THE TRIGONOMETRIC $E_8$ R-MATRIX

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Abstract. An expression for the $R$-matrix associated to $U_q(\hat{e}_8)$ in its 249-dimensional representation is given using the diagrammatic calculus of $U_q(\hat{e}_8)$ invariants.

1. Introduction

Quantized affine algebras are pseudo-triangular Hopf algebras [Dri88]; as such, given any pair of "generic" representations $V_1$ and $V_2$, there exists an intertwiner between $V_1 \otimes V_2$ and $V_2 \otimes V_1$, the so-called $R$-matrix. $R$-matrices are the central object in quantum integrable systems, and for most algebras and many low-dimensional representations, they are known explicitly [KRS81, ZF80, IK81, Baz85, Jim86, Jim89, Ma90, Kun90, DGZ94, Man14]. As far as the author knows, the only case that has not been studied in detail is that of (the quantized affine algebra of) $e_8$. The goal of this short paper is therefore rather modest: it is the explicit computation of the $R$-matrix associated to the quantized loop algebra $U_q(\hat{e}_8)$ in its lowest nontrivial representation, which is of dimension 249 (and is not irreducible for the nonaffine algebra $U_q(e_8)$, which is a source of complication). In order to do so, we develop a natural diagrammatic language for the theory of invariants of $U_q(e_8)$ based on tensors of the adjoint representation, which we then apply to the computation of the $R$-matrix. The latter is fixed (up to normalization) by solving the equations that require it to be an intertwiner [Jim85].

The main challenge of this paper is computational: for instance, the $R$-matrix is a matrix of size $249^2 = 62001$, so altogether an array of approximately 4 billion entries (albeit a sparse array, since only 0.05% of these are nonzero). This means that many computations in this paper cannot be performed by hand, and the help of a symbol computation program is necessary. (The author’s code is available on request.) In particular, quite a few results are presented below without proof; if so, the reader should assume that they are the result of a computer-assisted calculation.

The immediate motivation for this paper came from the work of the author in collaboration with A. Knutson [KZJ17, KZJ20] where the $R$-matrix of $U_q(\hat{e}_8)$ appeared unexpectedly in the computation of structure constants of the $K$-theory of 4-step flag varieties. Another possible source of interest is the fact that the scaling limit of the Ising model at the critical temperature in a magnetic field is known to be related to an $e_8$ integrable field theory [Zam89, BNW94], and a vertex lattice model based on $e_8$ might be desirable.

The paper is organized as follows. §2 contains basic definitions related to the Hopf algebra $U_q(\hat{e}_8)$. §3 presents the diagrammatic calculus for $U_q(e_8)$ invariants. §4 is the core of the paper: the 249-dimensional representation of $U_q(\hat{e}_8)$ is defined, the trigonometric $R$-matrix introduced and its main properties discussed. Finally, appendix A contain some cumbersome diagrammatic identities, appendix B is the full expression of the $R$-matrix, and appendix C discusses the rational limit of the $R$-matrix.

Intentionally, this paper has very concrete formulae which we hope will be useful in future studies of the quantum integrable systems associated to $e_8$.

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2. Preliminaries

2.1. Dynkin diagrams. The Dynkin diagram of $\mathfrak{e}_8$ is

![Dynkin diagram of $\mathfrak{e}_8$]

Its adjacency matrix $A$ uses the index set $I = \{0, 1, \ldots, 8\}$ as on the figure. Let $C = 2 - A$ be the corresponding Cartan matrix.

If one removes the (green) vertex labeled 0 in (1), one obtains the Dynkin diagram of $\mathfrak{e}_8$.

2.2. The $\mathfrak{e}_8$ root system. Let $\Phi = \Phi_+ \cup \Phi_-$ be the root system of $\mathfrak{e}_8$, divided into positive and negative roots; $|\Phi| = 240$. Let $Q$ be the root lattice (generated by $\Phi$). Let $\alpha_1, \ldots, \alpha_8 \in \Phi_+$ be the set of simple roots.

There is a nondegenerate scalar product $\langle \cdot | \cdot \rangle$ on $Q \otimes_\mathbb{Z} \mathbb{R} \cong \mathbb{R}^8$ (inverse of the Killing form on the Cartan subalgebra of $\mathfrak{e}_8$), such that $\Phi$ is exactly the subset of $Q$ of vectors of square length 2. More generally, one has, for $\beta, \gamma \in \Phi$, $\langle \beta | \gamma \rangle \in \{-2, -1, 0, +1, +2\}$ with $\langle \beta | \gamma \rangle = \pm 2$ iff $\beta = \pm \gamma$.

The dual basis of the simple roots $\alpha_i$ is that of fundamental weights $\omega_i$: $\langle \alpha_i | \omega_j \rangle = \delta_{i,j}$.

Define the sequence of integers

$$ (n_0, n_1, \ldots, n_8) = (1, 2, 3, 4, 5, 6, 4, 2, 3) $$

The highest root of $\mathfrak{e}_8$ (highest weight of the adjoint representation, and therefore a fundamental weight) is given by $\omega_1 = \sum_{i=1}^{8} n_i \alpha_i$. Alternatively, define $\alpha_0 = -\omega_1$ to be the lowest root of $\mathfrak{e}_8$; it then satisfies

$$ \sum_{i=0}^{8} n_i \alpha_i = 0 $$

The Cartan matrix of $\mathfrak{e}_8$ encodes the scalar products: $C_{ij} = \langle \alpha_i | \alpha_j \rangle$ for $i, j \in I$ (excluding the value 0).

We need three more definitions.

Given $\beta \in \Phi$, define $ht(\beta)$ to be the sum of its entries in the basis of simple roots, that is, $ht(\beta) = \langle \rho | \beta \rangle$ where $\rho = \sum_{i=1}^{8} \omega_i = \frac{1}{2} \sum_{\beta \in \Phi_+} \beta$.

Denote $\Phi_0^3$ the set of triples $(\beta, \gamma, -\beta - \gamma) \in \Phi^3$ that sum to zero. Define the binary relation $\rightarrow$ on $\Phi_0^3$ by $(\beta, \gamma, -\beta - \gamma) \rightarrow (\beta + \alpha_i, \gamma - \alpha_i, -\beta - \gamma)$, $(\beta, \gamma, -\beta - \gamma) \rightarrow (\beta, \gamma + \alpha_i, -\beta - \gamma - \alpha_i)$ and $(\beta, \gamma, -\beta - \gamma) \rightarrow (\beta + \alpha_i, \gamma - \beta - \gamma - \alpha_i)$ for $i = 1, \ldots, 8$. Given $x \in \Phi_0^3$, write $\ell(x)$ for the maximum of lengths $k$ of chains $x \rightarrow x_1 \rightarrow \cdots \rightarrow x_k$ starting from $x$.

Finally, define a map $\epsilon : \Phi^3 \rightarrow \{-1, +1\}$ by

$$ \epsilon(\beta, \gamma) = \prod_{1 \leq i < j \leq 8} (-1)^{\langle \alpha_i | \beta \rangle \langle \alpha_j | \gamma \rangle} $$

Note that it satisfies

$$ \epsilon(\beta, \gamma) \epsilon(\gamma, \beta) = (-1)^{\langle \beta | \gamma \rangle} $$

2.3. The quantized affine algebra $U_q(\hat{\mathfrak{e}}_8)$. In all that follows, $q$ is a “generic” nonzero complex number, i.e., not a root of unity.

Introduce the notation for $q$-numbers

$$ [n] = \frac{q^n - q^{-n}}{q - q^{-1}} $$
The quantized affine algebra $\mathcal{U}_q(\mathfrak{e}_8)$ is given by generators $\{e_i, f_i, k_i^\pm, i \in I\}$ and relations (see e.g. [CP94, Chap. 12])

\[
\begin{align*}
&k_i k_j = k_j k_i \\
&k_i e_j k_i^{-1} = q^{C_{ij}} e_j \\
&e_i e_j &= e_j e_i + \frac{1}{2} \xi_{ij} \delta_{ij} - \frac{1}{8} \sum_{k \neq i, j} (3) \delta_{ik} \delta_{jk} k_k^{-1} - \frac{1}{q - q^{-1}} q^{C_{ij}} f_j
\end{align*}
\]

for all $i, j \in I$.

The coproduct, antipode and counit are defined by

\[
\begin{align*}
\Delta(e_i) &= e_i \otimes 1 + k_i \otimes e_i & S(e_i) &= -k_i^{-1} e_i & \varepsilon(e_i) &= 0 \\
\Delta(f_i) &= f_i \otimes k_i^{-1} + 1 \otimes f_i & S(f_i) &= -f_i k_i & \varepsilon(f_i) &= 0 \\
\Delta(k_i) &= k_i \otimes k_i & S(k_i) &= k_i^{-1} & \varepsilon(k_i) &= 1
\end{align*}
\]

The Cartan matrix being degenerate has two consequences. Firstly, there is a central element $q^{\sum_i n_i}$ where the integers $n_i$ were defined in [2]. In all representations that we consider, this element is sent to 1.

Secondly, there is a nontrivial gradation on $\mathcal{U}_q(\mathfrak{e}_8)$: we choose $e_0$ to have degree +1, $f_0$ degree -1 and all other generators degree 0.

If one restricts the index set of the generators to $\{1, \ldots, 8\}$, one obtains the quantized algebra $\mathcal{U}_q(\mathfrak{e}_8)$; it is naturally a (horizontal, or degree 0) sub-Hopf algebra of $\mathcal{U}_q(\mathfrak{e}_8)$.

3. $\mathcal{U}_q(\mathfrak{e}_8)$ invariants

We first study the non-affine algebra $\mathcal{U}_q(\mathfrak{e}_8)$. We shall be interested in two representations of it: the trivial representation, which is simply given by the counit in [3], and the 248-dimensional representation (a deformation of the adjoint representation of $\mathfrak{e}_8$), which we discuss now.

3.1. The 248-dimensional representation. The definition is a straightforward $q$-deformation of the adjoint action of $\mathfrak{e}_8$. $V := V_{\omega_1}$ is a 248-dimensional vector space with basis $v_\beta$, $\beta \in \Phi$ and $u_j$, $j = 1, \ldots, 8$. The action of $\mathcal{U}_q(\mathfrak{e}_8)$ on it is given by

\[
\begin{align*}
k_i v_\beta &= q^{\langle \alpha_i, \beta \rangle} v_\beta \\
e_i v_\beta &= \begin{cases} 
\varepsilon(\alpha_i, \beta) v_{\beta + \alpha_i} & \beta + \alpha_i \in \Phi \\
u_i & \beta + \alpha_i = 0 \\
0 & \text{else}
\end{cases} \\
f_i v_\beta &= \begin{cases} 
\varepsilon(-\alpha_i, \beta) v_{\beta - \alpha_i} & \beta - \alpha_i \in \Phi \\
u_i & \beta - \alpha_i = 0 \\
0 & \text{else}
\end{cases}
\end{align*}
\]

for $i = 1, \ldots, 8$.

Because the representation ring of $\mathcal{U}_q(\mathfrak{e}_8)$ (for generic $q$) is isomorphic to that of $\mathfrak{e}_8$, various results are known. For example, the adjoint representation of $\mathfrak{e}_8$ has an invariant bilinear form (the Killing form) and an invariant trilinear form (built out of the bracket). The same is true of $V$:

**Lemma 1.** The following bilinear form $B$ is an $\mathcal{U}_q(\mathfrak{e}_8)$ invariant:

\[
\begin{align*}
B(v_\beta \otimes v_\gamma) &= \varepsilon(\beta, \gamma) q^{1 - \varepsilon(\beta, \gamma)} \\
B(u_i \otimes u_j) &= [C_{ij}]
\end{align*}
\]

all other entries being zero (as implied by their nonzero weight).
Proof. The $k_i$ invariance is trivial. Let us check the $e_i$ invariance (the $f_i$ invariance works similarly, and is in fact implied by $k_i, e_i$ invariance in finite dimension). There are three nontrivial cases to consider: either one starts with $v_\beta \otimes v_\gamma$, in which case for its image under $Be_i$ to be nonzero one must have $\gamma = -\beta - \alpha_i$; then

\[
B(e_i(v_\beta \otimes v_{-\beta - \alpha_i})) = B(e_i(v_\beta v_{\alpha_i} \otimes v_{-\beta - \alpha_i}) + q^{\alpha_i | \beta} e_i(v_\beta - \beta - \alpha_i)v_\beta \otimes v_\beta) = e_i(v_\beta) e_i(v_\beta + \alpha_i, \beta + \alpha_i)q^{1 - \text{ht}(\beta + \alpha_i)} + q^{\alpha_i | \beta} e_i(v_\beta - \beta - \alpha_i)e_i(\beta, \beta)q^{1 - \text{ht}(\beta)} = e_i(v_\beta) e_i(\beta, \beta)q^{-\text{ht}(\beta)}(1 - q^{1 + \alpha_i | \beta}) = e_i(v_\beta) e_i(\beta, \beta)q^{-\text{ht}(\beta)}(1 - q^{1 + \frac{1}{2}(|\beta + \alpha_i|^2 - |\beta|^2 - |\alpha_i|^2)}) = 0
\]

where in the last line we used the fact that all roots of $\Phi$ have square length 2.

Similarly, the other two cases are

\[
B(e_i(u_j \otimes v_{-\alpha_i})) = B([-C_{ij}v_{\alpha_i} \otimes v_{-\alpha_i}) + u_j \otimes u_i) = -[C_{ij}]q^{1 - \text{ht}(\alpha_i)} + [C_{ij}] = 0
\]

and

\[
B(e_i(v_{-\alpha_i} \otimes u_j)) = B(u_i \otimes u_j - q^{+\alpha_i | \alpha_i})v_{-\alpha_i} \otimes [C_{ij}]v_{\alpha_i}) = [C_{ij}] - q^{-2}q^{1 - \text{ht}(\alpha_i)}[C_{ij}] = 0
\]

The trilinear form is more complicated to define:

Lemma 2. Define $T \in V^* \otimes V^* \otimes V^*$ as follows.

Given a triple $(\beta, \gamma, -\beta - \gamma) \in \Phi^3_0$, write

\[
T(v_\beta \otimes v_\gamma \otimes v_{-\beta - \gamma}) = \epsilon(\beta, \gamma)e(\gamma, \beta)q^{41 + \epsilon(\beta, \gamma, -\beta - \gamma)} \quad \beta \in \Phi_+, \quad -\beta - \gamma \in \Phi_-
\]

and then use

\[
T(v_\beta \otimes v_\gamma \otimes v_\alpha) = T(v_\alpha \otimes v_\beta \otimes v_\gamma)q^{2\text{ht}(\alpha)}
\]

to reduce to the case above.

Also write

\[
\begin{align*}
T(u_i \otimes v_\alpha_i \otimes v_{-\alpha_i}) &= q \\
T(u_i \otimes v_{-\alpha_i} \otimes v_\alpha_i) &= -q \\
T(v_\alpha_i \otimes u_i \otimes v_{-\alpha_i}) &= -q^{-1} \\
T(u_i \otimes v_{-\alpha_i} \otimes u_i) &= q \\
T(v_{-\alpha_i} \otimes u_i \otimes v_\alpha_i) &= q^3 \\
T(v_{-\alpha_i} \otimes v_\alpha_i \otimes u_i) &= -q
\end{align*}
\]

and

\[
T(u_i \otimes u_j \otimes u_k) = (q - q^{-1}) \begin{cases} 
2(q^d(i) + q^{-d(i)}) & i = j = k \\
-[d(k)] & i = j \leftrightarrow k \text{ or permutations} \\
[d(k)] & i = j \rightarrow k \text{ or permutations}
\end{cases}
\]

where $i, j, k$ are viewed as nodes of the Dynkin diagram, $d(i)$ is the distance to the central node 5, and the Dynkin diagram is oriented away from the central node.

All other entries of $T$ are zero.

Then $T$ is an $U_q(t_8)$ invariant.
The proof is similar to that of the previous lemma, and is skipped.

More generally, the decomposition of $V^\otimes 2$ is known: one has

$$V_{\omega_1} \otimes V_{\omega_1} \cong \bigoplus_{\omega \in \Omega} V_\omega \quad \Omega = \{0, \omega_1, \omega_7, 2\omega_1, \omega_2\}$$

where $V_\omega$ is the irreducible representation of $U_q(e_8)$ of highest weight $\omega$; it is also the unique irreducible representation of dimension 1, 248, 3875, 27000, 30380 for $\omega = 0, \omega_1, \omega_7, 2\omega_1, \omega_2$, respectively.

This implies that $\text{End}(V^\otimes 2) U_q(e_8)$, the space of $U_q(e_8)$ intertwiners of endormorphisms of $V^\otimes 2$, is of dimension 5. Out of the invariants above we can build a basis for this space. It is convenient to do so using a diagrammatic language.

3.2. **Diagrammatic depiction of invariants.** We use here the graphical calculus which is standard in mathematical physics, in the particular form which is adapted to quantum groups [Kas95], namely our diagrams are planar. See also Cvitanovic [Cvi08] for a discussion of the non-$q$-deformed case (for all simple Lie algebras), so without the planarity requirement.

In general, to a graph embedded in the plane of the form

![Diagram](image)

we will associate an $U_q(e_8)$ intertwiner from $V^\otimes m$ to $V^\otimes n$, where tensor products are read from left to right. As usual, vertical concatenation corresponds to multiplication, whereas horizontal juxtaposition corresponds to tensor product.

The simplest graph is the identity from $V$ to $V$, which is

![Identity](image)

Next is the bilinear form $B$, which we draw as a cup:

![Bilinear Form](image)

Because it is nondegenerate, it possesses an inverse, drawn as a cap:

![Inverse](image)

with the identities

![Identity Diagram](image)
Gluing together cup and cap in the opposite way, we obtain our first nontrivial identity:

\[(6)\]

\[
\frac{[20][24][31]}{[6][10]} = \frac{[20][24][31]}{[6][10]}
\]

(we shall not prove this identity, or any that follow, but mention in passing that this is simply the principal specialization of the character of the adjoint representation of $\mathfrak{e}_8$).

Similarly, the trilinear form $T$ can be drawn as

Combining it with caps, we can get three more intertwiners, namely

In principle, there are several ways to glue caps to $T$ to produce these diagrams, but by Schur’s lemma and normalization condition these all produce the same result.

One also has two nontrivial identities:

\[(7)\]

\[
\frac{[10][15]^2[18][32]}{[5][9][16][30]} = \frac{[10][15]^2[18][32]}{[5][9][16][30]}
\]

\[(8)\]

\[
\frac{[6][10]^2[15][32]}{[2][5][30]} \left( \frac{[32]}{[3][16]} + \frac{[36]}{[9][12]} \right)
\]

At this stage we have all we need to produce a basis of intertwiners of endomorphisms of $V^{\otimes 2}$; we propose

It is a consequence of what follows that these intertwiners are indeed linearly independent.

Now let us work out the multiplication table in this basis. Note that the last basis element is simply the square of the fourth. Thanks to (6)–(8), we can compute all products except the product of the last two (which itself allows to compute the square of the last). This identity is not

\[\text{(Note that these identities do depend on the choice of normalization of $B$ and $T$, which is determined by imposing that it agree with the standard $\mathfrak{e}_8$ convention at $q = 1$, that it be the simplest possible (entries are coprime polynomials) and that constants occurring in the identities be palindromic in } q \text{ (which fixes the remaining freedom of multiplying by a power of } q).\]
particularly nice, and is given in Appendix A for the sake of completeness. We present in the next section a more pleasant alternative.

Before we proceed, let us point out that the trivial representation, being the same as $V^0$, is diagrammatically... invisible. However, in what follows it is sometimes useful to emphasize that in a tensor product some trivial factors are included, in which case we draw an empty vertex; e.g.,

is the natural map from $V \otimes \mathbb{C}$ to $\mathbb{C} \otimes V$.

3.3. The spectral parameter-independent $R$-matrix. As mentioned before, our diagrams are required to be planar. Let us informally discuss this point. In the undeformed case of $e_8$, it is natural to consider arbitrary diagrams (i.e., not embedded into the plane); however, one can always turn them into planar diagrams by projecting them onto the plane, thus creating new “virtual crossings” where two formerly nonintersecting lines are now on top of each other. These crossings simply correspond to the natural permutation of tensors $x \otimes y \mapsto y \otimes x$, which commutes with the $e_8$ action. The point is of course that such a permutation does not commute with the $U_q(e_8)$ action; instead one must use the $R$-matrix (of $U_q(e_8)$).

With a bit of foresight, we therefore define

\begin{equation}
(9) = f(q) + f(q^{-1}) + g(q) + g(q^{-1}) + h(q)
\end{equation}

where

\begin{align*}
f(q) &= \frac{[6][10][15]^2}{[3][5][30]^2} (q^{-24} - q^{-22} - q^{-18} + q^{-16} - q^{-14} + q^{-8} - 1 + q^6 - q^{12}) \\
g(q) &= \frac{[15]}{[3][30]} (-q^{-16} - q^{-14} - q^{-12} - q^{-8} + q^{-6} + q^{-2} - 1 + q^2 + q^6 - q^8 - q^{14} - q^{16} - q^{18}) \\
h(q) &= h(q^{-1}) = \frac{[5]}{6[10]}
\end{align*}

Rather than deriving this ($U_q(e_8)$, i.e., spectral parameter-independent) $R$-matrix now, we postpone its justification to §4.3, where the full ($U_q(\hat{e}_8)$, i.e., spectral parameter-dependent) $R$-matrix will be obtained. One can also define a similar diagram where overcrossing and undercrossing are switched, by a simple 90 degree rotation; equivalently, it corresponds to switching $q \leftrightarrow q^{-1}$ in the coefficients of the right hand side.
These two diagrams are inverses of each other so that one has the relations

\[ \begin{align*}
  & = \\
\end{align*} \]

which one can recognize as Reidemeister move II. We also mention for the sake of completeness the Reidemeister move III (or braid relation)

\[ = \]

We can now use the alternative basis of \( \text{End}(V^{\otimes 2})^{U_q(e_8)} \):

In order to complete the multiplication table, we need the following additional relations:

\[ \begin{align*}
  & = q^{60} \\
\end{align*} \]

which is nothing but Reidemeister move I, as well as

\[ \begin{align*}
  & = -q^{30} \\
\end{align*} \]

\[ \begin{align*}
  & = -q^{-30} \\
\end{align*} \]

\[ \begin{align*}
  & = -q^{30} \\
\end{align*} \]

\[ \begin{align*}
  & = -q^{-30} \\
\end{align*} \]
and

\[ \begin{align*}
=q^{-2} - q^{10} + q^{12} + q^{31}(q - q^{-1})^2 \frac{[6][10][60]}{[20][30]} \\
+ q^{18}(q - q^{-1}) \frac{[5][30]}{[10][15]} + (q^{-2} - q^{10} + q^{12}) - q^{10}
\end{align*} \]

(and the same identity with undercrossings ↔ overcrossings, \( q \leftrightarrow q^{-1} \)).

### 3.4. Primitive idempotents.

Because \( V^\otimes 2 \) is multiplicity-free, \( \text{End}(V^\otimes 2)^{U_q(e_8)} \) is a commutative algebra, and multiplication operators can be diagonalized simultaneously, with eigenvectors the primitive idempotents (projectors \( P_\omega \) onto irreducible subrepresentations \( V_\omega \)). The eigenvalues give the coefficients of the expansion:

\[ \begin{align*}
= P_0 + P_{\omega_1} + P_{\omega_7} + P_{2\omega_1} + P_{\omega_2} \\
= \frac{[20][24][31]}{[6][10]} P_0 \\
= \frac{[10][15]^2[18][32]}{[5][9][16][30]} P_{\omega_1} \\
= q^{60} P_0 - q^{30} P_{\omega_1} + q^{12} P_{\omega_7} + q^{-2} P_{2\omega_1} - P_{\omega_2}
\end{align*} \]

and the expansion of the inverse crossing is obtained by inverting all coefficients, i.e., \( q \leftrightarrow q^{-1} \) as usual.

The multiplication rule for the diagrams is equivalent to the statement

\[ P_\omega P_{\omega'} = \delta_{\omega,\omega'} P_\omega \quad \omega, \omega' \in \Omega \]
We also mention the expansion:

\[
\begin{split}
&= \frac{[10][15][18][32]}{[5][9][16][30]} P_0 
+ \frac{[10]^2[15]}{[30]} (q - q^{-1})^2 \left[ \frac{[7][12]}{4} + \frac{[6]^2}{[2][3][5]} \right] P_{w_1} \\
&+ \frac{[6][10][15][32]}{[5][16][30]} P_{w_2} 
- \frac{[10][15][18]}{[5][9][30]} P_{2w_1} 
+ \frac{(q - q^{-1})^2[6][10][15]}{[30]} P_{w_2}
\end{split}
\]

and the expansion of the “square” diagram is obtained by squaring all the coefficients above.

4. \( \mathcal{U}_q(\hat{c}_8) \) and its \( R \)-Matrix

4.1. The \((248 + 1)\)-dimensional representation. In this section we present explicit formulae for the one-parameter family of representations \( W_z \) of dimension 249 of \( \mathcal{U}_q(\hat{c}_8) \). Note that the parameter \( z \in \mathbb{C}^\times \), called “spectral parameter” comes automatically from the \( \mathbb{Z} \)-grading of the algebra.

The action of \( \mathcal{U}_q(\hat{c}_8) \subset \mathcal{U}_q(\hat{c}_8) \) on \( W_z \) is as follows: it is the direct sum of the 248-dimensional representation \( V \) defined in [5][1] and of the trivial representation. We use the same basis \( \{v_\beta, \beta \in \Phi \} \cup \{u_i, i = 1, \ldots, 8\} \) for the former and fix some nonzero vector \( w \) in the latter.

The action of the remaining generators of \( \mathcal{U}_q(\hat{c}_8) \) on \( W_1 \) is given by

\[
\begin{align*}
 k_0 v_\beta &= q^{(\alpha_0, \beta)} v_\beta \\
 e_0 v_\beta &= \left\{
\begin{array}{ll}
\epsilon(\alpha_0, \beta) v_{\beta + \alpha_0} & \beta + \alpha_0 \in \Phi \\
-\epsilon_0 \kappa^{-1} w - \sum_{i=1}^8 c_i u_i & \beta + \alpha_0 = 0
\end{array}\right.
\quad \quad e_0 u_j = -[C_{0j}] v_{\alpha_0} \\
 f_0 v_\beta &= \left\{
\begin{array}{ll}
\epsilon(-\alpha_0, \beta) v_{\beta - \alpha_0} & \beta - \alpha_0 \in \Phi \\
c_0 \kappa^{-1} w + \sum_{i=1}^8 c_i u_i & \beta - \alpha_0 = 0
\end{array}\right.
\quad \quad f_0 u_j = [C_{0j}] v_{-\alpha_0} \\
f_0 w &= -\kappa v_{-\alpha_0}
\end{align*}
\]

where the \( c_i \) are constants to be determined, and \( \kappa \) is a parameter related to the freedom in the relative normalization of \( w \) and of the basis of \( V \), which we do not fix for now.

Imposing the relations of the algebra is equivalent to the following system of equations:

\[
\sum_{j=1}^8 [C_{ij}] c_j = \delta_{i,1} \quad i = 1, \ldots, 8
\]

\[
c_0 + c_1 = [2]
\]

which can be readily solved:

\[
c_i \left( 1 \right) \left[ \frac{[6][10][15]}{[2][3][5][30]} \frac{[2][3][18]}{[6][9]} \frac{[2][3][12]}{[4][6]} \frac{[2][8]}{[4]} \frac{[5][2][3][2]}{[2][2][3]} \frac{[2]}{[3]} \right] \quad i = 1, \ldots, 8
\]

\[
c_0 = (q - q^{-1})^2 \left[ \frac{[6][10][15]}{[30]} \right]
\]

We then restore the \( z \)-dependence to obtain the general action on \( W_z \) by transforming generators according to their degree: \( e_0 \mapsto z e_0, f_0 \mapsto z^{-1} f_0 \), all others unchanged.

4.2. The spectral parameter-dependent \( R \)-matrix. Because \( \mathcal{U}_q(\hat{c}_8) \) is pseudotriangular, we know that for generic \( z, z' \in \mathbb{C}^\times \), \( W_z \otimes W_{z'} \) and \( W_{z'} \otimes W_z \) are isomorphic. Up to normalization, the (unique) intertwiner is given by the universal \( R \)-matrix in these representations; since the universal \( R \)-matrix is of degree 0, this intertwiner only depends on \( z'/z \).
Here we follow Jimbo’s approach [Jim85] which is to directly compute the intertwiner from the intertwining relations. The normalization that we obtain may be different from the one coming from the universal $R$-matrix.

We thus look for an operator $\hat{R}(z'/z) : W_z \otimes W_{z'} \to W_{z'} \otimes W_z$ that satisfies

$$\hat{R}(z'/z)(\rho_z \otimes \rho_{z'})(\Delta(x)) = (\rho_{z'} \otimes \rho_z)(\Delta(x))\hat{R}(z'/z)$$

for all $x \in \mathcal{U}_q(\hat{e}_8)$, where in this equation only, we write explicitly the representation map $\rho_z : \mathcal{U}_q(\hat{e}_8) \to \text{End}(W_z)$.

First we consider the commutation with $x \in \mathcal{U}_q(\hat{e}_8)$ only. We know that $W_z \cong V_{\omega_1} \oplus V_0$ as a $\mathcal{U}_q(\hat{e}_8)$ representation; using (5), we obtain

$$W_z \otimes W_{z'} \cong \mathbb{C}^2 \otimes V_0 \oplus \mathbb{C}^3 \oplus V_{\omega_1} \oplus V_{\omega_2} \oplus V_{\omega_7} \oplus V_{2\omega_1} \oplus V_{\omega_2}$$

This means that the space of $\mathcal{U}_q(\hat{e}_8)$ intertwiners from $W_z \otimes W_{z'}$ to $W_{z'} \otimes W_z$ is of dimension $2^2 + 3^2 + 1^2 + 1^2 + 1^2 = 16$. Using the diagrammatic language of §3.2 it is easy to produce a basis of this space, namely

Recall that filled (resp. empty) vertices correspond to $V_{\omega_1} = V$ (resp. $V_0 = \mathbb{C}$), so that when one concatenates diagrams, if the type of the vertices does not match, the resulting product is zero.

$\hat{R}(z)$ must therefore be a linear combination of these 16 intertwiners. Imposing (10) for the remaining generators $e_0, f_0, k_0$ of $\mathcal{U}_q(\hat{e}_8)$ results in a very large number of linear equations on the entries of $\hat{R}(z'/z)$. For practical purposes, since we know the system has a nontrivial solution, it is sufficient to extract a subset of 15 independent equations among these and then solve them. Such a subset can be found for any $z, z'$ (as opposed to, for generic $z, z'$), implying uniqueness of the intertwiner. Finding and solving these equations is best performed by computer, and we present the explicit solution in Appendix B, where the chosen normalization is

$$\hat{R}(z)v_{\omega_1} \otimes v_{\omega_1} = v_{\omega_1} \otimes v_{\omega_1}$$

(i.e., the highest-weight-to-highest-weight entry is 1).

\[\text{Note in particular that the “multiplicity-free” property that is crucial in e.g. [DGZ94] fails here.}\]
We now discuss properties of $\hat{R}(z)$. These properties should be checked using the diagrammatic algebra itself (which is only of dimension 16, so $16 \times 16$ matrices in the regular representation), rather than using the actual operators acting on copies of $W_z$. It is sometimes useful to represent graphically the $R$-matrix as

\[
\hat{R}(z'/z) = \begin{array}{c}
\end{array}
\]

where half-filled vertices correspond to the whole of $W_z \cong V_{\omega_1} \oplus V_0$, and correspondingly, dashed lines to identity operators on $W_z$, namely

\[
\begin{array}{c}
\end{array}
\]

4.3. **Basic properties of the $R$-matrix.** First, from Schur’s lemma one knows that $\hat{R}(z)\hat{R}(z^{-1})$ must be proportional to the identity. With the chosen normalization (11), one in fact has the so-called unitarity equation

\[
\begin{array}{c}
\end{array}
\]

As a check of the methods of this paper, the reader is invited to verify this equation by direct computation.

Similarly, one can check the Yang–Baxter equation

\[
\begin{array}{c}
\end{array}
\]

Next we come to the so-called crossing symmetry. Consider the 90 degree counterclockwise rotation of the $R$-matrix. Then we expect it to be related to the original $R$-matrix up to shift of the spectral parameter. The 90 degree rotation simply permutes our basis of $U_q(e_8)$ intertwiners, so it easy to compute the result. More specifically, because the dual representation of $W_z$ satisfies
$W^*_{z} \cong W_{q^{30}z}$, we expect the two diagrams below to be proportional:

For this to happen, it is clear from the expression of the $R$-matrix in appendix [13] that one must define dashed cups and caps as follows:

$$\begin{align*}
(14) & \quad = = + \frac{(q - q^{-1})^2 [6][10][15]}{[30] \kappa^2} \\
(15) & \quad = = + \frac{(q - q^{-1})^2 [6][10][15]}{[30] \kappa^2}
\end{align*}$$

Of course, it is very natural at this stage to set the free parameter $\kappa$ to the value

$$\kappa = (q - q^{-1}) \left( \frac{[6][10][15]}{[30]} \right)^{1/2}$$

and we do so in the remainder of this paper. The exact crossing relation is

$$\begin{align*}
(17) & \quad = q^4 \frac{(1 - z)(1 - q^{10}z)(1 - q^{18}z)(1 - q^{28}z)}{(1 - q^2z)(1 - q^{12}z)(1 - q^{20}z)(1 - q^{30}z)}
\end{align*}$$

Note that the prefactor can be absorbed in the redefinition

$$\hat{R}_{\text{poly}}(z) = z^{-2}(1 - q^2z)(1 - q^{12}z)(1 - q^{20}z)(1 - q^{30}z)\hat{R}(z)$$

(which turns entries into Laurent polynomials of $z$), but then it is the unitarity equation [12] which acquires a prefactor $^3$

There are three special values of the spectral parameter. First, at $z = 1$, one has $\hat{R}(1) = 1$. The other two values are 0 and $\infty$:

$$\begin{align*}
\hat{R}(0) = q^2 & \begin{pmatrix}
+ & + & + & + \\
+ & + & + & + \\
+ & + & + & + \\
+ & + & + & + 
\end{pmatrix} \\
\hat{R}(\infty) = q^{-2} & \begin{pmatrix}
+ & + & + & + \\
+ & + & + & + \\
+ & + & + & + \\
+ & + & + & + 
\end{pmatrix}
\end{align*}$$

where the crossing is defined in [9]. This gives an a posteriori justification for the introduction of the crossing in [3.3].

\[^3\text{Requiring both unitarity and crossing relations without a prefactor would take us beyond the realm of rational functions, and we do not pursue this here.}\]
At this stage it is natural to switch to the alternative basis which involves these crossings, and the expression of the $R$-matrix simplifies. We find, using (16) to further simplify,

\[
\hat{R}(z) = q^3(q - q^{-1})^2 \left[ 6 \right] \frac{z}{(1 - q^2 z)(1 - q^{12} z)} + q^3 \kappa^2 \left[ 5 \right] \frac{z(1 - z)}{(1 - q^2 z)(1 - q^{12} z)(1 - q^{20} z)(1 - q^{30} z)}
\]

\[+ q^7 \frac{1 - z}{(1 - q^2 z)(1 - q^{12} z)} \left( \begin{array}{cc} q^{-5} & -q^{5} z \\ \end{array} \right) \]

\[+ q^2(q - q^{-1}) \kappa^2 \frac{z(1 + q^{30} z)}{(1 - q^2 z)(1 - q^{12} z)(1 - q^{20} z)} \left( \begin{array}{c} \circ \\ \end{array} \right) + \left( \begin{array}{c} \circ \\ \end{array} \right)
\]

\[+ \left( \begin{array}{c} \circ \\ \end{array} \right) + q^7 \frac{z - z}{1 - q^2 z} \left( \begin{array}{cc} \circ & \circ \\ \end{array} \right) + \left( \begin{array}{cc} \circ & \circ \\ \end{array} \right)
\]

\[+ q^{17}(q - q^{-1}) \kappa \frac{z(1 - z)}{(1 - q^2 z)(1 - q^{12} z)(1 - q^{20} z)} \left( \begin{array}{cc} \circ & \circ \\ \end{array} \right) + \left( \begin{array}{cc} \circ & \circ \\ \end{array} \right) + \left( \begin{array}{cc} \circ & \circ \\ \end{array} \right) + \left( \begin{array}{cc} \circ & \circ \\ \end{array} \right)
\]

\[+ q^{32}(q - q^{-1}) \kappa^2 \frac{z(1 - z)}{(1 - q^2 z)(1 - q^{12} z)(1 - q^{20} z)(1 - q^{30} z)} \left( \begin{array}{cc} \circ & \circ \\ \end{array} \right) + \left( \begin{array}{cc} \circ & \circ \\ \end{array} \right)
\]

\[+ \left( \begin{array}{c} \circ \\ \end{array} \right) + \left( \begin{array}{c} \circ \\ \end{array} \right) + \left( \begin{array}{c} \circ \\ \end{array} \right) + \left( \begin{array}{c} \circ \\ \end{array} \right)
\]

\[+ q^{17}(q - q^{-1})^2 \left[ 6 \right] \left[ 15 \right] \frac{z(1 - z)}{(1 - q^2 z)(1 - q^{12} z)(1 - q^{20} z)}
\]

where we have reused certain coefficients by denoting them $\text{cf}[:].$
4.4. Fusion channels. In the three multiplicity-free channels, one can directly rewrite the \( R \)-matrix in terms of the idempotents:

\[
\hat{R}(z) = P_{\omega_1} + \frac{z - q^2}{1 - q^2 z} P_{\omega_2} + \frac{(z - q^2)(z - q^{12})}{(1 - q^2 z)(1 - q^{12} z)} P_{\omega_7} + \frac{1}{(1 - q^2 z)(1 - q^{12} z)(1 - q^{20} z)} \hat{R}_{\omega_1}(z) + \frac{1}{(1 - q^2 z)(1 - q^{12} z)(1 - q^{20} z)(1 - q^{30} z)} \hat{R}_0(z)
\]

where the last two terms are the restrictions of \( \hat{R}(z) \) to \( \mathbb{C}^3 \otimes V_{\omega_1} \) and \( \mathbb{C}^2 \otimes V_0 \), respectively, in which we have taken out the LCM of denominators (and the GCD of numerators is 1). Diagonalizing \( \hat{R}(z) \) in these remaining blocks results in an eigenvalue \( \frac{z^2 q^2}{1 - q^2 z} \) in the \( V_{\omega_1} \) block, and then pairs of eigenvalues that are solutions of quadratic equations in each block.

As a function of the spectral parameter \( z \), \( \hat{R}(z) \) is a rational function with poles at \( z = q^{-2}, q^{-12}, q^{-20}, q^{-30} \). Conversely, due to unitarity equation (12), at \( z = q^2, q^{12}, q^{20}, q^{30} \), \( \hat{R}(z) \) is noninvertible. More precisely, define

\[
f_a = \text{Res}_{z=q^{-a}} \hat{R}(z) \\
g_a = \hat{R}(z = q^a)
\]

Then from (12) we have \( f_a g_a = g_a f_a = 0 \), and in fact, expanding this equation around \( z = q^a \), diagonalizing \( \hat{R}(z) \) and noting that each zero eigenvalue of \( \hat{R}(z) \) at \( z = q^a \) corresponds to a pole eigenvalue of \( \hat{R}(z) \) at \( z = q^{-a} \), we conclude that their ranks are complementary, so that \( \text{Im}(f_a) = \text{Ker}(g_a) =: X_a \) and \( \text{Im}(g_a) = \text{Ker}(f_a) =: Y_a \). \( X_a \) is a submodule of \( W_z \otimes W_{q^{a}z} \) with quotient isomorphic to \( Y_a \), and vice versa for \( W_{q^a z} \otimes W_z \). Because \( \hat{R}(z')/z \) is the only intertwiner from \( W_{z} \otimes W_{z'} \) to \( W_{z'} \otimes W_{z} \) even at \( z'/z = q^{30} \), \( W_z \otimes W_{q^{30}z} \) is indecomposable.

If \( a = 30 \), \( X_{30} \) is of dimension 1: we recognize it as the trivial representation of \( U_q(\hat{e}_8) \). As an exercise, let us recover this by looking at possible embeddings of the trivial representation into \( W_z \otimes W_{z'} \). From the \( U_q(e_8) \) invariance, we know that this embedding must take the form

\[
\begin{aligned}
\alpha & \quad \circ \quad + \beta \quad \circ
\end{aligned}
\]

Impose the invariance under \( e_0 \) results in the relations

\[
z' = q^{30} z \quad \alpha = \beta
\]

which is consistent with the above. Of course, we recognize that (19) is (up to normalization) the dashed cap (14), and its invariance is once again nothing but the statement that \( W_z \cong W_{q^{30}z} \).

Dually, the invariance of the \( W_z \otimes W_{z'} \rightarrow \mathbb{C} \) diagram

\[
\begin{aligned}
\alpha & \quad \circ \quad + \beta \quad \circ
\end{aligned}
\]

leads to

\[
z' = q^{-30} z \quad \alpha = \beta
\]

(compare with the dashed cup (15)). Up to normalization, \( f_a \) is just the product of (19) and (20).

If \( a = 20 \), \( X_{20} \) is of dimension 249, and so it must be isomorphic to \( W_{z''} \) for some \( z'' \). Again, we look for an embedding of \( W_{z''} \) into \( W_z \otimes W_{z'} \); by \( U_q(e_8) \) invariance it is of the form

\[
\begin{aligned}
\alpha & \quad + \beta \quad + \gamma \quad + \delta \quad + \epsilon
\end{aligned}
\]
and requiring that it commute with the $e_0$ action leads to

$$z' = q^{20} z \quad z'' = q^{10} z \quad \frac{\alpha}{\epsilon} = \frac{5}{20} \sqrt{\frac{[10][30]}{[6][15]}} \quad \frac{\beta}{\epsilon} = \frac{\gamma}{\epsilon} = \frac{\delta}{\epsilon} = \frac{10}{20}$$

so that $W_z \otimes W_{q^{20} z} \supset X_{20} \cong W_{q^{10} z}$.

Dually, one finds

\[\text{(22)} \quad \alpha + \beta + \gamma + \delta + \epsilon\]

and requiring that it commute with the $e_0$ action leads to

$$z' = q^{-20} z \quad z'' = q^{-10} z$$

and the same ratios of parameters. Again, $f_a$ is proportional to the product of (21) and (22).

For $a = 2, 12$, the spaces $X_a$ are higher-dimensional, and are not so simple to describe in our diagrammatic language. In terms of $U_q(\mathfrak{e}_8)$, one has:

$$X_{12} = V_{\omega_2} \oplus V_{\omega_1} \oplus V_0$$
$$X_2 = V_{\omega_2} \oplus V_{\omega_1} \oplus \mathbb{C}^2 \otimes V_{\omega_1} \oplus V_0$$

By obvious complementation, one can get similar decompositions for the spaces $Y_a$, but again, their description is more complicated. For instance, we have

$$Y_2 \cong V_{2\omega_1} \oplus V_{\omega_1} \oplus V_0$$

and we can describe its embedding into $W_{q^{2} z} \otimes W_z$ as

$$\alpha + \beta + \gamma + \delta + \epsilon$$

where \(\text{ is the embedding of } V_{2\omega_1} \text{ into } V_{\omega_1}^{\otimes 2} \) (with some normalization). However, the coefficients $\alpha, \beta, \gamma$ can only be fixed by independently specifying the $U_q(\mathfrak{e}_8)$ action on $V_{2\omega_1} \oplus V_{\omega_1} \oplus V_0$. 
Appendix A. Some product rules

\[ = \frac{(q - q^{-1})[6][10][15]}{[5][30]} \]

\[ = q^{10} - q^{20} \]

\[ + q^{35} + q^{15} \]

\[ = -q^5 \]

\[ + \frac{(q^{32} - q^{30} + q^{28} - q^{26} + q^{22} - q^{20} + q^{18} - q^{16} + q^{14} - q^{12} + q^{10} - q^8 + q^6 - q^4 - q^2 + 1)}{q^{16}[4][5][9][16][30]^3} \]

\[ + \frac{(q^{20} + q^{18} + q^{16} + q^{12} + q^8 + q^4 + q^2 + 1)}{q^{10}[3][5][9][30]^2} \]

\[ - \frac{(q^{48} - q^{46} - q^{44} - q^{40} - q^{38} - q^{36} + q^{34} + q^{32} + q^{30} - q^{28} + q^{26} - 2q^{24} + q^{22} - q^{20} + q^{18} + q^{16} - q^{14} - q^{12} - q^{10} + q^8 - q^6 - q^4 - q^2 + 1)}{q^{44}[3][5][30]^2} \]

\[ + \frac{(q^{20} + q^{18} - q^{14} - q^{12} + q^{10} + q^8 - q^6 + q^4 + q^2 + 1)}{q^{10}[2][3][5][30]} \]
Appendix B. The main formula

\[ R(z) = \frac{[6][10][15]^2(q^{28}z^2 - q^{56}z^2 - q^{32}z^2 + q^{50}z^2 - q^{48}z^2 + q^{46}z^2 - q^{42}z^2 - q^{10}z + q^{38}z - q^{34}z^2 + q^{14}z - q^{32}}{q^{22}[3][5][30]z(q^2z - 1)(q^{12}z - 1)} \]

\[ \]
Appendix C. Rational limit

The rational $R$-matrix is obtained from the trigonometric one in the correlated limit $q, z \to 1$:

$$
q = e^{\epsilon h/2} \quad z = e^{\epsilon x} \quad \epsilon \to 0
$$

Representation-theoretically, it corresponds to the limit from the quantized loop algebra $\mathcal{U}_q(\mathfrak{e}_8[z^\pm])$ (i.e., the quotient of $\mathcal{U}_q(\mathfrak{e}_8)$ by $\prod_{i=0}^8 k_i^{n_i} = 1$) to the Yangian $\mathcal{Y}(\mathfrak{e}_8) \cong \mathcal{U}_q(\mathfrak{e}_8[z])$. In this appendix, we show briefly how to recover the results of [KNS99].

The main simplification in the diagrammatic calculus (see the related discussion at the start of §3.3) is that undercrossings and overcrossings become indistinguishable:

and both become equal to the naive permutation of tensors of $V \otimes V$. The reader is invited to write the simplified relations that diagrams satisfy in this limit; we only provide one example:
The expression of the $R$-matrix in \[\text{§3}\] becomes in the rational limit:

\[
\hat{R}_{\text{rat}}(x) = \frac{\hbar - x}{\hbar + x} - \frac{x(5h + x)(9h + x)(16h + x)}{(h + x)(6h + x)(10h + x)(15h + x)}
\]

\[- \frac{x(5h + x)(44h + 5x)}{6(h + x)(6h + x)(10h + x)} - \frac{x(31h + 5x)}{6(h + x)(6h + x)}
\]

\[+ \frac{x(5h + x)}{12(h + x)(6h + x)}
\]

\[+ \frac{60h^3}{(h + x)(6h + x)(10h + x)}\]

\[+ \frac{x(4h + x)(11h + x)}{(h + x)(6h + x)(10h + x)}\]

\[+ \frac{\sqrt{30}h^2x}{(h + x)(6h + x)(10h + x)}\]

\[- \frac{60h^3x}{(h + x)(6h + x)(10h + x)(15h + x)}\]

\[+ \frac{60h^3(30h + x) - (h - x)(6h + x)(10h + x)(15h + x)}{(h + x)(6h + x)(10h + x)(15h + x)}\]
The alternative basis of \([3.3]\) no longer makes sense because \(\otimes\) and \(\otimes\) are degenerate; we can however use an intermediate basis, in which the \(R\)-matrix becomes

\[
\tilde{R}_\text{rat}(x) = \frac{1}{(h + x)(6h + x)} \begin{pmatrix}
6h^2 & 6h^2x(5h + x) \\
& (10h + x)(15h + x)
\end{pmatrix} + \frac{hx(5h + x)}{10h + x}
\]

the other terms (involving empty vertices) remaining the same.

It is not hard to see that the latter result coincides with that of \([KNS99]\) (see the equation between (3.9) and (3.10)), being careful that the expression there is \(R(w) = P \tilde{R}_\text{rat}(x = -w)\) where \(P\) is permutation of factors, on condition that the following normalizing factor is used:

\[
f(w) = \frac{w(w - 5h)(w - 9h)}{(w - h)(w - 6h)(w - 10h)}
\]

with the substitution \(i\hbar \rightarrow \hbar\) and with the free parameter \(\alpha\) (which is related to our parameter \(\kappa\)) given the value

\[
\alpha = \frac{1}{2\sqrt{15}}
\]

References

[Baz85] V. Bazhanov, *Trigonometric solutions of triangle equations and classical Lie algebras*, Physics Letters B 159 (1985), no. 4, 321–324, doi:https://doi.org/10.1016/0370-2693(85)90259-X

[BNW94] V. Bazhanov, B. Nienhuis, and O. Warnaar, *Lattice Ising model in a field: \(E_8\) scattering theory*, Phys. Lett. B (1994), 198–206, arXiv:hep-th/9312169

[CP94] V. Chari and A. Pressley, *A guide to quantum groups*, Cambridge University Press, Cambridge, 1994. MR1300632

[Cvi08] P. Cvitanović, *Group theory: birdtracks, Lie’s, and exceptional groups*, Princeton University Press, 2008.

[DGZ94] G. Delius, M. Gould, and Y.-Z. Zhang, *On the construction of trigonometric solutions of the Yang–Baxter equation*, Nuclear Physics B 432 (1994), no. 1, 377–403, arXiv:hep-th/9405030, doi:10.1016/0550-3213(94)90607-6

[Dri88] V. Drinfeld, *Quantum groups*, J. Sov. Math. 41 (1988), 898–915, [Zap. Nauchn. Semin. LOMI 155 (1986) 18-49].

[IK81] A. Izergin and V. Korepin, *The inverse scattering method approach to the quantum Shabat–Mikhailov model*, Comm. Math. Phys. 79 (1981), no. 3, 303–316, http://projecteuclid.org/euclid.cmp/1103909051

[Jim85] M. Jimbo, *A \(q\)-difference analogue of \(U(g)\) and the Yang–Baxter equation*, Lett. Math. Phys. 10 (1985), no. 1, 63–69, doi:10.1007/BF00704588 MR797001

[Jim86] ______, *Quantum \(R\) matrix for the generalised Toda system*, Comm. Math. Phys. 102 (1986), no. 4, 537–547, http://projecteuclid.org/euclid.cmp/1104145597 MR824040

[Jim89] ______, *Introduction to the Yang–Baxter equation*, Internat. J. Modern Phys. A 4 (1989), no. 15, 3759–3777, doi:10.1142/S0217751X9901503, MR1017340

[Kas95] C. Kassel, *Quantum groups*, Graduate Texts in Mathematics, vol. 155, Springer–Verlag, New York, 1995, doi:10.1007/978-1-4612-0783-2 MR1321135

[KNS99] K. Koepsell, H. Nicolai, and H. Samtleben, *On the Yangian \(Y(e_8)\) quantum symmetry of maximal supergravity in two dimensions*, Journal of High Energy Physics 1999 (1999), no. 04, 023–023, arXiv:hep-th/9903111, doi:10.1088/1126-6708/1999/04/023

[KRS81] P. Kulish, N. Reshetikhin, and E. Sklyanin, *Yang–Baxter equations and representation theory. I*, Lett. Math. Phys. 5 (1981), no. 5, 393–403, doi:10.1007/BF02285311 MR649704
[Kun90] A. Kuniba, Quantum $R$ matrix for $G_2$ and a solvable 175-vertex model, Journal of Physics A: Mathematical and General 23 (1990), no. 8, 1349–1362, \texttt{doi:10.1088/0305-4470/23/8/011}

[KZJ17] A. Knutson and P. Zinn-Justin, Schubert puzzles and integrability I: invariant trilinear forms, 2017, \texttt{arXiv:1706.10019}

[KZJ20] Schubert puzzles and integrability II: motivic Segre classes, 2020, in preparation.

[Ma90] Z.-Q. Ma, The spectrum-dependent solutions to the Yang–Baxter equation for quantum $E_6$ and $E_7$, Journal of Physics A: Mathematical and General 23 (1990), no. 23, 5513–5522, \texttt{doi:10.1088/0305-4470/23/23/023}

[Man14] V. Mangazeev, On the Yang–Baxter equation for the six-vertex model, Nuclear Physics B 882 (2014), 70–96, \texttt{arXiv:1401.6494}, \texttt{doi:10.1016/j.nuclphysb.2014.02.019}

[Zam89] A. Zamolodchikov, Integrals of motion and $S$-matrix of the (scaled) $T = T_c$ Ising model with magnetic field, Internat. J. Modern Phys. A 4 (1989), no. 16, 4235–4248, \texttt{doi:10.1142/S0217751X8900176X}

[ZF80] A. B. Zamolodchikov and V. A. Fateev, Model factorized $S$ matrix and an integrable Heisenberg chain with spin 1. (in Russian), Sov. J. Nucl. Phys. 32 (1980), 298–303, [Yad. Fiz.32,581(1980)].

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