Affine invariant points and new constructions

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Abstract

In [2] Grünbaum asked if the set of all affine invariant points of a given convex body is equal to the set of all points invariant under every affine automorphism of the body. In [3] we have proven the case of a body with no nontrivial affine automorphisms. After some partial results ([7],[6]) the problem was solved in positive by Mordhorst [8]. In this note we provide an alternative proof of the affirmative answer, developing the ideas of [3]. Moreover, our approach allows us to construct a new large class of affine invariant points.

Keywords: affine invariant points, symmetry, convex geometry.

1 Introduction

Let $\mathbb{K}^n$ be the set of all convex bodies in $\mathbb{R}^n$ and let $P : \mathbb{K}^n \rightarrow \mathbb{R}^n$ be a function satisfying the following two conditions:

1. For every nonsingular affine map $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ and every convex body $K \in \mathbb{K}^n$ one has $P(\varphi(K)) = \varphi(P(K))$.
2. $P(K)$ is continuous in the Hausdorff metric.

Such a function $P$ is called an affine-invariant point. The centroid and the center of the John ellipsoid (the ellipsoid of maximal volume contained in a given convex body) are examples of affine-invariant points.

Let $\mathcal{P}$ be the set of all affine-invariant points in $\mathbb{R}^n$. It was shown in [7] that $\mathcal{P}$ is an affine subspace of the space of continuous functions on $\mathbb{K}^n$ with values in $\mathbb{R}^n$. Grünbaum [2] asked a natural question: how big is the set $\mathcal{P}$? In particular, how to describe the set $\mathcal{P}(K) = \{P(K) \mid P \in \mathcal{P}\}$ for a given $K \in \mathbb{K}^n$? Denote the set of points fixed under affine maps of $K$ onto itself by $\mathcal{F}(K)$. Grünbaum observed that $\mathcal{P}(K) \subset \mathcal{F}(K)$ and asked the following question:
Question 1.1. Is the set $\mathcal{P}$ big enough to ensure that $\mathcal{P}(K) = \mathcal{F}(K)$ for every $K \in \mathbb{K}^n$?

In [7], Meyer, Schütz and Werner proved that the set of convex bodies $K$ for which $\mathcal{P}(K) = \mathbb{R}^n$ is dense in $\mathbb{K}^n$. Then the author showed that if $\mathcal{F}(K) = \mathbb{R}^n$ then $\mathcal{P}(K) = \mathbb{R}^n$ [3]. Very recently, using a completely different approach, Mordhorst [8] has shown the affirmative answer to the Question [1,1] This proof used a previous development by P. Kuchment [4, 5]. The purpose of this note is to show that the method of [3] can be also used to answer Question [1,1] providing a new proof. Moreover, we construct a new large class of affine invariant points.

2 Definitions and Notation

Recall some basic notations from group theory.

The group of all invertible linear transformations of $\mathbb{R}^n$ is denoted by $GL(n, \mathbb{R})$. The group of all invertible linear transformations with the determinant equal to 1, i.e. the transformations which preserve volume and orientation is denoted by $SL(n, \mathbb{R})$.

For the purposes of the current paper we will use the group of all linear transformations preserving volume but not necessarily preserving orientation, i.e. the transformations with the determinant equal $\pm 1$ denoted by $SL_n$.

The group of all affine transformations of $\mathbb{R}^n$ is denoted by $Aff(n)$. It may be represented as $GL(n) \ltimes \mathbb{R}^n$ with the rule $(r, x)(a) = r(a) + x$ where $r \in GL(n)$, $x, a \in \mathbb{R}^n$.

The unit Euclidian ball in $\mathbb{R}^n$ is denoted by $B^2$. The Euclidian norm of a vector is denoted by $|x|$. The Lebesgue measure on $\mathbb{R}^n$ is denoted by $\mu$.

A right (left) Haar measure is a measure on a locally compact topological group that is preserved under multiplication by the elements of the group from the right (left). The Lebesgue measure is an example of a Haar measure on $\mathbb{R}^n$. Right and left Haar measures are unique up to multiplication however, not necessarily equal to each other. In this paper we always use a left Haar measure and denote the Haar measure of a set $X$ by $\text{meas}(X)$.

$SAff(n)$ is the group of all affine transformations of $\mathbb{R}^n$ preserving volume. This group may be represented as a semidirect product of the group of all matrices with determinants equal to $\pm 1$ and $\mathbb{R}^n$ with the rule $(r, x)(a) = r(a) + x$ for every $r \in GL(n)$ with $\det(r) = \pm 1$, $x \in \mathbb{R}^n$. $SAff(n)$ is equipped
with the Haar measure, which is the product of Haar measures on the group of all matrices with the determinant equal to ±1 and the group $\mathbb{R}^n$.

The Hausdorff metric is a metric on $\mathcal{K}_n$, defined as

$$d_H(K_1, K_2) = \min\{\lambda \geq 0 : K_1 \subset K_2 + \lambda B^n_2; K_2 \subset K_1 + \lambda B^n_2\}.$$  

By $\mathcal{K}_1^n$ we denote the set of all convex compact sets in $\mathbb{R}^n$ with volume 1.

3 Affine Invariant Points

For a given convex body $K \in \mathcal{K}^n$ a family of affine invariant points is constructed by taking an arbitrary point $v$ and averaging all possible affine transformations of this point with the weight

$$F = F_K : \mathcal{K}^n \to C(SAff(n))$$

defined by

$$F_K(L)(\varphi) = \mu(\varphi^{-1}(L) \cap K), L \in \mathcal{K}^n, \varphi \in SAff(x).$$

Let $k \geq 1$ be an integer. For $L \in \mathcal{K}_1^n$ define the affine invariant point $T_{k,K,v}$ by

$$T_{k,K,v}(L) = \left( \int_{SAff(n)} F^k(L)(\varphi)d\varphi \right)^{-1} \int_{SAff(n)} F^k(L)(\varphi)v(\varphi)d\varphi. \quad (1)$$

In general, for $L \in \mathcal{K}^n$ we set

$$T_{k,K,v}(L) = |L|^{1/n}T_{k,K,v}(L/|L|^{1/n}). \quad (2)$$

**Theorem 3.1.** For a given convex body $K$ and a vector $v \in \mathcal{F}(K)$, the function $T_{k,K,v} : \mathcal{K}^n \to \mathbb{R}^n$, defined in (1) has the following properties:

1. There exists $k_0 \in \mathbb{Z}_+$ such that for every $k \geq k_0$, $T_{k,K,v}(L)$ is defined for all $L \in \mathcal{K}^n$.
2. $T_{k,K,v}$ is an affine invariant point if defined.
3. $T_{k,K,v}(K) \to v, k \to \infty$.

Theorem 3.1 implies that for every $K \in \mathcal{K}^n$ and every $v \in \mathcal{F}(K)$ we can find an affine invariant point $F$ such that $F(K)$ is arbitrarily close to $v$. However, this implies that every point in $\mathcal{F}(K)$ can be obtained as an affine point of $K$ because the set of all affine points is an affine space [7].
4 Technical Part

To prove Theorem 3.1 we will require some tools for integration over the group $SAff(n)$.

For a matrix $A \in GL(n, \mathbb{R})$ the ordered sequence $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n > 0$ of the singular values of the matrix $A$, is the sequence of all eigenvalues of $\sqrt{AA^*}$ counting multiplicities; see e.g. [1]. In the case $A \in SL_n^-$ we have $1 = |\det(A)| = \prod_{i=1}^n \lambda_i$. For a matrix $A \in GL(n, \mathbb{R})$ we denote by $\|A\|$ its operator norm $\ell_2 \to \ell_2$, that is $\|A\| = \sup_{|x|=1} |Ax|.$

Note that singular values of $A$ give a convenient description of the norm $\|A\| = \lambda_1$.

For $R \geq 1$ the “ball” $S_R$ is the set of all matrices $A \in SL_n^-$ such that $\|A\| \leq R$.

Note that for $R_1, R_2 \geq 1$ the following equality holds: $S_{R_1}S_{R_2} = S_{R_1R_2}$. Indeed, by the property of the operator norm, $S_{R_1}S_{R_2} \subset S_{R_1R_2}$. On the other hand, according to the polar decomposition, every $A \in S_{R_1R_2}$ may be represented in the form $A = UP$, where $U$ is a unitary matrix and $P$ is positive Hermitian, see e.g. [9]. Then

\[ A = UP^{\ln R_1/\ln (R_1R_2)} P^{\ln R_2/\ln (R_1R_2)}, \]

with $U^{\ln R_1/\ln (R_1R_2)} \in S_{R_1}$, $P^{\ln R_2/\ln (R_1R_2)} \in S_{R_2}$.

**Lemma 4.1.** For every $\varepsilon > 0$ there exists a finite set $N \subset S_{2(1+\varepsilon)}$ such that for every integer $l \geq 0$ one has

\[ S_{2^l(1+\varepsilon)} \subset N^l S_{(1+\varepsilon)}. \]

**Proof.** Since the set $S_{2(1+\varepsilon)}$ is compact, it can be covered by some finite collection of balls:

\[ S_{2(1+\varepsilon)} \subset \bigcup_{N \in N_{1+\varepsilon}} N \cdot S = NS_{1+\varepsilon}. \]

We will show by induction that the set $N$ satisfies the condition of the proposition. The base case for $l = 0$ is trivial. Now we show the inductive step:

\[ S_{2^l(1+\varepsilon)} = S_{2^l(1+\varepsilon)} S_2 \subset N^l S_{1+\varepsilon} S_2 = N^l S_{2(1+\varepsilon)} \subset N^l N S_{1+\varepsilon}. \]

\[ \square \]
Proposition 4.2. For every \( n \geq 2, \alpha \geq 0 \) there exists \( p \geq 1 \) such that for any convex bodies \( K, L \) the integral

\[
\int \int_{SL_n^+ \mathbb{R}^n} \mu^p (L \cap (M(K) + x)) \|M\|^\alpha dx dM
\]

converges. Here \( dM \) is a Haar measure on \( SL_n^- \).

Proof. There exists a radius \( R > 0 \) such that the bodies \( K, L \) are simultaneously contained within the ball \( RB^n_2 \). Therefore,

\[
\int \int_{SL_n^+ \mathbb{R}^n} \mu^p (L \cap (M(K) + x)) \|M\|^\alpha dx dM \\
\leq \int \int_{SL_n^+ \mathbb{R}^n} \mu^p (RB^n_2 \cap (M(RB^n_2) + x)) \|M\|^\alpha dx dM \\
= R^{pn} \int \int_{SL_n^+ \mathbb{R}^n} \mu^p (B^n_2 \cap (M(B^n_2) + \frac{x}{R})) \|M\|^\alpha dx dM \\
= R^{pn+n} \int \int_{SL_n^+ \mathbb{R}^n} \mu^p (B^n_2 \cap (M(B^n_2) + x)) \|M\|^\alpha dx dM.
\]

It is enough to consider the convergence of the integral

\[
\int \int_{SL_n^+ \mathbb{R}^n} \mu^p (B^n_2 \cap (M(B^n_2) + x)) \|M\|^\alpha dx dM. \tag{3}
\]

Note that the lengths of semiaxes of the ellipsoid \( M(B^n_2) \) are defined by the singular values \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \) of \( M \) in particular, the diameter of \( M(B^n_2) \) equals \( 2\lambda_1 \) and its minimal width equals \( 2\lambda_n \). This means that for \( |x| > \lambda_1 + 1 \) the volume \( \mu (B^n_2 \cap (M(B^n_2) + x)) = 0 \). For all other \( x \) the ellipsoid \( MB^n_2 \) is contained within the slab \( L = \{ y \in \mathbb{R}^n : |\langle y, u \rangle| \leq \lambda_n \} \) for some vector \( u \). Therefore,

\[
\mu (B^n_2 \cap (M(B^n_2) + x)) \leq \mu (B^n_2 \cap (L + x)) \leq 2\lambda_n |B^n_2 - 1|.
\]
Summing up, the integral (3) is bounded by

\[
\int_{SL_n} \int_{\mathbb{R}^n} \mu^p (B_2^n \cap (M(B_2^n) + x)) \|M\|^\alpha \, dx \, dM
\]

\[
= \int_{SL_n} \int_{|x| \leq \lambda_1 + 1} \mu^p (B_2^n \cap (M(B_2^n) + x)) \|M\|^\alpha \, dx \, dM
\]

\[
\leq \int_{SL_n} \int_{|x| \leq \lambda_1 + 1} (2\lambda_n |B_2^{n-1}|)^p \|M\|^\alpha \, dx \, dM
\]

\[
\leq \int_{SL_n} (2\lambda_1)^n |B_2^n| (2\lambda_n |B_2^{n-1}|)^p \|M\|^\alpha \, dM
\]

\[
= 2^{n+p} |B_2^{n-1}|^p |B_2^n| \int_{SL_n} \lambda_1^{n+\alpha} \lambda_n^p \, dM.
\]

Keeping in mind that \( \prod_{i=1}^n \lambda_i = 1 \) one has

\[
\lambda_1^{n+\alpha} \lambda_n^p \leq \lambda_1^{n+\alpha} (\lambda_2 \lambda_3 \ldots \lambda_n)^p/(n-1) = \lambda_1^{n+\alpha} \left( \frac{1}{\lambda_1} \right)^{(n-1)} = \lambda_1^{n+\alpha - \frac{p}{n-1}}.
\]

Finally, putting \( q = -n - \alpha + \frac{p}{n-1} \) it is enough to show that there exists sufficiently big \( q > 0 \) such that the integral

\[
\int_{SL_n} \|M\|^{-q} \, dM
\]

is convergent. To prove this we split the group \( SL_n^- \) into smaller sets

\[
S_{2l} \setminus S_{2l-1}, l \geq 1.
\]

Then

\[
\int_{SL_n} \|M\|^{-q} \, dM = \sum_{l=1}^\infty \int_{S_{2l} \setminus S_{2l-1}} \|M\|^{-q} \, dM \leq \sum_{l=1}^\infty 2^{-lq} \text{meas}(S_{2l}).
\]
According to Lemma [111] there exists a set $N$ such that 
\[ \text{meas}(S_2) \leq |N| \text{meas}(S_{2(1+\varepsilon)}) \].
Therefore, the series (5) is bounded by a geometric series with the ratio $2^{-q} |N|$ which is convergent for $q > \log_2 |N|$.

Proposition 4.3. Let $G$ be a locally compact topological group and $dx$ be a Haar measure on $G$. Let continuous functions $f, g$ satisfy the following conditions:

1. For every $x \in G$: $0 \leq f(x) \leq 1$.
2. There exists $x_0 \in G$ such that $f(x_0) = 1$. Moreover, if $x_1 \in G$ is such that $f(x_1) = f(x_0) = 1$ then $g(x_1) = g(x_0)$.
3. There exists a constant $c < 1$ and a compact $K$ such that for every $x \in G \setminus K$, $f(x) < c$.
4. There exists $k_0 \geq 1$ such that for every $k \geq k_0$ the integrals 
\[
\int_G f^k(x) dx, \int_G f^k(x) |g(x)| dx
\]
are convergent.

Then 
\[
\lim_{k \to \infty} \frac{\int_G f^k(x)g(x) dx}{\int_G f^k(x) dx} = g(x_0).
\]

Proof. Note that the integral 
\[
\int_G f^k(x)|g(x) - g(x_0)| dx \leq \int_G f^k(x) |g(x)| dx + g(x_0) \int_G f^k(x) dx
\]
is convergent for for $k \geq k_0$. Passing to the new function $g - g(x_0)$ if needed, we may assume that $g(x_0) = 0$.

The set $N = f^{-1}(1) \subset K$ is closed and therefore compact. By the assumption of the proposition $g(N) = \{0\}$. Fix $\varepsilon > 0$ and consider a neighbourhood $U$ of $N$ such that $|g| < \varepsilon$ on $U$. There exists a positive constant $C < 1$ such that $f < C$ outside of $U$. Indeed, outside of $K$ the function $f$ is bounded from above by $c$, on the compact set $K \setminus U$ the function $f$ is separated from 1 by the compactness argument. By continuity of $f$, there exists a constant $D \in (C, 1)$ and a neighborhood $V \subset U$ of $N$ such that $D < f \leq 1$ on $V$. Then
\[ \frac{\int_G f^k(x) |g(x)| \, dx}{\int_G f^k(x) \, dx} \leq \frac{\int_U f^k(x) |g(x)| \, dx + \int_{G \setminus U} f^k(x) |g(x)| \, dx}{\int_U f^k(x) \, dx} \]
\[ \leq \varepsilon + \frac{\int_{G \setminus U} f^k(x) |g(x)| \, dx}{\int_V f^k(x) \, dx} \]
\[ \leq \varepsilon + \frac{C^{k-k_0} \int_{G \setminus U} f^{k_0}(x) |g(x)| \, dx}{D^{k-k_0} \int_V f^{k_0}(x) \, dx} \to \varepsilon, \ k \to \infty. \]

Sending \( \varepsilon \) to 0 we obtain the required statement.

\[ \square \]

**Proof of Theorem 3.1.** For fixed \( K \) and \( v \) we will shorten the notation by writing \( T_k \) instead of \( T_{k,K,v} \).

1. Proposition 4.2 applied with \( \alpha = 1 \) (respectively, \( \alpha = 0 \)) implies that the integral in the numerator (respectively, denominator) is convergent.

2. \( T_k(cK) = cT_k(K) \) by the definition of \( T_k \).

For every \( \tau \in SAff(n) \) and \( L \in \mathbb{K}_1^n : T_k(\tau(L)) = \tau(T_k(L)) \).

Denote
\[ c = \left( \int_{SAff(n)} F^k(L)(\varphi) \, d\varphi \right)^{-1}. \]

For arbitrary \( \tau \in SAff(n) \) we have
\[ T(\tau L) = c \int_{SAff(n)} F^k(\tau L)(\varphi) \varphi(v) \, d\varphi = c \int_{SAff(n)} F^k(L)(\tau^{-1} \varphi) \varphi(v) \, d\varphi. \]

Replacing \( \varphi \) by \( \tau \varphi \) we get
\[ T(\tau L) = c \int_{SAff(n)} F^k(L)(\varphi) \tau(\varphi(v)) \, d\varphi = \tau \left( c \int_{SAff(n)} F^k(L)(\varphi) \varphi(v) \, d\varphi \right). \]

The last equality holds because \( cF^k(L)(\varphi) \, d\varphi \) is a probabilistic measure. Therefore, for every affine \( \tau \) and every integrable function \( f \) one has
\[ \int_{SAff(n)} \tau(f(\varphi))cF^k(L)(\varphi) \, d\varphi = \tau \left( \int_{SAff(n)} f(\varphi)cF^k(L)(\varphi) \, d\varphi \right). \]
Note that the function
\[
\frac{1}{\int_{S_R \times \mathbb{R}^n} F^k(L)(\varphi)d\varphi} \int_{S_R \times \mathbb{R}^n} F^k(L)(\varphi(v))d\varphi
\]
is continuous as a function of \( L \) by the Lebesgue’s dominated convergence theorem because both integrals are uniformly bounded by a convergent integral by Proposition 4.2. Then
\[
T_k = \lim_{R \to \infty} \frac{1}{\int_{S_R \times \mathbb{R}^n} F^k(L)(\varphi)d\varphi} \int_{S_R \times \mathbb{R}^n} F^k(L)(\varphi(v))d\varphi
\]
is continuous.

3. Convergence is the direct application of the Proposition 4.3 where \( f(\varphi) = F(K)(\varphi) \) and \( g(\varphi) = \varphi(v) \) taken coordinatewise. Similarly to the proof of the Proposition 4.2 the function \( F(K)((A, x)) \) is separated from 1 when either \( \|A\| \) or \( |x| \) is big. Note that \( F(K)(id) = 1 \) and if \( F(K)(\varphi) = 1 \) then \( \varphi(K) = K \) which means \( \varphi(v) = v \) because \( v \in F(K) \). Therefore,
\[
\frac{1}{\int_{S_{Aff(n)}} F(K)^k(\varphi)d\varphi} \int_{S_{Aff(n)}} F(K)^k(\varphi(v))d\varphi \to id(v) = v, \ k \to \infty.
\]

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