Quantum emulation of classical dynamics

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In statistical mechanics, it is well known that finite-state classical lattice models can be recast as quantum models, with distinct classical configurations identified with orthogonal basis states. This mapping makes classical statistical mechanics on a lattice a special case of quantum statistical mechanics, and classical combinatorial entropy a special case of quantum entropy.

In a similar manner, finite-state classical dynamics can be recast as finite-energy quantum dynamics. This mapping translates continuous quantities, concepts and machinery of quantum mechanics into a simplified finite-state context in which they have a purely classical and combinatorial interpretation. For example, in this mapping quantum average energy becomes the classical update rate.

Interpolation theory and communication theory help explain the truce achieved here between perfect classical determinism and quantum uncertainty, and between discrete and continuous dynamics.

INTRODUCTION

In this paper we discuss a mapping between classical and quantum systems that lets us regard quantum dynamics as a generalization of finite state classical dynamics, and that allows us to identify equivalent quantities and concepts in classical and quantum systems.

A similar mapping has long been known in statistical mechanics that establishes classical lattice models and their combinatorial entropy as simple examples of quantum statistical mechanics.

There is an obvious candidate for the comparable dynamical mapping: classical computations are equivalent to a subset of quantum computations. Most work on quantum computation is, however, based on hybrid classical/quantum models in which macroscopic classical operations control the sequencing of quantum operations. Such systems do not provide a purely quantum target for a classical/quantum mapping. Instead, early work showing that autonomous quantum systems can perform classical computation forms the basis for the dynamical mapping presented here.

This mapping allows physical quantities such as energy and momentum to be identified with finite-state classical quantities, with the aid of classical interpolation theory. Related issues are addressed in [4], but a general dynamical mapping is not provided there.

As a preliminary to discussing dynamics we first review a canonical method for mapping classical lattice models onto quantum lattice models in statistical mechanics.

STATISTICAL MECHANICS

In statistical mechanics, it is well known that classical lattice models can be recast as quantum models, with distinct classical configurations identified with orthogonal basis states [1].

Consider, for example, the well known ferromagnetic 2D Ising model. In this model each of $M$ lattice sites in a square lattice is occupied by a classical two-state “spin,” and each state $S_n$ of the $N = 2^M$ possible configurations of the lattice is assigned a classical configurational energy $E_n^\text{classical}$ that depends only on how many pairs of adjacent lattice sites have the same spin value and how many have opposite values.

A quantum lattice model corresponding to such a classical lattice model can be constructed by identifying each of the $N$ distinct classical states $S_n$ with a distinct basis vector $|n⟩$ in an $N$ dimensional Hilbert space. A hamiltonian operator $H$ is defined by taking each configuration state $|n⟩$ to be an energy eigenstate of $H$ with energy eigenvalue $E_n^\text{classical}$:

$$H |n⟩ = E_n^\text{classical} |n⟩ .$$ (1)

In quantum statistical mechanics the energy eigenstates are also eigenstates of the density operator $ρ$, with eigenvalues that give the statistical weight to attach to each energy eigenstate. For example, for a canonical ensemble of quantum mechanical systems, $ρ$ is proportional to $e^{-βH}$. From this becomes the usual classical Boltzmann factor when applied to a configuration state $|n⟩$, and quantum statistical mechanics reduces to classical.

CLASSICAL DYNAMICS

Since the definition of $H$ used in the statistical mechanics mapping makes each classical configuration a time-invariant state under unitary time evolution, we use a different definition of $H$ to emulate classical dynamics.

Finite-state dynamics

An invertible classical finite-state dynamics is a discrete idealization of classical dynamics [5]. Perfect digital degrees of freedom are updated at discrete times according to a sequence of invertible transformations. The total amount of state in the system, including that used
to define the dynamics, is finite. Here we take the time between update events to always be $\tau$, so that the system is updated at the constant rate $\nu = 1/\tau$.

The finite set of possible configurations of the system is partitioned by the invertible dynamics into a collection of disjoint dynamical orbits, with each dynamical orbit consisting of a set of configurations that turn into each other under the dynamics (cf. \[6\] \cite[7]). For each dynamical orbit $d$ the number of configurations $N_d$ in the orbit determines the period $T_d = \tau N_d$ of the orbit. One configuration of each orbit is labeled with the integer 0. The configuration obtained from 0 by one update step is $\langle n, d \rangle = 1$ for integer $m$, where “$E$” is the name of the new basis. The inverse transformation is

$$\langle n, d \rangle = \frac{1}{\sqrt{N_d}} \sum_{m=0}^{N_d-1} e^{-2\pi im/N_d} \langle m, d \rangle.$$  \hspace{1cm} (3)

We define a hamiltonian $H$ by assigning the $\langle E : m, d \rangle$ states to be its energy eigenstates and $E_{m,d} = m \hbar / T_d$ to be the corresponding energy eigenvalues \cite{23}.

$$H \langle E : m, d \rangle = m \frac{\hbar}{T_d} \langle E : m, d \rangle.$$  \hspace{1cm} (4)

If we let $U = e^{-i H \tau / \hbar}$ be the time evolution operator for the time interval $\tau$, then

$$U \langle n, d \rangle = \frac{1}{\sqrt{N_d}} \sum_{m=0}^{N_d-1} e^{-2\pi i (m+n+1) \tau / T_d} \langle m, d \rangle = \langle n+1, d \rangle.$$  \hspace{1cm} (5)

**Average energy**

The configuration state $\langle n, d \rangle = 1$ is a uniform superposition of all $N_d$ energy eigenstates $\langle E : m, d \rangle$ with eigenvalues $m \hbar / T_d$, and so the average energy is

$$E = \frac{\hbar (N_d - 1)}{2 T_d}.$$  \hspace{1cm} (6)

We’ve taken $E_{0,d} = 0$ in the construction above, but the fact that the system has a harmonic-oscillator-like energy spectrum suggests that we should really add $\hbar / 2 T_d$ to all the energy eigenvalues. This is in fact the smallest energy allowed by quantum distinguishability bounds, assuming the ground state energy of a much larger system encompassing this one sets the zero of the energy scale \cite{9}. Adding $\hbar / 2 T_d$ makes the average energy \cite{6} independent of $T_d$,

$$E = \frac{\hbar \nu}{2}.$$  \hspace{1cm} (7)

This is the least possible average energy compatible with a dynamics that traverses distinct states at the average rate $\nu$ \cite{9}. Thus our construction is energetically ideal, and the average energy is identified with the classical update rate of the finite-state dynamics.

If a lattice dynamics is updated sequentially—one location at a time in a repeating cycle—the frequency with which a given location is updated determines a local energy. Total update frequency (total energy) is the sum of the local frequencies \cite{27}. Different kinds of updates (e.g., ones involving particle or bond motion, and ones that don’t \cite{12,13}) define different kinds of energy \cite{4}.

For a large system with a very long period, $\hbar / 2 T_d \approx 0$, and so for simplicity we will revert to taking $E_{0,d} = 0$ in the remainder of the discussion.

**BANDLIMTED STATES**

We have provided a prescription for constructing a continuous-time quantum hamiltonian description for any invertible classical finite-state dynamics—turning discrete-time models into continuous-time models. This construction can be regarded as an application of bandlimited interpolation theory \cite{14}.

**Bandlimited dynamics**

Let us choose our unit of time such that $\tau = 1$, so that our configuration basis states are simply the states seen in the dynamics at integer values of time starting from $\langle 0, d \rangle$. At a continuous moment of time $t$ the state is

$$\langle t, d \rangle = e^{-i H t / \hbar} \langle 0, d \rangle$$

$$= \frac{1}{\sqrt{N_d}} \sum_{m=0}^{N_d-1} e^{-2\pi i t m / N_d} \langle E : m, d \rangle.$$  \hspace{1cm} (8)
which is just (3) with $t$ replacing $n$. We can express the continuous-time state $|t,d\rangle$ as a function of the $N_d$ integer-time states $|n,d\rangle$ by replacing $|E: m, d\rangle$ with its definition (2):

$$|t,d\rangle = \sum_{n=0}^{N_d-1} S(N_d,n-t) |n,d\rangle$$

where

$$S(N,u) = \frac{1}{N} \sum_{m=0}^{N-1} e^{2\pi imu/N} .$$

The function $S(N_d,n-t)$ equals the Kronecker delta $\delta_{n,t}$ for integer values of $t$ between 0 and $N_d - 1$ but is also defined for non-integer values. $S(N,u)$ is a periodic version of the sinc function [15], which is the foundation of bandlimited interpolation theory: $S(N,u) = 1$ for integer values of $u$ that equal 0 modulo $N$ and $S(N,u) = 0$ for other integer values of $u$. In fact, if we sum the geometric series we recover sinc times a phase for large $N$,

$$\lim_{N \to \infty} S(N,u) = e^{i\pi u} \frac{\sin \pi u}{\pi u} .$$

A portion of the probability distribution $|S(N,u)|^2$ is shown in Figure 1 for $N = 100$ (solid). Near its center it is approximately gaussian (dashed).

**Reconstruction from samples**

Using $S(N,u)$, any periodic function $f(t)$ with period $T$ and a bandlimited Fourier spectrum with $N$ frequencies can be reconstructed from $N$ equally spaced samples. Because of the periodicity all frequencies must be integer multiples of $1/T$, and if the lowest frequency is 0, then

$$f(t) = \sum_{m=0}^{N-1} a_m e^{2\pi imt/T}$$

for some set of $a_m$. Using $\tau = T/N$, $f(t)$ is also given by

$$f(t) = \sum_{n=0}^{N-1} f(n\tau) S(N,\frac{t}{\tau} - n) .$$

This is obviously true at the $N$ sample times $t = n\tau$ and so it must be true at all times, since $S(N,\frac{t}{\tau} - n)$ is composed of the same frequency components as $f(t)$, and the $N$ coefficients $a_m$ are completely determined by the values of $f(t)$ at the $N$ sample times $t = n\tau$ (in fact, the $a_m$’s are the Fourier transform of the $f(n\tau)$’s). If the lowest frequency is $k/T$ rather than zero, use $S_k(N,u) = e^{2\pi iku/N} S(N,u)$ instead of $S(N,u)$ above.

Thus [9] can be regarded as an exact reconstruction of a continuous but bandlimited dynamics in Hilbert space from $N_d$ samples. The bandlimit on the energy spectrum erases the distinction between continuous-time and discrete-time dynamics (and field operators [16]), since a bandlimited periodic function is completely determined by a finite number of sample points.

If $g(t)$ has the same period and bandwidth as $f(t)$ (perhaps with a different lowest frequency) then [13] implies

$$\frac{1}{T} \int_0^T dt f(t) g(t) = \frac{1}{N} \sum_{n=0}^{N-1} f(n\tau) g(n\tau) ,$$

and so a bandlimit also erases some of the distinction between continuous and discrete analysis of the dynamics.

**CONTINUOUS ISOMORPHISM**

Rather than just have integer time states of a classical finite-state dynamics correspond to integer time states of a quantum finite-energy dynamics, we can also extend the classical finite-state dynamics to intermediate times and have the two systems be isomorphic at all times.

**Continuously extended dynamics**

In classical finite-state lattice dynamics it is often useful to imagine that, when a 1 representing a particle hops from one lattice site to another, it moves continuously in between. This extension of the dynamics allows us to extend classical-mechanical conservations associated with continuous spatial symmetries to discrete particle motion in order to define, for example, momentum conserving lattice gases [28].

Continuously extended lattice dynamics have a continuous evolution in both time and space but, at every
moment, only a finite amount of state: if there are \( n \) spots in space that can have a 1 or not at integer times, there are still only \( n \) spots that can at non-integer times. Since these \( n \) bits don’t change their values while they’re moving between integer locations, the non-integer-time states are really just a fixed sequence of rearrangements of the bits of the integer-time state. These extra intermediate states are distinct classically since the bits are in different places but they are redundant informationally.

Note that a continuously extended lattice dynamics can still be described as a repeated cycle of local updates, but in this case each update moves a bit only infinitesimally. After any finite interval of time all of the bits will have moved by equivalent amounts.

Continuously extended isomorphism

In a continuously extended classical lattice dynamics, any unit-time separated sequence of states provides a complete description of the logical dynamics: since the bits of state don’t change between integer times, exactly when we sample them doesn’t matter.

Similarly, any unit-time separated sequence of states from the continuous unitary evolution [14] constitute a complete orthonormal basis set, since [9] implies

\[
\langle t', | t, d \rangle = S(N_d, t' - t) .
\]

Thus we are free to define a distinguished basis at any time \( t \) consisting of the unit-time separated set of \( N_d \) states from the evolution [8] that includes the current state \( | t, d \rangle \). If we identify these basis states with corresponding unit-time separated classical configurations, then the classical and quantum dynamics are isomorphic at all times.

In analyzing finite-state dynamics, the \( | t, d \rangle \)’s act much like a complete continuous basis since, again from [9],

\[
\int_0^{T_d} dt \ | t, d \rangle \langle t, d | = \sum_{n=0}^{N_d-1} | n, d \rangle \langle n, d | = I .
\]

Moreover, the inner product [15] acts like a Dirac delta function in an integral with a bandlimited function \( f(t) \). From [14],

\[
\int_0^{T_d} dt \ f(t) \langle t', | t, d \rangle = f(t') .
\]

The continuously extended isomorphism can be used to compute average values for operators, such as momentum, defined on continuous sets of configurations.

CONTINUOUS HAMILTONIAN

Rather than use \( N_d \) orthonormal quantum states to describe a classical orbit with \( N_d \) informationally distinct configurations, it is sometimes convenient to use more. In the continuous-basis limit this yields a continuous-hamiltonian description.

Oversampled dynamics

Suppose that, starting with a classical finite-state dynamics, we add \( M - 1 \) redundant intermediate-time states in the unit interval between each pair of consecutive integer-time states. Each orbit \( d \) of the corresponding quantum dynamics (generated by the hamiltonian \( H_M \)) now visits \( MN_d \) basis states rather than just the \( N_d \) of the original dynamics (generated by \( H_1 \)), and the state of the new \( H_M \) dynamics at a continuous moment of time becomes, from [9],

\[
| t, d, M \rangle = \sum_{k=0}^{MN_d-1} S(MN_d, k - Mt) | \frac{k}{MN_d}, d, M \rangle ,
\]

where the basis state \( | \frac{k}{MN_d}, d, M \rangle \) has been labeled by the time \( k/M \) when it is reached in an evolution starting from \( |0, d, M \rangle \). Since this extended dynamics traverses distinct states at a rate \( \nu_M \) that is \( M \) times the original rate \( \nu \), it has \( M \) times the average energy. As the number of intermediate states added in a fixed time period goes to infinity, the hamiltonian \( H_M \) approaches a continuous hamiltonian \( H_\infty \) and the average energy of the state \( | t, d, M \rangle \) goes to infinity.

Bandlimited basis

By putting a bandlimit on the energy spectrum of the configuration basis states we can make the \( H_M \) dynamics isomorphic to the original \( H_1 \) dynamics, with the same average energy: a bandlimit on energy can correct for an oversampling of the underlying classical dynamics.

The Fourier transform relationship [2] between energy eigenstates and configurational basis states is left unchanged but we construct, in addition, a new set of bandlimited configurations \( | n, d, M \rangle_{N_d} \) which are the Fourier transforms of the lowest \( N_d \) energy eigenstates of \( H_M \),

\[
| n, d, M \rangle_{N_d} = \frac{1}{\sqrt{N_d}} \sum_{m=0}^{N_d-1} e^{-2\pi inm/N_d} | E: m, d, M \rangle ,
\]

with \( n \) an integer. These states constitute an orthonormal basis for bandlimited superpositions of configurations. They have the same average energy as the configuration basis states of the \( H_1 \) dynamics: the amount of time \( T_d \) taken for one period of the orbit is being kept constant, and so from [4] the first \( N_d \) energy eigenvalues \( m\hbar/T_d \) of \( H_M \) are the same as for \( H_1 \).

The continuous time states \( | t, d, M \rangle_{N_d} \) that evolve from \( |0, d, M \rangle_{N_d} \) are given by [19] with \( n \) replaced by \( t \). As in
they obey

$$|t, d, M\rangle_{N_d} = \sum_{n=0}^{N_d-1} S(N_d, n - t) |n, d, M\rangle_{N_d},$$

so the evolution of bandlimited states is isomorphic with that of $|t, d\rangle$. Moreover, from (19) with $n \rightarrow t$ and expressing $|E: m, d, M\rangle$ in terms of the $MN_d$ configuration basis states using (2),

$$|t, d, M\rangle_{N_d} = \frac{1}{\sqrt{M}} \sum_{k=0}^{M N_d-1} S(N_d, k/M - t) |k/M, d, M\rangle$$

$$= \frac{1}{\sqrt{M}} \sum_{m=0}^{M-1} \sum_{n=0}^{N_d-1} S(N_d, n + n/M - t) |n + n/M, d, M\rangle.$$

The bandlimited state is, at all times, an equally weighted superposition of $M$ equivalent states, each of which corresponds to the extended classical configuration at time $t$ represented in a different unit-time separated basis. Thus the correspondence of $|t, d, M\rangle_{N_d}$ to classical configurations is the same as for $|t, d\rangle$.

The state (21) is a sum over configurations separated in time by $du = 1/M$. If we normalize each configuration state to length $\sqrt{M}$ instead of to length 1, this becomes delta-function normalization in the limit $M \rightarrow \infty$ and

$$|t, d, \infty\rangle_{N_d} = \int_0^{T_d} du S(N_d, u - t) |u, d, \infty\rangle.$$

From this and (15),

$$\langle t', d, \infty|t, d, \infty\rangle_{N_d} = \langle t', d|t, d\rangle,$$

and so we can use the isomorphic $|t, d\rangle$ states to determine amplitudes in the continuous configuration basis.

**PARTICLE MOTION**

A classical finite-state lattice dynamics is naturally described as a repeating sequence of invertible gate operations [13]. In mapping this onto a quantum dynamics, the classical model can be implemented isomorphically as a sequence of local unitary operations.

Fundamental physics is, however, normally described as particle dynamics. To make contact with this viewpoint we can recast finite-state lattice dynamics as particle mechanics, following the motions of individual 1’s as if they were distinguishable particles.

**Single particle**

Consider a classical lattice dynamics in which a single particle, represented by a 1, hops in the $+x$ direction from lattice site to adjacent lattice site at a constant rate, with average speed $v = 1$. The motion is periodic in space, traversing $N$ lattice sites in a distance $L$ before repeating. At $t = 0$ the particle is at $x = 0$.

For this classical evolution, we can take the state of the system to be the integer position $n$ of the 1 at integer time $n$. In an isomorphic H$_1$ quantum evolution, the distinct classical configurations become integer-position basis states $|n\rangle$. From [9] we get a description of intermediate configurations in terms of integer-time ones,

$$|x\rangle = \sum_{n=0}^{N-1} S(N, n - x) |n\rangle,$$

where $|x\rangle$ is the configuration obtained by evolving for a time $t = x/v$ from the configuration $|0\rangle$. We identify the non-integer $|x\rangle$ with the non-integer positions of the continuously extended dynamics.

In the quantum description of a classical particle at a non-integer position $vt$ modulo $L$, there is some amplitude for the particle at more than one integer position. From (15) and using (24) we can interpret

$$\psi(x, t) = \langle x|vt\rangle = S(N, x - vt)$$

to be the amplitude to find the particle at any continuous position $x$ at time $t$, and compute the average momentum directly from $\psi(x, t)$.

Alternatively, we can instead start with an infinite-dimensional quantum hamiltonian that generates a continuous shift in space in the $+x$ direction at speed $v$:

$$H_\infty = vp, \quad \text{with } p = -i\hbar \frac{\partial}{\partial x}.$$
mechanical particle dynamics [5] [21] that obeys Hamilton’s equations,

\[ \frac{\partial H}{\partial q_j} = -\frac{dp_j}{dt}, \quad \frac{\partial H}{\partial p_j} = \frac{dq_j}{dt}. \]  

(29)

To make the lattice dynamics run faster by a factor \( \kappa \) we reduce the interval between the discrete events, \( \tau \rightarrow \tau / \kappa \). From [29], this can be accomplished by letting \( H \rightarrow \kappa H \), which is exactly the energy scaling required by (7).

We can’t just rescale \( \tau \) arbitrarily while keeping the \( p_j \)'s and \( q_j \)'s unchanged, however, because particle velocities are limited by the speed of light. We can, instead, run the dynamics faster by putting the discrete events closer together in both time and space, leaving velocities unchanged. If the distance between events \( \lambda \rightarrow \lambda / \kappa \), then the scale of the \( p_j \)'s must be multiplied by \( \kappa \) to get an overall scaling of \( H \) by \( \kappa \) in (29). This is exactly the momentum scaling required by (28).

**Indistinguishable particles**

Treating 1’s in a classical finite-state lattice dynamics as distinguishable particles—and keeping track of the discrete position and velocity of each 1—dramatically over-represents the number of distinct states: all states with the same spatial pattern of 1’s and velocities correspond to a single state of the original lattice model. We can fix this over-representation in a quantum description of the distinguishable particle dynamics by merging equivalent states, adding them together to form new occupation number basis states, and using only these to describe the evolution. If we antisymmetrize each sum under particle interchange, the new basis states will each have at most one 1 with a given position and velocity—we can symmetrize instead to allow more [19].

To describe a dynamics in which the number of ones changes with time, we can use creation and annihilation operators to add and remove particles from the state, while maintaining symmetrization. These field operators inherit fermionic or bosonic commutation rules from the symmetrization [31]. As we see from (25) (or from (20) for \( H_\infty \)), a finite set of bandlimited basis states allows a particle to be added centered at any continuous position in space. In one dimension with one velocity, for example, the creation operator \( \Psi^\dagger(x) \) for any \( x \) is a superposition of the creation operators \( \Psi^\dagger(n) \) for integer positions \( n \),

\[ \Psi^\dagger(x) = \sum_{n=0}^{N-1} S(N, n-x) \Psi^\dagger(n). \]  

(30)

Of course nothing essential is gained by using a continuous space and time description, since a bandlimited continuous state is completely determined by its values at discrete positions and times. Similarly, nothing essential is gained by introducing fermionic field operators: there would be no need to maintain the antisymmetry of equivalent states if the original dynamics were described isomorphically in terms of local unitary operations [32].

**UNCERTAINTY**

The particle described by [20] moves at a constant speed and is localized to a single position basis state of a finite-dimensional basis at all times (cf. [22]). This in no way conflicts with the uncertainty relations of quantum mechanics, which can be regarded as bounds on representing information using limited bandwidth.

**Bandwidth bounds**

Constraints on time or position determine the minimum width of the energy or momentum eigenfrequency distribution needed to describe a state that meets the constraints. In the usual uncertainty bounds we also associate a width with the time or position amplitude distribution [24], but in general other constraints on time or position can be used to determine a minimum width of energy or momentum eigenfrequencies.

For example, suppose we have an exactly periodic evolution with period \( T \). The state at time \( t \) can be written

\[ |t\rangle = \sum a_n e^{-2\pi i \nu_n t}|E_n\rangle. \]  

(31)

Exact periodicity requires that each \( \nu_n = E_n / \hbar \) be an integer multiple of \( 1 / T \). If this evolution passes through \( N \) mutually orthogonal states, then the superposition must involve at least \( N \) different \( |E_n\rangle \)'s (since you can’t construct \( N \) distinct states out of fewer than \( N \) distinct states). Moreover, there must also be at least \( N \) distinct frequencies (since groups of \( |E_n\rangle \)'s with the same frequency act like a single eigenstate in the construction). To have \( N \) distinct frequencies that are integer multiples of \( 1 / T \), the bandwidth \( B \) (highest frequency in the superposition minus lowest) must obey

\[ B \geq \frac{N-1}{T}. \]  

(32)

This is a version of the bandwidth-time theorem of communication theory [24]. If we let \( \tau = T / N \) be the average time between distinct states, we see that this is also a version of the time-energy uncertainty relation, using \( B \) directly rather than some other measure of the width of the energy eigenfrequency distribution. The definition [3] achieves this bound.

**Second-moment bounds**

In constructing uncertainty bounds, the standard deviation of the eigenfrequency distribution is traditionally
chosen to measure its width. This choice reflects both familiarity from statistics and (for position and momentum) a simple connection between the commutation relation and the standard-deviation bound \[2].\] This choice is often divergent, however, and so fails to provide a useful bound \[23].\] This is true in our case.

Consider the bandlimited state \[\langle x|\bar{x}\rangle = S(N, x - \bar{x})\] centered at \(\bar{x}\). Limiting ourselves to spatial frequencies \(m/L\) with \(m\) ranging from 0 to \(N - 1\), this state has the least possible information about what the momentum is, since all momentum eigenstates in the allowed range have equal amplitude. Correspondingly we might expect the position to be as well-defined as possible, given the limited bandwidth. It is clear from Figure 4 that the position localization of the probability distribution \(|S(N, x - \bar{x})|^2\) is similar to that of a gaussian (dotted line). This is not apparent in the mean square position deviation, however, which can be estimated for large \(N\) using \([11]\) as

\[
\langle (x - \bar{x})^2 \rangle \approx \int_{0}^{N} (x - \bar{x})^2 \sin^2 \frac{\pi(x - \bar{x})}{\pi \bar{x}} \ dx = \frac{N}{2\pi^2}, \tag{33}
\]

which diverges as \(N \to \infty\) (i.e., on an infinitely wide space) \[33\]. Thus \(S(N, x - \bar{x})\), which is perfectly distinct from a unit shift of itself, is not localized at all on the infinite line if we use the traditional second-moment measure of the width of the distribution. The unit-height gaussian, which looks so similar in the figure, has a mean square deviation of \(1/2\pi\). Other measures of the width have been proposed that avoid this disparity \[23\].

### First-moment bounds

For our purposes, a much better measure of the width of the eigenfrequency distribution is twice the average half-width: \(2(\bar{\nu} - \nu_0)\). Here \(\bar{\nu}\) is the average frequency \((e.g., E/h)\) and \(\nu_0\) the lowest frequency used \((e.g., E_0/h)\). In general \[9\],

\[
2(\bar{\nu} - \nu_0) \geq B_{\text{min}}, \tag{34}
\]

where \(B_{\text{min}}\) is the minimum bandwidth compatible with the temporal or spatial constraints on the system.

For example, if \(\tau_{\text{min}}\) is the minimum separation in time between two mutually orthogonal states in the evolution, then the minimum bandwidth needed is \(B_{\text{min}} = 1/2\tau_{\text{min}}\) if there must be at least two distinct frequencies and they must be separated by at least half of \(1/\tau_{\text{min}}\).

The \(B_{\text{min}} = 1/2\tau_{\text{min}}\) bound \[34\] is only achieved by the energy \(6\) for \(N = 2\). For \(N \gg 2\), the energy \(6\) is about twice as great as allowed by this bound. There is, however, the additional bandwidth constraint \(32\) required to have \(N\) distinct states in period \(T\). The energy \(6\) achieves \(34\) with this constraint.

### Uncertain states

We have seen examples where a quantum hamiltonian describes a classical finite-state dynamics, but also makes extra distinctions not present in the original dynamics: A many particle hamiltonian that keeps track of which identical 1-bit is where. A continuous-shift hamiltonian that adds distinct states between the discrete time steps.

We can eliminate over-representation and make the dynamics isomorphic to the original by adding together equivalent configurations with equal weight to construct truly distinct basis states. Starting from these, equivalent configurations will always have equal probability: \(\text{equivalence is represented as uncertainty} 34\).

In the construction of the occupation number basis states for identical 1-bits, a symmetrized or anti-symmetrized state represents equivalent states as being equally probable. In the case of over-representation of intermediate states, constructing a basis without the high frequency information needed to represent intermediate details also merges equivalent states \[21\], making them equally probable.

The continuous-hamiltonian representation of a discrete shift is an interesting limiting case of representing equivalence as uncertainty. A bandlimit with \(N\) distinct states yields \[22\] for finite \(M\). For a state centered at \(t = x/v\) and \(N \to \infty\) this becomes

\[
|x, \infty\rangle_N = \int_{0}^{1} du \left( \sum_{n=0}^{N-1} S(N, n + u - x) |n + u, \infty\rangle \right), \tag{35}
\]

which is a uniform superposition of all the equivalent ways to represent a classical particle at position \(x\) if only \(N\) equally-spaced positions are distinct.

The tradeoff between bandwidth and minimum separation in space determines the minimum uncertainty volume of phase space needed to represent each distinct state \[35\], and this is achieved by \(|x, \infty\rangle_N\).

### DISCUSSION

Classical finite-state dynamics that are invertible can be mapped isomorphically onto the discrete time behavior of finite-energy quantum dynamics. A quantum evolution mapping an infinite number of distinct states into a finite time period would have an infinite average energy.

Quantum-classical isomorphism challenges conventional wisdom about essential differences between quantum and classical systems: identical particles, amplitudes, frequencies, complementarity and uncertainty all play essential roles in describing and analyzing classical finite-state dynamics using continuous language.

Quantum-classical models also shed light on the foundations of classical mechanics. They provide a quantum
substrate where interesting classical behavior arises without approximation or decoherence. Physically meaningful energy and momentum scales are defined directly by the separation of classical events in time and space.

Finally, quantum-classical isomorphism may be helpful in understanding and teaching quantum mechanics. Just as it is useful to study classical information and classical computation as a preliminary to studying their quantum counterparts, it seems useful to study other aspects of the machinery and concepts of quantum mechanics in a simplified classical setting.

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[26] This is equivalent to phase and number [8, Eqn. 41], with $\theta[n,d] = (2\pi n/N_d)[n,d]$ and $N[E; m,d] = m = |E; m,d|$. In this example the local additivity of the energy derives from the locality of the sequence of update operations; H doesn’t need to be expressible in a manifestly local fashion (cf. [10, 11]).
[27] This idea is the basis of practical lattice gas models of classical hydrodynamics [12] and of quantum fluids [17] (both with continuous isotropy in the macroscopic limit).
[28] For motion in the $-x$ direction we would use $H_{\infty} = -v p$ instead. To represent the direction explicitly in the state we would multiply it by $|+\rangle$ or $|-\rangle$ and let $H_{\infty} = \sigma_z s p$.
[29] To make updates local, we can use a partitioning dynamics, in which each particle motion involves a single update that changes both old and new positions at once [11, 13].
[30] For example, if particle labels are generated sequentially as particles are created, then interchanging the order in which two particles are added to an antisymmetric state is equivalent to interchanging their particle labels, and so creation operators must anticommute [20].
[31] This should apply equally to quantum lattice gas simulations of non-classical systems.
[32] If $\bar{x}$ is near the middle of the periodic space, then the wavefunction goes to 0 at the boundary as $N \to \infty$ and so the usual uncertainty relations apply [8, Eqn. 22].
[33] Ignorance of differences between equivalent states doesn’t count toward entropy, which is one reason quantum probabilities must be kept separate from ordinary ones [1].
[34] In a periodic space of length $L$, momentum eigenfrequencies must be integer multiples of $1/L$. Thus to represent $N$ distinct states a bandwidth $B \geq (N - 1)/L$ is needed, and so the frequency-space volume per distinct state is $B/LN \geq (N - 1)/N$ (uncertainty tradeoff for $B$ vs. $L/N$).