Superselection rule for the cosmological constant in three-dimensional spacetime

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Efforts to understand the origin of the cosmological constant $\Lambda$ and its observed value have led to consider it as a dynamical field rather than as a universal constant. Then the possibility arises that the universe, or regions of it, might be in a superposition of quantum states with different values of $\Lambda$, so that its actual value would not be definite. There appears to be no argument to rule out this possibility for a generic spacetime dimension $D$. However, as proved herein, for $D = 3$ there exists a superselection rule that forbids such superpositions. The proof is based on the asymptotic symmetry algebra.

I. INTRODUCTION

Superselection rules [1] lay at the very foundation of quantum mechanics. One says that the hermitian operator $\hat{Q}$ obeys a superselection rule if the Hilbert space $\mathcal{H}$ of the system is a direct sum of subspaces $\mathcal{H}_j$ belonging to different eigenvalues of $\hat{Q}$,

$$\mathcal{H} = \bigoplus_j \mathcal{H}_j.$$ 

The $\mathcal{H}_j$ are called coherent subspaces or superselection sectors. In each of them the superposition principle is valid, but a linear combination,

$$\alpha \psi_1 + \beta \psi_2,$$

of states $\psi_1$ and $\psi_2$ from two distinct coherent subspaces is not physically realizable, except as a mixture with the density matrix,

$$|\alpha|^2 \psi_1 \otimes \psi_1^\dagger + |\beta|^2 \psi_2 \otimes \psi_2^\dagger.$$ 

An alternative equivalent description is that the relative phase $\beta$ and $\alpha$ in $\alpha \psi_1 + \beta \psi_2$ is not observable.

If $\hat{Q}$ obeys a superselection rule, then not every hermitian operator is observable, but only those with commute with $\hat{Q}$. This is because an eigenvector of an operator not commuting with $\hat{Q}$ would be a superposition of different eigenvectors of $\hat{Q}$.

Since the introduction of the concept in [1], superselection rules, such as the one for electric charge, have not been devoid of controversy (see for example [2]). Cases in which a superselection rule can be distinctly proven are somewhat scarce and precious. The proof is then often based on a symmetry principle, such as it is the case for the fermion-boson superselection rule and for the Bargmann superselection rule for the mass in non-relativistic quantum mechanics, which will be recalled below.

Spacetime symmetry-albeit asymptotic- will also be at the heart of the superselection rule for the cosmological constant in three-dimensional spacetime, which is the main new result reported in this paper.

The plan of the paper is the following: Section II recalls in some detail the superselection rule for the mass in non-relativistic quantum mechanics. This is done because the parallel between the concepts occurring in that simple case, and those appearing in the more involved situation of the cosmological constant, is an extremely close one. Section III is then devoted to the superselection for the cosmological constant, the proof of which is based on the asymptotic symmetry algebra of three-dimensional gravity with the cosmological constant $\Lambda$ treated as a dynamical variable. This algebra turns out to be the same as the one found in [3] for a fixed non-dynamical $\Lambda$.

The analysis of the asymptotic algebra when $\Lambda$ is dynamical is not devoid of subtleties, which are of the same nature of those previously found in four-dimensional spacetime [4]. In order not to interrupt the thread of the discussion of the superselection rule those subtleties are dealt with in the Appendix.

II. SUPERSELECTION RULE FOR THE MASS IN NON-RELATIVISTIC QUANTUM MECHANICS

A. Galilei Lie algebra

The superselection rule for the mass in non-relativistic quantum mechanics appears as a consequence of the transformation properties of the states under the Galilei algebra.

The Galilei group is characterized by the following set of transformations acting on space and time,

$$x'_i = R_{ij} x_j + v_i t + a_i, \tag{1}$$

$$t' = t + b, \tag{2}$$

where $R_{ij} = R_{ji}$ characterize the spatial rotations, $v_i$ the Galilei boosts and $a_i$ and $b$ translations in the space and the time respectively. From eqs. (1) and (2) one obtains directly the Lie algebra of the Galilei group when it acts

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on points \( x_i, t \) of spacetime,

\[ [J_i, J_j] = \epsilon_{ijk} J_k, \quad [J_i, P_j] = \epsilon_{ijk} P_k, \quad [J_i, K_j] = \epsilon_{ijk} K_k, \]

\[ [K_i, H] = P_i, \quad [P_i, P_j] = 0, \quad [K_i, K_j] = 0, \quad [J_i, H] = 0, \quad [P_i, H] = 0, \quad [K_i, P_j] = 0. \]  

(3)

Here \( H, \vec{P}, \vec{J} \) and \( \vec{K} \) are the generators of time translations, space translations, rotations and boosts respectively. The above algebra possesses no central charge.

**B. Canonical realization of the Galilei group algebra for a free particle**

If \( \vec{x} \) and \( \vec{p} \) respectively denote the position and its conjugate momentum, the generators are realized as,

\[ H = \frac{\vec{p}^2}{2m}, \quad \vec{P} = \vec{p}, \quad \vec{J} = \vec{x} \times \vec{p}, \quad \vec{K} = m\vec{x}. \]

In terms of Poisson brackets, these generators obey,

\[ \{J_i, J_j\} = \epsilon_{ijk} J_k, \quad \{J_i, P_j\} = \epsilon_{ijk} P_k, \quad \{J_i, K_j\} = \epsilon_{ijk} K_k, \]

\[ \{K_i, H\} = P_i, \quad \{P_i, P_j\} = 0, \quad \{K_i, K_j\} = 0, \quad \{J_i, H\} = 0, \quad \{P_i, H\} = 0, \quad \{K_i, P_j\} = m\delta_{ij}. \]

(4)

We see that the canonical realization of the algebra acquires a central charge.

**C. Quantum mechanics and superselection rule**

In passing to quantum mechanics one retains the algebra \( [\hat{J}_i, \hat{J}_j] = i\hbar\epsilon_{ijk} \hat{J}_k, \quad [\hat{J}_i, \hat{P}_j] = i\hbar\epsilon_{ijk} \hat{P}_k, \quad \hat{J}_i \hat{P}_j - \hat{P}_j \hat{J}_i = i\hbar \epsilon_{ijk} \hat{J}_k, \) of the Galilei algebra. This yields the Bargmann algebra.

\[ \hat{J}_i \hat{K}_j = i\hbar \epsilon_{ijk} \hat{K}_k, \quad \hat{K}_i \hat{H} = i\hbar \hat{P}_i, \]

\[ \hat{P}_i \hat{P}_j = 0, \quad \hat{K}_i \hat{K}_j = 0, \quad \hat{J}_i \hat{H} = 0, \quad \hat{P}_i \hat{H} = 0. \]

\[ \hat{m}, \hat{\dot{J}} = 0, \quad \hat{m}, \hat{P}_i = 0, \quad \hat{m}, \hat{K}_i = 0, \quad \hat{m}, \hat{H} = 0. \]  

The action of the Bargmann algebra on the quantum states has important physical consequence. It implies a superselection rule for the mass in non-relativistic quantum mechanics.

The key point is the following. One assumes that the observations are realized in spacetime. Therefore one postulates that a sequence of operations that has no effect on spacetime should have no physical effect, and hence should at most alter all quantum mechanical states by a common phase. Now, when the mass is allowed to be a dynamical variable, the action of the Galilei transformations on quantum mechanical states does not obey the Galilei algebra, it obeys the Bargmann algebra. Due to the central extension, a sequence of transformations which is the identity in the spacetime may alter the relative phase of different state vectors. This is the origin of the superselection rule.

Thus one considers a sequence of transformations with infinitesimal parameters \( \vec{v}, \vec{a}, -\vec{v}, -\vec{a} \). The action of this sequence on spacetime is the identity, so it should alter a permissible state vector \( \psi \) by,

\[ \delta \psi = i\theta \psi. \]

(5)

However, it follows from the Bargmann algebra that,

\[ \delta \psi = \frac{1}{\hbar^2} \left[ \vec{v} \cdot \hat{K}, \vec{a} \cdot \hat{P} \right] \psi = i (\vec{v} \cdot \vec{a}) \frac{m}{\hbar} \psi, \]

Now, if one has,

\[ \psi = \psi_1 + \psi_2, \]

(6)

where \( \psi_1 \) and \( \psi_2 \) are two different eigenstates of the mass operator one obtains,

\[ \delta \psi = \frac{1}{\hbar} \vec{v} \cdot \vec{a} (m_1 \psi_1 + m_2 \psi_2), \]

which is not of the form \( [\hat{J}_i, \hat{J}_j] = i\hbar\epsilon_{ijk} \hat{J}_k, \) Therefore the superposition is not allowed in the Hilbert space and the superselection rule is established.
III. SUPERSELECTION RULE FOR THE COSMOLOGICAL CONSTANT

A. Quantum mechanics and superselection rule

For gravity with a negative cosmological constant in three spacetime dimensions, the role of the Galilei algebra for the non-relativistic particle is played by its global symmetry algebra. This algebra is the asymptotic form at large distances of the algebra of deformations of a two-dimensional spacelike hypersurface embedded in the three-dimensional spacetime. It is of infinite dimension and it consists of two copies of the Witt algebra:

\[ [L^\pm_m, L^\pm_n] = (m - n) L^\pm_{m+n}. \]

These equations are the analog of (3).

When the symmetry is canonically realized in terms of Poisson brackets, the algebra is centrally extended and each copy of the Witt algebra becomes the Virasoro algebra,

\[ i \{ L^\pm_m, L^\pm_n \} = (m - n) L^\pm_{m+n} + \frac{c}{12} (m^3 - m) \delta_{m+n,0}. \]

The central charge \( c \) in (4) is expressed in term of Anti-de Sitter radius of curvature \( \ell \) and the gravitational constant \( G \) by,

\[ c = 3\ell/2G. \]  

The radius \( \ell \) is related to the cosmological constant by,

\[ \Lambda = -\frac{1}{\ell^2}. \]

To establish the desired superselection rule, one promotes the central charge to an operator \( \hat{c} \) and retains the algebra (7) in terms of operators, replacing the Poisson brackets by the commutator divided by \( i\hbar \).

\[ [\hat{L}^\pm_m, \hat{L}^\pm_n] = \hbar (m - n) \hat{L}^\pm_{m+n} + \frac{\hbar \hat{c}}{12} (m^3 - m) \delta_{m+n,0}, \]

\[ [\hat{L}^\pm_m, \hat{c}] = 0, \]

\[ [\hat{c}, \hat{c}] = 0. \]

Next, just as for the particle case, one looks for a sequence of infinitesimal symmetry operations which will be the identity when the central charge is turned-off, i.e., when acting on spacetime, but which will give a non-trivial result when the central charge is turned on.

It will be sufficient to consider one copy of the Virasoro algebra, and within it, the operators \( \hat{L}_n, \hat{L}_{-n} = \hat{L}_n^\dagger \) for any fixed \( n \geq 2 \), and \( \hat{L}_0 \). We may write,

\[ \hat{L}_n = \hat{K}_n + i\hat{P}_n, \]

where \( \hat{K}_n \) and \( \hat{P}_n \) are hermitian. It is then clear from (9) that the central charge appears only in the commutator of \( \hat{K}_n \) with \( \hat{P}_n \), which obeys,

\[ [\hat{K}_n, \hat{P}_n] - i\hbar n\hat{L}_0 = \frac{i\hbar \hat{c}}{24} n (n^2 - 1). \]

When the central charge is turned-off (Witt algebra), the left-hand side vanishes, which means that the Lie algebra element has no effect when acting on spacetime, and one should demand that it does not alter the physical state. If one compares this statement with the corresponding one for the non-relativistic particle, one sees that the operation involved is the composition of five elementary operations rather than four. If one calls \( v_n, a_n \) and \( b \) the infinitesimal parameters corresponding to \( \hat{K}_n, \hat{P}_n \) and \( \hat{L}_0 \), the sequence is given by \( v_n, a_n, -v_n, -a_n, b = -nv_n a_n \). It follows from (10) that the effect of the resulting transformation is given by,

\[ \delta\psi = \frac{in (n^2 - 1)}{24\hbar} (v_n a_n) \hat{c}\psi. \]

Now, if one has,

\[ \psi = \psi_1 + \psi_2, \]

where \( \psi_1 \) and \( \psi_2 \) are two different eigenstates of the central charge operator one obtains,

\[ \delta\psi = \frac{in (n^2 - 1)}{24\hbar} v_n a_n (c_1 \psi_1 + c_2 \psi_2) \neq i\theta\psi, \]

which proves the superselection rule (19).

B. Superselection rule is for a dynamical cosmological constant with a fixed universal gravitational constant

In the case of the non-relativistic particle one had a finite number of degrees of freedom and a finite number of symmetry generators in the Galilei algebra. Since after all the superselection rule is quantum mechanical, one might as well have skipped the classical mechanics altogether. It would have been sufficient to simply decree that the mass \( m \) was an additional hermitian operator, commuting with all the original degrees of freedom and therefore with all the original symmetry generators. We included the Poisson bracket algebra to emphasize the close analogy with the three-dimensional gravity case (which shows en passant that “classical central charges”, i.e., central charges arising in the canonical realization of spacetime symmetries, have been around for quite a while [20]).

However in the case of three-dimensional gravity the situation is not that simple for two reasons: (i) one is dealing with a local field theory and therefore it is unnatural to add by hand an overall single degree of freedom (the central charge), (ii) the symmetry is an asymptotic symmetry, whose definition and implementation critically depends on the boundary condition at large distances and on what is allowed to vary there.

Therefore it is necessary in this case to have an improved classical formalism in which what will become the central charge in the asymptotic symmetry algebra is a
dynamical variable to start with. In other words, the step described as optional for the particle in footnote [18] is now mandatory.

A simple mechanism exists in which the cosmological constant is treated as a dynamical field rather than as a universal constant [12] (see also [13]). The field then becomes independent of the point in spacetime by virtue of the equations of motion. To the knowledge of the present authors no analogous simple mechanism is available to achieve a similar goal for the gravitational constant. For this reason we will consider the superselection rule just proved for,

\[ c = \frac{3\ell}{2G}, \]

as a superselection rule for,

\[ \Lambda = -\frac{1}{\ell^2}, \]

and fixed \( G \).

The implementation of the mechanism will proceed along lines similar to the analysis performed for four spacetime dimensions with \( \Lambda < 0 \) in [10]. In that case, the finite dimensional \( so(3,2) \) algebra acquires an additional generator corresponding to the zero mode of the field \( \Lambda \), which commutes with \( so(3,2) \), but no central charge appears.

To make \( \Lambda \) dynamical one introduces a two-form abelian gauge potential \( A \) so that the action reads,

\[
I = \frac{1}{16\pi G} \int d^3x \sqrt{-\gamma} \left( R - 2\Lambda_0 \right) - \frac{1}{12} \int d^3x \sqrt{-\gamma} F_{\mu\nu\lambda} F^{\mu\nu\lambda}, \tag{11}
\]

where \( F = dA \) is the gauge invariant field strength of the 2-form.

The general solution of the field equation associated to the abelian gauge field can be written as,

\[
F^{\mu\nu\lambda} \bigg|_{\text{on-shell}} = (-\gamma)^{-1/2} Ee^{\mu\nu\lambda}, \tag{12}
\]

where \( E \) is an integration constant. Inserting (12) into the equation for the gravitational field yields the Einstein equations,

\[ G_{\mu\nu} + \Lambda g_{\mu\nu} = 0, \]

with,

\[ \Lambda = \Lambda_0 + 4\pi GE^2. \tag{13} \]

The Anti-de Sitter case which is of interest in the present work, is obtained by demanding that the “bare” cosmological constant \( \Lambda_0 \) be negative, and such that the bound,

\[ \Lambda_0 + 4\pi GE^2 < 0, \]

on \( E \) holds. One introduces \( \Lambda_0 \) in order to have a physical Lagrangian with positive energy density for the dynamical \( p \)-form gauge field.

The next step is to pass to the Hamiltonian form of the action (11) and to give boundary conditions at spacelike infinity for both the gravitational variables and the two-form gauge field which lead to well defined surface integrals as symmetry generators [14]. Since the cosmological constant has become a dynamical variable the central charge will automatically arise as a symmetry generator. As shown in the Appendix, after this is done, explicit forms for the Witt algebra generators \( L_n \) as surface integrals emerge and the Poisson bracket algebra (7) is established.

To conclude, a crucial step in the above derivation should be emphasized and commented upon: it has been assumed that, upon passing to quantum mechanics, the algebra (7) becomes (9), i.e., that its form remains unchanged. We do not know how to prove this assumption, but there is one remarkable fact that could be interpreted as supporting its validity. It is the observation [15] that, if taken literally, way beyond the scope of its derivation, the algebra (9) with the eigenvalue (8) reproduces the standard formula for the black hole entropy in the semiclassical limit [21]. Since that formula itself has also been obtained by applying a path integral expression for the entropy beyond the domain of validity of its derivation, one finds himself watching the consistency of two mysteries. This would appear strange enough to be of significance.

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Appendix A: Asymptotic symmetry algebra with a dynamical cosmological constant

The Hamiltonian corresponding to the action (11) in the main text has the form,

\[
H = \int d^2x \left[ N^+ \mathcal{H}_+ + N^i \mathcal{H}_i + A_i \mathcal{G}^i \right] + Q \left[ N^+, N^i, A_i \right],
\]

with,

\[
\mathcal{H}_+ = \frac{16\pi G}{\sqrt{\gamma}} \left[ \pi^{ij} \pi_{ij} - \pi^2 \right] - \frac{1}{16\pi G} \sqrt{\gamma} \left( R - 2\Lambda_0 \right),
\]

\[
\mathcal{H}_i = -2\nabla_j \pi^j_i,
\]

\[
\mathcal{G}^i = 2\partial_j P^{ij}.
\]
Here $\gamma_{ij}$ is the metric of the spatial section, $\pi^{ij}$ its conjugate momentum, and $P^{ij}$ the conjugate momentum to the spatial 2-form gauge potential $A_{ij}$. The functions $\mathcal{H}_1, \mathcal{H}_2, \mathcal{G}_I$ are the constraint-generators of surface deformations and gauge transformations of the 2-form gauge potential respectively. The lapse $N^\perp$, the shift $N^i$ and the component $A_{1i}$ are the Lagrange multipliers that parametrize the transformation generated by the corresponding constraints. The surface integral $Q \left[N^\perp, N^i, A_{1i}\right]$ is the boundary term necessary to make the Hamiltonian a well defined functional \cite{14}. It generates the asymptotic symmetries.

The variation of the boundary term is,

$$
\delta Q \left[N^\perp, N^i, \lambda_i\right] = \delta Q_g \left[N^\perp, N^i\right] + \int d\phi \left[2A_{t\phi}\delta P^\phi \right],
$$

(A1)

where $\delta Q_g \left[\epsilon^\perp, \epsilon^i\right]$ is the usual boundary term of the pure gravitational field, given by,

$$
\delta Q_g \left[N^\perp, N^i\right] = \int d\phi \left[\frac{1}{16\pi G} N^\perp G^{ijkl} \nabla_k \delta \gamma_{ij} - \frac{1}{16\pi G} \left(\nabla_k N^\perp\right) G^{ijkl} \delta \gamma_{ij} + 2 N^i \gamma_{k} \delta \pi^{kl} + \left(2N^k \pi^{jl} - N^j \pi^{lk}\right) \delta \gamma_{kj}\right],
$$

with $G^{ijkl} = \frac{1}{\sqrt{-g}} \left(\gamma^{ik} \gamma^{jl} + \gamma^{il} \gamma^{jk} - 2 \gamma^{ij} \gamma^{kl}\right)$. The boundary term (A1) becomes an “exact variation”, i.e., the variation symbol $\delta$ can be taken outside the integral, once boundary conditions are imposed to that effect.

### 1. Asymptotic conditions

In order to construct a consistent set of asymptotic conditions for the coupled system of the gravitational field with the 2-form gauge potential, we will proceed along the same lines of ref. [14]. The key new element is that we now need to be able to vary the cosmological radius $\ell$ through variation of $E$ in \cite{14}. Therefore it is natural to employ radial and time coordinates which are made dimensionless through division by $\ell$. This choice proves indeed useful because it facilitates the analysis of the surface terms arising in the variation of the Hamiltonian. In terms of the dimensionless $t$ and $\rho$, the fall-off conditions of ref. [3] read,

$$
\gamma_{\rho\rho} = \ell^2 \left[1 + \frac{f_{\rho\rho} (\phi)}{\rho^2} + O \left(\rho^{-5}\right)\right],
$$

$$
\gamma_{\rho\phi} = \ell^2 \left[\frac{f_{\rho\phi} (\phi)}{\rho^2} + O \left(\rho^{-4}\right)\right],
$$

$$
\gamma_{\phi\phi} = \ell^2 \left[\rho^2 + f_{\phi\phi} (\phi) + O \left(\rho^{-1}\right)\right],
$$

(A2)

and,

$$
\pi^{\rho\rho} = O \left(\rho^{-1}\right),
$$

$$
\pi^{\rho\phi} = \frac{1}{\ell^2} \left[\frac{p^{\rho\phi} (\phi)}{\rho^2} + O \left(\rho^{-4}\right)\right],
$$

$$
\pi^{\phi\phi} = O \left(\rho^{-5}\right).
$$

(A3)

In addition, we will demand the asymptotic behavior of the spatial component of the gauge field and its canonical momentum to be,

$$
A_{\rho\phi} = O \left(\rho^{-1}\right),
$$

$$
P^{\rho\phi} = \frac{E}{2} + O \left(\rho^{-5}\right).
$$

(A4)

The asymptotic behavior of the Lagrange multipliers $N^\perp, N^i$ and $\lambda_i$ is, in turn determined by requiring that the fall-off (A2)-(A4) be preserved under the symmetry transformations. Denoting with a prime the derivative with respect to the angle $\phi$, this gives,

$$
N^\perp = \ell^2 \rho \left(T^+ (\phi) - T^- (\phi)\right) + O \left(\rho^{-1}\right),
$$

$$
N^\rho = -\frac{\rho}{2} \left(T^+ (\phi) + T^- (\phi)\right) + O \left(\rho^{-1}\right),
$$

$$
N^\phi = \frac{1}{2} \left(T^+ (\phi) + T^- (\phi)\right) + O \left(\rho^{-1}\right),
$$

(A5)

just as in \cite{3}, whereas for the gauge field multipliers one finds,

$$
A_{\ell\rho} = O \left(\rho^{-1}\right),
$$

$$
A_{\ell\phi} = -\frac{\ell^3}{4} E \left[\rho^2 \left(T^+ (\phi) - T^- (\phi)\right) - \frac{1}{2} \left(T^+ (\phi) - T^- (\phi)\right) f_{\rho\rho} - 12\lambda + O \left(\rho^{-1}\right)\right].
$$

(A6)

As it also happens in the four dimensional case \cite{16}, eq. (A6) shows that the Lagrange multiplier $A_{\ell\phi}$ explicitly depends on the parameters $T^\pm$ that determine the asymptotic displacements in spacetime. This delicate interplay has two crucial effects: (i) The form of the $\rho^2$ piece cancels a divergence in the charges coming from the pure gravitational part, (ii) The form of the order $\rho^0$ piece makes (A1) an “exact differential”, which guarantees that the charges can be integrated.

### 2. Asymptotic symmetries and global charges.

Imposing the boundary conditions (A2)-(A4), and (A5)-(A6) we obtain for the surface integral which generates the asymptotic symmetries,

$$
Q \left[T^+, T^-, \eta\right] = \int d\phi \left\{T^+ (\phi) L^+ (\phi) - T^- (\phi) L^- (\phi)\right\} + \lambda c,
$$

where,

$$
L^\pm (\phi) = \frac{\ell}{32\pi G} \left(f_{\rho\rho} (\phi) + 2 f_{\rho\phi} (\phi)\right) \pm p^{\rho\phi} (\phi),
$$

$$
c = \frac{3\ell}{2G}.
$$

We see that indeed the central charge has become the generator of a global symmetry transformation (improper
gauge transformation in the terminology of [17]). A displacement generated by \( Q \) of magnitude \( \lambda \) corresponds to a global gauge transformation with,
\[
A_{t\phi} = 3 \ell^3 E \lambda,
\]
at infinity.

To determine the asymptotic symmetry algebra, one expresses the commutator of two asymptotic symmetry transformations in terms of their individual asymptotic parts. If we denote collectively by \( \xi = (\xi_+, \xi_-, \xi_\lambda) \) the parameter of an asymptotic symmetry transformation, the transformation law of \( L^\pm \) and \( c \) is given by,
\[
\delta_\xi L^\pm = \epsilon^\pm L^{\pm'} + 2L^\pm \epsilon^{\pm'} - \frac{c}{24\pi} T^{\pm''}, \\
\delta_\xi c = 0.
\]

One then finds for the parameter \( \zeta \) of the commutator \( \delta_\zeta = \delta_\xi \delta_\eta - \delta_\xi \delta_\eta \), of two transformations with parameters \( \xi \) and \( \eta \),
\[
\zeta_\pm (\phi) = \eta_\pm (\phi) \xi_\pm (\phi) - \xi_\pm (\phi) \eta_\pm (\phi), \\
\zeta_\lambda = 0.
\]

When expressed in terms of Fourier modes these equations read,
\[
\zeta^s_{\pm} = (m-n) \delta^s_{m+n} \epsilon^m_{\pm} n^0_{\pm}, \\
\zeta^s_{\lambda} = 0,
\]
which characterize two copies of the Witt algebra. There is no central charge when the algebra acts on spacetime through deformations of hypersurfaces. The central extension appears when one realizes the asymptotic symmetries in terms of Poisson brackets, and the Witt algebra becomes then the Virasoro algebra.

### 3. Canonical realiztion of the asymptotic symmetry algebra.

The realization of the asymptotic symmetry algebra in terms of Poisson brackets is obtained from the definition,
\[
\{Q[\xi], Q[\eta]\} = \delta_\eta Q[\xi],
\]
through direct evaluation of the asymptotic part of the variation on the right-hand side. Expanding in Fourier modes \( L^\pm (\phi) \propto \sum_n L^\pm_n e^{in\phi} \) one finds,
\[
i \{L^m_m, L^m_n\} = \{m-n\} L^\pm_{m+n} + \frac{c}{12} m^3 \delta_{m+n,0}, \\
i \{L^m_m, c\} = 0, \\
i \{c, c\} = 0,
\]
as just anticipated. (Replacing \( L_0 \) in the above equations by \( L_0 - c/24 \) yields equations (7) in the main text).

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[18] At the classical level the central charge in the Poisson bracket algebra [3] may be turned into a dynamical variable by introducing an additional canonical pair \( m(t), \tau(t) \) such that the action reads, \( I = \)}
\[ \int dt \left( \mathbf{p} \cdot \dot{x} + m \dot{\tau} - \frac{\mathbf{p}^2}{2m} \right). \] See [8] for elaborations on this simple procedure.

[19] The Galilei algebra may be deformed into the Poincaré algebra, of which it is the contraction when the speed of light goes to infinity. The superselection rule for the (rest) mass does not hold for the Poincaré case, which has no central charge (unstable relativistic particles do exist!). A natural question to ask is whether a similar phenomenon might occur for the cosmological constant, i.e., whether it would be possible to “decontract” the Virasoro algebra to one without central charge. The answer is in the negative since the Virasoro algebra is rigid [8]. We thank Prof. Marc Henneaux for informing us of this fact.

[20] It is curious to realize how many people, including the present authors, felt since the early days of string theory, that central charges were quantum mechanical and that their appearance in classical mechanics was exotic. This feeling is evident in references [9, 10] where a model for gravity in two spacetime dimensions was proposed which involved a classical central charge in the Hamiltonian generators, and also in [9] for gravity in three-spacetime dimensions. (Incidentally it follows from [11] that the two-dimensional model turns out to be the asymptotic dynamics of the three-dimensional one, given by the Liouville theory so the two classical central charges are one and the same).

[21] One would expect quantum mechanical corrections of the form \( c = \left(3\ell/2G\right) \left(1 + f \left(Gh/\ell\right)\right) \), where \( f \left(x\right) \) vanishes in the classical limit \( x \to 0 \). The assertion that there is a superselection rule for \( \ell \) remains valid because for fixed \( G \) (and \( h \)), \( c \) is just a function of \( \ell \).