Our initial goal was to study a generalization of the notion of exact operator space for which the operator space version of Grothendieck’s theorem obtained in joint work with Shlyakhtenko is still valid.
Let $E$ be a $d$-dimensional Banach space.

In *Remarques sur un résultat non publié de B. Maurey (1980/1981)*

I introduced the number

$$k_E(C) = \inf\{k \mid E^C \subset \ell_k^\infty\}$$

Gaussian random variables can be used to give a quick proof of the fact that if either $E = \ell_2^d$ or $E = \ell_1^d$ then $\exists \delta = \delta_C > 0$ such that

$$k_E(C) \geq \exp(\delta d).$$

In * it is shown same holds (with $\delta = \delta(C, C') > 0$) whenever $E^*$ has type $p > 1$ with constant at most $C'$.

**Open Problem (dichotomy conjecture):** Same holds whenever $E$ has finite cotype $q < \infty$.
Geometrically

Recall: Essentially **any (symmetric convex) body** is equivalent to a $d$-dimensional section of an $k$-cube of dimension $d \approx \log k$ (equivalently $k_E(C) \leq \exp(cd)$ for any $d$-dimensional $E$)

Consider a section of an $n$-cube of dimension $d >>> \log n$: does it admit a further section of large dimension (i.e. $>> 1$) that is equivalent to a cube?
Definition

(“Non-commutative Banach spaces”) An operator space is a subspace of $B(H)$, i.e. we are given

$$E \subset B(H)$$

“Operator space Theory" is now well developed after Ruan's 1987 thesis cf. Effros-Ruan, Blecher-Paulsen...
Let
\[ u : E \rightarrow F \]
be a linear map between operator spaces. We denote for any given \( N \geq 1 \)
\[ u_N = \text{Id} \otimes u : M_N(E) \rightarrow M_N(F) \]
\[ \begin{bmatrix} a_{ij} \end{bmatrix} \mapsto \begin{bmatrix} u(a_{ij}) \end{bmatrix} \]

\[ ( u_N = \text{Id} \otimes u : M_N \otimes E \rightarrow M_N \otimes F \]
\[ \sum a_k \otimes x_k \mapsto \sum a_k \otimes u(x_k) \]

Recall that
\[ \|u\|_{cb} = \sup_{N \geq 1} \|u_N\| \].
Given

\[ E \subset B(H) \quad F \subset B(\mathcal{H}) \]

we define

\[ E \otimes_{\text{min}} F \subset B(H \otimes_2 \mathcal{H}) \]

(“spatial" or “minimal" tensor product)

Note: This norm will be used everywhere!
\[ B = B(H) \quad \text{or} \quad B = M_N \quad \text{if} \quad \dim(H) = N \]

**Quantization:** Scalars replaced by operators

**Banach space structure on \( E \):** A norm on \( \mathbb{C} \otimes E \):

\[
\left\| \sum c_k \otimes x_k \right\|, \quad c_k \in \mathbb{C}, \quad x_k \in E
\]

**Operator space structure on \( E \):** A norm on \( B \otimes E \)

\[
\left\| \sum b_k \otimes x_k \right\|, \quad b_k \in B, \quad x_k \in E
\]

Here

\[
\| \cdot \| = \| \cdot \|_{\min}
\]
Matricial ("Quantized") Banach-Mazur distance

Assuming $E \cong F$, we set

$$d_N(E, F) = \inf \{ \|u_N\| \| (u^{-1})_N \| \}$$

where the inf runs over all the isomorphisms $u : E \to F$. We set $d_N(E, F) = \infty$ if $E, F$ are not isomorphic. Also, if $E, F$ are completely isomorphic, we set

$$d_{cb}(E, F) = \inf \{ \| u \|_{cb} \| u^{-1} \|_{cb} \}$$

where the inf runs over all the complete isomorphisms $u : E \to F$.

If $E, F$ finite dim by compactness:

$$d_{cb}(E, F) = \sup_{N \geq 1} d_N(E, F)$$
Recall (for $E$ finite dim. Banach space)

$$k_E(C) = \inf \{ k \mid C \subseteq \ell^k_\infty \}$$
Matricial version of $k_E(C)$

Let $E$ be a finite dimensional operator space. Fix $C > 0$. We denote by $k_E(N, C)$ the smallest integer $k$ such that there is an operator subspace $F \subset M_k$ such that

$$d_N(E, F) \leq C.$$ 

In short:

$$k_E(N, C) = \inf\{ k \mid E \subset M_k \}^{N,C}$$
Definition

We say that an operator space $X$ is $C$-subexponential if

$$\limsup_{N \to \infty} \frac{\log k_E(N, C)}{N} = 0.$$ 

for any finite dimensional subspace $E \subset X$.

Note: If $X$ itself is finite dimensional, it suffices to consider $E = X$.

We will denote by $C(X)$ the smallest $C$ such that $X$ is $C$-subexponential.
An operator space $X$ is called $C$-exact if for any finite dimensional subspace $E \subset X$ and any $c > C$ there is a $k$ and $F \subset M_k$ such that $d_{cb}(E, F) < c$. We denote by $ex(X)$ the smallest such $C$.

**Lemma**

An operator space $X$ is $C$-exact iff

$$\sup_{N \geq 1} k_E(N, C) < \infty.$$  

for any finite dimensional subspace $E \subset X$.

Proof: by a standard compactness argument
Let $Y^{(N)}$ be a random $N \times N$-matrix with i.i.d. complex Gaussian entries with mean zero and second moment equal to $N^{-1/2}$, let $(Y_j^{(N)})$ be a sequence of i.i.d. copies of $Y^{(N)}$.

Theorem (Implicit in Haagerup-Thorbjørnsen 1999)

For any $a_1, \cdots, a_r \in B(H)$, let $S = \sum_1^r a_j \otimes Y_j^{(N)}$. Assume $\max\{\|\sum a_j^* a_j\|^{1/2}, \|\sum a_j a_j^*\|^{1/2}\} \leq 1$.

For any integer $p \geq 1$, let $\Sigma = \mathbb{E}(S^* S)^p$. Then:

$$\mathbb{E}(S^* S)^p = \Sigma \otimes 1 \leq \mathbb{E}(Y^{(N)*} Y^{(N)})^p \otimes 1.$$

Consequently, if $\dim(H) = k$ we have

$$(\mathbb{E} \text{ tr}|S|^{2p})^{1/2p} \leq (k \mathbb{E} \text{ tr}|Y^{(N)}|^{2p})^{1/2p}. \quad (1)$$

where the trace on the left is on $\ell_2^N \otimes H$ and the one on the right is on $\ell_2^N$. 

Gilles Pisier  U. Pierre et Marie Curie (Paris VI) and Texas A&M Random Matrices and Subexponential Operator Spaces
Remark

Actually, (using Buchholz 2001) more generally, if 
\[ \max \{ \| (\sum a_j^* a_j)^{1/2} \|_{2p}, \| (\sum a_j a_j^*)^{1/2} \|_{2p} \} \leq 1, \]
then
\[ \mathbb{E} \| S \|_{2p}^{2p} \leq \mathbb{E} \| Y^{(N)} \|_{2p}^{2p}. \]

Obviously best possible.
Interpretation: Khintchine type inequality for Gaussian random matrices with best possible constant.
Lemma

For any $\varepsilon > 0$, there is a constant $\gamma_\varepsilon$ such that for any integer $k$, any $N \geq 1$ and any $a_1, \cdots, a_r \in M_k$ we have

$$\mathbb{E} \left\| \sum_{j=1}^{r} a_j \otimes Y_j^{(N)} \right\| \leq (1 + \varepsilon) \left( 2 + \gamma_\varepsilon \left( \frac{\log(ek)}{N} \right)^{1/2} \right) \max\{ \| (\sum a_j^* a_j)^{1/2} \|, \| (\sum a_j a_j^*)^{1/2} \| \}.$$
∀ \varepsilon > 0, \exists \gamma_\varepsilon \text{ such that } \forall a_1, \cdots, a_r \in M_k \text{ we have}

\[ \mathbb{E} \left\| \sum_{j=1}^{r} a_j \otimes Y_j^{(N)} \right\| \leq (1 + \varepsilon)(2 + \gamma_\varepsilon \left( \frac{\log(ek)}{N} \right)^{1/2}) \max \left\{ \left\| \left( \sum a_j^* a_j \right)^{1/2} \right\|, \left\| \left( \sum a_j a_j^* \right)^{1/2} \right\| \right\}.

This leads us to

**Theorem**

*Let E be a C-subexponential operator space i.e.*

\[ \frac{\log k_E(N, C)}{N} \to 0 \]

*Then for any a_1, \cdots, a_r \in E we have*

\[ \lim_{N \to \infty} \mathbb{E} \left\| \sum_{j=1}^{r} a_j \otimes Y_j^{(N)} \right\| \leq 2C \max \left\{ \left\| \left( \sum a_j^* a_j \right)^{1/2} \right\|, \left\| \left( \sum a_j a_j^* \right)^{1/2} \right\| \right\}. \]
Proof.

Fix $c > C$. Consider $u : E \to F$ with $F \subset M_k$, $k = k_E(N, C)$ and $\|u_N\|\|u^{-1}_{-1}\| \leq c$. By homogeneity we may assume

$$\max \{\| (\sum a_j^* a_j)^{1/2} \|, \| (\sum a_j a_j^*)^{1/2} \| \} \leq 1.$$  

Let $b_j = u(a_j)$. We may assume $r \leq N$. Then we have

$$\max \{\| (\sum b_j^* b_j)^{1/2} \|, \| (\sum b_j b_j^*)^{1/2} \| \} \leq \|r\| \leq \|N\|,$$

and also

$$\| \sum_{1}^{r} a_j \otimes Y_j^{(N)} \| \leq \|u^{-1}_{-1}\| \sum_{1}^{r} b_j \otimes Y_j^{(N)} \|.$$  

This gives us

$$\| \sum_{1}^{r} a_j \otimes Y_j^{(N)} \| \leq \|u^{-1}_{-1}\| \|u_N\|(1 + \varepsilon)(2 + \gamma \varepsilon(\frac{\log(k) + 1}{N})^{1/2},$$
Notation: For \( a = (a_1, \cdots, a_r) \) with \( a_j \in E \) we denote

\[
\|a\|_{RC} = \max\{\| \left( \sum a_j^* a_j \right)^{1/2} \|, \| \left( \sum a_j a_j^* \right)^{1/2} \| \},
\]

\[
\|a\|_R = \| \left( \sum a_j a_j^* \right)^{1/2} \|,
\]

and

\[
\|a\|_C = \| \left( \sum a_j^* a_j \right)^{1/2} \|,
\]

so that

\[
\|a\|_{RC} = \max\{\|a\|_R, \|a\|_C\}.
\]
Let $E, F$ be subexponential operator spaces with respective constants $C(E), C(F)$. Then any c.b. linear map $u : E \to F^*$ satisfies for any $r$, any $a = (a_1, \cdots, a_r) \in E^r$ and any $b = (b_1, \cdots, b_r) \in F^r$

$$|\sum \langle u(a_j), b_j \rangle| \leq 4C(E) C(F) \|u\|_{cb} \|a\|_{RC} \|b\|_{RC}.$$ 

It is easy to check that if $E$ is $C$-subexponential, the minimal tensor product $K(\ell_2) \otimes_{\min} E$ (of $E$ with the set $K(\ell_2)$ of all compact operators on $\ell_2$) is also $C$-subexponential. Indeed, this follows from the fact that, if $E$ is $C$-subexponential, $\forall n \ M_n(E)$ is $C$-subexponential because

$$k_{M_n(E)}(C, N) \leq k_E(C, N/n).$$
By known arguments (there is a very recent much simpler proof by Regev and Vidick RV), and combining that with JP, we obtain:

**Corollary**

Let $E$, $F$ be subexponential operator spaces with respective constants $C(E)$, $C(F)$. Then any c.b. linear map $u : E \to F^*$ with $\|u\|_{cb} \leq 1$ satisfies for any $r$, any $a = (a_1, \cdots, a_r) \in E^r$ and any $t_j > 0$ and any $b = (b_1, \cdots, b_r) \in F^r$

$$\left| \sum \langle u(a_j), b_j \rangle \right| \leq c(\| (a_j) \|_R \| (b_j) \|_C + \| (t_j a_j) \|_C \| (t_j^{-1} b_j) \|_R).$$

with

$$c = 4C(E) \ C(F).$$

Assuming $E \subset B(H)$ and $F \subset B(K)$ Hahn-Banach $\Rightarrow \exists$ states $f_1, f_2, g_1, g_2$ such that $\forall (a, b) \in E \times F$

$$| \langle u(a), b \rangle | \leq c \left( \sqrt{f_1(aa^*)} \sqrt{g_1(b^*b)} + \sqrt{f_2(a^*a)} \sqrt{g_2(bb^*)} \right)$$
We now compare the subexponential case with the general case. Unfortunately, there is a gap between the two estimates—subGaussian///subexponential—that we do not see how to fill.

**Lemma**

Let $E$ be a $d$-dimensional operator space, then for any $\delta > 0$ we have

$$k_E(N, 1 + \delta) \leq N(1 + \delta^{-1})^{dN^2}.$$  

Therefore, for any operator space $X$, any finite dimensional subspace $E \subset X$ we have

$$\forall C > 1 \limsup_{N \to \infty} \frac{\log k_E(N, C)}{N^2} < \infty.$$  

Underlying reason: $\dim(M_N(E)) = dN^2$!
Remark

A possible variant could be to define $X$ as $(C, C')$-subexponential if for any finite dimensional $E \subset X$ we have

$$\limsup_{N \to \infty} N^{-1} \log k_E(N, C) \leq C'.$$
We conclude by examining some examples. It turns out to be easy to show that the known examples of non-exact operator spaces are also not subexponential. Thus we unfortunately must leave as an open problem (and a conjecture) the existence of subexponential spaces that are not exact.
Minimal operator space $E \subset A \cong \ell_\infty \subset B(H)$
Equivalently $\forall F \forall u : F \to E$ we have $\|u\|_{cb} = \|u\|$
Example: $E = \ell_\infty$

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Maximal operator space (dual to minimal)
Equivalently $\forall F \forall u : E \to F$ we have $\|u\|_{cb} = \|u\|$
Example: $E = \text{span}\{U_j\}$ (universal sequence of unitaries)
Note: $E \cong \ell_1$ (with “maximal" operator space structure)
Proposition

Let $E$ be any $d$-dimensional space with its maximal operator space structure (e.g. $E = \ell^d_1$). Then

$$C(E) \geq c\sqrt{d}$$

where $c > 0$ is a constant independent of $d$.

Proof: We transplant from exact to subexponential an argument from [Junge-P 1994].
Remark

In the converse direction, for any $d$-dimensional operator space $E$ we have

$$C(E) \leq \text{ex}(E)$$

and it is known ([P 1996]) that

$$\text{ex}(E) \leq d^{1/2}$$
The Operator Hilbert space $OH$

$$OH_d = \text{span}[T_1, \ldots, T_d]$$

$$\forall a_j \in B(H) \quad \| \sum a_j \otimes T_j \| = \| \sum a_j \otimes \bar{a}_j \|^{1/2}$$

**Remark**

We claim that

$$d^{1/4}/2 \leq C(OH_d) \leq d^{1/4}.$$ 

$$\limsup_{N \to \infty} \mathbb{E} \left( \left\| \sum_{1}^{d} Y_j^{(N)} \otimes \overline{Y_j^{(N)}} \right\|^{1/2} \right) \leq 2C(OH_d)d^{1/4}, \text{ and}$$

since $$\left\| \sum_{1}^{d} Y_j^{(N)} \otimes \overline{Y_j^{(N)}} \right\| \geq \sum_{1}^{d} \text{tr}|Y_j^{(N)}|^2 \approx d$$ we obtain

$$d^{1/2} = \limsup_{N \to \infty} \mathbb{E} \left( \left( \sum_{1}^{d} \text{tr}|Y_j^{(N)}|^2 \right)^{1/2} \right) \leq 2C(OH_d)d^{1/4}.$$
Remark

Similarly

\[ \frac{d^{1/2}}{2} \leq C(R_d + C_d) \leq d^{1/2}. \]
Lemma

If $E$ is $\ell^d_1$ equipped with its maximal operator space structure then for any $C > 1$ there is $\delta > 0$ depending only on $C$ such that for any $d, N$ we have

$$k_E(N, C) \geq \exp \delta N d$$

(2)

If $E = R_d + C_d$ or $\ell^d_2$ equipped with its maximal operator space structure (resp. $E = OH_d$), this still holds (resp. we have $k_E(N, C) \geq \exp \delta N d^{1/2}$) for all $N \geq d$. 
O. Regev and T. Vidick have found a very quick, simple and more quantitative proof of the os version of GT proved by Haagerup-Musat for c.b. bilinear forms on $C^*$-algebras, and it applies also to the previous result on bilinear forms on exact or subexponential spaces.

Note:

Let $u : E \times F \to \mathbb{C}$

$$\|u\|_{cb} \leq 1$$

IFF

$$\forall n \forall \xi, \eta \in B_{\ell^2 \otimes 2 \ell^2_n} \quad \|\Phi_{\xi,\eta} \otimes u : M_n(E) \times M_n(F) \to \mathbb{C}\| \leq 1$$

where $\Phi_{\xi,\eta} : M_n \times M_n \to \mathbb{C}$ is defined by

$$\Phi_{\xi,\eta}(a, b) = \langle (a \otimes b)\xi, \eta \rangle$$
Regev and Vidick manage to deduce directly from

$$ († † †) \quad |\sum \langle u(a_j), b_j \rangle| \leq (\|a\|_R^2 + \|a\|_C^2)^{1/2}(\|b\|_R^2 + \|b\|_C^2)^{1/2} $$

assumed satisfied also on $M_n(E) \times M_n(F)$ for all $n$ that for any $t_j > 0$

$$ (⋆ ⋆ ⋆) \quad |\sum \langle u(a_j), b_j \rangle| \leq \|(a_j)\|_R \|(b_j)\|_C + \|(t_j a_j)\|_C \|(t_j^{-1} b_j)\|_R. $$

- Using this idea the OS version of GT of Haagerup-Musat (Invent. 2008) is reduced to Haagerup’s 1985 non-commutative GT (extending my own 1978 version)

- In the subexponential or exact case, the same idea reduces the version of P-Shlyakhtenko (Invent. 2002) to the Junge-P version (GAFA 1994)
Regev and Vidick’s key ingredient:

Just a family of $n \times n$ matrices $E(t)$ indexed by $t > 0$ with non-negative entries.

$\forall t > 0$

$$\sup_i \sum_j L(t)_{ij} \leq 1$$

$$\sup_j \sum_i L(t)_{ij} \leq t^2$$

and lastly

$$|t^{-1}\langle L(t)z_n, z_n \rangle - 1| \leq C(\log n)^{-1} \log(1 + \max\{t, t^{-1}\})$$

(3)

where $z_n = (Z_n)^{-1/2} \sum_1^n j^{-1/2} e_j$ with $Z_n \approx \log n$ defined so that $z_n$ is a unit vector in $\ell_2^n$.

**Note:** $E(t)$, $z$ above are what replaces the use of $q$-Gaussians for $q = -1$ in H-M and for $q = 0$ in P-S
The definition of $L(t)_{ij}$ is extremely simple: they just define it to be the length of the interval $[i - 1, i] \cap [(j - 1)t^2, jt^2]$. The verification of all the above properties is then entirely elementary.
Let \( \xi_n \in \ell_2 \otimes_2 \ell_2 \) be the diagonal operator with the same diagonal entries as \( z_n \) so that \( \|\xi_n\|_{\ell_2 \otimes_2 \ell_2} = 1 \).

Let \( \Phi_n : M_n \times M_n \to \mathbb{C} \) be the bilinear form defined by

\[
\Phi_n(a, b) = \langle (a \otimes b)\xi_n, \xi_n \rangle.
\]

Assume that \( \|u\|_{cb} \leq 1 \) then in particular

\[
\|\Phi_n \otimes u : M_n(E) \times M_n(F)\| \leq 1.
\]

Notation

\[
\|ta\|_C = \|(t_j a_j)\|_C, \quad \|t^{-1} a\|_C = \|(t_j^{-1} a_j)\|_C,
\]

and same with \( R \).
Theorem

Suppose $\Phi_n \otimes u : M_n(E) \times M_n(F) \to \mathbb{C}$ satisfies $\dagger \dagger \dagger$ (e.g. if $E, F$ are $C^*$-algebras $\| u \| \leq 1$ suffices) then $\forall a_j, b_j \in E, F$ and $\forall t_j > 0$

$$| \sum u(a_j, b_j) | \leq (\| a \|_R^2 + \| ta \|_C^2)^{1/2}(\| t^{-1} b \|_R^2 + \| b \|_C^2)^{1/2}$$

$$+ C (\log n)^{-1} \sum_j \log(1 + \max\{ t_j, t_j^{-1} \}) | u(a_j, b_j) |$$

Therefore if $n \to \infty$ we obtain

$$| \sum u(a_j, b_j) | \leq (\| a \|_R^2 + \| ta \|_C^2)^{1/2}(\| t^{-1} b \|_R^2 + \| b \|_C^2)^{1/2},$$

and after elementary manipulation we obtain the os-GT in the Haagerup-Musat form:

$$| \sum u(a_j, b_j) | \leq \| a \|_R \| b \|_C + \| ta \|_C \| t^{-1} b \|_R.$$
Proof.

∀ \( m = (p, q, j) \), let \( X_m \in M_n \otimes E \), \( Y_m \in M_n \otimes F \) defined by:

\[
X_m = L(t_j)_{p,q}e_{p,q} \otimes a_j \quad \text{and} \quad Y_m = t_j^{-1}L(t_j)_{p,q}^{1/2}e_{p,q} \otimes b_j.
\]

\[
\|X\|_C = \|ta\|_C, \quad \|X\|_R = \|a\|_R, \quad \|Y\|_C = \|b\|_C, \quad \|Y\|_R = \|t^{-1}b\|_R.
\]

\[
|\sum_m [\Phi_n \otimes u](X_m, Y_m) | \leq (\|X\|_R^2 + \|X\|_C^2)^{1/2} (\|Y\|_R^2 + \|Y\|_C^2)^{1/2}
\]

and hence

\[
|\sum_m [\Phi_n \otimes u](X_m, Y_m) | \leq (\|x\|_R^2 + \|tx\|_C^2)^{1/2} (\|t^{-1}y\|_R^2 + \|y\|_C^2)^{1/2}
\]

But now (4) allows us to conclude...

\[
|t^{-1}\langle L(t)z_n, z_n \rangle - 1| \leq C (\log n)^{-1} \log(1 + \max\{t, t^{-1}\}), \quad (4)
\]
THE END !!