Meson Scattering Identifying the BCS-BEC Crossover at Quark Level

Shijun Mao and Pengfei Zhuang
Physics Department, Tsinghua University, Beijing 100084, China
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Instead of the fermion-fermion scattering which identifies the BCS-BEC crossover in cold atom systems, boson-boson scattering is measurable and characterizes the BCS-BEC crossover in QCD superfluid. We study $\pi-\pi$ scattering in a pion superfluid described by the Nambu–Jona-Lasinio model. We found that the amplitude of the scattering length drops down monotonously with decreasing isospin density and finally vanishes at the boundary of the phase transition. This indicates clearly a BCS-BEC crossover in the pion superfluid.

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There are two kinds of condensed states in usual fermion gas, the Bardeen–Cooper–Shrieffer condensation (BCS) of fermions where the pair size is large and the pairs overlap each other, and the Bose–Einstein condensation (BEC) of molecules where the pair size is small and the pairs are distinguishable. The BCS wave function can be generalized to arbitrary attraction which leads to a smooth crossover from BCS to BEC [\ref{1}]. In cold atom systems, the experimental observable to identify the BCS-BEC crossover is the $s$-wave scattering length between fermions, which is positive in BEC state and negative in BCS state [\ref{2}].

Recently the study on quantum chromodynamics (QCD) phase structure is extended to finite isospin density. For a QCD system at finite temperature and baryon and isospin density, the phase transitions include not only color deconfinement [\ref{3}], chiral symmetry restoration [\ref{3}] and color superconductor [\ref{4}], but also pion superfluid [\ref{5}]. The increasing isospin density induces a phase transition from normal nuclear matter to pion superfluid, due to the spontaneous isospin symmetry breaking. The critical isospin chemical potential at zero temperature and baryon density is exactly the pion mass [\ref{6}]. By analogy with the usual superfluid, the BCS-BEC crossover in pion superfluid can be theoretically described [\ref{3}, \ref{7}] by the quark chemical potential which is positive in BCS and negative in BEC, the size of the Cooper pair which is large in BCS and small in BEC, and the scaled pion condensate which is small in BCS and large in BEC. However, unlike the fermion-fermion scattering in cold atom systems, quarks are unobservable degrees of freedom, and thus the quark-fermion scattering can not be measured or used to experimentally identify the BCS-BEC crossover.

In pion superfluid, the pairs themselves, namely the pion mesons, are observable objects. One can measure the $\pi-\pi$ scattering to probe the properties of the pion condensate and in turn the BCS-BEC crossover. Since pions are Goldstone modes corresponding to the chiral symmetry spontaneous breaking, the $\pi-\pi$ scattering provides a direct way to link chiral theories and experimental data and has been widely studied in many chiral models [\ref{8}, \ref{9}]. Note that pions are also the Goldstone modes of the isospin symmetry spontaneous breaking, the $\pi-\pi$ scattering should be a sensitive signature of the pion superfluid phase transition.

While the perturbative QCD can well describe the properties of the new phases at extremely high temperature and density, the study on the phase transitions at moderate temperature and density depends on lattice QCD calculations [\ref{10}] and effective models with QCD symmetries. One of the widely used effective models is the Nambu–Jona-Lasinio (NJL) model [\ref{11}], which is originally inspired by the BCS theory and its version at quark level [\ref{12}] gives simple and direct description of the dynamic mechanisms of spontaneous chiral symmetry breaking, color symmetry breaking and isospin symmetry breaking. The $s$-wave $\pi-\pi$ scattering length is calculated in the model is consistent with the Weinberg limit and the experimental data in vacuum. In this Letter, we extend the calculation to finite isospin chemical potential and focus on its relation to the BCS-BEC crossover in the pion superfluid.

The Lagrangian density of the two flavor NJL model at quark level is defined as [\ref{12}]

$$\mathcal{L} = \bar{\psi} (i\gamma^\mu\partial_\mu - m_0 + \gamma_0 \mu) \psi + G \left( (\bar{\psi}\psi)^2 + (\bar{\psi}\gamma_5 \tau_i \psi)^2 \right),$$

with scalar and pseudoscalar interactions corresponding to $\sigma$ and $\pi$ excitations, where $m_0$ is the current quark mass, $G$ is the four-quark coupling constant with dimension $\text{GeV}^{-2}$, $\tau_i$ ($i = 1, 2, 3$) are the Pauli matrices in flavor space, and $\mu = \text{diag}(\mu_u, \mu_d) = \text{diag}(\mu_B/3 + \mu_I/2, \mu_B/3 - \mu_I/2)$ is the quark chemical potential matrix with $\mu_u$ and $\mu_d$ being the $u$- and $d$-quark chemical potentials and $\mu_B$ and $\mu_I$ the baryon and isospin chemical potentials. At $\mu_I = 0$, the Lagrangian density has the symmetry of $U_B(1) \otimes SU_I(2) \otimes SU_A(2)$, corresponding to baryon, isospin and chiral symmetry. At $\mu_I \neq 0$, the symmetries $SU_I(2)$ and $SU_A(2)$ are firstly explicitly broken down to $U_I(1)$ and $U_A(1)$, and then the nonzero pion condensate leads to a spontaneous breaking of $U_I(1)$, with pions as the corresponding Goldstone modes. At $\mu_B = 0$, the Fermi surface of $u(d)$ and anti-$d(u)$ quarks coincide and hence the condensate of $u$ and anti-$d$ is favored at $\mu_I > 0$ and the condensate of $d$ and anti-$u$ quarks is favored at $\mu_I < 0$. Finite $\mu_B$ provides a mismatch between the two Fermi surfaces and will reduce the pion condensation.

Introducing the chiral and pion condensates $\sigma = \langle \bar{\psi}\psi \rangle$ and $\pi = \langle \bar{\psi}\gamma_5 \tau_i \psi \rangle$ and taking them to be real, the quark propagator $S$ in mean field approximation can be expressed as a matrix in the flavor space

$$S^{-1}(p) = \begin{pmatrix} \gamma^\mu p_\mu + \mu_u \gamma_0 - m_q & 2iG\pi \gamma_5 \\ 2iG\pi \gamma_5 & \gamma^\mu p_\mu + \mu_d \gamma_0 - m_q \end{pmatrix},$$

(2)
with the dynamical quark mass \( m_q = m_0 - 2G\sigma \) generated by the chiral symmetry breaking. By diagonalizing the propagator, the thermodynamic potential can be simply expressed as a condensation part plus a summation part of four quasiparticle contributions \([6]\). The gap equations to determine the condensates \( \sigma \) (or quark mass \( m_q \)) and \( \pi \) can be obtained by the minimum of the thermodynamic potential.

In the NJL model, the meson modes are regarded as quantum fluctuations above the mean field. The two quark scattering via meson exchange can be effectively expressed at quark level in terms of quark bubble summation in RPA \([13]\). The quark bubbles are defined as

\[
\Pi_{mn}(k) = i \int \frac{d^4p}{(2\pi)^4} Tr (\Gamma_m S(p+k)\Gamma_n S(p))
\]

with indexes \( m, n = \sigma, \pi_+, \pi_-, \pi_0, n_0 \), where the trace \( Tr = Tr_c Tr_F Tr_D \) is taken in color, flavor and Dirac spaces, the four momentum integration is defined as \( \int d^4p/(2\pi)^4 = i \int_0^\infty d\omega \int d^3p/(2\pi)^3 \) with fermion frequency \( p_0 = i\omega_j = i(2j+1)\pi T \) \((j = 0, \pm 1, \pm 2, \cdots)\) at finite temperature \( T \), and the meson vertices are from the Lagrangian density \([14]\).

\[
\Gamma_m = \begin{cases} 1 & m = \sigma \\ i\gamma_5 & m = \pi_+ \\ i\gamma_5 & m = \pi_- \\ i\gamma_5 \gamma_3 & m = \pi_0 . \end{cases}
\]

Since the quark propagator \( S \) contains off-diagonal elements, we must consider all possible channels in the bubble summation in RPA. Using matrix notation for the meson polarization function \( \Pi(k) \) in the 4 \times 4 meson space, the meson propagator can be expressed as

\[
D(k) = \frac{2G}{1 - 2G\Pi(k)}.
\]

Since the isospin symmetry is spontaneously broken in the pion superfluid, the original meson modes \( \sigma, \pi_+, \pi_-, \pi_0 \) with definite isospin quantum number are no longer the eigen modes of the Hamiltonian of the system, the new eigen modes \( \overline{\sigma}, \overline{\pi}_+, \overline{\pi}_-, \overline{\pi}_0 \) are linear combinations of the old ones, their masses \( M_i(i = \overline{\sigma}, \overline{\pi}_+, \overline{\pi}_-, \overline{\pi}_0) \) are determined by poles of the meson propagator at \( k_0 = M_i \) and \( k = 0, \) det\([1 - 2G\Pi(M_i, 0)] = 0 \), and their coupling constants \( \mu_{ij} \) are defined as the residues of the propagator at the poles.

The condition for a meson to decay into a quark and a \( \overline{\pi} \) is that its mass lies above the \( q - \overline{\pi} \) threshold. From the pole equation, the heaviest mode in the pion superfluid is \( \overline{\sigma} \) and its mass is beyond the threshold value. As a result, there exists no \( \overline{\sigma} \) meson in the pion superfluid, and the coupling constant \( \mu_{\overline{\pi}\overline{\sigma}} \) drops down to zero at the critical point \( \mu_I^2 \) and keeps zero at \( \mu_I > \mu_I^2 \) \([10]\).

We now study \( \pi - \pi \) scattering in the pion superfluid. To the lowest order in \( 1/N_c \) expansion, where \( N_c \) is the number of colors, the invariant amplitude \( T \) of the \( \pi - \pi \) scattering is calculated from the box diagrams in \( s, t \) and \( u \) channels, as shown in Fig.1 for the \( s \) channel. Different from the calculation in normal state \([10]\), the \( \sigma \)-exchange diagrams vanish here due to the disappearance of the \( \overline{\pi} \) meson in the pion superfluid. In the limit of the scattering at threshold \( \sqrt{s} = 2M_\pi \) and \( t = u = 0 \), the amplitude \( T \) approaches to the scattering length \( a \) \( \left( m_\pi^{-1} \right) \) in unit of the inverse of dynamical pion mass through the relation

\[
a = \frac{1}{32\pi} T(s = 4M_\pi^2, t = u = 0)
\]

Note that the threshold condition can be fulfilled by a simple choice of the pion momenta, \( k_\sigma = k_\rho = k_\pi = k_\rho = k = k^2 = M_\pi^2 = s/4 \), which facilitates a straightforward computation of the diagrams. Here the meson dynamical mass \( m_\pi \) is obtained from the quark and meson pole masses and the quark dynamical mass. For instance, we have \( m_\overline{\pi}_+ = M_\overline{\pi}_+ + \left( m_q - \sqrt{(m_q - \mu_I/2)^2 + 4G^2\mu_I^2} \right) \).

In the calculation in normal matter, people are interested in the \( \pi - \pi \) scattering length with definite isospin, \( a_{I=0,2} \), which can be measured in experiments due to isospin symmetry. However, finite isospin density breaks down the isospin symmetry and makes the scattering length \( a_{0,2} \) not well defined. In fact, the new meson modes in the pion superfluid do not carry definite isospin quantum numbers. Unlike the chiral dynamics in normal matter, where the three degenerated pions are all the Goldstone modes corresponding to the chiral symmetry spontaneous breaking, the pion mass splitting at finite \( \mu_I \) results in only one Goldstone mode \( \overline{\pi}_+ \) with \( M_\overline{\pi}_+ = 0 \) in the pion superfluid. In the following, we focus on the scattering process of \( \overline{\pi}_+ + \overline{\pi}_+ \rightarrow \overline{\pi}_+ + \overline{\pi}_+ \) and calculate the scattering length \( a_+ \).

The scattering amplitude for any channel of the box diagrams can be expressed as

\[
iT_{s,t,u} = -2\gamma_{\overline{\pi}s,\overline{\pi}t} \int \frac{d^4p}{(2\pi)^4} xTr \left( \gamma_5 \gamma_2 S_1 \gamma_5 \gamma_2 S_2 \gamma_5 \gamma_2 S_3 \gamma_5 \gamma_2 S_4 \right)
\]

with the quark propagators \( S_1 = S_3 = S(p) \), \( S_2 = S(p + k) \), and \( S_4 = S(p - k) \) for the \( s \) and \( t \) channels and \( S_1 = S_3 = S(p) \) for the \( u \) channel. In the limit of threshold, \( k = (M_\overline{\pi}_+, 0) = (0, 0) \) in

![FIG. 1: The lowest order diagram for \( \pi - \pi \) scattering in the pion superfluid. The solid and dashed lines are respectively quarks and pions, and the dots denote meson-quark vertices.](Image)
the pion superfluid leads to $S_1 = S_2 = S_3 = S_4 = S(p)$ for all the three channels. Making the fermion frequency summation over the internal quark lines, the total amplitude at the threshold is simplified as

$$\mathcal{T}_+ = 18g_{\pi \pi}^4 \int \frac{d^3p}{(2\pi)^3} \left\{ \frac{1}{E^+} \left( f(E^+) - f(-E^+) \right) - E^+ \left( f'(E^+) + f'(-E^+) \right) \right\} + \frac{1}{E^3} \left[ f(E^-) - f(-E^-) \right] - E^- \left( f'(E^-) + f'(-E^-) \right),$$

(8)

where $E^\pm = E \pm \mu_B/3$ are the energies of the four quasiparticles with $E = \sqrt{(E^\pm + \mu_B/2)^2 + 4G^2 \pi^2}$, $E = \sqrt{p^2 + m^2}$, and $f(x)$ is the Fermi-Dirac distribution function $f(x) = (e^{x/T} + 1)^{-1}$ with the derivative $f'(x) = df/dx$.

Since the NJL model is non-renormalizable, we can employ a hard three momentum cutoff $\Lambda$ to regularize the gap equations for quarks and pole equations for mesons. In the following numerical calculations, we take the current quark mass $m_0 = 5$ MeV, the coupling constant $G = 4.93$ GeV$^{-2}$ and the cutoff $\Lambda = 653$ MeV[17]. This group of parameters correspond to the pion mass $m_\pi = 134$ MeV, the pion decay constant $f_\pi = 93$ MeV and the effective quark mass $M_q = 310$ MeV in the vacuum.

The above $\mu_I$-dependence of the meson-meson scattering length in the pion superfluid can be understood well from the point of view of BCS-BEC crossover. We recall that the BCS and BEC states are defined in the sense of the degree of overlapping among the pair wave functions. The large pairs in BCS state overlap each other, and the small pairs in BEC state are individual objects. Therefore, the cross section between two pairs should be large in the BCS state and approach zero in the limit of BEC. From our calculation shown in Fig.2, the $\pi - \pi$ scattering length is a characteristic quantity for the BCS-BEC crossover in pion superfluid. The overlapped quark-antiquark pairs in the BCS state at high isospin density have large scattering length, while in the BEC state at low isospin density with separable pairs, the scattering length becomes small. This provides an experimental observable for the BCS-BEC crossover at QCD level, analogous to the fermion scattering length in cold atom systems.

![Fig. 2: The scattering length $|a_\pi|$ for the process $\pi^+ + \pi^- \rightarrow \pi^+ + \pi^-$ as a function of isospin chemical potential $\mu_I$ in the pion superfluid phase.](image)

With increasing temperature, the pairs will gradually melt and the coupling constant $g_{\pi \pi}$ drops down in the hot medium, leading to a smaller scattering length at $T = 100$ MeV in comparison with the case at $T = 0$, as shown in Fig.2. To see the continuous temperature effect on the scattering length in the BCS and BEC states, we plot in Fig.3 $|a_\pi|$ as a function of $T$ at $\mu_I = 160$ and $\mu_I = 400$ MeV, still keeping $\mu_B = 0$. While the temperature dependence is similar in both cases, the involved physics is different. In the BCS state at $\mu_I = 160$ MeV, $|a_\pi|$ is large and drops down with increasing temperature and finally vanishes at the critical temperature $T_c = 188$ MeV. Above $T_c$ the system becomes a fermion gas with weak coupling and without any pair. In the BEC state at $\mu_I = 160$ MeV, the scattering length is much smaller. At a lower critical temperature $T_c = 136$ MeV, the condensate melts but the strong coupling between quarks makes the system be a gas of free pairs with $|a_\pi| = 0$.

![Fig. 3: The scattering length $|a_\pi|$ for the process $\pi^+ + \pi^- \rightarrow \pi^+ + \pi^-$ as a function of temperature $T$ in the pion superfluid phase.](image)

There are some other quantities to describe the BCS-BEC crossover in superfluid or superconductor [7], which are difficult to be experimentally measured but can be used to confirm the BCS-BEC crossover picture characterized by the pion scattering length. We calculate the scaled binding energy $\epsilon/\mu_I$ as a function of $\mu_I$
in pion superfluid at $T=0$ and $T=100$ MeV, shown in Fig. 4. The binding energy of $\pi^+$ is defined as the mass difference between $\pi^+$ and the two quarks, $\epsilon = M_{\pi^+} - M_{\pi} - M_{\pi^-}$. With decreasing isospin chemical potential, the binding energy increases, indicating a BCS-BEC crossover in the pion superfluid. When the medium becomes hot, the condensate melts and the pairs are gradually dissociated. These results support our conclusion of BCS-BEC crossover identified by the measurable meson scattering length.

In summary, we proposed the meson-meson scattering as an experimentally measurable probe of the BCS-BEC crossover at quark level. Different from the fermion-fermion scattering which is often used to experimentally identify the BCS-BEC crossover in cold atom systems, quark scattering cannot be measured and its function to characterize the BCS-BEC crossover in QCD superfluid is replaced by the meson scattering. In the BCS quark superfluid, the large and overlapped pairs lead to large pair-pair cross section, but the small and individual pairs in the BEC superfluid interact weakly. In the frame of a two flavor NJL model at finite temperature and isospin density, we calculated the $\pi - \pi$ scattering length in the pion superfluid. It is large at high isospin chemical potential and drops down monotonously with decreasing isospin chemical potential and finally approaches zero at the border of the pion superfluid, indicating a BCS-BEC crossover.

The meson scattering length $|a_\pi|$ shown in Figs. 2 and 3 are obtained in a particular model, the NJL model, which has proven rather reliable in the study on chiral, color and isospin condensates at low temperature. Since there is no confinement in the model, one may ask the question to what degree the conclusions can be trusted. From the general picture for BCS and BEC states, the feature that the meson scattering length approaches to zero in the process of BCS-BEC crossover can be geometrically understood in terms of the degree of overlapping between the two pairs. Therefore, the qualitative conclusion of taking meson scattering as a probe of BCS-BEC crossover at quark level may survive any model dependence.

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