Symmetries and Universality Classes in Conservative Sandpile Models

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Abstract

The symmetry properties which determine the critical exponents and universality classes in conservative sandpile models are identified. This is done by introducing a set of models, including all possible combinations of abelian vs. non-abelian, deterministic vs. stochastic and isotropic vs. anisotropic toppling rules. The universality classes are determined by an extended set of critical exponents, scaling functions and geometrical features. Two universality classes are clearly identified: (a) the universality class of abelian models and (b) the universality class of stochastic models. In addition, it is found that non-abelian models with deterministic toppling rules exhibit non-universal behavior.
Sandpile models were introduced about a decade ago as a paradigm of self organized criticality (SOC) [1–3]. SOC provides a useful framework for the study of a large class of driven non-equilibrium systems which dynamically evolve into a critical state. At the critical state these systems exhibit long-range spatial and temporal correlations, which resemble the behavior at the equilibrium critical point. SOC was thus proposed as the mechanism underlying the appearance of fractal structures and $1/f$ noise, which exhibit power-law spatial and temporal correlations [1–3]. The critical state of SOC systems can be characterized by critical exponents, scaling functions and other geometric features. To examine these properties a variety of sandpile models have been introduced [4,5] and their scaling properties were studied both analytically [6,7] and numerically [8]. Numerical studies indicate the existence of a number of distinct universality classes of sandpile models [9,10]. However, a systematic classification relating the universality classes to the underlying symmetries has not been achieved.

In this letter we introduce a systematic framework for the classification of sandpile models into universality classes. This framework is based on the fundamental symmetries which can be identified in sandpile models: (a) the abelian symmetry; (b) the rotational symmetry (isotropic, uniaxial, directed) and (c) deterministic vs. stochastic update rules. Using this framework we identify the relevant parameters of sandpile models and the assignment into universality classes.

We will first introduce the models. Sandpile models are defined on a $d$-dimensional lattice of linear size $L$. Each site $i$ is assigned a dynamic variable $E(i)$ which represents some physical quantity such as energy, stress, etc. A configuration $\{E(i)\}$ is called stable if for all sites $E(i) < E_c$, where $E_c$ is a threshold value. The evolution between stable configurations is by the following rules: (i) Adding energy. Given a stable configuration $\{E(j)\}$ we select a site $i$ at random and increase $E(i)$ by some amount $\delta E$. When an unstable configuration is reached rule (ii) is applied. (ii) Relaxation rule. If $E(i) \geq E_c$, relaxation takes place and energy is distributed in the following way:
\[
E(i) \rightarrow E(i) - \sum_{e} \Delta E(e)
\]
\[
E(i + e) \rightarrow E(i + e) + \Delta E(e),
\]
where \(e\) are a set of (unit) vectors from the site \(i\) to some neighbors. As a result of the relaxation, \(E(i + e)\) for one or more of the neighbors may exceed the threshold \(E_c\). The relaxation rule is then applied until a stable configuration is reached. The sequence of relaxations is an avalanche which propagates through the lattice.

Since the parameters \(\delta E\) and \(E_c\) are irrelevant to the scaling behavior \([11, 12]\), the critical exponents depend only on the vector \(\Delta E\), to be termed relaxation vector. For a square lattice with relaxation to nearest neighbors (NN) it is of the form \(\Delta E = (E_N, E_E, E_S, E_W)\), where \(E_N, E_E, E_S\) and \(E_W\) are the amounts transferred to the northern, eastern, southern and western NN’s respectively. In the Bak-Tang-Wiesenfeld (BTW) model, \(E_c = 4, \delta E = 1\) and \(\Delta E = (1, 1, 1, 1)\). If an active site with \(E(i) > E_c\) is toppled, it would not become empty after the topple had occurred. In the Zhang model \([4]\), for which \(E_c = 1\) and \(0 < \delta E < 1\), the relaxation vector is given by \((b, b, b, b)\), where \(b = E(i)/4\) and \(E(i)\) is the amount of energy in the active site before the topple had occurred. Obviously, the site \(i\) remains empty after toppling. In the random relaxation models introduced by Manna \([5]\) a set of neighbors is randomly chosen for relaxation. Such models are specified by a set of relaxation vectors, each vector being assigned a probability for it application. For example, a possible realization of a two-state Manna model includes six relaxation vectors \((1, 1, 0, 0), (1, 0, 1, 0), (1, 0, 0, 1), (0, 1, 1, 0), (0, 1, 0, 1)\) and \((0, 0, 1, 1)\), each one applied with a probability of \(1/6\). A time step (of unit time) is defined as the relaxation of all the sites having \(E(i) \geq E_c\), after the completion of the previous time step.

To identify the relevant parameters of sandpile models we will consider the symmetries of the relaxation rules. The abelian symmetry: A model is said to be abelian if the configuration after an avalanche, is independent of the order in which the relaxation of the active sites was performed. The BTW model was shown to be abelian \([11]\). The Manna models \([5]\) are not abelian because they contain a random choice of the toppling direction. As a result,
they develop different scenes of toppling that depend on the order of relaxation of the active sites even within a single time step. The Zhang model is also non-abelian. This can be seen, when two active NN sites are toppled within the same time step: the site that was toppled last remains empty while the other one is non-empty. This shows that the final configuration depends on the order. The rotational symmetry: A model is said to be isotropic if in each topple the energy is divided equally between the NN's. The BTW and the Zhang models are isotropic by definition while the Manna models are non-isotropic. The deterministic vs. stochastic feature: A model is said to be deterministic if the toppling rule is deterministic, otherwise it is stochastic. The BTW and the Zhang models are deterministic while the Manna models are stochastic.

The three properties described above form eight possible combinations which can be graphically represented on the corners of a three dimensional cube (Fig. 1). The $x$ axis represents the rotational symmetry, where $x=1$ ($x=0$) for isotropic (anisotropic) models. The $y$ axis represents the deterministic vs. stochastic feature where $y=1$ ($y=0$) for deterministic (stochastic) models. The $z$ axis represents the abelian symmetry, where $z=1$ ($z=0$) for abelian (non-abelian) models. Each corner of the cube is specified by its coordinates $(x, y, z)$. The BTW model belongs to the $(1, 1, 1)$ corner, the Zhang model belongs to the $(1, 1, 0)$ corner and the Manna models belong to the $(0, 0, 0)$ corner. To systematically identify the relevant parameters we need to introduce models that belong to each of the eight corners and examine their scaling properties. We first note that an abelian model cannot be stochastic and vice versa. This is due to the fact that in a stochastic model (for a given seed of the random number generator) the actual moves to be performed in a given time step depend on the order in which they are performed. As a result, stochastic models cannot be abelian and the corners $(0, 0, 1)$ and $(1, 0, 1)$ in Fig. 1 remain vacant.

We will now introduce a new set of models which will later be used for a systematic study of the effects of the three properties described above on the critical behavior. These models fall into two groups: variations of the Zhang model and variations of the Manna model. We will first introduce variations of the Zhang model.
Generalized Zhang (GZ): in this model the Zhang relaxation vector is modified into $\Delta E = (b, b, b, b)$ where $b = pE(i)/4$ and $0 < p \leq 1$ is a pre-determined constant, such that only a fraction $p$ of the energy in site $i$ is distributed to its neighbors. Since none of the properties is changed this model remains in the $(1, 1, 0)$ corner in Fig. 1. In the limit $p \to 0$ the model becomes abelian as the order of relaxations becomes irrelevant. In this limit it is termed the Abelian Zhang (AZ) model, which shifts into the $(1, 1, 1)$ corner in Fig. 1. This resembles the situation in rotations of a rigid body in the three dimensional space, where infinitesimal rotations are abelian while finite rotations are not. Moreover, within this analogy, the BTW model may correspond to the group of rotations by $180^\circ$ around the $x$, $y$ and $z$ axes, which is also abelian.

Parallel-Update Zhang (PZ): this is a variation of the Zhang model, in which all the topplings in a given time step are performed simultaneously, namely each unstable site $i$ distributes $E(i)/4$ to each nearest neighbor, where $E(i)$ is its energy after the previous time step was completed. The model can thus be considered as abelian and assigned to the $(1, 1, 1)$ corner.

Stochastic Zhang (SZ): in this model the Zhang relaxation vector is modified into $\Delta E = (b, b, b, b)$ where $b = rE(i)/4$ and in each toppling event $r$ is chosen randomly in the range $0 < r \leq 1$. Therefore, the SZ model belongs to the $(1, 0, 0)$ corner.

Alternating Uni-Axial Zhang (AUZ): this model has two relaxation vectors: $(c, 0, c, 0)$ used in odd time steps and $(0, c, 0, c)$ used in even time steps, where $c = E(i)/2$. The basic features of the Zhang model are maintained except for the isotropy, and therefore it shifts to the $(0, 1, 0)$ corner.

We will now introduce models which are variations of the Manna models.

Uni-Axial Manna (UM): this is a restricted version of the Manna two-state model. It includes two equally probable relaxation vectors: $(1, 0, 1, 0)$ and $(0, 1, 0, 1)$ which are both uniaxial. The UM model has the same three properties as the ordinary Manna models, and it thus belongs to the $(0, 0, 0)$ corner. It only differs from the Manna models in the fact that is does not include directed moves.

Alternating Uni-Axial Manna (AUM): this model has the same relaxation vectors as the
UM model, but it is not stochastic, since the \((1,0,1,0)\) vector is used for all topplings in odd time steps and the \((0,1,0,1)\) vector is used for even time steps. This deterministic rule that decouples the horizontal and vertical directions makes the model abelian, and it thus belongs to the \((0,1,1)\) corner.

To obtain a complete characterization of the models introduced above we have performed extensive computer simulations of all the models and calculated an extended set of characterization measures. These measures include the distribution exponents, the geometric exponents, as well as scaling functions and geometric features of the avalanche \[10\]. The distribution exponents \(\tau_x\) characterize the distribution of various avalanche parameters. It is found that \(P(x) \sim x^{-1-\tau_x}\), where \(x\) may represent the avalanche size \((s)\), area \((a)\) or time \((t)\). The geometric exponents \(\gamma_{xy}\) relate the distribution of these quantities, and are defined in terms of the conditional expectation values \(E[x|y] \sim y^{\gamma_{xy}}\) where \(x, y \in \{s, a, t\}\) \[16,17\].

The scaling functions describe the time evolution of the avalanche size \(S(t)\) and area growth rate \(A(t)\) during the avalanche, averaged over a large number of avalanches. According to the dynamic scaling assumption, each one of these functions can be written in the general scaling form :

\[
X(t) = K_X \langle t \rangle_X^{-\alpha_X} f_X(\mu)
\]

where \(\mu = t/\langle t \rangle_X\), \(X \in \{S, A\}\) and

\[
\langle t \rangle_X = \frac{\sum t X(t)}{\sum X(t)}.
\]

The conditional expectation values \(E[s|a]\) vs. \(a\) are shown in Fig. 2 for the abelian models of class \(B\) (see Fig. 1) and the stochastic models of class \(C\). We find that the exponent \(\gamma_{sa}\) for the abelian models, namely the Alternating Uni-Axial Manna (AUM), Abelian Zhang (AZ) and Parallel-Update Zhang (PZ) models coincides with its value for the BTW model, \(\gamma_{sa} = 1.05 \pm 0.01\). For the stochastic models, namely the Uni-Axial Manna (UM) and Stochastic Zhang (SZ) models, \(\gamma_{sa}\) coincides with the Manna value \(\gamma_{sa} = 1.24 \pm 0.02\). The fact that the BTW and Manna models belong to different universality classes was pointed out
before [4]. However, the results presented here indicate a considerable degree of universality within each class and also attribute it to the abelian symmetry in class $B$ and to the stochastic dynamics in class $C$. The scaling functions for the abelian and stochastic models are shown in Fig. 3. In the stochastic class we observe very good coincidence between the Manna and Stochastic Zhang (SZ) models both for $f_S(\mu)$ and $f_A(\mu)$. The Uniaxial Manna (UM) model somewhat deviates from the other two, although it exhibits the same qualitative shape. In the abelian class we observe perfect coincidence between the BTW and the Abelian Zhang (AZ) for both $f_S(\mu)$ and $f_A(\mu)$. These results provide further evidence for Universality within each of the two classes.

To further characterize the avalanche structure we examined the function $f(i)$, that provides the number of toppling events at site $i$ during the avalanche [5]. For the abelian models, we observe a shell structure in which all sites which relaxed at least $n+1$ times form a connected cluster with no holes which is contained in the cluster of sites which relaxed at least $n$ times. The Stochastic models exhibit a random avalanche structure with many peaks and holes [5].

As we pointed out before, class $A$ is empty since a stochastic model cannot be abelian. We will now consider the remaining class, $D$. It turns out that systems in class $D$ exhibit non-universal critical behavior. The exponents $\gamma_{sa}$ and $\tau_s$ of the generalized Zhang (GZ) model vary continuously as the parameter $p$ is lowered from the Zhang values $\gamma_{sa} = 1.6 \pm 0.05$ and $\tau_s = 1.25 \pm 0.01$ at $p = 1$ to the BTW values $\gamma_{sa} = 1.05 \pm 0.01$ and $\tau_s = 1.09 \pm 0.01$ at $p \to 0$. For the models of class $D$, the avalanche structures are intermediate between those of the abelian and stochastic models and depend on the model parameters, such as the parameter $p$ in the GZ model. For example, as $p \to 0$ the avalanche structure converges to the ordered shell structure of the BTW model. Moreover, for this class the average size and area growth functions during the avalanche do not collapse into scaling functions. This indicates that models in class $D$ do not exhibit all the features of critical behavior which appear in classes $B$ and $C$.

In summary, we have performed a systematic study of critical behavior, relevant pa-
rameters and universality classes in sandpile models which exhibit self organized criticality. We introduced an extended set of models, including all possible combinations of abelian vs. non-abelian, deterministic vs. stochastic and isotropic vs. anisotropic toppling rules. To characterize the critical behavior we have used an extended set of critical exponents, particularly relying on the geometric exponents, which were found to be most useful, in addition to scaling functions and geometric features of the avalanche. Two universality classes were clearly identified and attributed to the underlying symmetry properties: the universality class of abelian models (which includes the BTW model) and the universality class of stochastic models (which includes the Manna models). In addition it was found that the class of deterministic models which are non-abelian (which includes the Zhang model) exhibits non-universal behavior.

A number of promising theoretical frameworks, based on the fixed scale transformation approach [6] and on the dynamic renormalization group approach [7] have been introduced in recent years for the study of universality in sandpile models. We believe that extending these approaches to include the relevant symmetry properties examined here would greatly improve our theoretical understanding of SOC.

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FIGURES

FIG. 1. Classification diagram of sandpile models into the corners of the unit three dimensional cube \((x, y, z)\), where \(x = 1\) (0) for isotropic (anisotropic), \(y = 1\) (0) for deterministic (stochastic) and \(z = 1\) (0) for abelian (non-abelian) models. Four classes are identified: A. An empty class of implausible models; B. The abelian universality class (BTW); C. The stochastic universality class (Manna); D. A class of non-universal models (Zhang).

FIG. 2. The conditional expectation values \(E[s|a]\) vs. \(a\) which yields the geometrical critical exponents \(\gamma_{sa}\) for the set of abelian and stochastic models. For the abelian models we find that \(\gamma_{sa} = 1.05 \pm 0.01\), while for the stochastic models \(\gamma_{sa} = 1.24 \pm 0.02\). The small parameter \(p\) in the AZ model is \(p = 0.005\).

FIG. 3. The scaling functions for the abelian and stochastic models. (a) \(f_S(t/\langle t\rangle)\) for the stochastic models, Manna (—), SZ (- -) and UM (⋯); (b) \(f_A(t/\langle t\rangle)\) for the same set of stochastic models. While the scaling functions for the first two models coincide, the third one has some deviation; (c) \(f_S(t/\langle t\rangle)\) for the abelian models, BTW (—) and AZ (- -); (d) \(f_A(t/\langle t\rangle)\) for the same set of abelian models. The scaling functions for these two models perfectly coincide.
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Fig. 3