WARNING PROPAGATION ON RANDOM GRAPHS

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ABSTRACT. Warning Propagation is a combinatorial message passing algorithm that unifies and generalises a wide variety of recursive combinatorial procedures. Special cases include the Unit Clause Propagation and Pure Literal algorithms for satisfiability as well as the peeling process for identifying the $k$-core of a random graph. Here we analyse Warning Propagation in full generality on the binomial random graph. We prove that under a mild stability assumption Warning Propagation converges rapidly. In effect, the analysis of the fixed point of the message passing process on a random graph reduces to analysing the process on a Galton-Watson tree. This result corroborates and generalises a heuristic first put forward by Pittel, Spencer and Wormald in their seminal $k$-core paper (JCTB 1996).

1. INTRODUCTION

1.1. Motivation and contributions. Warning Propagation is a message passing scheme that generalises several combinatorial algorithms that have been studied extensively in the combinatorics literature [19]. For instance, Unit Clause Propagation is an algorithm for tracing implications in a Boolean satisfiability problem [1, 12]. So long as the formula only contains clauses comprising two or more literals, Unit Clause Propagation sets a random variable to a random truth value. But once unit clauses emerge that contain one literal only, the algorithm pursues their direct implications. Clearly, a unit clause imposes a specific value on its sole variable, but the buck need not stop there. For setting the variable so as to satisfy the unit clause may effectively shorten other clauses where that variable appears with the opposite sign. Hence, additional unit clauses may emerge that demand further variables to take specific values. Thus, to understand the performance of Unit Clause Propagation on random $k$-SAT instances, we need to track this recursive process carefully. In particular, we need to get a grip on the fixed point of the propagation process to determine if it will lead to contradictions where different unit clauses end up imposing opposite values on a variable.

A second well known special case of Warning Propagation is the peeling process for finding the $k$-core of a random graph [23]. Once again this is a recursive process with a propensity to trigger avalanches. Launched on, for example, a sparse Erdős-Rényi graph, the peeling process recursively removes vertices of degree less than $k$. Of course, removing such a vertex may reduce the degrees of other vertices, thus potentially creating new low degree vertices. The $k$-core of the random graph is defined as the fixed point of this process, i.e., the final graph when no further removals occur. Similar recursive processes occur in many other contexts. Well-known examples include the study of sparse random matrices, the freezing phase transition in random constraint satisfaction problems, bootstrap percolation or decoding low-density parity check codes [2, 6, 9, 13, 21, 24].

Mathematical analyses of such recursive processes have thus far been conducted on a case-by-case basis by means of very different techniques. A classical tool has been the method of differential equations [26], used, for instance, in the original $k$-core paper [23] as well as in the analysis of Unit Clause Propagation [1]. Other tools that have been brought to bear include branching processes [25], enumerative methods [5], or birth-death processes [15, 17]. In any case, the theorems that these methods were deployed to prove always posit, in one way or another, that Warning Propagation converges quickly. In other words, the ultimate outcome of the recursive process can be approximated arbitrarily accurately by just running the recursive process for a bounded number of parallel iterations. This notion was actually explicated as a heuristic derivation of the $k$-core threshold [23].

The Warning Propagation message passing scheme unifies all of the aforementioned recursive procedures, and many more. An in-depth discussion in the context of general message passing algorithms can be found in [19]. But roughly speaking, Warning Propagation allows for a general set of rules defining how labels associated with the vertices of a graph can be updated recursively. These labels could be truth values as in the case of the satisfiability

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problem, or merely indicate whether a particular vertex has been peeled from the graph. They could also, in the context of graph colouring, represent a particular colour that a vertex has been assigned.

In this paper we analyse Warning Propagation in full generality on random graphs. We prove that under mild, easy-to-check assumptions Warning Propagation converges rapidly. Not only does this result corroborate the intuitive notion that running Warning Propagation for a bounded number of rounds suffices to approximate its ultimate fixed point arbitrarily well, the result also crystallises the key reasons for this principle. Specifically, we show that under a mild stability assumption the reason for the rapid convergence of Warning Propagation is that the changes that ensue after a sufficiently large but bounded number of iterations form a sub-critical branching process. Thus, late changes are, quite literally, few and far between; more formally, they are locally confined and essentially independent.

Apart from re-proving known results in a new, unified way, the main results of the paper facilitate new applications of Warning Propagation. Indeed, to analyse any specific recursive process that can be translated into the formalism of Theorem 1.2 below one just needs to investigate the recursion on a Galton-Watson tree that mimics the local structure of the respective random graph model. Typically this task boils down to a mundane fixed point problem in Euclidean space. Theorem 1.2 thus enables an easy and accurate analysis of generic recursive processes on random structures. A concrete example that actually inspired this work was our need to study a point problem in Euclidean space. Theorem 1.2 thus enables an easy and accurate analysis of generic recursive

1.2. Warning propagation. We proceed to define the Warning Propagation (WP) message passing scheme and its application to random graphs. Applied to a graph $G$, Warning Propagation will associate two directed messages $\mu_{v \to w}, \mu_{u \to w}$ with each edge $vw$ of $G$. These messages take values in a finite alphabet $\Sigma$. Hence, let $\mathcal{M}(G)$ be the set of all vectors $\{\mu_{v \to w} \mid (v, w) \in E(G), w \neq u\}$. The messages get updated in parallel according to some fixed rule. To formalise this, let $\left(\frac{\Sigma}{k}\right)$ be the set of all $k$-ary multisets with elements from $\Sigma$ and let

$$\phi : \bigcup_{k \geq 0} \left(\frac{\Sigma}{k}\right) \to \Sigma$$

be an update rule that, given any collection of input messages, computes an output message. Then we define the Warning Propagation operator on $G$ by

$$\text{WP}_G : \mathcal{M}(G) \to \mathcal{M}(G),$$

where $\|a_1, \ldots, a_k\|$ denotes the multiset whose elements (with multiplicity) are $a_1, \ldots, a_k$.

In words, to update the message from $v$ to $w$ we apply the update rule $\phi$ to the messages that $v$ receives from all its other neighbours $u \neq w$. In most applications of Warning Propagation the update rule $\phi$ enjoys a monotonicity property that ensures that for any graph $G$ and for any initialisation $\mu(0) \in \mathcal{M}(G)$ the pointwise limit $\text{WP}_G^\infty(\mu(0)) := \lim_{t \to \infty} \text{WP}_G^t(\mu(0))$ exists. If so, then clearly this limit is a fixed point of the Warning Propagation operator.

Our goal is to study the fixed points of WP and, particularly, the rate of convergence on the binomial random graph $G = G(n, d/n)$ for a fixed $d > 0$. Locally $G(n, d/n)$ has the structure of a Po(d) Galton-Watson tree. We are going to prove that under mild assumptions on the update rule the WP fixed point can be characterised in terms of this local structure only. To this end we need to define a suitable notion of WP fixed point on a random tree. Instead of actually engaging in measure-theoretic gymnastics by defining measurable spaces whose points are (infinite) trees, thanks to the recursive nature of the Galton-Watson tree we can take a convenient shortcut. Specifically, our fixed point will just be a probability distribution on $\Sigma$ such that, if the children of a vertex $v$ in the tree send messages independently according to this distribution, then the message from $v$ to its own parent will have the same distribution. The following definition formalises this idea.

**Definition 1.1.** For a probability distribution $p$ on $\Sigma$ let $\text{Po}(d,p)$ be the distribution of a multiset with elements from $\Sigma$ in which $\sigma \in \Sigma$ appears $\text{Po}(d,p(\sigma))$ times, independently for all $\sigma$. Further, let $\phi_p(p) = \phi_p(d,p) = \phi(\text{Po}(d,p))$ be image of the distribution $\text{Po}(d,p)$ under the Warning Propagation update rule $\phi$.

(i) We call $p$ a distributional WP fixed point if $\phi_p(p) = p$.

(ii) The fixed point $p$ is stable if $\phi_p$ is a contraction on a neighbourhood of $p$ with respect to total variation distance.

(iii) We say that $p$ is the stable WP limit of a distribution $q_0$ if $p$ is a stable fixed point, and furthermore the limit

$$\phi_p^*(q_0) := \lim_{t \to \infty} \phi_p^t(q_0)$$

exists and equals $p$. 

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Given a probability distribution $q_0$ on $\Sigma$, we ask how quickly Warning Propagation will converge on $G$ from a random initialisation where for all edges $vw$ the messages $\mu_{0,w,v}^{0}$, $\mu_{v,w}^{0}$ are chosen from $q_0$ independently. In many applications, the initialisation of the messages is actually deterministic, i.e., $q_0$ is concentrated on a single element of $\Sigma$. We will use $WP_{v,w}^{t}(\mu^{0})$ to denote the message from $v$ to $w$ in $G$ after $t$ iterations of $WP_{G}$ with initialisation $\mu^{0}$. Note that the graph $G = G(n,d/n)$ is implicit in this notation.

**Theorem 1.2.** Suppose that $p$, $q_0$ are probability distributions on $\Sigma$ such that $p$ is the stable limit of $q_0$. Suppose further that $\mu^{0} \in M(G(n,d/n))$ is an initialisation in which each message is chosen at random according to $q_0$ independently for each directed edge. Then for any $\delta > 0$ there exists $t_0 = t_0(\delta, d, p, q_0)$ such that w.h.p. for all $s, t > t_0$ we have

$$\sum_{v,w: v \rightarrow w \in E(G)} 1\{WP_{v,w}^{t}(\mu^{0}) \neq WP_{v,w}^{s}(\mu^{0})\} < \delta n.$$ 

Theorem 1.2 shows that under a mild stability condition Warning Propagation converges rapidly. Crucially, the number $t_0$ of steps before Warning Propagation stabilises does not depend on the order $n$ of the graph but only on the desired accuracy $\delta$, the average degree $d$, the Warning Propagation update rule and the initial distribution $q_0$.

1.3. **Discussion.** Theorem 1.2 implies a number of results that were previously derived by separate arguments. For instance, the theorem directly implies the main result from [23] on the $k$-core threshold as well as the typical number of vertices $k$.

**Proof outline.** A fundamental aspect of the proof is that we do not analyse $WP$ directly on $G$ and consider its effect after $t_0$ iterations, but instead define an alternative random model $\hat{G}_{t_0}$ (see Definition 2.5). Rather than generating the edges of the graph and then computing messages, this random model first generates half-edges with messages, and then matches up the half-edges in a consistent way. Thus in particular the messages are known a priori. The crucial observation is that the two models are very similar (Lemma 2.6). Among other things, it follows from this approximation that very few changes will be made when moving from $WP_{G}^{t_0}(\mu^{0})$ to $WP_{\hat{G}}^{t_0}(\mu^{0})$, but in principle these few changes could cause cascade effects later on.

To rule this out we define a branching process $\mathcal{F}$ which approximates the subsequent effects of a single change at time $t_0$. The crucial observation is that the stability of the distributional fixed point implies that this branching process is subcritical (Proposition 5.3), and is therefore likely to die out quickly. Together with the fact that very few changes are made at step $t_0$, this ultimately implies that there will be few subsequent changes.

1.5. **Paper overview.** The remainder of the paper is arranged as follows. In Section 2 we define the $\hat{G}_{t_0}$ model and state Lemma 2.6 which states that it is a good approximation for Warning Propagation on $G$. In Section 3 we present various preliminary results that will be used throughout the paper. In Section 4 we go on to prove Lemma 2.6.

In Section 5 we introduce the branching process $\mathcal{F}$, and prove that it is subcritical. In Section 6 we then draw together the results of previous sections to prove that after $t_0$ iterations of $WP$, very few further changes will be made, and thus prove Theorem 1.2.

In the appendices we present some technical proofs which are omitted from the main body for better readability.
2. AN ALTERNATIVE MODEL

Although our main result is primarily a statement about \( G = \mathcal{G}(n, d/n) \), a key method in this paper is to switch from this model to a second model which is easier to analyse. To introduce this second model, we need some more definitions.

2.1. Message histories. Let \( \mathcal{G}_n \) denote the set of \( \Sigma \)-message graphs on vertex set \([n]\), i.e. graphs on \([n]\) in which each edge \( uv \) comes equipped with directed messages \( \mu_{u \rightarrow v}, \mu_{v \rightarrow u} \in \Sigma \).

We will denote by \( \mu_{u \rightarrow v}(t) \) the message from \( u \) to \( v \) after \( t \) iterations of WP, and refer to this as the \( t \)-message from \( u \) to \( v \). Alternatively, we refer to the \( t \)-in-message at \( v \) or the \( t \)-out-message at \( u \) (this terminology will be especially helpful later when considering half-edges). In all cases, we may drop \( t \) from the notation if it is clear from the context.

In fact, we will need to keep track not just of the current Warning Propagation messages along each edge, but of the entire history of messages. For two adjacent vertices \( u, v \), define the \( t \)-story from \( u \) to \( v \) to be the vector

\[
\mu_{u \rightarrow v}^{(t)} := (\mu_{u \rightarrow v}(0), \ldots, \mu_{u \rightarrow v}(t)) \in \Sigma^t.
\]

We will also refer to \( \mu_{u \rightarrow v}^{(t)} \) as the \( t \)-in-message at \( v \), and as the \( t \)-out-message at \( u \). The \( t \)-story at \( v \) consists of the pair \((\mu_{u \rightarrow v}^{(t)}, \mu_{v \rightarrow u}^{(t)})\), i.e. the \( t \)-in-story followed by the \( t \)-out-story. It will sometimes be more convenient to consider the sequence consisting of the \( t \)-in-story followed by just the \( 0 \)-out-message, which we call the \( t \)-input. In all cases, we may drop \( t \) from the notation if it is clear from the context.

We denote by \( \mathcal{G}_n^{(t)} \) the set of \( \Sigma^t \)-message graphs on vertex set \([n] \) - the labels along each directed edge, which come from \( \Sigma^t \), will be the \( t \)-stories. (Note that the definition of \( \mathcal{G}_n^{(t)} \) makes no assumption that the stories along directed edges arise from running Warning Propagation – in principle, they could be entirely inconsistent – although of course in our applications, this will indeed be the case.)

**Definition 2.1.** For any \( t \in \mathbb{N} \) and probability distribution \( q_0 \) on \( \Sigma \), let \( G_t = G_t(n, d, q_0) \in \mathcal{G}_n^{(t)} \) be the random \( \Sigma^t \)-message graph produced as follows.

1. Generate the random graph \( G = \mathcal{G}(n, d/n) \).
2. Initialise all directed messages \( \mu_{*, \rightarrow *}(0) \) independently at random according to \( q_0 \).
3. Run Warning Propagation for \( t \) rounds according to update rule \( \varphi \).
4. Label each directed edge \((u, v)\) with the story \((\mu_{u \rightarrow v}(0), \ldots, \mu_{u \rightarrow v}(t))\) up to time \( t \).

We also define \( G : = \lim_{t \to \infty} G_t \), if this limit exists.

In fact, a key method in this paper is to move away from looking at \( G_t \) and instead consider a random graph model in which we first generate half-edges at every vertex, complete with stories in both directions, and only subsequently reveal which half-edges are joined to each other; thus we construct a graph in which the WP messages are known a priori. The trick is to do this in such a way that the resulting random message graph looks similar to \( G_t \).

In order to define this random model, we need a way of generating a story randomly, but accounting for the fact that the entries of a story are, in general, heavily dependent on each other.

**Definition 2.2.** Let \( T \) denote a Po\( (d) \) Galton-Watson branching process and let \( q \) be a probability distribution on \( \Sigma \). We define random variables \( X_0, X_1, X_2, \ldots \) as follows.

Let \( T \) be a randomly generated instance of \( T \), let \( u \) be the root of \( T \) and let us add an additional parent \( v \) of \( u \) to \( T \) to create \( T^* \). Initialise all messages in \( T^* \) independently at random according to \( q \). Now for each \( t \in \mathbb{N}_0 \), let \( X_t := \mu_{u \rightarrow v}(t) \) be the message from \( u \) to \( v \) after \( t \) iterations of Warning Propagation according to the update rule \( \varphi \).

Finally, let \( \Phi_{\varphi}^t(q) \) denote the distribution of \((X_0, \ldots, X_t)\), which is a probability distribution on \( \Sigma^t \).

Note that while it is intuitively natural to expect that \( X_t \) has the distribution of \( \Phi_{\varphi}^t(q) \), which motivates the similarity of the notation, this fact is not completely trivial. We will therefore formally prove this in Claim \[3.1\].

2.2. The random construction. We define the \( t \)-in-compilation at a vertex \( v \) to be the multiset of \( t \)-inputs at \( v \), and the \( t \)-in-compilation sequence is the sequence of \( t \)-in-compilations over all vertices of \([n]\). As before, we often drop the parameter \( t \) from the terminology when it is clear from the context.

We can now define the alternative random graph model to which we will switch our focus.
Definition 2.3. Given a probability distribution $q_0$ on $\Sigma$ and an integer $t_0$, let us define a random messaged graph $\hat{G}_{t_0} = G_{t_0}(n, d, q_0)$ on vertex set $[n]$ constructed by applying the following steps.

1. For each vertex in $[n]$ independently, generate an $in$-compilation by:
   (a) Generating Po($d$) half-edges;
   (b) Giving each half-edge a $t_0$-in-story according to $\phi_{\Sigma}^{t_0}(q_0)$ independently;
   (c) Giving each half-edge a 0-out-message according to $q_0$ independently of each other and of the in-stories.
2. Generate $t$-out-messages for each time $1 \leq t \leq t_0$ according to the rules of Warning Propagation based on the $(t-1)$-in-messages, i.e. if the $t_0$-in-stories at $v$, from dummy neighbours $u_1, \ldots, u_j$, are $\mu_{u_1 \to v}(\leq t_0)$, we set
   \[ \mu_{v \to u_i}(t) = \varphi(\{(\mu_{u_1 \to v}(t-1), \ldots, \mu_{u_{i-1} \to v}(t-1), \mu_{u_{i+1} \to v}(t-1), \ldots, \mu_{u_j \to v}(t-1))\}) . \]
3. Condition on the statistics matching, i.e. on having the same number of half-edges with in-story $\mu_1$ and out-story $\mu_2$ as half-edges with in-story $\mu_2$ and out-story $\mu_1$ for every $\mu_1, \mu_2 \in \Sigma^{t_0+1}$, and on the number of half-edges with both in-story and out-story $\mu$ being even for every $\mu \in \Sigma^{t_0+1}$.
4. Choose a matching of half-edges uniformly at random subject to the consistency condition that a half-edge with in-story $\mu_1^{(1)} \in \Sigma^{t_0+1}$ and out-story $\mu_2^{(2)} \in \Sigma^{t_0+1}$ is matched to a half-edge with in-story $\mu_1^{(2)}$ and out-story $\mu_2^{(1)}$, and subject to the condition that the resulting graph is simple.

We will show later (Claims 3.2) that the distribution of an out-story is identical to the distribution, which makes the conditioning on matching statistics seem at least plausible. We will also later prove a result (Lemma 3.3) which formalises this intuition, and implies that the event that we condition on is not too unlikely (in particular at least polynomial).

Remark 2.4. Note that Step (2) of the construction is an entirely deterministic one – the $t$-out-messages at time $t \geq 1$ are fixed by the incoming messages at earlier times. Therefore all in-stories and out-stories are in fact determined by the outcome of the random construction in Step (1), and in particular the conditioning in Step (3) may be viewed as a restriction on the random variables in Step (1) (although we needed Step (2) in order to describe this conditioning).

2.3. Main lemma. Observe that $\hat{G}_{t_0}$ and $G_{t_0}$ both define random variables in $\mathcal{G}_{t_0}$. With a slight abuse of notation, we also use $\hat{G}_{t_0}$ and $G_{t_0}$ to denote the distribution of the respective random variables.

Given a $\Sigma^{t+1}$-messaged graph $G \in \mathcal{G}_{t_0}$, we will denote by $\overline{G}$ the $\Sigma$-messaged graph in $\mathcal{G}_n$ obtained by removing all messages from each history except for the message at time $t$, i.e. the “current” message.

There are two main steps in the proof of Theorem 1.2.

1. Show that $\hat{G}_t$ and $G_t$ have similar distributions for any constant $t \in \mathbb{N}$, i.e. we can construct the random graph in such a way that the messages up to time $t$ are known a priori (Lemma 2.5).
2. Use this approximation to show that, for some large constant $t_0 \in \mathbb{N}$, the messaged graphs $\overline{\hat{G}}_{t_0}$ and $\overline{G}_{t_0}$ are also very similar, i.e. very few further changes are made after $t_0$ steps of Warning Propagation.

In particular, we must certainly choose $t_0$ such that $\phi_{\Sigma}^{t_0}(q_0)$ is very close to the fixed point $p$. It will follow that the distribution of a message along a randomly chosen directed edge in $\overline{\hat{G}}_{t_0}$ (and therefore also in $\overline{G}_{t_0}$) is approximately $p$ (see Claim 3.4).

We need a way of quantifying how “close” two graphs are to each other. Given sets $A$ and $B$, we use $A \Delta B := (A \setminus B) \cup (B \setminus A)$ to denote the symmetric difference.

Definition 2.5. Given two $\Sigma$-messaged graphs $G_1, G_2 \in \mathcal{G}_n$ and $\delta > 0$, we say that $G_1 \sim_{\delta} G_2$ if:

1. $E(G_1) \Delta E(G_2) \leq \delta n$;
2. The messages on $E(G_1) \cap E(G_2)$ in the two graphs agree except on a set of size at most $\delta n$.

We further say that $G_1 \approx_{\delta} G_2$ if in fact the underlying graphs are identical (i.e. $E(G_1) \Delta E(G_2) = \emptyset$).

Our main lemma is now the following.

Lemma 2.6. For any integer $t_0 \in \mathbb{N}$ and real number $\delta > 0$, the random $\Sigma^{t_0+1}$-messaged graphs $\hat{G}_{t_0}, G_{t_0}$ can be coupled in such a way that w.h.p. $\hat{G}_{t_0} \sim_{\delta} G_{t_0}$.
2.4. Message Terminology. We have introduced several pieces of terminology related to messages in the graph, which we recall and collect here for easy reference. For a fixed time parameter \( t \in \mathbb{N} \) and for a (half-)edge or set of (half-)edges incident to a specified vertex, we have the following terminology.

- The \( t \)-in-message is the incoming message at time \( t \).
- The \( t \)-out-message is the outgoing message at time \( t \).
- The \( t \)-in-story is the sequence of \( t' \)-in-messages for \( t' = 0, \ldots, t \).
- The \( t \)-out-story is the sequence of \( t' \)-out-messages at times \( t' = 0, \ldots, t \).
- The \( t \)-story consists of the pair \( t \)-in-story and \( t \)-out-story.
- The \( t \)-input consists of the pair \( t \)-in-story and 0-out-message.
- The \( t \)-in-compilation is the multiset of \( t \)-inputs over all half-edges at a vertex.
- The \( t \)-in-compilation sequence is the sequence of \( t \)-in-compilations over all vertices.

When the parameter \( t \) is clear from the context, we often drop it from the terminology.

3. Preliminary results

We begin with some fairly simple observations which help to motivate some of the definitions made so far, or to justify why they are reasonable. The first such observation provides a slightly simpler way of describing the individual "entries", i.e. the marginal probability distributions, of \( \phi^t_q \).

Claim 3.1. For any \( i, t \in \mathbb{N} \) with \( i \leq t \), the marginal distribution of \( \phi^t_q \) on the \( i \)-th entry is precisely \( \phi^t_{\mu_i}(q) \), i.e. for any \( \mu \in \Sigma \) we have

\[
\sum_{\mu=(\mu_0, \ldots, \mu_t) \in \Sigma^{t+1}} \mathbb{P}(\phi^t_{\mu_i}(q) = \mu) = \mathbb{P}(\phi^t_{\mu_i}(q) = \mu).
\]

Proof. By Definition 2.2, we have

\[
\sum_{\mu=(\mu_0, \ldots, \mu_t) \in \Sigma^{t+1}} \mathbb{P}(\phi^t_{\mu_i}(q) = \mu) = \sum_{\mu=(\mu_0, \ldots, \mu_t) \in \Sigma^{t+1}} \mathbb{P}(X_0 = \mu_0, X_1 = \mu_1, \ldots, X_t = \mu_t).
\]

Summing over all elements of vectors except \( \mu_i \), the above expression is equal to \( \mathbb{P}(X_i = \mu) \). We will prove by induction that \( \mathbb{P}(X_i = \mu) = \mathbb{P}(\phi^t_{\mu_i}(q) = \mu) \). For \( i = 0 \), again using Definition 2.2, the distribution of \( X_0 \) is simply \( q \), so suppose that \( i \geq 1 \) and that the result holds for \( 0, \ldots, i-1 \). Let \( x_1, \ldots, x_j \) be the children of \( u \) in the Po\((d)\) branching tree, so \( j \sim \text{Po}(d) \). By the recursive nature of the Po\((d)\) branching tree and the induction hypothesis, the message from any \( x_k \) to \( u \) at time \( i-1 \) has distribution \( \phi^{i-1}_{\mu_i}(q) \), and this is independent for all vertices. Thus the distribution of the message from \( u \) to \( v \) at time \( t = i \) is \( \phi_i \left( \text{Po}(d \phi^{i-1}_{\mu_i}(q)) \right) = \phi^t_{\mu_i}(q) \).

Claim 3.2. Given a half-edge in \( \hat{G}_t \), the distribution of its \( t \)-story is \( \phi^t_{\mu}(q_0) \), i.e. identical to the distribution of its \( t \)-in-story.

Proof. Given an arbitrary half-edge at a vertex \( u \), let us add a dummy vertex \( v \) to model the corresponding neighbour of \( u \). Apart from \( u \to v \), the vertex \( u \) has \( j \sim \text{Po}(d) \) half-edges connected to dummy variables \( c_1, \ldots, c_j \), and each half-edge receives \( t_0 \)-in-story according to \( \phi^t_{\mu}(q_0) \). This is equivalent to endowing each \( c_i \) with a Po\((d)\) tree independently, initialising the messages in these trees according to \( q_0 \) (and also initialising the messages from each \( c_i \) to \( u \) according to \( q_0 \)) and running \( t_0 \) rounds of Warning Propagation. Combining all the trees with \( u \), we have that \( u \) is again the root of a Po\((d)\) tree in which messages are all initialised according to \( q_0 \) with the same initialisation \( q_0 \). Then by Definition 2.2, \( \mu_{u \to v} (\leq t_0) \) is distributed as \( \phi^t_{\mu}(q_0) \).

Given an integer \( t_0 \) and \( \mu_1, \mu_2 \in \Sigma^{t_0+1} \), let \( m_{\mu_1, \mu_2} \) denote the number of half-edges in \( \hat{G}_t \) with in-story \( \mu_1 \) and out-story \( \mu_2 \). Furthermore, let

\[
q_{\mu_1, \mu_2} := \mathbb{P}(\phi^t_{\mu}(q_0) = \mu_1) \cdot \mathbb{P}(\phi^t_{\mu}(q_0) = \mu_2).
\]

We will also define \( m_{\mu, \mu} := d_n q_{\mu, \mu} \). Observe that at a single half-edge, since by Claim 3.2 both the in-story and out-story are distributed as \( \phi^t_{\mu} \) and they are independent of each other, \( q_{\mu_1, \mu_2} \) is precisely the probability that the
Corollary 3.4. \textbf{space, w.h.p. all statistics are close to their expectation s.} The condition is at least polynomially likely. Furthermore, the first equality also implies that in the conditioned For any \( \omega \in \mathbb{N} \), let us define \( S_0 \) to be the support of \( \phi_0^\omega(q_0) \times \phi_q^\omega(q_0) \), i.e. the set of \( (\mu_1, \mu_2) \in (\Sigma^{\omega+1})^2 \) such that \( \mathbb{P}(\phi_0^\omega(q_0) = \mu_1) \cdot \mathbb{P}(\phi_q^\omega(q_0) = \mu_2) > 0 \). We will partition \( S_0 \) into sets \( Q_0, R_0 \), where \( Q_0 \subseteq S_0 \) is the subset consisting of pairs \((\mu_1, \mu_2)\) where \( \mu_1 \neq \mu_2 \), and \( R_0 = S_0 \setminus Q_0 \). It will also be convenient to define the natural extensions of these sets to time \( t_0 = -1 \). At this time, the half-edges have no messages at all, and the stories are empty strings, which with a slight abuse of notation but in order to be clearer, we also denote by \( \emptyset \). So we set \( Q_{-1} = \emptyset \) and \( S_{-1} = R_{-1} := \{ (\emptyset, \emptyset) \} \). Similarly \( m_{(\emptyset, \emptyset)} \) is the total number of half-edges in \( \hat{s} \).

Now let us define the events
\[
A_1 = A_1(t_0) := \{ m_{\mu_1, \mu_2} = m_{\mu_2, \mu_1} \text{ for all } (\mu_1, \mu_2) \in Q_0 \}, \\
A_2 = A_2(t_0) := \{ m_{\mu_1, \mu_2} \text{ even for all } (\mu_1, \mu_1) \in R_0 \}, \\
A = A(t_0) := A_1 \cap A_2.
\]

Informally, we describe \( A \) as the event that “statistics match”. Let us observe that \( A(t_0) \subseteq A(t_0 - 1) \), i.e. if statistics match at time \( t_0 \), then they also match at time \( t_0 - 1 \), and inductively at any time \( t \in [t_0] \).

Furthermore, for \( 0 \leq x \in \mathbb{R} \) and \( y \in (0, 1] \), let us also define the following events.
\[
\mathcal{B}_x := \{ m_{\mu_1, \mu_2} = \overline{m}_{\mu_1, \mu_2} \pm x \sqrt{n} \text{ for all } (\mu_1, \mu_2) \in S_0 \}, \\
\mathcal{B}_y := \{ y \leq \overline{m}_{\mu_1, \mu_2} \leq 1/y \text{ for all } (\mu_1, \mu_2) \in S_0 \}.
\]

In other words, \( \mathcal{B}_x \) bounds the additive deviation of statistics from their expectation (scaled by \( \sqrt{n} \)), while \( \mathcal{B}_y \) bounds the multiplicative deviation. Naturally these events are very closely related, but in some situations it will be more convenient to use the one, in other situations to use the other.

For each \( t \in \mathbb{N} \), let us denote \( R_t := \bigcup_{t'<t} R_{t'} \), and observe that since the \( R_{t'} \) are necessarily disjoint for each \( t' \), we have \( |R_t| = \sum_{t'<t} |R_{t'}| \).

Lemma 3.3. For any \( \omega_0 \rightarrow \infty \), we have
\[
\mathbb{P}(A \cap \mathcal{B}_{\omega_0}) = (1 + o(1)) \mathbb{P}(A) \geq \Theta\left(n^{-[(Q_0) - 2|R_{t_0}|]/4}\right).
\]

In particular, this means that the conditioning in Step (3) of the construction of \( \hat{G}_{t_0} \) is not too restrictive, since the condition is at least polynomially likely. Furthermore, the first equality also implies that in the conditioned space, w.h.p. all statistics are close to their expectations.

Corollary 3.4. W.h.p. for all \( \mu_1, \mu_2 \in \Sigma^{\omega+1} \), the number of directed edges in \( \hat{G}_{t_0} \) with story \((\mu_1, \mu_2)\) is \( q_{\mu_1, \mu_2} d n \pm \sqrt{n} \ln n \).

Proof. For \( (\mu_1, \mu_2) \notin S_0 \), we have \( m_{\mu_1, \mu_2} = q_{\mu_1, \mu_2} = 0 \), so the statement trivially holds. For all \( (\mu_1, \mu_2) \in S_0 \), we apply Lemma 3.3 with \( \omega_0 = \sqrt{n} \) to deduce that
\[
\mathbb{P}(A \cap \mathcal{B}_{\omega_0} | \mathcal{B}_{d/\sqrt{n}} = (1 + o(1)) \mathbb{P}(A) = \Theta\left(n^{-[(Q_0) - 2|R_{t_0}|]/4}\right),
\]

and since the conditioning on \( A \) is precisely the conditioning in Step (3) of the construction of \( \hat{G}_{t_0} \), the result follows. \hfill \Box

Let us make a few remarks explaining the intuition behind Lemma 3.3. Intuitively, we imagine that each \( m_{\mu_1, \mu_2} \) is normally distributed with mean \( \overline{m}_{\mu_1, \mu_2} = \overline{m}_{\mu_2, \mu_1} \) and variance \( \Theta(n) \). For \( (\mu_1, \mu_2) \in R_0 \), the probability that the statistic is even is approximately 1/2. On the other hand, for \( (\mu_1, \mu_2) \in Q_0 \), in general \( m_{\mu_1, \mu_2} \) will be within \( O(\sqrt{n}) \) of its mean, and the probability that \( m_{\mu_1, \mu_2} \) will achieve the same value is \( \Theta(n^{-1/2}) \). If we could naively pretend
that this is independent for each dual pair \((\mu_1, \mu_2), (\mu_2, \mu_1) \in Q_{t_0}\), we would achieve a total success probability (i.e. the probability of \(\mathcal{A}\)) of \(n^{-1/2} |Q_{t_0}|^{1/2} = n^{-1/2} |Q_0|^{1/2}\), and furthermore w.h.p. all statistics would be close to their expectations.

However, this naive assumption is invalid for two reasons:

- The \(t_0\)-out-stories at a vertex \(v\) are dependent on each other.
- There are some intrinsic restrictions based on the fact that the sums of statistics whose in-story \(\mu_1\) and out-story \(\mu_2\) have the same initial segment \(\mu_1^{(t)} = \mu_2^{(t)}\) of length \(t\) must certainly match.

For these two reasons, proving Lemma 3.3 formally is a rather complicated task. However, since it is an intuitively natural result, particularly if we are not concerned with the precise exponent in the success probability, we defer the proof to Appendices A and B.

The first of these reasons is tricky to overcome, but is ultimately merely a technical difficulty; the second is precisely the reason for the appearance of \(|R_{t_0}|\) in the statement of the lemma, which intuitively represents the number of restrictions on the degrees of freedom. We note, however, that although we need to track the exponent quite precisely in order to prove Lemma 3.3, it is not actually important for subsequent applications of the lemma. More specifically, we will actually apply either the statement of Corollary 3.4, which does not mention \(\mathbb{P}(\mathcal{A})\) at all, or the fact that \(\mathbb{P}(\mathcal{A})\) is at least polynomial, i.e. at least \(n^{-c}\) for some sufficiently large constant \(c\).

Let \(Z\) denote the distribution of an input generated as in Step (1), i.e. generating Po\((d)\) half-edges and giving each a \(t_0\)-in-story according to \(\phi^{(t_0)}_{x}(q_0)\) and a \(0\)-out-message according to \(q_0\) independently. One very important observation is that if not too many inputs have been revealed, the distribution of each remaining input is approximately the same \(Z\). Of course, this statement would be trivially true were it not for the conditioning in Step (3).

For each set of vertices \(W \subseteq V\) let us further denote \(\mathcal{S}_W\) to be the (random) multiset of in-compilations of vertices in \(W\).

**Lemma 3.5.** Given a set \(W \subseteq V\) of size at most \(9n/10\) and a vertex \(v \notin W\), let \(Y\) be the distribution of the random variable \(\mathcal{S}_W\) conditioned on \(\mathcal{S}_W\) (so \(Y\) is also a random variable). Then w.h.p.

\[
d_{TV}(Y, Z) = o(1).
\]

This is also an intuitively obvious result whose proof we defer to Appendix C. We will implicitly use this fact throughout the proof by modelling the in-compilation of the vertices we reveal as being distributed according to \(Z\) – the lemma states that this is a reasonable approximation.

### 4. Contiguity: Proof of Lemma 2.6

The aim of this section is to prove Lemma 2.6, the first of our main results, which states that \(\mathcal{G}_{t_0}\) and \(\mathcal{G}_{t_0}\) have approximately the same distribution.

The overall strategy for the proof is to show that every step of the construction of \(\mathcal{G}_{t_0}\) closely reflects the situation in \(\mathcal{G}_{t_0}\). More precisely, the following are the critical steps in the proof. Recall from Definition 2.1 that \(\mathcal{G}\) is the underlying unmessaged random graph corresponding to \(\mathcal{G}_{t_0}\), and similarly let \(\hat{\mathcal{G}}\) denote the underlying unmessaged random graph corresponding to \(\hat{\mathcal{G}}_{t_0}\). We will show the following.

1. The local structure of \(\hat{\mathcal{G}}\) is approximately that of a Po\((d)\) branching process.
2. After initialising Warning Propagation on \(\mathcal{G}\) according to \(q_0\) and proceeding for \(t_0\) rounds, the distribution of the in-story along a random edge is approximately \(\phi^{(t_0)}_{x}(q_0)\).
3. Given a particular compilation sequence, i.e. multiset of stories (which consist of in-stories and out-stories) on half-edges at each vertex, each graph with this compilation sequence is equally likely to be chosen as \(\mathcal{G}\).
4. If we run Warning Propagation on \(\hat{\mathcal{G}}\), with initialisation identical to the constructed messages at time 0, for \(t_0\) steps, the message histories are identical to those generated in the construction of \(\hat{\mathcal{G}}_{t_0}\).

Let us observe that the first step is a very well-known fact about sparse random graphs, and the second step is a direct consequence of the first (see Proposition 4.5). One minor difficulty to overcome in these two steps is how to handle the presence of short cycles (of which we expect some bounded number to appear), which are the main reason the approximations are not exact. We will need to show that, while the presence of such a cycle close to a vertex may alter the distribution of incoming message histories at this vertex (in particular they may no longer be independent), it does not fundamentally alter which message histories are possible (Proposition 4.1). Therefore...
while the presence of a short cycle will change some distributions in its close vicinity, the fact that there are very few short cycles means that this perturbation will be masked by the overall random "noise".

Meanwhile, the third step is a basic observation about the Erdős-Rényi binomial random graph model (Claim 4.1) and the fourth step is an elementary consequence of the fact that we constructed the message histories in $\hat{G}_{t_0}$ to be consistent with Warning Propagation (Proposition 4.2).

We begin by showing that, if we initialise messages in a (deterministic) graph in a way which is admissible according to $q_0$, any message histories up to time $t_0$ produced by Warning Propagation have a non-zero probability of appearing under the probability distribution $\Phi_{\varphi}(q_0)$.

**Proposition 4.1.** Let $G$ be any graph and let $(u, v)$ be a directed edge of $G$. Suppose that messages are initialised in $G$ arbitrarily subject to the condition that each initial message has non-zero probability under $q_0$, and we run Warning Propagation with update rule $\varphi$ for $t_0$ steps. Let $\mu = \mu_{\mu \rightarrow v}(\leq t_0)$ be the corresponding $t_0$-input at $v$ along $(u, v)$, i.e. message history from $u$ to $v$ up to time $t_0$ as well as message from $v$ to $u$ at time 0.

Then

$$\Pr\{\Phi_{\varphi}(q_0) = \mu\} \neq 0.$$  

**Proof.** We first construct an auxiliary tree $G'$ as follows. First generate $u$ as the root of the tree, along with its parent $v$. Subsequently, recursively for each $t \in \{0\} \cup [t_0]$, for each vertex $x$ at distance $t$ below $u$, we generate children for all neighbours of the vertex corresponding to $x$ in $G$ except for the neighbour in $G$ corresponding to the parent $y$ of $x$ in $G'$.

Note that another way of viewing $G'$ is that we replace walks beginning at $u$ in $G$ (and whose second vertex is not $v$) by paths, where two paths coincide for as long as the corresponding walks are identical, and are subsequently disjoint. A third point of view is to see $G'$ as a forgetful search tree of $G$, where (apart from the parent) we don't remember having seen vertices before and therefore keep generating new children.

We will initialise messages in $G'$ from each vertex to its parent (and also from $v$ to $u$) according to the corresponding initialisation in $G$, and run Warning Propagation with update rule $\varphi$ for $t_0$ rounds. Let $\mu' = \mu'_{\mu' \rightarrow u}(\leq t_0)$ be the resulting $t_0$-input at $v$ along $(u, v)$ in $G'$, and recall that $\mu$ is the corresponding $t_0$-input in $G$. The crucial observation is the following.

**Claim 4.2.** $\mu' = \mu$.

We delay the proof of this claim until after the proof of Proposition 4.1 which we now complete. Recall that $\Phi_{\varphi}(q_0)$ was defined as the probability distribution of $(X_0, \ldots, X_{t_0})$, the message history in a Po$(d)$ tree in which messages are initialised according to $q_0$. Therefore the probability that $\Phi_{\varphi}(q_0) = \mu = \mu'$ is certainly at least the probability that a Po$(d)$ tree has exactly the structure of $G'$ (up to depth $t_0$) and that the initialisation chosen at random according to $q_0$ is precisely the same as the initialisation in $G'$. Since $G'$ is a finite graph, and each initial message has a positive probability under $q_0$, this probability is nonzero, as required.

We now go on to prove the auxiliary claim.

**Proof of Claim 4.2.** By construction the 0-out-message at $v$ along $(v, u)$ is identical in $\mu$ and $\mu'$, so it remains to prove that the $t_0$-in-stories are identical.

For any vertex $x \in G' \setminus \{v\}$, let $x^t$ denote the parent of $x$. In order to prove Claim 4.2 we will prove a much stronger statement from which the initial claim will follow easily. More precisely, we will prove by induction on $t$ that for all $x \in G' \setminus \{v\}$, $\mu'_{x \rightarrow x^t}(\leq t) = \mu_{x \rightarrow x^t}(\leq t)$ where for any vertex $y$, we use $s_y$ to denote the vertex corresponding to $y$ in $G$. For $t = 0$, by construction $\mu'_{x \rightarrow x^t}(0) = \mu_{x \rightarrow x^t}(0)$ for any $x \in G' \setminus \{v\}$ because each vertex to its parent messages in $G'$ are initialised according to the corresponding initialisation in $G$. Suppose that the statement is true for some $t \leq t_0 - 1$. Now, we want to prove that $\mu'_{x \rightarrow x^t}(\leq t + 1) = \mu_{x \rightarrow x^t}(\leq t + 1)$ for any $x \in G' \setminus \{v\}$. By the induction hypothesis, it remains to prove that $\mu'_{x \rightarrow x^t}(t + 1) = \mu_{x \rightarrow x^t}(t + 1)$. By the Warning Propagation update rule, we have $\mu'_{y \rightarrow x^t}(t + 1) = \varphi(\{\mu_{y \rightarrow x^t}(t) : y \in \partial_{G'} x \setminus \{x^t\}\})$. By the induction hypothesis, $\mu'_{y \rightarrow x^t}(t) = \mu_{y \rightarrow x^t}(t)$ for all $y \in \partial_{G'} x \setminus \{x^t\}$. Hence,

$$\{\mu'_{y \rightarrow x^t}(t) : y \in \partial_{G'} x \setminus \{x^t\}\} = \{\mu_{y \rightarrow x^t}(t) : y \in \partial_{G'} x \setminus \{x^t\}\}$$

and therefore also

$$\mu'_{x \rightarrow x^t}(t + 1) = \varphi(\{\mu_{y \rightarrow x^t}(t) : y \in \partial_{G'} x \setminus \{x^t\}\}) = \varphi(\{\mu_{y \rightarrow x^t}(t) : r \in \partial_{G'} x \setminus \{x^t\}\}) = \mu'_{x \rightarrow x^t}(t + 1),$$
Proposition 4.4 tells us that no matter how strange or pathological a messaged graph looks locally, there is still a positive probability that we will capture the resulting input (and therefore w.h.p. such an input will be generated a linear number of times in $\hat{G}_t$). In particular, within distance $t_0$ of a short cycle the distribution of an input may be significantly different from $(\Phi_c^d(q_0), q_0)$. However, we next show that there are unlikely to be many edges this close to a short cycle.

Claim 4.3. Let $C_0$ be the set of vertices which lie on some cycle of length at most $t_0$ in $G = G(n, \delta/n)$, and recursively define $C_t := C_{t-1} \cup \partial C_{t-1}$ for $t \in \mathbb{N}$.

Then w.h.p. $|C_0| \leq n^{1/3}$.

To prove this claim, one can simply use the first moment method to bound $|C_0|$ and then crudely use the maximum degree $\Delta(G) \leq \ln n$ to bound $|C_t| \leq \Delta(G)^{t_0}|C_0|$ - we omit the details.

We next show that the in-compilation sequence distribution in $G_{t_0}$ is essentially the same as that in $\hat{G}_{t_0}$.

Definition 4.4. Given integers $k, t \in \mathbb{N}_0$, a messaged graph $G \in \Gamma_n^{(t)}$ and a multisets $A \in \left( \binom{\Sigma^{t+2}}{k} \right)$, define $n_A = n_A(G)$ to be the number of vertices of $G$ which receive in- compilation $A$.

Further, let $\gamma_A = \gamma_A(t)$ denote the probability that when $Po(d)$ inputs are generated independently at random according to the probability distribution $(\Phi_c^d(q_0), q_0)$ (i.e. the in-compilation distribution at a vertex in the construction of $\hat{G}_t$), the resulting multiset is $A$.

Observe that for any $k, t \in \mathbb{N}_0$, the expression $\sum_{A \in \left( \binom{\Sigma^{t+2}}{k} \right)} n_A(G)$ is simply the number of vertices of degree $k$, and therefore if $G$ contains $n$ vertices, then for any $t \in \mathbb{N}_0$ we have $\sum_{k \in \mathbb{N}_0} \sum_{A \in \left( \binom{\Sigma^{t+2}}{k} \right)} n_A(G) = n$.

Proposition 4.5. Let $t_0$ be some (bounded) integer and define $k_0 := \frac{11 \ln \ln n}{\ln n}$ Then w.h.p. the following holds.

1. For every integer $k \leq k_0$ and for every $A \in \left( \binom{\Sigma^{t+2}}{k} \right)$ we have $n_A(G_{t_0}), n_A(\hat{G}_{t_0}) = \left( \gamma_A + \frac{1}{(\ln n)^2} \right)n$.
2. $\hat{G}_{t_0}, G_{t_0}$ each contains at most $\frac{n}{(\ln n)^{2}}$ vertices of degree at least $k_0$.

The proof is technical, but ultimately standard and we give only a short overview. Both statements for $\hat{G}_{t_0}$ can be proved using a simple Chernoff bound and union bound argument, together with Corollary 3.4 to show that the conditioning in the construction of $\hat{G}_{t_0}$ does not skew the distribution too much. (Alternatively, Statement 1 for $\hat{G}_{t_0}$ follows immediately from Corollary 3.4 which gives even finer information.) The second statement for $G_{t_0}$ can also be proved using a Chernoff bound and a union bound. The main difficulty is the first statement for $G_{t_0}$. For this we use the fact that $G_{t_0}$ has the local structure of a Po(d) branching tree, in the sense of local weak convergence: the proportion of vertices whose depth $t_0$ neighbourhood is isomorphic to a particular (rooted) graph $H$ converges in probability to the probability that a Po(d) branching tree is isomorphic to $H$. We refine this by further looking at $(H, e)$, the set of rooted graphs with a function $c$ from directed edges of $H$ to $\Sigma$, which models message initialisations along all directed edges. We show that the proportion of vertices in $G_{t_0}$ whose depth $t_0$ neighbourhood together with message initialisations is isomorphic to $(H, e)$ converges in probability to the probability that a Po(d) tree with directed messages initialised independently at random according to $q_0$ is isomorphic to $(H, e)$. Since the depth $t_0$ neighbourhood together with message initialisations in particular determines the $t_0$-in-compilation, this is sufficient.

One small detail to be aware of, and the reason that quoting standard results on local weak convergence of $G = G(n, \delta/n)$ as a black box is not quite sufficient, is that the statement of Proposition 4.5 includes an explicit $\frac{1}{\ln n}$ error term rather than simply $o(1)$. However, it is a simple if tedious exercise to check that the proof in [14], for example, does in fact give this statement if we track the error terms more carefully.

As a corollary of Proposition 4.5 we obtain the following result.

Corollary 4.6. After re-ordering vertices if necessary, w.h.p. the number of vertices whose in-compilations are different in $\hat{G}_{t_0}$ and $G_{t_0}$ is at most $\frac{n}{(\ln n)^{2}}$. 

**Proof.** Assuming the high probability event of Proposition 4.5 holds, the number of vertices with differing in- compilations is at most

\[
\left(\sum_{k=0}^{k_0} \sum_{l \leq t+2} \frac{2n}{(\ln n)^3} \right) + \frac{2n}{(\ln n)^{10}} \leq \frac{2n}{(\ln n)^{10}} \sum_{k=0}^{k_0} |\Sigma^{(k_0+2)}|^k + \frac{2n}{(\ln n)^{10}}
\]

\[
\leq \frac{2n}{(\ln n)^3} k_0 |\Sigma^{(k_0+2)}| k_0 + \frac{2n}{(\ln n)^{10}}
\]

\[
= \frac{2n}{(\ln n)^3} k_0 \exp(\Theta(k_0)) + \frac{2n}{(\ln n)^{10}}
\]

\[
\leq \frac{n}{(\ln n)^2},
\]

where the last line follows since \( \exp(\Theta(k_0)) = \Theta\left(\frac{\ln \ln n}{\ln \ln \ln n}\right) = (\ln n)^{o(1)} \).

Next, we show that choosing a uniformly random matching in the construction of \( \hat{G}_{t_0} \) is an appropriate choice. We need a definition which generalises the notion of the degree sequence of a graph.

**Definition 4.7.** For any \( \Sigma^{t+1} \)-messaged graph \( G \in \mathcal{G}_n \), let \( H_i = H_i(G) \) denote the in-compilation at vertex \( i \), for \( i \in [n] \) and let \( H(G) := (H_1, \ldots, H_n) \) be the in-compilation sequence.

**Claim 4.8.** Suppose that \( G_1, G_2 \) are two graphs on \([n]\) with \( H(G_1) = H(G_2) \). Then \( P(G = G_1) = P(G = G_2) \).

We omit the elementary proof. We also need to know that the message histories generated in the construction of \( \hat{G}_{t_0} \) match those that would be produced by Warning Propagation. Let \( \hat{G}_{WP} \) denote the graph with message histories generated by constructing \( \hat{G}_{t_0} \), stripping all the message histories except for the messages at time 0 and running Warning Propagation for \( t_0 \) steps with this initialisation.

**Proposition 4.9.** Deterministically we have \( \hat{G}_{WP} = \hat{G}_{t_0} \).

**Proof.** Since the two underlying unmessaged graphs are the same, we just need to prove that at any time \( t \in [0] \cup [t_0] \), the incoming and outgoing messages at a given vertex \( v \) are the same for \( \hat{G}_{t_0} \) and \( \hat{G}_{WP} \). We will proceed by induction on \( t \). At time \( t = 0 \), the statement is true by construction of \( \hat{G}_{WP} \). Now, suppose it is true up to time \( t \) for some \( t \in [0] \cup [t_0 - 1] \) and consider an arbitrary directed edge \((u, v)\). By Definition 2.1(2), the \((t + 1)\)-out-message from \( u \) in \( \hat{G}_{t_0} \) is produced according to the rules of Warning Propagation based on the \( t \)-in-messages to \( u \) at time \( t \). By the induction hypothesis, these \( t \)-in-messages are the same for \( \hat{G}_{t_0} \) and \( \hat{G}_{WP} \). Hence, the \((t + 1)\)-out-messages along \((u, v)\) are also the same in \( \hat{G}_{t_0} \) and \( \hat{G}_{WP} \).

We can now complete the proof of Lemma 2.6.

**Proof of Lemma 2.6.** We use the preceding auxiliary results to show that every step in the construction of \( \hat{G}_{t_0} \) closely mirrors a corresponding step in which we reveal partial information about \( G_{t_0} \). Let us first explicitly define these steps within \( G_{t_0} \) by revealing information one step at a time as follows.

1. First reveal the in-compilation at each vertex, modelled along half-edges.
2. Next reveal all out-stories along each half-edge.
3. Finally, reveal which half-edges together form an edge.

Corollary 4.4 shows that Step (1) in the construction of \( \hat{G}_{t_0} \) (together with the corresponding conditioning on Step (3), in view of Remark 2.4) can be coupled with Step (1) in revealing \( G_{t_0} \) in such a way that w.h.p. the number of vertices on which they produce different results is at most \( \frac{n}{(\ln n)^2} \), and therefore Condition (2) of Definition 2.5 is satisfied.

Furthermore, Proposition 4.9 shows that, for those vertices for which the in-compilations are identical in Step (1), the out-stories generated in Step (2) of the construction of both \( \hat{G}_{t_0} \) and \( G_{t_0} \) must also be identical (deterministically).

Now in order to prove that we can couple the two models in such a way that the two edge sets are almost the same (and therefore Condition (1) of Definition 2.5 is satisfied), we consider each potential story \( \mu \in \Sigma^{2(t_0+1)} \) in turn, and construct coupled random matchings of the corresponding half-edges. More precisely, let us fix \( \mu \) and let \( m \) be the number of half-edges with this story in \( \hat{G}_{t_0} \). Similarly, define \( m \) to be the corresponding number of
half-edges in \( G_{t_0} \). Furthermore, let \( r_1 \) be the number of half-edges with story \( \mu \) in \( G_{t_0} \setminus G_{t_1} \), let \( r_2 \) be the number of half-edges with the "dual story" \( \mu^* \), i.e. the story with in-story and out-story switched, and correspondingly \( r_1, r_2 \) in \( G_{t_0} \setminus G_{t_1} \).

For convenience, we will assume that \( \mu^* \neq \mu \) – the case when they are equal is very similar.

Let us call an edge of a matching good if it runs between two half-edges which are common to both models. Note that this does not necessarily mean it is common to both matchings, although we aim to show that we can couple in such a way that this is (mostly) the case. Observe that, conditioned on the number of good edges in a matching, we may first choose a matching of this size uniformly at random on the common half-edges, and then complete the matching uniformly at random (subject to the condition that we never match two common half-edges).

Observe further that the matching in \( G_{t_0} \) must involve at least \( m - r_1 - r_2 \) good edges, and similarly the matching in \( G_{t_0} \) must involve at least \( m - r_1 - r_2 \), and therefore we can couple in such a way that at least \( \min (m - r_1 - r_2, m - r_1 - r_2) \) edges are identical, or in other words, the symmetric difference of the matchings has size at most \( \max (r_1 + r_2, r_1 + r_2) \).

Repeating this for each possible \( \mu \), the total number of edges in the symmetric difference is at most twice the number of half-edges which are not common to both models. By the (standard) fact that w.h.p. the maximum degree is \( O(\ln n) \) and by Corollary 4.9 this number is \( O(\ln n) \cdot \frac{r}{(\ln n)^2} = o(n) \) which gives the desired bound. \( \square \)

5. Subcriticality: The idealised change process

With Lemma 2.6 in hand, which tells us that \( G_{t_0} \) and \( G_{t_1} \) look very similar, the proof of Theorem 1.2 consists of two main steps.

In this section, we describe an idealised approximation of how a change propagates when applying WP repeatedly to \( G_{t_0} \), and to show that this propagation is a subcritical process, and therefore quickly dies out. The definition of this idealised change process is motivated by the similarity to \( G_{t_0} \).

In the second step, in Section 6 we will use Lemma 2.6 to prove formally that the idealised change process closely approximates the actual change process, which therefore also quickly terminates.

**Definition 5.1.** Given a probability distribution \( q \) on \( \Sigma \), we say that a pair of messages \((\sigma_0, t_0)\) is a potential change with respect to \( q \) if there exist some \( t \in \mathbb{N} \) and some \( \mu = (\mu_0, \mu_1, \ldots, \mu_t) \in \Sigma^{t+1} \) such that

- \( \mu_{t-1} = \sigma_0; \)
- \( \mu_t = t_0; \)
- \( \mathbb{P} \left( \phi_{t_0}^t (q) = \mu \right) > 0. \)

We denote the set of potential changes by \( \mathcal{P}(q) \).

In other words, \((\sigma_0, t_0)\) is a potential change if there is a positive probability of making a change from \( \sigma_0 \) to \( t_0 \) at some point in the Warning Propagation algorithm on a Po\((d)\) branching tree when initialising according to \( q \). The following simple claim will be important later.

**Claim 5.2.** If \( p \) is a fixed point and \((\sigma_0, t_0) \in \mathcal{P}(p) \), then \( p(\sigma_0) > 0 \) and \( p(t_0) < 1 \).

**Proof.** The definition of \( \mathcal{P}(p) \) implies in particular that there exist a \( t \in \mathbb{N} \) and an \( \mu \in \Sigma^{t+1} \) such that \( \mu_{t-1} = \sigma_0 \) and \( \mathbb{P} (\phi_{t_0}^t (p) = \mu) > 0 \). Furthermore, by Claim 4.1 the marginal distribution of the \( t \)-th entry of \( \phi_{t_0}^t (p) \) is \( \phi_{t_0}^{t-1} (p) = p \) (since \( p \) is a fixed point), and therefore we have \( p(\sigma_0) \geq \mathbb{P} (\phi_{t_0}^t (p) = \mu) > 0 \).

On the other hand, clearly \( p(t_0) \leq 1 - p(\sigma_0) < 1 \). \( \square \)

5.1. The change branching process. Given a probability distribution \( p \) on \( \Sigma \) and a pair \((\sigma_0, t_0) \in \mathcal{P}(p) \), we define a branching process \( \mathcal{T}^* = \mathcal{T}^* (\sigma_0, t_0, p) \) as follows.

We begin with vertices \( u \) and \( v \), where \( u \) is the parent of \( v \), and generate the messages \( \mu_{u \rightarrow v}^{(1)} = \sigma_0 \) and \( \mu_{u \rightarrow v}^{(2)} = t_0 \). Intuitively, the notation includes a superscript \( (1) \) to indicate that this is the "original" message, while a \( (2) \) indicates that this is the "new, changed" message. We also choose some \( \omega_0 \in \Sigma \) at random according to \( p \) and generate the message \( \mu_{u \rightarrow v} = \omega_0 \).

During the process, whenever we generate a vertex \( y \) from a parent \( x \), we will also have generated messages \( \mu_{x \rightarrow y}^{(1)} = \sigma_1 \) and \( \mu_{x \rightarrow y}^{(2)} = t_1 \neq \sigma_1 \), and also a message \( \mu_{y \rightarrow x} = \omega_1 \), as is in particular the case for the initialisation, where \( y = v \) and \( x = u \). Whenever we have generated \( y \) from \( x \) with these messages, we generate further half-edges at \( y \) with in-messages according to Po\((dp)\), but conditioned on the event that they will produce the message \( \omega_1 \) from \( y \) to \( x \).
Now having generated all the in-messages, consider the out-messages at each half-edge when $\mu_{x \to y}^{(s)} \in \{\sigma_1, \tau_1\}$ for $s \in \{1, 2\}$. For each half-edge that produces differing results, generate a child $z$ with the appropriate messages $\mu_{z \to y}$ and also $\mu_{y \to z}^{(1)}$ and $\mu_{y \to z}^{(2)}$. For half-edges where no messages change, we do not generate children (and delete the half-edges).

5.2. Subcriticality. Intuitively, $\mathcal{T}$ approximates the cascade effect that a single change in a message from time $t_0 - 1$ to time $t_0$ subsequently causes (this is proved more precisely in Section 5). Therefore the following result is the essential heart of the proof of Theorem 1.2.

Proposition 5.3. If $p$ is a stable fixed point, then for any $(\sigma_0, \tau_0) \in \mathcal{P}(p)$, the branching process $\mathcal{T} = \mathcal{T}(\sigma_0, \tau_0, p)$ is subcritical.

Proof. Let us suppose for a contradiction that for some $(\sigma_0, \tau_0) \in \mathcal{P}(p)$, the branching process has survival probability $p > 0$. We fix parameters according to the following hierarchy:

$$0 < \varepsilon \ll \frac{1}{t_1} \ll \delta \ll p \ll \frac{1}{|\Sigma|} \ll 1.$$

In the following, given an integer $i$ and messages $\sigma_i, \tau_i \in \Sigma$, we will use the notation $\sigma_i := (\sigma_i, \tau_i)$. Let us define a new probability distribution $q$ on $\Sigma$ as follows.

$$q(\mu) := \begin{cases} p(\mu) - \varepsilon & \text{if } \mu = \sigma_0, \\ p(\mu) + \varepsilon & \text{if } \mu = \tau_0, \\ p(\mu) & \text{otherwise.} \end{cases}$$

Note that since $(\sigma_0, \tau_0) \in \mathcal{P}(p)$ is a potential change, for sufficiently small $\varepsilon$, this is indeed a probability distribution by Claim 5.2.

Let us generate the $t_1$-neighbourhood of a root vertex in a Po$(d)$ branching process and initialise messages at the leaves at depth $t_1$ according to both $p$ and $q$, where we couple in the obvious way so that all messages are identical except for some which are $\sigma_0$ under $p$ and $\tau_0$ under $q$. We call such messages changed messages.

We first track the messages where we initialise with $p$ through the tree (both up and down) according to the Warning Propagation rules, but without ever updating a message once it has been generated. Since $p$ is a fixed point of $\varphi$, every message either up or down in the tree has the distribution of $p$ (though clearly far from independently).

We then track the messages with initialisation according to $q$ through the tree, and in particular track where differences from the first set of messages occur. Let $x_i(\sigma_1)$ denote the probability that a message from a vertex at level $t_1 - i$ to its parent changes from $\sigma_1$ to $\tau_1$. Thus in particular we have

$$x_0(\sigma_1) = \begin{cases} \varepsilon & \text{if } \sigma_1 = \sigma_0, \\ 0 & \text{otherwise.} \end{cases}$$

Observe also that messages coming down from parent to child “don't have time” to change before we consider the message up (the changes from below arrive before the changes from above). Since we are most interested in changes which are passed up the tree, we may therefore always consider the messages down as being distributed according to $p$.

We aim to approximate $x_{i+1}(\sigma_1)$, so let us consider a vertex $u$ at level $t_1 - (i + 1)$ and its parent $v$. Let us define $C_j = C_j(u)$ to be the event that $u$ has precisely $j$ children. Furthermore, let us define the event $D_u(\sigma_2)$ to be the event that exactly one change is passed up to $u$ from its children, and that this change is of type $\sigma_2$. Finally, let $b_{u}(\sigma_1)$ be the number of messages from $u$ (either up or down) which change from $\sigma_1$ to $\tau_1$ (there may be more changes of other types).

The crucial observation is that given the neighbours of $u$, each is equally likely to be the parent. Therefore conditioned on the event $D_u(\sigma_2)$ and the values of $j$ and $b_{u}(\sigma_1)$, apart from the one child from which a change of type $\sigma_2$ arrives at $u$, there are $j$ other neighbours which could be the parent, of which $b_{u}(\sigma_1)$ will receive a change of type $\sigma_1$. Thus the probability that a change of type $\sigma_1$ is passed up to the parent is precisely $\frac{b_{u}(\sigma_1)}{j}$.
Therefore in total, conditioned on $C_j$ and $D_u(\sigma_2)$, the probability $a_{j,\sigma_1,\sigma_2}$ that a change of type $\sigma_1$ is passed on from $u$ to $v$ is

$$a_{j,\sigma_1,\sigma_2} = \sum_{k=1}^j \mathbb{P}\left( b_u(\sigma_1) = k \mid C_j \wedge D_u(\sigma_2) \right) \frac{k}{j} = \frac{1}{j} \mathbb{E}\left( b_u(\sigma_1) \mid C_j \wedge D_u(\sigma_2) \right).$$

Now observe that this conditional expectation term is exactly as in the change process. More precisely, in the $F$ process we know automatically that only one change arrives at a vertex, and therefore if we have a change of type $\sigma_2$, the event $D_u(\sigma_2)$ certainly holds. Therefore

$$\sum_{j=1}^\infty \mathbb{P}(\text{Po}(d) = j) a_{j,\sigma_1,\sigma_2} = T_{\sigma_1,\sigma_2},$$

where $T$ is the $|\Sigma|^2 \times |\Sigma|^2$ transition matrix associated with the $F$ change process, i.e. the entry $T_{\sigma_1,\sigma_2}$ is equal to the expected number of changes of type $\sigma_1$ produced in the next generation by a change of type $\sigma_2$.

On the other hand, defining $E_u$ to be the event that at least two children of $u$ send changed messages (of any type) to $u$, we also have

$$x_{i+1}(\sigma_1) \geq \sum_{j} \mathbb{P}(\text{Po}(d) = j) \sum_{\sigma_2} a_{j,\sigma_1,\sigma_2} \mathbb{P}(D_u(\sigma_2) | C_j)
\geq \sum_{j} \mathbb{P}(\text{Po}(d) = j) \sum_{\sigma_2} a_{j,\sigma_1,\sigma_2} \left( j x_{i}(\sigma_2) - \mathbb{P}(E_u | C_j) \right).$$

For each $i \in \mathbb{N}$, let $x_i$ be the $|\Sigma|^2$-dimensional vector whose entries are $x_i(\sigma)$ for $\sigma \in \Sigma^2$ (in some arbitrary order). We now observe that, since $p$ is a stable fixed point, for $\varepsilon$ small enough we have

$$\sum_{\sigma} x_i(\sigma) = \|x_i\|_1 = d_T \left( p, \phi_\sigma^i(q) \right) \leq d_T(p, q) = \|x_0\|_1 = \varepsilon,$$

and so we further have

$$\mathbb{P}(E_u | C_j) \leq \left( \frac{j}{2} \right)^2 \varepsilon.$$

Furthermore, we observe that since $a_{j,\sigma_1,\sigma_2}$ is a probability term by definition, we have

$$\sum_{\sigma_2} a_{j,\sigma_1,\sigma_2} \leq \sum_{\sigma_2} 1 = |\Sigma|^2.$$

Substituting (5.1), (5.3) and (5.4) into (5.2), we obtain

$$x_{i+1}(\sigma_1) \geq \sum_{\sigma_2} T_{\sigma_1,\sigma_2} x_i(\sigma_2) - |\Sigma|^2 \varepsilon^2 \sum_{j} \left( \frac{j}{2} \right) \mathbb{P}(\text{Po}(d) = j) = \sum_{\sigma_2} T_{\sigma_1,\sigma_2} x_i(\sigma_2) - d_T^2 |\Sigma|^2 \varepsilon^2,$$

and thus we have

$$x_{i+1} \geq T x_i - d_T^2 |\Sigma|^2 \varepsilon^2$$

(where the inequality is pointwise). As a direct consequence we also have $x_i \geq T^i x_0 - i d_T^2 |\Sigma|^2 \varepsilon^2$ (pointwise), and therefore

$$\|x_i\|_1 \geq \|T^i x_0\|_1 - i d_T^2 |\Sigma|^4 \varepsilon^2.$$

Now since the change process has survival probability $\rho > 0$ for the appropriate choice of $\sigma_0 = (\sigma_0, t_0)$, choosing $x_0 = \varepsilon e_{\sigma_0}$ (where $e_{\sigma_0}$ is the corresponding standard basis vector) we have

$$\|x_i\|_1 \geq \|T^i x_0\|_1 - i d_T^2 |\Sigma|^4 \varepsilon^2 \geq \rho \|x_0\|_1 - i d_T^2 |\Sigma|^4 \varepsilon^2 = \varepsilon (\rho - i d_T^2 |\Sigma|^4 \varepsilon).$$

On the other hand, since $p$ is a stable fixed point, there exists some $\delta > 0$ such that for small enough $\varepsilon$ we have $\|x_i\|_1 \leq (1 - \delta)^i \varepsilon$ for all $i$. In particular choosing $i = t_1$, we conclude that

$$\varepsilon (\rho - t_1 d_T^2 |\Sigma|^4 \varepsilon) \leq \|x_{t_1}\|_1 \leq (1 - \delta)^{t_1} \varepsilon.$$

However, since we have $\varepsilon \ll 1/t_1 \ll \rho, \delta, 1/|\Sigma|$, we observe that

$$(1 - \delta)^{t_1} \leq \rho/2 < \rho - t_1 d_T^2 |\Sigma|^4 \varepsilon,$$

which is clearly a contradiction. \qed
6. Applying subcriticality: Proof of Theorem 1.2

Our goal in this section is to use Proposition 5.3 to complete the proof of Theorem 1.2.

6.1. A consequence of subcriticality. Recall that during the proof of Proposition 5.3 we defined the transition matrix \( T \) of the change process \( \mathcal{F} \), which is an \(|\Sigma|^2 \times |\Sigma|^2\) matrix where the entry \( T_{\alpha_1, \alpha_2} \) is equal to the expected number of changes of type \( \alpha_1 \) that arise from a change of type \( \alpha_2 \). The subcriticality of the branching process is equivalent to \( T^n \xrightarrow{h \to \infty} 0 \) (meaning the zero matrix), which is also equivalent to all eigenvalues of \( T \) being strictly less than 1 (in absolute value). We therefore obtain the following corollary of Proposition 5.3.

**Corollary 6.1.** There exist a constant \( \gamma > 0 \) and a positive real \(|\Sigma|^2\)-dimensional vector \( \alpha \) (with no zero entries) such that

\[
T \alpha \leq (1 - \gamma) \alpha
\]

(where the inequality is understood pointwise). We may further assume that \( \|\alpha\|_1 = 1 \).

**Proof.** Given some \( \epsilon > 0 \), let \( T' = T'(\epsilon) \) be the matrix where every entry of \( T \) increases by \( \epsilon \). Thus \( T' \) is a strictly positive real matrix and we choose \( \epsilon \) to be small enough such that all the eigenvalues of \( T' \) are still less than 1 in absolute value. By the Perron-Frobenius theorem, there exists a positive real eigenvalue that matches the spectral radius \( \rho(T') < 1 \) of \( T' \). In addition, there exists a corresponding eigenvector to \( \rho(T') \), say \( \alpha \), all of whose entries are non-negative; since every entry of \( T' \) is strictly positive, it follows that in fact every entry of \( \alpha \) is also strictly positive. We have \( T' \alpha = \rho(T') \alpha \), and we also note that \( T \alpha < T' \alpha \) since every entry of \( T' \) is strictly greater than the corresponding entry of \( T \). Thus we deduce that \( T \alpha < \rho(T') \alpha \), and setting \( \gamma := 1 - \rho(T') > 0 \), we have the desired statement.

The final property that \( \|\alpha\|_1 = 1 \) can be achieved simply through scaling by an appropriate (positive) normalising constant, which does not affect any of the other properties of \( \alpha \). \( \square \)

However, let us observe that in fact the change process that we want to consider is slightly different – rather than having in-messages distributed according to \( p \), they should be distributed according to \( \phi_{\mu}^{-1}(\sigma_0) \). We therefore need the following.

**Corollary 6.2.** There exists \( \delta_0 > 0 \) sufficiently small that for any probability distribution \( q_1 \) on \( \Sigma \) which satisfies \( d_{TV}(p, q_1) \leq \delta_0 \), the following holds. Let \( \mathcal{F}_1 = \mathcal{F}(\sigma_0, \tau_0, q_1) \) denote the change branching process with \( q_1 \) instead of \( p \), and let \( T_1 \) be the transition matrix of \( \mathcal{F}_1 \). Then there exist a constant \( \gamma > 0 \) and a positive real \(|\Sigma|^2\)-dimensional vector \( \alpha \) (with no zero entries) such that

\[
T_1 \alpha \leq (1 - \gamma) \alpha
\]

(where the inequality is understood pointwise).

In other words, the same statement holds for \( T_1 \), the transition matrix of this slightly perturbed process, as for \( T \). In particular, \( \mathcal{F}_1 \) is also a subcritical branching process.

**Proof.** Observe that since \( d_{TV}(p, q_1) \leq \delta_0 \), for any \( \epsilon \) we may pick \( \delta_0 = \delta(\epsilon) \) sufficiently small such that \( T_1 \) and \( T \) differ by at most \( \epsilon \) in each entry. In other words, we have \( T_1 \leq T' \) pointwise, where \( T' = T'(\epsilon) \) is as defined in the proof of Corollary 6.1. Thus we also have \( T_1 \alpha \leq T' \alpha \leq \rho(T') \alpha = (1 - \gamma) \alpha \) as in the previous proof. \( \square \)

For the rest of the proof, let us fix \( \delta \) as in Theorem 1.2 and a constant \( \delta_0 < \delta \) small enough that the conclusion of Corollary 6.2 holds, and suppose that \( t_0 \) is large enough that \( q_1 := \phi_{\mu}^{-1}(\sigma_0) \) satisfies \( d_{TV}(p, q_1) \leq \delta_0 \) (this is possible since \( \phi_{\mu}^{-1}(\sigma_0) = p \)).

6.2. The marking process. We now use the idealised form \( \mathcal{F}_1 \) of the change process to give an upper bound on the (slightly messier) actual process. For an upper bound, we will simply simplify the process of changes made by WP to obtain WP\(^\ast\) \( [\mathcal{G}_0] \) = WP\(^\ast\) \( [\mathcal{G}_0] \) from WP\(\mathcal{G}_0\) \( [\mathcal{G}_0] \).

We will reveal the information in \( \mathcal{G}_0 \) little at a time as needed.

- Initialisation

\[\text{Note here that with a slight abuse of notation, we use WP to denote the obvious function on } \mathcal{G}_n \text{ which, given a graph } G \text{ with messages } \mu \in \mathcal{M}(G), \text{ maps } (G, \mu) \text{ to } \text{WP}(G, \mu) := (G, \text{WP}(G, \mu)).\]
We first reveal the \( t_0 \)-inputs at each vertex, and the corresponding out-stories according to the update rule \( \varphi \). We also generate the outgoing messages at time \( t_0 + 1 \). Any half-edge whose \( t_0 \)-out-message is \( \sigma_0 \) and whose \((t_0 + 1)\)-out-message is \( \tau_0 \neq \sigma_0 \) is called a change of type \( \sigma_0 \).

For each out-story which includes a change, this half-edge is marked.

We continue with a marking process:

- Whenever a half-edge at \( u \) is marked, we reveal its partner \( v \). The edge \( uv \) is marked.
- If \( v \) is a new vertex (at which nothing was previously marked), if the degree of \( v \) is at most \( k_0 \) and if the inputs are identical in \( G_{t_0} \) and \( \hat{G}_{t_0} \), we consider the remaining half-edges at \( v \) and apply the rules of Warning Propagation to determine whether any out-messages will change.
- If \( v \) does not satisfy all three of these conditions, we say that we have hit a snag. In particular:
  - If \( v \) is a vertex that we have seen before, it is called a duplicate vertex;
  - If \( v \) is a vertex of degree at most \( k_0 \) whose inputs are different according to \( G_{t_0} \) and \( \hat{G}_{t_0} \), it is called an error vertex;
  - If \( v \) is a vertex of degree larger than \( k_0 \), it is called a freak vertex.

In each case, all of the half-edges at \( v \) become marked. Such half-edges are called spurious edges, and are further classified as defective, erroneous and faulty respectively, according to the type of snag we hit. The corresponding messages can change arbitrarily (provided each individual change is in \( \mathcal{P}(p) \)).

Note that a duplicate vertex may also be either an error or a freak vertex. However, by definition, no snag is both an error and a freak vertex.

We first justify that this gives an upper bound on the number of changes made by Warning Propagation. Let \( \mathcal{E}_{WP} \) be the set of edges on which the messages are different in \( \hat{G}_{t_0} \) and in \( WP^* \left( G_{t_0} \right) \), and let \( \mathcal{E}_{mark} \) be the set of edges which are marked at the end of the marking process. Note that the set \( \mathcal{E}_{mark} \) is not uniquely defined, but depends on the arbitrary choices for the changes which are made at snags.

**Proposition 6.3.** There exists some choice of the changes to be made at snags such that \( \mathcal{E}_{WP} \subseteq \mathcal{E}_{mark} \).

**Proof.** We proceed in rounds indexed by \( t \in \mathbb{N}_0 \). We define \( \mathcal{E}_{WP}(t) \) to be the set of edges on which the messages are different in \( WP^* \left( G_{t_0} \right) \) compared to \( G_{t_0} \), while \( \mathcal{E}_{mark}(t) \) is the set of edges which are marked after \( t \) steps of the marking process. Since \( \mathcal{E}_{WP} = \lim_{t \to \infty} \mathcal{E}_{WP}(t) \) and \( \mathcal{E}_{mark} = \lim_{t \to \infty} \mathcal{E}_{mark}(t) \), it is enough to prove that for each \( t \in \mathbb{N}_0 \) we have \( \mathcal{E}_{WP}(t) \subseteq \mathcal{E}_{mark}(t) \), which we do by induction on \( t \).

The base case \( t = 0 \) is simply the statement that the set of initial marks contains the changes from \( \hat{G}_{t_0} \) to \( \hat{G}_{t_0+1} \), which is clearly true by construction.

For the inductive step, each time we reveal the incoming partner of a marked outgoing half-edge, if this is a vertex at which nothing was previously marked, i.e. a standard vertex, then we proceed with marking exactly according to Warning Propagation.

On the other hand, if at least one edge was already marked at this vertex we simply mark all the outgoing half-edges, and if we choose the corresponding changes according to the changes that will be made by Warning Propagation, the induction continues.

In view of Proposition 6.3 our main goal is now the following.

**Lemma 6.4.** At the end of the marking process, w.h.p. at most \( \sqrt{\delta_0 n} \) edges are marked.

During the proof of Lemma 6.4 we will make extensive use of the following fact.

**Claim 6.5.** W.h.p., for every \( \mu \in \Sigma_{k_0+1} \) such that \( \mathbb{P}(\Phi^0_{k_0}(q_0) = \mu) \neq 0 \), the total number of inputs of \( \mu \) over all vertices is at least \( \delta_0^{1/100} n \).

**Proof.** This follows from Proposition 6.5 since we choose \( \delta_0 \) sufficiently small, so in particular \( \delta_0^{1/100} < \gamma(\mu) \) for any \( \mu \) such that \( \mathbb{P}(\Phi^0_{k_0}(q_0) = \mu) > 0 \), meaning that w.h.p. there are certainly at least \( \gamma(\mu) n - o(n) \geq \delta_0^{1/100} n \) vertices which have degree one and receive input \( \mu \) along their only half-edge, which is clearly sufficient.

The following is the critical step for relating the marking process to the idealised branching process \( \mathcal{F} \).

**Proposition 6.6.** Whenever a standard vertex \( v \) is revealed in the marking process from a change of type \( \sigma_1 \), the further changes made at outgoing half-edges at \( v \) have asymptotically the same distribution as in the branching process \( \mathcal{F}(\sigma_0, \tau_0, q_1) \) below a change of type \( \sigma_1 \).
Proof. Given that \( v \) is a standard vertex, we may use \( G_{t_0} \) instead of \( G_{t_0} \) to model it. In particular, provided we have revealed at most \( 9n/10 \) vertices so far, Lemma 3.3 implies that there are \( Po_{\leq k_0}(d) \) further half-edges at \( v \), and this distribution tends asymptotically to the Po\( (d) \) distribution. Furthermore, by Claim 6.5, each of these half-edges has a \( t_0 \)-in-message distributed according to \( q_i \) independently. Since \( v \) was a new vertex, these in-messages have not changed, and therefore are simply distributed according to Po\( (d q_i) \), as in \( \mathcal{F}(\sigma_0, \tau_0, q_i) \).

Note that in the idealised process \( \mathcal{F}(\sigma_0, \tau_0, q_i) \) we additionally condition on these incoming messages producing \( \omega_0 \), the appropriate message to the parent. In this case we do not know the message that \( v \) sent to its "parent", i.e. its immediate predecessor, in the marking process. However, this message is distributed as \( q_i \), and letting \( X \) denote a Po\( (d q_i) \) variable, the probability that the multiset of incoming messages at \( v \) is \( A \) is simply

\[
P \{ q_i = \varphi(A) \} P \{ X = A \mid \varphi(X) = \varphi(A) \}.
\]

Since \( q_i \) is asymptotically close to the stable fixed point \( p \), we have that \( P \{ q_i = \varphi(A) \} \) is asymptotically close to \( P \{ \varphi(X) = \varphi(A) \} \) for each \( A \), and so the expression above can be approximated simply by \( P \{ X = A \} = P \{ X = A \} \), as required. \( \square \)

6.3. Three stopping conditions. In order to prove Lemma 6.4 we introduce some stopping conditions on the marking process. More precisely, we will run the marking process until one of the following three conditions is satisfied.

1. **Exhaustion** - the process has finished.
2. **Expansion** - there exists some \( \sigma_1 = (\sigma_1, \tau_1) \in \Sigma^2 \) such that at least \( \delta^{3/5} \alpha \sigma_1 n \) messages have changed from \( \sigma_1 \) to \( \tau_1 \) (where \( \alpha \) is the vector from Corollary 6.2).
3. **Explosion** - the number of spurious edges is at least \( \delta^{2/3} n \).

Lemma 6.4 will follow if we can show that w.h.p. neither expansion nor explosion occurs.

6.3.1. Explosion.

**Proposition 6.7.** W.h.p. explosion does not occur.

We will split the proof up into three claims, dealing with the three different types of spurious edges.

**Claim 6.8.** W.h.p., the number of defective edges is at most \( \delta^{2/3} n/2 \).

**Proof.** A vertex of degree \( i \) will contribute \( i \) defective edges if it is chosen at least twice as the partner of a marked half-edge. Using Claim 6.5, at each step there are at least \( \delta^{1/100} n \) possible half-edges to choose from, of which certainly at most \( i \) are incident to \( v \), and thus the probability that \( v \) is chosen twice in the at most \( \sqrt{\delta_0} n \) steps is at most

\[
\left( \frac{i}{\delta^{1/100} n} \right)^2 \left( \sqrt{\delta_0} n \right)^2 = \delta^{49/50} n^2.
\]

Thus setting \( S \) to be the number of defective edges, we have

\[
E(S) \leq \sum_{i=0}^{\infty} i (P(Po(d) = i) n) \delta^{49/50} n^2 = \delta^{49/50} n \sum_{i=0}^{\infty} i^3 P(Po(d) = i) = \delta^{49/50} n O(d^3) = O(\delta^{49/50} n^2) \leq \delta^{4/5} n.
\]

On the other hand, if two distinct vertices have degrees \( i \) and \( j \), then the probability that both become snags may be estimated according to whether or not they are adjacent to each other, and is at most

\[
\frac{ij}{\delta^{1/100} n} \cdot \frac{ij}{\delta^{1/100} n} \left( \sqrt{\delta_0} n \right)^3 \left( \frac{\delta^{1/100} n}{\delta^{1/100} n} \right)^4 \left( \sqrt{\delta_0} n \right)^4 \leq 2i^2 j^2 \delta^{24/25}.
\]

Therefore we have

\[
E(S^2) \leq E(S) + \sum_{i,j=0}^{\infty} ij P(Po(d) = i) n P(Po(d) = j) n2i^2 j^2 \delta^{49/25} \\
\leq \delta^{4/5} n + 2 \delta^{49/25} n^2 (E(Po(d)^3))^2 \\
\leq \delta^{9/5} n^2.
\]

Thus Chebyshev’s inequality shows that w.h.p. the number of spurious edges is at most \( \delta^{2/3} n/2 \), as claimed. \( \square \)
Claim 6.9. W.h.p., the number of erroneous edges is at most $\frac{n}{(\ln n)^{1/2}}$.

Proof. Observe that Corollary 4.6 implies in particular that the number of edges of $G_{k_0}$ which are attached to vertices of degree at most $k_0$ where the incoming message histories differ from those in $\hat{G}_{k_0}$ (i.e. which would lead us to an error vertex if chosen) is at most $k_0\frac{n}{(\ln n)^2} \leq \frac{1}{2\ln n}$. Furthermore, any time we meet an error we obtain at most $k_0$ erroneous edges, and since the marking process proceeds for at most $\delta_0 n$ steps, therefore the expected number of erroneous edges in total is at most

$$\delta_0^{3/5} n \cdot \frac{k_0}{\delta_0^{1/100} \ln n} \leq \frac{n}{\sqrt{\ln n}}.$$

An application of Markov’s inequality completes the proof. \qed

Claim 6.10. W.h.p. the number of faulty edges is at most $n/(\ln n)^6$.

Proof. This is similar to the proof of Claim 6.9. It is well-known that w.h.p. there are no vertices of degree larger than $O(\ln n)$. Therefore by Proposition 4.4, w.h.p. the number of edges adjacent to vertices of degree at least $k_0$ is at most $O(n/(\ln n)^3)$, so the probability of hitting a freak is at most $(\ln n)^{-8}$. If we hit a freak, at most $O(\ln n)$ half-edges become faulty, therefore the expected number of faulty edges is $\delta_0^{3/5} n \cdot O(n/(\ln n)^{-8}) = O(n/(\ln n)^{-7})$. An application of Markov's inequality completes the proof.

Combining all three cases we can prove Proposition 6.7.

Proof of Proposition 6.7. By Claims 6.8, 6.9 and 6.10, w.h.p. the total number of spurious edges is at most

$$\frac{\delta_0^{2/3} n}{2} + \frac{n}{(\ln n)^{1/3}} + \frac{n}{(\ln n)^6} \leq \delta_0 n$$

as required. \qed

6.3.2. Expansion.

Proposition 6.11. W.h.p. expansion does not occur.

Proof. By Proposition 6.7, we may assume that explosion does not occur, so we have few spurious edges. Therefore in order to achieve expansion, at least $\frac{1}{2\sqrt{\delta_0 n}}$ edges would have to be marked in the normal way, i.e. by being generated as part of a $\mathcal{T}$ branching process rather than as one of the $\delta_0 n$ initial half-edges or as a result of hitting a snag.

However, we certainly reveal children in $\mathcal{T}$ of at most $\delta_0^{3/5} a_{\sigma_2} n$ changes from $\sigma_2$ to $\tau_2$, for each choice of $\sigma_2 = (\sigma_2, \tau_2) \in \Sigma^2$, since at this point the expansion stopping condition would be applied. Thus the expected number of changes from $\sigma_1$ to $\tau_1$ is at most

$$\sum_{\sigma_2 \in \Sigma} \delta_0^{3/5} a_{\sigma_2} n T_{\sigma_1, \sigma_2} = (Ta)_{\sigma_1} \delta_0^{3/5} n \leq (1 - \gamma) a_{\sigma_1} \delta_0^{3/5} n.$$

We now aim to show that w.h.p. the actual number of changes is not much larger than this (upper bound on the) expectation, for which we use a second moment argument. Let us fix some $\sigma_2 \in \Sigma^2$. For simplicity, we will assume for an upper bound that we reveal precisely $s := \delta_0^{3/5} a_{\sigma_2} n$ changes of type $\sigma_2$ in $\mathcal{T}$.

Then the number of changes of type $\sigma_1$ that arise from these is the sum of $s$ independent and identically distributed integer-valued random variables $X_1, \ldots, X_s$, where for each $i \in [s]$ we have $E(X_i) = T_{\sigma_1, \sigma_2}$ and $E \left( X_i^2 \right) \leq E \left( Po(d_2)^2 \right) = d_2^2$. Therefore we have $\text{Var}(X_i) \leq d_2^2 = O(1)$, and the central limit theorem tells us that $\text{Var} \left( \sum_{i=1}^s X_i \right) = O \left( \sqrt{s} \right)$. Then the Chernoff bound implies that w.h.p.

$$\left| \sum_{i=1}^s X_i - E \left( \sum_{i=1}^s X_i \right) \right| \leq n^{1/4} O \left( \sqrt{s} \right) = O \left( n^{3/4} \right) \leq \frac{1}{2} \delta_0^{3/5} T_{\sigma_1, \sigma_2} a_{\sigma_2} n.$$

Taking a union bound over all $|\Sigma|^4$ choices of $\sigma_1, \sigma_2$, we deduce that w.h.p. the total number of changes of type $\sigma_1$ is at most

$$(1 - \gamma) a_{\sigma_1} \delta_0^{3/5} n + \sum_{\sigma_2} \frac{1}{2} \delta_0^{3/5} T_{\sigma_1, \sigma_2} a_{\sigma_2} n = (1 - \gamma/2) a_{\sigma_1} \delta_0^{3/5} n$$

for any choice of $\sigma_1$, as required. \qed
6.3.3. Exhaustion.

Proof of Lemma 6.4. By Propositions 6.7 and 6.11 neither explosion nor expansion occurs. Thus the process finishes with exhaustion, and (using the fact that \( \|\alpha\|_1 = 1 \)) the total number of edges marked is at most

\[
\sum_{\sigma_1 \in \Sigma^2} \delta_0^{3/5} \alpha_1 n + \delta_0^{2/3} n = (\delta_0^{3/5} + \delta_0^{2/3}) n \leq \sqrt{\delta_0} n
\]

as required. □

6.4. Proof of Theorem 1.2. We can now complete the proof of our main theorem.

Proof of Theorem 1.2. Recall from Proposition 6.3 that edges on which messages change when moving from WP\(^b\)(\(G_0\)) to WP\(^*\)(\(G_0\)), which are simply those in the set \(E_{\text{WP}}\), are contained in \(E_{\text{mark}}\). Furthermore, Lemma 6.4 states that \(|E_{\text{mark}}| \leq \sqrt{\delta_0} n\). Since we chose \(\delta_0 \ll \delta\), the statement of Theorem 1.2 follows. □

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APPENDIX A. PROOF OF LEMMA 3.3

Throughout this section, we assume the setup of Lemma 3.3 so in particular $\omega_0$ will be an arbitrary but fixed function of $n$ which tends to infinity as $n \to \infty$.

A.1. Three deviation propositions. We will prove the lemma with the help of three propositions. The first states that very large deviations from the (unconditional) are extremely unlikely.

**Proposition A.1.** There exists $\epsilon > 0$ sufficiently small such that

$$P \left( |\mathcal{A}| > n^{1/2} \right) \leq \exp \left( -n \right).$$

The second proposition states that statistics matching, but deviating moderately from the (unconditional) expectation is fairly unlikely.

**Proposition A.2.** For any $\epsilon > 0$, we have

$$P \left( |\mathcal{A}| \leq n^{-1/2} \right) = o \left( n^{-1/2} \right).$$

The third and final proposition gives a lower bound on the probability of statistics matching and also being close to their (unconditional) expectations.

**Proposition A.3.**

$$P \left( |\mathcal{A}| \leq n^{-1/2} \right) \geq \Theta \left( n^{-1/2} \right).$$

Let us first show how these three propositions together imply Lemma 3.3.

**Proof of Lemma 3.3.** Clearly by Proposition A.3 we have

$$P(\mathcal{A}) \geq P(\mathcal{A} \cap \overline{\mathcal{R}_{\omega_0}}) \geq \Theta \left( n^{-1/2} \right),$$

so it remains to prove that $P(\mathcal{A} \cap \overline{\mathcal{R}_{\omega_0}}) = (1 + o(1))P(\mathcal{A})$. Note that A.1 also implies that the probability bounds of Propositions A.1 and A.2 are both $o(P(\mathcal{A}))$, and therefore we have

$$P \left( |\mathcal{A}| \leq n^{-1/2} \right) = P \left( |\mathcal{A}| \leq n^{-1/2} \right) + P \left( |\mathcal{A}| \leq n^{-1/2} \right) \leq o \left( n^{-1/2} \right) + P \left( |\mathcal{A}| \leq n^{-1/2} \right) = o \left( P(\mathcal{A}) \right).$$

Therefore we have

$$P \left( |\mathcal{A}| \leq n^{-1/2} \right) = P(\mathcal{A}) - P(\mathcal{A} \cap \overline{\mathcal{R}_{\omega_0}}) = (1 + o(1))P(\mathcal{A})$$

as required.

A.2. Extreme deviations: Proof of Proposition A.1. To help prove Proposition A.1 we will use the following basic fact.

**Claim A.4.** With (unconditional) probability at least $1 - \exp \left( -\Theta(n) \right)$, the following holds.

1. The number $m$ of half-edges generated in Step (1) of the construction of $\hat{G}_{\omega_0}$ satisfies

   $$dn/2 \leq m \leq 2dn.$$  \hspace{1cm} (A.2)

2. If $\epsilon$ is a sufficiently small constant, then the number of half-edges attached to vertices of degree larger than $\hat{G}_{\omega_0}$ is at most $\epsilon^2 n$.

The proof is a basic exercise in the second moment method, which we omit.

We can now prove that it is also very unlikely that any story statistics are very far from their (unconditional) expectations.

**Proof of Proposition A.1.** We will assume that the high probability events of Claim A.4 hold. This introduces an error probability of $\exp \left( -\Theta(n) \right)$.

Recall that our aim is to bound the probability that $\frac{m_{\mu_1, \mu_2}}{m_{\mu_1, \mu_2}}$ is either smaller than $\epsilon$ or larger than $1/\epsilon$. First observe that if $\frac{m_{\mu_1, \mu_2}}{m_{\mu_1, \mu_2}} \geq 1/\epsilon$ then for sufficiently small $\epsilon$ we must have that $m_{\mu_1, \mu_2} \geq 2dn$, contradicting (A.2).
It therefore remains to bound the probability that \( \frac{m_{\mu_1, \mu_2}}{m_{\mu_1, \mu_2}} \leq \varepsilon \) for some \( \mu_1, \mu_2 \in S_0 \). For each \( t \in t_0 \), let \( \mu^{(t)}_1, \mu^{(t)}_2 \) denote the initial segment of the first \( t + 1 \) entries of \( \mu_i \), and let us set

\[
    r_t := \frac{m_{\mu_1^{(t)}, \mu_2^{(t)}}}{m_{\mu_1^{(t-1)}, \mu_2^{(t-1)}},} \quad s_t := \frac{q_{\mu_1^{(t)}, \mu_2^{(t)}}}{q_{\mu_1^{(t-1)}, \mu_2^{(t-1)}}}
\]

(where we interpret \( m_{\mu_1^{(t)}, \mu_2^{(t)}} = 1 \) and \( q_{\mu_1^{(t)}, \mu_2^{(t)}} = 1 \)). Observe that

\[
    2m_{\mu_1^{(t)}, \mu_2^{(t)}} \geq \frac{m_{\mu_1^{(t)}, \mu_2^{(t)}}}{\mu_1^{(t-1)}, \mu_2^{(t-1)}} \geq \sum_{t=0}^{t_0} \frac{r_t}{s_t}.
\]

Therefore in order for \( \frac{m_{\mu_1, \mu_2}}{m_{\mu_1, \mu_2}} \leq \varepsilon \) to hold, we must have \( \prod_{t=0}^{t_0} \frac{r_t}{s_t} \leq 2 \varepsilon \), and therefore there must be some \( t \in [t_0]_0 \) such that

\[
    \frac{r_t}{s_t} \leq (2\varepsilon)^\frac{1}{t_0 + 1}.
\]

If we pick the smallest such \( t \), then we certainly have

\[
    m_{\mu_1^{(t)}, \mu_2^{(t-1)}} = \left( \prod_{t=0}^{t-1} \frac{r_t}{s_t} \right) q_{\mu_1^{(t)}, \mu_2^{(t-1)}} \geq (2\varepsilon)^\frac{t}{t_0 + 1} q_{\mu_1^{(t)}, \mu_2^{(t-1)}} \frac{d_N}{2} \geq \varepsilon n,
\]

for \( \varepsilon \) sufficiently small.

We would now like to argue that it is very unlikely that fewer than a \( (2\varepsilon)^\frac{1}{t_0 + 1} \) proportion of these half-edges receive the \( t \)-story. While it is certainly true that the expected proportion is \( \bar{q} := \frac{q_{\mu_1^{(t)}, \mu_2^{(t-1)}}}{q_{\mu_1^{(t-1)}, \mu_2^{(t-1)}}} \), and this will certainly be much larger than \( (2\varepsilon)^\frac{1}{t_0 + 1} \) if \( \varepsilon \) is small enough, the problem is that the \( t \)-stories at the half-edges may not be independent of each other. More precisely, the \( t \)-in-messages will be independent of each other for each half-edge, but the \( t \)-out-messages may be dependent for two half-edges incident to the same vertex. Therefore we will first argue that we can find a large subset of the half-edges with \( (t-1) \)-story \( (\mu_1^{(t-1)}, \mu_2^{(t-1)}) \) which are all at distinct vertices.

We begin by applying the second statement of Claim A.4 which implies that we have a set of at least \((1-\varepsilon)m_{\mu_1^{(t-1)}, \mu_2^{(t-1)}}\) half-edges with this \((t-1)\)-story at vertices of degree at most \( e^{-\frac{\varepsilon}{d_N}} \). Subsequently, we simply greedily select one of these half-edges at every (low degree) vertex adjacent to one, and obtain a set of at least \( N := (1-\varepsilon)e^{-\frac{\varepsilon}{d_N}} m_{\mu_1^{(t-1)}, \mu_2^{(t-1)}} \) half-edges with this \((t-1)\)-story which are all at different vertices. Each of these then receives \( t \)-story \( m_{\mu_1^{(t)}, \mu_2^{(t)}} \) with probability \( \bar{q} = \frac{q_{\mu_1^{(t)}, \mu_2^{(t)}}}{q_{\mu_1^{(t-1)}, \mu_2^{(t-1)}}} \) independently, and the probability that at most \( e^{-\frac{\varepsilon}{d_N}} N \) \( m_{\mu_1^{(t)}, \mu_2^{(t-1)}} \) of them receive this \( t \)-story is at most

\[
    P\left( \text{Bin}(N, \bar{q}) \leq e^{-\frac{\varepsilon}{d_N} \frac{\varepsilon}{d_N} N} \right) \leq \exp\left(-\Theta(N)\right) \leq \exp\left(-\Theta\left(e^{-\frac{\varepsilon}{d_N}} \varepsilon n\right)\right) \leq \exp\left(-n^{3/4}\right).
\]

(Note that in the second inequality, we have used the fact that \( N = \Theta\left(e^{-\frac{\varepsilon}{d_N}} m_{\mu_1^{(t-1)}, \mu_2^{(t-1)}}\right) \geq \Theta\left(e^{-\frac{\varepsilon}{d_N}} \varepsilon n\right).\)

We now take a union bound over all times \( t \in [t_0]_0 \) and all possible \( \mu_1^{(t)}, \mu_2^{(t)} \in \Sigma_{t_0+1} \), and deduce that (conditioned on the high probability events of Claim A.4) the probability that \( \mathcal{B}_T^* \) holds is at most

\[
    (t_0 + 1)2^{t_0+1} \exp\left(-n^{3/4}\right) \leq \exp\left(-n^{2/3}\right).
\]

Combined with the error probability from Claim A.4, we have a total error probability of at most \( \exp(-\Theta(n)) + \exp(-n^{2/3}) \leq \exp(-n^{1/2}) \) as claimed.

**Remark A.5.** Note that the exponential error term in Proposition A.4 is smaller than any other probability terms that we will consider for the rest of this section. Therefore we may condition on \( \mathcal{B}_T^* \) without significantly changing any future calculations. In what follows, to simplify notation we will assume this conditioning implicitly. In particular, we will assume that

\[
    m_{\mu_1, \mu_2} \geq \varepsilon^2 n \quad \text{for any } (\mu_1, \mu_2) \in S_0.
\]

This will be important since it means that the partition classes are large enough to apply various concentration results.
A.3. Some preliminaries. We aim to prove Propositions A.2 and A.3 together using an inductive argument. The induction will be over $t_0$, starting from the (slightly unusual) base case $t_0 = -1$. In this case, the half-edges have no in-messages or out-messages at all, and therefore the statistics must match deterministically, so with probability 1. Furthermore, the total number $m$ of half-edges is distributed as $\text{Po}(dn)$, which has standard deviation $\Theta(\sqrt{n})$, so an application of Chebyshev’s inequality shows that w.h.p. $m = (1 \pm \frac{\omega_0}{\sqrt{n}}) dn$, and therefore

$$
\mathbb{P}(|B_+ \cap B_{00} \cap \omega|) \leq \mathbb{P}(B_{00}) = o(1),
$$

$$
\mathbb{P}(\omega \cap B_{00}) = \mathbb{P}(B_{00}) = 1 - o(1),
$$

which proves the base case of these two propositions.

To help us prove the inductive step, we introduce the following notations. Given $\mu_1, \mu_2 \in \Sigma^{t_0}$ and $\sigma_1, \sigma_2 \in \Sigma$, define

- $\mu^*_i := (\mu_i, \sigma_i)$ for $i = 1, 2$.
- Let $E_{\mu_1, \mu_2}$ be the class of half-edges with $(t_0 - 1)$-in-story $\mu_1$ and $(t_0 - 1)$-out-story $\mu_2$.
- Let $E_{\mu_1, \mu_2}^+$ be the class of half-edges with $(t_0 - 1)$-in-story $\mu_1$ and $t_0$-out-story $\mu_2^*$.
- Let $E_{\mu_1, \mu_2}^*$ be the class of half-edges with $t_0$-in-story $\mu_1^*$ and $t_0$-out-story $\mu_2^*$.
- Let $q_{\mu_i}^* := \mathbb{P}(\phi_{\mu_i}^{t_0}(q_0) = \mu_i^* \mid \phi_{\mu_i}^{t_0 - 1}(q_0) = \mu_i)$ for $i = 1, 2$.

Observe that $\bigcup_{(\mu_1, \mu_2) \in E_{\mu_1, \mu_2}} = E_{\mu_1, \mu_2}$. We aim to apply the second moment method to show that the size of each class of this refined partition is well-concentrated around its expectation.

Claim A.6. For any $\mu_1 \in \Sigma^{t_0}$ and $\mu_2^* \in \Sigma^{t_0+1}$, and for any $s \in \mathbb{N}$, we have

$$
\mathbb{E}\left(m_{\mu_1, \mu_2} \mid m_{\mu_1, \mu_2} = s\right) = s q_{\mu_2}^*.
$$

This result follows directly from Claim A.2, which gives us the distribution of an out-story and the fact that the in-story and out-story along any half-edge are independent of each other. We omit the details.

We will use $E = E(\sigma, \sigma')$ to denote the set of half-edges, and for half-edges $e_1, e_2 \in E$ we will use the notation $e_1 \sim e_2$ to denote that they are incident (i.e. they are half-edges at the same endvertex), and correspondingly $e_1 \not\sim e_2$ to denote that they are not incident.

Claim A.7. For any $\mu_1 \in \Sigma^{t_0}$ and $\mu_2^* \in \Sigma^{t_0+1}$, and for any $s \in \mathbb{N}$, we have $\text{Var}\left(m_{\mu_1, \mu_2} \mid m_{\mu_1, \mu_2} = s\right) = O(n)$.

Proof. We have

$$
\mathbb{E}\left(m_{\mu_1, \mu_2}^2 \mid m_{\mu_1, \mu_2} = s\right) = \sum_{e_1, e_2 \in E_{\mu_1, \mu_2}} \mathbb{P}\left(e_1, e_2 \in E_{\mu_1, \mu_2} \mid e_1, e_2 \in E_{\mu_1, \mu_2}\right)
$$

$$
\leq \sum_{e_1, e_2 \in E_{\mu_1, \mu_2}} 1 + \sum_{e_1, e_2 \in E_{\mu_1, \mu_2}} \mathbb{P}\left(e_1, e_2 \in E_{\mu_1, \mu_2}^+ \mid e_1, e_2 \in E_{\mu_1, \mu_2}\right)
$$

$$
\leq \sum_{e \in E} d(e)^2 + \left(\sum_{e \in E_{\mu_1, \mu_2}} \mathbb{P}\left(e \in E_{\mu_1, \mu_2}^* \mid e \in E_{\mu_1, \mu_2}\right)\right)^2
$$

$$
= O(n) + \mathbb{E}\left(m_{\mu_1, \mu_2} \mid m_{\mu_1, \mu_2} = s\right)^2,
$$

and the claim follows by re-ordering. \hfill \Box

A standard application of Chebyshev’s inequality now gives the following.

Corollary A.8. With probability at least 1/2, we have that $|m_{\mu_1, \mu_2} - m_{\mu_1, \mu_2} q_{\mu_2}^*| = O(\sqrt{n})$ for every $\mu_1 \in \Sigma^{t_0}$ and $\mu_2^* \in \Sigma^{t_0+1}$ simultaneously.

Furthermore, w.h.p. $|m_{\mu_1, \mu_2} - m_{\mu_1, \mu_2} q_{\mu_2}^*| = o(\omega_0 \sqrt{n})$ for every $\mu_1 \in \Sigma^{t_0}$ and $\mu_2^* \in \Sigma^{t_0+1}$ simultaneously.

Corollary A.8 will allow us to estimate the probability of the statistics deviating by certain amounts, but more difficult is to handle the probability that the statistics match as required.
A.4. The central concept: $a$-events. In this subsection we will introduce a concept which we call $a$-events. Intuitively, an $a$-event can be viewed as a success event if we have $a$ "degrees of freedom". We may think of each degree of freedom as representing a normal distribution with standard deviation $\sqrt{n}$, and the success event will require this variable to hit a specific value, which in general has probability of order $1/\sqrt{n}$. Thus an $a$-event will have fundamental probability of order $n^{-a/2}$. However it also accounts for deviations from the mean, either in terms of the "target" value or in terms of some output parameters. The formal definition is as follows.

**Definition A.9.** Suppose that we are given a discrete probability space $(\mathcal{Z}, \mathbb{P}_Z)$ (dependent on $n$) and an event $\mathcal{F}$ in that space. Suppose also that in the probability space we have parameters $\alpha, \beta \in \mathbb{Z}$, where $\alpha = \alpha(Z)$ is a parameter of the space and $\beta = \beta(Z)$ for $Z \in \mathcal{Z}$ is a parameter of each element in the space, which we call the input deviation and output deviation respectively. Given an integer $a$, we say that the event $\mathcal{F}$ is an $a$-event with input deviation $\alpha$ and output deviation $\beta$ if the following conditions hold.

\[
\begin{align*}
\mathbb{P}_Z(\mathcal{F}) &= O(n^{-a/2}) \\
\mathbb{P}_Z(\mathcal{F}) &= o(n^{-a/2}) \quad \text{if } \alpha = \Omega(\omega_0 \sqrt{n}) \\
\mathbb{P}_Z(\mathcal{F} \cap \{\beta = \Omega(\omega_0 \sqrt{n})\}) &= o(n^{-a/2}) \\
\mathbb{P}_Z(\mathcal{F} \cap \{\beta = O(\sqrt{n})\}) &= \Theta(n^{-a/2}) \quad \text{if } \alpha = O(\sqrt{n}).
\end{align*}
\]

As a very simple (and in fact degenerate) application of this concept, we can reformulate (a slightly weaker version of) Corollary A.8 in this language.

**Corollary A.10.** The event $\mathcal{F} = \top$ is a $0$-event with input deviation $0$ and output deviation

\[
\max_{\mu_1, \mu_2 \in \Sigma} \left| \frac{m_{\mu_1, \mu_2, \sigma^2}}{\sigma} - m_{\mu_1, \mu_2, \sigma^2} \right|.
\]

Before we continue with the formal proof, let us give an informal description of how we will use the concept of $a$-events.

We will have a total success probability which is given by a sequence of $a$-events (for different values of $a$). Intuitively, we will use (A.7) in each application to estimate the probability that we always succeed and that furthermore all statistics are close to their expectations. On the other hand, (A.5) and (A.6) show that if there is a significant deviation in any single step, we obtain a much smaller probability in that step, while (A.4) shows that this is never compensated by a much higher success probability in any later step. Altogether this will give a good estimate on the success probability and also show that conditioned on success, any large deviations are unlikely.

From this informal description, it is clear that we will need to concatenate $a$-events.

**Proposition A.11.** Suppose that $(Z_1, \mathbb{P}_{Z_1})$ is a probability space and for each possible outcome $Z_1 \in \mathcal{Z}_1$ we have a probability space $(Z_2, \mathbb{P}_{Z_2})$, where $Z_2 = Z_2(Z_1)$. Suppose $Z_1$ has input and output deviations $\alpha_1$ and $\beta_1$ respectively, while $Z_2$ has input and output deviations $\alpha'_2$ and $\beta_2$ respectively, and suppose that $\alpha_2' = \alpha_2' (Z_2(Z_1)) = \alpha_2 + O(\beta_1 (Z_1))$ for some $\alpha_2$ and for every $Z_1 \in \mathcal{Z}_1$.

Suppose that $\mathcal{F}_1 \subseteq \mathcal{Z}_1$ is an $a_1$-event with input deviation $\alpha_1$ and output deviation $\beta_1$, and that $\mathcal{F}_2 \subseteq \mathcal{Z}_2$ is an $a_2$-event with input deviation $\alpha_2'$ and output deviation $\beta_2$ (for any $Z_1$).

Then $\mathcal{F}_1 \cap \mathcal{F}_2$ is an $(a_1 + a_2)$-event with input deviation $\max(\alpha_1, \alpha_2')$ and output deviation $\max(\beta_1, \beta_2)$.

**Proof.** We check the four conditions for being an $(a_1 + a_2)$-event separately.

Property (A.4): By applying (A.4) for $\mathcal{F}_1$ and $\mathcal{F}_2$ individually, for any $\alpha_1, \alpha_2 \in \mathbb{Z}$ we have

\[
\mathbb{P}(\mathcal{F}_1 \cap \mathcal{F}_2) = \left( \sum_{Z_1 \in \mathcal{Z}_1} \mathbb{P}_{Z_1}(Z_1) \cdot \mathbb{P}_{Z_2(Z_1)}(\mathcal{F}_2) \right) = O(n^{-a_1/2}) \cdot O(n^{-a_2/2}) = O(n^{-(a_1 + a_2)/2}).
\]

Property (A.5): Suppose first that $\alpha_1 = \Omega(\omega_0 \sqrt{n})$. By applying (A.5) for $\mathcal{F}_1$ and (A.4) for $\mathcal{F}_2$, for any $\alpha_2 \in \mathbb{Z}$ we obtain

\[
\mathbb{P}(\mathcal{F}_1 \cap \mathcal{F}_2) = o(n^{-a_1/2}) \cdot O(n^{-a_2/2}) = o(n^{-(a_1 + a_2)/2}).
\]

On the other hand, if $\alpha_2 = \Omega(\omega_0 \sqrt{n})$ but $\alpha_1 = o(\omega_0 \sqrt{n})$, then we also have $\alpha_2' = \Omega(\omega_0 \sqrt{n})$, and applying (A.4) for $\mathcal{F}_1$ and (A.5) for $\mathcal{F}_2$, we obtain

\[
\mathbb{P}(\mathcal{F}_1 \cap \mathcal{F}_2) = O(n^{-a_1/2}) \cdot o(n^{-a_2/2}) = o(n^{-(a_1 + a_2)/2}).
\]
Property \textbf{A.6}: By applying \textbf{A.4} for \(\mathcal{S}_1\) and \textbf{A.6} for \(\mathcal{S}_2\), for any \(a_1, a_2 \in \mathbb{Z}\) we obtain
\[
P\left(\mathcal{S}_1 \cap \mathcal{S}_2 \cap \{\beta_2 = \Omega(\omega_0 \sqrt{n})\}\right) = O\left(n^{-a_1/2}\right)\cdot o\left(n^{-a_2/2}\right) = o\left(n^{-(a_1+a_2)/2}\right).
\]
Similarly, by applying \textbf{A.6} for \(\mathcal{S}_1\) and \textbf{A.4} for \(\mathcal{S}_2\), for any \(a_1, a_2 \in \mathbb{Z}\) we obtain
\[
P\left(\mathcal{S}_1 \cap \mathcal{S}_2 \cap \{\beta_1 = \Omega(\omega_0 \sqrt{n})\}\right) = O\left(n^{-a_1/2}\right)\cdot O\left(n^{-a_2/2}\right) = o\left(n^{-(a_1+a_2)/2}\right).
\]
Since \(\max(\beta_1, \beta_2) = \Omega(\omega_0 \sqrt{n}) = \{\beta_1 = \Omega(\omega_0 \sqrt{n})\} \cup \{\beta_2 = \Omega(\omega_0 \sqrt{n})\}\), a union bound completes the argument.

Property \textbf{A.7}: Let us first observe that if \(\alpha_2, \beta_2 = O(\sqrt{n})\), then also \(\alpha'_2 = \alpha_2 - O(1)/\sqrt{n}\). Therefore applying \textbf{A.6} for both \(\mathcal{S}_1\) and \(\mathcal{S}_2\), for any \(a_1, a_2 \in O(\sqrt{n})\) we obtain
\[
P\left(\mathcal{S}_1 \cap \mathcal{S}_2 \cap \{\beta_1, \beta_2 = O(\sqrt{n})\}\right) = \sum_{\beta_1 \in \mathcal{S}_1} P_{\mathcal{S}_1}(Z_1) \cdot P_{\mathcal{S}_2}(Z_2) \cdot P\left(\mathcal{S}_2 \cap \{\beta_2 = O(\sqrt{n})\}\right)
= P\left(\mathcal{S}_1 \cap \{\beta_1 = O(\sqrt{n})\}\right) \cdot \Theta\left(n^{-a_2/2}\right)
= \Theta\left(n^{-a_1/2}\right) \cdot \Theta\left(n^{-a_2/2}\right) = \Theta\left(n^{-(a_1+a_2)/2}\right).
\]
This proves the required lower bound, while the corresponding upper bound simply follows from \textbf{A.4}, which we have already proved for \(\mathcal{S}_1 \cap \mathcal{S}_2\).

We also need a similar proposition for \(\alpha\)-events which are independent of one another.

**Proposition A.12.** Suppose \(k \in \mathbb{N}\) and for each \(i \in [k]\) independently, let \((\mathcal{Z}_i, P_{\mathcal{Z}_i})\) be a probability space and \(\mathcal{S}_i \subseteq \mathcal{Z}_i\) be an \(\alpha_i\)-event with input and output deviations \(\alpha_i, \beta_i\) respectively. Then \(\mathcal{S} := \bigcap_{i \in [k]} \mathcal{S}_i\) is an \(\alpha\)-event, where
\[
\alpha := \sum_{i \in [k]} \alpha_i, \quad \text{with input and output deviations } \alpha := \max_{i \in [k]} \alpha_i \text{ and } \beta := \max_{i \in [k]} \beta_i.
\]
This proposition is elementary to prove directly, or can alternatively be deduced from Proposition \textbf{A.11}. For example in the case when \(k = 2\) we would apply Proposition \textbf{A.11} with \(\alpha'_2 = \alpha_2\) and all other parameters identical. We omit the details.

We need one final technical result about \(\alpha\)-events. In some situations we will want to slightly change what we consider to be the input and output deviations. The following claim says that, under certain reasonable conditions, this is allowed.

**Claim A.13.** Suppose that \(a \in \mathbb{N}\) is an integer and \(\mathcal{S}\) is an \(\alpha\)-event in a probability space \((\mathcal{Z}, P_{\mathcal{Z}})\) with input and output deviations \(\alpha, \beta_1\) respectively. Suppose further that \(\alpha_2 = \alpha_2(\mathcal{Z})\) is a parameter of the space and \(\beta_2(\mathcal{Z})\) is a parameter of each element in the space, and that deterministically \(\max(\alpha_2, \beta_2) = \Theta(\max(\alpha_1, \beta_1))\).

Then \(\mathcal{S}\) is also an \(\alpha\)-event with input and output deviations \(\alpha_2, \beta_2\) respectively.

**Proof.** The proof simply consists of checking properties \textbf{A.4}–\textbf{A.7} in the definition of an \(\alpha\)-event. We also need the following elementary observations:
\begin{itemize}
  \item If \(\alpha_2 = \Omega(\omega_0 \sqrt{n})\), then either \(\alpha_1 = \Omega(\omega_0 \sqrt{n})\) or \(\beta_1(\mathcal{Z}) = \Omega(\omega_0 \sqrt{n})\) for all \(Z \in \mathcal{Z}\);
  \item If \(\beta_2(\mathcal{Z}) = \Omega(\omega_0 \sqrt{n})\), then either \(\alpha_1 = \Omega(\omega_0 \sqrt{n})\) or \(\beta_1(\mathcal{Z}) = \Omega(\omega_0 \sqrt{n})\);
  \item If \(\alpha_2, \beta_2(\mathcal{Z}) = O(\omega_0 \sqrt{n})\), then also \(\alpha_1, \beta_1(\mathcal{Z}) = O(\omega_0 \sqrt{n})\).
\end{itemize}
We omit the remaining tedious details.

**A.5. Simplified setup.** We now describe a slightly more concrete setup which we will later apply to our setting. Note that this is an auxiliary setup with simplified notation, and we will later describe what all of the sets and parameters will represent in our application.

**Setup A.14.** Suppose we are given the following situation.
\begin{itemize}
  \item We have two constants \(k, \ell \in \mathbb{N}\);
  \item We have two disjoint (possibly empty) sets of elements \(X, Y\);
  \item \(X\) is partitioned into classes \(X_i\) for \(i \in [k]\) and \(Y\) is partitioned into classes \(Y_j\) for \(j \in [\ell]\);
  \item There are strictly positive constants \(p_i\) for \(i \in [k]\) and \(q_j\) for \(j \in [\ell]\) such that \(\sum_{i \in [k]} p_i = \sum_{j \in [\ell]} q_j = 1\).
\end{itemize}

We then refine each partition by assigning each \(x \in X_i\) to the subclass \(X_{i,j}\) with probability \(q_j\) (independently for each \(x\)), and similarly assigning each \(y \in Y_j\) to the subclass \(Y_{j,i}\) with probability \(p_i\) (independently for each \(y\)).
We will need to track some size deviations in this setup, some of which are determined by the input (i.e. deviations of the $|X_i|, |Y_i|$ from certain values), and some of which are a result of the random subpartitioning (dependent on the $|X_{i,j}|, |Y_{j,i}|$).

**Definition A.15.** Under Setup A.14 let us define the parameters

\[
\begin{align*}
s_X & := \max_i \{|X_i| - |X|p_i\} \\
s_Y & := \max_j \{|Y_j| - |Y|q_j\} \\
s_{X,Y} & := \max(s_X, s_Y).
\end{align*}
\]

Note that these parameters are not random, since they are determined by the input. We also need to track some random deviations.

**Definition A.16.** Under Setup A.14 let us define the parameters

\[
\begin{align*}
x_{i,j} & := |X_{i,j}| - |X|p_iq_j; \\
y_{j,i} & := |Y_{j,i}| - |Y|p_iq_j; \\
z_X & := \max_{i \in [k], j \in [\ell]} x_{i,j}; \\
z_Y & := \max_{i \in [k], j \in [\ell]} y_{j,i}; \\
z_{X,Y} & := \max(z_X, z_Y).
\end{align*}
\]

Note that these parameters are indeed random, since they are dependent on the random subpartitioning.

### A.6. Partitioning propositions.

The most common case we will have is when $X$ represents the set of half-edges with a particular $(t_0 - 1)$-story $\mu_1, \mu_2$, while $Y$ will represent the set of half-edges with the dual $(t_0 - 1)$-story $\mu_2, \mu_1$. To track whether statistics continue to match based on the further messages at time $t_0$, which are modelled by the subpartitioning into $X_{i,j}, Y_{j,i}$, we define the appropriate events as follows.

Under Setup A.14 let us define the events

\[
E_{i,j} := \{|X_{i,j}| = |Y_{j,i}|\} \quad \text{for all } i \in [k], j \in [\ell];
\]

\[
E := \bigcap_{i \in [k], j \in [\ell]} E_{i,j}.
\]

Note that if $E$ holds, then $z_X = z_Y$.

**Proposition A.17.** Under Setup A.14 suppose we also have the further properties that

- $|X| = |Y| = \Theta(n)$;
- $|X_i|, |Y_j| \geq \epsilon^2 n$ for all $i \in [k], j \in [\ell]$ for some sufficiently small constant $\epsilon$.

Then $E$ is a $(k\ell - 1)$-event with input and output deviations $s_{X,Y}$ and $z_{X,Y}$ respectively.

We will prove this Proposition in Appendix B. We also need a variant of Proposition A.17 which we will use when subpartitioning the diagonal classes of the partition (i.e. those in $R_{n-1}$), where we have to handle the further diagonal classes (which are in $R_n$) differently. In this special case, $Y$ is actually empty (which means that automatically $s_Y = z_Y = 0$), and we partition only $X$. For this case, we define the events

\[
\tilde{E}_{i,j} := \{|X_{i,j}| = |X_{j,i}|\} \quad \text{for } 1 \leq i < j \leq k;
\]

\[
\tilde{E}_{i,i} := \{|X_{i,i}| \text{ is even}\} \quad \text{for } i \in [k];
\]

\[
\tilde{E} := \bigcap_{i < j} \tilde{E}_{i,j}.
\]

**Proposition A.18.** Under Setup A.14 suppose we also have the further properties that

- $Y = \emptyset$
- $|X| = \Theta(n)$
- $|X_i| \geq \epsilon^2 n$ for all $i \in [k]$ for some sufficiently small constant $\epsilon$
- $\ell = k$
- $q_i = p_i$ for all $i \in [k]$.
Then $\bar{\varepsilon}$ is a $\left(\binom{t}{2} - 1\right)$-event with input and output deviations $s_x$ and $z_x$ respectively.

Proposition A.18 will also be proved in Appendix B.

A.7. A unifying lemma. Before proving the two Partitioning Propositions, we go on to show how to use them to prove Propositions A.2 and A.3.

In fact, both these results follow directly from the following lemma. Recall that $\mathcal{A}$ is the event that $|E_{\mu, \mu}| = |E_{\mu, \mu}|$ for all $(\mu_1, \mu_2) \in Q_0$ and that $|E_{\mu, \mu}|$ is even for all $(\mu_1, \mu_2) \in R_0$. For any $t \in \mathbb{N}$ and $\mu \in \Sigma^{t+1}$, let us define $\bar{\mu}$ to consist of the first $t$ entries of $\mu$.

Lemma A.19. $\mathcal{A}$ is a $\frac{|Q_0| - 2|R_0|}{2}$-event with input deviation 0 and output deviation

$$
\max_{t \in [t_0]} \left| m_{\mu_1, \mu_2} - m_{\mu_1, \mu_2} q_{\mu_1} q_{\mu_2} \right|.
$$

Before proving this lemma, we show how it proves our second and third main propositions. For convenience, we set

$$
\sigma_t := \max_{\mu_1, \mu_2 \in \Sigma^t} \left| m_{\mu_1, \mu_2, \mu_2} - m_{\mu_1, \mu_2} q_{\mu_1}^* q_{\mu_2}^* \right|,
$$
i.e. the largest deviation produced solely in step $t$, and also define

$$
s_{t-1} := |m - d_n|.
$$

We set

$$
s_{t-1} := \max_{-1 \leq r \leq t} s_r,
$$
i.e. the largest one-step deviation at any time up to $t$. It is important to observe that for any $\mu_1, \mu_2 \in \Sigma^{t+1}$,

$$
|m_{\mu_1, \mu_2} - m_{\mu_1, \mu_2}| = \left| m_{\mu_1, \mu_2} - d_n q_{\mu_1} q_{\mu_2} \right|
\leq \left( \sum_{0 \leq r \leq t} q_{\mu_1, \mu_2} \left| m_{\mu_1, \mu_2} - q_{\mu_1} q_{\mu_2} \right| \right) + q_{\mu_1} q_{\mu_2} |m - d_n|
\leq \left( \sum_{-1 \leq r \leq t} 1 \cdot s_r \right),
$$
(8)

Proof of Proposition A.3. The statement of this proposition simply follows from Lemma A.19 and in particular from Property A.7 of an a-event, along with the observation that if

$$
s_{t_0} = \max_{t \in [t_0]} \left| m_{\mu_1, \mu_2} - m_{\mu_1, \mu_2} q_{\mu_1} q_{\mu_2} \right| = O(\sqrt{n}),
$$
then for any $\mu_1, \mu_2 \in \Sigma^{t_0+1}$, by A.9 we have $|m_{\mu_1, \mu_2} - m_{\mu_1, \mu_2}| = O(\sqrt{n})$.

Proof of Proposition A.2. Similarly, the statement of this proposition follows from Lemma A.19 and in particular from Property A.6 of an a-event.

We now show how Lemma A.19 follows from the Partitioning Propositions.

Proof of Lemma A.19. Recall that we have already proved the base case $t_0 = -1$, so let us assume that the statement holds for $-1, 0, 1, \ldots, t_0 - 1$. In the proof we will use the following notation. Given $\mu \in \Sigma^{t_0}$, let EXT($\mu$) denote the set of messages $\sigma \in \Sigma$ such that $P(q_{\mu}^{t_0}(q_0) = (\mu, \sigma)) \neq 0$, i.e. the set of messages with which a $(t_0 - 1)$-story $\mu$ has a positive probability of being extended to a $t_0$-story, and let $\text{ext}(\mu) := |\text{EXT}(\mu)|$.

We will need to handle pairs $(\mu_1, \mu_2) \in S_{t_0-1}$ differently depending on whether they lie in $Q_{t_0-1}$ or $R_{t_0-1}$, or in other words, depending on whether they are off-diagonal or diagonal pairs.

Case 1: Off-diagonal pairs. Given a pair $\mu_1, \mu_2 \in Q_{t_0-1}$ let $\ell := \text{ext}(\mu_1)$ and $k := \text{ext}(\mu_2)$. Further, let $\sigma_1, \ldots, \sigma_\ell$ be the elements of EXT($\mu_1$) and $\tau_1, \ldots, \tau_k$ be the elements of EXT($\mu_2$) in some arbitrary order.
We now apply Setup A.14 with the $k$ and $\ell$ as just defined, and with the following further parameters.

$$X := E_{\mu_1, \mu_2},$$
$$X_i := E_{\mu_1, \mu_2, \tau_i},$$
$$Y := E_{\mu_2, \mu_1},$$
$$Y_j := E_{\mu_2, \sigma_j, \mu_1},$$
$$p_i := q^*_{(\mu_1, \tau_i)} = \Pr \left( \phi_{\delta, 0}^{\sigma_i} (q_0) = (\mu_1, \tau_i) \mid \phi_{\delta, 0}^{\sigma_i-1} (q_0) = \mu_1 \right),$$
$$q_j := q^*_{(\mu_2, \sigma_j)} = \Pr \left( \phi_{\delta, 0}^{\sigma_j} (q_0) = (\mu_2, \sigma_j) \mid \phi_{\delta, 0}^{\sigma_j-1} (q_0) = \mu_2 \right).$$

By Corollary A.8 and a union bound over all $i \in [k]$ and $j \in [\ell]$, we know that for each $i \in [k]$, $T$ is a $0$-event with input deviation 0 and output deviation $s_{X,Y}$. Meanwhile, Proposition A.17 tells us that, given the partitions of $X$ and $Y$ into $X_i$ and $Y_j$, the event $\mathcal{E}$ that $|X_i| = |Y_j|$ for all $i \in [k], j \in [\ell]$ is a $(k\ell - 1)$-event with input deviation $s_{X,Y}$ and output deviation $z_{X,Y}$. Therefore we may apply Proposition A.11 to deduce that, prior to revealing the partitions of $X$ and $Y$ into $X_i$ and $Y_j$, $\mathcal{E}$ is a $(k\ell - 1)$-event with input deviation 0 and output deviation $\max \{s_{X,Y}, z_{X,Y}\}$, and therefore also with output deviation $z_{X,Y}$ (this is easy to verify from the definition).

This is also independent for any further choice of $\mu_1, \mu_2$, except for the dual pair $(\mu_2, \mu_1)$ (where the statistics match if and only if they match for $(\mu_1, \mu_2)$). By applying Proposition A.12 we deduce that in total, the event that statistics continue to match when revealing the subpartition of all pairs in $R_{0_{-1}}$ is a

$$\left( \frac{1}{2k\ell} \sum_{(\mu_1, \mu_2) \in Q_{0_{-1}}} \left( \text{ext}(\mu_2) \text{ext}(\mu_1) - 1 \right) \right)$$

with input deviation 0 and output deviation

$$s^*_{0_{-1}} := \max_{(\mu_1, \mu_2) \in Q_{0_{-1}}} \left| m_{(\mu_1, \sigma_1), (\mu_2, \sigma_2)} - m_{\mu_1, \mu_2} q^*_{(\mu_1, \sigma_1)} q^*_{(\mu_2, \sigma_2)} \right|.$$ 

**Case 2: Diagonal pairs.** On the other hand, suppose that $(\mu_1, \mu_2) \in R_{0_{-1}}$, so in fact $\mu_2 = \mu_1$. Let $k := \text{ext}(\mu_1)$ and let $\tau_1, \ldots, \tau_k \in \Sigma$ be the elements of $\text{EXT}(\mu_1)$ in some arbitrary order. Let us first reveal which half-edges of $E_{\mu_1, \mu_1}$ are placed into $E_{\mu_1, (\mu_1, \tau_i)}$ for each $i \in [k]$.

Now we reveal the further partition of half-edges, and apply Proposition A.10 with the following parameters.

$$X := E_{\mu_1, \mu_2},$$
$$X_i := E_{\mu_1, (\mu_1, \tau_i)},$$
$$p_i := q^*_{(\mu_1, \tau_i)} = \Pr \left( \phi_{\delta, 0}^{\sigma_i} (q_0) = (\mu_1, \tau_i) \mid \phi_{\delta, 0}^{\sigma_i-1} (q_0) = \mu_1 \right).$$

Then Proposition A.10 tells us that $\mathcal{E}$ is a $\left( \frac{k}{2} - 1 \right)$-event with input deviation $s_X = \max_{i \in [k]} |X_i| - |X|/2$ and output deviation $z_X$. Similarly to the argument for diagonal pairs, observing that $s_X \leq \max_{(\mu_1, \mu_2) \in \Sigma_{0_{-1}}} \left| m_{(\mu_1, \mu_2), (\mu_2, \sigma_2)} - m_{\mu_1, \mu_2} q^*_{(\mu_1, \sigma_1)} q^*_{(\mu_2, \sigma_2)} \right|$, Corollary A.10 and Proposition A.11 tell us that prior to revealing which half edges of $E_{\mu_1, \mu_1}$ are placed into $E_{\mu_1, (\mu_1, \sigma_1)}$ for each $i \in [k]$, the event $\mathcal{E}$ is a $\left( \frac{k}{2} - 1 \right)$-event with input deviation 0 and output deviation $z_X$.

By applying Proposition A.12 in total, the event that statistics continue to match when revealing the subpartition of all pairs in $R_{0_{-1}}$ is a

$$\left( \sum_{(\mu_1, \mu_1) \in Q_{0_{-1}}} \left( \text{ext}(\mu_1) - 1 \right) \right)$$

with input deviation 0 and output deviation

$$s^*_{0_{-1}} := \max_{(\mu_1, \mu_1) \in Q_{0_{-1}}} \left| m_{(\mu_1, \sigma_1), (\mu_2, \sigma_2)} - m_{\mu_1, \mu_2} q^*_{(\mu_1, \sigma_1)} q^*_{(\mu_2, \sigma_2)} \right|.$$
Now to combine both cases, we need to observe that \( s'_0 + s''_0 = s_0 \), and that

\[
a_{i_0} := \frac{1}{2} \sum_{(\mu_1, \mu_2) \in Q_{b_0-1}} \left( \text{ext}(\mu_2) \text{ext}(\mu_1) - 1 \right) + \sum_{(\mu_1, \mu_2) \in R_{b_0-1}} \left( \frac{\text{ext}(\mu_1)}{2} - 1 \right)
= \frac{1}{2} \left( \sum_{(\mu_1, \mu_2) \in Q_{b_0-1}} \text{ext}(\mu_2) \text{ext}(\mu_1) + \sum_{(\mu_1, \mu_2) \in R_{b_0-1}} 2 \frac{\text{ext}(\mu_1)}{2} \right) - \frac{|Q_{b_0-1}|}{2} - |R_{b_0-1}|
= \frac{|Q_{b_0}| - |Q_{b_0-1}|-2|R_{b_0-1}|}{2}.
\]

Together with the induction hypothesis, this shows that in total the event that statistics match when revealing all the subpartitionings of \( S_0 - 1 \) is a \( b_0 \)-event, where

\[
b_{i_0} = b_{i_0-1} + a_{i_0} = \frac{|Q_{b_0-1}| - 2|R_{b_0-1}|}{2} + \frac{|Q_{b_0}| - |Q_{b_0-1}| - 2|R_{b_0-1}|}{4} = \frac{|Q_{b_0}| - 2|R_{b_0-1}|}{2},
\]
as required. \( \square \)

**Appendix B. Proofs of partitioning results**

B.1. **An even finer look: Partitioning claims.** To prove Proposition A.17 we will first reveal the subpartition of the \( X_i \) into \( X_{i,j} \), then reveal the subpartition of the \( Y_j \) into \( Y_{j,i} \) keeping track of the probabilities of various events. To simplify notation still further, we introduce two claims which will be applied to the two subpartitionings. The first will be applied to the subpartitioning of the \( X_i \) into \( X_{i,j} \). We will do this for each \( j \) one at a time for simplicity, which we model with the following setup.

**Setup B.1.** Suppose we are given the following.

- **Natural numbers** \( k \) and \( A_1, \ldots, A_k \geq \varepsilon^2 n \) and constants \( r_1, \ldots, r_k \in (0, 1) \);
- **Natural number** \( a^* \).

Then let us define \( a_i \sim \text{Bin}(A_i, r_i) \) for each \( i \in [k] \) independently.

The purpose of the parameter \( a^* \) is that in the application, we will eventually want \( |X_{i,j}| = |Y_{j,i}| \) for all \( i \) and \( j \). While the \( Y_{j,i} \) have not been revealed yet, once they are revealed, we will certainly need to have \( \sum_{i \in [k]} |X_{i,j}| = \sum_{i \in [k]} |Y_{j,i}| = |Y_j| \), whose value is known. In other words, while we do not have target values for the individual \( |X_{i,j}| \), we do have a target value for this sum (over \( i \) for fixed \( j \)), which will be modelled by \( a^* \) in this setup.

Under Setup B.1 let \( a := \sum_{i \in [k]} a_i \), and let \( \bar{a} := \sum_{i \in [k]} A_i r_i \), so \( a \) is a random variable with expectation \( \bar{a} \) and “target value” \( a^* \). We define the (deterministic) parameter \( s := |a^* - \bar{a}| \) and the (random) parameter \( z := \max_{i \in [k]} ||a_i - A_i r_i|| \).

**Claim B.2.** Under Setup B.1 the event \( \{a = a^*\} \) is a 1-event with input deviation \( s \) and output deviation \( z \).

Let us note that for the proof of Proposition A.17 we would actually need only the special case of this claim where all the \( r_i \) are equal – however, we will also use this claim in the proof of Proposition A.18 where we will need the more general statement.

The proof of Claim B.2 is an elementary exercise in checking that the four properties of Definition A.9 (the definition of an \( a \)-event) are satisfied, and we omit the details.

To handle the partitioning of the \( Y_j \) into \( Y_{j,i} \), we now have a slightly different situation since the \( X_{i,j} \) have already been revealed, so we already have “target values”. This is reflected in the following setup.

**Setup B.3.** Suppose we are given the following:

- **Natural numbers** \( k \) and \( b_1^*, \ldots, b_k^* \geq \varepsilon^2 n \);
- **Constants** \( p_1, \ldots, p_k \in (0, 1) \) with \( \sum_{i \in [k]} p_i = 1 \).

Then let us define \( b_1, \ldots, b_k \) to be the sizes of the classes of a random partition of \( B := \sum_{i \in [k]} b_i^* \) elements into \( k \) parts according to probabilities \( p_1, \ldots, p_k \).

The parameters \( b_i^* \) model \( |X_{i,j}| \), i.e. the “target size” for \( Y_{j,i} \).
Under Setup B.3 let us denote \( b := (b_1, \ldots, b_k) \) and \( b^* := (b_1^*, \ldots, b_k^*) \) and let us define the (deterministic) parameters
\[
s_i := |b_i^* - Bp_i|
\]
\[
s := \max_{i \in [k]} s_i.
\]
Note that there is no corresponding event for a random deviation since we have our target values \( b_i^* \) already as an input. In other words, we have no output deviation, and therefore simply set our output deviation to 0.

Claim B.4. Under Setup B.1 the event \( \{ b = b^* \} \) is a \((k - 1)\)-event with input deviation \( s \) and output deviation \( 0 \).

The proof of Claim B.4 is also an elementary exercise in checking the four properties of an \( a \)-event, which we omit. We remark only that the reason we obtain a \((k - 1)\)-event rather than a \( k \)-event is that if the statistics match (i.e. \( b_i = b_i^* \)) for \( 1 \leq i \leq k - 1 \), then we also have \( b_k = b_k^* \) deterministically, since we certainly have \( \sum_{i \in [k]} b_i = \sum_{i \in [k]} b_i^* \).

B.2. Proof of Proposition A.17

Proof of Proposition A.17. Let us first reveal which elements of \( X_i \) will be placed into \( X_{i,1} \) for each \( i \in [k] \), for which we will apply Setup B.1 with the same parameter \( k \) and further parameters
\[
A_i = A_i(1) := |X_i|,
\]
\[
r_i = r_i(1) := q_1 \quad \text{for all } i \in [k],
\]
\[
a^* = a^*(1) := |Y_1|.
\]
Then Claim B.2 tells us that the event \( \mathcal{E}(1) := \left\{ \sum_{i \in [k]} |X_{i,1}| = |Y_1| \right\} \) is a 1-event with input deviation
\[
s = s(1) = \left| a^* - \sum_{i \in [k]} A_i r_i \right| = |Y_1| - q_1 \sum_{i \in [k]} |X_i|
\]
and output deviation
\[
z = z(1) = \max_{i \in [k]} |a_i - A_i r_i| = \max_{i \in [k]} \left| |X_{i,1}| - q_1 |X_i| \right|.
\]

We now repeat this process for each \( j \in [\ell - 1] \) in turn. More precisely, for each \( j \in [\ell - 1] \) in turn, we apply Setup B.1 with parameters
\[
A_i = A_i(j) := |X_i| - \sum_{j' = 1}^{j-1} |X_{i,j'}|
\]
\[
r_i = r_i(j) := \frac{q_j}{1 - \sum_{j' = 1}^{j-1} q_{j'}} \quad \text{for all } i \in [k]
\]
\[
a^* = a^*(j) := |Y_j|.
\]
Then Claim B.2 tells us that \( \mathcal{E}(j) := \left\{ \sum_{i \in [k]} |X_{i,1}| = |Y_1| \right\} \) is a 1-event with input deviation
\[
s = s(j) = \left| a^* - \sum_{i \in [k]} A_i r_i \right| = |Y_j| - \frac{q_j}{1 - \sum_{j' = 1}^{j-1} q_{j'}} \sum_{i \in [k]} \left( |X_i| - \sum_{j' = 1}^{j-1} |X_{i,j'}| \right)
\]
and output deviation
\[
z = z(1) = \max_{i \in [k]} |a_i - A_i r_i| = \max_{i \in [k]} \left| |X_{i,j}| - \frac{q_j}{1 - \sum_{j' = 1}^{j-1} q_{j'}} \left( |X_i| - \sum_{j' = 1}^{j-1} |X_{i,j'}| \right) \right|.
\]
In order to apply Proposition A.11, it is important to observe that, defining \( z'(j) := \max_{i \in \{k\}} \max_{j' \in [j]} |X_i| - q_j'|X_i,j'| \), we have
\[
s(j) = |Y_j| - \frac{q_j}{1 - \sum_{j'=1}^{j-1} q_{j'}} \sum_{i \in \{k\}} \left( |X_i| - |X_i| \sum_{j'=1}^{j-1} q_{j'} \pm (j-1)z'(j-1) \right)
\]
\[
= |Y_j| - q_j \sum_{i \in \{k\}} |X_i| + O(z'(j-1))
\]
\[
= |Y_j| - q_j |X| + O(z'(j-1)),
\]
and a similar calculation shows that \( z'(j-1) = \Theta(\max_{j' \in \{j-1\}} z(j')) \), and therefore
\[
s(j) = |Y_j| - q_j |X| + O(\max_{j' \in \{j-1\}} z(j')).
\]
Thus we may inductively apply Proposition A.11 with
\[
\alpha_1 = \max_{j' \in \{j-1\}} s(j')
\]
\[
\alpha_2 = s(j)
\]
\[
\beta_1 = \max_{j' \in \{j-1\}} z(j')
\]
\[
\beta_j = z(j)
\]
to deduce that the event of success up to step \( j \) is a \( j \)-event with input deviation \( \max_{j' \in \{j\}} s(j') \) and output deviation \( \max_{j' \in \{j\}} z(j') \).

The case for \( j = \ell \) is different since then we would have \( r_i = \frac{q_i}{1 - \sum_{j'=1}^{\ell-1} q_{j'}} = 1 \), and Setup B.1 does not apply. This simply reflects the fact that all remaining elements of \( X_j \) must deterministically be put into \( X_{i,j} \). Therefore defining \( \mathcal{E}^{(\leq \ell)} := \bigcap_{j \in [\ell]} \mathcal{E}^{(j)} \) for \( j \in [\ell] \), we have the (deterministic) equality \( \mathcal{E} = \mathcal{E}^{(\leq \ell)} = \mathcal{E}^{(\leq \ell-1)} \).

Now let us define \( z_X' := \max_{i \in \{k\}, j \in [\ell]} |X_{i,j}| - |X_i| \). and \( z_X'' := \max_{i \in \{k\}, j \in [\ell-1]} |X_{i,j}| - |X_i| \). (Note that these parameters are similar to \( z_X \), which recall is defined as \( \max_{i \in \{k\}, j \in [\ell]} |X_{i,j}| - |X_i| \).) Observe that
\[
|X_{i,\ell} - |X_i| q_{\ell-1} | = \left| \left( |X_i| - \sum_{j=1}^{\ell-1} |X_{i,j}| \right) - |X_i| \left( 1 - \sum_{j=1}^{\ell-1} q_j \right) \right| \leq \sum_{j=1}^{\ell-1} z_X'' = (\ell-1)z_X',
\]
and therefore \( z_X' \leq z_X'' \leq (\ell-1)z_X' \), meaning that \( z_X'' = \Theta(z_X') \).

Now by concatenating the events \( \mathcal{E}^{(1)}, \ldots, \mathcal{E}^{(\ell-1)} \) and using the observation that \( \mathcal{E} = \mathcal{E}^{(\leq \ell-1)} \), we have that \( \mathcal{E} \) is an \( (\ell-1) \)-event with input deviation \( s_Y \) and output deviation \( z_X'' \). This completes our analysis of the subpartitioning of the \( X_i \) into \( X_{i,j} \).

Now let us fix some arbitrary \( j \in [\ell] \), and if \( \mathcal{E}^{(\leq \ell)} \) holds then we have \( |Y_j| = \sum_{i \in \{k\}} |X_{i,j}| \). For each \( i \in \{k\} \) in turn, we reveal which vertices of \( Y_j \) lie in \( Y_{i,j} \). We apply Setup B.3 with the same parameters \( k, p_1, \ldots, p_k \) and further parameters
\[
B := |Y_j|
\]
\[
b_j := |X_{i,j}| \quad \text{for each } i \in \{k\}
\]
and Claim B.4 tells us that, for each \( j \in [\ell] \) independently, the event \( \bigcap_{i \in \{k\}} \{ Y_{i,j} = X_{i,j} \} \) is a \( (k-1) \)-event with input deviation \( \max_{i \in \{k\}} |X_{i,j}| - |Y_j| p_i \) and output deviation 0, and therefore the event \( \mathcal{E}'' := \bigcap_{j \in [\ell], i \in \{k\}} \{ Y_{i,j} = X_{i,j} \} \) is a \( (k-1) \)-event with input deviation \( z_X', \max_{i \in \{k\}, j \in [\ell]} |X_{i,j}| - |Y_j| p_i \) and output deviation 0.

To complete the proof of Proposition A.12 (i.e. to show that \( \mathcal{E} = \mathcal{E}' \cap \mathcal{E}'' \) is a \( (k\ell-1) \)-event with the appropriate input and output deviations), we will concatenate the two events using Proposition A.12 and observe that \((\ell-1) + (k-1)\ell = k\ell - 1\).

However, the input and output deviations currently do not match as we would like. To complete the proof, we need to observe that
\[
\max \{ s_{X,Y}, z_{X,Y} \} = \Theta(\max \{ s_Y, z_Y', z_{X,Y}' \}),
\]
i.e. that the largest of the input and output deviations required by Proposition A.12 is of the same order as the largest of the input and output deviations given by Proposition A.12 which is sufficient by Claim A.13.
We give only an intuitive and informal overview here – the details are tedious, but elementary. Consider the four types of parameters:

1. \( p_i q_j |X| = q_i q_j |Y| \)
2. \( q_j |X_i| \)
3. \( p_i |Y_j| \)
4. \( |X_{i,j}| = |Y_{j,i}| \).

Each of \( s_{X,Y}, z_{X,Y}, s_x, z_{X,Y}' \) bounds the maximum deviation (over all \( i \) and \( j \)) between two such parameters. In particular, \( s_{X,Y} = \max (s_{X,Y}, s_x) \) and \( z_X \) provide relations between the following pairs.

- \( s_X \): (1) \& (2);
- \( s_y \): (1) \& (3);
- \( z_X = z_y \): (1) \& (4).

Transitivity is guaranteed by the triangle inequality, and therefore the maximum deviation between any pair of parameters, which we denote by \( m^* \), certainly satisfies \( x_1 := \max (s_{X,Y}, z_{X,Y}) \leq m^* = O(x_1) \).

Similarly, \( s_y, z_X', z_{X,Y}' \) give relations between the following pairs.

- \( s_y \): (1) \& (3);
- \( z_X' \): (2) \& (4);
- \( z_{X,Y}' \): (3) \& (4).

Therefore also \( x_2 := \max (s_y, z_{X,Y}', z_{X,Y}') \leq m^* = O(x_2) \), and we deduce that \( x_1 = \Theta(x_2) \) as required.

\[\Box\]

### B.3. Proof of Proposition A.10

#### B.3.1. Step 1: Revealing the diagonal

Let us first reveal which elements of \( X_i \) are placed into \( X_{i,i} \) for each \( i \in [k] \). The probability that \( \tilde{E}_{i,i} \) holds is \( 1/2 + o(1) = \Theta(1) \) for each \( i \). Furthermore, observe that \( \bigcap_{i \in [k]} \tilde{E}_{i,i} \) is a 0-event with input deviation 0 and output deviation \( z_{i,i}^\alpha := \max_{i \in [k]} |X_i,i| - |X_i,i| p_i | \).

#### B.3.2. Step 2: Revealing below the diagonal

Next for each \( j, k = j - 1, \ldots, 3 \) in this order, we will do the following. Reveal which of the elements of

\[ (X_i \setminus X_{i,i}) \setminus \bigcup_{j' \in [k] \setminus [j]} X_{j',i} \]

are placed into \( X_{j,i} \) for each \( i \in [j-1] \). In other words, we reveal the statistics below the diagonal. In order to still have the possibility that eventually \( \delta \) holds (i.e. \( |X_{i,j}| = |X_{j,i}| \) for all \( i, j \)), the relevant restriction in this step is that the row and column sums match, i.e. \( \sum_{i \in [k]} |X_{j,i}| = |X_j| \). However, since we do not reveal all of the statistics along a row, we can only go as far as the diagonal entry. However, since we have already revealed the statistics of the relevant column below the diagonal, we can subtract these from the total. In other words, the relevant restriction is that

\[ \sum_{i \in [j-1]} |X_{j,i}| = \left| X_j \setminus \bigcup_{j' \in [k] \setminus [j-1]} X_{j',j} \right| \]

Let us note that we do not need to consider the case \( j = 2 \) because in that case we have only one class \( X_{2,1} \) into which we are placing the remaining elements of \( X_2 \) if the target sum is attained for larger \( j \), it will also be attained for \( j = 2 \) deterministically.

We apply Setup B.1 with parameters

\[ k = k^{(j)} := j - 1 \]
\[ M_i = M_i^{(j)} := \left| X_i \setminus X_{i,i} \right| \setminus \bigcup_{j' \in [k] \setminus [j]} X_{j',i} \]
\[ \alpha_i = \alpha_i^{(j)} := \frac{p_i}{1 - p_i - \sum_{j' \in [k] \setminus [j]} p_{j'}} \]
\[ r = r^{(j)} := \left| X_j \setminus \bigcup_{j' \in [k] \setminus [j-1]} X_{j',j} \right| \]

and deduce by Claim B.2 that for each \( j \) the event that this restriction is satisfied is a 1-event with input deviation \( s^{(j)} = r^{(j)} - \sum_{i \in [j-1]} M_i^{(j)} \alpha_i^{(j)} \) and output deviation \( \max_{i \in [j-1]} \left| \left| X_{j,i} - M_i^{(j)} \alpha_i^{(j)} \right| \right| \).
We now aim to bound the input deviation $s^{(j)}$ in terms of the output deviations $z''_X, s^{(k)}, \ldots, s^{(j+1)}$ of previous events.

Let us define $s^{(z)} := \max \{z''_X, s^{(j)}, \ldots, s^{(k)}\}$. We now claim by backwards induction on $j$ that $||X_{j,i}|-|X_j||p_i| = O\left(s^{(z)}\right)$ for all $i \in [j]$. For $i = [j]$ this follows directly from the fact that $||X_{j,j}|-|X_j||p_j| \leq z''_X$, by definition of $z''_X$. This represents the (dummy) base case $j = k + 1$.

Assuming the statement holds for $j' \in [k] \setminus [j]$, we have

$$r^{(j)} = |X_j| - \sum_{j' \in [k] \setminus [j-1]} |X_{j',j}| = |X_j| - \sum_{j' \in [k] \setminus [j-1]} |X_j|p_j' + O\left(s^{(z)}\right)$$

while

$$\sum_{i \in [j-1]} M_i^{(j)}a_i^{(j)} = \sum_{i \in [j-1]} \left|X_i\right| - \left|X_{i,i}\right| - \sum_{j' \in [k] \setminus [j]} |X_{j',j}| \frac{p_i}{1 - p_i - \sum_{j' \in [k] \setminus [j]} p_{j'}}$$

$$= \sum_{i \in [j-1]} |X_i| \left(1 - p_i - \sum_{j' \in [k] \setminus [j]} p_{j'} + O\left(s^{(z)}\right)\right) \frac{p_i}{1 - p_i - \sum_{j' \in [k] \setminus [j]} p_{j'}}$$

$$= |X_j|p_j + O\left(s^{(z)}\right),$$

and the result follows.

Indeed, we have also shown here that $M_i^{(j)}a_i^{(j)} = |X_j|p_j + O\left(s^{(z)}\right)$. It follows that the input deviation $s^{(j)}$ for $j$ is $O\left(s^{(z+1)}\right)$, i.e. at most a constant times the output deviation for $j + 1$. This allows us to apply Proposition A.11 and deduce that in total, the event that all statistics on the diagonal are even and below the diagonal hit their targets is a $(k-2)$-event with input deviation 0 and output deviation $s^{(z)}$.

B.3.3. Step 3: Revealing above the diagonal. Finally we reveal the statistics above the diagonal, i.e. for each $j = 3, \ldots, k$ we reveal which elements of $X_j \setminus \bigcup_{j' \in [k] \setminus [j-1]} X_{j',j}$ are placed into $X_{i,j}$ for $i \in [j-1]$. (Observe that we do not need to do this for $j = 2$ since there is only one class in which the elements will be placed.)

For each $j$ in turn we apply Setup B.3 with parameters $k'$ and $b^*_i, p'_i$ given by

$$k' = j - 1$$

$$b^*_i = |X_{i,i}|$$

$$p'_i = \frac{p_i}{1 - p_j - \cdots - p_k}$$

and deduce from Claim B.4 that the event that the statistics match is a $(j-2)$-event with input deviation $s = s^{(j)} = \max_{i \in [j-1]} |b^*_i - p'_i \sum_{i \in [k']} b_{i'}|$. Let us observe that

$$\sum_{i \in [k']} b_{i'}^* = \sum_{i \in [j-1]} |X_{j,i'}| = |X_j| \sum_{i \in [j-1]} p_{i'} + O\left(s^{(z'_3)}\right),$$

and therefore

$$p'_i \sum_{i \in [k']} b_{i'}^* = \frac{p_i}{p_1 + \cdots + p_{j-1}} \cdot \left|X_j\right| \sum_{i \in [j-1]} p_{i'} + O\left(s^{(z'_3)}\right) = p_j|X_j| + O\left(s^{(z'_3)}\right) = s'_j + O\left(s^{(z'_3)}\right),$$

or in other words, the input deviation $s = O\left(s^{(z'_3)}\right)$. Repeating this for each $j \geq 3$ and applying Proposition A.12 and observing that $\sum_{j=3}^k (j-2) = \frac{(k-1)(k-2)}{2}$ we deduce that the event that all statistics above the diagonal hit their targets is a $(k-1)(k-2)$-event with input deviation $\max_{3 \leq j \leq k} s^{(j)} = O\left(s^{(z'_3)}\right)$ and output deviation 0.

We can now apply Proposition A.11 to concatenate the events below the diagonal (an $(k-2)$-event with input deviation 0 and output deviation $s^{(z'_3)}$) and above the diagonal (an $(k-1)(k-2)$-event with input deviation $O\left(s^{(z'_3)}\right)$ and output deviation 0) and deduce that the event is a $(k+1)(k-2)$-event with input deviation 0 and output deviation $s^{(z'_3)}$.

Finally, we observe that $\frac{(k+1)(k-2)}{2} = \binom{k}{2} - 1$ and that $\max_{i,j \in [k]} |X_{i,j}|-|X_{i,j}|p_j| = O\left(s^{(z'_3)}\right)$, which completes the proof of Proposition A.18.
Proof of Lemma 3.5. Let us fix $k_0 := \ln \ln n$ and consider any $k \in [k_0]_0$, denote $\left( \frac{x}{k} \right) := \bigcup_{k \in [k_0]} \left( \frac{x}{k} \right)$ and $\left( \frac{x}{\leq k_0} \right) := \left\{ \frac{x}{\leq k_0} \right\}$, and observe that

$$\left( \frac{\lfloor \Sigma \rfloor}{\leq k_0} \right) \leq k_0 \left( \frac{\lfloor \Sigma \rfloor}{k_0} \right) \leq k_0 |\Sigma|^{k_0} = (\ln n)^{O(1)}.$$  

Now let $k \in \mathbb{N}$ and let $A \in \left( \frac{x}{\leq k_0} \right)$, so $A$ is a potential in-compilation. Let $X = X(A)$ denote the number of vertices of $[n]$ which receive in-compilation $A$, and similarly let $X_W$ be the number of vertices of $W$ which receive in-compilation $A$. For any $x, x_W \in \mathbb{N}$, observe that

$$\mathbb{P} \{ Y = A \mid \mathcal{A} \cap \{ X = x \} \cap \{ X_W = x_W \} \} = \frac{x - x_W}{n - |W|}.$$  

Our aim is to show that w.h.p. this is close to $p_A := \mathbb{P}(Z = A)$ (and indeed the error probability should be low enough that we can apply a union bound over all $A$). To show this we first need to show that $X$ and $X_W$ are strongly concentrated around their expectations. In particular, defining the events

$$\mathcal{E}(A) := \left\{ |X - np_A| \leq n^{2/3} \right\},$$

$$\mathcal{E}_W(A) := \left\{ |X_W - |W||p_A| \leq n^{2/3} \right\},$$

$$\mathcal{E} := \bigcap_{k \in [k_0]} \bigcap_{A \in \left( \frac{x}{\leq k} \right)} \mathcal{E}(A),$$

$$\mathcal{E}_W := \bigcap_{k \in [k_0]} \bigcap_{A \in \left( \frac{x}{\leq k} \right)} \mathcal{E}_W(A),$$

for $A \in \left( \frac{x}{\leq k_0} \right)$ we have

$$\mathbb{P} \{ Y = A \mid \mathcal{A} \cap \mathcal{E}_W \} = \mathbb{P} \{ Y = A \mid \mathcal{A} \cap \mathcal{E}_W \} \cdot \mathbb{P} \{ \mathcal{E} \mid \mathcal{A} \cap \mathcal{E}_W \} + \mathbb{P} \{ Y = A \mid \mathcal{A} \cap \mathcal{E}_W \} \cdot \mathbb{P} \{ \mathcal{E} \mid \mathcal{A} \cap \mathcal{E}_W \}

= \frac{(np_A \pm n^{2/3} - (|W|p_A \pm n^{2/3})}{n - |W|} \cdot \mathbb{P} \{ \mathcal{E} \mid \mathcal{A} \cap \mathcal{E}_W \} \pm \mathbb{P} \{ \mathcal{E} \mid \mathcal{A} \cap \mathcal{E}_W \}.$$

We therefore need to show that $\mathcal{E}_W$ is a high probability event conditioned on $\mathcal{A}$, and that $\mathcal{E}$ is a high probability event conditioned on $\mathcal{A} \cap \mathcal{E}_W$.

Therefore observe that, without the conditioning on Step (3) (i.e. on $\mathcal{A}$), each vertex receives in-compilation $A$ with probability $p_A$ independently, and therefore by the Chernoff bound,

$$\mathbb{P} \{ \mathcal{E}(A) \} = \mathbb{P} \{ |X(A) - np_A| \geq n^{2/3} \} \leq \exp \left( \frac{-n^{4/3}}{2np_A + 2n^{2/3}} \right) \leq \exp \left( -n^{1/4} \right),$$

so

$$\mathbb{P} \{ \mathcal{E} \} \leq \sum_{A \in \left( \frac{x}{\leq k_0} \right)} \mathbb{P} \{ \mathcal{E}(A) \} \leq (\ln n)^{O(1)} \exp \left( -n^{1/4} \right) \leq \exp \left( -n^{1/5} \right).$$

Similarly we also have $\mathbb{P} \{ \mathcal{E}_W \} \leq \exp \left( -n^{1/5} \right)$. Together with the fact that $\mathbb{P}(\mathcal{A}) = n^{-O(1)}$ from Lemma 3.3, we deduce that

$$\mathbb{P} \{ \mathcal{E}_W \mid \mathcal{A} \} \leq \frac{\mathbb{P} \{ \mathcal{E}_W \}}{\mathbb{P}(\mathcal{A})} \leq \exp \left( -n^{1/5} \right) \cdot n^{-O(1)} \leq \exp \left( -n^{1/6} \right).$$

Furthermore, we have

$$\mathbb{P} \{ \mathcal{E} \mid \mathcal{A} \cap \mathcal{E}_W \} \leq \frac{\mathbb{P} \{ \mathcal{E} \}}{\mathbb{P}(\mathcal{A} \cap \mathcal{E}_W)} \leq \exp \left( -n^{1/5} \right) \cdot n^{O(1)} \leq \exp \left( -n^{1/6} \right).$$

Substituting these bounds into (C.1) we deduce that for $A \in \left( \frac{x}{\leq k_0} \right)$ we have

$$\mathbb{P} \{ Y = A \mid \mathcal{A} \cap \mathcal{E}_W \} = \frac{(p_A \pm n^{-1/4}) \cdot (1 \pm \exp \left( -n^{1/6} \right)) \pm \exp \left( -n^{1/6} \right)}{p_A \pm n^{-1/5}}.$$  

(C.2)
Now let us define
\[
d_{TV}^{(1)} = d_{TV}^{(1)}(Y, Z) := \sum_{A \in \{X : 0\}} \left| \mathbb{P}(Y = A \mid \mathcal{A} \cap \mathcal{E}_W) - \mathbb{P}(Z = A) \right|
\]
\[
d_{TV}^{(2)} = d_{TV}^{(2)}(Y, Z) := \sum_{A \in \{X : 0\}} \left| \mathbb{P}(Y = A \mid \mathcal{A} \cap \mathcal{E}_W) - \mathbb{P}(Z = A) \right|
\]
and observe that conditioned on $\mathcal{E}_W$ we have $d_{TV}(Y, Z) = d_{TV}^{(1)} + d_{TV}^{(2)}$. By (2) we have
\[
d_{TV}^{(1)} \leq \left( \frac{1}{k_0} \right) n^{-1/5} \leq n^{-1/6}.
\]
Furthermore, observe that
\[
d_{TV}^{(2)} \leq \sum_{A \in \{X : 0\}} \left( \mathbb{P}(Y = A \mid \mathcal{A} \cap \mathcal{E}_W) + \mathbb{P}(Z = A) \right)
\]
\[
= 2 - \sum_{A \in \{X : 0\}} \left( \mathbb{P}(Y = A \mid \mathcal{A} \cap \mathcal{E}_W) + \mathbb{P}(Z = A) \right)
\]
\[
\leq 2 - 2 \sum_{A \in \{X : 0\}} p_A + \left( \frac{1}{k_0} \right) n^{-1/5}
\]
\[
\leq 2 \left( 1 - \sum_{A \in \{X : 0\}} p_A \right) + n^{-1/6}.
\]
We note that
\[
1 - \sum_{A \in \{X : 0\}} p_A = \mathbb{P}(\text{Po}(d) > k_0) = o(1),
\]
so we have $d_{TV}^{(2)} = o(1)$ and therefore conditioned on $\mathcal{E}_W$ also $d_{TV}(Y, Z) = o(1)$. Since $\mathbb{P}(\mathcal{E}_W \mid \mathcal{A}) = 1 - o(1)$, this proves the lemma. 

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