Dualistic computational algebraic analyses of primal and dual minimum cost flow problems on acyclic tournament graphs

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Abstract

To integer programming problems, computational algebraic approaches using Gröbner bases or standard pairs via the discreteness of toric ideals have been studied in recent years. Although these approaches have not given improved time complexity bound compared with existing methods for solving integer programming problems, these give algebraic analysis of their structures. In this paper, we focus on the case that the coefficient matrix is unimodular, especially on the primal and dual minimum cost flow problems, whose structure is rather well-known, but new structures can be revealed by our approach. We study the Gröbner bases and standard pairs for unimodular programming, and give the maximum number of dual feasible bases in terms of the volume of polytopes. And for the minimum cost flow problems, we characterize reduced Gröbner bases in terms of graphs, and give bounds for the number of dual (resp. primal) feasible bases of the primal (resp. dual) problems: for the primal problems the minimum and the maximum are shown to be 1 and the Catalan number \( \frac{1}{d} \binom{2d-1}{d-1} \), while for the dual problems the lower bound is shown to be \( \Omega \left( 2^{\lfloor d/6 \rfloor} \right) \). To analyze arithmetic degrees, we use two approaches: one is the relation between reduced Gröbner bases and standard pairs, where the corresponding relation on the minimum cost flow — between a subset of circuits and dual feasible bases — has not been so clear, the other is the results in combinatorics related with toric ideals.

1 Introduction

Recently, some algebraic approaches to integer programming problems have been studied. The two main approaches are using Gröbner bases and standard pair decompositions. Although they neither give improved complexity bounds compared with existing methods nor have been demonstrated to solve hard practical instances which cannot be handled by existing methods, these approaches themselves are very interesting by applying computational algebraic methods to such hard problems, and give algebraic analysis of structure of integer programming problems. For an ideal over a polynomial ring, the reduced Gröbner basis and the set of standard pairs are dual in a sense that the complement of the monomials in the initial ideal, which is generated by initial terms of the reduced Gröbner basis, is the set of standard monomials, whose nice decomposition is the standard pair decomposition. This kind of duality may shed new light on duality in combinatorial optimization, and by considering a nice subclass of integer programming problems where the duality theorem holds, we might be able to obtain some complexity bounds by making use of the characteristics of the subclass, which could not be derived for general integer programming problems.

The problems whose coefficient matrices are unimodular form a nice subclass in a sense that the system \( yA \leq c \) becomes totally dual integral (TDI). Then each standard pair corresponds to a dual...
feasible basis, and the method using standard pairs is equivalent to calculate the reduced cost for each basis (Theorem \[7,4\]). Thus, the number of standard pairs, which is equal to that of dual feasible bases, gives the complexity of this approach. Additionally, the maximum number of standard pairs can be described by the normalized volume of another matrix (Theorem \[3,3\]).

Especially, the minimum cost flow problems form a well-known subclass of unimodular integer programming problems which can be solved in polynomial time. A Gröbner basis approach for the minimum cost flow problems is a variant of cycle-canceling algorithm. In the case of the strongly polynomial time algorithms \[8,12,13\], for any feasible flow they choose polynomial size of negative-cost cycles (by the selecting rules) from the set of negative-cost cycles in the residual network, which may be of exponential size, as many as possible. Similarly, the algorithm using Gröbner basis calculates the optimal flow by augmenting flows along the negative-cost cycles which correspond to the elements of Gröbner basis. Thus the cardinalities of reduced Gröbner bases may give some time bound for this algorithm. On the other hand, a standard pair approach for the minimum cost flow problems first finds the set of standard pairs, and solves linear system of equations for each standard pair until an integer and non-negative solution is obtained. For a network optimization problem, the duality between the reduced Gröbner basis and the set of standard pairs corresponds to the relation between circuits and dual feasible co-trees, dually, cutsets and primal feasible trees. Since such a relation has not been so clear, the computational algebraic duality may be interesting method for the analysis of network problems.

This paper is organized as follows. In Section 2, reduced Gröbner bases and standard pairs are defined, and their relations with integer programming problems, regular triangulations and dual polyhedra are introduced. The case that the coefficient matrix is unimodular is studied in Section 3. The maximum arithmetic degree (i.e. the maximum number of dual feasible bases) is shown to be the normalized volume of a polytope defined by homogenizing the coefficient matrix (Theorem \[3,3\]). In Section 4, we study the Gröbner bases and standard pairs on the primal minimum cost flow problems on acyclic tournament graphs with \(d\) vertices. We show that three types of reduced Gröbner bases can be characterized in terms of the circuits (Theorem \[4,6,4,8,4,10\]). These examples give the minimum and the maximum number of dual feasible bases of the minimum cost flow problems: the minimum is 1 (Theorem \[4,14\]), and the maximum is \(\frac{1}{d}(2^{(d-1)})\) (Theorem \[4,13\]). This maximum is shown using the result in Section 3 and the result about the hypergeometric system on unipotent matrices and related polytope \[4\]. In Section 5, we study the dual minimum cost flow problems. One reduced Gröbner basis is characterized in terms of cutsets (Theorem \[5,3\]). We also show that the lower bound for the number of primal feasible bases of the minimum cost flow problems is \(\Omega(2^{d/6})\).

|                  | Term on graph | Gröbner basis \[4\] | Standard pair \[8\] |
|------------------|---------------|---------------------|---------------------|
| **Primal**       |               |                     |                     |
| Algorithm        | Variant of    |                     |                     |
| cycle-canceling  |                |                     |                     |
| On acyclic       | min : \(d(d-1)/2\) | min : 1             |                     |
| tournament       | max : \(?\)   | max : \(\frac{1}{d}(2^{(d-1)})\) |                     |
| graph with \(d\) |               |                     |                     |
| Term on graph    | Set of        |                     |                     |
| cutsets          |                |                     |                     |
| **Dual**         |               |                     |                     |
| Algorithm        | Variant of    |                     |                     |
| cutset-canceling |                |                     |                     |
| On acyclic       | min : \(d-1\) | min : \(?\)        |                     |
| tournament       | max : \(?\)   | Lower bound \(\Omega(2^{d/6})\) |                     |
| graph with \(d\) |               |                     |                     |

Table 1: Dual algebraic approaches for primal and dual minimum cost flow problems
2 Toric ideals and Gröbner bases

For a matrix $A \in \mathbb{Z}^{d \times n}$ and a cost vector $c \in \mathbb{R}^n$, let $IP_{A,c}$ be the family of integer programming problems $IP_{A,c}(b) := \text{minimize } \{c \cdot x | Ax = b, x \in \mathbb{N}^n\}$ as $b$ varies in $\{Au | u \in \mathbb{N}^n\} \subseteq \mathbb{Z}^d$ ($\mathbb{N}$ is the set of non-negative integers). The cost vector $c$ is called generic if each program in $IP_{A,c}$ has the unique optimal solution.

Let $k$ be a field and $k[x] := k[x_1, \ldots, x_n]$ be the polynomial ring. For an exponent vector $a = (a_1, \ldots, a_n) \in \mathbb{N}^n$, we denote $x^a := x_1^{a_1}x_2^{a_2} \cdots x_n^{a_n}$. A total order on monomials in $k[x]$ is a term order if 1 is the unique minimal element, and $x^u > x^v$ implies $x^{u+w} > x^{v+w}$ for all $u, v, w \in \mathbb{N}^n$. For a fixed term order $\succ$, the refinement $\succ_c$ of $\succ$ by $c$ is a total order such that $x^u \succ_c x^v$ if either $c \cdot u > c \cdot v$ or “$c \cdot u = c \cdot v$ and $x^u > x^v$” holds. If $c \geq 0$, then $\succ_c$ becomes a term order.

The toric ideal $I_A$ of $A$ is a binomial ideal $I_A := \langle x^u - x^v \mid Au = Av, u, v \in \mathbb{N}^n \rangle$. For any $f \in I_A$, the initial term $\text{in}_{\succ_c}(f)$ of $f$ is the largest term in $f$ with respect to $\succ_c$. Then we define the initial ideal $\text{in}_{\succ_c}(I_A)$ of $I_A$ as $\text{in}_{\succ_c}(I_A) := \langle \text{in}_{\succ_c}(f) \mid f \in I_A \rangle$.

2.1 Gröbner bases and Conti-Traverso algorithm

A finite subset $\mathcal{G}_{\succ_c} = \{g_1, \ldots, g_s\} \subseteq I_A$ is a Gröbner basis for $I_A$ with respect to $\succ_c$ if $\text{in}_{\succ_c}(I_A) = \langle \text{in}_{\succ_c}(g_1), \ldots, \text{in}_{\succ_c}(g_s) \rangle$. In addition, Gröbner basis $\mathcal{G}_{\succ_c}$ is reduced if $\mathcal{G}_{\succ_c}$ satisfies that (i) for any $i$, the coefficient of $\text{in}_{\succ_c}(g_i)$ is 1, and (ii) for any $i$, any term of $g_i$ is not divisible by $\text{in}_{\succ_c}(g_j)$ ($i \neq j$). If $\succ_c$ is a term order, then the reduced Gröbner basis $\mathcal{G}_{\succ_c}$ exists uniquely, and is calculated by Buchberger algorithm (see [1]). Any Gröbner basis for $I_A$ is a basis of $I_A$ [1].

$I_A$ is called homogeneous with respect to the positive grading $\deg(x_i) = d_i > 0$ ($i = 1, \ldots, n$) if, for any $f = f_1 + f_2 + \cdots + f_m \in I_A$ ($f_i$ is the homogeneous component of degree $i$ in $f$), $f_i \in I_A$ for any $i$. Then $I_A$ is homogeneous if and only if $I_A$ is generated by homogeneous polynomials [1].

**Proposition 2.1** ([19]) If $I_A$ is a homogeneous with respect to some positive grading $\deg(x_i) = d_i > 0$, then $\succ_c$ becomes a term order for any $c \in \mathbb{R}^n \setminus \{0\}$, and the reduced Gröbner basis $\mathcal{G}_{\succ_c}$ exists.

The support $\text{supp}(u)$ of a vector $u$ is the index set $\{i \mid u_i \neq 0\}$. Any $u \in \mathbb{Z}^n$ can be written uniquely as $u = u^+ - u^-$ where $u^+, u^- \in \mathbb{N}^n$ and have disjoint support. Then $\mathcal{G}_{\succ_c}$ can be written as $\mathcal{G}_{\succ_c} = \{x^{u_1^+} - x^{u_1^-}, \ldots, x^{u_p^+} - x^{u_p^-}\}$ for some finite $u_1, \ldots, u_p \in \ker(A) \cap \mathbb{Z}^n$ [10].

**Example 2.2** Let $A = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ and consider the minimum cost flow problem $IP_{A,c}(b) = \text{minimize} \{c \cdot x \mid Ax = b, x = (x_{1,2}, x_{1,3}, x_{2,3}) \in \mathbb{N}^3\}$. Then the toric ideal is $I_A = \langle x_{1,2}x_{2,3} - x_{1,3} \rangle$.

![Figure 1: Acyclic tournament graph with 3 vertices.](image)

If $c = (c_{1,2}, c_{1,3}, c_{2,3}) = (3, 1, 2)$, then $\text{in}_c(I_A) = \langle x_{1,2}x_{2,3} \rangle$ and reduced Gröbner basis is $\mathcal{G}_{\succ_c} = \{x_{1,2}x_{2,3} - x_{1,3}\}$.

In the rest of this paper, we consider a cost vector $c$ which $\succ_c$ becomes a term order for some term order $\succ$. Let $IP_{A,\succ_c}(b)$ be the problem to find the unique minimal element in $\{x \in \mathbb{N}^n \mid Ax = b\}$ with
respect to $\succ_c$. Then the solution $u$ of $IPA_{\succ c}(b)$ is one of the optimal solutions of $IPA_{\succ c}(b)$. Conti and Traverso \cite{2} introduced an algorithm based on a Gröbner basis to solve $IPA_{\succ c}(b)$. We describe the condensed version of the Conti-Traverso Algorithm (see \cite{16}), which is useful for highlighting the main computational step involved. The normal form of $f \in k[x]$ by the reduced Gröbner basis $G$ is the unique remainder obtained upon dividing $f$ by $G$.

Algorithm 2.3 (Conti-Traverso Algorithm \cite{2,16})
1. Compute the reduced Gröbner basis $G_{\succ c}$ of $I_A$ with respect to $\succ_c$.
2. For any feasible solution $v$ of $IPA_{\succ c}(b)$, compute the normal form $x^u$ of $x^v$ by $G_{\succ c}$.
3. Output $u$. $u$ is the optimal solution of $IPA_{\succ c}(b)$.

Thus reduced Gröbner basis $G_{\succ c}$ is a test set for $IPA_{\succ c}$ \cite{20,21}.

Example 2.2 (continued) Let $b = (4, 5, -9)$. For a feasible solution $(4, 0, 9)$, the normal form of $x_{1,2}^4 x_{2,3}^3$ by $G_{\succ c}$ is $x_{1,3}^4 x_{2,3}^5$. Thus the optimal solution of $IPA_{\succ c}(b)$ is $(0, 4, 5)$.

2.2 Standard pair decompositions
Let $[n] := \{1, \ldots, n\}$. For a monomial $x^a \in k[x]$ and an index set $\sigma \subseteq [n]$, $(x^a, \sigma)$ is a standard pair of $in_{\succ c}(I_A)$ if (i) $\text{supp}(a) \cap \sigma = \emptyset$, (ii) every monomial in $x^a \cdot k[x_j | j \in \sigma] := \{x^a \cdot f | f \in k[x_j | j \in \sigma]\}$ is not an element of $in_{\succ c}(I_A)$, and (iii) there exists no other $(x^{a'}, \sigma')$, which satisfies (i) and (ii), such that $x^{a'}$ divides $x^a$ and $\text{supp}(x^{a'}/x^a) \cup \sigma \subseteq \sigma'$. We denote $S(in_{\succ c}(I_A))$ the set of all standard pairs of $in_{\succ c}(I_A)$. We use the same $(x^a, \sigma)$ to denote the set of all monomials in $x^a \cdot k[x_j | j \in \sigma]$. Then the above condition (iii) says that $(x^a, \sigma) \not\subset (x^{a'}, \sigma')$ for any other $(x^{a'}, \sigma')$ which satisfies the condition (i) and (ii). The standard pairs of $in_{\succ c}(I_A)$ induce a unique covering for the set of standard monomials of $in_{\succ c}(I_A)$, which we call the standard pair decomposition of $in_{\succ c}(I_A)$. $|S(in_{\succ c}(I_A))|$ is called the arithmetic degree of $in_{\succ c}(I_A)$ and denoted by $\text{arith-deg}(in_{\succ c}(I_A))$ \cite{18}.

Example 2.2 (continued) For $c = (3, 1, 2)$, the standard pairs of $in_{(3,1,2)}(I_A)$ are $(1, \{(1,2), (1,3)\})$ and $(1, \{(1,3), (2,3)\})$, thus the arithmetic degree of $in_{(3,1,2)}(I_A)$ is 2. On the other hand, for $c = (1, 4, 2)$, the standard pair of $in_{(1,4,2)}(I_A)$ is $(1, \{(1,2), (2,3)\})$, thus the arithmetic degree of $in_{(1,4,2)}(I_A)$ is 1.

![Figure 2: Two types of standard pair decompositions. A point $(p, q, r)$ in the figure corresponds to the monomial $x_{1,2}^p x_{1,3}^q x_{2,3}^r$.](image)

Let $c$ be a generic cost vector. Then $in_{\succ c}(I_A) = in_C(I_A)$. Let $\{a_1, \ldots, a_n\}$ be the column vectors of $A$ and cone($A$) the cone generated by $a_1, \ldots, a_n$. For $\sigma \subseteq [n]$, we denote $A_\sigma$ for the submatrix of $A$ whose columns are indexed by $\sigma$. For a cost vector $c$, we define the regular triangulation $\Delta_C$ of cone($A$) as follows: cone($A_\sigma$) is a face of $\Delta_C$ if and only if there exists a vector $y \in \mathbb{R}^d$ such that
\( \mathbf{y} \cdot \mathbf{a}_j = c_j \ (j \in \sigma) \) and \( \mathbf{y} \cdot \mathbf{a}_j < c_j \ (j \notin \sigma) \). If \( \text{cone}(A_\sigma) \) is a face of \( \Delta_\mathbf{c} \), \( \sigma \) also is called a face of \( \Delta_\mathbf{c} \). The genericity of \( \mathbf{c} \) implies that \( \Delta_\mathbf{c} \) is in fact a triangulation (i.e. each face of \( \Delta_\mathbf{c} \) is simplicial) \[7\].

**Lemma 2.4** \([16, 18]\)

(i) If \( \text{inc}(I_A) \) has \((*, \sigma)\) as a standard pair, then \( \sigma \) is a face of \( \Delta_\mathbf{c} \).

(ii) \( \text{inc}(I_A) \) has \((1, \sigma)\) as a standard pair if and only if \( \sigma \) is a maximal face of \( \Delta_\mathbf{c} \).

(iii) If \( a_1, \ldots, a_n \) span an affine hyperplane, then \( \Delta_\mathbf{c} \) is the same as the regular triangulation of \( \text{conv}(A) \) with respect to \( \mathbf{c} \), and the number of standard pairs \((*, \sigma)\) for a maximal face \( \sigma \) of \( \Delta_\mathbf{c} \) equals the normalized volume of \( \sigma \) in \( \Delta_\mathbf{c} \).

When vertices of \( \text{conv}(A) \) are in the \( m \)-dimensional lattice \( L \simeq \mathbb{Z}^m \), we define the normalized volume of a maximal face \( \sigma \) of \( \Delta_\mathbf{c} \) by the volume of \( \sigma \) with the normalization that the volume of the convex hull of \( 0, e_1, \ldots, e_m \) is 1. Here, \( \{e_i\}_{1 \leq i \leq m} \) are the basis of the lattice \( L \).

For a polyhedron \( P \subset \mathbb{R}^n \) and a face \( F \) of \( P \), the normal cone of \( F \) at \( P \) is the cone \( \text{NC}(F) := \{ \mathbf{w} \in \mathbb{R}^n \mid \mathbf{w} \cdot x' \geq \mathbf{w} \cdot x \ \text{for all} \ x' \in F \ \text{and} \ x \in P \} \). The set of normal cones for all faces of \( P \) is called the normal fan of \( P \).

**Lemma 2.5** \([9]\) \( \Delta_\mathbf{c} \) is the normal fan of the polyhedron \( P_\mathbf{c} := \{ y \in \mathbb{R}^d \mid yA \leq \mathbf{c} \} \).

We remark that \( P_\mathbf{c} \) is the dual polyhedron for the linear relaxation problem \( LP_{A,\mathbf{c}}(\mathbf{b}) := \text{minimize} \ \{ \mathbf{c} \cdot \mathbf{x} \mid A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq 0 \} \) of \( LP_{A,\mathbf{c}}(\mathbf{b}) \). Whenever \( A \) is row-full rank, this lemma shows that there is one-to-one correspondence between the dual feasible bases of \( LP_{A,\mathbf{c}}(\mathbf{b}) \) and the maximal faces of \( \Delta_\mathbf{c} \).

**Example 2.2** (continued) For \( \mathbf{c} = (3, 1, 2) \), \( \Delta_{(3,1,2)} = \{ \{1, 2\}, \{2, 3\}, \{1\}, \{2\}, \{3\}, \emptyset \} \).

![Figure 3: Dual polyhedron \( P_{(3,1,2)} \) and regular triangulation \( \Delta_{(3,1,2)} \).](image)

Let \( \mathbf{u} \) be the optimal solution to \( LP_{A,\mathbf{c}}(\mathbf{b}) \). Since standard pairs cover \( \text{inc}(I_A) \), \( \mathbf{x}^\mathbf{u} \) is covered by some standard pair \((\mathbf{x}^\mathbf{a}, \sigma)\). Thus \( \mathbf{u} = \mathbf{a} + \sum_{i \in \sigma} k_i e_i \) for some non-negative integers \( \{k_i\}_{i \in \sigma} \), and \( \mathbf{b} = A\mathbf{u} = A(a + \sum_{i \in \sigma} k_i e_i) = A\mathbf{a} + \sum_{i \in \sigma} k_i a_i \). Lemma 2.4 implies that \( \{a_i\}_{i \in \sigma} \) are linearly independent. Therefore \( \{k_i\}_{i \in \sigma} \) is the unique solution to the linear system \( \sum_{i \in \sigma} x_i a_i = \mathbf{b} - A\mathbf{a} \). This observation induces an algorithm to solve \( LP_{A,\mathbf{c}}(\mathbf{b}) \) using the standard pair decomposition of \( \text{inc}(I_A) \).

**Algorithm 2.6 (Solving \( LP_{A,\mathbf{c}}(\mathbf{b}) \) using \( S(\text{inc}(I_A)) \))**

(i) For \((\mathbf{x}^\mathbf{a}, \sigma) \in S(\text{inc}(I_A)) \), solve the linear system \( \sum_{i \in \sigma} x_i a_i = \mathbf{b} - A\mathbf{a} \). Let \( \{k_i\}_{i \in \sigma} \) be the solution.

(ii) If \( \{k_i\}_{i \in \sigma} \) are both integral and non-negative, output \( \mathbf{a} + \sum_{i \in \sigma} k_i e_i \) as the optimal solution. Otherwise, repeat (i) for another standard pair.

This algorithm solves at most \( \text{arith-deg}(\text{inc}(I_A)) \)-many linear systems of equations. Therefore \( \text{arith-deg}(\text{inc}(I_A)) \) is a measure of the complexity of \( LP_{A,\mathbf{c}} \).
3 Standard pairs for unimodular programming

Let $A \in \mathbb{Z}^{d \times n}$ be row-full rank and unimodular, i.e. each non-zero maximal minor is $\pm k$ for some $k \in \mathbb{N}$. Then $\text{in}_c(I_A)$ is minimally generated by square-free monomials for any $c \in \mathbb{R}$, and all standard pairs are obtained from all maximal faces of $\Delta_c$.

Lemma 3.1 ([9]) Let $\{m_1, \ldots, m_s\}$ be the minimal generators of $\text{in}_c(I_A)$. If $m_1, \ldots, m_s$ are all square-free then $S(\text{in}_c(I_A)) = \{(1, \sigma) \mid \sigma \text{ is the maximal faces of } \Delta_c\}$.

For a matrix $A \in \mathbb{Z}^{d \times n}$, the homogenized matrix $A' \in \mathbb{Z}^{(d+1) \times (n+1)}$ of $A$ is

$$A' := \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ A & a_1 & a_2 & \cdots & a_n \end{pmatrix} = \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ a_1 & a_2 & \cdots & a_n & 0 \end{pmatrix}. \quad (1)$$

Let $a_i' = \binom{a_i}{1}$ for $1 \leq i \leq n$ and $a_{n+1}'$ be the $(n+1)$-th column vector of $A'$. We remark that $a_1', \ldots, a_n', a_{n+1}'$ span an affine hyperplane.

We define another family $\text{IP}_{A',(c,0)}$ of integer programming problem

$$\text{IP}_{A',(c,0)}(b, \beta) := \text{minimize} \left\{ c \cdot x \mid A'(x_{n+1}^1) = \binom{\beta}{b}, (x_{n+1}^1) \in \mathbb{N}^{n+1} \right\}$$

as $(\binom{\beta}{b})$ varies in $\{A'u \mid u \in \mathbb{N}^{n+1}\}$. We remark that $(c,0)$ is generic if $c$ is generic.

The next proposition is due to Sturmfels et al. [18] for general ideals. We give another proof for the case of toric ideal.

Proposition 3.2 ([18]) $(x^a, \sigma) \in S(\text{in}_c(I_A)) (x^a \in k[x], \sigma \subseteq [n])$ if and only if $(x^a, \sigma \cup \{n+1\}) \in S(\text{in}_c(I_A'))$.

Proof: We first show that any monomial in $(x^a, \sigma)$ is standard for $\text{in}_c(I_A)$ if and only if any monomial in $(x^a, \sigma \cup \{n+1\})$ is standard for $\text{in}_c(I_A')$. Suppose that any monomial in $(x^a, \sigma)$ is standard for $\text{in}_c(I_A)$ and choose any $x^{p_{n+1}} \in (x^a, \sigma \cup \{n+1\})$. If there exist any other $\binom{v}{k} \in \mathbb{N}^{n+1}$ such that $A'(\binom{v}{k}) = A'(\binom{u}{k})$ and $\binom{v}{k} \neq \binom{u}{k}$, then $A\binom{u}{k} = A\binom{v}{k}$, and $c \cdot \binom{u}{k} < c \cdot \binom{v}{k}$ since $x^\binom{v}{k} \notin \text{in}_c(I_A)$. Therefore, $(\binom{u}{k})$ is the optimal solution to $\text{IP}_{A',(c,0)}(A\binom{u}{k}, \sum_{i=1}^n u_i + k)$. If there does not exist such $\binom{v}{k}$, then clearly $\binom{u}{k}$ is the optimal for $\text{IP}_{A',(c,0)}(A\binom{u}{k}, \sum_{i=1}^n u_i + k)$. This shows that any monomial in $(x^a, \sigma \cup \{n+1\})$ is standard for $\text{in}_c(I_A')$.

Conversely, suppose that any monomial in $(x^a, \sigma \cup \{n+1\})$ is standard for $\text{in}_c(I_A')$ and choose any $x^\binom{u}{k} \in (x^a, \sigma)$. If there exists some $\binom{v}{k} \in \mathbb{N}^n$ such that $A\binom{v}{k} = A\binom{u}{k}$, then $A'\binom{u}{k} = A'\binom{v}{k}$, and $c \cdot \binom{u}{k} < c \cdot \binom{v}{k}$, which implies that $c \cdot \binom{u}{k} < c \cdot \binom{v}{k}$. Therefore, $\binom{u}{k}$ is the optimal solution to $\text{IP}_{A,(c,0)}(A\binom{u}{k}, \sum_{i=1}^n u_i + k)$. If there does not exist such $\binom{v}{k}$, then clearly $\binom{u}{k}$ is the optimal for $\text{IP}_{A,(c,0)}(A\binom{u}{k}, \sum_{i=1}^n u_i + k)$. Thus any monomial in $(x^a, \sigma)$ is standard for $\text{in}_c(I_A)$.

Let $(x^a, \sigma \cup \{n+1\}) \in S(\text{in}_c(I_A'))$. If $(x^a, \sigma) \subset (x^a', \tau)$ for any other $(x^a', \tau)$ which satisfies (i) and (ii) in the definition of standard pairs for $\text{in}_c(I_A)$, then $(x^a, \sigma \cup \{n+1\})$ must be contained in $(x^a', \tau \cup \{n+1\})$, which contradicts the assumption. Thus $(x^a, \sigma) \in S(\text{in}_c(I_A))$. On the other hand, if $(x^a, \sigma \cup \{n+1\}) \notin S(\text{in}_c(I_A'))$, then there exists some $(x^a_k x^a_{n+1}^k, \tau')$ which contains $(x^a, \sigma \cup \{n+1\})$ and $(x^a_k x^a_{n+1}^k, \tau')$ satisfies (i) and (ii) in the definition of standard pairs for $\text{in}_c(I_A')$. Then $n+1 \in \tau'$ and therefore $k = 0$. Therefore, $(x^a', \tau' \setminus \{n+1\})$ contains $(x^a, \sigma)$ and satisfies (i) and (ii) in the definition of standard pairs. Thus $(x^a, \sigma) \notin S(\text{in}_c(I_A))$. This completes the proof. \[\square\]
Example 2.2 (continued) For this $A$, enlarged matrix $A'$ is

$$A' = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \end{pmatrix}.$$ 

We consider $I_{A'} \subset k[x_{1,2}, x_{1,3}, x_{2,3}, x_{1,4}]$. For $c = (3, 1, 2)$, the standard pairs of $\text{in}_{(3,1,2,0)}(I_{A'})$ are $(1, \{(1,2), (1,3), (1,4)\})$ and $(1, \{(1,3), (2,3), 4\})$. On the other hand, for $c = (1, 4, 2)$, the standard pairs of $\text{in}_{(1,4,2,0)}(I_{A'})$ are $(1, \{(1,2), (1,3), (2,3)\})$ and $(1, \{(1,2), (2,3), 4\})$. In this case, the only standard pair $(1, \{(1,2), (2,3)\})$ satisfies the condition in Proposition 3.2, which corresponds to the standard pair $(1, \{(1,2), (2,3)\})$ of $\text{in}_{(1,4,2)}(I_A)$.

Since $a_{i,1}, \ldots, a_{i,n+1}$ span an affine hyperplane, the normalized volume of $\text{conv}(A')$ gives the number of standard pairs of $\text{in}_{(c,k)}(I_{A'})$ which correspond to maximal faces of $\Delta_{(c,k)}$ by Lemma 2.3 (iii).

Theorem 3.3 [7] If $A$ is a unimodular matrix, then the maximum arithmetic degree of $\text{in}_c(I_A)$ equals the normalized volume of $\text{conv}(A')$.

Proof: For any $\tilde{c}$, the set of standard pairs of $\text{in}_c(I_A)$ is $\{(1, \sigma) \mid \sigma$ is a maximal face of $\Delta_c\}$, and each $(1, \sigma)$ corresponds to the standard pair $(1, \sigma \cup \{n+1\})$ of $\text{in}_{(c,0)}(I_{A'})$. Especially, $\sigma \cup \{n+1\}$ is a maximal face of $\Delta_{(c,0)}$. Therefore,

$$\text{arith-deg} (\text{in}_c(I_A)) = \left|\{(1, \sigma) \in S(\text{in}_c(I_A)) \mid \sigma \text{ is a maximal face of } \Delta_{(c,0)}\}\right|$$

Let $I_A \subset k[x]$ and $I_{A'} \subset k[x_1, \ldots, x_n, x_{n+1}]$. Then $x^a - x^b x_{n+1}^k \in I_{A'}$ $(x^a, x^b \in k[x])$ if and only if $\sum_{i=1}^n (a_i - b_i) = k$ and $x^a - x^b \in I_A$. We consider that $c = (1, 1, \ldots, 1)$ and $\succ$ is any reverse lexicographic term order such that $x_{n+1}$ is the smallest variable. Then for any $g$ in the reduced Gröbner basis $G$ for $I_{A'}$ with respect to $\succ_{(c,0)}$, $\text{in}_{\succ_{(c,0)}}(g)$ does not contain $x_{n+1}$ by the definition of the term order, and $\text{in}_{\succ_{(c,0)}}(g)$ is square-free since $\{\text{in}_{\succ_{(c,0)}}(g) \mid g \in G\}$ minimally generates $\text{in}_{\succ_{(c,0)}}(I_A)$ for some term order $\succ'$. Thus the corresponding triangulation $\Delta'_{(c,0)}$ is unimodular [10], and each facet of $\Delta'_{(c,0)}$ corresponds to a standard pair of $\text{in}_c(I_A)$ injectively. Then the arithmetic degree of $\text{in}_c(I_A)$ is equal to the number of facets of $\Delta'_{(c,0)}$, which is the normalized volume of $\text{conv}(A')$. \qed

We consider the primal problem which is equivalent with $LP_{A,c}(b)$:

$$P_{(M I), \tilde{c}(b)} : \text{ maximize } \{(-\tilde{c})^T x' \mid Mx' + I_d x'' = \tilde{b}_B, x', x'' \geq 0\},$$

which corresponds to some basis $B$, and its dual problem

$$D_{(I-M^T), \tilde{c}(b)} : \text{ minimize } \{b^T y'' \mid I_{n-d} y' - M^T y'' = \tilde{c}, y', y'' \geq 0\},$$

where $M \in \mathbb{Z}^{d \times (n-d)}$, $\tilde{b} = (\tilde{b}_B, \tilde{b}_N)$, $\tilde{b}_B = (\tilde{b}_i)_{i \in B} \in \mathbb{Z}^d$, $\tilde{b}_N = (\tilde{b}_i)_{i \notin B} = \mathbf{0} \in \mathbb{Z}^{n-d}$, $I_d \in \mathbb{Z}^{d \times d}$ and $I_{n-d} \in \mathbb{Z}^{(n-d) \times (n-d)}$ are identity matrices, $x''$ (resp. $x'$) is a basic (resp. non-basic) variable for $P_{(M I), \tilde{c}(b)}$, $y'$ (resp. $y''$) is a basic (resp. non-basic) variable for $D_{(I-M^T), \tilde{c}(b)}$, and $\tilde{c}$ is a reduced cost for $B$. 

7
For any standard pair \((1, \sigma)\) of \(\text{in}_C(I_A) = \text{in}_C(I_{(M_{11})})\), \(\sigma := \{1, \ldots, n\} \setminus \sigma\) forms a basis of the dual problem \(D_{(I - M^T)} \tilde{b}(\tilde{c})\) (Lemma 2.3). Let

\[
\sigma_1 := (\{1, \ldots, n\} \setminus B) \cap \sigma, \quad \sigma_2 := B \cap \sigma, \quad \sigma_1 := (\{1, \ldots, n\} \setminus B) \cap \sigma, \quad \sigma_2 := B \cap \sigma.
\]

Then the reduced cost of \(D_{(I - M^T)} \tilde{b}(\tilde{c})\) for the basis \(\sigma\) is

\[
\tilde{b}'_o = \tilde{b}_o - N^T_1 (B_1^{-1})^T \tilde{b}_\sigma,
\]

where \(B_1 = (I_{\sigma_1} (-M^T)_{\sigma_2}), \quad N_1 = (I_{\sigma_1} (-M^T)_{\sigma_2}).\)

**Theorem 3.4** The solution of the equation in Algorithm 2.4 (i) for a standard pair \((1, \sigma)\) is the reduced cost of \(D_{(I - M^T)} \tilde{b}(\tilde{c})\) for the basis \(\sigma\).

**Proof:** We show that \(\tilde{b}'_o\) is a solution of the linear system in Algorithm 2.4 (i) for \((1, \sigma)\), i.e. \((M_{\sigma_1} I_{\sigma_2}) \tilde{b}'_o = \tilde{b}\). This is because

\[
(M_{\sigma_1}, I_{\sigma_2}) \tilde{b}'_o = (M_{\sigma_1}, I_{\sigma_2}) \tilde{b}_o - (M_{\sigma_1}, I_{\sigma_2}) N_1^T (B_1^{-1})^T \tilde{b}_\sigma
\]

\[
= I_{\sigma_2} \tilde{b}_o - (M_{\sigma_1} (I_{\sigma_1})^T + I_{\sigma_2} ((-M^T)_{\sigma_2})^T (B_1^{-1})^T \tilde{b}_\sigma
\]

\[
= I_{\sigma_2} \tilde{b}_o - \{(M - M_{\sigma_1} (I_{\sigma_1})^T) + (-M - I_{\sigma_2} ((-M^T)_{\sigma_2})^T)\} (B_1^{-1})^T \tilde{b}_\sigma
\]

\[
= I_{\sigma_2} \tilde{b}_o + (M_{\sigma_1} I_{\sigma_1}) (I_{\sigma_1} (-M^T)_{\sigma_2})^T (B_1^{-1})^T \tilde{b}_\sigma
\]

\[
= I_{\sigma_2} \tilde{b}_o + (M_{\sigma_1} I_{\sigma_1}) B_1^T (B_1^{-1})^T \tilde{b}_\sigma
\]

\[
= I_{\sigma_2} \tilde{b}_o + (M_{\sigma_1} I_{\sigma_1}) \tilde{b}_\sigma
\]

\[
= I_{\sigma_2} \tilde{b}_o + I_{\sigma_2} \tilde{b}_\sigma
\]

\[
= \tilde{b}_o.
\]

the equation \((2)\) follows from the fact that \(M = MI = M_{\sigma_1} (I_{\sigma_1})^T + M_{\sigma_2} (I_{\sigma_2})^T\) and \(-M = I(-M) = I_{\sigma_2} ((-M^T)_{\sigma_2})^T + I_{\sigma_2} ((-M^T)_{\sigma_2})^T\).

\[\square\]

### 4 Gröbner bases and standard pairs of the primal minimum cost flow problems

Let \(G_d\) be the acyclic tournament graph with vertices \(1, 2, \ldots, d\) and \(n = \binom{d}{2}\) arcs, where each arc \((i, j)\) \((i < j)\) is directed from \(i\) to \(j\). We consider the following minimum cost flow problem \(P_{A,C}(b)\):

\[
P_{A,C}(b) := \text{minimize} \ \{c^T x \mid Ax = b, \ x \geq 0\},
\]

where \(A \in \mathbb{Z}^{d \times n}\) is the vertex-arc incidence matrix of \(G_d\).

A walk in \(G_d\) is a sequence \((v_1, v_2, \ldots, v_p)\) of vertices such that \((v_i, v_{i+1})\) or \((v_{i+1}, v_i)\) is an arc of \(G_d\) for each \(1 \leq i < p\). A cycle is a walk \((v_1, v_2, \ldots, v_p, v_1)\). A circuit is a cycle \((v_1, v_2, \ldots, v_p, v_1)\) such that \(v_i \neq v_j\) for any \(i \neq j\).

**Definition 4.1** Let \(C\) be a circuit in \(G_d\) and fix a direction of \(C\). If \(C\) passes an arc \((i, j)\) \(u_{ij}^+\) times forwardly and \(u_{ij}^-\) times backwardly, then we define \(u_C^+ = (u_{ij}^+)_{1 \leq i < j \leq d}\) and \(u_C^- = (u_{ij}^-)_{1 \leq i < j \leq d} \in \mathbb{R}^n\). The vector \(u_C := u_C^+ - u_C^-\) is called the incidence vector of \(C\). We identify a cycle \(C\) of \(G_d\) with the binomial \(f_C := x u_C^+ - x u_C^- \in I_A\).
Definition 4.2 A non-zero vector $u$ in $\ker(A)$ is a circuit if its support $\text{supp}(u)$ is minimal with respect to inclusion and the elements of $u$ are relatively prime. When $u \in \ker(A)$ is a circuit, we also call $x^u^+ - x^u^-$ a circuit of $I_A$. We denote $C_A$ a set of circuits of $I_A$.

Then $C_A$ corresponds to the set of all circuits in $G_d$. Let $U_A$ be the union of all reduced Gröbner bases for $I_A$ with respect to all term orders, which is called the universal Gröbner basis of $I_A$.

Proposition 4.3 ([16]) For the vertex-arc incidence matrix $A$ of $G_d$, $U_A = C_A$. Especially, any reduced Gröbner basis of $I_A$ is square-free, and the number of elements in $U_A$ is of exponential order with respect to $d$.

Proposition 4.4 $I_A$ is not homogeneous for the grading $\deg(x_{i,j}) = 1$, but is homogeneous for the grading $\deg(x_{i,j}) = j - i$.

Proof: For any $d$, $x_{1,2}x_{2,3} - x_{1,3} \in I_A$ and $x_{1,2}x_{2,3} \notin I_A$. This implies that $I_A$ is not homogeneous for the grading $\deg(x_{i,j}) = 1$.

Let $v_1, v_2, \ldots, v_p, v_1$ be a circuit in $G_d$, $C^+ := \{k \mid v_k < v_{k+1}\}$ and $C^- := \{k \mid v_k > v_{k+1}\}$ (we set $v_{p+1} := v_1$). The binomial $f_C$ corresponding to $C$ is $f_C = \prod_{k \in C^+} x_{v_kv_{k+1}} - \prod_{k \in C^-} x_{v_{k+1}v_k}$. Then $f_C$ is homogeneous for the grading $\deg(x_{i,j}) = j - i$ because

$$\deg \left( \prod_{k \in C^+} x_{v_kv_{k+1}} \right) - \deg \left( \prod_{k \in C^-} x_{v_{k+1}v_k} \right) = \sum_{k \in C^+} (v_{k+1} - v_k) - \sum_{k \in C^-} (v_k - v_{k+1}) = \sum_{k=1}^p (v_{k+1} - v_k) = 0.$$

Thus reduced Gröbner basis exists for any $c \in \mathbb{R}^n \setminus \{0\}$ by Proposition 2.1.

4.1 Some Gröbner bases for the primal problem

We first show that the elements in reduced Gröbner bases with respect to some specific term orders can be given in terms of graphs. As a corollary, we can show that there exist term orders for which reduced Gröbner bases remain in polynomial order. For other applications of the Gröbner bases found in this section, see our paper [11] and Proposition 4.5.

Proposition 4.5 Let $\succ$ be the purely lexicographic order induced by the variable ordering such that $x_{i,j} \succ x_{k,l}$ if and only if $i < k$ or ($i = k$ and $j < l$). Then the reduced Gröbner basis for $I_A$ with respect to $\succ$ is $\{g_{ijk} := x_{i,j}x_{j,k} - x_{i,k} \mid i < j < k\} \cup \{g_{ijkl} := x_{i,k}x_{j,l} - x_{i,l}x_{j,k} \mid i < j < k < l\}$. In particular, the number of elements in this Gröbner basis is equal to $\binom{d}{3} + \binom{d}{4}$.

The set $\{g_{ijk} \mid i < j < k\}$ corresponds to all of the circuits of length three, and $\{g_{ijkl} \mid i < j < k < l\}$ corresponds to some circuits of length four uniquely determined for each four vertices $i, j, k, l$ (Figure 3).

Proof: By Proposition 4.5, it suffices to show that any binomial which corresponds to a circuit in $G_d$ is $g_{ijk}$, $g_{ijkl}$ or whose initial term is divisible by some $\text{inv}_\succ(g_{ijk})$ or $\text{inv}_\succ(g_{ijkl})$.

Any binomial corresponding to a circuit of length 3 is contained in $\{g_{ijk}\}$.

The circuits defined by four vertices $i < j < k < l$ are $C_1 := (i, j, k, l, i)$, $C_2 := (i, j, l, k, i)$, $C_3 := (i, k, j, l, i)$ and their opposites. The binomial which corresponds to $C_1$ or its opposite is $\pm(x_{i,j}x_{j,k}x_{k,l} - x_{i,l})$, whose initial term $x_{i,j}x_{j,k}x_{k,l}$ is divisible by $\text{inv}_\succ(g_{ijk})$. Similarly, the initial term
of binomial which corresponds to $C_2$ or its opposite is divisible by $in_{>}(g_{ijl})$. The binomial which corresponds to $C_3$ or its opposite is $g_{ijkl}$.

Let $C$ be a circuit of length more than five. Let $v_1$ be the vertex whose label is minimum in $C$, and $C := (v_1, v_2, \ldots, v_p, v_1)$. Without loss of generality, we set $v_2 < v_p$. Let $f_C$ be the binomial corresponding to $C$, then $in_{>}(f_C)$ is the product of all variables whose associated arcs have the same direction as $(v_1, v_2)$ on $C$. If $v_2 < v_3$, then $(v_1, v_2)$ and $(v_2, v_3)$ have the same direction on $C$. Thus both $x_{v_1,v_2}$ and $x_{v_2,v_3}$ appear in $in_{>}(f_C)$, and $in_{>}(f_C)$ is divisible by $in_{>}(g_{v_1v_2v_3})$. If $v_2 > v_3$, then since $v_3 < v_2 < v_p$, there exists some $k$ ($3 \leq k \leq p - 1$) such that $v_1 < v_k < v_2 < v_{k+1}$. Then both $x_{v_1,v_2}$ and $x_{v_k,v_{k+1}}$ appear in $in_{>}(f_C)$, and $in_{>}(f_C)$ is divisible by $in_{>}(g_{v_1v_kv_{k+1}})$.

**Theorem 4.6** Let $>$ be any term order and $c = (c_{1,2}, \ldots, c_{d-1,d}) \in \mathbb{R}^n$ satisfy $c_{i,j} + c_{j,k} > c_{i,k}$ for any $i < j < k$ and $c_{i,j} + c_{j,l} > c_{i,l} + c_{j,k}$ for any $i < j < k < l$. Then the reduced Gröbner basis for $I_A$ with respect to $>c$ is the same as the basis in Proposition 4.5.

**Proof:** Let $>'$ be the term order defined in Proposition 4.5. Then $in_{>c}(g_{ijk}) = x_{i,j}x_{j,k} = in_{>'}(g_{ijk})$ since $c_{i,j} + c_{j,k} > c_{i,k}$, and $in_{>c}(g_{ijkl}) = x_{i,k}x_{j,l} = in_{>'}(g_{ijkl})$ since $c_{i,k} + c_{j,l} > c_{i,l} + c_{j,k}$. Thus $in_{>c}(I_A) = in_{>'}(I_A)$, which implies that the reduced Gröbner bases for $I_A$ with respect to $>c$ and $>'$ are the same. □

**Proposition 4.7** Let $>$ be the purely lexicographic order induced by the variable ordering such that $x_{i,j} > x_{i,l}$ if and only if $j - i < l - k$ or $(j - i = l - k$ and $i < k)$. Then the reduced Gröbner basis for $I_A$ with respect to $>$ is \{ $g_{ijk} := x_{i,j}x_{j,k} - x_{i,k}x_{j,l}$ $| i < j < k$ $\} \cup \{ g_{ijkl} := x_{i,j}x_{j,k} - x_{i,k}x_{j,l}$ $| i < j < k < l$ \}. In particular, the number of elements in this Gröbner basis is equal to $\binom{d}{3} + \binom{d}{4}$. The set \{ $g_{ijk} | i < j < k$ \} corresponds to all of the circuits of length three in $G_d$, and \{ $g_{ijkl} | i < j < k < l$ \} corresponds to the set of circuits of length four in Figure 3 but the direction is opposite.

**Proof:** Any binomial corresponds to a circuit of length 3 is contained in \{ $g_{ijk}$ \}. The circuits defined by four vertices $i < j < k < l$ are $C_1 := (i,j,k,l,i)$, $C_2 := (i,j,l,k,i)$, $C_3 := (i,k,j,l,i)$ and their opposites. The binomial which corresponds to $C_1$ or its opposite is $\pm(x_{i,j}x_{j,k}x_{k,l} - x_{i,k}x_{j,l})$, whose initial term $x_{i,j}x_{j,k}x_{k,l}$ is divisible by $in_{>}(g_{ijk})$. The binomial which corresponds to $C_2$ or its opposite is $\pm(x_{i,j}x_{j,k}x_{k,l})$. If its initial term is $x_{i,j}x_{j,k}x_{k,l}$, it is divisible by $in_{>}(g_{ijk})$. On the other hand, if initial term is $x_{i,k}x_{j,l}$, it is divisible by $in_{>}(g_{ijkl})$. The binomial which corresponds to $C_3$ or its opposite is $g_{ijkl}$.

Let $C$ be a circuit of length more than five. Let $(v_1, v_2)$ $(v_1 < v_2)$ be the arc in $C$ which the difference of labels is minimum, and $C := (v_1, v_2, \ldots, v_p, v_1)$. Let $f_C$ be the binomial corresponding to $C$, then $in_{>}(f_C)$ is the product of all variables whose associated arcs have the same direction with $(v_1, v_2)$ on $C$.

If $v_2 < v_3$, then both $x_{v_1,v_2}$ and $x_{v_2,v_3}$ appear in $in_{>}(f_C)$, and $in_{>}(f_C)$ is divisible by $in_{>}(g_{v_1v_2v_3})$. Similarly, if $v_p < v_1$, then $in_{>}(f_C)$ is divisible by $in_{>}(g_{v_pv_1v_2})$.

Let $v_3 < v_2$ and $v_1 < v_p$. Then $v_3 < v_1 < v_2 < v_p$ by the definition of $v_1$ and $v_2$. If there exists some $q$ such that $v_q < v_{q+1} < v_{q+2}$, then $in_{>}(f_C)$ is divisible by $in_{>}(g_{v_qv_{q+1}v_{q+2}})$. Consider the case
that there does not exist such $q$. For any $s$ such that $v_s < v_1 < v_{s+1} < v_2$, $v_{s+2} < v_{s+1}$ by assumption, and $v_{s+2} < v_1$ by the definition of $v_1$ and $v_2$. Thus there must be some $r$ $(3 \leq r \leq p-1)$ such that $v_r < v_1 < v_2 < v_{r+1}$ since $v_3 < v_1 < v_2 < i_p$. Then $i_{\prec}(f_C)$ is divisible by $i_{\prec}(g_{v_1,v_2,v_{r+1}})$.

\begin{proof}
Let $\succ$ be any term order and $c = (c_{1,2}, \ldots, c_{1,d}, c_{2,3}, \ldots, c_{d-1,d}) \in \mathbb{R}^n$ satisfy $c_{i,j} > c_{i,k}$ for any $i < j < k$ and $c_{i,l} + c_{j,k} > c_{i,k} + c_{j,l}$ for any $i < j < k < l$. Then the reduced Gröbner basis for $I_A$ with respect to $\succ_c$ is the same as the basis in Proposition 4.7.

Proof: Let $\succ'$ be the term order defined in Proposition 4.7. Then $i_{\prec_c}(g_{i,k}) = x_{i,j}x_{j,k} = i_{\prec'}(g_{i,k})$ since $c_{i,j} + c_{j,k} > c_{i,k}$, and $i_{\prec_c}(g_{i,k}) = x_{i,j}^\prime x_{j,k} = i_{\prec'}(g_{i,k})$ since $c_{i,l} + c_{j,k} > c_{i,k} + c_{j,l}$. Thus $i_{\prec_c}(I_A) = i_{\prec'}(I_A)$, which implies that the reduced Gröbner bases for $I_A$ with respect to $\succ_c$ and $\succ'$ are the same. \end{proof}

\begin{proposition}
Let $\succ$ be the purely lexicographic order induced by the variable ordering such that $x_{i,j} \succ x_{k,l}$ if and only if $i < k$ or $(i = k$ and $j > l)$. Then the reduced Gröbner basis for $I_A$ with respect to $\succ$ is $\{g_{ij} := x_{i,j} - x_{i,i+1}x_{i+1,i+2} \cdots x_{j-1,j} \mid i < j - 1\}$. In particular, the number of elements in this Gröbner basis is equal to $\binom{d}{2} - (d - 1)$.

The set $\{g_{ij} \mid i < j - 1\}$ corresponds to all of the fundamental circuits of $G_d$ for the spanning tree $T := \{(i,i+1) \mid 1 \leq i < d\}$.

Proof: Let $C$ be a circuit which is not a fundamental circuit of $T$. Let $v_1$ be the vertex whose label is minimum in $C$, and $C := (v_1, v_2, \ldots, v_p, v_1)$. Without loss of generality, we set $v_2 < v_p$. Then the variable $x_{v_1,v_p}$ appears in the initial term of the associated binomial $f_C$. Thus $i_{\prec}(f_C)$ is divisible by $i_{\prec}(g_{v_1,v_p})$. \end{proof}

\begin{proof}
Let $\succ'$ be the term order defined in Proposition 4.9. Then $i_{\prec_c}(g_{ij}) = x_{i,j}^\prime = i_{\prec'}(g_{ij})$ since $c_{i,j} > c_{i,j+1} + c_{i+1,i+2} + \cdots + c_{j-1,j}$. Thus $i_{\prec_c}(I_A) = i_{\prec'}(I_A)$, which implies that the reduced Gröbner bases for $I_A$ with respect to $\succ_c$ and $\succ'$ are the same. \end{proof}

\section{4.2 Bounds for the size of Gröbner bases}

Generally the degree of any reduced Gröbner basis for toric ideal is of exponential order with respect to the number of rows in the matrix $\mathbf{B}$, but the cardinality is not well understood. For the case of the toric ideals of acyclic tournament graphs, since those vertex-arc incidence matrices are unimodular, the cardinalities of the reduced Gröbner bases may be bounded.

\begin{proposition}
The minimum cardinality of the reduced Gröbner bases for $I_A$ is $\binom{d}{2} - (d - 1)$. The basis we have shown in Proposition 4.7 is the example achieving this cardinality.

Proof: Since the reduced Gröbner basis forms a basis for $I_A$, the cardinality of the reduced Gröbner basis is more than that of the basis for $I_A$. Since $I_A$ corresponds to the cycle space of $G_d$, the cardinality of the basis for $I_A$ is equal to the dimension of the cycle space, which is $\binom{d}{2} - (d - 1)$. \end{proof}

To analyze the upper bound for the cardinalities of the reduced Gröbner bases, we calculate all reduced Gröbner bases for small $d$ using TiGERS [10]. Table 2 is the result for $d = 4, 5, 6, 7$. 

\begin{table}[h] 
\centering
\begin{tabular}{|c|c|c|c|}
\hline
$d$ & $n$ & $r$ & $\mathcal{R}$ \\
\hline
4 & 4 & 0 & \\
5 & 5 & 0 & \\
6 & 6 & 0 & \\
7 & 7 & 0 & \\
\hline
\end{tabular}
\end{table}
Table 2: Number of reduced Gröbner bases, maximum and minimum of cardinality.

| $d$ | # GB | max cardinality | min cardinality |
|-----|------|-----------------|-----------------|
| 4   | 10   | 5               | 3               |
| 5   | 211  | 15              | 6               |
| 6   | 48312| 37              | 10              |
| 7   | $\geq 37665$ | $\geq 75$ | 15 |

For the case of $d = 7$, the number of reduced Gröbner bases and the maximum of the cardinality are both too large, so we could not know the correct values. For $d \leq 5$, the reduced Gröbner basis in Proposition 4.5 is the example achieving maximum cardinality, but for $d \geq 6$ the maximum cardinality is a little larger than the cardinality of Gröbner basis in Proposition 4.5. For $d = 6$, we do not know what cost vectors produce the Gröbner bases of cardinality 37. The reduced Gröbner bases which achieve the maximum cardinality seem to be complicated and difficult to characterize.

**Problem 4.12** Are the cardinalities of reduced Gröbner bases for $I_A$ of polynomial order with respect to $d$?

### 4.3 Standard pairs for primal problem

In this section, we assume that $c$ is generic. If $c$ is not generic, then we may use $c'$ which is obtained by perturbing $c$ such that $c'$ is generic and $in_c(f)$ contains the term $in_{c'}(f)$ for any $f \in I_A$. Since one constraint of $P_{A,c}(b)$ is redundant, we can consider the problem $P_{A,c}(\overline{b})$, which is obtained from $P_{A,c}(b)$ by deleting the last constraint. Then $in_{c}(I_A) = in_{c'}(I_A)$, and $\overline{A}$ is row-full rank. In addition, the regular triangulation of cone($A$) and that of cone($A$) by $c$ are the same as a simplicial complex, thus we denote both triangulations $\Delta_c$.

Since any initial ideal $in_c(I_A)$ is generated by square-free monomials (Proposition 4.3), the set of standard pairs $S(in_c(I_A))$ are $(1, \sigma)$ where $\sigma$ ranges among all maximal faces of $\Delta_c$.

Let $E$ be a set of arcs in $G_d$. For $S \subseteq E$, we denote $x^S := \prod_{(i,j) \in S} x_{i,j}$.

The arcs in the optimum flow of uncapacitated minimum cost flow problem define a forest [1]. Therefore, with the fact the dimension of cone($A$) equals $d - 1$, the next proposition is implied by Lemma 2.4, Proposition 4.3 and Lemma 3.1.

**Proposition 4.13** $(x^a, \sigma)$ is a standard pair of $in_c(I_A)$ if and only if $x^a = 1$ and $\sigma$ is a spanning tree of $G_d$ such that $x^\sigma \notin in_c(I_A)$.

Because of the result in Section 3, there is one-to-one correspondence between the standard pairs $(1, *)$ of $in_c(I_A)$ and the dual feasible bases of $P_{\overline{c}}(\overline{b})$. Therefore, Algorithm 2.4 for the minimum cost flow problem $P_{A,c}(b)$ is a variant of the enumeration of dual feasible bases.

Gröbner bases which have shown in the previous section give upper and lower bounds for the arithmetic degree (i.e. bounds for the number of vertices of the dual polyhedron). The genericity of $c$ implies that the arithmetic degree of $in_c(I_A)$ is equal to or greater than 1.

**Theorem 4.14** The minimum arithmetic degree of $in_c(I_A)$ which $c$ varies all generic cost vectors equals 1.

**Proof:** For a cost vector $c$ as in Theorem 4.10, $in_c(I_A) = (x_{i,j} \mid j - i > 1)$. Then $x^a \notin in_c(I_A)$ if and only if $a_{i,j} = 0$ for any $(i,j)$ such that $j - i > 1$. The set of all such monomials coincides $(1, \{(1,2), (2,3), \ldots, (d-1,d)\})$. Thus only this pair is a standard pair of $in_c(I_A)$. □
Theorem 4.15  The maximum arithmetic degree of $inc(I_A)$ which $c$ varies all generic cost vectors equals $C_{d-1} := \frac{1}{7}(2^{d-1})$, which is the $(d - 1)$-th Catalan number.

To show this theorem, we use the next result due to Gelfand et al. [3] which studies about some hypergeometric function.

Lemma 4.16 ([3]) Let $A'$ be the enlarged matrix (1) for the incidence matrix $A$ of the acyclic tournament graph with $d$ vertices, and $conv(A')$ be the convex hull of $a'_1, \ldots, a'_{n+1}$. Then the normalized volume of $conv(A')$ equals the $(d - 1)$-th Catalan number $C_{d-1}$.

Proof of Theorem 4.15: Since $A$ is unimodular, $\text{arith-deg}(inc(I_A)) \leq (\text{normalized volume of } conv(A')) = C_{d-1}$ by Theorem 3.3.

Because of Proposition 4.13 and Theorem 4.6, for $c$ as in Theorem 4.13, $(1, \sigma)$ is a standard pair of $inc(I_A)$ if and only if $\sigma$ is a spanning tree of the acyclic tournament graph which satisfies the following two conditions:

(a) there are no $1 \leq i < j < k \leq d$ such that both $(i, j)$ and $(j, k)$ are arcs in $\sigma$, and

(b) there are no $1 \leq i < j < l \leq d$ such that both $(i, k)$ and $(j, l)$ are arcs in $\sigma$.

The number of such spanning trees are known to be the $(d - 1)$-th Catalan number (e.g. see [4]). □

We remark that the Catalan number equals $\frac{4^n}{\sqrt{\pi n^2}} (1 + O(\frac{1}{n}))$ (e.g. see [3]).

5 Gröbner bases and standard pairs of dual minimum cost flow problems

In this section, we analyze Gröbner bases and standard pairs for the dual minimum cost flow problems.

As in Section 3, we study the problem which is equivalent with $P_{A,C}(b)$:

\[ P_{(M I),C}(\tilde{b}) := \text{maximize} \{ (\tilde{c})^T x' | Mx' + Ix'' = \tilde{b}_B, x', x'' \geq 0 \}, \]

which corresponds to the basis $\{(1, 2), (2, 3), \ldots, (d - 1, d)\}$, and its dual problem

\[ D_{(I - MT),\tilde{b}}(\tilde{c}) := \text{minimize} \{ \tilde{b}_B y'' | By' - MTy'' = \tilde{c}, y', y'' \geq 0 \}, \]

where $(M I)$ (resp. $(I - MT)$) is the fundamental cutset (resp. fundamental circuit) matrix which corresponds to the spanning tree $\{(1, 2), (2, 3), \ldots, (d - 1, d)\}$, $\tilde{c}$ is the reduced cost corresponding to the basis $\{(1, 2), (2, 3), \ldots, (d - 1, d)\}$, $\tilde{b} = (b_B, b_N) = (b_{ij})_{1 \leq i < j \leq d}, b_B = (b_{i+1})_{1 \leq i < d}, b_N = (b_{i,j})_{i < j - 1} = 0$, and

\[ x = (x', x''), \quad x' = (x_1, x_2, x_3, \ldots, x_{d-1}, d), \quad x'' = (x_1, x_2, x_3, \ldots, x_{d-2}, d), \]

\[ y = (y', y''), \quad y' = (y_1, y_2, y_3, \ldots, y_{d-2}, d), \quad y'' = (y_1, y_2, y_3, \ldots, y_{d-1}, d). \]

Then $P_{(M I),C}(\tilde{b})$ has $d - 1$ constraints (i.e. $(M I) \in \mathbb{Z}^{(d-1)\times n}$), $D_{(I - MT),\tilde{b}}(\tilde{c})$ has $n - d + 1$ constraints (i.e. $(I - MT) \in \mathbb{Z}^{(n-d+1)\times n}$).

Let $G_d = (V, E)$. $D \subseteq E$ is a cutset in $G_d$ if there exists a partition $(V_1, V_2)$ of $V$ (i.e. $V_1 \cap V_2 = \emptyset, V_1 \cup V_2 = V$) such that $D = \{(i, j) \in E | i \in V_1 \text{ and } j \in V_2, \text{ or } i \in V_2 \text{ and } j \in V_1\}$.

Definition 5.1 Let $D$ be a cutset in $G_d$ which corresponds to $V = (V^+, V^-)$. We define the vector $u_D \in \mathbb{R}^n$ as

\[ (u_D)_{ij} := \begin{cases} 
1 & (i \in V^+ \text{ and } j \in V^-) \\
-1 & (i \in V^- \text{ and } j \in V^+) \\
0 & (\text{otherwise})
\end{cases} \]

The vector $u_D$ is called the incidence vector of $D$. 

We identify a cutset \( D \) which corresponds to \((V^+, V^-)\) with the binomial \( f_D := x^D_u - x^D_u \). Since the rank of the fundamental circuit matrix \((I - M^T)\) is \( n - d + 1 \) and each row vector of the fundamental cutset matrix \((M I)\) is in \( \ker((I - M^T)) \), the set of row vectors of the fundamental cutset matrix \((M I)\) forms a basis of \( \ker(I - M^T) \).

For the fundamental circuit matrix \((I - M^T)\), the set of circuits \( \mathcal{C}_{(I - M^T)} \) corresponds to the set of all cutsets of \( G_d \). Since the fundamental circuit matrix \((I - M^T)\) is totally unimodular (e.g. see [14]), Proposition 4 implies \( \mathcal{C}_{(I - M^T)} = \mathcal{U}_{(I - M^T)} \).

**Proposition 5.2** For a cost vector \( \mathbf{b} \) such that the linear system \((M I)x = \mathbf{b}_B\) has a non-negative solution, \( I_{(I - M^T)} \) has a reduced Gröbner basis with respect to \( \mathbf{b} \).

**Proof:** Let \( \mathbf{a} \geq 0 \) be a solution of \((M I)x = \mathbf{b}_B\). We denote \( r_i \) the \( i\)-th row of \((M I)\), i.e. the row which corresponds to the fundamental cutset for the arc \((i, i + 1)\). For each cutset \( D \) corresponds to \((V^+, V \setminus V^+)\) \((V^+ \subseteq \{1, \ldots, d - 1\})\), since \( u_D = \sum_{i \in V^+, i+1 \notin V^+} r_i - \sum_{i \notin V^+, i+1 \in V^+} r_i \),

\[
\mathbf{a} \cdot u_D = \sum_{i \in V^+, i+1 \notin V^+} \mathbf{a} \cdot r_i - \sum_{i \notin V^+, i+1 \in V^+} \mathbf{a} \cdot r_i
\]

\[
= \sum_{i \in V^+, i+1 \notin V^+} \tilde{b}_{i,i+1} - \sum_{i \notin V^+, i+1 \in V^+} \tilde{b}_{i,i+1}
\]

Thus \( \text{in}_a(f_D) = \text{in}_b(f_D) \) for any cutset \( D \), and \( \text{in}_a(I_{(I - M^T)}) = \text{in}_b(I_{(I - M^T)}) \). Since \( \mathbf{a} \geq 0 \), \( I_{(I - M^T)} \) has a reduced Gröbner basis with respect to \( \mathbf{b} \). \( \square \)

**Example 2.2 (continued)** Let \( c = (3,1,2) \) and \( \mathbf{b} = (4,5,-9) \). Then the primal and dual problem which corresponds to the spanning tree \( \{(1,2), (2,3)\} \) are the following.

\[
\begin{align*}
\text{max} & \quad 4x_{1,3} \\
\text{s.t.} & \quad \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_{1,3} \\ x_{1,2} \\ x_{2,3} \end{pmatrix} = \begin{pmatrix} 4 \\ 9 \end{pmatrix} \\
& x_{1,2}, x_{1,3}, x_{2,3} \geq 0
\end{align*}
\]

\[
\begin{align*}
\text{min} & \quad 4y_{1,2} + 9y_{2,3} \\
\text{s.t.} & \quad \begin{pmatrix} 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} y_{1,3} \\ y_{1,2} \\ y_{2,3} \end{pmatrix} = -4 \\
& y_{1,2}, y_{1,3}, y_{2,3} \geq 0
\end{align*}
\]

Then \( I_{(1,-1,-1)} = \langle x_{1,2} - x_{2,3}, x_{1,2}x_{1,3} - 1, x_{1,3}x_{2,3} - 1 \rangle \) and reduced Gröbner basis for \( \mathbf{b} = (4,0,9) \) is \{\( x_{2,3} - x_{1,2}, x_{1,2}x_{1,3} - 1 \)\}.

### 5.1 Gröbner basis for dual problems

As for primal problems, we show that the elements in reduced Gröbner basis to some specific term order can be given in terms of graphs.

**Theorem 5.3** Let \( \mathbf{b} \) be the cost vector which satisfies the condition in Proposition 5.2, \( \tilde{b}_{i,i+1} > \tilde{b}_{j,j+1} \) \((1 \leq i < j \leq d)\) and \( \tilde{b}_{i,j} = 0 \) \((i, j \text{ such that } j > i + 1)\). Then the reduced Gröbner basis for \( I_{(I - M^T)} \) with respect to \( \mathbf{b} \) is \( \{g_i := \prod_{j \notin i} x_{j,i} - \prod_{k > i} x_{i,k} \mid i = 2, 3, \ldots, d\} \). In particular, the number of elements in this Gröbner basis is equal to \( d - 1 \).

**Proof:** For a cutset \( D \) which corresponds to \((V^+, V^-)\) such that \( 1 \in V^+ \), we define \( P^+ := \{i \in V^+ \mid i \neq d, i + 1 \in V^-\} \) and \( P^- := \{i \in V^- \mid i \neq d, i + 1 \in V^+\} \). Let \( P^+ = \{i_1, \ldots, i_p\} \) \((i_1 < i_2 < \cdots < i_p)\)
and $P^- = \{j_1, \ldots, j_q\}$ ($j_1 < j_2 < \cdots < j_q$). Then $p = q$ or $p = q + 1$, and $i_1 < j_1 < i_2 < j_2 < \cdots < i_k < j_k < i_{k+1} < j_{k+1} < \cdots$. Since $\bar{b} \cdot u_D^+ = \sum_{r=1}^{p} \bar{b}_{i_r,i_{r+1}} > \sum_{r=1}^{q} \bar{b}_{j_r,j_{r+1}} = \bar{b} \cdot u_D^-$, $in_b(f_D) = x u_b^+$. Since $in_b(g_{i+1}) = \prod_{j \leq i_1} x_{j,i_{j+1}}$, $in_b(f_D)$ can be reduced by $in_b(g_{i+1})$. □

5.2 Bounds for the size of Gröbner bases

**Proposition 5.4** The minimum cardinality of the reduced Gröbner bases for $I_{(I - M^T)}$ is $d - 1$. The basis we have shown in Theorem 5.3 is the example achieving this cardinality.

*Proof:* Since the reduced Gröbner basis forms a basis for $I_{(I - M^T)}$, the cardinality of the reduced Gröbner basis is more than that of the basis for $I_{(I - M^T)}$, which is $d - 1$. □

To analyze the upper bound for the cardinalities of the reduced Gröbner bases, we calculate all reduced Gröbner bases for small $d$ using TiGERS [10]. Table 3 is the result for $d = 4, 5, 6, 7$.

| $d$ | # GB | max cardinality | min cardinality |
|-----|------|-----------------|-----------------|
| 4   | 7    | 5               | 3               |
| 5   | 48   | 10              | 4               |
| 6   | 820  | 20              | 5               |
| 7   | 44288| 39              | 6               |

Table 3: Number of reduced Gröbner bases of dual problems, maximum and minimum of cardinality.

We do not know what cost vectors produce the Gröbner bases of maximum cardinality. The reduced Gröbner bases which achieve the maximum cardinality seem to be complicated and difficult to characterize.

**Problem 5.5** Are the cardinalities of reduced Gröbner bases for $I_{(I - M^T)}$ of polynomial order with respect to $d$?

5.3 Standard pairs for dual problem

In this section, we assume that $\bar{b}$ is generic same as Section 4.3. Since any initial ideal $in_b(I_{(I - M^T)})$ is generated by square-free monomials, any standard pair in $S(in_b(I_{(I - M^T)}))$ is of the form $(1, \star)$. Moreover, since the support of each optimal solution of $D_{(I - M^T),c}^b \bar{b}$ does not include a cutset, with the fact that $\dim cone((I - M^T)) = n - d + 1$, the next proposition is implied by Lemma 2.4 and Lemma 3.1.

**Proposition 5.6** $(x^\alpha, \sigma)$ is a standard pair of $in_b(I_{(I - M^T)})$ if and only if $x^\alpha = 1$ and $\sigma$ is a co-tree of $G_d$ such that $x^\sigma \notin in^*_b(I_{(I - M^T)})$.

**Example 2.2** (continued) For $c = (3, 1, 2)$ and $b = (4, 5, -9)$, the initial ideal $in_{(4,0,9)}(I_{(1|-1,-1)}) = \langle x_{2,3}, x_{1,2}x_{1,3} \rangle$ has two standard pairs $(1, \{(1, 2)\})$ and $(1, \{(1, 3)\})$.

**Theorem 5.7** For any $\bar{b}$ which satisfies the condition in Proposition 5.2, there exists $S \subset \{1, \ldots, d - 1\}$ with $|S| \geq [(d - 1)/6]$ such that, for any $\sigma \subseteq S$, there exists a spanning tree $T_\sigma$ of $G_d$ which satisfies the following:

(A) $T_\sigma$ contains the arc set $\{(i, i+1) \mid i \in S \setminus \sigma\}$ and does not contain any arc in $\{(j, j+1) \mid j \in \sigma\}$.
(B) $(1, T_\sigma)$ is a standard pair of $\text{in}_b(I_{(1 - M^\tau)})$, where $T_\sigma := E \setminus T_\sigma$ is a co-tree of $T_\sigma$.

Especially, since $T_\sigma \neq T_\tau$ for any $\sigma, \tau \subseteq S$ ($\sigma \neq \tau$), $\text{in}_b(I_{(1 - M^\tau)})$ has at least $\Omega(2^{d/6})$ standard pairs for any generic $b$ which satisfies the condition in Proposition 5.2.

Proof: We divide $\{1, \ldots, d - 1\}$ into the following subsets.

- $M_0 := \{i \in \{1, \ldots, d - 1\} \mid x_{i,i+1} \in \text{in}_b(I_{(1 - M^\tau)})\}$
- $M_1 := \{i \in \{1, \ldots, d - 1\} \mid i \notin M_0, \ i \equiv 0 \text{ (mod 3)}\}$
- $M_2 := \{i \in \{1, \ldots, d - 1\} \mid i \notin M_0, \ i \equiv 1 \text{ (mod 3)}\}$
- $M_3 := \{i \in \{1, \ldots, d - 1\} \mid i \notin M_0, \ i \equiv 2 \text{ (mod 3)}\}$

Lemma 5.8 $|M_0| \leq \lceil (d - 1)/2 \rceil$.

Proof of Lemma 5.8. We consider a cutset $D$ which corresponds to $(V^+, V^-)$ such that $f_D$ contains $x_{i,j}$ as a term of degree 1. Without loss of generality, we set $i \in V^+$. We assume that $j - i > 1$. Then for any $k$ ($i < k < j$), if $k \in V^+$ then $f_D$ contains $x_{k,j}$ and $x_{i,j}$ in the same term, otherwise $f_D$ contains $x_{i,k}$ and $x_{i,j}$ in the same term, which contradicts that $x_{i,j}$ is a term of $f_D$ of degree 1. Thus $j = i + 1$. In addition, $k \in V^-$ for any $k < i$ and $k \in V^+$ for any $k > i + 1$. Therefore, $V^+ = \{i, i+2, i+3, \ldots, d\}$ and $V^- = \{1, \ldots, i - 1, i + 1\}$.

We consider that $\text{in}_b(f) = x_{i,i+1}$ for some $f \in I_{(1 - M^\tau)}$. If $x_{i-1,i} \in \text{in}_b(I_{(1 - M^\tau)})$, then $f$ can be reduced by the binomial corresponding to the cutset between $\{i - 1, i + 1, \ldots, d\}$ and $\{1, \ldots, i - 2, i\}$ to

$$f' := x_{i,i+1} - \left( \prod_{k < i} x_{k,i} \right) \left( \prod_{k \geq i + 2} x_{i+1,k} \right) \left( \prod_{k < i - 1, i \geq k + 1} x_{k,i} \right) \left( \prod_{k \geq i} x_{k,i-1} \right) \left( \prod_{i-l \geq 1} x_{i,k} \right) \left( \prod_{k \geq i+1} x_{k,i} \right),$$

and its initial term is $x_{i,i+1}$. Since both terms of this binomial contain $x_{i,i+1}$, this implies that $\text{in}_b(f'/x_{i,i+1}) = 1$. Since $b$ defines a term order by Proposition 5.2, this is a contradiction.

Similarly, $x_{i+1,i+2} \notin \text{in}_b(I_{(1 - M^\tau)})$. Thus $|M_0| \leq \lceil (d - 1)/2 \rceil$. \[\square\]

Thus at least one of $M_1$, $M_2$, $M_3$ has at least $\lceil (d - 1)/6 \rceil$ elements. Let $S$ be one of such $M_i$ ($i = 1, 2, 3$). For any $\sigma := \{i_1 > i_2 > \cdots > i_r\} \subseteq S$, we construct desired spanning trees $T_\emptyset, T_{\{i_1\}}, T_{\{i_1, i_2\}}, \ldots, T_{\sigma}$ inductively.

- **Initial step:** Let $T_\emptyset := \{(1,2), (2,3), \ldots, (d-1, d)\}$. Clearly $T_\emptyset$ is a spanning tree. Since the reduced Gröbner basis corresponds to a subset of cutsets, the initial term of any elements of reduced Gröbner basis contains a variable $x_{i,i+1}$ for some $i$. Thus $x_{T_\emptyset} \notin \text{in}_b(I_{(1 - M^\tau)})$.

- **Induction step:** Let $T_{\sigma \backslash \{i_r\}}$ be the desired spanning tree for $\sigma \backslash \{i_r\}$. We define two edge set

$$T^1 := \{T_{\sigma \backslash \{i_r\}} \setminus \{(i_r, i_r + 1)\} \cup \{(i_r, i_r + 2)\}\}, \quad T^2 := \{T^1 \setminus \{(i_r + 1, i_r + 2)\}\} \cup \{(i_r - 1, i_r + 1)\}.$$

Then both $T^1$ and $T^2$ are spanning trees and satisfy the condition (A). We show that either $T^1$ or $T^2$ satisfies the condition (B).

(a) The case that $T^1$ satisfies the condition (B). Then $T^1$ is a desired spanning tree $T_\sigma$.

(b) The case that $T^1$ does not satisfy the condition (B).

In this case $x_{T^1} \in \text{in}_b(I_{(1 - M^\tau)})$. Let $G$ be the reduced Gröbner basis for $I_{(1 - M^\tau)}$ with respect to $b$. Then $x_{T^1}$ can be reduced some binomial $g \in G$, and such $g$ is one of the following form (See Figure 3).
(i) \( g^{(1)}_{(p)} \) which corresponds to the cutset for \((V^+, V^-)\), \( V^+ = \{p, p+1, \ldots, i_r, i_r + 2, i_r + 3, \ldots, d\} \) and \( V^- = \{1, 2, \ldots, p-1, i_r + 1\} \) for some \( p \leq i_r \), and its initial term is a product of variables corresponds to arcs from \( V^+ \) to \( V^- \), or

(ii) \((\text{The case of } r > 1)\) \( g^{(2)}_{(p,t)} \) which corresponds to the cutset for \((V^+, V^-)\), \( V^- = \{1, 2, \ldots, p-1, i_r + 1, i_q (1) + 1, \ldots, i_q (t) + 1\} \) and \( V^+ = V \setminus V^- \) for \( 1 \leq 3q(t) < \cdots < 3q(1) < r \) such that \((i_q (k) + 1, i_q (k) + 2) \in \mathcal{T}_{\sigma^r \setminus \{i_r\}}\) for \( k = 1, \ldots, t \) and \( 1 \leq 3p \leq i_r \), and its initial term is a product of variables corresponds to arcs from \( V^+ \) to \( V^- \).

![Figure 5: Cutsets corresponding to binomials (i) and (ii).](image)

**Lemma 5.9** \( g^{(1)}_{(p)} \in \mathcal{G} \) for some \( p \) and \( x^{\mathcal{T}} \) can be reduced by \( g^{(1)}_{(p)} \), i.e. the initial term of \( g^{(1)}_{(p)} \) corresponds to the set of arcs \( \{(k, i_r + 1) : k \leq i_r\} \).

**Proof of Lemma 5.9:** The case of \( r = 1 \) is trivial.

We suppose \( r > 1 \) and \( x^{\mathcal{T}} \) cannot be reduced by any \( g^{(1)}_{(p)} \). Then \( x^{\mathcal{T}} \) can be reduced by some \( g^{(2)}_{(p,t)} \) which is an element of \( \mathcal{G} \), and \( x^{\mathcal{T}} \) can be also reduced by \( g^{(2)}_{(1,t)} \) (otherwise, \( g^{(2)}_{(p,t)} \) is reduced by \( g^{(2)}_{(1,t)} \) and \( g^{(2)}_{(p,t)} \) cannot be an element in \( \mathcal{G} \)).

Suppose that \( x^{\mathcal{T}} \) can be reduced by \( g^{(2)}_{(1,t)} \) with \( t = 1 \). Let \( m_1 \) be the monomial obtained by reducing \( x^{\mathcal{T}} \) by \( g^{(2)}_{(1,t)} \), then \( m_1 \) can be reduced to the monomial \( m_2 \) by \( g^{(1)}_{(1)} \) (the initial term of \( g^{(1)}_{(1)} \) is a product of variables corresponds to arcs from \( V^- \) to \( V^+ \) by assumption).

| divided variables | multiplied variables | divided variables | multiplied variables |
|------------------|---------------------|------------------|---------------------|
| \( \{x_{k,i_r + 1} : k \leq i_r\} \), \( \{x_{k,i_q (1) + 1} : k \leq i_q (1), k \neq i_r + 1\} \) | \( \{x_{i_r + 1, l} : l \geq i_r + 2, l \neq i_q (1) + 1\}, \{x_{i_q (1) + 1, l} : l \geq i_q (1) + 2\} \) | \( \{x_{i_r + 1, l} : l \geq i_r + 2\} \) | \( \{x_{k,i_r + 1} : k \leq i_r\} \) |

Table 4: Divided and multiplied variables while reducing by \( g^{(2)}_{(1)} \) and \( g^{(1)}_{(1)} \).

For a binomial \( f_D \in I_{(p, t \mathcal{T})} \) which corresponds to the cutset \( D \) for \((V^+_D, V^-_D)\) such that \( V^-_D = \{i_q (1) + 1\} \) and \( V^+_D = V \setminus V^-_D \), \( \text{in}_b(f_D) \) corresponds to arcs from \( V^-_D \) to \( V^+_D \) (otherwise, \( x^{\mathcal{T} \sigma^r \setminus \{i_r\}} \) can be reduced by \( f_D \), which contradicts the assumption of the induction). Then \( m_2 \) can be reduced by \( f_D \),
and the resulting monomial is \( \mathbf{x}^{\mathbf{T}^2} \) (see Table 3), which contradicts to the definition of a term order by \( \mathbf{b} \).

Similarly, in the case that \( \mathbf{x}^{\mathbf{T}^1} \) can be reduced by \( g_{(1,t)}^{(2)} \) for some \( t > 1 \), using \( f_D \in I(\mathbf{I} - M^\mathbf{T}) \) which corresponds to the cutset \( D \) for \( (V_+^+, V_-^-) \) such that \( V^-_D = \{i_{(1)} + 1, i_{(2)} + 1, \ldots, i_{(t)} + 1\} \), and \( V^+_D = V \setminus V^-_D \), we can show a contradiction. Thus there exists some \( p \) such that \( g_{(p)}^{(1)} \in \mathcal{G} \).

If \( \mathbf{x}^{\mathbf{T}^2} \) cannot be reduced by \( g_{(1)}^{(1)} \), i.e. the initial term of \( g_{(1)}^{(1)} \) corresponds to the set of arcs \( \{(i_r + 1, l) : l \geq i_r + 2\} \), then \( g_{(1)}^{(1)} \) can be reduced by \( g_{(1)}^{(1)} \), which contradicts that \( g_{(p)}^{(1)} \) is an element of reduced Gröbner basis \( \mathcal{G} \). Thus the second statement follows.

We show that if \( \mathbf{x}^{\mathbf{T}^1} \in \text{in}_{\mathbf{b}}(I(\mathbf{I} - M^\mathbf{T})) \), then \( \mathbf{x}^{\mathbf{T}^2} \) cannot be reduced by any binomial in \( \mathcal{G} \). If \( \mathbf{x}^{\mathbf{T}^2} \) can be reduced by some \( g \in \mathcal{G} \), then such \( g \) is one of the following form.

(i) the binomial \( g_{(i_r)}^{(1)} \), and its initial term is \( x_{i_r,i_r+1} \),
(ii) any binomial which corresponds to the cutset for \( (V^+, V^-) \) such that \( i_r + 1 \in V^+ \) and \( 1, 2, \ldots, i_r, i_r + 2 \in V^- \), and its initial term is a product of variables correspond to arcs from \( V^+ \) to \( V^- \), or
(iii) (The case of \( r > 1 \)) \( g_{(i_r,t)}^{(2)} \), and its initial term is a product of variables correspond to arcs from \( V^+ \) to \( V^- \).

![Figure 6: Cutsets corresponding to binomials (i), (ii) and (iii).](image)

If the case (i) occurs, the initial term of \( g_{(i_r)}^{(1)} \) is \( x_{i_r,i_r+1} \), which contradicts that \( i_r \notin M(0) \). On the other hand, a binomial of type (ii) can be reduced by \( g_{(1)}^{(1)} \) by the above lemma, and cannot be contained in \( \mathcal{G} \).

Let us consider that the case (iii) occurs. If \( \mathbf{x}^{\mathbf{T}^2} \) can be reduced by \( g_{(i_r,t)}^{(2)} \) with \( t = 1 \). Then the monomial to which \( \mathbf{x}^{\mathbf{T}^2} \) are reduced by \( g_{(i_r,1)}^{(2)} \) can be reduced by a binomial \( f_D \in I(\mathbf{I} - M^\mathbf{T}) \), for the cutset \( D \) which corresponds to \( (V^+_D, V^-_D) \) where \( V^+_D = \{1, 2, \ldots, i_r - 1, i_r + 1\} \) and \( V^-_D = V \setminus V^+_D \), to some monomial \( m \) (the initial term of \( f_D \) is a product of variables corresponds to arcs from \( V^+_D \) to \( V^-_D \) since \( i_r \notin M(0) \)).

For a binomial \( f_{D'} \in I(\mathbf{I} - M^\mathbf{T}) \) which corresponds to the cutset \( D' \) for \( (V^+_D, V^-_D) \) such that \( V^-_D = \{i_{(1)} + 1\} \) and \( V^+_D = V \setminus V^-_D \), \( \text{in}_{\mathbf{b}}(f_{D'}) \) corresponds to arcs from \( V^-_D \) to \( V^+_D \) (otherwise, \( \mathbf{x}^{\mathbf{T}^2 \setminus \{i_r\}} \) can be reduced by \( f_{D'} \), which contradicts the assumption of the induction). Then \( m \) can be reduced by \( f_{D'} \), and the resulting monomial is \( \mathbf{x}^{\mathbf{T}^2} \) (see Table 4), which contradicts to the definition of a term order by \( \mathbf{b} \).
Table 5: Divided and multiplied variables while reducing by $g_{(i_r,1)}^{(2)}$ and $f_D$.

Similarly, in the case that $\mathbf{x}^{T_2}$ can be reduced by $g_{(i_r,t)}^{(2)}$ for some $t > 1$, using the same $f_D$ and $f_D' \in I(I-M_T)$ which corresponds to the cutset $D'$ for $(V_{D'}^+, V_{D'}^−)$ such that $V_{D'}^- = \{i_q(1)+1, i_q(2)+1, \ldots, i_q(t)+1\}$ and $V_{D'}^+ = V \setminus V_{D'}^-$, we can show the contradiction.

Therefore, $\mathbf{x}^{T_2} \notin \text{in}_b(I(I-M_T))$, and $T^2$ is a desired spanning tree $T_\sigma$. $\square$

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