Super congruences involving alternating harmonic sums modulo prime powers

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Abstract In 2014, Wang and Cai established the following harmonic congruence for any odd prime \( p \) and positive integer \( r \),
\[
\sum_{i+j+k=p^r, \ i,j,k \in \mathcal{P}_p} \frac{1}{ijk} \equiv -2p^{r-1}B_{p-3} \pmod{p^r},
\]
where \( \mathcal{P}_n \) denote the set of positive integers which are prime to \( n \).
In this note, we establish a combinational congruence of alternating harmonic sums for any odd prime \( p \) and positive integers \( r \),
\[
\sum_{i+j+k=p^r, \ i,j,k \in \mathcal{P}_p} (-1)^i \frac{1}{ijk} \equiv \frac{1}{2}p^{r-1}B_{p-3} \pmod{p^r}.
\]
For any odd prime \( p \geq 5 \) and positive integers \( r \), we have
\[
4 \sum_{\substack{i_1+i_2+i_3+i_4=2p^r \ 1 \leq i_1 < i_2 < i_3 < i_4 \leq p^r}} (-1)^{i_1} \frac{1}{i_1i_2i_3i_4} + 3 \sum_{\substack{i_1+i_2+i_3+i_4=2p^r \ 1 \leq i_1 < i_2 < i_3 < i_4 \leq p^r}} (-1)^{i_1+i_2} \frac{1}{i_1i_2i_3i_4}
\equiv \begin{cases} \frac{216}{5} pB_{p-5} \pmod{p^2}, & \text{if } r = 1, \\ \frac{36}{5} p^r B_{p-5} \pmod{p^{r+1}}, & \text{if } r > 1. \end{cases}
\]

This work is supported by the National Natural Science Foundation of China, Project (No.10871169) and the Natural Science Foundation of Zhejiang Province, Project (No. LQ13A010012).
For any odd prime \( p > 5 \) and positive integers \( r \), we have

\[
\sum_{i_1+i_2+i_3+i_4+i_5=2p} \frac{(-1)^{i_1}}{i_1i_2i_3i_4i_5} + 2 \sum_{i_1+i_2+i_3+i_4+i_5=2p'} \frac{(-1)^{i_1+i_2}}{i_1i_2i_3i_4i_5} = \begin{cases} 
12B_{p-5} \pmod{p}, & \text{if } r = 1, \\
6p^{r-1}B_{p-5} \pmod{p^r}, & \text{if } r > 1.
\end{cases}
\]

**Keywords** Bernoulli numbers, alternating harmonic sums, congruences, modulo prime powers

**MSC** 11A07, 11A41

1 **Introduction.**

At the beginning of the 21th century, Zhao (Cf.\([7]\)) first announced the following curious congruence involving multiple harmonic sums for any odd prime \( p > 3 \),

\[
\sum_{i+j+k=p} \frac{1}{ijk} \equiv -2B_{p-3} \pmod{p},
\]

which holds when \( p = 3 \) evidently. Here, Bernoulli numbers \( B_k \) are defined by the recursive relation:

\[
\sum_{i=0}^{n} \binom{n+1}{i} B_i = 0, n \geq 1.
\]

A simple proof of (1) was presented in \([1]\). This congruence has been generalized along several directions. First, Zhou and Cai \([8]\) established the following harmonic congruence for prime \( p > 3 \) and integer \( n \leq p - 2 \)

\[
\sum_{i_1+i_2+\cdots+i_n=p} \frac{1}{i_1i_2\cdots i_n} \equiv \begin{cases} 
-(n-1)!B_{p-n} \pmod{p}, & \text{if } 2 \nmid n, \\
-\frac{n(n!)}{2(n+1)}pB_{p-n-1} \pmod{p^2}, & \text{if } 2 \mid n.
\end{cases}
\]

Later, Xia and Cai \([5]\) generalized (1) to

\[
\sum_{i+j+k=p} \frac{1}{ijk} \equiv -\frac{12B_{p-3}}{p-3} - \frac{3B_{2p-4}}{p-4} \pmod{p^2},
\]

where \( p > 5 \) is a prime.

Recently, Wang and Cai \([4]\) proved for every prime \( p \geq 3 \) and positive integer \( r \),

\[
\sum_{\substack{i+j+k=p' \\text{mod } p \\text{ for any } i,j,k \in \mathbb{P}_p}} \frac{1}{ijk} \equiv -2p^{r-1}B_{p-3} \pmod{p^r},
\]
where $\mathcal{P}_n$ denote the set of positive integers which are prime to $n$.

Let $n = 2$ or $4$, for every positive integer $r \geq \frac{p}{2}$ and prime $p > n$, Zhao \[6\] generalized \[3\] to

\[
\sum_{i_1 + i_2 + \cdots + i_n = p^n, \ i_1, i_2, \cdots, i_n \in \mathcal{P}_p} \frac{1}{i_1 i_2 \cdots i_n} \equiv -\frac{n!}{n+1} p^r B_{p-n-1} \pmod{p^{r+1}}. \tag{4}
\]

For any prime $p > 5$ and integer $r > 1$, Wang \[2\] proved that

\[
\sum_{i_1 + i_2 + \cdots + i_n = p^n, \ i_1, i_2, \cdots, i_n \in \mathcal{P}_p} \frac{1}{i_1 i_2 \cdots i_5} \equiv -\frac{5!}{6} p^{r-1} B_{p-n-1} \pmod{p^r}.
\]

We consider the following alternating harmonic sums

\[
\sum_{i_1 + i_2 + \cdots + i_n = p^n, \ i_1, i_2, \cdots, i_n \in \mathcal{P}_p} \frac{(\sigma_1)^{i_1}(\sigma_2)^{i_2} \cdots (\sigma_n)^{i_n}}{i_1 i_2 \cdots i_n},
\]

where $\sigma_i \in \{1, -1\}$, $i = 1, 2, \cdots, n$. Given $n$, we only need to consider the following alternating harmonic sums,

\[
\sum_{i_1 + i_2 + \cdots + i_n = p^n, \ i_1, i_2, \cdots, i_n \in \mathcal{P}_p} (-1)^{i_1} \frac{1}{i_1 i_2 \cdots i_n}, \sum_{i_1 + i_2 + \cdots + i_n = p^n, \ i_1, i_2, \cdots, i_n \in \mathcal{P}_p} (-1)^{i_1+i_2} \frac{1}{i_1 i_2 \cdots i_n}, \cdots, \sum_{i_1 + i_2 + \cdots + i_n = p^n, \ i_1, i_2, \cdots, i_n \in \mathcal{P}_p} (-1)^{i_1+i_2+\cdots+i_n} \frac{1}{i_1 i_2 \cdots i_n}
\]

where $[x]$ denote the largest integer less than or equal to $x$.

In this paper, we consider the congruences involving the combination of alternating harmonic sums,

\[
\sum_{i_1 + i_2 + \cdots + i_n = p^n, \ i_1, i_2, \cdots, i_n \in \mathcal{P}_p} (-1)^{i_1} \frac{1}{i_1 i_2 \cdots i_n}, \sum_{i_1 + i_2 + \cdots + i_n = p^n, \ i_1, i_2, \cdots, i_n \in \mathcal{P}_p} (-1)^{i_1+i_2} \frac{1}{i_1 i_2 \cdots i_n}, \cdots, \sum_{i_1 + i_2 + \cdots + i_n = p^n, \ i_1, i_2, \cdots, i_n \in \mathcal{P}_p} (-1)^{i_1+i_2+\cdots+i_n} \frac{1}{i_1 i_2 \cdots i_n}
\]

we obtain the following theorems. Among them, Theorem 1 and Theorem 2 have been proved by Wang \[3\] using different method, but his method doesn’t for Theorem 3 and Theorem 4.

**Theorem 1.** Let $p$ be odd prime and $r$ positive integer, then

\[
\sum_{i+j+k=2p^r, \ i, j, k \in \mathcal{P}_{2p}} (-1)^i ijk \equiv p^{r-1} B_{p-3} \pmod{p^r}.
\]

**Remark 1** There is no solution $(i, j, k)$ for the equation $i + j + k = 2p^r$ with $i, j, k \in \mathcal{P}_{2p}$.
Theorem 2. Let $p$ be odd prime and $r$ positive integer, then
\[
\sum_{i+j+k=p^r \atop i,j,k \in \mathbb{P}_p} \frac{(-1)^i}{ijk} = \frac{1}{2} p^{r-1} B_{p-3} \pmod{p^r}.
\]

Theorem 3. Let $p \geq 5$ be a prime and $r$ positive integer, then
\[
4 \sum_{i_1+i_2+i_3+i_4=2p^r \atop i_1, i_2, i_3, i_4 \in \mathbb{P}_p} \frac{(-1)^{i_1}}{i_1 i_2 i_3 i_4} + 3 \sum_{i_1+i_2+i_3+i_4=2p^r \atop i_1, i_2, i_3, i_4 \in \mathbb{P}_p} \frac{(-1)^{i_1+i_2}}{i_1 i_2 i_3 i_4}
\]
\[
= \begin{cases} 
\frac{216}{p} B_{p-5} \pmod{p^2}, & \text{if } r = 1, \\
\frac{36}{p} p^r B_{p-5} \pmod{p^{r+1}}, & \text{if } r > 1.
\end{cases}
\]

Theorem 4. Let $p > 5$ be a prime and $r$ positive integer, then
\[
4 \sum_{i_1+i_2+i_3+i_4+i_5=2p^r \atop i_1, i_2, i_3, i_4, i_5 \in \mathbb{P}_p} \frac{(-1)^{i_1}}{i_1 i_2 i_3 i_4 i_5} + 2 \sum_{i_1+i_2+i_3+i_4+i_5=2p^r \atop i_1, i_2, i_3, i_4, i_5 \in \mathbb{P}_p} \frac{(-1)^{i_1+i_2}}{i_1 i_2 i_3 i_4 i_5}
\]
\[
= \begin{cases} 
12 B_{p-5} \pmod{p}, & \text{if } r = 1, \\
6 p^{r-1} B_{p-5} \pmod{p^r}, & \text{if } r > 1.
\end{cases}
\]

2 Preliminaries.

In order to prove the theorems, we need the following lemmas.

Lemma 1 ([1]). Let $p$ be odd prime and $r$, $m$ positive integers, then
\[
\sum_{i+j+k=mp^r \atop i,j,k \in \mathbb{P}_p} \frac{1}{ijk} \equiv -2mp^{r-1} B_{p-3} \pmod{p^r}.
\]

Lemma 2. Let $p$ be odd prime and $r$, $m$ positive integers, then
\[
\sum_{i+j+k=mp^r \atop i,j,k \in \mathbb{P}_p} \frac{1}{ijk} = \frac{6}{mp^r} \sum_{1 \leq j < l \leq mp^r} \frac{1}{jl}.
\]

Proof. It is easy to see that
\[
\sum_{i+j+k=mp^r \atop i,j,k \in \mathbb{P}_p} \frac{1}{ijk} = \frac{mp^r}{ijk} \sum_{i+j+k=mp^r \atop i,j,k \in \mathbb{P}_p} \frac{i+j+k}{ijk} = \frac{3}{mp^r} \sum_{i+j+k=mp^r \atop i,j,k \in \mathbb{P}_p} \frac{1}{ij}.
\]

Let $l = j + k$, then $1 \leq j < l \leq mp^r$ and $j$, $l$, $l-j \in \mathbb{P}_p$. By symmetry, we have
\[
\frac{3}{mp^r} \sum_{i+j+k=mp^r \atop i,j,k \in \mathbb{P}_p} \frac{1}{ij} = \frac{3}{mp^r} \sum_{1 \leq j < l \leq mp^r} \frac{1}{jl} = \frac{6}{mp^r} \sum_{1 \leq j < l \leq mp^r} \frac{1}{jl}.
\]

This completes the proof of Lemma 2. □
Lemma 3. Let \( p > 4 \) be a prime and \( r, m \) positive integers, then

\[
\sum_{i_1+i_2+i_3+i_4=mp} \frac{1}{u_1u_2u_3u_4} = \frac{24}{mp^r} \sum_{1 \leq u_1 < u_2 < u_3 \leq mp} \frac{1}{u_1u_2u_3}.
\]

Proof. The proof of Lemma 3 is similar to the proof of Lemma 2.

Lemma 4 ([8]). Let \( r, \alpha_1, \ldots, \alpha_n \) be positive integers, \( r = \alpha_1 + \cdots + \alpha_n \leq p-3 \), then

\[
\sum_{i_1 \leq i_2 \leq i_3 \leq \cdots \leq i_n} \frac{1}{i_1 \cdot i_2 \cdot \cdots \cdot i_n} = \begin{cases} (-)^n(n-1) \frac{r(r+1)}{2(r+2)} B_{p-r-2p^2} \pmod{p^3} & \text{if } 2 \mid r, \\ (-)^n(n-1) \frac{r}{r+1} B_{p-r-1p} \pmod{p^2} & \text{if } 2 \mid r. \end{cases}
\]

Lemma 5 ([2]). Let \( p \) be odd prime, and \( \alpha_1, \ldots, \alpha_n \) positive integers, where \( r = \alpha_1 + \cdots + \alpha_n \leq p-3 \), then

\[
\sum_{i_1 \leq i_2 \leq i_3 \leq \cdots \leq i_n} \frac{1}{i_1 \cdot i_2 \cdot \cdots \cdot i_n} = \begin{cases} (-)^n(n-1) \frac{2r(r+1)}{r+2} B_{p-r-2p^2} \pmod{p^3} & \text{if } 2 \mid r, \\ (-)^n(n-1) \frac{2r}{r+1} B_{p-r-1p} \pmod{p^2} & \text{if } 2 \mid r. \end{cases}
\]

Lemma 6. Let \( p > 4 \) be a prime, then

\[
\sum_{i_1+i_2+i_3+i_4=2p} \frac{1}{u_1u_2u_3u_4} \equiv -\frac{240}{5} pB_{p-5} \pmod{p^2}.
\]

Proof. By Lemma 2, we have

\[
\sum_{i_1+i_2+i_3+i_4=2p} \frac{1}{i_1i_2i_3i_4} = \frac{24}{2p} \sum_{1 \leq i_1 < i_2 < i_3 < i_4} \frac{1}{u_1u_2u_3u_4}. \tag{5}
\]

It is easy to see that

\[
\sum_{1 \leq u_1 < u_2 < u_3 \leq mp} \frac{1}{u_1u_2u_3} = \sum_{1 \leq u_1 < u_2 < u_3 \leq mp} \frac{1}{u_1u_2u_3} + \sum_{1 \leq u_1 < u_2 < u_3 \leq mp} \frac{1}{u_1u_2u_3}.
\]

By Lemma 4, we have

\[
\sum_{1 \leq u_1 < u_2 < u_3 \leq mp} \frac{1}{u_1u_2u_3} = \frac{p}{u_1} \sum_{1 \leq u_1 < u_3 < mp} \frac{1}{u_3} \equiv 0 \pmod{p^3}. \tag{6}
\]

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Hence
\[ \sum_{1 \leq u_1 < u_2 < u_3 \leq 2p} \frac{1}{u_1 u_2 u_3} \equiv \sum_{1 \leq u_1 < u_2 < u_3 \leq 2p} \frac{1}{u_1 u_2 u_3} \]
\[ = \sum_{1 \leq u_1 < u_2 < u_3 \leq 2p} \frac{1}{u_1 u_2 u_3} - \sum_{1 \leq u_1 < u_1 + p < u_3 \leq 2p} \frac{1}{u_1 (u_1 + p) u_3} \]
\[ - \sum_{1 \leq u_1 < u_2 < u_3 < 2p} \frac{1}{u_1 u_2 (u_2 + p)} \pmod{p^3}. \]

Replace \( u_3 = u_2 + p \), then
\[ \sum_{1 \leq u_1 < u_1 + p < u_3 \leq 2p} \frac{1}{u_1 (u_1 + p) u_3} = \sum_{1 \leq u_1 < u_1 + p < u_2 < 2p} \frac{1}{u_1 (u_1 + p) (u_2 + p)} \]
\[ = \sum_{1 \leq u_1 < u_2 < p} \frac{1}{u_2^2 u_2} (1 - \frac{p}{u_1} + \frac{p^2}{u_2}) (1 - \frac{p}{u_2} + \frac{p^2}{u_2}) \pmod{p^3}. \]

and
\[ \sum_{1 \leq u_1 < u_2 < 2p} \frac{1}{u_1 u_2 (u_2 + p)} \equiv \sum_{1 \leq u_1 < u_2 < p} \frac{1}{u_1 u_2^2} (1 - \frac{p}{u_2} + \frac{p^2}{u_2}) \pmod{p^3}. \]

Thus
\[ \sum_{1 \leq u_1 < u_2 < u_3 \leq 2p} \frac{1}{u_1 u_2 u_3} = \sum_{1 \leq u_1 < u_2 < u_3 \leq 2p} \frac{1}{u_1 u_2 u_3} - \sum_{1 \leq u_1 < u_2 < p} \left( \frac{1}{u_1^2 u_2} \right. \]
\[ + \frac{1}{u_1 u_2^2} - p \left( \frac{1}{u_1 u_2^2} + \frac{1}{u_1^2 u_2^2} + \frac{1}{u_1 u_2^3} \right) \]
\[ + p^2 \left( \frac{1}{u_1^2 u_2} + \frac{1}{u_1 u_2^3} + \frac{1}{u_1^2 u_2^3} + \frac{1}{u_1 u_2^4} \right) \]
\[ = \frac{1}{3} \left( \sum_{1 \leq u_1, u_2, u_3 \leq 2p} \frac{1}{u_1 u_2 u_3} \right. \]
\[ - \sum_{1 \leq u_1, u_2 < p} \left( \frac{1}{u_1^2 u_2} \right. \]
\[ - p \left( \frac{1}{u_1^2 u_2^2} + \frac{1}{2 u_1 u_2^3} \right) + p^2 \left( \frac{1}{u_1^2 u_2} + \frac{1}{u_1 u_2^3} \right) \pmod{p^3}. \] (7)

Using Lemma 5 in the first sum of the right hand in (7) and using Lemma 4 in
the second sum, we have
\[
\sum_{1 \leq u_1 < u_2 < u_3 \leq 2p} \frac{1}{u_1 u_2 u_3} \equiv \frac{1}{3!} (-1)^3 (3 - 1)! \frac{24}{5} B_{p-5} p^2 - (-1) \frac{12}{10} B_{p-5} p^2 \\
+ \frac{3p}{2} \left( -\frac{4}{5} B_{p-5} p \right) - p^2 \frac{30}{14} B_{p-7} p^2 \\
\equiv -\frac{20}{5} p^2 B_{p-5} \quad (\text{mod } p^3).
\]
(8)

Combining (5) with (8), we complete the proof of Lemma 6. □

**Lemma 7.** Let \( p > 4 \) be odd prime and \( r > 1 \) positive integer, then
\[
\sum_{i_1 + i_2 + i_3 = 2p} \frac{1}{i_1 i_2 i_3} \equiv -\frac{48}{5} p^r B_{p-5} \quad (\text{mod } p^{r+1}).
\]

**Proof.** The proof of Lemma 7 is similar to the proof method of (4) in [6]. □

**Lemma 8 ([2]).** Let \( p > 5 \) be a prime and \( r, m \) positive integers, \((m, p) = 1\), then
\[
\sum_{i_1 + i_2 + \cdots + i_5 = mp} \frac{1}{i_1 i_2 i_3 i_4 i_5} \equiv \left\{ \begin{array}{ll}
-4(5m + m^3) B_{p-5} \quad (\text{mod } p), & \text{if } r = 1, \\
-20mp^{r-1} B_{p-5} \quad (\text{mod } p^r), & \text{if } r > 1.
\end{array} \right.
\]

**Lemma 9.** Let \( p > 5 \) be a prime and \( r, m \) positive integers, then
\[
\sum_{i_1 + i_2 + \cdots + i_5 = mp} \frac{1}{i_1 i_2 i_3 i_4 i_5} = \frac{120}{mp^r} \sum_{1 \leq u_1 < u_2 < u_3 < u_4 \leq mp} \frac{1}{u_1 u_2 u_3 u_4}.
\]

**Proof.** The proof of Lemma 9 is similar to the proof of Lemma 2. □

3 Proofs of the Theorems.

**Proof of Theorem 1.** It is easy to see that
\[
\sum_{i+j+k=2p} \frac{(-1)^i}{ijk} = \frac{1}{2p^r} \sum_{i+j+k=2p} \frac{(-1)^i(i+j+k)}{ijk} \\
= \frac{1}{2p^r} \sum_{i+j+k=2p} \left( \frac{(-1)^i}{jk} + \frac{2(-1)^i}{ij} \right).
\]
(9)
Let \( l = j + k \), then \( 1 \leq j < l \leq 2p^r \) and \( j, l, l - j \in \mathcal{P}_p \), hence
\[
\sum_{i+j+k=2p^r \atop i,j,k \in \mathcal{P}_p} (-1)^i \left/ jk \right. = \sum_{1 \leq i < 2p^r \atop j, l, l-j \in \mathcal{P}_p} \frac{1}{l} (-1)^l(j+k) = \sum_{1 \leq i < 2p^r \atop j, l, l-j \in \mathcal{P}_p} \frac{2(-1)^i}{jl^2}. \tag{10}
\]

Let \( l' = i + j \), then \( 1 \leq i < l' \leq 2p^r \) and \( i, l', l' - i \in \mathcal{P}_p \), hence
\[
\sum_{i+j+k=2p^r \atop i,j,k \in \mathcal{P}_p} (-1)^i \left/ ij \right. = \sum_{1 \leq i < l' \atop j, l, l-j \in \mathcal{P}_p} \frac{2(-1)^i}{ij} = \sum_{1 \leq i < l' \atop j, l, l-j \in \mathcal{P}_p} \left( \frac{2(-1)^i}{jl'} + \frac{2(-1)^i}{il'} \right). \tag{11}
\]

Noting that \( i = l' - j \), \( (-1)^{l'-j} = (-1)^{l'+j} \) and we rename \( l' \) to \( l \), then
\[
\sum_{1 \leq i < l' \atop j, l, l-j \in \mathcal{P}_p} \frac{2(-1)^i}{jl'} = \sum_{1 \leq i < l \atop j, l, l-j \in \mathcal{P}_p} \frac{2(-1)^i}{jl}. \tag{12}
\]

Rename \( i \) to \( j \) and \( l' \) to \( l \), then
\[
\sum_{1 \leq i < l' \atop j, l' - i \in \mathcal{P}_p} \frac{2(-1)^i}{il'} = \sum_{1 \leq i < l \atop j, l - j \in \mathcal{P}_p} \frac{2(-1)^j}{jl}. \tag{13}
\]

Combining (9) - (13), we have
\[
\sum_{i+j+k=2p^r \atop i,j,k \in \mathcal{P}_p} (-1)^i \left/ ijk \right. = \frac{1}{p^r} \sum_{1 \leq i < l \atop j, l, l-j \in \mathcal{P}_p} \left( \frac{(-1)^l}{jl} + \frac{(-1)^{l+j}}{jl} + \frac{(-1)^j}{jl} \right)
\]
\[
= \frac{1}{p^r} \sum_{1 \leq i < l \atop j, l, l-j \in \mathcal{P}_p} \left( 1 + (-1)^l \right) \left( 1 + (-1)^j \right) - \frac{1}{p^r} \sum_{1 \leq i < 2p^r \atop j, l, l-j \in \mathcal{P}_p} \frac{1}{jl}
\]
\[
= \frac{1}{p^r} \sum_{1 \leq i < 2p^r \atop j, l, l-j \in \mathcal{P}_p} \frac{4}{jl} - \frac{1}{p^r} \sum_{1 \leq i < 2p^r \atop j, l, l-j \in \mathcal{P}_p} \frac{1}{jl}. \tag{14}
\]

Let \( j = 2j' \), \( l = 2l' \) in the first sum of (14) and noting that
\[
\sum_{1 \leq j < l' \atop j', l' - j' \in \mathcal{P}_p} \frac{1}{jl'} = \sum_{1 \leq j < l \atop j, l-j \in \mathcal{P}_p} \frac{1}{jl}.
\]

(14) is equal to
\[
\sum_{i+j+k=2p^r \atop i,j,k \in \mathcal{P}_p} (-1)^i \left/ ijk \right. = \frac{1}{p^r} \sum_{1 \leq i < 2p^r \atop j, l, l-j \in \mathcal{P}_p} \frac{1}{jl} - \frac{1}{p^r} \sum_{1 \leq i < 2p^r \atop j, l, l-j \in \mathcal{P}_p} \frac{1}{jl}. \tag{15}
\]
By Lemma 1, Lemma 2 and (15), we obtain
\[ \sum_{i+j+k=2p^r, i,j,k \in \mathcal{P}} (-1)^i \frac{1}{ijk} = \frac{1}{p^r} \sum_{j \leq i \leq 2p^r, j, i, j \in \mathcal{P}_p} \frac{1}{j} - \frac{1}{p^r} \sum_{i \leq j \leq 2p^r, j, i, j \in \mathcal{P}_p} \frac{1}{j} \]
\[ \equiv p^{r-1} B_{p-3} \pmod{p^r}. \]

This completes the proof of Theorem 1. \[ \square \]

Proof of Theorem 2. For every triple \((i, j, k)\) of positive integers which satisfies \(i + j + k = 2p^r, i, j, k \in \mathcal{P}\), we take it to 3 cases.

Cases 1. If \(1 \leq i, j, k \leq p^r - 1\) are coprime to \(pq\), \((i, j, k) \leftrightarrow (p^r - i, p^r - j, p^r - k)\) is a bijection between the solutions of \(i + j + k = 2p^r\) and \(i + j + k = p^r, i, j, k \in \mathcal{P}_p\), we have
\[ \sum_{i+j+k=2p^r, i,j,k \in \mathcal{P}_p} (-1)^i \frac{1}{ijk} \equiv \sum_{i+j+k=p^r, i,j,k \in \mathcal{P}_p} (-1)^{p^r-i} \frac{1}{(p^r - i)(p^r - j)(p^r - k)} \]
\[ \equiv \sum_{i+j+k=p^r, i,j,k \in \mathcal{P}_p} (-1)^i \frac{1}{ijk} \pmod{p^r}. \] \( (16) \)

Cases 2. If \(p^r + 1 \leq i \leq 2p^r - 1, 1 \leq j, k \leq p^r - 1\) are coprime to \(p\), \((i, j, k) \leftrightarrow (p^r + i, j, k)\) is a bijection between the solutions of \(i + j + k = 2p^r\) and \(i + j + k = p^r, i, j, k \in \mathcal{P}_p\), we have
\[ \sum_{i+j+k=2p^r, i,j,k \in \mathcal{P}_p} (-1)^i \frac{1}{ijk} \equiv \sum_{i+j+k=p^r, i,j,k \in \mathcal{P}_p} (-1)^{p^r+i} \frac{1}{(p^r + i)jk} \]
\[ \equiv - \sum_{i+j+k=p^r, i,j,k \in \mathcal{P}_p} (-1)^i \frac{1}{ijk} \pmod{p^r}. \] \( (17) \)

Cases 3. If \(p^r + 1 \leq j \leq 2p^r - 1, 1 \leq i, k \leq p^r - 1\) or \(p^r + 1 \leq k \leq 2p^r - 1, 1 \leq i, j \leq p^r - 1\) are coprime to \(p\), \((i, j, k) \leftrightarrow (i, p^r + j, k)\) in the former and \((i, j, k) \leftrightarrow (i, j, p^r + k)\) in the later are the bijections between the solutions
of \( i + j + k = 2p^r \) and \( i + j + k = p^r \), \( i, j, k \in \mathbb{P}_p \), we have

\[
\sum_{i+j+k=2p^r \atop i,j,k \in \mathbb{P}_p} (-1)^i \frac{1}{ijk} + \sum_{i+j+k=p^r \atop i,j,k \in \mathbb{P}_p} (-1)^i \frac{1}{ijk}
\]

\[
\equiv \sum_{i+j+k=2p^r \atop i,j,k \in \mathbb{P}_p} \frac{(-1)^i}{i(p^r+j)k} + \sum_{i+j+k=p^r \atop i,j,k \in \mathbb{P}_p} \frac{(-1)^i}{ij(p^r+k)}
\]

\[
\equiv 2 \sum_{i+j+k=2p^r \atop i,j,k \in \mathbb{P}_p} \frac{(-1)^i}{ijk} \pmod{p^r}.
\] (18)

Combining (16)-(18), we have

\[
\sum_{i+j+k=2p^r \atop i,j,k \in \mathbb{P}_p} (-1)^i \frac{1}{ijk} = \sum_{i+j+k=2p^r \atop i,j,k \in \mathbb{P}_p} \frac{(-1)^i}{ijk} + \sum_{i+j+k=p^r \atop i,j,k \in \mathbb{P}_p} (-1)^i \frac{1}{ijk}
\]

\[
+ \sum_{i+j+k=2p^r \atop i,j,k \in \mathbb{P}_p} \frac{(-1)^i}{ijk} + \sum_{i+j+k=p^r \atop i,j,k \in \mathbb{P}_p} \frac{(-1)^i}{ijk}
\]

\[
\equiv 2 \sum_{i+j+k=2p^r \atop i,j,k \in \mathbb{P}_p} \frac{(-1)^i}{ijk} \pmod{p^r}.
\]

By Theorem 1, we complete the proof of Theorem 2. \( \square \)

**Proof of Theorem 3.** By symmetry, it is easy to see that

\[
\sum_{i_1+i_2+i_3+i_4=2p^r \atop i_1, i_2, i_3, i_4 \in \mathbb{P}_p} (-1)^{i_1} \frac{1}{i_1i_2i_3i_4} = \frac{1}{2p^r} \sum_{i_1+i_2+i_3+i_4=2p^r \atop i_1, i_2, i_3, i_4 \in \mathbb{P}_p} (-1)^{i_1+i_2+i_3+i_4} \frac{1}{i_1i_2i_3i_4}
\]

\[
= \frac{1}{2p^r} \sum_{i_1+i_2+i_3+i_4=2p^r \atop i_1, i_2, i_3, i_4 \in \mathbb{P}_p} (-1)^{i_1} \frac{1}{i_2i_3i_4} + 3(-1)^{i_1} \frac{1}{i_1i_2i_3i_4}.
\] (19)

Let \( u_3 = i_2 + i_3 + i_4 \) in the first sum of the last equation in (19), then \( i_1 = 2p^r - u_3 \),
\(19\) equals to

\[
\begin{align*}
&= \frac{1}{2^{p'}} \left( \sum_{u_3 = i_2 + i_3 + i_4 < 2^{p'}} \frac{(-1)^{2^{p'}-u_3}(i_2 + i_3 + i_4)}{i_2 i_3 i_4 u_3} + 3 \sum_{u_3 = i_2 + i_3 + i_4 < 2^{p'}} \frac{(-1)^{i_1}(i_1 + i_3 + i_4)}{i_3 i_4 u_3} \right) \\
&= \frac{1}{2^{p'}} \left[ 3 \sum_{u_3 = i_2 + i_3 + i_4 < 2^{p'}} \frac{(-1)^{u_3}(i_3 + i_4)}{i_3 i_4 u_3} + 3 \sum_{u_3 = i_2 + i_3 + i_4 < 2^{p'}} \frac{(-1)^{u_3-u_2}(i_3 + i_4)}{i_3 i_4 u_3} \right] + 6 \sum_{u_3 = i_2 + i_3 + i_4 < 2^{p'}} \frac{(-1)^{i_1}(i_1 + i_3)}{i_1 i_3 u_3}. \\
&\quad \text{(20)}
\end{align*}
\]

Let \(u_2 = i_3 + i_4\) in the second sum of the last equation in (20), since \(u_3 = i_1 + i_3 + i_4\), then \(i_1 = u_3 - u_2\), (20) equals to

\[
\begin{align*}
&= \frac{1}{2^{p'}} \left[ 3 \sum_{u_2 = i_1 + i_3 < 2^{p'}} \frac{(-1)^{u_3}(i_3 + i_4)}{i_3 i_4 u_3} + 3 \sum_{u_2 = i_1 + i_3 < 2^{p'}} \frac{(-1)^{u_3-u_2}(i_3 + i_4)}{i_3 i_4 u_3} \right] + 6 \sum_{u_2 = i_1 + i_3 < 2^{p'}} \frac{(-1)^{i_1}(i_1 + i_3)}{i_1 i_3 u_3} \\
&= \frac{1}{2^{p'}} \left[ 6 \sum_{0 < u_1 < u_2 < u_3 < 2^{p'}} \frac{(-1)^{u_3}(i_1 + i_3)}{u_1 u_2 u_3} + 6 \sum_{0 < u_1 < u_2 < u_3 < 2^{p'}} \frac{(-1)^{u_3-u_2}(i_1 + i_3)}{u_1 u_2 u_3} \right] + 6 \sum_{0 < u_1 < u_2 < u_3 < 2^{p'}} \frac{(-1)^{u_2-u_1}(i_1 + i_3)}{u_1 u_2 u_3} \\
&= \frac{3}{2^{p'}} \sum_{0 < u_1 < u_2 < u_3 < 2^{p'}} \frac{(-1)^u_1 + (-1)^{u_2-u_1} + (-1)^{u_1+u_2} + (-1)^{u_1+u_3}}{u_1 u_2 u_3}.
\end{align*}
\]

Similarly, we have

\[
\begin{align*}
&= 4 \sum_{0 < u_1 < u_2 < u_3 < 2^{p'}} \frac{(-1)^{u_1+u_2} + (-1)^{u_1+u_2+u_3} + (-1)^{u_1+u_3}}{u_1 u_2 u_3}
\end{align*}
\]

\(11\)
By (4) and Lemma 7, if $r \equiv 0 \pmod{2}$, we have

$$1 + (\frac{-1}{2})^i \equiv 1 + \frac{(-1)^{i+1}}{i} \pmod{2}$$

By (2) and Lemma 6, we have

$$\sum_{i=1,2,3,4,14} (-1)^i \frac{1}{i_1i_2i_3i_4} + \sum_{i=1,2,3,4,14} \frac{(-1)^{i+1}}{i_1i_2i_3i_4} = \frac{12}{p^r} \sum_{0 < u_1 < u_2 < u_3 < 2p^r} \frac{1}{u_1u_2u_3} - \frac{12}{p^r} \sum_{0 < u_1 < u_2 < u_3 < 2p^r} \frac{1}{u_1u_2u_3}.$$

By Lemma 3, we have

$$\sum_{i=1,2,3,4,14} (-1)^i \frac{1}{i_1i_2i_3i_4} + \sum_{i=1,2,3,4,14} \frac{(-1)^{i+1}}{i_1i_2i_3i_4} = \frac{1}{2} \sum_{i=1,2,3,4,14} \frac{1}{i_1i_2i_3i_4} - \sum_{i=1,2,3,4,14} \frac{1}{i_1i_2i_3i_4}.$$

By (2) and Lemma 6, we have

$$\sum_{i=1,2,3,4,14} (-1)^i \frac{1}{i_1i_2i_3i_4} + \sum_{i=1,2,3,4,14} \frac{(-1)^{i+1}}{i_1i_2i_3i_4} = \frac{24}{5} pB_{p-5} + \frac{240}{5} pB_{p-5} = \frac{216}{5} pB_{p-5} \pmod{p^2}.$$

By (4) and Lemma 7, if $r \geq 2$, then

$$\sum_{i=1,2,3,4,14} (-1)^i \frac{1}{i_1i_2i_3i_4} + \sum_{i=1,2,3,4,14} \frac{(-1)^{i+1}}{i_1i_2i_3i_4} = \frac{12}{5} p^r B_{p-5} + \frac{48}{5} p^r B_{p-5} = \frac{36}{5} p^r B_{p-5} \pmod{p^{r+1}}.$$

This completes the proof of Theorem 3. 

**Proof of Theorem 4.** Similar to the proofs of Theorem 1 and Theorem 3, we have

$$\sum_{i=1,2,3,4,15} (-1)^i \frac{1}{i_1i_2i_3i_4i_5} = \frac{12}{p^r} \sum_{0 < u_1 < u_2 < u_3 < 2p^r} \frac{1}{u_1u_2u_3u_4}((-1)^{u_1} + (-1)^{u_1+u_2} + (-1)^{u_2+u_3} + (-1)^{u_3+u_4}).$$
and
\[\sum_{i_1+i_2+i_3+i_4+i_5 = 2p} \frac{(-1)^{i_1+i_2}}{i_1i_2i_3i_4i_5} = \frac{6}{p^r} \sum_{0 < u_1 < u_2 < u_3 < u_4 < 2p} \frac{1}{u_1u_2u_3u_4}((-1)^{u_2} + (-1)^{u_1+u_2} + (-1)^{u_1+u_3} + (-1)^{u_2+u_4} + (-1)^{u_1+u_2+u_3} + (-1)^{u_1+u_2+u_4} + (-1)^{u_1+u_3+u_4} + (-1)^{u_2+u_3+u_4} + (-1)^{u_1+u_2+u_3+u_4})\]

Hence
\[\sum_{i_1+i_2+i_3+i_4+i_5 = 2p} \frac{(-1)^{i_1+i_2}}{i_1i_2i_3i_4i_5} + 2 \sum_{i_1+i_2+i_3+i_4+i_5 = 2p} \frac{(-1)^{i_1+i_2}}{i_1i_2i_3i_4i_5} = \frac{12}{p^r} \sum_{0 < u_1 < u_2 < u_3 < u_4 < 2p} \frac{1}{u_1u_2u_3u_4} - \frac{12}{p^r} \sum_{0 < u_1 < u_2 < u_3 < u_4 < 2p} \frac{1}{u_1u_2u_3u_4}.
\]

By Lemma 9, we have
\[\sum_{i_1+i_2+i_3+i_4+i_5 = 2p} \frac{(-1)^{i_1}}{i_1i_2i_3i_4i_5} + 2 \sum_{i_1+i_2+i_3+i_4+i_5 = 2p} \frac{(-1)^{i_1+i_2}}{i_1i_2i_3i_4i_5} = \frac{1}{10} \sum_{i_1+i_2+i_3+i_4+i_5 = 2p} \frac{1}{i_1i_2i_3i_4i_5} - \frac{2}{10} \sum_{i_1+i_2+i_3+i_4+i_5 = 2p} \frac{1}{i_1i_2i_3i_4i_5}.
\]

By (2) and Lemma 8(1), we have
\[\sum_{i_1+i_2+i_3+i_4+i_5 = 2p} \frac{(-1)^{i_1}}{i_1i_2i_3i_4i_5} + 2 \sum_{i_1+i_2+i_3+i_4+i_5 = 2p} \frac{(-1)^{i_1+i_2}}{i_1i_2i_3i_4i_5} \equiv -\frac{24}{10} B_{p-5} + \frac{144}{10} B_{p-5} \equiv 12B_{p-5} \pmod{p}.
\]

By Lemma 8(2), if \(r \geq 2\), then
\[\sum_{i_1+i_2+i_3+i_4+i_5 = 2p} \frac{(-1)^{i_1}}{i_1i_2i_3i_4i_5} + 2 \sum_{i_1+i_2+i_3+i_4+i_5 = 2p} \frac{(-1)^{i_1+i_2}}{i_1i_2i_3i_4i_5} \equiv -2p^{r-1}B_{p-5} + 8p^{r-1}B_{p-5} \equiv 6p^{r-1}B_{p-5} \pmod{p^r}.
\]

This completes the proof of Theorem 4.

**Remark 2** Let \(p\) be odd prime and \(r\), \(m\) positive integers, \((m, p) = 1\), using Lemma 1 and Lemma 2, similar to the proof of Theorem 1, we can prove that
\[\sum_{i+j+k = 2mp^r} \frac{(-1)^i}{ijk} \equiv mp^{r-1}B_{p-3} \pmod{p^r}.
\]
In particular, if \( m = 1 \), it becomes Theorem 1.

Let \( p \) be odd prime and \( r, m \) positive integers, \( (m, p) = 1 \), similar to the proof of Theorem 2, we can prove that

\[
\sum_{i+j+k=mP} (-1)^i ijk \equiv \frac{m}{2} p^{r-1}B_{p-3} \pmod{p^r}.
\]

In particular, if \( m = 1 \), it becomes Theorem 2.

Let \( p > 4 \) be a prime and \( r, m \) positive integers, \( (m, p) = 1 \), we can deduce the congruence for

\[
4 \sum_{i_1 + i_2 + i_3 + i_4 = mP} (-1)^{i_1} i_1 i_2 i_3 i_4 + 3 \sum_{i_1 + i_2 + i_3 + i_4 = mP} (-1)^{i_1 + i_2} i_1 i_2 i_3 i_4 \pmod{p^{r+1}}.
\]

Let \( p > 5 \) be a prime and \( r, m \) positive integers, \( (m, p) = 1 \), we can deduce the congruence for

\[
\sum_{i_1 + i_2 + i_3 + i_4 + i_5 = mP} (-1)^{i_1} i_1 i_2 i_3 i_4 i_5 + 2 \sum_{i_1 + i_2 + i_3 + i_4 + i_5 = mP} (-1)^{i_1 + i_2} i_1 i_2 i_3 i_4 i_5 \pmod{p^r}.
\]

Similarly, we can consider the congruence

\[
\sum_{i_1 + i_2 + \cdots + i_n = mP} \frac{(\sigma_1)^{i_1} (\sigma_2)^{i_2} \cdots (\sigma_n)^{i_n}}{i_1 i_2 \cdots i_n} \pmod{p^{r+1}},
\]

where \( \sigma_i \in \{1,-1\}, i = 1, 2, \cdots, n \), but it seems much more complicated.

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