Chaotic dynamics size-dependent flexible rectangular flat shells

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Abstract. In this paper was constructed mathematical model nonlinear vibrations flexible size-dependent spherical rectangular shell. Sheath viewed as a continuum Cosserat with constrained rotation of the particles (pseudo-continuum). The equations of motion of the shell element and the boundary conditions are obtained from the Ostrogradski-Hamilton energy principle on the basis of Kirchhoff-Love kinematic hypotheses. The geometric nonlinearity is taken into account by the model of T. von Karman. The equations of motion of the shell element in the work are written in a mixed form. A system of nonlinear partial differential equations reduces to an ODE system by the Bubnov-Galerkin method in higher approximations. The system is regarded as a system with an infinite number of degrees of freedom. The Cauchy problems are solved by methods of Runge-Kutta type from the second to the eighth order of accuracy. The convergence of each method depending on the time step and the number of terms in the expansion series of functions in the Bubnov-Galerkin method. The influence of the size-dependent parameter to the nonlinear dynamics of the rectangular plane into a spherical shell. The largest Lyapunov exponent is determined using three methods: Wolf, Kantz, and Rosenstein to prove the truth of chaos.

1. Basic definitions of the modified moment theory
In the modified moment theory [1] stored strain energy $U_1$ in an elastic body, which occupies the area $\Omega = \{0 \leq x \leq a; 0 \leq y \leq b; -h/2 \leq z \leq h/2\}$, for infinitesimal deformations are written in the form

$$U_1 = \frac{1}{2} \int \Omega \left( \sigma_{ij} \varepsilon_{ij} + m_{ij} \chi_{ij} \right) dv,$$

where: $\varepsilon_{ij}$ - components of the strain tensor and $\chi_{ij}$ - symmetrical curvature tensor components of the gradient, which are defined as

$$\varepsilon_{ij} = \frac{1}{2} \left( u_{i,j} + u_{j,i} + u_m \delta_{ij} \right),$$

$$\chi_{ij} = \frac{1}{2} \left( \theta_{i,j} + \theta_{j,i} \right),$$

here $u_i$ represents the components of the displacement vector $\mathbf{u}$, $\theta$ is an infinitesimal rotation vector with components $\theta_i$ and $\delta_{ij}$ - Kronecker symbol. For a linear isotropic elastic material of stress,
caused by kinematic parameters, which are included in the expression for the energy density, are determined by the following equations of state [1]:
\[ \sigma_{ij} = \lambda \varepsilon_{mm} \delta_{ij} + 2 \mu \varepsilon_{ij}, \]
\[ m_{ij} = 2 \mu \chi_{ij}, \]
where \( \sigma_{ij}, \varepsilon_{ij}, m_{ij} \) and \( \chi_{ij} \) denote a classical component of the stress tensor \( \sigma \), strain tensor \( \varepsilon \), deviator part of the symmetric moment tensor of higher order \( m \) and the symmetric part of the curvature tensor \( \chi \), respectively; \( \lambda = \frac{E_v}{(1 + \nu)(1 - 2\nu)}, \mu = \frac{E}{2(1 + \nu)} \) - Lame parameters; \( E(x, y, z) \), \( \nu(x, y, z) \) - Young's modulus and Poisson's ratio, respectively, \( \rho(x, y, z) \) - the density of the shell material; \( \varepsilon_i \) - intensity of deformation.

The parameter \( l \), appearing in the moment of higher order \( m_{ij} \), it represents additional independent material length parameter associated with the rotation symmetric tensor gradient. In this model, in addition to the usual Lamé parameters, it is necessary to take into account another scale length parameter \( l \) [1]. This is a direct consequence of the fact that in stress elasticity, strain energy density function is the strain tensor and symmetric curvature tensor. It does not depend explicitly on the rotation (asymmetric part of the deformation gradient), and the asymmetric part of the curvature tensor [1].

In the framework of the modified moment theory, let us consider a flexible spherical shell on a rectangular plane with constant rigidity and density under the action of alternating external pressure. In a rectangular coordinate system, the three-dimensional region can be written as: \( \Omega = \{x_1, x_2, x_3 \mid (x_1, x_2) \in [0; a] \times [0, b], x_3 \in [-h, h] \}, \quad 0 \leq t < \infty \) (Fig. 1). The initial equations are the equations of the theory of shallow shells, recorded in a dimensionless form [2]:
\[ \nu \lambda^4 w - \frac{1}{12(1 + \nu)(1 - 2\nu)} + \nu^2 \frac{1}{2(1 + \nu)} \left( \nu^2 A \right) = -k_{x_2} \frac{\partial^2 F}{\partial x_1^2} - k_{x_1} \frac{\partial^2 F}{\partial x_2^2} - L(w, F) + \frac{\partial^2 w}{\partial t^2} + \varepsilon \frac{\partial w}{\partial t} - q(x_1, x_2, t) = 0, \]
\[ \nu \lambda^4 F + k_{x_2} \frac{\partial^2 w}{\partial x_1^2} + k_{x_1} \frac{\partial^2 w}{\partial x_2^2} + \frac{1}{2} L(w, w) = 0 \]
where \( \nu \lambda^4 = \frac{1}{2} \frac{\partial^4}{\partial x_1^4} + \lambda^2 \frac{\partial^4}{\partial x_1^2 \partial x_2^2} + \frac{\partial^4}{\partial x_1^2 \partial x_2^2}, \quad L(w, F) = \frac{\partial^2 w}{\partial x_1^2} \frac{\partial^2 F}{\partial x_2^2} + \frac{\partial^2 w}{\partial x_2^2} \frac{\partial^2 F}{\partial x_1^2} - 2 \frac{\partial^2 w}{\partial x_1 \partial x_2} \frac{\partial^2 F}{\partial x_1 \partial x_2} \)
known nonlinear operator, \( w \) and \( F \) - function of deflection and effort. System (1.1) is reduced to a dimensionless form using the following dimensionless parameters: \( \lambda = a/b; \quad x_1 = \tilde{x}_1, \quad x_2 = \tilde{x}_2; \quad k_{x_1} = a^2/R_{x_1} h, \quad k_{x_2} = b^2/R_{x_2} h \) - complex dimensionless shell parameter along the \( x_1 \) and \( x_2 \) respectively; \( w = 2h\tilde{w} \) - прогиб; \( F = E(2h)^3 \tilde{F} \) - function of effort; \( t = t_0 \tilde{t} \) - time; \( q = \frac{E(2h)^3}{a^2 b^2} q \) - external pressure; \( \varepsilon = (2h)^{-3} \) - medium damping coefficient, \( \gamma = \frac{1}{h} \) - size-dependent parameter. Dash over dimensionless parameters is omitted for simplicity. Here we introduce the following designations: \( a, b \) - shell sizes in terms of \( x_1 \) and \( x_2 \) respectively; \( \nu = 0.3 \) - Poisson's ratio.

To equations (1.1) we adjoin the boundary conditions:
1. Support for flexible incompressible (inextensible) ribs:
\[ w = 0; \quad \frac{\partial^2 w}{\partial x_1^2} = 0; \quad F = 0; \quad \frac{\partial^2 F}{\partial x_1^2} = 0 \quad \text{at} \quad x_1 = 0; 1; \quad w = 0; \quad \frac{\partial^2 w}{\partial x_2^2} = 0; \quad F = 0; \quad \frac{\partial^2 F}{\partial x_2^2} = 0 \quad \text{at} \quad x_2 = 0; 1 \]
\[ \text{Initial conditions } w(x_1, x_2) \big|_{t=0} = \phi_1(x_1, x_2), \quad \frac{\partial w}{\partial t} = \phi_2(x_1, x_2) \]
(1.3)
In what follows, we shall consider the case of applying a transverse external pressure, varying according to the harmonic law \( q(t) = q_0 \sin(\omega_p t) \), where \( \omega_p \) – frequency of the compelling force, \( q_0 \) – its amplitude.

2. The Bubnov-Galerkin method in higher approximations

We approximate the required functions which are the solution of the equations (1.1) the expression containing final number of any parameters, and we will present in the form performing two functions, one of which depends on time, and the other on coordinates satisfying the boundary condition (1.2):

\[
w(t) = \sum_{l=1}^{N} \sum_{j=1}^{N} A_{lj}(t) \varphi_{lj}(x_1, x_2), \quad F(t) = \sum_{l=1}^{N} \sum_{j=1}^{N} B_{lj}(t) \psi_{lj}(x_1, x_2)
\]  

(2.1)

This decision is based on the trial functions which are energetically orthonormalized such that

\[
\langle \nabla^4 \varphi_{lj}, \varphi_{mn} \rangle = \begin{cases} 0, & n \neq m, \quad i \neq j, \\ 1, & n = m, \quad i = j. \end{cases}
\]  

(2.2)

Coordinate systems \( \{ \varphi_{lj}(x_1, x_2), \psi_{lj}(x_1, x_2) \} \) we will choose so that functions \( \varphi_{lj}(x_1, x_2), \psi_{lj}(x_1, x_2) \) were for \( \forall i, j \) are linearly independent, continuous together with the private derivatives to the fourth order inclusive in the area \( \Omega \), and that \( \varphi_{lj}(x_1, x_2), \psi_{lj}(x_1, x_2) \) satisfied to one of the corresponding regional conditions (1.2)–(1.5), besides, it is required that \( \varphi_{lj}(x_1, x_2), \psi_{lj}(x_1, x_2) \) had property of completeness.

For convenience we will designate the left parts of the equations of system (1.1) for \( \Phi_1 \) and \( \Phi_2 \) respectively, then system (1.1) can be written in the form:

\[
\Phi_1(w, F, \frac{\partial^2 w}{\partial x_1^2}, \frac{\partial^2 w}{\partial x_1^2}, \ldots) + q(x_1, x_2, t) = 0, \quad \Phi_2(w, F, \frac{\partial^2 w}{\partial x_1^2}, \frac{\partial^2 w}{\partial x_1^2}, \ldots) = 0.
\]  

(2.3)

Applying the Bubnov-Galerkin procedure to (2.3), we obtain:

\[
\int_0^1 \int_0^1 \Phi_1 \varphi_{kl}(x_1, x_2) dx_1 dx_2 + \int_0^1 \int_0^1 q(x_1, x_2, t) \varphi_{kl}(x_1, x_2) dx_1 dx_2 = 0,
\]  

(2.4)

Using (2.4) into account, equations (2.3) are written in the form:

\[
\sum_{k,l} \sum_{j} A_{lj} I_{1,klij} - \sum_{k,l} \sum_{j} B_{lj} I_{2,klij} + \sum_{k,l} \sum_{j} q I_{3,klij} + \sum_{k,l} \sum_{j} A_{lj} \sum_{n} B_{nj} I_{4,klijn} - \sum_{k,l} \sum_{j} A_{lj} \sum_{n} A_{nj} I_{5,klijn} = 0,
\]  

(2.5)

Here the sign \( \sum \{ \} \) before each equation of the system (2.5) indicates that under this equation we mean the system kl of this kind of equations, and the integrals of the Bubnov-Galerkin procedure have the form:

\[
I_{4,klij} = \int_0^1 \int_0^1 L(\varphi_{lj}, \psi_{rs}) \varphi_{kl} dx_1 dx_2, \quad I_{6,klij} = \int_0^1 \frac{1}{2} L(\varphi_{lj}, \psi_{rs}) \psi_{kl} dx_1 dx_2.
\]  

(2.6)
\[ I_{4,6ij} = \int_0^1 \left\{ \frac{1 - \nu}{2(1 + \nu)(1 - 2\nu)} + \frac{v^2}{2(1 + \nu)} \right\} \] 

\[ \frac{1}{\lambda^2} \frac{\partial^2 \phi_{ji}}{\partial x_1^2} + \lambda^2 \frac{\partial^2 \phi_{kl}}{\partial x_2^2} + 2 \frac{\partial^2 \phi_{ji}}{\partial x_1 \partial x_2} \frac{\partial^2 \phi_{kl}}{\partial x_1 \partial x_2} \] 

\[ \nu_{kl} dx_1 dx_2 \]

The integrals (2.6), except perhaps, \( I_{3,klj} \). If a lateral load is applied not to the entire surface of the shell, calculated across the middle surface. After application of a method of Bubnov - Galerkin in the highest approximations on spatial coordinates we receive the system of the linear algebraic equations concerning coefficients \( B_{ij} \), which is solved by method of the return matrix which is solved by the inverse matrix method and the system of ordinary differential equations (ODE) of the second order with respect to the coefficients \( A_{ij} \), which is reduced to normal and is solved by a Runge-Kutta method:

\[ K_{ij} B_{ij} = F_1 (A_{ij}), \]  

\[ \frac{dA_{ij}}{dt} = X_{ij}, \]  

\[ \frac{dX_{ij}}{dt} + eX_{ij} = F_2 (A_{ij}, B_{ij}, t), \quad i = 1, N; j = 0, N, \]  

where \( K_{ij} \) in (2.7) – matrix of coefficients of linear algebraic system of the equations of rather unknown parameters \( B_{ij} \), \( F_1 (A_{ij}) \) – the column of free members depending on parameters \( A_{ij} \), (2.8) – the ODU normal system of the first order concerning unknown \( A_{ij} \) and \( X_{ij} \). The use of the inverse matrix method can significantly reduce the computation time, because there is no need to solve a system of linear algebraic equations at each time step of the Runge - Kutta method, since the matrix of the system (2.7) does not depend on the unknown coefficients \( A_{ij} \) and \( B_{ij} \), and remains constant. Only the column of free members changes. Therefore, at each step of the Runge - Kutta necessary to carry out a multiplication operation inverse matrix of the system (2.7) in the column of constant terms. This is an order of magnitude reduces the calculation time. The time step is chosen from the condition of stability of the solution by Runge’s rule.

3. Numerical experiment

As an example we will consider a spherical cover in the rectangular plan with uniform boundary conditions (1.2) and zero entry conditions (1.6). Geometric parameters of curvature \( k_{x_1} = 24, k_{x_2} = 24 \), \( \lambda = 1 \), dissipation factor \( \varepsilon = 1 \), Poisson's ratio \( \nu = 0.3 \). Represent \( \phi_{ij}, \psi_{ij} \) of (2.1) in the form of a product of two functions, each of which depends only on one argument satisfying the boundary conditions (1.2):
\[
\sum_{i=1}^{N} \sum_{j=1}^{N} A_{ij} (t) \sin(i\pi x_1) \sin(j\pi x_2), \quad F = \sum_{i=1}^{N} \sum_{j=1}^{N} B_{ij} (t) \sin(i\pi x_1) \sin(j\pi x_2)
\]

(2.9)

Cauchy problem is solved by several methods such as: Runge-Kutta 4th, 2nd-order Runge-Kutta-Felberg 4th order, cache-Karp method 4th order Runge-Kutta Prins Dormand eighth order, an implicit Runge-Kutta method of order 2nd and 4th order. As a result of the numerical experiment, the Runge-Kutta method of the fourth order was preferred, because when comparing the results obtained with this method with solutions obtained using higher-order methods, convergence was achieved, but the time for counting for the Runge-Kutta-Felberg methods 4th order, 4th order Kesh-Karp, eighth order Runge-Kutta Prince Dormand requires more, therefore the optimal method is the Runge-Kutta method of the 4th.

The convergence and comparison of each method, depending on the number of terms in the series expansion of functions in the Bubnov-Galerkin in higher approximations. By the definition of chaos given by Gulik [3] chaos exists when either there is an essential dependence on the initial conditions or the function has a positive largest Lyapunov exponent at each point of the domain of its definition and therefore is not ultimately periodic. In our studies below we follow the definition of chaos according to the work of Gulik [3]. The largest Lyapunov exponent is determined using three methods: Wolf [4], Kantz [5], and Rosenstein [6] to prove the truth of chaos. Figure 3.1 shows the convergence of the Bubnov-Galerkin method on the signal. Signals were calculated for the number of members of a number of decomposition of functions \( N = 3, 5, 7, 9, 11, 13, 15 \). For \( N \geq 7 \) chaotic signals completely coincide. Table 3.1 shows Fourier power spectrum. The frequency spectrum of the signal for \( N \geq 7 \) does not change. Thus, further studies were conducted for \( N = 7 \). All the Lyapunov exponents are positive, i.e. vibrations are chaotic.

![Fig. 3.1](image)

Table 3.1

| \( N \) | \( \omega_p \) |
|---|---|
| 3 | |
| 5 | |
| 7, 9, 11, 13, 15 | |
A study was conducted of the size-dependent parameter $\gamma = 0; 0.1; 0.5$. The effect of a dimension-dependent parameter on the nonlinear dynamics of a rectangular shell was studied. Consider the characteristics $w_{\text{max}}(q_0)$ to approximate the Bubnov-Galerkin method $N = 7$, where convergence is installed over the power spectrum and the signal at the center point. Along with dependencies $w_{\text{max}}(q_0)$ (fig. 3.2) it is necessary to give also scales of nature of fluctuations depending on the operating parameters. $\{q_0, \omega_p\}$, here $\omega_p = 25$ – excitation frequency, coincides with the frequency of natural vibrations. Identification of the type of a spherical shell vibrations in the construction of rectangular plan scales character vibrations $\{q_0, \omega_p\}$ for each signal $w(t)$ analysis was performed using the power spectrum $S(\omega)$ and largest Lyapunov exponents. The symbols is shown in the figure. As the results of the numerical experiments, an increase in the size-dependent parameter value decreases at intervals shell deflection $0 < q_0 \leq 10$ and $25 < q_0 \leq 165$. Scale character vibrations show that for $\gamma = 0; 0.1$ in the range from $0 < q_0 \leq 20$ vibrations - harmonic, curves on the graph $w_{\text{max}}(q_0)$ - smooth. From $20 < q_0 \leq 35 (\gamma = 0)$ and $20 < q_0 \leq 50$, there is a transition process to chaotic fluctuations. Further, under any load, the vibrations are chaotic. When $\gamma = 0.5$ zone of harmonic vibrations increases, the zone of chaotic vibrations begins with $q_0 \leq 100$.

![Graph](image)

**Symbols**

| Symbol   | Description                                      |
|----------|--------------------------------------------------|
| Harmonic vibrations on the frequency $\omega_p$ | $\omega_p$                                      |
| Independent frequencies and their linear combinations |                                               |
| Vibrations at frequencies $\omega_p/2^n$, where $n \in N$ | Chaos                                          |

Fig 3.2
4. Conclusions
1. A theory of size-dependent flexible spherical rectangular planar shells is constructed.
2. The convergence of the Bubnov-Galerkin method, depending on the number of terms of the series $N$.
3. It was found that an increase in the size-dependent parameter $\gamma$ vibrations have a smaller amplitude transition to chaotic vibrations occurs at higher loads than when $\gamma = 0$.

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