PAPER

Approximate expressions for solutions to two kinds of transcendental equations with applications

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Abstract
In a broad spectrum of physics and engineering applications, transcendental equations have to be solved in order to determine their roots. Exact and explicit algebraic expression of solutions to such equations is, in general, impossible. Analytical approximate solutions to two kinds of transcendental equations with wide applications are presented. These approximate root formulas are systematically established by using the Padé approximant and show high accuracy. As an application of the proposed approximations, a highly accurate expression of the effective mass of the spring for a spring-mass system is obtained. The method described in this paper is also applied to other transcendental equations in physics and engineering applications.

1. Introduction
The determination of roots of transcendental equations is a problem commonly encountered in a broad spectrum of physics and engineering applications. However, it is difficult to obtain analytical approximate root formulas for such equations. Though a wide variety of root finding algorithms are available to achieve the solutions to desired degree of accuracy, analytical approximate solutions, which provide explicit dependence of the roots on the physical parameters of problem compared with purely numerical solutions, are always desirable and preferable.

Consider the following two kinds of transcendental equations with various applications. The first kind of equation is

$$x \tan x = \alpha. \tag{1}$$

This equation arises from the solution of a longitudinal vibration problem in a uniform bar with one fixed and one attached mass boundary condition [1]. A similar equation comes from the solution of buckling problem of a uniform column which has one free and one elastically hinged supported boundary condition and is subjected an axial compression load [2]. The infinite series solution to the one-dimensional transient chemical diffusion problem under certain boundary conditions [3] is also related to this equation. The second kind of equation is

$$\frac{x}{\beta} = f(x) \equiv \begin{cases} \cos x & (n = 1, 3, 5, \cdots), \\ \sin x & (n = 2, 4, 6, \cdots), \end{cases} \tag{2}$$

where \(n\pi/2 < x \leq n\pi/2\). This equation comes from the problem of a particle moving in a finite square well potential where the energy eigenvalues are its roots [4]. After the fabrication of quantum wells [5], the experimental observation of revivals and super-revivals [6] and the progress of the so-called ‘ghost orbit spectroscopy’ [7], the square wells also describe realistic physical systems or phenomena. The need for an explicit solution exceeds the level of solving simple and relevant problems of quantum mechanics.

Researchers are interested in positive roots of these equations because only they are related to the physical quantities. Note that the analytical solutions to equations (1) and (2) are absent until now. A method for formulating an expression for the roots of any analytical transcendental function was presented [8]. The method...
is based on Cauchy’s integral theorem and uses only basic concepts of complex integration. Numerical evaluation of solutions requires a complex Fourier transform. However, the computational efficiency of this procedure would not be expected to rival that of traditional approximate root finding techniques [8]. Recently, Luo et al[9] constructed the analytical approximate solution to equation (1) by rewriting its series expansion solution [10] in the form of a ratio of polynomials by a second-order Padé approximant. However, their results showed large errors. Based on the algebraic approximations of trigonometric functions, it is possible to transform a class of transcendental equations in approximate, tractable algebraic equations [4, 11, 12]. As the algebraization used in those papers is, to a certain extent, an ad hoc procedure, this approximation must be used with a certain caution in order to avoid the appearance of spurious roots or of roots with too large errors [12].

In this paper, highly accurate approximate expressions for solutions to equations (1) and (2) are systematically constructed by exploring the periodic properties of functions \( \tan x \) and \( \sin x \), and using the Padé approximant [13–15] to them. These approximations are valid for small as well as large values of parameters. Furthermore, as an application of the proposed approximate expression, a highly accurate expression of the effective mass of the spring for the spring-mass system is also obtained.

2. Preliminaries

A Padé approximant is the ‘best’ approximation of a function by a rational function of given order—under this technique, the approximant’s power series agrees with the power series of the function it is approximating. The Padé approximant often gives better approximation of the function than truncating its Taylor series, and it may still work where the Taylor series does not converge and has thus abundant applications in physics and engineering.

Given a function \( f(x) \) and two integers \( p \geq 0 \) and \( q \geq 1 \), the Padé approximant of order \([p/q]\) is the rational function [13]

\[
R(x) = \frac{\sum_{j=0}^{p} a_j x^j}{1 + \sum_{k=1}^{q} b_k x^k}
\]

which satisfies \( f(x) - R(x) = O(x^{p+q+1}) \). The Padé approximant is unique for given \( p \) and \( q \), that is, the coefficients \( a_0, a_1, \ldots, a_p, b_1, b_2, \ldots, b_q \) can be uniquely determined. It is known that in many cases a higher accuracy of approximation is achieved for small integers \( p \) and \( q \); thus the degrees of both numerator and denominator are set to be small hereafter so that the analytical approximate roots of transcendental equations can be obtained.

3. Highly accurate approximate expressions of solutions to transcendental equation (1)

3.1. Derivation of the first approximate expressions for all roots

Based on the periodic property of function \( \tan x \), a rational approximate expression for function \( \tan x \) in interval \([-\pi/2, \pi/2]\) is first introduced, the simple approximate root formulas of equation (1) in terms of parameter \( \alpha \) are then established.

Note that from the graphs of functions \( z = \tan x \) and \( z = \alpha /x \), for each integer \( n \geq 0 \), equation (1) has a unique root in interval \([2n - 1]\pi/2, \,(2n + 1)\pi/2] \) and it is near \( n\pi \). Letting

\[
x = n\pi + y, \; n = 0, 1, \ldots,
\]

and substituting equation (4) into equation (1) results in

\[
[n\pi + y] \tan y = \alpha, \; n = 0, 1, \ldots.
\]

Constructing the Padé approximant of order \([1/2]\) [13] to function \( \tan y \) for \(-\pi/2 < y < \pi/2\) yields

\[
\tan y \approx \frac{y'}{1 - y^2/3}.
\]

Substituting equation (6) into equation (5) leads to

\[
\frac{(n\pi + y)y'}{1 - y^2/3} = \alpha, \; n = 0, 1, \ldots.
\]

Solving \( y \) from equation (7) and using equation (4) yields the first approximation to the \( (n + 1) \)th root of equation (1) in terms of parameter \( \alpha \).
\[ x_{n1} = n\pi + \frac{\sqrt{(n\pi)^2 + 4\alpha(1 + \alpha/3)} - n\pi}{2(1 + \alpha/3)}, \quad n = 0, 1, \ldots. \] (8)

Especially, for \( n = 0 \), equation (8) gives the first approximation to the first root of equation (1) as
\[ x_{01}^{a}(\alpha) = \frac{\alpha}{\sqrt{1 + \alpha/3}}. \] (9)

3.2. Derivation of the second approximate expressions for all roots

This section derives highly accurate approximations to the roots of equation (1).

For \(-\pi/2 < y < \pi/2\), the Padé approximant of order \([3/4]\) to the function \( \tan y \) is
\[ \tan y \approx \frac{y - 2y^3/21}{1 - 3y^2/7 + y^4/105}. \] (10)

Substitution of equation (10) into equation (5) produces
\[ \frac{[n\pi + y](y - 2y^3/21)}{1 - 3y^2/7 + y^4/105} = \alpha, \quad n = 0, 1, \ldots \]
which can be written as
\[ \left( \frac{2}{21} + \frac{\alpha}{105} \right)y^4 + \frac{2n\pi}{21}y^3 - \left( \frac{3}{7} + 1 \right)y^2 = n\pi y + \alpha = 0, \quad n = 0, 1, \ldots \] (11)

For \( n = 0 \), equation (11) reduces to
\[ \left( \frac{2}{21} + \frac{\alpha}{105} \right)y^4 - \left( \frac{3}{7} + 1 \right)y^2 = 0. \] (12)

Solving quadratic equation for \( y^2 \) in equation (12) and using equation (4) with \( n = 0 \) gives the second approximation to the first root of equation (1) as
\[ x_{02}^{a}(\alpha) = \sqrt{\frac{2\alpha}{1 + 3\alpha/7 + \sqrt{1 + 10\alpha/21 + 107\alpha^2/735}}} \] (13)

Equation (11) is a quartic one for \( n > 0 \), expressions of its roots are lengthy and complex. We will try to find approximate solutions as follows. Note that, \(-\pi/2 < y < \pi/2\), so higher powers of \( y \) could eventually be neglected in equation (11). Keeping the constant, linear and quadratic terms, and neglecting the cubic and quartic ones in equation (11) yields
\[ -\left( \frac{3}{7} + 1 \right)y^2 = 0, \quad n = 1, 2, \ldots \] (14)

Solving equation (14) for \( y \) gives the initial approximation to the positive root of equation (11)
\[ y_{20} = \sqrt{\frac{(n\pi)^2 + 4\alpha(1 + 3\alpha/7) - n\pi}{2(1 + 3\alpha/7)}}, \quad n = 1, 2, \ldots. \] (15)

Taking \( y_{20} \) as the initial guess value, applying the Newton method to equation (11), iterating one step and noticing that \( y_{20} \) satisfies equation (14), give the second approximations to the positive roots of equation (11) as
\[ y_{21} = y_{20} - \frac{(10 + \alpha)y_{20}^4 + 10n\pi y_{20}^3}{4(10 + \alpha)y_{20}^2 + 30n\pi y_{20}^2 - 30(3\alpha + 7)y_{20} - 105n\pi}, \quad n = 1, 2, \ldots. \] (16)

Finally, based on equation (4), the second approximate expression of the \((n + 1)\)th root of equation (1) is
\[ x_{n3}^{a} = n\pi + y_{21}, \quad n = 1, 2, \ldots. \] (17)

3.3. Results

For given value of parameter \( \alpha \), the roots of equation (1) can numerically be calculated by using the Newton method. The corresponding analytical approximations to these roots can be obtained by utilizing equations (8), (13) and (17), respectively. Relative errors are then calculated against these numerical exact roots. Here, the relative error of the \( i \)th analytical approximation to the \((n + 1)\)th root to equation (1) is defined as
\[ RE_{ni} = \frac{|x_{ni}^{a} - x_{n}^{N}|}{|x_{n}^{N}|}, \quad i = 1, 2, \] where \( x_{n}^{N} \) denotes the \((n + 1)\)th root obtained by using the Newton method.

For \( \alpha = 1 \), the relative errors of the approximate roots in equations (8), (13) and (17) computed by the proposed method are shown in table 1. For comparison purposes, the relative errors of the approximate roots,
Table 1. The first 11 roots of equation (1) with $\alpha = 1$ and the relative errors of the approximate roots proposed in this paper and in [9].

| $n$ | Exact roots | equation (8) | equation (13) ($n = 0$), equation (17) ($n \geq 1$) | equation (7) ($n = 0$), equation (12b) ($n \geq 1$) in [9] |
|-----|-------------|--------------|-----------------------------------------------|-----------------------------------------------|
| 0   | 0.860334    | $6.615 \times 10^{-1}$ | $1.7606 \times 10^{-4}$ | $3.8132 \times 10^{-3}$ |
| 1   | 3.42562     | $1.0912 \times 10^{-3}$ | $3.0997 \times 10^{-5}$ | $1.0591 \times 10^{-1}$ |
| 2   | 6.43730     | $2.9156 \times 10^{-3}$ | $2.9617 \times 10^{-7}$ | $2.4478 \times 10^{-3}$ |
| 3   | 9.52933     | $2.8750 \times 10^{-6}$ | $1.4082 \times 10^{-9}$ | $2.3364 \times 10^{-4}$ |
| 4   | 12.6453     | $5.3376 \times 10^{-7}$ | $1.5154 \times 10^{-9}$ | $4.2856 \times 10^{-3}$ |
| 5   | 15.7713     | $1.4273 \times 10^{-7}$ | $2.6303 \times 10^{-10}$ | $1.3394 \times 10^{-3}$ |
| 6   | 18.9024     | $4.8323 \times 10^{-8}$ | $6.2325 \times 10^{-11}$ | $3.8454 \times 10^{-4}$ |
| 7   | 22.0365     | $1.9290 \times 10^{-8}$ | $1.8354 \times 10^{-11}$ | $1.5320 \times 10^{-4}$ |
| 8   | 25.1724     | $8.6946 \times 10^{-8}$ | $6.3524 \times 10^{-12}$ | $6.8962 \times 10^{-4}$ |
| 9   | 28.3096     | $4.3014 \times 10^{-9}$ | $2.4919 \times 10^{-12}$ | $3.4091 \times 10^{-4}$ |
| 10  | 31.4477     | $2.2907 \times 10^{-9}$ | $1.0717 \times 10^{-12}$ | $1.8144 \times 10^{-4}$ |

Figure 1. Relative errors of the first and second approximations to the first root of equation (1).

equations (7) and (12b), in [9] are also listed in the same table. Note that Luo et al. [9] used the series expansion solutions [10] to equation (1) to construct the corresponding Padé approximants of order [2/2]. It can be seen from Table 1 that, except for the first approximation to the first root, the accuracy of the proposed approximate roots is much higher than that in [9].

In general, researchers are more interested in the case of $\alpha \leq 1$ [3]. Relative errors of the two approximate expressions given in equations (8), (13) and (17) for the first three roots of equation (1) are shown in figures 1–3, respectively. These figures indicate that, compared with the first approximation in equation (8), the second approximations in equations (13) and (17) can provide more accurate results. For $\alpha \leq 1$, the maximum relative errors of the first and second approximations to the first root are less than 0.662% and 0.000 176%, respectively; the maximum relative errors of the first and second approximations to the second and third roots are 0.001 09% and 0.000 0309%, and 0.000 0292% and 0.000 000 296%, respectively. For the nth root with $n \geq 4$, the accuracies of approximate roots in equations (8) and (17) are higher than those for the third root. It should be pointed that, for $\alpha \leq 1$, the expression in equation (8) can provide high quality approximation to the nth root of equation (1) for $n > 1$.

Letting $x_n^a(\alpha)$ denote the $n$th root of equation (1), we have

$$\lim_{\alpha \to 0} x_n^a(\alpha) = n\pi, \quad \lim_{\alpha \to +\infty} x_n^a(\alpha) = n\pi + \pi/2, \quad n = 0, 1, \cdots.$$  

(18)

Using equations (8), (13), (17) and (18) yields

$$\lim_{\alpha \to 0} x_n^a(\alpha) = n\pi, \quad \lim_{\alpha \to 0} x_n^a(\alpha) = n\pi, \quad \lim_{\alpha \to +\infty} x_n^a(\alpha) - x_n^a(\alpha) = \frac{\sqrt{3} - \pi/2}{(n + 1/2)\pi} = r_{nt}, \quad n = 0, 1, \cdots.$$
where \( r_{01} \approx 10.3\% \), \( r_1 \approx 3.42\% \), \( r_{11} \approx 2.05\% \) and \( \lim_{n \to +\infty} r_{n1} = 0 \). Furthermore, we have
\[
\lim_{\alpha \to +\infty} \frac{x_{02}^n(\alpha) - x_{02}^n(\alpha)}{x_{02}^n(\alpha)} = \frac{\sqrt{2/\left(3/7 + \sqrt{107/735}\right)} - \pi/2}{\pi/2} \approx 0.0278\% ,
\]
\[
\lim_{\alpha \to +\infty} \frac{x_{n2}^n(\alpha) - x_{n2}^n(\alpha)}{x_{n2}^n(\alpha)} = \frac{\left(1 + 7/262\right)\sqrt{7/3} - \pi/2}{(n + 1/2)\pi} \equiv r_{n2}, \quad n = 1, 2, \cdots ,
\]
where \( r_{n2} \approx -0.157\% \), \( r_{22} \approx -0.0522\% \), \( r_{32} \approx -0.0313\% \), \cdots , \( \lim_{n \to +\infty} r_{n2} = 0 \).

Based on the results above, the second approximate roots in equations (13) and (17) show excellent accuracy and they are valid for small as well as large value of parameter \( \alpha \).

### 3.4. Determination of the effective mass of the spring for a spring-mass system

As an application of the proposed approximation, the effective mass of the spring for a spring-mass system will be established. When a longitudinal vibration in a uniform bar with one fixed and one attached mass boundary condition \([1, 16, 17]\) is considered, equation (1) reads
\[
\omega \sqrt{\frac{P}{E}} \tan \left( \omega \sqrt{\frac{P}{E}} \right) = \frac{\rho A}{M} ,
\]

\[ (19) \]
where $\omega$, $l$, $A$, $\rho$ and $E$ are the natural frequency, length, cross-sectional area, density and modulus of elasticity of the bar, respectively, $M$ is the attached mass, and $\alpha \equiv m_i/M$ is the ratio of the mass $m_i = \rho A l$ of the bar to the attached mass.

The introduction of the correction to the spring oscillations due to including the mass $m_i$ of a spring has led to many researches, for example, we refer readers to [1, 16, 17] and cited therein. When the vibration of the bar is reduced to that of a spring-mass with a spring stiffness $K = EA/l$ and an end mass equal to $M + m_{\text{eff}}$, the effective mass $m_{\text{eff}}$ need to be determined. Based on equations (1), (9), (13) and (19), two approximations to the first frequency of the spring-mass system can be expressed as

$$\omega_{\text{eff}} = \frac{x_{\text{eff}}}{\sqrt{\rho/E}} \sqrt{\frac{K}{M + m_i/D_i}}, \quad i = 1, 2,$$

where the values of $D_i^w (i = 1, 2)$ are

$$D_1^w = 3, \quad D_2^w = \frac{2\alpha}{3\alpha/7 + \sqrt{1 + 10\alpha/21 + 107\alpha^2/735} - 1}.$$

Here,

$$\lim_{\alpha \to 0} D_1^w = 3, \quad \lim_{\alpha \to +\infty} D_2^w = \frac{2}{3/7 + \sqrt{107/735}} \approx 2.46877.$$

Note that, in [16, 17], the frequency of the spring-mass system was given by

$$\omega_i = \sqrt{\frac{K}{M + m_i/D}},$$

where $D$ depends on the solution to equation (19), and satisfies

$$\lim_{\alpha \to 0} D = 3, \quad \lim_{\alpha \to +\infty} D = \frac{\pi^2}{4} \approx 2.46740.$$

Based on the assumption that the velocity of a spring element located a distance from the fixed end to vary linearly with it and the use of Rayleigh method, the effective mass of the spring is found to be the one-third the mass of the spring [1, 16, 17]. Now, the proposed first approximate frequency $\omega_{\text{eff}}^w$ in equations (20) and (21) for $i = 1$ has the same effective mass $m_{\text{eff}} = (1/3) m$, as that in [1, 16, 17] which is reasonable up to $\alpha = 1$. Based on equations (22) and (24), the proposed second approximate frequency $\omega_{\text{eff}}^s$ in equations (20) and (21) for $i = 2$ provides explicit and highly accurate expression $m_{\text{eff}} = (1/D_i^s) m_i$ for the effective mass of the spring in the spring-mass system for any given mass ratio $\alpha$.

### 4. Highly accurate approximate expressions of solutions to transcendental equation (2)

#### 4.1. Derivation for highly accurate approximate expressions for all roots

Note that from the graphs of functions $z = f(x)$ in equation (2) and $z = x/\beta$, for each positive integer $n$, equation (2) has, at most, one root in interval $[n\pi/2 - \pi/2, n\pi/2]$ [4]. Setting

$$x = \frac{n\pi}{2} - y, \quad 0 \leq y \leq \frac{\pi}{2}, \quad n = 1, 2, \ldots,$$

and substituting equation (25) into equation (2) results in

$$\frac{n\pi/2 - y}{\beta} = \sin y, \quad 0 \leq y \leq \frac{\pi}{2}, \quad n = 1, 2, \ldots.$$  

Equation (26) has thus, at most, one root for $y$ in the interval $[0, \pi/2]$ for given positive integer $n$. Applying the Padé approximant of order $[3/2]13$ to function $\sin y$ for $0 \leq y \leq \pi/2$ yields

$$\sin y \approx \frac{y(1 - 7y^2/60)}{1 + y^2/20}.$$  

Substitution of equation (27) into equation (26) leads to

$$\frac{n\pi/2 - y}{\beta} = \frac{y(1 - 7y^2/60)}{1 + y^2/20},$$

which can be rewritten as

$$a_3 y^3 + a_2 y^2 + a_1 y + a_0 = 0,$$
where
\[ a_3 = \frac{1}{20} - \frac{7\beta}{60}, \quad a_2 = -\frac{n\pi}{40}, \quad a_1 = \beta + 1, \quad a_0 = -\frac{n\pi}{2}. \]

Setting
\[ y = z - \frac{a_2}{3a_3}, \]
we may transform equation (28) into
\[ z^3 + pz + q = 0, \]
where
\[ p = \frac{a_1}{a_3} - \frac{1}{3} a_2^2, \quad q = \frac{2}{27} a_2^3 + \frac{a_0}{a_3} - \frac{a_1 a_2}{3a_3}. \]

If \( \left( \frac{q}{2} \right)^2 + \left( \frac{p}{3} \right)^3 > 0 \), the positive root of equation (30) is given by
\[ z_n = \sqrt[3]{A} + \sqrt[3]{B}, \]
where
\[ A = -\frac{q}{2} + \left[ \left( \frac{q}{2} \right)^2 + \left( \frac{p}{3} \right)^3 \right]^{1/2}, \quad B = -\frac{q}{2} - \left[ \left( \frac{q}{2} \right)^2 + \left( \frac{p}{3} \right)^3 \right]^{1/2}. \]

If \( \left( \frac{q}{2} \right)^2 + \left( \frac{p}{3} \right)^3 < 0 \), the positive root of equation (30) is
\[ z_n = 2r^{1/3} \cos \left( \theta/3 + 4\pi/3 \right), \]
where
\[ r = \sqrt{-\frac{p}{3}}, \quad \theta = \arccos \frac{-\frac{q}{2}}{\sqrt{-\frac{p}{3}}}. \]

Using equations (25), (29), (31) and (32) may give the approximation to the nth root of equation (2):
\[ x_n^a = \frac{n\pi}{2} - z_n + \frac{a_2}{3a_3}, \quad n = 1, 2, \ldots. \] (33)

We mention that de Alcantara Bonfim and Griffiths [4] proposed to solve equation (2) with \( n = 1 \) by approximating the cosine:
\[ \cos x \approx f_{1/2}(x) = \frac{1 - (2x/\pi)^2}{1 + (2x/\pi)^2} \quad (0 \leq x \leq \pi/2). \] (34)

The numerator guarantees that \( f_{1/2}(\pi/2) = 0 \). The parameter \( c \) in the denominator can be chosen so as to match the first two terms in the Taylor expansion of \( \cos x \) at the origin, where \( c_1 = 1 - 8/\pi^2 \), or by a least-squares fit, \( c_2 = 0.2120126 \). Combining equation (34) with equation (2) for \( n = 1 \) yields the first approximate root
\[ x_1^{ag} = \left( 1 + 8\beta^2/\pi^2 - \sqrt{1 + 8\beta^2[c + (2/\pi)^2]} \right)^{1/2}. \] (35)

To obtain closed-form expression for roots with \( n > 1 \), they used the parabolic approximation: \( \cos x \approx 1 - (2x/\pi)^2 \quad \) (0 \leq x \leq \pi/2) and obtained
\[ x_n^{ag} = \frac{\pi}{8\beta} [4(n - 1)\beta - \pi + \sqrt{(4\beta + \pi)^2 - 8\pi n\beta}], \quad n = 2, 3, \ldots. \] (36)

4.2. Results
For given value of parameter \( \beta \), the roots of equation (2) can numerically be calculated by using the Newton method. The proposed analytical approximations to these roots can be obtained by utilizing equation (33). Relative errors are then calculated against these numerical exact roots. For \( \beta = 15 \), the relative errors of the proposed approximate roots in equation (33) are shown in table 2. For the purpose of comparison, the relative errors of the approximate roots in [4] as shown in equations (35) and (36) are also exhibited in the same table. It
can be observed from Table 2 that, compared with the approximate roots in [4], the proposed approximate roots in this paper show excellent accuracy.

Letting $b_n$ denote the $n$th root of equation (2), we have

\[ \lim_{\beta \to 0} x_1^n(\beta) = 0 \quad \text{(no root exists for } n \geq 2 \text{ as } \beta \to 0), \quad \lim_{\beta \to +\infty} x_n^n(\beta) = \frac{n\pi}{2}, \quad n \geq 1. \]

Using the approximate expression in equation (33) yields

\[ \lim_{\beta \to 0} x_1^n(\beta) = 0 \quad \text{(no root exists for } n \geq 2 \text{ as } \beta \to 0), \quad \lim_{\beta \to +\infty} x_n^n(\beta) = \frac{n\pi}{2}, \quad n \geq 1. \]

Based on the results above, the approximate expressions in equation (33) are valid for small as well as large value of parameter $\beta$.

5. Conclusion

In this paper, the analytical approximate expressions for solutions to two kinds of transcendental equations have been established by exploring the periodic properties of functions $\tan x$ and $\sin x$ and using the Padé approximant. As an application of the proposed approximations, the effective mass of the spring for a spring-mass system has also been obtained. The high accuracy of the proposed approximate expressions has been illustrated. The proposed method can be generalized to construct the approximate root formulas for other transcendental equations in physics and engineering applications.

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Table 2. The roots of equation (2) with $\beta = 15$ and relative errors of the approximate roots proposed in this paper and in [4].

| $n$ | Exact roots | equation (33) | equation (17) ($n = 1$), equation (24) ($n > 1$) in [4] |
|-----|-------------|---------------|--------------------------------------------------|
| 1   | 1.472 47    | 1.238 95 $\times 10^{-9}$ | 0.244 830 ($c = 1 - 8/x^3$), 0.139 491 ($c = 0.2120126$) |
| 2   | 2.944 04    | 8.286 03 $\times 10^{-8}$ | 1.122 89 |
| 3   | 4.413 72    | 1.017 80 $\times 10^{-6}$ | 1.012 63 |
| 4   | 5.880 35    | 6.389 14 $\times 10^{-6}$ | 0.905 316 |
| 5   | 7.342 47    | 2.859 52 $\times 10^{-6}$ | 0.799 872 |
| 6   | 8.798 01    | 1.041 23 $\times 10^{-4}$ | 0.693 757 |
| 7   | 10.2438     | 3.463 217 $\times 10^{-4}$ | 0.584 581 |
| 8   | 11.6744     | 1.133 88 $\times 10^{-3}$ | 0.468 236 |
| 9   | 13.0781     | 4.105 38 $\times 10^{-3}$ | 0.336 489 |
| 10  | 14.4169     | 2.360 44 $\times 10^{-2}$ | 0.163 854 |
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