Polynomial tau-functions of BKP and DKP hierarchies

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Abstract

We construct all polynomial tau-functions of the BKP, DKP and MDKP hierarchies.

1 Introduction

In his seminal paper [8] M. Sato introduced the KP hierarchy of evolution equations of Lax type
\[ \frac{dL}{dt_n} = [(L^n)_+, L], \quad n = 1, 2, \ldots, \] (1)
on the pseudodifferential operator \( L = \partial + u_1(t)\partial^{-1} + u_2(t)\partial^{-2} + \ldots \), where \( t = (t_1, t_2, \ldots) \) and \( \partial = \frac{\partial}{\partial t_1} \). He introduced the corresponding tau-function \( \tau(t) \) and showed that any solution \( u_1(t), u_2(t), \ldots \) of (1) can be expressed as a differential polynomial in \( \tau \). He also showed that the tau-functions form an infinite Grassmann manifold of type \( A \), and that the set of polynomial tau-functions includes all Schur polynomials \( s_\lambda(t) \), where \( \lambda \in \text{Par} \), the set of partitions. His ideas have been subsequently developed by his school in the series of papers, including [2], [3], [4], [5].

The totality of polynomial tau-functions of the Kp hierarchy is a disjoint union of Schubert cells \( \bigcup_{\lambda \in \text{Par}} C_\lambda \), such that \( s_\lambda(t) \) is the "center" of \( C_\lambda \). In our paper [7] we proved, using the boson-fermion correspondence, that each \( C_\lambda \) can be obtained from \( s_\lambda(t) \) by shifts of the \( t_i \) by certain constants.

In the papers [4], [5] the BKP and DKP hierarchies have been introduced along the lines proposed by Sato for KP. In his papers [9] and [10] Y. You proved that the Q-Schur polynomials are polynomial tau-functions of the BKP, DKP and MDKP hierarchies. As in the KP case, these tau-functions are "centers" of Schubert cells of the infinite-dimensional orthogonal Grassmann manifold. In the present paper we construct all polynomial tau-functions for the BKP, DKP and MDKP hierarchies in each of the Schubert cells, using the boson-fermion correspondence of types \( B \) and \( D \) [6].

Note that, as shown in [1], the CKP hierarchy has no polynomial tau-functions.
2 The spin representation of $b_\infty$ and $d_\infty$ and the BKP and DKP hierarchy in the fermionic picture

Consider the Lie algebra over $\mathbb{C}$,

$$gl_\infty = \{(a_{ij})_{i,j}\in\mathbb{Z} | \text{ all but a finite number of } a_{ij} \in \mathbb{C} \text{ are zero}\}.$$ 

The matrices $E_{ij}$ with $(i,j)$–th entry 1 and 0 elsewhere, for $i,j \in \mathbb{Z}$ form a basis. Define on $gl_\infty$ the following two linear anti–involutions:

$$\iota_B(E_{jk}) = (-1)^{j+k}E_{-k,-j} \quad \iota_D(E_{jk}) = E_{-k+1,-j+1}.$$

Using these anti–involutions we define the Lie algebras $b_\infty$ and $d_\infty$ as a subalgebra of $gl_\infty$:

$$b_\infty = \{a \in gl_\infty | \iota_B(a) = -a\}, \quad d_\infty = \{a \in gl_\infty | \iota_D(a) = -a\}.$$

The elements $F_{jk} = E_{-j,k} - (-1)^{j+k}E_{-k,j}$, with $j > k$, respectively $G_{j+\frac{1}{2},k-\frac{1}{2}} = E_{-j,k} - E_{-k+1,j+1} = -G_{k-\frac{1}{2},j+\frac{1}{2}}$, with $j \geq k$, form a basis of $b_\infty$, respectively $d_\infty$. We have the following root space decomposition of $b_\infty$:

$$b_\infty = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta^B} (\mathfrak{g}_{-\alpha} \oplus \mathfrak{g}_\alpha), \quad \text{where } \mathfrak{h} = \bigoplus_{i>0} \mathbb{C}F_{-i,i},$$

$$\Delta^B = \{\epsilon_i | i = 1, 2, \ldots\} \cup \{\epsilon_i \pm \epsilon_j | i > j > 0\}, \quad \mathfrak{g}_{\pm \epsilon_i} = \mathbb{C}F_{\pm i,0}, \quad \mathfrak{g}_{\epsilon_i \pm \epsilon_j} = \mathbb{C}F_{i,\pm j}.$$

The root space decomposition of $d_\infty$ is as follows:

$$d_\infty = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta^D} (\mathfrak{g}_{-\alpha} \oplus \mathfrak{g}_\alpha), \quad \text{where } \mathfrak{h} = \bigoplus_{i>0} \mathbb{C}G_{-i,i},$$

$$\Delta^D = \{\epsilon_i \pm \epsilon_j | i > j \geq 0\}, \quad \mathfrak{g}_{\epsilon_i \pm \epsilon_j} = \mathbb{C}G_{i,\pm (j+\frac{1}{2})}.$$

We now describe the spin representation of $b_\infty$ and $d_\infty$. For this purpose we introduce the Clifford algebras $BC\ell$ and $DC\ell$ as the associative algebras generated by the vector space $\mathbb{C}^\infty$ with basis $\phi_j$, $j \in \mathbb{Z}$, respectively $j \in \frac{1}{2} + \mathbb{Z}$, and symmetric bilinear forms

$$(\phi_i, \phi_j)_B = (-1)^i \delta_{i,-j}, \quad (\phi_i, \phi_j)_D = \delta_{i,-j}$$

with defining relations

$$vw + wv = (v, w), \quad v, w \in \mathbb{C}^\infty.$$  \hspace{1cm} (2)

We have a $\mathbb{Z}/2\mathbb{Z}$-gradation

$$BC\ell = BC\ell_{\mathbb{P}} \oplus BC\ell_{\mathbb{T}}, \quad \text{resp. } DC\ell = DC\ell_{\mathbb{P}} \oplus DC\ell_{\mathbb{T}}$$

where $BC\ell_{\nu}$ ($\nu \in \mathbb{Z}/2\mathbb{Z}$) is spanned by all products of $s$ elements of $\mathbb{C}^\infty$ with $s \equiv \nu \mod 2$. We shall identify $\mathbb{C}^\infty$ with its image in $BC\ell$ and $DC\ell$.  \hspace{1cm} (3)
We define the spin module $V_B$ over $BC\ell$ and $V_D$ over $DC\ell$ as the irreducible module with highest weight vector, the vacuum vector $|0\rangle \neq 0$, satisfying

$$\phi_j|0\rangle = 0 \quad \text{for } j > 0,$$

where in the $B$-case we also assume that

$$\phi_0|0\rangle = \frac{1}{\sqrt{2}}|0\rangle.$$

The elements $\phi_{j_1}\phi_{j_2} \cdots \phi_{j_p}|0\rangle$ with $j_1 < j_2 < \cdots < j_p < 0$ form a basis of $V_B$ (here all $j_k \in \mathbb{Z}$) and $V_D$ (here all $j_k \in \frac{1}{2} + \mathbb{Z}$). Then we obtain the representation of $b_\infty$, respectively $d_\infty$, by

$$\pi_B(F_{jk}) = (-1)^j \frac{1}{2}(\phi_j\phi_k - \phi_k\phi_j), \quad \text{resp. } \pi_D(G_{jk}) = \frac{1}{2}(\phi_j\phi_k - \phi_k\phi_j).$$

It is irreducible in the $B$-case. If we restrict to $d_\infty$, the module splits into two irreducible $d_\infty$-modules $V_D = V_0 \oplus V_1$, with highest weight vectors $|0\rangle$ and $|1\rangle = \phi_{-\frac{1}{2}}|0\rangle$.

From now on let $G = B$ or $D$, we will write e.g. $GC\ell$ for the corresponding Clifford algebra.

It will be convenient also to introduce the opposite $b_\infty$ and $d_\infty$-module with highest weight vector $<0|$, by

$$<0|\phi_0 = \frac{1}{\sqrt{2}}<0|, \quad <0|\phi_j = 0, \quad \text{for } j < 0.$$ 

For $d_\infty$ we also have $<1| = <0|\phi_{\frac{1}{2}}$. The vacuum expectation value of $a \in GC\ell$ is defined as $<0|a|0\rangle \in \mathbb{C}$.

Given a non-isotropic vector $\alpha \in \mathbb{C}^\infty$ (i.e. $(\alpha, \alpha) \neq 0$), the associated reflection $r_\alpha$ is defined by

$$r_\alpha(v) = v - \frac{2(\alpha, v)}{(\alpha, \alpha)}\alpha.$$

Let $GC\ell^\times$ denote the multiplicative group of invertible elements of the algebra $GC\ell$. We denote by $Pin_G\mathbb{C}^\infty$ the subgroup of $GC\ell^\times$ generated by all the elements $a$ such that $a\mathbb{C}^\infty a^{-1} = \mathbb{C}^\infty$ and let $Spin_G\mathbb{C}^\infty = Pin_G\mathbb{C}^\infty \cap GC\ell_\pi$.

If $\alpha \in \mathbb{C}^\infty$ is a non-isotropic vector, then, by (3):

$$\alpha^{-1} = \frac{2\alpha}{(\alpha, \alpha)}, \quad (4)$$

hence $\alpha \in GC\ell^\times$, and from (3) we obtain

$$\alpha v \alpha^{-1} = -r_\alpha(v), \quad (5)$$

hence $\alpha \in Pin_G\mathbb{C}^\infty$. We have a homomorphism $T : Pin_G\mathbb{C}^\infty \to B_\infty$ or $D_\infty$, $g \mapsto T_g$ defined by ($v \in \mathbb{C}^\infty$):

$$T_g(v) = gvg^{-1} \in \mathbb{C}^\infty.$$
Here $B_\infty$ or $D_\infty$ is the subgroup of the group $GL_\infty = \{(g_{ij})_{i,j \in \mathbb{Z}} \text{ which are invertible and all but a finite number of } g_{ij} - \delta_{ij} \text{ are 0}\}$, consisting of elements which preserve the bilinear form $\langle \cdot, \cdot \rangle$. Thus, we have a projective representation of $B_\infty$ on $V_B$ and $D_\infty$ on $V_D$.

The orthogonal $G$-Grassmannian is the collection of all linear subspaces $\text{Ann}_Gf = \{v \in \mathbb{C}^\infty \mid vf = 0\}$, for all $f \in O$. Each $\text{Ann}_Gf$ is a maximal isotropic subspace of $\mathbb{C}^\infty$. In fact, the orthogonal Grassmannian is the collection of all maximal isotropic subspaces $U$ of $\mathbb{C}^\infty$ such that $\phi_j \in U$ for all $j >> 0$.

Let us focus on the $B$-case first, which is well-known, see e.g. [4] or [9]. The group $B_\infty$ is the subgroup of $GL_\infty$ generated by all $-r_\alpha$. Let $O = (\text{Spin}_B \mathbb{C}^\infty)|0\rangle$. Since $\phi_0 \in \text{Pin}_B \mathbb{C}^\infty$ and $\phi_0|0\rangle = \frac{1}{\sqrt{2}}|0\rangle$, one has that $O = \text{Pin}_B \mathbb{C}^\infty|0\rangle$.

Let $S_B$ be the following operator on $V_B \otimes V_B$:

$$S_B = \sum_{j \in \mathbb{Z}} (-1)^j \phi_j \otimes \phi_{-j},$$

Then (see e.g. [4])

**Theorem 1** If $\tau \in V_B$ and $\tau \neq 0$, then $\tau \in O$ if only if $\tau$ satisfies the equation

$$S_B(\tau \otimes \tau) = \frac{1}{2} \tau \otimes \tau. \quad (6)$$

Equation (6) is called the fermionic BKP hierarchy.

In the $D$-case, let $O_0 = (\text{Spin}_D \mathbb{C}^\infty)|0\rangle$ and $O_1 = (\text{Spin}_D \mathbb{C}^\infty)|1\rangle$. In this case, let

$$S_D = \sum_{j \in \frac{1}{2} + \mathbb{Z}} \phi_j \otimes \phi_{-j},$$

Then (see e.g. [10] or [6])

**Theorem 2** (a) If $\tau \in V_\nu$ and $\tau \neq 0$, then $\tau \in O_\nu$ if only if $\tau$ satisfies the equation

$$S_D(\tau \otimes \tau) = 0 \quad (7)$$

(b) A pair of non-zero DKP tau-functions $\tau_0 \in O_0$ and $\tau_1 \in O_1$ satisfies the modified DKP hierarchy, i.e.,

$$S_D(\tau_0 \otimes \tau_1) = \tau_1 \otimes \tau_0. \quad (8)$$

if and only if

$$\dim ((\text{Ann } \tau_0 + \text{Ann } \tau_1)/\text{Ann } (\tau_0 \cap \text{Ann } \tau_1)) = 2.$$
If \( \tau \) is a BKP (resp. DKP) tau-function and \( \alpha \in \mathbb{C}^\infty \), then it is shown in [6], Lemmas 2.1 and 2.2, that \( \alpha \tau \) is again a BKP (resp. DKP) tau-function. Hence, since \( \langle 0 \rangle \) is a tau-function,

\[
\begin{align*}
&v_1 v_2 \cdots v_k |0\rangle \in \mathcal{O} \cup \{0\} \quad \text{for any } v_i \in \mathbb{C}^\infty, \\
&v_1 v_2 \cdots v_{2k} |0\rangle \in \mathcal{O}_0 \cup \{0\} \quad \text{for any } v_i \in \mathbb{C}^\infty, \\
&v_1 v_2 \cdots v_{2\ell+1} |0\rangle \in \mathcal{O}_1 \cup \{0\} \quad \text{for any } v_i \in \mathbb{C}^\infty.
\end{align*}
\]

Without loss of generality we may choose, in every \( v_j \), the coefficient of the lowest \( \phi \) equal to 1, in other words we assume that all \( v_j \) are of the form

\[
B: \quad v_j = \phi_{-\lambda_j} + \sum_{n<\lambda_j} a_{-n,j} \phi_{-n}, \quad \text{with } a_{-n,j} = 0 \text{ for } n << 0,
\]

\[
D: \quad v_j = \phi_{-\lambda_j - \frac{1}{2}} + \sum_{n<\lambda_j + \frac{1}{2}} a_{-n,j} \phi_{-n}, \quad \text{with } a_{-n,j} = 0 \text{ for } n << 0,
\]

where the \( \lambda_j \) are positive integers in the \( B \)-case and non-negative integers for \( D \). We may also assume that \( \lambda_1 > \lambda_2 > \cdots > \lambda_k \).

3 Schubert cell decomposition of the orbit

For \( G = B \), let \( W_0 \) be the subgroup of the Weyl group generated by reflections in the short roots \( \epsilon_j \), i.e. \( w \in W_0 \) if and only if \( w(\epsilon_j) = \pm \epsilon_j \), where only finitely many \( \epsilon_j \) are mapped to \( -\epsilon_j \). There is a one to one correspondence between the set of all strict partitions and the elements of \( W_0 \). Namely, we associate to a strict partition \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \), i.e., \( \lambda_i > \lambda_{i+1} \) the following element of \( W_0 \): \( w_\lambda(\epsilon_{\lambda_n}) = -\epsilon_{\lambda_n} \) for \( n = 1, 2, \ldots, k \) and all other \( \epsilon_j \) fixed.

For \( G = D \), let \( W_0 \) be the subgroup of the Weyl group consisting of elements \( w \) such that \( w(\epsilon_j) = \pm \epsilon_j \), now \( j \geq 0 \) where only a finite even number of \( \epsilon_j \) are mapped to \( -\epsilon_j \). The correspondence with strict partitions as in the \( B \)-case is almost the same, now we have an even number of parts, but the last one can be zero, i.e., \( \lambda_1 > \lambda_2 > \cdots > \lambda_{2k} \geq 0 \) corresponds to \( w_\lambda(\epsilon_{\lambda_n}) = -\epsilon_{\lambda_n} \) \( n = 1, 2, \ldots, 2k \) and all other \( \epsilon_j \) fixed. If we allow that a part is also 0, we call such a partition an extended strict partition.

Let \( P_B \) be the parabolic subgroup that corresponds to all long simple roots \( \epsilon_{i+1} - \epsilon_i \), for \( i = 1, 2, \ldots, \) i.e., \( P_B \) corresponds to the roots \( \epsilon_i - \epsilon_j, \epsilon_i + \epsilon_j \) and \( \epsilon_i \) for \( i \neq j > 0 \), thus the corresponding root vectors annihilate the vacuum. Then we have the Bruhat decomposition

\[
B_\infty = \bigcup_{w \in W_0} U_w w P_B \quad \text{disjoint union},
\]

where \( U_{w_\lambda} \) consists of all elements \( \exp \left( \sum_\alpha t_\alpha e_\alpha \right) \), where \( e_\alpha \) is a root vector corresponding to \( \alpha \in \Delta_+ \) such that \( w_\lambda(-\alpha) \in \Delta_+ \), and \( t_\alpha \in \mathbb{C} \).

For \( G = D \) we need two decompositions corresponding to a different parabolic subgroups, viz \( P_D^0 \) and \( P_D^1 \), where both correspond to the simple roots \( \epsilon_{i+1} - \epsilon_i \) for
\( i \geq \) together with \( \epsilon_1 - \epsilon_0 \) for \( P_D^0 \), respectively \( \epsilon_1 + \epsilon_0 \) for \( P_D^1 \). Hence, these parabolics correspond to the root vectors \( \epsilon_i - \epsilon_j \) and \( \epsilon_i + \epsilon_j \), where \( i \neq j \geq 0 \) for \( P_D^0 \), respectively \( \epsilon_i - \epsilon_j \), where \( i \neq j > 0 \), \( \epsilon_i + \epsilon_j \), where \( i > j \geq 0 \) and \( \pm \epsilon_j - \epsilon_0 \) for \( P_D \). We have the Bruhat decomposition

\[
D_\infty = \bigcup_{w \in W_0} U_w w P_D^i \quad \text{disjoint union, } i = 0 \text{ or } 1,
\]

where \( U_w \) consists of all elements \( \exp \left( \sum_\alpha t_\alpha e_\alpha \right) \), where \( e_\alpha \) is a root vector corresponding to \( \alpha \in \Delta_+ \) such that \( w_\lambda(-\alpha) \in \Delta_+ \), and \( t_\alpha \in \mathbb{C} \).

Using the Bruhat decompositions (11), respectively (12), and that \( P_G^i|0\rangle = \mathbb{C}^\times|0\rangle \), respectively \( P_G^i|\lambda\rangle = \mathbb{C}^\times|\lambda\rangle \), we obtain the Schubert cell decomposition of \( O \), \( O_0 \) and \( O_1 \):

\[
O = \bigcup_{w_\lambda \in W_0} U_w w_\lambda|0\rangle \quad \text{disjoint union},
\]

\[
O_i = \bigcup_{w_\lambda \in W_0} U_w w_\lambda|i\rangle \quad \text{disjoint union, } i = 0 \text{ or } 1.
\]

One has (see e.g. [9]):

\[
w_\lambda|0\rangle = |\lambda\rangle = \phi_{-\lambda_1} \phi_{-\lambda_2} \cdots \phi_{-\lambda_k}|0\rangle \quad \text{for } G = B
\]

and for \( G = D \):

\[
w_\lambda|0\rangle = |\lambda\rangle = \phi_{-\lambda_1} \phi_{-\lambda_2} \cdots \phi_{-\lambda_k}|0\rangle
\]

\[
w_\lambda|1\rangle = \begin{cases} 
|\lambda_1, \ldots, \lambda_{2k}, 0\rangle = \phi_{-\lambda_1} \phi_{-\lambda_2} \cdots \phi_{-\lambda_{2k}}|0\rangle & \text{if } \lambda_{2k} \neq 0 \\
|\lambda_1, \ldots, \lambda_{2k-1}\rangle = \phi_{-\lambda_1} \phi_{-\lambda_2} \cdots \phi_{-\lambda_{2k-1}}|0\rangle & \text{if } \lambda_{2k} = 0
\end{cases}
\]

Hence we have to calculate how an element of \( U_w \lambda \) acts on this vector. We describe the case \( G = B \), the case that \( G = D \) can be determined in a similar way. Note that we should only take vectors \( e_\alpha \) such that \( e_\alpha w_\lambda|0\rangle \) is non-zero. So we may remove all positive roots \( \epsilon_j + \epsilon_\lambda \) where \( j < \lambda_\lambda \) and \( j \not\in \lambda \). Hence, we may assume that an element of \( B_\infty \) is of the form \( \exp \left( \sum_{\alpha \in \Delta_\lambda} t_\alpha e_\alpha \right) \), where

\[
\Delta_\lambda = \{ \epsilon_\lambda, \epsilon_\lambda - j, \epsilon_\lambda + \epsilon_\lambda \mid n, m = 1, 2, \ldots, k, \ 0 < j < \lambda_\lambda, \ j \neq \lambda_{n+1}, \lambda_{n+2}, \ldots, \lambda_k \}
\]

We have:

\[
\exp \left( \sum_{\alpha \in \Delta_\lambda} t_\alpha e_\alpha \right) = \exp \left( \sum_{1 \leq m < n \leq k} t_{\lambda_m, \lambda_n} \phi_{\lambda_m} \phi_{\lambda_n} \right) \exp \left( \sum_{n=1}^{k} \sum_{0 < j < \lambda_n, j \not\in \lambda} t_{-j, \lambda_n} \phi_{-j} \phi_{\lambda_n} \right) \times
\]

\[
\exp \left( \sum_{n=1}^{k} t_{\lambda_n, \phi_0} \phi_{\lambda_n} \right),
\]

\[
6
\]
Note that:
\[
\exp \left( \sum_{1 \leq m < n \leq k} t_{\lambda_m, \lambda_n} \phi_{\lambda_m} \phi_{\lambda_n} \right) = \prod_{1 \leq m < n \leq k} (1 + t_{\lambda_m, \lambda_n} \phi_{\lambda_m} \phi_{\lambda_n}),
\]
\[
\exp \left( \sum_{n=1}^{k} \sum_{0 < j < \lambda_n, j \notin \lambda} t_{-j, \lambda_n} \phi_{-j} \phi_{\lambda_n} \right) = \prod_{n=1}^{k} \prod_{0 < j < \lambda_n, j \notin \lambda} (1 + t_{-j, \lambda_n} \phi_{-j} \phi_{\lambda_n}),
\]
\[
\exp \left( \sum_{n=1}^{k} t_{\lambda_n} \phi_0 \phi_{\lambda_n} \right) = 1 + \sum_{n=1}^{k} t_{\lambda_n} \phi_0 \phi_{\lambda_n}.
\]
Hence applying conjugation several times gives respectively
\[
\left( 1 + \sum_{n=1}^{k} t_{\lambda_n} \phi_0 \phi_{\lambda_n} \right) \phi_{-\lambda_i} \left( 1 - \sum_{n=1}^{k} t_{\lambda_n} \phi_0 \phi_{\lambda_n} \right) = \phi_{-\lambda_i} + (-1)^{\lambda} t_{\lambda_i} \phi_0,
\]
\[
\prod_{n=1}^{k} \prod_{0 < j < \lambda_n, j \notin \lambda} (1 + t_{-j, \lambda_n} \phi_{-j} \phi_{\lambda_n}) \left( \phi_{-\lambda_i} + (-1)^{\lambda_i} \right) \left( t_{\lambda_i} \phi_0 + \sum_{0 < j < \lambda_i, j \notin \lambda} t_{-j, \lambda_i} \phi_{-j} \right),
\]
and
\[
\exp \left( \sum_{\alpha \in \Delta} t_{\alpha} e_{\alpha} \right) \phi_{-\lambda_i} \exp \left( - \sum_{\alpha \in \Delta} t_{\alpha} e_{\alpha} \right) =
\]
\[
\prod_{1 \leq m < n \leq k} (1 + t_{\lambda_m, \lambda_n} \phi_{\lambda_m} \phi_{\lambda_n}) \left( \phi_{-\lambda_i} + (-1)^{\lambda_i} \left( t_{\lambda_i} \phi_0 + \sum_{0 < j < \lambda_i, j \notin \lambda} t_{-j, \lambda_i} \phi_{-j} \right) \right) \times
\]
\[
\prod_{1 \leq m < n \leq k} (1 - t_{\lambda_m, \lambda_n} \phi_{\lambda_m} \phi_{\lambda_n})
\]
\[
= \phi_{-\lambda_i} + (-1)^{\lambda_i} \left( t_{\lambda_i} \phi_0 + \sum_{0 < j < \lambda_i, j \notin \lambda} t_{-j, \lambda_i} \phi_{-j} + \sum_{0 < \ell < i} t_{\lambda_\ell, \lambda_i} \phi_{\lambda_\ell} - \sum_{i < \ell \leq k} t_{\lambda_i, \lambda_\ell} \phi_{\lambda_\ell} \right).
\]
So for a strict partition \( \lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k) \), let
\[
C_{\lambda} = U_{w_{\lambda}} w_{\lambda}|0\).
\]
From the observation on \( U_{w_{\lambda}} \) above, we find that the dimension of the corresponding Schubert cell \( C_{\lambda} \) is \( |\Delta_{\lambda}| = |\lambda| \). Moreover, from the calculations above, we find that the element \([9]\) with \( v_j \) given by \([10]\) must be an element of \( C_{\lambda} \). The calculations for \( G = D \) can be done in a similar way. We thus obtain

**Proposition 3** (a) Let \( SP \) be the set of strict partitions \( \lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k) \), i.e., \( \lambda_1 > \lambda_2 > \cdots > \lambda_k > 0 \). Then
\[
\mathcal{O} = \bigcup_{\lambda \in SP} C_{\lambda} \quad \text{disjoint union},
\]
where
\[ C_\lambda = \{ a \left( \phi_{-\lambda_1} + \sum_{j>\lambda_1} a_{j1} \phi_j \right) \left( \phi_{-\lambda_2} + \sum_{j>\lambda_2} a_{j2} \phi_j \right) \cdots \left( \phi_{-\lambda_k} + \sum_{j>\lambda_k} a_{jk} \phi_j \right) | 0 \} \]
and \( a, a_{j\ell} \in \mathbb{C}, \ a \neq 0 \).

(b) Let ESP be the set of extended strict partitions \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \), i.e., \( \lambda_1 > \lambda_2 > \cdots > \lambda_k \geq 0 \), and let ESP_0, respectively ESP_1, be the subsets of ESP, consisting of extended strict partitions \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \) with \( k \) even, respectively odd. Then
\[ \mathcal{O}_i = \bigcup_{\lambda \in ESP_i} D_\lambda \quad (i = 0, 1), \]
where
\[ D_\lambda = \{ a \left( \phi_{-\lambda_1/2} + \sum_{j>\lambda_1} a_{j1} \phi_j \right) \left( \phi_{-\lambda_2/2} + \sum_{j>\lambda_2} a_{j2} \phi_j \right) \cdots \left( \phi_{-\lambda_k/2} + \sum_{j>\lambda_k} a_{jk} \phi_j \right) | 0 \} \]
and \( a, a_{j\ell} \in \mathbb{C}, \) for all \( 1 \leq \ell \leq k \) and \( a \neq 0 \).

4 Vertex operators and the BKP hierarchy in the bosonic picture

In this section we will describe the case that \( G = B \), the case that \( G = D \), will be done in the next section.

Define the following two generating series:
\[ \phi(z) = \sum_{j \in \mathbb{Z}} \phi_j z^{-j}, \quad \alpha(z) = \sum_{k \in 2\mathbb{Z}+1} \alpha_k z^{-k} = : \phi(z) \frac{\phi(-z)}{z} :. \]

Then one has (see e.g. [6] for details):

**Theorem 4** ([6])
\[ \phi(z) = \frac{1}{\sqrt{2}} \exp\left( - \sum_{k<0, odd} \frac{\alpha_k}{k} z^{-k} \right) \exp\left( - \sum_{k>0, odd} \frac{\alpha_k}{k} z^{-k} \right). \]

The neutral (twisted) boson-fermion correspondence consists of identifying the space \( V \) with the space \( B = \mathbb{C}[t_1, t_3, t_5, \ldots] \) via the vector space isomorphism \( \sigma : F \to B \) given by
\[ \sigma(\alpha_{-m_1} \alpha_{-m_2} \ldots \alpha_{-m_s} | 0 \}) = m_1 m_2 \ldots m_s t_{m_1} t_{m_2} \ldots t_{m_s}. \]
The transported action of the operators \( \alpha_m \) is as follows
\[ \sigma \alpha_{-m} \sigma^{-1}(p(t)) = m t_m p(t), \]
\[ \sigma \alpha_m \sigma^{-1}(p(t)) = 2 \frac{\partial p(t)}{\partial t_m}. \]
Then
\[ \sigma \phi(z) \sigma^{-1} = \frac{1}{2} \sqrt{2} e^{\xi(t,z)} e^{-\eta(t,z)}, \quad \text{where} \]
\[ \xi(t, z) = \sum_{i=0}^{\infty} t_{2i+1} z^{2i+1}, \quad \eta(t, z) = \sum_{i=0}^{\infty} \frac{2}{2i + 1} \frac{\partial}{\partial t_{2i+1}} z^{-2i-1}. \]

We now rewrite the BKP hierarchy (16), using \( \text{Res}_{z=0} dz \sum_{i} f_{i} z^{i} = f_{-1} \). Namely, by Theorem 1, we have that \( 0 \neq \tau(t) \in \mathcal{B} \) is an element of \( \sigma(\mathcal{O}) \) if and only if
\[ \text{Res}_{z=0} \frac{dz}{z} \exp \left( \sum_{j > 0, \text{odd}} t_{j} z^{j} \exp(-2 \sum_{j > 0, \text{odd}} \frac{\partial}{\partial t_{j}} z^{-j}) \tau \right) \times \exp \left( \sum_{j > 0, \text{odd}} t_{j} z^{j} \exp(2 \sum_{j > 0, \text{odd}} \frac{\partial}{\partial t_{j}} z^{-j}) \tau \right) = \tau \otimes \tau \quad (16) \]

Equation (16) is called the BKP hierarchy in the bosonic picture. It is straightforward, using change of variables and Taylor’s formula, to rewrite (16) into a generating series of Hirota bilinear equations on the tau-function (see e.g. [4]). However, we will not do that here, but rather concentrate on obtaining polynomial tau-functions of this hierarchy.

Define \( H(t) = \sum_{j > 0, \text{odd}} t_{j} \alpha_{j} \). Since
\[ e^{\sum_{j > 0, \text{odd}} t_{j} \alpha_{j}} e^{-\sum_{k < 0, \text{odd}} \frac{\partial}{\partial t_{j}} z^{-k}} = e^{\sum_{j > 0, \text{odd}} t_{j} z^{j}} e^{-\sum_{k < 0, \text{odd}} \frac{\partial}{\partial t_{j}} z^{-k}} e^{\sum_{j > 0, \text{odd}} t_{j} \alpha_{j}}, \]

one finds, using the expression for \( \phi(z) \) of Theorem 4 that
\[ e^{H(t)} \phi(z) e^{-H(t)} = e^{\sum_{j > 0, \text{odd}} t_{j} z^{j}} \phi(z), \quad e^{H(t)} |0\rangle = |0\rangle. \quad (17) \]

Moreover,
\[ \tau(t) = \sigma(g|0\rangle) = \langle 0|e^{H(t)} g|0\rangle. \quad (18) \]

Let us calculate the following elements (cf. [9])
\[ \langle 0|e^{H(t)} v_{1} v_{2} \cdots v_{k}|0\rangle. \]

Define the elementary Schur polynomials \( s_{i}(t) \) by \( e^{\sum_{j=1}^{\infty} t_{j} z^{j}} = \sum_{i=0}^{\infty} s_{i}(t) z^{i} \) and let \( \bar{t} = (t_{1}, 0, t_{3}, 0, t_{5}, \ldots) \). Let \( v_{j} \) be of the form
\[ v_{j} = \phi_{-\lambda_{j}} + \sum_{n < \lambda_{j}} a_{-nj} \phi_{-n}, \quad \text{with} \ a_{-sj} = 0 \ \text{for} \ s << 0. \]

Since the map \((s_{1}(t), s_{2}(t), \ldots, s_{k}(t)) : \mathbb{C}^{k} \rightarrow \mathbb{C}^{k} \) is surjective (actually an isomorphism), one can find constants \( c_{j} = (c_{1j}, c_{2j}, \ldots) \) such that
\[ 1 + \sum_{n < \lambda_{j}} a_{-nj} z^{\lambda_{j} - n} = \sum_{k=0}^{\infty} s_{k}(c_{j}) z^{k} = e^{\sum_{j=1}^{\infty} c_{ij} z^{i}}. \]
Using this and (17), we have

\[ e^{H(t)v_je^{-H(t)}} = e^{H(t)} \left( \phi_{-\lambda_j} + \sum_{n<\lambda_j} a_{-nj} \phi_{-n} \right) e^{-H(t)} \]

\[ = \text{Res}_{z=0} e^{H(t)} \phi(z) e^{-H(t)} \left( z^{-\lambda_j-1} + \sum_{n<\lambda_j} a_{-nj} z^{-n-1} \right) \]

\[ = \text{Res}_{z=0} \phi(z) e^{\sum_{\ell,\text{odd}} t_{\ell} z^{\ell}} \left( z^{-\lambda_j-1} + \sum_{n<\lambda_j} a_{-nj} z^{-n-1} \right) \]

\[ = \text{Res}_{z=0} \phi(z) e^{\sum_{\ell,\text{odd}} t_{\ell} z^{\ell} + \sum_{i=j}^{\infty} c_{ij} z^{i} z^{-\lambda_j-1}} \]

\[ = \text{Res}_{z=0} \phi(z) e^{\sum_{i=1}^{\infty} (\tilde{t}_i + c_{ij}) z^{i} z^{-\lambda_j-1}} \]

\[ = \text{Res}_{z=0} \phi(z) \sum_{\ell=0}^{\infty} s_{\ell}(\tilde{t}_i + c_{ij}) z^{\ell-\lambda_j-1} \]

\[ = \sum_{\ell=0}^{\infty} s_{\ell}(\tilde{t}_i + c_{ij}) \phi_{-\lambda_j} := v_j(\tilde{t}_i + c_{ij}). \]

Hence the BKP tau-function that corresponds to an element of the form (9) is the vacuum expectation value

\[ \langle 0|v_1(\tilde{t}_i + c_1)v_2(\tilde{t}_i + c_2)\cdots v_k(\tilde{t}_i + c_k)|0 \rangle. \]  

(20)

If \( k = 2\ell \), this expectation value is equal to (cf. the Appendix of [4] or [9])

\[ \langle 0|v_1(\tilde{t}_i + c_1)v_2(\tilde{t}_i + c_2)\cdots v_{2\ell}(\tilde{t}_i + c_{2\ell})|0 \rangle = \text{Pf} \left( \langle 0|v_i(\tilde{t}_i + c_i)v_j(\tilde{t}_i + c_j)|0 \rangle \right)_{ij}. \]

(21)

Here Pf stands for the Pfaffian of a \( 2\ell \times 2\ell \) skewsymmetric matrix \( A = (a_{ij}) \):

\[ \text{Pf} (A) = \sum_{\sigma} \text{sgn}(\sigma) \prod_{i=1}^{\ell} a_{\sigma(2i-1),\sigma(2i)}, \]

where we take the sum over all permutations \( \sigma \) for which \( \sigma(2i-1) < \sigma(2i+1) \) and \( \sigma(2i-1) < \sigma(2i) \) for all \( 1 \leq i \leq \ell \). Note that in the Pfaffian of (21) we only take products of vacuum expectation values \( \langle 0|v_i(\tilde{t}_i + c_i)v_j(\tilde{t}_i + c_j)|0 \rangle \) with \( i < j \). If \( k \) is odd there is a problem, the vacuum expectation value is nonzero, but we cannot relate this to a Pfaffian, since the Pfaffian of an odd skewsymmetric matrix is 0. We solve this problem as follows. We can assume that \( k \) is always even; if \( k \) is odd, we add the element \( v_{k+1} = \phi_0 \), since this only changes the vacuum expectation value by a scalar factor:

\[ \langle 0|e^{H(t)}v_1v_2\cdots v_kv_{k+1}|0 \rangle = \langle 0|e^{H(t)}v_1v_2\cdots v_k\phi_0|0 \rangle = \frac{1}{\sqrt{2}} \langle 0|e^{H(t)}v_1v_2\cdots v_k|0 \rangle. \]
In this way we get a vacuum expectation value of an even number of \( v_j \)'s, which we can relate to a Pfaffian which is nonzero. Hence, from now on we assume, without loss of generality, that we have an even number of \( v_j \)'s. This means that that such an element is related to a strict partition \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{2\ell}) \) with an even number of parts, where the last part is allowed to be zero. We call such partition an extended strict partitions. Clearly there is a bijection between the this set and the collection of strict partitions.

Let

\[
\chi_{\lambda_i, \lambda_j}(c_i, c_j) = \langle 0 | v_i(c_i) v_j(c_j) | 0 \rangle .
\]

We have:

\[
\chi_{\lambda_i, \lambda_j}(c_i, c_j) = \sum_{m=0}^{\infty} s_m(c_i) \sum_{\ell=0}^{\infty} s_\ell(c_j) \langle 0 | \phi_{m-\lambda_i} \phi_{\ell-\lambda_j} | 0 \rangle
\]

\[
= \sum_{m=\lambda_i}^{\lambda_j} s_m(c_i) \sum_{\ell=0}^{\lambda_j} s_\ell(c_j) \langle 0 | \phi_{m-\lambda_i} \phi_{\ell-\lambda_j} | 0 \rangle
\]

\[
= \sum_{\ell=0}^{\lambda_j} s_{\lambda_i+\lambda_j-\ell}(c_i) s_\ell(c_j) \langle 0 | \phi_{\lambda_j-\ell} \phi_{\ell-\lambda_j} | 0 \rangle .
\]

Hence

\[
\chi_{\lambda_i, \lambda_j}(c_i, c_j) = \frac{1}{2} s_{\lambda_i}(c_i) s_{\lambda_j}(c_j) + \sum_{\ell=1}^{\lambda_j} (-1)^\ell s_{\lambda_i+\ell}(c_i) s_{\lambda_j-\ell}(c_j) .
\]

(22)

Above we assume that \( \lambda_i > \lambda_j \geq 0 \); in this case we let \( \chi_{\lambda_j, \lambda_i} = -\chi_{\lambda_i, \lambda_j} \). This function is zero if \( \lambda_j < 0 \).

As a result of Proposition 3 and the above considerations, we obtain the following

**Theorem 5** All polynomial tau-functions of the BKP hierarchy are, up to a scalar multiple, of the form

\[
Pf \left( \chi_{\lambda_i, \lambda_j}(\vec{t} + c_i, \vec{t} + c_j) \right)_{1 \leq i, j \leq 2n},
\]

where \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{2n}) \) is an extended strict partition, i.e., \( \lambda_1 > \lambda_2 > \cdots > \lambda_{2n} \geq 0 \), \( \vec{t} = (t_1, 0, t_3, 0, t_5, 0, \ldots) \), \( c_i = (c_{i1}, c_{i2}, c_{i3}, \ldots) \) are constants, and \( \chi_{\lambda_i, \lambda_j} \) is given by (22).

**Proof.** According to Proposition 3 an element in the group orbit \( O \) corresponds to an element in a \( C_\lambda \) for \( \lambda \) a strict partition. Thus, we have to determine the image of the elements appearing in such a \( C_\lambda \) under the map \( \sigma \) which appears in the neutral boson-fermion correspondence. According to (18) and the calculations (19), this is equal to (20). Since we always may assume that \( k \) is even (for odd \( k \) we add the element \( v_{k+1}(\vec{t} + c_{k+1}) = \phi_0 \)), using (21) we can express this in the Pfaffian. We thus obtain the desired result.

\[
\square
\]

If one puts all constants \( c_i \) and \( c_j \) in (23) equal to 0, one obtains the \( Q \)-Schur
functions, giving a result of You [9] that all Q-Schur functions are tau-functions of the BKP hierarchy.

**Example 6** One of the polynomial tau-functions corresponding to the partition \( \lambda_1 > \lambda_2 > \lambda_3 > 0 \), corresponds to

\[
v_1v_2v_3|0\rangle = \sqrt{2}v_1v_2v_3\phi_0|0\rangle.
\]

Add \( \lambda_4 = 0 \), then

\[
\sigma(v_1v_2v_3|0\rangle) = \sqrt{2}(0|e^{H(t)}v_1v_2v_3\phi_0|0\rangle
\]

\[
= \sqrt{2}(0|\sum_is_i(\tilde{t} + c_1)\phi_i-\lambda_1|\sum_js_j(\tilde{t} + c_2)\phi_j-\lambda_2|\sum_\ell s_\ell(\tilde{t} + c_3)\phi_\ell-\lambda_3\phi_0|0\rangle
\]

\[
= \sqrt{2}\text{Pf} \left( \chi_{\lambda_1,\lambda_2}(\tilde{t} + c_1, \tilde{t} + c_2) \right)_{1 \leq i, j \leq 4}
\]

\[
= \sqrt{2}(\chi_{\lambda_1,\lambda_2}(\tilde{t} + c_1, \tilde{t} + c_2)\chi_{\lambda_3,0}(\tilde{t} + c_1, \tilde{t} + c_3)\chi_{\lambda_2,0}(\tilde{t} + c_2, \tilde{t}) + + \chi_{\lambda_2,\lambda_3}(\tilde{t} + c_2, \tilde{t} + c_3)\chi_{\lambda_1,0}(\tilde{t} + c_1, \tilde{t}))
\]

\[
= \sqrt{2}(\chi_{\lambda_1,\lambda_2}(\tilde{t} + c_1, \tilde{t} + c_2)s_{\lambda_3}(\tilde{t} + c_3) - \chi_{\lambda_1,\lambda_3}(\tilde{t} + c_1, \tilde{t} + c_3)s_{\lambda_2}(\tilde{t} + c_2) + + \chi_{\lambda_2,\lambda_3}(\tilde{t} + c_2, \tilde{t} + c_3)s_{\lambda_1}(\tilde{t} + c_1)),
\]

since \( \chi_{\lambda_1,0}(\tilde{t} + c_i, \tilde{t}) = s_{\lambda_i}(\tilde{t} + c_i) \).

## 5 Vertex operators and the DKP hierarchy in the bosonic picture

In the spirit of [10], we embed \( b_\infty \) into \( d_\infty \) by viewing \( b_\infty \) as the fixed points of the involution \( \iota \) on \( DC\ell \) that sends \( \phi_{1/2} \) to \( \phi_{-1/2} \) and fixes all other \( \phi \)'s. Then,

\[
b_\infty = \{ x \in d_\infty | \iota(x) = x \}.
\]

Define as in the \( B \)-case

\[
\phi(z) = \sum_{j \in \mathbb{Z}} \phi_j z^{-j} := \frac{1}{\sqrt{2}}(\phi_{-1/2} + \phi_{1/2}) + \sum_{i=1}^{\infty} \left( \phi_{-i-1/2} z^i + (-1)^i \phi_{i+1/2} z^{-i} \right),
\]

\[
\alpha(z) = \sum_{k \in \mathbb{Z} + 1} \alpha_k z^{-k-1} =: \phi(z) \frac{\phi(-z)}{z}.
\]

There is a slight difference with the \( B \)-case in the sense that \( V_D = V_0 \oplus V_1 \) and that

\[
\sigma(V_D) = \mathbb{C}[\theta, t_1, t_3, \ldots], \text{ where } \theta \text{ ia a Grassmann variable,}
\]

i.e., \( \theta^2 = 0 \) and \( \theta \) commutes with all \( t_i \). Then,

\[
\sigma\phi(z)^{-1} = \frac{1}{\sqrt{2}} \left( \theta + \frac{\partial}{\partial \theta} \right) e^{H(t)z} e^{-H(t)z}, \quad \frac{\sigma\phi_{1/2} - \phi_{-1/2}}{\sqrt{2}} \sigma^{-1} = \frac{\theta - \partial}{\sqrt{2}}.
\]
Then similarly to the calculations in (19), we obtain

$$
S_D = S_B - \frac{1}{2} \left( \theta - \frac{\partial}{\partial \theta} \right) \otimes \left( \theta - \frac{\partial}{\partial \theta} \right)
$$

Hence equation (3) turns into (19) for \( \tau = \tau_0(t) \) or \( \tau = \tau_1(t) \theta \) and (8) turns into

$$
\text{Res}_{z=0} \frac{dz}{z} \exp \left( \sum_{j>0, \text{odd}} t_j z^j \right) \exp \left( -2 \sum_{j>0, \text{odd}} \frac{\partial}{\partial t_j} \frac{z^{-j}}{j} \right) \tau_0
$$

$$
\otimes \exp \left( - \sum_{j>0, \text{odd}} t_j z^j \right) \exp \left( 2 \sum_{j>0, \text{odd}} \frac{\partial}{\partial t_j} \frac{z^{-j}}{j} \right) \tau_1 = \tau_1 \otimes \tau_0.
$$

(24)

We now want to determine \( \tau_0(t) = \sigma(v_1 v_2 \cdots v_{2t-1}|0) \) and \( \tau_1(t) = \sigma(v_1 v_2 \cdots v_{2t+1}|0) \) where \( v_i \) is given by (10). As in the BKP case,

$$
\tau_0(t) = \langle 0 | e^{H(t)} v_1 v_2 \cdots v_{2t} | 0 \rangle, \quad \tau_1(t) = \langle 0 | \phi_2 e^{H(t)} v_1 v_2 \cdots v_{2t+1} | 0 \rangle =
$$

Now

$$
v_j = \phi_{-\lambda_j - \frac{1}{2}} + \sum_{n<\lambda_j} a_{-\frac{1}{2}, j} \phi_{n} - \frac{1}{2}
$$

$$
= \phi_{-\lambda_j} + \sum_{\lambda_j < n \in \mathbb{Z}} b_{-n, j} \phi_{-n} + b_j (\phi_{-\frac{1}{2}} - \phi_{\frac{1}{2}})
$$

$$
= \text{Res}_{z=0} \phi(z) \left( z^{-\lambda_j - 1} + \sum_{n<\lambda_j} b_{-n, j} z^{-n-1} \right) dz + b_j (\phi_{-\frac{1}{2}} - \phi_{\frac{1}{2}}),
$$

where for \( k \geq 1 \):

$$
b_{-k, j} = a_{-\frac{1}{2}, j}, \quad b_{k, j} = (-1)^k a_{\frac{1}{2}, j}, \quad b_{0, j} = \frac{1}{\sqrt{2}} (a_{\frac{1}{2}, j} + a_{-\frac{1}{2}, j}), \quad b_j = \frac{1}{2} (a_{\frac{1}{2}, j} - a_{-\frac{1}{2}, j}).
$$

Then similarly to the calculations in (19), we obtain

$$
e^{H(t)} v_j e^{-H(t)} = \text{Res}_{z=0} e^{H(t)} \phi(z) e^{-H(t)} \left( z^{-\lambda_j - 1} + \sum_{n<\lambda_j} b_{-n, j} z^{-n-1} \right) dz + b_j (\phi_{-\frac{1}{2}} - \phi_{\frac{1}{2}})
$$

$$
= \text{Res}_{z=0} \phi(z) e^{\sum_{\ell, odd} t_{\ell} z^\ell} \left( z^{-\lambda_j - 1} + \sum_{n<\lambda_j} b_{-n, j} z^{-n-1} \right) dz + b_j (\phi_{-\frac{1}{2}} - \phi_{\frac{1}{2}})
$$

$$
= \sum_{\ell=0}^\infty s_\ell (\tilde{\ell} + c_j) \phi_{-\lambda_j} + b_j (\phi_{-\frac{1}{2}} - \phi_{\frac{1}{2}}) := v_j (\tilde{\ell} + c_j ; b_j).
$$

(25)

Define, in a similar way as for \( G = B \) for \( i < j \) hence \( \lambda_i > \lambda_j \):

$$
\rho_{\lambda_i, \lambda_j} (c_i, c_j ; b_i, b_j) = \langle 0 | v_i (c_i ; b_i) v_j (c_j ; b_j) | 0 \rangle.
$$
We have:

\[
\rho_{\lambda, \lambda_j}(c_i, c_j; b_i, b_j) = \langle 0 | \left( \sum_{m=0}^{\infty} s_m(c_i) \phi_{m-\lambda} + b_i (\phi_{\frac{1}{2}} - \phi_{\frac{1}{2}}^\dagger) \right) \left( \sum_{\ell=0}^{\infty} s_\ell(c_j) \phi_{\ell-\lambda_j} + b_j (\phi_{\frac{1}{2}} - \phi_{\frac{1}{2}}^\dagger) \right) | 0 \rangle \\
= b_j s_{\lambda_j}(c_j) \langle 0 | \phi_{\frac{1}{2}} \phi_{\frac{1}{2}}^\dagger | 0 \rangle - b_i s_{\lambda_j}(c_j) \langle 0 | \phi_{\frac{1}{2}} \phi_{\frac{1}{2}} | 0 \rangle - b_i b_j \langle 0 | \phi_{\frac{1}{2}} \phi_{\frac{1}{2}}^\dagger | 0 \rangle \\
+ \sum_{m=\lambda_i}^{\infty} s_m(c_i) \sum_{\ell=0}^{\lambda_j} s_{\ell}(c_j) \langle 0 | \phi_{m-\lambda} \phi_{\ell-\lambda_j} | 0 \rangle \\
= \left( s_{\lambda_j}(c_j) - \sqrt{2} b_i \right) \left( s_{\lambda_j}(c_j) + \sqrt{2} b_j \right) \langle 0 | \phi_{\frac{1}{2}} \phi_{\frac{1}{2}} | 0 \rangle \\
+ \sum_{\ell=0}^{\lambda_j-1} s_{\lambda_j+\lambda_j-\ell}(c_i) s_{\ell}(c_j) \langle 0 | \phi_{\lambda_j-\ell} \phi_{\ell-\lambda_j} | 0 \rangle .
\]

Hence

\[
\rho_{\lambda, \lambda_j}(c_i, c_j; b_i, b_j) = \left( \frac{s_{\lambda_j}(c_j)}{\sqrt{2}} - b_i \right) \left( \frac{s_{\lambda_j}(c_j)}{\sqrt{2}} + b_j \right) + \sum_{\ell=1}^{\lambda_j} (-1)^\ell s_{\lambda_j+\ell}(c_i) s_{\lambda_j-\ell}(c_j) .
\]

(26)

Define

\[
\rho_{\lambda_j}(c_j; b_j) = \langle 0 | \phi_{\frac{1}{2}} v_j(c_j; b_j) | 0 \rangle ,
\]

then

\[
\rho_{\lambda_j}(c_j; b_j) = s_{\lambda_j}(c_j) \langle 0 | \phi_{\frac{1}{2}} \phi_{\frac{1}{2}} | 0 \rangle + b_j \langle 0 | \phi_{\frac{1}{2}} \phi_{\frac{1}{2}}^\dagger | 0 \rangle = \frac{s_{\lambda_j}(c_j)}{\sqrt{2}} + b_j .
\]

(27)

Thus we proved the following theorem.

**Theorem 7** Let \( \tilde{t} = (t_1, 0, t_3, 0, t_5, 0, \ldots) \) and \( c_i = (c_{1i}, c_{2i}, c_{3i}, \ldots) \) and \( b_i \) be constants. Denote by \( \tilde{c}_j \) \( \tilde{t} + c_j = (t_1 + c_{1j}, c_{2j}, t_3 + c_{3j}, c_{4j}, \ldots) \).

(a) All polynomial tau-functions in \( V_0 \) of the DKP hierarchy are, up to a scalar multiple, of the form

\[
\rho_{\lambda}(\tilde{t}; c_1, \ldots, c_{2n}; b_1, \ldots b_{2n}) := \text{Pf} \left( \rho_{\lambda_i, \lambda_j}(\tilde{c}_i, \tilde{c}_j; b_i, b_j) \right)_{1 \leq i, j \leq 2n} ,
\]

(28)

where \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{2n}) \) is an extended strict partition, i.e., \( \lambda_1 > \lambda_2 > \cdots > \lambda_{2n} \geq 0 \), and \( \rho_{\lambda_i, \lambda_j} \) is given by (26).

(b) All polynomial tau-functions in \( V_1 \) of the DKP hierarchy are, up to a scalar multiple, of the form

\[
\rho_{\lambda}(\tilde{t}; c_1, \ldots, c_{2n+1}; b_1, \ldots b_{2n+1}) := \text{Pf} \left( A \right) ,
\]

(29)

where \( A \) is a \((2n + 2) \times (2n + 2)\) skew-symmetric matrix, whose \( k \)-th row is equal to

\[
(-\rho_{\lambda_k}(\tilde{c}_k; b_k), -\rho_{\lambda_k, \lambda_k}(\tilde{c}_k; b_1, b_k), \ldots, -\rho_{\lambda_{k-1}, \lambda_k}(\tilde{c}_{k-1}; b_{k-1}, b_k), 0, \\
\rho_{\lambda_k, \lambda_{k+1}}(\tilde{c}_k, \tilde{c}_{k+1}; b_k, b_{k+1}), \ldots, \rho_{\lambda_k, \lambda_{2n+1}}(\tilde{c}_k, \tilde{c}_{2n+1}; b_k, b_{2n+1}))
\]

14
Again $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{2n+1})$ is an extended strict partition, i.e., $\lambda_1 > \lambda_2 > \cdots > \lambda_{2n+1} \geq 0$, $t = (t_1, 0, t_3, 0, t_5, 0, \ldots)$, $c_i = (c_{i1}, c_{i2}, c_{i3}, \ldots)$ and $b_i$ are constants, and $\rho_{\lambda_i, \lambda_j}$ is given by (26) and $\rho_{\lambda_i}$ is given by (27).

(c) Let $a_0, a_1 \in \mathbb{C}$. All polynomial tau functions of the MDKP hierarchy are pairs $(\tau_0, \tau_1)$, with either

\[
\tau_0 = a_0 \rho_{(\lambda_1, \ldots, \lambda_{j-1}, \lambda_{j+1}, \ldots, \lambda_{2n+1})}(\tilde{t}; c_1, \ldots, c_{j-1}, c_{j+1}, \ldots, c_{2n+1}; b_1, \ldots, b_{j-1}, b_{j+1}, \ldots, b_{2n+1}),
\]

\[
\tau_1 = a_1 \rho_{(\lambda_1, \ldots, \lambda_{2n+1})}(\tilde{t}; c_1, \ldots, c_{2n+1}; b_1, \ldots, b_{2n+1}),
\]  

or

\[
\tau_0 = a_0 \rho_{\lambda_1, \ldots, \lambda_{2n}}(\tilde{t}; c_1, \ldots, c_{2n}; b_1, \ldots, b_{2n}),
\]

\[
\tau_1 = a_1 \rho_{(\lambda_1, \ldots, \lambda_{j-1}, \lambda_{j+1}, \ldots, \lambda_{2n})}(\tilde{t}; c_1, \ldots, c_{j-1}, c_{j+1}, \ldots, c_{2n}; b_1, \ldots, b_{j-1}, b_{j+1}, \ldots, b_{2n}).
\]

(30)

(31)

Theorem 7 generalizes the results of Y. You [10], that $Q$-Schur functions are polynomial tau-functions of the DKP and MDKP hierarchies.

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