Cobimaximal lepton mixing from soft symmetry breaking

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Cobimaximal lepton mixing, i.e. \( \theta_{23} = 45^\circ \) and \( \delta = \pm 90^\circ \) in the lepton mixing matrix \( V \), arises as a consequence of \( SV = V^*P \), where \( S \) is the permutation matrix that interchanges the second and third rows of \( V \) and \( P \) is a diagonal matrix of phase factors. We prove that any such \( V \) may be written in the form \( V = URP \), where \( U \) is any predefined unitary matrix satisfying \( SU = U^* \). \( R \) is an orthogonal, i.e. real, matrix, and \( P \) is a diagonal matrix satisfying \( P^2 = P \). Using this theorem, we demonstrate the equivalence of two ways of constructing models for cobimaximal mixing— one way that uses a standard \( CP \) symmetry and a different way that uses a \( CP \) symmetry including \( \mu-\tau \) interchange. We also present two simple seesaw models to illustrate this equivalence; those models have, in addition to the \( CP \) symmetry, flavour symmetries broken softly by the Majorana mass terms of the right-handed neutrino singlets. Since each of the two models needs four scalar doublets, we investigate how to accommodate the Standard Model Higgs particle in them.

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1. Introduction

Cobimaximal lepton mixing [1–7], i.e. \( \theta_{23} = 45^\circ \) and \( \delta = \pm 90^\circ \), where \( \theta_{23} \) is the atmospheric mixing angle and \( \delta \) is the Dirac phase in the lepton mixing (PMNS) matrix, is still a viable option—for recent fits to the neutrino oscillation data see ref. [8]. Although the best-fit value of \( \theta_{23} \) is slightly off \( 45^\circ \), \( \theta_{23} = 45^\circ \) is still possible at the 3σ level. Moreover, the most recent data prefer \( \delta = -90^\circ \) to \( \delta = +90^\circ \). Cobimaximal mixing is equivalent to the condition

\[
|V_{\mu j}| = |V_{\tau j}| \quad \forall j = 1, 2, 3
\]  

(1)

in the lepton mixing matrix \( V \). The condition (1) does not restrict the mixing angles \( \theta_{12} \) and \( \theta_{13} \). In model-building practice, condition (1) is more easily achieved through the condition

\[
SV = V^*P,
\]  

(2)

where \( S \) is the permutation matrix given by

\[
S = \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix}
\]  

(3)

and \( P \) is a diagonal matrix of phase factors. Notice that \( S \) is real and unitary and that it satisfies \( S^2 = 1 \). (In this paper, 1 denotes the \( 3 \times 3 \) unit matrix.)

In the model-building literature one finds two approaches to cobimaximal mixing which can be distinguished by the \( CP \) transformation property of the Majorana mass Lagrangian of the three light neutrinos. The first approach [2–5] is based on the invariance of that mass Lagrangian under a non-standard \( CP \) transformation including the \( \mu-\tau \) interchange; the second approach [1, 6, 7] uses the standard \( CP \) transformation. In section 2 of this paper we demonstrate that the two approaches are completely equivalent in their consequences for lepton mixing. In sections 3 and 4 we present two very simple models for cobimaximal mixing that illustrate the equivalence of both approaches. Those models include several Higgs doublets; it is thus non-trivial to accommodate in them a Standard Model (SM)-like Higgs boson with mass \( m_h = 125 \text{ GeV} \) [9]; a discussion of this issue is found in section 5. Section 6 presents our conclusions. In order to facilitate the reading of sections 3 and 4, we display the general formulas for weak-basis changes in Appendix A. Appendix B discusses, in a general multi-Higgs-doublet model, the conditions for the alignment

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of the vacuum expectation values (VEVs) needed to accommodate the SM Higgs boson.

2. Two equivalent approaches to cobimaximal mixing

In order to fix the notation we write down the mass terms
\[ \mathcal{L}_{\text{mass}} = -\tilde{\ell} \ell M_{\ell} \ell_R + \frac{1}{2} v_{\ell}^2 C^{-1} M_{\nu} \nu_L + \text{H.c.} \] (4)
for the chiral charged-lepton and neutrino fields \( \ell \) and \( \nu \), respectively. In equation (4), \( C \) is the charge-conjugation matrix in Dirac space. The mass matrices in flavour space \( M_{\ell} \) and \( M_{\nu} \) are both \( 3 \times 3 \) in order to accommodate the three families; \( M_{\ell} \) is a symmetric but in general complex matrix since it corresponds to Majorana mass terms. The diagonalization of the mass matrices proceeds via
\[
U_{LL}^\dagger M_{\ell} U_{LR} = \text{diag}(m_1, m_2, m_3) = \hat{m}_\ell, \tag{5a}
\]
\[
U_{LL}^\dagger M_{\nu} U_{\nu} = \text{diag}(m_1, m_2, m_3) = \hat{m}_\nu. \tag{5b}
\]
The PMNS matrix is \( V = U_{LL}^\dagger U_{\nu} \).

We now prove

**Theorem 1.** Any two \( 3 \times 3 \) unitary matrices \( U_1 \) and \( U_2 \) that fulfill \( U_k^* = S U_k \) \( (k = 1, 2) \) differ only by an orthogonal matrix \( R \), i.e. \( U_2 = U_1 R \).

**Proof.** \( U_1^ \dagger U_2 = U_1^ \dagger (S S^\dagger) U_2 = (SU_1)^\dagger (SU_2) = U_1^ \dagger U_2 = (U_1^ \dagger U_2)^\dagger \). Thus, \( U_1^ \dagger U_2 \equiv R \) is a real matrix, hence it is orthogonal. \( \square \)

With Theorem 1 it is easy to understand that the following two approaches to cobimaximal mixing found in the literature are equivalent.

1. In the first approach \([2–5]\), the charged-lepton mass matrix is diagonal (or, more generally, of the form \( M_{\ell} = \hat{m}_\ell U_{LL}^\dagger U_{LR} \) with diagonal \( \hat{m}_\ell \) and unitary \( U_{LR} \)) while the Majorana neutrino mass term enjoys the generalized CP symmetry\(^2\)

\[ \nu_L \left( x^0, \bar{x} \right) \rightarrow i S \gamma_0 C \bar{V}_{LL}^T \left( x^0, -\bar{x} \right). \] (6)

Because of the symmetry (6), the mass matrix \( M_{\ell} \) fulfills the condition

\[ S M_{\ell} S = M_{\ell}^* \]. \tag{7}

It has been shown in ref. [3] that equation (7) entails \( U_1 = U P \), where \( U \) has the form of Theorem 1, i.e. \( S = U^* \), and \( P \) is a diagonal matrix of phase factors. (Moreover, if \( m_1 \neq 0 \) then \( P_{ij} \) can only be either \( \pm 1 \) or \( \pm i \). Thus, equation (7) leads to cobimaximal mixing plus a prediction for the Majorana phases.) Now, since the charged-lepton mass matrix is diagonal, the lepton mixing matrix \( V \) coincides—apart from multiplication from the left with a diagonal matrix of phase factors—with \( U_1 \), and is thus given by

\[ V = U P. \tag{8} \]

Equation (1) is then fulfilled and cobimaximal mixing obtained. Note that equation (8) together with \( SU = U^* \) lead to \( SV = V^* P^2 \), which is a special case of equation (2).

2. In the second approach \([1,6,7]\), \( M_{\nu} \) is real, i.e. the standard CP symmetry

\[ \nu_L \left( x^0, \bar{x} \right) \rightarrow i \gamma_0 C \bar{V}_{LL}^T \left( x^0, -\bar{x} \right) \] (9)

applies to the neutrino Majorana mass terms. Furthermore, there is some symmetry in the charged-lepton mass terms such that either \( U_{LL}^\dagger = U_{\omega} \), where

\[ U_{\omega} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix} \] \tag{10}

or \( U_{LL}^\dagger = U_{\rho} \), where \([7]^3\)

\[ U_{\rho} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \rho & -i \rho \\ 0 & i \rho & \rho \end{pmatrix} \] \tag{11}

with \( \rho \) orthogonal and \( \rho \) having the same properties as \( \omega \) in the previous paragraph. (The matrix \( U_{\nu} = R_{\alpha} P_{\alpha} \) diagonalizes the real matrix \( M_{\nu} \).)\(^4\) Since \( R_{\alpha} \) is real and the second and third rows in both \( U_{\omega} \) and \( U_{\rho} \) are the complex conjugates of each other, equation (1) holds and the matrix \( V \) of equation (12) displays cobimaximal mixing.

Now, Theorem 1 tells us that both approaches lead to the same predictions for the mixing matrix, namely cobimaximal mixing and Majorana phases in neutrinoless \( \beta \beta \)-decay which are either zero or \( 180^\circ \), while the mixing angles \( \theta_{12} \) and \( \theta_{13} \) are not restricted.\(^3\) Indeed, consider the first approach, leading to the mixing matrix of equation (8). Then, from \( SU = U^* \), we deduce with Theorem 1 that there are orthogonal matrices \( R \) and \( R' \) such that \( U = U_{\omega} R = U_{\rho} R' \). Thus, equation (8) may be converted into the form of equation (12) and both approaches are, therefore, equivalent.

If both approaches refer to the same model, then the mixing matrices in equations (8) and (12) are the same because they just correspond to the same \( CP \) symmetry in that model but written in different weak bases. For the general formulas of a weak-basis change, of a \( CP \) transformation, and of the transformation property under a weak-basis change of the \( CP \)-transformation matrix in flavour space, we refer the reader to equations (A1), (A3), and (A4), respectively, in Appendix A. In particular, if \( S_D \) is the matrix in flavour space that operates the \( CP \) transformation of the left-handed lepton doublets, we see that the \( CP \) transformation (6) corresponds to \( S_D = S \). According to equation (A4), a change of weak basis performed by a unitary matrix \( W_D \) satisfying \( W_D^* = S W_D \) changes \( S_D \) to

\[ S_D = W_D^\dagger S_D W_D = W_D^\dagger S W_D = W_D^\dagger S \left( S D \right) \left( S D \right) = W_D^\dagger W_D = \mathbb{1}, \] \tag{13}

i.e. the \( CP \) transformation becomes the standard \( CP \) transformation of equation (9). Now, since \( U_{\omega}^* = SU_{\omega} \) and \( U_{\rho}^* = SU_{\rho} \), both

\(^1\) A lemma demonstrated in ref. [5] states that if a unitary matrix \( U_1 \) fulfills \( U_1^* = S U_1 \), then the unitary matrix \( U_2 = U_1 R \), where \( R \) is an arbitrary real orthogonal matrix, fulfills \( U_2^* = S U_2 \). Our theorem is the converse of that lemma.

\(^2\) For the sake of clarity, we spell out the transformation (6) at length:

\[ \nu_L \left( x^0, \bar{x} \right) \rightarrow i \gamma_0 C \bar{V}_{LL}^T \left( x^0, -\bar{x} \right), \quad \nu_{\ell L} \left( x^0, \bar{x} \right) \rightarrow i \gamma_0 C \bar{V}_{LL}^T \left( x^0, -\bar{x} \right). \]

\[ \nu_{\ell L} \left( x^0, \bar{x} \right) \rightarrow i \gamma_0 C \bar{V}_{LL}^T \left( x^0, -\bar{x} \right). \]

\(^3\) In ref. [7] the notation \( U_2 \) is used for our \( U_{\rho} \).

\(^4\) The matrices \( P_{\alpha} \) are needed in order to obtain positive neutrino masses \( m_{1,2,3} \). Indeed, if \( M_{\nu} \) is real then it is diagonalized by the orthogonal matrix \( R_{\alpha} \), but the diagonal matrix resulting from that diagonalization may contain some negative diagonal entries; when \( m_1 < 0 \) one needs \( P_{\alpha} \) to be \( \pm i \) to correct for that.

\(^5\) We stress that this statement only refers to the conditions laid out in this section. In specific models the mixing matrix could be more predictive.
Both \( \mathcal{L}_M \) and \( \mathcal{L}_R^\prime \) are explicitly but softly broken through a non-diagonal \( M_R \). The CP symmetry \( (16c) \) is conserved.

Due to the CP symmetry one has

\[
M_R = SM_R^T S = \begin{pmatrix}
 r & c^* & c' \\
 c & c' & r^* \\
 c^* & r & c^*
\end{pmatrix},
\]

where \( r \) and \( r' \) are real while \( c \) and \( c' \) are complex. Equation \( (7) \) then holds, because \( M_R \sim M_R^{-1} \). One thus has a model corresponding to the first approach of section 2. Note that the neutrino masses are not predicted in this model.

We now seek to transform this model into an equivalent one that accords with the second approach of the previous section. In equation \( (14) \), the Yukawa-coupling matrices \( \Gamma_i \) responsible for the charged-lepton masses are

\[
\Gamma_e = \begin{pmatrix}
 y_1 & 0 & 0 \\
 0 & y_1 & 0 \\
 0 & 0 & y_1
\end{pmatrix}, \quad \Gamma_{\mu} = \begin{pmatrix}
 0 & 0 & 0 \\
 0 & y_1 & 0 \\
 0 & 0 & 0
\end{pmatrix},
\]

\[
\Gamma_\tau = \begin{pmatrix}
 0 & 0 & 0 \\
 0 & 0 & 0 \\
 0 & 0 & 0
\end{pmatrix}, \quad \Gamma_\nu = \begin{pmatrix}
 0 & 0 & 0 \\
 0 & 0 & 0 \\
 0 & 0 & 0
\end{pmatrix},
\]

(23)

in the notation

\[
\mathcal{L}_Y = - \sum_{\alpha=e, \mu, \tau} \sum_{i=e, \mu, \tau, \nu} \tilde{D}_{\alpha L} \left[ \phi_i (\Gamma_i)_{\alpha \beta} \beta_R + \phi_i (\Delta_i)_{\alpha \beta} \nu_{\beta R} \right] + \text{H.c.}
\]

(24)

The other Yukawa-coupling matrices are

\[
\Delta_e = \Delta_\mu = \Delta_\tau = 0, \quad \Delta_\nu = y_2 \Delta.
\]

Now we transform the fields into a basis where the CP symmetry has the standard form. Comparing equations \( (16) \) and \( (A.3) \), one sees that

\[
S_D = S_e = S_\nu = S, \quad S_\phi = \begin{pmatrix}
 S & 0_{3 \times 3} \\
 0_{3 \times 1} & 1
\end{pmatrix}.
\]

(26)

We want to use equation \( (A.4) \) to obtain \( S'_e = S'_\mu = S'_\tau = 1 \) and \( S'_\phi = \text{diag}(1, 1, 1, 1) \). Equation \( (13) \) suggests to choose\(^6\)

\[
W_D = W_e = W_\nu = U_{\omega}, \quad W_\phi = \begin{pmatrix}
 U_\omega & 0_{3 \times 1} \\
 0_{1 \times 3} & 1
\end{pmatrix}.
\]

(27)

with \( U_\omega \) of equation \( (10) \). One may now write the Yukawa-coupling matrices in the new basis by using equations \( (A.2) \). One obtains

\[
\Gamma'_e = y_1 \frac{1}{\sqrt{3}}, \quad \Gamma'_\mu = y_1 \frac{1}{\sqrt{3}} E^T, \quad \Gamma'_\tau = y_1 \frac{1}{\sqrt{3}} E, \quad \Delta' = y_2 \Delta
\]

where \( E \equiv \begin{pmatrix}
 1 & 0 & 0 \\
 0 & 0 & 1 \\
 0 & 1 & 0
\end{pmatrix} \).

(28)

\(^6\) In equation \( (27) \) we might have utilized \( U_\phi \) instead of \( U_\omega \). Indeed, we are free to utilize any matrix \( U \) that satisfies \( SU = U^* \) instead of \( U_\omega \) in equation \( (27) \). If we had utilized \( U_\phi \), we would have obtained the matrices \( \Gamma'_{\mu, \tau} \) in equation \( (33) \) below, with \( y_3 \mapsto y_1 \).
Note that $\Gamma_\epsilon' = \Delta_\epsilon' = \Delta_\mu' = 0$. The Yukawa-coupling matrices are real before and after the basis transformation because of the $CP$ symmetry. Obviously, the Yukawa-coupling matrices look simpler in the first approach—equation (23)—than in the second approach—equation (28). On the other hand, $M'_R = U_\rho M_R U_\rho$ is simply a real matrix in the second approach, and therefore the neutrino Majorana mass terms look simpler in the second approach than in the first one.

4. Another simple model for cobimaximal mixing

In our second model the Yukawa Lagrangian is

$$\mathcal{L}_Y = - \bar{\ell}_e e_R (\phi_1 + y_2 \tilde{\phi}_v v_R) - \sum_{\alpha = \mu, \tau} \bar{\ell}_\alpha (\phi_1 \alpha_R + y_4 \tilde{\phi}_v v_R) + \text{H.c.}$$

This Lagrangian may be enforced, for instance, through a $\mu - \tau$ permutation symmetry together with the $Z_2$ symmetries (15). We once again impose the $CP$ symmetry (16), making $y_{1,2,3,4}$ real. The charged-lepton mass matrix is

$$M_\ell = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & y_3 & y_\mu & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$ (30)

The breaking of $CP$ occurs via $\nu_\mu \neq \nu_\tau$.

We use once again the seesaw mechanism with the matrix $M_R$ of equation (22). We thus obtain another model for cobimaximal mixing, very similar to the one of the previous section but with a Yukawa Lagrangian with four coupling constants instead of just two.

We next transform this model into a form that accords with the second approach of section 2. The initial Yukawa-coupling matrices are

$$\Gamma_\epsilon = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \Gamma_\mu = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}, \quad \Gamma_{\nu} = \begin{pmatrix} 0 & y_3 & y_\mu \end{pmatrix},$$

$$\Gamma_\tau = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}, \quad \Delta_\nu = \begin{pmatrix} 0 & y_4 & 0 \end{pmatrix},$$

and $\Gamma_\nu = \Delta_\epsilon = \Delta_\mu = \Delta_\tau = 0$. We transform the fields into a basis where the $CP$ symmetry has the standard form by using

$$W_D = W_\ell = W_\phi = U_\rho,$$ (32)

with $U_\rho$ of equation (11). Note that in equation (32) we use the matrix $U_\rho$ instead of $U_\omega$, as in equation (27). This is just an arbitrary choice—we may use any unitary matrix $U$ such that $SU = U^*$. but using $U_\rho$ in this case yields simpler results. We write down the Yukawa-coupling matrices in the new basis: $\Gamma_\epsilon, \Gamma_\mu, \Gamma_\nu, \Gamma_\tau$ remain invariant while

$$\Gamma_\epsilon' = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}, \quad \Gamma_\mu' = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}, \quad \Gamma_{\nu}' = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix},$$

$$\Gamma_\tau' = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}.$$ (33)

The Yukawa-coupling matrices are still real, but now this follows from the standard $CP$ symmetry. The matrix $M'_R = U_\rho M_R U_\rho$ is real too.

5. Accommodation of the SM Higgs

We have two models, both of them with four scalar doublets, namely the $\phi_i$ with $i = e, \mu, \tau, \nu$. We must use those doublets to give mass to the quarks. It suggests itself to use $\phi_\nu$ for this purpose. Then $|\nu_\nu|$ should be large, because the top-quark Yukawa coupling cannot be much larger than 1. Consequently, it is natural to assume (for definiteness, in the model of section 3)

$$|\nu_\nu|^2 = \frac{2m_\nu^2}{y_1^2} \ll |\nu_\nu|^2 \approx (246 \text{ GeV})^2.$$ (34)

(In this section, as elsewhere in this paper, the indices $\alpha, \beta, \gamma$ vary in the range $\{e, \mu, \tau\}$.) Obviously, the inequalities $|\nu_\ell|^2 \ll |\nu_\mu|^2 \ll |\nu_\tau|^2$ hold due to the strong hierarchy among the charged-lepton masses.

We know that a scalar field $h$ has been discovered at LHC. We also know that the couplings of that scalar to the heavy fermions and to the gauge bosons are close to the couplings of the SM Higgs. The couplings of $h$ to the light fermions may be at variance with the SM.

We use the formalism in refs. [12–14] for the neutral scalars. The neutral components of the doublets $\phi_i$ are written

$$\phi_i^0 = \begin{pmatrix} 1 & \sum_{b=1}^{8} \nu_b S^0_b \end{pmatrix},$$ (35)

where the eight fields $S^0_b$ ($b = 1, 2, \ldots, 8$) are neutral eigenstates of mass. By definition, $S^0_b$ is the neutral “would-be” Goldstone boson and the $S^0_b$ for $b = 2, \ldots, 8$ are physical scalars. The 4 × 8 complex matrix $\nu$ has the property that

$$\nu = \begin{pmatrix} \text{Re} \nu \\ \text{Im} \nu \end{pmatrix},$$ (36)

is 8 × 8 orthogonal. The first column of $\nu$, which corresponds to the Goldstone boson, is given by $\nu_1 = i \nu_1 / v$, where

$$v = \sqrt{|\nu_\ell|^2 + |\nu_\mu|^2 + |\nu_\tau|^2 + |\nu_\nu|^2}.$$ (37)

The interaction of the neutral scalars with a pair of SM gauge bosons is given by the Lagrangian [14]

$$\mathcal{L} = - \frac{1}{2} \sum_{b=2}^{8} (m_B W^\mu W^\nu + 2 \kappa \bar{Z}_B \mu \mu Z^\mu) \sum_{b=2}^{8} S^0_b \text{ Im} (\bar{\nu}^\dagger \nu) b_1.$$ (38)

Equation (38) indicates that we should have $\text{Im} (\bar{\nu}^\dagger \nu) b_1 = 1$ for a neutral scalar with index $b > 1$ that has couplings to pairs of gauge bosons identical to the ones of the SM Higgs. Since $\nu_1 = i \nu_1 / v, \text{Im} (\bar{\nu}^\dagger \nu) b_1 = 1$ is equivalent to $\nu_1 = v / \nu$. Thus, in order to exactly reproduce the SM Higgs, the vector

$$\nu = \begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \\ \nu_\nu \end{pmatrix}$$ (39)

In the $CP$-conserving version of the two-Higgs-doublet model there are two physical neutral scalars named $h$ and $H$. Their couplings to pairs of gauge bosons are given by a Lagrangian similar to equation (38), with $\sum_{b=2}^{8} \text{Im} (\bar{\nu}^\dagger \nu) b_1 = h \sin (\beta - \alpha) + H \cos (\beta - \alpha)$, where $\beta - \alpha$ is a suitably defined angle [15]. The condition for the neutral scalar $h$ to have couplings to pairs of gauge bosons identical to the ones of the SM Higgs is thus $\sin (\beta - \alpha) = 1$. 

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must be one of the columns of the matrix $V$, namely the one corresponding to a neutral-scalar mass eigenstate with mass $m_h = 125\text{GeV}$. We discuss here the possibility that $v$ is an exact mass eigenstate. One may eventually consider an approximate mass eigenstate, taking into account the inequality (34).

The conditions that the scalar potential must satisfy in order for the vector $v$ to correspond to a neutral mass eigenstate with mass $m_h$ are given by equations (B.10). It is easy to fulfil equation (B.10a) by fine tuning. One simply has to assume for the matrix $\mu^2$ in the quadratic part of the scalar potential that

$$\mu^2 = -\frac{m_h^2}{2} v v^\dagger + \cdots,$$

where the dots indicate a part of the matrix $\mu^2$ operating in the space orthogonal to $v$.

We make the following simplifying assumptions concerning the scalar potential $V = V_2 + V_4$:

- In the quartic part $V_4$ we assume invariance under any permutation of $e, \mu, \tau$; in addition, we assume that all the doublets occur in pairs $\phi_i, \phi_i^\dagger$. Thus,

$$V_4 = \lambda \sum_i \left( \phi_i^\dagger \phi_i \right)^2 + \lambda_v \left( \phi_v^\dagger \phi_v \right)^2$$

$$+ \frac{1}{2} \sum_{i \neq j} \left[ \lambda' \left( \phi_i^\dagger \phi_j \right) \left( \phi_j^\dagger \phi_i \right) + \lambda'' \left( \phi_i^\dagger \phi_i \right) \left( \phi_i^\dagger \phi_i \right) \right]$$

$$+ \sum_i \left[ \lambda'' \left( \phi_i^\dagger \phi_i \right) \left( \phi_i^\dagger \phi_i \right) + \lambda''' \left( \phi_i^\dagger \phi_i \right) \left( \phi_i^\dagger \phi_i \right) \right] .$$

This may be achieved by observing that the Yukawa Lagrangian in equation (14) is invariant under

$$\left( \begin{array}{c} D_{el} \\ D_{\mu\tau} \\ D_{el} \end{array} \right) \rightarrow \left( \begin{array}{c} D_{el} \\ D_{\mu\tau} \\ D_{el} \end{array} \right), \quad \left( \begin{array}{c} e_R \\ \mu_R \\ \tau_R \end{array} \right) \rightarrow \left( \begin{array}{c} e_R \\ \mu_R \\ \tau_R \end{array} \right),$$

$$\left( \begin{array}{c} v_R \\ \nu_{\mu R} \\ \nu_{\tau R} \end{array} \right) \rightarrow \left( \begin{array}{c} v_R \\ \nu_{\mu R} \\ \nu_{\tau R} \end{array} \right), \quad \left( \phi_{\ell} \right) \rightarrow \left( \phi_{\ell} \right),$$

$$\left( \phi_{\nu} \right) \rightarrow \left( \phi_{\nu} \right),$$

and then by requiring $V_2$ to be invariant under this transformation too.

- We allow for a general quartic part $V_2$. This breaks softly both symmetries (15) and (42). The soft symmetry breaking also occurs in the matrix $M_R$ anyway, therefore we must assume its presence in $V_2$ too.

In order to use the notation of equation (B.1) we identify, in the $V_4$ of equation (41),

$$\lambda_{\alpha\alpha\alpha\alpha} = \lambda,$$

$$\lambda_{\nu\nu\nu\nu} = \lambda_v,$$

$$\lambda_{\alpha\alpha\beta\beta} = \lambda' \left( \alpha \neq \beta \right),$$

$$\lambda_{\alpha\beta\alpha\beta} = \lambda'' \left( \alpha \neq \beta \right),$$

$$\lambda_{\alpha\alpha\nu\nu} = \lambda'_v \left( \alpha \neq \beta \right),$$

$$\lambda_{\nu\nu\alpha\nu} = \lambda''_v \left( \alpha \neq \beta \right),$$

whence we find

$$\Lambda_{\alpha\alpha} = \lambda |v_{\alpha}|^2 + \frac{\lambda'}{2} \sum_{\nu \neq \alpha} |v_{\nu}|^2 + \frac{\lambda'_v}{2} |v_v|^2 .$$

$$\Lambda_{\nu\nu} = \lambda_v |v_{\nu}|^2 + \frac{\lambda''}{2} \sum_{\nu} |v_{\nu}|^2 .$$

$$\Lambda_{\alpha\beta} = \frac{\lambda''}{2} v_v^* v_{\nu} \left( \alpha \neq \beta \right),$$

$$\Lambda_{\nu\alpha} = \frac{\lambda''}{2} v_v^* v_{\alpha} .$$

Therefore,

$$\left( \Lambda v \right)_{\nu} = \left( \lambda_v |v_{\nu}|^2 + \frac{\lambda''}{2} \sum_{\nu} |v_{\nu}|^2 \right) \frac{v_{\nu}}{v} .$$

Equation (B.10b) then reads

$$m_h^2 = \frac{\lambda_v |v_{\nu}|^2 + \frac{\lambda''}{2} \sum_{\nu} |v_{\nu}|^2}{2} .$$

Equation (46b) separately holds for $\alpha = e, \mu, \tau$.

The right-hand side of equation (46b) has, in general, a dependence on $\alpha$, while its left-hand side is independent of $\alpha$. Consistency is achieved by assuming $\lambda = \lambda' + \frac{\lambda''}{2}$.

Equations (46) may be satisfied by assuming a custodial-type symmetry [16] in $V_4$. Since we have four Higgs doublets, we may choose $U(4)$. The $U(4)$-symmetric quartic potential is then

$$\tilde V_4 = a \left( \sum_i \phi_i^\dagger \phi_i \right)^2 + b \sum_{l e, \mu, \tau, v} \sum_{l e, \mu, \tau, v} \left( \phi_l^\dagger \phi_v \right) \left( \phi_l^\dagger \phi_v \right) .$$

Comparison of $\tilde V_4$ with $V_4$ of equation (41) yields

$$\lambda = \lambda_v = a + b, \quad \lambda' = \lambda'_v = 2a, \quad \lambda'' = \lambda''_v = 2b .$$

Equations (46a) and (46b) then merge into

$$m_h^2 = \frac{a + b}{2} v^2 .$$

6. Conclusions

Neutrino oscillation data indicate that cobimaximal mixing may be a viable scenario, at least as a first approximation, for lepton mixing. In the literature there are two approaches to cobimaximal mixing. They may be characterized by the following $CP$ symmetry of the mass Lagrangian of the light neutrinos. The first approach features a generalized $CP$ symmetry that includes a $\mu-\tau$ interchange, in the basis where the charged-lepton mass matrix is diagonal. In the second approach, the $CP$ symmetry is the standard one, but the charged-lepton mass matrix is non-diagonal and provides a factor $U_{\ell L}^\dagger U_{\ell L}^\dagger$ with $S$ given by equation (3). We have demonstrated
that the two approaches yield the same consequences for lepton mixing, on the one hand by using a simple mathematical theorem which proves the equivalence of the mixing matrices in both approaches, and on the other hand by explicitly stating the weak-basis transformation that transforms the generalized CP symmetry into the standard one.

Moreover, we have displayed two renormalizable models for cobimaximal mixing which illustrate the relationship between the two approaches. In these seesaw models, the mixing angles $\theta_{12}$ and $\theta_{13}$ as well as the neutrino masses are undetermined. They use not only the above-mentioned CP symmetry but also flavour symmetries which are softly broken in the Majorana mass terms of the right-handed neutrino singlets; in this way, cobimaximal mixing is obtained straightforwardly. Finally, since each of our models has four Higgs doublets, the accommodation of the SM Higgs boson is non-trivial. Using the general discussion of this issue for any number of Higgs doublets presented in Appendix B, we have formulated some necessary conditions for the existence of a neutral scalar with SM-like couplings in the two models.

Appendix A. Basis transformation

Let the Yukawa Lagrangian be as in equation (24). We perform the basis transformation

$$D_L = W_D D_L', \quad \ell_R = W_\ell \ell_R', \quad v_R = W_v v_R', \quad \phi = W_\phi \phi'. \quad (A.1)$$

The transformed Yukawa-coupling matrices are

$$\Gamma' = \sum_F (W_\phi)^T F \Gamma_F W_\phi, \quad \Delta = \sum_F (W_\phi)^T \Delta F W_\phi. \quad (A.2)$$

Let the CP symmetry be

$$D_L \to i S_D \gamma_5 CD_L^T, \quad \ell_R \to i S_\ell \gamma_0 C \ell_R^T, \quad v_R \to i S_v \gamma_0 C v_R^T, \quad \phi \to S_\phi \phi^*.$$  

The CP-transformation matrices in the new basis are

$$S_D' = W_D^T S_D W_D^*, \quad S_\ell' = W_\ell^T S_\ell W_\ell^*, \quad S_v' = W_v^T S_v W_v^*, \quad \phi' = W_\phi^T \phi W_\phi^*.$$  

Appendix B. VEV alignment for the SM Higgs

We use in this appendix the notation and results of appendix A of ref. [13]. Suppose there are $n_H$ Higgs doublets $\phi_i$ ($i = 1, 2, \ldots, n_H$). The scalar potential is

$$V = \sum_{i,j=1}^{n_H} \mu_{ij} \rho_{ij}^2 (\phi_i^* \phi_i) + \sum_{i,j,k,l=1}^{n_H} \lambda_{ijkl} \rho_{ij} \rho_{kl} (\phi_i^* \phi_j) (\phi_k^* \phi_l) \equiv V_2 + V_4. \quad (B.1)$$

where $V_2$ is the quadratic part of the potential and $V_4$ is the quartic part. From now on we employ the summation convention. We define the matrices $\Lambda$, $K$, and $K'$ through

$$\Lambda_{ij} \equiv \rho_{ij} \rho_{ij}, \quad K_{ik} \equiv \lambda_{ikl} \rho_{j} \rho_{j}, \quad K'_{ij} \equiv \lambda_{ijkl} \rho_{j} \rho_{j}.$$  

We also define the vector

$$\mathbf{v} = \left( \frac{\sum_{i=1}^{n_H} |\rho_{ii}|^2}{2} \right)^{-1/2} \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_{n_H} \end{pmatrix}.$$  

It is clear from these definitions that

$$\Lambda \mathbf{v} = K \mathbf{v} = K' \mathbf{v}.$$  

The expectation value of the potential in the vacuum state is

$$V_0 = \frac{1}{2} \mu_{ij}^2 \rho_{ij} v_j^2 + \frac{1}{4} \lambda_{ijkl} \rho_{ij} \rho_{kl} v_i v_j v_k v_l.$$  

Since the vacuum state is a stationary point of $V_0$,

$$0 = \frac{\partial V_0}{\partial v_i} = \frac{1}{2} \frac{1}{2} (\mu_{ij}^2 v_j + \lambda_{ijkl} v_i v_j v_k v_l)$$  

(note that $\lambda_{ijkl} = \lambda_{klji}$ by definition). We thus have

$$\left( \mu^2 + \Lambda \right) \mathbf{v} = 0.$$  

As explained after equation (39), we want $\mathbf{v}$ to be a column of the matrix $\mathbf{v}$ corresponding to a neutral scalar with mass $m_0$, therefore it has to fulfill (see equation (A18) of ref. [13])

$$\left( \mu^2 + \Lambda + K' \right) \mathbf{v} + K \mathbf{v} = m_0^2 \mathbf{v}.$$  

Because of equation (B.4), equation (B.8) may be rewritten

$$\left( \mu^2 + 2 \Lambda \right) \mathbf{v} = m_0^2 \mathbf{v}.$$  

We then employ equation (B.7) to obtain

$$\mu^2 \mathbf{v} = -\frac{m_0^2}{2} \mathbf{v}.$$  

$$\Lambda \mathbf{v} = \frac{m_0^2}{2} \mathbf{v}.$$  

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