Hawking radiation as tunneling from a Vaidya black hole in noncommutative gravity

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Abstract

In the context of a noncommutative model of coordinate coherent states, we present a Schwarzschild-like metric for a Vaidya solution instead of the standard Eddington-Finkelstein metric. This leads to the appearance of an exact \((t - r)\) dependent case of the metric. We analyze the resulting metric in three possible causal structures. In this setup, we find a zero remnant mass in the long-time limit, i.e. an instable black hole remnant. We also study the tunneling process across the quantum horizon of such a Vaidya black hole. The tunneling probability including the time-dependent part is obtained by using the tunneling method proposed by Parikh and Wilczek in terms of the noncommutative parameter \(\sigma\). After that, we calculate the entropy associated to this noncommutative black hole solution. However the corrections are fundamentally trifling; one could respect this as a consequence of quantum inspection at the level of semiclassical quantum gravity.

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I. INTRODUCTION

There have been many paradigms for the noncommutative field theory based on the Weyl-Wigner-Moyal $\ast$-product \cite{1} which fail to find a way for solving the subsequent problems, such as Lorentz invariance breaking, nonunitarity and UV divergences of quantum field theory. Recently, Smailagic and Spallucci \cite{2} suggested a noncommutative model of coordinate coherent states (CCS) which could be released from the above problems. Their findings were acquired by beginning with an innovative method to noncommutative geometry after a long period of time. Using the CCS approach, the authors in \cite{3} derived exact solutions of the Einstein equations for a static, spherically symmetric, asymptotically flat, minimal width, mass/energy distribution localized near the origin; as a result there is no curvature singularity at the origin. In this model, the pointlike structure of mass $M$, instead of being completely localized at a point, is portrayed by a smeared structure throughout a region of linear size $\sqrt{\sigma}$. The characteristic energy or inverse length scale related to the noncommutativity effects possibly and most rationally would have a natural value of order of the Planck scale. In fact, most of the phenomenological studies of the noncommutativity models expect that the noncommutative energy scale cannot lie far above the TeV scale \cite{4}. Since the fundamental Planck scale in models with large extra dimensions becomes as small as a TeV in order to solve the hierarchy problem \cite{5}, therefore, depending on the models, it is feasible to set the noncommutativity effects in a $1 - 10$ TeV regime, etc. \cite{6}.

A radiation spectrum of an evaporating black hole, which is closely comparable to the blackbody radiation spectrum, can be illustrated by a characteristic temperature known as the Hawking temperature \cite{7}. Hawking’s method unfortunately yields a nonunitarity of quantum theory, which maps a pure state to a mixed state, due to the purely thermal essence of the spectrum. In 2000, Parikh and Wilczek \cite{8} presented a new approach on the basis of null geodesics to draw out the Hawking radiation via tunneling through the quantum horizon. In this approach, the form of the black hole radiation spectrum is modified as a result of incorporation of backreaction effects. From another point of view, Shankaranarayanan et al. performed the tunneling process to get the Hawking temperature in different coordinates within a complex paths approach \cite{9}. The tunneling procedure illuminates the fact that the modified radiation spectrum is not accurately thermal and this yields the unitarity of underlying quantum theory \cite{10}.
In this paper, we would like to suggest a new formulation of noncommutativity of coordinates for a Vaidya black hole which is performed by a Gaussian distribution of coherent states. This type of black hole is considered as an illustration of a more practical case because it is a time-dependent lessening mass caused by the evaporation process. This trend continues to proceed the Parikh-Wilczek tunneling procedure through the event horizon of such a noncommutative-inspired Vaidya black hole.

The organization of this paper is as follows. In Sec. II, the influence of noncommutativity in the framework of coordinate coherent states for a Vaidya metric is investigated. In this manner, an exact $(t - r)$ dependence solution is obtained. In Sec. III, we study the Parikh-Wilczek tunneling for such a Vaidya solution. The tunneling amplitude at which massless particles tunnel across the event horizon is computed. Finally, the conclusions of the work in this paper are summarized in Sec. IV.

II. DISAPPEARANCE OF BLACK HOLE REMNANT

As mentioned briefly in the Introduction, the simple idea of a pointlike particle becomes physically irrelevant and should be replaced with a minimal width Gaussian distribution of mass/energy, corresponding to the principles of quantum mechanics [3]. The program we choose here is to perform an analysis which provides a solution in the case of a non-static, spherically symmetric, asymptotically flat, minimal width, Gaussian distribution of mass/energy whose noncommutative size is characterized by the parameter $\sqrt{\sigma}$. For this purpose, the mass/energy distribution should be displayed by a smeared delta function

$$\rho_\sigma = \frac{M}{(4\pi \sigma)^{3/2}} e^{-\frac{r^2}{4\sigma}},$$

(1)

where, in this approach, $\rho_\sigma = \rho_\sigma(t, r)$ and $M = M(t, r)$ are functions of both time and radius. The dynamics for the black hole mass with evaporation is a persistent problem. In the study of black hole evaporation, there is a significant point in which the black hole mass reduces as a backreaction of the Hawking radiation. Since a nonstatic and spherically symmetric spacetime depends on an arbitrary dynamical mass function, it can be properly demonstrated by a Vaidya solution [11] that seems to be the favored option. In this work, due to deriving an exact $(t - r)$ dependent case of the metric, we study the Schwarzschild-like
metric for the Vaidya solution instead of a standard Eddington-Finkelstein metric.

In this paper we want to generalize the Vaidya metric derived by Farley and D’Eath \[12\] to the noncommutative model of CCS. The general spherically symmetric Vaidya spacetime in \(\{x^\mu\} = \{t, r, \theta, \phi\}\) coordinates, \((\mu = 0, 1, 2, 3)\), in the presence of \((t-r)\) dependent mass sources can be written as \[12\]

\[
ds^2 = -e^{2\Psi(t,r)}F(t,r)dt^2 + F^{-1}(t,r)dr^2 + r^2d\Omega^2,
\]

where \(d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2\), and \(e^{2\Psi(t,r)} = \left(\frac{\dot{M}M}{\chi(M)\dot{M}}\right)^2\). Here \(\chi(M)\) is the arbitrary positive function of \(t\) and \(r\). In the following, due to the mathematical intricacy of Einstein field equations and in order to make the problem obedient, we frequently impose the particular cases, e.g. \(\chi(M) = -\dot{M} (\dot{M} < 0)\), the overdot abbreviates \(\frac{\partial}{\partial t}\). This metric looks like the Schwarzschild spacetime, except that the role of the Schwarzschild mass is taken by a mass function \(M(t,r)\), which changes exceedingly gradually concerning both \(t\) and \(r\) in the spacetime region containing the outgoing radiation \[13\]. The corresponding geometry in this region including the radially outgoing radiation is, therefore, the slowly varying Vaidya type.

In order to determine the mass function, we consider the covariant conservation condition \(T^{\mu\nu}_{\ ;\nu} = 0\), which yields the explicit result

\[
\partial_t T^t_t + \partial_r T^r_r + \frac{1}{2}g^{tt}\partial_t g_{tt}(T^r_r - T^t_t) + \frac{1}{2}g^{rr}\partial_t g_{rr}(T^r_r - T^t_t) + g^{\theta\theta}\partial_r g_{\theta\theta}(T^r_r - T^\theta_\theta) = 0.
\]

The Schwarzschild-like condition \(g_{tt} = -g_{rr}^{-1}\) will require that \(T^t_t = T^r_r = -\rho_\sigma\), and then the above relation leads to a solution for \(T^\theta_\theta\) which reads \[36\]

\[
T^\theta_\theta = \rho_\sigma \left(\frac{r^2}{4\sigma} - \frac{r}{2\dot{M}}\left(\dot{M} + M'\right) - 1\right),
\]

where the prime abbreviates \(\frac{\partial}{\partial r}\). The nonzero components of the Einstein field equations \(G_{\mu\nu} = 8\pi T_{\mu\nu}\) give the following equations:

\[
F'F + F + 8\pi r^2 \rho_\sigma - 1 = 0,
\]

\[
\dot{F} + 8\pi r F^2 T^r_r = 0,
\]

\[
rF''F^3 - 2r\dot{F}^2 + r\ddot{F}F + 2F^3F' - 16\pi r F^3 T^\theta_\theta = 0,
\]

\[
T^r_r = T^t_t, \quad \text{and} \quad T^\phi_\phi = T^\theta_\theta.
\]
Now, one can describe the self-gravitating, anisotropic matter source through a fluid-type $T_{\mu}^{\nu}$ of the following form:

$$T_{\mu}^{\nu} = \begin{pmatrix} T_{t}^{t} & T_{t}^{r} & 0 & 0 \\ T_{r}^{t} & T_{r}^{r} & 0 & 0 \\ 0 & 0 & T_{\theta}^{\theta} & 0 \\ 0 & 0 & 0 & T_{\phi}^{\phi} \end{pmatrix}.$$  \hspace{1cm} (9)

This type of energy-momentum tensor is slightly atypical due to the fact that $T_{\mu}^{\nu}$ deviates from the conventional perfect fluid form including the isotropic pressure terms. However, according to the slowly varying Vaidya form ($\dot{M} \ll 1$ and $M' \ll 1$) it is easy to show that the pressure terms $T_{r}^{r}$ and $T_{\theta}^{\theta}$ are different only surrounded by a few $\sqrt{\sigma}$ from the origin and the perfect fluid condition is recovered for larger distances.

To compute the mass function, we consider the situation of nonstatic in which the analytic mass solution is time dependent, $M = M(t, r)$. Using Eq. (4), the mass function is obtained by the constraint $T_{r}^{r} = T_{\theta}^{\theta}$, which gives

$$M = C e^{\left[\frac{r^2}{4\sigma} + \frac{t(r-t)}{2\sigma} \right]}.$$  \hspace{1cm} (10)

If we choose $C = M_I$ (initial black hole mass) to have physical meaningful solutions, then plugging the above $M$ into the relation (5), we find the line element:

$$ds^2 = -F(t, r)dt^2 + F^{-1}(t, r)dr^2 + r^2 d\Omega^2,$$  \hspace{1cm} (11)

with

$$F(t, r) = 1 - \frac{2M_\sigma(t, r)}{r},$$  \hspace{1cm} (12)

where the Gaussian-smeared mass distribution immediately reads

$$M_\sigma(t, r) = M_I \left( \mathcal{E} \left( \frac{r-t}{2\sqrt{\sigma}} \right) \left( 1 + \frac{t^2}{2\sigma} \right) - \frac{r}{\sqrt{\pi}\sigma} e^{-\frac{(r-t)^2}{4\sigma}} \left( 1 + \frac{t}{r} \right) \right).$$  \hspace{1cm} (13)

$\mathcal{E}(x)$ shows the Gauss error function defined as $\mathcal{E}(x) \equiv \frac{2}{\sqrt{\pi}} \int_{0}^{x} e^{-p^2} dp$. To find the $F(t, r)$ we have set the value of integration constant to zero. In the limit $\frac{t}{\sqrt{\sigma}} \gg 1$ and also $\frac{r}{\sqrt{\sigma}} \gg 1$, the expression given in Eq. (12) satisfies Eq. (7) with a good approximation. Depending on the different values of initial mass $M_I$, and upon a numerical solution, the metric displays three possible causal structures: (1) It is possible to have two distinct horizons when the initial mass of the black hole is larger than minimal nonzero mass $M_0$, i.e. $M_I > M_0$ (see Fig. 1).
It is possible to have one degenerate horizon (extremal black hole), for \( M_I = M_0 \) (see Fig. 2). (3) It is impossible to have a horizon at all (for \( M_I < M_0 \)), and this possibility is shown in Fig. 3.

**FIG. 1:** The temporal component of the metric versus \( \sqrt{\sigma} \) for different values of \( \sqrt{\sigma} \) with a sufficiently large amount of initial mass \( (M_I > M_0) \), e.g., \( M_I = 3.00\sqrt{\sigma} \). On the right-hand side of the figure, curves are marked from top to bottom by \( t = 0, 1.00\sqrt{\sigma}, 2.00\sqrt{\sigma}, 3.00\sqrt{\sigma}, \) and \( 4.00\sqrt{\sigma} \). This figure shows that, in the long-time limit, the distance between the horizons is increased.

As Fig. 1 shows, for a sufficiently large and fixed \( \frac{M_I}{\sqrt{\sigma}} \), e.g., \( M_I = 3.00\sqrt{\sigma} \), the distance between the horizons will increase as time progresses. The appearance of a naked singularity at \( r = 0 \) in a nonstatic case is natural which has not been supported by the cosmic censorship conjecture [14]. One of the main arguments in support of the censorship conjecture is the stability of black holes concerning small perturbations. Indeed there are various effects that point out that at least in its simplest form the censorship conjecture is dubious. Let us consider the possible blueshift unstableness at the inner horizon as an example which demonstrates the feasible formation of naked singularities in conditions which might be considered as physically sensible. Since an observer passing through the inner horizon would encounter an infinite blueshift of any entering emission, as he comes near the horizon, he surveys the total history of the outward region in a determinate interval of his individual proper time. So it would be possible for any small perturbation to upset the horizon and emerge as a naked singularity. In Ref. [15], it was illustrated that the inner (Cauchy) horizon
of some of the black holes (e.g. Reissner-Nordström black holes) is unsteady and their solutions are unsuccessful to be globally hyperbolic. If a spacetime fails to be globally hyperbolic, the weak censorship conjecture is destroyed \[16\]. Then it must include a naked singularity, i.e., there exists a future directed causal curve that arrives a distant observer, and in the past it ends at the singularity.

On the other hand when \( r \) is immoderately small, in the region where noncommutativity effects accurately commence to be perceived, the detailed nature of the sharpened mass distribution is not practically being scrutinized. Recently \[17\], we have reported some results about extraordinary thermodynamical behavior for Planck-scale black hole evaporation, i.e., when \( M_I \) is less than \( M_0 \), where there is the principal reactiveness to noncommutativity effects and the detailed form of the matter distribution. In this area, some unusual thermodynamical features, e.g., negative entropy, negative temperature, and abnormal heat capacity appeared. There are also the predominant differences between the Gaussian, Lorentzian, or some other forms of the smeared mass distribution at this extreme regime. In other words, the bases of these theories probably become as a result of the fractal nature of spacetime at very short distances. Theories such as \( E \) infinity \[18\] and scale relativity \[19\] which are on the basis of the fractal structure of spacetime at very short distances may provide a suitable framework to handle thermodynamics of these very short distance systems. Therefore, we really should not have credence to the details of our modeling when \( \frac{r}{\sqrt{\sigma}} \ll 1 \) and only apply the Gaussian-smeared mass distribution in our calculations just on the condition that \( M_I \geq M_0 \).

The plot presented in Fig. 2 shows, for several values of minimal nonzero mass \( M_0 \), the possibility of having an extremal configuration with one degenerate event horizon as time progresses. For more details, the numerical results for the remnant size of the black hole for different values of \( \frac{t}{\sqrt{\sigma}} \) are presented in Table I. According to Table II as time moves forward the minimal nonzero mass decreases but the minimal nonzero horizon radius increases which means that in the limit \( \frac{t}{\sqrt{\sigma}} \gg 1 \), the micro black hole can evaporate completely, i.e. \( M_0 \rightarrow 0 \). Therefore, the idea of a stable black hole remnant as a candidate to conserve information has failed. Note that, currently there are some proposals about what happens to the information that falls into a black hole. One of the main proposals is that the black hole never disappears completely, and the information is not lost, but would be stored in a Planck size stable remnant (for reviews on resolving the so-called information loss problem, see \[20\]).
FIG. 2: The temporal component of the metric versus $\sqrt{\sigma}$ for different values of $\sqrt{\sigma}$ under the condition that $M_I = M_0$. On the right-hand side of the figure, curves are marked from top to bottom by $t = 0$, $1.00\sqrt{\sigma}$, $2.00\sqrt{\sigma}$, $3.00\sqrt{\sigma}$, and $4.00\sqrt{\sigma}$. The figure shows the possibility of having an extremal configuration with one degenerate event horizon.

In fact, when one considers the time-varying mass, based on our model and preferred calculations, total evaporation of the black hole is possible in principle. This is in agreement with the original Bekenstein-Hawking approach [7, 21] and also our approach by using the time-varying speed of light model [22].

For a sufficiently small and fixed $M_I\sqrt{\sigma}$, e.g. $M_I = 0.40\sqrt{\sigma}$, there is no event horizon when the initial mass of the black hole is smaller than the minimal nonzero mass which is shown in Fig. 3.

Finally, the solution (12) can be substituted in (6) to obtain a solution for $T_t^r$ as the following expression:

$$T_t^r = \frac{-\sqrt{\sigma}e^{-\frac{(r+t)^2}{4\sigma}}(r^2 + 4\sigma) + 2\sqrt{\pi}\sigma t \mathcal{E}\left(\frac{r-t}{2\sqrt{\sigma}}\right)}{8\sqrt{\pi}M_I\left[2\sqrt{\pi}e^{-\frac{(r+t)^2}{4\sigma}}(r + t) - \sqrt{\pi}\mathcal{E}\left(\frac{r-t}{2\sqrt{\sigma}}\right)(t^2 + 2\sigma) + \sqrt{\pi}\sigma r\right]^2}. \quad (14)$$

For the time-independent case, $t = 0$, one recovers the noncommutative-Schwarzschild case [3], i.e.,

$$F(r) = 1 - \frac{2M_I}{r} \mathcal{E}\left(\frac{r}{2\sqrt{\sigma}}\right) + \frac{2M_I}{\sqrt{\pi}\sigma} e^{-\frac{r^2}{4\sigma}}, \quad (15)$$

and $T_t^r = 0$. In the commutative limit concerning the above equation, $\sigma \rightarrow 0$, the Gauss error function tends to 1 and the other term will exponentially be reduced to zero. Thus
TABLE I: The minimal nonzero mass of the black hole (remnant mass, $M_0^{\sqrt{\sigma}}$) and also the minimal nonzero horizon radius, $r_0^{\sqrt{\sigma}}$, for different values of $t^{\sqrt{\sigma}}$. In the long-time limit, i.e. $t^{\sqrt{\sigma}} \gg 1$, there is no black hole remnant.

| Extremal black hole | Time | Minimal nonzero mass | Minimal nonzero horizon radius |
|---------------------|------|----------------------|--------------------------------|
|                     | $t = 0$ | $M_0 \approx 1.90\sqrt{\sigma}$ | $r_0 \approx 3.02\sqrt{\sigma}$ |
|                     | $t = 1.00\sqrt{\sigma}$ | $M_0 \approx 1.68\sqrt{\sigma}$ | $r_0 \approx 4.49\sqrt{\sigma}$ |
|                     | $t = 2.00\sqrt{\sigma}$ | $M_0 \approx 0.99\sqrt{\sigma}$ | $r_0 \approx 5.34\sqrt{\sigma}$ |
|                     | $t = 3.00\sqrt{\sigma}$ | $M_0 \approx 0.62\sqrt{\sigma}$ | $r_0 \approx 6.14\sqrt{\sigma}$ |
|                     | $t = 4.00\sqrt{\sigma}$ | $M_0 \approx 0.43\sqrt{\sigma}$ | $r_0 \approx 7.18\sqrt{\sigma}$ |
|                     | $t = 5.00\sqrt{\sigma}$ | $M_0 \approx 0.32\sqrt{\sigma}$ | $r_0 \approx 8.32\sqrt{\sigma}$ |
|                     | $t = 10.00\sqrt{\sigma}$ | $M_0 \approx 0.13\sqrt{\sigma}$ | $r_0 \approx 13.27\sqrt{\sigma}$ |
|                     | $t = 100.00\sqrt{\sigma}$ | $M_0 \approx 0.01\sqrt{\sigma}$ | $r_0 \approx 105.05\sqrt{\sigma}$ |
|                     | $t \to \infty$ | $M_0 \to 0$ | $r_0 \to \infty$ |

FIG. 3: The temporal component of the metric versus $r^{\sqrt{\sigma}}$ for different values of $t^{\sqrt{\sigma}}$ with a sufficiently small amount of initial mass ($M_I < M_0$), e.g. $M_I = 0.40\sqrt{\sigma}$. On the right-hand side of the figure, curves are marked from top to bottom by $t = 0$, $1.00\sqrt{\sigma}$, $2.00\sqrt{\sigma}$, $3.00\sqrt{\sigma}$, and $4.00\sqrt{\sigma}$. The figure does not show any event horizon when the initial mass of the black hole is smaller than the minimal nonzero mass.
one retrieves the conventional result

\[ F(r) = 1 - \frac{2M_I}{r}. \]  
(16)

Therefore, the modified Vaidya solution is reduced to the ordinary Schwarzschild solution. It is clear that the line element (11) has a coordinate singularity at the event horizon as

\[ r_H = 2M_I(t, r_H). \]  
(17)

The analytical solution of Eq. (17) for \( r_H \) in a closed form is impossible, but it is possible to solve (17) to find \( M_I \), which provides the initial mass as a function of the horizon radius \( r_H \). This leads to

\[ M_I = \sqrt{\pi \sigma^4 r_H} \left[ \sqrt{\pi \sigma E} \left( \frac{r_H - t}{2\sqrt{\sigma}} \right) (2\sigma + t^2) - 2\sigma e^{-\frac{(r_H-t)^2}{4\sigma}}(r_H + t) \right]^{-1}. \]  
(18)

The results of the numerical solution of the initial mass as a function of the horizon radius are displayed in Fig. 4 which are comparable to Table I. As expected, from the initial mass equation (see Fig. 4), one acquires that noncommutativity indicates a minimal nonzero mass in order to have an event horizon. So, in the noncommutative case, for \( M_I < M_0 \) there is no event horizon (see Fig. 2 and also Table I).
The radiating behavior associated with this noncommutative black hole solution can now be investigated by the quantum tunneling process suggested by Parikh and Wilczek [8]. To portray the quantum tunneling method where a particle moves in dynamical geometry and passes through the horizon without singularity on the path, we should utilize a coordinate system that is not singular at the horizon. Painlevé coordinates [23] which are used to eliminate coordinate singularity are especially convenient choices in this analysis. Under the Painlevé time coordinate transformation,

$$\frac{dt}{dt} \rightarrow \frac{dt}{dt} - \frac{\sqrt{1 - F(t, r)}}{F(t, r)} dr,$$

the noncommutative Painlevé metric now takes the following form:

$$ds^2 = -F(t, r)dt^2 + 2\sqrt{1 - F(t, r)} dtdr + dr^2 + r^2 d\Omega^2$$

$$= -\left(1 - \frac{2M_\sigma(t, r)}{r}\right) dt^2 + 2\sqrt{2M_\sigma(t, r)} dtdr + dr^2 + r^2 d\Omega^2. \quad (20)$$

The metric is now stationary, and there is no coordinate singularity at the horizon. The outgoing motion of the massless particles (the outgoing radial null geodesics, $$ds^2 = d\Omega^2 = 0$$) takes the form

$$\frac{dr}{dt} = 1 - \sqrt{1 - F(t, r)}. \quad (21)$$

Since we just need an approximation value of $$F(t, r)$$ for short distances in the vicinity of the horizon, we can expand the coefficient $$F(t, r)$$ by using the Taylor series at a fixed time and just to first order. So, we have

$$F(t, r) \big|_t = F'(t, r_H) \big|_t (r - r_H) + O ((r - r_H)^2) \big|_t. \quad (22)$$

By this approximation at the neighborhood of the black hole horizon, the equation of radial null geodesic can be obtained by

$$\frac{dr}{dt} \simeq \frac{1}{2} F'(t, r_H) (r - r_H) \simeq \kappa(M_I) (r - r_H), \quad (23)$$

where $$\kappa(M_I) \simeq \frac{1}{2} F'(t, r_H)$$ is the surface gravity for the metric (20) at the horizon. Now we are ready to consider the Hawking temperature of such a black hole (see [24] for a more detailed discussion of the semiclassical methods to derive the Hawking temperature in the
Vaidya black hole). From the expression $T_H = \frac{\kappa}{2\pi} = \frac{1}{4\pi} F'(t, r_H)|_t$, the noncommutative Hawking temperature including the time-dependent part is given by

$$T_H = M_I \left[ \frac{E \left( \frac{r_H - t}{2\sqrt{\sigma}} \right)}{2\pi r_H^2} \left( 1 + \frac{t^2}{2\sigma} \right) - \frac{e^{-\frac{(r_H - t)^2}{4\sigma}}}{4(\pi\sigma)^{\frac{3}{2}}} \left( r_H + \frac{2\sigma}{r_H} + \frac{2\sigma t}{r_H^2} \right) \right]. \tag{24}$$

For the time-independent case, $t = 0$, one retrieves the Hawking temperature for the noncommutative Schwarzschild black hole that is consistent with the Ref. [3]. In the limit of $\sigma$ going to zero, we get the classical Hawking temperature, $T_H = \frac{1}{8\pi M_I}$. The numerical computation of the noncommutative Hawking temperature as a function of horizon radius (the outer horizon radius) is depicted in Fig. 5. As can be seen from Fig. 5 the black hole at the ultimate phase of evaporation ceases to radiate, its temperature reaches zero and the existence of a minimal nonzero mass is clear. In this modified version, there is no divergence at the final stage of the black hole evaporation because the temperature reaches a maximum definite value before cooling down to absolute zero, at the minimal nonzero value of the outer horizon radius $r_0$, that the black hole shrinks to (see Table I).

![Figure 5: The Hawking temperature, $T_H \sqrt{\sigma}$, as a function of horizon radius (the outer horizon radius), $r_H \sqrt{\sigma}$. We have set $M_I = 3.00\sqrt{\sigma}$. The existence of a minimal nonzero mass and disappearance of divergence are clear. On the right-hand side of the figure, curves are marked from bottom to top by $t = 0, 1.00\sqrt{\sigma}, 2.00\sqrt{\sigma}, 3.00\sqrt{\sigma},$ and $4.00\sqrt{\sigma}$.](image)

Let us come back to the tunneling procedure. In accordance with the original work proposed by Parikh and Wilczek [8], the WKB approximation is valid at the neighborhood
of the horizon. Then, the emission rate for the classically forbidden region as a function of the imaginary part of the action for a particle in a tunneling process is given by

\[ \Gamma \sim e^{-2\text{Im} I}. \]  
(25)

Now, we consider a spherical positive energy shell containing the components of massless particles each of which journeys on a radial null geodesic like an \( s \)-wave outgoing particle which crosses the horizon in the outward direction from \( r_{\text{in}} \) to \( r_{\text{out}} \). Hence, the imaginary part of the action takes the following form

\[ \text{Im} I = \text{Im} \int_{r_{\text{in}}}^{r_{\text{out}}} p_r dr = \text{Im} \int_{r_{\text{in}}}^{r_{\text{out}}} \int_{0}^{p_r} dp_r' dr. \]  
(26)

Utilizing Hamilton’s equation of motion \( \frac{dr}{dt} = \frac{dH}{dp_r}|_{r} \), the integral variable is changed from momentum to energy. So, we have

\[ \text{Im} I = \text{Im} \int_{r_{\text{in}}}^{r_{\text{out}}} \int_{0}^{H} \frac{dH'}{dr} dr. \]  
(27)

If we consider the particle’s self-gravitation effect, according to the original work by Kraus and Wilczek, then Eq. (23) should be modified. We retain the total Arnowitt-Deser-Misner mass \( M_I \) of the spacetime fixed, and allow the hole mass to fluctuate because we take into consideration the response of the background geometry which corresponds to an emitted quantum of energy \( E \) at a fixed time or a stationary phase. Therefore we should replace \( M_I \) by \( M_I - E \) in Eq. (23) and then (27). The imaginary action (27) now becomes

\[ \text{Im} I = \text{Im} \int_{r_{\text{in}}}^{r_{\text{out}}} \int_{M_I}^{M_I - E} \frac{d(M_I - E')}{\kappa(M_I - E') (r - r_H)} dr = -\text{Im} \int_{r_{\text{in}}}^{r_{\text{out}}} \int_{0}^{E} \frac{dE'}{\kappa(M_I - E') (r - r_H)} dr. \]  
(28)

The \( r \) integral can be performed first by a contour integration for the lower half \( E' \) plane due to the escape from the pole at the horizon. In this way, we acquire

\[ \text{Im} I = -\text{Im} \int_{0}^{E} \frac{dE'}{\kappa(M_I - E')} \int_{r_{\text{in}}}^{E} \frac{dr}{r - r_H} = \pi \int_{0}^{E} \frac{dE'}{\kappa(M_I - E')}, \]  
(29)

on the condition that \( r_{\text{in}} > r_{\text{out}} \). Using the first low of black hole thermodynamics, \( dM = \frac{\kappa}{2\pi} dS \), the expression \( \text{Im} I \) given by

\[ \text{Im} I = -\frac{1}{2} \int_{S_{\text{NC}}(M_I - E)}^{S_{\text{NC}}(M_I - E)} dS = -\frac{1}{2} \Delta S_{\text{NC}}, \]  
(30)
where $S_{NC}$ is the noncommutative black hole entropy. The tunneling amplitude in the high energy depends on the final and initial number of microstates available for the system (see also [31–34]). Thus, we have

$$\Gamma \sim \frac{e^{S_{final}}}{e^{S_{initial}}} = e^{\Delta S_{NC}} = e^{S_{NC}(M_f - E) - S_{NC}(M_i)}.$$ \hspace{1cm} (31)

From this viewpoint the emission rate is proportional to the difference in black hole entropies before and after emission which means that the emission spectrum cannot be accurately thermal at higher energies.

We should note that the tunneling amplitude can also be obtained by writing out the explicit metric in the tunneling calculation. To find the analytic form of the difference in black hole entropies before and after emission and then compute the expression $\Gamma$, we evaluate the integral (27) by writing the explicit form for the radial null geodesic, Eq. (21), which incorporates the backreaction effects. Therefore after performing the $r$ integration in Eq. (27) by deforming the contour [38], we find

$$\text{Im} I = \text{Im} \int_0^E 4\pi i M_\sigma(t, M_I - E')dE',$$

where

$$M_\sigma(t, M_I - E) = (M_I - E) \left[ \mathcal{E} \left( \frac{2(M_I - E) - t}{2\sqrt{\sigma}} \right) \left(1 + \frac{t^2}{2\sigma}\right) \right. \left. - \frac{2(M_I - E)}{\sqrt{\pi\sigma}} e^{-\left(\frac{2(M_I - E) - t}{2\sqrt{\sigma}}\right)^2} \left(1 + \frac{t}{2(M_I - E)}\right) \right],$$

so we can find the noncommutative-corrected tunneling amplitude as follows:

$$\Gamma \sim \exp(\Delta S_{NC}) = \exp \left[ \mathcal{E} \left( \frac{2(M_I - E) - t}{2\sqrt{\sigma}} \right) \left(4\pi(M_I - E)^2 \left(1 + \frac{t^2}{2\sigma}\right) - 6\pi \left(\sigma + t^2\right) - \frac{\pi t^4}{2\sigma}\right) + e^{-\left(\frac{2(M_I - E) - t}{4\sigma}\right)^2} \left[2\sqrt{\pi\sigma} (6 + 5t) + \sqrt{\frac{\pi}{\sigma}} t^2 (2(M_I - E) + t) \right] \right]$$

$$- e^{-\left(\frac{2M_I - t}{4\sigma}\right)^2} \left[4\pi M_I^2 \left(1 + \frac{t^2}{2\sigma}\right) - 6\pi \left(\sigma + t^2\right) - \frac{\pi t^4}{2\sigma}\right]$$

$$- e^{-\left(\frac{2M_I - t}{4\sigma}\right)^2} \left[2\sqrt{\pi\sigma} (6 + 5t) + \sqrt{\frac{\pi}{\sigma}} t^2 (2M_I + t) \right].$$ \hspace{1cm} (34)

In this situation, we would like to test our result approximately. It is adequate to acquire the analytic form of the noncommutative entropy $S_{NC}$, and then compute the difference
in black hole entropies before and after emission, \( \Delta S_{NC} = S_{NC}(M_I - E) - S_{NC}(M_I) \), to compare between the first law of black hole thermodynamics and the tunneling approaches.

For this purpose, we should note that our calculations to find Eq. (24) (noncommutative Hawking temperature) are accurate and no approximation has been performed. But there is no analytical solution for entropy from the first law of classical black hole thermodynamics \( dM = T_H dS \), even if we set \( r_H = 2M_I \) in Eq. (24). Now, we want to calculate the Hawking temperature in an approximate way to find the analytical form of the entropy as follows:

\[
T_H = \frac{1}{4\pi r_H},
\]

where \( r_H \) is given by

\[
r_H = 2M_I \left( \mathcal{E} \left( \frac{2(M_I - t)}{2\sqrt{\sigma}} \right) \left( 1 + \frac{t^2}{2\sigma} \right) - \frac{2M_I}{\sqrt{\pi\sigma}} e^{-\frac{(2M_I-t)^2}{4\sigma}} \left( 1 + \frac{t}{2M_I} \right) \right). \tag{36}
\]

Finally, the entropy of the black hole can be achieved as the analytical form by using the first low of classical black hole thermodynamics,

\[
S_{NC} = \int \frac{dM_I}{T_H} = \mathcal{E} \left( \frac{2(M_I - t)}{2\sqrt{\sigma}} \right) \left[ 4\pi M_I^2 \left( 1 + \frac{t^2}{2\sigma} \right) - 6\pi \left( \sigma + t^2 \right) - \frac{\pi t^4}{2\sigma} \right]
+ e^{-\frac{(2M_I-t)^2}{4\sigma}} \left[ 2\sqrt{\pi\sigma} (6 + 5t) + \sqrt{\frac{\pi}{\sigma}} t^2 (2M_I + t) \right]. \tag{37}
\]

It is clear that the two approaches mentioned above are accurately coincided, and it can be easily checked by computing \( \Delta S_{NC} \); however our approach has been approximated. We should stress that the total of our results in Ref. [35] are recovered by setting \( t = 0 \) into the above corresponding equations.

The question which arises here is the possible dependences between different modes of radiation during the evaporation and then the time evolution of these possible correlations which needs further investigation and probably sheds more light on the information loss problem [20]. This problem is currently under investigation.

IV. SUMMARY

In summary, we have analyzed a solution of the Einstein equations with a noncommutative distribution of mass/energy which is spherically symmetric, time-dependent and
localized near the origin of the spacetime, namely, the noncommutative-inspired Vaidya solution. In this setup, the proposal of stable black hole remnant as a candidate to store information has failed because the black hole evaporates completely in the long time limit. Finally, by using the semiclassical method, we have derived the Hawking temperature and the emission rate via a tunneling process which includes the corrections due to noncommutativity. The entropy for such a black hole is approximately computed in a closed form. These corrections would be significant at the level of semiclassical quantum gravity, specifically once the black hole mass becomes close to the Planck mass.

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contribution to the tunneling amplitude. When one includes this temporal contribution one gets exactly the correct temperature and exactly when one uses the canonically invariant tunneling amplitude.

Note that since there is no analytical solution for \( r_H \) versus \( M_I \), then one can approximately calculate the noncommutative horizon radius versus the initial mass by setting \( r_H = 2M_I \) into the smeared mass distribution \( M_\sigma(t, r_H) \).