Links between gravity and dynamics of quantum liquids

G.E. Volovik

Helsinki University of Technology, Low Temperature Laboratory, P.O. Box 2200, FIN-02015 HUT, Finland

L. D. Landau Institute for Theoretical Physics, RAS, Kosygin Str. 2, 117940 Moscow, Russia

We consider the Landau-Khalatnikov two-fluid hydrodynamics of superfluid liquid as an effective theory, which provides a self-consistent analog of Einstein equations for gravity and matter.

IV International Conference "Cosmology. Relativistic Astropohysics. Cosmoparticle Physics".
(COSMION-99)
In the Honor of 80-th Birthday of Isaak M. Khalatnikov

I. INTRODUCTION. PHYSICAL VACUUM AS CONDENSED MATTER.

In a modern viewpoint the relativistic quantum field theory is an effective theory. It is an emergent phenomenon arising in the low energy corner of the physical fermionic vacuum – the medium, whose nature remains unknown. Also it is argued that in the low energy corner the symmetry must be enhanced. If we neglect the very low energy region of electroweak scale, where some symmetries are spontaneously violated, then above this scale one can expect that the lower the energy, the better is the Lorentz invariance and other symmetries of the physical laws. The same phenomena occur in the condensed matter systems. If the spontaneous symmetry breaking at very low energy is neglected or avoided, then in the limit of low energy the symmetry of condensed matter is really enhanced. Moreover, there is one special universality class of Fermi systems, where in the low energy corner there appear almost all the symmetries, which we know today in high energy physics: Lorentz invariance, gauge invariance, elements of general covariance, etc (superfluid $^3$He-A is a representative of this class). The chiral fermions as well as gauge bosons and gravity field arise as fermionic and bosonic collective modes of such a system. The inhomogeneous states of the condensed matter ground state – vacuum – induce nontrivial effective metrics of the space, where the free quasiparticles move along geodesics. This conceptual similarity between condensed matter and quantum vacuum allows us to simulate many phenomena in high energy physics and cosmology, including axial anomaly, baryoproduction and magnetogenesis, event horizon and Hawking radiation, rotating vacuum, expansion of the Universe, etc., probing these phenomena in ultra-low-temperature superfluid helium, atomic Bose condensates and superconductors. Some of the experiments have been already conducted.

The quantum field theory, which we have now, is incomplete due to ultraviolet divergences at small scales, where the “microscopic” physics of vacuum becomes important. Here the analogy between quantum vacuum and condensed matter could give an insight into the transPlanckian physics. As in condensed matter system, one can expect that some or all of the known symmetries in Nature will be lost when the Planck energy scale is approached. The condensed matter analogue gives examples of the physically imposed deviations from Lorentz invariance. This is important in many different areas of high energy physics and cosmology, including possible CPT violation and black holes, where the infinite red shift at the horizon opens the route to the transPlanckian physics.

The low-energy properties of different condensed matter substances (magnets, superfluids, crystals, superconductors, etc.) are robust, i.e. they do not depend much on the details of microscopic (atomic) structure of these substances. The main role is played by symmetry and topology of condensed matter: they determine the soft (low-energy) hydrodynamic variables, the effective Lagrangian describing the low-energy dynamics, and topological defects. The microscopic details provide us only with the “fundamental constants”, which enter the effective phenomenological Lagrangian, such as speed of “light” (say, the speed of sound), superfluid density, modulus of elasticity, magnetic susceptibility, etc. Apart from these fundamental constants, which can be rescaled, the systems behave similarly in the infrared limit if they belong to the same universality and symmetry classes, irrespective of their microscopic origin. The detailed information on the system is lost in such acoustic or hydrodynamic limit. From the properties of the low energy collective modes of the system – acoustic waves in case of crystals – one cannot reconstruct the atomic structure of the crystal since all the crystals have similar acoustic waves described by the same equations of the same effective theory, in a given case the classical theory of elasticity. The classical fields of collective modes can be quantized to obtain quanta of acoustic waves – the phonons, but this quantum field remains the effective field which does not give a detailed information on the real quantum structure of the underlying crystal.
It is quite probable that in the same way the quantization of classical gravity, which is one of the infrared collective modes of quantum vacuum, will not add more to our understanding of the "microscopic" structure of the vacuum [6,7]. Indeed, according to this analogy, such properties of our world, as gravitation, gauge fields, elementary chiral fermions, etc., all arise in the low energy corner as a low-energy soft modes of the underlying "condensed matter". At high energy (of the Planck scale) these soft modes disappear: actually they merge with the continuum of the high-energy degrees of freedom of the "Planck condensed matter" and thus cannot be separated anymore from the others. Since the gravity appears as an effective field in the infrared limit, the only output of its quantization would be the quanta of the low-energy gravitational waves – gravitons.

The main advantage of the condensed matter analogy is that in principle we know the condensed matter structure at any relevant scale, including the interatomic distance, which corresponds to the Planck scale. Thus the condensed matter can suggest possible routes from our present low-energy corner of "phenomenology" to the "microscopic" physics at Planckian and trans-Planckian energies.

II. LANDAU-KHALATNIKOV TWO-FLUID MODEL HYDRODYNAMICS AS EFFECTIVE THEORY OF GRAVITY.

A. Superfluid vacuum and quasiparticles.

Here we consider the simplest effective field theory of superfluids, where only the gravitational field appears as an effective field. The case of the fermi superfluids, where also the gauge field and chiral fermions appear in the low-energy corner together with Lorentz invariance is discussed in [4,8].

According to Landau and Khalatnikov [1] a weakly excited state of the collection of interacting $^4$He atoms can be considered as a small number of elementary excitations – quasiparticles, phonons and rotons. In addition, the state without excitation – the ground state or vacuum – can have collective degrees of freedom. The superfluid vacuum can move without friction, and inhomogeneity of the flow serves as the gravitational and/or other effective fields. The matter propagating in the presence of this background is represented by fermionic (in Fermi superfluids) or bosonic (in Bose superfluids) quasiparticles, which form the so called normal component of the liquid. Such two-fluid hydrodynamics introduced by Landau and Khalatnikov [1] is the example of the effective field theory which incorporates the motion of both the superfluid background (gravitational field) and excitations (matter). This is the counterpart of the Einstein equations, which incorporate both gravity and matter.

One must distinguish between the particles and quasiparticles in superfluids. The particles describes the system on a microscopic level, these are atoms of the underlying liquid ($^3$He or $^4$He atoms). The many-body system of the interacting atoms form the quantum vacuum – the ground state. The conservation laws experienced by the atoms and their quantum coherence in the superfluid state determine the low frequency dynamics – the hydrodynamics – of the collective variables of the superfluid vacuum. The quasiparticles – fermionic and bosonic – are the low energy excitations above the vacuum state. They form the normal component of the liquid which determines the thermal and kinetic low-energy properties of the liquid.

B. Dynamics of superfluid vacuum.

In the simplest superfluid the coherent motion of the superfluid vacuum is characterized by two collective (hydrodynamic) variables: the particle density $n(r,t)$ of atoms comprising the liquid and superfluid velocity $v_s(r,t)$ of their coherent motion. In a strict microscopic theory $n = \sum_p n(p)$, where $n(p)$ is the particle distribution functions. The particle number conservation provides one of the equations of the effective theory of superfluids – the continuity equation

$$\frac{\partial n}{\partial t} + \nabla \cdot J = 0 .$$

The conserved current of atoms in monoatomic superfluid liquid is

$$J = \sum_p \frac{p}{m} n(p) ,$$

2
where \( m \) is the bare mass of the particle. Note that the liquids considered here are nonrelativistic and obeying the Galilean transformation law. In the Galilean system the momentum of particles and the particle current are related by Eq.(2). In the effective theory the particle current has two contributions

\[
J = n \mathbf{v}_s + \mathbf{J}_q , \quad J_q = \sum_{\mathbf{p}} \frac{\mathbf{p}}{m} f(\mathbf{p}) .
\] (3)

The first term \( n \mathbf{v}_s \) is the current transferred coherently by the collective motion of superfluid vacuum with the superfluid velocity \( \mathbf{v}_s \). If quasiparticles are excited above the ground state, their momenta also contribute to the particle current, the second term in rhs of Eq.(11), where \( f(\mathbf{p}) \) is the distribution function of quasiparticles. Note that under the Galilean transformation to the coordinate system moving with the velocity \( \mathbf{u} \) the superfluid velocity transforms as \( \mathbf{v}_s \rightarrow \mathbf{v}_s + \mathbf{u} \), while the momenta of particle and quasiparticle transform differently: \( \mathbf{p} \rightarrow \mathbf{p} + m \mathbf{u} \) for microscopic particles and \( \mathbf{p} \rightarrow \mathbf{p} \) for quasiparticles.

The second equation for the collective variables is the London equation for the superfluid velocity, which is curl-free in superfluid \( ^4 \text{He} \) \((\nabla \times \mathbf{v}_s = 0)\):

\[
m \frac{\partial \mathbf{v}_s}{\partial t} + \nabla \delta \mathcal{E} = 0 .
\] (4)

Together with the kinetic equation for the quasiparticle distribution function \( f(\mathbf{p}) \), the Eqs. (3) and (4) for collective fields \( \mathbf{v}_s \) and \( n \) give the complete effective theory for the kinetics of quasiparticles (matter) and coherent motion of vacuum (gravitational field) if the energy functional \( \mathcal{E} \) is known. In the limit of low density of quasiparticles, when the interaction between quasiparticles can be neglected, the simplest Ansatz satisfying the Galilean invariance is

\[
\mathcal{E} = \int d^3r \left( \frac{m}{2} n \mathbf{v}_s^2 + \epsilon(n) + \sum_{\mathbf{p}} \tilde{E}(\mathbf{p}, r) f(\mathbf{p}, r) \right) .
\] (5)

Here \( \epsilon(n) \) is the (phenomenological) vacuum energy as a function of the particle density; \( \tilde{E}(\mathbf{p}, r) = E(\mathbf{p}, n(r)) + \mathbf{p} \cdot \mathbf{v}_s(r) \) is the Doppler shifted quasiparticle energy in the laboratory frame; \( E(\mathbf{p}, n(r)) \) is the quasiparticle energy measured in the frame comoving with the superfluid vacuum. The Eqs. (3) and (4) can be also obtained from the Hamiltonian formalism using Eq.(2) as Hamiltonian and Poisson brackets

\[
\{ \mathbf{v}_s(r_1), n(r_2) \} = \frac{1}{m} \nabla \delta(r_1 - r_2) , \quad \{ n(r_1), n(r_2) \} = \{ \mathbf{v}_s(r_1), \mathbf{v}_s(r_2) \} = 0 .
\] (6)

Note that the Poisson brackets between components of superfluid velocity are zero only for curl-free superfluidity. In a general case it is

\[
\{ v_{si}(r_1), v_{sj}(r_2) \} = -\frac{1}{mn} \epsilon_{ijk}(\nabla \times \mathbf{v}_s)k \delta(r_1 - r_2) .
\] (7)

In this case even at \( T = 0 \), when the quasiparticles are absent, the Hamiltonian description of the hydrodynamics is only possible: There is no Lagrangian, which can be expressed in terms of the hydrodynamic variables \( \mathbf{v}_s \) and \( n \). The absence of the Lagrangian in many condensed matter systems is one of the consequences of the reduction of the degrees of freedom in effective field theory, as compared with the fully microscopic description. In ferromagnets, for example, the number of the hydrodynamic variables is odd: 3 components of the magnetization vector \( \mathbf{M} \). They thus cannot form the canonical pairs of conjugated variables. As a result one can use either the Hamiltonian description or introduce the effective action with the Wess-Zumino term, which contains an extra coordinate \( \tau \):

\[
S_{\text{WZ}} \propto \int d^3x \ dt \ d\tau \ \mathbf{M} \cdot (\partial_1 \mathbf{M} \times \partial_\tau \mathbf{M}) .
\] (8)

C. Normal component – “matter”.

In a local thermal equilibrium the distribution of quasiparticles is characterized by local temperature \( T \) and by local velocity \( \mathbf{v}_n \) called the normal component velocity.
where the sign + is for the fermionic quasiparticles in Fermi superfluids and the sign - is for the bosonic quasiparticles in Bose superfluids. Since \( \tilde{E}(p) = E(p) + \mathbf{p} \cdot \mathbf{v}_s \), the equilibrium distribution is determined by the Galilean invariant quantity \( \mathbf{v}_n - \mathbf{v}_s = \mathbf{w} \), which is the normal component velocity measured in the frame comoving with superfluid vacuum. It is called the counterflow velocity. In the limit when the counterflow velocity \( \mathbf{v}_n - \mathbf{v}_s \) is small, the quasiparticle (“matter”) contribution to the particle current is proportional to the counterflow velocity:

\[
J_{qi} = n_{nik}(v_{ik} - v_{sk}) ,
\]

where the tensor \( n_{nik} \) is the so called density of the normal component. In this linear regime the total current can be represented as the sum of the currents of the normal and superfluid components

\[
J_i = n_{sik}v_{sk} + n_{nik}v_{nk} ,
\]

where tensor \( n_{sik} = n\delta_{ik} - n_{nik} \) is the so called density of superfluid component. In the isotropic superfluids, \(^4\)He and \(^3\)He-B, the normal component is isotropic tensor, \( n_{nik} = n_{n}\delta_{ik} \), while in anisotropic superfluid \(^3\)He-A the normal component density is a uniaxial tensor \[3\]. At \( T = 0 \) there the quasiparticles are frozen out and one has \( n_{nik} = 0 \) and \( n_{sik} = n\delta_{ik} \).

D. Quasiparticle spectrum and effective metric

The structure of the quasiparticle spectrum in superfluid \(^4\)He becomes more and more universal the lower the energy. In the low energy corner the spectrum of these quasiparticles, phonons, can be obtained in the framework of the effective theory. Note that the effective theory is unable to describe the high-energy part of the spectrum – rotons, which can be determined in a fully microscopic theory only. On the contrary, the spectrum of phonons is linear, \( E(p, n) = c(n)|p| \), and only the “fundamental constant” – the speed of “light” \( c(n) \) – depends on the physics of the higher energy hierarchy rank. Phonons represent the quanta of the collective modes of the superfluid vacuum, sound waves, with the speed of sound obeying \( c^2(n) = (n/m)(d^2\epsilon/dn^2) \). All other information on the microscopic atomic nature of the liquid is lost. Note that for the curl-free superfluids the sound waves represent the only “gravitational” degree of freedom. The Lagrangian for these “gravitational waves” propagating above the smoothly varying background is obtained by decomposition of the superfluid velocity and density into the smooth and fluctuating parts: \( \mathbf{v}_s = \mathbf{v}_{s\,\text{smooth}} + \nabla\alpha \) \[10,11\]. The Lagrangian for the scalar field \( \alpha \) is:

\[
\mathcal{L} = \frac{m}{2\hbar} \left( (\nabla\alpha)^2 - \frac{1}{c^2} (\dot{\alpha} + (\mathbf{v}_s \cdot \nabla)\alpha)^2 \right) = \frac{1}{2} \sqrt{-g} g^{\mu\nu} \partial_\mu \alpha \partial_\nu \alpha .
\]

Thus in the low energy corner the Lagrangian for sound waves has an enhanced symmetry – the Lorentzian form, where the effective Riemann metric experienced by the sound wave, the so called acoustic metric, is simulated by the smooth parts of the hydrodynamic fields:

\[
g^{00} = -\frac{1}{mnc} , \quad g^{0i} = -\frac{v_i}{mnc} , \quad g^{ij} = \frac{c^2\delta^{ij} - v_i v_j}{mnc} ,
\]

\[
g_{00} = -\frac{mn}{c} (c^2 - \mathbf{v}_s^2) , \quad g_{0i} = -\frac{mnv_i}{c} , \quad g_{ij} = \frac{mn}{c}\delta_{ij} , \quad \sqrt{-g} = \frac{m^2n^2}{c} .
\]

Here and further \( \mathbf{v}_s \) and \( n \) mean the smooth parts of the velocity and density fields.

The energy spectrum of sound wave quanta, phonons, which represent the “gravitons” in this effective gravity, is determined by

\[
g^{\mu\nu} p_\mu p_\nu = 0 , \quad (\tilde{E} - \mathbf{p} \cdot \mathbf{v}_s)^2 = c^2 p^2 .
\]
E. Effective quantum field and effective action

The effective action Eq.(12) for phonons formally obeys the general covariance, this is an example of how the enhanced symmetry arises in the low-energy corner. In addition, in the classical limit of Eq.(13) corresponding to geometrical optics (in our case this is geometrical acoustics) the propagation of phonons is invariant under the conformal transformation of metric, \( g^{\mu\nu} \rightarrow \Omega^2 g^{\mu\nu} \). This symmetry is lost at the quantum level: the Eq.(12) is not invariant under general conformal transformations, however the reduced symmetry is still there: Eq.(12) is invariant under scale transformations with \( \Omega = \text{Const} \).

In superfluid \(^3\)He-A the other effective fields and new symmetries appear in the low energy corner, including also the effective \( SU(2) \) gauge fields and gauge invariance. The symmetry of fermionic Lagrangian induces, after integration over the quasiparticles degrees of freedom, the corresponding symmetry of the effective action for the gauge fields. Moreover, in addition to superfluid velocity field there are appear the other gravitational degrees of freedom with the spin-2 gravitons. However, as distinct from the effective gauge fields, whose effective action is very similar to that in particle physics, the effective gravity cannot reproduce in a full scale the Einstein theory: the effective action for the metric is contaminated by the noncovariant terms, which come from the “transPlanckian” physics \( \text{[4]} \). The origin of difficulties with effective gravity in condensed matter is probably the same as the source of the problems related to quantum gravity and cosmological constant.

The quantum quasiparticles interact with the classical collective fields \( \nu_s \) and \( n_\nu \) and with each other. In Fermi superfluid \(^3\)He the fermionic quasiparticles interact with many collective fields describing the multicomponent order parameter and with their quanta. That is why one obtains the interacting Fermi and Bose quantum fields, which are in many respect similar to that in particle physics. However, this field theory can be applied to a lowest orders of the perturbation theory only. The higher order diagrams are divergent and nonrenormalizable, which simply means that the effective theory is valid when only the low energy/momentum quasiparticles are involved even in their virtual states. This means that only those terms in the effective action can be derived by integration over the quasiparticle degrees of freedom, whose integral are concentrated solely in the low-energy region. For the other processes one must go beyond the effective field theory and consider the higher levels of description, such as Fermi liquid theory, or further the microscopic level of the underlying liquid with atoms and their interactions. In short, all the terms in effective action come from microscopic “Planck” physics, but only some fraction of them can be derived within the effective field theory.

In Bose superfluids the fermionic degrees of freedom are absent, that is why the quantum field theory there is too restrictive, but nevertheless it is useful to consider it since it provides the simplest example of the effective theory. On the other hand the Landau-Khalatnikov scheme is rather universal and is easily extended to superfluids with more complicated order parameter and with fermionic degrees of freedom (see the book \( \text{[1]} \)).

F. Vacuum energy and cosmological constant

The vacuum energy density \( \epsilon(n) \) and the parameters which characterize the quasiparticle energy spectrum cannot be determined by the effective theory: they are provided solely by the higher (microscopic) level of description. The microscopic calculations show that at zero pressure the vacuum energy per one atom of the liquid \(^4\)He is about \( \epsilon(n_0)/n_0 \sim -7K \) \( \text{[2]} \). It is instructive to compare this microscopic result with the estimation of the vacuum energy if we try to obtain it from the effective theory. In effective theory the vacuum energy is given by the zero point motion of phonons

\[
\epsilon_{eff} = (1/2) \sum_{E(p) < \Theta} c_p \quad \text{with} \quad c_p = \frac{1}{16\pi^2} \frac{\Theta^4}{\hbar^3} \frac{1}{v_s^3} = \frac{1}{16\pi^2} \sqrt{-g} (g^{\mu\nu} \Theta_\mu \Theta_\nu)^2.
\]

Eq.(16)

Here \( c \) is the speed of sound; \( \Theta \sim \hbar c/a \) is the Debye characteristic temperature with \( a \) being the interatomic space, \( \Theta \) plays the part of the “Planck” cutoff energy scale; \( \Theta_\mu = (-\Theta, 0, 0, 0) \).

We wrote the Eq.(14) in the form which is different from the conventional cosmological term \( \Lambda \sqrt{-g} \). This is to show that both forms and the other possible forms too have the similar drawbacks. The Eq.(16) is conformal invariant due to conformal invariance experienced by the quasiparticle energy spectrum in Eq.(15) (actually, since this term does not depend on derivatives, the conformal invariance is equivalent to invariance under multiplication of \( g_{\mu\nu} \) by constant factor). However, in Eq.(16) the general covariance is violated by the cutoff. On the contrary, the conventional cosmological term \( \Lambda \sqrt{-g} \) obeys the general covariance, but it is not invariant under transformation \( g_{\mu\nu} \rightarrow \Omega^2 g_{\mu\nu} \) with constant \( \Omega \). Thus both forms of the vacuum energy violate one or the other symmetry of the low-energy effective
Lagrangian Eq. (13) for phonons, which means that the vacuum energy cannot be determined exclusively within the low-energy domain.

Now on the magnitude of the vacuum energy. The Eq. (10) gives $\epsilon_{\text{eff}}(n_0)/n_0 \sim 10^{-2}\Theta \sim 10^{-1}\text{K}$. The magnitude of the energy is much smaller than the result obtained in the microscopic theory, but what is more important the energy has an opposite sign. This means again that the effective theory must be used with great caution, when one calculates those quantities, which crucially (non-logarithmically) depend on the “Planck” energy scale. For them the higher level “transPlanckian” physics must be used only. In a given case the many-body wave function of atoms of the underlying quantum liquid has been calculated to obtain the vacuum energy [12]. The quantum fluctuations of the phonon degrees of freedom in Eq.(16) are already contained in this microscopic wave function. To add the energy of this zero point motion of the effective field to the microscopically calculated energy $\epsilon(n_0)$ would be the double counting. Thus the proper regularization of the vacuum energy in the effective field theory must by equating it to exact zero.

This conjecture is confirmed by consideration of the equilibrium conditions for the liquid. The equilibrium condition for the superfluid vacuum is $(dc/dn)_{n_0} = 0$. Close to the equilibrium state one has

$$
\epsilon(n) = \epsilon(n_0) + \frac{1}{2} \frac{m c^2}{n_0} (n - n_0)^2.
$$

(17)

From this equation it follows that the variation of the vacuum energy over the metric determinant must be zero in equilibrium: $(dc/dg)_{n_0} = (dc/dn)_{n_0}/(dg/dn)_{n_0} = 0$. This apparently shows that the vacuum energy can be neither of the form of Eq.(10) nor in the form $\Lambda \sqrt{-g}$. The metric dependence of the vacuum energy consistent with the equilibrium condition and Eq.(17) could be only of the type $A + B (g - g_0)^2$, so that the cosmological term in Einstein equation would be $\propto (g - g_0)g_{\mu\nu}$. This means that in equilibrium, i.e. at $g = g_0$, the cosmological term is zero and thus the equilibrium vacuum is not gravitating. In relativistic theories such dependence of the Lagrangian on $g$ can occur in the models where the determinant of the metric is the variable which is not transformed under coordinate transformations, i.e. only the invariance under coordinate transformations with unit determinant represents the fundamental symmetry.

This probably has some relation to the problem of the cosmological constant in Einstein theory of gravity, where the estimation in Eq.(10) with $c$ being the speed of light and $\Theta = E_{\text{Planck}}$ gives the cosmological term by 100 orders of magnitude higher than its upper experimental limit. The gravity is the low-frequency, and actually the classical output of all the quantum degrees of freedom of the “Planck condensed matter”. So one should not quantize the gravity again, i.e. one should not use the low energy quantization for construction of the Feynman diagrams technique with diagrams containing the integration over high momenta. In particular the effective field theory is not appropriate for the calculation of the vacuum energy and thus of the cosmological constant. Moreover, one can argue that, whatever the real “microscopic” energy of the vacuum is, the energy of the equilibrium vacuum is not gravitating: The diverging energy of quantum fluctuations of the effective fields and thus the cosmological term must be regularized to zero as we discussed above, since these fluctuations are already contained in the “microscopic wave function” of the vacuum.

This however does not exclude the Casimir effect, which appears if the vacuum is not homogeneous. The smooth deviations from the homogeneous equilibrium vacuum are within the low-energy domain: they can be successfully described by the effective field theory, and their energy can gravitate.

G. Einstein action and higher derivative terms

In principle, there are the nonhydrodynamic terms in the effective action, which are not written in Eq.(3) since they contain space and time derivatives of the hydrodynamics variable, $n$ and $v_s$, and thus are relatively small. Only part of them can be obtained using the effective theory. As in the case of Sakharov effective gravity [13], the standard integration over the massless scalar field $\alpha$ propagating in inhomogeneous $n$ and $v_s$ fields, which provide the effective metric, gives the curvature term in Einstein action. It can also be written in two ways. The form which respects the general covariance of the phononic Lagrangian for $\alpha$ field in Eq.(12) is:

$$
L_{\text{Einstein}} = -\frac{1}{16\pi G} \sqrt{-g} R
$$

(18)

This form does not obey the invariance under multiplication of $g_{\mu\nu}$ by constant factor, which shows its dependence on the “Planck” physics. The gravitational Newton constant $G$ is expressed in terms of the “Planck” cutoff: $G^{-1} \sim \Theta^2$. Another form, which explicitly contains the “Planck” cutoff,
\[ \mathcal{L}_{\text{Einstein}} = -\frac{1}{16\pi} \sqrt{-g} R g^{\mu \nu} \Theta_{\mu} \Theta_{\nu} , \]  

(19)
is equally bad: the action is invariant under the scale transformation of the metric, but the general covariance is violated. Such incompatibility of different low-energy symmetries is the hallmark of the effective theories.

To give an impression on the relative magnitude of the Einstein action let us express the Ricci scalar in terms of the superfluid velocity field only

\[ \sqrt{-g} R = \frac{1}{c^3} \left( 2 \partial_t \nabla \cdot v_s + \nabla^2 (v_s^2) \right) . \]

(20)

In superfluids the Einstein action is small compared to the dominating kinetic energy term \( m_n v_s^2 / 2 \) in Eq.(5) by factor \( a^2/l^2 \), where \( a \) is again the atomic ("Planck") length scale and \( l \) is the characteristic macroscopic length at which the velocity field changes. That is why it can be neglected in the hydrodynamic limit, \( a/l \to 0 \). Moreover, there are many terms of the same order in effective actions which do not display the general covariance, such as \( (\nabla \cdot v_s)^2 \). They are provided by microscopic physics, and there is no rule in superfluids according to which these noncovariant terms must be smaller than the Eq.(18). But in principle, if the gravity field as collective field arises from the other degrees of freedom, different from the condensate motion, the Einstein action can be dominating.

There are the higher order derivative terms, which are quadratic in the Riemann tensor, such as

\[ \sqrt{-g} R^2 \ln \left( \frac{g^{\mu \nu} \Theta_{\mu} \Theta_{\nu}}{R} \right) . \]

(21)

They only logarithmically depend on the cut-off and thus their calculation in the framework of the effective theory is possible. Because of the logarithmic divergence (they are of the relative order \( (a/l)^4 \ln(l/a) \)) these terms dominate over the noncovariant terms of order \( (a/l)^4 \), which are obtained in fully microscopic calculations. Being determined essentially by the phononic Lagrangian in Eq.(12), these terms respect (with logarithmic accuracy) all the symmetries of this Lagrangian including the general covariance and the invariance under rescaling the metric. That is why they are the most appropriate terms for the self-consistent effective theory of gravity. The logarithmic terms also appear in the effective action for the effective gauge fields, which take place in superfluid \(^3\)He-A \[8\]. These terms in superfluid \(^3\)He-A have been obtained first in microscopic calculations, however it appeared that their physics can be completely determined by the low energy tail and thus they can be calculated using the effective theory. This is well known in particle physics as running coupling constants and zero charge effect.

Unfortunately in effective gravity of superfluids these logarithmic terms are small compared with the main terms – the vacuum energy and the kinetic energy of the vacuum flow. This means that the superfluid liquid is not the best condensed matter for simulation of Einstein gravity. In \(^3\)He-A there are other components of the order parameter, which also give rise to the effective gravity, but superfluidity of \(^3\)He-A remains to be an obstacle. One must try to construct the non-superfluid condensed matter system which belongs to the same universality class as \(^3\)He-A, and thus contains the effective Einstein gravity as emergent phenomenon, which is not contaminated by the superfluidity.

### III. “RELATIVISTIC” ENERGY-MOMENTUM TENSOR FOR “MATTER” MOVING IN “GRAVITATIONAL” SUPERFLUID BACKGROUND IN TWO FLUID HYDRODYNAMICS

#### A. Kinetic equation for quasiparticles (matter)

The distribution function \( f \) of the quasiparticles is determined by the kinetic equation:

\[ \dot{f} - \frac{\partial \mathcal{E}}{\partial \mathbf{r}} \cdot \frac{\partial f}{\partial \mathbf{p}} + \frac{\partial \mathcal{E}}{\partial \mathbf{p}} \cdot \frac{\partial f}{\partial \mathbf{r}} = J_{\text{coll}} . \]

(22)

The collision integral conserves the momentum and the energy of quasiparticles, i.e.

\[ \sum_{\mathbf{p}} p J_{\text{coll}} = \sum_{\mathbf{p}} \mathcal{E}(\mathbf{p}) J_{\text{coll}} = \sum_{\mathbf{p}} E(\mathbf{p}) J_{\text{coll}} = 0 , \]

(23)

but not necessarily the number of quasiparticle: as a rule the quasiparticle number is not conserved in superfluids.
B. Momentum exchange between superfluid vacuum and quasiparticles

From the Eq. (23) and from the two equations for the superfluid vacuum, Eqs. (1,4), one obtains the time evolution of the momentum density for each of two subsystems: the superfluid background (vacuum) and quasiparticles (matter). The momentum evolution of the superfluid vacuum is

$$m \partial_t (nv_s) = -m \nabla_i (J_i v_s) - n \nabla \left( \frac{\partial e}{\partial n} + \sum_p f \frac{\partial E}{\partial n} \right) + P_i \nabla v_{si} \tag{24}$$

where \( P = mJ_q \) is the momentum of liquid carried by quasiparticles (see Eq. (11)), while the evolution of the momentum density of quasiparticles:

$$\partial_t P = \sum_p p \partial_t f = -\nabla_i (v_s P) - \nabla_i \left( \sum_p p f \frac{\partial E}{\partial p_i} \right) - \sum_p f \nabla E - P_i \nabla v_{si} \tag{25}$$

Though the momentum of each subsystem is not conserved because of the interaction with the other subsystem, the total momentum density of the system, superfluid vacuum + quasiparticles, is conserved:

$$m \partial_t J_i = \partial_t (mnv_{si} + P_i) = -\nabla_i \Pi_{ik} \tag{26}$$

with the stress tensor

$$\Pi_{ik} = mJ_{iak} + v_{si} P_k + \sum_p p_k f \frac{\partial E}{\partial p_i} + \delta_{ik} \left( n \left( \frac{\partial e}{\partial n} + \sum_p f \frac{\partial E}{\partial n} \right) - \epsilon \right) \tag{27}$$

C. Covariance vs conservation.

The same happens with the energy. Energy and momentum can be exchanged between the two subsystems of quasiparticles and superfluid vacuum in a way similar to the exchange of energy and momentum between matter and the gravitational field. In the low energy limit, when the quasiparticles are “relativistic”, this exchange must be described in the general relativistic covariant form. The Eq. (25) for the momentum density of quasiparticles as well as the corresponding equation for the quasiparticle energy density can be represented as

$$T^\mu_{\nu\mu} = 0 \quad \text{or} \quad \frac{1}{\sqrt{-g}} \partial_\mu \left( T^\mu_{\nu} \sqrt{-g} \right) - \frac{1}{2} T^{\alpha\beta} g_{\alpha\beta} \partial_\nu T^\mu_{\mu} = 0 \tag{28}$$

This result does not depend on the dynamics of the superfluid condensate (gravity field), which is not “relativistic”. The Eq. (25) follows solely from the “relativistic” spectrum of quasiparticles.

The Eq. (25) does not represent any conservation in a strict sense, since the covariant derivative is not a total derivative. The extra term, which is not the total derivative, describes the force acting on quasiparticles (matter) from the superfluid condensate (an effective gravitational field). Since the dynamics of the superfluid background is not covariant, it is impossible to find such total energy momentum tensor, \( T^\mu_{\nu} \text{(total)} = T^\mu_{\nu} \text{(quasiparticles)} + T^\mu_{\nu} \text{(background)} \), which could have a covariant form and simultaneously satisfy the real conservation law \( \partial_\mu T^\mu_{\nu} \text{(total)} = 0 \). The total stress tensor in Eq. (27) is evidently noncovariant.

But this is impossible even in the fully covariant Einstein gravity, where one has an energy momentum pseudotensor for the gravitational background. Probably this is an indication that the Einstein gravity is really an effective theory. As we mentioned above, effective theories in condensed matter are full of such contradictions related to incompatible symmetries. In a given case the general covariance is incompatible with the conservation law; in cases of the vacuum energy (Sec. II F) and the Einstein action (Sec. II G) the general covariance is incompatible with the scale invariance; in the case of an axial anomaly, which is also reproduced in condensed matter (see e.g. [8]), the conservation of the baryonic charge is incompatible with quantum mechanics; the action of the Wess-Zumino type, which cannot be written in 3+1 dimension in the covariant form (as we discussed at the end of Sec. II B, Eq. (8)), is almost typical phenomenon in various condensed matter systems; the momentum density determined as variation of the hydrodynamic energy over \( \nu_s \) does not coincide with the canonical momentum in many condensed matter systems; etc., there are many other examples of apparent inconsistencies in the effective theories of condensed matter. All such paradoxes arise due to reduction of the degrees of freedom in effective theory, and they disappear completely (together with some symmetries of the low-energy physics) on the fundamental level, i.e. in a fully microscopic description, where all degrees of freedom are taken into account.
D. Energy-momentum tensor for “matter”.

Let us specify the tensor $T_{\mu \nu}$ which enters Eq. (28) for the simplest case, when the gravity is simulated by the superflow only, i.e. we neglect the space-time dependence of the density $n$ and of the speed of sound $c$. Then the constant factor $mc$ can be removed from the metric in Eqs. (13-14) and the effective metric is simplified:

$$
\begin{align*}
g^{00} &= -1, \\
g^{0i} &= -v^i, \\
g^{ij} &= c^2 \delta_{ij} - v^i v^j,
\end{align*}
\tag{29}
$$

$$
\begin{align*}
g^{00} &= - \left(1 - \frac{v^2}{c^2}\right), \\
g^{0i} &= -\frac{v^i}{c^2}, \\
g^{ij} &= \frac{1}{c^2} \delta_{ij}, \\
\sqrt{-g} &= \frac{1}{c^3}.
\end{align*}
\tag{30}
$$

Then the energy-momentum tensor of quasiparticles can be represented as [14]

$$
\sqrt{-g} T^\mu_\nu = \sum_p f v^\mu_g p^\nu, \quad v^\mu_g v^\nu_g = -1 + \frac{1}{c^2} \frac{\partial E}{\partial p_i} \frac{\partial E}{\partial p_i},
\tag{31}
$$

where $p_0 = -\tilde{E}$, $p^0 = E$; the group four velocity is defined as

$$
v^i_g = \frac{\partial \tilde{E}}{\partial p_i}, \quad v^0_g = 1, \quad v^i_g = \frac{1}{c^2} \frac{\partial E}{\partial p_i}, \quad v^0_g = - \left(1 + \frac{v^i_g \frac{\partial E}{\partial p_i}}{c^2}\right).
\tag{32}
$$

Space-time indices are throughout assumed to be raised and lowered by the metric in Eqs. (29-30). The group four velocity is null in the relativistic domain of the spectrum only: $v^i_g v^\mu_g = 0$ if $E = cp$. The relevant components of the energy-momentum tensor are:

$$
\begin{align*}
\sqrt{-g} T^0_i &= \sum_p f p_i = P_i \quad \text{momentum density in either frame}, \\
-\sqrt{-g} T^0_0 &= \sum_p f \tilde{E} \quad \text{energy density in laboratory frame}, \\
\sqrt{-g} T^k_i &= \sum_p f p_i v^k_g \quad \text{momentum flux in laboratory frame}, \\
-\sqrt{-g} T^i_0 &= - \sum_p f \tilde{E} \frac{\partial E}{\partial p_i} = \sum_p f \tilde{E} v^i_g \quad \text{energy flux in laboratory frame}, \\
\sqrt{-g} T^{00} &= \sum_p f p^0 = \sum_p f E \quad \text{energy density in comoving frame}.
\end{align*}
\tag{33}
$$

With this definition of the momentum-energy tensor the covariant conservation law in Eq. (28) acquires the form:

$$
(\sqrt{-g} T^\mu_\nu)_{,\mu} = \sum_p f \partial_\nu \tilde{E} = P_i \partial_\nu v^i_g + \sum_p f |p| \partial_\nu c.
\tag{34}
$$

The right-hand side represents “gravitational” forces acting on the “matter” from the superfluid vacuum.

E. Local thermodynamic equilibrium.

Local thermodynamic equilibrium is characterized by the local temperature $T$ and local normal component velocity $v_n$ in Eq. (9). In local thermodynamic equilibrium the components of energy-momentum for the quasiparticle system (matter) are determined by the generic thermodynamic potential (the pressure), which has the form

$$
\Omega = \mp T \frac{1}{(2\pi\hbar)^3} \sum_s \int d^3p \ln(1 \mp f),
\tag{35}
$$

with the upper sign for fermions and lower sign for bosons. For phonons one has
where the renormalized effective temperature $T_{\text{eff}}$ absorbs all the dependence on two velocities of liquid. The components of the energy momentum tensor are given as

$$ T^{\mu \nu} = (\varepsilon + \Omega) u^\mu u^\nu + \Omega g^{\mu \nu}, \quad \varepsilon = -\Omega + T \frac{\partial \Omega}{\partial T} = 3\Omega, \quad T^{\mu \mu} = 0. \quad (37) $$

where the four velocity of the “matter”, $u^\alpha$ and $u_\alpha = g_{\alpha \beta} u^\beta$, which satisfies the normalization equation $u_\alpha u^\alpha = -1$, is expressed in terms of superfluid and normal component velocities as

$$ u^0 = \frac{1}{\sqrt{1-w^2}}, \quad u^i = v_i^{(u)} \frac{1}{\sqrt{1-w^2}}, \quad u_i = \frac{w_i}{\sqrt{1-w^2}}, \quad u_0 = -1 + w \cdot v_s \frac{1}{\sqrt{1-w^2}}. \quad (38) $$

\section*{IV. HORIZONS, ERGOREGIONS, DEGENERATE METRIC, VACUUM INSTABILITY AND ALL THAT.}

\subsection*{A. Landau critical velocity, event horizon and ergoregion}

If the superfluid velocity exceeds the Landau critical value

$$ v_L = \min \frac{E(p)}{p}, \quad (42) $$

the energy $\tilde{E}(p)$ of some excitations, as measured in the laboratory frame, becomes negative. This allows for excitations to be nucleated from the vacuum. For a superfluid velocity field which is time-independent in the laboratory frame, the surface $v_s(r) = v_L$, which bounds the region where quasiparticles can have negative energy, the ergoregion, is called the ergosurface.
The behavior of the system depends crucially on the dispersion of the spectrum at higher energy. There are two possible cases. The spectrum bends upwards at high energy, i.e., $E(p) = cp + \gamma p^3$ with $\gamma > 0$. Such dispersion is realized for the fermionic quasiparticles in $^3$He-A. They are “relativistic” in the low energy corner but become “superluminal” at higher energy [10]. In this case the Landau critical velocity coincides with the “speed of light”, $v_L = c$, so that the ergosurface is determined by $v_s(r) = c$. In the Lorentz invariant limit of the energy much below the “Planck” scale, i.e., at $p^2 \ll \gamma/c$, this corresponds to the ergosurface at $g_{00}(r) = 0$, which is just the definition of the ergosurface in gravity. In case of radial flow of the superfluid vacuum towards the origin, the ergosurface also represents the horizon in the Lorentz invariant limit, and the region inside the horizon simulates a black hole for low energy phonons. Strictly speaking this is not a true horizon for phonons: Due to the nonlinear dispersion, their group velocity $v_g = dE/dp = c + 3\gamma p^2 > c$, and thus the high energy quasiparticles are allowed to leave the black hole region. It is, hence, a horizon only for quasiparticles living exclusively in the very low energy corner: they are not aware of the possibility of “superluminal” motion. Nevertheless, the mere possibility to exchange the information across the horizon allows us to construct the thermal state on both sides of the horizon (see Sec. IV D below) and to investigate its thermodynamics, including the entropy related to the horizon [14].

In superfluid $^4$He the negative dispersion is realized, with the group velocity $v_g = dE/dp < c$. In such superfluids the “relativistic” ergosurface $v_s(r) = c$ does not coincide with the true ergosurface, which is determined by $v_s(r) = v_L < c$. In superfluid $^4$He the Landau velocity is related to the roton part of the spectrum, and is about four times less than $c$. In case of radial flow inward, the ergosphere occurs at $v_s(r) = v_L < c$, while the inner surface $v_s(r) = c$ still marks the horizon. This is in contrast to relativistically invariant systems, for which ergosurface and horizon coincide for purely radial gravitational field. The surface $v_s(r) = c$ stays a horizon even for excitations with very high momenta up to some critical value, at which the group velocity of quasiparticle again approaches $c$.

B. Painlevé-Gullstrand metric in effective gravity in superfluids.

Let us consider the spherically symmetric radial flow of the superfluid vacuum, which is time-independent in the laboratory frame. Then dynamics of the phonon, propagating in this velocity field, is given by the line element provided by the effective metric in Eq. (30):

$$ds^2 = - \left(1 - \frac{v_s^2(r)}{c^2}\right)dt^2 + 2\frac{v_s(r)}{c^2}drdt + \frac{1}{c^2}(dr^2 + r^2d\Omega^2).$$

(43)

This equation corresponds to the Painlevé-Gullstrand line elements. It describes a black hole horizon if the superflow is inward (see refs. [10, 11], on the pedagogical review of Painlevé-Gullstrand metric see [17]). If $v_s(r) = -c(r_S/r)^{1/2}$ the flow simulates the black hole in general relativity. For the outward superflow with, say, $v_s(r) = +c(r_S/r)^{1/2}$ the white hole is reproduced. For the general radial dependence of the superfluid velocity, the Schwarzschild radius $r_S$ is determined as $v_s(r_S) = \pm c$; the “surface gravity” at the Schwarzschild radius is $\kappa_S = (1/2)dv^2_s/dr|_{r_S}$; and the Hawking temperature $T_H = h\kappa_S/2\pi$.

C. Vacuum resistance to formation of horizon.

It is not easy to create the flow with the horizon in the Bose liquid because of the hydrodynamic instability which takes place behind the horizon (see [13]). From Eqs. (6) and (11) of superfluid hydrodynamics at $T = 0$ (which correspond to conventional hydrodynamics of ideal curl-free liquid) it follows that for stationary motion of the liquid one has the relation between $n$ and $v_s$ along the streamline [13]:

$$\frac{\partial(nv_s)}{\partial v_s} = n \left(1 - \frac{v_s^2}{c^2}\right).$$

(44)

The current $J = nv_s$ has a maximal value just at the horizon and thus it must decrease behind the horizon, where $1 - (v_s^2/c^2)$ is negative. This is, however, impossible in the radial flow since, according to the continuity equation (11), one has $nv_s = Const/r^2$ and thus the current must monotonically increase across the horizon. This marks the hydrodynamic instability behind the horizon and shows that it is impossible to construct the time-independent flow with the horizon without the fine-tuning of an external force acting on the liquid [13]. Thus the liquid itself resists to the formation of the horizon.
Would the quantum vacuum always resist to formation of the horizon? Fortunately, not. In the considered case of superfluid $^4$He, the same “speed of light” $c$, which describes the quasiparticles (acoustic waves) and thus determines the value of the superfluid velocity at horizon, also enters the hydrodynamic equations that establish the flow pattern of the “black hole”. In $^3$He-A these two speeds are well separated. The “speed of light” $c$ for quasiparticles, which determines the velocity of liquid flow at the horizon, is much less than the speed of sound, which determines the hydrodynamic instabilities of the liquid. That is why there are no severe hydrodynamic constraints on the flow pattern, the hydrodynamic instability is never reached and the surface gravity at such horizons is always finite.

However, even in such superfluids another instability can develop due to the presence of a horizon [20]. Usually the “speed of light” $c$ for “relativistic” quasiparticles coincides with the critical velocity, at which the superfluid state of the liquid becomes unstable towards the normal state of the liquid. When the superfluid velocity with respect to the normal component or to the container walls exceeds $c$, the slope $\partial J/\partial v_s$ becomes negative the superflow is locally unstable.

Such superfluid instability, however, can be avoided if the container walls are properly isolated [21]. Then the reference frame imposed by the container walls is lost and the “inner” observer living in the superfluid does not know that the superfluid exceeded the Landau velocity and thus the threshold of instability. Formally this means that the superfluid instability is regulated not by the superfluid velocity field $v_s$ (as in the case of the hydrodynamic instability discussed above), but by the velocity $w = v_n - v_s$ of the counterflow between the normal and superfluid subsystems. A stable superfluid vacuum can be determined as the limit $T \to 0$ at fixed “subluminal” counterflow velocity $w < c$, even if the superfluid velocity itself is “superluminal”. This can be applied also to quasiequilibrium vacuum state across the horizon, which is locally the vacuum state as observed by comoving “inner” observer. The superfluid motion in this state is locally stable, though slowly decelerates due to the quantum friction caused by Hawking radiation and other processes related to the horizon and ergoregion [10].

D. Modified Tolman’s law across the horizon.

The realization of the quasiequilibrium state across the horizon at nonzero $T$ can be found in Ref. [14] for 1+1 case. In this state the superfluid velocity is “superluminal” behind the horizon, $v_s > c$, but the counterflow is everywhere “subluminal”: the counterflow velocity $w$ reaches maximum value $w = c$ at the horizon with $w < c$ both outside and inside the horizon. The local equilibrium with the effective temperature $T_{\text{eff}}$ in Eq.(36) is thus determined on both sides of the horizon. It is interesting that one has the modified form of the Tolman’s law, which is valid on both sides of the horizon:

$$T_{\text{eff}} = \frac{T_\infty}{\sqrt{1 - v_s^2/c^2}} = \frac{T_\infty}{\sqrt{|\mathcal{g}_{00}|}}.$$  \hspace{1cm} (45)

Here $T_\infty$ is the temperature at infinity. The effective temperature $T_{\text{eff}}$, which determines the local “relativistic” thermodynamics, becomes infinite at the horizon, with the cutoff determined by the nonlinear dispersion of the quasiparticle spectrum at high energy, $\gamma > 0$. The real temperature $T$ of the liquid is continuous across the horizon:

$$T = T_\infty \quad \text{at} \quad v_s^2 < c^2 \hspace{1cm} T = T_\infty \frac{c}{|v_s|} \quad \text{at} \quad v_s^2 > c^2.$$  \hspace{1cm} (46)

E. One more vacuum instability: Painlevé-Gullstrand vs Schwarzschild metric in effective gravity.

In the effective theory of gravity, which occurs in condensed matter systems, the primary quantity is the contravariant metric tensor $g^{\mu\nu}$ describing the energy spectrum. Due to this the two seemingly equivalent representations of the black hole metric, in terms of either the Schwarzschild or the Painlevé-Gullstrand line elements, are in fact not equivalent in terms of the required stability of the underlying superfluid vacuum.

An “equivalent” representation of the black or white hole metric is given by the Schwarzschild line element, which in terms of the same superfluid velocity reads

$$ds^2 = - \left(1 - v_s^2/c^2\right) dt^2 + \frac{dr^2}{c^2 - v_s^2} + \left(r^2/c^2\right) d\Omega^2.$$  \hspace{1cm} (47)
The Eqs. (47) and (43) are related by the coordinate transformation. Let us for simplicity consider the abstract flow with the velocity exactly simulating the Schwarzschild metric, i.e. \( v_s^2(r) = r_s / r \) and we put \( c = 1 \). Then the coordinate transformation is

\[
\tilde{t}(r, t) = t + \left( \frac{2}{v_s(r)} + \ln \frac{1 - v_s(r)}{1 + v_s(r)} \right), \quad \tilde{d}t = dt + \frac{v_s}{1 - v_s^2} dr.
\] (48)

What is the difference between the Schwarzschild and Painlevé-Gullstrand space-times in the effective gravity? The Painlevé-Gullstrand metric is determined in the “absolute” Newton’s space-time \((t, r)\) of the laboratory frame, i.e. as is measured by the external experimentalist, who lives in the real world of the laboratory and investigates the dynamics of quasiparticles using the physical laws obeying the Galilean invariance of the absolute space-time. The effective Painlevé-Gullstrand metric, which describes the quasiparticle dynamics in the inhomogeneous liquid, originates from

\[
E = v_s(r)p_r \pm cp,
\] (49)

or

\[
(E - v_s(r)p_r)^2 = c^2 p^2,
\] (50)

which determines the contravariant components of the metric. Thus the energy spectrum in the low-energy corner is the primary quantity, which determines the effective metric for the low-energy quasiparticles.

The time \( \tilde{t} \) in the Schwarzschild line element is the time as measured by the “inner” observer at “infinity” (i.e. far from the hole). The “inner” means that this observer “lives” in the superfluid background and uses “relativistic” coordinate transformation is

\[
\int t \approx (\tilde{R} - t) = r \approx \tilde{R} - t.
\]

Since from the point of view of the inner observer the speed of light (i.e. the speed of quasiparticles) is invariant (laboratory) velocities of radially propagating quasiparticles, moving outward and inward respectively

\[
v_\pm = \frac{dr}{dt} = \frac{dE}{dp_r} = \pm 1 + v_s.
\] (51)

Since from the point of view of the inner observer the speed of light (i.e. the speed of quasiparticles) is invariant quantity and does not depend on direction of propagation, for him the moment of arrival of pulse to \( r = 0 \) but \( t = (t_1 + t_2)/2 \), where \( t_2 \) is the time when the pulse reflected from \( r \) returns to observer at \( R \). Since \( t_2 - t_1 = \int_r^R dr/v_- + \int_r^R dr/v_+ \), one obtains for the time measured by inner observer as

\[
\tilde{t}(r, t) = \frac{t_1 + t_2}{2} = t + \frac{1}{2} \left( \int_r^R \frac{dr}{v_+} + \int_r^R \frac{dr}{v_-} \right) = t + \left( \frac{2}{v_s(r)} + \ln \frac{1 - v_s(r)}{1 + v_s(r)} \right) - \left( \frac{2}{v_s(R)} + \ln \frac{1 - v_s(R)}{1 + v_s(R)} \right),
\] (52)

which is just the Eq. (48) up to a constant shift.

In the complete absolute physical space-time of the laboratory the external observer can detect quasiparticles radially propagating into (but not out of) the black hole or out of (but not into) the white hole. The energy spectrum of the quasiparticles remains to be well determined both outside and inside the horizon. Quasiparticles cross the black hole horizon with the absolute velocity \( v_- = -1 - v_s = -2 \) i.e. with the double speed of light; \( r(t) = 1 - 2(t - t_0) \). In case of a white hole horizon one has \( r(t) = 1 + 2(t - t_0) \). On the contrary, from the point of view of the inner observer the horizon cannot be reached and crossed: the horizon can be approached only asymptotically for infinite time: \( r(t) = 1 + (r_0 - 1) \exp(-t) \). Such incompetence of the local observer, who ”lives” in the curved world of superfluid vacuum, happens because he is limited in his observations by the “speed of light”, so that the coordinate frame he uses is seriously crippled in the presence of the horizon and becomes incomplete.

The Schwarzschild metric naturally arises for the inner observer, if the Painlevé-Gullstrand metric is an effective metric in absolute space-time; however, in the presence of a horizon such metric indicates an instability of the underlying medium. To obtain a line element of Schwarzschild metric as an effective metric for quasiparticles, the quasiparticle energy spectrum in the laboratory frame has to be

\[
E^2 = c^2 \left( 1 - \frac{r_s}{r} \right)^2 p_r^2 + c^2 \left( 1 - \frac{r_s}{r} \right) p_\perp^2.
\] (53)
In the presence of a horizon such spectrum has sections of the transverse momentum $p_\bot$ with $E^2 < 0$. The imaginary frequency of excitations signals the instability of the superfluid vacuum if this vacuum exhibits the Schwarzschild metric as an effective metric for excitations: Quasiparticle perturbations may grow exponentially without bound in laboratory (Killing) time, as $e^{\imath t \omega}$, destroying the superfluid vacuum. Nothing of this kind happens in the case of the Painlevé-Gullstrand line element, for which the quasiparticle energy is real even behind the horizon. Thus the main difference between Painlevé-Gullstrand and Schwarzschild metrics as effective metrics is: The first metric leads to the slow process of the quasiparticle radiation from the vacuum at the horizon (Hawking radiation), while the second one indicates a crucial instability of the vacuum behind the horizon.

In general relativity it is assumed that the two metrics can be converted to each other by the coordinate transformation in Eq. (18). In condensed matter the coordinate transformation leading from one metric to another is not that innocent if an event horizon is present. The reason why the physical behaviour implied by the choice of metric formation in Eq. (48). In condensed matter the coordinate transformation leading from one metric to another is not everywhere regular coordinate transformation. Painlevé-Gullstrand metrics for black and white holes are determined everywhere, but belong to two different classes. The transition between these two metrics occurs via the singular transformation $t \rightarrow t + \int^r dr v_r/(c^2 - v_r^2)$, is singular on the horizon, and thus it can be applied only to a part of the absolute space-time. In condensed matter, only such effective metrics are physical which are determined everywhere in the real physical space-time. The two representations of the “same” metric cannot be strictly equivalent metrics, and we have different classes of equivalence, which cannot be transformed to each other by everywhere regular coordinate transformation. Painlevé-Gullstrand metrics for black and white holes are determined everywhere, but belong to two different classes. The transition between these two metrics occurs via the singular transformation $t \rightarrow t + 2 \int^r dr v_r/(c^2 - v_r^2)$ or via the Schwarzschild line element, which is prohibited in condensed matter physics, as explained above, since it is pathological in the presence of a horizon: it is not determined in the whole space-time and it is singular at horizon.

F. Incompleteness of space-time in effective gravity.

It is also important that in the effective theory there is no need for the additional extension of space-time to make it geodesically complete. The effective space time is always incomplete (open) in the presence of horizon, since it exists only in the low energy “relativistic” corner and quasiparticles escape this space-time to a nonrelativistic domain when their energy increase beyond the relativistic linear approximation regime. In the presence of a horizon such spectrum has sections of the transverse momentum $p_\bot$ with $E^2 < 0$. The imaginary frequency of excitations signals the instability of the superfluid vacuum if this vacuum exhibits the Schwarzschild metric as an effective metric for excitations: Quasiparticle perturbations may grow exponentially without bound in laboratory (Killing) time, as $e^{\imath t \omega}$, destroying the superfluid vacuum. Nothing of this kind happens in the case of the Painlevé-Gullstrand line element, for which the quasiparticle energy is real even behind the horizon. Thus the main difference between Painlevé-Gullstrand and Schwarzschild metrics as effective metrics is: The first metric leads to the slow process of the quasiparticle radiation from the vacuum at the horizon (Hawking radiation), while the second one indicates a crucial instability of the vacuum behind the horizon.

Another example of the incomplete space-time in effective gravity is provided by vierbein walls, or walls with the degenerate metric. The physical origin of such walls with the degenerate metric $g^{\mu\nu}$ in general relativity has been discussed by Starobinsky at COSMION-99. They can arise after inflation, if the inflaton field has a $Z_2$ degenerate vacuum. The domain walls separates the domains with 2 different vacua of the inflaton field. The metric $g^{\mu\nu}$ can everywhere satisfy the Einstein equations in vacuum, but at the considered surfaces the metric $g^{\mu\nu}$ cannot be diagonalized as $g^{\mu\nu} = \text{diag}(1,1,1,1)$. Instead, on such surface the metric is diagonalized as $g^{\mu\nu} = \text{diag}(1,0,0,0)$ and thus cannot be inverted. Though the space-time can be flat everywhere, the coordinate transformation cannot remove such a surface: it can only move the surface to infinity. Thus the system of such vierbein domain walls divides the space-time into domains which cannot communicate with each other. Each domain is flat and infinite as viewed by a local observer living in a given domain. In principle, the domains can have different space-time topology, as is emphasized by Starobinsky.

In $^3$He-A such walls appear in a film of the $^3$He-A, which simulates the 2+1 vacuum. The wall is the topological solitons on which one of the vectors (say, $e_1$) of the order parameter playing the part of the vierbein in general relativity, changes sign across the wall:

$$e_1(x) = \dot{x} c_0 \tanh x, \quad e_2 = \dot{y} c_0.$$  \hspace{1cm} (54)

The corresponding 2+1 effective metric experienced by quasiparticles, is

$$ds^2 = -dt^2 + \frac{1}{c_0^2} (dx^2 \tanh^{-2} x + dy^2).$$  \hspace{1cm} (55)

Here $c_0$ is “speed of light” at infinity. The speed of “light” propagating along the axis $x$ becomes zero at $x = 0$, and thus $g_{xx}(x = 0) = \infty$. This indicates that the low-energy quasiparticles cannot propagate across the wall. The coordinate singularity at $x = 0$ cannot be removed by the coordinate transformation. If at $x > 0$ one introduces a new coordinate $\tilde{x} = \int dx / \tanh x$, then the line element acquires the standard flat form

$$ds^2 = -dt^2 + d\tilde{x}^2 + dy^2.$$  \hspace{1cm} (56)
This means that for the “inner” observer, who measures the time and distances using the quasiparticles, his space-time is flat and infinite. But this is only half of the real (absolute) space-time: the other domain – the left half-space at $x < 0$ – which is removed by the coordinate transformation, remains completely unknown to the observer living in the right half-space. The situation is thus the same as discussed by Starobinsky for the domain wall in the inflaton field [22].

Thus the vierbein wall divides the bulk liquid into two classically separated flat “worlds”, when viewed by the local “inner” observers who use the low energy “relativistic” quasiparticles for communication. Such quasiparticles cannot cross the wall in the classical limit, so that the observers living on different sides of the wall cannot communicate with each other. However, at the “Planck scale” the quasiparticles have a superluminal dispersion in $^3$He-A, so that quasiparticles with high enough energy can cross the wall. This is an example of the situation, when the effective space-time which is complete from the point of view of the low energy observer appears to be only a part of the more fundamental underlying space-time. It is interesting that when the chiral fermionic quasiparticles of $^3$He-A crosses the wall, its chirality changes to the opposite [23]: the lefthanded particle viewed by the observer in one world becomes the righthanded particle in the hidden neighbouring world.

V. CONCLUSION.

We considered here only small part of the problems which arise in the effective gravity of superfluids. There are, for example, some other interesting effective metrics, which must be exploited. Quantized vortices with circulating superfluid velocity around them simulate the spinning cosmic strings, which experience the gravitational Aharonov-Bohm effect measured in superfluids as Iordanskii force acting on the vortex [24]: the superfluid vacuum around the rotating cylinder simulates [4] Zel’’dovich-Starobinsky effect of radiation by the dielectric object or black hole rotating in quantum vacuum [25]. The expanding Bose condensate in the laser manipulated traps, where the speed of sound varies in time, may simulate the inflation. The practical realization of the analogue of event horizon, the observation of the Hawking radiation and measurement of the Bekenstein entropy still remain a challenge for the condensed matter physics (see Ref. [27] for review of different proposals). However, even the theoretical consideration of the effective gravity in condensed matter can give insight into many unsolved problems in quantum field theory. We can expect that the analysis of the condensed matter analogues of the effective gravity, in particular, of the Landau-Khalatnikov two-fluid hydrodynamics [1] and its extensions will allow us to solve the longstanding problem of the cosmological constant.

[1] I.M. Khalatnikov: An Introduction to the Theory of Superfluidity, (Benjamin, New York, 1965).
[2] S. Weinberg, What is quantum field theory, and what did we think it is? hep-th/9702027.
[3] S. Chadha and H.B. Nielsen, Lorentz Invariance as a Low-Energy Phenomenon, Nucl. Phys. B 217, 125-144 (1983).
[4] G.E. Volovik, Field theory in superfluid $^4$He: What are the lessons for particle physics, gravity and high-temperature superconductivity?, Proc. Natl. Acad. Sci. USA 96, 6042 - 6047 (1999); G.E. Volovik, $^3$He and Universe parallelism, in “Topological Defects and the Non-Equilibrium Dynamics of Symmetry Breaking Phase Transitions”, Yu. M. Bunkov, H. Godfrin (Eds.), pp. 353-387 (Kluwer, 2000); cond-mat/9902171.
[5] R.B. Laughlin and D. Pines, The Theory of Everything, unpublished.
[6] B.L. Hu, General Relativity as Geometro-Hydrodynamics, Expanded version of an invited talk at 2nd International Sakharov Conference on Physics, Moscow, 20 - 23 May 1996, e-Print Archive: gr-qc/9607070.
[7] T. Padmanabhan, Conceptual issues in combining general relativity and quantum theory, hep-th/9812018.
[8] G.E. Volovik, Axial anomaly in $^3$He-A: Simulation of baryogenesis and generation of primordial magnetic field in Manchester and Helsinki, Physica B 255, 86-107 (1998).
[9] D. Vollhardt, and P. Wölfle, The superfluid phases of helium 3, Taylor and Francis, London - New York - Philadelphia, 1990.
[10] W. G. Unruh, Experimental Black-Hole Evaporation?, Phys. Rev. Lett. 46, 1351-1354 (1981); Sonic analogue of black holes and the effects of high frequencies on black hole evaporation, Phys. Rev. D 51, 2827-2838 (1995).
[11] M. Visser, Acoustic black holes: horizons, ergospheres, and Hawking radiation, Class. Quantum Grav. 15, 1767-1791 (1998).
[12] C.W. Woo, Microscopic calculations for condensed phases of helium, in: The Physics of Liquid and Solid Helium, Part I, eds. K.H. Bennemann and J.B. Ketterson (John Wiley & Sons, New York, 1976).
[13] A. D. Sakharov, Vacuum Quantum Fluctuations in Curved Space and the Theory of Gravitation, Sov. Phys. Dokl. 12, 1040-41 (1968).
[14] U.R. Fischer, and G.E. Volovik, Thermal quasi-equilibrium states across Landau horizons in the effective gravity of superfluids, gr-qc/0003017.
[15] R.C. Tolman, Relativity, Thermodynamics and Cosmology (Clarendon Press, Oxford, 1934).
[16] T. A. Jacobson and G. E. Volovik, Event horizons and ergoregions in $^3$He, Phys. Rev. D 58, 064021 (1998); Effective space-time and Hawking radiation from a moving domain wall in a thin film of $^3$He-A, JETP Lett. 68, 874-880 (1998).
[17] K. Martel and E. Poisson, Regular coordinate systems for Schwarzschild and other spherical spacetimes, gr-qc/0001069.
[18] S. Liberati, S. Sonego and M. Visser, Unexpectedly large surface gravities for acoustic horizons? gr-qc/0003105.
[19] L.D. Landau and E.M. Lifshitz, Fluid Mechanics, p. 317, Pergamon Press, 1989.
[20] N.B. Kopnin and G.E. Volovik, Critical velocity and event horizon in pair-correlated systems with "relativistic" fermionic quasiparticles, JETP Lett. 67, 140-145 (1998).
[21] G. E. Volovik, Simulation of Painlevé-Gullstrand black hole in thin $^3$He-A film, JETP Lett. 69, 705-713 (1999).
[22] A. Starobinsky, Plenary talk at Cosmion-99, Moscow, 17-24 October, 1999.
[23] G.E. Volovik, Vierbein walls in condensed matter, JETP Lett. 70, 711-716 (1999).
[24] G.E. Volovik, Vortex vs spinning string: Iordanskii force and gravitational Aharonov-Bohm effect, JETP Letters 67, 881 - 887 (1998).
[25] Ya.B. Zel’dovich, Amplification of cylindrical electromagnetic waves reflected from rotating body, Sov. Phys. JETP 35, 1085 (1971).
[26] A.A. Starobinskii, Amplification of waves during reflection from a rotating "black hole" Sov. Phys. JETP, 37, 28 (1973).
[27] T.A. Jacobson, Trans-Planckian redshifts and the substance of the space-time river, hep-th/0001083.