TI-bipolaron theory of superconductivity

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Consideration is given to thermodynamical properties of a three-dimensional Bose-condensate of translation-invariant bipolarons (TI-bipolarons). The critical temperature of transition, energy, heat capacity and the transition heat of TI-bipolaron gas are calculated. The generalization of theory on the case of magnetic field is made. Such values as maximum magnetic field, London penetration depth and their temperature dependencies are calculated. The results obtained are used to explain experiments on high-temperature superconductors.

Keywords: Bose-Einstein condensation; BEC; BCS; electron-phonon interaction; energy gap; crossover

1. Introduction.

Before the discovery of high-temperature superconductivity (HTSC) Bardeen-Cooper-Schrieffer theory [1] (BCS) played the role of fundamental microscopic theory of superconductivity having, in fact, no alternative. The discovery of HTSC revealed some problems which arose while trying to describe various properties of high-temperature superconductors within BCS. This gave birth to a great number of alternative theories aimed at resolving the problems. Review of various HTSC theories is presented in numerous papers. The current state-of-the-art can be found in [2–13]. All the approaches, are however, based on the same proposition the phenomenon of bosonization of Fermi-particles, or Cooper effect. This proposition straightforwardly leads to the conclusion that the phenomenon of superconductivity is related to the phenomenon of Bose-Einstein condensation (BEC). Presently the idea that superconductivity is based on BEC is generally recognized.

A great obstacle to the development of the theory which should be based on BEC was a statement made in BCS ([see comment in 1, p.1177] and [14]) about incompatibility of their theory with BEC.

Some evidence that this viewpoint is erroneous was first obtained in paper [15] whose authors, while studying the properties of high-density exciton gas, demonstrated an analogy between BCS theory and BEC. The results of [15] provided the basis for developing the idea of crossover passing on from the BCS theory which is appropriate for the limit of weak electron-phonon interaction (EPI) to BEC which is suitable for the limit of strong EPI [16–22]. It was believed that additional evidence in favor of this approach is Eliashberg theory of strong coupling [23]. According to [24], in the limit of infinitely strong EPI this theory leads to the regime of local

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pairs, though greatly different from the regime of BEC [25].

The attempts to develop a theory of crossover between BCS and BEC ran, however, into insurmountable obstacles. Thus, for example, an idea was put forward, to construct a theory with the use of T-matrix transition where T-matrix of the initial Fermion system would transform into T-matrix of Boson system as the force of EPI would increase [26–31]. The approach, however, turned out to be unfeasible even in the case of greatly diluted systems. Actually, the point is that when the system consists of only two fermions it is impossible to construct a one-boson state of them. In the EPI theory this problem is known as that of a bipolaron.

The reason why the crossover theory failed can be the following. The BCS theory, as the bipolaron theory, proceeds from Froehlich Hamiltonian. For this Hamiltonian, an important theorem of analyticity of the polaron and bipolaron energy on EPI constant is proved [32]. In the BCS theory a very important approximation is, however, made. Namely the actual matrix element of the interaction in Froehlich Hamiltonian is replaced by a model quantity a matrix element truncated from below and from above of the phonon momenta. This procedure is by no means fair. As is shown in [33], in the bipolaron theory it leads to ghost effects existence of a local energy level separated by a gap from the quasicontinuous spectrum (Cooper effect). This solution is isolated and does not possess the property of analyticity on coupling constant. In the BCS theory just this solution provides the basis for constructing the superconductivity theory.

As a result, the theory constructed and its analytic continuation (Eliashberg theory) greatly falsify the reality, in particular, they make impossible development of the superconductivity theory on the basis of BEC. Replacement of the actual matrix element by the model one enables one to make analytical calculations completely. Thus, substitution of local interaction for actual one in BCS enabled the authors of [34] to derive Ginzburg-Landau (GL) phenomenological model which is also a local model. Actually, the power of this approach can hardly be overestimated since it has enabled one to get a lot of statements consistent with the experiment.

This paper is an attempt to develop a HTSC theory on the basis of BEC of TI-bipolarons [33, 35–38], free of approximations made in [1].

We recall the main results of the theory of TI-polarons and bipolarons obtained in [33, 35–38]. Notice that consideration of just electron-phonon interaction is not essential for the theory and can be generalized to any type of interaction, for example the interaction of electrons with the spin subsystem [39].

In what follows we will deal only with the main points of the theory important for the HTSC theory. The main result of papers [33, 35–38] is construction of delocalized polaron and bipolaron states in the limit of strong electron-phonon interaction. The theory of TI-bipolarons is based on the theory of TI-polarons [40] and retains the validity of basic statements proved for TI-polarons. The chief of them is the theorem of analytic properties of the ground state of a TI-polaron (accordingly TI-bipolaron) depending on the constant of electron-phonon interaction $\alpha$. The main implication of this statement is the absence of a critical value of the EPI constant $\alpha_c$ below which the bipolaron state becomes impossible since it decays into independent polaron states. In other words, if there exists a value of $\alpha_c$ at which the TI-state becomes energetically disadvantageous with respect to its decay into individual polarons, then nothing occurs at this point but for $\alpha < \alpha_c$ the state becomes metastable. Hence, over the whole range of $\alpha$ variation we can consider TI-polarons as charged bosons capable of forming a superconducting condensate.

Another important property of TI-bipolarons is the possibility of changing the
correlation length over the whole range of \([0, \infty]\) variation depending on the Hamiltonian parameters \([36, 38]\). Hence, it can be both much larger (as is the case in metals) and much less than the characteristic size between the electrons in an electron gas (as happens with ceramics).

An outstandingly important property of TI-polarons and bipolarons is the availability of an energy gap between their ground and excited states (§3).

The above-indicated characteristics can be used to develop a microscopic HTSC theory on the basis of TI-bipolarons.

The paper is arranged as follows. In §2 we take Pekar-Froehlich Hamiltonian for a bipolaron as an initial Hamiltonian. The results of three canonical transformations, such as Heisenberg transformation, Lee-Low-Pines transformation and that of Bogolyubov-Tyablikov are briefly outlined. Equations determining the TI-bipolaron spectrum are derived. In §3 we analyze solutions of the equations for the TI-bipolaron spectrum. It is shown that the spectrum has a gap separating the ground state of a TI-bipolaron from its excited states which form a quasicontinuous spectrum. The concept of an ideal gas of TI-bipolarons is substantiated.

With the use of the spectrum obtained, in §4 we consider thermodynamic characteristics of an ideal gas of TI-bipolarons in the absence of a magnetic field. For various values of the parameters, namely phonon frequencies, we calculate the values of critical temperatures of Bose condensation, temperature of transition into the condensed state, heat capacity and heat capacity jumps at the point of transition.

In §5 we deal with the case when the external magnetic field differs from zero. It is shown that the current state in the system under discussion is caused by the existence of a constant quantity—the total momentum of the electron-phonon system in the magnetic field. Comparison of the value of the total momentum with that obtained within phenomenological approach enables us to determine the London penetration depth which is a very important characteristic. The results of the initial isotropic model are generalized to anisotropic case.

In §6 we investigate thermodynamic characteristics of an ideal TI-bipolaron gas in the presence of a magnetic field. It is shown that the availability of an energy gap in the TI-bipolaron spectrum makes possible their Bose-condensation in a magnetic field. A notion of a maximum value of the magnetic field intensity for which homogeneous Bose-condensation is possible is introduced. The temperature dependence of the value of the critical magnetic field and the dependence of the critical temperature on the magnetic field are found. It is shown that the phase transition of an ideal TI-bipolaron gas can be either of the 1-st kind or of infinite kind, depending on the magnetic field value. The theory is generalized to the case of anisotropic superconductor. This generalization enables us to compare the results obtained with experimental data (§7).

In §8 we consider scaling relations in superconductors. Alexandrov’s formula and Homes’s law are derived.

In §9 we deal with problems of extension of the theory which would enable us to make a more detailed comparison with experimental data on HTSC materials.

In §10 the results obtained are summed up.

2. Pekar-Froehlich Hamiltonian. Canonical transformations.

Following [33, 35–37], in describing bipolarons we will proceed from Pekar-Froehlich Hamiltonian with non zero magnetic field:
\begin{align}
H &= \frac{1}{2m^*} \left( \hat{p}_1 - \frac{e}{c} \vec{A}_1 \right)^2 + \frac{1}{2m^*} \left( \hat{p}_2 - \frac{e}{c} \vec{A}_2 \right)^2 + \sum_k \hbar \omega_k^0 a_k^+ a_k + \\
&\quad + \sum_k \left( V_k e^{i \vec{k} \vec{r}_1} a_k + V_k e^{i \vec{k} \vec{r}_2} a_k + H.c. \right) + U (|\vec{r}_1 - \vec{r}_2|),
\end{align}

where \( \hat{p}_1, \vec{r}_1, \hat{p}_2, \vec{r}_2 \) are momenta and coordinates of the first and second electrons, respectively; \( a_k^+, a_k \) are operators of the birth and annihilation of the field quanta with energy \( \hbar \omega_k^0 \); \( m^* \) is the electron effective mass; the quantity \( U \) describes Coulomb repulsion between the electrons; \( V_k \) is the function of the wave vector \( k \):

\begin{align}
V_k &= \frac{e}{k} \sqrt{\frac{2\pi \hbar \omega_0}{\epsilon V}} = \frac{\hbar \omega_0}{k u^{1/2}} \left( \frac{4\pi \alpha}{V} \right)^{1/2}, \\
u &= \left( \frac{2m^* \omega_0}{\hbar} \right)^{1/2}, \\
\alpha &= \frac{e^2 u}{2 \hbar \omega_0 \epsilon},
\end{align}

where \( e \) is the electron charge; \( \epsilon_\infty \) and \( \epsilon_0 \) are high-frequency and static dielectric permittivities; \( \alpha \) is the constant of electron-phonon interaction; \( V \) is the system’s volume.

The axis \( z \) is chosen along the direction of the magnetic field induction \( \vec{B} \) and use is made of symmetrical gauge:

\[ \vec{A}_j = \frac{1}{2} \vec{B} \times \vec{r}_j, \]

for \( j = 1, 2 \). For the bipolaron singlet state discussed below, the contribution of the spin term is equal to zero.

In the system of the center of mass Hamiltonian (1) takes the form:

\begin{align}
H &= \frac{1}{2M_e} \left( \hat{p} - \frac{e}{c} \vec{A}_R \right)^2 + \frac{1}{2\mu_e} \left( \hat{p}_r - \frac{e}{2c} \vec{A}_r \right)^2 + \sum_k \hbar \omega_k^0 a_k^+ a_k + \\
&\quad + \sum_k 2V_k \cos \frac{\vec{k} \vec{r}}{2} \left( a_k e^{i \vec{k} \vec{R}} + H.c. \right) + U (|\vec{r}|),
\end{align}

\[ \vec{R} = (\vec{r}_1 + \vec{r}_2)/2, \quad \vec{r} = \vec{r}_1 - \vec{r}_2, \quad M_e = 2m^*, \quad \mu_e = m^*/2, \]

\[ \vec{A}_r = \frac{1}{2} \vec{B}(-y, x, 0), \quad \vec{A}_R = \frac{1}{2} \vec{B}(-Y, X, 0), \]

\[ \hat{p}_R = \hat{p}_1 + \hat{p}_2 = -i\hbar \nabla_{\vec{r}}, \quad \hat{p}_r = (\hat{p}_1 - \hat{p}_2)/2 = -i\hbar \nabla_{\vec{r}}, \]

where \( x, y, \) and \( X, Y \) are components of vectors \( \vec{r}, \vec{R} \) respectively.
Let us subject Hamiltonian $H$ to Heisenberg canonical transformation [41, 42]:

$$S_1 = \exp i \left( \vec{G} - \sum_k \hbar a_k^+ a_k \right) \vec{R}. \quad (4)$$

$$\vec{G} = \hat{\vec{p}}_R + \frac{2e}{c} \vec{A}_R, \quad \hat{\vec{p}}_R = \hat{\vec{p}}_R + \sum_k \hbar a_k^+ a_k, \quad (5)$$

where $\vec{G}$ is the quantity commuting with the Hamiltonian, thereby being a constant, i.e. $c$-number, $\hat{\vec{p}}_R$ is the total momentum.

Action of $S_1$ on the field operator yields:

$$S_1^{-1} a_k S_1 = a_k e^{-i\vec{k} \vec{R}}, \quad S_1^{-1} a_k^+ S_1 = a_k^+ e^{-i\vec{k} \vec{R}}. \quad (6)$$

Accordingly, the transformed Hamiltonian $\tilde{H} = S_1^{-1} H S_1$ takes on the form:

$$\tilde{H} = \frac{1}{2M_e} \left( \vec{G} - \sum_k \hbar \vec{a}_k^+ \vec{a}_k - \frac{2e}{c} \vec{A}_R \right)^2 + \frac{1}{2\mu_e} \left( \hat{\vec{p}}_r - \frac{e}{2c} \vec{A}_r \right)^2 + \sum_k \hbar \omega_0 a_k^+ a_k + \frac{\sum_k 2V_k \cos \vec{k} \vec{r}}{2} (a_k + a_k^+) + U(|\vec{r}|). \quad (7)$$

In what follows we will believe:

$$\vec{G} = 0. \quad (8)$$

We will seek the solution of the stationary Schroedinger equation corresponding to Hamiltonian (7) in the form:

$$\Psi_H(r, R) = \phi(R) \Psi_{H=0}(r, R) \quad (9)$$

$$\phi(R) = \exp \left( -i \frac{2e}{\hbar c} \int_0^R \vec{A}_R \cdot d\vec{R} \right)$$

$$\Psi_{H=0}(r, R) = \psi(r) \Theta(r, R),$$

where $\Psi_{H=0}(r, R)$ is bipolaron wave function in the absence of magnetic field. The explicit form of functions $\psi(r)$ and $\Theta(r, R)$ is given in [35, 38].

Averaging of $\tilde{H}$ over the wave function $\phi(R)$ and $\psi(r)$ yields:

$$\tilde{H} = \frac{1}{2M_e} \left( \sum_k \hbar \vec{a}_k^+ \vec{a}_k \right)^2 + \sum_k \hbar \omega_k a_k^+ a_k + \sum_k \bar{V}_k (a_k + a_k^+) + \bar{T} + \bar{U} + \bar{\Pi}, \quad (10)$$

In what follows we will believe:

$$\vec{G} = 0.$$
where:

\[ T = \frac{1}{2\mu_e} \left\langle \psi \left| \left( \hat{p}_r - \frac{e}{2c} \vec{A}_r \right)^2 \right| \psi \right\rangle, \quad U = \left\langle \psi \left| U(r) \right| \psi \right\rangle, \quad V_k = 2V_k \left\langle \psi \left| \cos \frac{k\vec{r}}{2} \right| \psi \right\rangle, \quad (11) \]

\[ \bar{\Pi} = \frac{2e^2}{M_e c^2} \left\langle \phi \left| A_R^2 \right| \phi \right\rangle, \quad \hbar \tilde{\omega}_k = \hbar \omega_k^0 + \frac{2\hbar e}{M_e c} \left\langle \phi \left| \vec{k} \vec{A}_R \right| \phi \right\rangle. \]

In what follows in this section we will believe \( \hbar = 1, \omega_k^0 = 1, M_e = 1 \). Eq. (10) suggests that the bipolaron Hamiltonian differs from the polaron one in that in the latter the quantity \( V_k \) is replaced by \( \bar{V}_k \) and \( \bar{T}, \bar{U}, \bar{\Pi} \) are added to the polaron Hamiltonian.

With the use of Lee-Low-Pines canonical transformation [43]

\[ S_2 = \exp \left\{ \sum_k f_k (a_k^+ - a_k) \right\}, \quad (12) \]

where \( f_k \) are variational parameters having the sense of the distance by which the field oscillators are displaced from their equilibrium positions:

\[ S_2^{-1} a_k S_2 = a_k + f_k, \quad S_2^{-1} a_k^+ S_2 = a_k^+ + f_k, \quad (13) \]

for Hamiltonian \( \tilde{H} \):

\[ \tilde{H} = S_2^{-1} \tilde{H} S_2, \quad (14) \]

\[ \tilde{H} = H_0 + H_1, \]

we get:

\[ H_0 = 2 \sum_k \tilde{V}_k f_k + \sum_k f_k^2 \tilde{\omega}_k + \frac{1}{2} \left( \sum_k \tilde{k}^2 f_k^2 \right)^2 + \mathcal{H}_0 + \bar{T} + \bar{U} + \bar{\Pi}, \]

\[ \mathcal{H}_0 = \sum_k \omega_k a_k^+ a_k + \frac{1}{2} \sum_{k,k'} \tilde{k}^2 f_k f_{k'} (a_k a_{k'}^+ + a_k^+ a_{k'} + a_k^+ a_{k'}^+ + a_k a_{k'}^+), \quad (15) \]

where:

\[ \omega_k = \tilde{\omega}_k + \frac{k^2}{2} + \tilde{k} \sum_{k'} \tilde{k}' f_{k'}^2. \quad (16) \]

Hamiltonian \( H_1 \) contains terms linear, threefold and fourfold in the birth and annihilation operators. Its explicit form is given in [38, 40].

Then, as is shown in [38, 40], the use of Bogolyubov-Tyablikov canonical transformation [44] for passing on from operators \( a_k^+, a_k \) to new operators \( a_k^+, a_k \):

\[ a_k = \sum_{k'} M_{1kk'} a_{k'} + \sum_{k'} M_{2kk'}^a a_{k'}^+. \]
\[ a^+_k = \sum_{k'} M^*_{kk'} \alpha^+_{k'} + \sum_{k'} M_{2kk'} \alpha_{k'}, \]

(in which \( H_0 \) is a diagonal operator) makes mathematical expectation of \( H_1 \) equal to zero in the absence of magnetic field. The contribution of \( H_1 \) to the spectrum of transformed Hamiltonian when magnetic field is non-zero is discussed in 3.

In the new operators \( a^+_k, \alpha_k \) Hamiltonian (15) takes on the form

\[
\hat{H} = E_{bp} + \sum_k \nu_k \alpha^+_k \alpha_k,
\]

\[ E_{bp} = \Delta E_r + 2 \sum_k \tilde{V}_k f_k + \sum_k \tilde{\omega}_k f_k^2 + \tilde{T} + \tilde{U} + \tilde{\Pi}, \tag{18} \]

where \( \Delta E_r \) is the so-called recoil energy. The general expression for \( \Delta E_r = \Delta E_r \{ f_k \} \) was obtained in [40]. Actually, calculation of the ground state energy \( E_{bp} \) was performed in [38, 40] by minimization of (18) in \( f_k \) and in \( \Psi \) in the absence of a magnetic field.

Notice that in the theory of a polaron with broken symmetry a diagonalized electron-phonon Hamiltonian has the form of (18) [45]. This Hamiltonian can be treated as a Hamiltonian of a polaron and a system of its associated renormalized real phonons or as a Hamiltonian whose quasiparticle excitations spectrum is determined by (18) [46]. In the latter case excited states of a polaron are Fermi quasiparticles.

In the case of a bipolaron the situation is qualitatively different since a bipolaron is a boson quasiparticle whose spectrum is determined by (18). Obviously, a gas of such quasiparticles can experience Bose-Einstein condensation (BEC). Treatment of (18) as a bipolaron and its associated renormalized phonons does not prevent their BEC since maintenance of the number of particles required in this case takes place automatically due to commutation of the total number of renormalized phonons with Hamiltonian (18).

Renormalized frequencies \( \nu_k \) involved in (18), according to [38, 40, 47] are determined by the equation for \( s \):

\[
1 = 2 \sum_k \frac{k^2 f_k^2 \omega_k}{s - \omega_k^2}, \tag{19} \]

solutions of which yield the spectrum of \( s = \{ \nu_k^2 \} \) values.

3. **Energy spectrum of a TI-bipolaron.**

Hamiltonian (18) is conveniently presented in the form:

\[
\hat{H} = \sum_{n=0,1,2,...} E_n \alpha^+_n \alpha_n, \tag{20} \]
\[ E_n = \begin{cases} \nu_n = E_{bp}, & n = 0; \\ \nu_n = E_{bp} + \omega_{k_n}, & n \neq 0; \end{cases} \] (21)

where in the case of a three-dimensional ionic crystal \( \vec{k}_n \) is a vector with the components:

\[ k_{n_i} = \pm \frac{2\pi(n_i - 1)}{N_{ai}}, \quad n_i = 1, 2, \ldots, \frac{N_{ai}}{2} + 1, \quad i = x, y, z, \] (22)

\( N_{ai} \) is the number of atoms along the \( i \)-th crystallographic axis.

Let us prove the validity of the expression for the spectrum (20), (21). Since operators \( \alpha_n^+, \alpha_n \) obey Bose commutation relations:

\[ [\alpha_n, \alpha_n^+] = \alpha_n \alpha_n^+ - \alpha_n^+ \alpha_n = \delta_{n,n'}, \] (23)

they can be considered to be operators of birth and annihilation of TI-bipolarons. The energy spectrum of TI-bipolarons, according to (19), is determined by the equation:

\[ F(s) = 1, \] (24)

where:

\[ F(s) = \frac{2}{3} \sum_n \frac{k_{n}^2 f_{k_n}^2 \omega_{k_n}^2}{s - \omega_{k_n}^2}. \] (25)

It is convenient to solve equation (24) graphically (Fig. 1).

Fig. 1 suggests that the frequencies \( \nu_{k_n} \) lie between the frequencies \( \omega_{k_n} \) and \( \omega_{k_{n+1}} \). Hence, the spectrum \( \nu_{k_n} \) as well as the spectrum \( \omega_{k_n} \) are quazicontinuous: \( \nu_{k_n} - \omega_{k_n} = O(N^{-1}) \) which just proves the validity of (20), (21).

It follows that the spectrum of a TI-bipolaron has a gap between the ground state \( E_{bp} \) and the quasicontinuous spectrum, equal to \( \omega_0 \).

In the absence of an external magnetic field, functions \( f_k \) involved in expression for \( \omega_k \) (16) are independent of the direction of the wave vector \( \vec{k} \). When an external magnetic field is applied, \( f_k \) cannot be considered to be an isotropic quantity, accordingly, we cannot put the last term in equation (16) for \( \omega_k \) equal to zero. Besides, the angular dependence involved in the spectrum \( \omega_k \) in the magnetic field is also contained in the term \( \tilde{\omega}_k \) involved in the quantity \( \omega_k \). Since in the isotropic system discussed there is only one preferred direction determined by vector \( \vec{B} \), for \( \omega_k \) from (16) we will get:

\[ \omega_{k_n} = \omega_0 + \frac{\hbar k_n^2}{2M_e} + \frac{\eta}{M_e} \left( \vec{B} \vec{k}_n \right), \] (26)

where \( \eta \) is a scalar quantity. Notice that the contribution of \( H_1 \) to spectrum (26) will lead to the dependence of \( \eta \) from \( \vec{k} \) and \( \vec{k} \vec{B} \). For weak magnetic field in longitudinal limit (when Froehlich Hamiltonian is valid) we will neglect such dependence and consider \( \eta \) as a constant value.
For a magnetic field directed along the axis $z$, expression (26) can be written as:

$$\omega_{kn} = \omega_0 + \frac{\hbar^2}{2M_e} (k_{zn} + k_0)^2 + \frac{\hbar^2}{2M_e} (k_{xn} + k_{yn})^2 - \frac{\eta^2 B^2}{2\hbar^2 M_e}. \quad (27)$$

Notice, that formula (27) can be generalized to the anisotropic case when in the directions $k_x$ and $k_y$: $M_{ex} = M_{ey} = M_\parallel$, and in the direction $k_z$: $M_{ez} = M_\perp$ (§5). Formula (27) in this case takes on the form:

$$\omega_{kn} = \omega_0 + \frac{\hbar^2}{2M_\perp} (k_{zn} + k_0)^2 + \frac{\hbar^2}{2M_\parallel} (k_{xn} + k_{yn})^2 - \frac{\eta^2 B^2}{2\hbar^2 M_\perp}, \quad (27')$$

if the magnetic field is directed along the axis $z$ and:

$$\omega_{kn} = \omega_0 + \frac{\hbar^2}{2M_\perp} k_{zn}^2 + \frac{\hbar^2}{2M_\parallel} (k_{xn} + k_0)^2 + \frac{\hbar^2}{2M_\parallel} k_{yn}^2 - \frac{\eta^2 B^2}{2\hbar^2 M_\parallel}, \quad (27'')$$

if the magnetic field is directed along the axis $x$.

Below we will consider the case of low concentration of TI-bipolarons in a crystal. Then they can adequately be considered as an ideal Bose gas, whose properties are determined by Hamiltonian (20).
4. Statistical thermodynamics of low-density TI bipolarons without magnetic field.

Let us consider an ideal Bose-gas of TI-bipolarons which represents a system of \( N \) particles occurring in some volume \( V \). Let us write \( N_0 \) for the number of particles in the lower one-particle state and \( N' \) for the number of particles in higher states. Then:

\[
N = \sum_{n=0,1,2,...}^{\infty} \tilde{m}_n = \sum_n \frac{1}{e^{(E_n-\mu)/T} - 1},
\]

or:

\[
N = N_0 + N', \quad N_0 = \frac{1}{e^{(E_{bp}-\mu)/T} - 1}, \quad N' = \sum_{n \neq 0} \frac{1}{e^{(E_n-\mu)/T} - 1}.
\]

In expression \( N' \) (29) we will perform integration over quasicontinuous spectrum (instead of summation) (20), (21), (27) and assume \( \mu = E_{bp} \). As a result, from (28), (29) we get an equation for determining the critical temperature of Bose-condensation \( T_c \):

\[
C_{bp} = f_{\tilde{\omega}H} \left( \tilde{T}_c \right),
\]

\[
f_{\tilde{\omega}H} \left( \tilde{T}_c \right) = \tilde{T}_c^{3/2} F_{3/2} \left( \tilde{\omega}_H/\tilde{T}_c \right), \quad F_{3/2}(\alpha) = \frac{2}{\sqrt{\pi}} \int_0^\infty \frac{x^{1/2}dx}{e^{x+\alpha} - 1};
\]

\[
C_{bp} = \left( \frac{n^{2/3}2\pi \hbar^2}{M_e \omega^*} \right)^{3/2}, \quad \tilde{\omega}_H = \frac{\omega_0 - \eta^2 H^2/2M_e}{\omega^*}, \quad \tilde{T}_c = \frac{T_c}{\omega^*},
\]

where \( n = N/V \). In this section we will deal with the case when the magnetic field is lacking: \( H = 0 \). Fig. 2 shows a graphical solution of equation (30) for the values of parameters \( M_e = 2m^* = 2m_0 \), where \( m_0 \) is the mass of a free electron in vacuum, \( \omega^* = 5 \text{ meV} (\approx 58K) \), \( n = 10^{21} \text{ cm}^{-3} \) and the values: \( \tilde{\omega}_1 = 0.2; \tilde{\omega}_2 = 1; \tilde{\omega}_3 = 2; \tilde{\omega}_4 = 10; \tilde{\omega}_5 = 15; \tilde{\omega}_6 = 20; \tilde{\omega}_H = \tilde{\omega} = \omega_0/\omega^* \).

It is seen from Fig. 2 that the critical temperature grows with increasing phonon frequency \( \omega_0 \). The relations of critical temperatures \( T_{ci}/\omega_0 \) corresponding to the chosen parameter values are given in Table 1. Table 1 suggests that the critical temperature of a TI-bipolaron gas is always higher than that of ideal Bose-gas (IBG). It is also evident from Fig. 2 that an increase in the concentration of TI-bipolarons \( n \) will lead to an increase in the critical temperature, while a gain in the electron mass \( m^* \) to its decrease. For \( \tilde{\omega} = 0 \) the results go over into the limit of IBG. In particular, (30) for \( \tilde{\omega} = 0 \), yields the expression for the critical temperature of IBG:

\[
T_c = 3.31\hbar^2 n^{2/3}/M_e.
\]
Figure 2. Solutions of equation (30) with $C_{bp} = 331.35$ and $\tilde{\omega}_i = \{0.2; 1; 2; 10; 15; 20\}$, which correspond to $\tilde{T}_c = 30.0255; \tilde{T}_c = 32.1397; \tilde{T}_c = 41.8727; \tilde{T}_c = 46.1863; \tilde{T}_c = 49.9754$.

Table 1. Calculated characteristics of Bose-gas of TI-bipolarons with concentration $n = 10^{21}$ cm$^{-3}$.

| $\tilde{T}_c$ | $\omega_i$ | $q_i/T_{c,i}$ | $C_{v,i}$(|$T_{c,i}$|) | $C_s$ | $C_n$ | $n_{bpi}$·cm$^{-3}$ |
|-------------|------------|----------------|----------------|-------|-------|---------------------|
| $\tilde{\omega}_i$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| $\tilde{T}_c$ | $\tilde{\omega}_i$ | $q_i/T_{c,i}$ | $C_{v,i}$(|$T_{c,i}$|) | $C_s$ | $C_n$ | $n_{bpi}$·cm$^{-3}$ |
| $\tilde{\omega}_i$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |

It should be stressed, however, that (31) involves $M_e = 2m^*$, rather than the bipolaron mass. This resolves the problem of the low temperature of condensation which arises both in the small radius polaron theory and in the large radius polaron theory in which expression (31) involves the bipolaron mass [48–55]. Another important result is that the critical temperature $T_c$ for the parameter values considerably exceeds the gap energy $\omega_0$.

From (28), (29) follows that:

$$\frac{N'(\tilde{\omega})}{N} = \frac{\tilde{T}^{3/2}}{C_{bp}} F_{3/2} \left( \frac{\tilde{\omega}}{T} \right),$$  \hspace{1cm} (32)

$$\frac{N_0(\tilde{\omega})}{N} = 1 - \frac{N'(\tilde{\omega})}{N}.$$  \hspace{1cm} (33)

Fig. 3 shows temperature dependencies of the number of supracondensate particles $N'$ and the number of particles occurring in the condensate $N_0$ for the above-indicated parameter values $\tilde{\omega}_i$.

Fig. 3 suggests that, as could be expected, the number of particles in the condensate grows as the gap $\omega_i$ increases.
The energy of a TI-bipolaron gas $E$ is determined by the expression:

$$E = \sum_{n=0,1,2,...} \bar{m}_n E_n = \tilde{E}_{bp} N_0 + \sum_{n\neq 0} \bar{m}_n E_n. \quad (34)$$

With the use of (20), (21), (34) the specific energy (i.e. the energy per one TI-bipolaron) $\tilde{E}(\tilde{T}) = E/N\omega^*$, $\tilde{E}_{bp} = E_{bp}/\omega^*$ will be:

$$\tilde{E}(\tilde{T}) = \tilde{E}_{bp} + \frac{\tilde{T}^{5/2}}{C_{bp}} F_{3/2} \left( \frac{\tilde{\omega} - \tilde{\mu}}{\tilde{T}} \right) \left[ \frac{\tilde{\omega}}{\tilde{T}} + \frac{F_{5/2} \left( \frac{\tilde{\omega} - \tilde{\mu}}{\tilde{T}} \right)}{F_{3/2} \left( \frac{\tilde{\omega} - \tilde{\mu}}{\tilde{T}} \right)} \right], \quad (35)$$

$$F_{5/2}(\alpha) = \frac{2}{\sqrt{\pi}} \int_0^\infty \frac{x^{3/2} e^{-x}}{(e^{x+\alpha} - 1)} dx,$$

where $\tilde{\mu}$ is determined from the equation:

$$\tilde{T}^{3/2} F_{3/2} \left( \frac{\tilde{\omega} - \tilde{\mu}}{\tilde{T}} \right) = C_{bp}, \quad (36)$$

$$\tilde{\mu} = \begin{cases} 0, & \tilde{T} \leq \tilde{T}_c; \\ \tilde{\mu}(\tilde{T}), & \tilde{T} \geq \tilde{T}_c. \end{cases}$$

Relation of $\tilde{\mu}$ with the chemical potential of the system $\mu$ is given by the formula $\tilde{\mu} = (\mu - \tilde{E}_{bp})/\omega^*$. From (35)–(36) also follow expressions for the free energy: $F = -2E/3$ and entropy $S = -\partial F/\partial T$.

Fig. 4 illustrates temperature dependencies $\Delta E = \tilde{E} - \tilde{E}_{bp}$ for the above-indicated parameter values $\omega_i$. Break points on the curves $\Delta E_i(\tilde{T})$ correspond to the values of critical temperatures $T_{ci}$.

The dependencies obtained enable us to find the heat capacity of a TI-bipolaron.
gas: $C_V(\tilde{T}) = d\tilde{E}/d\tilde{T}$. With the use of (27) for $\tilde{T} \leq \tilde{T}_c$ we express $C_V(\tilde{T})$ as:

$$C_V(\tilde{T}) = \frac{\tilde{T}^{3/2}}{2C_{bp}} \left[ \tilde{\omega}^2 \frac{F_{1/2}}{T^2} \left( \frac{\tilde{\omega}}{T} \right) + 6 \left( \frac{\tilde{\omega}}{T} \right) \frac{F_{3/2}}{T^3} \left( \frac{\tilde{\omega}}{T} \right) + 5 \frac{F_{5/2}}{T^5} \left( \frac{\tilde{\omega}}{T} \right) \right], \quad (37)$$

$$F_{1/2}(\alpha) = \frac{2}{\sqrt{\pi}} \int_0^{\infty} \frac{1}{\sqrt{x}} \frac{dx}{e^{x+\alpha} - 1}. \quad (38)$$

Expression (37) yields a well-known exponential dependence of the heat capacity at low temperatures $C_V \exp(-\omega_0/T)$ caused by the availability of the energy gap $\omega_0$.

Fig. 5 shows temperature dependencies of the heat capacity $C_V(\tilde{T})$ for the above-indicated parameter values $\tilde{\omega}_i$. Table 1 lists the values $\tilde{\omega}_i$ of the heat capacity jumps:

$$\Delta \frac{\partial C_V(\tilde{T})}{\partial \tilde{T}} = \left. \frac{\partial C_V(\tilde{T})}{\partial \tilde{T}} \right|_{\tilde{T}=\tilde{T}_c+0} - \left. \frac{\partial C_V(\tilde{T})}{\partial \tilde{T}} \right|_{\tilde{T}=\tilde{T}_c-0}, \quad (38)$$

at the transition points.

The dependencies obtained will enable one to find the latent heat of transition $q = TS$, $S$ is the entropy of supracondensate particles. At the point of transition this value is: $q = 2T_cC_V(T_C - 0)/3$, where $C_V(T)$ is determined by formula (37). For the above-indicated parameter values $\omega_i$, it is given in Table 1.

5. **Current states of a TI-bipolaron gas.**

As is known, the absence of a magnetic field inside a superconductor is caused by the existence of surface currents compensating this field. Thus, from condition (8) it follows that:

$$\vec{P}_R = -\frac{2e}{c} \vec{A}_R, \quad (39)$$

i.e. in a superconductor there is a persistent current $\vec{j}$:

$$\vec{j} = 2en_0\vec{P}_R/M_e^* = -\frac{4e^2n_0}{M_e^*c} \vec{A}_R, \quad (40)$$
Figure 5. Temperature dependencies of the heat capacity for various values of the parameters $\omega_i$:

- $\omega_0 = 0; \ T_{c0} = 25.2445; \ C_v(T_{c0}) = 1.925$
- $\omega_1 = 0.2; \ T_{c1} = 27.325; \ C_v(T_{c1} - 0) = 2.162; \ C_v(T_{c1} + 0) = 1.868$
- $\omega_2 = 1; \ T_{c2} = 30.0255; \ C_v(T_{c2} - 0) = 2.465; \ C_v(T_{c2} + 0) = 1.812$
- $\omega_3 = 2; \ T_{c3} = 33.1397; \ C_v(T_{c3} - 0) = 2.699; \ C_v(T_{c3} + 0) = 1.777$
- $\omega_4 = 10; \ T_{c4} = 41.8727; \ C_v(T_{c4} - 0) = 3.740; \ C_v(T_{c4} + 0) = 1.6779$
- $\omega_5 = 15; \ T_{c5} = 46.1863; \ C_v(T_{c5} - 0) = 4.181; \ C_v(T_{c5} + 0) = 1.651$
- $\omega_6 = 20; \ T_{c6} = 49.9754; \ C_v(T_{c6} - 0) = 4.560; \ C_v(T_{c6} + 0) = 1.633$
(where $M^*_{e}$ is bipolaron effective mass), providing Meissner effect, where $n_0$ is a concentration of superconducting charge carriers: $n_0 = N_0/V$. Comparing (40) with the well-known phenomenological expression for the surface current $\vec{j}_S$ [56]:

$$\vec{j}_S = -\frac{e}{4\pi \lambda^2} \vec{A},$$

(41)

and putting $\vec{A} = \vec{A}_R$, with the use of (40), (41) and equality $\vec{j} = \vec{j}_S$ we will get a well-known expression for London penetration depth $\lambda$:

$$\lambda = \left(\frac{M^*_{e}c^2}{16\pi e^2 n_0}\right)^{1/2},$$

(42)

The equality of 'microscopic' expression for current (40) to its 'macroscopic' value cannot be exact. Accordingly, the equality $\vec{A} = \vec{A}_R$ is also approximate since $\vec{A}_R$ represents a vector-potential at the point where the center of mass of two electrons occurs, while in London theory $\vec{A}$ is a vector-potential at the point where a particle resides. For this reason these two quantities should better be considered as proportional. In this case the expression for the penetration depth has the form:

$$\lambda = \text{const} \left(\frac{M^*_{e}c^2}{16\pi e^2 n_0}\right)^{1/2},$$

(42′)

where the constant multiplier in (42′) should be determined from a comparison with the experiment.

Expression (39) was obtained in the case of isotropic effective mass of charge carriers. But actually, it has a more general character and does not change when anisotropy of effective masses is taken into account. Thus, for example in layered HTSC materials kinetic energy of charge carriers in Hamiltonian (1) should be replaced by the expression:

$$T_a = \frac{1}{2m_{||}^*} \left( \hat{P}_{1||} - \frac{e}{c} \hat{A}_1 \right)^2 + \frac{1}{2m_{||}^*} \left( \hat{P}_{2||} - \frac{e}{c} \hat{A}_2 \right)^2 +$$

$$\frac{1}{2m_{\perp}^*} \left( \hat{P}_{1\perp} - \frac{e}{c} \hat{A}_{1\perp} \right)^2 + \frac{1}{2m_{\perp}^*} \left( \hat{P}_{2\perp} - \frac{e}{c} \hat{A}_{2\perp} \right)^2,$$

(43)

where $\hat{P}_{1,2||}, \hat{A}_{1,2||}$ are operators of the momentum and vector-potential in the planes of layers (ab planes); $\hat{P}_{1,2\perp}, \hat{A}_{1,2\perp}$ are relevant quantities in the direction perpendicular to the planes (along c-axis); $m_{||}^*, m_{\perp}^*$ are effective masses in the planes and in the perpendicular direction.

As a result of transformation:

$$\tilde{x} = x, \quad \tilde{y} = y, \quad \tilde{z} = \gamma z$$

$$\tilde{A}_x = A_x, \quad \tilde{A}_y = A_y, \quad \tilde{A}_z = \gamma^{-1} A_z,$$

$$\tilde{\hat{P}}_x = \hat{P}_x, \quad \tilde{\hat{P}}_y = \hat{P}_y, \quad \tilde{\hat{P}}_z = \gamma^{-1} \hat{P}_z,$$

(44)

where $\gamma^2 = m_{\perp}^*/m_{||}^*$, $\gamma$ is the anisotropy parameter kinetic energy $\tilde{T}_a$ appears to be
isotropic. It follows that: \( \hat{\mathbf{P}}_R + (2e/c)\hat{\mathbf{A}}_R = 0 \). Then (44) suggests that relation (39) appears to be valid in the anisotropic case too. It follows that:

\[
\hat{\mathbf{P}}_{R||} = -\frac{2e}{c} \hat{\mathbf{A}}_{R||}, \quad \hat{\mathbf{P}}_{R\perp} = -\frac{2e}{c} \hat{\mathbf{A}}_{R\perp},
\]

The magnetic field directed perpendicular to the plane of layers will induce currents running in the plane of layers. Having penetrated into a sample such a field will decrease along the plane of layers. Let us write \( \lambda_{||} \) for the London penetration depth of the magnetic field perpendicular to the plane of layers (\( H_{\perp} \)) and \( \lambda_{\perp} \) for that of the magnetic field parallel to the plane of layers (\( H_{||} \)).

This suggests expressions for London depths of the magnetic field penetration into a sample:

\[
\lambda_{\perp} = \left( \frac{M_{e\perp}^* e^2}{16\pi e^2 n_0} \right)^{1/2}, \quad \lambda_{||} = \left( \frac{M_{e||}^* e^2}{16\pi e^2 n_0} \right)^{1/2}. \tag{46}
\]

For \( \lambda_{||} \) and \( \lambda_{\perp} \), designations \( \lambda_{ab} \) and \( \lambda_c \) are also used. From (46) it follows that:

\[
\frac{\lambda_{\perp}}{\lambda_{||}} = \left( \frac{M_{e\perp}^*}{M_{e||}^*} \right)^{1/2} = \gamma^* \tag{47}
\]

From (46) it also follows that the London penetration depth depends on temperature:

\[
\lambda^2(0)/\lambda^2(T) = n_0(T)/n_0(0). \tag{48}
\]

In particular, for \( \omega = 0 \), with the use of (31) we get: \( \lambda(T) = \lambda(0) \left( 1 - (T/T_C)^{3/2} \right)^{-1/2} \). Comparison of the dependence obtained with those derived within other approaches is given in §7.

It is generally accepted that Bose system became superconducting due to the inter-particle interaction. The existence of a gap in TI-bipolaron spectrum can drive their condensation and the Landau superfluidity condition:

\[
v < \hbar \omega_0 / \mathcal{P}. \tag{49}
\]

can be fulfilled even for noninteracting particles (§9). From condition (49) it follows the expression for maximum value of current density \( j_{\text{max}} = env_{\text{max}} \):

\[
j_{\text{max}} = en_0 \sqrt{\frac{\hbar \omega_0}{M_e^*}}. \tag{50}
\]

It should be noted that all the aforesaid refers to local electrodynamics. Accordingly, expressions obtained for \( \lambda \) are valid only on condition that \( \lambda >> \xi \), where \( \xi \).
is a correlation length determining the characteristic size of the pair, i.e. the characteristic scale of changes of the wave function $\psi(r)$ in (9). This condition is usually fulfilled in HTSC materials. In ordinary superconductors the inverse inequality is valid. Nonlocal generalization of superconductor electrodynamics was made by Pip-pard [57]. Within this approach relation between $j_S$ and $A$ in expression (41) can be written as:

$$j_S = \int \hat{Q}(z-r') A(r') d\mathbf{r'},$$

(51)

where $Q$ is a certain operator whose radius of action is usually believed to be equal to $\xi$. In the limit of $\xi \gg \lambda$ this leads to an increase in the absolute value of the length of the magnetic field penetration into a superconductor which becomes equal to $(\lambda^2 \xi)^{1/3}$ [56].

6. Thermodynamic properties of a TI-bipolaron gas in a magnetic field.

The fact that Bose-condensation in a magnetic field is impossible [58] does not mean that the mechanism of BEC cannot be used in describing superconductivity in a magnetic field. The reason is that in a superconductor the magnetic field is identically equal to zero. At the same time, leaving aside the problem of superconductivity, there is nothing to prevent us from considering Bose-gas placed in a magnetic field. It is of interest to investigate this problem as applied to a TI-bipolaron gas.

To start with, let us notice that expression for $\tilde{\omega}_H$ (30) suggests that for $\omega_0 = 0$ Bose-condensation appears to be impossible if $H \neq 0$. For an ordinary ideal charged Bose-gas, this conclusion was first made in [58]. In view of the fact that in the spectrum of TI-bipolarons there is a gap between the ground state of a TI-bipolaron gas and the excited one (§3), this conclusion becomes invalid for $\omega_0 \neq 0$.

Expression $\tilde{\omega}_H$ (30) suggests that there is a maximum value of the magnetic field $H_{\text{max}}$ equal to:

$$H_{\text{max}}^2 = \frac{2\omega_0 h^2 M_e}{\eta^2}.$$ 

(52)

For $H > H_{\text{max}}$ a homogeneous superconducting state is impossible. With the use of (52), $\tilde{\omega}_H$ (30) will be written as:

$$\tilde{\omega}_H = \tilde{\omega} \left(1 - \frac{H^2}{H_{\text{max}}^2}\right).$$ 

(53)

For a given temperature $T$, let us write $H_{cr}(T)$ for the value of the magnetic field at which the superconductivity disappears. According to (53), this value of the field

\footnote{As follows from (16), the value $\eta$ consists from two parts: $\eta = \eta' + \eta''$. The value $\eta'$ is determined by the integral entering into the expression for $\omega_k$ (11). For this reason $\eta'$ depends on the form of sample surface. The value $\eta''$ is determined by the sum entering into the expression for $\omega_k$ (16) and weakly depends on surface form. This leads to the conclusion that the value $\eta$ can be changed by changing sample surface and thus changing $H_{\text{max}}$.}
corresponds to $\tilde{\omega}_{H_{cr}}$:

$$\tilde{\omega}_{H_{cr}}(T) = \tilde{\omega} \left(1 - \frac{H_{cr}^2(T)}{H_{max}^2}\right).$$  \(54\)

The temperature dependence of the quantity $\tilde{\omega}_{H_{cr}}(T)$ can be found from equation (30):

$$C_{bp} = \tilde{T}^{3/2}F_{3/2} \left(\tilde{\omega}_{H_{cr}}(\tilde{T})/\tilde{T}\right).$$ \(55\)

It has the form given in Fig. 2 if we replace $\tilde{\omega}$ by $\tilde{\omega}_{H_{cr}}$ and $\tilde{T}_c$ by $\tilde{T}$.

Using (54) and the temperature dependence given in Fig. 2 we can find the temperature dependence of $H_{cr}(\tilde{T})$:

$$\frac{H_{cr}^2(\tilde{T})}{H_{max}^2} = 1 - \omega_{H_{cr}}(\tilde{T})/\tilde{\omega}.$$ \(56\)

For $\tilde{T} \leq \tilde{T}_{ci}$, these dependencies are given in Fig. 6.

As is seen from Fig. 6, $H_{cr}(\tilde{T})$ reaches its maximum at a finite temperature of $\tilde{T}_{c}(\tilde{\omega} = 0) \leq \tilde{T}_{c}(\omega_0)$. Fig. 6 suggests that at temperature below $\tilde{T}_{c}(\tilde{\omega} = 0) = 25.2445$ a further decrease of the temperature no longer changes the value of the critical field $H_{cr}(\tilde{T})$ irrespective of the gap value $\tilde{\omega}$.

Let us also introduce the notion of a transition temperature $T_{c}(H)$ in the magnetic field $H$. Fig. 7 illustrates the dependencies $T_{c}(H)$ resulting from Fig. 6 and determined by the relations:

$$C_{bp} = \tilde{T}^{3/2}F_{3/2} \left(\tilde{\omega}_{H_{cr}}(\tilde{T}_{C};(H))\right), \quad \tilde{\omega}_{H_{cr}} = \tilde{\omega}_{H=0,i} \left[1 - H^2/H_{max,i}^2\right].$$  \(57\)
Fig. 7 suggests that the critical transition temperature $T_{c}(H)$ changes stepwise as the magnetic field reaches the value $H_{\text{max},i}$.

To solve the problem of the type of the phase transition in a magnetic field let us proceed from the well-known expression which relates free energies in superconducting and normal states:

$$F_{S} + \frac{H^{2}}{8\pi} = F_{N},$$

(58)

where $F_{S}$ and $F_{N}$ are free energies of the unit volume of superconducting and normal states, respectively:

$$F_{S} = \frac{N}{V} E_{bp}(H = 0) - \frac{2}{3} \Delta E(\omega_{H=0}) \frac{N}{V},$$

(59)

$$F_{N} = \frac{N}{V} E_{bp}(H) - \frac{2}{3} \Delta E(\omega_{H}) \frac{N}{V},$$

(60)

where $\Delta E = E - E_{bp}$, $E = \omega^{*} \bar{E}$, where $\bar{E}$ is determined by formula (35). Differentiating (58) with respect to temperature and taking into account that $S = -\partial F/\partial T$, we express the heat of transition $q$ as:

$$q = T(S_{N} - S_{S}) = -T \partial(F_{N} - F_{S})/\partial T = -T \frac{H_{cr}}{4\pi} \frac{\partial H_{cr}}{\partial T},$$

(61)
Accordingly the difference of entropies $S_S - S_N$ will be written as:

$$S_S - S_N = \frac{H_{cr}}{4\pi} \left( \frac{\partial H_{cr}}{\partial T} \right) = \frac{H_{max}^2}{8\pi \omega^*} (\hat{S}_S - \hat{S}_N).$$ (62)

Fig. 8 shows the temperature dependence of the difference of entropies (62) for various values of critical temperatures ($\tilde{\omega}_i$) given in Fig. 2. These dependencies may seem strange in, at least, two respects: 1. In BCS and GL at the most critical point $T_c$ the difference of entropies becomes zero in accordance with Rutgers formula. In Fig. 8 entropy is a monotonous function $\tilde{T}$ which does not vanish for $T = T_c$. 2. Second, in absolute terms, the difference $|\tilde{S}_S - \tilde{S}_N|$, when approaching the limit point $\tilde{T}_c = 25.2$, which corresponds to the value $\tilde{\omega} = 0$, as it can be seen should decrease rather than increase vanishing at $\tilde{\omega} = 0$.

As for the second point, this is really the case for $|\tilde{S}_S - \tilde{S}_N|$, since the value of the maximum field $H_{max}$ and, accordingly, the multiplier $H_{max}^2/8\pi \omega^*$ relating the quantities $S_S - S_N$ and $\tilde{S}_S - \tilde{S}_N$ becomes zero for $\tilde{\omega} = 0$.

As for the first point, as will be shown below, Rutgers formula appears inapplicable for Bose-condensate of TI-bipolarons.

Table 2 lists the values of the quantity $\tilde{S}_S - \tilde{S}_N$ for critical temperatures corresponding to different values of $\tilde{\omega}_{H_{cr}}$.

The results obtained suggest some fundamental conclusions: 1. The curve of the dependence $H_{cr}(T)$ (Fig. 6) for $T = 0$ has a zero derivative, accordingly $dH_{cr}(T)/dT = 0$ for $T = 0$. This result is consistent with Nernst theorem which implies that entropy determined by (61) is equal to zero for $T = 0$.

2. According to Fig. 6, $H_{cr}(T)$ is a curve monotonously drooping with increasing $T$ for $T > T_c(\tilde{\omega} = 0)$, and a constant value for $T \leq T_c(\tilde{\omega} = 0)$. Hence $\partial H_{cr}(T)/\partial T < 0$ for $T > T_c(\tilde{\omega} = 0)$. Therefore on the temperature interval $[T_c(\tilde{\omega} = 0), T_c(\tilde{\omega})]$ $S_S < S_N$ and on the interval $[0, T_c(\tilde{\omega} = 0)]$ $S_S = S_N$. 

Figure 8. Temperature dependencies for the difference of entropies of superconducting and normal states for the values of parameters $\tilde{\omega}_i$ given in Fig. 6, 7.
This suggests some important conclusions:
1. Transition on the interval \([0, T_c(\tilde{\omega} = 0)]\) occurs without absorption or release of latent heat since in this case \(S_S = S_N\). Experimentally it will be seen as a phase transition of the second kind. Actually, in the region \([0, T_c(\tilde{\omega} = 0)]\), a phase transition into a superconducting state is a phase transition of infinite kind, since in this region, according to (58) and Fig. 6, any-order derivatives of the difference of free energies \(F_S - F_N\), become zero.

2. Passing in a magnetic field from a superconducting state to a normal one on the interval \([T_c(\tilde{\omega} = 0), T_c(\tilde{\omega})]\), which corresponds to \(S_S < S_N\), occurs with absorption of latent heat. On the contrary, passing from a normal state to a superconducting one takes place with release of latent heat. The phase transition on the interval \([0, T_c(\tilde{\omega} = 0)]\) is not attended by absorption or release of the latent heat being the phase transition of infinite kind.

With regard to the fact that the specific heat capacity of a substance is determined by the formula

\[
C = T \left( \frac{\partial S}{\partial T} \right),
\]

the difference of specific heat capacities of superconducting and normal states, according to (62) will be written as:

\[
C_S - C_N = \frac{T}{4\pi} \left[ \left( \frac{\partial H_{cr}}{\partial T} \right)^2 + H_{cr} \frac{\partial^2 H_{cr}}{\partial T^2} \right].
\]

This relation is usually used to get the well-known Rutgers formula. To do this one assumes the critical field in (63) to be \(H_{cr}(T_c) = 0\) for \(T = T_c\) and leaves in the brackets on the right-hand side of (63) only the first term:

\[
(C_S - C_N)_R = \frac{T_c}{4\pi} \left( \frac{\partial H_{cr}}{\partial T} \right)^2. \tag{64}
\]

It is easily seen, however, that at the point \(T = T_c\) the quantity \(\omega_{H_{cr}}\) determined by Fig. 2, for all the values of the temperature, has a finite derivative with respect to \(T\) and, therefore, according to (56), an infinite derivative \(\partial H_{cr}/\partial T\) for \(T = T_c\). Hence, the second term in the brackets (63) reduces to \(-\infty\), leaving this bracket a finite value. As a result, a proper expression for the difference of heat capacities of the considered model of Bose-gas should be determined by the formula:

\[
C_S - C_N = \frac{T}{8\pi} \frac{\partial^2}{\partial T^2} H_{cr}^2(T) = \frac{H_{max}^2}{8\pi\omega^*} (\tilde{C}_S - \tilde{C}_N), \tag{65}
\]

\[
\tilde{C}_S - \tilde{C}_N = T \frac{\partial^2}{\partial T^2} \left( \frac{H_{cr}^2(T)}{H_{max}^2} \right).
\]

Table 2 lists the values of quantity \(\tilde{C}_S - \tilde{C}_N\) for the values of critical temperatures corresponding to various values of \(\tilde{\omega}_{H_{cr}}\). Comparison of the jumps in the heat capacity presented in Fig. 5 with expression (65) enables us to calculate the value of \(H_{max}\). The values of \(H_{max}\) obtained by this means for various values of \(\tilde{\omega}_i\) are given in Table 2. These values unambiguously determine the values of constants \(\eta\) in formulae (27'), (27'').

It follows from what has been said, that Ginzburg-Landau temperature expansion for a critical field near the critical temperature \(T_c\) is not applicable for Bose-
Table 2. The values of $H_{\text{max}}$ entropy differences $\tilde{S}_S - \tilde{S}_N$ and heat capacities $\tilde{C}_S - \tilde{C}_V$ in superconducting and normal states determined by relations (62) and (65) are presented for transition temperatures $\tilde{T}_{C_i}$, for the same values of $\tilde{\omega}_{H_{\text{cr}},i}$ as in Fig. 2.

| $i$ | $\tilde{\omega}_{H_{\text{cr}},i}$ | $\tilde{T}_{C_i}$ | $\tilde{S}_S - \tilde{S}_N$ | $\tilde{C}_S - \tilde{C}_V$ | $\tilde{H}_{\text{max}} \cdot 10^{-3}$, Oe |
|-----|-------------------------------|-----------------|---------------------|---------------------|-----------------------------------|
| 0   | 0                             | 25.2145         | 0                   | 0                   | 0                                 |
| 1   | 0.2                           | 27.325          | -0.339995           | -11.5415            | 2.27                              |
| 2   | 1                             | 30.0255         | -0.39999            | -2.17594            | 7.78                              |
| 3   | 2                             | 32.1397         | -0.272182           | -0.939995           | 13.3                              |
| 4   | 10                            | 41.8727         | -0.106679           | -0.187327           | 47.1                              |
| 5   | 15                            | 46.1363         | -0.0831709          | -0.121063           | 64.9                              |
| 6   | 20                            | 49.9754         | -0.0694586          | -0.0888433          | 81.5                              |

condensate of TI-bipolarons. Since the temperature dependence $H_{\text{cr}}(T)$ determines the temperature dependencies of all thermodynamic quantities, this conclusion is valid for all such values. As was pointed out in the Introduction, this conclusion follows from the fact that BCS theory, in view of its nonanalyticity on coupling constant, on no condition passes on to the theory of bipolaron condensate.

Above we dealt with an isotropic case. In the anisotropic case formulae (27′), (27′′) yield:

$$H_{\text{max}}^2 = H_{\text{max}}^2 \perp = \frac{2\omega_0 M_\perp \hbar^2}{\eta^2}, \quad \vec{B} \parallel \vec{c},$$  

(66)

- i.e. in the case when the magnetic field is directed perpendicular to the plane of layers and:

$$H_{\text{max}}^2 = H_{\text{max}}^2 \parallel = \frac{2\omega_0 M_\parallel \hbar^2}{\eta^2}, \quad \vec{B} \perp \vec{c},$$  

(67)

-in the case when the magnetic field lies in the plane of layers. From (66), (67) it follows that:

$$\frac{H_{\text{max}}^2 \perp}{H_{\text{max}}^2 \parallel} = \sqrt{\frac{M_\perp}{M_\parallel}} = \gamma.$$  

(68)

With the use of (56), (67), (68), the critical field $H_{\text{cr}}(\tilde{T})$ (in the directions perpendicular and parallel to the plane of layers) will be:

$$H_{\text{cr}} (\tilde{T}) = H_{\text{max}} \parallel \sqrt{1 - \tilde{\omega}_{H_{\text{cr}}}(\tilde{T})/\tilde{\omega}}.$$  

(69)

From (69) it follows that the relations $H_{\text{cr}} (\tilde{T}) / H_{\text{cr}} (\tilde{T})$ are independent of temperature. The dependencies obtained are compared with experimental data in §7.

7. Comparison with the experiment.

Fig. 4 shows typical dependencies of $E(\tilde{T})$. They suggest that at the point of transition the energy is a continuous function of $\tilde{T}$. This means that the transition per
se occurs without energy expenditure being a phase transition of the 2-kind in complete agreement with the experiment. At the same time transition of Bose particles from a condensate state to a supracondensate one occurs with consumption of energy which is determined by the value $q$ (§4, Table 1), determining the latent heat of transition of a Bose gas which makes it a phase transition of the 1-st kind.

By way of example let us consider HTSC $YBa_2Cu_3O_7$ with the temperature of transition $90 \div 93$K, volume of the unit cell $0.1734 \cdot 10^{-21}$ cm$^3$, concentration of holes $n \cong 10^{21}$ cm$^{-3}$. According to estimates [59], Fermi energy is equal to: $\varepsilon_F = 0.37$ eV. Concentration of TI-bipolarons in $YBa_2Cu_3O_7$ is found from equation (29):

$$\frac{n_{bp}}{n}C_{bp} = f_{\tilde{\omega}}(\tilde{T}_c),$$

(70)

with $\tilde{T}_c = 1.6$. Table 1 lists the values of $n_{bp,i}$ for the values of parameters $\tilde{\omega}_i$ given in 4. It follows from Table 1 that $n_{bp,i} << n$. Hence, only a small part of charge carriers is in a bipolaron state which justifies the approximation of a low-density TI-bipolaron gas used by us. The energy levels of such TI-bipolarons lie near Fermi surface and are described by the wave function:

$$\psi(\vec{r}) = e^{i\vec{k}_F \vec{r}}\varphi(\vec{r}),$$

(71)

which leads to replacement of $\tilde{T}$ involved in (10), by:

$$\tilde{T} = -2 \langle \varphi | \Delta_r | \varphi \rangle + 2k_F^2,$$

(72)

that is to reckoning of the energy from Fermi level (the last term in the right-hand side of (72) in dimensional units is equal to $2\varepsilon_F$ where $\varepsilon_F = \hbar^2 k_F^2/2m^*$).

According to our approach, superconductivity arises when coupled electron states are formed. The condition for the formation of such states has the form:

$$|E_{bp}| \geq 2|E_p|,$$

(73)

where $E_p$ is the energy of a TI-polaron [33]. Condition (73) determines the value of a pseudogap:

$$\Delta_1 = |E_{bp} - 2E_p|.$$  

(74)

For $\Delta_1 >> \omega_0$, the value of a pseudogap can greatly exceed both $T_c$ and the energy of the gap (i.e. $\omega_0$). The expression for the spectrum $E_{bp}$ and $E_p$ (20)–(21) suggests that the angular dependence $\Delta_1(\vec{k})$ is completely determined by the symmetry of the isoenergetic surface of the phonon wave vector $\vec{k}$. Earlier this conclusion was made by Bennet [60] who proved that the main source of anisotropy of superconducting properties is the angular dependence of the phonon spectrum, though some contribution is also made by the anisotropy of Fermi surface.

It follows from what has been said that formation of a pseudogap is a phase transition preceding the phase transition to the superconducting state. Recent experiments [61] also testify in favor of this statement.
In paper [36] correlation length for TI-bipolarons was calculated. According to [36], in HTSC materials its value can vary from several angstroms to several tens of angstroms, which is also in agreement with the experiment.

Of special interest is to determine the characteristic energy of phonons responsible for the formation of TI-bipolarons and superconducting properties of oxide ceramics. To do this let us compare the calculated values of the heat capacity jumps with experimental data.

As is known, in BCS theory a jump in the heat capacity is equal to:

$$\left. \frac{C_S - C_n}{C_n} \right|_{T_c} = 1.43,$$

where $C_S$ is the heat capacity in the superconducting phase, and $C_n$ is the heat capacity in the normal phase, and is independent of the parameters of the model Hamiltonian. As it follows from numerical calculations shown in Fig. 9 and in Table 1, as distinct from the BCS theory, the value of the jump depends on the phonon frequency. Hence, the approach presented predicts the existence of an isotopy effect for the heat capacity jump.

As it is seen from Fig. 9, the heat capacity jump calculated theoretically (4) coincides with the experimental value in $YBa_2Cu_3O_7$ [62], for $\tilde{\omega} = 1.5$, i. e. for $\omega = 7.5$ MeV. This corresponds to the concentration of TI-bipolarons equal to $n_{bp} = 2.6 \cdot 10^{18} \text{cm}^{-3}$. Hence, in contrast to the widespread notion that in oxide ceramics superconductivity is determined by high-energy phonons (with energy $70 \div 80$ MeV [63]) actually, the superconductivity in HTSC materials should be determined by soft phonon modes.

Notice that in calculations of the temperature transition it was believed that the
effective mass $M_e$ in equation (30) is independent of the direction of the wave vector, i.e. isotropic case was dealt with.

In the anisotropic case, choosing principal axes of vector $\vec{k}$ as coordinate axes, we will get the quantity $(M_{ex} M_{ey} M_{ez})^{1/3}$ instead of the effective mass $M_e$. In complex HTSC materials the values of effective masses lying in the plane of layers $M_{ex}, M_{ey}$ are close in value. Assuming in this case $M_e = M_{ex} = M_{ey} = M_{||}$, $M_{ez} = M_{\perp}$, we will get instead of $C_{bp}$ determined by (30) the value $\tilde{C}_{bp} = C_{bp}/\gamma$, $\gamma^2 = M_{\perp}/M_{||}$ is the parameter of anisotropy. Hence anisotropy of effective masses gives for the concentration $n_{bp}$ the value $\tilde{n}_{bp} = \gamma n_{bp}$. Therefore taking account of anisotropy can order of magnitude and greater enhance the estimate of the concentration of TI-bipolarons. If for $YBa_2Cu_3O_7$ we take the estimate $\gamma^2 = 30$ [63], then for the concentration of TI-bipolarons we will get: $\tilde{n}_{bp} = 1.4 \cdot 10^{19}$ cm$^{-3}$, which holds valid the general conclusion: in the case under consideration only a small number of charge carriers are in TI-bipolaron state. The situation can change if the anisotropy parameter is very large. Thus, for example, in layered HTSC Bi-Sr-Ca-Cu-O the anisotropy parameter is $\gamma > 100$, accordingly, the concentration of TI-bipolarons in these compounds can have the same order of magnitude as the total concentration of charge carriers.

Another important conclusion emerging from taking account of the anisotropy of effective masses is that the temperature of the transition $T_c$ depends not on $n_{bp}$ and $M_{||}$ individually, but on their relation which straightforwardly follows from (30). This phenomenon is known as Uemura law. In the next section we will get a more general relation known as Alexandrov’s formula, for which Uemura law is a particular case.

Among experiments with the use of an external magnetic field, of importance are experiments concerned with measurements of London penetration depth $\lambda$. In $YBa_2Cu_3O_7$ for $\lambda$ for $T = 0$ the authors of [64] obtained $\lambda_{ab} = 150 \div 300$ nm, $\lambda_c = 800$ nm. The same order of magnitude of these quantities is given in a lot of papers [65–68]. The authors of [67] (see also references therein) demonstrate that anisotropy of lengths $\lambda_{a}$ and $\lambda_{b}$ in cuprate planes can be 30% depending on the type of the crystal structure. If we take the value $\lambda_a = 150$ nm and $\lambda_c = 800$ nm obtained on most papers, then, according to (47) the anisotropy parameter will be $\gamma \approx 30$, which is the value usually used for for $YBa_2Cu_3O_7$ crystals.

The temperature dependence $\lambda^2(0)/\lambda^2(T)$ was studied in many papers (see [68] and references therein).

Fig. 10 shows a comparison of various curves for $\lambda^2(0)/\lambda^2(T)$. In paper [68] it is shown that in high quality crystals of $YBa_2Cu_3O_7$ the temperature dependence $\lambda^2(0)/\lambda^2(T)$ is well approximated by a simple dependence $1 - t^2$, $t = T/T_c$.

Fig. 11 demonstrates a comparison of the experimental dependence $\lambda^2(0)/\lambda^2(T)$ [68] with the theoretical one:

$$\frac{\lambda^2(0)}{\lambda^2(T)} = 1 - \left(\frac{T}{T_c}\right)^{3/2} \frac{F_{3/2}(\omega/T)}{F_{3/2}(\omega/T_c)},$$

(75)

which follows from (48), (33), (32). Hence there is a good agreement between experimental and theoretical dependencies (75).

The theory developed enables us to compare the temperature dependence of the value of the critical magnetic field in $YBa_2Cu_3O_7$ with experimental data [70]. Since
Figure 10. Penetration depth of the magnetic field found with the use of BCS theory (a-local approximation, b-nonlocal approximation); on empirical law $\lambda^{-2} 1 - (T/T_c)^2$ (c) [69]; in $YBa_2Cu_3O_7$ (d) [68].

Figure 11. Comparison of the theoretical dependence $\lambda^2(0)/\lambda^2(\tilde{T})$ (solid line) obtained in the present article with the experimental one [68] (dotted line).

The theory constructed in 6 describes a homogeneous state of a TI-bipolaron gas, then the critical field under consideration corresponds to a homogeneous Meissner phase. In paper [70] this field is denoted by $H_{c1}$ which is related to denotations of 6 as: $H_{c1} = H_{cr}$, $H_{c1||} = H_{cr||}$, $H_{c1\perp} = H_{cr\perp}$. To make a comparison with the experiment we use parameter values obtained earlier for $YBa_2Cu_3O_7$: $\tilde{\omega} = 1.5$, $\tilde{\omega}_c = 1.6$. Fig. 12 shows a comparison of experimental dependencies $H_{c1\perp}(T)$ and $H_{c1||}(T)$ [70] with theoretical dependencies (69), where for $H_{max\perp\perp}(T)$, we
took the following experimental values: \(H_{max \parallel} = 240\), \(H_{max \perp} = 816\). The results presented in Fig. 12 confirm the conclusion (6) that relations \(H_{cr \perp}(T)/H_{cr \parallel}(T)\) are independent of temperature.

Relations (47), (66), (67) yield:

\[
(\gamma^*)^2 = \frac{M^*_{\perp}}{M^*_{\parallel}} \propto \frac{\lambda^2_{\perp}}{\lambda^2_{\parallel}}; \quad \frac{H_{max \perp}^2}{H_{max \parallel}^2} = \gamma^2 = 11.6.
\] (76)

The assessment of anisotropy parameters \(\gamma^2 = 11.6\) determined by relations (76) differs from the value \((\gamma^*)^2 = 30\) used above. This difference is probably caused by difference in anisotropy of polaron effective mass \(M^*_{\parallel,\perp}\) and electron band mass \(m^*_{\parallel,\perp}\).

8. Scaling relations.

Scaling relations play an important role in the theory of superconductivity assisting the search for new high-temperature superconductors with record parameters. These relations can emerge as a result of numerous experiments lacking any reliable theoretical substantiation. Or else they can be derived from insufficiently reliable theoretical considerations, but subsequently be supported by a lot of experiments. By way of example we refer to Uemura law considered in the previous section.

The theory presented here enables us to give a natural explanation to some important scaling relations. In particular, in this section we will derive Alexandrov’s
formula [71, 72] and Homes’s scaling law.

**Alexandrov’s formula.** As was mentioned above (7), in an anisotropic case formula (70) takes on the form:

\[
\tilde{T}_c = F_{3/2}^{-2/3} \left( \frac{\tilde{\omega}}{\tilde{T}_c} \right) \left( \frac{n_{bp}}{M_{||}} \right) \frac{2\pi h^2}{M^{1/3} \omega^*}.
\] (77)

It is convenient to pass on in formula (77) from quantities \(n_{bp}, M_{||}, M_\perp\) which can hardly be determined in experiments to quantities which are easily measured experimentally:

\[
\lambda_{ab} = \left[ \frac{M_{||}}{16\pi n_{bp} e^2} \right]^{1/2}, \quad \lambda_c = \left[ \frac{M_\perp}{16\pi n_{bp} e^2} \right]^{1/2}, \quad R_H = \frac{1}{2e n_{bp}},
\] (78)

where \(\lambda_{ab} = \lambda_{||}, \lambda_c = \lambda_\perp\) are London lengths of penetration into the planes of layers and in perpendicular direction, respectively; \(R_H\) is Hall coefficient. In expressions (78) the light velocity is assumed to be equal to unity: \(c = 1\). With the use of relations (78) and (77) we get:

\[
k_B T_c = \frac{2^{1/3}}{8} F_{3/2}^{-2/3} \left( \frac{\tilde{\omega}}{\tilde{T}_c} \right) \frac{h^2}{e^2} \left( \frac{e R_H}{\lambda_{ab}^4 \lambda_c^2} \right)^{1/3}.
\] (79)

In formula (79) the quantity \(e R_H\) is measured in cm\(^3\), \(\lambda_{ab}, \lambda_c\) in cm, \(T_c\) in Kelvin.

Taking into account that in most HTSC materials \(\tilde{\omega} \approx \tilde{T}_c\) and the function \(F_{3/2}(\tilde{\omega}/\tilde{T})\) changes near \(\tilde{\omega} = \tilde{T}_c\) only slightly, with the use of the value \(F_{3/2}(1) = 0.428\) and expression (78) we present \(T_c\) in the form:

\[
T_c \approx 8.7 \left( \frac{e R_H}{\lambda_{ab}^4 \lambda_c^2} \right)^{1/3}.
\] (80)

Formula (80) differs from Alexandrov’s formula [71, 72] only in numerical coefficient which in [71, 72] is equal to 1.64. As is shown in [71, 72], formula (80) practically always properly describes relation between the parameters for all known HTSC materials. In [71, 72] it is also shown that Uemura relation [73, 74] is a particular case of formula (80).

**Homes’s law.** Homes’s law holds that scaling relation are valid for superconducting materials [75, 76]:

\[
\rho_S = C \sigma_{DC}(T_c) T_c,
\] (81)

where \(\rho_S\) is the density of a superfluid component for \(T = 0\), \(\sigma_{DC}(T_c)\) is the conductivity of direct current for \(T = T_c\), \(C\) is a constant equal to \(\approx 35\text{cm}^{-2}\) for ordinary superconductors and HTSC materials for a current running in the plane of layers.
The quantity \( \rho_S \) involved in (81) is related to plasma frequency \( \omega_p = \sqrt{4\pi n_e e_S^2/m_S^*} \) (where \( n_S \) is a concentration of superconducting charge carriers; \( m_S^* \), \( e_S \) are a mass and charge of superconducting charge carriers) by a well-known expression \( \rho_S = \omega_p^2 \) [77]. Using this expression, relation \( \sigma_{DC} = e_n^2 n_n \tau/m_n^* \) (where \( n_n \) is the concentration of charge carriers for \( T = T_c \)), \( m_n^* \), \( e_n \) are the mass and the charge of charge carriers, relation \( \tau \sim \hbar/T_c \) (where \( \tau \) is the minimum Planck time for scattering of electrons at the critical point [77]) and also expression (81), on the assumption \( e_S = e_n \), \( m_S = m_n \), we get:

\[ n_S(0) \equiv n_n(T_c). \]  

(82)

In our scenario of Bose-condensation of TI-bipolarons, Home's law in the form of (82) becomes almost obvious. Indeed, for \( T = T_c \) TI-bipolarons are stable formations (they decay at temperature equal to the pseudogap energy which considerably exceeds \( T_c \)). Their concentration for \( T = T_c \) is equal to \( n_n \) and therefore these bipolarons for \( T = T_c \) start forming condensate whose concentration \( n_S(T) \) reaches its maximum \( n_S(0) = n_n(T_c) \) for \( T = 0 \) (i.e. when bipolarons fully pass on to condensed state) which corresponds to relation (82). Notice that in the framework of BCS theory Home's law can hardly be explained.

9. Essential generalizations of the theory.

In the foregoing we considered the case of an ideal TI-bipolaron gas. At small concentration of TI-bipolarons their Coulomb interaction will be greatly screened which justifies the use of the model of an ideal gas.

If the concentration of TI-bipolarons is large (for example \( n = 10^{21} \text{cm}^{-3} \), as was believed in §4,) then such Bose gas can no longer be considered to be ideal. Taking account of Coulomb interaction between bipolarons becomes necessary. For this purpose we can use Bogolyubov theory [78] for weakly imperfect Bose gas, which implies that the spectrum of elementary excitations with the momentum \( k \) will be determined by the expression:

\[ E(k) = \sqrt{k^2u^2(k) + (k^2/2m_B)^2}, \]  

(83)

where \( u(k) = \sqrt{n_B V(k)/m_B} \), \( V(k) \) is the Fourier component of the potential of pairwise interaction between charged bosons: \( V(k) = 4\pi e_B^2/\epsilon_0 k^2 \), \( e_B \) and \( m_B \) are the charge and mass of a boson, \( n_B \) is the concentration of bosons.

Whence it follows that:

\[ E(k) = \sqrt{(\hbar \omega_p)^2 + (k^2/2m_B)^2}, \quad \omega_p = \sqrt{4\pi e_B^2 n_B/\epsilon_0 m_B}, \]  

(84)

\( \omega_p \) is the plasma frequency. According to (83), the spectrum of excitations of quasiparticles of an imperfect gas is characterized by a finite energy gap which in the long-wavelength limit is equal to \( \omega_p \). The same result can be arrived at if the Coulomb interaction between the electrons is considered as a result of electron-plasmon interaction. According to [79], Hamiltonian of electron-phonon interaction coincides in structure with Froehlich Hamiltonian which involves plasmon frequency instead
of phonon frequency. This straightforwardly leads to the energy gap equal to the plasmon frequency $\omega_p$. If in (84) we put: $e_B = 2e$, $m_B = 2m^*$, $n_B = n/2$, where $n$ is the electron concentration.

Notice that according to Bogolyubov’s theory the inter-particle interaction is necessary for arising the gap in their spectrum.

Contrary this the single TI-bipolaron has the gap in its spectrum. Thus the non-interacting TI-bipolaron gas also has the gap. If Coulomb interaction of bipolarons is taken into account then Bogolyubov’s formula (83) can be generalized in the form:

$$E_k = \sqrt{k^2u^2(k) + (\hbar\omega_0 + k^2/2m_B)^2}.$$  (85)

It follows from (85) the next expression for the gap of Coulomb interacting TI-bipolaron gas:

$$\Delta = \sqrt{(\hbar\omega_p)^2 + (\hbar\omega_0)^2}.$$  (86)

Hence, for $\omega_p < \omega_0$, the energy gap will be determined by $\omega_0$ while for $\omega_p > \omega_0$ it will be determined by $\omega_p$.

Actually, in real HTSC, there are not only phonon and plasmon branches, but also some other elementary excitations which can take part in electron pairing. An example are spin fluctuations. Generalization of the theory to the case of interaction with various branches of excitations which will contribute into the ground state energy, the value of the gap and the dispersion law of a TI-bipolaron is a topical problem. Obviously, taking account of this interaction will lead to an increase in the coupling energy of both TI-bipolarons and TI-polarons. Therefore, a priori, without any particular calculations one cannot say anything of how the condition of stability of TI-bipolaron states determined by (73) will change.

Another important problem is generalization of the theory to the case of intermediate coupling of electron-phonon interaction. Formally, the expression for the functional of the ground state energy of a TI-bipolaron (18) is valid for any value of the electron phonon coupling constant. For this reason such a calculation will not change the spectrum of a TI-bipolaron, however it will alter the criteria of fulfillment of the conditions of a TI-bipolaron stability (73).

10. Summary.

It is generally accepted that super-flow, superfluidity and superconductivity are collective phenomena that are driven by inter-particle interactions. Here we state an opposite suggestion that the above phenomena are mainly determined by the specific properties of separate boson particles.

In this paper we have presented conclusions emerging from consistent translation-invariant consideration of EPI. It implies that, pairing of electrons, for any coupling constant, leads to a concept of TI-polarons and TI-bipolarons. Being bosons, TI-bipolarons can experience Bose condensation leading to superconductivity. Let us list the main results following from this approach.

First and foremost the theory resolves the problem of the great value of the bipolaron effective mass (§4). As a consequence, formal limitations on the value of the critical temperature of the transition are eliminated too. The theory quantitatively
explains such thermodynamic properties of HTSC-conductors as availability (§4) and value (§7) of the jump in the heat capacity lacking in the theory of Bose condensation of an ideal gas. The theory also gives an insight into the occurrence of a great ratio between the width of the pseudogap and $T_c$ (§6). It accounts for the small value of the correlation length [36] and explains the availability of a gap and a pseudogap ((§7) in HTSC materials. Accordingly, isotopic effect automatically follows from expression (30) where the phonon frequency $\omega_0$ acts as a gap. The conclusion of the dependence of the temperature of the transition $T_c$ on the relation $n_{bp}/M||$ (§7) correlates with Alexandrov-Uemura law (§8) universal for all HTSC materials. It is shown that Home's scaling law is a natural consequence of the theory presented (§8). The theory explains a wide variety of phenomena observed in a magnetic field (§6). In particular:

1. It is shown that the occurrence of a gap in the spectrum of TI-bipolarons makes possible their condensation in a magnetic field.
2. It is demonstrated that there exists a critical value of the magnetic field above which homogeneous Bose-condensation becomes impossible.
3. The temperature dependence of the critical magnetic field and London penetration depth obtained in the paper are in good agreement with the experiment.

At the same time the theory presented shows that: 1. Rutgers formula cannot be applied to describe Bose-condensation of TI-bipolarons. 2. Ginzburg-Landau expansions do not suit to describe Bose-condensation of TI-bipolarons.

The theory predict some phenomena such as:

1. Isotopic effect for a jump of heat capacity in passing from the normal phase to superconducting one.
2. A possibility of the existence of a phase transition of infinite kind in a magnetic field at low temperatures.
3. Identity of the energy gap with phonon frequency.
4. Existence of superconducting TI-bipolarons whose concentration is much less than the total concentration of charge carriers.

Application of the theory to 1D and 2D systems leads to qualitatively new results since the occurrence of a gap in the TI-bipolaron spectrum automatically removes divergences at small momenta, inherent in the theory of ideal Bose gas. An important consequence of this fact is the existence of a superconducting phase in homogeneous 1D and 2D systems.

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