Hidden supersymmetry and Berezin quantization of N=2, D=3 spinning superparticles

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Abstract

The first quantized theory of N=2, D=3 massive superparticles with arbitrary fixed central charge and (half)integer or fractional superspin is constructed. The quantum states are realized on the fields carrying a finite dimensional, or a unitary infinite dimensional representation of the supergroups OSp(2|2) or SU(1,1|2). The construction originates from quantization of a classical model of the superparticle we suggest. The physical phase space of the classical superparticle is embedded in a symplectic superspace $T^*(R^{1,2}) \times L^{1|2}$, where the inner Kähler supermanifold $L^{1|2} \cong \text{OSp}(2|2)/[\text{U}(1) \times \text{U}(1)] \cong \text{SU}(1,1|2)/[\text{U}(2|2) \times \text{U}(1)]$ provides the particle with superspin degrees of freedom. We find the relationship between Hamiltonian generators of the global Poincaré supersymmetry and the “internal” SU(1,1|2) one. Quantization of the superparticle combines the Berezin quantization on $L^{1|2}$ and the conventional Dirac quantization with respect to space-time degrees of freedom. Surprisingly, to retain the supersymmetry, quantum corrections are required for the classical N=2 supercharges as compared to the conventional Berezin method. These corrections are derived and the Berezin correspondence principle for $L^{1|2}$ underlying their origin is verified. The model admits a smooth contraction to the N=1 supersymmetry in the BPS limit.

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I. INTRODUCTION

In this paper we construct $N=2$, $D=1+2$ massive spinning superparticle model and study the symplectic supergeometry behind it. This supergeometry is compatible to the Berezin quantization method which is applied to construct the one-particle quantum theory. The main part of our consideration is based on the observation that the $N=2$ superextension of $D=3$ spinning particle results in the classical model which possesses simultaneously Poincaré supersymmetry and Lorentz supersymmetry of the superspin degrees of freedom. This “double” supersymmetry can be lifted to the quantum level and we obtain the realization of $N=2$, $D=3$ Poincaré supermultiplet on the fields carrying an irreducible representation of the supergroup $SU(1,1|2)$ (“Lorentz supergroup” whose even part is $SO^\uparrow(1,2) \times U(2) \times $ central charge). A nonlinear mutual involvement of the Hamiltonian generators of two supersymmetries requires the careful geometric quantization of the superparticle. At first, we try to explain the most important motivations of the problem.

In the hierarchy of all known entities, the particles living in three-dimensional space-time stand out mostly due to a possibility of fractional spin and statistics (anyons). Anyon excitations are actually presented in some planar physics phenomena \cite{1,2} and the relevant theoretical concept has both topological \cite{3,4,5} and group-theoretical \cite{6,7,8} grounds. It is well known that in the field theory fractional statistics originates usually from a coupling of the matter fields to the gauge field with the Chern-Simons mass term \cite{9}. The supersymmetric extension of this approach \cite{10} implies a direct interaction between anyon excitations.

The group-theoretical methods may give an alternative way to understand the anyon concept. One can start from the mechanical model of $D=3$ spinning particle, whose quantization leads to the one-particle quantum mechanics for the fractional spin state \cite{11,12,13,14,15,16}. It is established that $D=3$ spinning particles possess the following remarkable features: (i) the spinning particle carries as many physical degrees of freedom as a spinless one; (ii) there is the so-called canonical model \cite{17} of spinning particle, which implies a deformation of the canonical symplectic structure of spinless particle by the use of the Dirac monopole two-form, without extension of the phase space introducing any “spinning” variables; (iii) it is promising feature of the canonical model to be adapted for the construction of consistent couplings of the particle to external fields \cite{17,18,19,20,21} and self-interaction of anyons \cite{22,23,24}. In higher dimensions, the interaction problem for spinning particles becomes more involved, although some progress has recently been achieved there as well \cite{25,26}; (iv) the anyon wave equations may be formulated in analogy with the ones for bosons and fermions. An essential difference is that the fractional spin, in contrast to the (half)integer, is naturally described in terms of infinite component fields carrying infinite dimensional representations of the universal covering group $SO^\uparrow(1,2) \cong SU(1,1)$; (v) representations of fractional spin are multivalued.

There is no consistent quantum field theory of anyons up to now, nevertheless the
Chern-Simons and group-theoretical constructions are deemed to lead to a unified consistent theory. In this regard, it would be interesting to understand, how the supersymmetry may be included into a group-theoretical description of anyons in terms of the infinite component fields.

Another reason to investigate the D=3 superparticle is the exceptional fact that not only the Poincaré supersymmetry is possible in 1+2 dimensions, but the Lorentz one is too. The Lorentz group $SO^\uparrow(1,2)$ coincides with the D=2 anti-de Sitter group, the latter admits the superextension regardless of specific space-time dimension. Although the Lorentz and the Poincaré supersymmetries are not compatible with each other, surprisingly, we will show that the Lorentz supersymmetry of D=3 spinning superparticle (which is invariant by construction with respect to the global Poincaré SUSY transformations) manifests itself as a hidden supersymmetry of internal degrees of freedom associated to the particle superspin and to the underlying superextended monopole-like symplectic structure.

The hidden OSp(2|2) supersymmetry of N=1 superanyons has been found in Ref. [25] where the respective model is constructed. The presence of the OSp(2|2) supersymmetry already in the classical mechanics appears to be crucial for a consistent first quantization of N=1, D=3 superanyon. As a result, one obtains in quantum theory the realization of the N=1 Poincaré supermultiplet on the fields carrying an atypical unitary infinite dimensional representation of the OSp(2|2) [25]. It is a direct N=1 superextension of description in terms of infinite dimensional unitary representation of the D=3 Lorentz group [6, 7, 8] or the ones of the deformed Heisenberg algebra [26]. We argued in this manner the relevance of the group-theoretical approach for N=1 supersymmetric anyons. In this paper we suggest a nontrivial generalization of this construction to the case of D=3, N=2 massive spinning superparticle with arbitrary fixed central charge.

We construct a superparticle model, which gives N=2 superextension of the canonical description of the D=3 spinning particle mentioned above. It is essential for our consideration that the Hamiltonian formalism of the canonical model may be built either in terms of the minimal phase space, or in an extended phase space restricted by constraints [13, 15, 16, 25]. In both cases the reduced phase space could be thought of as a space of motion of a Souriau’s “elementary system” [27].

A general concept of elementary physical systems, including spinning particles and superparticles, is based on the so-called Kostant-Souriau-Kirillov construction [28, 27, 26]. The idea of the KSK construction is to identify the physical phase space (= space of motion) of any elementary system with a coadjoint orbit $\mathcal{O}$ of the symmetry group $G$. The symplectic action $G$ on $\mathcal{O}$ (classical mechanics) lifts to a representation of the group in a space of functions $\mathcal{H}$ on the classical manifold (prequantization). Then the quantization problem reduces to an appropriate choice of polarization, that is a global Lagrangian section in $T(\mathcal{O})$ being invariant under the action of the symmetry group.

In the special case of Kähler homogeneous spaces perfect results can be achieved in
the framework of the Berezin quantization method \[30, 31\], which implies one-to-one correspondence between the phase-space functions (covariant Berezin symbols) and linear operators in a Hilbert space. The latter is realized by holomorphic sections, because the Kähler homogeneous manifold admits a natural complex polarization \[32\]. Moreover, the multiplication of the operators in the Hilbert space induces a noncommutative binary ∗-operation for the covariant Berezin symbols and a correspondence principle can be proved \[30, 31\].

Physically speaking, it would not always be satisfactory to describe elementary systems in terms of the coadjoint orbits. In particular, the dynamics of relativistic particles and superparticles is usually supposed to evolve in a fibre bundle \(\mathcal{M}\) over a space-time manifold that is crucial for the interaction problem. Thus, the coadjoint orbit of the spinning (super)particle arises from embedding into evolution (super)space. The projection \(\pi : \mathcal{M} \to \mathcal{O}_G\), where \(G\) is a Poincaré (super)group, generates \(G\)-invariant constraints and gauge symmetries in \(\mathcal{M}\). The construction of interactions, being consistent with the gauge symmetries, and the quantization problem for \(\pi\) provide a subject of current interest in the problems of spinning particle and superparticle models \[11, 23, 24, 33, 34, 35\].

Concerning the D=3 spinning particle, we know \[13, 25\] that the quantization problem for the canonical model is naturally solved by means of an embedding of the maximal (four-dimensional) coadjoint orbit of the group \(\text{ISO}^\uparrow(1,2)\) into eight dimensional phase space (that is extended phase space) \(\mathcal{M}^8 \cong T^*(\mathbb{R}^{1,2}) \times \mathcal{L}\). Here \(\mathcal{L} \cong \text{SU}(1,1)/\text{U}(1)\) is a Lobachevsky plane and the character \(\cong\) denotes a symplectomorphism. The projection \(\pi : \mathcal{M}^8 \to \mathcal{O}_{m,s}\) onto co-orbit \(\mathcal{O}_{m,s}\) of the particle of mass \(m\) and spin \(s\) is provided by the constraints. The auxiliary variables parametrizing \(\mathcal{L}\) are used to describe the particle spin. One can interpret the (holomorphic) automorphisms of the Lobachevskian Kähler metric as a hidden symmetry of the internal particle’s structure, which is related to the spin. We observed in Refs. \[13, 25\] that the quantization of anyon could be achieved as a compromise of the conventional Dirac quantization on \(T^*(\mathbb{R}^{1,2})\) and of the geometric quantization in the Lobachevsky plane. Constraints of the classical mechanics are converted into wave equations of anyon according to the Dirac prescriptions.

The starting point of this paper is a mechanical model of \(N=2\) superparticle with arbitrary fixed mass \(m > 0\), superspin \(s \neq 0\) and central charge \(Z \equiv mb, |b| \leq 1\) briefly announced before \[37\]. For this elementary system the maximal coadjoint orbit \(\mathcal{O}_{m,s,b}\) of real dimension 4/4 is related to the case \(|b| < 1\). In our model, this orbit appears embedded into 8/4-dimensional extended phase superspace \(\mathcal{M}^{8/4}\) of a special geometry: \(\mathcal{M}^{8/4} \cong T^*(\mathbb{R}^{1,2}) \times \mathcal{L}^{1/2}\), where \(\mathcal{L}^{1/2} = \text{SU}(1,1)[2]/[\text{U}(2) \times \text{U}(1)] \cong \text{OSp}(2|2)/[\text{U}(1) \times \text{U}(1)]\) is an atypical Kähler coadjoint orbit of the supergroup \(\text{SU}(1,1|2)\) and the typical one of \(\text{OSp}(2|2)\). The inner supermanifold \(\mathcal{L}^{1/2}\), providing the particle model with a nonzero superspin, was studied originally in Refs. \[36, 38\] in relation to \(\text{OSp}(2|2)\) supercoherent states and called \(N=2\) superunit disc. The projection of \(\mathcal{M}^{8/4}\) onto physical subspace follows similarly to the nonsupersymmetric
model. In fact, introducing the supersymmetry for D=3 particle, we need to superextend only the inner submanifold $L$ of the extended phase space. The extended phase superspace $\mathcal{M}^{8|4} \cong T^*(R^{1,2}) \times \mathcal{L}^{1|2}$ carries “double supersymmetry”: one is related to the Poincaré supergroup and acts on the associated co-orbit $\mathcal{O}_{m,s,b} \subset \mathcal{M}^{8|4}$, another one lives in the inner subsupermanifold $\mathcal{L}^{1|2}$. Moreover, the model allows an extended hidden $N=4$ supersymmetry with special values of the central charges saturating the BPS bound.

We will quantize the theory similarly to quantization of the canonical model of the particle on $\mathcal{M}^8 \ [14, 23]$. Specifically, we combine the geometric quantization in the inner subsupermanifold $\mathcal{L}^{1|2}$ for the internal $SU(1,1|2)$ supersymmetry and the canonical Dirac quantization in $T^*(R^{1,2})$.

This quantization scheme implies from the outset that the mentioned “double supersymmetry” must survive in the quantum theory. The crucial point is to express the Hamiltonian generators of the Poincaré supersymmetry in $\mathcal{M}^{8|4}$ in terms of the ones of internal $SU(1,1|2)$ supersymmetry (as well as of space-time coordinates and momenta). These expressions appear to be nonlinear. As a consequence, some renormalization of the Poincaré supergenerators should be required for the closure of the Poincaré supersymmetry algebra. Roughly speaking, the corrections to generators could be treated as a manifestation of the ordering ambiguity for operators in quantum theory. We will see, that the origin of the corrections may also be clarified from the viewpoint of the Berezin quantization in $\mathcal{L}^{1|2}$ and the underlying correspondence principle. However, the Berezin method itself does not provide a regular technique of deriving the closing corrections which have to recover the representation of the Poincaré superalgebra in quantum theory. Moreover, it is unclear a priori whether the consistent corrections exist at all. Surprisingly, the problem is solved if a simple ansatz is taken for the renormalized Poincaré generators. Then the closing corrections, which appear in the order of $\mathcal{O}(s^{-2})$, can be exactly calculated.

We arrive eventually to the realization of the unitary representation of $N=2$, D=3 supermultiplet on the fields carrying atypical irreps of the supergroup $SU(1,1|2)$ and the typical ones of the subsupergroup $OSp(2|2)$. These irreps are certainly infinite dimensional for the case of fractional superspin, but for the habitual case of (half)integer superspin they may be chosen to be finite dimensional.

The model of N=2 superparticle reduces to the one of N=1 superparticle in the Bogomol’ny-Prassad-Sommerfield (BPS) limit for central charge, when $|b| = 1$. One can trace the BPS limit both at the classical and quantum levels. Classically, it corresponds to the degenerate coadjoint orbit of D=3, N=2 superparticle of dimension $4/2$. When $|b| = 1$, the extended phase superspace becomes degenerate and reduces to $\mathcal{M}^{8|2} \cong T^*(R^{1,2}) \times \mathcal{L}^{1|1}$ with inner supermanifold $\mathcal{L}^{1|1} \cong OSp(2|2)/U(1|1) \cong OSp(1|2)/U(1)$. $\mathcal{M}^{8|2}$ is exactly the extended phase superspace of N=1 superanyon [23]. In this exceptional case, the generators of N=1 Poincaré supersymmetry and the internal $OSp(2|2)$ one are linearly expressible to one another. Thus, the geometric quantization immediately gives the quantum theory of N=1 su-
per particle, without extra constructions and corrections. In particular, we don’t need the detailed Berezin correspondence principle for $L^{1\bar{1}}$.

The geometric quantization in the OSp(2|2) coadjoint orbits was constructed in Refs. [36, 38] and we follow these results. At the same time we have to clarify two important points, which have seemingly been unknown. First, we found out that the Kähler geometry of the regular co-orbit $L^{1\bar{1}}$ admits the symplectic holomorphic action of the supergroup SU(1, 1|2), which is larger than the supergroup OSp(2|2) in itself. We construct the geometric quantization on $L^{1\bar{1}}$ provided for this extended supersymmetry supergroup. Secondly, we perform Berezin quantization for $L^{1\bar{1}}$ to establish a correspondence principle and to explain the origin of quantum corrections to the N=2 Poincaré supercharges in $\mathcal{M}^{8|4}$.

The paper is organized as follows. In Sec. II we recall briefly the canonical model of D=3 spinning particle in terms of the minimal and extended phase spaces. Specifically, we focus at symplectic structure and symmetries of the minimal and extended spaces.

Then we are going to construct the superextension of the canonical model. The classical mechanics of N=2, D=3 massive spinning superparticle with arbitrary central charge is considered in Sec. III. Starting from a first order Lagrangian we study the supergeometry of the phase superspace and identify it with $\mathcal{M}^{8|4} = T^*(R^{1,2}) \times L^{1\bar{1}}$. We construct explicitly the embeddings of the N=2 Poincaré and Lorentz supergroup’s coadjoint orbits into $\mathcal{M}^{8|4}$ and find out the Hamiltonian generators of corresponding supersymmetries. The relation, being crucial for quantization, is established between the N=2 Poincaré and SU(1, 1|2) Hamiltonian generators. We also reveal a degenerate N=4 supersymmetry in the model and a special case of degenerate co-orbits, which appear in the BPS limit. Furthermore, a reduction of the model with respect to a part of constraints is shown to lead to a minimal 6/4-dimensional phase superspace with superextended symplectic monopole-like structure and with the mass-shell condition to be the only constraint. We obtain, in particular, N=2 superextension of the Dirac monopole two-form, which supplies the particle with superspin.

In Sec. IV we suggest a quantization procedure for the classical mechanics constructed in Sec. III. At first the Berezin quantization is considered on the regular OSp(2|2) coadjoint orbit. In particular, we construct the correspondence between symbols and operators on $L^{1\bar{1}}$ and prove the underlying correspondence principle. Then these results are applied to the consistent quantization of D=3, N=2 superparticle, which is the final object of construction.

The summary and a general outlook are given in Sec. V. Finally, the Appendix contains the calculation of N=2 Poincaré supercharge’s quantum anticommutator. The calculation provides manifest verification of consistency of the renormalization procedure for the Poincaré supercharges.
II. MINIMAL AND EXTENDED PHASE SPACES 
OF A CANONICAL MODEL OF SPINNING PARTICLE 

First consider the nonsupersymmetric canonical model of the particle (various formulations see [11, 6, 7, 13, 16]), which serves as an initial subject for further generalizations. The particle lives originally on six-dimensional phase space $\mathcal{M}^6$ with a symplectic two-form

$$\Omega_s = -dx^a \wedge dp_a + \Omega_m \quad \Omega_m = \frac{s}{2} \frac{\epsilon^{abc} p_a dp_b \wedge dp_c}{(-p^2)^{3/2}} \quad (p^2 < 0), \quad (2.1)$$

where $\Omega_m$ is known as the Dirac monopole form. The Poincaré transformations are generated by the following functions

$$P_a = p_a \quad J_a = \epsilon^{abc} x^b p^c - s \frac{p_a}{(-p^2)^{1/2}}, \quad (2.2)$$

which constitute D=3 Poincaré algebra with respect to Poisson brackets (PB’s)

$$\{P_a, P_b\} = 0 \quad \{J_a, P_b\} = \epsilon_{abc} P^c \quad \{J_a, J_b\} = \epsilon_{abc} J^c. \quad (2.3)$$

The fundamental PB’s read

$$\{x^a, x^b\} = s \frac{\epsilon^{abc} p_c}{(-p^2)^{3/2}} \quad \{x^a, p_b\} = \delta^a_b \quad \{p_a, p_b\} = 0. \quad (2.4)$$

The last two PB’s mean that $x^a$ and $p_a$ transform as coordinates and momenta by Poincaré translations. Moreover, they are Lorentz vectors because of $\{J_a, x_b\} = \epsilon_{abc} x^c$ and $\{J_a, p_b\} = \epsilon_{abc} p^c$.

Let us assume that the particle dynamics on $\mathcal{M}^6$ is governed by the mass shell constraint

$$p^2 + m^2 = 0, \quad (2.5)$$

whereas the canonical Hamilton function is identically zero. On the mass shell, the Casimir functions of the enveloping Poincaré algebra are identically conserved: $\mathcal{P}^2 = -m^2$, $(\mathcal{P}, \mathcal{J}) = ms$. We conclude that D=3 particle of mass $m$, spin $s$ and energy sign $p^0/|p^0|$ lives on mass shell. From now on, we take a further restriction $p^0 > 0$ bearing in mind the supersymmetric theory, when the energy is positive essentially. The mass shell constraint generates the reparametrization (gauge) invariance for every world line of the particle. The set of world lines, being considered modulo to the gauge equivalence, is named the particle history space, the latter is isomorphic

\footnote{We use Latin letters to denote D=3 Lorentz vectors and Greek letters for the SU(1,1) spinors; Minkowski metric is chosen to be $\eta_{ab} = \text{diag}(-1,1,1)$, totally antisymmetric tensor is normalized by condition $\epsilon_{012} = -\epsilon^{012} = 1$; the spinor indices are raised and lowered with the use of the spinor metric $\epsilon^{\alpha\beta} = -\epsilon_{\beta\alpha} = -\epsilon_{\alpha\beta} (\alpha, \beta = 0, 1)$, $\epsilon^{01} = -1$ by the rule $\psi_\alpha = \epsilon_{\alpha\beta} \psi^\beta, \psi^\alpha = \epsilon^{\alpha\beta} \psi_\beta$.}
to the physical state space $O_{m,s}$ of the spinning particle. The reduced symplectic manifold $O_{m,s}$ is symplectomorphic to the maximal coadjoint orbit $[27, 29, 32]$ of the D=3 Poincaré group.

There is a standard way to extend the canonical model to the Poincaré supergeometry. One may substitute $dx^a \to dx^a - i(\gamma^a)_{\alpha\beta} \theta^\alpha I d\theta^\beta I$ in Eq. (2.1) introducing real Grassmann variables $\theta^\alpha I, I = 1, \ldots, N$. The resulting symplectic superform appears to be invariant under the N-extended Poincaré supergroup without central charges. One may further generate central charges introducing some Wess-Zumino type terms $[39, 34]$ in Eq. (2.1). Then, imposing the mass shell constraint (2.5), one may build the classical model of D=3 superparticle of mass $m$, superspin $s$ and arbitrary fixed central charges in the $6/2N$-dimensional phase superspace. However, it is hardly possible to conceive satisfactory quantization of this model.

Even for the canonical model without supersymmetry the realization of the coordinate operators $\hat{x}^a$ is a nontrivial problem accounting for the complicated form of the first Poisson bracket in Eqs. (2.4). A detailed analysis of Ref. [13] shows that the manifest covariance of the canonical model, being formulated in terms of the “minimal” phase space $M^6$, is inevitably lost in quantum theory. The superextension of the canonical model makes the Poisson brackets, being quantized, much more complicated. In fact, the quantization problem in the reduced nonlinear phase superspace is not solved even for spinless D=3 superparticle. Thus we will reformulate from the outset the canonical model in an “extended” phase space, where a hidden symmetry of spinning particle becomes transparent and gives an efficient method for quantization making use of this symmetry. Moreover, the construction will be appropriate for intriguing superextension.

An adapted reformulation of the canonical model is suggested in Refs. [15, 27]. We observe, that the monopole two-form $\Omega_m$ in Eq. (2.1) is nothing else but the Kähler two form on the mass hyperboloid (2.3), which gives the realization of the Lobachevsky plane $\mathcal{L}$. It will be convenient to make use of another realization of $\mathcal{L} \cong \{ z \in \mathbb{C}^1, |z| < 1 \}$ by an open unit disc of complex plane $\mathbb{C}^1$. We rewrite the symplectic two-form (2.1) as follows

$$\Omega_s = -dx^a \wedge dp_a + \Omega_\mathcal{L} \quad \Omega_\mathcal{L} = -2is \frac{dz \wedge d\bar{z}}{(1-z\bar{z})^2}, \quad (2.6)$$

where (recall that $p^2 < 0$ and we have taken $p^0 > 0$)

$$p^a = \sqrt{-p^2} n^a \quad n^a = \left( \frac{1+z\bar{z}}{1-z\bar{z}}, -\frac{z+\bar{z}}{1-z\bar{z}}, i \frac{\bar{z}-z}{1-z\bar{z}} \right) \quad n^2 \equiv -1. \quad (2.7)$$

The unit timelike Lorentz vector $n^a$ parametrizes the points of the Lobachevsky plane.

Let us look at Eq. (2.6) from a different viewpoint. Consider a new phase space $\mathcal{M}^8 \cong T^* (\mathbb{R}^{1,2}) \times \mathcal{L}$ with a symplectic two-form (2.6) and an elementary system on $\mathcal{M}^8$, whose dynamics is subjected by three constraints

$$p^a = mn^a \quad (2.8)$$
Apparently these constraints project the extended phase space \( \mathcal{M}^8 \) into the same coadjoint orbit as the mass shell constraint (2.5) does for \( \mathcal{M}^6 \). Alternatively, one can solve explicitly only two constraints \( p^a = \sqrt{-p^2} n^a \) providing the reduction \( \pi_1 : \mathcal{M}^8 \to \mathcal{M}^6 \) of extended phase space to the minimal one. In other words, we have constructed the sequence of embeddings \( \mathcal{O}_{m,s} \subset \mathcal{M}^6 \subset \mathcal{M}^8 \). Hence we get an equivalent description of D=3 spinning particle in terms of the extended phase space \( T^*(R^{1,2}) \times \mathcal{L} \). The Hamiltonian generators of the canonical Poincaré transformations in \( \mathcal{M}^8 \) read

\[
\mathcal{P}_a = p_a \quad J_a = \epsilon_{abc} x^b p^c + J_a ,
\]

where the spin vector \( J_a \) is expressed in terms of the ‘inner’ space \( \mathcal{L} \):

\[
J_a = -sn_a .
\]

The Hamiltonians (2.9) generate the Poincaré algebra with respect to PB in \( \mathcal{M}^8 \), whereas the spin generators (2.10) span internal Lorentz algebra related to the (holomorphic) automorphism group of the Lobachevsky plane. The latter group can be recognized as a hidden symmetry of the internal structure of spinning particle. Although this concept looks may seem artificial at the moment, below, we will observe essentially nontrivial superextension of the hidden symmetry.

The Poincaré Casimir functions are identically conserved owing to constraints (2.8)

\[
p^2 + m^2 = 0 \quad (p, J) - ms = 0 .
\]

A crucial detail is that the equations (2.11) define the same surface in the extended phase space as the constraints (2.8) do.

The quantization of the model in \( \mathcal{M}^8 \) is almost transparent. We can combine the canonical Dirac quantization in \( T^*(R^{1,2}) \) and the Berezin quantization in the Lobachevsky plane \[25\]. Constraints (2.11) will be imposed in Hilbert space to separate the one-particle states.

Finally, write down the Lagrangian of the theory. One may choose the action functional as an integrand of the one-form \( \Theta \), where \( d\Theta = \Omega_s + V \) and \( V \) vanishes on shell. Let us take

\[
S = \int \Theta , \quad \Theta = p_a dx^a + is \frac{\bar{z}dz - zd\bar{z}}{1 - z\bar{z}} \equiv p_a dx^a + \Sigma_L , \quad d\Sigma_L = \Omega_L .
\]

It is implied here that the virtual paths lay in the constraint surface (2.8). Excluding the momenta accounting for constraints (2.8) and making pull back of \( \Theta \), one obtains the action functional

\[
S = \int_{\tau_1}^{\tau_2} L d\tau \quad L = m(\dot{x}, n) + is \frac{\bar{z}d\bar{z} - zdz}{1 - z\bar{z}}
\]

with the first order Lagrangian being invariant under reparametrizations. Notice that the Lagrangian is also strongly invariant under translations and spatial rotations, whereas the Lorentz boosts change it by a total derivative.
III. CLASSICAL MODEL OF D = 3 SPINNING SUPERPARTICLE

1. A first order Lagrangian

Introduce a N=2 superextension of the Lagrangian (2.13) providing both the superpoincaré invariance of the theory and other hidden supersymmetry as well. Introducing a pair of Majorana anticommuting spinors $\theta^\alpha = (\theta^\alpha, \chi^\alpha), I = 1, 2$ we suggest

$$L = m(\Pi, n) + mb(\theta^\alpha \dot{x}^\alpha - \chi^\alpha \dot{\chi}^\alpha) - mb\theta^\alpha n_\alpha \dot{n} \gamma^\beta \chi^\beta + is \frac{\ddot{z} \bar{z} - z \ddot{\bar{z}}}{1 - z \bar{z}} ,$$  \hspace{1cm} (3.1)

where $m, b, s$ are real parameters, $n^a$ is a unit Lorentz vector in the Lobachevsky plane, being defined by Eq. (2.7) and the reality condition for SU(1,1) read rather unusual:

$$n_{\alpha \beta} \equiv n^a \gamma_{a, \alpha \beta} \quad \Pi^a = \dot{x}^a - i \gamma_{a, \alpha \beta} \theta^\alpha \dot{\gamma}^\beta + \chi^\alpha \dot{\chi}^\beta .$$

The three dimensional Dirac matrices $\gamma_a$ are chosen in the form ²

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

The first term in the Lagrangian (3.1) is a conventional superextension of the respective expression in Eq. (2.13) and the second addend represents the Wess-Zumino type term generating the central charge for the supersymmetry [39]. At last, the third term accounts for the specific of D=3 spinning superparticle model. Owing to this addend the supertranslations, underlying Poincaré supersymmetry of the Lagrangian, read rather unusual:

$$\delta_\xi \dot{x}^a = i \gamma_{a, \alpha \beta} \theta^\alpha \dot{\theta}^\beta + ib e_{a, \alpha \beta} \gamma_{c, \alpha \beta} e^c \chi^\beta - m n^a \theta^\alpha \chi^\beta \delta_\xi \lambda^\alpha = 0 \quad \delta_\xi z = 0$$

$$\delta_\eta \dot{x}^a = i \gamma_{a, \alpha \beta} \eta^\alpha \chi^\beta - ib e_{a, \alpha \beta} \gamma_{c, \alpha \beta} \eta^\beta \theta^\alpha - m n^a \eta^\alpha \theta^\alpha \delta_\eta \lambda^\alpha = 0 \quad \delta_\eta \eta^\alpha = \eta^\alpha \quad \delta_\eta z = 0 .$$

Here $\epsilon^\alpha, \eta^\alpha$ are odd real parameters. For completeness, expose also the even infinitesimal Poincaré transformations and U(1) transformations as well

$$\begin{align*}
\delta_\omega \dot{x}^a &= \epsilon_{abc} \omega_b \dot{x}_c \\
\delta_\omega \theta^\alpha &= -\frac{i}{2} \omega^a \gamma_a^{\alpha \beta} \theta^\beta \\
\delta_\omega \lambda^\alpha &= -\frac{i}{2} \omega^a \gamma_a^{\alpha \beta} \chi^\beta \\
\delta_\omega z &= i \omega^a \xi_a \\
\delta_\xi \dot{x}^a &= f^a \\
\delta_\xi \theta^\alpha &= \delta_\xi \lambda^\alpha = 0 \\
\delta_\xi z &= 0 \\
\delta_\mu \dot{x}^a &= 0 \\
\delta_\mu \theta^\alpha &= -\mu \theta^\alpha \\
\delta_\mu \lambda^\alpha &= \mu \lambda^\alpha \\
\delta_\mu z &= 0
\end{align*}$$ \hspace{1cm} (3.4)

²One may wonder, why the $\gamma$-matrices are not Hermitian. It is instructive to note that the reality condition for SU(1,1) spinor formalism is not trivial, as for isomorphic SL(2,R) ones. For any $g \in SU(1,1)$ the complex conjugation reads $\tilde{g} = cg^c$, where $c = c^{-1} = \text{antidiag}(-1, -1)$. The matrices $c \gamma^a$ are truly Hermitian. The covariant Majorana (reality) condition looks like

$$c \tilde{\psi} = \psi$$ \hspace{1cm} (3.2)

for two-component SU(1,1)-spinor $\psi$. 

for two-component SU(1,1)-spinor $\psi$. 

9
with the even real parameters $\omega^a, f^a, \mu$ and the holomorphic object $\xi_a = -1/2(2z, 1 + z^2, i(1 - z^2))$. The infinitesimal transformations (3.3), (3.4) generate N=2 Poincaré superalgebra, which is discussed in Subsection 3.

2. Extended phase superspace

We show in this subsection that the superparticle being described by the Lagrangian (3.1) lives in a supersymplectic phase space $\mathcal{M}^8|4$ of very special supergeometry: $\mathcal{M}^8|4 \cong T^*(\mathbb{R}^{1,2}) \times L^{1/2}$. Then we identify $L^{1/2}$ with regular (when $|b| < 1$) or degenerate (when $|b| = 1$) coadjoint orbit of the OSp(2|2) supergroup. Having the goal to quantize the theory in $\mathcal{M}^8|4$ we will need for detailed information about SUSY’s and quantization in $L^{1/2}$. The supersymplectic geometry of $L^{1/2}$ is considered in Subsec. 4, while the Berezin quantization will be constructed in Subsec. IV.1.

The model (3.1) fits naturally into the formulation in symplectic language. The theory originates from the action functional

$$S = \int \Theta_{\text{SUSY}} \quad \Theta_{\text{SUSY}} = p_a dx^a + \Sigma_{L^{1/2}}$$

(3.5)

$$\Sigma_{L^{1/2}} = -imn_{\alpha\beta}\theta^\alpha d\theta^\beta - imn_{\alpha\beta}\chi^\alpha d\chi^\beta + mb\theta_a d\chi^a - mb\chi_a d\theta^a$$

$$- 2m \frac{z^\alpha \bar{z}^\alpha \theta_\alpha \chi_\beta d\bar{z} - \bar{z}^\alpha z^\alpha \bar{\theta}_\alpha \bar{\chi}_\beta dz}{(1 - z\bar{z})^2} + is \frac{\bar{z}dz - zd\bar{z}}{1 - z\bar{z}},$$

(3.6)

where the virtual paths belong the surface

$$p^a = mn^a,$$

(3.7)

as follows from the definition $p_a = \partial L/\partial \dot{x}^a$. Introduce the objects

$$z^\alpha \equiv (1, z) \quad \bar{z}^\alpha \equiv (\bar{z}, 1), \quad \alpha = 0, 1,$$

(3.8)

that simplifies Eq. (3.6) and many of the forthcoming formulae. In this section $z^\alpha, \bar{z}^\alpha$ are used for notation only. Remarkable origin and transformation properties of the objects $z^\alpha, \bar{z}^\alpha$ will be considered later in Subsec. IV.4.

Relation (3.5) shows that the particle dynamics is embedded in phase superspace $\mathcal{M}^8|4 \cong T^*(\mathbb{R}^{1,2}) \times L^{1/2}$ with some inner superspace of a real dimension 2/4 denoted by $L^{1/2}$. The symplectic two-superform in $\mathcal{M}^8|4$ reads

$$\Omega_s^{\text{SUSY}} = d\Theta_{\text{SUSY}} = -dx^a \wedge dp_a + \Omega_{L^{1/2}}, \quad \Omega_{L^{1/2}} = d\Sigma_{L^{1/2}},$$

(3.9)

The inner superspace is a N=2 superextension of the Lobachevsky plane. We show at first that $L^{1/2}$ coincides with a coadjoint orbit of the OSp(2|2) supergroup. Let us introduce new complex Grassmann variables ($m \neq 0, s \neq 0$)

$$\theta = \sqrt{\frac{m}{s}} (iz^\alpha \chi_\alpha - z^\alpha \theta_\alpha) \left[ 1 + m \frac{1 - b}{4s} (\theta^\alpha \theta_\alpha + \chi^\alpha \chi_\alpha) \right] \quad \bar{\theta} = \overline{(\theta)}$$

$$\chi = \sqrt{\frac{m}{s}} (iz^\alpha \theta_\alpha - \bar{z}^\alpha \chi_\alpha) \left[ 1 + m \frac{1 + b}{4s} (\theta^\alpha \theta_\alpha + \chi^\alpha \chi_\alpha) \right] \quad \bar{\chi} = \overline{(\chi)},$$

(3.10)
which are in one-to-one correspondence with the Majorana spinors $\theta^\alpha$, $\chi^a$ used before. It is easy to check that the symplectic two-superform $\Omega_{L^{1/2}}$ of inner superspace reads in new variables as

$$\Omega_{L^{1/2}} = -\frac{is}{2} \left( 2 - a_+ \frac{1 + z\bar{z}}{1 - z\bar{z}} \theta\bar{\theta} - a_- \frac{1 + z\bar{z}}{1 - z\bar{z}} \chi\bar{\chi} + a_+ a_- \frac{1 + 2z\bar{z}}{(1 - z\bar{z})^2} \theta\bar{\theta} \chi\bar{\chi} \right) \frac{dz \wedge d\bar{z}}{(1 - z\bar{z})^2}$$

$$+ \frac{isa_+ \bar{z}\theta}{2} \left( 1 - \frac{1 - \chi\bar{\chi}}{1 - z\bar{z}} \right) \frac{dz \wedge d\bar{\theta}}{(1 - z\bar{z})^2} + \frac{isa_- \bar{z}\bar{\theta}}{2} \left( 1 - \frac{1 - \chi\bar{\chi}}{1 - z\bar{z}} \right) \frac{dz \wedge d\bar{\chi}}{(1 - z\bar{z})^2}$$

$$- \frac{isa_+ z\theta}{2} \left( 1 - \frac{1 - \chi\bar{\chi}}{1 - z\bar{z}} \right) \frac{d\theta \wedge d\bar{\bar{z}}}{(1 - z\bar{z})^2} - \frac{isa_- z\bar{\theta}}{2} \left( 1 - \frac{1 - \chi\bar{\chi}}{1 - z\bar{z}} \right) \frac{d\bar{\theta} \wedge d\bar{z}}{(1 - z\bar{z})^2}$$

$$+ \frac{isa_+ \bar{a}_{- \chi\bar{\chi}}}{4} \frac{d\theta \wedge d\bar{\theta}}{(1 - z\bar{z})^2} + \frac{isa_- a_{- \chi\bar{\chi}}}{4} \frac{d\bar{\theta} \wedge d\bar{\theta}}{(1 - z\bar{z})^2}$$

$$a_+ = 1 + b \quad a_- = 1 - b \, .$$

This superform exactly coincides to the one deduced by Gradechi and Nieto in the supercoherent state’s approach for the $\text{OSp}(2|2)$ coadjoint orbits. $\Omega_{L^{1/2}}$ is nondegenerate iff $|b| \neq 1$. In the case $|b| < 1$, the supermanifold $L^{1/2}$ is the regular $\text{OSp}(2|2)$ coadjoint orbit $L^{1/2} \cong \text{OSp}(2|2)/[\text{U}(1) \times \text{U}(1)]$ and is called $N=2$ superunit disc. The degenerate orbit $\text{OSp}(2|2)/\text{U}(1|1)$, which is denoted usually by $L^{1|1}$ and called $N=1$ superunit disc, appears when $|b| = 1$. The other possibility $|b| > 1$ has no physical significance: neither the Poincaré supersymmetry, nor the internal $\text{OSp}(2|2)$ one admit unitary representations. It is seen from further consideration that the inequality $|b| > 1$ contradicts to the BPS bound.

3. Observables and the physical subspace

Consider in detail the realization of the Poincaré supersymmetry in the extended phase superspace $\mathcal{M}^{8|4} \cong T^*(\mathbb{R}^{1,2}) \times L^{1|2}$. The Poincaré supergroup is realized by a symplectic action leaving the coadjoint orbit (3.7) invariant. The vector superfields generating the transformations (3.3) and (3.4) are related to the corresponding canonical Hamiltonian generators by

$$X_H \lhd \Omega_s^{\text{SUSY}} = -(-1)^{\epsilon_H} dH \, ,$$

where $\epsilon_H$ is the Grassmann parity of the Hamiltonian $H$. Solving these equations one gets the following Hamiltonian generators (we denote the generator of isotopic $\text{U}(1)$

---

3 Explicit form of Eq. (3.11) depends on the grading conventions for the exterior superalgebra. We use $\mathbb{Z} \times \mathbb{Z}_2$ grading by analogy with Ref. [38]. The only difference, as compared to Ref. [38], is in convention for complex conjugation of the odd variables. We take $\bar{\theta}_1 \bar{\theta}_2 = \bar{\theta}_2 \bar{\theta}_1$, in particular $\theta\bar{\theta}$ is a real $c$-number, while in the Ref. [38] it is an imaginary.
rotations by \( P_3 \)
\[
\begin{align*}
\mathcal{P}_a &= p_a \\
\mathcal{J}_a &= \epsilon_{abc} x^b p^c - sn_a + \frac{1}{2} mn_a (\theta^a \theta_a + \chi^a \chi_a - 2i bn_{a\beta} \theta^\alpha \chi^\beta) \\
\mathcal{Q}_a^I &= ip_{a\beta}(\theta^\beta - ibn^\beta \chi^\gamma) + m(in_{a\beta} \theta^\beta + b \chi_a) \\
\mathcal{Q}_a^J &= ip_{a\beta}(\chi^\beta + ibn^\beta \chi^\gamma) + m(in_{a\beta} \chi^\beta - b \theta_a) \\
P_3 &= imn_{a\beta} \theta^\alpha \chi^\beta - \frac{mb}{2} (\theta^a \theta_a + \chi^a \chi_a). 
\end{align*}
\]

(3.13)

With respect to Poisson superbrackets on \( \mathcal{M}^{8|4} \) they generate the following superalgebra
\[
\begin{align*}
\{\mathcal{J}_a, \mathcal{J}_b\} &= \epsilon_{abc} \mathcal{J}^c \\
\{\mathcal{J}_a, \mathcal{P}_b\} &= \epsilon_{abc} \mathcal{P}^c \\
\{\mathcal{J}_a, \mathcal{Q}_a^I\} &= -\frac{i}{2} (\gamma_a)^{\beta} \mathcal{Q}_b^I \\
\{\mathcal{Q}_a^I, \mathcal{P}_3\} &= -\frac{1}{2} \epsilon^{IJ} \mathcal{Q}_a^J \\
\{\mathcal{Q}_a^I, \mathcal{Q}_a^J\} &\approx -2i \delta^{IJ} p_{a\beta} - 2 \epsilon^{IJ} \epsilon_{a\beta} Z \\
Z &= mb,
\end{align*}
\]

(3.14)

the other brackets being equal to zero and \( I, J = 1, 2, \epsilon^{IJ} = -\epsilon^{JI}, \epsilon^{01} = 1 \). We stress that the latter bracket \( \{\mathcal{Q}_a^I, \mathcal{Q}_b^J\} \) is closed only in a weak sense, that is modulo to constraints (3.7). What we have obtained is \( N=2, D=3 \) Poincaré superalgebra with central charge \( \mathcal{Z} = mb \) and isotopic charge \( P_3 \) acting on the internal indices of supercharges \( \mathcal{Q}_a^I \).

One can easily examine that the mass and the spin Casimir functions of the superalgebra (3.14) read \( C_1 \equiv \mathcal{P}^a \mathcal{P}_a = p^2 \) and \( C_2 \equiv \mathcal{P}^a \mathcal{J}_a + \frac{i}{8} \mathcal{Q}^I \mathcal{Q}_a^I - \mathcal{Z} P_3 = -s(p,n) \). On the constraint surface (3.7)
\[
p^2 + m^2 = 0 \quad (p,n) + m = 0
\]

(3.15)

the Casimirs are conserved identically. Eqs. (3.15) and (3.7) are completely equivalent to each other, in other words, they define one and the same surface in the phase superspace \( \mathcal{M}^{8|4} \). We conclude that the mechanical model describes \( N=2, D=3 \) superparticle of mass \( m \), superspin \( s \) and central charge \( mb \).

Regular and degenerate cases are essentially distinguished for the coadjoint orbit, being associated for the superparticle. Since the massless and spinless particles are not covered in our model, the Bogomol'nyi-Prassad-Sommerfield bound of central charge (see, for instance, [40]) assumes the only possibility for the degeneracy. The BPS bound \( m \geq |Z| \) provides, as is known, consistency of the quantum theory; the opposite inequality breaks the unitarity. As we have the goal to construct the quantum theory, we may restrict the consideration to the case of \( |b| \leq 1 \). Furthermore, the limiting point \( |b| = 1 \) corresponds to the multiplet-shortening [10]. It is the case \( m = |Z| \) when the massive multiplet contains the same number of particles as a massless one. These massive multiplets are called hypermultiplets. In the case of \( N=2, D=3 \) Poincaré superalgebra, a massive supermultiplet of superspin \( s \) describes a quartet of particles with spins \( s, s + \frac{1}{2}, s + \frac{1}{2}, s + 1 \) for \( m > |Z| \) and a doublet \( s, s + \frac{1}{2} \) for \( m = |Z| \). The shortening of the superparticle multiplet has the respective origin in the classical...
mechanics: the number of odd physical degrees of freedom of the superparticle halved in the BPS limit. Let us show that it is the case which is described by our model.

Reducing to the constraints (3.13) (or, equivalently, (3.7)) we come to the smaller 5/4-dimensional phase space $\mathcal{M}^{5/4} \subset \mathcal{M}^{8/4}$ with a degenerate symplectic two-superform

$$\Omega_{s}^{\text{SUSY}}|_{p_{a}=m_{a}} \equiv \Omega_{s}^{\text{red}} = -mdx^{a} \wedge dn_{a} + \Omega_{L^{1/2}} \quad dn_{a} \equiv \frac{2\xi_{a}dz + 2\bar{\xi}_{a}d\bar{z}}{(1-z\bar{z})^{2}}, \quad (3.16)$$

where $\Omega_{L^{1/2}}$ is defined by Eq. (3.11) and

$$\xi_{a} = -\frac{1}{2}(\gamma_{a})_{\alpha\beta}z^{\alpha}z^{\beta} = -\frac{1}{2}(2z,1+z^{2},i(z^{2}-1)) \quad \bar{\xi}_{a} = \overline{(\xi_{a})}. \quad (3.17)$$

The kernel of the two-superform (3.16) contains obviously the even one-dimensional null space $\text{Ker} \Omega_{s}^{\text{red}}$, related to the reparametrization invariance of the world lines. In the coset superspace $\mathcal{O}_{m,s,b} = \mathcal{M}^{5/4}/\text{Ker} \Omega_{s}^{\text{red}}$ the induced symplectic two-superform is nondegenerate when $|b| < 1$, the same is true in $L^{1/2}$ for the respective superform $\Omega_{L^{1/2}}$. Therefore, $\mathcal{O}_{m,s,b}$, $\dim \mathcal{O}_{m,s,b} = 4/4$, $|b| < 1$ is isomorphic to a regular coadjoint orbit of $\text{N}=2$, $D=3$ Poincaré supergroup. We have established both the embedding of the regular orbit into the original phase superspace and the underlying projection $\pi: \mathcal{M}^{8/4} \rightarrow \mathcal{O}_{m,s,b}$, provided by constraints (3.13).

In the BPS limit $|b| = 1$ the inner two-superform $\Omega_{L^{1/2}}$ generates 0/2-dimensional null-vector superspace. Thus, the full kernel $\text{Ker} \Omega_{s}^{\text{red}}$ of the symplectic two-superform on $\mathcal{M}^{8/4}$ becomes 1/2-dimensional if $|b| = 1$. The 4/2-dimensional coset superspace $\mathcal{O}_{m,s} = \mathcal{M}^{5/4}/\text{Ker} \Omega_{s}^{\text{red}}$ corresponds to a degenerate orbit of the $\text{N}=2$ Poincaré supergroup. Hence, the number of odd physical degrees of freedom of $\text{N}=2$, $D=3$ superparticle halved actually in the BPS limit and we observe an evident classical analogue of the multiplet-shortening. Some more peculiarities of the BPS limit for the superparticle model will be discussed in Subsec. 6.

We have described the embedding of coadjoint orbits of the $\text{N}=2$ superparticle in the phase superspace $\mathcal{M}^{8/4} \cong T^{\ast}(\mathbb{R}^{1,2}) \times L^{1/2}$. This description should be treated as a natural superextension of the $D=3$ spinning particle model with extended phase space $\mathcal{M}^{8} \cong T^{\ast}(\mathbb{R}^{1,2}) \times L$, where the particle spin is realized in terms of the Lobachevsky plane. One can also construct a different embedding of the superparticle dynamics generalizing the canonical model with the minimal six-dimensional phase space $\mathcal{M}^{6}$. This embedding is obtained by reduction $\pi_{1}: \mathcal{M}^{8/4} \rightarrow \mathcal{M}^{6/4}$ with respect to two second class constraints $p_{a} = \sqrt{-p^{a}p_{a}}$ of (3.7). If the coordinates in $\mathcal{M}^{6/4}$ are chosen to be $(x^{a},p_{a},\theta^{a}t)$, then the induced symplectic two-superform reads as

$$\Omega_{s}^{\text{SUSY}} = dp_{a} \wedge dx^{a} + \left(\frac{s}{2} + imb\right)\frac{p_{\alpha\beta}d\theta^{\alpha} \wedge d\theta^{\beta}}{\sqrt{-p^{2}}} \frac{\epsilon^{abc}p_{a}dp_{b} \wedge dp_{c}}{(-p^{2})^{3/2}} - 2im\epsilon_{\alpha\beta\gamma}d\theta^{\alpha} \wedge d\chi^{\beta} - \frac{im}{\sqrt{-p^{2}}}\left((\gamma_{a})_{\alpha\beta} - \left(b\frac{\epsilon_{abc}p_{c}}{\sqrt{-p^{2}}}ight)^{\alpha}\right)dp^{b} \wedge d\theta^{\beta} - im\frac{p_{\alpha\beta}d\theta^{\alpha} \wedge d\theta^{\beta}}{\sqrt{-p^{2}}}, \quad (3.18)$$
\[-\frac{im}{\sqrt{-p^2}}(\gamma^\alpha)_{\alpha\beta} \left( \Pi_{ab} \chi^\alpha + b \frac{\epsilon_{abc} \Pi^c}{\sqrt{-p^2}} \theta^\alpha \right) dp^b \wedge d\chi^\beta - \frac{im}{\sqrt{-p^2}} p_{\alpha\beta} \chi^\alpha \wedge d\chi^\beta, \]

(3.18)

where \( \Pi_{ab} = \eta_{ab} - p_a p_b / p^2 \). It is nondegenerate again if \(|b| \neq 1\). The superparticle dynamics on \( \mathcal{M}^{6|4} \) is governed by the mass shell constraint \( p^2 + m^2 = 0 \) only and provides the straightforward N=2 supergeneralization of the canonical description of spinning particle. Rel. (3.18) is an N=2 analogue of the monopole Dirac symplectic structure (2.1). It is likely to be interesting to invert the symplectic two-superform on \( \mathcal{M}^{6|4} \) and to represent the analogue of the fundamental Poisson brackets (2.4). The nonvanishing brackets are (we mark the PB’s on \( \mathcal{M}^{6|4} \) by the star)

\[
\begin{align*}
\{x^a, x^b\}^* &= s \frac{\epsilon_{abc} p_c}{(-p^2)^{3/2}} \left[ 1 - \frac{m}{2s} (\theta^\alpha \theta_\alpha + \chi^\alpha \chi_\alpha) + \frac{im p_{\alpha\beta} \theta^\alpha \chi^\beta}{s \sqrt{-p^2}} \right] \\
\{\theta^\alpha I, \theta^\beta J\}^* &= -\frac{1}{2m(1 - b^2)} \left( i\delta^{IJ} \frac{p_{\alpha\beta}}{\sqrt{-p^2}} + b \epsilon_{IJ} \epsilon^{\alpha\beta} \right) \\
\{x^a, p_b\}^* &= \delta^a_b \\
\{x^a, \theta^\alpha I\}^* &= -\frac{i}{2p^2} \epsilon_{abc} p_b (\gamma_c)^\alpha I^\beta \theta^\beta J
\end{align*}
\]

(3.19)

These nonlinear brackets defy the usual attempts of operator realization in a Hilbert space. An efficient alternative to the direct realization is in the use of the extended phase superspace \( \mathcal{M}^{8|4} \cong T^*(\mathbb{R}^{1,2}) \times \mathcal{L}^{1|2} \), which allows more supersymmetry, that affects on the quantization procedure drastically.

4. Hidden su(1,1|2) supersymmetry of the superspin degrees of freedom

We have shown that the superparticle dynamics is embedded in the phase superspace \( \mathcal{M}^{8|4} \cong T^*(\mathbb{R}^{1,2}) \times \mathcal{L}^{1|2} \). One can imply that the inner supermanifold \( \mathcal{L}^{1|2} \) carries internal (both even and odd) degrees of freedom of D=3 particle. Then the symplectomorphisms of \( \mathcal{L}^{1|2} \) should be treated as the hidden supersymmetry of the particle internal structure. Consider this supersymmetry in more detail. To be specific, let us assume that \(|b| < 1\). The degenerate case will be discussed separately in Subsec. 6.

We have already mentioned that \( \mathcal{L}^{1|2} \) is a homogeneous OSp(2|2) superspace. Introducing new odd complex variables (3.10) we established that the symplectic two-superform (3.11) reduces to the superform on the regular OSp(2|2) coadjoint orbit obtained earlier in Refs. [36, 38] in the framework of the supercoherent state technique. A crucial point is that \( \mathcal{L}^{1|2} \) reveals a Kähler supermanifold structure with the superpotential

\[
\Phi = -2s \ln(1 - z\bar{z}) - s(1 + b) \frac{\theta \bar{\theta}}{1 - z\bar{z}} - s(1 - b) \frac{\chi \bar{\chi}}{1 - z\bar{z}} + \frac{s(1 - b^2)}{2} \frac{\theta \bar{\theta} \chi \bar{\chi}}{(1 - z\bar{z})^2},
\]

(3.20)
so that
\[ \Omega_{L^1/2} = \frac{i}{2} \left( dz \frac{\partial}{\partial z} + d\theta \frac{\partial}{\partial \theta} + d\chi \frac{\partial}{\partial \chi} \right) \wedge \left( dz \frac{\partial}{\partial z} + d\bar{\theta} \frac{\partial}{\partial \bar{\theta}} + d\bar{\chi} \frac{\partial}{\partial \bar{\chi}} \right) \Phi, \]

and OSp(2|2) acts on N=2 superunit disc by the superholomorphic transformations. Moreover, the supergroup of the superholomorphic symplectomorphisms of L^{1/2} is in fact essentially larger than OSp(2|2) and it contains at least the supergroup SU(1, 1/2).

The corresponding infinitesimal transformations read
\[ \delta z = i \omega^a \partial_\xi \chi - \frac{\sqrt{1+b}}{2} \epsilon_\alpha z^\alpha \chi - \frac{1}{2} \eta_\alpha z^\alpha \chi \]
\[ \delta \theta = \frac{i}{2} \omega^a \partial_\xi \theta + \frac{i}{2} \sqrt{1-b} \mu_1 \chi - \frac{1}{2} (\mu_2 + \mu_3) \chi - \frac{1}{\sqrt{1+b}} \epsilon_\alpha z^\alpha - \frac{\sqrt{1-b}}{2} \eta_\alpha \partial z^\alpha \theta \chi \]
\[ \delta \chi = \frac{i}{2} \omega^a \partial_\xi \chi - \frac{i}{2} \sqrt{1+b} \mu_1 \theta - \frac{i}{2} (\mu_2 - \mu_3) \chi + \frac{1}{\sqrt{1+b}} \epsilon_\alpha z^\alpha \theta \chi - \frac{1}{1-b} \eta_\alpha \chi, \]

where \( \partial \equiv \partial / \partial z \), even parameters \( \omega^a, \mu_2, \mu_3 \) are real, even parameter \( \mu_1 \) is complex and the odd ones \( \epsilon_\alpha, \eta_\alpha \) are complex. Transformations (3.21) are generated by the following Hamiltonians, which may be obtained straightforwardly solving Eqs. (3.12).

There are seven (real) even Hamiltonians
\[ J_a = -sn_\alpha \left( 1 - \frac{1+b}{2} \frac{\theta \bar{\theta}}{1-zz} \right) \]
\[ P_1 = s \sqrt{1-b^2} \frac{\theta \bar{\theta} - \chi \bar{\chi}}{2} \]
\[ P_2 = is \sqrt{1-b^2} \frac{\theta \bar{\theta} + \chi \bar{\chi}}{2} \]
\[ P_3 = -s \left( \frac{1+b}{2} \frac{\theta \bar{\theta}}{1-zz} - \frac{1-b}{2} \frac{\chi \bar{\chi}}{1-zz} \right) \]
\[ P_4 = -s \left( \frac{1+b}{2} \frac{\theta \bar{\theta}}{1-zz} + \frac{1-b}{2} \frac{\chi \bar{\chi}}{1-zz} - \frac{4}{(1-zz)^2} \right) \]

and eight odd ones
\[ E^\alpha = s \sqrt{1+b} \left( \frac{z^\alpha \bar{\theta} - \bar{z}^\alpha \theta}{1-zz} \right) \left( 1 - \frac{1-b}{2} \frac{\chi \bar{\chi}}{1-zz} \right) \]
\[ F^\alpha = i n_\alpha^\beta E^\beta \]
\[ G^\alpha = s \sqrt{1-b} \left( \frac{z^\alpha \bar{\theta} - \bar{z}^\alpha \theta}{1-zz} \right) \left( 1 - \frac{1+b}{2} \frac{\bar{\theta} \theta}{1-zz} \right) \]
\[ H^\alpha = i n_\alpha^\beta G^\beta. \]

These Hamiltonians, together with one more even element \( Z \equiv s \), generate a closed superalgebra with respect to Poisson superbrackets on L^{1/2} (here \( I, J, K = 1, 2, 3 \)):
\[ \{ J_a, J_b \} = \epsilon_{abc} J^c \]
\[ \{ J_a, E^\alpha \} = \frac{i}{2} (\gamma_\alpha)^{\alpha \beta} E^\beta \]
\[ \{ J_a, F^\alpha \} = \frac{i}{2} (\gamma_\alpha)^{\alpha \beta} F^\beta \]
\[ \{ J_a, G^\alpha \} = \frac{i}{2} (\gamma_\alpha)^{\alpha \beta} G^\beta \]
\[ \{ J_a, H^\alpha \} = \frac{i}{2} (\gamma_\alpha)^{\alpha \beta} H^\beta \]
\[ \{ P_I, P_J \} = -\epsilon_{IJK} P_K \]
\[ \{ J_a, G^\alpha \} = \frac{i}{2} (\gamma_\alpha)^{\alpha \beta} G^\beta \]
\[ \{ J_a, H^\alpha \} = \frac{i}{2} (\gamma_\alpha)^{\alpha \beta} H^\beta \]

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\{E^\alpha, P_1\} = \frac{1}{2} H^\alpha \quad \{E^\alpha, P_2\} = -\frac{1}{2} G^\alpha \quad \{E^\alpha, P_3\} = -\frac{1}{2} F^\alpha \quad \{E^\alpha, P_4\} = -\frac{1}{2} E^\alpha
\{F^\alpha, P_1\} = -\frac{1}{2} G^\alpha \quad \{F^\alpha, P_2\} = -\frac{1}{2} H^\alpha \quad \{F^\alpha, P_3\} = \frac{1}{2} E^\alpha \quad \{F^\alpha, P_4\} = \frac{1}{2} E^\alpha
\{G^\alpha, P_1\} = \frac{1}{2} F^\alpha \quad \{G^\alpha, P_2\} = \frac{1}{2} E^\alpha \quad \{G^\alpha, P_3\} = \frac{1}{2} H^\alpha \quad \{G^\alpha, P_4\} = -\frac{1}{2} H^\alpha
\{H^\alpha, P_1\} = -\frac{1}{2} E^\alpha \quad \{H^\alpha, P_2\} = \frac{1}{2} F^\alpha \quad \{H^\alpha, P_3\} = -\frac{1}{2} G^\alpha \quad \{H^\alpha, P_4\} = \frac{1}{2} G^\alpha
\{E^\alpha, F^\beta\} = \epsilon^{\alpha\beta}(Z - P_3) \quad \{E^\alpha, G^\beta\} = -\epsilon^{\alpha\beta} P_2 \quad \{E^\alpha, H^\beta\} = \epsilon^{\alpha\beta} P_1
\{G^\alpha, H^\beta\} = \epsilon^{\alpha\beta}(Z + P_3) \quad \{F^\alpha, H^\beta\} = -\epsilon^{\alpha\beta} P_2 \quad \{F^\alpha, G^\beta\} = \epsilon^{\alpha\beta} P_1
\{E^\alpha, E^\beta\} = \{F^\alpha, F^\beta\} = \{G^\alpha, G^\beta\} = \{H^\alpha, H^\beta\} = i(\gamma_\alpha)^{\alpha\beta} J^a
\{J_a, P_1\} = 0 \quad \{P_1, P_4\} = 0 \quad \{J_a, P_4\} = 0 \quad \{Z, \text{anything}\} = 0 .

What we have obtained it is the explicit Poisson realization of the so-called su(1,1|2) superalgebra \[41\], whose even part is su(1,1|2)_0 = su(1,1) \oplus u(2) \oplus \mathbb{R} and the odd part constitutes an eight dimensional module of the even part; Z presents a central charge. The osp(2|2) subsuperalgebra found in Refs. \[36, 38\] is spanned by \(J_a, B, \sqrt{m} V^a, \sqrt{m} W^a\), where \(V^a, W^a\) are defined below by Eqs. (3.30) and \(B = P_3 - bZ\). We reveal that \(N = 2\) superunit disc is not only a typical coadjoint orbit of the OSp(2|2) supergroup, \(\mathcal{L}^{1|2} \cong \text{OSp}(2|2)/[U(1) \times U(1)]\), but it can be treated simultaneously as an atypical Kähler orbit of the supergroup SU(1,1|2): \(\mathcal{L}^{1|2} \cong \text{SU}(1,1|2)/[\text{U}(2|2) \times \text{U}(1)]\).

5. Hidden N=4 Poincaré supersymmetry

Subalgebra \(u(2)\) of the internal \(su(1,1|2)\) superalgebra acts on the odd variables, as is seen from Eqs. (3.24). It is exactly the subalgebra of the isotopic symmetry. However, the isotopic \(u(2)\) symmetry may now be involved in the Poincaré supersymmetry. The isotropic rotations together with the N=2 Poincaré transformations (3.3) generate (when \(|b| \neq 1\)) more wide D=3, N=4 Poincaré superalgebra. In addition to (3.3) there are the following supersymmetry transformations:

\[
\begin{align*}
\delta_\varepsilon x^a &= i b c^{\alpha\beta} \tilde{\epsilon}^\alpha \chi^\beta - i e^{abc} n^b \gamma_{a\beta} \tilde{e}^\alpha \theta_\alpha + n^a \tilde{e}^\alpha \theta_\alpha - i n^\alpha \beta \tilde{e}^\beta \\
\delta_\eta x^a &= -i b c^{\alpha\beta} \tilde{\eta}^\alpha \theta^\beta - i e^{abc} n^b \gamma_{a\beta} \tilde{\eta}^\alpha \chi^\beta + n^a \tilde{\eta}^\alpha \chi_\alpha - i n^\alpha \beta \tilde{\eta}^\beta \\
\delta_\varepsilon \chi^a &= -i n^\alpha \beta \tilde{\epsilon}^\beta \chi^\alpha \\
\delta_\eta \chi^a &= -i n^\alpha \beta \tilde{\eta}^\beta \chi^\alpha \\
\delta_\varepsilon z &= 0 \\
\delta_\eta z &= 0,
\end{align*}
\]

(3.24)

where \(\tilde{\epsilon}^\alpha, \tilde{\eta}^\alpha\) are odd infinitesimal parameters. The respective Hamiltonians on \(\mathcal{M}^{8|4}\) read

\[
\begin{align*}
\hat{Q}_{\alpha}^1 &= i p_{\alpha\beta}(n^\gamma \gamma^\theta + b \chi^\beta) - m(\theta_\alpha - i b n_{\alpha\beta} \chi^\beta) \\
\hat{Q}_{\alpha}^2 &= i p_{\alpha\beta}(n^\gamma \gamma^\theta - b \theta^\beta) - m(\chi_\alpha + i b n_{\alpha\beta} \theta^\beta) .
\end{align*}
\]

(3.25)

New supercharges together with \(N=2\) superpoincaré Hamiltonians (3.13) and isotopic U(2) Hamiltonians \(P_I, I = 1, 2, 3, 4\) generate closed N=4 Poincaré superalgebra with one central charge. It can be seen by introducing new basis for super-
The invariance of the original Lagrangian (3.1) under the transformations (3.24) can be examined straightforwardly. Thus, the model, being N=2 superpoincaré invariant by construction, allows the hidden N=4 supersymmetry. The appearance the enhanced supersymmetry is hardly surprising in the model. This N=4 supersymmetry is degenerate in a sense that the corresponding central charges equals to $m$ and, so, they saturate the BPS bound for N=4 Poincaré superalgebra. It reflects the degeneracy of N=4 supersymmetry and the shortening of the N=4 superparticle multiplet to the N=2 supermultiplet in quantum theory. Moreover, it is a general property of extended supersymmetry that some of the degenerate multiplets of a larger SUSY (those which saturate the BPS bound) have the same particle content, as is observed in the respective multiplets of a smaller SUSY. This fact provides a simple reason why some of supersymmetric theories may have the extended supersymmetries. The precedents are known both for D=4,6,10 superparticle models [42] and supersymmetric field theories (for example, the theories with non-trivial topological charge [43]). D=3, N=1 superparticle allows the hidden N=2 SUSY [25].

The degeneracy of the hidden N=4 supersymmetry can be observed already in the classical model. New supercharges $\tilde{Q}_I^\alpha$ are functionally dependent from the N=2 Hamiltonians. On the constraint surface (3.7) we have

$$\{\tilde{R}_I^\alpha, \tilde{R}_J^\beta\} \approx (1 - b)(-i\delta^{IJ}_\alpha p_{\alpha\beta} + m\epsilon^{IJ}_\alpha \epsilon_{\alpha\beta})$$

$$\{\tilde{R}_I^\alpha, \tilde{R}_J^\beta\} \approx (1 + b)(-i\delta^{IJ}_\alpha p_{\alpha\beta} + m\epsilon^{IJ}_\alpha \epsilon_{\alpha\beta})$$

$$\{R_I^\alpha, \tilde{R}_J^\beta\} \approx 0.$$  \hspace{1cm} (3.26)

We conclude that the hidden N=4 supersymmetry can be treated as an artifact of the embedding of N=4 Poincaré superalgebra into the universal enveloping algebra of N=2 one. The transformations (3.24) are, in fact, special linear combinations of the N=2 transformations (3.3) with the coefficients depending from the on shell conserved quantities.

6. Bogomol’ny-Prassad-Sommerfield limit

Let us briefly discuss the special case $|b| = 1$. To make the mentioned degeneracy more evident we introduce for a while new odd variables

$$\bar{\theta}^\alpha = \theta^\alpha - i n^\alpha_\beta \chi^\beta$$

$$\bar{\chi}^\alpha = \chi^\alpha - i n^\alpha_\beta \theta^\beta$$

instead of $\theta^\alpha, \chi^\alpha$. This change of the odd variables is one-to-one, and the original Lagrangian (3.1) reads in new variables as

$$L = m(\dot{x}, n) - im\frac{1 + b}{2} n_{\alpha\beta} \tilde{\theta}^\alpha \tilde{\theta}^\beta - im\frac{1 - b}{2} n_{\alpha\beta} \tilde{\chi}^\alpha \tilde{\chi}^\beta + is \frac{\bar{z} \ddot{z} - z \dot{\bar{z}}}{1 - z \bar{z}}.$$  \hspace{1cm} (3.1a)
It is seen immediately that half of the odd degrees of freedom of the superparticle drops out from the theory in the case of $|b| = 1$. Moreover, in the BPS limit expression (3.1a) reduces to the Lagrangian of $N=1$, $D=3$ superparticle [25] and does describe not a superquartet, but a supersymmetric doublet of particles of equal mass $m$ and spins $s$ and $s + \frac{1}{2}$ only.

The inner symplectic two-superform (3.11) turns out to be a Kähler superform on an atypical $\text{OSp}(2|2)$-coadjoint orbit $L^{1|1}$ of complex dimension $1/1$. Thus the phase superspace $\mathcal{M}^{8|4}$ reduces to $\mathcal{M}^{8|2} \cong T^*(R^{1,2}) \times L^{1|1}$ with internal $\text{OSp}(2|2)$ supersymmetry. The latter is realized by all superholomorphic transformations of $N=1$ superunit disc $L^{1|1}$. The respective model of $N=1$, $D=3$ superparticle on $\mathcal{M}^{8|2}$ was considered in detail in Ref. [25]. In the present paper we give an appropriate $N=2$ extension for the $N=1$ model retaining the hidden supersymmetries.

It is worth noting that the hidden $N=4$ supersymmetry vanishes if $|b| = 1$, whereas $N=2$ supersymmetry of the $N=1$ superparticle could be treated as the hidden one [25]. Almost all the equations of this Section still remain valid in the BPS limit if one takes formally $\tilde{\theta}^\alpha = \theta^\alpha$, $\chi^\alpha = 0$ and $b = 0$.

7. Relationship between Hamiltonian generators of the Poincaré and internal supersymmetries

We have observed that the model contains both the global Poincaré SUSY and the hidden $\text{SU}(1, 1|2)$, the latter is closely related to the superspin intrinsic structure. Thus, the relevant quantization procedure should make a provision for either symmetries to survive in quantum theory. This quantization can be based on a simple fact that the Hamiltonian generators (3.13) and (3.23) of the Poincaré supersymmetries, being the functions on $\mathcal{M}^{8|4} \cong T^*(R^{1,2}) \times L^{1|2}$, can be expressed in terms of the Minkowski-space co-ordinates and momenta $(x^a, p_a)$ and of the $\text{su}(1, 1|2)$ Hamiltonians $J_a, P_I, E^\alpha, F^\alpha, G^\alpha$ and $H^\alpha$ (3.22), which parametrize the coadjoint orbit $L^{1|2}$. We give here the explicit form of these expressions:

$$J_a = \epsilon_{abc}x^b p^c + J_a \quad P_a = p_a \quad Z = mb$$

$$Q^1_\alpha = (ip_{\alpha\beta}W^\beta + m\bar{W}_\alpha)[1 + q^d(bP_3 - \sqrt{1 - b^2}P_2 - P_4)] \quad (3.28)$$
$$Q^2_\alpha = (ip_{\alpha\beta}V^\beta + m\bar{V}_\alpha)[1 + q^d(bP_3 + \sqrt{1 - b^2}P_2 - P_4)]$$

$$\tilde{Q}^1_\alpha = (ip_{\alpha\beta}\bar{W}^\beta - mW_\alpha)[1 + q^d(bP_3 - \sqrt{1 - b^2}P_2 - P_4)]$$
$$\tilde{Q}^2_\alpha = (ip_{\alpha\beta}\bar{V}^\beta - mV_\alpha)[1 + q^d(bP_3 + \sqrt{1 - b^2}P_2 - P_4)]. \quad (3.29)$$

where

$$W^\alpha = \frac{1}{2\sqrt{ms}}(\sqrt{1 + b}E^\alpha + \sqrt{1 - b}H^\alpha) \quad \bar{W}^\alpha = \frac{1}{2\sqrt{ms}}(\sqrt{1 + b}F^\alpha - \sqrt{1 - b}G^\alpha)$$
$$V^\alpha = \frac{1}{2\sqrt{ms}}(\sqrt{1 + b}F^\alpha + \sqrt{1 - b}G^\alpha) \quad \bar{V}^\alpha = \frac{1}{2\sqrt{ms}}(\sqrt{1 - b}H^\alpha - \sqrt{1 + b}E^\alpha) \quad (3.30)$$
and constant $q^d$ reads as

$$q^d = \frac{1}{4s}.$$  \quad (3.31)

To construct an appropriate operator realization of these expressions we shall quantize $(x^a, p_a)$ canonically and extend simultaneously $\text{su}(1, 1|2)$ Hamiltonian vector fields to a representation by Hermitian operators in Hilbert space.

Notice at once some important details in relation to the quantization which should be compatible to the full symmetry of the superparticle. First, expressions (3.28) and (3.29) are essentially \textit{nonlinear} in the generators (3.22) of the inner $\text{su}(1, 1|2)$ superalgebra. Thus, even though the operator realization of the Poisson $\text{su}(1, 1|2)$ superalgebra (3.23) is found and the corresponding operators are substituted in Eq. (3.28), we may not be sure that the representation of the Poincaré superalgebra (neither $N=2$ nor $N=4$) is reproduced for certain in quantum theory. Because of the nonlinearity, the superalgebra of operators, corresponding to (3.28), might be disclosed, and it is the parameter $q$, which controls the possible disclosure of the Poincaré superalgebra. We will see that the parameter $q^d$ should be renormalized in quantum theory to reproduce a representation of the Poincaré supersymmetry.

Second, it is a matter of direct verification that the Hamiltonians $W^\alpha, \tilde{W}^\alpha$ have vanishing Poisson superbrackets with $bP_3 - \sqrt{1 - b^2} P_2 - P_4$, whereas $V^\alpha, \tilde{V}^\alpha$ commute to $bP_3 + \sqrt{1 - b^2} P_2 - P_4$. This point will be important for Hermitian properties of operators in quantum mechanics.

\textbf{IV. FIRST QUANTIZATION OF THE SUPERPARTICLE}

It is a primary objective of previous consideration to present the classical model of $N=2, D=3$ superparticle in the form, well-adapted for a quantizing procedure. We have obtained an embedding of the (maximal) coadjoint orbit of $N=2$ Poincaré supergroup in the extended phase superspace $\mathcal{M}^{8|4} \cong T^*(\mathbb{R}^{1,2}) \times \mathcal{L}^{1|2}$. Going to quantum theory we will combine the canonical Dirac quantization on $T^*(\mathbb{R}^{1,2})$ and the geometric quantization methods on the $\text{SU}(1, 1|2)$ co-orbit $\mathcal{L}^{1|2}$. In particular, a combination of the standard real polarization in $T^*(\mathbb{R}^{1,2})$ and the Kähler one in $\mathcal{L}^{1|2}$ will be used to construct the superparticle’s Hilbert space.

The quantization scheme implies from the outset that the internal $\text{SU}(1, 1|2)$ supersymmetry must survive at the quantum level. Mutual relation between Hamiltonians of $\text{SU}(1, 1|2)$ and Poincaré supersymmetries, being expressed by Eq. (3.28) and (3.29), is crucial in our approach. At first, we construct the operator realization for the Hamiltonians of $\text{su}(1, 1|2)$ superalgebra in the framework of Berezin quantization. Then the expressions (3.28) (possibly, together with (3.29)) are used to obtain the realization of a UIR for the $N=2$ (respectively, enhanced $N=4$) Poincaré superalgebra. We find that the classical meaning of the parameter $q$ (3.31) in the relations (3.28) and (3.29) should be accompanied by certain quantum corrections, referred to as a renormalization, for consistency of the quantum theory.
Eventually we obtain the straightforward N=2 supergeneralization of conventional realization of the unitary irreducible representations (UIR’s) of D=3 Poincaré group on the fields carrying representations of \( \text{SU}(1,2) \) \( [6] \). Two cases should be distinguished among these representations. The fields describing fractional superspin (superanyons) carry an atypical unitary infinite dimensional UIR’s of SU(1,1|2), whereas the UIR’s of (half)integer superspin can be realized on the spin-tensor fields carrying atypical finite dimensional non-unitary representations of SU(1,1|2). The realization of the superparticle Hilbert space is slightly different in these two cases.

1. Berezin quantization on \( \mathcal{L}^{1|2} \).

The Berezin technique \([30, 31, 44, 45]\) provides the perfect quantization method for the Kähler homogeneous spaces. We consider here briefly the application of this method to the supermanifold \( \mathcal{L}^{1|2} \cong \text{OSp}(2|2)/[\text{U}(1) \times \text{U}(1)] \cong \text{SU}(1,1|2)/[\text{U}(2|2) \times \text{U}(1)] \) with the nondegenerate symplectic structure when \( |b| < 1 \). The geometric quantization on \( \mathcal{L}^{1|2} \), being considered as a regular coadjoint orbit, is studied in Refs. \([36, 38]\) in detail. However, as we know, \( \mathcal{L}^{1|2} \) has not been considered as an irregular SU(1,1|2) co-orbit nor as a detailed Berezin quantization and the underlying correspondence principle is not explicitly established.

In the following Subsections we apply the obtained results for quantization of D=3 superparticle.

1.1 Antiholomorphic sections and an inner product

Let us consider the space \( \mathcal{O}_{s,b} \) of superantiholomorphic sections of the superholomorphic line bundle over \( \mathcal{L}^{1|2} \), whose elements are represented by functions

\[
f(\bar{\Gamma}) \equiv f(\bar{z}, \bar{\theta}, \bar{\chi}) = f_0(\bar{z}) + \sqrt{s(1+b)} \bar{\theta} f_1(\bar{z}) + \sqrt{s(1-b)} \bar{\chi} f_2(\bar{z}) + \sqrt{s(s+1/2)(1-b^2)} \bar{\theta} \bar{\chi} f_3(\bar{z}),
\]

where \( f_i(\bar{z}), i = 0, 1, 2, 3 \) are ordinary antiholomorphic functions on the unit disc of the complex plane. We denote by \( \Gamma \equiv \{ \Gamma^A \} = \{ z, \theta, \chi \} \) and \( \bar{\Gamma} \equiv \{ \bar{\Gamma}^A \} = \{ \bar{z}, \bar{\theta}, \bar{\chi} \} \) the sets of the superholomorphic and superantiholomorphic variables respectively. The space \( \mathcal{O}_{s,b} \) is equipped naturally by an inner product

\[
\langle f|g \rangle_{\mathcal{L}^{1|2}} = \int_{\mathcal{L}^{1|2}} \frac{f(\bar{\Gamma})}{\overline{g(\bar{\Gamma})}} e^{-\Phi(\Gamma, \bar{\Gamma})} d\mu(\Gamma, \bar{\Gamma}).
\]

(4.2a)

Here \( \Phi(\Gamma, \bar{\Gamma}) \) is the Kähler superpotential \((3.20)\) and \( d\mu(\Gamma, \bar{\Gamma}) \) is an SU(1,1|2) invariant Liouville supermeasure on \( \mathcal{L}^{1|2} \). Taking into account the definition of the
symplectic two-superform \( \Omega_{\mathcal{L}^{1|2}} \) \( \equiv \left. d\Gamma^A \Omega_{AB} d\Gamma^B \right\rangle \), one can derive the supermeasure explicitly \[38, 31\]

\[
d\mu(\Gamma, \bar{\Gamma}) = -\frac{1}{4\pi} \text{sdet}||\Omega_{AB}|| d\Gamma d\bar{\Gamma} = \frac{d\Gamma d\bar{\Gamma}}{i\pi s(1 - b^2)}, \quad \text{d\Gamma d\bar{\Gamma}} \equiv dz d\bar{z} d\theta d\bar{\theta} d\chi d\bar{\chi}.
\]

\text{(4.3)}

Using Eqs. (3.20), (4.1) and (4.3) we can integrate out the Grassmann variables in Eq. (4.2b), that reduces the inner product to the following form:

\[
\langle f|g \rangle_{\mathcal{L}^{1|2}} = \langle f_0|g_0 \rangle_{\mathcal{L}} + \langle f_1|g_1 \rangle_{\mathcal{L}}^{s+1/2} + \langle f_2|g_2 \rangle_{\mathcal{L}}^{s+1/2} + \langle f_3|g_3 \rangle_{\mathcal{L}}^{s+1}, \quad \text{(4.2b)}
\]

where

\[
\langle \varphi|\chi \rangle_{\mathcal{L}} = (2l - 1) \int_{|z| < 1} \frac{dzd\bar{z}}{2\pi i} (1 - z\bar{z})^{2l-2} \varphi(\bar{z}) \chi(z)
\]

\text{(4.4)}

is an inner product in the representation space \( D^l_+ \) of \( \text{SO}^+(1, 2) \) discrete series bounded below, being realized by antiholomorphic functions in the unit disc \(|z| < 1\). The inner product (4.4) is well defined and positive if \( l > 1/2 \). Moreover, for values \( 0 < l < 1/2 \) one can still use Eq. (4.4) if suitable analytic continuations are made. The case of \( l = 1/2 \) should be understood in the sense of the limit. We conclude that the inner product (4.2) in \( \mathcal{O}_{s,b} \) is well defined if \( s > 0 \) (and, of course, if \(|b| < 1\)).

In view of the transformation law for Kähler superpotential \( \Phi(\Gamma, \bar{\Gamma}) \) under the action of \( SU(1, 1|2) \) supergroup, the inner product (4.2) holds to be \( SU(1, 1|2) \) invariant, if an appropriate transformation law for \( f(\Gamma) \in \mathcal{O}_{s,b} \) is implemented. In other terms, the Hamiltonian action of \( SU(1, 1|2) \) on \( \mathcal{L}^{1|2} \) can be lifted to a unitary representation in \( \mathcal{O}_{s,b} \). We give below an infinitesimal form of this representation only, that is explicit representation of corresponding superalgebra \( \text{su}(1, 1|2) \). To obtain it, we first consider a conventional correspondence between linear operators in \( \mathcal{O}_{s,b} \) and Berezin’s symbols.

\textbf{1.2 Classical observables and operators}

Let \( A(\Gamma, \bar{\Gamma}) \) be a “classical observable”, that means it is a real function on \( \mathcal{L}^{1|2} \) to be continuously differentiable in \( z, \bar{z} \) that the integrals considered below do exist. We associate a linear operator \( \hat{A} \) in \( \mathcal{O}_{s,b} \) to the classical observable \( A(\Gamma, \bar{\Gamma}) \) by the rule

\[
(\hat{A}f)(\bar{\Gamma}) = \int_{\mathcal{L}^{1|2}} A(\Gamma, \bar{\Gamma}) f(\bar{\Gamma}) L_{s,b}(\Gamma, \bar{\Gamma}) e^{-\Phi(\Gamma, \bar{\Gamma})} d\mu(\Gamma, \bar{\Gamma}).
\]

\text{(4.5)}

where \( A(\Gamma, \bar{\Gamma}) \) serves only as an analytic continuation in \( \mathcal{L}^{1|2} \times \mathcal{L}^{1|2} \) for classical observable \( A(\Gamma, \bar{\Gamma}) \). The generating kernel \( L_{s,b}(\Gamma, \bar{\Gamma}) \) can be constructed by the use

\(^4\text{The monomials } \phi^n_{l} = [\Gamma(2l+n)/\Gamma(n-1)\Gamma(2l)]^{1/2}z^n, \ n \text{ is integer non-negative, serve as a standard orthonormal basis in } D^l_+.\)
of an arbitrary complete orthonormal basis \( f_k(\Gamma) \) in \( \mathcal{O}_{s,b} \), and appears to be related immediately to the analytic continuation in \( \mathcal{L}^{1|2} \times \mathcal{L}^{1|2} \) of the Kähler superpotential:

\[
L_{s,b}(\Gamma_1, \bar{\Gamma}) = \sum_{k=1}^{\infty} f_k(\bar{\Gamma}) f_k(\Gamma_1) = (1 - z_1 \bar{z})^{-2s} \left[ 1 - s(1 + b) \frac{\theta \bar{\theta}}{1 - z_1 \bar{z}} - s(1 - b) \frac{\chi \bar{\chi}}{1 - z_1 \bar{z}} \right] \\
+ s(s + 1) \left[ 1 - b^2 \right] \frac{\theta \bar{\theta} \chi \bar{\chi}}{(1 - z_1 \bar{z})^2} = \exp[\Phi(\Gamma_1, \bar{\Gamma})]. \tag{4.6}
\]

The state, being presented by the function \( \Phi_F(\bar{\Gamma}_1) = L_{s,b}(\Gamma, \bar{\Gamma}_1) \) with fixed \( \bar{\Gamma} \equiv \{ \bar{z}, \bar{\theta}, \bar{\chi} \} \) is denoted by \( |\bar{z}, \bar{\theta}, \bar{\chi}\rangle \), is called as an SU(1, 1|2) (or OSp(2|2)) supercoherent state. The analytic continuation in \( \mathcal{L}^{1|2} \times \mathcal{L}^{1|2} \) for any classical observable could be expressed in terms of the supercoherent states as follows

\[
A(\Gamma, \bar{\Gamma}_2) = \frac{\langle \Phi_F | \hat{A} | \Phi_F \rangle_{\mathcal{L}^{1|2}}}{\langle \Phi_F | \Phi_F \rangle_{\mathcal{L}^{1|2}}} . \tag{4.7}
\]

So, the symbol of the unit operator \( \hat{1} \) is just 1. Hence, the one-to-one correspondence between classical observables on \( \mathcal{L}^{1|2} \) and linear operators in \( \mathcal{O}_{s,b} \) is established. In view of Eq. (4.7) classical observables are also referred to as (covariant) Berezin symbols.

### 1.3 Atypical unitary and finite dimensional representations of the su(1, 1|2) superalgebra

Using equation (4.8), one can now obtain the operators which correspond to the Hamiltonian generators (3.22) of holomorphic transformations of N=2 superunit disc.

One gets

\[
\hat{J}_a = -\xi_a \partial - (\bar{\partial} \xi_a) \left( s + \frac{1}{2} \frac{\partial \theta}{\partial \theta} + \frac{1}{2} \bar{\chi} \frac{\partial \bar{\chi}}{\partial \bar{\chi}} \right) \quad \hat{Z} = s \hat{I} \\
\hat{P}_1 = -\frac{1}{\sqrt{1 - b^2}} \left( \frac{1 - b}{2} \frac{\bar{\chi}}{\partial \theta} + \frac{1 + b}{2} \frac{\partial \bar{\chi}}{\partial \bar{\chi}} \right) \\
\hat{P}_2 = \frac{i}{\sqrt{1 - b^2}} \left( \frac{1 - b}{2} \frac{\bar{\chi}}{\partial \theta} - \frac{1 + b}{2} \frac{\partial \bar{\chi}}{\partial \bar{\chi}} \right) \\
\hat{P}_3 = \frac{1}{2} \frac{\partial \theta}{\partial \partial \theta} - \frac{1}{2} \frac{\partial \bar{\chi}}{\partial \bar{\chi}} \\
\hat{P}_4 = \frac{1}{2} \frac{\partial \theta}{\partial \partial \theta} + \frac{1}{2} \frac{\partial \bar{\chi}}{\partial \bar{\chi}} . \tag{4.8}
\]

\[
\hat{E}^\alpha = \frac{\sqrt{1 + b}}{2} \left[ z^\alpha \bar{\partial} + (\bar{\partial} z^\alpha) \left( 2s + \bar{\chi} \frac{\partial}{\partial \bar{\chi}} \right) \right] \\
\hat{F}^\alpha = -i \frac{\sqrt{1 + b}}{2} \left[ z^\alpha \bar{\partial} + (\bar{\partial} z^\alpha) \left( 2s + \bar{\chi} \frac{\partial}{\partial \bar{\chi}} \right) \right] \\
\hat{G}^\alpha = \frac{\sqrt{1 - b}}{2} \left[ z^\alpha \bar{\partial} + (\bar{\partial} z^\alpha) \left( 2s + \bar{\theta} \frac{\partial}{\partial \bar{\theta}} \right) \right] \\
\hat{H}^\alpha = -i \frac{\sqrt{1 - b}}{2} \left[ z^\alpha \bar{\partial} + (\bar{\partial} z^\alpha) \left( 2s + \bar{\theta} \frac{\partial}{\partial \bar{\theta}} \right) \right] .
\]
where $\bar{\partial} \equiv \partial/\partial \bar{z}$, all the derivatives are left and $\bar{\zeta}_\alpha, \bar{z}^\alpha$ are defined by Eqs. (3.17) and (3.8) respectively. It is readily verified that the derived operators generate an irreducible representation of $\text{su}(1|1/2)$ superalgebra, and it is the same case for any values $s$ and $b$, not only for $s > 0$ and $|b| < 1$. The anticommutation relations for operators (4.8) completely correspond to the Poisson superbrackets (3.23), and it is sufficient to apply the correspondence rules, that is, to replace $\{, \} \rightarrow 1/i[\, , ]$ (anticommutator for two odd operators and commutator in the rest cases) in Rel. (3.23). By reduction to the orthosymplectic subsuperalgebra we reproduce just the typical UIR’s of the $\text{osp}(2|2)$ obtained in Refs. [32, 33].

The constructed representation is infinite dimensional for $s > 0$, $|b| < 1$ and unitary in the sense that the operators (4.8) are Hermitian with respect to inner product (4.2). It means, in particular, that $\langle f|\bar{J}_a|g\rangle_{L^{|2}} = \langle g|\bar{J}_a|f\rangle_{L^{|2}}, \langle f|\bar{P}_I|g\rangle_{L^{|2}} = \langle g|\bar{P}_I|f\rangle_{L^{|2}}$ for any $f, g \in \mathcal{O}_{s,b}$. The Hermitian selfconjugation conditions for the odd operators may reveal some subtlety. Any odd classical observable among (3.22) is the Majorana spinor and we have, for example, $\bar{E}^0 = -E^1, \bar{E}^1 = -E^0$ with respect to the reality condition (3.2). $\bar{E}^\alpha$ (and any odd operator with the spinor index) is Hermitian in the sense that $\langle f|\bar{E}^0|g\rangle_{L^{|2}} = -\langle g|\bar{E}^1|f\rangle_{L^{|2}}$.

We denote the UIR obtained by $\mathbf{D}_{s,b}^+$. With respect to the $\text{su}(1,1)$ subalgebra, it is decomposed into the direct sum $D^s_+ \oplus D^{s+1/2}_+ \oplus D^{s+1/2}_- \oplus D^{s+1}_+$ of the unitary representations of discrete series, and the components $f_0, f_{1,2}, f_3$ of the state (4.1) transform by the representations of higher weights $s, s + 1/2$ and $s + 1$ respectively.

The representations being obtained for $s \leq 0$ or $|b| > 1$ are non-unitary. The case of $s + 1 = -j$, $j$ is non-negative integer or half integer, is special. Then the operators (4.8) generate a finite dimensional representation $\mathbf{D}^j$ of dimension $8j + 8$. It is a superquartet of finite dimensional representations of $\text{su}(1,1)$, $\mathbf{D}^j = D^{j+1} \oplus D^{j+1/2} \oplus D^{j+1/2} \oplus D^j$, and the state’s components $f_0, f_{1,2}, f_3$ transform by the $2j + 3, 2j + 2$ and $2j + 1$ dimensional representations respectively.

It should be mentioned that the representations of the $\text{su}(1,1|2)$ being considered here correspond to an irregular coadjoint orbit $\mathcal{L}^{|1,2}$ of the supergroup $\text{SU}(1,1|2)$ and, hence, they are atypical representations. By reduction to the orthosymplectic subsuperalgebra we get just the typical representations of the $\text{osp}(2|2)$.

Keeping in mind the spinning superparticle we remember that the representations $D^s_+$ of the universal covering $\text{SO}^+(1,2)$ are commonly used for conventional realizations of the UIR’s of $D=3$ Poincaré symmetry of fractional spin [3, 4, 13], whereas the finite dimensional irreps $D^j$ serve the ones of integer or of half integer spin. It will be natural to extend these realizations to $N=2$ Poincaré supersymmetry by means of representations $\mathbf{D}_{s,b}^+$ and $\mathbf{D}^j$ of the inner $\text{su}(1,1|2)$ superalgebra.
1.4 The correspondence principle

To complete the quantization procedure on \( \mathcal{L}^{1|2} \) let us return to the relation between observables and linear operators. We have examined for the supersymmetry generators of su(1,1|2) superalgebra that there is an exact correspondence between supercommutators of the operators in \( O_{s,b} \) and the Poisson superbrackets of respective classical observables. In this sense we do have “the quantization” of a classical mechanics on the N=2 superunit disc. Consider now the correspondence between the algebras of arbitrary linear operators in the Hilbert space and their symbols.

The problem we concern with is thoroughly studied for Kähler homogeneous manifolds. Berezin proved the general “correspondence principle” [30, 31], which roughly consists in the following. The multiplication of operators induces a binary \(*\)-operation for corresponding symbols; \(*\)-multiplication is noncommutative. Furthermore, the theory contains a “Planck constant” \( h \) related to one of the quantum numbers, and in the limit when \( h \to 0 \) \(*\)-algebra transforms to the ordinary commutative algebra of functions on the manifold. Finally, the first order reset with respect to \( h \) of the commutator of symbols coincides with their Poisson bracket. The Lobachevsky plane has originally served as a test example for the Berezin technique [31]. The parameter \( s^{-1} \) plays the role of the Planck constant.

Similar principles hold true for N=2 superunit disc being a natural N=2 superextension of the Lobachevsky plane.

Let \( \hat{A}_1, \hat{A}_2 \) be two linear operators in \( O_{s,b} \) and \( A_1(\Gamma, \bar{\Gamma}) \), \( A_2(\Gamma, \bar{\Gamma}) \) being the respective Berezin covariant symbols. It follows from Eq. (4.5) that the symbol being corresponded to the product \( \hat{A}_2 \cdot \hat{A}_1 \) (and denoted by \( A_2 * A_1 \)) reads

\[
A_2 * A_1(\Gamma, \bar{\Gamma}) = \int_{\mathcal{L}^{1|2}} A_2(\Gamma_1, \bar{\Gamma}_1) A_1(\Gamma, \bar{\Gamma}_1) \frac{L_{s,b}(\Gamma, \bar{\Gamma}) L_{s,b}(\Gamma_1, \bar{\Gamma}_1)}{L_{s,b}(\Gamma_1, \bar{\Gamma}_1) L_{s,b}(\Gamma, \bar{\Gamma})} d\mu(\Gamma_1, \bar{\Gamma}_1). \quad (4.9)
\]

Hence the multiplication of the operators induces the \(*\)-multiplications of the symbols.

**Theorem** (the correspondence principle): The following estimations take place:

1) \( \lim_{s \to \infty} A_2 * A_1(\Gamma, \bar{\Gamma}) = A_2(\Gamma, \bar{\Gamma}) \cdot A_1(\Gamma, \bar{\Gamma}) \)

2) \( \lim_{s \to \infty} s \left( A_2 * A_1(\Gamma, \bar{\Gamma}) - A_1 * A_2(\Gamma, \bar{\Gamma}) \right) = i s \{ A_2, A_1 \} , \)

where \( \{ , \} \) is the Poisson superbracket on \( \mathcal{L}^{1|2} \).

To examine the correspondence principle we need for the explicit form of the fundamental Poisson superbrackets \( \Omega^{AB} = \{ \Gamma^A, \Gamma^B \} \). They are derived accounting for the condition \( \Omega^{AB} \Omega_{BC} = \delta^A_C \), where \( \Omega_{AB} \) is the supermatrix of the symplectic two-superform (3.11), \( \Omega_{\mathcal{L}^{1|2}} \equiv d \Gamma^A \Omega_{AB} d \Gamma^B \). A slightly cumbersome calculation leads to

\[
\{ z, \bar{z} \} = -\frac{i}{2s} (1 - z \bar{z})^2 \left( 1 + \frac{1 + b}{2} \frac{\theta \bar{\theta}}{1 - z \bar{z}} + \frac{1 - b}{2} \frac{\chi \bar{\chi}}{1 - z \bar{z}} \right)
\]
where \( \theta^l \equiv (\theta, \chi) \). It is seen, in particular, that the r.h.s. of the fundamental Poisson superbrackets contains the order \( s^{-1} \).

**Proof:** It is based on the asymptotic estimation

\[
A_2 \ast A_1(\Gamma, \bar{\Gamma}) = A_2(\Gamma, \bar{\Gamma}) \cdot A_1(\Gamma, \bar{\Gamma}) + iA_2(\Gamma, \bar{\Gamma}) \frac{\partial}{\partial \Gamma^A} \Omega^{AB} \frac{\partial}{\partial \Gamma^B} A_1(\Gamma, \bar{\Gamma}) + \mathcal{O}(s^{-2}) ,
\]

from which both propositions of the theorem are easily obtained. The validity of the latter relation is sufficient to prove when \( z = 0 \). If it is the case, Eqs. (4.11) hold true at any \( z \) in consequence of the SU(1, 1) invariance of the symplectic structure. Taking this fact into account the verification of Eq. (4.11) is made by means of an ordinary expansion of the symbols in (finite) series in the odd variables and the comparison of l.h.s. and r.h.s. of Eqs. (4.11) for the respective components. It is a trivial but cumbersome exercise, which may be successfully performed using the known estimation [31]

\[
\hat{T}_l[\varphi] \equiv \frac{2l - 1}{2\pi i} \int_{|z|<1} \varphi(z, \bar{z})(1 - z\bar{z})^{2l-2}dzd\bar{z} = \varphi(0, 0) + \frac{1}{2l} \Delta \varphi(z, \bar{z})|_{z=\bar{z}=0} + \mathcal{O}(l^{-2})
\]

and its consequence

\[
l(\hat{T}_l[\varphi] - \hat{T}_{l+1/2}[\varphi]) = \frac{1}{4l} \Delta \varphi(z, \bar{z})|_{z=\bar{z}=0} + \mathcal{O}(l^{-2}) .
\]

Here \( l > 1/2 \), \( \varphi(z, \bar{z}) \) is an arbitrary function to be continuously differentiable into the unit disc in a complex plane, and \( \Delta = (1 - z\bar{z})^2\partial\bar{\partial} \) is an invariant Laplace-Beltrami operator in \( \mathcal{L} \). It is exactly the estimation (4.12), which has been originally applied by Berezin for the proof of the correspondence principle in the Lobachevsky plane [31]. In this sense, we reduce the correspondence principle in \( \mathcal{L}^{1/2} \) to the one in \( \mathcal{L} \) by means of the expansion in the odd variables. ■
2. Operator realization of the Poincaré superalgebra.

Renormalization of the supercharges

Now we are in a position to proceed directly to the quantization of D=3 spinning superparticle. Consider the space $\mathcal{H}$ of functions of the form

$$F(p, \bar{\Gamma}) \equiv F(p, \bar{z}, \bar{\theta}, \bar{\chi}) = F_0(p, \bar{z}) + \sqrt{s(1+b)} \bar{\theta} F_1(p, \bar{z}) + \sqrt{s(1-b)} \bar{\chi} F_2(p, \bar{z}) + \sqrt{s(s+1/2)(1-b^2)} \bar{\theta} \bar{\chi} F_3(p, \bar{z}), \quad (4.13)$$

where $p \equiv p^a \in \mathbb{R}^{1,2}$, and $F_0(p, \bar{\Gamma}) \in \mathcal{O}_{s,b}$ at each fixed $p$. We would like to suppose that the Hamiltonians (3.28) (which are the same as in Rel. (3.13)) present “the classical symbols” of respective operators of the $N=2$ Poincaré superalgebra acting in $\mathcal{H}$. We take the following ansatz for these operators

$$\hat{J}_a = -i\epsilon_{abc} p^b \frac{\partial}{\partial p^c} + \hat{j}_a \quad \hat{P}_a = p_a \quad \hat{Z} = mb$$

$$\hat{Q}^1_\alpha = (ip_{\alpha\beta} \hat{W}^\beta + m\hat{W}_\alpha)[1 + q(b\hat{P}_3 - \sqrt{1-b^2}\hat{P}_2 - \hat{P}_4)]$$

$$\hat{Q}^2_\alpha = (ip_{\alpha\beta} \hat{V}^\beta + m\hat{V}_\alpha)[1 + q(b\hat{P}_3 + \sqrt{1-b^2}\hat{P}_2 - \hat{P}_4)]. \quad (4.14)$$

Here the operators $\hat{W}^\alpha, \hat{V}^\alpha$ are expressed as linear combinations of $\hat{E}^\alpha, \hat{F}^\alpha, \hat{G}^\alpha, \hat{H}^\alpha$ according to relations (3.30), whereas the latter, together with the operators $\hat{J}_a$ and $\hat{P}_I$, are defined by the expressions (4.8).

Recall that the classical observables (3.28) or (3.13) generate the Poincaré superalgebra on shell only, that is modulo to the constraints (3.15). The operator counterparts of the constraints are now imposed to annihilate the physical states according to Dirac quantization prescriptions. The linear operator in $\mathcal{O}_{s,b}$, which corresponds to the Berezin covariant symbol $-sn_a$, $(s > 0)$ reads

$$-sn_a = \hat{j}_a \left(1 - \frac{2}{2s+1}\hat{P}_4 + \frac{2}{(2s+1)(2s+2)} \bar{\theta} \frac{\partial}{\partial \bar{\theta}} \bar{\chi} \frac{\partial}{\partial \bar{\chi}} \right) = \hat{j}_a \left(1 + \frac{1}{s}\hat{P}_4 \right)^{-1}. \quad (4.15)$$

Thus, the wave equations for the superparticle are easily brought to the form

$$(p^2 + m^2) F^{\text{phys}}(p, \bar{z}, \bar{\theta}, \bar{\chi}) = 0$$

$$[(p, \hat{J}) - m\hat{P}_4 - ms] F^{\text{phys}}(p, \bar{z}, \bar{\theta}, \bar{\chi}) = 0. \quad (4.15)$$

Solutions of the wave equations generate a subspace $\mathcal{H}_{m,s,b}$ in $\mathcal{H}$. Furthermore, if $F \in \mathcal{H}_{m,s,b}$ and $\hat{S}$ is any one of the operators (4.14), then $\hat{S}F$ is a physical state again, regardless of the particular value of the parameter $q$. In this sense, the wave equations are superpoincaré invariant.

\footnote{It is worth noting that the second constraint equation (3.13) should be written in an equivalent form $(p, J) - mP_4 - ms = 0$.}
It is crucial now to examine explicitly whether the operators (4.14) actually generate the N=2 Poincaré superalgebra. One gets in \( \mathcal{H} \) (compare with Eq. (3.13))
\[
\hat{J}^a, \hat{J}^b] = i\epsilon^{abc} \hat{J}^c \\
\hat{J}^a, \hat{Q}^I_\alpha] = \frac{1}{2}(\gamma^a)_{\alpha\beta} \hat{Q}^I_\beta \\
\hat{P}^a, \hat{P}^b] = 0 \\
\hat{Q}^I_\alpha, \hat{P}_3] = -\frac{i}{2} \epsilon^{IJK} \hat{Q}^J_\alpha. \tag{4.16}
\]
These relations hold true with an arbitrary number taken for \( q \). However, the anticommutator of the supercharges is strongly dependent on the particular value of \( q \):
\[
\hat{Q}^I_\alpha, \hat{Q}^J_\beta] = 2\delta^{IJ} \rho_{\alpha\beta} - 2imbe^{IJ} \epsilon_{\alpha\beta} + c_{1IJ}^\alpha(p^2 + m^2) + c_{2IJ}^\alpha((p, \hat{J}) - m\hat{P}_4 - ms) + \mathcal{O}(s^{-2}), \tag{4.17}
\]
where \( c_{IJ}^{\alpha\beta} \) are some functions, and \( \mathcal{O}(s^{-2}) \) are the corrections of higher orders in \( s^{-1} \), which depend on \( q \) and the other parameters like \( m \) and \( b \). These corrections do not vanish if \( q = q^{cl} = 1/4s \).

Eq. (4.17) is presented in more detailed notation in the Appendix (Eqs. (A.5)) from the view of the closing of the operator superalgebra.

Notice that the quantum value of \( q \) is not uniquely determined. The value \( q^{cl} = 1/4s \) is derived from the expressions (3.28), when the relationship between the superpoincaré and su(1,1|2) generators (3.22) is taken into account. However, one can start immediately from the symbols (3.13) and restore the operators applying the correspondence rule (4.5). What is remarkable is that one obtains the same operators (4.14), but the parameter \( q \) changes, and \( q^{cl} \) appears to be \( q^{cl} = 1/(4s + 2) \). But the Poincaré superalgebra is disclosed by \( q = 1/(4s + 2) \); the same is true for \( q = 1/4s \) too.

Both the appearance of corrections \( \mathcal{O}(s^{-2}) \) in r.h.s. of Eq. (4.17) and the ambiguity in the definition of \( q \) have the same origin. That is, a nonlinearity of the Poincaré supercharge operators (4.14) in the generators (4.18) of the inner su(1,1|2) superalgebra. In consequence of the nonlinearity, different operator factor orderings may lead to the different forms for \( \hat{Q}^I_\alpha \), and the corrections appear in response to the correspondence principle in \( \mathcal{L}^{1|2} \).

We show that the disclosure of the Poincaré superalgebra at the quantum level has transparent mathematical ground in view of the Berezin correspondence principle. However, this disclosure is quite unsatisfactory from the physical viewpoint for the quantization of the elementary system. The latter is completely characterized by its inherent symmetries (in the present case it is the D=3 Poincaré SUSY). It is the representation of these symmetries in Hilbert space, which allows to identify the obtained quantum theory with the quantized elementary system. According to these reasons, to quantize D=3 superparticle we now have to provide an exact realization of the representation of the Poincaré superalgebra in the physical Hilbert space, without any corrections in the parameters of the model. To find the true quantum realization for the representation, we can try, starting from Eqs. (4.14) – (4.17), to introduce some renormalized terms in the observables (4.14), which should be sufficient for the closure of the anticommutators (4.17).
Certainly, we don’t have an a priori reason, which may ensure the consistency of the renormalization procedure; a structure of the possible higher order corrections to (4.14) is unclear in general. Surprisingly, the exact corrections may be obtained from the simplest ansatz (4.14) for the quantum observables. In other words, a true ordering exists for the \( \text{su}(1|2) \) superalgebra operators, entering in \( \hat{Q}_I^\alpha \) in Eq. (4.14), that allows to restore a representation of the Poincaré superalgebra by the renormalization of the only parameter \( q \).

It is examined by a direct calculation that the corrections \( \mathcal{O}(s^{-2}) \) in r.h.s. of Eqs. (4.17) vanish and the operators (4.14) generate the closed Poincaré superalgebra if, and only if

\[
q = q^{\text{quant}}_+ = 1 \mp \sqrt{1 - \frac{1}{2s + 1}}.
\]

Some details of calculations of the anticommutators (4.17) are given in the Appendix. The renormalized value \( q^{\text{quant}}_+ \) can be treated as a perturbative correction to the classical symbols of the supercharges. The other possible value \( q^{\text{quant}}_- \) emerges from the hidden \( \mathcal{N}=4 \) supersymmetry and could be understood from the following reasons.

Let \( q = q_- \). The operators of supercharges corresponding to the classical observables (3.29) (see also Eqs. (3.27), (3.25)) and providing the hidden \( \mathcal{N}=4 \) supersymmetry in \( H_{m,s,b} \) are presented by

\[
\hat{Q}_I^\alpha = -i m \rho_\alpha^\beta \hat{Q}_\beta^I = i \hat{K} \cdot \hat{Q}^I_{\alpha} \big|_{q \rightarrow 2 - q},
\]

where the parity operator \( \hat{K} \) is introduced. It acts on the components of the wave function (4.13) by the rule

\[
\hat{K} : F = (F_0, F_1, F_2, F_3) \rightarrow \hat{K} F = (F_0, -F_1, -F_2, F_3) \quad \hat{K}^2 = \hat{I},
\]

and \( \hat{Q}^I_{\alpha} \big|_{q \rightarrow 2 - q} \) denote the supercharges (4.14) being considered when the constant \( q \) is substituted for the expression \( \frac{1}{2 - q} \).

The same critical values (4.18) evidently provide the closure of the \( \mathcal{N}=4 \) Poincaré superalgebra. Moreover, the parity operator (4.20) possesses remarkable features: it commutes with the even generators of the \( \mathcal{N}=4 \) Poincaré superalgebra and anticommutates with the supercharges:

\[
[\tilde{J}_a, \hat{K}] = [\hat{P}_a, \hat{K}] = 0 \quad (I = 1, 2, 3, 4)
\]

\[
[\hat{Q}^I_{\alpha}, \hat{K}] = [\hat{Q}^I_{\alpha}, \hat{K}] = 0 \quad I = 1, 2.
\]

Therefore, the operators \( \hat{Q}^I_{\alpha} = -i \hat{K} \hat{Q}^I_{\alpha} = \hat{Q}^I_{\alpha} \big|_{q \rightarrow 2 - q} \) and \( \hat{Q}^I_{\alpha} = -i \hat{K} \hat{Q}^I_{\alpha} = \hat{Q}^I_{\alpha} \big|_{q \rightarrow 2 - q} \) satisfy the (anti)commutation relations being identical with Eqs. (4.16), (4.17) for the supercharges \( \hat{Q}^I_{\alpha} \) and \( \hat{Q}^I_{\alpha} \) themselves. This observation clarifies to some extent
the origin of the nonperturbative value \( q_{\text{quant}}^+ \) for the parameter \( q \). Notice that two representations of the Poincaré superalgebra corresponding to either possible value \( q \) are equivalent to each other. It is seen straightforwardly from relations

\[
\hat{Q}'_\alpha = \hat{U} \hat{Q}'_\alpha \hat{U}, \quad \hat{Q}''_\alpha = \hat{U} \hat{Q}''_\alpha \hat{U}, \quad [\hat{J}_a, \hat{U}]_+ = [\hat{P}_a, \hat{U}]_- = [\hat{P}_1, \hat{U}]_- = 0,
\]

where the operator \( \hat{U} \) reads

\[
\hat{U} = 1 - 2\tilde{\theta} \frac{\partial}{\partial \theta} \hat{\chi} = \hat{I}.
\]

We don’t observe, however, either any classical counterpart for the supercharges \( \hat{Q}'_\alpha, \hat{Q}''_\alpha \) or any algebraic construction (for instance, superalgebra) involving both sets of the \( N=4 \) supercharges \( \hat{Q}_I^\alpha, \hat{\tilde{Q}}_I^\alpha \) and \( \hat{Q}'_\alpha, \hat{\tilde{Q}}''_\alpha \) on equal footing.

To summarize briefly, the “double” SUSY of the classical mechanics of \( D=3 \) spinning superparticle can be lifted to the operator representation in the quantum theory. The key step of construction is the renormalization (4.18) for the Poincaré supercharges (4.16). Eq. (4.18) displays two exceptional values of the parameter \( q \) providing the closure for the anticommutator of supercharges (4.17) and recovering the consistent representation of the Poincaré superalgebra. We will suppose below that the parameter \( q \) is equal to either of two \( q_{\text{quant}}^\pm \). We will show below, that the representation space \( \mathcal{H}_{m,s,b} \) is endowed by a natural Hilbert space structure, thus the operators (4.14) and (4.19) of the Poincaré representation become Hermitian. To be specific, we will consider explicitly the operators (4.14) only, which give a representation of \( N=2 \) Poincaré superalgebra. The appearance of the hidden \( N=4 \) supersymmetry, being presented by additional supercharges (4.19), becomes thereby obvious.

### 3. Hilbert space for \( N=2 \) superanyon

For an arbitrary \( s > 0 \) and \( |b| < 1 \) the physical space \( \mathcal{H}_{m,s,b} \) is naturally endowed by an inner product

\[
(F|G) = \mathcal{N} \int \frac{dp}{p^0} (F|G)_{\mathcal{L}^{1/2}}, \quad p^0 = \sqrt{p^2 + m^2} > 0, \quad (4.22)
\]

where \( (F|G)_{\mathcal{L}^{1/2}} \) denotes the inner product (4.2) in \( \mathcal{O}_{s,b}, p^0 = (p^0, \mathbf{p}) \) and \( \mathcal{N} \) is an arbitrary normalization constant. The operators (4.14) of the \( N=2 \) Poincaré superalgebra are Hermitian with respect to the introduced inner product. This fact follows immediately from that the Hermitian property for the operators of the inner \( \text{su}(1,1|2) \) superalgebra with respect to \( \langle \cdot | \cdot \rangle_{\mathcal{L}^{1/2}} \) and

\[
[S^I_\pm, b\hat{P}_3 \pm \sqrt{1-b^2} \hat{P}_2 - \hat{P}_4]_- = 0 \quad S^I_+ \equiv (\hat{V}_\alpha, \tilde{\hat{V}}_\alpha) \quad S^I_- \equiv (\hat{W}_\alpha, \tilde{\hat{W}}_\alpha)
\]
Thus, we have constructed a unitary representation of \( N=2, D=3 \) superalgebra in the Hilbert space \( \mathcal{H}_{m,s,b} \). In terms of the expansion (4.13), the wave equations (4.15) for components reduce to

\[
(p^2 + m^2)F_{i}^{\text{phys}}(p, \bar{z}) = 0 \quad \text{and} \quad [(p, \hat{J}^l) - ml]F_{i}^{\text{phys}}(p, \bar{z}) = 0,
\]

where \( i = 0, 1, 2, 3; \ l = s \) for \( i = 0, \ l = s + 1/2 \) for \( i = 1, 2, \ l = s + 1 \) for \( i = 3 \) and \( \hat{J}_a^l = -\xi_a\partial - l(\partial\xi_a); \ \xi_a \) is defined by Eq. (3.17). It is well known [6, 13, 25] that the solutions of equations (4.23) generate the physical Hilbert space of \( D=3 \) particle of mass \( m \), arbitrary fixed fractional spin \( l > 0 \) and positive energy \( p^0 > 0 \).

In this realization the fields \( F_{i}(p, \bar{z}) \) carry an infinite dimensional UIR's \( D_{s}^{l+} \) of the group \( SO(1,2) \).

Now, we may conclude that the supersymmetric theory describes the irreducible superquartet of anyons of mass \( m \), superspin \( s \) and central charge \( |b| < 1 \). The corresponding representation is realized on the fields carrying the atypical UIR \( D_{s,b}^{l_{+}} \) of the superalgebra \( su(1,1|2) \) (or the typical one of \( osp(2|2) \)).

It is worth recalling that the Poincaré superalgebra admits unitary representations, when \( p^0 > 0 \) and the central charge satisfies the Bogomol'nyi-Prassad-Sommerfield bound \( m \geq |Z| \) (that is \( |b| \leq 1 \) in our case) [40]. In our realization, the wave equations (4.15), (4.23) admit, in fact, the positive energy solutions only, and the Hermitian conditions for operators (4.8) and (4.14) are broken when \( |b| > 1 \). In addition we observe a shortening of the massive \( N=2 \) supermultiplet to a hypermultiplet in the BPS limit \( |b| = 1 \); the latter case will be briefly discussed in the last subsection.

4. Hilbert space for \( N=2 \) superparticle of (half)integer superspin

In contrast to the anyon case, the ordinary states carrying (half)integer spin have conventional realization in terms of the finite component spin-tensor fields in the Minkowski space. We have mentioned above that the finite dimensional representations of the superalgebra \( su(1,1|2) \) may emerge at \( s + 1 = j, 2j = 0, 1, 2, \ldots \). These representations are non-unitary. In particular, the inner product (4.2) (and, thus, (4.22)) has the inherent singularity at \( |z| = 1 \), and the consideration of the previous subsection becomes inadequate in this case. We construct here a correct realization of the Hilbert space in the case of (half)integer (super)spin, which enables to reproduce the conventional description in terms of the spin-tensor fields.

Let us start from the spinning particle without supersymmetry living classically in the phase space \( \mathcal{M}^8 = T^*(\mathbb{R}^{1,2}) \times \mathcal{L} \), that is, the model of Sec. II. The Hilbert space of the particle of (half)integer spin \( j > 0 \) can be realized in the space \( \mathcal{H}_j \) of the fields on \( \mathcal{M}^8 \) of the following form:

\[
F(p, \bar{z}) = F_{\alpha_1 \alpha_2 \ldots \alpha_{2j}}(p) z^{\alpha_1} \bar{z}^{\alpha_2} \ldots z^{\alpha_{2j}} \quad \alpha_k = 0, 1,
\]
where the coefficients $F_{\alpha_1\alpha_2...\alpha_j}(p)$ are totally symmetric in their indices. $F(p, \tilde{z})$ appears to be polynomial of the degree $2j$ in the variable $\tilde{z}$.

The following consideration is based on remarkable transformation properties of the twistorlike objects $z^\alpha$ and $\tilde{z}^\alpha$ defined by Eqs. (3.8). The Lorentz group $\text{SO}^\uparrow(1, 2) \cong \text{SU}(1, 1)/\mathbb{Z}_2$ acts on $\mathcal{L}$ by fractional linear transformations

$$N : \ z \to z' = \frac{az - b}{\bar{a} - \bar{b}z}, \quad ||N_\alpha^\beta|| = \left( \begin{array}{cc} \frac{a}{b} & \bar{b} \\ \bar{a} & \frac{a}{b} \end{array} \right) \in SU(1, 1), \quad (4.25)$$

which may be rewritten identically as

$$N : \ z^\alpha \to \tilde{z}^\alpha = \left( \frac{\partial \tilde{z}'}{\partial z} \right)^{1/2} N^{-1}_{\beta}^\alpha z^\beta, \quad \tilde{z}^\alpha \to \tilde{z}^\alpha = \left( \frac{\partial \tilde{z}'}{\partial \bar{z}} \right)^{1/2} N^{-1}_{\beta}^\alpha \bar{z}^\beta, \quad (4.26)$$

or, in the infinitesimal form,

$$\delta z = \frac{i}{2} \omega_{\alpha\beta} z^\alpha \tilde{z}^\beta, \quad \delta \tilde{z} = -\frac{i}{2} \omega_{\alpha\beta} \tilde{z}^\alpha z^\beta,$$

where $\omega_{\alpha\beta} \equiv (\omega^\gamma_\alpha \gamma_\alpha)_{\alpha\beta}$ are the parameters of infinitesimal Lorentz transformations. As is seen, each of $z^\alpha$, $\tilde{z}^\alpha$ transforms simultaneously as a D=3 Lorentz spinor and as field density in $\mathcal{L}$. They are, in fact, the only independent twistorlike fields associated with the homogeneous space structure on the Lobachevsky plane. We have in particular

$$n^{\alpha\beta} = n^{\alpha} \gamma_\alpha = -\frac{z^\alpha \tilde{z}^\beta + \tilde{z}^\beta z^\alpha}{(\epsilon_\gamma \tilde{z}^\gamma \tilde{z}^\delta)} \Omega_{\mathcal{L}} = -2iz \frac{dz \wedge d\bar{z}}{(\epsilon_\gamma \tilde{z}^\gamma \tilde{z}^\delta)^2}, \quad (4.27)$$

for the unit timelike Lorentz vector $n^a$ (2.7) and the Kähler two-form (2.4) in the Lobachevsky plane.

Let us suppose the coefficients $F_{\alpha_1\alpha_2...\alpha_j}(p)$ in Eq. (4.24) to be Lorentz spin-tensor field of the type $j$. Then $F(p, \tilde{z})$ possesses the following transformation law with respect to the action of Lorentz group

$$N : \ F(p, \tilde{z}) \to F'(p, \tilde{z'}) = \left( \frac{\partial \tilde{z}'}{\partial \bar{z}} \right)^j F(p, \tilde{z}).$$

We have a standard realization of the finite dimensional representation $D_j$ [11, 14, 15]. Extending this construction to the representation of the Poincaré group in $\mathcal{H}_j$ one takes the following transformation law of the fields

$$F(p, \tilde{z}) \to F'(p', \tilde{z'}) = \left( \frac{\partial \tilde{z}'}{\partial \bar{z}} \right)^j F(p, \tilde{z}). \quad (4.28)$$

The Poincaré generators read

$$\hat{\mathcal{P}}_a = p_a, \quad \hat{\mathcal{J}}_a = -i \epsilon_{abc} p_b \frac{\partial}{\partial p_c} + \hat{J}_a^i, \quad (4.29)$$
where

\[ \hat{J}_a^j = -\bar{\xi}_a \partial_j + j(\partial \bar{\xi}_a) \]  

and we recall again that \( \bar{\xi}_a \) is defined by Eq. (3.17). Now, to separate a subspace of irreducible representation of the Poincaré group from \( \mathcal{H}_j \) it is sufficient to impose the operator counterparts of the constraints (2.11) to annihilate the physical states:

\[ (p^2 + m^2)F(p, \bar{z}) = 0 \quad [(p, \hat{J}_j) + mj]F(p, \bar{z}) = 0 \]  

The space \( \mathcal{H}_{m,j} \) of solutions of the wave equations (4.31) generates the irreducible one- or two-valued massive representation of ISO\(^{1}(1,2)\). Moreover, using the identity \( \bar{z}^\alpha \partial_j \bar{z}^\beta - \bar{z}^\beta \partial_j \bar{z}^\alpha = c^{\alpha\beta} \) one can verify that the irreducibility conditions are equivalent to the following set of equations for Lorentz spin-tensors:

\[ p_{\alpha_1 \beta} F_{\beta \alpha_2 ... \alpha_{2j}}(p) = mF_{\alpha_1 \alpha_2 ... \alpha_{2j}}(p) . \]  

On-shell, the only independent component survives among \( 2j+1 \) ones (for instance, in the rest system, where \( p^a = (m, 0, 0) \), the only nonvanishing component is \( F_{11 ... 1}(p) \)).

It is now easy to write down the well-defined Poincaré invariant inner product in the space \( \mathcal{H}_{m,j} \). For each two fields \( F(p, \bar{z}) \in \mathcal{H}_{m,j}, G(p, \bar{z}) \in \mathcal{H}_{m,j} \) it reads

\[ \langle F | G \rangle_{m,j} = m^{2\kappa+2j-1}N(\kappa, j) \int \frac{dp}{p^0} \int \frac{dz d\bar{z} (1 - z\bar{z})^{-2-2j}}{2\pi i [(p, n)]^{2\kappa+2j}} F(p, \bar{z})G(p, \bar{z}) , \]  

where \( \kappa > 1/2 \) is a real parameter and

\[ [N(\kappa, j)]^{-1} = \int_{|z|<1} dz d\bar{z} (1 - z\bar{z})^{-2-2j} = \frac{1}{2^{2j+1}} \sum_{k=0}^{2j} \frac{(2j)!}{k!(2j-k)!(k+2\kappa-1)} \]  

is a normalization constant. The operators (4.29) are Hermitian with respect to the inner product. Furthermore, parameter \( \kappa \) is, in fact, inessential, because the UIR’s, corresponding different \( \kappa \) in Eq. (4.33a) and the same \( m, j \), are unitary equivalent to each other. It is explicitly seen, if one takes the integral (4.33a) over \( \mathcal{L} \) with account of the expansion (4.24) for the wave functions. Then the inner product (4.33a) transforms to the following form

\[ \langle F_j | G_j \rangle_{m,j} = \frac{1}{m} \int \frac{dp}{p^0} F_{\alpha_1 \alpha_2 ... \alpha_{2j}}(p)G_{\alpha_1 \alpha_2 ... \alpha_{2j}}(p) , \]  

which is independent from \( \kappa \).

The superextension appears immediately. We need only to consider the space \( \mathcal{H}_{m,s,b} \) introduced in subsec. 2 and to put \( s + 1 = -j \leq 0 \) being a (half)integer
number. The wave function has the following component expansion

\[
F(p, \bar{\Gamma}) = F_0 \alpha_1 \alpha_2 \ldots \alpha_{2j+2} (p) \bar{z}^{\alpha_1} z^{\alpha_2} \ldots \bar{z}^{\alpha_{2j+2}}
\]

\[
+ i \sqrt{(j + 1)(1 + b)} \theta F_1 \alpha_1 \alpha_2 \ldots \alpha_{2j+1} (p) \bar{z}^{\alpha_1} z^{\alpha_2} \ldots \bar{z}^{\alpha_{2j+1}}
\]

\[
+ i \sqrt{(j + 1)(1 - b)} \chi F_2 \alpha_1 \alpha_2 \ldots \alpha_{2j+1} (p) \bar{z}^{\alpha_1} z^{\alpha_2} \ldots \bar{z}^{\alpha_{2j+1}}
\]

\[
+ \sqrt{(j + 1/2)(j + 1)(1 - b^2)} \theta \chi F_3 \alpha_1 \alpha_2 \ldots \alpha_{2j} (p) \bar{z}^{\alpha_1} z^{\alpha_2} \ldots \bar{z}^{\alpha_{2j}}
\]

number. The wave function has the following component expansion

\[
F(p, \bar{\Gamma}) = F_0 \alpha_1 \alpha_2 \ldots \alpha_{2j+2} (p) \bar{z}^{\alpha_1} z^{\alpha_2} \ldots \bar{z}^{\alpha_{2j+2}}
\]

\[
+ i \sqrt{(j + 1)(1 + b)} \theta F_1 \alpha_1 \alpha_2 \ldots \alpha_{2j+1} (p) \bar{z}^{\alpha_1} z^{\alpha_2} \ldots \bar{z}^{\alpha_{2j+1}}
\]

\[
+ i \sqrt{(j + 1)(1 - b)} \chi F_2 \alpha_1 \alpha_2 \ldots \alpha_{2j+1} (p) \bar{z}^{\alpha_1} z^{\alpha_2} \ldots \bar{z}^{\alpha_{2j+1}}
\]

\[
+ \sqrt{(j + 1/2)(j + 1)(1 - b^2)} \theta \chi F_3 \alpha_1 \alpha_2 \ldots \alpha_{2j} (p) \bar{z}^{\alpha_1} z^{\alpha_2} \ldots \bar{z}^{\alpha_{2j}}
\]

and it is subjected to the wave equations (4.15). The latter reduce to the “Dirac equations” (4.32) for each of the component \( F_i \alpha_1 \alpha_2 \ldots \alpha_{ji} \), \( i = 0, 1, 2, 3 \). The Hermitian inner product in \( H_{m,j,b} = H_{m,s,b} \) may be introduced by analogy with Eq. (4.2b) and for each two \( F(p, \bar{\Gamma}) \), \( G(p, \bar{\Gamma}) \in H_{m,j,b} \) is expressed in terms of the inner product (4.33)

\[
(F|G) = \langle F_0|G_0 \rangle_{m,j+1} + \langle F_1|G_1 \rangle_{m,j+1/2} + \langle F_2|G_2 \rangle_{m,j+1/2} + \langle F_3|G_3 \rangle_{m,j}.
\]

(4.36)

It is matter of direct verification to prove that the operators (4.14), which generate the representation of the N=2 Poincaré superalgebra, are truly Hermitian with respect to this inner product. Moreover, the BPS bound \( |b| \leq 1 \) and the reality of the renormalized value of \( q \) (4.18) by \( s \leq -1 \) provides the necessary and sufficient conditions for operators (4.14) to be Hermitian.

Thus, the quantization of the model reproduces the Hilbert space of the (half)integer superspin superparticle. Each component of the wave function (4.35) describes a particle with fixed spin. N=2 supersymmetry unifies four particles of the equal mass \( m \) and of the (half)integer spins \( j + 1, j + 1/2, j + 1/2, j, (j \geq 0) \) in the irreducible superquartet.

5. N=2 hypermultiplet

The theory is strongly simplified in the BPS limit when \( |b| = 1 \). The classical phase superspace appears to be \( M^{8/2} = T^* (R^{1,2}) \times L^{11} \), where the atypical OSp(2|2) co-orbit \( L^{11} \) of complex dimension 1/1 plays the role of the inner supermanifold associated to the superparticle superspin. The atypical coadjoint orbit of the OSp(2|2) substitutes the typical one in the BPS limit.

The supermanifold \( M^{8/2} \) serves the extended phase superspace of the N=1, D=3 superparticle allowing the hidden N=2 supersymmetry. Therefore, the quantization of the N=2 superparticle reduces in the BPS limit to the one of the N=1 superparticle. Following the same method we may combine the canonical Dirac procedure and the geometric quantization. The respective theory of N=1 superanyon has been considered earlier [23] and it results in the description of N=1, D=3 supersymmetric doublet of anyons in terms of the fields carrying the atypical UIR’s of the OSp(2|2). Moreover, it is shown in Ref. [25] that the N=1 superdoublet allows extended N=2 SUSY and it can be treated as the N=2 hypermultiplet of anyons. It is exactly N=2
fractional spin superparticle emerged in the BPS limit of the N=2 model suggested in the present paper. Let us briefly consider some details of this limiting case.

One can see, that the consideration of this Section is easily modified to the case, which saturates the BPS bound, when $|b| = 1$. Consider, for instance, $b = 1$. Then the wave function (4.13) does not depend on $\chi$. It is equivalent to the vanishing of half of the odd variables in the BPS limit mentioned at the classical level. The generators (4.14) of the N=2 Poincaré superalgebra reduce to

$$
\hat{J}_a = -i\epsilon_{abc}b^c \frac{\partial}{\partial p_c} + J_a, \quad \hat{p}_a = p_a, \quad \hat{Z} = m
$$

(4.37)

where

$$
\hat{J}_a = -\bar{\xi}_a \bar{\partial} - (\bar{\partial} \xi_a) \left(1 + \frac{1}{2} \bar{\partial} \frac{\partial}{\partial \bar{\theta}} \right)
$$

(4.38)

$$
\sqrt{ms}\hat{W}^\alpha = \bar{\theta}(\bar{z}^\alpha \bar{\partial} + 2s(\bar{\partial}z^\alpha)) - z^\alpha \frac{\partial}{\partial \bar{\theta}} \quad \sqrt{ms}\hat{\bar{W}}^\bar{\alpha} = -i\bar{\theta}(\bar{z}^\bar{\alpha} \bar{\partial} + 2s(\bar{\partial}z^\bar{\alpha})) - i\bar{z}^\bar{\alpha} \frac{\partial}{\partial \theta}.
$$

The operators (4.38) together with one more scalar U(1)-generator

$$
\hat{B} \equiv \hat{P}_4 - s = \frac{1}{2} \bar{\theta} \frac{\partial}{\partial \bar{\theta}} - s
$$

form an irreducible representation of the OSp(2|2), which is unitary for $s > 0$ (it is an atypical UIR of the OSp(2|2) mentioned above) and it is finite dimensional for $s = -j$, $j$ being positive (half)integer. The expressions (4.38) can also be obtained by straightforward geometric quantization on $L^{11}$ [36, 38].

The operators (4.37) are linear in the generators of the inner osp(2|2) superalgebra. Therefore, they generate N=2 Poincaré superalgebra with central charge $Z = m$ immediately, without any corrections in $1/s$, and the renormalization is not required for the case. The wave equations (4.13) hold their form in the BPS limit and the inner products are given by Eqs. (4.22), (4.36), where one should account for the vanishing of the last two components of the wave functions in the expansion (4.13).

The peculiarities of the model mentioned, mean that we obtain in the BPS limit an adequate description of the N=2 particle hypermultiplet. In particular, we have a natural smooth reduction for both the Poincaré supersymmetry and the internal one. The comparison of the classical mechanics given in subsec. III.6 and of the presented quantum theory demonstrates the direct relationship between the contraction of the classical phase superspace $T^*(R^{1,2}) \times L^{11}$ to $T^*(R^{1,2}) \times L^{11}$ and the shortening of the N=2 particle supermultiplet to the N=2 hypermultiplet in the BPS limit.

V. SUMMARY AND DISCUSSION

In the present paper we have constructed the consistent first quantized theory of N=2, D=3 superanyon as well as the one of massive superparticle of the habitual
integer superspin. The starting point for the quantization is the classical model of the superparticle in the nonlinear phase superspace $\mathcal{M}^{8|4} = T^*(\mathbb{R}^{1,2}) \times L^{1|2}$, that is different from the standard approach.

A traditional viewpoint in the construction of the spinning particle models \cite{34} is to describe the spinning degrees of freedom by some variables being simultaneously translation invariant and Lorentz covariant (as usual, those are Lorentz vectors or spinors). Such variables parametrize some linear space $L$ and then the extended phase space is chosen to be $\mathcal{M} = T^*(\mathbb{R}^{1,D-1}) \times L$ or $\mathcal{M} = T^*(\mathbb{R}^{1,D-1} \times L)$. The only difference for superparticles is to replace $D$-dimensional Minkowski space $\mathbb{R}^{1,D-1}$ by the respective superspace. The advantage of the covariant (super)space $\mathcal{M}$ is in the linear (“covariant”) action of the Poincaré supergroup. In this approach, however, an embedding of the (super)particle physical space $\mathcal{O}$ (that is the underlying coadjoint orbit) in the covariant phase (super)space may be ambiguous. Moreover, it is a common usage in this approach to give little attention to the geometry underlying the embedding $\mathcal{O} \to \mathcal{M}$.

We have demonstrated that the nonlinear phase superspace $\mathcal{M}^{8|4} = T^*(\mathbb{R}^{1,2}) \times L^{1|2}$ of $D=3$ spinning superparticle has the following remarkable features:

(i) The embedding of an appropriate coadjoint $\mathcal{O}_{m,s,b}$ orbit, being associated to the $N=2$, $D=3$ superparticle of arbitrary fixed mass $m > 0$, superspin $s \neq 0$ and central charge $mb$ ($|b| < 1$ in $\mathcal{M}^{8|4}$), is realized by two constraints, which provide the identical conservation of any Casimir function of the Hamiltonian Poincaré superalgebra. These constraints have transparent geometric origin and, after quantization, they are converted into wave equations of the superparticle in a natural way.

(ii) The ‘inner’ subsupermanifold $L^{1|2}$ of $\mathcal{M}^{8|4}$ appears to be in itself the coadjoint orbit for some supergroups. $L^{1|2}$ is shown to be symplectomorphic to the Kähler homogeneous superspace of the supergroup $SU(1,1|2)$ or its subsupergroup $OSp(2|2)$. In this sense the model admits the second supersymmetry ($SU(1,1|2)$ SUSY) along with the original Poincaré one.

(iii) The BPS limit of the model looks slightly different if compared to the standard picture for the superparticle \cite{39}. In the standard approach the extended superparticle model with central charge reveals the generation of new gauge degrees
of freedom in the BPS limit and it corresponds to an appropriate reduction for the physical phase space. Furthermore, it is usually impossible to impose the gauge in a covariant way and then to forget about the nonphysical variables. In contrast to the standard approach, the phase superspace \( T^* (R^{1,2}) \times \mathcal{L}^{1|2} \) can be explicitly truncated to \( T^* (R^{1,2}) \times \mathcal{L}^{1|1} \) in the BPS limit. In this case the inner \( N=2 \) Lobachevsky supergeometry reduces to the \( N=1 \) one, while the gauge variables drop out from the theory at all. The reduction of the phase superspace in the classical mechanics is directly related to the shortening of the particle supermultiplet in quantum theory considered in the BPS limit.

(iv) We suggest nontrivial quantization of the superparticle in the extended phase superspace \( \mathcal{M}^{8|4} \), which combines the canonical quantization in \( T^* (R^{1,2}) \) and the Berezin quantization in the inner phase superspace \( \mathcal{L}^{1|2} \). This quantization scheme leads naturally to the fields carrying infinite dimensional or finite dimensional representation of the supergroup \( SU(1,1|2) \) depending on the fractional or habitual (half)integer value of spin. The results are completely consistent with the previous known description of \( D=3 \) nonsupersymmetric particles as mechanical systems [6, 13].

Surprisingly, there are two, unitary equivalent to one another, series of \( N=2 \) supercharges in quantum theory, which correspond to different possibilities for the parameter \( q \) in Eq. (4.18). Only one of them, namely \( q_- \), is related directly to a conventional classical limit. Another possible value \( q_+ \) is shown to be related to the special properties (4.21) of the parity operator (4.20). However, the classical counterpart of this parity structure remains unclear, and the origin of the second possible value for \( q \) may seem enigmatic. Notice that the parity operator generates the structure of the deformed Heisenberg algebra in Hilbert space of anyon [26] or \( N=1 \) superanyon [25]. It would be interesting to understand, what is a geometry behind the parity operator for \( N=2 \) superanyon.

The significance of this one-particle theory may vary, in particular, depending on the possibility of an efficient second quantization of the model. One of the problems here is to construct a Lagrangian of the theory, which leads to the one-particle wave equation we have deduced from the classical mechanical action. The first step of construction may be to present two independent wave equations of superanyon (like Eqs. (4.13)) in the form of one spinor equation, when the mass and spin shell fixing conditions may emerge as integrability conditions. It is known that the similar construction for anyons gives a simple action functional [8, 20], which may be relevant for the second quantization of fractional spin particles. An adequate superextension (at least for \( N=1 \)) may be constructed probably using the representations of the \( su(1,1|2) \) superalgebra in the same way, as the spinor set of the anyon wave equations was constructed in Ref. [26] using the atypical UIR’s of \( osp(2|2) \). In this connection it should be noted that the exploitation of the atypical UIR’s of the \( osp(2|2) \) superalgebra and of the deformed Heisenberg algebra produces the linear set of spinor wave equations of \( N=1, D=3 \) superparticle only for special (half)integer \( j = 1/2 \) and \( j = 1 \) values of the superspin [27].
And, of course, the consistent interaction of (super)anyons remains an intriguing problem. Even in the first quantized theory the suggested approach to the description of anyon, being attempted for the extension to an interaction with an external field, implies (in the framework of minimal phase space) a perturbative representation for nonlinear commutation relations in terms of a series in powers of the field strengths [17]. In particular, it is unclear whether any consistent generalization exists for the wave equations of (super)anyons obtained in this paper in the presence of arbitrary external fields.

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Appendix. Anticommutation relations and renormalization for the N=2 Poincaré supercharges

We give here the calculation of the anticommutator (1.17) of N=2 Poincaré supercharges in more detail. The object is to show that the closure of the Poincaré superalgebra requires a renormalization of the parameter q entering in the definition of the supercharge’s (1.14) and the renormalized value is one of given by Eq. (1.18).

Before coming into explicit formulas, it is helpful to introduce a convenient notation, which is slightly different from that used in the paper. First, we redefine the coefficients in the expansion (1.13) and write down the wave function in the form

$$F(p, \bar{z}) = F_0(p, \bar{z}) + \bar{\theta}F_1(p, \bar{z}) + \bar{\chi}F_2(p, \bar{z}) + \bar{\theta}\bar{\chi}F_3(p, \bar{z}).$$

(A.1)

Hereafter, we shall represent the wave function $F \in \mathcal{H}$ as a four-column

$$F = \begin{pmatrix} F_0 \\ F_1 \\ F_2 \\ F_3 \end{pmatrix},$$

(A.2)

where the components $F_i, i = 0, 1, 2, 3$ depend on $p$ and $\bar{z}$. In these terms each linear operator $\hat{A}$ in $\mathcal{H}$ will be presented by a matrix $(\hat{A}^j_i)$ of dimension $4 \times 4$, whose elements are operators acting on the components. The matrix elements $(\hat{A}^j_i)$ are defined by
the rule \((\hat{A} F)_i = \sum_{j=0}^3 \hat{A}^j F_j\). The matrix representation gives a convenient tool for the explicit calculations performed below.

Second, we take \(q' = 1 - q\) to squeeze the notations.

Let us note that the Poincaré supercharge’s \((4.14)\) may be identically presented in the following form

\[
\hat{Q}^1 = i p^\alpha \hat{\nabla} \bar{\hat{W}}^\beta + m \hat{\nabla} \bar{\hat{W}}^\alpha, \quad \hat{Q}^2 = i p^\alpha \hat{\nabla} \bar{\hat{V}}^\beta + m \hat{\nabla} \bar{\hat{V}}^\alpha, \quad (A.3)
\]

where

\[
\hat{\nabla} \bar{\hat{W}}^\alpha = \frac{1}{2\sqrt{m^2}} \left( \begin{array}{cccc}
0 & -z^\alpha & -iz^\alpha & 0 \\
1 + b & \hat{Q}_0^\alpha & 0 & iq'z^\alpha \\
-1 - b & \hat{Q}_0^\alpha & 0 & -q'z^\alpha \\
0 & iq'1 - b \hat{Q}_1^\alpha & q'1 + b \hat{Q}_1^\alpha & 0 \\
\end{array} \right) \tag{A.4}
\]

\[
\hat{\nabla} \bar{\hat{V}}^\alpha = \frac{1}{2\sqrt{m^2}} \left( \begin{array}{cccc}
0 & -iz^\alpha & z^\alpha & 0 \\
1 + b & \hat{Q}_0^\alpha & 0 & q'z^\alpha \\
-1 + b & \hat{Q}_0^\alpha & 0 & -iq'z^\alpha \\
0 & -q'1 + b \hat{Q}_1^\alpha & -iq'1 - b \hat{Q}_1^\alpha & 0 \\
\end{array} \right) \tag{A.4}
\]

\[
\hat{\nabla} \bar{\hat{V}}^\alpha = \frac{1}{2\sqrt{m^2}} \left( \begin{array}{cccc}
0 & z^\alpha & -iz^\alpha & 0 \\
1 + b & \hat{Q}_0^\alpha & 0 & iq'z^\alpha \\
-1 + b & \hat{Q}_0^\alpha & 0 & q'z^\alpha \\
0 & iq'1 - b \hat{Q}_1^\alpha & -q'1 + b \hat{Q}_1^\alpha & 0 \\
\end{array} \right) \tag{A.4}
\]

and \(\hat{Q}^\alpha = z^\alpha \partial + (2s + \epsilon)(\partial z^\alpha) ; \epsilon = 0, 1\). When \(q' = 1\) we have \(\hat{W}_0^\alpha \equiv \hat{W}^\alpha\), \(\hat{W}_0^\alpha \equiv \hat{\nabla}^\alpha\) and similar identities for \(\hat{V}^\alpha\)’s.

The calculations of the anticommutators of N=2 supercharges give

\[
[\hat{Q}^1, \hat{Q}^1]_+ = 2p^\alpha \hat{X} + \hat{J}^\alpha \hat{J}^\beta (p^2 + m^2) + \frac{1}{2m} \hat{X} + p^\alpha \left( \hat{J}^\beta - m \hat{P} - m \hat{J} \right) + p^\alpha \hat{O}_+
\]

38
\[
[\hat{Q}^2_\alpha, \hat{Q}^2_\beta]_+ = 2p^{\alpha\beta} - \frac{1}{4ms} \hat{X}_- \hat{J}^{\alpha\beta}(p^2 + m^2) + \frac{1}{2ms} \hat{X}_- p^{\alpha\beta}((p, \hat{J}) - m\hat{P}_4 - ms) + p^{\alpha\beta}\hat{O}_- ,
\]

\[
[\hat{Q}^2_\alpha, \hat{Q}^1_\beta]_+ = -2imbe^{\alpha\beta} - \frac{i}{8ms}(\hat{X}_1\epsilon^{\alpha\beta} + i\hat{X}_0\hat{J}^{\alpha\beta})(p^2 + m^2)
\]

\[
\quad - \frac{i}{4ms}(m\hat{X}_2\epsilon^{\alpha\beta} + i\hat{X}_0p^{\alpha\beta})((p, \hat{J}) - m\hat{P}_4 - ms) + m\hat{O}^{\alpha\beta} ,
\]

where \( \hat{J}^{\alpha\beta} = (\gamma^a)^{\alpha\beta} \hat{J}_a \),

\[
\hat{X}_\pm = \begin{pmatrix} 2 & 0 \\ 0 & (1 + b) + q'^2(1 - b) \pm i(1 + b)(1 - q'^2) \end{pmatrix}
\]

\[
\hat{X}_0 = 2(1 - q'^2) \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 + b & 0 \\ 0 & 1 - b & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},
\]

\( \hat{X}_{1,2} \) are diagonal operators

\[
\hat{X}_1 F = 4bs\hat{F}_0 + [(1 + b)(2s - 1) - q'^2(1 - b)(2s + 1)]F_1
\]

\[
\quad - [(1 - b)(2s - 1) - q'^2(1 + b)(2s + 1)]F_2 + 4bsq'^2F_3
\]

\[
\hat{X}_2 F = 4b\hat{F}_0 + 2[(1 + b) - q'^2(1 - b)]F_1 - 2[(1 - b) - q'^2(1 + b)]F_2 + 4bq'^2F_3 ,
\]

and

\[
\hat{O}_\pm = \left( q'^2 \frac{2s + 1}{2s} - 1 \right) \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 - b & \mp i(1 + b) & 0 \\ 0 & \pm i(1 - b) & 1 + b & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}
\]

\[
\hat{O}^{\alpha\beta} = \left( q'^2 \frac{2s + 1}{2s} - 1 \right) \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & i(1 - b)\epsilon^{\alpha\beta} & - \frac{(1 + b)}{m}p^{\alpha\beta} & 0 \\ 0 & \frac{(1 - b)}{m}p^{\alpha\beta} & - i(1 + b)\epsilon^{\alpha\beta} & 0 \\ 0 & 0 & 0 & -2ib\epsilon^{\alpha\beta} \end{pmatrix} .
\]

What we have obtained in Eqs. (A.5) is in fact a detailed notation for Eqs. (4.15). In this notation the structure of the anticommutation relations becomes transparent and a number of helpful features stands out. Let us consider the physical subspace in \( \mathcal{H} \), which is generated by solutions of the wave equations (4.15). Then the relations (A.5) simplify themselves and we have

\[
[\hat{Q}^I_\alpha, \hat{Q}^J_\beta]_+ = 2\delta^{IJ}p_{\alpha\beta} - 2imbe^{IJ}\epsilon_{\alpha\beta} - m\delta^{IJ}\epsilon_{\alpha\beta}\hat{O}_\pm - me^{IJ}\hat{O}_{\alpha\beta} .
\]
The presence of the operators $\hat{O}_\pm$ and $\hat{O}_{\alpha\beta}$ breaks off N=2 Poincaré superalgebra in general. In the neighborhood of the value $q' = 1 - 1/4s$, being derived from the combination of the classical mechanics and of the correspondence rules, the operators $\hat{O}_\pm, \hat{O}_{\alpha\beta}$ should be treated as the corrections of order $s^{-2}$. However, it contains more. The explicit expressions (A.6) show immediately that in the case of $s \neq -1/2$ the renormalization of $q'$ is possible, which provides the identical vanishing of any corrections and the closure of the Poincaré superalgebra on shell. The renormalized values are presented by Eq. (4.18).

The following observation is that the anticommutators obtained are invariant under the substitution $q' \rightarrow -q'$ (that is $q \rightarrow 2 - q$). As mentioned in Subsec. 4.2, this invariance comes from the degenerate N=4 supersymmetry and from specific properties of the parity operator (4.20).

Finally, the structure of the anticommutators (A.5) changes drastically in the BPS limit $|b| = 1$. Consider, for instance, $b = 1$. In this case, the two latter components of the wave function (A.1) vanish, $F_2 = F_3 \equiv 0$ (see Eq. (4.13)). However, when $b = 1$, in the linear subspace $F_2 = F_3 \equiv 0$ the parameter $q$ becomes inessential and the corrections (A.6) vanish identically. Thus, we do not need any renormalization in the BPS limit, which corresponds to the N=1 superparticle, which was considered earlier in Ref. [25].

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