Hidden symmetries of integrable conformal mechanical systems

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We split the generic conformal mechanical system into a “radial” and an “angular” part, where the latter is defined as the Hamiltonian system on the orbit of the conformal group, with the Casimir function in the role of the Hamiltonian. We reduce the analysis of the constants of motion of the full system to the study of certain differential equations on this orbit. For integrable mechanical systems, the conformal invariance renders them superintegrable, yielding an additional series of conserved quantities originally found by Wojciechowski in the rational Calogero model. Finally, we show that, starting from any \( N=4 \) supersymmetric “angular” Hamiltonian system one may construct a new system with full \( N=4 \) superconformal \( D(1,2;\alpha) \) symmetry.

I. INTRODUCTION

Conformal invariance plays an important role in many areas of the quantum field theory and condensed matter physics, especially in string theory, the theory of critical phenomena, low-dimensional integrable models, spin and fermion lattice systems. Therefore, one-dimensional (mechanical) systems with conformal invariance are worth to be studied in detail and still require further investigation. Actually, the conformal group is not an exact symmetry for the conformal mechanical system. It does not commute with the Hamiltonian but, instead, is a symmetry of the action (a symmetry in the field theoretical context). The Hamiltonian itself forms an \( \mathfrak{so}(1,2) \) algebra together with generators of the dilatation and conformal boost, with respect to canonical Poisson brackets. It is interesting that, due to the conformal symmetry, the “angular” part of the Hamiltonian of conformal mechanics is a constant of motion. However, its relation with other constants of motion has not been investigated properly so far.

The (rational) Calogero model \([1,2,3]\), which is an integrable \( N \)-particle one-dimensional system with pairwise inverse-square interaction (and its various generalizations related with different Lie algebras and Coxeter groups \([4]\)) is a famous example of a conformal mechanical system (for the review, see \([5]\)). Usually, Lax-pair and matrix-model approaches are employed for the study of this system. These are common methods which are applied to other integrable models not related to the conformal group. At the same time, many properties of the rational Calogero model are due to its conformal invariance, and they are shared with other conformal mechanical models. For example, the “decoupling transformation” in the Calogero model \([6]\) can be formulated purely in terms of conformal transformations \([7]\) (see also \([8]\)). Note that the rational \( N \)-particle Calogero model is a maximally superintegrable system, i.e. it possesses \( N-1 \) additional functionally independent integrals apart from the Liouville integrals being in involution \([9]\). Despite an impressive list of references on this subject \([10]\), the superintegrability of the Calogero model still seems to be mysterious. Preliminary considerations have indicated a direct connection between the additional constants of motion and the “angular” part of the Calogero model \([11]\). This observation allows us to propose that conformal invariance provides any integrable conformal system with the superintegrability property. In the present paper we prove this statement.

The \( N=4 \) superconformal extension of the \( N \)-particle Calogero system based on the \( \mathfrak{su}(1,1|2) \) superalgebra \([12]\) is intimately related with solutions of the WDVV equation and therefore relevant for topological field theory. By looking at the \( N=4 \) superconformal three-particle rational Calogero system \([13]\), we find evidence that any \( N=4 \)
supersymmetric extension of the respective angular Hamiltonian system could be lifted to some \( \mathcal{N}=4 \) superconformal mechanics.

In present paper we clarify these issues. For this purpose we invariantly split the generic conformal mechanics into a “radial” and an “angular” part, where the latter is a Hamiltonian system on the orbit of conformal group, with the Casimir function playing the role of the Hamiltonian. We investigate how the constants of motion of the conformal mechanics are encoded in this “angular Hamiltonian system”. We reduce the analysis of the constants of motion to the study of certain differential equations in the angular variables. As one of the main results of this paper, we demonstrate that the additional series of constants of motion of the Calogero model found by Wojciechowski can be constructed as Poisson brackets of the Liouville constants of motion with the angular Hamiltonian of the Calogero model. Remarkably, these additional integrals exist for generic conformal mechanics. This means that any integrable conformal mechanics is also superintegrable. We give the explicit expression relating the constants of motion of the conformal mechanics with those of its angular part. All these observations may easily be extended to the case of superconformal systems. As an example, in the last Section we explicitly demonstrate that any \( \mathcal{N}=4 \) supersymmetric “angular” system can be lifted to some mechanics with full \( \mathcal{N}=4 \) superconformal \( D(1,2;\alpha) \) symmetry.

II. THE MODEL

Let us consider an arbitrary conformal mechanics defined by the Hamiltonian \( H \) which forms, together with some generators of dilatations \( D \) and conformal boost \( K \) the one-dimensional conformal algebra \( \text{so}(2,1) \)

\[
\{H, D\} = 2H, \quad \{K, D\} = -2K, \quad \{H, K\} = D. \tag{2.1}
\]

The Casimir element of (2.1) is given by the expression

\[
\mathcal{I} = 2KH - \frac{1}{2}D^2 \tag{2.2}
\]

If we now define the radial coordinate \( r \) and its conjugated momentum \( p_r \) as follows

\[
r \equiv \sqrt{2K}, \quad p_r \equiv \frac{D}{\sqrt{2K}} : \quad \{p_r, r\} = 1, \tag{2.3}
\]

then, taking into account that \( \mathcal{I} \) is the Casimir of of \( \text{so}(1,2) \) with the explicit form (2.2), we will arrive at the system

\[
H = \frac{p_r^2}{2} + \frac{\mathcal{I}}{r^2}, \quad D = rp_r, \quad K = \frac{r^2}{2}, \quad \{p_r, r\} = 1, \tag{2.4}
\]

where

\[
\{\mathcal{I}, H\} = 0, \quad \{p_r, \mathcal{I}\} = \{r, \mathcal{I}\} = 0. \tag{2.5}
\]

Thus, \( \mathcal{I} \) is indeed the constant of motion of the Hamiltonian \( H \). Therefore, although conformal symmetry is not a symmetry of the Hamiltonian, it equipped the system with the additional (to the Hamiltonian) constant of motion \( \mathcal{I} \).

Summarizing we conclude that the generic mechanical system with dynamical symmetry given by conformal algebra \( \text{so}(1,2) \) (2.1) is defined by the Hamiltonian system

\[
\Omega = dp_r \wedge dr + \omega_0(u), \quad H = \frac{p_r^2}{2} + \frac{\omega_0(u)}{r^2} \tag{2.6}
\]

where \( p_r \) and \( r \) are defined in (2.3), while \( (M_0, \omega_0, \mathcal{I}) \) is an arbitrary Hamiltonian system on the symplectic space \( (M_0, \omega_0) \) parameterized by some set of coordinates \( \{u_1, u_2, \ldots\} \).

It is seen from the above consideration that the whole information about conformal system is encoded, in some way, in the the lower-dimensional system \( (M_0, \omega_0, \mathcal{I}) \). In particular, it is obvious that if the system \( (M_0, \omega_0, \mathcal{I}) \) is integrable in the sense of Liouville then the corresponding conformal mechanics (2.6) is also integrable in this sense [14]. It is also obvious that the separation of variables of the \( (M_0, \omega_0, \mathcal{I}) \) immediately implies the separation of variables in the corresponding conformal mechanics. Vice versa, having at hand the exact solutions of the conformal mechanics (2.6), we immediately get the explicit solutions of the system \( (M_0, \omega_0, \mathcal{I}) \). Hence, the integrability of the
generic conformal mechanics leads to the integrability of \((M_0, \omega_0, \mathcal{I})\). However, it is unclear, how to extract the Liouville constants of motion of the angular part from ones of the underlying conformal mechanics. More generally, it is unclear, how the dynamical conformal symmetry impacts on the integrability properties of the system. In the next sections we present our preliminary observations on this subject.

An important particular case of the conformal system is the conformal mechanics on the Euclidean space \(\mathbb{R}^N\),

\[
\omega = dp \wedge dr, \quad H = \frac{p^2}{2} + V(r), \quad \text{where} \quad r \cdot \nabla V(r) = -2V(r), \quad (2.7)
\]

and \(r = (x_1, \ldots, x_N)\), \(p = (p_1, \ldots, p_N)\) are, respectively, the Cartesian coordinates of \(\mathbb{R}^N\) and their conjugated momenta. In this case the generators \(D\) and \(K\) looks as follows

\[
D = p \cdot r, \quad K = \frac{r^2}{2}. \quad (2.8)
\]

Extracting the radius \(r = |r|\), and introducing the conjugated momentum \(p_r = p \cdot r/r\), we can represent the above generators as in \((2.4)\). In this terms the Casimir \(\mathcal{I}\) is defined by the expression

\[
\mathcal{I} \equiv \sum_{i<j} \frac{l_{ij}^2}{2} + U(r), \quad U(r) \equiv r^2V(r), \quad (2.9)
\]

where \(l_{ij} = p_i x_j - p_j x_i\) is the angular momentum. The role of \(M_0\) plays the cotangent bundle of \((N-1)\)-dimensional sphere, \(T^* S^{N-1}\). In the spherical coordinate system with \(r, p_\alpha\) taken as radial coordinates, \(\mathcal{I}\) depends only on the \(N-1\) angular coordinates \(\phi^\alpha\) and their canonically conjugated momenta \(\pi_\alpha\). This particular Hamiltonian system can be presented in the form

\[
(T^* S^{N-1}, \omega_0 = d\pi_\alpha \wedge d\phi^\alpha, \mathcal{I}(u) = \frac{1}{2}g^{\alpha\beta}(\phi)\pi_\alpha\pi_\beta + U(\phi)), \quad (2.10)
\]

where \(g^{\alpha\beta}(\phi)\) is the inverse metrics of the hypersphere, and \(\alpha = 1, \ldots, (N-1)\). Its Hamiltonian, being quadratic on momenta, describes a particle moving on \(S^{N-1}\) in the presence of potential \(U(\phi)\).

An interesting example of such a spherical system is provided by the rational \(N\)-particle Calogero model with the excluded center-of-mass. Its spherical part defines the \(N(N-1)\)-center Higgs oscillator on the \((N-2)\)-dimensional sphere with the oscillator centers located in the roots of the \(A_{n-1}\) Lie algebra. In the three-particle case, the oscillator centers are located at the vertexes of hexagon while in the four-particle case, they correspond to the vertexes of cuboctahedron \([11]\).

### III. Integrals with Certain Conformal Dimension

The constant of motion of the conformal mechanics \((2.4)\), which is of the \(n\)-th order conformal dimension, \(\{D, I_n\} = -n I_n\), can be decomposed on \(p_r, r\) as follows

\[
I_n(p_r, r, u) = \sum_{k=0}^{n} \frac{p^{n-k}}{r^k} f_n^k(u), \quad (3.1)
\]

where the coefficients \(f_n^k(u)\) depend on the variables parameterizing \(M_0\), and do not depend on the radial coordinate \(r\) and momentum \(p_r\). Here and in the following we will use the notation \(\mathcal{X} \mathcal{f} \equiv \{X, f\}\) for the Poisson bracket action.

Note that for the conformal mechanics, any integral of motion can be presented as a sum of integrals with certain conformal dimension. This follows from the conformal invariance of the system \((2.7)\), or the zero dimensionality of the coupling constant. Indeed, for any integral presented in the form \(\mathcal{I} = \sum I_n\), we have \(\sum \mathcal{H} I_n = 0\). Acting by \(\mathcal{D}\) on both sides of the last equation and using the commutation relation between \(\mathcal{D}\) and \(\mathcal{H}\) inherited from \((2.7)\), we arrive at the another equation \(\sum \mathcal{H} I_n = 0\) for any \(k\). This implies that the quantities \(I_n\), conserve if their conformal dimensions differ. Therefore, the representation \((3.1)\) do not put any restriction on the constant of motion. Using the identity \(\mathcal{D} = 2(\mathcal{H}\mathcal{K} + \mathcal{K}\mathcal{H}) - 2\mathcal{D}\mathcal{D}\), we get

\[
\mathcal{D} I_n = (2\mathcal{H}\mathcal{K} + n\mathcal{D}) I_n. \quad (3.2)
\]
Inserting the decomposition (3.1), we observe that the operator \( \hat{I} \) acts only on the \( f^k_n(u) \), while the operator \( 2\hat{H} + nD \) acts only on the basic monomials
\[
\frac{p^{n-k}}{r^k}, \quad k = 0, 1, \ldots, n,
\] (3.3)
namely
\[
(2\hat{H} + nD)\frac{p^{n-k}}{r^k} = k\frac{p^{n-(k-1)}}{r^{k-1}} - 2(n-k)I\frac{p^{n-(k+1)}}{r^{k+1}}.
\] (3.4)
It is easy to see that it is invariant on the space formed by the monomials (3.3) with \( I \)-dependent coefficients. This enables us to obtain the following recursion relation from (3.2)
\[
\hat{I} f^k_n = (k + 1)f^{k+1}_n - 2(n - k)I f^{k-1}_n,
\] (3.5)
where the “boundary” conditions \( f^{-1}_n = f^{n+1}_n = 0 \) are supposed.

This recursion relation can be rewritten in the matrix form
\[
\hat{I} f_n = M_n f_n,
\] (3.6)
where
\[
M_n = \begin{pmatrix}
0 & -1 & 0 & \ldots & 0 & 0 \\
2nI & 0 & -2 & \ldots & 0 & 0 \\
0 & 2(n-1)I & 0 & \ldots & 0 & 0 \\
& \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & \ldots & 0 & -n \\
0 & 0 & 0 & \ldots & 2I & 0
\end{pmatrix}
\quad \text{and} \quad
f_n = \begin{pmatrix}
f^0_n \\
f^1_n \\
\vdots \\
f^{n-1}_n \\
f^n_n
\end{pmatrix}.
\] (3.7)
The matrix \( M_n \) (3.7) is conjugated to the \( x \)-projection of the spin-\( \frac{1}{2} \) operator:
\[
M_n = -2\sqrt{-2I} \, DL_x D^{-1}, \quad \text{where} \quad D = \text{diag}(d_0, \ldots, d_n), \quad d_k = \sqrt{\binom{n}{k}} (-2I)^{k/2},
\] (3.8)
and \( L_x \) is the matrix with the following nonzero elements
\[
(L_x)_{M,M-1} = (L_x)_{M-1,M} = \frac{1}{2} \sqrt{(L + M)(L - M + 1)},
\] (3.9)
i.e. is the the \( x \)-projection of spin \( L \) operator (12). So, the spectrum of \( M_n \) is proportional to the spectrum of \( L_x \).

The system of recursion relations (3.6) is overdefined. The compatibility condition is equivalent to the requirement
\[
\text{Det}(1 \cdot \hat{I} - M_n) f^i_n = 0,
\] (3.10)
where \( 1 \) is the \((n+1)\)-dimensional identity matrix. Joining pairwise the monomials with the same absolute value of the spin projection in order to avoid roots of \( I \), we arrive at the following expression for the determinant:
\[
\text{Det}(1 \cdot \hat{I} - M_n) = \begin{cases}
\prod_{k=0}^{n/2-1} (\hat{I}^2 + 2(n - 2k)^2 I) \hat{I} & \text{for even } n, \\
\prod_{k=0}^{(n-1)/2} (\hat{I}^2 + 2(n - 2k)^2 I) & \text{for odd } n.
\end{cases}
\] (3.11)
It is obvious that any integral integral (3.1) satisfies the similar equation \( \text{Det}(1 \cdot \hat{I} - M_n) I_n = 0 \). Since the operator (3.7) clearly commutes with (3.11), we conclude that the set of equations (3.10) is overcompleted: the fulfilling of this equation for the \( f^0_n(u) \),
\[
\text{Det}(1 \cdot \hat{I} - M_n) f^0_n = 0,
\] (3.12)
leads the fulfilling of this equation for any \( f^i_n(u) \).

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Hence, the coefficients \( f_n^k(u) \) in the decomposition of \( n \)-th order integral (3.11) are uniquely determined by \( f_n^0(u) \), which describes the asymptotical behavior of \( I_n \) in the large radius limit \( r \to \infty \). Note that in the above derivations, the form of the angular part \( I \) is unessential. From the (3.11) we see that the integral \( I \) is not in involution with any odd order integral \( I_{2k+1} \). Moreover, for any (even) integral of motion the condition \( \hat{T}^2 I_{2k} = 0 \) can be fulfilled only if \( I_{2k} \) is in involution with \( I \). In other words, a nontrivial action of the operator \( \hat{T} \) on the integrals of motion is not nilpotent.

Let us complete this section by the explicit solution of the recursion relations (3.5). For this purpose let us present (3.5) in the 2 \( \times \) 2 matrix form:

\[
\begin{pmatrix}
  f_n^k \\
  f_n^{k-1}
\end{pmatrix}
= \begin{pmatrix}
  \frac{1}{2} I & 2T \\
  1 & 0
\end{pmatrix}
\begin{pmatrix}
  f_n^{k-1} \\
  f_n^{k-2}
\end{pmatrix}.
\]

Then one can present the solution of this system of equations in terms of the matrix product:

\[
f_n^k = [T_k T_{k-1} \ldots T_1]_{11} f_n^0,
\]

where \( T_k \) is the 2 \( \times \) 2 transfer matrix defined in (3.13). From this equation, we see that \( f_n^k \) is the following polynomial on \( \hat{T} \) and \( I \):

\[
f_n^k = \sum_{i=0}^{[k/2]} c_{ki}(n) \hat{T}^i \hat{T}^{-2i} f_n^0, \quad \text{where} \quad c_{ki}(n) = \frac{1}{k!} \sum_{0 < k_1 < k \atop k_1 + 1 - k_i > 1} a_{k_1}(n) a_{k_2}(n) \ldots a_{k_i}(n), \quad a_k(n) = k(n - k + 1). \quad (3.15)
\]

Thus, the above equations determine entirely the constant of motion \( I_n \) with conformal dimension \( n \) from its higher order coefficient \( f_n^0(u) \) in \( p_r \). The last function must obey the condition (3.10). The equation (3.15) implies that the constant of motion \( I_n \) vanishes if it vanishes at the infinite boundary \( r = \infty \).

IV. WOJCIECHOWSKI’S CONSTANTS OF MOTION

By the use of operator \( \hat{T} \), we can get the additional serie of constants of motion of the conformal mechanics, associated with the known ones,

\[
F_k \equiv \hat{T} I_k = \{ I, I_k \} : \quad \{ F_k, \mathcal{H} \} = 0.
\]

These additional constants of motion precisely correspond to the hidden constants of motion of the Calogero model found by Wojciechowski [9]. For the self-consistency of the paper we present this construction in general terms.

Following [9], we introduce the auxiliary quantities

\[
J_n = \hat{K} I_n = - \sum_{k=0}^{n-1} (n - k) \frac{p_k^{n-k-1}}{\nu_{k-1}} f_k,
\]

which evolution is linear in time

\[
\dot{J}_n = \dot{H} J_n = \hat{H} \hat{K} I_n = \hat{D} I_n = -n I_n.
\]

Taking into account this expression, it is easy to check that the following expression also defines the constant of motion:

\[
L_{m,n} = m I_m J_n - n I_n J_m, \quad L_{n,m} = -L_{m,n}.
\]

For the Calogero system, \( L_{1,n} \) with \( n = 2, \ldots, N \) give rise to additional \( N - 1 \) integrals, which form, together with Liouville integrals, the functionally independent set of \( 2N - 1 \) constants of motion. This provides the Calogero model with the maximal superintegrability property. For the system with reduced center of mass, the first-order integral \( I_1 \) describing the total momentum of the initial system, vanishes. The Liouville integrals are obtained from the reduction of the Calogero integrals starting from \( n = 2 \). The additional \( N - 2 \) integrals correspond to \( L_{2,n} \) with \( n \geq 3 \).
Turning back to the equation (3.2), we see that it also defines a constant of motion of n-th conformal dimension, which, of course, possesses the similar decomposition in powers of \( p_r \). Take now \( H \) as a second order integral \( I_2 \), which implies \( J_2 = \{ K, H \} = -D \). Using the definition (4.3), one can rewrite (3.2) as follows:

\[
F_n = \hat{T}I_n = 2I_2J_n - nI_nJ_2 = L_{2,n}.
\]

(4.5)

So, the Hamiltonian action of \( I \) on the Liouville constants of motion of the Calogero model yields the additional serie of the constants of motion found by Wojciechowski in [9].

We proved that \( I \) is not in involution with any Liouville integrals except the Hamiltonian itself because the integrals \( L_{2,n}, n \neq 2 \) do not vanish and form together with Liouville ones a complete set of functionally independent constants of motion [9]. Using (4.3), one can derive a more general relation between \( I \) and other integrals of motion:

\[
mI_m\{I, I_n\} - nI_n\{I, I_m\} = 2HL_{m,n}.
\]

(4.6)

Note that, for any two constant of motion \( I_n \) and \( I_m \), the quantity \( I_{n,m} = \{I_n, I_m\} \) also conserves and has the conformal dimension \( n + m \), provided that it does not vanish since

\[
\hat{D}I_{n,m} = \{\hat{D}I_n, I_m\} + \{I_n, \hat{D}I_m\} = -(n + m)I_{n,m}.
\]

(4.7)

Therefore, the decomposition (3.1) with the replacement \( n \rightarrow n + m \) is valid for \( I_{n,m} \) too. The higher order coefficient in \( p_r \) is simply given by \( f_{n+m} = \{f_n^0, f_m^0\} \). The product of two integrals \( I_{n+m} = I_nI_m \) has the same property. It conserves, has the same conformal dimension, and its higher order coefficient coincides with the product \( f_{n+m}^0 = f_{n}^0f_{m}^0 \).

Thus, if some set of constants of motion \( \{I_n\} \) form closed algebra with respect to the Poisson brackets, then their leading asymptotes \( \{f_n^0\} \) will form the same algebra. The contraction map \( I_n \rightarrow f_n^0 = \lim_{r \rightarrow \infty} I_n \) defines a one-to-one correspondence conserving the Poisson structure. In particular, if the constants of motion are in involution, i.e.

\[
\{I_n, I_m\} = 0,
\]

(4.8)

then their limits at infinite radius are subjected to the similar relation, i.e. \( \{f_n^0, f_m^0\} = 0 \), and vice versa. The converse assertion can be also checked by direct calculation using the recursion relations (4.5). Hence, the analysis of the constants of motion of the conformal mechanical system can be reduced to the study of the equation (3.12).

**Example:** As simplest example, let us consider the three-particle Calogero model with excluded center of mass. The angular part of its Hamiltonian takes rather simple form [10]

\[
\mathcal{I} = \frac{p_r^2}{2} + \frac{9g}{2\cos^2 3\varphi},
\]

(4.9)

and describes particle motion on a circle interacting with three equidistant Higgs oscillator centers. The rest constants of motion in polar coordinates look as follows [11]:

\[
I_3 = \left( p_r^2 - \frac{6\mathcal{I}}{r^2} \right) p_r \sin 3\varphi + \left( 3p_r^2 - \frac{2\mathcal{I}}{r^2} \right) \frac{p_r \cos 3\varphi}{r}, \quad F_3 = \left( p_r^2 - \frac{6\mathcal{I}}{r^2} \right) 3p_r p_r \cos 3\varphi - \left( 3p_r^2 - \frac{2\mathcal{I}}{r^2} \right) \frac{6\mathcal{I} \sin 3\varphi}{r}.
\]

(4.10)

These four constants of motion \( \mathcal{I}, H, I_3, F_3 \) are functionally dependent with the following algebraic relation:

\[
F_3^2 + 2\mathcal{I}\mathcal{I}_2^2 = 8H^3(2\mathcal{I} - 9g).
\]

They generate the algebra with the following nonvanishing brackets:

\[
\{\mathcal{I}, I_3\} = 3F_3, \quad \{\mathcal{I}, F_3\} = -6IH_3, \quad \{F_3, I_3\} = 3(8H^3 - I_3^2).
\]

(4.11)

As was proved above for generic conformal mechanics (see (3.10), (3.11)), the operator

\[
M_3 = (\hat{T}^2 + 2\mathcal{I})(\hat{T}^2 + 18\mathcal{I})
\]

(4.12)

must annihilate any third order constant of motion. We see from (4.11) that just its factor monomial \( \hat{T}^2 + 18\mathcal{I} \) vanishes on both third order integrals \( I_3 \) and \( F_3 \), i.e., in fact, a stronger condition is fulfilled in this case. Since \((T^*S^1, \omega_0, \mathcal{I}) \) defines one-dimensional Hamiltonian system, it has no any nontrivial constants of motion.
V. CONSTANTS OF MOTION FOR \((M_0, \omega_0, \mathcal{I})\)

We demonstrated, in the previous section, that the information on the conformal mechanics is encoded in its angular part \((M_0, \omega_0, \mathcal{I})\), which could be considered, by itself, as some Hamiltonian system. It is obvious that the integrability of the initial conformal mechanics leads to the integrability of the “angular system” \((M_0, \omega_0, \mathcal{I})\), and vice versa. It is also evident that the constants of motion of the angular system are constants of motion for the conformal mechanics. Yet, the inverse statement is not true, though clearly, there should be some relation between the constants of motion of the conformal mechanics and those of its angular part.

So how to construct the constants of motion of \((M_0, \omega_0, \mathcal{I})\) from the ones of the initial conformal mechanics? For any integral \(I_n\) of the Hamiltonian \((2.6)\) with even conformal dimension, an integral of \(\dot{I}\) can be constructed from its coefficients in the decomposition \((3.1)\) easily. Indeed, the matrix \(M_n\) defined in \((3.7)\) is singular for even \(n\), as follows from \((3.8)\) and \((3.9)\). Denote by \(g_n\) the singular vector of the transposed matrix: \(M_n^T g_n = g_n^T M_n = 0\). Then, from \((3.10)\) we obtain

\[ \dot{\mathcal{I}}(g_n^T f_n) = g_n^T \dot{f}_n = g_n^T M_n f_n = 0, \]  

which means that \(G_n = g_n^T f_n\) is an integral of \((M_0, \omega_0, \mathcal{I})\) (if it does not vanish). Calculating the coordinates of the singular vector, we obtain the following explicit form of the integral:

\[ G_n = \sum_{i=0}^{n/2} (2i - 1)!!(n - 2i - 1)!!(2\mathcal{I})^{-i} f_n^0 = \sum_{i=0}^{n/2} \sum_{j=0}^{i} \alpha_{ij}(n) \mathcal{I}^{-i} \mathcal{I}^{-2(i-j)} f_n^0 \]  

where

\[ \alpha_{ij}(n) = 2^{-i}(2i - 1)!!(n - 2i - 1)!! c_{2i,j}(n), \quad (-1)!! = 1 \]  

and \(c_{2i,j}(n)\) is defined by \((3.11)\). As was proved above, the operator \((3.11)\) must annihilate \(f_n^0\). It is a \((n+1)\)th order polynomial on \(\dot{\mathcal{I}}\) while the expression of \(G_n\) contains \(n\) and less orders of \(\dot{\mathcal{I}}\). Therefore, the mentioned condition does not put any constraint on the terms, and there is a hope that this constant of motion does not vanish for concrete models.

The relation of the constants of motion of the \((M_0, \omega_0, \mathcal{I})\), which are associated with the ones of initial conformal mechanics, with odd conformal dimensions is more complicated.

VI. \(N=4\) SUPERCONFORMAL MECHANICS

In the previous sections we demonstrated that the analysis of conformal mechanics can be reduced to the study of its “angular part” given by the lower-dimensional Hamiltonian system \((M_0, \omega_0, \mathcal{I})\). Vise versa, such a system can be easily lifted to some conformal mechanics. In this section we will show that a similar correspondence exists between \(N=4\) superconformal mechanics with symmetry algebra \(D(1, 2|\alpha)\) its and \(N=4\) supersymmetric angular part. Namely, any \(N=4\) supersymmetric extention of the “angular” system \((M_0, \omega_0, \mathcal{I})\) can be lifted to the \(D(1, 2|\alpha)\) superconformally invariant one by coupling with super dilaton.

To describe this procedure let us, at first, consider the realization of the \(D(1, 2|\alpha)\) superalgebra on the \((2|4)\)-dimensional phase superspace parameterized by the coordinates \(r, p_r, \eta^\alpha \), and equipped with the canonical Poisson brackets

\[ \{p_r, r\} = 1, \quad \{\eta^{a\alpha}, \eta^{b\beta}\} = 2\varepsilon^{ab}\varepsilon^{\alpha\beta}, \]  

where all indices run from 1 to 2. We define, on this phase space, the following generators

\[ D = p_r r, \quad K = \frac{r^2}{2}, \quad \mathcal{H}_0 = \frac{p_r^2}{2} + \frac{(1 + 2\alpha)\eta^{a\alpha}\eta^{b\beta}\eta^{a\alpha}}{12r^2}, \quad V_0^{ab} = \eta^{a\alpha}\eta^{b\beta}, \quad W_0^{\alpha\beta} = \eta^{a\alpha}\eta^{b\beta}. \]  

\[ S^{a\alpha} = r\eta^{a\alpha}, \quad Q_0^{a\alpha} = p_r \eta^{a\alpha} - \frac{(1 + 2\alpha)\eta^{a\alpha}\eta^{b\beta}\eta^{a\alpha}}{3r}. \]
These generators form the superalgebra $D(1, 2; \alpha)$, given by the following non-zero relations

$$\{\mathcal{H}_0, D\} = 2\mathcal{H}_0, \quad \{K, D\} = -2K, \quad \{\mathcal{H}_0, K\} = D,$$

$$\{V_{0}^{ab}, V_{0}^{cd}\} = 2i(\varepsilon^{ac}V_{0}^{bd} + \varepsilon^{bd}V_{0}^{ac}), \quad \{W_{0}^{ab}, W_{0}^{cd}\} = 2i(\varepsilon^{\alpha\beta}W_{0}^{\beta\delta} + \varepsilon^{\beta\delta}W_{0}^{\alpha\gamma}),$$

$$\{Q_{0}^{\alpha}, Q_{0}^{\beta}\} = 2\varepsilon^{ab}\varepsilon^{\alpha\beta}\mathcal{H}_0, \quad \{S^{\alpha\alpha}, S^{\beta\beta}\} = 2\varepsilon^{ab}\varepsilon^{\alpha\beta}K,$$

$$\{Q_{0}^{\alpha}, S^{\beta\delta}\} = 2\varepsilon^{ab}\varepsilon^{\alpha\beta}D + \alpha\varepsilon^{ab}W_{0}^{\beta\delta} - (1 + \alpha)\varepsilon^{\alpha\beta}V_{0}^{ab},$$

$$\{S^{\alpha\alpha}, \mathcal{H}_0\} = -Q^{\alpha\alpha}, \quad \{D, Q^{\alpha\alpha}\} = -Q^{\alpha\alpha}, \quad \{Q^{\alpha\alpha}, K\} = S^{\alpha\alpha}, \quad \{D, S^{\alpha\alpha}\} = S^{\alpha\alpha},$$

$$\{V_{0}^{ab}, Q_{0}^{\alpha}\} = i(\varepsilon^{ac}Q_{0}^{ba} + \varepsilon^{ac}Q_{0}^{ab}), \quad \{W_{0}^{ab}, Q_{0}^{\alpha}\} = i(\varepsilon^{\alpha\gamma}Q_{0}^{\beta\delta} + \varepsilon^{\beta\gamma}Q_{0}^{\alpha\delta}),$$

$$\{V_{0}^{ab}, S^{\alpha\alpha}\} = i(\varepsilon^{ac}S^{ba} + \varepsilon^{bc}S^{aa}), \quad \{W_{0}^{ab}, S^{\alpha\alpha}\} = i(\varepsilon^{\alpha\gamma}S^{\beta\delta} + \varepsilon^{\beta\gamma}S^{\alpha\alpha}).$$

Notice that for the $\alpha = 0, -1$ we get the $su(1, 1/2)$ superalgebra while for $\alpha = -1/2$ emerges the $osp(4|2)$ one.

Now, we extend this system to the conformal mechanics with angular part $\mathcal{I}_{SUSY} = \mathcal{I} + \ldots$, which is the $\mathcal{N}=4$ supersymmetric “angular Hamiltonian” $\mathcal{I}$.

$$\{\Theta^{\alpha\alpha}, \Theta^{\beta\beta}\} = 2\varepsilon^{ab}\varepsilon^{\alpha\beta}\mathcal{I}_{SUSY}, \quad \{\Theta^{\alpha\alpha}, \mathcal{I}_{SUSY}\} = 0,$$

$$\{V_{1}^{ab}, \Theta^{\alpha\alpha}\} = -i(\varepsilon^{ac}Q_{1}^{ba} + \varepsilon^{ac}Q_{1}^{ab}), \quad \{W_{1}^{\alpha\beta}, \Theta^{\gamma\gamma}\} = -i(\varepsilon^{\alpha\gamma}Q_{1}^{\beta\delta} + \varepsilon^{\beta\gamma}Q_{1}^{\alpha\delta}),$$

$$\{V_{1}^{ab}, \mathcal{I}_{SUSY}\} = \{W_{1}^{\alpha\beta}, \mathcal{I}_{SUSY}\} = 0.$$

where $V_{1}^{ab}, W_{1}^{\alpha\beta}$ form $su(2) \times su(2)$ algebra

$$\{V_{1}^{ab}, V_{1}^{cd}\} = 2i(\varepsilon^{ac}V_{1}^{bd} + \varepsilon^{bd}V_{1}^{ac}), \quad \{W_{1}^{\alpha\beta}, W_{1}^{\gamma\delta}\} = 2i(\varepsilon^{\alpha\gamma}W_{1}^{\beta\delta} + \varepsilon^{\beta\delta}W_{1}^{\alpha\gamma}), \quad \{V_{1}^{ab}, W_{1}^{\alpha\beta}\} = 0.$$

Surely, $V_{1}^{ab}, W_{1}^{\alpha\beta}$ are precisely the $R$-symmetry generators of $\mathcal{N}=4$ supersymmetry. We give some particular realization of these generators via supercharges $\Theta^{\alpha\alpha}$ and Hamiltonian $\mathcal{I}_{SUSY}$:

$$V_{1}^{ab} = \frac{\Theta^{\alpha\alpha}\Theta^{ab} + \Theta^{\beta\beta}\Theta^{ab}}{2\mathcal{I}_{SUSY}}, \quad W_{1}^{\alpha\beta} = \frac{\Theta^{\alpha\alpha}\Theta^{ab} + \Theta^{\beta\beta}\Theta^{ab}}{2\mathcal{I}_{SUSY}}.$$

Let us assume that “radial” variables $p_r, r, \eta^{\alpha\alpha}$ commute with all elements of the above $\mathcal{N}=4$ superalgebra (6.3).

Then, we define the following odd generators

$$S^{\alpha\alpha} = r\eta^{\alpha\alpha}, \quad Q^{\alpha\alpha} = Q_{0}^{\alpha\alpha} + \frac{\Theta^{\alpha\alpha}}{r} - \frac{\varepsilon^{\alpha\beta}W_{0}^{\beta\delta} - \varepsilon^{\beta\delta}W_{0}^{\alpha\gamma}}{r} + i(1 + \alpha)\frac{\eta^{\alpha\alpha}V_{0}^{ab}}{r},$$

and the even ones

$$\mathcal{H} = \mathcal{H}_0 + \frac{\mathcal{I}_{SUSY}}{r^2}, \quad \frac{2\mathcal{I}_{SUSY}}{r^2} = \frac{i\Theta^{\alpha\alpha}\eta_{\alpha\alpha}}{r^2} + \frac{(1 + \alpha)V_{1}^{ab}V_{0}^{(ab)}}{2r^2} - \frac{\alpha W_{1}^{\alpha\beta}W_{0}^{\beta\delta}}{4r^2} + \frac{1 + 2\alpha}{12r^2} \left( V_{1}^{ab}V_{(1)ab} + W_{1}^{\alpha\beta}W_{1}^{(1)\alpha\beta} \right),$$

$$K = \frac{r^2}{2}, \quad D = p_r, \quad V^{ab} = V_{1}^{ab} + V_{0}^{ab}, \quad W^{\alpha\beta} = W_{1}^{\alpha\beta} + W_{0}^{\alpha\beta}.$$

It is easy to check that these new generators obey the relations (6.3), where the following replacement is made

$$Q_{0}^{\alpha\alpha} \rightarrow Q^{\alpha\alpha}, \quad \mathcal{H}_0 \rightarrow \mathcal{H}, \quad V_{0}^{ab} \rightarrow V^{ab}, \quad W_{0}^{\alpha\beta} \rightarrow W^{\alpha\beta}.$$

Hence, we presented the explicit construction which extends the arbitrary $\mathcal{N}=4$ supersymmetric Hamiltonian system $(M_0, \omega_0, \mathcal{I})$, to the superconformal mechanics with $D(1, 2; \alpha)$ superconformal algebra.

One should note that the proposed construction coupled “angular” part with $\mathcal{N}=4$ supermultiplet (1, 4, 3) \[18\]. The unique physical bosonic component of this supermultiplet is just a dilaton field. Clearly enough, one may construct other variants of superconformal systems by a proper coupling of the $\mathcal{N}=4$ supersymmetric “angular” part $\mathcal{I}_{SUSY}$ with any supermultiplet containing the dilaton field, i.e. with (2, 4, 2), (3, 4, 1) or (4, 4, 0) ones. The explicit construction of the supercharges and Hamiltonian for such systems will be present elsewhere.
VII. CONCLUSION

In present paper we split the generic conformal mechanics in to a “radial” and an “angular” part, where the latter is a Hamiltonian system on the orbit of $so(1,2)$, with the Hamiltonian given by the Casimir function of $so(1,2)$. We reduced the analysis of the constants of motion of the conformal mechanics to the study of some differential equations on the orbit. Moreover, we found that the separation of the system into radial and angular parts yields a transparent explanation of Wojciechowski’s construction of the hidden constants of motion of the rational Calogero model. More precisely, the hidden constants of motion are simply the Poisson brackets of the Casimir function of $so(1,2)$ with the Liouville constant of motion of the full Calogero model.

This construction is valid for any conformal mechanics. Consequently, any integrable conformal mechanical system is superintegrable. We also presented an explicit construction of the “angular” constants of motion from those of the full conformal mechanics. Finally, we demonstrated how to lift an arbitrary $\mathcal{N}=4$ supersymmetric “angular” Hamiltonian system to a superconformal mechanics model with $D(1,2;\alpha)$ symmetry algebra.

Various interesting issues remain to be tackled. Simplest among them, and probably most important, is the quantum counterpart of our construction, including the spectrum and wavefunctions of the “angular part” of the rational Calogero model from the data of the initial system.

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