Stable rationality of higher dimensional conic bundles

Hamid Ahmadinezhad and Takuzo Okada

Abstract. We prove that a very general nonsingular conic bundle $X \to \mathbb{P}^{n-1}$ embedded in a projective vector bundle of rank $3$ over $\mathbb{P}^{n-1}$ is not stably rational if the anti-canonical divisor of $X$ is not ample and $n \geq 3$.

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Titre. Rationalité stable des fibrés en coniques de grande dimension

Résumé. Nous démontrons qu’un fibré en coniques non-singulier très général $X \to \mathbb{P}^{n-1}$ plongé dans le projectivisé d’un fibré vectoriel de rang $3$ au dessus de $\mathbb{P}^{n-1}$ n’est pas stablement rationnel lorsque le diviseur anti-canonique de $X$ n’est pas ample et $n \geq 3$.

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1. Introduction

An important question in algebraic geometry is to determine whether an algebraic variety is rational; that is, birational to projective space. Two algebraic varieties are said to be birational if they become isomorphic after removing finitely many lower-dimensional subvarieties from both sides. The closest varieties to being rational are those that admit a fibration into a projective space with all fibres rational curves; so-called conic bundles.

In this article, we study stable (non-)rationality of conic bundles over a projective space of arbitrary dimension (greater than one). A non-rational variety $X$ may become rational after being multiplied by a suitable projective space, i.e., $X \times \mathbb{P}^m$ is birational to $\mathbb{P}^{n+m}$, where $n = \dim X$, in which case we say $X$ is stably rational.

Stable non-rationality of conic bundles in dimension 3 has been studied extensively in [1, 2] and [8], giving a satisfactory answer. In higher dimensions almost nothing is known except for a few examples of stably non-rational conic bundles over $\mathbb{P}^3$ given in [1] and [9].

Throughout this article, by a conic bundle we mean a Mori fibre space of relative dimension 1 (see Definition 2.5 for details). The following is our main result.

**Theorem 1.1.** Let $n \geq 3$ and $d$ be integers, and let $E$ be a direct sum of three invertible sheaves on $\mathbb{P}^{n-1}$. Let $X$ be a very general member of a complete linear system $|2D + dF|$ on $\mathbb{P}^{n-1}(E)$, where $D$ is the tautological divisor and $F$ is the pullback of the hyperplane on $\mathbb{P}^{n-1}$. Suppose that the natural projection $X \rightarrow \mathbb{P}^{n-1}$ is a conic bundle.

1. If $X$ is singular, then $X$ is rational.
2. If $X$ is non-singular and $-K_X$ is not ample, then $X$ is not stably rational.

This result covers the following varieties as a special case.

**Corollary 1.2.** Let $X$ be a very general hypersurface of bi-degree $(d, 2)$ in $\mathbb{P}^{n-1} \times \mathbb{P}^2$. If $2d \geq n \geq 3$, then $X$ is not stably rational.

This can be thought of as a higher dimensional generalisation of the main result of [2].

**Corollary 1.3.** Let $X$ be a double cover of $\mathbb{P}^{n-1} \times \mathbb{P}^1$ branched along a very general divisor of bi-degree $(2d, 2)$. If $2d \geq n \geq 3$, then $X$ is not stably rational.

By a result of Sarkisov [16], a conic bundle is birational to a standard conic bundle which is by definition nonsingular conic bundle flat over a smooth base. The following criterion for rationality in terms of the discriminant was conjectured by Shokurov [17] (see also [10, Conjecture I]). Remarkable progress toward this conjecture has been made in [10] and [13].

**Conjecture 1.4. ([17, Conjecture 10.3])** Let $X \rightarrow S$ be a 3-dimensional standard conic bundle and $\Delta \subset S$ the discriminant divisor. If $|2K_S + \Delta| \neq \emptyset$, then $X$ is not rational.

Although the statement becomes weaker than Theorem 1.1, we can restate our main result in terms of the discriminant:
Corollary 1.5. With notation and assumptions as in Theorem 1.1, assume in addition that $X$ is nonsingular and let $\Delta \subset \mathbb{P}^{n-1}$ be the discriminant divisor of the conic bundle $X \to \mathbb{P}^{n-1}$.

(1) If $|3K_{\mathbb{P}^{n-1}} + \Delta| \neq \emptyset$, then $X$ is not stably rational.

(2) If $n \geq 7$, $\pi : X \to \mathbb{P}^{n-1}$ is standard and $|2K_{\mathbb{P}^{n-1}} + \Delta| \neq \emptyset$, then $X$ is not stably rational.

This leads us to pose the following.

Conjecture 1.6. Let $\pi : X \to S$ be an $n$-dimensional standard conic bundle with $n \geq 3$. If $|2K_S + \Delta| \neq \emptyset$, then $X$ is not rational. If in addition $X$ is very general in its moduli, then $X$ is not stably rational.

The argument of stable non-rationality. It is known that a stably rational smooth projective variety is universally CH$_0$-trivial; see [5, Lemme 1.5] and [18, theorem 1.1] and references therein. Let $X \to B$ be a flat family over a complex curve $B$ with smooth general fibre. Then, by the specialisation theorem of Voisin [19, Theorem 2.1], the stable non-rationality of a very general fibre will follow if the special fibre $X_0$ is not universally CH$_0$-trivial and has at worst ordinary double point singularities. This was generalised by Colliot-Thélène and Pirutka [5, Théorème 1.14] to the case where

1. $X_0$ admits a universally CH$_0$-trivial resolution $\varphi : Y \to X_0$ such that $Y$ is not universally CH$_0$-trivial,

2. in mixed characteristic, that is, when $B = \text{Spec } A$ with $A$ being a DVR of possibly mixed characteristic.

Thus it is enough to verify the existence of such a resolution $\varphi : Y \to X_0$ over an algebraically closed field of characteristic $p > 0$. In view of [18, Lemma 2.2], the core of the proof of universal CH$_0$-nontriviality for $Y$ in our case is done by showing that $H^0(Y, \Omega^i) \neq 0$ for some $i > 0$, following Kollár [11] and Totaro [18]. This is done in Section 3.

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2. Embedded conic bundles

2.A. Weighted projective space bundles

In this subsection we work over a field $k$.

Definition 2.1. A toric weighted projective space bundle over $\mathbb{P}^n$ is a projective simplicial toric variety with Cox ring

$$\text{Cox}(P) = k[u_0, \ldots, u_n, x_0, \ldots, x_m],$$

which is $\mathbb{Z}^2$-graded as

$$\begin{pmatrix}
1 & \cdots & 1 & \lambda_0 & \cdots & \lambda_m \\
0 & \cdots & 0 & a_0 & \cdots & a_m
\end{pmatrix}$$

with the irrelevant ideal $I = (u_0, \ldots, u_n) \cap (x_0, \ldots, x_m)$, where $\lambda_0, \ldots, \lambda_m$ are integers and $n, m, a_0, \ldots, a_m$ are positive integers. In other words, $P$ is the geometric quotient

$$P = (\mathbb{A}^{n+m+2} \setminus V(I))/\mathbb{G}^2_m,$$

where the action of $\mathbb{G}^2_m = \mathbb{G}_m \times \mathbb{G}_m$ on $\mathbb{A}^{n+m+2} = \text{Spec } \text{Cox}(P)$ is given by the above matrix.
The natural projection $\Pi : P \to \mathbb{P}^n$ by the coordinates $u_0, \ldots, u_n$ realizes $P$ as a $\mathbb{P}(a_0, \ldots, a_m)$-bundle over $\mathbb{P}^n$. In this paper, we simply call $P$ the $\mathbb{P}(a_0, \ldots, a_m)$-bundle over $\mathbb{P}^n$ defined by

$$
\begin{pmatrix}
  u_0 & \cdots & u_n & x_0 & \cdots & x_m \\
  1 & \cdots & 1 & \lambda_0 & \cdots & \lambda_m \\
  0 & \cdots & 0 & a_0 & \cdots & a_m
\end{pmatrix}.
$$

In the following, let $P$ be as in Definition 2.1. Let $p \in P$ be a point and $q \in A^{n+m+2} \setminus V(I)$ a preimage of $p$ via the morphism $A^{n+m+2} \setminus V(I) \to P$. We can write $q = (\alpha_0, \alpha_n, \beta_0, \ldots, \beta_m)$, where $\alpha_i, \beta_j \in k$. In this case we express $p$ as $(a_0 : \cdots : a_n, \beta_0 : \cdots : \beta_m)$.

**Remark 2.2.** We will frequently use the following coordinate change. Consider a point $p = (a_0 : \cdots : a_n : \beta_0 : \cdots : \beta_m) \in P$ and suppose for example that $\alpha_0 \neq 0, \beta_j \neq 0$ and $a_j = 1$ for some $j$. Then for $l \neq j$ such that $\lambda_l/a_l \geq \lambda_j$, the replacement

$$
  x_l \mapsto \alpha_0 \beta_j \lambda_l/a_l x_l - \beta_l u_0 \lambda_l/a_l x_j
$$

induces an automorphism of $P$. By considering the above coordinate change, we can transform $p$ (via an automorphism of $P$) into a point for which the $x_l$-coordinate is zero for $l$ with $\lambda_l/a_l \geq \lambda_j$.

We have the decomposition

$$
\text{Cox}(P) = \bigoplus_{(a, \beta) \in \mathbb{Z}^2} \text{Cox}(P)_{(a, \beta)},
$$

where $\text{Cox}(P)_{(a, \beta)}$ consists of the homogeneous elements of bi-degree $(a, \beta)$. An element $f \in \text{Cox}(P)_{(a, \beta)}$ is called a (homogeneous) polynomial of bi-degree $(a, \beta)$.

The Weil divisor class group $\text{Cl}(P)$ is naturally isomorphic to $\mathbb{Z}^2$. Let $F$ and $D$ be the divisors on $P$ corresponding to $(1, 0)$ and $(0, 1)$, respectively, which generate $\text{Cl}(P)$. Note that $F$ is the class of the pullback of a hyperplane on $\mathbb{P}^n$ via $\Pi : P \to \mathbb{P}^n$. We denote by $\mathcal{O}_P(a, \beta)$ the rank 1 reflexive sheaf corresponding to the divisor class of type $(a, \beta)$, that is, the divisor $aF + \beta D$. More generally, for a subscheme $Z \subset P$, we set $\mathcal{O}_Z(a, \beta) = \mathcal{O}_P(a, \beta)|_Z$. We remark that there is an isomorphism

$$
H^0(P, \mathcal{O}_P(a, \beta)) \cong \text{Cox}(P)_{(a, \beta)}.
$$

**Definition 2.3.** For integers $k, l, m, n$ with $n \geq 3$, we define $P_n(k, l, m)$ (resp. $Q_n(k, l)$) to be the $\mathbb{P}^2$-bundle (resp. $\mathbb{P}^1$-bundle) over $\mathbb{P}^{n-1}$ defined by the matrix

$$
\begin{pmatrix}
  u_0 & \cdots & u_{n-1} & x & y & z \\
  1 & \cdots & 1 & k & l & m \\
  0 & \cdots & 0 & 1 & 1 & 1
\end{pmatrix}.
$$

**Remark 2.4.** Let $P$ be as in Definition 2.1. When $a_0 = \cdots = a_m = 1$, $P$ is a $\mathbb{P}^m$-bundle over $\mathbb{P}^n$. More precisely, we have an isomorphism

$$
P \cong \mathbb{P}_{\mathbb{P}^n}(\mathcal{O}_{\mathbb{P}^n}(-\lambda_0) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^n}(-\lambda_m)).
$$

Here, for a vector bundle $\mathcal{E}$ over $\mathbb{P}^n$, $\mathbb{P}(\mathcal{E}) = \mathbb{P}_{\mathbb{P}^n}(\mathcal{E})$ denotes the projective bundle of one-dimensional quotients of $\mathcal{E}$. Moreover, via the above isomorphism, the pullback of a hyperplane on $\mathbb{P}^{n-1}$ and the tautological divisor on $\mathbb{P}(\mathcal{E})$ are identified with the divisors on $P$ corresponding to $(1, 0)$ and $(0, 1)$, respectively.
2.B. Embedded conic bundles

In the rest of this section we work over \(\mathbb{C}\). By a splitting vector bundle, we mean a vector bundle which is a direct sum of invertible sheaves.

**Definition 2.5.** Let \(X\) be a normal projective \(\mathbb{Q}\)-factorial variety of dimension \(n\). We say that a morphism \(\pi: X \to \mathbb{P}^{n-1}\) is a conic bundle (over \(\mathbb{P}^{n-1}\)) if it is a Mori fibre space, that is, \(X\) has only terminal singularities, \(\pi\) has connected fibres, \(-K_X\) is \(\pi\)-ample and \(\rho(X) = 2\), where \(\rho(X)\) denotes the rank of the Picard group.

An embedded conic bundle \(\pi: X \to \mathbb{P}^{n-1}\) is a conic bundle such that \(X\) is embedded in a projective bundle \(\mathbb{P}(E) := \mathbb{P}_{\mathbb{P}^{n-1}}(\mathcal{E})\) as a member of \([dF + 2D]\) for some splitting vector bundle \(\mathcal{E}\) of rank 3 on \(\mathbb{P}^{n-1}\) and \(d \in \mathbb{Z}\), and \(\pi\) coincides with the restriction of \(\Pi: \mathbb{P}(\mathcal{E}) \to \mathbb{P}^{n-1}\) to \(X\). Here \(F\) and \(D\) denote the pullback of a hyperplane on \(\mathbb{P}^{n-1}\) and the tautological class \(D\) on \(\mathbb{P}(\mathcal{E})\), respectively.

In the following let \(\mathcal{E}\) be a splitting vector bundle of rank 3 on \(\mathbb{P}^{n-1}\) and \(X \in [dF + 2D]\) be a general member. We denote by \(\pi: X \to \mathbb{P}^{n-1}\) the restriction of \(\Pi: \mathbb{P}(\mathcal{E}) \to \mathbb{P}^{n-1}\) to \(X\). Without loss of generality we may assume that

\[\mathcal{E} \cong \mathcal{O}_{\mathbb{P}^{n-1}}(-k) \oplus \mathcal{O}_{\mathbb{P}^{n-1}}(-l) \oplus \mathcal{O}_{\mathbb{P}^{n-1}}(-m)\]

for some \(k \leq l \leq m\). Then, by Remark 2.4, we have \(\mathbb{P}(\mathcal{E}) \cong \mathbb{P}_n(k,l,m)\) and the linear system \([dF + 2D]\) on \(\mathbb{P}_n(k,l,m)\) corresponds to \([\mathcal{O}_{\mathbb{P}_n(k,l,m)}(d,2)]\). Here we do not assume that \(\pi: X \to \mathbb{P}^{n-1}\) is a conic bundle. We study conditions on \(k, l, m\) and \(d\) that make \(\pi: X \to \mathbb{P}^{n-1}\) a conic bundle.

**Lemma 2.6.** Let \(k, l, m, d\) be integers such that \(k \leq l \leq m\). Set \(P = P_n(k,l,m)\) and let \(X\) be a general member of \([\mathcal{O}_P(d,2)]\). Let \(P = P_n(k,l,m)\) and let \(X\) be a general member of \([\mathcal{O}_P(d,2)]\).

1. \(X\) is smooth if and only if \(d \geq 2m, d = l + m,\) or \(d = k + m\).
2. \(X\) is not smooth and has only terminal singularities if and only if \(2m > d > l + m\).
3. \(X\) is non-normal if and only if \(k + m > d\).

**Proof.** Suppose that \(d \geq 2m\). Then \([\mathcal{O}_P(d,2)]\) is base point free and its general member \(X\) is smooth. In the following we assume that \(2m > d \geq k + m\).

Suppose that \(2m > d > l + m\). Then \(X\) is defined in \(P\) by

\[ax^2 + by^2 + fxy + gxz + hyz = 0,\]

where \(a, b, f, g, h \in \mathbb{C}[u]\). We have \(\deg h = d - (l + m) > 0\) and \(\deg g = d - (k + m) > 0\). Then \(X\) is singular along \((x = y = g = h = 0) \neq \emptyset\). The singular locus of \(X\) is of codimension 3 in \(X\). Since \(X\) is general, the hypersurfaces in \(\mathbb{P}^{n-1}\) defined by \(g = 0\) and \(h = 0\) are both nonsingular and intersect transversally. It is then straightforward to check that the blowup \(\sigma: X' \to X\) along the singular locus is a resolution and we have \(K_{X'} = \sigma^*K_X + E\), where \(E\) is the exceptional divisor. Thus \(X\) has terminal singularities.

Suppose that \(2m > d = l + m\). Then \(X\) is defined in \(P\) by

\[ax^2 + by^2 + fxy + gxz + yz = 0.\]

Replacing \(y\) and \(z\) suitably, we can eliminate the terms \(by^2, fxy\) and \(gxz\), that is, \(X\) is defined by

\[ax^2 + yz = 0.\]

It is then clear that \(X\) is smooth, when \(a\) is general.

Suppose that \(l + m > d > k + m\). Then \(X\) is defined in \(P\) by

\[ax^2 + by^2 + fxy + gxz = 0.\]
We have \( \deg g = d - (k + m) > 0 \). Then \( X \) is singular along \((x = y = g = 0) \neq \emptyset\), and the singularity is not terminal since the singular locus is of codimension 2 in \( X \).

Suppose that \( l + m > d = k + m \). Then \( X \) is defined in \( P \) by

\[
ax^2 + by^2 + fxy + xz = 0.
\]

Replacing \( z \) suitably, we may assume that \( X \) is defined by

\[
by^2 + zx = 0.
\]

It is easy to see that \( X \) is smooth.

Finally suppose that \( k + m > d \). Then \( X \) is defined in \( P \) by

\[
ax^2 + by^2 + fxy = 0,
\]

where \( a, b, f \in \mathbb{C}[u] \). In this case \( X \) is singular along the divisor \((x = y = 0) \subset X \). Thus \( X \) is not normal. The above arguments prove (1), (2) and (3).

\( \square \)

**Lemma 2.7.** In the same setting as in Lemma 2.6, suppose that either \( d = l + m \) or \( d = k + m \). Then the variety \( X \) is rational. Moreover we have \( \rho(X) \geq 3 \) unless \( k = l = m \).

**Proof.** Suppose that \( d = l + m \), which implies \( 2m \geq d = l + m \). We claim that \( X \) is defined by an equation of the form \( ax^2 + yz = 0 \), where \( a \in \mathbb{C}[u] \). This is already proved in Lemma 2.6, when \( 2m > d \). Suppose that \( 2m = d = l + m \). Then \( l = m \) and \( X \) is defined by

\[
ax^2 + y^2 + z^2 + fxy + gyz + ayz = 0,
\]

where \( a \in \mathbb{C} \) and \( a, f, g \in \mathbb{C}[u] \). Replacing \( y \) and \( z \), the above equation can be transformed into \( ax^2 + yz = 0 \) and the claim is proved.

We consider the projection \( X \rightarrow Q := Q_n(k, l) \) Note that \( Q \cong \mathbb{P}(\mathcal{O}(-k) \oplus \mathcal{O}(-l)) \). Then the projection is birational, hence \( X \) is rational. The projection \( X \rightarrow Q \) is defined outside \((x = y = 0) \subset X \). Let \( p \in (x = y = 0) \) be a point. Then \( z \) does not vanish at \( p \) and we have

\[
y = \frac{yz}{z} = -\frac{ax^2}{z}.
\]

From this we deduce that \( X \rightarrow Q \) is everywhere defined. Now we assume that either \( k \neq l \) or \( l \neq m \). Then \( \deg a = d - 2k = l + m - k > 0 \). We see that \((y = a = 0) \subset X \) is a divisor and it is contracted by \( X \rightarrow Q \) to a codimension 2 subset of \( Q \). This shows \( \rho(X) \geq 3 \).

Next, suppose that \( d = k + m \). Note that \( l + m \geq d \). If in addition \( l + m > d \), then, by the proof of Lemma 2.6, the defining equation of \( X \) can be written as \( by^2 + xz = 0 \). The statement follows from the same argument as above. If \( l + m = d \), then \( k = l \) and we have \( d = l + m \). This case is already proved. \( \square \)

**Lemma 2.8.** In the same setting as in Lemma 2.6, \( \pi : X \rightarrow \mathbb{P}^{n-1} \) is a nonsingular conic bundle if and only if one of the following holds:

1. \( d > 2m \),
2. \( d = 2m \) and \( m > l \), or
3. \( d = 2m = 2l = 2k \).

**Proof.** This follows from Lemmas 2.6 and 2.7. \( \square \)
Proposition 2.9. Let $X$ be an embedded conic bundle over $\mathbb{P}^{n-1}$. If $X$ is general (in the linear system) and singular, then $X$ is rational.

Proof. We may assume that $X \in |\mathcal{O}_P(d,2)|$, where $P = P_n(k,l,m)$, for some $k \leq l \leq m$. By Lemma 2.6, we have $2m > d \geq k + m$. Then a general member $X$ is defined by an equation of the form

$$ax^2 + by^2 + fxy + gxz + hyz = 0,$$

where $a,b,f,g,h \in \mathbb{C}[u]$. Here, note that, if for example $l + m > d$, then we know that the term $hyz$ does not appear in the equation. The inequality $d \geq k + m$ implies that $g \neq 0$ since $X$ is general. Let $P \dasharrow Q = Q_n(k,l)$ be the natural projection. Now we can write the defining equation as

$$z(gx + hy) + ax^2 + by^2 + fxy = 0,$$

which implies that the restriction $X \dasharrow Q$ is birational. Therefore $X$ is rational. □

The following can be considered as a “normal form” of conic bundles, which describes nonsingular embedded conic bundles (see Proposition 2.11).

Definition 2.10. Let $(\lambda, \mu, \nu)$ be a triplet of integers $\lambda, \mu, \nu$. We say that $\pi: X \rightarrow \mathbb{P}^{n-1}$ (or $X$) is of type $[\lambda, \mu, \nu]$ if $X$ belongs to $|\mathcal{O}_P(\lambda + \mu + \nu, 2)|$, where $P = P_n(\lambda, \mu, \nu)$, and $\pi$ coincides with the restriction of $P \rightarrow \mathbb{P}^{n-1}$ to $X$.

Proposition 2.11. Let $\pi: X \rightarrow \mathbb{P}^{n-1}$ be a nonsingular embedded conic bundle. Then $X$ is either of type $[\lambda, \mu, \nu]$ for some $\lambda, \mu, \nu$ such that $0 < \lambda \leq \mu \leq \nu \leq \lambda + \mu$ or of type $[0,0,0]$.

Proof. We may assume that $X$ belongs to $|\mathcal{O}_P(k,l,m)(d,2)|$ for some $k \leq l \leq m$ and $d$. Since the family $X$ is non-singular, we have $d \geq 2m$ by Lemma 2.8 and $X$ is defined in $P_n(k,l,m)$ by an equation of the form

$$ax^2 + by^2 + cz^2 + fxy + gxz + hyz = 0,$$

where $a,b,c,f,g,h \in \mathbb{C}[u]$. We set $\alpha = \deg a, \beta = \deg b, \gamma = \deg c, \lambda = \deg h, \mu = \deg g$ and $\nu = \deg f$. By comparing the weights, we have

$$\alpha + 2k = \beta + 2l = \gamma + 2m = \nu + k + l = \mu + k + m = \lambda + l + m.$$

Now we have

$$P_n(k,l,m) \cong P_n(k + (\nu - m), l + (\nu - m), m + (\nu - m)) \cong P_n(\lambda, \mu, \nu) =: P,$$

and the linear system $|\mathcal{O}_P(k,l,m)(d,2)|$ is identified with $|\mathcal{O}_P(\lambda + \mu + \nu, 2)|$. Thus $X$ is of type $[\lambda, \mu, \nu]$. By applying Lemma 2.8 for $k = \lambda, l = \mu, m = \nu$ and $d = \lambda + \mu + \nu$, we get the desired result. □

Remark 2.12. In the language of [1, Definition 3.1], a conic bundle $\pi: X \rightarrow \mathbb{P}^{n-1}$ of type $[\lambda, \mu, \nu]$ with $\lambda \leq \mu \leq \nu \leq \lambda + \mu$ is a conic bundle of graded-free type over $\mathbb{P}^{n-1}$ corresponding to the triplet $(-\lambda + \mu + \nu, \lambda - \mu + \nu, \lambda + \mu - \nu)$.

3. Stable non-rationality

In this section we study stable (non-)rationality of nonsingular embedded conic bundles $\pi: X \rightarrow \mathbb{P}^{n-1}$. By Proposition 2.11, such a conic bundle is of type $[\lambda, \mu, \nu]$, where either $0 < \lambda \leq \mu \leq \nu \leq \lambda + \mu$ or $\lambda = \mu = \nu = 0$. In case $X$ is of type $[0,0,0]$, then $X \cong \mathbb{P}^{n-1} \times \mathbb{P}^1$ and it is obviously rational. We consider the remaining cases and thus we assume that

$$0 < \lambda \leq \mu \leq \nu \leq \lambda + \mu.$$
throughout this section. In addition we assume $v \geq 3$ throughout.

We set $P = P_n(\lambda, \mu, \nu)$, $\delta = \lambda + \mu + v$, and consider special members $X \in |O_P(\delta, 2)|$ defined in $P$ by an equation of the form
\[ ax^2 + by^2 + cz^2 + fxy = 0, \]
where $a, b, c, f$ are general polynomials in variables $u_0, \ldots, u_{n-1}$. Recall that $v = \deg f$ and $\deg a = -\lambda + \mu + v$, $\deg b = \lambda - \mu + v$ and $\deg c = \lambda + \mu - v$.

**Remark 3.1.** By the assumptions on $\lambda, \mu, \nu$, we have $\deg a = -\lambda + \mu + v \geq 3$, $\deg b = \lambda - \mu + v \geq 1$, $\deg c = \lambda + \mu - v \geq 0$ and $\deg f = v \geq 3$.

**Lemma 3.2.** If the ground field is an algebraically closed field of characteristic 0, then $X$ is smooth.

**Proof.** The variety $X$ is a general member of the base point free sub linear system of $|O_P(\delta, 2)|$ on the smooth variety $P$. Thus, by the Bertini theorem, a general $X$ is smooth. \qed

We use universal $\text{CH}_0$-triviality to test stable rationality of varieties.

**Definition 3.3.** Let $V$ be a projective variety defined over a field $k$. We denote by $\text{CH}_0(V)$ the Chow group of 0-cycles on $V$. We say that $V$ is universally $\text{CH}_0$-trivial if for any field $F$ containing $k$, the degree map $\text{CH}_0(V_F) \to Z$ is an isomorphism. A projective morphism $q: W \to V$ defined over $k$ is universally $\text{CH}_0$-trivial if for any field containing $k$, the push-forward map $q_*: \text{CH}_0(W_F) \to \text{CH}_0(V_F)$ is an isomorphism.

In the rest of this section we work over an algebraically closed field $\mathbb{k}$ of characteristic 2. Let $R$ be the $\mathbb{P}(1, 1, 2)$-bundle over $\mathbb{P}^1$ defined by
\[
\begin{pmatrix}
 u_0 & u_1 & \cdots & u_{n-1} & x & y & z \\
 1 & 1 & \cdots & 1 & \lambda & \mu & 2v \\
 0 & 0 & \cdots & 0 & 1 & 1 & 2
\end{pmatrix}
\]
and let $Z \subset R$ be the hypersurface defined by
\[ ax^2 + by^2 + cz + fxy = 0. \]

We have a natural morphism $P \to R$ which is a (purely inseparable) double cover branched along $(\bar{z} = 0) \subset R$. The image of $X$ under $P \to R$ is the hypersurface $Z \subset R$. Let $\tau: X \to Z$ be the induced morphism, which is a double cover branched along the divisor cut out on $Z$ by $\bar{z} = 0$. We set $L = O_Z(\nu, 1)$. Then $\bar{z}$ is a global section of $L^2$, and over the non-singular locus of $Z$, $\tau$ is the double cover obtained by taking the roots of $\bar{z} \in H^0(Z, L^2)$ in the sense of [11, Construction 8].

In Sections 3.3.A and 3.3.B below we will analyse the singularities of $X$ and $Z$, and finally we will show the existence of a universally $\text{CH}_0$-trivial resolution $q: Y \to X$ such that $H^0(Y, \Omega_Y^{n-1}) \neq 0$ under some conditions on $\lambda, \mu, \nu$. The latter implies that $Y$ is not universally $\text{CH}_0$-trivial by [18, Lemma 2.2].

### 3.A. Singularities

Recall that the ground field $\mathbb{k}$ is an algebraically closed field of characteristic 2 and $X$ is a hypersurface in $P = P_n(\lambda, \mu, \nu)$ defined by
\[ ax^2 + by^2 + cz^2 + fxy = 0 \]
for general $a, b, c, f \in \mathbb{k}[u_0, \ldots, u_{n-1}]$. Similarly $Z$ is the hypersurface in $R$ defined by
\[ ax^2 + by^2 + cz + fxy = 0. \]

We set
\[ \Xi = (x = y = 0) \subset R, \quad \Xi_Z = \Xi \cap Z = (x = y = c = 0), \]
and \( R^\circ = R \setminus \Xi, Z^\circ = Z \setminus \Xi_Z \).

In order to analyze singularities of \( Z^\circ \subset R^\circ \), we consider standard affine charts of \( R^\circ \). For \( i = 0, \ldots, n-1 \) and a coordinate \( w \in \{x, y\} \), we set \( U_{u_i,w} = (u_i \neq 0) \cap (w \neq 0) \subset R^\circ \). We have

\[
R^\circ = \bigcup_{i \in \{0, \ldots, n-1\}, w \in \{x, y\}} U_{u_i,w}.
\]

We remark that \( U_{u_i,w} \) is an affine \((n+1)\)-space and that the restriction of the sections

\[
\{u_0, \ldots, u_{n-1}, x, y, z\} \setminus \{u_i, w\}
\]

are affine coordinates of \( U_{u_i,w} \). We only treat \( U_{u_0,x} \) because the other open subsets can be understood by symmetry. We set

\[
u_i = u_i/u_0, \quad \tilde{y} = y/xu_0^{\mu-1}, \quad \tilde{z} = z/x^2 u_0^{\nu-2\lambda}.
\]

Then \( U_{u_0,w} \) is an affine \((n+1)\)-space with affine coordinates \( u_1, \ldots, u_{n-1}, \tilde{y}, \tilde{z} \). By a slight abuse of notation, the affine coordinates \( u_1, \ldots, u_{n-1}, \tilde{y}, \tilde{z} \) are simply denoted by \( u_1, \ldots, u_{n-1}, y, z \).

**Lemma 3.4.** \( Z^\circ \) is smooth.

**Proof.** If \( \deg c = 0 \), then \( c \) is a non-zero constant and thus \( \Xi_Z = \emptyset \). In this case \( Z = Z^\circ \) is a \( \mathbb{P}^1 \) bundle over \( \mathbb{P}^{n-1} \) and it is smooth.

In the following we assume that \( \deg c > 0 \) and set

\[
U_x = (x \neq 0), \quad U_y = (y \neq 0) \subset R,
\]

so that \( R^\circ = U_x \cup U_y \). We will show that for any point \( q \in R^\circ \), the condition that \( Z^\circ \) is singular at \( q \in Z \) imposes \( n+2 \) independent conditions on \( a, b, c, f \). Then the assertion will follow by a dimension count argument since \( \dim R^\circ = n+1 \). We note that \( \deg b = \lambda - \mu + \nu \geq 1, \deg c = \lambda + \mu - \nu \geq \lambda \geq 3 \) and \( \deg f = \lambda \geq 3 \) by Remark 3.1.

Let \( q \in U_x \). Replacing coordinates, we may assume \( q = (1:0:\cdots:0;1:0:0) \). Then \( U_{u_0,x} \subset Q \) is an affine space with coordinates \( u_1, \ldots, u_{n-1}, y, z \) and \( Z \cap U_{u_0,x} \) is defined by

\[
\tilde{a} + \tilde{b} y^2 + \tilde{c} z + \tilde{f} y = 0,
\]

where we set \( \tilde{h} = h(1, u_1, \ldots, u_{n-1}) \) for a polynomial \( h(u_0, \ldots, u_{n-1}) \). Note that \( q \) corresponds to the origin. The variety \( Z^\circ \) is singular at \( q \) if and only if \( \tilde{a}, \tilde{c}, \tilde{f} \) vanish at \( q \) and the linear part of \( \tilde{a} \) is zero. This imposes \( n+2 \) independent conditions since \( \deg a > 0 \) and \( \deg c, \deg f \geq 0 \) (cf. Remark 3.1).

Suppose that \( q \in U_y \). Replacing coordinates, we may assume \( q = (1:0:\cdots:0;0:1:0) \). Then \( U_{u_0,y} \subset Q \) is an affine space with coordinates \( u_0, \ldots, u_{n-1}, x, z \) and \( Z \cap U_{u_0,y} \) is defined by

\[
\tilde{a} x^2 + \tilde{b} + \tilde{c} z + \tilde{f} x = 0.
\]

The variety \( Z^\circ \) is singular at \( q \) if and only if \( \tilde{b}, \tilde{c}, \tilde{f} \) vanish at \( q \) and the linear part of \( \tilde{b} \) is zero. The latter imposes \( n+2 \) independent conditions since \( \deg b > 0 \) and \( \deg c, \deg f \geq 0 \) (cf. Remark 3.1), and the proof is complete.

We set \( X^\circ = \pi^{-1}(Z^\circ) \).

**Lemma 3.5.** \( X \) is smooth along \( X \setminus X^\circ \).

**Proof.** Note that \( X \setminus X^\circ = X \cap (x = y = 0) \). For a point \( p \in X \setminus X^\circ \), \( X \) is smooth at \( p \) if and only if the hypersurface \( (c = 0) \subset \mathbb{P}^{n-1} \) is smooth at the image of \( p \) under \( X \to \mathbb{P}^{n-1} \). Clearly the hypersurface \( (c = 0) \subset \mathbb{P}^{n-1} \) is smooth since \( c \) is general, and the assertion follows.
3.B. Analysis of critical points

We set $L^0 = L|_Z$, where we recall $L = O_Z(v, 1)$. By Lemma 3.4, $Z^\circ$ is non-singular and by Kollár's result [12, V.5] there exists an invertible sheaf $Q^\circ$ on $Z^\circ$ such that $M^\circ := \tau^*Q^\circ \subset (O^{-1}_X)^\vee$, where $\vee$ denotes the double dual. Let $M$ be the push-forward of the invertible sheaf $M^\circ$ via the open immersion $X^\circ \hookrightarrow X$. By Lemma 3.5, $M$ is an invertible sheaf on $X$.

Definition 3.6. Let $V$ be a nonsingular variety of dimension $n$ defined over an algebraically closed field $k$ of characteristic $2$, $N$ an invertible sheaf on $V$ and $s \in H^0(V, N^2)$ a section. Let $p \in V$ be a point, $\xi$ a local generator of $N$ at $p$ and $s = f(x_1, \ldots, x_n)\xi^2$ a local description of $s$ with respect to local coordinates $x_1, \ldots, x_n$ of $V$ at $p$. We say that $s$ has a critical point at $p$ if the linear term of $f$ is zero.

We say that $s$ has an admissible critical point at $p$ if for a suitable choice of coordinates $x_1, \ldots, x_n$,

$$f = \begin{cases} \alpha + x_1x_2 + x_3x_4 + \cdots + x_{n-1}x_n + g, & \text{if } n \text{ is even}, \\ \alpha + \beta x_1^2 + x_2x_3 + \cdots + x_{n-1}x_n + g, & \text{if } n \text{ is odd}, \end{cases}$$

where $\alpha, \beta \in k$, $g = g(x_1, \ldots, x_n) \in (x_1, \ldots, x_n)^3$ and, in case $n$ is odd, the coefficient of $x_1^3$ in $g$ is nonzero.

Lemma 3.7. The section $\tilde{z} \in H^0(Z, L^2)$ has only admissible critical points on $Z^\circ$.

Proof. We choose and fix a general $c \in k[u]$ so that the hypersurface $(c = 0) \subset \mathbb{P}^{n-1}$ is non-singular. Clearly $\tilde{z}$ does not have a critical point on $(c = 0) \subset Z^\circ$. On $Z^\circ \cap (c \neq 0)$, the section $c$ is invertible and thus the section $\tilde{z}$ has an admissible critical point if and only if the section

$$s := c(ax^2 + by^2 + fxy) = (c^2) \tilde{z}$$

has an admissible critical point. It is then enough to show that the section $s$, viewed as a section on $Q = Q_n(\lambda, \mu)$, has only admissible critical points on $U_\ell = (c \neq 0) \subset Q$ for general $a, b$ and $f$. We set $U_x = (x \neq 0) \subset Q$ and $\Pi_y = (x = 0) \cap (y \neq 0) \subset Q$ so that $Q = U_x \cup \Pi_y$.

We first show that $s$ does not have a critical point on $\Pi_y \cap U_\ell$. Let $p \in \Pi_y \cap U_\ell$ be a point. We may assume $p = (1 : 0 : \cdots : 0 : 1 : 0)$. We work on the open subset $U_{u_0, y} = (u_0 \neq 0) \cap (y \neq 0) \subset Q$ which is the affine space with coordinates $u_1, \ldots, u_{n-1}$ and $x$. For $c = c(u_0, \ldots, u_{n-1})$, we set $\tilde{c} = c(1, u_1, \ldots, u_{n-1})$. Moreover we denote by $\tilde{c}_i$ the degree $i$ part of $\tilde{c}$. Then the restriction of $s$ to $U_{u_0, y}$ is $c(\tilde{a}x^2 + \tilde{b}f + \tilde{f}x)$ and the point $p$ corresponds to the origin. Then $s$ has a critical point at $p$ if and only if

$$\tilde{c}_0(\tilde{b}_1 + \tilde{f}_0x) + \tilde{c}_1\tilde{b}_0 = 0.$$ 

Note that $\tilde{c}_0 \neq 0$. Since $\deg b \geq 1$, this imposes $n$ independent conditions on $a, b, f$. Thus, for any point $p \in \Pi_y$, $n$ conditions are imposed in order for $s$ to have a critical point at $p$. By counting dimensions we conclude that $s$ does not have a critical point on $\Pi_y \cap U_\ell$ since $\dim \Pi_y = n - 1$.

Let $p \in U_x \cap U_\ell$ be a point. We may assume $p = (1 : 0 : \cdots : 0 : 1 : 0)$. We work on the open subset $U_{u_0, x} = (u_0 \neq 0) \cap (x \neq 0) \subset R$ which is the affine space with coordinates $u_1, \ldots, u_{n-1}$ and $x$. We have $s|_{U_{u_0, y}} = c(\tilde{a}y^2 + \tilde{b}y^2 + \tilde{f}y)$. Let $\ell, q$ and $h$ be the linear, quadratic and cubic parts of $s|_{U_{u_0, y}}$, respectively. We have

$$\ell = \tilde{c}_0(\tilde{a}_1 + \tilde{f}_0y) + \tilde{c}_1\tilde{a}_0.$$ 

Since $\deg a \geq 1$, $n$ conditions are imposed in order for $s$ to have a critical point at $p$. It remains to show the existence of a section $s = c(ax^2 + by^2 + fxy)$ which has an admissible critical point at $p$. Now suppose that $s$ has a critical point at $p$, that is, $\ell = 0$. This implies that $\tilde{f}_0 = 0$ and $\tilde{a}_1 = \tilde{a}_0\tilde{c}_1/\tilde{c}_0$. Then, for the quadratic and cubic parts, we have

$$q = \tilde{c}_0(\tilde{a}_2 + \tilde{b}_0y^2 + \tilde{f}_1y) + \tilde{a}_0\tilde{c}_2/\tilde{c}_0 + \tilde{c}_2\tilde{a}_0,$$

$$h = \tilde{c}_0(\tilde{a}_3 + \tilde{b}_1y^2 + \tilde{f}_2y) + \cdots.$$
Since \( \deg a \geq 3 \) and \( \deg f \geq 3 \), we can choose \( a, b, f \) so that
\[
q = \begin{cases} 
y u_1 + u_2u_3 + u_4u_5 + \cdots + u_{n-2}u_{n-1}, & \text{if } n \text{ is even}, 
y u_1 + u_2u_3 + u_4u_5 + \cdots + u_{n-3}u_{n-2} + u_{n-1}^2, & \text{if } n \text{ is odd}.
\end{cases}
\]

In case \( n \) is even, the section \( s \) has a nondegenerate critical point at \( p \) and we are done. Suppose that \( n \) is odd. Since \( \deg a \geq 3 \), then we can choose \( a, b, f \) so that \( q \) is as above and the coefficient of \( u_{n-1}^3 \) in \( h \) is non-zero. For this choice of \( a, b, c, f \), the section \( s \) has an admissible critical point at \( p \) and the proof is completed by the dimension counting argument. \( \square \)

**Proposition 3.8.** Let the notation and assumption as above. Assume in addition that \( n \geq 1 \). Then there exists a universally \( CH_0 \)-trivial resolution \( q : Y \to X \) of singularities such that \( H^0(Y, \Omega^1_Y) \neq 0 \). In particular \( Y \) is not universally \( CH_0 \)-trivial.

**Proof.** By [15, Proposition 4.1] or [6], if the singularities of \( X \) correspond to admissible critical points of the section \( \zeta \), then there exists a universally \( CH_0 \)-trivial resolution \( q : Y \to X \) such that \( q^*\mathcal{M} \hookrightarrow \Omega^1_Y \) (in fact, \( q^* \) is just the composite of blowups at each isolated singular point). Thus, by Lemma 3.7, \( X \) admits such a resolution. The branch divisor (\( \zeta = 0 \)) is clearly reduced and, by [12, Lemma V.5.9], we have an isomorphism
\[
\mathcal{M}^0 \cong \tau^*(\omega_{\mathbb{Z}}^2 \otimes L^2) \cong O_X(v - n, 0),
\]
so that \( \mathcal{M} \cong O_X(v - n, 0) \). By the assumption we have \( v - n \geq 0 \), which implies \( H^0(X, \mathcal{M}) \neq 0 \). Thus \( H^0(Y, \Omega^1_Y) \neq 0 \) and by [18, Lemma 2.2], \( Y \) is not universally \( CH_0 \)-trivial. \( \square \)

### 3.C. Proof of theorems and corollaries

**Theorem 3.9.** Suppose that the ground field is \( \mathbb{C} \) and let \( (\lambda, \mu, \nu) \) be a triplet of integers such that \( 0 < \lambda \leq \mu \leq \nu \leq \lambda + \mu \). If in addition \( \nu \geq n \), then a very general embedded conic bundle \( \pi : X \to \mathbb{P}^{n-1} \) of type \( [\lambda, \mu, \nu] \) is not stably rational.

**Proof.** For a field (or more generally a ring) \( K \), we denote by \( P_K \) the \( \mathbb{P}^2 \)-bundle \( P_n(\lambda, \mu, \nu) \) over \( \mathbb{P}^{n-1} \) defined over \( K \). Let \( k \) be an algebraically closed field of characteristic 2 and let \( X \to \mathbb{P}^{n-1} \) be a very general hypersurface in \( P_k \) defined by an equation of the form \( (l) \). We take a mixed characteristic discrete valuation ring \( A \) whose residue field is \( k \), for example the Witt ring, and then we lift \( X \) to a hypersurface \( \tilde{X} \) of \( P_A \) defined by an equation of the form \( (l) \). We choose and fix an embedding of the quotient field of \( A \) into \( \mathbb{C} \) and set \( V = \tilde{X} \times_A \mathbb{C} \). Then \( V \) is a very general hypersurface of \( P_{\mathbb{C}} \) defined by an equation of the form \( (l) \). By Proposition 3.8, we can apply the specialization theorem [5, Théorème 1.14] and conclude that \( V \) is not universally \( CH_0 \)-trivial. Note that \( V \) is nonsingular by Lemma 3.2. Note also that \( V \) is not a very general conic bundle of type \( [\lambda, \mu, \nu] \). However a very general conic bundle of type \( [\lambda, \mu, \nu] \) degenerates (over a complex curve) to \( V \), hence the assertion follows from the specialization argument [19, Theorem 2.1] (or by [14, Theorem 4.2.10]). \( \square \)

Now we can prove the main theorem and corollaries in Section 1.

**Proof of Theorem 11.** The assertion \( (1) \) follows from Proposition 2.9.

Let \( \pi : X \to \mathbb{P}^{n-1} \) be a non-singular embedded conic bundles over \( \mathbb{P}^{n-1} \). By Proposition 2.11, we may assume that it is of type \( [\lambda, \mu, \nu] \), where either \( 0 < \lambda \leq \mu \leq \nu \leq \lambda + \mu \) or \( \lambda = \mu = \nu = 0 \). By adjunction we have \( O_X(-K_X) \cong O_X(n, 1) \). The complete linear system \( |O_p(n, 1)| \), where \( P = P_n(\lambda, \mu, \nu) \), is base point free if and only if \( n \geq \nu \). This shows that \( O_p(n, 1) \), and hence \( O_X(n, 1) \), is ample if \( n < \nu \). Since \( -K_X \) is not ample by assumption, we have \( n \geq \nu \). Therefore \( (2) \) follows from Theorem 3.9. \( \square \)
Proof of Corollaries 1.2 and 1.3. Let X be a very general hypersurface of bi-degree $(d, 2)$ in $\mathbb{P}^{n-1} \times \mathbb{P}^2$. Then $O_X(-K_X) \cong O_X(n-d, 1)$. By assumption $d \geq n$ and this implies that $-K_X$ is not ample. Thus X is not stably rational by Theorem 1.1.

Let X be a double cover of $\mathbb{P}^{n-1} \times \mathbb{P}^1$ branched along a very general divisor of bi-degree $(2d, 2)$, where $P = P_0(0, 0, d)$, and hence it is of type $[d,d,2d]$. By the assumption we have $2d \geq n$. Thus X is not stably rational by Theorem 3.9.

Proof of Corollary 1.5. Let $\pi : X \to \mathbb{P}^{n-1}$ be as in Corollary 1.5. Then we may assume that it is of type $[\lambda, \mu, \nu]$, where $0 < \lambda \leq \mu \leq \nu \leq \lambda + \nu$ or $\lambda = \mu = \nu = 0$. The discriminant divisor $\Delta$ is a hypersurface in $\mathbb{P}^{n-1}$ of degree $\lambda + \mu + \nu$. The condition $|3K_{\mathbb{P}^{n-1}} + \Delta| \neq \emptyset$ is equivalent to the condition $\lambda + \mu + \nu \geq 3n$. The latter implies $\nu \geq n$ since $\lambda \leq \mu \leq \nu$. Thus (l) follows from Theorem 3.9.

Now suppose in addition that $n \geq 7$ and $\pi : X \to \mathbb{P}^{n-1}$ is standard. Note that $X$ is defined in $P_n(\lambda, \mu, \nu)$ by an equation of the form

$$ax^2 + by^2 + cz^2 + fxy + gxyz + hxyz = 0,$$

where $a, \ldots, h \in \mathbb{C}[u]$. If $deg c = \lambda + \mu - \nu > 0$, then the system of equations $a = b = \cdots = h = 0$ has a non-trivial solution on $\mathbb{P}^{n-1}$ since $n \geq 7$. This implies that $\pi$ cannot be flat, in particular, not standard. Thus $\nu = \lambda + \mu$ and in this case the condition $|2K_{\mathbb{P}^{n-1}} + \Delta| = |O_{\mathbb{P}^{n-1}}(2(n - n))| \neq \emptyset$ is equivalent to $\nu \geq n$ which implies stable non-rationality of X again by Theorem 3.9. This proves (2).

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