Testing spherical evolution for modelling void abundances

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ABSTRACT

We compute analytical predictions for the volume function of voids based on the excursion set approach and the peaks formalism for random walks smoothed with a top-hat filter in real space and a large class of realistic barrier models. We test our prediction by comparing with voids identified in the dark matter density field in N-body simulations using the zobov void finder. This tests the extent to which the spherical evolution approximation, which forms the basis of the analytical predictions, models the highly aspherical voids that occur in the cosmic web, and are found by a watershed-based algorithm such as zobov. We show that the volume function returned by zobov is quite sensitive to the choice of treatment of sub-voids, a fact that has not been appreciated previously. For reasonable choices of sub-void exclusion, we find that the Lagrangian density \( \delta_v \) of the zobov voids, which is predicted to be a constant \( \delta_v \approx -2.7 \) in the spherical evolution model, is quite different from the predicted value, showing substantial scatter and scale dependence. Our analytical approximations are flexible enough to give a good description of the resulting volume function; however, this happens for choices of parameter values that are different from those suggested by the spherical evolution assumption. We conclude that analytical models for voids must move away from the spherical approximation in order to be applied successfully to observations, and we discuss some possible ways forward.

Key words: cosmology: theory, large-scale structure of Universe, voids – methods: N-body, numerical, analytical

1 INTRODUCTION

A visually striking aspect of all galaxy surveys to date is the presence of large, nearly empty regions known as voids (Kirshner et al. 1981; Kauffmann & Fairall 1991; Hoyle & Vogeley 2002, 2004; Croton et al. 2004; Patiri et al. 2006a; Pan et al. 2012; Sutter et al. 2012). There has been considerable interest in characterizing the observable properties of voids and understanding their origin and dynamics (Hoffman et al. 1983; Dubinski et al. 1993; van de Weygaert & van Kampen 1993; Sahni et al. 1994; Colberg et al. 2003; Patiri et al. 2006a; van de Weygaert & Platen 2011). While a void can be defined in many ways (see, e.g., Colberg et al. 2008, and references therein), the basic picture of a large, underdense, expanding region (Fillmore & Goldreich 1984; Bertschinger 1985) has stood the test of time. Typical void sizes depend on the type of galaxy used to define them; e.g., in the main sample of the Sloan Digital Sky Survey, they can range from \( \approx 15 h^{-1}\text{Mpc} \) to \( \approx 30 h^{-1}\text{Mpc} \) (e.g., Pan et al. 2012; Sutter et al. 2012), while there are also examples of voids as large as \( \sim 100 h^{-1}\text{Mpc} \) (e.g. Granett, Neyrinck, & Szapudi 2008).

The presence of voids in galaxy surveys leads to many questions: whether galaxies that reside in void environments are special (Goldberg et al. 2003; Hoyle et al. 2003); whether large, deep voids are a challenge to structure formation in ΛCDM cosmologies (Blumenthal et al. 1993; Hunt & Sarkar 2010), or whether they are a natural consequence of the well-understood dynamics of cold dark matter (Tinker & Croton 2009); whether voids can then be used as a cosmological tool to distinguish between models (Ryden 1992; Park & Lee 2005; Lam et al. 2009; Lavaux & Wandelt 2010; Kunz et al. 2005; Biswas et al. 2010; D’Amico et al. 2011; Lavaux & Wandelt 2012; Hamana et al. 2013; Melchior et al. 2013); and whether their dynamics and statistics can be modelled analytically (Sheth & van de Weygaert 2003; Furlanetto & Piran 2008).

Analytical models for isolated voids have been well-studied in the literature for decades (Bertschinger 1983; Blumenthal et al. 1992). A major advance in their statistical modelling was presented by Sheth & van de Weygaert (2003) and SvdW, in what follows), who demonstrated that voids obey a hierarchy similar to that of halos. In particular, their analysis led to a prediction for the size distribution of voids...
based on the excursion set approach \cite{Press&Schechter1974, Estein1983, Bond et al. 1991}. Essentially, voids are modelled as regions that are initially underdense enough to reach shell-crossing by the present epoch. The SvdW analysis had three shortcomings, however; (a) it was based on an excursion set model using random walks in the smoothed density field with uncorrelated rather than correlated steps, (b) it was entirely based upon the initial or “Lagrangian” dark matter density, and (c), the intrinsic averaging of the excursion set walks, (on randomly selected position) was not taken into account. Recently, these shortcomings were overcome. In \cite{Paranjape,Lam & Sheth 2012} the SvdW treatment was modified to account for both correlated steps in the random walks (which arise when using smoothing filters such as the real-space TopHat) as well as the fact that voids are identified in the evolved “Eulerian” field. In \cite{Achitouv et al. 2012}, it was shown that the consistency of the excursion set framework is preserved once the barrier threshold is extended to stochastic modelling and also shortcasing (a) was solved using an alternative path integral approach that we apply to voids in this work.

Despite these improvements, excursion set void models cannot be directly compared with the distribution of observed galaxy voids. This is because these models are meant to describe voids in the dark matter, whereas the galaxies used to define voids observationally are biased tracers of dark matter. \cite{Furlanetto & Piran 2006} showed how galaxies can be included in the analysis by combining the SvdW excursion set calculation with the Halo Model \cite{Peacock & Smith 2000, Seljak 2000}. As expected when using biased tracers, this increases the sizes of voids in a manner that is correlated with galaxy type (e.g., more luminous galaxies define larger voids on average). The size distributions of observed galaxy voids, e.g., those presented by \cite{Pan et al. 2013} or \cite{Sutter et al. 2012}, should therefore be compared with predictions such as those of \cite{Furlanetto & Piran 2006} and not with SvdW.

Before doing this, however, it is important to ask whether the SvdW model (or the improved version suggested by \cite{Paranjape et al. 2012}) gives a good description of voids identified in the dark matter density itself, which is possible in N-body simulations \cite{Colberg et al. 2005, Jennings et al. 2013}, and is one of the primary goals of this paper. Jennings et al. (2013) recently compared the SvdW predictions to the results of a void finder specifically built to identify spherical underdensities. Although it is plausible that this is the correct way of comparing the SvdW predictions with measurements, it ignores the highly aspherical, polyhedral shape that initial underdensities develop into as they form voids and the unrealistic voids volume defined by the sharp-p filter used in SvdW. It is therefore interesting to ask whether the voids identified by popular algorithms (we use ZOBV below) can be incorporated in an appropriate analytical framework that goes beyond the approximation of spherical evolution. Moreover, from a physical point of view, one also expects that voids tend to form near minima of the initial density field \cite[see Colberg et al. 2005]{who demonstrated this in N-body simulations}, and it is then interesting to ask whether including a peaks constraint \cite{Bardeen et al. 1986} in the excursion set calculation improves the comparison.

The plan of the paper is as follows. In section\footnote{2} we present analytical results for the void volume function based on path-integral calculations within the excursion set approach which we test against Monte Carlo simulations of random walks. We also present the results of including the peaks constraint in such a calculation, and extend both these results to the case of stochastic and scale-dependent void-formation thresholds. In section\footnote{3} we turn to voids identified in N-body simulations. We describe the simulations and discuss the ZOBV void finder. In particular, we explore the sensitivity of the latter to the choice of treatment of sub-structures within the identified voids, and compare our analytical results with the ZOBV voids. In section\footnote{4} we check whether ZOBV voids are consistent with the assumptions of the spherical evolution model by measuring the initial overdensity at an appropriately defined void center. We conclude in section\footnote{5} with a summary of our results and prospects for future work.

2 THE VOID HIERARCHY

If the statistics of voids carry cosmological information, then a successful theory should be able to predict void properties directly from the initial conditions once the cosmological background is known. The most naive idea is to link the site of a void to an underdense region in Lagrangian space. Assuming this initial depression evolves decoupled from the surrounding shear field, and is approximately spherical, then \cite{Sheth & van de Weygaert 2004} have shown that the linear critical underdensity required to form a void at \( z = 0 \) is approximately \( \delta_c = -2.7 \) in an Einstein-de Sitter universe. Unlike haloes, voids expand over time and repel matter. A spherical evolution model predicts that the Eulerian radius \((R_E)\) of a void is \( R_E \sim 1.7 R \), with \( R \) its Lagrangian size. This deterministic mapping is almost linear compared to the collapse of proto-halos. In fact the density within the void is \( \Delta_v(z = 0) \sim -0.8 \).

This rather simple analytical model is the building block which allows to pass from the statistical properties of voids in the matter density field to the statistical properties using biased tracers such as galaxies \cite{Furlanetto & Piran 2004}. The linear spherical threshold \( \delta_c \) can be used to predict the site of void formation from the Lagrangian field. However, the dynamics of voids have an additional constraint, the cloud-in-cloud void effect (SvdW). This corresponds to a region which is underdense on a large scale \( R_1 \) but has an overdensity sufficient to collapse in a smaller scale \( R_2 \). In such case one could expect halos at the site of the void. Note the opposite effect (void-in-cloud) still leads to the formation of a halo (SvdW) and thus is relevant only for the void hierarchy. Therefore, in addition to the void threshold, the prediction of the void abundance also requires the specification of a halo threshold. In what follows, we make predictions for the void abundance using a realistic volume prediction and halo threshold within the standard excursion-set approach, and using a modified peak-excision set approach.

2.1 Excursion set approach

The standard excursion-set theory \cite{Bond et al. 1991} is a useful framework to compute the abundance of dark matter halos, and can also be applied to voids. The key assumption is to equate the volume fraction in voids of radius \( R \) to an appropriate first-crossing distribution:

\[
V \frac{d n}{d \ln R} = f(\sigma) \frac{d \ln \sigma}{d \ln R}
\]
where \( F(S) \equiv 2\sigma^2 F(\sigma) \) is the so-called multiplicity function and \( F \equiv dF/dS \) is the derivative of the volume fraction arising from the first-crossing problem. Let us denote the probability density that an overdensity smoothed on a scale \( R(S) \) is below a critical threshold \( B \) by \( \Pi(\delta, S(R)) \). We have

\[
F(S(R)) = -\frac{d}{dS} \Pi(\delta, S(R)) + k,
\]

(2)

where \( k \) is a constant independent of the scale \( R \), and \( S \) is the variance of the associated field:

\[
S \equiv \langle \delta^2(R) \rangle = \sigma^2 = \frac{1}{2\pi} \int dk k^2 P(k)W^2(k, R)
\]

(3)

Once the filter \( W(k, R) \) is specified, there is a one-to-one mapping between the scale \( R \) and the variance \( S \).

This formalism can be extended in the void hierarchy where voids are characterized by their volume rather than the mass they encapsulate.

Another common quality of the excursion-set theory is that the so-called cloud-in-cloud issue is solved: a collapsing structure cannot be embedded in a larger one which would lead to miscounting the number of halos. This issue appears if the smoothed overdensity crosses the threshold at multiple smoothing scales and can be treated by adding an absorbing boundary condition: \( \Pi(\delta = B, S) = 0 \) such that the largest scale defines the mass \( M(R) \). In the case of void one should also include the cloud-in-void as it was done in SvdW using a sharp-\( k \) filter (SK). Therefore we must distinguish between the barrier associated with halos that we note \( B_v \) and the one associated with void: \( B_v \). Thus all the game is to compute the \( \Pi(\delta, S) \) under the condition that \( \Pi(\delta, S = 0) = \delta_D(\delta) \), \( \Pi(\delta = B_v, S) = \Pi(\delta = B_v) = 0 \) and compute the first-crossing \( F_S(\delta(S) < B_v(S')) \) with \( S < S' \). For a sharp-\( k \) filter and a constant barrier (e.g.: \( B_v = (-2.7, B_v = 1.686) \) the solution of this extension is given in SvdW. The extension to a linear moving barrier of the same slope: \( B_v = \delta - \beta S \), \( B_v = \delta - \beta S \) can be found in appendix C of SvdW, while the extension to positive slope has been worked out in the appendix A of Furlanetto & Pirani (2003). Note that for the halo barrier, ellipsoidal collapse predicts a positive slope. However, we will see in [4] that for the void threshold, it seems that a negative slope is in better agreement with the Lagrangian barrier. However, before jumping to the barrier criterion we should emphasize that all those predictions hold for a particular type of filter, a TopHat in Fourier space (i.e. SK). The volume encapsulated by such filter is given by

\[
V_{SK}(R) = \int d^3R W_{SK}(R) = 6\pi^2 R^3 - 12\pi R^3 \int_0^\infty \cos x dx.
\]

(4)

One could argue that the divergent integral part can be set to zero (see, e.g. Lacey & Cole 1994) however, it is more difficult to picture the shape associated with such a void and in addition, this volume is never used in observation or in \( N \)-body simulations to defined structures. Therefore, if we assume a spherical-shell evolution of the void, for consistency, the appropriate filter should be a TopHat in real space (SX filter), which defines a spherical volume. However, in this case there is no exact analytical solution to the first crossing. One could run Monte Carlo walks (Bond et al. 1991) and solve the exact associated first-crossing, which would be straightforward (Paranjape et al. 2012). Nevertheless, a path-integral approach [Maggiore & Riotto 2010a Corasaniti & Achitouv 2011b] can be used to compute analytically the correction induced by such a filter to the SK filter applied to halo formation in the excursion set framework. This method has been shown to be very accurate and to converge well, and the exact Monte Carlo solution corresponds well to the analytical approximation (Corasaniti & Achitouv 2011a Achitouv et al. 2012). In what follows, we investigate the pertinence of the void-in-cloud effect for realistic halo thresholds in the context of the excursion set theory, and show that a one-barrier threshold is a very good approximation to the exact Monte Carlo solution, providing a simple analytical formula for the SK filter. Finally we extend this prediction to the SX filter. Our results are consistent with previous work by SvdW for sharp-\( k \) filtering and Paranjape et al. 2012 for (SX) filtering.

Achitouv et al. (2012) found that within the excursion set framework, any consistent barrier should have an intrinsic scatter due to the randomness of the position that the excursion-set theory assumes to compute the fraction of collapsed regions. Over the range they investigate, they found that a Gaussian barrier with mean value \( B(S) = \delta + \beta S \) and variance \( \sigma_B^2 \) is consistent with the initial Lagrangian critical overdensity leading to halo formation, and predicts a mass function which is in very good agreement with \( N \)-body simulations (e.g. Achitouv & Corasaniti 2012b; Corasaniti & Achitouv 2011a). Therefore, in order to test the void-in-cloud effect on the abundance of void, we assume a realistic barrier for halo and void formation (a diffusive drifting barrier).

Following Bond et al. (1991), we perform Monte-Carlo random walks to solve the first-crossing associated with a generic filter and barrier, and we implement the condition that the Monte Carlo results for the SK filter are shown in Fig. 1. The light-blue histogram shows the Monte-Carlo result associated with the two-barrier condition, while the blue dots neglect the void-in-cloud effect. As investigated in SvdW, the void-in-cloud effect operates at low radius; for a drifting diffusive barrier and Markovian walks (SK filter), it appears at \( R < 3 \) Mpc/h. The blue solid line is the exact analytical prediction of the excursion-set theory for a diffusive drifting void threshold with \( B_v(S) = \delta_v - \beta_v S \) and \( B_v(S_1)B_v(S_2) = D_v\min(S_1, S_2) \). This solution neglects the void-in-cloud effect and is exact for the sharp-\( k \) filter. It was computed as in Corasaniti & Achitouv (2011b), leading to a Markovian multiplicity function (SK) for voids:

\[
\frac{f_0(\sigma)}{\sigma} = \frac{1}{(1 + D_v)} \left[ \frac{D_v(\delta_v - \beta_v S)^2}{\pi^2} \right]^{\frac{3}{2}},
\]

(5)

with \( a = 1/(1 + D_v) \). The original prediction of Sheth & van de Weygaert (2004) (SvdW) is shown in purple. Note also that Eq. (5) reproduces with high accuracy SvdW by simply setting \( \beta = 0 \), \( D_v = 0 \) \( (R > 2 \) Mpc/h). However, in order to have a coherent volume definition, we should consider walks smoothed with a (SX) filter to compute the multiplicity function. In this case, we use the same path integral technique as [Maggiore & Riotto 2010a, Corasaniti & Achitouv 2011b].
For the SX filters this Monte Carlo shows that the probability that an initial underdense patch of matter with Lagrangian radius greater than 1 Mpc/h is embedded in an overdense larger region is negligible. This result is also in agreement with [Jennings et al. 2013]. Overall those Monte Carlo tests shows that Eq. (4) is a good prediction for the void abundance as long as the excursion set assumptions (e.g. averaging the smoothed field over random positions) describe void statistics. If this is not the case, then a peaks approach could provide a better description, which we present in the next section.

2.2 Peaks approach

In addition to their 2-barrier sharp-k random-walk model, SvW also discussed alternative models based on counting density minima in the initial conditions. The model that they called “adaptive troughs”, which was based on previous work by [Appel & Jones 1990] for the halo mass function, is especially interesting for us, because recent work on the nature of random walks with correlated steps sheds new light on its interpretation.

The adaptive-troughs model states that the void multiplicity function can be written by using the [Bardeen et al. 1986, BBKS in what follows] result for counting density peaks/troughs and including the effect of a variable smoothing filter:

\[
\begin{align*}
  f(\sigma) &= e^{-\Delta^2/2\sigma^2} V \int_0^\infty dx x F(x) p_c(x - \gamma |\delta|/\sigma; 1 - \gamma^2) .
\end{align*}
\]

Here \( V = 4\pi R^3/3 \) is the Lagrangian volume of the void, \( p_c(y - \mu; \Sigma^2) \) is a Gaussian in the variable \( y \) with mean \( \mu \) and variance \( \Sigma^2 \), and \( \gamma \) and \( V \) are ratios of spectral integrals that appear when counting density peaks/troughs,

\[
\gamma \equiv \sigma^2_3/(\sigma_0^2\sigma_2) ; \quad V \equiv (6\pi)^{3/2} \sigma_1^3/\sigma_2^3 ,
\]

where

\[
\sigma_1^2 = \int \frac{d^3k}{(2\pi)^3} P(k) k^2 e^{-k^2R_c^2}.
\]

The Gaussian smoothing scale \( R_c \) depends approximately linearly on the Lagrangian radius \( R \) and is discussed below.

The integral in equation (10) is over the peak curvature \( x = -\nabla^2\delta/\sigma_2 \), and involves the weighting function \( F(x) \) given by

\[
\begin{align*}
  F(x) &= \frac{1}{2} \left( e^x - 3x + x^3 - \frac{3}{2} x^5 \right) + \text{erf} \left( x \sqrt{\frac{5}{2}} \right) + \text{erf} \left( x \sqrt{\frac{5}{8}} \right) \nonumber \\
  &\quad + \sqrt{\frac{2}{5\pi}} \left( \frac{31x^2}{4} + \frac{8x}{5} \right) e^{-5x^2/8} \nonumber \\
  &\quad + \left( \frac{x^2}{2} - \frac{8}{5} \right) e^{-5x^2/2} .
\end{align*}
\]

which is the result of integrating over peak shapes (equations A14–A19 in BBKS). While there is no closed form expression for the multiplicity (10), the integral involved is straightforward to compute numerically, and we also note that BBKS provide a very accurate analytical approximation in their equations (4.4, 4.5, 6.13, 6.14).

Recently, Paranjape & Sheth (2012) pointed out, based on results obtained by Musso & Sheth (2014), that the multiplicity in equation (10) is an excellent approximation to the first-crossing distribution of the constant barrier \( B = \delta_c \).
by peak-centered random walks with correlated steps. Moreover, as argued by Paranjape et al. (2013), accounting for the complications introduced by the fact that voids are identified in Eulerian rather than Lagrangian space does not lead to significant effects when dealing with walks that have correlated steps. In particular, Paranjape et al. (2012) showed (see their Figure 3) that the appropriate first-crossing distribution for Eulerian voids (under the assumption of spherical evolution) is indistinguishable from that of a single constant barrier of height $\delta_v$ for all but the smallest voids. In other words, taken together, the results of Paranjape et al. (2012) and Paranjape & Sheth (2013) suggest that equation (10) should be a good model of void abundance, if one expects voids to have formed near initial density minima.

There is a technical issue related to the choice of Gaussian filtering with scale $R_G$ in defining the spectral integrals in equation (12). Ideally one would use TopHat (SX) filtering to define these integrals. However, in this case the identification of peaks for the CDM power spectrum becomes ill-defined since, e.g., $\sigma_2$ is no longer well-defined. Gaussian filtering avoids this problem, and all results in BBKS assume this. In order to make the calculation consistent with the standard assumption of defining $\delta$ using TopHat filtering, Paranjape, Sheth, & Desjacques (2013) proposed the following: to identify peaks/troughs, one can use spatial derivatives of the Gaussian filtered density contrast $\delta_G(R_G)$ so that $\sigma_2^2 = \langle (\nabla \delta_G)^2 \rangle$ and $\sigma_5^2 = \langle (\nabla^5 \delta_G)^2 \rangle$ are well-defined. The heights of these density extrema, on the other hand, can be defined using the TopHat-filtered $\delta_T(R)$. The connection between the two smoothing scales $R_G$ and $R$ follows by demanding $\langle \delta_G(\delta_T) \rangle = \delta_T$. Since $\delta_G$ and $\delta_T$ are both Gaussian distributed, this amounts to requiring $\langle \delta_G \delta_T \rangle = \langle \delta_T^2 \rangle = \sigma_2^2$.

These results can also be extended to the case when the barrier relevant for void formation is stochastic and/or scale-dependent. Following Paranjape et al. (2013), for a barrier of the form

$$B_v = \delta_v - \beta p_k \sigma$$

with the slope $\beta p_k$ a stochastic quantity with distribution $p(\beta p_k)$ in general, the void multiplicity becomes

$$f(\sigma) = \int d\beta p_k p(\beta p_k) e^{-\left(\delta_v - \beta p_k \sigma\right)^2/(2\sigma^2)} \frac{V}{\sqrt{2\pi \gamma_m}} V_x \times \int_{\delta p_k \gamma_m}^{\infty} dx (x - \beta p_k \gamma_m) F(x) \times \rho_c(x - \beta p_k \gamma_m - \gamma_m | \delta_v | / \sigma ; 1 - \gamma_m^2).$$

The resulting multiplicity is shown in Figure 1E for different parameters. The dotted orange curve shows the prediction from equation (10) for the constant barrier $B_v = \delta_v = -2.7$. The dashed orange curve shows the effect of introducing a negative drift with constant slope $\beta p_k = 0.5$ (formally, equation (10) with $p(\beta p_k) = \delta_G(\beta p_k - 0.5)$), while the orange solid curve shows equation (10) setting $p(\beta p_k)$ to be Lognormal with mean 0.5 and variance 0.25. These numbers, strictly speaking, are motivated by the results of Paranjape et al. (2013) for the halo mass function (we made a similar choice in the previous section when discussing standard excursion sets). We return to this issue later.

First of all observe that the standard peak-based “adaptive troughs” prediction (orange dotted curve) leads to a higher amplitude for the abundance of voids compared to the standard excursion set approach with a diffusive drifting barrier (see also SvdW). In addition, introducing scatter in the barrier height in the peaks prediction also tends to increase the number of void but it is a subdominant effect compared to the negative drift which decreases the amplitude. The solid orange line implements both scatter and drift while the orange dashed line neglects the scatter. Finally we note that interestingly both predictions for stochastic barriers with a negative drift are close to each others. For comparison, we also show the original SvdW prediction as the dot-dashed curve in which the relation between the density variance $S$ and Lagrangian radius $R$ was computed using the SX filter. While this is the usual manner in which the SvdW result is used, we emphasize that doing so is technically inconsistent, since the derivation in SvdW assumed SK filtering.

3 COMPARISON WITH N-BODY SIMULATIONS

In order to test our theoretical predictions, we measure void abundances from N-body simulations using the ZOBOV void finder (Nevrínk et al. 2008), described in the next section. We use the dark-matter field of the DEUS N-body simulation2 described in (2013); Courtin et al. 2013 for two box sizes, of length 162 and 648 Mpc$/h$, both with 1024$^3$ particles, realized using the RAMSES code (Teyssier 2002) for a $\Lambda$CDM model calibrated to WMAP 5-year parameters ($\Omega_m, \Omega_b, n_s, h, \Omega_{\Lambda} = (0.26, 0.79, 0.67, 0.7, 0.0456)$. As we shall see, these simulations have a particle density sufficient to accurately resolve the void abundance for radius $R > 1$Mpc$/h$. The resulting void function is sufficiently well-resolved to compare with the previous theoretical modelling once we carefully post-process the ZOBOV output which we explain in the next section.

3.1 ZOBOV

The ZOBOV (ZOnes Bordering On Voidness) void finder (Nevrínk et al. 2003, Nevrínk 2008, N08) is designed to be parameter-free. ZOBOV uses the adaptive, parameter-free Voronoi tessellation to estimate the density (e.g. Schaap & van de Weygaert 2000) at every particle. A void is grown around each local-density-minimum particle using a watersheds transform (e.g. Platen et al. 2007): in an analogy to rain falling and flowing across a terrain, a particle $p$ gets associated with a density minimum $p_{\text{min}}$ if a particle-to-particle path on the tessellation down steepest density gradients from $p$ ends at $p_{\text{min}}$.

2 www.deus-consortium.org
ZOBOV forms a parameter-free partition of all particles into so-called ‘zones,’ each of which is a watershed-region flowing down into a single density minimum. It then returns a parameter-free void catalog, consisting of voids (sets of zones joined together) and subvoids. But we introduce two parameters to prune the raw void catalog to something physically corresponding to our theoretical model, in which there are no subvoids; i.e. ‘voids in voids’ are not double-counted.

To obtain a disjoint set of voids with boundaries that are likely not spurious, we apply the ‘specifying a significance level’ strategy described in N08 to the raw void catalog. In this strategy, a boundary between two adjacent zones is declared to be real if the ‘density-contrast ratio,’ i.e. the ratio between the ridge density (the lowest-density along the ridge separating the voids) and the maximum of the zones’ density minima, exceeds a threshold. Unfortunately, it is difficult to know a priori what value to use for this threshold. It can be calibrated by measuring abundances of density contrast ratios in Poisson point samples, but this describes its statistical, not physical, significance. If this density-contrast ratio is used to judge voidiness, the void catalog best corresponding to a physical set of voids would likely differ based on the mass resolution, or sampling level.

As proposed by Jennings et al. (2013), it might be more consistent with the spherical shell approximation to use voids defined via ‘spherical underdensities.’ Jennings et al. (2013) use density minima found by ZOBOV to get initial void and subvoid centers, but then they define the boundaries much differently: for each void and subvoid, they start from a large radius about the center, and decrease the radius it until the underdensity reaches $\Delta_\varsigma = -0.8$. While this procedure intuitively corresponds well to a spherical-shell approximation, it is not clear that it is optimal as a way to associate a spherical model to voids that are often quite aspherical. In full nonlinearity, voids run into each other and their boundaries typically depart substantially from a spherical shape; these arbitrary shapes will be picked up by ZOBOV. The ‘spherical underdensity’ approach adapts the data to the theoretical framework, while here we choose to do the inverse, and adjust the theory to the data: we extend the barrier threshold to a more sophisticated model than the spherical threshold. Note also that Jennings et al. (2013) compare the ‘spherical-underdensity’ voids with the excursion-set theory associated with the SK filter, and find that the data do not match the theory unless a factor $\sim 1/5$ is introduced. In fact since the main effect of correlated walks is to reduce the amplitude of the multiplicity function, it would be interesting to compare their data with Eq. (6).

To get a consistent void sample, the second post-processing we do is to remove zones with high densities. Those zones are likely not physical voids, but local density minima in the point sample. In a sufficiently well-sampled high-density structure, these will arise by chance. As shown by N08, removing low-density-contrast-ratio voids will typically also remove these high-density voids, but to be sure about this, we introduce another threshold, the minimum density found in the void, called the ‘core density.’ Happily, there is a natural value to use for this parameter theoretically, $\Delta_\varsigma = -0.8$. However, since the core density of a void may be much lower than the mean density in the void (this mean density is what we expect to correspond best to the critical $\Delta_\varsigma = -0.8$), we use an even lower value for this threshold, as we explain below.

One might wonder why we do not cut directly on the (volume-weighted) average density within the void. This average density is easily computed from quantities in the void catalog, but in fact it can be quite noisy and biased high. This is because a watershed transform does not give boundaries that necessarily correspond to density contours; all that is required for a particle to belong to a void is that the particle is up a steepest density gradient from a density minimum, so in fact entire haloes might be included at the edge of a void. Still, a void’s average density from ZOBOV gives additional information about it, and we will use it below.

Because we analyse cubes cut out from larger simulations, we have to deal with boundaries. As is typically done with the ZOBOV algorithm (Granett et al. 2008; Sutter et al. 2012), we surround the box with border particles, and remove any voids from the catalog that touch the border.

3.2 Results

For low-radius voids, we analyzed two sub-cubes of size $40.5 \ Mpc/h^3$ from the $(162 \ Mpc/h)^3$ box. To check convergence with the higher-radius, lower-mass-resolution box, we use three sub-cubes of size $(162 \ Mpc/h)^3$ from the $(648 \ Mpc/h)^3$ box. In order to avoid counting the same structure twice, we remove subvoids by declaring boundaries to be real at a density contrast corresponding to a two-sigma cut in a Poisson realization. We used a two sigma cut as a compromise between having an accurate sample of disjoint voids (without subvoids) and a large number of voids. If we reduce the cut to one-sigma, this would declare shallower boundaries separating subvoids within voids to be real, cutting up larger voids and shifting the distribution to low radius. Likewise, increasing the
threshold removes walls between voids, which would increase the number of larger voids.

When it joins zones together to form voids, ZOBOV has an optional core-particle threshold, that we use: the largest density that the core particle in each zone can have in a void made up of several zones. Large voids are composed of several zones (subvoids). While their density minima must be less than the threshold, subvoids on the outskirts of a void may have substantial high-density regions. Thus for large voids (648 Mpc/h box), we pick a core density threshold lower than the spherical expectation: \( \Delta_{v} = -0.9 \), since this is a threshold on the minimum, not the average, density. Nevertheless, we caution that this particular threshold is a bit arbitrary.

The results are shown in Fig. 2. To map the Eulerian effective void radius \( R_{E} = (3V_{E}/4\pi)^{1/3} \) of each ZOBOV void to a Lagrangian radius \( R \), we use the spherical model and set \( R = R_{E}/1.7 \). One could adopt a different perspective and use a different mapping which might arise from aspherical voids consistent with the Lagrangian underdensity (linear void threshold). We do not investigate this issue here. In the case of halos, this issue is not really relevant. Spherical overdensity halo-finders use the non-linear spherical collapse model, while in fact the halo linear threshold we plug into the excursion set theory is not the spherical collapse criterion. Therefore we adopt the same pragmatic approach for voids.

The black line in Fig. 2 is from Eq. (5), while the orange solid curve is from Eq. (10), with the same parameters as in Fig. 1. Both theory curves match the data reasonably well when the parameters used are the same as in Achitouv et al. (2012), Paranjape et al. (2013). This is a nice result since Eq. (10) correspond to the prediction of the SX filter, which assumes a spherical volume to map the variance to the radius of the void; this is one of the main differences compared to SvdW.

4 CONSISTENCY OF THE SPHERICAL EVOLUTION APPROXIMATION

Perhaps the best test of spherical evolution would define voids as the largest spherical structures encapsulated an underdensity \( \Delta_{v} = -0.8 \), as in Jennings et al. (2013). Then for all of those voids, we could go back in the initial conditions and test whether the proto-void corresponds to a linear underdensity of \( \delta_{v} = -2.7 \). However, as we already mentioned, including the edge of the void or not makes a significant difference for the void’s average density. Therefore, defining voids through a density criterion might be noisy. In this sense, ZOBOV may be more suitable to define voids and compare with observation. Nevertheless, from the theory modelling, the spherical shell evolution is generally assumed and in this section we propose to test whether this assumption is consistent or not. In order to perform this test, we first consider the volume centroid of the void at \( z = 0 \), i.e. averaging together particle positions, weighting each particle by its Voronoi volume (one divided by its density). This center can differ from the one defined using the particle which sits on the minimum density of the void (the ‘core particle’). Indeed, the minimum of the density profile might not correspond to the minimum of the potential if the surrounding shear field is asymmetric. Therefore for each void, we find the closest particle to its volume centroid. Going back to the initial conditions, we record the underdensity in a sphere centered on this center particle within the Lagrangian region leading to the void detected using ZOBOV.

\[^{3}\text{The offset at very large radius might be due to a resolution issue.}\]

\[^{4}\text{the } x \text{ subscript means that the barrier crosses the smoothed overdensity: } \delta_{x} \equiv \{B_{x} \cap \delta\}\]
dense regions. However, there is a significant scatter, indicating that the spherical threshold can not be exactly applied to model the abundance of voids defined as we find in our previous analysis. Secondly, the mean of the distribution varies for two different void sizes ($\bar{\delta}(R = 1.8) = -3.4$, $\bar{\delta}(R = 6.25) = -2.5$). While the mean value is close to the spherical prediction for the larger void size, at smaller sizes the implied threshold is significantly more underdense, with a larger scatter. In principle, the scale-dependence of the mean value can be modelled by a negative drift term. In the peaks-based model this would imply a mean value $\langle \delta\rangle \simeq 1$ when using equation [10]. For the DDB model, the value of $\beta_D$ can be infer using the mapping of Eq.(10) in [Achitouv et al. (2012)] once we built the PDF of $\delta$ at randomly selected positions. Note, however, that different sets of values would not lead to a good agreement with the measured volume function, indicating a possible breakdown of the spherical evolution assumption which we discuss further below.

In addition, we checked that the closest particle to the center void at $z = 0$ is at a distance $d$ well below the void radius $R$, to avoid a bias due to a wrong definition of the void center. Also, we checked that the PDF of $\delta_c$ is almost insensitive to the displacement of this center particle. This is important because if we assume that this particle sits at the minimum of the density, then tracking its displacement tells us if the void forms from an initial underdense peak in Lagrangian space. It also comforts us that we have picked the correct center of the void; if initially this particle is not at the minimum of the potential then it should be pulled out of the center at $z = 0$.

Note that alternatively to our protovoid center definition, we could have defined the center in Lagrangian space by averaging over initial positions of particles which belong to the void at $z = 0$, weighting each particle in the average by its $z = 0$ Voronoi volume. However, we expect this correction to be negligible since the PDF shown in Fig.4 does not change for the center particle which has a small displacement suggesting that the center particle really sits at the minimum of the potential. Note that we expect these displacements to be tiny only for the largest voids; large-scale flows can slightly move small voids and subvoids.

To summarize, this analysis shows that, as expected, voids form from initial underdense regions (see Fig. 4). Most of the void center displace over time from their initial location but the density criteria which leads to their formation is rather uncorrelated to this displacement. Furthermore, the deterministic spherical evolution is apparently not exactly achieved even for larger voids. This might be due to our void definition or indicate a break of the simple spherical model.

5 CONCLUSIONS

In this paper we compared analytical predictions for the void volume function and voids identified by the ZOBOV void finder in N-body simulations.

We presented simple analytical predictions for the void volume function based on the excursion set approach and the peaks formalism which consistently include the effects of filtering with a TopHat in real space. We extended traditional predictions that use a deterministic barrier threshold to a more general class of barriers which are stochastic and sensitive to the size of the voids. The analytical predictions were tested against Monte Carlo simulations of random walks crossing the diffusive drifting barrier and shows a very good agreement.

We then investigated the applicability of our results, finding voids in N-body simulations using the ZOBOV algorithm. ZOBOV picks up highly aspherical voids, as many initial underdensities are at the present epoch. While ZOBOV is well-suited to voids as they occur in the cosmic web, it is not obvious how to analyze its results in the context of the usual spherical evolution model. For example, one (as yet unappreciated) aspect of ZOBOV is that the measured void volume function can depend on the choices made when the in-principle parameter-free void and subvoid catalog is pruned to require disjoint voids. We introduce a parameter to do so, and we find that the results can depend sensitively on this parameter. This could have important consequences for observational results that have used ZOBOV in analysing galaxy surveys. Additionally, a direct measurement in the initial conditions shows that the usual linear-theory threshold is incompatible with the deterministic value predicted by the spherical model for ZOBOV voids defined with a core density $\Delta_c = -0.8$. This is an important result since current predictions for the abundances of voids defined using compact objects require the linear threshold criterion as an input [Furlanetto & Piran 2006].

Overall we find our analytical predictions are in reasonable agreement with the measured void volume functions when the former assume a stochastic void barrier that approaches the traditional value of $\Delta_c = -2.7$ for large voids. However, the specific parameter values needed to obtain this agreement (at least for the peaks-based model) do not appear to be consistent with our direct measurements of the overdensity threshold in the initial conditions. This suggests that analytical models for void evolution must likely move away from the spherical approximation in order to better describe voids identified by algorithms such as ZOBOV. It will be interesting to see whether more sophisticated treatments of void shapes can help improve
the comparison between theory and measurement for the volume function.

Our work can be extended in several directions: one could compare our prediction with a spherical-underdensity algorithm and see if the corresponding initial threshold tends to the deterministic spherical prediction, introduce a more general barrier as we present here, or perhaps a deeper investigation on the mapping between Lagrangian to Eulerian space could be performed by finding empirically a q factor (\(R_E/q = R\)) such that the proto-void underdensity distribution gets consistent with the linear barrier which enters in the modelling of the void abundance. If such a factor can be found we can already infer that it would have to be smaller than the spherical 1.7 value in order to get a less deep mean underdensity for small voids. We could also investigate the link between voids defined for different biased objects and the void densities. Those tests are crucial if we want to have analytical model prediction able to compare to current and future observational void catalogues made on biased tracers.

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