AdS/CFT endows gravity in anti-de Sitter (AdS) spacetime with a dual description in certain conformal field theories (CFTs) with matching symmetries. Tensor networks on regular discretizations of AdS space are natural toy models of AdS/CFT, but break the continuous bulk symmetries. We show that this produces a quasiregular conformal field theory (qCFT) on the boundary and rigorously compute its symmetries, entanglement properties, and central charge bounds, applicable to a wide range of existing models. An explicit AdS/qCFT model with exact fractional central charges is given by holographic quantum error correcting codes based on Majorana dimers. These models also realize the strong disorder renormalization group, resulting in new connections between critical condensed-matter models, exact quantum error correction, and holography.

FIG. 1. A time-slice of anti-de Sitter spacetime projected onto the Poincaré disk with radial coordinate $0 \leq \rho < 1$, shown in a continuous (left) and discretized form (right). The length of the geodesic $\gamma_A$ homologous to a region $A$ at the boundary $\rho \to 1$ determines the entanglement entropy $S_A$, but both $|A|$ and $|\gamma_A|$ are affected by discretizations. This example includes a massive deformation in the bulk, which does not affect the shape of $\gamma_A$ close to the boundary.

In this letter, we propose that tensor networks on regular discretizations of AdS time-slices produce states of a quasiregular conformal field theory (qCFT), a discretization of a CFT that breaks translation invariance equally on all scales, in what we call an AdS/qCFT correspondence. This extends an earlier observation that the boundary geometry of regular tilings resembles a quasicrystal [20]. By exploring the symmetries of tensor network states on such geometries, describing their renormalization group (RG) flow, and computing the effective central charges, we provide a physical realization of this connection. From this consistent discretization, we find that properties of continuum AdS/CFT – such as the Brown-Henneaux relation between bulk curvature and central charge [11] – need to be modified in a discrete setting. Finally, we introduce a concrete tensor network model that realizes the AdS/qCFT proposal.

Conformal boundary symmetries. In $2+1$-dimensions, a time-slice of AdS spacetime can be projected onto the Poincaré disk (Fig. 1) with the metric

$$ds^2 = 4\alpha^2 d\rho^2 + \rho^2 d\phi^2 \left(1 - \frac{\rho^2}{\alpha^2}\right)^2,$$

where $G$ is the gravitational constant in the AdS bulk. A further connection between AdS/CFT and QI, deeply related to the RT formula [5], is provided by quantum error correction, as the information of the bulk theory appears to be fault-tolerantly encoded on the boundary [6].

While the AdS/CFT conjecture was formulated in the continuum, it is tempting to combine it with a discrete language tailored for QI problems: That of tensor networks, which naturally incorporate the RT formula in the form of an upper bound on entanglement and yield boundary quantum states that can be efficiently computed. The multi-scale entanglement renormalization ansatz (MERA) [7], a tensor network that well approximates critical boundary states, was first identified as a possible realization of discrete holography [8, 9], but the bulk geometry of the MERA cannot be directly related to an AdS time-slice [10–12]. Instead, regular hyperbolic tilings have recently been used as the basis of numerous discrete holographic models [13–19], elucidating many aspects of AdS/CFT, including its connection to quantum error correction [13]. However, a clear interpretation of the resulting boundary states in terms of critical systems, as is possible for the MERA, remained elusive.

Fundamental to the AdS/CFT correspondence is the observation that $d+1$-dimensional anti-de Sitter (AdS) spacetime has the same SO($d$, 2) symmetry as a $d$-dimensional conformal field theory (CFT) [11]. AdS/CFT predicts that specific gravitational theories in asymptotically AdS spacetime are described by the same partition function as certain holographic CFTs, usually supersymmetric ones whose additional symmetries manifest themselves on the bulk side by additional compact dimensions [2, 3]. There exists an intimate connection between AdS/CFT and QI, deeply related to the RT formula [5], is provided by quantum error correcting codes based on Majorana dimers. These models also realize the strong disorder renormalization group, resulting in new connections between critical condensed-matter models, exact quantum error correction, and holography.

In this letter, we propose that tensor networks on regular discretizations of AdS time-slices produce states of a quasiregular conformal field theory (qCFT), a discretization of a CFT that breaks translation invariance equally on all scales, in what we call an AdS/qCFT correspondence. This extends an earlier observation that the boundary geometry of regular tilings resembles a quasicrystal [20]. By exploring the symmetries of tensor network states on such geometries, describing their renormalization group (RG) flow, and computing the effective central charges, we provide a physical realization of this connection. From this consistent discretization, we find that properties of continuum AdS/CFT – such as the Brown-Henneaux relation between bulk curvature and central charge [11] – need to be modified in a discrete setting. Finally, we introduce a concrete tensor network model that realizes the AdS/qCFT proposal.

Conformal boundary symmetries. In $2+1$-dimensions, a time-slice of AdS spacetime can be projected onto the Poincaré disk (Fig. 1) with the metric

$$ds^2 = 4\alpha^2 d\rho^2 + \rho^2 d\phi^2 \left(1 - \frac{\rho^2}{\alpha^2}\right)^2,$$
where the constant $\alpha$ is the AdS radius. The AdS boundary is located at $\rho \to 1$. In the continuum, the Poincaré disk is invariant under $PSL(2,\mathbb{R})$ transformations. Describing a coordinate point by the complex number $z = \rho e^{i\theta}$, these transformations have the form

$$z \mapsto z' = e^{i\theta} \frac{w + z}{1 + w^* z}$$

(3)

effectively shifting the origin of the Poincaré disk to the point $w$ with $|w| < 1$ and rotating by an angle $\theta$. We will refer to these as Möbius transformations, as $PSL(2,\mathbb{R})$ is a subgroup of the Möbius group $PG\text{L}(2,\mathbb{C})$. On the Poincaré disk boundary $\rho \to 1$, these transformations are equivalent to translations and local scale transformations, equivalent to the effect of conformal transformations in 1+1 dimensions restricted to a time-slice. Furthermore, as length scales diverge at $\rho \to 1$, our choice of a UV cutoff $\rho_0$ corresponds to a global scale transformation. Note that in AdS/CFT we often consider spacetimes that are only asymptotically AdS, i.e., have the form close to the AdS boundary but may contain massive deformations further in the bulk.

Discrete conformal transformations. Discretizing the Poincaré disk by a hyperbolic tiling breaks these continuous symmetries. We consider a regular $\{n, k\}$ tiling built of $n$-gons, $k$ of which are adjacent at each corner (the $k \to \infty$ case of ideal regular tilings and their symmetries was considered in [10]). First consider the global scale transformation: Instead of a radial cutoff $\rho_0$ in the Poincaré disk, the lattice is truncated after a finite number of inflation steps, a prescription for iterating growth the tiling. For finite $k$, a natural choice of such an inflation rule is given by vertex inflation, whereby all tiles around an open corner vertex are added in each iteration. This discrete cutoff does not follow a smooth curve at constant radius; the boundary geometry is in fact quasiregular, with self-similarity on all scales [20]. After sufficiently many inflation steps, the scaling of the total system size $L$ in each step approaches an asymptotic value depending on the tiling. Specifically,

$$L \mapsto L' = 2 + f(n, k) + \sqrt{4 + f(n, k)f(n, k)} L,$$

(4)

where $f(n, k) = nk - 2(n + k)$ is positive for any hyperbolic $\{n, k\}$ tiling (and vanishing for flat ones). This scaling law also holds for the length of any sufficiently fine-grained boundary region $A$. Similarly, we can compute the asymptotic scaling of the length of any discrete bulk geodesic $|\gamma_A|$ through each inflation step. Consider a tensor network of bond dimension $\chi$ embedded into this tiling, with each tile corresponding to a tensor and each edge between two tiles to a contraction of indices. For a minimal cut $\gamma_A$ through this network having the same endpoints as a boundary region $A$, its entanglement entropy is upper-bounded as

$$S_A \leq \frac{|\gamma_A|}{s} \log \chi,$$

(5)

where $s$ is the geodesic length of each edge, which is constant for regular tilings. We can thus bound the scaling of $S_A$ as well, and by extension, compute the maximal central charge $c^{max}$ [21] of the boundary state for any $\{n, k\}$ tiling [22]. Furthermore, we can relate $c^{max}$ to the Gaussian curvature $K = -1/\alpha^2$ of the Poincaré disk into which the tiling is embedded. The AdS radius $\alpha$ is then given by

$$\frac{s}{\alpha} = 2 \text{arccosh}(\frac{\cos \frac{\pi}{n}}{\sin \frac{\pi}{n}}) = 2 \log\left(\frac{2k}{\pi} \cos \frac{\pi}{n}\right) + O(k^{-2}).$$

(6)

We thus arrive at a discrete, tiling-dependent generalization of the Brown-Henneaux formula, examples of which are shown as dashed curves in Fig. [2]. At fixed $n$ and varied (small) $k$, we find the boundary states of $\{n, k\}$ tilings approximately following the continuum formula by a constant central charge offset [22].

Boundary translation invariance is also broken by the discretization. We can still center the tiling around either a vertex or a tile center in the Poincaré disk to produce a global $\mathbb{Z}_n$ or $\mathbb{Z}_k$ cyclic symmetry, but towards the boundary, these bulk rotations correspond to an asymptotically infinite translation. However, due to the quasiregular structure of the boundary we recover an approximate translation invariance: For example, the (vertex) inflation rule for the $\{5, 4\}$ tiling can be written as

$$a \mapsto abaab, \quad b \mapsto ab,$$

(7)

where $a$ and $b$ stand for boundary vertices connected to two and three edges, respectively. Starting from any arbitrary sequence of $as$ and $bs$ (i.e., an arbitrary convex bulk region), applying the inflation rule eventually leads to the same distribution of both letters, not only within the whole sequence, but within any sufficiently large sub-sequence. Thus, the boundary features of a regular tiling are equally distributed in any sufficiently large boundary region, leading to an approximate invariance under translations. We also need to discretize the Möbius transformations [3], which are broken down to shifts of lattice vectors plus suitable rotations. At a given inflation step, such a transformation changes the resolution of boundary lattice points in different angular regions of the Poincaré disk boundary, equivalent to a local rescaling.
Each arrow from Majorana mode $j$ to $k$ in this diagram corresponds to a term $i \gamma_j \gamma_k$ in the stabilizer Hamiltonian. We can use these states as the building blocks of a class of holographic models known as happy codes. Specifically, we construct the hyperbolic pentagon code (HyPeC), a tensor network embedded into a $[5,4]$ tiling with each six-leg tensor representing the $[5,1,3]$ encoding isometry: Five contracted legs represent the physical sites, while the sixth uncontracted leg represents the logical qubit encoded on the other five. Thus the contracted network represents a mapping of bulk qubits (one on each pentagon) to the physical sites on the boundary of the network. For a basis state bulk input, tensor network contraction is equivalent to simply connecting dimers across different tiles and multiplying their dimer parities. As each dimer carries an entanglement entropy of $\frac{1}{2} \log 2$, an inflation step is equivalent to an entanglement renormalization step [25], adding entanglement through the addition of dimers along the boundary. Again, relating the growth of boundary regions to the growth of entanglement over cuts under an inflation step allows us to analytically compute central charges; for example, for the $[5,4]$ HyPeC,

$$c_{[5,4]}^d = \frac{6 \log 2}{\log \frac{3 + \sqrt{5}}{2}} \approx 4.32,$$

$$c_{[5,4]}^e = \frac{9 \log 2}{\log (2 + \sqrt{3})} \approx 4.74,$$

for edge and vertex inflation, respectively [22]. On the level of the boundary states, a global inflation step results in a fine-graining transformation that leaves the correlation structure invariant (Fig. 3). This fulfills the first qCFT condition of invariance under discrete global scaling transformations. In fact, as shown in Fig. 2 the HyPeC (as well as its block-perfect generalizations) saturate the general central charge bound at large curvature, i.e., for $\{n,k\}$ tilings at large $k$. Local scaling transformations, which are produced by bulk Möbius transformations, also leave boundary states invariant. Consider the bulk shift of the contracted region shown in Fig. 4. This shift is equivalent to inflation (fine-graining) of one part of the boundary and deflation (coarse-graining) of another, leading to the same boundary state up to a translation. This implies self-similarity of the boundary states between scales: Large subsystems possess the same correlation structure as small ones, and local inflations only extend the resolution of these correlations to successively smaller scales. Importantly, discrete local scaling does not change the underlying inflation method, thus leaving the boundary central charge unchanged. Finally, we also find approximate translation invariance: As Fig. 5 shows, correlations on small scales follow the same pattern in any subsystem.

The renormalization group flow of these qCFT states relates them to a well-known class of critical states: Strongly
disordered spin chains. The ground states of a large family of these models – such as the Fibonacci XXZ chain – are given by a configuration of spin singlets, and the RG flow can be written as a replacement rule for singlets [26]. This procedure is completely analogous to the inflation rule of a holographic Majorana dimer model, except that we are considering pairs of fermionic Majorana modes instead of spins and allowing for crossings between these pairs. As shown in Fig. 6, the resulting entanglement scaling and correlation functions between the Fibonacci XXZ chain and the \( \{5, 4\} \) HyPeC are similar, as well: The entanglement entropy \( S_A \) grows linearly with subsystem size \( \ell = |A| \) in discrete intervals, but logarithmically on larger scales, with a coefficient of \( c/3 \) (with \( c_{\text{Fib}} \approx 0.796 \) [27]). The average decay of correlation functions, given by a histogram of the dimers/singlets over the boundary distance \( d \) over which they are paired up, also follows the \( 1/d \) falloff, as generally expected by a critical system. In the dimer case, this histogram appears “split” into two series of correlations both decaying at the same rate; this is the result of two types of dimer pairs that appear at each length scale (compare Fig. [5].

Discussion. The appearance of aperiodic, quasiregular symmetries is a natural consequence of discretizing AdS/CFT on time-slices. The continuous bulk symmetries of such a time-slice are replaced by discrete ones, invariance under which leads to what we termed a quasiregular CFT (qCFT). We expect qCFTs to appear on the boundary of all tensor network models on a regular bulk geometry [13, 14, 16, 17, 28], including p-adic AdS/CFT models [29, 30]. These models are instances of an AdS/qCFT correspondence in that the same symmetry constraints apply to all of them, with their central charge restricted by the bulk curvature through a generalized Brown-Henneaux relation. The hyperbolic pentagon code (HyPeC), studied here in its Majorana dimer form, shows that qCFTs are closely related to strongly disordered critical models, which have been extensively studied in the condensed matter literature [26, 31, 33]. As the HyPeC is also a model of quantum error correction, we find that AdS/qCFT includes exact holographic codes, rather than the approximate codes found in AdS/MERA models [36]. The similarity to strongly disordered models also suggests that boundary dynamics can be described by an effective local (though not nearest-neighbor) Hamiltonian, which would allow for dynamical AdS/qCFT models. Further work should also explore the role of qCFT excitations, where the regular symmetries of the tensor network are (locally) broken.

Acknowledgements. We thank Marek Gluza, Xiaoliang Qi, Sukhinder Singh, Tadashi Takayanagi, and Charlotte Verhoeven for helpful comments and discussions. This work has been supported by the Templeton Foundation, the DFG (CRC 183, EI 519/15-1), the NKFIH (K124351, K124152, K124176), and the FQXi. This work has also received funding from the European Unions Horizon 2020 research and innovation programme under grant agreement No. 817482 (PASQuanS). This research was supported in part by the Perimeter Institute for Theoretical Physics. Research at Perimeter Institute is supported by the Government of Canada through the Department of Innovation, Science, and Economic Development, and by the Province of Ontario through the Ministry of Research and Innovation.

FIG. 5. Approximate translation invariance of subsystems of the \( \{5, 4\} \) HyPeC dimers, projected onto the Poincaré half-plane.

FIG. 6. Entanglement entropy scaling (left) and correlation falloff (right) of the strongly disordered Fibonacci XXZ chain (top) and the \( \{5, 4\} \) HyPeC under vertex inflation (bottom). The corresponding translation-invariant form of entanglement and correlations is shown as a dashed line in each plot.

[1] J. D. Brown and M. Henneaux, Commun. Math. Phys. 104, 207 (1986)
[2] J. M. Maldacena, Adv. Theor. Math. Phys. 38, 1113 (1999) [Adv. Theor. Math. Phys. 2, 231 (1998)]
[3] E. Witten, Adv. Theor. Math. Phys. 2, 253 (1998) [arXiv:hep-th/9802150 [hep-th]]
[4] S. Ryu and T. Takayanagi, Phys. Rev. Lett. 96, 181602 (2006)
[5] D. Harlow, Commun. Math. Phys. 354, 865 (2017)
[6] A. Almheiri, X. Dong, and D. Harlow, JHEP 1504, 163 (2015).
[7] G. Vidal, Phys. Rev. Lett. 101, 110501 (2008).
[8] B. Swingle, Phys. Rev. D 86, 065007 (2012).
[9] S. Singh, Phys. Rev. D 97, 026012 (2018).
[10] C. Beny, New J. Phys. 15, 023020 (2013).
[11] N. Bao, C. J. Cao, S. M. Carroll, A. Chatwin-Davies, N. Hunter-Jones, J. Pollack, and G. N. Remmen, Phys. Rev. D91, 125036 (2015).
[12] A. Milsted and G. Vidal, (2018), arXiv:1812.00529 [hep-th].
[13] F. Pastawski, B. Yoshida, D. Harlow, and J. Preskill, JHEP 2015, 149 (2015).
[14] G. Evenbly, Phys. Rev. Lett. 119, 141602 (2017).
[15] A. Jahn, M. Gluza, F. Pastawski, and J. Eisert, Science Advances 5, eaaw0092 (2019).
[16] T. J. Osborne and D. E. Stiegemann, (2017), arXiv:1706.08823 [quant-ph].
[17] R. J. Harris, N. A. McMahon, G. K. Brennen, and T. M. Stace, Phys. Rev. A 98, 052301 (2018).
[18] T. Kohler and T. Cubitt, JHEP 08, 017 (2019) [JHEP 19, 017 (2020)].
[19] A. Jahn, M. Gluza, F. Pastawski, and J. Eisert, Phys. Rev. Research 1, 033079 (2019).
[20] L. Boyle, M. Dickens, and F. Flicker, Phys. Rev. X10, 011009 (2020) [arXiv:1805.02665 [hep-th]]
[21] F. Calabrese and J. L. Cardy, J. Stat. Mech. 0406, P06002 (2004) [arXiv:hep-th/0405152 [hep-th]]
[22] A. Jahn, Z. Zimboras, and J. Eisert, (2019), arXiv:1911.03485 [hep-th].
[23] C. H. Bennett, D. P. DiVincenzo, J. A. Smolin, and W. K. Wootters, Phys. Rev. A 54, 3824 (1996).
[24] R. Laflamme, C. Miquel, J. P. Paz, and W. H. Zurek, (1996), arXiv:quant-ph/9602019 [quant-ph].
[25] G. Vidal, Understanding Quantum Phase Transitions (2010), arXiv:0912.1651 [cond-mat.str-el].
[26] F. Igloi and C. Monthus, Eur. Phys. J. B 91, 290 (2018).
[27] R. Juhász and Z. Zimborás, J. Stat. Mech. 2007, P04004 (2007) [arXiv:cond-mat/0703527 [cond-mat.stat-mech]].
[28] P. Hayden, S. Nezami, X.-L. Qi, N. Thomas, M. Walter, and Z. Yang, JHEP 11, 009 (2016).
[29] S. S. Gubser, J. Knaute, S. Parikh, A. Samberg, and P. Witaszczyk, Commun. Math. Phys. 352, 1019 (2017).
[30] M. Heydeman, M. Marcolli, I. Säberi, and B. Stoica, Adv. Theor. Math. Phys. 22, 93 (2018).
[31] D. S. Fisher, Phys. Rev. B 51, 6411 (1995).
[32] G. Refael and J. E. Moore, Phys. Rev. Lett. 93, 260602 (2004).
[33] R. Vosk, D. A. Huse, and E. Altman, Phys. Rev. X 5, 031032 (2015).
[34] Z.-L. Tsai, P. Chen, and Y.-C. Lin, (2019), arXiv:1912.03529 [cond-mat.dis-nn].
[35] I. V. Protopopov, R. K. Panda, T. Parolini, A. Scardicchio, E. Demler, and D. A. Abanin, Phys. Rev. X 10, 011025 (2020).
[36] I. H. Kim and M. J. Kastoryano, JHEP 2017, 40 (2017).