A NOTE ON THE CODIMENSION OF THE LINEAR SECTION OF THE 
LAGRANGIAN-GRASSMANNIAN \( L(6, 12) \)

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Abstract. Consider a \( 2n \)-dimensional symplectic vector space \( E \) over an arbitrary field \( F \). Given a contraction map \( f : \Lambda^n E \to \Lambda^{n-2} E \) such that the Lagrangian–Grassmannian \( L(n, 2n) = G(n, 2n) \cap P(\ker f) \), where \( \Lambda^r E \) denotes the \( r \)-th exterior power of \( E \) and \( P(\ker f) \) is the projectivization of \( \ker f \). In this paper, for a symplectic vector space \( E \) of dimension \( n = 6 \), we prove that the surjectivity of the contraction map \( f : \Lambda^6 E \to \Lambda^4 E \) depends on the characteristic of the base field and we calculate the codimension of the linear section \( P(\ker f) \subseteq P(\Lambda^6 E) \) for any characteristic.

1. Introduction

Let \( E \) be a \( 2n \)-dimensional symplectic vector space over an arbitrary field \( F \) equipped with a non-degenerate symplectic form \( \langle , \rangle \). Consider the contraction map \( f : \Lambda^n E \to \Lambda^{n-2} E \) given by

\[
f(w_1 \wedge \cdots \wedge w_n) = \sum_{1 \leq s < t \leq n} \langle w_s, w_t \rangle w_1 \wedge \cdots \wedge \hat{w}_s \wedge \cdots \wedge \hat{w}_t \wedge \cdots \wedge w_n,
\]

where \( \hat{w} \) means that the corresponding term is omitted. Our main result shows that, in general, the map \( f \) is not surjective. Since, by [2] the Lagrangian-Grassmannian variety \( L(n, 2n) \) is cut out by the projectivization \( P(\ker f) \) of the kernel of \( f \), it follows that the codimension of \( L(n, 2n) \) in its Plücker embedding is not \( C_n^2 - C_n^2 \), where \( C_n^m \) denotes the binomial coefficient. Specifically, we prove that for \( n = 6 \), and a field of characteristic 3, the contraction map \( f : \Lambda^6 E \to \Lambda^4 E \) given by (1.1) is not surjective. To prove this, we use a combinatorial description in Lemma 1 of the set of indices that label the Plücker linear relations that is then used to describe the linear section \( P(\ker f) \) that cuts out the Lagrangian-Grassmannian \( L(6, 12) \) in the Grassmannian variety \( G(6, 12) \) in any characteristic. As a consequence we show that the codimension of \( L(6, 12) \) in its Plücker embedding depends on the characteristic of the base field.

The paper is organized as follows. In Section 2 we recall some results of the contraction map (1.1) and the Lagrangian–Grassmannian. In Section 3 we give an explicit example where the contraction map is not surjective and give all the details involved to obtain the linear section \( P(\ker f) \) that defines \( L(6, 12) \).

2. Preliminaries

Let \( E \) be a \( 2n \)-dimensional vector space over \( F \) equipped with a non-degenerate symplectic form \( \langle , \rangle \). Define the set \( I(\ell, m) = \{ \alpha = (\alpha_1, \ldots, \alpha_\ell) : 1 \leq \alpha_1 < \cdots < \alpha_\ell < m \} \).
\(\alpha \leq m\), such that \(\alpha_i \in \mathbb{N}\), and the support of \(\alpha = (\alpha_1, \ldots, \alpha_\ell) \in I(\ell, m)\) as the set \(\text{supp}(\alpha) := \{\alpha_1, \ldots, \alpha_\ell\}\). Thus, all indices \(\alpha\) are ordered sets of \(\ell\) different integers in the set \(\{1, 2, \ldots, m\}\). In what follows, all indices \(\alpha\) are sets with \(\ell\) different elements in the set \(\{1, 2, \ldots, m\}\), and up to permutation, we may (and do so) think of them in \(I(\ell, m)\).

Choose a basis \(\{e_1, \ldots, e_{2n}\}\) of the symplectic space \(E\) such that

\[
\langle e_i, e_j \rangle = \begin{cases} 1 & \text{if } j = 2n - i + 1, \\ 0 & \text{otherwise.} \end{cases}
\]

Then, for \(\alpha = (\alpha_1, \ldots, \alpha_n) \in I(n, 2n)\) write

\[
e_\alpha := e_{\alpha_1} \wedge \cdots \wedge e_{\alpha_n},
\]

\[
e_{\alpha, st} := e_{\alpha_1} \wedge \cdots \wedge e_{\alpha_{st}} \wedge \cdots \wedge e_{\alpha_n},
\]

\[
p_{i, \alpha, st}(2n-i+1) := p_{\alpha_1 \cdots \alpha_{st} \cdots \alpha_n}(2n-i+1),
\]

where \(\widehat{e}_\alpha\) and \(\widehat{\alpha}_k\) means that the corresponding term is omitted. Denote by \(\wedge^n E\) the \(n\)-th exterior power of \(E\), which is generated by \(\{e_\alpha : \alpha \in I(n, 2n)\}\). For \(w = \sum_{\alpha \in I(n, 2n)} p_\alpha e_\alpha \in \wedge^n E\), the coefficients \(p_\alpha\) are the Plücker coordinates of \(w\). In [2, Proposition 6] the kernel of the contraction map \(f\) is characterized as follows: For \(w = \sum_{\alpha \in I(n, 2n)} p_\alpha e_\alpha \in \wedge^n E\) written in Plücker coordinates, we have that

\[
w \in \ker f \iff \sum_{i=1}^n p_{i, \alpha, st}(2n-i+1) = 0, \text{ for all } \alpha_{st} \in I(n-2, 2n).
\]

In [2, Section 3] these linear forms were given the following description: For \(\alpha_{st} \in I(n-2, 2n)\) define the linear polynomials

\[
\Pi_{\alpha_{st}} := \sum_{i=1}^n c_{i, \alpha_{st}, 2n-i+1} X_{i, \alpha_{st}, 2n-i+1},
\]

with

\[
c_{i, \alpha_{st}, 2n-i+1} = \begin{cases} 1 & \text{if } |\text{supp}\{i, \alpha_{st}, 2n - i + 1\}| = n, \\ 0 & \text{otherwise,} \end{cases}
\]

hence \(\Pi_{\alpha_{st}}\) are polynomials in the ring \(\mathbb{F}[X_\alpha : \alpha \in I(n, 2n)]\). From the following formula in Plücker coordinates

\[
X_{1,2n} + X_{2,3(n-1)} + \cdots + X_{n,\overline{2n}+1} = 0,
\]

where the symbols \(\overline{\ldots}\) are to be replaced by elements \(\alpha_{st} \in I(n-2, 2n)\), we obtain homogeneous linear equations, that we call a Plücker linear relations in \(k\)-variables,

\[
\Pi_{\alpha_{st}} := X_{1, \alpha_{st}, 2n} + X_{2, \alpha_{st}, (2n-1)} + \cdots + X_{n, \alpha_{st}, (n+1)} = 0
\]

where the term \(X_{i, \alpha_{st}, (2n-i+1)}\) does not appear if \(|\text{supp}\{i, \alpha_{st}, (2n - i + 1)\}| < n\). When this happens we say \(\Pi_{\alpha_{st}}\) is a \(k\)-plane. For the system of homogeneous linear equations \(\Pi_{\alpha_{st}}\), \(\alpha_{st} \in I(n-2, 2n)\), we denote by \(B\) its associated matrix. Clearly the matrix \(B\) is of order \(C^{2n}_{n-2} \times C^n_n\). For example, if \(n = 6\) formula (2.2) becomes

\[
X_{1,10} + X_{2,11} + X_{3,12} + X_{4,13} + X_{5,14} + X_{6,15} = 0,
\]

where \(A = 10, B = 11, C = 12\).

Recall that a vector subspace \(W\) of \(E\) is isotropic iff for all \(x, y \in W\) we have that \(\langle x, y \rangle = 0\), and if \(W\) is isotropic its dimension is at most \(n\). The Lagrangian-Grassmannian
\(L(n, 2n)\) is the projective variety given by the isotropic vector subspaces \(W \subseteq E\) of maximal dimension \(n\):

\[
L(n, 2n) = \{ W \in G(n, 2n) : W \text{ is isotropic and } n\text{-dimensional} \},
\]

where \(G(n, 2n)\) denotes the Grassmannian variety of vector subspaces of dimension \(n\) of \(E\). The Plücker embedding is the regular map \(\rho : G(n, 2n) \rightarrow \mathbb{P}(\wedge^n E)\) given on each \(W \in G(n, 2n)\) by choosing a basis \(w_1, \ldots, w_n\) of \(W\) and then mapping the vector subspace \(W \in G(n, 2n)\) to the tensor \(w_1 \wedge \cdots \wedge w_n \in \wedge^n E\). Since choosing a different basis of \(W\) changes the tensor \(w_1 \wedge \cdots \wedge w_n\) by a nonzero scalar, this tensor is a well-defined element in the projective space \(\mathbb{P}(\wedge^n E) \simeq \mathbb{P}^{N-1}\), where \(N = C_n^{2n} = \dim \mathbb{F}(\wedge^n E)\). Under the Plücker embedding, the Lagrangian-Grassmannian is given by

\[
L(n, 2n) = \{ w_1 \wedge \cdots \wedge w_n \in G(n, 2n) : \langle w_i, w_j \rangle = 0 \text{ for all } 1 \leq i < j \leq n \}.
\]

Using the contraction map \(f : \wedge^n E \rightarrow \wedge^{n-2} E\) given by (1.1), if \(\mathbb{P}(\ker f)\) is the projectivization of \(\ker f\), in [2] it is proved that \(L(n, 2n) = G(n, 2n) \cap \mathbb{P}(\ker f)\). We call \(\mathbb{P}(\ker f)\) the linear section that defines \(L(n, 2n)\) in \(\mathbb{P}(\wedge^n E)\).

3. Non surjectivity of the contraction map in \(\text{char}(\mathbb{F}) = 3\)

The purpose of this section is twofold: First, to provide a description, completely explicit and self-contained of the linear space \(\mathbb{P}(\ker f)\) for \(n = 6\), for any field \(\mathbb{F}\), and then using this characterization we give an example of the non surjectivity of the contraction map for a field of characteristic 3.

Let \(P_1 = (1, C), P_2 = (2, B), P_3 = (3, A), P_4 = (4, 9), P_5 = (5, 8), P_6 = (6, 7)\), where \(A = 10, B = 11, C = 12\), as in Section 2, \(\Sigma_6 = \{ P_1, P_2, \ldots, P_6 \}\), and \(C_2(\Sigma_6)\) the set of all combinations of 6 objects taken 2 at a time. For \(1 \leq \alpha_1 < \alpha_2 \leq 12\) such that \(\alpha_1 + \alpha_2 \neq 13\), we define the following set

\[
\Sigma\{\alpha_1, \alpha_2\} = \{ (\alpha_1, \alpha_2, P_i) \in I(4, 12) : i + \alpha_j \neq 13, \alpha_j + 13 - i \neq 13, j = 1, 2 \}.
\]

Now, for \(1 \leq \alpha_1 < \alpha_2 < \alpha_3 < \alpha_4 \leq 12\) such that \(\alpha_i + \alpha_j \neq 13\), define

\[
\Sigma\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\} = \{ (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in I(4, 12) : \alpha_i + \alpha_j \neq 13 \text{ with } 1 \leq i, j \leq 4 \}.
\]

**Lemma 1.** With the notation above we have a partition of \(I(4, 12)\), given by

\[
C_2(\Sigma_6) \cup \bigcup_{1 \leq \alpha_1 < \alpha_2 \leq 12} \Sigma\{\alpha_1, \alpha_2\} \cup \bigcup_{1 \leq \alpha_1 < \alpha_2 < \alpha_3 < \alpha_4 \leq 12} \Sigma\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}.
\]

**Proof.** It is enough to show that every element in \(I(4, 12)\) is included in one and only one of the three different types of sets on the right hand side of the equality. Let \(\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in I(4, 12)\). For \(\alpha_1 + \alpha_2 \neq 13\), where \(i, j = 1, 2, 3, 4\), it follows that \(\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\} \in \Sigma\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}\). If for some \(\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)\) we have \(\alpha_1 + \alpha_2 \neq 13\), without loss of generality we may assume that \(\alpha_3 + \alpha_4 = 13\), and then \(\alpha \in \Sigma\{\alpha_1, \alpha_2\}\). Finally, if for some \(\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)\), \(\alpha_1 + \alpha_2 = \alpha_3 + \alpha_4 = 13\), then \(\alpha \in C_2(\Sigma_6)\). \(\square\)

**Remark 1.** For each \(1 \leq \alpha_1 < \alpha_2 \leq 12\) such that \(\alpha_1 + \alpha_2 \neq 13\), we have that \(\{|\Sigma\{\alpha_1, \alpha_2\}| = 4\}\) and \(|\Sigma\{\alpha_1, \alpha_2\}| = 1 \leq \alpha_1 < \alpha_2 \leq 12 | = 60\). Hence, in the second term of the displayed expression in Lemma 1 there are 240 indexes. Also, for each \(1 \leq \alpha_1 < \alpha_2 < \alpha_3 < \alpha_4 \leq 12\) such that \(\alpha_1 + \alpha_2 \neq 13\), we have that \(|\Sigma\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}| = 1\), \(|\Sigma\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}| = 1\), and \(|\Sigma\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}| = 1 \leq \alpha_1 < \alpha_2 < \alpha_3 < \alpha_4 \leq 12 | = 240\). Hence, in the third term of the displayed expression in Lemma 1 there are 240 indexes.
We translate now the combinatorial data of Lemma 1 in terms of the systems of linear equations associated to the contraction map \( f : \land^6 E \rightarrow \land^4 E \). For each \( \alpha_{rs} \in I(4, 12) \) consider the linear equation (2.2). Now, for the part \( C_2(\Sigma_0) \) in Lemma 1, writing

\[
\begin{align*}
C_2(\Sigma_0) &= \{(P_1, P_2), (P_1, P_3), (P_1, P_4), (P_1, P_5), (P_1, P_6), (P_2, P_3), (P_2, P_4), \\
& \quad (P_2, P_5), (P_2, P_6), (P_3, P_4), (P_3, P_5), (P_3, P_6), (P_4, P_5), (P_4, P_6), (P_5, P_6)\}. 
\end{align*}
\]

For the set \( C_2(\Sigma_0) \) ordered as above, filling the symbols \( \sqcup \) in (2.3) we obtain the system of linear equations

\[
\begin{align*}
\Pi_{(P_1, P_2)} &:= X_{P_1} P_1 P_2 + X_{P_2} P_1 P_2 + X_{P_3} P_1 P_2 + X_{P_4} P_1 P_2 = 0 \\
\Pi_{(P_1, P_3)} &:= X_{P_1} P_1 P_3 + X_{P_2} P_1 P_3 + X_{P_3} P_1 P_3 + X_{P_4} P_1 P_3 = 0 \\
\Pi_{(P_1, P_4)} &:= X_{P_1} P_1 P_4 + X_{P_2} P_1 P_4 + X_{P_3} P_1 P_4 + X_{P_4} P_1 P_4 = 0 \\
\Pi_{(P_1, P_5)} &:= X_{P_1} P_1 P_5 + X_{P_2} P_1 P_5 + X_{P_3} P_1 P_5 + X_{P_4} P_1 P_5 = 0 \\
\Pi_{(P_1, P_6)} &:= X_{P_1} P_1 P_6 + X_{P_2} P_1 P_6 + X_{P_3} P_1 P_6 + X_{P_4} P_1 P_6 = 0 \\
\Pi_{(P_2, P_3)} &:= X_{P_1} P_1 P_2 + X_{P_1} P_2 P_3 + X_{P_1} P_3 P_2 + X_{P_1} P_4 P_3 = 0. \\
\vdots \\
\Pi_{(P_5, P_6)} &:= X_{P_1} P_1 P_5 + X_{P_2} P_1 P_5 + X_{P_3} P_1 P_5 + X_{P_4} P_1 P_5 = 0. 
\end{align*}
\]

where we identify the variable \( X_{P_1} P_1 P_2 = X_{P_1} P_2 P_3 \) if \( \supp \{P_1 P_2 P_3\} = \supp \{P_1' P_2' P_3'\} \).

Similarly, for the second part in the partition of \( I(4, 12) \) in Lemma 1, for the sets \( \Sigma_{\{\alpha_1, \alpha_2\}} \), for each \( 1 \leq \alpha_1 < \alpha_2 \leq 12 \), consider the system of four homogeneous linear equations \( \Pi_{\alpha_{rs}} \) of (2.3), for each \( \alpha_{rs} \in \Sigma_{\{\alpha_1, \alpha_2\}} \), which have the form:

\[
\begin{align*}
X_1 + X_2 + X_3 &= 0 \\
X_1 + X_4 + X_5 &= 0 \\
X_2 + X_4 + X_6 &= 0 \\
X_3 + X_5 + X_6 &= 0.
\end{align*}
\]

and there are 60 such systems of linear equations (3.2). For example, for \( \Sigma_{\{1, 2\}} = \{12P_3, 12P_4, 12P_5, 12P_6\} \), setting \( A = 10, B = 11 \) and \( C = 12 \), as in (2.3), the system (3.2) is

\[
\begin{align*}
\Pi_{12P_3} &:= X_{12P_3} + X_{512P_3} + X_{612P_3} = 0 \\
\Pi_{12P_4} &:= X_{312P_4} + X_{512P_4} + X_{612P_4} = 0 \\
\Pi_{12P_5} &:= X_{312P_5} + X_{412P_5} + X_{612P_5} = 0 \\
\Pi_{12P_6} &:= X_{312P_6} + X_{412P_6} + X_{512P_6} = 0.
\end{align*}
\]

Finally, for the third part in the partition of \( I(4, 12) \) in Lemma 1, for each set \( \Sigma_{\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}} \), with \( 1 \leq \alpha_1 < \alpha_2 < \alpha_3 < \alpha_4 \leq 12 \), the matrix \( L_2 \) of the corresponding linear equation \( \Pi_{\alpha_1, \alpha_2, \alpha_3, \alpha_4} \) of (2.3) is a matrix (row vector) with the \( P_{i,\alpha_1, \alpha_2, \alpha_3, \alpha_4} \) and \( P_{j,\alpha_1, \alpha_2, \alpha_3, \alpha_4} \) components equal to one and all the other components equal to zero for \( 1 \leq i < j \leq 6 \). There are 240 such matrices \( L_2 \). The size of this vector is \( 1 \times C_6^{12} \). For example,

\[
\Pi_{1234} = X_{51234} + X_{61234} = 0
\]

with corresponding matrix \( L_2 := (0, \ldots, 0, 1, 0, \ldots, 0, 1, 0, \ldots, 0) \), where the first 1 is in the coordinate corresponding to 123458 and the second 1 is in the coordinate corresponding to the 123467.
The coefficient matrix associated to the system (3.1) is

\[
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

\[
\mathcal{L}_4 =
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

and the corresponding matrix associated to the system (3.2) is

\[
\mathcal{L}_3 = \begin{pmatrix}
1 & 1 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 0
\end{pmatrix}.
\]

For the matrix \(\mathcal{L}_4\), adding its 14 last rows to the first row, we obtain that \(\mathcal{L}_4\) is row-equivalent to the matrix \(\mathcal{L}_4 \cong \mathcal{L}_3\), where \(\mathcal{L}_4\) is the matrix obtained from \(\mathcal{L}_4\) deleting its first row, and \(\mathcal{L}_3\) is a row all whose entries are equal to 3. Thus, if \(\text{char}(\mathbb{F}) = 3\), the rank of \(\mathcal{L}_4\) is \(\leq 14\). Moreover, a direct computation shows that its rank is indeed 14 in characteristic 3. Similarly, for the matrix \(\mathcal{L}_3\), adding the last three rows to the first one we see that if \(\text{char}(\mathbb{F}) = 2\) then the rank of \(\mathcal{L}_3\) is \(\leq 3\), and again a computation shows that is exactly 3. Moreover, \(\text{rank}(\mathcal{L}_3) = 4\) if \(\text{char}(\mathbb{F}) \neq 2\), and \(\text{rank}(\mathcal{L}_4) = 15\) if \(\text{char}(\mathbb{F}) \neq 2, 3\).

**Proposition 2.** For any field \(\mathbb{F}\), the matrix \(B\), of size \(C_4^{12} \times C_6^{12}\), associated to the homogeneous system \(\Pi = \{\Pi_{\alpha rs} | \alpha_{rs} \in I(4,12)\}\) can be given by a block diagonal matrix as follows

\[
B = \mathcal{L}_4 \oplus \left( \bigoplus_{1 \leq \alpha_1 < \alpha_2 \leq 12} \mathcal{L}_3^{(\alpha_1, \alpha_2)} \right) \oplus \left( \bigoplus_{1 \leq \alpha_1 < \alpha_2 < \alpha_3 < \alpha_4 \leq 12} \mathcal{L}_2^{(\alpha_1, \alpha_2, \alpha_3, \alpha_4)} \right),
\]

\[
B = \begin{pmatrix}
\mathcal{L}_4 & 0 & \cdots & 0 \\
0 & \mathcal{L}_3 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & \mathcal{L}_2
\end{pmatrix},
\]

where there are 1 matrix \(\mathcal{L}_4\), 60 submatrices \(\mathcal{L}_3\), and 240 submatrices \(\mathcal{L}_2\).
Proof. It follows from the observation that \( I(4, 12) \) is a disjoint union of the sets described in Lemma 1 and the one-to-one relationship between those sets and their corresponding system of homogeneous linear equations. \( \square \)

For the contraction map \( f : \wedge^6 E \rightarrow \wedge^4 E \) given by (1.1), we obtain, from Proposition 2, the following consequences:

1. If \( \text{char}(F) = 3 \), then \( \text{rank}(B) = \text{rank}(L_4) + 60 \text{rank}(L_3) + 240 \text{rank}(L_2) = 494 \).
2. \( \dim_F(\ker f) = C_6^{12} - 494 = 430 \).
3. If \( \text{char}(F) = 3 \), then the map \( f \) is not surjective.

From Proposition 2 we calculate the codimension of the linear section \( \mathbb{P}(\ker f) \) in \( \mathbb{P}(\wedge^6 E) \) for any characteristic. That is

| char(F) | rank(B) | codimension of \( \mathbb{P}(\ker f) \) |
|---------|---------|-------------------------------------|
| 0       | 495     | 429                                 |
| 2       | 430     | 494                                 |
| 3       | 494     | 430                                 |
| \( p \geq 5 \) | 495     | 429                                 |

This computation shows that the codimension of \( \mathbb{P}(\ker f) \) in \( \mathbb{P}(\wedge^6 E) \) depends on the dimension \( 2n \) of the symplectic space \( E \) and the characteristic of the ground field \( F \). In a forthcoming paper the authors show that, in general, the contraction map \( f \) is surjective if and only if \( \text{char}(F) = 0 \) or \( \text{char}(F) \geq m \), for a certain integer \( m \).

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