A new family realizing saturated fusion systems

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Abstract

We construct a new group model for fusion systems related to Robinson’s models and study modules and a homology decomposition associated with it. Moreover we prove analogues of Glaubermann’s and Thompson’s theorems for $p$–local finite groups and a Künneth formula for fusion systems.

1 Introduction

In the topological theory of $p$–local finite groups introduced by Broto, Levi and Oliver one tries to approximate the classifying space of a finite group via the $p$–local structure of the group, at least up to $\mathbb{F}_p$–cohomology. In this article we introduce a new group model related to Robinson’s construction and study a homology decomposition and a family of modules associated to it. Building on the work of Díaz, Glesser, Mazza and Park we prove analogues of Glaubermann’s and Thompson’s theorems for $p$–local finite groups. Moreover we provide a Künneth formula independently of the existence of a classifying space.

2 Preliminaries

2.1 Fusion Systems

We review the basic definitions of fusion systems and centric linking systems and establish our notations. Our main references are [4], [5] and [12]. Let $S$ be a finite $p$-group. A fusion system $\mathcal{F}$ on $S$ is a category whose objects are all the subgroups of $S$, and which satisfies the following two properties for all $P,Q \leq S$: The set $\text{Hom}_\mathcal{F}(P,Q)$ contains injective group homomorphisms and amongst them all morphisms induced by conjugation of elements in $S$ and each element is the composite of an isomorphism in $\mathcal{F}$ followed by an inclusion. Two subgroups $P,Q \leq S$ will be called $\mathcal{F}$–conjugate if they are isomorphic in $\mathcal{F}$. Define $\text{Out}_\mathcal{F}(P) = \text{Aut}_\mathcal{F}(P)/\text{Inn}(P)$ for all $P \leq S$. A subgroup $P \leq S$ is fully centralized resp. fully normalized in $\mathcal{F}$ if $|C_S(P)| \geq |C_S(P')|$ resp. $|N_S(P)| \geq |N_S(P')|$ for all $P' \leq S$ which is $\mathcal{F}$-conjugate to $P$. $\mathcal{F}$ is called saturated if for all $P \leq S$ which is fully normalized in $\mathcal{F}$, $P$ is fully centralized in $\mathcal{F}$ and $\text{Aut}_S(P) \in \text{Syl}_p(\text{Aut}_\mathcal{F}(P))$ and moreover if $P \leq S$ and $\phi \in \text{Hom}_\mathcal{F}(P,S)$ are such that $\phi(P)$ is fully centralized, and if we set $N_\phi = \{g \in N_S(P)|\phi c_g \phi^{-1} \in \text{Aut}_S(\phi(P))\}$, then there is $\phi \in \text{Hom}_\mathcal{F}(N_\phi,S)$ such that $\phi|_P = \phi$. A subgroup $P \leq S$ will be called $\mathcal{F}$–centric if $C_S(P') \leq P'$ for all $P'$ which are $\mathcal{F}$–conjugate to $P$. Denote $\mathcal{F}^c$ the full subcategory of $\mathcal{F}$ with objects the $\mathcal{F}$–centric subgroups of $S$. Let $\mathcal{O}(\mathcal{F})$ be the orbit category of $\mathcal{F}$ with objects the same objects as $\mathcal{F}$ and morphisms the set $\text{Mor}_\mathcal{O}(\mathcal{F})(P,Q) = \text{Mor}_\mathcal{F}(P,Q)/\text{Inn}(Q)$. Let $\mathcal{O}^c(\mathcal{F})$ be the full subcategory of $\mathcal{O}(\mathcal{F})$ with objects the $\mathcal{F}$–centric subgroups of $\mathcal{F}$. A centric linking system associated to $\mathcal{F}$ is a category $\mathcal{L}$ whose objects are the $\mathcal{F}$-centric subgroups of $S$, together with a functor $\pi : \mathcal{L} \rightarrow \mathcal{F}^c$, and "distinguished"
monomorphisms $\delta_P : P \to Aut_\mathcal{L}(P)$ for each $\mathcal{F}$-centric subgroup $P \leq S$ such that the following conditions are satisfied: $\pi$ is the identity on objects and surjective on morphisms. More precisely, for each pair of objects $P, Q \in \mathcal{L}$, $Z(P)$ acts freely on $Mor_\mathcal{L}(P, Q)$ by composition (upon identifying $Z(P)$ with $\delta_P(Z(P)) \leq Aut_\mathcal{L}(P)$), and $\pi$ induces a bijection $\mathcal{M}or_\mathcal{L}(P, Q)/Z(P) \cong Hom_{\mathcal{F}}(P, Q)$. For each $\mathcal{F}$-centric subgroup $P \leq S$ and each $x \in P$, $\pi(\delta_P(x)) = c_x \in Aut_\mathcal{F}(P)$. For each $f \in Mor_\mathcal{L}(P, Q)$ and each $x \in P$, $f \circ \delta_P(x) = \delta_Q(f(x)) \circ f$. Let $\mathcal{F}$, $\mathcal{F}'$ be fusion systems on finite $p$-groups $S, S'$, respectively. A morphism of fusion systems from $\mathcal{F}$ to $\mathcal{F}'$ is a pair $(\alpha, \Phi)$ consisting of a group homomorphism $\alpha : S \to S'$ and a covariant functor $\Phi : \mathcal{F} \to \mathcal{F}'$ with the following properties: for any subgroup $Q$ of $S$ we have $\alpha(Q) = \Phi(Q)$ and for any morphism $\phi : Q \to R$ in $\mathcal{F}$ we have $\Phi(\phi) \circ \alpha|_Q = \alpha|_R \circ \phi$. Let $G$ be a discrete group. A finite subgroup $S$ of $G$ will be called a Sylow $p$-subgroup of $G$ if $S$ is a $p$-subgroup of $G$ and all $p$-subgroups of $G$ are conjugate to a subgroup of $S$. A group $G$ is called $p$-perfect if $H_1(BG; \mathbb{Z}_p) = 0$. Let $\mathcal{F}$ be a saturated fusion system over the finite $p$-group $S$. Let $G_1, G_2$ be groups with $Sylow$ $p$-subgroups and $\phi : G_1 \to G_2$ a group homomorphism. $\phi$ will be called fusion preserving if the restriction to the respective Sylow $p$-subgroups induces an isomorphism of fusion systems $\mathcal{F}_{S_1}(G_1) \cong \mathcal{F}_{S_2}(G_2)$. Let $S$ be a finite $p$-group and let $P_1, ..., P_n$, $Q_1, ..., Q_k$ be subgroups of $S$. Let $\phi_1, ..., \phi_r$ be injective group homomorphisms $\phi_i : P_i \to Q_i$, $\forall i$. The fusion system generated by $\phi_1, ..., \phi_r$ is the minimal fusion system $\mathcal{F}$ over $S$ containing $\phi_1, ..., \phi_r$. Let $\mathcal{F}$ be a fusion system on a finite $p$-group $S$. A subgroup $T \leq S$ is strongly closed in $S$ with respect to $\mathcal{F}$, if for each subgroup $P$ of $T$, each $Q \leq S$, and each $\phi \in Mor_\mathcal{F}(P, Q)$, $\phi(P) \leq T$. Fix any pair $S \leq G$, where $G$ is a (possibly infinite) group and $S$ is a finite $p$-subgroup. Define $\mathcal{F}_S(G)$ to be the category whose objects are the subgroups of $G$, and where $\mathcal{F}_S(G)(P, Q) = \{g \in G | gPq^{-1} \leq Q\} \cong N_G(P, Q)/C_G(P)$. Here $c_g$ denotes the homomorphism conjugation by $g (x \mapsto gxg^{-1})$, and $N_G(P, Q) = \{ g \in G | gPq^{-1} \leq Q \}$ (the transporter set). For each $P \leq S$, let $C_G(P)$ be the maximal $p$-perfect subgroup of $C_G(P)$. Let $\mathcal{L}_{S}(G)$ be the category whose objects are the $\mathcal{F}_S(G)$-centric subgroups of $S$, and where $\mathcal{F}_S(G)(P, Q) = N_G(P, Q)/C_G(P)$. Let $\pi : \mathcal{L}_{S}(G) \to \mathcal{F}_S(G)$ be the functor which is the inclusion on objects and sends the class of $g \in N_G(P, Q)$ to conjugation by $g$. For each $\mathcal{F}_S(G)$-centric subgroup $P \leq G$, let $\delta_P : P \to Aut_\mathcal{L}_S(P)$ be the monomorphism induced by the inclusion $P \leq N_G(P, Q)$. A triple $(S, \mathcal{F}, \mathcal{L})$ where $S$ is a finite $p$-group, $\mathcal{F}$ is a saturated fusion system on $S$, and $\mathcal{L}$ is an associated centric linking system to $\mathcal{F}$, is called a $p$-local finite group. It’s classifying space is $L_p^n$ where $(-)^n$ denotes the $p$-completion functor in the sense of Bousfield and Kan. This is partly motivated by the fact that every finite group $G$ gives canonically rise to a $p$-local finite group $(S, \mathcal{F}_S(G), \mathcal{L}_S)$ and $BG_p^n \simeq L_p^n[1]$. In particular, all fusion systems coming from finite groups are saturated. Let $\mathcal{F}$ be a fusion system on the the finite $p$-group $S$. $\mathcal{F}$ is called an Alperin fusion system if there are subgroups $P_1, P_2, \cdots, P_r$ of $S$ and finite groups $L_1, \cdots, L_r$ such that $P_i \cong O_p(L_i)$ (the largest normal $p$-subgroup of $L_i$) and $C_{L_i}(P_i) = Z(P_i), L_i/P_i \cong Aut_\mathcal{F}(P_i)$ for each $i$, $N_S(P_i)$ is a Sylow $p$-subgroup of $L_i$ for each $i$ and $P_1 = S$, for each $i$ $\mathcal{F}_{N_S(P_i)}(L_i)$ is contained in $\mathcal{F}$, $\mathcal{F}$ is generated by all the $\mathcal{F}_{N_S(P_i)}(L_i)$. Recall that every saturated fusion system is Alperin since let $\mathcal{F}$ be a saturated fusion system over a finite $p$-group $S$. Let $S = P_1, \cdots, P_n$ be subgroups of $S$ which are representatives of isomorphism classes of centric radicals in $\mathcal{F}$. From Section 4 it follows that we can find corresponding groups $L_i$, $i = 1, \ldots, n$ which have all the properties. One can define fusion systems and centric linking systems in a topological setting. We will need this when we make use of the fact that a group realizes a given fusion system if and only if its classifying space has a certain homotopy type. In particular we have for a $p$-local finite group $(S, \mathcal{F}, \mathcal{L})$ and a group $G$ such that $\mathcal{F}_S(G) = \mathcal{F}$ that there is a map from the one-skeleton of the nerve of $\mathcal{L}$ to the classifying space: $|\mathcal{L}|^{(1)} \to BG$. Fix a space $X$, a finite $p$-group $S$, and a map $f : BS \to X$. Define $\mathcal{F}_{S, f}(X)$ to be the category whose objects are the subgroups of $S$, and whose morphisms are given by $Hom_{\mathcal{F}_S,f}(X)(P, Q) = \{ \phi \in Inj(P, Q) | f|_{BP} \simeq f|_{BQ} \circ B\phi \}$ for each $P, Q \leq S$. Define $\mathcal{F}_{S,f}(X) \subseteq \mathcal{F}_{S,f}(X)$ to be the subcategory with the same objects as $\mathcal{F}_{S,f}(X)$, and where $\mathcal{M}or_{\mathcal{F}_{S,f}}(X)(P, Q)$ (for $P, Q \leq S$) is the set of all composites of restrictions of morphisms in $\mathcal{F}_{S,f}(X)$ between $\mathcal{F}_{S,f}(X)$-centric subgroups. Define $\mathcal{L}_{S,f}(X)$ to be the category whose objects are the $\mathcal{F}_{S,f}(X)$-centric subgroups of $S$, and whose morphisms are defined by $\mathcal{M}or_{\mathcal{L}_{S,f}}(P, Q) =$
\[
\{(\phi,[H])|\phi \in \text{Inj}(P,Q), H : BP \times I \to X, H|_{BP \times 0} = f|_{BP}, H|_{BP \times 1} = f|_{BQ} \circ B\phi\}. \]

The composite in \(L_{S,f}(X)\) of morphisms \(P (\phi,[H]) \xrightarrow{(\psi,[K])} Q \xrightarrow{R}\), where \(H : BP \times I \to X\) and \(K : BQ \times I \to X\) are homotopies as described above, are defined by setting \((\psi,[K]) \circ (\phi,[H]) = (\psi \circ \phi, [K \circ (B\phi \circ ID)] \ast H)\), where \(\ast\) denotes composition (juxtaposition) of homotopies. Let \(\pi : L_{S,f}(X) \to \mathcal{F}_{S,f}(X)\) be the forgetful functor: it is the inclusion on objects, and sends a morphism \((\phi,[H])\) to \(\phi\). For each \(\mathcal{F}_{S,f}(X)\)-centric subgroup \(P \leq S\), let \(\delta_P : P \to \text{Aut}_{L_{S,f}(X)}(P)\) be the "distinguished homomorphism" which sends \(g \in P\) to \((e_g,[f|_{BP} \circ H_g])\), where \(H_g : BP \times I \to BP\) denotes the homotopy from \(Id_{BP}\) to \(Be_g\) induced by the natural transformation of functors \(B(G) \to B(G)\) which sends the unique object \(\circ_G\) in \(BG\) to the morphism \(g\) corresponding to \(g\) in \(G\).

**Theorem 2.1** ([5], Theorem 2.1.) Fix a space \(X\), a \(p\)-group \(S\), and a map \(f : BS \to X\). Assume that \(f\) is Sylow; \(f|_{BP} \) is a centric map for each \(\mathcal{F}_{S,f}(X)\)-centric subgroup \(P \leq S\); and every \(\mathcal{F}_{S,f}(X)\)-centric subgroup of \(S\) is also \(\mathcal{F}_{S,f}(X)\)-centric. Then the triple \((S,\mathcal{F}_{S,f}(X),L_{S,f}(X))\) is a \(p\)-local finite group.

### 2.2 Groups Realizing a Given Fusion System

Given a fusion system \(\mathcal{F}\) on a finite \(p\)-group \(S\) it is not always true that there exists a finite group \(G\) such that \(\mathcal{F}_S = \mathcal{F}_S(G)\), (see [4], chapter 9 for example). However for every fusion system \(\mathcal{F}\) there exists an infinite group \(G\) such that \(\mathcal{F}_S(G) = \mathcal{F}\). We now describe the constructions by G. Robinson [14], and I. Leary and R. Stancu [8]. The groups of Robinson type are iterated amalgams of automorphism groups in the linking system, if it exists, over the \(S\)-normalizers of the respective \(\mathcal{F}\)-centric subgroups of \(S\). Note that these automorphism groups exist and are known regardless of whether \(L\) exists or not.

**Theorem 2.2** ([14], Theorem 2.) Let \(\mathcal{F}\) be an Alperin fusion system on a finite \(p\)-group \(S\) and associated groups \(L_1, \ldots, L_n\) as in the definition. Then there is a finitely generated group \(G\) which has \(S\) as a Sylow \(p\)-subgroup such that the fusion system \(\mathcal{F}\) is realized by \(G\). The group \(G\) is given explicitly by \(G = L_1 *_{N_S(P_2)} L_2 *_{N_S(P_3)} \cdots *_{N_S(P_n)} L_n\) with \(L_i, P_i\) as in the definition.

Corresponding to the various versions of Alperin’s fusion theorem (essential subgroups, centric radical subgroups) there exist several canonical choices for the groups generating \(\mathcal{F}\). The group constructed by I. Leary and R. Stancu is an iterated HNN-construction.

**Theorem 2.3** ([8], Theorem 2.) Suppose that \(\mathcal{F}\) is the fusion system on \(S\) generated by \(\Phi = \{\phi_1, \ldots, \phi_r\}\). Let \(T\) be a free group with free generators \(t_1, \ldots, t_r\), and define \(G\) as the quotient of the free product \(S * T\) by the relations \(t_i^{-1}u_t = \phi_i(u)\) for all \(i\) and for all \(u \in P_i\). Then \(S\) embeds as a \(p\)-Sylow subgroup of \(G\) and \(\mathcal{F}_S(G) = \mathcal{F}\).

### 2.3 Graphs of Groups

We give a short introduction to graphs of groups stating results we need. A finite directed graph \(\Gamma\) consists of two sets, the vertices \(V\) and the directed edges \(E\), together with two functions \(\iota, \tau : E \to V\). For \(e \in E\), \(\iota(e)\) is called the initial vertex of \(e\) and \(\tau(e)\) is the terminal vertex of \(e\). Multiple edges and loops are allowed in this definition. The graph \(\Gamma\) is connected if the only equivalence relation on \(V\) that contains all \((\iota(e), \tau(e))\) is the relation with just one class. A graph \(\Gamma\) may be viewed as a category, with objects the disjoint union of \(V\) and \(E\) and two non-identity morphisms with domain \(e\) for each \(e \in E\), one morphism \(e \to \iota(e)\) and one morphism \(e \to \tau(e)\). A graph \(\Gamma\) of groups is a connected graph \(\Gamma\) together with groups \(G_v, G_e\) for each vertex and edge and injective group homomorphisms \(f_{e,\iota} : G_e \to G_{\iota(e)}\) and \(f_{e,\tau} : G_e \to G_{\tau(e)}\) for each edge \(e\).
3 A new family realizing saturated fusion systems

We introduce a new group model realizing saturated fusion systems related to the construction of G. Robinson.

**Theorem 3.1** Let \((S, F, L)\) be a \(p\)-local finite group and
\[
\mathcal{G} = L_1 \ast_{N_S(P_2)} L_2 \ast_{N_S(P_3)} \cdots \ast_{N_S(P_n)} L_n
\]
a model of Robinson type for \(F\). For each of the \(L_i\), \(i = 1, \cdots, n\) choose subgroups \(K_1, \cdots, K_m\) of \(L_i\) such that each \(K_j\) contains (an isomorphic copy of) the group \(N_S(P_i)\) and \(K_1, \cdots, K_m\) generate the group \(L_i\). Assume as we can that \(S \in \text{Syl}_p(K_1)\) and after reindexing let \(G\) be the iterated amalgam
\[
G = K_1 \ast_{N_S(P_2)} K_2 \ast_{N_S(P_3)} \cdots \ast_{N_S(P_n)} K_n.
\]
Then \(G\) contains \(S\) as a Sylow \(p\)-subgroup and \(F_S(G) = F\).

**Theorem 3.2** Let \((S, F, L)\) be a \(p\)-local finite group and let \(\Phi = \{\phi_1, \cdots, \phi_n\}\) be a subset of the set of automorphisms of the fusion system which is chosen in a minimal way, i.e. we cannot omit any element without obtaining a proper subsystem. Moreover assume as we can that all the elements of \(\Phi\) have order coprime to \(p\). Then the group
\[
\mathcal{G} := S \ast F(\Phi)/\langle \phi u\phi^{-1} = \phi(u), \phi^{deg(\phi)} = 1 \rangle
\]
has the following properties. The group \(S \in \text{Syl}_p(\mathcal{G})\), \(F_S(\mathcal{G}) = F\), the classifying space \(BG\) is \(p\)-good and the cohomology of \(BG\) is \(F\)-isomorphic in the sense of Quillen to the stable elements.

### 3.1 Homology Decompositions

We investigate the cohomology of our models for a saturated fusion system \(F\) over a finite \(p\)-group \(S\). In the following \(G\) will always be a model for \(F\) of Robinson type, i.e. \(G = L_1 \ast_{N_S(P_2)} L_2 \ast_{N_S(P_3)} \cdots L_n\) where \(L_1, \cdots, L_n\) can be chosen such that \(L_1, \cdots, L_n\) are representatives of isomorphism classes of centric radicals, of \(F\)-centrics or of essential subgroups of \(F\), and \(L_1, \cdots, L_n\) are the corresponding automorphism groups of \(L_1, \cdots, L_n\) in the linking system if it exists. Note that these groups are known and are unique and do exist regardless of whether \(L\) exists or not, see [12, Theorem 4.6.] following the discussion in [2], Section 4.

**Theorem 3.3** Let \((S, F, L)\) be a \(p\)-local finite group and \(G\) a discrete group such that \(S \in \text{Syl}_p(G)\) and \(F = F_S(G)\). Then there exist a natural map of unstable algebras \(H^*(BG) \stackrel{\Delta}{\rightarrow} H^*(F)\) making \(H^*(F)\) a module over \(H^*(BG)\).

**Theorem 3.4** Let \((S, F, L)\) be a \(p\)-local finite group and \(G\) a model of our type for \(F\). Then there exist natural maps of unstable algebras over the Steenrod algebra \(H^*(BG) \stackrel{\Delta}{\rightarrow} H^*(|L|)\) and \(H^*(|L|) \stackrel{r}{\rightarrow} H^*(BG)\) such that we obtain a split short exact sequence of unstable modules over the Steenrod algebra \(0 \rightarrow H^*(|L|) \rightarrow H^*(BG) \rightarrow W \rightarrow 0\) where \(W \cong \ker(\text{Res}_S^G)\).
Proof: Let $\mathcal{C}$ be the following category.

Denote by $\phi_{i,j} : \bullet_i \to \bullet_j$ the unique morphism in $\mathcal{C}$ between $\bullet_i$ and $\bullet_j$ if it exists. Let $F : \mathcal{C} \to \text{Spaces}$ be a functor with $F(\bullet_i) = BL_i$ for $i = 1, \ldots, n$. Then $BN_S(P_i)$ for $i = n+1, \ldots, 2n-1$ and $F(\phi_{i,j}) = \text{Bincl} : F(\bullet_i) \to F(\bullet_j)$ for all $\phi_{i,j} : \bullet_i \to \bullet_j$ in $\mathcal{C}$, $i = n+1, \ldots, 2n-1$, $j = i-n+1, 1$. Note that $hocolim(F)$ is a $K(G, 1)$. Since $K_i \leq L_i = Aut_{\mathcal{C}}(P_i)$ for all $i = 1, \ldots, n$ we have a functor $BK_i \to \mathcal{C}$ which sends the unique object $\bullet$ to $P_i$ and a morphism $x$ to the corresponding morphism in $Aut_{\mathcal{C}}(P_i)$ for all $i = 1, \ldots, n$. Therefore we obtain a map $BK_i$ to $|\mathcal{C}|$ for all $i = 1, \ldots, n$. Note that all the diagrams $BN_S(P_i)$ commute up to homotopy since the third axiom from the definition of the linking system guarantees that we can find a compatible system of lifts of the inclusion $\iota_{N_S(P_i), S}$ in $\mathcal{C}$ for all $i = 1, \ldots, n$ such that all the diagrams $BN_S(P_i)$ commute up to the natural transformation which takes the object $\bullet \in \text{Obj}(BN_S(P_i))$ to $\iota_{N_S(P_i), S}$ for $i = 1, \ldots, n$. We obtain a map from the 1-skeleton of the homotopy colimit of the functor $F$ over the category $\mathcal{C}$ to $|\mathcal{C}|$. Since $\mathcal{C}$ is a 1-dimensional category we obtain a map from $BG$ to $|\mathcal{C}|$. This map will be denoted by $q$ inducing $H^*|\mathcal{C}| \xrightarrow{q^*} H^*(BG)$. Denote the kernel of the map $f$ by $W$. We have the following commutative diagram of unstable algebras over the Steenrod algebra where the maps $q^*$ and $\text{incl}$ are injective $H^*|\mathcal{C}| \xrightarrow{q^*} H^*(BG)$. Commutativity implies that $W \cong \text{Ker}(\text{Res}_S^G)$ in the category of unstable modules over the Steenrod algebra. □

**Theorem 3.5** Let $\mathcal{F}$ be an Alperin fusion system and $G$ a model of our type for it. Then $BG$ is $p$-good.

**Proof:** The group $G$ is a finite amalgam of finite groups. Note that each $K_i$ is generated by $N_S(P_i)$ and elements of $p'$-order. Therefore $G$ is generated by elements of $p'$-order and $S$. Let $K$ be the subgroup of $G$ generated by all elements of $p'$-order. Note that $K$ is normal in $G$ and $S$ surjects on $G/K$ and therefore $G/K$ is a finite $p$-group. We have $H^1(BK; \mathbb{F}_p) = 0$ and therefore $K$ is a $p$-perfect. Let $X$ be the cover of $BG$ with fundamental group $K$. Then $X$ is $p$-good and $X_p$ is simply connected since $\pi_1(X)$ is $p$-perfect as follows from [1] VII.3.2. Hence $X_p \to BG_p \to B(G/K)$ is a fibration sequence and $BG_p$ is $p$-complete by [1] II.5.2(iv). So $BG$ is $p$-good. □

**Theorem 3.6** Let $(S, \mathcal{F}, \mathcal{L})$ be a $p$-local finite group and $G$ be a model of our type for it. Then $H^*(BG)$ is finitely generated.

**Proof:** Note that we have a map $BG = hocolim(F) \to |\mathcal{C}|$ where $F$ and $C$ are as defined in the proof of Theorem 4.3, for the model of our type $G$. Note that $N_S(P_i) \in Syl_p(K_i)$ for all $i = 1, \ldots, n$. It follows from [2] Lemma 2.3 and [3] Theorem 4.4(a) that $H^*(B(P_i))$ is finitely generated.
over $H^*(|\mathcal{L}|)$ for all $i = 1, \ldots, n$, and $H^*(|\mathcal{L}|)$ is noetherian as follows from [4 Proposition 1.1. and Theorem 5.8.]. Therefore the Bousfield-Kan spectral sequence for $H^*(BG)$ is a spectral sequence of finitely generated $H^*(|\mathcal{L}|)$-modules, the $E_2$ term with $E_2^{s,t} = \lim^s H^t(F(-); \mathbb{F}_p)$ is concentrated in the first two columns and $E_2 = E_\infty$ for placement reasons. Therefore $H^*(BG)$ is a finitely generated module over $H^*(|\mathcal{L}|)$ and in particular noetherian. \(\square\)

### 3.1.1 A stable retract

**Theorem 3.7** Let $(S, \mathcal{F}, \mathcal{L})$ be a $p$-local finite group and $G$ a model of our type for $\mathcal{F}$. Then $|\mathcal{L}|^\wedge_p$ is a stable retract of $BG_p^\wedge$.

**Proof:** The diagram

$$
\begin{array}{c}
\Sigma^\infty B(\delta_2)^\wedge_p \ar[r] & \Sigma^\infty BS_p^\wedge \ar[l] \\
\Sigma^\infty |\mathcal{L}|^\wedge_p \ar[r] & \Sigma^\infty BG_p^\wedge
\end{array}
$$

commutes where $q$ is the map constructed in the proof of Theorem 4.1. By the work of K. Ragnarsson [3] there is a map $\sigma_\mathcal{F} : \Sigma^\infty |\mathcal{L}|^\wedge_p \to \Sigma^\infty BS_p^\wedge$ such that the composition of maps $\Sigma^\infty |\mathcal{L}|^\wedge_p \mathcal{F} \to \Sigma^\infty BS_p^\wedge \xrightarrow{\Sigma^\infty (\delta_2)_{\mathcal{F}}^\wedge_p} \Sigma^\infty |\mathcal{L}|^\wedge_p$ is the identity. Since $\Sigma^\infty B(\delta_2)^\wedge_p \circ \sigma_\mathcal{F} = \Sigma^\infty q^\wedge_p \circ \Sigma^\infty Bincl^\wedge_p \circ \sigma_\mathcal{F}$ we have $|\mathcal{L}|^\wedge_p$ is a stable retract of $BG_p^\wedge$. \(\square\)

### 4 Modules and Euler characteristic

Let $\mathcal{F}$ be an Alperin fusion system with associated groups $L_1, \ldots, L_r$ and associated subgroups $K_1, \ldots, K_n$ as described above and let

$$G = K_1 \underset{N_S(P_2)}{*} K_2 \underset{N_S(P_3)}{*} K_3 \cdots \underset{N_S(P_n)}{*} K_n$$

the group model discussed so far. Then we can see inductively that given a group $M$ and group homomorphisms $\phi_i : K_i \to M$ for $i = 1, \ldots, n$ with

$$\text{Res}_{N_S(P_i)}^{K_i}(\phi_i) = \text{Res}_S^{N_S(P_i)}(\phi_i)$$

for each $i$, there is a unique group homomorphism $\phi : G \to M$ which extends each $\phi_i$.

We now want to study the finite-dimensional $kG$-modules. Notice that if $H$ is a finite group generated by subgroups $M_i$ for $i = 1, \ldots, n$ such that for each $i$ there is a group epimorphism $\alpha_i : K_i \to M_i$ with $\text{Res}_{N_S(P_i)}^{K_i}(\alpha_i) = \text{Res}_S^{N_S(P_i)}(\alpha_i)$, then $H$ is an epimorphic image of $G$.

We allow the possibility that $U = O_p(G) \neq 1$. We remark that $U \leq E$, for each $i$, since

$$N_U(E_i)E_iC_G(E_i)/E_iC_G(E_i) \leq O_p(N_G(E_i))/E_iC_G(E_i) \cong O_p(\text{Out}_\mathcal{F}(E_i)) = 1.$$

**Theorem 4.1** For each $i$, let $H_i$ be a $p'$-subgroup of $K_i$, and let $t = \text{lcm}\{[K_i : H_i] : 1 \leq i \leq r\}$. Then there is a group homomorphism $\phi : G \to S_t$ whose kernel is a free group.

**Proof:** It is enough to construct the homomorphism $\phi$ so that $\ker \phi$ has trivial intersection with $S$. Let $\Omega = \{1, 2, \ldots, t\}$ and let each $K_i$ act as it would on the direct sum of $t/[K_i : H_i]$ copies of the permutation module of $K_i$ on the cosets of $H_i$. Since $S$ acts semi-regularly, and each $N_S(P_i)$ does, we may label the points so that each $N_S(P_i)$ acts in the manner determined by regarding it as a subgroup of $S$. By the remarks preceding the Theorem, $\Omega$ now has the structure of a $G$-set. Since the action of $P$ is free, the kernel of the action is a free normal subgroup of finite index. \(\square\)
Theorem 4.2 1. For $1 \leq i \leq r$ let $X_i$ be a finite-dimensional projective $kK_i$-module, and suppose that all $X_i$ have equal dimension. Then there is a $kG$-module $X$ such that $\text{Res}^G_{K_i}(X) \cong X_i$ for each $i$. Furthermore, $C_G(X)$ is a free normal subgroup of $G$ of finite index.

2. For each $i$, let $V_i$ be a simple $kK_i$-module. Then there is a finite dimensional projective $kG$-module $V$ such that for each $i$, $\text{soc} \left( \text{Res}^G_{K_i}(V) \right)$ is isomorphic to a direct sum of copies of $V_i$.

Proof:

1. Since $X_i$ is a free $kN_S(P_i)$-module for each $i$ we can suppose that $N_S(P_i)$ has the same action on $X_i$ as it does on $\text{Res}^S_{N_S(P_i)}(X_1)$ for each $i$. In that case, there is a unique way to extend the action of the $K_i$ on the underlying $k$-vector space to an action of $G$ on that space. We let $X$ denote the $kG$-module so obtained. Then $X$ is a free $kS$-module, so that $C_G(X) \cap S = 1$ and $C_G(X)$ is a free normal subgroup of $G$ of finite index.

2. Let $Y_i$ denote the projective cover of $V_i$ as $kK_i$-module, so that we also have $\text{soc}(Y_i) \cong V_i$. Let $V$ be a simple $kG$-module of $X$. Then $\text{soc}(\text{Res}^G_{K_i}(V))$ is a submodule of $\text{soc}(\text{Res}^G_{K_i}(X))$ for each $i$, so the result follows. □

Theorem 4.3 Let $\mathcal{F}$ be a saturated fusion system over the finite $p$-group $S$ and

$$G = K_1 \ast_{N_S(P_2)} K_2 \ast_{N_S(P_3)} \cdots \ast_{N_S(P_n)} K_n$$

a model of our type for $\mathcal{F}$. Then we have the following formula

$$\chi(G) = \left( \sum_{1 \leq i \leq n} \frac{1}{|K_i|} \right) - \left( \sum_{2 \leq i \leq n} \frac{1}{|N_S(P_i)|} \right)$$

for the Euler characteristic of $\chi(G)$ of $G$. Moreover we have

$$\chi(G) = \frac{d_F}{|S| \text{lcm}\{K_i : N_S(P_i), 1 \leq i \leq n\}}$$

for some negative integer $d_F$.

Proof: Proof;

Theorem 4.4 1. Any free subgroup of finite index of $X$ has index divisible by

2.

5 A Kuenneth Formula for Fusion Systems

We prove an analogue of the Kuenneth Formula for saturated fusion systems independently of the existence of a classifying space.

Theorem 5.1 Let $\mathcal{F}_1, \mathcal{F}_2$ be saturated fusion systems over the finite $p$-groups $S_1, S_2$ respectively. Then $H^*(\mathcal{F}_1 \times \mathcal{F}_2) \cong H^*(\mathcal{F}_1) \otimes H^*(\mathcal{F}_2)$.

The proof requires a lemma.
Lemma 5.1 Let $\mathcal{F}_1, \mathcal{F}_2$ be saturated fusion systems over the finite $p$–groups $S_1, S_2$ respectively. Let $\mathcal{F} = \mathcal{F}_1 \times \mathcal{F}_2$ be the saturated fusion system over $S = S_1 \times S_2$. Then $P = P_1 \times P_2$ is $\mathcal{F}$–centric if and only if $P_1$ is $\mathcal{F}_1$–centric and $P_2$ is $\mathcal{F}_2$–centric.

Proof: Assume $P_1$ is $\mathcal{F}_1$–centric and $P_2$ is $\mathcal{F}_2$–centric. Then we have for every $P' = P'_1 \times P'_2$ which is $\mathcal{F}$–conjugate to $P$ that $|C_S(P')| = |C_{S_1 \times S_2}(P_1 \times P_2)| = |C_{S_1}(P_1)| \cdot |C_{S_2}(P_2)| \geq |C_{S_1}(P'_1)| \cdot |C_{S_2}(P'_2)| = |C_S(P')|$. From the precedent inequality it can be seen that the converse holds as well. □

Proof of the Theorem: The following diagram commutes where the vertical isomorphisms are given through the Kuenneth Formula for topological spaces $H^*(P_1 \times P_2; \mathbb{F}_p) \cong H^*(P_1; \mathbb{F}_p) \otimes H^*(P_2; \mathbb{F}_p)$ and $H^*(Q_1 \times Q_2; \mathbb{F}_p) \cong H^*(Q_1; \mathbb{F}_p) \otimes H^*(Q_2; \mathbb{F}_p)$.

\[
\begin{array}{ccc}
H^*(P_1 \times P_2) & \cong & H^*(Q_1 \times Q_2) \\
\lim_{\mathcal{O}^*(\mathcal{F}_1 \times \mathcal{F}_2)} H^*(-) & \cong & H^*(\mathcal{F}_1 \times \mathcal{F}_2) \\
\phi_1 \circ \phi_2^* & \cong & \phi_1^* \circ \phi_2^* \\
\lim_{\mathcal{O}^*(\mathcal{F}_1)} H^*(-) \otimes \lim_{\mathcal{O}^*(\mathcal{F}_2)} H^*(-) & \cong & H^*\left(\mathcal{F}_1 \circ \mathcal{F}_2\right)
\end{array}
\]

Since there are no finiteness issues we obtain via the universal property of inverse limits that $H^*(\mathcal{F}_1 \times \mathcal{F}_2) \cong H^*(\mathcal{F}_1 \circ \mathcal{F}_2)$ in the category of unstable algebras over the Steenrod algebra. □

The Kuenneth formula is natural in the following way.

Theorem 5.2 Let $(S_1, \mathcal{F}_1, \mathcal{L}_1)$ and $(S_2, \mathcal{F}_2, \mathcal{L}_2)$ be two $p$–local finite groups respectively. Then $H^*(\mathcal{F}_1 \times \mathcal{F}_2) \cong H^*(\mathcal{F}_1 \circ \mathcal{F}_2) \cong H^*\left(\mathcal{L}_1 \times \mathcal{L}_2; \mathbb{F}_p\right) \cong H^*(\mathcal{L}_1; \mathbb{F}_p) \otimes H^*(\mathcal{L}_2; \mathbb{F}_p) \cong H^*(\mathcal{F}_1 \circ \mathcal{F}_2)$.

6 Glaubermann’s and Thompson’s theorems for $p$–local finite groups

In [7] Diaz, Glesser, Mazza and Park prove analogues of Glaubermann’s and Thompson’s theorems for fusion systems. We extend their results to $p$–local finite groups and give an algebraic criterion for the classifying space of a $p$–local finite group to be equivalent to $BS$ before completion.

Theorem 6.1 Let $(S, \mathcal{F}, \mathcal{L})$ be a $p$–local finite group. Then

\[(Z(S))^p \cap Z(N_{\mathcal{F}}(J(S))) \cap Z(N_{\mathcal{L}}(J(S))) \leq Z(\mathcal{F}).\]

If $p$ is odd or $\mathcal{F}$ is $S_4$-free, then $Z(\mathcal{F}) = Z(N_{\mathcal{F}}(J(S))) = Z(N_{\mathcal{L}}(J(S)))$.

Proof: In [7] the authors show that for $(S, \mathcal{F}, \mathcal{L})$ a $p$–local finite group we have $(Z(S))^p \cap Z(N_{\mathcal{F}}(J(S))) \leq Z(\mathcal{F})$ and if $p$ is odd or $\mathcal{F}$ is $S_4$-free, then $Z(\mathcal{F}) = Z(N_{\mathcal{F}}(J(S)))$. The statement follows from the fact that for every $p$–local finite group $(S, \mathcal{F}, \mathcal{L})$ we have an equality $Z(\mathcal{F}) = Z(\mathcal{L})$. □
Theorem 6.2 Let \((S, \mathcal{F}, \mathcal{L})\) be a \(p\)-local finite group. Assume that \(p\) is odd or that \(\mathcal{F}\) is \(S_4\)-free. If \(C_\mathcal{F}(Z(S)) = N_\mathcal{F}(J(S)) = \mathcal{F}_S(S)\), and \(N_\mathcal{L}(J(S)) = \mathcal{L}_S(S)\) then \(\mathcal{F} = \mathcal{F}_S(S)\) and \(|\mathcal{L}| \simeq BS\).

Proof:

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