On the relative surface density change of thermally unstable accretion disks

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ABSTRACT

The relations among the relative changes of surface density, temperature, disk height and vertical integrated pressure in three kinds of thermally unstable accretion disks were quantitatively investigated by assuming local perturbations. The surface density change was found to be very small in the long perturbation wavelength case but can not be ignored in the short wavelength case. It becomes significant in an optically thin, radiative cooling dominated disk when the perturbation wavelength is shorter than $15H$ ($H$ is the scale height of disk) and in a geometrically thin, optically thick and radiation pressure dominated disk when the perturbation wavelength is shorter than $50H$. In an optically thick, advection-dominated disk, which is thermally unstable against short wavelength perturbations, the relative surface density change is much larger. We proved the positive correlation between the changes of surface density and temperature in an optically thick, advection-dominated disk which was previously claimed to be the essential point of its thermal instability. Moreover, we found an anticorrelation between the changes of disk height and temperature in an optically thick, advection-dominated disk. This is the natural result of the absence of appreciable vertical integrated pressure change.

Subject headings: accretion, accretion disks — black hole physics — instabilities

1. INTRODUCTION

Shortly after the construction of the geometrically thin, optically thick accretion disk model in early 1970s (Shakura & Sunyaev 1973; Novikov & Throne 1973), the stability analyses indicated that the inner part of such a disk, where radiation dominates the pressure and electron scattering dominates the opacity, is thermally and secularly unstable (Shakura & Sunyaev 1976; Lightman & Eardley 1974). The optically thin disk model, proposed at first to account for the hard X-ray radiation of Cyg X-1 by Shapiro, Lightman & Eardley (1976), was also shown to be thermally unstable (Pringle, Rees & Pacholczyk 1976; Pringle 1976). The general criteria of the stability of accretion disks were derived by Piran (1978) who considered various kinds of cooling processes.

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in the disks. In last two decades, the thermal instability of accretion disks has been extensively studied with important applications to many time varying phenomena in dwarf novae, X-ray binaries and even AGNs, though its rapid growing also may lead to the breakdown of the disk equilibrium.

The early works we mentioned above all assumed that the surface density of the disk does not change when the thermal instability was addressed. This is a good approximation when the geometrically thin disk and the long wavelength perturbations were considered. In this case the effect of pressure force is minor and the radial balance is realized by the balance between the centrifugal and gravitational forces. A temperature change associated with the thermal mode occurs with no motion in the horizontal plane. The gas expands or shrinks only in the vertical direction so that the vertical hydrostatic balance is realized without changing the surface density. This is the essential point of the thermal instability in a geometrically thin disk (Kato, Abramowicz & Chen 1996, hereafter KAC). However, a hot optically thin disk may be not geometrically thin (Shapiro et al. 1976), the previous stability analyses assuming no surface density change are in doubt in such a disk. Moreover, an optically thick, advection-dominated disk, which is usually geometrically slim, was recently shown to be thermally unstable against short wavelength perturbations (KAC; Wu & Li 1996). Evidently most of the previous analyses, usually dealing with the long wavelength perturbations, are not suitable in this case. KAC also pointed out that in an advection-dominated disk the thermal instability occurs with large surface density change but with only small pressure change. This is quite different from the case of a geometrically thin disk where the thermal instability occurs with no appreciable surface density change. In addition, if the surface density change is strongly associated with the temperature change, some previous stability analyses, such as that simply assuming the surface density unchanged and comparing the cooling and heating rate near the equilibrium curve (usually the $\dot{M}(\Sigma)$ curve), would be not self-consistent (Chen et al., 1995). Therefore, a clear understanding of the relation between the surface density change and the temperature change is important.

In this paper, we will quantitatively investigate the relations among the changes of surface density, temperature, disk height and vertical integrated pressure in three kinds of thermally unstable accretion disks. Their dependences on the perturbation wavelength will be also calculated. Our present study serves as an extension of a recent work by Wu & Li (1996) (hereafter Paper I), where the linear stability properties of various kinds of disks were discussed by numerically solving a general dispersion relation. Our quantitative analyses will prove some qualitative results obtained previously and also reach some new conclusions.

2. BASIC EQUATIONS

The general time-dependent hydrodynamic equations which are usually involved to describe the viscous accretion models have been given by many authors. The vertical integrated equations in a cylindrical system of coordinates $(r, \varphi, z)$ were also summerized in the Eqs. (1)-(4) of Paper I
where the disk was assumed to be axisymmetric and non-self-gravitating. The radial perturbations were assumed of the form \((\delta V_r, \delta \Omega, \delta \Sigma, \delta T) \sim e^{i(\omega t - kr)}\), where \(V_r, \Omega, \Sigma\) and \(T\) are the radial velocity, angular velocity, surface density and temperature. \(k\) is the perturbation wavenumber defined by \(k = 2\pi/\lambda\), \(\lambda\) is the perturbation wavelength.

In this work we consider the local approximation, which means \(\lambda < r\). The validity of the vertically integrated equations requires \(kV_r < \Omega_k\), where \(\Omega_k\) is the Keplerian angular velocity. Since \(V_r \sim \alpha c_s^2/r\Omega_k\) where \(\alpha\) is the viscosity parameter and \(c_s\) is the local sound speed, the requirements above can be written as \(\frac{\lambda}{r} > \frac{\Omega_k}{\Omega} > 2\pi\alpha\frac{H}{r}\). We can see clearly that this inequality is well satisfied for a geometrically thin accretion disk even if we set \(\lambda/H\) and \(\alpha\) in a wide range, such as \(\lambda/H\) from 1 to 100 and \(\alpha\) from 0.001 to 1. However, for a geometrically slim disk, where \(H/r \leq 1\), it can be satisfied only when \(\alpha\) is sufficiently small. The range of \(\lambda/H\) also moves to the smaller value, such as from 0.01 to 2 if \(\alpha\) is about 0.001 and \(H/r\) is about 0.6. In addition, the validity of vertical integrated equations also requires the growth rates of unstable modes are less than the angular velocity (see also KAC). In order to get reasonable and self-consistent results, we present our discussions in this paper following all these restrictions.

If we consider the local approximation, the linearized perturbed equations can be written as:

\[
A \frac{\delta \Sigma}{\Sigma} + B \frac{\delta V_r}{\Omega_k r} + C \frac{\delta \Omega}{\Omega_k} + D \frac{\delta T}{T} = 0
\]

(1)

where \(A, B, C\) and \(D\) are four-component matrices and are functions of \(\tilde{\sigma}\) \((\tilde{\sigma} = \sigma/\Omega_k\) where \(\sigma = i(\omega - kV_r)\)) and other disk parameters (see Paper I for details). By setting the determinant of the coefficients in above perturbed equation to zero, we can get a dispersion relation:

\[
a_1 \tilde{\sigma}^4 + a_2 \tilde{\sigma}^3 + a_3 \tilde{\sigma}^2 + a_4 \tilde{\sigma} + a_5 = 0,
\]

(2)

where \(a_i (i = 1, ..., 5)\) is the coefficients given by the functions of disk parameters and perturbation wavelength. The four kinds of solutions of the dispersion relation are related with the stability properties of two inertial-acoustic modes, thermal and viscous modes in the disk. The real parts of the solutions correspond to the growth rates of the perturbation modes and the imaginary parts correspond to their propagating properties. If the real part corresponding to certain mode is positive, this mode will be unstable. Paper I has numerically solved the dispersion relation and obtained the stability properties of accretion disks with different disk structures. It also proved that there are three kinds of thermally unstable accretion disks, namely, the optically thin, radiative cooling dominated disk, the optically thick, radiative cooling and radiation pressure dominated disk and the optically thick, advection-dominated disk.

The relation between the relative changes of surface density and temperature in thermally unstable disks can be calculated by solving Eq. (1) if some terms such as \(\tilde{\sigma}\), \(\lambda\) and other disk parameters are specified. In this paper, we will obtain this relation based on the solutions given by Paper I to the thermal modes of three kinds of thermally unstable accretion disks.

Before solving the Eq. (1), we first mention some important relations among the changes of surface density and other disk parameters in accretion disks. For convenience we define some
quantities as follows:

\[ \delta \Sigma T = \frac{\delta \Sigma}{\Sigma} \frac{\delta T}{T}, \quad \delta \Sigma W = \frac{\delta \Sigma}{\Sigma} \frac{\delta W}{W}, \quad \delta \Sigma H = \frac{\delta \Sigma}{\Sigma} \frac{\delta H}{H} \]  

(3)

where \( W \) is the vertical integrated pressure. Above quantities describe the ratios of the relative surface density change and the relative changes of temperature, vertical integrated pressure and disk height respectively.

In an optically thin, radiative cooling dominated disk, we have 

\[ H = \frac{c_s}{\Omega_k} \propto T^{1/2}, \quad \rho = \frac{\Sigma}{2H} \propto \Sigma T^{-1/2} \quad \text{and} \quad W = 2pH \propto \Sigma T. \]

The local sound speed is given by \( c_s = p/\rho \) and \( p, \rho \) are the pressure and density respectively. Thus, the variation of \( W \) and \( H \) can be approximated by

\[ \frac{\delta H}{H} = \frac{1}{2} \frac{\delta T}{T} \quad \text{and} \quad \frac{\delta W}{W} = \frac{\delta \Sigma}{\Sigma} + \frac{\delta T}{T}. \]

That is,

\[ \delta \Sigma H = 2 \delta \Sigma T, \quad \delta \Sigma W = \frac{1}{1 + 1/\delta \Sigma T}. \]  

(4)

In an optically thick, radiation pressure dominated disk, we can write 

\[ H \propto \Sigma^{-1} T^4 \quad \text{and} \quad W \propto \Sigma^{-1} T^8. \]

Thus, we have:

\[ \delta \Sigma H = \frac{1}{4(\delta \Sigma T - 1)}, \quad \delta \Sigma W = \frac{1}{8(\delta \Sigma T - 1)}. \]  

(5)

These relations exist not only in a radiative cooling dominated disk but also in an advection-dominated disk. This is because only the equations of hydrostatic equilibrium and mass conservation and the equation of state were used to derive above relations, and these equations are the same for the two kinds of optically thick disks. The energy transport equation, which is different in these two kinds of optically thick disks, does not enter into above relations. Therefore from Eqs. (4) and (5) we can easily obtain the relative changes of surface density to disk height and vertical investigated pressure as long as \( \delta \Sigma T \) is given.

3. NUMERICAL RESULTS

Following we will solve Eq. (1) using the solution corresponding to thermal unstable mode obtained in Paper I, and derive the relative changes of surface density to temperature, disk height and vertical integrated pressure in three kinds of unstable accretion disks.

Fig. 1 shows the case of a hot optically thin, radiative cooling dominated disk. For completeness in (a) we show the stability properties of a disk where the cooling mechanism is bremsstrahlung (It is similar as Fig. 1 in Paper I but with a minor numerical correction). We can clearly see that the thermal mode is always unstable and the viscous mode is always stable. The inertial-acoustic modes are slightly unstable to long wavelength perturbations but stable to short wavelength perturbations. If the disk is a hot two temperature one and the Compton scattering dominates radiative cooling, the stability properties are quite similar with those in a bremsstrahlung disk (Pringle 1976). In (b) we show the the relative changes of surface density to
temperature, disk height and vertical integrated pressure corresponding to the unstable thermal mode. Evidently in long perturbation wavelength case, the surface density nearly does not change. But the surface density change becomes significant if the perturbation wavelength is shorter than 15H. This result shows that some previous stability analyses of hot optically thin disks where the surface density was assumed to be constant, such as Pringle, Rees & Pacholczyk (1976), Pringle (1976) and Piran (1978) are correct only in the long perturbation wavelength case. If the hot optically thin disk is not geometrically thin, the linear local stability analysis is valid only when the viscosity coefficient $\alpha$ is small and the perturbation wavelength is short (see KAC and Paper I). Thus, some previous analyses we mentioned above are not appropriate in this case where the surface density change is significant with the temperature perturbations. Fig. 1(b) also shows that $\delta \Sigma_T$, $\delta \Sigma_H$ and $\delta \Sigma_W$ are all negative. It means that with positive temperature perturbations the surface density decreases slightly but the disk height and vertical integrated pressure increase substantially. Actually in the long perturbation wavelength limit, we have $\delta \Sigma_T \sim 0$, $\delta H \sim \frac{1}{2} \delta T$ and $\delta W \sim \delta T$. In the short perturbation wavelength limit, we note that Fig. 1(b) remains valid even when $\alpha$ is very small and $H/r \leq 1$, and we have $\delta \Sigma_T \sim -\delta T$, $\delta H \sim \frac{1}{2} \delta T$ and $\delta W \sim 0$. We see that in a hot optically thin disk, the thermal instability in the long perturbation wavelength limit is different from that in the short perturbation wavelength limit. In the former case, the thermal instability occurs with no appreciable surface density change. In the later case, however, the thermal instability occurs with no appreciable vertical integrated pressure change.

Fig. 2 shows the case of an optically thick, radiative cooling and radiation pressure dominated disk, which is usually geometrically thin. From (a) we see that the disk is thermally and viscously unstable in long perturbation wavelength case but is acoustically unstable in the short perturbation wavelength case (see also Fig. 9 in Paper I). The non-traveling, unstable thermal mode exists when $\lambda/H > 32$ if $\alpha$ is taken as 0.01. In (b) we see that the relative change of surface density is small comparing with the relative changes of disk height and vertical integrated pressure, but it is significant comparing with the relative temperature change if the perturbation wavelength is shorter than 50H. Evidently the assumption that the surface density is constant is generally not appropriate in this case except in the long perturbation wavelength limit. When the perturbation wavelength is longer than 32H, the unstable thermal mode is not a traveling mode and we see that $\delta \Sigma_T$, $\delta \Sigma_H$ and $\delta \Sigma_W$ are all negative. It means that with a temperature increase the surface density decreases but the disk height and vertical integrated pressure increase, which is quite similar with the result in a hot optically thin disk. However, the increases of disk height and vertical integrated pressure in an optically thick, radiative cooling dominated disk are much larger than those in a hot optically thin disk if we assume a temperature increase. For example, in the long perturbation wavelength limit where $\delta \Sigma_T \sim 0$, we have $\delta H \sim 4 \delta T$ and $\delta W \sim 8 \delta T$. When the perturbation wavelength is shorter than 32H, the unstable thermal mode mixes with the viscous mode and becomes complex. It is a traveling mode which is different from the ordinary thermal mode. We will not discuss it in detail in this paper.

Fig. 3 shows the case of an optically thick, radiation pressure and advection dominated disk,
which is usually geometrically slim. Due to the local restriction, we will discuss the thermal instability against short wavelength perturbations and assume $\alpha$ is very small. From (a) we see that the thermal mode is always unstable and the viscous mode is stable (see also Fig. 12 in paper I). The disk is more thermally unstable if the thermal diffusion effect is considered. In (b) we clearly see that the surface density change is very important in this case. We have $\delta H \sim -2$ and $\delta W \sim +\infty$. Although the inclusion of thermal diffusion has a significant effect to enhance the thermal instability, it does not affect the surface density change very much. With a positive temperature increase associated with the thermal perturbation, the surface density increases significantly but the disk height decreases and the vertical integrated pressure nearly does not change. Here we also have $\delta T \sim -4\delta T$ and $\delta W \sim 0$. This is evidently quite different from the results in above two cases. It also implies that the essential point of thermal instability in an advection-dominated disk is different from that in a hot optically thin disk or a geometrically thin, optically thick and radiative cooling dominated disk.

Our calculations indicate that in the long perturbation wavelength limit $|\delta T|$, $|\delta H|$, and $|\delta W|$ either in a hot optically thin disk or in a geometrically thin, optically thick and radiation pressure dominated disk. However, in the short perturbation wavelength case, we have $|\delta T|$, $|\delta H|$, and $|\delta W|$ either in a hot optically thin disk, and $|\delta W|$. These are different from those in either the long perturbation wavelength case or short perturbation wavelength case of other two kinds of thermally unstable disks. Our results show that the surface density change can be ignored in investigating the thermal instability against long wavelength perturbations but can not be ignored in investigating the thermal instability against short wavelength perturbations especially in an optically thick, advection-dominated disk.

Another very interesting result of our investigation is the anticorrelation of the changes of the disk height and temperature in a thermally unstable advection-dominated disk. This has not been recognized in the analytic analysis of KAC. In fact, it is a natural result of the effect that the thermal instability occurs without significant vertical integrated pressure change in an optically thick, advection-dominated accretion disk where we have $W \propto T^4H$. The cause of thermal instability can be also understood as follows: Fig. 3(b) have indicated that $\delta H \sim 8$. If we assume an increase of temperature, it will lead to the much larger increase of surface density, which will result in the decrease of disk height since $H = c_s/\Omega_k \propto \Sigma^{-1}T^4$. Moreover, in an optically thick advection-dominated disk there is no appreciable vertical integrated pressure change. The decrease of disk height will act to amplify the increasement of temperature since $W \propto T^4H$. Therefore, we can clearly see that the positive correlation of the changes of surface density and temperature and the absence of the change of vertical integrated pressure are two essential points of thermal
instability in an advection-dominated disk. These properties are quite different from those in other two kinds of thermally unstable disks without advection.

4. DISCUSSIONS

Our quantitative investigations not only proved that the essential point of thermal instability in an advection-dominated disk is different from that in a disk without advection which has been previously indicated by KAC, but also derived some new conclusions. First, the dependence of surface density change on the perturbation wavelength is obtained for three kinds of thermally unstable accretion disks. We showed that the surface density changes are small in the long perturbation wavelength limit but are not ignorable in the short perturbation wavelength case. Especially in an optically thick, advection-dominated disk, the surface density change is much significant against short wavelength perturbations. Second, the relations among the changes of surface density and other disk parameters in an advection-dominated disk are different from those in other disks without advection. In later cases, there are anticorrelations among the surface density change and the changes of temperature, disk height and vertical integrated pressure. In former case, however, there are anticorrelation between the changes of surface density and disk height but positive correlation among the change of surface density and the changes of temperature and vertical integrated pressure. Third, we found an anticorrelation between the changes of temperature and disk height for an optically thick, advection-dominated disk. We have pointed out that it is a natural effect of the absence of the evident change of the vertical integrated pressure, which is also the essential point of the thermal instability of an advection-dominated disk.

Some early stability analyses, where the surface density was often assumed to be constant to derive the thermal instability criteria, were found to be valid only in the long wavelength perturbation limit. This is probably the case for a geometrically thin disk where the gravitational force is balanced by the centrifugal force and the radial pressure force is small. Thus, the variation of temperature only leads to the gas expansion or contraction in the vertical direction and the thermal instability occurs without appreciable surface density change. However, if the perturbation wavelength is not longer enough, the change of surface density must be taken into account and those previous thermal instability criteria are not appropriate. The change of surface density is the effect of radial expansion or contraction due to the non-neglectable radial pressure gradient. This usually happens in a geometrically slim or thick disk if the short wavelength perturbations are considered. Especially in an optically thick, advection-dominated disk, the thermal instability occurs with much more significant surface density change than the changes of vertical integrated pressure, temperature and disk height. We must consider the effects associated with the pressure, velocity and advection in the radial direction and take into account the surface density change in deriving the thermal instability criteria.

Together with the results in KAC and Paper I, our investigations indicate that the optically
thick, advection-dominated disk is thermally unstable against local short wavelength perturbations. This is quite different from some previous implications. For example, Abramowicz et al. (1995) and Chen et al. (1995) showed that the upper branch of their S shape curves in $\dot{M} - \Sigma$ diagram, which corresponds to the optically thick, advection-dominated equilibrium, is thermally stable. This result was obtained by comparing the heating and cooling rate near the equilibrium curve. As we have mentioned in Section 1, however, such kind of analysis usually considered only the temperature (or $\dot{M}$) change and assumed that the local surface density is constant. This is not appropriate because the surface density change associated with the temperature change is significant in an optically thick, advection dominated disk and can not be ignored. Thus, although the upper branch of the S shape curve has a positive slope, it only means that the optically thick, advection dominated disk is viscously stable. The thermal instability of this branch can be only validly obtained with the considerations of both the temperature and surface density changes. However, our analyses will not affect the results of numerous papers of Narayan and his collaborators (e.g. Narayan & Yi 1994; Narayan & Yi 1995; Narayan 1996) because most of them mainly considered the optically thin, advection dominated accretion disk which is both thermally and viscously stable. Such kind of disk has been adopted to explain the quiescent state of black hole X-ray binaries (Narayan, McClintock & Yi 1996; Yi et al., 1996).

Although the local thermal instability of an optically thick, advection dominated disk is probably related to some fluctuations and flickerings observed in black hole candidates and AGN (see KAC), we should mentioned that the global and non-linear evolution properties of a thermally unstable disk may be different from those in our analyses. We have restricted our analyses in the local approximation throughout this work. However, we think that the surface density change needs to be also seriously considered in investigating the properties of thermally unstable disks. We expect the future work will give us a more clearly understanding to the correlations among the global and time-dependent changes of the surface density and other disk parameters.

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REFERENCES

Abramowicz, M.A., Chen, X., Kato, S., Lasota, J.-P., Ragev, O. 1995, ApJ, 438, L37

Chen, X., Abramowicz, M.A., Lasota, J.-P., Narayan, R., Yi, I. 1995, ApJ, 443, L61

Kato, S., Abramowicz, M.A., Chen, X. 1996, PASJ, 48, 67 (KAC)

Lightman, A., Eardley, D. 1974, ApJ, 187, L1

Narayan, R. 1996, ApJ, 462, 136
Narayan, R., McClintock, J.E., Yi, I. 1996, ApJ, 457, 821
Narayan, R., Yi, I. 1994, ApJ, 428, L13
Narayan, R., Yi, I. 1995, ApJ, 444, 231
Novikov, I.D., Throne, K. 1973, in “Black Holes”, ed. DeWitt, C., DeWitt, B. (New York: Gordon and Breach), 343
Piran, T. 1978, ApJ, 221, 652
Pringle J.E., 1976, MNRAS, 177, 65
Pringle, J.E., Rees, M.J., Pacholczyk, A.G. 1973, A&A, 29, 179
Shakura, N.I., Sunyaev, R.A. 1973, A&A, 24, 337
———-. 1976, MNRAS, 175, 613
Shapiro, S.L., Lightman, A.P., Eardley, D.N. 1976, ApJ, 204, 187
Wu, X.B., Li, Q.B., 1996, ApJ, 469, 679 (Paper I)
Yi, I., Narayan, R., Barret, D., McClintock, J.E., 1996, A&AS, 120, 187
FIGURE CAPTIONS

Fig. 1.— The stability properties of an optically thin disk are shown in (a) and the relative changes of surface density associated with the unstable thermal mode are shown in (b). Here we take $\alpha = 0.01$, $\Omega = \Omega_k$, $H/r = 0.01$ and the Mach number $m = 0.01$. The solid, dashed and dotted lines in (a) correspond to the thermal mode, viscous mode and acoustic modes and those in (b) represent $\delta \Sigma_T$, $\delta \Sigma_W$ and $\delta \Sigma_H$ respectively.

Fig. 2.— The stability properties of an optically thick, radiative cooling and radiation pressure dominated disk and the relative changes of surface density associated with the unstable thermal mode. The lines have the same meanings and the parameters are taken the same values as in Fig. 1.

Fig. 3.— The stability properties of an optically thick, advection dominated disk and the relative changes of surface density associated with the unstable thermal mode (Note the quantities shown in the vertical axes in both (a) and (b) are different from those in Fig. 1 and Fig. 2). Here we take $\alpha = 0.001$, $\Omega = \Omega_k$, $H/r = 0.6$ and the Mach number $m = 0.01$. In (a) the solid and dashed lines correspond to the thermal and viscous modes in the disk without thermal diffusion and the dotted and dot-dashed lines correspond to the thermal and viscous modes in the disk with thermal diffusion. In (b) the solid, long-dashed and short dashed lines represent $\delta \Sigma_T$, $\delta \Sigma_W$ and $\delta \Sigma_H$ in the case without thermal diffusion respectively, and the dotted, dot-long dashed and dot-short dashed lines represent those in the case with thermal diffusion.
