HYPOELLIPTICITY FOR LINEAR DEGENERATE ELLIPTIC SYSTEMS IN CARNOT GROUPS AND APPLICATIONS

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Abstract. We prove that if $u$ is a weak solution to a constant coefficient system (with strong ellipticity assumed along the horizontal direction) in a Carnot group (no restriction on the step), then $u$ is actually smooth. We then use this result to develop blow-up analysis to prove a partial regularity result for weak solutions of certain non-linear systems.

1. Introduction

Carnot groups are relatively simple models of sub-Reimannian manifolds. In recent years there has been intensive study of the regularity theory for weak solutions of non-linear degenerate elliptic systems in this setting. This theory relies on the hypoellipticity for the corresponding constant coefficient system; the main purpose of this paper is to prove a regularity result for weak solutions of the constant coefficient system

$$
\sum_{\beta=1}^{N} \sum_{i,j=1}^{m} X_i(A^{\alpha \beta}_{ij} X_j u^\beta + f^\alpha_i) = f^\alpha \text{ in } \Omega \subset G,
$$

where $\Omega$ is an open set of a Carnot group $G$ and $A^{\alpha \beta}_{ij}$ satisfies the coercivity condition

$$
\sum_{\alpha, \beta=1}^{N} \sum_{i,j=1}^{m} A^{\alpha \beta}_{ij} \xi^\alpha_i \xi^\beta_j \geq \lambda |\xi|^2, \quad \text{for all } \xi \in \mathbb{R}^{mN},
$$

and $f^\alpha_i$, $f^\alpha$ are smooth functions defined in $\Omega$. Here $X_1, \ldots, X_m$ refer to the horizontal vector fields (or rather differentiation along the first layer of the Lie algebra stratification) in $G$. We consider weak solutions of (1.1) in the horizontal Sobolev space $S^{1,2}_{loc}(\Omega)$ which consists of $L^2_{loc}(\Omega)$ functions having horizontal derivatives of order one. In particular, we prove the following:

Theorem 1. (Main Theorem) Let $G$ be a Carnot group of step $r$ and $\Omega \subset G$ an open, bounded set. If $u \in S^{1,2}_{loc}(\Omega)$ is a weak solution to (1.1) and $f^\alpha$, $f^\alpha_i$ are smooth functions, then for any ball $B$ such that $2B \subset \Omega$ the following inequality holds:

$$
\| X^{I_1} X^{I_2} \cdots X^{I_r} u \|_{S^{1,2}(B)} \leq C \left( \| u \|_{S^{1,2}(2B)} + \| f^\alpha \|_{L^2(2B)} + \| f^\alpha_i \|_{L^2(2B)} \right).
$$

Here, $X^{I_1} \cdots X^{I_r}$ represent differentiation of indefinite order in each of the layers, $1, \ldots, r$. Also, we use the notation $f$, $f^\alpha_i$ to represent high order derivatives (possibly along every layer) on the original $f$ and $f_i$. 
The classical method to prove higher regularity of solutions consists in differentiating the system several times and applying $L^2$ energy estimates at each step. In Carnot groups this argument is somewhat different. Carnot groups are non-commutative groups; therefore, each time we differentiate the system we gain increasingly complex non-homogeneous terms, the commutators, whose $L^2$ norm we have to control. These new terms will not only involve differentiation on the original non-homogeneous terms but also on the solution $u$. However, since the commutators of two vector fields belong to a higher layer, one might expect the derivatives to eventually move to the highest layer, $r$, no matter in which order we differentiate the system. This is not always the case as the following example shows:

Suppose we are in a Carnot group, step 4. Let $X_4$ represent differentiation in the center of the group $V^4$, $X_3$ differentiation in $V^3$, and $X_2$ differentiation in $V^2$. Assume that we can differentiate indefinitely along $X_4$ and $X_3$. This is a reasonable assumption and is easily shown. We wish to then try to differentiate our system a total of three times, once along each of the directions $X_4$, $X_3$, and $X_2$, and then show that $X_2X_3X_4u^{\beta}$ still solves our system.

Without including all the details we will explain where the problem occurs. As we try to gain an $L^2$ estimate on $X_2X_3X_4u^{\beta}$, we will eventually get to an estimate where the right hand side includes the sum of the $L^2$ norm of the following terms\(^1\) (along with similar non-homogeneous terms):

$$Z(A^{\alpha\beta}_{ij}[X_3, X_j]X_4u + X_3X_4f_i), X_j \in V^1,$$

where $Z$ represents differentiation along each of the vector fields $X_2$, $X_3$, and $X_4$. However, the only direction that poses a problem is $Z = X_2$ (as the other directions are covered under the assumption). We are assuming that $f_i$ is smooth, so the second part of the sum, $Z(X_3X_4f_i)$, is bounded in $L^2_{loc}$. We need to prove an $L^2$ estimate on differentiation along $X_2$ yet we have a term where this derivative appears on both the right hand side and left hand sides. Therefore, assuming differentiation along the higher layers is not a sufficient hypothesis.

What we found is that unless we use the ”right” algorithm to differentiate the system, then phenomena like the one illustrated above may happen. Referring to this example, one sees that in the first term of the sum differentiation along the vector field $X_3$ is absorbed into the commutator term. Thus, differentiation has essentially moved to the next highest layer once the rules of commutators are applied. Throughout the paper we refer to this aspect as a shift in the derivatives to the right. Moreover, one will notice that since we no longer have differentiation along $X_3$ then all of the derivatives lie in only the first and last layers. This is not only true for the specific example above but it holds even in the general case; we show that as we apply our algorithm, the order of the derivatives begins to decrease in the middle layers until eventually all differentiation shifts to the first and last layers (see Theorem 12). The remaining term is then shown to be bounded above by the $L^2$ norm of a term with less derivatives than what we start with (see Theorem 13). Through an iteration argument we show that this is eventually bounded above by the $L^2$ norm of $u$. The difficulty in devising the algorithm is that it must work regardless of what layer one is differentiating along and regardless of the order of the derivatives.

\(^1\)This term is the non-homogeneous term that appears in place of $f_i$ when we differentiate along $X_3$ and $X_4$. 


As an immediate consequence of the main theorem and of Sobolev’s Embedding Theorem, we have the following:

**Corollary 2.** Let $G$ be a Carnot group of step $r$ and $\Omega \subset G$ an open, bounded set. If $u \in S^{1,2}_{loc}(\Omega)$ is a weak solution to (1.1) and $f^\alpha, f^i_\alpha$ are smooth functions, then $u$ is smooth.

In the case of scalar equations, Corollary 2 follows from a celebrated result of Hörmander. In 1967, Hörmander [H] studied the partial differential operator $P = \sum_{j=1}^r X_j^2 + X_0 + c$, where $X_0, \ldots, X_r$ are smooth vector fields in $\mathbb{R}^n$. He proved that if the vector fields and all of their commutators generate the whole space, then $P$ is hypoelliptic. This work, along with the papers of J.J. Kohn [K], Folland-Stein [FS], Folland [F], and Rothschild-Stein [RS], allow one to prove the $W^{2,2}$-estimates (and thus the hypoellipticity) of diagonal systems. We also refer the reader to the recent papers of Xu and Zuily [XZ] where quasilinear subelliptic systems are studied, and of Jost and Xu [JX] where subelliptic harmonic maps are studied. Whereas the above results address the diagonal case, they do not cover the non-diagonal case. In this regard, following a highly technical argument the $W^{2,2}$-estimates can be derived from the analysis of pseudo-differential operators on homogeneous groups developed in the papers of [CGGP], [T], and [G]. An advantage of the ideas presented in the present paper is they may be more familiar to those working in pde’s: We use fractional order difference quotients in order to establish differentiation once in any direction. Moreover, this method can be applied also to non-linear systems which cannot be reduced to linear systems (see, e.g., Theorem 3.9 in [CG]), whereas the method using pseudo-differential operators cannot.

Our main theorem is a generalization of some of the results proved in [CG]. In particular, we use a similar approach to establish hypoellipticity: Roughly speaking we first show that we can differentiate once in any direction, $Z$. The method of proof is analogous to the one in [C1], so the proof will be sketched only. In the step 2 case (see [CG]), once it is established that the system is differentiable once in any direction, then indefinite differentiation follows immediately by an iteration argument. This is not the case for Carnot groups of arbitrary step, and this is where most of the work in the present paper lies. The majority of the paper will be devoted to proving the main theorem, which will directly give us that $u$ has bounded Sobolev norm of any order. Once this is done we apply the Sobolev Embedding Theorem (see [F]), to conclude that $u$ is smooth.

The main motivation for our main result comes from non-linear regularity theory. We can prove a partial regularity result for weak solutions of the non-linear system

\begin{equation}
(1.2) \quad \sum_{\beta=1}^N \sum_{i,j=1}^m X_i(A_{i,j}^\alpha(x,u)X_ju^\beta) = 0, \quad \alpha = 1, \ldots, N, \quad x \in \Omega,
\end{equation}

where $\Omega \subset G$ is an open set, $u \in S^{1,2}_{loc}(\Omega, \mathbb{R}^N)$, and $A_{i,j}^\alpha(x,u)$ are bounded continuous or uniformly continuous functions satisfying for a.e. $x \in \mathbb{R}^N$, $u \in \mathbb{R}$

\[ \sum_{\alpha,\beta=1}^N \sum_{i,j=1}^m A_{i,j}^\alpha(x,u)\xi_i^\alpha \xi_j^\beta \geq \lambda|\xi|^2, \quad \xi \in \mathbb{R}^{mN}. \]

In fact, we have
Corollary 3. If \( u \in S^{1,2}_{\text{loc}}(\Omega, \mathbb{R}^N) \) is a weak solution to (1.2) then there exists an open set \( \Omega_0 \subset \Omega \) such that \( u \) is Hölder continuous in \( \Omega_0 \). Moreover, the Haar measure of \( \Omega \setminus \Omega_0 \) is zero.

This corollary extends to the Carnot group setting a celebrated result of Giusti and Miranda [GM]. Since much of the elliptic and degenerate elliptic non-linear regularity theory is based on elliptic linear estimates, we can consider this result as just a sample of what can actually be proven using the regularity of the constant coefficient system (see, for instance, [CG] and [Gi]). The proof relies on a blow up argument (see [Gi]) and is very similar to the work done for Carnot groups of step \( r = 2 \) in [CG]; we will briefly describe the argument in section 4, and we then refer the reader to the papers [CG] and [Gi] for further details of the proof.

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2. Preliminaries

A Carnot group of step \( r \geq 1 \) is defined to be a simply connected Lie Group \( G \) with a decomposition of its lie algebra \( g \) as a vector sum \( g = V^1 \oplus V^2 \oplus \cdots \oplus V^r \). This decomposition is called a stratification of \( g \) (of length \( r \)) if: \([V^j, V^\ell] = V^{j+\ell}\) for \( 1 \leq j < r \) and \([V^j, V^r] = 0 \) for \( j \geq r \). The length of the stratification then corresponds to the step of the group \( G \). In general, let \( X_{i,k} \) denote a left-invariant basis of \( V^k \), where \( 1 \leq k \leq r = \text{step of } G \) and \( 1 \leq i \leq m_k = \text{dimension of } V^k \).

For simplicity, set \( X_1 = X_{1,1} \) and \( m = m_1 \). By the horizontal layer we mean all of the vectors in the first layer \( V^1 \). Then we can let \( X = \{X_1, \ldots, X_m\} \) denote a left-invariant basis for \( V^1 \). For a function \( u = (u^1, \ldots, u^N) : G \rightarrow \mathbb{R}^N \) we set \( \{Xu\}_{i,j} = X_i u^j \) to denote the Jacobian of \( u \) with respect to the basis \( X \). We will say that \( \{Xu\} \) is the horizontal Jacobian of \( u \) since it refers to differentiation in the horizontal direction only. Following from the fact that the exponential map \( \exp : g \rightarrow G \) is a global diffeomorphism, we can use exponential coordinates on \( G \).

We say that \( P \in G \) has coordinates \((p_{i,k})\), for \( 1 \leq k \leq r \) and \( 1 \leq i \leq m_k \), where \( P = \exp(\sum_{k=1}^r \sum_{i=1}^{m_k} p_{i,k} X_{i,k}) \).

Carnot groups equipped with the Carnot-Carathéodory metric (see, e.g. [H]) behave like the Euclidean metric since natural dilations and translations can be defined. However, Euclidean spaces are abelian, and Carnot groups are non-abelian in general. In this setting, the formula for dilations is given by \( \delta_s(P) = \exp(\sum_{k=1}^r \sum_{i=1}^{m_k} s^k p_{i,k} X_{i,k}) \), for \( s > 0 \) and \( P \in G \). It is worth noting that Euclidean spaces are indeed abelian Carnot groups of step \( r = 1 \). The simplest example of a non-abelian Carnot group is the Heisenberg group, \( \mathbb{H}^1 \), which is a Carnot group of step 2 with \( \dim V^2 = 1 \) and \( \dim V^1 = 2 \).

Next we define the pseudo-distance and the gauge balls (see [F]). First, for \( P, Q \in g \) we let \( |P|^{2r} = \sum_{k=1}^r (\sum_{i=1}^{m_k} |p_{i,k}|^2)^{r/k} \). Then we have \( d(P, Q) = |Q^{-1} P| \).

In general \( d \) does not satisfy the triangle inequality and \( d \) is therefore not a metric. However, we refer to \( d \) as a gauge metric (or distance). Second, we use the pseudo-distance defined above to define the gauge balls. We have \( B(P, r) := \{ x \in G | d(P, x) < r \} \). We also have that \( |B(P, r)| = \omega_G R^Q, \) where \( \omega_G = |B(e, 1)| \), \( e \) is the group identity, and \( Q = \sum_{k=1}^r km_k \) is the so-called homogeneous dimension of \( G \) ([F]).
Next, we remind the reader of the definition of horizontal Sobolev spaces.

**Definition 4.** For \( k \in \mathbb{N} \), \( 1 \leq p \leq \infty \), and for \( \Omega \subset G \), we let \( S^{k,p}_{\text{loc}}(\Omega) \) represent the set of functions \( f : \Omega \to \mathbb{R}^N \) such that the components of \( f \) are in \( L^p_{\text{loc}}(\Omega) \) and all of the horizontal derivatives of the components of \( f \) of order up to \( k \) are in \( L^p_{\text{loc}}(\Omega) \).

The \( S^{k,p}_{\text{loc}}(\Omega) \) norm is then given by \( \| f \|_{S^{k,p}_{\text{loc}}} = \| f \|_{L^p_{\text{loc}}} + \sum_{i=1}^{k} \sum_{I \subseteq \{1, \ldots, m\}^i} \| X_{i_1}X_{i_2} \ldots X_{i_l}f \|_{L^p_{\text{loc}}} \). Note that if \( u \in S^{k,p}_{\text{loc}}(\Omega) \) for all \( k \) then we also have \( u \in W^{k,p}_{\text{loc}}(\Omega) \) for all \( k \), where \( W^{k,p}_{\text{loc}}(\Omega) \) represents the usual Euclidean Sobolev space.

**Definition 5.** A function \( u \in S^{1,2}_{\text{loc}}(\Omega) \) is a weak solution of (1.1) if we have the following identity for each \( \phi \in C^\infty(\Omega) \):

\[
(2.1) \quad \sum_{\beta=1}^{N} \sum_{i,j=1}^{m} \int_{\Omega} \left( A^\alpha_{ij} X_j u^\beta + f_i^{\alpha} \right)(p) X_i \phi^\alpha(p) \, dp = \int_{\Omega} f^{\alpha}(p) \phi^\alpha(p) \, dp.
\]

Before we prove the main theorem, we recall the following results. These show that we can differentiate our system (1.1) once in any direction, \( Z \), and that \( Zu \) is still a solution to the system.

**Theorem 6.** Let \( G \) be a Carnot group of step \( r \) and \( \Omega \subset G \) an open set, and \( u \in S^{1,2}_{\text{loc}}(\Omega) \) a weak solution to (1.1). If \( f^{\alpha}, X_{r,i_0}f^{\alpha}, f_i^{\alpha} \in L^2_{\text{loc}}(\Omega) \) for every \( 1 \leq i_0 \leq m_r \) and \( \alpha = 1, \ldots, N \), then for every \( X_{r,i_0} \in V^r \) one has

\[
X_{r,i_0} u \in S^{1,2}_{\text{loc}}(\Omega).
\]

Furthermore, for every pseudo-ball \( B(p_0,2R) \subset \Omega \), the following estimate holds for \( X_{r,i_0}u \)

\[
(2.2) \quad \| X_{r,i_0} u \|_{S^{1,2}_{\text{loc}}(B(p_0,R))} \leq C \left( \| u \|_{S^{1,2}_{\text{loc}}(B(p_0,2R))} + \| f \|_{L^2(B(p_0,2R))} + \| X_{r,i_0} f \|_{L^2(B(p_0,2R))} \right) + \sum_{i=1}^{m} \left( \| f_i \|_{L^2(B(p_0,2R))} + \| X_{r,i_0} f_i \|_{L^2(B(p_0,2R))} \right),
\]

for some constant \( C > 0 \) depending only on \( g \) and the coercivity condition. Moreover, \( X_{r,i_0} u \) is a weak solution to the system

\[
(2.3) \quad \sum_{\beta=1}^{N} \sum_{i,j=1}^{m} X_i \left( A^\alpha_{ij} X_j (X_{r,i_0} u)^\beta + X_{r,i_0} f_i^{\alpha} \right) = X_{r,i_0} f^{\alpha},
\]

for every \( \alpha = 1, \ldots, N \).

**Theorem 7.** Let \( u \in S^{1,2}_{\text{loc}}(\Omega) \) be a weak solution to (1.1), \( \Omega \subset G \) be an open set, and \( G \) a Carnot group, step \( r \). Assume \( \omega_k = X_{k,l} u^\alpha \in S^{1,2}_{\text{loc}}(\Omega) \) for every \( 1 \leq k_0 < \hat{k} \), and \( 1 \leq l \leq m_{\hat{k}} \) such that \( \omega_{\hat{k}} \) satisfies
\[ \| \omega_k^\alpha \|_{S^{1,2}_0(B(p_0, R))} \leq C \left[ \| u \|_{S^{1,2}_0(B(p_0, 2R))} + \| f \|_{L^2(B(p_0, 2R))} + \sum_{j=1}^{m_k} \sum_{k=0}^{r} \| X_{k,j} f \|_{L^2(B(p_0, 2R))} + \sum_{i=1}^{m} \| f_i \|_{L^2(B(p_0, 2R))} + \sum_{j=1}^{m_k} \sum_{k=0}^{r} \sum_{i=1}^{m} \| X_{k,j} f_i \|_{L^2(B(p_0, 2R))} \right] \]

for any ball \( B(p_0, 2R) \subset \Omega \). Further, for \( \alpha = 1, \ldots, N \), if \( f^\alpha, f_1^\alpha, X_{k_0,i_0} f^\alpha, X_{k_0,i_0} f_1^\alpha \in L^2_{0\text{loc}}(\Omega) \) then we have \( \omega_{k_0} = X_{k_0,i_0} u \in S^{1,2}_0(\Omega) \).

Moreover, \( \omega_{k_0} \) is a weak solution to

\[ \sum_{\beta=1}^{N} \sum_{i,j=1}^{m} X_i \left( A^{\alpha \beta}_{ij} X_j X_{k_0,i_0} u^\beta + A^{\alpha \beta}_{ij} [X_{k_0,i_0}, X_j] u^\beta + X_{k_0,i_0} f_1^\alpha \right) = X_{k_0,i_0} f^\alpha + \sum_{\beta=1}^{N} \sum_{i,j=1}^{m} [X_i, X_{k_0,i_0}] \left( A^{\alpha \beta}_{ij} X_j u^\beta + f_1^\alpha \right) \]

for \( \alpha = 1, \ldots, N \), and (2.4) holds for \( k = k_0 \).

**Sketch of proof.** The method of proof in [C1] still holds in this setting of systems, and we therefore refer the reader to that paper for the details of the proofs. We will need estimates on the Lebesgue norm of fractional derivatives of functions in the direction of commutators, so we must first introduce the following notation. Let \( \Omega \) be an open subset of \( G \), \( Z \in g \), \( \omega \in L^2(\Omega) \) with compact support in \( \Omega \), and \( \alpha \in (0, 1) \). We define the seminorm

\[ |\omega|_{2,\alpha}^2 = \sup_{|h|<\epsilon_0} \int_{\Omega} |h|^{-2\alpha} |\omega(ze^{hZ}) - \omega(z)|^2 \, dz, \]

where \( \epsilon_0 \) is chosen sufficiently small. Then we can express the \( L^2 \)-norm of the fractional derivative of \( \omega \) along the direction \( \partial_{p_{j,l}} \) in terms of exponential coordinates by the formula

\[ \| \partial_{p_{j,l}}^\alpha \omega \|_{L^2(G)} = \int_{G} |h|^{-2\alpha} |\hat{\omega}(p_{1,1}, \ldots, p_{j-1,l}, h, p_{j+1,l}, \ldots, p_{m,r})|^2 \, dp_{1,1} \cdots dp_{j-1,l} dh dp_{j+1,l} \cdots dp_{m,r}, \]

where we have denoted by \( \hat{\omega} \) the partial Fourier transform in the variable \( p_{j,l} \). Next, we use the following theorems of Peetre [P] and Hörmander (Theorem 4.3 in [Ho]), along with the Energy inequality:

**Theorem 8** (Peetre). Let \( G \) be a Carnot group of step \( r \), let \( 0 < \beta < \alpha < 1 \), and \( \omega \in C_0^\infty(g) \). Then there exists positive constant \( C = C(\alpha, \beta, N) \) such that

\[ C \| \partial_{p_{j,l}}^\beta \omega \|_{L^2(G)} \leq |\omega|_{\partial_{p_{j,l}}^\alpha} \leq C^{-1} \| \partial_{p_{j,l}}^\alpha \omega \|_{L^2(G)} \]

where \( p \in G \) has the coordinates \( p_{i,k} \), for \( 1 \leq i \leq m_k, 1 \leq k \leq r \).
Theorem 9 (Hörmander). Let \( \omega \in C^\infty_0(G) \). For \( 1 \leq k \leq r, 1 \leq i \leq m_k \) one has
\[
|\omega|_{X_{i,k},1} \leq C \sum_{j=1}^m |\omega|_{X_{j,1}} + \|\omega\|_{L^2(G)},
\]
for some positive constant \( C \), and for \( G \), a Carnot group of step \( r \).

Lemma 10 (Energy inequality). Let \( G \) be a Carnot group of step \( r \), \( \Omega \subset G \) be an open set (bounded). If \( u \in S^{1,2}_{loc}(\Omega) \) is a weak solution to (1.1) in \( \Omega \), with the assumption that there exists \( \lambda > 0 \) such that for every \( x \in \Omega \), one has
\[
\sum_{\alpha,\beta=1}^N \sum_{i,j=1}^m A_{i,j}^\alpha\beta(x,u) \xi_\alpha i \xi_\beta j \geq \lambda(M)|\xi|^2,
\]
then the following Caccioppoli-type inequality holds for \( 2B = B(p_0,2r) \subset \Omega \):
\[
(2.6) \quad \int_{B(p_0,r)} |Xu|^2 \, dp \leq C \int_{2B} |u|^2 \, dp + C \int_{2B} \left(|f|^2 + \sum_{i=1}^m |f_i|^2 \right) \, dp.
\]
Roughly speaking, to prove Theorem 9 we consider fractional difference quotients of \( u \) in the direction \( Z \) of order \( \alpha \in (0,1] \) and apply Theorems 8 and 9 in order to show that we can actually consider difference quotients of order 1. We express the fractional difference quotient using the formula
\[
u_{l-1}(Z,\alpha)(p) = \frac{u_{l-1}(pe^sZ) - u_{l-1}(p)}{|s|^\alpha}.
\]
Utilizing the Caccioppoli inequality gives us the result. Theorem 7 follows from Theorem 6 by using an iteration argument (based on the layer being differentiated in) to give us differentiation of order one in any direction. \( \square \)

3. Proof of Main Theorem

In this section we will show that if \( u \in S^{1,2}_{loc}(\Omega) \) is a weak solution to the constant coefficient system (1.1) then \( u \) is smooth.

3.1. Notation. For every \( h_k \in \mathbb{N} \), and for the multi-indices
\[
I_{h_k} = (i_1, i_2, \ldots, i_{h_k}) \in \{1, 2, \ldots m_k\}^{h_k},
\]
we define the following terms. Throughout the paper we let \( l-1 \) represent the lowest layer that we are differentiating with respect to. So to represent differentiating \( (h_{l-1}) + 1 \) times with respect to this layer only, we set
\[
X^{l_{h_{l-1}}-1} = X_{l-1,i_1}X_{l-1,i_2} \cdots X_{l-1,i_{h_{l-1}-1}}X_{l-1,i_{h_{l-1}}},
\]
Then for each \( k > l-1 \), i.e. each layer above \( l-1 \), the following definition represents taking \( h_k \) derivatives within each of the \( k \) layers:
\[
X^{l_{h_k}} = X_{k,i_1}X_{k,i_2}X_{k,i_3} \cdots X_{k,i_{h_k}}.
\]
We have set different notation for the lowest layer, \( k = l-1 \), that we are differentiating with respect to; this is simply a matter of convenience in order to make the computations more clear.

Whereas the above two definitions represent taking multiple derivatives in one layer at a time, the next two definitions give us notation to represent taking multiple derivatives in multiple layers. For \( k \geq l-1 \) we set
\[
V(k) = X^{l_{h_k}} X^{l_{h_{k+1}}} \cdots X^{l_{h_{r}}} u^\beta.
\]
The difference in the above two definitions is the second one keeps count of the order to simplify the computations. For every \( k \) homogeneous terms, with each one being defined in terms of the previous one due to the non-commutativity of the group structure. After only a few steps into differentiation, one can see that these terms are complicated and quickly become difficult to work with; we will use the following notation to define such non-homogeneous terms, with each one being defined in terms of the previous one in order to simplify the computations. For every \( k \geq l - 1 \), set

\[
\begin{align*}
 f_i^\alpha (k)^{k-1,i_{h_{k-1}}} & = A_{ij}^\alpha [X_{k-1,i_{h_{k-1}}}, X_j] V(k)^{k-1,i_{(h_{k-1})}-1} \\
 f^\alpha(k)^{k-1,i_{h_{k-1}}} & = X_{k-1,i_{h_{k-1}}} f^\alpha(k)^{k-1,i_{(h_{k-1})}-1}, \\
 f^\alpha(k)^{k-1,0} & = f^\alpha(k+1)^{k,i_{h_{k}}}, \\
 f^\alpha(k)^{k-1,0} & = f^\alpha(k+1)^{k,i_{h_{k}}}, \\
 f_i^\alpha(r)^{r-1,0} & = (f_i^\alpha)^{r-1,i_{h_r}} = X_{h_r} f_i^\alpha, \\
 f_i^\alpha(r)^{r-1,0} & = (f_i^\alpha)^{r-1,i_{h_r}} = X_{h_r} f_i^\alpha.
\end{align*}
\]

Last, we need a way to represent taking a different number of derivatives in any one layer than what we started with. We introduce the following notation that will represent taking \( b \) derivatives within the single layer \( V^k \), for \( b \leq h_k \) and \( l-1 \leq k \leq r \). We let

\[ X^{k,b} = X_{k,i_{h_{k}}} X_{k,i_{(h_{k})-1}} \cdots X_{i_{l}}. \]

**Note 11.** To simplify somewhat the heavy notation, we will set \( * = i_{h_{l-1}} \) throughout the paper.

### 3.2. Results

We are assuming that \( u \equiv (u^1, \ldots, u^N) : \Omega \to \mathbb{R}^N \in S^{1,2}_{loc}(\Omega) \) is a weak solution to (1.1) for every \( \beta = 1, \ldots, N \), and our first aim is to show that \( X^{l_{h_{l-1}}}_l X^{l_{h_{l}}}_l \cdots X^{l_{h_{r}}}_l u^\beta \in W^{1,2}_{loc}(\Omega) \) for every \( 1 \leq l - 1 \leq r \), for every \( \beta \), and that this is a weak solution to the system

\[
\sum_{\beta=1}^{N} \sum_{i=1}^{m} X_i \left( A_{ij}^{\alpha \beta} X_j V(l-1) + f_i^\alpha(l)^{l-1,i_{(h_{l-1})}+1} \right) = f^\alpha(l)^{l-1,i_{(h_{l-1})}+1}.
\]

In order to achieve this goal, our proof is divided into two main steps which are detailed in the following two theorems.

**Theorem 12.** Let \( u \in S^{1,2}_{loc}(\Omega) \) be a weak solution to the system (1.1) with \( f^\alpha, f_i^\alpha \in C^{\infty}(\Omega) \). If \( \bar{f}, \bar{f}_i \) are as in the statement of Theorem 1 then we have:

\[
\| X^{l_{h_{l-1}}} X^{l_{h_{l}}}_l \cdots X^{l_{h_{r}}}_l u^\beta \|_{S^{1,2}_{loc}(B)} \\
\leq \| X^{l_{h_{l-1}}} X^{l_{h_{l}}}_l \cdots X^{l_{h_{r}}}_l u^\beta \|_{S^{1,2}_{loc}(2B)} \\
+ \| \bar{f}^\alpha \|_{L^2(2B)} + \| \bar{f}_i^\alpha \|_{L^2(2B)},
\]
where either
\[ \sum_{i=l-1}^{r} |J_{hi}^i| < \sum_{i=l-1}^{r} |I_{hi}^i| \]
or
\[ \sum_{i=l-1}^{r} |J_{hi}^i| = \sum_{i=l-1}^{r} |I_{hi}^i|, \]
and one of the following two things occur:

(i.) \(|J_{hi-1}^i| < |I_{hi-1}^i|\) and consequently there exists at least one \(\beta > l - 1\) with \(|J_{hi}^\beta| > |I_{hi}^\beta|\)

(ii.) \(|J_{hi-1}^i| = |I_{hi-1}^i|\) and there exists \(\beta > l - 1\) with \(|J_{hi}^\beta| < |I_{hi}^\beta|\) and \(|J_{hi+1}^\beta| > |I_{hi+1}^\beta|\).

**Theorem 13.** Let \(u \in S_{\text{loc}}^{1,2}(\Omega)\) be a weak solution to the system (1.1) with \(f^\alpha\), \(f_i^\alpha \in C^\infty(\Omega)\). If \(\tilde{f}, \tilde{f}_i\) are as in the statement of Theorem 7 then we have:

\[
\|X^{J_{hi-1}}X^{J_{hr}}u^\beta\|_{S_{\text{loc}}^{1,2}(B)} \leq \|X^{J_{hi-1}}X^{J_{hi}} \cdots X^{J_{hr}}u^\beta\|_{S_{\text{loc}}^{1,2}(2B)} \quad \text{(3.2)}
\]

where
\[
\sum_{i=l-1}^{r} |J_{hi}^i| \leq \sum_{i=l-1}^{r} |I_{hi}^i|,
\]
with \(|J_{hi-1}^i|\) always being at least one less than \(|I_{hi-1}^i|\), and \(|J_{hi}^\beta| \geq |I_{hi}^\beta|\) for every other \(k\).

The crucial step to proving Theorems 12 and 13 is the estimates on the \(L^2\) norm of the terms \(f^\alpha(l)^{l-1,*}\) and \(f_i^\alpha(l)^{l-1,*}\). We start by showing that when we apply the definitions of \(f^\alpha(l)^{l-1,*}\) and \(f_i^\alpha(l)^{l-1,*}\) we have done one of two things. Either we have lessened the number of derivatives in the lowest layer (and thus lessened the total number of derivatives) or we have kept the same number in the lowest layer and shifted the derivatives on \(u^\beta\) somewhere to the right of the lowest layer, without adding to the total number of derivatives. Iterating Theorem 12 will eventually shift all of the derivatives to the first and last layers, \(l-1\) and \(r\), respectively. Theorem 13 tells us that derivatives in the first and last layer can be bounded above by an \(L^2\) estimate in which we have lessened the number of derivatives in the lowest layer yet added some to the higher layers. Iterating these two theorems will then give us that the \(S_{\text{loc}}^{1,2}\) norm of \(X^{I_{hi-1}}X^{I_{hi}} \cdots X^{I_{hr}}u^\beta\) is bounded above.

### 3.3. Auxiliary Lemmas

Recall the definitions of \(f^\alpha(k)^{k-1,i_{hi-1}}\) and \(f_i^\alpha(k)^{k-1,i_{hi-1}}\) above. The following two lemmas provide estimates on the \(L^2\) norms of \(f^\alpha(k)^{k-1,i_{hi-1}}\) and \(f_i^\alpha(k)^{k-1,i_{hi-1}}\) in terms of the \(L^2\) norms of our original \(f^\alpha\), \(f_i^\alpha\), and \(u^\beta\).
Lemma 14. If \( w^a, f^\alpha, \) and \( f_i^\alpha \) are the same as in Theorem 4 then we have the following:

\[
\| f^\alpha(l)^{l-1,*} \|_{L^2} \leq C \left( \| X_l^{l-1,*} X_l^{l_{h_1}} \ldots X_l^{l_{h_r}} f^\alpha \|_{L^2} + \sum_{q+k=s-1} \| X_l^{l-1,q} X_l^{l_{j_1}} V(l)^{-1,k} \|_{L^2} + \sum_{s=t+1}^{r} \| W(s) \|_{L^2} \right)
\]

where

\[
W(s) = \sum_{q+k=(l_{h_{s-1}})-1} X_l^{l-1,*} \ldots X_l^{s-1,q} X_l^{s_{1,k}} X_l^{s_{1,k}} \ldots X_l^{l_{k_r}} u^\beta + \sum_{q+k=(l_{h_{s-1}})-1} X_l^{l-1,*} \ldots X_l^{s-1,q} X_l^{s_{1,k}} f_i^\alpha(s)^{s-1,k}
\]

and the constant \( C \) depends on the coefficients.

Proof. Referring to the definition of \( f^\alpha(l)^{l-1,*} \) and using the notation\(^2\) \([X_i, X_{k, i_{h_k}}] = X^{k+1,1}\), we have:

\[
f^\alpha(l)^{l-1,*} = X_l^{l-1,*} f^\alpha(l)^{l-1,*-1} + [X_i, X_l^{l-1,*}] \left( A_{ij}^\alpha X_j V(l)^{-1,*-1} + f_i^\alpha(l)^{l-1,*-1} \right)
\]

(3.3)

Next, we rewrite \( f^\alpha(l)^{l-1,*-1} \) using this new representation. Doing so and then substituting the result back into (3.3) yields:

\[
f^\alpha(l)^{l-1,*} = X_l^{l-1,*} \left( X_l^{l-1,*-1} f^\alpha(l)^{l-1,*-2} + A_{ij}^\alpha X_l^{l_{j_1}} X_l V(l)^{-1,*-2} \right) + \left( A_{ij}^\alpha X_l^{l_{j_1}} X_l V(l)^{-1,*-1} + X_l^{l_{j_1}} f_i^\alpha(l)^{l-1,*-1} \right).
\]

Iterating this process \((* - 2)\) more times admits the following:

\(^2\)Suppose we have \([X_i, X_{m,b}]\). This is a linear combination of the vector field that one obtains by adding the subscripts together, i.e. \( X_{(i+m,b)} \). For simplicity, whenever we commute two vector fields, say \( X_{i,a} \) and \( X_{m,b} \), we will call the new term \( X^{i+m,1} \) and drop the second subscript.
f^α(l)^{-1,*} = X^{l-1,*} f^α(l)^{-1,0} + \sum_{q+k=s-1} A^{\alpha\beta}_{ij} X^{l-1,\alpha} X_j V(t)^{-1,k} \\
+ \sum_{q+k=s-1} X^{l-1,\alpha} X_j f^\alpha(t)^{-1,k} \\
= X^{l-1,*} f^\alpha(l+1)^{l,i,h_l} + \sum_{q+k=s-1} A^{\alpha\beta}_{ij} X^{l-1,\alpha} X_j V(t)^{-1,k} \\
+ \sum_{q+k=s-1} X^{l-1,\alpha} X_j f^\alpha(t)^{-1,k}. \\
(3.4)

So far we have rewritten f^α(l)^{-1,*} in terms of the next higher step with our eventual goal being to rewrite it based on differentiation along the original f^α. Once we finish rewriting this term there will still be other terms that appear, in particular ones similar to the f^α_l(l)^{-1,k} above. However, our next lemma will concern terms of this type, so for now we will leave these as is. What we do next is continue the following representation for f^α:

\begin{align*}
\sum_{s=l+1}^r \left( \sum_{q+k=\ell(h_{s-1})-1} A^{\alpha\beta}_{ij} X^{l-1,*} \cdots X^{s-1,\alpha} X_j V(s)^{-1,k} \right) \\
+ \sum_{s=l+1}^r \left( \sum_{q+k=\ell(h_{s-1})-1} X^{l-1,*} \cdots X^{s-1,\alpha} X_j f^\alpha(s)^{-1,k} \right) \\
+ \sum_{q+k=s-1} A^{\alpha\beta}_{ij} X^{l-1,\alpha} X_j V(t)^{-1,k} \\
+ \sum_{q+k=s-1} X^{l-1,\alpha} X_j f^\alpha(t)^{-1,k} \\
= X^{l-1,*} X^{l,i,h_l} \cdots X^{r,i,h_r} f^\alpha + \sum_{s=l+1}^r W(s) \\
+ \sum_{q+k=s-1} A^{\alpha\beta}_{ij} X^{l-1,\alpha} X_j V(t)^{-1,k} \\
+ \sum_{q+k=s-1} X^{l-1,\alpha} X_j f^\alpha(t)^{-1,k}.
\end{align*}

Lastly, take the L^2 norm of both sides to obtain the desired result. □
Lemma 15. If \( w^\alpha, f^\alpha, \) and \( f_i^\alpha \) are the same as in Theorem \( \text{14} \) then we have the following:

\[
\| f_i(l)^{l-1,*} \|_{L^2} \leq C \left( \| X^{l-1,*} X^{l,i_{h_1}} \ldots X^{r,i_{h_r}} f_i \|_{L^2} + \sum_{s=l+1}^{r} \| T(s) \|_{L^2} \right)
\]

where

\[
T(s) = \sum_{q+k=s-1}^{s} X^{l-1,*} \ldots X^{s-1,q} X^{s} V(s)^{s-1,k}.
\]

Proof. Referring to the definition of \( f_i(l)^{l-1,*} \) and using the notation \([X_i, X_{k,i_{h_k}}] = X^{k+1,1} \), we have:

\[
f_i(l)^{l-1,*} = A_{ij}^\alpha \left[ X_{l-1,*}, X_j \right] V(l)^{l-1,*-1} + X_{l-1,*} f_i(l)^{l-1,*-1} = A_{ij}^\alpha X^{l,1} V(l)^{l-1,*-1} + X_{l-1,*} f_i(l)^{l-1,*-1}.
\]

Next we apply this representation to \( f_i(l)^{l-1,*-1} \) and substitute the result into the equality above to obtain:

\[
f_i(l)^{l-1,*} = A_{ij}^\alpha X^{l,1} V(l)^{l-1,*-1} + X_{l-1,*} \left( A_{ij}^\alpha X^{l,1} V(l)^{l-1,*-2} \right) + X_{l-1,*} f_i(l)^{l-1,*-2} = A_{ij}^\alpha X^{l,1} V(l)^{l-1,*-1} + X_{l-1,*} f_i(l)^{l-1,*-2}.
\]

Iterating this \((* - 2)\) more times we get:

\[
f_i(l)^{l-1,*} = \sum_{q+k=s-1}^{s} A_{ij}^\alpha X^{l-1,q} X^{l,1} V(l)^{l-1,k} + X^{l-1,*} f_i(l)^{l-1,0} = \sum_{q+k=s-1}^{s} A_{ij}^\alpha X^{l-1,q} X^{l,1} V(l)^{l-1,k} + X^{l-1,*} f_i(l + 1)^{l,i_{h_1}}.
\]

Continue by first applying the argument above to \( f_i(l + 1)^{l,i_{h_1}} \), then to \( f_i(l + 2)^{l+1,i_{h_{i+1}}} \), etc, and last to \( f_i(l)^{l-1,*-1} \). Proceeding in this way, one obtains the following representation for \( f_i(l)^{l-1,*} \):

\[
f_i(l)^{l-1,*} = \sum_{q+k=s-1}^{s} A_{ij}^\alpha X^{l-1,q} X^{l,1} V(l)^{l-1,k}
\]

\[
+ \sum_{s=l+1}^{r} \left( \sum_{q+k=s-1}^{s} A_{ij}^\alpha X^{l-1,*} \ldots X^{s-1,q} X^{s,1} V(s)^{s-1,k} \right)
\]

\[
+ X^{l-1,*} X^{l,i_{h_1}} \ldots X^{r,i_{h_r}} f_i
\]

\[
= \sum_{q+k=s-1}^{s} A_{ij}^\alpha X^{l-1,q} X^{l,1} V(l)^{l-1,k} + \sum_{s=l+1}^{r} T(s)
\]

\[
+ X^{l-1,*} X^{l,i_{h_1}} \ldots X^{r,i_{h_r}} f_i.
\]

Last, take the \( L^2 \) norms of both sides to complete the proof.
We can apply lemma \[14\] to rewrite lemma \[13\]. As a direct consequence we have:

**Lemma 16.** If \(u^2\), \(f^n\), and \(f_1^n\) are the same as in Theorem \[17\] then we have the following:

\[
\| f(l)^{l-1,*} \|_{L^2} 
\leq 
C \left( \sum_{q+k=s-1} \| X^{l-1,q} X^{l,1} X_s V(l)^{l-1,k} \|_{L^2} 
+ \sum_{q+k=q+1} \left( \sum_{y+z=k-1} \| X^{l-1,q} X^{l,1} X^{l-1,k} X^{l,i_{h,v}} \|_{L^2} \right) 
+ \sum_{q+k=q+1} \left( \sum_{s=l+1} \left( \sum_{y+z=k-1} \| X^{l-1,q} X^{l,1} X^{l-1,k} X^{l,i_{h,v}} \|_{L^2} \right) \right) 
+ C(f) + C(f_1) \right),
\]

where \(C(f)\) and \(C(f_1)\) are terms corresponding to differentiation on the original \(f\) and \(f_1\), respectively, and \(C\) is a constant depending on the coefficients.

**Proof.** The two terms in lemma \[14\] to focus on are \(X^{l-1,q} X^{l,1} f_i(l)^{l-1,k}\) and \(X^{l-1,*} \cdots X^{l-1,q} X^{l,1} f_i(s)^{s-1,k}\), so begin by applying lemma \[14\] to the first term:

\[
\sum_{q+k=s-1} \| X^{l-1,q} X^{l,1} f_i(l)^{l-1,k} \|_{L^2} 
\leq \sum_{q+k=q+1} \| X^{l-1,q} X^{l,1} X^{l-1,k} \|_{L^2} 
+ \sum_{q+k=q+1} \left( \sum_{y+z=k-1} \| X^{l-1,q} X^{l,1} X^{l-1,k} X^{l,i_{h,v}} \|_{L^2} \right) 
+ \sum_{q+k=q+1} \left( \sum_{s=l+1} \left( \sum_{y+z=k-1} \| X^{l-1,q} X^{l,1} X^{l-1,k} X^{l,i_{h,v}} \|_{L^2} \right) \right) 
+ C(f) + C(f_1),
\]
Similarly, applying lemma 15 to the term \(X^{l-1,*} \ldots X^{s-1,q} X^{s,1} f_i(s)^{s-1,k}\), we have:

\[
\sum_{q+k=i(h_{n-1})-1} \|X^{l-1,*} \ldots X^{s-1,q} X^{s,1} f_i(s)^{s-1,k}\|_{L^2} \leq \sum_{q+k=i(h_{n-1})-1} \left( \sum_{y+z=k-1} \left( \sum_{s=l+1}^r \|A \alpha \beta X^{l-1,*} \ldots \right. \right. \\
\left. \left. X^{s-1,q} X^{s,1} X^{s,1,y} V(s)^{s-1,z} \|_{L^2} \right) \right) \\
+ \sum_{q+k=i(h_{n-1})-1} \left( \sum_{y+z=i(h_{p-1})-1} \left( \sum_{s=l+1}^r \|A \alpha \beta X^{l-1,*} \ldots \right. \right. \\
\left. \left. X^{s-1,q} X^{s,1} X^{s,1,k} X^{s,1,y} \ldots V(p+1) \|_{L^2} \right) \right) \\
+ \sum_{q+k=i(h_{n-1})-1} \left( \sum_{s=l+1}^r \|X^{l-1,*} \ldots X^{s-1,q} X^{s,1} X^{s,1,k} X^{s,1,y} \ldots X^{r,1} f_i\|_{L^2} \right).
\]

Substituting these estimates into the inequality derived in lemma 14 one arrives at the desired result.

In order to use the above lemmas in the proofs of Theorems 12 and 13 we need to modify them by grouping like terms together. Recall that differentiation along the last layer, \(r\), commutes with all other layers, so we can freely move these derivatives around. However, since each of the other layers do not commute we gain extra terms, called commutators, when we choose to switch the order of differentiation. The following lemma will be applied numerous times to these commutator terms that appear when we group like terms together.

**Lemma 17.** Applying commutator properties, we have:

\[
X^{l-1,q} X^{l,1} X^{l-1,k} X^{l,i_{h_i}} V(l+1) = X^{l-1,q+k} X^{l,i_{h_i}+1} V(l+1) \\
+ \sum_{s+t=q+k-1} X^{l-1,s} X^{2l-1,1} X^{l-1,t} X^{l,i_{h_i}} V(l+1).
\]

**Proof.** Begin with \(X^{l-1,q} X^{l,1} X^{l-1,k} X^{l,i_{h_i}} V(l+1)\); we will transfer \(X^{l,1}\) to its like terms. In order to do this, we need to shift it "\(k\)" times to the right to get it past all \(k\) derivatives in the "\(l-1\)" direction. To see how this process works, first move \(X^{l,1}\) just once to the right:

\[
X^{l-1,q} X^{l,1} X^{l-1,k} X^{l,i_{h_i}} V(l+1) = X^{l-1,q} X^{l-1,1} X^{l,1} X^{l-1,k-1} X^{l,i_{h_i}} V(l+1) \\
+ X^{l-1,q} \left( X^{l,1} X^{l-1,1} \right) X^{l-1,k-1} X^{l,i_{h_i}} V(l+1) \\
= X^{l-1,q} X^{l-1,1} X^{l,1} X^{l-1,k-1} X^{l,i_{h_i}} V(l+1) \\
+ X^{l-1,q} X^{2l-1,1} X^{l-1,k-1} X^{l,i_{h_i}} V(l+1).
\]

From here it is clear to see that if we apply this same technique "\(k-1\)" more times we eventually have the following:
\[ X^{l-1,q} X^{l-1,k} X^{l,i_h_l} V(t+1) = X^{l-1,q} X^{l-1,k} X^{l,i_h_l} V(l+1) + X^{l-1,q} X^{2l-1,k} X^{l-1,k-1} X^{l,i_h_l} V(l+1) + X^{l-1,q} X^{l-1,k} X^{l-1,k-1} X^{l,i_h_l} V(l+1) + \cdots + X^{l-1,q} X^{l-1,k} X^{l-1,k-1} X^{l,i_h_l} V(l+1) = X^{l-1,q+k} X^{l,i_{(h_l)+1}} V(l+1) + \sum_{s+t=q+k-1} X^{l-1,s} X^{2l-1,t} X^{l,i_h_l} V(l+1). \]

\[ \square \]

3.4. Proof of Theorems 9 and 10.

Proof. (Theorem 12)

Once again we let \( * = i_{h_l-1} \). We assume that \( u^\alpha \in S^1_{loc}(\Omega) \) is a weak solution to the system (11), and we want to show that the following inequality is finite:

\[
\| X^{l_i_{h_l-1}} X^{l_{h_l}} \ldots X^{l_{h_r}} u \|_{S^1_{loc}}^2 \leq \left[ \| X^{l-1,i_{h_l-1}} X^{l_{h_l}} \ldots X^{l_{h_r}} u^\alpha \|_{S^1_{loc}}^2 + \| f^{(l)} \|_{L^2} \right. \\
+ \sum_{j=1}^{r} \left( \sum_{k=1}^{m_j} \| X^{l,1} f^{(l)} \|_{L^2} \right) + \sum_{i=1}^{m} \| f^{(l)} \|_{L^2} \\
+ \left. \sum_{j=1}^{m} \left( \sum_{k=1}^{m_j} \| X^{l,j,k} f^{(l)} \|_{L^2} \right) \right]. 
\]

(3.6)

In order to show that \( X^{l_i_{h_l-1}} X^{l_{h_l}} \ldots X^{l_{h_r}} u \) is bounded from above, we need to show that each of the terms on the right hand side of (3.6) is bounded from above. The terms that require the most work are \( X^{l,j,k} f^{(l)} \) and \( X^{l,j,k} f^{(l)} \), so we will begin with these; this is where we use Lemmas 13 and 19 since they essentially show us exactly what we are looking at when we see \( f^{(l)} \) and \( f^{(l)} \). Set

\[
X^{l,j,k} f^{(l)} = X^{l,j,k} X^{l,i_{h_l}} \ldots X^{l,i_{h_{r-1}}} f^{(l)} + X^{l,j,k} \left( X^{l-1,s-1} X^{l,i_{h_l+1}} \ldots X^{l,i_{h_{r-1}}} f^{(l)} \right) \\
+ \sum_{q+k=i_{h_l-1}} X^{l-1,s} X^{l,i_{h_l}} f^{(l)} \ldots X^{l,i_{h_{r-1}}} f^{(l)} \right) \\
+ P_1 + P_2 + P_3 + P_4 + P_5 + P_6,
\]

where for simplicity we have let \( P_j, j = 1, \ldots, 6 \) equal the following:

...
\[ P_1 = \sum_{q+k=s-1} A_{ij}^{\alpha \beta} X_{j,k} X^{l-1,q} X^{l-1} X_j V(l)^l-1,k, \]
\[ P_2 = \sum_{s=l+1}^r \left( \sum_{q+k=i(h_{s-1})-1} A_{ij}^{\alpha \beta} X_{j,k} X^{l-1,s} \cdots X^{s-1,q} X^{s,1} X_j X^{s-1,k} X^{s,i_h} \cdots X^r,i_{hr} u^\beta \right), \]
\[ P_3 = C \sum_{q+k=s-1} X_{j,k} X^{l-1,q} X^{l,1} X^{l-1,y} X^{l,1} V(l)^l-1,z, \]
\[ P_4 = \sum_{s=l+1}^r \left( \sum_{q+k=i(h_{s-1})-1} \left( \sum_{s+1}^r A_{ij}^{\alpha \beta} X_{j,k} X^{l-1,s} \cdots X^{s-1,q} X^{s,1} X^{s-1,k} X^{s,1} V(s)^{s-1,z} \right) \right), \]
\[ P_5 = \sum_{q+k=i(h_{s-1})-1} \left( \sum_{s+1}^r \left( \sum_{s+1}^r A_{ij}^{\alpha \beta} X_{j,k} X^{l-1,s} \cdots X^{s-1,q} X^{s,1} X^{s-1,k} X^{s,1} V(s)^{s-1,z} \right) \right), \]
\[ P_6 = \sum_{q+k=i(h_{s-1})-1} \left( \sum_{s+1}^r \left( \sum_{s+1}^r A_{ij}^{\alpha \beta} X_{j,k} X^{l-1,s} \cdots X^{s-1,q} X^{s,1} X^{s-1,i_h} \cdots X^{p+1,q} X^{p,1} X^{p-1,z} X^{p,i_{hp}} V(p+1) \right) \right). \]

Before we begin bounding each of the \( P_i \) terms, recall from (3.6) that for \( X_{j,k} \) we are assuming \( j \geq l-1 \). If \( j > l-1 \), then we always have fewer derivatives in the lowest layer than when we began. When this is the case, the estimate (3.4) in Theorem 12 is satisfied. The only time that we may not lessen the number of derivatives in the lowest layer is if \( j = l-1 \), so this is what we will assume from here on.

When we apply Lemma 17 to the terms above that need rearranging, things quickly get complicated. However, the idea behind this theorem is not to necessarily have to keep track of each and every derivative, but instead to first count the total derivatives on \( u^\beta \) and second to count the number of derivatives on \( u^\beta \) in the lowest layer. We proceed by writing an estimate for each of the \( P_i \)'s above in relation to how many derivatives are attached to \( u^\beta \) in each layer. One item worth noting is that when you "move" derivatives around using lemma 17, you end up with numerous commutator terms, one for each time you move a derivative that does not commute. We do not need to actually keep track of each of these terms; we just note that when we commute two derivatives, we end up with one derivative in a higher layer, thus lessening the total number of derivatives. Therefore, we can group all of these terms together in one collective term that we will call "commutator" and be confident that this term has less total derivatives on \( u \) than what we started with.

For \( P_1 \):
\[
\| P_1 \|_{L^2} \leq \| \sum_{q+k=s-1} A_{ij}^{\alpha \beta} X_{j,k} X^{l-1,q} X^{l,1} X_j V(l)^l-1,k \|_{L^2} \\
\leq C \| X_j X_{j,k} X^{l-1,s} X^{l,h_{l+1}} V(l+1) \|_{L^2} \\
+ C \| \text{Commutator} \|_{L_{loc}^{2}} \\
\leq C \| J_{h_{l+1}} J_{h_l} \cdots J_{h_1} u^\beta \|_{S_{loc}^{1,2}}.
\]
where either $\sum_{i=l-1}^r J_h < \sum_{i=l-1}^r I_h$, (as is the case in the commutator term) or
$\sum_{i=l-1}^r J_h = \sum_{i=l-1}^r I_h$, so that $|J_{h_{i-1}}| < |I_{h_{i-1}}|$ and $|J_{h_0}| > |I_{h_0}|$ with $J_{h_k} = I_{h_k}$ elsewhere. Thus, $P_1$ satisfies estimate 8.1 in the statement of the theorem.

\textbf{For } $P_2$:

$$
\| P_2 \|_{L^2} \leq \sum_{s=l+1}^r \left( \sum_{q+k=(i_{h_{s-1}})-1}^r \| A^{\alpha \beta}_{ij} X_j X^l X^{l-1} \cdots X^{s-1,q} X^{s-1,k} X^{s,1} u^\beta \|_{L^2} \right)
$$

$$
\leq C \sum_{s=l+1}^r \| X_j X_j X^{l-1} \cdots X^{s-1,h_{s-1}-1} X^{s,1} X^{s,h_s+1} \cdots X^{r,i_k} u^\beta \|_{L^2}
$$

$$
+ \| \text{Commutator} \|_{L^2_{\text{loc}}}
$$

$$
\leq C \| J_{h_{i-1}} J_{h_{i-1}} \cdots J_{h_s} u^\beta \|_{S^1_{\text{loc}}}
$$

where either $\sum_{i=l-1}^r J_h < \sum_{i=l-1}^r I_h$, (as is the case in the commutator term) or
$\sum_{i=l-1}^r J_h = \sum_{i=l-1}^r I_h$, so that one of three things happens: Either $J_{h_{i-1}} = I_{h_{i-1}}$ and there exists $\beta_i > h_{i-1}$, $i \geq 1$, such that $J_{h_i} > I_{h_i}$ with $J_{h_k} < I_{h_k}$ for every
$h_k \neq h_{i-1}, \beta_i$ or $J_{h_{i-1}} = I_{h_{i-1}}$ with $J_{h_{i-1}} < I_{h_{i-1}}$ (actually $I_{h_{i-1}} - 1$), $h_s > I_{h_s}$ (actually $I_{h_s} + 1$) and $J_{h_k} < I_{h_k}$ for every $k \neq l-1, s-1, s$ or $J_{h_{i-1}} = I_{h_{i-1}}$ and there exists $\beta_i > h_{i-1}$, $i \geq 1$, such that $J_{h_i} > I_{h_i}$ with $J_{h_k} < I_{h_k}$ for every $h_k \neq h_{i-1}, \beta_i$. Thus, $P_2$ satisfies estimate 8.1 in the statement of the theorem.

\textbf{For } $P_3$:

$$
\| P_3 \|_{L^2} \leq C \sum_{q+k=s-1} \left( \sum_{y+z=k-1} \| X_j X^l X^{l-1,q} X^{l-1,y} X^{l-1,z} \|_{L^2} \right)
$$

$$
\leq C \| J_{h_{i-1}} J_{h_{i-1}} \cdots J_{h_s} u^\beta \|_{S^1_{\text{loc}}}
$$

where either $\sum_{i=l-1}^r J_h = \sum_{i=l-1}^r I_h$, so that $J_{h_{i-1}} < I_{h_{i-1}}$ (actually $I_{h_{i-1}} - 2$), $J_{h_i} > I_{h_i}$ (actually $I_{h_i} + 2$), and $J_{h_k} = I_{h_k}$ for every $k \neq l-1, l$ or $\sum_{i=l-1}^r J_h < \sum_{i=l-1}^r I_h$, with $J_{h_{i-1}}$ always at least 2 less than $I_{h_{i-1}}$ and there exists $\beta_i > h_{i-1}$ for $i \geq 1$ such that $J_{h_i} > I_{h_i}$ with $J_{h_k} < I_{h_k}$ for all $h_k \neq h_{i-1}, \beta_i$. Thus, $P_3$ satisfies estimate 8.1 in the statement of the theorem.

\textbf{For } $P_4$:

$$
\| P_4 \|_{L^2} \leq \sum_{q+k=s-1} \left( \sum_{y+z=i_{h_{s-1}}-1}^r \left( \sum_{s=l+1}^r \| X_j X^l X^{l-1,q} X^{l-1,k} X^{l,i_k} Y^{l-1,y} X^{s-1,z} V(s) X^{s-1,1} \|_{L^2} \right) \right)
$$

$$
\leq C \| J_{h_{i-1}} J_{h_{i-1}} \cdots J_{h_s} u^\beta \|_{S^1_{\text{loc}}}
$$

where either $\sum_{i=l-1}^r J_h = \sum_{i=l-1}^r I_h$, so that $J_{h_{i-1}} < I_{h_{i-1}}$ (actually $I_{h_{i-1}} - 1$), $J_{h_i} > I_{h_i}$ (actually $I_{h_i} + 1$), $J_{h_{i-1}} < I_{h_{i-1}}$ (actually $I_{h_{i-1}} - 1$), $J_{h_i} > I_{h_i}$ (actually $I_{h_i} + 1$), and $J_{h_k} = I_{h_k}$ for every $k \neq l-1, l, s-1, s$ or $\sum_{i=l-1}^r J_h < \sum_{i=l-1}^r I_h$, with $J_{h_{i-1}}$ and $J_{h_{i-1}}$ always at least 1 less than $I_{h_{i-1}}$ and $I_{h_{i-1}}$, respectively, and then there exists $\beta_i > h_{i-1}$ and/or $\beta_i > h_{i-1}$ for $i \geq 1$ such that
where either $\sum J_i$ with $J_{h_i} \leq I_{h_i}$ for all $h_i \neq h_{i-1}, h_{s-1}, \beta_i$. Thus, $P_3$ satisfies estimate (3.1) in the statement of the theorem.

For $P_5$:

$$\| P_5 \|_{L^2} \leq \sum \left( \sum \sum_{s=l+1}^r \| A^{i,j}_{ij} X_{j,k} X^{i-1,s} \cdots X^{s-1,q} X^{s-1,y} X^{s-1,y} V(s) \|_{L^2} \right)$$

$$\leq C \| J_{h_{i-1}} J_{h_i} \cdots J_{h_s} u^\beta \|_{S_{1,2}^{1,2}}$$

where either $\sum_{i=l-1}^r J_i = \sum_{i=l-1}^r I_i$, so that $J_{h_{i-1}} = I_{h_{i-1}}, J_{h_{s-1}} < I_{h_{s-1}}$ (actually $= I_{h_{s-1}} - 2$), $J_{h_s} > I_{h_s}$ (actually $= I_{h_s} + 2$), and $J_{h_k} = I_{h_k}$ for every $k \neq s - 1, s$ or $\sum_{i=l-1}^r J_i < \sum_{i=l-1}^r I_i$ with $J_{h_{s-1}}$ always at least 2 less than $I_{h_{s-1}}$, and then there exists $\beta_i > h_{s-1}$ for $i \geq 1$ such that $J_{\beta_i} > I_{\beta_i}$ with $J_{h_k} \leq I_{h_k}$ for all $h_k \neq h_{s-1}, \beta_i$. Thus, $P_3$ satisfies estimate (3.1) in the statement of the theorem.

For $P_6$:

$$\| P_6 \|_{L^2} \leq \sum \left( \sum \sum_{s=l+1}^r \left( \sum_{p=s+1}^r \left( \sum_{p=1}^r \left( \sum_{p=1}^r \| A^{i,j}_{ij} X_{j,k} X^{i-1,s} \cdots X^{s-1,q} X^{s-1,y} X^{s-1,y} V(p + 1) \|_{L^2} ) )\right) \right) \right)$$

$$\leq C \| J_{h_{i-1}} J_{h_i} \cdots J_{h_s} u^\beta \|_{S_{1,2}^{1,2}}$$

where either $\sum_{i=l-1}^r J_i = \sum_{i=l-1}^r I_i$, so that $J_{h_{i-1}} < I_{h_{i-1}}$ (actually $= I_{h_{i-1}} - 1$), $J_{h_s} > I_{h_s}$ (actually $= I_{h_s} + 1$), $J_{h_{p-1}} < I_{h_{p-1}}$ (actually $= I_{h_{p-1}} - 1$), $J_{h_p} > I_{h_p}$ (actually $= I_{h_p} + 1$), and $J_{h_k} = I_{h_k}$ for every $k \neq s - 1, s, p - 1, p$ or $\sum_{i=l-1}^r J_i < \sum_{i=l-1}^r I_i$ with $J_{h_{i-1}}$ and $J_{h_{p-1}}$ always at least 1 less than $I_{h_{i-1}}$ and $I_{h_{p-1}}$, respectively, and then there exists $\beta_i > h_{s-1}$ and/or $\beta_i > h_{p-1}$ for $i \geq 1$ such that $J_{\beta_i} > I_{\beta_i}$ with $J_{h_k} \leq I_{h_k}$ for all $h_k \neq h_{s-1}, h_{p-1}, \beta_i$. Thus, $P_6$ satisfies estimate (3.1) in the statement of the theorem.

Combining these estimates we have the following:
\[ \| X_{j,k}f^{\alpha}(l)^{l-1,*} \|_{L^2} \leq \| X_{j,k}X^{l-1,*}X^{r,i_1h_1} \cdots X^{r,i_rh_r} f^{\alpha} \|_{L^2} \]
\[ + \| X_{j,k}X^{l-1,*}X^{r,i_{(b)}h_{(b)+1}} \cdots X^{r,i_rh_r} f^{\alpha} \|_{L^2} \]
\[ + \| X_{j,k} \sum_{q+k=\gamma(ih,\gamma-1)} \left( \sum_{s=t+1} \cdots X^{s-1,1}X^{s-1,k}X^{s,i,h_s} \cdots X^{r,i_rh_r} f^{\alpha} I_i \right) \right) \]
\[ (3.7) \]
\[ + \| P_1 \|_{L^2} + \| P_2 \|_{L^2} + \| P_3 \|_{L^2} + \| P_4 \|_{L^2} + \| P_5 \|_{L^2} + \| P_6 \|_{L^2} \]
\[ \leq \| X_{j,k}X^{l-1,*}X^{r,i_1h_1} \cdots X^{r,i_rh_r} f^{\alpha} \|_{L^2} \]
\[ + \| X_{j,k}X^{l-1,*}X^{r,i_{(b)}h_{(b)+1}} \cdots X^{r,i_rh_r} f^{\alpha} \|_{L^2} \]
\[ + \| X_{j,k} \sum_{q+k=\gamma(ih,\gamma-1)} \left( \sum_{s=t+1} \cdots X^{s-1,1}X^{s-1,k}X^{s,i,h_s} \cdots X^{r,i_rh_r} f^{\alpha} I_i \right) \right) \]
\[ \]
Next, set
\[ X_{j,k}f^{\alpha}(l)^{l-1,*} = X_{j,k}X^{l-1,*}X^{r,i_1h_1} \cdots X^{r,i_rh_r} f^{\alpha} \]
\[ + \sum_{s=t+1} \left( \sum_{q+k=\gamma(ih,\gamma-1)} A_{ij}^{\alpha\beta} X_{j,k}X^{l-1,*} \cdots X^{s-1,1}X^{s,1}V(s)^{s-1,k} \right) \]
\[ + \sum_{q+k=\gamma(s-1)} A_{ij}^{\alpha\beta} X_{j,k}X^{l-1,*} \cdots X^{s-1,1}X^{s,1}V(l)^{l-1,k} \]
\[ = X_{j,k}X^{l-1,*}X^{r,i_1h_1} \cdots X^{r,i_rh_r} f^{\alpha} \]
\[ + Q_1 + Q_2, \]

where we let
\[ Q_1 = \sum_{s=t+1} \left( \sum_{q+k=\gamma(ih,\gamma-1)} A_{ij}^{\alpha\beta} X_{j,k}X^{l-1,*} \cdots X^{s-1,1}X^{s,1}V(s)^{s-1,k} \right), \]
\[ Q_2 = \sum_{q+k=\gamma(s-1)} A_{ij}^{\alpha\beta} X_{j,k}X^{l-1,*} \cdots X^{s-1,1}X^{s,1}V(l)^{l-1,k}. \]

Proceeding as we did before by applying the results of Lemma 17, we have the following:

For \( Q_1 \):

\[ \| Q_1 \|_{L^2} \leq \sum_{s=t+1} \left( \sum_{q+k=\gamma(ih,\gamma-1)} A_{ij}^{\alpha\beta} X_{j,k}X^{l-1,*} \cdots X^{s-1,1}X^{s,1}V(s)^{s-1,k} \right), \]

where either \( \sum_{i=1}^r J_{h_i} = \sum_{i=1}^r I_{h_i} \) such that \( J_{h_{i-1}} = I_{h_{i-1}}, J_{h_{i+1}} < I_{h_{i+1}} \)

(\text{actually} = I_{h_{i-1}} - 1, J_{h_{s}} > I_{h_{s}} \text{ (actually} = I_{h_{s}} + 1), \text{ and} J_{h_{k}} = I_{h_{k}} \text{ for every} \)

\( k \neq l - 1, s - 1, s \) or \( \sum_{i=1}^r J_{h_i} < \sum_{i=1}^r I_{h_i} \) with \( J_{h_{i-1}} = I_{h_{i-1}}, J_{h_{i+1}} \text{ always at} \)
least 1 less than $I_{h_{i-1}}$, and then there exists $\beta_i > h_{i-1}$ for $i \geq 1$ such that $J_{\beta_i} > I_{\beta_i}$ with $J_{\beta_i} \leq I_{h_k}$ for all $h_k \neq h_{i-1}, h_{i-1}, \beta_i$. Thus, $Q_1$ satisfies estimate 3.1 in the statement of the theorem.

For $Q_2$:

\[
\| Q_2 \|_{L^2} \leq \sum_{q+k=s-1}^{r} \| A_{qj}^{\beta} X_{j,k} X^{l-1, q} X^{l, 1} V(l)^{l-1, k} \|_{L^2} \\
\leq C \| J_{h_{i-1}} J_{h_{i}} \cdots J_{h_{r}} u^{\beta} \|_{S_{loc}^{1, 2}}
\]

where either $\sum_{t=1}^{r} J_{h_t} = \sum_{t=1}^{r} I_{h_t}$ such that $J_{h_{i-1}} < I_{h_{i-1}}$ (actually $= I_{h_{i-1}}$), $J_{h_{i-1}} > I_{h_{1}}$ (actually $= I_{h_{1}} + 1$), and $J_{h_{1}} = I_{h_{1}}$ for every $k \neq l - 1, l$ or $\sum_{t=1}^{r} J_{h_t} < \sum_{t=1}^{r} I_{h_t}$ with $J_{h_{i-1}} < I_{h_{i-1}}$, and then there exists $\beta_i > h_{i-1}$ for $i \geq 1$ such that $J_{\beta_i} > I_{\beta_i}$ with $J_{\beta_i} \leq I_{h_k}$ for all $h_k \neq h_{i-1}, \beta_i$. Thus, $Q_2$ satisfies estimate 3.1 in the statement of the theorem.

Combining these estimates, we have the following:

\[
\| X_{j,k} f^\alpha(l)^{l-1, *} \|_{L^2} \leq \| X_{j,k} X^{l-1, *} X^{l, i_{h_1}} \cdots X^{r, i_{h_{r}}} f^\alpha \|_{L^2} \\
+ \sum_{r=1}^{r} \left( \sum_{s=i_{h_{i-1}}-1}^{r} C \| X_{j,k} X^{l-1, *} \cdots X^{s-1, q} X^{s, 1} V(s)^{s-1, k} \|_{L^2} \right) \\
+ \sum_{q+k=s-1}^{r} \| X_{j,k} X^{l-1, q} X^{l, 1} V(l)^{l-1, k} \|_{L^2} \\
\leq \| X_{j,k} X^{l-1, *} X^{l, i_{h_1}} \cdots X^{r, i_{h_{r}}} f^\alpha \|_{L^2} \\
+ \| Q_1 \|_{L^2} + \| Q_2 \|_{L^2} \\
\leq \| X_{j,k} X^{l-1, *} X^{l, i_{h_1}} \cdots X^{r, i_{h_{r}}} f^\alpha \|_{L^2} \\
+ C \| J_{h_{i-1}} J_{h_{i}} \cdots J_{h_{r}} u^{\beta} \|_{S_{loc}^{1, 2}}
\]

What we have done is bound the terms $X_{j,k} f^\alpha(l)^{l-1, *}$ and $X_{j,k} f_i^\alpha(l)^{l-1, *}$ from above by counting and keeping track of the derivatives in each step. We will also need bounds for $f^\alpha(l)^{l-1, *}$ and $f_i^\alpha(l)^{l-1, *}$, but since we are counting derivatives, all of our estimates are just one less than what we calculated above. It is clear then, that the total number of derivatives in these terms is less than what we started with so they satisfy the estimate 3.1 in Theorem 3.1

If we look at the terms that involve differentiation on $f^\alpha$ and $f_i^\alpha$, we can group these together and collectively name them $\tilde{f}^\alpha$ and $\tilde{f}_i^\alpha$. Since our original $f^\alpha$ and $f_i^\alpha$ are in $C^\infty$, we know that these are bounded:
\[
\| \tilde{f}^\alpha \|_{L^2} \leq \| X^{l-1,s} X^l,i_{h_1} \ldots X^{r,i_{h_r}} f^\alpha \|_{L^2} \\
+ \sum_{j=l-1}^{r} \sum_{k=1}^{m_j} \| X_{j,k} X^{l-1,s} X^l,i_{h_1} \ldots X^{r,i_{h_r}} f^\alpha \|_{L^2},
\]
and
\[
\| \tilde{f}^\alpha \|_{L^2} \leq \| X^{l-1,s} X^l,i_{h_1} \ldots X^{r,i_{h_r}} f^\alpha_i \|_{L^2} + \| X^{l-1,s-1} X^{l,i(h_1)+1} \ldots X^{r,i_{h_r}} f^\alpha_i \|_{L^2} \\
+ \sum_{q+k=i(h_{s-1})-1}^{r} \left( \sum_{s=l+1}^{r} \| X^{l-1,s} X^{l,i_{h_1}} \ldots X^{r,i_{h_r}} f^\alpha_i \|_{L^2} \right) \\
+ \sum_{q+k=i(h_{s-1})-1}^{r} \left( \sum_{s=l+1}^{r} \| X^{l-1,s} X^{l,i_{h_1}} \ldots X^{r,i_{h_r}} f^\alpha_i \|_{L^2} \right).
\]

The following estimate follows from (3.7), (3.8), and (3.9):
\[
\| X^{h_{l-1}} X^{l_{h_1}} \ldots X^{l_{h_r}} u^\beta \|_{S^1_{loc}} \\
\leq \sum_{i=1}^{m} \| \tilde{f}^\alpha_i \|_{L^2} + \| \tilde{f}^\alpha \|_{L^2} \\
+ C \| J_{h_{l-1}} J_{h_1} \ldots J_{h_r} u^\beta \|_{S^1_{loc}},
\]
so that either
\[
\sum_{i=l-1}^{r} J_{h_i} < \sum_{i=l-1}^{r} I_{h_i},
\]
or
\[
\sum_{i=l-1}^{r} J_{h_i} = \sum_{i=l-1}^{r} I_{h_i},
\]
so that there exists $\beta_i > h_{l-1}$ such that $J_{h_i} > I_{h_i}$ and $J_{h_k} \leq I_{h_k}$ for every $h_k \neq h_{l-1}, \beta_i$.

Proof. (Theorem 13) We want an estimate for $\| X^{l_{h_{l-1}}} X^{l_{h_r}} u^\beta \|_{S^1_{loc}}$, so first consider what we have if we differentiate (11) in layer $r$. Since this layer commutes with everything, applying $X^{l_{h_r}}$ simply gives us
\[
\sum_{\beta=1}^{N} \sum_{i,j=1}^{m} X_i (A_{ij}^{\alpha\beta} X_j (X^{l_{h_r}} u^\beta) + X^{l_{h_r}} f^\alpha_i) = X^{l_{h_r}} f^\alpha_i.
\]

It is easy to see that $X^{l_{h_r}} u^\beta$ is indeed in $S^1_{loc}$, so next we need to show that we can differentiate (3.10) in the direction $l-1$:

By definition, we have that
\[
f^\alpha_i (r)^{l-1,s} = A_{ij}^{\alpha\beta} [X_{l-1,s}, X_j] V(r)^{l-1,s-1} + X_{l-1,s} f^\alpha_i (r)^{l-1,s-1}.
\]

\[
\sum_{\beta=1}^{N} \sum_{i,j=1}^{m} X_i (A_{ij}^{\alpha\beta} X_j (X^{l_{h_r}} u^\beta) + X^{l_{h_r}} f^\alpha_i) = X^{l_{h_r}} f^\alpha_i.
\]
We can make use of the proof of lemma (15) and our commutator result in lemma (17) to see that

\[ f^a_i (r)^{l-1,*} = \sum_{q+k=s-1} A_{ij}^{\alpha \beta} X^{l-1,q} X^{l,1} V(r)^{l-1,k} + X^{l-1,*} f^a_i (r)^{l-1,0} \]

\[ = \sum_{q+k=s-1} A_{ij}^{\alpha \beta} X^{l-1,q} X^{l,1} X^{l-1,k} X^{h_{l+1} \alpha} \beta + X^{l-1,*} X^{l,h_{l+1} \alpha} \beta \]

\[ = A_{ij}^{\alpha \beta} X^{l-1,s-1} X^{l,1} X^{h_{l+1} \alpha} \beta \]

\[ + \sum_{s+t=s-2} A_{ij}^{\alpha \beta} X^{l-1,s} X^{2t-1,1} X^{l-1,t} X^{h_{l+1} \alpha} \beta \]

\[ + X^{l-1,*} X^{l,h_{l+1} \alpha} \beta . \]

Continuing to apply Lemma (17) indefinitely in order to group all like terms together, one obtains the following:

\[ \| f^a_i (r)^{l-1,*} \|_{L^2} \leq C \| X^{l-1} X^{l-1} \ldots X^{l,1} \alpha \beta \|_{L^2} + \| X^{l-1,*} X^{l,h_{l+1} \alpha} \beta \|_{L^2} . \]

Counting derivatives at this step, we see that \( \sum_{l=1}^{n} J_{h_l} < \sum_{l=1}^{n} I_{h_l} \) with \( J_{h_{l-1}} \) always being at least 2 less than \( I_{h_{l-1}} \), \( J_{h_k} > I_{h_k} \) for every other \( k \). The key here is that the derivatives are definitely at least 2 less in the lowest layer; even though we are adding some derivatives to the right of this layer, we have that the sum of the derivatives in all layers is always less than what we started with.

Next, we need to apply this same technique to \( f^a(r)^{l-1,*} \). Referring back to the definition and using lemma (14) we have:

\[ f^a(r)^{l-1,*} = X^{l-1,*} f^a(r)^{l-1,*-1} \]

\[ + [X_i, X^{l-1,*}] (A_{ij}^{\alpha \beta} X^j V(r)^{l-1,*-1} + f^a_i (r)^{l-1,*-1}) \]

\[ = X^{l-1,*} f^a(r)^{l-1,*} + \sum_{q+k=s-1} A_{ij}^{\alpha \beta} X^{l-1,q} X^{l,1} X^j V(r)^{l-1,k} \]

\[ + \sum_{q+k=s-1} X^{l-1,q} X^{l,1} f^a_i (r)^{l-1,k} \]

\[ = X^{l-1,*} X^{l,h_{l+1} \alpha} \beta + \sum_{q+k=s-1} A_{ij}^{\alpha \beta} X^{l-1,q} X^{l,1} X^j X^{l-1,k} X^{l,h_{l+1} \alpha} \beta \]

\[ + \sum_{q+k=s-1} X^{l-1,q} X^{l,1} f^a_i (r)^{l-1,k} . \]

From here the idea is the same as before. Apply Lemma (17) to

\[ \sum_{q+k=s-1} A_{ij}^{\alpha \beta} X^{l-1,q} X^{l,1} X^j X^{l-1,k} X^{l,h_{l+1} \alpha} \beta \]

indefinitely in order to group all like terms together. However, what is important and can actually be seen already is that we will always have at least 2 derivatives less in the \( l-1 \) layer. We may once again add derivatives to the right of this layer, but we will always have that the sum of all derivatives is less than what we started with.
We still need to check \( \sum_{q+k=\ast-1} X^{l-1,q}X^{l,1} f_i^\alpha (r)^{l-1,k} \) to ensure it follows this same mode of thought. Applying previous calculations we see that

\[
\sum_{q+k=\ast-1} X^{l-1,q}X^{l,1} f_i^\alpha (r)^{l-1,k} = A_{ij}^{\alpha \beta} \sum_{q+k=\ast-1} X^{l-1,q}X^{l,1} X^{l-1,k-1} X^{l,1} X^{I_{hr}} u^\beta \\
+ A_{ij}^{\alpha \beta} \sum_{q+k=\ast-1} X^{l-1,q}X^{l,1} \left( \sum_{s+t=k-2} X^{l-1,s}X^{2l-1,t} X^{I_{hr}} u^\beta \right) \\
+ \sum_{q+k=\ast-1} X^{l-1,q}X^{l,1} X^{l-1,k} X^{I_{hr}} f_i^\alpha .
\]

Upon closer inspection of this equation, it is clear that we can argue the same as before. Counting the derivatives in every layer we see that the sum is at least 2 less than what we started with. We also have at least 3 less in the lower layer, but we are again adding derivatives to the higher layers. The key once again is that we are keeping less derivatives than what we started with and that we have less in the lower layer.

Hence, we can say that the following inequality holds:

\[
\| f^\alpha (r)^{l-1,*} \|_{L^2} \leq C \left[ \| X^{J_{h_{l-1}}} X^{J_{h_1}} \cdots X^{J_{h_r}} \|_{L^2} + \| X^{l-1,*} X^{I_{hr}} f_i^\alpha \|_{L^2} + \sum_{q+k=\ast-1} \| X^{l-1,q}X^{l,1} X^{l-1,k} X^{I_{hr}} f_i^\alpha \|_{L^2} \right] ,
\]

where we have that the \( \sum_{i=j-1}^{r} J_{h_i} < \sum_{i=l-1}^{r} I_{h_i} \) with \( J_{h_{l-1}} \) always being at least 2 less than \( I_{h_{l-1}} \) and \( J_{h_k} > I_{h_k} \) for every other \( k \).

We have now done the bulk of the calculations that we need in order to gain an estimate on \( X^{I_{h_{l-1}}} X^{I_{hr}} u^\beta \). Putting it all together we have the following:
\[
\|X^{I_{h_{l-1}}}_1 X^{I_{h_r}} u^\beta\|_{S_{loc}^{1,2}} \leq C \left( \|X^{l-1,*}_1 X^{I_{h_r}} u^\beta\|_{S_{loc}^{1,2}} + \|f^\alpha(r)^{l-1,*}\|_{L^2} \right.
\]
\[
+ \sum_{j=l-1}^{r} \sum_{k=1}^{m_j} \|X_{j,k} f^\alpha(r)^{l-1,*}\|_{L^2} + \sum_{i=1}^{m} \|f^\alpha_i(r)^{l-1,*}\|_{L^2} \\
+ \sum_{i=1}^{m} \sum_{j=l-1}^{r} \sum_{k=1}^{m_j} \|X_{j,k} f^\alpha_i(r)^{l-1,*}\|_{L^2} \\
\leq C \left( \|X^{J_{h_{l-1}}}_1 X^{J_{h_l}} \ldots X^{J_{h_r}} u^\beta\|_{S_{loc}^{1,2}} + \sum_{i=1}^{m} \|X^{l-1,*}_1 X^{I_{h_r}} f^\alpha_i\|_{L^2} \right)
\]
\[
+ \|X^{l-1,*}_1 X^{I_{h_r}} f^\alpha\|_{L^2} + \sum_{i=1}^{m} \left( \sum_{q+k=s-1} \|X^{l-1,q}_1 X^{l-1,k}_1 X^{I_{h_r}} f^\alpha_i\|_{L^2} \right)
\]
\[
+ \sum_{i=1}^{m} \sum_{q+k=s-1} \|X_{j,k} X^{l-1,q}_1 X^{l-1,k}_1 X^{I_{h_r}} f^\alpha_i\|_{L^2} \right)
\]
\[
\leq C \left( \|X^{J_{h_{l-1}}}_1 X^{J_{h_l}} \ldots X^{J_{h_r}} u^\beta\|_{S_{loc}^{1,2}} + \sum_{i=1}^{m} \|\tilde{f}_i^\alpha\|_{L^2} + \|\tilde{f}^\alpha\|_{L^2} \right)
\]

such that \( \sum_{i=l-1}^{r} J_{h_i} \leq \sum_{i=l-1}^{r} I_{h_i} \) with \( J_{h_{l-1}} \) always being at least 1 less than \( I_{h_{l-1}} \) and \( J_{h_k} \geq I_{h_k} \) for every other \( k \),

where we set

\[
\tilde{f} = X^{l-1,*}_1 X^{I_{h_r}} f^\alpha + X_{j,k} X^{l-1,*}_1 X^{I_{h_r}} f^\alpha
\]

\[
\tilde{f}_i = X^{l-1,*}_1 X^{I_{h_r}} f^\alpha_i + \sum_{q+k=s-1} X^{l-1,q}_1 X^{l-1,k}_1 X^{I_{h_r}} f^\alpha_i
\]

\[
+ X_{j,k} X^{l-1,*}_1 X^{I_{h_r}} f^\alpha_i + \sum_{q+k=s-1} X_{j,k} X^{l-1,q}_1 X^{l-1,k}_1 X^{I_{h_r}} f^\alpha_i.
\]

\( \square \)

**Corollary 18.** Let \( u \in S_{loc}^{1,2}(\Omega) \) be a weak solution to (1.1) such that the hypothesis of Theorems 12 and 13 hold. Then the following is true:

\[
\|X^{I_1} \ldots X^{I_r} u\|_{S_{loc}^{1,2}(B(0,1))} \leq C \left( \|X^{I_r} u\|_{S_{loc}^{1,2}(B(0,2))} + \|\tilde{f}\|_{L^2(B(0,2))} + \|\tilde{f}_i\|_{L^2(B(0,2))} \right),
\]

where \( \tilde{f}, \tilde{f}_i \) represent combinations of derivatives on \( f, f_i \), respectively.
Proof. (Main Theorem)

Iterating Theorem 6, we have the following estimate

$$\|X^r u\|_{S^{1,2}_{loc}(B(0,1))} \leq C \left( \|u\|_{S^{1,2}_{loc}(B(0,2))} + \|\hat{f}\|_{L^2(B(0,2))} + \|\hat{f}_i\|_{L^2(B(0,2))} \right)$$

where $\hat{f}$, $\hat{f}_i$ represent high order derivatives on the original $f$, $f_i$. The result follows once we apply Corollary 18.

Finally, as a direct consequence to Corollary 18, we have that $u \in W^{k,2}_{loc}(B(0,1))$ for every $k$, where $W^{k,2}_{loc}$ represents the usual Euclidean Sobolev space. Therefore, Corollary 2 is immediate by the Sobolev Embedding Theorem.

4. Sketch of Proof of Corollary 3

As mentioned in the introduction, this is only a sketch of the proof. For further details we refer the reader to [CG] and [Gi].

Proof. The following corollary (Corollary 19) and inequality that follows (see (4.1) below) are a direct consequence of the hypoellipticity result of the linear system in section 3 and will be used in the proof:

Corollary 19. Let $G$ be a Carnot group, step $r$, and $\Omega \subset G$ an open subset. If $u \in S^{1,2}_{loc}(\Omega)$ is a weak solution of the constant coefficient system

$$\sum_{i,j=1}^m \sum_{\beta=1}^N A^{\alpha \beta}_{ij} X_i X_j u^\beta = 0, \quad \alpha = 1, \ldots, N,$$

in $B(p_0, 3R) \subset \Omega$, then $u$ is smooth in $B(p_0, 3R)$. Moreover, there exists a positive constant $C$ such that

$$\sup_{B(p_0, R)} (|u|^2 + R^2 |Xu|^2 + R^4 \sum_{i,j=1}^m |X_i X_j u|^2) \leq C \frac{1}{|B(p_0, 2R)|} \int_{B(p_0, 2R)} |u|^2 \, dp.$$

Using Corollary 19 we can then prove the following inequality holds for each $0 < r < R < 2$, where $C$ is a positive constant and $G$, $\Omega$, and $u$ are the same as in Corollary 19.

$$\int_{B(p_0, r)} |u - u_{(0,r)}|^2 \, dp \leq C \left( \frac{r}{R} \right)^{Q+2} \int_{B(p_0, R)} |u - u_{0,R}|^2 \, dp.$$

We use the following notation and prove the inequalities that follow, assuming always that $u \in S^{1,2}_{loc}(\Omega)$ is a weak solution to (12). Set

$$U(p_0, R) = \frac{1}{|B(p_0, R)|} \int_{B(p_0, R) \cap \Omega} |u(p) - u_{p_0,R}|^2 \, dp.$$

The first step in the proof is to show that for each $M > 0$ and $0 < \tau < 1$, there exists $\epsilon_0$ and $R_0 > 0$ such that if one has $|u_{p_0,R}| \leq M$ and $U(p_0, R) < \epsilon_0^2$ for $R \leq \min(R_0, d(p_0, \partial \Omega))$ and for some $p_0 \in \Omega$, then the following inequality holds:

$$U(p_0, \tau R) \leq C \tau^2 U(p_0, R).$$
The argument is by contradiction. Set
\[ u_n(q) = \epsilon_n^{-1}[u^n(p_n\delta_R_n(q)) - u^n_{p_n,R_n}], \]
so that
\[ V^n(\epsilon, 1) = \frac{1}{|B_1|} \int_{B_1} |\epsilon u^n|^2 \, dq = 1, \]
and assume
\[ V^n(\epsilon, \tau) > 2C\tau^2. \]
Passing eventually to a subsequence and incorporating the continuity assumptions on \( A_{ij}^{\alpha\beta} \) along with the hypothesis of the corollary, we obtain
\[ A_{ij}^{\alpha\beta}(p_n\delta_R_n(q), \epsilon_n u^n_{p_n,R_n}) \to B_{ij}^{\alpha\beta}. \]
Arguing as in (4.10) pg. 25 in [CG], we then have that for every \( \phi \in C^\infty_0(B_1) \)
\[ \sum_{\beta=1}^N \sum_{i,j=1}^m \int_{B_1} B_{ij}^{\alpha\beta} X_i^\beta X_j^\alpha \phi \, dp = 0, \quad \alpha = 1, \ldots, N. \]
We can then apply inequality (4.1) to get
\[ V(0, \tau) \leq c\tau^2 V(0, 1). \]
But following from [FS] and [EM], we also must have that
\[ V(0, \tau) > 2C\tau^2, \]
which is a contradiction.

An induction argument is used to show that, for every integer \( k \),
\[ U(p_0, \tau^k R) \leq (2C\tau^2)^k U(p_0, R). \]
It then follows from this inequality that
\[ U(p, R) \leq CR^{2\alpha}, \]
for each \( R > 0 \) small enough and for \( 0 < \alpha < 1 \) Consequently, we now have that \( u \) is Hölder continuous with exponent \( \alpha \) outside of a certain set that can be shown to have Haar measure zero.

\[ \square \]

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