Adaptive Fast Smooth Second-Order Sliding Mode Control for Attitude Tracking of a 3-DOF Helicopter

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Abstract—This paper presents a novel adaptive fast smooth second-order sliding mode control for the attitude tracking of the three degree-of-freedom (3-DOF) helicopter system with lumped disturbances. Combining with a non-singular integral sliding mode surface, we propose a novel adaptive fast smooth second-order sliding mode control method to enable elevation and pitch angles to track given desired trajectories respectively with the features of non-singularity, adaptation to disturbances, chattering suppression and fast finite-time convergence. In addition, a novel adaptive-gain smooth second-order sliding mode observer is proposed to compensate time-varying lumped disturbances with the smoother output compared with the adaptive-gain second-order sliding mode observer. The fast finite-time convergence of the closed-loop system with constant disturbances and the fast finite-time uniformly ultimately boundedness of the closed-loop system with the time-varying lumped disturbances are proved with the finite-time Lyapunov stability theory. Finally, the effectiveness and superiority of the proposed control methods are verified by comparative simulation experiments.

Index Terms—Adaptive fast smooth second-order sliding mode control (AFSSOSMC), adaptive-gain smooth second-order sliding mode observer (ASSOSMO), 3-DOF helicopter.

I. INTRODUCTION

Due to the merits of vertical take-off and landing, air hovering as well as aggressive maneuver, the small unmanned helicopter has an extremely broad application prospect in military and civil fields [1]. However, small unmanned helicopters have the characteristics of high nonlinearity, strong coupling, under-actuated and extremely vulnerable to lumped disturbances during flight, which make it a challenge to design the high-performance attitude tracking controller [2].

Because of the similar dynamics with the real helicopter system, the three degree-of-freedom (3-DOF) lab helicopter from Quanser Consulting Inc., as shown in Figure 1, can serve as an ideal experimental platform for helicopter controller design with the advantage of testing various advanced control methods conveniently [3]. In recent years, researchers have proposed numerous methods to achieve the attitude tracking object of 3-DOF helicopter and verified these methods by the helicopter experimental platform. These methods can be mainly divided into three parts: linear control [4], [5], nonlinear control [2], [6]–[9] and intelligent control [10]–[12].

Among all the above-mentioned control methods, sliding mode control has attracted much attention because of its insensitivity and strong robustness to the disturbances. In the present sliding mode algorithms, the super-twisting algorithm is very popular and owns practical application value due to the features of finite-time convergence, strong robustness and solely requiring the information of sliding mode variables [13]. In [14], a fast super-twisting algorithm is proposed to solve the problem that the convergence speed becomes slow when the system states are far away from their equilibrium points existing in [13]. However, the bound of disturbance needing to be known in advance restricts the applications of these super-twisting algorithms. To deal with this problem, an adaptive super-twisting algorithm is proposed in [15], which can adapt to the disturbance of unknown boundary. Combining the merits of [14] and [15], an adaptive fast second-order sliding mode control method is proposed in [16]. An observer is also designed in the light of this method, but the output of the proposed observer is not smooth. In [17], a smooth second-order sliding mode control is proposed to further alleviate the chattering effect existing in [13]. In [18], a fast smooth second-order sliding mode method is present based on the method of [17]. However, this method cannot adapt to the unknown boundary disturbances.

In this paper, motivated by [18], we extend the adaptive fast second-order sliding mode control (AFSOSMC) method from [16], and propose a new proposition called the adaptive
fast smooth second-order sliding mode control (AFSSOSMC) method, which integrates the advantages of the former two. This new method solves the chattering problem of [16] as well as the trouble of manually tuning the parameters existing in [18]. Based on this new method, a novel adaptive-gain smooth second-order sliding mode observer (ASSOSMO) is also proposed to alleviate chattering effect and adjust the parameters automatically without a priori knowledge of the upper bound of the lumped disturbances derivative. The main contributions of this paper can be summarized as follows:

1) A novel adaptive fast smooth second-order sliding mode control (AFSSOSMC) method is proposed to enable elevation and pitch angles to track given desired trajectories respectively, which not only maintains the characteristics of high precision, fast finite-time convergence and adaptation to the disturbance of unknown boundary, but also enormously reduces the chattering effect. Combined with a non-singular integral sliding mode surface [19], this method can guarantee the fast finite-time convergence of the tracking errors with constant disturbances and the fast finite-time uniformly ultimately boundedness with the time-varying lumped disturbances.

2) A novel adaptive-gain smooth second-order sliding mode observer (ASSOSMO) is proposed to estimate the time-varying lumped disturbances of helicopter system control channels with smoother output than that of the adaptive-gain second-order sliding mode observer (ASOSMO) proposed in [16].

The fast finite-time convergence of the closed-loop system with constant disturbances and the fast finite-time uniformly ultimately boundedness of the closed-loop system with time-varying lumped disturbances will be proved with the corresponding finite-time Lyapunov stability theory. The effectiveness and superiority of the proposed control methods are verified by comparative experiments.

The rest of this paper is organized as follows. In Section II, the dynamic model and control objective of the 3-DOF helicopter system, some essential lemmas and the new proposition are given. The controller design process and stability analysis of the closed-loop system are presented in Section III. Section V provides contrastive experiment results and discussion. Section VI concludes this paper.

Notation: In this paper, we use $\|\bullet\|$ for the Euclidean norm of vectors and \text{sign}(\bullet) to denote the standard signum function.

II. PROBLEM FORMULATION AND PRELIMINARIES

A. The dynamics of the 3-DOF Helicopter

As shown in Fig.1, the 3-DOF helicopter system studied in this paper has elevation, pitch and travel motions, which are driven by two DC motors called the front motor and back motor. A positive voltage applied to each motor can generate the elevation motion, and positive pitch motion can be generated by applying a higher voltage on the front motor. The travel motion can be generated by thrust vectors when the helicopter body is pitching. Moreover, an active disturbance system (ADS) is installed on the arm, which can serve as an external disturbance or system uncertainty.

Due to the under-actuated mechanism of the 3-DOF helicopter system, only two of the three degree of freedoms can be controlled to track arbitrary trajectories in the operating domain. In this work, the elevation and pitch motions are investigated and the travel motion is set to move freely. The models of elevation and pitch can be formulated as follows [2]

$$
\dot{\alpha} = \frac{K_f L_a \cos(\beta)(V_f + V_b) - mgL_a \cos(\alpha)}{J_{\alpha}}
$$

$$
\dot{\beta} = \frac{K_f L_h (V_f - V_b)}{J_{\beta}}
$$

where $\alpha$ and $\beta$ represent the elevation and pitch angle respectively. Taking into account the mechanical constraints, the operating domain of the dynamical model of the 3-DOF helicopter is defined as follows

$$
-27.5^\circ \leq \alpha \leq +30^\circ
$$

$$
-45^\circ \leq \beta \leq +45^\circ
$$

The definitions and values of the relevant parameters of the 3-DOF helicopter system are shown in Table I. Denote $x_1 = \alpha, x_2 = \dot{\alpha}, x_3 = \beta, x_4 = \dot{\beta}$. Considering the system uncertainties and external disturbance, the model of 3-DOF helicopter can be rewritten as the following form

$$
\dot{x}_1 = x_2
$$

$$
\dot{x}_2 = \frac{L_a}{J_{\alpha}} \cos(x_3)u_1 + \frac{mgL_a}{J_{\alpha}} \cos(x_1) + d_1(x)
$$

$$
\dot{x}_3 = x_4
$$

$$
\dot{x}_4 = \frac{L_h}{J_{\beta}} u_2 + d_2(x)
$$

where $x = [x_1, x_2, x_3, x_4]^T$ represents the state vector of system (1), which is measurable and available, $d_1(x)$ and $d_2(x)$ represent the combination of the system uncertainties and external disturbance existing in the corresponding attitude control channels. In addition, $u_1$ and $u_2$ are defined by

$$
u_1 = K_f(V_f + V_b)$$

$$
u_2 = K_f(V_f - V_b)
$$

The control objective is to design the controllers such that the elevation and pitch angles can track the given desired trajectories respectively within small errors in finite time. For

| Symbol | Definition                | Value  |
|--------|--------------------------|--------|
| $J_{\alpha}$ | Moment of inertia of elevation axis | 1.0348kg·m² |
| $J_{\beta}$ | Moment of inertia of pitch axis | 0.0451kg·m² |
| $L_a$ | Distance from elevation axis to the center of helicopter body | 0.6600m |
| $L_h$ | Distance from pitch axis to either motor | 0.1780m |
| $m$ | Effective mass of the helicopter | 0.094kg |
| $g$ | Gravitational acceleration constant | 9.81m/s² |
| $K_f$ | Propeller force-thrust constant | 0.1188N/V |
| $V_f$ | Front motor voltage input | $[-24, 24]$V |
| $V_b$ | Back motor voltage input | $[-24, 24]$V |

TABLE I

THE PARAMETERS OF THE 3-DOF HELICOPTER SYSTEM
the design of the controllers, the following assumptions are required.

**Assumption 1:** The lumped disturbances and their first derivative of the 3-DOF helicopter system are assumed to be bounded, while the value of the bound is unknown, i.e., there exist unknown positive constants \(d_1, \delta_1, d_2, \delta_2\) such that \(|d_1(x)| \leq d_1, |\dot{d}_1(x)| \leq \delta_1, |d_2(x)| \leq d_2\) and \(|\dot{d}_2(x)| \leq \delta_2\), where \(d_1 > 0, \delta_1 \geq 0, d_2 > 0, \delta_2 \geq 0\).

**Assumption 2:** The desired trajectories given by \(x_{\text{d}}(t), x_{\beta d}(t)\) are assumed to be bounded and available up to their second derivative.

### B. Definitions and Lemmas

To better describe the following definitions and lemmas, we consider a general system

\[
\dot{x} = f(x(t)), x_0 = x(0)
\]

where \(f : U_0 \rightarrow \mathbb{R}^n\) is continuous on an open neighborhood \(U_0 \subset \mathbb{R}^n\) of the origin and assume that \(f(0) = 0\). The solution of (5) is denoted as \(x(t, x_0)\), which is understood in the sense of Filippov [20].

**Definition 1 ([21]):** If the origin of (5) is Lyapunov stable and any solution of (5) converges to the equilibrium point in finite time, i.e., \(\forall x_0 \in U_1 \subset U_0, x(t, x_0) \in U_1\{0\}\) when \(t \in [0, T(x_0)]\) with \(\lim_{t \rightarrow T(x_0)} x(t, x_0) = 0\), and \(x(t, x_0) = 0\) \(\forall t > T(x_0)\), then the system (5) is called finite-time stable.

**Definition 2 ([22]):** If any solution of (5) converges to a small region of the equilibrium point in a finite time, the system (5) is called finite-time uniformly ultimately bounded, i.e., for all \(x(0) = x_0\), there exists \(\varepsilon > 0\) and \(T(\varepsilon, x_0) < \infty\), such that \(|x(t)| < \varepsilon\) for all \(t \geq T\).

**Lemma 1 ([23]):** Suppose there exists a continuous and positive-definite function \(V : U_0 \rightarrow \mathbb{R}\) such that the following condition holds:

\[
\dot{V}(x) \leq -c_1 V(x)^p - c_2 V(x)
\]

where \(c_1 > 0, c_2 > 0, p \in (0, 1)\), then the trajectory of (5) is fast finite-time stable, and the settling time is given by:

\[
T \leq \frac{\ln[1 + c_2 V(x_0)^{1-p}/c_1]}{c_2 (1-p)}
\]

**Lemma 2 ([18]):** Suppose there exists a continuous and positive-definite function \(V : U_0 \rightarrow \mathbb{R}\) such that the following condition holds:

\[
\dot{V}(x) \leq -c_1 V(x)^{p_1} - c_2 V(x) + c_3 V(x)^{p_2}
\]

where \(c_1 > 0, c_2 > 0, c_3 > 0, p_1 \in (0, 1), p_2 \in (0, p_1)\), then the trajectory of (5) is fast finite-time uniformly ultimately bounded, and the settling time is given by:

\[
T \leq \frac{\ln[1 + (c_2 - \theta_2)V(x_0)^{1-p_1}/(c_1 - \theta_1)]}{(c_2 - \theta_2)(1-p_1)}
\]

where \(\theta_1\) and \(\theta_2\) are arbitrary positive constants holding \(\theta_1 \in (0, c_1), \theta_2 \in (0, c_2)\), then \(x(t, x_0)\) can converge to a region of equilibrium point in a finite time \(T\). In addition, the residual set of solution of (5) can be given by:

\[
D = \{x : \theta_1 V(x)^{p_1 - p_2} + \theta_2 V(x)^{1-p_2} < c_3\}
\]

Define an auxiliary variable \(\theta_3 \in (0, 1)\). If \(\theta_3\) is selected satisfying

\[
\theta_3^{1-p_2}\theta_2^{p_1-p_2}c_3^{1-p_2} = \theta_1^{1-p_2}(1-\theta_3)^{p_1-p_2}
\]

then (10) can be reduced to \(D = D_1 = D_2\), where

\[
D_1 = \{x : V(x)^{p_1 - p_2} < \theta_3 c_3 / \theta_1\}
\]

\[
D_2 = \{x : V(x)^{1-p_2} < (1-\theta_3) c_3 / \theta_2\}
\]

which means the state \(x\) can converge to \(D_1 = D_2\) in finite time \(T\).

**Remark 1:** The meaning of fast in Lemma 1 and Lemma 2 is that the solution of (5) can quickly converge to the origin or a small region of the origin regardless of the distance between the initial state and the equilibrium point, while the original finite-time stable converges slowly when the initial state is far from the equilibrium point.

**Lemma 3 ([19] [24]):** Considering the following system

\[
\dot{x}_1 = x_2
\]

\[
\dot{x}_2 = -\gamma_1 |x_1|^p sgn(x_1) - \gamma_2 |x_2|^{2p_1/(1+p_3)} sgn(x_2) + d
\]

where \(\gamma_1, \gamma_2, p_3\) are appropriate tuning parameters satisfying \(\gamma_1 > 0, \gamma_2 > 0, p_3 \in (0, 1)\). Then the following statements hold

(i) If \(d = 0\), then the trajectory of the system (13) is finite-time stable, i.e., \(x_1, x_2\) can converge to the origin in finite time.

(ii) If \(d \neq 0\) and \(d\) is a bounded disturbance, then the trajectory of the system (13) is finite-time uniformly ultimately boundedness, which illustrates \(x_1, x_2\) can converge to a region of the origin in finite time.

### C. A New Proposition

Motivated by [18], we extend the result in [16] to obtain the following new proposition and this generalization is non-trivial.

**Proposition 1:** Considering the following system

\[
\dot{s} = -L_1(t) |s|^{\frac{m-1}{m}} sgn(s) - L_2(t) s + \varphi
\]

\[
\dot{\varphi} = -L_3(t) |s|^{\frac{m-2}{m}} sgn(s) - L_4(t) s + d(t)
\]

where \(|d(t)| \leq \delta\), \(\delta\) is an unknown non-negative constant and the adaptive gains \(L_1, L_2, L_3, L_4\) are formulated as

\[
L_1(t) = k_1 L_0^m(t), L_2(t) = k_2 L_0^{2m-2}(t)
\]

\[
L_3(t) = k_3 L_0^{m}(t), L_4(t) = k_4 L_0^{4m-4}(t)
\]

where \(k_1, k_2, k_3, k_4, m\) are positive constants satisfying

\[
m^2 k_3 k_4 > \left(\frac{m^3 k_3}{m-1} + (4m^2 - 4m + 1) k_1^2\right) k_2^2, m > 2
\]
\( L_0(t) \) is a positive, time-varying, and scalar function. The \( L_0(t) \) satisfies
\[
\dot{L}_0(t) = \begin{cases} \kappa, & \text{if } |s| \neq 0 \\ 0, & \text{else} \end{cases}
\]  
(17)

where \( \kappa \) is a positive constant. Then the following statements hold

(i) If \( d = 0 \), then \( s, \dot{s} \) can fast converge to the origin in finite time.

(ii) If \( d \neq 0 \) and \( d \) is a bounded disturbance, then \( s, \dot{s} \) can fast converge to a region of the origin in finite time.

**Proof:** Define a new state vector
\[
\xi = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \varphi \end{bmatrix} = \begin{bmatrix} \frac{m-1}{L_0^m(t)} |s|^{m-1} \frac{m-1}{m} \text{sgn}(s) \\ \frac{2m-2}{L_0^m(t)} s \\ \frac{m-1}{L_0^m(t)} \xi_1 \\ \frac{m-1}{L_0^m(t)} \xi_2 \end{bmatrix}
\]  
(18)

After taking the derivative of \( \xi \), we can obtain
\[
\dot{\xi} = L_0^m \begin{bmatrix} -m-1k_1 & -m-1k_2 & m-1 \xi_1 \\ -k_3 & 0 & 0 \\ 0 & 0 & 0 \\ -k_1 & k_2 & 0 \end{bmatrix} \xi + L_0 \begin{bmatrix} 0 & 0 & 0 \\ -k_1 & -k_2 & 1 \\ 0 & -k_1 & 0 \end{bmatrix} \xi + \begin{bmatrix} \frac{m-1}{L_0^m} \frac{L_0(t)}{m} \xi_1 \\ \frac{m-1}{L_0^m} \frac{L_0(t)}{m} \xi_2 \end{bmatrix}
\]  
(19)

Then, choose a candidate Lyapunov function as \( V(\xi) = \xi^T P \xi \), where
\[
P = \frac{1}{2} \begin{bmatrix} \frac{2m-2}{k_1^2} k_1^2 + k_2^2 & k_1 k_2 & -k_1 \\ k_1 k_2 & 2k_1 k_2 & -k_2 \\ -k_1 & -k_2 & 2 \end{bmatrix}
\]  
(20)

where \( P \) is symmetric positive definite matrix because its leading principle minors are all positive. Taking the derivative of \( V(\xi) \) along the trajectories of system (14), we get
\[
\dot{V}(\xi) = -L_0 |\xi|^{m-1} \xi^T \Omega_1 \xi - L_0^m \xi^T \Omega_2 \xi + \sigma_1 d(t) \xi + \frac{2m-2}{L_0^m} \dot{L}_0(t) \sigma_2 P \xi
\]  
(21)

where \( \sigma_1 = [-k_1 - k_2, 2], \sigma_2 = [\xi_1, 2 \xi_2, 0] \) and
\[
\Omega_1 = \begin{bmatrix} k_3 m + k_1^2 (m-1) & 0 & -k_1 (m-1) \\ 0 & k_4 m + k_2^2 (3m-1) & -k_2 (2m-1) \\ -k_1 (m-1) & -k_2 (2m-1) & m-1 \end{bmatrix}
\]
\[
\Omega_2 = k_2 \begin{bmatrix} k_3 + k_1^2 (3m-2)/m & 0 & 0 \\ 0 & k_4 + k_2^2 & -k_2 \\ 0 & -k_2 & 1 \end{bmatrix}
\]  
(22)

It is easy to prove that the matrices \( \Omega_1 \) and \( \Omega_2 \) both are positive definite with (16). By using
\[
\lambda_{\min}(P) \|\xi\|^2 \leq V \leq \lambda_{\max}(P) \|\xi\|^2
\]  
(23)

(21) can be expressed as
\[
\dot{V} \leq -L_0(t) \lambda_{\min}(\Omega_1) |V|^{m-1} - L_0^m(t) \frac{2m-2}{m} \lambda_{\min}(\Omega_2) V + \frac{\sigma_1 d(t)}{\lambda_{\max}(P)} \frac{2m-2}{m} \frac{\dot{L}_0(t)}{L_0(t)} \sigma_2 P \xi
\]  
(24)

Due to \( \dot{L}_0(t) \geq 0 \), \( L_0(t) \) is positive in finite time. It follows from (28) that
\[
\dot{V} \leq -c_1 |V|^{p_1} - c_2 V
\]  
(29)

where \( c_1 \) and \( c_2 \) are positive constants, \( p_1 \in (0.5, 1) \). By using **Lemma 1**, \( \xi \) can converge to origin in finite time, then \( s, \dot{s} \) can fast converge to the origin in finite time and the proof of (i) is completed.

(ii) If \( d(t) \neq 0 \), with the same analysis of (i), it follows from (26) that
\[
\dot{V} \leq -c_4 |V|^{p_1} - c_5 V + c_3 V^\frac{1}{2}
\]  
(30)

where \( c_3, c_4, \text{and } c_5 \) are positive constants, \( p_1 \in (0.5, 1) \). By using **Lemma 2**, \( \xi \) can converge to a region of origin in finite time. In addition, the region can be given by
\[
D = \left\{ \xi : \theta_1 V(\xi)^{p_1-p_2} + \theta_2 V(\xi)^{1-p_2} < c_3 \right\}
\]  
(31)

where \( \theta_1 \in (0, c_4), \theta_2 \in (0, c_5), p_2 = 0.5 \).

Define an auxiliary variable \( \theta_3 \in (0, 1) \). If \( \theta_3 \) is selected satisfying
\[
\theta_3^{p_1-p_2} \theta_2^{p_1-p_2} c_3^{1-p_2} = \theta_1^{p_1-p_2} (1 - \theta_3)^{p_1-p_2}
\]  
(32)
\( \xi \) can converge to \( D = D_1 = D_2 \) in finite time \( T_1 \), where

\[
T_1 \leq \ln \left[ 1 + (c_5 - \theta_2) V(\xi_0)^{1-p_1} (c_4 - \theta_1) \right] / (c_5 - \theta_2)(1 - p_1) \tag{33}
\]

\[
D_1 = \left\{ \xi : V(\xi)^{p_1-p_2} < \theta_3 c_3 / \theta_1 \right\}
\]

\[
D_2 = \left\{ \xi : V(\xi)^{1-p_2} < (1 - \theta_3) c_3 / \theta_2 \right\}
\tag{34}
\]

Design a region \( D_4 \) as

\[
D_4 = \left\{ \xi : \lambda_{\min}(P) \| \xi \|^2 < (1 - \theta_3)^2 c_3 / \theta_2 \right\}
\]

\[
= \left\{ \xi : \| \xi \| < \Delta_2 \right\}
\tag{35}
\]

where \( \Delta_2 = \lambda_{\min}(P)^{-1/2} (1 - \theta_3) c_3 / \theta_2 \). In terms of (23), it follows from (34) and (35) that \( D_4 \) contains \( D_2 \). Considering the definition of \( \xi \), the following inequalities \( \| \xi_1 \| \leq \| \xi \|, \| \xi_2 \| \leq \| \xi \| \) and \( \| \xi_3 \| \leq \| \xi \| \) hold. Then the set

\[
D_5 = \left\{ \xi_1, \xi_2, \xi_3 : \| \xi_1 \| < \Delta_2, \| \xi_2 \| < \Delta_2, \| \xi_3 \| < \Delta_2 \right\}
\tag{36}
\]

contains the set \( D_4 \). Therefore, since \( \xi \) converges to \( D_1 \) in \( T_1 \), it will also converge to \( D_5 \) in \( T_1 \). Using (14), (15) and (18), we can obtain that \( s, \dot{s} \) converge to a region of the origin in \( T_1 \). The proof of (ii) is completed.

**Remark 2:** Motivated by [18], we extend the result in [16], and if \( m = 2 \), (14) can be transformed into the result in [16]. However, this generalization is non-trivial, and greatly reduces the chattering effect existing in [16]. The newly proposed proposition will be used in the design of controller and observer for the 3-DOF helicopter system in the next section, and the superiority of the proposed control method will be verified through the comparative simulation experiments.

**Remark 3:** Due to the influence of measurement noise, \( |s| = 0 \) of (17) cannot be achieved in practice. For practice application, the adaptive law (17) will be modified as [16]

\[
\dot{L}_0(t) = \begin{cases} \kappa, & |s| \geq \epsilon \\ 0, & \text{else} \end{cases}
\tag{37}
\]

where \( \epsilon \) is an arbitrary small positive value.

### III. Controller Design

In this section, the controller and observer design procedure will be given in detail. First, the system (3) is transformed into the tracking error system. Then a control scheme, consisting of a non-singular integral sliding mode surface and a novel adaptive fast smooth second-order sliding mode control method, is proposed for the tracking error system subject to constant disturbance and time-varying lumped disturbance. Finally, a novel adaptive-gain smooth second-order sliding mode observer is present to estimate disturbances.

#### A. System transformation

Defining the tracking errors \( e_1 = x_1 - x_{ad}(t), e_2 = x_2 - \dot{x}_{ad}(t), e_3 = x_3 - x_{\dot{a}d}(t), e_4 = x_4 - \ddot{x}_{\dot{a}d}(t) \). Then the tracking error system is given by

\[
\dot{e}_1 = e_2
\]

\[
\dot{e}_2 = \frac{L_a}{J_a} \cos(x_3)u_1 - \frac{g}{J_a} mL_a \cos(x_1) - \ddot{x}_{ad}(t) + d_1(x)
\]

\[
\dot{e}_3 = e_4
\]

\[
\dot{e}_4 = \frac{L_h}{J_b} u_2 - \ddot{x}_{\dot{a}d}(t) + d_2(x)
\tag{38}
\]

The control objective is then transformed into the design of finite-time controller so that \( e_1 \) and \( e_3 \) can converge to the origin or a small region of the origin. To facilitate the design of controllers, we express the state space model of the elevation and pitch channel in a unified form

\[
\dot{e}_i = e_{i+1}
\]

\[
e_{i+1} = g_i v_i + f_i + d_i
\tag{39}
\]

where \( f_i \) and \( g_i, i = 1, 2 \) represent the dynamics of corresponding channels, which can be express as follows

\[
f_1 = -\frac{g}{J_a} mL_a \cos(x_1) - \ddot{x}_{ad}(t)
\]

\[
g_1 = \frac{L_a}{J_a}
\tag{40}
\]

\[
f_2 = -\ddot{x}_{\dot{a}d}(t)
\]

\[
g_2 = \frac{L_h}{J_b}
\]

and the auxiliary control inputs \( v_1 \) and \( v_2 \) are defined as

\[
v_1 = \cos(x_3)u_1
\]

\[
v_2 = u_2
\tag{41}
\]

Then, we will take the elevation channel as an example to illustrate the design procedure and the controller for the pitch channel can be designed following a similar process. Afterwards, we can get the auxiliary control inputs \( v_1 \) and \( v_2 \). By using (4) and (41), we can easily obtain the true control inputs \( V_f \) and \( V_b \).

#### B. Controller design with constant disturbance

Considering the tracking error system of the elevation channel

\[
\dot{e}_1 = e_2
\]

\[
\dot{e}_2 = g_1 v_1 + f_1 + d_1
\tag{42}
\]

where \( d_1 \) is constant disturbance, i.e. \( \dot{d}_1 = 0 \). We adopt a non-singular integral sliding mode surface from [19], which is defined as follows

\[
s = e_2 + \int_0^t z d\tau
\tag{43}
\]

where \( z = \gamma_3 |e_1|^{p_4} \text{sgn}(e_1) + \gamma_4 |e_2|^{(2p_4)/(1+p_4)} \text{sgn}(e_2) \) and \( \gamma_3 > 0, \gamma_4 > 0, p_4 \in (0, 1) \).
Based on this sliding mode variable and Proposition 1, a novel adaptive fast smooth second-order sliding mode controller can be designed as
\[
v_1 = g_1^{-1}\left\{-L_1(t)|s|^\frac{m-1}{m}\text{sgn}(s) - L_2(t)s - z - f_1 - \int_0^t \left[L_3(t)|s|^\frac{m-2}{m}\text{sgn}(s) + L_4(t)s\right]d\tau\right\}
\]  
(44)

where \(L_1(t), L_2(t), L_3(t), L_4(t)\) are formulated the same as (15) and \(m > 2\).

**Theorem 1:** Considering the tracking error system (42) subject to constant disturbance, the proposed control law (44) guarantees that the tracking error can fast converge to the origin in finite time.

**Proof:** The proof will be divided into two steps. In **Step 1**, we will prove that the \(\dot{s}\) can fast converge to the origin in finite time. In **Step 2**, we prove that when \(\dot{s} = 0\) the states of system (42) will converge to origin in finite time.

**Step 1.** Taking the derivative of (43), we can obtain
\[
\dot{s} = \dot{e}_2 + z
\]  
(45)

Substituting (42) and (44) into (45) leads to
\[
\dot{s} = -L_1(t)|s|^\frac{m-1}{m}\text{sgn}(s) - L_2(t)s + d_1 - \int_0^t \left[L_3(t)|s|^\frac{m-2}{m}\text{sgn}(s) + L_4(t)s\right]d\tau + d_1
\]  
(46)

We define an intermediate variable
\[
\varphi = -\int_0^t \left[L_3(t)|s|^\frac{m-2}{m}\text{sgn}(s) + L_4(t)s\right]d\tau + d_1
\]  
(47)

Then (46) becomes
\[
\dot{s} = -L_1(t)|s|^\frac{m-1}{m}\text{sgn}(s) - L_2(t)s + \varphi
\]  
(48)

By using Proposition 1, \(\dot{s}\) can fast converge to the origin in finite time.

**Step 2.** When \(\dot{s} = 0\), (45) becomes
\[
\dot{e}_2 = -\gamma_3|e_1|^{p_3}\text{sgn}(e_1) - \gamma_4|e_2|^{2p_4/(1+p_4)}\text{sgn}(e_2)
\]  
(49)

By using Lemma 3 coupled with system (42), the states of system (42) can converge to origin in finite time, i.e. the tracking error can converge to the origin in finite time. The proof is completed.

**C. Controller design with time-varying lumped disturbance**

Considering the tracking error system of the elevation channel
\[
\dot{e}_1 = e_2
\]  
\[
\dot{e}_2 = g_1 e_1 + f_1 + d_1
\]  
(50)

where \(d_1\) is time-varying lumped disturbance satisfying Assumption 1, i.e. \(|d_1| \leq \delta_1, \delta_1 > 0\). We adopt the same non-singular integral sliding mode surface as (43), which is defined as follows
\[
s_v = e_2 + \int_0^t z_v d\tau
\]  
(51)

where \(z_v = \gamma_5|e_1|^{p_5}\text{sgn}(e_1) + \gamma_6|e_2|^{2p_5/(1+p_5)}\text{sgn}(e_2)\) and \(\gamma_5 > 0, \gamma_6 > 0, p_5 \in (0, 1)\).

Based on this sliding mode variable and Proposition 1, a novel adaptive fast smooth second-order sliding mode controller can be designed as
\[
v_1 = g_1^{-1}\left\{-L_{e1}(t)|s_v|^\frac{m-1}{m}\text{sgn}(s_v) - L_{e2}(t)s_v - z_v - f_1 - \int_0^t \left[L_{e3}(t)|s_v|^\frac{m-2}{m}\text{sgn}(s_v) + L_{e4}(t)s_v\right]d\tau\right\}
\]  
(52)

where \(L_{e1}(t), L_{e2}(t), L_{e3}(t), L_{e4}(t)\) are formulated the same as (15) and \(m > 2\).

**Theorem 2:** Considering the tracking error system (50) subject to time-varying lumped disturbance, the proposed control law (52) guarantees that the tracking error can fast converge to a region of the origin in finite time.

**Proof:** The proof will also be divided into two steps. In **Step 1**, we will prove that the \(\dot{s}\) can fast converge to a region of the origin in finite time. In **Step 2**, we prove that when \(|s_v| \leq \Delta\) the states of system (50) will converge to a region of the origin in finite time.

**Step 1.** Taking the derivative of (51), we can obtain
\[
\dot{s}_v = \dot{e}_2 + z_v
\]  
(53)

Substituting (50) and (52) into (53) leads to
\[
\dot{s}_v = -L_{e1}(t)|s_v|^\frac{m-1}{m}\text{sgn}(s_v) - L_{e2}(t)s_v + d_1 - \int_0^t \left[L_{e3}(t)|s_v|^\frac{m-2}{m}\text{sgn}(s_v) + L_{e4}(t)s_v\right]d\tau + d_1
\]  
(54)

We define an intermediate variable
\[
\varphi_v = -\int_0^t \left[L_{e3}(t)|s_v|^\frac{m-2}{m}\text{sgn}(s_v) + L_{e4}(t)s_v\right]d\tau + d_1
\]  
(55)

Then (54) becomes
\[
\dot{s}_v = -L_{e1}(t)|s_v|^\frac{m-1}{m}\text{sgn}(s_v) - L_{e2}(t)s_v + \varphi_v
\]  
(56)

According to Assumption 1, the disturbance \(|\dot{d}_1| \leq \delta_1\). By using Proposition 1, \(\dot{s}_v\) can fast converge to a region of the origin in finite time, and the region is defined as \(\Delta\).

**Step 2.** When \(|\dot{s}_v| \leq \Delta\), (51) becomes
\[
\dot{e}_2 + \gamma_5|e_1|^{p_5}\text{sgn}(e_1) + \gamma_6|e_2|^{2p_5/(1+p_5)}\text{sgn}(e_2) = \phi
\]  
(57)

where \(|\phi| \leq \Delta\).

Combined with system (50), (57) is rewritten as
\[
\dot{e}_1 = e_2
\]  
\[
\dot{e}_2 = -\gamma_3|e_1|^{p_3}\text{sgn}(e_1) - \gamma_4|e_2|^{2p_4/(1+p_4)}\text{sgn}(e_2) + \phi
\]  
(58)

By using Lemma 3, the states of system (50) can converge to a region of the origin in finite time, i.e. the tracking error can converge to a region of the origin in finite time. The proof is completed.
D. The design of an observer

For the elevation control channel

\[ \dot{e}_2 = g_1 v_1 + f_1 + d_1 \]  

where \( d_1 \) is a time-varying lumped disturbance satisfying Assumption 1. Then we will develop a novel sliding mode observer to estimate the disturbance based on Proposition 1, which is called adaptive-gain smooth second-order sliding mode observer (ASSOSMO).

First, an auxiliary estimation system is defined as

\[ \dot{\hat{e}}_2 = g_1 v_1 + f_1 + \hat{d}_1 \]  

Then a sliding mode variable is defined as

\[ s_d = e_2 - \hat{e}_2 \]  

Finally, the adaptive-gain smooth second-order sliding mode observer (ASSOSMO) can be designed as

\[ \begin{align*} 
\hat{d}_1 &= \dot{L}_{d1}(t)|s_d|^{m-1} \text{sgn}(s_d) + L_{d2}(t)s_d + \varphi_d \\
\varphi_d &= -L_{d3}(t)|s_d|^{m-2} \text{sgn}(s_d) + L_{d4}(t)s_d 
\end{align*} \]  

where \( L_{d1}(t), L_{d2}(t), L_{d3}(t), L_{d4}(t) \) are formulated the same as (15) and \( m > 2 \).

Theorem 3: Under the condition of Assumption 1, the proposed observer (62) can guarantee that the estimation disturbance \( \hat{d}_1 \) converges to a neighborhood of the true disturbance \( d_1 \).

Proof: Taking the derivative of (61), we obtain

\[ \dot{s}_d = \dot{e}_2 - \dot{\hat{e}}_2 = d_1 - \hat{d}_1 \]  

Substituting (62) into (63) leads to

\[ \begin{align*} 
\dot{s}_d &= -L_{d1}(t)|s_d|^{m-1} \text{sgn}(s_d) - L_{d2}(t)s_d + \varphi_d \\
\varphi_d &= -L_{d3}(t)|s_d|^{m-2} \text{sgn}(s_d) - L_{d4}(t)s_d + \hat{d}_1 
\end{align*} \]  

In terms of Assumption 1, the disturbance \( |\hat{d}_1| \leq \delta_1 \). Meanwhile, the \( L_{d1}(t), L_{d2}(t), L_{d3}(t), L_{d4}(t) \) are formulated the same as (15) and \( m > 2 \). By using the Proposition 1, we can get that \( s_d \) converges to a neighborhood of the origin. Then by using (63), the estimation disturbance \( \hat{d}_1 \) can converge to a neighborhood of the true disturbance \( d_1 \), and the proof is completed.

Remark 4: The adaptive-gain second-order sliding mode observer (ASOSMO) can also be developed following the same process by using the method in [16], which is selected as the comparing method. The effectiveness and smoothness of our proposed observer will be verified by the comparative simulation experiment.

IV. EXPERIMENTAL RESULTS

This section presents three sets of comparative simulation experiments to illustrate the effectiveness of the proposed control schemes. The adaptive fast second-order sliding mode control (AFSOSMC) method proposed in [16], combined with an integral non-singular terminal sliding mode (INTSM) surface, is implemented as the comparison method, where the whole control scheme is proposed and analyzed in [25]. The INTSM-AFSOSMC method can be specifically expressed as follows

(i) An INTSM surface is selected as

\[ s_c = e_2 + \int_0^t |z_c|^{\frac{1}{2}} \text{sgn}(z_c) \, d\tau \]  

where \( z_c = e_1 + |e_2|^{\frac{3}{2}} \text{sgn}(e_2) \)

(ii) The INTSM-AFSOSMC law is

\[ u_c = g_i^{-1} \left\{ -L_{c1}(t)|s_c|^{\frac{1}{2}} \text{sgn}(s_c) - L_{c2}(t)s_c - |z_c|^{\frac{1}{2}} \text{sgn}(z_c) - f_i - \int_0^t [L_{c3}(t) \text{sgn}(s_c) + L_{c4}(t)s_c] \, d\tau \right\} \]  

where \( i = 1, 2 \) and the time-varying gains are defined the same as (15) with \( m = 2 \).

The parameters of the 3-DOF helicopter system are given in Table I. The simulation solver is the fixed step 'ode4' and the sampling time for all the following experiments is set to 0.001 second. In these experiments, the system starts from the same initial position of elevation angle \( -24^\circ \) and pitch angle \( 0^\circ \). The desired trajectories of elevation angle and pitch angle are given as

\[ x_{ad}(t) = 0.2 \sin(0.08t - \frac{\pi}{2}), x_{pd}(t) = 0.1 \sin(0.06t) \]  

The purposes of three group experiments are summarized as follows

1) The first experiment is given to show the advantages of our proposed control scheme for the attitude tracking of the helicopter system with constant disturbance by comparing with the control method in [25].

2) The second experiment demonstrates the merits of our proposed control scheme for the attitude tracking of the system with time-varying disturbance, which is also compared with the control method in [25].

3) The third experiment is used to illustrate the smoother output of our proposed observer compared with the observer proposed from [16] in terms of estimating the time-varying lumped disturbances of helicopter system.

A. Experiment I: Attitude tracking control with constant disturbance

In experiment I, the parameters of our proposed control scheme for elevation angle and pitch angle tracking are set as: \( \gamma_1 = \gamma_1 = 5, p_3 = 0.6, m = 3, k_1 = 2, k_2 = 2.5, k_3 = 4, k_4 = 30, \kappa = 5 \). The INTSM-AFSOSMC law adopts the same gain parameters.

The results of experiment I are shown in Fig. 2: (a)-(d). Fig. 2: (a) and (c) demonstrate the responses of elevation angle tracking error by using INTSM-AFSOSMC and the proposed method, respectively. Fig. 2: (b) and (d) present the local magnification of corresponding tracking error to show the steady-state response more clearly. Fig. 2: (a) shows that elevation angle and pitch angle tracking errors converge to the origin in finite time by using our proposed control scheme as fast as the method in [25], which means our proposed method also possesses the characteristic of fast finite-time convergence. In addition, Fig. 2: (b) illustrates that the
proposed method can weaken the chattering effect existing in the AFSOSMC method. A similar conclusion can be drawn from the Fig. 2: (c) and (d) of the pitch angel tracking.

B. Experiment II: Attitude tracking control with time-varying lumped disturbances

In experiment II, the parameters of the both controller are as same as those in experiment I. The time-varying lumped disturbances are set as $d_1(t) = d_2(t) = 0.2 \sin(t)$.

The results of experiment II are given in Fig. 3: (a)-(d). Fig. 3: (a) and (c) illustrate the responses of elevation angle tracking error by using INTSM-AFSOSMC and the proposed method, respectively. Fig. 3: (b) and (d) present the local magnification of corresponding tracking error to show the steady-state response more clearly. From Fig. 3: (a) and (b), one can see that the proposed method can guarantee the fast finite-time uniformly ultimately boundedness coupled with large chattering suppression. However, the time-varying disturbance destroys the original property of finite-time convergence, which leads to the tracking errors only converging in a small neighborhood of the origin. A similar conclusion can be drawn from the Fig. 3: (c) and (d) of the pitch angel tracking.

C. Experiment III: Performance analysis of the observer

In experiment III, we compare the performance of our proposed observer with the adaptive-gain second-order sliding mode observer (ASOSMO). The parameters of the proposed observer are set as $m=3, k_1=2, k_2=2.5, k_3=4, k_4=30, \kappa=10$, and the ASOSMO adopts the same parameters except that $m$ is set as 2. To facilitate the comparison, we take the elevation angel tracking as an example and the time-varying disturbances is set as $0.2 \sin(t)$. Moreover, the conventional sliding mode control method is chosen as the control law, which is shown as follows

$$u_t = g_1^{-1} \left( -f_1 - c_t e_2 - \eta \text{sgn}(s_t) - k_t s_t \right) \quad (68)$$

where the sliding mode surface $s_t = e_2 + c_t e_1$ and $c_t = 2, k_t = 2, \eta$ is set sufficiently large to guarantee the stability of the closed-loop system.

The results of experiment III are given in Fig. 4: (a)-(b). Fig. 4: (a) illustrates the response of observer estimation error by using ASOSMO and the proposed observer and Fig. 4: (b) present the local magnification to show the response more clearly. From Fig. 4: (a) and (b), one can see that the estimation error can fast converge to a region of the origin in finite time by using our proposed observer. Furthermore, the proposed observer can effectively attenuate chattering existing in the ASOSMO and provide a more accurate and smoother output for the disturbance estimation.

V. CONCLUSION

In this paper, a novel adaptive fast smooth second-order sliding mode control algorithm has been proposed. According to the types of disturbances, the design process of helicopter attitude tracking controllers can be divided into two
For the system with constant disturbances, a control scheme, consisting of a non-singular integral sliding mode surface and a novel adaptive fast smooth second-order sliding mode control, is proposed to achieve the fast finite-time convergence. For the system with the time-varying lumped disturbances, the proposed control scheme can realize the fast finite-time uniformly ultimately boundedness. A novel adaptive-gain smooth second-order sliding mode observer is also present to estimate disturbances with a smooth output. The comparative simulation experiments are performed to demonstrate the effectiveness and superiority of the proposed control scheme with fast finite-time convergence, adaptation to disturbances, and chattering suppression for the attitude tracking of the 3-DOF helicopter system. In future, we will test our proposed methods in the 3-DOF helicopter system platform to examine their effectiveness.

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