Abstract

$C$-eigenvalues of piezoelectric-type tensors which are real and always exist, are introduced by Chen et al. [1]. And the largest $C$-eigenvalue for the piezoelectric tensor determines the highest piezoelectric coupling constant. In this paper, we give two intervals to locate all $C$-eigenvalues for a given Piezoelectric-type tensor. These intervals provide upper bounds for the largest $C$-eigenvalue. Numerical examples are also given to show the corresponding results.

Keywords: Piezoelectric tensors, $C$-eigenvalues, Interval.

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1. Introduction

Piezoelectric-type tensors are introduced by Chen et al. in [1] as a subclass of third order tensors which have extensive applications in physics and engineering [2, 3, 5, 6, 7, 9]. The class of Piezoelectric tensors, as the subclass of Piezoelectric-type tensors of dimension three, plays the key role in Piezoelectric effect and converse Piezoelectric effect [1].

Definition 1. [1, Definition 2.1] Let $A = (a_{ijk}) \in \mathbb{R}^{n \times n \times n}$ be a third-order $n$ dimensional real tensor. If the later two indices of $A$ are symmetric, i.e., $a_{ijk} = a_{ikj}$ for all $j \in N$ and $k \in N$ where $N := \{1, 2, \ldots, n\}$, then $A$ is called a piezoelectric-type tensor.
To explore more properties related to piezoelectric effect and converse piezoelectric effect in solid crystal, Chen et al. in [1] introduced $C$-eigenvalues and $C$-eigenvectors for Piezoelectric-type tensors, and shown that the largest $C$-eigenvalue corresponds to the electric displacement vector with the largest 2-norm in the piezoelectric electronic effect under unit uniaxial stress [1, 2, 8].

**Definition 2.** [1, Definition 2.2] Let $A = (a_{ijk}) \in \mathbb{R}^{n \times n \times n}$ be a piezoelectric-type tensor. If there exist a scalar $\lambda \in \mathbb{R}$, vectors $x \in \mathbb{R}^{n}$ and $y \in \mathbb{R}^{n}$ satisfying the following system

\[
Ayy = \lambda x, \quad xAy = \lambda y, \quad x^T x = 1 \text{ and } y^T y = 1,
\]

where $Ayy \in \mathbb{R}^{n}$ and $xAy \in \mathbb{R}^{n}$ with the $i$-th entry

\[
(Ayy)_i = \sum_{j,k \in N} a_{ijk} y_j y_k, \quad \text{and} \quad (xAy)_i = \sum_{j,k \in N} a_{jki} x_j y_k,
\]

respectively, then $\lambda$ is called a $C$-eigenvalue of $A$, $x$ and $y$ are called associated left and right $C$-eigenvectors, respectively.

For $C$-eigenvalues and associated left and right $C$-eigenvectors of a piezoelectric-type tensor, Chen et al. in [1] also provided several related results, such as:

**Property 1.** For a piezoelectric-type tensor $A$, there always exist $C$-eigenvalues of $A$ and associated left and right $C$-eigenvectors.

**Property 2.** Suppose that $\lambda$, $x$ and $y$ are a $C$-eigenvalue and its associated left and right $C$-eigenvectors of a piezoelectric-type tensor $A$. Then

\[
\lambda = xAy,
\]

where $xAy = \sum_{i,j,k \in N} a_{ijk} x_i y_j y_k$. Furthermore, $(\lambda, x, -y)$, $(-\lambda, -x, y)$ and $(-\lambda, -x, -y)$ are also $C$-eigenvalues and their associated $C$-eigenvectors of $A$.

**Property 3.** Suppose that $\lambda^*$ is the largest $C$-eigenvalue of a piezoelectric-type tensor $A$. Then

\[
\lambda^* = \max \left\{ xAy : x^T x = 1, y^T y = 1 \right\}.
\]

Property 2 and Property 3 provide theoretically the form to determine $C$-eigenvalues or the largest $C$-eigenvalue $\lambda^*$ of $A$. However, it is difficult to compute them in practice because determining $x$ and $y$ is not easy. So, we in this paper give some intervals to locate all $C$-eigenvalues of a piezoelectric-type tensor, and then give some upper bounds for the the largest $C$-eigenvalue. This can provide more information before calculating them out.
2. Main results

In this section, we give two intervals to locate all \(C\)-eigenvalues of a piezoelectric-type tensor. And the comparison of these two intervals are also established.

**Theorem 1.** Let \(A = (a_{ijk}) \in \mathbb{R}^{n \times n \times n}\) be a piezoelectric-type tensor, and \(\lambda\) be a \(C\)-eigenvalue of \(A\). Then

\[
\lambda \in [-\rho, \rho], \quad (2)
\]

where

\[
\rho := \max_{i,j \in \mathbb{N}} \left( R^{(1)}_i(A) R^{(3)}_j(A) \right)^{\frac{1}{2}},
\]

\[
R^{(1)}_i(A) := \sum_{l,k \in \mathbb{N}} |a_{ilk}| \quad \text{and} \quad R^{(3)}_j(A) := \sum_{l,k \in \mathbb{N}} |a_{lkj}|.
\]

**Proof.** Suppose that \(x = (x_1, x_2, \ldots, x_n)^T\) and \(y = (y_1, y_2, \ldots, y_n)^T\) are left and right \(C\)-eigenvectors corresponding to \(\lambda\) with \(x^T x = 1\) and \(y^T y = 1\). Let

\[
|x_p| = \max_{i \in \mathbb{N}} |x_i|, \quad \text{and} \quad |y_q| = \max_{i \in \mathbb{N}} |y_i|.
\]

Then \(0 < |x_p| \leq 1\) and \(0 < |y_q| \leq 1\) because \(x^T x = 1\) and \(y^T y = 1\).

By considering the \(p\)-th equation of \(Ay = \lambda x\) in (1), we have

\[
\lambda x_p = \sum_{j,k \in \mathbb{N}} a_{pjk} y_j y_k, \quad (3)
\]

and

\[
|\lambda||x_p| \leq \sum_{j,k \in \mathbb{N}} |a_{pjk}||y_j||y_k|
\]

\[
\leq \sum_{j,k \in \mathbb{N}} |a_{pjk}||y_q||y_q|
\]

\[
\leq \sum_{j,k \in \mathbb{N}} |a_{pjk}||y_q|. \quad \text{(by} \ |y_q| \leq 1)\]

Hence

\[
|\lambda||x_p| \leq R^{(1)}_p(A)|y_q|. \quad (4)
\]
On the other hand, by considering the $q$-th equation of $x \mathcal{A} y = \lambda y$ in (1), we have
\begin{equation}
\lambda y_q = \sum_{i,j \in N} a_{ijq} x_i y_j, \tag{5}
\end{equation}
and
\begin{align*}
|\lambda| |y_q| & \leq \sum_{i,j \in N} |a_{ijq}| |x_i| |y_j| \\
& \leq \sum_{i,j \in N} |a_{ijq}| |x_p| |y_q| \\
& \leq \sum_{i,j \in N} |a_{ijq}| |x_p|. \, (by \ |y_q| \leq 1)
\end{align*}

Hence
\begin{equation}
|\lambda| |y_q| \leq R_q^{(3)}(\mathcal{A}) |x_p|. \tag{6}
\end{equation}
Multiplying (1) with (6) yields
\begin{equation*}
|\lambda|^2 |x_p| |y_q| \leq R_p^{(1)}(\mathcal{A}) R_q^{(3)}(\mathcal{A}) |x_p| |y_q|,
\end{equation*}
consequently,
\begin{equation}
|\lambda| \leq \left( R_p^{(1)}(\mathcal{A}) R_q^{(3)}(\mathcal{A}) \right)^{\frac{1}{2}}. \tag{7}
\end{equation}

Note the facts that $\lambda$ is a $C$-eigenvalue of $\mathcal{A}$ if and only if $-\lambda$ is a $C$-eigenvalue of $\mathcal{A}$, and that a $C$-eigenvalue is real. Then
\begin{equation*}
\lambda \in \left[ -\left( R_p^{(1)}(\mathcal{A}) R_q^{(3)}(\mathcal{A}) \right)^{\frac{1}{2}}, \left( R_p^{(1)}(\mathcal{A}) R_q^{(3)}(\mathcal{A}) \right)^{\frac{1}{2}} \right] \subseteq [-\rho, \rho].
\end{equation*}
The conclusion follows. \hfill \square

From Theorem 1 we can obtain easily the following upper bound for the largest $C$-eigenvalue of a piezoelectric-type tensor.

**Corollary 1.** Let $\mathcal{A} = (a_{ijk}) \in \mathbb{R}^{n \times n \times n}$ be a piezoelectric-type tensor, and $\lambda^*$ be the largest $C$-eigenvalue of $\mathcal{A}$. Then
\begin{equation*}
\lambda^* \leq \rho.
\end{equation*}
Next we give another interval to locate all $C$-eigenvalues of a piezoelectric-type tensor. Before that some notation are given. For a subset $S$ of $N$, denote
\[
\Delta_S := \{(i, j) : i \in S \text{ or } j \in S\}
\]
and
\[
\bar{\Delta}_S := \{(i, j) : i \notin S \text{ and } j \notin S\}.
\]
Given a piezoelectric-type tensor $A = (a_{ijk}) \in \mathbb{R}^{n \times n \times n}$, let
\[
R_{j}^{\Delta_S,(3)}(A) = \sum_{(l,k) \in \Delta_S} |a_{lkl}|, R_{j}^{\Delta_S,(3)}(A) = \sum_{(l,k) \in \bar{\Delta}_S} |a_{lkl}|,
\]
where $R_{j}^{\Delta_S,(3)}(A) = 0$ if $S = \emptyset$, and $R_{j}^{\Delta_S,(3)}(A) = 0$ if $S = N$. Obviously, $R_{j}^{(3)}(A) = R_{j}^{\Delta_S,(3)}(A) + R_{j}^{\bar{\Delta}_S,(3)}(A)$ for each $j \in N$.

**Theorem 2.** Let $A = (a_{ijk}) \in \mathbb{R}^{n \times n \times n}$ be a piezoelectric-type tensor, and $\lambda$ be a $C$-eigenvalue of $A$. And let $S$ be a subset of $N$. Then
\[
\lambda \in [-\rho_S, \rho_S],
\]
where
\[
\rho_S := \max_{i,j \in N} \frac{1}{2} \left( R_{j}^{\Delta_S,(3)}(A) + \left( (R_{j}^{\Delta_S,(3)}(A))^2 + 4R_{i}^{(1)}(A)R_{j}^{\bar{\Delta}_S,(3)}(A) \right)^{1/2} \right).
\]
Furthermore,
\[
\lambda \in [-\rho_{\min}, \rho_{\min}],
\]
where $\rho_{\min} := \min_{S \subseteq N} \rho_S$.

**Proof.** Similarly to the proof of Theorem 1 (4) and (5) hold. Furthermore, by (5) we have
\[
|\lambda||y_q| \leq \sum_{i,j \in N} |a_{ijq}||x_p||y_q|
\]
\[
= R_q^{(3)}(A)|x_p||y_q|
\]
\[
= \left( R_q^{\Delta_S,(3)}(A) + R_q^{\bar{\Delta}_S,(3)}(A) \right) |x_p||y_q|
\]
\[
\leq R_q^{\Delta_S,(3)}(A)|y_q| + R_q^{\bar{\Delta}_S,(3)}(A)|x_p|
\]
Hence
\[ (|\lambda| - R_{q}^{\Delta S,(3)}(A)) |y_q| \leq R_{q}^{\Delta S,(3)}(A)|x_p|. \] (10)

Multiplying (4) with (10) yields
\[ |\lambda| (|\lambda| - R_{q}^{\Delta S,(3)}(A)) |x_p||y_q| \leq R_{p}^{(1)}(A) R_{q}^{\Delta S,(3)}(A)|x_p||y_q|, \]
consequently,
\[ |\lambda| (|\lambda| - R_{q}^{\Delta S,(3)}(A)) \leq R_{p}^{(1)}(A) R_{q}^{\Delta S,(3)}(A). \] (11)

Solving (11) for $|\lambda|$ gives
\[ |\lambda| \leq \frac{1}{2} \left( R_{q}^{\Delta S,(3)}(A) + \left( (R_{q}^{\Delta S,(3)}(A))^2 + 4R_{p}^{(1)}(A) R_{q}^{\Delta S,(3)}(A) \right)^{\frac{1}{2}} \right). \]

By an analogous way of Theorem 1, we have
\[ \lambda \in [-\rho_S, \rho_S]. \] (12)

Furthermore, since (12) holds for any $S \subseteq N$, it follows that
\[ \lambda \in \bigcap_{S \subseteq N} [-\rho_S, \rho_S] = \left[ -\min_{S \subseteq N} \rho_S, \min_{S \subseteq N} \rho_S \right] = [-\rho_{\min}, \rho_{\min}]. \]

The conclusion follows. \( \square \)

Note here that if $S = \emptyset$, then $R_{j}^{\Delta S,(3)}(A) = 0$ and $R_{j}^{\Delta S,(3)}(A) = R_{j}^{(3)}(A)$ for any $j \in N$, which implies
\[ \frac{1}{2} \left( R_{j}^{\Delta S,(3)}(A) + \left( (R_{j}^{\Delta S,(3)}(A))^2 + 4R_{j}^{(1)}(A) R_{j}^{\Delta S,(3)}(A) \right)^{\frac{1}{2}} \right) = \left( R_{j}^{(1)}(A) R_{j}^{(3)}(A) \right)^{\frac{1}{2}}, \]
consequently,
\[ \rho_S = \rho. \]

Hence,
\[ \rho_{\min} = \min_{S \subseteq N} \rho_S \leq \rho. \]

This gives the comparison of the intervals in Theorem 1 and Theorem 2 as follows.
Theorem 3. Let $A = (a_{ijk}) \in \mathbb{R}^{n \times n \times n}$ be a piezoelectric-type tensor, and $\lambda$ be a C-eigenvalue of $A$. Then

$$\lambda \in [-\rho_{\min}, \rho_{\min}] \subseteq [-\rho, \rho],$$

where $\rho$ is defined in Theorem 1 and $\rho_{\min}$ is defined in Theorem 2.

Remark 1. Theorem 1 shows that the interval $[-\rho_{\min}, \rho_{\min}]$ captures all C-eigenvalues of a piezoelectric-type tensor precisely than the interval $[-\rho, \rho]$, although $\rho_{\min}$ needs more computations than $\rho$.

Similarly to Corollary 1, we can obtain easily the following upper bound for the largest C-eigenvalue of a piezoelectric-type tensor by Theorem 2.

Corollary 2. Let $A = (a_{ijk}) \in \mathbb{R}^{n \times n \times n}$ be a piezoelectric-type tensor, and $\lambda^*$ be the largest C-eigenvalue of $A$. Then

$$\lambda^* \leq \rho_{\min}.$$

3. Numerical examples

In this section, we give some examples to show the results obtained above. Consider the eight piezoelectric tensors in [1];

(I) The piezoelectric tensor $A_{VFeSb}$ [1, 4], with its entries

$$a_{123} = a_{213} = a_{312} = -3.68180677,$$

and other elements are zeros;

(II) The piezoelectric tensor $A_{SiO2}$ [1, 2, 3], with its entries

$$a_{111} = -a_{122} = -a_{133} = -0.13685, \text{ and } a_{123} = -a_{213} = -0.009715,$$

and other elements are zeros;

(III) The piezoelectric tensor $A_{Cr2AgBiO8}$ [1, 4], with its entries

$$a_{123} = a_{213} = -0.22163, \text{ and } a_{113} = -a_{223} = 2.608665,$$

$$a_{311} = -a_{322} = 0.152485, \text{ and } a_{312} = -0.37153,$$

and other elements are zeros;
(IV) The piezoelectric tensor $A_{RbTaO_3}$\[^1\, 4\] with its entries 
\[a_{113} = a_{223} = -8.40955, \quad a_{222} = -a_{212} = -a_{211} = -5.412525, \]
\[a_{311} = a_{322} = -4.3031, \text{ and } a_{333} = -5.14766,\]
and other elements are zeros;

(V) The piezoelectric tensor $A_{NaBiS_2}$\[^1\, 4\] with its entries 
\[a_{113} = -8.90808, \quad a_{223} = -0.00842, \quad a_{311} = -7.11526, \]
\[a_{322} = -0.6222, \text{ and } a_{333} = -7.93831,\]
and other elements are zeros;

(VI) The piezoelectric tensor $A_{LiBiB_2O_5}$\[^1\, 4\] with its entries 
\[a_{123} = 2.35682, \quad a_{112} = 0.34929, \quad a_{211} = 0.16101, \quad a_{222} = 0.12562, \]
\[a_{233} = 0.1361, \quad a_{213} = -0.05587, \quad a_{323} = 6.91074, \text{ and } a_{312} = 2.57812,\]
and other elements are zeros;

(VII) The piezoelectric tensor $A_{KBi_2F_7}$\[^1\, 4\] with its entries 
\[a_{111} = 12.64393, \quad a_{122} = 1.08802, \quad a_{133} = 4.14350, \quad a_{123} = 1.59052, \]
\[a_{113} = 1.96801, \quad a_{112} = 0.22465, \quad a_{211} = 2.59187, \quad a_{222} = 0.08263, \]
\[a_{233} = 0.81041, \quad a_{223} = 0.51165, \quad a_{213} = 0.71432, \quad a_{212} = 0.10570, \]
\[a_{311} = 1.51254, \quad a_{322} = 0.68235, \quad a_{333} = -0.23019, \quad a_{323} = 0.19013, \]
\[a_{313} = 0.39030, \text{ and } a_{312} = 0.08381,\]
and other elements are zeros;

(VIII) The piezoelectric tensor $A_{BaNiO_3}$\[^1\, 4\] with its entries 
\[a_{113} = a_{223} = 0.038385, \quad a_{311} = a_{322} = 6.89822, \text{ and } a_{333} = 27.4628,\]
and other elements are zeros.

We now use the intervals in Theorem 11 and Theorem 12 to locate all $C$-eigenvalues of the eight tensors above, see Table 1. It is easy to see that for any $C$-eigenvalue $\lambda$,

$$\lambda \in [-\rho_{\text{min}}, \rho_{\text{min}}] \subseteq [-\rho, \rho].$$

|                | $A_{VFeSb}$ | $A_{SiO_2}$ | $A_{Cr_2AgBiO_8}$ | $A_{RbTaO_3}$ | $A_{NaBiS_2}$ | $A_{LiBiB_2O_5}$ | $A_{KBi_2F_7}$ | $A_{BaNiO_3}$ |
|----------------|-------------|-------------|-------------------|---------------|---------------|-----------------|----------------|--------------|
| $\rho$         | 7.3636      | 0.2882      | 5.6606            | 30.0911       | 17.3288       | 15.2911         | 22.6896        | 38.8162      |
| $\rho_{\text{min}}$ | 7.3636      | 0.2834      | 5.6606            | 23.5377       | 16.8548       | 12.3206         | 20.2351        | 35.3787      |
| $\lambda^*$    | 4.2514      | 0.1375      | 2.6258            | 12.4234       | 11.6674       | 7.7376          | 13.5021        | 27.4628      |

Table 1. The intervals $[-\rho, \rho]$ and $[-\rho_{\text{min}}, \rho_{\text{min}}]$, and $\lambda^*$ is the largest $C$-eigenvalue.
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