On the octonionic Bergman kernel

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Abstract: By introducing a suitable new definition for the inner product on the octonionic Bergman space, we determine the explicit form of the octonionic Bergman kernel, in the framework of octonionic analysis which is non-commutative and non-associative.

Keywords: octonions, octonionic analysis, Bergman kernel

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1 Introduction

In complex analysis the Szegő kernel and Bergman kernel are well-known, which had also been generalized into Clifford analysis (including quaternionic analysis as a special case, see [2]). But in octonionic analysis the existence of such kernels is still unknown, let alone the explicit expressions. The difficulty arises mainly because the octonion algebra (Cayley algebra) is non-associative.

The motivation for us to consider this kind of problem is that we want to unify the formulation of the analytic function theory in the largest normed division algebra over $\mathbb{R}$, namely, in octonions $\mathbb{O}$ (including complex numbers, quaternions as its special cases).

Recall that in complex analysis the Bergman space on the unit disc is defined to be the collection of functions that are holomorphic and square integrable on the unit disc. This definition can be naturally generalized to octonionic analytic functions. Since the Cayley algebra is non-commutative, there exist two different but symmetric octonionic analytic function theory. In this paper we focus on the left octonionic analytic functions, and we denote by $B^2(B)$ the corresponding octonionic Bergman space, where $B$ is the unit ball in $\mathbb{R}^8$ centered at origin. A nature problem comes: Does the octonionic Bergman kernel exist? and what is it? Of course this problem is closely related to the definition of the associated inner product. Usually the inner product of two Bergman functions $f$ and $g$ is defined to be the integral of $\overline{f}g$ on $B$. Since the octonions is non-associative, the usual definition is no longer valid to guarantee the existence of the kernel. We thus need to give a new definition.

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Definition 1.1 (inner product on $\mathcal{B}^2(B)$). Let $f, g \in \mathcal{B}^2(B)$, we define

$$(f, g)_B := \frac{1}{\omega_8} \int_B \left( \frac{\overline{g(x)}}{|x|} \right) \left( \frac{x}{|x|} f(x) \right) dV,$$

where $\omega_8$ is the surface area of the unit sphere in $\mathbb{R}^8$, $dV$ is the volume element on $B$.

Note that this modified inner product is real-linear and conjugate symmetric. The induced norm

$$\|f\|_B^2 := (f, f)_B = \frac{1}{\omega_8} \int_B |f|^2 dV$$

coincides with the norm induced by the usual inner product.

We can now state the main theorem of this paper.

Theorem 1.1. Let

$$B(x, a) = \frac{(6(1 - |a|^2|x|^2) + 2(1 - \overline{a}) (1 - \overline{a})}{|1 - \overline{a}|^8},$$

then $B(\cdot, a)$ is the desired octonionic Bergman kernel, i.e., $B(\cdot, a) \in \mathcal{B}^2(B)$, and for any $f \in \mathcal{B}^2(B)$ and any $a \in B$, there holds the following reproducing formula

$$f(a) = (f, B(\cdot, a))_B.$$

The rest of the paper is organized as follows. In Section 2 we give a brief review on the octonion algebra and octonionic analysis. In Section 3 we will exploit our new idea in defining the structure of the inner product to investigate the octonionic Szegő kernel for the unit ball in $\mathbb{R}^8$. Section 4 is then devoted to the proof of our main result Theorem 1.1. In the last section we point out that the Bergman kernel can be unified in one form in both complex analysis and hyper-complex analysis.

2 The octonions and the octonionic analysis

2.1 The octonions

If an algebra $\mathcal{A}$ is meanwhile a normed vector space, and its norm “$\| \cdot \|$” satisfies $\|ab\| = \|a\| \|b\|$, then we call $\mathcal{A}$ a normed algebra. If $ab = 0$ ($a, b \in \mathcal{A}$) implies $a = 0$ or $b = 0$, then we call $\mathcal{A}$ a division algebra. Early in 1898, Hurwitz had proved that the real numbers $\mathbb{R}$, complex numbers $\mathbb{C}$, quaternions $\mathbb{H}$ and octonions $\mathbb{O}$ are the only normed division algebras over $\mathbb{R}$ ($\mathbb{H}$), with the imbedding relation $\mathbb{R} \subseteq \mathbb{C} \subseteq \mathbb{H} \subseteq \mathbb{O}$.

As the largest normed division algebra, octonions, which are also called Cayley numbers or the Cayley algebra, were discovered by John T. Graves in 1843, and then by Arthur Cayley in 1845 independently. Octonions are an 8 dimensional algebra over $\mathbb{R}$ with the basis $e_0, e_1, \ldots, e_7$ satisfying

$$e_0^2 = e_0, \quad e_i e_0 = e_0 e_i = e_i, \quad e_i^2 = -1, \text{ for } i = 1, 2, \ldots, 7.$$
So $e_0$ is the unit element and can be identified with 1. Denote

$$W = \{(1, 2, 3), (1, 4, 5), (1, 7, 6), (2, 4, 6), (2, 5, 7), (3, 4, 7), (3, 6, 5)\}.$$ 

For any triple $(\alpha, \beta, \gamma) \in W$, we set

$$e_\alpha e_\beta = e_\gamma = -e_\beta e_\alpha, \quad e_\gamma e_\alpha = e_\alpha = -e_\alpha e_\beta, \quad e_\gamma e_\beta = e_\beta = -e_\beta e_\gamma.$$ 

Then by distributivity for any $x = \sum_{i=0}^7 x_i e_i, \ y = \sum_{j=0}^7 y_j e_j \in \mathbb{O}$, the multiplication $xy$ is defined to be

$$xy := \sum_{i=0}^7 \sum_{j=0}^7 x_i y_j e_i e_j.$$ 

For any $x = \sum_{i=0}^7 x_i e_i \in \mathbb{O}$, Re $x := x_0$ is called the scalar (or real) part of $x$ and $\mathcal{F} := x - \text{Re} x$ is called its vector part. $\mathcal{F} := \sum_{i=0}^7 x_i e_i = x_0 - \mathcal{F}$ and $|x| := (\sum_{i=0}^7 x_i^2)^{\frac{1}{2}}$ are respectively the conjugate and norm (or modulus) of $x$, they satisfy: $|xy| = |x||y|, \ x\mathcal{F} = \mathcal{F}x = |x|^2, \ \mathcal{F}\mathcal{F} = \overline{\mathcal{F}} \ (x, y \in \mathbb{O})$. So if $x \neq 0$, $x^{-1} = \mathcal{F}/|x|^2$ gives the inverse of $x$.

Octonionic multiplication is neither commutative nor associative. But the subalgebra generated by any two elements is associative, namely, The octonions are alternative. $[x, y, z] := (xy)z - x(yz)$ is called the associator of $x, y, z \in \mathbb{O}$, it satisfies $(\mathfrak{115})$

$$[x, y, z] = [y, z, x] = -[y, x, z], \quad [x, x, y] = [\mathcal{F}, x, y] = 0.$$

2.2 The octonionic analysis

As a generalization of complex analysis and quaternionic analysis to higher dimensions, the study of octonionic analysis was originated by Dentoni and Sce in 1973 ([3]), and it was not until 1995 that it began to be systematically investigated by Li et al ([6]). Octonionic analysis is a function theory on octonionic analytic (abbr. O-analytic) functions. Suppose $\Omega$ is an open subset of $\mathbb{R}^8$, $f = \sum_{j=0}^7 f_j e_j \in C^1(\Omega, \mathbb{O})$ is an octonion-valued function, if

$$Df = \sum_{i=0}^7 e_i \frac{\partial f}{\partial x_i} = \sum_{i=0}^7 \sum_{j=0}^7 \frac{\partial f_i}{\partial x_j} e_j e_i = 0,$$

$$\left( fD = \sum_{i=0}^7 \frac{\partial f_i}{\partial x_i} e_i = \sum_{i=0}^7 \sum_{j=0}^7 \frac{\partial f_j}{\partial x_i} e_i e_j = 0 \right),$$

then $f$ is said to be left (right) $\mathbb{O}$-analytic in $\Omega$, where the generalized Cauchy–Riemann operator $D$ and its conjugate $\overline{D}$ are defined by

$$D := \sum_{i=0}^7 e_i \frac{\partial}{\partial x_i}, \quad \overline{D} := \sum_{i=0}^7 \overline{e_i} \frac{\partial}{\partial x_i}.$$
respectively. A function \( f \) is \( \mathbb{O} \)-analytic means that \( f \) is meanwhile left \( \mathbb{O} \)-analytic and right \( \mathbb{O} \)-analytic. From

\[
\mathcal{D}(Df) = (\mathcal{D}D)f = \Delta f = f(D\mathcal{D}) = (fD)\mathcal{D},
\]

we know that any left (right) \( \mathbb{O} \)-analytic function is always harmonic. In the sequel, unless otherwise specified, we just consider the left \( \mathbb{O} \)-analytic case as the right \( \mathbb{O} \)-analytic case is essentially the same. A Cauchy-type integral formula and a Laurent-type series for this setting are:

**Lemma 2.1** (Cauchy’s integral formula, see [3, 8]). Let \( M \subset \Omega \) be an 8-dimensional, compact differentiable and oriented manifold with boundary. If \( f \) is left \( \mathbb{O} \)-analytic in \( \Omega \), then

\[
f(x) = \frac{1}{\omega_8} \int_{y \in \partial M} E(y - x)(d\sigma_y f(y)), \quad x \in M^o,
\]

where \( E(x) = \frac{x}{|x|^8} \) is the octonionic Cauchy kernel, \( d\sigma_y = n(y)dS \), \( n(y) \) and \( dS \) are respectively the outward-pointing unit normal vector and surface area element on \( \partial M \), \( M^o \) is the interior of \( M \).

**Lemma 2.2** (Laurent expansion, see [13, 14]). Let \( D \) be an annular domain in \( \mathbb{R}^8 \). If \( f \) is left \( \mathbb{O} \)-analytic in \( D \), then

\[
f(x) = \sum_{k=0}^{\infty} P_k f(x) + \sum_{k=0}^{\infty} Q_k f(x), \quad x \in D,
\]

where \( P_k f \) and \( Q_k f \) are respectively the inner and outer spherical octonionic-analytics of order \( k \) associated to \( f \).

Octonionic analytic functions have a close relationship with the Stein–Weiss conjugate harmonic systems. If the components of \( F \) consist a Stein–Weiss conjugate harmonic system on \( \Omega \subset \mathbb{R}^8 \), then \( \overline{F} \) is \( \mathbb{O} \)-analytic on \( \Omega \). But conversely this is not true ([7]). For more information and recent progress about octonionic analysis, we refer the reader to [6, 9–12, 14, 15].

### 3 The octonionic Szegö kernel

To see how our new definition works, let us check the octonionic Szegö kernel for the unit ball in \( \mathbb{R}^8 \).

Recall that on the unit ball the octonionic Hardy space \( \mathcal{H}^2(B) \) consists of the left octonionic analytic functions whose mean square value on the sphere is bounded for radius \( r \in [0, 1) \). For any \( f \in \mathcal{H}^2(B) \), according to the Cauchy’s integral formula, for all \( a \in B \) there holds

\[
f(a) = \frac{1}{\omega_8} \int_{x \in S^7} \frac{\overline{f(x)} - a}{|x - a|^8}(xf(x))dS.
\]
\[
= \frac{1}{\omega S} \int_{x \in S^7} \left( \frac{1 - \overline{x}a}{|1 - \overline{x}a|^8} \right) (xf(x))dS,
\]
where \(S^7 = \partial B\) is the unit sphere, \(dS\) is the area element on \(S^7\). If we define the inner product for \(\mathcal{H}^2(B)\) to be
\[
(f, g)_{S^7} := \frac{1}{\omega S} \int_{S^7} \overline{(g(\eta))}(\eta f(\eta))dS = \frac{1}{\omega S} \int_{S^7} \overline{(g(\eta))}(\eta f(\eta))dS,
\]
and let
\[
S(x, a) = \frac{1 - \overline{x}a}{|1 - \overline{x}a|^8},
\]
then \(S(\cdot, a) \in \mathcal{H}^2(B)\), and the Cauchy’s integral formula can be rewritten as
\[
f(a) = (f, S(\cdot, a))_{S^7}.
\]
We call \(S(\cdot, a)\) the octonionic Szegő kernel.

Denote by \(L^2(S^7)\) the space of square integrable (octonion-valued) functions on the unit sphere, for which we define its inner product to be the same as that in \text{[1]}. We have

**Proposition 3.1.** Let \(f, g \in L^2(S^7)\) be associated with the spherical octonionic-analytics expansions:
\[
f(\omega) = \sum_{k=0}^{\infty} (P_k f(\omega) + Q_k f(\omega)), \quad g(\omega) = \sum_{k=0}^{\infty} (P_k g(\omega) + Q_k g(\omega)), \quad \omega \in S^7.
\]
Then
\[
(f, g)_{S^7} = \sum_{k=0}^{\infty} ((P_k f, P_k g)_{S^7} + (P_k f, Q_k g)_{S^7}) + \sum_{k=0}^{\infty} ((P_k f, Q_{k+1} g)_{S^7} + (Q_{k+1} f, P_k g)_{S^7}).
\]

**Proof.** From
\[
\triangle(xP_k f(x)) = x\triangle(P_k f(x)) + 2D(P_k f(x)) = 0,
\]
we can easily see that the restriction of \(xP_k f(x)\) on \(S^7\) is a spherical harmonic of order \(k + 1\). Similarly, the restriction of \(xQ_k f(x)\) on \(S^7\) is a spherical harmonic of order \(k\). The proposition immediately follows by the fact that spherical harmonics of different orders are mutually orthogonal. \(\square\)

Thus we get

**Corollary 3.1.** Let \(f \in L^2(S^7)\) be associated with the spherical octonionic-analytics expansion
\[
f(\omega) = \sum_{k=0}^{\infty} (P_k f(\omega) + Q_k f(\omega)), \quad \omega \in S^7.
\]
Then
\[ \|f\|^2_{S^7} = \sum_{k=0}^{\infty} (\|P_k f\|^2_{S^7} + \|Q_k f\|^2_{S^7}) + \sum_{k=0}^{\infty} 2\text{Re}((P_k f, Q_{k+1} g)_{S^7}). \]

Remark: Proposition 3.1 is similar to the Parseval’s theorem. It is worthwhile to note that this version is a bit different from that in Clifford analysis where the second part in the summation vanishes ([2]), here \((P_k f, Q_{k+1} g)_{S^7}\) may not be zero. Below we give a counter-example. Let
\[ f(x) = x_1 - x_0 e_1, \]
\[ g(x) = \frac{7}{|x|^{12}}(x_1 x_2 e_4 + x_0 x_2 e_5 + x_0 x_1 e_6). \]
Then \(P_1 f = f, Q_2 g = g\), but
\[ (P_1 f, Q_2 g)_{S^7} = -\frac{2e_6}{\omega_8} \int_{S^7} x_0^2 x_1^2 dS \neq 0. \]

4 Derivation of the octonionic Bergman kernel

In this section we will prove Theorem 1.1. For the main idea we use in the proof one can also refer to [2].

Proof of Theorem 1.1 By definition it is straightforward that
\[ (f, g)_B = \int_0^1 r^7 (f_r, g_r)_{S^7} dr, \]
where \(f_r(\eta) = f(r\eta), \eta \in S^7\). Together with Proposition 3.1 we get
\[ (f, g)_B = \sum_{k=0}^{\infty} (P_k f, P_k g)_B = \sum_{k=0}^{\infty} \int_0^1 r^{2k+7} (P_k f, P_k g)_{S^7} dr = \sum_{k=0}^{\infty} (2k + 8)^{-1} (P_k f, P_k g)_{S^7}. \]
Therefore, \(f \in \mathcal{B}^2(B)\) if and only if \(f\) is left octonionic analytic in \(B\) and
\[ \|f\|^2_B = \sum_{k=0}^{\infty} (2k + 8)^{-1} \|P_k f\|^2_{S^7} < \infty. \]
From this viewpoint, if \(f \in \mathcal{H}^2(B)\), then
\[ \sqrt{T} f := \sum_{k=0}^{\infty} \sqrt{2k + 8} P_k f \in \mathcal{B}^2(B). \]
Similarly, if \( g \) is left octonionic analytic in \( B_R \) (the ball centered at the origin of radius \( R \), with \( R > 1 \)), then \( \sqrt{T} g \in B^2(B_{R'}) \), with \( 1 \leq R' < R \). Consequently,

\[
Tg := \sqrt{T^2} g = \sum_{k=0}^{\infty} (2k + 8) P_k g \in B^2(B_{R'}), \quad 1 \leq R' < R.
\]

Now, assume \( f \in B^2(B) \), when \( |a| < r \) we have

\[
f(a) = \frac{1}{\omega_8} \int_{\partial B_r} \frac{\pi - \pi}{|x - a|^8} d\mu_x f(x) = \frac{r^7}{\omega_8} \int_{S^7} \frac{r^7 - \pi}{|r\eta - a|^8} (\eta f(\eta)) dS
\]

\[
= \lim_{r \to 1^-} \frac{r^7}{\omega_8} \int_{S^7} \frac{r^7 - \pi}{|r\eta - a|^8} (\eta f(\eta)) dS = \lim_{r \to 1^-} r^7 (f_r, S^7(\cdot, a))_{S^7}, \tag{2}
\]

where

\[
S^7(\cdot, a) = \frac{r - \pi a}{|r - \pi a|^8}.
\]

Since \( S^7(\cdot, a) \) is left octonionic analytic in \( B_{r/|a|} \) \( (r/|a| > 1) \) with respect to \( x \), we have

\[
T S^7(\cdot, a) = \sum_{k=0}^{\infty} (2k + 8) P_k S^7(\cdot, a) \in B^2(B).
\]

So,

\[
(f_r, T S^7(\cdot, a))_B = \sum_{k=0}^{\infty} (2k + 8)^{-1} (P_k f_r, (2k + 8) P_k S^7(\cdot, a))_{S^7}
\]

\[
= \sum_{k=0}^{\infty} (P_k f_r, P_k S^7(\cdot, a))_{S^7}
\]

\[
= (f_r, S^7(\cdot, a))_{S^7}. \tag{3}
\]

By (2) and (3) we get

\[
f(a) = \lim_{r \to 1^-} r^7 (f_r, T S^7(\cdot, a))_B = (f, T S(\cdot, a))_B,
\]

where \( S(\cdot, a) \) is the octonionic Szegő kernel. We can now see that the octonionic Bergman kernel \( B(x, a) \) is

\[
B(x, a) = T S(x, a) = \sum_{k=0}^{\infty} (2k + 8) P_k S(x, a).
\]

The remaining thing we need to do is to evaluate the above summation. To this end, first note that

\[
S(x, a) = K(E(x, \pi)),
\]

where \( K \) is the octonionic Szegő kernel.
where $E(x, a) = \frac{x - a}{|x - a|^2}$ (|x| > 1), $Kf := E(x, 0)f(x^{-1})$ is the Kelvin inversion. So,

$$P_k S(x, a) = K(Q_k E(x, a)) = \overline{Q_k E(x, a)}|x|^{2k+6}.$$ 

Define the adjoint operator $A$ as follows:

$$(Af)(x) := D(|x|^{-6} \overline{f(x/|x|^2)}),$$

then it is easy to show that

$$A(Q_k E(x, a)) = (2k + 8)\overline{Q_k E(x, a)}|x|^{2k+6}.$$

Hence,

$$B(x, a) = \sum_{k=0}^{\infty} A(Q_k E(x, a))$$

$$= A \left( \sum_{k=0}^{\infty} Q_k E(x, a) \right)$$

$$= A(E(x, a))$$

$$= D_x \left( \frac{x - a|x|^2}{|1 - \overline{a}x|^8} \right)$$

$$= \frac{(6(1 - |a|^2|x|^2) + 2(1 - \overline{a}a))(1 - \overline{a}a)}{|1 - \overline{a}a|^{10}}.$$ 

The proof of Theorem 1.1 is complete.

5 Final remarks

By direct computation one can show that

$$B(x, a)a = \overline{D_a} \left( \frac{1 - |a|^2|x|^2}{|1 - \overline{a}x|^8} \right).$$

In fact, similar formulas also hold in both complex analysis and Clifford analysis. We therefore can unify the reproducing formulas in complex and hyper-complex contexts. Let $\mathcal{A}$ denote the complex algebra or hyper-complex algebra, i.e., $\mathcal{A}$ may refer to complex numbers $\mathbb{C}$, quaternions $\mathbb{H}$, octonions $\mathbb{O}$, or Clifford algebra $\mathcal{C}$. Assume that the dimension of $\mathcal{A}$ is $m$. Then for any function $f$ which belongs to the Bergman space $B^2(B_m)$ and any point $a \in B_m$ ($B_m$ is the unit ball centered at origin in $\mathbb{R}^m$), there holds

$$f(a) = (f, B(\cdot, a))_{B_m}$$

$$= \frac{1}{\omega_m} \int_{B_m} \left( B(x, a) \frac{\overline{x}}{|x|} \right) \left( \frac{x}{|x|} f(x) \right) dV$$

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\[
\begin{align*}
\omega_m \int_{B_m} \frac{1 - |a|^2 |x|^2}{|1 - a \overline{x}|^m} \left( \frac{x}{|x|^2} f(x) \right) dV \\
= \frac{1}{\omega_m} \int_{B_m} \frac{(m - 2)(1 - |a|^2 |x|^2) + 2(1 - \overline{a}x)}{|1 - \overline{x}a|^{m+2}} \left( \frac{x}{|x|^2} f(x) \right) dV,
\end{align*}
\]

where \( \omega_m \) is the surface area of the unit sphere in \( \mathbb{R}^m \), \( dV \) is the volume element on \( B_m \), and \( D \) is the generalized Cauchy–Riemann operator in the respective context.

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**References**

[1] J.C. Baez, The octonions, Bull. Amer. Math. Soc. 39 (2) (2002) 145–205.
[2] F. Brackx, R. Delanghe, F. Sommen, Clifford Analysis, Research Notes in Math., vol. 76, Pitman Advanced Publishing Program, Boston, 1982.
[3] P. Dentoni, M. Sce, Funzioni regolari nell’algebra di Cayley, Rend. Sem. Mat. Univ. Padova. 50 (1973) 251–267.
[4] A. Hurwitz, Über die Composition der quadratischen Formen von beliebig vielen Variablen, Nachr. Ges. Wiss. Göttingen (1898) 309–316.
[5] N. Jacobson, Basic Algebra I (2nd edition), W. H. Freeman and Company, New York, 1985.
[6] X.M. Li, Octonionic analysis, PhD Thesis, Peking University, 1998.
[7] X.M. Li, L.Z. Peng, On Stein–Weiss conjugate harmonic function and octonion analytic function, Approx. Theory & its Appl. 16 (2) (2000) 28–36.
[8] X.M. Li, L.Z. Peng, The Cauchy integral formulas on the octonions, Bull. Belg. Math. Soc. 9 (1) (2002) 47–64.
[9] X.M. Li, L.Z. Peng, T. Qian, Cauchy integrals on Lipschitz surfaces in octonionic spaces, J. Math. Anal. Appl. 343 (2) (2008) 763–777.
[10] X.M. Li, L.Z. Peng, T. Qian, The Paley–Wiener theorem in the non-commutative and non-associative octonions, Sci. China Ser. A 52 (1) (2009) 129–141.
[11] X.M. Li, J.X. Wang, Orthogonal invariance of the Dirac operator and the critical index of subharmonicity for octonionic analytic functions, Adv. Appl. Clifford Algebras 24 (1) (2014) 141–149.

[12] X.M. Li, K. Zhao, L.Z. Peng, Characterization of octonionic analytic functions, Complex Variables 50 (13) (2005) 1031–1040.

[13] X.M. Li, K. Zhao, L.Z. Peng, The Laurent series on the octonions, Adv. Appl. Clifford Algebras 11 (S2) (2001) 205–217.

[14] J.Q. Liao, X.M. Li, J.X. Wang, Orthonormal basis of the octonionic analytic functions, J. Math. Anal. Appl. 366 (1) (2010) 335–344.

[15] J.X. Wang, X.M. Li, J.Q. Liao, The quaternionic Cauchy–Szegö kernel on the quaternionic Siegel half space, arXiv:1210.5086v1.