Chern classes and generators

By Masaki KAMEKO
Department of Mathematical Sciences, Shibaura Institute of Technology,
307 Fukasaku Minuma-ku, Saitama 337-8570, Japan

(Communicated by Kenji FUKAYA, M.J.A., Dec. 12, 2011)

Abstract: We give a simple proof for the fact that certain algebra generators of the mod 2 cohomology of classifying spaces of exceptional Lie groups are given by Chern classes and Stiefel-Whitney classes of certain representations.

Key words: Classifying space; exceptional Lie group; characteristic class.

1. Introduction. One of the standard tools to compute the mod 2 cohomology of the classifying space $BG$ of a connected Lie group $G$ is the Rothenberg-Steenrod spectral sequence converging to the mod 2 cohomology of $BG$. In the case of $G = E_6$ or $E_7$, as an algebra over the Steenrod algebra, the $E_2$-term of the spectral sequence is generated by only two elements, one is of degree 4 and the other is of degree 32 or 64, respectively. To show the collapsing of the spectral sequence, it suffices to consider the $E_\infty$-term, which is of degree 32 or 64, respectively. It could be done by showing that the algebra generator is represented by a characteristic class of some representation of $G$. It is also conjectured that the mod 2 cohomology of $BE_8$ is also generated by two elements as an algebra over the mod 2 Steenrod algebra.

The case $G = E_6$ is first proved by Kono and Mimura in [3], see also [9]. The case $G = E_7$ is proved by Kono, Mimura and Shimada in [4]. The case $G = E_8$ of the following theorem is first proved by Mimura and Nishimoto in [6] with rather complicated calculation. Then, Kono, in [5], gives a simple proof for the following theorem by considering certain finite 2-groups in exceptional Lie groups and computing Stiefel-Whitney classes of their representations.

Theorem 1.1. For $G = E_6, E_7, E_8$, there exist representations $\rho_4, \rho_7, \rho_8$ such that $w_{16}(\rho_4), c_{10}(\rho_7), c_{22}(\rho_8)$ and $w_{128}(\rho_8)$ are indecomposable in $H^*(BG; \mathbb{Z}/2)$.

In this paper, we give another simpler proof for this theorem using Chern classes of representations of spinor groups. The method in this paper is used by Schuster, Yagita and the author in [8] and [2], where the degree 4 element in the integral cohomology of the above classifying spaces is studied in conjunction with Chern subrings of classifying spaces.

2. Chern classes. Let us recall complex representation rings of spinor groups. For $n \geq 6$, we consider $m = \lceil \frac{n}{2} \rceil$, so that $n = 2m$ or $2m + 1$. Let $T^m$ be a fixed maximal torus of Spin$(n)$ and denote by $f_m : T^m \to \text{Spin}(n)$ the inclusion. We also denote by $f_1 : T^1 \to \text{Spin}(n)$ the composition of the inclusion of the maximal torus and the inclusion of the first factor $T^1$ into $T^m$. The complex representation ring of Spin$(n)$ is given by

$$R(\text{Spin}(2m)) = \mathbb{Z}[\lambda_1, \ldots, \lambda_{m-2}, \Delta^+, \Delta^-],$$

and

$$R(\text{Spin}(2m + 1)) = \mathbb{Z}[\lambda_1, \ldots, \lambda_{m-1}, \Delta].$$

The image $f_m^* (\lambda_i)$ of the above generator $\lambda_i$ in

$$R(T^m) = \mathbb{Z}[z_1, z_1^{-1}, \ldots, z_m, z_m^{-1}],$$

is the $i$-th elementary symmetric function of $z_1^2 + \ldots, z_m^2$. The image $f_m^*(\Delta^+)$, $f_m^*(\Delta^-)$, $f_m^*(\Delta)$ in $R(T^m)$ are given by

$$f_m^*(\Delta^+) = \sum_{z_1 \cdot \ldots \cdot z_m = +1} z_1^2 \ldots z_m^2,$$

$$f_m^*(\Delta^-) = \sum_{z_1 \cdot \ldots \cdot z_m = -1} z_1^2 \ldots z_m^2.$$
\[ f'_m(\Delta) = \sum_{z_1^i \cdots z_m^i} z_1^i \cdots z_m^i. \]

Since \( z_1^2 + z_2^{-2} \) maps to 2, for \( 1 \leq i \leq m \), it is clear that
\[ f'_i(\Delta) = \alpha_i + \beta_i (z_1^2 + z_2^{-2}), \]
where \( \alpha_i = 2^{i(m-i)}, \beta_i = 2^{m-i}. \) It is also clear that
\[ f'_i(\Delta -) = f'_i(\Delta^-) = 2^{m-2}(z_1 + z_1^{-1}), \]
\[ f'_i(\Delta) = 2^{m-1}(z_1 + z_1^{-1}). \]
Therefore, the total Chern classes are
\[ c(f'_i(\Delta)) = \{(1 + 2u)(1 - 2u)\}^\beta_i \]
\[ = (1 - 4u^2)^\beta_i, \]
\[ c(f'_i(\Delta__)) = c(f'_i(\Delta^-)) \]
\[ = \{(1 + u)(1 - u)\}^{2^{m-2}} \]
\[ = (1 - u^2)^{2^{m-2}}, \]
\[ c(f'_i(\Delta)) = \{(1 + u)(1 - u)\}^{2^{m-1}} \]
\[ = (1 - u^2)^{2^{m-1}}. \]
in \( H^*(BT; \mathbb{Z}) \cong \mathbb{Z}[u] \), where \( u \) is the generator of \( H^2(BT; \mathbb{Z}) \cong \mathbb{Z} \).

Thus we have the following result on the mod 2 reduction of the total Chern classes:

**Proposition 2.1.** The mod 2 reduction of the total Chern classes of \( f'_i(\lambda_i) \), \( f'_i(\Delta^\pm) \), \( f'_i(\Delta) \) are given by
\[ c(f'_i(\lambda_i)) = 1, \]
\[ c(f'_i(\Delta^+)) = c(f'_i(\Delta^-)) \]
\[ = 1 + u^{2^{m-1}}, \]
\[ c(f'_i(\Delta)) = 1 + u^{2^m}. \]
in \( H^*(BT; \mathbb{Z}/2) \). In particular, \( c(f'_i(\mu)) = 1 + u^k \) where \( \deg u^k = 2k = 2 \dim \mu \) for \( \mu = \Delta^+, \Delta^-, \Delta \).

3. **Generators.** In this section, let \( H^*(X) \) be the mod 2 cohomology \( H^*(X; \mathbb{Z}/2) \) of \( X \) and \( \hat{H}^*(X) \) the reduced mod 2 cohomology \( H^*(X; \mathbb{Z}/2) \). Recall that if \( \xi \) is a real representation and if \( \xi_C \) is its complexification, then, the mod 2 Chern class \( c_i(\xi_C) \) is the square of the Stiefel-Whitney class \( w_i(\xi) \), that is, \( c_i(\xi_C) = w_i(\xi)^2 \).

We recall Quillen’s computation of the mod 2 cohomology of \( BSpin(n) \) in [7]. As in the previous section, let \( m = \lceil \frac{n}{2} \rceil \). Depending on the type of the representation of the spinor group \( Spin(n) \), we define \( h \) to be \( m - 1 \) or \( m \) in Table I.

Let \( p : Spin(n) \to SO(n) \) be the projection. Then, the mod 2 cohomology of \( BSpin(n) \) is given by
\[ H^*(BSpin(n)) = H^*(BSO(n))/J \otimes \mathbb{Z}/2[\tau], \]
where \( J \) is the ideal generated by \( w_2, Sq^4w_2, \ldots, Sq^{2^{m-1}}w_2 \) and \( z \) is an element of degree 2. Let \( f_1 : T^2 \to Spin(n) \) be the inclusion of the first factor of the maximal torus. Since \( T^2 \) is a closed subgroup of \( Spin(n) \), the induced homomorphism \( Bf_1 : H^*(BSpin(n)) \to H^*(BT^2) \) is integral. On the other hand, since \( c(f'_i(\lambda_i)) = 1 \), the induced homomorphism \( (Bp \circ Bf_1)^* : H^*(BSO(n)) \to H^*(BT^2) \) is zero. Hence, \( BF_1(z) \) must be non-zero, that is, \( BF_1(z) = u^{2^m} \) and it generates the image of \( BF_1 \).

Thus, we have the following proposition:

**Proposition 3.1.** The image of \( BF_1 \) is a polynomial ring generated by \( u^{2^m} \).

Now, we consider the following representations for exceptional Lie groups \( F_4, E_6, E_7, E_8 \):

| n | m | type | h | \( h \) |
|---|---|-----|---|-----|
| 8k | 4k | R | 4k - 1 | m - 1 |
| 8k + 1 | 4k | R | 4k - 1 | m - 1 |
| 8k + 2 | 4k + 1 | C | 4k + 1 | m |
| 8k + 3 | 4k + 1 | H | 4k + 1 | m |
| 8k + 4 | 4k + 2 | H | 4k + 2 | m |
| 8k + 5 | 4k + 2 | H | 4k + 2 | m |
| 8k + 6 | 4k + 3 | C | 4k + 3 | m |
| 8k + 7 | 4k + 3 | R | 4k + 2 | m - 1 |

The existence of such representations (and their construction) is proved in Adams’ book [1]. These representations are tightly connected with the construction of exceptional Lie groups.
By two elements and that one of these two generators is the degree 4 element, say $y$.

Table II.

| $G$   | Spin($n$) | $m$ | rep. dim of rep. | deg $z = 2^k$ |
|-------|-----------|-----|-----------------|---------------|
| $F_4$ | Spin(9)   | 4   | $\Delta$       | 16            |
| $E_6$ | Spin(10)  | 5   | $\Delta^+$     | 16            |
| $E_7$ | Spin(12)  | 6   | $\Delta^+$     | 32            |
| $E_8$ | Spin(16)  | 8   | $\Delta^+$     | 128           |

With the following Table II, we summarize the information we need in the proof of Theorem 1.1.

**Proof of Theorem 1.1.** For $G = E_6$, $E_7$, by Proposition 2.1, we have

$$c_{16}(f_1^* g_0^* p_0) = u^{16}, \quad c_{32}(f_1^* g_2^* p_1) = u^{32}.$$  

By Proposition 3.1, these elements are indecomposable in $\text{Im} Bf_1^*$. So, $c_{16}(p_0)$, $c_{32}(p_1)$ are indecomposable in $H^*(BE_6), H^*(BE_7)$.

For $G = F_4$, $E_8$, let $\rho_4, \rho_8$ be decompositions of $\rho_1, \rho_8$, respectively. Then, by Proposition 2.1, we have

$$c_{16}(f_1^* g_1^* \rho_4, c) = u^{16}, \quad c_{128}(f_1^* g_8^* \rho_8, c) = u^{128}.$$  

These elements are decomposable in $\text{Im} Bf_1^*$. However, since $\rho_4, \rho_8$ are complexification of $\rho_1, \rho_8$, we have $w_{16}(f_1^* g_1^* \rho_4) = w^8$, $w_{128}(f_1^* g_8^* \rho_8) = u^{64}$. By Proposition 3.1, these elements are indecomposable in $\text{Im} Bf_1^*$. Hence, $w_{16}(\rho_1), w_{128}(\rho_8)$ are indecomposable in $H^*(BF_4), H^*(BE_8)$.

**Remark 3.1.** If it is shown that, using the Rothenberg-Steenrod spectral sequence, the generators of the mod 2 cohomology of $BG$ is generated by two elements and that one of these two generators is the degree 4 element, say $y$, then since $f_1^* (g_1^* (y)) = 0$, we have that $f_1^* \circ g_1^*$ is generated by $f_1^* (g_1^* (w_{16} (\rho_1))), f_1^* (g_1^*(c_{16} (\rho_1))), f_1^* (g_1^* (c_{32} (\rho_1))), f_1^* (g_1^* (w_{128} (\rho_8)))$, respectively. Thus, we do not need to refer the reader to Quillen’s computation of the mod 2 cohomology of $BS\text{Spin}(n)$ to complete the proof of Theorem 1.1. This is the case for $G = F_4, E_6, E_7$ and, if, for some $r, E_r$-term of the spectral sequence converging to the mod 2 cohomology of $BE_8$ is also generated by two elements as an algebra over the Steenrod algebra, this argument is also applicable to the case $G = E_8$.

**Acknowledgement.** The author is partially supported by the Japan Society for the Promotion of Science, Grant-in-Aid for Scientific Research (C) 22540102.

**References**

[1] J. F. Adams, *Lectures on exceptional Lie groups*, Chicago Lectures in Mathematics, Univ. Chicago Press, Chicago, IL, 1996.

[2] M. Kameko and N. Yagita, Chern subrings, Proc. Amer. Math. Soc. 138 (2010), no. 1, 367–373.

[3] A. Kono and M. Mimura, Cohomology mod 2 of the classifying space of the compact connected Lie group mod 2, J. Pure Appl. Algebra 6 (1975), 61–81.

[4] A. Kono, M. Mimura and N. Shimada, On the cohomology mod 2 of the classifying space of the 1 connected exceptional Lie group $E_8$, J. Pure Appl. Algebra 8 (1976), no. 3, 267–283.

[5] A. Kono, A note on the Stiefel-Whitney classes of representations of exceptional Lie groups, J. Math. Kyoto Univ. 45 (2005), no. 1, 217–219.

[6] M. Mimura and N. Nishimoto, On the Stiefel-Whitney classes of the representations associated with Spin(15), in *Proceedings of the School and Conference in Algebraic Topology*, Geom. Topol. Monogr., 11, Geom. Topol. Publ., Coventry, 2007, pp. 141–176.

[7] D. Quillen, The mod 2 cohomology rings of extraspecial 2-groups and the spinor groups, Math. Ann. 194 (1971), 197–212.

[8] B. Schuster and N. Yagita, Transfers of Chern classes in BP-cohomology and Chow rings, Trans. Amer. Math. Soc. 353 (2001), no. 3, 1039–1054 (electronic).

[9] H. Toda, Cohomology of the classifying space of exceptional Lie groups, in *Manifolds—Tokyo 1973 (Proc. Internat. Conf., Tokyo, 1973)*, 265–271, Univ. Tokyo Press, Tokyo, 1975.