The ADM papers and part of their modern legacy: loop quantum gravity

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Abstract
We present a summary for non-specialists of loop quantum gravity as part of the modern legacy of the series of papers by Arnowitt, Deser and Misner circa 1960.

Keywords: ADM, loop quantum gravity, Arnowitt, Deser, Misner

1. Introduction: the ADM papers

The revolutionary character of Einstein’s general theory of relativity can be gauged by how long it took to appreciate fully its implications. Although the equations of the theory and some basic consequences were laid down in 1916, many of the concepts and ideas that we consider important today would have to wait several decades for their development. For instance, although the Schwarzschild solution was also written in 1916, the idea of a black hole was not properly understood until the 1960s. It is remarkable that many of the brightest minds of 20th century physics missed the concept or were quite confused by it. Gravitational waves, already discussed by Einstein himself in 1916, had a tortuous development (with Einstein himself coming to doubt their existence) until the binary pulsar put to rest the confusion about their existence [1].

Another of the concepts that remained in a confused state for many years was the dynamics of Einstein’s theory. This is not surprising. Einstein’s equations are a complex set of nonlinear partial differential equations. Unless handled properly, their character is unclear since they mix wave-like equations with elliptic equations. This created some confusion even as recently as the 1990s in the context of numerical relativity. Mainly, the Einstein equations are equations for a space–time and we are accustomed to describe physics as an evolution in time. However, general relativity is a background independent theory, implying that there is no preferred notion of time and therefore not a preferred notion of evolution. This was the source of confusion over the years. Early work on understanding the Einstein equations as an initial value problem was done by Darmois and Lichnerowicz (see the nice recent account of Choquet-Bruhat [2]). Experience with other theories suggested that further insights could be
obtained by casting the theory in canonical form. However, the elaborate symmetries of general relativity meant that progress in the canonical formulation was not achieved until the late 1950s by Dirac [3], and independent developments were done by Bergmann’s group in Syracuse [4].

Then, in an illuminating series of papers around the year 1960, Arnowitt, Deser and Misner (ADM) [5] provided remarkable insights concerning the canonical formulation and the identification of the true degrees of freedom, using techniques similar to those Schwinger had used in other field theories. This body of work led to ADM receiving the Dannie Heinemann Prize of the American Physical Society in 1994. Richard Arnowitt unfortunately passed away last year. The Einstein medal has recently been awarded to Stanley Deser and Charles Misner, also citing the same body of work.

A good summary of the results is still the paper by Arnowitt et al [6] in the volume edited by Louis Witten that presented an interesting snapshot of current research in relativity in the early 1960s. This summarizes a series of papers by the same authors in journals. It was reprinted as a ‘Golden oldie’ by the journal General Relativity and Gravitation [7]. A recent paper on the legacy of ADM was written by Stanley Deser [8].

The paper starts by recalling and casts in simple notation the Hamiltonian framework for parameterized systems, where the Hamiltonian is a constraint. Lanczos [9] already discusses this idea, which goes back to Jacobi, but the presentation in this paper is cleaner, almost as one would present it today. The idea is that in parameterized mechanical systems the action takes the form

\[
S = \int dt \left( \sum_{i=1}^{M} p_{q_i} \dot{q}_i - N H(q, p) \right),
\]

where \( N \) is a Lagrange multiplier and varying with respect to it yields \( H(q_i, p_i) = 0 \), a constraint. The next section of the paper goes on to show that the action of general relativity can be written in a way that it is manifestly in ‘already parameterized’ form.

Then the paper recalls the Palatini formulation of general relativity, where the connection is treated as an independent variable and writing the Ricci tensor as a function of the connection only

\[
S = \int d^4x \sqrt{-g} g^{\mu\nu} R_{\mu\nu}(\Gamma),
\]

so variation with respect to \( g^{\mu\nu} \) yields the Einstein equations and variation with respect to the connection gives the Christoffel formula. It then proceeds to consider a \( 3+1 \) decomposition of the metric

\[
g_{ij} = 4g_{ij}, \quad N = \left( -4 g^{00} \right)^{-1/2}, \quad N_i = 4g_{0i},
\]

\[
\pi^{ij} = \sqrt{-g} \left( 4T^0_{ki} - 4T^0_{kl} g^{lm} \right) g^{ik} g^{jl},
\]

with \( i, j, k \ldots \) going from one to three, and rewrite the Lagrangian as

\[
\mathcal{L} = g_{ij} \partial_i \pi^{ij} - NH - N_i H^i + \text{div}
\]

with \( \text{div} \) a divergence term and

\[
H = \sqrt{g} \left[ 3R + g^{-1} \left( \frac{1}{2} \pi^2 - \pi^{ij} \pi_{ij} \right) \right].
\]
\[ H^i = -2\pi^{ij}_i, \]  

where we recognize that it is already in a parameterized form as discussed previously and identify the Hamiltonian \( H \) and diffeomorphism \( H^i \) constraints. It goes on to notice that one can work out from the Lagrangian the equations of motion for \( g_{ij} \) and \( \pi^{ij} \), and together with the constraints, they are equivalent to the Einstein equations. The paper discusses the meanings of the various variables, how they are intrinsic to a surface \( t = \text{constant} \) and the roles of the multipliers \( N, N^i \) and the interpretation of the momentum \( \pi^{ij} \) and its relation to the extrinsic curvature of the surface.

The paper also uses the constraint structure and the Lagrange multipliers to elucidate the initial value problem of general relativity. It makes clear that if one specifies \( g_{ij}, \pi^{ij}, N \) and \( N^i \) at a given time \( t \), the equations determine the values of \( g_{ij} \) and \( \pi^{ij} \) at a later time. However, \( g_{ij} \) and \( \pi^{ij} \) cannot be chosen arbitrarily at an initial time \( t \), they are subject to constraints. It also notes that the constraints are preserved in time due to the Bianchi identities and therefore are satisfied at all future times if they are satisfied initially.

Moreover it is pointed out that the 12 dynamical variables \( g_{ij}, \pi^{ij} \) provide a complete Cauchy data set but not a minimal one. That will be further elucidated in the canonical section of the paper.

Then it casts the theory in canonical form, and starting from an analogy with linearized theory proceeds to fix the gauge of the theory to have a true Hamiltonian dictate the evolution. Having the true Hamiltonian it can proceed to define what we now call the ADM energy and ADM momentum of the gravitational field,

\[ E = \oint \mathcal{D}S \left( g_{ij,j} - g_{ij,i} \right), \]  
\[ P^i = -2 \oint \pi^{ij} dS_j, \]

These, together with the initial value formulation and the gauge fixings are widely used in numerical relativity, perhaps the other main legacy of the ADM papers. We will not cover the topic here but refer to the article by Sperhake [10] in this volume.

The paper then goes on to talk about gravitational radiation, in particular the definition of the wave-zone, a delicate concept in a nonlinear theory. This should be put in the context that at the time there was controversy in certain circles on the existence of gravitational radiation (see Kenneficck for a complete account of the controversy). It also considers coupling the theory to electromagnetism.

The paper concludes with some discussions on self-energy, in particular including an illuminating example about how non-perturbative effects could influence the self-energy. This example is cited in the introduction to one of the first books in loop quantum gravity by Ashtekar and Tate as motivation [11].

Attempts to use the canonical formulation described in this paper to quantize general relativity were done in the 1960s. The idea was to consider wavefunctions \( \Psi \) that depended on the spatial metric (e.g. [12]). The canonical momenta operated like functional derivatives and the constraints promoted to operatorial equations. These ran into several difficulties. Among them there was no success in constructing an inner product that would constitute a Hilbert space and that was compatible with the symmetries of the theory. There was little mathematical control on the space of wavefunctions of the metric. And the complicated non-polynomial nature of the Hamiltonian constraint made it difficult to promote it to a well defined quantum operator. Even in simplified contexts like quantum cosmology, where
one freezes all degrees of freedom but a finite number of them, the resulting quantum theories failed to achieve noteworthy results, in particular, the elimination of the singularity that appears in the classical theory. This led to a slowing of activity in canonical quantum gravity in the mid 1970s.

2. Loop quantum gravity: the beginnings

In 1986 Ashtekar [13] presented a reformulation of the canonical treatment of general relativity. Its original presentation was in terms of spinors, and part of the original calculations and motivations also came from spinors, but it turns out they are not essential to present the theory. The reformulation consisted of replacing the spatial metric with a (densitized) triad $\tilde{E}_i^a$, where $a$ corresponds to a spatial index and $i$ is a triad index that is raised and lowered with a flat ($SO(3)$ invariant) metric. The canonically conjugated momentum is related to the extrinsic curvature and the spin connection associated with the triad $A_i^a$. Such a variable had been invented by Sen in a different context [14]. The novelty is that it transforms as a (complex) $SO(3)$ connection. These are the same variables one would use to describe an $SO(3)$ Yang–Mills theory. Due to the additional number of variables there exist an additional set of constraints that are identical to the Gauss law that one has in Yang–Mills theories. Therefore one could view general relativity as a theory with the same phase space as Yang–Mills theories with additional constraints. The constraints took a very simple polynomial form

$$D_a E_i^a = 0, \quad (10)$$
$$\tilde{E}_i^a F_{ab} = 0, \quad (11)$$
$$\epsilon^{ijk} \tilde{E}_i^a \tilde{E}_j^b F_{ab} = 0, \quad (12)$$

where $D_a$ is the covariant derivative and $F_{ab}$ the curvature of the connection $A_i^a$. The new variables immediately led to a shift in point of view: the natural polarization one uses to treat Yang–Mills theories is to choose wavefunctions of the connection $\Psi[A]$, not of the electric field. That choice would be equivalent to considering wavefunctions of the extrinsic curvature rather than the metric. The wavefunctions have to be gauge invariant functions under $SO(3)$ transformations, plus satisfy four additional constraints. All this led to the hope that the many successful techniques that were used to treat Yang–Mills theories could be imported into gravity. In the end the latter hope proved naive: even if one could cast them as sharing an (unconstrained) phase space, gravity and Yang–Mills theories are vastly different theories. Some of the reasons that Yang–Mills theories can be treated, such as their asymptotic freedom in the case of QCD, and the possibility of putting them on a lattice, are just not available in the gravitational case. One technique, however, proved fruitful.

The technique in question is the use of loop variables. The key idea is contained in the familiar Stokes’ theorem,

$$\int_{\partial S} \tilde{A} \cdot d\tilde{r} = \int_S \tilde{B} \cdot d\tilde{S}, \quad (13)$$

where $d\tilde{S}$ is a loop given by the boundary of the surface $S$. Given a vector potential, the left-hand side of this equation is a function of a loop. If one knows the value of that function for all loops, for a vector potential, one knows the magnetic field. A simple way of seeing this is to consider infinitesimal loops where the right-hand side is just the value of the field times the area of the infinitesimal loops. A similar result holds for the vector potentials (connection) of Yang–Mills theories. So this opens the possibility to replace the wavefunctions of the
connection \( \Psi[A] \) with functions of loops \( \Psi[\gamma] \) with \( \gamma \) a closed curve. In the context of Yang–Mills theories (and even gravity [15]) this was first studied by Gambini and Trias [16]. In the context of gravity written in terms of Ashtekar’s new variables this was first studied by Rovelli and Smolin [17] and also Jacobson [18]. The resulting quantum representation is known as the loop representation. One can connect the connection representation with the loop representation via a ‘loop transform’,

\[
\Psi[A] = \int d\gamma W[\gamma] \Psi[\gamma],
\]

where the integral is a formal sum over loops (which has been made precise [19]) and \( W[\gamma] \) is the trace of the holonomy of the connection \( A_\mu \) along the loop \( \gamma \), the non-Abelian generalization of the circulation of the vector that appears in Stoke’s theorem.

Again, the loop representation does not create miracles, but it shifts the point of view on several problems. First of all, the loop variables are gauge invariant, so the Gauss law is automatically satisfied. Secondly, there is significant experience in writing functions of loops that are invariant under diffeomorphisms, it is the branch of mathematics known as knot theory. So one was left with the task of understanding the Hamiltonian constraint.

Several attempts were made to write the Hamiltonian constraint in terms of loop variables in the early 1990s [20]. It helped that the structure of the constraints in terms of the new variables is polynomial. However, the expression still had to be regularized, which is not easy to do in a diffeomorphism invariant way. Moreover, the loop variables are not free, they are constrained by a set of nonlinear identities known as Mandelstam identities [21]. Also, the complex nature of the Ashtekar variables (one was using complex coordinates to describe a real theory) required imposing extra ‘reality conditions’ on the framework. These problems hampered progress until new elements were put into place that address them.

3. The 1990s: quantum geometry and a well defined theory of quantum gravity

The situation started to improve when in 1995 Rovelli and Smolin [22] noted that one could use spin networks to solve the Mandelstam identities and provide a basis of (almost) independent loop states. Spin networks are graphs with intersections and ‘colors’ on their lines that had presciently been studied by Roger Penrose [23] in the late 1960s as potentially connected with quantum gravity. In terms of these states it was possible to write clean expressions for the operators representing the area of a surface and the volume of a region of space. A picture of quantum geometry started to emerge: surfaces were endowed with area when punctured by the lines of the spin networks and regions of space were endowed with volume when they contained vertices of the spin networks [24].

Also in the same year, Ashtekar and Lewandowski [25] showed how one could use spin networks to introduce a diffeomorphism invariant inner product in the space of connections modulo gauge transformations. The resulting inner product is remarkable as there were no known inner products in such a space known in closed form (the Yang–Mills theory provides an inner product perturbatively or on the lattice). The resulting inner product had a particularly simple form in terms of spin network states. Two spin-network states are orthogonal if their graphs or ‘colors’ are different. Otherwise their inner product is unity.

This inner product leads to unusual consequences. In particular it emerges that there is no well defined operator corresponding to the connection in the resulting Hilbert space. What has a well defined action is the holonomy. This may appear strange since the latter is the path ordered exponential of the former, but there is no well defined notion of ‘logarithm’ to invert the definition. This in particular implies a departure from the type of functional spaces that are
normally considered in quantum field theory. Some people have questioned the suitability of these spaces, in particular their non-separable nature [26] (see response here [27]). But it is clear that if something new was going to be obtained in quantum gravity one would have to relinquish structures that failed to do the job in the past. We will see that even in the simple context of cosmology this relinquishing has important consequences in the next section.

Then in 1996 Thomas Thiemann [28] surprised the community when he announced that using the quantum geometry techniques he could write a well defined quantum expression for the Hamiltonian constraint. The expression was free of infinities and free of anomalies. It could also be extended to matter couplings. In addition, it did not use the original Ashtekar variables (which are complex) but a real modification that Barbero [29] had put forward (more precisely a one-parameter family of modifications, the parameter is known as the Immirzi parameter [30]). So it did not have to deal with reality conditions. This was the first non-trivial, finite, well defined theory of quantum gravity. The remaining question, unsolved until today, is if it captures enough of the right physics of general relativity in the semi-classical limit.

In a separate development, Rovelli and Krasnov [31] in 1996 noted that one could make sense of the entropy of black holes in loop quantum gravity by counting the number of possible spin network configurations that would endow the horizon with a given area by piercing it. The result was further formalized by Ashtekar et al [32]. They noted that if one formulated the action classically on a manifold with a boundary given by the horizon, a Chern–Simons theory arises on the horizon and certain compatibility conditions have to be met at the ‘punctures’ the spin network generates on the horizon. The resulting entropy is proportional to the area, but the proportionality factor depends on the Immirzi parameter, whose value at the moment is not known. The status of the Immirzi parameter has parallels with the theta ambiguity in Yang–Mills theory. At the moment there are no other experimental predictions that could determine the parameter. At least for all calculations of black hole entropies including rotating, charged, deformed and other black holes the dependence on the Immirzi parameter is always the same, so a single value is compatible with them. The subject of black hole entropy in loop quantum gravity has been further developed by researchers in Spain [33] who used advanced combinatorial techniques to do the counting of configurations and discovered interesting structures in how eigenvalues for the area cluster. In particular they can compute higher corrections in terms of the area. There has been some dispute about the fact that the logarithmic corrections do not seem to agree with those worked out by more conventional Euclidean path integral techniques [34]. Many, however, share the view that it is unclear if the two calculations are computing the same thing since they are so different in nature. For instance, it should be remembered that in the Euclidean context there really is no horizon.

Another separate development was the observation that the discrete nature of quantum geometry could leave an imprint on the light that arrives from distant gamma ray bursts [35]. The discrete nature of space–time leads to dispersion of the light. The effect is minute, of the order of the Planck length divided by the wavelength, but for gamma ray bursts the light travels a very long distance so the effect can build up. It turns out the detailed (yet heuristic) calculation shows that the order of the effect in inverse powers of the wavelength depends on the parity chosen of the quantum state representing the universe. Only if there is a parity violation would one have a linear effect. And if the linear effect is absent then the effect becomes too small to be measured with current observations. It turns out that radio sources already put a stringent limit on the linear effects [36], suggesting that the state describing space—time does not violate parity. Many papers have been written on these issues, a good summary is in [37].
4. The 21st century: symmetry reduced models

An avenue that was chosen by several researchers to try to probe the physical content of Thiemann’s theory is to consider symmetry reduced models. In them one freezes most of the degrees of freedom of the theory, keeping only a small number of them dynamical, perhaps a finite number of them. The resulting models are usually too simple to apply ‘loop quantum gravity’ techniques to them, but one can use techniques inspired by loop quantum gravity.

A first set of such models comes from cosmology, where the first clean treatment was carried out by Bojowald [38]. One may consider a Friedmann–Robertson–Walker model where the only degree of freedom in the metric is the scale factor \( a(t) \). If one works out the Ashtekar variables for that model, a gauge can be chosen in which the connection takes the form \( A^i_j = c(t)\delta^i_j \) and the triad is related to the canonically conjugate variable to \( c(t) \), usually called \( p(t) \). Notice that the resulting model is a mechanical system, it has one degree of freedom, it is not a field theory. One then proceeds to quantize, however, mirroring the full theory, one constructs a Hilbert space that has some discreteness in it. Namely, instead of the usual inner product \( \langle p' | p \rangle = \delta^{\gamma'}_{\gamma p} \) one writes \( \langle p' | p \rangle = \delta^{\gamma'}_{\gamma p} \) with a Kronecker delta. This is tantamount to saying that the variable \( c \) is not well defined but its exponentiation is, \( \sin(pc)/\rho \). Here \( \rho \) is a free parameter that can be viewed as the remnant information of the loop, i.e. its length, in the context of a homogeneous space. In the limit \( \rho \to 0 \) one recovers the usual theory, but such a limit is not available in the Hilbert space chosen, inspired in the Ashtekar–Lewandowski measure. This implies that the quantum evolution equations one writes will depart from the classical behavior when \( \sin(pc) \sim 1 \). This occurs near where the big bang is in the classical theory. The picture that emerges is that of a universe that when it is large, is well approximated by general relativity, but if one runs it backwards, near where the big bang used to be it ‘bounces’ back into expansion into the past without a singularity developing. Some refinements on the picture were made, noticing that the role of \( \rho \) was related to the minimum value of the quantum of area around which to run a loop and therefore ought to depend on the dynamical variables. This led to an ‘improved dynamics’ by Ashtekar et al [39] that has by now been applied in a variety of models, and it appears that the feature of the bounce appears in all the models considered.

An important development has been the study of perturbations of the loop quantum cosmology models in search of potential imprints of the dynamics in the cosmic microwave background [40]. At first this may seem unlikely, the background gets constituted after inflation and during it one expects quantum gravity corrections to be of the order of \( 10^{-12} \). Although this is true, the dynamics of the bounce influences the initial state that is evolved through inflation and eventually gives rise to the cosmic microwave background. The result is that imprints are left in the very long wavelength modes that could potentially be observable. Unfortunately the observational data is very inaccurate for long wavelengths. Moreover the result depends on the value of the inflaton at the bounce, which we do not know. So it is not an unambiguous prediction of the theory that can be used to confirm or rule it out if it is not observed, one can simply adjust the value of the inflaton at the bounce to match the observed data. The ‘tilt’ of the tensor spectral index, and its consistency relation, however, differ from those predicted by standard inflation [40]. This has not been measured yet, but it is plausible that it could be measured in forthcoming years, leading to a prediction that can be used to confirm or rule out the theory. The fact that contact with experiment is so close is really an exciting possibility.

Another set of symmetry reduced models where the theory has been applied has been spherically symmetric models. The initial treatment was due to Bojowald and Swiderski [41]. There one faces more challenges than in the cosmological setting. The variables have spatial
dependence and therefore one in principle is dealing with field theories (even though at the end of the day the number of degrees of freedom may end up being finite in some cases, for instance in vacuum). One has a diffeomorphism and Hamiltonian constraint and one has to worry about their constraint algebra. Remarkably, it was found that for several of these models one can perform linear combinations of the constraints to yield a Hamiltonian constraint that is Abelian. One can then complete the Dirac quantization of them [42]. Remarkably, the constraints can be solved in closed form. One can write parameterized Dirac observables acting on the space of physical states that represent the components of the metric. The parameters play the role of choices of gauge. The resulting operator for the metric, in order for it to be self adjoint, does not have support on the point where the singularity used to be. So the black hole singularity is also resolved, and one can move through the region where it was into another region of space–time isometric to the exterior. Quantum fields have been studied on this quantum space–time background and Hawking radiation has been recovered with small corrections, as expected, and coinciding with a heuristic calculation that had simply assumed a cutoff a few years ago [43]. The Casimir effect between two shells living on the quantum space–time was computed [44], obtaining the correct result without having to resort to regularization nor renormalization, therefore opening new perspectives on the problem of back-reaction, which now appears treatable.

5. Other developments and conclusions

Given the brevity of this article, we cannot do justice to many other developments that have spun off from loop quantum gravity. The main one perhaps is the development of a covariant form of the theory using the path integral. The idea is to use the quantum geometry tools to address some of the usual problems in the definition of the path integral. The resulting theories are called spin foam models. There is now a book by Rovelli and Vidotto [45] discussing the topic. Closely related to these are a set of theories such that their Feynman diagrammatics coincide with the diagrams that spin foams produce [46]. This is analogous to what matrix models do in 1 + 1 dimensions. These field theories are called group field theories and can be viewed as generalizations of matrix models to four dimensions.

Summarizing, loop quantum gravity is now close to 30 years old. There is a proposal for a theory of quantum gravity based on it that is finite, anomaly free and well defined. It has proved challenging to confirm if it contains the correct physics in the semiclassical limit. However, steady progress in symmetry reduced models is producing attractive results that may lead to connections with experiment in the cosmological context and may open new lines of attack on the problem of back reaction in black hole evaporation in the spherically symmetric context. Important progress in these areas is likely to happen in a few short years. It can fairly be said that this quantum legacy of the ADM papers is a testimony of the profound influence they keep on having in gravitational physics.

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