A quasi-commutativity property of the Poisson and composition operators

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Abstract. Let Φ be a real valued function of one real variable, let $L$ denote an elliptic second order formally self-adjoint differential operator with bounded measurable coefficients, and let $P$ stand for the Poisson operator for $L$. A necessary and sufficient condition on Φ ensuring the equivalence of the Dirichlet integrals of $Φ \circ P h$ and $P(Φ \circ h)$ is obtained. We illustrate this result by some sharp inequalities for harmonic functions.

1 Introduction

In the present article we consider an elliptic second order formally self-adjoint differential operator $L$ in a bounded domain $Ω$. We denote by $P h$ the $L$-harmonic function with the Dirichlet data $h$ on $∂Ω$. The Dirichlet integral corresponding to the operator $L$ will be denoted by $\mathcal{D}[u]$. We also introduce a real-valued function $Φ$ on the line $\mathbb{R}$ and denote the composition of $Φ$ and $u$ by $Φ \circ u$.

We want to show that the Dirichlet integrals of the functions $Φ \circ P h$ and $P(Φ \circ h)$ are comparable. First of all, clearly, the inequality

$$\mathcal{D}[P(Φ \circ h)] \leq \mathcal{D}[Φ \circ P h]$$

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is valid. Hence we only need to check the opposite estimate

$$\mathcal{D}[\Phi \circ Ph] \leq C \mathcal{D}[P(\Phi \circ h)]. \quad (1)$$

We find a condition on $\Phi$ which is both necessary and sufficient for (1).

Moreover, we prove that the two Dirichlet integrals are comparable if and only if the derivative $\Psi = \Phi'$ satisfies the reverse Cauchy inequality

$$\frac{1}{b-a} \int_a^b \Psi^2(t) \, dt \leq C \left( \frac{1}{b-a} \int_a^b \Psi(t) \, dt \right)^2 \quad (2)$$

for any interval $(a, b) \subset \mathbb{R}$.

We add that the constants $C$ appearing in (1) and (2) are the same.

At the end of the paper this result is illustrated for harmonic functions and for $\Psi(t) = |t|^{\alpha}$ with $\alpha > -1/2$. In particular, we obtain the sharp inequalities

$$\int_\Omega |\nabla (|Ph| Ph)|^2 \, dx \leq \frac{3}{2} \int_\Omega |\nabla P(|h| h)|^2 \, dx \quad (3)$$

for any $h \in W^{1/2,2}(\partial \Omega)$ and

$$\int_\Omega |\nabla (P h^2)|^2 \, dx \leq \frac{4}{3} \int_\Omega |\nabla P(h^2)|^2 \, dx \quad (4)$$

for any nonnegative $h \in W^{1/2,2}(\partial \Omega)$. Here $P$ is the harmonic Poisson operator.

To avoid technical complications connected with non-smoothness of the boundary, we only deal with domains bounded by surfaces of class $C^\infty$, although, in principle, this restriction can be significantly weakened.

2 Preliminaries

All functions in this article are assumed to take real values and the notation $\partial_i$ stands for $\partial/\partial x_i$.

Let $L$ be the second order differential operator

$$Lu = -\partial_i (a_{ij}(x) \partial_j u)$$

defined in a bounded domain $\Omega \subset \mathbb{R}^n$. 

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The coefficients $a_{ij}$ are measurable and bounded. The operator $L$ is uniformly elliptic, i.e. there exists $\lambda > 0$ such that

$$a_{ij}(x)\xi_i\xi_j \geq \lambda |\xi|^2$$

for all $\xi \in \mathbb{R}^n$ and for almost every $x \in \Omega$.

Let $\Psi$ be a function defined on $\mathbb{R}$ such that, for any $N \in \mathbb{N}$, the functions

$$\Psi_N(t) = \begin{cases} \Psi(t) & \text{if } |\Psi(t)| \leq N \\ N \text{sign}(\Psi(t)) & \text{if } |\Psi(t)| > N \end{cases}$$

are continuous. We suppose that there exists a constant $C$ such that, for any finite interval $\sigma \subset \mathbb{R}^+$, we have

$$\Psi^2 \leq C(\Psi)^2$$

where $\overline{u}$ denotes the mean value of $u$ on $\sigma$.

Also let

$$\Phi(t) = \int_0^t \Psi(\tau) d\tau, \quad t \in \mathbb{R}.$$ 

Let $W^{1/2,2}(\partial \Omega)$ be the trace space for the Sobolev space $W^{1,2}(\Omega)$ and let $P$ denote the Poisson operator, i.e. the solution operator:

$$W^{1/2,2}(\partial \Omega) \ni h \rightarrow u \in W^{1,2}(\Omega)$$

for the Dirichlet problem

$$\begin{cases} Lu = 0 \quad \text{in } \Omega \\ \text{tr } u = h \quad \text{on } \partial \Omega \end{cases}$$

where $\text{tr } u$ is the trace of the function $u \in W^{1,2}(\Omega)$ on $\partial \Omega$.

We introduce the Dirichlet integral

$$\mathcal{D}[u] = \int_{\Omega} a_{ij} \partial_i u \partial_j u \, dx$$

and the bilinear form

$$\mathcal{D}[u,v] = \int_{\Omega} a_{ij} \partial_i u \partial_j v \, dx.$$

In the sequel we shall consider $\mathcal{D}[P(\Phi \circ h)]$ and $\mathcal{D}[\Phi \circ Ph]$ for $h \in W^{1/2,2}(\partial \Omega)$. Since $h \in W^{1/2,2}(\partial \Omega)$ implies neither $\Phi \circ h \in W^{1/2,2}(\partial \Omega)$ nor $\Phi \circ Ph \in W^{1,2}(\Omega)$, we have to specify what $\mathcal{D}[P(\Phi \circ h)]$ and $\mathcal{D}[\Phi \circ Ph]$ mean.
We define
\[ \mathcal{D}[P(\Phi \circ h)] = \liminf_{k \to \infty} \mathcal{D}[P(\Phi_k \circ h)]. \quad (9) \]
where
\[ \Phi_k(t) = \int_0^t \Psi_k(t) \, dt. \quad (10) \]
and \( \Psi_k \) is given by (6). Note that if \( h \) be in \( W^{1/2,2}(\partial \Omega) \), \( \mathcal{D}[P(\Phi_k \circ h)] \) makes sense. In fact
\[ |\Phi_k \circ h| \leq k \, |h| \quad \text{and} \quad |\Phi_k \circ h(x) - \Phi_k \circ h(y)| \leq k \, |h(x) - h(y)|, \]
imply that \( \Phi_k \circ h \) belongs to \( W^{1/2,2}(\partial \Omega) \).

In order to accept definition (9), we have to show that if the left hand side of (9) makes sense because \( \Phi \circ h \) belongs to \( W^{1/2,2}(\partial \Omega) \), then (9) holds. In fact, we have:

**Lemma 1** Let \( h \in W^{1/2,2}(\partial \Omega) \) be such that also \( \Phi \circ h \) belongs to \( W^{1/2,2}(\partial \Omega) \). Then
\[ \mathcal{D}[P(\Phi \circ h)] = \lim_{k \to \infty} \mathcal{D}[P(\Phi_k \circ h)]. \quad (11) \]

**Proof.** Obviously,
\[ \Phi_k \circ h(x) - \Phi \circ h(x) = \int_0^{h(x)} [\Psi_k(t) - \Psi(t)] \, dt \to 0 \quad \text{a.e.} \]
and
\[ |[\Phi_k \circ h(x) - \Phi \circ h(x)] - [\Phi_k \circ h(y) - \Phi \circ h(y)]| \leq 2 |\Phi \circ h(x) - \Phi \circ h(y)|. \]

In view of the Lebesgue dominated convergence theorem, these inequalities imply \( \Phi_k \circ h \to \Phi \circ h \) in \( W^{1/2,2}(\partial \Omega) \). Therefore, \( P(\Phi_k \circ h) \to P(\Phi \circ h) \) in \( W^{1,2}(\Omega) \) and (11) holds.

As far as \( \mathcal{D}[\Phi \circ Ph] \) is concerned, we remark that
\[ \mathcal{D}[\Phi_k \circ Ph] = \int_\Omega (\Psi_k(Ph))^2 a_{ij} \partial_i(Ph) \partial_j(Ph) \, dx \]
tends to
\[ \int_\Omega (\Psi(Ph))^2 a_{ij} \partial_i(Ph) \partial_j(Ph) \, dx \]
because of the monotone convergence theorem. Therefore, we set

\[ \mathcal{D}[\Phi \circ Ph] = \lim_{k \to \infty} \mathcal{D}[\Phi_k \circ Ph]. \]

Note that neither \( \mathcal{D}[\Phi \circ Ph] \) nor \( \mathcal{D}[\Phi(\Phi \circ h)] \) needs to be finite.

Let \( G(x, y) \) be the Green function of the Dirichlet problem (8) and \( \partial/\partial \nu \) the co-normal operator

\[ \frac{\partial}{\partial \nu} = a_{ij} \cos(n, x_j) \partial_i \]

where \( n \) is the exterior unit normal.

**Lemma 2** Let the coefficients \( a_{ij} \) of the operator \( L \) belong to \( C^\infty(\Omega) \). There exist two positive constants \( c_1 \) and \( c_2 \) such that

\[ \frac{c_1}{|x - y|^n} \leq \frac{\partial^2 G(x, y)}{\partial \nu_x \partial \nu_y} \leq \frac{c_2}{|x - y|^n} \]  \hspace{1cm} (12)

for any \( x, y \in \partial \Omega, x \neq y \).

**Proof.** Let us fix a point \( x_0 \) on \( \partial \Omega \). We consider a neighborhood of \( x_0 \) and introduce local coordinates \( y = (y', y_n) \) in such a way that \( x_0 \) corresponds to \( y = 0, y_n = 0 \) is the tangent hyperplane and locally \( \Omega \) is contained in the half-space \( y_n > 0 \). We may suppose that this change of variables is such that \( a_{ij}(x_0) = \delta_{ij} \).

It is known (see \[1\]) that the Poisson kernel \((\partial/\partial \nu_y)G(x, y)\) in a neighborhood of \( x_0 \) is given by

\[ 2 \omega_n^{-1} y_n |y|^{-n} + O \left( |y|^{2-n-\varepsilon} \right) \]

where \( \omega_n \) is the measure of the unit sphere in \( \mathbb{R}^n \) and \( \varepsilon > 0 \). Moreover the derivative of the Poisson kernel with respect to \( y_n \) is equal to

\[ 2 \omega_n^{-1} (|y|^2 - y_n^2) |y|^{-n-2} + O \left( |y|^{1-n-\varepsilon} \right) \]

which becomes

\[ 2 \omega_n^{-1} |y'|^{-n} + O \left( |y|^{1-n-\varepsilon} \right) \]  \hspace{1cm} (13)

for \( y_n = 0 \). Formula (13) and the arbitrariness of \( x_0 \) imply (12).

\[ \square \]
3 The main result

Theorem 1 If $\Psi: \mathbb{R} \to \mathbb{R}_+$ satisfies condition (7), then

$$\mathcal{D}[\Phi \circ Ph] \leq C \mathcal{D}[P(\Phi \circ h)]$$ (14)

for any $h \in W^{1/2, 2}(\partial \Omega)$, where $C$ is the constant in (7).

Proof. We suppose temporarily that $a_{ij} \in C^\infty$.

Let $u$ be a solution of the equation $Lu = 0$, $u \in C^\infty(\Omega)$. We show that

$$\mathcal{D}[u] = \frac{1}{2} \int_{\partial \Omega} \int_{\partial \Omega} (\text{tr } u(x) - \text{tr } u(y))^2 \frac{\partial^2 G(x, y)}{\partial \nu_x \partial \nu_y} d\sigma_x d\sigma_y. \quad (15)$$

In fact, since

$$L_x[(u(x) - u(y))^2] = 2a_{hk} \partial_h u \partial_k u,$$

the integration by parts in (15) gives

$$\int_{\partial \Omega} \int_{\partial \Omega} (\text{tr } u(x) - \text{tr } u(y))^2 \frac{\partial^2 G(x, y)}{\partial \nu_x \partial \nu_y} d\sigma_x d\sigma_y =
2 \int_\Omega a_{hk} \partial_h u \partial_k u \int_{\partial \Omega} \frac{\partial G(x, y)}{\partial \nu_y} d\sigma_y = 2 \mathcal{D}[u],$$

and (15) is proved.

Let $u \in W^{1, 2}(\Omega)$ be a solution of $Lu = 0$ in $\Omega$ and let $\{u_k\}$ be a sequence of $C^\infty(\bar{\Omega})$ functions which tends to $u$ in $W^{1, 2}(\Omega)$. Since $\text{tr } u_k \to \text{tr } u$ in $W^{1/2, 2}(\partial \Omega)$, we see that $P(\text{tr } u_k)$ tends to $P(\text{tr } u) = u$ in $W^{1, 2}(\Omega)$ and therefore $\mathcal{D}[P(\text{tr } u_k)] \to \mathcal{D}[u]$. This implies that (15) holds for any $u$ in $W^{1, 2}(\Omega)$ with $Lu = 0$ in $\Omega$.

Let now $u$ and $v$ belong to $W^{1, 2}(\Omega)$ and $Lu = Lv = 0$ in $\Omega$. Since

$$\mathcal{D}[u, v] = 4^{-1}(\mathcal{D}[u + v] - \mathcal{D}[u - v]),$$

we can write

$$\mathcal{D}[u, v] = \frac{1}{2} \int_{\partial \Omega} \int_{\partial \Omega} (\text{tr } u(x) - \text{tr } u(y))(\text{tr } v(x) - \text{tr } v(y)) \frac{\partial^2 G(x, y)}{\partial \nu_x \partial \nu_y} d\sigma_x d\sigma_y. \quad (16)$$
Note also that, if $h \in W^{1/2,2}(\partial \Omega)$ and $g \in W^{1,2}(\Omega)$, we have
\[
\mathcal{D}[Ph, P(\text{tr } g)] = \mathcal{D}[Ph, g]. \tag{17}
\]

Suppose now that $h \in W^{1/2,2}(\partial \Omega)$ is such that $\Phi \circ h \in W^{1/2,2}(\partial \Omega)$. We have
\[
\mathcal{D}[\Phi \circ Ph] = 
\int_{\Omega} a_{ij} \partial_i(\Phi \circ Ph) \partial_j(\Phi \circ Ph) \, dx = 
\int_{\Omega} a_{ij} (\Psi(Ph))^2 \partial_i(Ph) \partial_j(Ph) \, dx .
\]
The last integral can be written as
\[
\int_{\Omega} a_{ij} \partial_i(Ph) \partial_j \left( \int_0^{Ph} \Psi^2(\tau) \, d\tau \right) \, dx ,
\]
and we have proved that
\[
\mathcal{D}[\Phi \circ Ph] = \mathcal{D} \left( Ph, \int_0^{Ph} \Psi^2(\tau) \, d\tau \right)
\]
From (16) and (17), we get
\[
\mathcal{D}[\Phi \circ Ph] = \frac{1}{2} \int_{\partial \Omega} \int_{\partial \Omega} (h(y) - h(x)) \int_{h(x)}^{h(y)} \Psi^2(\tau) \, d\tau \frac{\partial^2 G(x, y)}{\partial \nu_x \partial \nu_y} \, d\sigma_x d\sigma_y \tag{18}
\]
In view of (12), $\partial^2 G(x, y)/\partial \nu_x \partial \nu_y$ is positive, and the condition (7) leads to
\[
\mathcal{D}[\Phi \circ Ph] \leq \frac{C}{2} \int_{\partial \Omega} \int_{\partial \Omega} \left( \int_{h(x)}^{h(y)} \Psi(\tau) \, d\tau \right)^2 \frac{\partial^2 G(x, y)}{\partial \nu_x \partial \nu_y} \, d\sigma_x d\sigma_y . \tag{19}
\]
Inequality (14) is proved, since the right-hand side in (19) is nothing but $C \mathcal{D}[P(\Phi \circ h)]$ (see (15)).

For any $h \in W^{1/2,2}(\partial \Omega)$, the inequality (14) follows from
\[
\mathcal{D}[\Phi \circ Ph] = \lim_{n \to \infty} \mathcal{D}[\Phi_k \circ Ph] \leq C \liminf_{n \to \infty} \mathcal{D}[P(\Phi_k \circ h)].
\]

Let us suppose now that $a_{ij}$ only belong to $L^{\infty}(\Omega)$. There exist $a_{ij}^{(k)} \in C^\infty(\mathbb{R}^n)$ such that $a_{ij}^{(k)} \to a_{ij}$ in measure as $k \to \infty$. We can assume that
\[
\|a_{ij}^{(k)}\|_{L^{\infty}(\Omega)} \leq K
\]
and that the operators $L^{(k)} = -\partial_t (a_{ij}^{(k)} \partial_j u)$ satisfy the ellipticity condition \([5]\) with the same constant $\lambda$.

Let $u$ be a solution of the Dirichlet problem \([8]\) and let $u_k$ satisfy
\[
\begin{aligned}
L^{(k)} u_k &= 0 \quad \text{in } \Omega \\
\text{tr} u_k &= h \quad \text{on } \partial \Omega .
\end{aligned}
\]

Denote the matrices $\{a_{ij}\}$ and $\{a_{ij}^{(k)}\}$ by $A$ and $A^{(k)}$ respectively.

Since we can write
\[
div A \nabla (u - u_k) = - div A \nabla u_k = - (A - A_k) \nabla u_k
\]
we find that
\[
\mathcal{D}[u - u_k] \leq \| (A - A_k) \nabla u_k \| \| \nabla (u - u_k) \|.
\]

Then there exists a constant $K$ such that
\[
\| \nabla (u - u_k) \| \leq K \| (A - A_k) \nabla u_k \|.
\]

Denoting by $D_k$ the quadratic form
\[
D_k[u] = \int_{\Omega} a_{ij}^{(k)} \partial_i u \partial_j u \, dx ,
\]
we have
\[
D_k[u_k] = \min_{u \in W^{1,2}(\Omega)} \| \nabla (u - u_k) \|
\]
and
\[
\lambda \| \nabla u_k \|^2 \leq D_k[u_k].
\]

This shows that the sequence $\| \nabla u_k \|$ is bounded and the right-hand side of \((20)\) tends to 0 as $k \rightarrow \infty$.

We are now in a position to prove \((14)\). Clearly, it is enough to show that
\[
\mathcal{D}[\Phi_m \circ Ph] \leq C \mathcal{D}[P(\Phi_m \circ h)] ,
\]
where $\Phi_m$ is given by \((10)\) for any $h \in W^{1,2}(\partial \Omega)$ such that $\Phi \circ h$ belongs to the same space.

Because of what we have proved when the coefficients are smooth, we may write
\[
D_k[\Phi_m \circ P_k h] \leq C D_k[P_k(\Phi_m \circ h)] ,
\]
where $P_k$ denotes the Poisson operator for $L^{(k)}$.

Formula (20) shows that $P_k(\Phi_m \circ h)$ tends to $P(\Phi_m \circ h)$ (as $k \to \infty$) in $W^{1,2}(\Omega)$ and thus

$$
\lim_{k \to \infty} \mathcal{D}_k[P_k(\Phi_m \circ h)] = \mathcal{D}[P(\Phi_m \circ h)].
$$

(22)

On the other hand, we have

$$
\nabla \Phi_m(P_k h) - \nabla \Phi_m(P h) = \Psi_m(P_k h) (\nabla P_k h - \nabla P h) + (\Psi_m(P_k h) - \Psi_m(P h)) \nabla P h.
$$

In view of the continuity of $\Psi_m$, we find

$$
\|\nabla \Phi_m(P_k h) - \nabla \Phi_m(P h)\|_{L^2(\Omega)} \to 0.
$$

This implies that $\mathcal{D}_k[\Phi_m \circ P_k h] \to \mathcal{D}[\Phi_m \circ P h]$, which together with (22), leads to (21).

Under the assumption that the coefficients of the operator are smooth, we can prove the inverse of Theorem 1.

**Theorem 2** Let the coefficients of the operator $L$ belong to $C^\infty(\Omega)$. If (14) holds for any $h \in W^{1/2,2}(\partial\Omega)$, then (7) is true with the same constant $C$.

**Proof.** Let $\Gamma$ be a subdomain of $\partial\Omega$ with smooth non-empty boundary. We choose a sufficiently small $\varepsilon > 0$ and denote the $\varepsilon$-neighborhood of $\Gamma$ by $[\Gamma]_\varepsilon$. We set $\gamma_\varepsilon = [\Gamma]_\varepsilon \setminus \Gamma$ and denote by $\delta(x)$ the distance of the point $x$ from $\Gamma$.

Let $a$ and $b$ be different real numbers and let $h$ be the function defined on $\partial\Omega$ by

$$
h(x) \begin{cases} 
= a & \text{if } x \in \Gamma \\
= a + (b - a) \varepsilon^{-1} \delta(x) & \text{if } x \in \gamma_\varepsilon \\
= b & \text{if } x \in \partial\Omega \setminus [\Gamma]_\varepsilon.
\end{cases}
$$

We know from (18) that $\mathcal{D}[\Phi \circ P h]$ is equal to

$$
\int_{\partial\Omega} d\sigma_x \int_{\partial\Omega} Q(x, y) d\sigma_y
$$

where

$$
Q(x, y) = \frac{1}{2} (h(y) - h(x)) \left( \int_{h(x)}^{h(y)} \Psi^2(\tau) d\tau \right) \frac{\partial^2 G(x, y)}{\partial \nu_x \partial \nu_y}.
$$
We can write
\[
\int_{\partial \Omega} d\sigma_x \int_{\partial \Omega} Q(x, y) d\sigma_y = I_1 + I_2 + I_3
\]
where
\[
I_1 = \int_{\gamma_{\varepsilon}} d\sigma_x \int_{\gamma_{\varepsilon}} Q(x, y) d\sigma_y,
\]
\[
I_2 = \int_{\partial \Omega \setminus \gamma_{\varepsilon}} d\sigma_x \int_{\gamma_{\varepsilon}} Q(x, y) d\sigma_y + \int_{\gamma_{\varepsilon}} d\sigma_x \int_{\partial \Omega \setminus \gamma_{\varepsilon}} Q(x, y) d\sigma_y,
\]
\[
I_3 = \int_{\Gamma} d\sigma_x \int_{\partial \Omega \setminus [\Gamma]_{\varepsilon}} Q(x, y) d\sigma_y + \int_{\partial \Omega \setminus [\Gamma]_{\varepsilon}} d\sigma_x \int_{\Gamma} Q(x, y) d\sigma_y.
\]
The right-hand estimate in (12) leads to
\[
I_1 \leq c \left( \frac{b - a}{\varepsilon} \right)^2 \int_{\gamma_{\varepsilon}} d\sigma_x \int_{\gamma_{\varepsilon}} (\delta(x) - \delta(y))^2 |x - y|^{-n} d\sigma_y. \quad (23)
\]
The integral
\[
\int_{\gamma_{\varepsilon}} (\delta(x) - \delta(y))^2 |x - y|^{-n} d\sigma_y
\]
with \( x \in \gamma_{\varepsilon} \) is majorized by
\[
c \int_0^\varepsilon (t - \delta(x))^2 dt \int_{\mathbb{R}^{n-2}} (|\eta| + (t - \delta(x)))^{-n} d\eta = \mathcal{O}(\varepsilon).
\]
This estimate and (23) imply \( I_1 = \mathcal{O}(1) \) as \( \varepsilon \to 0^+ \).

Since
\[
\left| \int_{\Gamma} d\sigma_x \int_{\gamma_{\varepsilon}} Q(x, y) d\sigma_y \right| \leq c \left( \frac{b - a}{\varepsilon} \right)^2 \int_{\gamma_{\varepsilon}} d\sigma_x \int_{\gamma_{\varepsilon}} \delta^2(y)|x - y|^{-n} d\sigma_y
\]
and the integral
\[
\int_{\gamma_{\varepsilon}} \delta^2(y) d\sigma_y \int_{\Gamma} |x - y|^{-n} d\sigma_x
\]
does not exceed
\[
c \int_0^\varepsilon t^2 dt \int_{\mathbb{R}^{n-2}} (|\eta| + t)^{-n+1} d\eta = \mathcal{O}(\varepsilon^2),
\]

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we find
\[ \int_{\Gamma} d\sigma_x \int_{\gamma_c} Q(x, y) d\sigma_y = O(1). \quad (24) \]

Analogously,
\[ \left| \int_{\partial \Omega \setminus [\Gamma]\varepsilon} d\sigma_x \int_{\gamma_c} Q(x, y) d\sigma_y \right| \leq \]
\[ c \left( \frac{b - a}{\varepsilon} \right)^2 \int_{\partial \Omega \setminus [\Gamma]\varepsilon} d\sigma_x \int_{\gamma_c} (\varepsilon - \delta(\varepsilon))^2 |x - y|^{-n} d\sigma_y = O(1). \quad (25) \]

Exchanging the roles of \( x \) and \( y \) in the previous argument, we arrive at the estimate
\[ \int_{\gamma_c} d\sigma_x \int_{\partial \Omega \setminus [\Gamma]\varepsilon} Q(x, y) d\sigma_y = O(1). \]

Combining this with (24) and (25), we see that
\[ I_2 = O(1). \]

Let us consider \( I_3 \). Since the two terms in the definition of \( I_3 \) are equal, we have
\[ I_3 = 2 (b - a) \int_a^b \Psi^2(\tau) d\tau \int_{\Gamma} d\sigma_x \int_{\partial \Omega \setminus [\Gamma]\varepsilon} \frac{\partial^2 G(x, y)}{\partial \nu_x \partial \nu_y} d\sigma_y. \]

By the left inequality in (22),
\[ I_3 \geq 2 c_1 (b - a) \int_a^b \Psi^2(\tau) d\tau \int_{\Gamma} d\sigma_x \int_{\partial \Omega \setminus [\Gamma]\varepsilon} |x - y|^{-n} d\sigma_y. \]

There exists \( \rho > 0 \) such that the integral over \( \Gamma \times (\partial \Omega \setminus [\Gamma]\varepsilon) \) of \( |x - y|^{-n} \) admits the lower estimate by
\[ c \int_{-\varepsilon}^{\varepsilon} dt \int_{-\varepsilon}^{\varepsilon} ds \int_{|\tau| \leq \rho} d\tau \int_{|s - t| \leq \rho} (|\eta - \tau| + |s - t|)^{-n} d\eta \]
which implies
\[ \int_{\Gamma} d\sigma_x \int_{\partial \Omega \setminus [\Gamma]\varepsilon} \frac{\partial^2 G(x, y)}{\partial \nu_x \partial \nu_y} d\sigma_y \geq c \log(1/\varepsilon). \quad (26) \]

Now we deal with \( \mathcal{O}[P(\Phi \circ h)] \). This can be written as \( J_1 + J_2 + J_3 \), where \( J_s \) are defined as \( I_s \) (\( s = 1, 2, 3 \)) with the only difference that \( Q(x, y) \) is replaced by
\[ \frac{1}{2} \left( \int_{h(x)}^{h(y)} \Psi(\tau) d\tau \right)^2 \frac{\partial^2 G(x, y)}{\partial \nu_x \partial \nu_y}. \]
As before, \( J_1 = \mathcal{O}(1) \), \( J_2 = \mathcal{O}(1) \) and
\[
J_3 = 2 \left( \int_a^b \Psi(\tau) d\tau \right)^2 \int_{\Gamma} d\sigma_x \int_{\partial\Omega \setminus [\Gamma]} \frac{\partial^2 G(x, y)}{\partial \nu_x \partial \nu_y} d\sigma_y.
\]
Inequality (14) for the function \( h \) can be written as
\[
I_3 + \mathcal{O}(1) \leq C \left( J_3 + \mathcal{O}(1) \right)
\]
i.e.
\[
\mathcal{O}(1) + (b-a) \int_a^b \Psi^2(\tau) d\tau \int_{\Gamma} d\sigma_x \int_{\partial\Omega \setminus [\Gamma]} \frac{\partial^2 G(x, y)}{\partial \nu_x \partial \nu_y} d\sigma_y \leq C \left( \mathcal{O}(1) + \left( \int_a^b \Psi(\tau) d\tau \right)^2 \int_{\Gamma} d\sigma_x \int_{\partial\Omega \setminus [\Gamma]} \frac{\partial^2 G(x, y)}{\partial \nu_x \partial \nu_y} d\sigma_y \right).
\]
Dividing both sides by
\[
\int_{\Gamma} d\sigma_x \int_{\partial\Omega \setminus [\Gamma]} \frac{\partial^2 G(x, y)}{\partial \nu_x \partial \nu_y} d\sigma_y
\]
and letting \( \varepsilon \to 0^+ \) we arrive at (2), referring to (26). The proof is complete.

\[\square\]

**Remark.** Inspection of the proofs of Theorems 1 and 2 shows that if the coefficients are smooth and Dirichlet data on \( \partial \Omega \) are nonnegative, we have also the following result:

The inequality
\[
\mathcal{D}[\Phi \circ P h] \leq C_+ \mathcal{D}[P(\Phi \circ h)]
\]
holds for any nonnegative \( h \in W^{1/2, 2}(\partial\Omega) \) if and only if
\[
\overline{\Psi^2} \leq C_+ (\Psi)^2
\]
for all finite intervals \( \sigma \subset \mathbb{R}_+ \).

**Example.** As a simple application of this Theorem, consider the case of the Laplace operator and the function \( \Psi(t) = |t|^\alpha \) \((\alpha > -1/2)\). Let \( C_\alpha \) and \( C_{\alpha, +} \) be the following constants:
\[
C_\alpha = \frac{(\alpha + 1)^2}{2\alpha + 1} \sup_{t \in \mathbb{R}} \frac{(1 - t)(1 - t^{2\alpha+1})}{(1 - |t|^\alpha t^2)}, \quad (27)
\]
\[
C_{\alpha, +} = \frac{(\alpha + 1)^2}{2\alpha + 1} \sup_{t \in \mathbb{R}_+} \frac{(1 - t)(1 - t^{2\alpha+1})}{(1 - t^{\alpha+1})^2}. \quad (28)
\]

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Theorem 1 shows that
\[
\int_\Omega |\nabla (|Ph|^\alpha Ph)|^2 dx \leq C_\alpha \int_\Omega |\nabla P(|h|^\alpha h)|^2 dx ,
\] (29)
for any \( h \in W^{1/2,2}(\partial \Omega) \), where \( P \) denotes the harmonic Poisson operator for \( \Omega \).

In view of the Remark above, we have also
\[
\int_\Omega |\nabla (Ph)^{\alpha+1}|^2 dx \leq C_{\alpha,+} \int_\Omega |\nabla P(h^{\alpha+1})|^2 dx ,
\] (30)
for any nonnegative \( h \in W^{1/2,2}(\partial \Omega) \). Because of Theorem 2, the constants \( C_\alpha \) and \( C_{\alpha,+} \) are the best possible in (29) and (30).

Hence the problem of finding explicitly the best constant in such inequalities is reduced to the determination of the supremum in (27) and (28).

One can check that \( C_{\alpha,+} = (\alpha + 1)^2/(2\alpha + 1) \). In the particular case \( \alpha = 1 \), \( C_1 \) can also be determined. This leads to the inequalities (3) and (4).

References

[1] Maz’ya, V., Plamenevski˘ı, On the asymptotics of the fundamental solutions of elliptic boundary value problems in regions with conical points (Russian), Probl. Mat. Anal. 7, 100–145 (1979), Engl. translation in: Sel. Math. Sov. 4, 363–397 (1985).