Covariant Transform

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Dedicated to the memory of Cora Sadosky

Abstract. The paper develops theory of covariant transform, which is inspired by the wavelet construction. It was observed that many interesting types of wavelets (or coherent states) arise from group representations which are not square integrable or vacuum vectors which are not admissible. Covariant transform extends an applicability of the popular wavelets construction to classic examples like the Hardy space \( H_2 \), Banach spaces, covariant functional calculus and many others.

Keywords: Wavelets, coherent states, group representations, Hardy space, Littlewood–Paley operator, functional calculus, Berezin calculus, Radon transform, Möbius map, maximal function, affine group, special linear group, numerical range, characteristic function, functional model.

A general group-theoretical construction [1, 3, 6, 7, 13, 17, 21] of wavelets (or coherent states) starts from an irreducible square integrable representation—in the proper sense or modulo a subgroup. Then a mother wavelet is chosen to be admissible. This leads to a wavelet transform which is an isometry to \( L_2 \) space with respect to the Haar measure on the group or (quasi)invariant measure on a homogeneous space.

The importance of the above situation shall not be diminished, however an exclusive restriction to such a setup is not necessary, in fact. Here is a classical example from complex analysis: the Hardy space \( H_2(\mathbb{T}) \) on the unit circle and Bergman spaces \( B^p_2(\mathbb{D}) \) in the unit disk produce wavelets associated with representations \( \rho_1 \) and \( \rho_n \) of the group \( SL_2(\mathbb{R}) \) respectively [11]. While representations \( \rho_n \) are from square integrable discrete series, the mock discrete series representation \( \rho_1 \) is not square integrable [20, § VI.5; 23, § 8.4]. However it would be natural to treat the Hardy space in the same framework as Bergman ones. Some more examples will be presented below.

1. Covariant Transform
To make a sharp but still natural generalisation of wavelets we give the following definition.

Definition 1. [15] Let \( \rho \) be a representation of a group \( G \) in a space \( V \) and \( F \) be an operator from \( V \) to a space \( U \). We define a covariant transform \( \mathcal{W} \) from \( V \) to the space \( L(G, U) \) of \( U \)-valued functions on \( G \) by the formula:

\[
\mathcal{W} : v \mapsto \hat{v}(g) = F(\rho(g^{-1})v), \quad v \in V, \ g \in G.
\]

Operator \( F \) will be called fiducial operator in this context.
We borrow the name for operator $F$ from fiducial vectors of Klauder and Skagerstam [17].

Remark 2. We do not require that fiducial operator $F$ shall be linear. Sometimes the homogeneity, i.e. $F(tv) = tF(v)$ for $t > 0$, alone can be already sufficient, see Example 12.

Remark 3. Usefulness of the covariant transform is in the reverse proportion to the dimensionality of the space $U$. The covariant transform encodes properties of $v$ in a function $Wv$ on $G$. For a low dimensional $U$ this function can be ultimately investigated by means of harmonic analysis. Thus $\dim U = 1$ (scalar-valued functions) is the ideal case, however, it is unattainable sometimes, see Example 9 below. We may have to use a higher dimensions of $U$ if the given group $G$ is not rich enough.

As we will see below covariant transform is a close relative of wavelet transform. The name is chosen due to the following common property of both transformations.

**Theorem 4.** The covariant transform (1) intertwines $\rho$ and the left regular representation $\Lambda$ on $L(G, U)$:

$$W\rho(g) = \Lambda(g)W.$$

Here $\Lambda$ is defined as usual by:

$$\Lambda(g) : f(h) \mapsto f(g^{-1}h).$$

(2)

**Proof.** We have a calculation similar to wavelet transform [13, Prop. 2.6]. Take $u = \rho(g)v$ and calculate its covariant transform:

$$[W(\rho(g)v)](h) = [W(\rho(g)v)](h) = F(\rho(h^{-1})\rho(g)v)$$

$$= F(\rho((g^{-1}h)^{-1})v)$$

$$= [Wv](g^{-1}h)$$

$$= \Lambda(g)[Wv](h).$$

The next result follows immediately:

**Corollary 5.** The image space $W(V)$ is invariant under the left shifts on $G$.

2. **Examples of Covariant Transform**

We start from the classical example of the group-theoretical wavelet transform:

**Example 6.** Let $V$ be a Hilbert space with an inner product $\langle \cdot, \cdot \rangle$ and $\rho$ be a unitary representation of a group $G$ in the space $V$. Let $F : V \to \mathbb{C}$ be a functional $v \mapsto \langle v, v_0 \rangle$ defined by a vector $v_0 \in V$. The vector $v_0$ is oftenly called the mother wavelet in areas related to signal processing or the vacuum state in quantum framework.

Then the transformation (1) is the well-known expression for a wavelet transform [1, (7.48)] (or representation coefficients):

$$W : v \mapsto \hat{v}(g) = \langle \rho(g^{-1})v, v_0 \rangle = \langle v, \rho(g)v_0 \rangle, \quad v \in V, \ g \in G.$$  

(3)

The family of vectors $v_g = \rho(g)v_0$ is called wavelets or coherent states. In this case we obtain scalar valued functions on $G$, thus the fundamental rôle of this example is explained in Rem. 3.

This scheme is typically carried out for a square integrable representation $\rho$ and $v_0$ being an admissible vector [1,3,6,7,21]. In this case the wavelet (covariant) transform is a map into the square integrable functions [5] with respect to the left Haar measure. The map becomes an isometry if $v_0$ is properly scaled.
However square integrable representations and admissible vectors does not cover all interesting cases.

**Example 7.** Let $G = \text{Aff}$ be the “$ax + b$” (or affine) group [1, § 8.2]: the set of points $(a, b)$, $a \in \mathbb{R}_+$, $b \in \mathbb{R}$ in the upper half-plane with the group law:

$$(a, b) * (a', b') = (aa', ab' + b)$$

and left invariant measure $a^{-2} da \, db$. Its isometric representation on $V = L_p(\mathbb{R})$ is given by the formula:

$$[\rho_p(g) f](x) = a^2 f(ax + b), \quad \text{where } g^{-1} = (a, b).$$

We consider the operators $F_\pm : L_2(\mathbb{R}) \to \mathbb{C}$ defined by:

$$F_\pm(f) = \frac{1}{2\pi i} \int_{x+i} f(t) \, dt.$$  

(6)

Then the covariant transform (1) is the Cauchy integral from $L_p(\mathbb{R})$ to the space of functions $\hat{f}(a,b)$ such that $a^{-\frac{1}{2}} \hat{f}(a,b)$ is in the Hardy space in the upper/lower half-plane $H_p(\mathbb{R}_\pm^2)$. Although the representation (5) is square integrable for $p = 2$, the function $\frac{1}{x+i}$ used in (6) is not an admissible vacuum vector. Thus the complex analysis become decoupled from the traditional wavelet theory. As a result the application of wavelet theory shall rely on an extraneous mother wavelets [9].

Many important objects in complex analysis are generated by inadmissible mother wavelets like (6). For example, if $F : L_2(\mathbb{R}) \to \mathbb{C}$ is defined by $F : f \mapsto F_+ f + F_- f$ then the covariant transform (1) reduces to the Poisson integral. If $F : L_2(\mathbb{R}) \to \mathbb{C}^2$ is defined by $F : f \mapsto (F_+ f, F_- f)$ then the covariant transform (1) represents a function $f$ on the real line as a jump:

$$f(z) = f_+(z) - f_-(z), \quad f_\pm(z) \in H_p(\mathbb{R}_\pm^2)$$

(7)

between functions analytic in the upper and the lower half-planes. This makes a decomposition of $L_2(\mathbb{R})$ into irreducible components of the representation (5). Another interesting but non-admissible vector is the Gaussian $e^{-x^2}$.

**Example 8.** For the group $G = SL_2(\mathbb{R})$ [20] let us consider the unitary representation $\rho$ on the space of square integrable function $L_2(\mathbb{R}_+^2)$ on the upper half-plane through the Möbius transformations:

$$\rho(g) : f(z) \mapsto \frac{1}{(cz+d)^2} f \left( \frac{az+b}{cz+d} \right), \quad g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$  

(8)

This is a representation from the discrete series and $L_2(\mathbb{D})$ and irreducible invariant subspaces are parametrised by integers. Let $F_k$ be the functional $L_2(\mathbb{R}_+^2) \to \mathbb{C}$ of pairing with the lowest/highest $k$-weight vector in the corresponding irreducible component $B_k(\mathbb{R}_+^2)$, $k \geq 2$ of the discrete series [20, Ch. VI]. Then we can build an operator $F$ from various $F_k$ similarly to the previous Example. In particular, the jump representation (7) on the real line generalises to the representation of a square integrable function $f$ on the upper half-plane as a sum

$$f(z) = \sum_k a_k f_k(z), \quad f_k \in B_n(\mathbb{R}_+^2)$$

for prescribed coefficients $a_k$ and analytic functions $f_k$ in question from different irreducible subspaces.

Covariant transform is also meaningful for principal and complementary series of representations of the group $SL_2(\mathbb{R})$, which are not square integrable [11].
Example 9. A straightforward generalisation of Ex. 6 is obtained if \( V \) is a Banach space and \( F : V \to \mathbb{C} \) is an element of \( \mathcal{V}^* \). Then the covariant transform coincides with the construction of wavelets in Banach spaces [13].

Example 10. The next stage of generalisation is achieved if \( V \) is a Banach space and \( F : V \to \mathbb{C}^n \) is a linear operator. Then the corresponding covariant transform is a map \( \mathcal{W} : V \to \mathbb{L}(G, \mathbb{C}^n) \). This is closely related to M.G. Krein’s works on directing functionals [18], see also multiresolution wavelet analysis [2], Clifford-valued Bargmann spaces [4] and [1, Thm. 7.3.1].

Example 11. Let \( F \) be a projector \( L_p(\mathbb{R}) \to L_p(\mathbb{R}) \) defined by the relation \( (Ff)(\lambda) = \chi(\lambda)f(\lambda) \), where the hat denotes the Fourier transform and \( \chi(\lambda) \) is the characteristic function of the set \([-2, -1] \cup [1, 2] \). Then the covariant transform \( L_p(\mathbb{R}) \to \mathbb{C}(\text{Aff}, L_p(\mathbb{R})) \) generated by the representation (5) of the affine group from \( F \) contains all information provided by the Littlewood–Paley operator [8, § 5.1.1].

Example 12. A step in a different direction is a consideration of non-linear operators. Take again the “\( ax + b \)” group and its representation (5). We define \( F \) to be a homogeneous but non-linear functional \( V \to \mathbb{R}_+ \):

\[
F(f) = \frac{1}{2} \int_{-1}^{1} |f(x)| \, dx.
\]

The covariant transform (1) becomes:

\[
[W_p f](a, b) = F(\rho_p(a, b)f) = \frac{1}{2} \int_{-1}^{1} |a^{\frac{1}{p}} f(ax + b)| \, dx = a^{\frac{1}{p}} \frac{1}{2a} \int_{b-a}^{b+a} |f(x)| \, dx.
\]

(9)

Obviously \( M_f(b) = \max_a [W_{\infty} f](a, b) \) coincides with the Hardy maximal function, which contains important information on the original function \( f \). From the Cor. 5 we deduce that the operator \( M : f \to M_f \) intertwines \( \rho_p \) with itself \( \rho_p M = M \rho_p \).

Of course, the full covariant transform (9) is even more detailed than \( M \). For example, \( ||f|| = \max_a [W_{\infty} f](\frac{1}{2}, b) \) is the shift invariant norm [10].

Example 13. Let \( V = L_2(\mathbb{R}^2) \) be the space of compactly supported bounded functions on the plane. We take \( F \) be the linear operator \( V \to \mathbb{C} \) of integration over the real line:

\[
F : f(x, y) \mapsto F(f) = \int_{\mathbb{R}} f(x, 0) \, dx.
\]

Let \( G \) be the group of Euclidean motions of the plane represented by \( \rho \) on \( V \) by a change of variables. Then the wavelet transform \( F(\rho(g)f) \) is the Radon transform.

Example 14. Let a representation \( \rho \) of a group \( G \) act on a space \( X \). Then there is an associated representation \( \rho_B \) of \( G \) on a space \( V = B(X, Y) \) of linear operators \( X \to Y \) defined by the identity:

\[
(\rho_B(g)A)x = A(\rho(g^{-1})x), \quad x \in X, \quad g \in G, \quad A \in B(X, Y).
\]

(10)

Following the Remark 3 we take \( F \) to be a functional \( V \to \mathbb{C} \), for example \( F \) can be defined from a pair \( x \in X, \ l \in Y^* \) by the expression \( F : A \mapsto \langle Ax, l \rangle \). Then the covariant transform is:

\[
W : A \mapsto \hat{A}(g) = F(\rho_B(g)A).
\]

This is an example of covariant calculus [13, 14].
Thus we approached the functional model where the characteristic function $X$ space $F$ accordance with Remark 3 the model is most fruitful for the case of operator $D$. Let the restriction of $G$ transform. Let $F$ The choice of fiducial operator

3. Induced Covariant Transform

The group $SU(1,1) \simeq SL_2(\mathbb{R})$ consists of $2 \times 2$ matrices of the form $\begin{pmatrix} \alpha & \beta \\ \beta & \bar{\alpha} \end{pmatrix}$ with the unit determinant [20, § IX.1]. Let $T$ be an operator with the spectral radius less than 1. Then the associated Möbius transformation

$$g: T \mapsto g \cdot T = \frac{\alpha T + \beta I}{\beta T + \bar{\alpha} I}, \quad \text{where } g = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \in SL_2(\mathbb{R}),$$

(11)

produces a well-defined operator with the spectral radius less than 1 as well. Thus we have a representation of $SU(1,1)$. A choice of an operator $F$ will define the corresponding covariant transform. In this way we obtain generalisations of Riesz–Dunford functional calculi [14].

Example 18. Consider again the action (11) of the Möbius transformations on operators from the previous Example. Let us introduce the defect operators $D_T = (I - T^* T)^{1/2}$ and $D_{T^*} = (I - TT^*)^{1/2}$. For the case $F = D_{T^*}$ the covariant transform is, cf. [22, § VI.1, (1.2)]:

$$[WT](g) = F(g \cdot T) = -e^{i\phi} \Theta_T(z) D_T,$$

for $g = \begin{pmatrix} e^{i\phi/2} & 0 \\ 0 & e^{-i\phi/2} \end{pmatrix} \begin{pmatrix} 1 & -z \\ \bar{z} & 1 \end{pmatrix}$,

where the characteristic function $\Theta_T(z)$ [22, § VI.1, (1.1)] is:

$$\Theta_T(z) = -T + D_{T^*} (I - z T^*)^{-1} z D_T.$$

Thus we approached the functional model of operators from the covariant transform. In accordance with Remark 3 the model is most fruitful for the case of operator $F = D_{T^*}$ being one-dimensional.

3. Induced Covariant Transform

The choice of fiducial operator $F$ can significantly influence the behaviour of the covariant transform. Let $G$ be a group and $H$ be its closed subgroup with the corresponding homogeneous space $X = G/H$. Let $\rho$ be a representation of $G$ by operators on a space $V$, we denote by $\rho_H$ the restriction of $\rho$ to the subgroup $H$.

Definition 19. Let $\chi$ be a representation of the subgroup $H$ in a space $U$ and $F: V \to U$ be an intertwining operator between $\chi$ and the representation $\rho_H$:

$$F(\rho(h)v) = F(v)\chi(h), \quad \text{for all } h \in H, \ v \in V.$$

Then the covariant transform (1) generated by $F$ is called the induced covariant transform.
The following is the main motivating example.

**Example 20.** Consider the traditional wavelet transform as outlined in Ex. 6. Chose a vacuum vector $v_0$ to be a joint eigenvector for all operators $\rho(h)$, $h \in H$, that is $\rho(h)v_0 = \chi(h)v_0$, where $\chi(h)$ is a complex number depending of $h$. Then $\chi$ is obviously a character of $H$.

The image of wavelet transform (3) with such a mother wavelet will have a property:

$$\hat{\psi}(gh) = \langle v, \rho(gh)v_0 \rangle = \langle v, \rho(g)\chi(h)v_0 \rangle = \chi(h)\hat{\psi}(g).$$

Thus the wavelet transform is uniquely defined by cosets on the homogeneous space $G/H$. In this case we can speak about the reduced wavelet transform [12]. A representation $\rho_0$ is square integrable mod $H$ if the induced wavelet transform $[Wf_0](w)$ of the vacuum vector $f_0(x)$ is square integrable on $X$.

The image of induced covariant transform have the similar property:

$$\hat{\psi}(gh) = F(\rho(gh)^{-1})v = F(\rho(h)^{-1}\rho(g^{-1})v) = F(\rho(g^{-1})v)\chi(h^{-1}).$$

(12)

Thus it is enough to know the value of the covariant transform only at a single element in every coset $G/H$ in order to reconstruct it for the entire group $G$ by the representation $\chi$. Since coherent states (wavelets) are now parametrised by points homogeneous space $G=H$ they are referred sometimes as coherent states which are not connected to a group [16], however this is true only in a very narrow sense as explained above.

**Example 21.** To make it more specific we can consider the representation of $SL_2(\mathbb{R})$ defined on $L^2(\mathbb{R})$ by the formula, cf. (8):

$$\rho(g) : f(z) \mapsto \frac{1}{(cx+d)^2} f \left( \frac{ax+b}{cx+d} \right), \quad g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$  

Let $K \subset SL_2(\mathbb{R})$ be the compact subgroup of matrices $h_t = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$. Then for the fiducial operator $F_\pm (6)$ we have $F_\pm \circ \rho(h_t) = e^{\pm it}F_\pm$. Thus we can consider the covariant transform only for points in $SL_2(\mathbb{R})/K$, however this set can be naturally identified with the $ax+b$ group. Thus we do not obtain any advantage of extending the group in Ex. 7 from $ax+b$ to $SL_2(\mathbb{R})$ if we will be still using the fiducial operator $F_\pm (6)$.

Functions on the group $G$, which have the property $\hat{\psi}(gh) = \hat{\psi}(g)\chi(h)$ (12), provide a space for the representation of $G$ induced by the representation $\chi$ of the subgroup $H$. This explains the choice of the name for induced covariant transform.

**Remark 22.** Induced covariant transform uses the fiducial operator $F$ which passes through the action of the subgroup $H$. This reduces information which we obtained from this transform in some cases.

There is also a simple connection between a covariant transform and right shifts:

**Proposition 23.** Let $G$ be a Lie group and $\rho$ be a representation of $G$ in a space $V$. Let $[Wf](g) = F(\rho(g^{-1})f)$ be a covariant transform defined by the fiducial operator $F : V \rightarrow U$. Then the right shift $[Wf](gg')$ by $g'$ is the covariant transform $[Wf'](g) = F'(\rho(g^{-1})f)$ defined by the fiducial operator $F' = F \circ \rho(g^{-1})$.

In other words the covariant transform intertwines right shifts with the associated action $\rho_B$ (10) on fiducial operators.

Although the above result is obvious, its infinitesimal version has interesting consequences.
Corollary 24. Let $G$ be a Lie group with a Lie algebra $\mathfrak{g}$ and $\rho$ be a smooth representation of $G$. We denote by $d\rho_B$ the derived representation of the associated representation $\rho_B$ (10) on fiducial operators.

Let a fiducial operator $F$ be a null-solution, i.e. $AF = 0$, for the operator $A = \sum_j a_j d\rho_B^X_j$, where $X_j \in \mathfrak{g}$ and $a_j$ are constants. Then the wavelet transform $[Wf](g) = F(\rho(g^{-1})f)$ for any $f$ satisfies:

$$DF(g) = 0,$$

where $D = \sum_j a_j g^{X_j}$.

Here $g^{X_j}$ are the left invariant fields (Lie derivatives) on $G$ corresponding to $X_j$.

Example 25. Consider the representation $\rho$ (5) of the $ax + b$ group with the $p = 1$. Let $A$ and $N$ be the basis of the corresponding Lie algebra generating one-parameter subgroups $(e^t, 0)$ and $(0, t)$. Then the derived representations are:

$$[d\rho^A f](x) = f(x) + x f'(x), \quad [d\rho^N f](x) = f'(x).$$

The corresponding left invariant vector fields on $ax + b$ group are:

$$\mathfrak{L}^A = a \partial_a, \quad \mathfrak{L}^N = a \partial_b.$$

The mother wavelet $\frac{1}{x+i}$ is a null solution of the operator $d\rho^A + i d\rho^N = I + (x + i) \frac{d}{dx}$. Therefore the covariant transform with the fiducial operator $F_+(6)$ will consist with the null solutions to the operator $\mathfrak{L}^A - i \mathfrak{L}^N = -ia(\partial_b + i \partial_a)$, that is in the essence the Cauchy-Riemann operator in the upper half-plane.

There is a statement which extends the previous Corollary from differential operators to integro-differential ones. We will formulate it for the wavelets setting.

Corollary 26. Let $G$ be a Lie group with a Lie algebra $\mathfrak{g}$ and $\rho$ be a unitary representation of $G$, which can be extended to a vector space $V$ of functions or distributions on $G$.

Let a mother wavelet $w \in V'$ satisfy the equation

$$\int_G a(g) \rho(g) w \, dg = 0,$$

for a fixed distribution $a(g) \in V$. Then any wavelet transform $F(g) = Wf(g) = \langle f, \rho(g)w_0 \rangle$ obeys the condition:

$$DF = 0,$$

where $D = \int_G \tilde{a}(g) R(g) \, dg$.

with $R$ being the right regular representation of $G$.

Clearly the Corollary 24 is a particular case of Corollary 26.

4. Inverse Covariant Transform

An object invariant under the left action $\Lambda$ (2) is called left invariant. For example, let $L$ and $L'$ be two left invariant spaces of functions on $G$. We say that a pairing $\langle \cdot, \cdot \rangle : L \times L' \to \mathbb{C}$ is left invariant if

$$\langle \Lambda(g)f, \Lambda(g)f' \rangle = \langle f, f' \rangle, \quad \text{for all } f \in L, f' \in L'.$$  \hspace{1cm} (13)

Remark 27. (i) We do not require the pairing to be linear in general.

(ii) If the pairing is invariant on space $L \times L'$ it is not necessarily invariant (or even defined) on the whole $C(G) \times C(G)$.
In a more general setting we shall study an invariant pairing on a homogeneous spaces instead of the group. However due to length constraints we cannot consider it here beyond the Example 30.

An invariant pairing on $G$ can be obtained from an invariant functional $l$ by the formula

$$\langle f_1, f_2 \rangle = l(f_1 \bar{f}_2).$$

For a representation $\rho$ of $G$ in $V$ and $v_0 \in V$ we fix a function $w(g) = \rho(g)v_0$. We assume that the pairing can be extended in its second component to this $V$-valued functions, say, in the weak sense.

**Definition 28.** Let $\langle \cdot, \cdot \rangle$ be a left invariant pairing on $L \times L'$ as above, let $\rho$ be a representation of $G$ in a space $V$, we define the function $w(g) = \rho(g)v_0$ for $v_0 \in V$. The **inverse covariant transform** $M$ is a map $L \to V$ defined by the pairing:

$$M : f \mapsto \langle f, w \rangle,$$

where $f \in L$. 

**Example 29.** Let $G$ be a group with a unitary square integrable representation $\rho$. An invariant pairing of two square integrable functions is obviously done by the integration over the Haar measure:

$$\langle f_1, f_2 \rangle = \int_G f_1(g) \bar{f}_2(g) \, dg.$$

For an admissible vector $v_0$ [5], [1, Chap. 8] the inverse covariant transform is known in this setup as a **reconstruction formula**.

**Example 30.** Let $\rho$ be a square integrable representation of $G$ modulo a subgroup $H \subset G$ and let $X = G/H$ be the corresponding homogeneous space with a quasi-invariant measure $dx$. Then integration over $dx$ with an appropriate weight produces an invariant pairing. The inverse covariant transform is a more general version [1, (7.52)] of the reconstruction formula mentioned in the previous example.

Let $\rho$ be not a square integrable representation (even modulo a subgroup) or let $v_0$ be inadmissible vector of a square integrable representation $\rho$. An invariant pairing in this case is not associated with an integration over any non singular invariant measure on $G$. In this case we have a **Hardy pairing**. The following example explains the name.

**Example 31.** Let $G$ be the “$ax + b$” group and its representation $\rho$ (5) from Ex. 7. An invariant pairing on $G$, which is not generated by the Haar measure $a^{-2}da\,db$, is:

$$\langle f_1, f_2 \rangle = \lim_{a \to 0} \int_{-\infty}^{\infty} f_1(a, b) \bar{f}_2(a, b) \, db.$$

For this pairing we can consider functions $\frac{1}{2a(a+b)}$ or $e^{-x^2}$, which are not admissible vectors in the sense of square integrable representations. Then the inverse covariant transform provides an **integral resolutions** of the identity.

Similar pairings can be defined for other semi-direct products of two groups. We can also extend a Hardy pairing to a group, which has a subgroup with such a pairing.

**Example 32.** Let $G$ be the group $SL_2(\mathbb{R})$ from the Ex. 8. Then the “$ax + b$” group is a subgroup of $SL_2(\mathbb{R})$, moreover we can parametrise $SL_2(\mathbb{R})$ by triples $(a, b, \theta)$, $\theta \in (-\pi, \pi]$ with the respective Haar measure [20, III.1(3)]. Then the Hardy pairing

$$\langle f_1, f_2 \rangle = \lim_{a \to 0} \int_{-\infty}^{\infty} f_1(a, b, \theta) \bar{f}_2(a, b, \theta) \, db \, d\theta.$$
is invariant on $SL_2(\mathbb{R})$ as well. The corresponding inverse covariant transform provides even a finer resolution of the identity which is invariant under conformal mappings of the Lobachevsky half-plane.

A further study of covariant transform shall be continued elsewhere.

Acknowledgement. Author is grateful to the anonymous referee for many helpful suggestions.

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