Characterization of the Imbalance Problem on Complete Bipartite Graphs

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Abstract. We study the imbalance problem on complete bipartite graphs. The imbalance problem is a graph layout problem and is known to be NP-complete. Graph layout problems find their applications in the optimization of networks for parallel computer architectures, VLSI circuit design, information retrieval, numerical analysis, computational biology, graph theory, scheduling and archaeology [2]. In this paper, we give characterizations for the optimal solutions of the imbalance problem on complete bipartite graphs. Using the characterizations, we can solve the imbalance problem in time polylogarithmic in the number of vertices, when given the cardinalities of the parts of the graph, and verify whether a given solution is optimal in time linear in the number of vertices on complete bipartite graphs. We also introduce a generalized form of complete bipartite graphs on which the imbalance problem is solvable in time quasilinear in the number of vertices by using the aforementioned characterizations.

Keywords: Imbalance Problem · Vertex layout · Complete bipartite graph · Proper interval bipartite graph.

1 Introduction

Graph layout problems are combinatorial optimization problems, where the goal is to find an ordering on the vertices that optimizes an objective function. A large number of problems from different domains can be formulated as graph layout problems [2]. The imbalance problem is a graph layout problem that has applications in 3-dimensional circuit design [7].

The imbalance problem was introduced by Biedl et al. [1]. Given an ordering of the vertices of a graph $G$, the imbalance of a vertex $v$ is the absolute difference in the number of neighbors to the left of $v$ and the number of neighbors to the right of $v$. The imbalance of an ordering is the sum of the imbalances of the vertices. An instance of the imbalance problem consists of a graph $G$ and an integer $k$. The problem asks whether there exists an ordering on the vertices of $G$ such that the imbalance of the ordering is at most $k$.

The imbalance problem is NP-complete for several graph classes, including bipartite graphs with degree at most 6, weighted trees [1], general graphs with degree at most 4 [6], and split graphs [4]. The problem becomes polynomial time
solvable on superfragile graphs [4]. The problem is linear time solvable on proper interval graphs [4], bipartite permutation graphs, and threshold graphs [3].

Gorzny showed that the minimum imbalance of a bipartite permutation graph $G = (V, E)$, which class is a superclass of complete bipartite graphs and proper interval graphs, can be computed in $O(|V| + |E|)$ time [3]. We give characterizations for the optimal solutions of the imbalance problem on complete bipartite graphs. Using the characterizations, we show that the imbalance problem is solvable in $O(\log(|V|) \cdot \log(\log(|V|)))$ time on complete bipartite graphs, when given the cardinalities of the parts of the graph. Additionally, using the characterizations, we can verify whether a given solution is optimal in $O(|V|)$ on complete bipartite graphs. We also introduce a generalized form of complete bipartite graphs, which we call chained complete bipartite graphs, on which the imbalance problem is solvable in $O(c \cdot \log(|V|) \cdot \log(\log(|V|)))$ time, where $c = O(|V|)$, by using the aforementioned characterizations. As chained complete bipartite graphs are a subclass of proper interval bipartite graphs, the result of Gorzny also applies to chained complete bipartite graphs.

2 Preliminaries

We only consider graphs that are finite, undirected, connected, and simple (i.e. without multiple edges or loops). A graph $G$ is denoted as $G = (V, E)$, where $V$ denotes the set of vertices and $E \subseteq V \times V$ denotes the set of undirected edges. For convenience, we abbreviate bipartite graph as bigraph.

For $n \in \mathbb{N}^+$, let us define $[n] = \{1, 2, \ldots, n\}$.

We define $\sigma_S : S \to |S|$ to be an ordering of $S$. For convenience, at times we denote an ordering $\sigma_S$ by $(\sigma^{-1}(1), \sigma^{-1}(2), \ldots, \sigma^{-1}(|S|))$, where $v = \sigma^{-1}(\sigma(v))$ for $v \in S$. We also say that $v \in S$ is at position $k$ in ordering $\sigma_S$ if $\sigma_S(v) = k$.

Let $S_1, S_2, \ldots, S_n$ be a collection of disjoint sets. Let $s \in S_i$, where $1 \leq i \leq n$, then the concatenation of orderings is defined as follows: $\sigma_{S_1} \sigma_{S_2} \ldots \sigma_{S_n} (s) = \sigma_{S_i}(s) + \sum_{j=1}^{i-1} |S_j|$. Additionally, we use the product notation to denote the concatenation over a set. That is, $\prod_{i=1}^{n} \sigma_{S_i} = \sigma_{S_1} \sigma_{S_2} \ldots \sigma_{S_n}$.

Given an ordering $\sigma_S$ we say that $v \in S$ occurs to the left of $u \in S$ in $\sigma_S$, if and only if $\sigma_S(v) < \sigma_S(u)$. We denote this as $v \prec_{\sigma_S} u$. We define $\succ_{\sigma_S}$ analogously.

Given an ordering $\sigma_S$ we say that $\sigma_{S'}$, where $S' \subseteq S$, is a subordering of $\sigma_S$ on $S'$ if $\sigma_{S'}$ preserves the relative ordering of the elements in $S'$. That is $\sigma_{S'}$ is a subordering of $\sigma_S$ if and only if $\forall u, v \in S', u \prec_{\sigma_S} v \iff u \prec_{\sigma_{S'}} v$.

For a graph $G = (V, E)$, the open neighborhood of a vertex $v \in V$, denoted by $N(v)$, is the set of vertices adjacent to $v$. We call the vertices in $N(v)$ the neighbors of vertex $v$. That is $N(v) = \{u \in V \mid \{u, v\} \in E\}$.

The imbalance of a vertex $v \in V$ on the ordering $\sigma_V$ in graph $G = (V, E)$, denoted by $I(v, \sigma_V, G)$, is defined to be the absolute difference in the number of neighbors of $v$ occurring to the left of $v$ and the number of neighbors of $v$
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3 Imbalance on Complete Bipartite graphs

Let $G = (X, Y, E)$ be a complete bigraph. We shall prove that the minimum imbalance of $G$ is $|X| \cdot |Y| + (|X| \mod 2) \cdot (|Y| \mod 2)$.

Lemma 1. If $G = (X, Y, E)$ is a complete bigraph, then there exists an ordering $\sigma_{X \cup Y}$ such that $I(\sigma_{X \cup Y}) = |X| \cdot |Y| + (|X| \mod 2) \cdot (|Y| \mod 2)$.

Proof. The proof constructs an ordering on $X \cup Y$ and considers two cases, either the cardinality one part is even or none of the cardinalities of the parts are even.

Case 1. $(|X| \mod 2 = 0) \vee (|Y| \mod 2 = 0)$.

W.l.og. assume that $|Y|$ is even. Let $Y_1$ and $Y_2$ partition $Y$ into two sets of equal size. Consider ordering $\sigma_{X \cup Y} = \sigma_{Y_1} \sigma_X \sigma_{Y_2}$. In this ordering, the imbalance of any vertex in $X$ is zero and the imbalance of any vertex in $Y$ is $|X|$. Thus the imbalance of ordering $\sigma_{X \cup Y}$ is $I(\sigma_{X \cup Y}) = |X| \cdot |Y|$.
σ_{Y_1, X_1, Y_2} = Y_1 \ X_1 \ Y_2 = X_2 \ Y_2

Fig. 1. Example of “sandwiched” ordering for G = K_{4,9}.

Case 2. (|X| \mod 2 = 1) \land (|Y| \mod 2 = 1).
Let y_m \in Y be an element of Y. Let X_1 and X_2 partition X into two sets such that ||X_1| - |X_2|| = 1 and let Y_1 and Y_2 partition Y \{y_m\} into two sets such that |Y_1| = |Y_2|. Consider the ordering σ_{X_\cup Y} = σ_Y_1 \sigma_{X_1} \sigma_{y_m} \sigma_{X_2} \sigma_{Y_2}. The imbalance of any vertex in X \cup \{y_m\} is 1 and the imbalance of any vertex in Y \{y_m\} is |X|. Thus the imbalance of the ordering σ_{X_\cup Y} is I(σ_{X_\cup Y}) = |X| \cdot |Y| + 1.

σ_{Y_1, X_1, (y_m) \sigma_{X_2} \sigma_{Y_2}} = Y_1 \ X_1 \ \{y_m\} \ X_2 \ Y_2

Fig. 2. Example of “pseudo-sandwich” ordering for G = K_{3,9}.

**Definition 1.** Let G = (X, Y, E) be a complete bigraph and σ_{X_\cup Y} be an arbitrary ordering on X \cup Y. We define L(Y, σ_{X_\cup Y}) \subseteq [|X| + |Y|] to be the positions of the elements of Y in σ_{X_\cup Y}. That is L(Y, σ_{X_\cup Y}) = \{σ_{X_\cup Y}(y) \mid y \in Y\}. Let us denote the elements in L(Y, σ_{X_\cup Y}) as L(Y, σ_{X_\cup Y}) = \{l_1^{σ_{X_\cup Y}}, l_2^{σ_{X_\cup Y}}, \ldots, l_{|Y|}^{σ_{X_\cup Y}}\} such that l_1^{σ_{X_\cup Y}} < l_2^{σ_{X_\cup Y}} < \cdots < l_{|Y|}^{σ_{X_\cup Y}}. Additionally, we define l_0^{σ_{X_\cup Y}} = 0 and l_{|Y|+1}^{σ_{X_\cup Y}} = |X| + |Y| + 1. We leave out the superscript in l_i^{σ_{X_\cup Y}}, when the ordering is clear from the context.

Additionally, we define L_i^{σ_{X_\cup Y}} to be the vertices of X between the positions l_i and l_{i+1} in ordering σ_{X_\cup Y}. Let Y = \{y_1, \ldots, y_{|Y|}\} be an arbitrary enumeration of the element in Y. We leave out the superscript in L_i^{σ_{X_\cup Y}}, when the ordering is clear from the context.

σ_{X_\cup Y} = l_0 \ l_1 \ l_2 \ l_3 \ l_4 \ l_5 \ l_6 \ l_7 \ l_8 \ l_9 \ l_{10} \ l_{11} \ l_{12} \ l_{13} \ l_{14}

σ_{X_\cup Y}(y) = 1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8 \ 9 \ 10 \ 11 \ 12 \ 13 \ 14

Fig. 3. Visualization of Definition 1.
Let $\sigma_{X\cup Y} = \sigma L_o \prod_{i=1}^{\lfloor \frac{|Y|}{2} \rfloor} \sigma_{\{y_i\}} \sigma L_i$ be an arbitrary ordering. Let us define

\begin{align*}
\text{shift}_L (\sigma_{X\cup Y}) &= \sigma_{\{y_{\lfloor \frac{|Y|}{2} \rfloor} \}} \sigma L_o \left( \prod_{i=1}^{\lfloor \frac{|Y|}{2} \rfloor} \sigma_{\{y_i\}} \sigma L_i \right) \sigma L_{\lfloor \frac{|Y|}{2} \rfloor+1} \left( \prod_{i=1}^{\lfloor \frac{|Y|}{2} \rfloor+1} \sigma_{\{y_i\}} \sigma L_i \right), \\
\text{shift}_R (\sigma_{X\cup Y}) &= \sigma L_o \left( \prod_{i=1}^{\lfloor \frac{|Y|}{2} \rfloor} \sigma_{\{y_i\}} \sigma L_i \right) \sigma L_{\lfloor \frac{|Y|}{2} \rfloor+1} \left( \prod_{i=1}^{\lfloor \frac{|Y|}{2} \rfloor+1} \sigma_{\{y_i\}} \sigma L_i \right) \sigma_{\{y_{\lfloor \frac{|Y|}{2} \rfloor+1} \}} 
\end{align*}

That is, $\text{shift}_L$ moves vertex $y_{\lfloor \frac{|Y|}{2} \rfloor}$ to the left most position and $\text{shift}_R$ moves vertex $y_{\lfloor \frac{|Y|}{2} \rfloor+1}$ to the right most position.

Fig. 4. Visualization of the $\text{shift}_L$ and $\text{shift}_R$ functions with an ordering $\sigma_{X\cup Y}$ on the vertices of graph $G = K_{10,4}$.

Lemma 2. Let $\sigma_{X\cup Y}$ be an arbitrary imbalance optimal ordering. Let $\sigma'_{X\cup Y} = \text{shift}_L (\sigma_{X\cup Y})$. We have that $I(\sigma_{X\cup Y}) = I(\sigma'_{X\cup Y})$.

Proof. We have that $I(\sigma_{X\cup Y}, y_{\lfloor \frac{|Y|}{2} \rfloor}) = \left| \left( \sum_{i=0}^{\lfloor \frac{|Y|}{2} \rfloor} |L_i^{\sigma_{X\cup Y}}| \right) - \left( \sum_{i=0}^{\lfloor \frac{|Y|}{2} \rfloor} |L_i^{\sigma_{X\cup Y}}| \right) \right|$ and $I(\sigma'_{X\cup Y}, y_{\lfloor \frac{|Y|}{2} \rfloor}) = \sum_{i=0}^{\lfloor \frac{|Y|}{2} \rfloor} |L_i^{\sigma_{X\cup Y}}| = |X|$. For each $0 \leq i \leq \lfloor \frac{|Y|}{2} \rfloor - 1$, the imbalance of all the vertices in $L_i^{\sigma_{X\cup Y}}$ is smaller by 2 in $\sigma'_{X\cup Y}$. The imbalance of the remaining vertices remain the same.

Case 3. $\left( \sum_{i=0}^{\lfloor \frac{|Y|}{2} \rfloor} |L_i^{\sigma_{X\cup Y}}| \right) > \left( \sum_{i=0}^{\lfloor \frac{|Y|}{2} \rfloor} |L_i^{\sigma_{X\cup Y}}| \right)$.

We can express the imbalance of $I(\sigma'_{X\cup Y})$ as follows:

$$I(\sigma'_{X\cup Y}) = I(\sigma_{X\cup Y}) - 2 \cdot \left( \sum_{i=0}^{\lfloor \frac{|Y|}{2} \rfloor} |L_i^{\sigma_{X\cup Y}}| \right) - I(\sigma_{X\cup Y}, y_{\lfloor \frac{|Y|}{2} \rfloor})$$
\[ + I(\sigma'_{X\cup Y}, y_{\frac{|Y|}{2}}) \]
\[ = I(\sigma_{X\cup Y}) - 2 \cdot \left( \sum_{i=0}^{\frac{|Y|-1}{2}} |L_i^{\sigma_{X\cup Y}}| \right) - \left( \sum_{i=0}^{\frac{|Y|-1}{2}} |L_i^{\sigma_{X\cup Y}}| \right) \]
\[ + \left( \sum_{i=\frac{|Y|}{2}}^{|Y|-1} |L_i^{\sigma_{X\cup Y}}| \right) + \sum_{i=0}^{|Y|} |L_i^{\sigma_{X\cup Y}}| \]
\[ < I(\sigma_{X\cup Y}). \]

Since \( \sigma_{X\cup Y} \) it is not possible that \( I(\sigma'_{X\cup Y}) < I(\sigma_{X\cup Y}) \). Thus this case can not occur.

**Case 4.** \( \left( \sum_{i=0}^{\frac{|Y|-1}{2}} |L_i^{\sigma_{X\cup Y}}| \right) \leq \left( \sum_{i=\frac{|Y|}{2}}^{|Y|-1} |L_i^{\sigma_{X\cup Y}}| \right). \)

We can express the imbalance of \( I(\sigma'_{X\cup Y}) \) as follows:

\[ I(\sigma'_{X\cup Y}) = I(\sigma_{X\cup Y}) - 2 \cdot \left( \sum_{i=0}^{\frac{|Y|-1}{2}} |L_i^{\sigma_{X\cup Y}}| \right) - \left( \sum_{i=0}^{\frac{|Y|-1}{2}} |L_i^{\sigma_{X\cup Y}}| \right) \]
\[ + I(\sigma'_{X\cup Y}, y_{\frac{|Y|}{2}}) \]
\[ = I(\sigma_{X\cup Y}) - 2 \cdot \left( \sum_{i=0}^{\frac{|Y|-1}{2}} |L_i^{\sigma_{X\cup Y}}| \right) - \left( \sum_{i=0}^{\frac{|Y|-1}{2}} |L_i^{\sigma_{X\cup Y}}| \right) \]
\[ + \left( \sum_{i=\frac{|Y|}{2}}^{|Y|-1} |L_i^{\sigma_{X\cup Y}}| \right) + \sum_{i=0}^{|Y|} |L_i| \]
\[ = I(\sigma_{X\cup Y}). \]

**Lemma 3.** Let \( \sigma_{X\cup Y} \) be an arbitrary imbalance optimal ordering. Let \( \sigma'_{X\cup Y} = shift_R(\sigma_{X\cup Y}) \). We have that \( I(\sigma_{X\cup Y}) = I(\sigma'_{X\cup Y}) \).

**Proof.** Analogous to Lemma 2.

**Remark 1.** Let both \(|X|\) and \(|Y|\) be odd. Let \( \sigma_{X\cup Y} \) be an arbitrary imbalance optimal ordering. Let \( y_m \in Y \) be the vertex at position \( l_{\frac{|Y|}{2}} \). We have that \( I(\sigma_{X\cup Y}, y_m) = 1 \). Otherwise \( \sigma_{X\cup Y} \) is not imbalance optimal. This can be shown by a proof by contradiction using Lemma 2 and Lemma 3.

**Theorem 1.** If \( G = (X, Y, E) \) is a complete bigraph, then the minimum imbalance of \( G \) is \(|X| \cdot |Y| + (|X| \mod 2) \cdot (|Y| \mod 2)\).

**Proof.** Let \( \sigma_{X\cup Y} \) be an arbitrary imbalance optimal ordering. Repeatedly apply the functions \( shift_L \) and \( shift_R \) on \( \sigma_{X\cup Y} \), until we have the same ordering as constructed in Lemma 1. Let us denote the obtained ordering as \( \sigma'_{X\cup Y} \). Since \( I(\sigma_{X\cup Y}) = I(\sigma'_{X\cup Y}) \) by Lemma 2 and Lemma 3, and \( I(\sigma'_{X\cup Y}) = |X| \cdot |Y| + (|X| \mod 2) \cdot (|Y| \mod 2) \), we have that \( I(G) = |X| \cdot |Y| + (|X| \mod 2) \cdot (|Y| \mod 2) \).
Corollary 1. By Lemma 2, Lemma 3, and Remark 1, any ordering \( \sigma_{X\cup Y} \) of a complete bigraph \( G = (X, Y, E) \) is imbalance optimal if and only if \( \sigma_{X\cup Y} \) has the following 3 properties:

1. \( \sum_{i=0}^{\left\lfloor \frac{|X|}{2} \right\rfloor} |L_i^{\sigma_{X\cup Y}}| \leq \left( \sum_{i=\left\lceil \frac{|Y|}{2} \right\rceil} |L_i^{\sigma_{X\cup Y}}| \right) \)

2. \( \sum_{i=0}^{\left\lfloor \frac{|Y|}{2} \right\rfloor} |L_i^{\sigma_{X\cup Y}}| \geq \left( \sum_{i=\left\lceil \frac{|X|}{2} \right\rceil + 1} |L_i^{\sigma_{X\cup Y}}| \right) \)

3. \( |X| \) and \( |Y| \) are odd \( \implies I(\sigma_{X\cup Y}, y_m) = 1 \), where \( y_m \in Y \) is the vertex at position \( \left\lfloor \frac{|V|}{2} \right\rfloor \).

These properties allow us to verify whether any ordering is imbalance optimal in \( O(|X| + |Y|) \).

Corollary 2. Let \( G = (X, Y, E) \) be a complete bigraph and let \( |X| + |Y| = n \). Given \( |X| \) and \( |Y| \), the minimum imbalance \( I(G) \) can be computed in \( O(\log(n) \cdot \log(\log(n))) \) time by using the formula of Theorem 1. This follows from the fact that the product of two \( k \)-bit integers can be computed in \( O(k \cdot \log(k)) \) time[5].

4 Imbalance on Chained Complete bipartite graphs

In this section we shall introduce the chained complete bigraph. We show that the chained complete bigraph is a subclass of PI-bigraphs and how to use the results of Section 3 to compute its minimum imbalance efficiently.

Definition 3. We define \( \mathcal{C} \) to be a family of maximal subsets of the vertices of graph \( G = (V, E) \) that induce a complete bigraph on \( G \). Additionally, for all edges \( e \in E \) there exists a vertex set \( C_i \in \mathcal{C} \) such that both endpoints of \( e \) are contained in \( C_i \). That is, \( \mathcal{C} \subseteq \mathcal{P}(V) \) such that:

\[
(\forall (u,v) \in E \exists C_i \in \mathcal{C} \{ u, v \} \subseteq C_i) \land (\forall C_i \in \mathcal{C} \exists j, k \in \mathbb{N} G[C_i] = K_{j,k}) \land
(\forall C_i \in \mathcal{C} \forall v \in V \setminus C_i \forall j, k \in \mathbb{N} G[C_i \cup \{ v \}] \neq K_{j,k}),
\]

where \( K_{i,j} \) denotes a complete bigraph. We call \( \mathcal{C} \) the maximal complete bigraph components, abbreviated by MCB-components, of \( G \).

Definition 4. A graph \( G = (X, Y, E) \) is a chained complete bigraph, if \( G \) has a MCB-component family \( \mathcal{C} \) such that we can label \( \mathcal{C} = \{ C_1, \ldots, C_n \} \) such that consecutive vertex sets \( C_i \) and \( C_{i+1} \) share exactly one vertex and non-consecutive vertex sets \( C_i \) and \( C_j \) share no vertices. Formally, \( (\forall 1 \leq i < n \ | C_i \cap C_{i+1} | = 1) \land (\forall 1 \leq i < j \leq n \ j - i > 1 \implies C_i \cap C_j = \emptyset) \). We call a vertex that is shared by two consecutive vertex sets of \( \mathcal{C} \) an overlapping vertex.

For any chained complete bigraph \( G \), with corresponding MCB-component family \( \mathcal{C} \), we can create a corresponding PI-bigraph interval representation \( I_G \).
For each \( C_i \in \mathcal{C} \) we create a staircase-shaped set of intervals with the overlapping vertices at the top and bottom.

Fig. 5. Example of a chained complete bigraph \( G = (X, Y, E) \), where \( X \) is represented by the red vertices \( x_i \) and \( Y \) by the blue vertices \( y_i \). The highlighted areas represent the vertex sets of \( \mathcal{C} \).

Fig. 6. Interval representation of the chained complete bigraph of Fig. 5.

Remark 2. By the definition of chained complete bigraph \( G = (X, Y, E) \) we have \( \forall v \in X \cup Y \ 1 \leq \{|C_i \in \mathcal{C} \mid v \in C_i\}| \leq 2 \).

Remark 3. By the definition of MCB-components \( \mathcal{C} \) of a chained complete bigraph \( \forall v \in X \cup Y \) \( N(v) \subseteq \bigcup_{C_j \in \{C_i \in \mathcal{C} \mid v \in C_i\}} C_j \).

Let \( G = (X, Y, E) \) be a chained complete bigraph with corresponding MCB-component family \( \mathcal{C} = \{C_1, \ldots, C_n\} \). Then we have that:

\[
I(G) = \sum_{i=1}^{n} |X_i| \cdot |Y_i| + (|X_i| \ \text{mod} \ 2) \cdot (|Y_i| \ \text{mod} \ 2) - \left( \sum_{i=1}^{n-1} g(s_i, C_i) + g(s_i, C_{i+1}) \right) + \left( \sum_{i=1}^{n-1} |g(s_i, C_i) - g(s_i, C_{i+1})| \right),
\]

where \( X_i, Y_i, s_i \), and the function \( g \) are defined below. First, we introduce additional definitions that are required to understand the proof.
Definition 5. Let $G = (X,Y,E)$ be a chained complete bigraph with corresponding MCB-component family $\mathcal{C} = \{C_1, \ldots, C_n\}$. We shall use the notation $G[C_i] = (X_i,Y_i,E_i)$ to denote the graph induced on $C_i \in \mathcal{C}$.

Definition 6. Let $G$ be a chained complete bigraph with corresponding MCB-component family $\mathcal{C} = \{C_1, \ldots, C_n\}$. We label the overlapping vertices of $\mathcal{C}$ as $s_i \in C_i \cap C_{i+1}$, where $1 \leq i \leq n - 1$. By the definition of chained complete bigraph, the overlapping vertex $s_i$ is unique. We define $S = \{s_i \mid 1 \leq i \leq n - 1\}$.

Definition 7. We define $g(s_i,C_j)$, where $s_i \in S$ and $C_j \in \mathcal{C}$, to be the number of neighbors of $s_i$ in $C_j$. Equivalently, $g(s_i,C_j)$ is the number of vertices in $C_j$ that do not belong to the same part as $s_i$. That is, $g(s_i,C_j) = |N(s_i) \cap C_j| = \begin{cases} |X_j| & \text{if } s_i \in Y \\ |Y_j| & \text{if } s_i \in X \end{cases}$

![Illustration of the additional definitions of Section 4 on the example graph of Fig. 5.](image)

4.1 Proof of the upper bound

Lemma 4. Let $G = (X,Y,E)$ be a chained complete bigraph with corresponding MCB-component family $\mathcal{C} = \{C_1, \ldots, C_n\}$. Let $C_i \in \mathcal{C}$ such that $|X_i| = 1$ or $|Y_i| = 1$. W.l.o.g. assume that $|X_i| = 1$, then neither overlapping vertices $s_{i-1}$ nor $s_i$ can be in $X_i$. Formally, $\forall C_i \in \mathcal{C}(|X_i| = 1 \implies s_{i-1} \notin X_i \land s_i \notin X_i \land (|Y_i| = 1 \implies s_{i-1} \notin Y_i \land s_i \notin Y_i))$.

Proof. Trivial proof by contradiction. (If not, then $C_i$ is not maximal.)

Lemma 5. Given a chained complete bigraph $G = (X,Y,E)$ with corresponding MCB-component family $\mathcal{C} = \{C_1, \ldots, C_n\}$, we have that

$$I(G) \leq \sum_{i=1}^{n} |X_i| \cdot |Y_i| + (|X_i| \mod 2) \cdot (|Y_i| \mod 2)$$
For the cases $(++)$ and $(-)$, thus, in the cases $(++)$ and $(-)$, the inequality as follows:

$$\left\lvert \sum_{i=1}^{n-1} g(s_i, C_i) + g(s_i, C_{i+1}) \right\rvert + \left\lvert \sum_{i=1}^{n-1} g(s_i, C_i) - g(s_i, C_{i+1}) \right\rvert.$$  

**Proof.** The lemma is proven by constructing an ordering $\sigma_{X \cup Y}$ whose imbalance is equivalent to the above expression. The ordering $\sigma_{X \cup Y}$ is constructed by creating a subordering for each $C_i \in \mathcal{C}$ separately and concatenating those suborderings. The suborderings are created in a similar fashion as the orderings in the proof of Lemma 1.

### 4.2 Proof of the lower bound

**Remark 4.** The imbalance of $v \in (X \cup Y) \setminus S$ is only influenced by the vertices in $C_i$. That is, $\forall v \in V \setminus S \ I(v, \sigma_{X \cup Y}, G) = I(v, \sigma_{X \cup Y}^C, G[C_i])$.

**Remark 5.** The imbalance of $s_i$ is only influenced by the vertices in $C_i \cup C_{i+1}$. That is, $\forall v \in S \ I(s_i, \sigma_{X \cup Y}, G) = I(s_i, \sigma_{X \cup Y}^C, G[C_i \cup C_{i+1}])$.

**Lemma 6.** Let $G = (X, Y, E)$ be a chained complete bigraph with corresponding MCB-component family $\mathcal{C} = \{C_1, \ldots, C_n\}$. For any arbitrary ordering $\sigma_{X \cup Y}$ it holds that

$$I(s_{n-1}, \sigma_{X \cup Y}^{C_{n-1} \cup C_n}, G[C_{n-1} \cup C_n]) - I(s_{n-1}, \sigma_{X \cup Y}^{C_{n-1}}, G[C_{n-1}])$$

$$- I(s_{n-1}, \sigma_{X \cup Y}^{C_{n-1}}, G[C_n])$$

$$\geq |g(s_{n-1}, C_{n-1}) - g(s_{n-1}, C_n)| - g(s_{n-1}, C_{n-1}) - g(s_{n-1}, C_n).$$

**Proof.** The expression $|g(s_{n-1}, C_{n-1}) - g(s_{n-1}, C_n)| - g(s_{n-1}, C_{n-1}) - g(s_{n-1}, C_n)$ takes two possible values depending on the sign of $g(s_{n-1}, C_{n-1}) - g(s_{n-1}, C_n)$. Either the above expression is equivalent to $-2g(s_{n-1}, C_n)$ or $2g(s_{n-1}, C_{n-1})$. To relate the expressions in the inequality, we shall denote the number of neighbors of $s_{n-1}$ to its left and to its right in $C_n$ and $C_{n-1}$ in ordering $\sigma_{X \cup Y}$ as:

- $l_1 = \{v \in C_{n-1} \cap N(s_{n-1}) \mid v <_{\sigma_{X \cup Y}} s_{n-1}\};$
- $l_2 = \{v \in C_n \cap N(s_{n-1}) \mid v <_{\sigma_{X \cup Y}} s_{n-1}\};$
- $r_1 = \{v \in C_{n-1} \cap N(s_{n-1}) \mid v >_{\sigma_{X \cup Y}} s_{n-1}\};$
- $r_2 = \{v \in C_n \cap N(s_{n-1}) \mid v >_{\sigma_{X \cup Y}} s_{n-1}\};$

Using the above definitions, we rewrite the expression on the left-hand side of the inequality as follows:

$$I(s_{n-1}, \sigma_{X \cup Y}^{C_{n-1} \cup C_n}, G[C_{n-1} \cup C_n]) - I(s_{n-1}, \sigma_{X \cup Y}^{C_{n-1}}, G[C_{n-1}])$$

$$- I(s_{n-1}, \sigma_{X \cup Y}^{C_{n-1}}, G[C_n])$$

$$= |l_1 - r_1 + l_2 - r_2| - |l_1 - r_1| - |l_2 - r_2|.$$  

According to the signs of $l_1 - r_1$ and $l_2 - r_2$, consider the following cases:

- $(++)$ $l_1 - r_1 \geq 0$ and $l_2 - r_2 \geq 0$; $(+-)$ $l_1 - r_1 \geq 0$ and $l_2 - r_2 < 0$;
- $(-+)$ $l_1 - r_1 < 0$ and $l_2 - r_2 \geq 0$; $(-)$ $l_1 - r_1 < 0$ and $l_2 - r_2 < 0$.

For the cases $(++)$ and $(-)$, we have that $|l_1 - r_1 + l_2 - r_2| - |l_1 - r_1| - |l_2 - r_2| = 0$. Thus, in the cases $(++)$ and $(-)$, it holds that

$$|l_1 - r_1 + l_2 - r_2| - |l_1 - r_1| - |l_2 - r_2| = 0.$$
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\[ \geq |g(s_{n-1}, C_{n-1}) - g(s_{n-1}, C_n)| - g(s_{n-1}, C_{n-1}) - g(s_{n-1}, C_n) . \]

This follows from the fact that \(-2 \cdot g(s_{n-1}, C_n) \leq 0\) and \(-2 \cdot g(s_{n-1}, C_{n-1}) \leq 0.\)

For the cases \((+-)\) and \((-+),\) we have that

\[ |l_1 - r_1 + l_2 - r_2| - |l_1 - r_1| - |l_2 - r_2| = \begin{cases} 
2(l_2 - r_2) & \text{if } (+-) \land l_1 - r_1 + l_2 - r_2 \geq 0 \\
2(r_2 - l_2) & \text{if } (-+) \land l_1 - r_1 + l_2 - r_2 < 0 \\
2(r_1 - l_1) & \text{if } (++) \land l_1 - r_1 + l_2 - r_2 < 0 \\
2(l_1 - r_1) & \text{if } (--) \land l_1 - r_1 + l_2 - r_2 \geq 0 
\end{cases} . \]

Observe that, by the definitions of \(l_1, r_1, l_2, r_2,\) and function \(g,\) we have \(l_1 + r_1 = g(s_{n-1}, C_{n-1})\) and \(l_2 + r_2 = g(s_{n-1}, C_n).\)

Using the above remark and case distinction, we derive that in the cases \((+-)\) and \((-+)\) it holds that

\[ |l_1 - r_1 + l_2 - r_2| - |l_1 - r_1| - |l_2 - r_2| \geq |g(s_{n-1}, C_{n-1}) - g(s_{n-1}, C_n)| - g(s_{n-1}, C_{n-1}) - g(s_{n-1}, C_n) . \]

**Lemma 7.** Given a chained complete bigraph \(G = (X, Y, E)\) with corresponding MCB-component family \(\mathcal{C} = \{C_1, \ldots, C_n\},\) we have that

\[
I(G) \geq \sum_{i=1}^{n} |X_i| \cdot |Y_i| + (|X_i| \mod 2) \cdot (|Y_i| \mod 2) \\
- \left( \sum_{i=1}^{n-1} |g(s_i, C_i) + g(s_i, C_{i+1})| \right) + \left( \sum_{i=1}^{n-1} |g(s_i, C_i) - g(s_i, C_{i+1})| \right) .
\]

**Proof.** We shall prove that the imbalance of any arbitrary ordering \(\sigma_{X U Y}\) on the vertex set \(X \cup Y\) is bounded from below by the above expression by induction on \(|\mathcal{C}| = n.\)

- **Base Case** \((n = 0 \lor n = 1):\)
  By the definition of MCB-components \(\mathcal{C},\) the graph \(G\) is an empty graph or a complete bigraph. Thus, by Theorem 1, the lemma holds for the base case.

- **Induction step** \((n > 1):\)
  Let \(G = (X, Y, E)\) be a chained complete bigraph with corresponding MCB-component family \(\mathcal{C} = \{C_1, \ldots, C_{k+1}\}.\)
  Let us define

\[ \mathcal{C} \setminus C_{k+1} = \bigcup_{C_i \in \mathcal{C} \setminus \{C_{k+1}\}} C_i . \]

We write the imbalance of \(\sigma_{X U Y}\) as follows:

\[
I(\sigma_{X U Y}) = I(\sigma_{X U Y}^{\mathcal{C} \setminus C_{k+1}}, G[\mathcal{C} \setminus C_{k+1}]) + I(\sigma_{X U Y}^{C_{k+1}}, G[C_{k+1}]) \\
- I(s_k, \sigma_{X U Y}^{C_{k+1}}, G[C_k]) - I(s_k, \sigma_{X U Y}^{C_{k+1}}, G[C_k]) \\
+ I(s_k, \sigma_{X U Y}^{C_k U C_{k+1}}, G[C_k U C_{k+1}])
\]

(1)
\[ I(G) = \sum_{i=1}^{n} |X_i| \cdot |Y_i| + (|X_i| \mod 2) \cdot (|Y_i| \mod 2) \]
\[ - \left( \sum_{i=1}^{n-1} g(s_i, C_i) + g(s_i, C_{i+1}) \right) + \left( \sum_{i=1}^{n-1} |g(s_i, C_i) - g(s_i, C_{i+1})| \right) \]

where Eq. (1) follows from Remark 4 and Remark 5, Eq. (2) follows from the induction hypothesis, and Eq. (3) follows from Lemma 6.

**Theorem 2.** Let \( G = (X, Y, E) \) be a chained complete bigraph with corresponding MCB-component family \( \mathcal{C} = \{C_1, \ldots, C_n\} \). We have that

\[ I(G) = \sum_{i=1}^{n} |X_i| \cdot |Y_i| + (|X_i| \mod 2) \cdot (|Y_i| \mod 2) \]

\[ - \left( \sum_{i=1}^{n-1} g(s_i, C_i) + g(s_i, C_{i+1}) \right) + \left( \sum_{i=1}^{n-1} |g(s_i, C_i) - g(s_i, C_{i+1})| \right) . \]

**Proof.** Follows from Lemma 5 and Lemma 7.

**Corollary 3.** Let \( G = (X, Y, E) \) be a chained complete bigraph with corresponding MCB-component family \( \mathcal{C} = \{C_1, \ldots, C_n\} \) and let \( |X| + |Y| = m \). Given \( \{|X_1|, \ldots, |X_n|\}, \{|Y_1|, \ldots, |Y_n|\}, \) and \( \{s_1 \in X, \ldots, s_{n-1} \in X\} \), the imbalance of \( G \) can be computed in \( O(n \cdot \log(m) \cdot \log(\log(m))) \) time. By applying a similar reasoning as in Corollary 2, we verify the correctness of this corollary.
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5 Appendix - Full Proof

5.4 Imbalance on Chained Complete bipartite graphs

Proof of the upper bound

Lemma 8. Let $G = (X,Y,E)$ be a chained complete bigraph with corresponding MCB-component family $\mathcal{C} = \{C_1, \ldots , C_n\}$. Let $C_i \in \mathcal{C}$ such that $|X_i| = 1$ or $|Y_i| = 1$. W.l.o.g. assume that $|X_i| = 1$, then neither overlapping vertices $s_{i-1}$ nor $s_i$ can be in $X_i$. Formally, for all $C_i \in \mathcal{C}$,

$$(|X_i| = 1 \implies s_{i-1} \notin X_i \land s_i \notin X_i) \land (|Y_i| = 1 \implies s_{i-1} \notin Y_i \land s_i \notin Y_i).$$

Proof. Assume that a vertex set $C_i \in \mathcal{C}$ exists such that

$$(|X_i| = 1 \land (s_{i-1} \in X_i \lor s_i \in X_i)) \lor (|Y_i| = 1 \land (s_{i-1} \in Y_i \lor s_i \in Y_i)).$$

W.l.o.g. assume that $|X_i| = 1 \land s_i \in X_i$. Then $G[C_i \cup Y_{i+1}]$ is a complete bigraph. This contradicts the maximality of $C_i$. Thus, there exists no $C_i \in \mathcal{C}$ such that

$$(|X_i| = 1 \land (s_{i-1} \in X_i \lor s_i \in X_i)) \lor (|Y_i| = 1 \land (s_{i-1} \in Y_i \lor s_i \in Y_i)).$$ (1)

Lemma 9. Given a chained complete bigraph $G = (X,Y,E)$ with corresponding MCB-component family $\mathcal{C} = \{C_1, \ldots , C_n\}$, we have that

$$I(G) \leq \sum_{i=1}^{n} |X_i| \cdot |Y_i| + (|X_i| \mod 2) \cdot (|Y_i| \mod 2)$$

$$- \left( \sum_{i=1}^{n-1} g(s_i, C_i) + g(s_i, C_{i+1}) \right) + \left( \sum_{i=1}^{n-1} |g(s_i, C_i) - g(s_i, C_{i+1})| \right).$$

We prove that the minimum imbalance of $G$ is bounded from above by the above expression, by constructing an ordering $\sigma_{X \cup Y}$ whose imbalance is equivalent to the above expression.

The ordering $\sigma_{X \cup Y}$ is constructed by creating a subordering for each $C_i \in \mathcal{C}$ separately and concatenating those suborderings.

Construction of ordering $\sigma_{X \cup Y}$ We shall first describe the construction of the subordering corresponding to the vertex set $C_i \in \mathcal{C}$, where $2 \leq i \leq n - 1$. Afterwards, we explain the additional steps required to apply the same construction for vertex sets $C_1$ and $C_n$. For $C_i \in \mathcal{C}$, with $2 \leq i \leq n - 1$, we have the following cases based on the signs of the cardinalities of the parts of $G[C_i]$:

Case 5. Both $|X_i|$ and $|Y_i|$ are even.
Consider the following cases based on the set membership of the overlapping vertices $s_{i-1}$ and $s_i$. 
1. Overlapping vertices $s_{i-1}$ and $s_i$ are contained in the same part:

W.l.o.g. assume that $s_{i-1}, s_i \in X_i$. Let us partition $X_i \setminus \{s_{i-1}, s_i\}$ into $X_1$ and $X_2$ such that $|X_1| = |X_2|$. We define $\sigma_{C_i \setminus \{s_{i-1}, s_i\}}$ as follows:

$$\sigma_{C_i \setminus \{s_{i-1}, s_i\}} = \sigma_{X_1} \sigma_{Y_1} \sigma_{X_2},$$

where $\sigma_{X_1}$, $\sigma_{Y_1}$, and $\sigma_{X_2}$ are arbitrary orderings on $X_1$, $Y_1$, and $X_2$ respectively.

![Fig. 8. Example of an ordering where $s_{i-1}, s_i \in X_i$ and $G[C_i] = K_{6,6}$.]

2. Overlapping vertices $s_{i-1}$ and $s_i$ are contained in the different parts:

W.l.o.g. assume that $s_{i-1} \in X_1 \land s_i \in Y_i$. Let us partition $X_i \setminus \{s_{i-1}\}$ into $X_1$ and $X_2$ such that $|X_1| + 1 = |X_2|$. We also partition $Y_i \setminus \{s_i\}$ into $Y_1$ and $Y_2$ such that $|Y_1| = |Y_2| + 1$. We define $\sigma_{C_i \setminus \{s_{i-1}, s_i\}}$ as follows:

$$\sigma_{C_i \setminus \{s_{i-1}, s_i\}} = \sigma_{X_1} \sigma_{Y_1} \sigma_{X_2} \sigma_{Y_2},$$

where $\sigma_{X_1}$, $\sigma_{Y_1}$, $\sigma_{X_2}$, and $\sigma_{Y_2}$ are arbitrary orderings on $X_1$, $Y_1$, $X_2$, and $Y_2$ respectively.

![Fig. 9. Example of an ordering where $s_{i-1} \in X_i \land s_i \in Y_i$ and $G[C_i] = K_{6,6}$.]

**Case 6.** Either $|X_i|$ is odd or $|Y_i|$ is odd.

W.l.o.g. assume that $|X_i|$ is odd and $|Y_i|$ is even. Consider the following cases based on the set membership of the overlapping vertices $s_{i-1}$ and $s_i$:

1. Overlapping vertices $s_{i-1}$ and $s_i$ are contained in $X_i$:

Let $x \in X_i \setminus \{s_{i-1}, s_i\}$ be chosen arbitrarily. By the fact that $|X_i|$ is odd, such a vertex $x$ must exist. Let us partition $X_i \setminus \{s_{i-1}, s_i, x\}$ into $X_1$ and $X_2$ such that $|X_1| = |X_2|$. We also partition $Y_i$ into $Y_1$ and $Y_2$ such that $|Y_1| = |Y_2|$. We define $\sigma_{C_i \setminus \{s_{i-1}, s_i\}}$ as follows:

$$\sigma_{C_i \setminus \{s_{i-1}, s_i\}} = \sigma_{X_1} \sigma_{Y_1} \sigma_{x} \sigma_{Y_2} \sigma_{X_2},$$
where $\sigma_{X_1}$, $\sigma_{Y_1}$, $\sigma_{(x)}$, $\sigma_{Y_2}$, and $\sigma_{X_2}$ are arbitrary orderings on $X_1$, $Y_1$, $\{x\}$, $Y_2$, and $X_2$ respectively.

$$\sigma_{G[C_i]} = s_{i-1} s_i$$

Fig. 10. Example of an ordering where $s_{i-1}, s_i \in X_i$ and $G[C_i] = K_{7,6}$.

2. Overlapping vertices $s_{i-1}$ and $s_i$ are contained in $Y_i$:
Let us partition $Y_i \setminus \{s_{i-1}, s_i\}$ into $Y_1$ and $Y_2$ such that $|Y_1| = |Y_2|$. We define $\sigma_{C_i \setminus \{s_{i-1}, s_i\}}$ as follows:

$$\sigma_{C_i \setminus \{s_{i-1}, s_i\}} = \sigma_{Y_1} \sigma_{X_1} \sigma_{Y_2}$$

where $\sigma_{Y_1}$, $\sigma_{X_1}$, and $\sigma_{Y_2}$ are arbitrary orderings on $Y_1$, $X_1$, and $Y_2$ respectively.

$$\sigma_{G[C_i]} = s_{i-1} s_i$$

Fig. 11. Example of an ordering where $s_{i-1}, s_i \in Y_i$ and $G[C_i] = K_{7,6}$.

3. Overlapping vertices $s_{i-1}$ and $s_i$ are contained in different parts:
W.l.o.g. assume that $s_{i-1} \in X_i$ and $s_i \in Y_i$. Let us partition $X_i \setminus \{s_{i-1}\}$ into $X_1$ and $X_2$ such that $|X_1| + 2 = |X_2|$. We also partition $Y_i \setminus \{s_i\}$ into $Y_1$ and $Y_2$ such that $|Y_1| = |Y_2| + 1$. We define $\sigma_{C_i \setminus \{s_{i-1}, s_i\}}$ as follows:

$$\sigma_{C_i \setminus \{s_{i-1}, s_i\}} = \sigma_{X_1} \sigma_{Y_1} \sigma_{X_2} \sigma_{Y_2}$$

where $\sigma_{X_1}$, $\sigma_{Y_1}$, $\sigma_{X_2}$, and $\sigma_{Y_2}$ are arbitrary orderings on $X_1$, $Y_1$, $X_2$, and $Y_2$ respectively.

$$\sigma_{G[C_i]} = s_{i-1} s_i$$

Fig. 12. Example of an ordering where $s_{i-1} \in X_i \land s_i \in Y_i$ and $G[C_i] = K_{7,6}$. 
Consider the following cases based on the set membership of the overlapping vertices $s_{i-1}$ and $s_i$.

1. Overlapping vertices $s_{i-1}$ and $s_i$ are contained in the same part:
   W.l.o.g. assume that $s_{i-1}, s_i \in X_i$. Let us choose $x_m \in X_i \setminus \{s_{i-1}, s_i\}$ arbitrarily. Let $X_1$ and $X_2$ partition $X_i \setminus \{s_{i-1}, s_i, x_m\}$ such that $|X_1| = |X_2|$. Additionally, let $Y_1$ and $Y_2$ partition $X$ such that $|Y_1| - |Y_2| = 1$. We define $\sigma_{C_i \setminus \{s_{i-1}, s_i\}}$ as follows:

   $$\sigma_{C_i \setminus \{s_{i-1}, s_i\}} = \sigma_{X_i} \sigma_{Y_1} \sigma_{\{x_m\}} \sigma_{Y_2} \sigma_{X_2},$$

   where $\sigma_{X_i}$, $\sigma_{Y_1}$, $\sigma_{Y_2}$, and $\sigma_{X_2}$ are arbitrary orderings on $X_1$, $Y_1$, $Y_2$, and $X_2$ respectively.

   ![Fig. 13. Example of an ordering where $s_{i-1}, s_i \in X_i$ and $G[C_i] = K_{7,7}$.](image)

2. Overlapping vertices $s_{i-1}$ and $s_i$ are contained in the different parts:
   W.l.o.g. assume that $s_{i-1} \in X_i$ and $s_i \in Y_i$. Let us partition $X_i \setminus \{s_{i-1}\}$ into $X_1$ and $X_2$ such that $|X_1| = |X_2|$. We also partition $Y_i \setminus \{s_i\}$ into $Y_1$ and $Y_2$ such that $|Y_1| = |Y_2|$. We define $\sigma_{C_i \setminus \{s_{i-1}, s_i\}}$ as follows:

   $$\sigma_{(s_{i-1})} \sigma_{C_i \setminus \{s_{i-1}, s_i\}} \sigma_{Y_i} = \sigma_{X_i} \sigma_{Y_1} \sigma_{\{s_i\}} \sigma_{Y_2} \sigma_{X_2},$$

   ![Fig. 14. Example of an ordering where $s_{i-1} \in X_i \land s_i \in Y_i$ and $G[C_i] = K_{7,7}$](image)

The vertex set $C_1$ contains exactly one overlapping vertex, namely $s_1$. We apply the above described construction to create $\sigma_{C_1}$ by arbitrarily picking a vertex in $C_1 \setminus \{s_1\}$ to be $s_0$. Similarly, for the vertex set $C_n$, we construct $\sigma_{C_n}$ by arbitrarily picking a vertex in $C_n \setminus \{s_{n-1}\}$ to be $s_n$ and applying the above construction.

Note that each constructed subordering $\sigma_{\{s_{i-1}\}} \sigma_{C_i \setminus \{s_{i-1}, s_i\}} \sigma_{\{s_i\}}$ is imbalance optimal for the graph induced by the vertices of $C_i$. That is, for all $G_i \in \mathcal{G}$, $I(G_i[C_i]) = I(G_i[C_i \setminus \{s_{i-1}\} \sigma_{C_i \setminus \{s_{i-1}, s_i\}} \sigma_{\{s_i\}}])$.

Using the above constructed suborderings, we define ordering $\sigma_{X \cup Y}$ as:
\[ \sigma_{X \cup Y} = \sigma_{C_1 \setminus \{s_0, s_1\}} \sigma_{C_2 \setminus \{s_1, s_2\}} \sigma_{C_3 \setminus \{s_2, s_3\}} \cdots \]

\[ \sigma_{C_{n-1} \setminus \{s_{n-2}, s_{n-1}\}} \sigma_{C_{n-1} \setminus \{s_{n-1}, s_n\}} \sigma_{C_n \setminus \{s_n, s_{n+1}\}} \]  

Fig. 15. Example of a chained complete bigraph with assigned \( s_0 \), and \( s_n \).

\[ \sigma_{C_1} = y_1 \ y_2 \quad \sigma_{C_2} = x_2 \ x_3 \quad \sigma_{C_3} = x_4 \ y_5 \ x_5 \]

\[ \sigma_{X \cup Y} = s_0 \ y_1 \ y_2 \ s_1 \ x_2 \ y_3 \ x_3 \ s_2 \ y_5 \ x_4 \ x_5 \]

Fig. 16. Constructed suborderings and final ordering of the example graph above.

**Proof of Lemma 9** To prove Lemma 9 we require several remarks on the constructed ordering \( \sigma_{X \cup Y} \).

**Remark 5.** From the definition of chained complete bigraphs, a vertex \( v \in (X \cup Y) \setminus S \) is contained in exactly one vertex set \( C_i \in \mathcal{C} \). Additionally, by Remark 3, the neighborhood of \( v \) is a subset of \( C_i \). Therefore, the imbalance of \( v \) is only influenced by the vertices in \( C_i \). That is, \( \forall C_i \in \mathcal{C} \ \forall v \in C_i \setminus S \ I(v, \sigma_{X \cup Y}, G) = I(v, \sigma_{C_i \cup C_i+1}, G[C_i]) \).

**Remark 6.** From the definition of chained complete bigraphs, an overlapping vertex \( s_i \in S \) is contained in exactly two vertex sets \( C_i, C_{i+1} \in \mathcal{C} \). Additionally, by Remark 3, the neighborhood of \( s_i \) is a subset of \( C_i \cup C_{i+1} \). Therefore, the imbalance of \( s_i \) is only influenced by the vertices in \( C_i \cup C_{i+1} \). That is, \( \forall s_i \in S \ I(s_i, \sigma_{X \cup Y}, G) = I(s_i, \sigma_{C_i \cup C_{i+1}}, G[C_i \cup C_{i+1}]) \). From the position of overlapping vertex \( s_i \) in ordering \( \sigma_{\{s_{i-1}\} \sigma_{C_i \setminus \{s_{i-1}, s_i\}} \sigma_{C_{i+1} \setminus \{s_i, s_{i+1}\}} \sigma_{s_{i+1}} \} \), we deduce that the imbalance of \( s_i \) in ordering

\[ \sigma_{\{s_{i-1}\} \sigma_{C_i \setminus \{s_{i-1}, s_i\}} \sigma_{s_i} \sigma_{C_{i+1} \setminus \{s_i, s_{i+1}\}} \sigma_{s_{i+1}}} \]

is \( |g(s_i, C_i) - g(s_i, C_{i+1})| \). That is,
\[ I(s_i, \sigma_{s_{i-1}}|\{s_{i-1},s_i\})\sigma_{\{s_i\}}|\{s_{i+1}\}|\{s_{i+1}\}, G[C_i \cup C_{i+1}]) = g(s_i, C_i) - g(s_i, C_{i+1}) \]

**Remark 7.** By applying a similar analysis as in Lemma 1 together with Lemma 4, we derive that for all \( C_i \in \mathcal{C} \) the ordering \( \sigma_{s_{i-1}}|\{s_{i-1},s_i\}\sigma_{\{s_i\}}|\{s_{i+1}\}|\{s_{i+1}\}, G[C_i] \), is an imbalance optimal ordering for complete bigraph \( G[C_i] \). That is, by Theorem 1 and the aforementioned analysis, for all \( C_i \in \mathcal{C} \) we have

\[ I(\sigma_{s_{i-1}}|\{s_{i-1},s_i\}\sigma_{\{s_i\}}|\{s_{i+1}\}, G[C_i]) = |X_i| \cdot |Y_i| + (|X_i| \mod 2) \cdot (|Y_i| \mod 2). \]

**Remark 8.** Since overlapping vertex \( s_i \in S \) is the rightmost vertex in ordering \( \sigma_{s_{i-1}}|\{s_{i-1},s_i\}\sigma_{\{s_i\}}|\{s_{i+1}\} \), we deduce that the imbalance of \( s_i \) in ordering \( \sigma_{s_{i-1}}|\{s_{i-1},s_i\}\sigma_{\{s_i\}}|\{s_{i+1}\} \) is the cardinality of the part of \( G[C_i] \) in which \( s_i \) is not contained. That is, for all \( s_i \in S \) we have

\[ I(s_i, \sigma_{s_{i-1}}|\{s_{i-1},s_i\}\sigma_{\{s_i\}}, G[C_i]) = g(s_i, C_i). \]

Similarly, since overlapping vertex \( s_i \in S \) is the leftmost vertex in ordering \( \sigma_{s_{i-1}}|\{s_{i-1},s_i\}\sigma_{\{s_i\}}|\{s_{i+1}\} \), we deduce that the imbalance of \( s_i \) in ordering \( \sigma_{s_{i-1}}|\{s_{i-1},s_i\}\sigma_{\{s_i\}}|\{s_{i+1}\} \) is the cardinality of the part of \( G[C_{i+1}] \) in which \( s_i \) is not contained. That is, for all \( s_i \in S \) we have

\[ I(s_i, \sigma_{s_{i-1}}|\{s_{i-1},s_i\}\sigma_{\{s_i\}}|\{s_{i+1}\}, G[C_{i+1}]) = g(s_i, C_{i+1}). \]

**Proof.** The imbalance of the ordering \( \sigma_{X \cup Y} =: \sigma_V \), constructed using the method of Section 5.4, can be written as follows:

\[
I(\sigma_V) = \sum_{v \in (X \cup Y) \setminus S} I(v, \sigma_{X \cup Y}, G) + \sum_{v \in S} I(v, \sigma_{X \cup Y}, G)
= \sum_{i=1}^{n} I(\sigma_{s_{i-1}}|\{s_{i-1},s_i\}\sigma_{\{s_i\}}, G[C_i])
- \sum_{i=1}^{n-1} I(s_i, \sigma_{s_{i-1}}|\{s_{i-1},s_i\}\sigma_{\{s_i\}}, G[C_i])
- \sum_{i=1}^{n-1} I(s_i, \sigma_{s_{i-1}}|\{s_{i-1},s_i\}\sigma_{\{s_i\}}, G[C_{i+1}])
+ \sum_{s_i \in S} I(s_i, \sigma_{s_{i-1}}|\{s_{i-1},s_i\}\sigma_{\{s_i\}}|\{s_{i+1}\}\sigma_{\{s_{i+1}\}}, G[C_i \cup C_{i+1}])
= \sum_{i=1}^{n} |X_i| \cdot |Y_i| + (|X_i| \mod 2) \cdot (|Y_i| \mod 2)
\]
\[
-n \sum_{i=1}^{n-1} g(s_i, C_i) + g(s_i, C_{i+1}) + \left( \sum_{i=1}^{n-1} g(s_i, C_i) - g(s_i, C_{i+1}) \right),
\]

where Eq. (2) follows from Remark 5, Remark 6 and the construction method of Section 5.4. Eq. (3) follows from Remark 7 and Remark 8.

Note that in the expression at Eq. (2) the first term twice adds an “incorrect imbalance” for each vertex in \( S \). This “incorrect imbalance” is removed by the second and third term of the expression.

Proof of the lower bound

**Lemma 7.** Given a chained complete bigraph \( G = (X, Y, E) \) with corresponding MCB-component family \( \mathcal{C} = \{C_1, \ldots, C_n\} \), we have that

\[
I(G) \geq \sum_{i=1}^{n} |X_i| \cdot |Y_i| + (|X_i| \mod 2) \cdot (|Y_i| \mod 2)
- \left( \sum_{i=1}^{n-1} g(s_i, C_i) + g(s_i, C_{i+1}) \right) + \left( \sum_{i=1}^{n-1} g(s_i, C_i) - g(s_i, C_{i+1}) \right).
\]

**Proof.** We shall prove that the imbalance of any arbitrary ordering \( \sigma_{X \cup Y} \) on the vertex set \( X \cup Y \) is bounded from below by the above expression by induction on \( |\mathcal{C}| = n \).

- **Base Case (n = 0):** By the definition of MCB-components \( \mathcal{C} \), the graph \( G \) is an empty graph. Thus,

\[
I(G) = 0 = \sum_{i=1}^{n} |X_i| \cdot |Y_i| + (|X_i| \mod 2) \cdot (|Y_i| \mod 2)
- \left( \sum_{i=1}^{n-1} g(s_i, C_i) + g(s_i, C_{i+1}) \right) + \left( \sum_{i=1}^{n-1} g(s_i, C_i) - g(s_i, C_{i+1}) \right).
\]

- **Base case (n = 1):** By the definition of MCB-components \( \mathcal{C} \), the graph \( G \) is a complete bigraph. Thus, by Theorem 1

\[
I(G) = |X| \cdot |Y| + (|X| \mod 2) \cdot (|Y| \mod 2)
= \sum_{i=1}^{n} |X_i| \cdot |Y_i| + (|X_i| \mod 2) \cdot (|Y_i| \mod 2)
- \left( \sum_{i=1}^{n-1} g(s_i, C_i) + g(s_i, C_{i+1}) \right) + \left( \sum_{i=1}^{n-1} g(s_i, C_i) - g(s_i, C_{i+1}) \right).
\]
• Induction hypothesis:
For any chained complete bigraph $G = (X, Y, E)$ with corresponding MCB-component family $\mathcal{C} = \{C_1, \ldots, C_k\}$, where $0 \leq k < n$, it holds that

$$I(G) \geq \sum_{i=1}^{k} |X_i| \cdot |Y_i| + (|X_i| \mod 2) \cdot (|Y_i| \mod 2)$$

$$- \left( \sum_{i=1}^{k-1} g(s_i, C_i) + g(s_i, C_{i+1}) \right) + \left( \sum_{i=1}^{k-1} |g(s_i, C_i) - g(s_i, C_{i+1})| \right).$$

• Induction step ($n > 1$):
Let $G = (X, Y, E)$ be a chained complete bigraph with corresponding MCB-component family $\mathcal{C} = \{C_1, \ldots, C_k\}$. Let us define $\mathcal{C} \setminus C_{k+1}$ as follows:

$$\mathcal{C} \setminus C_{k+1} = \bigcup_{C_i \in \mathcal{C} \setminus \{C_{k+1}\}} C_i.$$ 

Fig. 17. Illustration of the definitions of $\mathcal{C} \setminus C_{k+1}$ and $C_{k+1}$.

Using the above definition, we write the imbalance of $\sigma_{X \cup Y}$ as follows:

$$I(\sigma_{X \cup Y}) = I(\sigma_{X \cup Y}^{C_{k+1}}, G[\mathcal{C} \setminus C_{k+1}]) + I(\sigma_{X \cup Y}^{C_{k+1}}, G[C_{k+1}])$$

$$- I(s_k, \sigma_{X \cup Y}^{C_{k+1}}, G[C_k]) - I(s_k, \sigma_{X \cup Y}^{C_{k+1}}, G[C_{k+1}])$$

$$+ I(s_k, \sigma_{X \cup Y}^{C_{k+1}} \cup G[C_k \cup C_{k+1}])$$

$$\geq I(G[\mathcal{C} \setminus C_{k+1}]) + I(G[C_{k+1}]) - I(s_k, \sigma_{X \cup Y}^{C_{k+1}}, G[C_k])$$

$$- I(s_k, \sigma_{X \cup Y}^{C_{k+1}} \cup G[C_{k+1}]) + I(s_k, \sigma_{X \cup Y}^{C_{k+1}} \cup G[C_k \cup C_{k+1}])$$

$$\geq \left( \sum_{i=1}^{k} |X_i| \cdot |Y_i| + (|X_i| \mod 2) \cdot (|Y_i| \mod 2) \right)$$

$$- \left( \sum_{i=1}^{k-1} g(s_i, C_i) + g(s_i, C_{i+1}) \right) + \left( \sum_{i=1}^{k-1} |g(s_i, C_i) - g(s_i, C_{i+1})| \right)$$

$$+ |X_{k+1}| \cdot |Y_{k+1}| + (|X_{k+1}| \mod 2) \cdot (|Y_{k+1}| \mod 2).$$
\[- I(s_k, \sigma^{C_k}_{X\cup Y}, G[C_k]) - I(s_k, \sigma^{C_k+1}_{X\cup Y}, G[C_{k+1}]) \\
+ I(s_k, \sigma^{C_k\cup C_{k+1}}_{X\cup Y}, G[C_k \cup C_{k+1}]) \tag{5} \]
\[
= \left( \sum_{i=1}^{k+1} |X_i| \cdot |Y_i| + (|X_i| \mod 2) \cdot (|Y_i| \mod 2) \right) \\
- \left( \sum_{i=1}^{k-1} g(s_i, C_i) + g(s_i, C_{i+1}) \right) + \left( \sum_{i=1}^{k-1} |g(s_i, C_i) - g(s_i, C_{i+1})| \right) \\
- g(s_k, C_{k+1}) - g(s_k, C_k) + |g(s_k, C_k) - g(s_k, C_{k+1})| \tag{6} \]
\[
= \left( \sum_{i=1}^{k+1} |X_i| \cdot |Y_i| + (|X_i| \mod 2) \cdot (|Y_i| \mod 2) \right) \\
- \left( \sum_{i=1}^{k-1} g(s_i, C_i) + g(s_i, C_{i+1}) \right) + \left( \sum_{i=1}^{k-1} |g(s_i, C_i) - g(s_i, C_{i+1})| \right) \\
- \left( \sum_{i=1}^{k} g(s_i, C_i) + g(s_i, C_{i+1}) \right) + \left( \sum_{i=1}^{k} |g(s_i, C_i) - g(s_i, C_{i+1})| \right), \]

where Eq. (4) follows from Remark 5 and Remark 6, Eq. (5) follows from the induction hypothesis, and Eq. (6) follows from Lemma 6.

Similar to the analysis in Section 5.4, in the expression at Eq. (4), the first term twice adds an “incorrect imbalance” of vertex $s_k$. The “incorrect imbalance” is removed by the second and third term of the expression. The fourth term of the expression adds the correct imbalance of $s_k$. 