Optimal control of inverted pendulum on cart system

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Abstract. In this paper, a PID and PID-like fuzzy logic controller is designed for optimal control of inverted pendulum. An inverted pendulum is highly a nonlinear and unstable system. Thus, in this paper, the system is modeled, linearized and controlled. The control objective is to keep the pendulum in an upright position despite external disturbances. The stability analysis of the system is performed, which proves the instability of the system. Two separate controllers were developed to cater for the nonlinear behavior of the system. In the PID controller, the PID gains were fine-tuned through trial and error in MATLAB to obtain optimal control of the system. On the other hand, fuzzy logic is used to tune the PID in the PID-like fuzzy logic controller. Control performances of the system by PID and PID-like fuzzy control method were compared. The result indicated PID-like fuzzy control method performs better in respect of rise time, settling time, peak overshoot and steady-state error.

1. Introduction

An inverted pendulum is a nonlinear and unstable system which is very difficult to control; thus, researchers pose attention in that domain. According to [1], inverted pendulum control is grouped into three parts. The first part that received more interest by researchers is the swing-up inverted pendulum control. The second part is the inverted pendulum stabilization. The third one is tracking control of the inverted pendulum. In real sense, stabilization and tracking control are more suitable for the application. Control of inverted pendulum mounted on a motor-driven cart is highly researched because it has a similar model with an automatic aircraft landing system, satellite stabilization of a cabin in a ship, and that of the attitude control of a space booster on takeoff [2]. The overall system is highly unstable in such a way that it may fall over at any time in any direction unless an appropriate control force is applied.

Due to the dynamic nature of the system, and its wide range of application in the control world, researchers investigate the behavior of the system under different types of controllers. In control generally, a proportional-derivative (PD-type) controller can guarantee fast rise time, minimum overshoot, and shorter settling time but cannot eliminate the steady-state error. On the other hand, proportional-integral (PI-type) controller can reduce steady-state error but due to an integral part in the control variable, the response gets slower [3]. PID controllers are generally applied as part of the process control framework for improved control performance. Thus, this research aimed at using PID to control the pendulum angle and cart position of an inverted pendulum. In an attempt to obtain a better control performance in respect of rise time, settling time, peak-overshoot and steady-state error, a proportional-integral-derivative (PID)-like fuzzy logic controller (FLC) are further proposed. The fuzzy logic system is employed in this research because of its proven effectiveness in control systems. It provides an easier way to draw definite conclusions from uncertain, ambiguous or imprecise information [4]. Therefore, this study attempts to use the power of the fuzzy logic system to fine-tune the PID gains in controlling the pendulum and cart position of an inverted pendulum.
The rest of the paper is organized as follows. Modeling of the system is presented in section II, stability test of the system in section III, while controllers design is presented in section IV. Section V gives the results of the findings, while section VI concludes the paper.

2. Modeling of inverted pendulum-cart system

The inverted pendulum placed on a moving cart is shown in Figure 1. To model the system and linearize it, the following assumptions are made [5, 6]: (i) The pendulum rod is mass-less; thus, the pendulum mass \( m_p \) is concentrated at the center of gravity of the pendulum which is located at pendulum ball’s center, (ii) the hinge to which the pendulum is fixed at point \( P \) is frictionless, (iii) the system starts at equilibrium state i.e. the initial conditions are assumed to be zero, (iv) The angle to which the pendulum made with the vertical is very small such that \( \theta \approx 0 \)

The center of gravity of the pendulum is expressed as in Equation (1) and Equation (2).

\[
x_G = x + l \sin \theta \\
y_G = l \cos \theta
\]

(1)

(2)

The horizontal motion of the pendulum and cart is represented by Equation (3) and Equation (4) respectively while the vertical motion of the pendulum is described by Equation (5).

\[
m_p \frac{d^2(x+l \sin \theta)}{dt^2} = F_x
\]

(3)

\[
M_x \frac{d^2x}{dt^2} = u - F_x
\]

(4)

\[
m_p \frac{d^2(l \sin \theta)}{dt^2} = F_y - m_p g
\]

(5)

where \( F_x = m_p \frac{d^2x_G}{dt^2}, \quad F_y = m_p \frac{d^2y_G}{dt^2} \) are forces acting on the pendulum along x and y-direction respectively.

Since we assume \( \theta \) to be very small, \( \sin \theta \approx \theta \) and \( \cos \theta \approx 1 \), thus Equation (3) and Equation (4) reduces to Equation (6) and Equation (7) respectively.

![Figure 1. Inverted pendulum-cart system](image-url)
\[ m_p x + ml \dot{\theta} = F_x \]  
\[ M_c x = u - F_x \]  
\( m \)  

Substituting Equation (6) into Equation (7), leads to Equation (8).

\[
(M_c + m_p) x + ml \dot{\theta} = u
\]

The rotational motion of the rod about its center of gravity is described by Equation (9).

\[
I \ddot{\theta} = F_l l \sin \theta - F_l l \cos \theta
\]

The assumption is that the rod is massless \[7\]; thus its moment of inertia about the center of gravity \( I = 0 \) Thus, Equation (9) reduces to Equation (10) after substituting Equation (6).

\[
ml^2 \ddot{\theta} + ml \dot{x} = mgl \theta
\]

Eliminating \( \dot{x} \) in Equation (8) and \( \ddot{\theta} \) in Equation (10) leads to Equation (11) and Equation (12) respectively.

\[
M_c \dot{\theta} = (M_c + m_p) g \theta - u
\]

\[
M_c \ddot{x} = u - m_p g \theta
\]

The state variables are defined as in Equation (13):

\[
x_1 = \theta, \quad x_2 = \dot{\theta}, \quad x_3 = x \text{ and } x_4 = \dot{x}
\]

where \( \theta \) indicates the rotation of the pendulum about point P and \( x \) is the location of the cart; they also represent the output of the system.

The state-space of the system is described by Equation (14).

\[
\begin{bmatrix}
\theta \\
\dot{\theta} \\
x \\
\dot{x}
\end{bmatrix} =
\begin{bmatrix}
0 & 1 & 0 & 0 \\
\frac{M_c + m_p}{M_c l} g & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
\frac{m_p}{M_c} g & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\theta \\
\dot{\theta} \\
x \\
\dot{x}
\end{bmatrix} +
\begin{bmatrix}
0 \\
1 \\
0 \\
\frac{1}{M_c}
\end{bmatrix} u
\]

The output is described by Equation (15).

\[
\begin{bmatrix}
\theta \\
x
\end{bmatrix} =
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
\theta \\
\dot{\theta} \\
x \\
\dot{x}
\end{bmatrix}
\]
Table 1 shows the parameter values for inverted pendulum on the cart system used in this paper.

| Parameters                          | Symbol | Value   |
|-------------------------------------|--------|---------|
| Mass of the cart                    | $M_c$  | 2.4kg   |
| Mass of the pendulum                | $m_p$  | 0.23kg  |
| Bar length of the pendulum          | $l$    | 0.38m   |
| Path length                         | $L$    | 0.5m    |
| Acceleration due to gravity         | $g$    | 9.81m/s²|

3. **Stability test of the system**

The position of poles and zeros in an open-loop configuration of the linearized inverted pendulum model indicates that the model is unstable because there exists a pole/zero on the right-hand side of the S-plane. Figure 2 and Figure 3 shows the root locus plots for the pendulum and cart position respectively, which clearly indicates the instability of the system.

![Figure 2. Poles and Zeros of pendulum position](image1)

![Figure 3. Poles and Zeros of cart position](image2)
4. Controller design
To design the PID controller for a system that has poles and zeros, as shown in Figure 2 and Figure 3, the gains to be selected has to be positive so as to make the system stable.

4.1. PID controller
Two PID controllers are designed in this section. The first PID controller regulates the pendulum position while the other one takes care of the cart position. The PID equations for the pendulum and cart position are described by Equation (16) and Equation (17) respectively.

\[ u_p = K_{pe}e_\theta(t) + K_{ip}\int e_\theta(t)dt + K_{dp}\frac{de_\theta(t)}{dt} \]  \hspace{1cm} (16)

\[ u_c = K_{ce}e_x(t) + K_{ic}\int e_x(t)dt + K_{dc}\frac{de_x(t)}{dt} \]  \hspace{1cm} (17)

where \( u_p \) and \( u_c \) represents pendulum angle control signal and cart position control signal respectively while \( e_\theta(t) \) and \( e_x(t) \) represents angle error of the pendulum and position error of the cart respectively [8]. The PID gains are tuned in MATLAB through trial and error to obtain optimal control.

4.2. PID-like fuzzy logic controller
In this section, a proportional-integral-derivative (PID)-like Fuzzy Logic controller is developed. The basic idea of a principle behind PID-like FLC controller is to select a control law by considering the error \( e \), change of error \( \Delta e \) and integral of error \( \int e dt \) (or the sum of error, \( \sum e \)). The control law is the same with that described in equation (16) and equation (17) which has three inputs (error, change of error and integral of error). With these inputs, the number of linguistic rules to cater for all the possible input variations will be too much. For instance, if the number of membership function for each input is five, there will be a total of 5x5x5=125 linguistic rules. In real applications, the design and implementation of such a large number of rules is a tedious task, and it will consume a considerable amount of memory space and reasoning time [3]. Due to the long reasoning time, the response of such a type of controller will be too slow and therefore not appropriate for applications where a fast response is desired. In lieu of that, a fuzzy PD controller is designed separately while the integral action is realized by placing a conventional integral controller in parallel with the fuzzy PD controller. The block diagram of the control architecture is shown in Figure 4.

![Figure 4. Architecture of PID-like fuzzy logic controller.](image)

In the fuzzy PD controller designed, five membership functions (MFs) are selected for each of the two inputs (error and change in error), and similarly, five membership functions are selected for the output which resulted to twenty-five distinct rules. Triangular membership is chosen for this design because of its effectiveness and simplicity of implementation [3]. The rules are generated as shown in...
Table II for angle control of the pendulum. Similar rules are generated for position control of the cart system. The MFs are defined as negative big (NB), negative small (NS), zero (ZO), positive small (PS), and positive big (PB) for all inputs and outputs. Figure 5 shows the triangularly shaped membership functions for all the inputs and the output in pendulum angle control, and similar plots are also obtained for cart position control.

| e(t) | ∆e(t) | NB  | NS  | ZO  | PS  | PB  |
|------|-------|-----|-----|-----|-----|-----|
| NB   | NB    | NB  | NS  | ZO  | NS  | ZO  |
| NS   | NB    | NS  | ZO  | PS  | PS  | PS  |
| ZO   | NS    | NS  | ZO  | PS  | PS  | PS  |
| PS   | NS    | ZO  | PS  | PS  | PS  | PS  |
| PB   | ZO    | PS  | PS  | PS  | PB  | PB  |

Table 2. Rule base for the angle control of the inverted pendulum

![Membership function plots](image1)

(a)

![Membership function plots](image2)

(b)

![Membership function plots](image3)

(c)

Figure 5. Input/output Triangular MF plot for 5x5 FLC. (a) - Input 1 (error) MF plot. (b) - Input 2 (change in error) MF plot. (c) - Output (change in pendulum angle) MF plot.

5. Results

Figure 6 shows the PID and PID-like fuzzy logic controller response for pendulum position control. Both controllers performed to expectation because the system settles at less than 2.5 seconds. PID-like fuzzy performs better because its settling time is less compared to that of the PID controller. Figure 7 indicates the control response when the cart position is considered as the output of the system. Similarly, the PID-like fuzzy logic controller indicates a better control performance compared to the PID controller.
6. Conclusion
In this paper, the PID controller and PID-like fuzzy logic controller are developed to stabilize inverted pendulum on the cart system. Time response analysis of the controllers was performed; the PID-like fuzzy controller indicates better control performance in controlling the pendulum angle and cart position. Fuzzy logic is a good tool for handling highly non-linear and unstable systems.

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