On the Fundamental Solution of a Homogeneous Linearized Coagulation Equation.

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M. Escobedo\(^1\) and J. J. L. Velázquez \(^2\)

1 Introduction

Under rather general conditions on the kernel \(K(x,y)\), a symmetric homogeneous function in \(x\) and \(y\), the Cauchy problem for the Smoluchowski coagulation equation:

\[
\frac{\partial f}{\partial t} = Q[f] \tag{1.1}
\]

\[
Q[f] = \frac{1}{2} \int_0^x K(x-y,y) f(x-y) f(y) dy - \int_0^\infty K(x,y) f(x) f(y) dy \tag{1.2}
\]

\[
f(0,x) = f_{in}(x) \in L^1_1(\mathbb{R}), \tag{1.3}
\]

has a global solution \(f(t,x)\) for all initial data \(f_{in}(x)\) such that \(\int_0^\infty x f_{in}(x) dx < \infty\). Moreover, this solution satisfies the same estimate for all \(t > 0\).

Equation (1.1), (1.2) describes the aggregation process of particles of mass \(x\) and \(y\) with probability \(K(x,y)\), assuming that the distribution of particles are uncorrelated at all times. In this context the quantity:

\[
\int_0^\infty x f(t,x) dx \tag{1.4}
\]

represents the total mass of particles in the system.

On the other hand, it is known that, when the kernel is of the form \(K(x,y) = x^\alpha y^\beta + x^\beta y^\alpha\) with \(\alpha \geq 0\), \(\beta \geq 0\) and \(\alpha + \beta = \lambda > 1\), the solutions to the Cauchy problem for the Smoluchowski equation undergo the so called gelation phenomenon. This means that there exists a positive time \(T_g < \infty\) such that, for all \(t < T_g\),

\[
\int_0^\infty x f(t,x) dx = \int_0^\infty x f_{in}(x) dx \tag{1.5}
\]

\(^1\)Departamento de Matemáticas, Universidad del País Vasco, Apartado 644, E-48080 Bilbao, Spain. E-mail: mtpesam@lg.ehu.es

\(^2\)ICMAT (CSIC-UAM-UC3M-UCM) Facultad de Matemáticas, Universidad Complutense. E-28040 Madrid, Spain. E-mail: JJ_Velazquez@mat.ucm.es
and for all $t > T_g$,

$$\int_0^\infty x f(t, x) \, dx < \int_0^\infty x f_{in}(x) \, dx.$$  

(1.6)

It is also known that when $\lambda \leq 1$ gelation does not occur and mass is conserved for all time (cf. \[6, 10\]). Two interesting open questions are related with this phenomenon. One is to describe the solution $f$ as $t \to T_g$. We consider here the second one, which is to understand the behaviour of the solution $f$ after the gelling time $T_g$.

Although no general result is known, several partial results indicate that before the gelling time, at least for a large family of initial data, the solutions to the coagulation equation decay exponentially fast as $x \to +\infty$. The Smoluchowski equation (1.1) has a discrete counterpart: for which an explicit exact gelling solution was constructed in [13] for $K(k,j) = kj$. Such a solution decays exponentially fast before the gelling time, and as a power law after that time. The exponential decay, before the gelling time, was later shown in [7] for the continuous equation (1.1), $\lambda = 2$ and several initial data. Moreover, it has also been formally shown in [14] that, for several initial data and $\lambda \in (1,2]$, the solution of (1.1) decays, after gelling, like $x^{-(3+\lambda)/2}$ as $x \to +\infty$ (see [12] for more detailed references). On the other hand, it was proved in [8] that $x^{-(3+\lambda)/2}$ is the only possible power law decay for the solutions of (1.1) after gelation. Our main purpose is to prove that for the coagulation kernel

$$K(x,y) = (xy)^{\lambda/2}, \quad \lambda \in (1,2)$$

(1.7)

and any initial data $f_{in}$, regular near the origin and such that:

$$f_{in}(x) \sim x^{-(3+\lambda)/2} \quad \text{as} \quad x \to +\infty,$$

(1.8)

the problem (1.1)-(1.3) has a solution $f$ satisfying

$$f(t,x) \sim a(t) x^{-(3+\lambda)/2} \quad \text{as} \quad x \to +\infty.$$  

(1.9)

Moreover, this solution satisfies

$$\frac{d}{dt} \int_0^\infty x f(x,t) \, dx = -2\pi a^2(t) \quad \text{for all} \; t > 0$$

(1.10)

which was formally shown in [5] for the discrete equation. By (1.10) the total mass of the solution $f$ is decreasing. This loss of mass is a characteristic feature of the solutions of (1.1),(1.2) after the gelation time. The choice of exponents $\lambda < 2$ is natural, because $\lambda \leq 2$ excludes instantaneous gelation or non existence of solutions (3, 5, 15) and $\lambda = 2$ is one of the “explicit” cases which has been treated using the Laplace transform (cf. [7]).

In order to prove the existence of classical solutions of (1.1)-(1.3) after gelation we will use the same approach as in [9, 10]. The starting point of this approach is to linearize around an initial data $f_{in}$ satisfying $f_{in} \approx x^{-(3+\lambda)/2}$ for $x$ large and to derive detailed estimates on the solutions of the resulting linear equation.

$$\frac{\partial g}{\partial t} = \mathcal{L}(g,f_{in}).$$  

(1.11)

To this end we will need some rather delicate estimates on the asymptotics of the solutions as $x$ tends to infinity. Moreover, even to prove solvability of the linearized problem (1.11) is nontrivial.
We will obtain it treating this problem as a perturbation of the problem obtained replacing $f_{in}$ by its asymptotics as $x$ tends to infinity:

$$\frac{\partial g}{\partial t} = L(g).$$

In order to carry on this program we need to derive detailed estimates about the solutions of (1.12). This will be the main goal of this paper.

The linearised equation around the weak solution $x^{-(3+\lambda)/2}$ may be introduced more directly as follows. Consider a solution $f(t, x)$ of the coagulation equation with an initial data $f_{in}$ satisfying (1.8). If one is interested in the behaviour of the variables as follows: $x = R_\bar{\tau}, y = R_\bar{\gamma}, t = R^{-(\lambda-1)/2} \tau$ and $f(t, x) = R^{-(3+\lambda)/2} F_R(t, \bar{x})$. In these new variables, the equation (1.1) reads $\left(F_R\right)_{\bar{\tau}} = Q[F_R]$ and the initial data $F_{in}$ satisfies now: $F_R(0, \bar{x}) = R^{(3+\lambda)/2} f_{in}(R \bar{x}) \sim (\bar{x})^{-(3+\lambda)/2}$ as $R \to +\infty$. The limit of the function $F_R$ as $R \to +\infty$, if it exists, would then solve the same equation (1.1) with initial data $\bar{x}^{-(3+\lambda)/2}$. Therefore the linear problem (1.12) appears naturally as the linearisation of the coagulation equation (1.1) in the region $x >> 1$. Notice, however that in the region where $\bar{x}$ is small the function $\bar{f}$ is bounded and the approximation by means of the power law $\bar{x}^{-(3+\lambda)/2}$ cannot be valid. The analysis of that region would lead naturally to the study of a boundary layer whose description requires the analysis of the operator $L$. This will be made in a forthcoming work.

On the other hand, the linearised equation (1.12) has some interest by itself. It is indeed a simple model to describe a set of particles at equilibrium, whose density distribution is given by $x^{-(3+\lambda)/2}$, and where a small set of particles is introduced, whose distribution $\varphi(x)$ is considered as a small perturbation. The particles so introduced start to collide both between themselves and with the particles in the background. The equilibrium density distribution $x^{-(3+\lambda)/2}$ is then perturbed. The distribution density function of the resulting set of particles may then be seen at any time $t$ as the equilibrium distribution $x^{-(3+\lambda)/2}$ and a remaining perturbation $\varphi(t, x)$. The linear equation (1.12) only takes into account the collisions of the “particles in the perturbation” with the background and describes how the distribution of these particles evolves in time. It neglects the collisions between particles in the perturbation. This could be a reasonable approximation as long as the perturbation $\varphi(t, x)$ remains small. Notice that the number of clusters in the background as well as the number of particles (the total mass) are infinite (since nor $x^{-(3+\lambda)/2}$ nor $x^{1-(3+\lambda)/2}$ are integrable in $(0, +\infty)$), but the number of clusters and particles in the initial perturbation are finite. Our results show the following:

- There is instantaneously an infinite number of “perturbed clusters”, although their mass is finite.
- As $t \to +\infty$, the number of perturbed particles (the mass in the perturbation) tends to zero, but the number of perturbed clusters remains infinite.
- The total flux of particles is perturbed at $t$ finite but tends to the flux corresponding to the original equilibrium distribution as $t \to +\infty$.

Our results are obtained using classical Fourier analysis and the Wiener Hopf method, in a similar way as we did for the linearized Uehling Uhlenbeck operator in [9] although with an important difference. This is the regularising effect of the operator $L$, absent in the operator studied in [9], and coming from the fact that $L$ is similar to the half derivative operator. The fundamental solution of (1.12) has then very different properties than that obtained in [9].
In Section 2 we state our main results and transform the integro differential equation (1.12) to a Carleman equation in the complex plane. In Section 3 we state the fundamental properties of the auxiliary function Φ appearing in the Carleman equation. This equation is solved in Sections 4 and 5 using the classical Cauchy integral, which gives an explicit solution. In sections 6 and 7 the precise asymptotics of the solutions are obtained. The Section 8 is devoted to a brief mention of the initial value problem. Some properties of the fluxes of particles described by the solutions are considered in Section 9. We have finally added Appendices I, II and III where are collected some necessary technical results.

2 The Linearized Equation

We start this Section writing the precise expression of linearized equation (1.12).

Proposition 2.1 The linearised equation of (1.1)-(1.2) with \( K(x,y) = (xy)^{\lambda/2} \) around the solution \( G(x) = x - (3+\lambda)/2 \) is

\[
\frac{\partial g}{\partial t} = L(g),
\]

\[
L(g) = \int_0^{x/2} \left( (x-y)^{\lambda/2}G(x-y) - x^{\lambda/2}G(x) \right) y^{\lambda/2}g(y)dy
\]

\[
+ \int_0^{x/2} \left( (x-y)^{\lambda/2}g(x-y) - x^{\lambda/2}g(x) \right) y^{-3/2}dy
\]

\[-x^{-3/2} \int_{x/2}^\infty y^{\lambda/2}g(y)dy - 2\sqrt{2}x^{(\lambda-1)/2}g(x). \]

\( \Box \)

Proof. By the symmetry of the kernel \( K \):

\[
\frac{1}{2} \int_0^{x/2} K(y,x-y)f(y)f(x-y)dy = \frac{1}{2} \int_0^{x} K(y,x-y)f(y)f(x-y)dy
\]

Then we may then write the equation (1.1)-(1.2) as follows:

\[
\frac{\partial f}{\partial t} = \int_0^{x/2} \left[ (x-y)^{\lambda/2}f(x-y) - x^{\lambda/2}f(x) \right] y^{\lambda/2}f(y)dy - \int_{x/2}^\infty K(x,y)f(x)f(y)dy.
\]

If we linearize around the solution \( G(x) = x^{-(3+\lambda)/2} \), define \( f = G + g \) and neglect quadratic terms on \( g \) we obtain (2.1), (2.2).

Remark 2.2 The second term in the right hand side of (2.2) can be seen as some kind of half derivative operator applied to function \( x^{\lambda/2}g(x) \). This will appear again in the Fourier analysis that will be done later on the linearised equation.

Remark 2.3 In order for the first integral in the right hand side of (2.2) to be defined we need \( y^{1+\lambda/2}g(y) \) to be integrable at the origin. For the second integral we need some kind of regularity of \( g(x) \) with respect to \( x \). For example \( y^{\lambda/2}g(y) \) \( \gamma \)-Hölder continuous with \( \gamma > 1/2 \). Finally, for the
last one we need $y^{\lambda/2}g(y)$ to be integrable as $y \to \infty$. Assuming power like behaviours we then need bounds on $g$ of the form:

$$g(y) \leq C y^{-\lambda/2-r} \quad \text{as} \quad y \to +\infty,$$

$$g(y) \leq C y^{-\lambda/2-\rho} \quad \text{as} \quad y \to 0,$$

for some $r > 1$ and $\rho < 2$.

**Definition 2.4**  We will denote as $\mathcal{V}(r, \rho)$ the set of functions

$$\mathcal{V}(r, \rho) = \left\{ g \in L^\infty_{\text{loc}}(\mathbb{R}^+); \sup_{0 \leq x \leq 1} g(x)x^{\lambda/2+\rho} < \infty, \sup_{1 \leq x} g(x)x^{\lambda/2+r} < \infty \right\}$$

We state now the main results of this paper. The first one is an existence and uniqueness result of fundamental solutions for the equation (2.1), (2.2).

**Theorem 2.5**  For all $x_0 > 0$, there exists a unique solution $g(t, \cdot, x_0)$ of (2.1), (2.2) with initial data:

$$g(0, x, x_0) = \delta(x - x_0)$$

such that $g(t, \cdot, x_0) \in \mathcal{V}(3/2, (3 - \lambda)/2)$ for all $t > 0$. Moreover, $g(t, \cdot, x_0)$ has the self similar form

$$g(t, x, x_0) = \frac{1}{x_0} g\left(\frac{\lambda-1}{2} t x_0^2, \frac{x}{x_0}, 1\right).$$

The function $g(t, \cdot, 1)$ satisfies the following behaviours in the different regions of the $k,t$ space for some explicitly known constants $a_i, i = 1, \cdots, 5$ and arbitrarily small positive constants $\varepsilon$ and $\delta$.

We have the following representation formula for $t \geq 1$:

$$g(t, x, 1) = t^{\lambda-1} \varphi_1(\sigma) + \varphi_2(t, \sigma)$$

where $\sigma$ is the self similar variable:

$$\sigma = t^{\frac{2}{\lambda-1}} x,$$

and the functions $\varphi_1$ and $\varphi_2$ satisfy the following estimates:

$$\varphi_1(\sigma) = \begin{cases} a_1 \sigma^{-\frac{3}{2}} + \mathcal{O}\left(\sigma^{-\frac{4}{2}+\varepsilon}\right) & \text{for } 0 \leq \sigma < 1 \\ a_2 \sigma^{-\frac{3+\lambda}{2}} + \mathcal{O}\left(\sigma^{-(1+\lambda+\varepsilon)}\right) & \text{for } \sigma > 1 \end{cases}$$

where $a_1$ and $a_2$ are two explicit constants and for some positive constant $\varepsilon$ arbitrarily small,

$$\varphi_2(t, \sigma) = \begin{cases} b_1(t) \sigma^{-\frac{3}{2}} + \mathcal{O}\left(t^{\frac{2}{\lambda-1}-\delta_0} \sigma^{-\frac{3}{2}+\varepsilon}\right) & \text{for } 0 \leq \sigma < 1 \\ b_2(t) \sigma^{-\frac{3+\lambda}{2}} + \mathcal{O}\left(t^{\frac{2}{\lambda-1}-\delta_0} \sigma^{-\frac{3+\lambda}{2}-\varepsilon}\right) & \text{for } \sigma > 1 \end{cases}$$

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where \( b_1 \) and \( b_2 \) are two continuous functions such that \( |b_1(t)| + |b_2(t)| \leq Ct^{\frac{2}{\nu} - \delta_0}, \varepsilon > 0 \) and \( \delta_0 > 0 \). On the other hand, for all \( 0 < t < 1 \) fixed:

\[
g(t, x, 1) = \begin{cases} 
  a_3 t x^{-\frac{2}{\nu}} + b_3(t) x^{-\frac{3}{\nu}} + O(t x^{-\frac{5}{2} + \delta}) & \text{for } 0 \leq x \leq \frac{1}{2} \\
  a_4 t x^{-\frac{3+\lambda}{2}} + b_4(t) x^{-\frac{3+\lambda}{2}} + O(t x^{-\frac{3+\lambda}{2} - \delta}) & \text{for } x \geq \frac{3}{2},
\end{cases}
\]  

(2.14)

where \( b_3 \) and \( b_4 \) are continuous functions such that \( |b_3(t)| + |b_4(t)| \leq Ct^{1+\delta} \), with \( \delta > 0 \), \( \delta_1 > 0 \) and \( a_3, a_4 \) explicit numerical constants.

Finally, if \( t \to 0 \):

\[
g(t, x, 1) = \begin{cases} 
  t^{-2} \Psi \left( \frac{x-1}{t^2} \right) + O(t^2) & \text{for } x = 1 + O(t^2) \\
  O \left( \frac{t^{1-2\delta}}{|x-1|^{\frac{2}{\nu} - \delta}} \right) & \text{for } t^2 < |x-1| < \frac{1}{2},
\end{cases}
\]  

(2.15)

where the function \( \Psi \) is given by:

\[
\Psi(\chi) = \begin{cases} 
  \frac{2}{\pi} e^{-\frac{\chi^2}{2}}, & \text{for all } \chi \geq 0, \\
  0 & \text{for all } \chi < 0.
\end{cases}
\]  

(2.16)

**Remark 2.6** For \( t \geq 1 \) large the behaviour if \( g \) in terms of the variable \( x \) is:

\[
g(t, x, 1) = \begin{cases} 
  a_1 t x^{-\frac{2}{\nu}} \sigma^{-\frac{2}{\nu}} + b_1(t) \sigma^{-\frac{3}{\nu}} + O \left( t x^{-\frac{1+\lambda}{\nu} + \varepsilon} + t^{\frac{3+\lambda}{2} - \delta_0} \sigma^{-\frac{3}{\nu}} \right) & \text{for } 0 \leq \sigma < 1 \\
  a_2 t x^{-\frac{2}{\nu}} \sigma^{-\frac{3+\lambda}{2}} + b_2(t) \sigma^{-\frac{3+\lambda}{2}} + O \left( t x^{-\frac{1+\lambda}{\nu} + \varepsilon} + t^{\frac{3+\lambda}{2} - \delta_0} \sigma^{-\frac{3+\lambda}{2}} \right) & \text{for } \sigma \geq 1
\end{cases}
\]  

(2.17)

\[
g(t, x, 1) = \begin{cases} 
  a_1 t^{-\frac{1}{\nu}} x^{-\frac{3}{\nu}} + O \left( t^{-\frac{1+\lambda}{\nu} x^{-\frac{1}{2} + \varepsilon} + t^{-\frac{1}{\nu} - \delta} x^{-\frac{3}{2}} \right) & \text{for } 0 < x < 2 t^{-\frac{1}{\nu}} \\
  a_2 t^{-\frac{1+\lambda}{\nu} x^{-\frac{3+\lambda}{2}}} + O \left( t^{-\frac{1+\lambda+2\varepsilon}{\nu} x^{-\frac{3+\lambda}{2} - \varepsilon} + t^{-\frac{1+\lambda}{\nu} - \delta} x^{-\frac{3}{2}} \right) & \text{for } x > 2 t^{-\frac{1}{\nu}}
\end{cases}
\]  

(2.18)

Our second result is about the regularity of the fundamental solutions of (2.1), (2.2).

**Remark 2.7** Using the rescaling properties of the problem (2.1), (2.2), (2.3) has the self similar form (2.9). It is then enough to restrict our analysis to the case \( x_0 = 1 \).

Our strategy to solve the problem (2.1), (2.2), (2.3) is to use Fourier analysis. The resulting problem is explicitly solvable by means of the Wiener Hopf method [2]. Using the representation formula for the solution, we then prove Theorem (2.5) by deriving suitable a priori estimates. Similar arguments have been used in [9].
2.1 Fourier variables.

We make now a change of variables in order to have functions defined in all of the real line $\mathbb{R}$. To this end we define $x = e^X$, $X \in \mathbb{R}$, as well as the Fourier transform

$$\hat{G}(t, \xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-iX\xi} G(t, X) dX, \quad G(t, X) = g(t, e^X) \quad (2.19)$$

Then, the problem [2.1], [2.2], [2.8] reads in terms of the new variables:

$$\frac{\partial \hat{G}}{\partial t}(t, \xi) = \hat{G}(t, \xi + \lambda - \frac{1}{2}i) \Phi(\xi + \lambda - \frac{1}{2}i) \quad (2.20)$$

$$\hat{G}(0, \xi) = \frac{1}{\sqrt{2\pi}} \quad (2.21)$$

where the function $\Phi$ is given by:

$$\Phi(\xi) = -\frac{2\sqrt{\pi}}{\Gamma(i\xi + \frac{\lambda}{3})} \frac{\Gamma(i\xi + \frac{\lambda+1}{2})}{\Gamma(i\xi + \frac{\lambda}{2})} \quad (2.22)$$

as it is shown in Section 10. The fact that the function $g(t, \cdot, 1) \in \mathcal{V}(3/2, (3 + \lambda)/2)$ implies that the function $G(t, \cdot)$ is analytic in the strip: $S = \{\xi \in \mathbb{C}; \text{Im}\xi \in (3/2, (3 + \lambda)/2)\}$.

3 The auxiliary function.

The properties of the function $\Phi$, defined by (2.22), determine the properties of the equation (2.20). The set of its zeros is:

$$\xi_z(n) = i \left(n + \frac{1 + \lambda}{2}\right), \quad n = 0, 1, \cdots \quad (3.1)$$

The singularities of the function $\Phi$ are given by the family of poles:

$$\xi_p(n) = i \left(1 + \frac{\lambda}{2} + n\right), \quad n = 0, 1, \cdots \quad (3.2)$$

The asymptotic behaviour of the function $\Phi$ as $\text{Re}(\xi) \to \pm\infty$ is the following:

Proposition 3.1 For all $M > 0$ fixed:

$$\Phi(\xi) = -\sqrt{2\pi}(1 + iQ)\sqrt{Q}\xi - \frac{\sqrt{2\pi}(1 + iQ)i}{\xi}\sqrt{Q}\xi \left(\frac{1}{8} + \frac{\lambda}{4}\right) + \mathcal{O}\left(\frac{1}{|\xi|^{3/2}}\right)$$

as $\text{Re}(\xi) \to \infty$, uniformly on $\text{Im}(\xi) \in (-M, M)$ and where the function $Q$ is defined as:

$$Q = Q(\xi) = \text{sgn}(\text{Re}[\xi]). \quad (3.3)$$
Figure 1: Some relevant zeros and poles of the function $\Phi$.

4 **Solving \((2.20)-(2.21)\)**

Our goal is to solve the problem \((2.20)-(2.21)\) which we recall here:

\[
\frac{\partial \hat{G}}{\partial t}(t, \xi) = \hat{G}(t, \xi + \frac{\lambda - 1}{2}i) \Phi\left(\xi + \frac{\lambda - 1}{2}i\right) \tag{4.1}
\]

\[
\hat{G}(0^+, \xi) = \frac{1}{\sqrt{2\pi}} \tag{4.2}
\]

where the function $\hat{G}$ must be analytic in the strip $S$. This analyticity condition is a consequence of the decay required for $G(t, X)$ in order for the integral equation to make sense. Since we are interested in deriving a solution $G(t, X)$ in the sense of distributions, we want to obtain boundedness of $\hat{G}$ as $|Re(\xi)| \to \infty$. In this case we will obtain, for the particular solution $\hat{G}$ constructed here, exponential decay, something that means that $G(t, \cdot) \in C^\infty$ for $t > 0$.

The following transformation allows to reduce \((4.1), (4.2)\) to an equation for a function depending only in one variable $\xi$:

\[
\hat{G}(t, \xi) = -\frac{\sqrt{2}}{\sqrt{\pi i (\lambda - 1)}} \int_{Im(y) = \beta_0} \frac{V(\xi)}{V(y)} e^{\frac{2i(\xi - y)}{\lambda - 1}} \Gamma\left(-\frac{2i(\xi - y)}{\lambda - 1}\right) dy \tag{4.3}
\]

for some $\beta_0 \in (3/2, 2)$. We have:

**Lemma 4.1** Suppose that $V(\eta)$ is analytic in the strip $S$ and that $V$ satisfies

\[
\int_{Im(y) = \beta_0} \left| \frac{1}{\sqrt{V(y)}} e^{-(\pi - \eta)|y|} \sqrt{|y| + 1} |dy| < \infty \right. \tag{4.4}
\]
for any $\beta_0 \in \left(\frac{3}{2}, 2\right)$, as well as:

$$V(\eta) = -V\left(\eta + \frac{\lambda - 1}{2}i\right) \Phi\left(\eta + \frac{\lambda - 1}{2}i\right)$$  \hspace{1cm} (4.5)

for $\operatorname{Im}(\eta) \in \left(\frac{3}{2}, 2\right)$. Define $\widehat{G}(t, \xi)$ by means of (4.3) for $\operatorname{Im}(\xi) > \beta_0$. Then $\widehat{G}$ can be extended analytically to $S$ and it solves (4.1), (4.2) for $\operatorname{Im}(\eta) \in \left(\frac{3}{2}, 2\right)$.

**Proof.** It just follows by direct computation. Indeed, notice that Stirling’s formula, that is uniformly valid for $\Gamma(z)$, $\operatorname{arg}(z) \in (-\pi + \varepsilon_0, \pi - \varepsilon_0)$ with $\varepsilon_0 > 0$ (cf. [1]) implies:

$$\left|\Gamma\left(-\frac{2i(\xi - y)}{\lambda - 1}\right)\right| \leq C_R e^{-\frac{\pi}{2}y} \frac{|y|}{|y| + 1}$$

for $|\xi| \leq R, \operatorname{Im}(y) = \beta_0$. Therefore, the integral on the right-hand side of (4.3) converges for any $\xi \in S \cap \{\xi : \operatorname{Im}(\xi) \in (\beta_0, \frac{3+\lambda}{2})\}$ due to (4.4) and the function $\widehat{G}(t, \xi)$ satisfies:

$$\left|\widehat{G}(t, \xi)\right| \leq C_R \int_{\operatorname{Im}(y) = \beta_0} \left|\frac{1}{V(y)}\right| e^{-\frac{\pi}{2}y} \frac{dy}{|y| + 1} < \infty, \ |\xi| \leq R$$

Taking $\beta_0$ arbitrarily close to $\frac{3}{2}$ we obtain analyticity of $\widehat{G}$ in $S$.

Moreover, the derivative with respect to $t$ of $\widehat{G}$ in (4.3) can be computed by means of:

$$\frac{\partial \widehat{G}}{\partial t}(t, \xi) = \frac{\sqrt{2}}{\sqrt{\pi}i(\lambda - 1)} \int_{\operatorname{Im}(y) = \beta_0} \frac{V(\xi)}{V(y)} t^{\frac{2i(\xi - y)}{\lambda - 1} - 1} \Gamma\left(-\frac{2i(\xi - y)}{\lambda - 1} + 1\right) dy$$  \hspace{1cm} (4.6)

where we have used $z\Gamma(z) = \Gamma(z + 1)$. On the other hand, using (4.3) we obtain:

$$\widehat{G}\left(t, \xi + \frac{(\lambda - 1)}{2}i\right) = -\frac{\sqrt{2}}{\sqrt{\pi}i(\lambda - 1)} \int_{\operatorname{Im}(y) = \beta_0} \frac{V(\xi + \frac{(\lambda - 1)}{2}i)}{V(y)} t^{\frac{2i(\xi - y)}{\lambda - 1} - 1} \Gamma\left(-\frac{2i(\xi - y)}{\lambda - 1} + 1\right) dy$$

and using (4.5) and (4.6), (4.2) follows.

It only remains to check (4.2). To this end we use contour deformation and residue Theorem to transform (4.3) into:

$$\widehat{G}(t, \xi) = \frac{1}{\sqrt{2\pi}} - \widehat{G}_r(t, \xi)$$

$$\widehat{G}_r(t, \xi) = \frac{\sqrt{2}}{\sqrt{\pi}i(\lambda - 1)} \int_{\operatorname{Im}(y) = \beta_1} \frac{V(\xi)}{V(y)} t^{\frac{2i(\xi - y)}{\lambda - 1} - 1} \Gamma\left(-\frac{2i(\xi - y)}{\lambda - 1} + 1\right) dy$$

where $\beta_1 > \operatorname{Im}(\xi)$. Using (4.4) it follows that:

$$\left|\widehat{G}_r(t, \xi)\right| \leq C_R t^{\frac{2(\beta_1 - \operatorname{Im}(\xi))}{\lambda - 1}}, \ \xi \in S \cap \{||\xi|| \leq R\}$$

and therefore $\widehat{G}_r$ converges to zero uniformly in bounded sets of $\xi$, whence (4.2) follows. \hfill \Box

**Remark 4.2** A heuristic explanation for the formula (4.3) can be given using Laplace transform. Suppose that we define the Laplace transform of $\widehat{G}(t, \xi)$ in $t$ as:

$$\widehat{G}(z, \xi) = \int_0^\infty \widehat{G}(t, \xi) e^{-zt} dt$$

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Then, (4.1), (4.2) becomes:

\[ z \tilde{G}(z, \xi) = \tilde{G}\left(t, \xi + \frac{\lambda - 1}{2} i \right) \Phi\left(\xi + \frac{\lambda - 1}{2} i \right) + \frac{1}{\sqrt{2\pi}} \]  

(4.7)

The solution of this equation can be formally reduced to (4.5) by means of the transformation:

\[ \tilde{G}(z, \xi) = \exp\left(-\frac{2i}{\lambda - 1} \log(-z) \xi \right) \Phi(\xi) H(z, \xi) \] 

(4.8)

The reason for using \( \log(-z) \) instead of \( \log(z) \) is that \( \tilde{G}(\cdot, \xi) \) can be expected to be analytic for \( \text{Re}(z) > a \) for some \( a \in \mathbb{R} \) and in this form the function \( \log(-z) \) can be expected to be analytic in this region assuming that \( \text{arg}(z) \in (-\pi, \pi) \). The transformation (4.8) brings (4.7) to:

\[ H(z, \xi) - H\left(z, \xi + \frac{\lambda - 1}{2} i \right) = \frac{1}{\sqrt{2\pi}} e^{2i/\lambda - 1 \log(-z) \xi} \frac{\zeta^{\alpha(z)}}{zV(\xi)} \] 

(4.9)

Equation (4.9) can be solved using conformal mapping and Cauchy’s formula. Using the change of variables:

\[ H(z, \xi) = h(z, \zeta), \quad \zeta = e^{\frac{4\pi}{\lambda - 1} (\xi - \beta_0)} \] 

(4.10)

and where, for the sake of simplicity we will write, with some slight abuse of notation

\[ V(\xi) = V(\zeta) \]

we obtain:

\[ h(z, \zeta + i0) - h(z, \zeta - i0) = \frac{e^{\frac{4\pi}{\lambda - 1} \beta_0 \alpha(z)}}{\sqrt{2\pi}} \frac{\zeta^{\alpha(z)}}{zV(\zeta)} \zeta, \quad \zeta \in \mathbb{R}^+ \]  

(4.11)

with \( h \) analytic in \( \mathbb{C} \setminus \mathbb{R}^+ \) and:

\[ \alpha(z) = \frac{1}{2\pi i} \log(-z) \]

The solution of (4.11) can be obtained, assuming that \( \frac{\zeta^{\alpha(z)}}{V(\zeta)} \) satisfies suitable boundedness estimates for small and large \( \zeta \), and using Cauchy’s formula:

\[ h(z, \zeta) = \frac{1}{2\pi i} \frac{1}{\sqrt{2\pi}} \frac{e^{\frac{4\pi}{\lambda - 1} \beta_0 \alpha(z)}}{z} \int_0^\infty \frac{s^{\alpha(z)}}{V(s) (s - \zeta)} ds \]

and, using (4.10):

\[ H(z, \xi) = \frac{1}{2\pi i} \frac{1}{\sqrt{2\pi}} \frac{1}{z} \int_0^\infty \frac{e^{\frac{4\pi}{\lambda - 1} \beta_0 \alpha(z)}}{V(y) \frac{dy}{1 - e^{\frac{4\pi}{\lambda - 1} (\xi - y)}}} \] 

(4.12)

It then follows from (4.8) that:

\[ \tilde{G}(z, \xi) = \frac{1}{2\pi i} \sqrt{2\pi} \frac{1}{z} V(\xi) \int_0^\infty \frac{e^{\frac{4\pi}{\lambda - 1} (y - \xi)}}{V(y) \frac{dy}{1 - e^{\frac{4\pi}{\lambda - 1} (\xi - y)}}} \]

and inverting the Laplace transform we finally obtain (4.3).
5 On the solutions of (4.5).

Equation (4.5) admits infinitely many solutions. Indeed, given any solution $V_{\text{part}}(\xi)$ we can obtain any other one by means of:

$$V(\xi) = V_{\text{part}}(\xi) P(\xi)$$

where:

$$P(\xi) = P\left(\xi + \frac{\lambda - 1}{2} i\right)$$

(5.1)

Given such a non-uniqueness a natural and essential question is then how to chose one of them. We may state several sufficient conditions that would ensure that $\hat{G}$ is the Fourier transform of a tempered distribution. First we want the function $\hat{G}$ to be defined. This is guaranteed by the condition (4.4) above. However, this condition is not sufficient to prove that $\hat{G}(t,\xi)$ is globally bounded with respect to $\xi$. The difficulty comes from the fact that, if the behaviours of $V(\xi)$ are too disparate as $\text{Re}(\xi)$ tends to plus or minus infinity, the quotient $V(\xi)/V(\xi + \lambda - 1/2 i)$ may be strongly increasing in some regions of the integral in (4.3). A sufficient condition to avoid this difficulty is to have:

$$|V(\xi)| \approx e^{B_{\pm}|\xi|}, \quad |B_{\pm}| \leq \frac{\pi}{(\lambda - 1)}$$

(5.2)

as $\text{Re}(\xi) \to \pm\infty$. The decay rate of the Gamma function in (4.3) may then control the possible growth of the quotient $V(\xi)/V(\xi + \lambda - 1/2 i)$ uniformly on $\xi$.

Another requirement that we need for the function $V$ comes from the requirement that $\hat{G}$ must be analytic in the strip $\mathcal{S}$. This is ensured by imposing also that $V$ is also analytic in that same strip. We will then construct a function $V$ analytic in that strip, satisfying equation (4.5) for $\text{Im}(\xi) \in (3/2, 2)$, satisfying conditions (4.4) and (5.2).

A particular solution of (4.5) can be easily obtained using Cauchy’s formula. To this end we take the logarithm of both sides of (4.5) to obtain:

$$\log(V(\xi)) = \log\left(V\left(\xi + \frac{\lambda - 1}{2} i\right)\right) + \log\left(-\Phi\left(\xi + \frac{\lambda - 1}{2} i\right)\right)$$

(5.3)

or equivalently

$$\log\left(V\left(\xi - \frac{\lambda - 1}{2} i\right)\right) = \log\left(V(\xi)\right) + \log\left(-\Phi(\xi)\right).$$

(5.4)

Let us take any $\beta_1$ such that $\Phi(\xi)$ has no zeros nor poles along the line $\text{Im}(\xi) = \beta_1$. We define:

$$\psi(\zeta) = \log\left(V(\xi)\right) \quad \zeta = e^{\frac{4\pi}{\lambda - 1}(\xi - \beta_1 i)}, \quad Q(\zeta) = \log\left(-\Phi(\xi)\right)$$

Equation (5.4) then becomes

$$\psi(\zeta + i0) = \psi(\zeta - i0) + Q(\zeta - i0), \quad \zeta \in \mathbb{R}^+$$

(5.5)

with $\psi$ analytic in $\mathbb{C} \setminus \mathbb{R}^+$. Taking into account that $|Q(\zeta)| \leq C(1 + |\log(\zeta)|)$ we can obtain a particular solution of (5.5) as:

$$\psi(\zeta) = \frac{1}{2\pi i} \int_{\mathbb{R}^+} Q(s) \left[\frac{1}{s - \zeta} - \frac{1}{s + 1}\right] ds$$
where the term $1/(s + 1)$ has been added to the classical Cauchy integral in order to ensure the convergence of the integral. Then, returning to the variable $\xi$ we obtain:

$$V_{\text{part,} \beta_1} (\xi) = \exp \left( \frac{2}{(\lambda - 1)i} \int_{Im(\eta) = \beta_1} \log (-\Phi (\eta)) \left[ \frac{1}{1 - e^{\frac{4\pi}{\lambda - 1}(\xi - \eta)}} - \frac{1}{1 + e^{\frac{4\pi}{\lambda - 1}(\xi - \eta)}} \right] d\eta \right) \quad (5.6)$$

where $Im (\xi) \in (\beta_1 - \frac{\lambda - 1}{2}, \beta_1)$.

Formula $(5.6)$ provides a particular solution of $(4.5)$. On the other hand, we can obtain an infinite family of solutions of $(5.1)$ given by:

$$P (\xi) = e^{\frac{4\pi}{\lambda - 1} \ell \xi}, \quad \ell \in \mathbb{Z} \quad (5.7)$$

Let us define a family of solutions of $(4.5)$:

$$V (\xi) = e^{\frac{4\pi}{\lambda - 1} \xi} V_{\text{part,} \beta_1} (\xi) \quad (5.8)$$

Actually, using Fourier series, it can be seen that any solution of $(5.1)$ can be written as an infinite linear combination of the functions $V_{\ell} (\xi)$.

The formula $(5.6)$ does not define uniquely the function $V_{\text{part,} \beta_1}$ unless we prescribe the value of $\beta_1$ and the argument of the function $\ln(-\Phi(\eta))$. The different possible choices of this argument just differ by a factor $2\pi \ell i$ and therefore the resulting functions $V_{\text{part,} \beta_1}$ would differ by a multiplicative factor $(5.7)$. Proposition 3.1 implies that $\arg(-\Phi(\eta)) \to \pi/4 + 2\pi \ell i$ as $Re(\eta) \to +\infty$. In order to avoid exponential factors in some of the forthcoming formulas, we determine uniquely the function $\ln(-\Phi(\eta))$ by choosing:

$$\lim_{Re(\eta) \to +\infty} \arg(-\Phi(\eta)) = \frac{\pi}{4} \quad (5.9)$$

Notice that in the formula $(5.6)$ there exists an infinite possibility of choices of the constant $\beta_1$. These functions may be extended analytically moving $\xi$ and simultaneously the contour of integration in such a way that the condition $-(\lambda - 1)/2 < Im(\xi - \eta) < 0$ always holds. The only true obstruction to extend analytically the functions $V_{\text{part,} \beta_1}$ arises from crossing with the contour deformation the zeros or poles of the function $\Phi$. Suppose that $\xi_{\text{sing}}$ is a zero or a pole of $\Phi$ and $\beta_1$, $\beta_2$ are such that:

$$\xi_{\text{sing}} - \frac{1}{2} < \beta_1 < \xi_{\text{sing}} < \beta_2 < \xi_{\text{sing}} + \frac{1}{2}$$

Then:

$$\frac{V_{\text{part,} \beta_2} (\xi)}{V_{\text{part,} \beta_1} (\xi)} = - \left( \frac{e^{\frac{4\pi}{\lambda - 1} (\xi_{\text{sing}} - \xi)}}{1 + e^{\frac{4\pi}{\lambda - 1} \xi_{\text{sing}}}} \right)^{-n} \quad (5.10)$$

where:

$$n = \begin{cases} 
1 & \text{if } \xi_{\text{sing}} \text{ is a zero} \\
-1 & \text{if } \xi_{\text{sing}} \text{ is a pole}. 
\end{cases} \quad (5.11)$$

Combining $(5.10)$ with $(5.6)$ we can then extend any function $V_{\text{part,} \beta}$ to the whole complex plane as a meromorphic function. As it could be expected the different functions $V_{\text{part,} \beta}$ can be related to each other by means of linear combinations of functions of the form given in $(5.8)$.
In order to obtain the function \( V(\xi) \) with the properties requested above, it is sufficient to take
\[
V(\xi) = V_{\text{part}, \beta_1}(\xi), \quad \text{Im}(\xi) \in \left( \beta_1 - \frac{\lambda - 1}{2}, \beta_1 \right),
\]
with
\[
\beta_1 \in \left( \frac{2 + \lambda}{2}, \frac{3 + \lambda}{2} \right) \tag{5.13}
\]
Moving the contour of integration if needed, inside the strip \( \text{Im}(\eta) \in \left( 2 + \frac{\lambda}{2}, 3 + \frac{\lambda}{2} \right) \) we obtain
that the function \( V(\xi) \) has no zeros nor poles in the whole strip \( S \). The function \( V \) is then defined once and for all along the remainder of the paper. It only remains to check that this function satisfies the two conditions (4.4) and (5.2). It follows from Proposition 11.2 in Appendix 11 that:
\[
C_{\delta} e^{-\frac{1}{2} \left( \frac{\pi}{\lambda - 1} + \delta \right) |\xi|} \leq |V(\xi)| \leq C_{\delta} e^{-\frac{1}{2} \left( \frac{\pi}{\lambda - 1} - \delta \right) |\xi|} \quad \text{for} \quad \text{Im}(\xi) \in \left( \frac{3}{2}, \frac{3 + \lambda}{2} \right) \tag{5.14}
\]
for \( \delta > 0 \) arbitrarily small and \( C_{\delta} > 0 \) a constant depending on \( \delta \). This behaviour implies both (4.4) and (5.2).

Summarizing, we have shown:

**Proposition 5.1** The function \( V(\xi) \) defined by means of (5.6), (5.12) with \( \beta_1 \) as in (5.13) can be extended analytically to the strip \( S \) and meromorphically to the whole complex plane. It satisfies the equation (4.5) as well as the estimates (5.14). Moreover, \( V(\xi) \neq 0 \) in all the strip \( S \).

**Proposition 5.2** The function \( \hat{G}(t, \xi) \) defined by means of (4.3) with \( V \) defined in Proposition 5.1, solves (4.1) (4.2). Moreover we have the following representation formula:
\[
\frac{\mathcal{V}(\xi)}{\mathcal{V}(y)} = \exp \left[ \frac{2}{(\lambda - 1)i} \int_{\text{Im}(\eta) = \beta_1} \ln (-\Phi(\eta)) \times \left( \frac{1}{1 - e^{\frac{2\pi}{\lambda - 1}(\xi - \eta)}} - \frac{1}{1 - e^{\frac{2\pi}{\lambda - 1}(y - \eta)}} \right) d\eta \right] \tag{5.15}
\]
for \( \xi \) and \( y \) such that \( \beta_1 - (\lambda - 1)/2 < \text{Im}(\xi) < \beta_1 \) and \( \beta_1 - (\lambda - 1)/2 < \text{Im}(y) < \beta_1 \).

### 6 Decay estimates for the function \( \hat{G}(t, \xi) \).

Through all the following of this paper we shall use the following function \( A(z) \) that we define by means of:
\[
\Gamma(z) = \sqrt{2\pi} e^{-z - 1/2} A(z). \tag{6.1}
\]
Notice that by the Stirling’s formula:
\[
A(z) \to 1 \tag{6.2}
\]
uniformly as \( |z| \to \infty \) and \( \arg(z) \in (-\pi + \varepsilon_0, \pi - \varepsilon_0) \) for any \( \varepsilon_0 > 0 \) small.
6.1 The case $0 < t < 1$.

**Lemma 6.1** The function $\hat{G}$ defined in (4.3) satisfies the following.

\begin{align}
(1 + |\xi|^{1/2}) \left| \frac{\partial}{\partial \xi} \hat{G}(t, \xi) \right| &+ \left( 1 + |\xi|^{3/2} \right) \left| \frac{\partial^2}{\partial \xi^2} \hat{G}(t, \xi) \right| \leq \kappa t e^{-a \sqrt{|\xi| t}} \tag{6.3}
\end{align}

\begin{align}
|\hat{G}(t, \xi)| &\leq \kappa e^{-a \sqrt{|\xi| t}} \tag{6.4}
\end{align}

for all $t \in (0, 1)$ and for all $\xi \in \{ \xi \in \mathbb{C}; \text{Im} \xi \in (3/2, (3 + \lambda)/2) \} \cup \{ \xi \in \mathbb{C}; \text{Im} \xi \in [-L, L], |\xi| > 2L \}$ for any $L > 0$ sufficiently large and where $\kappa_1$ and $a$ are positive numerical numbers depending on $L$.

The proof of this Lemma requires the following result:

**Lemma 6.2** For any fixed constant $B > 0$ consider a function

\begin{align}
W(t, \xi) = \int_{\text{Im}(Y) = -\gamma_1} m(\xi, Y) e^{\Psi(\xi, Y, t)} dY \tag{6.5}
\end{align}

where the function $m(\xi, Y)$ is analytic with respect to $Y$ in the domain:

\[
\left| \text{Re} \left( Y + \frac{1}{8} \text{sign}(\text{Re}(\xi)) \xi \right) \right| \leq B|\text{Im}(Y)|, \text{sign}(\text{Re}(Y)) = \text{sign}(\text{Re}(\xi))
\]

and in the strip:

\[
\text{Im}(Y) \in \left[ -\gamma_1, \gamma_1 + \frac{\lambda - 1}{2} \right].
\]

Then, for any $L > 0$ sufficiently large, there exists $\xi_0 > 0$ sufficiently large, and depending on $L$, such that:

- If

\[
|m(\xi, Y)| \leq C \tag{6.6}
\]

then,

\[
|W(t, \xi)| \leq C e^{-a|\xi|^{1/2} t} \tag{6.7}
\]

for all $t \in [0, 1]$ and all $\xi$ such that $|\text{Re}(\xi)| \geq \xi_0$ and $\{ \xi \in \mathbb{C}; \text{Im}(\xi) \in (3/2, (3 + \lambda)/2) \} \cup \{ \xi \in \mathbb{C}; \text{Im} \xi \in [-L, L], |\xi| > 2L \}$.

- If

\[
|m(\xi, Y)| \leq C(1 + |Y|) \text{ and } m(\xi, 0) = 0 \tag{6.8}
\]

then,

\[
|W(t, \xi)| \leq C t |\xi|^{1/2} e^{-a|\xi|^{1/2} t} \tag{6.9}
\]

for all $t \in [0, 1]$ and all $\xi$ such that $|\text{Re}(\xi)| \geq \xi_0$ and $\{ \xi \in \mathbb{C}; \text{Im}(\xi) \in (3/2, (3 + \lambda)/2) \} \cup \{ \xi \in \mathbb{C}; \text{Im} \xi \in [-L, L], |\xi| > 2L \}$.
Proof of Lemma 6.2. It is convenient to introduce the new variable $Z$ as: $Y = \sqrt{|\xi|}Z$. Then the function $\Psi$ becomes:

$$\Psi(\xi, Y, t) = \Phi(\xi, Z, t) = 2\left(\lambda - 1\right) i \int_{Im\eta = \beta_0 + \frac{1}{2} - \varepsilon} \ln (-\Phi(\eta)) \Theta(\eta - \xi, \sqrt{|\xi|}Z) d\eta -$$

$$-\sqrt{|\xi|} \left(\frac{2iZ}{\lambda - 1} \ln(t) + \frac{2iZ}{\lambda - 1} - \left(\frac{2iZ}{\lambda - 1} - \frac{1}{2\sqrt{|\xi|}}\right) \ln \left(\frac{2iZ}{\lambda - 1}\right) -$$

$$- \left(\frac{2iZ}{\lambda - 1} - \frac{1}{2\sqrt{|\xi|}}\right) \ln |\xi|^{1/2}\right)$$

(6.10)

The possibility of extending the function $\Phi(\xi, Z, t)$ as a function of $Z$ to some suitable cones has been studied in Lemma 12.1. Moreover in Lemma 12.3 the existence of a critical point $Z_c$ of the function $\Phi(\xi, Z, t)$ with the asymptotics (12.12) has been proved.

Using the analyticity properties of the functions $\Phi(\xi, Z, t)$ and $A(z)$ as well as the change of variables $Y = \sqrt{|\xi|}Z$ we can obtain a new representation formula of $\hat{G}$

$$W(t, \xi) = \sqrt{|\xi|} \int_{C_t} e^{\Phi(\xi, Z, t)} m(\xi, \sqrt{|\xi|}Z) dZ$$

(6.11)

with the new integration contour $C_t$ given in Figure 2. Notice that we have moved the portion of the contour $ImZ = (\beta_0 - I\xi)|\xi|^{-1/2}$ where $|Re(Z)|$ is large to the line $Im(Z) = \gamma_1$ since in that deformation we do not cross any singularity of the integrand.

We now consider separately the two different cases $|\xi|^2 t \to +\infty$ and $|\xi|^2 t$ bounded.

The estimate of $W(t, \xi)$ as $|\xi|^2 t \to +\infty$. We write the function $W(t, \xi)$ as follows:

$$W(t, \xi) = I_1 + I_2$$

(6.12)

$$I_1 = \sqrt{|\xi|} \int_{C_t \cap \{Z: |Z - Z_0| \leq \delta_0 t\}} e^{\Phi(\xi, Z, t)} m(\xi, \sqrt{|\xi|}Z) dZ$$

(6.13)

$$I_2 = \sqrt{|\xi|} \int_{C_t \cap \{Z: |Z - Z_0| \geq \delta_0 t\}} e^{\Phi(\xi, Z, t)} m(\xi, \sqrt{|\xi|}Z) dZ$$

(6.14)
for some positive constant $\delta_0$ to be fixed.

We estimate first the integral $I_1$ using Lemmas 12.1–12.5 and Taylor’s expansion we obtain first:

$$
\Phi(\xi, Z, t) = \Phi(\xi, Z_c, t) + \frac{(Z - Z_c)^2}{2} \frac{\partial^2 \Phi}{\partial Z^2}(\xi, Z_c, t) + O\left(\frac{1}{\ell^2}(Z - Z_c)^3\right)
$$

(6.15)

for $|Z - Z_c| \leq \varepsilon_0 t$, and then

$$
\Phi(\xi, Z, t) = \Phi(\xi, Z_c, t) - \frac{1}{2} \frac{\sqrt{|\xi|}}{2\sqrt{2\pi} t (1 + i Q)} (1 + \delta(\xi, Z, t)) |Z - Z_c|^2,
$$

(6.16)

where $|\delta(\xi, Z, t)|$ can be made arbitrarily small if $|\xi|^2$ is large and $\varepsilon_0$ sufficiently small. Therefore:

$$
I_1 = \sqrt{|\xi|} e^{\Phi(\xi, Z, t)} \int_{C_t \cap \{Z : |Z - Z_c| \geq \delta_0 t\}} e^{\Phi(\xi, Z, t)} m(\xi, \sqrt{|\xi|} Z) dZ
$$

which gives, in the case (6.6):

$$
|I_1(t, \xi)| \leq C e^{-a\sqrt{|\xi|} t} \quad \text{as } |\xi|^2 \to +\infty, \quad 0 < t < 1,
$$

(6.17)

and in the case (6.8):

$$
|I_1(t, \xi)| \leq C t \sqrt{|\xi|} e^{-a\sqrt{|\xi|} t} \quad \text{as } |\xi|^2 \to +\infty, \quad 0 < t < 1.
$$

(6.18)

We must now estimate the integral $I_2$ given by (6.14). We estimate first the integral $I_2$ using Lemmas 12.1–12.5 and Taylor’s expansion we obtain first:

$$
I_2 = \sqrt{|\xi|} \int_{C_t \cap \{Z : |Z - Z_c| \geq \delta_0 t\}} e^{\Phi(\xi, Z, t)} m(\xi, \sqrt{|\xi|} Z) dZ
$$

In order to estimate the integral $I_2$ we use Lemma 12.8. To this end we split the integral as follows:

$$
I_2 = I_{2,1} + I_{2,2} + I_{2,3}
$$

$$
I_{2,1} = \sqrt{|\xi|} \int_{C_t \cap \{Z : |Z - Z_c| \geq \delta_0 t\} \cap \gamma(M, t)} e^{\Phi(\xi, Z, t)} m(\xi, \sqrt{|\xi|} Z) dZ
$$

$$
I_{2,2} = \sqrt{|\xi|} \int_{C_t \cap \{Z : |Z| \leq \varepsilon_1 \sqrt{|\xi|}\}} e^{\Phi(\xi, Z, t)} m(\xi, \sqrt{|\xi|} Z) dZ.
$$

$$
I_{2,3} = \sqrt{|\xi|} \int_{C_t \cap \{Z : |Z| \geq \varepsilon_1 \sqrt{|\xi|}\}} e^{\Phi(\xi, Z, t)} m(\xi, \sqrt{|\xi|} Z) dZ.
$$

where $\gamma(M, t)$ is the portion of the curve $C_t$ along which $\text{Im}(Z) < \gamma_1 t$. Using Lemma 12.8 it follows that, in the case (6.6):

$$
|I_{2,1}| \leq C e^{\Phi(\xi, Z, t)} e^{-\varepsilon_0 \sqrt{|\xi|} t} \sqrt{|\xi|} \int_{C_t \cap \{Z : |Z - Z_c| \geq \delta_0 t\} \cap \gamma(M, t)} m(\xi, \sqrt{|\xi|} Z) dZ
$$

$$
\leq C e^{\Phi(\xi, Z, t)} e^{-\varepsilon_0 \sqrt{|\xi|} t} \sqrt{|\xi|} t \leq C C e^{\Phi(\xi, Z, t)} e^{-\varepsilon_0 / 2 \sqrt{|\xi|} t},
$$

(6.19)

and in the case (6.8):

$$
|I_{2,1}| \leq C e^{\Phi(\xi, Z, t)} e^{-\varepsilon_0 \sqrt{|\xi|} t} (\sqrt{|\xi|} t)^2 \leq C (\sqrt{|\xi|} t) e^{\Phi(\xi, Z, t)} e^{-\varepsilon_0 / 2 \sqrt{|\xi|} t}.
$$

(6.20)
The estimate of $\mathcal{I}_{2,2}$ follows using Lemma 12.9. In the case (6.6) we obtain:

$$|\mathcal{I}_{2,2}| \leq C \sqrt{|\xi|} \int_{|Z| \geq M} e^{-a \sqrt{|\xi||Z|}} dZ = C e^{-aM \sqrt{|\xi|} t}. \quad (6.21)$$

and in case (6.8) we have:

$$|\mathcal{I}_{2,2}| \leq C \sqrt{|\xi|} \int_{|Z| \geq M} e^{-a \sqrt{|\xi||Z|}} \sqrt{|\xi|} Z dZ \leq C (\sqrt{|\xi|} t) e^{-aM \sqrt{|\xi|} t}, \quad (6.22)$$

since $|\xi|^2 t$ is large.

The third integral $\mathcal{I}_{2,3}$ is estimated using Proposition 11.2. To this end we use the variable $Y$ and the identity:

$$e^{\psi(\xi,Z,t)} = t^{-\frac{2iY}{\lambda-1}} \frac{V(\xi)}{V(Y + \xi)} \frac{\Gamma \left( \frac{2iY}{\lambda-1} \right)}{A \left( \frac{2iY}{\lambda-1} \right)}$$

where $A$ is defined by formula (6.1). This reduces the estimate of $\mathcal{I}_{2,3}$ to the estimate of the integral:

$$J = \int_{\text{Im}(Y) = \gamma_{1}, |Y| \geq \varepsilon_{1}|\xi|} \left| \frac{m(\xi,Y)}{A \left( \frac{2iY}{\lambda-1} \right)} \right| \frac{V(\xi)}{V(Y + \xi)} \left| t^{-\frac{2iY}{\lambda-1}} \right| \Gamma \left( \frac{2iY}{\lambda-1} \right) dY. \quad (6.23)$$

Using that $\gamma_{1} > 0$, $0 \leq t \leq 1$ and Stirling’s formula, it follows that, in both cases (6.6) and (6.8):

$$J \leq \int_{\text{Im}(Y) = \gamma_{1}, |Y| \geq \varepsilon_{1}|\xi|} (1 + |Y|) \left| \frac{V(\xi)}{V(Y + \xi)} \right| e^{-\frac{2iY}{\lambda-1} |Y|} dY.$$

Proposition 11.2 gives the following bounds

$$J \leq C \int_{\text{Im}(Y) = \gamma_{1}, |Y| \geq \varepsilon_{1}|\xi|} (1 + |Y|) e^{\delta \pi |\xi|} e^{-\pi \frac{|Y|}{\lambda-1}} dY \leq C e^{-\pi \frac{|\xi|}{\lambda-1}} \delta \pi |\xi|.$$

with $\varepsilon_{0} > 0$ if $\delta$ is sufficiently small. If we are in case (6.8), we estimate:

$$e^{-\varepsilon_{0}|\xi|} \leq e^{-\frac{\varepsilon_{0}}{2} |\xi|} e^{-\frac{\varepsilon_{0}}{2} |\xi|} \leq e^{-\frac{\varepsilon_{0}}{2} \sqrt{|\xi|} t} e^{-\frac{\varepsilon_{0}}{2} \sqrt{|\xi|} t}$$

which is estimated by the right hand side of (6.9).

This ends the proof of the estimate of $\mathcal{I}_{2}$ and then of the Lemma 6.1 in the domain where $|\xi| t^2$ large and $0 < t < 1$.

**The estimate of $W(t, \xi)$ for $|\xi| t^2$ bounded.** We suppose first that condition (6.6) holds. We split the integral in $\int_{6.32}$ in two pieces:

$$W(t, \xi) = J_{1} + J_{2} \quad (6.25)$$

$$J_{1}(t, \xi) = \int_{\text{Im} Y = -\gamma_{1}, |Y| \leq \varepsilon_{0}|\xi|} \frac{m(\xi,Y)}{A \left( \frac{2iY}{\lambda-1} \right)} \frac{V(\xi)}{V(Y + \xi)} t^{-\frac{2iY}{\lambda-1}} \Gamma \left( \frac{2iY}{\lambda-1} \right) dY.$$

$$J_{2}(t, \xi) = \frac{\sqrt{2}}{\sqrt{\pi i} (\lambda - 1)} \int_{\text{Im} Y = -\gamma_{1}, |Y| \geq \varepsilon_{0}|\xi|} \frac{m(\xi,Y)}{A \left( \frac{2iY}{\lambda-1} \right)} \frac{V(\xi)}{V(Y + \xi)} t^{-\frac{2iY}{\lambda-1}} \Gamma \left( \frac{2iY}{\lambda-1} \right) dY.$$

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The integral $J_2$ is estimated with the same argument used to bound the integral $J$ in (6.23).

$$|J_2| \leq C e^{-\varepsilon_0 |\xi|} \quad (6.26)$$

We rewrite $J_1$ as:

$$J_1(t, \xi) = \sqrt{|\xi|} t \int_{Im(\zeta) = -\frac{\pi}{\sqrt{|\xi|}} |\zeta| \leq \varepsilon_0 \frac{\sqrt{|\xi|}}{t}} e^{\Phi(\xi, \zeta, t)} m(\xi, \sqrt{|\xi|} \zeta t) d\zeta$$

where

$$\Phi(\xi, \zeta, t) = -\sqrt{|\xi|} t \frac{2i\zeta}{\lambda - 1} \left[ 1 - \ln \left( \frac{2i\zeta}{\lambda - 1} \right) + \ln \left( 2\sqrt{\pi} e^{iQ \zeta} \right) \right] - \frac{1}{2} \ln \left( t |\xi|^{1/2} \right) - \frac{1}{2} \ln \left( \frac{2i\zeta}{\lambda - 1} \right) + h(\xi, \zeta, t). \quad (6.27)$$

Notice that Lemma 12.2 implies

$$|h(\xi, \zeta, t)| \leq C \left( |\zeta|^2 t^2 + O \left( \frac{1}{|\xi|} \right) \right)$$

for $t |\xi| \leq \varepsilon_0 \sqrt{|\xi|}$, $\zeta \in D(\xi, B)$, $|\xi| \geq \varepsilon_0$ and for some positive constants $C = C(B)$ and $\varepsilon_0 = \xi_0(B)$ sufficiently large. Notice that

$$Re(\Phi(\xi, \zeta, t)) \leq \sqrt{|\xi|} t \left( C - \frac{\pi}{\lambda - 1} |\xi| \right) - \frac{1}{2} \ln \left( t |\xi|^{1/2} \right) + C \varepsilon_0 \sqrt{|\xi|} t |\xi|$$

for some positive constant $C$, for $\zeta$ such that $Im(\zeta) = -\frac{\pi}{\sqrt{|\xi|}}$ and $|\zeta| \leq \varepsilon_0 \sqrt{|\xi|}$, assuming that $\varepsilon_0$ is small. Using that $t \sqrt{|\xi|}$ is bounded it follows that, in the case (6.6):

$$|J_1(t, \xi)| \leq C \left( \sqrt{|\xi|} t \right)^{1/2} \int_{Im(\zeta) = -\frac{\pi}{\sqrt{|\xi|}} |\zeta| \leq \varepsilon_0 \frac{\sqrt{|\xi|}}{t}} e^{-\frac{\pi}{\lambda - 1} \sqrt{|\xi|} t |\zeta|} \frac{d|\zeta|}{\sqrt{|\zeta|}},$$

and computing the integral:

$$|J_1(t, \xi)| \leq C.$$

Combining this estimate with (6.26), estimate (6.7) of the Lemma follows.

In the case (6.8) we deform the contour integration in formula (6.5) using the analyticity properties of the function $m(\xi, Y)$ as well as the fact that $m(\xi, 0) = 0$ to obtain:

$$W(t, \xi) = -\frac{\pi (\lambda - 1)}{A(-1)} \int \frac{m(\xi, \lambda - 1 \frac{1}{2} i \frac{\lambda - 1}{2} i + \xi) t}{\lambda - 1}$$

$$+ \int_{Im(Y) = \gamma_1 + \frac{\lambda - 1}{2}} m(\xi, Y) e^{\Phi(\xi, Y, t)} dY$$

$$= \frac{\pi (\lambda - 1)}{A(-1)} \int m(\xi, \lambda - 1 \frac{1}{2} i \Phi \left( \frac{\lambda - 1}{2} i + \xi \right) t$$

$$+ \int_{Im(Y) = \gamma_1 + \frac{\lambda - 1}{2}} m(\xi, Y) e^{\Phi(\xi, Y, t)} dY = K_1 + K_2. \quad (6.28)$$
using \((4.5)\). By \((6.8)\), \(m(\xi, \frac{\lambda-1}{2} i)\) is uniformly bounded. Using then Proposition 3.1 we obtain that:

\[
|K_1| \leq C(1 + |\xi|^{1/2})t.
\]  

(6.29)

We are then left with the integral \(K_2\) which is formally very similar to \((6.5)\) although the integral contour is different. We split this integral in two terms \(J_1\) and \(J_2\) like in \((6.25)\). The term \(J_2\) is bounded as \((6.26)\) by a similar argument as before with one small difference which that the term \(m(\xi, Y)\) is now bounded as \((1 + |Y|)\). The term \(J_1\) can be written as:

\[
J_1(t, \xi) = \int_{\im Y = \gamma_1 + \frac{\lambda-1}{2}, |Y| \leq \epsilon|\xi|} m(\xi, Y) \frac{V(\xi)}{A \left( \frac{2iY}{\lambda-1} \right)} V(Y + \xi) t^{-2iY} \Gamma \left( \frac{2iY}{\lambda-1} \right) dY.
\]

(6.25)

Using Lemma 12.1, we may estimate \(|\frac{V(\xi)}{V(Y + \xi)}|\) by \(Ce^{\left(\frac{\pi}{2\lambda-1} + \delta_0\right)|Y|}\) in the domain of integration. On the other hand, since \(\im Y = \gamma_1 + \frac{\lambda-1}{2}\) we may estimate \(t^{-2iY}\) by \(t^{1 + \frac{2\gamma_1}{\lambda-1}}\). Therefore, using the decay of the Gamma function:

\[
|J_1(t, \xi)| \leq C t^{1 + \frac{2\gamma_1}{\lambda-1}} \int_{\im Y = \gamma_1 + \frac{\lambda-1}{2}} e^{\left(\frac{\pi}{2\lambda-1} + \delta_0\right)|Y|} e^{-\frac{\pi}{\lambda-1}|Y|} d|Y| 
\]

(6.30)

Combining \((6.29)\) and \((6.30)\) we deduce the estimate \((6.9)\) when \(|\xi| t^2\) is bounded. This concludes the proof of the Lemma 6.2.

\[\square\]

Proof of Lemma 6.1. Using the change of variables: \(y - \xi = Y\) in \((4.3)\) we obtain

\[
\hat{G}(t, \xi) = \frac{\sqrt{2}}{\sqrt{\pi i} (\lambda - 1)} \int_{\im Y = -\gamma_1} V(\xi) V(Y + \xi) t^{-2iY} \Gamma \left( \frac{2iY}{\lambda-1} \right) dY.
\]

(6.32)

where \(\gamma_1\) is a positive constant sufficiently small. We rewrite the function \(\hat{G}(t, \xi)\) as follows.

\[
\hat{G}(t, \xi) = \frac{\sqrt{2}}{\sqrt{\pi i} (\lambda - 1)} \int_{\im Y = -\gamma_1} e^{\Psi(\xi, Y, t)} A \left( \frac{2iY}{\lambda-1} \right) dY
\]

(6.33)

where

\[
\Psi(\xi, Y, t) = \frac{2}{(\lambda - 1) i} \int_{\im \eta = \beta_1} \ln(-\Phi(\eta)) \Theta(\eta - \xi, Y) d\eta - \frac{2iY}{\lambda - 1} \ln(t) - \frac{2iY}{\lambda - 1} + \left( \frac{2iY}{\lambda - 1} - \frac{1}{2} \right) \ln \left( \frac{2iY}{\lambda - 1} \right)
\]

(6.34)

\[
\Theta(\sigma, Y) = \frac{1}{1 - e^{-\frac{4\pi}{\lambda-1}}} - \frac{1}{1 - e^{-\frac{4\pi}{\lambda-1}} Y}.
\]

(6.35)

In order to obtain estimates \((6.3)\) and \((6.4)\) for bounded values of \(\xi\) we use contour deformation. In particular, crossing the pole at \(Y = 0\) in integral \((6.33)\) and using residue's Theorem we obtain:

\[
\hat{G}(t, \xi) = \frac{1}{\sqrt{2\pi}} + \hat{G}_1(t, \xi)
\]

(6.36)

\[
\hat{G}_1(t, \xi) = -\frac{\sqrt{2}}{\sqrt{\pi i} (\lambda - 1)} \int_{\im Y = \gamma_1} V(\xi) V(Y + \xi) t^{-2iY} \Gamma \left( \frac{2iY}{\lambda-1} \right) dY.
\]

(6.37)
with $\gamma_1 > 0$ sufficiently small. Using Proposition \[11.2\] it follows that

$$|\hat{G}_1(t, \xi)| \leq Ct^{\frac{2\gamma_1}{\lambda - 1}}$$

uniformly for $\xi$ in bounded sets. This yields estimate (6.3) for $\xi$ in bounded sets. If we differentiate in (6.37) with respect to $\xi$ we obtain:

$$\frac{\partial^\ell}{\partial \xi^\ell} \hat{G}(t, \xi) = -\sqrt{2} \left( \frac{\sqrt{\pi i}}{2} \right) m_\ell(\xi, Y) t^{-\frac{2\gamma_1}{\lambda - 1}} \Gamma \left( \frac{2iY}{\lambda - 1} \right) dY \quad (6.38)$$

$$m_\ell(\xi, Y) = \frac{\partial^\ell}{\partial \xi^\ell} \left( \frac{\psi(\xi)}{\psi(Y + \xi)} \right) \quad (6.39)$$

Using the analyticity properties of the functions $m_\ell(\xi, Y)$ we deform the integration contour in the integral (6.38). The first singularity that is met is the pole of function $\Gamma \left( \frac{2iY}{\lambda - 1} \right)$ located at $Y = \left( \frac{\lambda - 1}{i} \right)$.(This point is below the first zero of the function $V(\xi + Y)$ which is located at $Y = (2 + (\lambda/2) - \xi^2)$. (2 + (\lambda/2) - \xi > (\lambda - 1)/2.) We deduce

$$\frac{\partial^\ell}{\partial \xi^\ell} \hat{G}(t, \xi) = \sqrt{2} \left( \frac{\sqrt{\pi i}}{2} \right) m_\ell(\xi) t^{-\frac{2\gamma_1}{\lambda - 1}} \Gamma \left( \frac{2iY}{\lambda - 1} \right) dY \quad (6.40)$$

$$m_\ell(\xi) = \sqrt{2} \left( \frac{\sqrt{\pi i}}{2} \right) \left( \frac{\psi(\xi)}{\psi(Y + \xi)} \right) \quad (6.41)$$

for $\ell = 1, 2$. (Notice that $\frac{\partial m_\ell}{\partial \xi}(\xi, Y, t)$ and $\frac{\partial^2 m_\ell}{\partial \xi^2}(\xi, Y, t)$ are independent of $t$).

The function $m_\ell$ in (6.43) satisfies condition (6.6). On the other hand, using Lemma \[12.10\] it follows that for the choices (6.44), $\ell = 1, 2$: $|m(\xi, Y)| \leq C|Y|$. Moreover, by the definition of the function $\Psi$ (cf. (6.34))

$$\frac{\partial m_\ell}{\partial \xi}(\xi, Y) = \frac{\partial}{\partial \xi} \left( \ln \left( \frac{\psi(\xi)}{\psi(\xi + Y)} \right) \right)$$

and therefore $m(\xi, 0) = 0$ with the choice (6.44) and $\ell = 1$. This follows by a similar argument for the choice (6.44), $\ell = 2$. Applying then Lemma (6.2) the estimates (6.3) and (6.3) follow for $|\xi|$ sufficiently large and this concludes the proof of Lemma 6.1. \(\Box\)
6.2 The case \( t > 1 \).

**Lemma 6.3** The function \( \hat{G} \) defined in (4.3) satisfies that for any \( \varepsilon_0 > 0 \) arbitrarily small, there exists two positive constants \( \kappa_1 \) and a such that

\[
|\hat{G}(t, \xi)| \leq \kappa_1 t^{- \frac{1}{\lambda - 1} + \varepsilon_0} e^{-a\sqrt{|\xi|}}
\]  

(6.45)

for all \( \xi \) such that \( \text{Im}(\xi) \in (3/2, (3 + \lambda)/2) \) and all \( t > 1 \).

**Proof.** We deform the integration contour in the expression (6.32) to the line: \( \text{Im}(Y) = (\lambda - 1)/2 - \varepsilon_0 \) with \( \varepsilon_0 \) arbitrarily small in order to avoid the zero of the function \( V(\xi + Y) \) at \( \xi + Y = 1 \). Then we perform the change of variables \( Y = \sqrt{|\xi|} Z \), and perform a new deformation of the contour of integration in the \( Z \) variable to the curve \( D \) in Figure 3.

We argue now in the same way as in the previous case with \( t = 1 \) splitting the integral in the same pieces to obtain the estimate of the Lemma.

\( \square \)

7 Estimates on the function \( G(t, X) \).

We may now take the inverse Fourier transform of \( \hat{G}(t, \xi) \) to obtain the function:

\[
G(t, X) = \frac{1}{\sqrt{2\pi}} \int_{\text{Im}(\xi) = \beta_0} e^{iX\xi} \hat{G}(t, \xi) d\xi
\]  

(7.1)

where \( \beta_0 \) is in the interval \( (3/2, (3 + \lambda)/2) \).
Proposition 7.1 For all $t > 0$ the function $G(t, \cdot)$ defined by (7.1) belongs to $C^\infty(\mathbb{R})$ and, for $\ell = 0, 1, 2, \cdots$. Moreover, for any fixed $R > 0$, it satisfies:

$$\left| \frac{\partial^\ell G}{\partial X^\ell}(t, X) \right| \leq C_{t,R} \frac{t^{1+\ell}}{2^{2(1+\ell)}}, \quad \text{for } 0 < t \leq 1, \quad |X| \leq R. \tag{7.2}$$

and, for any $\varepsilon_0$ arbitrarily small and $R > 0$, there exists a positive constant $C_{\ell,\varepsilon_0,R}$ such that

$$\left| \frac{\partial^\ell G}{\partial X^\ell}(t, X) \right| \leq C_{\ell,\varepsilon_0,R} t^{1+\ell} - \varepsilon_0, \quad \text{for } t \geq 1, \quad |X| \leq R \tag{7.3}$$

Proof. This Proposition is just a consequence of (7.1) as well as from (6.3) for $t \in (0, 1)$ and (6.45) for $t > 1$.

The estimates (7.2) and (7.3) do not give any detailed description of the function $G$. We now proceed to obtain detailed descriptions of the function $G$ on the different regions in $X$ and $t$.

7.1 Behaviour as $t \to 0$, $0 \leq |X| \leq 1$.

We begin by showing that the function $G$ behaves like a mollifier of the Dirac measure when $t \to 0$ and for $X$ small. The asymptotic profile of the solution $G(t, X)$ can be heuristically guessed as follows. If we assume that the function $g$, solution of (2.1) with $g(0, x) \sim \delta(x - 1)$ is small, then equation (2.1) may be approximated, for $t$ and $x - 1$ small, as

$$\frac{\partial g}{\partial t} = \int_0^{1/2} \frac{g(x - y) - g(x)}{y^{3/2}} \, dy, \quad g(x, 0) = \delta(x - 1) \tag{7.4}$$

The equation in (7.4) describes the probability distribution for a system of particles whose size is initially equal to one and have a probability per unit of time $\chi_0 \leq y \leq 1/2 \, dy$ of coalescing with a particle of size in the interval $(y, y + dy)$. If we write the function $g$ as:

$$g(t, x) = \frac{1}{\sqrt{2\pi}} \int_\mathbb{R} \hat{g}(t, x) e^{ikx} \, dk$$

Applying Fourier transform we obtain:

$$\frac{\partial \hat{g}}{\partial t}(t, k) = m(k) \hat{g}(t, k), \quad \hat{g}(k, 0) = e^{-ik} \sqrt{2\pi}$$

$$m(k) = \int_0^{1/2} e^{-iky} \frac{1}{y^{3/2}} \, dy.$$ 

The multiplier $m(k)$ can be computed explicitly but its exact formula is not needed to compute the asymptotics of the function $g(t, x)$. The only relevant information that we really need is

$$m(k) \approx -2\sqrt{\pi k i}, \quad \text{for } |k| \to +\infty.$$ 

Then,

$$\hat{g}(t, k) \approx \frac{e^{-ik}}{\sqrt{2\pi}} e^{-2\sqrt{\pi k i} t}, \quad \text{for } |k| \to +\infty$$ 

and inverting the Fourier transform we obtain

$$g(t, x) \approx \frac{2}{t^2} \Psi \left( \frac{x - 1}{t^2} \right) \quad \text{as } t \to 0,$$
with
\[ \Psi(\chi) = \begin{cases} \frac{\pi}{\chi^{\frac{3}{2}}} e^{-\frac{\chi}{4}} & \text{for } \chi \geq 0, \\ 0 & \text{for } \chi < 0. \end{cases} \] (7.5)

Finally, since \( x = e^X, x - 1 = e^X - 1 \sim X \) when \( X \to 0 \) we obtain
\[ G(t, X) \approx \frac{2}{t^2} \Psi \left( \frac{X}{t^2} \right) \] as \( t \to 0. \) (7.6)

The fact that the support of \( G(t, X) \) is contained in \( \mathbb{R}^+ \) is a consequence of the interpretation of (7.4) in terms of coagulation of particles given above.

### 7.1.1 The region where \( X = O(t^2). \)

We now prove that the fundamental solution \( G(t, X) \) behaves in the self similar form (7.6) in the region where \( X = O(t^2) \) using the explicit representation formula given by (4.3), (5.15) and (7.1).

We are interested in the limit \( \lim_{t \to 0^+} t^2 G(t, t^2 \chi) = \Psi(\chi) \) in compact sets of \( \chi. \) To this end we rewrite (7.1) as follows:
\[ G(t, \chi) = \frac{1}{\sqrt{2\pi} t^2} \int_{\text{Im}(\eta) = \beta_0 t^2} e^{\eta \chi} \hat{G} \left( t, \frac{\eta}{t^2} \right) d\eta. \] (7.7)

Formula (7.7) suggests that in order to compute \( \Psi(\chi) \) we need to obtain \( \lim_{t \to 0^+} \hat{G} \left( t, \frac{\eta}{t^2} \right) \) for \( \eta \) such that \( \text{Im}(\eta) = \beta_0 t^2. \) To this end we use the expression (6.32) that implies:
\[ \hat{G}(t, \eta/t^2) = -\frac{\sqrt{2}}{\sqrt{\pi i} (\lambda - 1)} \int_{\text{Im}Y = -\gamma_1} \frac{V(\eta/t^2)}{V(Y + \eta/t^2)} t^{-\frac{3\lambda}{2\pi i} - \frac{\eta}{t^2}} \Gamma \left( \frac{2iY}{\lambda - 1} \right) dY. \] (7.8)

**Proposition 7.2**

\[ \lim_{t \to 0^+} \left( t^2 G(t, t^2 \chi) \right) = \psi(\chi) \]

uniformly for \( \chi \) in compact sets of \( \mathbb{R} \) where \( \psi(\chi) \) is as in (7.5).

The proof of Proposition 7.2 requires several Lemmas.

**Lemma 7.3** For all \( \varepsilon_0 > 0 \) and \( M > 0 \) there exists a function \( h_{\varepsilon_0, M}(t) \) such that
\[ \lim_{t \to 0^+} h_{\varepsilon_0, M}(t) = 0 \] (7.9)
and
\[ \left| \frac{V(\eta/t^2)}{V(Y + \eta/t^2)} t^{-\frac{2iY}{\lambda - 1}} - e^{-\frac{2iY}{\lambda - 1} \ln(2\sqrt{\pi}\sqrt{t\eta})} \right| \leq h_{\varepsilon_0, M}(t) \] (7.10)
for all \( Y \) such that \( |\text{Im}(Y)| \leq 1/4, |\text{Re}(Y)| \leq \frac{1}{\text{Im}(\eta)}, \) for \( \text{Im}(\eta/t^2) \) in compact subsets of \( (3/2, (3 + \lambda)/2) \) and \( \varepsilon_0 \leq |\eta| \leq M. \) Moreover there is \( \delta_0 > 0 \) sufficiently small (depending on \( \varepsilon_0 \) and \( M \)) such that:
\[ \left| \frac{V(\eta/t^2)}{V(Y + \eta/t^2)} t^{-\frac{2iY}{\lambda - 1}} \right| \leq C e^{\frac{\varepsilon_0}{\pi} \frac{|Y|}{\lambda - 1}} \] (7.11)
for all \( Y \) such that \( |\text{Im}(Y)| \leq 1/4, |\text{Re}(Y)| \leq \frac{\delta_0}{t^2}, \) for \( \text{Im}(\eta/t^2) \) in compact subsets of \( (3/2, (3 + \lambda)/2) \) and \( \varepsilon_0 \leq |\eta| \leq M. \)
Remark 7.4 In the Lemmas 7.3 until 7.6 and in Proposition 7.2 we choose the branch of the function square root as follows:
\[ \sqrt{z} = |z|^{1/2} e^{i\theta} \text{ with } \theta \in (-\pi, \pi]. \]

Proof. Lemma 7.3 is a consequence of Lemma 12.1 where \( \xi = \eta/t^2, Z = Y/\sqrt{\eta} \). By formula (5.15) we have:
\[
\frac{\mathcal{V}(\eta/t^2)}{\mathcal{V}(\eta/t^2 + Y)} = \exp \left[ \frac{2}{(\lambda - 1) i} \int_{\text{Im} Y = \beta_1} \ln (-\Phi(\eta)) \Theta(\eta - \eta/t^2, Y) d\eta \right]
\]
and using (12.4):
\[
\left| F \left( \eta/t^2, t Y/\sqrt{|\eta|} \right) + \frac{2 i Y}{\lambda - 1} \ln \left( 2 \sqrt{\pi} \sqrt{i \eta} \right) \right| \leq C t^2 |\eta|^{3/2}.
\]
Therefore, in the region where \( |\text{Re}(Y)| \leq 1/(t \ln t) \) we obtain (7.10). On the other hand, if \( |\text{Re}(Y)| \leq \delta_0/t^2 \) we deduce (7.11) using that:
\[
\left| \text{Re} \left( \frac{2 i Y}{\lambda - 1} \ln \left( 2 \sqrt{\pi} \sqrt{i \eta} \right) \right) \right| \leq \frac{\pi}{2(\lambda - 1)} |Y| + C
\]
as \( |\eta| \to +\infty \) and for \( \text{Im}(Y) = -\gamma_1 \).

Lemma 7.5 For all positive constants \( M, \varepsilon_0 \) such that \( M > \varepsilon_0 \):
\[
\lim_{t \to 0^+} \hat{G} \left( t, \frac{\eta}{t^2} \right) = -\frac{\sqrt{2}}{\sqrt{\pi} i (\lambda - 1)} \int_{\text{Im} Y = -\gamma_1, |\text{Re}(Y)| \leq \frac{1}{|\text{Im} t| \ln t}} \mathcal{V}(\eta/t^2) \mathcal{V}(Y + \eta/t^2) t^{-\frac{2iY}{\lambda - 1} \ln(2 \sqrt{\pi} \sqrt{i \eta})} \Gamma \left( \frac{2iY}{\lambda - 1} \right) dY,
\]
uniformly for \( \text{Im}(\eta/t^2) \) in compact subsets of \((3/2, (3 + \lambda)/2)\) and \( \varepsilon_0 \leq |\eta| \leq M \).

Proof. We split the integral in (7.8) as follows:
\[
\hat{G} \left( t, \frac{\eta}{t^2} \right) = \frac{\sqrt{2}}{\sqrt{\pi} i (\lambda - 1)} \int_{\text{Im} Y = -\gamma_1, |\text{Re}(Y)| \leq \frac{1}{|\text{Im} t| \ln t}} \mathcal{V}(\eta/t^2) \mathcal{V}(Y + \eta/t^2) t^{-\frac{2iY}{\lambda - 1} \ln(2 \sqrt{\pi} \sqrt{i \eta})} \Gamma \left( \frac{2iY}{\lambda - 1} \right) dY
\]
\[
- \frac{\sqrt{2}}{\sqrt{\pi} i (\lambda - 1)} \int_{\text{Im} Y = -\gamma_1, \frac{1}{|\text{Im} t| \ln t} \leq |\text{Re}(Y)| \leq \frac{\varepsilon_0}{|\text{Im} t| \ln t}} \mathcal{V}(\eta/t^2) \mathcal{V}(Y + \eta/t^2) t^{-\frac{2iY}{\lambda - 1} \ln(2 \sqrt{\pi} \sqrt{i \eta})} \Gamma \left( \frac{2iY}{\lambda - 1} \right) dY
\]
\[
+ \frac{\sqrt{2}}{\sqrt{\pi} i (\lambda - 1)} \int_{\text{Im} Y = -\gamma_1, |\text{Re}(Y)| \geq \frac{\varepsilon_0}{|\text{Im} t| \ln t}} \mathcal{V}(\eta/t^2) \mathcal{V}(Y + \eta/t^2) t^{-\frac{2iY}{\lambda - 1} \ln(2 \sqrt{\pi} \sqrt{i \eta})} \Gamma \left( \frac{2iY}{\lambda - 1} \right) dY
\]
\[
= J_1 + J_2 + J_3.
\]

\[
J_1 = -\frac{\sqrt{2}}{\sqrt{\pi} i (\lambda - 1)} \int_{\text{Im} Y = -\gamma_1, |\text{Re}(Y)| \leq \frac{1}{|\text{Im} t| \ln t}} \left( \mathcal{V}(\eta/t^2) \mathcal{V}(Y + \eta/t^2) t^{-\frac{2iY}{\lambda - 1} \ln(2 \sqrt{\pi} \sqrt{i \eta})} - e^{-\frac{2iY}{\lambda - 1} \ln(2 \sqrt{\pi} \sqrt{i \eta})} \right) \times
\]
\[
\times \Gamma \left( \frac{2iY}{\lambda - 1} \right) dY + \frac{\sqrt{2}}{\sqrt{\pi} i (\lambda - 1)} \int_{\text{Im} Y = -\gamma_1, |\text{Re}(Y)| \geq \frac{1}{|\text{Im} t| \ln t}} e^{-\frac{2iY}{\lambda - 1} \ln(2 \sqrt{\pi} \sqrt{i \eta})} \Gamma \left( \frac{2iY}{\lambda - 1} \right) dY
\]
\[
- \frac{\sqrt{2}}{\sqrt{\pi} i (\lambda - 1)} \int_{\text{Im} Y = -\gamma_1, |\text{Re}(Y)| \geq \frac{1}{|\text{Im} t| \ln t}} e^{-\frac{2iY}{\lambda - 1} \ln(2 \sqrt{\pi} \sqrt{i \eta})} \Gamma \left( \frac{2iY}{\lambda - 1} \right) dY.
\]
Using that:
\[
\left| e^{-\frac{2iY}{\lambda-1} \ln(\sqrt{\pi t})} \right| \leq C e^{|\frac{\pi}{\lambda-1} Y|} \quad \text{and} \quad \left| \Gamma\left( \frac{2iY}{\lambda-1} \right) \right| \leq C e^{-|\frac{\pi}{\lambda-1} Y|}
\]
the last term in (7.12) is estimated as:
\[
\left| \frac{\sqrt{2}}{\sqrt{\pi i} (\lambda - 1)} \int_{Im(Y) = \gamma_1, |Re(Y)| \geq \frac{1}{2\lambda |ln t|}} e^{-\frac{2iY}{\lambda-1} \ln(2 \sqrt{\pi Y})} \Gamma\left( \frac{2iY}{\lambda-1} \right) \; dY \right| \leq C e^{-\frac{a}{|\tau| |ln t|}}
\]
for some positive constant \(a\). In the first term of (7.12), using (7.10) in Lemma 7.3 we are led to estimate:
\[
h(t) \int_{Im(Y) = -\gamma_1} e^{-\frac{a}{(\lambda-1)^{|Y|}}} |dY|
\]
which tends to zero as \(t \to 0^+\) by (7.9).

We now consider the integral \(J_3\). This term is estimated using the estimates of the function \(V\) proved in Proposition 11.2 and Stirling’s formula:
\[
|J_3| \leq C \delta t^{-\frac{2\gamma_1}{\lambda-1}} e^{-\frac{\lambda}{12} \left( \frac{\pi^2}{45} - 26 \delta \right)} \leq C \delta t^{-\frac{2\gamma_1}{\lambda-1}} e^{-\frac{\lambda}{t^2}}
\]
for some positive constant \(\delta\) choosing \(\delta\) sufficiently small.

The integral \(J_2\) is estimated using (7.11) in Lemma 7.3
\[
|J_2| \leq C \int_{Im(Y) = -\gamma_1, \frac{1}{2\lambda |ln t|} \leq |Re(Y)| \leq \frac{\delta_0}{\tau^2}} e^{\frac{3\pi}{2(\lambda-1)^{|Y|}} Y} e^{-\frac{a}{(\lambda-1)^{|Y|}}} |dY| \leq C e^{-\frac{a}{|\tau^2| |ln t|}}
\]
for some positive constant \(a\).

\[\square\]

**Lemma 7.6**
\[
-\frac{\sqrt{2}}{\sqrt{\pi i} (\lambda - 1)} \int_{Im(Y) = -\gamma_1} e^{-\frac{2iY}{\lambda-1} \ln(2 \sqrt{\pi Y})} \Gamma\left( \frac{2iY}{\lambda-1} \right) \; dY = \sqrt{2 \pi} e^{-2\sqrt{\pi Y} i}
\]

**Proof.** We first perform the change of variable: \(2iY / (\lambda - 1) = s:\)
\[
-\frac{\sqrt{2}}{\sqrt{\pi i} (\lambda - 1)} \int_{Im(Y) = -\gamma_1} e^{-\frac{2iY}{\lambda-1} \ln(2 \sqrt{\pi Y})} \Gamma\left( \frac{2iY}{\lambda-1} \right) \; dY = \frac{1}{\sqrt{2 \pi}} \int_{Re(s) = \gamma_1} e^{-s \ln(2 \sqrt{\pi Y})} \Gamma(s) \; ds.
\]
Then we deform the integration contour and reduce the integral to the sum of the residues of the function \(e^{-s \ln(2 \sqrt{\pi Y})}\) at the poles \(s = -n\) of the Gamma function with residues \((-1)^n n!\). \[\square\]

**Proof of Proposition 7.2** We split the integral in (7.7)
\[
t^2 G(t, \chi) = I_1 + I_2 + I_3
\]
where
\[
I_1 = \frac{1}{\sqrt{2 \pi}} \int_{Im(\eta) = \beta_0 t^2, |\eta| \leq \epsilon_0} e^{i\eta x} \tilde{G} \left(t, \frac{\eta}{t^2}\right) d\eta
\]
\[
I_2 = \frac{1}{\sqrt{2 \pi}} \int_{Im(\eta) = \beta_0 t^2, \epsilon_0 \leq |\eta| \leq M} e^{i\eta x} \tilde{G} \left(t, \frac{\eta}{t^2}\right) d\eta
\]
\[
I_3 = \frac{1}{\sqrt{2 \pi}} \int_{Im(\eta) = \beta_0 t^2, |\eta| \geq M} e^{i\eta x} \tilde{G} \left(t, \frac{\eta}{t^2}\right) d\eta.
\]
For any Proposition 7.7, we obtain an estimate in self similar form in the domain where $t, X > 0$, as well as (6.4) we deduce:

$$|I_1| + |I_3| \leq C \int_{Im(\eta) = \beta_0 t^2, |\eta| \leq \epsilon_0, |\eta| \geq M} e^{-a|\eta|} d\eta \leq C (\epsilon_0 + e^{-aM}). \tag{7.13}$$

The integral $I_2$ is estimated using the previous Lemmas:

$$I_2 = \frac{1}{\sqrt{2\pi}} \int_{Im(\eta) = \beta_0 t^2, \epsilon_0 \leq |\eta| \leq M} \frac{e^{i\pi X}}{e^{2\pi} - e^{-2\pi \eta^2}} d\eta + \int_{Im(\eta) = \beta_0 t^2, \epsilon_0 \leq |\eta| \leq M} \frac{e^{i\pi X} e^{-2\pi \eta^2}}{e^{2\pi} - e^{-2\pi \eta^2}} d\eta = I_{2,1} + I_{2,2}$$

The first term is bounded as

$$|I_{2,1}| \leq C \int_{Im(\eta) = \beta_0 t^2, \epsilon_0 \leq |\eta| \leq M} \left| \frac{\hat{G}(t, \eta)}{e^{2\pi} - e^{-2\pi \eta^2}} \right| d\eta$$

and tends to zero by Lemma 7.5 and Lemma 7.6. And, where

$$\lim_{t \to 0^+} I_{2,1} = \int_{Im(\eta) = 0, \epsilon_0 \leq |\eta| \leq M} e^{i\pi X} e^{-2\pi \eta^2} d\eta.$$

Now taking the limit as $\epsilon_0 \to 0$, $M \to +\infty$ and using (7.13):

$$\lim_{t \to 0^+} t^2 G(t, \chi) = \int_{Im(\eta) = 0} e^{i\pi X} e^{-2\pi \eta^2} d\eta. \tag{7.14}$$

The integral in the right hand side of (7.14) may be calculated explicitly using contour deformation. For $\chi < 0$ the contour is sent to the region where $Im(\eta) \to -\infty$ and the integral gives zero. When $\chi > 0$ the deformation is made to the upper half plane in such a way that it avoids the cut along the half line $\eta \in i\mathbb{R}^+$ and the integral is reduced to:

$$2 \int_0^\infty \sin \left(2\sqrt{\pi \lambda} \right) e^{-\chi \lambda} d\lambda = \frac{\pi}{\chi^{3/2}} e^{-\frac{\pi}{\chi}}.$$

\[\square\]

### 7.1.2 Estimates of $G(t, X)$ for $t^2 \leq X \leq 1$.

We obtain an estimate in self similar form in the domain where $t^2 \leq X \leq C$ that is required to show that $G(t, X)$ converges to the Dirac measure as $t \to 0$.

**Proposition 7.7** For any $\delta \in (0, 1/2)$, there exists a positive constant $C_\delta$ such that

$$|G(t, X)| \leq C \frac{t^{1-2\delta}}{|X|^{1-\delta}} \text{ for } t^2 \leq |X| \leq 1.$$

**Proof.** We integrate by parts twice in formula (7.1) and obtain:

$$G(t, X) = \frac{1}{\sqrt{2\pi X^2}} \int_{Im(\xi) = \beta_0} \left( e^{iX\xi} - 1 \right) \frac{\partial^2}{\partial \xi^2} \hat{G}(t, \xi) d\xi \tag{7.15}$$

Using that $|e^{iX\xi} - 1| \leq C_\delta |X|^{1/2+\delta} |\xi|^{1/2+\delta}$ for $\delta \in [0, 1/2]$, as well as (6.4) we deduce:

$$|G(t, X)| \leq \frac{C_\delta t}{X^2} \int_{Im(\xi) = \beta_0} \frac{t}{1 + |\xi|^{3/2}} |X|^{1/2+\delta} |\xi|^{1/2+\delta} e^{-t \sqrt{|\xi|}} |d\xi|$$

$$\leq \frac{C_\delta t}{|X|^{2-\delta}} \int_{Im(\xi) = \beta_0} e^{-t \sqrt{|\xi|}} |d\xi|$$

and the result follows. \[\square\]
7.2 Behaviour of $G(t, X)$ for $t \geq 1$.

The behaviour of the function $G$ as $t \to +\infty$ has a self similar structure. This is seen by writing the function $G(t, X)$, given in (4.3) and (7.1), in terms of the variable

$$
\theta = X + \frac{2}{\lambda - 1} \ln(t)
$$

(7.16)

$$
G(t, X) = \frac{i}{\pi(\lambda - 1)} \int_{\Im(\xi) = \beta_0} d\xi e^{\xi \theta} \int_{\Im(y) = \beta_1} dy \frac{V(\xi)}{V(y)} t^{-\frac{\lambda + 1}{\lambda - 1}} \Gamma \left(-\frac{2i}{\lambda - 1}(\xi - y)\right)
$$

(7.17)

with $3/2 < \beta_0 < \beta_1 < (3 + \lambda)/2$.

Moving the integration contour of $y$ downward in the expression of $G(t, X)$, the first singularity to be met in the integrand of (7.17) is $y = i$ which is a zero of $V(y)$ (cf. Proposition 11.3). That gives:

$$
G(t, X) = t^{\frac{\lambda + 1}{\lambda - 1}} \Psi_1(\theta) + \frac{i}{\pi(\lambda - 1)} \int_{\Im(\xi) = \beta_0} d\xi e^{\xi \theta} \int_{\Im(y) = \beta_2} dy \frac{V(\xi)}{V(y)} t^{-\frac{\lambda + 1}{\lambda - 1}} \Gamma \left(-\frac{2i}{\lambda - 1}(\xi - y)\right)
$$

(7.18)

$$
\Psi_1(\theta) = \frac{2}{\lambda - 1} \int_{\Im(\xi) = \beta_0} d\xi e^{\xi \theta} \frac{V(\xi)}{V(i)} \Gamma \left(-\frac{2i}{\lambda - 1}(\xi - i)\right)
$$

(7.19)

where now $\beta_2 \in ((3 - \lambda)/2, 1)$.

Notice that $\Psi'(i) \neq 0$ by the following reason. If we differentiate the formula (4.5) we obtain:

$$
\Psi'(i) = -\Psi'(\frac{\lambda + 1}{2} i) \Phi'(\frac{\lambda + 1}{2} i) - \Psi'(\frac{\lambda + 1}{2} i) \Phi'(\frac{\lambda + 1}{2} i)
$$

At $\xi = \frac{\lambda + 1}{2} i$ the function $\Psi$ is analytic and $\Phi$ has a zero. Therefore,

$$
\Psi'(i) = -\Psi'(\frac{\lambda + 1}{2} i) \Phi'(\frac{\lambda + 1}{2} i)
$$

By Proposition 11.3 $\Psi'(\frac{\lambda + 1}{2} i) \neq 0$. Moreover, by Proposition 10.17

$$
\Phi(\xi) \sim -2\pi i (\xi - \frac{\lambda + 1}{2} i) \quad \text{as} \quad |Re(\xi)| \to +\infty
$$

and therefore $\Phi'(\frac{\lambda + 1}{2} i) = -2\pi i$. We then obtain the expression for $\Psi_1$:

$$
\Psi_1(\theta) = \frac{1}{\pi(\lambda - 1) i \Psi(\frac{\lambda + 1}{2} i)} \int_{\Im(\xi) = \beta_0} d\xi e^{\xi \theta} \frac{V(\xi)}{\Psi(i)} \Gamma \left(-\frac{2i}{\lambda - 1}(i - \xi)\right)
$$

(7.20)

We study now the function $\Psi(\theta)$ and give its behaviour as $\theta \to \pm \infty$.

**Proposition 7.8** The following estimates hold:

$$
\Psi_1(\theta) = C_1 e^{-\frac{3}{2} \theta} + O \left(e^{-(\frac{\lambda + 1}{2} + \varepsilon) \theta}\right) \quad \text{as} \quad \theta \to -\infty,
$$

(7.21)

$$
\Psi_1(\theta) = C_2 e^{-\frac{\lambda + 1}{2} \theta} + O \left(e^{-(\lambda + 1 - \varepsilon) \theta}\right) \quad \text{as} \quad \theta \to +\infty,
$$

(7.22)

for some positive constant $\varepsilon$ arbitrarily small and where

$$
C_1 = \frac{2i}{(\lambda - 1) \Psi((\frac{\lambda + 1}{2}) i)} \quad \text{and} \quad C_2 = \frac{\Gamma\left((\frac{\lambda + 1}{\lambda - 1}) i\right)}{2\pi i \Psi((\frac{\lambda + 1}{2}) i)}.
$$
Proof of Proposition 7.8. We use again contour deformation. In order to obtain the behaviour as \( \theta \to -\infty \) we deform the contour integration in \( \Psi \) downward. The first singularity of the integrand that we meet is \( \xi = 3i/2 \) which is a pole of \( V(\xi) \). Using (4.5) and (10.17) we obtain:

\[
\text{Res} \left( V, \xi = \frac{3i}{2} \right) = -i \frac{V(1 + \frac{\lambda}{2})}{i} \]

and

\[
\text{Res} \left( V, \xi = \frac{(3 + \lambda)i}{2} \right) = -\frac{V(2i)}{4\pi i}.
\]

Therefore

\[
\Psi_1(\theta) = \frac{2i}{(\lambda - 1)} \frac{V((1 + \lambda/2)i)}{V((\lambda + 1)/2)i} e^{-\frac{2i}{\lambda - 1} \theta} + \frac{1}{\pi(\lambda - 1)i V((\lambda + 1)/2)i} \int_{Im(\xi) = \beta_3} d\xi e^{i\xi \theta} V(\xi) \Gamma \left( -\frac{2i}{\lambda - 1} (\xi - i) \right)
\]

and

\[
\Psi_1(\theta) = -\frac{\Gamma((\lambda + 1)/2)}{2\pi i} \frac{V(2i)}{V((\lambda + 1)/2)i} e^{-\frac{2i\lambda}{\lambda - 1} \theta} + \frac{1}{\pi(\lambda - 1)i V((\lambda + 1)/2)i} \int_{Im(\xi) = \beta_4} d\xi e^{i\xi \theta} V(\xi) \Gamma \left( -\frac{2i}{\lambda - 1} (\xi - i) \right)
\]

where \( \beta_3 \in ((4 - \lambda)/2, 3/2), \beta_4 \in ((3 + \lambda)/2, 1 + \lambda) \). We have derived (7.25), (7.26) deforming the contour of integration upward and downward respectively.

Proposition 11.2 ensures that the function \( V(\xi) \Gamma \left( -\frac{2i}{\lambda - 1} (\xi - i) \right) \) is integrable and then, for \( \text{Re} \theta \leq 0 \):

\[
\left| \int_{Im(\xi) = \beta_3} d\xi e^{i\xi \theta} V(\xi) \Gamma \left( -\frac{2i}{\lambda - 1} (\xi - i) \right) \right| \leq C e^{-\beta_3 \text{Re}(\theta)}.
\]

while for \( \theta \geq 0 \):

\[
\left| \int_{Im(\xi) = \beta_4} d\xi e^{i\xi \theta} V(\xi) \Gamma \left( -\frac{2i}{\lambda - 1} (\xi - i) \right) \right| \leq C e^{-\beta_4 \text{Re}(\theta)}.
\]

Proposition 7.8 follows from (7.25), (7.26), (7.27) and (7.28).

7.2.1 Estimate of the remainder term in formula (7.18).

Let us call that term:

\[
G_1(t, \theta) = \frac{i}{\pi(\lambda - 1)} \int_{Im(\xi) = \beta_0} d\xi e^{i\xi \theta} \int_{Im(\xi) = \beta_2} dy \frac{V(\xi)}{V(y)} \Gamma \left( -\frac{2i}{\lambda - 1} (\xi - y) \right)
\]

We have then the following Lemma.
Lemma 7.9 For any positive constant \( C_1 \) there exist positive constants \( A, \delta_0 \) and \( C_2 \) such that, for all \( t > 1 \):

\[
|G_1(t, \theta)| e^{3\vartheta} \leq C_2 t^{2/\lambda-\delta_0} \quad \text{for all } \theta \leq 0 \quad (7.30)
\]

\[
|G_1(t, \theta)| e^{3\lambda\vartheta} \leq C_2 t^{2/\lambda-\delta_0} \quad \text{for all } \theta \geq 0. \quad (7.31)
\]

Proof of Lemma 7.9. The function \( G_1 \) may be written as:

\[
G_1(t, \theta) = \frac{i}{\pi(\lambda - 1)} \int_{l} d\xi e^{i\xi \theta} H_1(t, \xi) \quad (7.32)
\]

\[
H_1(t, \xi) = \int_{l} dy \frac{V(\xi)}{V(y)} t^{\frac{2}{\lambda-1} \Gamma \left( \frac{2i}{\lambda-1} (\xi - y) \right)} \quad (7.33)
\]

The function \( H_1 \) is estimated in the same way as the function \( \tilde{G}(t, \xi) \) in Lemma 6.3. To this end we first perform the change of variables: \( y = \xi + \sqrt{|\xi|} Z \) and obtain:

\[
H_1(t, \xi) = \sqrt{|\xi|} \int_{l} dy \frac{V(\xi)}{V(\sqrt{|\xi|} Z + \xi)} t^{\frac{2i(\sqrt{|\xi|} Z + \xi)}{\lambda-1} \Gamma \left( \frac{2i}{\lambda-1} \sqrt{|\xi|} Z \right)} \quad (7.34)
\]

Then we deform the contour of integration in (7.34) to the new contour \( D_1 \). Since along this new contour \( \text{Im}(Z) = \frac{\beta_2 - \beta_0}{\sqrt{|\xi|}} \), we have \( \text{Im}(\sqrt{|\xi|} Z + \xi) \leq \beta_2 \) and then

\[
|t^{-\frac{2i\vartheta}{\lambda-1}}| \leq C t^{\frac{2\beta_2}{\lambda-1}} = t^{\frac{2}{\lambda-1} - \delta}.
\]

We are then left with the term

\[
\sqrt{|\xi|} \int_{D_1} \left| \frac{V(\xi)}{V(\sqrt{|\xi|} Z + \xi)} t^{\frac{2i(\sqrt{|\xi|} Z + \xi)}{\lambda-1} \Gamma \left( \frac{2i}{\lambda-1} \sqrt{|\xi|} Z \right)} \right| dy
\]
Then we deform the integration contour in (7.41) to

$$|H_1(t, \xi)| \leq C t^{\frac{2}{\lambda - 1} - \delta} e^{-a\sqrt{|\xi|}}. \quad (7.35)$$

Using (7.1) we deduce that, for any positive constant $R$ there exists a constant $C_R > 0$ such that, for all $|\theta| \leq R$:

$$|G_1(t, \theta)| \leq C_R t^{\frac{2}{\lambda - 1} - \delta}$$

for some positive constant $C_R$. This shows (7.30) and (7.31) for $\theta$ bounded.

In order to prove (7.30) and (7.31) we deform the contour of the integral with respect to $\xi$ in formula (7.29). To obtain the estimate (7.31) as $\theta \to +\infty$ we deform the contour upward. The first singularity of $\hat{G}_1(t, \xi)$ is located at $\xi = (3 + \lambda) i/2$ (see Proposition 11.3). Therefore:

$$G_1(t, \theta) = b_2(t) e^{-\frac{2 + \lambda}{\lambda - 1} \theta} + Q_1(t, \theta) \quad (7.36)$$

$$Q_1(t, \theta) = \frac{i}{\pi(\lambda - 1)} \int_{\text{Im}(\xi) = \beta_5} d\xi e^{i\xi \theta} \int_{\text{Im}(y) = \beta_1} dy \frac{V(\xi)}{V(y)} t^{-\frac{2 + \lambda}{\lambda - 1}} \Gamma \left(-\frac{2i}{\lambda - 1} (\xi - y)\right) \quad (7.37)$$

where $\beta_5 \in ((3 + \lambda)/2, (4 + \lambda)/2)$, and

$$b_2(t) = \frac{V(2i)}{2\pi i(\lambda - 1)} \int_{\text{Im}(y) = \beta_1} dy \frac{t^{-\frac{2 + \lambda}{\lambda - 1}}}{V(y)} \Gamma \left(-\frac{2i}{\lambda - 1} \left(\frac{3 + \lambda}{2} - i - y\right)\right).$$

Using that again $\beta_1 \in ((3 - \lambda)/2, 1)$ and so in particular $\beta_1 < 1$, we deduce:

$$|b_2(t)| \leq C t^{\frac{2}{\lambda - 1} - \delta_0}, \quad \text{for } t > 1. \quad (7.38)$$

We are then left with the term $Q_1(t, \theta)$ that we write:

$$Q_1(t, \theta) = \int_{\text{Im}(\xi) = \beta_5} d\xi e^{i\xi \theta} H_2(t, \xi) \quad (7.39)$$

$$H_2 = \frac{i}{\pi(\lambda - 1)} \int_{\text{Im}(y) = \beta_1} dy \frac{V(\xi)}{V(y)} t^{-\frac{2 + \lambda}{\lambda - 1}} \Gamma \left(-\frac{2i}{\lambda - 1} (\xi - y)\right). \quad (7.40)$$

The function $H_2$ is estimated in the same way as the function $\hat{G}(t, \xi)$ in Lemma 6.3. We change of variables: $y = \xi + \sqrt{|\xi|} Z$ and obtain:

$$H_2(t, \xi) = \sqrt{|\xi|} \int_{\text{Im}(Z)} = \frac{\beta_1 - \beta_0}{\sqrt{|\xi|}} dy \frac{V(\xi)}{V(\sqrt{|\xi|} Z + \xi)} t^{-\frac{2 + i(\sqrt{|\xi|} + \xi)}{\lambda - 1}} \Gamma \left(\frac{2i}{\lambda - 1} \sqrt{|\xi|} Z\right) \quad (7.41)$$

Then we deform the integration contour in (7.41) to $D_2$. Since along this new contour $\text{Im}(Z) = \frac{\beta_1 - \beta_0}{\sqrt{|\xi|}}$, we have $\text{Im}(\sqrt{|\xi|} Z + \xi) \leq \beta_1$ and then

$$|t^{-\frac{2 + i}{\lambda - 1}}| \leq Ct^{\frac{2\beta_1}{\lambda - 1} - \delta} = t^{\frac{2}{\lambda - 1} - \delta}. \quad (7.42)$$
We are then left with the term

\[ \sqrt{\xi} \int_{D_1} \left| \frac{\mathcal{V}(\xi)}{\mathcal{V}(\sqrt{\xi}|Z + \xi)} t^{-\frac{2i(\sqrt{\xi}|Z + \xi)}{\lambda - 1}} \Gamma \left( \frac{2i}{\lambda - 1} \sqrt{\xi} |Z \right) \right| |dy| \]

that may be estimated following the same arguments as in the proof of Lemma 6.3. We obtain then the bound on \( H_2(t, \xi) \):

\[ |H_2(t, \xi)| \leq Ct^{\frac{2}{\lambda - 1} - \delta} e^{-a \sqrt{\xi}} \] (7.42)

Using now (7.39) we deduce that:

\[ |Q_1(t, \theta)| \leq Ct^{\frac{2}{\lambda - 1} - \delta} e^{-\beta_0 \theta} \] (7.43)

Combining this with (7.38), estimate (7.31) follows.

The estimate (7.30) for \( \theta \to -\infty \) is obtained in a very similar way. We deform downward the contour of the integral (7.33) and we continue the proof as for (7.31). This concludes the proof of Lemma 7.9. \( \square \)

### 7.3 Behaviour as \( 0 \leq t \leq 1, |X| \to +\infty \)

For the small values of time the solution is described in the \((t, X)\) variables as follows.

**Lemma 7.10** For \( 0 \leq t \leq 1 \) the following estimates hold:

\[
G(t, X) = \begin{cases} 
  e^{-\frac{3}{2}X} t + O \left( e^{-\left(\frac{3}{2} - \epsilon\right)X} t \right) & \text{as } X \to -\infty \\
  C_1 e^{-\frac{3+\lambda}{2}X} t + O \left( e^{-\left(\frac{3+\lambda}{2} + \epsilon\right)X} t \right) & \text{as } X \to +\infty.
\end{cases}
\] (7.44)

where:

\[
C_1 = \frac{\mathcal{V}(2i)}{4\pi \mathcal{V}((1 + \frac{3}{2})i)}
\]
Proof of Lemma 7.10. Using (7.1) we obtain:

\[ G(t, X) = \frac{1}{\sqrt{2\pi}} \int_{\Im(\xi)=\beta_0} e^{i\xi X} \hat{G}(t, \xi) \, d\xi = \]

\[ = \frac{i}{\pi(\lambda - 1)} \int_{\Im(\xi)=\beta_0} e^{i\xi X} \int_{\Im(Y)=-\gamma_1} dy \frac{\nu(\xi)}{\nu(\xi + Y)} t^{-\frac{2iY}{X}} \Gamma \left( \frac{2iY}{\lambda - 1} \right) \]

where \( \beta_0 \in \left( \frac{3}{2}, \frac{3+\lambda}{2} \right) \) and \( \gamma_1 \) small. Lemma 6.1 implies that \( \hat{G}(t, \xi) \) is exponentially bounded on \( |\xi| \) for large \( |\xi| \) and this yields convergence of the integral. Moreover, such exponential decay holds also for \( \xi \in \left( \frac{3}{2} - \delta_0, \frac{3+\lambda}{2} + \delta_0 \right) \) with \( \delta_0 > 0 \).

We can now deform the contour integration on \( \xi \) crossing the poles of \( \hat{G}(t, \xi) \) that are due to the poles of \( \nu(\xi) \). The closest poles are at \( \xi = \frac{3}{2} i, \xi = \frac{3+\lambda}{2} i \) respectively. We deform the contour upwards if \( X > 0 \) and downwards if \( X < 0 \). We then obtain using (7.23), (7.24):

\[ G(t, X) = \]

\[ -\frac{2\pi}{\pi(\lambda - 1)} \nu \left( \left( 1 + \frac{\lambda}{2} \right) i \right) e^{-\frac{3}{2} X} \int_{\Im(Y)=-\gamma_1} dy \frac{t^{-\frac{2iY}{X}}}{\nu(\frac{3}{2} i + Y)} \Gamma \left( \frac{2iY}{\lambda - 1} \right) + \]

\[ + \frac{i}{\pi(\lambda - 1)} \int_{\Im(\xi)=\beta_1} e^{i\xi X} d\xi \int_{\Im(Y)=-\gamma_1} dy \frac{\nu(\xi)}{\nu(\xi + Y)} t^{-\frac{2iY}{X}} \Gamma \left( \frac{2iY}{\lambda - 1} \right) \equiv J_1 + J_2 \]

(7.45)

\[ G(t, X) = -\frac{\nu(2i)}{2\pi(\lambda - 1)} e^{-\frac{3i+\lambda}{2} X} \int_{\Im(Y)=-\gamma_1} dy \frac{t^{-\frac{2iY}{X}}}{\nu(\frac{3+\lambda}{2} i + Y)} \Gamma \left( \frac{2iY}{\lambda - 1} \right) + \]

\[ + \frac{i}{\pi(\lambda - 1)} \int_{\Im(\xi)=\beta_2} e^{i\xi X} d\xi \int_{\Im(Y)=-\gamma_1} dy \frac{\nu(\xi)}{\nu(\xi + Y)} t^{-\frac{2iY}{X}} \Gamma \left( \frac{2iY}{\lambda - 1} \right) \equiv J_1 + J_2 \]

(7.46)

where \( \beta_1 = \frac{3}{2} - \delta \), \( \beta_2 = \frac{3+\lambda}{2} + \delta \) with \( \delta > 0 \) small.

The terms \( J_2 \) in (7.45), (7.46) can be estimated easily. Indeed, they can be written in the form:

\[ G(t, X) = \frac{1}{\sqrt{2\pi}} \int_{\Im(\xi)=\beta_{\ell}} e^{i\xi X} \hat{G}(t, \xi) \, d\xi \quad, \quad \ell = 1, 2 \]

Integrating by parts we obtain:

\[ G(t, X) = -\frac{1}{\sqrt{2\pi X^2}} \int_{\Im(\xi)=\beta_{\ell}} d\xi e^{i\xi X} \frac{\partial^2}{\partial\xi^2} \left( \hat{G}(t, \xi) \right) \]

We can now estimate \( \frac{\partial^2}{\partial\xi^2} \left( \hat{G}(t, \xi) \right) \) using Lemma 6.1. It then follows that:

\[ |J_2| \leq \frac{C}{X^2} \int_{\Im(\xi)=\beta_{\ell}} \left| e^{i\xi X} \right| \frac{t}{\left( 1 + |\xi|^{3/2} \right)} e^{-a\sqrt{\xi}|t|} |d\xi| \leq \frac{Ce^{\beta_{\ell}X}}{X^2} \int_{\Im(\xi)=\beta_{\ell}} \frac{t}{\left( 1 + |\xi|^{3/2} \right)} e^{-a\sqrt{\xi}|t|} |d\xi| \]

In order to estimate the integral we split it as follows:

\[ \int_{\Im(\xi)=\beta_{\ell}} \frac{t}{\left( 1 + |\xi|^{3/2} \right)} e^{-a\sqrt{\xi}|t|} |d\xi| = \int_{\Im(\xi)=\beta_{\ell}, |\xi| \leq \frac{1}{\ell^2}} [...] |d\xi| + \int_{\Im(\xi)=\beta_{\ell}, |\xi| \geq \frac{1}{\ell^2}} [...] |d\xi| \leq Ct + Ct^2 \]
|J_2| \leq Cte^{\beta X} \quad \text{for} \quad |X| \geq 1

where \( \ell = 1 \) for \( X < 0 \) and \( \ell = 2 \) for \( X > 0 \). In order to compute the terms \( J_1 \) in (7.45), (7.46) we deform the contour on \( Y \) upwards. We cross the first pole of the function \( \Gamma \left( \frac{2iY}{\lambda - 1} \right) \) for \( Y = 0 \). However, this point is not a pole of the integrand, because the functions \( V(\frac{3}{2}i + Y) \), \( V(\frac{3 + \lambda}{2}i + Y) \) have also a pole at \( Y = 0 \). Notice that \( V(\frac{3}{2}i + Y) \) does not have a zero before, since \( \frac{3}{2} + \frac{\lambda - 1}{2} = 1 + \frac{\lambda}{2} < \frac{3}{2} + \frac{\lambda}{2} \). On the other hand \( \frac{3 + \lambda}{2} + \frac{\lambda - 1}{2} = 1 + \lambda \) while the first zero of \( V(\eta) \) is at \( \eta = 2 + \frac{\lambda}{2} \) and \( 1 + \lambda < 2 + \frac{\lambda}{2} \). Then, after deforming the integral contour as indicated, the terms \( J_1 \) can be written as:

\[
J_1 = e^{-\frac{3}{2}X} t - \frac{2i}{\lambda - 1} V \left( \left( 1 + \frac{\lambda}{2} \right) i \right) e^{-\frac{3}{2}X} \int_{Im(Y) = \gamma_2} dy \frac{t^{\frac{2iY}{\lambda - 1}}}{V(\frac{3}{2}i + Y)} \Gamma \left( \frac{2iY}{\lambda - 1} \right)
\]

and

\[
J_1 = e^{-\frac{3 + \lambda}{2}X} \frac{V(2i)}{4\pi V \left( \left( 1 + \frac{\lambda}{2} \right) i \right)} - \frac{i V(2i)}{2\pi \left( \lambda - 1 \right)} e^{-\frac{3 + \lambda}{2}X} \int_{Im(Y) = \gamma_2} dy \frac{t^{\frac{2iY}{\lambda - 1}}}{V(\frac{3 + \lambda}{2}i + Y)} \Gamma \left( \frac{2iY}{\lambda - 1} \right)
\]

where \( \gamma_2 > (\lambda - 1)/2 \) is such that \( \gamma_2 - (\lambda - 1)/2 \) is small. In both cases there exist positive constants \( \delta \) and \( C_\delta \) such that for all \( t > 1 \):

\[
|J_3| \leq C_\delta t^{1+\delta},
\]

\[
|J_1| \leq Cte^{\beta X} \quad \text{for} \quad |X| \geq 1
\]

where, as before, \( \ell = 1 \) for \( X < 0 \) and \( \ell = 2 \) for \( X > 0 \).

It then follows that:

\[
|J_1| + |J_2| \leq Cte^{\beta X} \quad \text{for} \quad |X| \geq 1
\]

where, as before, \( \ell = 1 \) for \( X < 0 \) and \( \ell = 2 \) for \( X > 0 \), and Lemma 7.10 follows.

## 8 The initial value problem.

Using the fundamental solution \( g \) obtained in Theorem 2.5 we can obtain a solution of the initial value problem

\[
\frac{\partial h}{\partial t} = L[h] \quad \text{for} \quad t > 0 \quad \text{and} \quad x \in \Omega
\]

\[
h(0, x) = h_0(x)
\]
with $L \cdot$ defined in (2.2). Assuming that there are not difficulties with the integrals written below, we would expect, due to the linearity of the problem (8.1), (8.2) the following representation formula for their solutions (cf. Theorem 2.5):

$$h(t, x) = \int_0^\infty h_0(y) g\left( ty^{\frac{\lambda - 1}{2}}, \frac{x}{y}, 1 \right) \frac{dy}{y}$$  \hspace{1cm} (8.3)

We first precise sufficient conditions on $h_0$ that allow to define $h(t, x)$ in (8.3).

**Theorem 8.1** Suppose that the function $h_0 \in C(\mathbb{R}^+)$ satisfies

$$\int_0^1 |h_0(y)| y^\lambda dy + \int_1^\infty |h_0(y)| dy < \infty$$  \hspace{1cm} (8.4)

Then the function $h(t, x)$ defined for $t \geq 0, x > 0$ by means of (8.3) solves the initial value problem (8.1), (8.2).

The proof of this Theorem reduces to a detailed analysis of the conditions on $h_0$ yielding integrability of the right hand side of (8.3).

Under more stringent assumptions on $h_0$ it is possible to use Theorem 2.5 to derive more detailed information on the asymptotics of the solutions of (8.1), (8.2) for $x \to 0$ and $x \to \infty$. The meaning of this asymptotics will be explained in the next Section.

**Theorem 8.2** Suppose that

$$|h_0(x)| \leq C x^{-\frac{3+\varepsilon}{2}}, \quad 0 < x \leq 1, \quad \varepsilon > 0$$

$$|h_0(x)| \leq C x^{-(1+\varepsilon)}, \quad x \geq 1, \quad \varepsilon > 0$$  \hspace{1cm} (8.5), (8.6)

Then the function $h(t, x)$ given in (8.3) satisfies for any $t > 0$

$$\left| h(t, x) - A_- (t) x^{-\frac{3}{2}} \right| \leq B_- (t) x^{-\frac{3+\varepsilon}{2}} \text{ for } 0 < x \leq 1$$

$$\left| h(t, x) - A_+ (t) x^{-\frac{3+\lambda}{2}} \right| \leq B_+ (t) x^{-\frac{3+\lambda}{2} - \varepsilon} \text{ for } x \geq 1$$  \hspace{1cm} (8.7), (8.8)

for suitable functions $A_- (t), A_+ (t), B_- (t), B_+ (t)$.

Detailed proofs of these two results will be given in [11].

9 Particle fluxes for singular solutions of the coagulation equation.

9.1 Computing particle fluxes for the nonlinear equation.

It is well known that smooth solutions of the coagulation equation (1.1), (1.2) preserve the value of the quantity $\int_0^\infty x f(x, t) dx$. This could be expected given that the equation can be interpreted as a description of the coalescence of particles with sizes $x, y$ to form a particle of size $(x + y)$ in the time interval $[t, t + dt]$ taking place with a probability $K(x, y) dt$ for each particle pair $(x, y)$.

Notice that, in spite of this conservation property, we cannot expect to be able to rewrite (1.1), (1.2) as a conservation law with the form

$$\frac{\partial}{\partial t} (xf) + \frac{\partial}{\partial x} (j) = 0$$
with \( j \) depending only on local properties of \( f \) at the point \((t, x)\) due to the fact that the particles can change their size an amount of order one in an infinitesimal interval of time \([t, t + dt]\). However, we can expect to have equivalent ways of finding formulas with the form:

\[
\frac{d}{dt} \left( \int_{R_1}^{R_2} x f(x, t) \, dx \right) = J_{R_1, R_2}^+ - J_{R_1, R_2}^-(9.9)
\]

for arbitrary values \( 0 \leq R_1 < R_2 \), where \( J_{R_1, R_2}^+ \) denotes the number of monomers that enter in the interval \( x \in [R_1, R_2] \) coming from particles with sizes in the region \( \mathbb{R}^+ \setminus [R_1, R_2] \) for unit of time and \( J_{R_1, R_2}^- \) is the number of monomers that leave the set \( x \in [R_1, R_2] \) for unit of time.

A careful counting of the number of particles entering and leaving the interval \([R_1, R_2]\) yields

\[
J_{R_1, R_2}^+ = \frac{1}{2} \int_{D_{R_1, R_2}^+} f(y) f(z) K(y, z) (y + z) \, dydz + \int_{D_{R_1, R_2}^+} f(y) f(z) K(y, z) y \, dydz
\]

\[
(9.10)
\]

\[
J_{R_1, R_2}^- = \frac{1}{2} \int_{D_{R_1, R_2}^-} f(y) f(z) K(y, z) (y + z) \, dydz + \int_{D_{R_1, R_2}^-} f(y) f(z) K(y, z) y \, dydz
\]

\[
(9.11)
\]

\[
D_{1; R_1, R_2}^+ = \{(y, z) : y \leq R_1, z \leq R_1, R_1 \leq (y + z) \leq R_2\}
\]

\[
D_{2; R_1, R_2}^+ = \{(y, z) : y \leq R_1, R_1 \leq z \leq R_2, R_1 \leq (y + z) \leq R_2\}
\]

\[
D_{3; R_1, R_2}^- = \{(y, z) : R_1 \leq y \leq R_2, R_1 \leq z \leq R_2, R_1 \leq (y + z) \leq R_2\}
\]

There are two particular cases of \((9.9)\) that will be particularly relevant for our purposes, namely the cases

\[
R_1 = 0 \ , \ R_2 = R \in (0, \infty)
\]

\[
(9.12)
\]

and

\[
R_1 = R \in (0, \infty) \ , \ R_2 = \infty
\]

\[
(9.13)
\]

In the case \((9.12)\), \((9.9)\) reduces to:

\[
\frac{d}{dt} \left( \int_{0}^{R} x f(x, t) \, dx \right) = - J_{R}^- [f]
\]

\[
(9.14)
\]

\[
J_{R}^- [f] = \frac{1}{2} \int_{D_{1; R_1, R_2}^-} f(y) f(z) K(y, z) (y + z) \, dydz + \int_{D_{2; R_1, R_2}^-} f(y) f(z) K(y, z) y \, dydz
\]

\[
(9.15)
\]

\[
D_{1} (R) = \{(y, z) : y \leq R, z \leq R, R \leq (y + z)\}
\]

\[
(9.16)
\]
In the case (9.13), equation (9.9) becomes:
\[
\frac{d}{dt} \left( \int_R^\infty x f(x,t) \, dx \right) = J^+_{R}[f]
\]  
(9.17)

\[
J^+_{R}[f] = \frac{1}{2} \int_{D^+_1(R)} f(y) f(z) K(y,z) (y+z) \, dydz + \int_{D^+_2(R)} f(y) f(z) K(y,z) y \, dydz
\]

\[
D^+_1(R) = \{(y,z) : y \leq R, \, z \leq R, \, R \leq (y+z)\}
\]

\[
D^+_2(R) = \{(y,z) : y \leq R, \, R \leq z, \, R \leq (y+z)\}
\]

Using the formula (9.14) we can easily prove that for the kernel \(K(y,z) = (yz)^{\frac{3}{2}}\) the meaning of the solutions of (1.1), (1.2) having the asymptotics \(f(x) \sim \frac{A}{x^{\frac{3}{2}}}\) is that there is a flux of particles escaping to infinity. Indeed, it is readily seen that for this kernel:
\[
\lim_{R \to \infty} J^+_{R}[f] = 2\pi A^2
\]

Notice that the function \(f(x) = \frac{A}{x^{\frac{3}{2}}}\) can be thought as a stationary solution of (1.1), (1.2) characterized by a constant flux of particles propagating from the smaller to the larger values of \(x\).

### 9.2 Particle fluxes for the linearized equation.

Since equation (2.1), (2.2) has been obtained linearizing (1.1), (1.2) we can derive formulas for the particle fluxes associated to (2.1), (2.2) linearizing (9.9). We recall that (2.1), (2.2) has been obtained using that:
\[
f(x,t) = x^{-\frac{3+4\lambda}{2}} + g(x,t)
\]  
(9.18)
in (1.1), (1.2) and keeping just linear terms on \(g\). Plugging (9.18) into (9.12) and keeping just the linear term on \(g\) we obtain:
\[
\frac{d}{dt} \left( \int_0^R x g(t,x) \, dx \right) = -J^-_{R,lin}
\]  
(9.19)

where \(J^-_{R,lin}\) is:
\[
J^-_{R,lin} = \int_{D^-_1(R)} \frac{g(z)}{y^{3/2}} (yz)^{\frac{3}{2}} (y+z) \, dydz + \int_{D^-_2(R)} \frac{g(y)}{y^{3/2}} (yz)^{\frac{3}{2}} (y+z) \, dydz
\]

\[
+ \int_{D^-_3(R)} \frac{g(z)}{y^{3/2}} (y+z)^{\frac{3}{2}} z \, dydz + \int_{D^-_4(R)} \frac{g(y)}{y^{3/2}} (y+z)^{\frac{3}{2}} z \, dydz
\]

\[
= \int_{D^-_1(R)} \frac{z^{\frac{3}{2}}}{y^{3/2}} (y+z)g(z) \, dydz + \int_{D^-_2(R)} \frac{z^{\frac{3}{2}+1}}{y^{3/2}} g(z) \, dydz + \int_{D^-_3(R)} \frac{y^{\frac{3}{2}}}{} g(y) \, dydz
\]

\[
I_1(t,R) + I_2(t,R) + I_3(t,R)
\]  
(9.20)

\[
D^-_2(R) = \{0 \leq z \leq R, \, 0 \leq y \leq R, \, R \leq (y+z)\}
\]  
(9.21)

and \(D^-_1(R), D^-_2(R)\) defined by (9.15) and (9.16) respectively. Integrating (9.19) we obtain:
\[
\int_0^R x g(0,x) \, dx = \int_0^t J^-_{R,lin}(s) \, ds + \int_0^R x g(t,x) \, dx
\]  
(9.22)
The solution \( g(t, x) \) obtained in Theorem 2.5 satisfies
\[
g(t, x) \sim a(t) x^{-(3+\lambda)/2} \quad \text{as} \quad x \to +\infty.
\] (9.23)

Taking the limit \( R \to \infty \) in (9.22) and using the fact that the terms \( I_1(t, R), I_3(t, R) \) tend to zero as \( R \to \infty \) and that \( \lim_{R \to \infty} I_2(t, R) = a(t) \), it follows that:
\[
\int_0^\infty x g(0, x) dx = \int_0^t a(s) ds + \int_0^\infty x g(t, x) dx.
\] (9.24)

The self-similar asymptotics (2.10)-(2.13) implies that \( \lim_{t \to \infty} \int_0^\infty x g(t, x) dx = 0 \). Then:
\[
\int_0^\infty x g(0, x) dx = \int_0^\infty a(s) ds.
\] (9.25)

The left hand side of (9.25) is the initial total mass of the perturbation. The right hand side of (9.25) is the total amount of particles contained in clusters of infinite size. Equation (9.25) means that all the excess of particles initially introduced in the system move as \( t \to +\infty \) to an infinitely large cluster.

10 Appendix I.

We take now the Mellin transform on both hands of the equation. We recall that the Mellin transform of a function \( g(y) \) is defined as:
\[
\mathcal{M}(h)(s) = \int_0^\infty y^{s-1} h(y) dy.
\] (10.1)

Taking the Mellin transform of the right hand side of (2.1), (2.2) we obtain after straightforward calculations:
\[
\frac{\partial}{\partial s} \mathcal{M}(g)(s) = \mathcal{M}(g) \left( s + \frac{\lambda - 1}{2} \right) \mathcal{M} \left( -s + \frac{3}{2} \right),
\] (10.2)
\[
M(s) = \int_2^{\infty} \theta^{1/2-s} \left( (\theta - 1)^{-3/2} - \theta^{-3/2} \right) d\theta +
\int_{1/2}^{1} (1 - \theta)^{-3/2} (\theta^{s-1} - 1) d\theta + \frac{2^{-s}}{s} - 2\sqrt{2}
\] (10.3)
\[
= I_1(s) + I_2(s) + I_3(s) + I_4.
\] (10.4)

Remark 10.1 If the function \( g \) satisfies the estimates (2.5), (2.6) for some \( r > 1 \) and \( \rho < 2 \) such that \( \rho < r \) we will have that its Mellin transform is well defined in the strip \( \text{Re}s \in (\lambda/2 + \rho, \lambda/2 + r) \).
10.1 The function \( M(s) \).

We consider in this Section the auxiliary function \( M \) obtained by taking the Mellin transform of the equation (2.1), (2.2) and first rewrite it in terms of Gamma functions.

**Proposition 10.2** The function \( M(s) \) defined in (10.3) can be written as

\[
M(s) = -\frac{2 \sqrt{2} \Gamma(s)}{\Gamma(s-1/2)}. \tag{10.5}
\]

**Proof.** We can write the term \( I_1 \) of (10.4) as

\[
I_1(s) = \int_0^\infty \theta^{1/2-s} \theta^{-3/2} \left\{ \left( 1 - \frac{1}{\theta} \right)^{-3/2} - 1 \right\} d\theta
\]

Using Binomial Theorem to expand \( \left( 1 - \frac{1}{\theta} \right)^{-3/2} \) and integrating each term of the resulting series we obtain:

\[
I_1 = 2^{-s} \sum_{n=1}^\infty \left( -\frac{3}{2} \right) \frac{2^{-n}}{n} \frac{(-1)^n}{s+n}.
\]

Then,

\[
I_1 + I_3 = 2^{-s} \sum_{n=0}^\infty \left( -\frac{3}{2} \right) \frac{2^{-n}}{n} \frac{(-1)^n}{s+n}.
\]

Integrating by parts we obtain:

\[
I_2(s) = 2\sqrt{2} - 2\sqrt{2}(1/2)^{s-1} - 2(s-1) \int_{1/2}^1 (1 - \theta)^{-1/2} \theta^{s-2} d\theta. \tag{10.6}
\]

In order to compute the last term in (10.7) we write:

\[
\int_{1/2}^1 (1 - \theta)^{-1/2} \theta^{s-2} d\theta = \int_0^1 (1 - \theta)^{-1/2} \theta^{s-2} d\theta - \int_0^{1/2} (1 - \theta)^{-1/2} \theta^{s-2} d\theta, \tag{10.8}
\]

Expanding \( (1 - \theta)^{-1/2} \) using the Binomial Theorem and integrating each term of the resulting series:

\[
\int_0^{1/2} (1 - \theta)^{-1/2} \theta^{s-2} d\theta = 2^{-s} \left\{ \frac{2}{s-1} + \sum_{\ell=0}^{\infty} \frac{(-1/2)^{\ell+1}}{\ell+s} 2^{-\ell} \right\}. \tag{10.9}
\]

Moreover,

\[
\frac{2}{s-1} + \sum_{n=0}^\infty \left( -\frac{1}{n+1} \right) \frac{(-1)^{n+1}}{n+s} 2^{-n} = \frac{2(1 - 1/2)^{-1/2}}{s-1} - \frac{1}{2(s-1)} \sum_{n=0}^\infty \left( -\frac{3}{2} \right) \frac{(-1)^n}{n+1} 2^{-n} \tag{10.10}
\]

where we have used that \( \left( -\frac{1}{n+1} \right) = -\frac{1}{2(n+1)} \left( -\frac{3}{2} \right) \) as well as the Binomial Theorem. Combining (10.9) and (10.10) we deduce

\[
\int_0^{1/2} (1 - \theta)^{-1/2} \theta^{s-2} d\theta = 2^{-s} \left\{ \frac{2\sqrt{2}}{s-1} - \frac{1}{2(s-1)} \sum_{n=0}^\infty \left( -\frac{3}{2} \right) \frac{(-1)^n}{n+1} 2^{-n} \right\}. \tag{10.11}
\]

Therefore, using (10.3), (10.6), (10.7), (10.8) and (10.11),

\[
M(s) = -2(s-1) \int_0^1 (1 - \theta)^{-1/2} \theta^{s-2} d\theta = -\frac{2 \sqrt{2} \Gamma(s)}{\Gamma(s-1/2)}. \tag{10.12}
\]

\[\square\]
10.2 Zeros and Poles of $M(s)$

The function $M(s)$ is meromorphic on the whole complex plane. Its zeros are given by the poles of the function $\Gamma(s - 1/2)$:

$$s_z(n) = \frac{1}{2} - n, \quad n = 0, 1, 2, \ldots$$  \hspace{1cm} (10.13)

The second zero, $M(-1/2) = 0$ corresponds to the solution $x^{-(3+\lambda)/2}$.

On the other hand, the poles of $M(s)$ are those of the function $\Gamma(s)$, i.e.

$$s_p(n) = -n, \quad n = 0, 1, 2, \ldots$$  \hspace{1cm} (10.19)

10.3 Fourier variables.

We make now a change of variables in order to have functions defined in all of the real line $\mathbb{R}$. To this end we define:

$$\mathcal{M}(g)(s) = \int_0^\infty g(y)y^{s-1}dy = \int_{-\infty}^\infty g(e^X)e^{-iX\xi}dX \equiv \sqrt{2\pi}\hat{G}(\xi)$$

i.e., $\mathcal{M}(g)(s) = \sqrt{2\pi}\hat{G}(\xi)$ with $s = -i\xi$, then,

$$\mathcal{M}(g) \left(s + \frac{\lambda - 1}{2}\right) = \sqrt{2\pi}\hat{G} \left(\xi + \frac{\lambda - 1}{2}i\right).$$  \hspace{1cm} (10.14)

If we re-write the equation (10.12) in these new variables we obtain,

$$\frac{\partial G}{\partial t}(\xi) = G \left(\xi + \frac{\lambda - 1}{2}i\right)M \left(i\xi + \frac{3}{2}\right).$$  \hspace{1cm} (10.15)

It is more convenient to use instead of $M$ the function $\Phi$ defined as:

$$\Phi(\xi) = M \left(i\xi + 1 + \frac{\lambda}{2}\right).$$  \hspace{1cm} (10.16)

Using (10.15), we obtain $\Phi$:

$$\Phi(\xi) = -\frac{2\sqrt{\pi} \Gamma(i\xi + 1 + \frac{\lambda}{2})}{\Gamma(i\xi + \frac{\lambda + 1}{2})}.\hspace{1cm} (10.17)$$

The equation (10.15) reads then as equation (2.20). The set of zeros of $\Phi$ are:

$$\xi_z(n) = i \left(n + \frac{\lambda + 1}{2}\right) \quad n = 0, 1, 2, \ldots$$  \hspace{1cm} (10.18)

The set of poles is

$$\xi_p(n) = i \left(n + 1 + \frac{\lambda}{2}\right) \quad n = 0, 1, 2, \ldots$$  \hspace{1cm} (10.19)

Notice that by definition,

$$M \left(\frac{1}{2}\right) = \Phi \left(\frac{3 + \lambda}{2}i\right) = 0, \quad M \left(\frac{1}{2}\right) = \Phi \left(\frac{1 + \lambda}{2}i\right) = 0.$$
10.4 Behaviour of $\Phi$ at infinity.

**Theorem 10.3** Let us define $\Theta \equiv \Theta(s) = \text{sgn}(\text{Im}(s))$. For any fixed $L > 0$, the following asymptotic formulas hold:

$$M(s) = -2\sqrt{\pi s} \left(1 - \frac{3}{8s} + O\left(\frac{1}{s^2}\right)\right) \quad \text{as} \quad |\text{Im}(s)| \to \infty$$  \hspace{1cm} (10.20)

$$M(s) = -\sqrt{2\pi} (1 + i\Theta) \sqrt{-is\Theta} \frac{3\sqrt{2\pi}(1 + i\Theta) \sqrt{-is\Theta}}{s} + O\left(\frac{1}{s^{3/2}}\right) \quad \text{as} \quad |\text{Im}(s)| \to \infty$$  \hspace{1cm} (10.21)

uniformly in sets where $\text{arg}(s) \in (-\pi + \varepsilon_0, \pi - \varepsilon_0)$ for any $\varepsilon_0 > 0$.

**Proof** Formula (10.20) is a consequence of (10.5) as well as the asymptotic formula:

$$\Gamma(z) \sim \sqrt{2\pi} (s^{\frac{1}{2}} - 1 + \frac{1}{2s} + O\left(\frac{1}{s^2}\right)) \quad \text{as} \quad |z| \to \infty$$  \hspace{1cm} (10.22)

that is uniformly valid in sets $\text{arg}(z) \in (-\pi + \varepsilon_0, \pi - \varepsilon_0)$ for any $\varepsilon_0 > 0$. Then:

$$M(s) = -\frac{2\sqrt{\pi s} e^{s^{1/2}} (s - \frac{1}{2})^{1/2}}{(s - \frac{1}{2})^{s^{1/2}} e^{s^{1/2}}} \left(1 + O\left(\frac{1}{s^2}\right)\right) \quad \text{as} \quad |s| \to \infty,$$

uniformly in sets $\text{arg}(s) \in (-\pi + \varepsilon_0, \pi - \varepsilon_0)$ and $\text{arg}(s - 1/2) \in (-\pi + \varepsilon_0, \pi - \varepsilon_0)$ for any $\varepsilon_0 > 0$. Notice that:

$$\left(\frac{s}{s - \frac{1}{2}}\right)^{s^{1/2}} = \sqrt{e} \left[1 - \frac{1}{8(s - \frac{1}{2})} + O\left(\frac{1}{s^2}\right)\right] \quad \text{as} \quad |s| \to \infty,$$

uniformly in the same sets as above, whence:

$$M(s) = -2\sqrt{\pi s} \left(1 - \frac{3}{8s} + O\left(\frac{1}{s^2}\right)\right) \quad \text{as} \quad |s| \to \infty,$$

uniformly in sets $\text{arg}(s) \in (-\pi + \varepsilon_0, \pi - \varepsilon_0)$ for any $\varepsilon_0 > 0$. and (10.20) follows. Formula (10.21) is a consequence of (10.20). $\Box$

11 Appendix II: Some technical propositions.

We prove now some auxiliary results used to prove the results of the paper.

**Lemma 11.1** Suppose that $f$ and $h$ are two analytic functions in the cone

$$C(2\varepsilon_0) = \left\{\zeta \in \mathbb{C}; \zeta = |\zeta|e^{i\theta}, \theta \in (-2\varepsilon_0, 2\varepsilon_0)\right\}$$

for some $\varepsilon_0 > 0$ and real valued in $\mathbb{R}^+$. Let us also assume that

$$\int_0^\infty \frac{|f(re^{i\theta})| + |h(re^{i\theta})|}{1 + r^2} dr < +\infty, \quad \text{for any} \quad \theta \in (-2\varepsilon_0, 2\varepsilon_0)$$  \hspace{1cm} (11.1)

$$\lim_{|\zeta| \to 0} f(\zeta) = \theta_1 \quad \text{and} \quad \lim_{|\zeta| \to \infty} f(\zeta) = \theta_2$$  \hspace{1cm} (11.2)

$$|f'(\zeta)| = o\left(\frac{1}{|\zeta|}\right) \quad \text{as} \quad |\zeta| \to 0, |\zeta| \to \infty, \zeta \in C(2\varepsilon_0).$$  \hspace{1cm} (11.3)
Then, the function
\[ F(\zeta) = \frac{1}{2\pi i} \int_0^\infty (h(s) + i f(s)) \left( \frac{1}{s - \zeta} - \frac{1}{s + 1} \right) \, ds \]  (11.4)
is analytic in the domain:
\[ D(\varepsilon_0) = \left\{ \zeta \in \mathbb{C}; \zeta = |\zeta|e^{i\theta}, \theta \in (-\varepsilon_0, 2\pi + \varepsilon_0) \right\}. \]  (11.5)

Moreover,
\[ F(\zeta) = -\frac{\theta_1}{2\pi} \ln \zeta + iH(\eta) + o(\ln|\zeta|), \quad \text{as} \quad \zeta \to 0, \quad \zeta \in D(\varepsilon_0), \]  (11.6)
\[ F(\zeta) = -\frac{\theta_2}{2\pi} \ln \zeta + iH(\eta) + o(\ln|\zeta|), \quad \text{as} \quad |\zeta| \to +\infty, \quad \zeta \in D(\varepsilon_0). \]  (11.7)

where the function \( H(\zeta) \) is a real valued function defined by
\[ H(\zeta) = -\frac{1}{2\pi} \int_0^\infty h(s) \left( \frac{1}{s - \zeta} - \frac{1}{s + 1} \right) \, ds. \]  (11.8)

Proof of Lemma 11.1 Using the Lemma C.2 of [9] with the function \( f \) we obtain (11.6) and (11.8). On the other hand, the condition (11.1) ensures that the function \( F \) is well defined and of course real valued. \( \square \)

Proposition 11.2 Let \( V(\xi) \) defined by (5.6) and (7.7). Then, for any \( \delta > 0 \) arbitrarily small and all \( M > 0 \) arbitrarily large, there exist two positive constants \( C_{1,\delta,M}, C_{2,\delta,M} \) such that
\[ C_{1,\delta,M}e^{-\left(\frac{\xi}{\lambda - 1}\right)} \leq |V(\xi)| \leq C_{2,\delta,M}e^{-\left(\frac{\xi}{\lambda - 1}\right)} \]  (11.9)
uniformly for \( \text{Im}(\xi) \) in compact sets of \( (3/2, (3 + \lambda)/2) \) as well as for all \( \xi \) such that \( |\text{Re}(\xi)| \geq 1, |\text{Im}(\xi)| \leq M. \)

Proof of Proposition 11.2 Given \( \xi \) such that \( \text{Im}(\xi) \in (3/2, (3 + \lambda)/2) \) we can represent the function \( V \) by (5.6) and (7.7) for \( \beta_1 \) such that \( \beta_1 - (\lambda - 1)/2 < \xi \leq \beta_1 \). In order to simplify some of the calculations we use the following change of variables.
\[ \zeta = e^{\frac{4\pi}{\lambda-1}(-\beta_1)} \]  (11.10)
\[ \nu(\zeta) = V(\xi) \]  (11.11)
\[ \varphi(\zeta) = \Phi(\xi) \]  (11.12)

The function \( \ln(-\varphi(s)) \) may be written as
\[ \ln(-\varphi(s)) = \ln(|\varphi(s)|) + i \arg(-\varphi(s)). \]  (11.13)

The functions \( \ln(|\varphi(s)|) \) and \( \arg(\varphi(s)) \) satisfy the hypothesis required to \( h \) and \( f \) respectively in Lemma 11.2 In particular, by Proposition 3.1 and the fact that \( \text{Re}(-\Phi(y)) > 0 \) for all \( y \) such that \( \text{Im}(y) \in ((2 + \lambda)/2, (3 + \lambda)/2) \) we may normalize the argument of the function \( \ln(-\varphi(s)) \) such that:
\[ \lim_{\zeta \to 0} \arg(-\varphi(\zeta)) = -\frac{\pi}{4}, \quad \lim_{\zeta \to \infty} \arg(-\varphi(\zeta)) = \frac{\pi}{4}. \]  (11.14)
Applying Lemma 11.1 it follows that:

$$\frac{1}{2\pi i} \int_0^\infty \ln(-\varphi(s)) \left( \frac{1}{s - \zeta} - \frac{1}{s + 1} \right) \, ds = \frac{1}{8} \ln(-\zeta) + iH(\eta) + o(\ln|\zeta|)$$  \hspace{1cm} (11.15)

as $$\zeta \to 0$$, $$\zeta \in D(\varepsilon_0)$$,

$$\frac{1}{2\pi i} \int_0^\infty \ln(-\varphi(s)) \left( \frac{1}{s - \zeta} - \frac{1}{s + 1} \right) \, ds = -\frac{1}{8} \ln(-\zeta) + iH(\eta) + o(\ln|\zeta|)$$  \hspace{1cm} (11.16)

as $$\zeta \to \infty$$, $$\zeta \in D(\varepsilon_0)$$.

The two estimates in (11.9) follow, for $$\text{Im}(\xi)$$ in compact sets of $$\left(\frac{3}{2}, \frac{3 + \lambda}{2}\right)$$, by taking exponentials in both sides of (11.15) and (11.16) and inverting the change of variables (11.10)-(11.12).

In order to prove the estimate for $$\xi$$ in the region $$|\text{Re}(\xi)| \geq 1$$ and $$|\text{Im}(\xi)| \leq M$$, we extend analytically the function $$V(\xi)$$ to such regions using (4.5) as well as the fact that, by Proposition 3.1, we have for some positive constants $$C_1$$ and $$C_2$$:

$$C_1 |\xi|^{1/2} \leq |\Phi(\xi)| \leq C_2 |\xi|^{1/2}$$  \hspace{1cm} (11.24)

for $$|\text{Re}(\xi)| \geq 1$$ and $$|\text{Im}(\xi)| \leq M$$.

Proposition 11.3 The zeros of the function $$V$$ are:

$$\left(1 + \frac{\lambda}{2} + n + k\frac{\lambda-1}{2}\right)i, \quad n = 1, 2, \ldots, \quad k = 0, 1, \ldots$$  \hspace{1cm} (11.17)

$$\left(1 + \frac{\lambda}{2} - k\frac{\lambda-1}{2}\right)i, \quad k = 1, 2, \ldots$$  \hspace{1cm} (11.18)

The poles of the function $$V$$ are:

$$\left(1 + \frac{\lambda}{2} + n + k\frac{\lambda-1}{2}\right)i, \quad n = 1, 2, \ldots, \quad k = 0, 1, \ldots$$  \hspace{1cm} (11.19)

$$\left(1 + \frac{\lambda}{2} - k\frac{\lambda-1}{2}\right)i, \quad k = 1, 2, \ldots$$  \hspace{1cm} (11.20)

Proof. The proof follows from (4.5) and the distribution of zeros and poles of $$\Phi$$ given in (3.1) and (3.2). Consider first any $$\eta \in \mathbb{C}$$ such that $$\text{Im}(\eta) > \beta_0 + (\lambda - 1)/2$$ and let $$k$$ be the first natural integer such that

$$\text{Im}(\eta) - k\frac{\lambda-1}{2} \in \left(\beta_0, \beta_0 + \frac{\lambda-1}{2}\right).$$

By (4.5),

$$V(\eta) = (-1)^j \frac{V(\eta - j\frac{\lambda-1}{2}i)}{\Phi(\eta)\Phi(\eta - \frac{\lambda-1}{2}i) \cdots \Phi(\eta - j\frac{\lambda-1}{2}i)}$$  \hspace{1cm} (11.21)

as far as none of the points $$\eta - \ell\frac{\lambda-1}{2}i$$, $$\ell = 0, \ldots, j$$ is a zero nor a pole f $$\Phi$$. Therefore, if

$$\forall j = 0, 1, \ldots k: \quad \eta - j\frac{\lambda-1}{2}i \notin \{\xi_+(n), \xi_p(n), \quad n = 0, 1, \ldots\}$$  \hspace{1cm} (11.22)

then

$$V(\eta) = (-1)^k \frac{V(\eta - k\frac{\lambda-1}{2}i)}{\Phi(\eta)\Phi(\eta - \frac{\lambda-1}{2}i) \cdots \Phi(\eta - k\frac{\lambda-1}{2}i)}$$  \hspace{1cm} (11.23)
and therefore $V(\eta)$ is well defined and not zero. If for some $j \in \{0, 1, \cdots k\}$ and $n \in \{0, 1, \cdots \}$ we have $\eta - k\frac{\lambda-1}{2}i = \xi_z(n)$, then, by (11.26), the point $\eta$ would be a pole of $V$. Moreover, that is the only way for a given $\eta$ to be a pole of $V$. Therefore, the poles $\eta$ of $V$ such that $Im(\eta) > \beta_0 + (\lambda - 1)/2$ are:

$$\left( n + \frac{1 + \lambda}{2} + k\frac{\lambda-1}{2} \right)i, \quad n = 1, 2, \cdots, k = 0, 1, \cdots$$

(11.24)

If, on the other hand, for some $j \in \{0, 1, \cdots k\}$ and $n \in \{0, 1, \cdots \}$ we had $\eta - k\frac{\lambda-1}{2}i = \xi_p(n)$, then, by (11.26), the point $\eta$ would be a zero of $V$. Moreover, that would again be the only way for $\eta$ to be a zero of $V$. We deduce that the zeros $\eta$ of $V$ such that $Im(\eta) > \beta_0 + (\lambda - 1)/2$ are:

$$\left( 1 + n + \frac{\lambda}{2} + k\frac{\lambda-1}{2} \right)i, \quad n = 1, 2, \cdots, k = 0, 1, \cdots$$

(11.25)

We have thus proved that, above the strip $Im(\eta) \in (\beta_0, \beta_0 + (\lambda - 1)/2)$, the zeros and poles of $V$ are given by (11.17) and (11.19) respectively.

A similar argument may be done when $Im(\eta) < \beta_0$. If $k$ is the least natural integer such that $Im(\eta) + k\frac{\lambda-1}{2}i \in (\beta_0, \beta_0 + (\lambda - 1)/2)$, we have by (4.5),

$$V(\eta) = (-1)^j \frac{V(\eta + j\frac{\lambda-1}{2}i)}{\Phi(\eta)\Phi(\eta + \frac{\lambda-1}{2}i) \cdots \Phi(\eta + j\frac{\lambda-1}{2}i)}$$

(11.26)

as far as none of the points $\eta + \ell\frac{\lambda-1}{2}i, \ell = 0, \cdots j$ is a zero nor a pole of $\Phi$. Notice that by the definition of $k$ and the choice of $\beta_0$, the only zero of $\Phi$ that may be founded is $\xi_z(0) = (1 + \lambda)i/2$ and the only pole is $\xi_p(0) = (2 + \lambda)i/2$. The proof then follows as before.

$$\square$$

12 Appendix III: Stationary phase.

We must estimate in Sections 6 and 7 several integral expressions of the form

$$\int_{ImY = \beta_0 - Im\xi} e^{\Psi(\xi,Y,t)} A \left( \frac{2iY}{\lambda - 1} \right) dY$$

for a given function $\Psi$ but different functions $A$. This is done using the stationary phase method. We collect in this Section some technical results about the function $\Psi(\xi,Y,t)$ and its critical points.

12.1 The critical point.

We first compute the critical points of the function $\Psi(\xi,Y,t)$ defined by (6.34) or equivalently, that of $\Phi(\xi,Z,t)$ defined in (6.10). Define:

$$D(\xi, B) = \left\{ Z \in \mathbb{C}; \text{Im} Z < 0, |\text{Im} Z| \leq B \left| \text{Re} Z + \frac{Q}{8} \sqrt{|\xi|} \right|, \text{sign}(\text{Re} Z) = \text{sign}(\text{Re} \xi) \right\}$$

(12.1)

where $Q = \text{sign}(\text{Re} \xi)$.
12.1.1 The case $0 < t < 1$.

We start with the following:

**Lemma 12.1** Consider the function $F(\xi, Z)$ defined by means of

$$F(\xi, Z) = \frac{2}{(\lambda - 1)i} \int_{Im(\eta) = \beta_1} \ln(-\Phi(\eta)) \Theta(\eta - \xi, \sqrt{|\xi|} Z) d\eta$$

(12.2)

for $\beta_1 \in (3/2, (3 + \lambda)/2)$, $Im(\xi) \in (3/2, \beta_1)$ and

$$\frac{1}{\sqrt{|\xi|}}(\beta_1 - Im(\xi)) - \frac{\lambda - 1}{2} < Im(Z) \leq \frac{(\beta_1 - Im(\xi))}{\sqrt{|\xi|}}.$$  

(12.3)

For any constant $B > 0$ the function $F(\xi, \cdot)$ can be extended analytically on the variable $Z$ to the domain $|Z| \leq \frac{\sqrt{|\xi|}}{8}$, $Z \in D(\xi, B)$ if $|\xi| \geq \xi_0$ for $\xi_0 = \xi_0(B)$ sufficiently large. Moreover, there exists a positive constant $C = C(B)$ such that

$$\left| F(\xi, Z) + \frac{2i}{(\lambda - 1)} \ln(-\Phi(\xi)) \sqrt{|\xi|} Z \right| \leq C \left( Z^2 + O\left(\frac{1}{|\xi|}\right) \right)$$

(12.4)

for $|Z| \leq \frac{\sqrt{|\xi|}}{8}$, $Z \in D(\xi, B)$ and $|\xi| \geq \xi_0$.

**Proof.** The function $F$ is well defined in (12.3) since in that domain the variable $Z$ is in the region where the function $\Theta$ is analytic. The function $F(\xi, \cdot)$ given by (12.2) is then analytic in the strip $|ImZ| \leq \frac{\delta_0(Im\xi)}{\sqrt{|\xi|}}$ for some $\delta_0(Im\xi)$ sufficiently small. We now claim that for any fixed constant $B > 0$, and $Im\xi \in (\beta_0, \beta_0 + (\lambda - 1)/2)$, if $Re\xi$ is sufficiently large (depending on $B$), this function $F(\xi, \cdot)$ may be extended analytically to the region $D(\xi, B)$. To prove this we derive new representation formulas of the function $F$ performing suitable contour deformations in the variable $\eta$ in (12.2).

Notice that the singularities of the integrand in (12.2) are contained in the set $Re\eta = 0$, $\eta = \xi + (\lambda - 1)\ell/2$ and $\eta = \xi + \sqrt{|\xi|} Z + (\lambda - 1)\ell/2$ for $\ell \in Z$. Given $Z_0$ in the region above, let be $\hat{Z}_0$ such that:

$$Re\hat{Z}_0 = ReZ_0, \quad |Im\hat{Z}_0| \leq \frac{\delta_0(Im\xi)}{\sqrt{|\xi|}}$$
and the integral curve \( Im \eta = \beta_0 + \frac{\lambda-1}{2} - \varepsilon \) lies between the two points \( \xi + \hat{Z}_0 \sqrt{|\xi|} \) and \( \xi + \hat{Z}_0 \sqrt{|\xi|} + (\lambda - 1)i/2 \). (Notice that this is possible since \( \delta_0(Im\xi) \) may be made as small as we need and \( Im\eta > Im\xi \).)

Consider the vertical segment of the complex plane connecting \( Z_0 \) and \( \hat{Z}_0 \): \( Z_0 = (1-\theta)\hat{Z}_0 + \theta Z_0 \), \( \theta \in [0, 1] \). We then obtain an analytic extension of \( F(\xi, \cdot) \) varying \( \theta \) continuously from 0 to one and deforming continuously the contour \( Im\eta = \beta_0 + \frac{\theta-1}{2} - \varepsilon \) in such a way that:

- it always passes between \( \xi + \hat{Z}_0 \sqrt{|\xi|} \) and \( \xi + \hat{Z}_0 \sqrt{|\xi|} + (\lambda - 1)i/2 \)

- we do not change the original integration contour for

\[
\left| Re \left( \eta - \xi - Z_0 \sqrt{|\xi|} \right) \right| \geq \frac{|Re(Z_0)\sqrt{|\xi|}|}{2}
\]

The first condition ensures that the integration contour never crosses any of the singularities of the function \( \left( 1 - e^{-\frac{\lambda}{\lambda-1}(\eta-\xi)\sqrt{|\xi|}} \right)^{-1} \). The second one ensures that it does not cross neither any of the singularities of \( \left( 1 - e^{\frac{\lambda}{\lambda-1}(\xi-\eta)} \right)^{-1} \).

Finally, since \( sign(ReZ_0) = sign(Re\xi) \), the new integration contour never crosses the line \( Re\eta = 0 \) where the singularities of \( \ln(-\Phi(\eta)) \) are located.

To estimate this integral we write

\[
\frac{2}{(\lambda - 1)i} \int_{C_2} \ln(-\Phi(\eta)) \Theta(\eta - \xi, \sqrt{|\xi|}Z)d\eta = \frac{2}{(\lambda - 1)i} \int_{C_2} \Theta(\eta - \xi, \sqrt{|\xi|}Z)d\eta + \frac{2}{(\lambda - 1)i} \int_{C_2} \ln \left( \frac{\Phi(\eta)}{\Phi(\xi)} \right) \Theta(\eta - \xi, \sqrt{|\xi|}t Z)d\eta \equiv I_1 + I_2.
\]
The first integral, $I_1$ is computed explicitly:

$$I_1 = \frac{2 \ln(-\Phi(\xi))}{(\lambda - 1) i} \sqrt{\xi} Z = -\frac{2 i}{(\lambda - 1)} \ln(-\Phi(\xi)) \sqrt{\xi} Z. \quad (12.6)$$

In order to compute the second integral we have to distinguish the cases $Re \xi \to -\infty$ and $Re \xi \to +\infty$. Since both may be treated using similar arguments let us treat only the case $Re \xi \to +\infty$. In that case we decompose $I_2$ as follows

$$I_2 = \frac{2}{(\lambda - 1) i} \int_{C_2, Re \eta > 0, |\eta - \xi| \leq \frac{|\xi|}{4}} \cdots d\eta + \int_{C_2, Re \eta > 0, |\eta - \xi| > \frac{|\xi|}{4}} \cdots d\eta$$

$$+ \frac{2}{(\lambda - 1) i} \int_{C_2, Re \eta < 0} \cdots d\eta = I_{2,1} + I_{2,2} + I_{2,3} \quad (12.7)$$

In $I_{2,1}$ we use now Proposition 3.1 and Taylor’s expansion to obtain:

$$\Phi(\eta) = \frac{\sqrt{\eta} \left( 1 + \frac{(2\lambda + 1 + i)}{8\eta} + O(|\eta|^{-2}) \right)}{\sqrt{\xi} \left( 1 + \frac{(2\lambda + 1 + i)}{8\xi} + O(|\xi|^{-2}) \right)}$$

using now $\eta = \xi + (\eta - \xi)$ we can write:

$$\frac{\Phi(\eta)}{\Phi(\xi)} = 1 + O \left( \frac{\eta - \xi}{\xi} \right) + O \left( \frac{1}{\xi^2} \right), \text{ as } Re \xi > 1, |\eta - \xi| \leq \frac{|\xi|}{4}.$$ 

Therefore:

$$\ln \left( \frac{\Phi(\eta)}{\Phi(\xi)} \right) = O \left( \frac{\eta - \xi}{\xi} \right) + O \left( \frac{1}{\xi^2} \right), \text{ as } Re \xi > 1, |\eta - \xi| \leq \frac{|\xi|}{4}.$$ 

We now estimate the two following integrals for all $Z \in D(\xi, B)$ and $|Z| \leq \frac{\sqrt{|\xi|}}{8}$. The first one:

$$\int_{C_2, Re \eta > 0, |\eta - \xi| \leq \frac{|\xi|}{4}} \left( \frac{\eta - \xi}{\xi} \right) \Theta(\eta - \xi, \sqrt{|\xi|} Z) d\eta = \frac{1}{|\xi|} \int_{C_2, Re \sigma > -Re \xi, |\sigma| \leq \frac{|\xi|}{4}} \left| \sigma \Theta(\sigma, \sqrt{|\xi|} Z) \right| d\sigma$$

$$\leq \frac{1}{|\xi|} \int_{C_2, Re \sigma > -Re \xi, |\sigma| \leq 2|\xi|^{1/2}} \left| \sigma \Theta(\sigma, \sqrt{|\xi|} Z) \right| d\sigma + \frac{1}{|\xi|} \int_{C_2, Re \sigma > -Re \xi, |\sigma| \geq 2|\xi|^{1/2}} \left| \sigma \Theta(\sigma, \sqrt{|\xi|} Z) \right| d\sigma$$

$$\leq C \left( Z^2 + \frac{e^{-a|\xi|^{1/2} Z}}{|\xi|^{1/2}} \right) \text{ for } Re \xi > \xi_0(B)$$

the second one:

$$\int_{C_2, Re \eta > 0, |\eta - \xi| \leq \frac{|\xi|}{4}} \left( \frac{1}{|\xi|^2} \right) \left| \Theta(\eta - \xi, \sqrt{|\xi|} Z) \right| d\eta = \frac{1}{|\xi|^2} \int_{C_2, Re \sigma > -Re \xi, |\sigma| \leq \frac{|\xi|}{4}} \left| \Theta(\sigma, \sqrt{|\xi|} Z) \right| d\sigma$$

$$\leq \frac{1}{|\xi|^2} \int_{C_2, Re \sigma > -Re \xi, |\sigma| \leq 2|\xi|^{1/2}} \left| \Theta(\sigma, \sqrt{|\xi|} Z) \right| d\sigma + \frac{1}{|\xi|^2} \int_{C_2, Re \sigma > -Re \xi, |\sigma| \geq 2|\xi|^{1/2}} \left| \Theta(\sigma, \sqrt{|\xi|} Z) \right| d\sigma$$

$$\leq C \left( \frac{|Z|}{|\xi|^{1/2}} + \frac{e^{-a|\xi|^{1/2} Z}}{|\xi|^2} \right) \leq C \left( |Z|^2 + \frac{1}{|\xi|^3} \right) + C \frac{e^{-a|\xi|^{1/2} Z}}{|\xi|^2} \text{ for } Re \xi > \xi_0(B)$$
where in both cases $\tilde{C}_2 = C_2 - \xi$, $\xi_0(B)$ is a positive constant sufficiently large, depending on $B$ and $a$ and $C$ are positive constants which may depend on $B$ but are independent on $Re \xi$ and $Z \in D(\xi, B)$. From where we deduce that, for all $Re \xi > \xi_0(B)$:

$$|I_{2,1}| \leq C \left( |Z|^2 + \frac{1}{|\xi|^3} \right) + C\frac{e^{-a|\xi|^{1/2}|Z|}}{|\xi|}$$

(12.8)

In the integral $I_{2,2}$ we use the fact that when $|\eta - \xi| > \frac{|\xi|}{4}$ and $Re \xi > \xi_0(B)$, the function $\Theta(\eta - \xi, Y)$ has an exponential decay $Ce^{-a|\xi|}$ with $C$ and $a$ as above as well as the inequality $|\ln(-\Phi(\eta))| \leq C|\ln(|\eta - \xi| + |\xi|)| \leq C(|\ln(|\eta - \xi|)| + |\ln(|\xi|)|)$ for $\eta$ is large. For $\eta$ of order one we use that $\ln(-\Phi(\eta))$ is of order one to derive a similar estimate. Then

$$|I_{2,2}| \leq \int_{|\eta - \xi| > \frac{|\xi|}{4}} e^{-a|\eta - \xi|} (|\ln(|\eta - \xi|)| + |\ln(|\xi|)|) d\eta = O(1).$$

(12.9)

Finally, the estimate of $I_{2,3}$ follows using the same cut off properties of the function $\Theta$ since $Re \eta < 0$ and $Re \xi \to +\infty$ implies that $Re(\eta - \xi) > C|\xi|$. The final estimate of $I_2$, by (12.7), is then,

$$|I_2| \leq C \left( |Z|^2 + \frac{1}{|\xi|^3} \right) + C\frac{e^{-a|\xi|^{1/2}|Z|}}{|\xi|}.$$ 

(12.10)

Using Proposition 3.1, 6.10, 12.5, 12.6 and 12.10 the Lemma follows.

**Lemma 12.2** For any $B > 0$, the function $h$ defined by means of:

$$\Phi(\xi, Z, t) = -\sqrt{|\xi|} \frac{2iZ}{\lambda - 1} \left[ 1 + \ln t - \ln \left( \frac{2iZ}{\lambda - 1} \right) + \ln \left( 2\sqrt{\pi}e^{iQ^2} \right) \right] - \frac{1}{2} \ln \left( \frac{|\xi|^{1/2}}{\lambda - 1} \right) - \frac{1}{2} \ln \left( \frac{2iZ}{\lambda - 1} \right) + h(\xi, Z, t)$$

(12.11)

satisfies

$$|h(\xi, Z, t)| \leq C \left( Z^2 + O \left( \frac{1}{|\xi|} \right) \right)$$

for $|Z| \leq \frac{\sqrt{|\xi|}}{2}$, $Z \in D(\xi, B)$ and $|\xi| \geq \xi_0$ for some positive constants $C = C(B)$ and $\xi_0 = \xi_0(B)$ sufficiently large.

**Proof.** This Lemma is a direct consequence of Lemma (12.1). The only difficulty in order to estimate the function $\Phi$ defined in (6.10) comes from the term $\varphi(\xi)/\varphi(\xi + Y)$ which corresponds to the integral term. This term, given by formula (5.15) may also be written as:

$$\Lambda(\xi, Z) = \frac{\varphi(\xi)}{\varphi(\xi + \sqrt{|\xi|Z})} = \exp \left[ \frac{2}{(\lambda - 1)i} \int_{Im\eta = \beta_1} \ln (-\Phi(\eta)) \Theta(\eta - \xi, \sqrt{|\xi|Z}) d\eta \right].$$

As for the critical point $Z_c$ of $\Phi(\xi, \cdot, t)$, the precise result is the following.

**Lemma 12.3** Suppose that $B > 2\sqrt{\pi}$. Let us define $\delta_0 = (\lambda - 1)\sqrt{\pi}/4$. For all $0 \leq t \leq 1$ and $Im \xi \in (\beta_0, \beta_0 + (\lambda - 1)/2)$, $|\xi|^2 \geq \xi_0(B)$ with $\xi_0(B)$ a positive constant sufficiently large, there exists a unique point $Z_c \in D(\xi, B) \setminus B_{\delta_0}t(0)$ such that $\partial\Phi(\xi, Z_c, t)/\partial Z = 0$. Moreover the following asymptotics holds uniformly for $0 \leq t \leq 1$:

$$\frac{2iZ_c}{\lambda - 1} = \sqrt{2\pi t(1 + iQ)} \left( 1 + O \left( \frac{1}{\sqrt{|\xi|} t} \right) \right)$$

as $|Re \xi|^2 \to \infty$. 

(12.12)
Proof. Computing the derivative of the function $\Phi$ given by (6.10) gives for all $Z \in D(\xi, B)$:

$$
\frac{\partial \Phi}{\partial Z}(\xi, Z, t) = - \frac{8\pi \sqrt{|\xi|} i}{(\lambda - 1)^2} \int_{C_2} \ln (-\Phi(\eta)) \frac{e^{\xi_0}}{(1 - e^{\xi_0})^2} \frac{d\eta - 
\sqrt{|\xi|} + \left(2i \frac{|\lambda - 1|}{2}\right) \ln \left(\frac{2i (Z/t)}{\lambda - 1}\right) - \frac{1}{2\sqrt{|\xi|} Z} + \frac{2i \ln |\xi|^{1/2}}{\lambda - 1}. 
$$

(12.13)

We compute the leading term of the integral in the right hand side of (12.13) as $|\xi| \to \infty$:

$$
I(\xi, Z) = \int_{C_2} \ln (-\Phi(\eta)) \frac{e^{\xi_0}}{(1 - e^{\xi_0})^2} \frac{d\eta = 
\int_{C_2} \ln \left(-\Phi(\sigma + \xi + \sqrt{|\xi|} Z)\right) \frac{e^{-\xi_0}}{(1 - e^{-\xi_0})^2} \frac{d\sigma}
$$

where $\tilde{C}_2 = C_2 - \xi - \sqrt{|\xi|} Z$. Using Proposition 3.1 we have that, uniformly for $Z \in D(\xi, B)$, $|Z| \leq B$:

$$
\ln \left(-\Phi(\sigma + \xi + \sqrt{|\xi|} Z)\right) = \ln \left(-\Phi(\xi + \sqrt{|\xi|} Z)\right) + A(\xi) \frac{\sigma}{|\xi|^2} + O\left(\frac{\sigma^2}{|\xi|^2}\right)
$$

for $|\xi| \to +\infty$, where $A(\xi)$ is a bounded function of $\text{sign}(Re\xi)$. It then follows:

$$
I(\xi, Z) = \ln \left(-\Phi(\xi + \sqrt{|\xi|} Z)\right) \int_{\tilde{C}_2} \frac{e^{-\xi_0}}{(1 - e^{-\xi_0})^2} \frac{d\sigma + 
\frac{A(\xi)}{|\xi|} \int_{\tilde{C}_2} \frac{\sigma}{|\xi|^2} \frac{e^{-\xi_0}}{(1 - e^{-\xi_0})^2} \frac{d\sigma + \int_{\tilde{C}_2} O\left(\frac{\sigma^2}{|\xi|^2}\right) \frac{e^{-\xi_0}}{(1 - e^{-\xi_0})^2} \frac{d\sigma}
$$

(12.14)

uniformly for $Z \in D(\xi, B)$, $|Z| \leq B$, where $a > 0$ is independent on $\xi, Z$. Using:

$$
\frac{d}{d\sigma} \left(\frac{1}{1 - e^{-\frac{4\pi}{\lambda - 1}\sigma}}\right) = - \frac{4\pi}{\lambda - 1} \frac{e^{-\frac{4\pi}{\lambda - 1}\sigma}}{(1 - e^{-\frac{4\pi}{\lambda - 1}\sigma})^2},
$$

we obtain:

$$
\int_{\tilde{C}_2} \frac{e^{-\frac{4\pi}{\lambda - 1}\sigma}}{(1 - e^{-\frac{4\pi}{\lambda - 1}\sigma})^2} \frac{d\sigma = - \frac{\lambda - 1}{4\pi}}.
$$

On the other hand, in order to estimate the second term in the right hand side of (12.14) we deform the integration contour $\tilde{C}_2$ to a horizontal line at a bounded distance of the real axis. The resulting
integral can then be bounded by a positive constant independent of \( \xi, Z \). The third term in the right hand side of (12.14) can be bounded using the specific form of \( \hat{C}_2 \) as:

\[
C \left( \frac{|Z|^2}{|\xi|} + \frac{|Z|^3}{|\xi|^{1/2}} \right).
\]

We notice that by Proposition 3.1 we have:

\[
\ln \left( -\Phi(\xi + \sqrt{|\xi|} Z) \right) = \ln(-\Phi(\xi)) + O \left( \frac{|Z|}{\sqrt{|\xi|}} \right), \quad \text{as } |\xi| \to +\infty
\]

uniformly for \( Z \in D(\xi, B), |Z| \leq B \). Combining everything we deduce:

\[
I(\xi, Z) = -\frac{\lambda - 1}{4\pi} \ln(-\Phi(\xi)) + O \left( \frac{|Z|}{|\xi|^{1/2}} \right), \quad \text{as } |\xi| \to +\infty
\]

(12.15)

uniformly for \( Z \in D(\xi, B), |Z| \leq B \). Combining (12.13) and (12.15) it follows:

\[
\frac{\partial \Phi}{\partial Z}(\xi, Z, t) = \frac{2i\sqrt{|\xi|}}{(\lambda - 1)} \left( \ln(-\Phi(\xi)) + \ln \left( \frac{2i(Z/t)}{\lambda - 1} \right) + \frac{(\lambda - 1)i}{4\sqrt{|\xi|Z}} + \ln \left( |\xi|^{1/2} \right) \right) + O \left( \frac{|Z|}{|\xi|^{1/2}} \right)
\]

as \( |\xi| \to +\infty \)

uniformly for \( Z \in D(\xi, B), |Z| \leq B \).

Using Rouché’s Theorem it then follows that for \( |\xi|t^2 \) sufficiently large, \( \xi \in \text{Franja}, 0 \leq t \leq 1 \), there exists a unique root of \( (\partial \Phi/\partial Z)(\xi, Z, t) = 0 \) in \( Z \in D(\xi, B) \setminus B_{\delta t} \) satisfying the asymptotics (12.3) and the Lemma follows.

We now derive estimates for higher order derivatives of \( \Phi \).

**Lemma 12.4** Suppose that \( Z_c \) is as in Lemma 12.3. Then the following asymptotics holds:

\[
\frac{\partial^2 \Phi}{\partial Z^2}(\xi, Z_c, t) = \frac{2i\sqrt{|\xi|}}{(\lambda - 1)Z_c} \left( 1 + O \left( \frac{1}{\sqrt{|\xi|t}} \right) \right)
\]

as \( |\xi| t^2 \to +\infty \).

**uniformly in** \( 0 \leq t \leq 1 \).

**Proof.** The second derivative of \( \Phi \) with respect to \( Z \) is:

\[
\frac{\partial^2 \Phi}{\partial Z^2} = -\frac{8\pi|\xi|i}{(\lambda - 1)^2} \int_{C_2} \left[ \ln(-\Phi(\eta))' \right] \frac{e^{\frac{4\pi i}{\eta}(Z\sqrt{|\xi| - \eta + \xi})}}{(1 - e^{\frac{4\pi i}{\eta}(Z\sqrt{|\xi| - \eta + \xi})})^2} d\eta + \frac{2\sqrt{|\xi|} i}{(\lambda - 1) Z} + \frac{1}{2Z^2}
\]

\[
= -\frac{8\pi|\xi|i}{(\lambda - 1)^2} \int_{C_2} \frac{\Phi'(\eta)}{\Phi(\eta)} \frac{e^{\frac{4\pi i}{\eta}(Z\sqrt{|\xi| - \eta + \xi})}}{(1 - e^{\frac{4\pi i}{\eta}(Z\sqrt{|\xi| - \eta + \xi})})^2} d\eta + \frac{2i\sqrt{|\xi|}}{(\lambda - 1) Z} + \frac{1}{2Z^2}.
\]

Taking \( Z = Z_c \) and using Lemma 12.3, we deduce, uniformly in \( 0 \leq t \leq 1 \)

\[
\frac{\partial^2 \Phi}{\partial Z^2}(\xi, Z_c, t) = \frac{2i\sqrt{|\xi|}}{(\lambda - 1)Z_c} - \frac{8\pi|\xi|i}{(\lambda - 1)^2} \int_{C_2} \frac{\Phi'(\eta)}{\Phi(\eta)} \frac{e^{\frac{4\pi i}{\eta}(Z_c\sqrt{|\xi| - \eta + \xi})}}{(1 - e^{\frac{4\pi i}{\eta}(Z_c\sqrt{|\xi| - \eta + \xi})})^2} d\eta + O \left( \frac{1}{t} \right),
\]

(12.16)
Therefore, the first term in the right hand side of (12.17) is estimated as:

\[ J(\xi) = \int_{C_2} \frac{\Phi(\eta)}{\Phi(\eta)} e^{\frac{4\pi i}{Z_c \sqrt{|\xi| - \eta + \xi}}} \left(1 - e^{\frac{4\pi i}{Z_c \sqrt{|\xi| - \eta + \xi}}} \right)^2 d\eta = \frac{1}{2} \int_{C_2} \frac{1}{\eta} e^{\frac{4\pi i}{Z_c \sqrt{|\xi| - \eta + \xi}}} \left(1 - e^{\frac{4\pi i}{Z_c \sqrt{|\xi| - \eta + \xi}}} \right)^2 d\eta + \int_{C_2} \mathcal{O} \left( \frac{1}{|\xi|^2} \right) \frac{e^{\frac{4\pi i}{Z_c \sqrt{|\xi| - \eta + \xi}}}}{\left(1 - e^{\frac{4\pi i}{Z_c \sqrt{|\xi| - \eta + \xi}}} \right)^2} d\eta + \mathcal{O} \left( e^{-a|\xi|} \right) , \quad \text{as} \quad |\xi| \to +\infty \]  

(12.17)

where \( a \) is a positive constant independent on \( \xi \). The first term in the right hand side is estimated as:

\[ \int_{C_2} \frac{1}{\eta} e^{\frac{4\pi i}{Z_c \sqrt{|\xi| - \eta + \xi}}} \left(1 - e^{\frac{4\pi i}{Z_c \sqrt{|\xi| - \eta + \xi}}} \right)^2 d\eta = \frac{1}{\xi} \int_{C_2} \frac{e^{\frac{4\pi i}{Z_c \sqrt{|\xi| - \eta + \xi}}}}{\left(1 - e^{\frac{4\pi i}{Z_c \sqrt{|\xi| - \eta + \xi}}} \right)^2} d\eta + \int_{C_2} \mathcal{O} \left( \frac{\eta - \xi}{|\xi|^2} \right) \frac{e^{\frac{4\pi i}{Z_c \sqrt{|\xi| - \eta + \xi}}}}{\left(1 - e^{\frac{4\pi i}{Z_c \sqrt{|\xi| - \eta + \xi}}} \right)^2} d\eta + \mathcal{O} \left( e^{-a|\xi|} \right) , \quad \text{as} \quad |\xi| \to +\infty. \]

The first term in the right hand side can be computed explicitly and gives \(-(\lambda - 1)/4\pi \xi \). The second one is estimated using the form of the contour \( C_2 \) and is bounded by \( \mathcal{O} \left( |Z_c|/|\xi| \right) \). Therefore, the first term in the right hand side of (12.17) is estimated as \( \mathcal{O}(1/|\xi|) \) as \( |\xi| \to +\infty \). The second term is bounded using similar arguments by \( \mathcal{O}(1/|\xi|^3/2) \) as \( |\xi| \to +\infty \). It then follows that

\[ J(\xi) = \mathcal{O} \left( \frac{1}{|\xi|} \right) \quad \text{as} \quad |\xi| \to +\infty. \]

Using (12.16) we obtain

\[ \frac{\partial^2 \Phi}{\partial Z^2}(\xi, Z_c, t) = \frac{2i \sqrt{|\xi|}}{(\lambda - 1) Z_c} + \mathcal{O} \left( \frac{1}{t} \right) \quad \text{as} \quad |\xi| t^2 \to +\infty \]

whence Lemma 12.3 follows. \( \square \)

**Lemma 12.5** Suppose that \( Z_c, B \) and \( \delta_0 \) are as in Lemma 12.3. Then the following asymptotics holds:

\[ \left| \frac{\partial^3 \Phi}{\partial Z^3}(\xi, Z, t) \right| = \mathcal{O} \left( \frac{\sqrt{|\xi|}}{t^2} \right) \quad \text{as} \quad |\xi| t^2 \to +\infty. \]

uniformly in \( 0 \leq t \leq 1, Z \in D(\xi, B), |Z| \leq B. \)

**Proof of Lemma 12.5**

\[ \frac{\partial^3 \Phi}{\partial Z^3}(\xi, Z, t) = -\frac{8\pi |\xi|^{3/2} i}{(\lambda - 1)^2} \int_{C_2} \left( \frac{\Phi'(\eta)}{\Phi(\eta)} \right)' \frac{e^{\frac{4\pi i}{Z_c \sqrt{|\xi| - \eta + \xi}}}}{\left(1 - e^{\frac{4\pi i}{Z_c \sqrt{|\xi| - \eta + \xi}}} \right)^2} d\eta - \frac{2i \sqrt{|\xi|}}{(\lambda - 1) Z^3} - \frac{1}{Z^3} \]  

(12.18)
The last two terms of (12.18) are bounded as
\[ C \left( \sqrt{\frac{\xi}{t^2}} + \frac{1}{t^3} \right). \]
and this can be estimated as \( C \sqrt{\xi/t^2} \) for \( |\xi| t^2 \gg 1 \). On the other hand we may bound the first term in the right hand side of (12.18) using
\[
\left| \frac{\Phi'(\eta)}{\Phi(\eta)} \right|' \leq \frac{C}{1 + |\eta|^2}.
\]
the form of the contour \( C_2 \). The term under consideration is then bounded by a constant. Combining all these estimates for the terms in the right hand side of (12.18) the Lemma follows.

Combining Lemma 12.1 and Lemma 12.3 it follows

**Corollary 12.6** For all \( 0 < t < 1 \):
\[
\Phi(\xi, Z_c, t) = -\sqrt{\xi} \sqrt{2\pi t} (1 + iQ) - \frac{1}{2} \ln \left( |\xi|^{1/2} \right) - \frac{1}{2} \ln \left( \frac{2iZ_c}{\lambda - 1} \right) + O(1) \text{ as } |\xi| t^2 \to +\infty.
\]

**Proof.** Using (12.12) and (12.11) we deduce
\[
\Phi(\xi, Z_c, t) = -\sqrt{\xi} \sqrt{2\pi t} (1 + iQ) - \frac{1}{2} \ln \left( |\xi|^{1/2} \right) - \frac{1}{2} \ln \left( \frac{2iZ_c}{\lambda - 1} \right) + O(1)
\]
\[= -\sqrt{\xi} \sqrt{2\pi t} (1 + iQ) - \frac{1}{2} \ln \left( |\xi|^{1/2} \right) - \frac{1}{2} \ln \left( \frac{2iZ_c}{\lambda - 1} \right) + O(1)
\]
(12.19)
as \( |\xi| t^2 \to +\infty \). □

**Remark 12.7** We must emphasise that the sign of the real part of the main term in the asymptotic expansion of \( \Phi \) as \( t^2 |\xi| \to \infty \) is fundamental for the construction of our solutions.

In the next Lemma we prove that the function \( \Phi(\xi, Z, t) \) is “well behaved” in a region \( |Z - Z_c| \geq \delta t \) and \( |Z| \leq M t \) for \( M > 0 \) large.

**Lemma 12.8** For all \( \delta > 0 \) and \( M > 0 \) large, there exists \( \varepsilon_0 > 0 \) and \( L > 0 \) such that the function \( \Phi(\xi, Z, t) \) satisfies:
\[
\text{Re } \Phi(\xi, Z, t) \leq \text{Re } \Phi(\xi, Z_c, t) - \varepsilon_0 \sqrt{|\xi|} t
\]
when \( Z \) lies in the curve
\[
\gamma(M) = \gamma_1(M) \cup \gamma_2(M) \cup \gamma_3(M) \setminus \{ Z; |Z - Z_c| \leq \delta t \}
\]
where
\[
\gamma_1(M) = \{ Z; Z = Z_c + \lambda, \lambda \in \mathbb{R}, |\lambda| \leq M t \}
\]
\[
\gamma_2(M) = \{ Z; Z = Z_c + M t + \lambda i, \lambda \in [0, |ImZ_c| + \gamma_1] \}
\]
\[
\gamma_3(M) = \{ Z; Z = Z_c - M t + \lambda i, \lambda \in [0, |ImZ_c| + \gamma_1] \}
\]
for all \( \xi \) and \( t \) such that \( |\xi| t^2 > L \).
Proof. Let us consider the auxiliary function:

\[
\Theta(\xi, \Omega, t) = -2\sqrt{\pi} \sqrt{|\xi|} \Omega t \left[ 1 - \ln(\Omega) + \ln(\Omega_0) \right] \quad (12.20)
\]

\[
\Omega = \frac{iZ}{\lambda - 1} \frac{1}{\sqrt{\pi} t} \quad (12.21)
\]

\[
\Omega_0 = e^{iQ \frac{\pi}{4}}. \quad (12.22)
\]

By Lemma 12.1:

\[
\Phi(\xi, Z, t) = \Theta(\xi, \Omega, t) - \frac{1}{2} \ln \left( \frac{|\xi|}{1/2} \right) + h(\xi, Z, t).
\]

Moreover, it is easily checked that \( \Omega_0 \) is the critical point of the function \( \Theta(\xi, \Omega, t) \). By Lemma 12.3 we already know that \( Z_c \), the critical point of the function \( \Phi \), converges to \( \Omega_0 \) as \( |\xi| t^2 \to +\infty \). We are now going to study the behaviour of the function \( \Theta \) along the curve obtained from \( \gamma(M) \) using the change of variable (12.21). Due to the convergence properties of the function \( \Phi \) and its critical point \( Z_c \) when \( |\xi| t^2 \to +\infty \) this will be enough in order to prove the statement in Lemma 12.8.

We first consider the curve corresponding to \( \gamma_1(M) \). It is then enough to consider \( \Re \Theta(\xi, \Omega, t) \) along the points: \( \Omega = \Omega_0 + \frac{\lambda}{\sqrt{2}} i, \quad \lambda \in \mathbb{R} \). A straightforward calculation yields:

\[
\Re \Theta(\xi, \Omega, t) = -\sqrt{2\pi} \sqrt{|\xi|} t \psi(\sigma)
\]

\[
\psi(\sigma) = 1 - \frac{1}{2} \ln(1 + \sigma^2) + \frac{1}{2} \ln 2 - \sigma \left( \frac{\pi}{4} - \arctan(\sigma) \right).
\]

Since \( \psi'(\sigma) = -\frac{\pi}{4} + \arctan(\sigma) \) the point \( \sigma_0 = 1 \) is a strict minimum for the function \( \psi \). It is also easily checked that

\[
\psi(\sigma) \sim \frac{\pi}{4} \quad \text{as} \quad \sigma \to +\infty
\]

\[
\psi(\sigma) \sim -\frac{3\pi}{4} \quad \text{as} \quad \sigma \to -\infty.
\]

It follows that, for any \( \delta > 0 \) there exists \( \varepsilon > 0 \) such that if \( |\sigma - 1| > \delta \),

\[
\psi(\sigma) - \psi(1) \geq \varepsilon.
\]

Arguing by continuity this yields the statement of the Lemma when \( Z \) lies in the curve \( \gamma_1(M) \). The evolution of the function \( \theta \) along the two curves corresponding to \( \gamma_2(M) \) and \( \gamma_3(M) \) is studied with very similar arguments. We then only consider the case of the curve \( \gamma_2(M) \). It is then enough to consider \( \Re \Theta(\xi, \Omega, t) \) along the points:

\[
\Omega = r e^{i\varphi},
\]

\[
\Im(r e^{i\varphi} - \Omega_0) = \pm \frac{M}{\pi(\lambda - 1)}
\]

a straightforward calculation gives:

\[
\Re \Theta(\xi, \Omega, t) = -2\sqrt{\pi} \sqrt{|\xi|} \Omega t \left[ r \cos \varphi(1 - \ln r) - r \sin \varphi \left( \frac{\pi Q}{4} - \varphi \right) \right].
\]
For $Z \in \gamma_2(M)$ and the constant $M$ sufficiently large, we have that $\varphi > \pi/4 + \delta$, $r \cos \varphi \leq 2$ and $r > M/2$. Similarly, if $Z \in \gamma_3(M)$ and the constant $M$ sufficiently large, we have that $\varphi < -\pi/4 - \delta$, $r \cos \varphi \leq 2$ and $r > M/2$. Therefore, we have in both cases:

$$r \cos \varphi (1 - \ln r) - r \sin \varphi \left( \frac{\pi Q}{4} - \varphi \right) > \varepsilon > 0.$$ for $M$ large enough.

Using again the convergence properties of the function $\Phi$ and its critical point $Z_c$ when $|\xi|^t^2 \to +\infty$ this yields the statement in Lemma 12.8 when $Z$ lies in the curves $\gamma_2(M), \gamma_3(M)$. 

In the following Lemma we extend the behaviour of $\Phi(\xi,Z,t)$ to the region $|Z - Z_c| \geq \delta t$ and $|Z| \leq \varepsilon_1 \sqrt{|\xi|}$ for some $\varepsilon_1 > 0$ sufficiently small.

**Lemma 12.9** For all $\delta > 0$ and $M > 0$ large, there exists $a > 0$, $\varepsilon_1 > 0$ and $L > 0$ such that the function $\Phi(\xi,Z,t)$ satisfies:

$$Re \Phi(\xi,Z,t) \leq -a \sqrt{|\xi|} |Z|,$$

for all $Z \in C_1$ such that $M t \leq |Z| \leq \varepsilon_1 \sqrt{|\xi|}$ and all $\xi$ and $t$ such that $|\xi|^t^2 > L$.

**Proof.** We only need to check that the function $\Theta$ defined in the proof of the previous Lemma behaves linearly when $|\Omega| \to +\infty$ and $Re(\Omega)$ remains constant. This follows from

$$Re(\Theta(\xi,\Omega,t)) \leq -\frac{\pi^3}{2} \sqrt{|\xi|} t |\Omega|$$

uniformly for $\Omega = i |\Omega| + O(1)$. Using (12.21) we deduce

$$Re(\Theta(\xi,\Omega,t)) \leq -\frac{\pi}{2} \sqrt{|\xi|} \frac{|Z|}{\lambda - 1}$$

for all $Z \in C_1$ such that $|Z| \geq M t$ assuming that $M > 0$ is sufficiently large. Using Lemma 12.1 we have:

$$Re \Phi(\xi,Z,t) \leq Re(\Theta(\xi,\Omega,t)) + C \left( Z^2 + O \left( \frac{1}{|\xi|} \right) \right) \leq -\frac{\pi}{2} \sqrt{|\xi|} \frac{|Z|}{\lambda - 1} + C \left( Z^2 + O \left( \frac{1}{|\xi|} \right) \right) \leq -\sqrt{|\xi|} |Z| \left( \frac{\pi}{2(\lambda - 1)} - \varepsilon_1 + O \left( \frac{1}{|\xi|^{3/2} t} \right) \right)$$

for all $Z \in C_1$ such that $M t \leq |Z| \leq \varepsilon_0 \sqrt{|\xi|}$. The result follows for $\varepsilon_1$ small enough and $|\xi|^t^2 \to +\infty$.

**Lemma 12.10** For all $B > 0$ there exists $\xi_0$ and $C > 0$ such that

$$\left| \frac{\partial^\ell \Psi}{\partial \xi^\ell} (\xi,Y,t) \right| \leq C \frac{|Y|}{|\xi|^t}, \quad \ell = 1, 2$$

(12.23)

for $Y = Z \sqrt{|\xi|}$, $|Y| \leq \frac{|\xi|}{\varepsilon_0}$, $Z \in D(\xi,B)$ and $Re(\xi) > \xi_0$. 

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Proof. Differentiating the function $\Psi$:

$$\Psi(\xi, Y, t) = \frac{2}{(\lambda - 1)i} \int_{Im \eta = \beta_1} \ln (-\Phi(\eta)) \Theta(\eta - \xi, Y) d\eta - \frac{2iY}{\lambda - 1} \ln(t) - \frac{2iY}{\lambda - 1} + \left(\frac{2iY}{\lambda - 1} - \frac{1}{2}\right) \ln \left(\frac{2iY}{\lambda - 1}\right)$$

with respect to $\xi$ we obtain: then,

$$\frac{\partial \Psi}{\partial \xi}(\xi, Y, t) = \frac{2}{(\lambda - 1)i} \int_{Im \eta = \beta_1} \ln (-\Phi(\eta)) \frac{\partial \Theta}{\partial \xi}(\eta - \xi, Y) d\eta - \frac{2}{(\lambda - 1)i} \int_{Im \eta = \beta_1} \ln (-\Phi(\eta)) \frac{\partial \Theta}{\partial \eta}(\eta - \xi, Y) d\eta = \frac{2}{(\lambda - 1)i} \int_{Im \eta = \beta_1} \frac{\Phi'(\eta)}{\Phi(\eta)} \Theta(\eta - \xi, Y) d\eta$$

By Proposition 3.1 we have

$$\left|\frac{\Phi'(\eta)}{\Phi(\eta)}\right| \leq \frac{C}{1 + |\eta|}$$

in the domain $Y/\sqrt{|\xi|} \in D(\xi, B)$. We deform the contour of integration to $C_2$ defined in Figure 4. Then we split the integral in two pieces:

$$\left|\frac{\partial \Psi}{\partial \xi}(\xi, Y, t)\right| \leq C \int_{\eta \in C_2, |\eta - \xi| \leq \frac{|\xi|}{4}} \frac{1}{1 + |\eta|} |\Theta(\eta - \xi, Y)||d\eta| + C \int_{\eta \in C_2, |\eta - \xi| \geq \frac{|\xi|}{4}} \frac{1}{1 + |\eta|} |\Theta(\eta - \xi, Y)||d\eta| = J_1 + J_2.$$

By the exponential decay of the function $\Theta$:

$$J_2 \leq Ce^{-\alpha|\xi|}$$

for some positive constant $a$. On the other hand,

$$\Theta(\eta - \xi, Y) = \frac{1}{1 - e^{-\frac{\alpha}{\lambda - 1}(\eta - \xi)}} - \frac{1}{1 - e^{\frac{\alpha}{\lambda - 1}(-(\eta - \xi) + Y)}}.$$

The integral $J_1$ is then divided in two parts. The first, $J_{1,1}$ is the integral along the “vertical part of the curve” $C_2$. The second, $J_{1,2}$ is along the horizontal part of that curve, where $Im(\eta) = \beta_1$. In the integral $J_{1,1}$, $Re(\eta)$ is bounded and therefore $|\Theta(\eta - \xi, Y)| \leq C$.

Since the total length of the integration curve of $J_{1,1}$ is of order $|Y|$ we deduce that $J_{1,1} \leq C|Y|/(1 + |\xi|)$. We split the integral $J_{1,2}$ as follows:

$$J_{1,2} \leq \frac{C}{1 + |\xi|} \left( \int_{Im(\eta) = \beta_1, |\eta - \xi| \leq 2|Y|} |\Theta(\eta - \xi, Y)||d\eta + \int_{Im(\eta) = \beta_1, |\eta - \xi| \geq 2|Y|} |\Theta(\eta - \xi, Y)||d\eta \right)$$

$$+ \int_{Im(\eta) = \beta_1, |\eta - \xi| \geq 2|Y|} |\Theta(\eta - \xi, Y)||d\eta \leq \int_{Im(\eta) = \beta_1, |\eta - \xi| \geq 2|Y|} \left|\frac{e^{-\frac{\alpha}{\lambda - 1}(\eta - \xi)}}{(1 - e^{-\frac{\alpha}{\lambda - 1}(\eta - \xi)})^2} - \frac{e^{-\frac{\alpha}{\lambda - 1}(Y - \xi)}}{(1 - e^{-\frac{\alpha}{\lambda - 1}(Y - \xi)})^2} \right| d\eta.$$

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We use now that, if $\text{Re}(\sigma) > 2|Y|$:  

$$\left| \frac{e^{-\frac{4\pi}{x-1} \sigma} - e^{\frac{4\pi}{x-1} (Y-\sigma)}}{(1 - e^{-\frac{4\pi}{x-1} \sigma})(1 - e^{\frac{4\pi}{x-1} (Y-\sigma)})} \right| \leq C e^{-\frac{2\pi}{x-1} |\sigma|}$$

and if $\text{Re}(\sigma) < -2|Y|$:  

$$\left| \frac{e^{-\frac{4\pi}{x-1} \sigma} - e^{\frac{4\pi}{x-1} (Y-\sigma)}}{(1 - e^{-\frac{4\pi}{x-1} \sigma})(1 - e^{\frac{4\pi}{x-1} (Y-\sigma)})} \right| \leq C \frac{e^{\frac{4\pi}{x-1} Y}}{(e^{\frac{4\pi}{x-1} \sigma} - 1)(e^{\frac{4\pi}{x-1} (\sigma-Y)} - 1)} \leq C e^{\frac{4\pi}{x-1} Y} e^{-\frac{4\pi}{x-1} |\sigma|}.$$

The last remaining term is easily estimated by:  

$$\int_{I_m(\eta) = \beta_1, |\eta - \xi| \leq 2|Y|} |\Theta(\eta - \xi, Y)| |d\eta| \leq C \int_{I_m(\eta) = \beta_1, |\eta - \xi| \leq 2|Y|} |d\eta| \leq C |Y|.$$

It then follows that $J_{1,2} \leq C e^{-a|Y|}/(1 + |\xi|)$ for positive constant $C$ and $a$. This ends the proof of (12.23) for $\ell = 1$. Similarly,  

$$\frac{\partial^2 \Psi}{\partial \xi^2} (\xi, Y, t) = \frac{2}{(\lambda - 1) i} \int_{I_m(\eta) = \beta_1} \left( \frac{\Phi'(|\eta|)}{\Phi(|\eta|)} \right)' \Theta(\eta - \xi, Y) d\eta$$

with  

$$\left| \left( \frac{\Phi'(|\eta|)}{\Phi(|\eta|)} \right)' \right| \leq \frac{C}{(1 + |\eta|)^2}$$

for $Y/\sqrt{|\xi|} \in D(\xi, B)$, again by Proposition 3.1. The proof of (12.23) follows then from the same arguments as those of the proof of (12.23) for $\ell = 2$.  

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