Electronic and dynamic properties of a one-dimensional Thue-Morse chain

Gi-Yeong Oh
Department of Basic Science, Ansung National University, Kyonggi-do 456-749, Korea
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We study the electronic and dynamic properties of a one-dimensional Thue-Morse chain within the framework of the transfer model. By means of direct diagonalization of the Hamiltonian matrix we first show that the trace map of the transfer matrices is exactly the same as that in the diagonal model. Then, by calculating several quantities such as the wave function, the Lyapunov exponent, and the Landauer resistivity, we show that all the electronic states are extended despite the singular continuity of the energy spectrum. Our results indicate that the electronic properties of the Thue-Morse chain is independent of the kind of the model, which is contrary to the result of Chakrabarti et al. [Phys. Rev. Lett. 74, 1403 (1995)]. To deepen our understanding, we study the dynamics of an electronic wave packet and show that the wave packet spreads superdiffusively for long times with dynamic indices intermediate between periodic and quasiperiodic chains. Several features of dynamics distinctive from the case of the Fibonacci chain are discussed. We also study the effects of electron-phonon interaction on the dynamics of the wave packet by taking into account a kind of nonlinear interaction. Degree of dynamic localization is shown to be crucially dependent on the strength of the hopping energy and the nonlinear interaction parameter.

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1. INTRODUCTION

Since the discovery of the quasicrystalline phase\(^1\) much attention has been devoted to the quasiperiodic systems. The Fibonacci chain, a one-dimensional version of quasicrystal characterized by either the inflation rule \((L, S) \rightarrow (LS, L)\) or the substitution rule \(S_{l+1} = S_lS_{l-1}\) with \(S_1 = \{L\}\), is the most intensively studied example, and new concepts on the electronic properties such as the singular continuous energy spectrum and the critical states are now well established.\(^2\) Stimulated by unusual electronic properties of the Fibonacci chain as well as the development of layer-growth technique,\(^3\) there has been a variety of interest in one-dimensional deterministic aperiodic chains. Among them the most well-known example is the Thue-Morse chain characterized by either the inflation rule \((L, S) \rightarrow (LS, SL)\) or the substitution rule \(S_{l+1} = S_l\bar{S}_l\) with \(S_1 = \{L\}\).

Unusual electronic and Fourier spectral properties of the Thue-Morse chain are often compared with those of the Fibonacci chain. According to the nature of the Fourier spectrum of the sequence, the Thue-Morse sequence is more disordered than the Fibonacci sequence since the former has a singular continuous spectrum\(^4\) while the latter has a point-like spectrum with Bragg \(\delta\)-peaks. In the meanwhile, according to the nature of the electronic energy spectrum, both of the chains are located in the same category; both have the singular continuous energy spectrum, which is a general feature of one-dimensional deterministic aperiodic chains.\(^5\) However, according to the nature of the electronic states, the Thue-Morse chain is much closer to the periodic chain than the Fibonacci chain; all the electronic states of the latter are rigorously proven to be critical,\(^2\) while most of the electronic states of the former exhibit Bloch-like extended character.\(^6\)

In studying the electronic properties of one-dimensional deterministic aperiodic chains, it has been common to use the tight-binding equation

\[
t_{n+1,n} \psi_{n+1} + t_{n-1,n} \psi_{n-1} + v_n \psi_n = E \psi_n
\]

or

\[
\begin{pmatrix}
\psi_{n+1} \\
\psi_n
\end{pmatrix} = T_n
\begin{pmatrix}
\psi_{n-1} \\
\psi_n
\end{pmatrix}.
\]

Here \(v_n\) is the on-site energy of the site \(n\), \(t_{n\pm 1,n}\) is the nearest neighboring hopping energy between the sites \(n\) and \(n \pm 1\), and \(T_n\) is the transfer matrix.

When a sequence to be studied is introduced into \(v_n\) \((t_{nm})\) with setting \(t_{nm} = 1\) \((v_n = 0)\), it is called the diagonal (transfer) model. In the case of the Fibonacci chain, there is no preference in choosing the kind of the model and both of the models have been equally well studied. However, in the case of the Thue-Morse chain, most of the existing literatures have been devoted to the diagonal model rather than the transfer model because the transfer matrix \(M_l = \Pi_{n=1}^N T_n\) appeared in the transfer model is so complex that the trace map method which is the most powerful tool in studying quasiperiodic or aperiodic chains cannot be easily applied.\(^7\) Furthermore, the complexity of the transfer matrix in the transfer model also leads a literature that contains incorrect argument on the electronic properties of the Thue-Morse chain: Chakrabarti et al.\(^8\) argued that the transfer model does not support the extended states unlike to the case of the diagonal model. However, we argue that the electronic properties of the Thue-Morse chain are independent of the kind of the model. To certify our argument, we show that the trace map of the transfer matrices in the transfer...
model is exactly the same as that in the diagonal model and that all the electronic states except the two edge states are extended.

It has been generally believed that the electronic states corresponding to a singular continuous energy spectrum are critical. Of course, in some cases such as the generalized Fibonacci chain and the period-doubling chain, existence of extended states has been reported. However, even in these cases, the portion of the extended states to the critical states goes to zero in the infinite limit of the chain size, and the global electronic properties of the chains are qualitatively similar to those of the Fibonacci chain. In this context, the Thue-Morse chain is very exceptional since most of the electronic states exhibit extended nature despite of the singular continuity of the energy spectrum, and it may be interesting to investigate the phenomena resulting from these peculiar electronic properties. By the way, one of the interesting properties of the chains with a singular continuous energy spectrum lies in the quantum dynamics of an electronic wave packet. The spread of the wave packet in the Fibonacci chain is known to be anomalous power-law diffusion; the spread can be diffusive, subdiffusive, or superdiffusive, depending on the strength of the system parameters. Since the nature of the energy spectrum is known to play an important role in quantum dynamics of the wave packet, a chain with the singular continuous spectrum is expected to exhibit similar dynamic property to that of the Fibonacci chain. Then, it may be worthwhile to test whether the dynamic property of the Thue-Morse chain is similar to that of the Fibonacci chain or not. We try to answer this question and discuss several characteristics distinctive from the case of the Fibonacci chain.

Since Eq. (1) is the Schrödinger equation for a single electron, effects of electron-electron or electron-phonon interactions, that are inevitable in real systems, are neglected. Thus it may be questionable whether the single-electron properties obtained from Eq. (1) can be observable or not when these interactions are present. As a step to solve this question, we take into account a kind of nonlinear interaction and study the effect of the interaction on the dynamics of the electronic wave packet. We show that degree of dynamic localization is crucially dependent on the strength of the hopping energy and the interaction parameter.

This paper is organized as follows: In Sec. II we deduce the trace map of the transfer matrices in the transfer model by directly diagonalizing the Hamiltonian matrix of the Thue-Morse chain. Then we elucidate the nature of the electronic states by calculating the wave functions, the Lyapunov exponents, and the Landauer resistivities of the states. In Sec. III we illustrate the dynamic properties of an electronic wave packet by observing the evolution of the time-dependent Schrödinger equation. And, in Sec. IV we present the effect of the nonlinear interaction on the dynamics of the wave packet. Finally a summary is given in Sec. V.

II. ELECTRONIC PROPERTIES OF THE THUE-MORSE CHAIN

As a first step to examine the electronic properties in the framework of the transfer model, we deduce the trace map of the transfer matrices. To do this, we first consider the characteristics of the energy eigenvalues obtained by directly diagonalizing an $N \times N$ Hamiltonian matrix under a periodic boundary condition. Figure 1 shows a plot of $E$ versus $l$, where $l$ is the order of the chain size ($N = 2^l$) and $T (= t_{ls}/t_S)$ is the strength of the hopping energy. From this, we obtain the following characteristics: (a) The eigenvalues are symmetric with respect to $E = 0$ and there always exists a state with $E = 0$. (b) All the eigenvalues except the two outermost eigenvalues are doubly degenerate, and there are $(2^l - 1)$ distinctive eigenvalues at the $l$th-order Thue-Morse chain. (c) For $l \geq 2$, all the eigenvalues at the $l$th-order chain remain as those at the $(l + 1)$th-order chain. (d) Additional eigenvalues at the $(l + 1)$th-order chain are located between the eigenvalues at the $l$th-order chain.

The trace map of the transfer matrices can be deduced as follows: First, by noting that property (c) implies the fact that $(x_l - 1)$ is an ingredient of $(x_{l+1} - 1)$, we can write $x_{l+1} - 1 = f(l)(x_l - 1)$, where $x_l$ is the half of the trace of $M_l$. Second, by means of properties (c) and (d), we draw the eigenvalue tree, as in Fig. 2. Since the tree resembles that of the diagonal model, we assume that $f(l)$ is a power of $x_l$. In order to check this assumption, we calculate the energies $\{E_{l-1}^t\}$ satisfying $x_{l-1} = 0$ to compare them with the energies $\{E_{l+1}\}$ satisfying $x_{l+1} = 1$. Then, we find that $\{E_{l-1}^t\}$ becomes a half part of $\{E_{l+1}\}$, i.e., $\{E_{l+1}\} = \{E_{l-1}^t\}$. Thirdly, by combining this result with property (b), we find that $f(l)$ is a square function of $x_{l+1}$, i.e., $f(l) = 4x_{l+1}^2$. Thus, the trace map of the transfer matrices becomes

$$x_{l+1} = 4x_{l-1}^2(x_l - 1) + 1$$

with $x_1 = [E^2 - (t_{ls}^2 + t_S^2)]/2t_{ls}t_S$ and $x_2 = E^2(2E^2 - 2(t_{ls}^2 + t_S^2))/2t_{ls}^2t_S^2 + 1$.

Using Eq. (3), we can calculate energy eigenvalues for any chain size. For example, let us consider the case of $T = 0.5$. Then, since we can easily obtain the energies $E_1 = \{\pm \sqrt{5}\}$, $E_2 = \{\pm \sqrt{5} \pm \sqrt{17}\}$, $E_2 = \{0, \pm \sqrt{10}\}$ and $E_3 = \{\pm \sqrt{25 \pm \sqrt{497}}/2\}$, we can write the energy eigenvalues up to $l = 5$ exactly:

$$E_3 = \{0, \pm \sqrt{5}, \pm \sqrt{10}\}, E_4 = \{0, \pm \sqrt{5}, \pm \sqrt{10}, \pm \sqrt{5} \pm \sqrt{17}\}, \text{ and } E_5 = \{0, \pm \sqrt{5}, \pm \sqrt{10}, \pm \sqrt{5} \pm \sqrt{17}, \pm \sqrt{25 \pm \sqrt{497}}/2\}.$$

We have checked that, for higher $l$, the energy eigenvalues obtained numerically by Eq. (3) are exactly the same as those obtained from direct diagonalization method within the numerical accuracy.
Before going on further, we would like to mention some remarks. The first is that Eq. (3) is the same form as that of the diagonal model. This is a very surprising result because the transfer model has a much more complex pattern of the transfer matrices compared with those of the diagonal model. The second is that we deduced Eq. (3) by means of a numerical calculation. Thus, the problem of a mathematically rigorous derivation of Eq. (3) still remains as an open problem. However, our purpose in this paper is not to solve this problem exactly, but to elucidate the nature of the electronic energy eigenvalues and the eigenstates, let us remain it open. The third is that, by using the branching rule, the number of any desired subband can be indexed in the same way as in Ref. 6. This means that one can also perform a scaling analysis on the energy bandwidths as in the case of the diagonal model, which is in contrast to the argument of Ref. 8.

From now on, we discuss the nature of the eigenstates with energies obtained by Eq. (3). To this end, let us first classify the eigenstates into two groups - the degenerate states and the edge states. Since Eq. (3) is exactly the same as that of the diagonal model except for the initial conditions $x_1$ and $x_2$, we affirm that the same behavior as that of the diagonal model also appears in the transfer model; degenerate eigenstates at the $l$th-order chain will exhibit an $m$th-order-type chain-like extended behavior in the wave functions at the $(l + m)$th-order chain. Numerical calculations verify our affirmation. We present some examples in Fig. 3. The only exceptions are the edge states with energies satisfying $x_2 = 1$; the wave functions diverge algebraically with increasing the chain size, as in the edge states of the periodic chain.

A little cumbersome calculation enables us to write $M_l$ as

$$ M_l = I $$

for the degenerate states and

$$ M_l = \begin{pmatrix} 1 + aN & bN \\ cN & 1 - aN \end{pmatrix} $$

for the edge states. Here, we have set $a = \tau^2/4$, $b = \mp\sqrt{(\tau^2/8)t_L} \sqrt{2(t_L^2 + t_S^2)}$, $c = \mp\sqrt{(2t_L/\tau)t_S}b$, and $r \equiv t_L/t_S + t_S/t_L$. Using Eqs. (4) and (5), we can obtain the Lyapunov exponent $\zeta$ defined by

$$ \zeta = \lim_{N \to \infty} \frac{1}{N} \ln ||M_l||, $$

where $||\cdot||$ is the modulus of the matrix. Thus, the localization length $\xi_N = \xi_N^{-1}$ for a chain size $N$ is given by $\xi_N = N/\ln 2$ for the degenerate states and $\xi_N \approx N/2\ln N$ for the edge states; $\xi_N$ goes to infinity in the infinite limit of the chain size, which indicates the absence of exponentially localized states.

As another test of the nature of the eigenstates, we calculate the Landauer resistivity. Assuming that a Thue-Morse chain of size $N$ is embedded in the middle of a perfect conductor and that an electron with $E = 2T \cos k$ comes in from the left-hand side of the perfect conductor, the dimensionless resistivity can be written as

$$ \rho(k, N) = \frac{1}{N} |U_{21}(k, N)|^2, $$

where $U_{21}(k, N)$ is the $(2, 1)$ component of the transfer matrix relating the coefficients of the incoming and reflecting plane waves to that of the transmitted plane wave. [See Ref. 13 for the detailed formalism.] Using Eqs. (4) and (5), we obtain that $|U_{21}(k_0, N)|^2 = \cot^2 k_0$ for even $l$ and $|U_{21}(k_0, N)|^2 = (t_S/t_L)^2 \cot^2 k_0$ for odd $l$, while $|U_{21}(k_1, N)|^2 \sim N^2$. Here, $k_0$ and $k_1$ are the wave numbers that correspond to the degenerate and the edge eigenvalues. Thus, we have

$$ \rho(k, N) \sim \begin{cases} 1/N & \text{for } k = k_0 \\ N & \text{for } k = k_1 \end{cases} $$

which indicates that the degenerate (edge) states are extended (algebraically localized).

### III. Dynamic Properties of the Thue-Morse Chain

To study the dynamic properties of the Thue-Morse chain, we consider the time-dependent Schrödinger equation

$$ i \frac{\partial \psi_n}{\partial t} = H_0 \psi_n $$

where

$$ H_0 \psi_n = t_{n+1,n} \psi_{n+1} + t_{n-1,n} \psi_{n-1} + v_n \psi_n. $$

In integrating Eq. (9), we assume an electronic wave packet initially at $\psi_n = \delta_{n,n_0}$ and observe the time evolution of the wave packet by employing the fourth-order Runge-Kutta algorithm. Besides, we take the chain size as $N = 16384$ (i.e., $l = 14$) with $n_0 = 8192$ and the time step as $\Delta t = 0.05$. Note that, even though we use the finite chain size and the fixed boundary condition ($\psi_0 = \psi_N = 0$), they have no effect on the results of the integration since we perform the integration within time interval where the wave front does not reach the boundary region. We also check the accuracy of the numerical integration by monitoring the conservation of the total probability (i.e., $\sum_n |\psi_n|^2 = 1$).

In clarifying the dynamic nature of the wave packet, the most important quantity is the variance of the wave packet

$$ V(t) = \sum_n (n - n_0)^2 |\psi_n(t)|^2, $$

which gives a global estimate of the spread of the wave packet in space. The asymptotic long-time behavior of the variance follows the power-law
\[ V(t) \sim t^\gamma. \] (12)

It is well known that \( \gamma = 0 \) for the localization, \( 0 < \gamma < 1 \) for the subdiffusion, \( \gamma = 1 \) for the ordinary diffusion, \( 1 < \gamma < 2 \) for the superdiffusion, and \( \gamma = 2 \) for the ballistic motion, respectively.

Figures 4 and 5 show the results of \( V(t) \) for several values of the system parameters \( T \) and \( v (\equiv v_L = -v_S) \). The curves in Fig. 4 (5) are obtained within the framework of the transfer (diagonal) model. For the curves in Fig. 4, we obtain the dynamic index \( \gamma \) as \( \gamma = 1.942 \pm 0.394 \) for \( T = 0.125 \), \( \gamma = 1.931 \pm 0.103 \) for \( T = 0.2 \), \( \gamma = 1.924 \pm 0.039 \) for \( T = 0.5 \), \( \gamma = 1.998 \pm 0.001 \) for \( T = 1.0 \), \( \gamma = 1.853 \pm 0.028 \) for \( T = 1.25 \), \( \gamma = 1.924 \pm 0.044 \) for \( T = 2.0 \), and \( \gamma = 1.840 \pm 0.078 \) for \( T = 5.0 \), respectively. Note that the standard deviation in the calculated \( \gamma \)'s becomes larger as the value of \( T \) deviates from \( T = 1.0 \), which means that the oscillatory behavior becomes stronger as \( T \to \infty \) and \( T \to 0 \). However, from the trend of the curves, we expect that the oscillatory behavior will shrink to zero for sufficiently long times. Concerning with the oscillatory behavior, we would like to mention a point: Comparing Fig. 4 (5) with Fig. 2 (6) of Ref. 11, we can see that there is no strong hierarchical time evolution of \( V(t) \) unlike to the case of the Fibonacci chain. This feature may attribute to the weak self-similarity in the energy spectrum of the Thue-Morse chain.

Recently, de Brito et al.\textsuperscript{14} argued with the diagonal model of the Thue-Morse chain that \( \gamma \) is independent of the strength of \( v \) and is given by \( \gamma = 1.65 \). On the contrary, Katsanos et al.\textsuperscript{15} argued with the same model that \( \gamma \) is strongly dependent on the strength of \( v \) and given by \( \gamma = 1.5 \) for large values of \( v \). However, judging from our results, both of the arguments are partly correct. As can be seen in Figs. 4 and 5, \( \gamma \) depends on the system parameters \( T \) and \( v \) to a certain extent unlike to the result of Ref. 14. However, the dependence seems not to be strong unlike to the argument of Ref. 15. Note that this is another feature of the Thue-Morse chain distinctive from the Fibonacci chain: In the case of the Fibonacci chain, \( \gamma \) decreases continuously from 2 to 0 as the system parameters increase, and the diffusion of the wave packet is known to be either superdiffusive or subdiffusive, depending upon whether \( T^{-1} \leq 4 \) or \( T^{-1} \geq 4 \).\textsuperscript{11} However, there seems to be no such critical value of \( T \) in the Thue-Morse chain and the spread of the wave packet seems to be always superdiffusive: \( 1.8 \leq \gamma < 2.0 \) for \( T > 1 \) and \( 1.9 \leq \gamma < 2.0 \) for \( T < 1 \) in the transfer model and \( \gamma \approx 1.50 \sim 1.65 \) for the diagonal model. The most important point to be noted is the fact that the obtained \( \gamma \)'s are intermediate between the value in the perfect periodic chain (\( \gamma = 2 \)) and the value in the metallic regime of the Harper's model or three-dimensional disordered lattices (\( \gamma = 1 \)). This feature attributes to the unusual electronic properties of the Thue-Morse chain; \( \gamma > 1 \) reflects the extended nature of the electronic states, while \( \gamma < 2 \) reflects the singular continuity of the energy spectrum.

Another important quantity that characterizes the dynamics of the wave packet is the initial-site probability defined by \( R(t) = |\psi_{n}(t)|^2 \), which has been widely used in investigating the self-trapping transition of the wave packet.\textsuperscript{16} However, since the fluctuation in \( R(t) \) is too large to extract characteristic features of the dynamics, we calculate the temporal autocorrelation function\textsuperscript{17}

\[ C(t) = \frac{1}{t} \int_0^t R(t')dt', \] (13)

which is the time-averaged quantity of the initial-site probability. The decay of \( C(t) \) for long times is known to follow the power-law

\[ C(t) \sim t^{-\delta}. \] (14)

The exponent \( \delta \) approaches 1.0 for long times in the periodic chain, while it goes to 0 in the disordered chain. Figure 6 shows the results of \( C(t) \) for several values of \( T \) in the transfer model, where the power-law decaying behavior of \( C(t) \) is clearly seen; \( \delta \approx 0.897 \) for \( T = 1.0 \), \( \delta \approx 0.313 \) for \( T = 1.25 \), \( \delta \approx 0.214 \) for \( T = 2.0 \), and \( \delta \approx 0.114 \) for \( T = 5.0 \), respectively. The exponent \( \delta \) decreases with increasing \( T \), which implies that the spread of wave packet becomes slower as \( T \) increases.

We also calculate the participation number

\[ P(t) = \left\{ \sum_n |\psi_n(t)|^4 \right\}^{-1}, \] (15)

which gives a rough estimate of the number of sites where the wave packet has a significant amplitude; \( P(t) = 1 \) for a single-localized state and \( P(t) = N \) for a state uniformly extended over \( N \) sites. Figure 7 shows the results of \( P(t) \) for several values of \( T \) in the transfer model. \( P(t) \) increases almost monotonically for \( T = 1.0 \), while it increases with strong oscillation for \( T > 1.0 \). The oscillation in \( P(t) \) increases and the values of \( P(t) \) rapidly decreases with increasing \( T \). The former reflects the singular continuity of the energy spectrum and the latter implies that it becomes more difficult for a wave packet to propagate.

**IV. NONLINEAR INTERACTION AND THE DYNAMICS**

To study the effect of the electron-phonon interaction on the dynamics of the wave packet we take into account a kind of nonlinear interaction\textsuperscript{18} and consider the time-dependent Schrödinger equation

\[ i \frac{\partial \psi_n}{\partial t} = (H_0 + H_1) \psi_n, \] (16)

where

\[ H_1 = -\alpha |\psi_n|^2 \psi_n. \] (17)
Here $\alpha$ gives the strength of the nonlinear interaction. Note that the nonlinear interaction we consider describes a static short-range electron-phonon interaction, which appears in many situations such as, for example, the Holstein model for polaron theory.\textsuperscript{19}

Dynamics of an electronic wave packet is known to be sensitively dependent on the nonlinear term in Eq. (16). In the case of periodic chains,\textsuperscript{20} there occurs self-trapping transition when $\alpha$ becomes larger than a certain critical value $\alpha_c$. And, in disordered chains,\textsuperscript{21} degree of dynamic localization weakens with the increase of $\alpha$. In the meanwhile, in the case of the Thue-Morse chain, Johansson \textit{et al.}\textsuperscript{12} argued with the diagonal model that self-trapping occurs for arbitrarily small values of $\alpha$ and that the variance of the wave packet increase infinitely with time. Thus it may be questionable whether the dynamics of the wave packet within the transfer model is different from the case of the diagonal model or not. We integrate numerically Eq. (16) with setting $v = 0$.

Figure 8 shows the variance of the wave packet for several values of $\alpha$. As in the case of the linear model ($\alpha = 0$), the oscillatory behavior shrinks to zero for long times. Thus the asymptotic long-time behavior of the wave packet is also characterized by a superdiffusive movement in spite of the existence of the nonlinearity, which resembles the cases of the linear transfer model and the nonlinear diagonal model.\textsuperscript{12} Besides, the dynamic index $\gamma$ is nearly independent of the strength of $\alpha$, which also resembles the case of the nonlinear diagonal model. The most interesting point to be noted is that the dependence of the variance on the strength of $\alpha$ for $T > 1$ is qualitatively different from that for $T < 1$: When $T > 1$, the variance decreases with increasing $\alpha$ [Fig. 8(a)], which indicates that the nonlinearity resists the propagation of the wave packet. In the meanwhile, when $T < 1$, the variance increases with increasing $\alpha$ [Fig. 8(b)], which indicates that the nonlinearity assists the propagation of the electronic wave packet. Our results show that degree of dynamic localization of the wave packet in the Thue-Morse chain crucially depends on the strength of the interaction.

\section*{V. SUMMARY}

In summary, we study the spectral and dynamic properties of a one-dimensional Thue-Morse chain mainly within the framework of the transfer model. By using the method of direct diagonalization of the Hamiltonian matrix, we first illustrate the energy spectral properties and deduce the trace map of the transfer matrices. The trace map is shown to be exactly the same as that in the diagonal model. Then we illustrate the nature of the electronic states by calculating the wave functions, the Lyapunov exponents, and the resistivities of the states. All the doubly degenerate eigenstates are shown to be extended, while the edge states are algebraically localized. Next, in the study of the dynamics, we show that the asymptotic long-time behavior of an electronic wave packet is superdiffusive with the dynamic indices $\gamma$ satisfying $1 < \gamma < 2$; $\gamma > 1$ reflects the extended nature of the electronic states, while $\gamma < 2$ reflects the singular continuity of the energy spectrum. We also show that the dependence of the dynamic index on the system parameters is much weaker than that of the Fibonacci chain and that there is no hierarchical time evolution of the variance unlike to the Fibonacci chain. Finally, by taking into account a kind of nonlinear interaction, we study the effect of the electron-phonon interaction on the dynamics of the wave packet and show that degree of the localization of the wave packet crucially depends on the strength of the interaction.

In spite of fundamental importance of electron-electron interaction, we did not take into account the interaction in this paper. However, note that, in connection with the problem of whether degree of localization of a single-electron weakens or strengthens in the presence of this interaction, much attention has been devoted to the chains with disordered\textsuperscript{22} and quasiperiodic\textsuperscript{23} potential energies. Thus, it is worthwhile to investigate the effect of electron-electron interaction on the electronic and dynamic properties of the Thue-Morse chain. This problem is on working and will be published elsewhere.

\section*{ACKNOWLEDGMENTS}

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Fig. 1. $E$ versus $l$ in the transfer model with $T = 0.5$.

Fig. 2. Schematic representation of the eigenvalue tree.

Fig. 3. $|\psi_n|$ versus $n$ for states with (a) $E = \sqrt{5}$ ($x_1 = 0$) and (b) $E = \sqrt{5 + \sqrt{17}}$ ($x_2 = 0$) with $T = 0.5$, $(\psi_0, \psi_1) = (0.5, 1)$, and $l = 6$. $|\psi_n|$ exhibits a third-order-type (i.e., $LSSLSLSL$-type) chain-like behavior for (a) and a second-order-type (i.e., $LSSL$-type) chain-like behavior for (b).

Fig. 4. $\log_{10} V(t)$ versus $\log_{10} t$ within the transfer model: (a) $T = 1.0, 1.25, 2.0, 3.0$, and $5.0$, and (b) $T = 1.0, 0.5, 0.2$, and $0.125$ from top to bottom at $t = 1000$.

Fig. 5. $\log_{10} V(t)$ versus $\log_{10} t$ within the diagonal model, where $v = 0.1, 0.5, 1.0, 2.0$, and $3.0$ from top to bottom at $t = 3000$.

Fig. 6. $\log_{10} C(t)$ versus $\log_{10} t$ within the transfer model, where $T = 1.0, 1.25, 2.0$, and $5.0$ from bottom to top at $t = 1000$.

Fig. 7. $P(t)$ versus $t$ within the transfer model, where $T = 1.0, 1.25$, and $2.0$ from top to bottom at $t = 200$.

Fig. 8. $\log_{10} V(t)$ versus $\log_{10} t$ within the transfer model: (a) $T = 5.0$ with $\alpha = 0, 0.2$, and $0.8$, and (b) $T = 0.2$ with $\alpha = 0.6, 0.4, 0.1$, and $0$ from top to bottom at $t = 1000$. 

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Fig. 1: Gi-Yeong Oh
Fig. 2: Gi-Yeong Oh
Fig. 3(b) : Gi-Yeong Oh
Fig. 4: Gi-Yeong Oh
Fig. 5: Gi-Yeong Oh
Fig. 6: Gi-Yeong Oh
Fig. 7: Gi-Yeong Oh
Fig. 8: G. Yeong Oh