KURAMOTO ORDER PARAMETERS AND PHASE CONCENTRATION FOR THE KURAMOTO-SAKAGUCHI EQUATION WITH FRUSTRATION

SEUNG-YEAL HA*
Department of Mathematical Sciences and Research Institute of Mathematics
Seoul National University, Seoul 08826 and Korea Institute for Advanced Study
Hoegiro 87, Seoul, 130-722, Republic of Korea

JAVIER MORALES
Center for Scientific Computation and Mathematical Modeling
University of Maryland, College Park, MD 20742, USA

YINGLONG ZHANG
Stochastic Analysis and Application Research Center
Korea Advanced Institute of Science and Technology, Daejeon 34141, Republic of Korea

Dedicated to the celebration of the 80th birthday of Prof. Shuxing Chen

Abstract. We study phase concentration for the Kuramoto-Sakaguchi(K-S) equation with frustration via detailed estimates on the dynamics of order parameters. The Kuramoto order parameters measure the overall degree of phase concentrations. When the coupling strength is sufficiently large and the size of frustration parameter is sufficiently small, we show that the amplitude order parameter has a positive lower bound uniformly in time, and we also show that the total mass concentrates on the translated phase order parameter by a frustration parameter asymptotically, whereas the mass in the region around the antipodal point decays to zero exponentially fast.

1. Introduction. The purpose of this paper is to continue the study begun in [22, 35] on the phase concentration of the kinetic K-S type equations via the dynamics of order parameters. Since Christiaan Huygens’s novel observation on the asynchronous behavior of two pendulum clocks, collective behaviors of a large population of coupled oscillators is often observed in biology, control theory, physics, and social sciences, etc, see [38, 42]. However, its rigorous studies have been started a half century ago by two pioneers, Arthur Winfree and Yoshiki Kuramoto in [30, 41]. Recently, collective behaviors of multi-agent systems and networks have received a lot of attentions due to increasing needs from the control of sensor networks, power networks, multi-agent UAVs and internet networks. In this paper, we are interested

2020 Mathematics Subject Classification. Primary: 35Q70, 70F99; Secondary: 92B25.

Key words and phrases. Emergent dynamics, Kuramoto model, frustration, order parameters, synchronization.

The work of S. Y. Ha was supported by National Research Foundation of Korea(NRF-2020R1A2C3A01003881), and the work of Y. Zhang was supported by the National Research Foundation of Korea(NRF) grant funded by the Korea government(MSIT)(NRF-2019R1A5A1028324).

* Corresponding author.
in the collective behaviors of the Kuramoto model with frustration, which was discussed in the seminal work of Sakaguchi and Kuramoto [39] as a prototype example of synchronization of phase-coupled oscillators.

Let \( \theta_i = \theta_i(t) \in \mathbb{T} \) and \( \alpha \in (-\frac{\pi}{2}, \frac{\pi}{2}) \) be the phase of the \( i \)-th Kuramoto oscillator and the uniform frustration between oscillators, respectively. Then the dynamics of Kuramoto oscillators is governed by the following system [1, 29, 30, 39, 40]:

\[
\dot{\theta}_i = \nu_i + \frac{\kappa}{N} \sum_{j=1}^{N} \sin(\theta_j - \theta_i + \alpha), \quad t > 0,
\]

\[
\theta_i(0) = \theta_i^{\text{ini}}, \quad i = 1, \ldots, N, \quad (1.1)
\]

where \( \nu_i, \kappa > 0, \) and \( N \in \mathbb{N}_+ \) denote the natural frequency, the coupling strength, and the number of oscillators, respectively.

Since the right-hand side of (1.1) is Lipschitz continuous and bounded, a global well-posedness of equation (1.1) can be followed from the standard Cauchy-Lipschitz theory. Thus, one of remaining interesting issues is the large-time collective dynamics of (1.1) such as the existence of attractors, identification of basin of attractors and the relaxation process. The need of frustration is suggested from empirical facts in physics and biology [1, 5, 10, 11, 32, 36, 37, 43]. The presence of frustration can cause numerous mathematical difficulties in the analysis of synchronization behaviors underlying oscillatory systems. To name a few, total phase \( \sum_{j=1}^{N} \theta_j \) is not conserved, and system (1.1) is not a gradient flow any more. Thus, analytical tools employed in the study of the Kuramoto model with zero frustration cannot be used as it is, e.g., energy method breaks down for system (1.1) with \( \alpha \neq 0 \). Thus, analyzing large-time behavior of physical systems without conservation laws is one of challenging problems in nonlinear dynamics. For the Kuramoto model, detailed analysis for the emergence of synchronization has been extensively studied in [7, 9, 13, 14, 15, 19, 28, 34] and references therein, in contrast, existing nonzero frustration problems are mostly based on numerical approaches due to the aforementioned difficulties. As far as we know, there are only few analytical results studying on the nonzero frustration case. In [20, 21, 33], the authors showed synchronization of oscillators for (1.1) with initial configurations confined in a half circle. These results were further extended to a region a slightly larger than a half circle for identical case in [18]. Recently, the work [23] showed the emergence of phase-locking for system (1.1) with general initial data using the analysis of Kuramoto order parameters. At present, for an initial configuration which does not lie in a half circle, whether does system (1.1) with a distributed natural frequencies exhibit phase-locking in a large coupling regime is not known yet.

In this paper, we address synchronization phenomenon (or phase concentration) for the corresponding kinetic K-S equation which can be obtained from the particle model (1.1) in a mean-field limit. More precisely, we look for a sufficient framework leading to the phase concentration of the kinetic K-S equation. As the number of oscillators tends to infinity, i.e., \( N \to \infty \), the dynamics of a large particle system (1.1) can be effectively approximated by the corresponding mean-field kinetic equation. Below, we briefly introduce the mean-field kinetic equation which can be derived formally from the particle model (1.1) via the BBGKY hierarchy. Let \( f = f(t, \nu, \theta) \) be an one-oscillator probability density function at phase \( \theta \) with natural frequency \( \nu \) at time \( t \). Then, the dynamics of \( f \) is governed by the following Cauchy problem
to the K-S equation with frustration:

\[
\begin{align*}
\partial_t f + \partial_\nu (V[f]f) &= 0, & (\theta, \nu, t) &\in \mathbb{T} \times \mathbb{R} \times \mathbb{R}_+,
V[f](t, \nu, \theta) &:= \nu + \kappa \int_{\mathbb{T} \times \mathbb{R}} \sin(\theta_* - \theta + \alpha) f(t, \nu, \theta_*) d\theta_* d\nu_*,

f(0, \nu, \theta) &= f^{in}(\nu, \theta) \geq 0, & f^{in}(\nu, \theta + 2\pi) &= f^{in}(\nu, \theta),
\end{align*}
\]

A global existence of measure-valued solution for (1.2) was studied in [25], and the stability and instability analysis of the incoherent solution are studied in [17, 24]. For zero frustration case, we refer to [2, 3, 4, 6, 8, 12, 16, 26, 27, 31] and references therein for a global existence and asymptotic behaviors. In [22], the authors proved the amplitude order parameter has a positive lower bound near unity under the assumption that the coupling strength is sufficiently large, and they showed the total mass concentrates on the phase order parameter. Thus, following the same line of their study, we investigate the emergence of phase concentration for the K-S equation with frustration. More precisely, we study how the frustration affects the phase concentration dynamics for the K-S equation. In the presence of frustration, does the amplitude order parameter still has a lower bound near unity? If then under what condition? Does the mass still concentrate? If then, how does the concentration look like? Via a detailed analysis on the Kuramoto order parameters under what condition? Does the mass still concentrate? If then, how does the phase concentration dynamics for the K-S equation. In the presence of frustration, does the amplitude order parameter still has a lower bound near unity? If then under what condition? Does the mass still concentrate? If then, how does the concentration look like? Via a detailed analysis on the Kuramoto order parameters (R(t), \phi(t)), we show that the amplitude order parameter \( R(t) \) does have a positive lower bound \( R_\infty = R_\infty(\kappa, \alpha) \):

\[
\liminf_{t \to \infty} R(t) \geq R_\infty \approx 1,
\]

in a regime that the coupling strength is sufficiently large and the size of frustration is sufficiently small. We also present a positive lower bound \( R_\infty = R_\infty(\kappa) \) for the zero frustration case, which is somewhat different from the one given in [22], but it has the same property, i.e., it tends to unity as the coupling strength tends to infinity. Furthermore, we show the mass outside a time-dependent region \( \mathcal{D}_\infty(t) \) which is away from the translated average phase \( \phi(t) + \alpha \) decays exponentially fast: there exists positive constant \( \Lambda_\infty = \Lambda_\infty(R_\infty, \kappa, \alpha) \) such that

\[
\sup_{\nu \in [-\ell, \ell]} \sup_{\theta \in \mathbb{T} \setminus \mathcal{D}_\infty(t)} |f(t, \nu, \theta)| e^{-\kappa \Lambda_\infty t}, \quad t \geq 0.
\]

For a detailed main result, we refer to Section 3.

The rest of this paper is organized as follows. In Section 2, we provide several estimates on the order parameters \((R(t), \phi(t))\), masses around the average phase \( \phi(t) + \alpha \) and \( \phi(t) + \alpha + \pi \), and their relations. These estimates will be used crucially in later analysis. In Section 3, we provide phase concentration estimate for the K-S equation with frustration. In Section 4, we provide a proof of Proposition 3.1 which is used in section 3 without proof. In Appendix A, we provide the Lipschitz continuities of \( \hat{R}(t) \) and \( \mathcal{M}_t^\gamma(t) \).

2. Preliminaries. In this section, we study preliminary materials to be used in later sections. We first introduce real-valued Kuramoto order parameters and study their dynamics, and then we define two time-dependent intervals \( L_\ell^+(t) \) and \( L_\ell^-(t) \), with constant \( \gamma \) less than \( \pi/2 \) to be determined later, and study the evolution of masses on these intervals. Finally, we present several relations between \( R(t) \) and masses on \( L_\ell^+(t) \) and \( L_\ell^-(t) \).
First, we state a basic priori estimates on the positivity of $f$ and conservation law.

**Lemma 2.1.** Let $f = f(t, \nu, \theta)$ be a classical solution to (1.2). Then we have

$$f(t, \nu, \theta) \geq 0, \forall (t, \nu, \theta) \in \mathbb{T} \times \mathbb{R} \times \mathbb{R}_+,$$

and

$$\int_{\mathbb{T} \times \mathbb{R}} f(t, \nu, \theta) d\theta = \int_{\mathbb{T} \times \mathbb{R}} f^{in}(\theta, \nu) d\theta = g(\nu),$$

and

$$\int_{\mathbb{T} \times \mathbb{R}} f(t, \nu, \theta) d\theta d\nu = 1, \quad t \geq 0.$$

**Proof.** Since a proof of can be found in [17, 24], here we omit details here.

2.1. **The order parameters.** First, we introduce the real-valued order parameters $R(t) \in [0, 1]$ and $\phi(t) \in \mathbb{T}$ as follows:

$$R(t)e^{i\phi(t)} := \int_{\mathbb{T} \times \mathbb{R}} e^{i\theta} f(t, \nu, \theta) d\theta d\nu. \quad (2.1)$$

We divide the both sides of (2.1) by $e^{i\phi(t)}$, and compare the real and imaginary parts of the resulting relation to see

$$R(t) = \int_{\mathbb{T} \times \mathbb{R}} \cos(\theta - \phi) f(t, \nu, \theta) d\theta d\nu$$

and

$$0 = \int_{\mathbb{T} \times \mathbb{R}} \sin(\theta - \phi) f(t, \nu, \theta) d\theta d\nu. \quad (2.2)$$

If we divide the both sides of (2.1) by $e^{i(\theta - \alpha)}$, and compare the imaginary part of the resulting relation to get

$$\int_{\mathbb{T} \times \mathbb{R}} \sin(\theta_\star - \theta + \alpha) f(t, \nu_\star, \theta_\star) d\theta_\star d\nu_\star = -R \sin(\theta - \phi - \alpha).$$

Thus, equation (1.2) can be rewritten using order parameters:

$$\partial_t f + \partial_\theta \left( (\nu - \kappa R \sin(\theta - \phi - \alpha)) f \right) = 0, \quad (\theta, \nu, t) \in \mathbb{T} \times \mathbb{R} \times \mathbb{R}_+,$$

$$f(0, \nu, \theta) = f^{in}(\nu, \theta), \quad \int_{\mathbb{T} \times \mathbb{R}} f^{in}(\nu, \theta) d\theta d\nu = 1, \quad (2.3)$$

**Lemma 2.2.** Let $f = f(t, \nu, \theta)$ be a classical solution to (1.2), and let $R(t)$ and $\phi(t)$ be the order parameters defined in (2.2). Then we have

$$(i) \quad \dot{R} = -\int_{\mathbb{T} \times \mathbb{R}} (\nu - \kappa R \sin(\theta - \phi - \alpha)) \sin(\theta - \phi) f(t, \nu, \theta) d\theta d\nu,$$

$$(ii) \quad \dot{\phi} = \int_{\mathbb{T} \times \mathbb{R}} (\nu - \kappa R \sin(\theta - \phi - \alpha)) \cos(\theta - \phi) f(t, \nu, \theta) d\theta d\nu. \quad (2.4)$$

**Proof.** We first differentiate (2.1) with respect to $t$ to get

$$\dot{R} e^{i\phi} + iR \dot{\phi} e^{i\phi} = \int_{\mathbb{T} \times \mathbb{R}} e^{i\theta} \partial_t f d\theta d\nu. \quad (2.5)$$

Then we divide both sides of (2.5) by $e^{i\phi}$ to obtain

$$\dot{R} + iR \dot{\phi} = \int_{\mathbb{T} \times \mathbb{R}} e^{i(\theta - \phi)} \partial_t f d\theta d\nu.$$
\[
eq i \int_{\mathbb{T} \times \mathbb{R}} e^{i(\theta-\phi)} \left( \nu - \kappa R \sin(\theta - \phi - \alpha) \right) f d\theta d\nu. \quad (2.6)
\]

We compare the real and imaginary parts of equation (2.6) to get estimates (2.4).

Note that Lemma 2.2 yields
\[
\dot{R} = - \int_{\mathbb{T} \times \mathbb{R}} \nu \sin(\theta - \phi) f d\theta d\nu + \kappa R \cos \alpha \int_{\mathbb{T} \times \mathbb{R}} \sin^2(\theta - \phi) f d\theta d\nu
- \kappa R \sin \alpha \int_{\mathbb{T} \times \mathbb{R}} \sin(\theta - \phi) \cos(\theta - \phi) f d\theta d\nu.
\]

For the study of \( \dot{R}(t) \), we define
\[
S(t) = \kappa R \int_{\mathbb{T} \times \mathbb{R}} \sin^2(\theta - \phi) f d\theta d\nu. \quad (2.7)
\]

Remark 2.1. By definition of \( R(t) \) and Lemma 2.1, we have
\[
0 \leq S(t) \leq \kappa R(t).
\]

Lemma 2.3. Let \( f = f(t, \nu, \theta) \) be a classical solution to equation (1.2). Then we have
\[
\frac{d}{dt}(RS) \geq -3\kappa RS \cos \alpha - \frac{4(\ell^2 + \kappa^2 \sin |\alpha| \cos \alpha)}{\cos \alpha}, \quad t > 0.
\]

Proof. We use relation (2.7) to obtain
\[
\frac{d}{dt} RS = \dot{R}S + R \dot{S}
= \dot{R}S + \kappa R \dot{R} \int_{\mathbb{T} \times \mathbb{R}} \sin^2(\theta - \phi) f d\theta d\nu + \kappa R^2 \int_{\mathbb{T} \times \mathbb{R}} \sin^2(\theta - \phi) \partial_\nu f d\theta d\nu
= -2\dot{R}S + \kappa R^2 \int_{\mathbb{T} \times \mathbb{R}} \sin^2(\theta - \phi) \partial_\nu f d\theta d\nu. \quad (2.8)
\]

Recall that equation (2.3) gives
\[
\partial_\nu f = -\partial_\nu \left( \nu - \kappa R \sin(\theta - \phi - \alpha) f \right). \quad (2.9)
\]

We substitute (2.9) into (2.8) and integrate the resulting relation by parts to have
\[
\frac{d}{dt} RS = 2\dot{R}S + 2\kappa R^2 \int_{\mathbb{T} \times \mathbb{R}} \sin(\theta - \phi) \cos(\theta - \phi) \left( \nu - \kappa R \sin(\theta - \phi - \alpha) \right) f d\theta d\nu
= 2\dot{R}S + 2\kappa R^2 \int_{\mathbb{T} \times \mathbb{R}} \nu \sin(\theta - \phi) \cos(\theta - \phi) f d\theta d\nu
- 2\kappa^2 R^3 \cos \alpha \int_{\mathbb{T} \times \mathbb{R}} \sin^2(\theta - \phi) \cos(\theta - \phi) f d\theta d\nu
+ 2\kappa^2 R^3 \sin \alpha \int_{\mathbb{T} \times \mathbb{R}} \sin(\theta - \phi) \cos^2(\theta - \phi) f d\theta d\nu. \quad (2.10)
\]

By using the positivity of \( f \) and relation (2.7), we get
\[
2\kappa^2 R^3 \cos \alpha \int_{\mathbb{T} \times \mathbb{R}} \sin^2(\theta - \phi) \cos(\theta - \phi) f d\theta d\nu \leq 2\kappa R^2 S \cos \alpha \leq 2\kappa RS \cos \alpha, \quad (2.11)
\]

and
\[
2\kappa^2 R^3 \sin \alpha \int_{\mathbb{T} \times \mathbb{R}} \sin(\theta - \phi) \cos^2(\theta - \phi) f d\theta d\nu \leq 2\kappa^2 \sin |\alpha|. \quad (2.12)
\]
Thus, the remaining thing is to estimate \(2\dot{R}S\) and \(2\kappa R^2 \int_{T \times R} \nu \sin(\theta - \phi) \cos(\theta - \phi) f d\theta d\nu\).

- (Estimates on \(2\dot{R}S\)): By Lemma 2.1, Remark 2.1 and definition of \(R(t)\), one has
  \[
  \left| -2\kappa RS \sin \alpha \int_{T \times R} \sin(\theta - \phi) \cos(\theta - \phi) f d\theta d\nu \right| \leq 2\kappa RS \sin |\alpha| \leq 2\kappa^2 \sin |\alpha|. \tag{2.13}
  \]

We use Lemma 2.2 (i), relation (2.7) and estimate (2.13) to get
  \[
  2\dot{R}S = -2S \int_{T \times R} \nu \sin(\theta - \phi) f d\theta d\nu + 2\kappa RS \cos \alpha \int_{T \times R} \sin^2(\theta - \phi) f d\theta d\nu \\
  - 2\kappa RS \sin \alpha \int_{T \times R} \sin(\theta - \phi) \cos(\theta - \phi) f d\theta d\nu \\
  \geq -2S \int_{T \times R} \nu \sin(\theta - \phi) f d\theta d\nu + 2S^2 \cos \alpha - 2\kappa^2 \sin |\alpha|. \tag{2.14}
  \]

Then we use Relation (2.7), Cauchy inequality and Lemma 2.1 to get
  \[
  2S \int_{T \times R} \nu \sin(\theta - \phi) f d\theta d\nu \\
  \leq 2 \int_{T \times R} \sin^2(\theta - \phi) f d\theta d\nu \times \left\{ \frac{\cos \alpha}{2} \int_{T \times R} \kappa^2 R^2 \sin^2(\theta - \phi) f d\theta d\nu + \frac{1}{2 \cos \alpha} \int_{T \times R} \nu^2 f d\theta d\nu \right\} \\
  \leq S^2 \cos \alpha + \frac{\ell^2}{\cos \alpha}. \tag{2.15}
  \]

Hence, we substitute (2.15) into (2.14) to obtain
  \[
  2\dot{R}S \geq -\frac{\ell^2}{\cos \alpha} - 2\kappa^2 \sin |\alpha|. \tag{2.16}
  \]

- (Estimates on the second term): Direct calculation yields
  \[
  2\kappa R^2 \int_{T \times R} \nu \sin(\theta - \phi) \cos(\theta - \phi) f d\theta d\nu \\
  \leq 2R \left\{ \frac{\cos \alpha}{2} \int_{T \times R} \kappa^2 R^2 \sin^2(\theta - \phi) f d\theta d\nu + \frac{1}{2 \cos \alpha} \int_{T \times R} \nu^2 f d\theta d\nu \right\} \\
  \leq \kappa RS \cos \alpha + \frac{\ell^2}{\cos \alpha}. \tag{2.17}
  \]

Now we substitute estimates (2.11), (2.12), (2.16) and (2.17) into relation (2.10) to obtain
  \[
  \frac{d}{dt} RS \geq -3\kappa RS \cos \alpha - \frac{2\ell^2}{\cos \alpha} - 4\kappa^2 \sin |\alpha| \\
  \geq -3\kappa RS \cos \alpha - \frac{4(\ell^2 + \kappa^2 \sin |\alpha| \cos \alpha)}{\cos \alpha}, \quad t > 0.
  \]

**Lemma 2.4.** Let \(f = f(t, \nu, \theta)\) be a classical solution to equation (1.2). Then, as long as \(R\) has a positive lower bound \(R > 0\) such that
  \[
  R(t) > R,
  \]
we have the following estimates: for \(t \geq 0\),
  \[
  (i) \frac{\cos \alpha}{2} S(t) - \frac{\ell^2}{2\kappa R \cos \alpha} - \kappa \sin |\alpha| < \dot{R}(t) < \frac{3\cos \alpha}{2} S(t) + \frac{\ell^2}{2\kappa R \cos \alpha} + \kappa \sin |\alpha|.
  \]
\( \frac{\dot{\phi}(t)}{\dot{R}} \leq \frac{\ell}{R} + \kappa(1 - \frac{R}{R}) \cos \alpha + \kappa \sin |\alpha| \).

**Proof.** (i) Recall that
\[
\dot{R} = -\int_{\mathbb{T} \times \mathbb{R}} \nu \sin(\theta - \phi) f d\theta d\nu + \kappa R \cos \alpha \int_{\mathbb{T} \times \mathbb{R}} \sin^2(\theta - \phi) f d\theta d\nu \\
- \kappa R \sin \alpha \int_{\mathbb{T} \times \mathbb{R}} \sin(\theta - \phi) \cos(\theta - \phi) f d\theta d\nu.
\]
We use the Cauchy-Schwarz inequality and a priori assumption \( R(t) > R \) to get
\[
\left| -\int_{\mathbb{T} \times \mathbb{R}} \nu \sin(\theta - \phi) f(t, \nu, \theta) d\theta d\nu \right| \\
\leq \kappa R \cos \alpha \int_{\mathbb{T} \times \mathbb{R}} \sin^2(\theta - \phi) f d\theta d\nu + \frac{1}{2\kappa R \cos \alpha} \int_{\mathbb{T} \times \mathbb{R}} \nu^2 f d\theta d\nu \leq \frac{\cos \alpha}{2} S + \frac{\ell^2}{2\kappa R \cos \alpha}.
\]
Together with estimate
\[
\left| -\kappa R \sin \alpha \int_{\mathbb{T} \times \mathbb{R}} \sin(\theta - \phi) \cos(\theta - \phi) f d\theta d\nu \right| \leq \kappa R \sin |\alpha| \leq \kappa \sin |\alpha|,
\]
we have
\[
\dot{R} > -\frac{\cos \alpha}{2} S - \frac{\ell^2}{2\kappa R \cos \alpha} + \cos \alpha S - \kappa \sin |\alpha| \leq \frac{\cos \alpha}{2} S - \frac{\ell^2}{2\kappa R \cos \alpha} - \kappa \sin |\alpha|,
\]
\[
\dot{R} < \frac{\cos \alpha}{2} S + \frac{\ell^2}{2\kappa R \cos \alpha} + \cos \alpha S + \kappa \sin |\alpha| = \frac{3\cos \alpha}{2} S + \frac{\ell^2}{2\kappa R \cos \alpha} + \kappa \sin |\alpha|.
\]
(ii) We use Lemma 2.2 to know
\[
\dot{\phi} = \frac{1}{R} \int_{\mathbb{T} \times \mathbb{R}} \nu \cos(\theta - \phi) f d\theta d\nu - \kappa \cos \alpha \int_{\mathbb{T} \times \mathbb{R}} \sin(\theta - \phi) \cos(\theta - \phi) f d\theta d\nu \\
+ \kappa \sin \alpha \int_{\mathbb{T} \times \mathbb{R}} \cos^2(\theta - \phi) f d\theta d\nu.
\]
We claim:
\[
R - 1 \leq \int_{\mathbb{T} \times \mathbb{R}} \sin(\theta - \phi) \cos(\theta - \phi) f d\theta d\nu \leq 1 - R, \quad t \geq 0. \tag{2.19}
\]
**Proof of claim (2.19):** For the first inequality, we use relation (2.2), Lemma 2.1 and the fact \((\sin(\theta - \phi) - 1)(\cos(\theta - \phi) - 1) \geq 0\) to get
\[
\int_{\mathbb{T} \times \mathbb{R}} \sin(\theta - \phi) \cos(\theta - \phi) f d\theta d\nu \\
= \int_{\mathbb{T} \times \mathbb{R}} ((\sin(\theta - \phi) - 1)(\cos(\theta - \phi) - 1) + \sin(\theta - \phi) + \cos(\theta - \phi) - 1) f d\theta d\nu \\
\geq \int_{\mathbb{T} \times \mathbb{R}} \sin(\theta - \phi) f d\theta d\nu + \int_{\mathbb{T} \times \mathbb{R}} \cos(\theta - \phi) f d\theta d\nu - \int_{\mathbb{T} \times \mathbb{R}} f d\theta d\nu = R - 1. \tag{2.20}
\]
On the other side, we use relation (2.2), Lemma 2.1 and the fact \((\sin(\theta - \phi) + 1)(\cos(\theta - \phi) - 1) \leq 0\) to have
\[
\int_{\mathbb{T} \times \mathbb{R}} \sin(\theta - \phi) \cos(\theta - \phi) f d\theta d\nu \\
= \int_{\mathbb{T} \times \mathbb{R}} ((\sin(\theta - \phi) + 1)(\cos(\theta - \phi) - 1) + \sin(\theta - \phi) - \cos(\theta - \phi) + 1) f d\theta d\nu
Theorem 2.2. \( (2.20) \) will be verified in Section 4.

2.2. Evolution of masses on \( L^\pm_\gamma(t) \). In this subsection, we study temporal evolution of masses on time-dependent intervals \( L^+_\gamma(t) \) and \( L^-_\gamma(t) \) around \( \phi(t) + \alpha \) and \( \phi(t) + \alpha + \pi \):

\[
L^+_\gamma(t) := \left( \phi(t) - \frac{\pi}{2} + (\gamma + \alpha), \phi(t) + \frac{\pi}{2} - (\gamma - \alpha) \right),
\]

\[
L^-_\gamma(t) := \left( \phi(t) + \frac{\pi}{2} + (\gamma + \alpha), \phi(t) + \frac{3\pi}{2} - (\gamma - \alpha) \right),
\]

with \( \gamma \) less than \( \frac{\pi}{2} \) to be determined later. We define \( \ell^1 \) and \( \ell^2 \) functionals on \( L^+_\gamma(t) \) and \( L^-_\gamma(t) \) as follows:

\[
\mathcal{M}^+_\gamma(t) := \int_{L^+_\gamma \times \mathbb{R}} f(t, \nu, \theta) d\theta d\nu, \quad \widetilde{\mathcal{M}}^+_\gamma(t) := \int_{L^-_\gamma \times \mathbb{R}} f(t, \nu, \theta) d\theta d\nu,
\]

\[
\mathcal{M}^-_\gamma(t) := \int_{L^-_\gamma \times \mathbb{R}} f^2(t, \nu, \theta) d\theta d\nu.
\]

Lemma 2.5. Let \( f = f(t, \nu, \theta) \) be a classical solution to (1.2). Then, the functional \( \mathcal{M}^+_\gamma(t) \) in (2.23) satisfies

\[
\frac{d}{dt} \mathcal{M}^+_\gamma = \int_{L^+_\gamma \times \mathbb{R}} f(t, \nu, \theta) d\theta d\nu + \kappa R \cos \gamma \int_{\mathbb{R}} (B_+ + B_-) d\nu, \quad t > 0,
\]

where \( B_+ \) and \( B_- \) are functions as follows:

\[
B_+ := f(t, \nu, \phi + \frac{\pi}{2} - (\gamma - \alpha)), \quad B_- := f(t, \nu, \phi - \frac{\pi}{2} + (\gamma + \alpha)).
\]

Proof. We use definition of interval \( L^+_\gamma \) in (2.22) to obtain

\[
\frac{d}{dt} \mathcal{M}^+_\gamma = \int_{L^+_\gamma \times \mathbb{R}} f d\theta d\nu
\]

\[
= \phi \int_{\mathbb{R}} \left[ f(\phi + \frac{\pi}{2} - (\gamma - \alpha)) - f(\phi - \frac{\pi}{2} + (\gamma + \alpha)) \right] d\nu + \int_{L^+_\gamma \times \mathbb{R}} \partial_\nu f d\theta d\nu.
\]

Now, we use

\[
\partial_\nu f = -\partial_\nu ((\nu - \kappa R \sin(\theta - \phi - \alpha)) f)
\]

to see

\[
\frac{d}{dt} \mathcal{M}^+_\gamma
\]

\[
= \phi \int_{\mathbb{R}} (B_+ - B_-) d\nu - \int_{L^+_\gamma \times \mathbb{R}} \partial_\nu ((\nu - \kappa R \sin(\theta - \phi - \alpha)) f) d\theta d\nu.
\]
Thus, we have

\[
\int_\mathbb{R} \left( \phi - \nu \right) \left( B_+ - B_- \right) d\nu + \kappa R \cos \gamma \int_\mathbb{R} \left( B_+ + B_- \right) d\nu. \]

**Lemma 2.6.** Let \( f = f(t, \nu, \theta) \) be a classical solution to (1.2). Then, the functional \( \tilde{\mathcal{M}}_\gamma(t) \) in (2.23) satisfies

\[
\frac{d}{dt} \tilde{\mathcal{M}}_\gamma(t) = \kappa R \int_{L_\gamma^\times \mathbb{R}} \cos(\theta - \phi - \alpha) f^2(t, \nu, \theta) d\theta d\nu
\]

\[
+ \int_\mathbb{R} \left( \phi - \nu \right) \left( \tilde{B}_+^2 - \tilde{B}_-^2 \right) d\nu - \kappa R \cos \gamma \int_\mathbb{R} \left( \tilde{B}_+^2 + \tilde{B}_-^2 \right) d\nu, \quad t \geq 0,
\]

where \( \tilde{B}_+ \) and \( \tilde{B}_- \) are functions as follows:

\[
\tilde{B}_+ := f(t, \nu, \phi + \frac{3\pi}{2} - (\gamma - \alpha)), \quad \tilde{B}_- := f(t, \nu, \phi + \frac{\pi}{2} + (\gamma + \alpha)).
\]

**Proof.** By straightforward calculation, one has

\[
\frac{d}{dt} \tilde{\mathcal{M}}_\gamma(t) = \phi \int_\mathbb{R} \left( \tilde{B}_+^2 - \tilde{B}_-^2 \right) d\nu + 2 \int_{L_\gamma^\times \mathbb{R}} f \partial_t f d\theta d\nu. \tag{2.24}
\]

Now we come to estimate the second R.H.S. term of (2.24). Again we use relation

\[
\partial_t f = -\partial_\theta ((\nu - \kappa R \sin(\theta - \phi - \alpha)) f)
\]

to have

\[
2 \int_{L_\gamma^\times \mathbb{R}} f \partial_t f d\theta d\nu = -2 \int_{L_\gamma^\times \mathbb{R}} f \partial_\theta ((\nu - \kappa R \sin(\theta - \phi - \alpha)) f) d\theta d\nu
\]

\[
= 2 \int_{L_\gamma^\times \mathbb{R}} \kappa R \cos(\theta - \phi - \alpha)) f^2 d\theta d\nu - 2 \int_{L_\gamma^\times \mathbb{R}} (\nu - \kappa R \sin(\theta - \phi - \alpha)) f \partial_\theta f d\theta d\nu
\]

\[
= \int_{L_\gamma^\times \mathbb{R}} \kappa R \cos(\theta - \phi - \alpha)) f^2 d\theta d\nu - \int_{L_\gamma^\times \mathbb{R}} \partial_\theta ((\nu - \kappa R \sin(\theta - \phi - \alpha)) f^2) d\theta d\nu.
\]

Note that

\[
\int_{L_\gamma^\times \mathbb{R}} \partial_\theta ((\nu - \kappa R \sin(\theta - \phi - \alpha)) f^2) d\theta d\nu
\]

\[
= \int_\mathbb{R} \left[ (\nu - \kappa R \sin(\frac{3\pi}{2} - \gamma)) \tilde{B}_+^2 - (\nu - \kappa R \sin(\frac{\pi}{2} + \gamma)) \tilde{B}_-^2 \right] d\nu
\]

\[
= \int_\mathbb{R} \nu(\tilde{B}_+^2 - \tilde{B}_-^2) d\nu + \kappa R \cos \gamma \int_\mathbb{R} (\tilde{B}_+^2 + \tilde{B}_-^2) d\nu.
\]

Thus, we have

\[
2 \int_{L_\gamma^\times \mathbb{R}} f \partial_t f d\theta d\nu = \kappa R \int_{L_\gamma^\times \mathbb{R}} \cos(\theta - \phi - \alpha)) f^2 d\theta d\nu
\]

\[
- \int_\mathbb{R} \nu(\tilde{B}_+^2 - \tilde{B}_-^2) d\nu - \kappa R \cos \gamma \int_\mathbb{R} (\tilde{B}_+^2 + \tilde{B}_-^2) d\nu. \tag{2.25}
\]

Now we substitute (2.25) into (2.24) to get

\[
\frac{d}{dt} \tilde{\mathcal{M}}_\gamma(t) = \kappa R \int_{L_\gamma^\times \mathbb{R}} \cos(\theta - \phi - \alpha)) f^2 d\theta d\nu
\]
Hence, we can get
\[ \int_R (\phi - \nu)(\bar{B}_1^2 - \bar{B}_2^2) d\nu - \kappa R \cos \gamma \int_R (\bar{B}_1^2 + \bar{B}_2^2) d\nu. \]

2.3. Relations among \( R \) and \( M_t^\pm \). In this subsection, we study relations among \( R(t) \) and masses on \( L_t^\pm \).

**Lemma 2.7.** Let \( f = f(t, \nu, \theta) \) be a classical solution to equation (1.2). Then we have
\[ (1 + \sin(\gamma - |\alpha|)) M_t^+ (t) - 1 < R(t) < (1 - \sin(\gamma + |\alpha|)) M_t^+ (t) + \sin(\gamma + |\alpha|), \quad t \geq 0. \]

**Proof.** By definition of \( L_t^+ \) in (2.22), we have
\[ \cos(\theta - \phi) \geq \sin(\gamma - |\alpha|) \text{ in } L_t^+, \quad \cos(\theta - \phi) \leq \sin(\gamma + |\alpha|) \text{ in } T \setminus L_t^+. \]

Hence, we use relation (2.2) to get
\[
R = \int_{L_t^+ \times R} \cos(\theta - \phi) f d\theta d\nu + \int_{T \setminus L_t^+ \times R} \cos(\theta - \phi) f d\theta d\nu
\geq \sin(\gamma - |\alpha|) \int_{L_t^+ \times R} f d\theta d\nu - \int_{T \setminus L_t^+ \times R} f d\theta d\nu
= \sin(\gamma - |\alpha|) M_t^+ - (1 - M_t^+) = (1 + \sin(\gamma - |\alpha|)) M_t^+ - 1,
\]
and
\[
R = \int_{L_t^+ \times R} \cos(\theta - \phi) f d\theta d\nu + \int_{T \setminus L_t^+ \times R} \cos(\theta - \phi) f d\theta d\nu
\leq \int_{L_t^+ \times R} f d\theta d\nu + \sin(\gamma + |\alpha|) \int_{T \setminus L_t^+ \times R} f d\theta d\nu
= M_t^+ + \sin(\gamma + |\alpha|)(1 - M_t^-) = (1 - \sin(\gamma + |\alpha|)) M_t^- + \sin(\gamma + |\alpha|). \]

**Lemma 2.8.** Let \( f = f(t, \nu, \theta) \) be a classical solution to equation (1.2). Then for \( t \geq 0, \) we have
\[
(i) M_t^+ (t) \geq \frac{R(t) + 2 \sin \gamma \cos \alpha M_t^- (t) - \sin(\gamma + |\alpha|)}{1 - \sin(\gamma + |\alpha|)}, \\
(ii) M_t^+ (t) \leq \frac{R(t) + (1 - \sin(\gamma + |\alpha|)) M_t^- (t) + \sin(\gamma + |\alpha|)}{2 \sin \gamma \cos \alpha}.
\]

**Proof.** We use relation (2.2) to get
\[
R = \int_{L_t^+ \times R} \cos(\theta - \phi) f d\theta d\nu + \int_{L_t^- \times R} \cos(\theta - \phi) f d\theta d\nu + \int_{T \setminus (L_t^+ \cup L_t^-) \times R} \cos(\theta - \phi) f d\theta d\nu.
\]

(i) (Lower bound of \( M_t^+ \)): By definition of \( L_t^+ (t) \) in (2.22), we obtain
\[ \cos(\theta - \phi) \leq - \sin(\gamma - |\alpha|) \text{ in } L_t^-, \quad \cos(\theta - \phi) \leq \sin(\gamma + |\alpha|) \text{ in } T \setminus (L_t^+ \cup L_t^-). \]

Hence, we can get
\[
R \leq M_t^+ - \sin(\gamma - |\alpha|) M_t^- + \sin(\gamma + |\alpha|)(1 - M_t^+ - M_t^-)
= (1 - \sin(\gamma + |\alpha|)) M_t^+ + \sin(\gamma + |\alpha|) - (\sin(\gamma - |\alpha|) + \sin(\gamma + |\alpha|)) M_t^-
= (1 - \sin(\gamma + |\alpha|)) M_t^+ + \sin(\gamma + |\alpha|) - 2 \sin \gamma \cos \alpha M_t^-.
\]

This yields
\[ M_t^+ \geq \frac{R + 2 \sin \gamma \cos \alpha M_t^- - \sin(\gamma + |\alpha|)}{1 - \sin(\gamma + |\alpha|)}. \]
(ii) (Upper bound of $\mathcal{M}_+^\gamma$): We use definition of $L^\pm_\gamma(t)$ in (2.22) again to obtain
\[
\cos(\theta - \phi) \geq \sin(\gamma - |\alpha|) \sin\theta, \quad \cos(\theta - \phi) \geq -\sin(\gamma + |\alpha|) \sin\theta, \quad t, \nu, \theta \geq 0.
\]
Hence, we have
\[
R \geq \sin(\gamma - |\alpha|) \mathcal{M}_+^\gamma - \mathcal{M}_-^\gamma - \sin(\gamma + |\alpha|) \left(1 - \mathcal{M}_+^\gamma - \mathcal{M}_-^\gamma\right)
\]
\[
= \left(\sin(\gamma - |\alpha|) + \sin(\gamma + |\alpha|)\right) \mathcal{M}_+^\gamma - \sin(\gamma + |\alpha|) - \left(1 - \sin(\gamma + |\alpha|)\right) \mathcal{M}_-^\gamma
\]
\[
= 2 \sin \gamma \cos \alpha \mathcal{M}_+^\gamma - \sin(\gamma + |\alpha|) - \left(1 - \sin(\gamma + |\alpha|)\right) \mathcal{M}_-^\gamma.
\]
This yields
\[
\mathcal{M}_+^\gamma \leq \frac{R + \left(1 - \sin(\gamma + |\alpha|)\right) \mathcal{M}_-^\gamma + \sin(\gamma + |\alpha|)}{2 \sin \gamma \cos \alpha}.
\]

3. Emergence of phase concentration. In this section, we show the emergence of phase concentration for the K-S equation in a regime that the coupling strength is sufficiently large, the size of frustration is sufficiently small, and the frequency density function $g(\nu)$ is supported in $[-\ell, \ell]$, i.e.,
\[
\text{supp } g(\nu) \subset [-\ell, \ell], \quad \text{where } \ell > 0 \text{ is a constant. (3.1)}
\]
Note that $\theta$ is the only dynamic variable for $f$, whereas $\nu$ variable acts like a parameter in the dynamics of $f$, so $\nu$-support for $f$ is propagated in time without changing. Using this observation, we combine the equation (1.2), Lemma 2.1, and relation (3.1) to see
\[
\text{supp}_\nu f^{in}(\nu, \theta) \subset [-\ell, \ell], \quad \text{and } \supp_\nu f(t, \nu, \theta) \subset [-\ell, \ell],
\]
Now, we set
\[
R_{\infty} := \begin{cases}
\frac{\cos \kappa^{-\frac{1}{2}}}{2} + \frac{1}{2} \left(\cos^2 \kappa^{-\frac{1}{2}} - \frac{8\ell \cos \kappa^{-\frac{1}{2}}}{\kappa \sin^2 \kappa^{-\frac{1}{2}}}\right), & \alpha = 0, \\
\frac{\cos(2|\alpha| + |\alpha|^{-\frac{1}{2}})}{2} + \frac{1}{2} \left(\cos^2(2|\alpha| + |\alpha|^{-\frac{1}{2}}) - \frac{8(\ell + \kappa \sin |\alpha|) \cos(|\alpha| + |\alpha|^{-\frac{1}{2}})}{\kappa \sin^2 |\alpha|^{-\frac{1}{2}}}\right), & \alpha \neq 0,
\end{cases}
\]
and denote $D_\infty(t)$ by a time-dependent interval centered at $\phi(t) + \alpha$ with constant width $\arccos(\sqrt{1 - \xi_\infty^2}) + \epsilon$, where $\epsilon$ is a sufficiently small constant and $\xi_\infty$ is defined as
\[
\xi_\infty := \frac{1 + R_{\infty} \ell}{R_{\infty}^2} \frac{\sin |\alpha| + (1 - R_{\infty}) \cos \alpha}{R_{\infty}}.
\]
Then, our main result on phase concentration can be stated as follows.

**Theorem 3.1.** Let $f(t, \nu, \theta)$ be a classical solution to (1.2) with initial data $f^{in}$ continuously differentiable and $R_0 := R(0) > 0$. Suppose the coupling strength $\kappa$ is sufficiently large, the size of frustration $|\alpha|$ is sufficiently small, and $\text{supp}_\nu g(\nu) \subset [-\ell, \ell]$, then the following assertions hold:
\[
\liminf_{t \to \infty} R(t) \geq R_{\infty},
\]
\[
\sup_{\theta \in \mathbb{T} \setminus D_\infty(t)} |f(t, \nu, \theta)| \leq C \|f^{in}\|_{L^\infty e^{-\kappa(\sqrt{1 - \xi_\infty^2}) t}}, \quad t \geq 0. \quad (3.2)
\]
Remark 3.1. By analogue argument on nonidentical equation, our result holds for identical equation, i.e., \( g(\nu) = \delta(\nu) \) case, and we do not need to assume the coupling strength \( \kappa \) is sufficiently large in this case. The exponential decay of density function \( f \) on \( T \setminus D_\infty(t) \) means that \( f \) will concentrate to interval \( D_\infty(t) \) centered at \( \phi(t) + \alpha \) and with width \( \arccos(\sqrt{1 - \frac{\kappa^2}{\kappa^2}} + \epsilon) \). This coincide with the result in [23], where the authors show that if the frustration is small and the order parameter has some lower bound, the particles will be phase-locked to some point in \( [\phi(t) + \alpha - \sigma, \phi(t) + \alpha + \sigma] \) except at most one particle which will be phase-locked to some point in \( [\phi(t) + \alpha + \pi - \sigma, \phi(t) + \alpha + \pi + \sigma] \) with \( \sigma \) a constant. Hence, we can expect under the assumption that frustration \( |\alpha| \) is sufficiently small (the coupling strength \( \kappa \) is sufficiently large and the frustration \( |\alpha| \) is sufficiently small for nonidentical equation, respectively), there exists some interval \( D_\infty(t) \) centered at \( \phi(t) + \alpha \) such that the density function \( f \) will concentrate to \( D_\infty(t) \) and decay to zero outside the concentration interval (on \( T \setminus D_\infty(t) \)).

In the sequel, we present a proof of Theorem 3.1 by verifying each estimate in Theorem 3.1 one by one.

3.1. Derivation of the first estimate in (3.2). In this part, we show that there exists \( R_\infty \) which tends to 1 as \( \alpha \) tends to zero and \( |\alpha| \) tends to infinity such that
\[
\liminf_{t \to \infty} R(t) \geq R_\infty.
\]
Since the derivation of (3.2), is rather lengthy, we split the verification of the first estimate into several steps as follows.

bullet Step A.1: We first derive a Riccati type differential inequality (see Lemma 3.1):
\[
\dot{R} \geq C \left( - R^2 + CR - C(1 + \mathcal{M}_\gamma^{-1}) \right),
\]
where \( C \) is a positive constant depending on \( \gamma, \alpha \).

bullet Step A.2: Let \( \kappa \) be large enough and \( |\alpha| \) be small enough, we show that there exists a positive constant \( \mu \) such that
\[
R(t) > R \quad \text{and} \quad \limsup_{t \to \infty} \dot{R}(t) \leq \mu, \quad \text{for} \ t \geq t_0,
\]
where \( R \in (0, R_0) \) is a constant, and \( t_0 \) is any instant satisfying \( R(t_0) \geq R_0 > 0 \). (see Proposition 3.1).

bullet Step A.3: We derive an exponential decay of \( \mathcal{M}_\gamma^- \) (see Lemma 3.2):
\[
\mathcal{M}_\gamma^-(t)e^{-\frac{\sqrt{2(\ell + \kappa \sin |\alpha|)}}{2}}.
\]

bullet Step A.4: Finally, we use (3.3), (3.4) and comparison principle for ODE to derive a desired positive lower bound for \( R \).

To perform aforementioned steps, we introduce several quantities as follows. For any \( \eta > 0, T_0 > 0 \) and \( \gamma_0 \in \left( \frac{\pi}{3}, \frac{\pi}{2} - |\alpha| \right) \), we define \( g_{\eta, T_0} : [T_0, \infty) \to \mathbb{R} \) as a unique solution to the following ODE:
\[
\begin{cases}
\dot{g}_{\eta, T_0}(t) = \frac{\kappa \cos^2(\gamma_0 + |\alpha|)}{2 \sin \gamma_0} \\
\times \left\{ - g_{\eta, T_0}^2(t) + \sin(\gamma_0 - |\alpha|)g_{\eta, T_0}(t) - \frac{2(\ell + \kappa \sin |\alpha|) \sin \gamma_0}{\kappa \cos^2(\gamma_0 + |\alpha|)} - \eta \right\}, t > T_0, \\
g_{\eta, T_0}(T_0) = R(T_0),
\end{cases}
\]
and set \( g_{\pm}(\gamma_0) \) as the solutions of the following quadratic equation:
\[
x^2 - \sin(\gamma_0 - |\alpha|)x + \frac{2(\ell + \kappa \sin |\alpha|) \sin \gamma_0}{\kappa \cos^2(\gamma_0 + |\alpha|)} + \eta = 0.
\]
By direct calculation, we can obtain explicit representations:
\[
g_-(\gamma_0) = \frac{\sin(\gamma_0 - |\alpha|)}{2} - \frac{1}{2} \sqrt{\frac{\sin^2(\gamma_0 - |\alpha|) - \frac{8(\ell + \kappa \sin |\alpha|) \sin \gamma_0}{\kappa \cos^2(\gamma_0 + |\alpha|)} - 4\eta}{}}
\]
\[
g_+(\gamma_0) = \frac{\sin(\gamma_0 - |\alpha|)}{2} + \frac{1}{2} \sqrt{\frac{\sin^2(\gamma_0 - |\alpha|) - \frac{8(\ell + \kappa \sin |\alpha|) \sin \gamma_0}{\kappa \cos^2(\gamma_0 + |\alpha|)} - 4\eta}{}}.
\]
Note that if \( g_{\eta,T_0}(T_0) > g_- \), then one has
\[
\lim_{t \to \infty} g_{\eta,T_0}(t) = g_+.
\]
Next, we will present several useful lemmas as follows.

**Lemma 3.1.** Let \( f = f(t,\nu,\theta) \) be a classical solution to (1.2). Then, the order parameter \( R \) satisfies
\[
\dot{R} \geq \frac{\kappa \cos^2(\gamma + |\alpha|)}{2 \sin \gamma}
\times \left\{ - R^2 + \sin(\gamma - |\alpha|) R - \frac{2(\ell + \kappa \sin |\alpha|) \sin \gamma}{\kappa \cos^2(\gamma + |\alpha|)} - (1 + \sin(\gamma - |\alpha|)) \mathcal{M}_\gamma \right\}, \quad t > 0.
\]

**Proof.** Recall that Lemma 2.2 yields
\[
\dot{R} = - \int_{T \times \mathbb{R}} \nu \sin(\theta - \phi) f d\theta d\nu + \kappa R \cos \alpha \int_{T \times \mathbb{R}} \sin^2(\theta - \phi) f d\theta d\nu
- \kappa R \sin \alpha \int_{T \times \mathbb{R}} \sin(\theta - \phi) \cos(\theta - \phi) f d\theta d\nu.
\]
We use estimates
\[
\left| \int_{T \times \mathbb{R}} \nu \sin(\theta - \phi) f d\theta d\nu \right| \leq \ell, \quad \sin(\theta - \phi) \geq \cos(\gamma + |\alpha|), \quad \theta \in T \setminus (L^+_\gamma \cup L^-_\gamma),
\]
\[
\left| \kappa R \sin \alpha \int_{T \times \mathbb{R}} \sin(\theta - \phi) \cos(\theta - \phi) f d\theta d\nu \right| \leq \kappa \sin |\alpha|
\]
to obtain
\[
\dot{R} \geq - \ell - \kappa \sin |\alpha| + \kappa R \cos \alpha \int_{T \setminus (L^+_\gamma \cup L^-_\gamma)} \sin^2(\theta - \phi) f(t,\nu,\theta) d\theta d\nu
- \kappa \sin |\alpha| \left( 1 + \mathcal{M}_\gamma^+ - \mathcal{M}_\gamma^- \right).
\]
Then we Lemma 2.8 \((ii)\) to get
\[
\dot{R} \geq - \ell - \kappa \sin |\alpha| + \kappa R \cos \alpha \cos^2(\gamma + |\alpha|)
\times \left[ 1 - \frac{R + (1 - \sin(\gamma + |\alpha|)) \mathcal{M}_\gamma^- + \sin(\gamma + |\alpha|)}{2 \sin \gamma \cos \alpha} - \mathcal{M}_\gamma^- \right].
\]
Note that
\[
1 - \frac{R + (1 - \sin(\gamma + |\alpha|)) \mathcal{M}_\gamma^- + \sin(\gamma + |\alpha|)}{2 \sin \gamma \cos \alpha} - \mathcal{M}_\gamma^- = \frac{1}{2 \sin \gamma \cos \alpha} \left\{ - R + \sin(\gamma - |\alpha|) - (1 + \sin(\gamma - |\alpha|)) \mathcal{M}_\gamma^- \right\}.
\]
We substitute (3.7) into (3.6) to have
\[
\dot{R} \geq -\ell - \kappa \sin |\alpha| + \frac{\kappa R \cos^2 (\gamma + |\alpha|)}{2 \sin \gamma} \{-R + \sin (\gamma - |\alpha|) - [1 + \sin (\gamma - |\alpha|)] M_{\gamma}^r\}
\]
\[
\geq \frac{\kappa \cos^2 (\gamma + |\alpha|)}{2 \sin \gamma} \times \{-R^2 + \sin (\gamma - |\alpha|) R - \frac{2(\ell + \kappa \sin |\alpha|) \sin \gamma}{\kappa \cos^2 (\gamma + |\alpha|)} - (1 + \sin (\gamma - |\alpha|)) M_{\gamma}^r\}. \tag*{\Box}
\]

**Proposition 3.1.** For any constants \( R \in (0, R_0), \gamma \in [\frac{\pi}{3}, \frac{\pi}{2} - |\alpha|] \), and small \( \mu > 0 \), let \( \kappa \) be large enough and \( |\alpha| \) be small enough such that
\[
R_0 - E_1 > R, \quad \frac{R \mu}{9 \cos \alpha} > E_1 + E_2, \quad \kappa R \cos \gamma > \ell + \frac{\ell}{R} + \kappa \sin |\alpha| + \left(\frac{\kappa \cos \alpha}{R}\right)^\frac{1}{2} \left(\ell + \kappa \sin |\alpha| + \mu\right)^{\frac{1}{2}}.
\]
where \( E_1, E_2 \) are positive constants defined as
\[
E_1 := 1 - \sin (\gamma - |\alpha|) + \frac{2 \sin \gamma}{\cos^2 (\gamma + |\alpha|)} \left(\frac{\ell}{\kappa R} + \frac{1 - R}{R} \sin |\alpha|\right),
\]
\[
E_2 := \frac{1}{9 \kappa \cos \alpha} \left\{ \frac{\ell^2}{2 \kappa R \cos \alpha} - \frac{2 \ell^2 + \kappa^2 \cos \alpha \sin |\alpha|}{\kappa \cos \alpha} \right\}
\]
\[
+ \frac{\ell^2}{3 \kappa R \cos \alpha} + \frac{5 \ell^2 + 5 \kappa^2 \cos \alpha \sin |\alpha|}{3 \kappa \cos \alpha} \left(1 + \ln \left(\frac{\ell^2}{2 \kappa R \cos \alpha} + \frac{2 \ell^2 + 5 \kappa^2 \cos \alpha \sin |\alpha|}{3 \kappa \cos \alpha}\right)\right). \tag*{3.8}
\]
Let \( t_0 \geq 0 \) be an instant such that \( R(t_0) \geq R_0 = R(0) > 0 \). Then the following assertions hold:
\[
R(t) > R, \quad \text{and} \quad \limsup_{t \to \infty} \dot{R}(t) \leq \mu, \quad \text{for} \ t \geq t_0.
\]

**Proof.** Since the proof is very lengthy, we postpone its proof in Section 4. \( \Box \)

**Lemma 3.2.** Under the assumption of Proposition 3.1, the following assertions hold:
\[
(i) \tilde{M}_{\gamma}^r(t) \leq \tilde{M}_{\gamma}^r(t_1) e^{-\kappa R(t-t_1)} \sin \gamma, \quad t \geq t_0,
\]
\[
(ii) M_{\gamma}^r(t) \leq C(t) \left(\tilde{M}_{\gamma}^r(t_1)\right)^\frac{3}{2} e^{-\frac{\kappa R(t-t_1)}{2 \sin \gamma}} \sin \gamma, \quad t \geq t_0.
\]

**Proof.** (i) We use definition of \( L_{\gamma}^- \) in (2.22) to get
\[
\cos (\theta - \phi - \alpha) \leq -\sin \gamma, \quad \text{in} \ L_{\gamma}^- \tag*{3.8}
\]
On the other hand, by Lemma 2.6, assumption \( R(t) > R \) and relation (3.8), we have
\[
\frac{d}{dt} \tilde{M}_{\gamma}^r = \kappa R \int_{L_{\gamma}^-} \cos (\theta - \phi - \alpha) f^2 d\theta d\nu
\]
\[
+ \int_{\mathbb{R}} (\phi - \nu)(\tilde{B}_+^2 - \tilde{B}_-^2) d\nu - \kappa R \cos \gamma \int_{\mathbb{R}} (\tilde{B}_+^2 + \tilde{B}_-^2) d\nu.
\]
\[
\leq -\kappa R \sin \gamma \int_{L_{\gamma}^-} f^2 d\theta d\nu - \kappa R (\cos \gamma - |\phi| - \ell)(\tilde{B}_+^2 + \tilde{B}_-^2) d\nu. \tag*{3.9}
\]
We claim
\[ \kappa R \cos \gamma - |\dot{f}| - \ell \geq 0, \quad t \geq T_1. \] (3.10)

**Proof of claim (3.10):** Recall Lemma 2.2 (ii) implies
\[ \dot{\phi} = \frac{1}{R} \int_{\mathbb{T} \times \mathbb{R}} \nu \cos(\theta - \phi) f(t, \nu, \theta) d\theta d\nu - \kappa \cos \alpha \int_{\mathbb{T} \times \mathbb{R}} \sin(\theta - \phi) \cos(\theta - \phi) f d\theta d\nu 
+ \kappa \sin \alpha \int_{\mathbb{T} \times \mathbb{R}} \cos^2(\theta - \phi) f d\theta d\nu. \] (3.11)

By Lemma 2.1 and Proposition 3.1, we have
\[ \left| \frac{1}{R} \int_{\mathbb{T} \times \mathbb{R}} \nu \cos(\theta - \phi) f(t, \nu, \theta) d\theta d\nu \right| \leq \frac{\ell}{R}, \quad t \geq T_1, \]
\[ \left| \kappa \sin \alpha \int_{\mathbb{T} \times \mathbb{R}} \cos^2(\theta - \phi) f d\theta d\nu \right| \leq \kappa |\sin \alpha|, \quad t \geq T_1. \] (3.12)

Hence, the remaining thing is to estimate the second R.H.S term of (3.11). We use the Hölder inequality to obtain
\[ \left| \kappa \cos \alpha \int_{\mathbb{T} \times \mathbb{R}} \sin(\theta - \phi) \cos(\theta - \phi) f d\theta d\nu \right| \leq \left( \kappa \cos \alpha \right)^{\frac{1}{2}} \left( \kappa R \cos \alpha \int_{\mathbb{T} \times \mathbb{R}} \sin^2(\theta - \phi) f d\theta d\nu \right)^{\frac{1}{2}}. \] (3.13)

Recall that Lemma 2.2 (i) yields
\[ \dot{R} = - \int_{\mathbb{T} \times \mathbb{R}} \nu \sin(\theta - \phi) f d\theta d\nu + \kappa R \cos \alpha \int_{\mathbb{T} \times \mathbb{R}} \sin^2(\theta - \phi) f d\theta d\nu 
- \kappa R \sin \alpha \int_{\mathbb{T} \times \mathbb{R}} \sin(\theta - \phi) \cos(\theta - \phi) f d\theta d\nu 
> - \ell - \kappa |\sin \alpha| + \kappa R \cos \alpha \int_{\mathbb{T} \times \mathbb{R}} \sin^2(\theta - \phi) f(\theta, \nu) d\theta d\nu. \]

Thus, we use Proposition 3.1 to have, for \( t \geq T_1 \),
\[ \kappa R \cos \alpha \int_{\mathbb{T} \times \mathbb{R}} \sin^2(\theta - \phi) f(\theta, \nu) d\theta d\nu < \ell + \kappa |\sin \alpha| + \dot{R} \leq \ell + \kappa |\sin \alpha| + \mu. \] (3.14)

Now we substitute (3.14) into (3.13) to get
\[ \left| \kappa \cos \alpha \int_{\mathbb{T} \times \mathbb{R}} \sin(\theta - \phi) \cos(\theta - \phi) f d\theta d\nu \right| \leq \left( \frac{\kappa \cos \alpha}{R} \right)^{\frac{1}{2}} \left( \ell + \kappa |\sin \alpha| + \mu \right)^{\frac{1}{2}}. \] (3.15)

Now we combine relation (3.11) and estimates (3.12) and (3.15) to obtain
\[ |\dot{\phi}| < \frac{\ell}{R} + \kappa |\sin \alpha| + \left( \frac{\kappa \cos \alpha}{R} \right)^{\frac{1}{2}} \left( \ell + \kappa |\sin \alpha| + \mu \right)^{\frac{1}{2}}. \] (3.16)

Then (3.10) follows from estimate (3.16) and assumption \( \kappa R \cos \gamma > \ell + \frac{\ell}{R} + \kappa |\sin \alpha| + \left( \frac{\kappa \cos \alpha}{R} \right)^{\frac{1}{2}} \left( \ell + \kappa |\sin \alpha| + \mu \right)^{\frac{1}{2}} \). Now we use relation (3.9) and (3.10) to deduce
\[ \frac{d}{dt} \overline{\mathcal{M}}_\gamma^{-} \leq -\kappa R \overline{\mathcal{M}}_\gamma^{-} \sin \gamma, \quad t \geq T_1. \] (3.17)

We integrate (3.17) with respect to \( t \) to get
\[ \overline{\mathcal{M}}_\gamma^{-}(t) \leq \overline{\mathcal{M}}_\gamma^{-}(T_1) e^{-\kappa R(t-T_1) \sin \gamma}, \quad t \geq T_1. \]
(ii) We use estimate (i) and the Hölder inequality to obtain
\[
\mathcal{M}_\gamma(t) \leq \left( \int_{L_\gamma \times \mathbb{R}} 1 d\theta d\nu \right)^{\frac{1}{2}} \left( \int_{L_\gamma \times \mathbb{R}} f^2 d\theta d\nu \right)^{\frac{1}{2}} \\
\leq C(t) \left( \mathcal{M}_\gamma(T_1) \right)^{\frac{1}{2}} e^{-\frac{K(t-T_1)\sin\gamma}{2}}, \quad t \geq T_1.
\]

Now, we combine all the estimates in Lemma 3.1 - Lemma 3.2 to derive a positive lower bound for $R$.

**Proposition 3.2.** Under the assumption of Theorem 3.1, there exists constant $R_\infty$ such that
\[
\liminf_{t \to \infty} R(t) \geq R_\infty,
\]
where $R_\infty = R_\infty(\kappa, \alpha)$ is defined as follows:
\[
R_\infty := \left\{ \begin{array}{ll}
\frac{\cos \kappa^{-\frac{1}{2}}}{2} + \frac{1}{\lambda} \left( \cos^2 \kappa^{-\frac{1}{2}} - \frac{8\ell \cos \kappa^{-\frac{1}{2}}}{\kappa \sin^2 \kappa^{-\frac{1}{2}}} \right)^2, & \kappa = 0, \\
\frac{\cos(2|\alpha| + |\alpha|^\frac{1}{2})}{\kappa \sin^2 |\alpha|^\frac{1}{2}} + \frac{1}{2} \left( \cos^2(2|\alpha| + |\alpha|^\frac{1}{2}) - \frac{8(\ell + \kappa \sin |\alpha|) \cos(|\alpha| + |\alpha|^\frac{1}{2})}{\kappa \sin^2 |\alpha|^\frac{1}{2}} \right)^2, & \kappa \neq 0.
\end{array} \right.
\]

**Proof.** Suppose $\kappa$ is sufficiently large and $|\alpha|$ is sufficiently small, let $\frac{R}{2} > 0$ and $\gamma$ be near $\frac{\pi}{2} - |\alpha|$, then the following assertions hold:
\[
R_0 - E_1 > R, \quad \frac{R\mu}{9 \cos \alpha} > E_1 + E_2, \\
\kappa R \cos \gamma > \ell + \frac{\kappa \sin |\alpha| + \left( \frac{\kappa \cos \alpha}{R} \right)^2 (\ell + \kappa \sin |\alpha| + \mu)^\frac{1}{2}.\]

By Lemma 3.2 (ii), there exists $T > 0$ such that, for any $\eta > 0$,
\[
\mathcal{M}_\gamma(t) \leq \frac{\eta}{2}, \quad t \in [T, \infty).
\]

Thus, we use Lemma 3.1 to get that, for $t \in [T, \infty)$,
\[
\dot{R} \geq \frac{\kappa \cos^2(\gamma + |\alpha|)}{2 \sin \gamma} \\
\times \left\{ -R^2 + \sin(\gamma - |\alpha|) - \frac{2(\ell + \kappa \sin |\alpha|) \sin \gamma}{\kappa \cos^2(\gamma + |\alpha|)} - (1 + \sin(\gamma - |\alpha|)) \mathcal{M}_\gamma \right\} \\
\geq \frac{\kappa \cos^2(\gamma + |\alpha|)}{2 \sin \gamma} \left\{ -R^2 + \sin(\gamma - |\alpha|) - \frac{2(\ell + \kappa \sin |\alpha|) \sin \gamma}{\kappa \cos^2(\gamma + |\alpha|)} - \eta \right\}.
\]

Now we take $\gamma_0 = \frac{\pi}{2} - |\alpha| - |\alpha|^{\frac{1}{2}}$ for $\alpha \neq 0$, $\gamma_0 = \frac{\pi}{2} - \kappa^{-\frac{1}{2}}$ for $\alpha = 0$ and $T_0 = T$ in (3.5). Since $\kappa$ is sufficiently large and $|\alpha|$ is sufficiently small, it holds that
\[
R(t) > \frac{R}{\gamma} > g_-(\gamma_0).
\]

Thus, we have
\[
\liminf_{t \to \infty} R(t) \geq g_+(\gamma_0)
\]
Derivation of the second estimate in (3.2). In this part, we show the mass on \( T \setminus D \) tends to zero exponentially. For \( R \in (0, 1] \), we set
\[
\xi(R) := \frac{1 + R \ell \frac{\sin |\alpha|}{R} + \frac{\sin |\alpha| + (1 - R) \cos \alpha}{\ell}}{R^2 \kappa}.
\]
Then for \( \kappa \gg 1 \) and \( |\alpha| \ll 1 \), Proposition 3.2 yields that \( R_\infty \approx 1 \). Hence, we can get \( \xi_\infty := \xi(R_\infty) < 1 \). In particular, there exists \( \epsilon > 0 \) such that for any \( R_* \in |R_\infty - \epsilon, R_\infty| \), \( \xi := \xi(R_*) < 1 \). Now, we fix \( R_* \in |R_\infty - \epsilon, R_\infty| \). Then by Proposition 3.2, there exists time \( T_* \geq 0 \) such that
\[
R(t) \geq R_*, \quad \text{for all } t \geq T_*.
\]

Now for any \( 0 < \delta < \sqrt{1 - \xi_*^2} \), set
\[
D(t) := \left\{ \theta \in \mathbb{T} \mid \cos (\theta - \phi(t) - \alpha) > \sqrt{1 - \xi_*^2}, \ t \geq T_* \right\},
\]
\[
\overline{D}(t) := \left\{ \theta \in \mathbb{T} \mid \cos (\theta - \phi(t) - \alpha) \leq -\delta, \ t \geq T_* \right\}.
\]

We consider two equations as follows:
\[
\begin{cases}
\hat{\phi} = \nu - \kappa R \sin (\Theta - \phi - \alpha), \quad t \in [0, t_*), \\
\Theta|_{t=t_*} = \theta,
\end{cases}
\]
(3.18)

and
\[
\begin{cases}
\dot{P} = \kappa R_* \sqrt{1 - P^2} (\sqrt{1 - P^2} - \xi_*), \quad t \in [0, t_*), \\
P(t_*) = p,
\end{cases}
\]
(3.19)

where \( \theta \in \mathbb{T} \) and \( p \in (-\sqrt{1 - \xi_*^2}, \sqrt{1 - \xi_*^2}) \).

For the second estimate in Theorem 3.1, we take the following two steps:

\textbullet \textbf{Step B.1:} For any \( \nu \in \mathbb{R} \), we show that the mass on \( \overline{D}(t) \) decays to zero exponentially fast (see Proposition 3.3):
\[
\sup_{\theta \in \overline{D}(t)} |f(t, \nu, \theta)| \leq \sup_{\theta_0 \in \mathbb{T}} |f(T_*, \nu, \theta_0)| e^{-\kappa \delta R_* (t - T_*)}, \quad t \geq T_*.
\]

\textbullet \textbf{Step B.2:} For any \( \nu \in \mathbb{R} \), we derive
\[
\sup_{\theta \in \mathbb{T}, \overline{D}(t)} |f(t, \nu, \theta)| \leq C \sup_{\theta_0 \in \mathbb{T}} |f(T_*, \nu, \theta_0)| e^{-\kappa \delta R_* t}, \quad \text{for } t \geq T_* + D_*.
\]
(see Proposition 3.4).

Next, we perform the above outlined analysis one by one.

\textbullet \textbf{(Exponential decay of mass on} \( \overline{D}(t) \): In the sequel, we perform analysis for the mass on \( \overline{D}(t) \). For this, we begin with the estimate on \( \cos (\theta - \phi(t_* + \alpha)) \).

\textbf{Lemma 3.3.} Suppose the assumption of Theorem 3.1 holds, and let \( \Theta \) and \( P \) be solutions to (3.18) and (3.19), respectively. Then, the following assertions hold.

1. If we take \( t_* > T_* \), \( \theta \in \mathbb{T} \) and \( p \in (-\sqrt{1 - \xi_*^2}, \sqrt{1 - \xi_*^2}) \) such that
\[
\cos (\theta - \phi(t_* + \alpha)) \leq p,
\]
then one has
\[
\cos \left( \Theta(t) - \phi(t) - \alpha \right) \leq P(t), \quad \text{for all } t \in [T_*, t_*).
\]
(ii) If we take \( t_* > T_* \), \( \theta \in \mathbb{T} \) and \( \delta \in (0, \sqrt{1 - \xi_*^2}) \) such that
\[
\cos \left( \theta - \phi(t_*) - \alpha \right) \leq -\delta,
\]
then, we have
\[
\cos \left( \Theta(t) - \phi(t) - \alpha \right) \leq -\delta, \quad \text{for all } t \in [T_*, t_*).
\]
\text{Proof.} (i) We set
\[
Q(t) := \cos \left( \Theta(t) - \phi(t) - \alpha \right).
\]
Note that
\[
\dot{Q} = -\sin(\Theta - \phi - \alpha) \cdot (\dot{\Theta} - \dot{\phi})
\]
\[
= -\sin (\Theta - \phi - \alpha) \left( \nu - \kappa R \sin(\Theta - \phi - \alpha) - \dot{\phi} \right)
\]
\[
= \kappa R \sin^2(\Theta - \phi - \alpha) + (\dot{\phi} - \nu) \sin(\Theta - \phi - \alpha).
\]
Now we use Lemma 2.4 (ii) and \( R(t) \geq R_* \) for any \( t \geq T_* \) to get
\[
| (\dot{\phi} - \nu) \sin(\Theta - \phi - \alpha) | \leq (1 + \frac{1}{R_*}) \ell + \kappa (| \sin \alpha | + \cos \alpha (1 - R_*)).
\]
Hence, we can get
\[
\dot{Q} \geq \kappa R_* \sin^2(\Theta - \phi - \alpha) - (1 + \frac{1}{R_*}) \ell + \kappa (| \sin \alpha | + \cos \alpha (1 - R_*))
\]
\[
\geq \kappa R_* \sqrt{1 - \cos^2(\Theta - \phi - \alpha)}
\]
\[
\times \left\{ \sqrt{1 - \cos^2(\Theta - \phi - \alpha)} - \frac{1}{\kappa R_*} \left( (1 + \frac{1}{R_*}) \ell + \kappa (| \sin \alpha | + \cos \alpha (1 - R_*)) \right) \right\}
\]
\[
= \kappa R_* \sqrt{1 - \cos^2(\Theta - \phi - \alpha)}
\]
\[
\times \left\{ \sqrt{1 - \cos^2(\Theta - \phi - \alpha)} - \left( \frac{1 + R_* \ell}{R_*^2} \frac{\kappa \sin \alpha}{\kappa} + \cos \alpha (1 - R_*)) \right) \right\}
\]
\[
= \kappa R_* \sqrt{1 - Q^2(\sqrt{1 - \xi_*^2})}, \quad t \geq T_*.
\]
Hence, together with the assumption \( \cos \left( \theta - \phi(t_*) - \alpha \right) \leq p \), we get
\[
\cos \left( \Theta(t) - \phi(t) - \alpha \right) = Q(t) \leq P(t), \quad \text{for all } t \in [T_*, t_*).
\]
(ii) It follows from the standard ODE that the solution \( P(t) \) of (3.19) satisfies
\[
P(t) \in (-\sqrt{1 - \xi_*^2}; \sqrt{1 - \xi_*^2}), \quad \text{and} \quad \dot{P}(t) > 0, \quad \text{for } t \in [0, t_*).
\]
Now set \( p = -\delta \), we use (3.20) to obtain
\[
P(t) \leq P(t_*) = p = -\delta, \quad \text{for all } t \in [0, t_*).
\]
Recall that assertion (i) yields
\[
\cos \left( \Theta(t) - \phi(t) - \alpha \right) \leq P(t), \quad \text{for all } t \in [T_*, t_*),
\]
Thus, we combine relations (3.21) and (3.22) to get
\[
\cos \left( \Theta(t) - \phi(t) - \alpha \right) \leq -\delta, \quad \text{for all } t \in [T_*, t_*).
\]
Now, we show that the mass on the region \( \overline{D}(t) \) decays exponentially fast.
Proposition 3.3. Suppose the assumption of Theorem 3.1 holds, and let $\Theta$ be a solution to (3.18). Then for any $\nu \in \mathbb{R}$, one has
\[
\sup_{\theta \in \mathcal{D}(t)} |f(t, \nu, \theta)| \leq \sup_{\theta_0 \in \mathcal{T}} |f(T_*, \nu, \theta_0)|e^{-\kappa \delta R_*(t-T_*)}, \quad t \geq T_*.
\]

Proof. Take any $t > T_*$ and $\theta \in \mathcal{D}(t)$, we use Lemma 3.2 (ii) to obtain
\[
\cos(\Theta(s) - \phi(s) - \alpha) \leq -\delta, \quad \text{for all } s \in [T_*, t).
\]
Hence, we can obtain
\[
\frac{d}{ds} f(s, \nu, \Theta(s)) = \kappa R \cos(\Theta(s) - \phi(s) - \alpha) f(s, \nu, \Theta(s)) \leq -\kappa \delta R_* f(s, \nu, \Theta(s)), \quad s \in [T_*, t).
\]
(3.23)

Now we use relation (3.23) to get
\[
f(t, \nu, \theta) = f(t, \nu, \Theta(t)) \leq f(T_*, \nu, \theta_0)e^{-\kappa \delta R_*(t-T_*)}.
\]
Thus, we have
\[
\sup_{\theta \in \mathcal{D}(t)} |f(t, \nu, \theta)| \leq \sup_{\theta_0 \in \mathcal{T}} |f(T_*, \nu, \theta_0)|e^{-\kappa \delta R_*(t-T_*)}, \quad \text{for all } t > T_*.
\]

Proof. Take any $t > T_*$ and $\theta \in \mathcal{D}(t)$, we use Lemma 3.2 (ii) to obtain
\[
\cos(\Theta(s) - \phi(s) - \alpha) \leq -\delta, \quad \text{for all } s \in [T_*, t).
\]
Hence, we can obtain
\[
\frac{d}{ds} f(s, \nu, \Theta(s)) = \kappa R \cos(\Theta(s) - \phi(s) - \alpha) f(s, \nu, \Theta(s)) \leq -\kappa \delta R_* f(s, \nu, \Theta(s)), \quad s \in [T_*, t).
\]
(3.23)

Now we use relation (3.23) to get
\[
f(t, \nu, \theta) = f(t, \nu, \Theta(t)) \leq f(T_*, \nu, \theta_0)e^{-\kappa \delta R_*(t-T_*)}.
\]
Thus, we have
\[
\sup_{\theta \in \mathcal{D}(t)} |f(t, \nu, \theta)| \leq \sup_{\theta_0 \in \mathcal{T}} |f(T_*, \nu, \theta_0)|e^{-\kappa \delta R_*(t-T_*)}, \quad \text{for all } t > T_*.
\]

**Proof.** Take any $t > T_*$ and $\theta \in \mathcal{D}(t)$, we use Lemma 3.2 (ii) to obtain
\[
\cos(\Theta(s) - \phi(s) - \alpha) \leq -\delta, \quad \text{for all } s \in [T_*, t).
\]
Hence, we can obtain
\[
\frac{d}{ds} f(s, \nu, \Theta(s)) = \kappa R \cos(\Theta(s) - \phi(s) - \alpha) f(s, \nu, \Theta(s)) \leq -\kappa \delta R_* f(s, \nu, \Theta(s)), \quad s \in [T_*, t).
\]
(3.23)

Now we use relation (3.23) to get
\[
f(t, \nu, \theta) = f(t, \nu, \Theta(t)) \leq f(T_*, \nu, \theta_0)e^{-\kappa \delta R_*(t-T_*)}.
\]
Thus, we have
\[
\sup_{\theta \in \mathcal{D}(t)} |f(t, \nu, \theta)| \leq \sup_{\theta_0 \in \mathcal{T}} |f(T_*, \nu, \theta_0)|e^{-\kappa \delta R_*(t-T_*)}, \quad \text{for all } t > T_*.
\]

Lemma 3.4. Suppose the assumption of Theorem 3.1 holds, and let $\Theta$ be a solution to equation (3.18). If we take any $\theta \in \mathcal{T}$, $\delta \in (0, \sqrt{1-\xi_*^2})$ and
\[
t \geq T_* + D_* := T_* + \frac{2\delta}{\kappa R_\star \sqrt{1-\delta^2}} \frac{2\delta}{\sqrt{1-\delta^2}}
\]
such that
\[
\cos(\theta - \phi(t) - \alpha) \leq \delta,
\]
then one has
\[
\cos(\Theta(t - D_*) - \phi(t - D_*) - \alpha) \leq -\delta.
\]

Proof. We first claim that
\[
\exists \ d < D_* := \frac{2\delta}{\kappa R_\star \sqrt{1-\delta^2}} \frac{2\delta}{\sqrt{1-\delta^2}} \text{ such that } P(t - d) = -\delta. \quad (3.24)
\]

**Proof of claim (3.24).** Recall that the solution $P(t)$ of (3.19) satisfies
\[
P(t) \in (-\sqrt{1-\xi_*^2}, \sqrt{1-\xi_*^2}), \quad \text{and } \dot{P}(t) > 0, \quad \text{for } t \in [0, t_*).
\]
(3.25)

Hence, for any $\delta \in (0, \sqrt{1-\xi_*^2})$, there exists $d$ such that
\[
P(t - d) = -\delta. \quad (3.26)
\]
Furthermore, we use equation (3.19) to get
\[
p(t) - p(t - d) = \int_{t-d}^t \dot{P}(s)ds \geq \kappa R_\star \sqrt{1-\delta^2} \frac{2\delta}{\sqrt{1-\delta^2}} \frac{2\delta}{\sqrt{1-\delta^2}}.
\]
Thus, the constant $d$ has a upper bound as follows
\[
d \leq \frac{p(t) - p(t - d)}{D_*} = \frac{2\delta}{D_*},
\]
which yields claim (3.25). Now let \( p = \delta \), then by assumption \( \cos(\theta - \phi(t) - \alpha) < \delta = q \) and (i) of Lemma 3.3, we get

\[
\cos(\Theta(t - D_\ast) - \phi(t - D_\ast) - \alpha) \leq P(t - D_\ast).
\]

Furthermore, we use relation (3.25) to obtain

\[
P(t - D_\ast) \leq P(t - d) = -\delta.
\]

Hence, we have

\[
\cos(\Theta(t - D_\ast) - \phi(t - D_\ast) - \alpha) \leq -\delta.
\]

Proposition 3.4. Suppose the assumption of Theorem 3.1 holds, and let \( \Theta \) be a solution to equation (3.18). Then for any \( \nu \in \mathbb{R} \), we have

\[
\sup_{\theta \in T \backslash D(t)} |f(t, \nu, \theta)| \leq C \sup_{\theta_0 \in T} |f(T_\ast, \nu, \theta_0)| e^{-\kappa \delta R_\ast t}, \quad \text{for } t \geq T_\ast + D_\ast.
\]

Proof. Take any \( t > T_\ast + D_\ast \), and \( \theta \in T \backslash D(t) \), we use Lemma 3.4 to get

\[
\cos(\Theta(t - D_\ast) - \phi(t - D_\ast) - \alpha) \leq -\delta.
\]

Analogue analysis as in Proposition 3.3 yields

\[
f(t - D_\ast, \nu, \Theta(t - D_\ast)) \leq f(T_\ast, \nu, \theta_0) e^{-\kappa \delta R_\ast (t - T_\ast - D_\ast)}. \tag{3.27}
\]

For \( s \in [T_\ast, t) \), we have

\[
\frac{d}{ds} f(s, \nu, \Theta(s)) = \kappa R \cos(\Theta(s) - \phi(s) - \alpha) f(s, \nu, \Theta(s)) \leq \kappa f(s, \nu, \Theta(s)).
\]

Thus, we can get

\[
f(t, \nu, \theta) \leq e^{\kappa D_\ast} f(t - D_\ast, \nu, \Theta(t - D_\ast)), \quad \text{for } t \geq T_\ast + D_\ast. \tag{3.28}
\]

Now we combine (3.27) and (3.28) to obtain

\[
f(t, \nu, \theta) \leq f(T_\ast, \nu, \theta_0) e^{-\kappa \delta R_\ast (t - T_\ast - D_\ast) + \kappa D_\ast}, \quad \text{for } t \geq T_\ast + D_\ast.
\]

That is

\[
\sup_{\theta \in T \backslash D(t)} |f(t, \nu, \theta)| \leq \sup_{\theta_0 \in T} |f(T_\ast, \nu, \theta_0)| e^{-\kappa \delta R_\ast (t - T_\ast - D_\ast) + \kappa D_\ast}
\]

\[
\leq C \sup_{\theta_0 \in T} |f(T_\ast, \nu, \theta_0)| e^{-\kappa \delta R_\ast t}, \quad \text{for } t \geq T_\ast + D_\ast. \quad \square
\]

Remark 3.2. The second estimate in (3.2) can be get by the fact \( R_\ast \geq R_\infty - \epsilon \) and taking \( \delta = \sqrt{1 - \xi^2} - \epsilon \).

4. Proof of Proposition 3.1. In this section, we present a proof of Proposition 3.1. For this, we first show the existence of a positive lower bound of \( R(t) \).
4.1. Lower bounds of $R(t)$. Note that, if $\dot{R}(t) > 0$ for any $t \geq 0$, we can get the lower bound of $R(t)$ by only assuming $R_0 > 0$. However, the condition $\dot{R}(t) > 0$ is not always true. Thus, to prove existence of a positive lower bound of $R(t)$, we need to consider either the case $\dot{R}(t_0) \leq 0$, or the case $\dot{R}(t_0) \leq \mu$ with small enough $\mu$.

**Lemma 4.1.** For any constants $R > 0$ and $\gamma \in (\frac{\pi}{2}, \frac{\pi}{2} - |\alpha|)$, let $t_0 \geq 0$ be an instant such that

$$R(t_0) > R, \quad \text{and} \quad \dot{R}(t_0) \leq 0.$$ 

Then, $R(t_0)$ can be controlled by $M_\gamma^+(t_0)$, and vice versa. More precisely, the following estimates hold:

$$1 + \sin(\gamma - |\alpha|)M_\gamma^+(t_0) - 1 - E_0 < R(t_0) < (1 + \sin(\gamma - |\alpha|))M_\gamma^+(t_0) - 1 + E_1,$$

where $E_0, E_1$ are positive constants:

$$E_0 := \frac{\sin(\gamma + |\alpha|)}{\cos \alpha \cos^2(\gamma + |\alpha|)} \left( \frac{\ell}{\kappa R} + \frac{1 - R}{R} \sin |\alpha| \right),$$

$$E_1 := 1 - \sin(\gamma - |\alpha|) + \frac{2 \sin \gamma}{\cos^2(\gamma + |\alpha|)} \left( \frac{\ell}{\kappa R} + \frac{1 - R}{R} \sin |\alpha| \right).$$

**Proof.** We first claim that

$$1 - M_\gamma^+(t_0) - M_\gamma^-(t_0) \leq \frac{1}{\cos \alpha \cos^2(\gamma + |\alpha|)} \left( \frac{\ell}{\kappa R} + \frac{1 - R}{R} \sin |\alpha| \right). \quad (4.1)$$

**Proof of Claim (4.1).** We use Lemma 2.2 (i) to get

$$\dot{R}(t_0) = - \int_{T \times \mathbb{R}} \nu \sin(\theta - \phi) d\theta d\nu + \kappa R \cos \alpha \int_{T \times \mathbb{R}} \sin^2(\theta - \phi) f d\theta d\nu + \kappa R \sin \alpha \int_{T \times \mathbb{R}} \sin(\theta - \phi) \cos(\theta - \phi) f d\theta d\nu. \quad (4.2)$$

It is easy to have

$$\left| \int_{T \times \mathbb{R}} \nu \sin(\theta - \phi) d\theta d\nu \right| \leq \ell. \quad (4.4)$$

In the sequel, we will estimate the remaining R.H.S. terms of (4.2). Then we use estimate

$$\sin(\theta - \phi) > \cos(\gamma + |\alpha|), \quad \theta \in T \setminus (L_\gamma^+ \cup L_\gamma^-)$$

to obtain

$$\kappa R \cos \alpha \int_{T \times \mathbb{R}} \sin^2(\theta - \phi) f d\theta d\nu \geq \kappa R \cos \alpha \int_{T \times (L_\gamma^+ \cup L_\gamma^-) \times \mathbb{R}} \sin^2(\theta - \phi) f d\theta d\nu \geq \kappa R \cos \alpha \cos^2(\gamma - |\alpha|) \left( 1 - M_\gamma^+ - M_\gamma^- \right) \quad (4.5)$$

Then we use Claim (2.19) in proof of Lemma 2.4, $R(t_0) \geq R$ and $R \leq 1$ to get

$$\left| \kappa R \sin \alpha \int_{T \times \mathbb{R}} \sin(\theta - \phi) \cos(\theta - \phi) f(\theta, \nu) d\theta d\nu \right| \leq \kappa R \sin |\alpha| \int_{T \times \mathbb{R}} \sin(\theta - \phi) \cos(\theta - \phi) f(\theta, \nu) d\theta d\nu \leq \kappa R(1 - R) \sin |\alpha| \leq \kappa (1 - R) \sin |\alpha| \quad (4.6)$$

Now we combine relation (4.2), estimates (4.4), (4.6) and (4.5), and assumption $\dot{R}(t_0) \leq 0$ to have

$$0 \geq \dot{R}(t_0) > -\ell + \kappa R \cos \alpha \cos^2(\gamma + |\alpha|) \left( 1 - M_\gamma^+ - M_\gamma^- \right) - \kappa (1 - R) \sin |\alpha|.$$
Hence, it holds that
\[ 1 - \mathcal{M}_\gamma^+(t_0) - \mathcal{M}_\gamma^-(t_0) < \frac{1}{\cos \alpha \cos^2(\gamma + |\alpha|)} \left( \frac{\ell}{\kappa R} + \frac{1 - R}{R} \sin |\alpha| \right), \]
which yields Claim (4.1).

- (Proof of upper bound). We use relation (2.2) and estimates
\[ \cos(\theta - \phi) \leq \sin(\gamma - |\alpha|), \quad \theta \in L_\gamma^+ \], \quad \cos(\theta - \phi) \leq \sin(\gamma + |\alpha|), \quad \theta \in \mathbb{T}\backslash(L_\gamma^+ \cup L_\gamma^-) \]
to obtain
\[ R(t_0) = \int_{L_\gamma^+ \times \mathbb{R}} \cos(\theta - \phi) f d\theta d\nu + \int_{L_\gamma^- \times \mathbb{R}} \cos(\theta - \phi) f d\theta d\nu \]
\[ + \int_{\mathbb{T}\backslash(L_\gamma^+ \cup L_\gamma^-) \times \mathbb{R}} \cos(\theta - \phi) f d\theta d\nu \]
\[ \leq \mathcal{M}_\gamma^+ - \sin(\gamma - |\alpha|) \mathcal{M}_\gamma^- + \sin(\gamma + |\alpha|) \left( 1 - \mathcal{M}_\gamma^+ - \mathcal{M}_\gamma^- \right) \]
\[ = (1 + \sin(\gamma - |\alpha|)) \mathcal{M}_\gamma^+ - \sin(\gamma - |\alpha|) \]
\[ + (\sin(\gamma - |\alpha|) + \sin(\gamma + |\alpha|)) \left( 1 - \mathcal{M}_\gamma^+ - \mathcal{M}_\gamma^- \right) \]
\[ = (1 + \sin(\gamma - |\alpha|)) \mathcal{M}_\gamma^+ - \sin(\gamma - |\alpha|) + 2 \sin \gamma \cos \alpha \left( 1 - \mathcal{M}_\gamma^+ - \mathcal{M}_\gamma^- \right). \]

Now we use Claim (4.1) to get
\[ R(t_0) < (1 + \sin(\gamma - |\alpha|)) \mathcal{M}_\gamma^+ - \sin(\gamma - |\alpha|) + \frac{2 \sin \gamma}{\cos^2(\gamma + |\alpha|)} \left( \frac{\ell}{\kappa R} + \frac{1 - R}{R} \sin |\alpha| \right) \]
\[ = (1 + \sin(\gamma - |\alpha|)) \mathcal{M}_\gamma^+ - 1 \]
\[ + 1 - \sin(\gamma - |\alpha|) \]
\[ + \frac{2 \sin \gamma}{\cos^2(\gamma + |\alpha|)} \left( \frac{\ell}{\kappa R} + \frac{1 - R}{R} \sin |\alpha| \right) \]
\[ = (1 + \sin(\gamma - |\alpha|)) \mathcal{M}_\gamma^+ - 1 + E_1. \]

- (Proof of lower bound). Similar to estimates on the upper bound, we use estimates
\[ \cos(\theta - \phi) \geq \sin(\gamma - |\alpha|), \quad \theta \in L_\gamma^+ \], \quad \cos(\theta - \phi) \geq -\sin(\gamma + |\alpha|), \quad \theta \in \mathbb{T}\backslash(L_\gamma^+ \cup L_\gamma^-) \]
to get
\[ R(t_0) = \int_{L_\gamma^+ \times \mathbb{R}} \cos(\theta - \phi) f d\theta d\nu + \int_{L_\gamma^+ \times \mathbb{R}} \cos(\theta - \phi) f d\theta d\nu \]
\[ + \int_{\mathbb{T}\backslash(L_\gamma^+ \cup L_\gamma^-) \times \mathbb{R}} \cos(\theta - \phi) f d\theta d\nu \]
\[ \geq \sin(\gamma - |\alpha|) \mathcal{M}_\gamma^+ - \mathcal{M}_\gamma^- - \sin(\gamma + |\alpha|) \left( 1 - \mathcal{M}_\gamma^+ - \mathcal{M}_\gamma^- \right). \]

Then we use relation \( \mathcal{M}_\gamma^- \leq 1 - \mathcal{M}_\gamma^+ \) and Claim (4.1) to have
\[ R(t_0) > \sin(\gamma - |\alpha|) \mathcal{M}_\gamma^+ - 1 + \mathcal{M}_\gamma^+ - \frac{\sin(\gamma + |\alpha|)}{\cos \alpha \cos^2(\gamma + |\alpha|)} \left( \frac{\ell}{\kappa R} + \frac{1 - R}{R} \sin |\alpha| \right) \]
\[ = (1 + \sin(\gamma - |\alpha|)) \mathcal{M}_\gamma^+ - 1 - \frac{\sin(\gamma + |\alpha|)}{\cos \alpha \cos^2(\gamma + |\alpha|)} \left( \frac{\ell}{\kappa R} + \frac{1 - R}{R} \sin |\alpha| \right) \]
\[ = (1 + \sin(\gamma - |\alpha|)) \mathcal{M}_\gamma^+ - 1 - E_0. \]
\[ \square \]
Lemma 4.2. For any constants $R \in (0, R_0)$, $\gamma \in \left(\frac{\pi}{2}, \frac{\pi}{2} - |\alpha|\right)$, and small $\mu > 0$, suppose the following assumptions hold.

(a) Let $\kappa$ be large enough and $|\alpha|$ be small enough such that
\[
\kappa R \cos \gamma > \ell + \frac{\ell}{R} + \kappa \sin |\alpha| + \left(\frac{\kappa \cos \alpha}{R}\right)^{\frac{1}{2}} (\ell + \kappa \sin |\alpha| + \mu)^{\frac{1}{2}}.
\]
(b) Let $t = t_0$ be an instant such that
\[
R(t_0) > R, \quad \dot{R}(t_0) \leq \mu.
\]
Then we have
\[
\frac{d}{dt} \bigg|_{t=t_0} \mathcal{M}^\kappa_\gamma (t) > 0.
\]

Proof. Note that Lemma 2.5 yields
\[
\frac{d}{dt} \mathcal{M}^\kappa_\gamma (t) = \int_R \left( \dot{\phi} - \nu \right) (B_+ - B_-) d\nu + \kappa R \cos \gamma \int_R (B_+ + B_-) d\nu
\]
\[
\geq \int_R \left( \kappa R \cos \gamma - |\phi| - \ell \right) (B_+ + B_-) d\nu.
\]
We use the proof of Claim (3.10) in Lemma 3.2 to obtain
\[
|\phi(t_0)| < \frac{\ell}{R} + \kappa \sin |\alpha| + \left(\frac{\kappa \cos \alpha}{R}\right)^{\frac{1}{2}} (\ell + \kappa \sin |\alpha| + \mu)^{\frac{1}{2}}. \tag{4.7}
\]
Hence, we use relation (4.7) and assumption $\kappa R \cos \gamma > \ell + \frac{\ell}{R} + \kappa \sin |\alpha| + \left(\frac{\kappa \cos \alpha}{R}\right)^{\frac{1}{2}} (\ell + \kappa \sin |\alpha| + \mu)^{\frac{1}{2}}$ to get
\[
\kappa R \cos \gamma - |\phi(t_0)| - \ell > \kappa R \cos \gamma - \ell + \frac{\ell}{R} + \kappa \sin |\alpha| + \left(\frac{\kappa \cos \alpha}{R}\right)^{\frac{1}{2}} (\ell + \kappa \sin |\alpha| + \mu)^{\frac{1}{2}} > 0,
\]
which yields the desired estimate. \hfill \Box

Lemma 4.3. Suppose the following assumptions hold.

\[
\inf_{0 \leq t \leq T_0} R(t) > R, \quad \dot{R}(T_0) = \mu.
\]
Then we have
\[
\dot{R}(t) > 0 \text{ for } t \in [T_0, T_0 + d_0), \quad R(T_0 + d_0) - R(T_0) > \frac{R\mu}{9 \cos \alpha} - E_2,
\]
where $d_0$ and $E_2$ are positive constants defined as follows:
\[
d_0 := \frac{1}{3 \kappa \cos \alpha} \left\{ \ln \frac{\mu R}{2 \kappa R \cos \alpha} + \frac{\ell^2 + k^2 \cos \alpha \sin |\alpha|}{3 \kappa \cos \alpha} + \frac{\ell^2 + k^2 \cos \alpha \sin |\alpha|}{3 \kappa \cos \alpha} \right\},
\]
\[
E_2 := \frac{1}{9 \kappa \cos \alpha} \left\{ \frac{\ell^2}{2 \kappa R \cos \alpha} - \frac{2 \ell^2 + k^2 \cos \alpha \sin |\alpha|}{3 \kappa \cos \alpha} \right\},
\]
\[
+ \frac{\ell^2}{2 \kappa R \cos \alpha} + \frac{2 \ell^2 + 5 \kappa^2 \cos \alpha \sin |\alpha|}{3 \kappa \cos \alpha} \left\{ 1 + \ln \frac{\mu R}{2 \kappa R \cos \alpha} + \frac{2 \ell^2 + k^2 \cos \alpha \sin |\alpha|}{3 \kappa \cos \alpha} \right\}.
\]
Proof. We first claim
\[ R(t) \geq R(T_0), \quad t \in [T_0, T_0 + d_0). \] (4.8)

Proof of claim (4.8). Define 
\[ T_0 := \{ t \in [T_0, T_0 + d_0] \mid R(s) \geq R(T_0), \ s \in [T_0, t] \}, \quad T_0^* := \sup T_0. \]
Since \( T_0 \in T_0 \), so \( T_0 \) is not empty and it is suffice to prove \( T_0^* = T_0 + d_0 \). Assume \( T_0^* < T_0 + d_0 \), then by definition of \( T_0^* \), we have 
\[ \dot{R}(T_0^*) \leq 0, \quad R(t) \geq R(T_0) \text{ for } t \in [T_0, T_0^*]. \]
In the sequel, we will show \( \dot{R}(t) > 0 \) for any \( t \in [T_0, T_0^*] \), which yields a contradiction. We first use Lemma 2.3, Remark 2.1, and fact \( 0 \leq R \leq 1 \) to get
\[
S(t) \geq R(t)S(T_0) + \frac{4(\ell^2 + \kappa^2 \cos \alpha \sin |\alpha|)}{3\kappa \cos^2 \alpha} e^{-3\kappa(t-T_0) \cos \alpha} - \frac{4(\ell^2 + \kappa^2 \cos \alpha \sin |\alpha|)}{3\kappa \cos^2 \alpha}.
\]
For easy writing, we set \( a_0 := \frac{4(\ell^2 + \kappa^2 \cos \alpha \sin |\alpha|)}{3\kappa \cos^2 \alpha} \). Note that Lemma 2.4 yields
\[
S(T_0) > \frac{2}{3\cos \alpha} \left\{ \dot{R}(T_0) - \frac{\ell^2}{2R \cos \alpha} - \kappa \sin |\alpha| \right\}.
\]
Thus, it holds that, for \( t \in [T_0, T_0^*] \),
\[
S(t) > \frac{2R(T_0)}{3\cos \alpha} \left\{ \dot{R}(T_0) - \frac{\ell^2}{2R \cos \alpha} - \kappa \sin |\alpha| + a_0 \right\} e^{-3\kappa(t-T_0) \cos \alpha} - a_0
\]
\[
geq \frac{2}{3\cos \alpha} \left\{ \dot{R}(T_0) - \frac{\ell^2}{2R \cos \alpha} - \kappa \sin |\alpha| + \frac{3a_0 \cos \alpha}{2} \right\} e^{-3\kappa(t-T_0) \cos \alpha} - a_0.
\]
Hence, we use Lemma 2.4 again to have, for \( t \in [T_0, T_0^*] \),
\[
\dot{R}(t) > \frac{1}{3} \left\{ \dot{R}(T_0) - \frac{\ell^2}{2R \cos \alpha} - \kappa \sin |\alpha| + \frac{3a_0 \cos \alpha}{2} \right\} e^{-3\kappa(t-T_0) \cos \alpha}
\]
\[
- \frac{a_0 \cos \alpha}{2} - \frac{\ell^2}{2R \cos \alpha} - \kappa \sin |\alpha|
\]
\[
\geq \frac{1}{3} \left\{ \mu_R - \frac{\ell^2}{2R \cos \alpha} + \frac{2\ell^2 + \kappa^2 \cos \alpha \sin |\alpha|}{\kappa \cos \alpha} \right\} e^{-3\kappa d_0 \cos \alpha}
\]
\[
- \frac{\ell^2}{2R \cos \alpha} - \frac{2\ell^2 + 5\kappa^2 \cos \alpha \sin |\alpha|}{3\kappa \cos \alpha} = 0,
\]
where in the last inequality we use definition of \( d_0 \). Thus, claim (4.8) holds.
Recall that, for any \( t \in [T_0, T_0 + d_0] \),
\[
\dot{R}(t) > \frac{1}{3} \left\{ \mu_R - \frac{\ell^2}{2R \cos \alpha} + \frac{2\ell^2 + \kappa^2 \cos \alpha \sin |\alpha|}{\kappa \cos \alpha} \right\} e^{-3\kappa(t-T_0) \cos \alpha}
\]
\[
- \frac{\ell^2}{2R \cos \alpha} - \frac{2\ell^2 + 5\kappa^2 \cos \alpha \sin |\alpha|}{3\kappa \cos \alpha}.
\] (4.9)
For simplicity, we set

\[
\begin{align*}
    a &:= \frac{1}{3} \left\{ \mu R - \frac{\ell^2}{2\kappa R \cos \alpha} + \frac{2\ell^2 + \kappa^2 \cos \alpha \sin |\alpha|}{\kappa \cos \alpha} \right\}, \\
    b &:= \frac{\ell^2}{2\kappa R \cos \alpha} + \frac{2\ell^2 + 5\kappa^2 \cos \alpha \sin |\alpha|}{3\kappa \cos \alpha}.
\end{align*}
\]

Then by relation of \( d_0 \), we know \( d_0 = \frac{1}{3\kappa \cos \alpha} (\ln a - \ln b) \). Now we integrate (4.9) over \([T_0, T_0 + d_0]\) to obtain

\[
R(T_0 + d_0) - R(T_0) > \int_{T_0}^{T_0 + d_0} (ae^{-3\kappa(t-T_0) \cos \alpha} - b) dt = \frac{a}{3\kappa \cos \alpha} (1 - e^{-3\kappa d_0 \cos \alpha}) - bd_0 = \frac{a}{3\kappa \cos \alpha} - \frac{b}{3\kappa \cos \alpha} (1 + \ln a - \ln b).
\]

Hence, we have

\[
R(T_0 + d) - R(T_0) > \frac{R\mu}{9 \cos \alpha} - \frac{1}{9\kappa \cos \alpha} \left\{ \frac{\ell^2}{2\kappa R \cos \alpha} - \frac{2\ell^2 + \kappa^2 \cos \alpha \sin |\alpha|}{\kappa \cos \alpha} \right\} \\
- \frac{\ell^2}{2\kappa R \cos \alpha} + \frac{2\ell^2 + 5\kappa^2 \cos \alpha \sin |\alpha|}{3\kappa \cos \alpha} \left\{ 1 + \ln \frac{\mu R - \ell^2 \kappa R \cos \alpha + 2\ell^2 + \kappa^2 \cos \alpha \sin |\alpha|}{\kappa \cos \alpha} \right\} \\
- \ln \left( \frac{\ell^2}{2\kappa R \cos \alpha} + \frac{2\ell^2 + 5\kappa^2 \cos \alpha \sin |\alpha|}{3\kappa \cos \alpha} \right) \right\} = \frac{R\mu}{9 \cos \alpha} - E_2.
\]

Now we are ready to prove our first estimate on the positive lower bound as follows.

**Proposition 4.1.** For any constants \( R > 0 \), \( \gamma \in \left( \frac{\pi}{2}, \frac{\pi}{2} - |\alpha| \right) \), and small \( \mu > 0 \), let \( \kappa \) be large enough and \(|\alpha|\) be small enough such that

\[
R_0 - E_1 > R, \quad \frac{R\mu}{9 \cos \alpha} > E_1 + E_2,
\]

\[
\kappa R \cos \gamma > \ell + \frac{\ell^2}{R} + \kappa \sin |\alpha| + \left( \frac{\kappa \cos \alpha}{R} \right)^\frac{1}{2} \left( \ell + \kappa \sin |\alpha| + \mu \right)^\frac{1}{2},
\]

where \( E_1, E_2 \) are positive constants defined as

\[
E_1 := 1 - \sin(\gamma - |\alpha|) + \frac{2 \sin \gamma}{\cos^2(\gamma + |\alpha|)} \left( \frac{\ell}{\kappa R} + \frac{1 - R}{R} \sin |\alpha| \right),
\]

\[
E_2 := \frac{1}{9\kappa \cos \alpha} \left\{ \frac{\ell^2}{2\kappa R \cos \alpha} - \frac{2\ell^2 + \kappa^2 \cos \alpha \sin |\alpha|}{\kappa \cos \alpha} \right\} \\
+ \frac{\ell^2}{2\kappa R \cos \alpha} + \frac{2\ell^2 + 5\kappa^2 \cos \alpha \sin |\alpha|}{3\kappa \cos \alpha} \left\{ 1 + \ln \frac{\mu R - \ell^2 \kappa R \cos \alpha + 2\ell^2 + \kappa^2 \cos \alpha \sin |\alpha|}{\kappa \cos \alpha} \right\} \\
- \ln \left( \frac{\ell^2}{2\kappa R \cos \alpha} + \frac{2\ell^2 + 5\kappa^2 \cos \alpha \sin |\alpha|}{3\kappa \cos \alpha} \right) \right\}.
\]

Let \( t_0 \geq 0 \) be an instant such that \( R(t_0) \geq R_0 = R(0) > 0 \). Then the following assertion holds:

Either

\[
R(t) > 0 \quad \text{or} \quad \mathcal{M}_+^2(t) > \frac{1}{1 + \sin(\gamma - |\alpha|)} (R(t_0) + 1 - E_1), \quad \text{for } t \geq t_0.
\]

(4.10)
Proof. First, we set
\[ T := \left\{ s \in [t_0, \infty) \mid \text{either } \dot{R}(t) > 0 \text{ or } M^+_\gamma(t) > \frac{1}{1 + \sin(\gamma - |\alpha|)}(R(t_0) + 1 - E_1) \right\} \]
for any \( t \in [t_0, s] \}, \]
and
\[ T^* := \sup T. \]

Thus, to prove Proposition 4.1, we only need to show \( T \) is not empty and \( T^* = \infty \).

• Step A: We check \( T \) is not empty. Now we discuss the following two cases:
  ◦ Case A.1: If \( \dot{R}(t_0) > 0 \), then we use the continuity of \( \dot{R}(t) \) in Lemma A.1 to see that there exists a small constant \( \eta > 0 \) such that \( \dot{R}(t) > 0, \ t \in [t_0 - \eta, t_0 + \eta] \).

Thus, \([t_0 - \eta, t_0 + \eta] \subseteq T\).

  ◦ Case A.2: If \( \dot{R}(t_0) \leq 0 \), then we use Lemma 4.1 to conclude
\[ M^+_\gamma(t_0) > \frac{1}{1 + \sin(\gamma - |\alpha|)}(R(t_0) + 1 - E_1). \]
Then by the the continuity of \( M^+_\gamma(t) \) proved in Lemma A.2, we see that there exists a small constant \( \eta > 0 \) such that
\[ M^+_\gamma(t) > \frac{1}{1 + \sin(\gamma - |\alpha|)}(R(t_0) + 1 - E_1), \ t \in [t_0 - \eta, t_0 + \eta]. \]
That is, \([t_0 - \eta, t_0 + \eta] \subseteq T\).

• Step B (\( T^* = \infty \)): Suppose not, i.e., we assume \( T^* < \infty \).

(Proof of (4.12)): To prove (4.12), we first give two other claims:
  ◦ (Claim A): For any \( t_0 \geq 0 \) such that \( R(t_0) \geq R_0 \), it holds that
\[ R(t) > R(t_0) - E_1, \ \text{for all } t \in [t_0, T_*]. \] (4.13)
Proof of (4.13): We define
\[ S := \{ s \in [t_0, T^*) \mid R(t) > R(t_0) - E_1 \text{ for any } t \in [t_0, s] \}. \text{ and } S^* := \sup S. \]
Firstly, it is clear to see
\[ R(t_0) > R(t_0) - E_1. \]
By the continuity of \( R(t) \), we see that there exists a small constant \( \eta > 0 \) such that
\[ R(t) > R(t_0) - E_1, \ t \in [t_0, t_0 + \eta]. \]
Secondly, we want to prove
\[ S^* = T^*. \]
Now we assume $S^* < T^*$, then by the continuity of $R(t)$ and the definition of $S^*$, we have
\[ \dot{R}(S^*) \leq 0 \quad \text{and} \quad R(t) \geq R(t_0) - E_1, \quad \text{for} \quad t \in [t_0, S^*]. \]
Since $S^* < T^*$ and $\dot{R}(S^*) \leq 0$, then relation (4.10) holds at $t = S^*$, that is
\[ \mathcal{M}^+_\gamma(S^*) > \frac{1}{1 + \sin(\gamma - |\alpha|)} (R(t_0) + 1 - E_1). \]

Thus, we use Lemma 2.7 to get
\[ R(S^*) > (1 + \sin(\gamma - |\alpha|)) \mathcal{M}^+_\gamma(S^*) - 1 > R(t_0) - E_1. \]

By the continuity of $R(t)$, we know that there exists a small constant $\eta > 0$ such that
\[ R(t) > R_0 - E_1, \quad t \in [S^* - \eta, S^* + \eta], \]
which contradicts the definition of $S^*$. Thus, we conclude $S^* = T^*$.

○ (Claim B): We claim
\[ \frac{d}{dt} \mathcal{M}^+_\gamma(t) > 0, \quad \text{for all} \quad t \in [T^* - \eta, T^* + \eta]. \]  
(4.14)

Proof of (4.14): We first come to show
\[ \dot{R}(T^*) \leq 0. \]  
(4.15)
Suppose not, that is, $\dot{R}(T^*) > 0$, then by continuity of $\dot{R}(t)$, there exists time interval $[T^* - \eta, T^* + \eta]$ such that
\[ \dot{R}(t) > 0, \quad t \in [T^* - \eta, T^* + \eta]. \]
This contradicts to the definition of $T^*$. Thus $\dot{R}(T^*) \leq 0$. On the other hand, (Claim A) yields
\[ R(T^*) \geq R(t_0) - E_1 \geq R_0 - E_1 > R. \]
Thus, by the continuity of $R(t)$ and $\dot{R}(t)$, there exists time interval $[T^* - \eta, T^* + \eta]$ and a small constant $\mu$ such that
\[ R(t) > R \quad \text{and} \quad \dot{R}(t) \leq \mu, \quad \text{for all} \quad t \in [T^* - \eta, T^* + \eta]. \]
Recall that $\kappa R \cos \gamma > \ell + \frac{\kappa}{R} \sin |\alpha| + \left( \frac{\kappa \cos \alpha}{R} \right)^\frac{1}{2} (\ell + \kappa \sin |\alpha| + \mu)\frac{1}{2}$. Hence, by Lemma 4.2, we have
\[ \frac{d}{dt} \mathcal{M}^+_\gamma(t) > 0, \quad \text{for all} \quad t \in [T^* - \eta, T^* + \eta]. \]

Now we come to prove Claim (4.12) by the following cases.
○ Case B.1: Suppose that there exists $t_k \in [T^* - \eta, T^*)$ such that
\[ \dot{R}(t_k) \leq 0. \]
Then $t_k \in \mathcal{T}$ and by definition of $T^*$, we have
\[ \mathcal{M}^+_\gamma(t_k) > \frac{1}{1 + \sin(\gamma - |\alpha|)} (R(t_0) + 1 - E_1). \]
We use (Claim B) to have
\[ \mathcal{M}^+_\gamma(T^*) > \mathcal{M}^+_\gamma(t_k) > \frac{1}{1 + \sin(\gamma - |\alpha|)} (R(t_0) + 1 - E_1). \]
○ Case B.2: For any $t \in [T^* - \eta, T^*)$, $\dot{R}(t) > 0$, we set
\[ \mathcal{T}_s := \{ s \in [t_0, T^*) \mid \dot{R}(t) > 0 \quad \text{for any} \quad t \in [s, T^*) \} \quad \text{and} \quad \tau_s := \inf \mathcal{T}_s. \]
By assumption in this case, we have \([T_s - \eta, T_s) \subseteq T_s\), together with continuity of \(\dot{R}(t)\), we know \(\dot{R}(T_s) = 0\).

- Case B.2.1: Assume \(t_s = 0\). Thus, \(\dot{R}(t) \geq 0\), for any \(t \in [t_0, T^\ast)\). Then we have \(R(t) > R_0\), \(t \in (t_0, T^\ast)\).

We set \(M^+_1(T^\ast) > \frac{1}{1 + \sin(\gamma - |\alpha|)}(R(T^\ast) + 1 - E_1) \geq \frac{1}{1 + \sin(\gamma - |\alpha|)}(R(t_0) + 1 - E_1)\).

- Case B.2.2: Assume \(t_s \neq 0\). Then \(t_s \in (0, T^\ast - \eta) \subseteq (0, T_s)\). By the continuity of \(\dot{R}(t)\) and relation (4.15), we have

\[
\dot{R}(t_s) = 0 \quad \text{and} \quad \dot{R}(T_s) = 0.
\]

By the fact \(t_s \in (0, T_s)\) and \(\dot{R}(t_s) \leq 0\), one has

\[
M^+_1(t_s) > \frac{1}{1 + \sin(\gamma - |\alpha|)}(R(t_0) + 1 - E_1).
\]

We set \(T_c := \{s \in [t_s, T^\ast) \mid \dot{R}(t) < \mu \text{ for any } t \in [t_s, s]\} \quad \text{and} \quad t_c := \inf T_c.\)

Then we know that there exists \(\eta'' > 0\) such that \([t_s, t_s + \eta''] \subseteq T_c\) and \(t_c \in [t_s + \eta'', T_s]\).

- Case B.2.2.1: Assume \(t_c = T_s\). Then

\[
0 \leq \dot{R}(t) < \mu, \quad \text{for any } t \in [t_s, T_s].
\]

By (Claim A), \(R(t) > R\) for any \(t \in [t_s, T_s]\). By Lemma 4.2, \(\frac{d}{dt} M^+_1(t) > 0\) for any \(t \in [t_s, T_s]\). Hence, we have

\[
M^+_1(T^\ast) > M^+_1(t_s) > \frac{1}{1 + \sin(\gamma - |\alpha|)}(R(t_0) + 1 - E_1).
\]

- Case 2B.2. Assume \(t_c \neq T_s\), then \(t_c \in [t_s + \eta'', T_s] \subseteq (t_s, T_s)\), and we have

\[
\dot{R}(t_c) = \mu, \quad \dot{R}(t) < \mu \text{ for } t \in [t_s + \eta'', t_c).
\]

Hence, by Lemma 4.3, there exists \(d_0, E_0 > 0\) such that

\[
\dot{R}(t) > 0 \text{ for } t \in [t_c, t_c + d_0), \quad R(t_c + d_0) - R(t_c) > \frac{R\mu}{\frac{9}{\cos \alpha}} - E_2.
\]

Since \(t_s > t_c\) and \(\dot{R}(T_s) = 0\), we have \(t_c + d_0 < T_s\). In particular, \(t_s < t_c + d_0 < T_s\), and \(\dot{R}(t) > 0\) for any \(t \in (t_s, T_s)\). Then

\[
R(T_s) - R(t_c) > R(t_c + d_0) - R(t_c) > \frac{R\mu}{\frac{9}{\cos \alpha}} - E_2 \quad \text{and} \quad R(t_c) > R(t_0) - E_1.
\]

Hence, we use assumption \(\frac{R\mu}{\frac{9}{\cos \alpha}} - E_1 - E_2 > 0\) to have

\[
R(T_s) > R(t_0) + \frac{R\mu}{\frac{9}{\cos \alpha}} - E_1 - E_2 > R(t_0).
\]

Now we use Lemma 4.1 to get

\[
M^+_1(T_s) > \frac{1}{1 + \sin(\gamma - |\alpha|)}(R(T_s) + 1 - E_1) > \frac{1}{1 + \sin(\gamma - |\alpha|)}(R(t_0) + 1 - E_1). \quad \square
\]

Corollary 4.1. Under the assumption of Proposition 4.1, the following assertion holds:

\[
R(t) > R(t_0) - E_1, \quad t \geq t_0.
\]
Proof. By the proof of Proposition 4.1, we have already proved this assertion, so here we do not repeat. □

4.2. Proof of Proposition 3.1. Now we are ready to prove Proposition 3.1.

Proof of Proposition 3.1. We use Corollary 4.1 and assumption $R_0 - E_1 > R$ to get

$$R(t) > R(t_0) - E_1 \geq R_0 - E_1 > R, \quad t \geq t_0.$$ 

Thus, in the sequel, we only need to prove $\limsup_{t \to \infty} \dot{R}(t) \leq \mu$. Now suppose

$$\limsup_{t \to \infty} \dot{R}(t) > \mu.$$ 

Together with the fact $\liminf_{t \to \infty} \dot{R}(t) \leq 0$ (because $0 \leq R(t) \leq 1$) and the continuity of $\dot{R}(t)$, there exists a sequence $\{t_i\}$ such that

$$\dot{R}(t_i) = \mu \quad \text{and} \quad \lim_{i \to \infty} t_i = \infty. \quad (4.16)$$

Set

$$R_{inf} := \liminf_{t \to \infty} \dot{R}(t).$$

Then for any $\epsilon \in (0, E_1)$, there exists $t_\epsilon > 0$ such that

$$R(t) \geq R_{inf} - \epsilon, \quad \text{for all} \quad t \geq t_\epsilon. \quad (4.17)$$

Then by the continuity of $\dot{R}(t)$ and relations (4.16), (4.17), there exists $t_{i_0} \in \mathbb{N}$ such that

$$\dot{R}(t_{i_0}) = \mu \quad \text{and} \quad \dot{R}(t_{i_0}) \geq R_{inf} - \epsilon.$$ 

We use Lemma 4.3 to obtain

$$R(t_{i_0} + d_0) > R(t_{i_0}) + \frac{R_0 \mu}{9 \cos \alpha} - E_2 > R_0 + \frac{R_0 \mu}{9 \cos \alpha} - E_1 - E_2.$$ 

We use assumption $\frac{R_0 \mu}{9 \cos \alpha} > E_1 + E_2$ to have $R(t_{i_0} + d_0) > R_0$. Then we take $t_0 := t_{i_0} + d_0$ in Corollary 4.1 to get

$$R(t) > R(t_{i_0} + d_0) - E_1, \quad \text{for} \quad t \geq t_{i_0} + d_0.$$ 

Hence, we have

$$R(t) > R(t_{i_0}) + \frac{R_0 \mu}{9 \cos \alpha} - E_1 - E_2 \geq R_{inf} + \frac{R_0 \mu}{9 \cos \alpha} - E_1 - E_2 - \epsilon.$$ 

Now we can get

$$R_{inf} = \liminf_{t \to \infty} R(t) \geq R_{inf} + \frac{R_0 \mu}{9 \cos \alpha} - E_1 - E_2 - \epsilon > R_{inf},$$

which is a contradiction. Thus, the desired assertion holds.

Appendix A. Lipschitz continuities of $\dot{R}(t)$ and $\mathcal{M}_r^+(t)$. In this appendix, we show the Lipschitz continuities of $\dot{R}(t)$ and $\mathcal{M}_r^+(t)$ by establishing the uniform boundedness of their first derivatives.

Lemma A.1. (Uniform boundedness and Lipschitz continuity of $\dot{R}$) Let $f$ and $R$ be a classical solution to system (1.2), and the order parameter defined in (2.2), respectively. Then, the following assertions hold:

(i) The order parameter $R$ is Lipschitz continuous in $[0, \infty)$, and its Lipschitz constant is given by

$$|\dot{R}(t)| < \ell + \kappa, \quad t \in [0, \infty).$$
(ii) Suppose that the following a priori condition holds:
\[
\inf_{0 \leq t \leq T} R(t) \geq R \quad \text{for some } T \in (0, \infty) \text{ and } R > 0.
\]
Then there exists \( \eta > 0 \) such that \( \dot{R}(t) \) is Lipschitz continuous in \([0, T + \eta)\), and the Lipschitz constant is given by
\[
|\dot{R}| \leq \frac{1}{R} \left[ (\kappa + \ell)^2 + \frac{3}{2} (2\kappa^2 + 3\kappa\ell + \ell^2) \right].
\]

Proof. (i) It follows from Lemma 2.2 that
\[
\dot{R}(t) = -\int_{T \times R} \nu \sin(\theta - \phi) f(t, \nu, \theta) d\theta d\nu + \kappa R \int_{T \times R} \sin(\theta - \phi - \alpha) f(t, \nu, \theta) d\theta d\nu.
\]
Now, we use the facts \( R \leq 1 \) and \( \text{supp}_{\nu} f(t, \nu, \theta) \subseteq [-\ell, \ell] \) to get
\[
|\dot{R}(t)| < \ell + \kappa, \quad t \in [0, \infty).
\]
(ii) By relation (2.2), we have
\[
R^2(t) = \left\langle \int_{T \times R} e^{i\theta} f(t, \nu, \theta) d\theta d\nu, \int_{T \times R} e^{i\theta} f(t, \nu, \theta) d\theta d\nu \right\rangle, \quad \text{(A.1)}
\]
where \( \langle, \rangle \) denote the inner product in \( \mathbb{C} \).

Then, we differentiate (A.1) with respect to \( t \) to get
\[
\frac{d}{dt} R^2 = 2 \left\langle \int_{T \times R} e^{i\theta} f(t, \nu, \theta) d\theta d\nu, \int_{T \times R} e^{i\theta} \partial_t f(t, \nu, \theta) d\theta d\nu \right\rangle, \quad \text{(A.2)}
\]
\[
= 2 \left\langle \int_{T \times R} e^{i\theta} f(t, \nu, \theta) d\theta d\nu, i \int_{T \times R} e^{i\theta} \left( \nu - \kappa R \sin(\theta - \phi - \alpha) \right) f(t, \nu, \theta) d\theta d\nu \right\rangle,
\]
where we used
\[
\int_{T \times R} e^{i\theta} \partial_t f(t, \nu, \theta) d\theta d\nu = -\int_{T \times R} e^{i\theta} \partial_\theta \left[ \left( \nu - \kappa R \sin(\theta - \phi - \alpha) \right) f \right] d\theta d\nu
\]
\[
= i \int_{T \times R} e^{i\theta} \left( \nu - \kappa R \sin(\theta - \phi - \alpha) \right) f(t, \nu, \theta) d\theta d\nu.
\]
Now, we differentiate relation (A.2) with respect to \( t \) to obtain
\[
\frac{d^2}{dt^2} R^2 = 2 \left| i \int_{T \times R} e^{i\theta} \left( \nu - \kappa R \sin(\theta - \phi - \alpha) \right) f d\theta d\nu \right|^2
\]
\[
+ 2 \left\langle \int_{T \times R} e^{i\theta} f d\theta d\nu, i \int_{T \times R} e^{i\theta} \partial_t \left( \nu - \kappa R \sin(\theta - \phi - \alpha) \right) f d\theta d\nu \right\rangle
\]
\[
+ 2 \left\langle \int_{T \times R} e^{i\theta} f d\theta d\nu, i \int_{T \times R} e^{i\theta} \left( \nu - \kappa R \sin(\theta - \phi - \alpha) \right) \partial_t f d\theta d\nu \right\rangle
\]
\[
:= \mathcal{J}_{21} + \mathcal{J}_{22} + \mathcal{J}_{23}. \quad \text{(A.3)}
\]
In the sequel, we estimate the terms \( \mathcal{J}_{2i} \) one by one.

• (Estimate on \( \mathcal{J}_{21} \)): By direct calculation, one has
\[
|\mathcal{J}_{21}| \leq 2(\ell + \kappa)^2.
\]
• (Estimate on $J_{22}$): Note that

$$i \int_{T \times \mathbb{R}} e^{it} \partial_t (\nu - \kappa R \sin(\theta - \phi - \alpha)) f d\theta d\nu$$

$$= i \int_{T \times \mathbb{R}} e^{it} (\nu - \kappa R \sin(\theta - \phi - \alpha) - \kappa R \sin(\theta - \phi - \alpha)) f d\theta d\nu.$$ 

Thus, we use assertion (i) and estimate $|\dot{\phi}(t)| \leq \ell + \kappa$ to obtain

$$|J_{22}| \leq \kappa |\dot{R}| + \kappa |R \dot{\phi}| \leq \kappa (\ell + \kappa) + \kappa (\ell + \kappa) = 2 \kappa (\ell + \kappa).$$

• (Estimate on $J_{23}$): By direct calculation, we obtain

$$i \int_{T \times \mathbb{R}} e^{it} (\nu - \kappa R \sin(\theta - \phi - \alpha)) f d\theta d\nu$$

$$= - i \int_{T \times \mathbb{R}} e^{it} (\nu - \kappa R \sin(\theta - \phi - \alpha)) \partial_{\theta} [ (\nu - \kappa R \sin(\theta - \phi - \alpha)) f ] d\theta d\nu$$

$$= - \int_{T \times \mathbb{R}} e^{it} (\nu - \kappa R \sin(\theta - \phi - \alpha))^2 f d\theta d\nu$$

$$- i \int_{T \times \mathbb{R}} e^{it} \kappa R \cos(\theta - \phi - \alpha) (\nu - \kappa R \sin(\theta - \phi - \alpha)) f d\theta d\nu.$$ 

This yields

$$|J_{23}| \leq \ell^2 + \kappa^2 + 2\kappa \ell + n\ell + \kappa^2 = 2\kappa^2 + \ell^2 + 3\kappa \ell.$$ 

Now, we substitute the above estimates into relation (A.3) to yield

$$\left| \frac{d^2}{dt^2} R^2 \right| \leq 3(2\kappa^2 + 3\kappa \ell + \ell^2). \quad (A.4)$$

Note that

$$\left| \frac{d^2}{dt^2} R^2 \right| = 2 \ddot{R}^2 + 2 R \dddot{R}.$$ 

Thus, we use the assumption $R > \bar{R}$, the upper bound of $\dddot{R}$ get the estimate (i) and the upper bound of $\frac{d^2}{dt^2} R^2$ in (A.4) to get

$$2R \left| \dddot{R} \right| \leq 2 \left| R \dddot{R} \right| \leq \left| \frac{d^2}{dt^2} R^2 \right| + 2 \left| \dot{R} \right|^2 \leq 2(\kappa + \ell)^2 + 3(2\kappa^2 + 3\kappa \ell + \ell^2).$$

Hence, we obtain

$$\left| \dddot{R} \right| \leq \frac{1}{2R} \left( (\kappa + \ell)^2 + \frac{3}{2}(2\kappa^2 + 3\kappa \ell + \ell^2) \right).$$

Then we complete our proof.

**Lemma A.2.** Let $f = f(t, \nu, \theta)$ be a classical solution to (1.2) with the initial data $f^{in}$ satisfying $\|f^{in}\|_{L^\infty(T \times \mathbb{R})} < \infty$ and a priori condition:

$$\inf_{0 \leq t \leq T} R(t) > \bar{R}.$$ 

Then the following assertions hold.
(i) The time-dependent order parameter $\phi = \phi(t)$ is Lipschitz continuous in $[0, T + \eta]$, and the Lipschitz constant is given by

$$|\dot{\phi}(t)| \leq \frac{\ell}{R} + 2\kappa.$$ 

(ii) The function $M^+_\gamma(\cdot)$ is Lipschitz continuous in $[0, T + \eta]$, and the Lipschitz constant is given by

$$\left| \frac{d}{dt} M^+_\gamma(t) \right| \leq 3\ell\left(\frac{\ell}{R} + \kappa + \ell\right)\|f^{\mathrm{in}}\|_{L^\infty} e^{\kappa(T + \eta)}.$$ 

Proof. (i) We use Lemma 2.4 (ii) to know

$$|\dot{\phi}(t)| \leq \frac{\ell}{R} + \kappa \cos \alpha(1 - R) + \kappa \sin |\alpha| \leq \frac{\ell}{R} + 2\kappa, \quad t \in [0, T + \eta).$$

(ii) We first claim

$$\|f(t)\|_{L^\infty([\eta, T] \times \mathbb{R})} \leq \|f^{\mathrm{in}}\|_{L^\infty([\eta, T] \times \mathbb{R})} e^{\kappa t}, \quad t \in [0, \infty). \tag{A.5}$$

(Proof of claim (A.5)): Firstly, we define a forward characteristics $\Theta(t)$ issued from $\theta_0 \in \mathbb{T}$ at time $t = 0$ as a solution to the Cauchy problem as follows.

$$\begin{align*}
\dot{\Theta}(t) &= \nu - \kappa R \sin \left(\Theta(t) - \phi(t) - \alpha\right), \quad t > 0, \\
\Theta|_{t=0} &= \theta_0.
\end{align*}$$

Then we have

$$\frac{d}{dt} f(t, \nu, \Theta(t)) = \kappa R \cos(\Theta(t) - \phi(t) - \alpha) f(t, \nu, \Theta(t)) \leq \kappa f(t, \nu, \Theta(t)).$$

Thus, we use Gronwall's inequality to get

$$f(t, \nu, \Theta(t)) = e^{\kappa t} f^{\mathrm{in}}(\nu, \theta_0), \quad \text{for } t \geq 0,$$

which yields (A.5). Recall that Lemma 2.5 gives

$$\frac{d}{dt} M^+_\gamma = \int_{\mathbb{R}} (\dot{\phi} - \nu)(B_+ - B_-)d\nu + \kappa R \cos \gamma \int_{\mathbb{R}} (B_+ + B_-)d\nu, \quad t > 0,$$

where $B_+$ and $B_-$ are defined as

$$B_+ := f(t, \nu, \phi + \frac{\pi}{2} - (\gamma - \alpha)), \quad B_- := f(t, \nu, \phi + \frac{\pi}{2} + (\gamma + \alpha)).$$

Hence, we use assertion (i) and Claim (A.5) to obtain

$$\left| \frac{d}{dt} M^+_\gamma \right| \leq \int_{\mathbb{R}} (|\dot{\phi}| + \ell + \kappa)(B_+ + B_-)d\nu \leq 3\ell\left(\frac{\ell}{R} + \kappa + \ell\right)\|f^{\mathrm{in}}\|_{L^\infty} e^{\kappa(T + \eta)}.$$ 

REFERENCES

[1] J. A. Acebrón, L. L. Bonilla, C. J. P. Pérez Vicente, F. Ritort and R. Spigler, The Kuramoto model: A simple paradigm for synchronization phenomena, Rev. Modern. Phys., 77 (2005), 137–185.

[2] D. Amadori, S. Y. Ha and J. Park, On the global well-posedness of BV weak solutions for the Kuramoto-akaguchi equation, J. Differ. Equ., 262 (2017), 978–1022.

[3] D. Benedetto, E. Caglioti and U. Montemagno, On the complete phase synchronization for the Kuramoto model in the mean-field limit, Commun. Math. Sci., 13 (2015), 1775–1786.

[4] D. Benedetto, E. Caglioti and U. Montemagno, Exponential dephasing of oscillators in the kinetic Kuramoto model, J. Stat. Phys., 162 (2016), 813–823.

[5] M. Brede and A. C. Kalloniatis, Frustration tuning and perfect phase synchronization in the Kuramoto-Sakaguchi model, Phys. Rev. E, 93 (2016), 062315, 13 pp.
[6] J. A. Carrillo, Y. P. Choi, S. Y. Ha, M. J. Kang and Y. Kim, Contractivity of transport distances for the kinetic Kuramoto equation, J. Stat. Phys., 156 (2014), 395–415.
[7] Y. Choi, S. Y. Ha, S. Jung and Y. Kim, Asymptotic formation and orbital stability of phase-locked states for the Kuramoto model, Physica D, 241 (2012), 735–754.
[8] H. Chiba, Continuous limit of the moment system for the globally coupled phase oscillators, Discrete Contin. Dyn. Syst., 33 (2013), 1891–1903.
[9] N. Chopra and M. W. Spong, On exponential synchronization of Kuramoto oscillators, IEEE Trans. Autom. Control, 54 (2009), 353–357.
[10] H. Daido, Quasientrainment and slow relaxation in a population of oscillators with random and frustrated interactions, Phys. Rev. Lett., 68 (1992), 1073–1076.
[11] F. De Smet and D. Aeyels, Partial entrainment in the finite Kuramoto-Sakaguchi model, Physica D, 234 (2007), 81–89.
[12] H. Dietert, B. Fernandez and D. Gérard-Varet, Landau damping to partially locked states in the Kuramoto model, Commun. Pure Appl. Math., 71 (2018), 953–993.
[13] J. G. Dong and X. Xue, Synchronization analysis of Kuramoto oscillators, Commun. Math. Sci., 11 (2013), 465–480.
[14] F. Dorfler and F. Bullo, Synchronization and transient stability in power networks and nonuniform Kuramoto oscillators, SIAM J. Control Optim., 50 (2012), 1616–1642.
[15] F. Dorfler and F. Bullo, On the critical coupling for Kuramoto oscillators, SIAM J. Appl. Dyn. Syst., 10 (2011), 1070–1099.
[16] B. Fernandez, D. Grard-Varet and G. Giacomin, Landau damping in the Kuramoto model, Ann. Henri Poincaré, 17 (2016), 1793–1823.
[17] S. Y. Ha, D. Kim, J. Lee and Y. Zhang, Remarks on the stability properties of the Kuramoto-Sakaguchi-Fokker-Planck equation with frustration, Z. Angew. Math. Phys., 69 (2018), 25 pp.
[18] S. Y. Ha, H. K. Kim and J. Park, Remarks on the complete synchronization for the Kuramoto model with frustrations, Anal. Appl., 16 (2018), 525–563.
[19] S. Y. Ha, H. Kim and S. Ryoo, Emergence of phase-locked states for the Kuramoto model in a large coupling regime, Commun. Math. Sci., 14 (2016), 1073–1091.
[20] S. Y. Ha, Y. Kim and Z. Li, Large-time dynamics of Kuramoto oscillators under the effects of inertia and frustration, SIAM J. Appl. Dyn. Syst., 13 (2014), 466–492.
[21] S. Y. Ha, Y. Kim and Z. Li, Asymptotic synchronous behavior of Kuramoto type models with frustrations, Netw. Heterog. Media, 9 (2014), 33–64.
[22] S. Y. Ha, Y. H. Kim, J. Morales and J. Park, Emergence of phase concentration for the Kuramoto-Sakaguchi equation, Phys. D, 401 (2020), 24 pp.
[23] S. Y. Ha, D. Ko and Y. Zhang, Emergence of Phase-Locking in the Kuramoto Model for Identical Oscillators with Frustration, SIAM J. Appl. Dyn. Syst., 17 (2018), 581–625.
[24] S. Y. Ha, J. Lee and Y. Zhang, Robustness in the instability of the incoherent state for the Kuramoto-Sakaguchi-Fokker-Planck equation with frustration, Quart. Appl. Math., 77 (2019), 631–654.
[25] S. Y. Ha, H. Park and Y. Zhang, Nonlinear stability of stationary solutions to the Kuramoto-Sakaguchi equation with frustration, Netw. Heterog. Media, 15 (2020), 427–461.
[26] S. Y. Ha and Q. Xiao, Remarks on the nonlinear stability of the Kuramoto-Sakaguchi equation, J. Differential Equations, 259 (2015), 2430–2457.
[27] S. Y. Ha and Q. Xiao, Nonlinear instability of the incoherent state for the Kuramoto-Sakaguchi-Fokker-Plank equation, J. Stat. Phys., 160 (2015), 477–496.
[28] A. Jadbabaie, N. Motee and M. Barahona, On the stability of the Kuramoto model of coupled nonlinear oscillators, Proc. American Control Conf., 5 (2004), 4296–4301.
[29] Y. Kuramoto, Chemical Oscillations, Waves and Turbulence, Springer-Verlag, Berlin, 1984.
[30] Y. Kuramoto, International Symposium on Mathematical Problems in Mathematical Physics, Lecture notes in theoretical physics, 30 (1975), 420.
[35] J. Morales and D. Poyato, On the trend to global equilibrium for Kuramoto Oscillators, arXiv:1908.07657v1
[36] E. Oh, C. Choi, B. Kahng and D. Kim, Modular synchronization in complex networks with a gauge Kuramoto model, EPL, 83 (2008), 68003.
[37] K. Park, S. W. Rhee and M. Y. Choi, Glass synchronization in the network of oscillators with random phase shift, Phys. Rev. E, 57 (1998), 5030–5035.
[38] A. Pikovsky, M. Rosenblum and J. Kurths, Synchronization: A universal concept in nonlinear sciences, Cambridge University Press, Cambridge, 2001.
[39] H. Sakaguchi and Y. Kuramoto, A soluble active rotator model showing phase transitions via mutual entrainment, Progr. Theoret. Phys., 76 (1986), 576–581.
[40] S. H. Strogatz, From Kuramoto to Crawford: exploring the onset of synchronization in populations of coupled oscillators, Physica D, 143 (2000), 1–20.
[41] A. T. Winfree, Biological rhythms and the behavior of populations of coupled oscillators, J. Theor. Biol., 16 (1967), 15–42.
[42] A. T. Winfree. The Geometry of Biological Time, Springer, New York, 1980.
[43] Z. G. Zheng, Frustration effect on synchronization and chaos in coupled oscillators, Chin. Phys. Soc., 10 (2001), 703–707.

Received June 2020; revised December 2020.
E-mail address: syha@snu.ac.kr
E-mail address: javierm1@cscamm.umd.edu
E-mail address: yinglongzhang@kaist.ac.kr