A non-Fock fermion toy model

Boris Tsirelson

Abstract

Recent progress in mathematical theory of random processes provides us with non-Fock product systems (continuous tensor products of Hilbert spaces) used here for constructing a toy model for fermions. Some state vectors describe infinitely many particles in a finite region; the particles accumulate to a point. Electric charge can be assigned to the particles, the total charge being zero. Time dynamics is not considered yet, only kinematics (a single time instant).

Introduction

Non-interacting particles are quanta of free quantum fields. Corresponding Hilbert spaces, known as Fock spaces, are direct sums of $n$-particle spaces for $n = 0, 1, 2, \ldots$. Interacting particles are described by nonlinear quantized fields that do not fit into Fock spaces.

Fock spaces are also discussed by mathematicians in the context of product systems, that is, continuous tensor products of Hilbert spaces (see [1] and references therein) and white noise (see [1, 2]). Non-classical noises found recently in stochastic analysis (especially [10]; see also [9]) are a source of non-Fock product systems. It is natural to look at these systems from the viewpoint of quantum field theory. I try to do so in the present text by combining some ideas of Anatoly Vershik (the initiator of the trend), Jonathan Warren, and mine.

Toy models considered here are quite poor in several aspects. Especially, dynamics is neglected at all, and particles are treated as utterly local in space (see Sect. 3.3). Nevertheless, we may hope for some hints toward more useful rigorous non-Fock constructions.

1See [2, Chaps 1, 2].
2Or Fock-Cook representations, see [3, Sect. 1.9, p. 61].
3Or rather, should be; see [3, Sect. 8.1].
1 The idea

1.1 Lattice fermions

The model can be described at once using notions of stochastic analysis (see Sect. 2). However, for the beginning I prefer discrete approximations. So, we assume for now that our “physical space” is a one-dimensional lattice \( \varepsilon \mathbb{Z} = \{ \ldots, -2\varepsilon, -\varepsilon, 0, \varepsilon, 2\varepsilon, \ldots \} \); we ascribe a two-dimensional Hilbert space \( H_{k\varepsilon} \) to each lattice point \( k\varepsilon \); we introduce on \( H_{k\varepsilon} \) fermionic operators \( a_{k\varepsilon} \) (annihilation), \( a^*_\varepsilon \) (creation), satisfying CAR (canonical anticommutation relations) \( a_{k\varepsilon} a^*_{k\varepsilon} + a^*_{k\varepsilon} a_{k\varepsilon} = 1, a^2_{k\varepsilon} = 0 \); we consider the tensor product \( H_{\varepsilon \mathbb{Z}} \) of all \( H_{k\varepsilon} \) containing the vacuum vector \( \psi_0 \) such that \( a_{k\varepsilon}\psi_0 = 0 \) for all \( k \), treating now all \( a_{k\varepsilon} \) as operators on \( H_{\varepsilon \mathbb{Z}} \), so that \( a_{k\varepsilon} a^*_{l\varepsilon} + a^*_{l\varepsilon} a_{k\varepsilon} = 0 \) for \( k \neq l \), and \( a_{k\varepsilon} a_{l\varepsilon} + a_{l\varepsilon} a_{k\varepsilon} = 0 \) for all \( k, l \). Of course, \( H_{\varepsilon \mathbb{Z}} \) is just the usual (fermionic) Fock space; a non-Fock space will appear later in the limit \( \varepsilon \to 0 \), due to a non-classical limiting procedure. The total number of particles is the observable \( N_{\varepsilon \mathbb{Z}} = \sum_k a^*_{k\varepsilon} a_{k\varepsilon} \). Its eigenvalues are 0, 1, 2, \ldots. The eigenspace \( H_{\varepsilon \mathbb{Z}}^{(n)} \) corresponding to an eigenvalue \( n \) is the \( n \)-particle space, spanned by vectors \( a^*_{k_1\varepsilon} \ldots a^*_{k_n\varepsilon} \psi_0 \) for \( k_1 < \cdots < k_n \). The number of particles on an interval \([x, y]\) is the observable \( N_{x,y}^{\varepsilon \mathbb{Z}} = \sum_{k:x\leq k<y} a^*_{k\varepsilon} a_{k\varepsilon} \); clearly, \( N_{x,y}^{\varepsilon \mathbb{Z}} + N_{y,z}^{\varepsilon \mathbb{Z}} = N_{x,z}^{\varepsilon \mathbb{Z}} \) and \( N_{-\infty,\infty}^{\varepsilon \mathbb{Z}} = N^{\varepsilon \mathbb{Z}} \).

1.2 Quantum states and classical random variables

For each lattice point \( k\varepsilon \) we introduce a variable \( \tau_{k\varepsilon} \) with only two possible values \( \pm 1 \). Functions of these variables will be used as symbols of state vectors, as follows:

\[
\begin{align*}
\text{function} & \quad \text{state vector} \\
\tau_{k\varepsilon} & \quad \psi_0 \\
\tau_{k\varepsilon} \tau_{l\varepsilon} & \quad a^*_{k\varepsilon} \psi_0 \\
& \quad a^*_{k\varepsilon} a^*_{l\varepsilon} \psi_0 (k < l)
\end{align*}
\]

and so on; each factor \( \tau_{k\varepsilon} \) gives rise to a particle at \( k\varepsilon \). Extending the map by linearity we get a unitary operator from the Hilbert space \( L^2 \) of all square
integrable complex-valued functions of variables $\tau_{k\varepsilon}$ onto the Hilbert space $H_{\varepsilon Z}$ of all state vectors; $L_2 \ni f \mapsto \psi_f \in H_{\varepsilon Z}$.

Let $f \in L_2$, $\|f\| = 1$. Consider the probability $\|a_{k\varepsilon}\psi_f\|^2 = \langle a^*_{k\varepsilon}a_{k\varepsilon}\rangle\psi_f$ of finding a particle at $k\varepsilon$. In terms of $f$, it is the so-called influence of $\tau_{k\varepsilon}$ on $f$. For example, let $\varepsilon = 1$ and $f = f(\tau_1, \tau_2, \tau_3)$, then the influence of $\tau_1$ on $f$ is, by definition,

$$\frac{1}{4} \sum_{a,b \in \{-1,+1\}} \left(\frac{f(1,a,b) - f(-1,a,b)}{2}\right)^2;$$

a probabilist writes it as the expectation $\mathbb{E}$ of a conditional variance $\text{Var}(\cdot \mid \cdot)$, treating $\tau_{k\varepsilon}$ as independent random variables, each with two equiprobable values:

$$\mathbb{E} \text{Var}(f \mid \tau_2, \tau_3) = \frac{1}{4} \sum_{s_2,s_3 \in \{-1,+1\}} \left(\frac{f^2(1,s_2,s_3) + f^2(-1,s_2,s_3)}{2} - \left(\frac{f(1,s_2,s_3) + f(-1,s_2,s_3)}{2}\right)^2\right).$$

Similarly, the probability of finding at least one particle within the two-point set $\{k\varepsilon, l\varepsilon\}$ is the influence of $\{\tau_{k\varepsilon}, \tau_{l\varepsilon}\}$ on $f$. For example, the influence of $\{\tau_1, \tau_2\}$ on $f(\tau_1, \tau_2, \tau_3)$ is

$$\mathbb{E} \text{Var}(f \mid \tau_3) = \frac{1}{2} \sum_{s_3 \in \{-1,+1\}} \left(\frac{f^2(1,1,s_3) + f^2(1,-1,s_3) + f^2(-1,1,s_3) + f^2(-1,-1,s_3)}{4} - \left(\frac{f(1,1,s_3) + f(1,-1,s_3) + f(-1,1,s_3) + f(-1,-1,s_3)}{4}\right)^2\right).$$

In general, the probability of finding at least one particle on $[x, y]$ is

$$\mathbb{E} \text{Var}(f \mid \tau_{\mathbb{R}\setminus[x,y]}),$$

the influence of $\tau_{\mathbb{R}\setminus[x,y]}$ on $f$; here $\tau_{\mathbb{R}\setminus[x,y]}$ means the collection of $\tau_{k\varepsilon}$ for all $k$ such that $k\varepsilon \in \mathbb{R} \setminus [x, y]$, that is, $k\varepsilon < x$ or $k\varepsilon \geq y$.

The corresponding quantum observable is a function of $N_{x,y}^{\varepsilon Z}$. Consider the projection onto the eigenspace of $N_{x,y}^{\varepsilon Z}$ corresponding to the eigenvalue $0$; it may be written as $0^{N_{x,y}^{\varepsilon Z}}$ (it is means that $0^0 = 1$, $0^1 = 0$, $0^2 = 0$...; note that $0^{N_{x,y}^{\varepsilon Z}} = \lim_{\lambda \to 0} \lambda N_{x,y}^{\varepsilon Z}$). Its expectation $\langle 0^{N_{x,y}^{\varepsilon Z}} \rangle_\psi$ (denoted also by

$^8$Just classical random variables on some probability space, with (apriori) no relation to $H_{\varepsilon Z}$.}
\( \langle \psi | 0^{N_{x,y}} | \psi \rangle \) or \( (0^{N_{x,y}} | \psi, \psi \rangle ) \) is the probability of finding no particles on \( [x,y] \). So,

\[
\langle 0^{N_{x,y}} | \psi \rangle = 1 - \mathbb{E} \text{Var} \left( | \tau_{\mathbb{R}\setminus[x,y]} \psi \rangle \right)
\]

whenever \( f \in L_2, \| f \| = 1 \).

1.3 From a lattice to a continuum

The standard limiting procedure is based on functions of the form

\[
f_{\varphi, \varepsilon} = \sqrt{\varepsilon} \sum_{k \in \mathbb{Z}} \varphi(k \varepsilon) \tau_{k \varepsilon}
\]

(as well as polynomials of such functions); here \( \varphi : \mathbb{R} \rightarrow \mathbb{C} \) is a continuous compactly supported function ("test function"). Clearly,

\[
\| f_{\varphi, \varepsilon} \|^2 = \varepsilon \sum_k |\varphi(k \varepsilon)|^2 \xrightarrow{\varepsilon \to 0} \int_{-\infty}^{+\infty} |\varphi(x)|^2 \, dx = \| \varphi \|^2.
\]

The distribution of \( f_{\varphi, \varepsilon} \) tends (for \( \varepsilon \to 0 \)) to the normal distribution \( N(0, \| \varphi \|^2) \).

This way, the array of variables \( \tau_{k \varepsilon} \) over the lattice \( \varepsilon \mathbb{Z} \) turns (when \( \varepsilon \to 0 \)) into the white noise \( \sqrt{\varepsilon \tau_{k \varepsilon}} \) turns into \( dW(x) \). The function \( f_{\varphi, \varepsilon} \) turns into the linear functional \( \int \varphi(x) dW(x) \), the one-particle state with the wave function \( \varphi; \frac{1}{\sqrt{\varepsilon}} a_{k \varepsilon} \) turns into \( a(x) \). Similarly, for a function \( \varphi \) of two variables, the quadratic functional \( f_{\varphi} = \iint \varphi(x,y) dW(x) dW(y) \) corresponds to the two-particle state \( \psi_{f_{\varphi}} = \iint \varphi(x,y) a^*(x) a^*(y) \psi \, dx \, dy \), and so on.

The number-of-particles operator corresponding to \( \sum_k a_{k \varepsilon}^* a_{k \varepsilon} \) is naturally denoted by \( N = \int a^*(x) a(x) \, dx \). Accordingly, \( N_{x,y} = \int_x^y a^*(z) a(z) \, dz \). Still, \( N_{x,y} + N_{y,z} = N_{x,z} \) and \( N_{-\infty, \infty} = N \). The standard limiting procedure leads to the Fock space \( H^\text{Fock}_\mathbb{R} \), as usual.

1.4 Some specific random variables and corresponding quantum states

We introduce functions \( W_{k \varepsilon}^{\varepsilon \mathbb{Z}} \) of variables \( \tau_{k \varepsilon} \) as follows:

\[
W_{0 \varepsilon}^{\varepsilon \mathbb{Z}} = 0; \quad W_{(k+1) \varepsilon}^{\varepsilon \mathbb{Z}} - W_{k \varepsilon}^{\varepsilon \mathbb{Z}} = \sqrt{\varepsilon \tau_{k \varepsilon}} \quad \text{for} \ k \in \mathbb{Z}.
\]

\(^9\)Of course, \( a(x) \) is not an operator on \( H^\text{Fock}_\mathbb{R} \). Rather, \( a(\cdot) \) is a generalized function (Schwartz distribution) on \( \mathbb{R} \) taking on values in the space of linear operators on \( H^\text{Fock}_\mathbb{R} \).
Random variables $W_{k\varepsilon}^{\varepsilon\mathbb{Z}}$ form a random walk. Consider its maximum over $k\varepsilon \in [-1, +1]$:  

$$W_{X^{\varepsilon\mathbb{Z}}}^{\varepsilon\mathbb{Z}} = \max_{k\varepsilon \in \mathbb{Z} \cap [-1,1]} W_{k\varepsilon}^{\varepsilon\mathbb{Z}}, \quad X^{\varepsilon\mathbb{Z}} = \arg \max_{x \in \varepsilon\mathbb{Z} \cap [-1,1]} W_{x}^{\varepsilon\mathbb{Z}}.$$  

For almost all $\tau_{k\varepsilon}$ the maximum is reached only once, thus $X^{\varepsilon\mathbb{Z}}$ is a well-defined random variable (a function of random variables $\tau_{k\varepsilon}$) whose values belong to the finite piece $\varepsilon\mathbb{Z} \cap [-1,1]$ of the lattice. Also $W_{X^{\varepsilon\mathbb{Z}}}^{\varepsilon\mathbb{Z}}$ is a random variable. Both determine state vectors $\psi_{X^{\varepsilon\mathbb{Z}}}, \psi_{W^{\varepsilon\mathbb{Z}}}$; these are quite complicated linear combinations of various multi-particle states. In order to calculate them explicitly, one should evaluate Fourier-Walsh coefficients by averaging $\tau_{k_1 \varepsilon} \cdots \tau_{k_n \varepsilon} X^{\varepsilon\mathbb{Z}}$ and $\tau_{k_1 \varepsilon} \cdots \tau_{k_n \varepsilon} W_{X^{\varepsilon\mathbb{Z}}}$; fortunately we do not need it. The standard limiting procedure $\varepsilon \to 0$ gives the Wiener process $W(x)$, and $W(X) = \max_{x \in [-1,1]} W(x)$, $X = \arg \max_{[-1,1]} W(\cdot)$ (also reached only once, almost sure). Fourier-Walsh expansion of $X^{\varepsilon\mathbb{Z}}$ becomes Wiener chaos expansion of $X$ into Itô’s multiple stochastic integrals,

\begin{equation}
X = \hat{X}_0 + \int \hat{X}_1(x) dW(x) + \int \int \hat{X}_2(x, y) dW(x) dW(y) + \ldots
\end{equation}

The function $\hat{X}_k(x_1, \ldots, x_k)$ for $-1 < x_1 < \cdots < x_k < 1$ is the wave function of the $k$-particle component of $\psi_X$. The squared norm of $\psi_X$ is the average of $X^2$, equal to 1/2 since $\arcsin \psi_X$ is known to be distributed uniformly on $(-\pi/2, \pi/2)$. Still, it happens in the Fock space.

Trying to escape the Fock space, we consider such functions as $\exp(i\lambda X^{\varepsilon\mathbb{Z}})$ and $\exp(i\lambda W_{X^{\varepsilon\mathbb{Z}}})$; here $\lambda$ is a large parameter (a kind of cutoff parameter), it will ultimately tend to $\infty$. Clearly, both functions become senseless for $\lambda = \infty$. However, it is instructive to compare the behavior of $\psi_{\exp(i\lambda X^{\varepsilon\mathbb{Z}})}$ and $\psi_{\exp(i\lambda W_{X^{\varepsilon\mathbb{Z}}})}$ for large $\lambda$.

In order to estimate (via (1.2.2)) the density of particles in the state $\psi_{\exp(i\lambda W_{X^{\varepsilon\mathbb{Z}}})}$, consider the influence of a single variable $\tau_{k\varepsilon}$ on $\exp(i\lambda W_{X^{\varepsilon\mathbb{Z}}})$. It is an average over paths of the walk. Assume $k\varepsilon \in (0, 1)$. A path that reaches its maximum on $(-1, k\varepsilon)$ gives a small (in most cases, just 0) contribution to the influence. A path that reaches its maximum on $(k\varepsilon, 1)$ contributes (in most cases) roughly $(\lambda \sqrt{\varepsilon})^2$, assuming that $\varepsilon \ll 1/\lambda^2$. In the limit $\varepsilon \to 0$ the same argument, applied to the Wiener process, shows that the density of the particles on $(-1, +1)$ is roughly $\lambda^2$; too much for taking $\lambda \to \infty$.

Consider now the density of particles in the other state, $\psi_{\exp(i\lambda X^{\varepsilon\mathbb{Z}})}$. The influence of $\tau_{k\varepsilon}$ on $\exp(i\lambda X^{\varepsilon\mathbb{Z}})$ is an average over paths. Assume again $k\varepsilon \in (0, 1)$, and consider a path that reaches its maximum on $(k\varepsilon, 1)$, not too close
to $k\varepsilon$. Its contribution to the influence is in most cases exactly 0, irrespective of $\lambda$. A non-zero contribution appears only when the maximum on $(-1, k\varepsilon)$ is $\sqrt\varepsilon$-close to the maximum on $(k\varepsilon, 1)$). In that case, $\tau_{k\varepsilon}$ influences $X^{\varepsilon Z}$ by causing a jump of a size roughly 1. Also exp$(i\lambda X^{\varepsilon Z})$ makes a jump of a size roughly 1 irrespective of $\lambda$. The probability of a particle at a given point of $\varepsilon Z$ is $\sim \sqrt\varepsilon$ for large $\lambda$. The same argument, applied on the continuum ($\varepsilon = 0$) shows that the probability of at least one particle on $(x, x + \Delta x)$ is $\sim \sqrt\Delta x$ for large $\lambda$. This is why the limit $\lambda \to \infty$ is of interest for such a state.

Of course, the expression exp$(i\infty X)$ is not a well-defined function over the Wiener process, because of its infinite sensitivity to small changes of $W(\cdot)$. However, the sensitivity is concentrated near the maximizer. Accordingly, particles accumulate to a single point (though it does not follow from the simple argument of the preceding paragraph).

2 The model

2.1 Beyond the white noise

The Wiener process is the scaling limit of the random walk, which may be explained as follows. Consider the lattice $\varepsilon Z = \{\ldots, -2\varepsilon, -\varepsilon, 0, \varepsilon, 2\varepsilon, \ldots\}$, another lattice $\sqrt{\varepsilon} Z = \{\ldots, -2\sqrt{\varepsilon}, -\sqrt{\varepsilon}, 0, \sqrt{\varepsilon}, 2\sqrt{\varepsilon}, \ldots\}$, and maps $U_{x,y} : \sqrt{\varepsilon} Z \to \sqrt{\varepsilon} Z$ defined for $x, y \in \varepsilon Z$, $x < y$:

$$U_{k\varepsilon,l\varepsilon}(\sqrt{\varepsilon} m) = \sqrt{\varepsilon}(m + \tau_{k\varepsilon} + \tau_{(k+1)\varepsilon} + \cdots + \tau_{(l-1)\varepsilon});$$

here, as before, $(\tau_{k\varepsilon})_{k \in \mathbb{Z}}$ is an array of independent random signs. Note that $U_{y,z}(U_{x,y}(u)) = U_{x,z}(u)$, that is, $U_{y,z} \circ U_{x,y} = U_{x,z}$ for $x < y < z$; all $U_{x,y}$ are compositions of maps $U_{x,x+\varepsilon} (x \in \varepsilon Z)$, these being independent; each $U_{x,x+\varepsilon}$ is either $u \mapsto u + \sqrt{\varepsilon}$ or $u \mapsto u - \sqrt{\varepsilon}$. The lattice stochastic flow (Fig. 1b) results from random alternating the two simple transformations of $\sqrt{\varepsilon} Z$ (Fig. 1a). The standard limiting procedure for $\varepsilon \to 0$ leads to $U_{x,y}(u) = u + W(y) - W(x)$, $u \in \mathbb{R}$, $x, y \in \mathbb{R}$, $x \leq y$ (Fig. 1c); as before, $W(\cdot)$ is the Wiener process.

Now, breaking homogeneity on the $u$ axis, we consider another pair of transformations $\sqrt{\varepsilon} Z \to \sqrt{\varepsilon} Z$ (Fig. 2a). Alternating them at random we get another lattice stochastic flow (Fig. 2b) that has its scaling limit (Fig. 2c); we’ll call it the splitting flow.

The two-dimensional lattice $(\sqrt{\varepsilon} Z) \times (\varepsilon Z)$ decomposes into two sublattices (even and odd), closed under the flow. In order to get a scaling limit we restrict ourselves to one of the two sublattices.
Figure 1: Two simple transformations of the lattice $\sqrt{\varepsilon}Z$ (a), alternating at random, form the simple lattice stochastic flow (b), consisting of parallel copies of the random walk. Its scaling limit ($\varepsilon \to 0$) is the simple stochastic flow (c), consisting of parallel copies of the Wiener process.

Figure 2: Another pair of transformations of $\sqrt{\varepsilon}Z$ (a); their random alternation form another lattice stochastic flow (b). Its scaling limit ($\varepsilon \to 0$) is the splitting stochastic flow (c).

For large $u$, near $+\infty$, the splitting flow does not differ from the simple flow:

\begin{equation}
U_{x,y}(u) = u + W(y) - W(x) \quad \text{for } u > W(x) - \min_{[x,y]} W(\cdot);
\end{equation}

$W(\cdot)$ is still the Wiener process. Near $-\infty$ the situation is symmetric. Near 0 we have

\begin{equation}
U_{x,y}(u) = \pm (W(y) - \min_{[x,y]} W(\cdot)) \quad \text{for } |u| < W(x) - \min_{[x,y]} W(\cdot).
\end{equation}

Thus, for given $x, y \in \mathbb{R}$, $x < y$, the map $U_{x,y} : \mathbb{R} \to \mathbb{R}$ is chosen at random from a two-dimensional set $G$ of maps\footnote{The set $G$ is a semigroup, and a (two-dimensional) topological space. However, it is not a group (maps are not invertible), nor even a topological semigroup (maps are discontinuous, and the composition is a discontinuous operation on the semigroup).} determined by two parameters,
\( W(x) - \min_{[x,y]} W(\cdot) \) and \( \pm (W(y) - \min_{[x,y]} W(\cdot)) \). The first parameter is uniquely determined by \( W(\cdot) \), but the second is not; its sign is a random variable independent of \( W(\cdot) \). On the lattice, the sign is determined by the parity (evenness) of the number \( k \) of the point \( k\varepsilon \in [x,y] \) that minimizes \( W_{k\varepsilon} \). In terms of Sect. 1.4 the sign is just \( \exp(i\lambda X) \) for \( \lambda = \pi/\varepsilon \) (though, minimum is used now, rather than maximum). For \( \varepsilon \to 0 \) it turns into the ill-formed expression \( \exp(i\infty X) \). However, the sign itself is well-defined, and may be used as a well-formed substitute for the ill-formed expression!

The random map \( U_{x,y} \), that is, the family of random variables \( \{U_{x,y}(u) : u \in \mathbb{R}\} \) (for given \( x, y \)) reduces to independent components: a single random sign \( V_{x,y} = \text{sgn} U_{x,y}(0) \), and increments \( \{W(b) - W(a) : x < a < b < y\} \) of the Wiener process. The whole splitting flow \( \{U_{x,y}(u) : u \in \mathbb{R}, -\infty < x < y < +\infty\} \) boils down to random signs \( \{V_{x,y} : x < y\} \) and Wiener process increments \( \{W(b) - W(a) : a < b\} \). However, these are dependent;

\[
\mathbb{E} \left( V_{x,y} \mid W(\cdot) \right) = 0 ,
\]

\[
\mathbb{E} \left( V_{x_1,y_1}V_{x_2,y_2} \mid W(\cdot) \right) = \begin{cases} 1 & \text{if } \arg \min_{[x_1,y_1]} W(\cdot) = \arg \min_{[x_2,y_2]} W(\cdot), \\ 0 & \text{otherwise.} \end{cases}
\]

Given \( W(\cdot) \), random signs \( V_{x,y} \) depend on \( x, y \) only via \( \arg \min_{[x,y]} W(\cdot) \),

\[ V_{x,y} = V_{\arg \min_{[x,y]} W(\cdot)}. \]

Random signs \( V_x \) are ascribed to all points \( x \) of local minima of \( W(\cdot) \). Such points are a dense countable set. In fact,

\[ V_x = \lim_{\varepsilon \to 0} \text{sgn} U_{x-\varepsilon,x+\varepsilon}(0); \]

the limit exists if (and only if) \( W(\cdot) \) has a local minimum at \( x \).

### 2.2 From random processes to Hilbert spaces

The Wiener process \( (W(x))_{-\infty < x < \infty} \) is a Brownian motion (in the mathematical sense rather than physical) in the additive group \( \mathbb{R} \) of real numbers. The splitting flow \( (U_{x,y}(\cdot))_{-\infty < x < y < \infty} \) is a Brownian motion in the two-dimensional non-commutative semigroup \( G \). For the Wiener process, only increments are relevant, \( (W(y) - W(x))_{-\infty < x < y < \infty} \). For the splitting flow we cannot write \( U_{x,y} = U_{0,y}^{-1} \circ U_{0,x}^{-1} \) since \( U_{0,x} \) is not invertible; this is why we use \( U_{x,y} \) rather than \( U(x) = U_{0,x} \).

All square integrable measurable functionals of the Wiener process are a separable Hilbert space \( L_{2,\text{Wiener}}^{-}(-\infty, +\infty) \). Given \( (x, y) \subset \mathbb{R} \) we may consider
functionals of increments $W(b) - W(a)$ for all $(a, b) \subset (x, y)$; such functionals form $L^2_{\text{Wiener}}(x, y)$. Independence of increments gives

$$L^2_{\text{Wiener}}(x, y) \otimes L^2_{\text{Wiener}}(y, z) = L^2_{\text{Wiener}}(x, z).$$

All square integrable measurable functionals of the splitting flow are a separable Hilbert space $L^2_{\text{splitting}}(-\infty, +\infty)$. Given $(x, y) \subset \mathbb{R}$ we may consider functionals of $U_{a,b}^\ast(\cdot)$ for all $(a, b) \subset (x, y)$; such functionals form $L^2_{\text{splitting}}(x, y)$. Independence gives

$$L^2_{\text{splitting}}(x, y) \otimes L^2_{\text{splitting}}(y, z) = L^2_{\text{splitting}}(x, z).$$

The Wiener process is naturally embedded into the splitting flow by (2.1.1), thus,

$$L^2_{\text{Wiener}}(x, y) \subset L^2_{\text{splitting}}(x, y).$$

The space $L^2_{\text{Wiener}}(-\infty, +\infty)$ may be identified with the Fock space,

$$H^{\text{Fock}}_R = L^2_{\text{Wiener}}(-\infty, +\infty),$$

over the space $H^{(1)}(x, y)$ of all one-particle state vectors, identified with linear functionals $\int \varphi(x) dW(x)$. We have also local spaces

$$H^{(1)}_{(x,y)} = L^2_{\text{splitting}}(-\infty, +\infty),$$

and

$$H_{(x,y)}^{(1)} \oplus H_{(y,z)}^{(1)} = H_{(x,z)}^{(1)}.$$ 

Now we define non-Fock spaces of our model by

$$H_R = L^2_{\text{splitting}}(-\infty, +\infty),$$

$$H_{(x,y)} = L^2_{\text{splitting}}(x, y),$$

which gives

$$H_{(x,y)} \otimes H_{(y,z)} = H_{(x,z)}.$$ 

(2.2.1)

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12I mean independence between two families of random variables, $\{U_{a,b}(u) : u \in \mathbb{R}, (a, b) \subset (x, y)\}$ and $\{U_{a,b}(u) : u \in \mathbb{R}, (a, b) \subset (y, z)\}$.

13Recall Sect. 1.3.

14There are two equally canonical ways of identifying $H^{\text{Fock}}_{(x,y)} \otimes H^{\text{Fock}}_{(y,z)}$ with $H^{\text{Fock}}_{(x,z)}$. Namely, for $x_1 \in (x, y)$, $x_2 \in (y, z)$ we may identify $a(x_1) \otimes a(x_2)$ with $a(x_1)a(x_2)$ or alternatively with $a(x_2)a(x_1) = -a(x_1)a(x_2)$. See also \[ Th. 3.2].
2.3 Infinitely many particles

How to count particles in a non-Fock state $\psi \in H_\mathbb{R}$? The number-of-particles operator $N$ is defined on $H_\text{Fock}_\mathbb{R}$ only. Turn to $0^N$. This observable indicates the event “no particles”; it is the one-dimensional projection $|\psi_0\rangle\langle\psi_0|$ onto the vacuum $\psi_0$. It is natural to postulate that in the non-Fock model the vacuum is still the only state with no particles. That is, we extend $0^N$ from $H_\text{Fock}_\mathbb{R}$ onto $H_\mathbb{R}$ by $0^N = |\psi_0\rangle\langle\psi_0|$. In probabilistic terms $\psi_0$ is just 1 (recall (1.2.1)), and the corresponding one-dimensional projection is just $E$, the operator of mathematical expectation; $0^N f = (E f) \cdot \psi_0$, that is, $0^N f = E f$ for $f \in L^2_\text{splitting}(-\infty, +\infty)$.

We localize the argument to $(x, y) \subset \mathbb{R}$. Namely, we postulate that in the (local) non-Fock space $H_{(x,y)}$ the only state with no particles is the (local) vacuum state $\psi_{0(x,y)}$ (belonging to $H_{(x,y)}^\text{Fock}$) identified with the constant random variable 1. That is, we have

$$\psi_{0(x,y)} \otimes \psi_{0(y,z)} = \psi_{0(x,z)},$$

$$\psi_0 = \psi_{0(-\infty, +\infty)},$$

and we extend $0^{N_{x,y}}$ from $H_{(x,y)}^\text{Fock}$ onto $H_{(x,y)}$ by

$$0^{N_{x,y}} = |\psi_{0(x,y)}\rangle\langle\psi_{0(x,y)}|,$$

the one-dimensional projection onto the local vacuum. Alternatively, we may treat $0^{N_{x,y}}$ as an operator on the whole space:

$$H_\mathbb{R} = H_{(-\infty, x)} \otimes H_{(x,y)} \otimes H_{(y, +\infty)},$$

$$0^{N_{x,y}} = 1 \otimes |\psi_{0(x,y)}\rangle\langle\psi_{0(x,y)}| \otimes 1,$$

the projection onto the (infinite-dimensional) space $H_{(-\infty, x)} \otimes \psi_{0(x,y)} \otimes H_{(y, +\infty)}$. In probabilistic terms, the space consists of all functionals of $\{U_{x_1,x_2}(u) : u \in \mathbb{R}, (x_1, x_2) \subset (-\infty, x)\} \cup \{U_{x_1,x_2}(u) : u \in \mathbb{R}, (x_1, x_2) \subset (y, +\infty)\}$, and the projection is the conditional expectation,

$$0^{N_{x,y}} f = E \left( f \left| \mathcal{F}_{(-\infty, x) \cup (y, +\infty)} \right. \right);$$

$\mathcal{F}_E$ stands for the $\sigma$-field (on the probability space of the splitting flow) generated by the set $\{U_{a,b}(u) : u \in \mathbb{R}, (a, b) \subset E\}$ of random variables, whenever $E \subset \mathbb{R}$ is an interval or a union of finitely many intervals.

\textsuperscript{15}Thus, “to contain particles” means “to be orthogonal to $\psi_0$”. True, it is not a deep definition of “particles”. We’ll see soon that our “particles” can be localized in space (too much for physical particles, see Sect. 3.3). Further they will get an electric charge, see Sect. 3.2.
We have commuting observables $0^{N_{x,y}}$ on $H_\mathbb{R}$, satisfying
\begin{equation}
0^{N_{x,y}}0^{N_{y,z}} = 0^{N_{x,z}}.
\end{equation}
(2.3.1)

We may consider the corresponding joint quantum measurement. Its outcome assigns values (0 or 1) to all $N_{x,y}$ respecting (2.3.1), and is interpreted as a configuration of particles in the (one-dimensional) space. If $\psi \in H_\mathbb{R}^{\text{Fock}}$ then the outcome is always a finite set. For $\psi \in H_\mathbb{R}$ the outcome can be infinite. In fact, if $\psi$ is orthogonal to the Fock space (that is, $\psi \in H_\mathbb{R} \ominus H_\mathbb{R}^{\text{Fock}}$), then the outcome is always an infinite set.

More formally, every state vector $\psi \in H_\mathbb{R}$, $\|\psi\| = 1$, determines a probability measure $\mu_\psi$ on the space of compact subsets of $\mathbb{R}$, defined by
\begin{equation}
\forall E \quad \langle 0^{N_E} \rangle_\psi = \mu_\psi\{C : C \cap E = \emptyset\};
\end{equation}
here $E$ runs over finite unions of intervals; $0^{N_E}$ is defined naturally:
\begin{equation}
0^{N_E} = 0^{N_{x_1,y_1}} \ldots 0^{N_{x_n,y_n}} \quad \text{for } E = (x_1, y_1) \cup \ldots \cup (x_n, y_n);
\end{equation}
in fact, (2.3.1) is a special case of $0^{N_{E_1 \cup E_2}} = 0^{N_{E_1}} \cdot 0^{N_{E_2}}$. See [8, Sect. 2] and also [7] for a general theory of $\mu_\psi$.

If $\psi = \psi_0$ then $\mu_\psi$ is concentrated on $\{\emptyset\}$. That is, if we measure the configuration of particles in vacuum, we always get the empty set $\emptyset$.

If $\psi \in H_\mathbb{R}^{\text{Fock}}$ then $\mu_\psi$ is concentrated on (the set of) finite sets. That is, if we measure the configuration of particles in a Fock state, we get a random finite set.

If $\psi \in H_\mathbb{R} \ominus H_\mathbb{R}^{\text{Fock}}$ then $\mu_\psi$ is concentrated on (the set of) infinite sets. That is, if we measure the configuration of particles in a non-Fock state (orthogonal to the Fock space), we get a random infinite set. The general theory [8] states that $\mu_\psi\{C : C \ni x\} = 0$ for every $x$ (and $\psi$). It follows that $\mu_\psi$ is concentrated on (the set of) sets $C$ of zero Lebesgue measure. For some models, $C$ is always of cardinality continuum.

Return to our particular model, $H_\mathbb{R} = L_2^{\text{splitting}}$. Recall that $L_2^{\text{splitting}}$ is generated by $L_2^{\text{Wiener}}$ and random signs $V_{x,y}$. The simplest example of $\psi \in H_\mathbb{R} \ominus H_\mathbb{R}^{\text{Fock}}$ is $\psi = V_{0,1}$. The corresponding $\mu_\psi$ is described, rather explicitly, by Warren [10]. It is concentrated on (the set of) infinite, countable sets $C$ with a single accumulation point (which was roughly explained here, in Sect. 1.4). That is, if we measure the configuration of particles in such a state, we always get a random countable set with a single, random, accumulation point.
3 Toward more realistic models

3.1 Three-dimensional space

Recall the idea explained in Sect. 1.4: the maximizer \( X = \text{arg max}_{[-1,+1]} W(\cdot) \) of the Wiener process is rather insensitive to changes of \( W(\cdot) \) outside of a neighborhood of \( X \). The set of local maxima (or minima) of \( W(\cdot) \) is a (random) dense countable set. And, as explained in Sect. 2.1, the insensitivity emerges from locality of the notion of a local maximum.

Turning from \( \mathbb{R} \) to the 3-dimensional space \( \mathbb{R}^3 \), we turn from \( W(\cdot) \) to the white noise over \( \mathbb{R}^3 \); its sample function is a generalized function (Schwartz distribution) and gives a number, being integrated over some 3-dimensional domain. The integral is a random variable distributed normally with zero mean; its variance is equal to the volume of the domain.

A straightforward attempt to generalize \( \text{arg max} \, W(\cdot) \) for \( \mathbb{R}^3 \) meets many difficulties. However, we need only some dense countable set of (random) points, determined by a local property of a white noise sample function. I am unable to find such a property among natural, well-known properties. Instead, I am able to invent such a property.

Consider the integral \( S(x,r) \) of the 3-dimensional white noise over the 3-dimensional ball \( B(x,r) \subset \mathbb{R}^3 \) of radius \( r \in (0,\infty) \), centered at \( x \in \mathbb{R}^3 \). Given \( x_0 \) and \( r_0 \), we consider all \( x_1, r_1 \) such that \( B(x_1, r_1) \subset B(x_0, r_0) \) and \( r_1 \leq r_0/2 \). We choose \( x_1, r_1 \) maximizing \( |S(x_1,r_1)| \); the maximum is reached only once (with probability 1). Having \( x_1, r_1 \) we choose \( x_2, r_2 \) in the same way: \( B(x_2, r_2) \subset B(x_1, r_1) \), \( r_2 \leq r_1/2 \), and \( (x_2, r_2) = \text{arg max}_{(x_2,r_2)} |S(x_2,r_2)| \). And so on. The sequence \( (x_n) \) converges to a point: \( x_n \rightarrow x_{x_0,r_0} \). We have a family \( \{X_{x_0,r_0}\}_{x_0\in\mathbb{R}^3,r_0\in(0,\infty)} \) of \( \mathbb{R}^3 \)-valued random variables; they are functionals of the 3-dimensional white noise. A small change of \( (x_0, r_0) \) does not influence \( X_{x_0,r_0} \), therefore the random set \( \{X_{x_0,r_0} : x_0 \in \mathbb{R}^3, r_0 \in (0,\infty)\} \) is (at most) countable. The set is dense, since \( r_0 \) may be arbitrarily small.

Now we can mimick Warren’s construction as follows. We introduce random signs \( V_{x,r} \) such that (recall (2.1.2))

\[
\mathbb{E} \left( V_{x,r} \mid S(\cdot,\cdot) \right) = 0,
\]

\[
\mathbb{E} \left( V_{x_1,r_1} V_{x_2,r_2} \mid S(\cdot,\cdot) \right) = \begin{cases} 1 & \text{if } X_{x_1,r_1} = X_{x_2,r_2}, \\ 0 & \text{otherwise}. \end{cases}
\]

In other words, we ascribe new random signs \( V_x \) to all points of the dense countable random set.

Every smoothly bounded domain \( D \subset \mathbb{R}^3 \) determines its Hilbert space \( H_D^{\text{Fock}} \), generated by \( S(x,r) \) for all \( (x,r) \) such that \( B(x,r) \subset D \). Also, it
determines a larger space $H_D$ generated by $S(x, r)$ and $V_{x, r}$ for all $(x, r)$ such that $B(x, r) \subseteq D$. Similarly to (2.2.1),

\[
H_D^{\text{Fock}} \subset H_D,
\]

\[
H_D^{\text{Fock}} \otimes H_D^{\text{Fock}} = H_D^{\text{Fock}},
\]

\[
H_D \otimes H_D = H_D
\]

whenever $D$ is split into two domains $D_1, D_2$ by a smooth surface. 16

The random sign $V_{x, r}$ is identified with a state vector $\psi_{x, r} \in H_D \ominus H_D^{\text{Fock}}$, where $D = B(x, r)$. If we measure the configuration of particles in such a state, we always get a random countable subset of $D$ with a single, random, accumulation point.

### 3.2 Electric charge

Charged particles are quanta of complex fields. For the Fock model it may be interpreted via the complex-valued (rather than real-valued) white noise. In one-dimensional space, every square-integrable functional of the complex-valued Wiener process $Z(x) = W_1(x) + iW_2(x)$ (where $W_1, W_2$ are independent real-valued Wiener processes) has its Wiener chaos expansion similar to (1.4.1), but instead of $dW(x), dW(x)dW(y)$ and so on, we get $dZ(x), d\overline{Z}(x), dZ(x)dZ(y), dZ(x)d\overline{Z}(y), d\overline{Z}(x)dZ(y), d\overline{Z}(x)d\overline{Z}(y)$ and so on; here $\overline{Z}(x)$ is the complex conjugate to $Z(x)$. The term with $dZ(x)$ describes a single particle with the charge +1; the charge −1 is described by $d\overline{Z}(x); dZ(x)d\overline{Z}(y)$ describes two particles, one of positive charge (at $x$), the other of negative charge (at $y$); and so on. The sector of (total) charge 0 consists of functionals invariant under the global commutative gauge group $Z(\cdot) \mapsto e^{i\varphi}Z(\cdot)$. The corresponding one-parameter unitary group on $H_{\mathbb{R}}$ has its generator, the observable of total electric charge $Q$. Localization to $H_{x, y}$ gives $Q(x, y)$, the electric charge on $(x, y)$.

As far as I know, there is no interesting generalization to $Z(\cdot)$ of the idea of Sect. 1.4, based on $\arg \max_{[x, y]} W(\cdot)$. 17 However, the idea of Sect. 3.1 can be used easily. Still, $|S(x_1, r_1)|$ is maximized, but now $S(x_1, r_1)$ is a complex number. The maximizer is invariant under $Z(\cdot) \mapsto e^{i\varphi}Z(\cdot)$. We extend the gauge group from $H^{\text{Fock}}_D$ to $H$ by postulating that random signs $V_{x, r}$ are gauge invariant. Thus, $V_{x, r}$ is a state vector belonging to the zero-charge sector.

---

16Fock spaces $H^{\text{Fock}}_D$ can be defined also for non-smooth domains $D$ without loss of their natural properties, but non-Fock spaces $H_D$ cannot; see 1.

17We cannot place random signs at local maxima of $\text{Re}Z(\cdot)$, since $\text{Re}(e^{i\varphi}Z(\cdot))$ has quite different local minima; the union over all $\varphi$ is uncountable. Local minima of $|Z(\cdot)|$, that is, of $\int_0^\infty dZ(\cdot)$, lead to a theory treating the origin as a special point in the one-dimensional space.
Local gauge transformations $Z(x) \mapsto e^{i\varphi(x)}Z(x)$ determine the (infinite-dimensional, unitary) gauge group on $H_\mathbb{R}$ (still, $V_{x,r}$ are gauge-invariant by definition). The group commutes with the observables $\hat{0}^{N_{x,y}}$, which means that we can measure both the spatial configuration of particles and (simultaneously) the spatial distribution of electric charge. An outcome of such a measurement in the state $V_{x,r}$ is always a countable set of particles, each having a charge $\pm 1$. As was said, the set has a single accumulation point. The charge in a neighborhood of the accumulation point is determined by the fact that the total charge is equal to zero.

### 3.3 Dirac sea and dynamics

Non-interacting electrons (and positrons) are quanta of the free Dirac quantum field,\footnote{See [5, Sect. 7.2], [2, Sect. 6.5].} much more singular than our toy models. The distribution of the electric charge can be treated as a generalized function (Schwartz distribution) on 4-dimensional space-time rather than 3-dimensional space; otherwise fluctuations become infinite.\footnote{See [6, Sect. 7].} In terms of the lattice approximation of Sect. 1.1 we have, roughly speaking,\footnote{The space should be 3-dimensional; a spinor index should be added to the spatial index $k$; the infinite product is ill-defined.}

$$
\psi^\text{Dirac}_0 = \left( \prod_n \left( \sum_k \varphi_n(k\epsilon)a^*_k \right) \right) \psi_0;
$$

here $\psi_0$ is the simplistic vacuum of Sect. 1.1; $\psi^\text{Dirac}_0$ is the vacuum of the free Dirac quantum field; and functions $\varphi_n(\cdot)$ on the lattice $\epsilon\mathbb{Z}$ form an orthonormal basis of a special subspace of $L_2(\epsilon\mathbb{Z})$, so-called negative energy subspace. It is infinite-dimensional, moreover, it is a half of $L_2(\epsilon\mathbb{Z})$ in the sense that $\langle a^*_ka_k \rangle^\text{Dirac}_0 = 1/2$ for all $k$; each lattice point is occupied with probability $1/2$ (these events being correlated).

According to Dirac, the vacuum is such a state: all negative-energy states are occupied, while all positive-energy states are free. Positrons are holes in a sea of negative electrons. See [4, Sect. 6.5] for a rigorous definition of the relevant space that contains $\psi^\text{Dirac}_0$ (and does not contain $\psi_0$). Particles (electrons and positrons) are inherently delocalized, see [5, Sect. 1].

What about “filling in the Dirac sea” for non-Fock models considered here? For now it remains unclear. Maybe the “spatial” approach to product systems in quantum field theory is too naive; for a different, 4-dimensional approach see [2, Sect. 1].
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School of Mathematics, Tel Aviv Univ., Tel Aviv 69978, Israel
tsirel@math.tau.ac.il
http://www.math.tau.ac.il/~tsirel/