FOUR LECTURES ON SECANT VARIETIES

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Abstract. This paper is based on the first author’s lectures at the 2012 University of Regina Workshop “Connections Between Algebra and Geometry”. Its aim is to provide an introduction to the theory of higher secant varieties and their applications. Several references and solved exercises are also included.

1. Introduction

Secant varieties have travelled a long way from 19th century geometry to nowadays where they are as popular as ever before. There are different reasons for this popularity, but they can summarized in one word: applications. These applications are both pure and applied in nature. Indeed, not only does the geometry of secant varieties play a role in the study projections of a curve, a surface or a threefold, but it also in locating a transmitting antenna [Com00].

In these lectures we introduce the reader to the study of (higher) secant varieties by providing the very basic definition and properties, and then moving the most direct applications. In this way we introduce the tools and techniques which are central for any further study in the topic. In the last lecture we present some more advanced material and provide pointers to some relevant literature. Several exercises are included and they are meant to be a way for the reader to familiarize himself or herself with the main ideas presented in the lectures. So, have fun with secant varieties and their many applications!

The paper is structured as follows. In Section 2 we provide the basic definition and properties of higher secant varieties. In particular, we introduce one of the basic result in the theory, namely Terracini’s Lemma, and one main source of examples and problems, namely Veronese varieties. In Section 3 we introduce Waring problems and we explore the connections with higher secant varieties of Veronese varieties. Specifically, we review the basic result by Alexander and Hirschowitz. In Section 4, Apolarity Theory makes in its appearance with the Apolarity Lemma. We see how to use Hilbert functions and sets of points to investigate Waring problems and higher secant varieties to Veronese varieties. In Section 5 we give pointers to the literature giving reference to the topics we treated in the paper. We also provide a brief description and references for the many relevant topics which we were not able to include in the lectures because of time constraints. Finally, we provide solutions to the exercises in Section 6.

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2. Lecture One

In what follows, \( X \subset \mathbb{P}^N \) will denote an irreducible, reduced algebraic variety; we work over an algebraically closed field of characteristic zero, which we assume to be \( \mathbb{C} \).

The topic of this lecture are higher secant varieties of \( X \).

**Definition 2.1.** The \( s \)-th higher secant variety of \( X \) is

\[
\sigma_s(X) = \overline{\bigcup_{P_1, \ldots, P_s \in X} \langle P_1, \ldots, P_s \rangle},
\]

where the over bar denotes the Zariski closure.

In words, \( \sigma_s(X) \) is the closure of the union of \( s \)-secant spaces to \( X \).

**Example 2.2.** If \( X \subset \mathbb{P}^2 \) is a curve and not a line then \( \sigma_2(X) = \mathbb{P}^2 \), the same is true for hypersurfaces which are not hyperplanes. But, if \( X \subset \mathbb{P}^3 \) is a non-degenerate curve (i.e. not contained in a hyperplane), then \( \sigma_2(X) \) can be, in principle, either a surface or a threefold.

We note that the closure operation is in general necessary, but there are cases in which it is not.

**Exercise 2.3.** Show that the union of chords (secant lines) to a plane conic is closed. However, the union of the chords of the twisted cubic curve in \( \mathbb{P}^3 \) is not.

In general, we have a sequence of inclusions

\[
X = \sigma_1(X) \subseteq \sigma_2(X) \subseteq \ldots \subseteq \sigma_r(X) \subseteq \ldots \subseteq \mathbb{P}^N.
\]

If \( X \) is a linear space, then \( \sigma_i(X) = X \) for all \( i \) and all of the elements of the sequence are equal.

**Remark 2.4.** If \( X = \sigma_2(X) \) then \( X \) is a linear space. To see this consider a point \( P \in X \) and the projection map \( \pi_P : \mathbb{P}^N \dashrightarrow \mathbb{P}^{N-1} \). Let \( X_1 = \pi_P(X) \) and notice that \( \dim X_1 = \dim X - 1 \) and that \( \sigma_2(X_1) = X_1 \). If \( X_1 \) is a linear space also \( X \) is so and we are done. Otherwise iterate the process constructing a sequence of varieties \( X_2, \ldots, X_m \) of decreasing dimension. The process will end with \( X_m \) equal to a point and then \( X_{m-1} \) a linear space. Thus \( X_{m-2} \) is a linear space and so on up to the original variety \( X \).

**Exercise 2.5.** For \( X \subset \mathbb{P}^N \), show that, if \( \sigma_i(X) = \sigma_{i+1}(X) \neq \mathbb{P}^N \), then \( \sigma_i(X) \) is a linear space and hence \( \sigma_j(X) = \sigma_i(X) \) for all \( j \geq i \).

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\(^1\)Some authors use the notation \( S(X) \) for the (first) secant variety of \( X \), which corresponds to \( \sigma_2(X) \), and \( S_k(X) \) to denote the \( k \)-th secant variety to \( X \), which corresponds to \( \sigma_{k+1}(X) \). We prefer to reference the number of points used rather than the dimension of their span because this is often more relevant for applications because of its connection to rank.
Using this remark and Exercise 2.5, we can refine our chain of inclusions for $X$ a non degenerate variety (i.e. not contained in a hyperplane).

**Exercise 2.6.** If $X \subseteq \mathbb{P}^N$ is non-degenerate, then there exists an $r \geq 1$ with the property that
\[
X = \sigma_1(X) \subset \sigma_2(X) \subset \ldots \subset \sigma_r(X) = \mathbb{P}^N.
\]
In particular, all inclusions are strict and there is a higher secant variety that coincides with the ambient space.

It is natural to ask: what is the smallest $r$ such that $\sigma_r(X) = \mathbb{P}^N$? Or more generally: what is the value of $\dim \sigma_i(X)$ for all $i$?

As a preliminary move in this direction, we notice that there is an expected value for the dimension of any higher secant variety of $X$ that arises just from the naive dimension count. That is to say, if the secant variety doesn’t fill the ambient space, a point is obtained by choosing $s$ points from an $n$-dimensional variety and one point in the $\mathbb{P}^{s-1}$ that they span.

**Definition 2.7.** For $X \subset \mathbb{P}^N$, set $n = \dim X$. The *expected dimension* of $\sigma_s(X)$ is
\[
\text{expdim}(\sigma_s(X)) = \min\{sn + s - 1, N\}.
\]
Notice also that the expected dimension is also the maximum dimension of the secant variety. Moreover, if the secant line variety $\sigma_2(X)$ does not fill the ambient $\mathbb{P}^N$, then $X$ can be isomorphically projected into a $\mathbb{P}^{N-1}$. This interest in minimal codimension embeddings is one reason secants were classically studied.

**Exercise 2.8.** Let $X \subseteq \mathbb{P}^n$ be a curve. Prove that $\sigma_2(X)$ has dimension 3 unless $X$ is contained in a plane. (This is why every curve is isomorphic to a space curve but only birational to a plane curve.)

There are cases in which $\text{expdim}(\sigma_i(X)) \neq \dim(\sigma_i(X))$ and these motivate the following

**Definition 2.9.** If $\text{expdim}(\sigma_i(X)) \neq \dim(\sigma_i(X))$ then $X$ is said to be $i$-defective or simply defective.

**Remark 2.10.** Notice that $\dim(\sigma_{i+1}(X)) \leq \dim(\sigma_i(X)) + n + 1$, where $n = \dim X$. This means that if $\sigma_i(X) \neq \mathbb{P}^N$ and $X$ is $i$-defective, then $X$ is $j$-defective for $j \leq i$.

Let’s now see the most celebrated example of a defective variety, the Veronese surface in $\mathbb{P}^5$.

**Example 2.11.** Consider the polynomial ring $S = \mathbb{C}[x, y, z]$ and its homogeneous pieces $S_d$. The Veronese map $\nu_2$ is the morphism
\[
\nu_2 : \mathbb{P}(S_1) \rightarrow \mathbb{P}(S_2) \text{ defined by } [L] \mapsto [L^2].
\]
In coordinates this map can be described in terms of the standard monomial basis $\langle x, y, z \rangle$ for $S_1$ and the standard monomial basis $\langle x^2, 2xy, 2xz, y^2, 2yz, z^2 \rangle$ for $S_2$. Thus the Veronese map can be written as the map
\[
\nu_2 : \mathbb{P}^2 \rightarrow \mathbb{P}^5 \text{ defined by } [a : b : c] \mapsto [a^2 : ab : ac : b^2 : bc : c^2].
\]
The Veronese surface is then defined as the image of this map, i.e. the Veronese surface is $X = \nu_2(\mathbb{P}^2) \subseteq \mathbb{P}^5$. 
We now want to study the higher secant varieties of the Veronese surface $X$. In particular we ask: is $\dim \sigma_2(X) = \expdim(\sigma_2(X)) = 5$? In other words, does $\sigma_2(X)$ equal $\mathbb{P}^5$?

To answer this question, it is useful to notice that elements in $S_2$ are quadratic forms and, hence, are uniquely determined by $3 \times 3$ symmetric matrices. In particular, $P \in \mathbb{P}^5$ can be seen as $P = [Q]$ where $Q$ is a $3 \times 3$ symmetric matrix. If $P \in X$ then $Q$ also has rank equal one. Thus we have,

$$\sigma_2(X) = \bigcup_{P_1, P_2} \langle P_1, P_2 \rangle$$

$$= \{ [Q_1 + Q_2] : Q_1 \text{ is a } 3 \times 3 \text{ symmetric matrix and } \text{rk}(Q_1) = 1 \} \subseteq H,$$

where $H$ is the projective variety determined by the set of $3 \times 3$ symmetric matrices of rank at most two. Clearly $H$ is the hypersurface defined by the vanishing of the determinant of the general $3 \times 3$ symmetric matrix and hence $X$ is 2-defective.

**Exercise 2.12.** Let $M$ be an $n \times n$ symmetric matrix of rank $r$. Prove that $M$ is a sum of $r$ symmetric matrices of rank 1.

**Exercise 2.13.** Show that $H = \sigma_2(X)$.

**Exercise 2.14.** Repeat the same argument for $X = \nu_2(\mathbb{P}^3)$. Is $X$ 2-defective?

In order to deal with the problem of studying the dimension of the higher secant varieties of $X$ we need to introduce a celebrated tool, namely Terracini’s Lemma, see [Ter11].

**Lemma 2.15** (Terracini’s Lemma). Let $P_1, \ldots, P_s \in X$ be general points and $P \in \langle P_1, \ldots, P_s \rangle \subset \sigma_s(X)$ be a general point. Then the tangent space to $\sigma_s(X)$ in $P$ is

$$T_P(\sigma_s(X)) = \langle T_{P_1}(\sigma_s(X)), \ldots, T_{P_s}(\sigma_s(X)) \rangle.$$  

**Remark 2.16.** To get a (affine) geometric idea of why Terracini’s Lemma holds, we consider an affine curve $\gamma(t)$. A general point on $P \in \sigma_2(\gamma)$ is described as $\gamma(s_0) + \lambda_0[\gamma(t_0) - \gamma(s_0)]$. A neighborhood of $P$ is then described as

$$\gamma(s) + \lambda[\gamma(t) - \gamma(s)].$$

Hence the tangent space $T_P(\sigma_s(\gamma))$ is spanned by

$$\gamma'(s_0) - \lambda_0 \gamma'(s_0), \lambda_0 \gamma'(t_0), \gamma(t_0) - \gamma(s_0),$$

and this is the affine span of the affine tangent spaces $\{ \gamma(s_0) + \alpha \gamma'(s_0) : \alpha \in \mathbb{R} \}$ and $\{ \gamma(t_0) + \beta \gamma'(t_0) : \beta \in \mathbb{R} \}$.

As a first application of Terracini’s Lemma, we consider the twisted cubic curve.

**Example 2.17.** Let $X$ be the twisted cubic curve in $\mathbb{P}^3$, i.e. $X = \nu_3(\mathbb{P}^1)$ where $\nu_3$ is the map

$$\nu_3 : \mathbb{P}^1 \longrightarrow \mathbb{P}^3 \text{ defined by } [s : t] \mapsto [s^3 : st^2 : st : t^3].$$

We want to compute $\dim \sigma_2(X) = \dim T_P(\sigma_2(X))$ at a generic point $P$. Using Terracini’s Lemma it is enough to choose generic points $P_1, P_2 \in X$ and to study the linear span

$$\langle T_{P_1}(X), T_{P_2}(X) \rangle.$$
In particular, $\sigma_2(X) = \mathbb{P}^3$ if and only if the lines $T_{P_1}(X)$ and $T_{P_2}(X)$ do not intersect, that is, if and only if there does not exist a hyperplane containing both lines.

If $H \subset \mathbb{P}^3$ is a hyperplane, then the points of $H \cap X$ are determined by finding the roots of the degree three homogeneous polynomial $g(s, t)$ defining $\nu_3^{-1}(H) \subset \mathbb{P}^1$. If $H \supset T_{P_1}(X)$ then $g$ has a double root. However, the homogeneous polynomial is smooth and thus, in the general case, no hyperplane exists containing both tangent lines.

In conclusion, $\sigma_2(X) = \mathbb{P}^3$.

**Exercise 2.18.** Prove that if $H \supset T_{P}(X)$, then the polynomial defining $\nu_3^{-1}(H)$ has a double root.

We now introduce the Veronese variety in general.

**Definition 2.19.** Consider the polynomial ring $S = \mathbb{C}[x_0, \ldots, x_n]$ and its homogeneous pieces $S_d$. The $d$-th Veronese map $\nu_d$ is the morphism

$$\nu_d : \mathbb{P}(S_1) \rightarrow \mathbb{P}(S_d) \text{ defined by } [L] \mapsto [L^d].$$

In coordinates, using suitable monomial bases for $S_1$ and $S_d$, $\nu_d$ is the morphism

$$\nu_d : \mathbb{P}^n \rightarrow \mathbb{P}^N \text{ defined by } [a_0 : \ldots : a_n] \mapsto [M_0(a_0, \ldots, a_n) : \ldots : M_N(a_0, \ldots, a_n)]$$

where $N = \binom{n+d}{d} - 1$ and where $M_0, \ldots, M_N$ are monomials which form a basis for $S_d$.

We call $\nu_d(\mathbb{P}^n)$ a Veronese variety.

**Example 2.20.** A relevant family of Veronese varieties are the rational normal curves which are Veronese varieties of dimension one, i.e. $n = 1$. In this situation $S = \mathbb{C}[x_0, x_1]$ and $S_d$ is the vector space of degree $d$ binary forms. The rational normal curve $\nu_d(\mathbb{P}(S_1)) \subset \mathbb{P}(S_d)$ is represented by $d$-th powers of binary linear forms.

**Example 2.21.** The rational normal curve $X = \nu_2(\mathbb{P}^1) \subset \mathbb{P}^2$ is an irreducible conic. It is easy to see that $\sigma_2(X) = \mathbb{P}^2 = \mathbb{P}(S_2)$. This equality can also be explained by saying that any binary quadratic form $Q$ is the sum of two squares of linear forms, i.e. $Q = L^2 + M^2$.

**Exercise 2.22.** Consider the rational normal curve in $\mathbb{P}^3$, i.e. the twisted cubic curve $X = \nu_3(\mathbb{P}(S_1)) \subset \mathbb{P}(S_3)$. We know that $\sigma_2(X)$ fills up all the space. Can we write any binary cubic as the sum of two cubes of linear forms? Try $x_0x_1^2$.

**Exercise 2.23.** We described the Veronese variety $X = \nu_d(\mathbb{P}^n)$ in parametric form by means of the relation: $[F] \in X$ if and only if $F = L^d$. Use this description and standard differential geometry to compute $T_{[L^d]}(X)$ (describe this as a vector space of homogeneous polynomials). This can be used to apply Terracini’s Lemma, for example, to the twisted cubic curve.

3. Lecture Two

In the last lecture we spoke about higher secant varieties in general. Now we focus on the special case of Veronese varieties. Throughout this lecture we will consider the polynomial ring $S = \mathbb{C}[x_0, \ldots, x_n]$. 

An explicit description of the tangent space to a Veronese variety will be useful, so we give it here.

Remark 3.1. Let $X = \nu_d(\mathbb{P}^n)$ and consider $P = [L^d] \in X$ where $L \in S_1$ is a linear form. Then

$$T_P(X) = \langle [L^{d-1}M] : M \in S_1 \rangle.$$ 

We can use this to revisit the Veronese surface example.

Example 3.2. Consider the Veronese surface $X = \nu_2(\mathbb{P}^2) \subset \mathbb{P}^5$. To compute $\dim(\sigma_2(X))$ we use Terracini’s Lemma. Hence we choose two general points $P = [L^2], Q = [N^2] \in X$ and we consider the linear span of their tangent spaces

$$T = \langle T_P(X), T_Q(X) \rangle.$$ 

By applying Grassmann’s formula, and noticing that $T_P(X) \cap T_Q(X) = [LN]$ we get $\dim T = 3 + 3 - 1 - 1 = 4$ and hence $\sigma_2(X)$ is a hypersurface.

The study of higher secant varieties of Veronese varieties is strongly connected with a problem in polynomial algebra: the Waring problem for forms, i.e. for homogeneous polynomials, see [Ger96]. We begin by introducing the notion of Waring rank.

Definition 3.3. Let $F \in S$ be a degree $d$ form. The Waring rank of $F$ is denoted $\text{rk}(F)$ and is defined to be the minimum $s$ such that we have

$$F = L_1^d + \ldots + L_s^d$$

for some linear forms $L_i \in S_1$.

Remark 3.4. It is clear that $\text{rk}(L^d) = 1$ if $L$ is a linear form. However in general, if $L$ and $N$ are linear forms, $\text{rk}(L^d + N^d) \leq 2$. It is 1 if $L$ and $N$ are proportional and 2 otherwise. For more than two factors the computation of the Waring rank for a sum of powers of linear form is not trivial.

We can now state the Waring problem for forms, which actually comes in two fashions. The big Waring problem asks for the computation of

$$g(n, d)$$

the minimal integer such that

$$\text{rk}(F) \leq g(n, d)$$

for a generic element $F \in S_d$, i.e. for a generic degree $d$ form in $n + 1$ variables. The little Waring problem is more ambitious and asks us to determine the smallest integer

$$G(n, d)$$

such that

$$\text{rk}(F) \leq G(n, d)$$

for any $F \in S_d$.

Remark 3.5. To understand the difference between the big and the little Waring problem we can refer to a probabilistic description. Pick a random element $F \in S_d$, then $\text{rk}(F) \leq G(n, d)$ and with probability one $\text{rk}(F) = g(n, d)$ (actually equality holds). However, if the choice of $F$ is unlucky, it could be that $\text{rk}(F) > g(n, d)$.

Note also that these notions are field dependent, see [CO12] for example.
Remark 3.6. To make the notion of a generic element precise we use topology. Specifically, the big Waring problem asks us to bound the Waring rank for all elements belonging to a non-empty Zariski open subset of \( \mathbb{P}S_d \); since non-empty Zariski open subsets are dense this also explains the probabilistic interpretation.

The big Waring problem has a nice geometric interpretation using Veronese varieties — this interpretation allows for a complete solution to the problem. Also the little Waring problem has a geometric aspect but this problem, in its full generality, is still unsolved.

Remark 3.7. As the Veronese variety \( X = \nu_d(\mathbb{P}^n) \subset \mathbb{P}^N \) parameterizes pure powers in \( S_d \), it is clear that \( g(n, d) \) is the smallest \( s \) such that \( \sigma_s(X) = \mathbb{P}^N \). Thus solving the big Waring problem is equivalent to finding the smallest \( s \) such that secant variety \( \sigma_s(X) \) fills up \( \mathbb{P}^N \). On the other hand, as taking the Zariski closure of the set \( \bigcup_{p_1, \ldots, p_s \in X} \langle p_1, \ldots, p_s \rangle \) is involved in defining \( \sigma_s(X) \), this is not equivalent to solving the little Waring problem.

Remark 3.8. To solve the little Waring problem one has to find the smallest \( s \) such that every element \( [F] \in \mathbb{P}S_d \) lies on the span of some collection of \( s \) points of \( X \).

Let’s consider two examples to better understand the difference between the two problems.

Example 3.9. Let \( X = \nu_2(\mathbb{P}^1) \subset \mathbb{P}^2 \) be the rational normal curve in \( \mathbb{P}^2 \), i.e. a non-degenerate conic. We know that \( \sigma_2(X) = \mathbb{P}^2 \) and hence \( g(n = 1, d = 2) = 2 \). But we also know that each point of \( \mathbb{P}^2 \) lies on the span of two distinct points of \( X \) — every \( 2 \times 2 \) symmetric matrix is the sum of two rank-one symmetric matrices — thus \( G(n = 1, d = 2) = 2 \). In particular this means that the Waring rank of a binary quadratic form is always at most two.

Example 3.10. Let \( X = \nu_3(\mathbb{P}^1) \subset \mathbb{P}^3 \) be the rational normal curve. Again, we know that \( \sigma_2(X) = \mathbb{P}^3 \) and hence \( g(n = 1, d = 3) = 2 \). However, there are degree three binary forms \( F \) such that \( \text{rk}(F) = 3 \), and actually \( G(n = 1, d = 3) = 3 \).

To understand which the bad forms are, consider the projection map \( \pi_P \) from any point \( P = [F] \in \mathbb{P}^3 \). Clearly, if \( P \notin X \), \( \pi_P(X) \) is a degree 3 rational plane curve. Hence, it is singular, and being irreducible, only two possibilities arise. If the singularity is a node, then \( P = [F] \) lies on a chord of \( X \), and thus \( F = L^3 + N^3 \). But, if the singularity is a cusp, this is no longer true as \( P \) lies on a tangent line to \( X \) and not on a chord. Thus, the bad binary cubics lie on tangent lines to the twisted cubic curve. In other words, the bad binary cubics are of the form \( L^2N \).

Exercise 3.11. For binary forms, we can stratify \( \mathbb{P}S_2 \) using the Waring rank: rank one elements correspond to points of the rational normal curve, while all the points outside the curve have rank two. Do the same for binary cubics and stratify \( \mathbb{P}S_3 = \mathbb{P}^3 \).

We can produce a useful interpretation of Terracini’s Lemma in the case of Veronese varieties. We consider the Veronese variety \( X = \nu_d(\mathbb{P}^n) \subset \mathbb{P}^N \).

Remark 3.12. If \( H \subset \mathbb{P}^N \) is a hyperplane, then \( \nu_d^{-1}(H) \) is a degree \( d \) hypersurface. To see this, notice that \( H \) has an equation of the form \( a_0z_0 + \ldots + a_Nz_N \) where \( z_i \) are the coordinates of \( \mathbb{P}^N \). To determine an equation for \( \nu_d^{-1}(H) \) it is enough to substitute each \( z_i \) with the corresponding degree \( d \) monomial in the \( x_0, \ldots, x_n \).
Remark 3.13. If \( H \subset \mathbb{P}^N \) is a hyperplane and \([L^d] \in H\), then \( \nu_d^{-1}(H) \) is a degree \( d \) hypersurface passing through the point \([L] \in \mathbb{P}^n\). This is clearly true since \( \nu_d^{-1}([L^d]) = [L] \).

Remark 3.14. If \( H \subset \mathbb{P}^N \) is a hyperplane such that \( T_{[L^d]}(X) \subset H \), then \( \nu_d^{-1}(H) \) is a degree \( d \) hypersurface singular at the point \([L] \in \mathbb{P}^n\). This can be seen using apolarity or by direct computation choosing \( L^d = x_0^d \).

We illustrate the last remark in an example.

Example 3.15. Consider the Veronese surface \( X \subset \mathbb{P}^5 \), let
\[
P = [1 : 0 : 0 : 0 : 0] = [x^2] \in X,
\]
and let \( \mathbb{C}[z_0, z_1, \ldots, z_5] \) be the coordinate ring of \( \mathbb{P}^5 \). If \( H \) is a hyperplane containing \( P \), then \( H \) has equation
\[
a_1 z_1 + a_2 z_2 + a_3 z_3 + a_4 z_4 + a_5 z_5 = 0
\]
and hence \( \nu_2^{-1}(H) \) is the plane conic determined by the equation
\[
a_1 x y + a_2 x z + a_3 y^2 + a_4 y z + a_5 z^2 = 0,
\]
which passes through the point \( \nu^{-1}(P) = [1 : 0 : 0] \). The tangent space \( T_P(X) \) is the projective space associated to the linear span of the forms
\[
x^2, x y, x z,
\]
and hence it is the linear span of the points
\[
[1 : 0 : 0 : 0 : 0], [0 : 1 : 0 : 0 : 0], [0 : 0 : 1 : 0 : 0], [0 : 0 : 0 : 1 : 0], [0 : 0 : 0 : 0 : 1].
\]
Thus, if \( H \subset T_P(X) \) then \( a_1 = a_2 = 0 \) and the corresponding conic has equation
\[
a_3 y^2 + a_4 y z + a_5 z^2 = 0,
\]
which is singular at the point \([1 : 0 : 0]\).

Exercise 3.16. Repeat the argument above to prove the general statement: if \( T_{[L^d]}(\nu_d(\mathbb{P}^n)) \subset H \), then \( \nu_d^{-1}(H) \) is a degree \( d \) hypersurface singular at the point \([L] \in \mathbb{P}^n\).

We will now elaborate on the connection between double point schemes and higher secant varieties to Veronese varieties.

Definition 3.17. Let \( P_1, \ldots, P_s \in \mathbb{P}^n \) be points with defining ideals \( \varphi_1, \ldots, \varphi_s \) respectively. The scheme defined by the ideal \( \varphi_1^2 \cap \cdots \cap \varphi_s^2 \) is called a 2-fat point scheme or a double point scheme.

Remark 3.18. Let \( X = \nu_d(\mathbb{P}^n) \subset \mathbb{P}^N \). There is a bijection between
\[
\{H \subset \mathbb{P}^N \text{ a hyperplane : } H \supset \langle T_{P_1}(X), \ldots, T_{P_s}(X) \rangle \},
\]
and
\[
\{ \text{degree } d \text{ hypersurfaces of } \mathbb{P}^n \text{ singular at } P_1, \ldots, P_s \} = (\varphi_1^2 \cap \cdots \cap \varphi_s^2)_d.
\]

Using the double point interpretation of Terracini’s Lemma we get the following criterion to study the dimension of higher secant varieties to Veronese varieties.

Lemma 3.19. Let \( X = \nu_d(\mathbb{P}^n) \subset \mathbb{P}^N \) and choose generic points \( P_1, \ldots, P_s \in \mathbb{P}^n \) with defining ideals \( \varphi_1, \ldots, \varphi_s \) respectively. Then
\[
\dim \sigma_s(X) = N - \dim(\varphi_1^2 \cap \cdots \cap \varphi_s^2)_d.
\]
Example 3.20. We consider, again, the Veronese surface $X$ in $\mathbb{P}^5$. To determine $\dim \sigma_2(X)$ we choose generic points $P_1, P_2 \in \mathbb{P}^2$ and look for conics singular at both points, i.e. elements in $(\nu^2 \cap \nu^2)_2$. Exactly one such conic exists (the line through $P_1$ and $P_2$ doubled) and hence $\sigma_2(X)$ is a hypersurface.

Exercise 3.21. Solve the big Waring problem for $n = 1$ using the double points interpretation.

We now return to the big Waring problem. Notice that the secant variety interpretation and a straightforward dimension count yields an expected value for $g(n, d)$ which is

$$\left\lceil \frac{(d+n)}{n+1} \right\rceil.$$

This expectation turns out to be true except for a short list of exceptions. A complete solution for the big Waring problem is given by a celebrated result by Alexander and Hirschowitz, see [AH92].

Theorem 3.22 ([AH92]). Let $F$ be a generic degree $d$ form in $n+1$ variables. Then

$$\text{rk}(F) = \left\lceil \frac{(d+n)}{n+1} \right\rceil,$$

unless

- $d = 2$, any $n$ where $\text{rk}(F) = n + 1$.
- $d = 4, n = 2$ where $\text{rk}(F) = 6$ and not 5 as expected.
- $d = 4, n = 3$ where $\text{rk}(F) = 10$ and not 9 as expected.
- $d = 3, n = 4$ where $\text{rk}(F) = 8$ and not 7 as expected.
- $d = 4, n = 4$ where $\text{rk}(F) = 15$ and not 14 as expected.

Remark 3.23. A straightforward interpretation of the Alexander and Hirschowitz result in terms of higher secants is as follows. The number $g(n, d)$ is the smallest $s$ such that $\sigma_s(\nu_d(\mathbb{P}^n)) = \mathbb{P}^N$, unless $n$ and $d$ fall into one of the exceptional cases above.

Remark 3.24. Actually the Alexander and Hirschowitz result gives more for higher secant varieties of the Veronese varieties, namely that $\nu_d(\mathbb{P}^n)$ is not defective, for all $s$, except for the exceptional cases.

Let’s now try to explain some of the defective cases of the Alexander-Hirschowitz result.

Example 3.25. For $n = 2, d = 4$ we consider $X = \nu_4(\mathbb{P}^2) \subset \mathbb{P}^{14}$. In particular, we are looking for the smallest $s$ such that $\sigma_s(X) = \mathbb{P}^{14}$. We expect $s = 5$ to work and we want to check whether this is the case or not. To use the double points interpretation, we choose 5 generic points $P_1, \ldots, P_5 \in \mathbb{P}^2$ and we want to determine

$$\dim(\nu^2 \cap \ldots \cap \nu^2)_4.$$

To achieve this, we want to know the dimension of the space of quartic curves that are singular at each $P_i$. Counting conditions we have $15 - 5 \times 3 = 0$ and expect that

$$\dim(\nu^2 \cap \ldots \cap \nu^2)_4 = 0.$$
In fact, there exists a conic passing through the points $P_i$ and this conic doubled is a quartic with the required properties. Thus,
\[ \dim(\psi_i^2 \cap \ldots \cap \psi_5^2) \geq 1, \]
and $\dim \sigma_5(X) \leq 14 - 1 = 13$.

**Exercise 3.26.** Show that $\sigma_5(\nu_4(\mathbb{P}^2))$ is a hypersurface, i.e. that it has dimension equal to 13.

**Exercise 3.27.** Explain the exceptional cases $d = 2$ any $n$.

**Exercise 3.28.** Explain the exceptional cases $d = 4$ and $n = 3, 4$.

**Exercise 3.29.** Explain the exceptional case $d = 3$ and $n = 4$. (Hint: use Castelnuovo’s Theorem which asserts that there exists a (unique) rational normal curve passing through $n + 3$ generic points in $\mathbb{P}^n$.)

4. Lecture Three

In the last lecture we explained the solution to the big Waring problem and showed how to determine the Waring rank $\text{rk}(F)$ for $F$ a generic form. We now focus on a more general question: given any form $F$ what can we say about $\text{rk}(F)$?

The main tool we will use is *apolarity* and, in order to do this, we need the following setting. Let $S = \mathbb{C}[x_0, \ldots, x_n]$ and $T = \mathbb{C}[y_0, \ldots, y_n]$. We make $T$ act on $S$ via differentiation, i.e. we define
\[ y_i \circ x_j = \frac{\partial}{\partial x_i} x_j, \]
i.e. $y_i \circ x_j = 1$ if $i = j$ and it is zero otherwise. We then extend the action to all $T$ so that $\partial \in T$ is seen as a differential operator on elements of $S$; from now on we will omit $\circ$. If $A$ is a subset of a graded ring, we let $A_d$ denote the degree $d$ graded piece of $A$.

**Definition 4.1.** Given $F \in S_d$ we define the *annihilator*, or *perp ideal*, of $F$ as follows:
\[ F^\perp = \{ \partial \in T : \partial F = 0 \}. \]

**Exercise 4.2.** Show that $F^\perp \subset T$ is an ideal and that it also is Artinian, i.e. $(T/F^\perp)_i$ is zero for $i > d$.

**Exercise 4.3.** Let $S_i$ and $T_i$ denote the degree $i$ homogeneous pieces of $S$ and $T$ respectively. Show that the map
\[ S_i \times T_i \longrightarrow \mathbb{C} \]
\[ (F, \partial) \mapsto \partial F \]
is a perfect pairing, i.e.
\[ (F, \partial_0) \mapsto 0, \forall F \in S_i \implies \partial_0 = 0, \]
and
\[ (F_0, \partial) \mapsto 0, \forall \partial \in T_i \implies F_0 = 0. \]

**Remark 4.4.** Recall that Artinian Gorenstein rings are characterized by the property that they are all of the form $A = T/F^\perp$. Moreover, a property of such an $A$ is that it is finite dimensional, and the Hilbert function is symmetric.
Remark 4.5. Actually even more is true, and $A = T/F \perp$ is Artinian and Gorenstein with socle degree $d$. Using the perfect pairing $S_i \times T_i \rightarrow \mathbb{C}$ we see that $\dim A_d = \dim A_0 = 1$ and that $A_d$ is the socle of $A$.

In what follows we will make use of Hilbert functions, thus we define them here.

**Definition 4.6.** For an ideal $I \subset T$ we define the Hilbert function of $T/I$ as

$$HF(T/I, t) = \dim(T/I)_t.$$

**Example 4.7.** Let $F \in S^d$. We see that $HF(T/F \perp, t) = 0$ for all $t > d$, in fact all partial differential operators of degree $t > d$ will annihilate the degree $d$ form $F$ and hence $(T/F \perp)_t = 0$, for $t > d$. From the remark above we also see that $HF(T/F \perp, d) = 1$.

**Exercise 4.8.** Given $F \in S^d$ show that $HF(T/F \perp, t)$ is a symmetric function with respect to $\frac{d+1}{2}$ of $t$.

An interesting property of the ideal $F \perp$ is described by Macaulay’s Theorem (see [Mac1927]).

**Theorem 4.9.** If $F \in S_d$, then $T/F \perp$ is an Artinian Gorenstein ring with socle degree $d$. Conversely, if $T/I$ is an Artinian Gorenstein ring with socle degree $d$, then $I = F \perp$ for some $F \in S_d$.

Let’s now see how apolarity relates to the Waring rank. Recall that $s = \text{rk}(F)$ if and only if $F = \sum_1^s L_i^d$ and no shorter presentation exists.

**Example 4.10.** We now compute the possible Waring ranks for a binary cubic, i.e. for $F \in S^3$ where $S = \mathbb{C}[x_0, x_1]$. We begin by describing the Hilbert function of $F \perp$. There are only two possibilities:

| $t$ | 0 | 1 | 2 | 3 | 4 | 0 → |
|-----|---|---|---|---|---|-----|
| $HF(T/F \perp, t)$ | 1 | 1 | 1 | 1 | 0 |

| $t$ | 0 | 1 | 2 | 3 | 4 | 0 → |
|-----|---|---|---|---|---|-----|
| $HF(T/F \perp, t)$ | 1 | 2 | 1 | 0 |

We want to show that in case 1 we have $F = L^3$. From the Hilbert function we see that $(F \perp)_1 = \langle \partial_1 \rangle$. From the perfect pairing property we see that

$$\{L \in S_1 : \partial_1 L = 0\} = \langle L_1 \rangle.$$

Thus we can find $L_0 \in S_1$ such that $\partial_1 L_0 = 1$ and

$$S_1 = \langle x_0, x_1 \rangle = \langle L_0, L_1 \rangle.$$

We now perform a linear change of variables and we obtain a polynomial

$$G(L_0, L_1) = aL_0^3 + bL_0^2L_1 + cL_0L_1^2 + dL_1^3$$

such that

$$G(L_0, L_1) = F(x_0, x_1).$$

As $\partial_1 L_0 \neq 0$ and $\partial_1 L_1 = 0$ we get

$$0 = \partial_1 G = 2bL_0L_1 + cL_1^2 + 3dL_1^2,$$

and hence $G = F = aL_0^3$ thus $\text{rk}(F) = 1$. 
We want now to show that in case 2 we have \( \text{rk}(F) = 2 \) or \( \text{rk}(F) = 3 \). We note that \( \text{rk}(F) \neq 1 \), otherwise \((F^\perp)_1 \neq 0 \). As in this case \((F^\perp)_1 = 0 \), we consider the degree two piece, \((F^\perp)_2 = \langle Q \rangle \). We have to possibilities

\[
Q = \partial \vartheta', \text{ where } \partial \text{ and } \vartheta' \text{ are not proportional, or } Q = \partial^2.
\]

If \( Q = \partial \vartheta' \), where \( \partial \text{ and } \vartheta' \) are not proportional, we can construct a basis for \( S_1 = \langle L, L' \rangle \) in such a way that
\[
\partial L = \vartheta' L' = 1,
\]
and
\[
\vartheta' L = \partial L' = 0.
\]
Then we perform a change of variables and obtain
\[
F(x_0, x_1) = G(L_0, L_1) = aL_0^3 + bL_0^2L_1 + cL_0L_1^2 + dL_1^3.
\]
We want to show that \( F(x_0, x_1) = aL_0^3 + dL_1^3 \). To do this we define
\[
H(x_0, x_1) = G(L_0, L_1) - aL_0^3 - dL_1^3,
\]
and show that the degree 3 polynomial \( H \) is the zero polynomial. To do this, it is enough to show that \((H^\perp)_3 = T_3 \). We now compute that
\[
\partial^3 H = 6aL - 6aL = 0 \quad \text{and,}
\]
\[
\partial^3 H = 6dL' - 6dL' = 0.
\]
We then notice that \( \partial^2 \vartheta' = \partial Q \in F^\perp \) and \( \partial^2 \vartheta' H = 0 \); similarly for \( \partial \vartheta' \). Thus \( H = 0 \) and \( F(x_0, x_1) = aL_0^3 + dL_1^3 \). As \((F^\perp)_1 = 0 \) this means that \( \text{rk}(F) = 2 \).

Finally, if \( Q = \partial^2 \) we assume by contradiction that \( \text{rk}(F) = 2 \), thus \( F = N^3 + M^3 \) for some linear forms \( N \) and \( M \). There exist linearly independent differential operators \( \partial_N, \partial_M \in S_1 \) such that
\[
\partial_N N = \partial_M M = 1,
\]
and
\[
\partial_N M = \partial_M N = 0.
\]
And then \( \partial_N \partial_M \in F^\perp \) and this is a contradiction as \( Q \) is the only element in \((F^\perp)_2 \) and it is a square.

Remark 4.11. We consider again the case of binary cubic forms. We want to make a connection between the Waring rank of \( F \) and certain ideals contained in \( F^\perp \). If \( \text{rk}(F) = 1 \) then we saw that \( F^\perp \supset \langle \partial_1 \rangle \) and this is the ideal of one point in \( \mathbb{P}^1 \). If \( \text{rk}(F) = 2 \) then \( F^\perp \supset \langle \partial \vartheta' \rangle \) and this the ideal of two distinct points in \( \mathbb{P}^1 \); as \((F^\perp)_1 = 0 \) there is no ideal of one point contained in the annihilator. Finally, if \( \text{rk}(F) = 3 \), then \( F^\perp \supset \langle \partial^2 \rangle \) and there is no ideal of two points, or one point, contained in the annihilator. However, \((F^\perp)_3 = T_3 \) and we can find many ideals of three points.

There is a connection between \( \text{rk}(F) \) and set of points whose ideal \( I \) is such that \( I \subset F^\perp \). This connection is the content of the Apolarity Lemma, see [IK99].

Lemma 4.12. Let \( F \in S_d \) be a degree \( d \) form in \( n + 1 \) variables. Then the following facts are equivalent:

- \( F = L_1^d + \ldots + L_s^d \);
- \( F^\perp \supset I \) such that \( I \) is the ideal of a set of \( s \) distinct points in \( \mathbb{P}^n \).
Example 4.13. We use the Apolarity Lemma to explain the Alexander-Hirschowitz defective case $n = 2$ and $d = 4$. Given a generic $F \in S_4$ we want to show that $\text{rk}(F) = 6$ and not 5 as expected. To do this we use Hilbert functions. Clearly, if $I \subseteq F^\perp$ then $HF(T/I, t) \geq HF(T/F^\perp, t)$ for all $t$. Thus by computing $HF(T/F^\perp, t)$ we get information on the Hilbert function of any ideal contained in the annihilator, and in particular for ideal of sets of points.

| $t$ | $HF(T/F^\perp, t)$ |
|-----|---------------------|
| 0   | 1                   |
| 1   | 3                   |
| 2   | 6                   |
| 3   | 3                   |
| 4   | 1                   |
| 5   | 0                   |

In particular, $HF(T/F^\perp, 2) = 6$ means that for no set of 5 points its defining ideal $I$ could be such that $I \subseteq F^\perp$.

Exercise 4.14. Use the Apolarity Lemma to compute $\text{rk}(x_0x_1^2)$. Then try the binary forms $x_0^d_1$.

Exercise 4.15. Use the Apolarity Lemma to explain Alexander-Hirschowitz exceptional cases.

It is in general very difficult to compute the Waring rank of a given form and (aside from brute force) no algorithm exists which can compute it in all cases. Lim and Hillar show that this problem is an instance of the fact that, as their title states, “Most tensor problems are NP-Hard,” [HL09]. However, we know $\text{rk}(F)$ when $F$ is a quadratic form, and we do have an efficient algorithm when $F$ is a binary form.

Remark 4.16. There is an algorithm, attributed to Sylvester, to compute $\text{rk}(F)$ for a binary form and it uses the Apolarity Lemma. The idea is to notice that $F^\perp = (\partial_1, \partial_2)$, i.e. the annihilator is a complete intersection ideal, say, with generators in degree $d_1 = \deg \partial_1 \leq d_2 = \deg \partial_2$. If $\partial_1$ is square free, then we are done and $\text{rk}(F) = d_1$. If not, as $\partial_1$ and $\partial_2$ do not have common factors, there is a square free degree $d_2$ element in $F^\perp$. Hence, $\text{rk}(F) = d_2$.

Exercise 4.17. Compute $\text{rk}(F)$ when $F$ is a quadratic form.

Remark 4.18. The Waring rank for monomials was determined in 2011 in a paper of Carlini, Catalisano and Geramita, see [CCG12], and independently by Buczyńska, Buczyński, and Teitler, see [BBT13]. In particular, it was shown that

$$\text{rk}(x_0^{a_0} \ldots x_n^{a_n}) = \frac{1}{(a_0 + 1)} \Pi_{i=0}^n(a_i + 1),$$

where $1 \leq a_0 \leq a_1 \leq \ldots \leq a_n$.

We conclude this lecture by studying the Waring rank of degree $d$ forms of the kind $L_1^d + \ldots + L_r^d$. Clearly, $\text{rk}(L_1^d) = 1$ and $\text{rk}(L_1^d + L_2^d) = 2$, if $L_1$ and $L_2$ are linearly independent. If the linear forms $L_i$ are not linearly independent, then the situation is more interesting.

Example 4.19. Consider the binary cubic form $F = ax_0^3 + bx_1^3 + (x_0 + x_1)^3$. We want to know $\text{rk}(F)$. For a generic choice of $a$ and $b$, we have $\text{rk}(F) = 2$, but for special values of $a$ and $b \text{rk}(F) = 3$. The idea is that the rank three element of $\mathbb{P}S_3$ lie on the tangent developable of the twisted cubic curve, which is an irreducible surface. Hence, the general element of the plane

$$\langle [x_0^3], [x_1^3], [(x_0 + x_1)^3] \rangle$$

has rank two, but there are rank three elements.
Exercise 4.20. Prove that \( \text{rk}(L^d + M^d + N^d) = 3 \) whenever \( L, M \) and \( N \) are linearly independent linear forms.

5. Lecture Four

In the last lecture we introduced the Apolarity Lemma and used it to study the Waring rank of a given specific form. In this lecture we will go back to the study of higher secant varieties of varieties that are not Veronese varieties.

The study of higher secant varieties of Veronese varieties is connected to Waring's problems, and hence with sum of powers decompositions of forms. We now want to consider tensors in general and not only homogenous polynomials, which correspond to symmetric tensors.

Consider \( \mathbb{C} \)-vector spaces \( V_1, \ldots, V_t \) and the tensor product
\[
V = V_1 \otimes \ldots \otimes V_t.
\]

**Definition 5.1.** A tensor \( v_1 \otimes \ldots \otimes v_t \in V \) is called *elementary, or indecomposable* or *rank-one* tensor.

Elementary tensors are the building blocks of \( V \). More specifically, there is a basis of \( V \) consisting of elementary tensors, so any tensor \( T \in V \) can be written as a linear combination of elementary tensors; in this sense elementary tensors are analogous to monomials.

A natural question is: given a tensor \( T \) what is the minimum \( s \) such that \( T = \sum_i T_i \) where each \( T_i \) is an elementary tensor? The value \( s \) is called the *tensor rank* of \( T \) and is the analogue of Waring rank for forms. Of course we could state tensor versions of the Waring's problems and try to solve them as well.

In order to study these problems geometrically, we need to introduce a new family of varieties.

**Definition 5.2.** Given vector spaces \( V_1, \ldots, V_t \) the *Segre map* is the map
\[
\mathbb{P}V_1 \times \ldots \times \mathbb{P}V_t \longrightarrow \mathbb{P}(V_1 \otimes \ldots \otimes V_t)
\]
\[
([v_1], \ldots, [v_t]) \mapsto [v_1 \otimes \ldots \otimes v_t]
\]
and the image variety is called a *Segre variety* or the *Segre product* of \( \mathbb{P}V_1, \ldots, \mathbb{P}V_t \).

If the vector spaces are such that \( \dim V_i = n_i + 1 \), then we will often denote by
\[
X = \mathbb{P}^{n_1} \times \ldots \times \mathbb{P}^{n_t}
\]
the image of the Segre map. In particular, \( X \subset \mathbb{P}(V_1 \otimes \ldots \otimes V_t) = \mathbb{P}^N \) where \( N + 1 = \Pi(n_i + 1) \). Note that \( \dim X = n_1 + \ldots + n_t \).

By choosing bases of the vector spaces \( V_i \) we can write the Segre map in coordinates
\[
[a_0, \ldots, a_{n_1}, \ldots, 0] \times [a_0, \ldots, 0, a_{n_2}, \ldots, 0] \times \ldots \times [a_0, \ldots, 0, a_{n_t}, \ldots, 0] \mapsto [a_0 a_{n_2} \ldots a_{n_t}, a_0 a_{n_2} \ldots a_{n_t}, \ldots, a_0 a_{n_2} \ldots a_{n_t}].
\]

**Example 5.3.** Consider \( X = \mathbb{P}^1 \times \mathbb{P}^1 \), then \( X \subset \mathbb{P}^3 \) is a surface. The Segre map is
\[
\mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^3
\]
\[
[a_0 : a_1] \times [b_0 : b_1] \mapsto [a_0 b_0 : a_0 b_1 : a_1 b_0 : a_1 b_1].
\]
If $z_0, z_1, z_2, \text{ and } z_3$ are the coordinates of $\mathbb{P}^3$, then it is easy to check that $X$ has equation
\[ z_0 z_3 - z_1 z_2 = 0. \]
Thus $X$ is a smooth quadric in $\mathbb{P}^3$.

**Example 5.4.** Consider again $X = \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$. We identify $\mathbb{P}^3$ with the projectivization of the vector space of $2 \times 2$ matrices. Using this identification we can write the Segre map as
\[ [a_0 : a_1] \times [b_0 : b_1] \mapsto \begin{pmatrix} a_0 b_0 & a_0 b_1 \\ a_1 b_0 & a_1 b_1 \end{pmatrix}. \]
Thus $X$ represents the set of $2 \times 2$ matrices of rank at most one and the ideal of $X$ is generated by the vanishing of the determinant of the generic matrix \( \begin{pmatrix} z_0 & z_1 \\ z_2 & z_3 \end{pmatrix} \).

**Exercise 5.5.** Work out a matrix representation for the Segre varieties with two factors $\mathbb{P}^{n_1} \times \mathbb{P}^{n_2}$.

Before entering into the study of the higher secant varieties of Segre varieties, we provide some motivation coming from Algebraic Complexity Theory.

**Example 5.6.** The multiplication of two $2 \times 2$ matrices can be seen as bilinear map
\[ T : \mathbb{C}^4 \times \mathbb{C}^4 \rightarrow \mathbb{C}^4 \]
or, equivalently, as a tensor
\[ T \in \mathbb{C}^{4^*} \otimes \mathbb{C}^{4^*} \otimes \mathbb{C}^4. \]
It is interesting to try to understand how many multiplications over the ground field are required to compute the map $T$.

If we think of $T$ as a tensor, then we can write it as a linear combination of elementary tensors and each elementary tensor represents a multiplication. The naive algorithm for matrix multiplication, which in general uses $n^3$ scalar multiplications to compute the product of two $n \times n$ matrices, implies that $T$ can be written as $T = \sum_1^8 \alpha_i \otimes \beta_i \otimes c_i$. However, Strassen in [Str83] proved that
\[ [T] \in \sigma_7(\mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^3), \]
and, even more, that $T$ is the sum of 7 elementary tensors
\[ T = \sum_1^7 \alpha_i \otimes \beta_i \otimes c_i. \]

Strassen’s algorithm actually holds for multiplying matrices over any algebra. Thus by viewing a given $n \times n$ matrix in one with size a power of 2, one can use Strassen’s algorithm iteratively. So $2^m \times 2^m$ matrices can be multiplied using $7^m$ multiplications. In particular, this method lowers the upper bound for the complexity of matrix multiplication from $n^3$ to $n^{\log_2 7} \approx n^{2.81}$.

After Strassen’s result, it was shown that the rank of $T$ is not smaller than 7. On the other hand, much later, Landsberg proved that the border rank of $T$ is 7, that is to say that $T \notin \sigma_6(\mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^3)$, [Lan06].
The question of the complexity of matrix multiplication has recently been called one of the most important questions in Numerical Analysis [Tre12]. The reason for this is that the complexity of matrix multiplication also determines the complexity of matrix inversion. Matrix inversion is one of the main tools for solving a square system of linear ODE’s.

Williams, in 2012, improved the Coppersmith-Winograd algorithm to obtain the current best upper bound for the complexity of matrix multiplication [Wil12], but it would lead us to far afield to discuss this here. On the other hand, the current best lower bounds come from the algebra and geometry of secant varieties. These bounds arise by showing non-membership of the matrix multiplication tensor on certain secant varieties. To do this, one looks for non-trivial equations that vanish on certain secant varieties, but do not vanish on the matrix multiplication tensor. Indeed, the best results in this direction make use of representation theoretic descriptions of the ideals of secant varieties. For more, see [Lan12b, LO11a].

5.1. Dimension of secant varieties of Segre varieties. In the $2 \times 2$ matrix multiplication example, we should have pointed out the fact that $\sigma_7(P^3 \times P^3 \times P^3)$ actually fills the ambient space, so almost all tensors in $P(C^4 \otimes C^4 \otimes C^4)$ have rank 7. For this and many other reasons we would like to know the dimensions of secant varieties of Segre varieties.

Like in the polynomial case, there is an expected dimension, which is obtained by the naive dimension count. When $X$ is the Segre product $P^{n_1} \times \cdots \times P^{n_t}$, the expected dimension of $\sigma_s(X)$ is

$$\min \left\{ s \sum_{i=1}^t (n_i + 1) + s - 1, \prod_{i=1}^t (n_i + 1) - 1 \right\}.$$

We would like to have an analogue of the Alexander-Hirschowitz theorem for the Segre case, however, this is a very difficult problem. See [AOP09] for more details. There are some partial results, however. For example Catalisano, Geramita and Gimigliano in [CGG11] show that $\sigma_4(P^1 \times \cdots \times P^1)$ always has the expected dimension, except for the case of 4 factors.

Again, the first tool one uses to study the dimensions of secant varieties is the computation of tangent spaces together with Terracini’s lemma.

**Exercise 5.7.** Let $X = P V_1 \times \cdots \times P V_t$ and let $[v] = [v_1 \otimes \cdots \otimes v_t]$ be a point of $X$. Show that the cone over the tangent space to $X$ at $v$ is the span of the following vector spaces:

$$V_1 \otimes v_2 \otimes v_3 \otimes \cdots \otimes v_t,$$

$$v_1 \otimes V_2 \otimes v_3 \otimes \cdots \otimes v_t,$$

$$\vdots$$

$$v_1 \otimes v_2 \otimes \cdots \otimes v_{t-1} \otimes V_t.$$

**Exercise 5.8.** Show that $\sigma_2(P^1 \times P^1 \times P^1) = P^7$.

**Exercise 5.9.** Use the above description of the tangent space of the Segre product and Terracini’s lemma to show that $\sigma_3(P^1 \times P^1 \times P^1 \times P^1)$ is a hypersurface in $P^{15}$ and not the entire ambient space as expected. This shows that the four-factor Segre product of $P^1$’s is defective.

There are two main approaches to the study of the dimensions of secant varieties of Segre products: [CGG05a] and [AOP09]. In [CGG05a] the authors introduce...
and use what they call the affine-projective method. In this way, the study of the dimension of higher secant varieties of Segre, and Segre-Veronese, varieties reduces to the study of the postulation of non-reduced schemes supported on linear spaces. In [AOP09], the authors show that the “divide and conquer” method of Alexander and Hirschowitz can be used to set up a multi-step induction proof for the non-defectivity of Segre products. They are able to obtain partial results on non-defectivity by then checking many initial cases, often using the computer. On the other hand, for the remaining cases there are many more difficult computations to do in order to get the full result.

5.2. Flattenings. Often, the first tool used to understand properties of tensors is to reduce to Linear Algebra (when possible). For this, the notion of flattenings is essential. Consider for the moment the three-factor case. We may view the vector space $\mathbb{C}^a \otimes \mathbb{C}^b \otimes \mathbb{C}^c$ as a space of matrices in three essentially different ways as the following spaces of linear maps:

- $(\mathbb{C}^a)^* \to \mathbb{C}^b \otimes \mathbb{C}^c$
- $(\mathbb{C}^b)^* \to \mathbb{C}^a \otimes \mathbb{C}^c$
- $(\mathbb{C}^c)^* \to \mathbb{C}^a \otimes \mathbb{C}^b$

(A priori there are many more choices of flattenings, however, in the three-factor case the others are obtained by transposing the above maps.) When there are more than 3 factors the situation is similar, with many more flattenings to consider.

For $V_1 \otimes \cdots \otimes V_t$, a $p$-flattening is the interpretation as a space of matrices with $p$ factors on the left:

$$(V_i \otimes \cdots \otimes V_p)^* \to V_{j_1} \otimes \cdots \otimes V_{j_{t-p}}.$$ 

For a given tensor $T \in V_1 \otimes \cdots \otimes V_t$ we call a $p$-flattening of $T$ a realization of $T$ in one of the above flattenings. This naturally gives rise to the notion of multi-linear rank, which is the vector the ranks of the 1-flattenings of $T$, see [CK11].

Exercise 5.10. Show that $T$ has rank 1 if and only if its multilinear rank is $(1, \ldots, 1)$.

Recall that a linear mapping $T: (\mathbb{C}^a)^* \to \mathbb{C}^b$ has rank $r$ if the image of the map has dimension $r$ and the kernel has dimension $a - r$. Moreover, since the rank of the transpose is also $r$, after re-choosing bases in $\mathbb{C}^a$ and $\mathbb{C}^b$ one can find $r$-dimensional subspaces in $\mathbb{C}^a$ and $\mathbb{C}^b$ so that $T \in (\mathbb{C}^r)^* \otimes \mathbb{C}^r$. This notion generalizes to tensors of higher order. In particular, it is well known that $T \in V_1 \otimes \cdots \otimes V_t$ has multilinear rank $\leq (r_1, \ldots, r_t)$ if an only if there exist subspaces $\mathbb{C}^{r_i} \subset V_i$ such that $T \in \mathbb{C}^{r_1} \otimes \cdots \otimes \mathbb{C}^{r_t}$. The Zariski closure of all tensors of multilinear rank $(r_1, \ldots, r_t)$ is known as the subspace variety, denoted $\text{Sub}_{r_1, \ldots, r_t}$. See [LW07b] for more details.

The connection between subspace varieties and secant varieties is the content of the following exercise.

Exercise 5.11. Let $X = \mathbb{P}^V_1 \times \cdots \times \mathbb{P}^V_t$. Show that if $r \leq r_i$ for $1 \leq i \leq t$, then

$$(\sigma_r(X) \subset \text{Sub}_{r_1, \ldots, r_t}.$$ 

Notice that for the 2-factor case, $\sigma_r(\mathbb{P}^{a-1} \times \mathbb{P}^{b-1}) = \text{Sub}_{r,r}$. Aside from the case of binary tensors (tensor products of $\mathbb{C}^2$s), another case that is well understood is the case of very unbalanced tensors. (For the more refined
notion of “unbalanced” see [AOP09, § 4].) Again consider $\mathbb{C}^a \otimes \mathbb{C}^b \otimes \mathbb{C}^c$ and suppose that $a \geq bc$. Then for all $r \leq bc$ we have

$$\sigma_r(P^{a-1} \times P^{b-1} \times P^{c-1}) = \sigma_r(P^{a-1} \times P^{bc-1}) = \text{Sub}_{r,r}.$$ 

Therefore we can always reduce the case of very unbalanced 3-fold tensors to the case of matrices. More generally a tensor in $V_1 \otimes \cdots \otimes V_t$ is called very unbalanced if $(n_i + 1) \geq \prod_{j \neq i}(n_j + 1)$. In the very unbalanced case we can reduce to the case of matrices and use results and techniques from linear algebra.

5.3. Equations of Secant varieties of Segre products. Now we turn to the question of defining equations. Recall that a matrix has rank $\leq 5$.3. Therefore we can always reduce the case of very unbalanced 3-fold tensors to the case of matrices. More generally a tensor in $V_1 \otimes \cdots \otimes V_t$ is called very unbalanced if $(n_i + 1) \geq \prod_{j \neq i}(n_j + 1)$. In the very unbalanced case we can reduce to the case of matrices and use results and techniques from linear algebra.

Strassen noticed that a certain equation actually vanishes on $\sigma_3(P^2 \times P^2 \times P^2)$, and moreover, he shows that it is a hypersurface. Strassen’s equation was studied in more generality by [LM08], put into a broader context by Ottaviani [Ott09] generalized by Landsberg and Ottaviani [LO11b]. Without explaining the full generality of the construction, we can describe Ottaviani’s version of Strassen’s equation as follows.

Suppose $T \in V_1 \otimes V_2 \otimes V_3$ with $V_i \cong \mathbb{C}^3$ and consider the flattening $(V_1)^* \to V_2 \otimes V_3$.

Choose a basis $\{v_1, v_2, v_3\}$ for $V_1$ and write $T$ as a linear combination of matrices $T = v_1 \otimes T^1 + v_2 \otimes T^2 + v_3 \otimes T^3$. The $3 \times 3$ matrices $T^i$ are called the slices of $T$ in the $V_1$-direction with respect to the chosen basis. Now consider the following matrix:

$$\varphi_T = \begin{pmatrix} 0 & T^1 & -T^2 \\ -T^1 & 0 & T^3 \\ T^2 & -T^3 & 0 \end{pmatrix},$$

where all of the blocks are $3 \times 3$.

**Exercise 5.12.**

1. Show that if $T$ has rank 1 then $\varphi_T$ has rank 2.

2. Show that $\varphi$ is additive in its argument, i.e. show that $\varphi_{T+T'} = \varphi_T + \varphi_{T'}$.

The previous exercise together with the subadditivity of matrix rank implies that if $T$ has tensor rank $r$ then $\varphi_T$ has matrix rank $\leq 2r$. In particular, if $T$ has tensor rank 4, the determinant of $\varphi_T$ must vanish. Indeed det($\varphi_T$) is Strassen’s equation, and it is the equation of the degree 9 hypersurface $\sigma_4(P^2 \times P^2 \times P^2)$. 

$$\text{Sub}_{r,r}.$$
Remark 5.13. This presentation of Strassen’s equation \( \det(\varphi_T) \) is very compact yet its expansion in monomials is very large, having 9216 terms.

This basic idea of taking a tensor and constructing a large matrix whose rank depends on the rank of \( T \) is at the heart of almost all known equations of secant varieties of Segre products — see \([LO11b]\). One exception is that of the degree 6 equations in the ideal of \( \sigma_4(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^1) \). The only known construction of these equations comes from representation theoretic considerations. For more details see \([BO11a, LM04]\).

Despite this nice picture, we actually know surprisingly little about the defining equations of secant varieties of Segre products in general, and this is an ongoing area of current research.

6. Solution of the Exercises

In what follows if \( S \) is a subset of \( \mathbb{P}^n \), then \( \overline{S} \) denotes the smallest closed subset of \( \mathbb{P}^n \) containing \( S \) with the reduced subscheme structure. On the other hand, \( \langle S \rangle \) denotes the smallest linear subspace of \( \mathbb{P}^n \) containing \( S \).

Exercise 2.3. Show that the union of chords (secant lines) to a plane conic is closed. However, the union of the chords of the twisted cubic curve in \( \mathbb{P}^3 \) is not.

Solution. Let \( X \subseteq \mathbb{P}^2 \) be a plane conic. It suffices to show that \( \cup_{p,q \in X} \langle p, q \rangle = \mathbb{P}^2 \).

If \( y \in \mathbb{P}^2 \setminus X \) then there exists a line containing \( y \) and intersecting \( X \) in two distinct points \( p \) and \( q \). Thus, \( y \in \langle p, q \rangle \) so \( \cup_{p,q \in X} \langle p, q \rangle = \mathbb{P}^2 \).

Let \( X \subseteq \mathbb{P}^3 \) be the twisted cubic. Exercise 2.8 implies that \( \sigma_3(X) = \mathbb{P}^3 \). On the other hand, direct calculation shows that the point \([0 : 1 : 0 : 0]\) lies on no secant line of \( X \). Hence, \( \cup_{p,q \in X} \langle p, q \rangle \) is not equal to its closure and hence is not closed.

Exercise 2.5. For \( X \subset \mathbb{P}^N \), show that, if \( \sigma_i(X) = \sigma_{i+1}(X) \neq \mathbb{P}^N \), then \( \sigma_i(X) \) is a linear space and hence \( \sigma_j(X) = \sigma_i(X) \) for all \( j \geq i \).

Solution. It suffices to prove that, for \( k \geq 1 \), if \( \sigma_k(X) = \sigma_{k+1}(X) \) then \( \sigma_{k'}(X) = \langle X \rangle \) for \( k' \geq k \).

Note

\[
\sigma_{k+1}(X) = \bigcup_{p_i \in X} \langle p_1, \ldots, p_k \rangle
\]

while \( \sigma_k(X) = \bigcup_{p_i \in X} \langle p_1, \ldots, p_k \rangle \). Since the singular locus of \( \sigma_k(X) \) is a proper closed subset of \( \sigma_k(X) \) there exists a non-singular point \( z \in \sigma_k(X) \) such that \( z \in \bigcup_{p_i \in X} \langle p_1, \ldots, p_k \rangle \).

Now for all \( y \in X \) the line \( \langle y, z \rangle \) is contained in \( \bigcup_{x \in X} \langle x, z \rangle \) and passes through \( z \). Hence \( \bigcup_{x \in X} \langle x, z \rangle \subseteq T_z \bigcup_{x \in X} \langle x, z \rangle \). Using (1), we deduce

\[
X \subseteq \bigcup_{x \in X} \langle x, z \rangle \subseteq T_z \bigcup_{x \in X} \langle x, z \rangle \subseteq T_z \sigma_{k+1}(X).
\]

Hence \( \langle X \rangle \subseteq T_z \sigma_{k+1}(X) \). In addition, since \( \sigma_{k+1}(X) = \sigma_k(X) \) we deduce

\[
\sigma_k(X) \subseteq \langle X \rangle \subseteq T_z \sigma_k(X).
\]

Since \( z \) is non-singular, \( \dim T_z \sigma_k(X) = \dim \sigma_k(X) \). Finally, since \( T_z \sigma_k(X) \) is irreducible and \( \sigma_k(X) \) is reduced we conclude \( \sigma_k(X) = \langle X \rangle = T_z \sigma_k(X) \). If \( k' \geq k \) then \( \langle X \rangle \subseteq \sigma_k(X) \subseteq \sigma_{k'}(X) \). Since \( \sigma_{k'}(X) \subseteq \langle X \rangle \) we deduce \( \sigma_{k'}(X) = \sigma_k(X) \). \( \square \)
Exercise 2.6. If $X \subseteq \mathbb{P}^N$ is non-degenerate then there exists an $r \geq 1$ with the property that 

$$X = \sigma_1(X) \subsetneq \sigma_2(X) \subsetneq \ldots \subsetneq \sigma_r(X) = \mathbb{P}^N.$$ 

In particular, all inclusions are strict and there is a higher secant variety that coincides with the ambient space.

Solution. In the notation of Exercise 2.5, set $k_0 := \min\{k \mid \sigma_k(X) = \langle X \rangle\}$. It suffices to prove that there exists the following chain of strict inclusions

$$X = \sigma_1(X) \subsetneq \sigma_2(X) \subsetneq \ldots \subsetneq \sigma_{k_0}(X) = \langle X \rangle.$$ 

If $\sigma_k(X) = \sigma_{k+1}(X)$ then, by Exercise 2.5, $\sigma_k(X) = \langle X \rangle$ and hence $k \geq k_0$. \hfill \qed

Exercise 2.8. Let $X \subseteq \mathbb{P}^n$ be a curve. Prove that $\sigma_2(X)$ has dimension 3 unless $X$ is contained in a plane.

Solution. Let $X := \{((p,q), y) : y \in \langle p,q \rangle\} \subseteq G(1,3) \times \mathbb{P}^n$ be the incident correspondence corresponding to the (closure) of the secant line map $(X \times X) - \Delta_X \to G(1,3) \times \mathbb{P}^n$. Then $X$ is an irreducible closed subset of $G(1,3) \times \mathbb{P}^n$ and $\sigma_2(X) = p_2(X)$. Hence, $\dim \sigma_2(X) \leq 3$. Suppose now that $\dim \sigma_2(X) = 2$. Let us prove that $X$ is contained in a plane. First note that for a fixed $p \in X$, $\cup_{q \in X} \langle p,q \rangle$ is reduced, irreducible, has dimension $\dim X + 1$ and is contained in $\sigma_2(X)$. Hence, if $\sigma_2(X)$ has dimension $\dim X + 1$, $\sigma_2(X) = \cup_{q \in X} \langle p,q \rangle$. Hence, there exists a non-singular $x \in \sigma_2(X)$ such that $x \in \cup_{q \in X} \langle p,q \rangle$ and such that $X \subseteq \cup_{q \in X} \langle x,q \rangle = \cup_{q \in X} \langle p,q \rangle \subseteq T_x \cup_{q \in X} \langle x,q \rangle = T_x \sigma_2(X)$. Hence, $\langle X \rangle \subseteq T_x \sigma_2(X)$. Since $\sigma_2(X)$ has dimension 2 then $T_x \sigma_2(X)$ is a plane. Finally, note that if $\sigma_2(X)$ has dimension 1 then $\sigma_2(X) = \sigma_1(X)$. Hence $\sigma_2(X) = \langle X \rangle$ so $\sigma_2(X)$ and hence $X$ is a line. \hfill \qed

Exercise 2.12. Let $M$ be an $n \times n$ symmetric matrix of rank $r$. Prove that $M$ is a sum of $r$ symmetric matrices of rank 1.

Solution. Let $M$ be an $n \times n$ symmetric matrix of rank $r$. Prove that $M$ is a sum of $r$ symmetric matrices of rank 1.

More specifically, if $M$ has rank $r$ then there exists $\{i_1, \ldots, i_r\} \subseteq \{1, \ldots, n\}$ such that $\lambda_i \neq 0$ for $i \in \{i_1, \ldots, i_r\}$ and $\lambda_i = 0$ for $i \not\in \{i_1, \ldots, i_r\}$. For $k = 1, \ldots, n$, set $m^k_{ij} := \lambda_k p_{ik} p_{jk}$ and let $M_k$ be the matrix with $i,j$ entry $m^k_{ij}$. Since $m^k_{ij} = m^k_{ji}$, $M_k$ is symmetric. Moreover, $M = \sum_{i=1}^{n} M_k$. It remains to show that $M_k$ has rank at most 1. Let $P_k$ be the matrix with $i,j$ entry $p_{ik} p_{jk}$. Since $M_k$ is a scalar multiple of $P_k$ it suffices to show that $P_k$ has rank at most 1. For this, we show that all $2 \times 2$ minors of $P_k$ are zero. Indeed an arbitrary $2 \times 2$ minor of $P_k$ is the determinant of the $2 \times 2$ matrix formed by omitting all rows except row $\alpha, \beta$ and columns $\gamma, \delta$, that is, the determinant

$$\begin{vmatrix} p_{\alpha k} p_{\gamma k} & p_{\alpha k} p_{\delta k} \\ p_{\beta k} p_{\gamma k} & p_{\beta k} p_{\delta k} \end{vmatrix}$$

which is zero. \hfill \qed
Exercise 2.13. Let $X = \nu_2(\mathbb{P}^2)$. Let $H$ be the locus of $3 \times 3$ symmetric matrices of rank at most 2. Prove that $H = \sigma_2(X)$.

Solution. Using Exercise 2.12 we deduce that $X$ is the locus of $3 \times 3$ symmetric matrices of rank at most 1. On the other hand we know that $X \subseteq \sigma_2(X) \subseteq H$. Moreover, if $M$ is a symmetric matrix of rank 2 then, by Exercise 2.12, $M$ is a sum of two symmetric matrices of rank 1. Hence $M \in \sigma_2(X)$.

Exercise 2.14. Let $H$ be the locus of $4 \times 4$ symmetric matrices of rank 2. Let $X = \nu_2(\mathbb{P}^3)$. Prove that $H = \sigma_2(X)$. Is $X$ 2-defective?

Solution. Using Exercise 2.12 we deduce that $X$ is the locus of $4 \times 4$ symmetric matrices of rank at most 1. On the other hand we know that $X \subseteq \sigma_2(X) \subseteq H$. Moreover, if $M$ is a symmetric matrix of rank 2 then, by Exercise 2.12, $M$ is a sum of two symmetric matrices of rank 1. Hence $M \in \sigma_2(X)$. To see that $X$ is 2-defective note that $\text{expdim}(\sigma_2(X)) = 7$ while the locus of $4 \times 4$ symmetric matrices of rank at most 2 have dimension 6.

Exercise 2.18. Prove that if $H \supseteq T_P(X)$, then the polynomial defining $\nu_d^{-1}(H)$ has a double root.

Solution. This is a special case of the following more general claim which is also Exercise 3.16.

Claim: Let $L \in S_1$ be a linear form. If $\Lambda$ is a hyperplane in $\mathbb{P}^N$ containing $T_{[L]}(\nu_d(\mathbb{P}^n))$ then $\nu_d^{-1}(\Lambda)$ is a degree $d$ hypersurface singular at the point $[L] \in \mathbb{P}^n$.

Proof. We can find a basis for $S_1$ of the form $\{L, L_1, \ldots, L_n\}$ where $L_i$ are linear forms. With respect to this basis, in coordinates, and using the notation Lecture 1, the image of $[L]$ in $\mathbb{P}^N$ is the point $p = [1 : 0 : \cdots : 0]$. For $i = 0, \ldots, n$ let $p_i$ be the point of $\mathbb{P}^N$ with homogeneous coordinates $[Z_0 : \cdots : Z_N]$ given by $Z_j = \delta_{ij}$. Translating the intrinsic description of $T_{[L]}(\nu_d(\mathbb{P}^n))$ as the linear space $\langle L^d-1M \mid M \in S_1 \rangle$ into our coordinates for $\mathbb{P}^n$ and $\mathbb{P}^N$ with respect to the basis $\{L, L_1, \ldots, L_n\}$ for $S_1$ we conclude that $T_P(\nu(\mathbb{P}^n)) = \langle p_0, \ldots, p_n \rangle$. Let $F_\Lambda$ be a linear form defining $\Lambda$. Then $F_\Lambda = a_0Z_0 + \cdots a_NZ_N$. If $T_P(\nu_d(\mathbb{P}^n)) \subseteq \Lambda$ then $F_\Lambda(p_i) = 0$, for $i = 0, \ldots, n$. Hence $0 = a_0 = \cdots = a_n$. Let $f$ be the pullback of $F_\Lambda$ for $\mathbb{P}^n$. Then $f = a_{n+1}x_0^{d-2}x_1 + \cdots + a_Nx_n^d$. Clearly the partial derivatives of $f$ vanish at $q = [1 : 0 : \cdots : 0]$ which is identified with $[L]$. Hence the zero locus of $f$, a degree $d$ hypersurface, is singular at $q$. On the other hand, the zero locus of $f$ is $\nu_d^{-1}(\Lambda)$.

Exercise 2.22. Consider the rational normal curve in $\mathbb{P}^3$, i.e. the twisted cubic curve $X = \nu_3(\mathbb{P}(S_1)) \subset \mathbb{P}(S_3)$. We know that $\sigma_2(X)$ fills up all the space. Can we write any binary cubic as the sum of two cubes of linear forms? Try $x_0x_1^2$.

Solution. Direct calculation shows that we cannot write $x_0x_1^2$ as a linear combination of cubes of two linear forms.
Exercise 2.23. Let $X = \nu_d(\mathbb{P}^n)$. Recall that $[F] \in X$ if and only if $[F] = L^d$ for some linear form $L$ on $\mathbb{P}^n$. Use this description and standard differential geometry to compute $T_{[L^d]}(X)$.

Solution. Let $L$ be a linear form. Consider an affine curve passing through $L$. It will have the form $(L + tM)$ where $M$ is allowed to be any linear form. Considering the image in $\mathbb{P}^N$ we have

$$(L + tM)^d = L^d + \binom{d}{d-1} L^{d-1} t M + \text{terms containing higher powers of } t.$$ 

Taking the derivative with respect to $t$ and setting $t = 0$ we deduce that

$$T_{[F]} = \langle [L^d - 1 M] \mid M \in S_1 \rangle.$$ 

□

Exercise 3.11. For binary forms, we can stratify $\mathbb{P}S_2$ using the Waring rank: rank one elements correspond to points of the rational normal curve, while all the points outside the curve have rank two. Do the same for binary cubics and stratify $\mathbb{P}S_3 = \mathbb{P}^3$.

Solution. Let $X \subseteq \mathbb{P}^3$ be the rational normal curve. If $p$ is a point of $\mathbb{P}^3$ which does not lie on $X$, then considering the image of $X$ in $\mathbb{P}^2$ by projecting from $p$ we see that either $p$ lies on a tangent line to $X$ or that $p$ lies on a secant line to $X$. Suppose that $q$ is not a point of $X$ but that $q$ lies on a tangent line. If $X$ is the twisted cubic and $p = [L^3]$ then

$$T_p(X) = \langle [L^3], [L^2 M] \rangle$$

where $M$ is any linear form which is not a scalar multiple of $L$. Thus, without loss of generality, to show that $p$ can be written as a sum of 3 cubes it suffices to show that $x^2y$ is a sum of three cubes. For this, observe that

$$x^2y = \frac{1}{6}((x + y)^3 + (y - x)^3 - 2y^3).$$

Thus, $\mathbb{P}^3$ is stratified by Waring rank. Those points of rank 1 correspond to points of $X$. Those points of Waring rank 2 correspond to points which lie on no tangent line to $X$. Those points of Waring rank 3 correspond to points which lie on a tangent line and are not on $X$. All three of these sets are locally closed. □

Exercise 3.16. Prove the general statement. If $\Lambda$ is a hyperplane containing $T_{[L^d]}(\nu_d(\mathbb{P}^n))$ then $\nu_d^{-1}(\Lambda)$ is a degree $d$ hypersurface singular at the point $[L] \in \mathbb{P}^n$.

Solution. See the solution to Exercise 2.18. □

Exercise 3.21. Solve the big Waring problem for $n = 1$ using the double points interpretation.

Solution. We have $N = \binom{d+1}{d} - 1 = d$. On the other hand if $p_1, \ldots, p_s$ are general points of $\mathbb{P}^1$ then $\dim(\mathbb{P}^1 \cap \cdots \cap \mathbb{P}^1) = -2s + d + 1$ for $1 \leq s \leq d + 1$. So $N - 2s + d + 1 \geq N$ implies that $s \geq \lceil \frac{d+1}{2} \rceil$. In other words $g(1,d) = \lceil \frac{d+1}{2} \rceil$. □
Exercise 3.26. Show that $\sigma_5(\nu_4(\mathbb{P}^2))$ is a hypersurface, i.e. that it has dimension equal 13.

Solution. Since $\sigma_5(\nu_4(\mathbb{P}^2)) \neq \mathbb{P}^{13}$ we conclude $\dim \sigma_5(\nu_4(\mathbb{P}^2)) \geq \dim \sigma_4(\nu_4(\mathbb{P}^2)) + 2$. On the other hand we know $\dim \sigma_5(\nu_4(\mathbb{P}^2)) \leq 13$. Hence it suffices to show $\dim \sigma_4(\nu_4(\mathbb{P}^2)) \geq 11$. To see this we note that it costs at most 3 linear conditions for a plane curve to be singular at a point hence $\dim(p_2^2 \cap \ldots p_2^2)_{4} \geq 15 - 12 = 3$ for all collections of 4 points. Applying the double point lemma for a general collection of points, we deduce that $\dim \sigma_4(\nu_4(\mathbb{P}^2)) \geq 11$. \hfill $\square$

Exercise 3.27. Explain the exceptional cases $d = 2$ and any $n$.

Solution. Let us prove that $g(n, 2) = n + 1$. A general $n+1 \times n+1$ symmetric matrix has rank $n + 1$ and hence a linear combination of $n + 1$ symmetric matrices of rank 1 and is not a linear combination of any smaller number of rank 1 symmetric matrices. On the other hand, symmetric matrices of rank 1 are exactly those in the image of the quadratic Veronese map. This explains the exceptional case $d = 2$ for all $n$. \hfill $\square$

Exercise 3.28. Explain the exceptional cases $d = 4$ and $n = 3, 4$.

Solution. We want to show that $g(3, 4) = 10$ and not 9 and that $g(4, 4) = 15$ and not 14. In both cases a parameter count shows that we can find quadrics through 9 points and 14 points in $\mathbb{P}^3$ and $\mathbb{P}^4$ respectively. Squaring these forms produces a quartic singular at these points. Applying the double point lemma we conclude that $\sigma_9(\nu_4(\mathbb{P}^3))$ and $\sigma_{14}(\nu_4(\mathbb{P}^4))$ fail to fill up the space. On the other hand a parameter count also shows that $\sigma_{10}(\nu_4(\mathbb{P}^3))$ and $\sigma_{15}(\nu_4(\mathbb{P}^4))$ fill up the space. \hfill $\square$

Exercise 3.29. Explain the exceptional case $d = 3$ and $n = 4$. (Hint: use Castelnuovo’s Theorem which asserts that there exists a (unique) rational normal curve passing through $n + 3$ generic points in $\mathbb{P}^n$.)

Solution. By the double point lemma it suffices to pick 7 general points $p_1, \ldots, p_7 \in \mathbb{P}^4$ and prove that there exists a degree 3 hypersurface in $\mathbb{P}^4$ singular at $p_1, \ldots, p_7$. So choose $p_1, \ldots, p_7$ general points. By Castelnuovo’s Theorem, there exists a rational quartic curve $X$ in $\mathbb{P}^4$ passing through the points $p_1, \ldots, p_7$. It suffices to prove that $\sigma_2(X)$ is a degree 3 hypersurface singular along $X$.

To show that $\sigma_2(X)$ is singular along $X$ if $x \in X$, then we can show that $X \subseteq T_x \sigma_2(X)$ so that $\langle X \rangle \subseteq T_x \sigma_2(X)$. Since $X$ is non-degenerate we conclude that $T_x \sigma_2(X) = \mathbb{P}^4$, for all $x \in X$. Since $\sigma_2(X)$ has dimension 3 we conclude that $\sigma_2(X)$ is singular along $X$.

To compute the degree we project to $\mathbb{P}^2$ from a general secant line and count the number of nodes. Since the resulting curve is rational the number of nodes equals the arithmetic genus of a plane curve of degree 4 which is 3. \hfill $\square$

Exercise 4.2. Let $0 \neq F \in S_d$. Show that $F^\perp \subset T$ is an ideal and that it is also Artinian, i.e., $(T/F^\perp)_i = 0$ for all $i > d$.

Solution. If $\tilde{\partial} \in F^\perp$ and $\tilde{\partial} \in T$ then by definition of the $T$-action $\tilde{\partial} \delta F = (\tilde{\partial}) \delta F = 0$ so $\tilde{\partial} \in F^\perp$. On the other hand since $F$ has degree $d$, $F^\perp$ contains all differential operators of degree greater than $d$. Hence $T/F^\perp$ is a finite dimensional vector space and hence Artinian. \hfill $\square$
Exercise 4.3. Show that the map \( S_i \times T_i \rightarrow \mathbb{C} \), \((F, \partial) \mapsto \partial F\) is a perfect paring and that \( A = T/F^\perp \) is Artinian and Gorenstein with socle degree \( d \).

Solution. We use the standard monomial basis for \( S_i \) and \( T_i \). By definition of the action we have

\[
y_0^{a_0} \ldots y_n^{a_n} \circ x_0^{b_0} \ldots x_n^{b_n} = \begin{cases} 
a_0! \ldots a_n! & \text{iff } a_j = b_j, \text{ for } j = 0, \ldots, n \\
0 & \text{otherwise} \end{cases}
\]

from which the first assertion is clear.

It remains to show that \( A \) is Gorenstein with socle degree \( d \). Let \( \mathfrak{m} \) denote the homogeneous maximal ideal of \( A \). Then \( \text{Soc}(A) = \{x \in A \mid x\mathfrak{m} = 0\} \). Using this description we can check that

\[
\dim \text{Soc}(A)_i = \begin{cases} 
0 & \text{if } i < d \\
1 & \text{if } i = d.
\end{cases}
\]

Hence \( \text{Soc}(A) \) is 1 dimensional and nonzero only in degree \( d \) which implies that \( A \) is Gorenstein with socle degree \( d \). \( \square \)

Exercise 4.8. Given \( F \in S_d \) show that \( HF(T/F^\perp, t) \) is a symmetric function of \( t \).

Solution. Let \( A = T/F^\perp \) and let \( d \) be the socle degree of \( A \). It suffices to show that for \( 0 \leq l \leq d \), multiplication in \( A \) defines a perfect pairing \( A_{d-l} \times A_l \rightarrow A_d \).

If \( l = d \) or \( l = 0 \) the assertion is clear. Let \( 0 < l < d \). Let \( y \in A_l \). Suppose \( yx = 0 \) for all \( x \in A_{d-l} \). Let’s prove that \( y \in \text{Soc}(A) \). Since \( \text{Soc}(A) \) is zero in degrees less than \( d \) we will arrive at a contradiction.

To prove that \( y \in \text{Soc}(A) \) it suffices to prove that \( y \) annihilates every homogeneous element of \( A \) which has positive degree. If \( M \in A_n \) and \( n > d - l \) then \( yM \in A_{l+n} = 0 \) since \( l + n > d \). On the other hand we have by assumption that \( yA_{d-l} = 0 \). Descending induction with base case \( d - l \) proves that \( yA_l = 0 \) for \( 0 < l < d - l \). Indeed suppose \( 0 < n < d - l \) and let \( M \in A_n \). Suppose that \( yM \neq 0 \). We have \( Mx_i \in A_{n+1} \) for \( i = 0, \ldots, n \). By induction \( yMx_i = 0 \). Hence \( yM\mathfrak{m} = 0 \) so \( yM \in \text{Soc}(A) \). Since \( \deg yM < d \) this is a contradiction. Hence \( y \in \text{Soc}(A) \) which is also a contradiction. \( \square \)

Exercise 4.14. Use the Apolarity Lemma to compute \( \text{rk}(x_0x_1^2) \). Then try the binary forms \( x_0x_1^2 \).

Solution. \( F^\perp = \langle \partial_{x_0}^2, \partial_{x_1}^{d+1} \rangle \) so \( \text{rank } F = d+1 \). \( \square \)

Exercise 4.15. Use the Apolarity Lemma to explain the Alexander-Hirschowitz exceptional cases.

Solution. We explain the exceptional cases \( d = 4 \) and \( n = 2, 3 \) or 4. Then exceptional case \( d = 3 \) and \( n = 4 \) can be treated via syzygies.

Since \( \binom{2+2}{2} = 6 > 5 \) if \( I \) is the ideal of 5 points in \( \mathbb{P}^2 \), then \( I \) contains a quadric. Since \( \binom{2+3}{2} = 10 > 9 \) if \( I \) is the ideal of 9 points in \( \mathbb{P}^3 \), then \( I \) contains a quadric. Since \( \binom{2+4}{2} = 15 > 14 \) if \( I \) is the ideal of 14 points in \( \mathbb{P}^4 \), then \( I \) contains a quadric.

On the other hand, if \( F \) is a general form of degree 4, in \( \mathbb{C}[x_0, \ldots, x_n] \), for \( n = 2, 3 \) or 4, then \( F^\perp \) contains no quadrics. We work out the case \( n = 2 \) explicitly. The case \( n = 3 \) or 4 is similar.
Let $S = \mathbb{C}[x, y, z]$. Every form $F \in S_4$ determines a linear map from the vector space of differential operators of degree 2 to the space of degree 2 polynomials in $S$. This map is determined by applying the differential operators to $F$. Explicitly, if

$$F = ax^4 + bx^3y + cx^2z + dx^2y^2 + ex^2yz + fxz^2 + gy^3 + hy^2z + jyx^3 + ky^4 + ly^3z + mxyz + nx^2y + oz^3 + pz^4$$

then, using the basis $\partial_{xx}, \partial_{xy}, \partial_{xz}, \partial_{y^2}, \partial_{yz}, \partial_{z^2}$ for the source, and the basis $x^2, xy, xz, y^2, yz, z^2$ for the target space, the matrix for this map is given by

$$\begin{bmatrix}
12a & 3b & 3c & 2d & e & 2f \\
6b & 4d & 2e & 6g & 2h & 2i \\
6c & 2e & 4f & 2h & 2i & 6j \\
2d & 3g & h & 12k & 3l & 2m \\
2e & 2h & 2i & 6l & 4m & 6o \\
2f & i & 3j & 2m & 3o & 12p
\end{bmatrix}$$

Elements in the kernel of this map correspond to elements of $(F^⊥)^2$. The collection of forms for which this map is injective is given by the non-vanishing of the determinant of this matrix. We conclude that $(F^⊥)^2$ is zero for a general quartic form in $S_4$.

Note that we have shown that if $F$ is a general form of degree 4 in $\mathbb{C}[x_0, \ldots, x_n]$, for $n = 2, 3$ or 4, then

| $t$ | 0 | 1 | 2 | 3 | 4 |
|-----|---|---|---|---|---|
| $HF(T/F^⊥, t)$ | 1 | $n + 1$ | $\binom{n+2}{2}$ | $n + 1$ | 1 | 0 |

On the other hand, for every ideal $I$ of respectively, 5 points in $\mathbb{P}^2$, 9 points in $\mathbb{P}^3$, or 14 points in $\mathbb{P}^4$, we have shown that

$$HF(T/I, 2) < \binom{n+2}{2}.$$ 

Thus, using the Apolarity Lemma, a general quartic form in $\mathbb{C}[x_0, \ldots, x_n]$, for $n = 2, 3$ or 4, cannot have rank respectively, 5, 9, or 14.

To see an issue which arises in the case exceptional case $d = 3$ and $n = 4$, note that if $F$ is any cubic form in $\mathbb{C}[x_0, x_1, x_2, x_3, x_4]$ then

| $t$ | 0 | 1 | 2 | 3 | 4 |
|-----|---|---|---|---|---|
| $HF(T/F^⊥, t)$ | 1 | 5 | 5 | 1 | 0 |

Thus the Hilbert function $HF(T/F^⊥, t)$ is the same for all $F \in \mathbb{C}[x_0, x_1, x_2, x_3, x_4]$.
Nevertheless, the exceptional case $d = 3$ and $n = 4$ can still be explained using the Apolarity Lemma, although a more detailed study is needed to conclude that $F^\perp$ for a general $F \in \mathbb{C}[x_0, x_1, x_2, x_3, x_4]$ contains no ideal of 7 points in $\mathbb{P}^3$. See the paper [RS00] for a more detailed discussion.

**Exercise 4.17.** Compute $\text{rk}(F)$ when $F$ is a quadratic form.

**Solution.** $\text{rk}(F) = \text{rank } M_F$ where $M_F$ is the symmetric matrix associated to $F$. (See Exercise 2.12.)

**Exercise 4.20.** Prove that $\text{rk}(L^d + M^d + N^d) = 3$ whenever $L, M$ and $N$ are linearly independent linear forms.

**Solution.** It is clear that $\text{rk}(L^d + M^d + N^d) \leq 3$. To show that $\text{rk}(L^d + M^d + N^d)$ is not less than 3, without loss of generality it suffices to consider the case that $F = x_0^d + x_1^d + x_2^d$. In this case $F^\perp = (\partial_0^{d+1}, \partial_1^{d+1}, \partial_2^{d+1}, \partial_{x_3}, \ldots, \partial_{x_n})$. Now if $F^\perp$ contained the ideal of 1 or 2 distinct points in $\mathbb{P}^n$ then $(\partial_0^{d+1}, \partial_1^{d+1}, \partial_2^{d+1})$ would contain a linear form which it does not.

**Exercise 5.5.** Workout a matrix representation for the Segre varieties with two factors $\mathbb{P}^{n_1} \times \mathbb{P}^{n_2}$.

**Solution.** Let $A \cong \mathbb{C}^{n_1+1}$ and $B \cong \mathbb{C}^{n_2+1}$. By definition, points in $\text{Seg}(\mathbb{P}^{n_1} \times \mathbb{P}^{n_2})$ are of the form 

$$[a \otimes b] \in \mathbb{P}(A \otimes B).$$

Now consider $A \otimes B$ as a space of matrices $A^* \rightarrow B$. Then, by choosing bases of $A$ and $B$, we may represent $a$ as a column vector $(a_1, \ldots, a_{n_1+1})^t$ (an element of $A^*$) and $b$ as a row vector $(b_1, \ldots, b_{n_2+1})$ so that the tensor product $a \otimes b$ becomes the product of a column and a row:

$$a \otimes b = (a_1, \ldots, a_{n_1+1})^t \cdot (b_1, \ldots, b_{n_2+1}) = \begin{pmatrix} a_1 b_1 & a_1 b_2 & \cdots & a_1 b_{n_2+1} \\ a_2 b_1 & a_2 b_2 & \cdots & a_2 b_{n_2+1} \\ \vdots & \ddots & \ddots & \vdots \\ a_{n_1+1} b_1 & a_{n_1+1} b_2 & \cdots & a_{n_1+1} b_{n_2+1} \end{pmatrix}.$$

So we see that (up to scale) elements of $\text{Seg}(\mathbb{P}^{n_1} \times \mathbb{P}^{n_2})$ correspond to rank-one $(n_1 + 1) \times (n_2 + 1)$ matrices.

**Exercise 5.7.** Let $X = \mathbb{P}V_1 \times \cdots \times \mathbb{P}V_{\ell}$ and let $[v] = [v_1 \otimes \cdots \otimes v_{\ell}]$ be a point of $X$. Show that the cone over the tangent space to $X$ at $v$ is the span of the following vector spaces:

$$V_1 \otimes v_2 \otimes v_3 \otimes \cdots \otimes v_{\ell},$$

$$v_1 \otimes V_2 \otimes v_3 \otimes \cdots \otimes v_{\ell},$$

$$\vdots$$

$$v_1 \otimes v_2 \otimes \cdots \otimes v_{\ell-1} \otimes V_\ell.$$

**Solution.** The cone over the tangent space to a variety $X$ at a point $v$ may be computed by considering all curves $\gamma : [0, 1] \rightarrow \hat{X}$ such that $\gamma(0) = v$, and taking the linear span of all derivatives at the origin:

$$T_x \hat{X} = \left\{ \gamma'(0) \mid \gamma : [0, 1] \rightarrow \hat{X}, \gamma(0) = v \right\}.$$

Now take $X = \mathbb{P}V_1 \times \cdots \times \mathbb{P}V_{\ell}$ and let $[v] = [v_1 \otimes \cdots \otimes v_{\ell}]$ be a point of $X$. 
All curves $\gamma(t)$ on $\hat{X}$ through $v$ are of the form $\gamma(t) = v_1(t) \otimes \cdots \otimes v_t(t)$, where $v_i(t)$ are curves in $V_i$ such that $v_1(0) = v$.

Now apply the product rule, and for notational convenience, set $\gamma'_i = \gamma'_i(0) \in V_i$. We have

$$\gamma'(0) = \gamma'_1 \otimes \gamma'_2 \otimes \cdots \otimes \gamma'_t + v_1 \otimes \gamma'_2 \otimes v_3 \otimes \cdots \otimes v_t + \cdots + v_1 \otimes \cdots \otimes \gamma'_t.$$ 

Since $\gamma'_i$ can be anything in $V_i$ we get the result. □

**Exercise 5.8.** Show that $\sigma_2(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1) = \mathbb{P}^7$.

**Solution.** Let $a \otimes b \otimes c + a \otimes \tilde{b} \otimes \tilde{c}$ be a general point on $\sigma_2(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C)$, with $A \cong B \cong C \cong \mathbb{C}^2$.

By the previous exercise, we have

$$\widehat{T_{a\otimes b\otimes c}}(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1) = A \otimes b \otimes c + a \otimes B \otimes c + a \otimes b \otimes C,$$

and similarly

$$\widehat{T_{a\otimes \tilde{b} \otimes \tilde{c}}}(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1) = A \otimes \tilde{b} \otimes \tilde{c} + a \otimes B \otimes \tilde{c} + a \otimes \tilde{b} \otimes C.$$

Now by Terracini’s lemma we have

(2)

$$\widehat{T_{a\otimes b\otimes c + a\otimes \tilde{b} \otimes \tilde{c}}}(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1) = A \otimes b \otimes c + a \otimes B \otimes c + a \otimes b \otimes C + A \otimes \tilde{b} \otimes \tilde{c} + a \otimes B \otimes \tilde{c} + a \otimes \tilde{b} \otimes C.$$

Because we chose a general point, we have $\{a, \tilde{b}\} = A$ and similarly for $B$ and $C$. Now consider the linear space in (2). We see that we can get every tensor monomial in $A \otimes B \otimes C$ – all monomials with 0 or 1 occur in the first 3 summands, while all monomials with 2 or 3 occur in the second 3 summands.

So the cone over the tangent space at a general point is 8 dimensional, so the secant variety (being irreducible) fills the whole ambient space. □

**Exercise 5.9.** Use the above description of the tangent space of the Segre product and Terracini’s lemma to show that $\sigma_3(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1)$ is a hypersurface in $\mathbb{P}^{15}$ and not the entire ambient space as expected. This shows that the four-factor Segre product of $\mathbb{P}^1$s is defective.

**Solution.** This should be done on the computer. Take the following to be a general point:

$$a \otimes b \otimes c \otimes d + \tilde{a} \otimes \tilde{b} \otimes \tilde{c} \otimes \tilde{d} + (\alpha_1 a + \alpha_2 \tilde{a}) \otimes (\beta_1 b + \beta_2 \tilde{b}) \otimes (\gamma_1 c + \gamma_2 \tilde{c}) \otimes (\delta_1 d + \delta_2 \tilde{d}),$$

where $[\alpha_1, \alpha_2], [\beta_1, \beta_2], [\gamma_1, \gamma_2], [\delta_1, \delta_2] \in \mathbb{P}^1$.

Then compute the derivatives at the origin. Keep track of all the parameters and letting the $a(0)' \tilde{a}(0)'$ etc., vary produce 16 vectors that span the tangent space. Now compute the rank of the matrix with these vectors as columns. It gets hard to do by hand, but the next section produces an easier way to do the problem.

The “easier way” is to consider all 3 essentially different 2-flattening, and show that two of them are algebraically independent. □
Exercise 5.10. Show that $T$ has rank 1 if and only if its multilinear rank is $(1, \ldots, 1)$.

Solution. It is equivalent (by taking transposes) to consider the $n-1$ flattenings. Let $\varphi_{i,T}: (V_1 \otimes \cdots \otimes \hat{V}_i \otimes \cdots \otimes V_n)^* \to V_i$ denote the $(n-1)$-flattening to the $i$th factor. Denote by $A_i$ the image of $\varphi_{i,T}$.

If $\varphi_{i,T}$ has rank 1 then $\dim A_i = 1$. So, we must have $T \in A_1 \otimes \cdots \otimes A_n$. But every tensor in $A_1 \otimes A_2 \otimes \cdots A_n$ has rank 1 if all the factors have dimension 1.

Conversely, if $T = a_1 \otimes \cdots \otimes a_n \in V_1 \otimes \cdots \otimes V_n$ the image of $\varphi_{i,T}$ is the line through $a_i$, so the multilinear rank is $(1,1,\ldots,1)$.

Exercise 5.11. Let $X = \mathbb{P}V_1 \times \cdots \times \mathbb{P}V_t$. Show that if $r \leq r_i$ for $1 \leq i \leq t$,

$$\sigma_r(X) \subset \text{Sub}_{r_1,\ldots,r_t}.$$ 

Solution. A general point on $\sigma_r(X)$ is $p = \sum_{s=1}^{r} \bigotimes_{i=1}^{r} a_{i,s}$, where for fixed $i$ the $a_{i,s} \in V_i$ are linearly independent. Set $A_i = \{a_{i,1}, \ldots, a_{i,r}\}$. Then $p \in A_1 \otimes \cdots \otimes A_t$, so $p \in \text{Sub}_{r_1,\ldots,r_t}$. Now take the orbit closure of $p$ to obtain the result.

Exercise 5.12.

(1) Show that if $T$ has rank 1 then $\varphi_T$ has rank 2.

(2) Show that $\varphi$ is additive in its argument, i.e. show that $\varphi_{T+T'} = \varphi_T + \varphi_{T'}$.

Solution. If $T$ has rank 1, after change of coordinates we may assume that $T = v_1 \otimes w_1 \otimes x_1$. Then the matrix $\varphi_T$ has precisely two ones in different rows and columns, so clearly has rank 2.

For the second part, notice

$$\varphi_T + \varphi_{T'} = \begin{pmatrix} 0 & T^1 & -T^2 \\ -T^1 & 0 & T^3 \\ T^2 & -T^3 & 0 \end{pmatrix} + \begin{pmatrix} 0 & T'^1 & -T'^2 \\ -T'^1 & 0 & T'^3 \\ T'^2 & -T'^3 & 0 \end{pmatrix} = \begin{pmatrix} 0 & T^1 + T'^1 & -T^2 + T'^2 \\ -T^1 + T'^1 & 0 & T^3 + T'^3 \\ T^2 + T'^2 & -T^3 + T'^3 & 0 \end{pmatrix} = \varphi_{T+T'}.$$

7. Looking forward, further readings

This series of lectures draws from many sources from a large group of authors. We do our best to collect representative works here so that the reader may have some starting points for further study. A general introductory reference for the material we treated is the booklet is [Ger96]. Here we collect a few guiding questions and an extensive list of references.

Guiding questions. In our opinion, there are a few leading topics that are still driving current research in the area of secant varieties. These topics are dimension, identifiability, decomposition, and equations. More specifically here are 4 leading questions:

(1) For a variety $X$ what are the dimensions of the higher secant varieties to $X$? There is much interest when $X$ is the variety of elementary tensors of a given format (partially symmetric, skew symmetric, general).
(2) Suppose $p \in \sigma_s(X)$ is general, when does $p$ have a unique representation as a sum of $s$ points from $X$? When uniqueness occurs we say that $\sigma_s(X)$ is generically identifiable.

(3) Suppose $X \subset \mathbb{C}^N$ and $p \in \mathbb{C}^N$. If $X$ is not degenerate, then we know that there is some $s$ so that $p$ has a representation as the linear combination of $s$ points from $X$. For special $X$ (Segre, Veronese, etc.) find: (a) determine the minimal $s$ explicitly, and (b) find efficient algorithms when $s$ is relatively small to find such a decomposition of $p$.

(4) How do we find equations $\sigma_s(X)$ in general? What is the degree of $\sigma_s(X)$? Is $\sigma_s(X)$ a Cohen-Macaulay variety? Again, there is much interest when $X$ is the variety of elementary tensors of a given format.

**Background material.** Textbooks: [GKZ94, BV88, BCS97, CGH+05, CLO07, Eis05, Eis95, EC18, FOV99, FH91, Gan59, GW98, Har72, Har77, Lan12c, OSS80, PS05, Wey97, Wey03, Zak00].

Classical: [Ter11]. See also the nice overview in [Ger96] and the introduction of [RS00].

**Dimensions of secant varieties.** For Veronese varieties, the capstone result is that of Alexander and Hirschowitz, [AH95]. Ottaviani and Brambilla [BO08] provided a very nice exposition. Related work on polynomial interpolation: [AH92, BO11b].

Waring’s problem for binary forms: [IK99, CS11b, LT10] Waring’s problem for polynomials: [FOS12, Mel06, Mel09, Ott09, RS00, CM96, Chi04]. Waring’s problem for monomials was solved in [CCG12] and an alternative proof can be found also in [BBT13] where the apolar sets of points to monomials are described. See also [BCG11].

The polynomial Waring problem over the reals was investigated by Comon and Ottaviani, [CO12], with some solutions to their questions provided by [Ble12, CR11, Ball]. The case of real monomials in two variables is discussed in [BCG11]. For typical ranks of tensors see [CtBDLC09, Fri12, SSM13].

Recent algorithms for Waring decomposition: [IK99, OO13, BGI11, BCMT10, CGLM08].

The notion of Weak Defectivity: [CC02, CC06, Bal05].

Dimensions of Secant varieties of Segre-Veronese varieties: [CGG02b, CGG03, AOP09, CGG11, CGG05b, CGG08, Abr08, CGG05a, CGG02a, AB09, AB12, AB13, BCC11, BBC12]. Nice Summary of results on dimensions of secant varieties: [Cat].

**Equations of secant varieties.** See [Ott07, LM04, LW07b, Rai12, SS09, LW09, LO11b, Rai10, Str83, CEO12, LM08, Ber08, LW07a, Oed11, Kan99, Rai12, Rai11, SV11].

The Salmon Problem: [All, Fri13, BO11a, FG12].

**Applications.** The question of best low-rank approximation of tensors is often ill posed [dSL08], and most tensor problems are NP-Hard, [HL09].

Algebraic Statistics: [DSS07, DSS09, GS05, Stu09].

Phylogenetics: [AR03, AR08, CFS11, ERSS05].

Signal Processing: see [AFCC04, Com04, Com00, CR06, LdB08, DLFDMV02, DLCC07, DLC08, LMV04, Lat06, DBL07].

Matrix Multiplication: [Lan06, Lan08, Lan12a, Lan12b, LO11a].
Related varieties. Grassmannians: [CGG05c, AOP12, BB11b].
Discriminants and Hyperdeterminants are intimately related to secant varieties of Segre- Veronese varieties, see [CCD+11, GZK89, GKZ92, HSYY08, Oed12, BW00, WZ94, WZ96].
Chow varieties and monomials [AB11, Bri10, Car05].

Related concepts. Eigenvectors of Tensors: [Lim05, Qi05, BKP11, HHLQ13, OS13, NQWW07, CS11a, KM11].
Veronese reembeddings: [BB10]. Hilbert Schemes: [EV10].
Orbits: [Djo83, V`E78].
Ranks and Decompositions: [BB11a, BL13, BL11].
Asymptotic questions: [DK11].

Software. Symbolic computation: [DGPS10, CoC, GS, vLCL92].
Numerical Algebraic Geometry: [LLT, Ver, HSW11, BHSW08, BHSW, SW05, Li03].

REFERENCES

[AB09] H. Abo and M. C. Brambilla. Secant varieties of Segre-Veronese varieties $\mathbb{P}^n \times \mathbb{P}^n$ embedded by $\mathcal{O}(1,2)$. *Experiment. Math.*, 18(3):369–384, 2009.

[AB12] ———. New examples of defective secant varieties of Segre-Veronese varieties. *Collect. Math.*, 63(3):287–297, 2012.

[AB13] ———. On the dimensions of secant varieties of Segre-Veronese varieties. *Ann. Mat. Pura Appl. (4)*, 192(1):61–92, 2013.

[AOP09] H. Abo, G. Ottaviani, and C. Peterson. Induction for secant varieties of Segre varieties. *Trans. Amer. Math. Soc.*, 361(2):767–792, 2009.

[AOP12] ———. Non-defectivity of Grassmannians of planes. *J. Algebraic Geom.*, 21(1):1–20, 2012.

[Abr08] S. Abrescia. About the defectivity of certain Segre-Veronese varieties. *Canad. J. Math.*, 60(5):961–974, 2008.

[AFCC04] L. Albera, A. Ferreol, P. Comon, and P. Chevalier. Blind identification of over-complete mixtures of sources (BIOME). *Lin. Algebra Appl.*, 391:3–30, November 2004.

[AH92] J. Alexander and A. Hirschowitz. La méthode d’Horace éclatée: application à l’interpolation en degré quatre. *Invent. Math.*, 107(3):585–602, 1992.

[AH95] ———. Polynomial interpolation in several variables. *J. Algebraic Geom.*, 4(2):201–222, 1995.

[All] E. Allman. Open Problem: Determine the ideal defining Seca$(\mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^3)$, http://www.dms.uaf.edu/~eallman/salmonPrize.pdf, 2010.

[AR03] E. Allman and J. Rhodes. Phylogenetic invariants for the general Markov model of sequence mutation. *Math. Biosci.*, 186(2):113–144, 2003.

[AR08] ———. Phylogenetic ideals and varieties for the general Markov model. *Adv. in Appl. Math.*, 40(2):127–148, 2008.

[AB11] E. Arrondo and A. Bernardi. On the variety parameterizing completely decomposable polynomials. *Journal of Pure and Applied Algebra*, 215(3):201–220, 2011.

[BKP11] G. Ballard, T. Kolda, and T. Plantenga. Efficiently computing tensor eigenvalues on a GPU, 2011. CSRI Summer Proceedings.

[Bal05] E. Ballico. On the weak non-defectivity of Veronese embeddings of projective spaces. *Cent. Eur. J. Math.*, 3(2):183–187, 2005.

[Bal12] ———. On the typical rank of real bivariate polynomials. *ArXiv e-prints*, April 2012.

[BB11a] E. Ballico and A. Bernardi. Decomposition of homogeneous polynomials with low rank. *Mathematische Zeitschrift*, 271(3-4):1141–1149, 2011.

[BBC12] E. Ballico, A. Bernardi, and M. V. Catalisano. Higher secant varieties of $\mathbb{P}^n \times \mathbb{P}^1$ embedded in bi-degree $(a,b)$. *Comm. Algebra*, 40(10):3822–3840, 2012.
[BHSW08] D.J. Bates, J.D. Hauenstein, A.J. Sommese, and C.W. Wampler. Software for numerical algebraic geometry: a paradigm and progress towards its implementation. In Software for algebraic geometry, volume 148 of IMA Vol. Math. Appl., pages 1–14. Springer, New York, 2008.

[BHSW] Bertini: Software for Numerical Algebraic Geometry. Available at http://www.nd.edu/~sommese/bertini, 2010.

[BO11a] D. J. Bates and L. Oeding. Toward a salmon conjecture. Exp. Math., 20(3):358–370, 2011.

[Ber08] A. Bernardi. Ideals of varieties parameterized by certain symmetric tensors. J. Pure Appl. Algebra, 212(6):1542–1559, 2008.

[BCC11] A. Bernardi, E. Carlini, and M. V. Catalisano. Higher secant varieties of $P^n \times P^m$ embedded in bi-degree $(1, d)$. J. Pure Appl. Algebra, 215(12):2853–2858, 2011.

[BGI11] A. Bernardi, A. Gimigliano, and M. Idà. Computing symmetric rank for symmetric tensors. Journal of Symbolic Computation, 46(1):34–53, 2011.

[Ble12] G. Blekherman. Typical Real Ranks of Binary Forms. ArXiv e-prints, May 2012.

[BW00] G. Boffi and J. Weyman. Koszul complexes and hyperdeterminants. J. Algebra, 230(1):68–88, 2000.

[BCG11] M. Boij, E. Carlini, and A. V. Geramita. Monomials as sums of powers: the real binary case. Proc. Amer. Math. Soc., 139(9):3039–3043, 2011.

[BB11b] A. Boralevi and J. Buczyński. Secants of Lagrangian Grassmannians. Ann. Mat. Pura Appl. (4), 190(4):725–739, 2011.

[BCMT10] J. Brachat, P. Comon, B. Mourrain, and E. Tsigaridas. Symmetric tensor decomposition. Linear Algebra and its Applications, 433(11-12):1851–1872, 2010.

[BO08] M.C. Brambilla and G. Ottaviani. On the Alexander-Hirschowitz theorem. J. Pure Appl. Algebra, 212(5):1229–1251, 2008.

[BO11b] On partial polynomial interpolation. Linear Algebra and its Applications, 435(6):1415 – 1445, 2011.

[Bri10] E. Briand. Covariants vanishing on totally decomposable forms. In Liaison, Schottky problem and invariant theory, volume 280 of Progr. Math., pages 237–256. Birkhäuser Verlag, Basel, 2010.

[BB10] W. Buczyński and J. Buczyński. Secants of Lagrangian Grassmannians. Ann. Mat. Pura Appl. (4), 190(4):725–739, 2011.

[BL13] J. Buczyński and J.M. Landsberg. Ranks of tensors and a generalization of secant varieties. Linear Algebra Appl., 438(2):668–689, 2013.

[BCS97] W. Bruns and U. Vetter. Determinantal rings, volume 1327 of Lecture Notes in Mathematics. Springer-Verlag, Berlin, 1988.

[BV88] W. Bruns and C. Rotthaus. Monomial ideals and cohomology modules. J. Algebra, 109(2):481–502, 1987.

[BL11] J. Buczyński and J. M. Landsberg. On the third secant variety. ArXiv e-prints, November 2011. to appear: Journal of Algebraic Combinatorics.

[BL13] J. Buczyński and J.M. Landsberg. Ranks of tensors and a generalization of secant varieties. Linear Algebra Appl., 438(2):668–689, 2013.

[BCS97] P. Bürigisser, M. Clausen, and M. Shokrollahi. Algebraic complexity theory, volume 135 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Berlin: Springer-Verlag, 1997.

[Car05] E. Carlini. Codimension one decompositions and Chow varieties. In Projective varieties with unexpected properties, pages 67–79. Walter de Gruyter GmbH & Co. KG, Berlin, 2005.

[CCG12] E. Carlini, M.V. Catalisano, and A. V. Geramita. The solution to the Waring problem for monomials and the sum of coprime monomials. J. Algebra, 370:5–14, 2012.

[CK11] E. Carlini and J. Kleppe. Ranks derived from multilinear maps. J. Pure Appl. Algebra, 215(8):1999–2004, 2011.

[CEO12] D. Cartwright, D. Erman, and L. Oeding. Secant varieties of $P^2 \times P^n$ embedded by $O(1, 2)$. Journal of the London Mathematical Society, 85(1):121–141, 2012.

[CS11a] D. Cartwright and B. Sturmfels. The number of eigenvalues of a tensor. Linear Algebra and its Applications, 438(2):942–952, 2013.

[CF11] M. Casanellas and J. Fernández-Sánchez. Relevant phylogenetic invariants of evolution models. J. Math. Pures Appl. (9), 96(3):207–229, 2011.
M. V. Catalisano. Higher secant varieties of Segre, Segre-Veronese and Grassmann varieties. \textit{Interactions between Commutative Algebra and Algebraic Geometry the 22nd annual Route 81 conference — in honour of Tony Geramita [Conference],} Kingston, Ontario. 20 Oct. 2012. \url{http://www.mast.queensu.ca/~ggsmith/Route81/catalisano.pdf}.

M. V. Catalisano, A. V. Geramita, and A. Gimigliano. On the rank of tensors, via secant varieties and fat points. Zero-dimensional schemes and applications - Naples, 2000. 123:133–147, 2002.

M. V. Catalisano, A. V. Geramita, and A. Gimigliano. On the rank of tensors, via secant varieties and fat points. \textit{Linear Algebra Appl.}, 355:263–285, 2002.

Ranks of tensors, secant varieties of Segre varieties and fat points. \textit{Linear Algebra Appl.}, 367:347–348, 2003.

Higher secant varieties of Segre-Veronese varieties. In \textit{Projective varieties with unexpected properties}, pages 81–107. Berlin: Walter de Gruyter GmbH & Co. KG, 2005.

Higher secant varieties of the Segre varieties $P^1 \times \cdots \times P^1$. \textit{J. Pure Appl. Algebra}, 201(1-3):367–380, 2005.

Secant varieties of Grassmann varieties. \textit{Proc. Amer. Math. Soc.}, 133(3):633–642 (electronic), 2005.

On the ideals of secant varieties to certain rational varieties. \textit{J. Algebra}, 319(5):1913–1931, 2008.

Higher secant varieties of $P^1 \times \cdots \times P^1$ (n-times) are not defective for $n \geq 5$. \textit{J. Algebraic Geom.}, 20(2):295–327, 2011.

E. Cattani, M.A. Cueto, A. Dickenstein, S. Di Rocco, and B. Sturmfels. \textit{Mixed Discriminants}. \textit{Math. Z.} 274(3-4):761–778, 2013.

A. Causa and R. Re. On the maximum rank of a real binary form. \textit{Annali di Matematica Pura ed Applicata}, 190:55–59, 2011.

L. Chiantini and C. Ciliberto. Weakly defective varieties. \textit{Trans. Amer. Math. Soc.}, 354(1):151–178 (electronic), 2002.

On the concept of $k$-secant order of a variety. \textit{J. London Math. Soc. (2)}, 73(2):436–454, 2006.

J. Chipalkatti. The Waring loci of ternary quartics. \textit{Experiment. Math.}, 13(1):93–101, 2004.

C. Ciliberto, A. V. Geramita, B. Harbourne, R. M. Miró-Roig, and K. Ranestad, editors. \textit{Projective varieties with unexpected properties}. Berlin: Walter de Gruyter GmbH & Co. KG, 2005. A volume in memory of Giuseppe Veronese.

G. Comas and M. Seiguer. On the rank of a binary form. \textit{Signal Processing}, 86(9):2271–2281, 2006.

P. Comon. Symmetric tensors and symmetric tensor rank. \textit{SIAM J. Matrix Anal. Appl.}, 30(3):1254–1279, 2008.

P. Comon and B. Mourrain. Decomposition of quantics in sums of powers of linear forms. \textit{Signal Processing}, 53(2):93–107, September 1996. Special issue on High-Order Statistics.

P. Comon and G. Ottaviani. On the typical rank of real binary forms. \textit{Linear and Multilinear Algebra}, 60(6):657–667, 2012.

P. Comon, J. M. F. ten Berge, L. De Lathauwer, and J. Castaing. Generic and typical ranks of multi-way arrays. \textit{Linear Algebra Appl.}, 430(11-12):2997–3007, 2009.

D. Cox, J. Little, and D. O’Shea. \textit{Ideals, varieties, and algorithms: An introduction to computational algebraic geometry and commutative algebra}. Undergraduate Texts in Mathematics. New York: Springer, third edition, 2007.
FOUR LECTURES ON SECANT VARIETIES

[CoC] CoCoA: a system for doing Computations in Commutative Algebra. Available at http://cocoa.dima.unige.it.

[DBL07] CoCoATeam. CoCoA: a system for doing Computations in Commutative Algebra. Available at http://cocoa.dima.unige.it.

[DGPS10] W. Decker, G.-M. Greuel, G. Pfister, and H. Schönemann. Singular 3-1-1 — A computer algebra system for polynomial computations. http://www.singular.uni-kl.de.

[DLC08] L. De Lathauwer and J. Castaing. Blind identification of underdetermined mixtures by simultaneous matrix diagonalization. Signal Processing, IEEE Transactions on, 56(3):1096–1105, March 2008.

[DLCC07] L. De Lathauwer, C. Févotte, B. De Moor, and J. Vandevelle. Jacobi algorithm for joint block diagonalization in blind identification. In Proc. 23rd Symp. on Information Theory in the Benelux, pages 155–162, Louvain-la-Neuve, Belgium, May 2002.

[dSL08] V. de Silva and L.H. Lim. Tensor rank and the ill-posedness of the best low-rank approximation problem. SIAM J. Matrix Anal. Appl., 30:1084–1127, September 2008.

[Djo83] Dragomir Ž. Đoković. Closures of equivalence classes of trivectors of an eight-dimensional complex vector space. Canad. Math. Bull., 26(1):92–100, 1983.

[DK11] J. Draisma and J. Kuttler. Bounded-rank tensors are defined in bounded degree. ArXiv e-prints, March 2011.

[DSS07] M. Drton, B. Sturmfels, and S. Sullivant. Algebraic factor analysis: tetrads, pentads and beyond. Probab. Theory Related Fields, 138(3-4):463–493, 2007.

[DSS09] Lectures on algebraic statistics. Oberwolfach Seminars 39. Basel: Birkhäuser. viii, 271 p., 2009.

[Eis95] D. Eisenbud. Commutative algebra with a view toward algebraic geometry. Springer-Verlag, New York, 1995. Graduate Texts in Mathematics, No.150.

[Eis05] The geometry of syzygies, second course in commutative algebra and algebraic geometry. Springer-Verlag, New York, 2005. Graduate Texts in Mathematics, No. 229.

[EC18] F. Enriques and O. Chisini. Lezioni sulla teoria geometrica delle equazioni e delle funzioni algebriche. Zanichelli, Bologna, 1918.

[ERS05] N. Eriksson, K. Ranestad, B. Sturmfels, and S. Sullivant. Phylogenetic algebraic geometry. In Projective varieties with unexpected properties, pages 237–255. Walter de Gruyter GmbH & Co. KG, Berlin, 2005.

[EV10] E. Erman and M. Velasco. A syzygetic approach to the smoothability of zero-dimensional schemes. Adv. in Math., 224:1143–1166, 2010.

[FO19] H. Flenner, L. O’Carroll, and W. Vogel. Joins and intersections. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 1999.

[Fri12] S. Friedland. On the generic and typical ranks of 3-tensors. Linear Algebra Appl., 436(3):478–497, 2012.

[Fri13] On tensors of border rank $l$ in $C^m \times n \times l$, Linear Algebra Appl., 438(2):713–737, 2013.

[FG12] S. Friedland and E. Gross. A proof of the set-theoretic version of the salmon conjecture. J. Algebra, 356:374–379, 2012.

[FOS12] R. Fröberg, G. Ottaviani, and B. Shapiro. On the waring problem for polynomial rings. Proceedings of the National Academy of Sciences, 109(15):5600–5602, 2012.

[FH91] W. Fulton and J. Harris. Representation Theory, a First Course. New York: Springer-Verlag, 1991. Graduate Texts in Mathematics, No. 129.

[Gan59] F. R. Gantmacher. Applications of the theory of matrices. Translated by J. L. Brenner, with the assistance of D. W. Bushaw and S. Evanusa. Interscience Publishers, Inc., New York, 1959.

[GSS05] L. Garcia, M. Stillman, and B. Sturmfels. Algebraic geometry of Bayesian networks. J. Symbolic Comput., 39(3-4):331–355, 2005.

[GS] D. R. Grayson and M. E. Stillman. Macaulay2, a software system for research in algebraic geometry. Available at http://www.math.uiuc.edu/Macaulay2/.
[GKZ92] I. M. Gel’fand, M. M. Kapranov, and A. V. Zelevinsky. Hyperdeterminants. *Adv. Math.*, 96(2):226–263, 1992.

[Ger96] A. V. Geramita. Inverse systems of fat points: Waring’s problem, secant varieties of Veronese varieties and parameter spaces for Gorenstein ideals. In *The Curves Seminar at Queen’s, Vol. X (Kingston, ON, 1995)*, volume 102 of *Queen’s Papers in Pure and Appl. Math.*, pages 2–114. Queen’s Univ., Kingston, ON, 1996.

[GW98] R. Goodman and N. Wallach. *Representations and invariants of the classical groups*, volume 68 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, 1998.

[Har92] J. Harris. *Algebraic geometry: A First Course*. New York: Springer-Verlag, 1992. Graduate Texts in Mathematics, No. 133.

[Har77] R. Hartshorne. *Algebraic geometry*. Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52.

[HM04] J. M. Landsberg and L. Manivel. On the ideals of secant varieties of Segre varieties. *Found. Comput. Math.*, 4(4):397–422, 2004.

[LO11a] J. M. Landsberg and G. Ottaviani. New lower bounds for the border rank of matrix multiplication. *ArXiv e-prints*, December 2011.

[LO11b] J. M. Landsberg and G. Ottaviani. Equations for secant varieties of veronese and other varieties. *Annali di Matematica Pura ed Applicata*, pages 1–38, 2011.

[LT10] J. M. Landsberg and Z. Teitler. On the ranks and border ranks of symmetric tensors. *Found. Comput. Math.*, 10(3):339–366, 2010.

[LW07a] J. M. Landsberg and J. Weyman. On tangential varieties of rational homogeneous varieties. *J. Lond. Math. Soc. (2)*, 76(2):513–530, 2007.

[LW07b] J. M. Landsberg and J. Weyman. On the ideals and singularities of secant varieties of Segre varieties. *Bull. Lond. Math. Soc.*, 39(4):685–697, 2007.

[LW09] J. M. Landsberg and J. Weyman. On secant varieties of compact Hermitian symmetric spaces. *J. Pure Appl. Algebra*, 213(11):2075–2086, 2009.
[Lat06] L. De Lathauwer. A link between the canonical decomposition in multilinear algebra and simultaneous matrix diagonalization. *SIAM Journal on Matrix Analysis and Applications*, 28(3):642–666, 2006.

[LdB08] L. De Lathauwer and A. de Baynast. Blind deconvolution of DS-CDMA signals by means of decomposition in rank-\((1, L, L)\) terms. *IEEE Trans. Signal Processing*, 56(4):1562–1571, 2008.

[LMV04] L. De Lathauwer, B. De Moor, and Joos Vandewalle. Computation of the canonical decomposition by means of a simultaneous generalized Schur decomposition. *SIAM Journal on Matrix Analysis and Applications*, 26(2):295–327, 2004.

[LLT] T.L. Lee, T.Y. Li, and C.H. Tsai. HOM4PS-2.0: A software package for solving polynomials systems by the polyhedral homotopy continuation method. Available at http://www.math.nus.edu.sg/~li/, 2010.

[Li03] T.Y. Li. Numerical solution of polynomial systems by homotopy continuation methods. volume XI of *Handbook of Numerical Analysis, Special Volume: Foundations of Computational Mathematics*, pages 209–304. North-Holland, 2003.

[Lim05] L.H. Lim. Singular values and eigenvalues of tensors: a variational approach. In *Computational Advances in Multi-Sensor Adaptive Processing, 2005 1st IEEE International Workshop on*, pages 129–132, December 2005.

[Mac1927] F. S. Macaulay. Some Properties of Enumeration in the Theory of Modular Systems. *Proc. London Math. Soc.*, S2-26(1):531–555, 1927.

[Mel06] M. Mella. Singularities of linear systems and the Waring problem. *Trans. Amer. Math. Soc.*, 358(12):5523–5538 (electronic), 2006.

[Mel09] L. Mella. Base loci of linear systems and the Waring problem. *Proc. Amer. Math. Soc.*, 137(1):91–98, 2009.

[NQWW07] G. Ni, L. Qi, F. Wang, and Y. Wang. The degree of the E-characteristic polynomial of an even order tensor. *J. Math. Anal. Appl.*, 329(2):1218–1229, 2007.

[Oed11] L. Oeding. Set-theoretic defining equations of the tangential variety of the Segre variety. *Journal of Pure and Applied Algebra*, 215(6):1516 – 1527, 2011.

[Oed12] L. Oeding. Hyperdeterminants of polynomials. *Adv. Math.*, 231(3-4):1308–1326, 2012.

[OO13] L. Oeding and G. Ottaviani. Eigenvectors of tensors and algorithms for waring decomposition. *Journal of Symbolic Computation*, (54):9–35, 2013.

[OSS80] C. Okonek, M. Schneider, and H. Spindler. *Vector bundles on complex projective spaces*, volume 3 of *Progress in Mathematics*. Birkhäuser Boston, Mass., 1980.

[OS13] G. Ottaviani and B. Sturmfels. Matrices with Eigenvectors in a Given Subspace. *Proc. of the American Math. Soc.*, 141:1219–1232, 2013.

[Ott07] G. Ottaviani. Symplectic bundles on the plane, secant varieties and Lüroth quartics revisited. In *Vector bundles and low codimensional subvarieties: state of the art and recent developments*, volume 21 of *Abel Symp.*, pages 315–352. Dept. Math., Seconda Univ. Napoli, Caserta, 2007.

[Ott09] G. Ottaviani. An invariant regarding Waring’s problem for cubic polynomials. *Nagoya Math. J.*, 193:95–110, 2009.

[PS05] L. Pachter and B. Sturmfels, editors. *Algebraic statistics for computational biology*. New York: Cambridge University Press, 2005.

[Qi05] L. Qi. Eigenvalues of a real supersymmetric tensor. *J. Symbolic Comput.*, 40(6):1302–1324, 2005.

[Rai10] C. Raicu. 3 × 3 Minors of Catalecticants. *ArXiv e-prints*, November 2010.

[Rai11] C. Raicu. *Secant Varieties of Segre-Veronese Varieties*. ProQuest LLC, Ann Arbor, MI, 2011. Thesis (Ph.D.), University of California, Berkeley.

[Rai12] C. Raicu. Secant varieties of Segre–Veronese varieties. *Algebra and Number Theory*, 6-8:1817–1868, 2012.

[RS00] K. Ranestad and F.O. Schreyer. Varieties of sums of powers. *J. Reine Angew. Math.*, 525:147–181, 2000.

[SS09] J. Sidman and S. Sullivant. Prolongations and computational algebra. *Canad. J. Math.*, 61(4):930–949, 2009.

[SV11] J. Sidman and P. Vermeire. Equations defining secant varieties: geometry and computation. In *Combinatorial aspects of commutative algebra and algebraic geometry*, volume 6 of *Abel Symp.*, pages 155–174. Springer, Berlin, 2011.
Numerical solution of polynomial systems arising in engineering and science. World Scientific, Singapore, 2005.

T. Sumi, T. Sakata, and M. Miyazaki. Typical ranks for \( m \times n \times (m - 1)n \) tensors with \( m \leq n \). Linear Algebra Appl., 438(2):953–958, 2013.

V. Strassen. Rank and optimal computation of generic tensors. Linear Algebra Appl., (52-53):645–685, 1983.

B. Sturmfels. Open problems in algebraic statistics. In S. Sullivant and B. Sturmfels, editors, Emerging Applications of Algebraic Geometry, volume 149 of The IMA Volumes in Mathematics and its Applications, pages 1–13. Springer New York, 2009.

A. Terracini. Sulle \( \alpha \) per cui la varietà degli \( \alpha + 1 \)-secanti ha dimensione minore dell’ordinario. Rend. Circ. Mat. Palermo, Selecta vol. 1:392–396, 1911.

N. Trefethen. The smart money is on numerical analysts. 45(9), 2012.

M. A. A. van Leeuwen, A. M. Coehn, and B. Lisser. LiE, A Package for Lie Group Computations. Computer Algebra Nederland, 1992.

J. Verschelde. PHCpack: a general-purpose solver for polynomial systems by homotopy continuation. Available at http://www.math.uic.edu/~jan/PHCpack/phcpack.html, 2010.

E. B. Vinberg and A. G. Èlašvili. A classification of the three-vectors of nine-dimensional space. Trudy Sem. Vektor. Tenzor. Anal., 18:197–233, 1978.

H. Weyl. The classical groups; Princeton Landmarks in Mathematics. Princeton University Press, Princeton, NJ, 1997. Their invariants and representations, Fifteenth printing, Princeton Paperbacks.

J. Weyman. Cohomology of vector bundles and syzygies, volume 149 of Cambridge Tracts in Mathematics. Cambridge University Press, 2003.

J. Weyman and A. Zelevinsky. Multiplicative properties of projectively dual varieties. Manuscripta Math., 82(2):139–148, 1994.

F. L. Zak. Tangents and secants of algebraic varieties, volume 127 of Translations of Mathematical Monographs. Providence: American Mathematical Society, 1993. Translated from the Russian manuscript by the author.