Global uniqueness in inverse boundary value problems for the Navier–Stokes equations and Lamé system in two dimensions*

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Abstract
We consider inverse boundary value problems for the Navier–Stokes equations and the isotropic Lamé system in two dimensions. The question of global uniqueness for these inverse problems, without any smallness assumptions on unknown coefficients, has been a longstanding open problem for the Navier–Stokes equations and the isotropic Lamé system in two dimensions. We prove the global uniqueness for both inverse boundary value problems. Our methodology is the same for both systems. The key is the construction of complex geometric optics solutions after decoupling the systems into weakly coupling systems.

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1. Introduction and main results

In this article, we consider inverse boundary value problems for the two dimensional Navier–Stokes equations and the Lamé system, where we determine the spatially varying viscosity coefficient and two spatially varying Lamé coefficients in the Navier–Stokes

* The proofs of propositions in sections 2–7 are given in the supplementary data (available from stacks.iop.org/ip/31/035004/mmedia)
equations and the Lamé system, respectively, by the Dirichlet-to-Neumann maps on the whole boundary.

Throughout this paper, let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary $\partial \Omega$ and $\nu = (\nu_1, \nu_2)$ be the outward unit normal vector to $\partial \Omega$. We set $x = (x_1, x_2) \in \mathbb{R}^2$ and $\mathbb{N}_+ = \{0, 1, 2, 3, \ldots\}$. Let $\beta = (\beta_1, \beta_2) \in (\mathbb{N}_+)^2$, and $|\beta| = \beta_1 + \beta_2$. Let $(\cdot, \cdot)$ be the scalar product in $\mathbb{R}^2$.

We start with the formulation of the inverse boundary value problem for the Navier–Stokes equations. In the domain $\Omega$, we consider the stationary Navier–Stokes equations:

$$
G_{\rho}(x, D)(u, p) := \sum_{j=1}^2 \left( \frac{\partial}{\partial x_j} \left( \rho(x) \epsilon_{ij}(u) \right) + u_j \frac{\partial u_i}{\partial x_j} + \frac{\partial p}{\partial x_i} \right) = 0 \text{ in } \Omega,
$$

where $u = (u_1, u_2)$ is a velocity field, $p$ is a pressure and $\epsilon_{ij}(u) = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$. Assume that $\rho(x) > 0$ on $\Omega$, $\mu \in C^m(\overline{\Omega})$. \hfill (1.1)

Here and henceforth $m \in \mathbb{N}$ is sufficiently large (e.g., $m = 10$ is sufficient).

We define the Dirichlet-to-Neumann map $\Lambda_{\mu}$:

$$
\Lambda_{\mu}(f) = \left( \frac{\partial u}{\partial \nu}, p \right)_{\partial \Omega}, \quad (2.1)
$$

where $G_{\rho}(x, D)(u, p) = 0$ in $\Omega$, $u = f$ on $\partial \Omega$, $\text{div } u = 0$, $u \in W^2_0(\Omega)$, $p \in W^1_0(\Omega)$ and

$$
D(\Lambda_{\mu}) \subset X = \{ f \in W^2_0(\partial \Omega); \exists (w, q) \in W^2_0(\Omega) \times W^1_0(\Omega), \text{ div } w = 0, \text{ div } q = 0, \Delta w + \nabla q = 0 \}.
$$

The first subject of this paper is the following inverse boundary value problem: using the Dirichlet-to-Neumann map $\Lambda_{\mu}$, determine the coefficient $\mu$.

Our first main result is the global uniqueness.

**Theorem 1.1.** We assume that $\mu_1, \mu_2 \in C^{10}(\overline{\Omega})$ and $\partial^{\beta}_{\partial \Omega} \mu_1 = \partial^{\beta}_{\partial \Omega} \mu_2$ on $\partial \Omega$ for each multiindex $\beta$ with $|\beta| \leq 10$. If there exists a positive constant $\delta$ such that

$$
\Lambda_{\mu_1}(f) = \Lambda_{\mu_2}(f), \quad \forall f \in X \cap \{ \| f \|_{W^2_0(\partial \Omega)} \leq \delta \},
$$

then $\mu_1 = \mu_2$ in $\Omega$.

Here we note that for small boundary data $f$ there exists a unique solution, and so in the theorem we restricted the Dirichlet inputs $f$ to $X \cap \{ \| f \|_{W^2_0(\partial \Omega)} \leq \delta \}$.

The uniqueness result of the above theorem holds without any assumption on smallness of unknown coefficients or closeness of these coefficients to constants. We call such uniqueness the global uniqueness. To the authors’ best knowledge, there are no global
uniqueness results for such an inverse problem for the Navier–Stokes equations in two dimensions. On the other hand, in the three dimensional case, the global uniqueness was proved by Heck, Wang and Li [9] for the Stokes equations and by Li and Wang [17] for the Navier–Stokes equations.

Next we consider the inverse boundary value problem for the two dimensional Lamé system. Assume that

$$\mu(x) > 0, \ (\lambda + \mu)(x) > 0 \quad \text{on } \Omega.$$ 

We set

$$L_{\mu, \lambda}(x, D) u = \left( \sum_{j=1}^{2} \partial_{j} \left( \mu \left( \partial_{j} u_{1} + \partial_{j} u_{2} \right) \right), \sum_{j=1}^{2} \partial_{j} \left( \mu \left( \partial_{j} u_{2} + \partial_{j} u_{1} \right) \right) \right) + V \left( \lambda \text{ div } u \right)$$

$$= \mu \Delta u + (\mu + \lambda) V \text{ div } u + (\text{div } u) V \lambda$$

$$+ \left( (\nabla \mu, \nabla u_{1}), (\nabla \mu, \nabla u_{2}) \right) + \left( (V \mu, \partial_{1} u), (V \mu, \partial_{2} u) \right).$$

Here $u(x) = (u_{1}(x), u_{2}(x))$ describes displacement, and we call the functions $\lambda$ and $\mu$ the Lamé coefficients.

We define the Dirichlet-to-Neumann map $\Lambda_{\mu, \lambda}$ as follows:

$$\Lambda_{\mu, \lambda} f = \left( \mu \partial_{j} u_{1} + \mu \left( \partial_{j} u_{1}, \nu \right) + \lambda \left( \text{div } u \right) v_{1}, \mu \partial_{j} u_{2} + \mu \left( \partial_{j} u_{2}, \nu \right) + \lambda \left( \text{div } u \right) v_{2} \right)|_{\partial \Omega}, \quad (1.3)$$

where

$$L_{\mu, \lambda}(x, D) u = 0 \quad \text{in } \Omega, \quad u|_{\partial \Omega} = f \in W^{3/2}_{2}(\partial \Omega). \quad (1.4)$$

The second subject is the uniqueness in determining $\lambda, \mu$ by $\Lambda_{\mu, \lambda}$. The Dirichlet-to-Neumann map is well defined for each $f \in W^{3/2}_{2}(\partial \Omega)$ with value in $W^{1/2}_{2}(\partial \Omega)$. In fact, for boundary value problem (1.4) there exists a unique solution $u \in W^{3}_{2}(\Omega)$ for each $f \in W^{3/2}_{2}(\partial \Omega)$. The proof is done as follows: we prove the unique existence of the solution in $W^{3}_{2}(\Omega)$ by the variational method (see, e.g., chapter 3, section 3 of Duvaut and Lions [5], chapter 1, section 3 of Oleinik and Shamaev [22]). Then extending $f \in W^{3/2}_{2}(\partial \Omega)$ to $\tilde{f} \in W^{3}_{2}(\Omega)$ and considering $u - \tilde{f}$, we reduce the nonhomogeneous Dirichlet boundary value problem to the homogeneous boundary value problem. Finally we apply the argument for improving the $W^{3}_{2}(\Omega)$-regularity to the $W^{3}_{2}(\Omega)$-regularity (see, e.g., Giaquinta [8], pp 31–32).

Then we can prove the following.

**Theorem 1.2.** Assume that $\lambda_{1}, \lambda_{2}, \mu_{1}, \mu_{2} \in C^{10} \left( \Omega \right)$. Then $A_{\mu_{1}, \lambda_{1}} = A_{\mu_{2}, \lambda_{2}}$ implies that $\lambda_{1} = \lambda_{2}$ and $\mu_{1} = \mu_{2}$ in $\Omega$.

The global uniqueness for the Lamé system has been an open problem in spite of the physical significance, and theorem 1.2 affirmatively solves this problem for the two dimensional case. On the other hand, we can refer to non-global uniqueness as follows.

**Two dimensional case.**

In [19] Nakamura and Uhlmann proved that the Dirichlet-to-Neumann map uniquely determines the Lamé coefficients, assuming that $\lambda, \mu$ are sufficiently close to positive
constants. They assume that $\lambda, \mu \in W^{31}_0(\Omega)$, and our regularity assumption is $\lambda, \mu \in C^{10}(\Omega)$, although it may be more relaxed.

Imanuvilov and Yamamoto [13] proved the uniqueness by the Dirichlet-to-Neumann map limited to an arbitrary sub-boundary, provided that $\mu_1, \mu_2$ are some constants. These results require some assumptions on smallness or constants of unknown coefficients and the assumptions are quite restrictive. To our best knowledge, theorem 1.2 is the first result on the global uniqueness in two dimensions.

Three dimensional case.
We refer to Eskin and Ralston [6] and Imanuvilov, Uhlmann and Yamamoto [12]. In [6] the uniqueness for $\lambda$ and $\mu$ is proved provided that $V\mu$ is small in a suitable norm, while in [12] the uniqueness corresponding to the uniqueness [13] in two dimensions is proved. In [20], Nakamura and Uhlmann attempted to prove the global uniqueness for determination of Lamé coefficients in $C^{\infty}(\Omega)$. However, the global uniqueness in the three dimensional case remains a significant unsolved problem.

As for other types of inverse boundary value problem for the Lamé system, see Akamatsu, Nakamura and Steinberg [1], Ikehata [10] and Nakamura and Uhlmann [21].

The inverse boundary value problem was originally considered by Calderón [4], who studied the possibility of the determination of $\sigma$ in a conductivity equation $\text{div}(\sigma(x)\nabla u(x)) = 0$ by the Dirichlet-to-Neumann map. In the case of the higher dimensional case, that is, the spatial dimensions are equal to or greater than 3, the uniqueness for the conductivity equation and Schrödinger equation is proved by Sylvester and Uhlmann [24], and in two dimensions see Nachman [18] and Bukhgeim [3]. The inverse boundary value problems have attracted a lot of attention. Here we do not compose a complete list of references, but we refer to two surveys: Uhlmann [25], which is a survey as of 2009, and Imanuvilov and Yamamoto [15]. The latter mostly presents results on global uniqueness by the Dirichlet-to-Neumann map restricted on a sub-boundary which were published after 2009.

This paper is composed of seven sections. In section 2, we establish properties of integral operators in $\mathbb{C}$ and oscillatory integral operators, and prove a basic Carleman estimate for a first-order equation. In section 3 we consider second-order elliptic systems whose principal parts are the two dimensional Laplacian and construct special solutions (propositions 3.6 and 3.7), which are used for establishing adequate asymptotic behavior of some bilinear form. Section 4 is devoted to the proof of proposition 4.6, which yields integral–differential equalities from asymptotic behavior of some quadratic forms generated by a second-order elliptic operator and the special solutions considered in proposition 3.6. In section 5 we construct complex geometric optics solutions to the Navier–Stokes equations by reducing the equations to a weakly coupling elliptic system considered in section 3, and in section 6 we complete the proof of theorem 1.1. Section 7 is the proof of theorem 1.2 for the Lamé system in a way similar to one developed in sections 5 and 6.

2. Preliminary results

Throughout the paper, we use the following notations.

Notations. $i = \sqrt{-1}$, $x_1, x_2, \xi_1, \xi_2 \in \mathbb{R}^2$, $z = x_1 + ix_2, \zeta = \xi_1 + i\xi_2$, $\overline{z}$ denotes the complex conjugate of $z \in \mathbb{C}$. We identify $x = (x_1, x_2) \in \mathbb{R}^2$ with $z = x_1 + ix_2 \in \mathbb{C}$, $\partial_z = \frac{1}{2} \left( \partial_{x_1} - i \partial_{x_2} \right)$, $\partial_\zeta = \frac{1}{2} \left( \partial_{\xi_1} + i \partial_{\xi_2} \right)$, $\beta = (\beta_1, \beta_2)$, $|\beta| = \beta_1 + \beta_2$. $D = \left( \frac{1}{i} \frac{\partial}{\partial x_1}, \frac{1}{i} \frac{\partial}{\partial x_2} \right)$. 

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\[ \partial_{x_j} = \frac{\partial}{\partial x_j}, \quad j = 1, 2. \]

For \( \hat{x} \in \mathbb{R}^2 \) and \( r > 0 \), let \( B(\hat{x}, r) = \{ x \in \mathbb{R}^2 : |x - \hat{x}| < r \} \) and \( S(\hat{x}, r) = \{ x \in \mathbb{R}^2 : |x - \hat{x}| = r \} \).

Let \( \rho \) be a positive function in \( \Omega \). Denote \( \| \rho \| = \| \rho \|_2 < \infty \).

Let \( \Omega \) denote a usual Sobolev space and let \( \Omega^{1,1} \) denote the closure of \( \mathcal{C}^0(\Omega) \) in \( \Omega \). Let \( \mathbb{C} \) denote the Banach space of all bounded linear operators from a Banach space \( X \) to another Banach space \( Y \); \( I \) is the unit operator from a Banach space \( X \) into \( X \). Let \( E \) be the \( 3 \times 3 \) unit matrix. We set \( \tau = \| \| + \| \| \).

Let us introduce operators

\[ \int_{\Omega^T} f(z) \, d\sigma, \quad \text{where} \quad z = \xi_1 + i\xi_2. \]

Here \( \int_{\partial\Omega} \cdots \, d\sigma \) denotes a curvilinear integral in \( \zeta \).

**Proposition 2.1.**

(A) Let \( m \geq 0 \) be an integer number and \( \alpha \in (0, 1) \). Then \( \partial_{\xi_1}^{-1}, \partial_{\xi_2}^{-1} \in \mathcal{L}(C^{m+\alpha}(\Omega^T)) \).

(B) Let \( 1 \leq p \leq 2 \) and \( 1 < \gamma < \frac{2p}{2p-2} \). Then \( \partial_{\xi_1}^{-1}, \partial_{\xi_2}^{-1} \in \mathcal{L}(L^p(\Omega), L^\gamma(\Omega)) \).

(C) Let \( 1 < p < \infty \). Then \( \partial_{\xi_1}^{-1}, \partial_{\xi_2}^{-1} \in \mathcal{L}(L^p(\Omega), W^{1,1}_p(\Omega)) \) for all \( s \geq 0 \).

**Proof.** See, e.g., theorem 1.1 (p 15) of Bliev [2], pp 47, 56, 72 of Vekua [26].

Consider an operator

\[ T f(z) = \int_{\partial\Omega} \frac{f(\xi_1, \xi_2)}{z - \zeta} \, d\sigma, \quad \text{where} \quad \zeta = \xi_1 + i\xi_2. \]

**Proposition 2.2.** The operator \( T \) is continuous from \( L^p(\partial\Omega) \) to \( L^2(\Omega) \) for any \( p > 2 \).

**Proof.** See the supplementary data (available from stacks.iop.org/ip/31/035004/mmedia).
We set
\[ \Phi(z) = (z - \bar{z})^2 - \kappa, \]  \tag{2.1} \]
where \( \bar{z} = \bar{x} + i\bar{y} \in \Omega \) and \( \kappa \in \mathbb{R} \) is an arbitrary constant.

Assume in addition that a parameter \( \kappa \) satisfies
\[ \kappa \geq 1 + \sup_{z \in \Omega} |\Re(z - \bar{z})^2|, \quad - \min \Re \Phi(z) + 2 \max \Re \Phi(z) < -1 \]
\[ \forall (x, \bar{x}) \in \mathbb{R} \times \mathbb{R}. \]  \tag{2.2} \]

We have the following.

**Proposition 2.3.** Let \( \Phi(z) \) be given by (2.1) and \( u \in C_{0}^{5+a} (\Omega), \alpha \in (0, 1) \). Then the following asymptotic formula is true:
\[ \int_{\Omega} \nu e^{i(\phi - \sigma)} d\nu = \mathcal{F}_{\nu} u + o\left(\frac{1}{\tau}\right) \quad \text{as } \tau \to +\infty. \]  \tag{2.3} \]

**Proof.** See the supplementary data (available from stacks.iop.org/ip/31/035004/mmedia).

We introduce a functional
\[ \mathcal{I}_\nu u = \int_{\partial \Omega} \frac{(\nu_i - i\nu_2)}{2\tau \partial_z \Phi} e^{i(\phi - \sigma)} d\sigma = \int_{\partial \Omega} \frac{(\nu_i - i\nu_2)}{2\tau^2 \partial_z \Phi} d\sigma \left(\frac{u}{2\tau \partial_z \Phi}\right) e^{i(\phi - \sigma)} d\sigma. \]

We have the following.

**Proposition 2.4.** Let \( \Phi(z) \) be given by (2.1) and \( u \in C_{0}^{5+a} (\Omega), \alpha \in (0, 1) \). Then the following asymptotic formula is true:
\[ \int_{\Omega} \nu e^{i(\phi - \sigma)} d\nu = \mathcal{F}_{\nu} u + \mathcal{J}_\nu u + o\left(\frac{1}{\tau}\right) \quad \text{as } \tau \to +\infty. \]  \tag{2.4} \]

**Proof.** See the supplementary data (available from stacks.iop.org/ip/31/035004/mmedia).

Let
\[ \Phi(z) = e^{i\theta} z - \kappa, \]  \tag{2.5} \]
where \( \theta \in [0, 2\pi] \) and \( \kappa \) is a positive constant such that
\[ \kappa \geq 1 + \sup_{z \in \Omega, \theta \in [0, 2\pi]} |\Re(e^{i\theta} z)|, \quad - \min \Re \Phi(z) + 2 \max \Re \Phi(z) < -1 \]
\[ \forall (x, \theta) \in \mathbb{R} \times [0, 2\pi). \]  \tag{2.6} \]

We introduce a functional
\[ \mathcal{J}_\nu u = \int_{\partial \Omega} \frac{(\nu_i - i\nu_2)}{2\tau \partial_z \Phi} e^{i(\phi - \sigma)} d\sigma = \int_{\partial \Omega} \frac{(\nu_i - i\nu_2)}{2\tau^2 \partial_z \Phi} \left(\frac{u}{2\tau \partial_z \Phi}\right) e^{i(\phi - \sigma)} d\sigma. \]

In a way similar to proposition 2.4, we prove the following.
Proposition 2.5. Let \( u \in C^{\delta+\alpha}(\Omega) \), \( \alpha \in (0, 1) \). Then the following asymptotic formula is true:

\[
\int_{\Omega} u e^{\epsilon(\phi-x)} \, dx = \frac{\tau}{2} u + o\left(\frac{1}{\tau^{2}}\right) \quad \text{as} \quad \tau \to +\infty.
\] (2.7)

Next we show the following.

Proposition 2.6. Let \( \Phi(z) \) be given by (2.1) and \( u \in W^1_2(\Omega) \). Then

\[
\int_{\Omega} u e^{\epsilon(\phi-x)} \, dx = O\left(\frac{1}{\tau^{1-c}}\right) \quad \text{as} \quad \tau \to +\infty
\] (2.8)

for any \( c \in (0, 1) \).

Proof. See the supplementary data (available from stacks.iop.org/ip/31/035004/mmedia).

We define two operators:

\[
\mathcal{R}_\epsilon g = \frac{1}{2} e^{\epsilon(\phi-x)} \partial_x^{-1}\left( g e^{\epsilon(\phi-x)} \right), \quad \tilde{\mathcal{R}}_\epsilon g = \frac{1}{2} e^{\epsilon(\phi-x)} \partial_x^{-1}\left( g e^{\epsilon(\phi-x)} \right)
\] (2.9)

For any \( g \in L^2(\Omega) \) we have

\[
2\partial_x \mathcal{R}_\epsilon g + 2\tau \left( \partial_x \tilde{\mathcal{R}}_\epsilon g \right) = g \quad \text{and} \quad 2\partial_x \tilde{\mathcal{R}}_\epsilon g + 2\tau \left( \partial_x \Phi \right) \tilde{\mathcal{R}}_\epsilon g = g \quad \text{in} \quad \Omega.
\] (2.10)

We have the following.

Proposition 2.7. Let \( u \in W^1_2(\Omega) \). Then for any \( c \in (0, 1) \), there exist positive constants \( C(c) \) and \( \tau_0 \) such that

\[
\| \mathcal{R}_\epsilon u \|_{L^2(\Omega)} + \| \partial_x \mathcal{R}_\epsilon u \|_{L^2(\Omega)} \leq C(c) \| u \|_{W^1_2(\Omega)} / \tau^{1-c} \quad \forall \ |\epsilon| \geq \tau_0.
\] (2.11)

The constant \( C(c) \) is independent of \( \epsilon \).

Proof. See the supplementary data (available from stacks.iop.org/ip/31/035004/mmedia).

We conclude this section with Carleman estimates for some partial differential equations, which are fundamental in the arguments after the next sections.

Consider the boundary value problem

\[
\partial_x W + AW = f \quad \text{in} \quad \Omega, \quad W|_{\partial \Omega} = 0.
\] (2.12)

Let \( \tilde{\beta} \) be a smooth function on \( \Omega \) such that

\[
V \tilde{\beta}(x) \neq 0 \quad \text{on} \quad \Omega, \quad \tilde{\beta}(x) \neq 0 \quad \text{in} \quad \Omega, \quad \min_{x \in \partial \Omega} \tilde{\beta}(x) > \frac{3}{4} \max_{x \in \partial \Omega} \tilde{\beta}(x).
\] (2.13)

We set

\[
\phi_\epsilon = e^{\epsilon \tilde{\beta}}.
\]
Proposition 2.8. Let \( A \in C^0(\Omega) \) be a \( 3 \times 3 \) matrix. Then there exist constants \( s_0 \) and \( C \) independent of \( s \) such that
\[
\int_{\Omega} \phi W^2 e^{2 s \phi} \, dx \leq C \left\| e^{\phi} \right\|_{L^2(\Omega)}^2, \quad \forall \ W \in \mathcal{W}_2(\Omega),
\]
for all \( s \geq s_0 \).

**Proof.** Obviously it suffices to consider the case when \( A \equiv 0 \). Denote \( \tilde{v} = We^{\phi}, \tilde{f} = f e^{\phi} \), \( L_-(x, D, s) \tilde{v} = \frac{1}{2} \partial_x \tilde{v} + \frac{1}{2} s \left( \partial_s \phi \right) \tilde{v} \) and \( L_+(x, D, s) \tilde{v} = \frac{1}{2} \partial_x \tilde{v} - \frac{1}{2} s \left( \partial_s \phi \right) \tilde{v} \). In the new notations we rewrite equation (2.12) as
\[
L_-(x, D, s) \tilde{v} + L_+(x, D, s) \tilde{v} = \tilde{f} \quad \text{in} \quad \Omega.
\]
Taking the \( L^2 \)-norm of the left- and right-hand sides of (2.15), we obtain
\[
\| L_-(x, D, s) \tilde{v} \|_{L^2(\Omega)}^2 + 2 \text{ Re} \int_{\Omega} L_-(x, D, s) \tilde{v} L_+(x, D, s) \tilde{v} \, dx + \| L_+(x, D, s) \tilde{v} \|_{L^2(\Omega)}^2 = \| \tilde{f} \|_{L^2(\Omega)}^2.
\]
Integrating by parts the second term of (2.16), we obtain
\[
2 \text{ Re} \int_{\Omega} L_-(x, D, s) \tilde{v} L_+(x, D, s) \tilde{v} \, dx = \text{ Re} \int_{\Omega} \left( \left[ L_-, L_+ \right] \tilde{v} \right) \tilde{v} \, dx.
\]
Short computations yield
\[
\left[ L_-, L_+ \right] = \frac{s}{4} \Delta \phi = \left( \frac{s^3}{4} |V \hat{\beta}|^2 + \frac{s^2}{4} \Delta \phi \right) \phi.
\]
By (2.13) there exists a positive constant \( C \) independent of \( s \) such that
\[
\left[ L_-, L_+ \right] \geq Cs^3 \phi \quad \text{in} \quad \overline{\Omega}.
\]
From (2.16)–(2.19) we obtain (2.14). \( \square \)

Proposition 2.9. There exist constants \( s_0 \) and \( C \) independent of \( s \) such that
\[
\int_{\Omega} \phi W^2 e^{2 s \phi} \, dx \leq C \left\| e^{\phi} \right\|_{L^2(\Omega)}^2, \quad \forall \ W \in \mathcal{W}_2(\Omega),
\]
for all \( s \geq s_0 \).

**Proof.** See [7].

Corollary 2.1. There exist constants \( s_0 \) and \( C \) independent of \( s \) such that
\[
\int_{\Omega} \sum_{|\alpha| \leq 3} \phi^{2(3-|\alpha|)} W^2 e^{2 s \phi} \, dx \leq C \left\| e^{\phi} \right\|_{L^2(\Omega)}^2, \quad \forall \ W \in \mathcal{W}_2(\Omega),
\]
for all \( s \geq s_0 \).

3. Construction of special solutions to weakly coupling second-order elliptic systems

Let \( A_j(x), B_j(x), C_j(x), j = 1, 2 \), be smooth \( 3 \times 3 \) matrix functions in \( C^{5+\alpha}(\overline{\Omega}) \) with some \( \alpha \in (0, 1) \). Let \( U_0, V_0 \in C^{6+\alpha}(\overline{\Omega}) \) be nontrivial solutions to the differential equations:
\[ 2\partial_z U_0 + A U_0 = 0 \quad \text{in } \Omega \] (3.1)

and

\[ 2\partial_z V_0 + B_2 V_0 = 0 \quad \text{in } \Omega. \] (3.2)

We have the following.

**Proposition 3.1.** Let \( A, B_2 \in C^{5+\alpha} (\overline{\Omega}) \) for some \( \alpha \in (0, 1) \), \( \vec{n}_{0, k}, \ldots, \vec{n}_{2, k} \in \mathbb{C}^3 \) be arbitrary vectors and \( x_1, \ldots, x_k \in \Omega \) be arbitrary and mutually distinct. There exist a solution \( U_0 \in C^{5+\alpha} (\partial \Omega) \) to problem (3.1) and a solution \( V_0 \in C^{5+\alpha} (\partial \Omega) \) to problem (3.2) such that

\[ \partial_z U_0 (x_k) = \vec{n}_{j,k} \quad \text{and} \quad \partial_z V_0 (x_k) = \vec{n}_{j,k}, \quad j = 0, 1, 2. \] (3.3)

**Proof.** See [14].

Let \( B \) be a \( 3 \times 3 \) matrix whose elements are in \( C^{5+\alpha} (\overline{\Omega}) \) with \( \alpha \in (0, 1) \) and \( \vec{\chi} \) be some fixed point from \( \Omega \). Applying proposition 3.1 to the equation

\[ (2\partial_z + B) u = 0 \quad \text{in } \Omega, \] (3.4)

we can construct solutions \( U_{0, k} \) such that

\[ U_{0, k} (\vec{\chi}) = \vec{e}_k, \quad \forall \, k \in \{1, 2, 3\}. \]

Consider the matrix

\[ \Pi (x) = \left( U_{0,1} (x), U_{0,2} (x), U_{0,3} (x) \right). \]

Then

\[ \left( \partial_z + \frac{1}{2} \text{tr } B \right) \text{det} \Pi = 0 \quad \text{in } \Omega. \]

Hence there exists a holomorphic function \( q(z) \) such that \( \text{det} \Pi = q(z) e^{-\frac{1}{2} \text{tr } B} \) (see [26], p. 147). By \( Q \) we denote the set of the zeros of the function \( q \) on \( \overline{\Omega} : Q = \{ z \in \overline{\Omega}; \quad q(z) = 0 \} \). Obviously \( \text{card } Q < \infty \). By \( \kappa \) we denote the highest order of the zeros of the function \( q \) on \( \overline{\Omega} \).

Using proposition 3.1 we construct solutions \( \vec{U}_{0, k} \) to problem (3.4) such that

\[ \vec{U}_{0, k} (x) = \vec{e}_k, \quad \forall \, x \in \overline{\Omega}, \quad k \in \{1, 2, 3\}. \]

Set \( \overline{\Pi} = \left( \vec{U}_{0,1}, \vec{U}_{0,2}, \vec{U}_{0,3} \right) \). Then there exists a holomorphic function \( \vec{q} \) such that \( \text{det} \overline{\Pi} = \vec{q} (z) e^{-\frac{1}{2} \text{tr } B} \). Let \( \overline{Q} = \{ z \in \overline{\Omega}; \quad \vec{q}(z) = 0 \} \) and \( \vec{r} \) be the highest order of the zeros of the function \( \vec{q} \).

By \( \vec{U}_{0, k} (x) = \vec{e}_k \) for \( x \in \overline{Q} \), we see that

\[ \overline{Q} \cap Q = \emptyset. \]

Therefore there exists a holomorphic function \( r(z) \) such that

\[ r|_Q = 0 \quad \text{and} \quad (1 - r)|_Q = 0 \]

and the orders of the zeros of the function \( r \) on \( Q \) and the function \( 1 - r \) on \( \overline{Q} \) are greater than or equal to \( \max \{ \kappa, \vec{r} \} \).
We set
\[ P_0 f = \frac{1}{2} \Pi \partial_x^{-1} (\Pi^{-1} r f) + \frac{1}{2} \Pi \partial_{\bar{z}}^{-1} (\Pi^{-1} (1 - r) f). \]  
(3.5)

Then
\[ P_0^* f = -\frac{1}{2} r (\Pi^{-1})^* \partial_z^{-1} (\Pi^* f) - \frac{1}{2} (1 - r) (\Pi^{-1})^* \partial_{\bar{z}}^{-1} (\Pi^* f). \]

We have the following.

**Proposition 3.2.** The linear operators \( P_0, P_0^* \in \mathcal{L} (L^2(\Omega), W_2^1(\Omega)) \) solve the differential equations
\[ (-2 \partial_z + B^*) P_0^* g = g, \quad (2 \partial_z + B) P_0 g = g \quad \text{in} \quad \Omega. \]  
(3.6)

**Proof.** See the supplementary data (available from stacks.iop.org/ip/31/035004/mmedia).

Short computations imply
\[
\begin{aligned}
[\partial_z, P_0 f] &= \frac{1}{2} \partial_z \Pi \partial_x^{-1} (\Pi^{-1} r f) - \frac{\Pi}{8\pi} \int_{\partial \Omega} \frac{(\nu_1 - i \nu_2) \Pi^{-1} r f}{z - \zeta} \, d\sigma \\
&\quad + \frac{1}{2} \Pi \partial_{\bar{z}}^{-1} \left( \partial_z (\Pi^{-1} r f) \right) + \frac{1}{2} \partial_z \Pi \partial_{\bar{z}}^{-1} \left( \Pi^{-1} (1 - r) f \right) \\
&\quad - \frac{\Pi}{8\pi} \int_{\partial \Omega} \frac{(\nu_1 - i \nu_2) \Pi^{-1} (1 - r) f}{z - \zeta} \, d\sigma + \frac{1}{2} \Pi \partial_{\bar{z}}^{-1} \left( \partial_z (\Pi^{-1} (1 - r) f) \right).
\end{aligned}
\]  
(3.7)

In a similar way, we construct matrices \( \Pi_0, \bar{\Pi}_0 \), an antiholomorphic function \( r_0 (z) \) and operators
\[ T_0 f = \frac{1}{2} \Pi_0 \partial_x^{-1} (\Pi_0^{-1} r_0 (z) f) + \frac{1}{2} \bar{\Pi}_0 \partial_{\bar{z}}^{-1} (\bar{\Pi}_0^{-1} (1 - r_0 (z)) f) \]  
(3.8)

and
\[
\begin{aligned}
[\partial_z, T_0 f] &= \frac{1}{2} \partial_z \Pi_0 \partial_x^{-1} (\Pi_0^{-1} r_0 f) - \frac{\Pi_0}{8\pi} \int_{\partial \Omega} \frac{(\nu_1 + i \nu_2) \Pi_0^{-1} r_0 f}{z - \bar{z}} \, d\sigma \\
&\quad + \frac{1}{2} \Pi_0 \partial_{\bar{z}}^{-1} \left( \partial_z (\Pi_0^{-1} r_0 f) \right) + \frac{1}{2} \partial_z \Pi_0 \partial_{\bar{z}}^{-1} \left( \Pi_0^{-1} (1 - r_0) f \right) \\
&\quad - \frac{\Pi_0}{8\pi} \int_{\partial \Omega} \frac{(\nu_1 + i \nu_2) \Pi_0^{-1} (1 - r_0) f}{z - \bar{z}} \, d\sigma + \frac{1}{2} \Pi_0 \partial_{\bar{z}}^{-1} \left( \partial_z (\Pi_0^{-1} (1 - r_0) f) \right).
\end{aligned}
\]  
(3.11)
The formulae (3.11) and (3.7) and propositions 2.1 and 2.2 imply the following proposition.

**Proposition 3.3.** Let \( B \in C^{5+\alpha} (\Omega) \) for some \( \alpha \in (0, 1) \). Then

\[
[\partial_\tau, T_B], [\partial_\tau, P_B] \in \mathcal{L}(C^0(\Omega), L^2(\Omega)).
\]

Next we investigate the properties of the commutators of the operators \( P_B, T_B \) and \( \partial_\tau \).

**Proposition 3.4.** Let \( B \in C^{5+\alpha} (\Omega) \) for some \( \alpha \in (0, 1) \) and let \( q \in C^1 (\Omega) \) and \( q (\tilde{x}) = 0 \). Assume that the restriction on \( \partial \Omega \) of the function \( \psi := \text{Im} \Phi \) has a finite number of critical points on \( \partial \Omega \) and all these points are nondegenerate. Then there exist positive constants \( C \) and \( \tau_0 \) independent of \( \tau \) such that

\[
\| -\partial_\tau \Phi \left[ \partial_\tau, P_B \right] \left( q e^{i(\psi - \sigma)} \right) + \left[ \partial_\tau, P_B \right] \left( q \partial_\tau \Phi e^{i(\psi - \sigma)} \right) \|_{L^2(\Omega)} \leq \frac{C}{\tau^2} \quad \forall \tau \geq \tau_0.
\]

Moreover

\[
\left[ \partial_\tau, \left[ \partial_\tau, P_B \right] \right] \in \mathcal{L}(W^2_2(\Omega), L^2(\Omega)) \quad \forall s > \frac{1}{2}.
\]

**Proof.** See the supplementary data (available from stacks.iop.org/ip/31/035004/mmedia).

Next we introduce two operators

\[
\mathcal{R}_{r, b \Phi} = e^{i(\sigma - \psi)}T_B \left( e^{i(\sigma - \psi)} g \right), \quad \mathcal{R}_{r, b \Phi} = e^{i(\sigma - \psi)}P_B \left( e^{i(\sigma - \psi)} g \right).
\]

For any \( g \in L^2(\Omega) \), by (3.6) and (3.10) we see that the functions \( \mathcal{R}_{r, b \Phi} \) and \( \mathcal{R}_{r, b \Phi} \) solve the equations

\[
(2\partial_\tau + 2\tau \partial_\tau \Phi + B)\mathcal{R}_{r, b \Phi} = g, \quad \left( 2\partial_\tau + 2\tau \partial_\tau \Phi + B \right)\mathcal{R}_{r, b \Phi} = g \quad \text{in} \quad \Omega.
\]

We have the following.

**Proposition 3.5.** Let \( B \in C^{5+\alpha} (\Omega) \) for some \( \alpha \in (0, 1) \) and \( u \in W^2_2(\Omega) \). Then for any \( \epsilon \in (0, 1) \) there exist constants \( C(\epsilon) \) and \( \tau_0(\epsilon) \) such that

\[
\| \mathcal{R}_{r, b u} \|_{L^2(\Omega)} + \| \mathcal{R}_{r, b u} \|_{L^2(\Omega)} \leq C(\epsilon) \| u \|_{W^2_2(\Omega)} / r^{1-\epsilon} \quad \forall |r| \geq \tau_0.
\]

The constant \( C(\epsilon) \) is independent of \( \tau \).

**Proof.** See the supplementary data (available from stacks.iop.org/ip/31/035004/mmedia).

For \( 3 \times 3 \) matrix functions \( A_j, B_j, C_j \in C^{5+\alpha} (\Omega) \), \( j = 1, 2 \) with some \( \alpha \in (0, 1) \), consider the system of linear

\[
P_j (x, D) W := \Delta W + 2A_j \partial_\tau W + 2B_j \partial_\tau W + C_j W = 0 \quad \text{in} \quad \Omega, \quad j = 1, 2.
\]

We set

\[
Q_1 (j) = -2\partial_\tau A_j - B_j A_j + C_j, \quad Q_2 (j) = -2\partial_\tau B_j - A_j B_j + C_j, \quad j = 1, 2.
\]

We have the following.
Proposition 3.6. Let \( A_j, B_j \in C^{5+\alpha}(\Omega) \), \( j = 1, 2 \), the functions \( U_0, V_0 \) be given by proposition 3.1, the functions \( A_1, D_2 \) be some solutions to equations (3.1) and (3.2) respectively, and the functions \( Q_k(j) \) be defined by (3.18). Then the functions \( U \in W^2_2(\Omega) \) and \( V \in W^2_2(\Omega) \) are defined by

\[
U = e^{\phi}(U_0 - U_1) + \sum_{j=2}^{\infty} (-1)^j U_j e^{\phi_j}, \quad U_j = \mathcal{R}_{t,A_1}(TB_1(Q_1(1) U_{j-1})) \quad j \geq 3,
\]

\[
U_1 = \mathcal{R}_{t,B_1}(P_{A_1}(Q_1(1) U_0) - A_1), \quad U_2 = \mathcal{R}_{t,A_1}(TB_1(Q_2(1) e^{\phi - \phi_j} U_1)); \quad V = e^{-\tau \phi}(V_0 - V_1) + \sum_{j=2}^{\infty} (-1)^j V_j e^{-\tau \phi_j}, \quad V_j = \mathcal{R}_{t,B_2}(P_{A_2}(Q_2(2) V_{j-1})) \quad j \geq 3,
\]

\[
V_1 = \mathcal{R}_{t,A_1}(TB_1(Q_2(2) V_0) - D_2), \quad V_2 = \mathcal{R}_{t,B_2}(P_{A_2}(Q_2(2) e^{\phi - \phi_j} V_1))
\]

are solutions to the system (3.17) with \( j = 1, 2 \) respectively for all sufficiently large positive \( \tau \). Moreover, there exists a constant \( C \) independent of \( \tau \) such that

\[
\|U e^{-\tau \phi} \|_{W^2_2(\Omega)} + \|V e^{\tau \phi} \|_{W^2_2(\Omega)} \leq C(1 + \tau^2).
\]

Here \( \phi = \text{Re} \phi \).

Proof. See the supplementary data (available fromstacks.iop.org/ip/31/035004/mmedia).

Similarly to the operators \( \mathcal{R}_{t,B}, \mathcal{R}_{t,A} \), we introduce the operators \( \mathcal{P}_{t,B}, \mathcal{P}_{t,A} \) by

\[
\mathcal{P}_{t,B,2} = e^{\tau(\phi - \phi_j)} T_B(e^{\tau(\phi - \phi_j)} g), \quad \mathcal{P}_{t,A,2} = e^{\tau(\phi - \phi_j)} P_B(e^{\tau(\phi - \phi_j)} g),
\]

where the function \( \phi \) is introduced in (2.5).

We have the following.

Proposition 3.7. Let \( A_j, B_j \in C^{5+\alpha}(\Omega), \alpha \in (0, 1) \), \( j = 1, 2 \), the functions \( U_0, V_0 \) be given by proposition 3.1, the functions \( A_1, D_2 \) be some solutions to equations (3.1) and (3.2) respectively, and the functions \( Q_k(j) \) be defined by (3.18). Then the functions \( \tilde{U} \) and \( \tilde{V} \) defined by

\[
\tilde{U} = e^{\phi}(\tilde{U}_0 - \tilde{U}_1) + \sum_{j=2}^{\infty} (-1)^j \tilde{U}_j e^{\phi_j}, \quad \tilde{U}_j = \mathcal{P}_{t,A_1}(TB_1(Q_1(1) \tilde{U}_{j-1})) \quad j \geq 3,
\]

\[
\tilde{U}_1 = \mathcal{P}_{t,B_1}(P_{A_1}(Q_1(1) \tilde{U}_0) - A_1), \quad \tilde{U}_2 = \mathcal{P}_{t,A_1}(TB_1(Q_2(1) e^{\phi - \phi_j} \tilde{U}_1)); \quad \tilde{U}_0 = e^{\tau} U_0
\]

and

\[
\tilde{V} = e^{-\tau \phi}(\tilde{V}_0 - \tilde{V}_1) + \sum_{j=2}^{\infty} (-1)^j \tilde{V}_j e^{-\tau \phi_j}, \quad \tilde{V}_j = \mathcal{P}_{t,B_2}(P_{A_2}(Q_2(2) \tilde{V}_{j-1})) \quad j \geq 3
\]

\[
\tilde{V}_1 = \mathcal{P}_{t,A_2}(TB_1(Q_2(2) \tilde{V}_0) - D_2), \quad \tilde{V}_2 = \mathcal{P}_{t,B_2}(P_{A_2}(Q_2(2) e^{\phi - \phi_j} \tilde{V}_1)); \quad \tilde{V}_0 = e^{\tau} V_0
\]

are solutions to the system (3.17) with \( j = 1, 2 \) respectively for all sufficiently large positive \( \tau \). Here \( k \in \mathbb{N}_+ \) is a parameter which we choose later.
There exists a constant $C$ independent of $\tau$ such that
\[
\| \mathcal{U} e^{-i\phi} \|_{W^2_2(\Omega)} + \| \mathcal{V} e^{i\phi} \|_{W^2_2(\Omega)} \leq C (1 + \tau^3).
\] (3.25)

Here we set $\tilde{\phi} = \text{Re} \, \Phi$.

**Proof.** Proposition 3.7 is proved in exactly the same way as proposition 3.6.

The following proposition provides the a priori estimate for the functions $U_j, V_j, \mathcal{U}_j, \mathcal{V}_j$ which will be widely used in section 4.

**Proposition 3.8.** Let $A_j, B_j \in C^{5+\alpha} (\overline{\Omega})$, $\alpha \in (0, 1)$, the functions $U_j, V_j$ be determined by (3.19), (3.20), \( T_{B_j} (\mathcal{Q}_2 (2) V_0) - D_2 (\tilde{x}) = (P_{A_j} (\mathcal{Q}_1 (1) U_0) - \mathcal{A}_j)(\tilde{x}) = 0 \) and $\mathcal{U}_j, \mathcal{V}_j$ be given by (3.23) and (3.24). Then the following estimates hold true: For any $p > 2$ there exist constants $C(p)$ and $\tau_0(p)$ such that
\[
\| U_j \|_{L^p_2(\Omega)} + \| V_j \|_{L^p_2(\Omega)} + \| \mathcal{U}_j \|_{L^p_2(\Omega)} + \| \mathcal{V}_j \|_{L^p_2(\Omega)} \leq C(p) / \tau \quad \forall \tau \geq \tau_0(p).
\] (3.26)

For any $\epsilon \in (0, 1)$ there exist constants $C(\epsilon), \tau_0(\epsilon)$ such that
\[
\| U_j \|_{L^2(\Omega)} + \| V_j \|_{L^2(\Omega)} + \| \mathcal{U}_j \|_{L^2(\Omega)} + \| \mathcal{V}_j \|_{L^2(\Omega)} \leq C(\epsilon) / \tau^{1-\epsilon} \quad \forall \epsilon \in (0, 1)
\] (3.27)
and for $j \geq 2$
\[
\| V (U_j e^{i\phi}) \|_{L^2_{2,\text{reg}}(\Omega)} + \| V (V_j e^{-i\phi}) \|_{L^2_{2,\text{reg}}(\Omega)} \leq C(\epsilon) / \tau^{2-\epsilon} \quad \forall \tau \geq \tau_0(\epsilon).
\] (3.28)

There exist constants $C, \tau_0$ independent of $\tau$ such that
\[
\tau \| V (U_2 e^{i\phi}) \|_{L^2_{2,\text{reg}}(\Omega)} + \tau \| V (V_2 e^{-i\phi}) \|_{L^2_{2,\text{reg}}(\Omega)}
\] (3.29)
\[+ \| V (U_j e^{i\phi}) \|_{L^2_{2,\text{reg}}(\Omega)} + \| V (V_j e^{-i\phi}) \|_{L^2_{2,\text{reg}}(\Omega)} \leq C
\]
for all $\tau \geq \tau_0$.

**Proof.** See the supplementary data (available from stacks.iop.org/ip/31/035004/mmedia).

Consider a system of linear
\[
P_j (s, D) Z = \Delta Z + 2A_1 \partial_s Z + 2B_1 \partial_s Z + C_1 Z = e^{i\phi} f \quad \text{in } \Omega.
\] (3.30)

The following proposition establishes the solvability of equation (3.30).

**Proposition 3.9.** Let $A_1, B_1 \in C^{5+\alpha} (\overline{\Omega})$, $\alpha \in (0, 1)$ and $f \in L^2(\Omega)$. Then for large positive $\tau$ the functions $Z$ given by
\[
Z = \sum_{j=0}^{\infty} (\partial_s^j Z_0 e^{i\phi}) \quad Z_0 = \mathcal{R}_{\tau, h_1} \left( P_{A_1} \left( \mathcal{Q}_2 (1) Z_{j-1} \right) \right) \quad j \geq 2, \quad Z_0 = \mathcal{R}_{\tau, h_1} \left( P_{A_1} f \right)
\] (3.31)
is a solution to system (3.30). Moreover, there exist constants $\tau_0$ and $C$ independent of $\tau$ such that the following estimate holds true:
\[
\| e^{-i\phi} Z \|_{W^2_2(\Omega)} + \| r \|_{L^2(\Omega)} \| e^{-i\phi} Z \|_{L^2_2(\Omega)} \leq C (1 + \tau^2) \| f \|_{L^2_2(\Omega)} \quad \forall \tau \geq \tau_0,
\] (3.32)
where $\phi := \text{Re} \, \Phi$. 

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Proof. See the supplementary data (available from stacks.iop.org/ip/31/035004/mmedia).

Similarly we prove the following.

Proposition 3.10. Let \( A_1, B_1 \in C^{5+\alpha}(\Omega), \alpha \in (0, 1) \) and \( f \in L^2(\Omega) \). Then for large positive \( \tau \) the function

\[
Z = \sum_{j=0}^{\infty} (-1)^j Z_j e^{\Phi_j}, \quad Z_j = \mathcal{P}_{\tau, B_1}(P_{A_1}(Q_2(1)Z_{j-1})) \quad j \geq 2, \quad Z_0 = \mathcal{P}_{\tau, B_1}(P_{A_1} f) \quad (3.33)
\]

is a solution to the system (3.30). Moreover, there exist constants \( C \) and \( \tau_0 \) independent of \( \tau \) such that

\[
\| e^{-i\tau q} Z \|_{L^2(\Omega)} + \| \tau \| e^{-i\tau q} Z \|_{L^2(\Omega)} \leq C (1 + \tau^2) \| e^{-i\tau q + i\tau f} \|_{L^2(\Omega)} \quad \forall \tau \geq \tau_0, \quad (3.34)
\]

where we set \( \hat{q} = \text{Re} \Phi \) and \( q = \text{Re} \Phi \).

4. Derivation of differential equations with respect to unknown coefficients

Let \( H(x, \partial_z, \partial_{\bar{z}}) \) be a second-order differential operator whose coefficients are smooth and have compact supports in \( \Omega \). We write such an operator in the form

\[
H(x, \partial_z, \partial_{\bar{z}}) = C_1(x) \partial_z^2 + C_0(x) \partial_z + C_2(x) \partial_{\bar{z}}^2 + B_1(x) \partial_z + B_2(x) \partial_{\bar{z}} + B_0(x), \quad (4.1)
\]

where \( C_1, B_j \in C^{5+\alpha}(\Omega) \) with \( \alpha \in (0, 1) \) are complex 3 \times 3 matrices.

Let the functions \( U, V \) be solutions to equation (3.17) constructed in proposition 3.6 by formulae (3.19) and (3.20) respectively and let the functions \( \tilde{U}, \tilde{V} \) be solutions to equation (3.17) constructed in proposition 3.7 by formulae (3.23) and (3.24) respectively. Denote

\[
q_2 = (T_{B_1}(Q_2(2)U_0) - D_2), \quad q_1 = (P_{A_1}(Q_1(1)U_0) - \mathcal{A}_1) \quad (4.2)
\]

and

\[
\tilde{q}_2 = (T_{B_1}(Q_2(2)\tilde{V}_0) - z^{\kappa} D_2), \quad \tilde{q}_1 = (P_{A_1}(Q_1(1)\tilde{V}_0) - z^{\kappa} \mathcal{A}_1). \quad (4.3)
\]

Here \( \mathcal{A}_1 \) and \( D_2 \in C^{5+\alpha}(\Omega) \) satisfy

\[
\mathcal{A}_1 \in \text{Ker} \left( 2\partial_z + A_1 \right), \quad D_2 \in \text{Ker} \left( 2\partial_{\bar{z}} + B_2 \right) \quad (4.4)
\]

and \( k \in \mathbb{N}_+ \) is a parameter.

We recall that \( \Phi(z) = (z - \bar{z})^2 - \kappa \) and \( \overline{\Phi}(z) = e^{i\theta}z - \kappa \) with \( \theta \in [0, 2\pi) \). We have

Proposition 4.1. Let the functions \( U_j, V_j \) be defined in proposition 3.6 and \( q_j \) be given by (4.2) and \( q_j(\bar{x}) = q_j(\bar{x}) = 0 \). The following asymptotic formulae hold true:

\[
(U_0 e^{\Phi}(H(x, \partial_z, \partial_{\bar{z}})(V_0 e^{-\Phi})))_{L^2(\Omega)} = \delta_{z, \tau}(U_0, H(x, \partial_z, \partial_{\bar{z}} - \tau \partial_{\bar{z}} \overline{\Phi}) V_0)
\]

\[
+ o \left( \frac{1}{\tau} \right) \quad \text{as} \quad \tau \to +\infty, \quad (4.5)
\]
\[
(U_0 e^{\tau\Phi}, H(x, \partial_x, \partial_x) (V_0 e^{-\tau\Phi})_{L^2(\Omega)}) = \bar{\delta}_{x,r} (P^*_{B_1} H^* (x, \partial_x + \tau \partial \Phi, \partial_x) U_0, q_0) + o \left( \left( \frac{1}{\tau} \right) \right) \text{ as } \tau \rightarrow +\infty,
\]

(4.6)

\[
(U_2 e^{\tau\Phi}, H(x, \partial_x, \partial_x) (V_0 e^{-\tau\Phi})_{L^2(\Omega)}) = \bar{\delta}_{x,r} (q_1, T^*_{B_1} H (x, \partial_x, \partial_x - \tau \partial \Phi) V_0) + o \left( \left( \frac{1}{\tau} \right) \right) \text{ as } \tau \rightarrow +\infty,
\]

(4.7)

\[
(U_2 e^{\tau\Phi}, H(x, \partial_x, \partial_x) (V_2 e^{-\tau\Phi})_{L^2(\Omega)}) = \left( \frac{1}{2} \bar{\delta}_{x,r}, (q_1, T^*_{B_1} [Q_2 (1)^* T^*_{B_1} ((C_0 \partial_x + C_2 (\partial_x - \tau \partial \Phi) + b) U_0)] \right) + \left( \frac{1}{2} \bar{\delta}_{x,r}, (q_1, T^*_{B_1} [Q_2 (1)^* T^*_{B_1} ((C_0 \partial_x + C_2 (\partial_x - \tau \partial \Phi) + b) U_0)] \right) + o \left( \left( \frac{1}{\tau} \right) \right) \text{ as } \tau \rightarrow +\infty.
\]

(4.8)

where \( b = B_2 - \partial_x C_2 + \frac{1}{2} \partial_x C_2; \)

\[
(U_0 e^{\tau\Phi}, H(x, \partial_x, \partial_x) (V_2 e^{-\tau\Phi})_{L^2(\Omega)}) = \left( \frac{1}{2} \bar{\delta}_{x,r}, (P^*_{B_1} [Q_2 (1)^* P^*_{A_1} ((C_1^0 (\partial_x + \tau \partial \Phi) + C_0^* \partial_x + b) U_0)] \right) \right), q_2) + \left( \frac{1}{2} \bar{\delta}_{x,r}, (P^*_{B_1} [Q_2 (1)^* P^*_{A_1} ((C_1^0 (\partial_x + \tau \partial \Phi) + C_0^* \partial_x + b) U_0)] \right) \right), q_2) + o \left( \left( \frac{1}{\tau} \right) \right) \text{ as } \tau \rightarrow +\infty.
\]

(4.9)

where \( \tilde{b} = -B_1 + \frac{1}{2} B_1 \). \[\]

\[
(U_2 e^{\tau\Phi}, H(x, \partial_x, \partial_x) (V_2 e^{-\tau\Phi})_{L^2(\Omega)}) = \bar{\delta}_{x,r} (P^*_{B_1} (C_1^0 (\partial_x + \tau \partial \Phi + \frac{1}{2} B_1) = B^*_1 (q_1), q_2) + \left( \frac{1}{2} \bar{\delta}_{x,r}, (q_1, T^*_{B_1} (C_2 (\partial_x - \tau \partial \Phi + \frac{1}{2} A_2) + B_2) \right) + \left( \frac{1}{2} \bar{\delta}_{x,r}, (q_1, T^*_{B_1} (C_2 (\partial_x - \tau \partial \Phi + \frac{1}{2} A_2) + B_2) \right) + o \left( \left( \frac{1}{\tau} \right) \right) \text{ as } \tau \rightarrow +\infty.
\]

(4.10)

where \( \mathcal{B}_1 = -2 \partial_x C_1 + B^*_1 C_1 - \frac{1}{2} C_0 A_2 + B_1 \) \( \text{ and } \mathcal{B}_2 = -\frac{1}{2} B_1 C_0 - \partial_x C_0 - C_2 A_2 + B_2. \)

**Proof.** See the supplementary data (available from stacks.iop.org/ip/31/035004/mmedia).

Similarly to proposition 4.1, we prove proposition 4.2.
Proposition 4.2. Let the functions \( \tilde{U}, \tilde{V} \) be defined in proposition 3.6 and \( q \) be given by \((4.3)\). The following asymptotic formulae hold true:

\[
(\tilde{U}_0 e^{\tau \Phi}, H(x, \partial_x, \partial_z)(\tilde{V}_0 e^{-\tau \Phi}))_{L^2(\Omega)} = o\left(\frac{1}{\tau}\right) \quad \text{as} \quad \tau \to +\infty,
\]

\[
(\tilde{U}_0 e^{\tau \Phi}, H(x, \partial_x, \partial_z)(\tilde{V}_0 e^{-\tau \Phi}))_{L^2(\Omega)} = \gamma (P_{\Lambda_2}^n H^*(x, \partial_z + \tau \partial_x \Phi, \partial_z \tilde{U}_0, q_2) + o\left(\frac{1}{\tau}\right).
\]

Proof. See the supplementary data (available from stacks.iop.org/ip/31/035004/mmedia).

Henceforth we define

\[
\Psi_{1,k} = \begin{cases} \Phi & \text{for } k \in \{0, 1\}, \\ \Phi & \text{for } k \in \{2, \ldots\}, \end{cases}
\]

\[
\Psi_{2,k} = \begin{cases} \Phi & \text{for } k \in \{0, 1\}, \\ \Phi & \text{for } k \in \{2, \ldots\}. \end{cases}
\]

We have the following.

Proposition 4.3. Let \( k, \ell \in \mathbb{N}_+ \) and \( k + \ell \geq 3 \) and \( H(x, \partial_x, \partial_z) \) be a second-order differential operator with compactly supported smooth coefficients. Let the functions \( U \) and \( V \) be given by proposition 3.6 and \( q, q_1, q_2 \) be given by \((4.2)\). Assume that \( \psi := \text{Im} \Phi \) has a finite number of critical points on \( \partial \Omega \) and all these points are nondegenerate. If \( q_1(\tilde{x}) = q_2(\tilde{x}) = 0 \) and

\[
T_{\tilde{B}_1}^n (\partial_z \tilde{\Phi} C_2 V_0) = \partial_z \tilde{\Phi} T_{\tilde{B}_1}^n (C_2 V_0) \quad \text{and} \quad P_{\Lambda_2}^n \left( \partial_z \tilde{\Phi} C_1^n U_0 \right) = \partial_z \tilde{\Phi} P_{\Lambda_2}^n \left( C_1^n U_0 \right),
\]

(4.17)
then we have
\[
\left( U_k e^{y_k}, H(x, \partial_x, \partial_z) \left( V_k e^{y_k} \right) \right)_{L^2(\Omega)} = o \left( \frac{1}{\tau} \right) \text{ as } \tau \to +\infty. \tag{4.18}
\]

**Proof.** See the supplementary data (available from stacks.iop.org/ip/31/035004/mmedia).

Henceforth, we define
\[
\mathcal{Q}_1, k = \begin{cases} \Phi & \text{for } k \in \{0, 1\}, \\ \Phi & \text{for } k \in \{2, \ldots\}. \end{cases}
\]
\[
\mathcal{Q}_2, k = \begin{cases} \Phi & \text{for } k \in \{0, 1\}, \\ \Phi & \text{for } k \in \{2, \ldots\}. \end{cases}
\]

In a similar way to proposition 4.3, we prove the following.

**Proposition 4.4.** Let \( k, \ell \in \mathbb{N}_+ \) and \( k + \ell \geq 3 \) and \( H(x, \partial_x, \partial_z) \) be a second-order differential operator with compactly supported smooth coefficients. Let the functions \( U_k \) and \( V_\ell \) be given by proposition 3.7 and \( \mathcal{Q}_1, \mathcal{Q}_2 \) be given by (4.3). Then
\[
\left( U_k e^{y_k}, H(x, \partial_x, \partial_z) \left( V_\ell e^{y_\ell} \right) \right)_{L^2(\Omega)} = o \left( \frac{1}{\tau} \right) \text{ as } \tau \to +\infty.
\]

**Assumption.** Henceforth, without loss of generality we assume that the domain \( \Omega \) is a ball centered at the origin.

Denote
\[
\tilde{c}_1(x) = -P_{A^*} (2 \partial_\ell \Phi) \partial_\ell \left( C^*_1 U_0 \right) + \left( \partial_\ell \Phi \right) \partial_\ell \left( C^*_1 U_0 \right) + \left( \partial_\ell \Phi \right) B^*_1 U_0 \\
+ \frac{1}{2} Q_1 (2)^P \left( C^*_1 \left( \partial_\ell \Phi \right) U_0 \right) - \frac{C^*_1}{2} \left( \partial_\ell \Phi \right) \mathcal{Q}_1
\]
\[
\tilde{c}_2(x) = T^*_B \left( 2 C^*_2 \left( \partial_\ell \Phi \right) \partial_\ell V_0 + C^*_0 \left( \partial_\ell \Phi \right) \partial_\ell V_0 + \left( \partial_\ell \Phi \right) B^*_2 V_0 \\
+ \frac{1}{2} Q_2 (1)^P \left( C^*_2 \left( \partial_\ell \Phi \right) V_0 \right) - \frac{C^*_2}{2} \left( \partial_\ell \Phi \right) \mathcal{Q}_2 \right)
\]

We have the following.

**Proposition 4.5.** Let \( H(x, \partial_x, \partial_z) \) be a second-order differential operator given by (4.1) with compactly supported smooth coefficients in \( \Omega \). Suppose that for arbitrary solutions \( \mathcal{U}, \mathcal{V} \), which are dependent on \( \theta, A_0, D_2 \) and constructed in proposition 3.7, we have
\[
\left( \mathcal{U}, H(x, \partial_x, \partial_z) \mathcal{V} \right)_{L^2(\Omega)} = o \left( \frac{1}{\tau} \right) \text{ as } \tau \to +\infty. \tag{4.19}
\]

Then the following equalities hold true:
\[
P^*_A \left[ \mathcal{E} C^*_1 U_0 \right]_{\partial \Omega} = T^*_B \left[ \mathcal{E} C^*_2 V_0 \right]_{\partial \Omega} = 0 \quad k \in \{0, 1, 2\}, \tag{4.20}
\]
\( \mathbf{P}^n_{A_1} (2z^d \partial_z (C_0^n U_0) + z^d \partial_z (C_0^n U_0) - z^d \mathbf{B}_1 U_0 + \frac{1}{2} \mathbf{Q}_1 (2) \mathbf{P}^n_{A_1} (C_0^n z^d U_0) - \frac{1}{2} C_1 z^d q_i) \)

= 0 \text{ on } \partial \Omega \hspace{1cm} (4.21)

and

\( \mathbf{T}^n_{B_1} (2z^d C_2 \partial_z V_0 + C_0^2 \partial_z V_0 + z^d \mathbf{B}_2 V_0 + \frac{1}{2} \mathbf{Q}_2 (1) \mathbf{T}^n_{B_1} (C_2 z^d V_0) - \frac{1}{2} C_2 z^d q_i) \)

= 0 \text{ on } \partial \Omega. \hspace{1cm} (4.22)

**Proof.** From proposition 4.4 we obtain

\[
\begin{aligned}
\left( \bar{U}, \mathbf{H}(x, \partial_z, \partial_z) \bar{V} \right)_{L^2(\partial \Omega)} &= \left( (\bar{U}_0 - \bar{U}_1) e^{i \bar{\Phi}}, \mathbf{H}(x, \partial_z, \partial_z)((\bar{V}_0 - \bar{V}_1) e^{-i \bar{\Phi}}) \right)_{L^2(\partial \Omega)} \\
&+ \left( \bar{U}_2 e^{i \bar{\Phi}}, \mathbf{H}(x, \partial_z, \partial_z)(\bar{V}_2 e^{-i \bar{\Phi}}) \right)_{L^2(\partial \Omega)} \\
&+ O \left( \frac{1}{\tau} \right) \text{ as } \tau \to + \infty. \hspace{1cm} (4.23)
\end{aligned}
\]

Since the domain \( \Omega \) is assumed to be a ball, the linear function \( \bar{\Psi} := \text{Im} \bar{\Phi} \) has only two critical points on \( \partial \Omega \), that is, the point of minimum \( x_{\min} \) and the point of maximum \( x_{\max} \). Moreover after an appropriate choice of the function \( \bar{\Phi} \), we can assume that \( x_{\max} = \hat{x} \), where \( \hat{x} \) is an arbitrary fixed point from \( \partial \Omega \). Indeed, let \( \theta \) satisfy \( (\cos \theta, \sin \theta) = (x_{\max} - \hat{x}) / |x_{\max} - \hat{x}| \). We take \( \bar{\Phi} = e^{-i(\theta + \hat{x})} \) and the function \( \bar{\psi} = \text{Im} \bar{\Phi} = -\cos(\theta) x_1 - \sin(\theta) x_2 \) reaches the extremal values on \( \partial \Omega \) at the point \( \pm \hat{x} \). The second tangential derivatives of the function \( \bar{\psi} \) at the points \( x_{\max} \) and \( x_{\min} \) are not equal to zero since \( \partial \Omega \) is a circle centered at zero. Hence by (4.11)–(4.16) the first asymptotic term on the right-hand side of (4.23) is at most of the order \( \tau \). Therefore, by proposition 4.2 we can remove some terms and write (4.23) as

\[
\begin{aligned}
\left( \bar{U}, \mathbf{H}(x, \partial_z, \partial_z) \bar{V} \right)_{L^2(\partial \Omega)} &= - \int_{\partial \Omega} \left( \mathbf{P}^n_{A_2} \mathbf{H}^n (x, \partial_z + \tau \partial_z \bar{\Phi}, \partial_z) \bar{U}_0, \bar{q}_2 \right) \\
&- \int_{\partial \Omega} \left( \mathbf{T}^n_{B_2} \mathbf{H} (x, \partial_z, \partial_z - \tau \partial_z \bar{\Phi}) \bar{V}_0, \bar{q}_2 \right) + O \left( \frac{1}{\sqrt{\tau}} \right) \\
&- \int_{\partial \Omega} \left( \mathbf{P}^n_{A_1} \mathbf{H}^n (x, \partial_z + \tau \partial_z \bar{\Phi}, \partial_z) \bar{U}_1, \bar{q}_2 \right) \left( \frac{\nu_1 - \nu_2}{2 \tau \partial \bar{\Phi}} \right) e^{i \left( -\bar{\Phi} \right)} d\sigma \\
&- \int_{\partial \Omega} \left( \mathbf{T}^n_{B_1} \mathbf{H} (x, \partial_z, \partial_z - \tau \partial_z \bar{\Phi}) \bar{V}_1, \bar{q}_2 \right) \left( \frac{\nu_1 - \nu_2}{2 \tau \partial \bar{\Phi}} \right) e^{i \left( -\bar{\Phi} \right)} d\sigma \\
&+ O \left( \frac{1}{\sqrt{\tau}} \right) \text{ as } \tau \to + \infty. \hspace{1cm} (4.24)
\end{aligned}
\]

Then applying proposition 6 (p. 344) of [23], we obtain that there exists a function \( \chi \), which is not equal to zero at any point of \( \partial \Omega \), such that

\( \Box \)
\[
\left( \mathcal{U}, H(x, \partial_x, \partial_z) \overline{V} \right)_{L^2(\Omega)} = \left( \frac{(\nu_1 - i \nu_2) \sqrt{\mathcal{F}}}{2 \partial_z \phi} \right)(x_{\text{max}}) \\
\times \left\{ e^{-2i\theta} \left( \tilde{q}_1, T_{R\mathcal{H}}^{\nu} \left[ z^\nu C_2V_0 \right] \right) + e^{2i\theta} \left( P_{A\mathcal{H}}^{\nu} \left[ z^\nu C_2^* U_0 \right], \tilde{q}_2 \right) \right\} (x_{\text{max}}) e^{2ir\tilde{\psi}(x_{\text{max}})} \\
+ \left( \frac{(\nu_1 - i \nu_2) \sqrt{\mathcal{F}}}{2 \tau \partial_z \phi} \right)(x_{\text{min}}) \left\{ e^{-2i\theta} \left( \tilde{q}_1, T_{R\mathcal{H}}^{\nu} \left[ z^\nu C_2V_0 \right] \right) + e^{2i\theta} \left( P_{A\mathcal{H}}^{\nu} \left[ z^\nu C_2^* U_0 \right], \tilde{q}_2 \right) \right\} (x_{\text{min}}) e^{2ir\tilde{\psi}(x_{\text{min}})} \\
+ o(\sqrt{\tau}) \quad \text{as} \quad \tau \to +\infty, \quad (4.25)
\]

where \( \theta = \theta + \frac{\pi}{2} \). Since \( \tilde{\psi}(x_{\text{min}}) \neq \tilde{\psi}(x_{\text{max}}) \), the above equality implies
\[
\left\{ e^{-2i\theta} \left( \tilde{q}_1, T_{R\mathcal{H}}^{\nu} \left[ z^\nu C_2V_0 \right] \right) + e^{2i\theta} \left( P_{A\mathcal{H}}^{\nu} \left[ z^\nu C_2^* U_0 \right], \tilde{q}_2 \right) \right\} (x_{\text{max}}) = 0.
\]

This equality and proposition 3.1 imply (4.20). Indeed, the functions \( \tilde{q}_1, \tilde{q}_2 \) are given by equalities (4.3) with functions \( \mathcal{A}_1 \in \text{Ker} \left\{ 2 \partial_x + A_1 \right\} \) and \( \mathcal{D}_2 \in \text{Ker} \left\{ 2 \partial_x + B_2 \right\} \). By proposition 3.1 one can make the choice of the functions \( \mathcal{A}_1, \mathcal{D}_2 \) such that
\[
\mathcal{A}_1 \left( P_{A\mathcal{H}}^{\nu} \left[ z^\nu C_2^* U_0 \right], \tilde{q}_2 \right) + \mathcal{D}_2 \left( \tilde{q}_1, T_{R\mathcal{H}}^{\nu} \left[ z^\nu C_2V_0 \right] \right) = o \left( \frac{1}{\sqrt{\tau}} \right) \quad \text{as} \quad \tau \to +\infty. \quad (4.27)
\]

By (4.24), (4.25) and (4.27), we have
\[
\left( \mathcal{U}, H(x, \partial_x, \partial_z) \overline{V} \right)_{L^2(\Omega)} = \mathcal{A}_1 \left( P_{A\mathcal{H}}^{\nu} H^{\nu} \left( x, \partial_x + \tau \partial_x \phi, \partial_z \right) \mathbf{U}_0, \tilde{q}_2 \right) + \mathcal{D}_2 \left( \tilde{q}_1, T_{R\mathcal{H}}^{\nu} H \left( x, \partial_x, \partial_z - \tau \partial_z \phi \right) \mathbf{U}_0 \right) \\
\times \frac{\tau e^{-i\theta \delta}}{2} \delta \mathcal{A}_1 \left( \tilde{q}_1, T_{R\mathcal{H}}^{\nu} \left( \mathcal{Q}_1 (1) T_{R\mathcal{H}}^{\nu} \left( C_2V_0 \right) \right) \right) - \frac{\tau e^{i\theta \delta}}{2} \delta \mathcal{D}_2 \left( \tilde{q}_2, P_{A\mathcal{H}}^{\nu} \left( \mathcal{Q}_2 (2) P_{A\mathcal{H}}^{\nu} \left( C_2^* U_0 \right) \right) \right) \\
- \frac{\tau e^{-i\theta \delta}}{2} \delta \mathcal{A}_1 \left( \tilde{q}_1, T_{R\mathcal{H}}^{\nu} \left( C_2^* \tilde{q}_2 \right) \right) + \frac{\tau e^{i\theta \delta}}{2} \delta \mathcal{D}_2 \left( \tilde{q}_2, P_{A\mathcal{H}}^{\nu} \left( C_2^* \tilde{q}_1 \right) \right) + o \left( \frac{1}{\sqrt{\tau}} \right) \\
= \int_{\partial \Omega} \left( \overline{\mathcal{C}_1}, \tilde{q}_2 \right) e^{i\theta \delta} \left( \frac{(\nu_1 - i \nu_2) \sqrt{\mathcal{F}}}{2 \partial_z \phi} \right)(x_{\text{max}}) e^{2ir\tilde{\psi}(x_{\text{max}})} \\
+ \int_{\partial \Omega} \left( \tilde{q}_1, \tilde{C}_2 \right) e^{-i\theta \delta} \left( \frac{(\nu_1 - i \nu_2) \sqrt{\mathcal{F}}}{2 \partial_z \phi} \right)(x_{\text{min}}) e^{2ir\tilde{\psi}(x_{\text{min}})} + o \left( \frac{1}{\sqrt{\tau}} \right) \quad \text{as} \quad \tau \to +\infty. \quad (4.28)
\]

Applying the stationary phase argument to the right-hand side of (4.28), we obtain
\[
e^{-i\theta \delta} \left( \overline{\mathcal{C}_1}, \tilde{q}_2 \right)(x_{\text{max}}) + e^{i\theta \delta} \left( \tilde{q}_1, \tilde{C}_2 \right)(x_{\text{max}}) = 0.
\]

Then using proposition 3.1, we obtain (4.21) and (4.22).
Denote
\[ S_{U_0, V_0}(x) = \left\{ \left( U_0, C_i \partial_{x_i}^2 V_0 \right) - \left( \partial_z \left( C_i^0 U_0 \right), \partial_z V_0 \right) + \left( \partial_z^2 \left( C_i^0 U_0 \right), V_0 \right) + \left( U_0, B_i \partial_z V_0 \right) \right\}(x) \]

and
\[
\Omega_{U_0, V_0, q, q_2} (\tilde{x}) = - \left( P_A^s \left[ C_i^0 U_0 \right] \right) \frac{1}{2} \partial_z \left[ Q_i (2) V_0 \right] - \frac{1}{4} B_\tau Q_2 (2) V_0) (\tilde{x})
- \left( T_{B_i}^b \left[ C_i V_0 \right] \right) \frac{1}{2} \partial_z \left[ Q_i (1) U_0 \right] - \frac{1}{4} A \xi Q_1 (1) U_0 \right)(\tilde{x})
- 2 \left( \partial_z \left[ C_i^0 U_0 \right] \right) \frac{1}{2} Q_2 (2) V_0 \right) (\tilde{x}) - 2 \left( \partial_z \left[ C_i V_0 \right] \right) \frac{1}{2} Q_1 (1) U_0 \right)(\tilde{x})
+ \left( P_A^s \left[ \partial_z \left( C_i^0 U_0 \right) \right] + \partial_z \left( C_i^0 U_0 \right) + B_i \partial_z \right) \frac{1}{2} Q_2 (2) V_0 \right) (\tilde{x})
- \left( T_{B_i}^b \left[ \partial_z \left( C_i^0 U_0 \right) \right] + B_i \partial_z \right) \frac{1}{2} Q_1 (1) U_0 \right)(\tilde{x})
+ \left( P_A^s \left[ \partial_z \left( C_i^0 U_0 \right) \right] + B_i \partial_z \right) \frac{1}{2} Q_1 (2) \partial_z \left( C_i^0 U_0 \right) - \frac{C_i}{2} q_i \right)(\tilde{x}).
\]

The following proposition plays a crucial part in the proof of the uniqueness of the determination of coefficients for the Navier–Stokes equations and the Lamé system.

**Proposition 4.6.** Let \( H(x, \partial_z, \partial_z) \) be a second-order differential operator given by (4.1) with smooth coefficients which have compact supports in \( \Omega \). Assume that \( \psi = \text{Im} \Phi \) has a finite number of critical points on \( \partial \Omega \) and all these points are nondegenerate. Let \( U, V \) be the solutions constructed in proposition 3.6, \( q_1 (\tilde{x}) = q_2 (\tilde{x}) = 0 \) and (4.20)–(4.22) hold true. Suppose that
\[ (U, H(x, \partial_z, \partial_z) V)_{L^2(\Omega)} = o \left( \frac{1}{\tau} \right) \quad \text{as} \quad \tau \to +\infty. \]  

Then the following equality holds true:
\[ \Omega_{U_0, V_0, q, q_2} (\tilde{x}) + S_{U_0, V_0}(\tilde{x}) = 0. \]  

**Proof.** First, using (4.1), integrating by parts, noting that the coefficients of the operator \( H(x, \partial_z, \partial_z) \) are compactly supported and applying proposition 2.3, we obtain
\[
(U_0 e^{\Phi}, H(x, \partial_z, \partial_z)(V_0 e^{-\Phi}))_{L^2(\Omega)}
= \int_\Omega \left\{ \left( U_0, C_i \partial_{x_i}^2 V_0 \right) - \left( \partial_z \left( C_i^0 U_0 \right), \partial_z V_0 \right) + \left( \partial_z^2 \left( C_i^0 U_0 \right), V_0 \right) + \left( U_0, B_i \partial_z V_0 \right) \right\} e^{\Phi - \Phi} dx
- \int_\Omega \left\{ \left( \partial_z \left( B_i^2 U_0 \right), V_0 \right) + \left( U_0, B_i V_0 \right) \right\} e^{\Phi - \Phi} dx
+ o \left( \frac{1}{\tau} \right) = \frac{\pi}{2 \tau} S_{U_0, V_0}(\tilde{x}) + o \left( \frac{1}{\tau} \right) \quad \text{as} \quad \tau \to +\infty. \]  

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From proposition 4.3 we obtain
\[
(U, \mathbf{H}(x, \partial_x, \partial_z) V)_{L^2(\Omega)} = \left( U_0 - U_1, \mathbf{H}(x, \partial_x, \partial_z) \left( (V_0 - V_1) e^{-\tau \mathcal{F}_1} \right) \right)_{L^2(\Omega)} + \left( U_2 e^{\mathcal{F}_1}, \mathbf{H}(x, \partial_x, \partial_z) \left( V_0 e^{-\tau \mathcal{F}_1} \right) \right)_{L^2(\Omega)} + \left( U_0 e^{\mathcal{F}_1}, \mathbf{H}(x, \partial_x, \partial_z) \left( V_2 e^{-\tau \mathcal{F}_1} \right) \right)_{L^2(\Omega)} + o \left( \frac{1}{\tau} \right) \quad \text{as} \quad \tau \to +\infty.
\]
(4.32)

Using this equality, (4.1), (4.31) and proposition 4.1, we have
\[
(U, \mathbf{H}(x, \partial_x, \partial_z) V)_{L^2(\Omega)} = -\left( U_0 e^{\mathcal{F}_1}, \mathbf{H}(x, \partial_x, \partial_z) \left( V_0 e^{-\tau \mathcal{F}_1} \right) \right)_{L^2(\Omega)} + o (\tau)
\]
\[
= -\delta \chi_{x} \left( \left( \begin{array}{c} P_{\alpha_1}^{*} \mathbf{H}^{*} (x, \partial_x + \tau \partial \Phi, \partial_z) U_0, q_2 \end{array} \right) \right)_{L^2(\Omega)}
\]
\[
- \delta \chi_{x} \left( \left( \begin{array}{c} q_1, T_{R_{1}}^{*} \mathbf{H}(x, \partial_x, \partial_z - \tau \partial \mathcal{F}) V_0 \end{array} \right) \right)_{L^2(\Omega)}
\]
\[
- \delta \chi_{x} \left( \left( \begin{array}{c} P_{\alpha_2}^{*} \mathbf{H}^{*} (x, \partial_x + \tau \partial \Phi, \partial_z) U_0, q_2 \end{array} \right) \right)_{L^2(\Omega)} - \delta \chi_{x} \left( \left( \begin{array}{c} q_1, T_{R_{1}}^{*} \mathbf{H}(x, \partial_x, \partial_z - \tau \partial \mathcal{F}) V_0 \end{array} \right) \right) + o (\tau)
\]
\[
= -2\pi \tau \left( P_{\alpha_1}^{*} \left[ C_1^* (z - \bar{z})^2 U_0 \right] (\bar{x}), q_2 (\bar{x}) \right)
\]
\[
- 2\pi \tau \left( T_{R_{1}}^{*} \left[ C_2 (\tau - \bar{z})^3 V_0 \right] (\bar{x}), q_1 (\bar{x}) \right) + o (\tau).
\]
(4.33)

Here, in order to obtain the last equality, we used the fact that
\[
\delta \chi_{x} \left( \left( \begin{array}{c} P_{\alpha_2}^{*} \mathbf{H}^{*} (x, \partial_x + \tau \partial \Phi, \partial_z) U_0, q_2 \end{array} \right) \right) = o (\tau). \quad \text{(This follows from (4.20)).}
\]

Next we claim that
\[
P_{\alpha_1}^{*} \left[ z^k C_1^* U_0 \right] = z^k P_{\alpha_2}^{*} \left[ C_1^* U_0 \right] \quad \text{and} \quad T_{R_{1}}^{*} \left[ z^k C_2 V_0 \right] = z^k T_{R_{1}}^{*} \left[ C_2 V_0 \right],
\]
\[
\forall \quad k \in \{ 0, 1, 2 \}.
\]
(4.34)

Indeed, using the notations \(r_{1,k} = P_{\alpha_1}^{*} \left[ C_1^* z^k U_0 \right]\), \(r_{2,k} = z^k P_{\alpha_2}^{*} \left[ C_1^* U_0 \right]\) and applying proposition 3.2, we observe that
\[
\left( -2 \partial_z + A_2^* \right) r_{1,k} = z^k C_1^* U_0 \quad \text{in} \quad \Omega.
\]

Hence, using (4.20), we obtain
\[
\left( -2 \partial_z + A_2^* \right) \left( r_{1,k} - r_{2,k} \right) = 0 \quad \text{in} \quad \Omega, \quad (r_{1,k} - r_{2,k})_{\partial \Omega} = 0.
\]

By the uniqueness of the Cauchy problem for the \(\partial_z\)-equation, we obtain that \(r_{1,k} - r_{2,k} \equiv 0\). The proof of the second equality in (4.34) is the same.

We introduce the following notations:
\[
m_2 (x) = P_{\alpha_1}^{*} \left[ C_2^* (\partial_z \Phi)^3 U_0 \right], \quad m_1 (x) = T_{R_{1}}^{*} \left[ C_2 (\partial_z \mathcal{F})^3 V_0 \right].
\]
\[ \mathcal{C}_1(x) = -P_{A_1}^n (2(\partial \varphi) \partial_z (C_1^n U_0) + 2C_1^n U_0 + (\partial \varphi) \partial_z (C_0^n U_0) + (\partial \varphi) B_1^n U_0 + \frac{1}{2} Q_1 (2) P_{A_1}^n (C_1^n (\partial \varphi) U_0) - \frac{1}{2} C_1^n (\partial \varphi) q_i). \]

\[ \mathcal{C}_2(x) = T_{B_1}^n (C_2 (2 (\partial \varphi) \partial_z V_0 + 2V_0) + C_0 (\partial \varphi) \partial_z V_0 + (\partial \varphi) B_2 V_0 + \frac{1}{2} Q_2 (1) T_{B_1}^n (C_2 (\partial \varphi) V_0) - \frac{1}{2} C_2 (\partial \varphi) q_2). \]

In particular (4.34) implies

\[ m_2(x) = (\partial \varphi)^2 P_{A_1}^n [C_1^n U_0] \text{ and } m_1(x) = (\partial \varphi)^2 T_{B_1}^n [C_2 V_0] \text{ in } \Omega. \] (4.35)

Formula (4.35) implies the following equalities:

\[ \begin{align*}
  m_2(\bar{x}) &= \partial_{z} m_2(\bar{x}) = 0, \\
  \partial_{zz}^2 m_2(\bar{x}) &= 8P_{A_1}^n [C_1^n U_0](\bar{x}), \\
  \partial_{zzzz}^3 m_2(\bar{x}) &= 24 (\partial P_{A_1}^n [C_1^n U_0])(\bar{x}). \\
  m_1(\bar{x}) &= \partial_{z} m_1(\bar{x}) = 0, \\
  \partial_{zz}^2 m_1(\bar{x}) &= 8 T_{B_1}^n [C_2 V_0](\bar{x}), \\
  \partial_{zzzz}^3 m_1(\bar{x}) &= 24 (\partial T_{B_1}^n [C_2 V_0])(\bar{x}).
\end{align*} \] (4.36)

Moreover thanks to our assumption that \( q_i(\bar{x}) = q_2(\bar{x}) = 0 \), formulae (4.36) and (4.37) imply the following equalities:

\[ \begin{align*}
  \frac{1}{32} \partial_{zzzz}^4 (m_2, q_2)(\bar{x}) &= \frac{1}{32} \left\{ 4 \left( \partial_{zz}^3 m_2, \partial_{z} q_2 \right) + 6 \left( \partial_{zz}^2 m_2, \partial_{zz}^2 q_2 \right) \right\}(\bar{x}) \\
  &= 3 \left( \partial P_{A_1}^n [C_1^n U_0], \partial q_2 \right)(\bar{x}) + \frac{3}{2} \left( P_{A_1}^n [C_1^n U_0], \partial_{zz}^2 q_2 \right)(\bar{x}). \\
  \frac{1}{32} \partial_{zzzz}^4 (m_1, q_1)(\bar{x}) &= \frac{1}{32} \left\{ 4 \left( \partial_{zz}^3 m_1, \partial_q q_1 \right) + 6 \left( \partial_{zz}^2 m_1, \partial_{zz}^2 q_1 \right) \right\}(\bar{x}) \\
  &= 3 \left( \partial T_{B_1}^n [C_2 V_0], \partial_q q_1 \right)(\bar{x}) + \frac{3}{2} \left( T_{B_1}^n [C_2 V_0], \partial_{zz}^2 q_1 \right)(\bar{x}).
\end{align*} \] (4.38)

In addition, since the matrices \( C_j \) are compactly supported and the functions \( m_j \) satisfy the equations

\[ \begin{align*}
  - (2 \partial_z + A_z^n) m_1 = C_1^n U_0 \text{ in } \Omega, \\
  (2 \partial_z + B_z^n) m_2 = C_2 V_0 \text{ in } \Omega,
\end{align*} \]

we have

\[ \partial_{v}^k m_j |_{\partial \Omega} = 0 \quad \forall j \in \{1, 2\}, \] (4.40)

This implies that

\[ \partial_{v}^k m_j = 0 \quad \forall j \in \{1, 2\}. \] (4.41)
Repeating the above arguments and using (4.21) and (4.22), we obtain

$$C(x) = -\partial \Phi \partial x + \partial \Phi \partial x + \partial \Phi \partial x + \partial \Phi \partial x - \partial \Phi \partial x$$

and

$$C(x) = (\partial \Phi \partial x) T(x) (2C_0 \partial \Phi + B_2 V_0 + \frac{1}{2} Q_2 (2)^\partial (C_1 \partial \Phi + \frac{1}{2} C_2 q_2) - \frac{1}{2} C_2 q_2)$$

Indeed, let us prove that (4.42) holds true. We set

$$p_1 = -P_\lambda (2(\partial \Phi \partial x) + (\partial \Phi \partial x) B_2 U_0 + \frac{1}{2} Q_2 (2)^\partial (C_1 \partial \Phi \partial x) + \frac{1}{2} C_2 q_2).$$

Setting $f = -(2\partial \Phi \partial x) + \partial \Phi \partial x + B_2 U_0 + \frac{1}{2} Q_2 (2)^\partial (C_1 \partial \Phi \partial x) + \frac{1}{2} C_2 q_2)$ and $p_2 = (\partial \Phi \partial x) f$, by (4.21) we obtain

$$p_1 = p_2 = 0 \quad \text{on} \quad \partial \Omega$$

and

$$\left(-2\partial \Phi \partial x + A_1 \right)p_1 = 0 \quad \text{in} \quad \Omega \quad \forall \quad i \in \{1, 2\}.$$

Thanks to the uniqueness of the Cauchy problem for the $\partial \Phi \partial x$-operator, we have $p_1 = p_2$. Thus (4.42) is established. Formulae (4.42) and (4.43) imply that

$$C(x) = -2P_\lambda (C_1 U_0)(\bar{x}), \quad C(x) = 2T(x) (C_2 V_0)(\bar{x}).$$

Using

$$H(x, \partial \Phi \partial x - \tau \partial \Phi \partial x) V_0 = e^{\Phi} H(x, \partial \Phi \partial x) V_0 \left(2 \partial \Phi \partial x \right) V_0 = \tau^2 C_2 \partial \Phi \partial x V_0$$

and

$$H(x, \partial \Phi \partial x + \tau \partial \Phi \partial x) U_0 = e^{-\Phi} H(x, \partial \Phi \partial x) U_0 e^{\Phi} = \tau^2 C_1 \partial \Phi \partial x U_0$$

and

$$H^*(x, \partial \Phi \partial x + \tau \partial \Phi \partial x) U_0 = e^{-\Phi} H^*(x, \partial \Phi \partial x) U_0 e^{\Phi} = \tau^2 C_1 \partial \Phi \partial x U_0$$

and

$$H^*(x, \partial \Phi \partial x + \tau \partial \Phi \partial x) U_0 = e^{-\Phi} H^*(x, \partial \Phi \partial x) U_0 e^{\Phi} = \tau^2 C_1 \partial \Phi \partial x U_0$$
which are already used, in terms of (4.32) and (4.31), we write the left-hand side of (4.29) as

\[
(U, H(x, \partial_z, \partial \zeta)V)_{L^2(\Omega)} = \int_{\Omega} \{-\tau^2 (q_1, T_{R_1}^* (\partial_z \mathcal{F})^2 C_2 V_0) + \tau (q_2, \zeta_2) \\
+ (q_2, M_1) \} e^{x(\phi-\sigma)} dx \\
+ \int_{\Omega} \{-\tau^2 (q_2, P_{A_2}^* (C_1^* (\partial \Phi)^2 U_0)) + \tau (q_2, \zeta_1) \\
+ (q_1, M_2) \} e^{x(\phi-\sigma)} dx \\
+ \frac{\pi}{2c} \partial_{U_0} (\tilde{x}) + o \left( \frac{1}{\tau} \right) \quad \text{as} \quad \tau \to +\infty, \quad (4.45)
\]

where

\[
M_1 = -P_{A_2}^* (H^* (x, \partial_z, \partial \zeta) U_0) \\
- \frac{1}{2} P_{A_2}^* Q_1 (2) P_{A_2}^* \left( (C_1^* \partial_z + C_0^* \partial \zeta + \tilde{b}) U_0 \right) + P_{A_2}^* \left( (C_1^* \partial_z + \frac{1}{2} B_1) - B_1^* \right) q_1
\]

and

\[
M_2 = -T_{R_1}^* (H(x, \partial_z, \partial \zeta) V_0) - \frac{1}{2} T_{R_1}^* Q_2 (1) T_{R_1}^* \left( (C_0 \partial_z + C_2 \partial \zeta + b) V_0 \right) \\
- C_0 q_2 + T_{R_1}^* \left( (C_2 (\partial_z + \frac{1}{2} A_2) + B_2) q_2 \right).
\]

By proposition 2.4, we have

\[
\int_{\Omega} \left( (q_1, M_2) + (q_2, M_1) \right) e^{x(\phi-\sigma)} dx = o \left( \frac{1}{\tau} \right) \quad \text{as} \quad \tau \to +\infty.
\]

Then we rewrite (4.45) as

\[
(U, H(x, \partial_z, \partial \zeta)V)_{L^2(\Omega)} = \int_{\Omega} \{-\tau^2 (q_1, T_{R_1}^* (\partial_z \mathcal{F})^2 C_2 V_0) + \tau (q_1, \zeta_2) \} e^{x(\phi-\sigma)} dx \\
+ \int_{\Omega} \{-\tau^2 (q_2, P_{A_2}^* (C_1^* (\partial \Phi)^2 U_0)) + \tau (q_2, \zeta_1) \} e^{x(\phi-\sigma)} dx \\
+ \frac{\pi}{2c} \partial_{U_0} (\tilde{x}) + o \left( \frac{1}{\tau} \right) \quad \text{as} \quad \tau \to +\infty. \quad (4.46)
\]

Computing the next term in the asymptotic (4.29) and using the representation (4.46), by proposition 2.4 we obtain

\[
(U, H(x, \partial_z, \partial \zeta)V)_{L^2(\Omega)} = I_1 (\tilde{x}) + I_2 (\tilde{x}) + \tau \left( \mathcal{J}_1 (\zeta_1, q_2) + \mathcal{J}_1 (\zeta_2, q_1) \right) \\
+ o (1) \quad \text{as} \quad \tau \to +\infty,
\]

where

\[
I_1 (x) = \frac{\pi}{2} \left\{ \frac{1}{4} \left\{ \partial_{zz}^2 \left( P_{A_2}^* (C_1^* (\partial \Phi)^2 U_0), q_2 \right) (x) - \partial_{zz}^2 \left( P_{A_2}^* (C_1^* (\partial \Phi)^2 U_0), q_2 \right) (x) \right\} + (\zeta_1, q_2) (x) \right\}
\]
and
\[ I_2(x) = \frac{\pi}{2} \left\{ \frac{1}{4} \left\{ \partial_{zz}^2 \left( T_{B}^n \left( C_2 \left( \partial_z \Phi \right)^3 V_0 \right), q_1 \right) \right\} \left( x \right) - \partial_{zz}^2 \left( T_{B}^n \left( C_2 \left( \partial_z \Phi \right)^3 V_0 \right), q_1 \right) \right\} \]

\[ + \left( \mathcal{E}_2, q_1 \right) \left( x \right) \].

By (4.21) and (4.20) we have \( \mathcal{C}|_{\partial \Omega} = \mathcal{C}|_{\partial \Omega} = 0 \). Since the function \( \psi \) has a finite number of nondegenerate critical points on \( \partial \Omega \), by Proposition 2.4 of [11], we have
\[ \mathcal{E}_i \left( \mathcal{E}_1, q_2 \right) + \mathcal{E}_i \left( \mathcal{E}_2, q_1 \right) = o \left( \frac{1}{\tau^2} \right) \quad \text{as} \quad \tau \to +\infty. \] (4.47)

We claim that \( I_1(\tilde{x}) = I_2(\tilde{x}) = 0 \). Indeed, by (4.35), we obtain
\[ \partial_{zz}^2 \left( P_{A_1} (C_1^2 (\partial \mathcal{\Phi})^2 U_0), q_2 \right) (\tilde{x}) = 0 \quad \text{and} \quad \partial_{zz}^2 \left( T_{B_1}^n \left( C_2 \left( \partial_z \Phi \right)^3 V_0 \right), q_1 \right) (\tilde{x}) = 0. \]

Using the above equality, (4.2), (4.44) and (4.42), we can compute \( I_1(\tilde{x}) \) as
\[ I_1(\tilde{x}) = \frac{\pi}{8} \partial_{zz}^2 \left( \mathcal{P}_{A_1} \left( C_1^2 (\partial \mathcal{\Phi})^2 U_0 \right), q_2 \right) (\tilde{x}) \]
\[ - \frac{\pi}{2} \left( \mathcal{P}_{A_1} \left( 2C_1^2 U_0 \right), q_2 \right) (\tilde{x}) + \pi \left( \mathcal{P}_{A_1} \left( C_1^2 U_0 \right), q_2 \right) (\tilde{x}) \]
\[ - \frac{\pi}{2} \left( \mathcal{P}_{A_1} \left( 2C_1^2 U_0 \right), q_2 \right) (\tilde{x}) = 0. \]

Similarly, using (4.44) and (4.43), we compute \( I_2 \) at point \( \tilde{x} \):
\[ I_2(\tilde{x}) = - \frac{\pi}{8} \partial_{zz}^2 \left( T_{B_1}^n \left( C_2 \left( \partial_z \Phi \right)^3 V_0 \right), q_1 \right) (\tilde{x}) + \frac{\pi}{2} \left( q_1, T_{B_1}^n \left( 2C_2 q_2 \right) \right) (\tilde{x}) \]
\[ - \pi \left( T_{B_1}^n \left( C_2 V_0 \right), q_1 \right) (\tilde{x}) + \pi \left( q_1, T_{B_1}^n \left( C_2 V_0 \right) \right) (\tilde{x}) = 0. \]

Finally we compute the term of order \( \frac{1}{\tau} \) in the asymptotic of the left-hand side of (4.29). We recall our assumption
\[ q_1(\tilde{x}) = q_2(\tilde{x}) = 0. \] (4.48)

We introduce operators
\[ \mathcal{L} \mathcal{\Phi} = \frac{\pi}{8} \left( -\partial_{z\zeta}^2 \mathcal{\Phi} + \partial_{z\zeta}^2 \mathcal{\Phi} \right) (\tilde{x}) \quad \text{and} \quad \mathcal{\Phi} \mathcal{\Phi} = \frac{\pi}{2} \left( \frac{1}{32} \partial_{z\zeta z\zeta}^4 \mathcal{\Phi} - \frac{1}{16} \partial_{z\zeta z\zeta}^4 \mathcal{\Phi} + \frac{1}{32} \partial_{z\zeta z\zeta}^4 \mathcal{\Phi} \right) (\tilde{x}). \]

By (4.47) we see that
\[ \left( \mathcal{U}, H \right)(\tilde{x}, \partial_z, \partial_{zz}) \right)_{L^2(\Omega)} = \frac{1}{\tau} \left( \left( -\mathcal{B} (m_1, q_1) - \mathcal{B} (m_2, q_2) + L (\mathcal{E}_2, q_1) + L (\mathcal{E}_1, q_2) \right) \right) \]
\[ + \frac{\pi}{2 \tau} \mathcal{S}_{\mathcal{U}_0} \mathcal{V}_0 (\tilde{x}) + \tau \left( \mathcal{E}_1 (\mathcal{E}_2, q_2) + \mathcal{E}_1 (\mathcal{E}_2, q_1) \right) \]
\[ = \frac{1}{\tau} \left( -\mathcal{B} (m_1, q_1) - \mathcal{B} (m_2, q_2) \right) \]
\[ + L (\mathcal{E}_2, q_1) + L (\mathcal{E}_1, q_2) \] \[ + \frac{\pi}{2 \tau} \mathcal{S}_{\mathcal{U}_0} \mathcal{V}_0 (\tilde{x}) \]
\[ + o \left( \frac{1}{\tau} \right) \quad \text{as} \quad \tau \to +\infty. \] (4.49)
First we compute $\mathcal{B}(m_2, q_2)$. Observe that

$$m_2(\tilde{x}) = \partial_\tilde{x} m_2(\tilde{x}) = \partial_{\tilde{x}\tilde{x}}^2 m_2(\tilde{x}) = \partial_{\tilde{x}\tilde{x}\tilde{x}\tilde{x}} m_2(\tilde{x}) = 0. \quad (4.50)$$

Then

$$\partial_{\tilde{x}\tilde{x}\tilde{x}\tilde{x}} (m_2, q_2)(\tilde{x}) = 0. \quad (4.51)$$

Using (4.48) and (4.36), we obtain

$$\partial_{\tilde{x}\tilde{x}\tilde{x}\tilde{x}} (m_2, q_2)(\tilde{x}) = \left\{ \begin{array}{c}
\left( \partial_{\tilde{x}\tilde{x}}^3 m_2, q_2 \right) + 2 \left( \partial_{\tilde{x}\tilde{x}\tilde{x}} m_2, \partial_\tilde{x} q_2 \right) + 2 \left( \partial_{\tilde{x}\tilde{x}\tilde{x}} m_2, \partial_\tilde{x} q_2 \right) \\
+ \left( \partial_{\tilde{x}}^2 m_2, \partial_{\tilde{x}\tilde{x}}^2 q_2 \right) + 2 \left( \partial_{\tilde{x}}^2 m_2, \partial_{\tilde{x}\tilde{x}}^2 q_2 \right) + 4 \left( \partial_{\tilde{x}}^2 m_2, \partial_{\tilde{x}\tilde{x}}^2 q_2 \right) \\
+ \left( \partial_{\tilde{x}} m_2, \partial_{\tilde{x}\tilde{x}}^3 q_2 \right) + 2 \left( \partial_{\tilde{x}} m_2, \partial_{\tilde{x}\tilde{x}}^3 q_2 \right) + \left( m_2, \partial_{\tilde{x}\tilde{x}\tilde{x}}^4 q_2 \right) \end{array} \right\}(\tilde{x}) = 2 \left( \partial_{\tilde{x}\tilde{x}}^3 m_2, \partial_\tilde{x} q_2 \right)(\tilde{x}) + \left( \partial_{\tilde{x}\tilde{x}}^2 m_2, \partial_{\tilde{x}\tilde{x}}^2 q_2 \right)(\tilde{x}). \quad (4.52)$$

By (4.38), (4.51) and (4.52), we have

$$\mathcal{B}(m_2, q_2) = \frac{\pi}{2} \left( -\partial_{\tilde{x}} P_{\tilde{x}}^* \left[ C_\varepsilon U_0 \right], \partial_\tilde{x} q_2 \right)(\tilde{x}) - \frac{1}{2} \left( P_{\tilde{x}}^* \left[ C_\varepsilon U_0 \right], \partial_{\tilde{x}\tilde{x}}^2 q_2 \right)(\tilde{x}) + \frac{3}{2} \left( P_{\tilde{x}}^* \left[ C_\varepsilon U_0 \right], \partial_{\tilde{x}\tilde{x}}^2 q_2 \right)(\tilde{x}). \quad (4.53)$$

Next we similarly compute $\mathcal{B}(m_1, q_1)$. Observe that

$$m_1(\tilde{x}) = \partial_\tilde{x} m_1(\tilde{x}) = \partial_{\tilde{x}\tilde{x}}^2 m_1(\tilde{x}) = \partial_{\tilde{x}\tilde{x}\tilde{x}}^3 m_1(\tilde{x}) = \partial_{\tilde{x}\tilde{x}\tilde{x}\tilde{x}}^4 m_1(\tilde{x}) = 0. \quad (4.54)$$

Then, by (4.54) we have

$$\partial_{\tilde{x}\tilde{x}\tilde{x}\tilde{x}} (m_1, q_1)(\tilde{x}) = 0. \quad (4.55)$$

By short computations, (4.48) and (4.37) provide

$$\partial_{\tilde{x}\tilde{x}\tilde{x}\tilde{x}} (m_1, q_1)(\tilde{x}) = \left\{ \begin{array}{c}
\left( \partial_{\tilde{x}\tilde{x}}^4 m_1, q_1 \right) + 2 \left( \partial_{\tilde{x}\tilde{x}\tilde{x}}^3 m_1, \partial_\tilde{x} q_1 \right) + 2 \left( \partial_{\tilde{x}\tilde{x}\tilde{x}}^3 m_1, \partial_\tilde{x} q_1 \right) \\
+ \left( \partial_{\tilde{x}}^3 m_1, \partial_{\tilde{x}\tilde{x}}^2 q_1 \right) + \left( \partial_{\tilde{x}}^3 m_1, \partial_{\tilde{x}\tilde{x}}^2 q_1 \right) + 4 \left( \partial_{\tilde{x}}^3 m_1, \partial_{\tilde{x}\tilde{x}}^2 q_1 \right) \\
+ \left( \partial_{\tilde{x}} m_1, \partial_{\tilde{x}\tilde{x}}^3 q_1 \right) + 2 \left( \partial_{\tilde{x}} m_1, \partial_{\tilde{x}\tilde{x}}^3 q_1 \right) + \left( m_1, \partial_{\tilde{x}\tilde{x}\tilde{x}}^4 q_1 \right) \end{array} \right\}(\tilde{x}) = 2 \left( \partial_{\tilde{x}\tilde{x}}^3 m_1, \partial_\tilde{x} q_1 \right)(\tilde{x}) + \left( \partial_{\tilde{x}\tilde{x}}^2 m_1, \partial_{\tilde{x}\tilde{x}}^2 q_1 \right)(\tilde{x}). \quad (4.56)$$

By (4.37), (4.39), (4.55) and (4.56), we obtain

$$\mathcal{B}(m_1, q_1) = \frac{\pi}{2} \left( -\frac{1}{8} \left( \partial_{\tilde{x}\tilde{x}}^3 m_1, \partial_\tilde{x} q_1 \right)(\tilde{x}) - \frac{1}{16} \left( \partial_{\tilde{x}} m_1, \partial_{\tilde{x}\tilde{x}}^2 q_1 \right)(\tilde{x}) \right) + 3 \left( \partial_{\tilde{x}} T_{\tilde{x}}^* \left[ C_\varepsilon V_0 \right], \partial_\tilde{x} q_1 \right)(\tilde{x}) + \frac{3}{2} \left( T_{\tilde{x}}^* \left[ C_\varepsilon V_0 \right], \partial_{\tilde{x}\tilde{x}}^2 q_1 \right)(\tilde{x}). \quad (4.57)$$

Using (4.48) we have

$$L(q_1, \varepsilon_2) = \frac{\pi}{8} \left\{ -\left( \partial_{\tilde{x}}^2 q_1, \varepsilon_2 \right)(\tilde{x}) - 2 \left( \partial_\tilde{x} q_1, \partial_{\tilde{x}} \varepsilon_2 \right)(\tilde{x}) + \left( \partial_{\tilde{x}}^2 q_1, \varepsilon_2 \right)(\tilde{x}) \\
+ 2 \left( \partial_{\tilde{x}} q_1, \partial_{\tilde{x}} \varepsilon_2 \right)(\tilde{x}) \right\}. \quad (4.58)$$
and

\[
L \left( q_2, \xi_1 \right) = \frac{\pi}{4} \left\{ -\left( \partial_{x_2}^2 q_2, \xi_1 \right)(\tilde{x}) - 2 \left( \partial_q q_2, \partial_q \xi_1 \right)(\tilde{x}) + \left( \partial_{x_2}^2 q_2, \xi_1 \right)(\tilde{x}) + 2 \left( \partial_q q_2, \partial_q \xi_1 \right)(\tilde{x}) \right\}.
\]

(4.59)

By (4.58), (4.59), (4.53), (4.57) and (4.39), we obtain from (4.49)

\[
\frac{1}{4} \left( \xi_1 + \frac{1}{4} \partial_{x_2}^2 m_2, \partial_{x_2}^2 q_2 \right)(\tilde{x}) + \frac{1}{4} \left( -\xi_2 + \frac{1}{4} \partial_{x_2}^2 m_1, \partial_{x_2}^2 q_1 \right)(\tilde{x})
\]

\[
+ \left( \frac{1}{2} \partial_q \xi_1 + \frac{1}{8} \partial_{x_2}^3 m_2, \partial_q q_2 \right)(\tilde{x})
\]

\[
+ \left( -\frac{1}{2} \partial_q \xi_2 + \frac{1}{8} \partial_{x_2}^3 m_1, \partial_q q_1 \right)(\tilde{x})
\]

\[
- 3 \left( P_{A^*}[C^* U_0], \partial q_2 \right)(\tilde{x}) - \frac{3}{2} \left( P_{A^*}[C^* U_0], \partial_{x_2}^2 q_2 \right)(\tilde{x})
\]

\[
- 3 \left( \partial_{x_2} T^*_B[C_2 V_0], \partial_{x_2} q_1 \right)(\tilde{x}) - \frac{3}{2} \left( T^*_B[C_2 V_0], \partial_{x_2}^2 q_1 \right)(\tilde{x})
\]

\[
+ \frac{1}{4} \left( -\partial_{x_2}^2 (\xi_1, q_2) + \partial_{x_2}^2 (\xi_2, q_1) \right)(\tilde{x}) + \delta_{U_0 V_0}(\tilde{x}) = 0.
\]

By (4.44) and (4.36), we have

\[
\xi_1(\tilde{x}) + \frac{1}{4} \partial_{x_2}^2 m_2(\tilde{x}) = 0.
\]

(4.60)

By (4.44) and (4.37), we see

\[
\xi_2(\tilde{x}) - \frac{1}{4} \partial_{x_2}^2 m_1(\tilde{x}) = 0.
\]

(4.61)

Applying (4.42) and (4.36), we obtain

\[
\left( \frac{1}{2} \partial_q \xi_1 + \frac{1}{8} \partial_{x_2}^3 m_2 \right)(\tilde{x}) = 0.
\]

(4.62)

Applying (4.43) and (4.37), we obtain

\[
\left( -\frac{1}{2} \partial_q \xi_2 + \frac{1}{8} \partial_{x_2}^3 m_1 \right)(\tilde{x}) = 0.
\]

(4.63)

By (4.60)–(4.63), we rewrite (4.60) as

\[
- \left\{ 3 \left( \partial_{x_2} P_{A^*}[C^* U_0], \partial q_2 \right) + \frac{3}{2} \left( P_{A^*}[C^* U_0], \partial_{x_2}^2 q_2 \right) \right\}(\tilde{x})
\]

\[
- \left\{ 3 \left( \partial_{x_2} T^*_B[C_2 V_0], \partial_{x_2} q_1 \right) + \frac{3}{2} \left( T^*_B[C_2 V_0], \partial_{x_2}^2 q_1 \right) \right\}(\tilde{x})
\]

\[
+ \frac{1}{4} \left( -\partial_{x_2}^2 (\xi_1, q_2) + \partial_{x_2}^2 (\xi_2, q_1) \right)(\tilde{x}) + \delta_{U_0 V_0}(\tilde{x}) = 0.
\]

(4.64)
Now, using (4.42)–(4.44), we compute the last term in (4.63):

\[
\frac{1}{4} \left( -\partial_x^2 \left( \mathbf{c}_1, q_2 \right) + \partial^2_{\alpha \alpha} \left( \mathbf{c}_2, q_1 \right) \right)(\bar{x}) = \frac{1}{4} \left( -2 \left( \partial_x \mathbf{c}_1, \partial_x q_2 \right) \right)
\]

\[+ 2 \left( \partial_x \mathbf{c}_2, \partial_x q_1 \right) \right)(\bar{x}) = \frac{1}{4} \left( -2 \left( \partial_x \mathbf{c}_1, \partial_x q_2 \right) \right)
\]

\[+ 2 \left( \partial_x \mathbf{c}_2, \partial_x q_1 \right) \right)(\bar{x}) = \frac{1}{4} \left( -2 \left( \partial_x \mathbf{c}_1, \partial_x q_2 \right) \right)
\]

\[+ 2 \left( P_{A_1}^a \left[ C^a_1 U_0 \right], \partial_{\alpha \alpha} q_2 \right)(\bar{x}) + 2 \left( \partial_x \mathbf{c}_2, \partial_x q_1 \right) \right)(\bar{x}) + 2 \left( T_{R_1}^a \left[ C_2 V_0 \right], \partial_{\alpha \alpha} q_1 \right)(\bar{x}) \right)
\]

\[= \frac{1}{4} \left( 4 \left( \partial_x P_{A_1}^a \left[ C^a_1 U_0 \right], \partial_x q_2 \right)(\bar{x}) + 2 \left( P_{A_1}^a \left[ C^a_1 U_0 \right], \partial_{\alpha \alpha} q_2 \right)(\bar{x}) \right)
\]

\[+ 4 \left( \partial_x T_{R_1}^a \left[ C_2 V_0 \right], \partial_{\alpha \alpha} q_1 \right)(\bar{x}) \right)
\]

\[+ 4 \left( P_{A_1}^a \left( 2 \partial_x (C^a_0 U_0) + \partial_x (C^a_0 U_0) + B_{x}^a U_0 + \frac{1}{2} Q_1 (2) P_{A_1}^a \left( C^a_1 U_0 \right) \right)(\bar{x}) \right)
\]

\[+ 4 \left( T_{R_1}^a \left( 2 C_2 \partial_1 V_0 + C_0 \partial_1 V_0 + B_2 V_0 + \frac{1}{2} Q_2 (1) T_{R_1}^a \left( C_2 V_0 \right) \right)(\bar{x}) \right)
\]

\[+ 4 \left( C_4 (2) \right) (\bar{x}) \right).
\]

From (4.64) and (4.65), we obtain

\[- 2 \left( \partial_x P_{A_1}^a \left[ C^a_1 U_0 \right], \partial_x q_2 \right)(\bar{x}) - 2 \left( \partial_x T_{R_1}^a \left[ C_2 V_0 \right], \partial_x q_1 \right)(\bar{x}) \right)
\]

\[- \left( P_{A_1}^a \left[ C^a_1 U_0 \right], \partial_{\alpha \alpha} q_2 \right)(\bar{x}) - \left( T_{R_1}^a \left[ C_2 V_0 \right], \partial_{\alpha \alpha} q_1 \right)(\bar{x}) \right)
\]

\[+ \left( P_{A_1}^a \left( 2 \partial_x (C^a_0 U_0) + \partial_x (C^a_0 U_0) + B_{x}^a U_0 + \frac{1}{2} Q_1 (2) P_{A_1}^a \left( C^a_1 U_0 \right) \right)(\bar{x}) \right)
\]

\[+ \left( T_{R_1}^a \left( 2 C_2 \partial_1 V_0 + C_0 \partial_1 V_0 + B_2 V_0 + \frac{1}{2} Q_2 (1) T_{R_1}^a \left( C_2 V_0 \right) \right)(\bar{x}) \right)
\]

\[+ \partial_{\alpha V_0} (\bar{x}) \right) = 0.
\]

(4.66)

Observe that

\[2 \partial_x q_1 + A_1 q_1 = Q_1 (1) U_0 \quad \text{and} \quad 2 \partial_x q_2 + B_2 q_2 = Q_2 (2) V_0 \]

by (4.2). Then using (4.48), we have

\[\partial_x q_1 (\bar{x}) = \frac{1}{2} Q_1 (1) U_0 (\bar{x}) \quad \text{and} \quad \partial_x q_2 (\bar{x}) = \frac{1}{2} Q_2 (2) V_0 (\bar{x}).
\]

(4.67)

Taking into account that

\[2 \partial_x^2 q_1 + \partial_x A_1 q_1 + A_1 \partial_x q_1 = \partial_x [Q_1 (1) U_0],
\]

\[2 \partial_x^2 q_2 + \partial_x B_2 q_2 + B_2 \partial_x q_2 = \partial_x [Q_2 (2) V_0],
\]

we have

\[\partial_x^2 q_1 (\bar{x}) = \frac{1}{2} \partial_x [Q_1 (1) U_0] (\bar{x}) - \frac{1}{4} A_1 Q_1 (1) U_0 (\bar{x}).
\]

(4.68)
Using (4.67) and (4.68), we rewrite (4.66) as
\[
- (P^\alpha_{A_1} [C^\alpha_1 U_0]) - \frac{1}{2} \partial_1 [Q_2 (2) V_0] - \frac{1}{4} B_1 Q_2 (2) V_0) (\vec{x})
- (T^\alpha_{B_1} [C_2 V_0]) - \frac{1}{2} \partial_1 [Q_1 (1) U_0] - \frac{1}{4} A_1 Q_1 (1) U_0) (\vec{x})
- 2 (\partial_2 P^\alpha_{A_1} [C^\alpha_1 U_0]) - \frac{1}{2} Q_2 (2) V_0) (\vec{x}) - 2 (\partial_2 T^\alpha_{B_1} [C_2 V_0]) - \frac{1}{4} Q_1 (1) U_0) (\vec{x})
+ (P^\alpha_{A_1} (2\partial_1 (C^\alpha_1 U_0) + \partial_2 (C^\alpha_1 U_0) + B^\alpha_1 U_0) + \frac{1}{2} Q_1 (2) P^\alpha_{A_1} (C^\alpha_1 U_0) - \frac{1}{2} C^\alpha_1 q_1).
\]
\[
\frac{1}{2} Q_2 (2) V_0) (\vec{x}) + (T^\alpha_{B_1} (2C_2 \partial_2 V_0 + C_0 \partial_2 V_0 + B_2 V_0 + \frac{1}{2} Q_2 (1) T^\alpha_{B_1} (C_2 V_0) - \frac{1}{2} C_2 q_2).
\]
\[
\frac{1}{2} Q_1 (1) U_0) (\vec{x}) + s_{U_1} (\vec{x}) = 0.
\]
(4.69)

The proof of proposition 4.6 is complete. □

Remark 4.1. The equation (4.30) at each point \( \vec{x} \) depends on the choice of the functions \( q_1 \) and \( q_2 \) since they are supposed to satisfy the condition \( q_1 (\vec{x}) = q_2 (\vec{x}) = 0 \). Of course the choice of such functions for any \( \vec{x} \) is unique only modulo functions \( \partial + q^a \in \text{Ker} (z_1, z_1), \partial + q^b \in \text{Ker} (z_2, z_2) \) such that \( q_1 (\vec{x}) = q_2 (\vec{x}) = 0 \). On the other hand, any function \( q_1 \in \text{Ker} (2\partial_1 + A_1) \) satisfying \( q_1 (\vec{x}) = 0 \), can be represented in the form \( q_1 = (z - \vec{x}) q_1 \) with \( q_1 \in \text{Ker} (2\partial_1 + A_1) \), and any function \( q_2 \in \text{Ker} (2\partial_1 + B_1) \) satisfying \( q_2 (\vec{x}) = 0 \) can be represented in the form \( q_2 = (z - \vec{x}) q_2 \). Therefore by (4.35), as long as the point \( \vec{x} \) is fixed, the choice of the functions \( q_j \) does not affect equation (4.30).

We complete this section, presenting one of many possible choices of functions \( q_j \) which we will use later in sections 5 and 6. Let us fix some point \( \vec{x} \) in \( \Omega \) and consider a ball \( B (\vec{x}, \delta) \) centered at \( \vec{x} \) of the small positive radius \( \delta \). By proposition 3.1, there exist smooth functions \( q_{k,j} \) such that
\[
q_{k,j} (\vec{x}) = \delta_j, \quad q_{k,j} \in \text{Ker} (2\partial_1 + A_1), \quad q_{k,j} \in \text{Ker} (2\partial_1 + B_1), \quad k \in \{1, 2\}, \ j \in \{1, 2, 3\}.
\]
Then, provided that \( \delta > 0 \) is sufficiently small, there exist functions \( r_{k,j} (\vec{x}) \) such that
\[
\sum_{j=1}^{3} r_{k,j} (\vec{x}) q_{k,j} (\vec{x}) = - q_k^0 (\vec{x}) \quad \forall \vec{x} \in B (\vec{x}^0, \delta),
\]
where
\[
q_2^0 = T_{B_1} (Q_2 (2) V_0), \quad q_1^0 = P_{A_1} (Q_1 (1) U_0).
\]
Then we set
\[
q_k (\vec{x}) := q_k (x, \vec{x}) = q_k^0 (x) + \sum_{j=1}^{3} r_{k,j} (\vec{x}) q_{k,j} (x), \quad k = 1, 2.
\]
(4.70)

By (4.20), since one can take \( U_0 = q_{1,j} \) and \( V_0 = q_{2,j} \), we note that
\[
P_{A_1} (C^\alpha_1 q_{1,j}) = T_{B_1} (C_2 q_{2,j}) = 0 \quad \forall j \in \{1, 2, 3\}.
\]
Using the boundary conditions (4.71) and proposition 2.8, we obtain that there exist constants \( C \) and \( s_0 \) independent of \( s \).
5. Construction of complex geometric optics solutions for the Navier–Stokes equations

Let \( \mathbf{u} = (u_1, u_2) \) and \( p \) satisfy the Stokes equations

\[
L_p(x, D) (\mathbf{u}, p) = \left( \sum_{j=1}^{2} \partial_{x_j} \left( \mu \left( \partial_{x_j} u_1 + \partial_{x_j} u_2 \right) \right) + \partial_x p, \right. \\
\left. \sum_{j=1}^{2} \partial_{x_j} \left( \mu \left( \partial_{x_j} u_2 + \partial_{x_j} u_1 \right) \right) + \partial_x p \right) = 0. \tag{5.1}
\]

We construct the complex geometric optics solutions to the Stokes equations.

As the first step of such a construction we reduce the Stokes equations to a decoupled elliptic system.

**Proposition 5.1.** Let \( \mathbf{g} = (g_1, g_2) \in L^2(\Omega) \), and functions \( \mathbf{w} = (w_1, w_2) \) and \( f \) be some solutions to the elliptic system

\[
\Delta f = \text{div } \mathbf{w} \quad \text{in } \Omega, \tag{5.2}
\]

\[
\mu \Delta w_1 + 2\mu_{x_1} \partial_{x_1} w_1 + \mu_{x_2} \partial_{x_2} w_1 + \mu_{x_3} \text{div } \mathbf{w} \tag{5.3}
\]

\[
+ 2 \left( \partial^2_{x_3} \mu \right) \partial_{x_3} f + 2 \left( \partial^2_{x_7} \mu \right) \partial_{x_7} f = g_1 \quad \text{in } \Omega, \tag{5.4}
\]

\[
\mu \Delta w_2 + 2\mu_{x_2} \partial_{x_2} w_2 + \mu_{x_3} \partial_{x_3} w_2 + \mu_{x_1} \text{div } \mathbf{w} \tag{5.3}
\]

\[
+ 2 \left( \partial^2_{x_3} \mu \right) \partial_{x_3} f + 2 \left( \partial^2_{x_7} \mu \right) \partial_{x_7} f = g_2 \quad \text{in } \Omega.
\]

Then the pair \( (\mathbf{u}, p) = (\mathbf{w} - \nabla f, \frac{1}{2}\mu \Delta f + \left( \partial_{x_3} \mu \right) \partial_{x_3} f + \left( \partial_{x_7} \mu \right) \partial_{x_7} f) \) solves the Stokes equations

\[
L_p(x, D) (\mathbf{u}, p) = \mathbf{g} \quad \text{in } \Omega.
\]

**Proof.** See the supplementary data (available from stacks.iop.org/ip/31/035004/mmedia).

We set

\[
\mathbf{H} (x, \partial_{x}, \partial_{x}) \left( \begin{array}{c} \nabla f \\ g \end{array} \right) \tag{5.5}
\]
\[
\frac{\mu}{\mu_2} M_2(x, D)(v, g) = 2 (V\mu) \text{div} v - \left( \begin{array}{c}
(V\mu, Vv_1) \\
(V\mu, Vv_2)
\end{array} \right) - 2 \left( \begin{array}{c}
(V\partial_\mu, Vg) \\
(V\partial_\mu, Vg)
\end{array} \right)
\]

where the matrices
\[
M_2(x, D)(v, g)
\]
\[
= \frac{1}{2} \begin{pmatrix}
3\partial_\mu \mu_1 & \partial_\mu \mu_1 & 2\partial^2_{\mu_1 \mu_1} \\
\mu_1 & \mu_1 & \mu_1 \\
\partial_\mu \mu_1 & \partial_\mu \mu_1 & 2\partial^2_{\mu_1 \mu_1}
\end{pmatrix}
\]

and
\[
C_2 = \frac{1}{2} \begin{pmatrix}
\partial_\mu \mu_1 & \partial_\mu \mu_1 & 2\partial^2_{\mu_1 \mu_1} \\
\mu_1 & \mu_1 & \mu_1 \\
\partial_\mu \mu_1 & \partial_\mu \mu_1 & 2\partial^2_{\mu_1 \mu_1}
\end{pmatrix}
\]

A_1 = (C_1 + iC_2), \quad B_1 = (C_1 - iC_2).

The system (5.2)-(5.4) of proposition 5.1 with \( g \equiv 0 \) can be written in the form (3.17) with \( A_1, B_1 \) given by (5.8) and \( C_1 = 0 \). Therefore, by proposition 5.1 we can construct the complex geometric optics solutions \((u_1, p_1)\) to this system in the form
\[
u_1 = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}, \quad U = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}, \quad U = e^{i\Phi} (U_0 - U_1) + \sum_{j=2}^{\infty} (-1)^j U_j e^{i\Phi},
\]

\[
p_j = \frac{1}{2} \mu \Delta f + \partial_\mu \partial_\mu f, \quad p_1 = \frac{1}{2} \mu \Delta f + \partial_\mu \partial_\mu f,
\]

where the matrices \( A_1, B_1 \) are defined in (5.8) and the functions \( U_j \) are determined in proposition 3.6. Since the velocity field of a fluid is supposed to be described by a real-valued
vector function from the physical point of view, instead of the pair \((u_\tau, p_\tau)\) we work with the real and imaginary parts of this pair, which of course are also solutions to the Stokes system.

Let
\[
(w_\tau, q_\tau) = \text{Re} \left( u_\tau, p_\tau \right) \quad \text{or} \quad (w_\tau, q_\tau) = \text{Im} \left( u_\tau, p_\tau \right).
\] (5.10)

Next we construct complex geometric optics solutions for the stationary Navier–Stokes equations.

The stationary Navier–Stokes equations can be written in the form
\[
G_\mu(u, p) = L_\mu(x, D)(u, p) + (u, V) u = 0.
\] (5.11)

We construct a complex geometric optics solution for the Navier–Stokes equations using the Newton–Kantorovich iteration scheme. More precisely we use theorem 6 (1.XVIII) from [16] p 708.

We recall that \(\Phi\) is given by (2.5) and by (2.2) \(\Phi(x, \Omega) < -1 < 0\) \(\forall x \in \Omega\).

It is convenient for us to change the unknown function \(u = (r, q) + (w_\tau, q_\tau)\). By (5.11) the pair \((r, q)\) satisfies the equation
\[
L_\mu(x, D)(r, q) + (r + w_\tau, V)(r + w_\tau) = 0.
\] (5.13)

Denote \(X = \{ (u, p) \in W_2^{1,\tau}(\Omega) \times W_2^{1,\tau}(\Omega); \ \text{div} u = 0 \}\) with the norm \(\| (u, p) \|_X = \| e^{-\tau \Phi} (u, p) \|_{W_2^{1,\tau}(\Omega) \times W_2^{1,\tau}(\Omega)}\). Let \(L_\mu (x, D)^{-1}\) be the operator from \(L_2^{\tau}(\Omega)\) into the orthogonal complement of \(\text{Ker} L_\mu (x, D)\) in \(X\). Applying to both sides of equation (5.13) the operator \(L_\mu (x, D)^{-1}\), we obtain
\[
P (r, q) := (r, q) + L_\mu(x, D)^{-1} (r + w_\tau, V)(r + w_\tau) = 0.
\]

The mapping \(P\) is twice continuously differentiable as the mapping from \(X\) into \(X\). Set \(x_0 = (0, 0)\) and \(G_0 = \left[ P'(x_0) \right]^{-1}\), where \(P'(x_0) = I + L_\mu(x, D)^{-1} \left[ (w_\tau, V) \cdot + (r, V) w_\tau \right] \) denotes the Fréchet derivative at \(x_0\).

By propositions 3.9 and 5.1 there exist constants \(C\) and \(\tau_0\) independent of \(\tau\) such that
\[
\| G_0 \|_{L(X; X)} \leq C \quad \forall \ \tau \geq \tau_0.
\] (5.14)

Indeed, let \(g\) be an arbitrary function from \(L^2(\Omega)\). Then one of the possible solutions for the Stokes system by proposition 5.1 is given by system (5.2)–(5.4). Since system (5.2)–(5.4) can be written in the form (3.30), proposition 3.9 can be applied. We use this proposition in the following way. Let \(\Omega_1\) be a bounded domain in \(\mathbb{R}^2\) with smooth boundary such that \(\Omega \subset \subset \Omega_1\), and we extend the matrices \(A_\tau, B_\tau, C_1\) in \(\Omega_1 \setminus \Omega\) preserving the regularity, and we extend by zero the function \(g\) into \(\Omega_1 \setminus \Omega\). Proposition 3.9 guarantees the existence of the solution \(Z = (z_1, z_2, z_3)\) to system (3.30) such that
\[
\| e^{-\tau \Phi} Z \|_{W_2^{1,\tau}(\Omega)} \leq C \left( 1 + \tau^2 \right) \| e^{-\tau \Phi} g \|_{L^2(\Omega)}.
\]

This estimate immediately implies that
\[
\| e^{-\tau \Phi} w \|_{W_2^{1,\tau}(\Omega)} \leq C \left( 1 + \tau^2 \right) \| e^{-\tau \Phi} g \|_{L^2(\Omega)}.
\] (5.15)

In order to estimate the solution to the Stokes system in \(W_2^{2,\tau}(\Omega)\), we should obtain the estimate of \(z_3\) in \(W_2^{3,\tau}(\Omega)\). We recall that
$\Delta z_3 = \partial_{s_1} z_1 + \partial_{s_2} z_2$ in $\Omega_1$.

Let $\rho \in C_0^\infty(\Omega_1)$, $\rho|\Omega = 1$ and set $z_3 = \rho z_3$. The function $z_3$ solves a boundary value problem

$$
\Delta z_3 = \rho \left( \partial_{s_1} z_1 + \partial_{s_2} z_2 \right) + 2 \left( \nabla \rho , \nabla z_3 \right) + z_3 \Delta \rho \quad \text{in} \ \Omega, \quad z_3 |\partial \Omega = 0.
$$

Hence

$$
\Delta \left( e^{-\tau \rho} z_3 \right) = e^{-\tau \rho} \left( \rho \left( \partial_{s_1} z_1 + \partial_{s_2} z_2 \right) + 2 \left( \nabla \rho , \nabla z_3 \right) + z_3 \Delta \rho \right)
$$

$$
- \tau \left( \nabla \rho , \nabla z_3 \right) e^{-\tau \rho} + z_3 \Delta e^{-\tau \rho} \quad \text{in} \ \Omega.
$$

Since, by (5.15),

$$
\left\| e^{-\tau \rho} \left( \rho \left( \partial_{s_1} z_1 + \partial_{s_2} z_2 \right) + 2 \left( \nabla \rho , \nabla z_3 \right) + z_3 \Delta \rho \right) - \tau \left( \nabla \rho , \nabla z_3 \right) e^{-\tau \rho} + z_3 \Delta e^{-\tau \rho} \right\|_{W_{1,1}^2(\Omega)} \leq C \left( 1 + \tau^3 \right) \left\| e^{-\tau \rho} \rho \right\|_{L^1(\Omega)},
$$

using the classical a priori estimate for the Laplace operator we obtain

$$
\left\| e^{-\tau \rho} Z \right\|_{W_{1,1}^2(\Omega)} \leq C \left( 1 + \tau^3 \right) \left\| e^{-\tau \rho} \rho \right\|_{L^1(\Omega)}. \tag{5.16}
$$

Form (5.15) and (5.16) we have

$$
\left\| (u, p) \right\|_X = \left\| e^{-\tau \rho} (u, p) \right\|_{W_{1,1}^2(\Omega) \times W_{1,1}^2(\Omega)} \leq C \left( 1 + \tau^3 \right) \left\| e^{-\tau \rho} \rho \right\|_{L^1(\Omega)}.
$$

This inequality immediately provides the estimate for the norm of the operator $L_{\mu}(x, D)^{-1}$ as

$$
\left\| L_{\mu}(x, D)^{-1} \right\|_{L^1(\Omega; L^1(\Omega) \times X)} \leq C \left( 1 + \tau^3 \right). \tag{5.17}
$$

Next we consider an operator $R(u, p) = (u, \nabla) w_\tau + (w_\tau, \nabla) u$. Using (5.12) we estimate the norm of this operator as

$$
\left\| R(u, p) \right\|_{L_{1,2}^2(\Omega) \times L_{1,2}^2(\Omega)} \leq \left\| (u, \nabla) w_\tau \right\|_{L_{1,2}^2(\Omega)} + \left\| (w_\tau, \nabla) u \right\|_{L_{1,2}^2(\Omega)}
$$

$$
\leq C \left\| w_\tau \right\|_{W_{1,1}^2(\Omega)} \left\| (u, \nabla) \right\|_X = C \left\| e^{-\tau \rho} \rho \right\|_{L_{1,2}^2(\Omega)} \left\| (u, p) \right\|_X
$$

$$
\leq C e^{-\tau} \left\| e^{-\tau \rho} \rho \right\|_{W_{1,1}^2(\Omega)} \left\| (u, p) \right\|_X \leq C e^{-\tau} \left( 1 + \tau^3 \right) \left\| (u, p) \right\|_X. \tag{5.18}
$$

In order to obtain the last equality, we used (3.21). By (5.17) and (5.18) for all sufficiently large $\tau$ we have

$$
\left\| L_{\mu}(x, D)^{-1} R \right\|_{L^1(\Omega; L^1(\Omega) \times X)} \leq \frac{1}{2}.
$$

Hence inequality (5.14) is established.

From (5.14) and (5.12) we have

$$
\left\| T_{\rho} P (x_0) \right\|_X \leq \left\| T_{\rho} \right\|_{L(1,2)_{x,1}(\Omega; x)} \left\| P (x_0) \right\|_{L_{1,2}^2(x_0, \Omega)} \leq C \tau^3 \left\| (w_\tau, \nabla) \right\|_{L_{1,2}^2(x_0, \Omega)}
$$

$$
\leq C \tau^3 \left\| e^{-\tau \rho} \rho \right\|_{W_{1,1}^2(\Omega)} \left\| e^{-\tau \rho} w_\tau \right\|_{L_{1,2}^2(\Omega)} \leq C \tau^3 e^{-\tau} = \eta(\tau) \ \forall \ \tau \geq 1.
$$
We set \( \Omega = - \leq r \). Let \( P^* (x) \) be the second derivative of the mapping \( P \) at point \( x \). We have
\[
P^* (x) [u, v] = 2L \mu (x, D)^{-1} (u, V) v
\]
and
\[
\| P^* (x) [u, v] \|_x \leq \| L \mu (x, D)^{-1} \|_{L^2 (\Omega)} (u, V) v \|_{L^2 (\Omega)} \\
\leq C \left( 1 + \tau^3 \right) \| e^{-\tau \phi} u \|_{H^2 (\Omega)} \| v \|_{H^2 (\Omega)} 
\]
\[
\leq C \left( 1 + \tau^3 \right) \| e^{-\tau \phi} u \|_{H^2 (\Omega)} \| v \|_{H^2 (\Omega)} \\
\leq C \left( 1 + \tau^3 \right) e^{-\tau} \| (u, \rho) \|_X \| (\nu, q) \|_X
\]
by (5.17) and (5.12). By (5.14) and (5.19) for all sufficiently large \( \tau \) we have
\[
\sup_{x \in \Omega} \| L \mu (x, D)^{-1} (u, V) v \|_{L^2 (\Omega)} \leq C \tau^6 = K (\tau).
\]
Then \( h = K \eta \leq \tau^2 e^{-\tau} \) and \( n_0 (\tau) = \frac{1 - \sqrt{1 - 2 \eta}}{h} \leq 2 \tau^2 e^{-\tau} \leq \frac{1}{2} \) for all sufficiently large \( \tau \). Then there exists a solution \( x_0 \) to the equation \( P (x) = 0 \) such that \( \| x_0 \|_X \leq n_0 (\tau) \). The construction of a complex geometric optics solution for the Navier–Stokes equations is completed.

Then a solution to the Navier–Stokes equations can be represented in the form
\[
u = w_0 + w_{cor}, \quad \| e^{-\tau} \|_{H^2 (\Omega)} = o \left( \frac{1}{\tau} \right) \quad \text{as} \quad \tau \to +\infty.
\]
In a similar way, using propositions 3.7 and 3.10, we construct a complex geometric optics solution to the Navier–Stokes system of the form
\[
u = \bar{w}_0 + \bar{w}_{cor}, \quad \| e^{-\tau} \|_{H^2 (\Omega)} = o \left( \frac{1}{\tau} \right) \quad \text{as} \quad \tau \to +\infty.
\]
Here either \( \bar{w}_0 = \text{Re} \bar{u}_0 \) or \( \bar{w}_0 = \text{Im} \bar{u}_0 \), where
\[
\bar{u}_0 = \left( \begin{array}{c} w_1 \\ w_2 \\ f \end{array} \right), \quad \bar{U} = \left( \begin{array}{c} \bar{w}_1 \\ \bar{w}_2 \\ f \end{array} \right), \quad \bar{U} = e^{\varphi} \left( \bar{U}_0 - \bar{U}_f \right) + \sum_{j=2}^{\infty} (-1)^j (\bar{U}_f e^{\varphi})
\]
\[
\bar{R}_j = \frac{1}{2} \mu A \Delta f + (\partial \varphi \mu) \partial \varphi f + (\partial \varphi \mu) \partial \varphi f,
\]
and the matrices \( A_i, B_1 \) are defined in (5.8) and the functions \( \bar{U}_j \) are determined in proposition 3.7.

6. Completion of the proof of theorem 1.1

Suppose that the Dirichlet-to-Neumann maps (1.2) are the same for positive smooth functions \( \mu_1, \mu_2 \). Then

\[34\]
\[ \frac{\partial^j}{\partial u^j} = \frac{\partial^j}{\partial v^j} \quad \text{on} \quad \partial \Omega, \quad 0 \leq j \leq 10. \]  

Since the domain \( \Omega \) is assumed to be bounded, there exists a ball \( B(0, r) \) such that \( \overline{\Omega} \subset B(0, r) \). Thanks to (6.1), we can extend the coefficients \( \mu_i \) into \( B(0, r) \setminus \Omega \) in such a way that

\[ \mu_i \in C^1 \left( B(0, r) \right), \quad \mu_i > 0 \quad \text{on} \quad \overline{\Omega} \quad \text{and} \quad \mu_1 = \mu_2 \quad \text{in} \quad B(0, r) \setminus \Omega \]

and the functions \( \mu_i \) are constant in some neighborhood of \( S(0, r) \). Therefore, without loss of generality, we can assume that \( \Omega = B(0, r) \), equality (6.1) holds true and

\[ \mu_1 = \mu_2 \quad \text{on} \quad \partial \Omega. \]  

Let \((u_1, p_1)\) be the complex geometric optics solution for the operator \( G_{\mu_1}(x, D) \) given by formulae (5.20). Then there exists a pair \((u_2, p_2)\) such that

\[ G_{\mu_2}(x, D)(u_2, p_2) = 0 \quad \text{in} \quad \Omega, \quad (u_1 - u_2)|_{\partial \Omega} = \left( \frac{\partial u_1}{\partial v} - \frac{\partial u_2}{\partial v} \right)|_{\partial \Omega} = (p_1 - p_2)|_{\partial \Omega} = 0. \]

See proposition 5.1 for the representation of \( u_k, p_k, \quad k = 1, 2 \). We set

\[ L_{\mu_2}(x, D)(u, p) = G_{\mu}(x, D)(u, p) - (u, V)u, \]

which corresponds to the Stokes equations.

Setting \( u = u_1 - u_2 \) and \( p = p_1 - p_2 \), we see

\[ L_{\mu_2}(x, D)(u, p) + (u_1, V)u_1 - (u_2, V)u_2 = L_{\mu_2}(x, D)(u_1, 0) \quad \text{in} \quad \Omega, \]

\[ u|_{\partial \Omega} = \frac{\partial u}{\partial v}|_{\partial \Omega} = p|_{\partial \Omega} = 0. \]

Let \((\tilde{v}, \tilde{p})\) be a solution to the Stokes equations

\[ L_{\mu_2}(x, D)(\tilde{v}, \tilde{p}) = 0 \quad \text{in} \quad \Omega. \]

Taking the scalar product in \( L^2(\Omega) \) of equation (6.3) and the function \( \tilde{v} \), we obtain

\[ \left( L_{\mu_2}(x, D)(u, p) + (u_1, V)u_1 - (u_2, V)u_2, \tilde{v} \right)_{L^2(\Omega)} \]

\[ = \int_{\partial \Omega} \left( p(\nu, \tilde{v}) - 2 \sum_{i,j=1}^2 \mu_{ij} \nu_i \tilde{v}_j(u) \right) d\sigma \]

\[ + \int_{\partial \Omega} 2 \sum_{i,j=1}^2 \mu_{ij} u_i \tilde{v}_j(u) d\sigma + \int_{\Omega} \left\{ p \text{div} \tilde{v} + \left( u, L_{\mu_2}(x, D)(\tilde{v}, \tilde{p}) - V\tilde{p} \right) \right\} dx \]

\[ = \left( L_{\mu_2}(x, D)(u_1, 0), \tilde{v} \right)_{L^2(\Omega)} = \left( u_1, L_{\mu_2}(x, D)(\tilde{v}, 0) \right)_{L^2(\Omega)}. \]

Since \( (u_1, V)u_1 - (u_2, V)u_2 = (u_1, V)u_1 + (u_2, V)u_2 - u_2, V)u_2, u_2 \), one can write (6.3) as

\[ L_{\mu_2}(x, D)(u, p) + (u_1, V)u_1 + (u_2, V)u_2 = L_{\mu_2}(x, D)(u_1, 0) \quad \text{in} \quad \Omega, \]

\[ u|_{\partial \Omega} = \frac{\partial u}{\partial v}|_{\partial \Omega} = p|_{\partial \Omega} = 0. \]

Taking the scalar product of (6.6) with \(-u\) in \( L^2(\Omega) \), integrating by parts and taking into account that \( \int_{\Omega} \left( (u_2, V)u, u \right) dx = 0 \), we have

\[ 35 \]
\[
\int_{\Omega} 2 \mu \sum_{i,j=1}^{2} |e_{ij}(u)|^{2} dx = \int_{\Omega} - \left( (u, \nabla) u, u \right) dx + \left( L_{\mu_{2}-\mu_{1}}(x, D)(u, 0), u \right)_{L^{2}(\Omega)}, \quad (6.7)
\]

Let \( u \) be the complex geometric optics solution given by either (5.20) or (5.21).

Since by (3.25), (3.21), (5.20) and (5.21) there exists a constant \( C \) independent of \( \tau \) such that

\[
\| u \|_{W^{1}_{\infty}(\Omega)} \leq C \left( 1 + \tau^{2} \right) e^{-\tau} \quad \forall \tau \geq \tau_{2},
\]

identity (6.7) implies

\[
\| u \|_{W^{2}_{\infty}(\Omega)} \leq C \| u \|_{W^{1}_{\infty}(\Omega)} \quad \forall \tau \geq \tau_{2}.
\] (6.8)

Let \( u \) be defined by (5.20). Then let \((\nabla, \overline{p})\) be the complex geometric optics solution defined by the formula

\[
\nabla = v - Vg, \quad V = \begin{pmatrix} v_{1} \\ v_{2} \end{pmatrix}, \quad \overline{V} = e^{-i\phi} \left( V_{0} - V_{1} \right) + \sum_{j=2}^{\infty} (-1)^{j} V_{j} e^{-j\phi}, \quad \overline{\nabla} = e^{-i\overline{\phi}} \left( \overline{V}_{0} - \overline{V}_{1} \right) + \sum_{j=2}^{\infty} (-1)^{j} \overline{V}_{j} e^{-j\overline{\phi}},
\]

\[
\overline{p} = \frac{1}{2} \mu_{2} \Delta g + \left( \partial_{x} \mu_{2} \right) \partial_{x} g + \left( \partial_{x} \mu_{2} \right) \partial_{x} g,
\]

where the function \( V \) is constructed in proposition 3.6. For this case, we set \( \phi = \text{Re} \Phi \).

Let \( u \) be defined by (5.21). Then let \((\psi, \overline{p})\) be the complex geometric optics solution defined by the formula

\[
\psi = \begin{pmatrix} v_{1} \\ v_{2} \end{pmatrix} - Vg, \quad \overline{\psi} = \begin{pmatrix} v_{1} \\ v_{2} \end{pmatrix}, \quad \overline{V} = e^{-i\overline{\phi}} \left( \overline{V}_{0} - \overline{V}_{1} \right) + \sum_{j=2}^{\infty} (-1)^{j} \overline{V}_{j} e^{-j\overline{\phi}}, \quad \overline{\nabla} = e^{-i\overline{\phi}} \left( \overline{V}_{0} - \overline{V}_{1} \right) + \sum_{j=2}^{\infty} (-1)^{j} \overline{V}_{j} e^{-j\overline{\phi}},
\]

\[
\overline{p} = \frac{1}{2} \mu_{2} \Delta g + \left( \partial_{x} \mu_{2} \right) \partial_{x} g + \left( \partial_{x} \mu_{2} \right) \partial_{x} g,
\] (6.9)

where the function \( V \) is constructed in proposition 3.7. For this case we set \( \phi = \text{Re} \overline{\Phi} \).

By (3.21) and (3.25) there exists a constant \( C \) independent of \( \tau \) such that

\[
\| e^{i\phi} \psi \|_{W^{1}_{\infty}(\Omega)} \leq C \left( 1 + \tau^{2} \right). \quad (6.11)
\]

From (6.8) we have

\[
\| u \|_{W^{2}_{\infty}(\Omega)} \leq Cr^{2} e^{\tau \max_{x \in \partial \Omega}} \quad \forall \tau \geq \tau_{3}. \quad (6.12)
\]

By (5.20) there exists a constant \( C \) independent of \( \tau \) such that for all \( \tau \geq \tau_{0} \)

\[
\sum_{k=1}^{2} \| (u_{x}, \nabla u_{x}, \psi)_{L^{2}(\Omega)} \| \leq C \| (u_{x}, \nabla u_{x})_{L^{2}(\Omega)} \| \| \psi \|_{L^{2}(\Omega)} \leq C \| u_{x} \|_{L^{2}(\Omega)} \| \psi \|_{L^{2}(\Omega)} \leq C \| V_{k} u_{x} \|_{W^{2}_{\infty}(\Omega)} \| \psi \|_{L^{2}(\Omega)} \leq C \left( 1 + \tau^{2} \right) e^{-i\phi} e^{-i\overline{\phi}} e^{-2\tau \max_{x \in \partial \Omega}}. \quad (6.13)
\]

By (2.2) and (2.6), we obtain

\[-\min_{x \in \Omega} \phi + 2 \max_{x \in \Omega} \phi < -1 \quad \forall x \in \Omega.\]
Then this inequality, (6.13) and (6.5) imply

$$\left( u_1, L_{\mu_1, \mu_1}(x, D) \left( \tilde{v}, 0 \right) \right)_{L^2(\Omega)} = \left( (u_1, \nabla) \right)_{L^2(\Omega)} = o\left( \frac{1}{\tau} \right) \quad \text{as} \quad \tau \to +\infty.$$ 

Using (5.20), we rewrite the above equality as

$$\left( w_\tau, L_{\mu_1, \mu_1}(x, D) \left( \tilde{v}, 0 \right) \right)_{L^2(\Omega)} = o\left( \frac{1}{\tau} \right) \quad \text{as} \quad \tau \to +\infty.$$ 

This equality and the definition of \( w_\tau \) given in (5.10) imply

$$\left( u_\tau, L_{\mu_1, \mu_1}(x, D) \left( \tilde{v}, 0 \right) \right)_{L^2(\Omega)} = o\left( \frac{1}{\tau} \right) \quad \text{as} \quad \tau \to +\infty.$$ 

Then, thanks to (5.9), we can apply proposition 5.2 to transform the left-hand side of the above equality as

$$\left( (w, f), H(x, \partial_x, \partial_x)(v, g) \right)_{L^2(\Omega)} = o\left( \frac{1}{\tau} \right) \quad \text{as} \quad \tau \to +\infty.$$ 

(6.14)

We recall that the operator \( H(x, \partial_x, \partial_x) \) is given by (5.5) and (5.6). Since the coefficients \( \mu \) are constants near \( \partial \Omega \), the operator \( H(x, \partial_x, \partial_x) \) is compactly supported in \( \Omega \). Moreover, since the domain \( \Omega \) is the ball and equality (6.14) is true with \( \mu \) given by (3.23) and \( \tilde{v} \) given by (6.9) and (6.10), all the conditions of proposition 4.5 hold true. Hence we have equalities (4.20) – (4.22) and all the conditions of proposition 4.3 hold true. This in turn implies that all the conditions in proposition 4.6 hold true. By proposition 4.6, equality (4.30) holds true.

Consider equality (4.30). For the matrix differential operator \( H(x, \partial_x, \partial_x) \), we denote the \((i, j)-entry by \( H_{i,j}(x, \partial_x, \partial_x) \) with either \( i \neq j \) or \( i \neq 3 \) are the second-order differential operators with respect to \( \mu \), we are interested in the equation which corresponds to the matrix entry (3, 3).

Let \( C_{j, \ell, m} \) denote the \((k, \ell)-entry of the matrix \( C_j \). From (5.5) and (5.6) we compute

$$H_{33}(x, \partial_x, \partial_x)(g) = -2 \sum_{i,j=1}^2 \left( \partial_{x_i x_j} \mu \right) \partial_{x_i x_j} g - 2 \sum_{k=1}^2 \left( \nabla \partial_{x_k x_k} \mu, \nabla g \right) + \text{div} \left( \mu \mu_{x_k} \partial_{x_k} \right).$$

(6.15)

Here we recall that \( \mu_{x_k} \) is the Hessian matrix. Then by (4.1) we have

$$C_{333} = -2 \left( \partial_{x_3} \mu + 2i \partial_{x_3} \mu - \partial_{x_2} \mu \right), \quad C_{233} = -2 \left( \partial_{x_2} \mu - 2i \partial_{x_2} \mu - \partial_{x_1} \mu \right) = -8 \partial_{x_3} \mu,$$

$$C_{3333} = -4 \left( \partial_{x_3} \mu \right), \quad B_{133} = -2 \left( \partial_{x_3} \mu + i \partial_{x_3} \mu \right), \quad B_{233} = -2 \left( \partial_{x_3} \mu + i \partial_{x_3} \mu \right).$$

Then we have

$$\partial_{x_3} C_{233} = -4 \partial_{x_3} \mu = 8 \Delta \mu \quad \text{in} \quad \Omega.$$ 

(6.16)

For the fixed functions \( U_0, V_0 \), the function \( \delta U_0, V_0 \) can be considered as the fourth-order operator applied to the function \( \mu \). The principal part of this operator is \( 8 V_0, U_0 \partial^2 \mu \). By proposition 3.1, for each point \( \tilde{x} \in \text{supp} \mu \) we can choose functions \( U_{0, \tilde{x}} = (U_{0,1, \tilde{x}}, U_{0,2, \tilde{x}}, U_{0,3, \tilde{x}}) \) and \( V_{0, \tilde{x}} = (V_{0,1, \tilde{x}}, V_{0,2, \tilde{x}}, V_{0,3, \tilde{x}}) \) satisfying (3.1) and (3.2) respectively and \( U_{0,3, \tilde{x}}(\tilde{x}) = U_{0,3, \tilde{x}}(\tilde{x}) = 1 \). Therefore, for any \( \tilde{x} \in \text{supp} \mu \) there exists \( \delta(\tilde{x}) > 0 \) such that on the ball \( B(\tilde{x}, \delta(\tilde{x})) \) we have

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Since by (6.2) the function $\mu$ has a compact support in $\Omega$, for $\tilde{x} \in \mathcal{B}\setminus \text{supp } \mu$, there exists positive $\delta(\tilde{x})$ such that
\[ \Delta^2 \mu = 0 \quad \text{in } B(\tilde{x}, \delta(\tilde{x})). \] (6.18)

Since $\mathcal{B}$ is covered by $\bigcup_{j=1}^J B(x_j, \delta(x_j))$, from such a covering one can choose a finite subcovering $\bigcup_{j=1}^J B(x_j, \delta(x_j))$, $j \in \{1, \ldots, J\}$.

From (4.30), (6.16) and (6.17), there exist functions $c_j, p_k \in L^\infty(\Omega)$ such that
\[ 8\Delta^2 \mu + \sum_{j=1}^J c_j \partial_j^2 \mu + \sum_{k=1}^J p_k \Omega_{x_j,0,3,0,3,0,3} = 0 \quad \text{in } \Omega. \] (6.19)

We claim that there exist constants $C$ and $s_0$ independent of $s$ such that
\[ \left\| \Omega \sum_{j=1}^J \Delta \mu \right\|_{L^2(\Omega)} \leq C \left\| \sum_{j=1}^J \sum_{k=1}^J p_k \Omega_{x_j,0,3,0,3,0,3} \right\|_{L^2(\Omega)} \quad \forall s \geq s_0. \] (6.20)

Indeed in order to prove (6.20), observe that as we mentioned the above equalities (4.20) hold true. Proposition 2.8 and (4.20) yield
\[ \left\| \Omega \sum_{j=1}^J \Delta \mu \right\|_{L^2(\Omega)} \leq C \left\| \sum_{j=1}^J \sum_{k=1}^J p_k \Omega_{x_j,0,3,0,3,0,3} \right\|_{L^2(\Omega)} , \] (6.21)
and
\[ \left\| \Delta \mu \right\|_{L^2(\Omega)} \leq C \left\| \sum_{j=1}^J \sum_{k=1}^J p_k \Omega_{x_j,0,3,0,3,0,3} \right\|_{L^2(\Omega)} \] (6.22)

Let the functions $q_j$ be given by (4.70). By (6.21), (4.72), proposition 2.8 and (4.21), we have
\[ \left\| \Omega \sum_{j=1}^J \Delta \mu \right\|_{L^2(\Omega)} \leq C \left\| \sum_{j=1}^J \sum_{k=1}^J p_k \Omega_{x_j,0,3,0,3,0,3} \right\|_{L^2(\Omega)} \leq C \left\| \sum_{j=1}^J \sum_{k=1}^J p_k \Omega_{x_j,0,3,0,3,0,3} \right\|_{L^2(\Omega)} + \frac{1}{2} \sum_{j=1}^J \Omega_{x_j,0,3,0,3,0,3} \left( \sum_{k=1}^J p_k \right) e^{\| \Omega \sum_{j=1}^J \Delta \mu \|_{L^2(\Omega)} } \] (6.25)
By (6.25), (4.73), proposition 2.8 and (4.22), we obtain

\[ \| s\phi^2 T_{\beta_i} (2C_2 \partial_\omega V_0 + C_0 \partial_\omega V_0 + B_2 V_0 + \frac{1}{2} Q_2 (1)^* T_{\beta_i} (C_2 V_0) - \frac{1}{2} C_2 q_2^0) e^{\phi} \|_{L^2(\Omega)} \]

\[ \leq \| s\phi^2 T_{\beta_i} \left( 2C_2 \partial_\omega V_0 + C_0 \partial_\omega V_0 + B_2 V_0 + \frac{1}{2} Q_2 (1)^* T_{\beta_i} (C_2 V_0) - \frac{1}{2} C_2 q_2^0 \right) e^{\phi} \|_{L^2(\Omega)} + \frac{1}{2} \| \sum_{j=1}^{3} (\partial_\omega T_{\beta_i} (C_2 q_2^0) e^{\phi} \|_{L^2(\Omega)} \]

\[ \leq C \left\| \left( 2C_2 \partial_\omega V_0 + C_0 \partial_\omega V_0 + B_2 V_0 - \frac{1}{2} C_2 q_2^0 \right) e^{\phi} \right\|_{L^2(\Omega)} + C \left\| C_2 e^{\phi} \right\|_{L^2(\Omega)}. \quad (6.26) \]

Finally, observing that

\[ \sum_{j=0}^{2} \left( \left| \nabla C_j (x) \right| + \left| C_j (x) \right| \right) + \sum_{j=1}^{2} \left| B_j (x) \right| \leq C \sum_{j \in \mathbb{R}^3} \left| \partial_\omega e^{\psi} (x) \right| \quad \forall x \in \Omega \]

from (6.21)–(6.26), we obtain (6.20).

Applying the Carleman estimate (2.21) to equation (6.19) and using (6.20), we obtain \( \mu \equiv 0 \). Thus the proof of theorem 1.1 is complete.

7. Construction of complex geometric optics solutions for the Lamé system and proof of theorem 1.2

Denote

\[ Z_\alpha (x, D) v = \sum_{j=1}^{2} \partial_\alpha \left( \alpha \left( \partial_\alpha v_1 + \partial_\alpha v_2 \right) \right) + \sum_{j=1}^{2} \partial_\alpha \left( \alpha \left( \partial_\alpha v_2 + \partial_\alpha v_1 \right) \right) \]

\[ = \alpha \Delta v + \alpha \nabla \nabla \left( \left( \nabla \alpha, \nabla v_1 \right), \left( \nabla \alpha, \nabla v_2 \right) \right) \]

\[ + \left( \left( \nabla \alpha, \partial_\alpha v_1 \right), \left( \nabla \alpha, \partial_\alpha v_2 \right) \right). \]

Then we note

\[ L_{\mu, \lambda} (x, D) w = Z_\alpha (x, D) w + \nabla (\lambda \div w). \]

**Proposition 7.1.** Let \( w = (w_1, w_2) \) and \( f \) satisfy the elliptic system

\[ (\lambda + 2\mu) \Delta f + (\lambda + \mu) \div w + 2 (\nabla \mu, \nabla f) = 0 \quad \text{in} \ \Omega \]

(7.1)

and

\[ \mu \Delta w + M (x, D) (w, f) = 0 \quad \text{in} \ \Omega, \]

(7.2)
where

\[ M(x, D)(w, f) = -(\nabla \mu) \text{div } w + \left( (\nabla \mu, \nabla w_1), (\nabla \mu, \nabla w_2) \right) \]
\[ + \left( (\nabla \mu, \partial_1 w), (\nabla \mu, \partial_2 w) \right) + 2\frac{\lambda + \mu}{\lambda + 2\mu} (\nabla \mu) \text{div } w \]
\[ -2 \left( (\partial_1 \nabla \mu, \nabla f), (\partial_2 \nabla \mu, \nabla f) \right) + 4(\nabla \mu)\overline{(\nabla \mu, \nabla f)} \lambda + 2\mu. \quad (7.3) \]

Then the function \( u = w + \nabla f \) solves the Lamé system (1.4).

**Proof.** See the supplementary data (available from stacks.iop.org/ip/31/035004/mmedia).

Let \( \alpha = \mu_1 - \mu_2 \) and \( \beta = \lambda_1 - \lambda_2 \). Consider the operator

\[ P(x, D)(v, g) = -2 \left( (\partial_1 \nabla \alpha, \nabla g), (\partial_2 \nabla \alpha, \nabla g) \right) \]
\[ + \left\{ \frac{2\lambda_2 + 2\mu_2}{\lambda_2 + 2\mu_2} \text{div } v + 4\frac{(\nabla \mu_2, \nabla g)}{\lambda_2 + 2\mu_2} \right\} (\nabla \alpha + K_\alpha(x, D)v - \frac{\alpha}{\mu_2} M_2(x, D)(v, g), \]

where \( K_\alpha(x, D)v = \left( (\nabla \alpha, \nabla v_1), (\nabla \alpha, \nabla v_2) \right) \) and \( P_\beta(x, D)(v, g) = -\nabla \left\{ \frac{\beta \mu_1}{\lambda_1 + 2\mu_1} \left( \frac{\mu_2}{\lambda_2 + 2\mu_2} \text{div } v - 2\frac{(\nabla \mu_2, \nabla g)}{\lambda_2 + 2\mu_2} \right) \right\}. \]

We have the following.

**Proposition 7.2.** Let \( \alpha, \beta \in C_0^{10}(\Omega) \) and let \( (w, f), (v, g) \) be some smooth solutions to the system (7.1)–(7.2) with the Lamé coefficients \( (\mu_1, \lambda_1) \) and \( (\mu_2, \lambda_2) \) respectively. Then

\[ (w + \nabla f, L_{\alpha, \beta}(x, D)(v + \nabla g))_{L^2(\Omega)} = (w, f), H(x, \partial_1, \partial_2)(v, g))_{L^2(\Omega)}, \quad (7.4) \]

where

\[ H(x, \partial_1, \partial_2)(v, g) = -\nabla \left\{ \alpha \left( \frac{\lambda_2}{\lambda_2 + 2\mu_2} \text{div } v + 4\frac{(\nabla \mu_2, \nabla g)}{\lambda_2 + 2\mu_2} \right) \right\}. \]
Here the operator $M(x, D)$ is given by (7.3) with the functions $\lambda_2, \mu_2$ instead of $\lambda$ and $\mu$.

**Proof.** See the supplementary data (available from stacks.iop.org/ip/31/035004/mmedia).

Now we complete the following.

**Proof of theorem 1.2.** Suppose that the Dirichlet-to-Neumann maps (1.3) are the same for positive smooth functions $\mu_j, \lambda_j$. In [1] it is proved that

$$\frac{\partial^j \mu_j}{\partial \nu^j} = \frac{\partial^j \lambda_j}{\partial \nu^j} = 0 \quad \text{on} \quad \partial \Omega, \quad \forall \ell \in \{0, \ldots, 10\}. \quad (7.6)$$

Since the domain $\Omega$ is assumed to be bounded, there exists a ball $B(0, r)$ such that $\overline{\Omega} \subset B(0, r)$. Thanks to (7.6) we can extend the coefficients $\mu_j, \lambda_j$ into $B(0, r) \setminus \Omega$ such that

$$\mu_j, \lambda_j \in C^{10}(B(0, r)), \quad \mu_1 - \mu_2 = \lambda_1 - \lambda_2 = 0 \quad \text{in} \quad B(0, r) \setminus \Omega$$

and the functions $\mu_j, \lambda_j$ are constant in some neighborhood of $S(0, r)$. Therefore, we can assume that $\Omega = B(0, r)$, that (7.6) holds true, and in particular that

$$\mu_1(x) - \mu_2(x) = \lambda_1(x) - \lambda_2(x) = 0 \quad \text{for all} \quad x \quad \text{from some neighborhood of} \quad \partial \Omega. \quad (7.7)$$

Let $u_i$ be some solution for the Lamé system (1.4) in $\Omega = B(0, r)$ with coefficients $\lambda_1$ and $\mu_1$. Then there exists a function $u_2$ such that

$$L_{\mu_2, \lambda_2}(x, D) u_2 = 0 \quad \text{in} \quad \Omega, \quad (u_i - u_2)|_{\partial \Omega} = \left(\frac{\partial u_i}{\partial \nu} - \frac{\partial u_2}{\partial \nu}\right)|_{\partial \Omega} = 0.$$

We set $u = u_1 - u_2, \mu = \mu_2 - \mu_1$ and $\lambda = \lambda_2 - \lambda_1$. Then

$$L_{\mu_2, \lambda_2}(x, D) u = L_{\mu, \lambda}(x, D) u_1 \quad \text{in} \quad \Omega, \quad u|_{\partial \Omega} = \frac{\partial u_1}{\partial \nu}|_{\partial \Omega} = 0. \quad (7.8)$$

Let $\tilde{v}$ be a complex geometric optics solution to the system

$$L_{\mu_2, \lambda_2}(x, D) \tilde{v} = 0 \quad \text{in} \quad \Omega. \quad (7.9)$$
Taking the scalar product in $L^2(\Omega)$ of equation (7.8) and the function $\tilde{v}$ we obtain

$$
\left( L_{\mu_2,\lambda_2} (x, D) u, \tilde{v} \right)_{L^2(\Omega)} = \int_{\partial \Omega} \left( \lambda_2 + \mu_2 \right) \text{div} u \left( \nu, \tilde{v} \right) + 2 \sum_{i,j=1}^{2} \mu_2 \nu_i \nu_j \left( u \right) \, d\sigma \\
- \int_{\partial \Omega} 2 \sum_{i,j=1}^{2} \mu_2 \nu_i \nu_j \left( \tilde{v} \right) d\sigma = \int_{\partial \Omega} \left( \lambda_2 + \mu_2 \right) \left( \nu, u \right) \text{div} \tilde{v} \, d\sigma \\
+ \int_{\Omega} \left( u, L_{\mu_2,\lambda_2} (x, D) \tilde{v} \right) \, dx = \left( L_{\mu_1,\lambda_1} (x, D) \tilde{v}, \tilde{v} \right)_{L^2(\Omega)} = 0.
$$

(7.10)

Let $u_i$ be the complex geometric optics solution for the operator $L_{\mu_2,\lambda_2} (x, D)$ given by

$$
u_i = \begin{cases} w_1 \\ w_2 \end{cases}, \quad U = \begin{cases} w_1 \\ w_2 \end{cases}, \quad U = e^{\varphi} \left( U_0 - U_1 \right) + \sum_{j=2}^{\infty} (-1)^j U_j e^{i\varphi},
$$

(7.11)

where the function $\tilde{U}$ is constructed in proposition 3.7.

Let $\tilde{v}$ be the complex geometric optics solution for the operator $L_{\mu_2,\lambda_2} (x, D)$ given by the formula

$$
\tilde{v} = \begin{cases} v_1 \\ v_2 \end{cases}, \quad \tilde{\nu} = \begin{cases} v_1 \\ v_2 \end{cases}, \quad \tilde{V} = e^{-i\varphi} \left( \tilde{V}_0 - \tilde{V}_1 \right) + \sum_{j=2}^{\infty} (-1)^j \tilde{V}_j e^{-i\varphi},
$$

where the function $\tilde{V}$ is constructed in proposition 3.7.

We plug the formulae for $u_i$, $\tilde{v}$ into (7.10).

Then, thanks to (7.7), we can apply proposition 7.2 to transform the left-hand side of the above equality as

$$
\left( (w, f), H \left( x, \partial_x, \partial_x \nu, \nu \right) \right)_{L^2(\Omega)} = o \left( \frac{1}{\tau} \right) \quad \text{as} \quad \tau \to +\infty.
$$

(7.12)

We recall that the operator $H \left( x, \partial_x, \partial_x \nu \right)$ is given by (7.5) with $\beta = \lambda_1 - \lambda_2$ and $\alpha = \mu_1 - \mu_2$.

Since the coefficients $\mu_1$, $\lambda_1$ are equal to the same constants near $\partial \Omega$, the coefficients of the operator $H \left( x, \partial_x, \partial_x \nu \right)$ are compactly supported in $\Omega$. Since all the assumptions of proposition 4.5 hold true, we have equalities (4.20)–(4.22).

Let $u_i$ be the complex geometric optics solution for the operator $L_{\mu_2,\lambda_2} (x, D)$ given by

$$
u_i = \begin{cases} w_1 \\ w_2 \end{cases}, \quad U = \begin{cases} w_1 \\ w_2 \end{cases}, \quad U = e^{\varphi} \left( U_0 - U_1 \right) + \sum_{j=2}^{\infty} (-1)^j U_j e^{j\varphi},
$$

(7.13)

where the function $U$ is constructed in proposition 3.6.

Let $\tilde{v}$ be the complex geometric optics solution for the operator $L_{\mu_2,\lambda_2} (x, D)$ given by formula

$$
\tilde{v} = \begin{cases} v_1 \\ v_2 \end{cases}, \quad \tilde{V} = \begin{cases} v_1 \\ v_2 \end{cases}, \quad \tilde{v} = e^{-i\varphi} \left( \tilde{V}_0 - \tilde{V}_1 \right) + \sum_{j=2}^{\infty} (-1)^j \tilde{V}_j e^{-i\varphi},
$$

where the function $V$ is constructed in proposition 3.6. We plug the formulae for $u_i$, $\tilde{v}$ in (7.10) and obtain (7.12).

Since equalities (4.20)–(4.22) are already established, by proposition 4.6, equality (4.30) holds true.
Computing the coefficients of the operator \( H(x, \partial_x, \partial_x) \), we have

\[
C_2 = \begin{pmatrix}
\frac{\lambda \mu_2}{(\lambda_1 + 2\mu_1)(\lambda_2 + 2\mu_2)} & -\frac{i\lambda \mu_2}{(\lambda_1 + 2\mu_1)(\lambda_2 + 2\mu_2)} & \frac{-4i\lambda \mu_2}{(\lambda_1 + 2\mu_1)(\lambda_2 + 2\mu_2)} \\
\frac{-i\lambda \mu_2}{(\lambda_1 + 2\mu_1)(\lambda_2 + 2\mu_2)} & \frac{\lambda \mu_2}{(\lambda_1 + 2\mu_1)(\lambda_2 + 2\mu_2)} & \frac{4i\lambda \mu_2}{(\lambda_1 + 2\mu_1)(\lambda_2 + 2\mu_2)} \\
\frac{-4\lambda \mu_2}{(\lambda_1 + 2\mu_1)(\lambda_2 + 2\mu_2)} & \frac{4i\lambda \mu_2}{(\lambda_1 + 2\mu_1)(\lambda_2 + 2\mu_2)} & \frac{8\lambda \mu_2}{(\lambda_1 + 2\mu_1)(\lambda_2 + 2\mu_2)}
\end{pmatrix} + l.\ o.\ t. \tag{7.14}
\]

and

\[
B_2 = \begin{pmatrix}
\frac{\lambda \mu_2}{(\lambda_1 + 2\mu_1)(\lambda_2 + 2\mu_2)} & -\frac{i\lambda \mu_2}{(\lambda_1 + 2\mu_1)(\lambda_2 + 2\mu_2)} & \frac{-4i\lambda \mu_2}{(\lambda_1 + 2\mu_1)(\lambda_2 + 2\mu_2)} \\
\frac{-i\lambda \mu_2}{(\lambda_1 + 2\mu_1)(\lambda_2 + 2\mu_2)} & \frac{\lambda \mu_2}{(\lambda_1 + 2\mu_1)(\lambda_2 + 2\mu_2)} & \frac{4i\lambda \mu_2}{(\lambda_1 + 2\mu_1)(\lambda_2 + 2\mu_2)} \\
-\frac{2}{(V_2, V_2)_{\mu_2}} & \frac{2}{(V_2, V_2)_{\mu_2}} & \frac{8}{(V_2, V_2)_{\mu_2}} - 16\mu_2^2
\end{pmatrix} + l.\ o.\ t. \tag{7.15}
\]

By l.o.t. in (7.14) we mean smooth matrices which are independent of \( \lambda \) and possibly dependent on the derivatives of \( \mu \) up to the order one, and in (7.15) we mean smooth matrices which are possibly dependent of \( \lambda \) and possibly dependent on the derivatives of \( \mu \) up to the order two. In the succeeding arguments, especially in applying the Carleman estimates, the lower-order terms do not play essential roles. From (7.14) and (7.15), noting \((V_2, V_\mu_1) = 2(V_\lambda)(V_\mu_1) + 2(V_\mu)(V_\mu_1)\), we have

\[
\partial_x^2 C_2 - \partial_x B_2 = \begin{pmatrix}
\frac{-\partial_x^2 \lambda \mu_2}{(\lambda_1 + 2\mu_1)(\lambda_2 + 2\mu_2)} & \frac{\partial_x^2 \lambda \mu_2}{(\lambda_1 + 2\mu_1)(\lambda_2 + 2\mu_2)} & \frac{-4\partial_x^2 \lambda \mu_2}{(\lambda_1 + 2\mu_1)(\lambda_2 + 2\mu_2)} \\
\frac{-\partial_x^2 \lambda \mu_2}{(\lambda_1 + 2\mu_1)(\lambda_2 + 2\mu_2)} & \frac{\partial_x^2 \lambda \mu_2}{(\lambda_1 + 2\mu_1)(\lambda_2 + 2\mu_2)} & \frac{4\partial_x^2 \lambda \mu_2}{(\lambda_1 + 2\mu_1)(\lambda_2 + 2\mu_2)} \\
\frac{-4\partial_x^2 \lambda \mu_2}{(\lambda_1 + 2\mu_1)(\lambda_2 + 2\mu_2)} & \frac{4\partial_x^2 \lambda \mu_2}{(\lambda_1 + 2\mu_1)(\lambda_2 + 2\mu_2)} & \frac{8\partial_x^2 \lambda \mu_2}{(\lambda_1 + 2\mu_1)(\lambda_2 + 2\mu_2)}
\end{pmatrix} + l.\ o.\ t. \tag{7.16}
\]

Using (7.16) we write (4.30) as

\[
-\begin{pmatrix}
\Delta \lambda \left( \frac{\mu_1}{\lambda_1 + 2\mu_1} U_{0.1} + i U_{0.2} \right) \\
4 \left( \partial_x \mu_1 \right) U_{0.3} \left( \frac{\lambda_1 + 2\mu_1}{\lambda_1 + 2\mu_1} \right) \left( \frac{\mu_2}{\lambda_2 + 2\mu_2} \right) - 4 \left( \partial_x \mu_2 \right) V_{0.3} \left( \frac{\lambda_2 + 2\mu_2}{\lambda_2 + 2\mu_2} \right)
\end{pmatrix}
\]

\[
+ 8 \Delta^2 \mu V_{0.3} U_{0.3} + P_1(x, D) \mu + P_1(x, D) \lambda + \Omega_{V_0, V_0} = 0, \tag{7.17}
\]

where \( P_1(x, D), P_1(x, D) \) are differential operators with smooth coefficients depending on \( U_0, V_0 \) of orders three and one respectively. For fixed functions \( U_0 \) and \( V_0 \), the function \( S_{V_0, V_0} \)
can be considered as a fourth-order differential operator applied to the function $\mu$ and a second-order differential operator applied to the function $\lambda$ respectively. Let $O = \text{supp } \mu \cup \text{supp } \lambda$. By (7.7), we have

$$O \subset \Omega.$$  (7.18)

By proposition 3.1 for each point $\tilde{x} \in O$ we can choose functions $U_{0,\tilde{x}} = (U_{0,1,\tilde{x}}, U_{0,2,\tilde{x}}, U_{0,3,\tilde{x}})$ and $V_{0,\tilde{x}} = (V_{0,1,\tilde{x}}, V_{0,2,\tilde{x}}, V_{0,3,\tilde{x}})$ satisfying (3.1) and (3.2) respectively and $V_{0,3,\tilde{x}}(\tilde{x}) = U_{0,3,\tilde{x}}(\tilde{x}) = 1$. Applying again proposition 3.1 for each point $\tilde{x}$, we can choose functions $\hat{U}_{0,1,\tilde{x}} = (\hat{U}_{0,1,1,\tilde{x}}, \hat{U}_{0,1,2,\tilde{x}}, \hat{U}_{0,1,3,\tilde{x}})$ and $\hat{V}_{0,1,\tilde{x}} = (\hat{V}_{0,1,1,\tilde{x}}, \hat{V}_{0,1,2,\tilde{x}}, \hat{V}_{0,1,3,\tilde{x}})$ satisfying (3.1) and (3.2) respectively and $\hat{V}_{0,1,1,\tilde{x}}(\tilde{x}) = \hat{U}_{0,1,1,\tilde{x}}(\tilde{x}) = 1$ and $\hat{V}_{0,1,j,\tilde{x}}(\tilde{x}) = 0$ for $j = 2, 3$.

Therefore, for some $\epsilon > 0$ and any $\tilde{x} \in O$ there exists $\delta(\tilde{x}) > 0$ such that

$$V_{0,3,\tilde{x}}(x)U_{0,3,\tilde{x}}(x) \geq \frac{1}{2} \quad \forall x \in B(\tilde{x}, \delta(\tilde{x})).$$

(7.19)

Then there exist partial differential operators $\hat{P}_j(x, D)$ and $\bar{P}_j(x, D)$ of orders three and one respectively such that

$$A(x) \begin{pmatrix} \Delta j \mu \\ \Delta j \lambda \end{pmatrix} + \hat{P}_j(x, D)\mu + \bar{P}_j(x, D)\lambda = \begin{pmatrix} \Omega_{\nu_{1,j}, q_{1,j}, q_{2,j}} \\ \Omega_{\nu_{2,j}, q_{1,j}, q_{1,j}} \end{pmatrix} \quad \forall x \in B(\tilde{x}, \delta(\tilde{x})).$$

(7.20)

Taking a parameter $\epsilon$ sufficiently small, we obtain that $\det A(x) \neq 0$ in $B(\tilde{x}, \delta(\tilde{x}))$. If $\tilde{x} \in \bar{\Omega} \setminus O$, then there exists positive $\delta(\tilde{x})$ such that

$$\Delta^2 \mu = 0 \quad \text{in } B(\tilde{x}, \delta(\tilde{x})) \quad \text{and} \quad \Delta \lambda = 0 \quad \text{in } B(\tilde{x}, \delta(\tilde{x})).$$

(7.21)

Since $\bar{\Omega}$ is covered by $\bigcup_{\tilde{x} \in \partial O} B(\tilde{x}, \delta(\tilde{x}))$, from such a covering one can take a finite subcovering $B(\tilde{x}_j, \delta(\tilde{x}_j))$, $j \in \{1, \ldots, J\}$.

From (7.20) there exist constants $c^{\rho}_{j,1,\tilde{x}}, b^{\rho}_{j,1,\tilde{x}}, p_{j,1,\tilde{x}} \in L^\infty(\Omega)$ such that

$$\Delta^2 \mu + \sum_{|\rho| \leq 1} c^{\rho}_{j,1,\tilde{x}} \partial^\rho \mu + \sum_{|\rho| \leq 1} b^{\rho}_{j,1,\tilde{x}} \partial^\rho \lambda$$

$$+ \sum_{k=1}^J \left( p_{k,1,1,\tilde{x}} \Omega_{\nu_{1,j}, q_{1,j}, q_{2,j}} + p_{k,1,2,\tilde{x}} \Omega_{\nu_{2,j}, q_{1,j}, q_{1,j}} \right) = 0 \quad \text{in } \Omega,$$

$$\Delta \lambda + \sum_{|\rho| \leq 3} c^{\rho}_{j,2,\tilde{x}} \partial^\rho \mu + \sum_{|\rho| \leq 1} b^{\rho}_{j,2,\tilde{x}} \partial^\rho \lambda$$

$$+ \sum_{k=1}^J \left( p_{k,2,1,\tilde{x}} \Omega_{\nu_{1,j}, q_{1,j}, q_{2,j}} + p_{k,2,2,\tilde{x}} \Omega_{\nu_{2,j}, q_{1,j}, q_{1,j}} \right) = 0 \quad \text{in } \Omega.$$

From (4.30), (4.20), (4.21), (4.72), (4.73) and propositions 2.8 and 4.4, there exist constants $C$ and $s_0$ such that
\[ \left\| \nabla \phi_{\mu} \right\|_{L^2(\Omega)} \leq C \left( \left\| \sum_{P \in \mathbb{L}} \phi_{\mathbb{L}}^{2} \nabla \phi_{\mathbb{L}}^{2} \right\|_{L^2(\Omega)} + \left\| \sum_{P \in \mathbb{L}} \phi_{\mathbb{L}}^{2} \nabla \phi_{\mathbb{L}}^{2} \right\|_{L^2(\Omega)} \right) \quad \forall \ s \geq s_0. \] (7.24)

Applying the Carleman estimate (2.21) to the system (7.22) and (7.23) and using (7.24) and (7.6), we obtain \( (\mu, \lambda) \equiv 0 \). Thus the proof of the theorem is complete.

\[ \square \]

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