We consider integer programming problems in standard form $\max\{c^T x : Ax = b, \ x \geq 0, \ x \in \mathbb{Z}^n\}$ where $A \in \mathbb{Z}^{m \times n}$, $b \in \mathbb{Z}^m$, and $c \in \mathbb{Z}^n$. We show that such an integer program can be solved in time $(m \cdot \Delta)^O(m) \cdot \|b\|_\infty^2$, where $\Delta$ is an upper bound on each absolute value of an entry in $A$. This improves upon the longstanding best bound of Papadimitriou [27] of $(m \cdot \Delta)^O(m^2)$, where in addition, the absolute values of the entries of $b$ also need to be bounded by $\Delta$. Our result relies on a lemma of Steinitz that states that a set of vectors in $\mathbb{R}^m$ that is contained in the unit ball of a norm and that sum up to zero can be ordered such that all partial sums are of norm bounded by $m$.

We also use the Steinitz lemma to show that the $\ell_1$-distance of an optimal integer and fractional solution, also under the presence of upper bounds on the variables, is bounded by $m \cdot (2m \cdot \Delta + 1)^m$. Here $\Delta$ is again an upper bound on the absolute values of the entries of $A$. The novel strength of our bound is that it is independent of $n$.

We provide evidence for the significance of our bound by applying it to general knapsack problems where we obtain structural and algorithmic results that improve upon the recent literature.

CCS Concepts: • Theory of computation → Discrete optimization;

Additional Key Words and Phrases: Integer programming, dynamic program

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1 INTRODUCTION

Many algorithmic problems, most notably problems from combinatorial optimization and the geometry of numbers, can be formulated as an integer linear program. This is an optimization problem of the form

$$\max\{c^T x : Ax = b, \ x \geq 0, \ x \in \mathbb{Z}^n\}, \tag{1}$$

where $A \in \mathbb{Z}^{m \times n}$, $b \in \mathbb{Z}^m$, and $c \in \mathbb{Z}^n$. An integer program as we describe it above is in (equation) standard form. Any integer program in inequality form, i.e., $\max\{c^T x : Ax \leq b, \ x \in \mathbb{Z}^n\}$, can be
transformed into an integer program in standard form by duplicating variables and introducing slack variables. Unlike linear programming, integer programming is NP-complete [9].

Lenstra [24] has shown that an integer program in inequality form, with a fixed number of variables, can be solved in polynomial time. A careful analysis of his algorithm shows a time bound of \(2^{O(n^2)}\) times a polynomial in the length of the input that contains binary encodings of numbers. This has been improved by Kannan [22] to \(2^{O(n \log n)}\), which is the best asymptotic upper bound on the exponent of 2 in 30 years. The question whether this can be improved to \(2^{O(n)}\) belongs to one of the most prominent mysteries in the theory of algorithms. The current record on the constant hidden in the \(O\)-notation in the exponent is held by Dadush [13].

Papadimitriou [27] has provided an algorithm for integer programs in standard form that is, in some sense, complementary to the result of Lenstra and its improvement of Kannan. He considered the case of an integer program (1) in which the entries of \(A\) and \(b\) are bounded by \(\Delta\) in absolute value. His algorithm is pseudopolynomial if \(m\) is fixed and is thus a natural generalization of pseudopolynomial time algorithms to solve unbounded knapsack problems [18].

The algorithm is based on dynamic programming and can be briefly described as follows. First, one shows that, if Equation (1) is feasible and bounded, then Equation (1) has an optimal solution with components bounded by \(U = (n + 1)(m \cdot \Delta)^m\). The dynamic program is a maximum weight path problem on the (acyclic) graph with nodes

\[ V = \{0, \ldots, n\} \times \{-n \cdot \Delta \cdot U, \ldots, n \cdot \Delta \cdot U\}^m, \]

where one has an arc from \((j, b')\) to \((j + 1, b'')\) if \(b'' - b' = k \cdot a^{(j+1)}\) for some \(k \in \mathbb{N}_0\) and where \(a^{(j+1)}\) is the \(j + 1\)-st column of \(A\). The weight of this arc is \(k \cdot c_{j+1}\). The optimum solution corresponds to a longest path to the vertex \((n, b)\). The running time of this algorithm is linear in the size of the graph. The number of nodes of this graph is equal to \((n + 1) \cdot (2n\Delta U + 1)^m\) and this is at least \((m\Delta)^m\). The upper bound on the running time, as stated in [27] is

\[ O\left(n^{2m+2} \cdot (m\Delta)^m(m+1)(2m+1)\right). \]  

1.1 Contributions of This Article

We present new structural and algorithmic results concerning integer programs in standard form (1) using the Steinitz lemma (see Section 1.1 below).

(a) We show that the integer program (1) can be solved in time

\((m \cdot \Delta)^O(m) \cdot \|b\|^2_\infty\),

where \(\Delta\) is an upper bound on the entries of \(A\) only. This improves upon the \((m \cdot \Delta)^\Omega(m^2)\) running time of the algorithm of Papadimitriou. Recall that in the setting of Papadimitriou the entries of \(b\) are bounded by \(\Delta\) as well. This improvement addresses an open problem raised by Fomin et al. [15, 25].

We then consider integer programs of the form

\[ \max \{c^T x : Ax = b, \ 0 \leq x \leq u, \ x \in \mathbb{Z}^n\}, \]

where \(A \in \mathbb{Z}^{m \times n}, b \in \mathbb{Z}^m, u \in \mathbb{N}^n, \) and \(c \in \mathbb{Z}^n\) and \(|a_{ij}| \leq \Delta\) for each \(i, j\). Thus, we allow the variables of integer program (1) to be bounded from above by \(0 \leq x \leq u\) for some \(u \in \mathbb{N}^n\). In this setting, we show the following.

(b) We provide new bounds on the distance of an optimal vertex \(x^*\) of the LP-relaxation and an optimal solution of the integer program itself. More precisely, we show that there exists
Faster Algorithms for Integer Programming

Fig. 1. An example of a re-ordering satisfying the Steinitz bound for the $\ell_\infty$-norm. The vectors on the left have $\ell_\infty$-norm at most one and sum up to zero. These vectors are rearranged on the right such that the partial sums have $\ell_\infty$-norm bounded by 2.

an optimal solution $z^*$ of the integer program such that

$$\|z^* - x^*\|_1 \leq m \cdot (2 \cdot m \cdot \Delta + 1)^m$$

holds. A classical bound of Cook et al. [11] implies, in the standard-form setting, $\|z^* - x^*\|_\infty \leq n \cdot (\sqrt{m} \cdot \Delta)^m$ and thus $\|z^* - x^*\|_1 \leq n^2 \cdot (\sqrt{m} \cdot \Delta)^m$. Thus, our bound, which is independent of $n$, is an improvement by a factor of $n^2$ for integer programs in standard form and fixed $m$.

(c) We use this to generalize a recent bound on the absolute integrality gap for the case $m = 1$ by Aliev et al. [2] that states that $c^T(x^* - z^*) \leq \|c\|_\infty \cdot 2 \cdot \Delta$. Our distance bound shows that the absolute integrality gap is bounded by $\|c\|_\infty \cdot O(m^{m+1} \cdot O(\Delta)^m$.

(d) Our new distance bound yields an algorithm for integer programs in standard form that runs in time

$$n \cdot O(m)^{(m+1)^2} \cdot O(\Delta)^{m \cdot (m+1)} \cdot \log^2 (m \cdot \Delta).$$

For the unbounded and bounded knapsack problems where all items are of weight $\Delta_\alpha$ at most, we obtain algorithms that run in time $O(n \cdot \Delta_\alpha^2)$ and $O(n^2 \cdot \Delta_\alpha^2)$, respectively. This is an improvement by a factor of $n$ to the so far best bounds for this problem by Tamir [34].

Our techniques have been recently refined by Jansen and Rohwedder [21] who obtained better constants in the exponent of the running time of integer programs without upper bounds. We also want to mention a recent tight lower bound for integer programming. Knop et al. [23] prove that even for $\{0,1\}$-matrices, the running time of our algorithm is probably optimal. In a nutshell, an algorithm with better asymptotic running time in the exponent for unbounded integer programs would contradict the exponential time hypothesis. This improves the lower bounds of Fomin et al. [15].

1.2 The Steinitz Lemma

Our algorithms and structural results rely on a Lemma of Steinitz [33] that we now describe. Here $\| \cdot \|$ denotes an arbitrary norm of $\mathbb{R}^m$ (Figure 1).
Theorem 1.1 (Steinitz (1913)). Let \( x_1, \ldots, x_n \in \mathbb{R}^m \) such that
\[
\sum_{i=1}^{n} x_i = 0 \quad \text{and} \quad \|x_i\| \leq 1 \quad \text{for each} \ i.
\]
There exists a permutation \( \pi \in S_n \) such that all partial sums satisfy
\[
\left\| \sum_{j=1}^{k} x_{\pi(j)} \right\| \leq c(m) \quad \text{for all} \ k = 1, \ldots, n.
\]
Here \( c(m) \) is a constant depending on \( m \) only.

Steinitz showed \( c(m) \leq 2m \) (see also [7, 32]). It was later shown by Sevast’anov [31, 32] that the constant \( c(m) \leq m \). This is tight for asymmetric norms, i.e., general gauge functions [16]. However, this bound is not optimal for symmetric norms. In particular, Banaszczyk proved \( c(m) \leq m - 1 + 1/m \) (see [32]). It is a wide open question to understand the Steinitz constant for \( \ell_p \)-norms for \( p > 2 \). It is conjectured that the Steinitz constant should be \( O(\sqrt{m}) \) for the \( \ell_\infty \)-norm [7]. A proof of this conjecture or any asymptotic improvement would directly improve the bounds provided in this article and would provide tightness results in a variety of settings. The proof of the Steinitz lemma with constant \( c(m) = m \) is based on LP-techniques [16] and can be quickly summarized as follows. One constructs sets \( A_n \supset A_{n-1} \supset \cdots \supset A_m \) where \( A_n = \{1, \ldots, n\} \) and \( |A_k| = k \) for each \( k \) such that the following linear system, which is described by \( A_k \) with variables \( \lambda_i, \ i \in A_k \), is feasible for each \( k \):
\[
\sum_{i \in A_k} \lambda_i x_i = 0,
\]
\[
\sum_{i \in A_k} \lambda_i = k - m,
\]
\[
0 \leq \lambda_i \leq 1, \quad i \in A_k.
\]
For any permutation \( \pi \) with \( \{\pi(i)\} = A_i \setminus A_{i-1} \) for \( i = n, \ldots, m + 1 \), one has then for any \( k \geq m \)
\[
\left\| \sum_{i=1}^{k} x_{\pi(i)} \right\| = \left\| \sum_{i \in A_k} x_i \right\| = \left\| \sum_{i \in A_k} (1 - \lambda_i) x_i \right\|
\leq \sum_{i \in A_k} (1 - \lambda_i) = m.
\]
In the inequality, we used \( \|x_i\| \leq 1 \) for each \( i \) and in the first and second equations we used Equation (3). The sets \( A_k \) are constructed inductively as follows. \( A_n = \{1, \ldots, n\} \). If \( A_k \) has been constructed, where \( k > m \), one first notes that the system (3) is of course also solvable if the right-hand side \( k - m \) of the second constraint is replaced by \( k - 1 - m \). Once this replacement has been done, one observes that Equation (3) consists of \( m + 1 \) equations and the inequalities \( 0 \leq \lambda \leq 1 \). A vertex solution of Equation (3) has thus at most \( m + 1 \) fractional entries that sum up to a value less than \( m + 1 \). A vertex solution of Equation (3) must therefore have one entry equal to zero. Otherwise, the components of the vertex sum up to a value larger than \( k - 1 - m \). The set \( A_{k-1} \) is now the set \( A_k \) from which the index corresponding to the zero in the vertex solution has been removed.

The reader will notice some resemblance in spirit to the proof of the Beck-Fiala theorem in Discrepancy Theory [8, 26]. Discrepancy techniques have given improvements to the Steinitz and Beck-Fiala problem, when one allows a weak dependence on the number of vectors. In particular, Banaszczyk [3] proved a Beck-Fiala bound of \( O(\sqrt{k \log n}) \) for set systems on \( n \) elements with sparsity \( t \). He also derived an \( \ell_\infty \) Steinitz bound of \( O(\sqrt{m \log n}) \) for \( n \) vectors in dimension \( m \) [4]. For constructive versions of these bounds, we refer the interested reader to [5, 6].
We are not the first to apply the Steinitz lemma in the context of integer programming. Dash et al. [14] have shown that an integer program (1) can be solved in pseudopolynomial time if a certain parameter of the number of rows \( \tau \) is a function of \( m \), i.e., \( \tau = \tau(m) \). The interesting aspect of their algorithm is that it relies on linear programming techniques only. The number of inequalities in their linear program is bounded by an exponential in \( \tau(m) \). Buchin et al. [10] have shown that \( m^{m/2-o(m)} \leq \tau(m) \leq m^{m+o(m)} \), which then yields an algorithm for integer programming that is pseudopolynomial for fixed \( m \) but doubly exponential in \( m \). Their upper bound on \( \tau(m) \) is proved via the Steinitz lemma. We take a different path in applying the Steinitz lemma. We use it to derive more efficient dynamic programming formulations directly and indirectly via new proximity results between integer and linear programming optimal solutions.

2 A FASTER DYNAMIC PROGRAM

We now describe a dynamic programming approach to solve Equation (1) that is based on the Steinitz-type-lemma (Theorem 1.1) and which is more efficient than the original algorithm of Papadimitriou [27]. Let us first consider the feasibility problem, i.e., we have to decide whether there exists a non-negative integer vector \( z^* \in \mathbb{Z}_{\geq 0}^m \) such that \( Az^* = b \) holds. The solution \( z^* \) gives rise to a sequence of vectors \( v_1, \ldots, v_t \) such that each \( v_i \) is a column of \( A \) and

\[
v_1 + \cdots + v_t = b. \tag{4}
\]

The \( i \)-th column of \( A \) appears \( z_i^* \) times on the left of Equation (4) and \( t = \|z^*\|_1 \). This equation can be re-written as

\[
(v_1 - b/t) + \cdots + (v_t - b/t) = 0. \tag{5}
\]

Observe that the infinity norm of each \( v_i - b/t \) is at most \( 2 \Delta \). The Steinitz-type lemma implies that there exists a permutation \( \pi \) of the numbers \( 1, \ldots, t \) such that all partial sums of the sequence

\[
v_{\pi(1)} - b/t, \ldots, v_{\pi(t)} - b/t \tag{6}
\]

have infinity norm at most \( 2 \cdot m \cdot \Delta \). In other words, for each \( j \in \{1, \ldots, t\} \), one has

\[
\|v_{\pi(1)} + \cdots + v_{\pi(j)} - (j/t) \cdot b\|_{\infty} \leq 2 \cdot m \cdot \Delta. \tag{7}
\]

This implies that each partial sum of the sequence

\[
v_{\pi(1)}, \ldots, v_{\pi(t)} \tag{8}
\]

is contained in the set \( \mathcal{S} \subseteq \mathbb{Z}^m \) that consists of all points \( x \in \mathbb{Z}^m \) for which there exists a \( j \in \{1, \ldots, t\} \) with

\[
\|x - (j/t) \cdot b\|_{\infty} \leq 2 \cdot m \cdot \Delta. \tag{9}
\]

This set \( \mathcal{S} \) is as large as the number of integer points at distance at most \( 2 \cdot m \Delta \) from the line segment connecting \( 0 \) and \( b \). We now argue that this number is bounded from above by

\[
|\mathcal{S}| \leq (4m \cdot \Delta + 1)^m \cdot \|b\|_1. \tag{10}
\]

Let \( f \in \mathbb{R}^m \) be any point. Since \( 2m\Delta \) is an integer, the integer points at distance at most \( 2m\Delta \) from \( f \) are contained in the set of integer points at distance at most \( 2m\Delta \) from \( \lfloor f \rfloor \). Therefore, an upper bound on \( |\mathcal{S}| \) is the number of different integer vectors that can be obtained by rounding a point on the line-segment \((0, b)\) times \((4m\Delta + 1)^m\). The number of rounded integer points is at most \( \|b\|_1 \).

The partial sums of

\[
v_{\pi(1)}, \ldots, v_{\pi(t)} \tag{11}
\]

correspond to the nodes of a directed walk from \( 0 \) to \( b \) in the digraph \( D = (\mathcal{S}, A) \) where one has a directed arc \( xy \in A \) from \( x \in \mathcal{S} \) to \( y \in \mathcal{S} \) if \( y - x \) is a column of \( A \). If there exists a path from \( 0 \) to \( b \) in this digraph \( D \) on the other hand, then the arcs of the path define a multiset of columns of \( A \) summing up to \( b \).
How fast is this approach to solve the integer feasibility problem? The number of vertices $|\mathcal{V}|$ of the digraph is equal to $(4m \cdot \Delta + 1)^m \cdot ||b||_1$. The number of arcs $|\mathcal{A}|$ is bounded by $|\mathcal{V}| \cdot n$. The integer feasibility problem is an unweighted single-source shortest path problem that can be solved with breadth-first search in linear time \cite{1, 12}. Consequently, the integer feasibility problem in standard form (1) can be solved in time

$$|\mathcal{V}| \cdot n = O(m \cdot \Delta)^m \cdot ||b||_1 \cdot n.$$  

**Theorem 2.1.** Let $A \in \mathbb{Z}^{m \times n}$ and $b \in \mathbb{Z}^m$ be given and suppose that each absolute value of an entry of $A$ is bounded by $\Delta$. In time $O(m \cdot \Delta)^m \cdot ||b||_1 \cdot n$ one can compute a solution of 

$$Ax = b, \; x \in \mathbb{Z}^n_{\geq 0}$$

or assert that such a solution does not exist.

We next describe how to tackle the optimization problem (1). We introduce weights on the arcs of the digraph $D = (\mathcal{V}, A)$. The weight of the arc $xy$ is $c_i$ if $y - x$ is the $i$-th column of $A$. Down below, we will argue that the longest path in the thereby weighted digraph from 0 to $b$ corresponds to an optimal solution of Equation (1). The longest path problem in $D$ can be solved in time $O(|\mathcal{V}| \cdot |\mathcal{A}|)$ with the Bellman-Ford algorithm \cite{1}. Since 

$$|\mathcal{A}| \leq |\mathcal{V}| \cdot n,$$

our discussion below implies that the integer program (1) can be solved in time $O(n \cdot |\mathcal{V}|^2)$ provided that there do not exist positive cycles reachable from 0. The next lemma clarifies that such a positive cycle exists if and only if the feasible integer program (1) is unbounded.

**Lemma 2.2.** Suppose that (1) is feasible. The integer program (1) is unbounded if and only if $D$ contains a cycle of strictly positive length that is reachable from 0.

**Proof.** It follows from the theory of integer linear programming \cite{30} that Equation (1) is unbounded if and only if there exists an integer solution of $Ax = 0, x \geq 0, c^T x > 0$. Let $r^* \in \mathbb{Z}^n_{\geq 0}$ be such a solution. Using the Steinitz-type lemma in the spirit of the rearrangement (6) but with $b = 0$, $r^*$ corresponds to a (not necessarily simple) cycle in $D$ of positive length starting at 0. This proves the lemma. \hfill $\Box$

**Remark 1.** The reader might have noticed that $D$ contains a positive simple cycle that is reachable from 0 if and only if there exists a positive simple cycle in $D$ containing 0. The two cycles, however, might not be a translation of each other.

The algorithm to solve Equation (1) is now as follows. We first check the integer feasibility of Equation (1). Then we run a single-source longest path algorithm from 0 to the other nodes of $D$, in particular to $b$. If the algorithm detects a cycle of positive weight, we assert that Equation (1) is unbounded. Otherwise, the longest path from 0 to $b$ corresponds to an optimal solution of Equation (1). We therefore have proved the following theorem.

**Theorem 2.3.** The integer program (1) can be solved in time 

$$n \cdot O(m \cdot \Delta)^2 \cdot m \cdot ||b||_1^2,$$

where $\Delta$ is an upper bound on all absolute values of entries in $A$.

**Remark 2.**

(a) For an integer program in standard form, without upper bounds on the variables, we can assume that $A$ does not have repeated columns. Hence, $n = O(\Delta^m)$. The running time bound in Theorem 2.3 is $(m \cdot \Delta)^2 \cdot ||b||_1^2$.  

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(b) The longest path problem runs in linear time if the digraph $D$ does not have any cycles at all. This is, for example, the case when $A$ has only non-negative entries. In this case one has a running time of $O(m \cdot \Delta)^m \cdot \|b\|_1 \cdot n$ for the integer program (1). A well known example of such an integer program is the configuration IP for scheduling (see, e.g., [17, 19, 20]).

Remark 3. For the case in which $\Delta$ is an upper bound on the absolute values of the entries of both $A$ and $b$, the set $\mathcal{S}$ contains at most $(4m \cdot \Delta + 1)^m$ elements and the integer program (1) can be solved in time $n \cdot O(m \cdot \Delta)^2 m$ and in time $n \cdot O(m \cdot \Delta)^m$ if the digraph does not have any cycles.

3 PROXIMITY IN THE $\ell_1$-NORM

In this section, we provide the results (b) and (c). From now on we consider integer programs in standard form with upper bounds on the variables, where the absolute values of $A$ only need to be bounded by some integer $\Delta$. In other words, we consider a problem of the form

$$\max \{ c^T x : Ax = b, \ 0 \leq x \leq u, \ x \in \mathbb{Z}^n \},$$

where $A \in \mathbb{Z}^{m \times n}$, $b \in \mathbb{Z}^m$, $c \in \mathbb{Z}^n$, and $u \in \mathbb{N}^n$ such that $|a_{ij}| \leq \Delta$ for each $i, j$. We are interested in the distance between an optimal vertex of the LP-relaxation of Equation (10) and a closest integer optimum $z^*$ in the $\ell_1$-norm.

A previous bound, that has been useful in many algorithmic applications (see, for example [29]), was shown by Cook et al. [11]. In its full generality, it is concerned with the distance in the $\ell_\infty$ norm in the setting of an integer program in inequality form

$$\max \{ c^T x : Ax \leq b, \ x \in \mathbb{Z}^n \}.$$ (11)

We suppose that $A$ and $b$ are integral and that Equation (11) is feasible and bounded. Cook et al. [11] show that for any optimal solution $x^*$ of the linear programming relaxation there exists an optimal solution $z^*$ of the integer program with

$$\|x^* - z^*\|_\infty \leq n \cdot \delta,$$ (12)

where $\delta$ is the largest absolute value of the determinant of any square submatrix of $A$. By the Hadamard bound, see, e.g., [30], $\delta$ is bounded by $n^{n/2} \cdot \Delta^n$, where $\Delta$ is, as before, an upper bound on the absolute values of the entries of $A$.

Applied to an integer program in standard form (1), this result implies that, for a given optimal linear solution $x^*$ there exists an integer optimal solution $z^*$ such that $\|z^* - x^*\|_1 \leq n^2 \delta$. Since the Hadamard bound implies $\delta \leq m^{m/2} \Delta^m$,

$$\|z^* - x^*\|_1 \leq n^2 \cdot m^{m/2} \Delta^m.$$ (13)

Using the Steinitz lemma, we show next that

$$\|z^* - x^*\|_1 \leq m \cdot (2 \cdot m \cdot \Delta + 1)^m.$$ (14)

We will see in a later section how this leads to algorithms for integer programs in standard form with upper bounds on the variables. In the following, let $x^*$ and $z^*$ be optimal solutions of the linear programming relaxation of Equation (10) and of the integer program (10), respectively. A vector $y \in \mathbb{Z}^n$ is called a cycle of $(z^* - x^*)$ if $Ay = 0$ and

$$|y_i| \leq |(z^* - x^*)_i| \quad \text{and} \quad y_i \cdot (z^* - x^*)_i \geq 0 \quad \text{for each} \ i.$$ (14)
Lemma 3.1. Let $y$ be a cycle of $(z^* - x^*)$, then the following assertions hold.

(i) $z^* - y$ is a feasible integer solution of Equation (10).

(ii) $x^* + y$ is a feasible solution of the linear programming relaxation of Equation (10).

(iii) One has $c^T y \leq 0$.

Proof. We show (i) and (ii). Since $Ay = 0$ and $y$ is integral, we only need to verify that the bounds on the variables are satisfied.

If $(z^* - x^*)_i < 0$, then $y_i \leq 0$, and since $z^*$ and $x^*$ are feasible, one has

$$0 \leq z_i^* - y_i \leq z_i^* - (z^* - x^*)_i = x_i^* \leq u_i$$

and

$$u_i \geq x_i^* + y_i \geq x_i^* + (z^* - x^*)_i = z_i^* \geq 0,$$

if $(z^* - x^*)_i > 0$ is analogous.

To see (ii), note that $y_i > 0$ implies that $z_i^* > x_i^*$ and thus $x_i^*$ is not at the upper bound $u_i$. If $y_i < 0$, then $z_i^* < x_i^*$, which means that the lower bound $0 \leq x_i$ is not tight at $x^*$. Therefore, there exists an $\epsilon > 0$ such that $x^* + \epsilon y$ is a feasible solution of the linear program.

The assertion (iii) follows from the optimality of $x^*$ and (ii). \hfill \Box

Lemma 3.2. Let $x^*$ be an optimal solution of the linear programming relaxation of Equation (10) and let $z^*$ be an optimal integer solution of Equation (10) such that $\|z^* - x^*\|_1$ is minimal. There does not exist a cycle of $z^* - x^*$.

Proof. Suppose that $y$ is a cycle of $z^* - x^*$. By (i) and (iii) of Lemma 3.1, $z^* - y$ is also an optimal solution of the integer program (10). But $\|z^* - y - x^*\|_1 < \|z^* - x^*\|_1$ contradicts the minimality of $\|z^* - x^*\|_1$. \hfill \Box

We are now ready to apply the Steinitz-type lemma to derive a new bound on the $\ell_1$-distance between $x^*$ and $z^*$.

Theorem 3.3. Let $x^*$ be an optimal vertex solution of the linear programming relaxation of Equation (10). There exists an optimal solution $z^*$ of the integer program (10) such that

$$\|z^* - x^*\|_1 \leq m \cdot (2m \cdot \Delta + 1)^m.$$

Here, $\Delta$ is an upper bound on the absolute values of the entries in $A$.

Proof. Let $z^*$ be an optimal integer solution such that $\|z^* - x^*\|_1$ is minimal. In the following, we use the notation $[x_i^+]$ for the vector that one obtains from $x^*$ by rounding each component towards the corresponding component of $z^*$. More precisely, the $i$-th component of $[x_i^+]$ is set to

$$[x_i^+] = \begin{cases} 
[x_i^+]_i & \text{if } z_i^* > x_i^* \text{ and } \\
[x_i^+]_i & \text{if } z_i^* \leq x_i^* 
\end{cases}$$

and we denote the rest by $[x_i^+] = x^* - [x_i^+]$. Clearly, one has

$$A(z^* - [x_i^+]) - A(x_i^+) = 0.$$  \hfill (15)

We are now again in the setting of the Steinitz lemma where we have a sequence of vectors

$$v_1, \ldots, v_{t}, -A(x_i^+)$$  \hfill (16)

that sum up to zero. More precisely, this sequence is constructed as follows. Start with the empty sequence. For each column index $i$ append $|z_i^* - [x_i^+]_i|$ copies of $\text{sign}(z_i^* - [x_i^+]_i) \cdot a_i$ to the list, where $a_i$ is the $i$-th column of $A$. Finally, append $-A(x_i^*)$ to the list. Since $x^*$ has at most $m$ positive
entries, we conclude that \( \| -A[x^*] \|_\infty \leq \Delta \cdot m \) and that there are integer vectors \( w_1, \ldots, w_m \) of \( \ell_\infty \)-norm at most \( \Delta \) with
\[
-A[x^*] = w_1 + \cdots + w_m.
\]
This means that the sequence of vectors (16) can be expanded to a sequence
\[
v_1, \ldots, v_t, w_1, \ldots, w_m,
\]
where each vector is at most of \( \ell_\infty \)-norm \( \Delta \) and that sum up to the zero vector. Observe that
\[
t = \| z^* - \lfloor x^* \rfloor \|_1
\]
and that \( t + m \geq \| z^* - x^* \|_1 \). The Steinitz lemma implies that the sequence (17) can be re-arranged in such a way
\[
u_1, \ldots, u_{t+m},
\]
that for each \( 1 \leq k \leq t + m \) the partial sum \( p_k = \sum_{i=1}^{k} u_i \) satisfies
\[
\| p_k \|_\infty \leq m\Delta.
\]
We will now argue that there cannot be indices \( 1 \leq k_1 < \cdots < k_{m+1} \leq t + m \) with
\[
p_{k_1} = \cdots = p_{k_{m+1}},
\]
which implies that \( t + m \) is bounded by \( m \) times the number of integer points of norm at most \( m \cdot \Delta \) and therefore
\[
\| z^* - x^* \|_1 \leq m \cdot (2 \cdot m \cdot \Delta + 1)^m.
\]
Assume to this end that there exist \( m + 1 \) indices \( 1 \leq k_1 < \cdots < k_{m+1} \leq t + m \) satisfying Equation (20). This yields a partition of the sequence into \( m + 1 \) nonempty pieces that sum up to zero, namely:
\[
u_1, \ldots, u_{k_1}, u_{k_{m+1}+1}, \ldots, u_{t+m}
\]
and
\[
u_{k_{j+1}}, \ldots, u_{k_{j+1}}, j = 1, \ldots, m.
\]
One of these subsequences does not contain an element from \( \{ w_1, \ldots, w_m \} \), and hence are columns of \( A \) or negatives thereof. This corresponds to a cycle \( y \) of \( z^* - x^* \) which, by the minimality of \( \| z^* - x^* \|_1 \) and Lemma 3.2 is impossible.

3.1 Integrality Gaps of Integer Programs

Our bound of Theorem 3.3 directly leads to a bound on the (absolute) integrality gap of integer programs. This gap is \( c^T (x^* - z^*) \) and can, via Theorem 3.3, be bounded by
\[
c^T (x^* - z^*) \leq \| c \|_\infty \| z^* - x^* \|_1 \leq \| c \|_\infty m \cdot (2 \cdot m \cdot \Delta + 1)^m.
\]
An integer program (1) is called an unbounded knapsack problem if \( m = 1 \). In this case, Aliev et al. [2] show that one has
\[
c^T (x^* - z^*) \leq 2 \cdot \| c \|_\infty \cdot \Delta,
\]
which is asymptotically our bound for \( m = 1 \). They derived their bound using methods from the geometry of numbers. A careful analysis of our proof in the case \( m = 1 \) also yields the bound (22) exactly. More precisely, this follows since we can choose \( u_1 = w_1 \) in the Steinitz sequence (18). What is special about the one-dimensional case, is that any vector can be chosen as the first element. Clearly, \( w_1 \) cannot be re-visited as a partial sum. This implies \( \| z^* - x^* \|_1 \leq (2\Delta + 1) - 1 = 2\Delta \).
4 ALGORITHMIC IMPLICATIONS

We now devote our attention to dynamic programming algorithms for integer programs in standard form with upper bounds on the variables and where \(|a_{ij}| \leq \Delta\) for each \(i, j\). This setting has received considerable attention in the approximation algorithm community, especially for scheduling problems and the respective configuration LPs (see, e.g., [19, 20, 29]).

Our proximity result can now be used in a dynamic programming approach to solve an integer program in standard form with upper bounds on the variables (10). We first compute an optimal basic solution \(x^*\) of the LP-relaxation of Equation (10). In the following, we denote our bound on \(\|z^* - x^*\|_1\) by \(L'_1 = m \cdot (2 \cdot m \cdot \Delta + 1)^m\). Theorem 3.3 reveals that there exists an optimal integer solution \(z^*\) with

\[
\|z^* - [x^*]\|_1 \leq \|z^* - x^*\|_1 + \|x^* - [x^*]\|_1 \leq L'_1 + m =: L_1.
\]

After the variable transformation \(y = z - \lceil x^* \rceil\), one has to solve an integer program of the form

\[
\max c^T y \quad \text{s.t.} \quad Ay = A \cdot \{x^*\}, \\
-\ell_x^* \leq y \leq u_x^*, \\
\|y\|_1 \leq L_1, \\
y \in \mathbb{Z}^n,
\]

where \(\ell_x^* = \min\{L_1, \lceil x^* \rceil\}\) and \(u_x^* = \min\{L_1, u - \lceil x^* \rceil\}\). Notice that \(\|\ell_x^*\|_\infty \leq L_1\) and \(\|u_x^*\|_\infty \leq L_1\). The potential of the new proximity bound lies in the constraint on the \(\ell_1\)-norm. For \(y \in \mathbb{Z}^n\) that satisfies \(\|y\|_1 \leq L_1\), one has for each \(1 \leq k \leq n\)

\[
\left\| \sum_{i=1}^k y_i \cdot a_{i} \right\|_\infty \leq \Delta \cdot L_1. \quad (24)
\]

Let \(U \subseteq \mathbb{Z}^m\) be the set of integer vectors of infinity norm at most \(\Delta \cdot L_1\). The cardinality of \(U\) is equal to

\[
|U| = (2 \cdot \Delta L_1 + 1)^m = O(m \cdot \Delta)^m \cdot (m+1). \quad (25)
\]

To find the optimal \(y^*\), we build the following acyclic directed graph (see Figure 2). The nodes of the graph consist of a starting node \(s = 0\) and a target node \(t = A \cdot \{x^*\}\). Furthermore, we have \(n-1\) copies of the set \(U\) that we denote by \(U_1, \ldots, U_{n-1}\). The arcs are as follows.

There is an arc from \(s\) to a node \(v \in U_1\) if there exists an integer \(y_1\) such that

\[
y = y_1 \cdot a_1 \quad \text{and} \quad -\ell_1^* \leq y_1 \leq u_1^*
\]

holds. Again, \(a_1\) denotes the first column of \(A\). The weight of the arc is \(c_1 \cdot y_1\). There is an arc from a node \(u \in U_{i-1}\) to a node \(v \in U_i\) if there exists an integer \(y_i\) such that

\[
v = y_i \cdot a_i \quad \text{and} \quad -\ell_i^* \leq y_i \leq u_i^*
\]

holds. The weight of this arc is \(c_i \cdot y_i\). Finally, there is an arc from \(u \in U_{n-1}\) to \(t\) of weight \(y_n \cdot c_n\) if

\[
A\{x^*\} - u = y_n \cdot a_n \quad \text{and} \quad -\ell_n^* \leq y_n \leq u_n^*
\]

Fig. 2. An illustration of the directed acyclic graph to solve the integer program (10).
holds for some integer $y_m$. Clearly, a longest path in this graph corresponds to an optimal solution $y^*$ of the integer program (10). The out-degree of each node is bounded by $u_i^* + l_i^* \leq 2 \cdot L_1 + 1$. Therefore, the number of arcs is bounded by

$$n \cdot |U| \cdot (2 \cdot L_1 + 1) = n \cdot O(m)^{(m+1)^2} \cdot O(\Delta)^{m \cdot (m+2)},$$

which would lead to a corresponding running time of $n \cdot O(m)^{(m+1)^2} \cdot O(\Delta)^{m \cdot (m+2)}$ since longest path in an acyclic digraph can be computed in linear time in the number of nodes and arcs.

However, a standard technique can be applied to significantly decrease the number of arcs. This idea is based on the binary representation of an integer and is as follows. Imagine that, for each integer program of the form $(\Sigma_i \leq s_i, \Sigma_i c_i \leq L)$, an inductive argument applies and the conditions (i) and (ii) are satisfied.

Let us briefly comment on how to find these integers $s_1, \ldots, s_k$ such that

(i) for each $z \in [-L, U]$ there exist $y_1, \ldots, y_k \in \{0, 1\}$ such that

$$\sum_{j=1}^k y_j \cdot s_j = z.$$

(ii) For each choice of $y_1, \ldots, y_k \in \{0, 1\}$ one has

$$\sum_{j=1}^k y_j \cdot s_j \in [-L, U].$$

We can then replace the part of the digraph connecting $U_{i-1}$ and $U_i$ with $O(\log^2 (l_i^* + u_i^*))$ copies of $U$. Each copy is associated to a binary variable $y_j^i$ and an integer $s_j^i$ corresponding to the construction for the interval $[-l_i^*, u_i^*]$. We order them arbitrarily and have an arc from a node $u$ from one copy of $U$ to the node $v$ of its successor of weight zero, if $u = v$ and of weight $c_j \cdot s_j^i$ if the successor copy is associated to the variable $y_j^i$ and $v = u + c_j \cdot s_j^i$. In this way, the out-degree of each node is at most two and the total number of nodes and arcs is

$$n \cdot O(\log^2 L_1) \cdot |U| = n \cdot m \cdot O(m \cdot \Delta \cdot (m \Delta)^{m \cdot (m+1)}),$$

where we assume $\Delta \geq 2$.

We therefore have the following result.

**Theorem 4.1.** An integer program of the form (10) can be solved in time

$$n \cdot O(m)^{(m+1)^2} \cdot O(\Delta)^{m \cdot (m+1)} \cdot \log^2 (m \cdot \Delta)$$

if each component of $A$ is bounded by $\Delta$ in absolute value.

Let us briefly comment on how to find these integers $s_1, \ldots, s_k$ satisfying (i) and (ii). It is enough to show how to find them for an interval of the form $[0, U]$, since

$$[-L, U] = \{-x + y : x \in [0, L], y \in [0, U]\}.$$

Thus, let $[0, U]$ be a given interval. If $U = 2^k - 1$, one lets $s_j = 2^{j-1}$ for $j = 0, \ldots, k - 1$. If $U$ is not of this form, then let $p$ be the largest power of two less than or equal to $U$. For the interval $[0, p - 1]$, we use the construction from above. Now we are left with representing the interval $[0, U - p + 1]$ and concatenate the sequence of integers $s_j$ from both constructions. Since the interval $[0, U - p + 1]$ is half as long as $[0, U]$, an inductive argument applies and the conditions (i) and (ii) are satisfied.
4.1 Faster Algorithms for Integer Knapsack

The bounded knapsack problem is of the following kind:

\[ \max \{ c^T x : a^T x = \beta, \ 0 \leq x \leq u, \ x \in \mathbb{Z}^n \}, \tag{27} \]

where \( c, a, u \in \mathbb{Z}_{\geq 0}^n \) and \( \beta \in \mathbb{Z}_{>0} \). If the upper bound is \( u = \beta \cdot 1 \), then the knapsack problem is called unbounded. We let \( \Delta_a \) be an upper bound on the entries of \( a \).

Tamir [34] has shown that the unbounded and bounded knapsack problem can be solved in time \( O(n^2 \Delta_a^2) \) and in time \( O(n^3 \Delta_a^2) \), respectively. These running times were obtained by applying the proximity result of Cook et al. [11]. We now use our proximity bound to save a factor of \( n \) in each case.

Unbounded Knapsack. We begin with the unbounded knapsack problem. An optimal fractional vertex \( x^* \) has only one positive entry, \( x_1^* \) lets say, and by Theorem 3.3 there exists an optimal integer solution \( z^* \) with \( \| z^* - x^* \|_1 \leq 2 \cdot \Delta_a + 1 \). We can assume that \( x_1^* \geq 2 \cdot \Delta_a + 1 \) since otherwise \( \beta = O(\Delta_a^2) \) and an \( O(n \cdot \Delta_a^2) \) algorithm is obvious (see Remark 3). If \( y^* \) is an optimal solution of

\[ \max \{ c^T y : a^T y = (2 \cdot \Delta_a + 1 + \{ x_1^* \}) a_1, \ y \geq 0, \ y \in \mathbb{Z}^n \}, \tag{28} \]

then \( (y_1^* + \{ x_1^* \}) - (2 \cdot \Delta_a + 1, y_2^*, \ldots, y_n^*) \) is an optimal solution of the unbounded knapsack problem. Since all entries of \( a \) and \( (2 \cdot \Delta_a + 1) a_1 \) are positive and bounded by \( O(\Delta_a^2) \) one can solve the knapsack problem (28) in time \( O(n \cdot \Delta_a^2) \) (see again Remark 3 and notice that the digraph has no cycles as all integers are positive). Consequently, we have the following theorem.

**Theorem 4.2.** An unbounded knapsack problem (27) can be solved in time \( O(n \cdot \Delta_a^2) \).

Bounded Knapsack. Setting \( m = 1 \) in Theorem 4.1, we obtain a running time of

\[ O(n \cdot (\log \Delta)^2 \cdot \Delta^2), \]

which is already an improvement over the running time of Tamir’s algorithm if \( \log \Delta \leq n \). A running time of \( O(n^2 \cdot \Delta^2) \) can be obtained as follows. Again, we solve the linear programming relaxation of Equation (27) and obtain an optimal vertex solution \( x^* \). Following the notation from Section 4, we now have to solve an integer program of the form

\[ \max \{ c^T x : a^T x = \beta^*, \ -l^* \leq x \leq u^*, \ x \in \mathbb{Z}^n \}, \tag{29} \]

where \( \beta^* \) is an integer with \( 0 \leq \beta^* \leq \Delta_a \) and \( \| l^* \|_{\infty}, \| u^* \|_{\infty} \leq 2 \cdot \Delta_a + 1 \). This is equivalent to the bounded knapsack problem

\[ \max \{ c^T x : a^T x = \beta^* + \sum_i a_i \cdot l_i^*, \ 0 \leq y \leq l_i^* + u^*, \ x \in \mathbb{Z}^n \}. \tag{30} \]

The new right-hand side of this problem is \( O(n \cdot \Delta_a^2) \). Pferschy [28] has shown that a bounded knapsack problem in \( n \) variables and right-hand side \( \gamma \) can be solved in time \( O(n \cdot \gamma) \). Thus, the bounded knapsack problem can be solved in time \( O(n^2 \cdot \Delta_a^2) \).

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REFERENCES

[1] A. V. Aho, J. E. Hopcroft, and J. D. Ullman. 1974. The Design and Analysis of Computer Algorithms. Addison-Wesley, Reading.

[2] I. Aliev, M. Henk, and T. Oertel. 2017. Integrality gaps of integer knapsack problems. In International Conference on Integer Programming and Combinatorial Optimization. Springer, 25–38.

[3] W. Banaszczyk. 1998. Balancing vectors and Gaussian measures of n-dimensional convex bodies. Random Structures and Algorithms 12, 4 (1998), 351–360.

[4] W. Banaszczyk. 2012. On series of signed vectors and their rearrangements. Random Structures and Algorithms 40, 3 (2012), 301–316.

[5] N. Bansal, D. Dadush, and S. Garg. 2016. An algorithm for Komlos conjecture matching Banaszczyk’s bound. In IEEE 57th Annual Symposium on Foundations of Computer Science (FOCS’16). IEEE, 788–799.

[6] N. Bansal and S. Garg. 2017. Algorithmic discrepancy beyond partial coloring. In Proceedings of the 49th Annual ACM SIGACT Symposium on Theory of Computing. ACM, 914–926.

[7] I. Bárány. 2008. On the power of linear dependencies. In Building Bridges. Springer, 31–45.

[8] J. Beck and T. Fiala. 1981. “Integer-making” theorems. Discrete Applied Mathematics 3, 1 (1981), 1–8.

[9] I. Borosh and L. B. Treybig. 1976. Bounds on positive integral solutions of linear diophantine equations. Proceedings of the American Mathematical Society 55, 2 (1976), 299–304.

[10] K. Buchin, J. Matoušek, R. A. Moser, and D. Pálvölgyi. 2012. Vectors in a box. Mathematical Programming 135, 1–2 (2012), 323–335.

[11] W. Cook, A. M. H. Gerards, A. Schrijver, and E. Tardos. 1986. Sensitivity theorems in integer linear programming. Mathematical Programming 34 (1986), 251–264.

[12] T. H. Cormen, C. E. Leiserson, R. L. Rivest, and C. Stein. 2001. Introduction to Algorithms (2nd ed.). MIT Press, Cambridge, MA.

[13] D. N. Dadush. 2012. Integer Programming, Lattice Algorithms, and Deterministic Volume Estimation. Georgia Institute of Technology.

[14] S. Dash, R. Fukasawa, and O. Günlük. 2012. The master equality polyhedron with multiple rows. Mathematical Programming 132, 1–2 (2012), 125–151.

[15] P. F. Fomin, F. Panolan, M. Ramanujan, and S. Saurabh. 2016. Fine-grained complexity of integer programming: The case of bounded branch-width and rank. arXiv preprint arXiv:1607.05342 (2016).

[16] V. S. Grinberg and S. V. Sevast’yanov. 1980. Value of the Steinitz constant. Functional Analysis and Its Applications 14, 2 (1980), 125–126.

[17] D. S. Hochbaum and D. B. Shmoys. 1987. Using dual approximation algorithms for scheduling problems: Theoretical and practical results. Journal of the ACM 34, 1 (Jan. 1987), 144–162.

[18] O. H. Ibarra and C. E. Kim. 1975. Fast approximation algorithms for the knapsack and sum of subset problems. Journal of the Association for Computing Machinery 22, 4 (1975), 463–468.

[19] K. Jansen. 2010. An EPTAS for scheduling jobs on uniform processors: Using an MILP relaxation with a constant number of integral variables. SIAM Journal on Discrete Mathematics 24, 2 (2010), 457–485.

[20] K. Jansen, K.-M. Klein, and J. Verschae. 2016. Closing the gap for makespan scheduling via sparsification techniques. In 43rd International Colloquium on Automata, Languages, and Programming (ICALP ’16), volume 55 of LIPIcs. Schloss Dagstuhl-Leibniz-Zentrum fuer Informatik.

[21] K. Jansen and L. Rohwedder. 2018. On integer programming and convolution. In 10th Innovations in Theoretical Computer Science Conference (ITCS’19). Schloss Dagstuhl-Leibniz-Zentrum fuer Informatik.

[22] R. Kannan. 1987. Minkowski’s convex body theorem and integer programming. Mathematics of Operations Research 12, 3 (1987), 415–440.

[23] D. Knop, M. Pilipczuk, and M. Wrochna. 2018. Tight complexity lower bounds for integer linear programming with few constraints. arXiv preprint arXiv:1811.01296 (2018).

[24] H. W. Lenstra. 1983. Integer programming with a fixed number of variables. Mathematics of Operations Research 8, 4 (1983), 538–548.

[25] M. Lewenstein, S. Pettie, and V. Vassilevska Williams. 2017. Structure and hardness in p (Dagstuhl seminar 16451). In Dagstuhl Reports, volume 6:11. Schloss Dagstuhl-Leibniz-Zentrum fuer Informatik.

[26] J. Matoušek. 2009. Geometric Discrepancy: An Illustrated Guide, Vol. 18. Springer Science & Business Media.

[27] C. H. Papadimitriou. 1981. On the complexity of integer programming. Journal of the Association for Computing Machinery 28, 4 (1981), 765–768.

[28] U. Pferschy. 1999. Dynamic programming revisited: Improving knapsack algorithms. Computing 63, 4 (1999), 419–430.

[29] P. Sanders, N. Sivadasan, and M. Skutella. 2009. Online scheduling with bounded migration. Mathematics of Operations Research 34, 2 (2009), 481–498.

[30] A. Schrijver. 1986. Theory of Linear and Integer Programming. John Wiley.
[31] S. Sevast’janov. 1978. Approximate solution of some problems of scheduling theory. *Metody Diskretnogo Analiza* 32 (1978), 66–75.

[32] S. Sevast’janov and W. Banaszczyk. 1997. To the Steinitz lemma in coordinate form. *Discrete Mathematics* 169, 1–3 (1997), 145–152.

[33] E. Steinitz. 1913. Bedingt konvergente Reihen und konvexe Systeme. *Journal für die reine und angewandte Mathematik* 143 (1913), 128–176.

[34] A. Tamir. 2009. New pseudopolynomial complexity bounds for the bounded and other integer knapsack related problems. *Operations Research Letters* 37, 5 (2009), 303–306.

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