Progress in studying small-scale turbulence using ‘exact’ two-point equations

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\textbf{Abstract.} The analytical framework based on the similarity hypotheses of Kolmogorov and Obukhov arguably provides an adequate description of homogeneous isotropic turbulence at very large Reynolds numbers. In the flows normally encountered in the laboratory, the Reynolds number is finite and other influences, for example those due to a mean shear or, more generally, inhomogeneities associated with the larger scales, are present. In this paper, we review and assess some of the current progress in using ‘exact’ two-point equations for analysing the manner in which small-scale turbulence is affected by different types of inhomogeneities that may be present. There is strong support for this approach from experimental and/or numerical data for decaying homogeneous isotropic turbulence and along the axis of a round jet where the Reynolds number remains constant. In each of these flows, the major source of inhomogeneity is the streamwise decay of energy. Overall implications are discussed in the context of results obtained in physical space, although the correspondence to the spectral domain is also commented on briefly.
1. Introductory background

There is continuous progress in understanding the physics of turbulence, at least for non-complex flows and relatively simple boundary conditions. However, there is as yet no reliable and general predictive theory. One of the difficulties is the existence of a wide range of length scales, the smallest and largest being nominally the Kolmogorov length scale \( \eta \equiv \nu^{3/4}/\langle \epsilon \rangle^{1/4} \), where \( \nu \) is the kinematic viscosity of the fluid, \( \langle \epsilon \rangle \) the mean energy dissipation rate and angular brackets denote time averaging) and the integral length scale \( L \) which characterizes the energy containing eddies. Considerable attention is being given by physicists and engineers to the study of the so-called small scale motion, which typically encompasses the dissipative range (DR) where the influence of \( \nu \) cannot be ignored and the inertial range (IR) where the effect of \( \nu \) can be discarded. Provided the Reynolds number is sufficiently large, these two ranges are expected to be unaffected by the large scale motion. The interest in the small scale motion is understandable since, as noted by Sreenivasan and Antonia [1], ‘a proper theory of turbulence, if one were to emerge, may well relate to the small scale, which has the best prospect of being universal or quasi-universal’.

The appropriate starting point for describing the behaviour of small scale velocity and scalar fluctuations in turbulent flows is the analytical framework established by the similarity hypotheses proposed by Kolmogorov [2], hereafter K41 and Obukhov [3] or O49. The first hypothesis contained in K41 states that, in the DR, the pdf of the velocity increment \( \Delta \alpha \equiv \alpha(x + r) - \alpha(x) \) (\( r \) is the separation between the two points, \( \alpha \equiv u_1, u_2 \) or \( u_3 \) the velocity fluctuations in the \( x_1 \) (longitudinal), \( x_2 \) (spanwise) and \( x_3 \) (inhomogeneous) directions respectively) is uniquely determined by \( \langle \epsilon \rangle \) and \( \nu \) so that

\[
\langle (\Delta \alpha^*)^n \rangle = f_{\alpha n}(r^*),
\]

where the asterisk denotes normalization by the Kolmogorov velocity scale \( u_K \equiv (\nu \langle \epsilon \rangle)^{1/4} \) and/or the Kolmogorov length scale \( \eta \). The second hypothesis states that, in the IR (\( \eta \ll r \ll L \)),

\[
\langle (\Delta \alpha^*)^n \rangle = C_{\alpha n} r^{*n/3},
\]
where $C_{an}$ can be identified with the Kolmogorov ‘constants’ (which may depend on the flow macrostructure). Underpinning (1) and (2) are the following assumptions:

(i) the Reynolds number is very large, and
(ii) the small scales are isotropic.

(i) and (ii) are not unrelated since local isotropy becomes more plausible as the Reynolds number increases. It is common to use the longitudinal Taylor microscale Reynolds number $R_{\lambda}$ ($\equiv u_1' \lambda_{u1}/\nu$, where $\lambda_{u1} \equiv u_1'/(\partial u_1/\partial x_1)'$ is the longitudinal Taylor microscale; the prime denotes an rms value) as the appropriate Reynolds number. Hereafter, we identify the Taylor microscale $\lambda$ with $\lambda_{u1}$.

Expressions similar to (1) and (2) can be readily written for a passive temperature field which is advected and mixed by locally isotropic turbulence (the hypotheses in O49 are discussed in [4]). We shall not write these here; the reader is referred to chapter 7 of Chassaing et al [5] which reviews the behaviour of scalar structure functions within a framework which allows meaningful comparison with energy structure functions.

To account for the small scale intermittency (or fluctuations in time and space of $\epsilon$), Kolmogorov [6], hereafter K62, modified the dimensional arguments of K41 by replacing $\langle \epsilon \rangle$ by the variable $\epsilon_r$, where the subscript $r$ denotes averaging over a length scale $r$. In K62, (2) is replaced by

$$\langle (\Delta \alpha^*)^n \rangle = C_{an}^+ (\epsilon_r^{sn/3}) r^{sn/3},$$

(3)

where the superscript $+$ is introduced to allow a possible distinction from $C_{an}$. The extension of K62 to a passive scalar ($Pr \simeq 1$) has been considered by several authors [7]–[11] and will not be discussed here (again, we refer the reader to chapter 7 of [5]).

Relation (3) now allows for possible departures from the $r^{sn/3}$ predictions of K41, as given by relation (2). These departures clearly depend on the behaviour of $\langle \epsilon_r^{sn/3} \rangle$. If this quantity has a power-law variation in the IR, (3) can be rewritten as

$$\langle (\Delta \alpha^*)^n \rangle = C_{an}^+ r^{\zeta_{an}}.$$

(4)

There is significant evidence (see e.g. [1, 12, 13]) to indicate that $\zeta_{an}$ departs from K41, the departure increasing with $n$. Whilst this suggests that K62 is a more realistic descriptor than K41, the validity of all these frameworks hinges on assumptions (i) and (ii). For the majority of turbulent flows studied in laboratory, the magnitude of $R_{\lambda}$ is only moderate and isotropy and homogeneity are generally violated by large scales and, at best, only imperfectly satisfied by the small scales. Attempts to generate isotropic turbulence without a mean flow, thus reducing the inhomogeneity associated with the large scales, have been recently reported in the literature [14, 15].

A general idea of the departure from K41 of the second-order moments of velocity increments is given in figure 1. The data were collected [16] in a number of different flows (grid turbulence, wake, boundary layer, circular and plane jets, fully developed channel and pipe flows, atmospheric surface layer) over a two-decade increase in $R_{\lambda}$ (40 to 10 000) using an X-wire probe (oriented so as to measure either $u_1$, $u_2$ or $u_1$, $u_3$ fluctuations). The exponents $\zeta_{u1}$ and $\zeta_{u2}$ were estimated using the ESS method [17], i.e. by plotting $\langle |\Delta u_1|^2 \rangle$ or $\langle |\Delta u_2|^2 \rangle$ versus $\langle |\Delta u_1|^3 \rangle$. It is evident that $\zeta_{u1}$ approaches its asymptotic value much more rapidly than
\( \zeta_{u_2} \). The two exponents are not quite equal at \( R_\lambda = 10000 \) (here, the atmospheric boundary layer of Sreenivasan and Dhruva [18] were used), implying that even higher Reynolds numbers may be needed before either (2) or (3) is satisfied (K41 and K62 do not distinguish between exponents for \( u_1, u_2 \) and \( u_3 \)).

Caution is needed when interpreting the dependence of \( \zeta_{u_1} \) on \( R_\lambda \), as displayed in figure 1. This is because ESS provides a relative, rather than absolute, measure of the scaling exponents, e.g., [19]. These latter authors showed that, when ESS is avoided, \( \zeta_{u_1} \) tends to increase with \( R_\lambda \) and asymptotes to the K62 value at sufficiently large \( R_\lambda \). Different growth rates of \( \zeta_{u_2} \), denoted by the letters \( A, B, C, D \) are identified in figure 1. \( A \) corresponds to data \( (R_\lambda \simeq 205) \) on the axis of a cylinder wake where the anisotropy ratio \( (\langle u_2^2 \rangle/\langle u_1^2 \rangle) \) is about 0.89. \( B \) refers to off-centreline pipe data, where \( (\langle u_2^2 \rangle/\langle u_1^2 \rangle) \) is in the range 0.36–0.99 and the shear parameter \( S^* = (\eta/u_K)\partial U_1/\partial x_3 \) is in the range 0.06 to 0.08. The increased anisotropy dramatically enhances the inequality \( \zeta_{u_2} < \zeta_{u_1} \). Within the regions marked with the letters \( C \) and \( D \), \( \zeta_{u_2} \) increases systematically with \( R_\lambda \), even though \( (\langle u_2^2 \rangle/\langle u_1^2 \rangle) \) is approximately constant (in the range 0.71–0.77).

A few points emerge from figure 1. One cannot discard the relationship between anisotropy and the departure from K41 or K62, at least for the values of \( R_\lambda \) covered by the data. The approach of \( \zeta_{u_2} \) to its K41 (or K62) value may well depend on the nature of the flow, or indeed differences between different regions of the same flow. One may also need to distinguish between different sources of anisotropy, e.g., the difference between the anisotropy in grid turbulence, caused mainly by the wakes generated by the grid, and that associated with the vicinity of a solid wall or the presence of a mean shear. In each case, the anisotropy is closely linked to the presence of organized structures. Near a wall, the presence of quasi-streamwise vortices results in a significant departure of \( \zeta_{u_1} \) from its K41 (or K62) value, e.g., [20, 21].

It is now well established that for a wake or a jet, different states of self-preservation can exist, depending on the initial conditions [22]. This can affect inertial range scales, as illustrated in figure 2, where values of \( (\langle \Delta u_1 \rangle^2) \) and \( (\langle \Delta u_2 \rangle^2) \) are plotted for wakes generated by five different bluff bodies [23]. In each case, \( x/d \) (\( \simeq 70 \)) and \( R_\lambda \) (\( \simeq 200 \)) are the same. The effect
Figure 2. Values of $\langle (\Delta u_1^*)^2 \rangle$ and $\langle (\Delta u_2^*)^2 \rangle$ for wakes generated by five different bluff bodies (from [23]).

is larger for $\langle (\Delta u_1^*)^2 \rangle$, the oscillations reflecting the larger degree of organization for the two porous bodies (screen strip and screen cylinder) wakes; this is consistent with the observations of Cimbala et al [24], Cannon et al [25] and Zhou and Antonia [26].

Figure 2 implies that the magnitude of $\zeta u_2$ may reflect the manner in which each wake is initiated. It is worth commenting on the behaviour of $\langle (\Delta u_1^*)^2 \rangle$ and $\langle (\Delta u_2^*)^2 \rangle$ in the DR ($r^* \lesssim 10$). Although the collapse is better for $\langle (\Delta u_1^*)^2 \rangle$, it is somewhat contrived since the mean energy dissipation rate was estimated using 

$$\langle \epsilon \rangle_{iso} = 15\nu\langle (\partial u_1/\partial x_1)^2 \rangle.$$ (5)

The imperfect collapse across the DR, for $\langle (\Delta u_2^*)^2 \rangle$ reflects small departures from the locally isotropic relation $\langle (\partial u_2/\partial x_1)^2 \rangle = 2\langle (\partial u_1/\partial x_1)^2 \rangle$. In this context, the seemingly good collapse reported for Kolmogorov-normalized spectra of $u_2$ (e.g., [27]) cannot be interpreted as providing unambiguous support for relation (1) given that (5) was used to estimate $\langle \epsilon \rangle$.

The effect of initial conditions can also be observed in grid turbulence (figure 3, from [28]). Although both circular and square bars (which make up the grid) shed periodic wakes, the periodicity persists throughout the decay region in the former case. In each case, quite satisfactory similarity of the structure functions is achieved, using the variables $\langle q^2 \rangle \equiv \langle u_i u_i \rangle$ (repeated indices imply summation) and $\lambda$ advocated by George (1992) [29]. Whilst $\langle (\Delta q)^2 \rangle$ increases monotonically towards $2\langle q^2 \rangle$ at large $\tilde{r} \equiv r/\lambda$ for the square bars, this limiting value is overshot for the circular rods.

The above observations and considerations highlight the importance of accounting, preferably explicitly, for the non-negligible correlations between the scales at which energy is injected, those which dominate the transfer of this energy down the cascade and those responsible for dissipating this energy. This is especially important when $R_\lambda$ is moderate and various sources
of anisotropy are present. In this regard, an equation which represents an energy balance for any given length scale would be a desirable analytical platform for assessing the relationship between \(\langle (\Delta u_1)^3 \rangle\) and \(\langle (\Delta u_1)^2 \rangle\) in physical space or between \(T(k)\) and \(E(k)\) in the spectral domain.

In particular, we shall pay attention to departures from Kolmogorov’s equation

\[
-\langle (\Delta u_1)^3 \rangle + 6\nu \frac{d}{dr} \langle (\Delta u_1)^2 \rangle = \frac{4}{5} \langle \epsilon \rangle r, \tag{6}
\]

which reduces to the well-known ‘four-fifths’ law

\[
-\langle (\Delta u_1)^3 \rangle = \frac{4}{5} \langle \epsilon \rangle r, \tag{7}
\]

when the effect of viscosity is negligible. The mean energy dissipation rate \(\langle \epsilon \rangle\) is defined as

\[
\langle \epsilon \rangle = \frac{1}{2} \nu \left( \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)^2 \right). \tag{8}
\]

Equation (6) is derived within the framework of Kolmogorov [2, 30], which assumes a cascade that is universal and locally isotropic for small enough scales and large enough Reynolds numbers. Writing equation (6) as \(A + B = C\), term \(C\) (directly proportional to \(\langle \epsilon \rangle\)) is associated with the total transfer of energy at a scale \(r\). Equation (6) indicates that, at scale \(r\), the mean energy is transferred by both turbulent advection (term \(A\)) and molecular diffusion (term \(B\)).

At small Reynolds numbers, the sum \(A + B\) cannot be expected to balance \(C\), except at the smallest scales (e.g., for grid turbulence, this equality is satisfied only for \(r/\eta \lesssim 5\) at \(R_\lambda = 66\) [31]. For intermediate Reynolds numbers (100 < \(R_\lambda\) < 500), Kolmogorov’s four-fifths law is not satisfied for moderate to large scales (e.g., [32, 33]). This equation is only satisfied up to a maximum separation which depends on the Reynolds number. Obviously, it cannot be satisfied at large \(r\) when the contributions from the two terms on the left side of (6) become negligible.

Figure 3. Distributions of \(\langle (\Delta q)^2 \rangle\) for two different grids. The mesh size \(M\) is the same for both grids. – – –, square bars, \(x/M = 80\); – - –, grid with circular rods, \(x/M = 80\) (from [28]).
Given that, at Reynolds numbers normally encountered in the laboratory, the third-order moment (term $A$) cannot be proportional to $r$ over a large range of scales, it is worth enquiring if term $A$ can continue to be used for determining the inertial range. The scales over which $A \propto r$ are then ‘restricted’ and we thus refer to a restricted scaling range RSR. As will be shown in section 2, further contributions are required to restore the balance between the left and right sides of (6). They are associated with the effect of large-scale inhomogeneities which may differ intrinsically between different flows. Indeed, the difference may be such that, for the same $R_\lambda$, the inhomogeneous contributions may vary from flow to flow or possibly different regions of the same flow. In order to better appreciate the imbalance of equation (6) at large scales, it is appropriate to consider the limiting form at small separations. Given that

$$\lim_{r \to 0} 6v \frac{d}{dr} \langle (\Delta u_1)^2 \rangle = \lim_{r \to 0} 6v \frac{d}{dr} \left( \frac{(\Delta u_1)^2}{r^2 - r^2} \right) = 6v \left( \frac{\partial u_1}{\partial x_1} \right)^2 \frac{d}{dr} (r^2) = \frac{4}{5} \langle \epsilon \rangle_{iso} r,$$  

(9)

equation (6) reduces to $B = C$ in the limit $r \to 0$. In this case, term $A$ is two orders of magnitude smaller and therefore negligible. Consequently, at the smallest scales, equation (6) is consistent with the expression for $\langle \epsilon \rangle_{iso}$. This result is expected since the Kolmogorov equation was derived by assuming isotropy of the small scales. Alternatively, the Kolmogorov equation will be verified for very small scales if and only if $\langle \epsilon \rangle = \langle \epsilon \rangle_{iso}$. Analytical considerations [34] and experimental results [35, 36] indicate that, in the presence of a mean shear, the small-scale structure is not isotropic, so that $\langle \epsilon \rangle \neq \langle \epsilon \rangle_{iso}$. A more general relation between second- and third-order moments of velocity increments is given by [37]

$$-\langle \Delta u_1 (\Delta u_i)^2 \rangle + 2v \frac{d}{dr} \langle (\Delta u_i)^2 \rangle = \frac{4}{3} \langle \epsilon \rangle r.$$  

(10)

The analogy between equation (10) and Yaglom’s equation for temperature increments was also discussed in [37]. This more general form of Kolmogorov’s equation has the same characteristics as equation (6), i.e., it is valid only for relatively small scales when the magnitude of the Reynolds number is moderate. The range over which equation (10) is satisfied is identical to that for equation (6) [37]. Equation (10) actually represents an extended form (for all velocity components) of Kolmogorov’s equation. For very small scales, it complies with the homogeneous form of $\langle \epsilon \rangle$, namely

$$2v \lim_{r \to 0} \frac{d}{dr} \langle (\Delta u_i)^2 \rangle = 4v \left( \left( \frac{\partial u_i}{\partial x_1} \right)^2 \right) r = \frac{4}{3} \langle \epsilon \rangle r \Rightarrow \langle \epsilon \rangle = 3v \left( \left( \frac{\partial u_i}{\partial x_1} \right)^2 \right) = \langle \epsilon \rangle_{hom}.$$  

(11)

In this paper, we are mainly interested in the manner in which departures from equation (10) occur for different flows. The flows we have chosen are representative for different kinds of inhomogeneities. Generalized forms of (6) have already been presented in [31], [38]–[43].

The material is organized as follows. Following the presentation of the analytical framework (section 2) for different (homogeneous regions of) flows, equilibrium similarity for structure functions is discussed and results are then presented in section 3 for grid turbulence (section 3.1) and for the axial region of a round jet (section 3.2). A further discussion is provided in section 4.
2. Generalized forms of the Kolmogorov equation in homogeneous flows

The major objective here is to gain some insight into the flow physics which results in an imbalance between the left and right sides of equation (6) and more especially equation (10), in nearly homogeneous flows. We will mainly reassess previously published results [31], by drawing attention to the originality, merit and limitation of our approach.

Different flows are characterized by inhomogeneities along different spatial directions, reflected in the variation of quantities such as the mean longitudinal velocity, turbulent kinetic energy \( \langle q^2 \rangle \), as well as the mean energy dissipation rate \( \langle \epsilon \rangle \). Relatively simple flows usually present inhomogeneities along only one spatial direction, e.g., grid turbulence (along the flow direction), or the centreline of a turbulent channel flow (along the direction normal to the wall). For more complex flows, different large-scale effects could coexist, e.g., away from the centreline in a channel flow, where shear and lateral diffusion effects could act simultaneously. The jet axis is characterized mainly by a streamwise decay of energy, whereas several extra effects are present (shear, lateral diffusion, pressure) away from the axis. We restrict our attention here to the nearly homogeneous flows which contain only one large-scale effect. Specifically, we will attempt to provide an overview of the large-scale effects in grid turbulence and the axis of a turbulent round jet.

The inhomogeneities of these flows could be taken into account, although local isotropy is still maintained for all the other (turbulent advection, molecular diffusion and pressure diffusion) terms. From a mathematical point of view, the extra inhomogeneous terms are introduced and manipulated within a quasi-isotropic context. With this latter qualification, the equations given in this section can be regarded as ‘exact’.

2.1. Scale-by-scale energy budget equation for grid turbulence

In this subsection, we recall very briefly the salient steps in deriving the generalized form of Kolmogorov’s equation (10) [37], since a detailed derivation has been given in [31]. The presence of only one inhomogeneous direction leads naturally to the use of Cartesian coordinates when writing gradient and Laplacian operators.

Using the same procedure as that presented in [4, 31, 39] we write the incompressible Navier–Stokes equations at the two points \( \vec{x} \) and \( \vec{x}^+ \), which are separated by the increment \( \vec{r} = \vec{x}^+ - \vec{x} \), as follows:

\[
\partial_t u_i + u_\alpha \partial_\alpha u_i = -\partial_i p/\rho + \nu \partial^2_\alpha u_i, \tag{12}
\]

\[
\partial_\alpha u_i^+ + u_\alpha^+ \partial_\alpha u_i^+ = -\partial_\alpha^+ p^+/\rho + \nu \partial^2_\alpha^+ u_i^+. \tag{13}
\]

The superscript + refers to \( \vec{x}^+ \) and \( \rho \) is the fluid density. In (12) and (13), \( u_i \) is the instantaneous velocity vector, with \( \langle u_i \rangle = U_1 \delta_{ij} \) (\( \delta_{ij} \) is the Kronecker delta symbol, \( \rho \) is the mean velocity); \( p \) is the instantaneous pressure, \( \partial_i \equiv \partial/\partial t \), \( \partial_\alpha \equiv \partial/\partial x_\alpha \), and \( \partial^2_\alpha \) is the Laplacian \( \partial^2/\partial x_\alpha^2 \) (hereafter, the notation \( \partial_\alpha \) and \( \partial^2_\alpha \) will be used to denote derivatives with respect to \( x_\alpha \) and \( x_\alpha^+ \); when other spatial variables are involved, the derivatives will be written explicitly, e.g., \( \partial/\partial X_\alpha \) or \( \partial/\partial X_\alpha^+ \)).

We then consider that the two points \( \vec{x} \) and \( \vec{x}^+ \) are independent, i.e. \( u_i \) depends only on \( \vec{x} \), and \( u_i^+ \) depends only on \( \vec{x}^+ \), so that subtraction of (12) from (13) yields an equation for the velocity increment \( \Delta u_i = u_i^+ - u_i \), namely

\[
U_1 \partial_\chi_1 \Delta u_i + u_\alpha^+ \partial_\alpha (\Delta u_i) + u_\alpha \partial_\alpha (\Delta u_i) = - (\partial_\alpha^+ + \partial^2_\alpha^+)(\Delta p)/\rho + \nu (\partial^2_\alpha + \partial^2_\alpha^+)(\Delta u_i). \tag{14}
\]
At this stage, it is important to underline the presence of the first term in the LHS of (14), which represents the decay. Hereafter, for simplicity, $u_i$ will denote the fluctuating velocity (i.e. $\langle u_i \rangle = 0$). By subtracting and adding the term $u_a \partial_a^+ (\Delta u_i)$ to the LHS of (14), we obtain

$$U_1 \frac{\partial}{\partial x_1} \Delta u_i + \Delta (u_a) \partial_a^+ (\Delta u_i) + u_a \partial_a (\Delta u_i) + u_a \partial_a (\Delta u_i) = -(\partial_i + \partial_i^+) (\Delta p)/\rho + v(\partial_a^2 + \partial_a^{2+}) (\Delta u_i).$$

(15)

By multiplying (15) with $2 \Delta u_i$ and averaging, we finally obtain

$$U_1 \frac{\partial}{\partial x_1} \langle (\Delta u_i)^2 \rangle (\vec{r}) + \frac{\partial}{\partial r} \langle \Delta u_a (\Delta u_i)^2 \rangle (\vec{r}) = -2(\partial_i + \partial_i^+) \langle \Delta p \cdot \Delta u_i \rangle (\vec{r}) + 2v \frac{\partial^2}{\partial r^2} \langle (\Delta u_i)^2 \rangle (\vec{r}) - 2(\langle \epsilon \rangle + \langle \epsilon^+ \rangle).$$

(16)

In grid turbulence, local isotropy is satisfied by those terms which are present in a restricted scaling range (RSR) only, i.e. for the molecular diffusion and the advection terms. These terms exist only at small and moderate scales and they would depend therefore on $r$ (the modulus of the separation $\vec{r}$) only. The divergence and Laplacian operators assume particular forms. Note that exactly the same assumptions are made in the classical equations (6) or (10). Moreover, the pressure–velocity correlation term is negligible, also because of local isotropy. Finally, the decay term is isotropic [31], and therefore depends on $r$ only.

Since all the terms in (16) depend on $r$ only, we finally obtain by following the same procedure as in [4]:

$$-\langle \Delta u_i (\Delta u_i)^2 \rangle + 2v \frac{d}{dr} \langle (\Delta u_i)^2 \rangle - \frac{1}{r^2} \int_0^r s^2 \left[ U_1 \frac{\partial}{\partial x_1} \langle (\Delta u_i)^2 \rangle \right] ds = \frac{4}{3} \langle \epsilon \rangle r,$$

(17)

where $s$ is a dummy variable.

Equation (17) could be written as

$$A + B + D = C,$$

(18)

where $A$ represents the third-order structure function, $B$ is the dissipation term, $C = 4/3 \langle \epsilon \rangle r$ and $D$ is the decay term.

The limiting form, at very large scales, of this equation can be obtained by applying the same method as described in [31]. The result is

$$\lim_{r \to \infty} D = \lim_{r \to \infty} \frac{2}{r^2} \int_L y^2 \left[ \frac{1}{2} U_1 \frac{\partial}{\partial x_1} \langle (\Delta u_i)^2 \rangle \right] dy = \frac{2r}{3} U_1 \frac{\partial}{\partial x_1} \langle u_i^2 \rangle = \frac{4}{3} \langle \epsilon \rangle r.$$

(19)

We now introduce another definition of $\langle \epsilon \rangle$. This is a kinetic energy balance equation, using homogeneity and only local isotropy (pressure diffusion is neglected), namely

$$\langle \epsilon \rangle^{LS} = -\frac{1}{2} U_1 \frac{\partial}{\partial x_1} \langle u_i^2 \rangle,$$

(20)

where the superscript ‘LS’ indicates that it is a one-point approach, so that it concerns the large-scales of the flow. Equation (19) then reduces to $\langle \epsilon \rangle = \langle \epsilon \rangle^{LS}$. The generalized form of the
extended Kolmogorov equation (17) is therefore consistent, at very large scales, with the equality \( \langle \epsilon \rangle = \langle \epsilon \rangle_{LS} \).

2.2. Scale-by-scale energy budget equation on the axis of a round jet

We now turn our attention to the axis of a round jet flow, for which axisymmetry holds. The inhomogeneity of this flow at both \( \mathbf{x} \) and \( \mathbf{x}^+ \) is along \( x_1 \) (the flow direction) and \( \rho \), the radial direction. We further assume in the specific context of the jet that

- the flow is stationary at a given position \( x_1 \), so that time derivatives of all statistical quantities are zero;
- the flow is inhomogeneous along \( x_1 \) and \( \rho \). However, on the jet axis, the only inhomogeneity of the flow is along \( x_1 \). For those terms reflecting large-scale effects, cylindrical coordinates will be used. For the classical terms (advection and molecular diffusion terms) spherical coordinates are more adequate, since these terms only appear at small scales, which may be supposed to be locally isotropic, on the axis. The same assumption has to be made with more caution out of the jet axis, because of the shear which automatically creates (small-scale) anisotropy.
- on the jet axis, pressure containing terms are negligible, because of local isotropy.

Therefore, the gradient vector could be written in cylindrical coordinates

\[
\nabla \equiv \left( \partial_1, \frac{\partial}{\partial \rho}, \frac{1}{\rho} \frac{\partial}{\partial \theta} \right),
\]

where \( \partial_1 = \partial / \partial x_1 \), \( \theta \) is the azimuthal direction. We further note that \( \vec{u} = (u_1, u_\rho, u_\theta) \) and \( \vec{U} = (U_1, U_\rho, U_\theta) \). The property of axisymmetry leads to: \( U_\theta \equiv 0 \) (everywhere in the jet) and \( \partial(\cdot) / \partial \theta \equiv 0 \). In addition, by taking into account that on the jet axis, \( \langle u_1 u_\rho \rangle = 0, U_\rho = 0 \) and both diffusion terms could be neglected [44], the same type of calculations as in section 2.1 could be carried out.

We next consider the spatial dependence of the large-scale terms. The decay term \( U_1 \partial_1 \langle (\Delta u_j)^2 \rangle (r) \) depends on the second-order structure function \( \langle (\Delta u_j)^2 \rangle (r) \).

We will show that

\[
U_1 \partial_1 \langle (\Delta u_j)^2 \rangle (r_j) \approx U_1 \partial_1 \langle (\Delta u_k)^2 \rangle (r_k),
\]

with \( k \neq j \), which is equivalent to \( \langle (\Delta u_j)^2 \rangle (r_j) \approx \langle (\Delta u_k)^2 \rangle (r_k) \). The validation of this approximation requires numerical investigations or rather special experimental measurements. For small scales, local isotropy requires that

\[
\langle (\partial_j u_i)(\partial_j u_i) \rangle \approx \langle (\partial_k u_i)(\partial_k u_i) \rangle.
\]

Relation (22) only holds in locally isotropic regions of the flow. Therefore, our analysis is well-adapted to such regions, in particular to the jet axis. The large-scale behaviour of the second-order structure function is given by

\[
\lim_{r \to \infty} \langle (\Delta u_j)^2 \rangle (r) = \langle u_j^2 \rangle (\mathbf{x}) + \langle u_j^2 \rangle (\mathbf{x} + \mathbf{r}).
\]
If the flow is homogeneous, equation (23) becomes
\[
\lim_{\vec{r} \to \infty} \langle (\Delta u_{i})^{2} \rangle(\vec{r}) = \langle u_{i}^{2} \rangle(\vec{x}) + \langle u_{i}^{2} \rangle(\vec{x} + \vec{r}) = 2 \langle u_{i}^{2} \rangle(\vec{x}).
\] (24)

The region near the axis can be considered homogeneous along $\rho$ [44], while the decay along $x_1$ is small. This last effect is qualitatively similar to that characterizing decaying grid turbulence, where the scales over which the decay is significant are much larger than the characteristic scales of the flow. Therefore, the axial region of a round jet could, to a first approximation, be compared to grid turbulence. Conversely, grid turbulence could be considered analytically as axisymmetric.

The second-order structure function and hence the decay term do not depend on the spatial orientation. This is valid at both small separations (in locally isotropic regions of the flow), and at large scales, when the flow is homogenous. Note that second-order structure functions are also included in the dissipative term, for which local isotropy has already been supposed. As this dissipative term depends explicitly on the second-order derivative with respect to $r$ of the second-order structure function, it contains no information about the large-scale behaviour of this structure function. In other words, the requirement that the decay term does not depend on the spatial orientation of the separation vector $\vec{r}$ is much stronger than imposing isotropy for the dissipative term, since the decay and dissipative terms act over non-overlapping ranges of scales.

We finally obtain
\[
-\langle (\Delta u_{1}(\Delta u_{1})^{2} \rangle + 2\nu \frac{d}{dr} \langle (\Delta u_{1})^{2} \rangle - \frac{U_1}{r^2} \int_{0}^{r} s^2 \partial_{1} \langle (\Delta u_{1})^{2} \rangle \, ds

\] 
\[
- 2 \frac{\partial U_{1}}{\partial x_{1}} \frac{1}{r^2} \int_{0}^{r} s^2 ((\Delta u_{1})^{2} - (\Delta u_{1})^{2}) \, ds = \frac{4}{3} \langle \varepsilon \rangle r,
\] (25)

where $s$ is a dummy variable. We emphasize that equation (25) holds only on the jet axis.

Equation (25) could be written in a dimensionless form as (by dividing equation (25) by $\langle \varepsilon \rangle \lambda$):
\[A^* + B^* + D^* + P^* = C^*,\]

where $C^* = 4/3\tilde{r}$, $D^*$ is the inhomogeneous (‘decay’) term along the streamwise direction $x_1$ and $P^*$ is the production term.

3. Use of similarity for comparing with experimental and numerical data

In equations (17) and (25), which can be interpreted as scale-by-scale energy budget equations for different types of flows, the large-scale terms reflect the presence of inhomogeneities along the flow direction. Our aim in this section is to test these equations against experimental data. Naturally, the large-scale terms need to be evaluated as accurately as possible. The assumption of similarity, or self-preservation, facilitates the evaluation considerably, even though we recognize that it is not necessarily rigorous.

Self-preservation is a well-established idea, which simply implies that distributions of different quantities (mean velocity profiles, spectra, structure functions, etc) at different spatial
locations (or different times) are similar when non-dimensionlized by appropriate velocity or length scales.

For our purpose, two main types of similarity are most relevant:

– *local similarity*, which implies that the small-scale behaviour of spectra and structure functions is similar when using Kolmogorov variables. This similarity was introduced by Kolmogorov [2] and has received satisfactory experimental support in a variety of flows. The region over which this similarity holds increases with increasing $R_\lambda$.

– *equilibrium similarity*, which supposes that all scales evolve in a similar way, for a prescribed set of initial conditions. This idea was developed by George, who applied it to several flows: grid turbulence [29] (hereafter, G92); a jet flow [45] and a homogeneous shear flow [46]. George’s approach has focused on the spectral domain. Following [29], Antonia et al [47] extended this analysis to structure function equations (scale-by-scale energy budget equations) in grid turbulence. The advantage of using structure functions is that these show departures (in physical space) that are masked in the spectral domain (where there is an integral constraint).

### 3.1. Similarity for grid turbulence. Correspondence between structure functions and spectral equations

In this section, we restrict ourselves to the simplest case of decaying homogeneous isotropic turbulence. The structure function equation derived and discussed in detail by [31] is given by

$$
-\langle (\Delta u_1)^3 \rangle + 6v \frac{d}{dr} \langle (\Delta u_1)^2 \rangle + \frac{3U_1}{r^4} \int_0^r s^4 \frac{d}{ds} \langle (\Delta u_1)^2 \rangle \, ds = \frac{4}{5} \langle \varepsilon \rangle r,
$$

which is a simpler form of equation (17).

We briefly recall how equilibrium similarity can be used to properly estimate the large-scale (decay) term in equation (17). For more details, the reader should refer to [47]. As already shown [47], a possible similarity solution requires that the mean energy $\langle q^2 \rangle$ decays with a power-law behaviour

$$
\langle q^2 \rangle \sim x_1^m.
$$

We first briefly recall the conditions under which equation (17) can satisfy similarity. Following G92 [29],

$$
\langle (\Delta q)^2 \rangle = Vf \left( \frac{r}{\mathcal{L}} \right)
$$

and

$$
-\langle (\Delta u_1)(\Delta q)^2 \rangle = Hg \left( \frac{r}{\mathcal{L}} \right),
$$

where $\mathcal{L}$ is a characteristic length scale, $V$ (with dimensions of velocity squared) and $H$ (with dimensions of velocity cubed) are scales that characterize $\langle (\Delta q)^2 \rangle$ and $\langle (\Delta u_1)(\Delta q)^2 \rangle$; $\mathcal{L}$, $V$ and $H$ depend only on $x_1$. The dimensionless functions $f$ and $g$ depend not only on $r/\mathcal{L}$ but also on initial conditions (for simplicity of expression, this latter dependence is not indicated). After
substituting (27) and (28) into (17), we obtain

$$Hg + 2\frac{vV}{L} f' + \frac{U_1L^2}{r^2} V \frac{dL}{dx_1} \Gamma_1 - \frac{U_1L^3}{r^2} \frac{dV}{dx_1} \Gamma_2 = \frac{-2}{3} \frac{U_1L^2 d\langle q^2 \rangle}{dx_1} \frac{r}{L},$$  \hspace{1cm} (29)

where a prime denotes differentiation with respect to \(r/L\) and \(\Gamma_1\) and \(\Gamma_2\) are given by

$$\Gamma_1 \equiv \int_0^{r/L} \left( \frac{s}{L} \right)^3 f' d\left( \frac{s}{L} \right),$$  \hspace{1cm} (30)

$$\Gamma_2 \equiv \int_0^{r/L} \left( \frac{s}{L} \right)^2 f d\left( \frac{s}{L} \right).$$  \hspace{1cm} (31)

In (29), \(\langle \epsilon \rangle\) was replaced by its corresponding value for decaying isotropic turbulence, given by (20), and the dependence of both \(f\) and \(g\) on \(\tilde{r} \equiv r/L\) is implicitly assumed. After multiplication by \((L/vV)\), equation (29) could be written as

$$\left[ \frac{H \lambda}{vV} \right] g + [2] f' + \left[ \frac{U_1L \frac{dL}{dx_1}}{vV} \right] \tilde{r}^{-2} \Gamma_1 - \left[ \frac{U_1L^2 \frac{dV}{dx_1}}{vV} \right] \tilde{r}^{-2} \Gamma_2 = \frac{-2}{3} \left[ \frac{U_1L^2 d\langle q^2 \rangle}{dx_1} \right] \tilde{r}.$$  \hspace{1cm} (32)

By imposing equilibrium similarity (e.g., G92 [29]), all the terms within square brackets must evolve in the streamwise direction in exactly the same way. Since the second term of these is constant, all the other terms must also be constant. The analysis leads to the characteristic length scale \(L\), which can be identified with \(\lambda\); that \(V \equiv \langle q^2 \rangle\) and

$$H \sim v \frac{\langle q^2 \rangle}{\lambda} \text{ or } H \sim R_\lambda^{-1} \langle q^2 \rangle^{3/2}.$$

With \(V \equiv \langle q^2 \rangle\), \(H \equiv 3^{-1/2} R_\lambda^{-1} \langle q^2 \rangle^{3/2}\) and \(L \equiv \lambda\), equation (32) can be rewritten as

$$g = \frac{20}{3} \tilde{r} + 5 \frac{\tilde{r}^{-2}}{m} \Gamma_1 - 10 \tilde{r}^{-2} \Gamma_2 - 2 f'.$$  \hspace{1cm} (33)

This indicates that \(g\) can be uniquely determined once \(f\), the normalized second-order structure function, and \(m\), the power-law exponent describing the rate of decay of \(\langle q^2 \rangle\), are known. The dependence of \(g\) on initial conditions occurs implicitly through \(f\) and \(m\), both of which depend on the initial conditions.

Using experimental data described in [47] (where details about the experimental setup and measurements are given), the distributions of \(\langle (\Delta q)^2 \rangle\), when normalized by \(\langle q^2 \rangle\) and \(\lambda\), satisfy similarity reasonably well over a significant range of scales [47], thus providing support for the analysis. Kolmogorov-normalized distributions of \(\langle (\Delta q)^2 \rangle\) collapse only at small \(r\) for the present small value of \(R_\lambda \approx 66\). Despite the small Reynolds number, this collapse is consistent, in spirit, with Kolmogorov’s 1941 [48] first similarity hypothesis mainly because the small scales are nearly isotropic in this flow. Kolmogorov-normalization breaks down at sufficiently large \(r\) since the ratio \(\langle q^2 \rangle^2 \equiv \langle q^2 \rangle / u_\lambda^2\) evolves linearly with \(R_\lambda\); this local sensitivity may be regarded as one advantage of structure functions over spectra since, for the latter, it is the integral under the spectrum which is equal to \(\langle q^2 \rangle\). The range of \(r\) for which Kolmogorov normalization applies should dilate as \(R_\lambda\) increases.
The collapse of the λ-normalized $\langle (\Delta q)^2 \rangle$ data provides an important platform for the purpose of calculating $g \equiv -3^{1/2} R_3 \langle (\Delta \tilde{u})(\Delta \tilde{q})^2 \rangle$, the normalized third-order structure function. Agreement between calculated and measured distributions of $g$ is satisfactory, allowing for the uncertainty in this measurement, the fact that homogeneity and isotropy are satisfied only approximately in the experiment and the possible influence of the tunnel walls. All the terms of equation (33) are shown if figure 4. It is evident that the magnitude of the $\Gamma_1$ term increases rapidly as $\tilde{r}$ increases and exceeds all the other terms when $\tilde{r} \gg 2$. The $\Gamma_2$ term is the next largest term for $\tilde{r} \gg 1$. We have also included in figure 4 the contribution from $(g - 5\Gamma_1 \tilde{r}^{-2}/m + 10\Gamma_2 \tilde{r}^2 + 2f')$ (solid circles) and $(g + 10\Gamma_2 \tilde{r}^{-2} + 2f')$. The latter, where $\Gamma_1$ is ignored, does not provide as good an agreement with the line $20\tilde{r}/3$ as the former.

The level of agreement is an improvement over that previously reported between calculated and measured third-order correlation functions.

Although the focus in this paper is on the structure function equation, it is of interest to comment on the correspondence between results obtained from this equation and those deduced from the spectral energy equation which represents an energy balance for any given wavenumber. Equation (26) corresponds to the well-known spectral energy equation:

$$ \frac{\partial E(k)}{\partial t} = T(k) - 2\nu k^2 E(k), \tag{34} $$

where $T(k)$ is the spectral transfer function ($\int_0^\infty T(k) \, dk = 0$), and the 3-D spectrum $E(k)$ is defined such that $\int_0^\infty E(k) \, dk = \langle q^2 \rangle/2$. The integral term in equation (26) corresponds to $\partial E(k)/\partial t$ in equation (34), the third-order velocity structure function corresponds to the non-linear energy transfer function while both terms $6\nu (d/dr) \langle (\Delta u_1)^2 \rangle$ and $\frac{1}{2} \langle \epsilon \rangle r$ correspond to $2\nu k^2 E(k)$.

The energy budget equation is recovered by integrating (34) with respect to $k$ (recall that $\langle \epsilon \rangle = 2\nu \int_0^\infty k^2 E(k) \, dk$), or by taking the limit of equation (26) as $r \to \infty$ (in this case, $\langle (\Delta u_1)^3 \rangle \to 0$ and $\langle (\Delta u_1)^2 \rangle \to 2u_1^2$).
The vorticity budget [49]

\[- \frac{1}{35 \nu} \frac{\partial \langle \epsilon_{iso} \rangle}{\partial t} = \left\langle \left( \frac{\partial u_1}{\partial x_1} \right)^3 \right\rangle + 2 \nu \left\langle \left( \frac{\partial^2 u_1}{\partial x_1^2} \right)^2 \right\rangle \]  

(35)

is recovered either by integrating equation (34), after first multiplying by \( k^2 \) (using that \( \frac{\partial}{\partial t} \int_0^\infty k^2 E(k) \, dk = \frac{1}{2 \nu} \frac{\partial \langle \epsilon \rangle}{\partial t} \); \( \int_0^\infty k^2 T(k) \, dk \propto \langle (\partial u_1/\partial x_1)^2 \rangle \) and \( \int_0^\infty k^4 E(k) \, dk \propto \langle (\partial^2 u_1/\partial x_1^2)^2 \rangle \)), or by taking the limit of equation (26) as \( r \to 0 \) (see Antonia et al [50], for details).

Whilst both equations (26) and (34) comply with (35), the manner in which this is achieved differs in each case. This difference needs to be kept in mind when interpreting results for equations (26) or (34). For example, \( \langle (\Delta \epsilon_1)^2 \rangle \) and \( E(k) \) display different sensitivities to the effect of \( \lambda R \) when normalized by Kolmogorov variables. In the first case, \( \langle (\Delta \epsilon_1)^2 \rangle \sim \langle (\Delta q^2) \rangle \sim 2 \langle q^2 \rangle \sim R_\lambda \), i.e. it is the actual value of \( \langle (\Delta q^2) \rangle \) that is directly proportional to \( R_\lambda \) when \( r^* \) is sufficiently large. For the second case, \( K_{41} \) is satisfied at sufficiently large \( k \) when \( \partial E/\partial t \approx 0 \), so that \( T \approx 2 \nu k^2 E(k) \). This would be the spectral equivalent of Kolmogorov’s equation (6).

Equation (35) could be written as [49]:

\[ U_1 \frac{d \langle \epsilon_{iso} \rangle}{dx_1} = \frac{7 \langle \epsilon_{iso} \rangle^{3/2}}{3 \sqrt{15} \nu^{1/2}} \left( S - \frac{2G}{R_\lambda} \right) \]  

(36)

where

\[ S = -\left\langle \left( \frac{\partial u_1}{\partial x_1} \right)^3 \right\rangle / \left\langle \left( \frac{\partial u_1}{\partial x_1} \right)^{2.5} \right\rangle \quad \text{and} \quad G = \langle u_1^2 \rangle \left\langle \left( \frac{\partial^2 u_1}{\partial x_1^2} \right)^2 \right\rangle / \left\langle \left( \frac{\partial u_1}{\partial x_1} \right)^2 \right\rangle \]  

(37)

Equation (36) was recently generalized [51] to

\[ U_1 \frac{d \langle \epsilon_{hom} \rangle}{dx_1} = \langle \epsilon_{hom} \rangle^{1/2} \left( \frac{5}{3} \right)^{1/2} S_q - \frac{10 G_q}{9 R_\lambda} R^{-1/2} \]  

(38)

where

\[ R = \frac{\langle \epsilon_{iso} \rangle}{\langle \epsilon_{hom} \rangle} \]  

(39)

is the ratio of the isotropic and homogeneous values of \( \langle \epsilon \rangle \), with

\[ S_q = -\left( \frac{\partial u_1}{\partial x_1} \right)^{2.5} \left\langle \left( \frac{\partial u_i}{\partial x_1} \right)^2 \right\rangle \quad \text{and} \quad G_q = \frac{\langle u_i u_i \rangle \left\langle \left( \frac{\partial^2 u_i}{\partial x_1^2} \right)^2 \right\rangle}{\left\langle \left( \frac{\partial u_i}{\partial x_1} \right)^2 \right\rangle ^2} \]  

(40)
$S_q$ is a production of $\langle \epsilon \rangle_{hom}$ through vortex stretching and $G_q$ is a destruction of $\langle \epsilon \rangle_{hom}$ via the molecular viscosity.

While both equations (38) and (36) are well verified against experimental data [28, 51] similarity implies that $SR_\lambda$ or $S_q R_\lambda$ should stay constant for different positions behind the grid. This is not confirmed by measurements, presumably because of the fact that the equilibrium similarity is not well verified for the very small scales.

A recent study [52] based on DNS box turbulence data (with well resolved small scales), indicated that the high wavenumber part of the velocity spectrum collapsed better when normalized by $\eta$ and $u_K$ than with $\lambda$ and $\langle q^2 \rangle$. However, the latter scales provide good collapse of the remainder of the spectrum and are especially appropriate for calculating the non-linear transfer of energy from large to small scales. For decaying homogeneous isotropic turbulence at low/moderate values of $R_\lambda$, it has been shown that the similarity solution of George cannot be strictly correct for all scales. For the (very) small scales, Kolmogorov similarity (K41) appears to collapse the high wavenumber energy spectrum better than G92. This is presumably due to the fact that, whilst G92 is rigorous, it is too constraining in the sense that all the terms in the budget equation are required to satisfy similarity, irrespectively of the wavenumber. Note that the two types of similarity are fully consistent with each other when $R_\lambda$ remains constant (as is, for example, the case for the far field of a round jet) or when $R_\lambda$ is infinitely large.

3.2. Equilibrium similarity on the axis of a round jet

It is of interest to inquire into the similarity of all the scales on the axis of a round jet, since $R_\lambda$ is expected to remain constant along the axis in the far field of a jet [53]. This can be done by considering either velocity power spectra or second-order velocity structure functions, since both sift out the distribution of the energy among the different scales. In this section, we investigate the equilibrium similarity of the velocity structure function equation, (25). Once more, we prefer to focus on the physical space primarily because the results are more amenable to testing with experimental data.

In order to examine the conditions under which Eq. (25) satisfies similarity, we need to assume functional forms for the terms in this equation. Following [47], we take

\[
\langle (\Delta u_i)^2 \rangle = Q(x_1) f(\xi), \quad (41)
\]
\[
\langle (\Delta u_1)^2 \rangle = M(x_1) e(\xi), \quad (42)
\]
\[
\langle (\Delta u_\rho)^2 \rangle = R(x_1) h(\xi), \quad (43)
\]
\[
-(\Delta u_1 (\Delta u_i)^2) = T(x_1) g(\xi), \quad (44)
\]

where $\xi = r/L$ and $L$ is a characteristic length scale, to be determined. A possible dependence on the initial conditions, as explained in [29], is also plausible, but is not explicitly considered here. $Q(x_1), M(x_1), R(x_1)$, and $T(x_1)$ are scales that characterize the second-order structure functions of $u_1, u_\rho$ and the third-order structure function, respectively. The lower-case functions represent the shape of the structure functions. The separation between functions of $x_1$ and $\xi$ allows solutions of the transport equation for which a relative balance among all of the terms is maintained as...
the flow progresses downstream to be obtained. Upon substituting equations (44) into equation (25), we obtain (after differentiating and rearranging)

\[
T(x_1)g(\xi) + 2\nu Q(x_1) \frac{1}{L} f'(\xi) - \frac{U_1 Q'(x_1)}{r^2} L^3 \Gamma_1 + \frac{U_1 Q(x_1)}{r^2} \frac{d\xi}{dx_1} L^2 \Gamma_2
\]

\[
- 2\frac{dU_1 M(x_1)}{dx_1} L^3 \Gamma_3 + 2\frac{dU_1 R(x_1)}{dx_1} L^3 \Gamma_4 = \frac{4}{3} \langle \epsilon \rangle_{\text{hom}} r.
\]

(45)

where

\[
\Gamma_1 = \int_0^{r/L} \frac{s^2}{L^2} f(\xi) d\left(\frac{s}{L}\right), \quad \Gamma_2 = \int_0^{r/L} \frac{s^3}{L^3} f'(\xi) d\left(\frac{s}{L}\right),
\]

\[
\Gamma_3 = \int_0^{r/L} \frac{s^2}{L^2} e(\xi) d\left(\frac{s}{L}\right), \quad \Gamma_4 = \int_0^{r/L} \frac{s^2}{L^2} h(\xi) d\left(\frac{s}{L}\right).
\]

Note that the following relation has been used

\[
\frac{d\xi}{dx_1} = -r L^{-2} \frac{d\xi}{dx_1}. \tag{45}
\]

After separating the terms which depend on \( x_1 \) from those which depend on \( \xi \), multiplying by \( L/\nu Q(x_1) \) and requiring that, for equilibrium similarity, all terms function of \( x_1 \) must evolve in the streamwise direction in the same way, we obtain

\[
\frac{T(x_1) L}{\nu Q(x_1)} = \text{const.}, \tag{46}
\]

\[
\frac{U_1 Q'(x_1) L^2}{Q(x_1)} = \text{const.}, \tag{47}
\]

\[
U_1 \frac{d\xi}{dx_1} L = \text{const.}, \tag{48}
\]

\[
\frac{\partial U_1 M(x_1)}{\partial x_1} \frac{L^2}{Q(x_1)} = \text{const.}, \tag{49}
\]

\[
\frac{\partial U_1 R(x_1)}{\partial x_1} \frac{L^2}{Q(x_1)} = \text{const.}, \tag{50}
\]

\[
\frac{\langle \epsilon_{\text{hom}} \rangle L^2}{Q(x_1)} = \text{const.}. \tag{51}
\]

From equation (51), a characteristic scale can be identified with the Taylor microscale \( \lambda \). Similar to what was found formally for grid turbulence [29, 47] and the homogeneous shear...
flow [46], the Taylor microscale emerges as a relevant length scale for the whole of the energy spectrum, possibly reflecting the fact that it combines information from both the large- and small-scale quantities, apparently negating the criticism that the Taylor microscale has no direct physical meaning.

Energy structure functions and spectra (not shown here), measured at a number of locations (30 \( \leq x_1/D \leq 90 \), where \( D \) is the jet diameter) along the jet axis, support the requirement of self-similarity at all separations to a good approximation. This is shown in figure 5, where the second-order structure functions, normalized by \( \lambda \) and \( \langle q^2 \rangle \) are given. Since the Taylor microscale Reynolds number is approximately constant along the axis, the structure functions and spectra also collapse reasonably well when the normalization uses either the Kolmogorov or integral length scales.

It is worth mentioning that the scale-by-scale budget equation reduces to the mean kinetic energy equation, in the limit of very large separations.

The use of similarity forms for the structure functions turns out to be advantageous in evaluating the terms in equation (45). This can be rewritten in a dimensionless form

\[
A^* + B^* + D^* + D_\lambda^* + P^* = C^*,
\]

where \( C^* = 4/3\bar{r} \). Here, the decay term has been split to highlight the dual role of \( \lambda \). Equation (52) is subsequently tested against experimental data measured in the self-preserving region of a circular jet.

These measurements provided the streamwise and radial velocity components in the self-preserving region (\( x_1/D = 40 \)) of a round jet at \( R_{\lambda} = 450 \). For this purpose, an X-wire probe was used, after calibration using a look-up-table approach. The length-to-diameter ratio of the sensitive part of each wire was nearly 200, with a diameter of 2.54 \( \mu \text{m} \). For the jet data, the length of the wire was typically 2.5 times larger than the estimated Kolmogorov length scale. Low pass filters, set at half the sampling frequency, were also used to attenuate high-frequency components.
noise. The frequency response of the anemometers was of the order of 20 kHz. Sampling duration was long enough to allow converge of the structure functions at large separations. The signals were digitized with a resolution of 16 bit. More details on the experimental setup can be found in [44].

The scale-by-scale budget equation, (52), is satisfied reasonably well by the experimental data in figure 6. Although the sum $A^* + B^*$ balances the term $C^*$ at small $r/\lambda$, the inclusion of the new terms leads to a satisfactory balance for the whole range of scales. For $r/\lambda \gtrsim 10$, the energy-decay term $D^*$ is dominant, lending support to the earlier suggestion that, along the axis, the round jet tends to behave similarly to grid turbulence. Nonetheless, other effects associated with the production (term $P^*$) or the decay of the length scale (term $D^*_\lambda$), are not entirely negligible for $r/\lambda \gtrsim 10$. In order to illustrate further the satisfactory balance displayed in figure 6, term $A^*$ is compared with that ‘calculated’ from equation (52) using the remaining terms. Figure 7 compares the dimensionless third-order structure function $A^*$ divided by $\tilde{r}$, as obtained from experiments, with the calculated value of this term, $A^*/\tilde{r}$. The very good agreement between measured and calculated values of $A^*/\tilde{r}$ leads to the conclusion that the third-order structure function can be satisfactorily predicted from the scale-by-scale energy budget equation. Note that the asymptotic value of $4/3$, expected to apply for high Reynolds number homogeneous isotropic turbulence, is significantly higher than the maximum measured (or calculated) value of $A^*/\tilde{r}$. The maximum is located near $r = \lambda$ (this observation is also valid for grid turbulence). Although $R_\lambda$ is not small, figure 7 indicates that the RSR is indeed quite restricted. Nonetheless, one may be justified in expecting the velocity structure functions to exhibit an approximate power-law behaviour over a limited range centred about $r = \lambda$. This justification is based on the observation (figure 6) that terms $B^*$ and more especially $D^*$ (the major source of inhomogeneity) remain small compared to $A^*$ over this range. Clearly, the identification of this range would not be possible without a knowledge of $D^*$.

**Figure 6.** Measured distributions of terms in equation (52) along the axis of a round jet. $\Delta, A^*$; $\times, B^*$; $\bigcirc, P^*$; $\square, D^*_\lambda$; $\nabla, D^*$; solid line, $C^*$; $+$, LHS of equation (52).
4. Concluding discussion

More general forms of the Kolmogorov equation have been proposed and verified for decaying grid turbulence and on the axis of a round jet. The new equations take into account the large-scale inhomogeneity, which acts along the streamwise direction. The equations are important in that they essentially reflect the budget for the energy at any particular scale.

Taking account of the large-scale inhomogeneity results in a significant improvement of the budget, the level of agreement being typically about $\pm 10\%$ in the RSR for isotropic turbulence or on the jet axis. Thus, viscous effects, turbulent advection and large-scale inhomogeneities need to be considered properly in the transport equations of second-order structure functions. For the two flows we have focused on, the streamwise decay of energy is quite important and one may expect a certain analogy to exist between the characteristics of the small scales in these two flows.

The equations proposed here do not contradict Kolmogorov’s theoretical framework; they simply extend it in the context of two relatively simple flows. Indeed, relation (6) is retrievable from these equations when the Reynolds number is sufficiently large. For very small scales, equations (17) or (25) lead to the homogeneous expression for $\langle \epsilon \rangle$ given by (11). More importantly, the equations, when analysed using one-point measurements, are, at very large scales, consistent with different expressions of $\langle \epsilon \rangle$ derived from the Navier–Stokes equations, e.g., equation (20). These well known relations represent the kinetic energy budget equations in the flows considered.

The generalized equations (17) and (25) may be considered to represent a ‘bridge’ between the already known forms of these equations, e.g., equation (10), and the (also already known) one-point energy budget equations. Note also that the success of the new equations is very good in flows or flow regions where the isotropic forms of $\langle \epsilon \rangle$ (i.e. the small-scale limits of our equations) are very close to the values of $\langle \epsilon \rangle$ given by the one-point energy budget (i.e., the...
large-scale limits of our equations). A lower quality of agreement is expected when a mean shear is present.

Our results indicate that, for moderate Reynolds numbers, the third-order structure function is not well-adapted (as traditionally thought) for defining an inertial range since it does not strictly vary with \( r \). Its magnitude does not balance \( 4/5 \langle \epsilon \rangle r \) and it cannot therefore be used for estimating \( \langle \epsilon \rangle \). The effect of the large-scale inhomogeneity also needs to be considered. This shortcoming of the third-order structure function has been associated with the so-called ‘finite Reynolds number’ (FRN) effect by Qian [54] who quantified it using the ‘exact spectral equations’ for freely decaying isotropic turbulence and shear flow turbulence.

In the RSR, it may be possible to determine the scaling exponents \( \zeta_{u,n} \) and compare them with Kolmogorov’s predictions (either K41 or K62). It is now generally accepted that an unambiguous estimation of \( \zeta_{u,n} \) is fraught with difficulties. The present equations help in this regard since they allow the lower and, more especially, the upper limit of the RSR to be identified with more rigour than would be possible if the contribution from the inhomogeneity were not known.

The large-scale inhomogeneity has to be assessed with care when discussing intermittency [42]; in the context of K62, intermittency is strictly associated with very large Reynolds numbers and the absence of any form of inhomogeneity. Under such conditions, \( \langle (\Delta u_1)^3 \rangle \) is given by the ‘four-fifths’ law, equation (7). This result is not affected by intermittency; indeed, intermittency theories and models have complied with this requirement in the limit of infinite Reynolds number. As can be inferred from the results in section 3, the position is somewhat more delicate in flows at finite Reynolds numbers with a non-negligible inhomogeneity associated with the large scales. For the smallest scales, the effect of the inhomogeneity remains small, lending support to the notion of quasi-universality in this range [1]. In the RSR, the departure from the four-fifths law may be significant and, correspondingly, the departures of the scaling from K41 and K62 are also likely to be important. One may loosely ascribe these departures to the large-scale inhomogeneity. A precise breakdown of the causes for these departures could be difficult, especially in flows more complex than the ones considered here. Near a wall, one expects the quasi-streamwise vortices to be responsible for the strong inhomogeneity and anisotropy of the fluctuations in the instantaneous energy dissipation rate. One expects the large-scale inhomogeneity to play a role in the budget equations for \( n \)th order moments. The correct behaviour of \( S_n \) is described by a more complete equation which should also consider the inhomogeneity specific to the flow. Our aim has been to describe the relationship between the third-order and second-order structure functions in two simple flows as accurately as possible, by ensuring that the appropriate physical phenomena specific to these flows are considered. It is worth mentioning that the large-scale effects are not restricted to laboratory realizations of isotropic turbulence. Forcing of the flow, which is applied typically at low wavenumbers, has to be taken into account explicitly when studying the energy budget in the physical domain from direct numerical simulations of forced isotropic turbulence (e.g., [55], and references therein). Fukayama et al [56] found that the departures from K41 or K62 persist in forced box turbulence, although they are smaller than that for decaying turbulence.

While the present approach presented here has the main merit to improve the energy budget equation essentially in the RSR (and for the larger scales), we are aware that there is no improvement at all in the energy budget equation for the (very) small scales. As previously discussed, the small-scale limits of our equations are consistent with the classical Kolmogorov equations (6) and (10). The role played by the large-scale effects on the energy transfer at (very) small scales would be an interesting question to pursue in the future.

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An important effort is currently being made in modelling the small scales in the context of large eddy simulations (LES). The use of structure functions for sub-grid scale models is being considered [57]–[59]. The objective is to determine the (non-resolved) small scale characteristics from the larger (resolved) scales. In this context, a good knowledge of the small-scale properties in each flow becomes an important outcome. As previously stated, our generalized equations are ‘exact’ (allowing for the assumptions that have been made) for the two simple flows that are considered. For example, equation (33) (grid turbulence) is an ‘exact’ relation between the second- and third-order structure functions. It can therefore predict the third-order SF, once the second-order SF is known. This could be exploited in the LES approach by starting with the calculated values of the second-order SF for the large resolved scales. Using the modelled second-order SF for the small (unresolved) scales (Batchelor’s model [60] etc), equation (33) would then provide a means for estimating the third-order SF for the small scales. This direction needs to be pursued further.

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