Gromov-Witten Invariants in Algebraic Geometry

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Abstract

Gromov-Witten invariants for arbitrary projective varieties and arbitrary genus are constructed using the techniques from [K. Behrend, B. Fantechi. The Intrinsic Normal Cone.]

Introduction

In [2] the problem of constructing the Gromov-Witten invariants of a smooth projective variety $V$ was reduced to defining a ‘virtual fundamental class’

$$[\mathcal{M}_{g,n}(V,\beta)]^{\text{virt}} \in A_{1-g}(\dim V - 3 - \beta(\omega_V) + n)(\mathcal{M}_{g,n}(V,\beta))$$

in the Chow group of the algebraic stack $\mathcal{M}_{g,n}(V,\beta)$ of stable maps of class $\beta \in H_2(V)$ from an $n$-marked prestable curve of genus $g$ to $V$.

If $g = 0$ and $V$ is convex (i.e. $H^1(\mathbb{P}^1, f^*T_V) = 0$, for all $f : \mathbb{P}^1 \to V$), then $\mathcal{M}_{0,n}(V,\beta)$ is smooth of the expected dimension $\dim V - 3 - \beta(\omega_V) + n$ and the usual fundamental class

$$[\mathcal{M}_{g,n}(V,\beta)]$$

will work. This was proved in [2].

In this paper we treat the general case using the construction from [1]. Recall from [ibid.] that virtual fundamental classes are constructed using an obstruction theory, and the intrinsic normal cone. The obstruction theory serves to give rise to a vector bundle stack $\mathcal{E}$, into which the intrinsic normal cone $\mathcal{C}$ can be embedded as a closed subcone stack. The virtual fundamental class is then obtained by intersecting $\mathcal{E}$ with the zero section of $\mathcal{E}$.
In our context, this process works as follows. Let $\mathcal{M}_{g,n}$ be the algebraic stack of $n$-marked prestable curves of genus $g$. This is an algebraic stack, not of Deligne-Mumford (or even finite) type, but smooth of dimension $3(g-1)+n$. There is a canonical morphism

$$\overline{\mathcal{M}}_{g,n}(V,\beta) \to \mathcal{M}_{g,n},$$

given by forgetting the map, retaining the curve (but not stabilizing). Then $\overline{\mathcal{M}}_{g,n}(V,\beta) \to \mathcal{M}_{g,n}$ is an open substack of a stack of morphisms, and as such has a relative obstruction theory, which in this case is $(R\pi_* f^* T_V)^\vee$, where $\pi: C \to \overline{\mathcal{M}}_{g,n}(V,\beta)$ is the universal curve and $f: C \to V$ is the universal stable map. Saying that $(R\pi_* f^* T_V)^\vee$ is a relative obstruction theory means that there is a homomorphism

$$\phi: (R\pi_* f^* T_V)^\vee \to L^*_{\overline{\mathcal{M}}_{g,n}(V,\beta)/\mathcal{M}_{g,n}},$$

(where $L^*$ is the cotangent complex) such that $h^0(\phi)$ is an isomorphism and $h^{-1}(\phi)$ is surjective.

The homomorphism $\phi$ induces a closed immersion

$$\phi^\vee: \mathfrak{N}_{\overline{\mathcal{M}}_{g,n}(V,\beta)/\mathcal{M}_{g,n}} \to h^1/h^0(R\pi_* f^* T_V)$$

of abelian cone stacks (see [1]) over $\overline{\mathcal{M}}_{g,n}(V,\beta)$, where $\mathfrak{N}$ is the relative intrinsic normal sheaf. The relative intrinsic normal cone $\mathfrak{N}_{\overline{\mathcal{M}}_{g,n}(V,\beta)/\mathcal{M}_{g,n}}$ is a closed subcone stack of $\mathfrak{N}_{\overline{\mathcal{M}}_{g,n}(V,\beta)/\mathcal{M}_{g,n}}$, and so we get a closed immersion of cone stacks

$$\mathfrak{N}_{\overline{\mathcal{M}}_{g,n}(V,\beta)/\mathcal{M}_{g,n}} \to h^1/h^0(R\pi_* f^* T_V).$$

Now since $R\pi_* f^* T_V$ has global resolutions (see Proposition [3]), we may intersect $\mathfrak{N}_{\overline{\mathcal{M}}_{g,n}(V,\beta)/\mathcal{M}_{g,n}}$ with the zero section of the vector bundle stack $h^1/h^0(R\pi_* f^* T_V)$ to get the virtual fundamental class $[\overline{\mathcal{M}}_{g,n}(V,\beta)]^{\text{virt}}$.

The fundamental axioms (see [4]) Gromov-Witten invariants need to satisfy to deserve their name are reduced in [5] to five basic compatibilities between the virtual fundamental classes. These follow from the basic properties proved in [6]. The dimension axiom, for example, follows from the basic fact that the intrinsic normal cone always has dimension zero.

We also show that if $V = G/P$, for a reductive group $G$ and a parabolic subgroup $P$, there is an alternative construction of the virtual fundamental classes avoiding the intrinsic normal cone. We construct a cone $C$ in the vector bundle $R^1 \pi_* \mathcal{O} \otimes \mathfrak{g}$ on $\overline{\mathcal{M}}_{g,n}(V,\beta)$, which may then be intersected with
the zero section of $R^1\pi_*\mathcal{O} \otimes g$ to obtain the virtual fundamental class. This cone $C$ is constructed as the normal cone of an embedding of $\mathcal{M}_{g,n}(V,\beta)$ into a certain stack of principal $P$-bundles (which is smooth, but not of Deligne-Mumford type).

A construction of Gromov-Witten invariants using a cone inside a vector bundle has also been announced by J. Li and G. Tian. Their methods differ from ours in that they use analytic methods, including the Kuranishi map.

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Preliminaries on Prestable Curves

Let $k$ be a field. We shall work over the category of locally noetherian $k$-schemes (with the fppf-topology). For a modular graph $\tau$ (see [2], Definition 1.5) let $\mathcal{M}(\tau)$ denote the $k$-stack of $\tau$-marked prestable curves (which are defined in [2], Definition 2.6).

**Lemma 1** The algebraic $k$-stack $\overline{\mathcal{M}}(\tau)$ of stable $\tau$-marked curves is an open substack of $\mathcal{M}(\tau)$.

**Proof.** Let $C_v \to \mathcal{M}(\tau)$ be the universal curve corresponding to the vertex $v \in V_\tau$. Let $\overline{C_v}$ be the stabilization. Then $\overline{\mathcal{M}}(\tau)$ is the substack of $\mathcal{M}(\tau)$ over which all $p_v : C_v \to \overline{C_v}$ are isomorphisms. This is open because the $C_v$ are proper over $\mathcal{M}(\tau)$. □

Now consider a modular graph $\tau'$ obtained from $\tau$ by adding some tails. We get an induced morphism of $k$-stacks $\mathcal{M}(\tau') \to \mathcal{M}(\tau)$ which simply forgets the markings corresponding to the tails $S_{\tau'} - S_\tau$. If $S_{\tau'} - S_\tau$ has cardinality 1, then $\mathcal{M}(\tau') \to \mathcal{M}(\tau)$ is a smooth curve, hence representable and smooth of relative dimension 1. So by induction, $\mathcal{M}(\tau') \to \mathcal{M}(\tau)$ is representable and smooth of relative dimension $(S_{\tau'} - S_\tau)$. By Lemma 1 the same is true for $\overline{\mathcal{M}}(\tau') \to \mathcal{M}(\tau)$.

**Proposition 2** The stack $\mathcal{M}(\tau)$ is a smooth algebraic $k$-stack of dimension

$$\dim(\tau) = \#S_\tau - \#E_\tau - 3\chi(\tau).$$
Proof. For the definition of $\dim(\tau)$ and $\chi(\tau)$ see [2], Definitions 6.1 and 6.2. Note that for every point of $\mathcal{M}(\tau)$ there exists a $\tau'$ as above such that the induced morphism $\overline{\mathcal{M}}(\tau') \to \mathcal{M}(\tau)$ contains this given point in its image. Thus $\bigsqcup_{\nu'} \overline{\mathcal{M}}(\tau')$ is a presentation of $\mathcal{M}(\tau)$ showing that $\mathcal{M}(\tau)$ is algebraic. \qed

Now let $\tau^s$ be the stabilization of $\tau$. Stabilization defines a morphism of algebraic $k$-stacks

$$s : \mathcal{M}(\tau) \longrightarrow \overline{\mathcal{M}}(\tau^s).$$

If $\tau'$ is obtained as above by adjoining tails to $\tau$ such that $\tau'$ is stable, we have a commutative diagram

$$\begin{array}{ccc}
\overline{\mathcal{M}}(\tau') & \longrightarrow & \overline{\mathcal{M}}(\tau^s) \\
\downarrow & & \downarrow \\
\mathcal{M}(\tau) & \longrightarrow & \overline{\mathcal{M}}(\tau^s).
\end{array}$$

Here $\phi : \tau' \to \tau^s$ is the canonical morphism of stable modular graphs. In fact, one may define $s$ locally by using such diagrams.

**Proposition 3** The morphism $s : \mathcal{M}(\tau) \to \overline{\mathcal{M}}(\tau^s)$ is flat.

Proof. This follows by descent since the morphism $\overline{\mathcal{M}}(\phi)$ for various $\phi : \tau' \to \tau^s$ as above are flat. \qed

**The Virtual Fundamental Classes**

Over $\mathcal{M}(\tau)$ there is a family $(\mathcal{C}_v)_{v \in V_\tau}$ of universal curves, with sections $x_i : \mathcal{M}(\tau) \to C_{\partial_\tau(i)}$. Let $\mathcal{C}(\tau) \to \mathcal{M}(\tau)$ be the curve obtained from $\bigsqcup_{v \in V_\tau} \mathcal{C}_v$ by identifying $x_i$ and $x_j$, for every edge $\{i, j\} \in E_\tau$. The curve $\mathcal{C}(\tau)$ has markings $x_i : \mathcal{M}(\tau) \to \mathcal{C}(\tau)$, for each $i \in S_\tau$. In fact, $\mathcal{C}(\tau)$ is a $\overline{\tau}$-marked prestable curve, where $\overline{\tau}$ is the graph obtained from $\tau$ by contracting all edges of $\tau$. Let us denote the structure morphism by

$$\pi : \mathcal{C}(\tau) \longrightarrow \mathcal{M}(\tau).$$

We shall also denote any base change of $\pi$ by $\pi$.

Now let $V$ be a smooth projective $k$-variety, $(\tau, \beta)$ a stable $V$-graph and let $\text{Mor}_{\mathcal{M}(\tau)}(\tau, V)$ be the $\mathcal{M}(\tau)$-space of morphisms from $\mathcal{C}(\tau)$ to $V$. Denote the universal morphism by

$$f : \mathcal{C}(\tau) \times \text{Mor}_{\mathcal{M}(\tau)}(\tau, V) \longrightarrow V.$$
By the stack $\text{Mor}_{(\tau)}(\tau, V)$ is an algebraic $k$-stack and the structure morphism

\[ \text{ Mor}_{(\tau)}(\tau, V) \to \mathfrak{M}(\tau) \]

is representable.

**Proposition 4** The proper Deligne-Mumford stack $\overline{M}(V, \tau, \beta)$ of stable maps is an open substack of $\text{Mor}_{(\tau)}(\tau, V)$.

**Proof.** The set of points where stabilization is an isomorphism is open. □

To define the virtual fundamental class on $\overline{M}(V, \tau, \beta)$ we consider the morphism $\overline{M}(V, \tau, \beta) \to \mathfrak{M}(\tau)$ and denote the relative intrinsic normal cone (see [1]) by

\[ \mathfrak{C}(V, \tau, \beta) = \mathfrak{C}_{\overline{M}(V, \tau, \beta)/\mathfrak{M}(\tau)} \]

The intrinsic normal sheaf [ibid.] of $\overline{M}(V, \tau, \beta)$ over $\mathfrak{M}(\tau)$ we shall denote by $\mathfrak{N}(V, \tau, \beta)$.

By the relative version of [1] Proposition 6.2 we have a perfect relative obstruction theory [ibid.]

\[ \pi^*(e^\vee \vee) : R\pi^*(f^*T_V) \to L^*_{\text{Mor}_{(\tau)}(\tau, V)/\mathfrak{M}(\tau)} \]

Restricting to the open substack $\overline{M}(V, \tau, \beta)$ we get a perfect relative obstruction theory

\[ \pi^*(e^\vee \vee) : R\pi^*(f^*T_V) \to L^*_{\overline{M}(V, \tau, \beta)/\mathfrak{M}(\tau)} \]

which we shall also denote by $E^*_{\mathfrak{N}}(V, \tau, \beta)$. Thus $\mathfrak{C}(V, \tau, \beta)$ is embedded as a closed subcone stack in the vector bundle stack

\[ \mathfrak{C}(V, \tau, \beta) = h^1/h^0(R\pi^*f^*T_V) \]

Note that the relative virtual dimension of $\overline{M}(V, \tau, \beta)$ over $\mathfrak{M}(\tau)$ with respect to the obstruction theory $R\pi^*(f^*T_V)^\vee$ is equal to

\[ \text{rk } R\pi^*(f^*T_V)^\vee = \chi(f^*T_V) = \text{deg } f^*T_V + \text{dim } V \cdot \chi(C(\tau)) = \chi(\tau) \text{dim } V - \beta(\tau)(\omega_V) \]

Essential is the following result.
Proposition 5 Let \((C, x, f)\) be a stable map over \(T\) to \(V\), where \(T\) is a finite type algebraic \(k\)-stack. Let \(E\) be a vector bundle on \(C\). Then \(R\pi_*E\) has global resolutions, where \(\pi : C \to T\) is the structure map.

**Proof.** Let \(M\) be an ample invertible sheaf on \(V\) and let
\[
L = \omega_{C/T}(x_1 + \ldots + x_n) \otimes f^*M \otimes 3.
\]
By Proposition 3.9 of [2] the sheaf \(L\) is ample on the fibers of \(\pi\). So for sufficiently large \(N\) we have that
1. \(\pi^*\pi_*(E \otimes L^{\otimes N}) \to E \otimes L^{\otimes N}\) is surjective,
2. \(R^1\pi_*(E \otimes L^{\otimes N}) = 0,\)
3. for all \(t \in T\) we have that \(H^0(C_t, L_t^{\otimes -N}) = 0.\)

Let
\[
F = \pi^*\pi_*(E \otimes L^{\otimes N}) \otimes L^{\otimes -N}
\]
and let \(H\) be the kernel of the map \(F \to E\). Thus we have a short exact sequence
\[
0 \longrightarrow H \longrightarrow F \longrightarrow E \longrightarrow 0
\]
of vector bundles on \(C\). Note that for every \(t \in T\) we have
\[
H^0(C_t, F) = H^0(C_t, \pi_*(E \otimes L_t^{\otimes N}) \otimes L_t^{\otimes -N}) = H^0(C_t, L_t^{\otimes -N}) \otimes \pi_*(E \otimes L_t^{\otimes N}) = 0
\]
and hence \(H^0(C_t, H) = 0\), also. Therefore, \(\pi_*H\) and \(\pi_*F\) are zero and \(R^1\pi_*H\) and \(R^1\pi_*F\) are locally free. This implies that
\[
R\pi_*E \cong [R^1\pi_*H \to R^1\pi_*F].
\]

\(\Box\)

As shown in [1], by Proposition 5 the obstruction theory \(R\pi_*(f^*T_V)^\vee\) gives rise to a virtual fundamental class
\[
[M(V, \tau, \beta), R\pi_*(f^*T_V)^\vee] \in A_{\dim(V, \tau, \beta)}(\overline{M}(V, \tau, \beta)),
\]

since
\[
\dim \mathcal{M}(\tau) + \operatorname{rk} R\pi_*(f^*T_V)^\vee
= \chi(\tau)(\dim V - 3) - \beta(\tau)(\omega_V) + \# S_\tau - \# E_\tau
= \dim(V, \tau, \beta).
\]
(See Definition 6.2 in [2] for the definition of \(\dim(V, \tau, \beta)\).)

**Theorem 6** The system of virtual fundamental classes
\[
J(V, \tau, \beta) = [M(V, \tau, \beta), R\pi_*(f^*T_V)^\vee]
\]
is an orientation of \(\overline{M}\) over \(\mathfrak{G}_s(V)\). If \(V\) is convex, on the tree level subcategory \(T_s(V)\), we get back the orientation of [2], Theorem 7.5.

**Proof.** If \(V\) is convex and \(\tau\) a forest, then \(R^1\pi_*(f^*T_V) = 0\), so that the virtual fundamental class is the usual fundamental class by [1] Proposition 7.3. Thus the virtual fundamental class agrees with the orientation of [2], Theorem 7.5. To check that \(J\) is an orientation, we need to check the five axioms listed in [2], Definition 7.1. This shall be done in the next Section. \(\square\)

**Remark** As shown in [2], we get an associated system of Gromov-Witten classes for \(V\).

**Checking the Axioms**

**Axiom I.** Mapping to a point

Let \(\tau\) be a stable \(V\)-graph of class zero such that \(|\tau|\) is non-empty and connected. As noted in [2] Section 7 we have
\[
\overline{M}(V, \tau, 0) = V \times \overline{M}(\tau)
\]
which is obviously smooth over \(\mathcal{M}(\tau)\). In fact, the morphism \(\overline{M}(V, \tau, 0) \to \mathcal{M}(\tau)\) is just the composition
\[
V \times \overline{M}(\tau) \longrightarrow \overline{M}(\tau) \longrightarrow \mathcal{M}(\tau)
\]
of projection followed by inclusion. If \(\bar{\pi} : C(\tau) \to \overline{M}\tau\) is the universal curve over \(\overline{M}(\tau)\), then \(C(V, \tau, 0) = V \times C(\tau)\) and \(\pi : C(V, \tau, 0) \to \overline{M}(V, \tau, 0)\) is identified with \(\text{id} \times \bar{\pi} : V \times C(\tau) \to V \times \overline{M}(\tau)\). Hence
\[
R^1\pi_* f^*T_V = T_V \boxtimes R^1\bar{\pi}_* O_{C(\tau)} = T^{(1)}
\]
is locally free. So by \[\text{Proposition 7.3}\] we have

\[
J(V, \tau, 0) = c_{1k R^1 \pi_* f^* T_V} (R^1 \pi_* f^* T_V) \cdot [\mathcal{M}(V, \tau, 0)] \\
= c_{g(\tau) \dim V(T(1))} \cdot [\mathcal{M}(V, \tau, 0)],
\]

which is Axiom I.

**Axiom II. Products**

Let \((\sigma, \alpha)\) and \((\tau, \beta)\) be stable \(V\)-graphs and denote the ‘product’ by \((\sigma \times \tau, \alpha \times \beta)\). Note that

\[
E^*(V, \sigma \times \tau, \alpha \times \beta) = E^*(V, \sigma, \alpha) \boxplus E^*(V, \tau, \beta),
\]

so by \[\text{Proposition 7.4}\] we have

\[
J(V, \sigma \times \tau, \alpha \times \beta) = [\mathcal{M}(V, \sigma \times \tau, \alpha \times \beta), E^*(V, \sigma, \alpha) \boxplus E^*(V, \tau, \beta)] \\
= [\mathcal{M}(V, \sigma, \alpha), E^*(V, \sigma, \alpha)] \times [\mathcal{M}(V, \tau, \beta), E^*(V, \tau, \beta)] \\
= J(V, \sigma, \alpha) \times J(V, \tau, \beta),
\]

which is the product axiom.

**Axiom III. Cutting Edges**

Use notation as in \[\text{[2]}\], Section 7, modified as necessary to avoid confusion. Let \(\beta\) denote the \(H_2(V)^+\)-structure on both \(\sigma\) and \(\tau\). Write \(\mathfrak{M} = \mathfrak{M}(\tau) = \mathfrak{M}(\sigma)\). Consider the cartesian diagram

\[
\begin{array}{ccc}
\mathcal{M}(V, \sigma, \beta) & \overset{\mathcal{M}(\Phi)}{\longrightarrow} & \mathcal{M}(V, \tau, \beta) \\
g \downarrow & & \downarrow \\
\mathfrak{M} \times V & \overset{\Delta}{\longrightarrow} & \mathfrak{M} \times V \times V
\end{array}
\]

of stacks over \(\mathfrak{M}\). Let us show that the obstruction theories \(E^*(V, \tau, \beta)\) and \(E^*(V, \sigma, \beta)\) are compatible over \(\Delta\) (see \[\text{[1]}\]).

Over \(\mathcal{M}(V, \sigma, \beta)\) let us consider the following two curves. First the curve \(\mathcal{C} = \mathcal{C}(V, \sigma, \beta)\) obtained from the universal curves \((C_v)_{v \in V_\sigma}\) by gluing according to the edges of \(\sigma\). Secondly, we have the curve \(\mathcal{C}'\), which we obtain from \((C_v)_{v \in V_\sigma}\) by gluing according to the edges of \(\tau\). In other words, \(\mathcal{C}' = \mathcal{M}(\Phi)^* \mathcal{C}(V, \tau, \beta)\). Moreover, \(\mathcal{C}\) is obtained from \(\mathcal{C}'\) by identifying the two sections \(x_1\) and \(x_2\) of \(\mathcal{C}'\), corresponding to the edge \(\{i_1, i_2\}\) of \(\sigma\) which
is cut by $\Phi$. Thus there is a structure morphism $p : C' \to C$ fitting into the commutative diagram

$$
\begin{array}{ccc}
C' & \xrightarrow{p} & C \\
\pi' \downarrow & & \downarrow \pi \\
\mathcal{M}(V, \sigma, \beta). & & \\
\end{array}
$$

We shall also use the diagram

$$
\begin{array}{ccc}
C' & \xrightarrow{p} & C \\
\pi' \downarrow & & \downarrow \pi \\
\mathcal{M}(V, \sigma, \beta). & & \\
\end{array}
$$

where $f : C \to V$ is the universal map. Let $x = p \circ x_1 = p \circ x_2$.

If $E$ is any locally free sheaf on $C$, then for $i = 1, 2$ we have the evaluation homomorphism

$$
u_i : p^* E \longrightarrow x_i^* x_1^* p^* E = x_i^* E.
$$

Applying $p_*$ we get

$$
p_*(\nu_i) : p_* p^* E \longrightarrow x_* x^* E.
$$

Letting $u = p_*(u_2) - p_*(u_1)$ we have a short exact sequence

$$0 \longrightarrow E \longrightarrow p_* p^* E \overset{u}{\longrightarrow} x_* x^* E \longrightarrow 0
$$

of coherent sheaves on $C$. Applying $R\pi_*$ we get a distinguished triangle

$$
R\pi_* E \longrightarrow R\pi'_* p^* E \overset{R\pi_*(u)}{\longrightarrow} x^* E \longrightarrow R\pi_* E[1]
$$

in $D(\mathcal{O}_M(V, \sigma, \beta))$. Taking $E = f^* T_V$ we get the distinguished triangle

$$
R\pi_* f^* T_V \longrightarrow R\pi'_* f^* T_V \overset{R\pi_*(u)}{\longrightarrow} x^* f^* T_V \longrightarrow R\pi_* f^* T_V[1],
$$

or dually,

$$
x^* f^* \Omega_V \xrightarrow{R\pi_*(u)} (R\pi'_* f^* T_V)^\vee \longrightarrow (R\pi_* f^* T_V)^\vee \longrightarrow x^* f^* \Omega_V[1]. \quad (1)
$$

Note that we have $E^*(V, \sigma, \beta) = (R\pi_* f^* T_V)^\vee$ and $\mathcal{M}(\Phi)^*(E^*(V, \tau, \beta)) = (R\pi'_* f^* T_V)^\vee$. Moreover, $L^*_\Delta = \Omega_V[1]|_{\mathbb{R} \times V}$, so that $g^* L_\Delta = x^* f^* \Omega_V[1]$, since $f \circ x = p_V \circ g$. So (1) gives the distinguished triangle

$$
g^* L_\Delta[-1] \xrightarrow{R\pi_*(u)} \mathcal{M}(\Phi)^* E^*(V, \tau, \beta) \longrightarrow E^*(V, \sigma, \beta) \longrightarrow g^* L_\Delta,
$$

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which we may shuffle around to give
\[
\overline{M}(\Phi)^* E^*(V, \tau, \beta) \rightarrow E^*(V, \sigma, \beta) \rightarrow g^* L_\Delta \rightarrow \overline{M}(\Phi)^* E^*(V, \tau, \beta)[1].
\]
Now we have the obstruction morphisms \(E^*(V, \tau, \beta) \rightarrow L_{\overline{M}(V, \tau, \beta)/\mathfrak{M}}^e\) and \(E^*(V, \sigma, \beta) \rightarrow L_{\overline{M}(V, \sigma, \beta)/\mathfrak{M}}^e\). Moreover, we have the natural homomorphism \(g^* L_\Delta \rightarrow L_{\overline{M}(\Phi)}^e\). These give rise to a homomorphism of distinguished triangles
\[
\begin{array}{cccc}
\overline{M}(\Phi)^* E^*(V, \tau, \beta) & \rightarrow & E^*(V, \sigma, \beta) & \rightarrow \quad g^* L_\Delta \rightarrow \overline{M}(\Phi)^* E^*(V, \tau, \beta)[1] \\
\overline{M}(\Phi)^* L_{\overline{M}(V, \tau, \beta)/\mathfrak{M}}^e & \rightarrow & L_{\overline{M}(V, \sigma, \beta)/\mathfrak{M}}^e & \rightarrow \quad L_{\overline{M}(\Phi)}^e & \rightarrow \quad \overline{M}(\Phi)^* L_{\overline{M}(V, \tau, \beta)/\mathfrak{M}}^e[1],
\end{array}
\]
showing that \(E^*(V, \tau, \beta)\) and \(E^*(V, \sigma, \beta)\) are compatible over \(\Delta\). Hence by Proposition 7.5 we have
\[
\Delta^! J(V, \tau, \beta) = J(V, \sigma, \beta)
\]
which is Axiom III.

**Axiom IV. Forgetting Tails**

Let us deal with the incomplete case, leaving the tripod losing cases to the reader. Letting \(C \rightarrow \mathfrak{M}(\tau)\) be the universal curve corresponding to the vertex \(w \in V_\tau\) (notation from [2], Section 7). We have a cartesian diagram of algebraic \(k\)-stacks
\[
\begin{array}{ccc}
\overline{M}(V, \sigma, \beta) & \xrightarrow{\Delta} & \overline{M}(V, \tau, \beta) \\
\downarrow & & \downarrow \\
C & \xrightarrow{\Delta} & \mathfrak{M}(\tau).
\end{array}
\]
By Proposition 7.2 we have
\[
\overline{M}(\Phi)^* J(V, \tau, \beta) = [\overline{M}(V, \sigma, \beta), \overline{M}(\Phi)^* E^*(V, \tau, \beta)].
\]
Here the class on the right hand side is the virtual fundamental class defined by the relative intrinsic normal cone of the morphism \(d\) and the relative obstruction theory \(\overline{M}(\Phi)^* E^*(V, \tau, \beta)\). Note that the structure morphism \(\overline{M}(V, \sigma, \beta) \rightarrow \mathfrak{M}(\sigma)\) factors through \(d: \overline{M}(V, \sigma, \beta) \rightarrow C\).
\[
\begin{array}{ccc}
\overline{M}(V, \sigma, \beta) & \xrightarrow{d} & C \\
\downarrow & & \downarrow \\
& \mathfrak{M}(\sigma).
\end{array}
\]
The morphism \( d : \overline{M}(V, \sigma, \beta) \to \mathcal{C} \) associates to the stable map \((C, x, h)\) the pair \(( (C', x'), y) \), where \((C', x', h')\) is the image of \((C, x, h)\) under \( \overline{M}(\Phi) \) and \((C', x')\) the underlying \( \tau \)-marked prestable curve. Letting \( x_f \) be the section of \( C_v \) corresponding to the flag \( f \), we obtain \((C', x', h')\) by forgetting \( x_f \) and stabilizing. Moreover, \( y \) is the image of the forgotten section \( x_f \) in \( C'_w \).

The morphism \( \mathcal{C} \to \mathfrak{M}(\sigma) \) associates to the pair \(((C, x), y)\), where \((C, x)\) is a \( \tau \)-marked prestable curve and \( y \) a section of \( C_w \), the \( \sigma \)-marked prestable curve \((\tilde{C}, \tilde{x})\) obtained as follows. For \( v' \neq v \) we have \( C_{v'} = C_{w'} \), where \( w' \) is the vertex of \( \tau \) corresponding to \( v' \). The curve \((\tilde{C}_v, (\tilde{x}_j)_{j \in F_\sigma(v)})\) is obtained from \(((C_w, (x_j)_{j \in F_\tau(w)}), y)\) by ‘prestabilizing’ (i.e. separating the special points) as in [4], Definition 2.3.

**Lemma 7** The morphism \( \mathcal{C} \to \mathfrak{M}(\sigma) \) is étale.

**Proof.** We will use the formal criterion for étaleness. Without loss of generality assume that \( w \) is the only vertex of \( \tau \). So let \(((C, x), y)\) be a \( \tau \)-marked prestable curve with section over the scheme \( T, T \to T' \) a square zero extension and \((C', x')\) a \( \sigma \)-marked prestable curve over \( T' \) such that \((C', x')|T \) is the prestabilization of \(((C, x), y)\). We may assume that we may choose additional sections \( s \) of \( C \) over \( T \), making \((C, x, s)\) a stable marked curve. Then we extend the sections \( s \) to sections \( s' \) of \( C' \) over \( T' \). Taking the stabilization of \((C', x', s')\) after forgetting the section \( x'_f \) gives an extension of \(((C, x), y)\) to \( T' \) whose prestabilization is \((C', x')\). \( \square \)

Consider the natural morphism \( p : \mathcal{C}(V, \sigma, \beta) \to \overline{M}(\Phi)^* \mathcal{C}(V, \tau) \), which fits into the two commutative diagrams

\[
\begin{align*}
\mathcal{C}(V, \sigma, \beta) \xrightarrow{p} & \overline{M}(\Phi)^* \mathcal{C}(V, \tau, \beta) \\
\pi \downarrow & \downarrow \pi' \\
\mathcal{M}(V, \sigma, \beta) & \overline{M}(V, \sigma, \beta)
\end{align*}
\]

and

\[
\begin{align*}
\mathcal{C}(V, \sigma, \beta) \xrightarrow{p} & \overline{M}(\Phi)^* \mathcal{C}(V, \tau, \beta) \\
f \downarrow & \downarrow f' \\
\mathcal{V} & \mathcal{V}
\end{align*}
\]

Whenever \( E \) is a locally free sheaf on \( \overline{M}(\Phi)^* \mathcal{C}(V, \tau, \beta) \) the canonical homomorphism \( E \to p_* p^* E \) is an isomorphism. Applying this principle to \( E = f'^* T_V \) we get an isomorphism

\[ f'^* T_V \to p_* f^* T_V. \]
Applying $R\pi_\tau'$ to this, gives an isomorphism
\[ R\pi_\tau' f^{*}T_V \longrightarrow R\pi_\tau f^{*}T_V. \]
Noting that $R\pi_\tau' f^{*}T_V = \overline{M}(\Phi)^* E^*(V, \tau, \beta)$ we get an isomorphism
\[ \overline{M}(\Phi)^* E^*(V, \tau, \beta) \longrightarrow E^*(V, \sigma, \beta) \]
and whence an isomorphism
\[ \mathcal{E}(V, \sigma, \beta) \longrightarrow \overline{M}(\Phi)^* \mathcal{E}(V, \tau, \beta). \]
By Proposition 7.1 there is a natural isomorphism
\[ \mathcal{E}_{\overline{M}(V, \sigma, \beta)/\mathcal{C}} \longrightarrow \overline{M}(\Phi)^* \mathcal{E}_{\overline{M}(V, \tau, \beta)/\mathfrak{M}(\tau)}. \]
By Lemma 7 we have a canonical isomorphism
\[ \mathcal{E}_{\overline{M}(V, \sigma, \beta)/\mathcal{C}} \longrightarrow \mathcal{E}_{\overline{M}(V, \sigma, \beta)/\mathfrak{M}(\sigma)}, \]
such that the diagram
\[
\begin{array}{ccc}
\mathcal{E}_{\overline{M}(V, \sigma, \beta)/\mathfrak{M}(\sigma)} & \xleftarrow{\sim} & \mathcal{E}_{\overline{M}(V, \sigma, \beta)/\mathcal{C}} \\
\cap & & \cap \\
\mathcal{E}(V, \sigma, \beta) & \xrightarrow{\sim} & \overline{M}(\Phi)^* \mathcal{E}(V, \tau, \beta)
\end{array}
\]
commutes. So finally, we have
\[ \overline{M}(\Phi)^* J(V, \tau, \beta) = [\overline{M}(V, \sigma, \beta), \overline{M}(\Phi)^* E^*(V, \tau, \beta)] = [\overline{M}(V, \sigma, \beta), E^*(V, \sigma, \beta)] = J(V, \sigma, \beta), \]
which is Axiom IV.

**Axiom V. Isogenies**

Before we start with the proof, some general remarks. Let $\Phi : \tau \to \sigma$ be an elementary contraction of stable modular graphs, contracting the edge $\{f, \overline{f}\}$ of $\tau$. Let $a : \tau \to \tau'$ and $b : \sigma \to \sigma'$ be combinatorial morphisms of modular graphs identifying $\tau$ and $\sigma$ as the stabilizations of $\tau'$ and $\sigma'$, respectively. Finally, let $\Phi' : \tau' \to \sigma'$ be as follows. We require $\{a(f), a(\overline{f})\}$ to be an edge of $\tau'$ and $\Phi' : \tau' \to \sigma'$ to be the elementary contraction contracting the edge
\{a(f), a(\overline{f})\}. Moreover, we require \(\Phi\) to be the stabilization of \(\Phi'\). To fix notation, denote the vertex onto which \(\Phi'\) contracts the edge \(\{a(f), a(\overline{f})\}\) by \(v_0 \in V_{\tau'}\) and let \(v_1 = \partial_{\tau'}(a(f))\) and \(v_2 = \partial_{\tau'}(a(\overline{f}))\).

In this situation we get a commutative diagram of algebraic stacks

\[
\begin{array}{ccc}
\mathcal{M}(\tau') & \xrightarrow{\mathcal{M}(\Phi')} & \mathcal{M}(\sigma') \\
\downarrow s & & \downarrow s \\
\overline{\mathcal{M}}(\tau) & \xrightarrow{\overline{\mathcal{M}}(\Phi)} & \overline{\mathcal{M}}(\sigma).
\end{array}
\]

Define \(\mathfrak{P}\) to be the fibered product

\[
\begin{array}{ccc}
\mathfrak{P} & \longrightarrow & \mathcal{M}(\sigma') \\
\downarrow & & \downarrow s \\
\overline{\mathcal{M}}(\tau) & \xrightarrow{\overline{\mathcal{M}}(\Phi)} & \overline{\mathcal{M}}(\sigma).
\end{array}
\]

Consider the induced morphism \(l : \mathcal{M}(\tau') \to \mathfrak{P}\).

**Proposition 8** We have \(l_*[\mathcal{M}(\Phi')] = s^*[\overline{\mathcal{M}}(\Phi)]\).

**Proof.** First note that \(\mathcal{M}(\tau')\) is irreducible, since \(\mathcal{M}(\tau')\) is a product of stacks of the form \(\mathcal{M}_{g,n}\), which are irreducible since the stacks \(\overline{\mathcal{M}}_{g,n}\) are. Moreover, \(\mathcal{M}(\tau') \to \mathfrak{P}\) is surjective, so that \(\mathfrak{P}\) is irreducible, too.

Secondly, let us remark that there exist non-empty (hence dense) open substacks \(\mathcal{M}(\tau')^0 \subset \mathcal{M}(\tau')\) and \(\mathfrak{P}^0 \subset \mathfrak{P}\) such that \(l\) induces an isomorphism \(l^0 : \mathcal{M}(\tau')^0 \cong \mathfrak{P}^0\). In fact, let \(\mathcal{M}(\tau')^0\) be the open substack of \(\mathcal{M}(\tau')\) characterized by the requirement that the marked curves \(C_{v_1}\) and \(C_{v_2}\) be stable. To construct \(\mathfrak{P}^0\), let \(\mathcal{M}(\sigma')^0\) be the open substack of \(\mathcal{M}(\sigma')\) where the marked curve \(C_{v_0}\) is stable. Then set

\[
\mathfrak{P}^0 = \overline{\mathcal{M}}(\tau) \times_{\overline{\mathcal{M}}(\sigma)} \mathcal{M}(\sigma')^0.
\]

These facts imply the claim. \(\square\)

Now let \((\Phi, m) : \tau \to \sigma\) be an elementary isogeny of type forgetting a tail. Let \(f \in F_{\tau}\) be the forgotten tail. Let \(a : \tau \to \tau'\) and \(b : \sigma \to \sigma'\) be as above. Finally, let \(\Phi' : \tau' \to \sigma'\) be the ‘adjoint’ of a combinatorial morphism of graphs, such that there exists a tail map \(m'\), a semigroup \(A\) and \(A\)-structures on \(\tau'\) and \(\sigma'\) making \((\Phi', m')\) the elementary isogeny of stable \(A\)-graphs forgetting the tail \(a(f)\). Moreover, we require \(\Phi\) to be the stabilization of \(\Phi'\).
Let $\mathcal{P}$ be the fibered product
\[
\begin{array}{c}
\mathcal{P} \\
\downarrow \\
\underline{M}(\tau)
\end{array}
\rightarrow
\begin{array}{c}
M(\sigma') \\
\downarrow \ s
\end{array}
\]
and $\mathcal{C}$ the universal curve over $M(\sigma')$ corresponding to $w \in V_{\sigma'}$, where $w$ is the vertex of the forgotten tail. (If $w$ does not exist, i.e. if $\Phi'$ is complete, then $\mathcal{C} = M(\sigma')$.) As in the proof of Axiom IV we have a morphism $\mathcal{C} \rightarrow M(\tau')$ giving rise to a commutative diagram
\[
\begin{array}{c}
\mathcal{C} \\
\downarrow
\end{array}
\rightarrow
\begin{array}{c}
M(\sigma') \\
\downarrow \ s
\end{array}
\]
and hence to a morphism $l : \mathcal{C} \rightarrow \mathcal{P}$.

**Proposition 9** We have $l_*(\pi') = s^* [\underline{M}(\Phi)]$.

**Proof.** Again, $\mathcal{C}$ and $\mathcal{P}$ are irreducible and $l$ induces an isomorphism $l^0 : C^0 \rightarrow \mathcal{P}^0$, where $C^0$ is the restriction of $\mathcal{C}$ to $M(\sigma')^0$ and $\mathcal{P}^0 = \underline{M} \times \underline{M}(\sigma)$ $M(\sigma')^0$. Here $M(\sigma')^0 \subset M(\sigma')$ is the open substack where $C_w$ is stable. □

Now let us prove Axiom V. According to [3], Remark 7.2, it suffices to do this for the case that $\Phi : \tau \rightarrow \sigma$ is an elementary isogeny, $\# I = 1$ and $(\alpha, \tau_i, \Phi_i)_{i \in I}$ a pullback. So we shall use notation as in the Definition of pullback ([2], Definition 6.10). We shall include the $H_2(V)^+$-structures on $\sigma'$ and $\tau_i$ ($i \in I$) in the notation. They shall be denoted by $\beta'$ and $\beta_i$ ($i \in I$), respectively. The underlying graph of $(\tau_i, \beta_i)$ is the same for all $i \in I$. Let us call it simply $\tau'$.

Let us first consider the case where $\Phi$ is a contraction.

**Lemma 10** We have a cartesian diagram
\[
\begin{array}{c}
\prod_{i \in I} \underline{M}(V, \tau', \beta_i) \\
\downarrow \\
\underline{M}(\tau')
\end{array}
\rightarrow
\begin{array}{c}
\underline{M}(V, \sigma', \beta') \\
\downarrow \ s
\end{array}
\]
\[
\rightarrow
\begin{array}{c}
\underline{M}(\sigma') \\
\downarrow \ s
\end{array}
\]
of algebraic k-stacks. Moreover,
\[ \mathfrak{M}(\Phi')^{1} J(V, \sigma', \beta') = \sum_{i \in I} J(V, \tau', \beta'). \]

**Proof.** The first fact follows immediately from the definitions. The second fact is Proposition 7.2. 

Axiom V will follow by putting Lemma 10 and Proposition 8 together as follows. By Lemma 10 all squares in the following diagram are cartesian.

\[
\begin{array}{ccc}
\prod_{i \in I} \mathcal{M}(V, \tau', \beta_i) & \xrightarrow{h} & \mathcal{M}(\tau) \times_{\mathcal{M}(\sigma)} \mathcal{M}(V, \sigma', \beta') \\
\downarrow & & \downarrow \\
\mathcal{M}(\tau') & \xrightarrow{l} & \mathcal{M}(\tau) \times_{\mathcal{M}(\sigma)} \mathcal{M}(\sigma') \\
\downarrow s & & \downarrow s \\
\mathcal{M}(\tau) & \xrightarrow{j} & \mathcal{M}(\Phi) \\
\end{array}
\]

So we may calculate as follows.

\[ \mathcal{M}(\Phi)^{1} J(V, \sigma', \beta') = a^{*} s^{*} [\mathcal{M}(\Phi)] \cdot J(V, \sigma', \beta') \]

(by Proposition 8)

\[ = h_{*} \mathfrak{M}(\Phi)^{1} J(V, \sigma', \beta') \]

\[ = h_{*} \sum_{i \in I} J(V, \tau', \beta_i) \]

by Lemma 10. This is the context of Axiom V.

The case that \( \Phi \) is of type forgetting a tail is similar. Instead of Lemma 10 one uses Axiom IV, and Proposition 8 is replaced by Proposition 9.

This finishes the proof of Axiom V and hence the proof of Theorem 6.

**Homogeneous Spaces**

In the case where \( V \) is a generalized flag variety, we can give a more explicit construction of Gromov-Witten invariants as follows.

**Curves and Principal Bundles**

For a smooth algebraic k-group \( G \) with Lie algebra \( \mathfrak{g} \), we denote by

\[ \mathfrak{S}^{1}(\tau, G) \]
the $k$-stack of $G$-torsors on $\tau$-marked prestable curves. More precisely, for a $k$-scheme $T$, the category $\mathcal{H}_{1}(\tau, G)(T)$ is the category of pairs $(C, E)$, where $C = (C_v)_{v \in V_\tau}$ is a $\tau$-marked prestable curve over $T$, giving rise to a morphism $f : T \to \mathfrak{M}(\tau)$, and $E$ is a $G$-torsor on $f^*C(\tau)$.

Let $(C, E)$ be such a pair. Denote by $E_v$, for $v \in V_\tau$, the $G$-bundle induced by $E$ on $C_v$. We call $\deg_v(E) = \deg(E_v) = \deg(E_v \times_{G, \text{Ad}} \mathfrak{g})$ the degree of $E$ at the vertex $v \in V_\tau$. The degree thus defines a $\mathbb{Z}_{\geq 0}$-structure on $\tau$, which is locally constant on $T$. (See [2], Definition 1.6, for $\mathbb{Z}_{\geq 0}$-structures.)

In this way, we get for every $\mathbb{Z}_{\geq 0}$-structure $\alpha$ on $\tau$ an open and closed substack $\mathcal{H}_{1, \alpha}(\tau, G) \subset \mathcal{H}_{1}(\tau, G)$, the substack of $G$-torsors of degree $\alpha$.

**Proposition 11** For every $\mathbb{Z}_{\geq 0}$-structure $\alpha$ on $\tau$ the stack $\mathcal{H}_{1, \alpha}(\tau, G)$ is an algebraic $k$-stack. The canonical morphism

$$\mathcal{H}_{1, \alpha}(\tau, G) \to \mathfrak{M}(\tau)$$

is smooth of relative dimension

$$-\chi(\tau) \dim G - \alpha(\tau),$$

where $\alpha(\tau) = \sum_{v \in V_\tau} \alpha(v)$.

**Proof.** To prove that $\mathcal{H}_{1}(\tau, G)$ is algebraic, choose a suitable embedding $G \hookrightarrow GL_n$ to reduce the case of $G$-bundles to the case of vector bundles, for which it is well-known. The smoothness of $\mathcal{H}_{1}(\tau, G)$ follows from the fact that $H^2(C, E \times_{G, \text{Ad}} \mathfrak{g}) = 0$ for any $G$-torsor $E$ on a $\tau$-marked prestable curve $C$. The dimension of $\mathcal{H}_{1}(\tau, G)$ is equal to

$$-\chi(E \times_{G, \text{Ad}} \mathfrak{g}) = -\deg(E \times_{G, \text{Ad}} \mathfrak{g}) - \chi(O_C) \operatorname{rk}(E \times_{G, \text{Ad}} \mathfrak{g})$$

$$= -\alpha(\tau) - \chi(\tau) \dim G$$

by Riemann-Roch. \(\square\)

**Maps to $G/P$**

Now let $G$ be a reductive algebraic group over $k$ and $P$ a parabolic subgroup of $G$. Then $G/P$ is a smooth projective variety over $k$. Let us assume for
simplicity that \( G \) is split over \( k \). The morphism \( G \to G/P \) is a principal \( P \)-bundle, which we shall denote by \( F \).

Let \( U_1, \ldots, U_r \) be the elementary representations of \( P \) over \( k \), \( V_1, \ldots, V_r \) the corresponding vector bundles on \( G/P \) and \( L_1, \ldots, L_r \) their determinants. For every \( i = 1, \ldots, r \) we have

\[
V_i = F \times_P U_i.
\]

Note that \( \operatorname{Pic}(G/P) \otimes \mathbb{Q} \) is spanned by \( L_1, \ldots, L_r \) and that \( L_1^{-1} \otimes \ldots \times L_r^{-1} \) is ample.

Let \( H_2(G/P)^+ \) be the set of homomorphisms of abelian groups \( \psi : \operatorname{Pic}(G/P) \to \mathbb{Z} \), which are non-negative on ample line bundles. Then we get a canonical injection

\[
H_2(G/P)^+ \longrightarrow (\mathbb{Z}_{\geq 0})^r
\]

\[
\psi \longmapsto (\psi(L_1^{-1}), \ldots, \psi(L_r^{-1})).
\]

Using this injection we shall think of classes in \( H_2(G/P)^+ \) as \( r \)-tuples of non-negative integers.

Let \( \mathfrak{g} \) and \( \mathfrak{p} \) be the Lie algebras of \( G \) and \( P \), respectively. We will consider these only as adjoint representations, ignoring the Lie algebra structure. Denote by \( \mathfrak{p} \) also the induced vector bundle

\[
F \times_{P, \operatorname{Ad}} \mathfrak{p}
\]

on \( G/P \). Evaluating on the inverse of its determinant defines a morphism

\[
\deg : H_2(G/P)^+ \longrightarrow \mathbb{Z}_{\geq 0}
\]

\[
\psi \longmapsto \psi(\det(\mathfrak{p})^{-1}).
\]

This morphism has the property that \( \deg(\psi) = 0 \) implies \( \psi = 0 \).

**Remark** We have \( \det \mathfrak{p} \cong \omega_{G/P} \). In particular, \( \deg \psi = -\psi(\omega_{G/P}) \).

Now fix an \( H_2(G/P)^+ \)-graph \((\tau, \beta)\), with underlying modular graph \( \tau \). Let \((\bar{\tau}, \bar{\beta})\) be the \( H_2(G/P)^+ \)-graph obtained by contracting all edges of \( \tau \).

Consider the algebraic \( k \)-stacks \( \mathcal{H}^1(\tau, G) \) and \( \mathcal{H}^1(\tau, P) \). Since \( G \) is reductive, any \( G \)-torsor on a curve has degree zero, and thus

\[
\mathcal{H}^1(\tau, G) \longrightarrow \mathcal{M}(\tau)
\]

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is smooth of relative dimension

\[-\chi(\tau) \dim G.\]

If \(E\) is a \(P\)-torsor, then associated to \(U_1, \ldots, U_r\), we have associated vector bundles \(E_i = E \times P U_i\), for \(i = 1, \ldots, r\), and thus we may associate to \(E\) the multi-degree

\[
\text{mult-deg}(E) = (-\deg(E_1), \ldots, -\deg(E_r)).
\]

Let \(\mathcal{H}_{\beta}(\tau, P)\) be the open and closed substack of \(\mathcal{H}(\tau, P)\) of \(P\)-torsors whose multi-degree is equal to \(\beta\).

Let \(\alpha = \deg \beta\) be the \(\mathbb{Z}_{\geq 0}\)-structure on \(\tau\) associated to \(\beta\). Then we have

\[
\mathcal{H}_{\beta}(\tau, P) \subset \mathcal{H}_{-\alpha}(\tau, P),
\]

so that by Proposition 11 the stack \(\mathcal{H}_{\beta}(\tau, P)\) is smooth of relative dimension

\[-\chi(\tau) \dim P - \beta(\tau)(\omega_{G/P})\]

over \(\mathcal{M}(\tau)\).

Now let \(\mathcal{M}(G/P, \tau, \beta)\) be the stack of maps from \(\tau\)-marked prestable curves to \(G/P\) of class \(\beta\). More precisely, for a \(k\)-scheme \(T\), the objects of \(\mathcal{M}(G/P, \tau, \beta)(T)\) are triples \((C, x, f)\), where \((C, x)\) is a \(\tau\)-marked prestable curve over \(T\) and \(f = (f_v)_{v \in V_\tau}\) is a family of \(k\)-morphisms \(f_v : C_v \to G/P\) such that

1. for all \(i \in F_\tau\) we have \(f_{\theta(i)}(x_i) = f_{\theta(j_r(i))}(x_{j_r(i)})\),
2. for all \(v \in V_\tau\) we have \(f_{v, \ast}[C_v] = \beta(v)\).

**Remark** If \((\tau, \beta)\) is stable, then \(\overline{\mathcal{M}}(G/P, \tau, \beta)\) is an open substack of \(\mathcal{M}(G/P, \tau, \beta)\).

Note that \(G^{\text{V}_\tau}\) acts on \(\mathcal{M}(G/P, \tau, \beta)\) as follows. An element \((g_w)_{w \in V_\tau}\) of \(G^{\text{V}_\tau}\) takes \((C, x, (f_v)_{v \in V_\tau})\) to \((C, x, (g_{\phi(v)} \circ f_v)_{v \in V_\tau})\), where \(\phi : \tau \to \tilde{\tau}\) is the structure contraction. Let

\[
\mathcal{M}(G/P, \tau, \beta)/G^{\text{V}_\tau}
\]

be the stack-theoretic quotient of this action. This is an abuse of notation, since this is a left and not a right action.
We shall let $G^V$ act trivially on $\mathcal{M}(\tau)$ and denote by 
$$\mathcal{M}(\tau)/G^V$$
the quotient.

**Proposition 12** There is a natural cartesian diagram of algebraic $k$-stacks

\[
\begin{array}{ccc}
\mathcal{M}(G/P, \tau, \beta)/G^V & \xrightarrow{\kappa} & H^1(\tau, P) \\
\eta \downarrow & & \downarrow \\
\mathcal{M}(\tau)/G^V & \xrightarrow{\iota} & H^1(\tau, G).
\end{array}
\]

The vertical maps are representable, the horizontal maps are local immersions.

**Proof.** This is essentially the fact that a map to $G/P$ is the same as a principal $P$-bundle with a trivialization of the associated $G$-bundle. □

The morphism $\iota$ is a local regular immersion with normal bundle $R^1\pi_*\mathcal{O} \otimes g$. Thus the normal cone $C(\tau, \beta)$ of $\mathcal{M}(G/P, \tau, \beta)/G^V$ in $H^1(\tau, P)$ is a cone in

$$n(\tau, \beta) = \eta^*R^1\pi_*\mathcal{O} \otimes g.$$

Pulling back to $\mathcal{M}(G/P, \tau, \beta)$ and, if $(\tau, \beta)$ is stable, to $\overline{\mathcal{M}}(G/P, \tau, \beta)$ defines $G^V$-equivariant cones, which we shall still denote $C(\tau, \beta)$, inside equivariant vector bundles, which we shall still denote by $n(\tau, \beta)$.

Let us now assume that $(\tau, \beta)$ is stable. Then we may intersect the cone $C(\tau, \beta)$ over $\overline{\mathcal{M}}(G/P, \tau, \beta)$ with the zero section of the vector bundle $n(\tau, \beta)$, to define a cycle class

$$J(\tau, \beta) \in A_{\dim(G/P, \tau, \beta)}(\overline{\mathcal{M}}(G/P, \tau, \beta))$$

with rational coefficients. Note that $C(\tau, \beta)$ is pure of the correct dimension, since it is constructed as a normal cone inside a smooth stack of the correct dimension.

**Proposition 13** The collection of cycle classes $J(\tau, \beta)$ is the orientation of $\overline{\mathcal{M}}$ over $\mathfrak{g}_s(G/P)$ defined using the intrinsic normal cone.

**Proof.** This follows from [1] Example 7.6, since

$$(R\pi_*f^*T_{G/P})^\vee = \kappa^*L^\vee_{\mathcal{H}^1_\beta(\tau, P)/\mathcal{H}^1(\tau, G)}.$$

□

**Remark** As a corollary we get that the orientation classes $J(\tau, \beta)$ are $G^V$-invariant. The same is then true for the Gromov-Witten invariants.
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