Superanalogs of symplectic and contact geometry and their applications to quantum field theory.

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Abstract

The paper contains a short review of the theory of symplectic and contact manifolds and of the generalization of this theory to the case of supermanifolds. It is shown that this generalization can be used to obtain some important results in quantum field theory. In particular, regarding $N$-superconformal geometry as particular case of contact complex geometry, one can better understand $N = 2$ superconformal field theory and its connection to topological conformal field theory. The odd symplectic geometry constitutes a mathematical basis of Batalin-Vilkovisky procedure of quantization of gauge theories.

The exposition is based mostly on published papers. However, the paper contains also a review of some unpublished results (in the section devoted to the axiomatics of $N = 2$ superconformal theory and topological quantum field theory). The paper will be published in Berezin memorial volume.

Introduction.

It is a great pleasure for me to publish my paper in this volume. F. Berezin made extremely important contribution to mathematical physics in many different directions. However, the most significant part of his heritage is related to the idea that the theory of fermions becomes very similar to the theory of bosons, if the usual functions ("functions of commuting variables") are replaced by the "functions of anticommuting variables" (elements of a Grassmann algebra). He was the first who realized that along with standard algebra, analysis and geometry one can construct algebra and analysis of functions depending not only on commuting, but also on anticommuting variables, and develop geometry of manifolds with commuting and anticommuting coordinates. These ideas found very important applications to physics. They were used to analyze a new kind

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of symmetry. This symmetry, mixing bosons and fermions, is called supersymmetry; therefore corresponding mathematical concepts are also provided with the prefix "super".

The present paper is based completely on such concepts. It begins with a brief introduction to the ideas of supergeometry. The main part of the paper contains a short review of the theory of symplectic and contact manifolds and of the generalization of this theory to the case of supermanifolds. It is shown that this generalization can be used to obtain some important results in quantum field theory. In particular, regarding $N$-superconformal geometry as a particular case of contact complex geometry, one can better understand $N = 2$ superconformal field theory and its connection to topological conformal field theory. The odd symplectic geometry constitutes a mathematical basis of the Batalin-Vilkovisky procedure of quantization of gauge theories.

Our exposition is based mostly on the papers [1]-[7]. However, the paper contains also a review of some unpublished results (in the section devoted to the axiomatics of $N = 2$ superconformal theory and topological quantum field theory).

Supergeometry.

A smooth $m$-dimensional manifold can be defined as an object obtained from domains in $\mathbb{R}^m$ pasted together by means of smooth transformations. This definition can be formulated in a purely algebraic way. Namely, one can identify a domain $U \subset \mathbb{R}^m$ with the algebra $C^\infty(U)$ of all smooth functions on $U$ and a smooth map of $U$ into $V$ with a homomorphism of $C^\infty(V)$ into $C^\infty(U)$. Such an algebraic construction can be generalized as follows. By definition, we identify an $(m|n)$-dimensional superdomain $U_n$ with the $\mathbb{Z}_2$-graded algebra $C^\infty(U) \otimes \Lambda_n$ where $U$ is a domain in $\mathbb{R}^m$ and $\Lambda_n$ is a Grassmann algebra with $n$ generators $\xi_1, \ldots, \xi_n$. In particular if $U = \mathbb{R}^m$ we obtain a superdomain denoted by $\mathbb{R}^{m|n}$.

One says that $\mathbb{R}^{m|n}$ is an $(m|n)$-dimensional linear superspace. A map of $U_n$ into $V_{n'}$, where $U$ is a domain in $\mathbb{R}^m$, $V$ is a domain in $\mathbb{R}^{m'}$, is defined as an even homomorphism of $C^\infty(V) \otimes \Lambda_{n'}$ into $C^\infty(U) \otimes \Lambda_n$. (We say that an operator acting on $\mathbb{Z}_2$-graded spaces is even if it is parity preserving and odd if it is parity reversing.) Elements of the algebra $C^\infty(U) \otimes \Lambda_n$ can be written as formal linear combinations

$$F = \sum_k \sum_{i_1, \ldots, i_k} f_{i_1, \ldots, i_k}(x) \xi^{i_1} \ldots \xi^{i_k},$$

where $f_{i_1, \ldots, i_k}(x)$ are smooth functions on $U$ and $\xi^i \xi^j = -\xi^j \xi^i$. Without loss of generality we assume that the coefficients $f_{i_1, \ldots, i_k}$ are antisymmetric with respect to a permutation of $i_1, \ldots, i_k$. It is convenient to consider the elements of $C^\infty(U) \otimes \Lambda_n$ as functions depending on commuting variables $(x^1, \ldots, x^m) \in U$ and anticommuting variables $\xi^1, \ldots, \xi^n$. (In other words elements $C^\infty(U) \otimes \Lambda_n$ are considered as functions on the superdomain $U_n$ with $m$ commuting coordinates $x^1, \ldots, x^m$ and $n$ anticommuting coordinates $\xi^1, \ldots, \xi^n$). A map of a
superdomain $U_n$ with coordinates $x^1, \ldots, x^m, \xi^1, \ldots, \xi^n$ into a superdomain $V_{n'}$ with coordinates $\tilde{x}^1, \ldots, \tilde{x}^{m'}, \tilde{\xi}^1, \ldots, \tilde{\xi}^{n'}$ can be specified by the formulas

$$\tilde{x}^i = a^i(x^1, \ldots, x^m, \xi^1, \ldots, \xi^n)$$

$$\tilde{\xi}^j = \alpha^j(x^1, \ldots, x^m, \xi^1, \ldots, \xi^n)$$

where $a^i$, $1 \leq i \leq m'$, and $\alpha^j$, $1 \leq j \leq n'$ are correspondingly even and odd elements of $C^\infty(U) \otimes \Lambda_n$. (It is easy to check that the substitution of $a^i$ and $\alpha^j$ into the functions of variables $\tilde{x}^i, \tilde{\xi}^j$ determines a homomorphism from $C^\infty(V) \otimes \Lambda_{n'}$ into $C^\infty(U) \otimes \Lambda_n$ if the functions $a^i(x^1, \ldots, x^m, 0, \ldots, 0)$, $1 \leq i \leq m'$ determine a map of the domain $U \subset \mathbb{R}^m$ into the domain $V \subset \mathbb{R}^{m'}$.)

Now one can define an $(m|n)$-dimensional supermanifold as an object pasted together from $(m|n)$-dimensional superdomains by means of invertible maps. The definition of superdomain and supermanifold given above is completely algebraic. In this definition a supermanifold "has no points." However, if $M$ is a supermanifold and $\Lambda$ is an arbitrary Grassmann algebra, one can construct the set $M_\Lambda$ of $\Lambda$-points of $M$. In the particular case when $M$ is a superdomain $U_n$ we define $M_\Lambda$ as the set of all rows $(x^1, \ldots, x^m, \xi^1, \ldots, \xi^n)$ where $\xi^1, \ldots, \xi^n$ are arbitrary odd elements of $\Lambda$ and $x^1, \ldots, x^m$ are even elements of $\Lambda$ obeying $(m(x^1), \ldots, m(x^m)) \in U$ (here $m$ is the standard homomorphism of $\Lambda$ onto $\mathbb{R}$.) It is easy to check that a map of superdomains generates a map of corresponding sets of $\Lambda$-points. Using this remark one can define the set $M_\Lambda$ for an arbitrary supermanifold $M$. To every parity preserving homomorphism $\rho : \Lambda \to \Lambda'$ of Grassmann algebras one can assign a map $\tilde{\rho} : M_\Lambda \to M_{\Lambda'}$ in natural way; if $\rho : \Lambda \to \Lambda'$ and $\rho' : \Lambda' \to \Lambda''$ are two parity preserving homomorphism of Grassmann algebras then $\rho \rho' = \tilde{\rho} \cdot \tilde{\rho}'$. In other words a supermanifold $M$ determines a functor on the category of Grassmann algebras with values in the category of sets. (If $M$ is an $(m|n)$-dimensional manifold and $\Lambda$ is a Grassmann algebra having $l$ generators, one can consider $M_{\Lambda}$ as a smooth manifold of dimension $2^{l-1}(m+n)$. Therefore one can say also that a supermanifold determines a functor on the category of Grassmann algebras with values in the category of smooth manifolds.) The language of $\Lambda$-points is often very convenient.

We define an algebraic operation on the supermanifold $M$ as a map $M \times M \to M$. It is easy to see that such an operation in $M$ determines naturally a map $M_\Lambda \times M_\Lambda \to M_\Lambda$, i.e an operation on $M_{\Lambda}$. (We are talking here about binary operations; however one can consider in the same way operations with arbitrary number of arguments.) Let us suppose that $M$ is equipped with a binary operation and a unary operation in such a way that $M_\Lambda$ is a Lie group with respect to corresponding operations in $M_{\Lambda}$ for every $\Lambda$. (The operations are considered as multiplication and taking the inverse respectively.) We say then that $M$ is provided with a structure of Lie supergroup. (In such a way a Lie supergroup determines a functor on the category of Grassmann algebras
with values in the category of Lie groups.) The notions of super Lie algebra, supercommutative algebra etc. can be defined in similar way.

We will define a superspace as a functor on the category of Grassmann algebras taking values in the category of sets. (Morphisms in the category of Grassmann algebras are parity preserving homomorphism.) Similarly, a supergroup can be defined as a functor on the category of Grassmann algebras taking values in the category of groups. One can define in natural way the notion of action of a supergroup on a superspace and the notion of orbit superspace of such an action. It is clear that a supermanifold can be considered as a superspace and a Lie supergroup can be regarded as a supergroup. However if a Lie supergroup acts on a supermanifold the corresponding orbit superspace is not necessarily a supermanifold.

Almost all notions of algebra, geometry and analysis can be formulated for supermanifolds. For example, the (Berezin) integral of the function \((1)\) over \(U_n\) is defined by the formula
\[
\int_{U_n} F d^n x d^n \xi = n! \int_U f_1, \ldots, n(x) d^n x
\]
(2)
A (left) derivation in \(C^\infty(U) \otimes \Lambda_n\) can be defined as a linear operator \(D\) satisfying the condition
\[
D(\xi^i) = \delta^i_1, \quad D f(x) = 0.
\]
It is easy to check that derivations in \(C^\infty(U) \otimes \Lambda_n\) can be identified with first order differential operators (operators of the form \(A(x, \xi) \frac{\partial}{\partial x^i} + \alpha^i(x, \xi) \frac{\partial}{\partial \xi^j}\). The definitions of right derivation and right derivative \(\frac{\partial}{\partial \xi^j}\) are similar. To define a differential form on a superdomain \(U_n\) with commuting coordinates \((x^1, \ldots, x^m) \in U\) and anticommuting coordinates \(\xi^1, \ldots, \xi^n\) we consider a function of commuting variables \(x^1, \ldots, x^m, \tilde{\xi}^1, \ldots, \tilde{\xi}^n\) and anticommuting variables \(\xi^1, \ldots, \xi^n, \tilde{x}^1, \ldots, \tilde{x}^m\). Such a function determines a \(k\)-form if it is a polynomial of degree \(k\) with respect to the variables \(\tilde{x}^1, \ldots, \tilde{x}^m, \tilde{\xi}^1, \ldots, \tilde{\xi}^n\). The variables \(\tilde{x}^i, \tilde{\xi}^j\) can be identified with differentials \(dx^i, d\xi^j\) (note that the parity of the differential is opposite to the parity of corresponding variable.) The generalization of the Rham differential is defined by the formula
\[
d = \sum_i \tilde{x}^i \frac{\partial}{\partial x^i} + \sum_j \tilde{\xi}^j \frac{\partial}{\partial \xi^j}.
\]
All notions defined above for superdomains are invariant with respect to invertible transformations and therefore our definitions work also for supermanifolds. The only exception is the Berezin integral. As in the case of the usual
integral, the integrand acquires a factor equal to the determinant of the Jacobian matrix by the change of variables. Of course, the determinant here should be understood as the superdeterminant (Berezinian).

For a supermanifold $M$ one can introduce the notion of tangent bundle $TM$ and cotangent bundle $T^*M$. One can consider also the tangent and cotangent bundles with the parity of fibers reversed; we will use the notations $\Pi TM$ and $\Pi T^*M$ in this case. Note, that a section of the bundle $TM$ or of the bundle $\Pi TM$ can be considered as a vector field on $M$; a section of $T^*M$ or of $\Pi T^*M$ is a 1-form on $M$. A $k$-form on $M$ can be considered as a function on $\Pi TM$.

We will use freely the supergeneralizations of standard notions of algebra and analysis.

**Symplectic supermanifolds.**

We can apply the standard definition of symplectic manifold to the case of supermanifolds. Namely, a symplectic structure on a supermanifold $M$ can be specified by means of an even non-degenerate closed 2-form

$$\omega = d\bar{z}^a \omega_{ab}(z) d\bar{z}^b.$$  (3)

Here $z^a$ are (even and odd) coordinates on $M$. As always $\omega$ is closed if $d\omega = 0$ and non-degenerate if the (super) matrix $\omega_{ab}$ is invertible. If the form $\omega$ in the definition above is odd we say that it specifies an odd symplectic structure; the manifolds equipped with an odd symplectic structure (odd symplectic manifolds or $P$-manifolds) will be considered at the end of the paper. Symplectomorphisms (canonical transformations) of a symplectic manifold $M$ are defined as transformations of $M$ preserving the form $\omega$ (i.e. $f$ is a symplectomorphism if $f^* \omega = \omega$). Locally a 2-form $\omega$ specifying a symplectic structure in an appropriate coordinate system can be written as

$$\omega_0 = \sum_{1 \leq i \leq n} dp_idx^i + \sum_{1 \leq j \leq m} (d\xi^j)^2$$  (4)

(Here $x^1, ..., x^n, p_1, ..., p_n$ are even coordinates, $\xi^1, ..., \xi^m$ are odd coordinates.) Therefore one can define a symplectic supermanifold as a supermanifold pasted together from superdomains in $R^{2n|m}$ by means of transformations preserving $\omega_0$ (canonical transformations).

One can define the Poisson bracket of two functions $F, G$ on the symplectic manifold $M$ by the formula

$$\{F, G\} = \frac{\partial F}{\partial z^a} \omega^{ab}(z) \frac{\partial G}{\partial \bar{z}^b}$$  (5)

where $\omega^{ab}$ stands for the matrix inverse to $\omega_{ab}$.

The space $C(M)$ of functions on $M$ can be considered as a $Z_2$-graded Lie algebra (super Lie algebra) with respect to this operation. We can assign to
every function $H$ a vector field $K_H$ on $M$ by the formula
\[ K_H^a = \omega^{ab}(z) \frac{\partial H}{\partial z^b}. \] (6)

The corresponding first order differential operator $\hat{K}_H$ acts in the following way:
\[ \hat{K}_H F = \{F,H\}. \] (7)

Vector fields obtained by means of this construction (Hamiltonian vector fields) preserve the form $\omega$; in other words they can be considered as infinitesimal symplectomorphisms. The map $H \rightarrow K_H$ is a homomorphism of the (super)Lie algebra $C(M)$ into the (super)Lie algebra of vector fields (of first order differential operators).

If we omit the requirement of non-degeneracy of the form $\omega$ in the definition of symplectic (super)manifold we obtain the definition of presymplectic manifold. If $\omega$ is degenerate there exist vectors $\xi^l$ satisfying $\omega_{kl}(z)\xi^l = 0$. Such vectors are called null vectors of the form $\omega$. The set $W_z$ of null vectors at the point $z \in M$ can be considered as linear subspace of the tangent space $T_z$ to $M$ at the point $z \in M$.

We want to construct a symplectic manifold $M'$ corresponding to the presymplectic manifold $M$. With this aim we identify the point $z \in M$ with the point $z + dz$ if $dz$ is a null vector of $\omega$ (i.e. $\omega_{kl}dz^l = 0$). To be more rigorous, we will suppose that $W_z$ is a linear superspace whose dimension does not depend on $z \in M$; this dimension will be denoted by $(d_1|d_2)$. By means of Frobenius’ theorem for every point $z \in M$ we can construct at least locally a $(d_1|d_2)$-dimensional manifold $R$ satisfying 1) $z \in R$, 2) the tangent space to $R$ at an arbitrary point $u \in R$ coincides with $W_u$. (The subspaces $W_z$ determine $(d_1|d_2)$-dimensional distribution $W$ on the manifold $M$. This distribution is integrable; this means that for every two vector fields $\xi_1(z) \in W_z$, $\xi_2(z) \in W_z$ their commutator also belongs to $W_z$. By the Frobenius’ theorem we can construct for every point $z \in M$ an integral manifold $R$ of the distribution $W$ containing the point $z$.)

We will identify two points of $M$ belonging to the same manifold $R$. Then for every point $z \in M$ we can find a neighborhood $U$ in such a way that by this identification we obtain from $U$ a symplectic manifold $U'$ (the form $\omega'$ specifying the symplectic structure on $U'$ can be determined by relation $\pi^*\omega' = \omega$ where $\pi$ denotes the natural projection of $U$ onto $U'$). In global consideration we can meet topological troubles. (The global behavior of an integral manifold can be very complicated. Therefore by the identification of points belonging to the same integral manifold we can obtain from $M$ a complicated topological space $M'$. The space is not necessarily a manifold; moreover one can’t assert the points of $M'$ are closed sets.) However in physics we do not usually meet these troubles.

**Contact manifolds.**

We will say that a 1-form $\alpha$ on the domain $U \subset \mathbb{R}^{2n+1}$ determines a precontact structure on $U$. If $\alpha' = F\alpha$, where $F$ denotes a non-vanishing function on
we say that the form $\alpha$ and $\alpha'$ determine the same precontact structure. If the 1-form $\alpha$ is non-degenerate (i.e. the $(2n+1)$-form $\alpha \wedge (d\alpha)^n$ does not vanish anywhere) we say that $\alpha$ determines a contact structure on $U$. For instance, the form

$$\alpha = dz + \frac{1}{2} (pdq - qdp), \quad z \in \mathbb{R}, \quad p \in \mathbb{R}^n, \quad q \in \mathbb{R}^n \quad (8)$$

determines a contact structure on $\mathbb{R}^{2n+1}$. This example is universal in some sense: locally every non-degenerate 1-form takes the form (8) in an appropriate coordinate system.

A transformation $\varphi$ is called a contactomorphism if

$$\varphi^* \alpha = G \alpha \quad (9)$$

where $G$ is a non-vanishing function on $U$. In other words a contactomorphism can be defined as a map preserving (pre)contact structure. If a precontact structure is defined on the domain $U \in \mathbb{R}^{2n+1}$ by means of the 1-form $\alpha = \alpha_i(z) dz^i$ we can determine a presymplectic structure on the domain $\mathbb{R}^* \times U \subset \mathbb{R}^{2n+2}$ by means of the closed 2-form $\omega = d\lambda$ where

$$\lambda = dt + t\alpha = dt + t\alpha_i(z) dz^i \quad (10)$$

(here $\mathbb{R}^*$ denotes the set of non-zero real numbers: $\mathbb{R}^* = \{t \mid t \in \mathbb{R}, \ t \neq 0\}$). The presymplectic structure on $\mathbb{R}^* \times U$ is called a symplectization of the precontact structure in $U$. It is easy to check that the 2-form

$$\omega = d\lambda = dt \wedge \alpha + t d\alpha \quad (11)$$

is non-degenerate if and only if the form $\alpha$ is non-degenerate. In other words one can say that a contact structure in $U$ can be defined as a precontact structure giving a symplectic structure in $\mathbb{R}^* \times U$ after symplectization.

Let us consider a contactomorphism $\varphi$ of the domain $U$. We will construct a transformation $\tilde{\varphi}$ of $\mathbb{R}^* \times U$ by the formula

$$\tilde{\varphi}(t, u) = (G^{-1}t, \varphi(u)) \quad (12)$$

where $G$ is defined by (9). It is easy to check that $\tilde{\varphi}$ is a symplectomorphism of $\mathbb{R}^* \times U$; one can say that $\tilde{\varphi}$ is obtained from $\varphi$ by means of symplectization. The transformation $\tilde{\varphi}$ is homogeneous of degree 1 with respect to $t$: if $\tilde{\varphi}(t, u) = (t', u')$ then $\tilde{\varphi}(\lambda t, u) = (\lambda t', u')$. One can verify that every symplectomorphism $\tilde{\varphi}$ of $\mathbb{R}^* \times U$ satisfying this condition generates a contactomorphism $\varphi$ of $U$ by the formula (12). This assertion permits us to give a description of infinitesimal contactomorphisms. Let us suppose for simplicity that $u = (z, p, q), \ z \in \mathbb{R}, \ p \in \mathbb{R}^n, \ q \in \mathbb{R}^n$ and $\alpha$ is given by (8). It is evident that infinitesimal contactomorphisms correspond to infinitesimal symplectomorphisms (to Hamiltonian vector fields) in $\mathbb{R}^* \times U$ having a function of the form
\[ H(t, u) = H(t, z, p, q) = t K(z, p, q) \]  \hspace{1cm} \text{(13)}

as a Hamiltonian. We see that an infinitesimal contactomorphism can be written in the form

\[ \delta z = K - \frac{p}{2} \frac{\partial K}{\partial p} - \frac{q}{2} \frac{\partial K}{\partial q}, \quad \delta p = -\frac{\partial K}{\partial q} + \frac{p}{2} \frac{\partial K}{\partial z}, \quad \delta q = \frac{\partial K}{\partial p} + \frac{q}{2} \frac{\partial K}{\partial z} \]  \hspace{1cm} \text{(14)}

where \( K \) is an arbitrary function on \( U \).

Let us define a \((2n + 1)\)-dimensional contact manifold as a manifold pasted together from domains in \( \mathbb{R}^{2n+1} \) provided with contact structure, by means of a contactomorphism. Without loss of generality one can assume that the contact structure on these domains is standard (determined by the form \((8)\)). Given an arbitrary \( n \)-dimensional manifold \( M \), one can construct a \((2n - 1)\)-dimensional contact manifold \( PT^* M \) in the following way. Let us consider the space \( T^* M \) of all covectors in \( M \) and the form \( \omega = dp_1 \wedge dq^1 \) specifying a symplectic structure on \( T^* M \) (here \( q^1, \ldots, q^n \) are local coordinates on \( M \) and \( p_1, \ldots, p_n \) denote coordinates of the covector). Let us identify the points \((p_1, \ldots, p_n, q^1, \ldots, q^n)\) and \((\lambda p_1, \ldots, \lambda p_n, q^1, \ldots, q^n)\) in \( T^* M \setminus M \) (here \( \lambda \neq 0 \)). The space obtained by means of this identification will be denoted by \( PT^* M \). (One can say that \( PT^* M \) is the space of the projectivized cotangent bundle.) The space \( PT^* M \) can be provided with a contact structure in natural way; the symplectization of this contact structure gives the natural symplectic structure on \( T^* M \). The contact structure in \( PT^* M \) can be specified by means of a 1-form \( F p_i dq^i \). More precisely, let us consider the case when \( M \) is a domain in \( \mathbb{R}^n \) and \( q^1, \ldots, q^n \) are coordinates on \( M \). Then \( T^* M \setminus M \) can be covered by sets \( \tilde{U}_1, \ldots, \tilde{U}_n \), where \( \tilde{U}_i \) is singled out by the condition \( p_i \neq 0 \). By identification we obtain from \( \tilde{U}_1, \ldots, \tilde{U}_n \) the charts \( U_1, \ldots, U_n \) covering \( PT^* M \). We can introduce coordinates in \( U_i \) by the formula \( \pi_k = p_k/p_i \). In other words each point of \( U_i \) can be obtained from a point of \( \tilde{U}_i \) satisfying \( p_i = 1 \); the coordinates of this point of \( \tilde{U}_i \) are considered as coordinates of the corresponding point in \( U_i \). The 1-form specifying the contact structure in \( U_i \) can be written as

\[ \alpha^{(i)} = dq^i + \sum_{k \neq i} \pi_k dq^k = p_i dq^i. \]  \hspace{1cm} \text{(15)}

It is easy to check that the forms \( \alpha^{(i)} \) specify a contact structure in \( PT^* M \) (i.e. \( \alpha^{(i)} \) and \( \alpha^{(j)} \) are proportional in \( U_i \cap U_j \)). The definition of precontact structure and the definition of contactomorphism can be applied also in the case when the domain \( U \subset R^{2n+1} \) is replaced by a superdomain \( U \subset R^{2n+1|m} \).

The symplectization of precontact structure can be defined in the supercase and gives a presymplectic structure on the superdomain \( R^* \times U \subset R^{2n+2|m} \).

As usual, we say that the 1-form \( \alpha \) determines a contact structure, if \( \alpha \) is non-degenerate, but the definition of non-degeneracy should be changed. One can
say, for example, that $\alpha$ is non-degenerate if the 2-form (11) is non-degenerate. (In other words a contact structure in $U$ can be defined as a precontact structure giving a symplectic structure in $R^* \times U$ after symplectization.)

**Symplectic and contact complex (super)manifolds.**

Let $M$ be a complex (super)manifold. A presymplectic structure on $M$ can be specified by a closed holomorphic (2,0)-form $\omega$ on $M$; if this form is non-degenerate we say that $M$ is a symplectic complex manifold. In local coordinates $(z^1, ..., z^n)$ the form $\omega$ can be represented by the formula (3) where $\omega_{ab}(z)$ are holomorphic functions.

In such a way the definition of symplectic structure on a complex manifold repeats the definition in the real case, but $\omega$ has to be considered as a holomorphic (2,0)-form. The definition of contact structure on a complex manifold also repeats the definition of contact structure on a real manifold (the form $\alpha$ in this definition must be considered as a non-degenerate holomorphic (1,0)-form).

If $M$ is an arbitrary real or complex ($m|n$)-dimensional supermanifold then the cotangent space $T^* M$ can be considered as a ($2m|2n$)-dimensional symplectic supermanifold and the projectivized cotangent space $PT^* M$ as a ($2m-1|2n$)-dimensional contact supermanifold. (The construction of symplectic structure on $T^* M$ and of contact structure in $PT^* M$ is quite similar to the construction described above for the case when $M$ is a real $m$-dimensional manifold).

Complex contact geometry is closely related to superconformal geometry. Recall that an $N$-superconformal transformation of a ($1|N$)-dimensional complex superdomain $U$ can be defined as a complex analytic transformation preserving up to multiplier the 1-form

$$\alpha = dz + \sum_i \theta_i d\theta_i$$

(16)

Here $(z_1, \theta_1, ..., \theta_n)$ denote complex coordinates in $U$ ($z$ is even, $\theta_1, ..., \theta_n$ are odd). The 1-form $\alpha$ is non-degenerate and therefore specifies complex contact structure. Superconformal transformations can be interpreted as complex contact transformations of the contact manifold $U$. An $N$-superconformal manifold can be defined as a manifold pasted together from ($1|N$)-dimensional complex superdomains by means of $N$-superconformal transformations. We see that an $N$-superconformal manifold can be considered as a ($1|N$)-dimensional complex contact manifold. Conversely, every ($1|N$)-dimensional complex contact manifold can be considered as an $N$-superconformal manifold. The proof is based on the fact that locally every non-degenerate holomorphic form $\alpha_0(z, \theta)dz + \sum \alpha_i(z, \theta)d\theta_i$ on a ($1|N$)-dimensional superdomain in appropriate complex coordinate system can be identified (up to holomorphic multiplier) with the form (16). We see that a ($1|N$)-dimensional complex contact manifold can be covered with local coordinates $(z, \theta_1, ..., \theta_N)$ in such a way that transition functions are $N$-superconformal transformations. (These coordinates are called superconformal coordinates.) Using these coordinates one can introduce covariant
derivatives

\[ D_i = \frac{\partial}{\partial \theta_i} + \theta_i \frac{\partial}{\partial z}. \]

It is easy to check that covariant derivatives in superconformal coordinate systems \((z, \theta_1, ..., \theta_N)\) and \((\tilde{z}, \tilde{\theta}_1, ..., \tilde{\theta}_N)\) are connected by the relation

\[ \tilde{D}_i = \sum_j F_{ij}(z, \theta) D_j \]

(17)

(This fact follows immediately from the remark that the orthogonal complement to the covector corresponding to the 1-form (16) is spanned by the vectors \(D_1, ..., D_N\).) In the case \(N = 2\) it is convenient to introduce coordinates

\[ \theta_+ = \frac{1}{\sqrt{2}}(\theta_1 + i\theta_2), \quad \theta_- = \frac{1}{\sqrt{2}}(\theta_1 - i\theta_2) \]

and covariant derivatives

\[ D_+ = \frac{1}{\sqrt{2}}(D_1 + iD_2) = \frac{\partial}{\partial \theta_-} + \frac{1}{2} \theta_+ \frac{\partial}{\partial z}, \quad D_- = \frac{1}{\sqrt{2}}(D_1 - iD_2) = \frac{\partial}{\partial \theta_+} + \frac{1}{2} \theta_- \frac{\partial}{\partial z} \]

(18)

One can verify that the behavior of \(D_+, D_-\) under an \(N = 2\) superconformal transformation is given by the formulas

\[ \tilde{D}_+ = F_+ D_+, \quad \tilde{D}_- = F_- D_- \]

(19)

(untwisted case) or by the formulas

\[ \tilde{D}_+ = G_+ D_-, \quad \tilde{D}_- = G_- D_+ \]

(20)

(twisted case). An \(N = 2\) superconformal manifold \(M\) is called untwisted if a local coordinate system in \(M\) can be chosen in such a way that the transition functions are untwisted superconformal transformations (covariant derivatives in different patches are related by (19)). Let us denote by \(M^{k|l}\) the moduli space of \((k|l)\)-dimensional complex supermanifolds, i.e. the space of classes of such manifolds with respect to holomorphic equivalence. This definition is not rigorous. More precisely one can also define the moduli space using as a starting point the superspace of all complex structures on a fixed real supermanifold and factorizing this superspace with respect to an appropriate equivalence relation. The moduli space defined in such a way is not necessarily a supermanifold. However it can be considered as a superspace.

The notion of moduli space of complex manifolds is closely related to the notion of family of complex manifolds. In some sense the moduli space can be considered as a base of a universal family. Recall that a holomorphic family
of compact complex $(k|l)$-dimensional supermanifolds can be defined as holomorphic map of a complex supermanifold $E$ onto a complex supermanifold $S$ having $(k|l)$-dimensional compact complex manifolds as fibers. One says that $S$ is the base of the family or that the family is parametrized by the points of $S$. The definition of a continuous family is analogous. (Speaking about families of manifolds we have in mind continuous families.) For every continuous map $\rho : S' \to S$ of a space $S'$ into the base $S$ of a family $F$ one can construct in natural way a family $F'$ with the space $S'$ as a base (the pullback of $F$).

In similar way we can define other moduli spaces, for example the moduli space $M_N$ of $N$-superconformal complex manifolds. (This moduli space is related to families of $N$-superconformal manifolds.)

For every $(1|n)$-dimensional complex supermanifold $M$ we can construct a $(1|2n)$-dimensional complex contact supermanifold $PT^*M$. Considering $PT^*M$ as $2n$-superconformal manifold we obtain an embedding of the moduli space $M^{1|n}$ into the moduli space $M_{2n}$. Let us consider more carefully the case $n = 1$. In this case we see that to every $(1|1)$-dimension complex supermanifold $M$ we can assign the $N = 2$ superconformal manifold $\hat{M} = PT^*M$. If $(z, \theta)$ are complex coordinates in $M$ then the points of $T^*M$ can be specified by coordinates $(p, \rho, z, \theta)$ where $(p, \rho)$ are the coordinates of the covector on $M$.

The behavior of these coordinates under the transformation $(z, \theta) \to (\tilde{z}, \tilde{\theta})$ is given by

$$\tilde{p} = \frac{\partial z}{\partial \tilde{z}} p + \frac{\partial z}{\partial \tilde{\theta}} \rho,$$

$$\tilde{\rho} = \frac{\partial \theta}{\partial \tilde{z}} b + \frac{\partial \theta}{\partial \tilde{\theta}} b.$$

A point of $PT^*M$ can be specified by a triple $(p, z, \theta)$ and the behavior of this triple by the transformation $(z, \theta) \to (\tilde{z}, \tilde{\theta})$ is given by

$$\tilde{z} = \tilde{z}(z, \theta), \quad \tilde{\theta} = \tilde{\theta}(z, \theta), \quad \tilde{\rho} = (\frac{\partial \theta}{\partial z} + \frac{\partial \theta}{\partial \theta} \rho)(\frac{\partial z}{\partial z} + \frac{\partial z}{\partial \theta} \rho)^{-1}$$

Using the general rule one can write the 1-form specifying the contact ($N = 2$ superconformal) structure in $PT^*M$ as

$$\alpha = dz + \rho d\theta$$

(we consider the form $pdz + \rho d\theta$ for $p = 1$).

In appropriate coordinates this form coincides with (16). Using these coordinates one can check that the transformation (23) is an untwisted $N = 2$ superconformal transformation and therefore $PT^*M$ is an untwisted $N = 2$ superconformal manifold. It is easy to check that every untwisted $N = 2$ superconformal manifold can be obtained by means of this construction (see ref. [2]). The proof can be based on the remark that for every untwisted $N = 2$ superconformal manifold $N$ one can construct a $(1|1)$-dimensional complex manifold.
α(N) factorizing N with respect to the vector field $D_+$. (In other words, we assume that there exists a natural map $\pi : N \to \alpha(N)$ and that the set of functions on N having the form $\Phi(\pi(x))$, where $\Phi$ is a function on $\alpha(N)$, consists of functions annihilated by the operator $D_+$). One can check that $PT^*\alpha(N)$ is isomorphic to $N$. In such a way we have proved that there is one-to-one correspondence between classes of (1|1)-dimensional complex manifolds and classes of $N = 2$ untwisted superconformal manifolds. In other words, the moduli space $M^{1|1}$ coincides with the part of $M_2$ corresponding to untwisted $N = 2$ superconformal manifolds; we denote this part by $M_2$. Note that all moduli spaces at hand have natural complex structures; the isomorphism $\alpha$ is compatible with the complex structures in $M_2$ and $M^{1|1}$. One can replace $D_+$ by $D_-$ in the construction of the isomorphism $\alpha : M_2 \to M^{1|1}$, we obtain another isomorphism $\beta : M_2 \to M^{1|1}$. This means that there exists a natural involution $\iota : M^{1|1} \to M^{1|1}$ defined by the formula $\beta = \alpha \iota$. This involution can be described geometrically in the following way: if $M$ is a (1|1)-dimensional complex manifold, we define $\iota(M)$ as the manifold consisting of all (0|1)-dimensional submanifolds of $M$.

The superMumford form in superstring theory can be considered as an object defined on $M^{1|1}$; using this fact and the connection between $M^{1|1}$ and $M_2$ one can discover hidden $N = 2$ superconformal symmetry of the superMumford form [2]. A similar consideration leads to the explanation of hidden $N = 2$ superconformal symmetry of $B - C$ system discovered in [8].

**Axiomatics of $N = 2$ superconformal theory and topological quantum field theory**.

Let us recall briefly Segal’s axiomatics of conformal field theory (CFT).

Let us consider a compact complex one-dimensional manifold $M$ with $m + n$ embedded non-overlapping parametrized disks $D_1, ..., D_m, D'_1, ..., D'_n$. (The set of disks is divided in two parts: ”incoming” disks $D_1, ..., D_m$ and ”outgoing” $D'_1, ..., D'_n$.) The moduli space of such objects will be denoted by $P_{m,n}$. There exist natural maps

$$P_{m_1,n_1} \times P_{m_2,n_2} \to P_{m_1 + m_2,n_1 + n_2},$$

(25)

$$P_{m,n} \to P_{m-1,n-1}.$$  

(26)

(To construct the first map we take the disjoint union of manifolds. The second map corresponds to sewing of $D_m$ with $D'_n$. In other words we identify the points $P \in D_m$ and $P' \in D'_n$ if corresponding coordinates are related by the formula $z \cdot z' = 1/4$ and delete the points with $|z| < \frac{1}{2}$, $|z'| < \frac{1}{2}$. We assume that all disks are parametrized by the points of the unit disk $D \subset C$.) In Segal’s axiomatics we should fix a linear space $H$. The main object is a linear map $H^m \to H^n$ assigned to every point $M \in P_{m,n}$; this map must have a trace. In other words we should have a map $\alpha_{m,n} : P_{m,n} \to B_{m,n} = Map(H^m,H^n)$ of $P_{m,n}$ into the space $B_{m,n}$ of trace class linear maps of $H^m$ into $H^n$. (Here $H^m$ stands for the tensor product of $m$ copies of $H$). Axioms are the conditions of compatibility
of maps $\alpha_{m,n}$ and maps (25), (26). These axioms can be formulated in terms of commutativity of diagrams containing the maps $\alpha_{m,n}$, the maps (25), (26) and the natural maps

$$B_{m_1,n_1} \otimes B_{m_2,n_2} \rightarrow B_{m_1+m_2,n_1+n_2},$$

(27)

$$B_{m,n} \rightarrow B_{m-1,n-1}.$$  

(28)

One should require also the compatibility of the maps $\alpha_{m,n}$ with the natural action of $S_m \times S_n$ on $P_{m,n}$ and $B_{m,n}$. (Here $S_m$ denotes the symmetric group.) The maps $\alpha_{11}$ can be used to define the representation of Virasoro algebra on $H$; more precisely we obtain two copies of the Virasoro algebra with generators $L_n, \bar{L}_n$ obeying $[L_n, \bar{L}_m] = 0$. (The axioms above describe CFT with central charge $c = 0$. The maps $\alpha_{11}$ give in this case representations of the Lie algebra $diff(S^1)$ of vector fields on the circle. To describe CFT with non-vanishing central charge one should assume that the maps $\alpha_{m,n}$ are defined only up to a multiplier. Then the maps $\alpha_{11}$ give rise to projective representations of $diff(S^1)$, i.e. to representations of the Virasoro algebra, the central extension of $diff(S^1)$.)

It is important to emphasize that one can reformulate the Segal’s axiomatics using families of one-dimensional complex manifolds with embedded disks instead of moduli spaces of such manifolds. We will use the term $(m, n)$-family for a family of one-dimensional compact complex manifolds with $m$ embedded ”incoming” disks and $n$ embedded ”outgoing” disks. The base of an $(m, n)$-family $F$ will be denoted by $B_F$. One should consider as a basic object a map $\alpha_{m,n}^F : B_F \rightarrow B_{m,n} = \text{Map}(H^m, H^n)$ defined for every $(m, n)$-family $F$. Repeating the definitions of the maps (25), (26) we can obtain corresponding maps for families. (For example, the construction of the map (26) permits us to assign an $(m-1, n-1)$-family to every $(m, n)$-family.) It is easy to translate the axioms above into the language of families. One should add the following axiom:

If the $(m, n)$-family $F'$ can be obtained as a pullback of an $(m, n)$-family $F$ by the map $\rho : B' \rightarrow B_F$, then $\alpha_{m,n}^{F'} = \alpha_{m,n}^F \rho$.

Similar axiomatics for superconformal theory can be obtained if we replace one-dimensional complex manifolds with $N$-superconformal manifolds. (The space $H$ should be $Z_2$-graded in this case.) It follows from the consideration above that $N = 2$ superconformal theory has also another description: one should replace one-dimensional complex manifolds with $(1|1)$-dimensional complex manifolds and disks with superdisks. The simplicity of this description leads to a conclusion that probably $N = 2$ superconformal symmetry is more fundamental than $N = 1$ superconformal symmetry.

To give an axiomatic description of topological conformal field theory we introduce a notion of a $T$-manifold. One can define a $T$-manifold as a $(1|1)$-dimensional complex manifold equipped with a non-degenerate even 1-form $\omega$ (non-degeneracy means here that $\omega$ can be represented in the form $\omega =$
$d\varphi$ where locally the odd function $\varphi$ can be written as $\alpha(z)+a(z)\theta$ with invertible $a(z)$.) The notion of a $\mathcal{T}$-manifold is closely related to the notion of semirigid super Riemann surface [9]. We define topological conformal field theory (TCFT) replacing complex manifolds with $\mathcal{T}$-manifolds in Segal’s axiomatics. Following the construction of operators $L_n, L_n$ in CFT we can construct operators in $\mathbb{Z}_2$-graded space $H$. We obtain even operators $L_n, L_n$, generating two commuting copies of the Virasoro algebra, and odd operators $b_n, b_n, Q, Q$ obeying $L_n = [b_n, Q]_+,$ $L_n = [b_n, Q]_+$ (all other anticommutators of odd operators vanish). For every $N = 2$ superconformal theory we can construct a TCFT in an obvious way utilizing the fact that there is a natural map $\tau$ of the moduli space $M_{\mathcal{T}}$ of $\mathcal{T}$-manifolds into $M^{1|1}$ and that this map can be extended to a map of the moduli space of manifolds with embedded disks. (Every $\mathcal{T}$-manifold can be considered as a space consisting of points obeying this involution). The construction above can be considered as a geometric counterpart of Witten’s twisting. The two TCFT corresponding to $N = 2$ superconformal theory are known as the $A$-model and $B$-model.

Note that in the definition of TCFT it is convenient to work with the version of Segal’s axiomatics based on the consideration of families of manifolds. (The moduli space of $\mathcal{T}$-manifolds is not a supermanifold.) If the map $p : E \to B$ determines a holomorphic family $F$ of one-dimensional complex manifolds, then one can construct a family $\tilde{F}$ of $\mathcal{T}$-manifolds in the following way. Let us consider a supermanifold $\tilde{B} = \Pi T B$ (the space of the tangent bundle with reversed parity) and the natural map $\Phi : \Pi T B \times C^{0|1} \to B$. (In local coordinates $\Phi$ transforms the point $(m, \mu, \nu)$ where $m \in B$, $\mu$ is a tangent vector at the point $m$, and $\nu \in C^{0|1}$ into $m + \mu \nu$.) Then we can define $\tilde{E}$ as a space consisting of points $(e, m, \mu, \nu) \in E \times \tilde{B} \times C^{0|1}$ obeying $p(e) = \Phi(m, \mu, \nu)$. The maps $\tilde{p} : \tilde{E} \to \tilde{B}$ and $\pi : \tilde{E} \to C^{0|1}$ are defined by means of projections of the direct product $E \times \tilde{B} \times C^{0|1}$ onto the factors $\tilde{B}$ and $C^{0|1}$. The map $\tilde{p} : \tilde{E} \to \tilde{B}$ determines a family of $(1|1)$-dimensional complex manifolds; the map $\pi$ determines a structure of $\mathcal{T}$-manifold on the fibers of $\tilde{p}$. We obtained therefore holomorphic family $\tilde{F}$ of $\mathcal{T}$-manifolds. Applying this construction to the family corresponding to the moduli space $P_{m,n}$ we obtain a family of $\mathcal{T}$-manifolds with embedded superdisks; the base of this family is $\tilde{P}_{m,n} = \Pi T P_{m,n}$. The functions on $\tilde{P}_{m,n}$ are forms on $P_{m,n}$; this fact permits us to relate the axiomatics based on $\mathcal{T}$-manifolds with axiomatics of TCFT formulated in [10].
factorization with respect to the group of diffeomorphisms and Weyl group we obtain the moduli space $M$ of complex structures. Similarly, after appropriate factorization we obtain $\Pi T M$ from $Q = \Pi T Q$. Using this remark one can construct a $\mathcal{T}$-manifold for every two-dimensional manifold equipped with tensors $g_{\alpha\beta}, \varphi_{\alpha\beta}$.

**Odd symplectic geometry and BV-formalism.**

We mentioned already that odd symplectic structure on a supermanifold $M$ is determined by a non-degenerate closed odd 2-form $\omega$. Odd symplectic manifolds ($P$-manifolds) play an important role in the Batalin-Vilkovisky quantization procedure; we will formulate here some results about the notion of $P$-manifold and related notions.

One can check that a $P$-manifold can be covered by local coordinate systems in such a way that in every chart the form $\omega$ is represented as $dx^i d\xi_i$ (here $x^1, ..., x^n$ are even and $\xi_1, ..., \xi_n$ are odd coordinates). In other words a $P$-manifold can be pasted together from superdomains by means of transformations preserving the form $dx^i d\xi_i$. It is important to emphasize that the transformations preserving the form $dx^i d\xi_i$ are not necessarily volume preserving. (By definition, a transformation of a superdomain is volume preserving if the determinant of Jacobian matrix is equal to 1. Of course we are talking here about the superdeterminant, or Berezinian). It is well known that there exists a natural volume element in an even symplectic manifold and symplectic transformations are volume preserving; we see that in the case of $P$-manifolds the situation is different. We define an $SP$-manifold as a manifold $M$ pasted together from $(n|n)$-dimensional superdomains by means of transformations preserving the form $dx^i d\xi_i$ and the volume. An operator $\Delta$ acting on functions on an $SP$-manifold can be defined by the formula

$$\Delta = \frac{\partial^2}{\partial x^i \partial \xi_i}.$$  \hspace{1cm} (29)

It is easy to check that this formula gives a globally defined operator obeying $\Delta^2 = 0$. Conversely, let

$$\Delta = \omega^{ij}(z) \frac{\partial^2}{\partial z^i \partial z^j} + \alpha^i(z) \frac{\partial}{\partial z^i} + \beta(z)$$  \hspace{1cm} (30)

be an odd second order differential operator on supermanifold $M$ satisfying $\Delta^2 = 0$. One can prove that in the case when the matrix $\omega^{ij}(z)$ is non-degenerate the operator $\Delta$ can be represented locally in the form (29). This means that such an operator determines an $SP$-structure in the manifold $M$. In particular, it determines a $P$-structure and a volume element on $M$.

Let us say that a submanifold $L$ of a $P$-manifold $M$ is Lagrangian if in a neighborhood of every point of $L$ one can introduce such a coordinate system $x^1, ..., x^n, \xi_1, ..., \xi_n$ such, that $\omega$ has the standard form $dx^i d\xi_i$ and $L$ is singled out by the equations $\xi_1 = 0, ..., \xi_n = 0$. (Here $x^i$ can be even or odd, the parity of $\xi_i$
is opposite to the parity of \( x^i \). Therefore \( \dim L = (k|n-k) \) where \( 0 \leq k \leq n \).

In a more invariant way one can characterize Lagrangian manifold as a maximal submanifold of \( M \) where the form \( \omega \) vanishes. If \( M \) is an \( SP \)-manifold one can introduce a volume element in a Lagrangian submanifold \( L \subset M \). For example one can use a local coordinate system \( x^1, ..., x^n, \xi_1, ..., \xi_n \) where \( \Delta \) is standard and \( L \) is singled out by equations \( \xi_1 = ... = \xi_n = 0 \); the volume element \( dx^1...dx^n \) in \( L \) does not depend on the choice of coordinates up to a sign. (One can prove the existence of the coordinate system that we used.) In Batalin-Vilkovisky formalism physical quantities are represented by integrals of \( \exp(\frac{1}{\hbar} S) \) over Lagrangian submanifolds. Here \( S \) is the extended classical action that should satisfy the so called quantum master equation \( \Delta \exp(\frac{1}{\hbar} S) = 0 \).

The following theorem constitutes a mathematical basis of Batalin-Vilkovisky formalism.

\[ \int_L H \, d\lambda = \int_{L'} H' \, d\lambda' \]  

(31)

Here \( d\lambda \) and \( d\lambda' \) denote volume elements induced by the \( SP \)-structure in \( M \). Compactness and homology are defined in terms of bodies of supermanifolds. \( H \) and \( H' \) are even, \( K \) is odd. This theorem is proved in [4], the proof is based on the classification of \( P \)-manifolds, \( SP \)-manifolds and Lagrangian submanifolds given in this paper.

Let us formulate also a theorem that permits us to give a description of symmetry transformations in BV-formalism.

Let us consider two operators \( \tilde{\Delta} \) and \( \Delta \) on a supermanifold \( M \) that determine different \( SP \)-structures, but the same \( P \)-structure on \( M \). Let us suppose that corresponding volume elements \( d\tilde{\mu} \) and \( d\mu \) are connected by the formula: \( d\tilde{\mu} = e^\sigma d\mu \). Then

a) \( \Delta \exp(\frac{\sigma}{2}) = 0 \),

b) If \( S \) obeys the master equation \( \Delta \exp S = 0 \) then \( \tilde{S} = S - \frac{1}{2} \sigma \) satisfies the master equation \( \tilde{\Delta} \exp \tilde{S} = 0 \).

c) The action functional \( \tilde{S} \) in the \( SP \)-structure determined by \( \tilde{\Delta} \) describes the same physics as the action functional \( S \) in the \( SP \)-structure determined by \( \Delta \). In particular,

\[ \int_L e^{\tilde{S}} d\tilde{\lambda} = \int_L e^{\hat{S}} d\lambda \]  

(32)

for every Lagrangian submanifold \( L \subset M \).

It follows immediately from this statement that every theory is physically equivalent to the theory with an action functional \( \tilde{S} = 0 \). In this representation
a symmetry is simply an automorphism of corresponding \(SP\)-structure. For an arbitrary \(SP\)-structure we obtain:

Every function \(H\) satisfying \(\Delta H + \{H, S\} = 0\) (every quantum observable) determines a symmetry in the following sense. Neither the the action functional \(S\) nor the volume element \(d\mu\) are invariant with respect to an infinitesimal transformation with the Hamiltonian \(H\), however the new action functional

\[
\tilde{S} = S + \epsilon \{H, S\}
\]

and the new volume element

\[
d\tilde{\mu} = d\mu(1 + 2\epsilon \Delta H)
\]
describe the same physics as the old action functional \(S\) and the old volume element \(d\mu\).

We will not discuss here interesting questions arising in the analysis of the asymptotic behavior of \(\int_L \exp(\hbar^{-1}S)d\lambda\) in the \(\lim \hbar \to 0\) (semiclassical approximation in BV1formalism). Such an analysis was performed in [5]; the answer was expressed in terms a generalization of Reidemeister torsion.

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