi \) is closed, consider the set \( S := \bigcup_{x \in [0,1] \setminus M_\phi} N_x \), where \( N_x \) again represents a neighborhood of \( x \) on which \( \text{ess sup}_{N_x} \phi < \alpha \). So clearly \( S \) is open, and \( S \cap M_\phi = \emptyset \) since otherwise some \( N_{x'} \) for \( x' \in [0,1] \setminus M_\phi \) would contain an essential point of maximum. Hence \( S = [0,1] \setminus M_\phi \) and \( M_\phi \) is closed.

For (ii), suppose that \( \text{ess sup}_{[0,1] \setminus V_{\varepsilon'}} \phi = \alpha \) for some \( \varepsilon' > 0 \). Then the argument in the proof of (i) applied to the compact set \([0,1] \setminus V_{\varepsilon'}\), shows that it contains an essential point of maximum.

Finally for (iii), it follows from (ii) that for every \( \varepsilon > 0 \) there exists a \( \delta_0 > 0 \) such that \( \text{ess sup}_{[0,1] \setminus V_\varepsilon} \phi < \alpha - \delta_0 < \alpha \) and hence

\[
\phi^{-1}([\alpha - \delta, \alpha]) = \{ x \in [0, 1] : \alpha - \delta \leq \phi(x) \leq \alpha \} \subset V_\varepsilon
\]

except possibly for a subset of measure 0, whenever \( 0 < \delta < \delta_0 \). \( \square \)

We recall that the left and right derivatives of \( \phi \) at the point \( y \) are defined as

\[
D^-_i(y) = \liminf_{t \to y^-} \frac{\phi(y) - \phi(t)}{y - t}
\]

\[
D^+_i(y) = \liminf_{t \to y^+} \frac{\phi(y) - \phi(t)}{y - t}
\]

\[
D^-_s(y) = \limsup_{t \to y^-} \frac{\phi(y) - \phi(t)}{y - t}
\]

\[
D^+_s(y) = \limsup_{t \to y^+} \frac{\phi(y) - \phi(t)}{y - t}
\]

respectively.
Suppose \( \phi : [0, 1] \to [0, 1] \) is a Borel function such that \( \alpha = \text{ess sup}_{[0, 1]} \phi < 1 \). Then it is easy to show that the measure defined by \( \mu = \phi^* m \) has support in \( [0, \alpha] \). In fact
\[
\mu((\alpha, 1]) = \int_{(\alpha, 1]} d(\phi^* m) = \int_{\phi^{-1}((\alpha, 1])} dm = m(\phi^{-1}(\alpha, 1]) = 0.
\]

Hence in this case \( i^2 \mu \in S_q \) by Theorem 1.7(i). Therefore \( C_\phi \in S_q \) by Lemma 1.8 so from here onwards we assume that \( \alpha = \text{ess sup}_{[0, 1]} \phi = 1 \).

Since changing the values of \( \phi \) on a set of measure zero does not effect \( \mu = \phi^* m \), whenever \( m(\mathcal{M}_\phi) = 0 \), one may take \( \phi \equiv 1 \) on \( \mathcal{M}_\phi \). This will be assumed in the rest of the paper.

**Lemma 3.4.** Suppose \( \phi \) is a Borel function with \( \mathcal{M}_\phi = \{x_1, \ldots, x_k\} \) and \( \mu = \phi^* m \). If for some \( s \geq 1 \) there exists an \( \varepsilon > 0 \) and a constant \( c > 0 \) such that
\[
|x - x_i| \leq c|\phi(x) - 1|^s \quad \text{whenever} \quad |x - x_i| < \varepsilon
\]
for all \( i = 1, \ldots, k \), then there exists a \( \delta_0 > 0 \) such that \( \mu(J_\delta) \leq 2kc\delta^s \) whenever \( 0 < \delta < \delta_0 \).

**Proof.** By Lemma 3.3(iii), there exists a \( \delta_0 > 0 \) such that \( \phi^{-1}(J_\delta) \subseteq V_\varepsilon \) almost everywhere whenever \( 0 < \delta < \delta_0 \). Since \( \sup_{\phi^{-1}(J_\delta)}|\phi(x) - 1| \leq \delta \), we get
\[
m(\phi^{-1}(J_\delta)) \leq \sum_{i=1}^{k} m(\phi^{-1}(J_\delta) \cap \{\text{dist}(x, x_i) < \varepsilon\}) \leq 2 \sum_{i=1}^{k} \sup_{\phi^{-1}(J_\delta) \cap \{|x - x_i| < \varepsilon\}} |x - x_i| |\phi(x) - 1|^s \leq 2kc\delta^s.
\]
Therefore we get
\[
\mu(J_\delta) = \int_{J_\delta} d\mu = \int_{J_\delta} d(\phi^* m) = \int_{\phi^{-1}(J_\delta)} dm = m(\phi^{-1}(J_\delta)) \leq 2kc\delta^s \quad \text{whenever} \quad 0 < \delta < \delta_0.
\]

We arrive at the main theorem that gives necessary conditions for composition operators on \( M^2_A \) to be bounded, compact or in \( S_q \).

**Theorem 3.5.** Let \( \Lambda \) be lacunary and \( \mathcal{M}_\phi = \{x_1, \ldots, x_k\} \).

(i) If \( D_- > 0 \) and \( D_+ < 0 \) on \( \mathcal{M}_\phi \), then \( C_\phi : M^2_A \to L^2 \) is bounded.

(ii) If \( D_- = +\infty \) and \( D_+ = -\infty \) on \( \mathcal{M}_\phi \), then \( C_\phi : M^2_A \to L^2 \) is compact.

(iii) If for some \( \varepsilon > 0 \), \( \beta > 1 \) and constant \( c \) we have
\[
|x - x_i| \leq c|\phi(x) - 1|^\beta \quad \forall \quad |x - x_i| < \varepsilon \tag{3.1}
\]
for \( i = 1, \ldots, k \), then \( C_\phi \in S_q(M^2_A, L^2) \forall \ q > 0 \).
We conclude by presenting some results that serve as converses to the boundedness and compactness theorems given above for composition operators on \( M^2 \). Whenever \( |x - x_i| < \varepsilon \) for all \( i = 1, \ldots, k \). Hence by Lemma 3.4 we get \( \mu(J_\delta) \leq 2kM^{-1}\delta \) for \( 0 < \delta < \delta_0 \). Therefore \( \mu \) is sublinear and \( \imath_\mu^2 \) is bounded by Theorem 1.6(i). So Lemma 1.8 implies that \( C_\phi : M^2_\Lambda \to L^2 \) is bounded.

By our hypothesis, for any \( M > 0 \) there exists an \( \varepsilon > 0 \) such that

\[
\frac{|\phi(x) - 1|}{|x - x_i|} \geq M \quad \iff \quad |x - x_i| \leq M^{-1}|\phi(x) - 1|
\]

whenever \( |x - x_i| < \varepsilon \) for all \( i = 1, \ldots, k \). Hence by Lemma 3.4 we get \( \mu(J_\delta) \leq 2kM^{-1}\delta \) for \( 0 < \delta < \delta_0 \). Therefore \( \mu \) is sublinear and \( \imath_\mu^2 \) is bounded by Theorem 1.6(ii), and so is \( C_\phi = J \circ \imath_\mu^2 \).

Applying Lemma 3.4 directly to condition (3.1), we get \( \mu(J_\delta) \leq 2k\delta^\beta \) whenever \( 0 < \delta < \delta_0 \). Hence by Theorem 1.7(ii), \( \imath_\mu \in S_q(M^2_\Lambda, L^2(\mu)) \) for all \( q > 0 \). So \( C_\phi \in S_q(M^2_\Lambda, L^2) \) for all \( q > 0 \).

**Remark 3.6.** If \( \psi \in L^\infty \) then these results still hold true for the weighted composition operator \( M_\psi \circ C_\phi \) where \( M_\psi \) is the multiplication operator with symbol \( \psi \), which is a bounded operator on \( L^2 \).

### 4. Composition Operators on \( M^2_\Lambda \): Inverse results

We conclude by presenting some results that serve as converses to the boundedness and compactness theorems given above for composition operators on \( M^2_\Lambda \). We shall need the following two lemmas.

**Lemma 4.1.** Let \( \mu \) be a positive measure on \([0, 1]\). Then the following hold:

(i) If \( \imath_\mu^2 \) is bounded, then \( \lim \inf_{\delta \to 0} \mu(J_\delta) < \infty \)

(ii) If \( \imath_\mu^2 \) is compact, then \( \lim \inf_{\delta \to 0} \frac{\mu(J_\delta)}{\delta} = 0 \).

**Proof.** (i) Suppose \( \mu \) is \( \Lambda_2 \)-embedding. Since \( \lim_{n \to \infty}(1 - \frac{1}{\lambda_n})^{\lambda_n} = \frac{1}{e} \), there exists an integer \( N \) such that, for all \( n \geq N \) and for all \( x \in [1 - \frac{1}{\lambda_n}, 1] \), we have \( x^{\lambda_n} \geq \frac{1}{3} \). It follows that for all \( n \geq N \)

\[
\frac{1}{3^2} \mu(J_{1/\lambda_n}) \leq \int_{J_{1/\lambda_n}} x^{2\lambda_n} d\mu \leq ||\imath_\mu^2||^2 \int_0^1 x^{2\lambda_n} dx = \frac{||\imath_\mu^2||^2}{2\lambda_n + 1}.
\]

Therefore for all \( n \geq N \), we have

\[
\mu(J_{1/\lambda_n}) \leq \frac{3^2||\imath_\mu^2||^2}{\lambda_n} \quad \iff \quad \mu(J_{1/\lambda_n}) \leq 9||\imath_\mu^2||^2.
\]

This implies that \( \lim \inf_{\delta \to 0} \frac{\mu(J_\delta)}{\delta} < \infty \).
(ii) Choosing \( f_n(x) = \lambda_n^{1/2} x^{\lambda_n} \), we see that
\[
\langle f_n, x^{\lambda_k} \rangle = \int_{[0,1]} \lambda_n^{1/2} x^{\lambda_n + \lambda_k} \, dx = \frac{\lambda_n^{1/2}}{\lambda_n + \lambda_k + 1} \to 0
\]
as \( n \to \infty \) for all \( k \in \mathbb{N} \). Noting that \( \| f_n \|_{L^2} \) is bounded and the linear span of the sequence \( (x^{\lambda_k})_k \) is dense in \( M^2_{\Lambda} \), it follows that \( f_n \to 0 \) weakly in \( M^2_{\Lambda} \), as \( n \to \infty \). If \( i_\mu^2 \) is compact, this implies that \( (i_\mu^2 f_n)_n \) converges strongly to 0 in \( L^2(\mu) \) and hence \( \| f_n \|_{L^2(\mu)} \to 0 \) as \( n \to 0 \). Therefore
\[
\| f_n \|_{L^2(\mu)}^2 = \int_{[0,1]} \lambda_n x^{2\lambda_n} \, d\mu \geq \int_{J_{1/\lambda_n}} \lambda_n x^{2\lambda_n} \, d\mu \geq (1 - \frac{1}{\lambda_n})^{2\lambda_n} \frac{\mu(J_{1/\lambda_n})}{1/\lambda_n}.
\]
Since \( (1 - \frac{1}{\lambda_n})^{2\lambda_n} \to e^{-2} \) as \( n \to \infty \), we get
\[
\frac{\mu(J_{1/\lambda_n})}{1/\lambda_n} \to 0 \text{ as } n \to \infty
\]
and the result follows.

The next lemma might be compared to Lemma 3.4.

**Lemma 4.2.** Suppose \( \phi : [0,1] \to [0,1] \) is a Borel function and \( \mu = \phi^* m \). If for some \( x_0 \in [0,1] \) with \( \phi(x_0) = 1 \) and \( \eta > 0 \), there exists an \( \varepsilon > 0 \) such that
\[
x_0 - x > \frac{1}{\eta} (1 - \phi(x)) \text{ whenever } 0 < x_0 - x < \varepsilon,
\]
then \( \mu(J_\delta) \geq \frac{\delta}{\eta} \) for \( 0 < \delta < \eta \varepsilon \).

**Proof.** Since there exists an \( \varepsilon > 0 \) such that for \( 0 < x_0 - x < \varepsilon \), we have
\[
\frac{1 - \phi(x)}{x_0 - x} < \eta \iff 1 - \phi(x) < \eta(x_0 - x).
\]
Then suppose \( 0 < \delta < \delta_0 = \eta \varepsilon \). If \( 0 < x_0 - x < \frac{\delta}{\delta_0} \varepsilon = \frac{\delta}{\eta} \) then \( 1 - \phi(x) < \eta \frac{\delta}{\eta} = \delta \) which implies \( \phi(x) > 1 - \delta \). So \( \phi^{-1}(J_\delta) \) contains the interval \( (x_0 - \frac{\delta}{\eta}, x_0) \) of Lebesgue measure \( \frac{\delta}{\eta} \). Therefore
\[
m(\phi^{-1}(J_\delta)) \geq \frac{\delta}{\eta} \Rightarrow \mu(J_\delta) \geq \frac{\delta}{\eta} \Rightarrow \frac{\mu(J_\delta)}{\delta} \geq \frac{1}{\eta}.
\]

For the partial converses to parts (i) and (ii) of Theorem 3.6 we need neither lacunarity nor any assumption on \( \mathcal{M}_\phi \):

**Theorem 4.3.** Suppose \( \phi : [0,1] \to [0,1] \) is a Borel function, and \( \phi(x_0) = 1 \) for some \( x_0 \in [0,1] \).

(i) If \( C_\phi \) is bounded, then \( D^+(x_0) > 0 \) and \( D^i_+(x_0) < 0 \).

(ii) If \( C_\phi \) is compact, then \( D^+(x_0) = +\infty \) and \( D^i_+(x_0) = -\infty \).
Proof. (i) Suppose on the contrary that either $D_s^- (x_0) = 0$ or $D_i^+ (x_0) = 0$. We shall deduce a contradiction for one of these cases since both are analogous. So suppose $D_s^- (x_0) = 0$, that is
\[
\lim_{x \to x_0^-} \frac{1 - \phi(x)}{x_0 - x} = 0.
\]
For each $\eta > 0$ there exists an $\varepsilon > 0$ such that for $0 < x_0 - x < \varepsilon$, we have
\[
\frac{1 - \phi(x)}{x_0 - x} < \eta.
\]
Therefore by Lemma 4.2 we get $\frac{\mu(J_\delta)}{\delta} \geq \frac{1}{\eta}$ whenever $\delta < \eta \varepsilon$. So $\frac{\mu(J_\delta)}{\delta} \to +\infty$ as $\delta \to 0$. Since $C_\phi$ is bounded we get that $\mu$ is $\Lambda_2$-embedding by Lemma 1.8. This leads to a contradiction since Lemma 4.1 gives $\lim_{\delta \to 0} \frac{\mu(J_\delta)}{\delta} < \infty$.

For (ii), suppose to the contrary that either $D_s^- (x_0) < +\infty$ or $D_i^+ (x_0) > -\infty$ for some $x_0 \in \mathcal{M}_\phi$. Again due to similarities we shall deal with one case. So suppose $D_s^- (x_0) < \infty$, that is
\[
\lim_{x \to x_0^-} \frac{1 - \phi(x)}{x_0 - x} < \infty.
\]
So there exists a $\zeta > 0$ and an $\varepsilon > 0$ such that for $0 < x_0 - x < \varepsilon$, we have
\[
\frac{1 - \phi(x)}{x_0 - x} < \zeta.
\]
Therefore by Lemma 4.2 we get $\frac{\mu(J_\delta)}{\delta} \geq \frac{1}{\zeta}$ for $\delta < \zeta \varepsilon$. This contradicts Lemma 4.1 because $i^2_\mu$ is compact by Lemma 1.8.

Corollary 4.4. Suppose $\phi$ is a polynomial with $\phi^{-1}(1)$ non-empty. Then $C_\phi$ is not compact, and if it is bounded then $\phi^{-1}(1) \subset \{0, 1\}$.

Proof. If $C_\phi$ is bounded and some $x_0 \in \phi^{-1}(1)$ is an interior point of $[0, 1]$, then clearly $x_0$ must be a local maximum and hence $\phi'(x_0) = 0$. This contradicts Theorem 4.3 (i) and hence $\phi^{-1}(1) \subset \{0, 1\}$. Similarly, by part (ii) of the theorem we get the conclusion that $C_\phi$ can never be compact because $\phi$ is differentiable everywhere.

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