EXTREMAL ORDERS OF THE ZECKENDORF SUM OF DIGITS OF POWERS

THOMAS STOLL

Abstract. Denote by $s_F(n)$ the minimal number of Fibonacci numbers needed to write $n$ as a sum of Fibonacci numbers. We obtain the extremal minimal and maximal orders of magnitude of $s_F(n^h)/s_F(n)$ for any $h \geq 2$. We use this to show that for all $N > N_0(h)$ there is a $n$ such that $n$ is the sum of $N$ Fibonacci numbers and $n^h$ is the sum of at most $130h^2$ Fibonacci numbers. Moreover, we give upper and lower bounds on the number of $n$'s with small and large values $s_F(n^h)/s_F(n)$. This extends a problem of Stolarsky to the Zeckendorf representation of powers, and it is in line with the classical investigation of finding perfect powers among the Fibonacci numbers and their finite sums.

1. Introduction

Denote by $s_q(n)$ the sum of digits in the usual $q$-ary digital expansion of $n$. Stolarsky [11] studied the maximal and minimal order of magnitude of the ratio $s_2(n^h)/s_2(n)$ for fixed $h \geq 2$. It is reasonable to expect that the quantities $s_2(n^h)$ and $s_2(n)$ are independent in the sense that the lim sup of the ratio tends to $\infty$ and the lim inf to 0 as $n$ tends to infinity. It is an interesting question to find the extremal orders of magnitude of this ratio. In a recent work, Hare, Laishram and the author [9] were able to settle an open question of Stolarsky, so to finally get a complete picture of the maximal and minimal order of magnitude of the ratio $s_q(n^h)/s_q(n)$.

Theorem 1.1 ([11, 9]). There exist $c_1$ and $c_2$, depending at most on $q$ and $h$, such that for all $n \geq 2$,

$$\frac{c_2}{\log n} \leq \frac{s_q(n^h)}{s_q(n)} \leq c_1(\log n)^{1-1/h}.$$

This is best possible in that there exist $c'_1$ and $c'_2$, depending at most on $q$ and $h$, such that

$$s_q(n^h)/s_q(n) > c'_1(\log n)^{1-1/h},$$

1991 Mathematics Subject Classification. Primary 11B39. Secondary 11N56, 11A63.

Key words and phrases. Sequences and sets; Digital expansions.
respectively,

\begin{equation}
\frac{s_q(n^h)}{s_q(n)} < \frac{c_2}{\log n}
\end{equation}

infinitely often.

In the present paper we find the maximal and minimal order of magnitude of the ratio $s_F(n^h)/s_F(n)$, where $s_F$ denotes the Zeckendorf sum of digits function, and we give a Diophantine application. Let

\begin{equation}
x = \sum_{2 \leq j \leq n} \varepsilon_j F_j,
\end{equation}

with $\varepsilon_n = 1$ and $\varepsilon_j \in \{0, 1\}$ be the (greedy) Zeckendorf expansion of $x \in \mathbb{Z}^+$ with respect to the Fibonacci numbers $F_j$. Recall that in this expansion we do not allow adjacent 1 digits [13, 1]. Hence $x$ can have at most $\lfloor n/2 \rfloor$ digits 1 in its expansion. We write $x = (\varepsilon_n \varepsilon_{n-1} \ldots \varepsilon_2)_F$ to refer to this expansion. Denote by $s_F$ the Zeckendorf sum of digits function defined by

\begin{equation}
s_F(x) = \sum_{2 \leq j \leq n} \varepsilon_j.
\end{equation}

This function can also be interpreted as the minimal number of Fibonacci numbers needed to write $n$ as a sum of Fibonacci numbers. $s_F$ shares many properties with the ordinary sum of digits function $s_q$. For instance, $s_F$ is also subadditive (i.e., $s_F(a + b) \leq s_F(a) + s_F(b)$ for all $a, b \geq 1$), has fractal summatory behaviour [6] and satisfies a Newman phenomenon [8]. Contrary to $s_q$ [9], the function $s_F$ is not submultiplicative, as the example

\[ 2 \cdot 3 = (10)_F \cdot (100)_F = (1001)_F = 6 \]

shows. Therefore, there is a priori no obvious relation between $s_F(n^h)$ and $s_F(n^h)$. Drmota and Steiner [7], extending a result of Bassily and Kátai [2], showed that $s_F(n^h)$ properly renormalized is asymptotically normally distributed. The mean value of $s_F(n^h)$ is asymptotically $h$ times the mean value of $s_F(n)$ which is $c_F \log n$ with a suitable constant $c_F$ [10]. This means that we expect $n^h$ to have roughly $h$ times as many 1’s in the Zeckendorf expansion compared to $n$, thus the ratio $s_F(n^h)/s_F(n)$ should be roughly $h$. Our main result is as follows.

**Theorem 1.2.** There exist $c_3$ and $c_4$, depending at most on $h$, such that for all $n \geq 2$,

\begin{equation}
\frac{c_4}{\log n} \leq \frac{s_F(n^h)}{s_F(n)} \leq c_3 \log n.
\end{equation}
This is best possible in that there exist $c'_3$ and $c'_4$, depending at most on $h$, such that

\begin{equation}
\frac{s_F(n^h)}{s_F(n)} > c'_3 \log n
\end{equation}

respectively,

\begin{equation}
\frac{s_F(n^h)}{s_F(n)} < c'_4 \frac{1}{\log n},
\end{equation}

infinitely often. Moreover, possible values for the constants are

\begin{equation}
c_3 = 2h, \quad c'_3 = 1, \quad c_4 = \frac{1}{2}, \quad c'_4 = 120h^2.
\end{equation}

This is strongly related to the classical investigation of finding perfect powers among Fibonacci numbers and their finite sums. A recent deep result of Bugeaud, Mignotte and Siksek [4] says that the only powers $n^h$ that are Fibonacci numbers (or equivalently, with $s_F(n^h) = 1$), are 1, 8 and 144. From [3], [7] and our construction we obtain the following Diophantine result.

**Theorem 1.3.** For any $h \geq 2$ there exists $N_0(h)$, only depending on $h$, such that for all $N > N_0$ there exists an integer $n$ with the following two properties:

(i) $n$ is the sum of $N$ distinct, non-adjacent Fibonacci numbers.

(ii) $n^h$ is the sum of at most $130h^2$ Fibonacci numbers.

Recently, Bugeaud, Luca, Mignotte and Siksek [3] found all powers which are at most one away from a Fibonacci number. In our context, this is the investigation of finding powers with very large and very small sum of digits values. A refinement of our construction yields that $s_F(n^h)$ is small and large indeed quite often compared to $s_F(n)$.

**Theorem 1.4.** For $\varepsilon > 0$ there exists

\begin{equation}
\alpha > \frac{1}{\max(36h^2/\varepsilon + 18, 8h + 1)}
\end{equation}

such that

\begin{equation}
\#\{n < N : \frac{s_F(n^h)}{s_F(n)} < \varepsilon\} \gg N^\alpha.
\end{equation}

**Theorem 1.5.** For $\delta > 0$ there exists

\begin{equation}
\beta > \frac{1}{h(\delta + 1) + 2}
\end{equation}

such that

\begin{equation}
\#\{n < N : \frac{s_F(n^h)}{s_F(n)} > \delta\} \gg N^\beta.
\end{equation}
2. Preliminaries

Since \( F_n = \lfloor \phi^n / \sqrt{5} \rfloor \), where \( \phi = \frac{1}{2}(\sqrt{5} + 1) \) is the golden ratio, we have by (3) that

\[
\frac{\phi^{n-3/2}}{\sqrt{5}} < \lfloor \frac{\phi^n}{\sqrt{5}} \rfloor \leq \frac{\phi^{n+1}}{\sqrt{5}}
\]

for \( n \geq 2 \). Therefore,

(10) \[ n = \frac{\log x}{\log \phi} + \gamma_n, \]

where \( \gamma_n \) lies in the interval

\[ (\delta, \delta') := \left( \frac{\log \sqrt{5}}{\log \phi} - 1, \frac{\log \sqrt{5}}{\log \phi} + \frac{3}{2} \right) \approx (0.672, 3.172). \]

This already implies (11) with \( c_3 = 2h \) and \( c_4 = \frac{1}{2} \).

In the following we show that subtracting a “small” number from a Fibonacci number gives rise to a large number of digits 1 in the Zeckendorf expansion.

**Lemma 2.1.** Let \( k \geq 1 \).

(i) For \( 0 < z \leq F_{2k+1} \) we have

\[ s_F(F_{2k+1} - z) = k - l + s_F(F_{2l+1} - z) \geq k - \frac{\log z}{2 \log \phi} - \frac{\delta'}{2}, \]

where \( l \) is such that \( F_{2l} < z \leq F_{2l+1} \).

(ii) For \( 0 < z \leq F_{2k} \) we have

\[ s_F(F_{2k} - z) = k - l + s_F(F_{2l} - z) \geq k - \frac{\log z}{2 \log \phi} - \frac{\delta'}{2}, \]

where \( l \) is such that \( F_{2l-1} < z \leq F_{2l} \).

**Proof.** Part (i) follows at once from the identity

\[ F_{2k+1} - z = \left( \sum_{i=l+1}^{k} F_{2i} + F_{2l+1} \right) - z = \sum_{i=l+1}^{k} F_{2i} + (F_{2l+1} - z) \]

and (10). The second case is similar. \( \square \)
Denote by $L_k$ the Lucas numbers defined by

\[(11)\]

\[L_k = F_{k-1} + F_{k+1} = \lfloor \phi^k \rfloor.\]

Powers and products of Lucas numbers are given by the following formulae.

**Lemma 2.2.** For all $k > l \geq 1$ and $h \geq 2$ we have

\[(12)\]

\[L_h^k = \frac{1}{2} \sum_{i=0}^{h} \binom{h}{i} (-1)^i L_{(h-2i)k},\]

\[(13)\]

\[L_k L_l = L_{k+l} + (-1)^l L_{k-l}.\]

**Proof.** See for example [12]. \hfill \Box

Formula (12) shows that powers of odd indexed Lucas numbers can be written as linear sum of Lucas numbers having positive coefficients. Furthermore, from (13) we have that products of two even indexed Lucas numbers can be rewritten as sums of two single Lucas numbers. We will further need the fact that fixed multiples of Lucas numbers have bounded sum of digits values.

**Lemma 2.3.** For $m \geq 1$ there exists $k_0 = k_0(m)$ such that for all $k \geq k_0$,

\[s_F(mL_k) < \frac{\log m}{\log \phi} + 3.\]

**Proof.** Since $F_1 L_k = F_{l+k} - (-1)^l F_{k-l}$ we have that

\[F_{2l+1} L_k = F_{k+2l-1} + F_{k-2l+1},\]

\[F_{2l} L_k = F^2 - F_{k-2l} = F_{k-2l+3} + \cdots + F_{k+2l-1}.\]

Hence, by writing $m$ in Zeckendorf representation we get that for all $m$ with $F_{2l} < m < F_{2l+1}$ the Zeckendorf representation of $mL_k$ involves a block of $4l + 2$ digits (k sufficiently large) and a following block of zeros only, and for all $m$ with $F_{2l+1} \leq m \leq F_{2l+2}$ a block of $4l + 3$ digits with a block of zeros appended. This yields that for each $m \geq 1$ and $k \geq k_0$ a block of length at most

\[(14)\]

\[\frac{2 \log \sqrt{5} m}{\log \phi} + 2\]

appears in the representation. Thus,

\[s_F(mL_k) \leq \frac{\log \sqrt{5} m}{\log \phi} + 1 < \frac{\log m}{\log \phi} + 3,\]

which proves the claim. \hfill \Box
3. Proof of the extremal upper bound

We use a construction of an extremal sequence based on the power expansion of Lucas numbers ([12]). Set \( n_k = L_{2k-1} \) for \( k \geq 1 \). Then by (11) we have \( s_F(n_k) = 2 \). For the proof of (11) it suffices to show that \( s_F(n_k^h) = 2k + O_h(1) \), where the implied constant depends only on \( h \). We have

\[
n_k^h = F_{h(2k-1)+1} + F_{h(2k-1)-1}
\]

(15)

The last sum is positive since

\[
\frac{1}{2} \sum_{i=1}^{h-1} \binom{h}{i} (-1)^{(i+1)(2k-1)} L_{(h-2i)(2k-1)} = \left\lfloor \frac{\phi^{h(2k-1)}}{\sqrt{5}} \right\rfloor - \left\lfloor \frac{\phi^{2k-1}}{\sqrt{5}} \right\rfloor^h
\]

\[
\geq \phi^{h(2k-1)} \left( \frac{1}{\sqrt{5}} - \frac{1}{\sqrt{5}} \right) - 1 > 0.
\]

Moreover, this quantity is small with respect to the leading term. In fact, we get by a trivial estimate

\[
\frac{1}{2} \sum_{i=1}^{h-1} \binom{h}{i} (-1)^{(i+1)(2k-1)} L_{(h-2i)(2k-1)} \leq 2^{h-1} L_{(h-2)(2k-1)} \leq 2^{h-1} \phi^{(h-2)(2k-1)}
\]

which is smaller than \( F_{h(2k-1)-1} \) for sufficiently large \( k \). Therefore, using Lemma 2.1 we get

\[
s_F(n_k^h) \geq 1 + \left\lfloor \frac{h(2k-1) - 1}{2} \right\rfloor - \frac{\log(2^{h-1} \phi^{(h-2)(2k-1)})}{2 \log \phi} - \frac{\delta'}{2}
\]

\[
\geq 2k - \frac{h-1}{2} \cdot \frac{\log 2}{\log \phi} - \frac{3}{2} \cdot \frac{\delta'}{2}
\]

\[
\geq 2k - \frac{3h}{4} - 3,
\]

for \( k \) sufficiently large. Therefore, as \( k \) tends to infinity,

\[
\frac{s_F(n_k^h)}{s_F(n_k)} \geq k - \frac{3h}{8} - \frac{3}{2}
\]

\[
\geq \frac{\log n_k}{2 \log \phi} + \frac{1}{2} - \frac{3h}{8} - \frac{3}{2} \gg \log n_k.
\]

Hence, we can put \( c_3' = 1 \) and get (11). \( \square \)
4. Proof of the extremal lower bound

Here, we use a construction which uses (13). Let \( k \geq 1 \) and set

\[
n_k = L_{8k} + L_{6k} + L_{4k} + L_{2k} - 1.
\]

We have

\[
s_F(n_k) = 6 + s_F(L_{2k} - 1) = 6 + s_F(F_2 + F_4 + \cdots + F_{2k-2} + F_{2k+1}) = 6 + k.
\] (16)

First we calculate the Zeckendorf expansion of \( n_{2k} = (L_{8k} + L_{6k} + L_{4k} + L_{2k} - 1)^2 \). We expand the square by employing (13) and use the special value \( L_0 = 2 \) to get

\[
n_{2k}^2 = L_{16k} + 2L_{14k} + 3L_{12k} + 4L_{10k} + L_{8k} + 2L_{6k} + 3L_{4k} + 4L_{2k} + 9.
\]

We replace all appearances of multiples of Lucas numbers by the corresponding linear sum in Fibonacci numbers. In this case, we use

\[
\begin{align*}
2L_k &= F_{k+3} + F_{k-3}, \\
3L_k &= F_{k+3} + F_{k+1} + F_{k-1} + F_{k-3}, \\
4L_k &= F_{k+4} + F_{k+1} + F_{k-2} + F_{k-5}.
\end{align*}
\]

It is now a straightforward calculation to write down the expansion of \( n_{2k}^2 \). In order to simplify notation, denote by \( (e_p e_{p-1} \ldots e_0)_l \) the sum of Fibonacci numbers \( e_p F_{p+1} + e_{p-1} F_{p-1+l} + \cdots + e_0 F_1 \). We get

\[
n_{2k}^2 = (101)_{16k-1} + (1000001)_{14k-3} + (1010101)_{12k-3} \\
+ (1001001001)_{10k-5} + (101)_{8k-1} + (1000001)_{6k-3} \\
+ (1010101)_{4k-3} + (1001001001)_{2k-5} + (10001)_2.
\] (17)

Thus, we have \( s_F(n_{2k}^2) = 26 \) for all \( k \geq 7 \).

In a similar style we obtain the expansion for \( n_{3k}^3 \). This time we use (13) twice to rewrite all products of three Lucas numbers as sums of four Lucas numbers. We have

\[
n_{3k}^3 = L_{24k} + 3L_{22k} + 6L_{20k} + 10L_{18k} + 9L_{16k} + 9L_{14k} + 10L_{12k} \\
+ 12L_{10k} + 27L_{8k} + 28L_{6k} + 27L_{4k} + 24L_{2k} + 11.
\]
Similarly as before we replace multiples of Lucas numbers by sums of Fibonacci numbers. We get

\[ n_k^3 = (101)_{24k-1} + (1010101)_{22k-3} + (10001010001)_{20k-5} \\
+ (10010000001001)_{18k-7} + (10000100101001)_{16k-7} \\
+ (10000100101001)_{14k-7} + (10010000001001)_{12k-7} \\
+ (1010010010001)_{10k-7} + (10010001010001)_{8k-9} \\
+ (1001010000101001)_{6k-9} + (100100010100001001)_{4k-9} \\
+ (100001010100101001)_{2k-9} + (10100) \\
\]  
(18)

For \( k \geq 10 \) the summands in (18) are noninterfering. This yields \( s_F(n_k^3) = 60 \) for \( k \geq 10 \). Note that (17) and (18) already prove (6) in the case of \( h = 2 \) and \( h = 3 \).

The general case follows from (17), (18) and Lemma 2.3. For that purpose set \( h = 2h_1 + 3h_2 \) with \( h_1, h_2 \geq 0 \) and consider \( n_k^h = (n_k^{h_1})(n_k^{h_2}) \). Since both \( n_k^{h_1} \) and \( n_k^{h_2} \) are linear forms in Lucas numbers with fixed positive coefficients, the powers \( (n_k^{h_1}) \) and \( (n_k^{h_2}) \) are linear forms with positive coefficients, too, that are independent of \( k \). Thus we have that \( n_k^h \) is a linear form in 4h Lucas numbers with positive coefficients independent of \( k \) (plus an additive constant). This means that there exists \( k_0 = k_0(h) \) such that for all \( k \geq k_0 \) the terms in the Lucas sum are noninterfering. All coefficients in this sum are bounded by 9h. Therefore, by Lemma 2.3

\[ s_F(n_k^h) \leq \left( h \frac{\log 9}{\log \phi} + 3 \right) \cdot (4h + 1). \]

Since \( \phi^{8k} < n_k < \phi^{8k+1} \) we also get

\[ s_F(n_k) = k + 6 \geq \frac{\log n_k}{8 \log \phi} - \frac{1}{8} + 6 \gg \frac{1}{4} \log n_k. \]

This shows that for sufficiently large \( k \),

\[ \frac{s_F(n_k^h)}{s_F(n_k)} < \frac{4(5h + 3)(4h + 1)}{\log n_k} < \frac{120h^2}{\log n_k}. \]

This completes the proof of (6) with \( c'_4 = 120h^2 \). \( \square \)

**Proof of Theorem 1.3** This follows at once from (16) and

\[ s_F(n) \leq \frac{\log n}{2 \log \phi} + 2 \ll \frac{13}{12} \log n. \]

\( \square \)
5. Proofs of Theorems 1.4 and 1.5

Proof of Theorem 1.4: For $m \geq 1$ set

$$n_k = n_k(m) = m(L_{8k} + L_{6k} + L_{4k} + L_{2k} - 1).$$

As before, we have that $n_k^h$ is a linear sum of Lucas numbers with positive coefficients independent of $k$. Suppose now

$$k > \frac{h \log(9m)}{\log \phi} + O(1).$$

Then the blocks in the expansion of $n_k$ respectively $n_k^h$ are noninterfering. Using (14) we have

$$s_F(n_k) \geq k - \frac{2 \log m}{\log \phi} + O(1)$$

and

$$s_F(n_k^h) \leq \left(\frac{h \log(9m)}{\log \phi} + 3\right)(4h + 1).$$

Let $k_0$ be sufficiently large such that

$$\left(\frac{h \log(9m)}{\log \phi} + 3\right)(4h + 1) < \varepsilon \left(k_0 - \frac{2 \log m}{\log \phi} + O(1)\right)$$

and set $m = \phi^\gamma$. Then for any $\gamma$ sufficiently large we find $k_0$ such that $n_{k_0} < m\phi^{8k_0+1}$ satisfies

$$\frac{s_F(n_{k_0}^h)}{s_F(n_{k_0})} < \varepsilon.$$

By a direct calculation one can check that each $k = k_0$ with (20) also satisfies (19) provided

$$\varepsilon < \frac{h(4h + 1)}{h - 2},$$

where (21) is empty for $h = 2$. By construction, each distinct $m$ will give rise to a distinct $n$. We therefore have for $\gamma$ sufficiently large,

$$\alpha > \frac{\gamma}{8k_0 + \gamma + O(1)} > \frac{\gamma}{2(4h + 1)(h \log \phi + h\gamma + 3) + 16\gamma + \gamma + O(1)} > \frac{1}{36h^2/\varepsilon + 18}.$$

Now, suppose $\varepsilon \geq h(4h + 1)/(h - 2)$. Then we conclude

$$\alpha > \frac{\gamma}{8k_0 + \gamma + O(1)} > \frac{\gamma}{8h \log \phi + 8h\gamma + O(1)} > \frac{1}{8h + 1}.$$

This completes the proof of Theorem 1.4. □

Proof of Theorem 1.5: Let $n_k = mL_{2k-1}$. With the help of (15) we see that $n_k^h$ can be written as the difference of $m^hL_{h(2k-1)}$ and a positive number that
is bounded by $m^h2^{h-1}\phi^{(h-2)(2k-1)}$. In order to have terms noninterfering we suppose that $k$ is such that
\[m^h2^{h-1}\phi^{(h-2)(2k-1)} \leq F_h(2k-1)-1,\]
or equivalently,
\[
(22) \quad k > \frac{h \log(2m^2)}{4 \log \phi} + O(1).
\]
Lemma 2.3 shows that for all such $k$ we have
\[s_F(n_k) \leq \frac{h \log m}{\log \phi} + O(1).
\]
On the other hand, a similar calculation as in Section 3 gives
\[s_F(n_k^h) \geq 1 + \left[ \frac{h(2k-1) - 1 - h \log m/\log \phi + O(1)}{2} - \frac{\log \left(m^h2^{h-1}\phi^{(h-2)(2k-1)}\right)}{2 \log \phi} - \frac{\delta'}{2} \right]
\[\geq 2k - \frac{h}{\log \phi} \left( \log m + \frac{\log 2}{2} \right) + O(1).
\]
We now choose $k_0$ in a way that
\[2k_0 - \frac{h}{\log \phi} \left( \log m + \frac{\log 2}{2} \right) + O(1) > \delta \left( \frac{h \log m}{\log \phi} + O(1) \right).
\]
Observe that for any $\delta > 0$ each such $k = k_0$ automatically satisfies (22). Put $m = \phi^\gamma$. Similarly as above we get for $\gamma$ sufficiently large,
\[\beta > \frac{\gamma}{2k_0 + \gamma + O(1)} > \frac{\gamma}{\delta h + O(1)} + h\gamma + \frac{h \log 2}{2 \log \phi} + \gamma + O(1)
\[> \frac{1}{\delta h + h + 2}.
\]
This completes the proof of Theorem 1.5. \hfill \Box

References

[1] J.-P. Allouche, J. Shallit, Automatic sequences: Theory, applications, generalizations, Cambridge University Press, Cambridge, 2003.
[2] N. L. Bassily, I. Kátaí, Distribution of the values of $q$-additive functions on polynomial sequences, *Acta Math. Hung.* 68 (1995), 353–361.
[3] Y. Bugeaud, F. Luca, M. Mignotte, S. Siksek, Fibonacci numbers at most one away from a perfect power, *Elem. Math.* 63 (2008), no. 2, 65–75.
[4] Y. Bugeaud, M. Mignotte, S. Siksek, Classical and modular approaches to exponential Diophantine equations. I. Fibonacci and Lucas perfect powers, *Ann. of Math. (2)* 163 (2006), no. 3, 969–1018.
[5] R. C. Bose, S. Chowla, Theorems in the additive theory of numbers, *Comm. Math. Helv.* 37 (1962/63), 141–147.
THE ZECKENDORF SUM OF DIGITS OF POWERS

[6] J. Coquet, P. Van Den Bosch, A summation formula involving Fibonacci digits, *J. Number Theory* **22** (1986), no. 2, 139–146.

[7] M. Drmota, W. Steiner, The Zeckendorf expansion of polynomial sequences, *J. Théor. Nombres Bordeaux* **14** (2002), no. 2, 439–475.

[8] M. Drmota, M. Skalba, The parity of the Zeckendorf sum-of-digits function, *Manuscripta Math.* **101** (2000), no. 3, 361–383.

[9] K. G. Hare, S. Laishram, T. Stoll, Stolarsky’s conjecture and the sum of digits of polynomial values, *Proc. Amer. Math. Soc.*, to appear (2010), arXiv:1001.4169.

[10] A. Pethő, R. F. Tichy, On digit expansions with respect to linear recurrences, *J. Number Theory* **33** (1989), no. 2, 243–255.

[11] K. B. Stolarsky, The binary digits of a power, *Proc. Amer. Math. Soc.* **71** (1978), 1–5.

[12] S. Vajda, *Fibonacci and Lucas numbers, and the Golden Section: Theory and Applications*, Dover Press (2008).

[13] E. Zeckendorf, Représentation des nombres naturels par une somme de nombres de Fibonacci ou de nombres de Lucas, *Bull. Soc. Roy. Sci. Liège* **41** (1972), 179–182.

Institut de Mathématiques de Luminy, Université de la Méditerranée, 13288 Marseille Cedex 9, France.

E-mail address: stoll@iml.univ-mrs.fr