Lattice Formulation of Supersymmetric Yang-Mills Theories without Fine-Tuning

Nobuhito Maru$^a$ and Jun Nishimura$^b$

Department of Physics, Nagoya University
Chikusa-ku, Nagoya 464-01, Japan

$^a$ e-mail: maru@eken.phys.nagoya-u.ac.jp
$^b$ e-mail: nisimura@eken.phys.nagoya-u.ac.jp

Abstract

We present a lattice formulation which gives super Yang-Mills theories in any dimensions with simple supersymmetry as well as extended supersymmetry in the continuum limit without fine-tuning. We first formulate super Yang-Mills theories with simple supersymmetry in 3,4,6 and 10 dimensions, incorporating the gluino on the lattice using the overlap formalism. In 4D, exact chiral symmetry forbids gluino mass, which ensures that the continuum limit is supersymmetric without fine-tuning. In 3D, exact parity invariance plays the same role. 6D and 10D theories being anomalous, we formulate them as anomalous chiral gauge theories as they are. Dimensional reduction within lattice formalism is then applied to the theories in 3,4,6 and 10D to obtain super Yang-Mills theories in arbitrary dimensions with either simple or extended supersymmetry.

PACS: 11.15.Ha; 11.30.Pb
Keywords: Lattice gauge theory, Supersymmetry
I. Introduction

Lattice formulation of supersymmetric gauge theories will provide a numerical method to obtain non-perturbative results in these theories and would complement the recent analytical developments [1,2]. Since the lattice formulation breaks the continuous rotational and translational invariance, it is also expected to break supersymmetry. A practical strategy might be, therefore, to give up manifest supersymmetry on the lattice and to recover it in the continuum limit by fine-tuning, as was advocated by Curci and Veneziano [3] some time ago. They showed within lattice perturbation theory that four-dimensional $\mathcal{N} = 1$ super Yang-Mills theory can be obtained by using the Wilson-Majorana fermion for the gluino and fine-tuning the hopping parameter to the chiral limit. Based on this work, some numerical simulations have been started [4]. An extension of Ref. [3] to $\mathcal{N} = 2$ case has also been discussed [5]. Fine-tuning necessary in either case, however, might be a practical obstacle in extracting interesting non-perturbative features due to supersymmetry through numerical simulations.

In general, thanks to the universality of the field theory, one can hope to obtain a supersymmetric theory by fine-tuning as many parameters as the relevant operators that breaks supersymmetry around the supersymmetric ultraviolet fixed point. If we have some symmetry that forbids those supersymmetry breaking operators, we can avoid fine-tuning by imposing the symmetry on the lattice theory. Indeed, this might be the best we can hope for concerning the supersymmetry on the lattice, considering that even the continuous rotational and translational invariance can be restored only in the continuum limit, but without any fine-tuning, so long as we maintain the discrete rotational and translational invariance on the lattice.

In four-dimensional $\mathcal{N} = 1$ super Yang-Mills theory, the gluino is Majorana fermion in the adjoint representation of the gauge group. There are two relevant operators, which correspond to the gauge coupling and the gluino mass. We can therefore obtain the supersymmetric continuum limit by fine-tuning the gluino mass to zero, as was shown in Ref. [3]. Alternatively, one can impose chiral symmetry, which forbids the gluino mass, so that one can avoid fine-tuning. Unfortunately the chiral symmetry is not easy to impose on the lattice. There is even a no-go theorem under some modest assumptions [6]. We have, however, a formalism which preserves exact chiral symmetry; that is the overlap formalism [7], which was originally developed to deal with chiral gauge theories on the lattice. The key to circumvent the no-go theorem lies in the fact that the formalism can be thought of as involving an infinite number of fermion fields at each space-time point [8], which is beyond the assumptions of the no-go theorem. In fact, in section 9 of [7], it
has been suggested that the overlap formalism can be used to formulate four-dimensional $\mathcal{N} = 1$ super Yang-Mills theory without fine-tuning. A method in the same spirit using the domain-wall approach [9] has been proposed in [10].

In this paper, we show that using the overlap formalism, not only the four-dimensional $\mathcal{N} = 1$ case, but also any other super Yang-Mills theories in arbitrary dimensions with simple or extended supersymmetry can be accessible on the lattice without fine-tuning. We first consider super Yang-Mills theories in 3, 4, 6 and 10 dimensions [11], in which the dynamical degrees of freedom of the gluon and the gluino exactly balance. The gluino should be Majorana, Majorana (or equivalently Weyl), Weyl and Majorana-Weyl, respectively. As in Ref. [12], by means of dimensional reduction, one can obtain all the other super Yang-Mills theories in any dimensions with either simple or extended supersymmetry. Although the parent theories in 6D and 10D suffer from gauge anomaly [13] since the gluino is chiral, the descendant theories are anomaly free since the gluino, after dimensional reduction, is no longer chiral. Our strategy is to formulate the parent theories in 3, 4, 6 and 10 dimensions using the overlap formalism and to perform dimensional reduction on the lattice to obtain all the descendant theories.

The overlap formalism, as a regularization of chiral gauge theories, gives the gauge anomaly correctly residing only in the phase of the fermion determinant. The gauge dependence is expected to disappear in the continuum limit if and only if the fermion content is chosen to be anomaly free. Indeed this seems to be natural enough to be true. An obvious undesirable point of the formalism, however, is that the gauge invariance of the phase for anomaly-free case is not manifest on the lattice, although we do not know at present whether it is possible at all to define anomaly-free chiral gauge theories in a non-perturbative way with manifest gauge invariance. Apart from this, numerical simulation of chiral gauge theories is rather difficult in any case, since the fermion determinant is complex in general. When we apply the overlap formalism to the present case, however, we are almost free from these problems, as we will see, essentially because the theories we aim at are vector-like.

Let us start with three-dimensional $\mathcal{N} = 1$ theory. We should note here that the gluon can acquire mass without violating gauge invariance in odd dimensions through Chern-Simons term. Supersymmetry only requires the gluon and the gluino mass to be equal. The massive case is referred to as “supersymmetric Yang-Mills Chern-Simons theory” in the literature [14]. In this paper, we consider only the massless case, which can be obtained by imposing the parity invariance since it prohibits both the gluon and the gluino mass. Parity invariance of three-dimensional gauge-fermion system has been elucidated in Ref. [15], where it is emphasized that the well-known parity anomaly is regularization dependent,
and that one can even preserve the parity invariance within gauge-invariant regularizations. While the Wilson fermion breaks the parity invariance because of the Wilson term and gives rise to parity anomaly in the massless limit [16], the overlap formalism applied to three-dimensional massless Dirac fermion is shown to be parity invariant [15]. Here we extend the formalism to massless Majorana fermion and show that it is parity invariant. Using this formalism, we can obtain three-dimensional $\mathcal{N} = 1$ super Yang-Mills theory in the continuum limit without fine-tuning.

Two-dimensional super Yang-Mills theory can be obtained through dimensional reduction of the three-dimensional theory [17]. The theory now contains a scalar field in the adjoint representation, which comes from the gauge field in the reduced direction. This theory cannot be constructed directly in two dimensions, since there are infinitely many relevant operators that break supersymmetry, due to the existence of the scalar field which is dimensionless, and we do not have any symmetry on the lattice that can prohibit all of them. We can, however, exploit the fact that it can be obtained through dimensional reduction of the three-dimensional theory. We discuss how this can be done within the lattice formalism.

The anomalous six- and ten-dimensional parent theories can be formulated using the overlap formalism, in which only the phase of the fermion determinant is gauge dependent as it should. After dimensional reduction, the gauge dependence is expected to disappear, thus resulting in anomaly-free supersymmetric theories. It is interesting to examine whether there exist non-trivial ultraviolet fixed points in more than four dimensions, thanks to supersymmetry. Also of particular interest is that four-dimensional $\mathcal{N} = 2$ and $\mathcal{N} = 4$ super Yang-Mills theories, for example, can be accessible in this way without fine-tuning.

This paper is organized as follows. In section II, we consider four-dimensional $\mathcal{N} = 1$ super Yang-Mills theory. We stress that, in this case, the fermion determinant obtained through the overlap formalism is real and the gauge invariance on the lattice is guaranteed up to the sign. In section III, we consider three-dimensional $\mathcal{N} = 1$ super Yang-Mills theory. We extend the overlap formalism constructed for three-dimensional massless Dirac fermion to massless Majorana fermion. We prove that the formalism preserves the parity invariance. We also show that the fermion determinant is real, and the gauge invariance on the lattice is guaranteed up to the sign. In section IV, we consider how we can obtain two-dimensional $\mathcal{N} = 1$ super Yang-Mills theory from the three-dimensional theory constructed above, through dimensional reduction on the lattice. In section V, we formulate the anomalous parent theories in six and ten dimensions on the lattice, from which we obtain anomaly-free super Yang-Mills theories through the lattice dimensional reduction. Section VI is devoted to summary and future prospects.
II. 4D $\mathcal{N} = 1$ Super Yang-Mills theory

Four-dimensional $\mathcal{N} = 1$ super Yang-Mills theory can be obtained using the overlap formalism without fine-tuning [7], where the gluino is treated as Weyl fermion. Note that in four dimensions, in general, Weyl fermion in a real representation is equivalent to Majorana fermion. Unlike in the Wilson’s formalism applied to Majorana fermion, chiral symmetry is exact when the overlap formalism is used regarding the Majorana fermion as Weyl fermion. Thus it provides a way to recover supersymmetry in the continuum limit without fine-tuning.

Here we note that when 4D Weyl fermion is in a real representation, as in the present case, the fermion determinant should be real. This is because 4D Weyl fermion in a real representation is equivalent to Majorana fermion, whose fermion determinant should be the square root of that for the Dirac fermion, which is real positive. In this section, we review that the overlap formalism indeed gives a real value for the fermion determinant of 4D Weyl fermion in a real representation. The statement is commented at the footnote on page 315 of Ref. [7] as a corollary of the lemma 4.5 in it. We stress that this is important practically, since it means that the gauge invariance is guaranteed up to the overall sign even on the lattice. The gauge dependence of the sign is expected to be very small on the lattice and to disappear in the continuum limit. Thus the overlap formalism gives almost gauge invariant regularization in this case. This reflects the fact that the theory is actually vector-like.

In the overlap formalism [7], the determinant of a Weyl fermion is given by an overlap of two many-body states. The two many-body states are ground states of two many-body Hamiltonians describing non-interacting fermions. Explicitly, the two many-body Hamiltonians on the lattice are

$$\mathcal{H}_\pm = a^\dagger \mathcal{H}_\pm a, \quad (2.1)$$

where

$$\mathcal{H}_\pm = \begin{pmatrix} B \pm m_o & C \\ C^t & -B \mp m_o \end{pmatrix}, \quad (2.2)$$

$$C(x\alpha i, y\beta j; U) = \frac{1}{2} \sum_{\mu=1}^{4} (\sigma_\mu)_{\alpha\beta} \left[ \delta_{y,x+\tilde{\mu}_x}(U_\mu(x))_{ij} - \delta_{x,y+\tilde{\mu}_y}(U^\dagger_\mu(y))_{ij} \right], \quad (2.3)$$

$$B(x\alpha i, y\beta j; U) = \frac{1}{2} \delta_{\alpha\beta} \sum_{\mu=1}^{4} [2\delta_{xy}\delta_{ij} - \delta_{y,x+\tilde{\mu}_x}(U_\mu(x))_{ij} - \delta_{x,y+\tilde{\mu}_y}(U^\dagger_\mu(y))_{ij}], \quad (2.4)$$

and

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_4 = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}. \quad (2.5)$$
0 < m_0 < 1 is a number that is kept fixed as the lattice spacing is taken to zero. Note that B^† = B. We define |±⟩^WB as the ground states of the two Hamiltonians in (2.1) that obey the Wigner-Brillouin phase choice [7], namely, \( \langle \pm | \pm \rangle^U \) are real and positive for all U. Then the determinant of the Weyl fermion is given by \( \langle - | + \rangle^U \). Note that the above condition on the phases of |±⟩^WB completely fixes the U dependent part of them.* Thus the fermion determinant is uniquely defined including its phase up to an irrelevant constant through the overlap formula for every gauge configuration.

In order to obtain the ground states of the many-body Hamiltonians, we first diagonalize the single-particle Hamiltonians H_± as

\[ H_± = V_±^† \Lambda_± V_±, \]  
(2.6)

where \( \Lambda_± \) are real diagonal matrices and V_± are unitary matrices. Let us define the operators \( a'_± \) through \( a'_± = V_± a. \) Then the ground states of the many-body Hamiltonians can be obtained by operating all the elements of \( a'_±^† \) that correspond to negative eigenvalues on the kinematical vacuum \(|0⟩ \) annihilated by a. We denote the states thus obtained as \(|±⟩_U \). We define

\[ |±⟩^U = e^{iχ_±(U)}|±⟩_U, \]  
(2.7)

where \( χ_±(U) \) is chosen such that \( \langle \pm | \pm ⟩^U \) are real and positive for all U. The undetermined U independent part of the phase is irrelevant and can be fixed as one likes.

When the fermion is in a real representation, namely, \( U^*_μ = U_μ \), we have

\[ ΣH_± Σ^† = [H_±]^*, \]  
(2.8)

where Σ is a unitary matrix defined as

\[ Σ = \begin{pmatrix} σ_2 & 0 \\ 0 & -σ_2 \end{pmatrix}. \]  
(2.9)

In deriving eq.(2.8), we have used the property \( -σ_2 σ_μ σ_2 = σ_μ^* \). Using the identity, we can rewrite the many-body Hamiltonians as

\[ \mathcal{H}_± = a^† H_± a = b^† [H_±]^* b, \]  
(2.10)

where \( b = Σ a. \) Note that \([H_±]^* \) can be diagonalized as follows.

\[ [H_±]^* = V_±^* Λ_± V_±^*. \]  
(2.11)

* Strictly speaking, the exceptions are the gauge configurations for which \( \langle \pm | \pm ⟩^U \) vanishes. We can neglect them, however, since the measure of the set of those configurations is zero.
Let us define the operators $b'_\pm$ through $b'_\pm = V'_\pm b$ and construct the ground states of the many-body Hamiltonians by operating all the components of $b'_\pm$ that correspond to negative eigenvalues on $|0\rangle$ in the same order as we constructed $|\pm\rangle_U$. We denote the states thus obtained as $|\bar{\pm}\rangle_U$. As before, we define

$$|\bar{\pm}\rangle_U = e^{i\tilde{\chi}_\pm(U)}|\bar{\pm}\rangle_U,$$  \hspace{1cm} (2.12)

where $\tilde{\chi}_\pm(U)$ is chosen such that $WB_1\bar{\langle\pm|\bar{\pm}\rangle}_U$ are real and positive for all $U$. Since the Wigner-Brillouin phase choice fixes the $U$ dependent part of the phase of the ground states completely, we can write as

$$|\bar{\pm}\rangle_U^{WB} = e^{i\theta_\pm} |\pm\rangle_U^{WB},$$  \hspace{1cm} (2.13)

where $\theta_\pm$ are constants independent of the gauge configuration $U$.

As is obvious from the above construction, we have

$$1\langle\pm|\pm\rangle_U = \left[WB_1\langle\bar{\pm}|\bar{\pm}\rangle_U\right]^*$$  \hspace{1cm} (2.14)

$$U\langle-|+\rangle_U = \left[U\langle\bar{\bar{\pm}}|\bar{\pm}\rangle_U\right]^*. $$  \hspace{1cm} (2.15)

Let us rewrite the above relations in terms of the ground states with the Wigner-Brillouin phase choice. From eq. (2.14), we obtain

$$e^{-i(\chi_\pm(U)-\chi_\pm(1))}WB_1\langle\pm|\pm\rangle_U^{WB} = e^{-i(\tilde{\chi}_\pm(U)-\tilde{\chi}_\pm(1))}WB_1\langle\bar{\pm}|\bar{\pm}\rangle_U^{WB}\left[WB_1\langle\pm|\pm\rangle_U^{WB}\right]^*,$$  \hspace{1cm} (2.16)

Since $WB_1\langle\pm|\pm\rangle_U^{WB}$ is real and positive, we have

$$e^{-i(\chi_\pm(U)-\chi_\pm(1))} = e^{i(\tilde{\chi}_\pm(U)-\tilde{\chi}_\pm(1))}.$$  \hspace{1cm} (2.17)

From eq.(2.15), we have

$$e^{-i(\chi_+(U)-\chi_-(U))}WB_1\langle-|+\rangle_U^{WB} = e^{-i(\tilde{\chi}_+(U)-\tilde{\chi}_-(U))}WB_1\langle\bar{\pm}|\bar{\pm}\rangle_U^{WB}\left[WB_1\langle-|+\rangle_U^{WB}\right]^*,$$  \hspace{1cm} (2.18)

where we used eq.(2.13) at the second equality. Using eq.(2.17), we obtain

$$WB_1\langle-|+\rangle_U^{WB} = e^{i(\chi_+(1)-\chi_-(1))}e^{i(\tilde{\chi}_+(1)-\tilde{\chi}_-(1))}e^{-i(\theta_+-\theta_-)}\left[WB_1\langle-|+\rangle_U^{WB}\right]^*$$  \hspace{1cm} (2.19)

$$= e^{i\theta}\left[WB_1\langle-|+\rangle_U^{WB}\right]^*,$$
where $\theta$ is a constant independent of the gauge configuration $U$. Rewriting the above equation as

$$e^{-i \theta/2} \cdot \mathbf{W} \langle - | + \rangle \mathbf{W} = \left[ e^{-i \theta/2} \cdot \mathbf{W} \langle - | + \rangle \mathbf{W} \right]^*,$$

we find that the overlap is real up to the irrelevant constant phase factor.

Thus we have proved that the fermion determinant defined by the overlap formula is real when the 4D Weyl fermion is in a real representation, which is indeed the case with the gluino in 4D $\mathcal{N} = 1$ super Yang-Mills theory.

### III. 3D $\mathcal{N} = 1$ Super Yang-Mills theory

In this section, we will see that 3D $\mathcal{N} = 1$ Super Yang-Mills theory can be obtained without fine-tuning using the overlap formalism. The overlap formalism has been applied to odd dimensions for the first time in Ref. [15]. In it, it was shown that the overlap formalism gives a parity invariant lattice regularization of massless Dirac fermion in three dimensions. Here we extend the formalism to massless Majorana fermion and prove the parity invariance. The parity invariance prohibits both gluon and gluino mass, thus enabling a lattice formulation of 3D $\mathcal{N} = 1$ Super Yang-Mills theory without fine-tuning. We also show that the fermion determinant is real as in the 4D case, which is practically important since it guarantees the gauge invariance up to the sign on the lattice.

The determinant of a single 3D massless Dirac fermion can be given by an overlap of two many-body states [15]. The two many-body states are ground states of two many-body Hamiltonians describing non-interacting fermions, which can be given on the lattice as

$$\mathcal{H}_\pm = a^\dagger \mathbf{H}_\pm a,$$

where

$$\mathbf{H}_\pm = \begin{pmatrix} \mathbf{B} \pm m_o & \mathbf{D} \\ -\mathbf{D} & -\mathbf{B} \mp m_o \end{pmatrix},$$

$$\mathbf{D}(x\alpha i, y\beta j; U) = \frac{1}{2} \sum_{\mu=1}^3 (\sigma_{\mu})_{\alpha\beta} \left[ \delta_{y,x+\mu}(U_\mu(x))_{ij} - \delta_{x,y+\mu}(U_\mu^\dagger(y))_{ij} \right],$$

$$\mathbf{B}(x\alpha i, y\beta j; U) = \frac{1}{2} \delta_{\alpha\beta} \sum_{\mu=1}^3 \left[ 2\delta_{xy}\delta_{ij} - \delta_{y,x+\mu}(U_\mu(x))_{ij} - \delta_{x,y+\mu}(U_\mu^\dagger(y))_{ij} \right].$$

$0 < m_o < 1$ is a number that is kept fixed as the lattice spacing is taken to zero. Note that $B^\dagger = B$ and $D^\dagger = -D$.

When the fermion is in a real representation, namely, $U_\mu^* = U_\mu$, we have

$$\left[ \mathbf{D}\sigma_2 \right]^t = -\mathbf{D}\sigma_2, \quad B^t = B^* = B.$$
We will see that in this case, the many-body Hamiltonians split into two identical terms corresponding to two Majorana fermions. The steps involved are similar to the ones in [18]. We write out
\[ a = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \] (3.6)
and define \( \xi, \eta \) by the following Bogoliubov transformation.
\[ a_1 = \frac{1}{\sqrt{2}} [\xi - i\eta] \]
\[ a_2 = \frac{1}{\sqrt{2} \sigma_2} [\xi^t - i\eta^t] \] (3.7)
\( \xi \) and \( \eta \) obey canonical anticommutation relations. Substitution of (3.7) in (3.1) and the use of (3.5) results in
\[ (a_1^\dagger \ a_2^\dagger) \begin{pmatrix} B \pm m_0 & D \\ -D & -B \mp m_0 \end{pmatrix} (a_1 \ a_2) = \frac{1}{2} \begin{pmatrix} \xi^t & \xi^t \end{pmatrix} \begin{pmatrix} B \pm m_0 & D\sigma_2 \\ -\sigma_2 D & -B \mp m_0 \end{pmatrix} \begin{pmatrix} \xi \\ \xi^t \end{pmatrix} \]
\[ + \frac{1}{2} \begin{pmatrix} \eta^t & \eta^t \end{pmatrix} \begin{pmatrix} B \pm m_0 & D\sigma_2 \\ -\sigma_2 D & -B \mp m_0 \end{pmatrix} \begin{pmatrix} \eta \\ \eta^t \end{pmatrix}. \] (3.8)
Both the many-body Hamiltonians on the righthand side of the above equation are identical and each one of them stands for a single Majorana fermion. The many-body Hamiltonians for a single Majorana fermion are
\[ \mathcal{H}_{\pm}^{(maj)} = \frac{1}{2} \begin{pmatrix} \xi^t & \xi^t \end{pmatrix} \begin{pmatrix} B \pm m_0 & D\sigma_2 \\ -\sigma_2 D & -B \mp m_0 \end{pmatrix} \begin{pmatrix} \xi \\ \xi^t \end{pmatrix}. \] (3.9)
We define \( |\pm\rangle^\text{WB}_U \) as the ground states of the two Hamiltonians in (3.9) that obey the Wigner-Brillouin phase choice, namely, \( ^\text{WB}_U |\pm| \pm \rangle^\text{WB}_U \) is real and positive for all \( U \). Then the determinant of a single Majorana fermion is given by \( ^\text{WB}_U |-| + \rangle^\text{WB}_U \). We comment that the decoupling of 3D massive Dirac fermion [15] in a real representation into two Majorana fermions can be derived in a similar way.

Let us address the issue of parity invariance in the context of the overlap formalism of massless Majorana fermion [19]. Consider the parity transformed gauge field on the lattice, namely,
\[ U'_\mu(x) = U_\mu^\dagger(-x - \hat{\mu}) = U_\mu^t(-x - \hat{\mu}). \] (3.10)
It follows from the definition of \( D \) and \( B \) in (3.3) and (3.4) that
\[ D(x\alpha i, y\beta j; U') = -D(-x\alpha i, -y\beta j; U) \] (3.11)
\[ B(x_\alpha, y_\beta; U') = B(-x_\alpha, -y_\beta; U). \]  
(3.12)

If we now define
\[ \xi(x_\alpha) = i\eta(-x_\alpha) \]  
(3.13)
in (3.9), we have
\[ \frac{1}{2} \left( \begin{array}{cc} \xi^\dagger & \xi^t \end{array} \right) \left( \begin{array}{cc} B(U') \pm m_0 & D(U')\sigma_2 \\ -\sigma_2 D(U') & -B(U') \mp m_0 \end{array} \right) \left( \begin{array}{c} \xi \\ \xi^t \end{array} \right) = \]
\[ \frac{1}{2} \left( \begin{array}{cc} \eta^\dagger & \eta^t \end{array} \right) \left( \begin{array}{cc} B(U) \pm m_0 & D(U)\sigma_2 \\ -\sigma_2 D(U) & -B(U) \mp m_0 \end{array} \right) \left( \begin{array}{c} \eta \\ \eta^t \end{array} \right). \]  
(3.14)

Since the many-body Hamiltonians for the gauge configuration \( U \) and the ones for the parity transformed configuration \( U' \) can be written in the same form by an appropriate unitary transformation of the fermion operators, we can construct the many-body ground states for \( U \) and \( U' \) as
\[ |\pm\rangle_U = f_\pm(\xi^\dagger)|0\rangle \]  
(3.15)
\[ |\pm\rangle_U' = f_\pm(\eta^\dagger)|0\rangle, \]  
(3.16)
where \(|0\rangle\) is the kinematical vacuum annihilated by \( \xi \) (or equivalently by \( \eta \)) and \( f_\pm(x) \) represent polynomials of \( x \). We have the following relations.
\[ i\langle\pm|\pm\rangle_U = \tilde{\langle}\pm|\pm\rangle_{U'} \]  
(3.17)
\[ u(-|+\rangle_U = u\langle\sim|-\sim\rangle_{U'}. \]  
(3.18)
Note that the gauge configuration \( U = 1 \) remains the same under the parity transformation.

We define the ground states with the Wigner-Brillouin phase choice as follows.
\[ |\pm\rangle_{WB}^{U} = e^{i\chi_\pm(U)}|\pm\rangle_U, \]  
(3.19)
\[ |\pm\rangle_{WB}^{U'} = e^{i\tilde{\chi}_\pm(U)}|\pm\rangle_U, \]  
(3.20)
where \( \chi_\pm(U) \) and \( \tilde{\chi}_\pm(U) \) are chosen such that \( |\pm\rangle_{WB}^{U} \) and \( |\pm\rangle_{WB}^{U'} \) are real and positive for all \( U \). As before, we can write as
\[ |\pm\rangle_{WB}^{U'} = e^{i\theta_\pm}|\pm\rangle_{WB}^{U}, \]  
(3.21)
where \( \theta_\pm \) are constants independent of the gauge configuration \( U \). From eq.(3.17),
\[ e^{-i(\chi_\pm(U)-\chi_\pm(1))|\pm\rangle_{WB}^{U}} = e^{-i(\tilde{\chi}_\pm(U')-\tilde{\chi}_\pm(1))|\pm\rangle_{WB}^{U'}}, \]  
(3.22)
Since \( \text{WB}_1(\pm|\pm)_{U}^{\text{WB}} \) and \( \text{WB}_1(\pm|\pm)_{U'}^{\text{WB}} \) are real positive, we have
\[
e^{-i(\chi_{\pm}(U)-\chi_{\pm}(1))} = e^{-i(\tilde{\chi}_{\pm}(U')-\tilde{\chi}_{\pm}(1))}.
\]
(3.23)

From eq.(3.18), we have
\[
e^{-i(\chi_{+}(U)-\chi_{-}(U))} \text{WB}_{U(-|+)}^{\text{WB}} = e^{-i(\tilde{\chi}_{+}(U')-\tilde{\chi}_{-}(U'))} \text{WB}_{U'(-|+)^{\text{WB}}}
\]
\[
= e^{-i(\tilde{\chi}_{+}(U')-\tilde{\chi}_{-}(U'))} e^{i(\theta_{+}-\theta_{-})} \text{WB}_{U'(-|+)^{\text{WB}}}
\]
(3.24)

Using eq.(3.23), we obtain
\[
\text{WB}_{U(-|+)}^{\text{WB}} = e^{i(\chi_{+}(1)-\chi_{-}(1))} e^{-i(\tilde{\chi}_{+}(1)-\tilde{\chi}_{-}(1))} e^{i(\theta_{+}-\theta_{-})} \text{WB}_{U'(-|+)^{\text{WB}}}
\]
\[
= e^{i\theta} \text{WB}_{U'(-|+)^{\text{WB}}},
\]
(3.25)

where \( \theta \) is a constant independent of the gauge configuration \( U \). By putting \( U = 1 \), one finds \( \theta = 0 \). Therefore, we have
\[
\text{WB}_{U(-|+)}^{\text{WB}} = \text{WB}_{U'(-|+)^{\text{WB}}}.
\]
(3.26)

Thus the overlap formalism for 3D massless Majorana fermion preserves the parity invariance. We can therefore use it to obtain 3D \( \mathcal{N} = 1 \) super Yang-Mills theory in the continuum limit without fine-tuning.

Let us next show that the fermion determinant is real. We first comment that this is formally expected in the continuum. The fermion determinant of concern is actually the Pfaffian (The word “fermion determinant” might, therefore, be confusing in this sense, but we keep on using this term.) of the antisymmetric operator \( D\sigma_2 \). We note that
\[
(D\sigma_2)^* = \sigma_2(D\sigma_2)\sigma_2.
\]
(3.27)

Taking the Pfaffian on both sides, we obtain
\[
\left[\text{Pf}(D\sigma)\right]^* = \text{Pf}(\sigma_2(D\sigma_2)\sigma_2)
\]
\[
= -\det(\sigma_2)\text{Pf}(D\sigma_2)
\]
\[
= \text{Pf}(D\sigma_2),
\]
(3.28)

which means that the Pfaffian is real. From the first line to the second line, we used the formula
\[
\text{Pf}(X^tAX) = \text{Pf}(A)\det(X),
\]
(3.29)
where $A$ is an antisymmetric matrix and $X$ is an arbitrary matrix. We will see that the fermion determinant defined through the overlap formula is indeed real.*

Although the proof goes in a similar way as in the previous section, things are a little more complicated due to the fact that the many-body Hamiltonians (3.9) for the Majorana fermion violate fermion number and we have to make Bogoliubov transformation instead of simple unitary transformation in order to obtain the ground states. Let us write the many-body Hamiltonians for a single Majorana fermion as

$$H_{\pm}^{(maj)} = \frac{1}{2} (\xi^\dagger \xi^t) H_{\pm} \left( \begin{array}{c} \xi \\ \xi^\dagger \end{array} \right).$$

(3.30)

Due to the hermiticity of $H_{\pm}^{(maj)}$ and the anticommutation relations among the $\xi$ operators, the hermite matrix $H_{\pm}$ must have the following particular form.

$$H_{\pm} = \begin{pmatrix} h & \lambda \\ -\lambda^* & -h^* \end{pmatrix},$$

(3.31)

where $h$ is hermitian and $\lambda$ is antisymmetric. This is indeed satisfied with the explicit form of $H_{\pm}$ given through (3.9). Suppose $(x, y)^t$ is an eigenvector of $H_{\pm}$ with the eigenvalue $\omega$, then one can easily see that $(y^*, x^*)^t$ is an eigenvector of $H_{\pm}$ with the eigenvalue $-\omega$, due to (3.31). One can therefore diagonalize the $H_{\pm}$ as follows.

$$H_{\pm} = V_\pm^\dagger \Lambda_\pm V_\pm$$

(3.32)

$$V_\pm = \begin{pmatrix} x^\dagger & y^\dagger \\ \vdots & \vdots \\ y^t & x^t \\ \vdots & \vdots \end{pmatrix},$$

(3.33)

where $\Lambda_\pm=\text{diag}(\omega, \cdots, -\omega, \cdots)$. Without loss of generality, we can take the first half of the diagonal elements to be positive. Due to the particular form of the unitary matrix $V_\pm$, the $\xi_{\pm}'$ operators, which satisfy canonical anticommutation relations, can be consistently defined through

$$\begin{pmatrix} \xi_{\pm}' \\ \xi_{\pm}'^\dagger \end{pmatrix} = V_\pm \begin{pmatrix} \xi \\ \xi^\dagger \end{pmatrix}.$$

(3.34)

This gives the desired Bogoliubov transformation. The ground states of the many-body Hamiltonians can be given by the states $|\pm\rangle_U$ annihilated by $\xi_{\pm}'$. Note that the above states are different from the kinematical vacuum $|0\rangle$ annihilated by $\xi$, since $\xi_{\pm}'$ and $\xi$ are connected through the Bogoliubov transformation instead of a simple unitary transformation.

* We thank R. Narayanan for pointing this out to the authors.
In order to show that the fermion determinant is real, we start with the following identity.

\[ \Gamma H_{\pm} \Gamma^\dagger = [H_{\pm}]^*, \]  

(3.35)

where \( \Gamma \) is a unitary matrix defined by

\[ \Gamma = \begin{pmatrix} i\sigma_2 & 0 \\ 0 & i\sigma_2 \end{pmatrix}. \]  

(3.36)

By defining the \( \eta \) operators through \( \eta = i\sigma_2 \xi \), one can rewrite the many-body Hamiltonian as

\[ H_{\pm}^{(maj)} = \frac{1}{2} \begin{pmatrix} \xi^\dagger & \xi^t \end{pmatrix} H_{\pm} \begin{pmatrix} \xi \\ \xi^t \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \eta^\dagger & \eta^t \end{pmatrix} [H_{\pm}]^* \begin{pmatrix} \eta \\ \eta^t \end{pmatrix}, \]  

(3.37)

which can be diagonalized in terms of

\[ \begin{pmatrix} \eta_{\pm}^l \\ \eta_{\pm}^t \end{pmatrix} = V_{\pm}^* \begin{pmatrix} \eta^l \\ \eta^t \end{pmatrix}. \]  

(3.38)

The ground states can now be given by the states \( \widetilde{|\pm}\rangle \) annihilated by \( \eta_{\pm}^l \). Due to eqs. (3.34) and (3.38), one can take the following particular forms for the many-body ground states.

\[ |\pm\rangle_U = f_{\pm}(\xi^\dagger)|0\rangle \]  

(3.39)

\[ \widetilde{|\pm\rangle}_U = f_{\pm}^*(\eta^t)|0\rangle, \]  

(3.40)

where \( f_{\pm}(x) \) represent polynomials of \( x \) and \( f_{\pm}^*(x) \) represent polynomials with the coefficients which are complex conjugate of those of \( f_{\pm}(x) \). From this, we have the following relations.

\[ 1\langle\pm|\pm\rangle_U = \left[ 1\langle\pm|\pm\rangle_U \right]^* \]  

(3.41)

\[ U\langle-|+\rangle_U = \left[ U\langle-|+\rangle_U \right]^*. \]  

(3.42)

The rest of the proof goes exactly in the same way as in the previous section.

Thus we find that the overlap for the 3D massless Majorana fermion is real. The gauge invariance on the lattice is therefore guaranteed up to the sign of the fermion determinant. Also, as a corollary, we find that the overlap formula for massless Dirac fermion in a real representation is real positive, since it is the square of that for massless Majorana fermion.

**IV. 2D Super Yang-Mills theory through the lattice dimensional reduction**

Super Yang-Mills theory in two dimensions can be obtained by dimensional reduction of the three-dimensional super Yang-Mills theory [17]. The gauge field in the reduced
Dimension becomes a scalar field in two dimensions. The key point of obtaining the supersymmetric continuum limit in three and four dimensions is that there exists a symmetry that prohibits the supersymmetry breaking relevant operators. Now in two dimensions, we have a scalar field and due to this, there are infinitely many relevant operators that break supersymmetry. We can, however, exploit the fact that two-dimensional super Yang-Mills theory can be obtained by dimensional reduction of the three-dimensional super Yang-Mills theory. The main issue here is how to perform dimensional reduction on the lattice.

Dimensional reduction has been discussed intensively in the context of finite temperature field theory [20,21]. Finite temperature field theory can be formulated by keeping the physical extent in one direction finite, which corresponds to the inverse temperature, while sending those in the other directions to infinity corresponding to the infinite volume limit. In the zero temperature limit, one obtains the original field theory in the infinite volume naturally. In the high temperature limit, after integrating out the oscillating modes in the inverse temperature direction, one obtains an effective field theory with one dimension lower than the original theory. The effective field theory contains only local interactions [20].

The boundary condition in the inverse temperature direction is relevant to physics, since the physical extension in this direction is kept finite in contrast to the other directions where the boundary condition does not affect the physics due to the infinite volume limit. In finite temperature field theory, the boundary condition in the inverse temperature direction should be periodic for bosonic fields and anti-periodic for fermionic fields. The dimensionally reduced theory obtained in the high temperature limit is composed of bosonic fields only [20], since fermionic fields do not have zero modes in the inverse temperature direction due to the boundary condition. When we consider the dimensional reduction in the context of supersymmetric theories, we have to take periodic boundary condition also for the fermionic fields in order to preserve supersymmetry. Hence, fermionic fields as well as bosonic fields appear in the resulting dimensionally reduced theory, which should be a local theory with supersymmetry.

Dimensional reduction on the lattice can be done for supersymmetric theories as well as for high-temperature limit of ordinary field theories [21]. We consider the dimensional reduction of three-dimensional theory down to two-dimensional theory. In order to achieve the dimensional reduction, we have to make the physical extent in one direction, say \( l_3 \), finite, while we take the physical extent in the other directions, say \( l \), to infinity, corresponding to the infinite volume limit. Let us denote the typical scale of the theory (e.g.; the inverse of the lambda parameter) as \( r \). In order to realize the dimensional reduction, we have to take \( l_3 \ll r \ll l \).
In addition to this, we have to take the continuum limit $a \to 0$, since we are working on the lattice. If we take the lattice size to be $L \times L \times L_3$, we have $l = aL$ and $l_3 = aL_3$. A typical correlation length $\xi$ is related to $r$ through $r = a\xi$. The dimensional reduction on the lattice can therefore be realized by taking $1 \ll L_3 \ll \xi \ll L$.

In this way, one can obtain 2D $\mathcal{N} = 1$ super Yang-Mills theory from 3D $\mathcal{N} = 1$ super Yang-Mills theory in Section III. The point in avoiding fine-tuning which seemed to be inevitable when we consider 2D theory directly is that we have the 3D rotational and translational invariance restored in the continuum limit at the scale much smaller than the extent of the reduced direction. Similarly, one can dimensionally reduce the 4D $\mathcal{N} = 1$ super Yang-Mills theory in Section II, to obtain 3D $\mathcal{N} = 2$ and 2D $\mathcal{N} = 2$ super Yang-Mills theories.

V. Anomalous 6D and 10D theories and their dimensional reductions

In this section, we discuss the application of the overlap formalism to anomalous 6D and 10D theories, from which we obtain anomaly-free super Yang-Mills theories by the lattice dimensional reduction described in the previous section. Before that, let us briefly review how we get the dimensions 3, 4, 6 and 10.

Super Yang-Mills theory requires the balance between the bosonic and fermionic degrees of freedom. In $D$ dimensions the physical degrees of freedom of the gauge field is $(D-2)$, while the physical degrees of freedom of a Dirac fermion is $2^{[D/2]}$, where $[x]$ denotes the largest integer not more than $x$.

In the following, we summarize the relevant properties of the spinors in Minkowski space with arbitrary dimensions. In even dimensions, Dirac fermion is decomposed into two Weyl fermions. In 2, 3 and 4 dimensions, Dirac fermion in a real representation decomposes into two Majorana fermions. In 8 and 9 dimensions, massless Dirac fermion in a real representation decomposes into two pseudo Majorana fermions. In 4 dimensions, Weyl fermion in a real representation is equivalent to Majorana fermion. In 8 dimensions, Weyl fermion in a real representation is equivalent to pseudo Majorana fermion. In 2 dimensions, Weyl fermion in a real representation further decomposes into two Majorana-Weyl fermions. All the above statements hold for the dimensions up to modulo 8.

It is now easy to find that the dimensions in which the physical degrees of freedom of the gauge field and the gluino balance are 3, 4, 6 and 10. The gluino in each dimension is, Majorana in 3D, Majorana (or equivalently Weyl) in 4D, Weyl in 6D, Majorana-Weyl in 10D. Note that in 6 and 10 dimensions the gluino is chiral, giving rise to gauge anomaly, which has been studied in Ref. [13]. Thus in these dimensions, super Yang-Mills theory cannot be considered as a consistent quantum field theory with unitarity. It is not even su-
persymmetric actually, since the gauge mode does not decouple, which adds to the bosonic degrees of freedom. Still we can define it as a statistical system by naively integrating out the gauge mode. This theory can be formulated using the overlap formalism. The overlap formalism for Majorana-Weyl fermion, which is necessary in formulating the 10D theory, is studied in Ref. [18], where its application to 10D super Yang-Mills theory as a regularization of 4D $\mathcal{N} = 4$ super Yang-Mills theory is suggested. Including this as a special case, we give a general prescription to obtain anomaly-free super Yang-Mills theories by dimensional reduction from the anomalous parent theories in 6D and 10D.

We apply the lattice dimensional reduction considered in the previous section to the 6D and 10D theories. After reducing one direction, we obtain 5D and 9D theories with the gluino being Dirac and pseudo Majorana respectively. Since now the fermions are no longer chiral, we do not have gauge anomaly. This means that although the chiral determinant calculated through the overlap formalism in 6D and 10D is generally complex and the phase is gauge dependent, the gauge dependence disappears after the dimensional reduction by taking the continuum limit. Moreover, we know in the continuum that the fermion determinant for the above theories in 5D and 9D should be real, which can be used as useful information in numerical simulations. Note also that since the gauge mode decouples, the resulting dimensionally reduced theories must be supersymmetric.

Of course, it remains to be seen whether there exists a non-trivial ultra-violet fixed point which allows a continuum limit in more than four dimensions, since gauge theory in more than four dimensions is perturbatively unrenormalizable. This is an issue that has to be explored nonperturbatively. In non-supersymmetric Yang-Mills theory, numerical simulations in more than four dimensions have given negative conclusions [22,23]. However, we might be able to have non-trivial ultra-violet fixed points in supersymmetric theories, thanks to supersymmetry. This is an interesting issue also in the context of string theory [23,24].

Apart from this, we can further dimensionally reduce the theories down to four dimensions. From six dimensions, we obtain four-dimensional $\mathcal{N} = 2$ super Yang-Mills theory, while from ten dimensions, we obtain four-dimensional $\mathcal{N} = 4$ super Yang-Mills theory. These theories should exist from perturbative point of view, and therefore it is no doubt that we can obtain a continuum limit, in contrast to the theories in more than four dimensions. We can further dimensionally reduce the theories even lower.

VI. Summary and future prospects

In this paper, we presented a method to deal with supersymmetric Yang-Mills theory on the lattice in any dimensions with either simple or extended supersymmetry without
fine-tuning. Instead of preserving supersymmetry manifestly on the lattice, we impose other symmetries on the lattice that ensure the continuum limit to be supersymmetric automatically. This is quite analogous to how we deal with continuous translational and rotational symmetries on the lattice. Although we break them on the lattice, we can restore them in the continuum limit without fine-tuning, so long as we maintain the discrete translational and rotational symmetry on the lattice. As for supersymmetry, we have to prohibit the gluon and the gluino mass. The former is prohibited in even dimensions by the gauge invariance. In four dimensions, the chiral symmetry further prohibits the gluino mass, while in six and ten dimensions, it is prohibited by the chiral nature of the gluino. In three dimensions, we can prohibit both the gluon and the gluino mass by imposing the parity invariance. Note that all the necessary ingredients to restore supersymmetry automatically, namely the chiral symmetry in 4D, the parity invariance in 3D, and the chiral nature in 6D and 10D, are what the standard lattice formalism of fermions fails to deal with. Surprisingly enough, the overlap formalism deals with all of these features quite nicely, thus enabling a lattice formulation of super Yang-Mills theories without fine-tuning. We also argued that the dimensional reduction technique within the lattice formalism, which has been developed in the context of finite-temperature field theory, can be used to obtain all the other super Yang-Mills theory in arbitrary dimensions with either single or extended supersymmetry. Although the 6D and 10D theories are anomalous and not supersymmetric actually, the theories after dimensional reduction should be both anomaly-free and supersymmetric. We stressed that all the super Yang-Mills theories in the above sense are vector-like and the fermion determinant should be real. Hence we are almost free from the subtlety as with general anomaly-free chiral gauge theories, in which the phase can take arbitrary values and its gauge dependence within the overlap formalism is not restricted at all kinematically.

Let us comment on some possible applications of our formalism. In four-dimensional $\mathcal{N} = 1$ super Yang-Mills theory, the gluino condensation is an important issue in phenomenology. The argument of Witten index [25] and the instanton calculations [26] suggest that the condensation indeed occurs [2]. It would be interesting to examine this in numerical simulations. In four-dimensional $\mathcal{N} = 2$ super Yang-Mills theory, the scalar fields will have undetermined VEV’s, which give rise to a non-trivial moduli space. We will have to fix the VEV’s by hand in numerical simulations. It would be interesting to examine the conjectures given in Ref. [1] numerically. Also an issue which deserves further study is the possible existence of non-trivial ultra-violet fixed points in more than four dimensions in the supersymmetric case. Considering the variety of the applications, an efficient algorithm for the overlap formalism is highly desired.
Finally we should comment that we cannot put additional matters in a supersymmetric way into the present formalism. Thus the interesting conjectures on the infrared fixed points in supersymmetric QCD [2] cannot be examined within the formalism as it stands. This might be possible, if we could extract some remnant of the supersymmetry in the overlap formalism, which is of course an interesting issue itself worth studying in future.

Acknowledgement

The authors would like to thank R. Narayanan for helpful communication throughout this work. J.N. is also grateful to T. Eguchi, N. Haba and P. Pouliot for valuable comments.
References

1. N. Seiberg and E. Witten, Nucl. Phys. B426 (1994) 19; B431 (1994) 484.
2. N. Seiberg, Nucl. Phys. B435 (1995) 129; K. Intriligator and N. Seiberg, Nucl. Phys. Proc. Suppl. 45BC (1996) 1.
3. G. Curci and G. Veneziano, Nucl. Phys. B292 (1987) 555.
4. I. Montvay, Nucl. Phys. B466 (1996) 259; A. Donini and M. Guagnelli, Phys. Lett. B383 (1996) 301.
5. I. Montvay, Phys. Lett. B344 (1995) 176; I. Montvay, Nucl. Phys. B445 (1995) 399.
6. H.B. Nielsen and M. Ninomiya, Nucl. Phys. B185 (1981) 20; Nucl. Phys. B193 (1981) 173; B195 (1982) 541(E).
7. R. Narayanan and H. Neuberger, Nucl. Phys. B443 (1995) 305.
8. R. Narayanan and H. Neuberger, Phys. Lett. B302 (1993) 62; Nucl. Phys. B412 (1994) 574.
9. V. Furman and Y. Shamir, Nucl. Phys. B439, 54 (1995), T. Blum and A. Soni, preprint hep-lat/9611030 (1996).
10. J. Nishimura, preprint, hep-lat/9701013, to appear in Phys. Lett. B.
11. See, for example, Appendix 4.A in Superstring Theory, vol. 1, M.B. Green, J.H. Schwarz and E. Witten, Cambridge University Press (1987).
12. L. Brink, J.H. Schwarz and J. Scherk, Nucl. Phys. B121 (1977) 77.
13. P.H. Frampton and T.W. Kephart, Phys. Rev. Lett. 50 (1983) 1343,1347; P.K. Townsend and G. Sierra, Nucl. Phys. B222 (1983) 493; B. Zumino and Y.-S. Wu and A. Zee, Nucl. Phys. B239 (1984) 477.
14. See, for example, F. Ruiz Ruiz and P. van Nieuwenhuizen, Nucl. Phys. B486 (1997) 443.
15. R. Narayanan and J. Nishimura, preprint, hep-th/9703109.
16. A. Coste and M. Lüscher, Nucl. Phys. B323 (1989) 631.
17. S. Ferrara, Lett. Nuovo Cim. 13 (1975) 629.
18. P. Huet, R. Narayanan and H. Neuberger, Phys. Lett. B380 (1996) 291.
19. R. Narayanan, private communication.
20. T. Reisz, Z. Phys. C53 (1992) 169.
21. P. Lacock, D.E. Miller and T. Reisz, Nucl. Phys. B369 (1992) 501; L. Kärkkäinen, P. Lacock, D.E. Miller, B. Petersson and T. Reisz, Phys. Lett. B282 (1992) 121; L. Kärkkäinen, P. Lacock, B. Petersson and T. Reisz, Nucl. Phys. B395 (1993) 733.
22. H. Kawai, M. Nio, and Y. Okamoto, Prog. Theor. Phys. 88, (1992) 341.
23. J. Nishimura, Mod. Phys. Lett. A11 (1996) 3049.
24. T. Banks, W. Fischler, S. Shenker and L. Susskind, preprint, \texttt{hep-th/9610043} (1996); N. Ishibashi, H. Kawai, Y. Kitazawa and A. Tsuchiya, preprint, \texttt{hep-th/9612115} (1996).

25. E. Witten, Nucl. Phys. B202 (1982) 253.

26. D. Amati, K. Konishi, Y. Meurice, G.C. Rossi and G. Veneziano, Phys. Rep. 162 (1988) and references therein.