A Følner Invariant for Type $II_1$ Factors

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Abstract

In this article we introduce an isomorphism invariant for type $II_1$ factors using the Connes-Følner condition. We compute bounds of this number for free group factors.

Key words: Functional Analysis, Operator Algebras, Type $II_1$ Factors

Introduction

In a series of papers [11]-[15], Murray and von Neumann introduced “rings of operators”, known nowadays as von Neumann algebras. To them it was clear that what they were developing was a theory of quantized groups. Many of the examples in their original paper come from group algebras. Subsequently, concepts and results in group theory have been a major source of motivation for the development of operator algebras. Many of the important operator algebra concepts, such as amenability, property $T$, etc., come directly from properties of various groups. In this paper, we are concerned with a certain characterization of amenability for groups due to Følner. Our main aim is to introduce an isomorphism invariant, motivated by Følner’s characterization, for an important class of von Neumann algebras called type $II_1$ factors.

Von Neumann himself showed that any von Neumann algebra is a direct sum of “simple objects”, called factors. These are weak-operator closed self-adjoint subalgebras of $B(\mathcal{H})$, the algebra of all bounded operators on a Hilbert space $\mathcal{H}$, whose centers consist of only scalar multiples of the identity operator. Factors are called finite if there is a faithful tracial state on them. Those finite factors which are finite-dimensional as vector spaces are full matrix algebras.
Those finite factors which are infinite-dimensional are called factors of type $II_1$. In order to complete the classification of all factors up to $*$-isomorphism, it remains to classify the factors of type $II_1$ (cf. [5]).

A factor $\mathcal{M}$ is injective if it is the range of a Banach space projection $\Phi \in B(B(\mathcal{H}))$, for some Hilbert space $\mathcal{H}$. There are few computable nontrivial invariants for type $II_1$ factors in general, but the classification of injective factors is complete [4]. It stands to reason that we should try to use tools from the classification of injective factors to define isomorphism invariants for general type $II_1$ factors. In this paper, we define an invariant $F_\text{ø}(\mathcal{M})$ that will measure how badly a separable type $II_1$ factor $\mathcal{M}$ fails to satisfy Connes’ Følner-type condition (Theorem 5.1 in [4]). We compute explicit bounds for $F_\text{ø}(\mathcal{M})$ in the case where $\mathcal{M}$ is the free group factor $L(F_n)$.

The layout of the paper is as follows. In the first section, we present some background on the Følner condition for groups in order to provide some motivation. In the second section we discuss a Følner invariant for groups. In the third section we give some examples of factors and some questions that will provide further context. In the fourth and final section we define the pre-invariant $F_\text{ø}(\mathcal{M}, X)$ for a finite subset $X$ of unitary elements in $\mathcal{M}$, and the invariant $F_\text{ø}(\mathcal{M})$. We then prove that $F_\text{ø}(\bigotimes_{j=1}^{\infty}(L(F_2)_j)) > 0$, and that for any type $II_1$ factor $\mathcal{M}$, $F_\text{ø}(\mathcal{M}) \leq 2$. Finally, we prove that $F_\text{ø}(L(F_n), X) \leq \sqrt{2 - \frac{2}{n^2}}$, where $X = \{L_{a_1}, L_{a_2}, ..., L_{a_n}\}$ is the set of standard generators of $L(F_n)$.

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1 Følner Conditions

Let $G$ be a discrete group with identity $e$. Let $\mathbb{C}G$ denote the complex group algebra of formal linear combinations of elements from $G$. This is a unital $*$-algebra, with involution given by the conjugate-linear extension of the map $g \mapsto g^{-1}$. A faithful trace state $\tau_0$ is defined on $\mathbb{C}G$ by

$$\tau_0(\sum \lambda_g g) = \lambda_e.$$

Performing the GNS construction using $\tau_0$, we faithfully embed $\mathbb{C}G$ as a $*$-subalgebra $\text{span}\{L_g : g \in G\}$ of $B(L^2(\mathbb{C}G, \tau_0))$, where the action of $L_g$ on $h \in G \subseteq \mathbb{C}G$ is given by left-translation in the group $L_g h = gh$. We define the (left)
The classification of injective factors gives us that any two injective type II$_1$ factors are $\ast$-isomorphic. Furthermore, there are myriad invariant properties (see [8]) that are equivalent to injectivity of a type II$_1$ factor $\mathcal{M} \subseteq B(\mathcal{H})$ with trace $\tau$. One such property is Connes’ Følner-type condition, found in the statement of Theorem 5.1 in [4]: Given $\{x_1, x_2, \ldots, x_n\} \subseteq M$ and $\varepsilon > 0$, there exists a nonzero finite-rank projection $e \in B(\mathcal{H})$ such that $\forall j \in \{1, 2, \ldots, n\}$

$$\frac{\#((g_jU \cup U) \setminus (g_jU \cap U))}{\#U} \leq \varepsilon.$$ 

In [16], I. Namioka was able to prove, using functional analysis, that an amenable group satisfies Følner’s condition. The key ingredient in Namioka’s proof is a theorem of Day (see [16], Theorem 2.2).

A discrete group $G$ is amenable if there exists a state on $l^\infty(G)$ which is invariant under the left action of $G$ on $l^\infty(G)$. Such a state will be called an invariant mean on $l^\infty(G)$. In [7], Følner used combinatorial methods to find the following condition on a countable discrete group $G$, and to prove that this condition holds if and only if $G$ is amenable: Given $\{g_1, g_2, \ldots, g_n\} \subseteq G$ and $\varepsilon > 0$, there exists a finite, non-empty set $U \subseteq G$ such that $\forall j \in \{1, 2, \ldots, n\}$

$$\frac{\#((g_jU \cup U) \setminus (g_jU \cap U))}{\#U} \leq \varepsilon.$$ 

To elucidate the origin of this condition, note that if $\mathcal{M}$ is injective then $\rho = \tau \circ \Phi$ defines a state on $B(\mathcal{H})$ with the property $\rho|_\mathcal{M} = \tau$. Such a state $\rho$ is called a hypertrace on $\mathcal{M}$. In the case of the von Neumann algebra $L(G)$ of a discrete group $G$ we have that $l^\infty(G)$ is embedded in $B(l^2(G))$ as multiplication operators. We see that if $L(G)$ is injective then $\tau \circ \Phi|_{l^\infty(G)}$ is an invariant mean. Conversely, given an invariant mean on $l^\infty(G)$, an averaging process over $R(G)(= L(G))$ can be used to construct a conditional expectation of $B(l^2(G))$ onto $L(G)$, and hence a hypertrace (see 8.7.24 and 8.7.29 of [18]). This suggests that for a general type II$_1$ factor, we may think of a hypertrace as analogous to an invariant mean. Connes exploited this analogy to prove that when a type II$_1$ factor admits a hypertrace then the factor satisfies the above Følner-type condition. The proof of this follows Namioka’s method of obtaining Følner’s condition from an invariant mean on a group.
2 A Følner Invariant for Groups

In [3], Arzhantseva, Burillo, Lustig, Reeves, Short and Ventura have defined a group invariant $Føl(G)$ that measures how badly a finitely-generated discrete group $G$ fails to satisfy the classical Følner condition. In particular this number satisfies, for a group $G$ generated by $n$ elements, the inequality $0 \leq Føl(G) \leq \frac{4n-2}{2n-1}$. Also $Føl(G) = 0$ whenever $G$ is amenable and $Føl(G) = \frac{4n-2}{2n-1}$ if and only if $G = \mathbb{F}_n$. The notion of boundary of a subset of a finitely generated group $G$ generally depends on a given finite generating subset $X$. Arzhantseva et. al. define

$$Føl(G, X) = \inf_{A \subseteq G \text{ finite}} \frac{\#\partial_X A}{\#A}$$

where $\partial_X A = \{ a \in A | ax \not\in A \text{ for some } x \in X^{\pm 1}\}$ is the interior boundary of $A$ with respect to $X$ in $G$. They go on to define the universal Følner invariant

$$Føl(G) = \inf_X Føl(G, X)$$

where the infimum is taken over all finite generating subsets $X$ of $G$. They prove that if $Føl(G, X) = 0$ for some finite generating set $X$ of $G$, then $Føl(G, X') = 0$ for any other finite generating set $X'$, and this happens only if $G$ is amenable. Non-amenable discrete groups for which $Føl(G) = 0$ are called weakly amenable and those for which $Føl(G) \neq 0$ are called uniformly non-amenable. In [3] it is also proven that groups of both types exist.

In light of the above results, we define the invariant $Føl(M)$ for a type $II_1$ factor with separable predual. We note that the analogy is not entirely straightforward with the group case. The first major difference is that we exclusively use unitary elements in the computation of $Føl(M)$, to avoid blowing up due to scaling by a constant in the Connes-Følner condition. The second major difference is that in a type $II_1$ factor we can find unitary elements arbitrarily norm-close to the identity, which implies that the second infimum taken in the group case would always be zero in the new setting. This, in particular, means that the invariant we introduce will not provide a satisfactory notion of weak-amenability for type $II_1$ factors.

3 Some Related Examples of Factors

For the basics of the theory of operator algebras, we refer the reader to [18].

The first classification result in the theory of type $II_1$ factors is the following, due to Murray and von Neumann[13]. It remains one of the deepest results in the subject.
Theorem 1 Let $\Pi$ denote the group of those permutations of $\mathbb{Z}$ each of which permutes only finitely many integers, and let $\mathbb{F}_n$ be the nonabelian free group on $n$ generators. Both of these groups are i.c.c., and give rise to non-isomorphic type $II_1$ factors.

The number $Fol(\mathcal{M})$ will be zero if and only if the factor $\mathcal{M}$ is injective. The main problem is to determine whether or not the invariant can distinguish between a pair of non-injective type $II_1$ factors. We are particularly interested in computing the number in the following two cases.

Example 2 Let $B(m,n) = \langle a_1, \ldots, a_m \mid g^n = e \rangle$ denote the free Burnside group on $m$ generators with exponent $n$. If $m > 1$ and $n \geq 665$ is odd, then the centralizer of any nonidentity element in $B(m,n)$ is a cyclic group of order $n$ (cf. [1]). It follows in this case that $L(B(m,n))$ is a type $II_1$ factor. Also, in [2] it is shown that if $m > 1$ and $n \geq 665$ is odd then $B(m,n)$ is not amenable. It follows from our earlier discussion that $L(B(m,n))$ cannot be an injective factor.

Example 3 Consider R. Thompson’s group $F = \langle x_0, x_1, x_2, \ldots \mid x_i^{-1}x_nx_i = x_{i+1}, 0 \leq i \leq n \forall n \in \mathbb{N} \rangle$. It is proven in [17] that $F$ is i.c.c., and hence that $L(F)$ is a type $II_1$ factor. A famous conjecture of Geoghegan in 1979 asks if $F$ is a non-amenable group which contains no non-abelian free subgroup. It was proven by Brin and Squier in 1983 that $F$ contains no non-abelian free subgroup, but it is still unknown whether or not $F$ is an amenable group (cf. [17]).

It should be noted that distinguishing the $*$-isomorphism classes of the above type $II_1$ factors is an open problem. The last example is interesting, since finding a single finite subset $X \subseteq F$ with respect to which the pre-invariant $Fol(L(F), X) \neq 0$ amounts to showing that $F$ is not amenable.

4 Main Results

4.1 The $Følner$ Invariant

We first collect some basic facts about the Hilbert-Schmidt class.

Let $\mathcal{H}$ be a separable Hilbert space. For a positive operator $T \in B(\mathcal{H})$, let $Tr(T) = \sum_{i=1}^{\infty} \langle Te_i, e_i \rangle$, where $\{e_i\}_{i=1}^{\infty}$ is any orthonormal basis for $\mathcal{H}$. The Hilbert-Schmidt norm of an operator $T \in B(\mathcal{H})$ is given by

$$||T||_{H.S.} = Tr(T^*T)^{1/2}.$$
We say that \( T \in B(H) \) is in the Hilbert-Schmidt class when \( \|T\|_{\text{H.S.}} < \infty \). The class of all such operators in \( B(H) \) may be regarded as a Hilbert space when equipped with the inner product \( \langle A, B \rangle_{\text{H.S.}} = \text{Tr}(B^*A) \).

Let \( \mathcal{M} \) be a factor of type \( \text{II}_1 \) with trace \( \tau \) acting standardly on \( H (= L^2(\mathcal{M}, \tau)) \), and let \( U(\mathcal{M}) \) be the unitary group of \( \mathcal{M} \). Suppose throughout that \( \mathcal{M} \) has separable predual. Connes proves in [4] that \( \mathcal{M} \) is injective if and only if the following condition holds:

Given \( \{x_1, x_2, ..., x_n\} \subset U(\mathcal{M}) \) and \( \varepsilon > 0 \), there exists a nonzero finite-rank projection \( e \in B(H) \) such that for all \( j \in \{1, 2, ..., n\} \)

\[
\| [x_j, e] \|_{\text{H.S.}} \leq \varepsilon \|e\|_{\text{H.S.}} \text{ and } |\tau(x_j) - \langle x_j e, e \rangle_{\text{H.S.}}| \leq \varepsilon.
\]

We call this the Connes-Følner condition.

**Definition 1** Let \( \mathcal{M} \) be a factor of type \( \text{II}_1 \), and \( X = \{x_1, x_2, ..., x_n\} \) be a finite subset of \( U(\mathcal{M}) \). We define the property \( Q(X, \varepsilon) \) to be “there exists a nonzero finite-rank projection \( e \in B(H) \) such that for all \( j \in \{1, 2, ..., n\} \), \( \| [x_j, e] \|_{\text{H.S.}} \leq \varepsilon \|e\|_{\text{H.S.}} \) and \( |\tau(x_j) - \langle x_j e, e \rangle_{\text{H.S.}}| \leq \varepsilon.\)”

**Definition 2** Let \( \mathcal{M} \) be a factor of type \( \text{II}_1 \), and \( X \) be a finite subset of \( U(\mathcal{M}) \). Define

\[
\text{Føl}(\mathcal{M}, X) = \inf \{ \varepsilon > 0 : Q(X, \varepsilon) \}.
\]

**Definition 3** Let \( \mathcal{M} \) be a factor of type \( \text{II}_1 \). We define the universal Følner invariant \( \text{Føl}(\mathcal{M}) = \sup_X \text{Føl}(\mathcal{M}, X) \), where the supremum is taken over all finite sets \( X \subset U(\mathcal{M}) \).

**Remark 1** By Theorem 5.2 in [4], \( \mathcal{M} \) is injective if and only if \( \text{Føl}(\mathcal{M}) = 0 \).

We include the following basic observation about monotonicity.

**Proposition 1** Let \( \mathcal{M} \) be a factor of type \( \text{II}_1 \). If \( X_1 \) and \( X_2 \) are finite subsets of \( U(\mathcal{M}) \) that generate \( \mathcal{M} \) as a von Neumann algebra, and \( X_1 \subseteq X_2 \), then \( \text{Føl}(\mathcal{M}, X_1) \leq \text{Føl}(\mathcal{M}, X_2) \).

**Proof.** We have that for any \( \varepsilon > 0 \) that \( Q(X_2, \varepsilon) \Rightarrow Q(X_1, \varepsilon) \), hence

\[
\inf \{ \varepsilon > 0 : Q(X_1, \varepsilon) \} \leq \inf \{ \varepsilon > 0 : Q(X_2, \varepsilon) \}.
\]
4.2 Lower Bounds

4.2.1 Positivity of $F \phi (\bigotimes_{j=1}^{\infty} (L(F_2))_j)$

We review the construction of the ultraproduct of finite factors (cf. [4]).

Let $\mathcal{M}^{(n)}$ be finite factors with traces $\tau_n$, and let $\prod \mathcal{M}^{(n)}$ denote their $C^*$-product, i.e. the $C^*$-algebra of uniformly norm-bounded sequences equipped with coordinatewise operations and the supremum norm. Viewing the Stone-Šećek compactification $\beta \mathbb{N}$ as the maximal ideal space of $l^\infty(\mathbb{N}, \mathbb{C})$, for each $\omega \in \beta \mathbb{N}$ there corresponds a multiplicative linear functional $\rho \in (l^\infty(\mathbb{N}, \mathbb{C}))^\#$. Given $f \in l^\infty(\mathbb{N}, \mathbb{C})$, we define $\lim_{n \to \omega} f \equiv \rho(f)$. Consider a free ultrafilter $\omega \in \beta \mathbb{N} \setminus \mathbb{N}$. We have that $I_\omega = \{(A_i)_i \in \prod \mathcal{M}^{(n)} : \lim_{i \to \omega} \tau_i(A_i^* A_i) = 0\}$ is a closed two-sided ideal in $\prod \mathcal{M}^{(n)}$, and by a result of Sakai [19], the quotient $(\prod \mathcal{M}^{(n)}) / I_\omega$ is a factor von Neumann algebra $\prod_\omega \mathcal{M}^{(n)}$ with a faithful, normal trace $\tau_\omega$ defined by $\tau_\omega((A_i)_i + I_\omega) = \lim_{i \to \omega} \tau_i(A_i)$. The factor $\prod_\omega \mathcal{M}^{(n)}$ will be called an ultraproduct of the $\mathcal{M}^{(n)}$ with respect to the free ultrafilter $\omega$, or simply an ultraproduct of the $\mathcal{M}^{(n)}$. If $\mathcal{M}$ is a finite factor and $\mathcal{M}^{(n)} = \mathcal{M}$ for all $n$, then the ultraproduct is called an ultrapower, and is written as $\mathcal{M}^\omega$.

In this case, we embed $\mathcal{M}$ in $\mathcal{M}^{\omega}$ as constant sequences.

In what follows, let $\tau_k$ denote the normalized trace on the appropriate type $I_k$ factor.

**Lemma 2** Suppose that $\mathcal{M}$ is a type $II_1$ factor with trace $\tau$. If $X$ is a finite subset of $U(\mathcal{M})$ and $F \phi (\mathcal{M}, X) = 0$, then for every $U \in X$, $M \in \mathbb{N}$ and $\delta > 0$ there exists $m \in \mathbb{N}$ such that $m \geq M$, $Q(X, \frac{1}{m})$ via a projection $e_m$ of some finite rank $l(m)$ and there exists a unitary element $W_m \in e_m B(L^2(\mathcal{M}, \tau))e_m$ satisfying

$$||e_m U e_m - W_m||_{\tau(m)} \leq \delta.$$ 

**Proof.** Let $M > 0$. If $F \phi (\mathcal{M}, X) = 0$, then there is an increasing sequence of positive integers $(n_k)_{k \in \mathbb{N}}$ such that $Q(X, \frac{1}{n_k})$, and hence there is a projection $e_{n_k}$ of some finite rank $l(n_k)$ onto the span of an orthonormal set $\{S^{(n_k)}_i\}_{i=1}^{l(n_k)}$ of vectors in $L^2(\mathcal{M})$ satisfying

$$0 \leq \frac{||[U, e_{n_k}]||_{H.S.}}{||e_{n_k}||_{H.S.}} \leq \sqrt{2} \sqrt{1 - ||e_{n_k} U e_{n_k}||_{\tau(n_k)}^2} \leq \frac{1}{n_k}.$$
for all \( U \in X \). With \( e_{nk} U e_{nk} = A_{nk} = [(\xi^{(nk)}_q, U \xi^{(nk)}_p)]_{q,p=1}^{l(nk)} \), we have

\[
1 - \frac{1}{2n_k^2} \leq \tau_{l(n_k)}(A^*_{nk} A_{nk}) = \tau_{l(n_k)}(A_{nk} A^*_{nk}) = ||e_{nk} U e_{nk}||^2_{\tau_{l(n_k)}}.
\]

Furthermore, since \( e_{nk} \) is a projection, \( ||e_{nk}|| \leq 1 \) and hence

\[
||A_{nk}|| = ||e_{nk} U e_{nk}|| \leq ||U|| ||e_{nk}||^2 \leq 1,
\]

and hence \( ||A_{nk} A^*_{nk}|| = ||A_{nk}||^2 \leq 1 \). Let \( \omega \) be a free ultrafilter, and \( \prod M_{l(n_k)}(\mathbb{C}) \) denote the ultraproduct factor as defined above. We have a sequence \( (A_{nk}) = (A_{nk})_{n_k \geq M} \) of matrices satisfying

\[
\tau_\omega((A^*_{nk} A_{nk}) + I_\omega) = \tau_\omega((A_{nk} A^*_{nk}) + I_\omega) = 1
\]

so by faithfulness of \( \tau_\omega \) and the fact that \( (I_{nk} - A_{nk} A^*_{nk})_{nk} \geq 0 \) for all \( n \),

\[
\tau_\omega((I_{nk} - A_{nk} A^*_{nk})_{nk} + I_\omega) = 0,
\]

so indeed \( (A_{nk}) \) represents a unitary element in the ultraproduct \( \prod M_{l(n_k)}(\mathbb{C}) \). Recall that if

\[
(A_{nk}) + I_\omega \text{ and } (B_{nk}) + I_\omega
\]

represent distinct elements of \( \prod M_{l(n_k)}(\mathbb{C}) \), then the 2-norm distance between them is given by

\[
||(A_{nk} - B_{nk}) + I_\omega||_2
\]

\[
= \tau_\omega(((A^*_{nk} - B^*_{nk}) + I_\omega)((A_{nk} - B_{nk}) + I_\omega))^{1/2}
\]

\[
= \lim_{l(n_k) \rightarrow \omega} \tau_{l(n_k)}((A^*_{nk} - B^*_{nk})(A_{nk} - B_{nk}))^{1/2}
\]

\[
= \lim_{l(n_k) \rightarrow \omega} ||A_{nk} - B_{nk}||_{\tau_{l(n_k)}}^2^{1/2}.
\]

Suppose that \( \delta > 0 \) and that for every unitary \( l(n_k) \times l(n_k) \) matrix \( W_{nk} \),

\[
||A_{nk} - W_{nk}||_{\tau_{l(n_k)}} > \delta,
\]

it then follows that \( ||(A_{nk} - W_{nk}) + I_\omega||_2 > \delta \) in \( L^2(\prod \hat{M}_{l(n_k)}(\mathbb{C}), \tau_\omega) \). Since every sequence \( (W_{nk}) \) represents a unitary element in \( \prod \hat{M}_{l(n_k)}(\mathbb{C}) \), and from the polar decomposition and the fact that the ultraproduct is a finite factor every unitary element is represented by such a sequence, a contradiction follows, since \( (A_{nk}) \) represents a unitary element in \( \prod M_{l(n_k)}(\mathbb{C}) \). Therefore, for all \( \delta > 0 \) there exists a unitary \( l(n_k) \times l(n_k) \) matrix \( W_{nk} \) so that \( ||A_{nk} - W_{nk}||_{\tau_{l(n_k)}} \leq \delta \), hence we may view \( W_{nk} \) as a unitary element of \( e_{nk} B(L^2(M)) e_{nk} \) (i.e. a unitary operator on \( \text{span}\{\xi_i^{(nk)}\}_{i=1}^{l(n_k)} \cong \mathbb{C}^{l(n_k)} \)).
We recall the construction of the infinite tensor product of a collection of finite factors. Let \( \{M_i\}_{i \in \mathbb{N}} \) be a countable collection of finite factors with faithful normal traces \( \tau_i \), and let \( A_n \equiv \bigotimes_{i=1}^{n} M_i \) denote an algebraic tensor product. The map \( T_1 \otimes \ldots \otimes T_n \mapsto T_1 \otimes \ldots \otimes T_n \otimes I \) on simple tensors extends to a unital embedding of \( A_n \) into \( A_{n+1} \). Let \( A \) be the direct limit algebra obtained via these embeddings. We have that \( A \) obtains a unital \( * \)-algebra structure and a faithful normal trace \( \tau_0 \) from the \( M_i \). Let \( \pi \) denote the GNS representation obtained from \( A \) and \( \tau_0 \). We define \( \bigotimes_{i=1}^{\infty} M_i \equiv \pi(A)' \). It is easy to see that this is a factor. The state \( \tau_0 \) extends uniquely to a faithful normal trace on \( \bigotimes_{i=1}^{\infty} M_i \), so we obtain that the factor is finite.

The central sequence algebra \( M_\omega = M' \cap M^\omega \) is the algebra of all elements in \( M^\omega \) that commute with \( M \) (see [6], [10], [4]). If \( M_\omega \neq C_1 \), then we say that \( M \) has property \( \Gamma \). It is a straightforward exercise to show that every infinite tensor product factor \( \bigotimes_{i=1}^{\infty} M_i \) has property \( \Gamma \).

In the next theorem, let \( M \) denote the type \( II_1 \) factor \( \bigotimes_{j=1}^{\infty} (L(F_2))^j \), and let \( U = L_a \otimes I \otimes I \ldots \) and \( V = L_b \otimes I \otimes I \ldots \) in \( M \). We now compute an explicit lower bound for \( F\phi_l(M) \).

**Theorem 3** If \( X = \{U, V\} \), then \( F\phi_l(M, X) > 0 \).

**Proof.** Suppose that \( F\phi_l(M, X) = 0 \), so by the lemma, there exists a positive integer \( n \) and a rank \( n \) projection \( e \in B(L^2(M)) \) such that

\[
0 \leq \frac{||[U, e]||_{H.S.}}{||e||_{H.S.}} = \sqrt{2} \sqrt{1 - ||eUe||^2_{\tau_n}} \leq \frac{1}{7}.
\]

We have that \( ||eUe - Ue||^2_{\tau_n} = 1 - ||eUe||^2_{\tau_n} \leq \frac{1}{98} \). By the above lemma, there is an \( n \times n \) unitary matrix \( W \in eB(L^2(M))e \) such that

\[
||eUe - W||_{\tau_n} \leq (1 - \frac{1}{\sqrt{2}}) \frac{1}{7}.
\]

By the triangle inequality, we have that

\[
||Ue - W||_{\tau_n} \leq \frac{1}{7}.
\]

Let \( \{\xi_1, \ldots, \xi_n\} \subseteq L^2(M) \) be an orthonormal basis for the range of \( e \). Since \( W \in eB(L^2(M))e \) we have

\[
||Ue - W||^2_{\tau_n} = \frac{1}{n} \sum_{i=1}^{n} ||(Ue - W)e_i||^2.
\]
Writing \((g_1, g_2, \ldots) \in \mathbb{F}_2 \times \mathbb{F}_2 \times \mathbb{F}_2 \ldots\) in place of \(\chi_{\{g_1\}} \otimes \chi_{\{g_2\}} \otimes \ldots\), we may view \(\mathbb{F}_2^\infty = \mathbb{F}_2 \times \mathbb{F}_2 \times \mathbb{F}_2 \times \mathbb{F}_2 \ldots\) as an orthonormal basis for \(L^2(\mathbb{F}_2^\infty) \cong L^2(\mathcal{M})\). Consider the action of \(\mathbb{F}_2\) on \(\mathbb{F}_2^\infty\) in the first coordinate, that is, the action \(g \in \mathbb{F}_2\) given by \(g(g_1, g_2, \ldots) = (gg_1, g_2, \ldots)\). For \(i \in \{1, 2, \ldots, n\}\), if

\[
\xi_i = \sum_{(g_1, g_2, \ldots) \in \mathbb{F}_2^\infty} \lambda_{(g_1, g_2, \ldots)}^{(i)}(g_1, g_2, \ldots)
\]

then

\[
\| (U - W) \xi_i \|^2 = \| (U - W) \sum_{(g_1, g_2, \ldots) \in \mathbb{F}_2^\infty} \lambda_{(g_1, g_2, \ldots)}^{(i)}(g_1, g_2, \ldots) \|^2 = \sum_{(g_1, g_2, \ldots) \in \mathbb{F}_2^\infty} |\lambda_{(a^{-1}g_1, g_2, \ldots)}^{(i)} - \sum_{k=1}^n W_{ik} \lambda_{(g_1, g_2, \ldots)}^{(k)}|^2.
\]

For \(S\) a non-empty subset of \(\mathbb{F}_2^\infty\) and

\[
\eta = \sum_{(g_1, g_2, \ldots) \in \mathbb{F}_2^\infty} \mu_{(g_1, g_2, \ldots)}(g_1, g_2, \ldots) \in L^2(\mathbb{F}_2^\infty),
\]

define

\[
\| \eta \|^2_S = \sum_{(g_1, g_2, \ldots) \in S} |\mu_{(g_1, g_2, \ldots)}|^2.
\]

It follows that

\[
\| (U - W) \xi_i \|^2_S = \sum_{(g_1, g_2, \ldots) \in S} |\lambda_{(a^{-1}g_1, g_2, \ldots)}^{(i)} - \sum_{k=1}^n W_{ik} \lambda_{(g_1, g_2, \ldots)}^{(k)}|^2.
\]

We have that

\[
\| \|U \xi_i\|_S - \|W \xi_i\|_S\| \leq \| (U - W) \xi_i \|_S \leq \| (U - W) \xi_i \|.
\]

and using the inequality \((x_1 + \ldots + x_n)^2 \leq n(x_1^2 + \ldots + x_n^2)\) and the triangle inequality, we get
\[
\frac{1}{n} \sum_{i=1}^{n} ||U \xi_i||_S^2 - \frac{1}{n} \sum_{i=1}^{n} ||W \xi_i||_S^2 \leq \frac{n}{n^2} \sum_{i=1}^{n} (||U \xi_i||_S^2 - ||W \xi_i||_S^2) \\
\leq \frac{1}{n} \sum_{i=1}^{n} (||U \xi_i||_S - ||W \xi_i||_S)(||U \xi_i||_S + ||W \xi_i||_S) \\
\leq \frac{4}{n} \sum_{i=1}^{n} (||U \xi_i||_S - ||W \xi_i||_S) \\
\leq \frac{4}{n} \sum_{i=1}^{n} ||(U - W) \xi_i||_S^2 \\
\leq \frac{4}{n} \sum_{i=1}^{n} ||(U - W) \xi_i||^2 \\
\leq \frac{4}{49}.
\]

With \(\eta = \sum_{(g_1, g_2, \ldots) \in \mathbb{F}_2^\infty} \mu_{(g_1, g_2, \ldots)}(g_1, g_2, \ldots) \in L^2(\mathbb{F}_2^\infty)\), define
\[
\eta|_S \equiv \sum_{(g_1, g_2, \ldots) \in S} \mu_{(g_1, g_2, \ldots)}(g_1, g_2, \ldots) \in L^2(S) \subseteq L^2(\mathbb{F}_2^\infty).
\]

Note that \(||\eta|_S||_{L^2(S)} = ||\eta||_S\). We have that
\[
W \xi_i|_S = \sum_{k=1}^{n} W_{ik} \xi_k|_S = \sum_{g \in S} (\sum_{k=1}^{n} W_{ik} \lambda^{(k)}_g)g = (W \xi_i)|_S,
\]

We may conclude, since \(W\) is a unitary operator on \(\mathbb{C}^n\), that
\[
\sum_{i=1}^{n} ||W \xi_i||_S^2 = \sum_{i=1}^{n} ||(W \xi_i)|_S||_{L^2(S)}^2 \\
= \sum_{i=1}^{n} ||W \xi_i||_S^2 \\
= \sum_{i=1}^{n} ||\xi_i||_S^2 = \sum_{i=1}^{n} ||\xi_i||_S^2.
\]

We also have that for each \(i\),
\[
(U \xi_i)|_S = \sum_{(g_1, g_2, \ldots) \in S} \lambda^{(i)}_{(a^{-1} g_1, g_2, \ldots)}(g_1, g_2, \ldots) = U \xi_i|_S.
\]
It follows that
\[
\sum_{i=1}^{n} \|U\xi_i\|_{L^2(S)}^2 = \sum_{i=1}^{n} \|(U\xi_i)\|_{L^2(S)}^2 = \sum_{i=1}^{n} \|U\xi_i\|^2_S.
\]

Notice that
\[
\|U\xi_i\|^2_S = \|(U\xi_i)\|_{L^2(S)}^2
= \sum_{(g_1,g_2,\ldots)\in S} |\lambda^{(i)}_{(a^{-1}g_1,g_2,\ldots)}|^2
= \sum_{(g_1,g_2,\ldots)\in a^{-1}S} |\lambda^{(i)}_{(g_1,g_2,\ldots)}|^2
= \|\xi_i|_{a^{-1}S}\|^2_{L^2(a^{-1}S)} = \|\xi_i\|^2_{a^{-1}S}.
\]

We have that
\[
\left| \frac{1}{n} \sum_{i=1}^{n} (\|\xi_i\|^2_{a^{-1}S} - \|\xi_i\|^2_S) \right| = \left| \frac{1}{n} \sum_{i=1}^{n} \|U\xi_i\|^2_S - \frac{1}{n} \sum_{i=1}^{n} \|\xi_i\|^2_S \right|^2
\leq \frac{4}{49}.
\]

Now we shall choose a subset $S$ for which the above inequality will give us a contradiction. For simplicity of notation, let us define
\[
c_S \equiv \frac{1}{n} \sum_{i=1}^{n} \|\xi_i\|^2_S.
\]

The above inequality becomes
\[
|c_{a^{-1}S} - c_S| \leq \frac{4}{49}.
\]

If we carry out the above analysis using $V$ in place of $U$, we obtain
\[
|c_{b^{-1}S} - c_S| \leq \frac{4}{49}.
\]

Since $S$ was arbitrary, we could replace $S$ by $aS$ (resp. $bS$) to get
\[
|c_S - c_{aS}| \leq \frac{4}{49}
\]
(resp. $|c_S - c_{bS}| \leq \frac{4}{49}$).

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Choose the set $S$ to be $S' \times \mathbb{F}_2^\infty$, where $S'$ is the set of all reduced words in $\mathbb{F}_2$ that begin with $a^{-1}$. Then $S \cup aS = \mathbb{F}_2^\infty$ and also $S$, $bS$ and $b^{-1}S$ are pairwise disjoint. Since $S \cup aS = \mathbb{F}_2^\infty$, we have that $c_S$ or $c_{aS}$ exceeds $\frac{1}{2}$. Since $S$, $bS$ and $b^{-1}S$ are pairwise disjoint, at least one of $c_S$, $c_{bS}$ or $c_{b^{-1}S}$ must be $\leq \frac{1}{3}$. With no loss of generality, we may assume that $\frac{1}{3} \leq c_{aS}$. It follows that

$$\frac{1}{2} \leq c_{aS} \leq |c_S - c_{aS}| + |c_S| \leq \frac{4}{49} + c_S$$

so that

$$\frac{1}{2} - \frac{4}{49} \leq c_S.$$

Let us assume, again with no loss of generality, that $c_{bS} \leq \frac{1}{3}$, then

$$c_S \leq |c_S - c_{bS}| + c_{bS} \leq \frac{4}{49} + \frac{1}{3}.$$

It follows that

$$\frac{5}{12} < \frac{1}{2} - \frac{4}{49} \leq c_S \leq \frac{1}{3} + \frac{4}{49} < \frac{5}{12},$$

which is a contradiction. ■

**Remark 2** The above proof, slightly modified, gives that

$$F\phi(L(\mathbb{F}_2), \{L_a, L_b\}) > 0.$$

### 4.3 Upper Bounds

We begin this section by proving that the universal Følner constant of any given type $II_1$ factor cannot exceed $2$. We then move on to compute specific upper bounds for $F\phi(L(\mathbb{F}_n), X)$, with $X$ the set of standard generators.

**Proposition 4** For any type $II_1$ factor $\mathcal{M}$, $F\phi(\mathcal{M}) \leq 2$.

**Proof.** First suppose that $X$ is a finite set of unitary elements in $M$, such that $\varepsilon > 2$ and the negation of $Q(X, \varepsilon)$ holds. If $k \in \mathbb{N}$ and $e$ is a rank $k$ projection such that $\sqrt{2\sqrt{1 - ||eUe||^2_{\tau_k}}} > \varepsilon$ then $||eUe||^2_{\tau_k} < 0$, which cannot happen. It follows that for every $k \in \mathbb{N}$ and rank $k$ projection $e$ in $B(L^2(M))$, there exists $U \in X$ such that

$$|\tau(U) - \tau_k(eUe)| > \varepsilon.$$

However, using the triangle and Cauchy-Schwartz inequalities,

$$2 < \varepsilon < |\tau(U) - \tau_k(eUe)| \leq |\tau(U)| + |\tau_k(eUe)| \leq 2,$$

a contradiction. ■
Proposition 5 \( F\varnothing(L(\mathbb{F}_n), X) \leq \sqrt{2 - \frac{2}{n^2}} \), where

\[ X = \{ L_{a_1}, L_{a_2}, ..., L_{a_n} \} \]

is the set of standard generators of \( L(\mathbb{F}_n) \).

**Proof.** For \( i \in \{1, 2, ..., n\} \) and \( \varepsilon \in \{\pm 1\} \), define

\[ S_{a_i}^\varepsilon = \{ g \in \mathbb{F}_n | g \text{ begins with } a_i^\varepsilon \} \]

Given \( i \in \{1, 2, ..., n\} \), let \( w \) be the least positive integer equivalent to \( (i - 1) \) modulo \( n \), and \( \{g_j^{(i)} | j \in \mathbb{N}\} \) be the list of elements in \( S_{a_{w-1}} \). For \( m \in \{1, 2, ..., k\} \), let

\[ \xi_m = \sum_{t=1}^{\infty} \frac{1}{\sqrt{(n+1)^t}} \sum_{i=1}^{n} a_{i}^m g_i^{(i)} \in L^2(\mathbb{F}_n). \]

We have \( \{\xi_m\}_{m=1}^k \) is an orthonormal set.

Let \( e \) be the projection onto \( \text{span}\{\xi_m\}_{m=1}^k \). We have that for all \( j \in \{1, 2, ..., n\} \) and \( m, s \in \{1, 2, ..., k\} \) that

\[ \langle L_{a_j} \xi_s, \xi_m \rangle = 0 \]

unless \( m > 1 \) and \( s = m - 1 \), in this case

\[ \langle L_{a_j} \xi_{m-1}, \xi_m \rangle = \frac{1}{n}. \]

It follows that \( ||eL_{a_j}e||_{\tau_k}^2 = \frac{1}{k} \sum_{m=2}^{k} |\langle L_{a_j} \xi_{m-1}, \xi_m \rangle|^2 = \frac{k-1}{kn^2} \), and hence

\[ \sqrt{2} \sqrt{1 - ||eL_{a_j}e||_{\tau_k}^2} = \sqrt{2} \sqrt{1 - \left(\frac{k-1}{k}\right) \frac{1}{n^2}}. \]

It follows that

\[ F\varnothing(L(\mathbb{F}_n), X) \leq \inf_{k \in \mathbb{N}} \{ \sqrt{2} \sqrt{1 - \left(\frac{k-1}{k}\right) \frac{1}{n^2}} \} = \sqrt{2 - \frac{2}{n^2}}. \]

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