I. INTRODUCTION

Data replication consists of maintaining multiple copies of data, called replicas, on separate computing entities. It is a critical enabling technology in distributed systems, improving system performance, reliability, and scalability [1], [2], [3]. Practically, it is desirable for a data replication system to achieve three properties simultaneously, namely data consistency (C), availability (A), and partition-tolerance (P) [4]. However, this has been theoretically proved impossible by the CAP theorem [5], [6]. The impossibility result leads to multiple balance options, among which modern commercial data replication systems often choose to sacrifice consistency under network partitions and certain failure scenarios for high availability. Thus, researchers have developed various weak consistency models such as PRAM consistency (Pipelined RAM) [7], cache consistency [8] (a.k.a. memory coherence [9]), causal consistency [10], processor consistency [11], and eventual consistency [12], besides the strong ones such as linearizability [13] (a.k.a. atomicity [14]) and sequential consistency [15]. [16]. For example, Yahoo!’s PNUTS [2] provides per-record timeline consistency (similar to the processor consistency). Amazon’s Dynamo [1] only promises eventual consistency. Nowadays, weak consistency is playing a more and more important role, with the prevalence of cloud data storage services, mobile devices, and wireless communications.

In this work, we focus on PRAM consistency [7], one of the well-known weak consistency models. Informally, a read/write trace satisfies PRAM consistency if and only if write operations performed by a single process are observed by all the other processes in the order they were issued, whereas write operations from different processes may be observed in different orders by different processes [11]. To illustrate its practical usefulness, let us consider the photo sharing application described in [2]. In this application, users can post photos and control their accesses. Now Alice wishes to share some photos with her classmates but not with her mother. She does a sequence of updates to her album; adds her classmates to and removes her mother from the album access list, and then posts photos. Under PRAM consistency, the updates from Alice are guaranteed to be seen by any user in the order they were issued.

Different protocols can be designed to guarantee PRAM consistency. However, theoretically correct protocols can suffer from buggy implementations and unexpected runtime failures. Furthermore, the implementations of such systems, when they are published as commercial web services, are often inaccessible to users. Thus, the users can only test the system by observing and analyzing its logs (i.e., read/write traces of operations) to verify whether it is delivering promised consistency [17]. Though weak consistency models are regarded important, to the best of our knowledge, their verification problems have not been sufficiently studied yet. In this work, we systematically study the problem of verifying PRAM consistency over read/write traces (VPC, for short). Specifically,

- First, we identify four variants of VPC according to whether (a) there are Multiple shared variables (or one Single variable), (b) whether write operations can assign Duplicate values (or only Unique values) for each shared variable; the four variants are labeled VPC-SU, VPC-MU, VPC-SD, and VPC-MD. Second, we present a simple VPC-MU algorithm, called RW-CLOSURE. It constructs an operation graph $G$ by iteratively adding edges according to three rules. Its time complexity is $O(n^2)$, where $n$ is the number of operations in the trace. Third, we present an improved VPC-MU algorithm, called READ-CENTRIC, with time complexity $O(n^4)$. Basically it attempts to construct the operation graph $G$ in an incremental and efficient way. Its correctness is based on that of RW-CLOSURE. Finally, we prove that VPC-SD (so is VPC-MD) is NP-complete by reducing the strongly NP-complete problem 3-PARTITION to it.

Index Terms—Consistency, PRAM, Replication, Verification.
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- Second, we present a simple VPC-MU algorithm, called RW-CLOSURE. It constructs an operation graph \( G \) by iteratively adding edges according to three rules. Its time complexity is \( O(n^5) \), where \( n \) is the number of operations in the trace.

- Third, we present an improved VPC-MU algorithm, called READ-CENTRIC, with time complexity \( O(n^3) \). Basically it attempts to construct the operation graph \( G \) in an incremental and efficient way. It is incremental in that it processes, one at a time, the read operations. It is efficient because for each read operation, it applies the three rules in a Read-induced subgraph and organize them in a reverse topological order of the subgraph. Its correctness is based on that of RW-CLOSURE.

- Finally, we prove that VPC-SD (so is VPC-MD) is NP-complete by reducing the strongly NP-complete problem 3-PARTITION \([18],[19]\) to it.

The rest of this paper is organized as follows. Section \[II\] discusses the related work. Section \[III\] defines the problem of verifying PRAM consistency over read/write traces and its four variants. Sections \[IV\] and \[V\] present the two VPC-MU algorithms: RW-CLOSURE and READ-CENTRIC respectively. Section \[VI\] gives the NP-completeness proof of VPC-SD (so is VPC-MD). Section \[VII\] concludes the paper including suggestions for future work.

II. RELATED WORK

Many efforts have been made on the verification problems with respect to other consistency models than PRAM. In their seminal work, Gibbons and Korach \([20]\) study the verifying sequential consistency (VSC) and the verifying linearizability (VL) problems. Both problems are proved to be NP-complete in general. In addition, they define the VSC-read problem, in which a read-mapping is known, and prove that it remains NP-complete. Here a read-mapping is a function mapping each read operation to a write operation which was responsible for the value read. Cantin et al. \([21]\) show that the verifying memory coherence problem (VMC) is NP-complete. They also prove that the problem of verifying sequential consistency for executions that are memory coherent (VSCL) remains NP-complete. Golab et al. \([17]\) study the verification problems with respect to safety, atomicity, regularity, and sequential consistency. Beyond a yes/no answer, they seek online algorithms to detect a consistency violation as soon as it appears. They also consider how to quantify the severity of violations. More recently, Golab et al. \([22]\) solve the verification problem of 2-atomicity (2-AV) and show that the weighted \( k \)-AV problem is NP-complete. In this work we investigate the verifying PRAM consistency (VPC) problem. As far as we know, we are the first to systematically solve this problem.

In the context of shared memory multiprocessor, some relaxed memory consistency models have been studied \([24],[25]\). Specifically, Hangal et al. \([23]\) develop TSOtool to verify the traces of programs against Total Store Order model when a read-mapping is known (VTSO-read). The time complexity of their algorithm is \( O(n^5) \), where \( n \) is the number of operations in the trace. Roy et al. \([24]\) also deal with the VTSO-read problem and present a fully parallelized algorithm with \( O(n^3) \) time complexity. Baswana et al. \([25]\) identify a graph problem called implied-set-closure as the abstraction of the bottleneck of the VTSO-read problem, and further reduce its time complexity to \( O(n^3) \). However, all the above algorithms only do approximate checking because the problem itself is NP-complete \([23]\). In contrast, we show that the VPC problem for traces in which write operations do not assign duplicate values (thus a read-mapping is known) can be completely solved in polynomial time. Although its basic idea is simple and resembles that of \([23]\), its correctness proof is one of our key contributions. On the other hand, we prove its NP-completeness for other traces.

III. PROBLEM DEFINITION

In this section, we first define read/write traces of data replicas and PRAM consistency, and then define the problem of verifying PRAM consistency over read/write traces.

A. Read/Write Trace

We model the data replicas as a collection of read/write shared variables supporting read/write operations, and the separate computing entities as a collection of processes.

Definition III.1 (Operation \((o)\)). An operation \( o \) is a quadruple \((t,p,v,d)\) \(\in \{R,W\} \times P \times V \times D\) where,

- \( t \in \{R,W\} \) is the type of operation (R for read and W for write).
- \( p \in P \) is the process issuing the operation.
- \( v \in V \) is the variable to which the operation is applied.
- \( d \in D \) is a valid value for the variable \( v \).

We adopt the following notational conventions for operation \( o = (t,p,v,d) \). The process is denoted by \( p(o) \). The variable and the value involved are denoted by \( var(o) \) and \( val(o) \) respectively. Generally, we use \( o \) for any operation, \( r \) for any read operation, \( w \) for any write operation, \( O \) for the set of all operations, \( R \) for the set of all read operations, \( W \) for the set of all write operations, and \( W_o \) for the set of all write operations on the same variable \( v \).

There are two basic partial orders between operations. Program order, denoted \( <_PO \), is the order in which operations are issued by each process. Write-to order, denoted \( <_WR \), defines which write is read by each read.

Definition III.2 (Program Order \((<_PO)\)). \( (o_1,o_2) \in <_PO \) if and only if \( p(o_1) = p(o_2) \) and \( o_1 \) is issued (and completed) before \( o_2 \). We employ \( >=_PO \) to denote the reflexive closure of \( <_PO \).

Definition III.3 (Write-to Order \((<_WR)\)). \( (o_1,o_2) \in <_WR \) if and only if \( o_1 \in W \land o_2 \in R \), and \( var(o_1) = var(o_2) \land val(o_1) = val(o_2) \).
We can now define the read/write traces as follows. Figure [1] in Section IV-C shows an example of a read/write trace consisting of four processes.

**Definition III.4 (Read/Write Trace (**T**)). A read/write trace **T** of data replicas comprises multiple process histories, each of which consisting of a finite sequence of read and write operations in program order.

**B. PRAM Consistency Model**

The PRAM consistency model is one of the well-known weak consistency models [2], [11]. It takes into account both program order and write-to order. Informally, a read/write trace satisfies PRAM consistency if and only if write operations performed by a single process are observed by all other processes in the order they were issued (i.e., program order), whereas write operations from different processes may be observed in different orders by different processes [11]. There are two key points to explain. First, PRAM consistency is weak in that it does not require all the processes to agree on the same view of the order in which operations occur. It implies that each process can be checked against PRAM consistency separately. Second, the operations visible to each process **p** are all write operations and its own read operations, while ignoring read operations from other processes (formally, it is the set of \{ \omega | \omega \in W \} \cup \{ p(o) = p \land o \in R \}). Note that, for process **p**, its visible read operations are all on the same process (i.e., \( p \) itself).

To state PRAM consistency formally, we first give some basic definitions on schedule. A schedule (denoted \( \pi \)) is just a sequence of operations. Given a schedule, the precedence relation between any two operations is denoted by ‘\( \prec \)’.

We employ \( \leq \) to denote the reflexive closure of \( \prec \). Moreover, we define \( \min(o_1, o_2) = o_1 \) and \( \max(o_1, o_2) = o_2 \) if \( o_1 \leq o_2 \).

A schedule \( \pi \) of a set of operations \( O \) is said to respect some partial order \( P \) (denoted \( (\pi, P) \)) if and only if the schedule is a linearization of the partial order. Formally,

\[
(\pi, P) \iff \forall o_1, o_2 \in O ((o_1, o_2) \in P \Rightarrow o_1 \prec o_2).
\]

Intuitively, the notion of respect enforces a schedule to satisfy specified partial orders. Furthermore, the following notion of legal schedule is considered a fundamental correctness requirement for all consistency models [11].

**Definition III.5 (Legal Schedule).** A schedule \( \pi \) of operations is legal if and only if each read reads the value from the latest preceding write on the same variable in the schedule. Predicate \( LS(\pi) \) is evaluated true if and only if the schedule \( \pi \) is legal.

**Definition III.6 (PRAM Consistency).** A read/write trace satisfies PRAM consistency if and only if for each process \( p \), there exists a legal schedule \( \pi \) of its visible operations, respecting both program order and write-to order. Formally,

\[
\forall p \in P \exists \pi (LS(\pi) \land (\pi, \prec_{PO} \cup \prec_{WR})).
\]

According to Definition III.6, we can verify each process against PRAM consistency separately. In the remainder of this paper, we thus focus on the verification problem with respect to some particular process and distinguish it with \( p_0 \).

**C. The Problem of Verifying PRAM Consistency**

The problem of Verifying PRAM Consistency (VPC, for short) over read/write traces is defined as a decision problem.

**Definition III.7 (Verifying PRAM Consistency Problem).**

- **INSTANCE:** A read/write trace \( T \). Its size (denoted \( n \)) is defined as the total number of the operations in it.
- **QUESTION:** Does \( T \) satisfy PRAM consistency?

Following the terminology in [20], we identify four variants of the general VPC problem from two orthogonal dimensions: \( a \) whether there are Multiple shared variables (or one Single variable), and \( b \) whether write operations can assign Duplicate values (or only Unique values) for each shared variable.

As summarized in Table I, the VPC-SU variant can be solved in polynomial time, following from [17]. In this paper, we address the other three variants. Specifically, we show that VPC-MU can also be solved in polynomial time by presenting two algorithms: the READ-CLOSURE algorithm with \( O(n^5) \) time complexity and the READ-CENTRIC algorithm with \( O(n^4) \) time complexity. On the other hand, we prove that VPC-SD (so is VPC-MD) is NP-complete by reducing the strongly NP-complete problem 3-PARTITION [18], [19] to it.

| TABLE I | A summary of complexity results for VPC problem ([•] : New results). |
|-----------------|-----------------|-----------------|
| write (U)nique value | VPC-SU (P) [17] | VPC-MU (P) [•] |
| write (D)uplicate values | VPC-SD (NPC) [•] | VPC-MD (NPC) [•] |

**IV. The RW-CLOSURE Algorithm**

In this section, we present a VPC-MU algorithm, called RW-CLOSURE. Note that in the trace of VPC-MU instance, for each read operation \( r \), there is at most one write (denoted \( D(r) \) for dictating write) from which \( r \) reads the value. In practice, each write operation can be tagged with a globally unique identifier, e.g., by combining its process id and a local sequence number [17].

**A. Overview**

The RW-CLOSURE algorithm models the read/write trace as a directed graph with operations as nodes and precedence relations between operations as directed edges. PRAM consistency is captured by three kinds of edges. The RW-CLOSURE algorithm keeps adding such edges to the transitive closure of the graph until no more edges can be added. Then the trace \( T \) satisfies PRAM consistency if and only if the resulting graph \( G \) is acyclic (i.e., DAG).

Specifically, at least two kinds of edges are necessary to meet PRAM consistency: edges for program order and edges...
Algorithm 1 The RW-Closure algorithm.

1: apply Rule A to add edges for program order
2: apply Rule B to add edges for write-to order
3: if \( \exists w \text{ has no } D(r) \) then return false
4: compute the transitive closure of \( G \)
5: foreach read operation \( r \) in program order do
6: \( w \leftarrow D(r) \), \( v \leftarrow var(r) \)
7: foreach \( w' \neq w \) s.t., \( \text{opMatrix}[w'][r] = 1 \) do
8: if \( \text{var}(w') = v \) \&\& \( \text{opMatrix}[w'][w] = 0 \) then
9: \( \text{opMatrix}[w'][w] \leftarrow 1 \)
10: if any edges are added by Rule C then goto Line 4
11: if \( G \) is a DAG then return true else return false

for write-to order. The third kind of edges can be derived from the legal schedule notion in Definition 11.3 [11], [24]. In a legal schedule, between each read operation \( r \) on variable \( v \) and its dictating write operation \( w = D(r) \), there cannot be any other write (denoted \( w' \)) on the same variable \( v \). This observation results in two cases: (1) if \( w' \not\prec r \), we have \( w' \not\prec w \); and (2) if \( w \not\prec w' \), we have \( r \not\prec w' \). Thus we get the following four rules for adding edges in \( G \):

- **(Rule A: program order)** For any pair of operations \( o_1 \) and \( o_2 \), if \( o_1 \not\prec_{PO} o_2 \), then add an edge from \( o_1 \) to \( o_2 \).
- **(Rule B: write-to order)** For any pair of operations \( w \) and \( r \), if \( w \not\prec_{WR} r \), then add an edge from \( w \) to \( r \).
- **(Rule C: \( w'w \) order)** For any triple of operations \( w, r \) and \( w' \) on the same variable, if \( w = D(r) \) \&\& \( w' \not\prec r \), then add an edge from \( w' \) to \( r \), leading to \( w' \not\prec_{WR} w \not\prec_{WR} r \). Note that we denote the precedence relation between such \( w' \) and \( w \) by \( \not\prec_{WR} \).
- **(Rule D: \( wrw' \) order)** For any triple of operations \( w, r \) and \( w' \) on the same variable, if \( w = D(r) \) \&\& \( w' \not\prec w \), then add an edge from \( r \) to \( w' \), leading to \( w \not\prec_{WR} r \not\prec_{RW} w' \).

As shown in the following, the first three rules are sufficient for the VPC-MU problem.

B. Detailed Design

In Algorithm 1 Rule A for program order edges and Rule B for write-to order edges are first applied (Lines 1–5). To apply Rule C, it is expected to first identify the triples conformed to it. To this end, the algorithm checks each pair of \( r \) and \( w = D(r) \), and find out all potential \( w' \) such that there is a path from \( w' \) to \( r \) (i.e., \( w' \not\prec r \)) (Lines 5–9). The reachability relation between \( w' \) and \( r \) is computed by transitive closure algorithm (Line 3) based on an \( n \times n \) Boolean operation matrix (opMatrix). If any edges are added by Rule C, new triples conformed to Rule C can emerge due to updated reachability relation. Therefore, the algorithm keeps applying Rule C and computing the transitive closure, until no more edges are added (Line 10). Finally, it concludes that the trace satisfies PRAM consistency if the resulting graph is acyclic (Line 11).

D. Correctness Proof

If the resulting graph \( G \) of Algorithm 1 is a DAG, we expect to construct some legal schedule (denoted \( \pi_G \)) as a witness to PRAM consistency. To this end, a specific topological sorting on \( G \) is performed. It is based on the following two notations.

Intuitively, \( r \)-downset consists of all the operations which must be scheduled before \( r \), plus \( r \) itself.

Definition IV.1 \((r \text{-downset} (r_0))\), \( r \)-downset of a read operation \( r \) is a set \( r_0 \) of operations such that:

- \( r \in r_0 \);
- \( o \in r_0 \) \&\& \( o' \not\prec o \Rightarrow o' \in r_0 \).

Let \( r \) be a read operation and \( r' \) be \( r \)'s previous read operation. We use \( r \)-delta to refer to the “extra” operations which are also scheduled before \( r \), besides those in \( r' \)-downset.

In other words, \( r \)-delta (denoted \( r_0 \)) of a read operation \( r \) is a set (of operations) which equals the relative complement of \( r_0' \) with respect to \( r_0 \) (i.e., \( r_0 \setminus r_0' \)). For the first read operation \( r \) on process \( p_0 \), we define \( r_0 = r_0 \). In terms of \( r \)-delta, we can now describe the construction of the legal schedule \( \pi_G \).

Definition IV.2 \((\text{DAG-schedule} (\pi_G))\). Given the resulting DAG \( G \) of Algorithm 1, the legal schedule \( \pi_G \) (initially, it is an empty sequence) is constructed as follows:

- Repeatedly take each read operation \( r \) on process \( p_0 \) in program order, perform any topological sorting on \( r_0 \)-induced subgraph, and append it to \( \pi_G \).
Lemma IV.1. If the resulting graph $\mathcal{G}$ of Algorithm 1 is acyclic, the schedule $\pi_G$ constructed in Definition IV.2 is legal.

Proof: By contradiction. If the resulting graph $\mathcal{G}$ is not a DAG, there exists some operation scheduled before itself.

The correctness of the RW-CLOSURE algorithm is stated in the following theorem.

Theorem IV.1. The VPC-MU instance satisfies PRAM consistency if and only if the resulting graph $\mathcal{G}$ of the RW-CLOSURE algorithm is acyclic.

Proof: (⇒) If the resulting graph $\mathcal{G}$ is not a DAG, there exists some operation scheduled before itself. (⇐) If the resulting graph $\mathcal{G}$ is acyclic, Lemma IV.1 shows that the schedule $\pi_G$ constructed in Definition IV.2 is legal.

E. Time and Space Complexity

The worst-case time complexity of the RW-CLOSURE algorithm is dominated by the cost for Step 3 (Lines 4 - 9 in Algorithm 1). The transitive closure of $\mathcal{G}$ (Line 4) can be computed in $\Theta(n^3)$ time using Floyd-Warshall’s algorithm [26]. Applying Rule C costs $O(n^2)$ to explore all potential pairs of nodes (Lines 5 - 9). The iteration over Step 3 and Step 4 may loop at most $O(n^2)$ times, adding one edge by Rule C in each iteration. In total, the worst-case time complexity of the RW-CLOSURE algorithm is $O(n^5)$.

Its space complexity is $\Theta(n^2)$, for the Boolean operation matrix (opMatrix).

V. THE READ-CENTRIC ALGORITHM

In this section, we present an improved VPC-SD algorithm, called READ-CENTRIC, with worst-case time complexity $O(n^4)$. Its correctness proof is based on the previous RW-CLOSURE algorithm.

A. Overview

In Theorem IV.1, we have shown that the trace $T$ satisfies PRAM consistency if and only if the resulting graph $\mathcal{G}$ of the RW-CLOSURE algorithm is acyclic. Generally speaking, the READ-CENTRIC algorithm attempts to construct graph $\mathcal{G}$ in an incremental and efficient way. It is incremental in that it processes the read operations on process $p_0$ sequentially. It is efficient because for each read operation, it applies Rule C locally and in a well-organized order. Algorithm 2 sketches its basic idea.

Let $r$ be the current read operation under scrutiny, $r'$ be its previous read operation, and $v$ be the variable of $r$. Upon read operation $r$, the READ-CENTRIC algorithm first initializes...
Algorithm 2 The Read-Centric algorithm (sketch).

1: apply Rule A to add edges for program order
2: apply Rule B to add edges for write-to order
3: if \( \exists r (r \text{ has no } D(r) \text{ and } r \prec_{po} D(r)) \) then return false
4: foreach read operation \( r \) in program order do
5: Let \( r' \) be \( r \)'s previous read operation
6: \( v \leftarrow \text{var}(r) \)
7: INIT-REACHABILITY\((r', r)\)
8: foreach \( w' \) s.t., \( w' \neq D(r) \land w' \in LW[v] \) do
9: add edge \( w' \rightarrow D(r) \)
10: if CYCLE-Detection\((w', D(r))\) then return false
11: return
12: UPDATE-REACHABILITY\((w', D(r), r)\)
13: if \( D(r) \notin r'_y \) then continue // case 1)
14: cycle \( \leftarrow \text{TOPO-SCHEDULE}(r'_y) \) // case 2)
15: if cycle then return false
16: return true

the reachability relation concerning the incrementally new operations in \( r_s = r_y \setminus r'_y \) (Line 7). (Here both \( r_y \) and \( r'_y \) are obtained according to Definition \([V.1]\) with respect to the dynamic graph \( G \) till that time.) It then attempts to schedule locally on the \( r_y \)-induced subgraph. Specifically, the schedule procedure starts with a simple observation that \( r \) must read from its dictating write operation \( D(r) \) (Lines 8 - 12). According to Rule C, any write operation \( w' \) in \( r \)'s downset on the variable \( v \) other than \( D(r) \) must be scheduled before \( D(r) \). Thus the edges like \( w' \rightarrow D(r) \) are added, updating the reachability relation between operations. Consequently, more applications of Rule C may be triggered. There are two cases to consider: 1) \( D(r) \notin r'_y \) and 2) \( D(r) \in r'_y \). In the former case (Line 13), the new added edges like \( w' \rightarrow D(r) \) have no effect on the reachability relation between the operations from \( r'_y \). In the latter one (Line 14), the operations in \( r'_y \) should be locally scheduled. This involves a serial of applications of Rule C. Contrast to that of the RW-CLOSURE algorithm, the applications of Rule C here are carried out in a reverse topological order of the \( r_y \)-induced subgraph. Once some cycle is created, the algorithm aborts and outputs “no”. If all the read operations are processed and no cycles arise, the algorithm terminates and outputs “yes”.

Before describing the algorithm in detail, we first introduce some terminology and notations.

B. Terminology and Notations

During the course of TOPO-SCHEDULE, the \( r_y \)-induced subgraph is dynamic in that edges are added on demand due to Rule C. To capture the dynamic reachability relation, two kinds of information are dynamically maintained.

First, ReachableRead maintains, for each write operation, the first read operation it can reach via the precedence relation (i.e., \( \prec \)). Recall that read operations are all program ordered on the process \( p_0 \).

Definition V.1 (ReachableRead (RR)). ReachableRead is a dictionary composed of a collection of \( (w, r) \in W \times R \) pairs such that

\[ RR[w] = r \leftrightarrow w \prec r \land \nexists r' \prec_{po} (r \prec r') \]

Being complementary to ReachableRead, PrecedingWrite maintains, for each operation, the last write operation for each variable preceding it. Strictly speaking,

Definition V.2 (PrecedingWrite (PW)). PrecedingWrite is a two-dimensional dictionary. For each operation \( o \), \( PW[o] \) is a dictionary composed of a collection of \( (v, w) \in V \times W \) pairs with the following conditions:

1) \( w \prec o \land \text{var}(w) = v \);
2) \( \exists r \in R (w \prec_R o \prec r) \);
3) for any other \( w' \) satisfying 1) - 2), we have \( w' \prec w \).

Condition 1) focuses on the preceding write operations on the same variable. Condition 2) concerns only the ones which have dictated read operations. Condition 3) requires that all write operations satisfying 1) - 2) be totally ordered. This is justified due to Rule C and the fact that all read operations are program ordered. Moreover, the precedence relation between them is determined by the program order of their respectively first dictated read operations.

Initially, PrecedingWrite associates each \( PW[o][v] \) with a dummy write operation NILWRITE which precedes all write operations. It provides procedure PW-UPDATE\((o, o')\) to update \( PW[o'] \) based on \( PW[o] \) for each variable.

1: procedure PW-UPDATE\((o, o')\) // update to the latter write
2: foreach \( v \in V \) do
3: \( PW[o'][v] \leftarrow \max(PW[o'][v], PW[o][v]) \)
4: if \( o \) has dictated read operations then // consider \( o \)
5: \( PW[o'][\text{var}(o)] \leftarrow o \)

Both ReachableRead and PrecedingWrite are used in procedure APPLY-RULE-C (more specifically, in its sub-procedures IDENTIFY-RULE-C and CYCLE-Detection, respectively). They are updated once some edge is added.

Besides, we use LocalWrites to simply keep record of, for each variable, the write operations locally in \( r_y \). Formally, LocalWrites (denoted \( LW \)) is a dictionary composed of a collection of \( (v, LW_v) \in V \times 2^{W_v} \) pairs. Recall that \( W_v \) stands for the set of write operations on the same variable \( v \).

C. Detailed Design

In this section, we first describe INIT-REACHABILITY (called in Line 7 of Algorithm 2) preparing for the key procedure TOPO-SCHEDULE. We then describe procedure APPLY-RULE-C and its sub-procedures. Particularly, during the course of TOPO-SCHEDULE we will show how to perform APPLY-RULE-C locally and in a well-organized order.
1) Procedure INIT-REACHABILITY: Upon each read operation \( r \) and its previous read operation \( r' \), the procedure INIT-REACHABILITY initializes the reachability relation, in terms of ReachableRead and PrecedingWrite, concerning the operations in \( r_\delta = r_y \setminus r_y' \) (Algorithm [3]). Here both \( r_y \) and \( r_y' \) are obtained according to Definition IV.1 with respect to the dynamic graph \( G \) until the time when INIT-REACHABILITY is called. On the one hand, the first reachable read operation (i.e., \( RR \)) of each write operation in \( r_\delta \) is now \( r \) (Line 3). On the other hand, we initialize \( PW \) of each operation in program order. Specifically, the operations in \( r_\delta \) (except \( r \)) are partitioned into two groups (both could be empty): 1) the \( rr \)-group (denoted \( grp_{rr} \)) consists of all the write operations between \( r' \) and \( r \) on process \( p_0 \) (both exclusive); and 2) the \( wv \)-group (denoted \( grp_{wv} \)) consists of the rest on the same process with that of \( D(r) \). Both groups are scanned through to initialize \( PW \) of each operation in the same manner (Lines 5-13).

2) Procedure APPLY-RULE-C: Procedure APPLY-RULE-C is called once the reachability relation has been dynamically updated. Basically it applies Rule C if necessary and returns false if some cycle is created (Algorithm [4]). In the following, we refer to the three operations involved in Rule C as “the \( w' \), \( w \), and \( r \) parts of Rule C” or simply “\( w'w \), \( w \), and \( r \)”. We also use the term “\( w'w \)-triple”.

First, to identify the \( w'w \)-triple of Rule C (procedure IDENTIFY-RULE-C): For some \( w'w \), it is sufficient to check whether new paths like from \( w' \) to \( r \) arise. The notation ReachableRead (Definition IV.1) serves the purpose. For \( w' \) (on variable \( v \)) in check, suppose that its first reachable read operation \( RR[w'] \) has been changed from \( r_{old} \) to \( r_{new} \). It means that \( w' \) can now reach the read operations in \( R[r_{new} \ldots r_{old}] \) which denotes the set of read operations between \( r_{new} \) and \( r_{old} \) on process \( p_0 \) (formally, \( R[r_{new} \ldots r_{old}] \equiv \{ r \in R \mid r_{new} \preceq po r \preceq po r_{old} \} \) (Lines 2-4). For each read operation \( r \) on variable \( v \) in \( R[r_{new} \ldots r_{old}] \), a triple of \( w'w', w = D(r), r \) is identified. If there are more than one such \( r \), we take the first one (in program order) and its corresponding triple (Line 7). This choice is justified in Lemma [V.1].

Second, cycle detection (procedure CYCLE-DETECTION): After identifying a \( w'w \)-triple of Rule C and adding the edge \( w' \rightarrow w \), procedure CYCLE-DETECTION is called to check whether some cycle involving \( w' \rightarrow w \) is created. To complete a cycle with the new edge \( \overrightarrow{w'w} \), an existing path from \( w \) to \( w' \) (denoted \( w \sim w' \)) is needed. The notation PrecedingWrite (Definition V.2) serves the purpose. Note that \( w \) (on variable \( v \)) concerned here has dictated read operations. \( PW[w'][v] \) maintains the last write operation on variable \( v \) which precedes \( w' \) and also has dictated read operations. Thus cycle detection amounts to figuring out whether or not \( w \) precedes (or is) \( PW[w'][v] \) (Line 2).

Third, to update the reachability relation (procedure UPDATE-REACHABILITY): If no cycle is created, UPDATE-REACHABILITY is called to update the reachability relation, namely ReachableRead of \( w' \) and PrecedingWrite of \( w \) and its successors. The ReachableRead of \( w' \) is updated to the read operation \( RR[w] \) if \( RR[w] \preceq po RR[w'] \). Note that ReachableRead of \( w' \)'s predecessors will be updated in procedure TOPO-SCHEDULE. The PrecedingWrite of \( w \) and its successors (in \( r_{loop} \)) are updated to integrate that of \( w' \).

3) Procedure TOPO-SCHEDULE: Recall that procedure TOPO-SCHEDULE mainly involves a serial of applications of Rule C and returns false once some cycle is created. The key is that the applications of Rule C are carried out locally and in a well-organized order. First, the operations which may
Algorithm 5 Procedure TOPO-SCHEDULE.
1: procedure TOPO-SCHEDULE(r)
2: // data structures for reverse topological sorting
3: \( G_{D(r)} \leftarrow D(r)_u \) - induced subgraph
4: traverse \( G_{D(r)} \) to compute for each \( o \in D(r)_u \):
5: (a) COUNT: number of direct successors
6: (b) SUCLIST: list of direct successors
7: (c) PRELIST: list of direct predecessors
8: // queue to maintain “sink” operations
9: QZERO \leftarrow empty queue
10: enqueue(QZERO, \( D(r) \)) // start from \( D(r) \)
11: // schedule in a reverse topological order of \( G_{D(r)} \)
12: while QZERO is not empty do
13: \( w' \leftarrow \) dequeue(QZERO)
14: // apply Rule C if necessary
15: if \( w' \in W \land w'.DONE = false \) then
16: foreach \( o \in w'.SUCLIST \)
17: \( RR[w'] \leftarrow \min(RR[w'], RR[o]) \)
18: cycle \leftarrow APPLY-RULE-C\((w', r)\)
19: if cycle then return false
20: // Rule C is applied; edge \( w' \rightarrow w \) is added
21: if \( w \in D(r)_u \land (w.DONE = false) \) then
22: insert \( w' \) into \( w.PRELIST \)
23: insert \( w \) into \( w.SUCLIST \)
24: \( w'.COUNT \leftarrow w'.COUNT + 1 \)
25: if \( w'.COUNT = 0 \) then
26: \( w'.DONE \leftarrow true \)
27: foreach \( o \in w'.PRELIST \)
28: \( o.COUNT \leftarrow o.COUNT - 1 \)
29: if \( o.COUNT = 0 \) then enqueue(QZERO,o)
30: return true

In the following, we show how to organize the applications of Rule C (Algorithm 5). The basic idea is to integrate the applications of Rule C with a (reverse) topological sorting algorithm [20]. In such a reverse topological sorting algorithm, a queue is used to maintain the sink operations that have no successors (Lines 8 - 10). Each time we pick up (and remove) one of the sink operations (denoted \( w' \)), update its ReachableRead based on its direct successors, and apply Rule C if necessary (Lines 11 - 19). After \( w' \) has been processed, it is marked DONE and the dependencies on it are erased. The new sink operations are put into the queue (Lines 25 - 29).

In procedure APPLY-RULE-C, for \( w' \), only the first \( r \) in \( R[\{new \ldots fold\}] \) is considered for Rule C (sub-procedure IDENTIFY-RULE-C). This choice does not act as the \( w' \) parts of Rule C are all locally in \( D(r)_u \)-downset. Second, they are carried out in a reverse topological order of the \( D(r)_u \)-induced subgraph. The former claim follows from a simple argument: a) whether to apply Rule C is determined by ReachableRead of its \( w' \) part (procedure IDENTIFY-RULE-C); and b) ReachableRead of \( w' \) is updated only due to its successors; and c) the procedure TOPO-SCHEDULE is called immediately after some Rule C edges to \( D(r) \) are added (Lines 8 - 12 of Algorithm 2).

In procedure TOPO-SCHEDULE (Algorithm 5), the operations which may act as the \( w' \) parts of Rule C are in \( W_{2-x} \)-downset (in a rectangle dotted box). Suppose in the course of reverse topological sorting, \( W_{z2} \) is processed before \( W_{y2} \) and \( W_{z1} \). By Rule C, an edge \( W_{z2} \rightarrow W_{z1} \) (label 2.3) is added. Since \( W_{z1} \) is not DONE, we have to process \( W_{z1} \) first before marking \( W_{z2} \) DONE (Lines 20 - 24). According to the reverse topological order, \( W_{y2} \) is processed and an edge \( W_{y2} \rightarrow W_{y1} \) (label 2.4) is added. Then it calls the procedure TOPO-SCHEDULE in the case of \( W_{z2} \in R_{b1} \) (Line 14).

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reduce any reachability relation of the resulting graph of the RW-Closure algorithm.

Proof: It is sufficient to show that each missing edge for w'w order is implied by other existing edges. This is illustrated in Figure 4 in which all operations perform on the same variable and w = D(r), w'' = D(r''). For w' there exists a path w' → r (label 2). By Rule C, both the edge w' → w (label 3) and the edge w'' → w (label 4) should be added. However, the latter one is implied by: 1) a path w → w'' (label 1) whose existence is guaranteed by r ∼ pO r''; and 2) the edge w' → w (label 3).

Hence, the correctness of the Read-Centric algorithm follows from that of the RW-Closure algorithm.

Theorem V.1. The VPC-MU instance satisfies PRAM consistency if and only if the Read-Centric algorithm terminates with a DAG.

F. Time and Space Complexity

The worst-case time complexity of the Read-Centric algorithm is dominated by the cost of TOPO-SCHEDULE. The efficiency of the latter is justified by the following lemma.

Lemma V.2. Let r be the read operation under scrutiny. For each w' ∈ D(r), procedure TOPO-SCHEDULE applies Rule C at most once with it as the w' part.

Proof: In procedure TOPO-SCHEDULE, the only case in which w' will be checked for Rule C is that an edge w' → w is added, w is in D(r), and w has not been marked DONE yet (Line 4 in Algorithm 5). If this is not applicable when w' is checked again. This is illustrated in Figure 5 in which all operations perform on the same variable v and w = D(r), w'' = D(r''). The first application of Rule C to triple w, v, and r have introduced the edge w' → w (label 2.1). Assume, by contradiction, that Rule C is applicable when w' is checked again. It requires that via w a new read operation r" on variable v with r" ∼ pO r be now reachable. Back to the time when w was checked, r" was reachable from w (label 3). An edge w → w' (label 2.2) was added, closing a cycle with the edge w'' → w (label 1) whose existence is guaranteed by r'' ∼ pO r. The procedure TOPO-SCHEDULE would abort then.

The following theorem gives the overall worst-case time complexity of the Read-Centric algorithm.

Theorem V.2. The worst-case time complexity of the Read-Centric algorithm is O(n^4).

Proof: Suppose that read operation r is under scrutiny. There are at most n operations in r, and m = O(n^2) edges between them. The time complexity of procedure TOPO-SCHEDULE comprises 1) O(n + m) for reverse topological sorting; 2) O(n · capply) for at most n applications of Rule C (Lemma V.2), each of which costs:

\[ c_{\text{apply}} = O(n) + O(1) \]

Thus procedure TOPO-SCHEDULE costs O(n^3) in the worst case. Then the worst-case time complexity of the Read-Centric algorithm is O(n^4):

\[ O(n) \cdot \left( O(n^2) + O(n^3) \right) = O(n^4). \]

The space complexity of the Read-Centric algorithm is O(n^2):

\[ O(n) + O(n^2) + O(n^2) = O(n^2). \]

VI. The VPC-SD AND VPC-MD PROBLEMS ARE NP-complete

In this section, we show that the VPC-SD problem (so is VPC-MD) is NP-complete by reducing the strongly NP-complete problem 3-Partition [18], [19] to it.

Definition VI.1 (3-Partition).

- **INSTANCE:** Set A of 3m elements, a bound B ∈ Z^+,

  and a size \( s(a) \in \mathbb{Z}^+ \) for each \( a \in A \) such that \( B/4 < s(a) < B/2 \) and \( \sum_{a \in A} s(a) = mB \).

- **QUESTION:** Can A be partitioned into m disjoint sets \( A_1, A_2, \ldots, A_m \), such that, for \( 1 \leq i \leq m \), \( \sum_{a \in A_i} s(a) = B \) (note that each \( A_i \) must therefore contain exactly three elements from A)?

We choose to reduce from 3-Partition because it is NP-complete even if the inputs \( a \in A \) and B are provided in unary [19]. We use the Unary 3-Partition problem.

Theorem VI.1. VPC-SD is NP-complete.

Proof: VPC-SD is in NP: Given a schedule of the VPC-SD instance, it is straightforward to check whether it is legal by scanning it in polynomial time.
Fig. 6. The VPC-SD trace corresponding to an instance of UNARY3-PARTITION (reading) value a the corresponding VPC-SD instance, we assume that integers constitute an arbitrary instance of UNARY3-PARTITION. We then add three auxiliary processes $P_{c_1}, P_{c_2}, P_{c_3}$. Specifically, $P_{c_1}$ comprises 3m write operations $Wxa$. $P_{c_2}$ comprises $mB = \sum_{a \in A} s(a)$ write operations $Wxb$. $P_{c_3}$ comprises 3m write operations $Wxc$.

Now we construct the process $P_0$ made only of read operations by concatenating $m$ slot sequences; each slot sequence is made of:

- a leading open subsequence $Rxa$ $Rxa'$ $Rxa'$ $Rxa'$, that forces to pop three operations from three distinct $P_a$, and open those processes;
- followed by a sum subsequence $Rxb$ $Rxb'$ repeated $B$ times, that forces to pop $B$ operations $Rxb'$ from the processes that are currently open;
- followed by a trailing close subsequence $Rxc$ $Rxc'$ $Rxc'$ $Rxc'$, that forces to pop three operations $Wxc'$ from the end of the processes that are currently open.

Figure 6 shows an example of the VPC-SD instance equivalent to the UNARY3-PARTITION instance in which $A = \{3, 3, 2, 2, 2\}$, $m = 2$, $B = 7$.

The reduction is polynomial: The size (i.e., total number of operations) of the VPC-SD instance is

$$\left(6 + 2B + 6\right)m + \left(6m + Bm\right) + 3m + Bm + 3m$$

$$= 24m + 4Bm.$$

The $a_i$’s and $B$ are given in unary, so it is polynomial in $m$ and $B$ and the reduction is polynomial.

We now prove that the UNARY3-PARTITION instance has a solution if and only if the VPC-SD instance has a solution.

(⇒) If the UNARY3-PARTITION instance has a solution $A_1, A_2, \ldots, A_m$, we construct a legal schedule $\pi$ for the VPC-SD instance. Let the elements of $A_i$ be $a_{i1}, a_{i2}, a_{i3}$ (in unary). Each $A_i$ corresponds to a subsequence $\pi_i$ of $\pi$ in the following way: $P_i$ use the open leading subsequence of its $i^{th}$ slot sequence to open each process of $P_{a_{i1}}, P_{a_{i2}},$ and $P_{a_{i3}}$ by using its $Wxa'$, meanwhile “consuming” three $Wxa$ from process $P_{c_1}$. The following sum sequence completes the B operation $Wxb'$ from the three currently open processes and B write operations $Wxb$ from $P_{c_2}$. Finally, the trailing close sequence is scheduled together with B write operations $Wxc'$ from the three currently open processes and B write operations $Wxc$ from $P_{c_3}$. It is straightforward to ensure that the schedule is legal during this construction.

(⇐) If the VPC-SD instance has a legal schedule $\pi$, we show that it is possible to construct a solution to the UNARY3-PARTITION instance. Note that in $\pi$, read operations and write operations must be scheduled alternately; otherwise write operations would run out and some read operations were left unscheduled. Thus for each slot sequence of $P_0$, $P_0$ has to first use its leading open subsequence to open three processes of the $m$ unary $P_a$. We claim that the total number of $Wxb'$ in the three opened processes equals $B$. Otherwise, there are two cases: 1) the total number of $Wxb'$ is greater than $B$. This means that a process is opened, the corresponding sum subsequence of $P_0$ is consumed, and some $Wxb'$ are still there.
In order to complete the current trailing close subsequence, we pop them (without corresponding $Rx'b'$) to reach the final $Wxb'$. However, in one of the next slot sequences there will be not enough $Wxb'$ to schedule and to reach its close subsequence. 2) the total number of $Wxb'$ is less than $B$. This means that we are in the middle of a sum subsequence and we need a $Wxb'$, but we have already reached the end of all the currently opened processes. We cannot open another process to recover a $Wxb'$ to complete the sum subsequence. Otherwise in one of the next slot sequences there will be not enough $Wxa'$ to complete an open subsequence.

Thus, VPC-SD is NP-hard and in NP. Therefore VPC-SD is NP-complete.

Note that the largest integer value assigned to the variables in the VPC-SD instance can be constant (e.g., $a = 1, a' = 2, b = 3, b' = 4, c = 5, c' = 6$), so it is trivially polynomially bounded by the instance size. Therefore we can further conclude that VPC-SD is NP-complete in the strong sense [19].

Because VPC-MD is a generalization of VPC-SD, we have:

**Corollary VI.1.** VPC-MD is NP-complete.

VII. CONCLUDING REMARKS

In this work, we have studied the problem of verifying PRAM consistency over read/write traces (VPC, for short). Specifically, we proposed two polynomial algorithms for its VPC-MU variant, namely RW-CLOSURE and READ-CENTRIC with the time complexity $O(n^3)$ and $O(n^3)$, respectively. We also proved that both its VPC-SD and VPC-MD variants are NP-complete.

The verification problems with respect to other weak consistency models, e.g., causal consistency [10], are also worth investigation. Because PRAM is a weakening of causal consistency, our NP-complete result also applies to the general problem of verifying causal consistency. However, it remains open to solve its restricted variant when writes can only assign unique values for each shared variable. Moreover, it would be interesting to further study the complexity issues of evaluating the severity of consistency violations [17], [22].

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