Hydrodynamic Limits of non-Markovian Interacting Particle Systems on Sparse Graphs

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Abstract

Consider an interacting particle system indexed by the vertices of a (possibly random) locally finite graph whose vertices and edges are equipped with marks representing parameters of the model such as the environment and initial conditions. Each particle takes values in a countable state space and evolves according to a pure jump process whose jump intensities depend only on its own state (or history) and marks, and states (or histories) and marks of particles and edges in its neighborhood. Under mild conditions on the jump intensities, it is shown that if the sequence of (marked) interaction graphs converges locally in probability to a limit (marked) graph that satisfies a certain finite dissociability property, then the corresponding sequence of empirical measures of the particle trajectories converges weakly to the law of the marginal dynamics at the root vertex of the limit graph. The proof of this hydrodynamic limit relies on several auxiliary results of potentially independent interest. First, such interacting particle systems are shown to be well-posed on (almost surely) finitely dissociable graphs, which include graphs of maximal bounded degree and any Galton-Watson tree whose offspring distribution has a finite first moment. A counterexample is also provided to show that well-posedness can fail for dynamics on graphs outside this class. Next, the dynamics on a locally convergent sequence of graphs are shown to converge in the local weak sense to the dynamics on the limit graph when the latter is finitely dissociable. Finally, the dynamics are also shown to exhibit an (annealed) asymptotic correlation decay property. These results complement recent work that establishes hydrodynamic limits of locally interacting probabilistic cellular automata and diffusions on sparse random graphs. However, the analysis of jump processes requires very different techniques, including percolation arguments and notions such as consistent spatial localization and causal chains.

Contents

1 Introduction 2
2 Preliminaries and Notation 5
3 Model Description 8
4 Statement of Main Results 11
5 Spatial Localization and the Proof of Well-Posedness 17

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1 Introduction

We consider interacting particle systems (IPS) indexed by the vertices of a (possibly random) locally finite graph that encodes the interaction structure between the particles and whose vertices and edges are equipped with marks representing parameters of the model such as the environment and initial conditions. Particles take values in a countable state space and evolve according to a pure jump process, with the jump intensities of each particle depending only on its own history and marks, and the histories and marks of particles and edges in its neighborhood in the graph. The framework covers standard IPS like the voter model and Glauber dynamics for the Ising and Potts models, as well as heterogeneous models like the contact process in a random environment and non-Markovian processes such as the renewal contact process. It can also be extended to cover models with directed interactions such as (the non-Markovian) neuronal Hawkes models (see Remark 2.2).

An important quantity of interest is the empirical measure of the particle trajectories. When the underlying interaction graph is the complete graph, it is a classical result that under general assumptions, neighboring particles become asymptotically independent as the number of particles goes to infinity (a phenomenon referred to as propagation of chaos) and the corresponding sequence of empirical measure processes converges to a deterministic limit whose evolution is described by a measure-valued equation, often referred to as the mean-field limit [20, 28]. Recent work has established similar convergence results in the case of certain (sufficiently) dense graph sequences (see, e.g., [7]), for which the minimum (or average) degree of the graph goes to infinity with the size of the graph, in which case the strength of the interaction between neighboring particles still decays to zero as the number of particles goes to infinity and propagation of chaos still holds.

The focus of this article is on the complementary case when the underlying graph is truly sparse (i.e., with uniformly bounded average degree). Such IPS arise as models in a wide variety of fields including statistical physics [26], epidemiology [3, 19, 30], neuroscience [34], opinion dynamics [8, 18, 25], engineering and operations research [7, 32]. Unlike in the case of complete or dense graphs, for sparse graph sequences one cannot expect the empirical measure process to have a limit simply by sending the number of particles to infinity since the topologies of the graphs in the sequence also matter. Instead, a suitable alternative that respects the topology is the notion of local convergence of sparse graphs introduced by Benjamini and Schramm [2]. For a broad class of IPS we show that when the sequence of finite interaction graphs $G_n$, additionally marked with initial conditions, converges to a limit (finite) marked almost surely finitely dissociable graph $G$ in probability in the local weak sense (as specified precisely in Definition 4.4), then the sequence of empirical measure processes of the IPS on $G_n$ converges weakly to a deterministic limit, referred to as the hydrodynamic limit (see Corollary 4.7). The class of finitely dissociable graphs, specified in Definition 5.11 includes graphs with bounded maximal degree and Galton-Watson trees whose offspring distributions have finite mean (see Proposition 5.15), which are of particular interest since they arise naturally as local weak limits of many random graph sequences such as Erdös-Rényi graphs and configuration models [6, Theorems 3.12 and 3.15]. In fact, we show that the
hydrodynamic limit coincides with the law of the trajectory of a typical (or root) particle of the IPS on the limit (marked) graph $G$. When the limit graph is a tree, an autonomous description of the dynamics of the hydrodynamic limit (which is non-Markovian even when the original IPS is Markovian) is provided in forthcoming work.

The first step of the proof entails establishing strong well-posedness (in the sense of Definition 3.7) for IPS on finitely dissociable graphs, which is of potential independent interest. Indeed, although several recent works studying IPS on random graphs provide intuitive descriptions of the IPS [3, 17, 19, 27, 31], there appears to be no general result that rigorously establishes well-posedness of even Markovian IPS on a general class of random graphs. While well-posedness of IPS on finite graphs is standard under our assumptions, on infinite graphs the issue is more subtle and well-posedness can in fact fail to hold even for simple Markovian IPS, as illustrated by the simple example in Appendix A. Previous well-posedness results for IPS on infinite graphs have almost exclusively focused on graphs with uniformly bounded maximum degree. For example, on lattices, an analytical proof of well-posedness of a large class of Feller IPS via examination of their semi-groups can be found in the seminal paper of Liggett [24] (see also [25]), and a probabilistic proof of well-posedness of IPS with nearest-neighbor interactions using percolation arguments can be found in the classical work of Harris [15]. The latter argument can be extended to locally interacting IPS on translation invariant graphs, but crucially relies on the graph having uniformly bounded maximum degree. Another approach to well-posedness involves a standard Picard iteration argument applied to the (jump) stochastic differential equation (SDE) representation of the IPS dynamics. This works when the jump rates of any individual particle satisfy a strong Lipschitz property, that is, when they are uniformly Lipschitz with respect to the state (or trajectory, in the non-Markovian setting) of each of the neighboring particles with the (single-neighbor) Lipschitz coefficient being inversely proportional to the degree of the vertex (see also [10] for a slightly weaker averaged version of this Lipschitz condition). However, for even standard Markovian IPS such as the majority process or the contact process [25], the Lipschitz constant of the jump rate function with respect to the states (or trajectories and marks) of each neighboring particle does not decrease, but rather increases with the degree of the vertex of the particle, or at best remains constant. In particular, strong Lipschitz continuity of the jump rates (in the sense described above) fails on infinite (random) graphs that have unbounded maximum degree such as Galton-Watson (GW) trees with Poisson offspring distribution. In Theorem 4.2 we establish strong well-posedness of a general class of possibly non-Markovian IPS on (almost surely) finitely dissociable graphs, which includes the GW trees mentioned above.

Our proof of strong well-posedness consists of three main ingredients. First, we introduce the notion of spatial localization of the IPS dynamics (see Definition 5.1). Roughly speaking, an IPS with interaction graph $G$ is said to be spatially localized if the IPS dynamics on any finite subset $U$ of the graph $G$ coincides with the marginal dynamics on $U$ of the same IPS model on (the induced subgraph of $G$ on) a finite random subset that contains $U$. Leveraging strong well-posedness of the IPS on finite graphs, we show that the IPS is strongly well-posed on any graph $G$ that spatially localizes the IPS SDE (see Proposition 5.7). Then, under mild conditions on the jump rates, we show that the IPS SDE is spatially localized by any finitely dissociable graph (see Proposition 5.19). This proof entails the analysis of so-called causal chains that capture the propagation of influence of the dynamics at a vertex in the IPS, as well as a suitably defined (inhomogeneous) site percolation (see Section 5.4). In addition, we also show in Section 5.3 that finitely dissociable graphs include all bounded degree graphs and GW trees whose offspring distributions have finite first moments. To the best of our knowledge, the only other work that proves well-posedness of a (jump) IPS on a graph with unbounded maximal degree appears to be the recent work of Gantert and Schmidt [13], which establishes well-posedness of the simple exclusion process on a GW tree whose offspring
distribution has finite mean by crucially exploiting the special structure of the exclusion process to reduce the problem to the study of a standard bond percolation problem. Our result does not subsume that of [13], but is applicable to a wide class of possibly non-Markovian models and does not rely on specific features of the IPS.

The second step in the proof of the hydrodynamic limit is to show that the local weak convergence of a sequence of interaction graphs (marked with initial conditions) implies local weak convergence of the trajectories of the corresponding IPS (see Theorem 4.3). This result is proved via coupling arguments that entail establishing a certain consistent spatial localization property of the sequence of interaction graphs (see Definition 5.4). The final ingredient of the proof is an asymptotic spatial decay of correlations of the trajectories of the IPS that is annealed (or averaged over the randomness of the graph). Specifically, in Theorem 4.3 we show that although neighboring vertices remain strongly correlated for sparse graph sequences (in contrast to dense graph sequences), finite neighborhoods of two independent randomly chosen vertices become asymptotically independent as the number of particles goes to infinity. The proof of the asymptotic correlation decay property involves suitable coupling arguments and also exploits the local convergence result.

The present article complements recent work by Oliveira et al [29], which establishes local convergence of interacting diffusions with (possibly random) pairwise interactions on locally convergent sequences of finite graphs, and the works of Lacker et al [22,23,33], which establish hydrodynamic limits for homogeneously interacting cellular automata and diffusions with general (not necessarily pairwise) symmetric interactions. The hydrodynamic limit in [22] is also shown by first establishing local weak convergence and then asymptotic correlation decay, but the proofs of these results rely crucially on the previously mentioned strong Lipschitz continuity conditions on the drift and diffusion coefficients, which makes well-posedness trivial. As mentioned above, for jump processes, even simple examples require weaker assumptions on the jump rates, and well-posedness need not hold for IPS on all graphs. In addition, our framework also allows for heterogeneities, which are of relevance in many applications, and the nature of correlation decay established is different (see the discussion in Section 4.3.1 for an elaboration of this point). Thus, several new tools are required for the analysis in comparison with the diffusion setting. Finally, even though the class of IPS we focus on already covers a large class of models (see the examples in Section 4.4), many auxiliary results are established under still weaker assumptions so as to facilitate further extensions of the main results to IPS described by more general Poisson-driven SDE, in particular those with simultaneous jumps (see Remarks 5.6, 5.8, 7.3 and 7.4).

The article is organized as follows. In Section 2 we introduce notation that will be applied in the sequel. In Section 3 we introduce the class of IPS we consider, state the basic assumptions on the jump rates and also properly define notions of strong and weak solutions for IPS on random graphs. The main results are stated in Section 4 with Section 4.4 containing several examples covered by our results. The rest of the article is devoted to proofs of the main results: Section 5 introduces the notion of (consistent) spatial localization, and contains the proof of well-posedness; the local convergence result is proved in Section 6 and asymptotic correlation decay is established in Section 7. Appendix A contains an example of a simple IPS that fails to be well-posed, Appendix B contains auxiliary technical results related to canonical measurable representatives of random (marked) graph isomorphism classes, Appendix C addresses strong well-posedness on finite graphs and Appendix D establishes an alternative characterization of strong well-posedness.
2 Preliminaries and Notation

2.1 Graph Notation

Given an undirected graph \( G = (V, E) \) with vertex set \( V \) and edge set \( E \), for \( v \in V \), let \( \mathcal{N}_v = \mathcal{N}_v(G) := \{ u \in V : (u, v) \in E \} \) denote the neighbors of \( v \) in \( G \) and let \( \text{cl}_v = \text{cl}_v(G) := \{ v \} \cup \mathcal{N}_v \).

For any \( U \subseteq V \), set \( \mathcal{N}_U := \cup_{v \in U} (\mathcal{N}_v \setminus U) \) and \( \text{cl}_U := U \cup \mathcal{N}_U \). For clarity, we may write \( \text{cl}_U(G) \) to emphasize that the closure is taken with respect to edges in \( G \). We define \( \Lambda_G := \{ U \subseteq V : |U| < \infty \} \) to be the set of finite subsets of the vertices in \( G \). Recall that the degree of a vertex \( v \) is equal to \( |\mathcal{N}_v| \), where for any set \( A \), \( |A| \) denotes its cardinality. The graph \( G \) is said to be locally finite if each of its vertices has finite degree. We always assume graphs are simple (i.e., they do not have self-loops or multi-edges) and locally finite.

A graph \( G = (V, E) \) equipped with a distinguished vertex \( \emptyset \in V \), denoted the root, is called a rooted graph, and denoted by \( (G, \emptyset) := (V, E, \emptyset) \), although when the root is clear from context, we will simply write \( G \) instead of \( (G, \emptyset) \). For \( U \subseteq V \), we denote by \( G[U] \) the induced subgraph of \( G \) on \( U \), that is, \( G[U] = (U, E[U]) \) where \( E[U] = E \cap \{ \{ x, y \} : x, y \in U \} \). For \( u, v \in V \), a path between \( u \) and \( v \) in \( G \) is defined to be a sequence of vertices \( \Gamma = (u = v_0, v_1, \ldots, v_{n-1}, v_n = v) \) such that for all \( i \in \{1, \ldots, n\}, \{v_{i-1}, v_i\} \in E \) and \( v_i \neq v_j \) whenever \( i \neq j \) except possibly when \( \{i, j\} = \{n, 0\} \), in which case the path is said to be a cycle. A graph is said to be acyclic if it has no cycles. The length of the path, denoted \( |\Gamma| \), is the number of edges in the path. We let \( d(G, u, v) \) denote the usual graph distance, which is the length of the shortest path between \( u \) and \( v \) in \( G \). When \( G \) is a finite rooted graph, its radius is the maximal distance from any vertex to the root. Let \( \mathcal{S}_{k,1} \) denote the set of rooted graphs of radius \( k \).

2.2 Configurations and Path Space Notation

Given a Polish space \( Z \) and \( U \subseteq V \), we define the configuration space

\[ Z^U = \{(z_v)_{v \in U} : z_v \in Z \text{ for all } v \in U \}. \tag{2.1} \]

The space \( Z^U \) is equipped with the product topology. For any \( z \in Z^V \), we write \( z_U = (z_v)_{v \in U} \in Z^U \) to mean the restriction of \( z \) to \( Z^U \). Given two vertex sets \( V_1 \) and \( V_2 \), a map \( \varphi : V_1 \to V_2 \), a subset \( U \subseteq V_1 \), and configurations \( x \in Z^{V_1}, y \in Z^{V_2} \), we write \( x_U = y_{\varphi(U)} \) to mean \( x_v = y_{\varphi(v)} \) for all \( v \in U \). Vertex set indices are assumed to be ordered.

Let \( \mathcal{X} \) denote a countable state space, which we identify with a subset of \( Z \) and equip with the discrete topology. For any \( U \subseteq V \) and \( t \in (0, \infty] \), let \( \mathcal{D}^U := \mathcal{D}([0, t]; \mathcal{X}^U) \) (respectively, \( \mathcal{D}_-^U := \mathcal{D}([0, t); \mathcal{X}^U) \)) be the space of càdlàg functions from \([0, t]\) (respectively, \([0, t)\)) to \( \mathcal{X}^U \), equipped with the product J1 topology, which makes it a Polish space [36, Section 11.5]. Also, let \( \mathcal{D}^U := \mathcal{D}^U_{\infty} \) denote the space of càdlàg functions from \([0, \infty)\) to \( \mathcal{X}^U \), equipped with the topology such that \( x^n \) converges to \( x \) in \( \mathcal{D}^U \) if and only if for each \( t \in [0, \infty) \), the restriction of \( x^n \) to \([0, t]\) converges to the restriction of \( x \) to \([0, t]\) in \( \mathcal{D}^U \). When \( |U| = 1 \), we will denote \( \mathcal{D}^U \) or \( \mathcal{D}^U_{\infty} \) simply by \( \mathcal{D}_t \) or \( \mathcal{D} \), respectively. If \( x \in \mathcal{D}^U \) and \( v \in U \), then \( x_v(t) \) denotes the value of the \( v \)th component of \( x \) at time \( t \). The restrictions of \( x \) to \([0, t]\) and \([0, t)\) are denoted by \( x[t] \in \mathcal{D}^U_t \) and \( x[t] \in \mathcal{D}^U_{\infty} \) respectively. For \( 0 \leq s \leq t \leq \infty, W \subseteq U \subseteq V \) and \( x \in \mathcal{D}^U_t \), define the sets of jump times:

\[ \text{Disc}_s(x_W) := \{ s' \in [0, s] : x_W(s') \neq x_W(s'-) \}. \tag{2.2} \]

2.3 Measure Notation and Point Processes

For any Polish space \( Z \), let \( \mathcal{B}(Z) \) be the Borel \( \sigma \)-algebra on \( Z \), and let \( \mathcal{P}(Z) \) be the space of probability measures on \((Z, \mathcal{B}(Z))\) equipped with the topology of weak convergence, that is, \( \mu_n \)
converges to µ weakly if and only if \( \lim_{n\to\infty} \int_Z f d\mu_n = \int_Z f d\mu \) for every bounded, continuous function \( f \) on \( Z \). Given any \( \zeta \in \mathcal{P}(Z) \) and \( Z \)-valued random element \( Z \) and \( Y \), let \( \mathcal{L}(Z) \) denote the distribution (equivalently, law) of \( Z \), and we write \( Z \sim \zeta \) to mean \( \mathcal{L}(Z) = \zeta \), and \( Y \overset{(d)}{=} Z \) to mean \( \mathcal{L}(Y) = \mathcal{L}(Z) \). We additionally define \( \mathcal{M}_\ast(Z) \) to be the space of locally finite, non-negative integer-valued measures on \( (Z, \mathcal{B}(Z)) \). We equip \( \mathcal{M}_\ast(Z) \) with the weak topology. As is well known, \( \mathcal{P}(Z) \) and \( \mathcal{M}_\ast(Z) \) are Polish spaces (see [1] Theorem 6.8 and [3] Proposition 9.1IV (iii), respectively).

For any measure \( \zeta \in \mathcal{P}(I) \) for an interval \( I \subseteq \mathbb{R} \), we will write \( \zeta(a,b) \) and \( \zeta(a,b) \) etc. for \( (a,b) \) and \( (\zeta(a,b)) \) etc. whenever these intervals lie in \( I \).

A random element \( P \) taking values in \( \mathcal{M}_\ast(Z) \) is called a point process. For every compact set \( \hat{K} \subseteq \mathbb{R} \), there exists an almost surely finite set of points \( \{z_i\}_{i=1}^N \subseteq \hat{K} \), referred to as events, such that \( P(\{z_i\}) > 0 \) for all \( i = 1, \ldots, N \), and \( P(\hat{K} \setminus \{z_i\}_{i=1}^N) = 0 \). In this paper, we assume all point processes are simple, that is, \( \sup_{z \in \mathbb{Z}} P(\{z\}) \in \{0,1\} \). Given any measure \( \zeta \in \mathcal{P}(I) \) that is finite on each compact set \( K \in \mathcal{B}(Z) \), a Poisson point process on \( Z \) with intensity measure \( \zeta \) is a point process \( P \) such that for any disjoint sets \( A, B \in \mathcal{B}(Z) \), \( P(A) \) and \( P(B) \) are independent and \( \mathbb{E}[P(A)] = \zeta(A) \).

If \( \hat{Z} = I \times \mathbb{R} \), where \( I \subseteq \mathbb{R} \) is an interval and \( Z \) is a Polish space, then we refer to a point process \( P \) on \( \hat{Z} \) as a marked point process on \( I \) with marks in \( Z \). If \( P \) has events \( \{(t_i, \kappa_i)\}_{i=1}^N \), then we call \( \{\kappa_i\}_{i=1}^N \) the marks of \( P \). We say a marked point process \( P \) on \( I \) defined on the filtered probability space \( (\Omega, \mathcal{H}, \mathbb{H} = \{\mathcal{H}_t\}, \mathbb{P}) \) is \( \mathbb{H} \)-adapted if for every \( t \in I \) and \( A \in \mathcal{B}([0,t] \cap I \times \mathbb{R}) \), \( P(A) \) is \( \mathcal{H}_t \)-measurable. Furthermore, an \( \mathbb{H} \)-adapted marked Poisson point process \( P \) on \( I \) with marks in \( Z \) is said to be a \( \mathbb{H} \)-Poisson marked point process if for every \( t \in I \) and \( A \in \mathcal{B}((t,\infty) \cap I \times \mathbb{R}) \), \( P(A) \) is independent of \( \mathcal{H}_t \).

### 2.4 Local Convergence

Since we represent our IPS as marked graphs we briefly review the notion of local convergence of graphs and marked graphs, which was introduced in [2]. Let \( G_i = (V_i, E_i) \), \( i = 1,2 \), be (unrooted) graphs. A mapping \( \varphi : V_1 \rightarrow V_2 \) is said to be an isomorphism from \( G_1 \) to \( G_2 \) if it is a bijection and \( e = \{u,v\} \in E_1 \) if and only if \( \varphi(e) := \{\varphi(u),\varphi(v)\} \in E_2 \). Given roots \( \phi_i \in V_i \), \( i = 1,2 \), \( \varphi \) is an isomorphism from the rooted graph \( (G_1, \phi_1) \) to the rooted graph \( (G_2, \phi_2) \) if, in addition, \( \varphi(\phi_1) = \phi_2 \). Recall that when denoting the rooted graph, we often omit the explicit dependence on the root. Given locally finite rooted graphs \( G_1 \) and \( G_2 \), let \( I(G_1,G_2) \) denote the collection of isomorphisms from \( G_1 \) to \( G_2 \). If \( I(G_1,G_2) \) is non-empty, then \( G_1 \) and \( G_2 \) are said to be isomorphic, which is denoted \( G_1 \cong G_2 \). Let \( \mathcal{G}_s \) be the space of isomorphism classes of connected, locally finite, rooted graphs. Then for any connected, locally finite rooted graph \( G \), we let \( \langle G \rangle \in \mathcal{G}_s \) denote the isomorphism class of \( G \), namely \( \langle G \rangle \) is the collection of connected locally finite rooted graphs isomorphic to \( G \). Conversely, we refer to \( G \) as a representative graph of \( \langle G \rangle \). Clearly, if \( H \cong G \) then \( H \in \langle G \rangle \). For each \( m \in \mathbb{N} \), let \( B_m(G) \) be the induced subgraph of \( G \) consisting of all vertices within (graph) distance \( m \) of the root. We equip \( \mathcal{G}_s \) with the topology of local convergence in which \( \{\langle G_n \rangle\}_{n \in \mathbb{N}} \subseteq \mathcal{G}_s \) converges to \( \langle G \rangle \in \mathcal{G}_s \) if for every \( m \in \mathbb{N} \), there exists \( n_m < \infty \) such that \( B_m(G_n) \cong B_m(G) \) for all \( n \geq n_m, G_n \in \langle G_n \rangle \) and \( G \in \langle G \rangle \).

Next, fix any two Polish spaces \( \mathcal{K} \) and \( \mathcal{K} \) that represent the edge and vertex mark spaces, respectively, and consider a (not necessarily connected) marked rooted graph \( G = (V, E, \phi, \overline{\kappa}, \kappa) \), where \( (V, E, \phi) \) is a rooted graph, \( \overline{\kappa} \in \mathcal{K}^E \) and \( \kappa \in \mathcal{K}^V \). Also, let \( [G_s] = (V, E, \phi) \) denote the marked rooted graph \( G \) with its marks removed, and let \( [G] \) denote \( G \) with its root and marks removed. For \( m \in \mathbb{N} \), let \( B_m(G) \) be the induced marked rooted subgraph of \( G \) consisting of all vertices within (graph) distance \( m \) of the root, equipped with the same marks and root. We slightly abuse
notation at times by allowing $B_m(G)$ to also denote the set of vertices within graph distance $m$ of the root. We say the marked graphs $G := (V, E, \phi, \pi, \kappa)$ and $G' := (V', E', \phi', \pi', \kappa')$ with edge and vertex marks in $\mathcal{K}$ and $\mathcal{K}$, respectively, are isomorphic, and write $G \cong G'$ if there exists an isomorphism $\varphi \in I([G_*], [G'_*])$ such that $\kappa_v = \kappa'_{\varphi(v)}$ and $\pi_e = \pi'_{\varphi(e)}$ for all $v \in V$ and $e \in E$. We let $I(G, G')$ denote the set of isomorphisms between $G$ and $G'$, and let $\mathcal{G}_*[\mathcal{K}, \mathcal{K}]$ denote the collection of isomorphism classes of graphs with edge and vertex marks in $\mathcal{K}$ and $\mathcal{K}$, respectively. Once again, for any such marked graph $G$, $\langle G \rangle \in \mathcal{G}_*[\mathcal{K}, \mathcal{K}]$ denotes the isomorphism class of $G$. Likewise, for any $\langle H \rangle \in \mathcal{G}_*[\mathcal{K}, \mathcal{K}]$, the marked graph $H$ (with edge and vertex marks in $\mathcal{K}$ and $\mathcal{K}$, respectively) denotes an arbitrary representative of $\langle H \rangle$. Also, given a (possibly marked or rooted) graph $G$, we let $V_G$ and $E_G$, respectively, denote the vertex and edge sets of $G$. We also occasionally abuse notation by letting $G$ denote its own vertex set.

We equip $\mathcal{G}_*[\mathcal{K}, \mathcal{K}]$ with the topology of local convergence, defined as follows.

**Definition 2.1** (Local convergence). The sequence $\langle G_n \rangle \in \mathcal{G}_*[\mathcal{K}, \mathcal{K}]$, $n \in \mathbb{N}$, is said to converge locally to $\langle G \rangle \in \mathcal{G}_*[\mathcal{K}, \mathcal{K}]$ if for every sequence of representatives $G_n = (V_n, E_n, \phi_n, \pi_n, \kappa_n)$, $n \in \mathbb{N}$, and $G = (V, E, \phi, \pi, \kappa)$, and for every $m \in \mathbb{N}$ there exists $n_m < \infty$ and a sequence $\varphi_{n,m} \in I(B_m([G_*]), B_m([G'_*]))$, $n > n_m, n \in \mathbb{N}$, such that for every $v \in V_{B_m([G_*])}$ and $e \in E_{B_m([G_*])}$, $\kappa^n_{\varphi_{n,m}(v)} \rightarrow \kappa_v$ and $\pi^n_{\varphi_{n,m}(e)} \rightarrow \pi_e$ as $n \rightarrow \infty$.

We also let $\mathcal{G}_{s,1}[\mathcal{K}, \mathcal{K}] \subset \mathcal{G}_*[\mathcal{K}, \mathcal{K}]$ denote the (closed) space of isomorphism classes of graphs in $\mathcal{G}_{s,1}$ with edge and vertex marks in $\mathcal{K}$ and $\mathcal{K}$, respectively, equipped with the topology induced by $\mathcal{G}_*[\mathcal{K}, \mathcal{K}]$. Note that one can also view $\mathcal{G}_* = \mathcal{G}_s[\{1\}, \{1\}]$ as a space of (isomorphism class of) marked graphs with trivial marks, that is, when both mark spaces are equal to the trivial Polish space $\{1\}$, in which case the local convergence defined above coincides with the notion defined earlier on $\mathcal{G}_s$.

It is well known that $\mathcal{G}_s[\mathcal{K}, \mathcal{K}]$ and hence, $\mathcal{G}_{s,1}[\mathcal{K}, \mathcal{K}]$ and $\mathcal{G}_s$ (equipped with the above local topology), are Polish spaces (see [3] Lemma 3.4). If $\{\zeta_n\}_n \subset \mathcal{P}(\mathcal{G}_s[\mathcal{K}, \mathcal{K}])$ converges in distribution to $\zeta$, then we say that $\zeta_n$ converges to $\zeta$ locally weakly, and denote it by $\zeta_n \Rightarrow \zeta$. If $G_n \sim \zeta_n$ for every $n \in \mathbb{N}$ and $G \sim \zeta$, we may also write $G_n \Rightarrow G$ to denote the local weak convergence of $G_n$ to $G$. We refer the reader to [6, 35] and [22] Appendix A] for general results on local weak convergence, and to [22] Section 2.2.3 for several standard examples of local weak convergence of sequences of random graphs, which include sequences of Erdős-Rényi graphs, configuration models and random regular graphs, all of which converge to UGW trees (which are defined in Section 5.14.1). As mentioned in Section 2.3, we will assume all $\mathcal{G}_s[\mathcal{K}, \mathcal{K}]$-random elements are measurable with respect to the Borel $\sigma$-algebra defined in terms of the local topology.

**Remark 2.2.** While the above discussion focused on local convergence of undirected graphs, there exist several frameworks for working with local convergence of directed graphs. One possible approach is to define a space of isomorphism classes of marked directed graphs in which all isomorphisms additionally respect edge orientation (in the spirit of [35] Exercise 2.17]). Using an argument similar to that used in the proof of [6 Lemma 3.4]), it is possible to then show that this space is Polish and, furthermore, one can construct a random map that “lifts” marked directed graphs from this new space to marked undirected graphs in a suitably continuous way. This construction can be combined with the results of this paper to obtain convergence and hydrodynamic limit results for a large class of IPS on directed sparse graphs, including neuronal Hawkes processes [33]. A fully rigorous justification is somewhat technical, and hence postponed to future work.
3 Model Description

3.1 A Standing Assumption on the Jump Rate Functions

We consider IPS in which each particle takes values in a countable state space \( \mathcal{X} \), which we identify with a subset of \( \mathbb{Z} \) and equip with the discrete topology. We let \( \mathcal{J} \subseteq \{i - j : i, j \in \mathcal{X}, i \neq j \} \) denote the set of allowable transitions or jumps of any particle, and equip \( \mathcal{J} \) with a finite measure \( \zeta \in \mathcal{P}(\mathcal{J}) \) that assigns strictly positive mass to all elements of \( \mathcal{J} \) and satisfies \( \sum_{j \in \mathcal{J}} |j| \zeta(j) < \infty \), where we write \( \zeta(j) \) for \( \zeta(\{j\}) \). Let \( \mathcal{K} \) and \( \mathcal{K} \) be two Polish spaces that specify static parameters of the model such as random environments (see Example 1.19), histories before time zero for non-Markovian processes (see Example 1.10), or heterogeneities and other graph attributes. The model is specified by a family of jump rate functions \( \mathbf{r} := \{r_{j}^{G,v} : \mathbb{R}^{+} \times D^{V}G \rightarrow \mathbb{R}^{+}, j \in \mathcal{J}, G \in [\mathcal{K}, \mathcal{K}] \} \)-graph and \( v \in V_{G} \), where recall from Section 2.2 that \( D \) is the space of càdlàg functions taking values in \( \mathcal{X} \).

We now define a notion of regularity of jump rate functions. Recall that \( \mathcal{G}_{s,1} \) denotes the set of finite rooted graphs \( H = (V_{H}, E_{H}, \phi_{H}) \) of radius 1.

**Definition 3.1** (Regularity of local rate functions). Given \( j \in \mathcal{J} \), the family of functions \( \tilde{r}_{j}^{H} : \mathbb{R}^{+} \times \mathcal{D}^{V}H \times \mathcal{K}^{E}H \times \mathcal{K}^{V}H \rightarrow \mathbb{R}^{+} \), \( H = (V_{H}, E_{H}, \phi_{H}) \in \mathcal{G}_{s,1} \), is said to be regular if \( \tilde{r}_{j}^{H} \) is Borel measurable for each \( H \in \mathcal{G}_{s,1} \), and in addition,

1. for every \( t > 0 \), \( H \mapsto \tilde{r}_{j}^{H}(t, \cdot, \cdot, \cdot) \) is a class function in the sense that for any \( (x, \pi, \kappa) \in \mathcal{D}^{V}H \times \mathcal{K}^{E}H \times \mathcal{K}^{V}H \), \( \tilde{H} \in \mathcal{G}_{s,1} \) with \( \tilde{H} \equiv H \), and \( \varphi \in \mathbb{I}(\tilde{H}, H) \),
   \[
   \tilde{r}_{j}^{H}(t, (x_{v})_{v \in V_{H}}, (\pi_{e})_{e \in E_{H}}, (\kappa_{v})_{v \in V_{H}}) = \tilde{r}_{j}^{H}(t, (x_{\varphi(v)})_{v \in V_{\tilde{H}}}, (\pi_{e})_{e \in E_{\tilde{H}}}, (\kappa_{v})_{v \in V_{\tilde{H}}});
   \]
2. for every \( H \in \mathcal{G}_{s,1} \), \( (x, \pi, \kappa) \in \mathcal{K}^{E}H \times \mathcal{K}^{V}H \), \( (t, x, \pi, \kappa) \) is predictable in the sense that for every \( t > 0 \) and \( x, y \in \mathcal{D}^{V}H \),
   \[
   x(s) = y(s) \quad \forall s \in [0, t) \quad \Rightarrow \quad \tilde{r}_{j}^{H}(t, x, \pi, \kappa) = \tilde{r}_{j}^{H}(t, y, \pi, \kappa).
   \]

**Standing Assumption.** For every \( j \in \mathcal{J} \), there exists a family of regular (in the sense of Definition 3.1) functions \( \{\tilde{r}_{j}^{H} : \mathbb{R}^{+} \times \mathcal{D}^{V}H \times \mathcal{K}^{E}H \times \mathcal{K}^{V}H \rightarrow \mathbb{R}^{+} \}_{H=(V_{H}, E_{H}, \phi_{H}) \in \mathcal{G}_{s,1}} \) such that given any \( [\mathcal{K}, \mathcal{K}] \)-graph \( G = (V, E, \emptyset, \pi, \kappa) \) and \( v \in V \),

\[
\tilde{r}_{j}^{G,v}(t, x) = \tilde{r}_{j}^{H_{v}}(t, x_{H_{v}}, (\pi_{e})_{e \in E_{H_{v}}}, (\kappa_{u})_{u \in V_{H_{v}}}), \quad \text{for every } (t, x) \in \mathbb{R}^{+} \times \mathcal{D}^{V},
\]

where \( H_{v} := (G[cl_{v}], v) \) is the induced subgraph of \( G \) on the closure \( cl_{v} \) of \( v \), with \( v \) as root.

**Remark 3.2.** As a consequence of the class function property of \( \tilde{r}_{j}^{H} \) specified in Property 1 of Definition 3.1, the jump rate function \( r_{j}^{G'} \) satisfies a similar class property. Given any \( [\mathcal{K}, \mathcal{K}] \)-graphs \( G, G' \) with \( G \equiv G' \) and \( \varphi \in \mathbb{I}(G', G) \), for any \( v \in V_{G} \), \( t \in \mathbb{R}^{+} \) and \( x \in \mathcal{D}^{G} \),

\[
r_{j}^{G,v}(t, (x_{v})_{v \in V_{G}}) = r_{j}^{G', \varphi^{-1}(v)}(t, (x_{\varphi(v)})_{v \in V_{G'}}).
\]

3.2 Dynamics and Notions of Solutions

Fix a family of jump rate functions \( \mathbf{r} \) that satisfy the Standing Assumption introduced in the previous section. Also fix a random possibly unrooted \( [\mathcal{K}, \mathcal{K} \times \mathcal{X}] \)-graph \( (G, \xi) \), henceforth referred to as the initial data, which encodes both the random \( [\mathcal{K}, \mathcal{K}] \)-graph describing the interaction
structure of the IPS as well as the $X$-valued initial conditions encoded by $\xi$. Then the dynamics of
the associated IPS are described by the following SDE:

$$X_{v}^{G,\xi}(t) = \xi_{v} + \int_{[0,t]} \sum_{j} j \mathbb{1}_{r \leq \tau_{j}^{G,v}(s, X_{v}^{G})} N_{v}^{G}(ds, dr, dj), \quad v \in V, t \in [0, \infty),$$

(3.3)

where $N_{v}^{G}$ is the so-called driving noise, comprised of a collection of i.i.d Poisson point processes
described in Definition 3.3 below. When $(G, \xi)$ is deterministic, the driving noise $N_{v}^{G} = \{N_{v}^{G}\}_{v \in V_{G}}$
is simply a collection of i.i.d. adapted Poisson processes and (3.3) reduces to a standard Poisson-driven SDE. When $(G, \xi)$ is random, analogous to what is done with initial conditions, it is more natural to think of the driving Poisson processes as marks on the graphs and their measurability and adaptedness properties have to be described with some care, as spelled out in Definition 3.3.

**Definition 3.3 (Driving Noise).** Given a complete, filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ such that
$\mathbb{F}$ satisfies the usual conditions and a (possibly random) $\mathcal{F}_{0}$-measurable $[\mathcal{K}, \mathcal{K} \times \mathcal{X}]$-graph $(G, \xi)$, an $\mathbb{F}$-driving noise (compatible with $G$) is a $[\mathcal{K}, \mathcal{K} \times \mathcal{M}(\mathbb{R}_{+}^{2} \times \mathcal{J})]$-random graph $(G, N_{G}^{G})$ that satisfies the following properties:

1. conditioned on $\mathcal{F}_{0}$, $N_{G}^{G} = \{N_{v}^{G}\}_{v \in V_{G}}$ is a collection of i.i.d. Poisson processes on $\mathbb{R}_{+}^{2} \times \mathcal{J}$ with intensity measure $\text{Leb}^{2} \otimes \varsigma$, indexed by the vertex set $V_{G}$ of $G$;
2. for any $t > 0$ and $A \in \mathcal{B}([0, t] \times \mathbb{R}_{+} \times \mathcal{J})$, the $[\mathcal{K}, \mathcal{K} \times \mathcal{N}(\mathbb{R}_{+} \times \mathcal{J})]$-random graph $(G, N_{G}^{G}(A))$ is conditionally independent of $\mathcal{F}_{t}$ given $\mathcal{F}_{0}$;
3. for any $\mathcal{F}_{0}$-measurable $v \in V_{G}$, $N_{v}^{G}$ is an $\mathbb{F}$-adapted point process in the sense described in Section 2.3.

With a slight abuse of notation, we often denote the driving noise $(G, N_{G}^{G})$ just by $N_{G}^{G}$.

When $(G, \xi)$ is random, analogous to what is done with initial conditions and the driving noise, it is natural to also encode the trajectories of the IPS, or equivalently any solution to (3.3), as additional marks on the random graph. This leads to the following definitions of weak and strong solutions to (3.3).

Given a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ that supports a $[\mathcal{K}, \mathcal{K} \times \mathcal{X} \times \mathcal{M}(\mathbb{R}_{+}^{2} \times \mathcal{J})]$-random graph $(G, \xi, N_{G}^{G})$, define $\mathcal{H}_{t} := \sigma((G, \xi, N_{G}^{G}(A)) : A \in \mathcal{B}([0, t] \times \mathbb{R}_{+} \times \mathcal{J}))$ for $t \geq 0$, and define $\mathbb{F}^{G,\xi,N_{G}^{G}}$ to be the augmentation of the filtration $\mathcal{H} = \{\mathcal{H}_{t}\}_{t \in \mathbb{R}_{+}}$, that is, $\mathbb{F}^{G,\xi,N_{G}^{G}}$ is the smallest complete, right-continuous filtration such that $\mathcal{F}_{t}^{G,\xi,N_{G}^{G}} \supseteq \mathcal{H}_{t}$ for every $t \geq 0$, and $\mathcal{F}_{0}^{G,\xi,N_{G}^{G}}$ contains all sets $N \subset A \in \mathcal{F}$ with $\mathbb{P}(A) = 0$. When $(G, \xi)$ is deterministic, we denote $\mathbb{F}^{G,\xi,N_{G}^{G}}$ simply by $\mathbb{P}^{N_{G}^{G}}$.

**Definition 3.4 (Weak and Strong Solutions).** Given (possibly random) initial data $(G, \xi)$, a weak solution to (3.3) is a tuple $((G, X^{G,\xi}, N_{G}^{G}), (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}))$ such that

1. $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is a complete, filtered probability space such that $\mathbb{F}$ satisfies the usual conditions;
2. $(G, N_{G}^{G})$ is an $\mathbb{F}$-driving noise compatible with $G$ in the sense of Definition 3.3.
3. $(G, X^{G,\xi})$ is a random $[\mathcal{K}, \mathcal{K} \times \mathcal{D}]$-graph with $\mathbb{F}$-adapted vertex marks (i.e., for any $\mathcal{F}_{0}$-measurable $v \in V_{G}$, $X_{v}^{G,\xi}$ is an $\mathbb{F}$-adapted process) such that $X_{v}^{G,\xi}$ satisfies (3.3) $\mathbb{P}$-a.s..

Given a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and an $\mathcal{F}_{G,\xi,N_{G}^{G}}$-driving noise $(G, N_{G}^{G})$ compatible with $G$ in the sense of Definition 3.3, a $N_{G}^{G}$-strong solution $(G, X^{G,\xi})$ is a $[\mathcal{K}, \mathcal{K} \times \mathcal{D}]$-random graph such that $((G, X^{G,\xi}, N_{G}^{G}), (\Omega, \mathcal{F}, \mathbb{P}^{G,\xi,N_{G}^{G}}, \mathbb{P}))$ is a weak solution to (3.3).
Remark 3.5. For conciseness, we often omit mention of the whole probability space \( (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}) \) and simply refer to \( (\mathbb{F}, \mathbb{N}^G) \) as a filtration-Poisson process pair, and say \((G, X^{G, \xi})\) is a \((\mathbb{F}, \mathbb{N}^G)\)-weak solution to (3.3) if \(((G, X^{G, \xi}, N^G), (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}))\) is a weak solution to (3.3). We also say that \((G, X^{G, \xi})\) is a weak solution to (3.3) if there exists a filtration-Poisson process pair \((\mathbb{F}, \mathbb{N}^G)\) such that \((G, X^{G, \xi})\) is a \((\mathbb{F}, \mathbb{N}^G)\)-weak solution to (3.3). We would also like to emphasize that if \(X\) and \(X'\) are both \((\mathbb{F}, \mathbb{N}^G)\)-weak solutions, then they are implicitly defined on the same filtered probability space with the same driving noise.

### 3.3 Notions of Well-Posedness

Given the definitions of weak and strong solutions in the previous section, the definitions of uniqueness and well-posedness we employ are mostly standard, except for the notion of strong well-posedness which requires pathwise uniqueness of solutions instead of just uniqueness in law.

**Definition 3.6 (Uniqueness notions).** The SDE (3.3) is said to be **unique in law** for the initial data \((G, \xi)\) if given any other initial data \((G', \xi')\), possibly defined on a different probability space, that satisfies \((G', \xi') \overset{d}{=} (G, \xi)\), any weak solutions \((G, X)\) and \((G', X')\) to the SDE for the respective initial data \((G, \xi)\) and \((G', \xi')\) satisfy \((G, X) \overset{d}{=} (G', X')\). The SDE (3.3) is said to be **pathwise unique** for \((G, \xi)\) if for any filtration-Poisson process pair \((\mathbb{F}, \mathbb{N}^G)\) and any two \((\mathbb{F}, \mathbb{N}^G)\)-weak solutions \((G, X^1)\) and \((G, X^2)\) to (3.3), \((G, X^1) = (G, X^2)\) a.s.

Solutions to the SDE (3.3) for the initial data \((G, \xi)\) will typically be denoted by \((G, X^{G, \xi})\).

**Definition 3.7 (Well-posedness).** We say that the SDE (3.3) is **well-posed** for the initial data \((G, \xi)\) if there exists at least one weak solution to (3.3) and the SDE (3.3) is unique in law for \((G, \xi)\). We say the SDE (3.3) is **strongly well-posed** for the initial data \((G, \xi)\) if there exists at least one weak solution to (3.3) and the SDE (3.3) is pathwise unique for \((G, \xi)\).

The next lemma establishes an intuitive alternative formulation of strong well-posedness for random initial data. Its proof is deferred to Appendix D.

**Lemma 3.8.** The SDE (3.3) is strongly well-posed for the random initial data \((G, \xi)\) if it is strongly well-posed for a.s. every realization of \((G, \xi)\).

In order to discuss local convergence of solutions, it will be convenient to work with isomorphism classes of initial data and solutions. To this end, note that (strong) well-posedness of (3.3) depends only on the isomorphism class of the initial data. More precisely, given \(((G, \xi)) \in \mathcal{G}_\ast [\mathbb{K}, \mathcal{K} \times \mathcal{X}]\), let \((G_i, \xi_i), i = 1, 2\), be two different representatives of \(((G, \xi))\) and fix some isomorphism \(\varphi \in I((G_2, \xi_2), (G_1, \xi_1))\). Let \((\mathbb{F}, \mathbb{N}^{G_1})\) be a filtration-Poisson process pair compatible with \(G_1\) and let \(X^{G_1, \xi_1}\) be a \((\mathbb{F}, \mathbb{N}^{G_1})\)-weak solution to (3.3) for the initial data \((G_1, \xi_1)\). Define,

\[
N_{v}^{G_2} := N_{\varphi(v)}^{G_1} \quad \text{and} \quad X_{v}^{G_2, \xi_2} := X_{\varphi(v)}^{G_1, \xi_1}, \quad \text{for all } v \in V_{G_2}.
\]

Then \((\mathbb{F}, \mathbb{N}^{G_2})\) is a filtration-Poisson process pair compatible with \(G_2\) and by Remark 3.2 and the form of the SDE (3.3), it follows that \(X^{G_2, \xi_2}\) is a \((\mathbb{F}, \mathbb{N}^{G_2})\)-weak solution to (3.3). Thus, (3.3) is (strongly) well-posed for \((G_1, \xi_1)\) if and only if it is (strongly) well-posed for \((G_2, \xi_2)\), showing that (strong) well-posedness is a class property.

**Definition 3.9 (Strong well-posedness for isomorphism classes).** We say the SDE (3.3) is (strongly) well-posed for the (possibly random) initial data \(((G, \xi))\) taking values in \(\mathcal{G}_\ast [\mathbb{K}, \mathcal{K} \times \mathcal{X}]\) if there exists
a (possibly random) $[\mathcal{K}, \mathcal{K} \times \mathcal{X}]$-graph $\langle G, \xi \rangle$ a.s. such that (3.3) is (strongly) well-posed for $(G, \xi)$. Furthermore, we say that $(G, X^{G, \xi})$ is a strong (resp. weak) solution to (3.3) for $(G, \xi)$ if there exists a (random) representative $(G, \xi)$ that lies in $\langle G, \xi \rangle$ a.s. and a strong (resp. weak) solution $(G, X^{G, \xi})$ to (3.3) for $(G, \xi)$ such that $(G, X^{G, \xi}) \in \langle (G, X^{G, \xi}) \rangle$ a.s..

We conclude this section with a Yamada-Watanabe type result.

Lemma 3.10. For the initial data $\langle (G, \xi) \rangle$, if the SDE (3.3) is strongly well-posed, then it is also well-posed. Furthermore, (3.3) is strongly well-posed for $\langle (G, \xi) \rangle$ if and only if the set of weak solutions to (3.3) for $\langle (G, \xi) \rangle$ is non-empty and coincides with the set of strong solutions to (3.3).

Proof. Fix the $\sigma(\langle (G, \xi) \rangle)$-measurable $[\mathcal{K}, \mathcal{K} \times \mathcal{X}]$-random graph $(G, \xi)$ such that $(G, \xi) \in \langle (G, \xi) \rangle$ a.s. By Definition 3.9 and Lemma 3.8 it suffices to prove the lemma for any deterministic $[\mathcal{K}, \mathcal{K} \times \mathcal{X}]$-graph initial data $(G, \xi)$ instead of $\langle (G, \xi) \rangle$. The lemma can be deduced by showing that pathwise uniqueness as defined in Definition 3.6 matches the definition of pathwise uniqueness in [21]. Then by [21, Theorem 3.14], strong well-posedness is equivalent to all weak solutions being strong as desired. The same theorem also shows that strong well-posedness implies well-posedness. \(\square\)

Remark 3.11. Definition 3.4 and Lemma 3.10 imply that under strong well-posedness of the SDE, $(G, X^{G, \xi})$ is characterized by the initial data $(G, \xi)$ and a compatible driving noise $(G, N^G)$. Hence, a coupling of the initial data and driving noises of a sequence of IPS immediately yields a coupling of the respective solutions to (3.3) (see Sections 6 and 7). It is worth emphasizing that strong well-posedness is key to facilitating the construction of such couplings.

4 Statement of Main Results

We assume throughout that the IPS satisfies the Standing Assumption from Section 3.1. We state the strong well-posedness, local convergence and hydrodynamic limit results for our IPS model in Sections 4.1, 4.2 and 4.3, respectively. Several motivating examples are introduced in Section 4.4 and shown to satisfy all stated assumptions.

4.1 Well-Posedness Results

We start by imposing a fairly mild boundedness condition on the jump rates of the IPS model.

Assumption 1. There exists a non-decreasing family of constants $\{C_{k, T}\}_{k \in \mathbb{N}, T \in \mathbb{R}_+} \in (0, \infty)$ such that for any $[\mathcal{K}, \mathcal{K}]$-graph $G$, $v \in V_G$, $j \in J$, $x \in D^G$, $T \in \mathbb{R}_+$ and $t \in [0, T]$, $r^G_{j, v}(t, x) \leq C_{|\chi(v)|, T}$.

In the case of finite initial data, that is, when the initial data consists of a finite (possibly unrooted) graph, Assumption 1 implies the rates are uniformly bounded and strong well-posedness is easily established via a simple recursive construction. For completeness, this result is summarized in Proposition 4.1 and its proof provided in Appendix C.

Proposition 4.1. Suppose Assumption 1 holds. Then the SDE (3.3) is strongly well-posed for all finite initial data, and all solutions to the SDE for finite initial data are a.s. proper.

We now turn to the main case of infinite graphs, with possibly unbounded maximal degree. On such graphs, since Assumption 1 allows the jump rates to grow with the degree of the vertex, which is typical of most IPS of interest as the examples in Section 4.4 demonstrate, jump rates can be unbounded. In this case well-posedness of even Markovian SDEs of the form (3.3) is subtle, and may in fact fail to hold. Indeed, in Appendix D we construct a simple Markovian IPS satisfying
Assumption 1 that admits multiple strong solutions (with different laws) on certain graphs with super-exponential growth. Nevertheless, the following main result of this section shows that strong well-posedness does hold for IPS on a large class of so-called a.s. finitely dissociable graphs, whose precise definition is deferred to Section 5.3.

**Theorem 4.2.** Suppose Assumption 1 holds and fix an a.s. finitely dissociable \( G^* \)-random element \( \langle (G, \xi) \rangle \) (as specified in Definition 5.11 and Remark 5.13). Then the SDE (3.3) is strongly well-posed for \( \langle (G, \xi) \rangle \) in the sense of Definition 3.9.

As shown in Proposition 5.15, Corollary 5.16 and Proposition 5.17, the class of a.s. finitely dissociable graphs includes GW trees with offspring distributions that have finite first moment, and hence, UGW trees whose offspring distributions have finite second moments, in addition to graphs with bounded maximal degree. Thus, in particular, when applied to the contact process, Theorem 4.2 provides rigorous justification that the contact process on GW trees, which was for example studied in [3, 17, 27, 31], is well defined.

Theorem 4.2 is a direct consequence of more general results established in Section 5, which are broad enough to cover applications for which Assumption 1 may fail, but the conclusion of Proposition 4.1 nevertheless holds, that is, for which the following condition is satisfied.

**Assumption 1′.** The family of jump rate functions \( r = \{ r^G,v, j \in J, G \text{ a } [K,K]-\text{graph and } v \in V_G \} \) is such that the SDE (3.3) is strongly well-posed for all finite initial data.

Under Assumption 1′ (and hence, under Assumption 1), Proposition 5.7 establishes well-posedness of the SDE (3.3) for all initial data that “spatially localize” the SDE in the sense of Definition 5.1. Together with Proposition 5.19, which shows that (under Assumption 1) any finitely dissociable graph spatially localizes the SDE, this proves Theorem 4.2.

### 4.2 Local Weak Convergence of the Dynamics

We now address local weak convergence of processes. Given well-posedness, this is equivalent to establishing continuity (in the local weak topology) of the law of the isomorphism class \( \langle (G, X^{G,\xi}) \rangle \) of the graph marked with the trajectory of the unique strong solution to the SDE (3.3) with respect to the \( \mathcal{S}_s[K,K \times X] \)-valued initial data \( \langle (G, \xi) \rangle \). This requires the following additional continuity assumption on the local jump rates with respect to the “environment” marks, which holds trivially when the mark spaces \( K \) and \( \mathcal{K} \) are discrete.

**Assumption 2.** Let \( (G, X^{G,\xi}) = (V, E, \phi, \pi, \kappa, X^{G,\xi}) \) be any representative of a strong solution \( \langle (G, X^{G,\xi}) \rangle \) to (3.3) for the initial data \( \langle (G, \xi) \rangle \). Then a.s. for every \( (j, v) \in J \times V_G \) and Lebesgue-a.e. \( t \in \mathbb{R}_+ \), \( (\pi, \kappa) \) is a continuity point of the map:

\[
\mathcal{K}^{EG} \times K^{V_G} \ni (\phi, \theta) \mapsto r^G_r(V_G,E_G,\phi,\theta,\nu)(t, X^{G,\xi}).
\]

It is worth noting that for the remaining results in this section, we only apply Assumption 2 under conditions that also imply strong well-posedness of (3.3), in which case the strong solution \( \langle (G, X^{G,\xi}) \rangle \) in the statement of the assumption has a well defined law.

**Theorem 4.3.** Suppose Assumption 1 holds, and \( \{(G_n, \xi^n)\}_{n \in \mathbb{N}} \) is a sequence of a.s. finitely dissociable \( \mathcal{S}_s[K,K \times X] \)-random elements (in the sense of Definition 5.11) such that \( \langle (G, \xi) \rangle \) satisfies Assumption 2. If \( \langle (G_n, \xi^n) \rangle \Rightarrow \langle (G, \xi) \rangle \) in \( \mathcal{S}_s[K,K \times X] \), then \( \langle (G_n, X^{G_n,\xi^n}) \rangle \Rightarrow \langle (G, X^{G,\xi}) \rangle \) in \( \mathcal{S}_s[K,K \times D] \).
Theorem 4.3 follows from a more general almost sure version of this statement proved in Proposition 6.11 under weaker assumptions that only require Assumption 1′, a consistent spatial localization condition introduced in Section 5.1 and a weaker finite convergence condition, Assumption 2′, in place of Assumption 2, which is useful for several applications.

4.3 Hydrodynamic Limit and Correlation Decay

Given a Polish space $Z$, and a finite, unrooted $[K, K]$-graph $G$, define the (global) empirical measure of the finite, unrooted $[K, K \times Z]$-graph $(G, z)$ by

$$\pi^{G, z}(\cdot) := \frac{1}{|G|} \sum_{v \in V_G} \delta_{z_v}(\cdot),$$

where $\delta_{z_v}(\cdot)$ is the Dirac delta measure concentrated at $z_v$. Note that for $(G', z') \cong (G, z)$, $\pi^{G, z} = \pi^{G', z'}$, so the empirical measure is determined only by the isomorphism class $\langle (G, z) \rangle$ of $(G, z)$. More generally, consider the so-called neighborhood empirical distribution given by

$$\tilde{\pi}^{G, z}(\cdot) := \frac{1}{|G|} \sum_{v \in V_G} \delta_{\langle B_1(\mathcal{C}_v(G, z)) \rangle}(\cdot),$$

where $\mathcal{C}_v(G, z)$ denotes the rooted $[K, K \times Z]$-graph obtained by restricting $(G, z)$ to the connected component of $v$ in $[G]$, equipped with $v$ as its root. Note that $\tilde{\pi}^{G, z}$ is a $\mathcal{P}(\mathcal{S}_s)$-valued random element that describes the empirical distribution of the isomorphism class of a uniformly distributed root in $(G, z)$ and its neighborhood.

Since $\pi^{G, z}$ and $\tilde{\pi}^{G, z}$ are global quantities, their asymptotic behavior cannot be deduced from the local convergence result established in Theorem 4.3. Moreover, as discussed in the introduction, unlike in the case of IPS on dense graphs, states of neighboring vertices of IPS on sequences of converging sparse graphs remain strongly correlated and do not become asymptotically independent, that is, propagation of chaos typically fails. Hence, the analysis of the convergence of $\pi^{G, z}$ is more subtle for IPS on sparse (as opposed to dense) graph sequences, and due to this strong dependence between neighboring vertices, more complex empirical quantities such as $\tilde{\pi}^{G, z}$ are also of interest (see also Example 4.8 for additional motivation). Nevertheless, Theorem 4.6 and Corollary 4.7 of Section 4.3.2 show that under a slightly stronger convergence condition on the initial data than that imposed in Theorem 4.3 these empirical quantities do have a deterministic limit. A key ingredient of the proof is a certain asymptotic correlation decay property, which is first stated in Section 4.3.1 (see Theorem 4.5).

4.3.1 An Annealed Correlation Decay Property

We start by introducing a slightly stronger notion of local convergence that applies to graphs that are not necessarily connected, and which in a sense has a more global flavor.

**Definition 4.4** (Convergence in probability in the local weak sense). Consider a sequence $\{G_n\}_{n \in \mathbb{N}}$ of finite, unrooted $[\mathcal{K}, \mathcal{K}]$-random graphs. Then $G_n$ converges to a $\mathcal{S}_s[\mathcal{K}, \mathcal{K}]$-random element $\langle G \rangle$ in probability in the local weak sense if for every bounded, continuous mapping $f : \mathcal{S}_s[\mathcal{K}, \mathcal{K}] \to \mathbb{R}$ and as $n \to \infty$,

$$\frac{1}{|G_n|} \sum_{v \in V_{G_n}} f(\langle \mathcal{C}_v(G_n) \rangle) \to \mathbb{E}[f(\langle G \rangle)]$$

in probability. (4.1)
We now state a correlation decay property that holds for sequences of unrooted \([\mathcal{K}, \mathcal{K} \times \mathcal{X}]\)-random graphs \(\{(G_n, \xi^n)\}_{n \in \mathbb{N}}\) with finite vertex sets \(\{V_n\}_{n \in \mathbb{N}}\).

**Theorem 4.5.** Suppose Assumption 1 holds, and let \(\langle (G, \xi) \rangle\) be a \([\mathcal{K}, \mathcal{K} \times \mathcal{X}]\)-random element that satisfies Assumption 2, and is a.s. finitely dissociable in the sense of Definition 5.11. Suppose there exists a countable deterministic set \(\mathcal{S}\) and a sequence \(\{(G_n, \xi^n)\}_{n \in \mathbb{N}}\) of finite, unrooted \([\mathcal{K}, \mathcal{K} \times \mathcal{X}]\)-random graphs, each of whose vertex sets a.s. lie in \(\mathcal{S}\). If \(|G_n| \to \infty\) in probability, \((G_n, \xi^n)\) converges in probability in the local weak sense to \(\langle (G, \xi) \rangle\) and for each \(n \in \mathbb{N}\), \(\alpha^n_i\), \(i = 1, 2\), are independent, uniformly distributed vertices of \(G_n\), then for any bounded continuous functions \(f_i : \mathcal{S} \times [\mathcal{K}, \mathcal{K} \times \mathcal{D}] \to \mathbb{R}\), \(i = 1, 2\),

\[
\lim_{n \to \infty} \text{Cov} \left( f_1 \left( \langle \xi_{n}^i \rangle (G_n, X^{G_n, \xi^n}) \right), f_2 \left( \langle \xi_{n}^i \rangle (G_n, X^{G_n, \xi^n}) \right) \right) = 0,
\]

(4.2)

where \(\text{Cov}\) represents the covariance functional and for each \(n \in \mathbb{N}\), \((G_n, X^{G_n, \xi^n})\) is the strong solution to (3.3) for the initial data \((G_n, \xi^n)\).

The assumption that the vertex sets of each \(G_n\) a.s. lie in \(\mathcal{S}\) is merely imposed for technical convenience. It allows us to construct couplings of the driving noise on the graph sequence \(\{G_n\}\) in a measurable way and thereby apply the results of Section 6.1 in the proofs of Lemma 7.2 and Theorem 4.5 in Section 7. The assumption is not restrictive because it is satisfied by most common random graph sequences of interest including Erdős-Rényi graphs, configuration models and the Barabási-Albert model.

When the jump rate functions of the SDE (3.3) are strongly Lipschitz continuous in the sense described in the introduction, then arguments similar to those used for diffusions in [22] Lemma 5.2 can be applied to obtain stronger quantitative quenched (i.e., conditioned on the graph) bounds on the decay of correlation of IPS that are uniform with respect to graphs, and only depend on the cardinality of the sets of particles being compared and the graph distance between the sets. Under Assumption 1 such a strong Lipschitz condition holds for Markov IPS on graphs with bounded maximal degree, but fails to hold for many interesting IPS on graphs with unbounded maximal degree. In such situations, one does not expect there to be a similar quenched correlation bound that is uniform over all graphs (or even all finitely dissociable graphs), and thus the arguments we use to prove Theorem 4.5 are crucially different from those used in [22]. Specifically, to establish the annealed asymptotic correlation decay property (1.2), which is averaged over the randomness of the initial data, we first use the fact that the initial data satisfies an analogous asymptotic correlation decay property (due to the assumed convergence in probability in the local weak sense) and then carefully construct an appropriate coupling and the stronger almost sure local convergence result of Proposition 6.11 mentioned in Section 4.2 to extend the correlation decay property to the solution process.

### 4.3.2 Hydrodynamic Limits

The existence of the hydrodynamic limit follows as a simple consequence of well-posedness of the limit, local convergence and asymptotic correlation decay.

**Theorem 4.6.** Suppose Assumption 1 holds and the sequence of finite, unrooted \([\mathcal{K}, \mathcal{K} \times \mathcal{X}]\)-random graphs \(\{(G_n, \xi^n)\}_{n \in \mathbb{N}}\) and the \(\mathcal{S}\) \([\mathcal{K}, \mathcal{K} \times \mathcal{X}]\)-random element \(\langle (G, \xi) \rangle\) satisfy the conditions of Theorem 4.5. Let \((G_n, X^{G_n, \xi^n})\) and \((G, X^{G, \xi})\) be weak solutions to (3.3) for \((G_n, \xi^n)\) and \((G, \xi)\), respectively. Then the sequence \(\{(G, X^{G_n, \xi^n})\}_{n \in \mathbb{N}}\) converges in probability in the local weak sense to \(\langle (G, X^{G, \xi}) \rangle\).
Proof. First note that, given any driving noise, \( \langle (G, X^{G,\xi}) \rangle \) and \( (G_n, X^{G_n,\xi_n}) \), \( n \in \mathbb{N} \), are well defined unique strong solutions to (3.3) by Theorem 4.2, Proposition 4.1, and Lemma 3.10. For each \( n \in \mathbb{N} \), let \((\omega_n^1, \omega_n^2)\) be two i.i.d. uniformly distributed vertices in \( G_n \). Then, by Theorem 4.3 and Theorem 4.4 for any bounded, continuous functions \( f_1, f_2 : \mathcal{S}_* [\mathcal{K}, \mathcal{K} \times \mathcal{D}] \rightarrow \mathbb{R} \),

\[
\lim_{n \to \infty} \mathbb{E} \left[ \pi_{G_n, X^{G_n,\xi_n}} \left( f_1 \left( \langle \mathcal{E}_{\omega_n^1} (G_n, X^{G_n,\xi_n}) \rangle \right) \right) \right] = \lim_{n \to \infty} \mathbb{E} \left[ f_1 \left( \langle \mathcal{E}_{\omega_n^1} (G_n, X^{G_n,\xi_n}) \rangle \right) \right] = \mathbb{E} \left[ f_1 \left( \langle (G, X^{G,\xi}) \rangle \right) \right].
\]

That this implies the desired result follows from [22, Lemma 2.12].

We now show that Theorem 4.6 implies the convergence in probability of both the empirical measure and the empirical neighborhood distribution to corresponding deterministic limits.

Corollary 4.7. Under the conditions of Theorem 4.6, the \( \mathcal{P}(\mathcal{D}) \)-valued random empirical measure sequence \( \{ \pi_{G_n, X^{G_n,\xi_n}} \}_{n \in \mathbb{N}} \) converges in probability to \( \mathcal{L}(X_0^{G,\xi}) \in \mathcal{P}(\mathcal{D}) \), and the \( \mathcal{P}(\mathcal{S}_*[\mathcal{K}, \mathcal{K} \times \mathcal{D}]) \)-valued random empirical neighborhood distribution sequence \( \{ \pi_{G_n, X^{G_n,\xi_n}} \}_{n \in \mathbb{N}} \) converges in probability to \( \mathcal{L}(\langle (B_1(G, X^{G,\xi})) \rangle) \in \mathcal{P}(\mathcal{S}_*[\mathcal{K}, \mathcal{K} \times \mathcal{D}]) \).

Proof. By Theorem 4.6 the sequence \( \{ (G_n, X^{G_n,\xi_n}) \}_{n \in \mathbb{N}} \) converges in probability in the local weak sense to \( \langle (G, X^{G,\xi}) \rangle \). By [22, Lemma 2.11], this directly implies that the sequence \( \pi_{G_n, X^{G_n,\xi_n}} \) of \( \mathcal{P}(\mathcal{D}) \)-valued random measures converges in probability to \( \mathcal{L}(X_0^{G,\xi}) \).

Furthermore, note that the map \( \mathcal{S}_*[\mathcal{K}, \mathcal{K} \times \mathcal{D}] \ni \langle (G, x) \rangle \mapsto \langle (B_1(G, x)) \rangle \in \mathcal{S}_*[\mathcal{K}, \mathcal{K} \times \mathcal{D}] \) is continuous. Thus, for any bounded, continuous function \( f : \mathcal{S}_*[\mathcal{K}, \mathcal{K} \times \mathcal{D}] \rightarrow \mathbb{R} \), the function \( g : \mathcal{S}_*[\mathcal{K}, \mathcal{K} \times \mathcal{D}] \rightarrow \mathbb{R} \) given by \( g(\langle (G, x) \rangle) := f(\langle (B_1(G, x)) \rangle) \) is also bounded and continuous. Hence, as \( n \to \infty \),

\[
\int_{\mathcal{S}_*[\mathcal{K}, \mathcal{K} \times \mathcal{D}]} f(\langle (G, x) \rangle) \pi_{G_n, X^{G_n,\xi_n}}(d(\langle (G, x) \rangle)) = \frac{1}{|G_n|} \sum_{v \in V_{G_n}} f(\langle B_1(\mathcal{E}_v((G_n, X^{G_n,\xi_n})) \rangle)
\]

\[
= \frac{1}{|G_n|} \sum_{v \in V_{G_n}} g(\langle \mathcal{E}_v(G_n, X^{G_n,\xi_n}) \rangle)
\]

\[
\to \mathbb{E} \left[ g(\langle (G, X^{G,\xi}) \rangle) \right]
\]

\[
= \mathbb{E} \left[ f(\langle (B_1(G, X^{G,\xi})) \rangle) \right],
\]

where the convergence \( \to \) in the penultimate line is in probability and justified by (4.4). Thus, \( \pi_{G_n, X^{G_n,\xi_n}} \to \mathcal{L}(\langle (B_1(G, X^{G,\xi})) \rangle) \) in probability in \( \mathcal{P}(\mathcal{S}_*[\mathcal{K}, \mathcal{K} \times \mathcal{D}]) \).

As was already observed in [22, Theorem 6.4 and Proposition 7.7] in the context of interacting diffusions, the stronger convergence in probability in the local weak sense of the initial data is in general necessary to obtain deterministic hydrodynamic limits that coincide with \( \mathcal{L}(X_0^{G,\xi}) \) because if one only has convergence in the local weak sense of the initial data as in Theorem 4.3, the hydrodynamic limit can fail to be deterministic, or fail to coincide with \( \mathcal{L}(X_0^{G,\xi}) \) even when deterministic.

4.4 Motivating Examples

We now present illustrative examples of IPS that fall within our framework.
Example 4.8 (Finite-state Markovian IPS). Consider the graph $G = (V, E, o)$ (i.e., $\overline{K}$ and $K$ are trivial) and suppose that for every $j \in J$, there exist functions $\tilde{r}_j^H : \mathcal{X}^V \setminus \mathbb{R}_+$, $H \in \mathbb{S}_{+, 1}$, such that the functions $\tilde{r}_j^H : \mathcal{X}^V \setminus \mathbb{R}_+$, $H \in \mathbb{S}_{+, 1}$, from the Standing Assumption satisfy

$$\tilde{r}_j^H(t, x) = \tilde{r}_j^H(x(t-)) \quad \forall (t, x) \in \mathbb{R}_+ \times \mathcal{D}^V.$$  

If there exist non-decreasing constants $\{C_k\}_{k \in \mathbb{N}}$ such that $\sup_{x \in \mathcal{X}^V} |\tilde{r}_j^H(x)| \leq C_k$ for all $j \in J$ and $H \in \mathbb{S}_{+, 1}$, then Assumption 1 holds with $C_{k, T} := C_k$. Moreover, Assumption 2 holds trivially for all initial data because the mark spaces $\overline{K} = K = \{1\}$ are trivial, and the solution $X^{G, \xi}$ is a homogeneous Markov process (and Feller when $\mathcal{X}$ is finite and $G$ has bounded maximal degree [25, Chapter I, Theorem 3.9]). This class of processes includes many IPS such as the contact process, voter model, majority process etc.

We will use the contact process introduced by Harris [16] to illustrate the variety of models supported by our framework. The contact process has the state space $\mathcal{X} = \{0, 1\}$, where 0 represents the healthy state and 1 represents the infected state, and the rates $\{\tilde{r}_j^H\}$ from (4.3) are given by

$$\tilde{r}_1^H(z) = \mathbb{I}_{\{z_{o, u} = 0\}} \lambda \sum_{u \in N_{o, u}(H)} z_u \quad \text{and} \quad \tilde{r}_{-1}^H(z) = \mathbb{I}_{\{z_{o, u} = 1\}} \quad \text{for } z \in \mathcal{X}^{V, H},$$

where $\lambda > 0$ is a parameter indicating the rate of spread of infection. Then it is easy to see that the contact process satisfies both assumptions as described above with $C_k = 1 + (k-1)\lambda$. Therefore, the conditions of Theorem 4.6 and hence, Corollary 4.7 apply. In particular, Corollary 4.7 can be used to shed insight on the proportion of infected vertices at a given time or the maximum fraction of infected vertices over a time interval, as well as more complicated quantities such as the proportion, at any time, of infected vertices with a high number of infected neighbors, which can be used to detect clustering of the infection.

Example 4.9 (The Contact Process in a Heterogeneous Environment). Suppose each individual $v$ has an associated recovery rate $\kappa_v \geq 0$ and the level of transmission between two neighboring individuals $u$ and $v$ is proportional to $\kappa_{\{u, v\}} > 0$. The model is then described by the $[\overline{K}, K] := [\mathbb{R}^+ \times \mathbb{R}_+]$-graph $G = (V, E, o, \overline{\kappa}, \kappa)$ and, as before, $\mathcal{X} = \{0, 1\}$, $J = \{-1, 1\}$, $\varsigma(\{1\}) = 1$, but now the Standing Assumption holds with $\tilde{r}_j^H(t, x, \overline{\kappa}, \kappa) = \tilde{r}_j^H(x(t-), \overline{\kappa}, \kappa)$, for

$$H = (V_H, E_H, o_H) \in \mathbb{S}_{+, 1},$$

where $\tilde{r}_j^H : \mathcal{X}^{V, H} \times \overline{\mathcal{K}}^{E_H} \times \mathcal{K}^{V, H} \rightarrow \mathbb{R}$ is given by

$$\tilde{r}_1^H(z, \overline{\kappa}, \kappa) = \mathbb{I}_{\{z_{o, u} = 0\}} \sum_{u \in N_{o, u}(H)} \overline{\kappa}_{\{u, v\}} z_u \quad \text{and} \quad \tilde{r}_{-1}^H(z) = \kappa_{o, u} \mathbb{I}_{\{z_{o, u} = 1\}} \quad \text{for } z \in \mathcal{X}^{V, H}.$$  

Then the jump rate function is continuous with respect to the initial marks $(\overline{\kappa}, \kappa)$, so Assumption 1 holds for all initial data. This model satisfies Assumption 2 when there exist deterministic constants $\hat{\kappa}$ and $\overline{\kappa}$ such that $\sup_{v \in V} \kappa_v \leq \hat{\kappa} < \infty$ and $\sup_{\xi \in \mathcal{E}} \overline{\kappa}_\xi \leq \overline{\kappa} < \infty$, in which case $C_{k, T} = \hat{\kappa} + (k-1)\overline{\kappa}$ for all $k \in \mathbb{N}$, $T \in \mathbb{R}^+_+$. 

Example 4.10 (Non-Markovian Contact Process). We now introduce a class of non-Markovian contact processes that fall within our framework. (Such processes have recently been considered on the lattice in [12] under the rubric of renewal contact processes.) As before, let $\mathcal{X} = \{0, 1\}$ but now consider $G = (V, E, o, \{1\}, \kappa) \in \mathbb{S}_{+, 1}, \kappa]$ where $\mathcal{K} := \mathcal{D}([-T, 0], \mathcal{X})$ for some fixed $T \in [0, \infty]$. The mark $\kappa$ represents initial data that encodes the history of the process from time $-T$ up to time 0. Let $f_I$ and $f_R$ be the probability density functions of two positive continuous random variables that represent the respective times required for a particle to infect another particle or to recover.
from infection, and let $h_I$ and $h_R$ be the respective hazard rates: $h_I(t) = f_I(t) / \int_t^\infty f_I(s) \, ds$ and $h_R(t) = f_R(t) / \int_t^\infty f_R(s) \, ds$. In the Markov case, $f_I$ and $f_R$ are both exponential densities with respective means $1/\lambda$ and 1. Also, recall that $\Delta g(s) = g(s) - g(s^-)$ for any càdlàg function $g$, and for any $U \subseteq V$ and $(x_U, \kappa_U) \in (D \times K)^U$ where we define $\Delta x_v(s) = 0$ for $s \in [-T, 0]$ and $\Delta \kappa_v(s) = 0$ for $s > 0$, consider the functional $\tau_U : [-T, \infty) \times D^U \times K^U \to [-T, \infty)$ given by

$$\tau(t, x_u, \kappa_U) := \text{sup}\{s \in (-T, t), v \in U : \Delta \kappa_v(s) \neq 0 \text{ or } \Delta x_v(s) \neq 0\}, \quad t \geq -T, (x_U, \kappa_U) \in D^U \times K^U,$$

where we apply the convention that if the above set is empty then the corresponding supremum is $-T$. For any $U \subseteq V$, and at any time $t > 0$, when $x$ is a realization of the actual process, $\tau_U(t, x, \kappa)$ represents the last time that a particle in $U$ experienced an event (infection or cure), and it takes the value $-T$ if no such events have occurred prior to time $t$. Then, the jump rate functions $r_{\pm 1}^{G,v} : \mathbb{R}_+ \times D^G \to \mathbb{R}_+$ are given by

$$r_{1}^{G,v}(t, x) := \mathbb{I}_{\{x_v(t^-) = 0\}} \sum_{u \in N_G(v): x_u(t^+) = 1} h_I(\tau(t, x_{\{u,u\}}, \kappa_{\{u,u\}})), \quad r_{-1}^{G,v}(t, x) := \mathbb{I}_{\{x_v(t^-) = 1\}} h_R(\tau(t, x, \kappa_v)).$$

If $T < \infty$, then Assumption 1 holds, for example, when the distributions have infinite support and the hazard rate functions are locally bounded, i.e., $\sup_{s \in [0,t]} |h_I(s) + h_R(s)| < \infty$ for all $t \in \mathbb{R}_+$. When $T = \infty$, Assumption 1 holds under the more stringent conditions that the distributions have infinite support, the hazard rate functions are uniformly bounded on $\mathbb{R}_+$ and $\lim_{t \to \infty} h_I(t)$ and $\lim_{t \to \infty} h_R(t)$ both exist. Again note that in the Markov case, $h_I \equiv \lambda$ and $h_R \equiv 1$ are both constant. Finally, if both $h_I$ and $h_R$ are also continuous, then Assumption 2 holds on all initial data.

## 5 Spatial Localization and the Proof of Well-Posedness

Throughout this section we continue to assume, without explicit mention, that the Standing Assumption holds. In Section 5.1 we introduce the notions of spatial localization and consistent spatial localization (see Definitions 5.1 and 5.4) and in Section 5.2 show that the SDE \[
\text{(3.3)}
\] is strongly well-posed for any marked graph that “spatially localizes” the IPS. In Section 5.3, we define the class of finitely dissociable graphs and provide examples of graphs in this class. In Section 5.4 we show that sequences of such graphs consistently spatially localize the SDE \[
\text{(3.3)}
\].

### 5.1 Spatial Localization and Consistent Spatial Localization

We now introduce the notion of spatial localization for a graph, which roughly speaking says that on any finite time interval the marginal of the associated IPS dynamics on any finite (random) subset $U$ of the graph coincides with the marginal dynamics (on $U$) of an IPS on the induced (random) subgraph of $G$ on a finite subset that contains $U$. When combined with Assumption 1 this will allow us in Section 5.2 to establish well-posedness for the class of graphs that spatially localize the SDE \[
\text{(3.3)}
\]. The measurability and consistency properties required to state this property precisely are carefully spelled out in the following definition. Recall the notion of filtration-Poisson process pairs from Remark 3.5 and recall from Section 2.1 that $\Delta_G := \{U \subset V_G : |U| < \infty\}$. Now, supposing that Assumption 1 holds, we introduce the notion of spatial localization. In what follows, we will deal with deterministic initial data $(G, \xi)$, and recall the notion of a filtration-Poisson process pair $(\mathcal{F}, \mathcal{N}_G)$ from Remark 3.5 and its associated augmented filtration $\mathcal{F}^{NG} = \{\mathcal{F}_t^{NG}\}_{t \geq 0}$ introduced
prior to Definition 5.3. Also recall from Section 2.2 that given $f \in \mathcal{D}$ and $t > 0$, $f[t]$ represents the truncated path that lies in $\mathcal{D}_t$.

**Definition 5.1** (Spatial localization). A deterministic $[\bar{K}, K]$-graph $G$ is said to spatially localize the SDE (3.3) if for any filtration-Poisson process pair $(\mathbb{P}, \mathbb{N}^G)$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$ and $T \in \mathbb{R}_+$, there exists a mapping $S_T(\cdot; G, \mathbb{N}^G) : \Lambda_G \times \Omega \to 2^{2^{\mathbb{N}^G}}$ such that for every $U, U' \in \Lambda_G$,

$$\{S_T(U; G, \mathbb{N}^G) \subseteq U'\} := \{\omega \in \Omega : S_T(U; G, \mathbb{N}^G)(\omega) \subseteq U'\} \in \mathcal{F}_T^{\mathbb{N}^G},$$

and the following properties hold:

1. for each $\mathcal{O} \in \Lambda_G$, $S_T(\mathcal{O}; G, \mathbb{N}^G)(\omega) \supseteq \mathcal{O}$ for every $\omega \in \Omega$;

2. given any $\ell \in \mathbb{N}, \xi \in \mathcal{X}^G$ and $U, W \subset V$ such that $U \subseteq B_\ell(G) \subseteq W$, every $(\mathbb{F}, \mathbb{N}^G)$-weak solution $X^{G[W], \xi_W}_U$ to the SDE (3.3) for the initial data $(G[W], \xi_W)$ satisfies

$$X^{G[W], \xi_W}_U[T] = X^{B_\ell(G), \xi_{B_\ell(G)}}_U[T] \quad \text{a.s. on the event} \ \{S_T(U; G, \mathbb{N}^G) \subseteq B_\ell(G)\},$$

where $X^{B_\ell(G), \xi_{B_\ell(G)}}$ is the unique $\mathbb{N}^G_{B_\ell(G)}$-strong solution to (3.3) for $(B_\ell(G), \xi_{B_\ell(G)})$. In this case $S_T(\cdot; G, \mathbb{N}^G)$ is said to be a localizing map for the SDE (3.3) on $(G, \mathbb{N}^G)$.

**Remark 5.2.** Note that Assumption 11 ensures that $X^{B_\ell(G), \xi_{B_\ell(G)}}$ in Definition 5.1 is well defined. Also, recall from Proposition 4.1 that Assumption 11 is implied by Assumption 11 but is strictly more general and may hold even when Assumption 11 fails.

**Remark 5.3.** In Definition 5.1, we may omit one or both of the last two arguments of $S_T$ when the graph and/or Poisson processes are clear from the context.

Given a fixed model (equivalently, a family of local jump rate functions $\bar{f} = \{\bar{f}_j^H\}_{H \in \mathcal{H}, j \in J}$ as specified in the Standing Assumption), we now introduce the concept of consistent spatial localization of a sequence of graphs, which will be used in the proofs of local convergence and the hydrodynamic limit in Sections 6 and 7 respectively.

**Definition 5.4** (Consistent Spatial Localization). A sequence of $[\bar{K}, K]$-graphs $\{G_n\}_{n \in \mathbb{N}}$ defined on some common probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is said to consistently spatially localize the SDE (3.3) if for every $n \in \mathbb{N}$ and corresponding filtration-Poisson process pair $(\mathbb{F}^n, \mathbb{N}^{G_n})$ on $(\Omega, \mathcal{F}, \mathbb{P})$, there exists a mapping $S_T(\cdot; G_n, \mathbb{N}^{G_n}) : \Lambda_{G_n} \times \Omega \to 2^{\mathbb{N}^{G_n}}$ that satisfies the following properties:

1. for each $n \in \mathbb{N}$, $S_T(\cdot; G_n, \mathbb{N}^{G_n})$ is a localizing map for the SDE (3.3) on $(G_n, \mathbb{N}^{G_n})$ in the sense of Definition 5.1;

2. for every $n, n', \ell \in \mathbb{N}$ such that there exists an isomorphism $\varphi \in I(B_\ell([G_{n'}], B_\ell([G_n])))$, the following property holds: for each pair of sets $U \subseteq B_\ell(G_n)$ and $U' := \varphi^{-1}(U) \subseteq B_\ell(G_{n'})$,

$$S_T(U; G_n, \mathbb{N}^{G_n}) = \varphi(S_T(U'; G_{n'}, \mathbb{N}^{G_{n'}})) \quad \text{a.s. on the event} \ \{S_T(U; G_n, \mathbb{N}^{G_n}) \subseteq B_\ell(G_n)\} \cap \mathcal{I}_\varphi,$$

where $\mathcal{I}_\varphi$ is the $\mathcal{F}_T^{\mathbb{N}^{G_{n'}}} \vee \mathcal{F}_T^{\mathbb{N}^{G_n}}$ measurable event given by

$$\mathcal{I}_\varphi := \{\varphi \in I(B_\ell([G_{n'}], \mathbb{N}^{G_{n'}}), B_\ell([G_n], \mathbb{N}^{G_n}))\}.$$
In this case \( \{S_T(\cdot; G_n, N^{G_n})\}_{n \in \mathbb{N}} \) is said to be a consistent sequence of localizing maps for the SDE (3.3) on \( \{(G_n, N^{G_n})\}_{n \in \mathbb{N}} \).

In Proposition 5.19 we show that any sequence of graphs that is a subset of the class of so-called finitely dissociative graphs (see Definition 5.11) consistently spatially localizes the SDE (3.3).

It is not hard to show that (consistent) spatial localization is a property of isomorphism classes. We omit the proof as it follows easily from the definition.

**Definition 5.5.** An isomorphism class \( \langle G \rangle \in G_A [\mathcal{K}, \mathcal{X}] \) is said to spatially localize the SDE (3.3) whenever each representative \( [\mathcal{K}, \mathcal{X}] \)-graph \( G \) does the same. A sequence of isomorphism classes \( \{(G_n)\}_{n \in \mathbb{N}} \in G_A [\mathcal{K}, \mathcal{X}] \) is said to consistently spatially localize the SDE (3.3) if every sequence of representative graphs \( G_n \in \langle G_n \rangle, n \in \mathbb{N} \), does the same.

Note that while spatial localization is a class property, the notion of a localizing map given in Definition 5.1 applies to a given graph, and not its isomorphism class.

**Remark 5.6.** Definitions 5.1 and 5.4 are abstract properties that easily extend to a more general class of graph-indexed jump processes \( X^{G, \xi} \) that satisfy a different Poisson-driven SDE from (3.3), as long as Assumption 1 still holds for that SDE. For instance, they could apply to IPS such as the exclusion process in which particles may experience simultaneous jumps, where it would be more natural to index \( N^{G} \) by the edges, rather than the vertices, of \( G \), or processes with both types of jumps.

### 5.2 Well-posedness on spatially localizing graphs

**Proposition 5.7.** Suppose that Assumption 1 holds, and that \( \langle (G, \xi) \rangle \) is a \( G_A [\mathcal{K}, \mathcal{X} \times \mathcal{X}] \)-valued random element such that \( \langle G \rangle \) a.s. spatially localizes the SDE (3.3). Then the SDE (3.3) is strongly well-posed for the initial data \( \langle G, \xi \rangle \).

**Proof.** By Lemma B.7, there exists a \( \sigma((G, \xi)) \)-measurable \( [\mathcal{K}, \mathcal{K} \times \mathcal{X}] \)-random graph \( (G, \xi) \) such that \( (G, \xi) \in \langle (G, \xi) \rangle \) a.s.. By Definition 3.9 and Lemma 3.8, it suffices to prove that (3.3) is strongly well-posed for a.s. every realization of \( (G, \xi) \). Therefore, for the remainder of the proof we assume without loss of generality that \( (G, \xi) \) is a deterministic \( [\mathcal{K}, \mathcal{K} \times \mathcal{X}] \)-graph where \( G \) spatially localizes the SDE (3.3).

We now explicitly construct a strong solution to the SDE (3.3) for \( (G, \xi) \). Let \( N^G = \{N^G_v\}_{v \in V_G} \) be a collection of i.i.d. Poisson processes on \( \mathbb{R}_+^2 \times \mathcal{J} \) with intensity \( \text{Leb}^2(\cdot) \). Let \( \mathbb{F} := \{\mathcal{F}_t\} := \mathbb{F}^{N^G} \) be the associated filtration (as defined prior to Definition 3.4). It is clear from Definition 3.3 that \( (G, N^G) \) is an \( \mathbb{F} \)-driving noise. So by Remark 3.5 \( \mathbb{F}, N^G \) is a filtration-Poisson process pair. Fix \( T \in \mathbb{R}_+ \), and let \( S_T(\cdot) := S_T(\cdot; G, N^G) \) be a corresponding localizing map, which exists due to our assumption that \( G \) spatially localizes the SDE. For notational conciseness, let \( B_m := B_m(G) \) for each \( m \in \mathbb{N} \), and additionally fix \( \xi \) and omit the dependence on \( \xi \) from the superscript, with the understanding that the mark is always the restriction of \( \xi \) to the corresponding graph in the superscript. Furthermore, for any \( \ell \in \mathbb{N} \), recalling that by Assumption 1 the SDE (3.3) is strongly well-posed for the finite data \( (B_\ell, \xi_{B_\ell}) \), we let \( X^{B_\ell} \) denote the corresponding \( N_{B_\ell}^G \)-strong solution to (3.3). By Definition 5.1 for each \( m \in \mathbb{N} \), there exists a random finite set \( \mathcal{O}_m := S_T(B_m) \) such that for any \( \ell, \ell' \in \mathbb{N} \), \( \ell' \geq \ell \geq m \), by (5.1) and applying (5.5) with \( W = B_{\ell'} \) and \( U = B_m \),

\[
X^{B_{\ell}}_{B_m}[T] = X^{B_{\ell'}}_{B_m}[T] \quad \text{on the event } \{\mathcal{O}_m \subseteq B_\ell\} \in \mathcal{F}_T.
\]

For \( m \in \mathbb{N} \), we define the \( \mathcal{D}^{B_m}_T \)-valued random element \( X^m \) and the random integer \( M_m \) as follows:

\[
X^m[T] := \lim_{\ell \to \infty} X^{B_\ell}_{B_m}[T] \quad \text{and} \quad M_m := \inf\{\ell \geq m : \mathcal{O}_m \subseteq B_\ell\}.
\]
By (3.1) of the Standing Assumption and (5.8), it follows that for \( X \) since

\[
X^m[T] = \lim_{\ell \to \infty} X^{B_{m \ell}}_m[T] = X^{B_{m \ell}}_m[T].
\] (5.5)

Furthermore, clearly the sequence \((X^m[T])_{m \in \mathbb{N}}\) is consistent: for any \( m' > m \), a.s.,

\[
X^{B_{m \ell}}_m[T] = \lim_{\ell \to \infty} (X^{B_{m \ell}}_{m'})_{m \in \mathbb{N}} = \lim_{\ell \to \infty} X^{B_{m \ell}}_m[T] = X^m[T] = X^{m'}_{B_{m \ell}}[T],
\] (5.6)

where the first and third equalities invoke (5.5) and the remaining equalities hold trivially. Now, for every \( v \in V_G \), there exists an integer \( m_v \in \mathbb{N} \) such that \( v \in B_{m_v} \), and the last display shows that \( X^m_{B_{m \ell}}[T] = X^{m'}_{B_{m \ell}}[T] \) a.s. when \( m' \geq m_v \). Because \( V_G \) is countable and \( \mathbb{F} \) is complete, we can define the \( \mathbb{F} \)-adapted \( D_G \)-random element \( X[T] \) by setting

\[
X_v[T] := \lim_{m \to \infty} X^m_v[T] = X^{m_v}_v[T], \quad v \in V_G,
\] (5.7)
on the set of measure one where the latter limits exist, and setting \( X_v[T] \equiv \xi_v, v \in V_G \), on the complement.

To show that the \( X[T] \) thus constructed is a \( N^G \)-strong solution to the SDE \((3.3)\) on \([0, T] \), fix \( v \in V_G \) and define \( m_v := \max\{m_u : u \in cl_v\} \). Then \( cl_v \subseteq B_{m_v} \) and from (5.5)–(5.7), it follows that for \( \ell \in \mathbb{N} \),

\[
X_{cl_v}[T] = X^{m_v}_{cl_v}[T] = X^{B_{B_{m \ell}}}_{cl_v}[T] \quad \text{on} \quad A_\ell(v) := \{ \ell \geq M_{m_v} \}.
\] (5.8)

Since \( X^{B_{m \ell}}_m \) is a \( N^G_{B_{m \ell}} \)-strong solution to the SDE \((3.3)\), we obtain a.s., for \( t \in [0, T] \),

\[
X_v(t)_{A_\ell(v)} = X^{B_{B_{m \ell}}}_m(t)_{A_\ell(v)}
\]

\[
= \left( \xi_v + \int_{[0, t] \times \mathbb{R}_+ \times \mathcal{J}} j^\mathbb{I}_{\{r \in \mathbb{I}^G_{B_{m \ell}}(s, X^{B_{m \ell}}_m)\}} \mathbb{N}^G_v(ds, dr, dj) \right)_{A_\ell(v)}.
\]

By \((3.1)\) of the Standing Assumption and \((5.8)\), it follows that with \( H := G[cl_v] \),

\[
r^G_{B_{m \ell}, v}(s, X^{B_{m \ell}}_m) = r^H_j(s, X_{cl_v}, (\mathcal{R}_e)_{e \in \mathcal{E} H, (k_u)_{u \in cl_v}}) = r^H_j(s, X_{cl_v}, (\mathcal{R}_e)_{e \in \mathcal{E} H, (k_u)_{u \in cl_v}}) = r^G_{B_{m \ell}, v}(s, X),
\]

for every \( j \in \mathcal{J} \) and \( s \in \mathbb{R}_+ \) on the event \( A_\ell(v) \). Hence, we have

\[
X_v(t)_{A_\ell(v)} = \left( \xi_v + \int_{[0, t] \times \mathbb{R}_+ \times \mathcal{J}} j^\mathbb{I}_{\{r \leq r^G_{B_{m \ell}, v}(s, X)\}} \mathbb{N}^G_v(ds, dr, dj) \right)_{A_\ell(v)}.
\]

Since \( M_{m_v} \) is finite, taking the limit as \( \ell \to \infty \) shows that a.s.,

\[
X_v(t) = \xi_v + \int_{[0, t] \times \mathbb{R}_+ \times \mathcal{J}} j^\mathbb{I}_{\{r \leq r^G_{B_{m \ell}, v}(s, X)\}} \mathbb{N}^G_v(ds, dr, dj).
\]

Thus, we have proved the existence of a \( N^G \)-strong solution to \((3.3)\) on any interval \([0, T] \).

We now turn to the proof of pathwise uniqueness. Suppose that \( \tilde{X} \) and \( \tilde{X}' \) are any two \((\mathbb{F}, N^G)\)-weak solutions to the SDE \((3.3)\) for \((G, \xi)\). Since \( G \) spatially localizes the SDE, for any \( \ell > M \in \mathbb{N} \), and \( M_m \) defined as in (5.4), invoking (5.1) and applying (5.3), with \( W = V \) and \( U = B_m \), to both
weak solutions $\tilde{X}$ and $X'$, we obtain a.s. on the event \( \{ \ell \geq M_m \} = \{ S_T(U; G, N^G) \subseteq B_\ell(G) \} \in \mathcal{F}^{N^G}_T \),
\[
\tilde{X}_{B_m}[T] = X^{B_\ell(G)}_{B_m}[T] = \tilde{X}'_{B_m}[T],
\]
where recall from Remark 5.1 that $X^{B_\ell(G)}$ is the unique $N^{B_\ell(G)}$-strong solution to (3.3) for the initial data $B_\ell(G, \xi)$. Taking the limit as $\ell \to \infty$ and using the almost sure finiteness of $M_m$, it follows that $\tilde{X}_{B_m}[T] = X_{B_m}[T]$ a.s. for every $m \in \mathbb{N}$, which in turn shows that $X[T] = \tilde{X}[T]$ a.s.

Since $T$ is arbitrary for both existence and pathwise uniqueness, $X[T], T > 0$, are consistent. So there exists a.s. a unique pathwise extension $X$ of the strong solution to all of $[0, \infty)$. This concludes the proof.

\[\square\]

**Remark 5.8.** Most of the proof of Proposition 5.7 also extends to more general IPS. For instance, if an IPS has simultaneous jumps, then the proof will hold given Assumption 1 and spatial localization (see Remark 5.6) except for the verification that $X := \lim_{m \to \infty} X^m$ solves (3.3). Instead of the latter, one would have to prove that if $X^{B_\ell}$ satisfies the SDE defining the new model on the finite graph $B_\ell(G)$, and $X^m$ is as defined in (5.5), then the limit $X = \lim_{m \to \infty} X_m$ also satisfies that SDE on the infinite graph $G$.

### 5.3 Finitely Dissociable Graphs

#### 5.3.1 Definition of Finitely Dissociable Graphs

We now introduce the class of finitely dissociable graphs, which are defined in terms of an inhomogeneous site percolation on the graph. Recall the definition of the measure space $\mathcal{M}_N(0, T)$ from Section 2.3.

**Definition 5.9.** For any $0 < T < \infty$, let $(G, \zeta)$ be a $[[1], \mathcal{M}_N(0, T)]$-random graph. Fix $0 \leq t_1 < t_2 \leq T$, and set $R_v = R_v(t_1, t_2) := \mathbb{1}_{\{v \mid (t_1, t_2) > 0 \}}$ for $v \in V_G$. Then the percolated graph $\text{perc}_{t_1, t_2}(G, \zeta)$ is defined to be the (possibly disconnected and random) subgraph of $G$ induced by the vertex set $\{v \in V_G : R_v = 1\}$. When $t_1 = 0$, we write $\text{perc}_{t_2}(G, \zeta) := \text{perc}_{0, t_2}(G, \zeta)$.

In the percolation we will refer to vertices $v \in V_G$ with $R_v = 1$ as active, and those with $R_v = 0$ as inactive. In our application of Definition 5.9 the vertex marks $\zeta$ of the graph $G$ will be realizations of (Poisson) point processes.

**Definition 5.10.** Given $0 < \Delta \leq T < \infty$, we say a $[[1], \mathcal{M}_N(0, T)]$-graph $(G, \zeta)$ $\Delta$-dissociates if all connected components of $\text{perc}_\Delta(G, \zeta)$ are finite.

For any graph $G$, let $(G, N^G)$ be a driving noise in the sense of Definition 3.3. Also, for $T \in (0, \infty)$, let $N^{G, T} = (N^G_v)_{v \in V_G}$ be the collection of point processes on $[0, T]$ defined by
\[
N^G_v(t_1, t_2) := N^G_v \left( (t_1, t_2) \times (0, C_{[t_1, t_2]} \times \mathcal{J}) \right), \quad \text{for } 0 < t_1 < t_2 \leq T < \infty, v \in V_G, \quad (5.9)
\]
where $\{C_{k,T}\}_{k \in \mathbb{N}, T \in \mathbb{R}}$ is the family of constants from Assumption 1. Note that the union of the events of $N^G_v(t_1, t_2), v \in V_G$, almost surely contains the set of discontinuities of any weak solution $X^{G, \xi}[T]$ of the SDE (3.3).

**Definition 5.11** (Finite dissociability). A deterministic graph $G$ is said to be finitely dissociable if for any $T \in (0, \infty)$, there exists $\Delta > 0$ such that $(G, N^{G, T})$ $\Delta$-dissociates a.s. We call $\Delta$ a $G$-dissociation number. If $G$ is a marked graph, we will say $G$ is finitely dissociable if and only if the corresponding unmarked graph $\tilde{G}$ is finitely dissociable.
Remark 5.12. In the percolated graph $\text{perc}_\Delta(G, \mathbb{N}^{G,T})$ described in Definition 5.11 setting
\[ \bar{C}_{k,T} := \varsigma(J)C_{k,T}, \quad \forall k, T \in \mathbb{R}_+, \] (5.10)
each vertex $v$ is removed from $G$ independently with a probability $\exp(-\bar{C}_{\{v\},T}\Delta)$, which is decreasing in the degree of the vertex $v$.

Remark 5.13. Finite dissociability is a “class property” in that it depends only on the isomorphism class $(G) \in \mathcal{G}_*$ of the graph $G$, and not on the particular representative or choice of the driving noise. Indeed, if $G_1 \cong G_2$, then for any fixed $T \in (0, \infty)$ and Poisson processes $\mathbb{N}^{G_1,T}_i$, $i = 1, 2$, constructed as in (5.9), it is easy to construct a coupling $(\mathbb{N}^{G_1,T}, \mathbb{N}^{G_2,T})$ of $(\mathbb{N}^{G_1,T}, \mathbb{N}^{G_2,T})$ in which $\mathbb{N}^{G_1,T} \equiv \mathbb{N}^{G_2,T}$ for $i = 1, 2$, such that $(G_1, \mathbb{N}^{G_1,T}) \cong (G_2, \mathbb{N}^{G_2,T})$ a.s.. Hence, for $\Delta \in [0, T]$,
\[ \text{perc}_\Delta(G_1, \mathbb{N}^{G_1,T}) \equiv \text{perc}_\Delta(G_2, \mathbb{N}^{G_2,T}), \]
with the equivalence holding a.s.. Since the finite dissociability of $G_i$ only depends on the probability that $\text{perc}_\Delta(G_i, \mathbb{N}^{G,T})$ has an infinite component for some $\Delta > 0$, and the existence of an infinite component is isomorphism invariant, this shows that the finite dissociability property is also invariant with respect to graph isomorphisms. Thus, the statement “$(G) \in \mathcal{G}_*$ is (or is not) finitely dissociable” is well defined.

5.3.2 Examples of Finitely Dissociable Graphs

We now show that the class of (almost surely) finitely dissociable graphs encompasses several interesting classes of graphs of interest in applications, including lattices and regular, GW and UGW trees.

We start by recalling that a rooted tree is a connected, rooted acyclic graph $T = (V, E, \rho)$. Any pair of vertices $u, v \in V$ has a unique path connecting them. A vertex $v \in V$ is said to be in the $k$th generation of $T$ if $d_T(\rho, v) = k$. For $v \in V \setminus \{\rho\}$, the parent of $v$, denoted $\pi_v(T)$, is the unique neighbor $u$ of $v$ such that $d_T(\rho, u) < d_T(\rho, v)$. We also set $c_v(T) := \mathbb{N}_v(T) \setminus \pi_v(T)$ to be the children of $v$. We will write $\pi_v$ and $c_v$ when $T$ is clear from context. For any vertex set $U \subseteq V$, we will let $c_U(T) = \bigcup_{u \in U} c_u(T)$ in addition, given $v, w \in V$, $w$ is said to be a descendant of $v$ if there exists $n \in \mathbb{N}$ and a path $(v = u_0, u_1, \ldots, u_n = w)$ in $T$ such that for every $i = 1, \ldots, n$, $u_i = c_{u_{i-1}}(T)$. We now introduce the definition of UGW trees.

Definition 5.14 (UGW trees). Given a probability $\rho \in \mathcal{P}(\mathbb{N}_0)$, the UGW tree, denoted $\text{UGW}(\rho)$, is a random rooted tree $(T, \rho)$ such that $|c_\rho(T)| \sim \rho$ and for each $k \in \mathbb{N}_0$, conditioned on $B_k(T)$, \[ \{|c_\rho(T)|\}_{v \in V : d_T(\rho, v) = k} \] is an i.i.d. collection of random variables with distribution $\tilde{\rho} \in \mathcal{P}(\mathbb{N}_0)$, where
\[ \tilde{\rho}_{k+1} := \frac{(k + 1)\rho_{k+1}}{\sum_{n=0}^{\infty} n \rho_n}, \quad k \in \mathbb{N}_0. \] (5.11)

Proposition 5.15. If $\rho \in \mathcal{P}(\mathbb{N}_0)$ has a finite first moment, that is, $\sum_{k \in \mathbb{N}} k \rho_k < \infty$, then the GW tree $T := \text{GW}(\rho)$ is a.s. finitely dissociable.

Proof. Fix $T < \infty$, $\Delta \in (0, T)$, and let $T^\Delta := \text{perc}_\Delta(T, \mathbb{N}^{T,T})$ and $R_v = R^{\Delta}_v := \mathbb{I}_{\{\mathbb{N}^{T,T}_v \cup \{v\} \cap (0, \Delta] > 0\}}$, $v \in V$, be as in Definition 5.9. Recall that $T^\Delta$ is precisely the subgraph of $T$ induced by active vertices in $V$. Also, recall that for $v \in V_{T^\Delta}$, $c_v(T^\Delta)$ denotes the connected component of $T^\Delta$ containing $v$, with $v$ as its root. With a small abuse of notation, we extend the definition of $c_v(T^\Delta)$ to all $v \in V$ by setting $c_v(T^\Delta)$ to be the 1-vertex graph $\{v\}$ for $v \in V \setminus V_{T^\Delta}$. In addition, for $v \in V$, let
$T_v$ and $T^\Delta_v$ be the restrictions of $T$ and $T^\Delta$, respectively, to the set containing $v$ and its descendants in $T$. By the self-similarity of the GW tree, for each $v \in V$, $L(T, (R_w)_{w \in V}) = L(T_v, (R_w)_{w \in V_{T_v}})$, and hence, $L(T^\Delta_v|R_\emptyset = 1) = L(T^\Delta_v|R_v = 1)$. For $v \in V$, $R_v = 0$ implies $C_v(T^\Delta_v)$ consists of a single isolated vertex. Hence, $|C_v(T^\Delta_v)| < \infty$ a.s. if and only if $|C_v(T^\Delta_v)| < \infty$ a.s. for all $v \in V$, in which case all connected components of $T^\Delta$ must be a.s. finite. Thus, it suffices to prove that $|C_v(T^\Delta_v)| < \infty$ a.s.

Since the percolation probability at a site or vertex depends on its degree via the dependence on $C_{k,T}$ in (5.3), to bound the size of $C_v(T^\Delta_v)$, we couple $C_v(T^\Delta_v)$ to a larger set obtained from a simpler percolation that only removes vertices from the odd generations of $T$. To this end, for any rooted tree $T$ and $n \in \mathbb{N}$, let $L_n(T) := \{v \in V : d_T(v, o) = n\}$ denote the set of vertices in the $n$th generation. Define the half-percolated forest $\hat{T}^\Delta := \text{hperc}_\Delta(T, \mathbb{N}^{T,T})$ to be the subgraph of $T$ induced by the vertex set $\{v \in V : R_v = 1 \text{ or } \exists T(o, v) \text{ is even}\}$. Then let $\hat{T}^\Delta_v := C_v(\hat{T}^\Delta)$ be the subtree of $\hat{T}^\Delta$ that contains the root (note that the root always belongs to $\hat{T}^\Delta$). Clearly, $|C_v(T^\Delta_v)| \leq |\hat{T}^\Delta_v|$. Thus, to prove the proposition it clearly suffices to show that for all sufficiently small $\Delta > 0$,

$$\lim_{n \to \infty} \mathbb{E} \left[|L_{2n}(\hat{T}^\Delta_v)|\right] = 0,$$

(5.12)

since then $\mathbb{P}(|\hat{T}^\Delta_v| = \infty) = \mathbb{P}\left(|L_{2n}(\hat{T}^\Delta_v)| > 0 \text{ for all } n \in \mathbb{N}\right) \leq \inf_{n \in \mathbb{N}} \mathbb{P}\left(|L_{2n}(\hat{T}^\Delta_v)| > 0\right) = 0.

To prove (5.12), choose $n \in \mathbb{N}_0$ with $L_{2n}(\hat{T}^\Delta_v) \neq \emptyset$, and fix $v \in L_{2n}(\hat{T}^\Delta_v)$. Recall that for any tree $T$ and $v \in V$, $c_v(T)$ denotes the collection of children of $v$ in $T$ and $\pi_v(T)$ is the parent of $v$ in $T$ whenever $v \neq o$, and note that $c_v(\hat{T}^\Delta_v) = c_v(\hat{T}^\Delta) = \{u \in c_v(T) : R_u = 1\}$. Also, observe that

$$c_{c_v(\hat{T}^\Delta_v)}(\hat{T}^\Delta_v) := \bigcup_{u \in c_v(\hat{T}^\Delta_v)} c_u(\hat{T}^\Delta_v) = \{w \in c_v(T) : u \in c_v(T) \text{ and } R_u = 1\},$$

(5.13)

where the equality uses the fact that the half-percolation does not remove any vertices from even generations of the tree. Now, for each $w \in L_{2n}(T) = L_{2n}(\hat{T}^\Delta_v)$, let $Z_{\Delta,w}$ be the number of grandchildren of $w$ in $\hat{T}^\Delta_v$. Then, since $v \in L_{2n}(\hat{T}^\Delta_v)$, $Z_{\Delta,w}$ is also the number of grandchildren of $v$ in $\hat{T}^\Delta_v$ and hence, by (5.13),

$$Z_{\Delta,w} := |c_{c_v(\hat{T}^\Delta_v)}(\hat{T}^\Delta_v)| = |c_{c_v(\hat{T}^\Delta_v)}(T)| \quad \text{is } \sigma\left(\{c_u(T)\}_{u \in \{v\} \cup c_v(T)}\right) \text{-measurable. \quad (5.14)}$$

Furthermore, $L_{2n}(T)$ and $L_{2n}(\hat{T}^\Delta_v)$ are both measurable with respect to $\mathcal{H}_n := \sigma(B_{2n}(T), \{R_w\}_{w \in B_{2n-1}(T)})$. Let $A \in \mathcal{H}_n$ be an atomic event, that is, $A = \{B_{2n}(T) = \hat{T}, \{R_w\}_{w \in B_{2n-1}(T)} = \{r_w\}_{w \in B_{2n-1}(T)}\}$ for some $k$-generation tree $\hat{T}$, with $k \leq 2n$ and some $(r_w)_{w \in B_{2n-1}(T)} \in \{0,1\}^{B_{2n-1}(T)}$. Then, conditioned on $A$, the collection of random variables

$$\left(\{c_u(T)\}_{u \in \{v\} \cup c_v(T)}, \{R_u\}_{u \in c_v(T)}\right)_{v \in L_{2n}(T)}$$

is equal in distribution to $|L_{2n}(T)|$ independent copies of $\left(\{c_u(T)\}_{u \in \{v\} \cup c_v(T)}, \{R_u\}_{u \in c_v(T)}\right)$. Together with (5.13), this implies that $\gamma = \gamma_{\Delta,v} := L(Z_{\Delta,w}|A)$ does not depend on $A$ or the specific choice of $v$ in $L_{2n}(T)$. Moreover, conditioned on $A$, (5.13) implies

$$Z_{\Delta,v} = \sum_{u \in c_v(T)} R_u |c_u(T)| \overset{(d)}{=} \sum_{u \in c_v(T)} R_u |c_u(T)| = Z_{\Delta,0}.$$
Thus, conditioned on \( \mathcal{H}_n \), \( \{ Z_{\Delta, v} \}_{v \in L_{2n}(T)} \) is equal in distribution to \( L_{2n}(T) \) i.i.d. copies of \( Z_{\Delta, \phi} \). Also, for \( v \in L_{2n}(T) \), by the assumption that \( \rho \) has finite mean

\[
\mathbb{E} \left[ Z_{\Delta, v} \mid \mathcal{H}_n \right] = \mathbb{E} \left[ Z_{\Delta, \phi} \right] \leq \mathbb{E} \left[ \sum_{v \in \mathcal{V}(T)} |c_u(T)| \right] = \left( \sum_{k=0}^{\infty} k \rho(k) \right)^2 < \infty.
\]

Since \( L_{2n}(\tilde{T}_\phi) \) is \( \mathcal{H}_n \)-measurable and \( \gamma_{\Delta, v} = \gamma_\Delta \) for all \( v \in L_{2n}(\tilde{T}_\phi) \), a recursive calculation shows

\[
\mathbb{E} \left[ |L_{2n+2}(\tilde{T}_\phi)| \right] = \mathbb{E} \left[ \sum_{v \in L_{2n}(\tilde{T}_\phi)} \mathbb{E} \left[ Z_{\Delta, v} \mid \mathcal{H}_n \right] \right] = \mathbb{E} \left[ |L_{2n}(\tilde{T}_\phi)| \right] \mathbb{E} \left[ Z_{\Delta, \phi} \right] = \mathbb{E} \left[ Z_{\Delta, \phi} \right]^{n+1}.
\]

Thus, to show (5.12), it suffices to prove that for sufficiently small \( \Delta > 0 \), \( \mathbb{E} \left[ Z_{\Delta, \phi} \right] < 1 \). However, note that for all \( \Delta > 0 \), \( \mathbb{E} \left[ Z_{\Delta, \phi} \right] < \infty \). Furthermore, the definition of \( \Delta \)-dissociation clearly implies \( \lim_{\Delta \to 0} Z_{\Delta, \phi} = 0 \) a.s., so the Lebesgue dominated convergence theorem implies \( \lim_{\Delta \to 0} \mathbb{E} \left[ Z_{\Delta, \phi} \right] = 0 \), which concludes the proof.

In light of Definition 5.14 this immediately implies the corresponding result for UGW trees (see Definition 5.14).

**Corollary 5.16.** If \( T \sim \text{UGW}(\rho) \), where \( \rho \in \mathcal{P}(\mathbb{N}_0) \) has a finite second moment, then \( T \) is a.s. finitely dissociable.

**Proof.** The fact that \( \rho \) has a finite second moment implies that \( \hat{\rho} \) in Definition 5.14 has a finite first moment. Let \( \tilde{T}_v \) denote the descendant tree of \( T \) rooted at \( v \). Then for each \( v \neq \phi \), \( \tilde{T}_v \) is a GW(\( \hat{\rho} \))-tree, so by Theorem 5.13 for sufficiently small \( \Delta > 0 \), \( \mathbb{E} \left[ \hat{\rho}^k \right] \mathbb{E} \left[ \text{perc}(\tilde{T}_v, N(\tilde{T}_v, T)) \right] < \infty \) a.s. finite. Since the \( \Delta \)-percolated descendant tree of every non-root vertex is finite, it follows that \( T \) a.s. \( \Delta \)-dissociates.

For completeness, we also show that graphs with bounded maximum degree (such as infinite lattices) are also finitely dissociable.

**Proposition 5.17.** Let \( G = (V, E) \) be a graph with finite maximum degree: \( d^* := \sup_{v \in V} |N_v| < \infty \). Then \( G \) is finitely dissociable.

**Proof.** Fix \( 0 < T < \infty \) and \( \Delta \in (0, T] \). Then by Definitions 5.9 and 5.9 each vertex in \( G \) is inactive independently with probability

\[
p_v = \exp \left( -\Delta \bar{C}_{|cl_v|, T} \right) \geq \exp \left( -\Delta \bar{C}_{d^*+1, T} \right) := p_{\Delta, T},
\]

where the inequality invokes (5.10) and the definition of \( d^* \). Therefore, the probability that \( (G, N^{G, T}) \) \( \Delta \)-dissociates is greater than or equal to the probability that \( G \) fails to percolate with respect to a standard site percolation in which each vertex is independently removed with probability \( p_{\Delta, T} \). For any \( T < \infty \), \( \lim_{\Delta \to 0} p_{\Delta, T} = 1 \), but as is well known, the critical probability (i.e., the probability \( p_c \) such that \( G \) fails to percolate a.s. when vertices are independently removed with probability \( p > p_c \)) is strictly less than 1 (see [14, Equation (0.3)]). Thus, for all \( T \), there exists a sufficiently small \( \Delta \) such that \( (G, N^{G, T}) \) \( \Delta \)-dissociates a.s. By Definition 5.14 this shows that \( G \) is finitely dissociable.

As demonstrated in Appendix 4 even Markovian IPS with very regular jump rate functions, finite dissociability and well-posedness can fail on some graphs.
5.4 Consistent Spatial Localization on Finitely Dissociative Graph Sequences

The main result of this section is Proposition 5.19 which states that sequences of finitely dissociative graphs consistently spatially localize the SDE (3.3). A key challenge is to find an explicit and consistent representation of the localizing map \( S_T(\cdot; G, N^G) \) on finitely dissociative graphs. To do this, we start by introducing the notion of a \textit{causal chain}. Fix \( G \) and let

\[
\mathcal{E}_{v,T} := \{ t \in (0, T] : N^G_v(\{t\}) = 1 \}, \quad v \in V_G, T \in \mathbb{R}_+ ,
\]

where \( N^G_v \) is defined as in (5.9).

**Definition 5.18 (Causal chains).** Given \( T \in (0, \infty) \), an interval \( I := [t_1, t_2] \subseteq [0, T] \) and vertices \( u, v \in V_G \), a \((G, N^G)\)-causal chain from \( v \) to \( u \) during \( I \) is a path \( \Gamma := (v = u_0, u_1, \ldots, u_n = u) \) in \( G \) for some \( n \in \mathbb{N}_0 \) such that if \( n \neq 0 \), there exists an increasing sequence \( t_1 := s_0 < s_1 < \cdots < s_n \leq t_2 \) for which \( s_i \in \mathcal{E}_{u_i, T}, i = 1, \ldots, n \). We write \( v \sim_{t_1, t_2} u \) if there exists a \((G, N^G)\)-causal chain from \( v \) to \( u \) during \( I = [t_1, t_2] \), and for any \( U \subset V_G \), we write \( v \sim_{t_1, t_2} U \) if \( v \sim_{t_1, t_2} u \) for some \( u \in U \).

Intuitively, causal chains describe long-range interactions over the graph that can develop over an interval \( I \), even though the instantaneous evolution of the state of a vertex is only influenced by the states of neighboring vertices. Specifically, given a graph \( G, O \in \Lambda_G \) and \( 0 \leq t_1 < t_2 < \infty \), define

\[
\mathcal{A}_{t_1, t_2}^G(O) := \{ v \in V_G : v \sim_{t_1, t_2} O \} \quad \text{and} \quad \mathcal{A}_t^G(O) := \mathcal{A}_{0, t}(O).
\]

Then \( \mathcal{A}_{t_1, t_2}^G(O) \) represents the set of vertices in \( G \) that are “seen” by vertices in \( O \) through a causal chain during some time interval \([t_1, t_2]\). We now state the main result of this section.

**Proposition 5.19.** Suppose Assumption \( \text{A} \) holds, and suppose the deterministic graph \( G \) is finitely dissociable. Then \( G \) spatially localizes the SDE (3.3) in the sense of Definition 5.7. Moreover, any sequence of deterministic finitely dissociable \((\mathcal{K}, \mathcal{K})\)-graphs \( \{G_n\}_{n \in \mathbb{N}} \) consistently spatially localizes the SDE (3.3).

We first show why the proposition directly implies Theorem 4.2 and subsequently present its proof, which proceeds by showing that the (random) map that takes \( O \in \Lambda_G \) to the closure of \( \mathcal{A}_t^G(O) \) for finitely dissociable graphs \( G \) defines a consistent family of localizing maps.

**Proof of Theorem 4.2 given Proposition 5.19:** Given Assumption \( \text{A} \) Proposition 5.19 and Definition 5.5 show that \((G)\) a.s. spatially localizes (3.3), and Proposition 4.3 shows that Assumption \( \text{A} \) holds. Therefore, the theorem follows from Proposition 5.7.

**Proof of Proposition 5.19:** We start with the proof of the first statement of the proposition. Fix a deterministic \((\mathcal{K}, \mathcal{K})\)-graph \( G \) and let \((\mathcal{F}, \mathcal{N}^G)\) be a filtration-Poisson process pair. For each (not necessarily finite) \( W \subseteq V_G \), let \( X^{G[W], \xi_W} \) be an arbitrary \((\mathcal{F}, \mathcal{N}^G)\)-weak solution to (3.3) for \((G[W], \xi_W)\) (assuming one exists, which is always the case when \( W \) is finite by Proposition 3.1). Let \( O \in \Lambda_G \), and for \( T \in \mathbb{R}_+ \), set \( S_T(O; G, N^G) := \mathcal{A}_T(O) := \mathcal{A}_{t_1, t_2}^G(O) \), with the latter defined as in (5.16). Note that by the definition of causal chains, for any \( U' \in \Lambda_G \), \( \{A_T(O) \subset U' \} \) lies in \( \mathcal{F}^N_{G,T} \), and hence \( S_T(\cdot; G, N^G) = \mathcal{A}_T \) satisfies (5.1). Moreover, since \( u \sim_{t_1, t_2} u \) for all \( u \in O \), \( \mathcal{A}_T(O) \supseteq O \), thus verifying that \( S_T(\cdot; G, N^G) \) satisfies property 1 of Definition 5.1. To prove that finitely dissociable graphs spatially localize (3.3), we argue below that it suffices to establish the following claims for any \( T \in (0, \infty) \):

**Claim 1:** If \( T \) is a \( G \)-dissociation number, then \( |\mathcal{A}_T(O)| < \infty \) a.s. for every \( O \in \Lambda_G \).
Claim 2: If \( G \) is finitely dissociable, then \( |A_T(O)| < \infty \) a.s. for every \( O \in \Lambda_G \).

Claim 3: Fix any \( \ell \in \mathbb{N} \) and (not necessarily finite) \( W \subseteq V_G \) such that \( B_t(G) \subseteq W \) and a weak solution \( X^{G,W}_{\ell,W} \). Then for each \( O \in \Lambda_G \), \( X^{B_t(G),\xi_{B_t(G)}}_{O} [T] = X^{G,W}_{\ell,W} [T] \) a.s. on the event \( \{A_T(O) \subseteq B_t(G)\} \).

Claim 2 shows that \( S_T(\cdot;G,N^G) \) satisfies (5.2) when \( G \) is finitely dissociable. Suppose Claims 1-3 hold, and let \( W \supseteq B_t(G) \) be any (not necessarily finite) vertex set for which there exists a weak solution \( X^{G,W}_{\ell,W} \) to the SDE (3.3). Then clearly \( B_t(G[W]) = B_t(G) \) for \( O \in \Lambda_G \), and on the event \( \{B_t(G) \supseteq A_T(O)\} \), the \((G[W],N^G_W)\)-causal chains ending at \( O \) at time \( T \) are the same as the \((G,N^{G,T})\)-causal chains ending at \( O \) at time \( T \), that is, \( A^G_T(O) = A^G_T(O) = A_T(O) \). Thus, we can apply Claim 3 twice (first directly and then when \( W \) is replaced with \( V \)) to conclude that if there exists a weak solution \( X^{G,\xi}_O \) to the SDE (3.3), then

\[
X^{G,W}_{\ell,W} [T] = X^{B_t(G),\xi_{B_t(G)}}_{O} [T] = X^{G,\xi}_O [T] \text{ a.s. on } \{A_T(O) \subseteq B_t(G)\}.
\]

This implies that \( S_T(\cdot;G,N^G) \) satisfies (5.3) and thus, Property 2 of Definition 5.1 holds. Since we have verified all properties of Definition 5.1 when \( G \) is dissociable, the first assertion of the proposition follows.

We now turn to the proofs of the claims.

Proof of Claim 1. Fix \( O \in \Lambda_G \). Define \( \widehat{G} := \text{perc}_T(G,N^{G,T}) \) as in Definition 5.9 where the point process \( N^{G,T}_W \) is given by (5.9). If \( T \) is a dissociation number, then each of the connected components of \( \widehat{G} \) is a.s. finite, and \( N^{G,T}_W(0,T) = 0 \) for all \( w \in \mathcal{N}_O \), where \( \widehat{\mathcal{O}} := \mathcal{O} \cup \bigcup_{v \in \mathcal{O}} \mathcal{E}_v(\widehat{G}) \).

Since \( \mathcal{O} \) is finite, \( \widehat{\mathcal{O}} \) is finite. Now suppose \( u \in \mathcal{O} \) and \( v \in A_T(u) \). Let \( \{v = u_0, u_1, \ldots, u_n = u\} \) be a \((G,N^{G,T})\)-causal chain with respect to \([0,T]\). Then, for any \( i = 1, \ldots, n \), \( u_i \) must be active, that is, \( N^{G,T}_W(0,T) > 0 \). Thus, for each \( i = 1, \ldots, n \), \( u_i \in \widehat{\mathcal{O}} \), and so \( v \in \text{cl}_{\widehat{\mathcal{O}}}(G) \). Hence, \( A_T(O) = \bigcup_{u \in \mathcal{O}} A_T(u) \subseteq \text{cl}_{\widehat{\mathcal{O}}}(G) \), which is a.s. finite since \( \widehat{\mathcal{O}} \) is a.s. finite and \( G \) is locally finite.

Proof of Claim 2. Fix \( u \in U \) and \( v \in A_T(U) \). Because \( G \) is finitely dissociable, there must exist a \( G \)-dissociation number \( \Delta > 0 \). If \( \Delta \geq T \), the result follows from Claim 1. If \( \Delta \in (0,T) \), then Claim 1 implies that a.s.,

\[
|A_\Delta(U)| < \infty \quad \text{for every } U \in \Lambda_G. \tag{5.17}
\]

To complete the proof, for every \( t \in [0,T-\Delta] \), we will show that if \( |A_t(U)| < \infty \) a.s. for every \( U \in \Lambda_G \), then we also have \( |A_{t+\Delta}(O)| < \infty \) a.s. for every \( O \in \Lambda_G \).

To this end, fix \( t \in [0,T-\Delta] \), and suppose \( |A_t(O)| < \infty \) a.s. for all \( O \in \Lambda_G \). Fix \( O \in \Lambda_G \), \( u \in O \) and \( v \in A_{t+\Delta}(u) \). Then there exists a \((G,N^{G,T})\)-causal chain \( \Gamma = (v = u_0, u_1, \ldots, u_n = u) \) during the interval \([0,t+\Delta]\), with the corresponding sequence of times \( 0 = s_0 < s_1 < \cdots < s_n \leq t + \Delta \).

Let \( i_\ast \) be the largest integer \( i \in \{0, \ldots, n\} \) such that \( s_i \leq \Delta \). Then \( v \in A_\Delta(u_{i_\ast}) \). Furthermore, by considering the path \((u'_0 = u_{i_\ast}, u'_1 = u_{i_\ast+1}, \ldots, u'_{n-i_\ast} = u) \) with times \( s'_0 = \Delta \) and \( s'_i = s_{i+i'_\ast} \), \( i = 1, \ldots, n-i_\ast \), it follows that \( u_{i_\ast} \sim_{\Delta,t+\Delta} u \). Thus, \( v \in A_\Delta(A_{\Delta,t+\Delta}(u)) \), which implies \( A_{t+\Delta}(O) \subseteq A_\Delta(U) \), where \( U := A_{\Delta,t+\Delta}(O) \). In view of (5.17), it suffices to show that \( U \) is a.s. finite, but this holds because by time homogeneity of Poisson processes, \( U^{(d)} = A_t(O) \), which is a.s. finite by assumption.

Proof of Claim 3. For notational conciseness, let \( H := G[W] \), where we recall that \( W \) is a deterministic vertex set such that \( B_t(G) \subseteq W \). Additionally, let \( H_t := B_t(G) \), and note that
\( H_t = B_t(H) \). Fix \( \mathcal{O} \subseteq A_H \) with \(|\mathcal{A}_H^H(\mathcal{O})| < \infty \) and note that \( \mathcal{A}_H^H(\mathcal{O}) = \mathcal{A}_T^G(\mathcal{O}) = \mathcal{A}_T(\mathcal{O}) \) on the event \( \{H_t \supseteq \mathcal{A}_H^H(\mathcal{O})\} \). To prove Claim 3, we need to show that

\[
X_{H_t}^{H_t,\xi_U}[T] = X_{H_t}^{H_t,\xi_{H_t}}[T] \text{ a.s. on } \{\mathcal{A}_T(\mathcal{O}) \subseteq V_{H_t}\}. \tag{5.18}
\]

Assume \( N_{G,T}^G(\{T\}) = 0 \), which holds a.s. because \( N_{G,T}^G \) is a countable collection of homogeneous Poisson processes. We prove this claim in a recursive fashion by iterating over events in the driving noise \( \mathcal{N}_G \), and relating them to the dynamics of the SDE \( (5.3) \). For each \( v \in W \), let \( \{t^v_i, i \in \mathbb{N}\} \) be an enumeration of the (a.s. finite) set \( \mathcal{E}_{v,T} \) from \( (5.15) \) of events of \( N_{v,T}^G \) in \([0,T]\), arranged in increasing order. Below, we use the conventions that \( \max \emptyset = 0 \), \( \inf \emptyset = \infty \) and \( \{0\} = \emptyset \). Define \( U_0 := \mathcal{O}, \tau_0 := T \), choose an arbitrary vertex \( v_0 \in \mathcal{O} \) and for \( k \in \mathbb{N} \), recursively define

\[
\tau_k := \begin{cases} 
\max \{t^v_i : v \in U_{k-1}, t^v_i < \tau_{k-1}\} & \text{if } \tau_{k-1} > 0, \\
0 & \text{if } \tau_{k-1} = 0,
\end{cases} \tag{5.19}
\]

\[
v_k := \begin{cases} 
v \in U_{k-1} \text{ s.t. } N_{v,T}^G(\{\tau_k\}) = 1 & \text{if } \tau_k > 0, \\
v_{k-1} & \text{if } \tau_k = 0,
\end{cases} \tag{5.20}
\]

\[
U_k := \begin{cases} 
U_{k-1} \cup \text{cl}_{v_k} & \text{if } \tau_k > 0, \\
U_{k-1} & \text{if } \tau_k = 0.
\end{cases} \tag{5.21}
\]

Also, set \( K := \inf \{k \in \mathbb{N} : \tau_k = 0\} \). Note that the above construction is well defined even if \( K = \infty \), the sequence \( \{\tau_k\}_{k \in \mathbb{N}} \) is strictly decreasing, and the sequence \( \{U_k\}_{k \in \mathbb{N}} \) is non-decreasing, but with possible repetitions (for example, if \( v_1 \) lies in the interior of \( U_{k-1} \)).

The set \( U_K \) is specifically constructed so that \( U_K \subseteq A_T(\mathcal{O}) \) and so that \( X_{U_K}^{H_t,\xi_U}[T] = X_{H_{\tau_k},\xi_{H_{\tau_k}}}[T] \) on the event \( \{H_t \supseteq U_K\} \), which immediately implies Claim 3. In fact, we will show that the recursive construction \( (5.19)-(5.21) \) is such that the following two claims are true.

**Claim 3A:** For every \( k \in \mathbb{N}_0 \), \( U_k \subseteq A_T(\mathcal{O}) \).

**Claim 3B:** For every \( k \in \mathbb{N}_0 \),

\[
X_{U_k}^{H_t,\xi_U}[\tau_k] = X_{U_k}^{H_{\tau_k},\xi_{H_{\tau_k}}}[\tau_k] \Rightarrow X_{U_k}^{H_t,\xi_U}[T] = X_{U_k}^{H_{\tau_k},\xi_{H_{\tau_k}}}[T] \text{ a.s. on } \{U_k \subseteq V_{H_t}\}. \tag{5.22}
\]

We first show how Claim 3 follows from the auxiliary claims. On the event \( \{A_T(\mathcal{O}) \subseteq V_{H_t}\} \), Claim 3A and \( (5.19) \) together show that \( \{\tau_k\}_{k \in \mathbb{N}} \setminus \{0\} \) is contained in the events of \( N_{G,T}^G(0,T) \); since \( H_t \) is finite, this implies \( K \leq 1 + \sum_{v \in V_{H_t}} N_{G,T}^G(0,T) < \infty \) a.s.. By the definition of \( K \), \( (5.19) \) and \( (5.21) \) this implies that a.s. on the event \( \{A_T(\mathcal{O}) \subseteq V_{H_t}\} \), \( \tau_K = 0 \), \( U_K = \bigcup_{k \in \mathbb{N}_0} U_k \) and (applying Claim 3A with \( k = K \)) \( U_K \subseteq A_T(\mathcal{O}) \subseteq V_{H_t} \subseteq W \). Together, these properties imply, \( X_{U_K}^{H_t,\xi_U}[\tau_K] = \xi_{U_K} = X_{U_K}^{H_t,\xi_{H_t}}[\tau_K] \). Invoking Claim 3B with \( k = K \), we see that a.s. on \( \{U_K \subseteq V_{H_t}\} \supseteq \{A_T(\mathcal{O}) \subseteq V_{H_t}\} \), we have \( X_{U_K}^{H_t,\xi_U}[T] = X_{U_K}^{H_t,\xi_{H_t}}[T] \). This proves Claim 3.

We first provide a rough idea of the proof of the auxiliary claims. When \( k = 0 \), \( U_k = \mathcal{O} \) and it is easy to see that claims 3A and 3B will hold trivially (due to the stipulation that \( \mathcal{O} \subseteq A_T(\mathcal{O}) \) and the assumption that there is no jump at \( T \)). If there is any jump in the driving processes \( \mathcal{N}_O^{G,T} \) in \([0,T]\), then by the construction \( (5.19)-(5.21) \), the most recent event before \( \tau_0 = T \) that could have influenced the value of the process \( X_{U_K}^{H_t,\xi_U} \) at \( \tau_0 \) occurs at \( \tau_1 \) corresponding to a transition at the vertex \( v_1 \). The local nature of the dynamics implies that this transition is influenced by the particles in \( \text{cl}_{v_1} \), so that the trajectory of \( X_{U_k}^{H_t,\xi_U} \) up to time \( \tau_1 \) is influenced by the trajectories of
the particles in $O \cup \text{cl}
v = U_0 \cup \text{cl}
u = U_1$ before time $\tau_1$. Any vertex $u$ in $U_1 \setminus O$ belongs to $N_{\nu_1}$ and so $(u, \nu_1)$ forms a $(G, N_{\nu_1}^G)$ causal chain, showing that $U_1 \subset \mathcal{A}_T(0)$. Claims 3A and 3B follow by proceeding inductively in this manner, tracing backwards in time the $(G, N_{\nu_1}^G)$-causal chains that end in $O$. We now provide fully rigorous proofs of the auxiliary claims.

**Proof of Claim 3A.** We prove the following assertion using induction: For any $k \in \mathbb{N}_0$ and $v \in U_k$, $v \sim_{\tau_k+1,T} O \cup \text{cl}
u_1 = U_1$ before time $\tau_1$. Any vertex $u$ in $U_1 \setminus O$ belongs to $N_{\nu_1}$ and so $(u, \nu_1)$ forms a $(G, N_{\nu_1}^G)$ causal chain, showing that $U_1 \subset \mathcal{A}_T(0)$. Claims 3A and 3B follow by proceeding inductively in this manner, tracing backwards in time the $(G, N_{\nu_1}^G)$-causal chains that end in $O$. We now provide fully rigorous proofs of the auxiliary claims.

**Proof of Claim 3B.** We will again use an argument by induction. First, note that the base case $k = 0$ in (5.22) is true because $N_{O}^G, \tau_0 = T$ and both $X_{O}^{H,\xi_W}$ and $X_{O}^{H,\xi_H_t}$ a.s. do not have a jump at $T$ because $N_{O}^G(\{T\}) = 0$ a.s. Now, suppose (5.22) holds for $k = m - 1$, for some $m \in \mathbb{N}$. To show it holds for $k = m$, we argue that it suffices to show that

$$X_{U_m}^{H,\xi_W}[\tau_m] = X_{U_m}^{H,\xi_H_t}[\tau_m] \Rightarrow X_{U_{m-1}}^{H,\xi_W}[\tau_{m-1}] = X_{U_{m-1}}^{H,\xi_H_t}[\tau_{m-1}] \text{ a.s. on } \{U_m \subseteq V_{H_t}\}. \tag{5.23}$$

Indeed, $U_{m-1} \subseteq U_m$ implies $U_{m-1} \subseteq V_{H_t}$ on the event $\{U_m \subseteq V_{H_t}\}$, and so (5.23) and (5.22), with $k = m - 1$, shows that (5.22) holds for $k = m$. Claim 3B then follows by induction.

To establish (5.23), assume $U_m \subseteq V_{H_t}$ and $X_{U_m}^{H,\xi_W}[\tau_m] = X_{U_m}^{H,\xi_H_t}[\tau_m]$. Since $U_{m-1} \subseteq U_m$, this implies $X_{U_{m-1}}^{H,\xi_W}[\tau_{m-1}] = X_{U_{m-1}}^{H,\xi_H_t}[\tau_{m-1}]$. Moreover, note from (5.19) that $\tau_m$ is the largest time prior to $\tau_{m-1}$ that there is an event for any of the Poisson processes in $U_{m-1}$ and hence, $\sum_{v \in U_{m-1}} N_{U_m}^G(\tau_m, \tau_{m-1}) = 0$. The form of the SDE (5.23) then implies that both $X_{U_{m-1}}^{H,\xi_W}$ and $X_{U_{m-1}}^{H,\xi_H_t}$ are constant on $(\tau_m, \tau_{m-1})$. Thus, to establish (5.23), it suffices to show that $X_{U_{m-1}}^{H,\xi_W}(\tau_m) = X_{U_{m-1}}^{H,\xi_H_t}(\tau_m)$ a.s. Now, by (5.19)-(5.20), $v_m$ is the only vertex in $U_{m-1}$ such that $N_{v_m}^G(\{\{\tau_m\}\}) = 1$. Thus,

$$X_{U_{m-1}}^{H,\xi_W}(v_m) = X_{U_{m-1}}^{H,\xi_W}(v_m)(\tau_m) = X_{U_{m-1}}^{H,\xi_H_t}(v_m)(\tau_m) = X_{U_{m-1}}^{H,\xi_H_t}(v_m)(\tau_m), \tag{5.24}$$

and so it only remains to show that $X_{U_m}^{H,\xi_W}(\tau_m) = X_{U_m}^{H,\xi_H_t}(\tau_m)$ a.s. Since for $m \in \mathbb{N}$, $\text{cl}_{v_m} \subset U_m$ by (5.21), and $U_m \subseteq V_{H_t} \subseteq W$ by assumption, $\text{cl}_{v_m}(H_t) = \text{cl}_{v_m}(G) = \text{cl}_{v_m}$. Then the assumption $X_{U_m}^{H,\xi_W}(\tau_m) = X_{U_m}^{H,\xi_H_t}(\tau_m)$ implies $X_{\text{cl}_{v_m}}^{H,\xi_W}(\tau_m) = X_{\text{cl}_{v_m}}^{H,\xi_H_t}(\tau_m)$ a.s. The locality and predictability (see Definition 3.1) of the jump rates stated in the Standing Assumption then imply that for $j \in \mathcal{I}$, denoting by $(\bar{\Gamma}, \kappa)$ the marks of $G|\text{cl}_{v_m}$ we have

$$r_j^{H,\xi_W}(s, X_{\text{cl}_{v_m}}^{H,\xi_W}) = r_j^{H,\xi_H_t}(s, X_{\text{cl}_{v_m}}^{H,\xi_H_t}), \quad s \in [0, \tau_m].$$

28
Due to the form of the SDE (3.3), this shows \( X^{H,\xi_{W}}_{\tau_{m}}(\tau_{m}) = X^{H,\xi_{H_{t}}}_{\tau_{m}}(\tau_{m}) \), as desired. \( \square \)

We now turn to the proof of the second assertion of the proposition. Let \( \{G_{n}\} \) be a sequence of deterministic finitely dissociable \( [K,K] \)-graphs, and for each \( n \in \mathbb{N} \), let \( N^{G_{n}} \) be a driving noise compatible with \( G_{n} \). Then define \( A_{t_{1},t_{2}}^{n} \) analogously to \( A_{t_{1},t_{2}} \), but with \( G \) and \( N^{G} \) replaced by \( G_{n} \) and \( N^{G_{n}} \) respectively. By the first assertion of the proposition just established above, for each \( n \), \( A_{T}^{n} \) is a localizing map of the SDE (3.3) on \( (G_{n},N^{G_{n}}) \). It follows that \( S_{T}(\cdot;G_{n},N^{G_{n}}) := cl_{A_{T}^{n}}(\cdot)(G_{n}) \) is likewise a localizing map of the SDE (3.3) on \( (G_{n},N^{G_{n}}) \). Note that \( S_{T}(\cdot;G_{n},N^{G_{n}}) \) satisfies (5.1) as a consequence of the fact that \( \{A_{T}^{n}(U)\} \) for every \( U,U' \in \Lambda_{G_{n}} \). With this choice of localizing maps \( S_{T}(\cdot;G_{n},N^{G_{n}}) \), clearly Property 1 of Definition 5.4 holds.

It only remains to show that Property 2 of Definition 5.4 is also satisfied. Fix \( n,n',\ell \in \mathbb{N} \) such that there exists an isomorphism \( \varphi \in I(B_{\ell}([G_{n},s]),B_{\ell}([G_{n},s])) \). Define the \( F_{0} \)-measurable event

\[
I_{\varphi} := \{ \varphi \in I(B_{\ell}([G_{n},s],N^{G_{n}}),B_{\ell}([G_{n},s],N^{G_{n}})) \}
\]

and for notational conciseness, set \( W := B_{\ell-1}(G_{n}) \) and \( W' := B_{\ell-1}(G_{n'}) \). Let \( U_{n} \subset V_{G_{n}} \) and \( U_{n'} \subset V_{G_{n'}} \) be any pair of sets such that \( \varphi(U_{n}) = U_{n} \). Then it suffices to prove that on the event \( I_{\varphi} \cap \{ W \supseteq A_{T}^{n}(U_{n}) \} = I_{\varphi} \cap \{ B_{\ell}(G_{n}) \supseteq S_{T}(U_{n};G_{n},N^{G_{n}}) \} \cap A_{T}^{n}(U_{n}) = \varphi(A_{T}^{n}(U_{n})) \) a.s..

Let \( \Gamma = (v = u_{0},u_{1},\ldots,u_{m} \in U_{n}) \) be a \( (G_{n},N^{G_{n},T}) \)-causal chain up to time \( T \) which we will refer to as a \( G_{n} \)-causal chain for now. Let \( (0 = t_{0} < t_{1} < \cdots < t_{n} \leq T) \) be the corresponding times associated with \( \Gamma \). By definition of \( A_{T}^{n}(U_{n}) \), \( \Gamma \subseteq W \) on the event \( I_{\varphi} \cap \{ W \supseteq A_{T}^{n}(U_{n}) \} \). Moreover, \( \varphi^{-1}|_{W} \) is an isomorphism in \( I([G_{n},s][W],N^{G_{n}}),([G_{n}',s][W'],N^{G_{n}'}) \) so \( \varphi^{-1}(\Gamma) \) is a path in \( G_{n}' \). Furthermore, for every \( k \in \{ 1,\ldots,m \} \), \( N^{G_{n}'}(\{ t_{k} \}) = 1 \), so \( N^{G_{n}'}(\{ t_{k} \}) = 1 \). It follows that \( \varphi^{-1}(\Gamma) \) is also a \( (G_{n}',N^{G_{n}'}) \)-causal chain which we will refer to as a \( G_{n}' \)-causal chain.

We do not claim that for any \( G_{n}' \)-causal chain \( \Gamma' \), \( \varphi(\Gamma') \) is a \( G_{n} \)-causal chain. To see why this claim holds, let \( \Gamma' = (v' = u'_{0},u'_{1},\ldots,u'_{m} \in U_{n'}) \). By an identical argument to what we applied in the previous paragraph, if \( \Gamma' \subseteq W' \), then \( \varphi(\Gamma') \) is a \( G_{n} \)-causal chain. We now argue by contradiction to justify that this is the only possible case. Indeed, suppose there exists \( k \in \{ 0,1,\ldots,m \} \) such that \( u'_{k} \notin W' \). Note that \( u'_{m} \in U_{n'} \subseteq W' \), so \( k < m \). Choose the maximal such \( k \) so that \( u'_{k+1},\ldots,u'_{m} \subseteq W' \). Then because subpaths of causal chains are causal chains, \( \Gamma' := (u'_{k},\ldots,u'_{m}) \) is also a \( G_{n}' \)-causal chain. Furthermore, \( u'_{k} \in N_{W'}(G_{n}') \), so on the event \( I_{\varphi}, \varphi(\Gamma') \) is a path and a causal chain in \( G_{n} \). Moreover, \( \varphi(\Gamma') \notin W \) which contradicts our assumption that \( A_{T}^{n}(U_{n}) \subseteq W \). This proves the claim.

Because \( \varphi \) a.s. induces a bijection between causal chains in \( G_{n} \) and \( G_{n}' \) that end in \( U_{n} \) and \( U_{n'} \) respectively, it follows that \( A_{T}^{n}(U_{n}) = \varphi(A_{T}^{n}(U_{n})) \) on the event \( I_{\varphi} \cap \{ A_{T}^{n}(U_{n}) \subseteq W \} \). Equivalently, \( S_{T}(U_{n};G_{n},N^{G_{n}}) = \varphi(S_{T}(U_{n};G_{n},N^{G_{n}'})) \) on the event \( \{ S_{T}(U_{n};G_{n},N^{G_{n}}) \subseteq B_{\ell}(G_{n}) \} \cap I_{\varphi} \). It follows that Property 2 of Definition 5.4 also holds for the sequence \( \{G_{n}\}_{n \in \mathbb{N}} \).

This proves the second assertion and hence, concludes the proof of the proposition. \( \square \)

6 Local Convergence of the IPS

The main goal of this section is to prove the local convergence result of Theorem 4.3. When the initial data have finite, deterministic and equal unmarked representatives \( [G_{n},s] = [G,s] \), \( n \in \mathbb{N} \), \( |V_{G}| < \infty \), and the corresponding SDEs in (3.3) are driven by the same Poisson processes for different \( n \), Theorem 4.3 is essentially a consequence of the continuity condition in Assumption 2 which implies pathwise continuity of the dynamics of the SDE (3.3) with respect to the initial data. For more general finite initial data, the proof entails carefully constructed couplings, and when dealing
with infinite graphs, it will also involve applications of consistent spatial localization to reduce the analysis to the finite graph case. While the localizing maps associated with consistent spatial localization (see Definition 5.4) only make sense on graphs, rather than their equivalence classes, local convergence results are defined in terms of equivalence classes. To bridge this gap, we need to carefully select suitable representatives of equivalence classes, as well as establish correspondences between statements about convergence of equivalence classes and statements about representative graphs.

Section 6.1 defines canonical representatives, with a view to constructing suitable couplings. These are then used in Section 6.2 to establish a more general almost sure local convergence result in Proposition 6.11. The latter result is used in Section 6.3 to prove Theorem 4.3 and also in Section 7 in the proof of the hydrodynamic limit.

### 6.1 Canonical Representative Graphs and Consistent Extensions

Let $\mathcal{Z}$ be Polish spaces. We begin by defining a space of $[\mathcal{Z}, \mathcal{Z}]$-graphs:

$$\mathcal{G}_s[\mathcal{Z}, \mathcal{Z}] = \{[\mathcal{Z}, \mathcal{Z}]\text{-graphs } G := (V, E, \phi, \vartheta) \text{ s.t. } V \subseteq \mathbb{N}\}. \quad (6.1)$$

It follows from Lemmas B.5, B.6 and B.7 that $\mathcal{G}_s[\mathcal{Z}, \mathcal{Z}]$ can be equipped with a Polish topology that is compatible with the topology of $\mathcal{G}_s[\mathcal{Z}, \mathcal{Z}]$, and that for any $\mathcal{G}_s[\mathcal{Z}, \mathcal{Z}]$-random element $\langle G \rangle$, there exists a $\sigma((G))$-measurable $\mathcal{G}_s[\mathcal{Z}, \mathcal{Z}]$-random element $G$ such that $G \in \langle G \rangle$ almost surely. In the latter case, $G$ is referred to as a $\mathcal{G}_s[\mathcal{Z}, \mathcal{Z}]$-random representative of $\langle G \rangle$, and thus, $\mathcal{G}_s[\mathcal{Z}, \mathcal{Z}]$ can be viewed as a canonical space of measurable representative graphs compatible with the local topology. We begin with the definition of a representative convergent sequence.

**Definition 6.1 (Representative convergent sequences).** Let $\langle (G_n, \xi^n) \rangle$ be a random sequence converging a.s. to $\langle (G, \xi) \rangle$ in $\mathcal{G}_s[\mathcal{K}, \mathcal{K} \times X]$ on some complete probability space. Then a representative convergent sequence (henceforth abbreviated to rep-con sequence) of $\{(G_n, \xi^n)\}_{n \in \mathbb{N}} \cup \{(G, \xi)\}$ is a $\sigma(\{(G_n, \xi^n)\}_{n \in \mathbb{N}}, \{\langle (G, \xi) \rangle\})$-measurable tuple $\{(G_n, \xi^n), M_n\}_{n \in \mathbb{N}}, (G, \xi), \{\varphi_{n,m}\}_{n \in \mathbb{N}, m \leq M_n}$, defined on the same probability space, that satisfies the following properties:

1. for each $n \in \mathbb{N}$, $(G_n, \xi^n) = (V_n, E_n, \varphi_n, \kappa^n, \varphi^n, \xi^n)$ is a $\mathcal{G}_s[\mathcal{K}, \mathcal{K} \times X]$-random representative of $\langle (G_n, \xi^n) \rangle$;
2. $(G, \xi) = (V, E, \phi, \sigma, \kappa, \vartheta, \xi)$ is a $\mathcal{G}_s[\mathcal{K}, \mathcal{K} \times X]$-random representative of $\langle (G, \xi) \rangle$;
3. $\lim_{n \to \infty} M_n = \infty$ a.s. and $B_m([G], \xi) \equiv B_m([G_n], \xi^n)$ on the event $\{m \leq M_n\}$;
4. for each $n \in \mathbb{N}, m \leq m' \leq M_n, \varphi_{n,m} \in I(B_m([G], \xi), B_m([G_n], \xi^n))$ and $\varphi_{n,m'}|_{B_m(G)} = \varphi_{n,m}$;
5. for every $n', m \in \mathbb{N}$ such that $m \leq M_n' \text{ and } e \in E_{B_m(G)}, \lim_{n \to \infty, n > n'} \kappa_{n', \varphi_{n,m}(e)} = \kappa_e$;
6. for every $n', m \in \mathbb{N}$ such that $m \leq M_n'$ and $v \in B_m(G), \lim_{n \to \infty, n > n'} \kappa_{n', \varphi_{n,m}(v)} = \kappa_v$.

The next lemma guarantees the existence of rep-con sequences. A constructive proof of the lemma, which leverages the existence of suitably measurable representative graph sequences, isomorphisms and driving maps is deferred to in Appendix B.2.

**Lemma 6.2.** Fix a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ that supports the random sequence $\langle (G_n, \xi^n) \rangle$ converging a.s. to $\langle (G, \xi) \rangle$ in $\mathcal{G}_s[\mathcal{K}, \mathcal{K} \times X]$. Then there exists an $\mathcal{F}$-measurable sequence of $\mathcal{G}_s[\mathcal{K}, \mathcal{K} \times X]$-random elements $\{(G_n, \xi^n)\}_{n \in \mathbb{N}} \cup \{(G, \xi)\}$ satisfying Properties 1 and 2 of Definition 6.1. In addition, given any sequence $\{(G_n, \xi^n)\}_{n \in \mathbb{N}}, (G, \xi)$ satisfying Properties 1 and 2 of Definition 6.1 the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ also supports a rep-con sequence $\langle (G_n, \xi^n), M_n \rangle, (G, \xi), \{\varphi_{n,m}\}$.
To extend the notion of rep-con sequences from initial data consisting of \( \hat{\mathcal{G}}_n[\hat{\mathcal{K}}, \mathcal{K} \times \mathcal{X}] \)-random elements to the corresponding IPS characterized by \( \hat{\mathcal{G}}_n[\hat{\mathcal{K}}, \mathcal{K} \times \mathcal{D}] \)-random elements, we will need a common probability space on which we can define both the driving noise and a rep-con sequence.

**Definition 6.3** (Consistent representative convergent extensions). Given a complete probability space \((\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}})\) that supports a sequence \(\{(G_n, \xi^n)\}\) converging a.s. to \((G, \xi)\) in \(\hat{\mathcal{G}}_n[\hat{\mathcal{K}}, \mathcal{K} \times \mathcal{X}]\), a consistent representative convergent extension (henceforth abbreviated to consistent rep-con extension) of \((\Omega, \mathcal{F}, \mathbb{P}), \{(G_n, \xi^n)\}_{n \in \mathbb{N}}, \{(G, \xi)\}\) is a 4-tuple \((\Omega, \mathcal{F}, \mathbb{P}), \{(G_n, \xi^n, \mathcal{N}^{G_n}), M_n\}_{n \in \mathbb{N}}, (G, \xi, \mathcal{N}^G), \{\varphi_{n,m}\}_{n \in \mathbb{N}, m \leq M_n}\) such that

1. \(\{(G_n, \xi^n), M_n\}, (G, \xi), \{\varphi_{n,m}\}\) is a rep-con sequence of \(\{(G_n, \xi^n)\}_{n \in \mathbb{N}}, \{(G, \xi)\}\);
2. \((\Omega, \mathcal{F}, \mathbb{P})\) is a complete extension of the probability space \((\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}})\) such that \(\mathbb{P}\) satisfies the usual conditions and \((G_n, \mathcal{N}^{G_n}), n \in \mathbb{N}\), and \((G, \mathcal{N}^G)\), respectively, are \(\mathbb{P}\)-driving noises that are compatible with \(G_n, n \in \mathbb{N}\), and \(G\), as defined in Definition 3.3.
3. for each \(n \in \mathbb{N}\) and \(v \in B_{M_n}(G)\), \(\mathcal{N}^{G_n}_{\varphi_{n,M_n}}(v) = \mathcal{N}^G_v\) a.s.

The construction of consistent rep-con extensions is facilitated by the use of so-called *driving maps* defined below. Let \(\mathcal{M}\) be the space of maps from subspaces \(W \subseteq \mathbb{N}\) to \(\mathbb{N}\), which can be equipped with a Polish topology by Remark 6.8.

**Definition 6.4.** Given a \(\hat{\mathcal{G}}_n[\hat{\mathcal{K}}, \mathcal{K} \times \mathcal{X}]\)-random element \((G, \xi) := (V, E, \varnothing, \mathcal{K}, \kappa, \xi)\), a driving map is a random injective map \(\psi : V \to \mathbb{N}\), that is, \(\psi\) is a \(\sigma(G, \xi)\)-measurable random element taking values in \(\mathcal{M}\). Suppose \((\Omega, \mathcal{F}, \mathbb{P})\) is a filtered probability space supporting a collection of i.i.d. \(\mathbb{F}\)-Poisson processes \(\{N_k\}_{k \in \mathbb{N}}\) on \(\mathbb{R}_+^2 \times \mathcal{J}\) with intensity \(\text{Leb}^2 \otimes \varsigma\). Suppose also that \(\mathbb{F}\) satisfies the usual conditions and \((G, \xi)\) is \(\mathcal{F}_0\)-measurable. Then the \([\mathcal{K}, \mathcal{K} \times \mathcal{M}^\mathbb{N}(\mathbb{R}_+^2 \times \mathcal{J})]\)-random graph \((G, \mathcal{N}^G)\) defined by \(\mathcal{N}^G_v = \mathcal{N}(\psi_v)\) is said to be an \(\mathbb{F}\)-driving noise generated by \(\psi\).

**Remark 6.5.** We now justify our reference to \((G, \mathcal{N}^G)\) as a \(\mathbb{F}\)-driving noise in Definition 6.4.

It is easy to see that on any complete, filtered probability space \((\Omega, \mathcal{F}, \mathbb{P})\) supporting the \(\mathcal{F}_0\)-measurable initial data \((G, \xi)\) and \(\mathbb{F}\)-Poisson processes \(\{N_k\}_{k \in \mathbb{N}}\), the \(\sigma(G, \xi)\)-measurability (and thus \(\mathcal{F}_0\)-measurability) of driving maps ensures that for any \(\mathcal{F}_0\)-measurable \(v \in V_G\), \(\mathcal{N}^G_v\) is \(\mathbb{F}\)-adapted and therefore supported on \((\Omega, \mathcal{F}, \mathbb{P})\). In this case, Condition 1 of Definition 3.3 is satisfied.

Furthermore, conditioned on \(\mathcal{F}_0\), \(\mathcal{N}^G\) is a collection of i.i.d. Poisson processes indexed by \(V_G\). Thus, Condition 1 of Definition 3.3 is satisfied. Lastly, because \(\{N_k\}_{k \in \mathbb{N}}\) are \(\mathbb{F}\)-Poisson processes, for any \(t > 0\) and \(A \in \mathcal{B}(\mathcal{D}((t, \infty) \times \mathcal{R}_+ \times \mathcal{J}))\), \((G, \mathcal{N}^G(A))\) is conditionally independent of \(\mathcal{F}_1\) given \(\mathcal{F}_0\). Therefore, Condition 2 of Definition 3.3 is satisfied. Assuming \(\mathbb{F}\) satisfies the usual conditions, all three conditions of Definition 3.3 are satisfied.

The following lemma shows that there always exists a consistent rep-con extension of any convergent sequence of random elements in \(\hat{\mathcal{G}}_n[\hat{\mathcal{K}}, \mathcal{K} \times \mathcal{X}]\). We prove this at the end of Appendix B.

**Lemma 6.6.** Let \(\{(G_n, \xi^n)\}_{n \in \mathbb{N}}\) be a random sequence on \((\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}})\) that converges a.s. to \((G, \xi)\) in \(\hat{\mathcal{G}}_n[\hat{\mathcal{K}}, \mathcal{K}]\). Then there exists a consistent rep-con extension

\[ (\Omega, \mathcal{F}, \mathbb{P}), \{(G_n, \xi^n, \mathcal{N}^{G_n}), M_n\}_{n \in \mathbb{N}}, (G, \xi, \mathcal{N}^G), \{\varphi_{n,m}\}_{n \in \mathbb{N}, m \leq M_n} \]

of

\[ (\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}}), \{(G_n, \xi^n)\}_{n \in \mathbb{N}}, (G, \xi) \]

such that \((\Omega, \mathcal{F}, \mathbb{P})\) supports a collection of i.i.d. \(\mathbb{F}\)-Poisson processes \(\mathcal{N} := \{N_k\}_{k \in \mathbb{N}}\) and driving maps \(\psi_n, n \in \mathbb{N}\), and \(\psi\) that generate the respective \(\mathbb{F}\)-driving maps \(\mathcal{N}^{G_n}, n \in \mathbb{N}\), and \(\mathcal{N}^G\).
Consistent rep-con extensions are useful because, as shown in Proposition 6.11 of the next section, under the conditions of Theorem 4.3 a sequence of IPS will converge almost surely if it is generated by a consistent rep-con extension in the following sense.

Definition 6.7. A sequence of IPS \( \{(G_n, X^{G_n} \xi^n)\}_{n \in \mathbb{N}} \) is said to be generated by a consistent rep-con extension \( ((\Omega, F, F, \mathbb{P}), \{(G_n, \xi^n, N^{G_n}), M_n\}_{n \in \mathbb{N}}, (G, \xi, N^G), \{\varphi_{n,m}\}_{n \in \mathbb{N}, m \leq M_n}) \) if for each \( n \in \mathbb{N} \), \( (G, X^{G_n} \xi^n) \) is a \( N^{G_n} \)-strong solution to (3.3) for \( (G_n, \xi^n) \) and, likewise, \( (G, X^{G} \xi) \) is a \( N^G \)-strong solution to (3.3) for \( (G, \xi) \).

Not every sequence of IPS can be generated by a consistent rep-con extension of its initial data. For example, this would fail if \( \{(G_n, \xi^n)\}_{n \in \mathbb{N}}, (G, \xi) \) is a deterministic sequence of isomorphism classes such that \( (G_n, \xi^n) \to (G, \xi) \) in \( S_n(\mathcal{F}, K \times \mathcal{A}) \), and \( (G, X^{G_n} \xi^n), (G, X^{G} \xi) \) are mutually independent. However, as noted below, consistent rep-con extensions are preserved under conditioning on \( F_0 \).

Remark 6.8. If \( \{(G_n, X^{G_n} \xi^n)\}_{n \in \mathbb{N}}, (G, X^G \xi) \), are IPS generated by a consistent rep-con extension of \( ((\Omega, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}), \{(G_n, \xi^n)\}_{n \in \mathbb{N}}, (G, \xi)) \), then for almost every \( \omega \in \Omega \), \( L(\{(G_n, X^{G_n} \xi^n)\}_{n \in \mathbb{N}}, (G, X^{G} \xi)|F_0)(\omega) \) describes the law of a sequence of IPS generated by a consistent rep-con extension of a tuple that contains the terms \( \{(G_n, \xi^n)(\omega)\}_{n \in \mathbb{N}} \) and \( (G, \xi)(\omega) \). A rigorous justification of this would follow along the same lines as Lemma 3.8 and is thus omitted.

### 6.2 Proof of Local Convergence in the almost sure setting

We start by stating a “finite convergence” condition on IPS defined on finite truncations of graphs (equipped with the associated marks).

Assumption 2. Given any consistent rep-con extension \( ((\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}), \{(G_n, \xi^n, N^{G_n}), M_n\}_{n \in \mathbb{N}}, (G, \xi, N^G), \{\varphi_{n,m}\}_{n \in \mathbb{N}, m \leq M_n}) \) of \( (\Omega, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}), \{(G_n, \xi^n)\}_{n \in \mathbb{N}}, (G, \xi)) \), \( T \in \mathbb{R}_+ \) and \( m \in \mathbb{N} \), there exists an a.s. finite, \( \mathcal{F}_T \)-measurable random variable \( \bar{N}_m := \bar{N}_{m,T} \) such that for every \( n \in \mathbb{N} \),

\[
(B_m([G_n]), X^{m,n}[T]) \overset{a.s.}{=} (B_m([G_n]), X^{m,\infty}[T]) \text{ a.s. on the event } \{n \geq \bar{N}_m\} \cap \{m \leq M_n\},
\]

where \( X^{m,n} \) and \( X^{m,\infty} \) are the respective \( N^{G_n}_{B_m(G_n)} \)- and \( N^G_{B_m(G)} \)-strong solutions to (3.3) for the initial data \( B_m(G_n, \xi^n) \) and \( B_m(G, \xi) \).

Remark 6.9. Given Assumption 1 as a consequence of Remark 6.8, if Assumption 2 holds for a tuple \( ((\Omega, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}), \{(G_n, \xi^n)\}, (G, \xi)) \) then it holds for a.s. every realization of the isomorphism classes \( \{(G_n, \xi^n)\}, (G, \xi)) \).

The following lemma shows that the finite convergence property of Assumption 2 holds under our basic Assumptions 1 and 2 together with consistent spatial localization.

Lemma 6.10. Suppose the family of jump rate functions \( r \) satisfies Assumption 1 and \( (G, \xi) \) satisfies Assumption 2. If the sequence \( \{(G_n)\}_{n \in \mathbb{N}}, (G) \) defined on \( (\Omega, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}) \) consistently spatially localizes the SDE (3.3), then the tuple \( ((\Omega, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}), \{(G_n, \xi^n)\}_{n \in \mathbb{N}}, (G, \xi)) \) satisfies Assumption 2.

The weaker Assumptions 1 and 2 have been introduced in place of Assumptions 1 or Assumption 2 with a view to future extensions of our results to more general IPS for which the latter assumptions may not hold but the former assumptions and consistent spatial localization can nevertheless be established. Indeed, as we now show, the main convergence result of this section holds under these weaker assumptions. We defer the proof of Lemma 6.10 to after the proof of this main result.

32
Proposition 6.11. Suppose that Assumption 1 holds, the tuple \((\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}), \{(G_n, \xi^n)\}_{n \in \mathbb{N}}, \langle (G, \xi) \rangle\) satisfies Assumption 3 and \{(G_n)\}_{n \in \mathbb{N}}, \langle (G) \rangle\) a.s. consistently spatially localizes the SDE (3.3). Also, let \((G_n, X_n^{G_n, \xi^n})\)\(n \in \mathbb{N}\), \((G, X^G, \xi)\) be a collection of IPS generated by a consistent rep-con extension of the tuple. If \(\langle (G_n, \xi^n) \rangle \to \langle (G, \xi) \rangle\) a.s. in \(\mathcal{G}_x[\mathcal{K} \times \mathcal{X}]\), then \(\langle (G_n, X_n^{G_n, \xi^n}) \rangle \to \langle (G, X^G, \xi) \rangle\) a.s. in \(\mathcal{G}_x[\mathcal{K} \times \mathcal{D}]\).

Proof. Let \(\mathbb{N}_\infty := \mathbb{N} \cup \{\infty\}\) and let \(\langle (G^\infty, \xi^\infty) \rangle := \langle (G, \xi) \rangle\), and \((G^\infty, X^\infty, \xi^\infty) := (G, X^G, \xi)\) respectively. We first prove the proposition under the additional assumption that \((G_n, \xi^n)\)\(n \in \mathbb{N}_\infty\) are deterministic. In this case, let \((\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}), \{(G_n, \xi^n, N_n^{G_n}), M_n\}_{n \in \mathbb{N}}, \langle (\varphi_{n,m})_{n,m \in \mathbb{N}, m \leq M_n}\rangle\) be a consistent rep-con extension of \((\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}), \{(G_n, \xi^n)\}_{n \in \mathbb{N}_\infty}\), where \((G^\infty, \xi^\infty) := (G, \xi)\) holds. In addition, let \(N_n^{G^\infty} := N^G\). Due to Assumption 1 and the assumption of consistent spatial localization, there exists a consistent sequence of localizing maps \(\{S_T(\cdot; G_n, N_n^{G_n})\}_{n \in \mathbb{N}_\infty}\) for the SDE (3.3) on \((G_n, N_n^{G_n})\)\(n \in \mathbb{N}_\infty\), and by Proposition 5.7 for each \(n \in \mathbb{N}_\infty\), the SDE (3.3) is strongly well-posed for \((G_n, \xi^n)\). Hence, the \(N_n^{G_n}\)-strong solution \(X_n^{G_n, \xi^n}\) to (3.3) for the initial data \((G_n, \xi^n)\) is well defined.

Fix \(n', m \in \mathbb{N}\) such that \(m \leq M \leq M_{n'}\). By Property 3 of Definition 6.3, for every \(n \in \mathbb{N}\), \(m \leq M_n\), and \(v \in B_m(G), N_n^{G_n, \xi^n} = N^G\). Thus, by an application of Property 2 of the consistent spatial localization property in Definition 5.4 with \(\ell = M\), \(n = \infty\), \(n' = n\), \(\varphi = \varphi_{n', M}\), and \(U = B_m(G)\), and noting that then \(\mathcal{I}_\varphi = \{M \leq M_{n'}\} = \tilde{\Omega}\), we see that

\[ S_T(U; G, N^G) = \varphi(S_T(\varphi^{-1}(U); G', N_n^{G_n})) = S_T(\varphi_{n', M}(U); G_n, N_n^{G_n}) = \varphi_{n', M}(S_T(U; G, N^G)) \]

a.s. on the event \(\{S_T(U; G, N^G) \subseteq B_M(G)\}\). Because \(S_T(U; G, N^G)\) is a.s. finite, there must exist a \(\mathcal{F}_T\)-measurable a.s. finite random variable \(M_{n'}\) such that \(S_T(U; G, N^G) \subseteq B_{M_{n'}}(G)\) a.s.

Furthermore, by (5.3) this implies that

\[ X_n^{G, \xi}[T] = X_n^{B_M(G), \xi_B M(G)}[T] \text{ and } X_n^{G_{n'}, \xi_{n'}}[T] = X_n^{B_M(G_{n'}), \xi_{B_M(G_{n'})}}[T] \text{ a.s. on } \{M \geq M_{n'}\}. \]

By (6.2) of Assumption 1 there exists an a.s. finite, \(\mathcal{F}_T\)-measurable random variable \(N_{M}\) such that

\[ X_n^{B_M(G), \xi_B M(G)}[T] = X_n^{B_M(G_{n'}), \xi_B M(G_{n'})}[T] \text{ a.s. on the event } \{n' \geq N_M\}. \]

The last two displays together show that a.s. on the event \(\{M \geq M_{n'}\} \cap \{n' \geq N_M\}\),

\[ X_n^{G, \xi}[T] = X_n^{B_M(G), \xi_B M(G)}[T] = X_n^{B_M(G_{n'}), \xi_B M(G_{n'})}[T] = X_n^{G_{n'}, \xi_{n'}}[T]. \]

Applying the last display for each \(M\) satisfying \(m \leq M \leq M_{n'}\) and noting by Property 1 of Definition 5.4 that \(\overline{M}_m \geq m\), it follows that

\[ X_n^{G, \xi}[T] = X_n^{G_{n'}, \xi_{n'}}[T] \text{ a.s. on the event } \{\overline{M}_m \leq M_{n'}\} \cap \{n' \geq N_{\overline{M}_m}\}. \]

Sending \(n' \to \infty\) and noting that then \(M_{n'} \to \infty\) by Property 3 of Definition 6.3 and \(\overline{M}_m\) is a.s. finite it follows that

\[ \lim_{n' \to \infty} X_n^{G_{n'}, \xi_{n'}}[T] = X_n^{G, \xi}[T] \text{ a.s. } \]

This concludes the proof for deterministic sequences.
The random case can be obtained by conditioning on the initial data. More precisely, suppose \( \{ (G_n, \xi^n) \}_{n \in \mathbb{N}} \), \( (G, \xi) \) is random and let \( \{ (G_n, X^{G_n, \xi^n}) \}_{n \in \mathbb{N}} \), \( (G, X^{G, \xi}) \) be the sequence of IPS generated by a consistent rep-con extension of the tuple \( \{ \langle \Omega, \bar{F}, \bar{P} \rangle, \{ (G_n, \xi^n) \}_{n \in \mathbb{N}}, \langle (G, \xi) \rangle \} \). Then by Remark 6.9, for almost every \( \omega \in \Omega \), \( \mathcal{L}(\{ (G_n, X^{G_n, \xi^n}) \}, \langle G, X^{G, \xi} \rangle | \mathcal{F}_0)(\omega) \) describes the law of a sequence of IPS generated by a consistent rep-con extension of a tuple that contains the terms \( \{ (G_n, \xi^n)(\omega) \}_{n \in \mathbb{N}}, \langle (G, \xi) \rangle(\omega) \). Moreover, by Remark 6.9, Assumption 3 holds a.s. conditioned on \( \mathcal{F}_0 \). Therefore, by the proof of the proposition for deterministic initial data, \( \mathbb{P}(\{ (G_n, X^{G_n, \xi^n}) \} \rightarrow \langle (G, X^{G, \xi}) \rangle | \mathcal{F}_0) = 1 \) a.s.. Therefore, \( \{ (G_n, X^{G_n, \xi^n}) \} \rightarrow \langle (G, X^{G, \xi}) \rangle \) a.s., as desired. \( \square \)

We finish the section with a proof of Lemma 6.10.

**Proof of Lemma 6.10**: Fix a consistent rep-con extension of the given tuple, write \( G = (V, E, \phi, \pi, \kappa) \), \( G_n = (V_n, E_n, \phi_n, \pi_n, \kappa_n) \), \( n \in \mathbb{N} \), and set \( M := \max_{v \in V} d_G(v, \phi) \). We first consider the case when \( \{ G_n \}_{n \in \mathbb{N}} \) and \( G \) are a.s. finite graphs and additionally assume that \( \max_{v \in V_n} d_{G_n}(v, \phi_n) \leq M \) for all \( n \in \mathbb{N} \), and show that for any \( T \in \mathbb{R}_+ \), there exists an a.s. finite, \( \mathcal{F}_T \)-measurable random variable \( \bar{N} := \bar{N}_{M,T} \) such that,

\[
\left( [G_n, X^{G_n, \xi^n}[T]] \right) \cong \left( [G, X^{G, \xi}[T]] \right) \text{ a.s. on the event } \{ n \geq \bar{N} \} \cap \{ M \leq M_n \}. \quad \tag{6.3}
\]

Note that on the event \( A_n := \{ M \leq M_n \} \), \( [G_n, \xi^n] \cong [G, \xi^n] \). To show \( (6.3) \), fix \( T \in \mathbb{R}_+ \) and for each \( n \in \mathbb{N} \), let \( \phi_n : V \rightarrow V_n \) be the \( \mathcal{F}_0 \)-measurable map given by \( \phi_n := \phi_{n,M} \) on the event \( A_n \) (on \( A_n^c \), we may define \( \phi_n \) to be any measurable function with the appropriate domain and range, for instance, the function that maps all vertices of \( G \) to the root of \( G_n \)). For each \( n \in \mathbb{N} \) and \( v \in V_G \), recall from Property 3 of Definition 4.3 that \( \mathbb{N}^{G_n}_{\phi_n(v)} = \mathbb{N}^G_v \) on the event \( A_n \). In terms of the family of constants \( \{ C_{k,T} \}_{k \in \mathbb{N}, T \geq 0} \) from Assumption 4, define

\[
\mathcal{E} = \mathcal{E}_T := \{(v, t, r, j) \in V \times [0, T] \times (0, C_{\pi(G), T}] \times \mathcal{J} : \mathbb{N}^G_v \left( \{ (t, r, j) \} \right) = 1 \}.
\]

Since \( |\mathcal{E}| < \infty \) a.s., we can a.s. order the elements \( \{(v_k, \tau_k, r_k, j_k) \} \) of \( \mathcal{E} \) such that \( \{ \tau_k \}_{k \in \mathbb{N}} \) is strictly increasing. Note that \( \{ \bar{\tau}_k \}_{k \in \mathbb{N}} \) is the sequence of points in a time-homogeneous Poisson process, which implies that each \( \bar{\tau}_k \) is an absolutely continuous (Gamma distributed) non-negative random variable. Thus, for any Borel set \( O \subset [0, T] \) such that \( \text{Leb}(O) = 0 \), \( \mathbb{P}(\bar{\tau}_k \in O \text{ for any } k) = \sum_{k=1}^{\infty} \mathbb{P}(\bar{\tau}_k \in O, k \leq |\mathcal{E}|) = 0 \). Let \( R := \inf \left\{ r \in \mathbb{R} : r^G_v(\tau_k, X^{G, \xi}) : k = 1, \ldots, |\mathcal{E}| \right\} \). By the predictability of the jump rate function \( r_{jk}^{G,v} \), the random variables \( r_k \) and \( r_{jk}^{G,v}(\tau_k, X^{G, \xi}) \) are independent and therefore \( r_k \neq r_{jk}^{G,v}(\tau_k, X^{G, \xi}) \) a.s. which implies that \( R > 0 \) a.s. On the event \( A_n \), let

\[
R_n := \sup \left\{ r_{jk}^{G,v}(\tau_k, X^{G, \xi}) - r_{jk}^{G,\phi(v)}(\tau_k, X^{G, \xi}_{\phi(v)}) \left( \tau_k, X^{G, \xi}_{\phi(v)} \right) : k = 1, \ldots, |\mathcal{E}| \right\},
\]

where \( X^{G, \xi}_{\phi(v)} = (X^{G, \xi}_{\phi(v)}(v))_{v \in V_n} \). By Property 3 of Definition 6.1, \( M_n \) a.s. diverges to infinity, so \( \{ A_n \}_{n \in \mathbb{N}} \) is a sequence of events such that \( \mathbb{P}(\cap_{n \geq 1} A_{n'}) = 1 \) so that \( \lim_{n \rightarrow \infty} \mathbb{I}(A_n) = 1 \) a.s.. By Properties 5 and 6 of Definition 6.1 it follows that for each \( v \in V \) and \( e \in E \),

\[
\lim_{n \rightarrow \infty} \mathbb{I}_{A_{n}}(\mathbb{K}^n_{\phi_n(e)} \mathbb{K}^n_{\phi_n(v)}) = (\mathbb{K}_e, \mathbb{K}_v) \quad \text{a.s..} \quad \tag{6.4}
\]

Let \( \mathbb{K}_{\phi(E)}^n = (\mathbb{K}^n_{\phi(e)})_{e \in E} \) and \( \mathbb{K}^n_{\phi(v)} = (\mathbb{K}^n_{\phi(v)})_{v \in V} \). Then using the fact that the local rate functions are class functions (specifically, applying Remark 3.2 with \( G_1 = (V, E, \phi, \mathbb{K}^n_{\phi(v)}), G_2 = G_n \)}
and $\varphi = \varphi_n^{-1}$ in the first equality below, and combining (6.4) with the absolute continuity of $\tau_k$ and the fact that $(G, \xi)$ satisfies Assumption 2 we have for every $k \in \mathbb{N}$,

$$\lim_{n \to \infty} r_{jk} G_n \varphi_n(v_k) \left( \tau_k, X^G_\xi(\varphi^{-1}_n(V_n)) \right) I_{A_n} = \lim_{n \to \infty} r_{jk} G_n \varphi_n(v_k) \left( \tau_k, X^G_\xi \right) I_{A_n} = r_{jk} G_n \varphi_n(v_k) \left( \tau_k, X^G_\xi \right) a.s. \text{ on the event } \{ k \leq |\bar{E}| \}.$$  

Because $|\bar{E}|$ is a.s. finite, it follows that $\lim_{n \to \infty} R_n = 0$ a.s.. Moreover, the fact that $R > 0$ a.s. implies the existence of an a.s. finite random variable $\bar{N}$ such that,

$$R_n < \frac{R}{2} \text{ on the event } \{ n \geq \bar{N} \} \cap A_n. \quad (6.5)$$

We now argue that $X^G_\xi[T] = X^{G_n, \xi^n}[T]$ on the event $\{ n \geq \bar{N} \} \cap A_n$ by making use of the fact that for each $n \in \mathbb{N}$ and $v \in V$, $X^G_\xi$ and $X^{G_n, \xi^n}$ are driven by the same Poisson processes on the event $A_n$. Fix $n \in \mathbb{N}$ and note that by the SDE (3.3), $X^G_\xi$ and $X^{G_n, \xi^n}$ are both a.s. continuous on the random set $\{ t \in [0, \infty) \setminus \{ \tau_k \}_{k \in \mathbb{N}} \} \cap A_n$. Furthermore, at time $\tau_k$, the processes $X^G_\xi$ and $X^{G_n, \xi^n}$ may either remain constant or experience a jump of size $j_k$ at the respective vertices $v_k$ and $\varphi_n(v_k)$. It follows from the SDE (3.3) that the processes will either simultaneously jump or both fail to jump if and only if

$$\text{sgn} \left( r_k - r_{jk} G_n \varphi_n(v_k) (\tau_k, X^G_\xi) \right) = \text{sgn} \left( r_k - r_{jk} G_n \varphi_n(v_k) (\tau_k, X^{G_n, \xi^n}) \right),$$

where $\text{sgn}: \mathbb{R} \to \mathbb{R}$ is the càdlàg function given by $\text{sgn}(a) = I_{\{a \geq 0\}} - I_{\{a < 0\}}$. Suppose that $X^G_\xi[\tau_k] = X^{G_n, \xi^n}[\tau_k]$. Then, using the predictability of the jump rates (by the Standing Assumption and Definition 3.1) in the first equality below, we have

$$\text{sgn} \left( r_k - r_{jk} G_n \varphi_n(v_k) (\tau_k, X^G_\xi) \right) = \text{sgn} \left( r_k - r_{jk} G_n \varphi_n(v_k) (\tau_k, X^{G_n, \xi^n}) \right) = \text{sgn} \left( r_k - r_{jk} G_n \varphi_n(v_k) (\tau_k, X^G_\xi) \right) + r_{jk} G_n \varphi_n(v_k) (\tau_k, X^{G_n, \xi^n}) (\tau_k, X^{G_n, \xi^n}) \right).$$

However, the last line of the above display is a.s. equal to $\text{sgn} \left( r_k - r_{jk} G_n \varphi_n(v_k) (\tau_k, X^G_\xi) \right)$ on the event $\{ n \geq \bar{N} \}$ by (6.5). Thus, $X^G_\xi[\tau_k] = X^{G_n, \xi^n}[\tau_k]$ a.s. on the event $\{ n \geq \bar{N} \} \cap A_n$. It follows that $X^G_\xi[\tau_k] = X^{G_n, \xi^n}[\tau_k]$ a.s. if $|\bar{E}| > k$ and $X^G_\xi[T] = X^{G_n, \xi^n}[T]$ a.s. if $|\bar{E}| = k$. Applying induction, we see that $X^G_\xi[T] = X^{G_n, \xi^n}[T]$ a.s. on the event $\{ n \geq \bar{N} \} \cap A_n$, and so (6.3) follows.

To see why this implies Lemma 6.10 in the general case of possibly infinite graphs, note that (6.3) implies that for any $m \in \mathbb{N}$ and any sequence $\{ (G_n) \}_{n \in \mathbb{N}}$, $(G, \xi)$ we may replace $(G_n, \xi^n), n \in \mathbb{N}$ by $B_m(G_n, \xi^n), n \in \mathbb{N}, (G, \xi)$ by $B_m(G, \xi), \bar{N}_m$ by $\bar{N}$ and $M$ by $m$ in (6.3) to get (6.2). This concludes the proof of the lemma.

### 6.3 Proof of Theorem 4.3

We now show how Theorem 4.3 follows from Proposition 6.11. The proof uses a simple argument involving the Skorokhod representation theorem and Proposition 5.19.

**Proof of Theorem 4.3** Set $\langle (G, \xi) \rangle := \langle (G, \xi) \rangle$ and $\mathbb{N}_\infty := \mathbb{N} \cup \{ \infty \}$. By Assumption 1 and the a.s. finite dissociability of $\{ (G_n) \}_{n \in \mathbb{N}_\infty}$, the conditions of Proposition 5.19 are satisfied a.s.. Hence, the collection $\{ (G_n) \}_{n \in \mathbb{N}_\infty}$ a.s. consistently spatially localizes the SDE (3.3). Assumption
Proposition 4.4 and Proposition 5.7 then imply that the SDE (3.3) is strongly well-posed for all initial data in \((\mathbb{(G_n, \xi^n)})_{n \in \mathbb{N}}\). Moreover, since \(((G_n, \xi_n)) \Rightarrow (G, \xi)\) in \(\mathbb{S}_s[\mathcal{K}, \mathcal{K} \times \mathcal{X}]\), by the Skorokhod representation theorem there exists a (complete) probability space \((\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})\) that supports random elements \((\tilde{G}_n, \tilde{\xi}^n) \overset{d}{=} (G_n, \xi_n), n \in \mathbb{N}_\infty\) such that \((\tilde{G}_n, \tilde{\xi}^n) \Rightarrow (\tilde{G}, \tilde{\xi}) \overset{d}{=} (G, \xi)\) in \(\mathbb{S}_s[\mathcal{K}, \mathcal{K} \times \mathcal{X}]\) \(\tilde{\mathbb{P}}\text{-a.s.}\).

By Lemma 6.4 there exists a consistent rep-con extension of \((\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}), \mathbb{(\tilde{G}_n, \tilde{\xi}^n)})_{n \in \mathbb{N}_\infty}\) given by \((\Omega, \mathcal{F}, \mathbb{P}), \mathbb{(G_n, X^{\tilde{G}_n, \tilde{\xi}^n})_{n \in \mathbb{N}}, (\tilde{G}, \xi, N^{\tilde{G}}), \mathbb{\{\tilde{\mathcal{F}}_n\}_{n \in \mathbb{N}, m \leq M_n}}\). For each \(n \in \mathbb{N}_\infty\), let \((G_n, X^{\tilde{G}_n, \tilde{\xi}^n})\) be the resulting \(\mathbb{N}^{\tilde{G}_n}\)-strong solution to (3.3) for \((\tilde{G}_n, \tilde{\xi}^n)\), where \((\tilde{G}_\infty, \tilde{\xi}_\infty) = (\tilde{G}, \tilde{\xi})\) and let \((\tilde{G}_n, X^{\tilde{G}_n, \tilde{\xi}^n})\) denote its isomorphism class. Such strong solutions are well defined by Lemma 3.10 which establishes the existence of a pathwise unique strong solution for the initial data \((\tilde{G}_n, \tilde{\xi}^n), n \in \mathbb{N}_\infty\). Because \(\{\tilde{G}_n\}_{n \in \mathbb{N}_\infty}\) spatially localizes (3.3), by Assumption 1 and the fact that \((G, \xi)\) (and therefore \((\tilde{G}, \tilde{\xi})\)) satisfies Assumption 2 Lemma 6.4 implies that the tuple \((\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}), \mathbb{(\tilde{G}_n, \tilde{\xi}^n)})_{n \in \mathbb{N}_\infty}, (\tilde{G}, \tilde{\xi})\) satisfies Assumption 2. Proposition 6.11 then implies

\[
\lim_{n \to \infty} \mathbb{(\tilde{G}_n, X^{\tilde{G}_n, \tilde{\xi}^n})} = \mathbb{(\tilde{G}_\infty, X^{\tilde{G}_\infty, \tilde{\xi}_\infty})} \quad \text{a.s.}
\]

By well-posedness of (3.3), \(\mathbb{(\tilde{G}_n, X^{\tilde{G}_n, \tilde{\xi}^n})} \overset{d}{=} \mathbb{(G_n, X^{G_n, \xi^n})}\) for every \(n \in \mathbb{N}_\infty\). Thus, \(\mathbb{(G_n, X^{G_n, \xi^n})} \overset{d}{=} \mathbb{(\tilde{G}_n, X^{\tilde{G}_n, \tilde{\xi}^n})} \Rightarrow \mathbb{(\tilde{G}, X^{\tilde{G}, \tilde{\xi}}) \overset{d}{=} \mathbb{(G, X^{G, \xi})}}\) in \(\mathbb{S}_s[\mathcal{K}, \mathcal{K} \times \mathcal{X}]\).

7 Proof of Asymptotic Correlation Decay

This section is devoted to the proof of Theorem 4.5. Recall that for an unrooted \([\mathcal{K}, \mathcal{K}]-\text{graph } G\) and a vertex \(v \in V_G, \mathcal{E}_v(G)\) is the connected component of \(G\) equipped with \(v\) as its root. For the remainder of the section, we fix a sequence of finite, (possibly disconnected) unrooted \([\mathcal{K}, \mathcal{K} \times \mathcal{X}]-\text{random graphs } (G_n, \xi^n), n \in \mathbb{N}, \text{ and a } \mathcal{S}_s[\mathcal{K}, \mathcal{K} \times \mathcal{X}]-\text{random element } (\mathcal{E}_v(G, \xi))\) (henceforth, denoted just \((G, \xi))\), all defined on a common complete probability space \((\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})\). We additionally assume that, by extending the probability space if necessary, \((\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})\) also supports an i.i.d. pair of vertices \((o^n_1, o^n_2)\), each uniformly distributed on \(G_n\), for all \(n \in \mathbb{N}\). The proof of Theorem 4.5 is comprised of two steps. The first step is to establish an asymptotic independence property stated in Lemma 7.1 below.

**Lemma 7.1.** Suppose \((G_n, \xi^n)\) converges in probability in the local weak sense to \((G, \xi)\). Then

\[
((\mathcal{E}_v^{o^n_1}(G_n, \xi^n)), (\mathcal{E}_v^{o^n_2}(G_n, \xi^n))) \Rightarrow ((G^{(1)}, \xi^{(1)})), (G^{(2)}, \xi^{(2)})),
\]

where \(\Rightarrow\) represents convergence in distribution in \(\mathbb{S}_s[\mathcal{K}, \mathcal{K} \times \mathcal{X}]^2\) and \((G^{(i)}, \xi^{(i)}), i = 1, 2, \text{ are two independent copies of } (G, \xi))\).

**Proof.** Let \(f_1, f_2 : \mathcal{S}_s[\mathcal{K}, \mathcal{K} \times \mathcal{X}] \to \mathbb{R}\) be bounded, continuous functions. Then by [22] Lemma 2.12,

\[
\lim_{n \to \infty} \mathbb{E} \left[ f_1((\mathcal{E}_v^{o^n_1}(G_n, \xi^n)))f_2((\mathcal{E}_v^{o^n_2}(G_n, \xi^n))) \right] = \mathbb{E} \left[ f_1((G, \xi)) \right] \mathbb{E} \left[ f_2((G, \xi)) \right] = \mathbb{E} \left[ f_1((G^{(1)}, \xi^{(1)})))f_2((G^{(2)}, \xi^{(2)}))) \right].
\]

Since \(\mathcal{S}_s[\mathcal{K}, \mathcal{K} \times \mathcal{X}]\) is a metric space, the algebra of separable bounded functions on \(\mathbb{S}_s[\mathcal{K}, \mathcal{K} \times \mathcal{X}]^2\) strongly separates points and hence, is convergence determining [11] Theorem 3.4.5(b)]. Thus, (7.2) implies (7.1).
Invoking the Skorokhod representation theorem, the second and main step of the proof assumes joint local convergence of the initial data \( \{(\mathcal{C}_{o_h}^i(G_n, \xi^n)), (\mathcal{C}_{o_h}^i (G_n, \xi^n))\}_{n \in \mathbb{N}} \) to the i.i.d. pair \( (\langle (G^{(1)}, \xi^{(1)}), (G^{(2)}, \xi^{(2)}) \rangle) \) and proves convergence in probability of the corresponding pairs of strong solutions \( \{(\mathcal{C}_{o_h}^i (G_n, X^{G_n, \xi^n})), (\mathcal{C}_{o_h}^i (G_n, X^{G_n, \xi^n}))\}_{n \in \mathbb{N}} \) to \( (\langle (G^{(1)}, X^{G^{(1)}, \xi^{(1)}}, (G^{(2)}, X^{G^{(2)}, \xi^{(2)}}) \rangle) \). The coupling proof proceeds as follows. For each \( i = 1, 2 \), we first construct a sequence of driving noises \( \{(G, N^{n,i})\}_{n \in \mathbb{N}} \) such that the isomorphism class of the corresponding \( \{N^{n,i}\}- \) strong solution \( \{(\mathcal{C}_{o_h}^i (G_n, X^{G_n, \xi^n}))\}_{n \in \mathbb{N}} \) converges a.s. as \( n \to \infty \) to \( \langle (G^{(i)}, X^{G^{(i)}, \xi^{(i)}}) \rangle \). From this, we construct a single sequence of common driving noises \( \{(G, N^{G_n})\}_{n \in \mathbb{N}} \) such that for each \( i = 1, 2 \), the corresponding \( \{N^{G_n}\}_{n \in \mathbb{N}} \)-strong solutions \( \mathcal{C}_{o_h}^i (G_n, X^{G_n, \xi^n}), n \in \mathbb{N} \), and \( (G^{(i)}, X^{G^{(i)}, \xi^{(i)}}) \) satisfy

\[
(\mathcal{C}_{o_h}^i (G_n, X^{G_n, \xi^n})) \to \langle (G^{(i)}, X^{G^{(i)}, \xi^{(i)}}) \rangle \text{ in probability.}
\]

By the independence of \( \{(G^{(i)}, X^{G^{(i)}, \xi^{(i)}})\}_{i=1,2} \), this would imply the desired correlation decay result.

**Lemma 7.2.** Given \( \{(G_n, \xi^n)\}_{n \in \mathbb{N}} \), suppose there exists a countable set \( S \) such that for \( \mathbb{P} \)-a.s. every \( \omega \in \Omega \), \( V_{G_n(\omega)} \subset S \) for all \( n \in \mathbb{N} \). Suppose also that Assumption 7 holds and that the collection of isomorphism classes \( \{(\mathcal{C}_{o_h}^i (G_n, \langle (G^{(i)}, \xi^{(i)}) \rangle)\}_{n \in \mathbb{N}, i=1,2} \) a.s. consistently spatially localizes the SDE (3.3). In addition, assume that for each \( i = 1, 2 \),

\[
(\mathcal{C}_{o_h}^i (G_n, \xi^n)) \to \langle (G^{(i)}, \xi^{(i)}) \rangle
\]

and the tuple \( \{(\mathcal{C}_{o_h}^i (G_n, \xi^n))\}_{n \in \mathbb{N}}, \langle (G^{(i)}, \xi^{(i)}) \rangle \rangle \) satisfies Assumption 2. Then it is possible to define a filtered probability space \( (\Omega, \mathcal{F}, \mathcal{F}^i, \mathbb{P}) \) supporting solutions \( (G_n, X^{G_n, \xi^n}), n \in \mathbb{N} \), and \( \langle (G^{(i)}, X^{G^{(i)}, \xi^{(i)}}) \rangle, i = 1, 2 \), to the SDE (3.3) for the respective initial data \( (G_n, \xi^n), n \in \mathbb{N} \), and \( \langle (G^{(i)}, \xi^{(i)}) \rangle, i = 1, 2 \), such that as \( n \to \infty \),

\[
(\mathcal{C}_{o_h}^i (G_n, X^{G_n, \xi^n})) \to \langle (G^{(i)}, X^{G^{(i)}, \xi^{(i)}}) \rangle, i = 1, 2, \text{ in probability.}
\]

**Proof.** Fix any deterministic injection \( \hat{\psi} : S \to \mathbb{N} \). For each \( n \in \mathbb{N} \), let \( (G_n, \hat{\xi}^n) \) be the unique \( \hat{\mathcal{G}}_n[K, \mathcal{K} \times \mathcal{X}] \)-random element such that \( \hat{\psi} \mid V_{G_n} \in I((G_n, \hat{\xi}^n), (\hat{G}_n, \hat{\xi}^n)) \). To be precise, if for each \( n \in \mathbb{N}, (G_n, \xi^n) = (V_n, E_n, \phi_n, \kappa^n, \xi^n) \), then

\[
(G_n, \hat{\xi}^n) := (\hat{V}_n, \hat{E}_n, \hat{\phi}_n, \hat{\kappa}^n, \hat{\xi}^n) = (\hat{\psi}(V_n), \hat{\psi}(E_n), \hat{\psi}(\phi_n), \hat{\kappa}^n_{\hat{\psi}(e)}(\xi^n_{\hat{\psi}(e)}))_{e \in \hat{V}_n}.
\]

Noting that all the remaining statements of the lemma depend only on \( (\mathcal{C}_{o_h}^i (G_n, \xi^n)) = (\mathcal{C}_{\hat{\psi}(o_h)}^i (G_n, \hat{\xi}^n)) \), we may assume without loss of generality that \( S \subseteq \mathbb{N} \) and therefore that \( \mathcal{C}_{o_h}^i (G_n, \xi^n) \) is a \( \hat{\mathcal{G}}_n[K, \mathcal{K} \times \mathcal{X}] \)-random element for each \( n \in \mathbb{N} \) and \( i = 1, 2 \).

Now, by Assumption 7 and the spatial localization assumption, Proposition 5.7 implies that the SDE (3.3) is strongly well-posed for all initial data in the collection of marked graph representatives \( \{(\mathcal{C}_{o_h}^i (G_n, \xi^n)), (G^{(i)}, \xi^{(i)})\}_{n \in \mathbb{N}, i=1,2} \), where for \( i = 1, 2 \), \( (G^{(i)}, \xi^{(i)}) \) is a random representative of \( (G^{(i)}, \xi^{(i)}) \) (whose existence is guaranteed by Lemma B.7). Also, by assumption, the collection of \( \hat{\mathcal{G}}_n[K, \mathcal{K} \times \mathcal{X}] \)-random elements \( (\mathcal{C}_{o_h}^i (G_n, \xi^n)) \}_{n \in \mathbb{N}, i=1,2} \) is \( \hat{\mathcal{F}} \)-measurable. For each \( i = 1, 2 \), Lemma 6.6 implies the existence of an associated consistent rep-con extension \( \{(\Omega^i, \mathcal{F}^i, \mathbb{P}^i), (\mathcal{C}_{o_h}^i (G_n, \xi^n, N^{n,i}), M_{n}^{i})\}_{n \in \mathbb{N}}, (G^{(i)}, \xi^{(i)}, N^{G_{n}}), \{\tau^{n}_{m}\}_{n \in \mathbb{N}, m \leq M_{n}} \rangle \) where \( (\Omega^i, \mathcal{F}^i, \mathbb{P}^i) \) supports a collection of i.i.d. \( \mathbb{P} \)-Poisson processes \( \mathbb{N}^i := \{\mathbb{N}^i_k\}_{k \in \mathbb{N}} \) which together
with the driving maps $\overline{\psi}^{n,i}$, $n \in \mathbb{N}$ and $\overline{\psi}^{(i)}$ generate the respective $\mathbb{F}^i$-driving noises $N^{n,i}$, $n \in \mathbb{N}$ and $N^{G^{(i)}}$. We may also identify $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}) := (\Omega^1, \mathcal{F}^1, \mathbb{F}^1) = (\Omega^2, \mathcal{F}^2, \mathbb{F}^2)$ and assume that $\mathbb{N}$ is independent of $\mathbb{N}^d$ so that both collections consist of $\mathbb{F}$-Poisson processes. On this space, define for each $k \in \mathbb{N}$ and $i = 1, 2$, $N_{2k-2+i} := N_k$. Then for each $n \in \mathbb{N}$ and $i = 1, 2$, $N^{n,i}$ and $N^{G^{(i)}}$ are generated from $\{N_k\}_{k \in \mathbb{N}}$ by the driving maps $\psi^{n,i} := 2\overline{\psi}^{n,i} = 2\overline{\psi}^{(i)} - 2 + i$ and $\psi^{(i)} := \overline{\psi}^{(i)} - 2 + i$, respectively.

For each $n$, let $L_n := d_{G_n} (o_n^1, o_n^2)$ and let $M_n$ be the maximal $\mathcal{F}_0$-measurable random variable such that $M_n < \frac{2}{\sqrt{M}}$ and $M_n \leq \min_{i=1,2} M_n^{(i)}$ a.s.. Define the mapping $\psi_n : V_n \rightarrow \mathbb{N}$ by
\[
\psi_n (v) := \begin{cases} 
\psi^{n,1} (v) & \text{on the event } \{d_{G_n} (v, o_n^1) \leq M_n\}, \\
\psi^{n,2} (v) & \text{otherwise.}
\end{cases}
\]

Since for $i = 1, 2$, the driving maps $\psi^{n,i}$ have disjoint images and $M_n < \frac{1}{\sqrt{M}} d_{G_n} (o_n^1, o_n^2)$, it follows that $\psi_n$ is $\mathcal{F}_0$-measurable and injective and is therefore also a driving map. For each $n \in \mathbb{N}$, let $N^{G_n}$ be the $\mathcal{F}$-driving noise generated by the driving map $\psi_n$. Then $N^{G_n}$ is compatible with $\mathcal{C}_{o_n^i} (G_n)$, $i = 1, 2$. Because $M_n \leq M_n^{(i)}$, $i = 1, 2$, it follows that for any $i = 1, 2$, $n \in \mathbb{N}$ and $v \in B_m (\mathcal{C}_{o_n^i} (G_n))$,
\[
N^{G_n}_v = N^{n,i}_v = N^{G^{(i)}}_{(\varphi_{n,m})^{-1} (v)} \text{ on the event } \{m \leq M_n\}. \tag{7.5}
\]

Let $\{n_k\}_{k \in \mathbb{N}}$ be any deterministic sequence such that $L_{n_k}$ (and therefore $M_{n_k}$) diverges to infinity a.s. as $k \rightarrow \infty$. Such a sequence exists because $L_n \rightarrow \infty$ in probability. Fix $i \in \{1, 2\}$ and consider the tuple $T_i := ((\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}), \mathcal{C}_{o_n^i} (G_{n_k}, G^{(i)}, N^{G^{(i)}}), M_{n_k})_{k \in \mathbb{N}}$, $(G^{(i)}, \xi^{(i)}, N^{G^{(i)}}, \varphi^{(i)}_{n_k,m}, \mathbb{F}^{n_k}_m, o_n^i, o_n^2)$. Using (7.5), the definition of $M_n$ and directly checking the three properties of Definition 6.3, it is easy to see that this tuple is a consistent rep-con extension of $R_i := ((\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}), \mathcal{C}_{o_n^i} (G_{n_k}, \xi_{n_k}))$, $((G^{(i)}, \xi^{(i)}))$. For completeness the verification is provided below.

**Property 1:** By construction, $\{(\mathcal{C}_{o_n^i} (G_n, \xi^n)), M_n^{(i)}\}_{n \in \mathbb{N}}$, $(G^{(i)}, \xi^{(i)}, \varphi^{(i)}_{n,m}, m \in \mathbb{M}_{n_k})$ is a rep-con sequence of $\{(\mathcal{C}_{o_n^i} (G_n, \xi^n)), \varphi^{(i)}_{n,m}, m \in \mathbb{M}_{n_k}\}$. Since $M_n \leq M_n^{(i)}$ for every $n \in \mathbb{N}$, the tuple $\mathbb{R}_i := \{(\mathcal{C}_{o_n^i} (G_{n_k}, \xi_{n_k}), M_{n_k}), (G^{(i)}, \xi^{(i)}, \varphi^{(i)}_{n_k,m}), m \in \mathbb{M}_{n_k}\}$ satisfies Properties 1, 2 and 4-6 of Definition 6.1. Because $M_{n_k} \rightarrow \infty$ a.s. and $M_{n_k} \leq M_{n_k}^{(i)}$ for each $k \in \mathbb{N}$, it follows that Property 3 of Definition 6.1 holds. Thus, $\mathbb{R}_i$ is a rep-con sequence of $\{(\mathcal{C}_{o_n^i} (G_n, \xi^n))\}_{n \in \mathbb{N}}$, $((G^{(i)}, \xi^{(i)}))$.

**Property 2:** This holds because $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is part of the consistent rep-con extension of $R_i$ and $N^{G_{n_k}}$ is an $\mathcal{F}$-driving noise.

**Property 3:** This holds by (7.5).

Given that $T_i$ is a consistent rep-con extension of $R_i$, by Proposition 6.11,
\[
(\mathcal{C}_{o_{n_k}^i} (G_{n_k}, X_{G_{n_k}}} \xi_{n_k})) \Rightarrow ((G^{(i)}, \xi^{(i)})) \text{ a.s..}
\]

Lastly, note that for any deterministic subsequence $\{n_k\}_{k \in \mathbb{N}} \subseteq \mathbb{N}$, there exists a further deterministic subsequence $\{n_{k_{\ell}}\}_{\ell \in \mathbb{N}}$ such that $\lim_{\ell \rightarrow \infty} L_{n_{k_{\ell}}} = \infty$ a.s.. By the above result, this implies that
\[
\lim_{\ell \rightarrow \infty} (\mathcal{C}_{o_{n_{k_{\ell}}}^i} (G_{n_{k_{\ell}}}, X_{G_{n_{k_{\ell}}}} {\xi}_{n_{k_{\ell}}})) = ((G^{(i)}, \xi^{(i)})) \text{ a.s.,}
\]
which immediately implies (7.4) and completes the proof. \qed
Remark 7.3. Note that the proof of Lemma 7.2 does not make direct use of the form of (3.3) but only relies on spatial localization of the SDE, Assumptions 1 and 2, and Proposition 6.11, all of which could potentially be verified for solutions $X^{G,\xi}$ of more general Poisson-driven SDEs.

Applying Lemmas 7.1 and 7.2, we now prove Theorem 4.5.

**Proof of Theorem 7.3.** Lemma 7.1 implies that
\[ ((\mathcal{C}_{\Omega_n} \langle G_n, \xi^n \rangle), \langle \mathcal{C}_{\Omega_n} \langle G_n, \xi^n \rangle \rangle) \Rightarrow \langle (G^{(1)}, \xi^{(1)}), (G^{(2)}, \xi^{(2)}) \rangle, \]
where $\langle (G^{(i)}, \xi^{(i)}) \rangle$, $i = 1, 2$, are i.i.d. copies of $\langle (G, \xi) \rangle$. By the Skorokhod representation theorem and Lemma 3.7, there exist $\langle (\tilde{G}^{(i)}, \tilde{\xi}^{(i)}) \rangle _{i=1, 2} \overset{\text{d}}{=} \langle (G^{(i)}, \xi^{(i)}) \rangle _{i=1, 2}$, and finite, $\mathbb{G}_s[K, K \times X]$-random elements $\{\tilde{G}_n, \tilde{\xi}_n\}_{n \in \mathbb{N}, i=1, 2} \overset{\text{d}}{=} \{G_n, \xi^n\}_{n \in \mathbb{N}, i=1, 2}$ such that
\[ \langle \mathcal{C}_{\Omega_n} \tilde{G}_n, \tilde{\xi}_n \rangle \rightarrow \langle (\tilde{G}^{(i)}, \tilde{\xi}^{(i)}) \rangle, \quad i = 1, 2, \text{ a.s.} \]
By assumption, $\tilde{G}_n \overset{\text{d}}{=} G_n$ and $\tilde{\xi}^{(i)} \overset{\text{d}}{=} G$ are a.s. finitely dissociable for $i = 1, 2$, and $n \in \mathbb{N}$. So by Assumption 1 and Proposition 5.19, $\{\{G_n, \tilde{\xi}_n\}, \{\tilde{G}^{(i)}, \tilde{\xi}^{(i)}\}\}_{n \in \mathbb{N}, i=1, 2}$ consistently spatially localizes the SDE (3.3). Assumptions 1 and 2 imply by Lemma 6.10 that the tuple $\{(\hat{\Omega}, \hat{F}), \{\mathcal{C}_{\Omega_n} \tilde{G}_n\}\}_{n \in \mathbb{N}}, \langle (\tilde{G}^{(i)}, \tilde{\xi}^{(i)}) \rangle$ satisfies Assumption 2 for each $i$. Thus, by Lemma 7.2, it is possible to construct a collection of driving noises $\{N^{\tilde{G}_n}, N^{\tilde{G}^{(i)}}\}_{n \in \mathbb{N}, i=1, 2}$ such that for each $n \in \mathbb{N}$ and $i = 1, 2$, the respective $N^{\tilde{G}_n}$ and $N^{\tilde{G}^{(i)}}$-solutions $X^{\tilde{G}_n, \tilde{\xi}_n}$ and $X^{\tilde{G}^{(i)}, \tilde{\xi}^{(i)}}$ satisfy
\[ \langle \mathcal{C}_{\Omega_n} \tilde{G}_n, X^{\tilde{G}_n, \tilde{\xi}_n} \rangle \rightarrow \langle (\tilde{G}^{(i)}, X^{\tilde{G}^{(i)}, \tilde{\xi}^{(i)}}) \rangle \text{ in probability as } n \rightarrow \infty. \]
By the bounded convergence theorem, for any bounded, continuous function $f : (\mathbb{G}_s[K, K \times D])^2 \rightarrow \mathbb{R}$,
\[ \lim_{n \rightarrow \infty} \mathbb{E} \left[ f \left( \langle \mathcal{C}_{\Omega_n} \tilde{G}_n, X^{\tilde{G}_n, \tilde{\xi}_n} \rangle _{i=1, 2} \right) \right] = \mathbb{E} \left[ f \left( \langle (\tilde{G}^{(i)}, X^{\tilde{G}^{(i)}, \tilde{\xi}^{(i)}}) \rangle _{i=1, 2} \right) \right]. \tag{7.6} \]
By well-posedness of the SDE (3.3) on each of the graphs $(\tilde{G}_n, \tilde{\xi}_n)$, $n \in \mathbb{N}$, and $(\tilde{G}, \tilde{\xi})$, which holds by Theorem 4.2, it follows that for every $n \in \mathbb{N}$,
\[ \langle \mathcal{C}_{\Omega_n} \tilde{G}_n, X^{\tilde{G}_n, \tilde{\xi}_n} \rangle _{i=1, 2} \overset{\text{d}}{=} \langle \mathcal{C}_{\Omega_n} (G_n, X^{G_n, \xi^n}) \rangle _{i=1, 2} \]
and $(\tilde{G}^{(1)}, X^{\tilde{G}^{(1)}, \tilde{\xi}^{(1)}}) \overset{\text{d}}{=} (\tilde{G}^{(2)}, X^{\tilde{G}^{(2)}, \tilde{\xi}^{(2)}}) \overset{\text{d}}{=} (G, X^{G, \xi})$. Together with (7.5) and the independence of $\langle (\tilde{G}^{(1)}, X^{\tilde{G}^{(1)}, \tilde{\xi}^{(1)}}) \rangle$ and $\langle (\tilde{G}^{(2)}, X^{\tilde{G}^{(2)}, \tilde{\xi}^{(2)}}) \rangle$, for any bounded, continuous functions $f_1, f_2 : \mathbb{G}_s[K, K \times D] \rightarrow \mathbb{R}$, it follows that
\[ \lim_{n \rightarrow \infty} \mathbb{E} \left[ f_1 \left( \langle \mathcal{C}_{\Omega_n} (G_n, X^{G_n, \xi^n}) \rangle \right) f_2 \left( \langle \mathcal{C}_{\Omega_n} (G_n, X^{G_n, \xi^n}) \rangle \right) \right]
= \lim_{n \rightarrow \infty} \mathbb{E} \left[ f_1 \left( \langle \mathcal{C}_{\Omega_n} (G_n, X^{G_n, \xi^n}) \rangle \right) f_2 \left( \langle \mathcal{C}_{\Omega_n} (G_n, X^{G_n, \xi^n}) \rangle \right) \right]
= \mathbb{E} \left[ f_1 \left( \langle \tilde{G}^{(1)}, X^{\tilde{G}^{(1)}, \tilde{\xi}^{(1)}} \rangle \right) \right] \mathbb{E} \left[ f_2 \left( \langle \tilde{G}^{(2)}, X^{\tilde{G}^{(2)}, \tilde{\xi}^{(2)}} \rangle \right) \right]
= \mathbb{E} \left[ f_1 \left( \langle (G, X^{G, \xi}) \rangle \right) \right] \mathbb{E} \left[ f_2 \left( \langle (G, X^{G, \xi}) \rangle \right) \right]. \]
This allows us to conclude the desired result:
\[ \lim_{n \rightarrow \infty} \text{Cov} \left( f_1 \left( \langle \mathcal{C}_{\Omega_n} (G_n, X^{G_n, \xi^n}) \rangle \right), f_2 \left( \langle \mathcal{C}_{\Omega_n} (G_n, X^{G_n, \xi^n}) \rangle \right) \right) = 0. \]

**Remark 7.4.** From the proof of Theorem 4.5, it is not hard to see that in fact the conclusion of Theorem 4.5 holds under Assumptions 1 and 2 and the spatial localization property in place of Assumptions 1 and 2 and the finite dissociability property respectively.
A Counterexample: When the SDE is not Well-Posed

There are two ways the SDE (3.3) can fail to be well-posed. Either there exist no solutions, or there exist multiple solutions. To construct an IPS with no solutions, it is a simple matter of constructing unbounded jump rate functions that increase fast enough that all stochastic processes \( X \) for which (3.3) holds a.s. with \( X^{G,\xi} := X \) explode (have infinitely many discontinuities on some compact interval in \( \mathbb{R}_+ \)) with positive probability. In this section, we show that well-posedness is non-trivial even if we restrict ourselves to Markov processes with bounded jump rate functions by constructing a graph and a collection of jump rate functions with respect to which the SDE (3.3) has multiple strong solutions with different laws.

**Proposition A.1.** Let \( T \) be the tree such that for every \( k \in \mathbb{N} \), all vertices \( v \) in the \( k \)th generation of \( T \) have \( 4^k \) children. Using the notation of Example 4.8 suppose \( X = \{0,1\} \), \( J = \{1\} \) and for \( z \in X^V \) and \( v \in V \), let \( \tilde{r}_1^{T[cl],v}(z) = 1 \) if both \( z_u = 0 \) and \( \sum_{u \in X_v} z_u > 0 \), and let \( \tilde{r}_1^{T[cl],v}(z) = 0 \) otherwise. Then the corresponding SDE (3.3) has multiple strong solutions for the initial state \( \xi_v = X_v(0) = 0, \forall v \in V \) and these solutions also have different laws.

The proof of this proposition requires us to introduce the notion of infinite causal chains.

**Definition A.2** (Infinite causal chains). For any locally finite graph \( G, T \in (0, \infty) \) and associated Poisson processes \( N^{G,T} \) defined as in (5.9), an infinite \((G,N^{G,T})\)-causal chain ending at a vertex \( u \) is a path \( \Gamma = (u = u_0, u_1, u_2, \ldots) \) such that for every \( n \in \mathbb{N} \), \((u_n, u_{n-1}, \ldots, u_1, u_0) \) is a \((G,N^{G,T})\)-causal chain (see Definition 5.18).

To prove Proposition A.1, we first show that the tree \( T \) has an infinite causal chain almost surely.

**Lemma A.3.** Let \( T = (V,E,\phi) \) be the tree defined in Proposition 4.7 and let \( N^T \) be the corresponding driving noise. Fix any \( T \in (0, \infty) \) and a strictly positive, non-decreasing family of constants \( \{C_{k,T}\}_{k \in \mathbb{N}} \), and let \( N^{T,T} \) be the associated Poisson processes defined in (5.9). Then there exists an infinite \((T,N^{T,T})\)-causal chain with probability 1.

**Proof.** Define the path \( \Gamma := \{ \phi := v_0, v_1, \ldots \} \) using the following recursive construction. For \( n \geq 1 \), first check if there exists \( u \in c_{v_n}(T) \) such that \( N^{T,T}_u(2^{-n-2}, 2^{-n-1}) > 0 \). If this condition is satisfied, then set \( v_{n+1} = u \) (if multiple children satisfy this then we choose one arbitrarily from amongst them). If not, then we choose any child of \( v_n \) arbitrarily and set it to be \( v_{n+1} \).

Define \( U := \{ v_i, i \in I \} \subseteq \Gamma \), where \( I := \{ n \in \mathbb{N} : N^{T,T}_{v_n}(2^{-n-1}, 2^{-n}) > 0 \} \). We now claim that \( U \) contains an infinite path a.s. To see why, we prove the equivalent claim that \( U \) a.s. contains all but finitely many vertices in \( \Gamma \). Let \( \alpha_n := \mathbb{P}(v_{n+1} \notin U) \). Note that \( \sum_{k,T} C := \sum_{1,T} C > 0 \) for all \( k \).

Thus, for each \( u \in U \),
\[
\mathbb{P}(N^{T,T}_{u}(2^{-n-1}, 2^{-n}) = 0) = e^{-2^{-n-1}C_{|u|,T}} \leq e^{-2^{-n-1}C}.
\]

The independence of the Poisson processes \( (N^{T,T}_{u})_{u \in V} \) then implies that
\[
\alpha_n = \prod_{u \in c_{v_n}(T)} \mathbb{P}(N^{T,T}_{u}(2^{-n-1}, 2^{-n}) = 0) \leq \left( e^{-2^{-n-1}C} \right)^4 = e^{-2^{n-1}C}.
\]

Because \( \alpha_n \) decreases super-exponentially fast, \( \sum_{n=1}^{\infty} \alpha_n < \infty \). By the Borel-Cantelli lemma, it follows that with probability 1, all but finitely vertices in \( \{v_n\}_{n \in \mathbb{N}} \) are contained in \( U \), thus proving the claim. Note that for any path \( \Gamma' = (u_0, u_{n-1}, \ldots, u_0) \subseteq U \) where \( u_0 \) is closest to the root, there exists a decreasing sequence of times \( \{t_k \in (2^{-(\delta_T(u_0,\phi)+k)}, 2^{-(\delta_T(u_0,\phi)+k)+1})\} \) such that \( N^{T,T}_{u_k}(\{t_k\}) = 1 \). Thus, \( \Gamma' \) is a \((T,N^{T,T})\)-causal chain, and we have shown that \( U \) contains an infinite \((T,N^{T,T})\)-causal chain a.s..
We can now prove Proposition [A.1].

Proof of Proposition [A.1]: Although there are infinitely many strong solutions for the initial state \( \xi \equiv 0 \), we explicitly construct two of them. First, the process \( X_v^{T,\xi}(t) = 0 \) for all \( v \in V \) and \( t \in \mathbb{R}_+ \) is clearly a strong solution to the SDE (3.3) for the initial data \((T, \xi)\) with \( r_1^{T,v}(t,x) := \tilde{r}_1^{[\cdot \cdot]}(x(t-)) \), with \( \tilde{r} \) as given in the proposition.

Fix \( T < \infty \). Note that for each \( v \in V \), by (3.9), \( N_v^{T,T} \) is a unit rate Poisson process. Then for \((v,t) \in V \times (0,T)\), we say \((v,t)\) has an infinite causal chain if there exists an infinite \((T,N_v^{T,T})\)-causal chain ending at \( v \) on the interval \([0,t]\). Then for \( v \in V \) and \( t \in [0,T]\), define

\[
\tilde{X}_v^{T,\xi}(t) := \begin{cases} 1 & \text{if } (v,t) \text{ has an infinite causal chain,} \\ 0 & \text{otherwise.} \end{cases} \tag{A.1}
\]

We claim this is also a solution to the same SDE. Fix any \( v \in V \) and define

\[
\tilde{X}_v(t) := \xi_v + \int_{(0,t] \times \mathbb{R}_+} \mathbb{I}_{\{r \leq r_1^{T,v}(s,\tilde{X}_v,\tilde{X}_v^{T,\xi}(T))\}} N_v^{T}(ds,dr,dj), \quad t \in \mathbb{R}_+.
\tag{A.2}
\]

If \((v,t) \in V \times \mathbb{R}_+\) has no infinite causal chain, then by Definition [A.2] for any \( s \leq t \), either \( N_v^{T,T}([s,t]) = N_v^{T}([s,t] \times (0,1] \times \{1\}) = 0 \), or there does not exist a \( u \in N_v(T) \) for which \((u,s)\) has an infinite causal chain. It immediately follows that for every event \( \tau \) in \( N_v^{T,T} \) in the interval \((0,t]\), \( r_1^{T,v}(\tau,\tilde{X}_v,\tilde{X}_v^{T,\xi}(N_v(T))) = 0 \) so that \( \tilde{X}_v(t) = \tilde{X}_v^{T,\xi}(t) = 0 \). If \((v,t)\) has an infinite causal chain, then there exists a minimal event \( \tau \leq t \) in \( N_v^{T,T} \) such that there exists a neighbor \( u \) of \( v \) for which \((u,\tau)\) has an infinite causal chain. By minimality of \( \tau \), \((v,s)\) does not have an infinite causal chain on the event \( \{s \leq \tau\} \), and hence, \( \tilde{X}_v(\tau^-) = 0 \) by the previous case. Thus, \( r_1^{T,v}(\tau,\tilde{X}_v,\tilde{X}_v^{T,\xi}(N_v(T))) = 1 \), so \( \tilde{X}_v(\tau) = 1 \). It follows that \( \tilde{X}_v(t) = \tilde{X}_v^{T,\xi}(t) \).

Thus, \( \tilde{X}_v = \tilde{X}_v^{T,\xi} \). Because \( v \) and \( t \) were chosen arbitrarily, it follows that \( \tilde{X}_v^{T,\xi} \) satisfies (3.3).

Finally, because \((T,N_v^{T,T})\)-causal chains are \( \mathbb{F}^{T,\xi,N_v^{T}} \)-adapted, \( \tilde{X}_v^{T,\xi} \) must also be \( \mathbb{F}^{T,\xi,N_v^{T}} \)-adapted by (A.1). Thus, \( \tilde{X}_v^{T,\xi} \) is also a strong solution to (3.3), and by Lemma [A.3], \( X_v^{T,\xi} \neq \tilde{X}_v^{T,\xi} \) a.s.. \( \square \)

B Measurable Representatives of Graph Isomorphism Classes

In this section, we establish measurability of marked graph representatives of random isomorphism classes, culminating in the proofs of Lemmas 6.2 and 6.6. Along the way, we introduce a Polish space of canonical representative graphs that may be of independent interest. We start with preliminaries in Section 3.4.1 and then establish the main measurable selection results in Section 3.4.2.

Throughout the section, we let \( \mathcal{Z} \) and \( \Xi \) be Polish spaces denoting the respective spaces in which the edge and vertex marks lie, and let \( d_{\Xi} \) and \( d_{\Xi} \) be associated metrics that induce the respective topologies. Let \( \odot \) be an arbitrary point not lying in \( \overline{\mathcal{Z}} \cup \Xi \), define the spaces \( \overline{\mathcal{Z}}_{\odot} := \overline{\mathcal{Z}} \cup \{\odot\} \) and \( \Xi_{\odot} := \Xi \cup \{\odot\} \), and endow them with the corresponding Polish topologies from \( \overline{\mathcal{Z}} \) and \( \Xi \), respectively, with \( \odot \) being an isolated point. Throughout the section, we often implicitly denote the components of (possibly random) \([\mathcal{Z},\Xi]\)-graphs by \( G := (V,E,\odot,\overline{\mathcal{Z}},\odot) \) and \( G_n := (V_n,E_n,\odot_n,\overline{\mathcal{Z}}^n,\odot^n) \).

Definition B.1. Given a Polish space \( \mathcal{Z}' \) and a measurable space \((\Omega,\mathcal{F})\), a mapping \( F : \Omega \to \text{clo}(\mathcal{Z}') \), where \( \text{clo}(\mathcal{Z}') \) denotes the set of closed subsets of \( \mathcal{Z}' \), is said to be an \( \mathcal{F} \)-random closed subset of \( \mathcal{Z}' \) if it is weakly measurable in the following sense: for every open set \( U \subseteq \mathcal{Z}' \), the set \( \{\omega \in \Omega : U \cap F(\omega) \neq \emptyset\} \) lies in \( \mathcal{F} \). Moreover, \( F \) is said to be non-empty if \( F(\omega) \neq \emptyset \) for every \( \omega \in \Omega \).
Several results in this appendix make use of the following measurable selection theorem.

**Theorem B.2** (Kuratowski & Ryll-Nardzewski Measurable Selection Theorem). *Suppose \( Z' \) is a Polish space, \((\Omega, \mathcal{F})\) is a measurable space and \( F : \Omega \mapsto \text{clo}(Z') \) is a non-empty \( \mathcal{F} \)-random closed subset of \( Z' \). Then there exists an \( \mathcal{F} \)-measurable function \( Z : \Omega \mapsto Z' \) such that \( Z(\omega) \in F(\omega) \) for all \( \omega \in \Omega \).*

**Proof.** This is simply a restatement of [5, Theorem 6.9.3] in our notation, in particular with \( X, B \) and \( \Psi \) in [5] replaced by \( Z', \mathcal{F} \) and \( F \), respectively. \( \square \)

### B.1 A Canonical Subspace of Rooted Graphs and its Properties

In this section, we construct a canonical subspace of the space of \([\mathbb{Z}, \mathbb{Z}]\)-graphs introduced in Section 2.4 and equip it with a topology that is compatible with the topology of \( \mathcal{G}_s[\mathbb{Z}, \mathbb{Z}] \). In the ensuing definition, we use the following standard notion of convergence of subsets of \( \mathbb{N} \). Given \( S_n \subseteq \mathbb{N}, n \in \mathbb{N}, \) and \( S \subseteq \mathbb{N} \), we write \( S_n \to S \) if and only if

\[
S = \bigcup_{n \in \mathbb{N}} S_n' = \bigcap_{n \in \mathbb{N}} S_n'.
\]

Equivalently, \( S_n \to S \) if and only if for every \( k \in S \), there exist only finitely many \( n \in \mathbb{N} \) such that \( k \notin S_n \) and for every \( k' \notin S \) there exist only finitely \( n' \in \mathbb{N} \) such that \( k' \in S_n' \). In addition, we equip \( \mathbb{N} \) with the discrete topology. Lastly, recall the definition of \( \text{cl}_o(G) \) from Section 2.1.

**Definition B.3.** We equip the canonical space \( \widehat{\mathcal{G}}_s[\mathbb{Z}, \mathbb{Z}] \) of rooted \([\mathbb{Z}, \mathbb{Z}]\)-graphs defined in (6.1) by

\[
\widehat{\mathcal{G}}_s[\mathbb{Z}, \mathbb{Z}] := \{ [\mathbb{Z}, \mathbb{Z}] \text{-graphs } G = (V, E, \varnothing, \vartheta, \vartheta) \text{ s.t. } V \subseteq \mathbb{N} \},
\]

with the following notion of convergence: \( G_n \to G \) in \( \widehat{\mathcal{G}}_s[\mathbb{Z}, \mathbb{Z}] \) if and only if

1. \( \lim_{n \to \infty} V_n = V \);
2. \( \lim_{n \to \infty} E_n = E \);
3. \( \lim_{n \to \infty} \varnothing_n = \varnothing \);
4. \( \lim_{e \in E_n} \vartheta^n_e = \vartheta_e \) for all \( e \in E \);
5. \( \lim_{v \in V_n} \vartheta^n_v = \vartheta_v \) for all \( v \in V \).
6. for each \( v \in V \), \( \lim_{n \to \infty} \max\{u \in \text{cl}_o(G_n)\} = \max\{u \in \text{cl}_o(G)\} \).

For \( G, G' \in \widehat{\mathcal{G}}_s[\mathbb{Z}, \mathbb{Z}] \), notions such as graph distance \( d_G(\cdot, \cdot) \), truncations \( B_m(G) \), sets of isomorphisms \( I(G, G') \) and isomorphism classes \( \langle G \rangle \) are all defined as in Sections 2.1 and 2.4.

**Remark B.4.** The least intuitive condition is perhaps Condition 6 of Definition B.3, but it is necessary for the topology on \( \widehat{\mathcal{G}}_s[\mathbb{Z}, \mathbb{Z}] \) to be compatible with the topology of \( \mathcal{G}_s[\mathbb{Z}, \mathbb{Z}] \) (in the sense made precise in Lemma B.6). This is best illustrated via an example of how compatibility could fail without Condition 6. Let \( \mathbb{Z} = \mathbb{Z} = \{1\} \) be trivial and suppose that \( V \subseteq \{2k : k \in \mathbb{N}\}, \varnothing := 2 \) and \( E \) is any set of distinct pairs of \( V \) such that \( G = (V, E, \varnothing) \) is locally finite. If for each \( n \in \mathbb{N}, G_n \) is defined by \( \varnothing_n := 2, V_n := V \cup \{2n + 1\} \) and \( E_n := E \cup \{2, 2n + 1\} \), then Conditions 1-5 of Definition B.3 are all satisfied, and only Condition 6 fails. However, note that \( G_n \cong G_{n'} \not\equiv G \) for all \( n, n' \in \mathbb{N} \), so \( \langle G_n \rangle \to \langle G_1 \rangle \neq \langle G \rangle \) in \( \mathcal{G}_s[\mathbb{Z}, \mathbb{Z}] \).
In order to apply Theorem B.2 to find measurable representatives of isomorphism classes, it is necessary to prove that the space $\widehat{S}_s[\mathcal{Z}, \mathcal{Z}]$ is Polish. This can be done by direct verification using Definition B.3. We fill in the details for completeness.

**Lemma B.5.** The space $\widehat{S}_s[\mathcal{Z}, \mathcal{Z}]$ is Polish.

**Proof.** Define the Polish space $\mathcal{R} := (\mathbb{N} \times \mathcal{Z} \times (\mathcal{Z}^2)^\mathbb{N})^\mathbb{N} \times \mathbb{N}$, equipped with the product topology, and consider the map $\psi : \widehat{S}_s[\mathcal{Z}, \mathcal{Z}] \to \mathcal{R}$ defined by $\psi(G) := ((\psi_k(G))_{k \in \mathbb{N}}, \emptyset)$, where for each $k \in \mathbb{N}$, $\psi_k(G) = (c'_k, \vartheta'_k, (\mathcal{V}(k,k'))_{k', \in \mathbb{N}})$, with

$$c'_k := \begin{cases} \max\{\vartheta_k(G)\} & \text{if } k \in V, \\ k & \text{otherwise,} \end{cases} \quad \vartheta'_k := \begin{cases} \vartheta_k & \text{if } k \in V, \\ \emptyset & \text{otherwise,} \end{cases} \quad \mathcal{V}(k,k') := \begin{cases} \mathcal{V}(k,k') & \text{if } \{k, k'\} \in E, \\ \emptyset & \text{otherwise.} \end{cases}$$

We then have the following observations:

(i) The map $\psi$ is a bijection from $\widehat{S}_s[\mathcal{Z}, \mathcal{Z}]$ to $\widehat{\mathcal{R}}$, where $\widehat{\mathcal{R}}$ is the subset of elements $\zeta = ((c'_k, \vartheta'_k, (\mathcal{V}(k,k'))_{k' \in \mathbb{N}})_{k \in \mathbb{N}}, \emptyset)$ in $\mathcal{R}$ that satisfy the following constraints:

(a) $\emptyset \in V_\zeta := \{k \in \mathbb{N} : \vartheta'_k \neq \emptyset\}$;
(b) for every $k \notin V_\zeta$, $c'_k = k$;
(c) One has $\{k, k'\} \subseteq V_\zeta$ for every $\{k, k'\} \in E_\zeta := \bigcup_{k \in \mathbb{N}} E_{\zeta}(k)$, where $E_{\zeta}(k) := \{\{k, k'\} \subseteq \mathbb{N} : k \neq k', \mathcal{V}(k,k') \neq \emptyset\}$; and condition (b) above, $\vartheta_\zeta = \emptyset$, $\mathcal{V}(k,\zeta) = \vartheta'_k \in \mathcal{Z}$ for $k \in V_\zeta$, and $\mathcal{V}(\{k, k'\}, \zeta) = \mathcal{V}(k, k') \in \mathcal{Z}$ for $\{k, k'\} \in E_\zeta$.

(d) for every $k \in \mathbb{N}$, $c'_k = \max\{\} \cup \{k' \in \mathbb{N} : \{k, k'\} \in E_{\zeta}(k)\}$.

It is trivial to check that for every $G \in \widehat{S}_s[\mathcal{Z}, \mathcal{Z}]$, $\psi(G)$ must satisfy conditions (a)–(d) above. Thus, the image of $\widehat{S}_s[\mathcal{Z}, \mathcal{Z}]$ under $\psi$ is contained in $\widehat{\mathcal{R}}$. On the other hand, note that any $\zeta = ((c'_k, \vartheta'_k, (\mathcal{V}(k,k'))_{k' \in \mathbb{N}})_{k \in \mathbb{N}}, \emptyset)$ in $\widehat{\mathcal{R}}$ has a unique inverse under $\psi$, described by $\psi^{-1}(\zeta) = G_{\zeta} := (V_\zeta, E_{\zeta}, 0_{\zeta}, \mathcal{V}(\zeta), (\mathcal{V}(k,k'))_{\zeta})$, where $V_\zeta$ and $E_{\zeta}$ are respectively defined as in (a) and (c) above, $0_{\zeta} = \emptyset$, $\mathcal{V}(k,\zeta) = \vartheta'_k \in \mathcal{Z}$ for $k \in V_\zeta$, and $\mathcal{V}(\{k, k'\}, \zeta) = \mathcal{V}(k, k') \in \mathcal{Z}$ for $\{k, k'\} \in E_{\zeta}$.

Condition (c) also ensures that $\zeta \in \widehat{\mathcal{R}}$ implies $|E_{\zeta}(k)| < \infty$ for every $k \in V_\zeta$, and so the resulting graph $G_{\zeta}$ is locally finite. Moreover, conditions (a) and (c) together ensure that the edge and vertex marks lie in $\mathcal{Z}$ and $\mathcal{Z}$, respectively, thus showing that $G_{\zeta}$ is a $[\mathcal{Z}, \mathcal{Z}]$-graph. Lastly, it is easy to see that for any $G \in \widehat{S}_s[\mathcal{Z}, \mathcal{Z}]$ and $\zeta \in \widehat{\mathcal{R}}$, $\psi \circ \psi^{-1}(\zeta) = \zeta$ and $\psi^{-1} \circ \psi(G) = G$, thus proving $\psi$ is a bijection between $\widehat{S}_s[\mathcal{Z}, \mathcal{Z}]$ and $\widehat{\mathcal{R}}$.

(ii) $\widehat{\mathcal{R}}$ is a closed subset of $\mathcal{R}$ under componentwise convergence: suppose the sequence $\zeta^n = ((c^n_k, \vartheta^n_k, (\mathcal{V}(k,k'))_{k' \in \mathbb{N}})_{k \in \mathbb{N}}, \emptyset^n)$ converges to $\zeta = ((c'_k, \vartheta'_k, (\mathcal{V}(k,k'))_{k' \in \mathbb{N}})_{k \in \mathbb{N}}, \emptyset)$ pointwise. Then $\zeta \in \mathcal{R}$ because $\mathcal{R}$ is Polish. To show $\zeta \in \widehat{\mathcal{R}}$, it suffices to show that $\zeta$ satisfies the constraints (a)–(d) in (i) above. To prove condition (a) we argue by contradiction. Suppose $\vartheta' = \emptyset$. Then since $\vartheta^n \to \emptyset$ it follows that there exists $N < \infty$ such that $\vartheta^n = \emptyset$ for all $n \geq N$. Since $\vartheta^n \to \vartheta'$ and $\emptyset$ is isolated, this implies that for all sufficiently large $n$, $\vartheta^n = \emptyset = \vartheta'$, which implies $\vartheta_n \notin V_{\zeta^n}$ and thus contradicts the assumption that $\zeta^n \in \widehat{\mathcal{R}}$. Thus, this proves that $\zeta$ satisfies condition (a). Condition (c) can be established in an exactly analogous fashion. Next, suppose $k \notin V_\zeta$. Then since $\zeta$ satisfies (a) as shown above, $\vartheta_k = \emptyset$ and since $\emptyset$ is isolated and $\vartheta^n \to \vartheta_k$, it follows that $\vartheta_k^n = \emptyset$ for all sufficiently large $n$. In turn, since $\zeta^n \in \widehat{\mathcal{R}}$, this implies $k \notin V_{\zeta^n}$ and hence that $c_k^n = k$. Since $c_k^n \to c_k$, this implies $c_k = k$ and condition (b) follows for $\zeta$. Finally, condition (d) for $\zeta$, which implies each $E_{\zeta}(k), k \in \mathbb{N}$, is a finite set, can
be deduced by similarly observing that $c^n_k = c_k$ and therefore $E_k^n(k) = E_k(k)$ for all $k \in \mathbb{N}$ and all sufficiently large $n$, and the fact that each $\zeta^n$ satisfies condition (d).

(iii) The map $\psi$ is a homeomorphism from $\mathcal{G}_s[\bar{Z}, \bar{Z}]$ to $\mathcal{R}$: For each $n \in \mathbb{N}$, let $G_n = (V_n, E_n, \vartheta_n, \vartheta_n) \in \mathcal{G}_s[\bar{Z}, \bar{Z}]$ and $\psi(G_n) = ((c^n_k, \vartheta^n_k), \{\vartheta^n_{k,k'}\}_{k', k \in \mathbb{N}}) \in \mathcal{R}$. Then we wish to show that $G_n \rightarrow G = (V, E, \vartheta, \vartheta)$ in $\mathcal{G}_s[\bar{Z}, \bar{Z}]$ if and only if $\psi(G_n) \rightarrow \psi(G) := ((c^n_k, \vartheta^n_k), \{\vartheta^n_{k,k'}\}_{k', k \in \mathbb{N}}) \in \mathcal{R}$.

(a) If $G_n \rightarrow G$, then $\lim_{n \to \infty} \psi(G_n) = \psi(G)$: for any $k, k' \in \mathbb{N}$, Conditions 1 and 2 of Definition [B.3] imply that if $k \in V$ and $\{k, k'\} \in E$, then $k \in V_n$ and $\{k, k'\} \in E_n$ for $n$ sufficiently large. Then for any $k \in V$, Condition 6 implies that $\lim_{n \to \infty} c^n_k \to \lim_{n \to \infty} \max\{\text{cl}_k(G_n)\} = \max\{\text{cl}_k(G)\} = c_k'$ and for $k \notin V$, $k \notin V_n$ for $n$ sufficiently large so $\lim_{n \to \infty} c^n_k = c_k'$. Conditions 4 and 5 imply that for $k \in V$ and $\{k, k'\} \in E$, $\lim_{n \to \infty} \vartheta^n_k, \vartheta^n_{k,k'} = \lim_{n \to \infty} (\vartheta^n_{k,k'}, \vartheta_{k,k'}) = (\vartheta_k, \vartheta_{k,k'})$. If $k \notin V$, then $k \notin V_n$ for $n$ sufficiently large and because $\circ$ is an isolated point, this implies that $\lim_{n \to \infty} \vartheta^n_k = \circ = \vartheta_k$. By the same argument, if $\{k, k'\} \notin E$, then $\lim_{n \to \infty} \vartheta^n_{k,k'} = \circ = \vartheta_{k,k'}$. Lastly, $\circ \rightarrow \circ$ by Condition 3 of Definition [B.3]. Thus, $\psi(G_n) \rightarrow \psi(G)$, which establishes the continuity of $\psi$.

(b) If $\psi(G_n) \rightarrow \psi(G)$, then $G_n \rightarrow G$: Conditions 1 and 2 of Definition [B.3] follow from the convergence of $\vartheta^n_k$ and the convergence of $\vartheta^n_{k,k'}$ to $\vartheta_k$ and the fact that $\circ$ is isolated in $\mathcal{G}_s[\bar{Z}, \bar{Z}]$. This directly implies that every vertex $k \in V$ and edge $\{k, k'\} \in E$ is in $V_n$ and $E_n$ respectively for all sufficiently large $n$, and likewise every non-vertex $k \in \mathbb{N} \setminus V$ and non-edge $\{k, k'\} \in \mathbb{N} \setminus E$ is not in $V_n$ or $E_n$ respectively for all sufficiently large $n$. Condition 3 follows from the convergence of $\vartheta^n_k$ to $\vartheta_k$. To prove Condition 4, recall that we have already shown that any $k \in V$ is in $V_n$ for $n$ sufficiently large. Thus, $\lim_{n \to \infty} \vartheta^n_k = \lim_{k \in V_n} \vartheta^n_k = \lim_{k \in \mathbb{N}} \vartheta^n_k = \vartheta_k = \text{cl}_k$. The proof of Condition 5 is exactly analogous except we apply the fact that $\{k, k'\} \in E$ implies $\{k, k'\} \in E_n$ for $n$ sufficiently large and then apply the convergence of $\vartheta^n_{k,k'}$ to $\vartheta_{k,k'}$. Lastly, for each $k \in V$, because $c^n_k = c_k'$ for sufficiently large $n$, this implies that $E_k^n(k) \cap \{c_k' + 1, \ldots\} = \emptyset$ for $n$ sufficiently large. Thus, the fact that $\vartheta_k \in E_k^n$ for $n$ sufficiently large and that $c_k' \geq \text{cl}_k(k)$ implies $\{k, c_k'\} \in E_k^n$ for all $n$ sufficiently large, and because $\{k, c_k'\} = \text{cl}_k$ and $c_k' = c_k$ for sufficiently large $n$, this implies that $\{k, c_k'\} \in E$ so $c_k' = \text{cl}_k(G)$. Thus, Condition 6 holds.

Because $\psi$ is a one-to-one map onto $\mathcal{R}$, this proves the claim.

Since $\mathcal{G}_s[\bar{Z}, \bar{Z}]$ is homeomorphic to the closed subset $\mathcal{R}$ of the Polish space $\mathcal{R}$, it is also Polish. \qed

Next, in Lemma [B.6] below, we show that the map $G \mapsto \langle G \rangle$ is continuous and its set-valued inverse map $\langle G \rangle \mapsto \{G \in \mathcal{G}_s[\bar{Z}, \bar{Z}]: G \in \langle G \rangle\}$ is lower semicontinuous in the sense of [I] Definition 1.4.2.

**Lemma B.6.** If $G_n \rightarrow G$ in $\mathcal{G}_s[\bar{Z}, \bar{Z}]$, then the isomorphism classes also converge locally, that is, $\langle G_n \rangle \rightarrow \langle G \rangle$ in $\mathcal{G}_s[\bar{Z}, \bar{Z}]$. Moreover, given the limit $(G_n) \rightarrow \langle G \rangle$ in $\mathcal{G}_s[\bar{Z}, \bar{Z}]$ and any representative $G \in \langle G \rangle$ such that $G \in \mathcal{G}_s[\bar{Z}, \bar{Z}]$, there exists a sequence of representatives $G_n \in \langle G_n \rangle$, $n \in \mathbb{N}$, such
that $G_n \to G$ in $\hat{G}_n[\mathbb{Z}, \mathbb{Z}]$. In other words, the correspondence $\hat{G}_n[\mathbb{Z}, \mathbb{Z}] \ni \langle G \rangle \mapsto \{ G \in \hat{G}_n[\mathbb{Z}, \mathbb{Z}] : G \in \langle G \rangle \}$ is lower semicontinuous.

Proof. For convenience of notation, let $[n] := \{1, \ldots, n\}$. We start by proving the first statement. Fix any $m \in \mathbb{N}$, and let $M_m := \max\{ v \in B_m(G) \}$. Then Conditions 1 and 2 of Definition [B.3] imply that for sufficiently large $n$, $V \cap [M_m] = V_n \cap [M_m]$ and $E \cap [M_m]^2 = E_n \cap [M_m]^2$. Condition 3 implies that $\omega_n = \omega$ for $n$ sufficiently large. Thus, the subgraphs of $B_m([G_s])$ and $B_m([G_{n,s}])$ induced by the set $[M_m]$ exactly match for sufficiently large $n$. Then by Condition 6, max$\{ c_l(G_n) \} = \max\{ c_l(G) \}$ for all $v \in V \cap [M_m]$ and $n$ sufficiently large. Because $B_m(G) \subseteq [M_m]$, these statements imply that all of the vertices in $B_m(G_n)$ also fall inside $[M_m]$, and hence, $B_m([G_s]) = B_m([G_{n,s}])$ for all sufficiently large $n$, that is $N_m := \min\{ n \in \mathbb{N} : B_m([G_{n,s}]) = B_m([G_s]) \}$ for all $n' \geq n$ is finite (where the minimum of an empty set is taken to be $\infty$). For $n > N_m$, let $\varphi_{n,m} : B_m(G) \to B_m(G_n)$ be the identity isomorphism. Then for each $v \in B_m(G)$ and $e \in E_{B_m(G)}$, Conditions 4 and 5 of Definition [B.3] imply $\lim_{n \to \infty} \varphi_{n,m}(v) = \lim_{n \to \infty} \varphi_{n,m}(e) = \varphi_v$, and $\lim_{n \to \infty} \varphi_{n,m}(e) = \varphi_{n,m}(e) = \lim_{n \to \infty} \varphi_{n,m}(e)$ = $\varphi_{n,m}(e) = \varphi_{n,m}(e)$. By Definition [B.1] this proves that $\langle G_n \rangle \to \langle G \rangle$.

To prove the second statement, first consider the case when the representative $G$ in $\langle G \rangle$ has a vertex set that is canonical in the sense that $V = \{ |V| \} = \{1, \ldots, |V| \}$. It is easy to see that one can always choose representatives $G'_n = (V'_n, E'_n, \varphi'_n, \varphi'_n, \varphi'_n)$ of $\langle G_n \rangle$, $n \in \mathbb{N}$, such that

$$V'_n = \{ |V'_n| \} = \{1, \ldots, |V'_n| \} \text{ for } n \in \mathbb{N}, \tag{B.2}$$

where we interpret $\{1, \ldots, \infty\}$ as $\mathbb{N}$. By construction, each $V_n$ is also in canonical form. Then by Definition [B.1] for each $m \in \mathbb{N}_0$, there exist $m < \infty$ and a collection of isomorphisms $\varphi_{n,m} \in I(B_m([G_s]), B_m([G_{n,s}]))$, $m \in \mathbb{N}_0$, $n > m$, such that for each $m \in \mathbb{N}$, the inclusions $v \in B_m(G)$ and $e \in E_{B_m(G)}$ imply

$$\lim_{n \to \infty} d_{\varphi_{n,m}(e), \varphi_{n,m}(v)} = 0 \text{ and } \lim_{n \to \infty} d_{\varphi_{n,m}(e), \varphi_{n,m}(v)} = 0. \tag{B.3}$$

Hence, there exists a sequence of non-decreasing integers $M_n, n \in \mathbb{N}$, converging to infinity such that for each $n \in \mathbb{N}$, the inclusions $v \in B_{M_n}(G)$ and $e \in E_{B_{M_n}(G)}$ imply $d_{\varphi_{n,m}(v), \varphi_{n,m}(v)} < 2^{-M_n}$ and $d_{\varphi_{n,m}(e), \varphi_{n,m}(e)} < 2^{-M_n}$ when $M_n > 0$. Set $\varphi_n := \varphi_{n,M_n}$, and for each $n \in \mathbb{N}$, define $\varphi_n : \mathbb{N} \to \mathbb{N}$ by

$$\varphi_n(v) = \begin{cases} \varphi_{n,M_n}(v) & \text{if } v \in B_{M_n}(G), \\ w^n & \text{otherwise,} \end{cases}$$

where if $v$ is the $k$th smallest element of $\mathbb{N} \setminus B_{M_n}(G)$, then $w^n$ is the $k$th smallest element of $\mathbb{N} \setminus B_{M_n}(G)$. Now for each $n \in \mathbb{N}$, define

$$G_n := (V_n, E_n, \varphi_n, \varphi_n, \varphi_n) := \left( \varphi_n^{-1}(V'_n), \varphi_n^{-1}(E'_n), \varphi_n^{-1}(\varphi'_n), \varphi_n^{-1}(\varphi'_n), \varphi_n^{-1}(\varphi'_n) \right)_{v \in V_n, \varphi'_n(e) \in E_n, \varphi'_n(e) \in V_n}.$$

By construction, for each $n \in \mathbb{N}$, $V_n = V'_n = \{1, \ldots, |V_n| \}$, just as in (B.2). Then by definition, for each $m \in \mathbb{N}$ and noting that (i) when $m \leq M_n$, $\varphi_n|B_m(G) = \varphi_{n,M_n}|B_m(G)$ (which implies $\varphi_n^{-1}|B_m(G) = \varphi_{n,M_n}^{-1}|B_m(G)$) and (ii) $M_n$ increases to infinity, one has

$$B_m([G_{n,s}]) = \varphi_{n,M_n}^{-1}(B_m([G_{n,s}])) = B_m([G_s]) \text{ for } n \text{ sufficiently large.} \tag{B.4}$$

For every $v \in V$ and $e \in E$, there exists $m \in \mathbb{N}$ such that $v \in B_m(G), e \in E_{B_m(G)}$. Then by (B.4), $v \in V_n$ and $e \in E_n$ for $n$ sufficiently large. On the other hand, suppose $v \notin V$. Then because $V$
Thus, Conditions 4 and 5 of Definition B.3 also hold, proving that \( t \) is in canonical form. Thus, \( G \) has radius \( M < \infty \). Setting \( m = M + 1 \) in (B.4), \( B_{M+1}([G_{n,*}]) = B_{M+1}([G_*)] = [G_*] \) for \( n \) sufficiently large. Furthermore, because \( B_{M+1}([G_{n,*}]) \) has radius \( M \), it follows that \( [G_{n,*}] = B_{M+1}([G_{n,*}]) = [G_*] \). Thus, \( v \notin V_n \) for \( n \) sufficiently large which implies that \( V_n \rightarrow V \). Now, suppose \( e = \{u,v\} \notin E \). Then either (without loss of generality) \( u \notin V \), in which case \( u \notin V_n \) for sufficiently large \( n \) and \( \{u,v\} \notin E_n \) for such \( n \), or \( u, v \in V \). In the latter case, we can fix \( m \) so that \( u, v \in B_m(G) \). Then for \( n \) sufficiently large, \( B_m([G_{n,*}]) = B_m([G_*]) \) by (B.4), so \( u, v \notin E_n \). Thus, \( E_n \rightarrow E \). Setting \( m = 0, e_n = \overline{\varphi}_0^{-1}(\varphi'_n) = \emptyset \) for all \( n \in \mathbb{N} \). Lastly, for each \( v \in V \), fix \( m \) such that \( v \in B_{m-1}(G) \). Then by (B.4), cl\(_e\)(G\(_n\)) = cl\(_e\)(G) for \( n \) sufficiently large. Thus, Conditions 1-3 and 6 of Definition B.3 hold. Now for each \( v \in E \) and \( e \in \epsilon E_m(G) \). Then by the definition of \( M_n \), it follows that

\[
\lim_{n \to \infty} d_Z(\vartheta_n^0, \vartheta_v) = \lim_{n \to \infty} d_Z(\vartheta_n^{\varphi_{n,M_n}(v)} \vartheta_v) = \lim_{n \to \infty} 2^{-M_n} = 0
\]

Thus, Conditions 4 and 5 of Definition B.3 also hold, proving that \( G_n \rightarrow G \) in \( \mathcal{G}_*([Z, \mathbb{Z}]) \).

We finish the proof of the second assertion by considering the general case in which \( G \) is an arbitrary representative of \( \langle G \rangle \) with no restriction on the vertex set. In this case, there exists \( G' := (V', E', \varphi', \mathcal{E}, \varphi') \cong G \) whose vertex set \( V' := V_{G'} \) is in canonical form. Using the argument above, construct the sequence \( \{G'_n\} \) whose vertex sets \( V'_n := V_{G'_n} \) are in canonical form and such that \( G'_n \rightarrow G' \) in \( \mathcal{G}_*([Z, \mathbb{Z}]) \). Given any isomorphism \( \varphi \in I(G', G) \), define

\[
\overline{\varphi}(v) := \begin{cases} 
\varphi(v) & \text{if } v \in V_{G'}, \\
w_v & \text{otherwise,}
\end{cases}
\]

where if \( v \) is the \( k \)th smallest element of \( \mathbb{N} \setminus V' \), then \( w_v \) is the \( k \)th smallest element of \( \mathbb{N} \setminus V \). Then \( \overline{\varphi} : \mathbb{N} \rightarrow \mathbb{N} \) is a bijection, so for each \( n \in \mathbb{N} \),

\[
G_n := \overline{\varphi}(G_n) := (\overline{\varphi}(V'_n), \overline{\varphi}(E'_n), \overline{\varphi}(\varphi'_n), (\overline{\varphi}_n^{-1}(e) \in \overline{\varphi}(E'_n)), (\overline{\varphi}_n^{-1}(e) \in \overline{\varphi}(E'_n)), (\overline{\varphi}_n^{-1}(v) \in \overline{\varphi}(V'_n)),
\]

is isomorphic to \( G'_n \) and \( G_n \rightarrow G \). Thus, the correspondence \( \langle G \rangle \mapsto \{G' \in \mathcal{G}_*([Z, \mathbb{Z}]) : G' \in \langle G \rangle \} \) is lower semicontinuous by [1, Definition 1.4.2].

### B.2 Existence of Measurable Selections

The goal of this section is to establish Lemmas 6.2 and 6.6. We begin this section by showing that every random isomorphism class of rooted graphs has a measurable random representative.

**Lemma B.7.** Given any \( \mathcal{G}_*[\mathbb{Z}, \mathbb{Z}] \)-random element \( \langle G \rangle \), there exists a \( \sigma(\langle G \rangle) \)-measurable representative \( \mathcal{G}_*[\mathbb{Z}, \mathbb{Z}] \)-random graph \( G \) in \( \mathcal{G}_*[\mathbb{Z}, \mathbb{Z}] \).

**Proof.** For every \( \langle H \rangle \in \mathcal{G}_*[\mathbb{Z}, \mathbb{Z}] \), define

\[
\Psi_*(\langle H \rangle) := \{H' \in \mathcal{G}_*[\mathbb{Z}, \mathbb{Z}] : H' \in \langle H \rangle\}.
\]

Then the set \( \Psi_*(\langle H \rangle) \) is non-empty since it contains \( H \), and is also closed due to the continuity of the map \( H \mapsto \langle H \rangle \) established in Lemma B.6. Thus \( \Psi_* \) maps any isomorphism class in \( \mathcal{G}_*[\mathbb{Z}, \mathbb{Z}] \) to the closed set in \( \text{clo}(\mathcal{G}_*[\mathbb{Z}, \mathbb{Z}]) \) that contains all graphs in \( \mathcal{G}_*[\mathbb{Z}, \mathbb{Z}] \) that lie in that isomorphism class.
class. Given the $\mathcal{G}_s[\mathbb{Z}, \mathbb{Z}]$-random element $\langle G \rangle$, we first argue that to prove the lemma it suffices to prove the claim that for every open set $U \subseteq \mathcal{G}_s[\mathbb{Z}, \mathbb{Z}]$, the following set is open in $\mathcal{G}_s[\mathbb{Z}, \mathbb{Z}]$:

$$\Lambda^U_s := \{ \langle H \rangle \in \mathcal{G}_s[\mathbb{Z}, \mathbb{Z}] : \Psi_s(\langle H \rangle) \cap U \neq \emptyset \} = \bigcup_{H \in U} \{ \langle H \rangle \}.$$  

Indeed, since $\langle G \rangle$ is a $\mathcal{G}_s[\mathbb{Z}, \mathbb{Z}]$-random element, the claim implies that $\psi_s(\langle G \rangle)$ is a non-empty $\sigma(\langle G \rangle)$-random closed subset of $\mathcal{G}_s[\mathbb{Z}, \mathbb{Z}]$ in the sense of Definition [B.1] and the lemma follows on applying Theorem [B.2] with $\mathcal{Z}' = \mathcal{G}_s[\mathbb{Z}, \mathbb{Z}]$, $\psi = \psi_s(\langle G \rangle)$ and $\mathcal{F} = \sigma(\langle G \rangle)$.

We now turn to the proof of the claim. If $U$ is empty then so is $\Lambda^U_s$. Now, fix $U \subseteq \mathcal{G}_s[\mathbb{Z}, \mathbb{Z}]$ non-empty. Then $\Lambda^U_s$ is trivially non-empty as well. For $\langle H \rangle \in \Lambda^U_s$, suppose there exists a sequence $\{ \langle H_n \rangle \}_{n \in \mathbb{N}}$ of (deterministic) elements of $\mathcal{G}_s[\mathbb{Z}, \mathbb{Z}]$ converging to $\langle H \rangle$. Select $H' \in \Psi_s(\langle H \rangle) \cap U$ (which is non-empty since $\Lambda^U_s \neq \emptyset$). By Lemma [B.6], there exists a sequence of representative graphs $H'_n \in \langle H_n \rangle$, $n \in \mathbb{N}$, that converges to $H'$ in $\mathcal{Z}'$. Because $H' \in U$ and $U$ is open, $H'_n \in U$ for all but finitely many $n$. By the definition of $\Lambda^U_s$, this implies $\langle H_n \rangle \in \Lambda^U_s$ for all but finitely many $n$. Because this is true for any sequence converging to an element in $\Lambda^U_s$, and because $\mathcal{G}_s[\mathbb{Z}, \mathbb{Z}]$ is a Polish space (in which convergence is equivalent to sequential convergence) $\Lambda^U_s$ is open. This concludes the proof. □

**Remark B.8.** For $W_1, W_2 \subseteq \mathbb{N}$, let $\mathcal{M}(W_1, W_2)$ be the space of mappings from $W_1$ to $W_2$, and define $\mathcal{M} := \bigcup_{W_1, W_2 \subseteq \mathbb{N}} \mathcal{M}(W_1, W_2)$. Then any map $f \in \mathcal{M}(W_1, W_2) \subset \mathcal{M}$ can be embedded in $\mathbb{N}^0$ (equipped with the discrete product topology) via the bijective map

$$\beta(f) := (\beta_n(f))_{n \in \mathbb{N}}, \quad \text{where} \quad \beta_n(f) = \begin{cases} f(n) & \text{if } n \in W_1, \\ 0 & \text{otherwise.} \end{cases} \quad \text{for} \quad f \in \mathcal{M}(W_1, W_2). \quad \text{(B.6)}$$

We equip $\mathcal{M}$ with the topology induced by $\beta$, that is, we define a subset $U \subseteq \mathcal{M}$ to be open if and only if $\beta(U) \subseteq \mathbb{N}^0$ is open. With this definition, $\beta$ is automatically a homeomorphism and $\mathcal{M}$ is a Polish space. We now apply Theorem [B.2] to select the isomorphisms in Lemma [6.2] in a measurable manner.

**Lemma B.9.** Fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ that supports finite, $\mathcal{F}$-measurable $\mathcal{G}_s[\mathbb{Z}, \mathbb{Z}]$-random elements $G_i$, $i = 1, 2$, that satisfy $\mathbb{P}(G_1 \equiv G_2) > 0$. Then given any $\mathcal{F}$-random closed subset $A$ of $\mathcal{M}$ such that $A$ is a non-empty subset of $I(G_1, G_2)$ on the event $\{G_1 \equiv G_2\}$, there exists an $\mathcal{F}$-measurable map $\varphi \in \mathcal{M}(V_1, V_2)$ such that $\varphi \in A$ on the event $\{G_1 \equiv G_2\}$.

**Proof.** Given any $\mathcal{F}$-random closed subset $A$ of $\mathcal{M}$ such that $A \subseteq I(G_1, G_2)$, consider the set-valued mapping $B(\omega) := \mathcal{M}(V_1, V_2)$ if $G_1 \not\equiv G_2$ and $B(\omega) := A$ otherwise. Note that by finiteness of $G_1$ and $G_2$, $B$ is also finite because $B \subseteq \mathcal{M}(V_1, V_2)$, and thus $B(\omega)$ is closed for all $\omega \in \Omega$. Let $B = \{ \beta(f) : f \in B \}$, with $\beta$ as in (B.6), and note that $B(\omega)$ is similarly finite and therefore closed and non-empty for all $\omega \in \Omega$. We claim that $B(\omega)$ is weakly $\mathcal{F}$-measurable. That is, for every open $U \subseteq \mathbb{N}^0$, the set $\{ \overline{B} \cap U \neq \emptyset \}$ is $\mathcal{F}$-measurable. If the claim holds, then $\overline{B}$ is a non-empty $\mathcal{F}$-random closed subset, so by Theorem [B.2] with $\mathcal{Z}' = \mathbb{N}^0$ and $\mathcal{F} = \overline{B}$, there exists an $\mathcal{F}$-measurable random variable $b$ such that $b(\omega) \in \overline{B}(\omega)$ for every $\omega \in \Omega$. The lemma follows on setting $\varphi = \beta^{-1}(b)$.

To prove the claim, first note that the set $\{ \omega : G_1 \equiv G_2 \}$ is $\mathcal{F}$-measurable by assumption. Moreover, fixing any open set $U \subseteq \mathbb{N}^0$, note that $\{ G_1 \equiv G_2 \} \cap \{ \overline{B} \cap U \neq \emptyset \} = \{ G_1 \equiv G_2 \} \cap \{ A \cap \beta^{-1}(U) \neq \emptyset \}$, and also that $U' := \beta^{-1}(U) \subset \mathcal{M}$ is open since $\beta$ is continuous. Because $A$ is an $\mathcal{F}$-random closed subset, this implies $\{ G_1 \equiv G_2 \} \cap \{ \overline{B} \cap U \neq \emptyset \}$ is $\mathcal{F}$-measurable. Next,
observe that \( \{G_1 \not\approx G_2\} \cap \{\overline{B} \cap U \neq \emptyset\} = \{G_1 \not\approx G_2\} \cap \{\mathcal{M}(V_1, V_2) \cap \beta^{-1}(U) \neq \emptyset\} = \{G_1 \not\approx G_2\} \cap \{\mathcal{M}(V_1, V_2) \cap U' \neq \emptyset\}. \) However, note that

\[
\mathcal{M}(V_1, V_2) \cap U' \neq \emptyset = \bigcup_{\text{finite } W_1, W_2 \subseteq \mathbb{N}} \{V_i(\omega) = W_i, i = 1, 2\} \cap \{U' \cap \mathcal{M}(W_1, W_2) \neq \emptyset\},
\]

which is a countable union of \( \mathcal{F} \)-measurable sets since \( U' \) is open. Thus, \( \{\overline{B} \cap U \neq \emptyset\} \) is \( \mathcal{F} \)-measurable for every open \( U \), and the claim follows.

The following lemma ensures driving maps on different graphs can be constructed so as to be consistent with isomorphisms between those graphs, which is required in Sections 6 and 7.

**Lemma B.10.** Fix a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) that supports the \( \mathcal{F} \)-measurable \( \hat{\mathcal{G}}_n[\mathbb{Z}, \mathbb{Z}] \)-random elements \( G_i, i = 1, 2 \). Let \( M \) be an \( \mathcal{F} \)-measurable random variable such that \( M(\omega) \in \{m \in \mathbb{N}_0 : B_m(G_1(\omega)) \cong B_m(G_2(\omega))\} \) for every \( \omega \in \Omega \). Then for any \( \mathcal{F} \)-measurable isomorphism \( \varphi \in I(B_M(G_1), B_M(G_2)) \) and \( \mathcal{F} \)-measurable driving map \( \psi_1 : V_1 \to \mathbb{N} \) such that \( \mathbb{N} \setminus \psi_1(V_1) \) is infinite, there exists an \( \mathcal{F} \)-measurable driving map \( \psi_2 : V_2 \to \mathbb{N} \) such that \( \psi_1(v) = \psi_2(\varphi(v)) \) for every \( v \in B_M(G_1) \) and \( \psi_1(G_1) \cap \psi_2(G_2 \setminus B_M(G_2)) = \emptyset \).

**Proof.** Fix an \( \mathcal{F} \)-measurable random variable \( M \) and \( \mathcal{F} \)-measurable isomorphism \( \varphi \in I(B_M(G_1), B_M(G_2)) \) as in the statement of the lemma. Let \( \hat{\mathbb{Z}} := (\mathbb{N}_0)^2 \), equipped with the product topology, and consider the \( \hat{\mathbb{Z}} \)-random element \( b = (b_1, b_2) \) with \( b_i = (b_{i,k})_{k \in \mathbb{N}} \) given, for \( i = 1, 2 \) and \( k \in \mathbb{N} \), by

\[
b_{i,k} = \begin{cases} 
0 & \text{ if } k \notin G_i, \\
\psi_1(k) & \text{ if } k \in V_{G_1}, \\
\psi_1(\varphi^{-1}(k)) & \text{ if } k \in B_M(G_2) \text{ and } i = 2, \\
\alpha(k) & \text{ if } k \in V_2 \setminus B_M(G_2) \text{ and } i = 2,
\end{cases}
\]

where \( \alpha : \mathbb{N} \to \mathbb{N} \) is an injection mapping each \( k \) to the \( k \)th smallest element of \( \mathbb{N} \setminus \psi_1(V_1) \). Observe that \( \alpha \) is well defined because \( \mathbb{N} \setminus \psi_1(V_1) \) is infinite and is also an \( \mathcal{F} \)-measurable element of \( M \) because \( \psi_1 \) is \( \mathcal{F} \)-measurable. Thus, \( b \) is clearly also \( \mathcal{F} \)-measurable. Define \( \psi_i := \beta^{-1}(b_i), \ i = 1, 2 \), with \( \beta \) as in (1.6), and note that since \( \beta^{-1} \) is Borel measurable, each \( \psi_i \) is also \( \mathcal{F} \)-measurable. Because \( \psi_1(V_1) \) and \( \alpha(\mathbb{N}) \) are disjoint, \( \psi_2 \) is also injective and therefore a driving map. Furthermore, for each \( k \in B_M(G_1) \), \( \psi_1(k) = \psi_1(\varphi^{-1}(\varphi(k))) = \psi_2(\varphi(k)) \), and \( \psi_1(G_1) \subseteq \psi_1(V_1) \) and \( \psi_2(G_2 \setminus B_M(G_2)) \subseteq \alpha(\mathbb{N}) \), so \( \psi_1(G_1) \) and \( \psi_2(G_2 \setminus B_M(G_2)) \) are disjoint. This completes the proof.

We finish the appendix with proofs of Lemmas 6.2 and 6.6.

**Proof of Lemma 6.2.** The first assertion follows from Lemma 3.1 and the fact that \( \sigma((\{G_n, \xi^n\})_{n \in \mathbb{N}}) \subseteq \hat{\mathcal{F}}. \) For the second assertion, let \( \{(G_n, \xi^n)\}_{n \in \mathbb{N}}, (G, \xi) \) be any sequence of measurable representatives satisfying Properties 1 and 2 of Definition 6.1. By Definition 2.1 the convergence of \( ((G_n, \xi^n)) \) to \( ((G, \xi)) \) in \( \mathcal{G}[\mathcal{K}, \mathcal{K} \times \mathcal{X}] \) implies that for every \( m \in \mathbb{N} \), \( B_m((G_n^*) \cong B_m((G_n^*, \xi^n)) \) for sufficiently large \( n \). Since \( \mathcal{X} \) is discrete, this further implies that there exists an a.s. finite random variable \( N_m \) such that \( B_m((G_n^*, \xi^n) \cong B_m((G_n^*, \xi^n)) \) for all \( n \geq N_m \) and \( N_m \) is \( \hat{\mathcal{F}} \)-measurable. Then for any \( \varphi \in \mathcal{M} \), define the following random variable:

\[
\Psi_{n,m}(\varphi) := \begin{cases} 
\sum_{v \in B_m(G)} d_{\mathcal{K}}((\kappa^i_{\varphi(v)}), \kappa_v) + \sum_{v \in E_{B_m(G)}} d_{\mathcal{K}}((\tau_{\varphi(v)}, \tau_v) & \text{ if } \varphi \in \mathcal{M}, \\
0 & \text{ otherwise},
\end{cases}
\]

48
where $I_{n,m} := I(B_m([G^*_n], \xi_n), B_m([G^*_{n+1}], \xi_n^*)))$. Let $\mathcal{M}_{n,m}^\Psi = \arg\min_{\varphi \in \mathcal{M}} \Psi_{n,m}(\varphi)$, where we define $\mathcal{M}_{n,m}^\Psi$ to be empty if $\Psi_{n,m}(\varphi) = \infty$ for all $\varphi \in \mathcal{M}$. On the $\hat{\mathcal{F}}$-measurable set $\{ n \geq N_n \}$, $I_{n,m}$ is non-empty and finite, and so $\mathcal{M}_{n,m}^\Psi$ is also non-empty and finite. We first show that to prove the following claim:

**Key claim:** $\mathcal{M}_{n,m}^\Psi$ is a $\hat{\mathcal{F}}$-random closed subset of $\mathcal{M}$ that is non-empty on the event $\{ I_{n,m} \neq \emptyset \}$.

Deferring the proof of the claim, first note that given the claim, applying Lemma 13.3 with $G_1 = B_m([G^*_n], \xi_n)$, $G_2 = B_m([G^*_{n+1}], \xi_n^*)$ and $A = \mathcal{M}_{n,m}^\Psi$, there exists a sequence of $\hat{\mathcal{F}}$-measurable maps $\{ \hat{\varphi}_{n,m} \}_{n,m \in \mathbb{N}}$ such that $\hat{\varphi}_{n,m} \in \mathcal{M}_{n,m}^\Psi$ for every $n, m \in \mathbb{N}$ on the event $\{ n \geq N_m \}$. Hence, by the definition of $\mathcal{M}_{n,m}^\Psi$, the fact that $\hat{\varphi}_{n,m} \in I_{n,m}$ and by Definition 2.1 for any $m \in \mathbb{N}$,

$$\lim_{n \to \infty, n > N_m} \min_{\varphi \in I_{n,m}} \Psi_{n,m}(\varphi) = \lim_{n \to \infty, n > N_m} \Psi_{n,m}(\hat{\varphi}_{n,m}) = 0.$$

This implies that for every $m \in \mathbb{N}$ there exists a $\hat{\mathcal{F}}$-measurable integer $\hat{N}_m$ such that $\Psi_{n,m}(\hat{\varphi}_{n,m}) < 2^{-m}$ for all $n \geq \hat{N}_m$. Moreover, $\hat{N}_m$ is non-decreasing and a.s. finite and so $M_n := \max\{ m \in \mathbb{N} : n \geq \hat{N}_m \}$ is $\hat{\mathcal{F}}$-measurable and increases to infinity. Therefore, Property 3 of Definition 6.1 is satisfied. For each $n \in \mathbb{N}$ and $m \leq M_n$, define $\varphi_{n,m} := \hat{\varphi}_{n,M_n}|_{B_m(G)}$. It follows that $\{ \varphi_{n,m} \}_{n \in \mathbb{N}, m \leq M_n}$ is also an $\hat{\mathcal{F}}$-measurable sequence and satisfies Property 4 of Definition 6.1. Furthermore, Properties 5 and 6 follow directly from the fact that for $v \in B_m(G)$ and $e \in E_{B_m(G)}$,

$$\max\{ d_K(K_{\varphi_{n,m}(v)}, v_e), d_K(K_{\varphi_{n,m}(e)}, e_e) \} \leq \Psi_{n,M_n}(\hat{\varphi}_{n,n,M_n}) < 2^{-M_n} \to 0 \quad \text{as} \quad n \to \infty.$$

We now turn to the proof of the key claim. Fix $n, m \in \mathbb{N}$. We first prove the following:

**Sub-Claim 1:** For each $\varphi \in \mathcal{M}$, $\Psi_{n,m}(\varphi)$ is $\hat{\mathcal{F}}$-measurable.

**Proof of Sub-Claim 1:** Fix $\varphi \in \mathcal{M}$. Define $\mathcal{A}_{n,m}^\varphi := \mathcal{A}_{n,m,1} \cap \mathcal{A}_{n,m,2}$, where

$$\mathcal{A}_{n,m,1} := \left\{ (H_1, H_2) \in \widehat{G}_n[S, K \times X]^2 : \max_{i=1,2, v \in V_{H_i}} d_{H_i}(v, v) \leq m \right\},$$

$$\mathcal{A}_{n,m,2} := \left\{ (H_1, H_2) \in \widehat{G}_n[S, K \times X]^2 : \varphi \in I(\theta(H_1), \theta(H_2)) \right\},$$

with $\theta : \widehat{G}_n[S, K \times X] \to \widehat{G}_n[S, 1, X]$ being the mapping that takes $H = (V_H, E_H, \varphi_H, K_H, \xi_H) \in \widehat{G}_n[S, K \times X]$ to $(V_H, E_H, \varphi_H, \xi_H) \in \widehat{G}_n[S, 1, X]$, which is the rooted representative graph $H$ with only the $X$-valued vertex marks retained. Then consider the mapping $\Theta_{n,m}^\varphi : (\widehat{G}_n[S, K \times X]^2) \to \mathbb{R}_+ \cup \{ \infty \}$ given by

$$\Theta_{n,m}^\varphi(H_1, H_2) := \sum_{v \in V_{H_1}} d_{K}(K_{\varphi(v)}, v_{H_1}) + \sum_{e \in E_{H_1}} d_{K}(K_{\varphi(e)}, e_{H_1}) \quad \text{if} \quad (H_1, H_2) \in \mathcal{A}_{n,m}^\varphi,$$

$$\Theta_{n,m}^\varphi(H_1, H_2) := \infty \quad \text{otherwise}.$$

Since $\Psi_{n,m}(\varphi) = \Theta_{n,m}^\varphi(B_m(G), \xi_n, B_m(G_n, \xi_n^*))$, to prove Sub-Claim 1, it suffices to show the following:

**Sub-Claim 2:** The map $\Theta_{n,m}^\varphi$ is continuous.

**Proof of Sub-Claim 2:** Fix $\varphi \in \mathcal{M}(W_1, W_2)$ and $m \in \mathbb{N}$. If $W_1$ is infinite, $\Theta_{n,m}^\varphi \equiv \infty$, and so is trivially continuous. Next, suppose $|W_1| < \infty$. Then, since $\theta$ and $H \mapsto \max_{v \in V_{H}} d_{H}(v, v)$ are
Since this is a countable union of \( \hat{\mathcal{A}}_{m}^{n} \) is closed, so \( \hat{\mathcal{A}}_{m}^{n} \) is likewise closed. Let \( \{(H_{1}^{n}, H_{2}^{n})\}_{n \in \mathbb{N}} \subseteq \hat{\mathcal{A}}_{m}^{n} \cap \hat{\mathcal{A}}_{m}^{n} \) be any sequence that is convergent in \( \hat{\mathcal{A}}_{m}^{n} \), and let \( (H_{1}^{\infty}, H_{2}^{\infty}) \) denote its limit. Then \( (H_{1}^{\infty}, H_{2}^{\infty}) \in \hat{\mathcal{A}}_{m}^{n} \) since \( \hat{\mathcal{A}}_{m}^{n} \) is closed. Moreover, by Conditions 1 and 2 of Definition 3.3, \( V_{H_{1}^{1}} = V_{H_{1}^{\infty}} = W_{1} \) and \( E_{H_{1}^{1}} = E_{H_{1}^{\infty}} \) for all sufficiently large \( n \). Then the convergence of \( \Theta_{m}^{n}(H_{1}^{1}, H_{2}^{1}) \) to \( \Theta_{m}^{n}(H_{1}^{\infty}, H_{2}^{\infty}) \) is an immediate consequence of the definition of \( \Theta_{m}^{n} \) and Conditions 4 and 5 of Definition 3.3. Lastly note that the map \( \theta_{2} : (\hat{\mathcal{A}}_{m}^{n}, \mathcal{K} \times \mathcal{X}) \rightarrow (\hat{\mathcal{A}}_{m}^{n}, \mathcal{X}) \) defined by \( \theta_{2}(H_{1}, H_{2}) := (\theta(H_{1}), \theta(H_{2})) \) is continuous and that \( \hat{\mathcal{A}}_{m}^{n} = \theta_{2}^{-1}(\hat{\mathcal{A}}_{m}^{n}) \) where \( \hat{\mathcal{A}}^{n} \subseteq (\hat{\mathcal{G}}_{m}[1, \mathcal{X}])^{2} \) is an isolated point in \( \hat{\mathcal{A}}_{m}^{n}, \mathcal{X} \mathcal{A} \mathcal{F} \mathcal{M} \mathcal{R} \) which implies that \( \hat{\mathcal{A}}_{m}^{n} \) is open. Sub-Claim 2 then follows on noting that \( \Theta_{m}^{n} \) is identically equal to infinity and thus trivially continuous on the closed set \( (\hat{\mathcal{A}}_{m}^{n})^c \).

Next, define \( \Psi_{n,m}^{\min} := \min_{\varphi \in \mathcal{M}} \Psi_{n,m}(\varphi) \) where \( \Psi_{n,m}^{\min} = \infty \) when \( I_{n,m} = \emptyset \). Then \( \Psi_{n,m}^{\min} \) always exists because \( I_{n,m} \) is finite. Note that since \( |B_{m}(G)| + |B_{m}(G_{n})| < \infty \), it follows that

\[
\Psi_{n,m}^{\min} = \min_{\varphi \in \mathcal{M}(W_{1}, W_{2})} \Psi_{n,m}(\varphi),
\]

is a minimum over a countable collection of \( \hat{\mathcal{F}} \)-measurable random variables. Sub-Claim 1 then shows that \( \Psi_{n,m}^{\min} \) is also \( \hat{\mathcal{F}} \)-measurable. For any open \( U \subseteq \mathcal{M} \),

\[
\{ \mathcal{M}_{n,m}^{\psi} \cap U \neq \emptyset \} = \bigcup_{\varphi \in \mathcal{M}(W_{1}, W_{2}) \cap U} \{ \Psi_{n,m}(\varphi) = \Psi_{n,m}^{\min} \} \cap \{ \Psi_{n,m}^{\min} < \infty \}.
\]

Since this is a countable union of \( \hat{\mathcal{F}} \)-measurable sets, the key claim follows from Definition 3.1.

**Proof of Lemma 6.6.** Let \( \{(G_n, \xi_n)\}_{n \in \mathbb{N}} \) be a random sequence on \( (\Omega, \hat{\mathcal{F}}, \hat{\mathbb{P}}) \) that converges a.s. to \( (G, \xi) \) in \( \hat{\mathcal{G}}_{m}[\mathcal{K}, \mathcal{K}] \). By Lemma 6.2 there exists a rep-con sequence \( \{(G_n, \xi_n), M_n\}_{n \in \mathbb{N}} \) of \( \{(G_n, \xi_n)\}_{n \in \mathbb{N}} \) defined on the same probability space \( (\Omega, \hat{\mathcal{F}}, \hat{\mathbb{P}}) \). Let \( \psi \) be a \( \hat{\mathcal{F}} \)-measurable driving map \( \psi : V_{G} \rightarrow \mathbb{N} \) such that \( \mathbb{N} \setminus \psi(G) \) is infinite (e.g., consider the map \( \psi(k) = 2k \)). Then, invoking Properties 3 and 4 of Definition 6.1 and repeatedly applying Lemma 3.10 with \( M = M_n, G_1 = (G_n, \xi_n), G_2 = \xi_n, \psi = \psi_n, \varphi = \varphi_n, \psi_1 = \psi, \) for each \( n \in \mathbb{N} \) we can construct a \( \hat{\mathcal{F}} \)-measurable driving map \( \psi_n : G_n \rightarrow \mathbb{N} \) such that for every \( m \leq M_n \) and \( v \in B_{m}(G_n) \), \( \psi_n(\varphi_{m,v}(v)) = \psi(v) \). Then extending the space \( (\Omega, \hat{\mathcal{F}}, \hat{\mathbb{P}}) \) to add a countable sequence of i.i.d. Poisson processes \( \{N_n\}_{n \in \mathbb{N}} \) and using the driving maps \( \psi_n, n \in \mathbb{N} \), and \( \psi \) to generate the respective \( \mathcal{F} \)-driving noise \( (G_n, N^{G_n}, n \in \mathbb{N}), (G, N^G) \) as in Definition 6.4 we obtain a consistent rep-con extension.

**C Well-Posedness for Finite Initial Data**

Under Assumption A well-posedness of \( (\hat{\mathcal{G}}_{m}[\mathcal{K}, \mathcal{K}]) \) is common knowledge when the initial data is finite, but we establish it here for completeness. In this case, we also show that the trajectories also satisfy the following additional regularity property. Recall the definition of the discontinuity set \( \text{Disc}_t(x) \) given in (2.2).

**Definition C.1** (Proper trajectories). Given a countable set \( W \) and \( t \in [0, \infty) \), we say \( x \in \mathcal{D}^W \) is proper if \( \text{Disc}_t(x_w) \cap \text{Disc}_t(x_v) = \emptyset \) for all distinct \( v, w \in W \). Moreover, we say a trajectory \( x \in \mathcal{D}^W \) is proper if its restriction \( x|_t \) to \( [0, t] \) is proper for all \( t \in \mathbb{R}_+ \).
Proof of Proposition 4.1 and trajectories being a.s. proper: Let \((G, \xi) = (V, E, \omega, \pi, \kappa, \xi)\) any a.s. finite \([K, K \times \mathcal{X}]\)-random graph. By Lemma 3.3, it suffices to prove that (3.3) is strongly well-posed for \((G, \xi)\) with a.s. proper solutions under the additional assumption that \((G, \xi)\) is deterministic.

Let \((\mathcal{F}, \mathcal{N}^G)\) be a filtration-Poisson process pair in the sense of Remark 4.5. First note that for any \((\mathcal{F}, \mathcal{N}^G)\)-weak solution \((G, X)\) to (3.3), any distinct vertices \(u, v \in V\) and any \(T > 0\),

\[
\text{Disc}_T (X_u) \cap \text{Disc}_T (X_v) \subseteq \{ s \in [0, T]: \mathcal{N}^G_u (\{ s \times (0, C_k, T] \times \mathcal{J} \}) = 1 \} = \emptyset \text{ a.s.,}
\]

by (3.3) and Assumption \(\mathbf{1}\) where \(\{ C_k, T \}\) is the family of constants from Assumption \(\mathbf{1}\) and \(k := \max \{|d_u|, |d_v|\}\). Since this holds for all \(T\), \((G, X)\) is a.s. proper.

Next, fix a filtration-Poisson process pair \((\mathcal{F}, \mathcal{N}^G)\). Define the finite collection,

\[
\mathcal{E} := \{ (\tau_n, r_n, j_n, v_n) \in [0, T] \times [0, C_K, T] \times \mathcal{J} : \mathcal{N}^G_{v_n} (\{ (\tau_n, r_n, j_n) \}) = 1 \},
\]

\(K = \max_{v \in V} |d_v|\) and \(\{ \tau_n \}_{n=1}^{\infty}\) is increasing. Note that \(\{ \tau_n \}\) is, in fact, strictly increasing. Define the \(\mathcal{D}^V_T\)-random element \(X[T]\) by,

\[
X_v(t) = \begin{cases} 
\xi_u & \text{if } 0 \leq t < \tau_1, \\
X_v(\tau_n) & \text{if } \tau_n \leq t < \tau_{n+1}, n < |\mathcal{E}|, \\
X_v(\tau_n) & \text{if } \tau_n \leq t \leq T, n = |\mathcal{E}|, \\
X_v(\tau_n) & \text{if } t = \tau_{n+1}, r_n = \tau_{n+1} > \tau_{n+1}, (t_n+1, X), \\
X_v(\tau_n) + j_{n+1} & \text{if } t = \tau_{n+1}, r_n = \tau_{n+1} > \tau_{n+1}, (t_n+1, X).
\end{cases}
\]

Then note that for \(t \in [0, T]\), \(X(t)\) is clearly \(\mathcal{F}^G, \mathcal{N}^G\)-measurable so \(X\) is a \(\mathcal{N}^G\)-strong solution to (3.3) for \((G, \xi)\) on the interval \([0, T]\). Furthermore, it is clear that any \((\mathcal{F}, \mathcal{N}^G)\)-weak solution to (3.3) must satisfy the above display, so all \((\mathcal{F}, \mathcal{N}^G)\)-weak solutions equal \(X\) on the interval \([0, T]\). Because \(T\) is arbitrary, and for any \(T' > T\) the corresponding solution \(X'\) satisfies \(X'[T] = X[T]\) a.s., it follows that there exists a \(\mathcal{N}^G\)-strong solution to (3.3) and that solution is pathwise unique. Therefore (3.3) is strongly well-posed for \((G, \xi)\).

\[
\square
\]

D Characterization of Strong Well-Posedness on Random Graphs

Proof of Lemma 3.3. The key issue here is to show that conditioning on the initial data does not change the driving noise structure. Note that this is slightly non-standard as the driving noise is indexed by the vertices of the graph and thus is not completely independent of the initial data. Let \((\Omega, \mathcal{F}, \mathcal{F}, \mathbb{P})\) be a complete, filtered probability space that supports \((G, \xi)\) and a filtration-Poisson process pair \((\mathcal{F}, \mathcal{N}^G)\) such that \((G, X)\) and \((G, Y)\) are two \((\mathcal{F}, \mathcal{N}^G)\)-weak solutions of (3.3) for \((G, \xi)\).

To prove the lemma, it suffices to prove the following claim: for \(\mathbb{P}\)-a.s. \(\omega \in \Omega\), setting \((H, \xi^H) := (G(\omega), \xi(\omega))\), there exists a filtration-Poisson process pair \((\mathcal{F}^H, \mathcal{N}^H)\) on some probability space \((\Omega^H, \mathcal{F}^H, \mathbb{P}^H)\) and two \((\mathcal{F}^H, \mathcal{N}^H)\)-weak solutions \((H, X^H)\) and \((H, Y^H)\) to (3.3) for \((H, \xi^H)\) such that \(\mathcal{L}((G, X, Y, \mathcal{N}^G) | \mathcal{F}_0)(\omega) = \mathcal{L}((H, X^H, Y^H, \mathcal{N}^H))\). Indeed, if the claim holds, then a.s. strong well-posedness of (3.3) for every realization of the random graph \((G, \xi)\) implies that \(\mathbb{P}(X = Y | \mathcal{F}_0)(\omega) = \mathbb{P}^H(X^H = Y^H) = 1\) for \(\mathbb{P}\)-a.s. \(\omega \in \Omega\). Hence \(\mathbb{P}(X = Y) = 1\), which proves strong well-posedness. This claim can be proved via direct verification of Definitions 3.3, 3.4. We include the details for completion.

To prove the claim, let \((H, X^H, Y^H, \mathcal{N}^H)\) be a random element with law \(\mathcal{L}((G, X, Y, \mathcal{N}^G) | \mathcal{F}_0)(\omega)\) and fix a complete, filtered probability space \((\Omega^H, \mathcal{F}^H, \mathbb{P}^H)\) that
supports \((H, X^H, Y^H, N^H)\) where \(F^H\) is the minimal filtration satisfying the usual conditions such that \(X^H, Y^H\) and \(N^H\) are all \(F^H\)-adapted in the sense that for any \(v \in V_H, X^H_v, Y^H_v\) and \(N^H_v\) are all \(F^H\)-adapted (point) processes. Since by assumption \(N^G\) is a \(F\)-driving noise, Condition 1 of Definition 3.3 immediately implies \(N^H\) is a collection of i.i.d. Poisson processes on \(\mathbb{R}_+^\times \mathcal{J}\) with intensity \(\text{Leb}^2 \otimes \mathcal{J}\), indexed by the vertices of \(H\), and hence that \((H, N^H)\) is a \(F^H\)-driving noise.

Since (3.3) holds a.s., for \(\mathbb{P}\)-a.s. \(\omega \in \Omega\), \((H, X^H)\) and \((H, Y^H)\) are both graphs with random càdlàg marks that, together with \((H, N^H)\), \(\mathbb{P}((\mathcal{F}_0(\omega))\text{-a.s. solve (3.3) for } (H, \xi^H).\) Thus, \((H, X^H)\) and \((H, Y^H)\) satisfy Conditions 1 and 3 of Definition 3.3 \(\mathbb{P}\)-a.s. with respect to the filtered probability space \((\Omega^H, \mathcal{F}^H, \mathcal{P}^H, \mathbb{P}^H)\) (noting that any \(\mathbb{P}((\mathcal{F}_0(\omega))\text{-null event is also } \mathbb{P}^H\text{-null})\), and so it suffices to prove that \(N^H\) is \(\mathbb{P}\)-a.s. a collection of i.i.d. \(F^H\)-Poisson processes. To do this, we note that Condition 2 of Definition 3.3 implies that for any \(t > 0\) and \(A \in \mathcal{B}(t, \infty) \times \mathbb{R}_+ \times \mathcal{J}\), the random element \((G(N^G(A))\) is conditionally independent of the \(\mathcal{F}_t\)-measurable random element \((G, X^H[t], Y^H[t], N^G)\) given \(\mathcal{F}_0\), where \(N^G_t = N^G|[0, t] \times \mathbb{R}_+ \times \mathcal{J}\). Thus, \(\mathbb{P}\)-a.s., \((H, N^H(A))\) is independent of \((H, X^H[t], Y^H[t], N^H)\). Then using a standard approximation argument exploiting the fact that Borel sigma algebras of subsets of Polish spaces are countably generated and that \(\mathbb{P}^H\) is complete, it follows that \(\mathbb{P}\)-a.s., \(N^H(A)\) is independent of \(\mathcal{F}^H\) for all \(v \in V_H, t \in \mathbb{R}_+\) and \(A \in \mathcal{B}(t, \infty) \times \mathbb{R}_+ \times \mathcal{J}\). Thus, \(\{N^H_t\}_{t \in V_H}\) is \(\mathbb{P}\)-a.s. a collection of \(F^H\)-Poisson processes. Therefore, \(N^H\) is \(\mathbb{P}\)-a.s. an \(F^H\)-driving noise in the sense of Definition 3.3 so Condition 2 of Definition 3.3 is also \(\mathbb{P}\)-a.s. satisfied. This concludes the proof of the claim. 

\[\square\]

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