ASYMPTOTIC BEHAVIOR OF POSITIVELY CURVED
STEADY RICCI SOLITONS, II

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Abstract. In this note, we show that any $n$-dimensional $\kappa$-noncollapsed
steady Kähler-Ricci soliton with nonnegative bisectional curvature must
be flat. The result is an improvement to our former work in [7].

1. Introduction

In [7], we prove

Theorem 1.1. There is no any $\kappa$-noncollapsed steady Kähler-Ricci soliton
with nonnegative sectional curvature and positive Ricci curvature.

In this note, we improve Theorem 1.1 in sense of nonnegative bisectional
curvature. Namely, we have

Theorem 1.2. There is no any $\kappa$-noncollapsed steady Kähler-Ricci soliton
with nonnegative bisectional curvature and positive Ricci curvature.

As an application of Theorem 1.1, we get the following rigidity result for
the steady Kähler-Ricci solitons and Kähler-Ricci flow.

Theorem 1.3. Any $\kappa$-noncollapsed steady Kähler-Ricci soliton $(M, g, f)$
with nonnegative bisectional curvature must be flat. More generally, any
$\kappa$-noncollapsed noncompact and eternal Kähler-Ricci flow with nonnegative
bisectional curvature and uniformly bounded curvature must be a flat flow.

In the proof of Theorem 1.1 one main step is to use the blow-down argu-
ment to analysis the structure of limit Kähler-Ricci flow of a sequence of
rescaled Kähler-Ricci flows. The nonnegative sectional curvature condition
is used to guarantee the existence of lines on the limit flow by using To-
ponogov comparison Theorem (cf. [9]) and consequently the limit flow can

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be split off a line by Cheeger-Gromoll splitting theorem [4]. The splitting property is crucial in our proving the curvature decay:

\[(1.1) \quad R(p, t) \to 0, \text{ as } t \to -\infty.\]

Here \(R(p, t)\) is a scalar curvature of corresponding Ricci flow of steady Kähler-Ricci soliton \((M, g, f)\).

The new ingredient at present is to prove a local splitting result for \(\kappa\)-noncollapsed steady Kähler-Ricci solitons with nonnegative bisectional curvature (cf. Lemma 3.1). Then we can generalize the argument in the proof of Theorem 1.1 to Theorem 1.2.

Theorem 1.3 can be regarded as a generalization of Ni’s rigidity theorem for ancient solution of Kähler-Ricci flow with maximal volume growth [10]. Ni’s result is a complex version of Perelman’s Theorem for ancient solution of Ricci flow in [11], Section 11.4.

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2. **Primary results in [7]**

In this section, we recall some results proved in [7], which will be used to complete the proof of Theorem 1.2 in the next section.

Let \((M, g, f)\) be a complete \(\kappa\)-noncollapsed steady Kähler-Ricci soliton with nonnegative bisectional curvature. It is proved that there exists a quilibrium point \(o\) of \(M\) such that \(\nabla f(o) = 0\) in [6]. Let \(\phi_t\) be a family of biholomorphisms generated by \(-\nabla f\). Let \(g(t) = \phi_t^*(g)\). Then \(g(t)\) satisfies the Ricci flow

\[(2.1) \quad \frac{\partial g}{\partial t} = -2\text{Ric}(g), \quad t \in (-\infty, \infty).\]

Let \(p_i \in M\) be any sequence with \(\text{dist}(o, p_i) \to \infty\) as \(i \to \infty\). We consider a sequence of rescaled flows \((M, R(p_i)g(R^{-1}(p_i)t), p_i)\). By using the compactness theorem, Theorem 3.3 in [7], we prove the following convergence result.

**Theorem 2.1.** \((M, R(p_i)g(R^{-1}(p_i)t), p_i)\) converges subsequently to a pseudo \(\kappa\)-solution \((M_\infty, g_\infty(t)) (t \in (-\infty, \infty))\) of Kähler-Ricci flow in the Cheeger-Gromov topology.

Theorem 2.1 is in fact a part of Theorem 1.5 in [7]. Since \((M, g, f)\) is weakened to have nonnegative bisectional curvature condition, we could not
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get the splitting property of limit \((M_\infty, g_\infty(t))\) as in Theorem 1.5 under non-negative sectional curvature condition. Here a pseudo \(\kappa\)-solution of Kähler-Ricci flow means that a non-flat Kähler-Ricci flow with \(\kappa\)-noncollapsed condition satisfies the Harnack inequality,

\[
\frac{\partial R}{\partial t} + \nabla_i R \nabla^i + \nabla_i R \nabla^i + R_{ij} \nabla^i \nabla^j \geq 0, \quad \forall \, V \in T^{(1,0)} M.
\]

For a steady Kähler-Ricci soliton \((M, g, J)\) with positive Ricci curvature, which admits an equilibrium point, Bryant in [1] proves that there exist global holomorphic coordinates (Poincaré coordinates) \(z : M \to \mathbb{C}^n\) which linearize

\[
Z = \sum_{i=1}^n h_i z_i \frac{\partial}{\partial z_i},
\]

where \(h_1, \ldots, h_n\) are positive constants. As in Corollary 5.2 in [7], we choose points \(p_k = (k, 0, \ldots, 0, 0 \cdot \ldots, 0) \in M\). By computation of lengths of integral curves of \(J\nabla f\) starting from \(p_k\), we have

**Lemma 2.2.** Suppose that \((M, g, f)\) is an \(n\)-dimensional \(\kappa\)-noncollapsed steady Kähler-Ricci soliton with nonnegative bisectional curvature and positive Ricci curvature. Then there exists a positive constant \(C\) such that \(R(p_k) > C\), where \(C\) is independent of \(p_k\).

Lemma 2.2 is in fact from Lemma 5.4 in [7] while the nonnegative sectional curvature condition is replaced by nonnegative bisectional curvature condition. This can be done since we only use the convergence result, Theorem 2.1 in the proof Lemma 5.4.

3. Proof of Theorem 1.2

The following proposition is a key lemma in the proof of Theorem 1.2

**Lemma 3.1.** Let \((M, g, f)\) be a \(\kappa\)-noncollapsed steady Kähler-Ricci soliton with nonnegative bisectional curvature and positive Ricci curvature. Then, for any \(p \in M \setminus \{o\}\), at least one of the following two properties holds:

i) \(R(p, t) \to 0\), as \(t \to -\infty\).

ii) For any \(\tau_i \to -\infty\), \((M, R(p, \tau_i)g(R^{-1}(p, \tau_i)t), \phi_{\tau_i}(p))\) converges subsequentially to a flow of \(\kappa\)-noncollapsed steady Kähler-Ricci soltions \((M_\infty, g_\infty(t), p_\infty)\) with nonnegative bisectional curvature which splits locally off a complex line.

**Proof.** First we note

\[
|\nabla f|^2 + R = R_{\text{max}} = \sup_{x \in M} R(x).
\]
and $R(p) \geq 0$. Then by the uniqueness of quilibrium points, we have

$$0 < R(p, t) < R_{\text{max}}.$$ 

Thus by relation

$$\frac{\partial R(p, t)}{\partial t} = 2\text{Ric}(\nabla f, \nabla f)(\phi_t(p)) > 0,$$

we see that $\lim_{t \to -\infty} R(p, t)$ exists. Now we suppose that the property i) in Lemma 3.1 does not hold. Then there exists a point $p \in M$ such that

$$\lim_{t \to -\infty} R(p, t) = C_0 > 0.$$

Since $o$ is an unique equilibrium point, by (3.2), we have $C_0 < R_{\text{max}}$.

Consider any sequence $(M, g_i(t), p_{\tau_i})$, where $g_i(t) = R(p, \tau_i)g(R^{-1}(p, \tau_i)t)$ and $p_{\tau_i} = \phi_t(p)$ with $\tau_i \to -\infty$. Then $(M, g_i(t))$ is $\kappa$-noncollapsed and curvature of $(M, g_i(t))$ is uniformly bounded. By Hamilton compactness theorem [8], $(M, g_i(t), p_{\tau_i})$ converges subsequently to a pseudo $\kappa$-solution $(M_\infty, g_\infty(t), p_\infty)$, where $t \in (-\infty, +\infty)$. Moreover, by (3.3), we have

$$R_\infty(p_\infty, t) = \lim_{\tau_i \to -\infty} \frac{R(p_{\tau_i}, R^{-1}(p, \tau_i)t)}{R(p, \tau_i)} = 1, \forall t \in (-\infty, +\infty).$$

and consequently,

$$\frac{\partial}{\partial t} R_\infty(p_\infty, t) \equiv 0.$$

Since $(M_\infty, g_\infty(t); p_\infty)$ is not flat by (3.4), we may assume that $(M_\infty, g_\infty(t))$ has positive Ricci curvature by Cao’s dimension reduction theorem [3]. By the Harnack inequality (2.2) together with condition (3.5), following the argument in the proof of Theorem 4.1 in [2], we can further prove that $(M_\infty, g_\infty(t), p_\infty)$ is in fact a steady Kähler–Ricci soliton, which is $\kappa$-noncollapsed and has nonnegative bisectional curvature and positive Ricci curvature (also see Proposition 2.2, [5]). More precisely, there is a smooth vector field $V$ on $M_\infty$ such that

$$R^{(\infty)}_{ij} = \frac{1}{2}(\nabla_i^{(\infty)}V_j + \nabla_j^{(\infty)}V_i)$$

and

$$\nabla_i^{(\infty)}V_j = \nabla_j^{(\infty)}V_i.$$

Thus

$$\frac{\partial R^{(\infty)}(p_\infty, t)}{\partial t}|_{t=0} = 2\text{Ric}^{(\infty)}(V, V).$$

By (3.5), it follows $V(p_\infty) = 0$.

Let $X^{(i)} = R^{-1}(p, \tau_i)\nabla f$. Then

$$|\nabla_i X^{(i)}|_{\hat{g}_t} \leq C_0^{-1}|\text{Ric}^{(i)}|_{\hat{g}_t} \leq C,$$
where \( \hat{g}_i = R(p, \tau_i)g \) and \( \text{Ric}^{(i)} \) is Ricci curvature of \( \hat{g}_i \). By Shi’s higher order estimate [12] and the soliton equation, we also get

\[
|(|\nabla|^m X_{(i)}|_\hat{g}_i| \leq C(n)|(|\nabla|^{m-1}\text{Ric}^{(i)}|_\hat{g}_i) \leq C(m).
\]

Thus by taking a subsequence, we may assume that

\[
X_{(i)} \to X \text{ as } i \to \infty.
\]

Since

\[
\nabla^{(i)}_{j} X_{(i)k} = \nabla^{(i)}_{k} X_{(i)j} = R^{(i)}_{jk},
\]

we get

\[
(3.8) \quad \nabla^{(\infty)}_{j} X_{k} = \nabla^{(\infty)}_{k} X_{j} = R^{(\infty)}_{jk}.
\]

Moreover, by (3.1),

\[
|X|_{g_{\infty}}(p_{\infty}) = \lim_{i \to \infty} |X_{(i)}|_{\hat{g}_i}(p_i) = \lim_{i \to \infty} \sqrt{\frac{R_{\text{max}}}{R(p, t_i)}} - 1.
\]

Hence

\[
(3.9) \quad |X|_{g_{\infty}}(p_{\infty}) = \sqrt{C_0^{-1} R_{\text{max}}} - 1 > 0.
\]

Let \( W = V - X \). Then by (3.6), (3.7) and (3.8), we have

\[
(3.10) \quad \nabla^{(\infty)} W = \nabla^{(\infty)}(V - X) \equiv 0.
\]

On the other hand, \( V(p_{\infty}) = 0 \). By (3.9), we see that \( W \) is nonzero everywhere. Thus the Kähler manifold \((M_{\infty}, g_{\infty}(0))\) splits locally off a line along \( W \). Note that \( J_{\infty}W(p_{\infty}) \neq 0 \) and

\[
\nabla^{(\infty)}(J_{\infty}W) = 0,
\]

where \( J_{\infty} \) is the complex structure of \( M_{\infty} \). Hence \((M_{\infty}, g_{\infty}(0))\) also splits locally off a line along \( J_{\infty}W \). As a consequence, the steady soliton \((M_{\infty}, g_{\infty}(0))\) splits locally off a complex line. Therefore, the property ii) in Lemma 3.1 holds. The lemma is proved. \(\square\)

**Remark 3.2.** Lemma [3.7] is still true for the steady Kähler-Ricci soliton with nonnegative bisectional curvature if the \( \kappa \)-noncollapsing condition is replaced by the existence of uniform injective radius of soliton.

**Corollary 3.3.** Let \((M, g, f)\) be a 2-dimensional \( \kappa \)-noncollapsed steady Kähler-Ricci soliton with nonnegative bisectional curvature and positive Ricci curvature. Then for any \( p \neq o \),

\[
(3.11) \quad R(p, t) \to 0, \text{ as } t \to -\infty.
\]
Proof. We prove the corollary by contradiction. Then by Lemma 3.1 there exist a point $p \in M$ and a sequence of $t_i \to -\infty$ such that

$$\lim_{t \to -\infty} R(p, t_i) = C > 0$$

and rescaled $(M, R(p, t_i)g(R^{-1}(p, t_i)t), \phi_{t_i}(p))$ converges subsequently to a flow of 2-dimensional complete $\kappa$-noncollapsed, steady Kähler-Ricci solitons $(M_\infty, g_\infty(t), p_\infty)$ ($t \in (-\infty, \infty)$) which splits locally off a complex line. Moreover

$$R_\infty(p_\infty, t) = 1, \text{ for } t \in (-\infty, +\infty).$$

Let $(\tilde{M}_\infty, \tilde{g}(0))$ be the universal covering of $(M_\infty, g(0))$. Then it is easy to see that $(\tilde{M}_\infty, \tilde{g}(0))$ still satisfies the $\kappa$-noncollapsing condition. Moreover, the parallel vector field $W$ in Lemma 3.1 can be lifted on $(\tilde{M}_\infty, \tilde{g}(0))$ and this vector field generates a trivial homotopy group of $(\tilde{M}_\infty, \tilde{g}(0))$. Thus by Wu’s de Rham decomposition Theorem [13], the steady soliton $(\tilde{M}_\infty, \tilde{g}(0))$ splits off a complex line. Hence the corresponding steady solitons flow $(\tilde{M}_\infty, \tilde{g}(t))$ splits off a complex line, and so it splits out a real 2-dimensional flow $(N, g_N(t))$ of complete $\kappa$-noncollapsed steady Ricci solitons with positive curvature. On the other hand, by Lemma 4.4 in [7], any complete pseudo-$\kappa$-solution on a surface is a shrinking flow of round spheres and so $(N, g_N(t))$ does. In particular, $(N, g_N(0))$ is compact. But this is impossible since any compact gradient steady Ricci soliton should be flat. The corollary is proved.

Choosing $p \neq o \in M$ and a sequence of $t_k \to -\infty$. Let $p_k = \phi_{t_k}$. Then by Corollary 3.3 $R(p, t_k) \to 0$ as $t_k \to -\infty$. Moreover, one can check $p_k = (e^{-t_k h_1}, ..., 0)$ under Poincaré coordinates. Thus applying Lemma 2.2 to 2-dimensional steady Kähler-Ricci soliton together with Cao’s dimension reduction theorem, we prove

**Proposition 3.4.** Let $(M, g)$ be a 2-dimensional $\kappa$-noncollapsed steady Kähler-Ricci soliton with nonnegative bisectional curvature. Then $(M, g)$ is flat.

Now we complete the proof of Theorem 1.2.

**Proof of Theorem 1.2.** By Proposition 3.4 we can use the induction argument as in the proof of Theorem 1.3, [7]. Suppose that there is no $k$-dimensional $\kappa$-noncollapsed steady Kähler-Ricci soliton with nonnegative bisectional curvature and positive Ricci curvature for all $k < n$. We claim

**Claim 3.5.** Under the induction hypothesis, for any fixed $p \in M \setminus \{o\}$, $R(p, -t) \to 0$ as $t \to \infty$. 


The proof of Claim 3.5 is similar to one of Corollary 3.3. In fact, if the claim is not true, then by Lemma 3.1 there exist a point \( p \in M \) and a sequence of \( t_i \to -\infty \) such that

\[
\lim_{t \to -\infty} R(p, t_i) = C > 0
\]

and rescaled \((M, R(p, t_i)g(R^{-1}(p, t_i)t), \phi_{t_i}(p))\) converges subsequently to a flow \((M_\infty, g_\infty(t), p_\infty)\) \((t \in (-\infty, \infty))\) of complete \(\kappa\)-noncollapsed, steady Kähler-Ricci solitons which splits locally off a complex line. Moreover

\[
R_\infty(p_\infty, t) = 1, \text{ for } t \in (-\infty, +\infty).
\]

(3.15)

Now we can consider the universal covering \((\tilde{M}_\infty, \tilde{g}(0))\) of \((M_\infty, g(0))\). As in Corollary 3.3, \((\tilde{M}_\infty, \tilde{g}(0))\) can be split out an \(n-1\)-dimensional complete \(\kappa\)-noncollapsed steady Kähler-Ricci soliton \((N, g_N)\) with nonnegative bisectional curvature. Note that \((N, g_N)\) is not flat by (3.15). Thus by Cao’s dimension reduction theorem, we may assume that \((N, g_N)\) has positive Ricci curvature. On the other hand, by the induction assumption, \((N, g_N)\) should be flat. This is a contradiction! The claim is proved.

By Claim 3.5, we can choose \(p_k\) as in the proof of Proposition 3.4 to apply Lemma 2.2 to finish the proof of Theorem 1.2.

\[\square\]

Theorem 1.3 is an application of Theorem 1.2 by using an argument as in proof of Theorem 1.4 in [7].

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