THE NON-PERTURBATIVE EQUATION OF STATE
FOR THE GLUON MATTER

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In order to derive equation of state for the pure SU(3) Yang-Mills fields from first principles, it is proposed to generalize the effective potential approach for composite operators to non-zero temperatures. It is essentially non-perturbative by construction, since it assumes the summation of an infinite number of the corresponding contributions. There is no dependence on the coupling constant, only a dependence on the mass gap, which is responsible for the large-scale structure of the QCD ground state. The equation of state generalizes the Bag constant at non-zero temperatures, while its nontrivial Yang-Mills part has been approximated by the generalization of the free gluon propagator to non-zero temperatures, as a first necessary step. Even in this case we were able to show explicitly that the pressure may almost continuously change its regime at \( T^* = 266.5 \) MeV. All the other thermodynamical quantities such as energy density, entropy, etc. are to be understood to have drastic changes in their regimes in the close vicinity of \( T^* \). All this is in qualitative and quantitative agreement with thermal lattice QCD results for the pure Yang-Mills fields. We have firmly established the behavior of all the thermodynamical quantities in the region of low temperatures, where thermal lattice QCD calculations suffer from big uncertainties.

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I. INTRODUCTION

The prediction of a possible existence of the Quark-Gluon Plasma (QGP) was one of the best theoretical achievements of Quantum Chromodynamics (QCD) at non-zero temperatures and densities (the rather full list of the corresponding references can be found in the text-book on finite-temperature field theory in Ref. [1] and in Ref. [2] as well). The equation of state (EoS) for the QGP has been derived analytically up to the order \( g^6 \ln(1/g^2) \) by using the perturbation theory (PT) expansion for the evaluation of the corresponding thermodynamical potential term by term [1, 2, 3] and references therein).

However, the most characteristic feature of this PT expansion is its non-analytical dependence on the QCD coupling constant \( g^2 \). In fact, this means that the PT QCD is not applicable at finite temperatures, apart from maybe at very high temperatures. The problem is not in poor convergence of this series [1, 2, 3] (in mathematics there exist methods how to improve convergence). The problem is that in the case of the above-mentioned non-analytical dependence one cannot even define the radius of convergence, so any next calculated term can be bigger than the previous one. This is the principle problem which can be resolved by no means. From the strictly mathematical point of view the four-dimensional QCD at non-zero temperatures effectively becomes the three-dimensional theory. At the same time, the three-dimensional QCD has much more severe infrared singularities [4] and its coupling constant becomes dimensional. That is the reason why the dependence becomes not analytical, while using the dimensionless coupling constant \( g^2 \) (one needs to introduce three different scales, \( T, gT \) and \( g^2T \), where \( T \) is the temperature in order to somehow understand the dynamics of the QGP within the thermal PT QCD approach). Thus there is an exact indication that the analytical EoS derived by thermal PT QCD is wrong.

At present, the only method to be used in order to investigate thermal QCD is the lattice QCD at finite temperature and baryon density which underwent a rapid recent progress [1, 2, 3, 6, 7] and references therein). However, the lattice QCD, being a very specific regularization scheme, first of all is aimed at obtaining the well-defined corresponding expressions in order to get correct numbers from them. So, one gets numbers, but not understanding on what is going on. Such kind of understanding can only come from the dynamical theory which is continuous QCD. For example, any description of the QGP is to be formulated in the framework of the dynamical theory. The lattice thermal QCD is useless in this. The need in the analytical EoS remains, but, of course it should be essentially non-perturbative (NP),

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reproducing the thermal PT QCD results at a very high temperature only. Thus analytic NP QCD and lattice QCD approaches to finite-temperature QCD do not exclude each other, but contrary they should complement each other. Especially this is true for low temperatures where lattice QCD calculations suffer from big uncertainties [1, 2, 3, 6, 7]. There already exist an interesting analytic approaches based on quasi-particle and liquid model pictures [8] to analyze results of SU(3) lattice QCD calculations for the QGP EoS.

The formalism we are going to use in order to generalize it to non-zero temperature is the effective potential approach for composite operators [9]. It is essentially NP from the very beginning, since it is dealing with the expansion of the corresponding skeleton loop contributions (for more detail description see section 2 and Ref. [10] as well, where it has been generalized on quark degrees of freedom, but not using the confinement-type solution for the quark propagator). The main purpose of this paper is to derive EoS for the gluon matter by introducing the temperature dependence into the effective potential approach in a self-consistent way, in particular by using the confinement-type solution for the full gluon propagator (see below).

II. THE VED

The quantum part of the vacuum energy density (VED) is determined by the effective potential approach for composite operators [9]. In the absence of external sources the effective potential is nothing but the VED. It is given in the form of the skeleton loop expansion, containing all the types of the QCD full propagators and vertices, see Fig. 1. So each vacuum skeleton loop itself is a sum of an infinite number of the corresponding PT vacuum loops, i.e., it contains the point-like vertices and free propagators (see Fig. 2, where one term only in each lower order is shown, for simplicity). The number of the vacuum skeleton loops is equal to the power of the Planck constant, $h$.

Here we are going to formulate a general method of numerical calculation of the quantum part of the truly NP Yang-Mills (YM) VED in the covariant gauge QCD. The gluon part of the VED to leading order (the so-called log-loop level $\sim h$, the first skeleton loop diagram in Fig. 1, and which PT expansion is shown explicitly in Fig. 2) is analytically given by the effective potential for composite operators as follows [9]:

$$V(D) = \frac{i}{2} \int \frac{d^4 q}{(2\pi)^4} \text{Tr} \{ \ln(D^{-1} D) - (D_0^{-1} D) + 1 \},$$

where $D(q)$ is the full gluon propagator and $D_0(q)$ is its free counterpart (see below). Traces over space-time and color group indices are assumed. Evidently, the effective potential is normalized to $V(D_0) = 0$. Next-to-leading and higher order contributions (two and more vacuum skeleton loops) are suppressed at least by one order of magnitude in powers of $h$. They reproduce very small numerical corrections to the log-loop terms, and thus are not important for the numerical calculation of the VED to leading order.

![FIG. 1: The skeleton loop expansion for the effective potential. The wavy lines describe the full gluon propagators $D$, while the solid lines – the full quark propagators $S$. $\Gamma$ is the full quark-gluon vertex, while $T_3$ and $T_4$ are the full three- and four-gluon vertices, respectively. The ghost skeleton loops are not shown explicitly.](image1)

![FIG. 2: Infinite series for the gluon part of the VED (the first skeleton diagram in Fig. 1)](image2)
The two-point Green’s function, describing the full gluon propagator, is

\[ D_{\mu\nu}(q) = -i \{ T_{\mu\nu}(q)d(-q^2, \xi) + \xi L_{\mu\nu}(q) \} \frac{1}{q^2}, \]

(2.2)

where \( \xi \) is the gauge-fixing parameter and

\[ T_{\mu\nu}(q) = g_{\mu\nu} - \frac{g_{\mu\nu}q_{\mu}}{q^2} = g_{\mu\nu} - L_{\mu\nu}(q). \]

(2.3)

Its free PT counterpart \( D_0 \equiv D_{\mu\nu}^0(q) \) is obtained by putting the full gluon form factor \( d(-q^2, \xi) \) in Eq. (2.2) simply to one, i.e.,

\[ D_{\mu\nu}^0(q) = -i \{ T_{\mu\nu}(q) + \xi L_{\mu\nu}(q) \} \frac{1}{q^2}. \]

(2.4)

In order to evaluate the effective potential (2.1), on account of Eqs. (2.2) and (2.4), we use the well-known expression

\[ Tr\ln(D_0^{-1} D) = 8 \times 4 \ln \det(D_0^{-1} D) = 32 \ln[(3/4)d(-q^2, \xi) + (1/4)]. \]

(2.5)

It becomes zero indeed when equating \( d(-q^2, \xi) = 1. \)

Going over to four-dimensional Euclidean space in Eq. (2.1), one obtains \((\epsilon_g = V(D))\)

\[ \epsilon_g = -16 \int \frac{d^4q}{(2\pi)^4} \left[ \ln[1 + 3d(q^2, \xi)] - \frac{3}{4}d(q^2, \xi) + a \right], \]

(2.6)

where the constant \( a = (3/4) - 2 \ln 2 = -0.6363 \) and the integration from zero to infinity is assumed. The VED \( \epsilon_g \)
derived in Eq. (2.6) is already a colorless quantity, since it has been already summed over color indices. Also, only the transversal ("physical") degrees of freedom of gauge bosons contribute to this equation, so there is no need for ghosts to cancel their longitudinal (unphysical) counterparts.

However, the derived expression (2.6) remains rather formal, since it suffers from different types of the PT contributions ("contaminations"). In order to define the truly NP VED free of all the above-mentioned problems, let us make first the identical transformation of the full effective charge in Eq. (2.6) as follows:

\[ d(q^2, \xi) = d(q^2, \xi) - d^{PT}(q^2, \xi) + d^{PT}(q^2, \xi) = d^{NP}(q^2) + d^{PT}(q^2, \xi), \]

(2.7)

where \( d^{PT}(q^2, \xi) \) correctly describes the PT structure of the full effective charge \( d(q^2, \xi) \), including its behavior in the ultra-violet (UV) limit, compatible with asymptotic freedom (AF) phenomenon in QCD \[11\], otherwise remaining arbitrary. On the other hand, \( d^{NP}(q^2) \) defined by the above-made subtraction, is assumed to reproduce correctly the NP structure of the full effective charge, including its asymptotic in the deep infrared (IR) limit. This underlines the strong intrinsic influence of the IR properties of the theory on its NP dynamics. Evidently, both terms are valid in the whole energy/momentum range, i.e., they are not asymptotics. Let us also emphasize the principle difference between \( d(q^2, \xi) \) and \( d^{NP}(q^2) \). The former is NP quantity ”contaminated” by the PT contributions, while the latter one, being also NP, is, nevertheless, free of them. Thus the separation between the truly NP effective charge \( d^{NP}(q^2) \) and its nontrivial PT counterpart \( d^{PT}(q^2, \xi) \) is achieved. For example, if the full effective charge explicitly depends on the scale responsible for the truly NP dynamics in QCD, say \( \Lambda_{NP}^2 \), then one can define the subtraction

\[ d^{NP}(q^2, \Lambda_{NP}^2) = d(q^2, \Lambda_{NP}^2) - d(q^2, \Lambda_{NP}^2) = 0 = d(q^2, \Lambda_{NP}^2) - d^{PT}(q^2), \]

(2.8)

which is obviously equivalent to the decomposition (2.7). In this way the above-mentioned separation becomes exact and unique as well (for such concrete example see below). Let us emphasize that the dependence of the full effective charge \( d(q^2, \Lambda_{NP}^2) \) on \( \Lambda_{NP}^2 \) can be only regular. Otherwise it is impossible to assign to it the above-mentioned physical meaning, since \( \Lambda_{NP}^2 \) can be only zero (the formal PT limit) or finite, i.e., it cannot be infinitely large. In principle, in some special models of the QCD vacuum, for example such as the Abelian Higgs model \[12, 13\], the NP scale is to be identified with the mass of the dual gauge boson. Let us note that if there is no exact criterion how to distinguish between the truly NP and the nontrivial PT parts in the full effective charge as described above, then it is possible from the full effective charge to subtract its UV asymptotic only. Evidently, in this case the separation between the truly NP and the nontrivial PT parts may not be unique.
III. GENERALIZATION TO NON-ZERO TEMPERATURES

Substituting the above-discussed exact decomposition (2.7) into Eq. (2.6), introducing further the effective scale squared, separating the NP region from the PT one (soft momenta from hard momenta), and omitting some algebraic rearrangements (see Refs. [14, 15] and especially recent paper [16] for details), one obtains

$$\epsilon_Y(T) = -B_Y + B_Y(T) + P_Y(T).$$  \hspace{0.6cm} (3.1)

Here evidently $\epsilon_\rho \equiv \epsilon_Y$ and $B_Y$ is the Bag constant at zero temperature [16]. Also, $B_Y(T)$ and $P_Y(T)$ are explicitly given by the following expressions

$$B_Y(T) = 16 \int_0^{\frac{\epsilon_Y}{2\Lambda}} \frac{d^4q}{(2\pi)^4} \left[ \ln[1 + 3\alpha_s^{NP}(q^2)] - \frac{3}{4}\alpha_s^{NP}(q^2) \right]$$ \hspace{0.6cm} (3.2)

and $P_Y(T)$ has more complicated form, namely

$$P_Y(T) = -16 \int \frac{d^4q}{(2\pi)^4} \left[ \ln[1 + 3\alpha_s^{PT}(q^2) + 3\alpha_s^{NP}(q^2)] - \frac{3}{4}[\alpha_s^{PT}(q^2) + \alpha_s^{NP}(q^2)] + a \right],$$ \hspace{0.6cm} (3.3)

respectively, since it depends on both effective charges. In all these equations

$$\alpha_s^{NP}(q^2) \equiv d^{NP}(q^2), \quad \alpha_s^{PT}(q^2) \equiv d^{PT}(q^2),$$ \hspace{0.6cm} (3.4)

because $d^{NP}(q^2)$ and $d^{PT}(q^2)$ are the truly NP and the nontrivial PT effective charges, respectively, as it follows from above. Precisely these expressions should be generalized to non-zero temperatures in order to get EoS for the pure YM fields. That is why we introduce the dependence on the temperature $T$ in advance. Evidently, Eq. (3.2) will reproduce the temperature-dependent Bag constant. In the expression for $P_Y(T)$ the integration is from zero to infinity, while in the integral for $B_Y(T)$ it is from zero to the effective scale squared $q^{2\ M}_\epsilon$, which just symbolically shown in Eq. (3.2).

It is worth emphasizing that a so defined Bag constant (3.2) is free of all types of PT contributions ("contaminations"), as it is required (this was a reason for the above-mentioned algebraic rearrangements and subtractions, see Ref. [16] and references therein).

The problem remaining to solve is to choose the truly NP effective charge $\alpha_s^{NP}(q^2)$. For the different truly NP effective charges we will get different analytical and numerical results. That is why the choice for its explicit expression should be physically and mathematically well justified. Let us choose the truly NP effective charge as follows:

$$\alpha_s^{NP}(q^2) = \frac{\Lambda_{NP}^2}{q^2},$$ \hspace{0.6cm} (3.5)

where $\Lambda_{NP}$ is the mass scale parameter (the mass gap) responsible for the large-scale structure of the true QCD vacuum. It is well known that in continuous QCD it leads to the linear rising potential between heavy quarks, “seen” by lattice QCD [17, 18] as well (($q^2)^{-1}$-type behavior for the full gluon propagator). Moreover, in Ref. [19] it has been explicitly shown that it is a direct nonlinear iteration solution of the transcendental equation for the full gluon propagator in the presence of a renormalized mass gap. The separation between the truly NP and the nontrivial PT effective charges is both exact and unique, since the PT effective charge is always regular at zero, while the truly NP effective is singular at the origin (in the formal PT limit ($\Lambda_{NP}^2 \to 0$) the truly NP effective charge vanishes, while its nontrivial PT counterpart will survive). Let us also note that the chosen effective charge (3.5) does not depend explicitly on the gauge choice. It has been already used [14, 15, 16] in order to calculate the Bag constant, which turned out to be in a very good agreement with such important phenomenological parameter as the gluon condensate. It leads to many other desirable properties for the Bag pressure at zero temperature [16]. Thus, our choice (3.5) is physically justified and mathematically confirmed, as required above.

In the imaginary time formalism [11, 20], these expressions can be easily generalized to non-zero temperatures $T$ according to the prescription (let us remind that there is already Euclidean signature)

$$\int \frac{d\omega}{(2\pi)} \to T \sum_{n=-\infty}^{+\infty} \omega_n = 2n\pi T,$$ \hspace{0.6cm} (3.6)
i.e., each integral over \( q_0 \) of the loop momentum is to be replaced by the sum over Matsubara frequencies labelled by \( n \), which obviously assumes the replacement \( q_0 \to \omega_n = 2n\pi T \) for bosons (gluons). In frequency-momentum space the truly NP effective charge becomes

\[
\alpha^N_P(q^2) = \alpha^{NP}_N(q^2, \omega^2_n) = \frac{\Lambda^2_{NP}}{q^2 + \omega^2_n}, \quad \alpha^{PT}_P(q^2) = \alpha^{PT}_N(q^2, \omega^2_n) = \alpha^{PT}(\omega^2, \omega^2_n). \tag{3.7}
\]

It is also convenient to introduce the following notations

\[
T^{-1} = \beta, \quad \omega = \sqrt{q^2}, \tag{3.8}
\]

where, evidently, in all expressions here and below \( q^2 \) is the three-dimensional loop momentum squared in complete agreement with the relations (3.6).

IV. THE DERIVATION OF \( B_{YM}(T) \)

In frequency-momentum space the Bag pressure (3.2) after the substitution of the relations (3.6) becomes

\[
B_{YM}(T) = 16 \int \frac{d^3q}{(2\pi)^3} T \sum_{n=-\infty}^{+\infty} \left[ \ln[1 + 3\alpha^{NP}_N(q^2, \omega^2_n)] - \frac{3}{4} \alpha^{NP}_N(q^2, \omega^2_n) \right], \tag{4.1}
\]

where the truly NP effective charge is given in Eq. (3.7), and notations of Eqs. (3.7)-(3.8) are also valid, of course. After its substitution into Eq. (4.1), one yields

\[
B_{YM}(T) = 16 \int \frac{d^3q}{(2\pi)^3} T \sum_{n=-\infty}^{+\infty} \left[ \ln[3\Lambda^2_{NP} + q^2 + \omega^2_n] - \ln[\omega^2_n] - \frac{3}{4} \Lambda^2_{NP} \frac{1}{q^2 + \omega^2_n} \right]. \tag{4.2}
\]

The summation over the Matsubara frequencies squared \( \omega^2_n = (2\pi T)^2 n^2 \) can be easily done, and the dependence on the effective scale \( \omega_{\text{eff}} \) (see Appendix) is omitted, for simplicity. Here it is also convenient to introduce the following notation

\[
\omega' = \sqrt{q^2 + m_{\text{eff}}^2} = \sqrt{q^2 + 3\Lambda^2_{NP}} = \sqrt{\omega^2 + 3\Lambda^2_{NP}} = \omega \sqrt{1 + \frac{3}{\omega^2} \Lambda^2_{NP}}, \tag{4.3}
\]

So it is possible to say that within our approach to non-zero temperatures we have two sorts of gluons: massless \( \omega \) and massive \( \omega' \) with the effective mass

\[
m_{\text{eff}}' = \sqrt{3} \Lambda_{NP}. \tag{4.4}
\]

In the second term the summation over Matsubara frequencies can be done explicitly, namely

\[
\sum_{n=-\infty}^{+\infty} \frac{1}{q^2 + \omega^2_n} = \sum_{n=-\infty}^{\infty} \frac{1}{\omega^2 + (2\pi T)^2 n^2} = (2\pi/\beta)^{-2} \sum_{n=-\infty}^{+\infty} \frac{n}{n^2 + (\beta \omega/2\pi)^2}

\quad = (2\pi/\beta)^{-2} (2\pi^2/\beta \omega) \left( 1 + \frac{2}{e^{\beta \omega} - 1} \right) = \frac{\beta}{2\omega} \left( 1 + \frac{2}{e^{\beta \omega} - 1} \right). \tag{4.5}
\]
A. The summation of logarithms

In terms of the above-introduced parameters the sums in Eq. (4.2), containing the corresponding logarithms, look like

$$
\sum_{n=-\infty}^{+\infty} \ln[3\Lambda N_{\nu} + q^2 + \omega_n^2] = \ln \omega^2 + 2 \sum_{n=1}^{\infty} \ln(2\pi/\beta)^2[n^2 + (\beta\omega'/2\pi)^2] \quad (4.6)
$$

and

$$
\sum_{n=-\infty}^{+\infty} \ln[q^2 + \omega_n^2] = \ln \omega^2 + 2 \sum_{n=1}^{\infty} \ln(2\pi/\beta)^2[n^2 + (\beta\omega/2\pi)^2]. \quad (4.7)
$$

It is convenient to introduce the notations as follows:

$$
L(\omega') = \sum_{n=1}^{\infty} \ln[n^2 + (\beta\omega'/2\pi)^2] = \sum_{n=1}^{\infty} \ln n^2 + \sum_{n=1}^{\infty} \ln \left[1 - \frac{x'^2}{n^2\pi^2}\right] \quad (4.8)
$$

and equivalently

$$
L(\omega) = \sum_{n=1}^{\infty} \ln[n^2 + (\beta\omega/2\pi)^2] = \sum_{n=1}^{\infty} \ln n^2 + \sum_{n=1}^{\infty} \ln \left[1 - \frac{x^2}{n^2\pi^2}\right]. \quad (4.9)
$$

Calculating explicitly the first one, we can calculate automatically the second by simply replacing \( \omega' \rightarrow \omega \) and vice-versa. Evidently, in these expressions we introduce the following notations:

$$
x'^2 = -\left(\frac{\beta\omega'}{2}\right)^2, \quad x^2 = -\left(\frac{\beta\omega}{2}\right)^2. \quad (4.10)
$$

So the difference \( L(\omega') - L(\omega) \) becomes

$$
L(\omega') - L(\omega) = \sum_{n=1}^{\infty} \ln \left[1 - \frac{x'^2}{n^2\pi^2}\right] - \sum_{n=1}^{\infty} \ln \left[1 - \frac{x^2}{n^2\pi^2}\right] = \ln \sin x' - \frac{1}{2} \ln x'^2 - \ln \sin x + \frac{1}{2} \ln x^2, \quad (4.11)
$$

or equivalently

$$
L(\omega') - L(\omega) = -\frac{1}{2} \ln \left(\frac{x'^2}{x^2}\right) + \ln \left(\frac{\sin x'}{\sin x}\right). \quad (4.12)
$$

From the relation (4.10) it follows

$$
x' = \pm i \left(\frac{\beta\omega'}{2}\right), \quad x = \pm i \left(\frac{\beta\omega}{2}\right), \quad (4.13)
$$

so the previous equation (4.12) finally becomes

$$
L(\omega') - L(\omega) = \frac{1}{2} \ln \left(\frac{\omega'^2}{\omega^2}\right) + \frac{1}{2} \beta(\omega'-\omega) + \ln \left(\frac{1 - e^{-\beta\omega'}}{1 - e^{-\beta\omega}}\right). \quad (4.14)
$$
B. The explicit expressions for the integrals

Substituting all our results of the summations into Eq. (4.2), dropping a $\beta$-independent terms [1], and performing almost trivial integration over angular variables, one obtains

$$B_{YM}(T) = -\frac{8}{\pi^2} \int d\omega \omega^2 \left[ \frac{3}{4} \Lambda_{NP}^2 \frac{1}{\omega} \frac{1}{e^{\beta\omega} - 1} - 2\beta^{-1} \ln \left( \frac{1 - e^{-\beta\omega'}}{1 - e^{-\beta\omega}} \right) \right].$$

(4.15)

It is convenient to present the integral (4.15) as a sum of a few terms

$$B_{YM}(T) = -\frac{6}{\pi^2} \Lambda_{NP}^2 B_{YM}^{(1)}(T) - \frac{16}{\pi^2} T \left[ B_{YM}^{(2)}(T) - B_{YM}^{(3)}(T) \right],$$

(4.16)

where the explicit expressions of all these integrals are given below

$$B_{YM}^{(1)}(T) = \int_0^{\omega_{eff}} d\omega \frac{\omega}{e^{\beta\omega} - 1};$$

(4.17)

$$B_{YM}^{(2)}(T) = \int_0^{\omega_{eff}} d\omega \omega^2 \ln \left( 1 - e^{-\beta\omega} \right);$$

(4.18)

$$B_{YM}^{(3)}(T) = \int_0^{\omega_{eff}} d\omega \omega^2 \ln \left( 1 - e^{-\beta\omega'} \right).$$

(4.19)

In all these integrals the upper limit $\omega_{eff}$ is explicitly shown now and $\beta^{-1} = T$, while

$$\omega' = \sqrt{\omega^2 + 3\Lambda_{NP}^2} = \omega \sqrt{1 + \frac{3\Lambda_{NP}^2}{\omega^2}}.$$  

(4.20)

V. THE DERIVATION OF $P_{YM}(T)$

The term which contains the information about the nontrivial YM part (3.3) of the future gluon plasma EoS is

$$P_{YM}(T) = -16 \int \frac{d^3q}{(2\pi)^3} \left[ \ln[1 + 3\alpha_s^{PT}(q^2) + 3\alpha_s^{NP}(q^2)] - \frac{3}{4} \alpha_s^{NP}(q^2) + a \right],$$

(5.1)

where $\alpha_s^{PT}(q^2)$ is the nontrivial PT effective charge. Due to the above-mentioned normalization of the effective potential approach in Eq. (2.1), the investigation of this part makes sense to begin with the approximation of the nontrivial PT part by its free PT counterpart, i.e., to put $\alpha_s^{PT}(q^2) = 1$, as a first necessary step. Then Eq. (5.1) for the YM pressure in frequency-momentum space becomes

$$P_{YM}(T) = -16 \int \frac{d^3q}{(2\pi)^3} T \sum_{n=-\infty}^{+\infty} \left[ \ln[1 + \frac{3}{4} \alpha_s^{NP}(q^2, \omega_n^2)] - \frac{3}{4} \alpha_s^{NP}(q^2, \omega_n^2) \right],$$

(5.2)

and after substituting of the relations (3.7) into it, one obtains

$$P_{YM}(T) = -16 \int \frac{d^3q}{(2\pi)^3} T \sum_{n=-\infty}^{+\infty} \left[ \ln[\frac{3}{4} \Lambda_{NP}^2 + q^2 + \omega_n^2] - \ln[q^2 + \omega_n^2] - \frac{3}{4} \Lambda_{NP}^2 \frac{1}{q^2 + \omega_n^2} \right],$$

(5.3)
and the summation over the Matsubara frequencies squared $\omega_n^2 = (2\pi T)^2 n^2$ can be easily done. For this purpose it is also convenient to introduce the following notation

$$\bar{\omega} = \sqrt{q^2 + \bar{m}_{eff}^2} = \sqrt{q^2 + \frac{3}{4} \Lambda_{NP}^2} = \sqrt{\omega^2 + \frac{3}{4} \Lambda_{NP}^2} = \omega \sqrt{1 + \frac{3}{4\omega^2} \Lambda_{NP}^2},$$  \hspace{1cm} (5.4)

so it is possible to say that within our approach to non-zero temperatures at this intermediate stage we have two sorts of gluons: massless $\omega$ and massive $\bar{\omega}$ with the effective mass

$$\bar{m}_{eff} = \sqrt{\frac{3}{2} \Lambda_{NP}} = \frac{1}{2} m'_{eff} \hspace{1cm} (5.5)$$

Comparing Eqs.(4.2) and (5.3) one can write down the final result directly. For this purpose, in the final system of Eqs. (4.16)-(4.19) one must change the overall sign, replace $\omega'$ by $\bar{\omega}$ and integrate from zero to infinity. Thus, one obtains

$$P_Y M(T) = \frac{6}{\pi^2} \Lambda_{NP}^2 P_1(T) + \frac{16}{\pi^2} T \left[ P_2(T) + P_3(T) - P_4(T) \right], \hspace{1cm} (5.6)$$

where the explicit expressions of all these integrals are given below

$$P_1(T) = \int_0^\infty d\omega \frac{\omega}{e^{\beta \omega} - 1}, \hspace{1cm} (5.7)$$

$$P_2(T) = \int_{\omega_{eff}}^\infty d\omega \omega^2 \ln \left(1 - e^{-\beta \omega'}\right), \hspace{1cm} (5.8)$$

$$P_3(T) = \int_0^{\omega_{eff}} d\omega \omega^2 \ln \left(1 - e^{-\beta \omega}\right), \hspace{1cm} (5.9)$$

VI. THE GLUON MATTER EOS

Denoting further $\epsilon_Y M(T) + B_Y M = P_{GM}(T)$ in the left-hand-side of our EoS (3.1), one obtains

$$P_{GM}(T) = B_Y M(T) + P_Y M(T), \hspace{1cm} (6.1)$$

and in this equation $B_Y M(T)$ and $P_Y M(T)$ are given in Eqs. (4.16) and (5.6), respectively. Summing up all the integrals (4.17)-(4.19) and (5.7)-(5.9), one obtains that the gluon matter EoS (6.1) finally becomes,

$$P_{GM}(T) = \frac{6}{\pi^2} \Lambda_{NP}^2 P_1(T) + \frac{16}{\pi^2} T[P_2(T) + P_3(T) - P_4(T)], \hspace{1cm} (6.2)$$

where the dependence on the thermodynamical variable $T$ is only shown explicitly and

$$P_1(T) = \int_{\omega_{eff}}^\infty d\omega \frac{\omega}{e^{\beta \omega} - 1}, \hspace{1cm} (6.3)$$

while

$$P_2(T) = \int_{\omega_{eff}}^\infty d\omega \omega^2 \ln \left(1 - e^{-\beta \omega}\right),$$

$$P_3(T) = \int_0^{\omega_{eff}} d\omega \omega^2 \ln \left(1 - e^{-\beta \omega'}\right),$$

$$P_4(T) = \int_0^\infty d\omega \omega^2 \ln \left(1 - e^{-\beta \omega}\right). \hspace{1cm} (6.4)$$
Let us recall once more that in all integrals $\beta = T^{-1}$ and $\omega_{\text{eff}}$ is fixed (see Appendix), while

$$\tilde{\omega} = \sqrt{\omega^2 + \frac{3}{4} \Lambda_{NP}^2}, \quad \omega' = \sqrt{\omega^2 + 3 \Lambda_{NP}^2}. \quad (6.5)$$

In the formal PT limit ($\Lambda_{NP}^2 = 0$) from these relations it follows that $\tilde{\omega} = \omega' = \omega$ and the combination $P_2(T) + P_3(T) - P_4(T)$ becomes identical zero. Thus the gluon matter pressure (6.2) in this limit vanishes, i.e., it is truly NP, indeed.

The effective potential has been normalized to zero in the $D \to D_0$ limit, which reproduces the so-called Stefan-Boltzmann (SB) non-interacting (ideal) gas of massless particles (gluons) at high temperatures [1]. So the SB limit can be added (if necessary) to the truly NP pressure (6.2) in the $T \to \infty$ limit only, i.e.,

$$P_{GM}(T) \to P_{SB} = \frac{8}{45} \pi^2 T^4, \quad T \to \infty (\beta \to 0). \quad (6.6)$$

In the same way, the corresponding SB limits should be added (if necessary) to all other thermodynamical quantities considered below.

**VII. THERMODYNAMICAL POTENTIAL AND OTHER THERMODYNAMICAL QUANTITIES**

In quantum statistics the thermodynamical potential $\Omega(T)$ is nothing but the pressure $P(T)$ apart from the sign, i.e, in our case we can put

$$\Omega(T) = -P_{GM}(T). \quad (7.1)$$

In quantum statistical theory all the important quantities such as energy density, entropy, etc., are to be expressed in terms of the thermodynamical potential. However, in the truly NP approach we cannot use the trivial relations between them which traced back to the PT even at non-zero temperatures. So the general formulae which to be used are [1]

$$\begin{align*}
\epsilon(T) &= -T \left( \frac{\partial \Omega(T)}{\partial T} \right) + \Omega, \\
\sigma(T) &= -\frac{\partial \Omega(T)}{\partial T}
\end{align*} \quad (7.2)$$

for the pure YM fields, i.e., when the chemical potential is equal to zero. Evidently, here and everywhere below $\epsilon$ and $\sigma$ are energy density and entropy, respectively, of the pure NP gluon matter.

**A. The energy density**

From Eqs. (7.1)-(7.2) it follows that

$$\epsilon(T) = Ts(T) - P(T), \quad (7.3)$$

so substituting the corresponding explicit expressions (7.1) and (6.2) and doing some algebra, one obtains

$$\epsilon(T) = \frac{6}{\pi^2} \Lambda_{NP}^2 \left( T \frac{dP_1(T)}{dT} - P_1(T) \right) + \frac{16}{\pi^2} T^2 \frac{d}{dT} M(T), \quad (7.4)$$

where obviously $d \equiv \partial$, since the only dependence on $T$ is present. Here and everywhere below we introduced the following notation:

$$M(T) = P_2(T) + P_3(T) - P_4(T). \quad (7.5)$$
Also, here and below all the integrals and their derivatives can be explicitly obtained from the expressions (6.3)-(6.5). Again the SB energy density

$$\epsilon_{SB}(T) = \frac{24}{45} \pi^2 T^4$$  \hspace{1cm} (7.6)$$

should be added to our expression (7.4) in the high temperature \( T \to \infty \) (\( \beta \to 0 \)) limit only.

**B. The entropy**

In the same way the entropy (7.2) becomes

$$s(T) = \frac{6}{\pi^2} \Lambda_{NP}^{2} \frac{dP_1(T)}{dT} + \frac{16}{\pi^2} M(T) + \frac{16}{\pi^2} T \frac{d}{dT} M(T),$$  \hspace{1cm} (7.7)$$

and again the SB entropy

$$s_{SB}(T) = \frac{32}{45} \pi^2 T^3.$$  \hspace{1cm} (7.8)$$

should be added to our expression (7.7) in the high temperature \( T \to \infty \) (\( \beta \to 0 \)) limit only.

**C. The heat capacity**

One of the interesting thermodynamical characteristics of the QGP is the heat capacity \( c_V \), which is defined as the derivative of the energy density. Then from the thermodynamical relations (7.1)-(7.2) it follows

$$c_V(T) = \frac{\partial \epsilon(T)}{\partial T} = T \left( \frac{\partial s(T)}{\partial T} \right).$$  \hspace{1cm} (7.9)$$

Using the explicit expression for the energy density (7.4), one finally obtains

$$c_V(T) = \frac{6}{\pi^2} \Lambda_{NP}^{2} T \frac{d^2 P_1(T)}{dT^2} + \frac{32}{\pi^2} T \frac{d}{dT} M(T) + \frac{16}{\pi^2} T^2 \frac{d^2}{dT^2} M(T).$$  \hspace{1cm} (7.10)$$

As in previous cases, the SB heat capacity

$$c_{SB}^V(T) = \frac{96}{45} \pi^2 T^3$$  \hspace{1cm} (7.11)$$

should be added to our expression (7.10) in the high temperature \( T \to \infty \) (\( \beta \to 0 \)) limit only.

**VIII. NUMERICAL RESULTS AND DISCUSSION**

All our numerical results are present in Fig. 3. It is seen explicitly that the NP gluon pressure may almost continuously change its regime in the close neighborhood of a maximum at \( T^* = 266.5 \text{ MeV} \) in order to achieve the thermodynamical SB limit at high temperatures. For the displayed quantities in Fig. 3 the SB limits are the corresponding constants. At the same time, for all other thermodynamical quantities such as the energy density, entropy and heat capacity this is impossible as it follows from the curves shown in Fig. 3 (none of their power-type fall off at this point can be smoothly transformed into the constant behavior at high temperatures). In order to achieve the thermodynamical SB limits at high temperatures their full counter-parts should undergo drastic changes in their regimes in the close neighborhood of this point. As we already know from thermodynamics of \( SU(3) \) lattice QCD [1, 2, 21] the energy and entropy densities have a discontinuity at a point \( T_c = 260 \text{ MeV} \), while the pressure
remains continuous. Our characteristic temperature $T^* = 266.5$ MeV is, surprisingly, very close to the same value. A clear evidence that something nontrivial in the behavior of the thermodynamical quantities in the vicinity of our characteristic temperature $T^* = 266.5$ MeV should actually take place follows from the fact that at this point $\epsilon = 3P$, which should be valid at a very high temperatures only (SB limit). In other words, in order to derive EoS valid above $T^*$, and thus to provide a correct picture of thermodynamics of the gluon matter in the whole range of temperature, one needs the nontrivial approximation of the YM part (5.1), compatible with AF phenomenon in QCD [11]. Just this will be subject of the subsequent paper.

If we were not aware of the thermal lattice QCD results then we would be able to predict them. But we aware of them, so lattice results confirm our expectations of a sharp changes in the behavior of the entropy and energy densities in the region where the pressure is continuous. At the same time, it is worth emphasizing that we have no any problems in describing the behavior of all the important thermodynamical quantities at low temperatures below $T^*$ (see Fig. 3). Moreover, apparently for the first time it is possible to predict their behavior in the region of low temperatures within our approach (there are no convincing lattice data for this region). We do not expect any serious changes in the behavior of the thermodynamical quantities in this region (exponential fall off or rise when the temperature goes down or up, respectively) even after taking into account the above-mentioned nontrivial approximation of the YM part (5.1), apart from ”non-physical” maximums which should disappear, of course. However, whatever changes may occur they will be under our control.

The confinement dynamics (3.5) generalized on zero-temperatures in Eq. (3.7) is still important especially in the region of low temperatures even up to the temperature at which all the important thermodynamical quantities may undergo drastic changes in their behavior (apart from the pressure). From the structure of our EoS (see Eqs. (6.2)-(6.5)), it clearly follows that below $\omega_{eff}$ (which fixes $T^*$, there is no explicit dependence between them, but rather a correspondence) and thus below $T^*$ we have mainly the massive gluon excitations $\omega'$ of the dynamical origin, which can be interpreted as the glueballs with masses $m'_{eff} = \sqrt{3}\Lambda_{NP} = 1.17$ GeV. Above $T^*$ the gluon matter consists mainly of the free gluons $\omega$ ($\bar{\omega}$ gluons are artifacts due to the approximation of the nontrivial PT effective charge by its free PT counterpart in Eq. (5.1), as well as the above-mentioned ”non-physical” maximums and hence their ”tails” at high temperatures in Fig. 3). Just the confinement dynamics determines the phase transition from glueballs to ”free” gluons and vice-versa in the case of $SU(3)$ YM fields within our approach, indeed.

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APPENDIX A: THE SCALE-SETTING SCHEME

From the relations (3.6) it follows that in frequency-momentum space a possible free parameter of our approach is the effective scale

$$\omega_{\text{eff}} = \sqrt{q_{\text{eff}}^2 - \omega_c^2},$$  \hfill (A1)

where we introduced the constant Matsubara frequency $\omega_c$, which is always positive. So $\omega_{\text{eff}}$ is always less or equal to $q_{\text{eff}}$ of the four-dimensional QCD, i.e.,

$$\omega_{\text{eff}} \leq q_{\text{eff}}.$$  \hfill (A2)

One then can conclude that $q_{\text{eff}}$ is a very good upper limit for $\omega_{\text{eff}}$. In this connection, let us recall now that the Bag constant $B_{YM}$ at zero temperatures has been successfully calculated at a scale $q_{\text{eff}}^2 = 1 \text{ GeV}^2$, in fair agreement with other phenomenological quantities such as gluon condensate [16]. So let us fixed the effective scale $\omega_{\text{eff}}$ as follows:

$$\omega_{\text{eff}} = q_{\text{eff}} = 1 \text{ GeV}.$$  \hfill (A3)

The mass gap squared $\Lambda_{NP}^2$ calculated just at this scale is equal to [16]

$$\Lambda_{NP}^2 = 0.4564 \text{ GeV}^2.$$  \hfill (A4)

Thus, we have no free parameters in our approach. The confinement dynamics is nontrivially taken into account directly through the mass gap, and not through the Bag constant itself.