THE DISTRIBUTION OF THE FREE PATH LENGTHS IN THE PERIODIC TWO-DIMENSIONAL LORENTZ GAS IN THE SMALL-SCATTERER LIMIT

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ABSTRACT. We study the free path length and the geometric free path length in the model of the periodic two-dimensional Lorentz gas (Sinai billiard). We give a complete and rigorous proof for the existence of their distributions in the small-scatterer limit and explicitly compute them. As a corollary one gets a complete proof for the existence of the constant term \( c = 2 - 3 \ln 2 + \frac{27}{2} \zeta(3) \) in the asymptotic formula \( h(T) = -2 \ln \varepsilon + c + o(1) \) of the KS entropy of the billiard map in this model, as conjectured by P. Dahlqvist.

1. Introduction and main results

A periodic two-dimensional Lorentz gas (Sinai billiard) is a billiard system on the two-dimensional torus with one or more circular regions (scatterers) removed. This model in classical mechanics was introduced by Lorentz [31] in 1905 to describe the dynamics of electrons in metals. The associated dynamical system is simple enough to allow a comprehensive study, yet complex enough to exhibit chaos. According to Gutzwiller [26], “The original billiard of Sinai was designed to imitate, in the most simple-minded manner, a gas of hard spherical balls which bounce around inside a finite enclosure. The formidable technical difficulties of this fundamental problem were boiled down to the shape of a square for the enclosure, and the collisions between the balls were reduced to a single point particle hitting a circular hard wall at the center of the enclosure.”

The model was intensively studied from the point of view of dynamical systems [10, 13, 14, 21, 22, 24, 34]. Our primary goal here is to estimate the free-path length (first return time) in this periodic two-dimensional model in the small-scatterer limit. We solve the following three open problems:

1. the existence and computation of the distribution of the free path length, previously considered in [9, 11, 16].
2. the existence and computation of the distribution of the geometric free path length, previously shown, but not fully proved, in [14].
3. the existence and computation of the second (constant) term in the asymptotic formula of the KS entropy \( h(T_\varepsilon) \) of the billiard map in this model, previously studied in [12, 13, 14, 21].

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For each \( \varepsilon \in (0, \frac{1}{2}) \) let
\[
Z_\varepsilon = \{ x \in \mathbb{R}^2 \mid \text{dist}(x, Z^2) \geq \varepsilon \},
\]
denote by \( \partial Z_\varepsilon \) the boundary \( Z^2 + \varepsilon \mathbb{T} \) of \( Z_\varepsilon \), and define the free path length (also called first exit time) as the Borel map given by
\[
\tau_\varepsilon(x, \omega) = \inf \{ \tau > 0 \mid x + \tau \omega \in \partial Z_\varepsilon \}, \quad x \in Z_\varepsilon, \quad \omega \in \mathbb{T}.
\]
If \( \tan \omega \) is irrational, then \( \tau_\varepsilon(x, \omega) < \infty \) for every \( x \in Z_\varepsilon \). We consider the probability space \((Y_e, \mu_e)\), with \( Y_e = Z_e/\mathbb{Z}^2 \subseteq [0,1)^2 \) and \( \mu_e \) the normalized Lebesgue measure on \( Y_e \). Let \( \epsilon_t = \epsilon(t, \infty) \) denote the characteristic function of \((t, \infty)\). For every \( t > 0 \) the probability that \( \tau_\varepsilon(x, \omega) > \frac{4}{2\varepsilon} \) is given by
\[
\mathbb{P}_\varepsilon(t) = \mu_e(\{(x, \omega) \in Y_e \times (0, 2\pi) \mid 2\varepsilon \tau_\varepsilon(x, \omega) > t\}) = \int_{Y_e \times \mathbb{T}} \epsilon_t(2\varepsilon \tau_e) \, d\mu_e.
\]
Lower and upper bounds for \( \mathbb{P}_\varepsilon \) of correct order of magnitude were established by Bourgain, Golse and Wennberg [9], using the rational channels introduced by Bleher [3]. More recently, Caglioti and Golse [11] have proved the existence of the Cesaro \( \limsup \) and \( \liminf \) means, proving for large \( t \) that
\[
\limsup_{\delta \to 0^+} \frac{1}{\ln \delta} \int_\delta^{1/4} \mathbb{P}_\varepsilon(t) \frac{dx}{\varepsilon} = \frac{2}{\pi^2 t} + O \left( \frac{1}{t^2} \right) = \liminf_{\delta \to 0^+} \frac{1}{\ln \delta} \int_\delta^{1/4} \mathbb{P}_\varepsilon(t) \frac{dx}{\varepsilon}.
\]
(1.1)

In Sections 2-7 below we prove the existence of the limit \( \mathbb{P}(t) \) of \( \mathbb{P}_\varepsilon(t) \) as \( \varepsilon \to 0^+ \) and explicitly compute it.

**Theorem 1.** For every \( t > 0 \) and \( \delta > 0 \)
\[
\mathbb{P}_\varepsilon(t) = \mathbb{P}(t) + O_\delta(\varepsilon^{1/8 - \delta}) \quad (\varepsilon \to 0^+),
\]
with
\[
\mathbb{P}(t) = \frac{6}{\pi^2} \left\{ \begin{array}{ll}
\frac{\pi^2}{6} \left( 1 - t + \frac{t^2}{2} \right) & \text{if } 0 < t \leq 1; \\
\int_0^{t-1} \psi(x, t) \, dx + \int_{t-1}^1 \phi(x, t) \, dx & \text{if } 1 < t \leq 2; \\
\int_0^1 \psi(x, t) \, dx & \text{if } t > 2,
\end{array} \right.
\]
\[
\psi(x, t) = \frac{(1-x)^2}{x} \left( 2 \ln \frac{t-x}{t-2x} - \frac{t-x}{t} \ln \frac{(t-x)^2}{(t-2x)^2} \right),
\]
\[
\phi(x, t) = \frac{1}{x} \ln \frac{1}{1-x} + \frac{(t-x)(x-t+1)}{x} + \frac{(1-x)^2}{x} \left( 2 \ln \frac{t-x}{1-x} - \frac{t-x}{1-x} \ln \frac{t-x}{t(1-x)} \right).
\]

After a direct computation the above formula for \( \mathbb{P}(t) \) yields
\[
\mathbb{P}(t) = \frac{24}{\pi^2} \sum_{n=1}^{\infty} \frac{2^n - 1}{n^2(n+1)^2(n+2)t^n}, \quad t \geq 2,
\]
and thus for large $t$ we find 
\[ P(t) = \frac{2}{\pi t^2} + O\left(\frac{1}{t^2}\right), \]
which agrees with \(1.1\).

The related “homogeneous” problem when the trajectory starts at the origin $O$ and the phase space is a subinterval of the velocity range $[0, 2\pi)$ was studied by Gologan and the authors. The limit distribution 
\[ H(t) = \lim_{\varepsilon \to 0^+} \frac{1}{2\pi} |\{ \omega \in [0, 2\pi) \colon \varepsilon \tau_\varepsilon(O, \omega) > t\}| = \lim_{\varepsilon \to 0^+} \int_T e_t(\varepsilon \tau_\varepsilon(O, \omega)) d\omega, \]
where $| \cdot |$ denotes the Lebesgue measure, was shown to exist and explicitly computed in \([6, 7]\). Unlike $P$, the function $H$ is compactly supported on the interval $[0, 1]$. Interestingly, in the particular situation where the scatterers are vertical segments, this case is related to some old problems in diophantine approximation investigated by Erdős, Szüsz and Turán \([17, 18]\), Friedman and Niven \([20]\), and by Kesten \([28]\).

The main tools used to prove Theorem 1 are a certain three-strip partition of $[0, 1)^2$ and the Weil-Salié estimate for Kloosterman sums \([19, 27, 35]\). The latter is used in infinitesimal form with respect to the parameter $\omega$ to count the number of solutions of equations of form $xy = 1 \pmod{q}$ in various regions in $\mathbb{R}^2$. This approach, somehow reminiscent of the circle method, produces good estimates, allowing us to keep under control the error terms. It was developed and used recently in many situations to study problems related to the spacing statistics of Farey fractions and lattice points in $\mathbb{R}^2$ \([1, 4, 5, 6, 7]\). A possible source for getting better estimates for the error terms might come from further cancellations in certain sums of Kloosterman sums, of the form \([15, 23, 29]\).

The three-strip partition of $\mathbb{T}^2$ is related to the continued fraction decomposition of the slope of the trajectory. Following work of Blank and Krikorian \([2]\) on the longest orbit of the billiard, Caglioti and Golse explicitly introduced this partition and used it in conjunction with ergodic properties of the Gauss map \([11]\) to prove \([1.1]\). We will use it in Section 3 in a suitable setting for our computations.

One can also consider the phase space $\Sigma^+_\varepsilon = \{(x, \omega) \in \partial Y_\varepsilon \times \mathbb{T} \colon \omega \cdot n_x > 0\}$ with $n_x$ the inward unit normal at $x \in \partial Y_\varepsilon$ and the probability measure $\nu_\varepsilon$ on $\Sigma^+_\varepsilon$ obtained by normalizing the Liouville measure $\omega \cdot n_x dx d\omega$ to mass one. Consider also the distribution 
\[ G_\varepsilon(t) = \nu_\varepsilon(\{(x, \omega) \in \Sigma^+_\varepsilon \colon 2\varepsilon \tau_\varepsilon(x, \omega) > t\}) = \int_{\Sigma^+_\varepsilon} e_t(2\varepsilon \tau_\varepsilon) d\nu_\varepsilon \]
of the geometric free path length $\tau_\varepsilon(x, \omega)$. The first moment (geometric mean free path length) of $\tau_\varepsilon$ with respect to $\nu_\varepsilon$ can be expressed as 
\[ \int_{\Sigma^+_\varepsilon} \tau_\varepsilon d\nu_\varepsilon = \frac{\pi |Y_\varepsilon|}{|\partial Y_\varepsilon|} = \frac{1}{2\varepsilon} \left(1 - \frac{\pi \varepsilon^2}{2\varepsilon}\right). \]
Equality \((1.2)\) is a consequence of a more general formula of Santaló \[33\] who extended earlier work of Pólya on the mean visible distance in a forest \[32\]. The formulation from \((1.2)\) appears in \[12, 13, 16\]. Knowledge of the mean free path does not give however any information on other moments or on the limiting distribution of the free path in the small-scatterer limit. Our number theoretical analysis leads to the following solution of this limiting distribution problem, proved in Sections 8-11 below.

**Theorem 2.** For every \(t > 0\) and \(\delta > 0\)

\[
G_\varepsilon(t) = G(t) + O_\delta(\varepsilon^{1/8-\delta}) \quad (\varepsilon \to 0^+),
\]

with

\[
G(t) = \frac{6}{\pi^2} \left\{ \begin{array}{ll}
\frac{\pi^2}{6} - t & \text{if } 0 < t \leq 1; \\
-2 + t + (t-1) \ln \frac{1}{t-1} & \text{if } 1 < t \leq 2; \\
\int_0^1 \bar{\psi}(x,t) \, dx + \int_{t-1}^1 \bar{\phi}(x,t) \, dx & \text{if } t > 2,
\end{array} \right.
\]

\[
\bar{\psi}(x,t) = \frac{(1-x)^2}{x^2} \ln \frac{(t-x)^2}{t(t-2x)}, \quad \bar{\phi}(x,t) = \frac{1}{x} \ln \frac{1}{t-x} + \frac{(1-x)^2}{x^2} \ln \frac{t-x}{t(1-x)}.
\]

**Figure 1.** The graphs of \(P(t)\), \(G(t)\), and respectively \(g(t)\)

We note the equalities

\[
G(t) = -P'(t), \quad t > 0,
\]

and

\[
g(t) := -G'(t) = P''(t)
\]

\[
= \frac{6}{\pi^2} \left\{ \frac{1}{t} + 2 \left(1 - \frac{1}{t}\right)^2 \ln \left(1 - \frac{1}{t}\right) - \frac{1}{2} \left(1 - \frac{2}{t}\right)^2 \ln \left|1 - \frac{2}{t}\right| \right\} \quad \text{if } 0 < t \leq 1; \quad (1.4)
\]

The latter also yields

\[
g(t) = \frac{24}{\pi^2 t^2} \sum_{n=1}^{\infty} \frac{2^n - 1}{n(n+1)(n+2)t^n}, \quad t \geq 2. \quad (1.5)
\]
Remarkably, formulas (1.4) and (1.5) were found by Dahlqvist [14]. That approach however does not provide a rigorous proof for the existence of the limit distribution, because it fails to control in a quantitative way the uniform distribution of his variable $\Delta$ (see the comments after formulas (75) and (86) in [14]).

In the final section we use some standard analysis arguments and properties of the dilogarithm and trilogarithm to estimate

$$C_\varepsilon := \ln \int_{\Sigma^+} \tau_\varepsilon \, d\nu - \int_{\Sigma^+} \ln \tau_\varepsilon \, d\nu.$$ 

It was conjectured by Friedman, Kubo and Oono [21] that $C_\varepsilon$ is convergent as $\varepsilon \to 0^+$. Its hypothetical limit $C$ was estimated to be $0.44 \pm 0.001$ in [21] and $\approx 0.43$ in [8]. This conjecture was known to imply [13, 21] the asymptotic formula

$$h(T_\varepsilon) = -2 \ln \varepsilon + 2 - C + o(1) \quad \text{as } \varepsilon \to 0^+$$

for the KS entropy of the associated billiard map. In [12] Chernov proved that $C_\varepsilon$ remains bounded when $\varepsilon \to 0^+$, without giving however any estimate for the bounds. The constant $C$ was identified by Dahlqvist [14, formula (73)] as being

$$3 \ln 2 - \frac{9\zeta(3)}{4\zeta(2)} = 0.43522513609...$$

The conjecture of Friedman, Kubo and Oono, in the more precise form provided by Dahlqvist, follows now from Theorem 2.

**Theorem 3.** In the small scatterer limit $\varepsilon \to 0^+$ the following holds:

(i) $C_\varepsilon = 3 \ln 2 - \frac{9\zeta(3)}{4\zeta(2)} + o(1)$.

(ii) $h(T_\varepsilon) = -2 \ln \varepsilon + 2 - 3 \ln 2 + \frac{9\zeta(3)}{4\zeta(2)} + o(1)$.

These methods work for any convex scatterer due to the good error control they give when integrating over the velocity in very short intervals. To keep the presentation of the paper neat we have chosen to only consider circular scatterers.

In dimension $\geq 3$ the problem of the existence of the limiting distribution of the free-path length in the small-scatterer limit remains open and is manifestly difficult. Partial results in this direction have appeared in [9, 24, 25].

2. Farey fractions and summation over primitive lattice points

In this section we collect some basic properties of Farey fractions and outline the summation method that will allow us to estimate the limit distribution of the free path length when the size of scatterers tends to zero.

For each positive integer $Q$, let $\mathcal{F}_Q$ denote the set of Farey fractions of order $Q$. These are the rational numbers $\gamma = \frac{a}{q}$ with coprime integers $a, q$ such that $1 \leq a \leq q \leq Q$. For each interval $I \subseteq [0, 1]$ the number of elements in the set

$$\mathcal{F}_Q(I) = I \cap \mathcal{F}_Q$$

can be expressed, using elementary arguments on Möbius and Euler-Maclaurin summation, as

$$\#\mathcal{F}_Q(I) = \frac{Q^2 |I|}{2\zeta(2)} + O(Q \ln Q).$$
If \( \gamma = \frac{a}{q} < \gamma' = \frac{a'}{q'} \) are two consecutive elements in \( F_Q \), then
\[
a'q - aq' = 1 \quad \text{and} \quad q + q' > Q.
\] (2.1)

This shows on the one hand that the denominators of consecutive Farey fractions of order \( Q \) are exactly the primitive integer points in the set
\[
Q_T = \{(Qx, Qy) : 0 < x, y \leq 1, x + y > 1\},
\]
and on the other hand that denominators uniquely determine consecutive Farey fractions. For instance, \( a \) is the unique integer in \([0, q]\) for which \((q - a)q' = 1 \pmod{q}\).

In many instances in this paper we will seek to estimate sums of type
\[
S_{f, \Omega, I}(Q) = \sum_{\gamma \in F_Q(T)} f(q, q', a),
\]
where \( I \subseteq [0, 1] \) is an interval, \( \Omega \subseteq T \) a region, and \( f \) a \( C^1 \) function. These kinds of sums can be roughly approximated by some integrals, with control on error terms given by the following two results which will be systematically used in this process. The first one is a standard fact and is a plain consequence of the Möbius summation (for a proof see [4, Lemma 2.3]).

**Lemma 1.** Let \( 0 < a < b \) and \( f \) be a \( C^1 \) function on \([a, b]\). Then
\[
\sum_{a < k \leq b} \frac{\varphi(k)}{k} f(k) = \frac{1}{\zeta(2)} \int_a^b f(x) \, dx + O \left( \ln b \left( \|f\|_\infty + \int_a^b |f'| \right) \right),
\]
where \( \varphi \) denotes Euler's totient function.

The second one is a consequence of Weil's type bounds for Kloosterman sums (cf. [7, Lemma 2.2]).

**Lemma 2.** Let \( q \geq 1 \) be an integer, \( I \) and \( J \) intervals with \(|I|, |J| < q\), \( f \) a \( C^1 \) function on \( I \times J \), and \( T \geq 1 \) an integer. Then for all \( \delta > 0 \)
\[
\sum_{a \in I, b \in J, \atop ab \equiv 1 (\text{mod } q)} f(a, b) = \frac{\varphi(q)}{q^2} \iint_{I \times J} f(x, y) \, dx \, dy + \mathcal{E},
\]
with
\[
\mathcal{E} = \mathcal{E}(q, T, f, |I|, |J|, \delta) \ll \delta T^2 q^{\frac{1}{2} + \delta} \|f\|_\infty + Tq^{\frac{3}{2} + \delta} \|Df\|_\infty + \frac{|I| \, |J| \, \|Df\|_\infty}{T},
\]
where we denote \( \| \cdot \|_\infty = \| \cdot \|_{\infty, I \times J} \) and \( Df = |\frac{\partial f}{\partial x}| + |\frac{\partial f}{\partial y}| \).

When \( \Omega = \{(x, y) : \alpha < x \leq \beta, \xi(x) \leq y \leq \eta(x)\} \) is a subset of \( T = \{(x, y) : 0 < x, y \leq 1, x + y > 1\} \), the above mentioned properties of Farey fractions lead to
\[
S_{f, \Omega, I}(Q) = \sum_{\alpha Q < q \leq \beta Q} \sum_{\alpha \in \mathcal{Q}_J(q/Q) \atop \alpha q \equiv 1 (\text{mod } q)} f(q, q', a) = \sum_{\alpha Q < q \leq \beta Q} \sum_{\alpha \in \mathcal{Q}_J(q/Q) \atop \alpha q \equiv 1 (\text{mod } q)} f(q, q', q - a),
\]
where we denote
\[ J_{\Omega}(x) = \{ y; (x, y) \in \Omega \} = [\xi(x), \eta(x)] \subseteq (1 - x, 1), \quad x \in (\alpha, \beta). \]

The inner sum above is mastered by Lemma 2, being approximated by
\[ \int_{QJ_{\Omega}(q/Q) \times q(1-I)} f(q, q', a) \, dq' \, da = \int_{QJ_{\Omega}(q/Q) \times qI} f(q, q', a) \, dq' \, da. \]

Thus
\[ S_{f,\Omega,I}(Q) = \sum_{\alpha Q < q \leq \beta Q} \frac{\varphi(q)}{q} V(q) + \text{error term}, \]
where we take
\[ V(q) = \frac{1}{q} \int_{QJ_{\Omega}(q/Q) \times qI} f(q, q', a) \, dq' \, da = Q \int_{J_{\Omega}(q/Q) \times I} f(q, Qy, q\gamma) \, dy \, d\gamma. \]

When the error term is small enough, this sum is mastered by Lemma 3 giving
\[ S_{f,\Omega,I}(Q) = \frac{1}{\zeta(2)} \int_{\alpha Q}^{\beta Q} V(q) \, dq + \text{error} \]
\[ = \frac{Q}{\zeta(2)} \int_{\alpha Q}^{\beta Q} dq \int_{J_{\Omega}(q/Q) \times I} dy \, d\gamma \, f(q, Qy, q\gamma) + \text{error} \]
\[ = \frac{Q^2}{\zeta(2)} \int_{\alpha}^{\beta} dx \int_{J_{\Omega}(x) \times I} dy \, d\gamma \, f(Qx, Qy, Qx\gamma) + \text{error} \]
\[ = \frac{1}{\zeta(2)} \iint_{Q\Omega \times I} f(v, w, v\gamma) \, dv \, dw \, d\gamma + \text{error}. \]

3. A partition of the unit square

In this section we give an account on the three-strip partition mentioned in the introduction. This approach, slightly different from that in [11], is suitable for computations involving Farey fraction partitions of the unit interval.

In the first part of this section we shall consider a fixed (small) \( \varepsilon > 0 \) and let \( Q = \lfloor \frac{1}{\varepsilon} \rfloor \) be the integer part of \( \frac{1}{\varepsilon} \). For each \( \gamma = \frac{a}{q} \in F_Q \), consider the points \( N_{\gamma}(q, a + \varepsilon) \) and \( S_{\gamma}(q, a - \varepsilon) \). Consider also the points \( N_0(0, \varepsilon) \) and \( S_0(0, -\varepsilon) \), and denote by \( \mathcal{S}_{\gamma} \) the strip determined by the lines \( N_0N_{\gamma} \) and \( S_0S_{\gamma} \).

A segment does not interfere with an open strip when their intersection is empty. Throughout this section \( \gamma = \frac{a}{q} < \gamma' = \frac{a'}{q'} \) will be two consecutive fractions in \( F_Q \), so that (2.1) is fulfilled. In particular this gives
\[ 2\varepsilon \max\{q, q'\} \leq 2\varepsilon Q \leq 1 \quad \text{and} \quad 2\varepsilon(q + q') > 2\varepsilon(Q + 1) > 1. \quad (3.1) \]

The slope of a segment \( AB \) is denoted by \( t_{AB} \). Set \( t_P = t_{OP} \).

**Lemma 3.** The segment \( N_{\gamma}S_{\gamma} \) does not interfere with the strip \( \mathcal{S}_{\gamma'} \), and the segment \( N_{\gamma'}S_{\gamma'} \) does not interfere with the strip \( \mathcal{S}_{\gamma} \).
Figure 2. The strips $S_\gamma$ and $S_{\gamma'}$

Proof. First, we show that $S_{\gamma'}$ lies above the line $N_0N_\gamma$ of equation $y - \varepsilon - \frac{aq}{q} = 0$, which amounts to $a' - 2\varepsilon - \frac{aq'}{q} \geq 0$. The latter is equivalent to $1 - 2\varepsilon q \geq 0$, which is true by (3.1).

Furthermore, $N_\gamma$ lies below the line $S_0S_{\gamma'}$ of equation $y + \varepsilon - \frac{a'q}{q} = 0$, as a result of $a + 2\varepsilon < \frac{a'q'}{q}$ being equivalent to $2\varepsilon q' < 1$. □

For each $k \in \mathbb{N}_0 = \{0, 1, 2, \ldots \}$ set
\[
q_k = q' + kq, \quad a_k = a' + ka, \quad q'_k = q + kq', \quad a'_k = a + ka',
\]
\[
\gamma_k = \frac{a_k}{q_k}, \quad t_k = \frac{a_k - 2\varepsilon}{q_k}, \quad u_k = \frac{a'_k + 2\varepsilon}{q'_k},
\]
\[
\alpha_k = \arctan t_k, \quad \beta_k = \arctan u_k.
\]

The following three relations hold for every $k \in \mathbb{N} = \{1, 2, \ldots \}$
\[
a_{k-1}q_k - a_k q_{k-1} = 1 = a'_k q'_k - a'_{k-1} q'_k; \quad (3.2)
\]
\[
a_{k-1} q - a q_{k-1} = 1 = a' q'_k - a'_{k-1} q'_k; \quad (3.3)
\]
\[
\min_{k \geq 1} \{2\varepsilon q_k, 2\varepsilon q'_k \} \geq 2\varepsilon (q' + q) > 1. \quad (3.4)
\]

As a result of (2.1) and (3.1)–(3.4), it is seen that
\[
\gamma' = \gamma_0 = \frac{a'}{q'} > \gamma_1 > \gamma_2 > \ldots > \gamma_k \xrightarrow{k \to \infty} \frac{a}{q} = \gamma,
\]
and that
\[
\gamma = \frac{a}{q} \xrightarrow{\infty} t_k \leq t_{k-1} \leq \ldots \leq t_1 \leq t_0 = \frac{a' - 2\varepsilon}{q'} < \frac{a + 2\varepsilon}{q} = u_0 \leq u_1 \leq \ldots \leq u_{k-1} \leq u_k \xrightarrow{k \to \infty} \frac{a'}{q'} = \gamma'.
\]

So putting
\[
I_{\gamma,0} = (t_0, u_0), \quad I_{\gamma,k} = (t_k, t_{k-1}), \quad I_{\gamma,-k} = (u_{k-1}, u_k), \quad k \in \mathbb{N},
\]
we end up with a partition $(I_{\gamma,k})_{k \in \mathbb{Z}}$ of the interval $(\gamma, \gamma')$.

Next we consider the points $N_{q_k}(a_k + \varepsilon)$ and $S_{q_k}(a_k - \varepsilon)$, proving...
Lemma 4. The following inequalities hold for every $k \geq 1$:

(i) $t_{N_0 N_k} > t_{N_0 N_k} > t_{N_0 S_k} \geq t_{N_0 S_k} \geq t_{N_0 N_k}$.

(ii) $t_{S_k S_k} \geq t_{S_k N_k} > t_{S_k S_k} \geq t_{S_k S_k}$.

Proof. The inequalities in (i) are equivalent to

$$\frac{a_{k-1}}{q_{k-1}} > \frac{a_k}{q_k} > \frac{a_{k-1} - 2\varepsilon}{q_{k-1}} \geq \frac{a_k - 2\varepsilon}{q_k} \geq \frac{a}{q},$$

which follow from (3.2), (3.3), (3.1), and from (3.4).

The inequalities in (ii) are equivalent to

$$\frac{a'}{q'} \geq \frac{a + 2\varepsilon}{q} > \frac{a_1}{q_1} = \frac{a'}{q'} + \frac{a - a}{q_k},$$

which follow from (2.1), (3.2), (3.4), and $a_1 q_k - a q_1 = k - 1$. □

Consider the half-infinite strip

$$\mathcal{S} = \mathcal{S}_\omega = \{(x, y + x \tan \omega) : x > 0, -\varepsilon \leq y \leq \varepsilon\}$$

of direction $\omega$, top line passing through $N_0$, and bottom line passing through $S_0$. Assume that $\gamma < \tan \omega < \gamma'$. For each $y_0 \in [-\varepsilon, \varepsilon]$ we wish to find the first vertical segment of form $\{m\} \times [n - \varepsilon, n + \varepsilon]$, $m, n \in \mathbb{N}$, that intersects the line of slope $\tan \omega$ passing through $(0, y_0)$. In other words, we wish to calculate

$$q(\omega, y_0) = \inf \{n \in \mathbb{N} : \|y_0 + n \tan \omega\| \leq \varepsilon\},$$

where we denote $\|x\| = \text{dist}(x, \mathbb{Z})$, $x \in \mathbb{R}$. We shall assume that $\tan \omega$ is irrational and split the discussion according to the three cases where the slope of $\omega$ belongs to one of the intervals $\left(\frac{a}{q}, \frac{a' - 2\varepsilon}{q'}\right], \left(\frac{a' - 2\varepsilon}{q'}, \frac{a + 2\varepsilon}{q}\right]$ or $\left(\frac{a + 2\varepsilon}{q}, \frac{a'}{q'}\right]$.

Proposition 1. Let $\gamma = \frac{a}{q} < \gamma' = \frac{a'}{q'}$ be consecutive fractions in $\mathcal{F}_Q$. Suppose $\tan \omega \in \left(\frac{a}{q}, \frac{a' - 2\varepsilon}{q'}\right]$ is irrational and $\tan \omega \in \mathcal{I}_Q(k) = (t_k, t_{k-1})$ for some $k \in \mathbb{N}$. Set

$$w_{B_k} = w_{B_k}(\omega) = q_k \tan \omega - a_k + 2\varepsilon,$$

$$w_{C_k} = w_{C_k}(\omega) = -q_{k-1} \tan \omega + a_{k-1} - 2\varepsilon,$$

$$w_{A_k} = w_{A_k}(\omega) = -q \tan \omega + a + 2\varepsilon,$$

$$I_{A_k} := [\varepsilon, -\varepsilon + w_{A_k}],$$

$$I_{B_k} := [\varepsilon + w_{A_k} + w_{C_k}, \varepsilon] = [\varepsilon - w_{B_k}, \varepsilon],$$

$$I_{C_k} := [-\varepsilon + w_{A_k} + \varepsilon + w_{A_k} + w_{C_k}],$$

and

$$L(\omega, y_0) = \begin{cases} L_{A_k}(\omega) := q & \text{if } y_0 \in I_{A_k}; \\ L_{B_k}(\omega) := q_{k+1} & \text{if } y_0 \in I_{B_k}; \\ L_{C_k}(\omega) := q_k & \text{if } y_0 \in I_{C_k}. \end{cases}$$

Then for any $* \in \{A_k, B_k, C_k\}$ we have $0 \leq w_* \leq 2\varepsilon$ and

$$q(\omega, y_0) = L(\omega, y_0) = L_*(\omega), \quad y_0 \in I_*.$$
Moreover, if $S_*$ denotes the parallelogram of height $\{0\} \times I_*$, angle $\omega$ between its side and the horizontal direction, and side length $\frac{L_*(\omega)}{\cos \omega}$, then

$\text{area}(S_{A+}) + \text{area}(S_{B_k}) + \text{area}(S_{C_k}) = w_{A+} L_{A+} + w_{B_k} L_{B_k} + w_{C_k} L_{C_k} = 1$.

Moreover, $\{S_{A+}, S_{B_k}, S_{C_k}\} \mod \mathbb{Z}^2$ provides a partition of the unit square $[0,1)^2$ (we allow the boundaries of these three sets to intersect).

**Figure 3.** The case $\tan \omega \in I_{\gamma,k}$, $k \in \mathbb{N}$

**Proof.** Taking stock on Lemma 4, we notice that the line of slope $\tan \omega$ through $S_0$ intersects the vertical line $N_\gamma S_\gamma$ at a point between $N_\gamma$ and $S_\gamma$ (see Figure 3). Also, because $t_{N_0 S_{\gamma-k}} = t_k < \tan \omega \leq t_{k-1} = t_{N_0 S_{\gamma-k-1}} < t_{N_0 N_{\gamma-k}}$, the line of slope $\tan \omega$ through $N_0$ (respectively through $N_\gamma$) intersects the line $N_{\gamma-k} S_{\gamma-k}$ (respectively $N_{\gamma-k+1} S_{\gamma-k+1}$) between $N_{\gamma-k}$ and $S_{\gamma-k}$ (respectively between $N_{\gamma-k+1}$ and $S_{\gamma-k+1}$). The segment $N_{\gamma-k} S_{\gamma-k-1}$ is placed above these two parallel lines because $\tan \omega \leq t_{k-1} = t_{N_0 S_{\gamma-k-1}}$.

Next, we find that the intersections with the vertical axis of the lines $y - a - \varepsilon = (x - q) \tan \omega$ and $y - a + \varepsilon = (x - q) \tan \omega$ which have slope $\tan \omega$ and pass through $N_\gamma$ and respectively $S_{\gamma-k}$, are $(0, \varepsilon + a - q \tan \omega)$ and respectively $(0, -\varepsilon + a - q_k \tan \omega)$, whence the required values of $w_{A+}, w_{B_k}$ and $w_{C_k}$ follow. Notice that

\[
2\varepsilon > w_{A+} = 2\varepsilon + a - q \tan \omega \geq 2\varepsilon + a - q \frac{a' - 2\varepsilon}{q'} = \frac{2\varepsilon (q + q') - 1}{q'} > 0,
\]

\[
2\varepsilon > \frac{1 - 2q}{q_k - 1} k_{k-1} - 2\varepsilon a_k + 2\varepsilon \geq w_{B_k} = q_k \tan \omega - a_k + 2\varepsilon > 0,
\]

\[
2\varepsilon > \frac{1 - 2q}{q_k} = q_k a_{k-1} - 2\varepsilon a_k - 2\varepsilon a_{k-1} = a_{k-1} - 2\varepsilon - q_{k-1} \tan \omega = w_{C_k} \geq 0.
\]

Besides one clearly has

\[w_{A+} + w_{B_k} + w_{C_k} = 2\varepsilon,\]

and it is easy to check by a direct calculation that

\[
\sum_{* \in \{A_+, B_k, C_k\}} \text{area}(S_*) = \sum_{* \in \{A_+, B_k, C_k\}} w_\star L_\star = 1.
\]
It remains to check that the interiors of the subsets $S_*$ mod $\mathbb{Z}^2 \subseteq [0,1)^2$, $* \in \{A_*, B_k, C_k\}$, are disjoint. If not, there exist two points $P, P'$ inside $\cup_* S_*$ such that $P - P' \in \mathbb{Z}^2$. The latter is preserved by translating the segment $PP'$ to a parallel segment. Owing to the shape of $\cup_* S_*$ we may thus assume that, say, $P$ lies on the $y$-axis; hence $P = P'(0,y_0)$ and $P' = P'(n, m + y_0)$ for some $y_0 \in [-\varepsilon, \varepsilon]$, $m, n \in \mathbb{N}^*$. The line of slope $\tan \omega$ which passes through $P'$ intersects the $y$-axis at $(0, m + y_0 - n \tan \omega)$. Hence $-\varepsilon \leq m + y_0 - n \tan \omega \leq \varepsilon$, which shows that $||y_0 - n \tan \omega|| = |-y_0 + n \tan \omega| \leq \varepsilon$. By the first part of the proposition this gives $n \geq L(\omega, -y_0)$, thus $P'$ must belong to the boundary, which is a contradiction. \qed

**Proposition 2.** Let $\gamma = \frac{a}{q} < \gamma' = \frac{a'}{q}$ be consecutive fractions in $\mathcal{F}_Q$. Suppose $\tan \omega \in (\frac{a'-2\varepsilon}{q'}, \frac{a'}{q'})$ is irrational.

(i) If $\tan \omega \in I_{\gamma, 0} = (t_0, u_0]$, then the analog of Proposition 1 holds true, with\(^1\)

\[
\begin{align*}
w_{B_0} &= w_{B_0}(\omega) = q' \tan \omega - a' + 2\varepsilon \in (0, 2\varepsilon), \\
w_{C_0} &= w_{C_0}(\omega) = -(q' - q) \tan \omega + a' - a - 2\varepsilon \in [0, 2\varepsilon), \\
w_{A_0} &= w_{A_0}(\omega) = w_{A_0}(\omega) = q \tan \omega + a + 2\varepsilon \in [0, 2\varepsilon), \\
I_{A_0} &= \{-\varepsilon, \varepsilon + w_{A_0}\}, \\
I_{B_0} &= \{-\varepsilon + w_{A_0}, -\varepsilon + w_{A_0} + w_{C_0}\}, \\
I_{C_0} &= \{-\varepsilon + w_{A_0}, -\varepsilon + w_{A_0} + w_{C_0}\}, \\
L(\omega, y_0) &= \begin{cases} 
L_{A_0}(\omega) := q & \text{if } y_0 \in I_{A_0}; \\
L_{C_0}(\omega) := q' + q & \text{if } y_0 \in I_{C_0}; \\
L_{B_0}(\omega) := q' & \text{if } y_0 \in I_{B_0}.
\end{cases}
\end{align*}
\]

(ii) If $k \in \mathbb{N}$ and $\tan \omega \in I_{\gamma, -k} = (u_{k-1}, u_k]$, then the analog of Proposition 1 holds true, with

\[
\begin{align*}
w_{B_-} &= w_{B_-}(\omega) = w_{B_0}(\omega) = q' \tan \omega - a' + 2\varepsilon \in (0, 2\varepsilon), \\
w_{C_{-k}} &= w_{C_{-k}}(\omega) = q'_{k-1} \tan \omega - a'_{k-1} - 2\varepsilon \in (0, 2\varepsilon), \\
w_{A_{-k}} &= w_{A_{-k}}(\omega) = -(q'_k - q') \tan \omega + a'_k + 2\varepsilon \in [0, 2\varepsilon), \\
I_{A_{-k}} &= \{-\varepsilon, \varepsilon + w_{A_{-k}}\}, \\
I_{C_{-k}} &= \{-\varepsilon + w_{A_{-k}}, -\varepsilon + w_{A_{-k}} + w_{C_{-k}}\}, \\
I_{B_{-k}} &= \{-\varepsilon + w_{A_{-k}} + w_{C_{-k}}, \varepsilon\} = (\varepsilon - w_{B_{-k}}, \varepsilon], \\
L(\omega, y_0) &= \begin{cases} 
L_{A_{-k}}(\omega) := q'_k & \text{if } y_0 \in I_{A_{-k}}; \\
L_{C_{-k}}(\omega) := q'_{k+1} & \text{if } y_0 \in I_{C_{-k}}; \\
L_{B_{-k}}(\omega) := q' & \text{if } y_0 \in I_{B_{-k}}.
\end{cases}
\end{align*}
\]

**Proof.** (i) follows as in the proof of Proposition 1 using 
\[
\varepsilon > a' - \varepsilon - q' \tan \omega \geq a + \varepsilon - q \tan \omega \geq -\varepsilon, \quad \tan \omega \in I_{\gamma, 0}.
\]

---

\(^1\)Note that in both cases $q < q'$ or $q'< q$ we get $0 \leq w_{C_0} < 2\varepsilon$. 

---
(ii) follows as in the proof of Proposition 1 using
\[ \varepsilon > a' - \varepsilon - q' \tan \omega > a'_k + \varepsilon - q'_k \tan \omega \geq -\varepsilon, \quad \tan \omega \in I_{\gamma,-k}. \]
\[ \square \]

We now start investigating the case where the scatterers are vertical slits. Propositions 1 and 2 will only be applied for \( \varepsilon = \frac{1}{2Q} \), corresponding to the case of vertical slits of height \( \frac{1}{Q} \). The Lebesgue measure of a Borel set \( A \) in \( \mathbb{R}^d \), \( d = 1, 2, 3 \), will be denoted by \( |A| \).

Throughout the paper \( \bar{\tau}_3(x, \omega) \) will denote the free path length in the periodic two-dimensional Lorentz gas with vertical slits of height \( 2\delta \) as scatterers centered at all integer lattice points. Given \( \lambda > 0 \), \( I = [\tan \omega_0, \tan \omega_1] \subseteq [0, 1] \) with \( 0 \leq \omega_0 \leq \omega_1 \leq \frac{\pi}{4} \), and \( Q \geq 1 \) integer, we denote
\[ \bar{P}_{I,Q}(\lambda) = |\{(x, \omega) : x \in [0, 1]^2, \ \omega_0 \leq \omega \leq \omega_1, \ \bar{\tau}_{I/(2Q)}(x, \omega) > \lambda\}|. \]

Although the cases \( 0 < t < 1, 1 < t < 2, t > 2 \), will be considered separately, applying Propositions 1 and 2 to \( 2\varepsilon = \frac{1}{Q} \), can write for all \( t, \varepsilon > 0 \)
\[ \bar{P}_{I,Q} \left( \frac{t}{2\varepsilon^*} \right) = \sum_{\gamma \in P_Q(I)} \sum_{k=1}^{\infty} \int_{\alpha_k}^{\alpha_{k-1}} w_{C_\gamma}(\omega) \max \left\{ q_{k+1} - \frac{t \cos \omega}{2\varepsilon^*}, 0 \right\} d\omega 
+ \text{eight other similar terms where} \ \frac{t \cos \omega}{2\varepsilon^*} \ \text{appears.} \] (3.5)

**Lemma 5.** For any interval \( I = [\tan \omega_0, \tan \omega_1] \subseteq [0, 1] \) such that \( |I| \asymp \varepsilon^c \) with fixed \( 0 < c < 1 \) and small \( \varepsilon > 0 \), and any (large) integer \( Q = \frac{\cos \omega_1}{2\varepsilon^*} + O(\varepsilon^{-1}) \), the
estimate
\[ P_{1,Q}(t) = P_{1,Q}(t) + O(\varepsilon^2), \] (3.6)
holds uniformly in \( t \) on compact subsets of \((0, \infty)\). Here \( P_{1,Q}(t) \) is obtained by substituting \( tQ \) in place of \( \frac{\cos \omega}{2\varepsilon} \) in (3.3), that is
\[
P_{1,Q}(t) := \sum_{\gamma \in \mathcal{F}_Q(t)} \sum_{k=1}^{\infty} \int_{\alpha_k}^{\alpha_{k-1}} \left( w_{C_k}^\alpha(\omega) \max\{q_{k+1} - tQ, 0\} \right.
+ w_{B_k}(\omega) \max\{q_k - tQ, 0\} + w_{A_+}(\omega) \max\{q - tQ, 0\} \bigg) \, d\omega
+ \sum_{\gamma \in \mathcal{F}_Q(t)} \int_{\alpha_0}^{\alpha_0} \left( w_{C_0}^\alpha(\omega) \max\{q + q' - tQ, 0\} \right.
+ w_{B_0}(\omega) \max\{q' - tQ, 0\} + w_{A_0}(\omega) \max\{q - tQ, 0\} \bigg) \, d\omega
+ \sum_{\gamma \in \mathcal{F}_Q(t)} \sum_{k=1}^{\infty} \int_{\beta_{k-1}}^{\beta_k} \left( w_{C_{-k}}^\alpha(\omega) \max\{q_{k+1} - tQ, 0\} \right.
+ w_{A_{-k}}(\omega) \max\{q_k - tQ, 0\} + w_{B_{-k}}(\omega) \max\{q' - tQ, 0\} \bigg) \, d\omega,
\]
with
\[
w_{A_+}(\omega) = w_{A_0}(\omega) = Q^{-1} + a - q \tan \omega,
\]
\[
w_{B_-}(\omega) = w_{B_0}(\omega) = q' \tan \omega - a' + Q^{-1},
\]
\[
w_{C_0}(\omega) = Q^{-1} - w_{A_0}(\omega) - w_{B_0}(\omega),
\]
\[
w_{B_k}(\omega) = q_k \tan \omega - a_k + Q^{-1}, \quad w_{C_k}(\omega) = a_{k-1} - Q^{-1} - q_{k-1} \tan \omega,
\]
\[
w_{C_{-k}}(\omega) = q_{k-1} \tan \omega - a'_{k-1} - Q^{-1}, \quad w_{A_{-k}}(\omega) = Q^{-1} + a'_k - q'_k \tan \omega,
\]
\[
\alpha_k = \arctan \frac{a_k - 1/Q}{q_k}, \quad \beta_k = \arctan \frac{a'_k + 1/Q}{q'_k}, \quad k \in \mathbb{N}.
\]

Proof. Using the inequality \( \max\{w_A, w_B, w_C\} \leq Q^{-1} \ll \varepsilon \), which is a consequence of \( w_{A_+} + w_{B_k} + w_{C_k} = Q^{-1} \) and of the similar relations for \( k = 0 \) and \( k \leq -1 \), the estimate (see also (7.2))
\[
\sup_{\omega \in I} \left| Q - \frac{\cos \omega}{2\varepsilon} \right| \ll \varepsilon^{-1} + \frac{\left| \cos \omega_0 - \cos \omega \right|}{2\varepsilon} \leq \varepsilon^{-1} + \frac{\left| \tan \omega_1 - \tan \omega_0 \right|}{2\varepsilon} \ll \varepsilon^{-1},
\]
and the inequalities
\[
\left| \max(x, 0) - \max(y, 0) \right| \leq |x - y| \quad (3.9)
\]
and
\[
\sum_{\gamma \in \mathcal{F}_Q(t)} \frac{1}{qq'} \ll |I| + \frac{1}{Q} \ll \varepsilon,
\]
it follows that we can replace \( \frac{t \cos \omega}{2c} \) by \( tQ \) in (3.5) at a cost which is

\[
\ll \sum_{\gamma \in \mathcal{F}_Q(I)} \frac{1}{Q} \varepsilon^{-1} \frac{1}{qq'} \ll \varepsilon^c \sum_{\gamma \in \mathcal{F}_Q(I)} \frac{1}{qq'} \ll \varepsilon^{2c}.
\]

Equality (3.7) will be at the center of most of the forthcoming computations because it shows how the estimation of distribution of the free path length reduces to estimates on sums involving Farey fractions.

There is an alternative approach to estimating \( \tilde{\mathbb{P}}_{I,Q} \), by using a monotonicity argument instead of the continuity argument which based on (3.9). Such an argument will be used in the proof of Theorem 2.

In the remainder of the paper given \( I = [\tan \omega_0, \tan \omega_1] \subseteq [0,1] \) we denote

\[
c_I = \int_I \frac{du}{1 + u^2} = \omega_1 - \omega_0.
\]

(3.10)

4. The case \( 0 < t \leq 1 \)

The aim of this section is to prove the following result

Proposition 3. Suppose \( I \) is a subinterval of \([0,1]\) of size \( |I| \approx Q^{-c} \) for some \( 0 < c < 1 \). Then for every \( c_1 > 0 \) with \( c + c_1 < 1 \) and \( \delta > 0 \)

\[
P_{I,Q}(t) = \left( 1 - t + \frac{t^2}{2(2)} \right) c_I + O_\delta(E_{c,c_1,\delta}(Q)) \quad (Q \to \infty),
\]

with

\[
E_{c,c_1,\delta}(Q) = Q^{\max\{2c_1-1/2+\delta,-c-c_1\}}.
\]

The estimate is uniform in \( t \in (0,1] \).

Before starting to estimate \( P_{I,Q} \), the following remark is in order.

Remark 1. If \( I \subseteq [0,1] \) is an interval with \( |I| \geq \frac{1}{Q} \), then as a consequence of \( \gamma' - \gamma = \frac{1}{qq'} \leq \frac{1}{Q} \leq |I| \) we have

\[
\sum_{\gamma \in \mathcal{F}_Q} f(\gamma) = \sum_{\gamma' \in \mathcal{F}_Q} f(\gamma') + O(Q^{-1}||f||_\infty).
\]

As a result, replacing the condition \( \gamma \in I \) by \( \gamma' \in I \) only produces an error of order \( Q^{-1} \), which has no impact in any of the forthcoming estimates. Thus in Propositions 3 4 5 6 7 8 the assumption \( |I| \approx Q^{-c} \) can be replaced by the weaker assumption \( |I| \ll Q^{-c} \) and \( |I| \geq Q^{-1} \).
Then we notice that since \( \min \{ q_k, q'_k \} \geq q + q' > tQ \) for all \( k \geq 1 \), we can write, according to (3.7) and (3.8),

\[
P_{I,Q}(t) = \sum_{\gamma \in \mathcal{F}_Q(t)} \sum_{k=1}^{\infty} \int_{\alpha_k}^{\alpha_k + 1} S_{Q,\gamma,k}(\omega) \, d\omega + \sum_{\gamma \in \mathcal{F}_Q(t)} \int_{\alpha_0}^{\beta_0} S_{Q,\gamma,k}^{(0)}(\omega) \, d\omega
\]

\[
+ \sum_{\gamma \in \mathcal{F}_Q(t)} \sum_{k=1}^{\infty} \int_{\beta_k}^{\beta_k + 1} S_{Q,\gamma,k}^{+}(\omega) \, d\omega,
\]

with

\[
S_{Q,\gamma,k}^{-}(\omega) = w_{C_k}(\omega)(q_{k+1} - tQ) + w_{B_k}(\omega)(q_k - tQ) + w_{A_+}(\omega) \max\{q - tQ, 0\},
\]

\[
S_{Q,\gamma,k}^{(0)}(\omega) = w_{C_0}(\omega)(q + q' - tQ) + w_{A_+}(\omega) \max\{q - tQ, 0\} + w_{B_-}(\omega) \max\{q' - tQ, 0\},
\]

\[
S_{Q,\gamma,k}^{+}(\omega) = w_{C_-(\omega)}(q_{k+1} - tQ) + w_{A_-} (q'_k - tQ) + w_{B_-}(\omega) \max\{q' - tQ, 0\}.
\]

Here the formulas for the width of the strips are as in (3.8), and we take

\[
\alpha_k = \arctan \frac{a - 1/Q}{q_k}, \quad \beta_k = \arctan \frac{a' + 1/Q}{q'_k},
\]

\[
\alpha_\infty = \arctan \frac{a}{q}, \quad \beta_\infty = \arctan \frac{a'}{q'}.
\]

Taking into account the equalities

\[
q_{k+1}w_{C_k} + q_kw_{B_k} + qw_{A_+} = 1 = q'_k w_{C_-} + q'_k w_{A_-} + qw_{B_-}, \quad (4.2)
\]

\[
w_{C_k} + w_{B_k} + w_{A_+} = Q^{-1} = w_{C_-} + w_{A_-} + w_{B_-}, \quad (4.3)
\]

and

\[
w_{C_0}(\omega) = Q^{-1} - w_{A_+}(\omega) - w_{B_-}(\omega),
\]

we can write

\[
P_{I,Q}(t) = \sum_{\gamma \in \mathcal{F}_Q(t)} (T_{Q,\gamma}^{(1)} + \cdots + T_{Q,\gamma}^{(5)}) \, d\omega, \quad (4.4)
\]

with

\[
T_{Q,\gamma}^{(1)} = \max\{q - tQ, 0\} \int_{\alpha_\infty}^{\beta_0} w_{A_+}(\omega) \, d\omega, \quad T_{Q,\gamma}^{(2)} = \max\{q' - tQ, 0\} \int_{\alpha_0}^{\beta_\infty} w_{B_-}(\omega) \, d\omega,
\]

\[
T_{Q,\gamma}^{(3)} = (q + q' - tQ) \int_{\alpha_0}^{\beta_0} \left( \frac{1}{Q} - w_{A_+}(\omega) - w_{B_-}(\omega) \right) \, d\omega,
\]

\[
T_{Q,\gamma}^{(4)} = \int_{\alpha_\infty}^{\beta_\infty} ((tQ - q) w_{A_+}(\omega) + 1 - t) \, d\omega, \quad T_{Q,\gamma}^{(5)} = \int_{\beta_0}^{\beta_\infty} ((tQ - q) w_{B_-}(\omega) + 1 - t) \, d\omega.
\]

Rewriting the terms in a convenient way we arrive at

\[
P_{I,Q}(t) = A_0 + A_1 + A_2 + A_3,
\]
where

\[ A_0 = (1 - t) \sum_{\gamma \in \mathcal{F}_Q(t)} (\beta_\infty - \alpha_\infty), \]

\[ A_1 = \sum_{\gamma \in \mathcal{F}_Q(t)} \left( \max\{q - tQ, 0\} + tQ - q \right) \int_{\alpha_\infty}^{\beta_\infty} w_{A_+}(\omega) \, d\omega \]

\[ = - \sum_{\gamma \in \mathcal{F}_Q(t)} \min\{q - tQ, 0\} \int_{\alpha_\infty}^{\beta_\infty} w_{A_+}(\omega) \, d\omega, \]

\[ A_2 = \sum_{\gamma \in \mathcal{F}_Q(t)} \left( \max\{q' - tQ, 0\} + tQ - q' \right) \int_{\alpha_0}^{\beta_\infty} w_{B_-}(\omega) \, d\omega \]

\[ = - \sum_{\gamma \in \mathcal{F}_Q(t)} \min\{q' - tQ, 0\} \int_{\alpha_0}^{\beta_\infty} w_{B_-}(\omega) \, d\omega, \]

\[ A_3 = \sum_{\gamma \in \mathcal{F}_Q(t)} \int_{\alpha_0}^{\beta_0} \left( \frac{q + q'}{Q} - 1 - q'w_{A_+}(\omega) - qw_{B_-}(\omega) \right) \, d\omega. \]

Remark first that \( A_3 = 0 \), as a result of

\[ \frac{q + q'}{Q} - 1 - q'w_{A_+}(\omega) - qw_{B_-}(\omega) \]

\[ = \frac{q + q'}{Q} - 1 - q' \left( \frac{1}{Q} + a - q \tan \omega \right) - q \left( \frac{1}{Q} + q' \tan \omega - a' \right) \]

\[ = a'q - aq' - 1 = 0. \] (4.5)

The next elementary statement will be repeatedly used.

**Lemma 6.** For any \( \lambda, \mu \in \mathbb{R} \) we have, uniformly in \( c \in [0, 1] \) as \( h \to 0^+ \),

\[ \int_{\arctan(c + h)}^{\arctan(c + h)} \left( \lambda \tan \omega + \mu \right) \, d\omega = \left( \frac{h}{1 + c^2} + \frac{h^2 c}{(1 + c^2)^2} \right) \left( \lambda c + \mu \right) \]

\[ + \frac{h^2 \lambda}{2(1 + c^2)} + O(h^3(|\lambda| + |\mu|)), \] (4.6)

\[ \int_{\arctan(c - h)}^{\arctan(c - h)} \left( \lambda \tan \omega + \mu \right) \, d\omega = \left( \frac{h}{1 + c^2} + \frac{h^2 c}{(1 + c^2)^2} \right) \left( \lambda c + \mu \right) \]

\[ - \frac{h^2 \lambda}{2(1 + c^2)} + O(h^3(|\lambda| + |\mu|)). \] (4.7)

**Proof.** Applying to our situation Taylor’s formula

\[ \int_a^{a + \xi} f(x) \, dx = \xi f(a) + \frac{\xi^2}{2} f'(a) + O(||f''||_{\infty} |\xi|^3) \]
together with
\[ \xi = \arctan(c + h) - \arctan c = \frac{h}{1 + c^2} - \frac{h^2 c}{(1 + c^2)^2} + O(h^3), \]  
we get
\[
\int_{\arctan c}^{\arctan(c+h)} \frac{\tan \omega + \mu}{\lambda} \, d\omega = \left( \frac{h}{1 + c^2} - \frac{h^2 c}{(1 + c^2)^2} + O(h^3) \right) \left( c + \frac{\mu}{\lambda} \right)
\]
\[+ \frac{1}{2} \left( \frac{h}{1 + c^2} - \frac{h^2 c}{(1 + c^2)^2} + O(h^3) \right)^2 \left( 1 + c^2 \right) + O(h^3)\]
whence (4.6) follows for \( \lambda \neq 0 \). The case \( \lambda = 0 \) is a direct consequence of (4.8), while (4.7) is derived from (4.6) by changing \( h \) into \( -h \).

This result will only be applied in cases where \( \lambda c + \mu = 0 \). We shall also use the following weaker form of (4.8):
\[ \arctan(c + h) - \arctan c = \frac{h}{1 + c^2} + O(h^2) \]  
It remains to estimate \( A_0, A_1 \) and \( A_2 \). By (4.9) it is immediate that
\[
A_0 = (1 - t) \sum_{\gamma \in \mathcal{F}_Q(I)} \left( \frac{1}{qq'(1 + \gamma^2)} + O\left( \frac{1}{q^2q'^2} \right) \right).
\]  
This shows in conjunction with
\[
\sum_{\gamma \in \mathcal{F}_Q(I)} \frac{1}{qq'(1 + \gamma^2)} \ll Q \sum_{q = 1}^{Q} \frac{1}{q^2q'^2} \ll \frac{1}{Q} \sum_{q = 1}^{Q} \frac{1}{q^2} \ll \frac{1}{Q}
\]  
and with the subsequent Lemma 7 that
\[
A_0 = c_I (1 - t) + O_\delta(E_{c,c_1}(Q)).
\]  

**Lemma 7.** Let \( c, c_1 > 0 \) such that \( c + c_1 < 1 \). Then for any interval \( I \subseteq [0,1] \) with \( |I| \gg Q^{-c} \) and \( \delta > 0 \)
\[
\sum_{\gamma \in \mathcal{F}_Q(I)} \frac{1}{qq'(1 + \gamma^2)} = c_I + O_\delta(E_{c,c_1}(Q)).
\]

**Proof.** We decompose the sum above as \( S_1 + S_2 \), according to whether \( q' > q \) or \( q > q' \). Thus we can write
\[
S_1 = \sum_{q = 1}^{Q} \sum_{q' \in \mathcal{I} := \{ \max(Q - q, q) \}} f_q(q', a),
\]  
where
where we put
\[ f_q(q', a) = \frac{1}{qq'(1 + a^2/q^2)}, \quad a \in I, \ q' \in J, \ q \in [1, Q - 1]. \]

The inclusion \( I \subseteq (\frac{Q}{2}, Q] \) gives
\[ 0 \leq f_q(q', a) = \frac{q}{q'(q^2 + a^2)} \leq \frac{1}{qq'}, \]
\[ 0 \leq Df_q(q', a) = \left| \frac{\partial f_q}{\partial q'} (q', a) \right| + \left| \frac{\partial f_q}{\partial a} (q', a) \right| = \frac{q}{q'(q^2 + a^2)} \left( \frac{1}{q'} + \frac{2a}{q^2 + a^2} \right) \leq \frac{1}{qq'} \left( \frac{2}{Q} + \frac{2}{q} \right) \leq \frac{4}{q^2 q'} \leq \frac{8}{q^2 Q}. \]

Applying Lemma 2 with \( T = \left[ \frac{Q}{c}, 1 \right] \), the inner sum in (4.13) can be expressed as
\[ \varphi(q) \int_I \frac{dq'}{qq'} \int_q^{Q} \frac{da}{1 + a^2/q^2} + O_\delta \left( Q^{2c_1} q^{1/2 + \delta} \frac{1}{qQ} + Q^{c_1} q^{3/2 + \delta} \frac{1}{q^2 Q} + \frac{q^2 Q^{-c}}{Q^2 q^2 Q} \right) \]
\[ = c_I \frac{\varphi(q)}{q} V(q) + O_\delta(Q^{2c_1-1} q^{1/2 + \delta} + Q^{c_1-1} q^{-1/2 + \delta} + Q^{-1-c-c_1}), \]

where
\[ V(q) = \int_I \frac{dq'}{qq'} = \frac{1}{q} \ln \max\{q, Q - q\}, \quad q \in (0, Q]. \]

The function
\[ W(x) := \begin{cases} \frac{1}{x} \ln \frac{1}{\max\{x, 1-x\}} & \text{if } x \in (0, 1]; \\ 1 & \text{if } x = 0, \end{cases} \]

is bounded and has finite total variation on \([0, 1]\), hence
\[ M := \sup_{x \in [0, 1]} |W(x)| + \int_0^1 |W'(x)| \, dx = O(1). \]

Since \( V(Qx) = \frac{W(x)}{Q} \), Lemma 2 yields
\[ \sum_{q=1}^{Q} \frac{\varphi(q)}{q} V(q) = \frac{1}{\zeta(2)} \int_0^Q V(q) \, dq + O \left( \ln Q \left( \sup_{q \in [0, Q]} |V(q)| + \int_0^Q |V'(q)| \, dq \right) \right) \]
\[ = \frac{1}{\zeta(2)} \int_0^1 W(x) \, dx + O(Q^{-1} \ln Q). \]

Hence
\[ S_1 = c_I \sum_{q=1}^{Q} \frac{\varphi(q)}{q} V(q) + O_\delta(E_{c, c_1}(Q)) = \frac{c_I}{\zeta(2)} \int_0^1 W(x) \, dx + O_\delta(E_{c, c_1}(Q)). \]

(4.14)
Using a familiar identity of Euler (cf. formula (1.8) in [30]) we find that
\[
\int_0^1 W(x) \, dx = - \int_0^{1/2} \frac{\ln(1-x)}{x} \, dx - \int_{1/2}^1 \frac{\ln x}{x} \, dx
\]
\[
= - \int_0^{1/2} \frac{\ln(1-x)}{x} \, dx - \frac{\ln^2 2}{2} = \frac{\ln^2 2}{2} - \frac{\ln^2 2}{2} = \frac{\zeta(2)}{2},
\]
which we combine with (4.14) to get
\[
S_1 = \frac{c_1}{2} + O_\delta(E_{c,c_1,\delta}(Q)). \tag{4.15}
\]

Finally we employ
\[
\frac{1}{1 + \gamma'^2} = \frac{1}{1 + \gamma^2} + O(\gamma' - \gamma) = \frac{1}{1 + \gamma^2} + O\left(\frac{1}{qq'}\right) \tag{4.16}
\]
and (4.11) to write
\[
S_2 = \sum_{\gamma \in F_Q(1) \atop q > q'} \frac{1}{qq'(1 + \gamma^2)}.
\]

Using (4.16) and (4.11) we see that
\[
S_2 = \sum_{\gamma \in F_Q(1) \atop q > q'} \frac{1}{qq'(1 + \gamma^2)} = \frac{1}{qq'(1 + \gamma^2)}.
\]

Changing \(a'\) to \(q' - a'\), reversing the roles of \(q\) and \(q'\), and using
\[
\int_{q'(1-\alpha)} q'(1 - x/q')^2 = c_{1}q',
\]
it follows that \(S_2\) is given by the same expression as in (4.15).

Next we estimate \(A_1\) and find, taking \(c = \frac{a+1/Q}{q}\), \(h = \frac{1}{qq'}\), \(\lambda = -q\), \(\mu = a + \frac{1}{Q}\) in (4.7), that
\[
\begin{align*}
\int_{\alpha_{\infty}} \omega_{A_{+}}(\omega) \, d\omega &= \int_{\arctan \frac{a+1/Q}{q}} \frac{1}{Q} + a - q \tan \omega \, d\omega \\
&= \frac{1}{2qQ^2(1 + (a + 1/Q^2)/q^2)} + O\left(\frac{1}{q^2Q^3}\right) \tag{4.17} \\
&= \frac{1}{2qQ^2(1 + \gamma^2)} + O\left(\frac{1}{q^2Q^3}\right).
\end{align*}
\]

Since
\[
\sum_{\gamma \in F_Q} \frac{Q}{q^2Q^3} \leq \frac{1}{Q^2} \sum_{q=1}^{Q} \frac{\varphi(q)}{q^2} \ll \frac{\ln Q}{Q^2} = O(Q^{-1}),
\]
we infer from (4.17) and the definition of $A_1$ that
\begin{equation}
A_1 + O(Q^{-1}) = \sum_{\gamma \in F_Q(t)} \frac{tQ - q}{2qQ^2(1 + a^2/q^2)} = \sum_{1 \leq q \leq tQ} \frac{tQ - q}{2qQ^2} \sum_{q \leq q' \leq Q, a \equiv q' \equiv 1 (\text{mod } q)} \frac{1}{1 + a^2/q^2}.
\end{equation}

Applying Lemma 2 to $I = (Q - q, Q)$, $J = qI$, $f_q(q', a) = \frac{1}{1 + a^2/q^2}$, and taking $T = [Q^{c_1}]$, the inner sum above becomes
\begin{equation}
\frac{\varphi(q)}{q} \int_{q}^{tQ} \frac{1 + a^2/q^2}{da} + O_6 \left( \frac{Q^{2c_1}q^{1/2+\delta}}{q} + Q^{c_1}q^{3/2+\delta} \frac{1}{q} + \frac{q^2|I|}{Q^{c_1}q} \right)
= c_I \varphi(q) + O_6(Q^{2c_1}q^{1/2+\delta} + Q^{-c_1}q),
\end{equation}
which inserted back into (4.18) gives that $A_1 + O(Q^{-1})$ may be written as
\begin{equation}
\frac{c_I}{2Q^2} \sum_{1 \leq q \leq tQ} \frac{\varphi(q)}{q} (tQ - q) + O_6 \left( \sum_{q=1}^{Q} \frac{Q}{qQ^2} (Q^{2c_1}q^{1/2+\delta} + Q^{-c_1}q) \right)
= \frac{c_I}{2Q^2} \sum_{1 \leq q \leq tQ} \frac{\varphi(q)}{q} (tQ - q) + O_6(E_{c,c_1,\delta}(Q)).
\end{equation}

Applying now Lemma 1 to the main term above with $V(q) = tQ - q$, $q \in [1, tQ]$, we find that
\begin{equation}
A_1 = \frac{c_I t^2}{2Q^2 \zeta(2)} \int_{0}^{tQ} (tQ - q) dq + O_6(Q^{-1+\delta} + E_{c,c_1,\delta}(Q))
= \frac{c_I t^2}{2 \zeta(2)} \int_{0}^{t} (t - x) dx + O_6(E_{c,c_1,\delta}(Q)) = \frac{c_I t^2}{4 \zeta(2)} + O_6(E_{c,c_1,\delta}(Q)).
\end{equation}
In a similar way we find
\begin{equation}
A_2 = \frac{c_I t^2}{4 \zeta(2)} + O_6(E_{c,c_1,\delta}(Q)),
\end{equation}
and therefore
\begin{equation}
P_{L,Q}(t) = \left( 1 - t + \frac{t^2}{2 \zeta(2)} \right) c_I + O_6(E_{c,c_1,\delta}(Q)),
\end{equation}
which proves Proposition 3.

5. The case $t > 2$

In this section we shall evaluate the contribution of the integrals on $[\alpha_k, \alpha_{k-1}]$ in (3.7) when $k \geq 1$ and $t > 2$. In this situation there is a unique nonnegative integer, given by
\begin{equation}
K = K(\gamma, t) = \left[ \frac{tQ - q'}{q} \right] \geq 0,
\end{equation}
for which

\[ q_K \leq tQ < q_{K+1}. \] (5.1)

When \( t \geq 2 \) it follows that \( K \geq 1 \), and we prove

**Proposition 4.** Suppose \( I \) is a subinterval of \([0, 1]\) of size \(|I| \approx Q^{-c}\) for some \( 0 < c < 1 \). Then for every \( c_1 > 0 \) with \( c + c_1 < 1 \) and \( \delta > 0 \)

\[
P_{I, Q}(t) = \frac{c_I}{\zeta(2)} \int_0^1 \psi(x, t) \, dx + O_\delta(E_{c, c_1, \delta}(Q)) \quad (Q \to \infty),
\]

with \( \psi \) as in Theorem 11 and \( E_{c_1, \delta} \) as in Proposition 8. The estimate is uniform in \( t \) on compacts of \((2, \infty)\).

Next, \( \alpha_k \) and \( \beta_k \) will be as in (4.1) and the widths \( w \) as in (3.8). Since \( t > 2 \), then \( q + q' < tQ \) and the second sum in (3.7) is zero.

In the beginning we fix \( \gamma = \frac{q}{q} \in F_Q(I) \) and estimate

\[ S_2(\gamma, t) := \sum_{k=K+1}^{\infty} \int_{\alpha_k}^{\alpha_{k-1}} \left( q_{k+1} w_{c_k}(\omega) + q_k w_{B_k}(\omega) - tQ w_{c_k}(\omega) + w_{B_k}(\omega) \right) d\omega. \]

Using (4.2) and (4.3), taking \( c = \frac{a}{q} \), \( h = \frac{a_k - 1}{q} - \frac{a}{q} = \frac{1}{q} q/Q - \frac{1}{q} q/Q \), \( \lambda = q \), \( \mu = -a \) in (4.10), and also owing to

\[ \alpha_K - \alpha_\infty = \arctan \frac{a_K - 1/q}{q_K} - \arctan \frac{a}{q} = \frac{1}{q} \frac{1}{q_k q_k (1 + a^2/q^2)} + O \left( \frac{1}{q^2 q_k^2} \right), \]

we infer that

\[
S_2(\gamma, t) = \int_{\alpha_\infty}^{\alpha_K} \left( 1 - q w_{A_1}(\omega) - tQ \left( \frac{1}{Q} - w_{A_1}(\omega) \right) \right) d\omega \\
= \int_{\alpha_\infty}^{\alpha_K} \left( q^2 \tan \omega + 1 - \left( \frac{1}{Q} + a \right) q \right) d\omega - tQ \int_{\alpha_\infty}^{\alpha_K} (q \tan \omega - a) d\omega \\
= \left( 1 - \frac{q}{Q} \right) (\alpha_K - \alpha_\infty) + (q - tQ) \int_{\alpha_\infty}^{\alpha_K} (q \tan \omega - a) d\omega \\
= \left( 1 - \frac{q}{Q} \right) (\alpha_K - \alpha_\infty) + (q - tQ) \left( \frac{(1 - q/Q)^2 q}{2q^2 q_k^2 (1 + \gamma^2)} + O \left( \frac{1}{q^2 q_k^2} \right) \right) \\
= \frac{1}{q} \frac{1}{q_k q_k (1 + \gamma^2)} + (q - tQ) \frac{(1 - q/Q)^2}{2q^2 q_k^2 (1 + \gamma^2)} + O \left( \frac{1}{q^2 q_k^2} \right)
\]

On the other hand, taking \( c = \frac{a_k - 1/q}{q_k - 1/q_k} \), \( h = \frac{a_k - 1/q}{q_k - 1/q_k} - \frac{a}{q} = \frac{1}{q} q/Q - \frac{1}{q} q/Q \), \( \lambda = -q_{-1} \), \( \mu = a_{k-1} - \frac{1}{\gamma} \) in (4.17), and also using

\[ 0 \leq \frac{1}{1 + c^2} - \frac{1}{1 + \gamma^2} \leq \frac{a_{k-1} - \frac{1}{\gamma}}{q_{k-1}} - \frac{a}{q} \leq \frac{1}{q} q_{k-1}, \]
we estimate
\[
\int_{\alpha_K}^{\alpha_{K-1}} w_{C_k}(\omega) \, d\omega = \int_{\alpha_K}^{\alpha_{K-1}} \left( \alpha_{K-1} - \frac{1}{Q} - q_{K-1} \tan \omega \right) \, d\omega
\]
\[
= \frac{q_{K-1}(1 - q/Q)^2}{2q_{K-1}q_K^2(1 + \gamma^2)} + O\left( \frac{q_{K-1}}{q_{K-1}q_K^2} \right)
\]
\[
= \frac{(1 - q/Q)^2}{2q_{K-1}q_K^2(1 + \gamma^2)} + O\left( \frac{1}{q_{K-1}^2} + \frac{1}{q_{K-1}q_K} \right)
\]
\[
= \frac{(1 - q/Q)^2}{2q_{K-1}q_K^2(1 + \gamma^2)} + O\left( \frac{1}{q_{K-1}q_K^2} \right).
\]
Using also \(0 < q_{K+1} - tQ \leq q\), this gives whenever \(t > 2\) (so \(K \geq 1\))
\[
S_1(\gamma, t) := (q_{K+1} - tQ) \int_{\alpha_K}^{\alpha_{K-1}} w_{C_k}(\omega) \, d\omega
\]
\[
= \frac{(q_{K+1} - tQ)(1 - q/Q)^2}{2q_{K-1}q_K^2(1 + \gamma^2)} + O\left( \frac{1}{q^2(q + q')^2} \right).
\]
Since \(t > 2\), the sum of integrals on \([\alpha_k, \alpha_{k-1}]\) in 3.4 becomes
\[
P_{t,Q}^+(t) := \sum_{\gamma \in F_{Q(t)}} \left( S_1(\gamma, t) + S_2(\gamma, t) \right).
\]
Making use of
\[
\sum_{\gamma \in F_{Q}} \frac{1}{q^2(q' + q')^2} \leq \sum_{q = 1}^{Q} \frac{1}{q^2} \sum_{q' = Q}^{Q} \frac{1}{(q' + q')^2} \leq \sum_{q = 1}^{Q} \frac{1}{q^2} \sum_{k = Q+1}^{\infty} \frac{1}{k^2} < \frac{1}{Q}
\]
and of
\[
\frac{(1 - q/Q)^2(q_{K+1} - tQ)}{2q_{K-1}q_K^2(1 + \gamma^2)} + \frac{(1 - q/Q)^2(q + 2q_K - tQ)}{2qq_K^2(1 + \gamma^2)}
\]
\[
= \frac{(1 - q/Q)^2(q(q_{K+1} + q_{K-1}) + 2q_{K-1}q_K - tQ(q + q_{K-1}))}{2qq_{K-1}q_K^2(1 + \gamma^2)}
\]
\[
= \frac{(1 - q/Q)^2(2qq_K + 2q_{K-1}q_K - tQq)}{2qq_{K-1}q_K^2(1 + \gamma^2)} = \frac{(1 - q/Q)^2(2q_K - tQ)}{2qq_{K-1}q_K(1 + \gamma^2)},
\]
we find
\[
P_{t,Q}^+(t) = \sum_{\gamma \in F_{Q(t)}} \frac{(1 - q/Q)^2(2q_K - tQ)}{2qq_{K-1}q_K(1 + \gamma^2)} + O(Q^{-1}).
\]
Next for each integer \(k \geq 1\) consider the sets
\[
\Omega_k = \left\{ (x, y) \in \mathbb{R}^2 : \left[ \frac{t - y}{x} \right] = k \right\} \quad \text{and} \quad I_k = \left[ \frac{t - 1}{k}, \frac{t - 1}{k - 1} \right) \cap [0, 1),
\]
and for \( q \in QI_k \) and \( k \geq 1 \), respectively \( k \geq 2 \), the intervals (see Figure 6)

\[
J^{(0)}_{k,q} = \left( t - \frac{kq}{Q}, 1 \right] = \left\{ \frac{q}{Q} : \left( \frac{q}{Q}, \frac{q'}{Q} \right) \in \Omega_{k-1} \cap \mathcal{T} \right\} \subseteq \left( 1 - \frac{q}{Q}, 1 \right],
\]

\[
J^{(1)}_{k,q} = \left( 1 - \frac{q}{Q}, t - \frac{kq}{Q} \right] = \left\{ \frac{q'}{Q} : \left( \frac{q}{Q}, \frac{q'}{Q} \right) \in \Omega_k \cap \mathcal{T} \right\} \subseteq \left( 1 - \frac{q}{Q}, 1 \right].
\]

Note that \( |QJ^{(0)}_{k,q}|, |QJ^{(1)}_{k,q}| < q \), that \( \min\{k : |\Omega_k \cap \mathcal{T}| > 0\} = [t] - 1 \geq 1 \), and that \( |I_k| = 0 \) unless \( k \geq [t] \geq 2 \).

![Figure 6. The set \( \Omega_k \cap \mathcal{T} \)](image)

We also consider the function

\[
Q\Omega_k \times [0,q] \ni (q,q',a) \mapsto f_k(q,q',a) = \frac{(1-q/Q)^2(2q_k - tQ)}{2qq_kq_{k-1}(1 + \gamma^2)} = \frac{(1-q/Q)^2(2q' + 2kq - tQ)}{2q(q'+kq)(q' + (k-1)q)(1 + a^2/q^2)}.
\]

Using the one-to-one correspondence between the primitive integer points in \( Q(\Omega_k \cap \mathcal{T}) \) and the set of consecutive Farey fractions \( \gamma = \frac{a}{q} \) and \( \gamma' = \frac{a'}{q'} \) in \( \mathcal{F}_Q \) with
We aim to estimate \( S_k(q) \) and \( T_k(q) \) applying Lemma 2 to the intervals \( I = QJ_{t, q}^{(1)} \), \( J = qI \) and the function \( f \) defined as \( f_k = f_k(q, \cdot, \cdot) \), and respectively to \( I = QJ_{t, q}^{(0)} \), \( J = qI \) and \( f = f_k(q, \cdot, \cdot) \). For \((q, q') \in Q(\Omega_k \cap T)\) we have \( q_k \leq tQ < q_{k+1} \), or equivalently

\[
q_k - 1 < 2q_k - tQ \leq q_k.
\]

As a result, we see that (here \( k \geq 2 \))

\[
\| f_k(q, \cdot, \cdot) \|_{\infty} \leq \sup_{q' \in QJ_{t, q}^{(1)}} \frac{q_k}{qq_k q_{k-1}} \leq \sup_{q' \in Q(Q - q, q]} \frac{1}{q(q + q')} < \frac{1}{q Q},
\]

\[
\| f_{k-1}(q, \cdot, \cdot) \|_{\infty} \leq \sup_{q' \in QJ_{t, q}^{(0)}} \frac{q_{k-1}}{qq_{k-1} q_{k-2}} \leq \sup_{q' \in Q(Q - tQ, q]} \frac{1}{q q'} \leq \frac{1}{(t - 2)q Q} \ll_t \frac{1}{q Q}.
\]

The last estimate holds without the factor \( \frac{1}{t-2} \) whenever \( k \geq [t] \). In the remainder of this section we will simply write \( \frac{1}{t-2} \ll_t 1 \) with the understanding that this holds uniformly in \( t \) on compacts of \((2, \infty)\).

We also need to estimate the \( L_{\infty} \)-norm of \( D f_k \). It is easily seen that

\[
\left\| \frac{\partial f_k}{\partial a}(q, \cdot, \cdot) \right\|_{\infty} \leq \frac{2q}{q^2} \| f_k(q, \cdot, \cdot) \|_{\infty} \ll \frac{1}{q^2 Q},
\]

\[
\left\| \frac{\partial f_{k-1}}{\partial a}(q, \cdot, \cdot) \right\|_{\infty} \leq \frac{2q}{q^2} \| f_{k-1}(q, \cdot, \cdot) \|_{\infty} \ll \frac{1}{(t - 2)q^2 Q} \ll \frac{1}{q^2 Q},
\]

\[
\left\| \frac{\partial f_k}{\partial q}(q, \cdot, \cdot) \right\|_{\infty} \leq \frac{1}{2q} \sup_{q' \in QJ_{t, q}^{(1)}} \frac{\left| 2qq_k q_{k-1} - (2q_k - tQ)(q_k + q_{k-1}) \right|}{q_k q_{k-1}}
\]

\[
\ll \sup_{q' \in QJ_{t, q}^{(1)}} \left( \frac{1}{qq_k q_{k-1}} + \frac{q_k (q_k + q_{k-1})}{qq_k q_{k-1}^2} \right) \ll \sup_{q' \in QJ_{t, q}^{(1)}} \frac{q_k + q_{k-1}}{qq_k q_{k-1}^2},
\]

\[
\ll \sup_{q' \in QJ_{t, q}^{(0)}} \frac{1}{qq_k^2} \ll \sup_{q' \in Q(Q - q, q]} \frac{1}{q(q + q')^2} \leq \frac{1}{q Q^2} \leq \frac{1}{q^2 Q},
\]

and similarly

\[
\left\| \frac{\partial f_{k-1}}{\partial q}(q, \cdot, \cdot) \right\|_{\infty} \ll \frac{1}{q} \sup_{q' \in [t-2)Q, q]} \frac{1}{q^2} \leq \frac{1}{(t - 2)^2 q Q^2} \ll_t \frac{1}{q Q^2} \leq \frac{1}{q^2 Q}.
\]
Applying now Lemma 2 to this situation with \( T = [Q^{c_1}] \), where \( 0 < c_1 < \frac{1}{2} \) is to be determined later, we approximate \( S_k(q) + T_k(q) \) within error
\[
E_k(q) \ll \delta Q^{2c_1 - 1}q^{-1/2 + \delta} + Q^{c_1}q^{3/2 + \delta} + Q^{-c - c_1}q^2 + 1 \\
\ll \delta \sum Q^{2c_1 - 1}q^{-1/2 + \delta} + Q^{-c - c_1}
\]
by
\[
\frac{\varphi(q)}{q^2} \int_{Q J_{2\delta}^{(1)}(a)} f_k(q, q', a) dq' da + \frac{\varphi(q)}{q^2} \int_{Q J_{2\delta}^{(1)}(a)} f_{k-1}(q, q', a) dq' da \\
c_l \frac{\varphi(q)}{q} \frac{(1 - q/Q)^2}{2q} W_k(q),
\]
where \( c_l \) is as in (3.10) and
\[
W_k(q) = \int_{Q J_{2\delta}^{(1)}(a)} g_k(q, q') dq' + \int_{Q J_{2\delta}^{(1)}(a)} g_{k-1}(q, q') dq',
\]
g_k(q, q') = \frac{2q_k - tQ}{q_k q_{k-1}}, (q, q') \in Q(\Omega_k \cap T).

By a direct computation we find that
\[
W_k(q) = W(q) = \int_{Q - q}^{tQ - kq} 2q_k - tQ dq' + \int_{tQ - kq}^{tQ} \frac{2q_{k-1} - tQ}{q_k q_{k-2}} dq' \\
= \int_{Q + (k-1)q}^{tQ} \frac{2y - tQ}{y(y - q)} dy + \int_{tQ - q}^{tQ} \frac{2y - tQ}{y(y - q)} dy \\
= \int_{tQ - q}^{tQ} \frac{2y - tQ}{y(y - q)} dy \left(2 \ln(y - q) - \frac{tQ}{q} \frac{y - q}{y}\right) \bigg|_{y = tQ - q}^{tQ} \\
= 2 \ln \frac{tQ - q}{tQ - 2q} - \frac{tQ}{q} \ln \left(\frac{(tQ - q)^2}{tQ(tQ - 2q)}\right)
\]
is independent of \( k \). Since the error terms sum up to
\[
\sum_{k=2}^{\infty} \sum_{q \in Q \cap T} (Q^{2c_1 - 1}q^{-1/2 + \delta} + Q^{-1 - c - c_1}) = Q^{2c_1 - 1} \sum_{q=1}^{Q} q^{-1/2 + \delta} + Q^{1 - 1 - c - c_1} \ll E_{c, c_1, \delta}(Q),
\]
we arrive at
\[
P_{1, Q}(t) = c_l \sum_{q=1}^{Q} \frac{\varphi(q)}{q} V(q) + O_\delta(E_{c, c_1, \delta}(Q)), \quad \text{for} \quad t > 2
\]
with
\[
V(q) = \frac{(1 - q/Q)^2}{2q} W(q), \quad q \in (0, Q).
\]
For \( t > 2 \) consider the function
\[
f_t(x) = \frac{\psi(x, t)}{2} = (1 - x^2) \left(2 \ln \left(1 + \frac{x}{t - 2x}\right) - \frac{t}{x} \ln \left(1 + \frac{(t - x)^2}{t(t - 2x)}\right)\right), \quad x \in (0, 1].
\]
Using the Taylor series of the logarithm we obtain for small $x$

$$f_t(x) = \frac{(1-x)^2}{2x} \left( \frac{2x}{t-2x} - \frac{x^2}{(t-2x)^2} - \frac{t}{x} \cdot \frac{x^2}{t(t-2x)} + O(x^3) \right)$$

$$= (1-x)^2 \left( \frac{1}{2(t-2x)} - \frac{x}{2(t-2x)^2} + O(x^2) \right),$$

which shows that $f$ extends to a $C^1$ function on $[0,1]$, and so

$$\int_0^1 |f'_t(x)| \, dx \ll 1,$$

uniformly for $t$ in compacts of $(2, \infty)$.

The equality $V(Qx) = Q^{-1} f(x), \, x \in (0,1]$, implies now that both $\|V\|_{\infty}$ and the total variation of $V$ on $(0,Q]$ are $\ll Q^{-1}$. Thus we may apply Lemma 1 to (5.2) and conclude, also using $c + c_1 < 1$, that

$$P_{I,Q}(t) = \frac{c_I}{\zeta(2)} \int_0^Q V(q) \, dq + O_\delta(E_{c,c_1,\delta}(Q)) = \frac{c_I}{\zeta(2)} \int_0^1 f_t(x) \, dx + O_\delta(E_{c,c_1,\delta}(Q))$$

$$= \frac{c_I}{2\zeta(2)} \int_0^1 \psi(x,t) \, dx + O_\delta(E_{c,c_1,\delta}(Q)).$$

(5.3)

One can see in a similar way that the contribution of integrals on the intervals $[\beta_{k-1}, \beta_k]$ in (3.1) for $k \geq 1$ and $t > 2$ is

$$P_{I,Q}^-(t) = \frac{c_I}{2\zeta(2)} \int_0^1 \psi(x,t) \, dx + O(E_{c,c_1,\delta}(Q)).$$

(5.4)

Proposition 4 now follows from (5.3) and (5.4).

6. The case $1 < t < 2$

In this section we prove

**Proposition 5.** Suppose $I$ is a subinterval of $[0,1]$ of size $|I| \asymp Q^{-c}$ for some $0 < c < 1$. Then for any $c_1$ with $c + c_1 < 1$ and $\delta > 0$

$$P_{I,Q}(t) = \frac{c_I}{\zeta(2)} \left( \int_0^{t-1} \psi(x,t) \, dx + \int_{t-1}^1 \phi(x,t) \, dx \right) + O_\delta(E_{c,c_1,\delta}(Q)) \quad (Q \to \infty),$$

with $\phi$ and $\psi$ as in Theorem 1. The estimate holds uniformly in $t$ on compacts of $(1,2)$. 
In this case (3.7) gives

\[
P_{I,Q}(t) = \sum_{\gamma \in F_Q(I)} \int_{\alpha_0}^{\beta_0} w_{C_0}(\omega) \max\{q + q' - tQ, 0\} \, d\omega \\
+ \sum_{\gamma \in F_Q(I)} \sum_{k=1}^{\infty} \int_{\alpha_k}^{\alpha_k-1} \left( w_{C_k}(\omega) \max\{q_{k+1} - tQ, 0\} + w_{B_k}(\omega) \max\{q_k - tQ, 0\} \right) \, d\omega \\
+ \sum_{\gamma \in F_Q(I)} \sum_{k=1}^{\infty} \int_{\beta_{k-1}}^{\beta_k} \left( w_{C_{-k}}(\omega) \max\{q_{k+1} - tQ, 0\} + w_{A_{-k}}(\omega) \max\{q_k - tQ, 0\} \right) \, d\omega.
\]

We break the main term above according as to whether \(q + q' > tQ\) or \(q + q' \leq tQ\). Thus we first estimate

\[
P^{>}_{I,Q}(t) := \sum_{\gamma \in F_Q(I)} \int_{\alpha_0}^{\beta_0} \left( \frac{1}{Q} - w_{A_+}(\omega) - w_{B_+}(\omega) \right) (q + q' - tQ) \, d\omega \\
+ \sum_{\gamma \in F_Q(I)} \sum_{k=1}^{\infty} \int_{\alpha_k}^{\alpha_k-1} \left( w_{C_k}(\omega)q_{k+1} + w_{B_k}(\omega)q_k - tQ(w_{C_k}(\omega) + w_{B_k}(\omega)) \right) \, d\omega \\
+ \sum_{\gamma \in F_Q(I)} \sum_{k=1}^{\infty} \int_{\beta_{k-1}}^{\beta_k} \left( w_{C_{-k}}(\omega)q_{k+1} + w_{A_{-k}}(\omega)q_k - tQ(w_{C_{-k}}(\omega) + w_{A_{-k}}(\omega)) \right) \, d\omega.
\]

Using (4.2) and (4.3) we may also write

\[
P^\omega_{I,Q}(t) = \sum_{\gamma \in F_Q(I)} \int_{\alpha_0}^{\alpha_0} \left( \frac{1}{Q} - w_{A_+}(\omega) - w_{B_+}(\omega) \right) (q + q' - tQ) \, d\omega \\
+ \sum_{\gamma \in F_Q(I)} \int_{\alpha_\infty}^{\alpha_\infty} \left( 1 - w_{A_+}(\omega)q - tQ(\frac{1}{Q} - w_{A_+}(\omega)) \right) \, d\omega \\
+ \sum_{\gamma \in F_Q(I)} \int_{\beta_\infty}^{\beta_\infty} \left( 1 - w_{B_-}(\omega)q' - tQ(\frac{1}{Q} - w_{B_-}(\omega)) \right) \, d\omega \\
= A_0 + A_1 + \bar{A}_2 + \bar{A}_3,
\]
with
\[ \tilde{A}_0 = (1 - t) \sum_{\gamma \in F_Q(I)} (\beta_\infty - \alpha_\infty), \quad \tilde{A}_1 = \sum_{\gamma \in F_Q(I)} (tQ - q) \int_{\alpha_\infty}^{\beta_0} w_{A_+}(\omega) \, d\omega, \]
\[ \tilde{A}_2 = \sum_{\gamma \in F_Q(I)} (tQ - q) \int_{\alpha_0}^{\beta_\infty} w_{B_-}(\omega) \, d\omega, \]
\[ \tilde{A}_3 = \sum_{\gamma \in F_Q(I)} \int_{\alpha_0}^{\beta_0} \left( \frac{q + q'}{Q} - 1 - q' w_{A_+}(\omega) - q w_{B_-}(\omega) \right) \, d\omega. \]

**Figure 7.** The set \( \bigcup_{k=1}^\infty \Omega_k \cap \mathcal{T} \) when \( 1 < t < 2 \)

We proceed to estimate \( \tilde{A}_0, \tilde{A}_1, \tilde{A}_2 \) and \( \tilde{A}_3 \) by noticing that (4.5) yields
\[ \tilde{A}_3 = 0. \] (6.1)

Next \( \tilde{A}_1 \) is estimated in a similar way as \( A_1 \) was in Section 4, only with the difference that the summation over \( \gamma \in F_Q(I) \) is being done under the additional requirement \( q + q' > tQ \). This is not going to produce any change in the error, and will only affect the main terms. As in (4.10) and (4.11) we obtain
\[ \tilde{A}_0 = (1 - t) \sum_{\gamma \in F_Q(I)} \frac{1}{q q'(1 + \gamma^2)} + O(Q^{-1}). \]

Then, as in the proof of Lemma 7, we find that
\[ \tilde{A}_0 = (1 - t) \sum_{q=1}^{\infty} \sum_{q' \in \mathcal{I} : (tQ-q,q)} f_I(q',a) = c_I(1 - t) \sum_{(t-1)Q < q \leq Q} \frac{\varphi(q)}{q} V(q) + O_{\delta}(E_{c,c_1,\delta}(Q)), \]
where this time we take
\[ V(q) = \frac{1}{q} \ln \frac{Q}{tQ - q}, \quad q \in ((t - 1)Q, Q]. \]

But \( V(Qx) = Q^{-1} \tilde{V}(x) \) and the function
\[ \tilde{V}(x) = \frac{1}{x} \ln \frac{1}{t - x}, \quad x \in [t - 1, 1], \]
is \( C^1 \) on \([t - 1, 1]\). Hence both the \( L^\infty \)-norm and the total variation of \( V \) on the interval \([(t - 1)Q, Q]\) are \( \ll Q^{-1} \), uniformly in \( t \) on compacts of \((1, 2)\). Lemma 1 applies now and yields
\[
\tilde{A}_0 = c_I \int_{(t-1)Q}^Q V(q) \, dq + O_{c,c_1}(E_{c,c_1}(Q))
\]
\[ = c_I \frac{(1 - t)}{\zeta(2)} \int_{t-1}^1 \frac{1}{x} \ln \frac{1}{t - x} \, dx + O_{c,c_1}(E_{c,c_1}(Q)). \quad (6.2)\]

Proceeding as in Section 4 (see (4.18)–(4.20)) we find
\[
\tilde{A}_1 = \sum_{(t-1)Q < q \leq Q} \frac{tQ - q}{2qQ^2} \sum_{tQ/q < q' \leq Q} \frac{1}{1 + a^2/q^2} + O(Q^{-1})
\]
\[ = \frac{c_I}{2Q^2} \sum_{(t-1)Q < q \leq Q} \frac{\varphi(q)}{q^2} \left( tQ - q \right) \left( q - (t - 1)Q \right) + O_{c,c_1}(E_{c,c_1}(Q))
\]
\[ = \frac{c_I}{2Q^2 \zeta(2)} \int_{(t-1)Q}^Q \frac{(tQ - q)(q - (t - 1)Q)}{q} \, dq + O_{c,c_1}(E_{c,c_1}(Q)). \quad (6.3)\]

This immediately gives
\[
\tilde{A}_1 = c_I \frac{(1 - t)}{\zeta(2)} \int_{t-1}^1 \frac{(t - x)(x - t + 1)}{x} \, dx + O_{c,c_1}(E_{c,c_1}(Q)). \quad (6.4)\]

In a similar way we find
\[
\tilde{A}_2 = c_I \frac{(1 - t)}{\zeta(2)} \int_{t-1}^1 \frac{(t - x)(x - t + 1)}{x} \, dx + O_{c,c_1}(E_{c,c_1}(Q)). \quad (6.5)\]

From (6.1)–(6.5) we now collect
\[
P_{I,Q}^>(t) = c_I \frac{(1 - t)}{\zeta(2)} \int_{t-1}^1 \frac{1}{x} \ln \frac{1}{t - x} \, dx + c_I \frac{(1 - t)}{\zeta(2)} \int_{t-1}^1 \frac{(t - x)(x - t + 1)}{x} \, dx + O_{c,c_1}(E_{c,c_1}(Q)).
\]

It remains to estimate the contribution of Farey fractions of order \( Q \) with \( q + q' \leq tQ \) to \( P_{I,Q}(t) \), which is
\[
P_{I,Q}^>(t) := B_1 + B_2,
\]
where $B_1$ denotes
\[
\sum_{\gamma \in \mathcal{F}_Q(I)} \sum_{k=1}^{a_{k-1}} \int_{a_{k-1}} \left( w_{C_k}(\omega) \max\{q_{k+1} - tQ, 0\} + w_{B_k}(\omega) \max\{q_k - tQ, 0\} \right) d\omega,
\]
and $B_2$ denotes
\[
\sum_{\gamma \in \mathcal{F}_Q(I)} \sum_{k=1}^{b_{k-1}} \int_{b_{k-1}} \left( w_{C_{-k}}(\omega) \max\{q_{k+1} - tQ, 0\} + w_{A_{-k}}(\omega) \max\{q_k' - tQ, 0\} \right) d\omega.
\]
In this case one also has
\[
K = \left\lceil \frac{tQ - q'}{q} \right\rceil \geq 1,
\]
and as in Section 5 we find $B_1 + O_\delta(E_{c,1,\delta}(Q))$
\[
B_1 = \frac{c_I}{2\zeta(2)} \int_0^{t-1} \psi(x,t) \, dx + \frac{c_I}{2\zeta(2)} \int_{t-1}^{1} \frac{(1-x)^2}{x} \left( 2 \ln \frac{t-x}{1-x} - \frac{t}{x} \ln \frac{t-x}{t(1-x)} \right) \, dx
\]
and thus
\[
B_1 = \frac{c_I}{2\zeta(2)} \int_0^{t-1} \psi(x,t) \, dx + O_\delta(E_{c,1,\delta}(Q)).
\]
In a similar way we find that $B_2$ can too be expressed as in (6.6), and thus
\[
P^{<}_{I,Q}(t) = \frac{c_I}{\zeta(2)} \int_{t-1}^{1} \frac{(1-x)^2}{x} \left( 2 \ln \frac{t-x}{1-x} - \frac{t}{x} \ln \frac{t-x}{t(1-x)} \right) \, dx
\]
and Proposition 5 follows now from (6.6) and (6.7).
7. Proof of Theorem 1

We may assume without loss of generality that \( \omega \in [0, \frac{\pi}{4}] \), thus estimate for small \( \varepsilon > 0 \) the quantity

\[
P_\varepsilon(t) = \frac{4}{\pi} \left\{ (x, \omega) \in Y_\varepsilon \times \left[0, \frac{\pi}{4}\right] : \tau_\varepsilon(x, \omega) > \frac{t}{2\varepsilon} \right\}.
\]

We partition the interval \([0, 1]\) as a union of \(N\) intervals \(I_j = [\tan \omega_j, \tan \omega_{j+1}]\) of equal size, with \(N = [\varepsilon^{-c}]\), thus with \(|I_j| = \frac{1}{\pi} \approx \varepsilon^c\), where \(0 < c < 1\) is to be chosen later. For each \(j\) we set

\[
Q^-_j = \left[\frac{\cos \omega_j}{2\varepsilon + 2\varepsilon^{c+1}}\right], \quad Q^+_j = \left[\frac{\cos \omega_j}{2\varepsilon - 2\varepsilon^{c+1}}\right] + 1.
\]

Since \(\omega_j \in [0, \frac{\pi}{4}]\), we have \(Q^\pm_j \approx \varepsilon^{-1}\), and thus \(|I_j| \approx \varepsilon^c\). Moreover, for \(\omega \in [\omega_j, \omega_{j+1}]\) we have

\[
\frac{1}{2Q_j^+} < \frac{\varepsilon - \varepsilon^{c+1}}{\cos \omega_j} \leq \frac{\varepsilon}{\cos \omega_j} \leq \frac{\varepsilon}{\cos \omega_j} \leq \frac{\varepsilon + \varepsilon^{c+1}}{\cos \omega_j + 1} \leq \frac{1}{2Q_j^-}.
\]

From the definition of \(Q^\pm_j\) and from

\[
|x - y| = |\sin(x - y)| \leq |\tan x - \tan y|, \quad x, y \in [0, \pi/4],
\]

we infer that

\[
Q^+_j - Q^-_j \ll \frac{\cos \omega_j}{2\varepsilon - 2\varepsilon^{c+1}} = \frac{\varepsilon^{c+1}}{\varepsilon} = \frac{\cos \omega_j - \cos \omega_{j+1}}{2\varepsilon} \ll \varepsilon^{c-1}
\]

and

\[
Q^+_j = \frac{\cos \omega_j}{2\varepsilon} + O(\varepsilon^{c-1}).
\]

Remark 2. If \(\omega \in [0, \frac{\pi}{4}]\) and \(\lambda^\pm\) are such that \(\lambda^- < \frac{\cos \omega}{2\varepsilon} < \lambda^+\), then for all \(x \in Y_\varepsilon\) we have

\[
\tilde{\tau}_{1/(2\lambda^+)}(x, \omega) + \varepsilon > \tau_\varepsilon(x, \omega) > \tilde{\tau}_{1/(2\lambda^-)}(x, \omega) - \varepsilon.
\]

This shows in turn that if for each interval \(I = [\tan \omega_0, \tan \omega_1] \subseteq [0, 1]\) we denote

\[
\mathbb{P}_{\varepsilon, I} := \left\{ (x, \omega) : x \in Y_\varepsilon, \tan \omega \in I, \tau_\varepsilon(x, \omega) > \frac{t}{2\varepsilon} \right\},
\]

then for any integers \(Q^\pm\) such that \(Q^- < \frac{\cos \omega_j}{2\varepsilon} < \frac{\cos \omega_{j+1}}{2\varepsilon} < Q^+\) we have

\[
\mathbb{P}_{I, Q^-} \left(\frac{t + \varepsilon}{2\varepsilon}\right) - \pi \varepsilon^2 \leq \mathbb{P}_{\varepsilon, I}(t) \leq \mathbb{P}_{I, Q^+} \left(\frac{t - \varepsilon}{2\varepsilon}\right).
\]

By the previous remark we infer

\[
\mathbb{P}_{I_j, Q^-_j} \left(\frac{t + \varepsilon}{2\varepsilon}\right) - \pi \varepsilon^2 \leq \mathbb{P}_{\varepsilon, I_j}(t) \leq \mathbb{P}_{I_j, Q^+_j} \left(\frac{t - \varepsilon}{2\varepsilon}\right), \quad j = 1, \ldots, N.
\]

For small \(\varepsilon > 0\) we also have

\[
\frac{t}{2\varepsilon + 2\varepsilon^{c+1}} < \frac{t - \varepsilon}{2\varepsilon} < \frac{t + \varepsilon}{2\varepsilon} < \frac{t}{2\varepsilon - 2\varepsilon^{c+1}}.
\]
uniformly in $t$ on compacts of $(0, \infty)$. Thus \((7.4), (7.3),\) and Lemma 5 yield
\[
\mathbb{P}_{\varepsilon,I_j}(t) \leq \mathbb{P} I_j,Q_j^\varepsilon \left( \frac{t}{2\varepsilon + 2\varepsilon^{c+1}} \right) = P_{I_j,Q_j^\varepsilon} \left( \frac{t}{1 + \varepsilon^c} \right) + O(\varepsilon^{2c}) \tag{7.5}
\]
and
\[
\mathbb{P}_{\varepsilon,I_j}(t) \geq \mathbb{P} I_j,Q_j^\varepsilon \left( \frac{t}{2\varepsilon - 2\varepsilon^{c+1}} \right) - \pi \varepsilon^{2} = P_{I_j,Q_j^\varepsilon} \left( \frac{t}{1 - \varepsilon^c} \right) + O(\varepsilon^{2c}). \tag{7.6}
\]

By the definition of $\mathbb{P}$ we see that for any compact interval $K \subset (0, \infty) \setminus \{1, 2\}$, there exists a constant $C_K > 0$ such that
\[
|\mathbb{P}(t_1) - \mathbb{P}(t_2)| \leq C_K|t_1 - t_2|, \quad t_1, t_2 \in K. \tag{7.7}
\]
Now by Propositions 3, 4, 5 we know that for any $j \in \{1, 2, \ldots, N\}$ we have for small $\varepsilon > 0$
\[
P_{I_j,Q_j^\varepsilon} \left( \frac{t}{1 \pm \varepsilon^c} \right) = P_{I_j,Q_j^\varepsilon} (t (1 + O(\varepsilon^c))) = c_{I_j} (\mathbb{P}(t) + O(\varepsilon^c)) + O_\delta(\varepsilon^{2c} + \varepsilon^{c+c_1} + \varepsilon^{1/2-2c_1-\delta}),
\]
uniformly in $t$ on compacts of $(0, \infty) \setminus \{1, 2\}$. Here $\mathbb{P}(t)$ is defined as in Theorem 1.1. Summing over $j$ the inequalities (7.5) and (7.6), and using also
\[
\sum_{j=1}^{N} c_{I_j} = \int_{0}^{1} \frac{du}{1 + u^2} = \frac{\pi}{4}, \tag{7.8}
\]
$N \leq \varepsilon^{-c}$, and (7.7), we gather
\[
\sum_{j=1}^{N} \mathbb{P}_{\varepsilon,I_j}(t) = \frac{\pi}{4} \mathbb{P}(t) + O_\delta(\varepsilon^{c} + \varepsilon^{c_1} + \varepsilon^{1/2-2c_1-c-\delta}).
\]

For obvious symmetry reasons we can only consider $\omega \in [0, \frac{\pi}{4}]$. Thus, after normalizing the Lebesgue measure $\mu_\varepsilon$ on $Y_\varepsilon$ by dividing by $\frac{\pi}{4}$ area($Y_\varepsilon$) = $\frac{\pi(1-\varepsilon^2)}{4}$, we get
\[
\mathbb{P}_\varepsilon(t) = \mathbb{P}(t) + O_\delta(\varepsilon^{c} + \varepsilon^{c_1} + \varepsilon^{1/2-2c_1-c-\delta}).
\]
The proof of Theorem 1 is completed by taking $c = c_1 = \frac{1}{8}$.

8. The geometric free path length in the case $0 < t \leq 1$

In this and the next two sections we shall take $\omega \in [0, \frac{\pi}{4}]$, and analyze the geometric free path length in the case of vertical scatterers of height $2\delta$ centered at integer lattice points. In this setup we will consider the phase space $(\Sigma_{\delta,I}, \frac{d\mu}{\mu})$, where $\delta > 0$, $I = [\tan \omega_0, \tan \omega_1] \subset [0, 1]$ is an interval, $\Sigma_{\delta,I} = [-\delta, \delta] \times [\omega_0, \omega_1]$ and $d\mu$ is the (non-normalized) Lebesgue measure on $\Sigma_{\delta,I}$. The trajectory will
Remark 3. If $I = [\tan \omega_0, \tan \omega_1] \subseteq [0, 1]$ and $0 < \lambda_1 \leq \frac{\cos \omega_0}{2\varepsilon} < \frac{\cos \omega_1}{2\varepsilon} \leq \lambda_+$, then owing to (8.2), (8.3) and to the fact that $x \mapsto \Delta_\lambda(x)$ is monotonically decreasing.
we have
\[ G_{I,Q} \left( \frac{1}{Q} \right) \leq \tilde{G}_{I,Q} \left( \frac{t}{2} \right) \leq \hat{G}_{I,Q} \left( \frac{1}{Q} \right). \]

The argument, based on inequality (8.3) used to compare \( \tilde{F}_{I,Q}(\frac{1}{t}) \) with \( F_{I,Q}(t) \) in Lemma 3, is not going to apply here because \( \Delta_{\lambda} \) is not a Lipschitz function. Nevertheless, we can overcome this problem by appealing again to a soft monotonicity argument, based on Remark 3 and on the fact (which can be seen directly from the definition of the function \( G(t) \)) that for any compact \( K \subset (0, \infty) \setminus \{1, 2\} \), there exists a constant \( C_K > 0 \) such that
\[ |G(t_1) - G(t_2)| \leq C_K |t_1 - t_2|, \quad t_1, t_2 \in K. \tag{8.4} \]

In this and the next two sections we will analyze the asymptotic of the quantity \( G_{I,Q}(t) \) for large integers \( Q \) and short intervals \( I \) such that \( |I| \approx Q^{-1/8} \). We note at this point that the relation (1.3) is hinted by formula (8.3) and by
\[
\frac{d}{dt} \max\{q - tQ, 0\} = -Q \Delta_q(tQ), \quad t \neq \frac{q}{Q}.
\]

For the sake of space, the error estimates which are similar to the ones already derived in the first part of the paper are going to be more sketchy.

**Proposition 6.** For every interval \( I \subseteq [0, 1] \) of size \( |I| \approx Q^{-1/8} \) and every \( \delta > 0 \)
\[ G_{I,Q}(t) = \left( 1 - \frac{t}{\zeta(2)} \right) c_I + O_{\delta}(Q^{-1/4 + \delta}) \quad (Q \to \infty). \]
The estimate holds uniformly in \( t \in (0, 1] \).

**Proof.** Since \( 0 < t \leq 1 \), we have \( \min\{q_k, q_{k}'\} \geq q + q' > tQ \) for all \( k \geq 1 \). Thus we infer from (8.2), as in formula (4.4), that
\[ G_{I,Q}(t) = G^{(1)}_{I,Q}(t) + G^{(2)}_{I,Q}(t) + G^{(3)}_{I,Q}(t), \]
with
\[
G^{(1)}_{I,Q}(t) := Q \sum_{\gamma \in F_Q(t)} \sum_{k=1}^{\infty} \int_{\alpha_k}^{\alpha_{k-1}} (w_{C_k}(\omega) + w_{B_k}(\omega)) \, d\omega + Q \sum_{\gamma \in F_Q(t)} \int_{\beta_0}^{\beta_0} w_{C_0}(\omega) \, d\omega
\]
\[ + Q \sum_{\gamma \in F_Q(t)} \sum_{k=1}^{\infty} \int_{\beta_k}^{\beta_{k-1}} (w_{C_{-k}}(\omega) + w_{A_{-k}}(\omega)) \, d\omega
\]
\[ = Q \sum_{\gamma \in F_Q(t)} \int_{\alpha_0}^{\alpha_0} \left( \frac{1}{Q} - w_{A_+}(\omega) \right) \, d\omega + Q \sum_{\gamma \in F_Q(t)} \int_{\beta_0}^{\beta_0} \left( \frac{1}{Q} - w_{B_-}(\omega) \right) \, d\omega
\]
\[ + Q \sum_{\gamma \in F_Q(t)} \int_{\alpha_0}^{\beta_0} \left( \frac{1}{Q} - w_{A_+}(\omega) - w_{B_-}(\omega) \right) \, d\omega
\]
\[ = \sum_{\gamma \in F_Q(t)} (\beta_0 - \alpha_0) - Q \sum_{\gamma \in F_Q(t)} \int_{\alpha_0}^{\beta_0} w_{A_+}(\omega) \, d\omega - Q \sum_{\gamma \in F_Q(t)} \int_{\alpha_0}^{\beta_0} w_{B_-}(\omega) \, d\omega,
\]
G_{1,Q}^{(2)}(t) := Q \sum_{\gamma \in \mathcal{F}_Q(I)} \sum_{q > tQ} \sum_{q^2 \leq Q} \int_{\alpha_k}^{\alpha_{k-1}} w_{A_+}(\omega) \, d\omega + Q \sum_{\gamma \in \mathcal{F}_Q(I) \cap \alpha_0} \int_{q > tQ} w_{A_+}(\omega) \, d\omega = Q \sum_{\gamma \in \mathcal{F}_Q(I) \cap \alpha_0} \int_{\alpha_\infty}^{\beta_0} w_{A_+}(\omega),

G_{1,Q}^{(3)}(t) := Q \sum_{\gamma \in \mathcal{F}_Q(I)} \sum_{q' > tQ} \sum_{q'^2 \leq Q} \int_{\beta_k}^{\beta_{k-1}} w_{B_-}(\omega) \, d\omega + Q \sum_{\gamma \in \mathcal{F}_Q(I) \cap \gamma_0} \int_{q' > tQ} w_{B_-}(\omega) \, d\omega

= Q \sum_{\gamma \in \mathcal{F}_Q(I) \cap \gamma_0} \int_{\alpha_0}^{\beta_\infty} w_{B_-}(\omega) \, d\omega.

From (4.12) we gather

\sum_{\gamma \in \mathcal{F}_Q(I)} \left( \beta_\infty - \alpha_\infty \right) = \sum_{\gamma \in \mathcal{F}_Q(I)} \left( \arctan \frac{a'}{q'} - \arctan \frac{a}{q} \right) = c_I + O_\delta(Q^{-1/4+\delta}). \quad (8.5)

On the other hand, (4.17) gives

Q \int_{\alpha_\infty}^{\beta_0} w_{A_+}(\omega) \, d\omega = \frac{1}{2qQ(1 + \gamma^2)} + O\left( \frac{1}{q^2Q^2} \right). \quad (8.6)

We can show in a similar way that

Q \int_{\alpha_0}^{\beta_\infty} w_{B_-}(\omega) \, d\omega = \frac{1}{2q'Q(1 + \gamma'^2)} + O\left( \frac{1}{q'^2Q^2} \right). \quad (8.7)

From the formulas for $G_{1,Q}^{(1)}$, $G_{1,Q}^{(2)}$, $G_{1,Q}^{(3)}$ and from (8.5)–(8.7) we infer

$G_{I,Q}(t) = c_I - \sum_{\gamma \in \mathcal{F}_Q(I) \cap \alpha_\infty} \left( \frac{1}{2qQ(1 + \gamma^2)} + O\left( \frac{1}{q^2Q^2} \right) \right) - \sum_{\gamma \in \mathcal{F}_Q(I) \cap \gamma_0} \left( \frac{1}{2q'Q(1 + \gamma'^2)} + O\left( \frac{1}{q'^2Q^2} \right) \right) + O_\delta(Q^{-1/4+\delta})

= c_I - \sum_{\gamma \in \mathcal{F}_Q(I) \cap \alpha_\infty} \frac{1}{2qQ(1 + \gamma^2)} - \sum_{\gamma \in \mathcal{F}_Q(I) \cap \gamma_0} \frac{1}{2q'Q(1 + \gamma'^2)} + O_\delta(Q^{-1/4+\delta}). \quad (8.8)$
Finally we show as at the end of Section 4 that
\[
\sum_{\gamma \in \mathcal{F}_Q(t), q \leq t Q} \frac{1}{2q Q (1 + \gamma^2)} = \frac{1}{2Q} \sum_{1 \leq q \leq t Q} \frac{1}{q} \sum_{Q - q < q' \leq Q \atop a \in \mathcal{I}} \frac{1}{1 + a^2/q^2} = \frac{1}{2Q} \sum_{1 \leq q \leq t Q} \frac{1}{q} \frac{\varphi(q)}{q^2} q^2 c_I + O_\delta(Q^{-1/4+\delta})
\]
\[
= \frac{c_I}{2Q} \sum_{1 \leq q \leq t Q} \frac{\varphi(q)}{q} + O_\delta(Q^{-1/4+\delta})
\]
\[
= \frac{c_I t}{2\zeta(2)} + O_\delta(Q^{-1/4+\delta}).
\]
A similar formula holds for the second sum in (9.8), and therefore we get
\[
G_{I, Q}(t) = c_I - \frac{c_I t}{\zeta(2)} + O_\delta(Q^{-1/4+\delta}).
\]
It is clear that these estimate hold uniformly in \(t \in [0, 1]\). \(\square\)

9. The geometric free path in the case \(t > 2\)

In this section we prove in the setting of Section 8 the following result

**Proposition 7.** For every interval \(I \subseteq [0, 1]\) of size \(|I| = Q^{-\frac{1}{2}}\) and every \(\delta > 0\)
\[
G_{I, Q}(t) = \frac{c_I t}{\zeta(2)} + O_\delta(Q^{-1/4+\delta}) \quad (Q \to \infty).
\]
The estimate holds uniformly in \(t\) on compacts of \([2, \infty)\).

**Proof.** We proceed as in Section 5, estimating first
\[
\tilde{S}_2(\gamma, t) := Q \sum_{k=K+1}^{\infty} \int_{\alpha_k}^{\alpha_{k-1}} (w_{\mathcal{C}_k}(\omega) + w_{\mathcal{B}_k}(\omega)) d\omega
\]
\[
= Q \int_{\alpha_\infty}^{\alpha_K} \left( \frac{1}{Q} - w_{A_\infty}(\omega) \right) d\omega = Q \int_{\arctan \frac{a_{K-1}}{q_k}}^{\arctan \frac{a_{K-1}/Q}{q_k}} (q \tan \omega - a) d\omega \quad (9.1)
\]
and then
\[
\tilde{S}_1(\gamma, t) = Q \int_{\alpha_K}^{\alpha_{K-1}} w_{\mathcal{C}_K}(\omega) d\omega = Q \int_{\arctan \frac{a_{K-1}/Q}{q_{K-1}}}^{\arctan \frac{a_{K-1}-1/Q}{q_{K-1}}} \left( a_{K-1} - q_{K-1} \tan \omega - \frac{1}{Q} \right) d\omega
\]
\[
= \frac{Q(1 - q/Q)^2 q_{K-1}}{2(1 + \gamma^2)^2 q_{K-1}^2 q_{K-1}^2} + O \left( \frac{Q q_{K-1}}{q^2 q_{K-1}^2} \right) = \frac{Q(1 - q/Q)^2}{2(1 + \gamma^2) q_{K-1}^2 q_{K-1}^2} + O \left( \frac{1}{q^2 q_{K-1}^2} \right). \tag{9.2}
\]
In this case we find from (8.3), (9.1) and (9.2), as in Section 5, that
\[ G_{t, Q}(t) = 2 \sum_{\gamma \in \mathcal{F}_{Q}(t)} \frac{Q(1 - q/Q)^2(q_{K-1} + q)}{2(1 + \gamma^2)q_k q_{K-1}^k} + O(Q^{-1} \ln Q) \]
\[ = \sum_{\gamma \in \mathcal{F}_{Q}(t)} \frac{Q(1 - q/Q)^2}{(1 + \gamma^2)q_k q_{K-1}^k} + O(Q^{-1} \ln Q) \]
\[ = \sum_{k=2}^{\infty} \sum_{q \in \mathcal{Q}_{k}} (\overline{S}_k(q) + \overline{T}_k(q)) + O(Q^{-1} \ln Q), \]
with
\[ \overline{S}_k(q) = \sum_{q' \in QJ_{k, q}, a \in \mathcal{I}} f_k(q, q', a), \quad \overline{T}_k(q) = \sum_{q' \in QJ_{k, q}, a \in \mathcal{I}} f_{k-1}(q, q', q - a), \]
\[ f_k(q, q', a) = \frac{Q(1 - q/Q)^2}{(1 + a^2/q^2)q_k q_{K-1}^k}, \quad q' \in \mathcal{I} = QJ_{k, q}, \quad a \in \mathcal{J} = \mathcal{I}Q. \]

Employing the same technique as in Section 5 and the fact that one gets a similar result while integrating between \( \beta_{k-1} \) and \( \beta_k \), we find that \( G_{t, Q}(t) \) can be expressed, up to an error term of order \( O_3(Q^{-1/4+\delta}) \), as
\[ \sum_{k=2}^{\infty} \sum_{q \in \mathcal{Q}_{k}} \frac{\varphi(q)}{q^2} \left( \int \int_{QJ_{k, q} \times \mathcal{Q}_{k}} f_k(q, q', a) \, dq' \, da + \int \int_{QJ_{k, q} \times \mathcal{Q}_{k}} f_{k-1}(q, q', a) \, dq' \, da \right) \]
\[ = c_1 \sum_{k=2}^{\infty} \sum_{q \in \mathcal{Q}_{k}} \frac{\varphi(q)}{q} \left( \frac{(1 - q/Q)^2}{q_k q_{K-1}^k} \left( \int_{Q-q}^{tQ-kq} \frac{dq'}{q_{k-1}q_k} + \int_{Q-kq}^{Q} \frac{dq'}{q_{k-2}q_{k-1}} \right) \right) \]
\[ = c_1 \sum_{k=2}^{\infty} \sum_{q \in \mathcal{Q}_{k}} \frac{\varphi(q)}{q} \left( \frac{(1 - q/Q)^2}{q_k q_{K-1}^k} \left( \int_{Q+(k-1)q}^{tQ} \frac{dy}{y(y-q)} + \int_{Q-kq}^{Q} \frac{dy}{y(y-q)} \right) \right) \]
\[ = c_1 \sum_{k=2}^{\infty} \sum_{q \in \mathcal{Q}_{k}} \frac{\varphi(q)}{q} \left( \frac{(1 - q/Q)^2}{q_k q_{K-1}^k} \ln \frac{(tQ - q)^2}{tQ(tQ - 2q)} \right). \]

This is further equal to
\[ \frac{c_1}{\zeta(2)} \int_{0}^{Q} \frac{Q(1 - q/Q)^2}{q^2} \ln \frac{(tQ - q)^2}{tQ(tQ - 2q)} \, dq = \frac{c_1}{\zeta(2)} \int_{0}^{Q} \frac{Q(1 - q/Q)^2}{q^2} \ln \frac{(tQ - q)^2}{tQ(tQ - 2q)} \, dq \]
\[ = \frac{c_1}{\zeta(2)} \int_{0}^{1} \frac{(1 - x)^2}{x^2} \ln \frac{(t - x)^2}{t(t - 2x)} \, dx, \]
which is the desired conclusion. \( \square \)
10. The geometric free path in the case \(1 < t < 2\)

In this section we prove in the setting of Section 8 the following result

**Proposition 8.** For every interval \(I \subseteq [0, 1]\) of size \(|I| \approx Q^{-\frac12}\) and \(\delta > 0\)

\[
G_{I,Q}(t) = \frac{c_I}{\zeta(2)} \left( \int_{t-1}^{1} \frac{1}{x} \ln \frac{1}{t-x} \, dx - 2 + t + (t-1) \ln \frac{1}{t-1} + \int_{t-1}^{1} \frac{(1-x)^2}{x^2} \ln \frac{t-x}{t(1-x)} \, dx + \int_{0}^{t-1} \frac{(1-x)^2}{x^2} \ln \frac{(t-x)^2}{t(t-2x)} \, dx \right) + O(Q^{-1/4+\delta}) \quad (Q \to \infty).
\]

The estimate holds uniformly in \(t\) on compacts of \((1,2)\).

**Proof.** Since \(1 < t < 2\), we have \(\max\{q,q'\} \leq tQ\) and we infer from (8.2) that

\[
G_{I,Q}(t) = G_{I,Q}^R(t) + G_{I,Q}^C(t),
\]

where \(G_{I,Q}^R(t)\), respectively \(G_{I,Q}^C(t)\), contains the contribution of Farey fractions in \(F_Q(I)\) with \(q + q' > tQ\), respectively with \(q + q' \leq tQ\).

When \(q + q' > tQ\) we have \(\min\{q_k,q_k'\} > tQ, k \geq 1\), and therefore

\[
G_{I,Q}^R(t) := Q \sum_{\gamma \in F_Q(I)} \sum_{k=1}^{\infty} \int_{\alpha_k}^{\alpha_k-1} \left(w_{C_k}(\omega) + w_{B_k}(\omega)\right) d\omega + Q \sum_{\gamma \in F_Q(I)} \sum_{k=1}^{\infty} \int_{\beta_{k-1}}^{\beta_k} \left(w_{C_{-k}}(\omega) + w_{A_{-k}}(\omega)\right) d\omega
\]

\[
= Q \sum_{\gamma \in F_Q(I)} \int_{\alpha_\infty}^{\alpha_0} \left(\frac{1}{Q} - w_{A_+}(\omega)\right) d\omega + Q \sum_{\gamma \in F_Q(I)} \int_{\alpha_0}^{\beta_0} \left(\frac{1}{Q} - w_{A_+}(\omega) - w_{B_-}(\omega)\right) d\omega + Q \sum_{\gamma \in F_Q(I)} \int_{\alpha_0}^{\beta_0} \left(\frac{1}{Q} - w_{B_-}(\omega)\right) d\omega
\]

\[
= \sum_{\gamma \in F_Q(I)} (\beta_\infty - \alpha_\infty) - Q \sum_{\gamma \in F_Q(I)} \int_{\alpha_\infty}^{\alpha_0} w_{A_+}(\omega) \, d\omega - Q \sum_{\gamma \in F_Q(I)} \int_{\alpha_0}^{\beta_0} w_{A_+}(\omega) \, d\omega - Q \sum_{\gamma \in F_Q(I)} \int_{\alpha_0}^{\beta_0} w_{B_-}(\omega) \, d\omega.
\]

Standard considerations as in Sections 6 and 8 show that, uniformly in \(t\) on compacts of \((1,2)\) and up to an error term of order \(O(Q^{-1} \ln Q)\), \(G_{I,Q}^C(t)\) can be expressed as
We conclude the proof by adding the formulas for
\[ G_q \sim O(\gamma^2) + 2q' (1 + \gamma) \]
Next we estimate
\[ G^{\approx}_Q = c \sum_{q < Q} \frac{1}{q} \sum_{Q < q' < Q} \frac{Q - q'}{q' (1 + a^2 / q^2)} \]
Next we estimate
\[ G^{\approx}_Q = \frac{c \zeta(2)}{Q} \int_{(t-1)Q}^Q \frac{1}{q} \int_{Q-q}^Q \frac{Q - q'}{q'} dq' dq + O_\delta(Q^{-1/4 + \delta}) \]
Next we estimate
\[ G^{\approx}_Q(t) := Q \sum_{\gamma \in F_Q(t)} \sum_{k=1}^\infty \int_{\alpha_k}^{\beta_k} (w_{C_k} + w_{B_k}) d\omega \]
and find as in Sections 5, 6 and 9 that \( G^{\approx}_Q(t) \) can be expressed, up to an error term of order \( O_\delta(Q^{-1/4 + \delta}) \), as
\[ 2 \sum_{k=2}^{\infty} \sum_{q \in Q_t_k} (S_k(q) + \tilde{T}_k(q)) + 2 \sum_{q \in Q_{t_1}} \sum_{q' \in Q_{t_2}, a \in Q_{t_3}, -aq' = 1 (mod q)} \tilde{f}_k(g, q', a) \]
We conclude the proof by adding the formulas for \( G^{\approx}_Q(t) \) and \( G^{\approx}_Q(t) \).
Identifying $\Sigma^+ = \{ (\varepsilon e^{i\alpha}, \omega); -\omega - \pi/2 \leq \alpha \leq \omega + \pi/2 \}$ with
$$\left\{ (\varepsilon e^{i(\omega+\beta)}, \omega); \beta \in [-\pi/2, \pi/2] \right\},$$
the (non-normalized) Liouville measure on the phase space $\Sigma^+$ is expressed as
$$d\lambda_\varepsilon = \varepsilon \langle \cos \alpha, \sin \alpha \rangle \langle \cos \omega, \sin \omega \rangle \, d\alpha \, d\omega = \varepsilon \cos(\omega - \alpha) \, d\alpha \, d\omega = \varepsilon \cos \beta \, d\beta \, d\omega.$$

Figure 8. The parametrization of $\Sigma^+$

Next we shall consider a fixed interval $I = [\tan \omega_0, \tan \omega_1] \subseteq [0, 1]$, define
$$\Sigma^{+, \varepsilon}_I := \left\{ (\varepsilon e^{i(\omega+\beta)}, \omega); |\beta| \leq \pi/2, \omega_0 \leq \omega \leq \omega_1 \right\},$$
and estimate
$$G_{\varepsilon, I}(t) := \frac{1}{\varepsilon} \lambda_\varepsilon \left( \left\{ (x, \omega) \in \Sigma^{+, \varepsilon}_I; \tau_\varepsilon(x, \omega) > \frac{t}{2\varepsilon} \right\} \right).$$

To each point $P = \varepsilon e^{i(\omega+\beta)}$ we associate (see Figure 8) the point $P'(0, y)$, where $y = \frac{\varepsilon \sin \beta}{\cos \omega} \in \left[ -\frac{\varepsilon}{\cos \omega}, \frac{\varepsilon}{\cos \omega} \right]$. Note that
$$\lambda_\varepsilon(\Sigma^{+, \varepsilon}_I) = \varepsilon \int_{\omega_0}^{\omega_1} \int_{-\pi/2}^{\pi/2} \cos \beta \, d\beta \, d\omega = 2\varepsilon c_I.$$

Since $PP'$ has slope $\tan \omega$, we have the obvious inequality
$$\left| \tau_\varepsilon(\varepsilon e^{i(\omega+\beta)}, \omega) - \tau_{\varepsilon/\cos \omega}(\varepsilon \frac{\sin \beta}{\cos \omega}, \omega) \right| < 2\varepsilon.$$

and as a consequence we can write
\[ G_{\varepsilon,I}(t) = \int_{\omega_0}^{\omega_1} \int_{-\pi/2}^{\pi/2} e^{i/2} \left( \tau_{\varepsilon} \left( \varepsilon \Omega^{(\omega+\beta)} \right), \omega \right) \cos \beta d\beta d\omega \]
\[ \leq \int_{\omega_0}^{\omega_1} \int_{-\pi/2}^{\pi/2} e^{i/2} \left( \tau_{\varepsilon} / \cos \omega \left( \varepsilon \sin \beta \right) \right) \cos \beta d\beta d\omega \]
\[ = \int_{\omega_0}^{\omega_1} \int_{-\pi/2}^{\pi/2} \frac{\cos \omega}{\varepsilon} e^{i(\pi/2)/(2\varepsilon)} \left( \tau_{\varepsilon} / \cos \omega(y, \varepsilon) \right) dy d\omega. \]

When \( 0 < \lambda_- \leq \frac{\cos \omega_1}{2\varepsilon} < \frac{\cos \omega_0}{2\varepsilon} \leq \lambda_+ \), obvious monotonicity properties yield
\[ G_{\varepsilon,I}(t) \leq 2\lambda_+ \int_{\omega_0}^{\omega_1} \int_{1/(2\lambda_-)}^{1/(2\lambda_+)} e^{i(4\varepsilon^2)/(2\varepsilon)} \left( \tilde{\tau}_{1/(2\lambda_+)}(y, \omega) \right) dy d\omega \]
\[ = 2\lambda_+ \int_{\omega_0}^{\omega_1} \int_{1/(2\lambda_+)}^{1/(2\lambda_-)} e^{i(4\varepsilon^2)/(2\varepsilon)} \left( \tilde{\tau}_{1/(2\lambda_+)}(y, \omega) \right) dy d\omega + O \left( \lambda_+ \left( \frac{1}{\lambda_-} - \frac{1}{\lambda_+} \right) \right) \]
\[ = 2\tilde{G}_{1/(2\lambda_+),I} \left( \frac{t - 4\varepsilon^2}{2\varepsilon} \right) + O \left( \frac{\lambda_+}{\lambda_-} - 1 \right), \quad (11.1) \]

with \( \tilde{G}_{\delta,I} \) as defined in [84]. Using similar arguments we infer
\[ G_{\varepsilon,I}(t) \geq 2\tilde{G}_{1/(2\lambda_-),I} \left( \frac{t + 4\varepsilon^2}{2\varepsilon} \right) + O \left( 1 - \frac{\lambda_-}{\lambda_+} \right). \quad (11.2) \]

Take now \( \varepsilon > 0 \) small, and suppose that \(|I| \approx \varepsilon^{1/8}\) and \( Q^\pm \) are two integers such that
\[ Q^- \leq \frac{\cos \omega_1}{2\varepsilon} \leq \frac{\cos \omega_0}{2\varepsilon} \leq Q^+, \quad Q^\pm = \frac{\cos \omega_0}{2\varepsilon} + O(\varepsilon^{1/8} - 1), \quad \frac{Q^+}{Q^-} = 1 + O(\varepsilon^{1/8}). \]

Such integers can be chosen for instance as at the beginning of Section 7 with \( c = \frac{1}{8} \). Fix also a compact \( K \subset (0, \infty) \setminus \{1, 2\} \). Applying successively [114], Remark 3, Propositions [64][68] and inequality [84], we infer that
\[ G_{\varepsilon,I}(t) \leq 2\tilde{G}_{1/(2Q_+),I} \left( \frac{t - 4\varepsilon^2}{2\varepsilon} \right) + O \left( \frac{Q^+}{Q^-} - 1 \right) \]
\[ = 2\tilde{G}_{I,Q_+} \left( \frac{t - 4\varepsilon^2}{2\varepsilon} \right) + O \left( \frac{Q^+}{Q^-} - 1 \right) \]
\[ = 2G_{I,Q_+} \left( \frac{(t - 4\varepsilon^2)Q^-}{Q^+} \right) + O \left( \frac{Q^+}{Q^-} - 1 \right) \]
\[ = 2G_{I,Q_+} \left( (t - 4\varepsilon^2)(1 + O(\varepsilon^{1/8})) \right) + O(\varepsilon^{1/8}) \]
\[ = 2G_{I,Q_+} \left( t + O(\varepsilon^{1/8}) \right) + O(\varepsilon^{1/8}) \]
\[ = 2c_I G(t) + O_\delta(\varepsilon^{1/8 - \delta}) \quad \text{uniformly in} \ t \in K. \]
In a similar way we infer from (11.2) and the previous arguments that
\[ G_{\varepsilon,I}(t) \geq 2c_I G(t) + O_\delta(\varepsilon^{1/8-\delta}) \] uniformly in \( t \in K \). (11.4)

Consider now a partition of \([0,1]\) with intervals \( \{I_j\}_{j=1}^N \), where \( N = [\varepsilon^{-1/8}] \) and \( |I_j| = \frac{1}{N} \approx \varepsilon^{1/8} \). Summing over \( j \) we find as a result of (11.3), (11.4) and (7.8) that
\[ G_{\varepsilon,[0,1]}(t) = \sum_{j=1}^N G_{\varepsilon,I_j}(t) = \frac{\pi}{2} G(t) + O_\delta(\varepsilon^{1/8-\delta}) \]
and thus
\[ \frac{G_{\varepsilon,[0,1]}(t)}{\lambda_c(\Sigma_{\varepsilon,[0,1]}^+)} = \frac{\lambda_c(\{(x,\omega) \in \Sigma_{\varepsilon,[0,1]}^+; 2\varepsilon\tau(\omega) > t\})}{\lambda_c(\Sigma_{\varepsilon,[0,1]}^+)} = \frac{\pi}{2} G(t) + O_\delta(\varepsilon^{1/8-\delta}) \]

For obvious symmetry reasons we can only consider \( \omega \in [0,\frac{\pi}{4}] \), therefore
\[ G_{\varepsilon}(t) = G(t) + O_\delta(\varepsilon^{1/8-\delta}) \]
which ends the proof of Theorem 2.

12. Estimates of \( C_\varepsilon = \ln(\tau_\varepsilon) - \langle \ln \tau_\varepsilon \rangle \)

In this section we prove Theorem 3(i). Part (ii) then follows from (i) and from relation (2.8) in [13].

We consider the probability measures \( \nu_0 \) and \( \bar{\nu}_\varepsilon \) on \([0,\infty)\) defined by
\[ \int_0^\infty f(u) \, d\nu_0(u) = \int_0^\infty f(u)g(u) \, du, \]
\[ \int_0^\infty f(u) \, d\bar{\nu}_\varepsilon(u) = \int_{\Sigma_{\varepsilon}^+} f(2\varepsilon\tau) \, d\nu_\varepsilon, \quad f \in C_c([0,\infty)). \]

As a result of Theorem 2
\[ \lim_{\varepsilon \to 0^+} \int_t^\infty d\bar{\nu}_\varepsilon(u) = \int_t^\infty d\nu_0(u), \quad t > 0, \]
which implies
\[ \lim_{\varepsilon \to 0^+} \int_0^\infty f(u) \, d\bar{\nu}_\varepsilon(u) = \int_0^\infty f(u) \, d\nu_0(u), \quad f \in C_c([0,\infty)), \]
meaning that \( \bar{\nu}_\varepsilon \to \nu_0 \) vaguely as \( \varepsilon \to 0^+ \). Since
\[ \lim_{\varepsilon \to 0^+} \frac{1}{x} \int_x^\infty d\bar{\nu}_\varepsilon(u) = \frac{1}{x} \int_x^\infty d\nu_0(u), \quad x \geq 1, \]
and the map
\[ x \mapsto \frac{1}{x} \int_x^\infty d\nu_0(u) = \frac{1}{x} \int_x^\infty g(u) \, du \]
The periodic Lorentz gas in the small-scatterer limit

belongs to $L^1([1, \infty), dx)$ because $g(u) = O(u^{-3}), u \geq 1$, the Lebesgue Dominated Convergence theorem yields

$$\lim_{\varepsilon \to 0^+} \int_1^\infty \frac{1}{x} \int_x^\infty d\nu_\varepsilon(u) \ dx = \int_1^\infty \frac{1}{x} \int_x^\infty d\nu_0(u) \ dx < \infty.$$ 

Using Fubini’s theorem, these double integrals can also be expressed as

$$\int_1^\infty \int_1^\infty \frac{1}{x} e_{[1,u]}(x) d\nu_\varepsilon(u) \ dx = \int_1^\infty \int_1^\infty \frac{1}{x} e_{[1,u]}(x) d\nu_\varepsilon(u) = \int_1^\infty \int_1^u \frac{dx}{x} d\nu_\varepsilon(u) = \int_1^\infty \ln u \ d\nu_\varepsilon(u),$$

and respectively as

$$\int_1^\infty \int_1^\infty \frac{1}{x} e_{[1,u]}(x) d\nu_0(u) \ dx = \int_1^\infty \ln u \ d\nu_0(u).$$

It follows that for any (small) $\varepsilon > 0$

$$\int_1^\infty \ln u \ d\nu_\varepsilon(u) < \infty,$$ \hspace{1cm} (12.1)

and also that

$$\lim_{\varepsilon \to 0^+} \int_1^\infty \ln u \ d\nu_\varepsilon(u) \ du = \int_1^\infty g(u) \ln u \ du.$$ \hspace{1cm} (12.2)

We show in a similar way that

$$\lim_{\varepsilon \to 0^+} \int_0^1 \ln u \ d\nu_\varepsilon(u) = \int_0^1 g(u) \ln u \ du = \frac{6}{\pi^2} \int_0^1 \ln u \ du = -\frac{6}{\pi^2}$$

by using Fubini’s theorem which gives in turn

$$\int_0^1 \ln u \ d\nu_\varepsilon(u) = -\int_0^1 \int_u^1 \frac{1}{x} dx \ d\nu_\varepsilon(u) = -\int_0^1 \int_0^1 \frac{1}{x} e_{[u,1]}(x) dx \ d\nu_\varepsilon(u)$$

$$= -\int_0^1 \int_0^1 \frac{1}{x} e_{[u,1]}(x) d\nu_\varepsilon(u) dx = -\int_0^1 \int_0^x d\nu_\varepsilon(u) dx.$$

By (12.1) and (12.2) we get

$$-C := \int_0^\infty g(u) \ln u \ du = \lim_{\varepsilon \to 0^+} \int_0^\infty \ln u \ d\nu_\varepsilon(u) = \lim_{\varepsilon \to 0^+} \int_{\Sigma^+} \ln(2\varepsilon \tau_\varepsilon) \ d\nu_\varepsilon$$

$$= \ln 2 + \lim_{\varepsilon \to 0^+} \left( \ln \varepsilon + \int_{\Sigma^+} \ln \tau_\varepsilon \ d\nu_\varepsilon \right).$$

Since (12.2) yields

$$\lim_{\varepsilon \to 0^+} \left( \ln \int_{\Sigma^+} \tau_\varepsilon \ d\nu_\varepsilon + \ln \varepsilon + \ln 2 \right) = \lim_{\varepsilon \to 0^+} \left( \ln \int_{\Sigma^+} \tau_\varepsilon \ d\nu_\varepsilon - \ln \frac{1}{2\varepsilon} \right) = 0,$$

we collect

$$\lim_{\varepsilon \to 0^+} \left( \ln \int_{\Sigma^+} \tau_\varepsilon \ d\nu_\varepsilon - \int_{\Sigma^+} \ln \tau_\varepsilon \ d\nu_\varepsilon \right) = C.$$
Finally we outline the proof of the identity
\[ C = 3 \ln 2 - \frac{9 \zeta(3)}{4 \zeta(2)}. \]  
(12.3)

First we note that
\[ \int_0^1 g(t) \ln t \, dt = \frac{6}{\pi^2} \int_0^1 \ln t \, dt = -\frac{6}{\pi^2}, \]
so we may write
\[ -C = -\frac{6}{\pi^2} + C_1 + C_2 + C_3, \]  
(12.4)

where
\[ C_1 = \frac{6}{\pi^2} \int_1^\infty \left( \frac{2}{t} + 2 \left( 1 - \frac{1}{t} \right)^2 \ln \left( 1 - \frac{1}{t} \right) \right) \ln t \, dt, \]
\[ C_2 = \frac{6}{\pi^2} \int_1^2 \left( -\frac{1}{t} - \frac{1}{2} \left( 1 - \frac{2}{t} \right)^2 \ln \left( \frac{2}{t} - 1 \right) \right) \ln t \, dt, \]
\[ C_3 = \frac{6}{\pi^2} \int_2^\infty \left( -\frac{1}{t} - \frac{1}{2} \left( 1 - \frac{2}{t} \right)^2 \ln \left( 1 - \frac{2}{t} \right) \right) \ln t \, dt. \]

The substitution \( t = 2u \) leads to
\[ C_3 = -\frac{1}{2} C_1 - \frac{6}{\pi^2} \ln 2 \int_1^\infty \left( \frac{1}{u} + \left( 1 - \frac{1}{u} \right)^2 \ln \left( 1 - \frac{1}{u} \right) \right) \, du. \]

By a direct computation, the integral above is equal to \( 2(\frac{\pi^2}{6} - 1) \), thus
\[ C_3 = -\frac{C_1}{2} - \left( \frac{6}{\pi^2} \ln 2 \right) 2 \left( \frac{\pi^2}{6} - 1 \right) = -\frac{C_1}{2} - (2 \ln 2) \left( 1 - \frac{6}{\pi^2} \right). \]  
(12.5)

Next, a direct computation shows that
\[ C_1 = \frac{12}{\pi^2} (2 \zeta(3) - 1). \]  
(12.6)

The relations \( 12.4 - 12.6 \) provide
\[ -C = C_2 + \frac{12}{\pi^2} (2 \zeta(3)) - \frac{12}{\pi^2} \frac{1}{2} \left( 1 - \frac{6}{\pi^2} \right) \ln 2. \]  
(12.7)

But
\[ C_2 = -\frac{6}{\pi^2} \cdot \frac{\ln^2 2}{2} - \frac{3}{\pi^2} \int_1^2 \left( 1 - \frac{2}{t} \right)^2 \ln \left( \frac{2}{t} - 1 \right) \ln t \, dt, \]
thus we get
\[ -C = \frac{12}{\pi^2} \zeta(3) - \frac{12}{\pi^2} \frac{1}{2} \left( 1 - \frac{6}{\pi^2} \right) \ln 2 - \frac{6}{\pi^2} \cdot \frac{\ln^2 2}{2} - \frac{3}{\pi^2} C_4, \]  
(12.8)

with
\[ C_4 = \int_1^2 \left( 1 - \frac{2}{t} \right)^2 \ln \left( \frac{2}{t} - 1 \right) \ln t \, dt = C_5 - C_6 + C_7. \]
where
\[ C_5 = \ln 2 \int_1^2 \left(1 - \frac{2}{t}\right)^2 \ln t \, dt, \quad C_6 = \int_1^2 \left(1 - \frac{2}{t}\right)^2 \ln^2 t \, dt, \]
\[ C_7 = \int_1^2 \left(1 - \frac{2}{t}\right)^2 \ln \left(1 - \frac{t}{2}\right) \ln t \, dt. \]

By a direct computation we find
\[ C_5 = \ln 2 - 2 \frac{\ln 2}{3}, \quad C_6 = 6 - 8 \ln 2 - \frac{4}{3} \ln^3 2. \]

As a result we gather
\[ C_4 = \ln 2 - 2 \ln^2 2 - 6 + 8 \ln 2 + \frac{4}{3} \ln^3 2 + C_7, \]
and so
\[ -C = \frac{12}{\pi^2} \zeta(3) - \frac{12}{\pi^2} \ln 2 + \frac{12 \ln 2}{\pi^2} - \frac{3}{\pi^2} \ln^2 2 - \frac{3}{\pi^2} \ln 2 + \frac{6}{\pi^2} \ln^3 2 \]
\[ + \frac{18}{\pi^2} - \frac{24 \ln 2}{\pi^2} - \frac{4}{\pi^2} \ln^3 2 - \frac{3}{\pi^2} C_7 \]
\[ = \frac{12}{\pi^2} \zeta(3) + \frac{6}{\pi^2} - 2 \ln 2 - \frac{15}{\pi^2} \ln 2 - \frac{3}{\pi^2} \ln^2 2 + \frac{2}{\pi^2} \ln^3 2 - \frac{3}{\pi^2} C_7. \]

(12.9)

By a careful computation we find
\[ C_7 = -\ln^2 2 - 5 \ln 2 + \frac{2 \pi^2 \ln 2}{3} + 4 \text{Li}_3 \left(\frac{1}{2}\right) - 4 \zeta(3) + 2, \]
where \( \text{Li}_3 \) denotes the trilogarithm function
\[ \text{Li}_3(z) = \sum_{m=1}^{\infty} \frac{z^m}{m^3}, \quad |z| \leq 1. \]

Using the equality (cf. [30, formula (6.12)])
\[ \text{Li}_3 \left(\frac{1}{2}\right) = \frac{7}{8} \zeta(3) - \frac{\pi^2}{12} \ln 2 + \frac{\ln^2 2}{6} \]
we infer
\[ C_7 = -\ln^2 2 - 5 \ln 2 + \frac{\pi^2}{3} \ln 2 - \frac{\zeta(3)}{2} + \frac{2}{3} \ln^3 2 + 2. \]

Inserting this back into (12.9) we finally find
\[ -C = \frac{12}{\pi^2} \zeta(3) + \frac{6}{\pi^2} - 2 \ln 2 - \frac{15}{\pi^2} \ln 2 - \frac{3}{\pi^2} \ln^2 2 + \frac{2}{\pi^2} \ln^3 2 + \frac{3}{\pi^2} \ln^2 2 \]
\[ + \frac{15}{\pi^2} \ln 2 - \ln 2 + \frac{3}{2 \pi^2} \zeta(3) - \frac{2}{\pi^2} \ln^3 2 - \frac{6}{\pi^2} \]
\[ = -3 \ln 2 + \frac{27}{2 \pi^2} \zeta(3) = -3 \ln 2 + \frac{9 \zeta(3)}{4 \zeta(2)}. \]
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