Efficient Deterministic Distributed Coloring with Small Bandwidth

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Abstract. We show that the \((\text{degree} + 1)\)-list coloring problem can be solved deterministically in \(O(D \cdot \log n \cdot \log^3 \Delta)\) in the CONGEST model, where \(D\) is the diameter of the graph, \(n\) the number of nodes, and \(\Delta\) is the maximum degree. Using the network decomposition algorithm from [RG19] this implies the first efficient deterministic, that is, \(\text{poly} \log n\)-time, CONGEST algorithm for the \(\Delta + 1\)-coloring and the \((\text{degree} + 1)\)-list coloring problem. Previously the best known algorithm required \(2^{O(\sqrt{\log n})}\) rounds and was not based on network decompositions.

Our results also imply deterministic \(O(\log^3 \Delta)\)-round algorithms in MPC and the CONGESTED CLIQUE.

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1 Introduction

In the distributed message passing model, a communication network is abstracted as an $n$-node graph $G = (V, E)$. The nodes of $G$ host processors that communicate with each other through the edges of the graph. In the context of distributed graph algorithms, the objective is to solve some graph problem on $G$ by a distributed message passing algorithm. On of the most important and most extensively studied problems in the area is the distributed graph coloring problem, where we need to compute a proper (vertex) coloring of the communication graph $G$. Typically, at the beginning of an algorithm, the nodes of $G$ do not know anything about $G$, except maybe the names of their immediate neighbors and at the end of an algorithm, each node of $G$ needs to know its local part of the solution of the given graph problem (e.g., for distributed coloring, at the end, each node must know its own color). The two classic models in which distributed graph algorithms have been studied are the LOCAL and the CONGEST model [Lin92, Pe00]. In both models, time is divided into synchronous rounds and in each round, each node can perform arbitrary internal computations and send a message to each of its neighbors. The time complexity of an algorithm is the number of rounds required for the algorithm to terminate. In the LOCAL model, messages can be of arbitrary size, whereas in the CONGEST model, all messages have to be of size $O(\log n)$ bits.

Distributed Graph Coloring. Computing a coloring with the optimal number of colors $\chi(G)$ was one of the first problems known to be NP-complete [Kar72]. In the distributed setting, one therefore aims for a more relaxed goal and the usual objective is to color a given graph $G$ with $\Delta + 1$ colors, where $\Delta$ is the maximum degree of the $G$ [BE13]. Note that any graph admits such a coloring and it can be computed by a simple sequential greedy algorithm. Despite the simplicity of the sequential algorithm, the question of determining the distributed complexity of computing a $(\Delta + 1)$-coloring has been an extremely challenging question. In particular, while efficient $O(\log n)$-time randomized distributed $(\Delta + 1)$-coloring algorithms have been known for more than 30 years, the question whether a similarly efficient (i.e., polylogarithmic time) deterministic distributed coloring algorithm exists, remained one of the most important open problems in the area [BE13, GKM17] until it was resolved very recently by Rozhoň and Ghaffari [RG19]. In [RG19] the question was answered in the affirmative for the LOCAL model by designing an efficient deterministic distributed method to decompose the communication graph into a logarithmic number of subgraphs consisting of connected components (clusters) of small, polylogarithmic (weak) diameter, a structure known as a network decomposition. It was known before that an efficient deterministic algorithm for network decomposition would essentially imply efficient deterministic LOCAL model algorithms for any problem for which efficient randomized algorithms exist [Lin87, AGLP89a, BE13, GKM17, GHK18].

Note that due to the unbounded message size any (solvable) problem can be solved in diameter time in the LOCAL model by simply collecting the whole graph topology at one node, solving the problem locally (potentially using unbounded computational power) and redistributing the solution to the nodes. Thus the small diameter components of a network decomposition almost immediately give rise to an efficient $(\Delta + 1)$-coloring in the LOCAL model. In light of the breakthrough Rozhoň and Ghaffari [RG19], which implies that a polylogarithmic-time deterministic $(\Delta + 1)$-coloring algorithm exists in the LOCAL model, it is natural to ask whether a polylogarithmic-time deterministic algorithm also exists in the more restricted, but seemingly also more realistic CONGEST model. As the main result of this paper, we answer this question in the affirmative.

1.1 Our Contributions in CONGEST

The main technical contribution of this work is to provide an efficient deterministic CONGEST model algorithm for the $(\Delta + 1)$-coloring problem, and more generally the $(\deg + 1)$-list coloring problem in time proportional to the diameter of the graph. By using network decompositions, the result then also implies efficient algorithms for general graphs even if the diameter is large.

Given a graph $G = (V, E)$, a color space $|C|$ and color lists $\mathcal{L}(v) \subseteq |C|$ for each $v \in V$, a list-coloring of $G$ is a proper $C$-coloring such that each node is colored with a color from its list. In this paper we consider the $(\deg + 1)$-list coloring problem, where the list sizes are $|\mathcal{L}(v)| = \deg(v) + 1$. Note that just as the $(\Delta + 1)$-coloring problem the $(\deg + 1)$-list coloring problem admits a simple sequential greedy algorithm.

While in the LOCAL model the diameter of the graph is an trivial upper bound on the time needed to solve any graph problem, it is not clear per se whether a small diameter also helps for solving problems in the
CONGEST model. So far, there are a few examples where a small diameter helps in the CONGEST model. In particular we know that the problems of computing a maximal independent set problem, a sparse spanner, and an \((1 + \varepsilon)\log \Delta\)-approximation of minimum dominating set can all be solved deterministically in time \(D \cdot \text{polylog} \ n\) in the CONGEST model [CPS17, GK18, DKM19]. Other problems cannot profit from small diameter, e.g., verifying or computing a minimum spanning tree requires \(\Omega(D)\) communication graph is a complete graph on the nodes or almost exactly (maximum independent set) requires \(\tilde{\Omega}(s)\) solving many optimization problems exactly (minimum dominating set, vertex cover, chromatic number) or almost exactly (maximum independent set) requires \(\Omega(n^s)\) rounds in the CONGEST model even if the diameter of the graph is constant [CKP17, BCD+19].

We prove the following theorem.

**Theorem 1.1 (simplified).** There is a deterministic CONGEST algorithm that solves the list-coloring problem for instances \(G = (V, E)\) with \(|L(v)| \geq \deg(v) + 1\) in time \(O(D \cdot \text{log} n \cdot \text{log}^3 \Delta)\).

The algorithm to obtain Theorem 1.1 is based on a similar basic strategy as the algorithms in [CPS17, GK18, DKM19], which achieve similar time complexities for other graph problems. More specifically, the proof of Theorem 1.1 relies on designing a 0-round randomized process that in expectation colors a constant fraction of the vertices. Then, we derandomize this process and iterate \(O(\text{log} n)\) times until all nodes are colored. In Section 1.4 we elaborate on the challenges that occur with this approach and how we approach them.

The most important implication of the result in Theorem 1.1 is that it can be lifted to an efficient, i.e., \(\text{polylog} n\) time algorithm, for general graphs, even for graphs with large diameter.

**Corollary 1.2.** There is a deterministic CONGEST algorithm that solves any \((\text{degree} + 1)\)-list-coloring problem in \(O(\text{log}^8 n + \text{log}^7 n \cdot \text{log}^3 \Delta)\) rounds.

The proof of Corollary 1.2 is based on iterating through the color classes of a suitable network decompositions and applying Theorem 1.1 on the clusters of the same color [AGLP89a, RG19]. Many network decomposition algorithms, in particular the one in [RG19], only compute so called weak-diameter network decompositions in which the diameter of components is only small if edges and vertices outside the component can be used for communication. One needs additional care to use these decompositions in the CONGEST model. For more details on the definition of a suitable network decomposition we refer to Section 3 and [RG19]. We want to point out that improvements in computing such network decompositions immediately carry over to Corollary 1.2.

Corollary 1.2 is a drastic improvement over the state of the art even for the standard \((\Delta + 1)\)-coloring problem: Surprisingly until the beginning of 2019 the best deterministic CONGEST algorithm for the \((\Delta + 1)\) coloring problem was the \(O(\text{log}^{3/4} \Delta + \text{log}^2 n)\) algorithm by Barenboim [Bar15, BEG18]. Even though the objective of [Bar15] was to optimize the runtime mainly as a function of the maximum degree \(\Delta\), the paper also provided the fastest known algorithm if the runtime is solely expressed as a function of the number of nodes, i.e., it provided an \(O(n^{3/4})\) round CONGEST algorithm. In 2019, the runtime for \((\text{degree} + 1)\)-list coloring was improved to \(2^{O(\text{log} \Delta)} \cdot \text{log} n = 2^{O(\text{log} \text{log} n)}\) rounds [Kuh19]. The algorithm in [Kuh19] does work in the CONGEST model but it does not rely on network decompositions. Hence its runtime does not improve with the recent breakthrough result by Rozhoň and Ghaffari [RG19].

### 1.2 Our Contributions in the CONGESTED CLIQUE and MPC

The techniques of Theorem 1.1 can also be adapted to yield the best known deterministic algorithms in the CONGESTED CLIQUE and the MPC model.

**The CONGESTED CLIQUE model [LPPP03]:** The CONGESTED CLIQUE model differs from the CONGEST model in the way that the input graph might be different from the communication network. Given an input graph \(G = (V, E)\) consisting of \(n\) nodes, in each round, each node \(u \in V\) can send a message of size \(O(\text{log} n)\) to each other node \(v \in V\) in the graph (i.e., although \(G\) might be an arbitrary graph, the communication graph is a complete graph on the nodes \(V\)). Initially, each node only knows its neighbors in
G. More specifically, we consider the UNICAST CONGESTED CLIQUE model, in which nodes are allowed to send different messages to each other node in each round.

Since the CONGESTED CLIQUE allows direct all-to-all communication, we can avoid the diameter term in the runtime of Theorem 1.1.

**Corollary 1.3 (CONGESTED CLIQUE).** There is a deterministic CONGESTED CLIQUE algorithm that solves the (degree + 1)-list-coloring problem in time $O(\log^3 \Delta)$.

The $O(\log n)$ factor is turned into a $O(\log \Delta)$ factor as we can send a subgraph to a single vertex of the clique and solve the problem locally as soon as the subgraph consists of at most $n/\Delta$ vertices. The additional $\log \Delta$ factor that we save in the runtime is due to compressing $\Omega(\log \Delta)$ rounds of our derandomization procedure into $O(1)$ rounds in the CONGESTED CLIQUE model.

We emphasize that Corollary 1.3 does not improve the state of the art as [Par18] provides a $O(\log \Delta)$ deterministic algorithm. However, as the result immediately follows from our CONGESTED algorithm we still like to mention it.

**The MPC model [KSV10, ANOY14]:** In the MPC model, an input graph $G = (V, E)$ is distributed in a worst case manner among $M = \tilde{O}(\frac{|V| + |E|}{S})$ machines, each having a memory of $S$ words. $S$ is a parameter of the model, a word consists of $O(\log |V|)$ bits and $\tilde{O}$ hides $\text{poly} \log |V|$ factors that can be chosen arbitrarily by the designer of an algorithm. Time progresses in synchronous rounds in which machines exchange messages. In one round, every machine is allowed to send a (different) message to each other machine such that the size of all messages sent and received by a machine does not exceed its local memory. Additionally, each machine can perform an arbitrary local computation on its stored data. At the end of computation, each machine outputs a part of the solution which may not exceed its local memory. So for the coloring problem each machine is responsible for the output of some of the nodes and has to output their colors. The complexity of a deterministic MPC algorithm is the number of synchronous rounds until the problem is solved. We say that the MPC model works with linear memory if $S = \tilde{O}(|V|)$ and with sublinear memory if there exists some constant $0 < \delta < 1$ and $S = \tilde{O}(n^\delta)$.

The algorithm of Corollary 1.3 can also directly be implemented in the deterministic MPC model with linear memory; however, additional care is needed to reason that none of the steps of the algorithm ever exceed the memory of a machine, i.e., a machine can never send, receive nor store more than $S = \tilde{O}(n)$ words.

**Theorem 1.4 (MPC, linear memory).** There is a deterministic MPC algorithm that solves the (degree + 1)-list-coloring problem in $O(\log^3 \Delta)$ rounds with linear memory.

We remark that a similar result can also be proven in the deterministic MPC model with sublinear memory, but we leave a formal proof of this result for a later version of this paper.

### 1.3 Related Work

We continue with a short overview on the most important distributed graph coloring algorithms in the LOCAL and CONGEST model. For a more detailed discussion of previous work on distributed graph coloring we refer to [BE13, Kuh19] for deterministic algorithms and [CLP18] for randomized algorithms.

**Deterministic Distributed Coloring:** The work on distributed coloring started more than 30 years ago with work on several symmetry breaking problems in the parallel setting [ABI86, CV86, GPS88, Lub86] and a seminal paper by Linial [Lin87] that introduced the LOCAL model for solving graph problems in a distributed setting. Linial showed that a graph can be colored (deterministically) in $O(\log^* N)$ rounds with $O(\Delta^2)$ colors and that $\Omega(\log^* N)$ rounds are necessary even for graphs of degree 2 where $N$ is the size of the identifier space. Even though these algorithms and lower bounds were devised for the LOCAL model, all of them also directly apply to the CONGEST model. If randomization is allowed, already the early

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2The function $\log^* x$ denotes the number of iterated logarithms needed to obtain a value at most 1, that is, $\forall x \leq 1 : \log^* x = 0$, $\forall x > 1 : \log^* x = 1 + \log^* \log x$. As most results in the area assume that the identifier space $N$ is of size $\text{poly}(n)$ Linial’s lower bound and upper bounds are usually used as $\Omega(\log^* n)$ and $O(\log^* n)$; we do the same.
works imply that a \((\Delta + 1)\)-coloring can always be computed in \(O(\log n)\) rounds, even in the CONGEST model [ABI86, Lub86, Lin87, BKM19].

The focus of our current work is to express the complexity of distributed graph coloring in the CONGEST model as a function of \(n\), i.e., the number of nodes of the network. Despite ample attention to the problem in the LOCAL model, the \(2^{O(\sqrt{\log n})}\)-time solution of [PS95] has been the state of the art until the recent breakthrough by Rozhoň and Ghaffari [RG19], which provided the first polylog \(n\)-time algorithm. However, to the best of our knowledge and somewhat surprisingly there are no published results on the \((\Delta + 1)\)-coloring problem in the CONGEST model, when runtime is expressed solely as a function of \(n\). But there has been extensive work on determining the problem’s round complexity in terms of the maximum degree \(\Delta\) and most of these algorithms also work in CONGEST: A simple single-round color elimination scheme combined with Linial’s \(O(\Delta^2)\)-coloring algorithm provided an \(O(\Delta \log \Delta + \log^* n)\) algorithm [SV93, KW06]. This was improved to \(O(\Delta + \log^* n)\) rounds by using a divide-and-conquer approach based on defective colorings [BE09][Kuh09a]. The first and only sublinear in \(\Delta\) algorithm in the for \((\Delta + 1)\)-coloring was obtained by Barenboim and used \(O(\Delta^{3/4} + \log^* n)\) rounds [Bar15, BEG18]. The current state of the art in the LOCAL model uses \(O(\sqrt{\Delta} + \log^* n)\) rounds, but it does not extend to CONGEST [FHK16, BEG18]. There are faster algorithms if the final coloring can use more than \(\Delta + 1\) colors: [BE10] shows that coloring with \(O(\Delta^{1+\epsilon}) \gg \Delta + 1\) colors for some constant \(\epsilon > 0\) can be done in \(O(\log \Delta \log n)\) rounds. If one desires to reduce the allowed communication from \(O(\log n)\) bits per round to a single bit per round, Barenboim, Elkin and Goldenberg provided algorithm that uses \(O(\Delta + \log n + \log^* n)\) rounds and colors with \(O(\Delta)\) colors [BEG18]. In its original version the algorithm computes an \(O(\Delta)\)-coloring in \(O(\Delta + \log^* n)\) rounds in the standard CONGEST model.

While there has been extensive progress on upper bounds, the original \(\Omega(\log^* n)\) lower bound by Linial is still the best known lower bound on the \((\Delta + 1)\)-coloring problem. It was recently shown that in a weak version of the LOCAL model (called the SET-LOCAL model), the \((\Delta + 1)\)-coloring problem has a lower bound of \(\Omega(\Delta^{1/3})\) [HMKS16].

Randomized Distributed Coloring: There has also been a lot of work on understanding the randomized complexity of the distributed coloring problem [KSOS06, SW10, BEPS12, PS13, HSS16, GHKM18, CLP18]. Most of the work focuses on the LOCAL model and a particularly important contribution was provided by Barenboim, Elkin, Pettie, and Schneider [BEPS12], who introduced the so-called graph shattering technique to the theory of distributed graph algorithms. The paper shows that a \((\Delta + 1)\)-coloring can be computed in time \(O(\log \Delta) + 2^{O(\sqrt{\log \log n})}\) in the LOCAL model. [Gha19] showed that the same can be done in the CONGEST model and in both algorithms the \(2^{O(\sqrt{\log \log n})}\) term can most likely be reduced with the result from [RG19]. The state of the art for randomized \((\Delta + 1)\)-coloring in the LOCAL model is the graph shattering based polylog log \(n\) algorithm by Chang et al. [CLP18, RG19] that does not translate to the CONGEST model.

\((\Delta + 1)\)-Coloring in the CONGEST CLIQUE and MPC: In the special case of \(\Delta = O(n^{1/3})\) the MIS algorithm in [CPS17] combined with a well-known reduction from the \((\Delta + 1)\)-coloring problem to the MIS problem [Lub86, Lin87] can be used to deterministically compute a \((\Delta + 1)\)-coloring in \(O(\log \Delta)\) rounds. Building up on the result from [CPS17] Parter provided an \(O(\log \Delta)\) deterministic algorithm [Par18]. All other results in the CONGEST CLIQUE and the MPC model are randomized: [CFG+19] provides a \(O(1)\)-round algorithm in the CONGEST CLIQUE and a \(O(\sqrt{\log n})\) round randomized algorithm for the MPC model with sublinear memory. Previously Assadi et al. [ACK19] provided an \(O(1)\) randomized algorithm in the MPC model in the linear memory regime, i.e., with memory \(O(n)\).

Derandomization in the CONGEST Model: As pointed out earlier [CPS17, GK18, DKM19] use a similar derandomization strategy for other graph problems. [CPS17] computes an MIS in \(O(D)\) rounds and uses the same strategy to obtain deterministic algorithms for spanners. [GK18] computes a \(O(2^2 n)\) approximation for the minimum dominating set problems and [DKM19] computes a \((1 + \epsilon) \log \Delta\)-approximation for the minimum dominating set problem for any constant \(\epsilon > 0\). We cite [CPS17]:

*This work opens a window to many additional intriguing questions. First, we would like to see many more local problems being derandomized despite congestion restrictions.*
Despite the fact that [CPS17, GK18, DKM19] and the current paper use the same derandomization strategy, applying it to further graph problems is unfortunately non-black-box.

Kawarabayashi and Schwartzman [KS18] also use the concept of derandomization in the CONGEST model. Their approach relies on iterating through the color classes of a given coloring; by only considering a suitably chosen subgraphs and defective colorings they obtain several algorithms for various cut problems including a deterministic \(O(\varepsilon^{-2} \log \Delta + \log^* n)\) round algorithm for an \((1/2 - \varepsilon)\)-approximation of the max cut problem.

### 1.4 Our Derandomization in a Nutshell

**Randomized \((1+\varepsilon)\Delta\)-coloring.** We start by sketching a simple algorithm to compute a coloring with \((1+\varepsilon)\Delta\) colors, where \(\varepsilon > 0\) is some arbitrarily small positive constant. In this case, we can use a simple divide-and-conquer approach to color the graph. We first give a randomized algorithm, which we afterwards derandomize. If the maximum degree \(\Delta\) of a given graph is at most \(\text{polylog} n\), we can use a standard algorithm to deterministically color the graph with \(\Delta + 1\) colors in polylogarithmic time (see, e.g., [BE13]). If \(\Delta\) is sufficiently large, there is a trivial randomized algorithm to partition the set of nodes into two parts such that the maximum degree of the induced subgraph of each part is at most \((1 + \varepsilon/(2 \log \Delta)) \cdot \Delta\), with high probability; just assign each node to one of the two parts with probability \(1/2\) each. We can then divide the original color space into two equal subspaces and recursively color the two parts of the graph by using disjoint sets of colors. In order to reduce the maximum degree of each subgraph to polylog \(n\), we need to run \(\log \Delta - O(\log \log n)\) phases of this degree halving step. In order to still have sufficiently many colors once the remaining graphs have polylogarithmic maximum degree, the initial number of colors has to be at least \((1 + \varepsilon/(2 \log \Delta)) \cdot \Delta\). Note that a single phase of the algorithm fixes a single bit of the color of each node. For a randomized implementation of this algorithm, it is of course not necessary to fix the bits of the colors one after another. Each node can directly fix the first \(\log \Delta\) bits of its color to reduce the problem to a problem on graph of polylogarithmic degree. However, we need this bitwise fixing of the colors to be able to efficiently derandomize the above algorithm.

**Derandomization strategy.** It is straightforward to efficiently derandomize each of the degree halving steps of the above algorithm by using the method of conditional expectations (see e.g., [MU05, Chapter 6.3]) and by employing recent distributed derandomization techniques developed in [CPS17, GHK18, RG19]. Consider one degree halving step, where each node \(v \in V\) participates in some subgraph with maximum degree \(\Delta'\). Let \(X_v \in \{0,1\}\) be a random variable, which is set to 1 if the degree of \(v\) after the halving step is more than \((1 + \varepsilon/(2 \log \Delta)) \cdot \Delta\) and let \(X = \sum_{v \in V} X_v\). If \(\Delta'\) is sufficiently large, a straightforward application of a Chernoff bound shows that \(E[X_v] < 1/n\) and thus \(E[X] < 1\). Note that this still works if the random choices of the nodes are not completely independent. Specifically, if the random choices are \(k\)-wise independent for some \(k = \text{polylog} n\) (i.e., for any subset of \(k\) nodes, the choices are independent), we can still apply a Chernoff bound and obtain \(E[X_v] < 1/n\) [SSS95]. It is well-known that a set of \(n\) polylog \(n\)-wise independent random bits can be generated from only polylog \(n\) independent random bits [Vad12]. The algorithm can therefore be implemented with a shared random seed of polylog \(n\) bits and we can use the method of conditional expectations to deterministically find an assignment to these polylog \(n\) shared bits for which the algorithm is successful. When implementing the algorithm in the CONGEST model, we deterministically fix the polylog \(n\) bits of the random seed one-by-one, where fixing a single bit involves a global aggregation for computing the conditional expectations of \(E[X]\) based on setting the bit to 0 or 1. One bit of the random seed can therefore be fixed in \(O(D)\) time in the CONGEST model. In combination with the new polylogarithmic-time network decomposition algorithm of [RG19], the running time of the degree halving algorithm can be reduced from \(D \cdot \text{polylog} n\) to only \(\text{polylog} n\). The idea of getting an \(\tilde{O}(D)\)-time deterministic CONGEST algorithm by derandomizing a randomized algorithm that only uses \(\text{polylog} n\) shared random bits has been introduced by Censor-Hillel, Parter, and Schwartzman in [CPS17] and it has afterwards also been used in [GK18, DKM19].

**The need of list coloring.** The above algorithm is simple, it however inherently requires some slack in the number of colors and cannot be used to color with \(\Delta + 1\) colors. Even if we are able to compute a perfect split of the maximum degree in each recursion step, we need an additive slack of 1 to color each of the \(\Delta/\text{polylog} n\) subgraphs on the lowest recursion level and we thus need at least \(\Delta + \Delta/\text{polylog} n\) colors. By using techniques developed in [Bar15, FHK16, Kuh19], it is possible to efficiently turn an algorithm that computes a coloring with slack into a \((\Delta + 1)\)-coloring algorithm. However, in order to apply these
techniques, the \((1 + \epsilon)\Delta\)-coloring (or also even just \(2\Delta\)-coloring) algorithm has to solve the more general list coloring problem. In an instance of an \(2\Delta\)-list coloring problem, each node \(v \in V\) is initially given an arbitrary list \(L_v\) of \(|L_v| = 2\Delta\) colors. The objective is to compute a proper vertex coloring of the graph such that every node \(v \in V\) is assigned a color from its list \(L_v\). It is formally shown in [Kuh19] that if a distributed \(2\Delta\)-list coloring algorithm is given, it can be turned into a \((\text{degree} + 1)\)-list coloring algorithm at the cost of only an \(O(\log \Delta)\) factor in the running time (moreover, this reduction is deterministic).

Our approach. Note that it is not clear how one should apply the recursive degree halving approach with list coloring. We would need to be able to divide the global color space into two parts such that up to a small \((1 + o(1))\)-multiplicative error, each list is divided into two equal parts, a problem that cannot be solved locally.\(^3\) Instead, we try to apply a different approach, which is inspired by the algorithm of [Kuh19]. We still use a divide and conquer approach, where we divide the global color space in two equal parts in each iteration and each node \(v\) decides to participate in one of the parts (which reduces \(v\)'s list size). However, we do not try to also equally divide the degrees of the nodes. Instead, we roughly aim to make sure that for most nodes the ratio between its list size and its degree does not decrease.

More formally, let the global color space be \([C]\) for some integer \(C\) (i.e., for each \(v \in V\), we have \(L_v \subseteq [C]\)). Each color thus has a binary representation of \(\log C\) bits. We recursively divide the color space \([C]\) by using the bit representation of the colors. That is, for each node \(v \in V\), we fix the color of \(v\) bit by bit. Note that in that sense, the approach can be seen as a natural extension of the simple \((1 + \epsilon)\Delta\)-coloring algorithm sketched at the beginning of Section 1.4. After each vertex has fixed all \(\log C\) bits it has divided its list until it consists of a single color, i.e., it has merely chosen one color from its list. When each node chooses a random color from its list, the expected number of conflicts \(X_v\) of a node \(v\) can be computed as

\[
E[X_v] = \sum_{u \in V} \frac{|L_u \cap L_v|}{|L_u| \cdot |L_v|} \leq \Delta \cdot \frac{1}{2\Delta} = \frac{1}{2}
\]  

(1)

Note that for (1) to hold, we need only pairwise independent choices of the colors. It would be natural to use the expected number of conflict \(\sum_{v \in V} E[X_v]\) as a “potential function” to measure the progress of the algorithm. The derandomization by the method of conditional expectations would then keep the value of this potential function upper bounded by its initial value. As we have \(\sum_{v \in V} E[X_v] \leq \frac{n}{2}\), this would imply that when all node have chosen a single color, at least half the nodes can keep their color. The whole process can then be repeated for \(O(\log n)\) iterations to color all vertices. In the CONGEST model, we can however not directly use \(\sum_{v \in V} E[X_v]\) for the method of conditional expectations. In order to compute the conditional probability of \(E[X_v], v\) needs to know the lists of its neighbors, and this information is too expensive to acquire in the CONGEST model. Instead, we replace the exact value \(E[X_v]\) by an upper bound and define a potential value \(\Phi(v) := \sum_{u \in \Gamma_v} 1/|L_u|\) (after fixing some prefix of the color, this sum is only over the neighbors with the same prefix). We still have \(\sum_{v \in V} \Phi(v) \leq \frac{n}{2}\) and we can thus again use the method of conditional expectations to guarantee that at least half the nodes have no conflict and can thus keep their color.

Shorter Random Seeds. In [CPS17, GK18, DKM19] the length of the shared random seed is \(\text{polylog} n\). As the seed length appears as a factor in the runtime of the resulting deterministic algorithms one wishes to keep it as short as possible. In our algorithms we manage to reduce the seed length to \(O(\log \Delta + \log \log C)\) bits, in particular the seed length is independent of \(n\). The main ingredient for a shorter seed length is the observation that pairwise independent random coins for adjacent nodes are sufficient in our randomized algorithms and these coins can be produced from a random seed whose length does not depend on \(n\). This observation might be helpful for derandomizing other algorithms.

\(^3\)It is in fact possible to solve this problem locally if shared randomness is available and it is even sufficient to have only \(\text{polylog} n\) bits of shared randomness. Such a list splitting algorithm with shared randomness can be derandomized in time \(O(D)\). When combined with a network decomposition in a proper way, this would indeed lead to an alternative \(\text{polylog} n\)-time deterministic CONGEST algorithm to solve the \((\text{degree} + 1)\)-coloring problem. The algorithm is however not simpler than the approach we are following, and it would result in a higher polylogarithmic complexity.
2 Degree+1 List Coloring in Diameter Time

Throughout, let $D$ denote the diameter of a graph. Often we run algorithms on subgraphs of a graph; however, all our algorithms can be implemented such that $D$ always refers to the diameter of the original communication graph. For $C \in \mathbb{N}$ we introduce the notation $[C] := \{0, \ldots, C - 1\}$. A $C$-coloring of a graph $G = (V, E)$ is a map $\phi : V \to [C]$. A coloring therefore defines a partition of $V$ into color classes $V_1 \cup \ldots \cup V_C = V$. We call the coloring proper if adjacent nodes have different colors, i.e., for any edge $\{u, v\} \in E$ we have $\phi(u) \neq \phi(v)$. Whenever we talk about colorings, we always refer to proper colorings unless explicitly stated otherwise. One important class of improper colorings are so-called defective colorings.

**Definition 2.1 (Defective Coloring).** The defect of a (possibly improper) $C$-coloring is the maximum degree of any graph induced by one of the $C$ color classes. A $C$-coloring with defect $d$ is said to be a $d$-defective $C$-coloring.

Note that a 0-defective coloring is a proper coloring. We will later need the following well known results.

**Theorem 2.1 ([Lin92, Kuh09b]).** Let $\Delta$ be an upper bound on the maximum degree of the network graph. There are deterministic CONGEST algorithms that

- compute an $O(\Delta^2)$-coloring in time $O(\log^* n)$,
- given a $k$-coloring with $k \leq \Delta^3$, compute an $O(\Delta^2)$-coloring in $O(1)$ rounds,
- for any $d \in \{1, \ldots, \Delta\}$ compute an $O(\Delta^2/d^2)$-coloring with defect at most $d$ in time $O(\log^* n)$. If a proper k-coloring is given this runtime can be reduced to $O(\log^* (k/\Delta))$.

Given a graph $G = (V, E)$, a color space $[C]$ and color lists $L(v) \subseteq [C]$ for each $v \in V$, a list-coloring of $G$ is a proper $C$-coloring such that each node is colored with a color from its list. In this paper we consider the $(\text{degree} + 1)$-list-coloring problem, where $|L(v)| = \deg(v) + 1$ for each $v \in V$. Throughout the paper we assume that each color from each node's list fits in $O(1)$ messages in the CONGEST model, i.e., we only consider list-coloring instances with $C \leq \text{poly } n$, where $[C]$ is the color space of the instance and $n$ is the number of nodes of the communication network $G$, even if we—and we do so often—consider list-coloring instances on subgraphs of $G$. Whenever we do not explicitly mention the color space we assume $C = \text{poly } \Delta$.

As our main technical contribution, we show that we can list-color a constant fraction of the nodes of a graph if each node has a multiplicative slack, i.e., the size of each node's color list exceeds the maximum number of neighbors by a constant factor. More formally, we prove the following theorem.

**Theorem 2.2 (List Coloring with Slack).** There is a deterministic CONGEST algorithm that given a list-coloring instance $G = (V, E)$ with color space $[C]$, lists $L(v) \subseteq [C]$ for which $|L(v)| \geq 2\Delta$ for all $v \in V$ and an initial $O(\Delta^2)$-coloring of $G$, list-colors at least $1/4$ of the nodes in $O(D \cdot \log C \cdot (\log \Delta + \log \log C))$.

When the result is applied to a subgraph of a communication graph $G$, the maximum degree $\Delta$ refers to the maximum degree of the subgraph but the diameter $D$ refers to the diameter of $G$.

Note that the given initial $O(\Delta^2)$-coloring is only used for symmetry breaking purposes and does not relate to the lists of the nodes. In the algorithm of Theorem 2.2, every node deterministically (and tentatively) selects a color from its list such that at least $1/4$ of the nodes do not have a neighbor that selected the same color, i.e., at least $1/4$ of the nodes can permanently keep their selected color. We now give a description how the nodes select colors from their list: All colors in the color space $[C]$ can be represented by bitstrings of length $\lceil \log C \rceil$. The color selection process consists of $\lceil \log C \rceil$ phases $1, \ldots, \lceil \log C \rceil$, where the bits of the color of each node $v \in V$ are fixed one by one. That is, each node $v \in V$ starts with an empty prefix and after phase $\ell \in \{1, \ldots, \lceil \log C \rceil\}$, each node $v$ has a prefix $s_\ell(v)$ of length $\ell$. After phase, $\ell$, all colors of $v$’s list that have $s_\ell(v)$ as a prefix in their bit representation are still candidates for $v$ in the selection process. At the beginning, when $v$ starts with an empty prefix, all colors in $v$’s list are still candidates. Then, the space of possible colors is decreased until after $\lceil \log C \rceil$ phases, the prefix $s_\ell(v)$ coincides with a single color of $v$’s list; this is the color that the node tentatively selects and keeps if no neighbor selected the same color.

To measure the quality of this coloring process, we introduce a potential function that captures the ‘usefulness’ of the prefixes for our purpose; informally, the potential relates (but does not equal!) to the
expected number of monochromatic edges if each node chooses a random of the candidate colors from its list (i.e., a random color that is consistent with the current prefix). The initial potential, i.e., when prefixes are of length zero, is small and a small potential at the end, i.e., when prefixes are of length \( \lceil \log C \rceil \) and determine a single color, implies that a large fraction of nodes can keep their selected color. We show that using biased coin flips to determine the next bit of the prefix yields a zero round randomized algorithm that does not increase the potential in expectation where the bias of the coin of a node depends on the current list of candidate colors of that node. Then we use the method of conditional expectation to deterministically choose good coin flips.

We use a technique that was introduced in [FHK16] and also used in [Kuh19]. It reduces the list coloring problem without multiplicative slack to list coloring instances with multiplicative slack such that we can apply Theorem 2.2. In order to generate list coloring instances with slack, one first reduces the maximum degree by computing a defective coloring. One then iterates through the color classes of this defective coloring and one always extends the list-coloring computed on already processed color classes; coloring only nodes of one color class incurs enough slack—at least for high degree nodes—to apply Theorem 2.2. In Section 2.2 we formally prove the following theorem.

**Theorem 1.1** (List Coloring without Slack). There is a deterministic CONGEST algorithm that solves the list-coloring problem for instances \( G = (V,E) \) with \( L(v) \subseteq [C] \) and \( |L(v)| \geq \deg(v) + 1 \) in time \( O(D \cdot \log n \cdot \log \Delta \cdot \log C(\log \Delta + \log \log C)) \).

In the introduction we presented a simplified version of Theorem 1.1 by setting \( C = \text{poly} \Delta \). The \( O(\Delta^2) \)-coloring needed to apply Theorem 2.2 can be computed with Linial’s algorithm (cf. Theorem 2.1) in \( O(\log^* n) \) rounds.

**Remark.** When applying Theorem 1.1 to disconnected subgraphs the diameter term in the runtime can be substituted by the maximum diameter of the connected components.

Theorem 2.2 is proven in Section 2.1 and Theorem 1.1 is proven in Section 2.2.

### 2.1 Proof of Theorem 2.2: List Coloring with Slack

We quickly (with the purpose of introducing notation) review the general algorithmic idea and we encourage the reader to also consult the high level paragraph after Theorem 2.2 in Section 2.

**General Algorithmic Idea:** In our algorithm, each node tentatively selects a color. Nodes not having a neighbor with the same tentative color keep their color permanently, all other nodes discard it. Choosing the tentative colors works as follows: Given a list-coloring instance \( G = (V,E) \) with color space \([C]\), each color can be represented by a bitstring of length \( \lceil \log C \rceil \). Our algorithm operates in phases in which we determine one further bit of each nodes color. That is, each \( u \in V \) maintains a bitstring \( s(u) \) with the property that \( s(u) \) is the prefix of some color(s) in \( L(u) \), starting with \( s(u) = s_0(u) \) being the empty string and successively extending the prefix \( s(u) \) by one bit per phase. We write \( s_\ell(u) \) for the bitstring that node \( u \) has chosen after phase \( \ell \). The string \( s[\log C](u) \) corresponds to the tentative color that \( u \) selects. For \( \ell = 1, \ldots, \lceil \log C \rceil \) we define

\[
L_\ell(u) := \{ x \in L(u) \mid \text{prefix of } x \text{ is } s_\ell(u) \}
\]

as the set of colors in \( L(u) \) that start with \( s_\ell(u) \). This is the set of \( u \)'s candidate colors at the end of phase \( \ell \). To measure the ‘usefulness’ of prefixes we introduce the following potential function

\[
\Phi_\ell(u) := \sum_{v \in V} \mathbb{1}_{s_\ell(v) = s_\ell(u)} \cdot \frac{1}{|L_\ell(v)|}.
\]

At the beginning, when all prefixes are empty, we have \( |L_0(u)| = |L(u)| \geq 2\Delta \) for all \( u \in V \). It follows that \( \Phi_0(u) \leq 1/2 \) for all \( u \in V \) and thus \( \sum_{u \in V} \Phi_0(u) \leq n/2 \). We will give an algorithm that extends all prefixes bitwise while keeping the overall increase of the potential small. In particular, we show that when all \( \lceil \log C \rceil \) bits are fixed, the sum of the potentials of all nodes is at most \( 3n/4 \). It follows that at least \( 1/4 \) of the nodes \( u \) have \( \Phi_\ell(u) < 1 \) with \( \ell = \lceil \log C \rceil \). At this stage all nodes have selected a single candidate color,
i.e., $|L_{\ell}(u)| = 1$ for all $u \in V$. Thus $\Phi_{\ell}(u)$ equals the number of neighbors $v$ of $u$ with the same candidate color. So each node $u$ with $\Phi_{\ell}(u) < 1$ in fact has $\Phi_{\ell}(u) = 0$, i.e., it has no neighbor with the same candidate color and so it can permanently keep its own candidate color.

In Section 2.1.1 we provide a 0-round randomized algorithm for choosing the next bit of each node’s prefix such that, in expectation, there is no increase in the node’s potential; then we show that the increase of the node’s potential can be kept very small (in expectation) if the 0-round randomized algorithm is not run with (independent) randomness in each node but only with very little shared randomness, i.e., with a shared short random seed. In Section 2.1.2 we use the method of conditional expectation and a BFS tree to derandomize this algorithm, i.e., to pick a seed that only incurs a minimal increase in the potential.

### 2.1.1 Extending Prefixes: Zero Round Randomized Algorithms

To fix bit $\ell$ of all prefixes (that we describe by Algorithm 1) each node flips a coin to determine its $\ell$-th bit. The bit and also the coin equals 1 with probability $p_{\ell} := \frac{k_1}{|L_{\ell-1}(u)|}$ where $L_{\ell-1}(u)$ is the list of current candidate colors and $k_1 := |\{x \in L_{\ell-1}(u) \mid x[\ell] = 1\}|$ is the number of candidate colors whose $\ell$-th bit equals 1. Thus $p_{\ell}$ is the fraction of candidate colors whose $\ell$-th bit equals 1. This process can be seen as a slowed down version of selecting a color from the initial list uniformly at random as iterating this process for $\lceil \log C \rceil$ times yields the same probability for each color to be selected. However, we do not know how to immediately derandomize the non slowed down process.

#### Algorithm 1 Randomized One Bit Prefix Extension

**Input:** Bitstring $s_{\ell-1}(u)$ of length $\ell - 1$ for all $u \in V$

1. **for** each node $u$ in parallel **do**
   2. $p_u := \frac{k_1}{|L_{\ell-1}(u)|}$ where $k_1 := |\{x \in L_{\ell-1}(u) \mid x[\ell] = 1\}|$
   3. **Coin Flip:** Set $s_\ell(u) = s_{\ell-1}(u) \odot 1$ with probability $p_u$ and $s_\ell(u) = s_{\ell-1}(u) \odot 0$ otherwise ($\odot$ represents the concatenation of strings)

**Lemma 2.3.** Let $\ell \leq \lceil \log C \rceil$ and assume the prefixes $s_{\ell-1}(v)$ are fixed for all nodes $v$. Let all nodes choose the $\ell$-th bit according to Algorithm 1. Then we obtain

$$E[\Phi_{\ell}(u)] \leq \Phi_{\ell-1}(u)$$

if the coin flips in Algorithm 1 of adjacent nodes are independent. Furthermore, the list of candidate colors of each node never becomes empty.

**Intuition for Lemma 2.3:** To bound $E[\Phi_{\ell}(u)]$ let us first condition on the $\ell$-th bit of node $u$ being 1. Then, if a neighbor $v$ of $u$ with the same $(\ell - 1)$-length prefix as $u$ also flips its coin such that its $\ell$-th bit is 0, its contribution to the potential $\Phi_{\ell}(u)$ is zero as $u$ and $v$ will have different prefixes. If the neighbor $v$’s $\ell$-th bit turns out to be 1, $v$ will have a larger contribution to the potential $\Phi_{\ell}(u)$ than it had to $\Phi_{\ell-1}(u)$ because $v$’s list might have gotten shorter — it now consists of the sublist of colors that have a 1 as the $\ell$-th bit — and $v$’s contribution to the potential of $u$ is the reciprocal of the list length. The expected potential of $u$ does not increase because the probability $p_v$ that $v$ also chooses bit 1 cancels exactly with its increased contribution to the potential. However, to formally prove the result we need to introduce some notation.

**Proof of Lemma 2.3.** Let $B_u$ denote the random variable that equals the value of the $\ell$-th bit of $u$’s prefix. As we assume that the $(\ell - 1)$-length prefixes $s_{\ell-1}(v)$ are already fixed for all nodes $v$, we introduce the following shorter notations that contains all information that is relevant for the current proof:

$$A_u := \{v \in \Gamma(u) \mid s_{\ell-1}(u) = s_{\ell-1}(v)\}$$  

$$L_u^0 := \{x \in L_{\ell-1}(u) \mid x[\ell] = 0\} \text{ and } L_u^1 := \{x \in L_{\ell-1}(u) \mid x[\ell] = 1\}$$  

$$A_u^0 := \{v \in A_u \mid L(v) \cap L_u^0 \neq \emptyset\} \text{ and } A_u^1 := \{v \in A_u \mid L(v) \cap L_u^1 \neq \emptyset\}$$

$A_u$ contains all neighbors of $u$ that chose the same prefix as $u$ so far. $L_u^0$ contains those colors in $u$’s list starting with $s_{\ell-1}(u) \odot 0$. The set $A_u^0$ consists of those neighbors of $u$ that chose the same prefix as $u$ so far...
and have at least one color in their list starting with \( s_{\ell - 1}(u) \circ 0 \). The sets \( L^1_u \) and \( A^1_u \) are defined analogously. Note that we have \( L^1_{\ell - 1}(u) = L^0_u \cup L^1_u \) and \( A_u = A^0_u \cup A^1_u \) where the first union is always disjoint and the second one might not be disjoint.

These notations are helpful and natural because only nodes in \( A_u \) might contribute to \( \Phi_{\ell}(u) \). In particular, when we condition on \( \mathbb{E}[\Phi_{\ell}(u)] \) and, e.g., \( B_u = 1 \), then only nodes in \( A^1_u \) possibly contribute to \( \Phi_{\ell}(u) \) and their contribution equals either 0 or \( 1/|L^1_v| \).

We want to point out that \( L^1_u \), \( L^0_u \), \( A^1_u \), and \( A^0_u \) are fixed sets of nodes after prefixes of length \( \ell - 1 \) have been fixed; in particular the sets do not depend on the choice of the \( \ell \)-th bit.

We next compute the expectation of \( \Phi_{\ell}(u) \) conditioned on the two choices for \( B_u \). We obtain

\[
E[\Phi_{\ell}(u) \mid B_u = 1] = \sum_{v \in \Gamma(u)} E\left[ \mathbb{1}_{s_{\ell - 1}(u) \circ B_u = s_{\ell - 1}(v) \circ B_v} \frac{1}{|L^1_v|} \left| B_u = 1 \right. \right]
\]

\[
= \sum_{v \in A_u} \left[ \mathbb{1}_{B_u = B_v \mid L^1_v} \frac{1}{|L^1_v|} \left| B_u = 1 \right. \right]
\]

\[
= \sum_{v \in A_u} \Pr(B_v = 1 \mid B_u = 1) \cdot \frac{1}{|L^1_v|} + \sum_{v \in A_u \setminus A^1_u} 0 \quad \text{(ii)}
\]

\[
= \sum_{v \in A_u \setminus A^1_u} p_v \cdot \frac{1}{|L^1_v|} = \Phi_{\ell - 1}(u) \quad \text{(iii)}
\]

At (i) we used that \( \Pr(B_v = 1 \mid B_u = 1) = 0 \) if \( v \in A^0_u \setminus A^1_u \), at (ii) we used the pairwise independence of the events \( B_u = 1 \) and \( B_v = 1 \) and at (iii) we used the fact that \( p_v > 0 \) (due to \( v \in A^1_u \)) and \( |L^1_v| = p_v |L_v| \).

Analogously we obtain \( E[\Phi_{\ell}(u) \mid B_u = 0] \leq \Phi_{\ell - 1}(u) \). Combining both claims and using the law of total expectation we obtain

\[
E[\Phi_{\ell}(u)] = \Pr(B_u = 1) \cdot E[\Phi_{\ell}(u) \mid B_u = 1] + \Pr(B_u = 0) \cdot E[\Phi_{\ell}(u) \mid B_u = 0]
\]

\[
\leq p_u \cdot \Phi_{\ell - 1}(u) + (1 - p_u) \cdot \Phi_{\ell - 1}(u) = \Phi_{\ell - 1}(u)
\]

To prove that the list of candidate colors of each node \( u \) does not become empty, we show that for each \( \ell \leq \lceil \log \mathcal{C} \rceil \), \( s_\ell(u) \) is the prefix of at least one color in \( L(u) \) when all bits of all prefixes where chosen by iteratively applying Algorithm 1. This obviously holds for \( s_0(u) \) (as this is the empty string). Assume that it holds for \( s_{\ell - 1}(u) \), i.e., \( L_{\ell - 1}(u) \neq \emptyset \). As \( \ell \leq \lceil \log \mathcal{C} \rceil \), we know that \( |x| > \ell - 1 \) for all \( x \in L_{\ell - 1}(u) \), i.e., each color in \( L_{\ell - 1}(u) \) has an \( \ell \)-th bit.

There are three possible cases: (1) There is a color \( x \in L_{\ell - 1}(u) \) with \( x[\ell] = 0 \) and a color \( y \in L_{\ell - 1}(u) \) with \( y[\ell] = 1 \). In this case, \( s_\ell(u) \) will be either the prefix of \( x \) or the prefix of \( y \). (2) All colors \( x \in L_{\ell - 1}(u) \) have \( x[\ell] = 0 \). In this case we have \( p_u = 0 \) and so Algorithm 1 will set \( s_\ell(u) = s_{\ell - 1}(u) \circ 0 \) which is the prefix of all colors in \( L_{\ell - 1}(u) \). (3) All \( x \in L_{\ell - 1}(u) \) have \( x[\ell] = 1 \). This goes along similar lines as case (2).

We will not immediately derandomize the described procedure but a very similar one where nodes produce their biased random coins to represent the probabilities \( p_v \) from a common (short) random seed. Then, in the derandomization process, we will find a good seed from which nodes can determine the values of their coins.

Producing biased coins from the common random seed implies that not all probabilities can be produced, in fact, instead of having a coin that equals 1 with an arbitrary probability \( p_v \) we can only produce probabilities of the type \( k/2^b \) for some large enough \( b \) that we will chose later. That is, each \( p_v \) can only be approximated up to some \( \varepsilon = \Theta(2^{-b}) \). The next lemma shows that the expected increase of the potential can be kept small with these inaccurate probabilities.

**Lemma 2.4.** Let \( \ell \leq \lceil \log \mathcal{C} \rceil \) and assume the prefixes \( s_{\ell - 1}(v) \) are fixed for all nodes \( v \). Let all nodes choose the \( \ell \)-th bit according to Algorithm 1, but with the following adjustment: For each node \( v \), if \( p_v = 0 \) or \( p_v = 1 \), then \( v \) chooses 1 as its \( \ell \)-th bit with probability 0 or 1 respectively. For all other values of \( p_v \), \( v \) chooses 1 as its \( \ell \)-th bit with some probability in the interval \( \left[ \max\{0, p_v - \varepsilon\}, \min\{1, p_v + \varepsilon\} \right] \). Then we obtain

\[
E[\Phi_{\ell}(u)] \leq \Phi_{\ell - 1}(u) + \varepsilon \cdot \Delta
\]

(6)
if the coin flips of adjacent nodes are independent. Furthermore, the list of candidate colors of each node never becomes empty.

Proof. We use the same notations as in the proof of Lemma 2.3. We compute the expectation of $\Phi_i(u)$ conditioned on the two choices for $B_u$. We only mention those steps that either differ from the ones in Lemma 2.3 or need additional explanation why they also hold here.

\[
E[\Phi_i(u) \mid B_u = 1] \overset{(i)}{=} \sum_{v \in A_u} E\left[ \frac{1}{L_v^B} \mid B_u = 1 \right] \leq \sum_{v \in A_u} (p_v + \varepsilon) \cdot \frac{1}{p_v |L_v|} \\
\leq \Phi(u) + \varepsilon \sum_{v \in A_u} \frac{1}{p_v |L_v|} \overset{(ii)}{\leq} \Phi_{\ell-1}(u) + \varepsilon \cdot |A_u| \leq \Phi_{\ell-1}(u) + \varepsilon \cdot \Delta
\]

For equation $(i)$ we have to argue that the random variables on the right hand side are well defined, i.e., that there is no $v$ with $L_v^1 = \emptyset$ but $\Pr(B_v = 1) > 0$ (or $L_v^0 = \emptyset$ but $\Pr(B_v = 0) > 0$). This can not happen due to the assumption that nodes can represent the probabilities 0 and 1 exactly (without $\varepsilon$-slack). At $(ii)$ we used that $p_v = i/|L_v|$ for some $i \in \{1, \ldots, |L_v|\}$ if $v \in A_u$, so $p_v \geq 1/|L_v|$.

Analogously we obtain $E[\Phi_i(u) \mid B_u = 0] \leq \Phi_{\ell-1}(u) + \varepsilon \cdot \Delta$ and we deduce

\[
E[\Phi_i(u)] = \Pr(B_u = 1) \cdot E[\Phi_i(u) \mid B_u = 1] + \Pr(B_u = 0) \cdot E[\Phi_i(u) \mid B_u = 0] \\
\leq p_u \cdot (\Phi_{\ell-1}(u) + \varepsilon \cdot \Delta) + (1 - p_u) \cdot (\Phi_{\ell-1}(u) + \varepsilon \cdot \Delta) \\
= \Phi_{\ell-1}(u) + \varepsilon \cdot \Delta
\]

Because nodes can represent the probabilities 0 and 1 exactly, the proof that the list of candidate colors of each node never becomes empty goes along similar lines as in Lemma 2.3.

\[\square\]

2.1.2 Extending Prefixes Deterministically through Derandomization

To fix the $\ell$-th bit of all prefixes deterministically (cf. the general algorithmic idea in Section 2.1) we produce the nodes’ coins for Algorithm 1 from a short random seed, such that (1) the coins of each two adjacent nodes are independent, (2) the coins can represent the probabilities $p_v$ with a sufficient accuracy and (3) the common random seed is short enough to find a good seed deterministically and efficiently in the CONGEST model. To this end, we need the following result on how to compute biased coins from a random seed.

Definition 2.2 ([Vad12]). For $N, M, k \in \mathbb{N}$ such that $k \leq N$, a family of functions $\mathcal{H} = \{h : [N] \rightarrow [M]\}$ is $k$-wise independent if for all distinct $x_1, \ldots, x_k \in [N]$, the random variables $h(x_1), \ldots, h(x_k)$ are independent and uniformly distributed in $[M]$ when $h$ is chosen uniformly at random from $\mathcal{H}$.

Theorem 2.5 ([Vad12]). For every $a, b, k$, there is a family of $k$-wise independent hash functions $\mathcal{H} = \{h : \{0, 1\}^a \rightarrow \{0, 1\}^b\}$ such that choosing a random function from $\mathcal{H}$ takes $k \cdot \max\{a, b\}$ random bits.

Theorem 2.5 can be used to produce biased random coins. E.g., to produce $n$ pairwise independent random coins $C_0, \ldots, C_{n-1}$ with probabilities $p_0, \ldots, p_{n-1}$ with accuracy $1/\Delta$, i.e., the probabilities are a multiples of $1/\Delta$, we set $a = \log n$ and $b = \log \Delta$. Then, by Theorem 2.5, one can efficiently select a function $h_S : [n] \rightarrow [\Delta]$ from a uniformly chosen random seed $S$ of length $2 \cdot \max\{a, b\} = O(\log n)$. Here, for all $i \in [n]$ the random variable $h_S(i)$ (over the randomness of the random seed) is uniformly distributed in $[\Delta]$; further the random variables $h_S(0), \ldots, h_S(n-1)$ are pairwise independent. We obtain the desired $n$ biased coins, i.e., random variables over the randomness of the seed by defining

\[
C_i = \begin{cases} 
1, & \text{if } h_S(i)/\Delta < p_i \\
0, & \text{otherwise}
\end{cases}
\]

To prove the next lemma we want to produce $n$ coins, that is, one coin per node of the graph, such that the coins of adjacent nodes are independent. Thus adjacent nodes use random variables $h_S(i)$ and $h_S(j)$ with $i \neq j$ to define their coins. However, any two non-adjacent nodes $u$ and $v$ might use the same $i$, or more formally the same random variable $h_S(i)$, to define their respective coins $C_v$ and $C_u$ even if the probabilities $p_u$ and $p_v$ are not equal. The next lemma shows how to use the method of conditional expectation to find a good random seed that only incurs a small increase of the potential.
Lemma 2.6. There is a deterministic CONGEST algorithm that given an $O(\Delta^2)$-coloring of the graph, fixes the $\ell$-th bit of all prefixes in time $O(D \cdot (\log \Delta + \log \log C))$ such that

$$ \sum_{u \in V} \Phi_\ell(u) \leq \sum_{u \in V} \Phi_{\ell-1}(u) + \frac{n}{4|\log C|} $$

and the list of candidate colors of each node does not become empty for $\ell \leq |\log C|$. 

Proof. Let $K = O(\Delta^2)$ be the number of colors of the initial coloring of the graph and denote this coloring by $\psi$. We derandomize a variant of Algorithm 1 that relies on shared randomness. In particular, the biased coins that nodes need to flip to execute the algorithm are produced from a shared short random seed. Concretely, within Algorithm 1 each node $v$ wishes to produce a coin $C_v$ that equals 1 with probability $p_v := k_1/|L_v|$ where $k_1$ is the number of colors in $L_v$ whose $\ell$-th bit equals one. To produce these coins from a shared random seed we apply Theorem 2.5 with $a = |\log K|$, $b = |\log(4\Delta|\log C)|$ and $k = 2$. Then a random seed $S$ of length $d = 2 \cdot \max\{a, b\} = O(\log \Delta + \log \log C)$ defines $K$ random variables $h_S(0), \ldots, h_S(K-1)$ such that each of the random variables is uniformly distributed in $[2^b]$. Given the probability $p_v$ of node $v$ and its color $\psi(v)$ in the initial $O(\Delta^2)$-coloring we define a coin $C_v$ for $v$ with the help of the random variable $h_S(\psi(v))$ as follows.

$$ C_v = \begin{cases} 1 & \text{if } h_S(\psi(v)) < p_v \\ 0 & \text{otherwise} \end{cases} $$

Node $v$ fixes the $\ell$-th bit of its prefix to the value of $C_v$. Note that this algorithm per se is not a distributed algorithm but uses shared randomness in the form of the shared random seed.

We show that we can bound the expected increase of the potential of each node $u$ by

$$ E[\Phi_\ell(u)] \leq \Phi_{\ell-1}(u) + \frac{1}{4|\log C|}. $$

To this end, we show that the described random process satisfies all properties needed to apply Lemma 2.4, with $\epsilon = \frac{1}{4\Delta|\log C|}$. As $h_S(\psi(v))$ only assumes values in $[2^b]$, we always have $h_S(\psi(v))/2^b < 1$ and never $h_S(\psi(v))/2^b < 0$. Hence, if $p_v = 0$ or $p_v = 1$, then $C_v = 1$ with probability 0 or 1 respectively. Generally, as $h_S(\psi(v))$ is uniformly distributed in $[2^b]$, we have $Pr(C_v = 1) = i/2^b$ with $i = |\{k \in [2^b] \mid k/2^b < p_v\}|$. That is, $Pr(C_v = 1) = p_v$ rounded up to the next multiple of $1/2^b$. By the choice of $b$, we have $2^b \geq 4\Delta|\log C|$ and hence

$$ p_v \leq Pr(C_v = 1) = p_v + \frac{1}{2^b} \leq p_v + \frac{1}{4\Delta|\log C|}. $$

Adjacent nodes have distinct colors in the coloring $\psi$ and do not use the same random variable to define their coins. As Theorem 2.5 implies that the random variables $h_S(0), \ldots, h_S(K-1)$ are pairwise independent we can deduce that the produced coins of adjacent nodes are also pairwise independent.

Derandomization: For $j \in \{1, \ldots, d\}$, let $R_j$ be the random variable that describes the value of the $j$-th bit of the random seed. To derandomize the aforementioned algorithm we iterate through the bits of the random seed and deterministically determine a good bit $r_j$ for each $R_j$ using the method of conditional expectation—we will later define the notion of a good bit. The computed good seed $s = r_1, \ldots, r_d$ will be such that the potential does not increase by much if coins are flipped and prefixes are extended according to the seed $s$. We use a BFS tree with a designated root as a leader that gathers all the necessary information to find and distribute good bits for the seed. We obtain a deterministic distributed algorithm to extend the prefixes by one bit without increasing the potential by much. The runtime to find a single good bit for some $R_j$ will be $O(D)$ due to communication over the BFS tree. The total runtime of extending prefixes by one bit is $O(D \cdot \text{seedlength}) = O(D \cdot d) = O(D \cdot (\log \Delta + \log \log C))$.

Finding a good bit for $R_j$: Assume we already chose good values $R_1 = r_1, \ldots, R_{j-1} = r_{j-1}$ for a $1 \leq j \leq b$ and we want to find a good bit $r_j$ for $R_j$. By the law of total expectation there must be an $r_j \in \{0, 1\}$ such that
where the randomness is over the non-determined random bits of $R_{j+1}, \ldots, R_d$ and $R_j, \ldots, R_d$ of the random seed, respectively. We call a value $r_j$ satisfying Equation (11) a good bit; note that the property of $r_j$ being good depends on the choice of $r_1, \ldots, r_{j-1}$.

To let the leader of the BFS tree find a good bit $r_j$, each node $v \in V$ computes

$$x^0_v = E\left[ \Phi_t(v) \mid R_1 = r_1, \ldots, R_{j-1} = r_{j-1}, R_j = 0 \right]$$

and

$$x^1_v = E\left[ \Phi_t(v) \mid R_1 = r_1, \ldots, R_{j-1} = r_{j-1}, R_j = 1 \right].$$

In order to compute $x^0_v$ and $x^1_v$, $u$ needs to know the following two properties for all its neighbors $v \in A_u$:

1. $p_v | L_v|$, (1) $1 - p_v | L_v|$ and (2) $\Pr(C_v = 1 \mid R_1 = r_1, \ldots, R_{j-1} = r_{j-1})$. Note that each node $u$ is aware which of its neighbors belong to the set $A_u = \{v \in \Gamma(u) \mid s_{t-1}(u) = s_{t-1}(v)\}$ when nodes always exchange the latest chosen bit of their prefix each time a new prefix bit is fixed.

To compute the values in (1), $u$ learns the two integers $|L_v|$ and $k_1(v)$ (the number of colors in $L_v$ with 1 as the $\ell$-th bit) for each $v \in A_u$ in one CONGEST round. Node $u$ can locally compute the value in (2) for a neighbor $v \in A_u$ after learning $\psi(v)$ in one CONGEST round and using the already fixed good bits $r_1, \ldots, r_{j-1}$. The initial color $\psi(v)$ determines which random variable $v$ uses to produce $C_v$ and all nodes know how to generate the random variable $h_S(t)$ for any color $t \in [K]$ from a seed $S$.

Next step, we aggregate $\sum_{v \in V} x^0_v$ and $\sum_{v \in V} x^1_v$ at the leader, choose a good bit

$$r_j := \arg\min_{i \in \{0, 1\}} \sum_{v \in V} x^i_v$$

and broadcast $r_j$ to all nodes. $r_j$ is a good bit, as we know there is a bit which fulfills (11) and $r_j$ is chosen as the bit which minimizes the left hand side in (11).

After $d$ iterations, we found a good seed $s_1, \ldots, s_d$. By iteratively applying (11) we obtain

$$E\left[ \sum_{v \in V} \Phi_t(v) \mid R_1 = r_1, \ldots, R_d = r_d \right] \leq E \left[ \sum_{v \in V} \Phi_t(v) \right]^{(s)} \leq \sum_{v \in V} \Phi_{t-1}(v) + \frac{n}{4 \log C}.$$  

(12)

At $(s)$ we used linearity of expectation and Equation (10). As we have fixed the complete random seed as $s = r_1 \circ \ldots \circ r_d$, the left hand side of Equation (12) does not contain any randomness. The claim of the lemma (Equation (8)) follows if each node deterministically extends its prefix by one bit according to $s$.

In the same way as in Lemma 2.4 we can show that by applying our randomized algorithm, the list of candidate colors of each node does not become empty (this holds for any possible outcome of the algorithm, i.e., for any choice of the random seed). Hence, this also holds for our deterministic algorithm as its output equals one possible outcome of the randomized one. 

**Proof of Theorem 2.2.** We apply the algorithm from Lemma 2.6 $\ell = \lceil \log C \rceil$ times. As $|L(u)| \geq 2\Delta$, for the initial potential of each node $u$ we have $\Phi_0(u) = \sum_{v \in \Gamma(u)} \frac{1}{|L_v|} \leq \frac{1}{2}$. So for the sum of all (initial) potentials we obtain

$$\sum_{v \in V} \Phi_0(v) \leq \frac{n}{2}. \quad (13)$$

Together with Lemma 2.6 it follows

$$\sum_{v \in V} \Phi_t(v) \leq \sum_{v \in V} \Phi_0(v) + \frac{n}{4 \log C} \cdot \lceil \log C \rceil \leq \frac{3n}{4}.$$  

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It follows that at least 1/4 of the nodes \( u \) have \( \Phi_i(u) < 1 \). For each node \( v \) we know that \( s_i(v) \) is the prefix of all colors in \( L_i(v) \neq \emptyset \). But as \( |s_i(v)| = \lceil \log C \rceil \), the prefix \( s_i(v) \) describes a single color from \( L(v) \) and is thus in fact the only remaining color in \( L_i(v) \). So we have \( |L_i(v)| = 1 \) for all \( v \in V \) and hence \( \Phi_i(u) \) equals the number of neighbors \( v \) of \( u \) with \( s_i(u) = s_i(v) \). So each node \( u \) with \( \Phi_i(u) < 1 \) in fact has \( \Phi_i(u) = 0 \) and thus can color itself with \( s_i(u) \) without getting a conflict with one of its neighbors.

The runtime equals the time needed for \( \lceil \log C \rceil \) iterations of the algorithm from Lemma 2.6, i.e., \( O(\log C \cdot D \cdot (\log \Delta + \log \log \log C)) \).

\[ \square \]

2.2 Proof of Theorem 1.1: List Coloring without Slack

Theorem 2.2 colors a constant fraction of the nodes in a list coloring instance if all lists are large enough, i.e., if all lists are twice as large as the maximum degree; let us denote this algorithm by \( A \). In this section we show how algorithm \( A \) can be used as a subroutine to also solve the \((\text{degree} + 1)\)-list coloring problem with small polylogarithmic overhead; note that unlike the instances that can be solved with algorithm \( A \) there is no guaranteed multiplicative slack between a node’s list size and its degree in the \((\text{degree} + 1)\)-list coloring problem. A similar argument has been used in [FHK16, Kuh19]. Formally we prove the following lemma.

**Lemma 2.7.** Assume there is an algorithm \( A \) that given a list-coloring instance on a subgraph \( G' = (V', E') \) of some graph \( G = (V, E) \) where each node \( v \in V' \) has a list \( L(v) \) of size at least \( 2\Delta_{G'} \) where \( \Delta_{G'} \) is an upper bound on the maximum degree of \( G' \) and all lists are subsets of \([C]\) for some \( C \), colors at least a quarter of the nodes of \( G' \) in time \( T(G) \).

Then there is an algorithm that solves the list coloring problem for any instance on an \( n \)-node graph \( G = (V, E) \) where each node \( v \in V \) has a list \( L(v) \subseteq [C] \) of size at least \( \deg(v) + 1 \) in time \( O(T(G) \cdot (\log(D) \cdot \log(n))) \).

Before we prove Lemma 2.7 we show that Lemma 2.7 and Theorem 2.2 imply Theorem 1.1. **Proof of Theorem 1.1.** The algorithm of Theorem 2.2 can be plugged into Lemma 2.7 to obtain the desired result. The resulting runtime is

\[ O(T_A(G) \log \Delta \log n) = D \log n \log \Delta \log C(\log \Delta + \log \log C) \cdot \]

If the color space is of size \( C = \text{poly} \Delta \) the runtime simplifies to \( O(D \log n \log^3 \Delta) \).

**Proof of Lemma 2.7.** Detailed pseudocode can be found in Algorithm 2. The algorithm consists of \( O(\log \Delta) \) phases and in each phase the maximum degree of the graph induced by all still uncolored nodes of \( G \) is halved. Assume that the uncolored nodes induce a graph \( R \) with maximum degree \( \Delta \) at the start of phase \( i \). Phase \( i \) begins with computing a \( \Delta_i/4 \)-defective \( O(1) \)-coloring of \( R \); let \( G_j \) denote the graph induced by uncolored nodes with color \( j \) in the defective coloring. We iterate through the color classes of the defective coloring and when handling \( G_j \) we color some of its nodes to ensure that each node of \( G_j \) that remains uncolored has at most \( \Delta_i/2 \) uncolored neighbors in \( G \). This is done by repeatedly applying algorithm \( A \) to the graph \( H_j \subseteq G_j \) induced by all nodes that do not satisfy the condition—after \( O(\log n) \) iterations all nodes of \( G_j \) will satisfy it. Here the list of available colors of a node \( v \in V(H_j) \) consists of all colors of its initial list that are not used by any of its already colored neighbors in \( G \).

More formally, denote \( H_j = G_j[V_{>\Delta_i/2}] \) where \( V_{>\Delta_i/2} \) consists of the uncolored nodes of \( G_j \) that have at least \( \Delta_i/2 \) uncolored neighbors in \( G \) with regard to the current partial list-coloring of \( G \); the set \( V_{>\Delta_i/2} \) and the graph \( H_j \) change after each call of algorithm \( A \) as the partial coloring of \( G \) changes: In one application of algorithm \( A \) at least a quarter of the nodes in \( V_{>\Delta_i/2} \) are colored and after \( O(\log n) \) iterations of recomputing \( V_{>\Delta_i/2} \) and applying algorithm \( A \) the set \( V_{>\Delta_i/2} \) will be empty where nodes leave \( V_{>\Delta_i/2} \) because they get colored or because sufficiently many of their neighbors get colored; here *sufficiently* means that their ‘uncolored degree’ in \( G \) drops below \( \Delta_i/2 \). It remains to show that graph \( H_j \) always forms a feasible instance for algorithm \( A \), that is, the list of each node is at least twice as large as the maximum degree of \( H_j \): This is the case due to (1) the maximum degree of a node \( v \in V(H_j) \) is at most \( \Delta_i/4 \) because \( H_j \) is contained in one color class of a \( \Delta_i/4 \)-defective coloring and (2) each \( v \in V(H_j) \) has at least \( \Delta_i/2 + 1 \) available colors in its list as the list size of a node is always larger than the number of its uncolored neighbors (this is true initially and for each color deleted from a node’s list, at least one of its neighbors got colored.)
The runtime follows as each of the $O(\log \Delta)$ phases consists of $O(\log n)$ iterations. Computing the $O(\Delta_{H_j}^2)$-coloring of $R$ can be done in $O(1)$ rounds given such a coloring of the previous iteration, or in time $O(\log^3 n)$ for the first iteration. Computing the defective coloring on $R$ takes $O(\log^* \Delta)$ rounds given the $O(\Delta_{H_j}^2)$-coloring (Theorem 2.1). Hence, assuming $T_A(G) \geq 1$, one phase takes $O(T_A(G) \log n)$ rounds.

If applying $A$ to $H_j$ requires a $O(\Delta_{H_j}^2)$-coloring of $H_j$, this can be computed in $O(1)$ rounds given a $O(\Delta_{H_j}^2)$-coloring of $G_j$, which again can be computed using the $O(\Delta_{H_j}^2)$-coloring of $R$. One cannot simply use the $O(\Delta_{H_j}^2)$ coloring of $R$ as a $O(\Delta_{H_j}^2)$-coloring of $H_j$ as the constant in the $O$-notation needs to be the same.

\[ \square \]

Algorithm 2 (\textit{degree} + 1)-List Coloring using a Slack List Coloring Algorithm

| Line | Description |
|------|-------------|
| 1:   | Set $R = G$ and $\Delta_R = \Delta$ |
| 2:   | for $i = 1$ to $\log \Delta$ do |
| 3:   | Compute an $O(\Delta_{H_j}^2)$-coloring of $R$. |
| 4:   | Compute a defective coloring $\psi$ on $R$ with defect $\leq \Delta_R/4$ and $c = O(1)$ colors |
| 5:   | for $j = 1$ to $c$ do |
| 6:   | Let $G_j = (V_j, E_j)$ be the graph induced by the nodes of color $j$ in $\psi$ |
| 7:   | for $\log n$ times do |
| 8:   | $H_j = G_j[\hat{V}]$ where $\hat{V} = \{v \in V_j \mid v$ is uncolored and has $\geq \Delta_R/2$ uncolored neighbors\} |
| 9:   | Compute an $O(\Delta_{H_j}^2)$-coloring of $H_j$ |
| 10:  | Run $A$ on $H_j$ |
| 11:  | Every uncolored node in $V$ updates its color list |
| 12:  | $U = \{v \in V \mid v$ uncolored\} |
| 13:  | $R = G[U]$ |
| 14:  | $\Delta_R = \Delta_R/2$ |

3 Efficient (\textit{degree} + 1)-List Coloring in CONGEST

For solving the (degree + 1)-list-coloring problem deterministically in poly log $n$ rounds in CONGEST, we first compute an $(\alpha, \beta)$-network decomposition with $\alpha, \beta = \text{poly} \log n$ (see Definition 3.1), that is, a decomposition of the network graph into poly log $n$ classes such that connected components (clusters) in each color class have small diameter, i.e., diameter poly log $n$. Then we iterate through the poly log $n$ classes and apply the algorithm from Theorem 1.1 on the small diameter clusters of one class in parallel. The concept of network decomposition was introduced in [AGLP89b] and was later differentiated into weak and strong decompositions [LS93]. However, both concepts are not suitable for our purpose. In a weak decomposition, when solving the list-coloring on a cluster, it might be necessary to communicate also via edges outside the cluster. Hence, it is not possible to run the algorithm from Theorem 1.1 on $\omega(\log n)$ clusters in parallel as there might be an edge being involved in all these computations. In contrast, a strong decomposition would be sufficient, however, we do not know how to compute it in poly log $n$ rounds in CONGEST. We therefore introduce a slightly more general definition of a network decomposition, which also includes a congestion parameter $\kappa$. A similar definition was for example used previously in [GK19].

Definition 3.1 (Network decomposition with congestion). An $(\alpha, \beta)$-network decomposition with congestion $\kappa$ of a graph $G = (V, E)$ is a partition of $V$ into clusters $C_1, \ldots, C_p$ together with associated subtrees $T_1, \ldots, T_p$ of $G$ and a color $\gamma_i \in \{1, \ldots, \alpha\}$ for each cluster $C_i$ such that

(i) the tree $T_i$ of cluster $C_i$ contains all nodes of $C_i$ (but it might contain other nodes as well)

(ii) each tree $T_i$ has diameter at most $\beta$

(iii) clusters that are connected by an edge of $G$ are assigned different colors

(iv) each edge of $G$ is contained in at most $\kappa$ trees of the same color
When we assume to have a network decomposition on a graph, we require that each node knows the color of the cluster it belongs to and for each of its incident edges e the set of associated trees e is contained in. Note that a decomposition according to this definition has weak degree 1 and a strong network decomposition is a decomposition with congestion 1 where the tree $T_i$ of each cluster $C_i$ contains exactly the nodes in $C_i$.

**Theorem 3.1** ([RG19]). There is a deterministic algorithm that computes an $(O(\log n), O(\log^3 n))$-network decomposition with congestion $O(\log^2 n)$ in $O(\log^8 n)$ rounds in the CONGEST model.4

We use Theorem 3.1 and Theorem 1.1 to list-color graphs efficiently in the CONGEST model. The corollary is formulated under the assumption that all lists are from a color space $[C]$ with $C = \text{poly} \Delta$.

**Corollary 1.2.** There is a deterministic CONGEST algorithm that solves any $(\text{degree} + 1)$-list-coloring problem in $O(\log^8 n + \log^7 n \cdot \log^3 \Delta)$ rounds.

**Proof.** Given a graph $G$, we compute an $(O(\log n), O(\log^3 n))$-network decomposition with congestion $O(\log^2 n)$ in $O(\log^8 n)$ rounds using the algorithm from Theorem 3.1. To list-color $G$ we iterate through the color classes of the network decomposition. When handling a single color class we need to solve a list-coloring problem on each cluster which is done by applying Theorem 1.1 on all clusters of the color class in parallel. We continue with a detailed description and runtime analysis of the process: Let $L_G(v)$ be the initial list of $v \in V$ in $G$. Assume all clusters with colors 1, …, $k - 1$ are already list-colored. To list-color the clusters of color $k$, every node in such a cluster $C$ updates its list, i.e., it deletes all colors from its list $L_G(v)$ taken by an already colored neighbor in $G$ and obtains a new list $L_C(v)$. Let $\deg_C(v)$ denote the degree of $v$ in the graph induced by the nodes in cluster $C$ and $\deg_G(v)$ the degree of $v$ in $G$. We obtain $|L_C(v)| \geq \deg_C(v) + 1$ because initially we have $|L_G(v)| \geq \deg_G(v) + 1 = \deg_C(v) + |\Gamma_G(v) \setminus C| + 1$ and we remove at most one color from $|L_G(v)|$ for each neighbor in $|\Gamma_G(v) \setminus C|$.

Thus we can apply the algorithm from Theorem 1.1 for each cluster $C$ of color $k$ in parallel where all aggregation and broadcast in the derandomization part is done over the associated tree of the cluster. This tree has diameter $O(\log^3 n)$ (note that $n$ is the size of $G$ and not of the tree). Messages over edges that are contained in more than one tree are pipelined and as each edge of $G$ is contained in at most $O(\log^2 n)$ trees of the same color, the number of rounds to list-color all clusters of color $k$ is at most $O(\log^2 n)$ times the number of rounds required to list-color a single cluster (without pipelining) which is $O(\log^4 n \log^3 \Delta)$ according to Theorem 1.1. So in total $O(\log^6 n \log^3 \Delta)$ rounds are needed to list-color a single color class of the network decomposition. After iterating through all $O(\log n)$ colors, each node chose a color from its list and has no conflict with a neighbor. Hence we computed a $(\text{degree} + 1)$-list-coloring of $G$ in time $O(\log^8 n + \log^7 n \cdot \log^3 \Delta)$.

\[ \square \]

## 4 List Coloring in the CONGESTED CLIQUE and MPC

In this section we show how to adapt the algorithm from Theorem 1.1 to obtain faster algorithms in the CONGESTED CLIQUE and MPC model with linear memory. As in these models we have direct communication between the nodes/machines, the derandomization from Theorem 2.2 can be sped up in two ways:

1. Communication with a leader can be done directly, removing the dependence on the diameter in the runtime
2. Due to the large routing capabilities of the respective models we can derandomize $\log n$ bits of the seed in constant time (as described in the following proofs), reducing the influence of the seed-length in the runtime to a constant factor.

**Corollary 1.3** (CONGESTED CLIQUE). There is a deterministic CONGESTED CLIQUE algorithm that solves the $(\text{degree} + 1)$-list-coloring problem in time $O(\log^3 \Delta)$.

**Proof Sketch.** We use the same algorithm as in Theorem 1.1 but speed up the derandomization step (cf. Lemma 2.6) through fixing $\Omega(\log \Delta)$ bits of the random seed in $O(1)$ rounds: Assume we want to fix the $\ell$-th bit of all prefixes. As the random seed of the randomized algorithm has length $O(\log \Delta)$, we can split it into $O(1)$ segments of length at most $\left\lfloor \log n \right\rfloor$. For the first segment we can choose among $n$ different

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4In on-going unpublished work [RGG19] it is shown that diameter and runtime can be improved. These improvements carry over to our results.
partial seeds. The idea is to make each node \( u \) responsible for one partial seed \( R(u) \) and let \( u \) collect the values \( E[\phi_1(v) \mid R_1 \cdots \cdots R_{\log n} = R(u)] \) from all \( v \in V \). To this end, the leader chooses a bijection \( R : V \rightarrow \{0,1\}^{\log n} \) and sends \( R(v) \) to node \( v \). Afterwards all nodes exchange the tuples \( (v, R(v)) \) such that each node knows which node is responsible for which partial seed. Next, each node \( u \) learns \( |L_u|, k_1(v) \) and \( \psi(v) \) from each of its neighbors \( v \). With these information, \( u \) can compute \( E[\phi_1(u) \mid R_1 \cdots \cdots R_{\log n} = R(w)] \) for any \( w \in V \) and send this value to \( w \). Now each node \( u \) has collected all the values for its seed, i.e., it can compute \( \sum_{v \in V} E[\phi_1(v) \mid R_1 \cdots \cdots R_{\log n} = R(u)] \) and send it to the leader. The leader chooses the partial seed minimizing this sum and broadcasts it to every node. This way we proceed with all other segments and thus the derandomization of the whole seed is done in \( O(1) \) rounds.

In the proof of Theorem 1.1 (or more specific in the proof of Lemma 2.7) we iterate this algorithm for \( O(\log n) \) rounds to reduce the maximum degree of the graph induced by uncolored nodes by a factor two. In the CONGESTED CLIQUE \( O(\log \Delta) \) iterations are sufficient to reduce the number of participating vertices to \( n/\Delta \). Then \( O(1) \) rounds are sufficient to send the whole subgraph of participating vertices to one leader vertex which locally solves the problem.

Next we sketch how the same runtime can be obtained in the MPC model. However, there are several subtleties that need to be taken care of to ensure that all steps fit into the memory of the machines, e.g., when the input is a \((\text{degree} + 1)\)-list coloring problem we assume that the nodes’ lists are part of the input and thus the lists fit into the total memory of all machines. However, if a \((\text{degree} + 1)\)-list coloring problem only occurs as a subroutine of a \((\Delta + 1)\)-coloring and the maximum degree \( \Delta \) is significantly larger than the average degree of the graph there is not enough total storage to explicitly store a list with up to \( \Delta + 1 \) colors for each vertex. In our algorithm each node can instead store one forbidden color for each already colored neighbor and sublists of the lists can be succinctly represented by prefixes.

**Theorem 1.4 (MPC, linear memory).** There is a deterministic MPC algorithm that solves the \((\text{degree} + 1)\)-list-coloring problem in \( O(\log^3 \Delta) \) rounds with linear memory.

**Proof sketch.** Similarly as in Corollary 1.3, we parallelize the derandomization (cf. Lemma 2.6) such that fixing one bit of the prefix can be done in \( O(1) \) rounds. We describe the idea of the algorithm. Let \( G \) be the input graph on which we want to solve \(((\text{deg} + 1)\)-list-coloring and let \( H \) be the graph on which we want to list-color a fraction of the nodes (cf. Line 10 of Algorithm 2). Assume we want to fix the \( \ell \)-th bit of all prefixes. We split the random seed (which has length \( O(\log \Delta_H + \log \log C) \)) into \( O(1) \) segments of length at most \( k := \log \Delta_H + \log \log C \). There are at most \( 2^k = \Delta_H \log C \) different bitstrings which the random seed can assume on the first segment. We call each of these a partial seed. For each node \( u \in V(H) \) and each partial seed \( R \) we choose machines \( M_u \) and \( M_R \) being responsible for \( u \) and \( R \) respectively. There are two main steps:

1. For each node \( u \), machine \( M_u \) computes \( E[\phi_1(u) \mid R_1 \cdots \cdots R_k = R'] \) for all partial seeds \( R' \).
2. For each partial seed \( R \), machine \( M_R \) collects \( E[\phi_1(v) \mid R_1 \cdots \cdots R_k = R] \) for all nodes \( v \).

Finally, for each partial seed \( R \), machine \( M_R \) computes \( \sum_{v \in V} E[\phi_1(v) \mid R_1 \cdots \cdots R_k = R] \) and sends it to a leader machine. The leader chooses the partial seed minimizing this sum and broadcasts it to every node. The same way we proceed with the other segments.

This algorithm does the derandomization in \( O(1) \) rounds. We show that it can be implemented such that it satisfies the memory restrictions of the MPC model. Let \( n_H \) be the number of nodes in \( H \). For all \( v \in V(H) \) we have \((\text{deg}_G(v) \geq \Delta_H/2) \). It follows that \( G \) has \( \Omega(n_H \cdot \Delta_H) \) edges and thus we have \( \Omega(n_H \cdot \Delta_H) \) memory available (\( \Omega \) hides polylog factors where \( n \) is the size of \( G \)). More specifically, as \( \log C = O(\log n) \), we have \( \Omega(n_H \Delta_H \log C) \) memory available in total and \( \Omega(n) \) on each machine. Hence, for each \( u \in V(H) \) and each partial seed \( R \) we can choose machines \( M_u \) and \( M_R \) being responsible for \( u \) and \( R \) respectively such that each machine has \( \Omega(\Delta_H \log C) \) extra memory for each node and \( \Omega(n_H) \) extra memory for each partial seed it is responsible for. So the machines have enough memory to first exchange the necessary information (if a machine stores an edge \( \{u,v\} \), it sends \( |L_u|, k_1(u) \) and \( \psi(v) \) to \( M_u \) and \( |L_v|, k_1(v) \) and \( \psi(v) \) to \( M_v \) and if a machine stores \( L(u) \) it sends it to \( M(u) \)) and then compute all the values in step (1). Furthermore, the data each machine sends and receives in step (2) does not exceed the local memory.
The $O(\log n)$ factor can be turned into an $O(\log \Delta)$ factor analogously to the argument in the proof of Corollary 1.3 by sending the subgraph of all participating vertices to one machine as soon as the number of participating vertices drops below $n/\Delta$.

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