ISOTROPIC MEASURES AND MAXIMIZING ELLIPSOIDS: BETWEEN JOHN AND LOEWNER

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Abstract. We define a one parameter family of positions of a convex body which interpolates between the John position and the Loewner position: for $r > 0$, we say that $K$ is in maximal intersection position of radius $r$ if $\Vol_n(K \cap rB^2_n) \geq \Vol_n(K \cap rTB^2_n)$ for all $T \in SL_n$. We show that under mild conditions on $K$, each such position induces a corresponding isotropic measure on the sphere, which is simply a normalized Lebesgue measure on $r^{-1}K \cap S^{n-1}$. In particular, for $r_M$ satisfying $r_M^2\kappa_n = \Vol_n(K)$, the maximal intersection position of radius $r_M$ is an $M$-position, so we get an $M$-position with an associated isotropic measure. Lastly, we give an interpretation of John’s theorem on contact points as a limit case of the measures induced from the maximal intersection positions.

1. Introduction and main results

Given a convex body (that is, a compact convex set with non-empty interior) in $\mathbb{R}^n$, the John ellipsoid $J(K)$ is the maximum-volume ellipsoid contained in $K$. The body $K$ is in John position if $J(K) = B^2_n$, the Euclidean unit ball. Dually, the Loewner ellipsoid $L(K)$ is the minimum-volume ellipsoid containing $K$, and $K$ is in Loewner position if $L(K) = B^2_n$. The John and Loewner positions always exist and are unique up to orthogonal transformations. They are dual in the sense that $J(K^\circ) = (L(K))^\circ$ where $L^\circ = \{ y : \langle x, y \rangle \leq 1 \ \forall x \in L \}$ is the dual body of $L$ (see [1] for more details).

A finite Borel measure $\mu$ on $S^{n-1}$ is isotropic if

$$\int_{S^{n-1}} \langle x, \theta \rangle^2 d\mu(x) = \frac{\mu(S^{n-1})}{n}$$

for all $\theta \in S^{n-1}$. In 1948, Fritz John [6] showed the following:

Theorem 1.1 (John). Let $K \subset \mathbb{R}^n$ be a convex body in John position. Then there exists an isotropic measure whose support is contained in $\partial K \cap S^{n-1}$. Moreover, there exists such a measure whose support is at most $n(n+1)/2$ points.

A reverse result was given by K. Ball [2], who showed that if $B^2_n \subseteq K$ and there is an isotropic measure supported on $\partial K \cap S^{n-1}$, then $K$ is in John position. By duality, the same result holds for a body in Loewner position.

John’s theorem is a special case of a general phenomenon: the family $\{TK : T \in SL_n\}$ of a convex body $K$ is called the family of positions of $K$. Giannopoulos and Milman [3] showed that solutions to extremal problems over the positions of a convex body often give rise to isotropic measures, and demonstrated this fact for, among others, the John position, the isotropic position, the minimal surface area position, and an $M$-position.

In this work, we consider a one-parametric family of extremal positions which seems not to have been considered before:

**Definition 1.2.** For a centrally symmetric convex body $K \subset \mathbb{R}^n$, the ellipsoid $E_r$ of volume $r^n\kappa_n$ is a maximum intersection ellipsoid of radius $r$, if

$$\Vol_n(K \cap E_r) \geq \Vol_n(K \cap E)$$

for all ellipsoids $E$ of volume $r^n\kappa_n$, where $\kappa_n = \Vol_n(B^2_n)$. We say that $K$ is in maximal intersection position of radius $r$ if $rB^2_n$ is a maximum intersection ellipsoid of radius $r$.

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In the following, $\mathcal{E}_r$ will always denote a maximum intersection ellipsoid of radius $r$. The set of maximal intersection positions interpolates between the John and Loewner positions: indeed, let $r_J$ be a positive number satisfying $\text{Vol}_n(\partial K) = r_J^n \kappa_n$, and let $r_L$ be such that $\text{Vol}_n(L(K)) = r_L^n \kappa_n$. It can be easily shown that $K$ is in maximal intersection position of radius $r_J$ if and only if $r_J^{-1} K$ is in John position, and similarly for the Loewner position. In other words, up to a scaling, the maximal intersection position of radius $r_J$ is the John position, and the maximal intersection position of radius $r_L$ is the Loewner position.

Our first result is the following:

**Theorem 1.3.** Let $K \subset \mathbb{R}^n$ be a centrally symmetric convex body such that $\text{Vol}_{n-1}(\partial K \cap \partial \mathcal{E}) = 0$ for all but finitely many ellipsoids $\mathcal{E}$, $\text{Vol}_{n-1}(\partial K \cap r S^{n-1}) = 0$, and $\text{Vol}_{n-1}(K \cap r S^{n-1}) > 0$. If $K$ is in maximal intersection position of radius $r$, then the restriction of the surface area measure on the sphere to $S^{n-1} \cap r^{-1} K$ is an isotropic measure.

**Remark 1.4.** Note that the condition $\text{Vol}_{n-1}(\partial K \cap r S^{n-1}) = 0$ cannot be omitted. As an example, consider the convex hull of a ball and two points, e.g., $K = \text{conv}\{B^2_2 \cup (\pm \sqrt{2}, 0)\} \subset \mathbb{R}^2$. Here one may check that $K$ is in John position, and so it is in maximal intersection position of radius $1$. However, the restriction of the surface area measure to $K \cap S^{n-1}$ is clearly not isotropic, as it has more weight in the direction of the $y$ axis than in the direction of the $x$ axis.

We will denote the surface area measure on the sphere by $\sigma$, and for a Borel set $A \subset \mathbb{R}^n$ with $\sigma(A \cap S^{n-1}) > 0$ we let $\mu_A$ denote the restriction of $\sigma$ to $A$, i.e.,

$$\mu_A(B) = \frac{\sigma(B \cap A \cap S^{n-1})}{\sigma(A \cap S^{n-1})}.$$

Note that if $\mu_A$ is isotropic and $\sigma(S^{n-1} \setminus A) > 0$, then $\mu_{S^{n-1} \setminus A}$ is also isotropic.

Theorem 1.3 shows that as in [5], an extremal position induces an isotropic measure. Contrary to John’s Theorem 1.3, in our case we have an explicit description of the isotropic measure, which is uniform on $r^{-1} K \cap S^{n-1}$, namely it is $\mu_{r^{-1} K}$.

Theorem 1.3 does not formally include the result of Theorem 1.4, in the case $r = r_J = 1$, since for $K$ in John position we have $S^{n-1} \subset K$, so Theorem 1.3 merely states that $\sigma$ is isotropic, a triviality. Nevertheless, our second result gives a new interpretation to John’s Theorem. We show that when $K$ is in John position, the isotropic measure which is guaranteed to exist by Theorem 1.3 may be constructed as a limit of the isotropic measures from Theorem 1.3. In other words, as $r$ approaches $r_J$, the corresponding induced measures approach a measure of the type described in John’s theorem:

**Theorem 1.5.** Let $K \subset \mathbb{R}^n$ be a centrally symmetric convex body in John position such that $\text{Vol}_{n-1}(\partial K \cap \partial \mathcal{E}) = 0$ for all but finitely many ellipsoids $\mathcal{E}$. For every $r > 1$, denote by $\mu_r$ the uniform probability measure on $S^{n-1} \cap r^{-1} T_r K$, where $T_r K$ is in maximal intersection position of radius $r$. Then there exists a sequence $r_J \downarrow 1$ such that the sequence of measures $\mu_{r_J}$ weakly converges to an isotropic measure whose support is contained in $\partial K \cap S^{n-1}$.

A similar result holds for the Loewner position:

**Theorem 1.6.** Let $K \subset \mathbb{R}^n$ be a centrally symmetric convex body in Loewner position such that $\text{Vol}_{n-1}(\partial K \cap \partial \mathcal{E}) = 0$ for all but finitely many ellipsoids $\mathcal{E}$. For every $r < 1$, denote by $\nu_r$ the uniform probability measure on $S^{n-1} \cap r^{-1} T_r K$, where $T_r K$ is in maximal intersection position of radius $r$. Then there exists a sequence $r_J \uparrow 1$ such that the sequence of measures $\nu_{r_J}$ weakly converges to an isotropic measure whose support is contained in $\partial K \cap S^{n-1}$.

In the range $[r_J, r_L]$ there is a special radius which we denote $r_M$, defined so that $\text{Vol}_n(K) = r_M^n \kappa_n$, and for this special radius the maximal intersection position of radius $r_M$ is an $M$-position. To explain what this means we need a few more definitions and background.

In the mid-80s, Vitali Milman [9] discovered the existence of a position for convex bodies which enabled him, and the researchers following, to prove many new results, and had a major influence on the field. This position, now called $M$-position, can be described in many different and equivalent ways. We choose one such way, and for an extensive description and the many equivalences see [1].
Theorem 1.7 (Milman). There exists a universal constant $C > 0$ such that for every $n \in \mathbb{N}$ and any centrally symmetric convex body $K \subset \mathbb{R}^n$, there exists a centrally symmetric ellipsoid $E$ with $\text{Vol}_n(E) = \text{Vol}_n(K)$ such that

$$\frac{\text{Vol}_n(K^{o} + E^{o})}{\text{Vol}_n(K^{o} + E^{o})} \geq C^n.$$  

(1.1)

In fact, one may show that if an ellipsoid of the same volume as $K$ satisfies any of the four inequalities

$$\text{Vol}(K^{o} + E^{o}) \leq c_1^n \text{Vol}_n(K), \quad \text{Vol}_n(K^{o} \cap E^{o}) \geq c_1^{-n} \text{Vol}_n(K),$$

$$\text{Vol}(K + E) \leq c_0^n \text{Vol}_n(K), \quad \text{Vol}_n(K \cap E) \geq c_0^{-n} \text{Vol}_n(K),$$

then it must satisfy inequality (1.1) with some constant $C = C(c_1)$ depending only on $c_1$ and not on the body $K$ or on the dimension. For this reason, we shall use the following simple definition for $M$-position:

Definition 1.8. A centrally symmetric convex body $K$ is in $M$-position with constant $C$ if the centrally symmetric Euclidean ball of radius $\lambda = \left(\frac{\text{Vol}(K)}{\kappa_n}\right)^{1/n}$ satisfies

$$\text{Vol}_n(K \cap \lambda B^n_2) \geq C^{-n} \text{Vol}_n(K).$$

Since Milman’s theorem implies that there exists some universal $C$ for which any body has an affine image in $M$-position with constant $C$, we shall usually omit the words “with constant $C$” and talk simply of “$M$-position”, by which we mean an $M$-position with respect to the constant $C$ guaranteed by Milman’s Theorem 1.7.

Clearly, when we maximize the volume of the intersection of $K$ and an ellipsoid of volume $\text{Vol}_n(K)$, we get an $M$-ellipsoid, and when it is a Euclidean ball we get that $K$ is in $M$-position. We have then:

Corollary 1.9. Let $K \subset \mathbb{R}^n$ be a centrally symmetric convex body such that $\text{Vol}_{n-1}(\partial K \cap \partial E) = 0$ for all but finitely many ellipsoids $E$, $\text{Vol}_{n-1}(\partial K \cap r_M S^{n-1}) = 0$, and $\text{Vol}_{n-1}(K \cap r_M S^{n-1}) > 0$, where $r_M = \left(\frac{\text{Vol}(K)}{\kappa_n}\right)^{1/n}$. If $K$ is in maximal intersection position of radius $r_M$, then $K$ is in $M$-position, and the restriction of the surface area measure on the sphere to $S^{n-1} \cap r_M^{-1} K$ is an isotropic measure.

This paper is organized as follows: in Section 2 we provide some basic results regarding the maximal intersection position. The section concludes with a detailed proof of the main ingredient for the proof of Theorem 1.3. In Section 3 we prove the main theorems 1.5 and 1.6. The last section discusses the question of uniqueness of the maximum intersection position, a question that is still open. We show that uniqueness follows from a variant of the (B) conjecture.

2. Preliminaries

In this section we provide some results needed for the proof of the main theorems. We start by showing that for $r > 0$, the maximal intersection position of radius $r$ does in fact exist. We will make frequent use of the following function:

Definition 2.1. For a centrally symmetric convex body $K = -K \subset \mathbb{R}^n$, define for every $r > 0$,

$$m(r) = \sup \{ \text{Vol}_n(K \cap E) : E \text{ is an ellipsoid of volume } r^n \kappa_n \}.$$  

(2.1)

Our first lemma shows that a maximal intersection ellipsoid always exists:

Lemma 2.2. For every centrally symmetric convex body $K \subset \mathbb{R}^n$ and every $r > 0$, the supremum in (2.1) is attained.

Proof. First note that since $K = -K$, the Brunn-Minkowski inequality implies that for every $x \in \mathbb{R}^n$ and every $T \in SL_n$, we have

$$\text{Vol}_n(K \cap (TB^n_2 + x)) \leq \text{Vol}_n(K \cap TB^n_2)$$

(2.2)

and so if the supremum is attained, it is attained on a centrally symmetric ellipsoid. Note that the supremum may also be attained on a non-centrally symmetric ellipsoid only if we have equality in (2.2), which is only possible if $K \cap (TB^n_2 + x)$ and $K \cap (TB^n_2 - x)$ are homothetic. This occurs, for instance, in the case $(TB^n_2 + x) \subset K$ or $K \subset (TB^n_2 + x)$, i.e., when $r < r_J$ or $r > r_L$. 


Let $\mathcal{E}_j = T_jB_2^n$ be a sequence of centrally symmetric ellipsoids where $T_j$ is positive definite with $\det T_j = r^n$ and $\text{Vol}_n(K \cap T_jB_2^n) \to m(r)$. If the sequence defined by the maximum eigenvalue of $T_j$ grows to infinity then $\text{Vol}_n(K \cap T_jB_2^n) \to 0 \neq m(r)$, so the set of eigenvalues of $\{T_j\}_{j=1}^\infty$ must be bounded, which implies that the ellipsoids $T_jB_2^n$ are all contained in a compact set. It now follows from Blaschke’s selection theorem that there exists a subsequence of ellipsoids converging in the Hausdorff distance to a centrally symmetric ellipsoid $\mathcal{E}$ of volume $r^n\kappa_n$ with $\text{Vol}_n(K \cap \mathcal{E}) = m(r)$.

Note the following properties of $m(r)$:

**Lemma 2.3.** Let $K \subset \mathbb{R}^n$ be a centrally symmetric convex body. We have that

1. For $0 < r \leq r_j$ we have $m(r) = r^n\kappa_n$ and for $r \geq r_L$ we have $m(r) = \text{Vol}_n(K)$.
2. The function $m(r)$ is strictly monotone increasing in $[r_J, r_L]$.
3. $m(r)$ is continuous, and satisfies for $t \leq s$ that
   \[ m(t) \leq m(s) \leq \left(\frac{s}{t}\right)^n m(t). \]

**Proof.** Fact (1) is trivial. For (2) let $r_J \leq t < s \leq r_L$ and choose some intersection maximizing ellipsoid $\mathcal{E}_i$. Then
   \[ m(t) = \text{Vol}_n(K \cap \mathcal{E}_i) \leq \text{Vol}_n \left( K \cap \frac{s}{t} \mathcal{E}_i \right) \leq \text{Vol}_n \left( K \cap \mathcal{E}_s \right). \]

If the last inequality is an equality then $K \cap \mathcal{E}_i = K \cap \frac{s}{t} \mathcal{E}_i$ which is only possible if $K \subset \mathcal{E}_i$ (which is impossible since $t < r_L$) or if $\frac{s}{t} \mathcal{E}_i \subset K$ (which is impossible since $s > r_J$).

To prove (3) it is enough to show the right hand side inequality and to this end simply note that
   \[ m(t) = \text{Vol}_n(K \cap \mathcal{E}_i) \geq \text{Vol}_n \left( K \cap \frac{t}{s} \mathcal{E}_s \right) \geq \text{Vol}_n \left( \frac{t}{s} K \cap \frac{t}{s} \mathcal{E}_s \right) = \left( \frac{t}{s} \right)^n \text{Vol}_n \left( K \cap \mathcal{E}_s \right) = \left( \frac{t}{s} \right)^n m(s). \]

By continuity of $m(r)$, we have:

**Lemma 2.4.** Let $K \subset \mathbb{R}^n$ be a centrally symmetric convex body. As $r \searrow r_J$ the ellipsoids $\mathcal{E}_r$ converge to $\mathcal{E}_{r_J} = J(K)$ in the Hausdorff distance.

**Proof.** Since $\text{Vol}_n(K \cap J(K)) = \text{Vol}_n(J(K))$ then by the continuity of $m(r)$, both $\text{Vol}_n(K \cap \mathcal{E}_r)$ and $\text{Vol}_n(\mathcal{E}_r)$ approach $m(r_J) = r_J^n\kappa_n$ as $r \searrow r_J$. Let $T_r$ be a sequence of transformations such that $T_r \mathcal{E}_r = B_2^n$. As before, since $\text{Vol}_n(K \cap T_r^{-1}B_2^n) \to m(r_J)$ then the set $\mathcal{E}_r$ is contained in a compact set. We thus have a converging subsequence $\mathcal{E}_{r_j} \to \mathcal{E}$ with $\text{Vol}_n(\mathcal{E}) = \text{Vol}_n(K \cap K) = r_J^n\kappa_n$, so $\mathcal{E}$ is an ellipsoid contained in $K$ with the same volume as $J(K)$, which is unique. It follows that $\mathcal{E} = J(K)$. Since this was true for any converging subsequence, we get that $\mathcal{E}_r$ converges to $J(K)$ as $r \searrow r_J$.

We will make use of the following fact. The proof is a simple exercise, see e.g. [1]:

**Lemma 2.5.** A Borel measure $\mu$ on $S^{n-1}$ is isotropic if and only if every $A \in M_n(\mathbb{R})$ such that $trA = 0$ has
   \[ \int_{S^{n-1}} \langle x, Ax \rangle d\mu(x) = 0. \]

Lastly, the following theorem is essential for the proof of Theorem 2.3:

**Theorem 2.6.** Let $K \subset \mathbb{R}^n$ be a centrally symmetric convex body such that $\text{Vol}_{n-1}(\partial K \cap \partial \mathcal{E}) = 0$ for all but finitely many ellipsoids $\mathcal{E}$, $\text{Vol}_{n-1}(\partial K \cap S^{n-1}) = 0$ and $\text{Vol}_{n-1}(K \cap S^{n-1}) > 0$. Let $A \in M_n(\mathbb{R})$ with $trA = 0$, and let $V(t) : \mathbb{R} \to \mathbb{R}$ be defined by $V(t) = \text{Vol}_n(K \cap e^{tA}B_2^n)$. If $K$ is in maximal intersection position of radius $1$, then
   \[ \frac{dV(t)}{dt} \bigg|_{t=0} = \int_{S^{n-1} \cap K} \langle x, Ax \rangle dS(x) \]
where $S = \text{Vol}_{n-1}$ is the surface area measure.
We will see in the next section that Theorem 1.3 is almost a direct corollary of Theorem 2.6. However, Remark 1.4 shows that some caution is needed, and especially, the use of the assumption \( \text{Vol}_{n-1}(\partial K \cap S^{n-1}) = 0 \) should be identified. Therefore, while the following proof is basically a direct application of some fundamental results in calculus, we provide full details.

**Proof of Theorem 2.6.** Let \( \{\phi_j\}_{j=1}^\infty \) be a sequence of continuous functions from \( \mathbb{R}^n \) to \( \mathbb{R} \) approximating \( 1_{\text{int}B_2} \), chosen as

\[
\phi_j(x) = \begin{cases} 
1 & |x| \leq 1 - \frac{1}{j} \\
g_j(x) & 1 - \frac{1}{j} \leq |x| \leq 1 \\
0 & |x| \geq 1
\end{cases}
\]

where \( g_j(x) : \mathbb{R}^n \to [0,1] \) is chosen such that \( \phi_j(x) \) is continuously differentiable and there is a constant \( c \) such that \( 0 < |\nabla \phi_j(x)| < j c \) for all \( x \). For instance we may take \( g_j(x) = \frac{1}{j} - \frac{1}{2} \cos j \pi (|x| - 1) \) to have \( \nabla g_j(x) = \frac{j \pi}{2} \sin j \pi (|x| - 1) \).

Similarly, let \( \psi_j(x) \) be a family of functions approximating \( 1_{\text{int}K} \), chosen as

\[
\psi_j(x) = \begin{cases} 
1 & \|x\|_K \leq 1 - \frac{1}{\sqrt{j}} \\
h_j(x) & 1 - \frac{1}{\sqrt{j}} \leq \|x\|_K \leq 1 \\
0 & \|x\|_K \geq 1
\end{cases}
\]

where \( h_j(x) : \mathbb{R}^n \to [0,1] \), \( \psi(x) \) is continuously differentiable, and \( 0 < |\nabla \psi_j(x)| < c \sqrt{j} \) for all \( x \).

As \( j \to \infty \), \( \phi_j(x) \) converges pointwise to \( 1_{\text{int}B_2} \) and \( \psi_j(x) \) converges pointwise to \( 1_{\text{int}K} \). We have then:

\[
\frac{d}{dt} \big|_{t=0} V(t) = \frac{d}{dt} \big|_{t=0} \int_{\mathbb{R}^n} 1_{\text{int}B_2}(e^{-tA}x) \, 1_{\text{int}K}(x) dx = \frac{d}{dt} \big|_{t=0} \int_{\mathbb{R}^n} \phi_j(e^{-tA}x) \psi_j(x) dx.
\]

We will show that the following hold in a neighborhood of \( t = 0 \):

(2.3) \[ \frac{d}{dt} \int_{\mathbb{R}^n} \lim_{j \to \infty} \phi_j(e^{-tA}x) \psi_j(x) dx = \lim_{j \to \infty} \int_{\mathbb{R}^n} \phi_j(e^{-tA}x) \psi_j(x) dx \]

(2.4) \[ \frac{d}{dt} \int_{\mathbb{R}^n} \phi_j(e^{-tA}x) \psi_j(x) dx = \lim_{j \to \infty} \frac{d}{dt} \int_{\mathbb{R}^n} \phi_j(e^{-tA}x) \psi_j(x) dx \]

(2.5) \[ \frac{d}{dt} \int_{\mathbb{R}^n} \phi_j(e^{-tA}x) \psi_j(x) dx = \int_{\mathbb{R}^n} \langle \nabla \phi_j(x), -\psi_j(e^{-tA}x)A x \rangle dx \]

(2.6) \[ \lim_{j \to \infty} \int_{\mathbb{R}^n} \langle \nabla \phi_j(x), -\psi_j(e^{-tA}x)A x \rangle dx = \int_{S^{n-1} \cap e^{-tA}K} \langle x, Ax \rangle dS \]

The equality (2.3) is a direct consequence of Lebesgue’s dominated convergence theorem. For (2.5), note that

\[
\frac{d}{dt} \phi_j(e^{-tA}x) \psi_j(x) dx = \langle \nabla \phi_j(e^{-tA}x), -\psi_j(x)A e^{-tA} x \rangle
\]

and that by Leibniz’s integral rule,

\[
\frac{d}{dt} \int_{\mathbb{R}^n} \phi_j(e^{-tA}x) \psi_j(x) dx = \int_{\mathbb{R}^n} \frac{d}{dt} \phi_j(e^{-tA}x) \psi_j(x) dx.
\]

It follows that for every fixed \( j \in \mathbb{N} \) (recall \( \text{tr} A = 0 \)),

\[
\frac{d}{dt} \int_{\mathbb{R}^n} \phi_j(e^{-tA}x) \psi_j(x) dx = \int_{\mathbb{R}^n} \langle \nabla \phi_j(e^{-tA}x), -\psi_j(x)A e^{-tA} x \rangle dx
\]

\[
= \int_{\mathbb{R}^n} \langle \nabla \phi_j(x), -\psi_j(e^{-tA}x)A x \rangle dx,
\]

proving (2.5).

To prove (2.4) and (2.6), it is enough to show the following:
Claim 2.7. There is a neighborhood of \( t = 0 \) where the function \( \frac{d}{dt} f_j(t) = \frac{d}{dt} \int_{\mathbb{R}^n} \phi_j(e^{-tA}x) \psi_j(x) dx \) converges uniformly to \( \int_{S^{n-1}} e^{-tA}K(x, Ax) dS \).

Proof. Denote

\[ M_j := \left\{ x : 1 - \frac{1}{j} \leq |x| \leq 1 \right\} \supset \text{supp} \nabla \phi_j(x). \]

Then

\[ \int_{\mathbb{R}^n} \langle \nabla \phi_j(x), -\psi_j(e^{tA}x)Ax \rangle dx = \int_{M_j} \langle \nabla \phi_j(x), -\psi_j(e^{tA}x)Ax \rangle dx. \]

The functions \( \phi_j(x), -\psi_j(e^{tA}x)Ax \) are continuously differentiable, \( \partial M_j \) is smooth, and so we may integrate by parts to have

(2.7)

\[ \int_{M_j} \langle \nabla \phi_j(x), -\psi_j(e^{tA}x)Ax \rangle dx = \]

\[ = \int_{\partial M_j} \phi_j(x) \langle \bar{n}, -\psi_j(e^{tA}x)Ax \rangle dS + \int_{M_j} \phi_j(x) \text{div}(\psi_j(e^{tA}x)Ax) dx \]

where \( \bar{n} \) is the outward unit normal of \( M_j \). Note that

\[ \text{div}(\psi_j(e^{tA}x)Ax) = \langle \nabla \psi_j(e^{tA}x), Ax \rangle + \psi_j(e^{tA}x)\text{div}Ax = \]

\[ = \langle \nabla \psi_j(e^{tA}x), Ax \rangle + \psi_j(e^{tA}x)\text{tr}A = \langle \nabla \psi_j(e^{tA}x), Ax \rangle \]

and so

\[ \left| \int_{M(j)} \phi_j(x) \text{div}(\psi_j(e^{tA}x)Ax) dx \right| \leq \int_{M(j)} \left| \langle \nabla \psi_j(e^{tA}x), Ax \rangle \right| dx. \]

There is a constant \( c \) such that

\[ \int_{M_j} \left| \langle \nabla \psi_j(e^{tA}x), Ax \rangle \right| dx \leq c \sqrt{j} \text{Vol}_n(M(j)) = c \sqrt{j} \left( 1 - \left( \frac{j-1}{j} \right)^n \right) \to 0 \]

as \( j \to \infty \).

Going back to (2.7), we have shown that \( \int_{M_j} \phi_j(x) \text{div}(\psi_j(e^{tA}x)Ax) dS \) converges uniformly to 0. As for \( \int_{\partial M_j} \phi_j(x) \langle \bar{n}, -\psi_j(e^{tA}x)Ax \rangle dS \), note that

\[ \partial M_j = S^{n-1} \cup \frac{\bar{j}}{j} S^{n-1} \]

where \( \phi_j(x) = 0 \) on \( S^{n-1} \), and \( \phi_j(x) = 1 \) on \( \frac{\bar{j}}{j} S^{n-1} \). For every \( x \in \frac{\bar{j}}{j} S^{n-1} \), the outer unit normal \( \bar{n} \) of \( M(j) \) is \(-\frac{\bar{j}}{j-1}x\), and so:

\[ \int_{\partial M_j} \phi_j(x) \langle \bar{n}, -\psi_j(e^{tA}x)Ax \rangle dS = \int_{\frac{\bar{j}}{j-1} S^{n-1}} \psi_j(e^{tA}x) \langle \bar{n}, -Ax \rangle dS \]

\[ \hspace{1cm} = \frac{j}{j-1} \int_{\frac{\bar{j}}{j} S^{n-1}} \psi_j(e^{tA}x) \langle x, Ax \rangle dS = \left( \frac{j-1}{j} \right)^n \int_{\frac{\bar{j}}{j} S^{n-1}} \psi_j \left( \frac{j-1}{j} e^{tA}x \right) \langle x, Ax \rangle dS. \]

We will show that there is some sequence \( \xi(j) \to 0 \) and some \( \delta > 0 \) such that for every \( |t| < \delta \),

\[ \left| \int_{S^{n-1}} \left( \frac{j-1}{j} \right)^n \psi_j \left( \frac{j-1}{j} e^{tA}x \right) \langle x, Ax \rangle dS - \int_{S^{n-1}} 1_K(e^{tA}x) \langle x, Ax \rangle dS \right| \leq \xi(j). \]

Denote \( \nu_j(x) = \left( \frac{j-1}{j} \right)^n \psi_j \left( \frac{j-1}{j} e^{tA}x \right) \), and consider

\[ \left| \int_{S^{n-1}} \nu_j(e^{tA}x) \langle x, Ax \rangle dS - \int_{S^{n-1}} 1_K(e^{tA}x) \langle x, Ax \rangle dS \right| \leq c \int_{S^{n-1}} |\nu_j(e^{tA}x) - 1_K(e^{tA}x)| dS \]
The set $S^{n-1}$ is a union of the following three sets:

$$S_1(j, t) = \left\{ x \in S^{n-1} : \|e^T x\|_K \geq \frac{j}{j-1} \right\}$$

$$S_2(j, t) = \left\{ x \in S^{n-1} : \|e^T x\|_K \leq 1 - \frac{1}{\sqrt{j}} \right\}$$

$$S_3(j, t) = \left\{ x \in S^{n-1} : \frac{\sqrt{j} - 1}{\sqrt{j}} \leq \|e^T x\|_K \leq \frac{j}{j-1} \right\}.$$

On $S_1$ we have that $\nu_j(x) = \mathbf{1}_K(e^T x) = 0$ and so $\int_{S_1(j, t)} |\nu_j(e^T x) - \mathbf{1}_K(e^T x)| \, dS = 0$ for all $j, t$. On $S_2$ we have that $\nu_j(e^T x) = \left(\frac{j-1}{j}\right)^n \mathbf{1}_K(e^T x) = 1$ and so:

$$\int_{S_2(j, t)} |\nu_j(e^T x) - \mathbf{1}_K(e^T x)| \, dS = \left| \left(\frac{j-1}{j}\right)^n - 1 \right| \text{Vol}_{n-1}(S_2(j, t)).$$

There is a constant $c$ such that $\text{Vol}_{n-1}(S_2(j, t)) \leq c$ for all $j \in \mathbb{N}$ and for all $t \in [1, -1]$. It follows that

$$\left| \left(\frac{j-1}{j}\right)^n - 1 \right| \text{Vol}_{n-1}(S_2(j, t)) \leq \left| \left(\frac{j-1}{j}\right)^n - 1 \right| c \to 0.$$ 

Finally, on $S_3$ we have that

$$\int_{S_3(j, t)} |\nu_j(e^T x) - \mathbf{1}_K(e^T x)| \, dS \leq \xi_j(t)$$

where

$$\xi_j(t) = \text{Vol}_{n-1}(S_3(j, t)) = \text{Vol}_{n-1}\left\{ x \in S^{n-1} : \frac{\sqrt{j} - 1}{\sqrt{j}} \leq \|e^T x\|_K \leq \frac{j}{j-1} \right\}$$

is monotonically decreasing in $j$ for every fixed $t$. By Dini’s theorem, $\xi_j(t)$ converges uniformly to $\xi(t) = \text{Vol}_{n-1}\left(S^{n-1} \cap \partial e^T K\right)$. Assuming $\text{Vol}_{n-1}\left(S^{n-1} \cap \partial K\right) = 0$ and $\text{Vol}_{n-1}\left(\partial \mathcal{E} \cap \partial K\right) = 0$ for all but finitely many ellipsoids, there is some $\delta > 0$ such that $\xi(t) = \text{Vol}_{n-1}\left(S^{n-1} \cap \partial e^T K\right) = 0$ for all $|t| < \delta$, and so on the set $|t| < \delta$, the sequence $\int_{S_3} |\nu_j(e^T x) - \mathbf{1}_K(e^T x)| \, dS$ converges uniformly to 0. This proves Claim 2.7 and with it Theorem 2.6.

**Remark 2.8.** Note that the proof above shows that the conditions of Theorem 2.6 (and therefore of Theorem 1.3) may be slightly relaxed: in fact, we do not need $\text{Vol}_{n-1}(K \cap \mathcal{E}) = 0$ for all but finitely many ellipsoids. It is enough to have a neighborhood $N \subset SL_n$ of $I_n$ such that $\text{Vol}_{n-1}(K \cap T \mathcal{E}) = 0$ for all $T \in N$.

3. PROOF OF THE MAIN THEOREMS

In this section we use the results of Section 2 to provide short proofs to the three main Theorems 1.3, 1.5, and 1.6.

As we mentioned, the proof of Theorem 1.3 follows almost directly from Theorem 2.6.

**Proof of Theorem 1.3.** First note that $K$ is in maximal intersection position of radius $r$ if and only if $r^{-1} K$ is in maximal intersection position of radius 1, and so it is enough to prove the theorem in the case $r = 1$.

Let $W : SL_n \rightarrow \mathbb{R}$, $W(T) = \text{Vol}_{n}(K \cap T B^n_2)$. If $I_n$ is a local maximum of $W$, then for any $A \in M_n(\mathbb{R})$ such that $\text{tr} A = 0$, the derivative $\frac{dW(e^T)}{dt} \bigg|_{t=0} = \frac{dW(t)}{dt} \bigg|_{t=0}$ is either zero or does not exist. Theorem 2.6 states that the derivative does exist for all $A$, and it equals $\int_{S^{n-1} \cap K} \langle x, Ax \rangle \, dS(x)$. It follows that

$$\int_{S^{n-1}} \langle x, Ax \rangle \, d\mu_K = \frac{1}{\text{Vol}_{n-1}(S^{n-1} \cap K)} \int_{S^{n-1} \cap K} \langle x, Ax \rangle \, dS = 0$$

for all $A$ such that $\text{tr} A = 0$, and by Lemma 2.5, $\mu_K$ is isotropic.

\[ \square \]
As we have mentioned, the result of Theorem 1.3 resembles that of John’s Theorem (Theorem 1.1), but does not include it. However, Theorem 1.3 provides a family of isotropic measures which are used in the proof of Theorem 1.5.

**Proof of Theorem 1.4** Let \( r < 1 \). By Lemma 2.2, we may choose an intersection maximizing ellipsoid \( E_r \) for each \( r \). By Lemma 2.3, \( E_r \to B_2^n \) and so we may choose a sequence of positive definite transformations \( T_r \to I_n \) such that \( B_2^n = T_r E_r \). Then \( T_r K \) is in maximal intersection position of radius \( r \) and \( \text{Vol}_{n-1}(\partial T_r K \cap S^{n-1}) = 0 \) for almost all \( r \). By Theorem 1.3, the probability measures on the sphere

\[
\mu_r(A) = \mu_{S^{n-1}\setminus T_r K}(A) = \frac{\sigma(A \setminus T_r K)}{\sigma(S^{n-1} \setminus T_r K)}
\]

are isotropic.

Note that \( S^{n-1} \) is a compact metric space, and so the family of measures \( \mu_r \) has a weakly converging subsequence \( \mu_j \to \mu \) where \( \mu \) is a probability measure on \( S^{n-1} \). We will show that the limit measure \( \mu \) is an isotropic measure whose support lies in \( \partial K \cap S^{n-1} \).

First, weak convergence implies

\[
\int_{S^{n-1}} (x, \theta)^2 \, d\mu_j(x) \to \int_{S^{n-1}} (x, \theta)^2 \, d\mu(x)
\]

and

\[
\frac{1}{n} \to \frac{\mu_j(S^{n-1})}{n} \to \frac{\mu(S^{n-1})}{n}
\]

so \( \int_{S^{n-1}} (x, \theta)^2 \, d\mu(x) = \frac{\mu(S^{n-1})}{n} = \frac{1}{n}, \) i.e., \( \mu \) is isotropic.

Second, let

\[
U_k = \left\{ x \in S^{n-1} : d(x, \partial K) > \frac{1}{k} \right\}
\]

where \( d(\cdot, \cdot) \) is a metric on \( S^{n-1} \). The measure \( \mu_j \) is supported on \( S^{n-1} \setminus T_r K \) where \( T_r K \to K \), and so there is \( M \) such that for any \( k > M \) there is some \( N(k) \) such that \( \mu_j(U_k) = 0 \) for all \( j > N(k) \). Since \( U_k \) is open, weak convergence implies \( \mu(U_k) \leq \lim \inf \mu_j(U_k) = 0 \), so \( \mu(U_k) = 0 \) for all \( k > M \). It follows that \( \mu(\bigcup_{k=M}^{\infty} U_k) = \lim_{k \to \infty} \mu(U_k) = 0 \), where

\[
\bigcup_{k=M}^{\infty} U_k = \left\{ x \in S^{n-1} : d(x, \partial K) > \frac{1}{k} \right\} = S^{n-1} \setminus \partial K = S^{n-1} \setminus \partial K.
\]

It follows that \( \mu(S^{n-1} \setminus \partial K) = 0 \) and so \( \text{supp} \mu \subset S^{n-1} \cap \partial K \).

The proof of Theorem 1.6 is analogous to that of Theorem 1.4, only here we use

\[
\nu_j(A) = \mu_{T_r K}(A) = \frac{\sigma(A \cap T_r K)}{\sigma(S^{n-1} \setminus T_r K)}
\]

which is isotropic by Theorem 1.3. In this case, it is the measures \( \nu_j \) that satisfy \( \nu_j(U_k) = 0 \) for all \( j > N(k) \), rather than the measures \( \mu_j \). In other words, for a John-type measure we use a sequence of uniform measures “outside” \( T_r K \), whereas for a Loewner-type measure we use a sequence of uniform measures “inside” \( T_r K \).

4. **Remarks about uniqueness following from the (B) property**

Throughout this text we discussed maximal intersection positions of a body \( K \). While Lemma 2.2 shows that such a position always exists, we did not show that this measure is unique. If \( 0 < r < r_J \) or \( r > r_L \) then the maximum intersection ellipsoid \( E_r \) of radius \( r \) is clearly not unique. If \( r = r_J \) or \( r = r_L \) then \( E_r \) is unique, by John’s theorem. The question of uniqueness remains open for the case \( r_J < r < r_L \), but it is implied by a variant of a well known conjecture which we next discuss:

**Conjecture 4.1.** For a convex body \( K \subset \mathbb{R}^n \) and a diagonal \( n \times n \) matrix \( \Lambda \), the function

\[
\phi(t) = \text{Vol}_n(e^{t \Lambda} K \cap B_2^n)
\]
is log-concave in $t$, i.e.

$$\text{(4.1)} \quad \text{Vol}_n \left( e^{\frac{\lambda}{2} K \cap B_2^n} \right)^2 \geq \text{Vol}_n \left( e^{\lambda K \cap B_2^n} \right) \text{Vol}_n \left( K \cap B_2^n \right)$$

for all $t \in \mathbb{R}$ and all diagonal $\Lambda$. Furthermore, equality is attained if and only if one of the following hold: $K \subset B_2^n$, $B_2^n \subset K$, or $\Lambda = \lambda I_n$ for some $\lambda \in \mathbb{R}$.

**Proposition 4.2.** Assuming Conjecture 4.1 is true, if $K$ is a centrally symmetric convex body, the maximum intersection ellipsoid of radius $r$ is unique for $r_J < r < r_L$.

**Proof.** Letting $r_J < r < r_L$, assume there are two distinct maximum intersection ellipsoids of radius $r$. We may assume that one of these ellipsoids is $B_2^n$, and the other is of the form $e^\Lambda B_2^n$ where $\Lambda$ is a diagonal matrix with $\text{tr}\Lambda = 0$. Conjecture 4.1 now gives

$$\text{Vol}_n \left( K \cap e^{\frac{\Lambda}{2} B_2^n} \right) \geq \text{Vol}_n \left( K \cap B_2^n \right)$$

where maximality of $B_2^n$ implies equality in the above. Since $r_J < r < r_L$, we have $K \not\Subset B_2^n$ and $B_2^n \not\subset K$. It follows that $\Lambda$ is a traceless scalar matrix, i.e. $\Lambda$ is the zero matrix and $e^\Lambda = I_n$.

Conjecture 4.1 describes a (B)-type property on the Lebesgue measure on $B_2^n$, under the following terminology:

**Definition 4.3.** Given a measure $\mu$ on $\mathbb{R}^n$ and a measurable set $K \subset \mathbb{R}^n$, we say that $\mu$ and $K$ have the weak (B) property if the function

$$t \mapsto \mu(e^t K)$$

is log-concave on $\mathbb{R}$. Denoting $\text{diag}(t_1, \ldots, t_n)$ the diagonal matrix with diagonal entries $t_1, \ldots, t_n$, we will say that $\mu$ and $K$ have the strong (B) property if the function

$$(t_1, \ldots, t_n) \mapsto \mu(e^{\text{diag}(t_1, \ldots, t_n)} K)$$

is log-concave on $\mathbb{R}^n$.

The notion of the (B) property arises from a problem proposed by Banaszczyk and described by Latala [2] known as the (B) conjecture (now the (B) theorem), where, in the terminology above, it was conjectured that the standard Gaussian probability measure $\gamma$ on $\mathbb{R}^n$ and any centrally symmetric convex body $K \subset \mathbb{R}^n$ have the weak (B) property. The (B) conjecture was solved by Cordero-Erausquin, Fradelizi, and Maurey [3], where it was shown that $\gamma$ and $K$ have in fact a strong (B) property.

Conjecture 4.1 proposes that the uniform Lebesgue measure on $B_2^n$ and any centrally symmetric convex body have the strong (B) property, with further assumptions on the equality case.

Unfortunately not a lot is known about the (B) property of general measures, and even less about the equality case. We will briefly mention what is currently known: Livne Bar-on [8] showed that in $\mathbb{R}^2$, the uniform Lebesgue measure on a centrally symmetric convex body $L \subset \mathbb{R}^2$ has the weak (B) property with any centrally symmetric convex body $K \subset \mathbb{R}^2$. This result was generalized by Saroglou [10], where it was shown that if the log-Brunn-Minkowski inequality holds in dimension $n$, then the uniform probability measure on the $n$-dimensional cube has the strong (B) property, and the uniform probability measure of every centrally symmetric convex body has the weak (B) property, with any centrally symmetric convex body $K$.

The log-Brunn-Minkowski inequality states that for two centrally symmetric convex bodies $K, L \subset \mathbb{R}^n$ and $\lambda \in [0, 1]$,

$$\text{(4.2)} \quad \text{Vol}_n \left( (1 - \lambda) K +, \lambda L \right) \geq \text{Vol}_n(K)^{1-\lambda} \text{Vol}_n(L)^\lambda$$

where

$$(1 - \lambda) K +, \lambda L = \bigcap_{u \in S^{n-1}} \left\{ x : \langle x, u \rangle \leq h_K(u)^{1-\lambda} h_L(u)^\lambda \right\}.$$
the uniform measure on $B^n_2$ is unconditional log-concave. It follows that Conjecture 4.1 (without the equality case) holds whenever $K$ is unconditional, i.e. $(x_1,\ldots,x_n) \in K$ implies $(\delta_1 x_1,\ldots,\delta_n x_n) \in K$ for any choice of $\delta_i \in \{-1,1\}$ where $i = 1,\ldots,n$.

Still not a lot is known on equality cases in inequalities such as (4.1). In [11], Saroglou expands further on the relationship between the (B) property and the log-Brunn-Minkowski, and conjectures that equality in (4.2) is attained if and only if $K = K_1 \times \ldots \times K_m$ for some convex sets $K_1,\ldots,K_m$ that cannot be written as cartesian products of lower dimensional sets, and $L = c_1 K_1 \times \ldots \times c_m K_m$ for some positive numbers $c_1,\ldots,c_m$. We have not found similar conjectures or results regarding the equality case in (4.1).

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