Abstract

In this paper we give an explicit construction of a representing system generated by the Szegö kernel for the Hardy space. Thus we answer an open question posed by Fricain, Khoi and Lefèvre. We use frame theory to prove the main result.

Keywords: representing system, frame, reproducing kernel Hilbert space, Szegö kernel, Hardy space.

1 The main result

The aim of this paper is to give an affirmative answer to the following question raised by Fricain, Khoi and Lefèvre [1]. Here and subsequently, \( K_{\lambda_n}(\cdot) = K(\cdot, \lambda_n), n = 1, 2, \ldots \) is a sequence generated by the Szegö kernel \( K(z, \lambda) = (1 - \overline{\lambda z})^{-1} \).

**Question 1.** *Can we construct a sequence of points \( \{\lambda_n\}_{n=1}^{\infty} \) in the open unit disk \( \mathbb{D} \) so that \( \{K_{\lambda_n}\}_{n=1}^{\infty} \) forms a representing system for \( H^2(\mathbb{D}) \)?*

It is well-known that the sequence \( \{K_{\lambda_n}\}_{n=1}^{\infty} \) is not a basis for the Hardy space \( H^2(\mathbb{D}) \) for any set of points \( \{\lambda_n\}_{n=1}^{\infty} \). Moreover, the normalized sequence \( \{(1 - |\lambda_n|^2)^{1/2}K_{\lambda_n}\}_{n=1}^{\infty} \) can not be a Duffin-Schaeffer frame for \( H^2(\mathbb{D}) \). Nevertheless, we will show that question \( \Box \) has a positive answer.
Let \( \{ \lambda_n \}_{n=1}^\infty \) be a sequence of points on the unit disk \( \mathbb{D} \). We partition \( \{ \lambda_n \}_{n=1}^\infty \) into groups, so each group consists of \( k^{th} \) roots of unity placed on a circle with a radius \( r_k = 1 - \frac{1}{k} \)

\[
\lambda_n = \lambda_{k,j} = (1 - \frac{1}{k})e^{\frac{2\pi ij}{k}}, \quad j = 0, \ldots, k-1, \quad k = 1, 2, \ldots \tag{1}
\]

**Theorem 1.** Let \( \{ \lambda_n \}_{n=1}^\infty \) be given by (1). Then the sequence \( \{ K_{\lambda_n} \}_{n=1}^\infty \) is a representing system for the Hardy space \( H^2(\mathbb{D}) \).

Note that any representing system is obviously a complete sequence. By Szegő’s zero-point theorem, the completeness of the sequence \( \{ K_{\lambda_n} \}_{n=1}^\infty \) is equivalent to the Blaschke condition being false, i.e.

\[
\sum_{n=1}^\infty (1 - |\lambda_n|) = \infty. \tag{2}
\]

At the same time, by the recovery theorem of Totik [2], if condition (2) holds then there exist polynomials \( P_{n,k} \), where \( k = 1, \ldots, n \) and \( n = 1, 2, \ldots \), such that for every \( f \in H^2(\mathbb{D}) \) we have

\[
f = \lim_{n \to \infty} \sum_{k=1}^{n} f(\lambda_k) P_{n,k}.
\]

Of course, this approximation does not provide the representation by series of the form

\[
f = \sum_{n=1}^{\infty} x_n K_{\lambda_n}.
\]

Speaking informally, we must take “too many” points \( \{ \lambda_n \}_{n=1}^\infty \) to get the representation above and (1) is one such choice of points.

## 2 Representing systems and frames

This section is devoted to the study of the relationship between representing systems and frames.

**Definition 1.** A sequence \( \{ f_n \}_{n=1}^\infty \subset F \) is a representing system for a Banach space \( F \) if for every \( f \in F \) there are coefficients \( x_n \), \( n = 1, 2, \ldots \) such that

\[
f = \sum_{n=1}^{\infty} x_n f_n.
\]
The following notion of a frame for a Hilbert space was introduced by Duffin and Schaeffer [3].

**Definition 2.** A sequence \( \{f_n\}_{n=1}^{\infty} \subset H \setminus \{0\} \) is a Duffin-Schaeffer frame for a Hilbert space \( H \) if there exist constants \( 0 < A \leq B < \infty \) such that for all \( f \in H \)

\[
A \|f\|^2_H \leq \sum_{n=1}^{\infty} |\langle f, f_n \rangle|^2 \leq B \|f\|^2_H.
\]

It is known that every Duffin-Schaeffer frame \( \{f_n\}_{n=1}^{\infty} \) is a representing system for a Hilbert space \( H \) (see [4, Theorem 5.1.6]).

Usually, when the notion of a Duffin-Schaeffer frame is generalized to the case of a Banach space \( F \), the duality \( \langle f, g_n \rangle \) is considered as the values of the functionals \( g_n \in G := F^* \) at \( f \in F \). Using this approach, the notions of an atomic decomposition and a Banach frame were introduced by Gröchenig [5].

For our purposes, it is more convenient to introduce the dual definitions by considering the Fourier coefficients \( \langle f_n, g \rangle \) of a functional \( g \in G \) with respect to a sequence \( \{f_n\}_{n=1}^{\infty} \) in the original space \( F \).

Let \( X \) be a sequence Banach space with a natural basis \( \{e_n\}_{n=1}^{\infty} \) (\( e_n = \{\delta_{mn}\}_{m=1}^{\infty} \) where \( \delta_{mn} \) is the Kronecker delta). Therefore, the dual space \( X^* \) is isomorphic to some sequence Banach space \( Y \).

As before, \( F \) is a Banach space and \( G := F^* \) is its dual space.

**Definition 3.** We say that a sequence \( \{f_n\}_{n=1}^{\infty} \subset F \setminus \{0\} \) of elements of a Banach space \( F \) is a frame for \( F \) with respect to \( X \) if there exist constants \( 0 < A \leq B < \infty \) such that for all bounded linear functionals \( g \in G \) the following inequalities are satisfied

\[
A \|g\|_G \leq \|\{\langle f_n, g \rangle\}_{n=1}^{\infty}\|_Y \leq B \|g\|_G. \tag{3}
\]

If we take \( F = G = H \) to be a Hilbert space and \( X = Y = \ell^2 \), we get a Duffin-Schaeffer frame.

**Lemma 1.** A sequence \( \{f_n\}_{n=1}^{\infty} \subset F \setminus \{0\} \) is a frame for \( F \) with respect to \( X \) if and only if the following two assertions hold

(i) for all \( x \in X \) the series \( \sum_{n=1}^{\infty} x_n f_n \) converges in \( F \),

(ii) for all \( f \in F \) there is an \( x \in X \) such that \( f = \sum_{n=1}^{\infty} x_n f_n \).
In particular, any frame is a representing system. The converse is also true: any representing system \( \{ f_n \}_{n=1}^\infty \subset F \setminus \{0\} \) is a frame for \( F \) with respect to its coefficients space \( X(f_n) \) consisting of all sequences \( \{ x_n \}_{n=1}^\infty \) for which the series \( \sum_{n=1}^\infty x_n f_n \) converges in \( F \). The coefficients space \( X(f_n) \) is equipped with the norm

\[
\| x \|_{X(f_n)} = \sup_{N=1,2,\ldots} \left\| \sum_{n=1}^N x_n f_n \right\|_F.
\]

In general, the same representing system can be a frame with respect to various sequence spaces \( X \).

Definition 3 was introduced in [6]. For more details about proofs in this section we refer the reader to [7] and [8].

3 Proof of the theorem

We have divided the proof into a sequence of lemmas.

**Lemma 2.** For each \( k \in \mathbb{N} \), let \( \omega_k^j \) be a \( k \)-th root of unity

\[
\omega_k^j = e^{\frac{2\pi i j}{k}}, \quad j = 0, \ldots, k - 1,
\]

and

\[
\| f \|_k := \left( \frac{1}{k} \sum_{j=0}^{k-1} \| f(\omega_k^j) \|^2 \right)^{1/2}.
\]

For each polynomial \( P(z) = \sum_{j=0}^{k-1} c_j z^j \) of degree less than \( k \) one has

\[
\| P \|_k = \| P \|_{H^2}.
\]

**Proof.** As the inverse discrete Fourier transform

\[
\hat{c}_l = \frac{1}{\sqrt{k}} \sum_{j=0}^{k-1} c_j \omega_k^j = \frac{P(\omega_k^l)}{\sqrt{k}}, \quad l = 0, \ldots, k - 1,
\]

is unitary, it follows that

\[
\| P \|_k = \left( \sum_{l=0}^{k-1} |\hat{c}_l|^2 \right)^{1/2} = \left( \sum_{j=0}^{k-1} |c_j|^2 \right)^{1/2} = \| P \|_{H^2}.
\]
Lemma 3. Let $$\sigma_r f(z) = f(rz), \quad 0 < r < 1.$$ The following inequality holds

$$\|\sigma_r f\|_k \leq \frac{\|f\|_{H^2}}{(1 - r^{2k})^{1/2}}.$$ 

Proof. We can expand $$f \in H^2, \quad f = \sum_{n=0}^{\infty} c_n z^n$$ into an orthogonal series

$$f(z) = \sum_{l=0}^{\infty} z^{kl} P_l(z),$$

where $$P_l(z) = \sum_{j=0}^{k-1} c_{j+kl} z^j$$ is a polynomial of degree less than $$k$$. Then

$$\|\sigma_r f\|_k \leq \sum_{l=0}^{\infty} r^{kl} \|\sigma_r P_l\|_k = \sum_{l=0}^{\infty} r^{kl} \|\sigma_r P_l\|_{H^2} \leq \sum_{l=0}^{\infty} r^{kl} \|P_l\|_{H^2}$$

$$\leq \left( \sum_{l=0}^{\infty} r^{2kl} \right)^{1/2} \left( \sum_{l=0}^{\infty} \|P_l\|_{H^2}^2 \right)^{1/2} = \frac{\|f\|_{H^2}}{(1 - r^{2k})^{1/2}}.$$ 

Lemma 4. The following inequalities hold true for all $$f \in H^2$$

$$\|f\|_{H^2} \leq \sup_{k \in \mathbb{N}} \|\sigma_{1-1/k} f\|_k \leq \frac{\|f\|_{H^2}}{(1 - e^{-2})^{1/2}}. \quad (4)$$ 

Proof. Since $$(1 - \frac{1}{k})^k \uparrow e^{-1}$$, we can easily obtain the upper estimate by using lemma 3

$$\|\sigma_{1-1/k} f\|_k \leq \frac{\|f\|_{H^2}}{(1 - (1 - \frac{1}{k})^{2k})^{1/2}} \leq \frac{\|f\|_{H^2}}{(1 - e^{-2})^{1/2}}.$$ 

To prove the lower estimate, we initially check it for an arbitrary polynomial $$P(z), \deg P = N$$. According to lemma 2, we have

$$\sup_{k \in \mathbb{N}} \|\sigma_{1-1/k} P\|_k \geq \sup_{k > N} \|\sigma_{1-1/k} P\|_{H^2} = \|P\|_{H^2}.$$ 

Now assume that $$f \in H^2$$ is an arbitrary function and select a polynomial $$P$$ such that $$\|f - P\|_{H^2} < \varepsilon$$. Using the triangle inequality and the proof
above, we can obtain
\[
\sup_{k \in \mathbb{N}} \|\sigma_{1/k} f\|_k \geq \sup_{k \in \mathbb{N}} \|\sigma_{1/k} P\|_k - \sup_{k \in \mathbb{N}} \|\sigma_{1/k} (f - P)\|_k \\
\geq \|P\|_{H^2} - \frac{\varepsilon}{(1 - e^{-2})^{1/2}} \geq \|f\|_{H^2} - \varepsilon - \frac{\varepsilon}{(1 - e^{-2})^{1/2}}.
\]

The proof is complete as \(\varepsilon\) tends to 0. \(\square\)

Now we have all the ingredients to prove theorem 1.

Let us denote by \(\ell_k^2\) a \(k\)-dimensional Hilbert space equipped with the norm
\[
\|c\|_2 := \left(\sum_{j=0}^{k-1} |c_j|^2\right)^{1/2}.
\]

Throughout the proof, \(X\) stands for a space with a mixed norm
\[
X = \ell^1(\ell_k^2) = \left(\bigoplus_{k=1}^{\infty} \ell_k^2\right)_{\ell^1}.
\]

Let
\[
I := \{(k, j) : j = 0, 1, \ldots, k - 1 \text{ and } k = 1, 2, \ldots\}
\]
be an index set. So \(X = \ell^1(\ell_k^2)\) is the space of families \(x = \{x_{k,j}\}_{(k,j) \in I}\) such that
\[
\|x\|_{1,2} := \sum_{k=1}^{\infty} \left(\sum_{j=0}^{k-1} |x_{k,j}|^2\right)^{1/2} < \infty.
\]

Clearly, the dual space of \(X\)
\[
Y = \ell^\infty(\ell_k^2) = \left(\bigoplus_{k=1}^{\infty} \ell_k^2\right)_{\ell^\infty}
\]
is the space of families \(y = \{y_{k,j}\}_{(k,j) \in I}\) satisfying
\[
\|y\|_{\infty,2} := \sup_{k \in \mathbb{N}} \left(\sum_{j=0}^{k-1} |y_{k,j}|^2\right)^{1/2} < \infty
\]
with the standard duality
\[
\langle x, y \rangle = \sum_{k=1}^{\infty} \sum_{j=0}^{k-1} x_{k,j} y_{k,j}.
\]
Inequality (4) of lemma 4 implies that the normalized sequence
\[
\hat{K}_{\lambda_n} = (1 - |\lambda_n|^2)^{1/2} K_{\lambda_n}, \quad n = 1, 2, \ldots,
\]
fulfills the frame inequalities
\[
A \|g\|_{H^2} \leq \|\{\langle \hat{K}_{\lambda_n}, g \rangle\}_{n=1}^{\infty}\|_{\infty,2} \leq B \|g\|_{H^2}
\]
when \(\{\lambda_n\}_{n=1}^{\infty}\) is defined by (1), because in this case we have
\[
(1 - |\lambda_{k,j}|^2)^{1/2} = (1 - (1 - \frac{1}{k})^2)^{1/2} \approx \frac{1}{\sqrt{k}}, \quad k = 1, 2, \ldots,
\]
and by definition
\[
\sup_{k \in \mathbb{N}} \|\sigma_{1-1/k} g\|_k = \sup_{k \in \mathbb{N}} \left(\frac{1}{k} \sum_{j=0}^{k-1} |g(\lambda_{k,j})|^2\right)^{1/2} \lesssim \|\{\langle \hat{K}_{\lambda_n}, g \rangle\}_{n=1}^{\infty}\|_{\infty,2}.
\]

Applying lemma 1 we conclude that for each function \(f \in H^2(\mathbb{D})\) there exist coefficients \(x_n = x_{k,j}\) such that
\[
\sum_{k=1}^{\infty} \left(\sum_{j=0}^{k-1} |x_{k,j}|^2\right)^{1/2} < \infty
\]
and the representation is valid
\[
f = \sum_{n=1}^{\infty} x_n \hat{K}_{\lambda_n} = \sum_{n=1}^{\infty} x_n (1 - |\lambda_n|^2)^{1/2} K_{\lambda_n}.
\]
This completes the proof.

**Remark 1.** Theorem 1 was announced without proof in our paper [9].

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