Renormalization Group and Grand Unification with 331 Models

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Abstract

By making a renormalization group analysis we explore the possibility of having a 331 model as the only intermediate gauge group between the standard model and the scale of unification of the three coupling constants. We shall assume that there is no necessarily a group of grand unification at the scale of convergence of the couplings. With this scenario, different 331 models and their corresponding supersymmetric versions are considered, and we find the versions that allow the symmetry breaking described above. Besides, the allowed interval for the 331 symmetry breaking scale, and the behaviour of the running coupling constants are obtained. It worths saying that some of the supersymmetric scenarios could be natural frameworks for split supersymmetry. Finally, we look for possible 331 models with a simple group at the grand unification scale, that could fit the symmetry breaking scheme described above.

PACS: 11.10Hi, 12.10.-g, 12.60.Jv, 11.15.Ex, 11.30.Ly

Keywords: renormalization group equations, grand unification, 331 models, supersymmetric unification.

1 Introduction

Since the birth of the Standard Model (SM) many attempts have been done to go beyond it, and solve some of the problems of the model such as the charge quantization and the unification of the gauge couplings. In some cases the unification is done by taking a simple group of grand unification, arising the so called Grand Unification Theories (GUT), where the three interactions described by SM are treated as only one [1]-[3], the most common GUT’s are $SO(10)$ and $E_6$. The first condition for these kind of theories is an equal value for the three couplings at certain scale of energy, $M_U$. This condition cannot be fulfilled by the simplest grand unification schemes with the minimal SM particle content and taking the precision low-energy data. However, the minimal supersymmetric SM can achieve this scenario for the coupling constants [4]-[6]. Other possibilities for unification are the introduction of more degrees of freedom like fermions and scalar fields that lead the three couplings to converge at a high energy scale [7]. Polychromatic extensions of the SM i.e. $SU(N)_C \otimes SU(2)_L \otimes U(1)_Y$ have been considered where the unification of gauge couplings is achieved with $N = 7, 5$ and with two, three Higgs doublets, respectively [8]. Alternative proposals of unification of quarks and leptons at TeV scale were considered too [9]. Finally, another interesting alternative consists of enlarging the electroweak sector of the SM gauge group such that the renormalization group equations (RGE) could lead to unification of the three gauge couplings at certain scale $M_U$, in which there is no necessarily a group of grand unification at the scale of convergence of the couplings. In particular, the model based on the $SU(3)_C \otimes SU(3)_L \otimes U(1)_X$ gauge group (hereafter 331 models) is an interesting choice that could address problems like the charge quantization [10]-[11] and the existence of three families based on cancellation of anomalies [12]-[17].

One way to look for new Physics in 331 models is to check for $tT$ production with $t$ denoting the ordinary top quark and $T$ being an exotic quark with an exotic charge $2/3$ or $4/3$ according to the model considered. In LHC the production channels would be $pp \rightarrow X^0 \rightarrow tT$, when the charge of $T$ is $2/3$, and $pp \rightarrow X^{++} \rightarrow tT$, in the case in which $T$ has an exotic charge $4/3$. In the first case, in order to identify the signals, the decay of $t$ is known and identified by the energy spectrum and angular distribution of the final fermions [18]-[20], then this signal is correlated with the one of an exotic particle with the same charge of the top, but with a totally different decay in the final state. The decays of $T$ would be of the form $T \rightarrow X^0 t \rightarrow \nu Et$, $T \rightarrow K^+ b \rightarrow \nu Eb$, which could be easily identifiable if exotic leptons have already been produced at LHC. In the second case in which $T$ has an exotic

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of the type $T \to X^{++}t$ where $X^{++}$ is a doubly charged gauge field that should be easily identified if this model is correct. The other possible channel is $T \to K^+ b \to \nu Eb$.

In the scenario of SUSY 331 models, different channels can be searched. A good perspective is $pp \to gg \to g \to g\bar{g}$ where $\bar{g} \to t\bar{T}, t\bar{T}, T\bar{T}$ in the case in which $T$ possesses an ordinary charge $2/3$. Another production mechanism in the gaugino sector is could be $pp \to X^{++} \to \chi^{++}\chi^0, pp \to \chi^{++} \to X^{++}\chi^0, pp \to Z, Z^{++}\chi^0$ [21]-[25].

In looking for unification of the coupling constant by passing through a 331 model, we shall assume that 1) The 331 gauge group is the only extension of the SM before the unification of the running coupling constants. 2) The hypercharge associated with the 331 gauge group is adequately normalized such that the three gauge couplings unify at certain scale $M_U$, and 3) There is no necessarily a unified gauge group at the scale of convergence of the couplings $M_U$. In the absence of a grand unified group, there are no restriction on $M_U$ coming from proton decay[3].

Under our scheme, we have three characteristic energy scales: $M_{U}$ where the three gauge couplings converge, $M_X$ where the 331 symmetry is broken, and $M_Z$ where the SM breaking occurs. We are going to consider different scenarios for 331 models with one and three families, i. e., one family models. We also introduce Supersymmetric versions of the 331 models with different scalar Higgs multiplets. As for the SUSY breaking, we shall consider two different scenarios: when SUSY is broken at the electroweak scale, or when SUSY breaks at the scale of $M_X$.

In our scheme we have four parameters to take into account, and to look for a possible unification of the coupling constants (UCC). They are the scales $M_X, M_U$, the value of the coupling constants at the unification convergence point, $\alpha_U$, and the parameter associated with the normalization of the hypercharge (denoted by $a$). Since we are interested mostly in possible phenomenological scenarios, the relevant parameter will be the gauge symmetry breaking scale $M_X$; and the parameters $M_U, a$ can be viewed as functions of this one.

If the unification came from a grand unified symmetry group $G$, the normalization of the hypercharge $Y$ would be determined by the group structure. However, under our assumptions, this normalization factor is free and the problem could be addressed the opposite way, since the values obtained for $a$ could in turn suggest possible groups of grand unification in which the 331 group is embedded, we shall explore this possibility as well.

In the present work, we study six different versions of 331 models with non-SUSY and SUSY particle content, and find which models could lead to a unification at certain scale $M_U$ with only one symmetry breaking between $M_Z$ and $M_U$ scales. Some of the SUSY versions studied, could provide a quite natural scenario for split supersymmetry. Finally, we also consider the possibility of embedding those 331 versions into a grand unified theory (GUT) in which a gauge group at the unification scale appears.

### 2 Running Coupling Constants

The evolution for the Running Coupling Constants (RCC) at one-loop order is ruled by the solution of the Renormalization Group Equations (RGE), which can be written in the form [28]:

$$\frac{1}{\alpha_i(\mu_2)} = \frac{1}{\alpha_i(\mu_1)} - \frac{b_i}{2\pi} \ln \left( \frac{\mu_2}{\mu_1} \right),$$

(2.1)

where $\alpha_i = g_i^2/4\pi$, and the coefficients $b_i$ are given by [29]:

$$b_i = \frac{2}{3} \sum_f T_Ri(f) + \frac{4}{3} \sum_s T_Ri(s) - \frac{11}{3} C_{2i}(G).$$

(2.2)

The summations run over Weyl fermions and scalars, respectively. The coefficient $T_R$ is the Dynkin index

$$Tr(T^a T^b)_R = T_R \delta^{ab},$$

(2.3)

with the generators in the representation $R$. The last term is the quadratic Casimir for the adjoint representation

$$f^{acd} f^{bcd} = C_2(G) \delta^{ab},$$

(2.4)

with $C_2(G) = N$ for $SU(N)$. On the other hand, the respective supersymmetric versions are

$$b_i^{SUSY} = \sum_f T_Ri(f) + \sum_s T_Ri(s) - 3C_{2i}(G).$$

(2.5)

where the usual non-supersymmetric degrees of freedom are counted.

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1Notwithstanding, proton decay could be induced even in the absence of a group of grand unification. It may occurs via six dimensional 331 invariant effective operators, that violates baryonic and leptonic numbers [27].

2331 models with three identical fermion multiplets are usually call one family models.
2.1 Matching Conditions

The general expression for the electromagnetic charge operator will be a linear combination of diagonal generators for the gauge group 331:

\[ Q = T_3 + Y = T_3 + \frac{2}{\sqrt{3}} b T_8 + X \]  

(2.6)

with \( T_i \) the Gell-Mann matrices normalized as \( \text{Tr}(T_i T_j) = \frac{1}{2} \delta_{ij} \). The \( X \) operator for the abelian group \( U(1)_X \) is proportional to the identity matrix \( 3 \times 3 \). The hypercharge will be given by

\[ Y = \frac{2}{\sqrt{3}} b T_8 + X. \]  

(2.7)

\( b \) is a known parameter that determines the class of 331 models to be considered [30].

The renormalization group analysis compares the couplings for different gauge groups at given energy scales, and models with symmetry breakings need relations for couplings at different energy regions which are called the matching conditions; they are extracted from the way in which the unbroken group is embedded into a bigger broken group. Also, in order to have all couplings in the same ground, all the generators should be normalized in the same way, and well normalized couplings are those which will converge in an unification point.

Calling \( \tilde{Y} \) the well-normalized hypercharge operator, it will be proportional to the original one,

\[ Y = a\tilde{Y}, \]  

(2.8)

where \( a \) is the normalizing parameter so that the convergence of the running coupling constants at certain scale \( M_U \) is guaranteed. In the same way the operator \( X \) has a well-normalized \( \tilde{X} \), i.e. \( X = c\tilde{X} \), and the normalizing parameter for it, is given by Eq. (2.7) requiring the same normalization for \( \tilde{Y} \), \( T_8 \) which satisfies the following relation

\[ a^2 = \frac{4}{3} b^2 + c^2. \]  

(2.9)

Therefore, the parameter \( a \) should be such that

\[ a^2 \geq \frac{4}{3} b^2. \]  

(2.10)

Then, the well-normalized hypercharge operator can be written as a function of the unknown parameter \( a \), as

\[ a\tilde{Y} = \frac{2}{\sqrt{3}} b T_8 + \sqrt{\left(a^2 - \frac{4}{3} b^2\right)} \tilde{X}. \]  

(2.11)

And from this equation, we obtain the following matching condition for the corresponding couplings [33]:

\[ a^2 \tilde{\alpha}^{-1}_Y = \frac{4b^2}{3} \alpha_{3L}^{-1} + \left(a^2 - \frac{4}{3} b^2\right) \tilde{\alpha}^{-1}_X, \]

(2.12)

where \( \tilde{\alpha}_Y \), \( \tilde{\alpha}_X \), \( \alpha_{3L} \) are related with \( U(1)_Y \), \( U(1)_X \) and \( SU(3)_L \), respectively.

The following relations must also be satisfied

\[ \tilde{\alpha}_Y = a^2 \alpha_Y, \quad \tilde{\alpha}_X = (a^2 - \frac{4b^2}{3}) \alpha_X, \]

\[ \alpha_s = \alpha_{3C}, \quad \alpha_{2L} = \alpha_{3L}. \]  

(2.13)

where \( \alpha_X \), \( \alpha_Y \) and \( \alpha_{2L} \) are related with \( U(1)_X \), \( U(1)_Y \) and \( SU(2)_L \), respectively. The third relation corresponds to the strong interaction where \( \alpha_s \) is associated with the Standard Model and \( \alpha_{3C} \) is related with the color part in the 331 model. Finally, the last relation corresponds to the embedding of \( SU(2)_L \) into \( SU(3)_L \).
2.2 RGE analysis

By replacing the relations described by Eqs. (2.12, 2.13) into Eq. (2.1), we can write the evolution for the RCC from the $Z$ boson-pole $M_Z$ passing through a 331 symmetry breaking scale $M_X$, up to a certain Scale of unification $M_U$,

$$
\alpha^{-1}_U = \frac{1}{a^2 - \frac{4b^2}{3}} \left\{ \alpha_{EM}(M_Z)^{-1} - \frac{4b^2}{3} \alpha_{2L}(M_Z)^{-1} \right. \\
- \frac{b_Y - \frac{4b^2}{3} b_{2L}}{2\pi} \ln\left(\frac{M_X}{M_Z}\right) - \frac{b_X}{2\pi} \ln\left(\frac{M_U}{M_X}\right) \right\},
$$

(2.14)

$$
\alpha^{-1}_U = \alpha_{2L}(M_Z)^{-1} - \frac{b_{2L}}{2\pi} \ln\left(\frac{M_X}{M_Z}\right) - \frac{b_{3L}}{2\pi} \ln\left(\frac{M_U}{M_X}\right),
$$

(2.15)

$$
\alpha^{-1}_U = \alpha_s(M_Z)^{-1} - \frac{b_s}{2\pi} \ln\left(\frac{M_X}{M_Z}\right) - \frac{b_{3C}}{2\pi} \ln\left(\frac{M_U}{M_X}\right).
$$

(2.16)

where the coefficients $b_s$, $b_{2L}$ and $b_Y$ are related with $SU(3)_C, SU(2)_L, U(1)_Y$, respectively and they are calculated at energies in the range $M_Z \leq \mu \leq M_X$. The coefficients $b_{3C}, b_{3L}$, and $b_X$ are related with $SU(3)_C, SU(3)_L$, and $U(1)_X$ respectively; they are calculated for energies in the range $M_X \leq \mu \leq M_U$. For our study, we need $b_i$ coefficients for energy scales below (and above) the symmetry breaking scale $M_X$, which will be given by SM (and 331) degrees of freedom. Then for different models, we have different $b_i$’s for the intervals of energy scales which can change the running of the coupling constants. The input parameters from precision measurements are [31]

$$
\alpha_{EM}(M_Z) = 127.934 \pm 0.027,
$$

$$
\sin^2\theta_W(M_Z) = 0.23113 \pm 0.00015,
$$

$$
\alpha_s(M_Z) = 0.1172 \pm 0.0020,
$$

$$
\alpha_{2L}^{-1}(M_Z) = 29.56938 \pm 0.00068.
$$

(2.17)

The $M_U$ scale, where all the well-normalized couplings have the same value, can be calculated from (2.15) and (2.16) as a function of the symmetry breaking scale $M_X$

$$
M_U = M_X \left(\frac{M_X}{M_Z}\right)^{-\frac{b_s-b_{2L}}{b_{3C}-b_{3L}}} \exp \left\{ 2\pi \frac{\alpha_s(M_Z)^{-1} - \alpha_{2L}(M_Z)^{-1}}{b_{3C} - b_{3L}} \right\}.
$$

(2.18)

The hierarchy condition $M_X \leq M_U \leq M_{Planck}$, must be satisfied. We shall however impose a stronger condition of $M_U \lesssim 10^{17}$GeV, in order to avoid gravitational effects. Hence, the hierarchy condition becomes

$$
M_X \leq M_U \leq 10^{17}$GeV
$$

(2.19)

Such condition can establish an allowed range for the symmetry breaking scale $M_X$ in order to obtain grand unification for a given normalizing parameter $a$.

With a similar procedure, the expression for $a^2$ is found, and is given by

$$
a^2 = \frac{4b^2}{3} + \left\{ \alpha_{EM}(M_Z)^{-1} - \frac{4b^2}{3} \alpha_{2L}(M_Z)^{-1} \right. \\
- \frac{b_Y - \frac{4b^2}{3} b_{2L}}{2\pi} \ln\left(\frac{M_X}{M_Z}\right) \\
+ \frac{b_X}{2\pi} \ln\left(\frac{M_X}{M_Z}\right) \right\}
$$

(2.20)

$$
\times \left\{ \alpha_{2L}(M_Z)^{-1} - \frac{b_{2L}}{2\pi} \ln\left(\frac{M_X}{M_Z}\right) \right. \\
+ \frac{b_{3L}}{2\pi} \ln\left(\frac{M_X}{M_Z}\right) \right\}
$$

In order to analyze the possibility of having unification at certain scale $M_U$ with 331 as the only intermediate gauge group between the SM and $M_U$ scales, we could distinguish three scenarios of unification pattern
1. The scenario with \( b_{3C} \neq b_{3L} \) and \( (b_{3C} - b_{3L}) \neq (b_s - b_{2L}) \). We shall call it the first unification pattern (1UP). In that case, we obtain an allowed region for the scale \( M_X \). It is carried out by combining the results of Section 2.2. The procedure to get the allowed interval for \( M_X \) is described in detail in Section 3.1 for the so-called model A.

2. The scenario with \( b_{3C} = b_{3L} \). We call it (2UP). In such a case Eq. (2.18) is not valid anymore, and we should go back to Eqs. (2.15, 2.16). For an arbitrary value of \( M_X \) the couplings \( \alpha_{3C} \) and \( \alpha_{3L} \) parallel each other for energies larger than \( M_X \). Therefore, the only possible way to still obtain unification is by setting \( \alpha_{3C} = \alpha_{3L} \) at the scale \( M_X \) such that both couplings go together for scales above \( M_X \). Unification with the third coupling could occur at any scale bigger than \( M_X \). By equating Eqs (2.15, 2.16), we find a single value of \( M_X \) that makes the couplings \( \alpha_{3C} \) and \( \alpha_{3L} \) to converge. This convergence occurs at the scale

\[
M_X = M_Z \exp \left[ \frac{2\pi \left( \alpha_{2L}(M_Z)^{-1} - \alpha_s(M_Z)^{-1} \right)}{(b_{2L} - b_s)} \right] \tag{2.21}
\]

It worths emphasizing that this scenario leads to a unique value of \( M_X \) and not to an allowed range. Finally, Eq. (2.20) for \( a^2 \) must also be recalculated to find

\[
a^2 = \frac{F_1(M_X) - \frac{b_s}{b_{2L}} \ln \left( \frac{M_X}{M_Z} \right)}{F_2(M_X) - \frac{b_s}{b_{2L}} \ln \left( \frac{M_X}{M_Z} \right)} + \frac{4b_s^2}{3} \tag{2.22}
\]

3. The 3UP with \( (b_{3C} - b_{3L}) = (b_s - b_{2L}) \neq 0 \); according to Eq. (2.18), the unification scale \( M_U \) becomes independent of \( M_X \).

The case \( (b_{3C} - b_{3L}) = (b_s - b_{2L}) = 0 \), does not lead to unification as can be seen by trying to equate Eqs. (2.16, 2.17). Since the first scenario is the most common one, we shall only indicate when the other two scenarios appear. We will study non-SUSY and SUSY versions of the 331 models. In the case of SUSY models we shall consider two scenarios for the SUSY breaking pattern

1. The SUSY Breaking Scenario at the \( Z \)-pole (ZSBS), in which the SUSY breaking scale is taken as \( \Lambda_{SUSY} \sim M_Z \). Although this is not a very realistic scenario, numerical results do not change significantly with respect to the more realistic scenario with \( \Lambda_{SUSY} \) lying at some few TeV’s.

2. The SUSY Breaking Scenario at the \( M_X \) scale (XSBS), with \( \Lambda_{SUSY} \sim M_X \) i.e. SUSY breaking at the 331 breaking scale.

3 331 Models

Analogously to the SM, fermions will transform as singlets or in the fundamental representation of \( SU(3)_L \), and gauge fields in the adjoint representation. Assignment of \( U(1)_X \) quantum numbers should be done ensuring a model free of anomalies. There are models with three different families necessary to cancel out the anomalies, and there are models with only one family and the other two are a copy of the first. We will take into account six different versions of the 331 model for the analysis of the unification scheme.

The minimal spectrum necessary for symmetry breaking and generation of masses is given by

\[
\begin{align*}
\phi_1 & \sim (1, 3^*, -1/3), \\
\phi_2 & \sim (1, 3^*, -1/3), \\
\phi_3 & \sim (1, 3^*, 2/3).
\end{align*} \tag{3.1}
\]

Where the quantum numbers are associated with \( SU(3)_C \), \( SU(3)_L \), and \( U(1)_X \) respectively. The first multiplet acquires a vacuum expectation value (VEV) at \( M_X \) scale, breaking the symmetry as \( 331 \rightarrow 321 \); the other two will
be decomposed as singlets and doublets of $SU(2)_L$, which will break the 321 symmetry. The spectrum transforms like

$$
\begin{align*}
\phi_2 & \rightarrow \begin{cases} 
\phi_{2SM} & \sim (1, 2^*, -1/2), \\
\phi_2^0 & \sim (1, 1, 0).
\end{cases} \\
\phi_3 & \rightarrow \begin{cases} 
\phi_{3SM} & \sim (1, 2^*, 1/2), \\
\phi_3^2 & \sim (1, 1, 1).
\end{cases}
\end{align*}
$$

(3.2)

(3.3)

This is applied for models with $b = 1/2$, with tiny changes that do not affect our analysis \[13\] [44]. For models with $b = 3/2$ this spectrum should be varied in order to get a phenomenological mass spectrum. For this case we have the following transformation for the triplets \[26\] [34] [35] [36]:

$$
\begin{align*}
\eta & \sim (1, 3^*, 0), \\
\rho & \sim (1, 3^*, 1), \\
\chi & \sim (1, 3^*, -1).
\end{align*}
$$

(3.4)

the 331 symmetry is broken when $\chi$ acquires a VEV in the third component. The remaining scalars will transform in the 321 symmetry as

$$
\begin{align*}
\eta & \rightarrow \begin{cases} 
\eta_{SM} & \sim (1, 2^*, -1/2), \\
\eta_2^+ & \sim (1, 1, 1).
\end{cases} \\
\rho & \rightarrow \begin{cases} 
\rho_{SM} & \sim (1, 2^*, 1/2), \\
\rho^{++} & \sim (1, 1, 2).
\end{cases}
\end{align*}
$$

(3.5)

(3.6)

The Model that we will call Model E needs also another scalar boson transforming under 6 representation of $SU(3)_L$. Such an structure is required in order to give masses to the neutrinos \[26\] [34] [35],

$$
S \sim (1, 6, 0)
$$

(3.7)

which has the following representation in the 321 symmetry:

$$
S \rightarrow \begin{cases} 
\psi_{3SM} & \sim (1, 3, 1), \\
\phi_{2SM} & \sim (1, 2, 1/2), \\
\phi^- & \sim (1, 1, -2).
\end{cases}
$$

(3.8)

### 3.1 Model A

The simplest anomaly-free structure for this gauge group is generated by taking $b = 1/2$; and the following spectrum is obtained \[37\]

$$
\begin{align*}
\psi_{1L} & = (e^-, \nu_e, N_1^T)_{L} \sim (1, 3^*, -1/3), \\
\psi_{2L} & = (E^-, N_2^0, N_3^0)_{L} \sim (1, 3^*, -1/3), \\
\psi_{3L} & = (N_1^0, E^+, e^+)_{L} \sim (1, 3^*, 2/3), \\
Q_L & = (u, d, D)_{L}^T \sim (3, 3, 0), \\
u_L^c & \sim (3^*, 1, -2/3), \\
d_L & \sim (3^*, 1, 1/3), \\
D_L^c & \sim (3^*, 1, 1/3).
\end{align*}
$$

(3.9)

The other two families are copies of the first one, and each family is free of anomalies. This particular 331 model could be embedded in the $E_6$ gauge theory, but we shall assume that not necessarily such embedding occurs.

From the general expression for coefficients $b_i$ Eq. (2.2), and using the quantum numbers assigned for each representation in this model, they can be expressed at energies below $M_X$ in the following form

$$
\begin{align*}
b_Y & = \frac{20}{9} N_g + \frac{1}{6} N_H + \frac{1}{3} \sum_{sing-s} Y^2(s), \\
b_{2L} & = \frac{4}{3} N_g + \frac{1}{6} N_H - \frac{22}{3}, \\
b_s & = \frac{4}{3} N_g - 11.
\end{align*}
$$

(3.10)
Figure 1: Allowed interval for $M_X$ in model A. The solid line represents $M_U$ as a function of $M_X$. The horizontal short dashed line is the graphics for $M_U = 10^{17}$ GeV. The long dash line represents the function $M_U = M_X$.

Figure 2: Normalization parameter $a^2$ as function of the symmetry breaking scale $M_X$ for model A. The intersection of this function with the constant function $\frac{4}{3}b^2$ gives a lower limit for $M_X$ owing to the condition described by Eq. (2.10).

Figure 3: Running Coupling Constants for model A with $M_X = 1.63 \times 10^{16}$ GeV. The unification scale appears at $M_U \simeq 10^{17}$ GeV.
where \( N_g \) is the number of families, \( N_H \) the number of scalar doublets in \( SU(2)_L \), and the sum runs over scalar singlets.

In the same way, the \( b_i \)'s for energies above the \( M_X \) are given by

\[
\begin{align*}
  b_X &= \frac{8}{3} N_g - \frac{2}{3}, \\
  b_{3L} &= 2 N_g + \frac{1}{2} - 11, \\
  b_{3C} &= 2 N_g - 11.
\end{align*}
\]

Therefore, from the spectrum showed in Eq. (3.9), we obtain

\[
(b_Y, b_{2L}, b_s) = \left( \frac{22}{3}, -3, -7 \right)
\]

for energies below \( M_X \), and

\[
(b_X, b_{3L}, b_{3C}) = \left( \frac{26}{3}, \frac{9}{2}, -5 \right)
\]

for energies above \( M_X \).

First of all, by using the hierarchy condition Eq. (2.19), an allowed region for \( M_X \) can be found. In Fig. 1 we plot \( M_U \) as a function of \( M_X \), Eq. (2.18). We also plot the line \( M_U = M_X \), and the line \( M_U = 10^{17} \text{GeV} \), the intersection of these lines with the first plot, provides the allowed region given by the constraints \( M_X \leq M_U \leq 10^{17} \text{GeV} \) according to Eq. (2.19). The intersection between the curve \( M_U(M_X) \) from Eq. (2.18) with \( M_U = 10^{17} \text{GeV} \) gives the lower bound for \( M_X \), and the intersection with \( M_U = M_X \) gives the upper limit for \( M_X \). Then from Fig. 1 and the hierarchy condition, it is obtained that

\[
1.63 \times 10^{16} \text{GeV} \leq M_X \leq 2.05 \times 10^{16} \text{GeV}.
\]

In Figure 2 we plot \( a^2 \) as a function of \( M_X \) from Eq. (2.20). For this model \( b = 1/2 \) and from Eq. (2.10) the restriction \( a^2 \geq 1/3 \) is obtained. The horizontal line \( a^2 = 1/3 \) is also showed and the allowed values are above of this line. In this case the constraint for \( M_X \) is given by \( M_X \leq 2 \times 10^{14} \text{GeV} \) which is weaker than the previous restriction.

From Eq. (2.18) and the allowed interval for \( M_X \) of Eq. (3.14) we get the region permitted for \( M_U \) i.e. \( 2 \times 10^{16} \text{GeV} \leq M_U \leq 10^{17} \text{GeV} \).

Now, for \( M_X = 2.05 \times 10^{16} \text{GeV} \) (the highest allowed value) we obtain from Eq. (2.18) that \( M_U \approx M_X \) which is not a natural hierarchy. Instead, for \( M_X = 1.63 \times 10^{16} \text{GeV} \) (the lowest allowed value) the unification occurs at a more natural scale of \( 10^{17} \text{GeV} \). Then, we shall take the latter to plot the evolution of the running coupling constants.

The evolution of the RCC described by Eqs. (2.1), requires the results for the matching conditions Eqs. (2.12 2.13). Moreover, the following input parameters are necessary: \( \oplus \) The \( b_i \) coefficients Eqs. (3.12 3.13), \( \ominus \) The value of the 331 breaking scale \( M_X \), and \( \otimes \) the values of the couplings at Z-pole Eqs. (2.17). Assuming the lowest allowed value for the 331 breaking scale \( M_X = 1.63 \times 10^{16} \text{GeV} \) we plot the evolution of the running coupling constants in Fig. 3.

The unification scale appears to be \( M_U = 10^{17} \text{GeV} \), in agreement with our previous results.

### 3.1.1 SUSY Extension

One possible SUSY extension in which the superpartners cancel the anomalies of their partners, arises by introducing new scalars which transform in the conjugate representations of the original ones. Under these considerations, and using Eqs. (2.5) we obtain the supersymmetric \( b_i \) coefficients

\[
\begin{align*}
  (b_Y, b_{2L}, b_s) &= \left( 14, 2, -3 \right), \\
  (b_X, b_{3L}, b_{3C}) &= \left( 16, 3, 0 \right).
\end{align*}
\]

We take first the ZSBS in which SUSY breaking is assumed to occur at EW scale\(^8\). Since in that case SUSY is not broken by going from EW to the unification scale, we should use the supersymmetric \( b_i \) coefficients, at scales

\(^8\)Our results are also valid for SUSY breaking at scales of the order of 1 TeV.
below and above the 331 breaking scale. Taking it into account, and using the hierarchy condition \[ \text{(2.19)}, \] as it was explained in the previous section, we get an allowed range for \( M_X \), of
\[ 1.27 \times 10^8 \text{GeV} \leq M_X \leq 2.76 \times 10^{13} \text{GeV}. \] (3.17)

The bound coming from the \( a \) parameter Eq. \[ \text{(2.10)}, \] yields \( 6.43 \times 10^6 \text{GeV} \leq M_X \), which gives no further restriction. This allowed region for \( M_X \) leads to an allowed interval for \( M_U \) obtained from Eq. \[ \text{(2.13)}, \] getting\(^4\)
\[ 2.76 \times 10^{13} \text{GeV} \leq M_U \leq 10^{17} \text{GeV} \]

On the other hand, we shall consider the XSBS with SUSY breaking at the \( M_X \) scale. In that case, we should use the non-SUSY \( b_i \) coefficients for scales below \( M_X \), and the SUSY \( b_i \) coefficients for energies above \( M_X \). From this we obtain the following allowed interval for \( M_X \)
\[ 1.76 \times 10^{14} \text{GeV} \leq M_X \leq 2.05 \times 10^{16} \text{GeV}, \] (3.18)
for which the restriction \( a^2 > 1/3 \) is also satisfied. This allowed region provides a unification scale between \( 2.05 \times 10^{16} \text{GeV} \) and \( 10^{17} \text{GeV} \) which correspond to the values calculated from Eq. \[ \text{(2.15)}, \] for the lower and upper values for \( M_X \), respectively. Therefore, this is a possible scenario of unification with 331 as the unique intermediate gauge group, for both schemes of SUSY breaking.

It worths noting that SUSY scenarios leads to much lower breaking scales for the 331 model, especially for low energy SUSY breaking.

In another possible supersymmetric extension of model A, the scalar fields are not introduced explicitly. Instead, they appear as superpartners of the leptons, since they have the same representations and quantum numbers.\(^4\)

The supersymmetric \( b_i \) coefficients read
\[ b_Y, b_{2L}, b_s = (10, 0, -3), \] (3.19)
\[ (b_X, b_{3L}, b_{BC}) = (12, 0, 0). \] (3.20)

For both schemes of SUSY breaking, the model correspond to the 2UP explained in Sec. \[ \text{(2.2)}, \] so that \( M_X \) acquires a single value. The 331 scale appears at \( M_X \simeq 10^{21} \text{GeV} \). It discards this SUSY version of model A, to get UCC under our scheme.

### 3.2 Model B

Again the parameter \( b \) takes the value 1/2, and in this case the 331 gauge theory is a family-symmetric model. For each family the spectrum is \[ \Sigma_{\text{331}} \]:
\[ \psi_{1L} = (e^{-}, \nu_e, E_{1}^-_{L}) \sim (1, 3^*, -2/3), \]
\[ \psi_{2L} = (N_1^0, E_{2}^+_{L}, \nu_e) \sim (1, 3^*, 1/3), \]
\[ \psi_{3L} = (E_{3}^-_{L}, N_2^0, E_{3}^-_{L}) \sim (1, 3^*, 1/3), \]
\[ e_{L}^+ \sim (1, 1, 1), \quad E_{1L}^+ \sim (1, 1, 1), \quad E_{3L}^+ \sim (1, 1, 1), \]
\[ Q_L = (u, d, U)_{L} \sim (3, 3, 1/3), \]
\[ u_{L}^c \sim (3^*, 1, -2/3), \quad d_{L}^c \sim (3^*, 1, 1/3), \quad U_{L}^c \sim (3^*, 1, -2/3). \] (3.21)

In this case, the 331 gauge group could be embedded into the \( SU(6) \otimes U(1)_X \) gauge theory, but we suppose it is not mandatory. This model has the same spectrum of model A at low energies. Therefore, it has the same values for the \( b_i \) coefficients at low energies given by Eq. \[ \Sigma_{\text{331}} \]
\[ (b_Y, b_{2L}, b_s) = \left( \frac{22}{3}, -3, -7 \right). \] (3.22)

For energies larger than \( M_X \), we have that \( b_{3L} \) and \( b_{BC} \) are the same as in model A, since their values are ruled by the number of triplets, which coincide for both models. For \( b_X \), there is a difference coming from the additional singlets in the lepton sector. In this case, we obtain
\[ b_X = \frac{20}{3} N_g + \frac{2}{3} = \frac{62}{3}. \]

\(^4\)When a group of grand unification is present, protection from proton decay leads to a stronger bound for \( M_X \) by demanding \( M_U \gtrsim 2 \times 10^{16} \text{GeV} \), in that case we find \( 1.27 \times 10^8 \text{GeV} \leq M_X \leq 1.4 \times 10^9 \text{GeV} \).
Hence,
\[
(b_X, b_{3L}, b_{3C}) = \left( \frac{62}{3}, -9, -5 \right).
\]

Following the same procedure as for the Model A, we find the allowed range for \(M_X\)
\[
1.63 \times 10^{16}\text{GeV} \leq M_X \leq 2.05 \times 10^{16}\text{GeV}.
\]
In this interval, \(a^2 > 1/3\) yields \(M_X \geq 3.42 \times 10^{15}\) GeV, and \(M_U\) lies in the interval \([2.05 \times 10^{16}, 10^{17}]\) GeV.

### 3.2.1 SUSY Extension

As a first attempt, we take the model in Ref. \[39\], where scalars are considered as sleptons and two new superfields are introduced in order to give mass to the supersymmetric particles and cancel out the anomalies. They transform as
\[
\psi_{4L} \sim (1, 3, 1/3), \quad \psi_{5L} \sim (1, 3^*, -1/3).
\]
The spectrum is the same at low energies as SUSY model A, Eq. \(3.19\). The supersymmetric \(b_i\) coefficients read
\[
(b_Y, b_{2L}, b_3) = (10, 0, -3),
\]
\[
(b_X, b_{3L}, b_{3C}) = (24, 3, 0).
\]
For the ZSBS (\(\Lambda_{SUSY} \approx M_Z\)), we are in the 3UP explained in Sec. \(2.2\) so that \(M_U\) acquires a single value
\[
M_U = 1.24 \times 10^{21}\text{GeV}.
\]
which discards this SUSY version of model B with \(\Lambda_{SUSY} \approx M_Z\), to achieve unification with our breaking scheme.

In the XSBS (\(\Lambda_{SUSY} \approx M_X\)), the allowed region for the model is
\[
1.76 \times 10^{14}\text{GeV} \leq M_X \leq 2.05 \times 10^{16}\text{GeV}.
\]
the bound from \(a^2\) is \(M_X \geq 1.44 \times 10^{11}\text{GeV}\) giving no further restriction. \(M_U\) belongs to the interval \([2.05 \times 10^{16}, 10^{17}]\) GeV. Hence, this SUSY version of model B with \(\Lambda_{SUSY} \approx M_X\) permits UCC under our assumptions.

### 3.3 Model C

In this model, the parameter \(b\) is also equal to 1/2, but the cancellation of anomalies is obtained with a number of families multiple of three \[40\]. Its fermionic spectrum is
\[
\psi_L^\alpha = (\nu, \alpha^-, E^-_L) \sim (1, 3, -2/3), \quad \alpha_L^+ \sim (1, 1, 1), \quad E^+_{3L} \sim (1, 1, 1),
\]
\[
Q_L^i = (d^i, u^i, U^i) \sim (3^*, 1, 1/3), \quad u^i_{3L} \sim (3^*, 1, -2/3), \quad d^i_{3L} \sim (3^*, 1, 1/3), \quad U^i_{3L} \sim (3^*, 1, -2/3),
\]
\[
Q_L^i = (u_3, d_3, D^i L) \sim (3, 3, 0), \quad u_3^* \sim (3^*, 1, -2/3), \quad d_3^* \sim (3^*, 1, 1/3), \quad D^i_L \sim (3^*, 1, /3),
\]
where \(\alpha = 1, 2, 3\) label the three lepton families; while \(i = 1, 2\) refer to only two of the quark families because the third one transforms in a different way. The spectrum is the same of the Model A at low energies, so that the \(b_i\) coefficients are given by Eq. \(3.12\). For energies above \(M_X\) there are new contributions and the \(b_i\)'s yield
\[
(b_Y, b_{2L}, b_3) = \left( \frac{22}{3}, -3, -7 \right),
\]
\[
(b_X, b_{3L}, b_{3C}) = \left( \frac{42}{3}, -\frac{13}{2}, -5 \right).
\]
Following the same procedure explained before, we find that the allowed region for \(M_X\) is
\[
2.05 \times 10^{16}\text{GeV} \leq M_X \leq 3.15 \times 10^{16}\text{GeV}.
\]
In this case \(a^2 \geq 1/3\), leads to \(M_X \leq 9.57 \times 10^{14}\text{GeV}\) and provides an upper (and not a lower) limit because \(a^2\) is a decreasing function of \(M_X\). Of course, this constraint gives no further restrictions. Taking values for \(M_X\) from the interval in Eq. \(3.31\), the permitted interval for \(M_U\) is \([2.05 \times 10^{16}, 10^{17}\text{GeV}]\).
3.3.1 SUSY Extension

The extension is done adding new scalars transforming in the conjugate representation in order to cancel the anomalies. With this spectrum we obtain

\[
(b_Y, b_{2L}, b_s) = (14, 2, -3) \\
(b_X, b_{3L}, b_{3C}) = (24, 0, 0).
\]

At low energy scales, the coefficients coincide with the analogous in model A, Eq. (3.15). For both schemes of SUSY breaking we lie in the 2UP getting a unique \(M_X\) from Eq. (2.21). For the ZSBS, \(M_X\) becomes

\[
M_X = 2.76 \times 10^{13} \text{GeV}.
\]

If we fit \(M_U = 2.76 \times 10^{13} (= 1 \times 10^{17}) \text{ GeV}\), we need \(a^2 = 3.26 (= 1.78)\) to obtain unification of the three couplings according to Eq. (2.22). In general, for \(M_U\) lying in the interval \([2.76 \times 10^{13} \text{GeV}, 10^{17} \text{GeV}]\), we find that the \(a^2\) parameter lies in the interval \([1.78, 3.26]\); and the condition \(a^2 \geq (4/3)b^2\) is satisfied. Therefore, this model gives a consistent unification with 331 as the only intermediate gauge group.

For the XSBS, \(M_X = 2.05 \times 10^{16} \text{GeV}\). From \(M_U \in [2.05 \times 10^{16}, 1 \times 10^{17}] \text{GeV}\), we get \(a^2 \in [1.84, 1.97]\) and \(a^2 \geq (4/3)b^2\) is fulfilled in all cases and the model is viable as well.

3.4 Model D

This model differs from model C just in its spectrum [41][42], which is given by

\[
\psi_L = (\alpha^-, \nu, N_{\alpha}^0_L \sim (1, 3, -1/3),
\alpha^+_L \sim (1, 1, -1),
Q'_L = (d^c_i, -u^i, D^i)^T \sim (3, 3^*, 0),
\alpha^+_L \sim (3, 1, +2/3),
\alpha^c_L \sim (3, 1, -1/3),
Q'_L = (d^c_i, u^i, U^c)^T \sim (3, 3, 1/3),
\alpha^c_L \sim (3, 1, 2/3),
\alpha^c_L \sim (3, 1, -1/3),
U^c_L \sim (3, 1, 2/3).
\]

with \(\alpha = 1, 2, 3\), running over the three families while \(i = 1, 2\) runs over two families. The model differs from the others above in its spectrum at high energies. Therefore for the 331 gauge group the \(b_i\) functions coincide with the values in the Eq. (3.15).

\[
(b_Y, b_{2L}, b_s) = \left(\frac{22}{3}, -3, -7\right).
\]

For energies larger than the symmetry breaking scale \(M_X\) we have

\[
(b_X, b_{3L}, b_{3C}) = \left(\frac{26}{3}, -\frac{13}{2}, -5\right).
\]

These values give the following allowed region

\[
2.05 \times 10^{16} \text{GeV} \leq M_X \leq 3.15 \times 10^{16} \text{GeV},
\]

and \(a^2 > 1/3\) predicts \(M_X \leq 1.54 \times 10^{20} \text{GeV}\), yielding no further restriction. The scale of grand unification lies in the interval \(2.05 \times 10^{16} \text{GeV} \leq M_U \leq 10^{17} \text{GeV}\).

3.4.1 SUSY Extension

Treating the model in the same way as model C, the \(b_i\) coefficients at low energies are given in Eq. (3.15), and for high energy scales they are given by

\[
(b_X, b_{3L}, b_{3C}) = (16, 0, 0).
\]

Both schemes of SUSY breaking are in the 2UP. For the ZSBS we find \(M_X = 2.76 \times 10^{13} \text{GeV}\). Taking \(M_U \in [2.76 \times 10^{13}, 1 \times 10^{17}] \text{GeV}\) we obtain \(a^2 \in [2.27, 3.26]\), and the basic restriction on \(a^2\) is always accomplished. With XSBS, it is found that \(M_X = 2.05 \times 10^{16}\), and we get \(M_U \in [2 \times 10^{16}, 1 \times 10^{17}] \text{GeV}\) for \(a^2 = [1.88, 1.97]\), respectively.

Thus the SUSY version provides a possible scenario of unification for both schemes of SUSY breaking.
3.5 Model E

In this model \( b = 3/2 \) and the cancellation of anomalies is obtained by the interplay of three families \([26, 34, 35]\). Its fermionic spectrum is

\[
\begin{align*}
\psi^\alpha_L &= (e^\alpha, \nu^\alpha, e^{c\alpha})^T \sim (1, 3^*, 0), \\
q^i_L &= (u^i, d^i, j^i)^T_L \sim (3, 3, -1/3), \\
qu^\alpha_L &= (d^\alpha, u^\alpha, j^\alpha)^T_L \sim (3, 3^*, 2/3), \\
u^\alpha_L &\sim (3, 1, -2/3), \\
d^\alpha_L &\sim (3, 1, 1/3), \\
s^\alpha_L &\sim (3, 1, -5/3), \\
\end{align*}
\]

where \( \alpha = 1, 2, 3 \) labels the families, and \( i = 2, 3 \) are related with two of them.

Taking into account the whole possible scalar spectrum at low energy scales, including the scalars in the 6-dimensional representation, we can write the \( b_i \) coefficients as

\[
\begin{align*}
b_Y &= \frac{20}{9} N_g + \frac{1}{6} N_H + \frac{1}{3} 3Y^2(\text{triplet}) + \frac{1}{3} \sum \text{singl}-s Y^2(s), \\
b_{2L} &= \frac{4}{3} N_g + \frac{1}{6} N_H + \frac{1}{3} T_R(\text{triplet}) - 22/3, \\
b_s &= \frac{4}{3} N_g - 11. \\
\end{align*}
\]

(3.40)

where \( Y(\text{triplet}) \) and \( T_R(\text{triplet}) \) mean the quantum numbers of the \( SU(2)_L \) scalar triplet in the 6-dimensional representation. Therefore the values of the \( b_i \) coefficients read

\[
(b_Y, b_{2L}, b_s) = \left( \frac{23}{2}, -\frac{13}{6}, -7 \right),
\]

(3.41)

\[
(b_X, b_{3L}, b_{3C}) = \left( 22, -\frac{17}{3}, -5 \right).
\]

(3.42)

With these \( b_i \) coefficients we only find a lower limit from the hierarchy condition,

\[
6.87 \times 10^{13}\text{GeV} \leq M_X
\]

(3.43)

An upper limit is obtained from Eq. (2.10) with 3 \( \leq a^2 \), which is satisfied for \( M_X \leq 6.03 \times 10^{12}\text{GeV} \), but both constraints are inconsistent and rule out the model under our assumptions.

3.5.1 SUSY Extension

The supersymmetric model is constructed adding scalar fields in the complex representation in order to avoid chiral anomalies from their superpartners. With this spectrum we obtain the following values for the \( b_i \) coefficients

\[
\begin{align*}
(b_Y, b_{2L}, b_s) &= (28, 6, -3), \\
(b_X, b_{3L}, b_{3C}) &= (42, 5, 0). \\
\end{align*}
\]

(3.44)

We also consider an extended spectrum for the scalar sector. With these values we get only an upper limit for \( M_X \) in the ZSBS

\[
M_X \leq 2.18 \times 10^8\text{GeV}.
\]

(3.45)

But the restriction in the normalization parameter \( a \) in Eq. (2.10), requires \( M_X \geq 8.1 \times 10^8 \). This situation discards the model with only one intermediate breaking.

For the XSBS, the hierarchy condition gives only a lower limit \( 6.87 \times 10^{13}\text{GeV} \leq M_X \). But \( a^2 \geq 3 \) gives \( M_X \geq 1.35 \times 10^{22}\text{GeV} \), discarding the model.
3.6 Model F

It is also possible to obtain cancellation of anomalies by a tiny variation of the fermionic spectrum from the model D of Pleitez-Tonasse [30]. This new spectrum will be given by

\[
\begin{align*}
\psi^\alpha_L & = (\nu^\alpha, e^\alpha, E^{\alpha+})^T \sim (1, 3, 0), \\
e^{\alpha+}_L & \sim (1, 1, -1), \quad E^\alpha_L \sim (1, 1, 1).
\end{align*}
\]  

where an exotic lepton has been introduced in the third entry of the triplet instead of the electron right-handed part, such that there are also leptonic singlets of \(SU(3)_L\).

The \(b_i\) coefficients at low energy scales differ from those for the model E in the contribution due to the sextuplet. Therefore the values are

\[
\begin{align*}
(b_Y, b_{2L}, b_s) & = (26/3, -3, -7), \\
(b_X, b_{3L}, b_{3C}) & = \left(26, -\frac{13}{2}, -5\right).
\end{align*}
\]  

With these coefficients, the allowed region for \(M_X\) is given by

\[2.05 \times 10^{16} \text{GeV} \leq M_X \leq 3.15 \times 10^{16} \text{GeV}.\]  

The restriction from Eq. (2.10) concerning the parameter \(a^2\), gives an upper bound of \(M_X \leq 4.1 \times 10^{17}\text{GeV}\), i.e. no additional restriction.

3.6.1 SUSY Extension

The extension is done analogously to the model E, obtaining the following values for the \(b_i\) coefficients

\[
\begin{align*}
(b_Y, b_{2L}, b_s) & = (22, 2, -3), \\
(b_X, b_{3L}, b_{3C}) & = (48, 0, 0).
\end{align*}
\]  

The SUSY version is in the 2UP for both breaking schemes. In the ZSBS, \(M_X\) reads

\[M_X = 2.76 \times 10^{13}\text{GeV}.\]  

In this case, the condition \(a^2 \geq 1/3\) imposes a strong bound on the \(M_U\) scale. From

\[2.76 \times 10^{13}\text{GeV} \leq M_U \leq 1.13 \times 10^{15}\text{GeV}\]

we get \(a^2 = [1/3, 1.67]\). Larger values of \(M_U\) leads to non allowed values of \(a^2\) (less than 1/3).

In the XSBS we have \(M_X = 2.05 \times 10^{10}\text{GeV}\). Using \(M_U \in [2 \times 10^{16}, 10^{17}]\ \text{GeV}\) we get \(a^2 \in [1.55, 1.82]\). They satisfy the condition \(a^2 \geq 1/3\). Hence, SUSY versions give a possible scenario of unification in our scheme.

4 Some possible scenarios with groups of grand unification

One of our main assumptions was that there is no necessarily a group of grand unification at the scale in which the coupling constants converge. In this way, the \(a^2\) parameter becomes free, and indeed we could reverse the problem in a certain way, since we have in many cases an allowed region for \(a^2\) and we could ask what groups of grand unification (if any) could lead to values of \(a^2\) admitted by our scheme. Let us elaborate about this possibility.

After working some 331 models with one or three families, we can see that in some of them is possible to find a scale \(M_X\) and a normalization factor \(a^2\) that gives unification of the coupling constants. In the absence of a group of grand unification (GUT) \(a^2\) is not fixed by the group structure. Notwithstanding, UCC imposes some restrictions on \(a^2\), and by using these values we can look for simple groups containing a 331 group and fixing an \(a^2\) belonging to the allowed interval mentioned above. Some good examples of GUT are \(E_6\), and \(SU(7)\).

The group \(E_6\) can be broken into \(SU(3)_C \otimes SU(3)_L \otimes SU(3)_R\) or \(SU(6)_L \otimes SU(2)\). Models A and B shown in this paper could be embedded in this scheme. On the other hand, the 331 models of three families can be embedded into \(SU(7)\), thus, we shall study the latter scenario in more detail.
There are different versions of the \( SU(7) \) GUT that lead to 331 models according to the irreducible representations (irrep) that cancel anomalies, and the definition of the electric charge or equivalently the linear combination of diagonal generators that define the hypercharge (i.e. the normalizing factor \( a^2 \)). The combination of irreducible representations free of anomalies, permits to accomodate those 331 models with three families by the branching rules. The additional singlets or exotic fields acquire masses over the \( M_X \) scale, and do not affect the RGE. There are different combinations of anomaly free irreps. Taking into account the irreps \( \Psi^{(1)}_{(3)}[7] \), \( \Psi^{(3)}_{(2)}[21] \) and \( \Psi^{\alpha\beta\gamma}[35] \), where the subindex means the anomaly coefficient, the bracket coefficient means the dimension and the labels \( \alpha, \beta, \gamma = 1, \cdots, 7 \). Models can be classified if no irrep appears more than once i.e. a form like \( \Psi^{(-1)}_{(1)} \oplus \Psi^{(0)}_{(3)} \oplus \Psi^{(2)}_{(-2)} \); but in general the same irrep can be repeated such as \( 5 \times \Psi^{(-1)}_{(1)} \oplus \Psi^{(3)}_{(2)} \oplus \Psi^{(3)}_{(2)} \). On the other hand, the branching rules according to \( (SU(3)_{c}, SU(2)_{L}) \) are given by

\[
\Psi^{\alpha} \oplus \Psi^{\alpha\beta} \oplus \Psi^{\alpha\beta\gamma} = \left[ [(1^{*},1) + (1,2) + (1,1) + (1,1)]_{\gamma} \oplus [(1^{*},2) + (3,2) + (1,1)]_{21} + (3,1) + (3,1) + (1,2) + (1,2) + (1,1) \right]_{35}
\]

(4.1)

The electromagnetic charge of the particles can be chosen by defining the hypercharge which is a linear combination of the \( U(1) \) factors of the \( SU(7) \), and imposing the condition that the singlet of color have electromagnetic charges \( q = \pm 1/3, \pm 2/3, \pm 4/3, \pm 5/3 \) or that the electromagnetic charge of the leptonic singlets are \( \pm 1, 0 \).

Assuming an unification scheme with the simple group \( SU(7) \) and passing through a 331 model with three families, by one step of symmetry breaking, we get the scheme

\[
SU(7) \xrightarrow{M_{\chi}} SU(3)_{c} \otimes SU(3)_{L} \otimes U(1)_{X} \xrightarrow{M_{X}} SU(3)_{c} \otimes SU(2)_{L} \otimes U(1)_{Y}
\]

(4.2)

the assignment of the electromagnetic charge is of the form

\[
Q = T_{3L} + \alpha Y^{a} + \beta Y^{b} + \gamma Y^{c} \equiv T_{3L} + a \tilde{Y}
\]

\[
a^2 = \alpha^2 + \beta^2 + \gamma^2
\]

where \( Y^{a} \), \( Y^{b} \), \( Y^{c} \) have the same normalization as the \( T_{3L}, \tilde{Y} \) generators, and they correspond to the \( U(1) \) abelian subgroups induced from different subalgebras of \( SU(7) \supset SU(n)_{L} \otimes SU(m)_{C} \otimes U(1)^{a} \), where \( SU(n)_{L} \supset SU(2)_{L} \otimes U(1)^{b} \) and \( SU(m)_{C} \supset SU(3)_{c} \otimes U(1)^{c} \). If the fundamental representation is decomposed as in Eq. (4.3), the most general assignment is

\[
Q = diag(q, q, b, a, a - 1, -1, 1 - 3q - 2a - b).
\]

(4.3)

And we can identify the charge of (3, 1) to be \(-1/3\) and those of (1, 2) as \((1, 0)\), then

\[
Q = diag(-1/3, -1/3, -1/3, b, 1, 0, -b)
\]

(4.4)

where \( b = 0, \pm 1 \) if we have singlet leptons or \( b = \pm 1/3, \pm 2/3, \pm 4/3, \pm 5/3 \) if we have singlet quarks.

From the following possible \( SU(7) \) maximal subalgebras

- **Model I**: \( SU(4)_{C} \otimes SU(3)_{L} \otimes U(1)^{a} \)
- **Model II**: \( SU(3)_{C} \otimes SU(4)_{L} \otimes U(1)^{a} \)
- **Model III**: \( SU(6)_{L} \otimes U(1)^{a} \rightarrow SU(3)_{C} \otimes SU(3)_{L} \otimes U(1)^{c} \otimes U(1)^{a} \)

we can settle three different assignments for the hypercharge according to the scheme in Eqs. (4.3) (4.4). For example, Model I is known in the literature as the Pati Salam model. The generators are defined by

\[
Y^{a} = \sqrt{\frac{6}{7}} \ diag(1/4, 1/4, 1/4, 1/4, -1/3, -1/3, -1/3)
\]

\[
Y^{b} = \sqrt{\frac{1}{3}} \ diag(0, 0, 0, 1/2, 1/2, -1)
\]

\[
Y^{c} = \sqrt{\frac{1}{8}} \ diag(1/3, 1/3, 1/3, -1, 0, 0)
\]

\(^{a}\text{The values 4/3, 5/3 correspond to exotic quark charges that arise in some 331 models, like model E.}\)
Renormalization group in $331$ models

| $b$ | Model I | Model II | Model III |
|-----|---------|----------|-----------|
| $\sqrt{\frac{2}{3}}\alpha$ | $\sqrt{\frac{2}{3}}\beta$ | $\sqrt{\frac{2}{3}}\gamma$ | $\sqrt{\frac{2}{3}}\alpha$ | $\sqrt{\frac{2}{3}}\beta$ | $\sqrt{\frac{2}{3}}\gamma$ | $\sqrt{\frac{1}{3}}\alpha$ | $\sqrt{\frac{1}{3}}\beta$ | $\sqrt{\frac{1}{3}}\gamma$ |
| 0 | $-1$ | $+\frac{1}{2}$ | $-\frac{1}{2}$ | $-1$ | $+\frac{1}{2}$ | $+\frac{1}{2}$ | 0 | $-\frac{1}{2}$ | $+\frac{1}{2}$ |
| 1 | 0 | 1 | $-1$ | 1 | $-\frac{1}{2}$ | $+\frac{1}{2}$ | 1 | $-\frac{1}{2}$ | $+\frac{1}{2}$ |
| $-1$ | $-2$ | $-\frac{1}{2}$ | $\frac{1}{2}$ | $-1$ | $-\frac{1}{2}$ | $+\frac{1}{2}$ | 1 | $-\frac{1}{2}$ | $-\frac{1}{2}$ |
| $+\frac{1}{2}$ | $-\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $-1$ | $+\frac{1}{2}$ | $-\frac{1}{2}$ | $-\frac{1}{2}$ | $\frac{1}{2}$ | $+\frac{1}{2}$ |
| $-\frac{1}{2}$ | $-\frac{1}{2}$ | $\frac{1}{2}$ | $0$ | $-1$ | $+\frac{1}{2}$ | $\frac{1}{2}$ | $-\frac{1}{2}$ | $+\frac{1}{2}$ |
| $+\frac{1}{2}$ | $-\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $-1$ | $+\frac{1}{2}$ | $-\frac{1}{2}$ | $-\frac{1}{2}$ | $\frac{1}{2}$ | $+\frac{1}{2}$ |
| $-\frac{1}{2}$ | $-\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $-1$ | $+\frac{1}{2}$ | $-\frac{1}{2}$ | $-\frac{1}{2}$ | $\frac{1}{2}$ | $+\frac{1}{2}$ |
| $+\frac{1}{2}$ | $-\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $-1$ | $+\frac{1}{2}$ | $-\frac{1}{2}$ | $-\frac{1}{2}$ | $\frac{1}{2}$ | $+\frac{1}{2}$ |
| $-\frac{1}{2}$ | $-\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $-1$ | $+\frac{1}{2}$ | $-\frac{1}{2}$ | $-\frac{1}{2}$ | $\frac{1}{2}$ | $+\frac{1}{2}$ |

Table 1: Hypercharge definition using the embedding of $U(1)$ into subalgebras of $SU(7)$ for models I, II, III

In this case $Y^b$ corresponds to the generator $T_8$ of $SU(3)_L$; and a linear combination of $Y^a$, $Y^c$ corresponds to $U(1)_X$ of the $331$ models. For model II the generators are defined by

$$Y^a = \sqrt{\frac{6}{7}} \text{diag} (1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)$$

$$Y^b = \sqrt{\frac{1}{3}} \text{diag} (0, 0, 0, 0, 1, 2, 1, 2, -1)$$

$$Y^c = \sqrt{\frac{3}{8}} \text{diag} (0, 0, 0, -1, 1, 3, 1, 3, 1, 3) \quad (4.6)$$

And for model III the generators are defined by

$$Y^a = \sqrt{\frac{3}{7}} \text{diag} (1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)$$

$$Y^b = \sqrt{\frac{1}{12}} \text{diag} (1, 1, 1, 0, -1, -1, -1)$$

$$Y^c = \sqrt{\frac{1}{3}} \text{diag} (0, 0, 0, 0, 1, 2, 1, 2, -1) \quad (4.7)$$

By choosing the values for the free parameter $b = 0, \pm 1$ for leptons, and $b = \pm 1/3, \pm 2/3, \pm 4/3, \pm 5/3$ for quarks, the coefficients $\alpha, \beta, \gamma$ that determine the hypercharge can be obtained, and they are displayed in table 1. On the other hand, table 2 shows the $a^2$ normalization factor of the hypercharge according to the $b$ factor.

The next step is to check whether the three family models consider here (models C, D, E, F) can be accommodated properly in a $SU(7)$ GUT. It can be done by comparing the allowed region for $a^2$ in table 3 with the values of $a^2$ in table 2. The second column shows the allowed interval of $a^2$ obtained in previous sections in the absence of a grand unification group, the third column shows the allowed interval by taking into account additional restrictions coming from proton decay ($M_U \gtrsim 2 \times 10^{16}$GeV), that arise when a GUT theory is introduced.

The restrictions displayed in table 3 for non SUSY versions of models C, D, F; show that they can be embedded in $SU(7)$ since the bound $a^2 \geq 1/3$ is accomplished by all the values of $a^2$ in table 2. Model E (the only one with exotic charges studied here) has been ruled out from phenomenological grounds under our scheme. On the other hand, from table 3, we see that the $a^2$ parameter is strongly restricted in the SUSY versions of these models, and not all of them can be accommodated in $SU(7)$ according to table 2.
For our analysis of the RGE, the following conditions are taken into account: of the order of \( M \) scale to arrive at the SM gauge group. We are also assuming that all particles beyond the SM are getting masses different \( 331 \) models of one and three families as well as their supersymmetric extensions.

Specifically, we are supposing that the three different couplings unify at certain scale \( U \) into subalgebras of \( SU(7) \) for models I, II, III.

### Table 2: Hypercharge definition using the embedding of \( U(1) \) into subalgebras of \( SU(7) \) for models I, II, III

| \( b \) | \( a^2 \) |
|---|---|
| \( 0 \) | \( \frac{3}{2} \approx 1.67 \) |
| \( +1 \) | \( \frac{4}{2} \approx 2.11 \) |
| \( +1 \) | \( \frac{6}{2} \approx 3.44 \) |
| \( +2 \) | \( \frac{6}{2} \approx 8.78 \) |
| \( +2 \) | \( \frac{12}{2} \approx 12.78 \) |

Table 3: Allowed interval for the normalization \( a^2 \) for models with three families. In the second column, no restrictions from proton decay are taken into account, while in the third column they are. Models E (SUSY and non SUSY) were discarded from phenomenological grounds.

### 5 Conclusions

We have studied the possibility of having a \( 331 \) model as the only intermediate group between the electroweak scale and the scale \( M_U \) in which the three coupling constants unify. We assume that there is no necessarily a simple gauge group at the \( M_U \) scale. From the analysis of the renormalization group equations (RGE) we examine different \( 331 \) models of one and three families as well as their supersymmetric extensions.

Specifically, we are assuming that the three different couplings unify at certain scale \( M_U \), and a symmetry breaking to a gauge group \( SU(3)_L \otimes U(1)_X \) occurs at a lower scale \( M_X \). Then, a second breaking occurs at the \( M_Z \) scale to arrive at the SM gauge group. We are also assuming that all particles beyond the SM are getting masses of the order of \( M_X \). Other analyses can be done by supposing that the particle contents are in different thresholds. For our analysis of the RGE, the following conditions are taking into account: ❶ The hierarchy condition given by Eq. (2.19) i.e. \( M_X \leq M_U \leq 1 \times 10^{17} \) GeV, should be satisfied. ❷ The condition described by Eq. (2.10) should be satisfied.

### Table 4: Summary of the results obtained in the paper for the six different \( 331 \) models studied, as well as their corresponding SUSY extensions

| Model | No SUSY | NSBS | SBS |
|---|---|---|---|
| A | \([1.63 \times 10^{16}, 2.05 \times 10^{16}]\) | \([1.27 \times 10^6, 2.76 \times 10^{13}]\) | \([1.76 \times 10^{14}, 2.05 \times 10^{16}]\) |
| B | \([1.63 \times 10^{16}, 2.05 \times 10^{16}]\) | No allowed | \([1.76 \times 10^{14}, 2.05 \times 10^{16}]\) |
| C | \([2.05 \times 10^{16}, 3.15 \times 10^{16}]\) | \(= 2.76 \times 10^{13}\) | \(= 2.05 \times 10^{16}\) |
| D | \([2.05 \times 10^{16}, 3.15 \times 10^{16}]\) | \(= 2.76 \times 10^{13}\) | \(= 2.05 \times 10^{16}\) |
| E | No allowed | No allowed | No allowed |
| F | \([2.05 \times 10^{16}, 3.15 \times 10^{16}]\) | \(= 2.76 \times 10^{13}\) | \(= 2.05 \times 10^{16}\) |

\[ \text{6There is another correction coming from the new spectrum introduced, but we have assumed that the new heavy modes do not alter the RGE significantly.} \]
fulfilled, where \( a \) is the normalization parameter of the hypercharge \( Y \), that leads to the unification of the couplings at certain scale. There is no necessarily a grand unification gauge group at the \( M_U \) scale.

In the case of the supersymmetric extensions, the RGE analysis depends on the specific supersymmetric version but also on the supersymmetry breaking scale. In particular, we assume two possible SUSY breaking scenarios: SUSY breaking at electroweak scale and SUSY breaking at the scale of 331 breaking. Although SUSY breaking at electroweak scale is not a realistic scenario, numerical results do not change significantly with respect to the more realistic framework with SUSY breaking at some few TeV's.

Based on the criteria explained above, we can either find an allowed interval for the 331 breaking scale \( M_X \) or rule out the model as a possible grand unified theory under the scheme described above. We summarize the results found in this paper in table [1] for the six different 331 models studied here as well as their SUSY extensions (some SUSY extensions that are ruled out have not been included). In many cases, we see that SUSY versions tend to give lower allowed values for the 331 breaking scales than their non SUSY counterparts, it could make SUSY extensions easier to test from the phenomenological point of view. It worths emphasizing that taking into account that the spectrum out of the SM lies at the \( M_X \) scale, when assuming that the SUSY breaking occurs at the \( M_X \) scale we are in a natural scenario for split Supersymmetry [46]-[47].

On the other hand, we see that models \( C \) and \( D \) predicts the same allowed intervals for \( M_X \) in SUSY and non SUSY versions. However, SUSY versions predict different values for the normalization parameter \( a^2 \).

In addition, by finding the allowed values for \( a^2 \) we can proceed to see what groups of grand unification could give a value of \( a^2 \) lying in the allowed range. Although the introduction of a group at \( M_U \) scale introduces new singlets that could lead to tiny changes in the RGE analysis from \( M_X \) to \( M_U \), this procedure helps us to figure out what scenarios of grand unification (if any) are possible under our scheme. In particular, we find that some 331 models with three families, can be properly embedded in a grand unification scenario with \( SU(7) \), especially in non susy frameworks.

As a matter of perspectives, two loops analysis for RGE can be carried out [18] in order to fit the \( M_U \) scale better, especially taking into account the uncertainty in the starting coupling constants at electroweak scale. However, significant changes in the allowed regions obtained here are not expected for a well behaved perturbative regime. On the other hand, it is possible to unify the coupling constant associated to \( U(1) \) at string scale instead of the \( M_U \) scale [49]-[50].

Finally, it worths saying that the non allowed models are not necessarily ruled out. We only can say that they cannot produce UCC under the scheme in which 331 is the only gauge symmetry between the \( M_U \) and SM scales. For instance, it could be possible that they achieve unification by either introducing new physics, or requiring extra breaking steps from the \( M_U \) scale to SM scale. By introducing new physics, it is possible to get a lower \( M_X \) scale of the order of TeV accessible to the LHC.

6 Acknowledgements

We thank Colciencias, Fundación Banco de la República, and High Energy Latin American European Network for its financial support.

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