A NOTE ON THE LEAST SQUAREFREE NUMBER IN AN ARITHMETIC PROGRESSION

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ABSTRACT. We prove an asymptotic formula for squarefree in arithmetic progressions with squarefree moduli, improving previous results by Prachar. The main tool is an estimate for counting solutions of a congruence inside a box that goes beyond what can be obtained by using the Weil bound.

1. INTRODUCTION

Let \( \mu \) denote the Möbius function, i.e. \( \mu \) is the multiplicative function such that for every prime number \( p \) and every positive integer \( j \), one has,

\[
\mu(p^j) = \begin{cases} 
-1, & \text{if } j = 1, \\
0, & \text{otherwise.}
\end{cases}
\]

We remark that \( \mu^2(n) = 1 \) if \( n \) is squarefree and \( \mu^2(n) = 0 \) otherwise. In this paper we are concerned with the distribution of squarefree numbers in arithmetic progressions. By the above discussion, this is equivalent to studying the distribution of the \( \mu^2 \) function in arithmetic progressions.

In this direction, a result of Prachar [7], subsequently improved by Hooley [4] says that

\[
\sum_{n \leq x \atop n \equiv a \pmod{q}} \mu^2(n) = \frac{1}{\varphi(q)} \sum_{n \leq x \atop (n,q) = 1} \mu^2(n) + O \left( \frac{X^{1/2}}{q^{1/2}} + q^{1/2+\epsilon} \right).
\]

Here and throughout the article, \( \epsilon \) denotes a small constant that might vary from line to line and the implied constants in the symbols \( O \) and \( \ll \) are allowed to depend on \( \epsilon \).

It follows from (1) that the sequence of squarefree numbers \( \leq X \) is well distributed in arithmetic progressions modulo \( q \) whenever

\[
q \leq X^{2/3-\epsilon}.
\]

Even though it is largely believed that one should be able to replace \( 2/3 \) by \( 1 \) in the above inequality, this constant has resisted any improvement until very recently.

The author [5] proved, using more sophisticated techniques than those contained here, that if one restricts to prime values of \( q \), the exponent in (2) can be improved to \( 13/19 \). In the present paper we show how to further improve this constant and at the same time relax the condition on \( q \). Our main result is the following:

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Theorem 1.1. Let \( \epsilon > 0 \). Then there exists \( \delta = \delta(\epsilon) > 0 \) such that, uniformly for \( X \geq 2 \), integers \( a \) and squarefree numbers \( q \) coprime with \( a \) satisfying
\[
q \leq X^{\frac{25}{36} - \epsilon},
\]
we have
\[
\sum_{\substack{n \leq X \ n \equiv a \pmod{q}}} \mu^2(n) = \frac{1}{\varphi(q)} \sum_{\substack{n \leq X \ (n,q) = 1}} \mu^2(n) + O \left( \frac{X^{1-\delta}}{q} \right).
\]
In other terms, the value \( \Theta = \frac{25}{36} \) is an exponent of distribution for the characteristic function of the sequence of squarefree numbers \( \mu^2 \) restricted to squarefree moduli.

Alternatively, one can ask the simpler question of when is the left-hand side of (1) nonzero. This is equivalent to study the least squarefree number in an arithmetic progression. We let \( n(q,a) \) denote the least positive squarefree number which is congruent to \( a \) modulo \( q \). Prachar’s result implies
\[
n(q,a) \ll q^{3/2 + \epsilon}.
\]
This result was improved by Erdös [2], who essentially proved that \( n(q,a) = o(q^{1/2}) \) and then later by Heath-Brown [3], who showed the upper bound
\[
n(q,a) \ll q^{13/9 + \epsilon}.
\]
It is a direct consequence of Theorem 1.1 that we can also improve this inequality for squarefree values of \( q \). Indeed we have the following:

Corollary 1.2. For every \( \epsilon > 0 \), we have the inequality
\[
n(q,a) \ll q^{\frac{25}{36} + \epsilon},
\]
uniformly for \( q \) squarefree and \( a \) coprime with \( q \).

The key input comes from estimates for the number of solutions to a congruence inside a dyadic box that follow from the work of Pierce [6].

Let \( q \) be a positive integer and let \( a \in (\mathbb{Z}/q\mathbb{Z})^* \). Let \( u > 0 \) and \( v \) be nonzero integers and let \( M \) and \( N \) be real numbers such that \( M, N \geq 1 \). We consider the counting function
\[
S_{u,v}(M,N,q,a) := \# \{ m \leq M, n \leq N; m^u \equiv an^v \pmod{q} \},
\]
where, if \( v \) is negative, then \( n^v \) stands for \( \bar{n}^{\nu} \). Moreover, \( \bar{n} \) denotes the multiplicative inverse of \( n \) modulo \( q \).

It is not hard to see that one has the upper bound
\[
S_{u,v}(M,N,q,a) \ll \frac{MN}{q} + \min(M,N).
\]
In certain cases, this can even be improved by making use of the Weil bound for exponential sums over curves. For example, suppose \( q \) is squarefree and \( (u,v) = 1, u \neq v \). Then we have the inequality
\[
S_{u,v}(M,N,q,a) \ll q^\epsilon \left( MNq^{-1} + Mq^{-\frac{1}{2}} + Nq^{-\frac{1}{2}} + q^\frac{1}{2} \right).
\]
Unfortunately, when \( M \approx N \approx q^{1/2} \), both (6) and (7) give the same bound
\[
S_{u,v}(M, N, q, a) \ll q^{1/2 + \varepsilon}.
\]
This is an important threshold when trying to improve (1) or (4). Indeed, one of the main achievements in [3] is giving an upper bound for
\[
S_{1-2}(M, N, q, a)
\]
that improves on (8) in the range where \( M \) and \( N \) are close to \( q^{1/2} \) in logarithmic scale. The following lemma is a particular case of a result by Pierce [6] generalizing the main argument in [3]. Both of these results are inspired by work of Burgess [1].

Lemma 1.3. (see [6, Theorem 4]) We have, uniformly for
\[
a \in (\mathbb{Z}/q\mathbb{Z})^*, \quad 1 \leq M \leq q^{3/4} \quad \text{and} \quad 1 \leq N < q/2
\]
the inequality
\[
S(M, N, q, a) \ll M^{1/2}N^{1/4}q^{\varepsilon}.
\]

We will use Lemma 1.3 with \((u, v) = (1, -2)\) and \((u, v) = (2, -1)\). For the first of these pairs, the work of Heath-Brown suffices and if we only had Lemma 1.3 for this value of \((u, v)\), we could prove a version of Theorem 1.1 with the exponent \(25/36\) replaced by \(9/13\). Hence Corollary 1.2 would be just a particular case of [3, Theorem 2]. It is thanks to the more powerful result from [6] and the simple symmetry relation
\[
S_{u,v}(M, N, q, a) = S_{-v,-u}(N, M, q, a),
\]
that we can obtain the improved exponent \(25/36\).

2. Initial steps

Let \( q \) be a squarefree number, let \( a \) be coprime with \( q \) and \( X \geq q \). We consider \( E = E(X, q, a) \) given by
\[
E := \sum_{n \leq X \atop n \equiv a \mod q} \mu^2(n) - \frac{1}{\varphi(q)} \sum_{n \leq X \atop (n, q) = 1} \mu^2(n).
\]
Our goal is to prove that we have the inequality \( E \ll X^{1-\delta}/q \) uniformly for \( q \leq X^{25/36 - \varepsilon} \).

If \( q \leq X^{1/2} \), then this already follows from (11). Therefore we may suppose
\[
q \geq X^{1/2}.
\]
We recall the classical identity
\[
\mu^2(n) = \sum_{n_1, n_2 \geq 1 \atop n_1n_2 = n} \mu(n_2).
\]
This gives
\[
E = \sum_{n \leq X^{1/2} \atop (n, q) = 1} \mu(n) \Delta(X/n^2, q, an^2),
\]
where for every \( x \geq 1 \), \( q \) integer and \( a \in \mathbb{Z}/q\mathbb{Z} \),
\[
\Delta(x, q, a) := \sum_{m \leq x \atop m \equiv a \mod q} 1 - \frac{1}{\varphi(q)} \sum_{m \leq x \atop (m, q) = 1} 1.
\]
It is clear that for any $x, q, a$, we have
$$\Delta(x, q, a) \ll 1.$$  
Let $N_0$ be a parameter to be chosen optimally later and such that $1 \leq N_0 \leq X^{1/2}$. The previous inequality shows us that
$$E = \sum_{N_0 < n \leq X^{1/2}} \mu(n) \Delta(X/n^2, q, a\bar{n}^2) + O(N_0^\epsilon).$$  
Notice that
$$\frac{1}{\varphi(q)} \sum_{N_0 < n \leq X^{1/2}} \mu(n) \sum_{m \leq X/n} \sum_{(m, q) = 1} 1 \ll \frac{X^{1+\epsilon}}{N_0 q},$$
This and (11) combined give
$$|E| \leq \sum_{N_0 < n \leq X^{1/2}} \sum_{m \leq X/n} \sum_{(m, q) = 1} 1 + O \left( X^\epsilon \left( N_0 + \frac{X^{1+\epsilon}}{N_0 q} \right) \right).$$
3. Division in dyadic boxes

We now proceed by means of a dyadic decomposition. If we put
$$S(M, N, q, a) = \sum_{m \sim M, n \sim N} \sum_{mn^2 \equiv a \pmod{q}} 1,$$
we deduce from (12) the upper bound
$$E \ll (\log X)^2 \cdot \sup_{M, N} S(M, N, q, a) + N_0 + \frac{X^{1+\epsilon}}{N_0 q},$$
where the supremum is taken over all $M$ and $N$ such that
$$M, N \geq 1, N_0 \leq N \leq 2X^{1/2}, MN^2 \leq 8X.$$  
Let $M_0 \geq 1$ be a parameter to be chosen optimally later. Suppose that $M \leq M_0$ and that $M, N$ satisfy the conditions (14). Then, by the crude estimate
$$\sum_{n \sim N} \sum_{n \equiv \alpha \pmod{q}} 1 \ll \frac{N}{q} + 1,$$
we see that
$$S(M, N, q, a) \ll MQ^\epsilon \left( \frac{N}{q} + 1 \right) \ll X^\epsilon \left( \frac{X}{N_0 q} + M_0 \right).$$
Thus
$$E \ll (\log X)^2 \sup_{M, N} S(M, N, q, a) + X^\epsilon \left( M_0 + N_0 + \frac{X}{N_0 q} \right),$$
where now the supremum is taken over all $M$ and $N$ satisfying
$$M \geq M_0, N \geq N_0, MN^2 \leq 8X.$$
4. Using Lemma 1.3

We notice that
\[(17) \quad S(M, N, q, a) \leq S_{1,-2}(M, N, q, a).\]

Suppose that \(M_0\) and \(N_0\) satisfy
\[(18) \quad M_0 > Xq^{-3/2}, N_0 > X^{1/2}q^{-3/8}.\]

This readily implies that every \(M, N\) satisfying \((16)\) we have \(1 \leq M, N \leq q^{3/4}.\) Lemma 1.3, (17) and (9) now give the upper bound
\[(19) \quad S(M, N, q, a) \ll q^\epsilon \min\left(M^{1/3}N^{1/3}, M^{1/3}N^{1/3}\right)\]
for every \(M, N\) satisfying \((16).\)

It is not hard to see that for every \(0 < \alpha < 1,\) It follows from \((19)\) that we have the inequality
\[(20) \quad S(M, N, q, a) \ll q^\epsilon \min\left(M^{1/3}N^{1/3}\right)^\alpha \left(M^{1/3}N^{1/3}\right)^{1-\alpha}.\]

Taking \(\alpha = 2/15,\) we get
\[S(M, N, q, a) \ll q^\epsilon \left(MN\right)^{\frac{11}{36}} \leq X^{\frac{11}{36}+\epsilon}.\]

Now by \((18)\) we see that
\[(21) \quad E \ll X^{\frac{11}{36}+\epsilon} + M_0 + N_0 + N_0^{-1}Xq^{-1}.\]

5. Conclusion

We make the choices
\[(22) \quad M_0 = 2 \max(Xq^{-3/2}, 1), N_0 = 2X^{1/3}q^{-3/8}.\]

Note that these choices clearly satisfy \((18).\) We also notice that we have \(1 \leq M_0 \leq X\) and \(1 \leq N_0 \leq X^{1/2}.\) With the choices \((22),\) the upper bound \((21)\) becomes.
\[E \ll X^{\epsilon} \left(X^{\frac{11}{36}} + Xq^{-\frac{3}{2}} + X^{\frac{1}{3}}q^{-\frac{3}{8}}\right).\]

It is now straightforward to verify that for every \(\epsilon > 0,\) there exists \(\delta = \delta(\epsilon) > 0\) such that whenever \(q \leq X^{\frac{25}{36}-\epsilon},\) we have the inequality
\[E \ll \frac{X^{1-\delta}}{q}.\]

This concludes the proof of Theorem 1.1.

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