Mean index for non-periodic orbits in Hamiltonian systems

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Abstract

In this paper, we define mean index for non-periodic orbits in Hamiltonian systems and study its properties. In general, the mean index is an interval in \( \mathbb{R} \) which is uniformly continuous on the systems. We show that the index interval is a point for a quasi-periodic orbit. The mean index can be considered as a generalization of rotation number which defined by Johnson and Moser in the study of almost periodic Schrödinger operators. Motivated by their works, we study the relation of Fredholm property of the linear operator and the mean index at the end of the paper.

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1 Introductions

In this paper, we consider the following linear Hamiltonian system

\[
\dot{z} = J B(t) z, \quad t \in \mathbb{R},
\]

where \( J := \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} \) denotes the standard symplectic matrix, \( B(t) \in \text{Sym}(d, \mathbb{R}) \) the set of all \( d \times d \) symmetric matrices. Throughout of the paper, we assume

(L1) \( B \in E_K := \{ B \in C^0(\mathbb{R}, \text{Sym}(2d, \mathbb{R})) | \|B\| \leq K \} \) for some \( K > 0 \).

Let \( \gamma(t) \) be the fundamental solution matrix of (1.1), that is \( \dot{\gamma}(t) = J B(t) \gamma(t), \gamma(0) = I \). It is well known that \( \gamma(t) \in \text{Sp}(2d) := \{ M \in GL(\mathbb{R}^{2d}) | M^TJM = J \} \). In the periodic case, that is \( B(t) = B(t + T) \), the Maslov-type index

\[
i_{\omega}(\gamma, [0, T]) \in \mathbb{Z}, \quad \omega \in U
\]

is well defined, and the mean index per period is defined by

\[
\hat{i}(\gamma) = \lim_{k \to \infty} \frac{i_1(\gamma, [0, kT])}{k}.
\]

It is an important tool in study the multiplicity and stability of periodic orbits in Hamiltonian systems [Eke90, Lon02].

In the case of non-periodic, Ekeland [Eke90] defined the mean index \( \mathcal{I} \) by the limit of \( \frac{i_1(\gamma, [0,T])}{T} \), but as pointed out by him, no reason for \( \frac{i_1(\gamma, [0,T])}{T} \) converges unless \( B(t) \) happens to be periodic. Ekeland proved that, \( \mathcal{I} \) exists almost every where on the energy level of convex Hamiltonian system. There are few results about the mean index of non-periodic trajectory. Only recently, Zhou, Wu and Zhu [ZWZ18] gave a generalization of Ekeland’s almost existence theorem. In general, we almost know nothing about the mean index. Motivated by their works, we give the following definition.

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Definition 1.1. We define the positive upper mean index $J_U(\gamma)$ and lower mean index $J_L(\gamma)$ of the linear system (1.1) as follows:

\[
\begin{align*}
J_U(\gamma) &= \lim_{t \to +\infty} \frac{i_t(\gamma, [0,l])}{t} \\
J_L(\gamma) &= \lim_{t \to -\infty} \frac{i_t(\gamma, [0,l])}{t},
\end{align*}
\]

Similarly, we define

\[
\begin{align*}
J_{U,-}(\gamma) &= \lim_{t \to +\infty} \frac{i_t(\gamma, [-l,0])}{t} \\
J_{L,-}(\gamma) &= \lim_{t \to -\infty} \frac{i_t(\gamma, [-l,0])}{t},
\end{align*}
\]

where $\gamma(t), [0, -l]$ is the path $\gamma(-t), t \in [0, l]$.

Since the system (1.1) is determined by $B(t)$, we can also denote the upper and lower mean index by $J_U(B)$ and $J_L(B)$ respectively.

We give Example 4.5 to show that it is possible $J_U(\gamma) \neq J_L(\gamma)$. In this case, we proved that for any $\alpha \in [J_L(\gamma), J_U(\gamma)]$, there exists a sequence $t_n \to \infty$, such that

\[
\lim_{n \to \infty} \frac{i_t(\gamma, [0, t_n])}{t_n} = \alpha.
\]

Please refer to Lemma 2.6 for the detail. Then we define positive mean index set and negative mean index set by

\[
J(B) = [J_L(\gamma), J_U(\gamma)], \quad J^-(B) = [J_{L,-}(\gamma), J_{U,-}(\gamma)].
\]

Obviously, in the $T$-periodic case, both $J(B)$ and $J^-(B)$ are points and satisfy

\[
\hat{\gamma}([\gamma, 0, T]) = T\gamma(0) = TJ^-(B).
\]

Please note that $J(B), J^-(B)$ are invariant under translation of time, and they do not depend on the value of $B(t)$ at any finite interval. Roughly speaking, they are only depend on the value of $B(t)$ at infinity. Please refer to Corollary 2.5 for the detail. Moreover, we proved that $J(B)$ and $J^-(B)$ are uniformly continuous on $E_K$.

Theorem 1.2. $J_U, J_L, J_{U,-}, J_{L,-}$ are uniformly continuous on $E_K$.

Let $x$ be an orbit of a $C^2$ Hamiltonian systems, that is $\dot{x} = JH'(x(t)), \forall x \in C^2(\mathbb{R}^{2n}, \mathbb{R})$. Let $B(t) = 3t^2(x(t))$, we define

\[J(x) = J(B), \quad J^-(x) = J^-(B),\]

and $J(x)$, where $x(0) = \xi$. It is obvious that $J, J^-$ are constant along the orbit. Assume $\Lambda$ is an invariant set of the Hamiltonian flow. We define

\[J(\Lambda) = \cup_{\xi \in \Lambda} J(x), \quad J^-(\Lambda) = \cup_{\xi \in \Lambda} J^-(x).\]

An invariant set is uniquely ergodic if there is precisely one invariant probability measure with the Hamiltonian flow. A special case is the quasi-periodic orbit. More precisely, let $\Lambda(x) := \{x(t), t \in \mathbb{R}\}$ which is diffeomorphic to torus $T^n$ and let $D : T^n \to \Lambda(x)$ be the homeomorphism, then $x(t) = D(D^{-1}x(0) + \omega t)$ with $\omega = (\omega_1, \cdots, \omega_n)$ which are independent over $\mathbb{Q}$. In this case, $\Lambda(x)$ is uniquely ergodic. The following theorem shows that the mean index for a quasi-periodic orbit is a point.

Theorem 1.3. For a quasi-periodic orbit $x$, then

\[\lim_{T \to \infty} \frac{i_t(\gamma, [0,T])}{T} \text{ exists and}
\]

\[J(\Lambda_x) = J^-(\Lambda_x) = \lim_{T \to \infty} \frac{i_T(\gamma, [0,T])}{T}.\]

For a bounded orbit $x$, the $\omega$-limit set ($\alpha$-limit set) of $x$ is denoted by $\Lambda_\omega(x)$ ($\Lambda_\alpha(x)$). $\Lambda_\omega(x)$ ($\Lambda_\alpha(x)$) is a compact invariant set. Obviously, for a quasi-periodic orbit $x, \Lambda_\omega(x) = \Lambda_\alpha(x) = \Lambda(x)$.
**Corollary 1.4.** Assume $\Lambda_\omega(x) = \Lambda(\tilde{x})$ for some quasi periodic orbit $\tilde{x}$, then we have

$$\mathcal{J}(x) = \mathcal{J}(\Lambda_\omega(x)) = \mathcal{J}(\Lambda(\tilde{x})).$$

Same result holds for $\Lambda_\alpha(x)$.

We say an orbit $x$ is heteroclinic to quasi-periodic if the $\omega$-limit set and $\alpha$-limit set are invariant torus of some quasi-periodic orbits respectively. This theorem shows that for a heteroclinic orbit $x$ to quasi-periodic orbits, then $\mathcal{J}(x)$ and $\mathcal{J}^-(\lambda)$ is independent of $\lambda$ small enough.

This paper is organized as follows. In Section 2, we prove some basic properties of the mean index. In Section 3, we study the quasi-periodic orbits and study the relation with Fredholm property at Section 4. We give an appendix for Maslov-type index at Section 5.

2 The property of Lower and upper index

In this section, we will prove some fundamental properties of upper and lower mean index. We will only consider $\mathcal{J}_U(B), \mathcal{J}_L(B)$, since every property will also hold for $\mathcal{J}_U(B), \mathcal{J}_L(B)$ if we change $B(t)$ with $B(-t)$.

In Definition 1.1, we use $i_\gamma$ to define mean index. In fact, it can be defined by Maslov-type index $i(M, \gamma)$ with any $M \in \text{Sp}(2n)$. Please refer to Section 5 for the detail of Maslov index.

**Lemma 2.1.** For any $M \in \text{Sp}(2n)$, We have

$$\mathcal{J}_L(B) = \lim_{\gamma \to \pm \infty} \frac{i(M, \gamma(t), [0, l])}{l},$$

$$\mathcal{J}_U(B) = \lim_{\gamma \to \pm \infty} \frac{i(M, \gamma(t), [0, l])}{l}.$$

**Proof.** This is from the fact that $i_\gamma(\gamma) + d = i(\gamma)$ and the comparison theorem 5.3. Please refer to (5.1). \qed

**Lemma 2.2.** The upper and lower mean index is monotone for $B$. If $B_1(t) \geq B_2(t), t \in \mathbb{R}$, then $\mathcal{J}_U(B_1) \geq \mathcal{J}_U(B_2)$ and $\mathcal{J}_L(B_1) \geq \mathcal{J}_L(B_2)$

**Proof.** It is a direct consequence of the monotone property of Maslov-type index. \qed

Use this lemma, we see that $\mathcal{J}(Kt) \geq \mathcal{J}_U(B), \mathcal{J}_L(B) \geq \mathcal{J}(-Kt)$. We can calculate $\mathcal{J}(\pm Kt)$ directly, and we get the bound of upper and lower mean index.

**Lemma 2.3.** $dK/\pi \geq \mathcal{J}_U(B)$ and $\mathcal{J}_L(B) \geq -dK/\pi$. 

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We only need to prove the lemma for $I,\gamma(\gamma(a)^{-1},t)\in [a,b]$. Similarly we have $I,\gamma(\gamma(a)^{-1},t)\in [a,b]$ and $I,\gamma(\gamma(a)^{-1},t)\in [a,b]$.

We have

$$-2d(K(b-a)/(2\pi)) \leq I,\gamma(t)\gamma(a)^{-1},t\in [a,b]) \leq 2d[K(b-a)/(2\pi)] + 2d.$$ 

The lemma then follows.

Now we can prove that the upper and lower mean index are invariant under translation.

**Corollary 2.5.** Let $B(Bs(t) = B(s + t)$, we have $\mathcal{J}_V(B) = \mathcal{J}_V(B_s)$ and $\mathcal{J}_L(B) = \mathcal{J}_L(B_s)$.

**Proof.** The fundamental solution related to $B_s$ is $\gamma(s + t)\gamma(s)^{-1}$. By Lemma 2.1, we have

$$\mathcal{J}_V(B_s) = \lim_{l \to +\infty} I,\gamma(s),\gamma(s+t),t\in [0,l]/l = \lim_{l \to +\infty} I,\gamma(s),\gamma(s+t),t\in [0,l]/l$$

By Lemma 2.4, we have

$$|I_1(\gamma(t),t\in [0,l)] - I_1(\gamma(s+t),t\in [0,l])| = |I_1(\gamma(t),[0,s]) - I_1(\gamma(t),[l+s])| \leq 2dKs/\pi + 4d.$$ 

It follows that

$$\mathcal{J}_V(B) = \lim_{l \to +\infty} I,\gamma(t),[0,l])/l = \lim_{l \to +\infty} I,\gamma(s+t),t\in [0,l])/l = \mathcal{J}_V(B_s).$$

Similarly we have $\mathcal{J}_L(B) = \mathcal{J}_V(B_s)$.

**Lemma 2.6.** For each $v \in [\mathcal{J}_L(B),\mathcal{J}_V(B)]$ and $u > 0$, there is a series of integer $m_k \to +\infty$ such that

$$\lim_{k \to +\infty} I,\gamma([0,um_k])/um_k = v.$$

**Proof.** We only need to prove the lemma for $u = 1$. Consider the series $A_k = I,\gamma([0,k])/k$. For each $t \in [k,k+1]$, we have

$$I,\gamma(\gamma(t),[0,l])/l - A_k = I,\gamma(\gamma(t),[0,l])/l - I,\gamma(\gamma(t),[0,k])/k.$$ 

By Lemma 2.4, we have

$$|I_1(\gamma(t),[0,l])/l - A_k| \leq \frac{nK/\pi + 2d}{l} + \frac{nkK/\pi + 2d}{kl} \leq \frac{nK/\pi + 2d}{k} + \frac{dkK/\pi + 2d}{k^2}. \quad (2.1)$$

It follows that there is a series of integers $p_k$ such that

$$\lim_{l \to +\infty} I,\gamma(\gamma(t),[0,l])/l = \lim_{k \to +\infty} A_{p_k}.$$ 

Similarly there is a series of integers $q_k$ such that

$$\lim_{l \to +\infty} I,\gamma(\gamma(t),[0,l])/l = \lim_{k \to +\infty} A_{q_k}.$$ 

Then we get that $\mathcal{J}_V(B) = \lim_{k \to +\infty} A_k$ and $\mathcal{J}_L(B) = \lim_{k \to +\infty} A_k$.

Then we only need to show that $\lim_{k \to +\infty} |A_k + 1 - A_k| = 0$. It is a direct consequence of equation (2.1). The Lemma then follows.
We will show that if $\mathcal{I}_L(B)$ and $\mathcal{I}_U(B)$ are considered as functionals of $B$, then they are both uniformly continuous on $E_K$. We need some lemmas to prove it.

We define the functions
\[
\begin{align*}
f(B, n) &:= \iota(I, \gamma_B(t), [0, n]) \\
g(B, n) &:= \iota(\gamma(n]), \gamma_B(t), [0, n]) \\
h(B, n) &:= \tilde{\iota}(\gamma_B(t), [0, n]),
\end{align*}
\]
and let $S$ be the shift operator $SB(t) = B(t + 1)$.

Recall that for any path of symplectic matrices $\gamma(t)$, we have
\[
\iota(\gamma(b)), \gamma(t), [a, b]) \leq \iota(\Lambda, \gamma(t), [a, b]) \leq \iota(\gamma(b), \gamma(t), [a, b]) - 2d.
\]

**Lemma 2.7.** We have
\[
\begin{align*}
f(B, n) &\geq h(B, n) \geq g(B, n) \geq f(B, n) - 2d, \\
f(B, n + m) &\leq f(B, n) + f(S^n B, m), \\
g(B, n + m) &\geq g(B, n) + g(S^n B, m).
\end{align*}
\]

**Proof.** We have the formula $\tilde{\iota}(\gamma_B(t), [0, n]) = \frac{1}{2\pi i} \int_0^{2\pi} i_{\gamma_B(t)} B(t), [0, n]) d\theta$. By (2.2), we have
\[
f(B, n) \geq i_{\gamma_B(t)} B(t), [0, n]) \geq g(B, n).
\]

It follows that $f(B, n) \geq \tilde{\iota}(\gamma_B(t), [0, n]) = h(B, n) \geq g(B, n)$.

By path additivity of Maslov index, we have
\[
f(B, n + m) = \iota(I, \gamma_B, [0, n]) + \iota(I, \gamma_B, [n, n + m])
\]

Use symplectic invariance of Maslov index, we have
\[
\iota(I, \gamma_B, [n, n + m]) = \iota(\gamma_B(n) - 1, \gamma_B(t) \gamma_B(n) - 1, [n, n + m])
\]
\[
\quad = \iota(\gamma_B(n) - 1, \gamma_S^n B, [0, m]) \leq \iota(I, \gamma_S^n B, [0, m])
\]
\[
\quad = f(S^n B, m).
\]

Then we get $f(B, n + m) \leq f(B, n) + f(S^n B, m)$. Similarly, we have $g(B, n + m) \geq g(B, n) + g(S^n B, m)$.

Let
\[
\begin{align*}
F_{k,n}(B) &= \sum_{l=0}^{n-1} f(S^{2l} B, 2^k)/n, \\
F_k(B) &= \lim_{n \to \infty} F_{k,n}(B), \\
F_k(B) &= \lim_{k \to +\infty} F_{k,n}(B),
\end{align*}
\]
\[
\begin{align*}
G_{k,n}(B) &= \sum_{l=0}^{n-1} g(S^{2l} B, 2^k)/n, \\
G_k(B) &= \lim_{n \to \infty} G_{k,n}(B), \\
G_k(B) &= \lim_{k \to +\infty} G_{k,n}(B),
\end{align*}
\]
\[
\begin{align*}
H_{k,n}(B) &= \sum_{l=0}^{n-1} h(S^{2l} B, 2^k)/n, \\
H_k(B) &= \lim_{n \to \infty} H_{k,n}(B), \\
H_k(B) &= \lim_{k \to +\infty} H_{k,n}(B).
\end{align*}
\]

Then we have a formula to calculate $\mathcal{I}_U(B)$.

**Lemma 2.8.** The limits of $H_k(B)$ and $\underline{H}_k(B)$ exist. We have
\[
\mathcal{I}_U(B) = \lim_{k \to +\infty} H_k(B)/2^k, \quad \mathcal{I}_L(B) = \lim_{k \to +\infty} \underline{H}_k(B)/2^k.
\]
Then we get
\[ f(B_{t+2^k}) \leq f(B_{t+2^{k-1}}) + f(B_{t+2^k}) = f(B_{t+2^{k-1}} + B_{2^k}). \]

It follows that
\[ \lim_{k \to \infty} f(B_{t+2^k}) \leq \lim_{k \to \infty} f(B_{t+2^{k-1}}) + \lim_{k \to \infty} f(B_{2^k}). \]

Then we can conclude that
\[ \lim_{k \to \infty} f(B_{t+2^k}) \leq \lim_{k \to \infty} f(B_{t+2^{k-1}}) \geq \lim_{k \to \infty} f(B_{2^k}). \]

Then we get
\[ \lim_{k \to \infty} f(B_{t+2^k}) \leq \lim_{k \to \infty} f(B_{t+2^{k-1}}) \geq \lim_{k \to \infty} f(B_{2^k}). \]

Now we will show that
\[ H_k(B) = \lim_{k \to \infty} f(B_{t+2^k}). \]

By Lemma 2.7, we have
\[ G_k,n(B) \leq g(B, n2^k)/n \leq f(B, n2^k)/n \leq F_k,n(B). \]

Similar with Lemma 2.6, for any fixed \( k \), we have
\[ J_u(B) = \lim_{n \to \infty} f(B, n2^k)/(n2^k). \]

By (2.5), we have
\[ F_k(B)/2^k = \lim_{n \to \infty} F_k,n(B)/2^k \geq J_u(B) \geq \lim_{n \to \infty} G_k,n(x)/2^k = G_k(B)/2^k. \]

Take limit, then we get
\[ \lim_{k \to \infty} H_k(B)/2^k = J_u(B). \]

**Proof of Theorem 1.2.**

**Proof.** We only prove the first equation, since the proof of the other one is similar.

By Lemma 2.7, we have
\[ F_k,n(B) \geq H_k,n(B) \geq G_k,n(B) \geq F_k,n(B) - 2d. \]

It follows that
\[ F_k(B)/2^k \geq H_k(B)/2^k \geq G_k(B)/2^k \geq F_k(B)/2^k - 2d/2^k. \]

Also by Lemma 2.7, we have
\[ f(B_{t+2^k}) \leq f(B_{t+2^{k-1}}) + f(B_{t+2^k}) = f(B_{t+2^{k-1}} + B_{2^k}). \]

It follows that
\[ F_k,B,2^k = \sum_{l=0}^{n-1} f(B_{t+2^k})/n \leq \sum_{l=0}^{n-1} f(B_{t+2^{k-1}})/n = 2F_k-1,2n(B). \]

Then we can conclude that
\[ \lim_{k \to \infty} F_k-1,2n(B)/2^{k-1} \geq F_k,n(B)/2^k. \]

Then we get
\[ \lim_{k \to \infty} H_k(B)/2^k = \lim_{k \to \infty} F_k(B)/2^k = \lim_{k \to \infty} G_k(B)/2^k. \]

Now we will show that
\[ J_u(B) = \lim_{k \to \infty} H_k(B)/2^k. \]

By Lemma 2.7, we have
\[ G_k,n(B) \leq g(B, n2^k)/n \leq f(B, n2^k)/n \leq F_k,n(B). \]

\[ \begin{align*} 
\| \gamma_1(t) - \gamma_2(t) \| & \leq \sup_{0 \leq s \leq t} \| \gamma_1(t) \| (\exp \int_0^t \| B_2(s) \| ds) \times \int_0^t \| B_1(s) - B_2(s) \| ds \\
& \text{(2.7)} \\
\| \gamma_1(t) - I \| & \leq (\exp \int_0^t \| B_1(s) \| ds) \int_0^t \| B_1(s) \| ds. \\
\text{Then we get} \\
\| \gamma_1(t) \| & \leq 1 + Ke^{K}. \end{align*} \]

(2.8)

Substitute it to (2.7), then we see that there is a constant \( C(K) \) such that
\[ \| \gamma_1(t) - \gamma_2(t) \| \leq C(K) \| B_1 - B_2 \|_{C^0} \text{ for } t \in [0, n]. \]

Let \( B_s = (1 - s)B_1 + sB_2 \). Let \( \gamma_{1+s}(t) \) be the associated fundamental matrix solution.

Then by homotopy invariance of Maslov index, we have
\[ i_{e^{s}}(\gamma_2,[0,1]) - i_{e^{s}}(\gamma_1,[0,1]) = i_{e^{s}}(\gamma_s(1), s \in [0, 1]). \]
By the definition of \( \tilde{\gamma} \), we have
\[
\tilde{i}(\gamma_2, [0, 1]) - \tilde{i}(\gamma_1, [0, 1]) = \frac{1}{2\pi} \int_0^{2\pi} i_{\epsilon,s} (\gamma_s(1), s \in [0, 1]) d\theta.
\] 
(2.9)

Since \( B_s \in E \), like (2.8), we have \( \| \gamma_s(1) \| \leq 1 + Ke^K \). Note that the set of eigenvalues of matrix \( M \) is continuous as a function of \( M \). Then it is also uniformly continuous on \( \{ M \in R^{2d \times 2d} | \| M \| < 1 + Ke^K \} \). By (2.7), \( \| \gamma_s(1) - \gamma_1(1) \| \leq C(K)s \| B_1 - B_2 \|_{C^0} \). So for any \( \epsilon > 0 \), there is \( \delta > 0 \) such that if \( \| B_1 - B_2 \|_{C^0} < \delta \), the measure of the set \( F := \{ \theta \in [0, 2\pi] | e^{i\theta} \in \sigma(\gamma_s(1)) \text{for some } s \in [0, 1] \} \) is less than \( \epsilon \).

Then we can conclude that
\[
\left| \frac{1}{2\pi} \int_0^{2\pi} i_{\epsilon,s} (\gamma_s(1), s \in [0, 1]) d\theta \right| \leq \frac{1}{2\pi} \int_{\theta \in F} i_{\epsilon,s} (\gamma_s(1), s \in [0, 1]) d\theta.
\]

Since \( \gamma_s(1) \) in a small neighbourhood of \( \gamma_1(1) \), we have \( i_{\epsilon,s} (\gamma_s(1), s \in [0, 1]) \leq 2d \). It follows that
\[
\left| \frac{1}{2\pi} \int_0^{2\pi} i_{\epsilon,s} (\gamma_s(1), s \in [0, 1]) d\theta \right| \leq \epsilon / \pi.
\]

Then by (2.9), for any \( B_1, B_2 \in E \), if \( \| B_1 - B_2 \|_{C^0} < \delta \), we have
\[
|\tilde{i}(\gamma_2, [0, 1]) - \tilde{i}(\gamma_1, [0, 1])| \leq \epsilon / \pi.
\]

Then we get the uniform continuity for \( h(B, 1) \). Similarly, we also get the uniform continuity for \( h(B, m) \) for any integer \( m \).

Step 2.

By (2.4) and (2.6), we have
\[
F_k(B)/2^k \geq J_{U}(B) \geq G_k(B)/2^k \geq F_k(B)/2^k - 2d/2^k
\]
\[
F_k(B)/2^k \geq H_k(B) \geq G_k(B)/2^k \geq F_k(B)/2^k - 2d/2^k.
\]

It follows that \( |J_{U}(B) - H_k(B)| \leq 2d/2^k \). Then for each \( \epsilon > 0 \), there is \( k \in \mathbb{N} \) such that \( |J_{U}(B) - H_k(B)| \leq \epsilon \). Note that this \( k \) is independent with \( B \).

Since \( h(B, 2^k) \) is uniformly continuous on \( E \), for \( B_1, B_2 \in E \), there is \( \delta > 0 \) such that if \( \| B_1 - B_2 \|_{C^0} < \delta \), then \( |h(B_1, 2^k) - h(B_2, 2^k)| < \epsilon \).

Note that \( |TB_1 - TB_2|_{C^0} = \| B_1 - B_2 \|_{C^0} \). Then we have
\[
|H_k,n(B_1) - H_k,n(B_2)| \leq \sum_{i=0}^{n-1} |h(T^{2^i}B_1, 2^k) - h(T^{2^i}B_2, 2^k)|/n \leq \epsilon.
\]

It follows that
\[
|H_k(B_1) - H_k(B_2)| = \lim_n H_k,n(B_1) - \lim_n H_k,n(B_2) \leq \epsilon.
\]

Then we have \( |H_k(B_1)/2^k - H_k(B_2)/2^k| \leq \epsilon / 2^k \). Finally we can conclude that \( |J_{U}(B_1) - J_{U}(B_2)| \leq |J_{U}(B_1) - H_k(B_1)| + |J_{U}(B_2) - H_k(B_2)| + |H_k(B_1) - H_k(B_2)| \leq 2\epsilon / 2^k \). The theorem then follows.

Using the continuity of \( J(B) \) we have some asymptotic result.

**Corollary 2.9.** Assume that \( \tilde{B} \in E_K \) with \( \lim_{t \to +\infty} (\tilde{B}(t) - B(t)) = 0 \). Then \( J(B) = J(\tilde{B}) \).

**Proof.** Let \( B_s(t) = B(s + t) \), \( \tilde{B}_s(t) = \tilde{B}(s + t) \). We have
\[
J_L(B) = J_L(B_s)
\]
By Theorem 1.2, for each \( \epsilon > 0 \), there is \( \delta > 0 \) such that if \( B_s(t) - \tilde{B}_s(t) \leq \delta \) on \( \mathbb{R}^+ \), we have \( |J_L(B_s) - J_L(B_s)| < \epsilon \). So for \( \delta \) large enough, we have \( |J_L(B_s) - J_L(\tilde{B}_s)| < \epsilon \). By the corollary 2.5, we see that \( J_L \) and \( J_U \) are invariant under translation. It follows that \( |J_L(B) - J_L(\tilde{B})| < \epsilon \) for each \( \epsilon > 0 \). Then we have \( J_L(B) = J_L(\tilde{B}) \). Similarly, we have \( J_U(B) = J_U(\tilde{B}) \).
3 The mean index of quasi-periodic system

The upper and lower mean index are the same for any important orbits in Hamiltonian system. For example, they are the same for periodic orbits. In this cases, $\mathcal{J}(x) \in \mathbb{R}$ is a point. In this section, we will show that $\mathcal{J}_u$ and $\mathcal{J}_l$ are the same for quasi-periodic orbits.

Now we only consider the linear equation of quasi-periodic orbit. Assume $B(t) = S(p + qt)$ with $S \in C^0(\mathbb{T}^m) \rightarrow \text{Sym}(2n, \mathbb{R})$, $p, q \in \mathbb{R}^m$ and $q = (q_1, q_2, \cdots, q_m) \in \mathbb{R}^m$ with $q_1, q_2, q_3, \cdots, q_m$ are independent over the rationals. Choose $u > 0$ such that $q_1, q_2, \cdots, q_m, 1/u$ are independent over the rationals. Then $\sum_{k=1}^m n_k u q_k \notin \mathbb{Z}$ if integers $\{n_k\}$ are not all zero. Let $P : \mathbb{T}^m \rightarrow \mathbb{T}^m$ be the map $t \rightarrow t + u q$. Then for each nonzero integer $l$, $P^l$ is an irrational rotation on torus and it has a unique ergodic measure which is the Lebesgue measure on $\mathbb{T}^m$.

To study the mean index of quasi-periodic system, we need an ergodic theorem.

**Theorem 3.1.** ([Walter82, Theorem 6.19]) Let $T$ be a continuous transformation of a compact metrisable space $X$. Assume that $T$ is uniquely ergodic. Let $\mu$ be the unique ergodic measure. Let $f \in C(X)$. Then $\sum_{i=1}^n f(T^i x)/n$ converge uniformly to $\int_X f \, d\mu$.

**Theorem 3.2.** For quasi-periodic system, $\mathcal{J}_u(B) = \mathcal{J}_L(B) = \mathcal{J}_u^T(B) = \mathcal{J}_L^T(B)$.

**Proof.** Let $p \in \mathbb{T}^m$, $B_p(t) = S(p + qt)$. For simplicity, we modify the notations used in Section 2. Let

$$f(p, n) := \iota(Gr(I), Gr\gamma_{B_p}(t), [0, un]),$$

$$g(p, n) := \iota(Gr\gamma(u n)), Gr\gamma_{B_p}(t), [0, un]),$$

$$h(p, n) := \hat{\iota}(\gamma_{B_p}(t), [0, un]),$$

and

$$F_{k,n}(p) = \sum_{l=0}^{n-1} f(P^{2k+1}p, 2^k) / n, F_k(p) = \lim_{n \rightarrow \infty} F_{k,n}(p), F_k(p) = \lim_{n \rightarrow \infty} F_{k,n}(p)$$

$$G_{k,n}(p) = \sum_{l=0}^{n-1} g(P^{2k+1}p, 2^k) / n, G_k(p) = \lim_{n \rightarrow \infty} G_{k,n}(p), G_k(p) = \lim_{n \rightarrow \infty} G_{k,n}(p)$$

$$H_{k,n}(p) = \sum_{l=0}^{n-1} h(P^{2k+1}p, 2^k) / n, H_k(p) = \lim_{n \rightarrow \infty} H_{k,n}(p), H_k(p) = \lim_{n \rightarrow \infty} H_{k,n}(p).$$

Like Lemma 2.8, we have

$$\mathcal{J}_u(B_p) = \lim_{k \rightarrow +\infty} H_k(p)/(u 2^k), \quad \mathcal{J}_L(B_p) = \lim_{k \rightarrow +\infty} H_k(p)/(u 2^k).$$

So we only need to show that $\lim_n H_{k,n}(p)$ exists. To use Theorem 3.1, we need to show $h(p, n)$ is continuous on $\mathbb{T}^m$. Let $d$ be the distance on $\mathbb{T}^m$. Since $\mathbb{T}^m$ is compact, then $S$ is uniformly continuous on $\mathbb{T}^m$. Note that $d(p_1 + qt, p_2 + qt) = d(p_1, p_2)$. So for each $\epsilon > 0$, there is $\delta > 0$ such that if $d(p_1, p_2) < \delta$ then $\|S(p_1 + qt) - S(p_2 + qt)\| < \epsilon$.

It follows that $\|B_{p_1}(t) - B_{p_2}(t)\| < \epsilon$ if $d(p_1, p_2) < \delta$. By step 1 in the proof of Theorem 1.2, we see that $h(p, n)$ is continuous on $\mathbb{T}^m$, then $H_{k,n}(p)$ is continuous. Then by Theorem 3.1, $\lim_n H_{k,n}(p)$ uniformly converge to a constant. We have

$$\lim_n H_{k,n}(p) = \int_{\mathbb{T}^m} \sum_{l=0}^{n-1} h(P^{2l+1}p, 2^k) / nd\mu = \int_{\mathbb{T}^m} h(p, 2^k) d\mu.$$  

Now we calculate $\mathcal{J}_u(B)$ and $\mathcal{J}_L(B)$. Let $h^- (p, n) = \iota(I, \gamma_{B_p}(t), [0, -un])$. Replace $P$ by $P^{-1}$, we can define $H_{k,n}(p)$. With the same method, we get

$$\mathcal{J}_u(B) = \mathcal{J}_L(B) = \lim_{k \rightarrow +\infty} \int_{\mathbb{T}^m} h^-(p, 2^k)/(-u 2^k) d\mu.$$
By the path additivity of Maslov index, we have \( h^{-}(p,n) = -\iota(I,\gamma_{B}\gamma_{p}(t),[un,0]) \). It follows that \( h^{-}(p,n) = -\iota(I,\gamma_{B_{-}u}(t),[0,un]) = -h(p-u,n) \). Then we have

\[
\mathcal{I}_{U}(B) = \mathcal{I}_{L}(B) = -\lim_{k \to +\infty} \int_{\mathbb{T}^{m}} h(p-u2^{k},2^{m})/(-u2^{k})d\mu = \lim_{k \to +\infty} \int_{\mathbb{T}^{m}} h(P-2^{k}p,2^{k})/(u2^{k})d\mu.
\]

Since the measure \( \mu \) is invariant under transform \( P \), we have \( \mathcal{I}_{L}(B) = \mathcal{I}_{U}(B) = \mathcal{I}_{L}(B) = \mathcal{I}_{U}(B) \).

With this theorem we immediately get Theorem 1.3.

Furthermore, we have

**Corollary 3.3.** For quasi-periodic system, \( \mathcal{I}(B) = \lim_{k \to +\infty} h(p,2^{k})/(u2^{k}) \) and \( \lim_{k \to +\infty} h(p,2^{k})/(u2^{k}) \) converge uniformly for \( p \).

**Proof.** Note that \( \mathcal{I}(B) = \lim_{k \to +\infty} H_{k}(p)/(u2^{k}) \) where \( H_{k}(p) = \lim_{n \to +\infty} H_{k,n}(p) \) is independent with \( p \). Then for each \( \epsilon > 0 \), there is \( k_{0} \) such that \( |\mathcal{I}(B) - H_{k}(p)/(u2^{k})| < \epsilon \) for each \( k > k_{0} \).

Fix some \( k > k_{0} \) such that \( 2d/2^{k} < \epsilon \). By Theorem 3.2, \( \lim_{n \to +\infty} H_{k,n}(p) \) converge uniformly for \( p \). It follows that for each \( \epsilon > 0 \) there is \( n_{0} \) for each \( n > n_{0} \),

\[
|h_{k}(p) - H_{k,n}(p)| < \epsilon.
\]  

(3.1)

By Lemma 2.7, similar with (2.3) (2.5), we have

\[
F_{n,k}(p) \geq H_{k,n}(p) \geq G_{k,n}(p) \geq F_{n,k}(p) - 2d
\]

\[
G_{k,n}(p) \leq g(p,n2^{k})/n \leq h(p,n2^{k})/n \leq f(p,n2^{k})/n \leq F_{k,n}(p)
\]

It follows that

\[
|h(p,n2^{k})/n - H_{k,n}(p)| \leq 2d.
\]

By (3.1), we have \( |h(p,n2^{k})/n - H_{k}(p)| \leq 2d + \epsilon \). It follows that

\[
|h(p,n2^{k}) - \mathcal{I}(B)| \leq |\mathcal{I}(B) - H_{k}(p)/(u2^{k})| + |h(p,n2^{k})/(u2^{k}) - H_{k}(p)/(u2^{k})| \leq 2d/2^{k} + \epsilon/2^{k} \leq 3\epsilon.
\]

Choose some \( l \) such that \( 2^{l} > n_{0} \). Then for \( m > l + k \), we have

\[
|h(p,2^{m})/(u2^{m}) - \mathcal{I}(B)| < 3\epsilon.
\]

The corollary then follows.

**Proof of Corollary 1.4.**

**Proof.** Let \( \phi(t,p) \) be the flow of Hamiltonian system \( \dot{x} = J\mathcal{H}'(x) \). We can denote the Low and upper mean index of \( \phi(t,p) \) by \( \mathcal{L}(p), \mathcal{U}(p) \) respectively, and denote \( \iota_{\omega}(\phi,\gamma_{p}(t)) \) by \( \iota_{\omega}(p) \). Then we have \( \iota_{\omega}(p) = \iota_{\omega}(\phi(t_{0},p)) \) for each \( t_{0} \in \mathbb{R} \). By Corollary 2.5, we have \( \mathcal{I}(p) = \iota(\phi(t_{0},p)) \). Let \( p_{1} = \bar{x}(0), p_{2} = x_{2}(t) \). Since \( \bar{x} \) is a quasi-periodic orbit, \( \iota_{\omega}(p_{1}) \) is the invariant torus. We simply denote \( M = \iota_{\omega}(p_{1}) \).

Let \( B_{p} = \mathcal{H}'(\phi,\gamma_{p}(t)) \), then we have \( \mathcal{I}(B_{p}) = \iota(\phi,\gamma_{p}(t)) \). Denote \( P : \mathbb{R}^{2d} \mapsto \mathbb{R}^{2d} \) be the map \( p \mapsto \phi(u,p) \). We use notations in Theorem 3.2 with such \( P \). Then by Lemma 2.8, we have

\[
\mathcal{I}_{U}(p) = \lim_{k \to +\infty} H_{k}(p)/(u2^{k})
\]

By Theorem 3.2, \( \mathcal{I}(p) = \lim_{k \to +\infty} H_{k}(p)/(u2^{k}) \) is a constant for \( p \in M \) and it converge uniformly on \( M \). Since \( \iota_{\omega}(p_{2}) = M \), we have

\[
\lim_{t \to +\infty} d(\phi(t,p_{2}),M) = 0.
\]

Then there is a compact neighborhood \( W \) of \( M \) such that \( \phi(t,p_{2}) \in W \) for \( t \geq 0 \). 

By Corollary 3.3, for each \( \epsilon > 0 \) there is \( k_0 \) for each \( k > k_0 \),
\[
|h(r_m, 2^k)/(u2^k) - J(p_2)| < \epsilon. \tag{3.2}
\]
Choose some \( k > k_0 \). Note that \( \phi \) is continuous on \([0, 2^k] \times W\). Let So \( H''(\phi(\cdot, \cdot)) \) is continuous on \([0, 2^k] \times W\). Then it is uniformly continuous on \([0, 2^k] \times M\) by the compactness of \( W\).

Note that \( h(p, 2^k) = h(B_p, u2^k) \). Then by the continuity of \( h(B, 2^k) \) for \( B \), we see that \( h(p, 2^k) \) is uniformly continuous on \( W\).

Then we can conclude that for each \( \epsilon > 0 \), there is \( \delta > 0 \) such that \( |h(u, 2^k) - h(v, 2^k)| < \epsilon \), for \( u, v \in W \) with \( d(u, v) < \delta \).

For each \( \delta > 0 \) there is \( m_0 > 0 \) such that \( d(\phi(t, p_2), M) < \delta \) for each \( t > m_0 2^k \).

Let \( r_m = P \phi(0, p_2) = \phi(m2^k, p_2) \). Then for each \( m > m_0 \), there is \( r'_m \in M \) such that \( d(r_m, r'_m) < \delta \). It follows that \( |h(r_m, 2^k) - h(r'_m, 2^k)| < \epsilon \). Then by (3.2), we get
\[
|h(r_m, 2^k)/(u2^k) - J(p_1)| \leq 2\epsilon, m \geq m_0.
\]

It follows that
\[
|H_{u,k}(p_2)/(u2^k) - J(p_1)| \leq \sum_{m=1}^{m_0} |h(r_m, 2^k)/(u2^k) - J(p_1)|/n + \sum_{m=m_0+1}^{n} |h(r_m, 2^k)/(u2^k) - J(p_1)|/n
\]
\[
\leq 2\epsilon + \sum_{m=1}^{m_0} |h(r_m, 2^k)/(u2^k) - J(p_1)|/n.
\]

Take upper limit for \( n \), then we get \( |H_u(p_2)/(u2^k) - J(p_1)| \leq 2\epsilon \). Take limit for \( k \), then we get \( |J_u(p_2) - J(p_1)| \leq 2\epsilon \) for each \( \epsilon > 0 \). Then we get \( J_u(p_2) = J(p_1) \). Similarly we get \( J_l(p_2) = J(p_1) \).

In [JMS82], Johnson and Moser define the rotation number for almost periodic Schrödinger operator \( \mathcal{L} = -d^2/dt^2 + q(t) \). We will generalize the definition of rotation number to 2-dimensional quasi-periodic system and prove that it is proportional to the mean index defined in this paper.

**Definition 3.4.** Let \( B \in C(\mathbb{R}, \text{Sym}(2, \mathbb{R})) \), \( z = (u, v)^T \) be a nonzero solution of (1.1). Then \( t \to \text{arg}(u + iv) \) is a map from \( \mathbb{R} \) to the unit circle \( \mathbb{T} \). Let \( \theta_0 \) be an argument of \( u(0) + iv(0) \). By the homotopy lifting property, there is a unique continuous function \( \theta : \mathbb{R} \to \mathbb{R} \) such that \( \theta(t) \) is the argument of \( u(t) + iv(t) \) and \( \theta(0) = \theta_0 \). We define the rotation number by
\[
R(B) = \lim_{t \to +\infty} \theta(t)/t.
\]

**Theorem 3.5.** For two dimensional quasi-periodic system, \( \pi J(B) = R(B) \).

**Proof.** Note that \( z(t) = \gamma(t)z(0) \), where \( \gamma(t) \) is the fundamental matrix solution. Use polar decomposition, we have \( \gamma(t) = M(t)U(t) \) with positive definite symplectic matrix \( M(t) \) and orthogonal symplectic matrix \( U(t) \) for each \( t \) and \( M(t), U(t) \) is continuous for \( t \). Note that 2-dimensional orthogonal symplectic matrix has the form \( e^{J\theta} \) with \( J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \). So \( t \to S(t) \) is a map from \( \mathbb{R} \) to the unit circle. Using homotopy lifting property, there is a continuous function \( \phi(t) \) such that \( \phi(0) = 0 \) and \( U(t) = e^{J\phi(t)} \). Let \( f_u(t) = M(st)U(t)z(0)/|M(st)U(t)z(0)| \). It is a map from \([0, 1] \times \mathbb{R} \) to unit circle. Note that \( f_0(t) = U(t)z(0)/|z(0)| \). Let \( \theta_0 \) be an argument of \( u(0) + iv(0) \) Then \( f_0(t) = (\cos(\theta_0 + \phi(t)), \sin(\theta_0 + \phi(t))^T \).

Use homotopy lifting property, there is a continuous function \( \theta(s, t) \) such that
\[
f_u(t) = (\cos(\theta(s, t)), \sin(\theta(s, t))^T \)
\]
and \( \theta(0, t) = \theta_0 + \phi(t) \).
Then we can conclude that the rotation number is

\[ R(B) = \lim_{t \to +\infty} \frac{\theta(1, t) - \theta(0)}{t} = \lim_{t \to +\infty} \frac{\theta(1, t) - \theta(0, t) + \phi(t)}{t}. \]

For a fixed \( t \), \( M(st) \) is a positive definite matrix and \( (Mv, v) > 0 \) for any \( v \neq 0 \). So the angle between \( Mv \) and \( v \) is between \(-\pi/2\) and \( \pi/2 \). It follows that \( |\theta(1, t) - \theta(0, t)| \leq \pi/2 \). Then we can conclude that

\[ R(B) = \lim_{t \to +\infty} \phi(t)/t. \]  \hspace{1cm} (3.3)

Now we calculate the mean index. Note that \( \mathfrak{A}(B) = \lim_{t \to +\infty} = i_1(\gamma(t), [0, l])/l \). By the homotopy invariance of Maslov index, we have

\[ i_1(\gamma(t), [0, l]) = i_1(M(t)U(t), [0, l]) = i_1(U(t), [0, l]) + i_1(M(sl)U(l), s \in [0, 1]). \]

We also have

\[ i(U(I, M(sl)U(l), [0, 1]) = \iota(U^{-1}, M(sl), [0, 1]). \]

Note that \( |\iota(U^{-1}, M(sl), [0, 1])| \leq 2 \) for some \( \omega \neq 1 \) on unit circle. Since all the eigenvalues of \( M(sl) \) are real, we have \( i_\omega(M(sl), [0, 1]) = 0 \). Finally we can conclude that

\[ \lim_{t \to +\infty} i_1(\gamma(t), [0, l])/l = \lim_{t \to +\infty} i_1(U(t), [0, l])/l = \lim_{t \to +\infty} \frac{2\phi(t)}{2\pi}/l. \]

By (3.3), we have \( \pi \mathfrak{A}(B) = R(B) \). \( \square \)

4 The relation with essential spectrum

In this section we study the relation of mean index and essential spectrum. First we need a theorem from [Palmer88].

Let

\[ A := -J \frac{d}{dt} - B(t) : W^{1, 2}(\mathbb{R}, \mathbb{R}^{2d}) \subset L^2(\mathbb{R}, \mathbb{R}^{2d}) \to L^2(\mathbb{R}, \mathbb{R}^{2d}) \]
\[ A_+ := -J \frac{d}{dt} - B(t) : W^{1, 2}(\mathbb{R}^+, \mathbb{R}^{2d}) \subset L^2(\mathbb{R}^+, \mathbb{R}^{2d}) \to L^2(\mathbb{R}^+, \mathbb{R}^{2d}) \]
\[ A_- := -J \frac{d}{dt} - B(t) : W^{1, 2}(\mathbb{R}^-, \mathbb{R}^{2d}) \subset L^2(\mathbb{R}^-, \mathbb{R}^{2d}) \to L^2(\mathbb{R}^-, \mathbb{R}^{2d}) \]

**Theorem 4.1.** Let \( \gamma(t) \) be the fundamental solution of (1.1). Then the operator \( A_+ \) are Fredholm if and only if there is a projection \( P : \mathbb{R}^{2d} \to \mathbb{R}^{2d} \) and \( C, \beta > 0 \) such that the inequalities

\[ |\gamma(t)P\gamma^{-1}(s)| \leq Ce^{-\beta(t-s)}(s \leq t) \]
\[ |\gamma(t)(I - P)\gamma^{-1}(s)| \leq Ce^{-\beta(s-t)}(s \geq t). \]

hold on \( \mathbb{R}^+ \) respectively.

If \( B \) is periodic with period \( T \), we have a simple criterion to determine Fredholmness.

**Lemma 4.2.** \( A, A_+, A_- \) are Fredholm if and only if \( \sigma(\gamma(T)) \cap \mathbb{U} = \emptyset \) where \( \mathbb{U} \) is the unit circle.

**Proof.** Let \( M = \gamma(T) \). Since \( B(t) \) is periodic, we have \( \gamma(t) = \gamma(t - KT)M^k \) for \( t \in [kT, (k + 1)T], \)

(\text{\( \leftarrow \))} We assume that \( \sigma(M) \cap \mathbb{U} = \emptyset \) . Then \( \mathbb{R}^{2d} = V \oplus W \) such that \( V, W \) are invariant subspace of \( M \) and \( |M|_V < c \), \( |M^{-1}|_W < c \) for some \( 0 < c < 1 \). Let \( P \) be the projection to \( W \) with ker \( P = W \). Then \( I - P \) is the projection to \( W \) and \( MP = PM \).

Assume that \( t \in [kT, (k + 1)T], s \in [(t - l)T, (t+ 1)T]. \) We have

\[ |\gamma(t)P\gamma^{-1}(s)| = |\gamma(t - kT)M^kPM^{-1}\gamma(s - lT)| = |\gamma(t - kT)M^{k-l}P\gamma(s - lT) \]
\[ \leq c^{k-l}|\gamma(t - kT)||\gamma(s - lT)| \]

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Let $Q = \max \{ |\gamma(t)|, t \in [0, T] \}$. We have
\[
|\gamma(t)(I - P)^{-1}(s)| = |\gamma(t - kT)M^k(I - P)M^{-1}\gamma(s - lT)| = |\gamma(t - kT)M^{k-1}(I - P)\gamma(s - lT)| \\
\leq e^{-k}|\gamma(t - kT)||\gamma(s - lT)|.
\]

By theorem 4.1, we see that $A, A_+, A_-$ are all Fredholm operators.

(⇒) Assume that $A$ is Fredholm. For integer $k > 0$, let $t = kT$, $s = 0$, then we have
\[
|M^k P| = |\gamma(kT)P| \leq Ce^{-\beta kT}.
\]
Similarly we have $|(I - P)M^{-k}| \leq Ce^{-\beta kT}$. Assume that there is $\omega \in U$ such that $\omega \in \sigma(M)$. Then there is $x \in \mathbb{R}^{2M}$, $|x| = 1$ such that $Mx = \omega x$. It follows that $|(I - P)x| = |(I - P)\omega^{-k}x| = |(I - P)M^{-k}x| \leq Ce^{-\beta kT}$.

Let $k \rightarrow +\infty$, then we get $x = Px$. It follows that
\[
|x| = |\omega^k x| = |M^k x| = |M^k P x| \leq Ce^{-\beta kT}
\]
Let $k \rightarrow +\infty$, then we get $|x| = 0$. It is a contradiction. So $\sigma(M) \cap U = \emptyset$. The results for $A_-, A_+$ are similar.

Now we can show the relation of mean index and Fredholmness for periodic system. Recall that for periodic system, $T\mathcal{J}(B) = \mathcal{J}(B)$, where
\[
\mathcal{J}(B) = \mathcal{J}(\gamma(t), [0, T]) = \frac{1}{2\pi} \int_0^{2\pi} i_{\epsilon^s}(\gamma(t), [0, T])d\theta.
\]

So we can prove the theorem for $\mathcal{J}$ instead.

**Theorem 4.3.** $A, A_-, A_+$ are Fredholm if and only if $\mathcal{J}(B + \lambda I) = \mathcal{J}(B)$ for $|\lambda|$ small enough.

**Proof.** Let $B_s(t) = B(t) + sI$ and $\gamma_s$ be the associated fundamental matrix solution, then
\[
\mathcal{J}(B_s) = \frac{1}{2\pi} \int_0^{2\pi} i_{\epsilon^s}(\gamma(t), [0, T])d\theta.
\]
(⇒) Assume that $A$ is not Fredholm. By Lemma 4.2, there is $\omega_0 \in U \cap \sigma(M)$, where $M = \gamma_0(T)$. Let $s_0 > 0$, using homotopy invariance of Maslov index, we have
\[
\mathcal{J}(B_{s_0}) - \mathcal{J}(B_{-s_0}) = \frac{1}{2\pi} \int_0^{2\pi} (i_{\epsilon^s}(\gamma(t), [0, T]) - i_{\epsilon^s}(\gamma(t), [0, T]))d\theta.
\]

We will show that $\gamma_s(T)$ is a positive path for $s$, that is, $-\int_0^T \frac{\partial \gamma_s(T)}{\partial s} \gamma_s^{-1}(T) > 0$. Direct compute show that
\[
-\gamma_s^{-1}(T)\frac{\partial \gamma_s(T)}{\partial s} = \int_0^T \gamma_s^{-1}(T)\gamma_s(t)dt > 0,
\]
which implies the result. Then we have
\[
\Delta(\omega) = i_{\omega}(\gamma_s(T), s \in [-s_0, s_0]) = \sum_{\xi \in [-s_0, s_0]} \dim \ker(\gamma_\xi(T) - \omega I) \geq 0, \quad \forall \omega \in U.
\]

By the homotopy invariance of Maslov index and spectral flow, we have
\[
\Delta(\omega) = i_{\omega}(\gamma_s(t), t \in [0, T]) - i_{\omega}(\gamma_{-s}(t), t \in [0, T])
\]
Since $\omega_0 \in U \cap \sigma(M)$, we have $\dim \ker(M - \omega_0 I) \neq 0$. Since $\gamma_s(T)$ is positive path, we have
\[ \Delta(\omega_0) \geq \dim \ker (M - \omega_0 I) > 0, \]
\[ \Delta(\omega) \geq \dim \ker ((M - \omega I)) \geq 0, \forall \omega \in \mathbb{U}, \]
\[ \ker (\gamma_{\pm s_0}(T) - \omega_0 I) = 0, \text{ for } s_0 \text{ small enough.} \]

Then \( \omega_0 \) has a neighborhood \( V \) on \( \mathbb{U} \), such that
\[ \ker (\gamma_{\pm s_0}(T) - \omega I) = 0, \quad \forall \omega \in V. \]

By homotopy invariance of Maslov index, we have
\[ i_{\omega}(\gamma_s(T), s \in [-s_0, s_0]) = i_{\omega_0}(\gamma_s(T), s \in [-s_0, s_0]) \]

Then for \( \omega \in V \), we have \( \Delta(\omega) = \Delta(\omega_0) > 0 \). By (4.1), we have
\[ \tilde{i}(B_{s_0}) - \tilde{i}(B_{-s_0}) = \frac{1}{2\pi} \int_{\Gamma} i_{e^{\theta}}(\gamma_{s_0}(t), [0, T]) - i_{e^{\theta}}(\gamma_{-s_0}(t), [0, T]) d\theta = \frac{1}{2\pi} \int_{\Gamma} \Delta(e^{\theta}) d\theta \]
\[ \geq \frac{1}{2\pi} \int_{c e^{\theta} \in V} \Delta(e^{\theta}) d\theta > 0. \]

So for any \( s_0 > 0 \), \( \tilde{i}(B + \lambda I) \) is not invariant for \( \lambda \in [-s_0, s_0] \).

\((\Rightarrow)\) Assume that \( A \) is Fredholm, then by Lemma 4.2, \( \sigma(\gamma(T)) \cap \mathbb{U} = \emptyset \). It follows that there is \( s_0 > 0 \) such that \( \sigma(\gamma_{s}(T)) \cap \mathbb{U} = \emptyset \) for \( s \in [-s_0, s_0] \). Then we have
\[ \Delta(\omega) = i_{\omega}(\gamma_{s}(T), s \in [-s_0, s_0]) = 0, \forall \omega \in \mathbb{U}. \]

It follows that
\[ \tilde{i}(B_{s_0}) - \tilde{i}(B_{-s_0}) = \frac{1}{2\pi} \int_{c e^{\theta} \in V} \Delta(e^{\theta}) d\theta = 0. \]

By Lemma 2.2, we see that \( \tilde{i}(B_s) \) is increasing for \( s \). So \( \tilde{i}(B_s) = \tilde{i}(B) \) for \( s \in [-s_0, s_0] \). Similarly, the theorem also hold for \( A_- , A_+ \). \(\blacksquare\)

Now we consider perturbation of periodic system.

**Corollary 4.4.** Assume that \( B(t) \) is periodic with period \( T \). Assume that \( A \) is Fredholm. Then there is \( \delta > 0 \) such that for \( \tilde{B} \in C(\mathbb{R}, \mathbb{R}^{2d}) \) if \( |B(t) - \tilde{B}(t)| < \delta \) for \( t \in \mathbb{R} \) then \( \mathcal{J}_L(\tilde{B}) = \mathcal{J}_L(B) \) and \( \mathcal{J}(\tilde{B} + \lambda I) = \mathcal{J}(\tilde{B}) \) for \( |\lambda| \) small enough.

**Proof.** By Theorem 4.3, there is \( \delta > 0 \) such that \( \mathcal{J}(B + \lambda I) = \mathcal{J}(B) \) for \( |\lambda| \in [-\delta, \delta] \). By Lemma 2.2, we have
\[ \mathcal{J}(B + \delta I) \geq \mathcal{J}_L(B) \geq \mathcal{J}_L(\tilde{B}) \geq \mathcal{J}(B - \delta I). \]

It follows that \( \mathcal{J}_L(\tilde{B}) = \mathcal{J}_L(\tilde{B}) = \mathcal{J}(B) \). For \( |\lambda| \) small enough, we have \( B - \delta I < \tilde{B} + \lambda I < B + \delta I \). Then similarly, we have \( \mathcal{J}(\tilde{B} + \lambda I) = \mathcal{J}(B) \). \(\blacksquare\)

We will give a non periodic example. It shows that the upper and lower mean index may not be the same for Fredholm operator.

**Example 4.5.** Let \( \gamma(t) = e^{J\psi(t)} \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix}, t \in \mathbb{R}^+ \) with \( \psi(t) = t \frac{\sin 1}{t+1} \).

Let \( B(t) = -J \frac{d}{dt} \gamma(t) \gamma(t)^{-1} \). We have
\[ -J \frac{d}{dt} \gamma(t) \gamma(t)^{-1} = -JJ \psi \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} e^{-t} & 0 \\ 0 & e^t \end{bmatrix} e^{-J\psi(t)} - J e^{J\psi(t)} \begin{bmatrix} e^t & 0 \\ 0 & -e^{-t} \end{bmatrix} \begin{bmatrix} e^{-t} & 0 \\ 0 & e^t \end{bmatrix} e^{-J\psi(t)} \]
\[ = \left( \sin \frac{1}{t+1} - \frac{t}{(t+1)^2} \cos(t+1) \right) e^{-J\psi(t)} - J e^{J\psi(t)} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} e^{-J\psi(t)} \]
Then $|B(t)| < 3$ for $t \in \mathbb{R}^+$, and $\gamma(t)$ is the fundamental solution of $-J \frac{d}{dt} z - B(t)z = 0$. Similar with the method in theorem 3.5, we have
\[
J_L(B) = \lim_{t \to +\infty} \psi(t)/t = -1, \quad J_U(B) = \lim_{t \to +\infty} \psi(t)/t = 1.
\]

5 Appendix

In this section, we briefly review the index theory for symplectic path. The detail could be found in [Lon02, LZ00a, LZ00b, HS09].

Following [Lon02], we define the following hypersurface of codimension one in $Sp(2n)$:
\[
Sp(2n)_0^\omega = \{ M \in Sp(2n) | \det(M - \omega I) = 0 \}.
\]

For $M \in Sp(2n)_0^\omega$, we define a co-orientation of $Sp(2n)_0^\omega$ at $M$ by the positive direction $\frac{d}{dt} Me^{tJ}|_{t=0}$ of the path $Me^{tJ}$ with $|t|$ sufficiently small.

**Definition 5.1.** For $\omega \in U$, $\gamma \in C([0, T], Sp(2n))$ with $\gamma(0) = I$, we define
\[
i_\omega(\gamma) = [e^{-\epsilon J} \gamma : Sp(2n)_0^\omega] + \frac{1}{2} \dim \ker(I - \omega I),
\]
for $\epsilon$ small enough, where $[\cdot, \cdot]$ is the intersection number.

The mean index of periodic system on $[0, T]$ can be defined as
\[
\hat{i}(\gamma, [0, T]) = \lim_{n \to \infty} \frac{i_1(\gamma, [0, nT])}{n} = \frac{1}{2\pi} \int_0^{2\pi} i_{\epsilon, \omega}(\gamma, [0, T])d\theta.
\]

Since we will use it in this paper, we give a small generalization of the standard Maslov index.

**Definition 5.2.** For $M \in Sp(2n)$, $\gamma \in C([0, T], Sp(2n))$, $\omega \in U$, we define
\[
i(\omega M, \gamma) = [e^{-\epsilon J} M^{-1} : Sp(2n)_0^\omega],
\]
for $\epsilon$ small enough.

We use $i$ instead $i$ to avoid misunderstanding. Some property of the Maslov-type index can be found in [LT15]. It is obvious that for $\gamma \in C([0, T], Sp(2n))$ with $\gamma(0) = I$
\[
i_1(\gamma) + n = i(I, \gamma).
\]

The Maslov-type index can be explained by the Maslov index theory. We now briefly reviewing the Maslov index theory [Arn67, CLM94, RS93]. Let $(\mathbb{R}^{2n}, \omega)$ be the standard symplectic space and $Lag(2n)$ the Lagrangian Grassmannian. For two continuous paths $L_1(t), L_2(t)$, $t \in [a, b]$ in $Lag(2n)$, the Maslov index $i(L_1, L_2)$ is an integer. Here we use the definition from [CLM94]. We list several properties of the Maslov index. The details could be found in [CLM94].

**Reparametrization invariance** Let $\phi: [c, d] \to [a, b]$ be a continuous and piecewise smooth function with $\phi(c) = a$, $\phi(d) = b$, then
\[
\mu(L_1(t), L_2(t)) = \mu(L_1(\phi(t)), L_2(\phi(t))).
\]

**Homotopy invariant with end points** For two continuous families of Lagrangian path $L_1(s, t)$, $L_2(s, t)$, $0 \leq s \leq 1$, $a \leq t \leq b$ which satisfy that $\dim L_1(s, a) \cap L_2(s, a)$ and $\dim L_1(s, b) \cap L_2(s, b)$ are constant, we have
\[
\mu(L_1(0, t), L_2(0, t)) = \mu(L_1(1, t), L_2(1, t)).
\]
(Path additivity) If \( a < c < b \), then

\[
\mu(L_1(t), L_2(t)) = \mu(L_1(t), L_2(t)|_{[a,c]}) + \mu(L_1(t), L_2(t)|_{[c,b]}).
\]

(Symplectic invariance) Let \( \gamma(t), t \in [a, b] \) be a continuous path in \( \text{Sp}(2n) \), then

\[
\mu(L_1(t), L_2(t)) = \mu(\gamma(t)L_1(t), \gamma(t)L_2(t)).
\]

(Monotony property) Suppose for \( j = 1, 2 \), \( L_j(t) = \gamma_j(t)V \), where \( \gamma_j(t) = JB_j(t)\gamma_j(t) \) with \( \gamma_j(0) = I_{2n} \). If \( B_1(t) \geq B_2(t) \), then for any \( V_0, V \in \text{Lag}(2n) \), we have

\[
\mu(V_0, \gamma_1 V) \geq \mu(V_0, \gamma_2 V).
\]

We have comparision results of Maslov index.

**Theorem 5.3 ([ZWZ18, Corollary 3.16]).** Let \( \lambda \in C([a, b], \text{Lag}(2n)) \) be a Lagrangian path. Then for any \( V_1, V_2 \in \text{Lag}(2n) \), we have

\[
\mu(\lambda(b), \lambda) \leq \mu(\lambda_1, \lambda) \leq \mu(\lambda(a), \lambda)
\]

\[
|\mu(V_1, \lambda) - \mu(V_2, \lambda)| \leq n
\]

We will express the Maslov-type index by Maslov index. Please note that \( \mathbb{R}^{2n} \times \mathbb{R}^{2n}, -\Omega \times \Omega \) is a \( 4n \)-dimensional symplectic space. For \( M \in \text{Sp}(2n) \), let \( \text{Gr}(M) := \{ (x, Mx) | x \in \mathbb{R}^{2n} \} \in \text{Lag}(4n) \). We have

\[
\iota(M, \gamma) = \mu(\text{Gr}(M), \text{Gr}(\gamma)).
\]

Then the Maslov-type index has the same properties of Maslov index. From Theorem 5.3, we have for \( M_1, M \in \text{Sp}(2n) \), \( \gamma \in C([a, b], \text{Sp}(2n)) \),

\[
\iota(\gamma(b), \gamma) \leq \iota(M, \gamma) \leq \iota(\gamma(a), \gamma).
\]

\[
|\iota(M_1, \gamma) - \iota(M, \gamma)| \leq 2n.
\]

(5.1)

For \( \gamma \in C^1([a, b], \text{Sp}(2n)) \) satisfied \( \dot{\gamma}(t) = JB(t)\gamma(t) \). We say \( \gamma \) is a positive path if \( B(t) > 0 \) for \( t \in [a, b] \). In this case, we have

\[
\iota(\omega M, \gamma) = \sum_{\xi \in [a,b]} \dim \ker(\gamma(\xi) - \omega M).
\]

**Remark 5.4.** In [LZ00b], the authors generalized the Maslov index to complex symplectic space. Let \( (\cdot, \cdot) \) be the standard inner product of \( \mathbb{C}^{2n} \). Define \( \omega(x, y) = (Jx, y) \) as the symplectic form on \( \mathbb{C}^{2n} \). Then the Lagrangian Grassmannian can also be defined. Then the Maslov index can also be defined for a pair of continuous paths in complex Lagrangian Grassmannian and each property also holds for complex Maslov index.

They also define the complex symplectic group as \( \text{Sp}(2n, \mathbb{C}) = \{ M \in \text{GL}(2n, \mathbb{C}) \mid M^*JM = J \} \). Then for real symplectic matrix \( M \) and \( \omega \in \mathbb{U}, \omega M \) is a complex symplectic matrix.

So \( \iota(\omega M, \gamma) = \mu(\text{Gr}(\omega M), \text{Gr}(\gamma)) \) is well defined and theorem 5.3 is also proved for complex Maslov index in [ZWZ18].

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