The effective Lagrangian for nonrelativistic systems

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The effective Lagrangian for Nambu–Goldstone bosons (NGBs) in systems without Lorentz invariance has a novel feature that some of the NGBs are canonically conjugate to each other, hence describing one dynamical degree of freedom by two NGB fields. We develop explicit forms of their effective Lagrangian up to the quadratic order in derivatives. We clarify the counting rules of NGB degrees of freedom, and completely classify possibilities of such canonically conjugate pairs based on topology of the coset spaces. Its consequence on the dispersion relations of the NGBs is clarified. We also present simple scaling arguments when interactions among NGBs are marginal, or irrelevant, which justifies lore in the literature about the stability of long-range order.

I. INTRODUCTION

In studies of any macroscopic physical systems, behavior of the system at low temperatures, small energies, and long distances is determined predominantly by microscopic excitations with small or zero gap. It is hence important to develop a general theory to discuss gapless excitations. Barring special reasons, however, we generally do not expect any gapless degrees of freedom in a given system. The important exceptions are (1) Fermi liquid with the Fermi level within a continuous band, (2) second-order phase transitions with scale (and often conformal) invariance, (3) states protected by topological reasons such as edge states of topological insulators or quantum Hall states, and (4) Nambu–Goldstone bosons (NGB) of spontaneous symmetry breaking. The first three cases are discussed extensively in the literature. We focus on the last case in this paper because a general theory so far has surprisingly been lacking despite its importance and long history.

Spontaneously broken symmetry is a common theme through all areas of physics. The examples are numerous: Bose–Einstein condensate of cold atoms, superfluid of He\textsuperscript{4} or He\textsuperscript{3}, crystal lattice, neutron star, ferromagnets, anti-ferromagnets, liquid crystal, chiral symmetry in QCD, and cosmic inflation. The universal feature is that it guarantees the existence of gapless excitations when the relevant symmetries are continuous. Once promoted to gauge symmetries, it is the basis to discuss superconductivity, Englert–Bour–Higgs mechanism, and cosmic strings. The crucial question is what is the general theory that can describe the number of NGB degrees of freedom, their dispersion relations, their interactions among each other, and to other degrees of freedom. Ideally, the theory does not depend on specifics of a given system or perturbation theory, but rather determined by symmetries alone so that it is applicable even when the system is strongly coupled or we lack understanding of the microscopic description.

In systems with Lorentz invariance, the general theory has been established already back in 1960’s by the celebrated Nambu–Goldstone theorem \cite{Nambu:1960xd, Goldstone:1961eq}, and later with “Phenomenological Lagrangians” by Callan, Coleman, Wess, and Zumino \cite{Callan:1969sn, Coleman:1969sm}. It is important to formulate the theory using Lagrangians, because a Lagrangian is a self-contained package to describe a system. It determines the degrees of freedom, equations of motion, Noether currents for symmetries, commutation relations, provides the basis for perturbation theory using Feynman diagrams, and many non-perturbative methods based on path integrals. In comparison, Hamiltonian formalism requires additional input of degrees of freedom and their commutation relations. Especially when the degrees of freedom are not clear at the beginning of the discussion, which turns out to be the case for our purpose, the Lagrangian formulation is essential.

However, many systems we are interested in are not Lorentz-invariant. A finite temperature violates Lorentz invariance because the Boltzmann weight depends on the energy, which is the time component of the energy-momentum four-vector, and hence requires a specific choice of the reference frame. A chemical potential needed to describe systems with finite densities couples to the charge density, which is also the time component of a conserved four-current. Often the surrounding environment violates Lorentz invariance as well. In all of these cases, rotational invariance may still be present, while Lorentz invariance is certainly not there.

It is, therefore, of foremost importance to develop a general theory of NGBs based on symmetry principles alone without assuming Lorentz invariance. This is what we do in this paper.

NGBs without Lorentz invariance have been discussed for their obvious importance as discussed above. The nonrelativistic analog of one aspect of the NG theorem, that which ensures the appearance of at least one NGB, was discussed already back in the 1970s \cite{Srednicki:1978}. However, the number and the dispersion of the NGBs had been only studied on case by case basis until quite recently.
The Nambu–Goldstone theorem says there must be one gapless excitation for every broken symmetry generator, assuming Lorentz invariance. Moreover, Lorentz invariance constrains the dispersion relation for gapless excitation to be \( \omega = ck \) where \( c \) is the speed of light.

However these predictions are known to be false in systems without Lorentz invariance. A classic example is a ferromagnet. When spins line up macroscopically due to the nearest neighbor interaction, it spontaneously breaks the \( \text{SO}(3) \) spin rotational symmetry with three generators down to the unbroken \( \text{SO}(2) \) axial symmetry with only one generator. Despite the two spontaneously broken symmetries, the ferromagnet exhibits only one NGB. Moreover its dispersion is quadratic, rather than linear.

In contrast an antiferromagnet supports two NGBs with a linear dispersion, although it shows the same symmetry breaking pattern \( \text{SO}(3) \rightarrow \text{SO}(2) \).

More recent examples appeared in relativistic field theories with non-zero chemical potentials, where examples of an “abnormal” number of Nambu–Goldstone bosons are identified in many contexts \([10–16]\). Also, spinor Bose-Einstein condensates in cold atom systems added a number of new examples and realized some of them in the actual experiment \([17, 18]\). The dispersion of the softest NGB immediately modifies the thermodynamic property of the system at a low temperature. For example, the low-temperature heat capacity behaves as \( C(T) \propto T^d/z \) for the NGB with the dispersion \( \omega \propto k^z \) in \( d + 1 \) dimensions. In general, the low-energy dynamics of systems with SSB is governed by NGBs and hence it is clearly important to establish a general theorem which predicts the correct number, dispersion, and interactions of NGBs.

In the pioneering work \([19]\), Nielsen and Chadha established an inequality that relates the number of NGBs to their dispersion relations. In their approach, NGBs are classified as type-I (type-II) if their dispersion in the long wavelength limit behaves as \( \omega \propto k^{2n-1} \) (\( \omega \propto k^{2n} \)). Based on the analytic property of correlation functions, Nielsen and Chadha proved that the number of type-I NGBs plus twice the number of type-II NGBs is greater than or equal to the number of broken symmetry generators. Note that their conclusion is merely an inequality, and hence it does not give any lower nor upper bound for each type of NGBs. Also, their classification breaks down when the actual experiment \([20, 21]\) pointed out the importance of expectation values of the commutators of the broken generators in reducing the number of NGBs. They showed that the number of NGBs must be equal to the number of broken generators if \( \langle [Q_a, Q_b] \rangle = 0 \) for all combinations of broken generators. Although their argument is physically plausible, it contains a few questionable points. They identified the NG state associated with the charge \( Q_a \) as \( Q_a |\Psi_0 \rangle \) (\( |\Psi_0 \rangle \) is the quantum many-body ground state) and discussed the possibility of linear dependence among such vectors. However, it is well-known that, once symmetries are spontaneously broken, broken generators themselves are ill-defined. We should rather use commutation relations of generators with other local quantities.

Nambu \([20, 21]\) was probably the first to obtain the correct insight in this problem. He observed that the non-zero expectation value \( \langle [Q_a, Q_b] \rangle \) makes zero modes associated with these generators canonically conjugate to each other and hence the number of NGBs is reduced by one per such a pair. However, he did not prove this claim on general grounds.

With these previous works in mind, the current authors unified all of the above observations into a simple and well-defined form by proving them using field theory \([22]\):

\[
\begin{align*}
 n_A &= \dim G/H - \text{rank}\rho, \\
n_B &= \frac{1}{2} \text{rank}\rho, \\
n_{\text{NGB}} &= \dim G/H - \frac{1}{2} \text{rank}\rho, \\
n_A + 2n_B &= \dim G/H, \\
n_{\rho, A} &= \langle [Q_a, j^0_a(0)] \rangle.
\end{align*}
\]

Equation (3) was conjectured earlier in Ref. \([23]\) and was also obtained independently in Ref. \([24]\). Here, \( n_A, n_B \) represent the number of type-A, B NGBs, respectively, and \( n_{\text{NGB}} = n_A + n_B \) is the total number of NGBs. Equations (3) and (4) follow from Eqs. (1) and (2). \( j^\mu_a(x) \) is the conserved current associated with a broken charge \( Q_a = \int d^d x j^0_a(x) \). The Lie group \( G \) represents the original symmetry of the system and \( H \) is its unbroken subgroup, so that \( \dim G/H \) represents the number of broken symmetry generators. Clearly, the symmetry breaking pattern \( G \rightarrow H \) is not sufficient to fix the number of NGBs and we need an additional information, the matrix \( \rho \) in Eq. (5), of the ground state.

The definition of type-A, B NGBs are not based on their dispersion relations but on their symplectic structure, as we will discuss in detail later. For now, we just note that generically type-A NGBs have a linear dispersion and type-B NGBs have a quadratic dispersion but there are exceptions. Therefore, Eq. (4) can be understood as the equality version of the Nielsen-Chadha theorem for most cases.

The above-explained theorem by Schäfer et al. can also be understood as the special case where the matrix \( \rho \) vanishes and hence \( n_{\text{NGB}} = \dim G/H \) from Eq. (3).

In order to prove the counting rule of NGBs and clarify their dispersion relations, we develop the nonrelativistic analog of the “Phenomenological Lagrangian” à la Refs. \([4, 5]\), following Leutwyler’s works \([25, 26]\). We derive an explicit expression of the effective Lagrangian for a general symmetry breaking pattern \( G \rightarrow H \). In this process, we find a set of terms that have not been taken into account in the literature.
This fully-nonlinear effective Lagrangian contains only a few parameters that play the role of coupling constants between NGBs. By analyzing the scaling law of the dominant interaction, we discuss the stability of the symmetry-broken ground state. In sufficiently high dimensions, the system is essentially free as expected. However, it turns out that in general internal symmetries can be spontaneously broken even in 1 + 1 dimensions. This is one of the aspects enriched by the absence of Lorentz invariance — in a Lorentz invariant theory, the well-known Coleman theorem\[27\] prohibits that possibility.

The explicit form of the effective Lagrangian leads to another nontrivial prediction: that is, a no-go theorem for a certain number of type-A,B NGBs. One might think that the any combinations of $n_A$ and $n_B$ subject to Eq. (4) should be possible. However, for given $G$ and $H$, possibilities are quite restricted. This is because type-B NGBs are described by symplectic homogeneous spaces, which are special types of coset spaces that admit so-called Kähler structure, if $G$ is semi-simple. We will discuss how the possible numbers for type-A and type-B can be completely enumerated for any given $G$ and $H$.

This paper is organized as follows. In Sec. II we discuss the most general form of the effective Lagrangian for nonrelativistic systems and derive differential equations for the coefficients appearing in the effective Lagrangians by paying careful attention to gaugeability of the symmetry $G$. We present an analytic solution of the differential equations in terms of the Maurer–Cartan form in Sec. III. We also clarify the obstacle to gauge Wess–Zumino-Witten terms and algebras with central extension. Analyzing the free part of our effective Lagrangian, we prove the counting rule in Sec. IV and derive their dispersion in Sec. V. We discuss the interaction effect and SSB in 1 + 1 dimensions in Sec. VI.

In Sec. VII, we present the mathematical foundation of the canonically-conjugate (presymplectic) structure among some NGBs. With this preparation, we completely classify the presymplectic structure and prove a no-go theorem that prohibits a certain combination of type-A,B NGBs in Sec. VIII. It followed by concrete demonstration thorough familiar examples in Sec. IX.

We will not discuss the counting of NGBs associated to space-time symmetries. For those symmetries, the number of NGBs is reduced not only by forming canonically conjugate pairs but also by other mechanisms, e.g., linear dependence among conserved currents. Hence the above counting rule does not hold. See Refs. [28–31] for more details. Nevertheless we explain how to impose the Galilean symmetry, it it exists, on the effective Lagrangian in Sec. X.

For reader’s convenience, we present a pedagogical introduction to the cohomology of Lie algebra in Appendix A. We also review how to couple matter fields to NGBs in Appendix B. Finally, we clarify a confusion in existing literature on the relation between type-B NGBs and the time-reversal symmetry in Appendix C.

II. EFFECTIVE LAGRANGIAN FOR NONRELATIVISTIC SYSTEMS

In this section, we describe the general effective Lagrangian for the NGBs on the coset space $G/H$. We assume rotational invariance of space, but no Lorentz invariance. There are terms that had not been considered traditionally. The Lagrangian is considered to be an expansion in the number of derivatives to study long-range and low-energy excitations of the system. We restrict ourselves to terms up to second order in derivatives because they are sufficient to read off number and dispersion relation of NGBs for most purposes. To work out symmetry requirements on the functional forms of each term in the Lagrangian, differential forms turn out to be very useful.

A. Coset space

Suppose that the symmetry group $G$ of a microscopic Lagrangian is spontaneously broken down to its subgroup $H$. The set of the degenerate ground states form the coset space $G/H$. The low-energy effective Lagrangian is the nonlinear sigma model with the target space $G/H$. We consider only exact symmetries (i.e., without anomalies or explicit breaking). We also set $\hbar = 1$ throughout the paper. Except in Sec. X and few examples in VIIA, we assume $G$ and $H$ are compact Lie groups for internal symmetries.

Let $\pi^a (a = 1, \ldots, \dim G/H)$ be a local coordinate of $G/H$. By definition, the number of fields always equals the number of broken generators $\dim G/H$. Every point on this space is equivalent and we pick the origin $\pi^a = 0$ as our ground state. The NG field $\pi^a(\vec{x}, t)$ is a map $\pi: \mathbb{R}^{d+1} \rightarrow G/H$ ($d$ is the spatial dimensions).

$\pi^a$’s form a nonlinear realization of $G$. They transform under $\epsilon^i Q_i$ as

$$\delta_i \pi^a = \epsilon_i^a \dot{h}_i^a (\pi).$$

(6)

Generators $h_i^a (\pi)$ can be viewed as vector fields on $G/H$,

$$h_i (\pi) = h_i^a (\pi) \partial_a, \quad \partial_a \equiv \partial / \partial \pi^a,$$

(7)

and their Lie bracket is identified with the commutation relation,

$$[h_i, h_j] \equiv (h_i^k \partial_k h_j^a - h_j^k \partial_k h_i^a) \partial_a = f_{ijk}^k h_k.$$

(8)

Here, $i, j, k, \cdots$ refer to generators of $G$.

In general, we will look for the most general Lagrangian $L_{\text{eff}}(\pi, \dot{\pi}, \nabla_\pi \pi, \dot{\pi}^a, \nabla_\pi \dot{\pi}^a, \nabla_\pi \nabla_\pi \pi^a, \cdots)$ that only changes by total derivatives under the transformation in Eq. (6). A particularly useful choice of the nonlinear realization is given by the Callan–Coleman–Wess–Zumino coset construction [4, 5], which we introduce in Sec. IIIA.
If the symmetry can be gauged, parameters of symmetry transformations are local $\epsilon^i(x)$, and we may introduce gauge fields that transform as

$$\delta \epsilon A^i_\mu(x) = [D_\mu \epsilon(x)]^i = \nabla_\mu \epsilon^i(x) + f^i_{jk} A^j_\mu \epsilon^k(x),$$

where $A^i_\mu = (A^i_t, \tilde{A}^i)$ and $\nabla_\mu = (\nabla_t, \tilde{\nabla})$. However, not all symmetries can be gauged. Such examples are discussed in IIIE. In order to keep the full generality, we first proceed without gauging the symmetry. We will then discuss the local symmetry and clarify the obstruction.

**B. Derivative expansion and symmetry requirements**

We employ the derivative expansion to extract the low-energy, long-distance physics of the system. Namely, we expand the Lagrangian in the power series of the time derivative $\nabla_t$ and the spatial derivative $\nabla_r$ ($r, s = 1, \ldots, d$). We do not require Lorentz invariance but we do require spatial rotational symmetry. Due to the lack of the Lorentz invariance, the space and the time derivative may scale differently. For example, $O(\nabla^2_t)$ and $O(\nabla^2_r)$ may not be of the same order in derivative expansion. We also assume the broken symmetries are internal symmetries, and hence the NG fields are spacetime scalars.

To avoid possible confusion, we use $\nabla_r$ to represent the spatial derivative and $\nabla_t$ or “dot” to the time derivative. $\partial_a = \partial/\partial \pi^a$ ($a = 1, \ldots, \dim G/H$) refers to the derivatives with respect to internal coordinates of $G/H$.

With these cautions in mind, we find the most general form of the effective Lagrangian [25] up to the second order in derivatives in 3 + 1 dimensions and above,

$$\mathcal{L}_{\text{eff}} = c_a(\pi) \dot{\pi}^a + \frac{1}{2} g_{ab}(\pi) \dot{\pi}^a \dot{\pi}^b - \frac{1}{2} g_{ab}(\pi) \nabla \pi^a \cdot \nabla \pi^b.$$  (10)

In 1 + 1 dimensions, there is no spatial rotation and therefore we can add three more terms:

$$\tilde{c}_a(\pi) \nabla_x \pi^a + \tilde{g}_{ab}(\pi) \dot{\pi}^a \nabla_x \pi^b + \tilde{b}_{ab}(\pi) \dot{\pi}^a \nabla_x \pi^b.$$  (11)

Also, in 2 + 1 dimensions, there is an invariant antisymmetric tensor $\epsilon^{rs}$ and therefore

$$-\frac{1}{2} b_{ab}(\pi) \epsilon^{rs} \nabla_r \pi^a \nabla_s \pi^b.$$  (12)

is allowed. $g_{ab}$, $\tilde{g}_{ab}$ and $\tilde{b}_{ab}$ are symmetric and $b_{ab}$ and $\tilde{b}_{ab}$ are antisymmetric with respect to $a, b$. Terms that contain $\dot{\pi}^a$, $\nabla_r \dot{\pi}^a$, $\nabla_r \nabla_x \pi^a$ can be brought to the above form by integration by parts.

We discuss that the $c_\epsilon(\pi)$ term can be interpreted as the Berry phase in Sec. IIIF. The terms in Eqs. (11) and (12) have not been taken into account in Ref. [25]. However, they preserve the assumed rotational invariance in 1 + 1 or 2 + 1 dimensions and therefore are allowed in general. We present an example of them in Sec. IIIB4.

There are two subtleties about terms $\tilde{c}_a(\pi)$ and $b_{ab}(\pi)$. First, the energy functional derived by the Lagrangian Eq. (10) plus the terms in Eq. (11) is

$$\int d^d x \left[ \frac{1}{2} g_{ab} \dot{\pi}^a \dot{\pi}^b + \frac{1}{2} g_{ab} \nabla_x \pi^a \nabla_x \pi^b - \tilde{c}_a \nabla \pi^a \right].$$  (13)

In the Fourier space, the second term is $O(k_x^2)$ and the last term is $O(k_x)$. Thus, the energy is minimized by a non-zero $k_x$ and the translational symmetry will be spontaneously broken. Although the $O(k_x)$ term and the $O(k_x^2)$ term balance against each other, this solution may still be consistent with the derivative expansion if the coefficient of the $O(k_x)$ term is somehow small. Since our main interest is in the situation with unbroken translational symmetry, we will not discuss the consequence of this term any further.

Second, $\tilde{c}_a(\pi)$ and $b_{ab}(\pi)$ cannot be a Wess-Zumino-Witten type term (see Sec. IIIE). They appear in the energy functional, unlike terms $\tilde{c}_a(\pi)$ and $\tilde{b}_{ab}(\pi)$, which are linear in the time derivative. In order for the energy to be well-defined, $\int dx \tilde{c}_a(\pi) \nabla_x \pi^a$ and $\int d^d x (1/2) b_{ab}(\pi) \epsilon^{rs} \nabla_x \pi^a \nabla_x \pi^b$ cannot possess the ambiguity of $2\pi k (k \in \mathbb{Z})$.

Our task is to determine coefficients $c_a(\pi)$, $\tilde{c}_a(\pi)$, $g_{ab}(\pi)$, $\tilde{g}_{ab}(\pi)$, $\tilde{g}_{ab}(\pi)$, $b_{ab}(\pi)$, and $\tilde{b}_{ab}(\pi)$ by imposing the global symmetry $G$.

Under global transformation Eq. (6), the first term of the Lagrangian Eq. (10) transforms as

$$\delta_\epsilon (c_a \dot{\pi}^a) = (h^b \partial_b c_a + c_a \partial_b h^b) \dot{\pi}^a.$$  (14)

By requiring that this combination is a total derivative $\nabla_t (e_i + c_a h_a^i)$, we find

$$(\partial_b c_a - \partial_a c_b) h_i^b = \partial_a e_i.$$  (15)

Similarly, for $\tilde{c}_a(\pi)$, $b(\pi)$, and $\tilde{b}(\pi)$, we have

$$(\partial_b \tilde{c}_a - \partial_a \tilde{c}_b) h_i^b = \partial_b e_i,$$  (16)

$$(\partial_a h_{bc} + \partial_b h_{ac} + \partial_c h_{ab}) h_i^c = \partial_a e_i^c - \partial_b e_i^c,$$  (17)

$$(\partial_a h_{bc} - \partial_b h_{ac} + \partial_c h_{ab}) h_i^c = \partial_b e_i^c - \partial_a e_i^c.$$  (18)

Here, $\tilde{e}_i(\pi)$, $e_i(\pi)$, and $\tilde{e}_i(\pi)$ are also related to the change of the Lagrangian by total derivatives $\nabla_t (e_i + \tilde{c}_a h_i^a)$, $\nabla_t [\epsilon^{rs} (e_i^b + b_{ab} h_a^i) \nabla_x \pi^b]$, and $\nabla_t [(\epsilon_i^b + \tilde{b}_{ab} h_i^a) \nabla_x \pi^b] - \nabla_x [(\epsilon_i^b + \tilde{b}_{ab} h_i^a) \nabla_x \pi^b]$. In contrast, the second term of Eq. (10) must be invariant by itself; i.e., they cannot change by a surface term.

$$\delta_\epsilon \left( g_{ab} \nabla \pi^a \cdot \nabla \pi^b \right) = (h_i^c \partial_c g_{ab} + g_{ac} \partial_c h_i^c + g_{ac} \partial_b h_i^c) \nabla \pi^a \cdot \nabla \pi^b = 0.$$  (19)

If this were a total derivative $\nabla_t A_t^i$, $A_t^i$ would take the form $f_{ia}(\pi) \nabla_t \pi^a$. However, $\nabla_t A_t^i$ then contains a term $\nabla^2 \pi^a$, which was absent in Eq. (19). Thus $A_t^i$ has to be zero. Therefore,

$$h_i^c \partial_c g_{ab} + g_{cb} \partial_a h_i^c + g_{ac} \partial_b h_i^c = 0.$$  (20)
The same equation holds for $\tilde{g}_{ab}(\pi)$ and $\tilde{g}_{ab}(\pi)$:

$$h_i^c \partial_c g_{ab} + \tilde{g}_{cb} \partial_a h_i^c + \tilde{g}_{ac} \partial_b h_i^c = 0,$$

$$h_i^c \partial_c \tilde{g}_{ab} + \tilde{g}_{cb} \partial_a h_i^c + \tilde{g}_{ac} \partial_b h_i^c = 0.$$  

(21) (22)

In summary, coefficients in the effective Lagrangian must obey the differential equations Eqs. (15)–(18) and (20)–(22) in order that the Lagrangian has the symmetry $G$. We also have to derive the differential equations for $e_i(\pi)$, $\tilde{e}_i(\pi)$, $\tilde{e}_i'(\pi)$, and $\tilde{e}_i''(\pi)$ and it can be easily done by using the mathematical technique we introduce in the next section.

C. Geometric derivation

1. Equations on $c(\pi)$’s and $g(\pi)$’s

Here we rederive above differential equations by using differential geometry, to set up notations and introduce useful mathematical tools for later calculation. The terms in the effective Lagrangian can be viewed as one-forms

$$c(\pi) = c_a(\pi) d\pi^a,$$

$$\tilde{c}(\pi) = \tilde{c}_a(\pi) d\pi^a,$$

(23) (24)

symmetric tensors

$$g(\pi) = g_{ab}(\pi) d\pi^a \otimes d\pi^b,$$

$$\tilde{g}(\pi) = \tilde{g}_{ab}(\pi) d\pi^a \otimes d\pi^b,$$

(25) (26)

and two-forms

$$b(\pi) = b_{ab}(\pi) d\pi^a \wedge d\pi^b,$$

$$\tilde{b}(\pi) = \tilde{b}_{ab}(\pi) d\pi^a \wedge d\pi^b,$$

(27) (28)

on the manifold $G/H$. Note that $c(\pi)$, $\tilde{c}(\pi)$, $b(\pi)$, and $\tilde{b}(\pi)$ do not necessarily exist globally.

In the following, we use Cartan’s magic formula that relates the Lie derivative $\mathcal{L}_X$, the exterior derivative $d$, and the interior product $i_X$:

$$\mathcal{L}_X \omega = (d i_X + i_X d) \omega.$$  

(30)

This is true for arbitrary forms $\omega$ and vector fields $X$ [32, 33].

We require that the Lie derivative of the effective Lagrangian along a vector $h_i$ to be a total derivative,

$$\mathcal{L}_h \mathcal{L}_{\text{eff}} = d\Lambda_i.$$  

(31)

Let us first focus on the one-form $c$. To fulfill the symmetry requirement,

$$\mathcal{L}_h c = d(i_h c) + i_{h_i} dc = d(e_i + i_{h_i} c),$$

(32)

we need

$$i_{h_i} dc = de_i.$$  

(33)

This is nothing but the Eq. (15). In the same way, one can obtain

$$i_{h_i} d\bar{c} = d\bar{e}_i, \quad i_{h_i} db = d\bar{e}_i', \quad i_{h_i} d\bar{e} = d\bar{e}_i',$$

(34)

which correspond to Eqs. (16)–(18). Note that the definition of $e_i$, $\bar{e}_i$, $e_i'$ and $\bar{e}_i'$ in Eqs. (33) and (34) fix them only unto a constant or a closed one-form. We will come back to this ambiguity shortly.

Finally, Eqs. (20)–(22) are nothing but the Killing equation for $G$-invariant metrics,

$$\mathcal{L}_h g = 0, \quad \mathcal{L}_h \tilde{g} = 0, \quad \mathcal{L}_h \tilde{g} = 0.$$  

(35)

If $\pi^a$ transforms irreducibly under the unbroken symmetry $H$, the invariant metric on $G/H$ is unique and $g$, $\tilde{g}$, and $\tilde{g}$ may differ only by an overall factor. In general, they may differ by overall factors for each irreducible representation [see Eq. (84)].

2. Equations on $e_i(\pi)$’s and $\tilde{e}_i'(\pi)$’s

In order to solve Eqs. (15)–(18), we have to specify functions $e_i(\pi)$, $\tilde{e}_i(\pi)$ and one-forms $e_i'(\pi) = e_{ia}(\pi) d\pi^a$, $\tilde{e}_i'(\pi) = \tilde{e}_{ia}(\pi) d\pi^a$. Now we show that they obey the differential equations,

$$\mathcal{L}_h e_j = f_{ij}^k e_k + z_{ij},$$

$$\mathcal{L}_h \tilde{e}_j = f_{ij}^k \tilde{e}_k + \tilde{z}_{ij},$$

$$\mathcal{L}_h e_j' = f_{ij}^k e_k' + d z_{ij}',$$

$$\mathcal{L}_h \tilde{e}_j' = f_{ij}^k \tilde{e}_k' + d \tilde{z}_{ij}',$$

(36) (37) (38) (39)

where $z_{ij}$, $\tilde{z}_{ij}$ are constants and $z_{ij}'(\pi)$, $\tilde{z}_{ij}'(\pi)$ are functions. For example, given the initial condition $e_i(0)$ and the constants $z_{ij}$, we can solve Eq. (36) to find $e_i(\pi)$.

If possible, we always remove $z_{ij}$, $\tilde{z}_{ij}$, $z_{ij}'(\pi)$, and $\tilde{z}_{ij}'(\pi)$ from Eqs. (36)–(39) by shifting $e_i(\pi)$, $\tilde{e}_i(\pi)$ by constants and $e_i'(\pi)$, $\tilde{e}_i'(\pi)$ by closed one-forms using the above-mentioned ambiguity. However, they can not always be completely removed. For example, $z_{ij}$ cannot be eliminated when the second cohomology of the Lie algebra $H^2(q)$ is nontrivial. (See Appendix A for a brief review of this subject.) In Sec. IIIE, we show that the nontrivial $z_{ij}$ corresponds to a central extension of the Lie algebra.

To derive Eq. (36), we first note that the Lie derivative of the two-form $dc$ vanishes,

$$\mathcal{L}_h dc = d^2 e_i + i_{h_i} d^2 c = 0.$$  

(40)

We also use the commutativity $\mathcal{L}_h d = d \mathcal{L}_h$, and a property of the interior product,

$$\mathcal{L}_h i_{h_j} = f_{ij}^k i_{h_k} + i_{h_j} \mathcal{L}_h i_{h_j}.$$  

(41)

Combining these with Eqs. (33), we obtain

$$d(\mathcal{L}_h e_j) = \mathcal{L}_h (d e_j) = \mathcal{L}_h (i_{h_j} dc) = f_{ij}^k (i_{h_k} dc) + i_{h_j} (\mathcal{L}_h dc) = d(f_{ij}^k e_k).$$  

(42)
This proves Eq. (36). Exactly the same derivation applies to Eqs. (37)–(39).

**D. Local symmetry**

Here we discuss the case where the symmetry \( G \) can be gauged. Since gauge fields appear in covariant derivatives, it is natural to assume that \( A^i_{\mu} = (A^i_1, \tilde{A}^i) \) is of the same order as \( \nabla_{\mu} = (\nabla_{\mu}, \mathbf{V}) \) in derivative expansion. Equation (10) is then replaced by the sum of the following terms [25, 26]:

\[
L^{(0,1)}_{\text{eff}} = c_k(\pi) \pi^a + c_i(\pi) A^i_1,
\]

\[
L^{(0,2)}_{\text{eff}} = \frac{1}{2} g_{ab}(\pi) \bar{\pi}^a \bar{\pi}^b - \bar{h}_{ia}(\pi) A^i_1 \bar{\pi}^a + \frac{1}{2} \bar{k}_{ij}(\pi) A^i_1 A^j_1,
\]

\[
L^{(2,0)}_{\text{eff}} = \frac{1}{2} g_{ab}(\pi) \nabla^a \pi^b \cdot \nabla^b \pi^b + h_{ia}(\pi) \bar{A}^i \cdot \nabla^a \pi^b - \frac{1}{2} k_{ij}(\pi) \bar{A}^i \cdot \bar{A}^j.
\]

Here, \( k_{ij}(\pi) \) and \( \bar{k}_{ij}(\pi) \) are symmetric with respect to \( i, j \). As discussed before, one can add

\[
L^{(1,1)'}_{\text{eff}} = \bar{e}_a(\pi) \nabla_x \pi^a + \bar{c}_i(\pi) A^i_x,
\]

\[
L^{(1,1)''}_{\text{eff}} = \bar{g}_{ab}(\pi) \bar{\pi}^a \nabla_x \pi^b - \bar{h}_{ia}(\pi) \left( A^i_1 \nabla_x \pi^a + A^a_1 \bar{\pi}^b \right) + \bar{k}_{ij}(\pi) A^i_1 A^j_1,
\]

\[
L^{(1,1)'''}_{\text{eff}} = \bar{b}_{ab}(\pi) \bar{\pi}^a \nabla_x \pi^b + \bar{e}'_{ia}(\pi) \left( A^i_1 \nabla_x \pi^a - A^a_1 \bar{\pi}^b \right) + \bar{a}_{ij}(\pi) A^i_1 A^j_1
\]

in 1 + 1 dimensions, and

\[
L^{(2,0)'}_{\text{eff}} = \frac{-1}{2} b_{ab}(\pi) \bar{c}_e A^a_1 A^b_1 - \bar{a}_{ij}(\pi) \bar{c}_e A^a_1 A^b_1
\]

in 2 + 1 dimensions. Here, \( \bar{h}_{ij}(\pi) \) is symmetric and \( \bar{a}_{ij}(\pi) \), \( \bar{a}_{ij}(\pi) \) are antisymmetric.

We require that the action \( S_{\text{eff}}[\pi, A] = \int d^d x dt L_{\text{eff}} \) is invariant under the local transformation \( \pi(x) = \pi_0(x) + \delta_\pi \pi(x) \) and \( A'(x) = A(x) + \delta_\pi A(x) \), where \( \delta_\pi \pi^a \) and \( \delta_\pi A(x) \) are defined in Eqs. (6) and (9). Here we assume that the infinitesimal parameters \( e^i(x) \) vanish as \( |x| \to 0 \). The invariance of the action can be reexpressed as

\[
0 = \delta_\pi S_{\text{eff}}[\pi, A] = \int d^d x dt \left[ \frac{\delta S_{\text{eff}}}{\delta \pi^a} \delta_\pi \pi^a + \frac{\delta S_{\text{eff}}}{\delta A^a_{\mu}} \delta_\pi A^a_{\mu} \right] = \int d^d x dt \bar{e}'(x) \left[ \frac{\delta S_{\text{eff}}}{\delta \pi^a} h^a_{ij} - (D^a_{\mu})^j_i \frac{\delta S_{\text{eff}}}{\delta A^a_{\mu}} \right].
\]

where \( (D^a_{\mu})^j_i = \delta^j_\mu \nabla^a_{\mu} + f_{ik}^j A^k_{\mu} \). Therefore, the effective Lagrangian must satisfy

\[
h^a_{ij}(\pi) \frac{\delta S_{\text{eff}}}{\delta \pi^a} = (D^a_{\mu})^j_i \frac{\delta S_{\text{eff}}}{\delta A^a_{\mu}}.
\]

This condition leads to the differential equations we have derived above. For example, Eq. (51) for \( L^{(0,1)}_{\text{eff}} \) is

\[
0 = \delta_\pi \left[ \pi^a \partial_\mu c_b - \partial_\mu c_a - \partial_i e_i \right] + A^a_{ij} [h^a_{ij} \partial_\mu e_j - f_{ik}^j e_k],
\]

which leads to the differential equations for \( c_a(\pi) \) and \( e_i(\pi) \),

\[
h^a_{ij} (\partial_\mu c_b - \partial_\mu c_a) = \partial_\mu c_i,
\]

\[
h^a_{ij} \partial_\mu e_j = f_{ij}^k e_k.
\]

Similarly for \( L^{(0,1)'}_{\text{eff}} \),

\[
h^a_{ij} (\partial_\mu \bar{c}_b - \partial_\mu \bar{c}_a) = \partial_\mu \bar{c}_i,
\]

\[
h^a_{ij} \partial_\mu \bar{e}_j = f_{ij}^k \bar{e}_k.
\]

We can easily work out for other terms in the effective Lagrangian in the same way. Especially, \( L^{(0,2)}_{\text{eff}}, L^{(2,0)}_{\text{eff}}, \) and \( L^{(1,1)'}_{\text{eff}} \) can be compactly expressed as

\[
L^{(0,2)}_{\text{eff}} = \frac{1}{2} g_{ab}(\pi) D^a_{\mu} \pi^b D^b_{\mu},
\]

\[
L^{(2,0)}_{\text{eff}} = \frac{1}{2} g_{ab}(\pi) \bar{D}^a_{\mu} \bar{D}^b_{\mu},
\]

\[
L^{(1,1)'}_{\text{eff}} = \bar{g}_{ab}(\pi) D^a_{\mu} \pi^b D^b_{\mu}. \]

Here \( D_{\mu} = \nabla_{\mu} - h^a_{ij} A^i_{\mu} \) is the covariant derivative and \( g_{ab}(\pi), \bar{g}_{ab}(\pi) \) are \( G \) invariant metrics of \( G/H \), obeying the Killing equation Eq. (35). To verify Eqs. (57)–(59), one has to use the Lie bracket Eq. (8) several times.

Finally, for the term \( L^{(2,0)'}_{\text{eff}} \), Eq. (51) leads to

\[
i_{h} \delta \bar{d} = d \bar{e}'_i,
\]

\[
L_{\text{eff}} \bar{e}'_i = f_{ij}^k \bar{e}_k,
\]

\[
i_{h} e'_i + i_{h} \bar{e}'_i, \quad a_{ij} = i_{h} e'_j
\]

and, for \( L^{(1,1)'''}_{\text{eff}} \),

\[
i_{h} \delta \bar{d} = d \bar{e}'_i,
\]

\[
L_{\text{eff}} \bar{e}'_i = f_{ij}^k \bar{e}_k,
\]

\[
i_{h} e'_i + i_{h} \bar{e}'_i, \quad a_{ij} = i_{h} e'_j.
\]

These differential equations are almost identical to those we derived before, except for the following two constraints:

1. \( z_{ij}, \tilde{z}_{ij}, \tilde{z}_{ij}(\pi), z'_{ij}(\pi) \) in Eqs (36)–(39) have to vanish.

2. Additional constrains (62) and (66) must be satisfied.
Thus, the requirement of the local invariance is stronger than the global symmetry. If these additional constraints are not fulfilled, the symmetry cannot be gauged. See Sec. III E for detailed discussion on examples which violate at least one of these conditions.

### III. Solution with Maurer–Cartan Form

In this section, we present the exact analytic solutions to the differential equations derived in the previous section. We initially assume the two conditions listed in Sec. II D, namely when the symmetry is gaugeable. Since the end result can be understood without technical details, readers without interest in the derivation can directly go to Sec. III C, where we summarize our result. We obtain the same result using an alternative formalism of gauging the right-translation by $H$ in Sec. III D. Finally in Sec. III E we discuss the additional terms allowed when the symmetry is not gaugeable.

#### A. Preliminaries

The Callan–Coleman–Wess–Zumino coset construction is a famous and useful formalism to achieve a nonlinear realization and building blocks of the effective Lagrangian [4, 5].

The coset space $G/H$ can be parametrized as $U(\pi) = e^{i\Pi}$ with $\Pi = \pi^aT_a$. Here, $T_i$ is a faithful representation of the Lie algebra $g$. Throughout this paper, we use the following notation:

- $i, j, k, \cdots$ refer to generators $g$, including both broken and unbroken.
- $a, b, c, \cdots$ refer to broken generators $g/h$.
- $\rho, \sigma, \lambda, \cdots$ refer to unbroken generators $h$.

If $G$ is compact, we can always find an unitary representation of $G$ such that $T_i$s are Hermitian and orthogonal, $\text{tr}(T_iT_j) = \lambda\delta_{ij}$. As a result, the structure constants become fully antisymmetric; i.e., $f_{ij}^k = -f_{jk}^i = 0$. However, it is not always convenient to work in this orthogonal basis, especially when $G$ is not semisimple, and in this section we only use $f_{ij}^k$, which follows just by the antisymmetric property of commutators.

The transformation law of NG fields under the action of $g \in G$ is defined through the decomposition of the product $gU(\pi)$ into the form

$$gU(\pi) = U(\pi')h_g(\pi), \quad h_g(\pi) \in H. \quad (68)$$

Now we define an important $g$-valued one-form on $G/H$, the so-called Maurer–Cartan one-form:

$$\omega(\pi) = -iU(\pi)U(\pi) = \sum_{n=0}^{\infty} \frac{(-i)^n}{(n+1)!}[\Pi, [\Pi, \ldots, [\Pi, d\Pi] \ldots]]. \quad (69)$$

In the following, we use the notation $\omega(\pi) = \omega_a(\pi)d\pi^a = \omega^i(\pi)T_i = \omega^i_a(\pi)d\pi^a$ and $A = A^\mu T_\mu dx^\mu$.

Infinitesimal transformation $h_a^\pi(\pi)$ is defined by $\pi'^a = \pi^a + \epsilon h_a^\pi(\pi) + O(\epsilon^2)$ for $g = e^{\epsilon \tau_i^Tr_i}$. To find their explicit expression, we compare at the order one $\epsilon$ terms in Eq. (68):

$$i_h\omega = h'_a(\pi)\omega_a(\pi) = \nu_j^i(\pi)T_j - T_\mu k^i_\mu(\pi), \quad (70)$$

where $k^i_\mu(\pi, g)$ is defined by $h_g(\pi) = e^{i\epsilon \nu_j^i(\pi)T_j}$ and

$$\nu_j^i(\pi)T_j \equiv U(\pi)^\dagger U(\pi) = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!}[[\Pi, [\Pi, \ldots, [\Pi, T_i] \ldots]]. \quad (71)$$

By solving Eq. (70), we can compute $h_a^\pi(\pi)$ around the origin as

$$h_a^\pi(\pi) = -i\nu_j^i(\pi)h_j^\pi - ih_\mu h_\mu^\pi, \quad (72)$$

$$h_a^\pi(\pi) = \delta_a^b + \frac{1}{2}\pi^c f_{cb}^a + O(\pi^2). \quad (73)$$

Note in particular that $h_a^\pi(0) = \delta_a^b$ and $h_a^\pi(0) = 0$ at $\pi = 0$, meaning that the broken generator $h_a$ shifts $\pi^a$ and that the unbroken generator $h_\mu$ does not change the ground state.

The transformation law of the Maurer–Cartan form follows from the definition (68),

$$\omega(\pi') = -i(h_gU^\dagger g')d(gU h_g^\dagger) = h_g\omega(\pi)h_g^\dagger - ih_\mu h_\mu^\pi. \quad (74)$$

It is convenient to decompose the Maurer–Cartan forms $\omega = \omega_\perp + \omega_\parallel$, where $\omega_\perp = \omega^aT_a$ are in $g/h$, while $\omega_\parallel = \omega^\pi T_\pi$ are in $h$. Since $h_g^\dagger h_\mu h_\mu^\pi$ in $h$, we have

$$\omega_\perp(\pi') = h_g\omega_\perp(\pi)h_g^\dagger, \quad (75)$$

$$\omega_\parallel(\pi') = h_g\omega_\parallel(\pi)h_g^\dagger - ih_\mu h_\mu^\pi. \quad (76)$$

Their infinitesimal versions are

$$L_{h} \omega^\pi(\pi) = -f_{j\rho} k^j_\rho(\pi)\omega^{\rho}(\pi), \quad (77)$$

$$L_{h} \omega_\parallel(\pi) = -f_{j\rho} k^j_\rho(\pi)\omega^\rho(\pi) - dh^\pi(\pi). \quad (78)$$

When we gauge the symmetry $G$ by introducing gauge fields that obey the transformation rule in Eq. (9), the Maurer–Cartan form no longer transforms covariantly i.e., does not obey Eq. (75) for local transformation. Instead, the combination

$$(\omega_\perp)_a D\pi^a \equiv (\omega_\perp)_a (d\pi^a - h_\mu^\pi A^\mu) \equiv [-iU^\dagger (d - iA)U]_\perp \quad (79)$$

transforms covariantly.

It is also straightforward to verify following useful relations.

$$d\omega^k(\pi) = \frac{1}{2}f_{ijk} \omega^i(\pi) \wedge \omega^j(\pi), \quad (80)$$

$$L_{h} \nu_j^k(\pi) = f_{ij}^l \nu_j^l(\pi) - f_{j\rho} k^j_\rho(\pi)\nu_j^l(\pi), \quad (81)$$

$$d\nu_j^k(\pi) = f\nu_j^k \omega^j(\pi)\nu_j^i(\pi). \quad (82)$$
Finally, we note that the last line of Eqs. (69) and (71) is written in terms of commutation relations. Therefore, the Maurer–Cartan form \( \omega(\pi) \) and generators \( h_i(\pi) \) do not fundamentally depend on a specific choice of the representation of \( T_i \).

With these preparations, we now present our analytic solutions to the differential equations derived in Sec. II one by one.

**B. Explicit solutions**

1. \( g(\pi) \)'s

As the first example, here we show that

\[
g(\pi) = g_{ab}(0)\omega^a(\pi) \otimes \omega^b(\pi)
\]

is the solution to the Killing equation (35). If NGBs transform irreducibly under the unbroken subgroup \( H \), constants \( g_{cd}(0) \) must be proportional to \( \delta_{cd} \). In the most general case, \( g_{cd}(0) \) has to be invariant under unbroken symmetries; namely,

\[
f_{pa} \epsilon g_{cb}(0) + f_{pb} \epsilon g_{ac}(0) = 0,
\]

which can be derived from the Killing equation (35) at the origin \( \pi = 0 \) with the help of Eq. (72).

To see that \( g(\pi) \) in Eq. (83) is the solution of Eq. (35), we use Eq. (77):

\[
\mathcal{L}_{h_i} g = \mathcal{L}_{h_i} [g_{cd}(0)\omega^c(\pi) \otimes \omega^d(\pi)]
\]

\[
= g_{cd}(0)[(\mathcal{L}_{h_i} \omega^c) \otimes \omega^d + \omega^c \otimes (\mathcal{L}_{h_i} \omega^d)]
\]

\[
= -k^d_{i}[g_{cd}(0)f_{pc} \epsilon + g_{ce}(0)f_{pd} \epsilon] \omega^c \otimes \omega^d.
\]

The combination in \([\cdots]\) vanishes thanks to Eq. (84). Equation (83) also respect the initial value since \( \omega^a = \delta^a_3 \) at \( \pi = 0 \). Hence this is the unique solution of Eq. (35).

The same is true for \( \tilde{g}_{ab}(\pi) \) and \( \tilde{g}_{ab}(\pi) \); i.e.,

\[
\tilde{g}(\pi) = \tilde{g}_{ab}(0)\omega^a(\pi) \otimes \omega^b(\pi),
\]

\[
\tilde{g}(\pi) = \tilde{g}_{ab}(0)\omega^a(\pi) \otimes \omega^b(\pi)
\]

with

\[
f_{pa} \epsilon \tilde{g}_{cb}(0) + f_{pb} \epsilon \tilde{g}_{ac}(0) = 0,
\]

\[
f_{pa} \epsilon \tilde{g}_{cb}(0) + f_{pb} \epsilon \tilde{g}_{ac}(0) = 0.
\]

2. \( e_i(\pi) \)'s

We now prove that

\[
e_i(\pi) = \nu_i(\pi)e_j(0)
\]

is the solution of Eq. (36) when \( \zeta_{ij} = 0 \). By multiplying \( e_k(0) \) to Eq. (81), we get

\[
\mathcal{L}_{\nu_j} \nu_j = f_{ij} [\nu_j(\pi)e_k(0)] = 0.
\]

The second term vanishes, because Eq. (36) at \( \pi = 0 \) implies

\[
f_{pi} \epsilon e_k(0) = 0.
\]

Therefore, Eq. (90) satisfies the differential equation Eq. (36). Combined with \( \nu_j(0) = \delta_i^j \) [see Eq. (71)], we conclude that this is the unique solution that is consistent with the initial value.

Similarly,

\[
c_i(\pi) = \nu_i(\pi)e_j(0),
\]

\[
f_{pi} \epsilon e_k(0) = 0
\]

is the solution of Eq. (37).

3. \( e_c(\pi) \)'s

Next, we claim that

\[
c(\pi) = -\omega^a(\pi)e_i(0) + d\chi
\]

with \( \chi \) a smooth function is a solution of Eq. (33). First, we multiply \( e_k(0) \) to Eq. (80) and Eq. (82) to get

\[
d[\omega^k(\pi)e_k(0)] = \frac{1}{2} f_{ij} k^j(\pi) \omega^i(\pi)e_k(0),
\]

\[
de_i(\pi) = d[\nu_i(\pi)e_j(0)] = f_{ji} k^j(\pi)\nu_i(\pi)e_k(0).
\]

Further operating \( i_h \) to the former equation, we have

\[
i_h \cdot dc(\pi)
\]

\[
= i_h \cdot d[-\omega^k(\pi)e_k(0)]
\]

\[
= -f_{ij} k^j(\pi) [\omega^i(\pi)]e_k(0),
\]

\[
= f_{ji} k^j(\pi) \omega^i(\pi)e_k(0) - [f_{jk} k^j(\pi)] e_i(\pi)
\]

\[
= de_i(\pi).
\]

In the derivation, we used Eqs. (70), (92), and (96). Therefore, \( c(\pi) \) in Eq. (94) indeed obeys the differential equation. The undetermined part \( d\chi \) is a total derivative term in the Lagrangian.

Similarly, \( \tilde{c}(\pi) = -\omega^a(\pi)e_i(0) \) up to a closed one-form.

4. \( e_t(\pi) \)'s and \( b(\pi) \)'s

In the same way, it is not difficult to verify that

\[
e_i(\pi) = e_i(\pi)\nu_i(\pi)e_j(0),
\]

\[
e_i(\pi) = e_i(\pi)\nu_i(\pi)e_j(0)
\]

are the solution of Eqs. (38) and (39) and

\[
b(\pi) = -e_{cd}(0)\omega^i(\pi) \otimes \omega^d(\pi) + d\chi',
\]

\[
b(\pi) = -e'_{cd}(0)\omega^i(\pi) \otimes \omega^d(\pi) + d\chi'
\]

are the solution of Eq. (34). Constants \( e'_t(a)(0), \tilde{e}'_t(a)(0) \) have to satisfy

\[
e'_{pa}(0), e'_{ab}(0) + e'_{pa}(0) = 0,
\]

\[
f_{pa} \epsilon e'_t(0) + f_{pb} \epsilon c'_1(0) = 0
\]

\[
f_{pa} \epsilon e'_t(0) + f_{pb} \epsilon c'_1(0) = 0
\]
\[ \tilde{e}_{\alpha} (0) = 0, \quad \tilde{e}_{\alpha} (0) + \tilde{e}_{\beta} (0) = 0, \]
\[ \frac{\partial}{\partial x^\alpha} \tilde{e}_{\alpha} (0) + \frac{\partial}{\partial x^\beta} \tilde{e}_{\beta} (0) = 0. \]  
(103)

One can see that a condition for the gaugeability (62) is indeed fulfilled since
\[ i_\mu e_\nu (\pi) = e_\nu (0) \nu^{\nu}_\mu (i_\mu, \omega^\nu (\pi)) = e_\nu (0) \nu^{\nu}_\mu (\pi) \nu^{\nu}_\mu (\pi) \]
(104)
is antisymmetric with respect to \( i, j \), thanks to the second relation of Eq. (102).

Among constants \( e_i (0) \) that satisfy above conditions, those which can be written as
\[ e_i (0) = f_\alpha k C_k, \quad f_\alpha k C_k = 0 \]
give only a total derivative term in the Lagrangian. Indeed, from Eq. (80) and \( f_\alpha k C_k = 0 \), it follows that
\[ d[C_k \omega^k] = f_\alpha k C_k \omega^\alpha \wedge \omega^k. \]
(106)

For example, for \( G/H = \text{SO}(3)/\text{SO}(2) = S^2 \), the choice \( e_i (0) = \epsilon_{ab} \) satisfies all conditions in Eq. (102). In this case, \( \epsilon_{ab} \omega^\alpha \wedge \omega^b \) is nothing but the \( \theta \)-term:
\[ \frac{\theta}{4\pi} \tilde{n} \cdot \nabla_\tilde{n} \times \nabla_\tilde{n} \]
(107)
up to an overall factor. This is expected since \( \epsilon_{ab} \) can be written as \( f_{\alpha \beta} = \epsilon_{ab} \) (\( C_x = 1 \) and \( C_y = C_z = 0 \)).

An example of \( b_{ab} (\pi) \) terms that is not a total derivative is given by the coset \( SU(3)/U(1) \times U(1) \). We use the standard notation of Gell-Mann matrices \( \lambda_i \) (\( i = 1, \ldots, 8 \)) and set \( T_i = \lambda_{i/2} \). In this case,
\[ \omega^i \wedge \omega^j, \quad \omega^4 \wedge \omega^5, \quad \omega^6 \wedge \omega^7 \]
are candidates for \( b_{ab} (\pi) d\omega^a \wedge d\omega^b \), but we have to pay attention to
\[ d\omega^3 = \omega^1 \wedge \omega^2 + \frac{1}{2} (\omega^4 \wedge \omega^5 - \omega^6 \wedge \omega^7), \]
\[ d\omega^8 = \frac{\sqrt{3}}{2} (\omega^4 \wedge \omega^5 + \omega^6 \wedge \omega^7). \]

Therefore, only one of the three in Eq. (108) is not a total derivative and affects the equation of motion.

C. Summary of the Lagrangian

Let us summarize what we have shown above. We found explicit analytic solutions for differential equations derived in Sec. II under the assumptions that the symmetries can be gauged (see conditions discussed in Sec. II D).

In \( 3 + 1 \) dimensions, the most general effective Lagrangian that has the internal symmetry \( \delta \pi^a = \epsilon^i (x) h_i^a (\pi), \delta A_{\mu}^i = \nabla_\mu \epsilon^i (x) + f_{j k} A_{\mu}^j \epsilon^k (x) \) as well as the spatial rotation is given by
\[ \mathcal{L}_{\text{eff}} = c_a (\pi) \tilde{\pi}^a + c_i (\pi) A_{i}^a + \frac{1}{2} g_{ab} (\pi) D_{\mu} \pi^a D_{\nu} \pi^b - \frac{1}{2} g_{ab} (\pi) \tilde{D}_{\mu}^a \cdot \tilde{D}_{\nu}^b. \]  
(111)
to the quadratic order in derivatives. Here, \( D_{\mu} = \nabla_{\mu} - h_i^a (\pi) A_{i}^a \) is the covariant derivative. The coefficients \( c_a (\pi), c_i (\pi), g_{ab} (\pi), \tilde{g}_{ab} (\pi) \) are given by
\[ c_a (\pi) = -\omega_i^a (\pi) c_i (0), \]
\[ e_i (\pi) = \nu_i^f (\pi) e_i (0), \]
\[ g_{ab} (\pi) = g_{cd} (0) \omega_i^a (\pi) \omega_i^b (\pi), \]
\[ \tilde{g}_{ab} (\pi) = -\tilde{g}_{cd} (0) \omega_i^a (\pi) \omega_i^b (\pi). \]

Here, \( \omega_i^a (\pi) T_i = -i U (\pi)^{1/2} \tilde{D}_{\mu}^a \cdot \tilde{D}_{\nu}^b. \]

The Lagrangian contains only few parameters (coupling constants) \( c_i (0), g_{ab} (0) \) and \( \tilde{g}_{ab} (0) \). They have to be invariant under unbroken symmetry transformation; i.e.,
\[ \frac{f_{\alpha i j}}{4} e_j (0) = 0, \]
\[ f_{\alpha i} \tilde{e}_i (0) = 0, \]
\[ f_{\alpha i} \tilde{e}_i (0) + f_{\alpha j} \tilde{e}_j (0) = 0, \]
\[ f_{\alpha i} \tilde{e}_i (0) = 0, \]
\[ f_{\alpha i} \tilde{e}_i (0) + f_{\alpha j} \tilde{e}_j (0) = 0. \]

If we further demand the Lorentz invariance, \( \tilde{g}_{ab} (0) = e^{c_2 g_{ab} (0)} \) and \( e_i (0) = 0 \) so that the Lagrangian is reduced to
\[ \mathcal{L}_{\text{eff}} = \frac{1}{2} g_{ab} (\pi) D_{\mu} \pi^a D_{\nu} \pi^b. \]
(119)

This is the leading order term of the standard chiral perturbation theory. Therefore, our effective Lagrangian equally applies to Lorentz invariant systems.

In \( 2 + 1 \) dimensions, one can add
\[ -\frac{1}{2} b_{ab} \epsilon^r s \nabla_r \pi^a \nabla_s \pi^b - \epsilon_i^r \epsilon^s \epsilon^a \pi^i \nabla_s \pi^a - \frac{1}{2} a_{i j} \epsilon^r \epsilon^s A_{i}^a \]
to the effective Lagrangian Eq. (111), where
\[ b_{ab} (\pi) = -\epsilon_i^r \epsilon^s \epsilon^a \pi^i \nabla_s \pi^a, \]
\[ e_i^r (\pi) = \epsilon_{bc} (0) \nu_i^f (\pi) \omega_i^s (\pi), \]
\[ e_i^r (\pi) = \epsilon_{bc} (0) \nu_i^f (\pi) \omega_i^s (\pi), \]
\[ a_{i j} (\pi) = h_i^a (\pi) \nu_j^b (\pi) \]
with constraints Eq. (102) on \( e_i^r (0) \).

Similarly, in \( 1 + 1 \) dimensions, the following terms are allowed:
\[ \tilde{c}_a \nabla_x \pi^a + \tilde{c}_i A_{i}^a + \tilde{g}_{ab} D_{\nu} \pi^a D_{\mu} \pi^b + \tilde{b}_{ab} \tilde{\pi}^a \nabla_x \pi^b \]
\[ + \epsilon_i^r (\pi) \nabla_x \pi^a - A_{i}^a \pi^a + \tilde{a}_{i j} A_{i}^a A_{j}^b \]
(124)
where
\[ \tilde{c}_a (\pi) = -\omega_i^a (\pi) \tilde{c}_i (0), \]
\[ \tilde{c}_i (\pi) = \nu_i^f (\pi) \tilde{c}_i (0), \]
\[ \tilde{g}_{ab} (\pi) = \tilde{g}_{cd} (0) \omega_i^a (\pi) \omega_i^b (\pi), \]
\[ \tilde{b}_{ab} (\pi) = -\epsilon_i^r \epsilon^s \epsilon^a \pi^i \nabla_s \pi^a, \]
\[ \tilde{e}_i^r (\pi) = \epsilon_{bc} (0) \nu_i^f (\pi) \omega_i^s (\pi), \]
\[ \tilde{a}_{i j} (\pi) = h_i^a (\pi) \tilde{a}_{i j} (\pi), \]
with constraints Eqs. (89), (93), and (103) on coupling constants.
D. Gauging $\mathcal{H}$ rather than modding

It is well-known (see [34] for a review) that the coset construction on $G/H$ is equivalent to that on $G$ with the right-translation by $\mathcal{H}$ gauged. Here we use the notation $\mathcal{H}$ that commutes with the left-translation by $G$, as opposed to $H \subset G$ that does not commute with $G$. The gauging of the unbroken $\mathcal{H}$ symmetry eliminates unwanted NG bosons. Using this method, it is now somewhat more transparent to derive the action in the differential-geometric method above because the transformation laws are linear.

We first consider $U = e^{i\Pi}$ with $\Pi = \pi^a T_a + \pi^\rho T_\rho$ for all generators of $\mathcal{g}$. Namely, $T_\rho \in \mathfrak{h}$ and $T_a \in \mathfrak{g}/\mathfrak{h}$. Under the global symmetry $G$, $U$ transforms as left-translation,

$$U(\pi) \rightarrow gU(\pi) = U(\pi').$$

On the other hand, we require a local symmetry under the right-translation by $\mathcal{H}$,

$$U(\pi) \rightarrow U(\pi)h(x).$$

Note that this is different from the gauging we studied in the previous sections that corresponds to the left-translation.

The point here is that one can always take the gauge $\pi^\rho = 0$. In order for $U$ to stay in this gauge, the global transformation needs to be accompanied by a gauge transformation

$$U(\pi) \rightarrow gU(\pi)h_\rho(\pi) = U(\pi')$$

with a suitable choice of $h_\rho \in \mathcal{H}$. The end result is therefore equivalent to writing the theory on $G/H$.

We introduce a gauge field $A = A^\rho T_\rho = A_\mu dx^\mu$ for the right-translation gauge group $\mathcal{H}$ so that the Lagrangian is invariant under both the global $G$ and the local $\mathcal{H}$. Note that we use a different symbol from the gauge field $A'$ in the previous section [see, e.g., Eq. (79)] for the left-translation under $G$. The Maurer–Cartan form $\omega = -i U^\dagger dU$ is invariant under the global $G$, while it transforms as

$$\omega \rightarrow -ih^\dagger U^\dagger d(Uh) = h^\dagger \omega h - ih^\dagger dh.$$  \hspace{1cm} (134)

On the other hand, the gauge field transforms as usual,

$$A \rightarrow h^\dagger Ah + ih^\dagger dh.$$  \hspace{1cm} (135)

Then the combination

$$\omega + A$$

is gauge covariant. As before, we decompose the Maurer–Cartan forms $\omega = \omega_\perp + \omega_\parallel$, where $\omega_\perp = \omega^a T_a$ are in $\mathfrak{g}/\mathfrak{h}$, while $\omega_\parallel = \omega^\rho T_\rho$ are in $\mathfrak{h}$. Then, the inhomogeneous transformation occurs only on $\omega_\parallel$,

$$\omega_\perp \rightarrow h^\perp \omega_\perp h,$$

$$\omega_\parallel + A \rightarrow h^\parallel (\omega_\parallel + A)h.$$  \hspace{1cm} (138)

Therefore, we can build invariant Lagrangian just by focusing on local $\mathcal{H}$ invariance on $\omega_\perp$ and $\omega_\parallel + A$.

We introduce the notation for the pull-back of Maurer–Cartan forms to space and time,

$$\pi^a \omega = \partial^a t U(\hat{\pi}^a dt + \nabla \pi^i \cdot d\vec{x}).$$

They are decomposed as

$$\omega = \omega^a T_a + \omega^\rho T_\rho,$$  \hspace{1cm} (140)

$$\omega = \omega^a T_a + \omega^\rho T_\rho.$$  \hspace{1cm} (141)

The general Lagrangian at the second order in time-derivative is

$$\mathcal{L}_{\text{eff}} = \frac{1}{2} \bar{g}_{ab}(0) \bar{\omega}^a \bar{\omega}^b + \frac{1}{2} \bar{g}_{\rho\sigma}(0) (\bar{\omega}^\rho + A^\rho_\parallel) (\bar{\omega}^\sigma + A^\sigma_\parallel)$$

$$- \frac{1}{2} g_{ab}(0) \bar{\omega}^a \cdot \bar{\omega}^b - \frac{1}{2} g_{\rho\sigma}(0) (\bar{\omega}^\rho + A^\rho_\parallel) \cdot (\bar{\omega}^\sigma + A^\sigma_\parallel).$$  \hspace{1cm} (142)

$\bar{g}_{ab}(0)$, $\bar{g}_{\rho\sigma}(0)$, $g_{ab}(0)$, $g_{\rho\sigma}(0)$ are all constants subject to $H$-invariance as in the previous section [see Eqs. (117) and (118)].

Because the Lagrangian is quadratic in $A$, we can integrate it out and find

$$A = -\omega_\parallel.$$  \hspace{1cm} (143)

In addition, we can perform a gauge transformation in $\mathcal{H}$ to remove all $\pi^\rho$ without a loss of generality. Then the Lagrangian reduces to the form

$$\mathcal{L}_{\text{eff}} = \frac{1}{2} \bar{g}_{ab}(0) \bar{\omega}^a \bar{\omega}^b - \frac{1}{2} g_{ab}(0) \bar{\omega}^a_\perp \cdot \bar{\omega}^b_\perp,$$  \hspace{1cm} (144)

which can easily verified to be the same as what we derived in earlier sections.

So far, everything is well-known. Now come the new terms we discussed in previous sections.

We first discuss terms with a single derivative. If the generator $T_a \in \mathfrak{g}/\mathfrak{h}$ commutes with $\mathcal{H}$, $\omega^a \rightarrow h^\perp \omega^a_\parallel h = \omega^a$ and hence is invariant. Therefore, we can add it to the Lagrangian. On the other hand, if the generator $T_\rho \in \mathfrak{h}$ commutes with $\mathcal{H}$, it generates a $U(1)$ subgroup, and hence

$$\omega^\rho \rightarrow h^\perp \omega^\rho h - i(h^\perp dh)^\rho = \omega^\rho - id(\log h)^\rho.$$  \hspace{1cm} (145)

Namely, the shift is a total derivative. It is also allowed as a term of the Lagrangian. In addition, the combination $(\omega^a + A^a)$ is invariant. Therefore, the following terms are allowed,

$$\mathcal{L}_{\text{eff}}^{(0,1)} = -e_a(0) \omega^a - e_\rho(0) \omega^\rho - \bar{e}_\rho(0) (\omega^\rho + A^\rho).$$  \hspace{1cm} (146)

The last term is removed after integrating over $A^\rho$ together with the quadratic terms. Therefore, we only need to consider the first two terms, which are nothing but

$$\mathcal{L}_{\text{eff}}^{(0,1)} = -e_i(0) \omega^a_\parallel \hat{\pi}^a.$$  \hspace{1cm} (147)
we derived in Eq. (94).

The anti-symmetric tensor can also be included in the same fashion,
\[
L_{\text{eff}}^{(2,0)'} = -\frac{1}{2} e'_{ab}(0) \omega^a \times \omega^b_{\perp} - \frac{1}{2} \epsilon'_{\rho(0)}(\omega^\rho + \vec{A}^\rho) \times (\omega^\sigma + \vec{A}^\sigma). \tag{148}
\]
e'_{ab}(0) is invariant under \( \mathcal{H} \) [see Eq. (102)]. The second line is again eliminated by integrating out \( \vec{A}^\rho \), and the first line again can be shown to be the same as the previous result.

The central extension or Wess–Zumino–Witten terms, however, cannot be written using Maurer–Cartan forms because they are not gaugeable as we discuss in the following section.

The advantage of this formulation is that the only question is to find \( \mathcal{H} \)-invariant tensors. It is therefore easier to generalize to higher-derivative terms than solving the differential equations. In that case, integration over the gauge field needs to be done by an order-by-order basis because the Lagrangian is no longer quadratic in the gauge field.

Note that we integrated out the gauge fields \( A \) to show the equivalence to the results in the previous sections. For some applications such as the equivalence to the results in the previous sections.

E. Central extensions and Wess–Zumino–Witten term

We have presented our analytic expressions of the effective Lagrangian in terms of Maurer–Cartan forms assuming that the symmetry \( G \) is gaugeable. The conditions for the gaugeability are summarized in Sec. II D. In this section, we discuss examples in which at least one of these conditions is violated, making it impossible to gauge the symmetry.

1. Central extensions

Let us consider the case \( G = U(1) \times U(1) \) and \( H = \{ \epsilon \} \). The NG fields \( \varphi^a \) \((a = 1, 2)\) independently changes by a constant under \( G \). In such a case, the effective Lagrangian may contain
\[
e_a(\varphi) \varphi^a = \frac{C}{2} \epsilon_{ab} \varphi^a \varphi^b. \tag{149}
\]
with \( C \) a constant.

Here we explain that the one-form \( \epsilon = (C/2) \epsilon_{ab} \varphi^a \varphi^b \) ends up with nonzero \( \tilde{z}_{ij}(\pi) \) in Eq. (36). To that end, we first compute \( e_a(\varphi) \) following the definition in Eq. (33),
\[
de c = \frac{C}{2} \epsilon_{ab} \varphi^a \wedge d\varphi^b, \tag{150}
\]
\[
i_{h_a} dc = C \epsilon_{ab} d\varphi^b = d\epsilon_a, \tag{151}
\]
where \( h_a = \partial_a \). Therefore, \( \epsilon_a = C \epsilon_{ab} \varphi^b \) up to a constant. Their Lie derivative is
\[
\mathcal{L}_{h_a} e_b = \partial_a e_b = -C \epsilon_{ab}. \tag{152}
\]
Comparing this with Eq. (36), we see \( z_{ab} = -C \epsilon_{ab} \neq 0 \). Therefore, the symmetry \( G \) cannot be gauged. The Lagrangian
\[
L_{\text{eff}}^{(0,1)} = \frac{C}{2} \epsilon_{ab} \varphi^a \varphi^b + C \epsilon_{ab} \varphi^b A_0^a \tag{153}
\]
changes not only by a surface term \( \nabla e(C \epsilon_{ab} \varphi^a)/2 \) but also by \( C \epsilon_{ab} A_0^b = -C \epsilon_{ab} A_0^b \).

To make a connection to central extensions, we note that conserved charges of the internal symmetry \( G \) are dominated by \( Q_a = \int d^d x j_a^0 = \int d^d x C \epsilon_{ab} \varphi^b \).

Their commutation relation can be computed by using the commutation relation \( [\varphi^a(\vec{x}, t), \varphi^b(\vec{x}, t)] = -C^{-1} \delta^a_{\parallel} (\vec{x} - \vec{x}') \) as
\[
[Q_a, Q_b] = -i \epsilon_{ab} C \Omega, \tag{154}
\]
where \( \Omega \) is the volume of the system. Naively the shift symmetries \( \varphi^a \rightarrow \varphi^a + c^a \) for \( a = 1 \) and \( a = 2 \) commute with each other, but Noether charges do not. This is the central extension of the \( g = u(1) \times u(1) \) algebra.

The shift symmetry \( \psi = \psi + c \) \((c \in \mathbb{C})\) of the free boson Schrödinger field theory,
\[
\mathcal{L} = \frac{i}{2} (\psi^\dagger \psi - \psi \psi^\dagger) - \frac{1}{2m} \vec{\nabla} \psi^\dagger \cdot \vec{\nabla} \psi \tag{155}
\]
cannot be gauged due to the same reason, although the phase rotation \( \psi \rightarrow e^{i \epsilon} \psi \) can be gauged.

The central extension is possible only when the second cohomology of the Lie algebra \( H^2(g) \) is non-trivial. Namely, \( G \) must to have at least two abelian generators that commute with all the other generators. [See Appendix A for a brief review of \( H^2(g) \).] Therefore, the corresponding terms in the Lagrangian are always of the form \(-(1/2) \epsilon_{ab} \varphi^a \varphi^b\), where \( \varphi^a \) are the NG fields for such abelian generators, which leads to the extended algebra \([Q_a, Q_b] = i \epsilon_{ab} \Omega\).

Note that the coefficient \( C \) is quantized when \( G/H \) is compact. See the discussion at the end of Sec. VII D.

2. Example of \( \tilde{z}_{ij}(\pi) \)

We now give an example of nonzero \( \tilde{z}_{ij}(\pi) \) in Eq. (39). We take \( G = U(1)^3 = \{(\varphi^1, \varphi^2, \varphi^3) | \varphi^i \in [0, 2\pi) \} \) and \( H = \{ \epsilon \} \). The effective Lagrangian may contain
\[
L_{\text{eff}}^{(1,1)''} = b_{ab}(\varphi) \varphi^a \nabla_x \varphi^b = \frac{k}{3(2\pi)^2} \epsilon_{abc} \varphi^a \varphi^b \nabla \varphi^c, \tag{156}
\]
which can be regarded as the two-from \( b = (k/3!)(2\pi)^{-2} \epsilon_{abc} \varphi^a \varphi^b \wedge d\varphi^c \). The one-form \( \tilde{e}_a'(\varphi) \) can
be computed as
\[ d\tilde{b} = \frac{k}{3(2\pi)^2} \epsilon_{abc} d\varphi^a \wedge d\varphi^b \wedge d\varphi^c, \]  
\[ i_{h_a} d\tilde{b} = d \left( \frac{k}{2(2\pi)^2} \epsilon_{abc} \varphi^b d\varphi^c \right) = d\tilde{e}_b^c. \]  
(157)
(158)
Therefore, \( \tilde{e}_b^c(\varphi) = (k/2)(2\pi)^{-2} \epsilon_{abc} \varphi^b d\varphi^c \) up to a closed one-form.

Let us check conditions for gaugeability summarized in Sec. II D one by one. First, Eq. (62) is satisfied since
\[ i_{h_a} \tilde{e}_b^c = \frac{k}{2(2\pi)^2} \epsilon_{abc} \varphi^c \]  
is antisymmetric with respect to \( a, b \). However,
\[ \mathcal{L}_{h_a} \tilde{e}_b^c = d \left( -\frac{k}{2(2\pi)^2} \epsilon_{abc} \varphi^c \right) = dz_{ab}. \]  
(159)
(160)
Hence \( z_{ab}(\varphi) = -(k/2)(2\pi)^{-2} \epsilon_{abc} \varphi^c \neq 0 \) up to a constant. This is the obstruction to gauge the symmetry \( G \).

Note that the coefficient \( k \) must be an integer to ensure that the Lagrangian changes only by integer multiples of \( 2\pi \) under the periodic shift \( \varphi^a \rightarrow \varphi^a + 2\pi \). This is because the integrand \( e^{iS} \) in the path integral must be single-valued even though the action \( S \) itself is multi-valued. See the discussion at the end of Sec. VII D.

On the other hand, this type of term is not allowed in \( \mathcal{L}_{\text{eff}}^{(2,0)'} = -(1/2)\tilde{h}_{ab}(\pi) \tau^x \nabla_x \pi^a \nabla_x \pi^b \) in Eq. (49) because Hamiltonian must be single-valued.

3. Wess–Zumino–Witten term

In general, we can write a similar term whenever \( H_{\text{dR}}^n(G/H) [32, 33] \) is non-trivial. (Here and below, \( H^n_{\text{dR}} \) refers to de Rham cohomology, the space of closed but not exact n-forms.) Then there is a non-trivial closed three-form \( \omega_3 \) on \( G/H \). Because \( \omega_3 \) is locally exact \( \omega_3 = d\tilde{b} \), we can take the 1 + 1 dimensional spacetime Wick-rotated and compactified to Euclidean space \( S^2 = \partial B_3 \) as a boundary of a three-ball \( B_3 \), and we can have
\[ \int_{B_3} \omega_3 = \int_{S^2} \tilde{b} \]  
(161)
as a part of a Lagrangian or a Hamiltonian.

Note that there are in general more than one \( B_3 \) in \( G/H \) whose boundary is \( S^2 = \partial B_3 \). Therefore, the action is defined only up to an integral of \( \omega_3 \) over a closed three-surface in \( G/H \). To ensure that \( e^{iS/h} \) in path integral is single-valued, the difference may only be integer multiples of \( 2\pi h \) [35]. It requires quantization condition on the coefficient of terms of this type. The same quantization condition can be obtained from the requirement of the associativity of the group elements [36].

An important example is the Wess–Zumino–Witten term [37]. This term exists for any compact simple \( G \) and \( H = \{ e \} \), because \( H^3_{\text{dR}}(G) = \mathbb{R} \). It is defined with
\[ \omega_3 = \frac{k}{12\pi} \text{tr}[(U^{-1}dU)^3] = \frac{k\lambda}{24\pi} f_{abc} \omega^a \wedge \omega^b \wedge \omega^c \]  
(162)
with \( k \) an integer, which is sometimes referred to as the level. Here, we normalized \( T_a \) as \( \text{tr}[T_a T_b] = \delta_{ab} \) so that the structure constant is completely antisymmetric. In order for the path integral \( e^{iS} \) to be single-valued, \( k \) must be an integer in \( 1 + 1 \) dimension. (See the discussion at the end of Sec. VII D.) Also, because of this ambiguity of \( 2\pi k \), Wess–Zumino–Witten term cannot be used to construct a \( b(\pi) \) term since it takes part in the energy functional, as noted before.

Consider the transformation \( U(\pi) \rightarrow U(\pi') = gU(\pi) \). Obviously for a global \( g \), \( \omega_3 \) does not change. However, \( \tilde{b} \) can change. To see this, let us temporarily regard \( g = e^{iv} \) to be local, and consider infinitesimal change up to the linear order in \( idv = g^{-1}dg \).
\[ \frac{12\pi k}{k} \delta(\tilde{b}) = \text{tr}[(U^{-1}dU + U^{-1}(g^{-1}dg)U)^3] - (U^{-1}dU)^3 \]
\[ = 3\text{tr}[g^{-1}dg(UdU^{-1})^2] \]
\[ = 3id\text{tr}[v(UdU^{-1})^2], \]  
(163)
and hence
\[ \delta\tilde{b} = \frac{ik}{4\pi} \text{tr}[v(UdU^{-1})^2]. \]  
(164)
Now we can set \( v \) to be constant. Then we see
\[ \delta\tilde{b} = \frac{ik}{4\pi} \text{tr}[v(UdU^{-1})^2] = -\frac{ik}{4\pi} d\text{tr}[vUdU^{-1}] \]  
(165)
is indeed a total derivative.

There is no compact way to write \( \omega_3 = d\tilde{b} \), but the following trick works for a power series expansion in \( \pi \). By defining \( U_\tau = e^{\tau\Pi} \) for a real parameter \( \tau \), it is easy to show
\[ \frac{\partial}{\partial \tau} U_\tau^{-1}dU_\tau = iU_\tau^{-1}(d\Pi)U_\tau, \]  
(166)
and therefore
\[ \frac{\partial}{\partial \tau} \text{tr}[(U_\tau^{-1}dU_\tau)^3] = -3id\text{tr}[\Pi dU_\tau \wedge dU_\tau^{-1}]. \]  
(167)
We can integrate the both sides and find
\[ \tilde{b} = -\frac{ik}{4\pi} \int_0^1 d\tau \text{tr}[\Pi dU_\tau \wedge dU_\tau^{-1}] \]  
(168)
to obtain an explicit form in a power series expansion in \( \Pi \). To the leading order in \( \pi \), we find
\[ \tilde{b} = \frac{k\lambda}{24\pi} f_{abc} \pi^a d\pi^b \wedge d\pi^c + O(\pi^4). \]  
(169)
Since \( \pi^a \) shifts under the \( G \) transformation, we can see that \( \tilde{b} \) changes by a total derivative.
It is well-known that the Wess–Zumino–Witten term cannot be gauged. To clarify the obstruction we now compute $\tilde{c}_a^e(\pi)$.

\[
i_{h_a} -db = \frac{k\lambda}{4\pi} (i_{h_a} \omega^a) \left( \frac{1}{2} f_{abc} \omega^b \wedge \omega^c \right)
= \frac{k\lambda}{4\pi} \nu_a \omega^a = -\frac{k\lambda}{4\pi} d(\nu_a \omega^a),
\]
where we used Eqs. (70) and (80). (Since we assume all generators are broken, terms with indices $\rho, \sigma, \cdots$ should be neglected.) The last equality can be shown backwards:

\[
d(\nu_a^b \omega^c) = (d\nu_a^b) \wedge \omega^c + \nu_a^b d\omega^c
= (f_{abc} \omega^b \nu_a^c) \wedge \omega^a + \nu_a^b d\omega^a
= -\nu_a^b f_{abc} \omega^c \wedge \omega^b + \nu_a^b d\omega^a
= -2\nu_a^b d\omega^c + \nu_a^b d\omega^a = -\nu_a^b d\omega^c,
\]
where we used Eq. (82) in the first line. Comparing Eq. (170) with Eq. (34), we find

\[
\tilde{c}_a^e = -\frac{k\lambda}{4\pi} \nu_a \omega^c
\]
up to an exact one-form.

Having obtained $\tilde{c}_a^e(\pi)$, let us now check the gauge-invariance condition. First, the Lie derivative of $\tilde{c}_a^e(\pi)$ satisfies

\[
\mathcal{L}_{\xi} \tilde{c}_a^e = -\frac{k\lambda}{4\pi} \left( (\mathcal{L}_{\xi} \nu_a^b) \omega^c + \nu_a^b (\mathcal{L}_{\xi} \omega^c) \right)
= -\frac{k\lambda}{4\pi} \left( f_{abc} \nu_a^b \omega^c + 0 \right) = f_{abc} \tilde{c}_c^e,
\]
meaning that $\tilde{c}_a^e(\pi)$ does vanish according to Eq. (38). However, since

\[
i_{h_a} \tilde{c}_b^e = -\frac{k\lambda}{4\pi} \nu_b^e (i_{h_a} \omega^c) = \frac{k\lambda}{4\pi} \nu_b^e \nu_a^c = -\frac{k}{4\pi} \delta_{ab},
\]
$i_{h_a} \tilde{c}_b^e$ is symmetric, rather than antisymmetric, with respect to $a, b$, and therefore does not satisfy Eq. (66). This is why the Wess–Zumino–Witten term cannot be made gauge-invariant.

In the derivation of Eqs. (173) and (174), we used Eqs. (70), (71), (77), and (81).

Another example of this type is $G/H = U(1) \times SO(3)/SO(2) = S^1 \times S^2$ with $H^3_{\text{DR}}(G/H) = H^3_{\text{DR}}(S^1) \times H^2_{\text{DR}}(S^2) = \mathbb{R}$. Parameterizing the coset space with $\varphi$ for $S^1$ and the unit vector $\vec{n}$ for $S^2$, we can write

\[
\mathcal{L}_{\text{eff}}^{(1,1)^t} = k \frac{\varphi}{4\pi} \vec{n} \cdot (\vec{n} \times \nabla_{\vec{x}} \vec{n}).
\]
Under a constant shift of $\varphi$ by $2\pi$, the change is a total derivative in space and hence the Lagrangian is $U(1)$ invariant. However for a local shift of $\varphi$, it changes the Lagrangian and hence is not an invariant.

Note that the shift of $\varphi$ by $2\pi$ does not change $e^{iS}$ in the path integral because

\[
\int \! dt \! dx \frac{1}{4\pi} \vec{n} \cdot (\vec{n} \times \nabla_{\vec{x}} \vec{n}) \in \mathbb{Z}
\]
is the winding number of $S^2 \to S^2$, as long as $k \in \mathbb{Z}$.

### F. Berry’s phase

Finally, we discuss the interpretation of the linear time derivative term of the effective Lagrangian as the Berry phase. Terms of our interest are

\[
\mathcal{L} = c_a(\pi) \dot{\pi}^a + e_i(\pi) A_i^a
= -\omega_a^a(\pi) \dot{\pi}^a e_i(0) + e_i(0) \pi^a(\pi) A_i^a.
\]

We apply a set of infinitesimal external fields $A_i^a = \mu^i(\pi)$ that slowly depend on time. NG fields $\pi^a$ condense in such a way that $\{\pi^a\}_{a=1}^{\dim G/H}$ minimize the potential,

\[
V(t) \equiv -e_i(0) \pi^a(\pi) \mu^i(t),
\]

at each time. Now we consider a closed path $\mu^i(t)$ in the parameter space $\{\mu^i\}_{i=1}^{\dim \mathcal{G}}$. NG fields adiabatically depend on time through external fields, i.e., $\pi^a = \pi^a(\mu(t))$. Under this process, the ground state $|\Psi_0\rangle$ evolves as

\[
|\Psi(t)\rangle = e^{i\pi^a(\mu(t))} Q_a |\Psi_0\rangle,
\]

where $Q_a = \int dt d^dx f_{ji}^a(\vec{x}, t) \dot{\pi}^j(\vec{x}, t)$ are broken generators. Note that $\pi^a$ here is a c-number, not an operator, that is fixed by $\mu^i(t)$.

The Berry phase acquired under this cyclic process is

\[
\Theta_{\text{BP}} = \int dt \frac{i}{\hbar} \langle \Psi(t) | \frac{d}{dt} |\Psi(t)\rangle
= -\int dt \frac{d\pi^a(\mu(t))}{dt} \langle \Psi_0 | Q_a |\Psi_0\rangle
= -\int dt d^dx \omega_a^a(\pi) \dot{\pi}^a e_i(0),
\]

where $e_i(0) = \langle \Psi_0 | Q_a |\Psi_0\rangle / \Omega = \langle \Psi_0 | j_0^a(\vec{x}, t) |\Psi_0\rangle$ due to the translational invariance of the ground state. Again, we have used the fact that the Maurer–Cartan form $\omega_a^e(\pi)$ only depends on the commutation relation and it does not depend on the specific representation. Equation (180) reproduces the $c_a(\pi) \dot{\pi}^a$ term of the effective Lagrangian, except for the $\xi$ dependence of $\pi^a$.

To treat the coordinate dependence properly, we introduce external fields $\mu^i = \mu^i(\vec{x}, t)$ that are slowly varying over both space and time. In this case, the ground state is given by

\[
|\Psi(t)\rangle = e^{i\Pi(t)} |\Psi_0\rangle,
\]

\[
\Pi(t) = \int dt d^dx j^a_{\mu}(\vec{x}, t) \pi^a(\mu, \vec{x}, t).
\]

To compute the Berry phase, we have to evaluate commutation relations

\[
\langle \Psi_0 | \Pi, \cdots, \Pi, \nabla_{\vec{x}} \Pi, \cdots \rangle |\Psi_0\rangle = \int dt d^dx \left[ \hat{\pi}^a(\Psi_0) |\Pi, \cdots, \Pi, j^a_{\mu}(\vec{x}, t), \cdots \rangle |\Psi_0\rangle \right.
\]

\[
-\pi^a \nabla_{\vec{x}} \cdot (\dot{\Psi}_0 |\Pi, \cdots, \Pi, j^a_{\mu}(\vec{x}, t), \cdots \rangle |\Psi_0\rangle).
\]
Here we used the current conservation, \( \nabla f_\alpha^0(\vec{x}, t) + \vec{v} \cdot j_\alpha(\vec{x}, t) = 0 \). The second line vanishes since we assume the rotational symmetry of the ground state. Also, due to the translational symmetry of the ground state, the expectation value of the commutator in the first line does not actually depend on \( \vec{x} \) or \( t \). Using the current algebra \( [j_\alpha^0(\vec{x}, t), j_\beta^0(\vec{x}', t)] = i f_{ij}^k j_\alpha^0(\vec{x}) \delta^k(\vec{x} - \vec{x}') \), one can easily show Eq. (180) with the proper coordinate dependence of \( \pi^a \).

\[ \pi_i = \delta_{ij} x_j. \]

\section{IV. Number of Nambu–Goldstone Bosons}

In the next two sections, we will make use of the effective Lagrangian developed in the previous section to derive several rigorous results on the number of NGBs. To be consistent with the assumed broken symmetries, in this and next sections we assume 2 + 1 or higher dimensions.

In order to discuss the number and the dispersion relation of NGBs, we focus on the free part of the Lagrangian. We will justify ignoring the interaction terms in Sec. VI A. Keeping only the quadratic terms in \( \pi \) in Eq. (111) and setting \( A^i_\mu = 0 \), we find

\[ \mathcal{L}_{\text{eff}} = \frac{1}{2} \sigma_{ab} \dot{\pi}^a \dot{\pi}^b + \frac{1}{2} \bar{g}_{ab}(0) \dot{\pi}^a \pi_b - \frac{1}{2} \bar{g}_{ab}(0) \vec{\nabla} \pi^a \cdot \vec{\nabla} \pi^b. \]  

Note that \( b(\pi) \) term does not contribute to the free part.

When \( z_{ij} \) in Eq. (36) does not vanish, \( e_i(\pi) \) and \( e_i(\pi) \) receives contribution from \( z_{ij} \):

\[ e_i(\pi) = \frac{1}{2} f_{ab}^k e_k(0) \dot{\pi}^a \pi_b + \frac{1}{2} \bar{g}_{ab}(0) \dot{\pi}^a \pi_b - \frac{1}{2} \bar{g}_{ab}(0) \vec{\nabla} \pi^a \cdot \vec{\nabla} \pi^b. \]  

The condition Eq. (92) should also be replaced by \( f_{\rho \lambda}^b e_b(0) + z_{\rho i} \rho = 0 \). Including this contribution, we have

\[ \mathcal{L}_{\text{eff}} = \frac{1}{2} \sigma_{ab} \dot{\pi}^a \dot{\pi}^b + \frac{1}{2} \bar{g}_{ab}(0) \dot{\pi}^a \pi_b - \frac{1}{2} \bar{g}_{ab}(0) \vec{\nabla} \pi^a \cdot \vec{\nabla} \pi^b. \]  

A. Derivation 1

The parameter \( e_i(\pi) \) is related to the expectation value of the conserved charge density. From the Noether’s theorem, the conserved current associated with \( \delta_i \pi^a = h_i^a \) can be derived as

\[ j_\alpha^0(x) = e_i(\pi) - \bar{g}_{ab}(\pi) h_i^a(\pi) \pi^b. \]  

Note that the conserved current operators are free of anomalous dimensions even in the presence of interactions because \( j_\alpha^0 \rightarrow Z j_\alpha^0 \) would violate the commutation relations \( [j_\alpha^0(x), j_\beta^0(y)] = i f_{\alpha\beta\gamma} j_\gamma(x) \delta(x - y) \). This is the non-renormalization theorem of conserved currents. Therefore, its expectation value is that of the origin,

\[ \langle j_\alpha^0(x) \rangle = e_i(0). \]  

We present explicit calculation in Sec. VI B and an alternative argument in Sec. VII C to support this point.

Now, let us define a real and antisymmetric matrix \( \rho \) by

\[ i \rho_{ab} = \langle [Q_{a}, j_b(x)] \rangle. \]  

Assuming the translational invariance of the ground state, \( \rho_{ab} \) is independent of \( x \). We see that \( \rho_{ab} \) is related to the first term in the effective Lagrangian:

\[ \rho_{ab} = -i \langle [Q_{a}, j_b^0(x)] \rangle = f_{ab}^c (j_c^0(x)) + z_{ab} = \sigma_{ab}. \]  

One can always block diagonalize \( \rho \) by an orthogonal matrix as

\[ \rho = \begin{pmatrix} i \sigma_y \lambda_1 & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ i \sigma_y \lambda_m & \cdots & \cdots & \cdots \end{pmatrix}, \lambda_\alpha \neq 0 (\alpha = 1, \ldots, m). \]  

Here, \( \sigma_y \) is the Pauli matrix and \( m = (1/2) \text{rank} \rho \). On this basis, the first term of the effective Lagrangian becomes

\[ \sum_{\alpha=1}^m \lambda_\alpha \pi^{2\alpha} \pi^{2\alpha-1} = \lambda_1 \pi^{2} \pi^{1} + \cdots + \lambda_m \pi^{2m} \pi^{2m-1}. \]  

In the presence of these single time-derivative terms, one can neglect \( O(\nabla^2) \) terms at a sufficiently low energy. Therefore, \( \lambda_\alpha \pi^{2\alpha} \) (no sum) is in fact a canonically conjugate valued to \( \pi^{2\alpha-1} \). They together represent one low-energy degree of freedom, rather than two. We call those NGBs that are generated by a pair of canonically conjugate generators type-B, while the rest type-A. By definition, the number of type-A and type-B NGBs are given by

\[ n_A = \dim G/H - \text{rank} \rho, \quad n_B = \frac{1}{2} \text{rank} \rho. \]  

This proves the counting rules in Eqs. (1) and (2). As a corollary, the number of NGBs always falls into the range,

\[ \frac{1}{2} \dim G/H \leq n_{\text{NGB}} \leq \dim G/H. \]  

This is obvious since \( 0 \leq \text{rank} \rho \leq \dim G/H \).

Note that our definition of type-A, B NGBs is not based on the dispersion relation. They are instead classified based on the structure of time-derivatives that defines the \textit{presymplectic structure} (see, e.g., [38]) as we discuss in Sec. VII.
Another way of deriving the same result is to make use of the canonical commutation relation. Let us go back to the first term of Lagrangian \( \frac{1}{2} \sigma_{ab} \pi^a \pi^b \). Here we assume that \( \sigma \) is block diagonalized as
\[
\sigma = \begin{pmatrix}
i\sigma_y \lambda_1' & \cdots & 0 \\
0 & \ddots & 0 \\
\vdots & \ddots & \ddots \\
i\sigma_y \lambda_m' & \cdots & i\sigma_y \lambda_m'
\end{pmatrix},
\]
where \( \lambda'_\alpha \neq 0 (\alpha = 1, \ldots, m) \).

We denote by \( \sigma' \) the \( 2m \times 2m \) upper left part of the matrix \( \sigma \), which has the full rank.

When we neglect the \( O(\nabla^2) \) term of the effective Lagrangian, there are \( m \) constraints of the second class in the system. By following Dirac’s quantization procedure, one can derive the equal-time commutation relation,
\[
[\pi^a(\vec{x},t), \pi^b(\vec{x'},t)] = i(\sigma'^{-1})^{ab}_{cd}(\vec{x} - \vec{x'}).
\]

for \( 0 \leq a, b \leq 2m \). By definition, \( n_A = \dim G/H - \text{rank} \sigma' \) and \( n_B = (1/2)\text{rank} \sigma'' \). In this approach, we have to prove that \( \text{rank} \sigma' = \text{rank} \rho \).

The Noether current in Eq. (188) can be expanded around the origin as
\[
j_\rho^a(x) = e_a(0) + \sigma_{ab} \pi^b(x) + O(\pi^2).
\]

By neglecting the contribution from higher order terms,
\[
\rho_{ab} \equiv -i\langle [Q_a, j_b^0(\vec{x},t)] \rangle = -i \int d^4x' \langle j_a^0(\vec{x'},t), j_b^0(\vec{x},t) \rangle = \sigma'^{ac}(\sigma'^{-1})_{cd} \sigma'^{db} = \sigma'_{ab}.
\]

Therefore, \( \text{rank} \sigma' = \text{rank} \rho \).

V. DISPERSION RELATION

In this section, we discuss the dispersion relation of NGBs. In particular, we show that type-A NGBs generically have linear dispersions, while type-B quadratic, in the absence of long-range interactions. We will then discuss the impact of long-range interactions on the dispersion relations.

A. No long-range interaction

The linearized effective Lagrangian in Eq. (187) leads to the equation of motion, \( g_{ab} \pi^b = i\omega \sigma_{ab} \pi^a \), where
\[
G = i\omega + \bar{g}(0)\omega^2 - g(0)k^2.
\]

The dispersion relations of NGBs is determined by solving \( \det G = 0 \). If type-A and type-B NGB do not coexist, the situation is pretty simple. When \( \sigma = 0 \) (only type-A), the dispersion is always linear since \( \omega^2 \) has to balance with \( k^2 \). In contrast, when \( \sigma \) has the full rank (only type-B), we can ignore \( \bar{g}(0)\omega^2 \ll i\sigma \omega \) in the low-energy limit, and the dispersion is quadratic by the same argument.

Note that \( g(0) \) must always be full-rank as long as we consider an internal symmetry group \( G \). This is because the field transformation rule in Eq. (6) does not explicitly depend on coordinates and thus there are no symmetries that prohibit the appearance of \( O(k^2) \) term. In Sec. VI A we explain examples of NGBs associated with spacetime symmetries which lack \( O(k^2) \) term, but for now let us focus on internal symmetries.

When type-A and type-B NGBs do coexist, and especially when there are NGBs of the same representation under \( H \), the metrics \( g(0) \) and \( \bar{g} \) may mix them and the discussion of the dispersion becomes complicated. To discuss the dispersion even in such a general situation, here we develop a perturbation theory for small \( \omega \).

Assuming that \( g(0) \) is positive and non-singular, we can always write it as \( g(0) = Z^2 \) with \( Z \) a symmetric, positive, and non-singular matrix. Substituting this into \( G \), we have
\[
G' \equiv Z^{-1}GZ^{-1} = i\Sigma \omega + Z^{-1}\bar{g}(0)Z^{-1}\omega^2 - k^2,
\]
where \( \Sigma = Z^{-1}\sigma Z^{-1} \). Because \( \Sigma \) is still real and anti-symmetric, one can always find an orthogonal matrix \( O \) such that
\[
\Sigma = O\Lambda O^T, \quad \Lambda = \begin{pmatrix}
i\sigma_y \lambda'_1 & \cdots & 0 \\
0 & \ddots & 0 \\
\vdots & \ddots & \ddots \\
i\sigma_y \lambda'_m & \cdots & i\sigma_y \lambda'_m
\end{pmatrix}.
\]

Here, \( \lambda'_\alpha > 0 \) for \( \alpha = 1, \ldots, m = (1/2)\text{rank} \rho \). Now \( \det G = 0 \) is equivalent to \( \det G'' = 0 \), where
\[
G'' \equiv O^T G'O = i\Lambda \omega + \bar{G}\omega^2 - k^2
\]
and \( \bar{G} = O^T Z^{-1}\bar{g}(0)Z^{-1}O \).

We regard \( O(\omega^2) \) terms as a small perturbation. Following the standard procedure for the degenerate perturbation theory, we diagonalize the bottom right \( n \times n \) block of \( \bar{G} \):
\[
\bar{G} = \begin{pmatrix}
\ast & \ast & \ast & \ast & \ast \\
\vdots & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast \\
\vdots & \ast & \ast & \ast & 0 \\
\ast & \ast & \ast & \ast & \ast\ast
\end{pmatrix}
\]

Asterisks stand for unknown elements. This diagonalization is compatible with above transformation of \( \Sigma \), since all relevant components of \( \Lambda \) vanish.
The upper left $2m \times 2m$ block has a non-zero unperturbed term, which reads
\[ \lambda_n^\alpha \omega \sigma_y + k^2 \sigma_0 = 0 \Leftrightarrow \omega_n(k) = \frac{k^2}{\lambda_n^\alpha} \quad (205) \]
for $\alpha = 1, \ldots, m$. The off-diagonal component $\omega \sigma_y$ is reminiscent of the presymplectic structure in Eq. (193).

Therefore, these modes with quadratic dispersion may still be called type-B NGBs, although strictly speaking fields describing these modes are in general a mixture of type-A and B NG fields according to the definition in Sec. IV A.

On the other hand, in the bottom right $n \times n$ block, where the zero-th order term vanishes, the linear order correction gives
\[ s_\xi \omega^2 - k^2 = 0 \Leftrightarrow \omega_\xi(k) = \pm \frac{k}{\sqrt{s_\xi}} \quad (206) \]
for $\xi = 1, \ldots, n$. Because there is no presymplectic structure in this block, these linear dispersions can be regarded as type-A NGBs. Our ground state is stable only when all of $s_\xi > 0$. Note that the mixing between upper and lower blocks induces only negligible corrections of $O(\omega^3)$.

We have shown here that generically type-A NGBs have a linear dispersion and type-B NGBs have a quadratic dispersion. Therefore, the equality version of the Nielsen-Chadha theorem is now proven. When $O(\nabla^2)$ term of the effective Lagrangian is somehow absent, type-A NGBs may have a quadratic dispersion and type-B NGBs may have a quartic dispersion. As explained above, that never happens for internal symmetries, but there are examples of NGBs originated from spacetime symmetries that lack the $O(\nabla^2)$ term. See Sec. VI A for more details.

\section*{B. With long-range interaction}

So far we assumed the locality of the effective Lagrangian. However, there are examples in condensed matter physics that inherit long-range interactions. In such cases, NGBs may be gapped or may have a gapless dispersion with a fractional power.

As a concrete example, let us take a superfluid of bosons with a long-range density interaction. We take a model Lagrangian,
\[ L = \int d^d x \left[ \hat{\theta} - \frac{1}{2m} (\nabla \hat{\theta})^2 \right] n(\vec{x}, t) - \frac{1}{2} \int d^d x d^d y \delta n(\vec{x}, t)V(\vec{x} - \vec{y})\delta n(\vec{y}, t) \quad (207) \]
where $\theta$ is the superfluid phase, $n$ is the $U(1)$ density, and $\delta n \equiv n - n_0$ is the fluctuation of the density. After integrating out $n$, we find the effective Lagrangian in terms NG field $\theta$
\[ L_{\text{eff}} = \int \frac{d^d k}{(2\pi)^d} \left[ \frac{1}{2} \hat{\theta}(\vec{k}, t)V(\vec{k})^{-1}\hat{\theta}(\vec{k}, t) - \frac{n_0 k^2}{2m} \theta(\vec{k}, t)\theta(\vec{k}, t) \right] (208) \]
to the leading order in $k$, where $V(\vec{k})$ is the Fourier transformation of $V(\vec{x})$. As long as we trust this Lagrangian seriously, it describes a superfluid phonon with the dispersion $\omega = \sqrt{(n_0/m)k^2V(\vec{k})}$.

When the interaction is local $V(\vec{x}) = \delta^d(\vec{x})$, or more generally damped fast enough, e.g. $V(\vec{x}) \sim e^{-\lambda|\vec{x}|^{1}} \sim \lambda^0$, $V(k)$ is analytic around $k = 0$ and $V(k) = 0$ exists. Therefore, the phonon dispersion is linear in the long-wavelength limit. However, for a Coulomb interaction $V(\vec{x}) \propto |\vec{x}|^{-1}$, $V(k) \propto k^{1-d}$ for $d = 2, 3$. Therefore, the phonon is gapped in $d = 3$ regardless of the spontaneous symmetry breaking. In $d = 2$, it has a dispersion with a fractional power, $\omega \propto k^{1/2}$.

In this example, once Eq. (207) has the term $\hat{\theta}^2$ from the beginning, integrating out produces only sub-leading corrections and the phonon recover gapless linear dispersion in the long-wavelength limit. Also, since $\hat{\theta}^2$ is allowed by symmetry it may be added to Eq. (208) by renormalization process. Thus from the effective Lagrangian point of view, this is not really a good example.

The case of the longitudinal phonon in crystals of charged particles (Wigner crystal) is slightly different. It can be described by the Lagrangian,
\[ L_{\text{eff}} = \int d^d x \left[ \frac{mn_0}{2} \ddot{u}^2 - \frac{mn_0}{2} (\nabla \times \ddot{u})^2 \right] - \frac{n_0^2}{2} \int d^d x d^d y \ddot{u}(\vec{x}, t)V(\vec{x} - \vec{y}) \ddot{u}(\vec{y}, t) (209) \]
where $\ddot{u}$ is the displacement field and $-qn_0 \ddot{u}$ represents the charge excess. In the Fourier space, the Lagrangian for the longitudinal mode $u_L(\vec{k}) \equiv \ddot{u} / |\vec{k}|$ reads
\[ \frac{mn_0}{2} \dddot{u}_L^2 - \frac{n_0^2 k^2 V(k)}{2} u_L^2, \quad (210) \]
describing the same dispersion $\omega = \sqrt{(n_0/m)k^2V(k)}$ as above. In this case, there is no renormalization that can recover the linear gapless dispersion unlike the above superfluid example.

Whether NGBs remain to be gapless or not depend on the detailed form of the long-range interaction. As long as the Lagrangian is local in time direction, type-A,B NGBs are well-defined and the counting rule works, but some of them might be gapped or may have a weird dispersion.

\section*{VI. STABILITY OF THE SYMMETRY BREAKING GROUND STATE}

In identifying the degrees of freedom and read off their dispersion relations, we used the perturbation theory and...
studied the quadratic part of the effective Lagrangian. One may be concerned that the interactions may upset the conclusion. Namely the question is whether the cubic and higher terms can modify the dynamics at long distances, which is equivalent to the question about the stability of a long-range order.

A. Scaling of interactions among NGBs

Here we examine the scaling law of the most relevant interactions among NGBs to see the stability of the symmetry-breaking ground state.

We start with the situation when there are only type-A NGBs. In order to keep the free action

$$\int d^4x dt \left( \frac{\bar{g}_{ab}(0)}{2} \bar{\pi}^a \pi^b - \frac{g_{ab}(0)}{2} \nabla \bar{\pi}^a \cdot \nabla \pi^b \right)$$

invariant, NG fields $\pi^a$ should transform as $\pi'^a(\alpha \vec{x}, \alpha t) = \alpha^{1-d} \pi^a(\vec{x}, t)$. In 1 + 1 dimensions we should include $\bar{g}_{ab}(0) \bar{\pi}^a \nabla \pi^b$ in the free action but it does not change the scaling law. Note again that $b(\pi)$ and $b(\pi)$ terms do not have the free part, and $\bar{c}_a(\pi)$ term causes an instability to a translational symmetry broken phase as discussed before and hence we do not consider them here. The most relevant interactions, $d^4x d^2\nabla^2 \pi^3$ and $d^4x d^2 \nabla^2 \pi^3$, then scale with $\alpha^{1-d/2}$. Therefore, if the spatial dimension $d$ is greater than one, all interactions are irrelevant and the system flows into the free fixed point. In this case, the symmetry-breaking ground state is stable and one can understand the property of the system via the standard perturbation theory. On the other hand, when $d = 1$, the interaction is marginal so that broken symmetries are restored and the low-energy spectrum may get gapped.

This result is consistent with the Coleman theorem that guarantees the absence of continuous symmetry breaking in 1 + 1 dimensions for Lorentz invariant case $g_{ab} = g_{ab}$ [27]. Superfluids in 1 + 1 dimensions are in the Kosterlitz-Thouless phase, which possesses only a quasi-long range order (power-law decay) and has a gapless density wave. The $S = 1/2$ antiferromagnetic chain also shows a quasi-long range order and supports gapless excitations called the Cloizeaux-Pearson modes [39]. In contrast, the $S = 1$ antiferromagnetic chain is believed to be in the Haldane phase and be gapped.

We can easily extend our analysis for other types of dispersion. Although spacetime symmetries are not the main focus of the current paper, type-A NGBs originated from spontaneously-broken spacetime symmetries sometimes have weird dispersions. In such a case, the criteria we derived for internal symmetries may be violated. For example, in a rotating superfluid in 2 + 1 dimensions, a vortex lattice breaks the magnetic translation. The NG bosons, so-called Tkachenko mode, is described by the effective Lagrangian,

$$\int d^2x dt \left[ \frac{A}{2} \dot{\varphi}^2 - B \left( \nabla \varphi \right)^2 \right]. \tag{212}$$

Note that the term $(\nabla \varphi)^2$ is prohibited by symmetry transformation $\delta \varphi \propto \vec{x} \varphi [29]$. In this case, it is easy to see that the dominant interaction is marginal, which destroys the long-range phase correlation even at $T = 0$ [40]. This makes contrast with usual superfluids or crystals in 2 + 1 dimensions, which are stable at $T = 0$. Another example is a helical magnet. Due to the spin-orbit coupling, the spin rotation must be accompanied by the spatial one. The helical (spiral) order breaks some combination of the rotation and translation. It turns out that there is only one gapless mode [41], which is described by

$$\int d^3x dt \left[ \frac{A}{2} \dot{\varphi}^2 - \frac{B}{2} (\nabla \varphi)^2 - \frac{C}{2} (B_2 + B_4 \varphi) \right]. \tag{213}$$

Again, terms $(\nabla \varphi)^2$ and $(\nabla \varphi)^2$ are prohibited by symmetry. As a result, the dispersion of the NGB is anisotropic, $\omega = \sqrt{(B/A)k_x^2 + (C/A)(k_y^2 + k_z^2)}$, which is an example of NGBs that cannot classified as type-I nor type-II, although it can be classified as type-A unambiguously. All interactions are irrelevant at $T = 0$, but there are marginal interactions at a finite temperature, despite the fact that usually broken symmetries are stable at a finite temperature in three dimensions.

Let us go back to the usual case $z = 1$ and instead consider a finite temperature. When $T > 0$, all imaginary-time dependences drop out at a sufficiently long-distance and low-energy scale, leaving only $n = 0$ component of the Matsubara frequency. Then the free part of the action is just $-T \int d^4x [g_{ab}(0)/2] \nabla \pi^a \cdot \nabla \pi^b$ and fields transform as $\pi'^a(\alpha \vec{x}, \alpha t) = \alpha^{2-d/2} \pi^a(\vec{x}, t)$. The most relevant interaction $d^4x \nabla^2 \pi^3$ scale as $\alpha^{2-d/2}$, so that the stability condition is given by $d > 2$. This is nothing but the Mermin-Wagner theorem.

Next, we discuss the case where only type-B NGBs are present. To keep the free action

$$\int d^4x dt \left[ \frac{\sigma_{ab}}{2} \dot{\bar{\pi}}^a \pi^b + \frac{g_{ab}(0)}{2} \nabla \bar{\pi}^a \cdot \nabla \pi^b \right] \tag{214}$$

invariant, NG fields should obey the scaling law $\varphi'^a(\alpha \vec{x}, \alpha^2 t) = \alpha^{-d/2} \varphi(\vec{x}, t)$. We could add $g_{ab}(0) \pi^a \nabla \pi^b$ in 1 + 1 dimensions but it is clearly higher-order in derivatives. In this case, the most relevant interactions, $d^4x d^2 \nabla \pi^3$ and $d^4x d^2 \nabla \pi^3$, scale as $\alpha^{-d/2}$. Therefore, the theory is essentially free in all dimensions and hence broken symmetries can never be restored. This conclusion might sound surprising for high-energy theorists, but actually it is a well-known fact in condensed matter physics [42]. We will come back to this point in Sec. V. C.

Type-A NGBs with a quadratic dispersion $\omega \propto k^2$ ($z = 2$) and type-B NGBs with the same dispersion have a
TABLE I. The stability condition for the symmetry breaking ground state in d spatial dimensions, obtained by evaluating the scaling law of interactions and the infrared divergence for NGBs.

| d | T = 0 | T > 0 |
|---|---|---|
| only type-A NGBs | d > 1 | d > 2 |
| only type-B NGBs | d > 0 | d > 2 |

completely different effect on broken symmetries. The former destroys the order parameter if d \( \leq 2 \), while the latter does not do anything if d > 0.

The discussion for a finite temperature for type-B NGBs is identical to the type-A case, since all imaginary-time dependences drop out. We summarize our result in Table I.

### B. Fluctuation of order parameters

The stability of the symmetry-breaking ground state can also be discussed by evaluating the quantum correction to the expectation value of order parameters. The infrared divergence originated from gapless NGBs tends to destroy the symmetry-breaking order parameters in lower dimensions.

Again assuming that the free theory is a good starting point, we express the expectation value of order parameters in terms of the free Green functions \( G^{ab}(x - y) = \langle T \pi^a(x) \pi^b(y) \rangle \). For example, the Noether charge density \( j_i^0(\vec{x}, t) \) plays the role of the order parameter for charges \( Q_a \) for which \( C_{ab}^0(x) = 0 \) for some b. The current density \( j_i^0(\vec{x}, t) \) in Eq. (188) can be expanded in terms of NG fields as

\[
j_i^0 = e_k(0) \left[ \delta_i^k + \pi^b f_{bi}^k + \frac{1}{2} f_{ai}^j f_{jb}^k \pi^a \pi^b + O(\pi^3) \right]
- g_{ab}(0) \left[ \delta_i^a \delta^b + C_{abcd}^a \delta^c \delta^d + O(\nabla_i \pi^3) \right],
\]

where \( C_{abcd}^a = \frac{1}{2} C_{cd} \frac{1}{2} \delta_i^a \) for unbroken currents (i = \( \rho \)) and \( C_{cd} = \frac{1}{2} C \frac{1}{2} \delta_i^a \) for broken currents (i = \( \epsilon \)). Therefore, the dominant contribution to the expectation value is given by

\[
\langle j_i^0 \rangle \simeq e_k(0) \left[ \delta_i^k + \frac{1}{2} f_{ai}^j f_{jb}^k G^{ab}(0) + \cdots \right].
\]

For superfluids, \( \psi(\vec{x}, t) \) \( \simeq \sqrt{n_0} e^{i\theta(\vec{x}, t)} \) is the order parameter and its expectation value with quantum fluctuation is

\[
\langle e^{i\theta(\vec{x}, t)} \rangle = e^{-\frac{1}{2} \langle (\theta(\vec{x}, t))^2 \rangle} = e^{-\frac{1}{2} \theta^2(0)}.
\]

(Note that \( \theta \) itself is not a good quantity to look at since it does not have the assumed periodicity of 2\pi.) As one can see, we need \( |G^{ab}_0(\vec{x} = 0, t = 0)| \ll 1 \) in order for the quantum correction to be small compared to the classical value.

We can easily evaluate \( G^{ab}(0) \) by scaling. When only type-A NGBs appear, \( i(G_0^{-1})_{ab}(\vec{k}, \omega) = \bar{g}_{ab} \omega^2 - g_{ab} k^2 \) and

\[
\int d^d k \omega G^{ab}(\vec{k}, \omega) \propto \frac{k^d}{\Lambda} dk^{d-2},
\]

\[
\sum_n \int d^d k \omega G^{ab}(\vec{k}, i\omega_n) \propto T \int_0^\Lambda \frac{dk}{k^{d-3}},
\]

for \( T = 0 \) and \( T > 0 \), respectively. We introduced the ultraviolet cutoff \( \Lambda \). Therefore, for the convergence of the infrared contribution, we need \( d > 1 \) at the zero temperature and \( d > 2 \) at a finite temperature. Similarly, when only type-B NGBs appear, \( i(G_0^{-1})_{ab}(\vec{k}, \omega) = -i\sigma_{ab} \omega - g_{ab} k^2 \) and

\[
\int d^d k \omega G^{ab}(\vec{k}, \omega) \propto \frac{k^d}{\Lambda} dk^{d-1},
\]

\[
\sum_n \int d^d k \omega G^{ab}(\vec{k}, i\omega_n) \propto T \int_0^\Lambda \frac{dk}{k^{d-3}}.
\]

Therefore, there is no infrared divergence even at 1 + 1 dimensions at the zero temperature. These results are consistent with those summarized in Table I.

In Sec. IV A, we discussed the non-renormalization theorem of \( \langle j_i^0(0) \rangle \). However, Eqs. (216) and (220) may appear to indicate that \( \langle j_i^0(0) \rangle \) receives a finite correction due to quantum fluctuations. Now we show it is not the case by explicitly evaluating the magnetization of ferromagnets at the 1-loop level. The effective Lagrangian (187) for the coset \( G/H = SO(3)/SO(2) \) reads

\[
\mathcal{L} = \frac{i}{2} e_0 (\bar{z} \gamma - z \gamma) + \bar{g}_0 \bar{z} \gamma - g_0 \bar{\gamma} \bar{z} \gamma - \gamma \bar{z} \gamma
\]

to the quadratic order in \( z = (\pi^1 + i\pi^2)/\sqrt{2} \). According to Eq. (215), the magnetization including the fluctuation is \( j_z^0 = e_0 - e_0 \bar{z} \gamma - i\bar{g}_0 (\bar{\gamma} \bar{z} - \bar{z} \gamma) \). Therefore,

\[
\langle j_z^0 \rangle = e_0 - \sum_n \int \frac{d^d k}{(2\pi)^d} e_0 + 2\bar{g}_0 \omega_n + \bar{g}_0 \bar{\omega}_n + g_0 k^2
\]

We can perform the Matsubara summation using the standard trick and find

\[
\langle j_z^0 \rangle = e_0 - n(\omega) + n(\omega'),
\]

where \( n(\epsilon) = (e^{\beta \epsilon} - 1)^{-1} \) is the Bose distribution function,

\[
\omega = \frac{\sqrt{\bar{e}_0^2 + 4g_0\bar{g}_0 k^2} - e_0}{2g_0} = \frac{\bar{g}_0 k^2 + O(k^4)}{e_0}
\]

is the dispersion of the gapless Goldstone mode (magnon) and

\[
\omega' = \frac{\sqrt{\bar{e}_0^2 + 4g_0\bar{g}_0 k^2} + e_0}{2g_0} = \frac{e_0}{g_0} + O(k^2).
\]

is the dispersion of the gapped mode. (The existence of the gapped mode is questionable since this solution balances the \( (\nabla_i \pi^3) \) term and the \( (\nabla_i \pi^3) \) term of the effective Lagrangian. It is easily eliminated from calculation by taking the limit \( \bar{g} \to 0 \).) Since \( n(\omega) = n(\omega') = 0 \) at \( T = 0 \), the 1-loop correction to the expectation
value of the magnetization vanishes in the ground state. Clearly the finite temperature correction is dominated by magnons and is proportional to $T^{d/2}$ at low-temperature, which is known as Bloch’s law [43].

So far we have only considered the case where only one type of NGBs appear, since both of our above arguments are essentially based on scaling. However, in general, type-A and type-B NGBs can coexist. In such a case, there is no field transformation that keeps all of the free part invariant unless type-A, B NGBs are somehow completely decoupled. In the next section we present some arguments that can be used in type-A, B coexisting cases.

C. SSB in 1+1 dimensions

The usual argument for ferromagnets in 1 + 1 dimensions is as follows [42]. As the ferromagnetic order parameter $S_z$ commutes with the Hamiltonian $H$, one can simultaneously diagonalize $H$ and $S_z$ and obtain quantum many-body eigenstates $|\Psi_{E,M}\rangle$ labeled by the eigenvalue of $H$ and $S_z$. Since $|\Psi_{E,M}\rangle$ is an eigenstate, there is no quantum fluctuation of order parameter, $\langle \Psi_{E,M}|S_z|\Psi_{E,M}\rangle = \langle E,M|S_z|E,M\rangle$. From the translational invariance of the ground state, it follows that $\langle S_{x},j_{\alpha}^{0}(x,t)|\Psi_{E,M}\rangle = i\mathcal{M}/\Omega$ where $\Omega$ is the volume of the system. As usual, applying the magnetic field $-B_z S_z$ to pick up a particular state, taking the large volume limit first and then switching off the field, one finds the definition of symmetry-breaking of $S_x$, $\langle S_x,j_{\alpha}^{0}(x,t)\rangle = im \neq 0$ with $m$ the magnetization density.

This argument can be easily generalized to an arbitrary case. As discussed above, only Cartan generators, which commute with each other by definition, can have non-zero expectation values. We can thus simultaneously diagonalize all of them (except for Abelian invariant algebra of $G$ that never plays the role of an order parameter) and the Hamiltonian. This is an alternative proof of the non-renormalization theorem of the expectation value of the current operator at $T = 0$, discussed in Sec. IV.A. (At a finite temperature, we no longer use a pure quantum eigenstate but take an ensemble over all states and the expectation value gets a finite temperature correction.)

In Ref. [44], it has been proved that continuous symmetry breaking in 1 + 1 dimensions is possible only when uniform susceptibilities of broken charges diverge. Indeed, we can show the divergence of uniform susceptibility whenever type-B NGBs appear. Equation (215) tells us that the current-current correlation function of charges associated to type-B NGBs is dominated by

$$\langle \delta j_{\alpha}^{0}(x,t)\delta j_{\beta}^{0}(0)\rangle = c_{\epsilon}(0)c_{\epsilon}(0)f_{\alpha}^{\epsilon}f_{\beta}^{\epsilon}\langle \pi^{a}(x,t)\pi^{a}(0)\rangle.$$  

Therefore, the uniform susceptibility

$$\chi_{ab} = \lim_{|k| \rightarrow 0} \left[\delta j_{\alpha}^{0}(k, i\omega_n)\delta j_{\beta}^{0}(-k, -i\omega_n)\right]_{\omega_n = 0}$$  

diverges due to poles of Green’s functions corresponding to type-B NGBs.

In contrast, when type-B NGBs do not exist, all $c_{\epsilon}(0)$s in Eq. (215) vanish and the correlation function is dominated by

$$\langle \delta j_{\alpha}^{0}(x,t)\delta j_{\beta}^{0}(0)\rangle = g_{ac}(0)g_{bd}(0)\langle \pi^{c}(x,t)\pi^{d}(0)\rangle.$$  

Additional time-derivatives cancel the divergence and the uniform susceptibility converges.

An example of continuous symmetry-breaking at 1 + 1 dimensions, which supports both a linear and a quadratic dispersion is given by spinor BECs [45–49]. The model is defined by

$$\mathcal{L} = \frac{i}{2}(\psi^{\dagger}\gamma^0\psi - c.c.) - \frac{\nabla\psi}{2m} - \frac{g}{2}(\psi^{\dagger}\psi - n_0)^2.$$  

Here $\psi = (\psi_1, \psi_2)^T$ is a two component complex scaler field and $n_0 = N/L$ ($N$ is the number of bosons and $L$ is the system size). The dimensionless coupling constant is given by $\gamma = mg/n_0$.

At the tree level (mean field approximation), the system exhibits a long-range order $\langle \psi \rangle = (0, 0)^T$ and $U(2)$ symmetry is spontaneously broken into $U(1)$. There are two NGBs, a type-A NGB (sound wave) with a linear dispersion $\omega_{ph}(k) = (n_0/m)^{1/2}$ and a type-B NGB (spin wave) with a quadratic dispersion $\omega_{sw}(k) = k^2/(2m)$ as $|k| \rightarrow 0$. However, the strong fluctuation caused by the linear dispersion invalidates this simple analysis.

Surprisingly, there exists an exact solution of this model based on Bethe-anzatz [46]. The solution exhibits the ferromagnetic long-range order, showing the spontaneous breaking of spin rotation. Correspondingly, there is a well-defined spinwave excitation with the dispersion $\omega_{sw}(k) = \left[1 - (2\sqrt{3}/3π) + \cdots\right] (k^2/2m)$ in the weak coupling limit $\gamma \ll 1$ and $\omega_{sw}(k) = \left[(2\pi^2/3Γ) + \cdots\right] (k^2/2m)$ in the strong coupling limit $\gamma \gg 1$ [46].

On the other hand, the phase-phase correlation is not truly long-ranged. As a result, the sound wave should be understood as Tomonaga-Luttinger liquid rather than a type-A NGB [47–49].

VII. TOPOLOGY

In this section, we discuss the geometry behind the type-B NGBs that do not appear in Lorentz-invariant theories. There is an underlying geometrical foundation called a presymplectic structure. Understanding geometry of NGBs turns out to be important for classifying possible division between type-A and type-B NGBs in the next section.

A. Presymplectic structure

We have seen that the one form $\epsilon = c_{a}d\pi^{a}$ on the cotangent space $T^{*}(G/H)$ is in general not invariant
under \( G \), while the two form \( \omega = dc \) is [see Eq. (40)]. Therefore, we should focus on \( \omega \), which is a closed and \( G \)-invariant two-form on \( G/H \). If the anti-symmetric matrix \( \omega = \omega_{ab} (\pi) \mathrm{d} \pi^a \wedge \mathrm{d} \pi^b \) has a non-zero determinant \( \det \omega_{ab}(\pi) \neq 0 \), it defines a symplectic structure on \( G/H \). The combination of a manifold and a non-degenerate closed two-form \((M, \omega)\) is called a symplectic manifold. In physics terminology, it is nothing but a phase space of a dynamical system with well-defined canonical commutation relations among its coordinates given by \([\pi^a, \pi^b] = i(\omega^{-1})^{ab}\). It is obvious that it requires \( G/H \) to be even-dimensional. If \( G/H \) is compact, its second cohomology \( H^2(G/H) \) must be non-trivial. Note that many coset spaces do not satisfy these requirements.

If \( \omega \) is degenerate, namely if \( \det \omega_{ab} = 0 \), it is called a presymplectic structure, or partially symplectic, because only a subset of the coordinates \( \pi^a \) participates in the matrix \( \omega_{ab} \). Recall that a symplectic structure on a manifold is what defines the canonical commutation relation on a phase space \([\pi^a, \pi^b] = i(\omega^{-1})^{ab}\). If it is only partially symplectic, \( \omega^{-1} \) is singular. Then the coset space \( G/H \) is partially phase space, partially coordinate space. Only a subset of the coordinates participates in the canonical conjugate pairs, while the remainder does not. The former correspond to type-B NGBs, while the latter to type-A.

### B. Compact Semi-Simple Case

It is important to ask the question what kind of coset spaces support a presymplectic structure. We have a definite answer to this question when \( G \) is compact semi-simple.

As we have seen, \( c(\pi) = e_i(0)\omega^i \) is completely specified in terms of constants \( e_i(0) \) where the generator \( T_i \) commutes with the entire \( H \) [see Eq. (92)]. Therefore, we can enlarge \( H \) to include all generators that commute with \( T_i \) to define the subgroup \( U \) such that \( U^i e_i(0) T_i U = e_i(0) T_i \) in \( G \). Mathematically, \( T_i \)'s generate an abelian group \( T \) which is called a torus. Then \( U \) is called a centralizer of the torus \( T \) in \( G \). Then the following theorem was proven by Borel [50]:

Let \( G \) be compact semisimple and \( U \) be the centralizer of a torus. Then \( G/H \) is homogeneous Kählerian and algebraic.

A torus \( T \) in this context means an Abelian subgroup of \( G \). For example, an \( SU(N) \) group is simple and has many possible Abelian subgroups, \( T = U(1), U(1)^2, \ldots, U(1)^{N-1} \). In general, a simple group admits a torus up to \( T_{\text{max}} = U(1)^r \), where \( r \) is the rank of its Lie algebra, called the maximal torus \( T_{\text{max}} \). An Abelian subgroup is called a torus because it is a manifold of coordinates with periodic boundary conditions for each, just like the surface of a doughnut (a two-torus). A centralizer \( U \) of a torus \( T \) is defined by the collection of elements in \( G \) that commute with every element of \( T \),

\[
\pi^a \omega = dc \quad G/H \xrightarrow{\pi} F = U/H
\]

**FIG. 1.** Fibration responsible for the presymplectic structure. \( U \subset G \) is the subgroup that commutes with all Cartan generators \( T_i \) with non-vanishing \( e_i(0) \). The base manifold \( B = G/U \) is symplectic, which describes the type-B NGBs, while the fiber \( F = U/H \) describes the type-A NGBs. The symplectic form \( \omega \) on \( B \) is pulled back to \( \pi^* \omega = dc \) on \( G/H \).

\[ i.e., \quad U = \{ \{ u \in G \} | u t u^{-1} = t, \forall t \in T \}. \]

For instance for

\[ T = \{ \{ \mathrm{e}^{i \mathrm{diag}(a_1, \ldots, a_1, a_2, \ldots, a_2, \ldots, a_k, \ldots, a_k) \} \}, \]

where \( \sum_{i=1}^k n_i = N \) and \( \sum_{i=1}^k n_i a_i = 0 \) (traceless), \( T = U(1)^{k-1} \subset SU(N) \), and its centralizer is \( U = U(1)^{k-1} \times \prod_{i=1}^k \mathbb{R} \). The Borel’s theorem states then \( G/U = SU(N)/(U(1)^{k-1} \times \prod_{i=1}^k \mathbb{R}) \), is Kähler. A Kähler manifold always allows for a symplectic structure.

Therefore, this kind of a partially symplectic structure is possible on the coset space by considering the following fiber bundle, \( F \rightarrow G/H \xrightarrow{\pi_2} B \), where the base space \( B = G/U \) is symplectic. (Note we used the bold face \( \pi \) here to avoid a possible confusion with the NG field \( \pi \)). The fiber is \( F = U/H \). The symplectic structure \( \omega \) on \( B \) is pulled back by the projection \( \pi \) as \( \pi^* \omega \) on the entire coset space \( G/H \). Since the closedness \( d\omega = 0 \) on \( B \) implies the closedness \( d(\pi^* \omega) = 0 \) on \( G/H \), we can always find a one-form \( c \) such that \( dc = \pi^* \omega \) locally on \( G/H \). Therefore, what we see in the Lagrangian at the first order in time derivative is this pullback \( \pi^* \omega \) (further pulled back to spacetime by \( \pi \)).

The projection on a symplectic manifold makes sense from physics point of view. In the long-distance limit, the modes with quadratic dispersion, typically type-B, have much lower energies than those with linear dispersion, typically type-A. Therefore, keeping only the type-B modes, namely those with canonically-conjugate pairs, would make sense in this limit. It corresponds to the projection on the symplectic base manifold that describes type-B NGBs while eliminating the fiber that describes type-A NGBs.

The symplectic structure on \( B = G/U \) is specified by parameters \( e_i(0) \). Going back to the example of \( G/U = SU(N)/(U(1)^{k-1} \times \prod_{i=1}^k \mathbb{R}) \), using the exact sequence of the homotopy groups, it is seen that

\[
\pi_2(G/U) = \{ \pi_{1}(U(1)^{k-1}) \} / \pi_1(G) = Z^{k-1},
\]

while Hurewicz theorem says \( H_2(G/U) = \pi_2(G/U) \) when \( \pi_1(G/U) = 0 \). In addition, because \( G/U \) is compact without a boundary, \( H^2_{dR}(G/U, \mathbb{R}) \) (de Rham theorem). Therefore, there are \( k-1 \) generators of \( H^2_{dR}(G/U) \), \( \omega_1, \omega_2, \ldots, \omega_{k-1} \), that can be used for the symplectic form \( \omega = \sum_{i=1}^{k-1} \omega_i \) on \( G/U \). These numbers \( a_i(i = 1, \ldots, k - 1) \) specify...
\( \omega = dc \), and hence \( c = c_ao^a \) in the Lagrangian. This is precisely the same number of parameters as \( e_i(0) \) for this coset space.

In general, \( \dim H^2_{\text{IR}}(G/U) \) is the same as the number of \( U(1) \) factors in \( U \) when \( G \) is semisimple (i.e., no \( U(1) \) factors in \( G \)). Pulled back to \( G/H \), the possibilities of presymplectic structure corresponds to the number \( N_C \) of Cartan generators in \( G \) that commute with \( H \). We will use this fact extensively when we present the classification of possible presymplectic structures in the next section.

Note, however, that the linear combination \( \omega = \sum_{i=1}^{k} a_i \omega_i \) may be degenerate for a certain choice of the parameters \( a_i \). For instance, \( G/H = SU(3)/U(1) \times U(1) \) is Kähler, has \( H^2(G/H) = \mathbb{Z}^2 \), and it supports a symplectic structure. There are two linearly independent closed invariant two-forms in Eq. (108), \( \omega^3 \) (109) and \( \omega^8 \) (110). Note that \( \omega^3 \) and \( \omega^8 \) are not globally defined as they transform inhomogeneously under the group transformations [see Eq. (76)]. Therefore these two two-forms are closed but not exact, generate \( H^2_{\text{IR}}(SU(3)/U(1) \times U(1)) \), and are candidates for the symplectic structure. Indeed, \( \omega^3 = d\pi^1 \wedge d\pi^2 + (1/2)(d\pi^3 \wedge d\pi^5 + d\pi^6 \wedge d\pi^7) + O(\pi)^3 \) and hence is non-degenerate. On the other hand, if we pick \( \omega^8 = (\sqrt{3}/2)(d\pi^3 \wedge d\pi^5 + d\pi^6 \wedge d\pi^7) + O(\pi)^3 \), it does not provide canonical structure between \( \pi^1 \) and \( \pi^2 \), and hence it is degenerate. There is actually a larger symmetry that preserves this choice, because the torus is \( U(1) \) generated by \( T_s \), and its centralizer is \( U(2) \). Then it can be projected down to \( SU(3)/U(2) = \mathbb{CP}^2 \), where the fiber is \( U(2)/U(1) \times U(1) = S^2 \). This is an example where the fiber is not a group [51].

C. Case with Central Extensions

So far, we assumed \( G \) is compact semi-simple. If \( G \) is not semi-simple, especially if it has more than one \( U(1) \) factors, its second cohomology \( H^2(g) \) is non-trivial and it allows for a central extension. See Sec. A for more discussions on the central extension.

In this case, \( G/H \) may not necessarily be projected down to a symplectic manifold. Consider \( G = U(1)^3 \), \( H = \{ e \} \) for example, parameterized by three angles \( T^3 = G/H = \{ \vec{\varphi} \in [0, 2\pi] | a = 1, 2, 3 \} \) is a three-torus. We can introduce a presymplectic structure [52]

\[
\omega = d\varphi^1 \wedge (d\varphi^2 + rd\varphi^3).
\]

If \( r \) is a rational number \( r = p/q \) for \( p \) and \( q \) relatively prime, the orbit winds around \( T^3 \) \( q \)-times and closes on itself. Then there is a well-defined projection down to \( T^2 \). On the other hand if \( r \) is an irrational number, there is no well-defined projection because the orbit winds around \( T^3 \) infinite times without closing on itself.

We suspect such a pathological case would not arise in physical systems. Yet we do not have a concrete proof what goes wrong in such a case.

D. Quantization Condition

The normalization of the presymplectic structure may be quantized. All discussions above so far concerned with the invariance of the action up to a surface term. In classical physics, the action itself does not have a physical meaning while its variation leads to the equations of motion. In quantum physics, however, the action itself goes into the path integrals as \( e^{iS/\hbar} \) and hence its value matters. Yet, change in the action by an integer multiples of \( 2\pi \hbar \) does not change the path integral. Recall that we use the unit \( \hbar = 1 \) in this paper and henceforth we drop \( \hbar \) in expressions.

When \( \omega = dc \) is closed but not exact, namely an element of \( H^2_{\text{IR}}(G/H) \), its coefficient is quantized. Considering time integral to be a periodic loop \( L^1 \) on \( G/H \), the loop can be viewed as a boundary of a two-disk. (Here we assumed \( \pi_1(G/H) = 0 \) so that every loop on \( G/H \) is contractible to a point.) However non-trivial \( H^2_{\text{IR}} \) implies non-trivial \( H_2 \), and hence there are non-contractible two-cycles on \( G/H \). Namely there are non-trivial closed two-dimensional surfaces \( C_2 \) in \( G/H \). Then \( C_2 = C_2^+ \cup C_2^- \) is a union of two surfaces that share the same boundary \( L^1 = \partial C_2^- = -\partial C_2^+ \). The simplest example is \( C_2 \simeq S^2 \), where \( L^1 \) is the equator, \( C_2^+ \) the northern hemisphere, and \( C_2^- \) the southern hemisphere. For the action

\[
S \equiv \int d^4x \int_{L^1} c \tag{232}
\]

to give a single-valued \( e^{iS} \), its ambiguity

\[
\Delta S = \int d^4x \int_{C_2^+} dc - \int d^4x \int_{C_2^-} dc = \int d^4x \int_{C_2} dc \tag{233}
\]

must be quantized in units of \( 2\pi \). This is the same as the discussion on Wess–Zumino–Witten terms in Sec. III E 3.

When the system is finite \( \Omega = \int d^4x < \infty \), the quantization condition restricts the normalization of \( c \). In other words, \( \Omega dc \) is an element of \( H^2(G/H, \mathbb{Z}) \), rather than \( H^2_{\text{IR}}(G/H) = H^2(G/H, \mathbb{R}) \).

The same consideration applies to central extensions. When the target space is compact, the (pre)symplectic form is quantized. For example for \( U(1)^2 = \{ (\varphi^1, \varphi^2) | \varphi^i \in [0, 2\pi] \} \), \( \Omega dc = k(2\pi)^{-1}d\varphi^1 \wedge d\varphi^2 \) with \( k \in \mathbb{Z} \). On the other hand, when the target space is non-compact, such as \( \mathbb{R}^2 = \mathbb{C} \) in the case of the free Schrödinger field mentioned in Sec. III E and Appendix C, the coefficient is not quantized.

VIII. CLASSIFICATION OF POSSIBLE PRESYMPLECTIC STRUCTURES

As we have seen in Sec. VII, a presymplectic structure on a coset space \( G/H \) is characterized by its fibration on a symplectic base space \( B = G/U \) with the fiber \( F = U/H \), when \( G \) and \( H \) are compact semi-simple. \( U \subset G \) is
the subgroup that commutes with generators with non-zero $e_i(0)$. Since $e_i(0)$'s need to be invariant under $H$ [Eq. (92)], $H \subset U$. The base space describes type-B NGBs while the fiber type-A NGBs. In this section, we show how such structures can be completely classified.

A. Preliminary discussions

The number of type-A, B NGBs is given by the counting rule in Eqs. (1) and (2). If the rank of $\rho$ explores all the possible integral values in the range

$$0 \leq \text{rank } \rho \leq \dim G/H,$$

(234)

the number of type-A,B NGBs can be any combinations between $(n_A, n_B) = (\dim G/H, 0)$ and $(0, \frac{1}{2}\dim G/H)$. Indeed, in the case of Heisenberg magnets $G/H = SO(3)/SO(2)$ ($\dim G/H = 2$), antiferromagnets and ferromagnets respectively realize the case rank$\rho = 0, 1$. However, in this section we discuss that in general allowed values of rank$\rho$ are strongly constrained.

In general, we can always choose the basis of generators in such a way that only Cartan generators [53] of $G$ that commute with all generators of $H$ may have a non-zero expectation value $\langle \xi^0(x, t) \rangle \neq 0$ [23] as we have discussed in previous sections. Their expectation values specify $e_i(0)$, and the corresponding generators generate the torus $T$. Each nonzero expectation value of conserved charge densities defines a presymplectic structure on $G/H$ by $e = -e_i(0)\omega^i$ [Eq. (94)]; namely, it makes NG fields associated to broken generators $Q_{\alpha}$ and $Q_{\beta}$ canonically conjugate to each other as discussed in Sec. VII.

For a given $G$ and $H$, let $N_C$ be the number of Cartan generators of $\mathfrak{g}$ that commute with $\mathfrak{h}$. Based on the above considerations, we know that these generators are the only ones that are allowed to have non-vanishing $e_i(0)$. Therefore, there are $N_C$ parameters to specify the possible presymplectic structure on $G/H$. This counting takes into account only the connected component $G_0$ of the identity, and the discrete subgroup $G/G_0$ might further restricts allowed presymplectic structures.

Therefore, we first consider the case when $H$ is generated by Cartan generators alone, so that all Cartan generators commute with $\mathfrak{h}$ to maximize $N_C$.

B. Flag Manifolds

To study the case of maximum $N_C$ for a given $G$, let us consider the flag manifolds $G/U(1)^r$, where $N_C = r \geq 1$ is the rank of the simple group $G$. We can enumerate all possibilities of presymplectic structures systematically for them. It turns out that this list allows us to also classify possibilities for other $G/H$ as well. In this sense, the discussion here is the basis of all other cases. For concreteness we discuss SU($n + 1$)/U(1)$^n$ first.

A flag manifold is Kähler thanks to the Borel theorem [50] and hence symplectic. Indeed for SU($n + 1$)/U(1)$^n$, $\dim G/H = n(n + 1)$ is always even. Since all Cartan generators of $G$ remain unbroken, $N_C = n$ and there are many presymplectic structures that can control the number of type-A, B NGBs. The simplest case of SU(3)/U(1)$^2$ with $N_C = 2$ is shown in Table II.

The two limiting cases can be easily understood. Any symplectic manifold is endowed with an associated symplectic two-form, which always realizes the case rank$\rho = \dim G/H$ (unless discrete the subgroup puts an obstacle). Thus we know that $(n_A, n_B) = (0, n(n + 1)/2)$ is possible. Also, by setting all expectation values of charge densities to be zero, one can realize the case where rank$\rho = 0$ and hence $(n_A, n_B) = (n(n + 1), 0)$.

The question is whether it is possible to realize combinations of $(n_A, n_B)$ between these two limiting cases. Although there are $N_C = n$ parameters to control, the number of integers in the range Eq. (234) grows as $n^2$, so obviously it is not possible to realize all of these values for a large $n$. For example, there is a minimum value of rank$\rho$ (except for 0), which is achieved by the presymplectic structure that appeared in the above discussion of SU($n + 1$)/U($n$) = CP$^n$ model. This gives rank$\rho = n$ and $0 < \text{rank } \rho < n$ is prohibited.

The case for simple classical groups is straightforward to work out. The smallest possible $H$ that makes $G/H$ symplectic is the flag manifold $H = U(1)^r$, where $r$ is the rank of $G$. All Cartan generators commute with U(1)$^r$, and hence $N_C = r$. Therefore, this case allows for the most number of possible choices for $U$.

Because $e_i(0)$ belong to the adjoint representation, the corresponding generators $T_i$ generate a torus $T$, and its centralizer $U$ is generated by all generators of $\mathfrak{g}$ that leave $e_i(0)$ invariant. Such symmetry breaking patterns have been studied extensively in the literature (see, e.g., Ref. [54]).

For SU($n$) groups, the possible form of $e_i(0)T_i$ is

$$e_i(0)T_i = \text{diag}(\alpha_1, \cdots, \alpha_1, \alpha_2, \cdots, \alpha_2, \cdots, \alpha_k, \cdots, \alpha_k),$$

(235)

and the corresponding centralizer is

$$U = U(1)^{k-1} \times \prod_k \text{SU}(n_k), \quad n = \sum_k n_k, \quad \sum_k n_k \alpha_k = 0.$$  

(236)

In this expression, SU(1) counts as a trivial group.

For SO($n$) groups, any element of the adjoint representation is an anti-symmetric matrix that can be skew-

| $n_A$ | $n_B$ | $F = U/H$ | $B = G/U$ |
|-------|-------|----------|----------|
| 6     | 0     | SU(3)/U(1) × U(1) |         |
| 2     | 2     | SU(2)/U(1)         | SU(3)/SU(2) × U(1) |
| 0     | 3     | SU(3)/U(1) × U(1)  |         |

TABLE II. Possible of number of type-A and type-B NGBs for SU(3)/U(1) × U(1).
diagonalized. Therefore, the possible form of $e_i(0)T_i$ is
\[ e_i(0)T_i = \text{diag}(0, \cdots, 0, \alpha_1, \cdots, \alpha_{n_k}) \otimes i\sigma_3, \]
and we find the centralizer
\[ U = \text{SO}(m) \times \prod_k U(n_k), \quad n = m + 2 \sum n_k. \tag{237} \]

Finally for $\text{Sp}(n)$ groups (we use the notation that the rank is $n$ for $\text{Sp}(n)$), every element $g \in \text{Sp}(n)$ preserves
\[ J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}, \quad gJg^T = J. \tag{239} \]

Therefore, the adjoint representation is $2n \times 2n$ matrix of the form
\[ S = \begin{pmatrix} A & B \\ C & -A^T \end{pmatrix}, \quad SJ + JS^T = 0. \tag{240} \]

Here, $B^T = B$ and $C^T = C$ are symmetric matrices. The Cartan generators are given by the diagonal matrices in $A$ with $B = C = 0$, and therefore has the form $S = A_{\text{diag}} \otimes \sigma_3$. In general,
\[ e_i(0)T_i = \text{diag}(0, \cdots, 0, \alpha_1, \cdots, \alpha_{n_k}) \otimes i\sigma_3, \tag{241} \]
and we find
\[ U = \text{Sp}(m) \times \prod_k U(n_k), \quad n = m + \sum n_k. \tag{242} \]

The problem is basically listing up partition of integers. Once all possibilities $U$ are listed, it is easy to count $n_A = \dim U/H$ and $n_B = \dim G/U$. We present all possible cases for rank 5 groups in tables: $\text{SU}(6)$ (Table III), $\text{SO}(10)$ (Table IV), $\text{SO}(11)$ and $\text{Sp}(5)$ (Table V).

Looking at Table IV, one might think that $U = \text{SO(6)} \times \text{U(1)}^2$ and $U = \text{U(4)} \times \text{U(1)}$ are the same because $\text{so(6)}$ and $\text{su(4)}$ are identical Lie algebras. They are not. The spectrum of the type-B NGBs on $\text{SO}(10)/(\text{SO(6)} \times \text{U(1)}^2) = \text{SO}(10)/(\text{SU(4)}/\mathbb{Z}_2 \times \text{U(1)}^2)$ consists of $14 = 6 + 4 + 6 + 1 + 1$ under $\text{SO(6)}$, while those on $\text{SO}(10)/(\text{U(4)} \times \text{U(1)}) = \text{SO}(10)/(\text{SU(4)} \times \text{U(1)})/\mathbb{Z}_4 \times \text{U(1)}$ of $14 = 4 + 4 + 6$ under $\text{SO(4)}$. The same comment applies to $\text{SO(4)} \times \text{U(1)}^3$ vs $\text{U(2)}^2 \times \text{U(1)}$ as $\text{so(4)} = \text{su(2)} + \text{su(2)}$. On $\text{SO}(10)/(\text{SU(4)} \times \text{U(1)}^3)$, type-B spectrum is $18 = 4 \times 3 + 1 \times 6$ under $\text{SO(4)}$, while for $\text{SO}(10)/(\text{U(2)}^2 \times \text{U(1)})$, it is $18 = (2, 2) \times 2 + (2, 1) \times 2 + (1, 2) \times 2 + (1, 1) \times 2$ under $\text{U(2)} \times \text{U(2)}$. Therefore, one has to be careful about not identifying local isomorphisms among groups.

On the other hand, in the case of $\text{SO}(n)$ with $n$ even, it can break to $U = \text{SO(2)} \times \prod_k U(n_k)$. Turning $e_i(0)$ for the $\text{SO(2)}$ generator would “break” it further to $\text{U(1)}$ with no difference in the group structure or representations of NGBs. Namely two cases are continuously connected without an order parameter that distinguishes them. Therefore, we can identify $\text{SO(2)}$ and $\text{U(1)}$ and we have eliminated duplicates from Table IV.

Note that there is a duality between $\text{Sp}(n)$ and $\text{SO}(2n+1)$ groups in each symmetry breaking pattern because the dimensions of the group match: $(1/2)(2n+1)2n = n(2n+1)$ for $\text{SO}(2n+1)$, and $(1/2)2n(2n+1) = n(2n+1)$ for $\text{Sp}(n)$.

It should be possible to enumerate possibilities for exceptional groups $G_2$, $F_4$, and $E_{6,7,8}$ as well, but we do not attempt it here.

C. General $H$

For more general $G/H$, we start with the list of possible $U$ for $G/U(1)^r$, and remove those that do not commute with $H$. It gives all possible presymplectic structures. The number of type-B NGBs is given by

\begin{table}[h]
\begin{tabular}{|c|c|c|}
\hline
$n_A$ & $n_B$ & $U$ \\
\hline
30 & 0 & . \\
20 & 5 & $\text{SU}(5) \times \text{U(1)}$ \\
14 & 8 & $\text{SU(4)} \times \text{SU(2)} \times \text{U(1)}$ \\
12 & 9 & $\text{SU(4)} \times \text{U(1)}^2$ \\
12 & 9 & $\text{SU(3)}^2 \times \text{U(1)}$ \\
8 & 11 & $\text{SU(3)} \times \text{SU(2)} \times \text{U(1)}^2$ \\
6 & 12 & $\text{SU(3)} \times \text{U(1)}^4$ \\
6 & 12 & $\text{SU(2)}^2 \times \text{U(1)}^2$ \\
4 & 13 & $\text{SU(2)}^2 \times \text{U(1)}^3$ \\
2 & 14 & $\text{SU(2)} \times \text{U(1)}^4$ \\
0 & 15 & $\text{U(1)}^5$ \\
\hline
\end{tabular}
\caption{Possible number of type-A and type-B NGBs for $\text{SU(6)}/\text{U(1)}^5$.}
\end{table}
| $n_A$ | $n_B$ | $U \subset \text{SO}(11)$ | $U \subset \text{Sp}(5)$ |
|-------|-------|--------------------------|--------------------------|
| 50    | 0     | $\text{SO}(9) \times \text{U}(1)$ | $\text{Sp}(4) \times \text{U}(1)$ |
| 32    | 9     | $\text{SO}(7) \times \text{U}(2)$ | $\text{Sp}(3) \times \text{U}(2)$ |
| 20    | 15    | $\text{U}(5)$ | $\text{U}(5)$ |
| 18    | 16    | $\text{SO}(7) \times \text{U}(1)^2$ | $\text{Sp}(3) \times \text{U}(1)^2$ |
| 14    | 18    | $\text{SO}(5) \times \text{U}(3)$ | $\text{Sp}(2) \times \text{U}(3)$ |
| 14    | 18    | $\text{SO}(3) \times \text{U}(4)$ | $\text{Sp}(1) \times \text{U}(4)$ |
| 12    | 19    | $\text{U}(4) \times \text{U}(1)$ | $\text{U}(4) \times \text{U}(1)$ |
| 10    | 20    | $\text{SO}(5) \times \text{U}(2) \times \text{U}(1)$ | $\text{Sp}(2) \times \text{U}(2) \times \text{U}(1)$ |
| 8     | 21    | $\text{SO}(5) \times \text{U}(1)^3$ | $\text{Sp}(2) \times \text{U}(1)^3$ |
| 8     | 21    | $\text{SO}(3) \times \text{U}(3) \times \text{U}(1)$ | $\text{Sp}(1) \times \text{U}(3) \times \text{U}(1)$ |
| 8     | 21    | $\text{U}(3) \times \text{U}(2)$ | $\text{U}(3) \times \text{U}(2)$ |
| 6     | 22    | $\text{SO}(3) \times \text{U}(2)^2$ | $\text{Sp}(1) \times \text{U}(2)^2$ |
| 6     | 22    | $\text{U}(3) \times \text{U}(1)^2$ | $\text{U}(3) \times \text{U}(1)^2$ |
| 4     | 23    | $\text{SO}(3) \times \text{U}(2) \times \text{U}(1)^2$ | $\text{Sp}(1) \times \text{U}(2) \times \text{U}(1)^2$ |
| 4     | 23    | $\text{U}(2)^2 \times \text{U}(1)$ | $\text{U}(2)^2 \times \text{U}(1)$ |
| 2     | 24    | $\text{SO}(3) \times \text{U}(1)^4$ | $\text{Sp}(1) \times \text{U}(1)^4$ |
| 2     | 24    | $\text{U}(2) \times \text{U}(1)^3$ | $\text{U}(2) \times \text{U}(1)^3$ |
| 0     | 25    | $\text{U}(1)^5$ | $\text{U}(1)^5$ |

| $\dim G/U$ | $\dim U/H$ |
|------------|------------|
| 24          | 11         |

Table V. Possible number of type-A and type-B NGBs for $\text{SO}(11)/\text{U}(1)^5$ and $\text{Sp}(5)/\text{U}(1)^5$.

$n_B = (1/2)\dim G/U$, while $n_A = \dim U/H$. Let us discuss a few examples below.

For instance, one can consider $\text{SU}(6)/\text{SU}(5)$, whose dimension is $35 - 24 = 11$. Note that $\text{SU}(6)/\text{SU}(5) = \text{U}(6)/\text{U}(5) = S^{11}$ which was discussed in Sec. IX C. Looking at the list in Table III, the only $U$ that commutes with $\text{SU}(5)$ is the top two. Therefore there are two types of presymplectic structures possible on $\text{SU}(6)/\text{SU}(5)$. If $U$ is trivial, all 11 are type-A NGBs. If $U = \text{SU}(5) \times \text{U}(1)$, $B = \text{SU}(6)/\text{SU}(5) \times \text{U}(1)$ and there are 5 type-B NGBs for $(1/2)\dim B = 5$. There is only one type-A NGB.

If the same $\text{SU}(6)$ is broken by an order parameter in rank-three anti-symmetric tensor, the unbroken group is $H = \text{SU}(3) \times \text{SU}(3)$. In this case, there is no $U$ that commutes with $H$ except for the trivial one. Namely this coset space allows for no presymplectic structure and hence $n_A = 19, n_B = 0$. However if one of the $\text{SU}(3)$ is further broken completely by order parameters in fundamental representations (at least two of them), $H = \text{SU}(3)$ commutes with first seven choices of $U$ in Table III, and there are seven possibilities of $(n_A, n_B)$ accordingly.

This way, one can work out all possibilities of $(n_A, n_B)$ for a given $G$ and $H$ if compact and simple. Then we look at discrete subgroups if $G$ or $H$ have more than one connected components to further eliminate some possibilities. It is also straightforward to study examples with additional $\text{U}(1)$ factors, paying attention to possible central extensions.

This way, one can enumerate all possible presymplectic structures for a given $G/H$, and write down the most general effective Lagrangian using the explicit forms we found in Sec. III.

**IX. EXAMPLES**

Having developed a complete classification of presymplectic structures, we revisit popular examples of coset spaces in the literature and show what effective Lagrangians are possible for them.

**A. $O(n+1)/O(n) = S^n$**

For $O(n+1)/O(n) = S^n$, $SO(n+1)/SO(n) = S^n$, and $O(n+1)/O(n) \times \mathbb{Z}_2 = \mathbb{R}P^n$, there is no possible presymplectic structure for $n \geq 3$. As seen in Tables IV and V, there is no non-trivial $U$ that commutes with $SO(n)$ subgroup within $SO(n+1)$ and hence $N_C = 0$. Therefore we can only have $n$ type-A NGBs. The most general Lagrangian is hence

$$L_{\text{eff}} = \frac{1}{2} g_0 \tilde{n}_i \dot{n}_i - \frac{1}{2} g_0 \tilde{\nabla} n_i \cdot \tilde{\nabla} n_i$$

up to the second order in derivatives, where $\tilde{n}$ is a normalized $(n+1)$-component vector.

When $n = 2$, all of these examples have $N_C = 1$ and the coset $SO(3)/SO(2) = S^2$ indeed describes both ferro- and antiferromagnets. However, for $O(3)/O(2) = S^2$, there is no presymplectic structure that is consistent with the discrete subgroup $\{+1, -1\}$ at least when we realize it as an internal symmetry. To see this, let us parametrize the coset $S^2$ by the spherical coordinate $(\theta, \phi)$. The candidate of one-form that is associated with the would-be symplectic structure is $\cos \theta \phi$, but it changes sign under $-\xi : \theta \rightarrow \pi - \theta$ and $\phi \rightarrow \phi + \pi$ unless the discrete symmetry incorporates with the time-reversal $t \rightarrow -t$. The coset $O(3)/O(2) \times \mathbb{Z}_2 = \mathbb{R}P^2$ can be discussed in a similar fashion, but since $\mathbb{R}P^2$ is not even orientable, there is obviously no symplectic structure that is consistent with the global topology of $G/H$.

**B. $\text{SU}(n+1)/\text{U}(n) = \mathbb{C}P^n$**

The $\mathbb{C}P^n$ ($n \geq 1$) model is a natural generalization of ferromagnets based on $S^2 = \mathbb{C}P^1$. For $G/H = SU(n+1)/U(n) = \mathbb{C}P^n$, $N_C = 1$ because there is a unique Cartan generator diag($n, -1, \cdots, -1$) that commutes with $H = U(n)$. Therefore, there is a unique symplectic structure on $G/H$ (up to an overall normalization). The effective Lagrangian can be most conveniently expressed in terms of $n$-component complex field $z(x, t) \in \mathbb{C}^n$ and the most general effective Lagrangian

$$L_{\text{eff}} = \frac{1}{2} g_0 \tilde{n}_i \dot{n}_i - \frac{1}{2} g_0 \tilde{\nabla} n_i \cdot \tilde{\nabla} n_i$$

up to the second order in derivatives, where $\tilde{n}$ is a normalized $(n+1)$-component vector.
to the quadratic order in derivatives is given by
\[ \mathcal{L}_{\text{eff}} = i \frac{z_1 \dot{z}_2 - \dot{z}_1 z_2}{1 + z_1 z_2} + G_{ab} \left( \bar{g}_0 z^a \dot{z}^b - g_0 \nabla \bar{z} \cdot \nabla z \right), \]
where
\[ G_{ab}(\bar{z}, z) = \frac{\delta_{ab}(1 + \bar{z} z) - \bar{z}^a z^b}{(1 + \bar{z} z)^2} \]
is the Fubini–Study metric on \( \mathbb{C}P^n \) [32, 33]. In \( 2 + 1 \) dimensions, we can add a topological term (\( \theta \)-term) \( (i/2\pi)G_{ab}(\bar{z}, z) e^{i\theta} \partial \bar{z} \partial_z \bar{z} \). The \( n = 1 \) case is identical to fermomagnets (recall that \( \mathbb{C}P^1 = S^2 \)). The coefficient of the first term \( s_0 \) is the charge density of the ground state \( \langle j_{\text{gh}}^0(x) \rangle \), where \( j_{\text{gh}}^0 \) is the U(1) part of the unbroken subgroup \( H = U(n) \). This term \( s_0 \Omega \) must be quantized to a half integer, where \( \Omega \) is the volume of the system, as discussed in Sec. VII.D. When \( s_0 \neq 0 \), the system resembles ferromagnets: the real and imaginary part of \( z^a \) become canonically conjugate to each other and there are \( n \) type-B NGBs. On the other hand, when \( s_0 = 0 \), the ground state is antiferromagnetic and there are \( 2n \) type-A NGBs. Other possibilities, \( (n_A, n_B) = (2, n - 1), (4, n - 2), \ldots, (2n - 2, 1) \), cannot be realized.

### C. \( U(n+1)/U(n) = S^{2n+1} \)

\( U(n+1)/U(n) = S^{2n+1} (n \geq 1) \) is topologically the same as \( \text{SO}(2n + 2)/\text{SO}(2n + 1) \), yet its field theory is very different because \( N_C = 1 \) for the generator diagonal \( n, -1, \ldots, -1 \). It is closely related to the \( \mathbb{C}P^n \) model since it admits a fibration \( S^1 \to S^{2n+1} \xrightarrow{\pi} \mathbb{C}P^n \), where type-B NGBs live on the base manifold \( \mathbb{C}P^n \) and a type-A NGB is in the fiber \( S^1 \). Therefore, there are only two possibilities, \( (n_A, n_B) = (1, n) \) and \( (2n + 1, 0) \), which is expected from \( N_C = 1 \). The case \( n = 1 \) of this model describes the physics of Kaon condensation [10, 11]. The generalization to \( n \geq 1 \) is discussed in Ref. [55].

As a concrete example, let us consider a \( U(n+1) \) symmetric Schrödinger field,
\[ \mathcal{L} = i \psi^\dagger \dot{\psi} - \frac{1}{2m} \nabla \psi^\dagger \cdot \nabla \psi - \frac{\lambda}{2} (\psi^\dagger \psi - n_0)^2, \]
where \( \psi(x) \) is a complex \( n+1 \)-dimensional column vector. A similar model was discussed in Refs. [10, 11]. At the tree level, it has the vacuum
\[ \langle \psi \rangle = \sqrt{n_0}(1, 0, \ldots, 0)^T. \]
In this case, the original \( U(n+1) \) symmetry is broken to \( U(n) \) symmetry. The coset space \( U(n+1)/U(n) = S^{2n+1} \) does not admit a symplectic structure.

Therefore, we have to carefully parameterize the coset space. Since \( U(n+1)/U(n) \times U(1) \cong \mathbb{C}P^n \), which does admit a symplectic structure, we view \( S^{2n+1} \) as a U(1) bundle on \( \mathbb{C}P^n \). The symplectic two-form lives on \( \mathbb{C}P^n \). We parametrize the field \( \psi(x) \) as
\[ \psi = \sqrt{n} e^{-i\theta} \frac{1}{\sqrt{1 + z_1 z_2}} \]
where \( z(x) \) is an \( n \) dimensional column vector. Substituting the above parametrization, we find
\[ \mathcal{L}_{\text{eff}} = n_0 \left( \dot{\theta} + \frac{i}{2} \frac{z_1 \dot{z}_2 - \dot{z}_1 z_2}{1 + z_1 z_2} \right)^2 - \frac{n_0}{2m} \left( \nabla \theta + \frac{i}{2} \frac{z_1 \nabla \bar{z}_2 - \bar{z}_2 \nabla z_1}{1 + z_1 z_2} \right)^2 \]
\[ - \frac{n_0}{2m} \left( \nabla \bar{z}_2 \nabla \bar{z}_1 - (\bar{z}_2 \nabla z_1)(z_1 \nabla \bar{z}) \right) + \cdots. \]

The second term arises from integrating out \( n \) at the tree level, and looks the same as the terms \( O(\nabla^2) \) except for the overall normalization because of the irreducible nature of \( \theta \) and \( z^1 \) under \( H = U(n) \).

The terms in the last parentheses above are nothing but the Fubini–Study metric on \( \mathbb{C}P^n \) which is Kähler. On the other hand, the first term defines a one-form
\[ c = i z_1 dz - d z_1 z, \]
while its exterior derivative
\[ dc = i \frac{(1 + z_1 z) dz_1 \wedge dz - (dz_1 z)^2 \wedge (z_1 dz)}{(1 + z_1 z)^2} \]
is the Kähler form on \( \mathbb{C}P^n \) associated with the Fubini–Study metric. The coordinate \( \theta \) represents the U(1), which is orthogonal to the tangent vectors of \( \mathbb{C}P^n \).

In Sec. X, we derive the effective Lagrangian for \( n = 1 \) based purely on the Galilean symmetry and the U(2) internal symmetry. We should be able to rewrite the Lagrangian Eq. (249) in terms of the Galilean-covariant derivatives,
\[ \mathcal{D}_\theta \dot{\theta} = \dot{\theta} - \frac{\nabla \theta^2}{2m}, \]
\[ \mathcal{D}_z \dot{z} = \dot{z} - \frac{\nabla \theta \cdot \nabla z}{m}, \quad \mathcal{D}_z = \nabla z, \]

neglecting higher-order derivatives. Comparing Eq. (246) with Eq. (298), we notice that the Lagrangian lacks the term that contains \( \mathcal{D}_z \mathcal{D}_z \mathcal{D}_z \). In general, if we start from a particular microscopic model and work only at tree levels, the effective Lagrangian may not include all possible terms allowed by symmetries. Missing terms are often generated by higher corrections [56].

### X. GALILEAN INVARIANCE

So far our discussions focused on spontaneous breaking of internal symmetries. However in many interesting
physical systems, space-time symmetries are also spontaneously broken. For the sake of clarity of our discussions, we restrict ourselves to translationally and rotationally invariant systems in this paper. Therefore we discuss spontaneously broken Galilean invariance as an illustrative example in this section. We demonstrate how space-time symmetries can be discussed within our effective Lagrangian formalism, and see how they provide additional constraints on the parameters in the theory. So-called inverse Higgs mechanism provides a heuristic method to show how would-be NGB degrees of freedom can be consistently removed from the physical spectrum in accordance with observations. This method was discussed mostly in Lorentz-invariant systems, and our presentation here shows how it can be extended to Lorentz-non-invariant systems successfully.

A. Coset construction with spacetime symmetries

In condensed matter physics, superfluid Helium and various types of Bose-Einstein condensate often spontaneously break the Galilean symmetry as well as the U(1) phase rotation. In such a situation, one has to make sure that the effective Lagrangian has the Galilean symmetry.

Here we discuss how to incorporate spacetime symmetries in our effective Lagrangian. Spacetime symmetries are those which change coordinates $x^\mu = (t, \vec{x})$ in addition to the fields. For example, the transformation rule of the superfluid phase under the Galilean transformation is

$$\vec{x}' = \vec{x} + \vec{v}_0 t, \quad t' = t, \quad \theta'(\vec{x}', t') = \theta(\vec{x}, t) - m \vec{v}_0 \cdot \vec{x} - \frac{mv_0^2}{2} t, \quad (254)$$

for a constant vector $\vec{v}_0 \in \mathbb{R}^3$. Since $\vec{z}$ changes, Galilean symmetry is a spacetime symmetry.

For simplicity, here we discuss the situation where the spacetime translation $P_\mu = (H, -P)$ is not broken, and unbroken generators $Q_\rho$ are internal symmetries, while broken generators $Q_a$ may contain spacetime symmetries such as the Galilean boost generator.

Following Ref. [57], we use 

$$U(x, \pi(x)) = e^{ix^\mu P_\mu} e^{i\pi(x)Q_a} \quad (256)$$

to define the Maurer–Cartan form $\omega$.

$$\omega(x, \pi(x)) = -i U^{-\dagger} U = e^{\pi^a P_a} + \omega_\perp + \omega_\parallel, \quad (257)$$

Again, $\omega_\perp = \omega^a Q_a$ is the broken part and $\omega_\parallel = \omega^\rho Q_\rho$ is the unbroken part. $e^{\pi^a P_a}$ is called vielbein and $G_{\mu\nu} = g_{\rho\sigma} e^\rho_{\mu} e^\sigma_{\nu}$ gives a space-time metric that transforms nicely. Especially, the space-time invariant volume-form is given by $d^4x dt \sqrt{|\text{det} G|}$.

The symmetry transformation of $x$ and $\pi(x)$ under the action of $g$ is defined by [c.f. Eq. (68)]

$$gU(x, \pi(x)) = U(x', \pi'(x')) h_g(x, \pi(x)). \quad (258)$$

Since $P_a$ is unbroken, one may be confused by the $e^{ix^\mu P_\mu}$ factor of $U$, but thanks to this factor we can realize the spacetime symmetry in this way. Analogously to Eqs. (75) and (76), we have

$$e^{\pi^a P_a} (x, \pi(x)) = e^{\pi^a (x, \pi(x))}, \quad (259)$$

$$\omega_\perp (x', \pi(x')) = h_g \omega_\perp (x, \pi(x)) h^\dagger_g, \quad (260)$$

$$\omega_\parallel (x', \pi(x')) = h_g \omega_\parallel (x, \pi(x)) h^\dagger_g - i h_g dh^\dagger_g. \quad (261)$$

Here we used the assumption that unbroken generators are internal.

Let us first discuss the broken part of the Maurer–Cartan form. We define the spacetime-covariant derivative $\mathcal{D}_\mu \pi^a$ through

$$e^{\mu} \mathcal{D}_\mu \pi^a = \omega^a. \quad (262)$$

According to Eq (260), it indeed transforms covariantly:

$$\mathcal{D}_\mu \pi^a = h_g (\mathcal{D}_\mu Q_a) h^\dagger_g, \quad (263)$$

thanks to the covariance of the vielbein $e^{\mu}(x, \pi(x))$ [see Eq. (259)]. If we had defined the covariant derivative by

$$d\pi^a \tilde{\mathcal{D}}_\mu \pi^a = \omega^a \quad (264)$$

instead of Eq. (262), $\tilde{\mathcal{D}}_\mu \pi^a$ would not transform covariantly, since $dx^\mu$ is not covariant, i.e., $dx^\mu \neq d\pi^a$.

For the same reason, the unbroken part $\partial_{\pi^a} \omega_\parallel^\rho$ does not transform covariantly. From Eq (261), we have

$$\left(\partial_{\pi^a} \omega_\parallel^\rho\right) Q_\rho = \frac{\partial x^\nu}{\partial x'^\mu} \left[ h_g (\partial_{\pi^a} \omega_\parallel^\rho) Q_\rho + i h_g \partial_{\pi^a} h^\dagger_g \right]. \quad (265)$$

If the factor $\partial x^\nu/\partial x'^\mu$ were absent, as in the case for internal symmetries, the unbroken part would transform covariantly up to the inhomogeneous term $-i h_g \partial_{\pi^a} h^\dagger_g$, which may be just a total derivative. In such a case the unbroken part can be added to the effective Lagrangian, as discussed in Sec. III D. However, nontrivial $\partial x^\nu/\partial x'^\mu$ poses an obstacle as we shall see shortly.

Covariant derivatives in Eq. (262) are the building blocks of the effective Lagrangian. The case considered in Sec. III, where only internal symmetries are broken, can be understood as the spacial case of $e^{\pi^a (x, \pi(x))} = dx^\mu$. In the following, we will demonstrate what we have said here using a concrete example.

B. Example

In this section, we discuss the effective Lagrangian for the microscopic model,

$$\mathcal{L} = \frac{i}{2} (\bar{\psi} \gamma^\dagger \psi - \text{c.c.}) - \frac{\nabla \psi \dagger \cdot \nabla \psi}{2m} - \frac{g}{2} (\psi \dagger \psi - n_0)^2. \quad (266)$$

This model can be seen as the nonrelativistic version of the model for the Kaon condensation discussed in Ref. [10, 11]. Here $\psi = (\psi_1, \psi_2)^T$ is a two component
complex scalar field. The ground state expectation value $\langle \psi \rangle = \sqrt{\pi_0(0,1)^2}$ breaks the U(2) symmetry down to U(1) symmetry. Broken symmetry generators are $\sigma_1, \sigma_2,$ and $\sigma_3 - \sigma_0$, where $\sigma_{1,2,3}$ are Pauli matrices and $\sigma_0$ is the identity matrix.

The Lagrangian Eq. (266) possesses the Galilean symmetry,

$$\mathcal{L} = \mathcal{L}(\pi, \dot{\pi}, \pi') = e^{i\pi_0 T_0} + \frac{1}{2} \pi^2 + \frac{1}{2} \pi_0^2 \pi',$$

in addition to the internal U(2) symmetry. The low-energy effective Lagrangian must respect it.

Note that our discussion below is solely based on the internal U(2) symmetry and the Galilean symmetry so that it applies to any microscopic Lagrangians as long as they respect these symmetries and show the same symmetry breaking pattern.

1. Without Galilean symmetry

Before going into the detailed discussion on the consequence of Galilean invariance, let us first review what we developed in Sec. III without paying attention to the Galilean symmetry for comparison. We parametrize the coset as

$$U = e^{i\pi_0 T_0} = \cos(\pi_0) + [\pi_1 \sin(\pi_0) + \pi_2 \cos(\pi_0)]$$

We compute the Maurer–Cartan form

$$\omega = -iU_0 U_1,$$

where $U_0 = e^{-i\theta \sigma_3}$ and $U_1 = e^{i\pi_0 \sigma_0} \sigma_0 \cos \rho + i \rho \sigma_3 \sin \rho$

with the constraint $\pi^3 = \theta$. Here, $a = 1, 2, 3$ and $\rho \equiv \sqrt{\pi_3 \pi_0}$. Using the property of Pauli matrices, the Maurer–Cartan form for $U_0$ and $U_1$,

$$-iU_0^\dagger \dot{U}_0 = \sigma_0 \Omega^0,$$

$$-iU_1^\dagger \dot{U}_1 = \sigma_1 \Omega^1 + \sigma_2 \Omega^2 + \sigma_3 \Omega^3.$$ 

can be easily evaluated as

$$\Omega^0 = -d\theta,$$

$$\Omega^a = d\alpha^b \left[ \left( \delta^{ab} - \frac{\pi_0 \pi_b}{\rho^2} \right) \sin 2\rho \right. + \frac{\pi_0 \pi_b}{\rho^2} - e^{abc} \frac{\sin \rho}{\rho} \right].$$

The full Maurer–Cartan form $\omega = -iU_0^\dagger \dot{U}_0 - iU_1^\dagger \dot{U}_1$ is given by

$$\omega^1 = \Omega^1, \quad \omega^2 = \Omega^2,$$

$$\omega^3 = \Omega^3.$$  

2. With Galilean symmetry

To implement the Galilean symmetry, we introduce the boost operator $\tilde{B}$ as well as the spacetime translation $P_\mu = (H, -\tilde{P})$. Their nonzero commutation relations are $[Q_a, Q_b] = 2i\delta_{ab} Q_c, [\tilde{B}, H] = -i\tilde{P}$, and $[B^i, P_j] = -i\delta^{ij}\delta_{ab}$ is centrally extended (see Appendix A). $Q_1, Q_2, Q_3$ and $\tilde{B}$ are spontaneously broken. The unbroken generator $Q + Q_3$ is internal so that the assumption in the previous section is fulfilled. Therefore, we use

$$\tilde{U} = e^{i\pi_0 \sigma_0} e^{i\pi_0 (\tilde{x}, t) Q_a \cdot \theta (\tilde{x}, t) Q - i\tilde{P} (\tilde{x}, t) \cdot \tilde{B}}.$$ 

Here we introduced a new vector field $\tilde{v}(\tilde{x}, t)$ that does not describe any physical modes and will be eliminated later in favor of real NG fields $\pi^1, \pi^2,$ and $\pi^3 = \theta$.

The Maurer–Cartan form $\tilde{\omega} = -i\tilde{U}_0^\dagger \dot{U}_0$ is given by

$$\tilde{\omega} = \tilde{\omega}^0 (Q_3 + Q) + [\omega^1 Q_1 + \omega^2 Q_2 + \omega^3 (Q_3 - Q)]$$

$$+ e^{-\theta \sigma_3} P_\mu - \tilde{B} \cdot \tilde{d}\tilde{x},$$

where $\omega^{0,1,2,3}$ stands for those defined in Eq. (270).

$$\tilde{\omega}^0 = \omega^0 + \frac{1}{2} \left( \frac{mv^2}{2} dt - m\tilde{v} \cdot \tilde{d}\tilde{x} \right),$$

$$\tilde{\omega}^3 = \omega^3 - \frac{1}{2} \left( \frac{mv^2}{2} dt - m\tilde{v} \cdot \tilde{d}\tilde{x} \right),$$

and

$$\tilde{e}^0 (\tilde{x}, t) = dt, \quad \tilde{e}(\tilde{x}, t) = d\tilde{x} - \tilde{v}(\tilde{x}, t) dt.$$ 

$\tilde{e}$ is indeed covariant:

$$\tilde{e}^a (\tilde{x}, t) = d(\tilde{x} + \tilde{v} t) - [\tilde{v}(\tilde{x}, t) + \tilde{v}_0] dt = d\tilde{x} - \tilde{v}(\tilde{x}, t) dt = \tilde{e}(\tilde{x}, t).$$

In this case, $detG$ is trivial and $d^4xd t$ by itself is an invariant volume-form.
Following the definition in Eq. (262), covariant derivatives are given by

\[ \tilde{\mathcal{D}}\pi^1 = \bar{\omega}^1, \]
\[ \tilde{\mathcal{D}}\pi^2 = \bar{\omega}^2, \]
\[ \tilde{\mathcal{D}}\pi^3 = \bar{\omega}^3 + \frac{m\bar{v}}{2}, \]
\[ \mathcal{D}_t\pi^1 = \bar{\omega}^1 + \bar{\omega} \cdot \tilde{\mathcal{D}}\pi^1, \]
\[ \mathcal{D}_t\pi^2 = \bar{\omega}^2 + \bar{\omega} \cdot \tilde{\mathcal{D}}\pi^2, \]
\[ \mathcal{D}_t\pi^3 = \bar{\omega}^3 - \frac{mv^2}{4} + \bar{\omega} \cdot \tilde{\mathcal{D}}\pi^3. \]

Let us now focus on \[ \tilde{\mathcal{D}}\pi^3. \] It contains a linear term of \( \bar{v} \) without derivatives. Thus we can impose a covariant constraint \( \tilde{\mathcal{D}}\pi^3 = 0 \), so-called the inverse Higgs constraint [57], to eliminate the unphysical field \( \bar{v} \) in terms of true NG fields,

\[ \bar{v} = -\frac{2\bar{\omega}^3}{m}. \]

This is a heuristic way to get rid of unphysical fields in the coset construction with spacetime symmetries. See Refs. [31, 58, 59] for more details.

After imposing this constraint, covariant derivatives become

\[ \tilde{\mathcal{D}}\pi^1 = \bar{\omega}^1, \]
\[ \tilde{\mathcal{D}}\pi^2 = \bar{\omega}^2, \]
\[ \mathcal{D}_t\pi^1 = \bar{\omega}^1 - \frac{2m}{m} \bar{\omega}^3 \cdot \tilde{\mathcal{D}}\pi^1, \]
\[ \mathcal{D}_t\pi^2 = \bar{\omega}^2 - \frac{2m}{m} \bar{\omega}^3 \cdot \tilde{\mathcal{D}}\pi^2, \]
\[ \mathcal{D}_t\pi^3 = \bar{\omega}^3 - \frac{1}{m} \bar{\omega}^3 \cdot \bar{\omega}^3. \]

These are the Galilean-covariant building blocks of the effective Lagrangian.

For the usual superfluid, the inverse Higgs constraint is \( \tilde{\mathcal{D}}\theta = \tilde{\mathcal{D}}\mathcal{D}_t\theta = -\frac{m\bar{v}}{2} \) and the combination in Eq. (296) corresponds to \( \tilde{\mathcal{D}}\theta = \theta - (\tilde{\mathcal{D}}\tilde{\mathcal{D}}\theta)^2/2m \). Quantities in Eqs. (294) and (295) correspond to the second term in Eq. (12) of Ref. [60] for supersolids.

According to Eq. (265), \( \bar{\omega}^0 = \omega^0 \delta^a_a \) transforms as

\[ (\bar{\omega}^0)(\bar{x} + \bar{v}_0 t, t) \]
\[ = \omega^0(\bar{x}, t) + \bar{v}_0 \cdot \bar{\omega}^0(\bar{x}, t) + (\tilde{\mathcal{D}}_t + \bar{v}_0 \cdot \tilde{\mathcal{D}})\Lambda \]

for some \( \Lambda \). Therefore, the change of \( \omega^0 \) is more than a surface term and it cannot be added to the effective Lagrangian.

In summary, the most general form of the effective Lagrangian which respects the Galilean symmetry is

\[ \mathcal{L}_{\text{eff}} = -e_0(0) \mathcal{D}_t\pi^3 \]
\[ + \frac{g_{11}(0)}{2} [(\mathcal{D}_t\pi^1)^2 + (\mathcal{D}_t\pi^2)^2] + \frac{g_{33}(0)}{2} (\mathcal{D}_t\pi^3)^2 \]
\[ - \frac{g_{11}(0)}{2} (\tilde{\mathcal{D}}\pi^1 \cdot \tilde{\mathcal{D}}\pi^1 + \tilde{\mathcal{D}}\pi^2 \cdot \tilde{\mathcal{D}}\pi^2), \]

which now contains only four parameters. Compared to Eq. (271), we have two restrictions:

\[ e_0(0) = 0, \quad g_{33}(0) = -\frac{2e_3(0)}{m} (> 0). \]

Since \( e_0(0) \) represents the classical expectation value of \( (Q_3 + Q)/\Omega \), the spin must be fully polarized and \( e_3(0) = (Q_3 - Q)/\Omega = -2n < 0 \), where \( n \) is the number density of the particles. This is consistent with the rigorous result in Ref. [45].

Galilean invariant combinations contain mixed powers of derivatives and one can drop higher order derivative terms as it does not affect the physics to the aimed order of the derivative expansion.

One may think that introducing the unphysical field \( \bar{v}(\bar{x}, t) \) first and eliminate it by imposing a covariant condition is just a complicated useless way of deriving the effective Lagrangian. However, as we demonstrated here, it is actually a convenient way to systematically generate terms with proper spacetime symmetries.

Finally, let us discuss the power counting of the derivative expansion. In this paper, we assign \( \pi^a = O(1) \) so that \( \mathcal{N}_a \pi^a = O(k_n) \) and expand the Lagrangian in the series of derivatives. However, Ref. [60] introduced an alternative way of power counting, which assigns \( \mathcal{N}_a \pi^a = O(1) \) provided that the Lagrangian does not depend on \( \pi^a \) without derivatives. In this power counting method, the lowest order term is the sum of all invariant combinations with one derivative per a field. This counting has an advantage that it can deal with the situation with large fluctuation \( \pi^a = O(k^{-1}) \) from the ground state, but it works only for Abelian groups \( G \), otherwise the effective Lagrangian depends on fields without derivative as one can see from the example discussed in this section.

XI. CONCLUSION

In this paper, we derived the explicit form of the most general nonrelativistic Lagrangian of NGBs in terms of Maurer–Cartan form, which must be quite useful to systematically discuss quantum corrections. By using the free part of the effective Lagrangian, we proved the counting rule NGBs and clarified the dispersion relation of NGBs for a general set up. We also completely classified possible numbers of type-A and type-B NGBs for a given choice of \( G/H \).

To discuss additional constraints on the effective Lagrangian from spacetime symmetries, we showed explicitly the consequence of Galilean invariance. In addition, we presented an intuitive interpretation of the presymplectic structure as the Berry’s phase of the ground state.

Having derived the most general effective Lagrangian, we could develop a simple scaling arguments, and showed why a long-range order is stable in \( 1+1d \) when only type-B NGBs are present, while the stability requires \( 2+1d \) and above for type-A NGBs. It remains an interesting
question whether there is a general rule of thumb when both types of NGBs coexist.

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Appendix A: Lie Algebra Cohomology

Cohomology of Lie algebra was introduced by Chevalley and Eilenberg [61] as a way to compute de Rham cohomology of compact connected Lie groups using their Lie algebra. The work of HM was supported by the U.S. DOE under Contract DE-AC03-76SF00098, by the NSF under grants PHY-1002399 and PHY-1316783, by the JSPS grant (C) 23540289, and by WPI, MEXT, Japan.

A form on a Lie algebra \( \omega_k \in \Omega^k(\mathfrak{g}) \) is a map from \( \wedge^k \mathfrak{g} \) to \( \mathbb{R} \)

\[ \omega_k(g_1,\ldots,g_k) \in \mathbb{R}, \tag{A3} \]

anti-symmetric among arguments,

\[ \omega_k(g_1,\ldots,g_i,\ldots,g_j,\ldots,g_k) = -\omega_k(g_1,\ldots,g_j,\ldots,g_i,\ldots,g_k). \tag{A4} \]

A two-form \( \omega_2 \) is exact if it can be obtained from a one-form \( \omega_1 \) as a way to compute de Rham cohomology.

\[ d\omega_1(g_1, g_2) = \omega_1([g_1, g_2]). \tag{A5} \]

On the other hand, it is closed if

\[
\omega_2(g_1, g_2, g_3) + \omega_2(g_1, [g_2, g_3]) + \omega_2([g_1, g_2], g_3) = 0
\tag{A6}
\]

for any \( g_1, g_2, g_3 \). This is called the cocycle condition. For an exact two-form, it is nothing but the Jacobi identity, and hence it is automatically closed.

Therefore, the possibility of \( \omega_2(g_1, g_2) \) that cannot be written as the original commutation relation yet satisfies Jacobi identity is the central extension, and hence can be described by the second cohomology \( H^2(\mathfrak{g}) \).

According to the theorem by Chevalley and Eilenberg, \( H^2(\mathfrak{g}) = H^2_{dR}(G) \) if \( G \) is the compact connected group generated by \( \mathfrak{g} \). Since all compact simple Lie groups have trivial second cohomology, central extensions are not possible for their Lie algebras. On the other hand, if there are \( U(1) \) factors,

\[ \dim H^2_{dR}(U(1)^n) = \frac{n(n-1)}{2}, \tag{A7} \]

generated by \( d\phi^a \wedge d\phi^b \). Therefore, the Lie algebra cohomology \( H^2(U(1)^n) \) is also non-trivial and hence a central extension is possible.

Note that the Lie algebra knows only about the local information, and hence it makes no distinction between \( U(1) \) and \( \mathbb{R} \). For instance, consider the Galilean group of rotations \( M_{ij} \), translations \( P_i \), and Galilean boosts \( B_i \)

\[
[M_{ij}, P_k] = i(\delta_{ik}P_j - \delta_{jk}P_i), \tag{A8}
\]

\[
[M_{ij}, B_k] = i(\delta_{ik}B_j - \delta_{jk}B_i), \tag{A9}
\]

\[
[M_{ij}, M_{kl}] = i(\delta_{ik}M_{jl} - \delta_{il}M_{jk} - \delta_{jk}M_{il} + \delta_{jl}M_{ik}), \tag{A10}
\]

\[ [P_i, B_j] = 0. \tag{A11} \]

\( \vec{P} \) and \( \vec{B} \) form \( \mathbb{R}^d \) individually, which allows for a central extension

\[ [P_i, B_j] = i\delta_{ij}M, \tag{A12} \]

where the eigenvalue of the operator \( M \) is the mass of the particle and a center of the Lie algebra (i.e., commutes with everything else). The rotational invariance restricts the form to be proportional to \( \delta_{ij} \). The exception is for the \( 2+1 \) dimension, where \( \epsilon_{ij} \) allows for alternative extensions, \( [P_x, P_y] \propto \epsilon_{xy} = 1 \) [62].

Another example of central extension based on \( \mathbb{R} \) is the shift symmetry of the Schrödinger field mentioned in Sec. III E and Appendix C. It has a central extension thanks to \( H^2(\mathbb{R}^2) = \mathbb{R} \neq 0 \).

Appendix B: Matter Fields

In this paper we established the effective Lagrangian of NGBs for systems without Lorentz invariance. The effective Lagrangian can also describe the situation where
other low-energy degrees of freedom (matter fields) couple to NGBs. In this appendix, we review how to write down such low-energy theory for reader’s convenience. Such matter fields are important in many physical systems, e.g., fermions coupled to a spin system, and nucleons coupled to pions.

1. Approach 1: Modding $H$

As discussed originally in Ref. [5], any representation of $H \psi \rightarrow \rho(h)\psi$, where $\rho(h)$ is a representation matrix, can be promoted to transform under the full $G$ by

$$\psi \rightarrow \psi' = \rho(h_g(\pi))\psi.$$  

(B1)

Since $h_g(\pi)$ is an element of $H$, the above expression is well-defined. To see that it is a consistent transformation law, we perform two successive transformations

$$U(\pi) \rightarrow g_2 g_1 U(\pi) = g_2 U(\pi') h_{g_1}(\pi') = U(\pi'') h_{g_2} U(\pi'),$$  

(B2)

while

$$\psi \rightarrow \rho(h_{g_2}(\pi') h_{g_1}(\pi))\psi = \rho(h_{g_2}(\pi')) \rho(h_{g_1}(\pi))\psi,$$  

(B3)

given that $\rho$ is a representation of $H$.

Note that this transformation law is local in the sense that $\rho(h_g(\pi(\vec{x},t)))$ is position-dependent. As a result, $d\psi$ does not transform in the same way as $\psi$ does.

$$(d\psi)' = d[\rho(h_g(\psi)] = \rho(h_g)[d + \rho(h_g' \cdot dh_g)]\psi.$$  

(B4)

However, the inhomogeneous part can be exactly compensated by the unbroken component of the Maurer–Cartan form [see Eq. (78)],

$$\rho(\omega_{\parallel}) = \rho(h_g \omega_{\parallel} h_g^{-1}) - i\rho(h_g dh_g).$$  

(B5)

Therefore, the combination

$$D\psi = [d + i\rho(\omega_{\parallel})]\psi$$  

(B6)

is covariant. [This also means that $D^n\psi$ ($n \geq 0$) is covariant.] Then, the question is how to write down $H$-invariant combinations out of these $H$-covariant building blocks. For example, $\psi^\dagger \psi$, $i(\psi^\dagger D_\mu \psi - c.c.)$, $D_\mu \psi^\dagger D^\mu \psi$ are all invariant combinations. Since $D$ contains $\omega_{\parallel}$, they describe interactions between NGBs and matter fields. We can also multiply invariants such as $g_{\alpha\beta}(0)\partial^\alpha \omega^\beta$ to them. Since all Maurer–Cartan forms come with at least one derivative acting on NG fields, all interactions become smaller and smaller in the low-energy limit.

What may be surprising is that the matter fields need to be only in linear representations of $H$, not $G$. For instance, when electrons are coupled to ferromagnets, $G = SO(3)$, $H = SO(2)$, and the electrons need to transform only under $U(1)$ representation with a particular charge $q$, namely, $\rho(T_z) = q$ and $\psi' = e^{iq\varphi} \psi$. Then, the low-energy effective Lagrangian for the interacting system of electrons and magnons (the NGB in ferromagnets) are given by $\mathcal{L}_{\text{eff}} = \mathcal{L}_{\text{mag}} + \mathcal{L}_{\text{el+int}}$, where

$$\mathcal{L}_{\text{mag}} = -s\bar{\omega}^2 - \frac{1}{2}g_0 \left[(\bar{\omega}^x)^2 + (\bar{\omega}^y)^2\right],$$  

(B7)

$$\mathcal{L}_{\text{el+int}} = \frac{i}{2} \left(\psi^\dagger D_t \psi - c.c.\right) - \mu \psi^\dagger \psi - \frac{\bar{D} \psi^\dagger \cdot \bar{D} \psi}{2m} - \lambda \left[(\bar{\omega}^x)^2 + (\bar{\omega}^y)^2\right] \psi^\dagger \psi$$  

(B8)

to the order $O(\nabla^2, \bar{\nabla}^2)$. Here, $D_t = \nabla_t + iq\bar{\omega}^z$ and $\bar{D} = \bar{\nabla} + iq\bar{\omega}^z$, $s$ is the magnetization density, $m$ is the effective mass and $\mu$ is the chemical potential of electrons.

The interaction Lagrangian Eq. (B8) may be derived from a microscopic model,

$$\mathcal{L} = \frac{i}{2} \left[\Psi^\dagger \nabla_t \Psi - c.c.\right] - \mu \Psi^\dagger \Psi - \frac{\bar{\nabla} \Psi^\dagger \cdot \bar{\nabla} \Psi}{2m} - J\bar{n} \cdot \sigma \Psi,$$  

(B9)

where $\Psi(\vec{x},t)$ is a two component spinor and $\bar{n}(\vec{x},t)$ represents the magnetization of the ferromagnet including the fluctuation. At this moment, the interaction term $\lambda\bar{n} \cdot \vec{s}$ does not contain any derivatives and the weakness of the interaction at long-distance is less apparent. To get the effective Lagrangian Eq. (B8), we define locally a unitary transformation $U(\bar{n}(\vec{x},t))$ [42, 63] such that

$$U^\dagger \bar{n} \cdot \sigma U = \sigma_z,$$  

(B10)

and rewrite Eq. (B9) in terms of $(\psi, \psi')^T \equiv U \Psi$. Then $\lambda\bar{n} \cdot \vec{s}$ becomes just a constant $\lambda \sigma_z/2$, giving different chemical potentials to $\psi$ and $\psi'$. The derivative of $\Psi$ now contains the Maurer–Cartan form

$$d\Psi = U(d + i\omega)(\psi, \psi')^T.$$  

(B11)

Since $\psi'$ electrons have a gap $J$, we can integrate them out, ending up with (B8) with $q = 1/2$ and $\lambda = 1/8m$ to the current order of the derivative expansion.

2. Approach 2: Gauging $H$

The translation law in the previous section is often awkward to deal with because it is non-linear. Using the formalism to gauge the right-translation of $U$ by $H$, we can identify the above transformation law as the gauge transformation with the gauge $\pi^a = 0$. Therefore, we consider $U = e^{i(\pi^a T_a + \pi^c T_c)}$ for the entire $G$ and its global transformation

$$U(\pi) \rightarrow gU(\pi), \quad \psi \rightarrow \psi,$$  

(B12)

while the local $H$ transformation is

$$U(\pi) \rightarrow U(\pi)h(x), \quad \psi \rightarrow \rho(h^\dagger(x))\psi.$$  

(B13)
Then we can construct invariant Lagrangian using $\psi$ and its covariant derivatives

$$D\psi = [d - i\rho(A)]\psi,$$

(B14)

where $A = A^a T_a$ transforms as in Eq. (135). This is indeed covariant under the right translation:

$$\langle D\psi \rangle = [d - i\rho(h^1 A h + i h^1 dh)]\rho(h^1)\psi = \rho(h^1)[d + \rho(h^1)]\rho(h^1)\psi = \rho(h^1)[d - i\rho(A)]\psi = \rho(h^1)D\psi.$$

At the end of the day, we integrate the gauge fields out and stick to the gauge $\pi^\rho = 0$. Within this gauge, $h(x) = h^1_x(\pi)$, and $\rho(h^1(x)) = \rho(h_2(x))$ as desired.

**Appendix C: Time reversal symmetry**

In this appendix, we clarify a confusion on discrete symmetries in the existing literature [64, 65]. Contrary to the claim made in these references, we argue that type-B NGBs can appear without breaking any discrete symmetries such as the time-reversal symmetry (TRS).

In the case of ferromagnets, the expectation value

$$\langle [S_x, S_y] \rangle = i\langle S_z \rangle \neq 0$$

spontaneously breaks not only the spin rotational symmetry but also TRS, since under the time reversal the spin operator $\vec{S}$ flips its sign, $\vec{S} \to -\vec{S}$. However, in general

$$\langle [Q_a, Q_b] \rangle = i\epsilon_{abc} \langle Q_c \rangle \neq 0$$

(C2)

does not necessarily mean that TRS is broken. In order to respect TRS, all generators that have a nonzero expectation value $\langle Q_a \rangle$ have to be even under the time-reversal. Then, Eq. (C2) dictates that either $Q_a$ or $Q_b$ that appear in the commutator must be even and the other one must be odd, since TRS is anti-unitary and flips the sign of the right hand side.

The simplest example is again given by the free boson model in Eq. (155). In this model, we identify the free bosons with the dispersion $\omega = k^2/2m$ as the type-B NGB corresponding to the spontaneously broken shift symmetry $\psi \to \psi + c$ ($c \in \mathbb{C}$) [9]. The Noether charge for shifting the real and imaginary part of $\psi$ is given by $Q_R = i \int d^4 x (\psi - \psi^\dagger)$ and $Q_I = \int d^4 x (\psi + \psi^\dagger)$, respectively. Due to the commutation relation $[\psi(x, t), \psi^\dagger(\vec{x}, t)] = \delta^4(\vec{x} - \vec{x}')$, $Q_R$ and $Q_I$ do not commute and $[Q_R, Q_I] = 2i\Omega$. In this case, the field $\psi$ transforms under TRS as $\mathcal{T}\psi(x, t)\mathcal{T}^{-1} = \psi(\vec{x}, -t)$ and hence $Q_R$ is odd and $Q_I$ is even under TRS.

A more non-trivial example is the model discussed in Sec. X, which exhibits the symmetry breaking pattern $U(2) \to U(1)$. The field $\psi$ transforms as $\psi = e^{i\sigma^3_1} \psi$ under the SU(2) symmetry and corresponding conserved charges are given by $Q_1 = \int d^4 x \psi^\dagger \sigma_1 \psi$.

There are several consistent definitions of the time-reversal symmetries for this model. If $\psi$ is a scalar, $\mathcal{T}$ acts as

$$\mathcal{T}\psi(\vec{x}, t)\mathcal{T}^{-1} = \psi(\vec{x}, -t).$$

(C3)

In this case, $Q_1$ and $Q_3$ are even and $Q_2$ is odd since $\sigma_2$ is imaginary. Thus, $\langle [Q_1, Q_2] \rangle = 2i\langle Q_3 \rangle \neq 0$ does not break this TRS while a type-B NGB appears in this model. $\vec{Q}/2$ represents a pseudo spin. Another way of defining $\mathcal{T}$ symmetry is

$$\mathcal{T}\psi(\vec{x}, t)\mathcal{T}^{-1} = i\sigma_2 \psi(\vec{x}, -t).$$

(C4)

This time, all of $Q_1$s are odd under $\mathcal{T}$ and $\vec{Q}/2$ represents the real spin. $\langle [Q_1, Q_2] \rangle = 2i\langle Q_3 \rangle \neq 0$ breaks this TRS.

Other discrete symmetries, such as the parity $P$ and the charge conjugation $C$, if they exists, can be discussed in the same way.

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