Secure and Private Source Coding with Private Key and Decoder Side Information

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Abstract—The problem of secure source coding with multiple terminals is extended by considering a remote source whose noisy measurements are the correlated random variables used for secure source reconstruction. The main additions to the problem include 1) all terminals noncausally observe a noisy measurement of the remote source; 2) a private key is available to all legitimate terminals; 3) the public communication link between the encoder and decoder is rate-limited; and 4) the secrecy leakage to the eavesdropper is measured with respect to the encoder input, whereas the privacy leakage is measured with respect to the remote source. Exact rate regions are characterized for a lossy source coding problem with a private key, remote source, and decoder side information under security, privacy, communication, and distortion constraints. By replacing the distortion constraint with a reliability constraint, we obtain the exact rate region also for the lossless case. Furthermore, the lossy rate region for scalar discrete-time Gaussian sources and measurement channels is established.

I. INTRODUCTION

Consider multiple terminals that observe correlated random sequences and wish to reconstruct these sequences at another terminal, called a decoder, by sending messages through noiseless communication links, i.e., the distributed source coding problem [1]. A sensor network, where each node observes a correlated random sequence that should be reconstructed at a distant node is a classic example for this problem [2, pp. 258]. Similarly, function computation problems in which a fusion center observes messages sent by other nodes to compute a function are closely related problems and can be used to model various recent applications [3], [4]. Since the messages sent over the communication links can be public, security constraints are imposed on these messages against an eavesdropper in the same network [5]. If all sent messages are available to the eavesdropper, then it is necessary to provide an advantage to the decoder over the eavesdropper to enable secure source coding. Providing side information, which is correlated with the sequences that should be reconstructed, to the decoder can provide such an advantage over the eavesdropper that can also have side information, as in [6]–[8]. Allowing the eavesdropper to access only a strict subset of all messages is also a method to enable secure distributed source coding, considered in [9]–[11]; see also [12] in which a similar method is applied to enable secure remote source reconstruction. Similarly, also a private key that is shared by legitimate terminals and hidden from the eavesdropper can provide such an advantage, as in [13], [14].

Source coding models in the literature commonly assume that dependent multi-letter random variables are available and should be compressed. For secret-key agreement [15], [16] and secure function computation problems [17], [18], which are instances of the source coding with side information problem [19, Section IV-B], the correlation between these multi-letter random variables is posted in [20], [21] to stem from an underlying ground truth that is a remote source such that its noisy measurements are these dependent random variables. Such a remote source allows to model the cause of correlation in a network, so we also posit that there is a remote source whose noisy measurements are used in the source coding problems discussed below, which is similar to the models in [22, pp. 78] and [23, Fig. 9]. Furthermore, in the chief executive officer (CEO) problem [24], there is a remote source whose noisy measurements are encoded such that a decoder can reconstruct the remote source by using the encoder outputs. Our model is different from the model in the CEO problem, since in our model the decoder aims to recover encoder observations rather than the remote source that is considered mainly to describe the cause of correlation between encoder observations. Thus, we define the secrecy leakage as the amount of information leaked to an eavesdropper about encoder observations. Since the remote source is common for all observations in the same network, we impose a privacy leakage constraint on the remote source because each encoder output observed by an eavesdropper leaks information about unused encoder observations, which might later cause secrecy leakage when the unused encoder observations are employed [25], [26]; see [27]–[29] for joint secrecy and joint privacy constraints imposed due to multiple uses of the same source.

We characterize the rate region for a lossy secure and private source coding problem with one private key, remote source, encoder, decoder, eavesdropper, and eavesdropper and decoder side information. Requiring reliable source reconstruction, we characterize the rate region also for the lossless case. A Gaussian remote source and independent additive Gaussian noise measurement channels are considered to establish their
lossy rate region under squared error distortion.

II. SYSTEM MODEL

We consider the lossy source coding model with one encoder, one decoder, and an eavesdropper (Eve), depicted in Fig. 1. The encoder $Enc(\cdot, \cdot)$ observes a noisy measurement $\tilde{X}^n$ of an i.i.d. remote source $X^n$ through a memoryless channel $P_{X|X}$ in addition to a private key $K \in [1 : 2^nR_0]$. The encoder output is an index $W$ that is sent over a link with limited communication rate. The decoder $Dec(\cdot, \cdot, \cdot)$ observes the index $W$, as well as the private key $K$ and another noisy measurement $Y^n$ of the same remote source $X^n$ through another memoryless channel $P_{Y|X}$ in order to estimate $\tilde{X}^n$ as $\tilde{X}^n$. The other noisy output $Z^n$ of $P_{Y|X}$ is observed by Eve in addition to the index $W$. Suppose $K$ is uniformly distributed, hidden from Eve, and independent of the source output and its noisy measurements. The source and measurement alphabets are finite sets.

We next define the rate region for the lossy secure and private source coding problem defined above.

**Definition 1.** A lossy tuple $(R_w, R_s, R_\ell, D) \in \mathbb{R}_{\geq 0}^4$ is achievable, given a private key with rate $R_0 \geq 0$, if for any $\delta > 0$ there exist $n \geq 1$, an encoder, and a decoder such that

\begin{align}
\log |W| &\leq n(R_w + \delta) \quad \text{(storage)} \quad (1) \\
I(\tilde{X}^n; W|Z^n) &\leq n(R_s + \delta) \quad \text{(secrecy)} \quad (2) \\
I(X^n; W|Z^n) &\leq n(R_\ell + \delta) \quad \text{(privacy)} \quad (3) \\
\mathbb{E}\left[d\left(\tilde{X}^n, \tilde{X}\left(Y^n, W, K\right)\right)\right] &\leq D + \delta \quad \text{(distortion)} \quad (4)
\end{align}

where $d(\tilde{x}^n, \tilde{x}^n) = \frac{1}{n} \sum_{i=1}^{n} d(x_i, \tilde{x}_i)$ is a per-letter bounded distortion metric. The lossy secure and private source coding region $\mathcal{R}_D$ is the closure of the set of all achievable lossy tuples.

Note that in (2) and (3) we consider conditional mutual information terms to take account of unavoidable privacy and secrecy leakages due to Eve’s side information; see also [21], [30]. Furthermore, considering conditional mutual information terms rather than corresponding conditional entropy terms, the latter of which is used in [6], [14], [31]–[33], to characterize the secrecy and privacy leakages simplifies our analysis.

We next define the rate region for the lossless secure and private source coding problem.

**Definition 2.** A lossless tuple $(R_w, R_s, R_\ell) \in \mathbb{R}_{\geq 0}^3$ is achievable, given a private key with rate $R_0 \geq 0$, if for any $\delta > 0$ there exist $n \geq 1$, an encoder, and a decoder such that we have (1)-(3) and

\begin{align}
\Pr\left[\tilde{X}^n \neq \tilde{X}\left(Y^n, W, K\right)\right] &\leq \delta \quad \text{(reliability)} \quad (5)
\end{align}

The lossless secure and private source coding region $\mathcal{R}_L$ is the closure of the set of all achievable lossless tuples.

III. SECURE AND PRIVATE SOURCE CODING REGIONS

A. Lossy Source Coding

The lossy secure and private source coding region $\mathcal{R}_D$ is characterized next; see [34, Section V] for its proof and below for a proof sketch.

Define $[a]^- = \min\{a, 0\}$ for $a \in \mathbb{R}$ and denote

\begin{equation}
R' = I(U; Z|V, Q) - I(U; Y|V, Q) - D.
\end{equation}

**Theorem 1.** For given $P_X, P_{X|X}, P_{Y|X}$, and $R_0$, the region $\mathcal{R}_D$ is the set of all rate tuples $(R_w, R_s, R_\ell, D)$ satisfying

\begin{align}
R_w &\geq I(U; \tilde{X}|Y) \quad (7) \\
\text{and if } R_0 < I(U; \tilde{X}|Y, V), \text{ then } R_s &\geq I(U; \tilde{X}|Z) + R' - R_0 \quad (8) \\
\text{and if } I(U; \tilde{X}|Y, V) \leq R_0 < I(U; \tilde{X}|Y), \text{ then } R_\ell &\geq I(V; \tilde{X}|Z) \quad (9) \\
\text{if } R_0 \geq I(U; \tilde{X}|Y), \text{ then } R_s &\geq 0 \quad (10) \\
\text{if } R_0 \geq I(U; \tilde{X}|Y), \text{ then } R_\ell &\geq 0 \quad (11)
\end{align}

for some

\begin{equation}
P_{QUV; \tilde{X}XYZ} = P_Q|V|P_U|U|P_Y|X|P_{\tilde{X}|X}P_XP_{YZ|X}
\end{equation}

such that $\mathbb{E}\left[d(\tilde{X}, \tilde{X}\left(U, Y\right)\right)\right] \leq D$ for some reconstruction function $\tilde{X}(U, Y)$. The region $\mathcal{R}_D$ is convexified by using the time-sharing random variable $Q$, required due to the $\lfloor a \rfloor^-$ operation. One can limit the cardinalities to $|Q| \leq 2$, $|V| \leq |\tilde{X}| + 3$, and $|U| \leq (|\tilde{X}| + 3)^2$.

**Proof Sketch:** For the achievability proof, we leverage the output statistics of random binning (OSRB) method [16], [35], [36] by following the steps described in [37, Section I.6].

Three private key rate ranges are considered separately. For small private key rates, we assign two random bin indices to auxiliary sequences $V^n = v^n$ and three random bin indices to
Thus, the reliability constraint in (5) is satisfied because \( d \) can be reliably reconstructed at the decoder. Applying the OSRB method consecutively, six different recoverability cases, indicating which single-letter bound can be used, are analyzed to obtain the secrecy and privacy leakage rates. A time-sharing key rates, two random bin indices are assigned to \( R_w \) for some \( \mathcal{U} \) random variable \( Q \) to obtain the secrecy and privacy leakage rates. A time-sharing random variable \( Q \) is used to enlarge the rate region via convexification. The converse proof follows by using standard properties of the Shannon entropy.

We remark that (12) and (13) show that one can simultaneously achieve strong secrecy and strong privacy, i.e., the conditional mutual information terms in (2) and (3), respectively, are negligible, by using a large private key \( K \), which is a result missing in some recent works on secure source coding with private key.

B. Lossless Source Coding

The lossless secure and private source coding region \( \mathcal{R} \) is characterized next; see below for a proof sketch.

Denote

\[
R'' = [I(\tilde{X}; Z|V, Q) - I(\tilde{X}; Y|V, Q)].
\]

**Lemma 1.** For given \( P_X, P_{X|X}, P_{Y|X}, \) and \( R_0 \), the region \( \mathcal{R} \) is the set of all rate tuples \( (R_s, R_t, R_e) \) satisfying

\[
R_w \geq H(\tilde{X}|Y)
\]

and if \( R_0 < H(\tilde{X}|Y, V) \), then

\[
\begin{align*}
R_s \geq & H(\tilde{X}|Z) + R'' - R_0 \\
R_t \geq & I(\tilde{X}; Z) + R'' - R_0 \quad \text{if } H(\tilde{X}|Y, V) \leq R_0 < H(\tilde{X}|Y) \\
R_s \geq & I(V; \tilde{X}|Z) \\
R_t \geq & I(V; X|Z) \quad \text{if } R_0 \geq H(\tilde{X}|Y) \\
R_s \geq & 0 \\
R_t \geq & 0
\end{align*}
\]  

for some

\[
P_{QYXZ} = P_{Q|V} P_{V|X} P_{X} P_{Y|X}.  
\]

One can limit the cardinalities to \(|Q| \leq 2\) and \(|V| \leq |\tilde{X}| + 2\).

**Proof Sketch:** The proof for the lossless region \( \mathcal{R} \) follows from the proof for the lossy region \( \mathcal{R}_D \), given in Theorem 1 above, by choosing \( U = \tilde{X} \) such that we have the reconstruction function \( \tilde{X}(X, Y) = \tilde{X} \) so we achieve \( D = 0 \). Thus, the reliability constraint in (5) is satisfied because \( d(\cdot, \cdot) \) is a distortion metric.

IV. GAUSSIAN SOURCES AND CHANNELS

We evaluate the lossy rate region for a Gaussian example with squared error distortion by finding the optimal auxiliary random variable in the corresponding rate region. Consider a special lossy source coding case in which (i) there is no private key; (ii) the eavesdropper’s channel observation \( Z^n \) is less noisy than the decoder’s channel observation \( Y^n \) such that we obtain a lossy source coding region with a single auxiliary random variable that should be optimized. We next define less noisy channels, considering \( P_{Y|X} \).

**Definition 3** ([38]). \( Z \) (or eavesdropper) is less noisy than \( Y \) (or decoder) if

\[
I(L; Z) \geq I(L; Y)
\]

holds for any random variable \( L \) such that \( L \rightarrow X \rightarrow (Y, Z) \) form a Markov chain.

**Corollary 1.** For given \( P_X, P_{X|X}, P_{Y|X}, \) and \( R_0 = 0 \), the region \( \mathcal{R}_D \) when the eavesdropper is less noisy than the decoder is the set of all rate tuples \( (R_s, R_t, R_e, D) \) satisfying

\[
\begin{align*}
R_w \geq & I(\bar{U}; \bar{Y}|Y) = I(\bar{U}; \bar{X}) - I(\bar{U}; Y) \\
R_e \geq & I(U; \tilde{X}|Z) = I(U; \bar{X}) - I(U; Z) \\
R_t \geq & I(U; \tilde{X}) - I(U; Z) \quad \text{for some}
\end{align*}
\]

\[
P_{U\bar{X}YZ} = P_{U|X} P_{\bar{X}|X} P_{X} P_{Y|X} 
\]

such that \( \mathbb{E}[d(\tilde{X}, \tilde{X}(U, Y))] \leq D \) for some reconstruction function \( \tilde{X}(U, Y) \). One can limit the cardinality to \(|U| \leq |\tilde{X}| + 3\).

**Proof Sketch:** The proof for Corollary 1 follows from the proof for Theorem 1 by considering the bounds in (7)-(9) since \( R_0 = 0 \). Furthermore, \( R'' \) defined in (6) is 0 for the less noisy condition considered, which follows because \( \langle Q, V \rangle \rightarrow U \rightarrow X \rightarrow (Y, Z) \) form a Markov chain.

Suppose the following scalar discrete-time Gaussian source and channel model for the lossy source coding problem depicted in Fig. 1

\[
\begin{align*}
X & = \rho_x \tilde{X} + N_x \\
Y & = \rho_y X + N_y \\
Z & = \rho_z X + N_z 
\end{align*}
\]

where we have the remote source \( X \sim \mathcal{N}(0, 1) \), fixed correlation coefficients \( \rho_x, \rho_y, \rho_z \in (-1, 1) \), and additive Gaussian random noise variables \( N_x \sim \mathcal{N}(0, 1 - \rho_x^2) \), \( N_y \sim \mathcal{N}(0, 1 - \rho_y^2) \), \( N_z \sim \mathcal{N}(0, 1 - \rho_z^2) \) such that \( (X, N_x, N_y, N_z) \) are mutually independent, and we consider the squared error distortion, i.e.,

\[
d(\tilde{X}, X) = (\tilde{X} - X)^2. \]

We remark that (29) is an inverse measurement channel \( P_{X|\tilde{X}} \) that is a weighted sum of two independent Gaussian random variables, imposed to be able to apply the conditional entropy power inequality (EPI) [39, Lemma II]; see [20, Theorem 3] and [40, Section V] for binary symmetric inverse channel assumptions imposed to apply Mrs. Gerber’s
lemma [41]. Suppose \( |\rho_z| > |\rho_y| \) such that \( Y \) is stochastically-degraded than \( Z \) since then there exists a random variable \( Y \) such that \( P_{Y|X} = P_{Y|X} \) and \( P_{Z|X} = P_{Z|X} P_{Y|Z} \) [42, Lemma 6], so \( Z \) is also less noisy than \( Y \) since less noisy channels constitute a strict superset of the set of stochastically-degraded channels and both channel sets consider only the conditional marginal probability distributions [2, pp. 121].

We next take the liberty to use the lossy rate region in Corollary 1, characterized for discrete memoryless channels, for the model in (29)-(31) without a private key is given below.

For Gaussian sources and channels, we use differential entropy and eliminate the cardinality bound on the auxiliary random variable. The lossy source coding region for the model in (29)-(31) without a private key is given below.

**Lemma 2.** For the model in (29)-(31) such that \( |\rho_z| > |\rho_y| \) and \( R_0 = 0 \), the region \( \mathcal{R}_D \) with squared error distortion is the set of all rate tuples \((R_u, R_t, R_e, D)\) satisfying, for \( 0 < \alpha \leq 1 \),

\[
R_u \geq \frac{1}{2} \log \left( \frac{1 - \rho_z^2 \rho_y^2 (1 - \alpha)}{1 - \rho_z \rho_y + \alpha} \right) \tag{32}
\]

\[
R_t \geq \frac{1}{2} \log \left( \frac{1 - \rho_z^2 \rho_y^2 (1 - \alpha)}{1 - \rho_z \rho_y + \alpha} \right) \tag{33}
\]

\[
R_e \geq \frac{1}{2} \log \left( \frac{1 - \rho_z^2 \rho_y^2 (1 - \alpha)}{1 - \rho_z \rho_y + \alpha} \right) \tag{34}
\]

\[
D \geq \frac{\alpha(1 - \rho_z^2 \rho_y^2)}{1 - \rho_z \rho_y + \alpha}. \tag{35}
\]

**Proof Sketch:** For the achievability proof, let \( U \sim \mathcal{N}(0, 1 - \alpha) \) and \( \Theta \sim \mathcal{N}(0, \alpha) \), as in [43, Eq. (34)] and [44, Appendix B], be independent random variables for some \( 0 < \alpha \leq 1 \) such that \( \tilde{X} = U + \Theta \) and \( U - \tilde{X} - X - (Y, Z) \) form a Markov chain. Choose the reconstruction function \( \tilde{X}(U, Y) \) as the minimum mean square error (MMSE) estimator, and given any fixed \( D > 0 \) auxiliary random variables are chosen such that the distortion constraint is satisfied. We then have for the squared error distortion

\[
D = \mathbb{E} \left[ (\tilde{X} - \tilde{X}(U, Y))^2 \right] \overset{(a)}{=} \frac{1}{2} \pi e^{2h(\tilde{X}|U, Y)} \tag{36}
\]

where equality in \((a)\) is achieved because \( \tilde{X} \) is Gaussian and the reconstruction function is the MMSE estimator [45, Theorem 8.6.6]. Define the covariance matrix of the vector random variable \( [\tilde{X}, U, Y] \) as \( K_{\tilde{X}UV} \) and of \([U, Y] \) as \( K_{UY} \), respectively. We then have

\[
h(\tilde{X}|U, Y) = h(\tilde{X}, U, Y) - h(U, Y)
\]

\[
= \frac{1}{2} \log \left( \frac{\det(K_{\tilde{X}UV})}{\det(K_{UY})} \right) \tag{37}
\]

where \( \det(\cdot) \) is the determinant of a matrix; see also [12, Section F]. Combining (36) and (37), and calculating the determinants, we obtain

\[
D = \frac{\alpha(1 - \rho_z^2 \rho_y^2)}{1 - \rho_z \rho_y + \alpha}. \tag{38}
\]

One can also show that

\[
I(U; \tilde{X}) = h(\tilde{X}) - h(\tilde{X}|U) = \frac{1}{2} \log \left( \frac{1}{\alpha} \right) \tag{39}
\]

\[
I(U; X) = h(X) - h(X|U) = \frac{1}{2} \log \left( \frac{1}{1 - \rho_z^2 (1 - \alpha)} \right) \tag{40}
\]

\[
I(U; Z) = h(Z) - h(Z|U) = \frac{1}{2} \log \left( \frac{1}{1 - \rho_z^2 \rho_y^2 (1 - \alpha)} \right) \tag{41}
\]

\[
I(U; Z) = h(Z) - h(Z|U) = \frac{1}{2} \log \left( \frac{1}{1 - \rho_z^2 \rho_y^2 (1 - \alpha)} \right) \tag{42}
\]

Thus, by calculating (25)-(27), the achievability proof follows.

For the converse proof, one can first show that

\[
I(U; \tilde{X}) = h(\tilde{X}) - h(\tilde{X}|U) \tag{43}
\]

\[
I(U; X) = h(X) - h(X|U) \tag{44}
\]

\[
I(U; Z) = h(Z) - h(Z|U) \tag{45}
\]

which follow since \( h(\tilde{X}) = h(X) = h(Y) = h(Z) \). Suppose

\[
h(\tilde{X}|U) = \frac{1}{2} \log(2\pi e \alpha) \tag{46}
\]

for any \( 0 < \alpha \leq 1 \) that represents the unique variance of a Gaussian random variable; see [20, Lemma 2] for a similar result applied to binary random variables. Thus, by applying the conditional EPI, we obtain

\[
e^{2h(Y|U)} \overset{(a)}{=} e^{2h(\rho_z \rho_y \tilde{X}|U)} + e^{2h(\rho_y N_x + N_y)}
\]

\[
= 2\pi e (\rho_z^2 \rho_y^2 \alpha + \rho_y^2 (1 - \rho_z^2) + 1 - \rho_y^2)
\]

\[
= 2\pi e (1 - \rho_z^2 \rho_y^2 (1 - \alpha)) \tag{47}
\]

where \((a)\) follows because \( U - \tilde{X} = (N_x, N_y) \) form a Markov chain and \((N_x, N_y)\) are independent of \( \tilde{X} \), so \((N_x, N_y)\) are independent of \( U \), and equality is satisfied since, given \( U, \rho_z \rho_y \tilde{X} \) and \((\rho_y N_x + N_y)\) are conditionally independent and they are Gaussian random variables, as imposed in (46) above; see [20, Lemma 1 and Eq. (28)] for a similar result applied to binary random variables by extending Mrs. Gerber’s lemma. Similarly, we have

\[
e^{2h(Z|U)} = 2\pi e (1 - \rho_z^2 \rho_y^2 (1 - \alpha)) \tag{48}
\]

which follows by replacing \((Y, \rho_y, N_y)\) with \((Z, \rho_z, N_z)\) in (47), respectively, because the channel \( P_{Y|X} \) can be mapped to \( P_{Z|U} \) with these changes due to (29)-(31) and the Markov chain \( U - \tilde{X} - X - (Y, Z) \). Furthermore, we have

\[
e^{2h(X|U)} \overset{(a)}{=} e^{2h(\rho_z \tilde{X}|U)} + e^{2h(N_z)}
\]

\[
= 2\pi e (\rho_z^2 \alpha + 1 - \rho_z^2) = 2\pi e (1 - \rho_z^2 (1 - \alpha)) \tag{49}
\]

where \((a)\) follows because \( N_x \) is independent of \( U \), and equality is achieved since, given \( U, \rho_z \tilde{X} \) and \( N_x \) are conditionally independent and are Gaussian random variables. Therefore, by
applying (43)-(49) to (25)-(27), the converse proof for (32)-(34) follows. Next, consider
\[
\begin{align*}
h(\tilde{X}|U,Y) &= -I(U;\tilde{X}|Y) + h(\tilde{X}|Y) \\
&= -h(Y|U) + h(\tilde{X}|U) + h(Y|\tilde{X}) \\
&\leq \frac{1}{2} \log \left( \frac{\alpha}{1 - \rho_{xy}^2 \rho_{x}^2 (1 - \alpha)} \right) + h(\rho_x \rho_N + \rho_y N_x + N_y, \tilde{X}) \\
&\leq \frac{1}{2} \log \left( \frac{\alpha}{1 - \rho_{xy}^2 \rho_{x}^2 (1 - \alpha)} \right) + h(\rho_y N_x + N_y) \\
&= \frac{1}{2} \log \left( \frac{2 \pi e}{\alpha(1 - \rho_{xy}^2 \rho_{x}^2 (1 - \alpha))} \right) \\
&= \frac{1}{2} \log \left( \frac{2 \pi e}{\alpha(1 - \rho_{xy}^2 \rho_{x}^2 (1 - \alpha))} \right)
\end{align*}
\] (50)

where (a) follows by (25) and (43), and since \(h(Y) = h(\tilde{X})\), (b) follows by (46) and (47), and (c) follows because \((N_x, N_y)\) are independent of \(\tilde{X}\). For any random variable \(\tilde{X}\) and reconstruction function \(\tilde{Y}(U, Y)\), we have [45, Theorem 8.6.6]
\[
\mathbb{E}\left[ (\tilde{X} - \tilde{Y}(U, Y))^2 \right] \geq \frac{1}{2 \pi e} e^{2h(\tilde{X}|U, Y)}.
\] (51)
Combining the distortion constraint given in Corollary 1 with (50) and (51), the converse proof for (35) follows.

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