Optimal liquidation for a risk averse investor in a one-sided limit order book driven by a Lévy process

Arne Løkka*, Junwei Xu†

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Abstract

In a general one-sided limit order book where the unaffected price process follows a Lévy process, we consider the problem for an investor with constant absolute risk aversion to optimally liquidate a given large position of shares. Since liquidation normally takes place within a short period of time, modelling the risk as a Lévy process should provide a realistic model with good statistical fit to observed market data, thus providing a realistic reflection of the investor’s market risk. We can reduce the optimisation problem to a deterministic two-dimensional singular problem, to which we are able to derive an explicit solution in terms of the model data. In particular we find an expression for the optimal intervention boundary, which completely characterise the optimal liquidation strategy.

1 Introduction

This paper is concerned with how a large investor should go about selling (or purchasing) a large position of shares. This kind of problem has attracted considerable interest over the past few years following the introduction of electronic trading platforms. In the model we consider, we specify the limit order book and how this recovers over time. Thus the optimal liquidation strategy that we want to find will explicitly specify the orders we submit to the market, as opposed to just specifying the optimal speed at which to trade that is the case for

*Department of Mathematics Columbia House London School of Economics Houghton Street, London WC2A 2AE United Kingdom (a.lokka@lse.ac.uk)
†Department of Mathematics Columbia House London School of Economics Houghton Street, London WC2A 2AE United Kingdom (j.xu19@lse.ac.uk)
the popular impact models (see e.g. Lehall and Laruelle (2013), Alvaro Cartea and Penalva (2015) and Guéant (2010) for an introduction to optimal execution and the most common models).

More precisely, in this paper we study the optimal liquidation problem in the context of a limit order book where the aim for the investor is to maximise expected utility of his cash position. We assume that the utility function is of the CARA type (constant absolute risk aversion), that is an exponential utility function. Moreover, we assume that the market risk of the stock price is represented by a Lévy process. A number of studies demonstrate that Lévy processes are able to capture the essential statistical properties of stock price movements over short time horizons (see e.g. Madan and Seneta (1990), Eberlein and Keller (1995), Barndorff-Nielsen (1997) and Cont and Tankov (2004)). Since the majority of liquidation tend to finish within a short period of time, this model should provide a reasonable reflection of the market risk faced by the investor. We consider a bid limit order book with general shape with a general resilience function. It is assumed that the unaffected bid price provides a lower bound for the best ask price and that the bid limit order book is unaffected by the large investor’s buy orders. These assumptions allow us to exclude any buy orders in the optimal trading strategy, and they also exclude price manipulation in our model in the sense of Huberman and Stanzl (2004). The available number of bid limit orders are assumed to be finite, which limits the large investor’s trading strategy in the sense that he cannot make a block sale larger than currently available number of bid orders. With an infinite time horizon, we solve the problem of maximising the expected utility of the large investor’s the final cash. We do this by showing that the problem can be reduced to a two-dimensional deterministic singular optimisation problem to which we can obtain an explicit solution in terms of the characteristics of the limit order book and the investors risk aversion. With reference to Løkka (2014), we guess that the optimal strategy consists of either a block sale or a period of waiting at the beginning, and the large investor sells continuously along an intervention boundary afterwards. The intervention boundary is associated with the Hamilton-Jacobi-Bellman (HJB) variational inequalities corresponding to the optimisation problem. This intervention boundary might have discontinuities as well as constant parts. The discontinuities corresponds to waiting while the order book recovers, while the constant part corresponds to selling at the same rate as the resilience rate. Moreover, the intervention boundary is non-increasing. This means that when the large investor is continuously selling shares, it is never optimal to implement a speed which is greater than the resilience rate. Following the same idea as in Løkka (2014), the value function in our problem can be expressed in an explicit way, and we characterise
the intervention boundary via the HJB variational inequalities. The strategy associated with this intervention boundary is shown to be optimal by a verification argument.

The model we use is a version of the model introduced in Obizhaeva and Wang (2013), which later was generalised in Alfonsi et al. (2010), and then further in Predoiu et al. (2011). However, these papers did not consider risk-aversion, but assumed the investor wanted to maximising the expected value of the cash position. The problem we consider in this paper is an extension of Løkka (2014) in the sense that unaffected price process, shape of limit order book and resilience function is more general.

This paper is structured as follows. In Section 2 we introduce the limit order book model and the large investor’s optimisation problem. We simplify the problem and also obtain results related to price manipulation strategies in Section 3. The simplified deterministic optimisation problem is solved in Section 4. Then proofs omitted in the previous sections are contained in Section 5.

2 Problem formulation

Let \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\) be a complete filtered probability space satisfying the usual conditions and supporting a one-dimensional non-trivial Lévy process \(L\).

**Assumption 2.1.** We assume that \(L\) is an \((\mathcal{F}_t)\)-martingale, and that there exists some \(\delta > 0\) such that \(\mathbb{E}[e^{\theta L_1}] < \infty\), for \(|\theta| < \delta\).

Let \(\kappa\) denote the cumulant generating function of \(L_1\), i.e.

\[
\kappa(\theta) = \ln \left( \mathbb{E}[e^{\theta L_1}] \right), \quad \theta \in \mathbb{R}.
\]

Assumption 2.1 guarantees that the cumulant generating function \(\kappa\) is continuously differentiable in a neighbourhood of 0. With reference to Assumption 2.1, we notice that the Lévy process \(L\) is square integrable, hence admits the representation

\[
L_t = \sigma W_t + \int_{\mathbb{R}\setminus\{0\}} z \left( N(t, dz) - t\nu(dz) \right), \quad t \geq 0,
\]

where \(W\) is a standard Brownian motion, \(N\) is a Poisson random measure which is independent of \(W\) with compensator \(\pi(t, dz) = t\nu(dz)\), where \(\nu\) denotes the Lévy measure associated with
The cumulant generating function $\kappa$ can then be expressed as
\[
\kappa(\theta) = \frac{1}{2} \sigma^2 \theta^2 + \int_{\mathbb{R} \setminus \{0\}} \left( e^{\theta z} - 1 - \theta z \right) \nu(dz), \quad |\theta| < \delta. \tag{2.1}
\]
In particular,
\[
\kappa'(0) = 0 \quad \text{and} \quad \kappa''(0) = \sigma^2 + \int_{\mathbb{R} \setminus \{0\}} z^2 \nu(dz).
\]
Moreover, $\kappa$ is lower semi-continuous (see Ganesh et al., 2004, Lemma 2.3). With reference to (2.1), one can check that $\kappa$ is strictly convex and continuously differentiable on its effective domain, and satisfies $\kappa(0) = 0$. Therefore, $\kappa(\theta)$ is strictly decreasing for $\theta < 0$ and strictly increasing for $\theta > 0$. Set
\[
\mathbb{R}^+ = [0, \infty) \quad \text{and} \quad \mathbb{R}^- = (-\infty, 0].
\]
For any given $A > 0$ define the function $\kappa_A : \mathbb{R}^+ \to [0, \infty]$ by
\[
\kappa_A(y) = \kappa(-Ay), \quad y \geq 0,
\]
and set
\[
\bar{y}_A = \sup \{ y \geq 0 \mid \kappa_A(y) < \infty \}.
\]
Then $\kappa_A$ is strictly increasing, strictly convex, lower semi-continuous and continuously differentiable on $[0, \bar{y}_A]$ with $\kappa_A(0) = 0$. With reference to (2.1), one can deduce that there exist $\epsilon, C_1, C_2 > 0$ such that
\[
C_1 y^2 \leq \kappa_A(y) \leq C_2 y^2, \quad 0 \leq y \leq \epsilon. \tag{2.2}
\]
The function $\kappa_A$ will play a predominant role in the sequel, where the number $A$ will be a parameter describing the large investor’s risk aversion.

We consider an investor who aims to sell a large amount of shares of a single stock in an infinite time horizon. Let $Y_t$ denote the number of shares held by the investor at time $t$. We refer to a process $Y$ as a liquidation strategy if $Y_t$ tends to 0 as $t$ tends to infinity. We consider the following set of admissible liquidation strategies.
**Definition 2.2.** For $y \in \mathbb{R}^+$, let $\mathcal{A}(y)$ denote the set of all $(\mathcal{F}_t)_{t \geq 0}$-adapted, predictable, decreasing, càdlàg processes $Y$, satisfying $Y_0 = y$ and

$$\int_0^\infty \kappa_A(\|Y_t\|_{L^\infty(P)}) \, dt < \infty. \tag{2.3}$$

Moreover, let $\mathcal{A}_D(y)$ denote the set of all deterministic strategies in $\mathcal{A}(y)$.

To describe the investor’s execution price, we explicitly model a bid limit order book. We assume that the unaffected bid price process $B^0_0$, which is the process describing the best bid prices in the market if the investor does not trade, is given by

$$B^0_t = b + L_t, \quad t \geq 0,$$

where $b > 0$ is the best bid price at the initial time. This Bachelier-Lévy price model may seem simplistic, but this kind of modelling of the unaffected price process is widely used in the optimal liquidation literature (see e.g. Almgren and Chriss, 2000; Kissell and Malamut, 2005; Schied and Schöneborn, 2009; Gatheral, 2010, etc). There are studies which show that a liquidation model with linear price processes provide a good approximation for models with exponential price processes (see e.g. Gatheral and Schied, 2011, and the references there in). In our model, the unaffected bid price is assumed to provide a lower bound for the best ask price and that the best bid price as well as all bid prices are unaffected by the large investor’s buy orders (if he is allowed to buy back). These assumptions are satisfied throughout the whole chapter, and they allow us to exclude any buy orders in the optimal trading strategy (see Remark 3.1), and they also exclude price manipulation in our model (see Remark 3.1).

In order to describe the bid limit order book, we consider a measure $\mu$ defined on the Borel $\sigma$-algebra on $\mathbb{R}^-$, denoted by $\mathcal{B}(\mathbb{R}^-)$. If $S \in \mathcal{B}(\mathbb{R}^-)$, then $\mu(S)$ represents the number of bid orders with prices in the set $B^0 + S = \{B^0 + s \mid s \in S\}$, provided that the investor did not make any trades before time $t$. Notice that the undisturbed bid order book described by $\mu$ is relative to the unaffected bid prices in the sense that it shifts together with the movement of the unaffected price (see Figure 3.1 and Figure 3.2 for illustrations). We impose the following assumptions on $\mu$.

**Assumption 2.3.** We assume that

(i) there exists some $\bar{x} \in (-\infty, 0)$ such that $\mu([\bar{x}, 0]) = \mu(\mathbb{R}^-) < \infty$;
Figure 3.1: An illustration of an undisturbed bid limit order book at time $0$.

Figure 3.2: An illustration of an undisturbed bid limit order book at time $t$.

(ii) $\mu$ is absolutely continuous with respect to Lebesgue measure, and is non-zero on any interval properly containing the origin;

(iii) the function $x \mapsto \mu((x, 0])$ is concave in $x$, for $x \in \mathbb{R}^-$.

The first assumption means that there are finitely many bid orders available in the order book and the finite number $\bar{x}$ is equal to the smallest bid price in the book. We know from (ii) that the right end of the bid order book coincides with the best bid price in the undisturbed bid order book; in other words, one can always sell some amount of shares at the unaffected bid price in an undisturbed bid order book. The concavity of $\mu((x, 0])$ tells that if we look at the undisturbed bid order book, there are less bid orders placed at a price which is farther away from the best bid price.

Set $\bar{\varepsilon} = -\mu(\mathbb{R}^-)$, and introduce the functions $\phi : [-\infty, 0] \to \mathbb{R}^-$ and $\psi : \mathbb{R}^- \to [-\infty, 0]$.
by

\[ \phi(x) = -\mu((x,0)) \quad \text{and} \quad \psi(z) = \phi^{-1}(z), \]

where \( \phi(\psi(z)) = z \), for all \( z \in [\bar{z}, 0] \), and \( \psi(z) = -\infty \), for all \( z < \bar{z} \). As direct consequences of Assumption 2.3, \( \phi \) is convex, \( \psi \) is concave, and they are both continuous and strictly increasing on their effective domain. They also satisfy

\[ \phi(0) = \psi(0) = 0; \quad (2.4) \]

\[ \int_0^{\bar{z}} \psi(u) \, du < \infty \quad \text{and} \quad \psi(\bar{z}) > -\infty. \quad (2.5) \]

The state of the limit order book changes during trading. The book recovers as new limit orders arrives. In order to model the dynamic of the bid order book during trading, we need to introduce one more process. For a given strategy \( Y \), let \( Z_Y \) be an \( \mathbb{R}^- \)-valued process such that \( -Z_Y^t \) represents the volume spread at time \( t \). That is \( -Z_Y^t \) is equal to the total number of bid orders which have already been executed subtracted by the total amount of limit orders which have arrived to refill the book up to time \( t \). We call \( Z_Y \) the state process of the bid limit order book associated with a trading strategy \( Y \). Let \( Z_0 = z \), where \( z \geq \bar{z} \) is the initial state of our bid order book. Therefore, we have \( \psi(Z_Y^t) = B_Y^t - B_0^t \) where \( B_Y^t \) is the best bid price at time \( t \) corresponding to \( Y \), and \( \psi(Z_Y^t) \) can be understood as the extra price spread at time \( t \), caused by the investor who implements a strategy \( Y \) (see Figure 3.3). Note that we have defined \( \psi(z) = -\infty \), for all \( z < \bar{z} \). This implies that the best bid price drops down to \( -\infty \), if one sells more share than available bids in the book. The rate at which bid orders are refilling the order book is described by a resilience function \( h : \mathbb{R}^- \to \mathbb{R}^- \) which satisfies the following.
Assumption 2.4. We assume the resilience function $h : \mathbb{R}^- \to \mathbb{R}^-$ is increasing and locally Lipschitz continuous. It satisfies $h(0) = 0$ and $h(x) < 0$, for all $x < 0$. We also assume that $1/h$ is a concave function.

Then, we consider the state process $Z_t^Y$ following the dynamic

$$dZ_t^Y = -h(Z_t^Y) \, dt + dY_t, \quad Z_0^Y = z \in \mathbb{R}^-.$$  \hspace{1cm} (2.6)

For any admissible strategy $Y$, we refer to Predoiu et al. (2011) Appendix A, for the existence and uniqueness of a negative, càdlàg and adapted solution to this dynamic. Combining Assumption 2.4 and equation (2.6), we see that the further the best bid price is away from the unaffected bid price, the larger the resilience speed of the best bid price. If the investor make no trades from time $t_1$ to $t_2$, then $(Z_t^Y)_{t_1 < t < t_2}$ satisfies

$$dZ_t^Y = -h(Z_t^Y) \, dt.$$  \hspace{1cm} (2.7)

Now define a strictly decreasing function $H : \mathbb{R}^- \to \mathbb{R} \cup \{-\infty\}$ by

$$H(x) = \int_{-1}^x \frac{1}{h(u)} \, du.$$  \hspace{1cm} (2.8)

Let $H^{-1}$ denote the inverse of $H$, satisfying $H^{-1}(H(x)) = x$ for all $x \leq 0$ and $H^{-1}(u) = 0$ for $u \in (-\infty, \lim_{x \to 0^-} H(x)]$. Then, it can be verified that the process $Z$ given by

$$Z_t = H^{-1}(H(Z_0) - t)$$  \hspace{1cm} (2.9)
has dynamic (2.7). Hence, for any $t$ between time $t_1$ and $t_2$, $Z_t^Y = H^{-1}(H(Z_{t_1}^Y) - t + t_1)$; and if $Z_{t_2}^Y < 0$, then

$$t_2 - t_1 = H(Z_{t_1}^Y) - H(Z_{t_2}^Y).$$ \hspace{1cm} (2.10)

Suppose the investor’s initial cash position is $c$ and that he implements a strategy $Y \in \mathcal{A}(y)$. Then his cash position at time $T > 0$ is

$$C_T(Y) = c - \int_0^T B_Y^Y dY^c - \sum_{0 \leq t \leq T} \int_{0}^{\Delta Y_t} \{ B_{t_-}^{0} + \psi(Z_t^Y + x) \} dx,$$ \hspace{1cm} (2.11)

which corresponds to the best bids offered at all times being executed first so as to match the investor’s sell orders, where the first integral is the cost from the continuous component of the liquidation strategy and the sum of integrals gives out total cost due to all block sales.

We also suppose the investor has a constant absolute risk aversion (CARA). With initial cash position $c$, an initial share position $y$ and infinite-time horizon, he wants to maximise the expected utility of his cash position at the final time. Mathematically, the investor’s optimal liquidation problem is

$$\sup_{Y \in \mathcal{A}(y)} \mathbb{E}[U(C_\infty(Y))],$$ \hspace{1cm} (2.12)

where the utility function $U$ is given by

$$U(c) = -e^{-Ac}, \quad A > 0.$$ 

This can be seen as a generalisation of the problem considered in Løkka (2014), and a risk-averse version of the problem considered by Predoiu et al. (2011).

If $Z_t^Y < \bar{z}$, then $B_t^Y = B_t^0 + \psi(Z_t^Y) = -\infty$. Clearly the value minus infinity of the best bid price would be unfavourable to the investor. Indeed, (2.11) shows that this brings the investor an infinite cost. Due to this consideration, we will from now on restrict ourselves to admissible strategies $Y$ with $Z_t^Y \geq \bar{z}$, for all $t \geq 0$. 

9
3 Problem simplification

In this section, we show that the utility maximisation problem in (2.12) can be reduced to a deterministic optimization problem. This reduction was first explored in Schied et al. (2010), who proved that with a certain market structure and a CARA investor, the optimal liquidation strategy is deterministic.

Let $Y \in \mathcal{A}(y)$. Then it follows from (2.11) that

$$C_T(Y) = c + by - (b + L_T)Y_T + \int_0^T Y_{t-} \, dL_t + \sum_{0 \leq t \leq T} \Delta L_t \Delta Y_t - F_T(Y),$$

where $F_T$ is given by

$$F_T(Y) = \int_0^T \psi(Z^Y_{t-}) \, dY^c_t + \sum_{0 \leq t \leq T} \int_0^{\Delta Y_t} \psi(Z^Y_{t-} + x) \, dx. \tag{3.1}$$

The condition (2.3) implies that any admissible strategy $Y \in \mathcal{A}(y)$ satisfies

$$\lim_{T \to \infty} T \kappa_A(\|Y_T\|_{L^\infty(\mathbb{P})}) = 0,$$

and with reference to (2.2),

$$\lim_{T \to \infty} \mathbb{E}[|L_T Y_T|^2] \leq \lim_{T \to \infty} \kappa''(0) T\|Y_T\|_{L^\infty(\mathbb{P})}^2 \leq \lim_{T \to \infty} \kappa''(0) C_1^{-1} T \kappa_A(\|Y_T\|_{L^\infty(\mathbb{P})}) = 0.$$

We conclude that $B^0_T Y_T$ tends to 0 in $L^2(\mathbb{P})$ as $T \to \infty$. Set

$$t_\epsilon = \inf \{ t \geq 0 \mid \|Y_t\|_{L^\infty(\mathbb{P})} \leq \epsilon \}.$$

From (2.2) and (2.3), it follows that

$$\mathbb{E} \left[ \left( \int_0^\infty Y_{t-} \, dL_t \right)^2 \right] \leq \kappa''(0) \left( y^2 \|t_\epsilon\|_{L^\infty(\mathbb{P})} + \int_{t_\epsilon}^\infty \|Y_t\|_{L^\infty(\mathbb{P})}^2 \, dt \right) \leq \kappa''(0) \left( y^2 \|t_\epsilon\|_{L^\infty(\mathbb{P})} + C_1^{-1} \int_{t_\epsilon}^\infty \kappa_A(\|Y_t\|_{L^\infty(\mathbb{P})}) \, dt \right) < \infty.$$
Hence, $\int_0^\infty Y_t^- dL_t$ is well-defined in $L^2(\mathbb{P})$. Due to the predictability of $Y$, we also have that
\[
\mathbb{E}\left[\left(\sum_{0 \leq t \leq T} \Delta L_t \Delta Y_t\right)^2\right] = \mathbb{E}\left[\int_0^T (\Delta Y_t)^2 dt\right] \left(\int_{\mathbb{R}\setminus\{0\}} z^2 \nu(dz)\right) = 0, \quad \text{a.s.,}
\]
for all $T > 0$, which shows that the quadratic covariation of the jumps of $L$ and $Y$ is almost surely 0. Moreover, note that $F_T(Y) \geq 0$ is an increasing function of $T$. Therefore, $F_\infty$ is a well defined function from the set of càdlàg non-increasing functions into the extended positive real numbers. The final cash position is hence given by
\[
C_\infty(Y) = c + by + \int_0^\infty Y_t^- dL_t - F_\infty(Y), \tag{3.2}
\]
where $c + by$ gives out the mark-to-market value of the total wealth of the large investor’s positions at the beginning of liquidation, $\int_0^\infty Y_t^- dL_t$ represents the cost due to the market volatility risk, and $F_\infty(Y)$ represents the cost inherited due to the price impact resulted from the limited liquidity.

**Remark 3.1.** Suppose intermediate purchases are allowed, and consider a càdlàg adapted strategy that can be decomposed into a pure buy strategy $X$ and a pure sell strategy $Y$. We assume $X + Y$ satisfies (2.3), $\lim_{t \to \infty} t \kappa_A(||X_t + Y_t||_{L^\infty(\mathbb{P})}) = 0$ and $0 \leq X_t + Y_t < \bar{y}_A$, for all $t \geq 0$. Then the cash position associated with the strategy $X + Y$ at time infinity, denoted by $C_\infty(X,Y)$, is well-defined similar to (3.2). We then have that
\[
C_\infty(X,Y) \leq \lim_{T \to \infty} C_T(Y) - \int_0^T B^0_t^- dX_t \tag{3.3}
\]
\[
= c + by + \int_0^\infty (X_t^- + Y_t^-) dL_t - F_\infty(Y)
\]
\[
\leq c + by + \int_0^\infty (X_t^- + Y_t^-) dL_t, \tag{3.4}
\]
where the first inequality is due to the assumption that the unaffected bid price is a lower bound for the best ask price. Taking $y = 0$ shows that the expected cost associated with any round-trip strategy is always positive. Hence, the model doesn’t allow for price manipulation in the sense of Huberman and Stanzl (2004).

Note that the derivation of (3.4) does not involve the resilience function. The only property of the order book that contributes to the absence of price manipulation is the assumption...
that the best ask price provides an upper bounded for the best bid price and that the bid limit order book is unaffected by buy trades. We refer to Alfonsi et al. (2012) and Gatheral et al. (2012) for different model settings which do require conditions on decay of price impact in order to avoid price manipulations.

Let $Y \in \mathcal{A}(y)$ and define the process $M^Y$ by

$$M^Y_t = \exp\left( -A \int_0^t Y_s^- dL_s - \int_0^t \kappa_A(Y_s^-) \, ds \right), \quad t \geq 0.$$ 

Then it follows from Theorem 3.2 in Kallsen and Shiryaev (2002) that $M^Y$ is a uniformly integrable martingale. We can therefore define a probability measure $\tilde{P} = P^Y$ by

$$\frac{d\tilde{P}}{dP} = M^Y_\infty.$$ 

Based on the idea of Theorem 2.8 in Schied et al. (2010), we calculate that

$$\sup_{Y \in \mathcal{A}(y)} \mathbb{E}\left[U(C_\infty(Y))\right] = -e^{-A(c+by)} \inf_{Y \in \mathcal{A}(y)} \mathbb{E}\left[ \exp\left( -A \int_0^\infty Y_t^- dL_t + AF_\infty(Y) \right) \right]$$

$$= -e^{-A(c+by)} \inf_{Y \in \mathcal{A}(y)} \mathbb{E}\left[ M_\infty \exp\left( \int_0^\infty \kappa_A(Y_t^-) \, dt + AF_\infty(Y) \right) \right]$$

$$= -e^{-A(c+by)} \inf_{Y \in \mathcal{A}(y)} \mathbb{E}\left[ \exp\left( \int_0^\infty \kappa_A(Y_t^-) \, dt + AF_\infty(Y) \right) \right]$$

$$\leq -e^{-A(c+by)} \inf_{Y \in \mathcal{A}(y)} \mathbb{E}\left[ \exp\left( \int_0^\infty \kappa_A(Y_t^-) \, dt + AF_\infty(Y) \right) \right]$$

$$= -e^{-A(c+by)} \exp\left( \inf_{Y \in \mathcal{A}(y)} \left\{ \int_0^\infty \kappa_A(Y_t^-) \, dt + AF_\infty(Y) \right\} \right), \quad (3.5)$$

which shows that in order to solve the optimal liquidation problem, it is sufficient to solve the problem in (3.5).

**Lemma 3.2.** Let $F$ be given by (3.7). Then for every $Y \in \mathcal{A}_D(y)$ and $z \in [\bar{z}, 0]$,

$$F_\infty(Y) = \int_z^0 \psi(s) \, ds + \int_0^\infty h(Z^-_t) \psi(Z^-_t) \, dt. \quad (3.6)$$

With reference to Lemma 3.2 and (3.5), the optimal liquidation problem amounts to
solving

\[ V(y, z) = \inf_{Y \in A_D(y)} \int_0^\infty \left( \kappa_A(Y_t^-) + Ah(Z_t^-)\psi(Z_t^-) \right) dt, \tag{3.7} \]

with \( y = Y_{0-} \) and \( z = Z_{0-}^Y \). Since \( h \) and \( \psi \) are both negative-valued and \( \kappa_A \geq 0 \), we have \( V \geq 0 \). Suppose \( y > \bar{y}_A \), which is the upper bound for which \( \kappa_A \) is finite (\( \bar{y}_A \) might be \( +\infty \)). In this case, the investor will make an immediate block sale so that \( Y_0 \leq \bar{y}_A \), otherwise \( Y \) doesn’t satisfy (2.3) and \( V(y, z) = \infty \). However, the investor cannot sell more than \( z - \bar{z} \) amount of shares, otherwise \( V(y, z) \) will be infinite as well. We therefore define the solvency region to be

\[ D = \{ (y, z) \in \mathbb{R}^+ \times [\bar{z}, 0] \mid z > y - \bar{y}_A + \bar{z} \} , \]

and for the remainder of the paper focus on this region. For technical reasons, we don’t consider \( z = y - \bar{y}_A + \bar{z} \), as the value function may explode also along this line.

4 Solution to the problem

Our next aim is to derive a solution to the problem in (3.7). The derivation will be based on applying a time-change, and the principle of dynamic programming. With reference to the results in Løkka (2014) and the general theory of optimal control (see e.g. Fleming and Soner, 2006), it is natural to think that there exists a decreasing càdlàg function \( \beta = \beta^* : \mathbb{R}^+ \rightarrow [\bar{z}, 0] \) which separates the \((y, z)\) domain into two different regions; a region where the large investor makes an immediate block sale and another where he waits. Let \( \beta_* \) denote the càdlàg version of \( \beta^* \), and set

\[
\mathcal{S}\beta = \{ (y, z) \in D \mid z \geq \beta_*(y) \}
\]

\[
\mathcal{W}\beta = \{ (y, z) \in \mathcal{D} \mid z \leq \beta^*(y) \} \cup \{ (y, z) \mid y = 0 \}
\]

\[
\mathcal{G}\beta = \mathcal{S}\beta \cap \mathcal{W}\beta.
\]

1Intuitively, when the volume spread is small but the stock position is large, it might be optimal to sell rapidly; on the other hand, if the volume spread is large but the stock position is small, then it might be optimal to wait for a while. This motivates us to make a guess of a decreasing free boundary in the \((y, z)\) domain.
$S^\beta$ represents the immediate sales region, $W^\beta$ is the waiting region, and $G^\beta$ is the continuous sales region. For $y > 0$, the Hamilton-Jacobi-Bellman equation corresponding to $V$ given by (3.7) takes the form

$$D_y v(y, z) + v_z(y, z) = 0, \quad \text{for } (y, z) \in S^\beta,$$

$$h(z)v_z(y, z) - \kappa_A(y) - Ah(z)\psi(z) \leq 0, \quad \text{for } (y, z) \in S^\beta \setminus G^\beta,$$

and

$$h(z)v_z(y, z) - \kappa_A(y) - Ah(z)\psi(z) = 0, \quad \text{for } (y, z) \in W^\beta,$$

$$D_y^{-} v(y, z) + v_z(y, z) \leq 0, \quad \text{for } (y, z) \in W^\beta \setminus G^\beta,$$

with associated boundary condition $v(0, z) = A \int_0^z \psi(u)\,du$ for all $z \in [\bar{z}, 0]$, where

$$D_y^{-} v(y, z) = \lim_{\epsilon \to 0^-} \frac{1}{\epsilon} \left( v(y + \epsilon, z) - v(y, z) \right).$$

The equations (4.1)–(4.4) can be motivated as follows. When the large investor is trying to optimise over deterministic strategies, he basically has two options. He can either sell a certain number $\Delta > 0$ of shares or wait. Given a state $(y, z)$, it may or may not be optimal to sell $\Delta$ amount of shares, thus

$$v(y, z) \leq v(y - \Delta, z - \Delta),$$

because the share position is decreased from $y$ to $y - \Delta$, due to $\Delta$ number of shares being sold, while at the same time the state of the bid order book changes from $z$ to $z - \Delta$. This inequality should hold for all $0 < \Delta \leq y$, therefore

$$\max_{0 < \Delta \leq y} \left\{ v(y, z) - v(y - \Delta, z - \Delta) \right\} \leq 0.$$  (4.5)

---

2 The value function turns out to be continuously differentiable in $z$, but is only continuous with a one-sided derivative in $y$ (see Proposition 4.5).
On the other hand, during a period of time $\Delta t > 0$, it may or may not be optimal to wait, hence

$$v(y, z) \leq v(y, Z_{\Delta t}) + \int_{0}^{\Delta t} \left( \kappa_A(y) + Ah(Z_{u-})\psi(Z_{u-}) \right) du$$

$$= v(y, z) + \int_{0}^{\Delta t} \left( \kappa_A(y) + Ah(Z_{u-})(Z_{u-}) - v_z(y, Z_{u-})h(Z_{u-}) \right) du,$$

where $dZ_u = -h(Z_u) du$, for $0 \leq u \leq \Delta t$. Multiplying the above inequality by $(\Delta t)^{-1}$ and sending $\Delta t$ to 0, we obtain

$$h(z)v_z(y, z) - \kappa_A(y) - Ah(z)\psi(z) \leq 0. \quad (4.6)$$

Since one of these strategies should be optimal, equality should hold in either (4.5) or (4.6). We therefore get

$$\max \left\{ \max_{0 < \Delta \leq y} \left\{ v(y, z) - v(y - \Delta, z - \Delta) \right\}, \ h(z)v_z(y, z) - \kappa_A(y) - Ah(z)\psi(z) \right\} = 0,$$

from which (4.1)–(4.4) follow.

We define the liquidation strategy $Y_\beta$ corresponding to an intervention boundary $\beta$ as the càdlàg function with the following properties:

(i) If $(y, z) \in \mathcal{S}_\beta$, then the investor initially makes a block trade of size $\Delta$ such that $(Y_0^\beta, Z_0^\beta) = (y - \Delta, z - \Delta) \in \mathcal{G}_\beta$, and set $t_w = 0$.

(ii) If $(y, z) \in \mathcal{W}_\beta$, then wait until the time $t_w = \inf\{ t \geq 0 \mid Z_t^\beta = \beta(y) \}$, where

$$Z_t^\beta = z - \int_{0}^{t} h(Z_u^\beta) du, \quad 0 \leq t \leq t_w.$$

(iii) For $t \geq t_w$, continuously sell shares in such a way that $(Y_t^\beta, Z_t^\beta) \in \mathcal{G}_\beta$, where

$$Z_t^\beta = Z_{t_w}^\beta - \int_{t_w}^{t} h(Z_u^\beta) du + Y_t^\beta - Y_{t_w}^\beta, \quad t \geq t_w.$$

(iv) Stop (take no further action) once $Y_t^\beta = 0$.

Figure 3.4 provides an illustration of such a strategy. We will later characterise an optimal
Note that $\beta$ exists, and is admissible and optimal. The following functions related to liquidation strategy $Y^{\beta}$ correspond to a boundary $\beta$. We will consider any intervention boundary $\beta : \mathbb{R}^+ \to [\bar{z}, 0]$ which is decreasing, càdlàg, and satisfies $\beta(y) < 0$, for all $y > 0$. We also require that $\lim_{y \to \infty} \beta(y) = \bar{z}$ and $\beta(0) = 0$. Now, given an intervention boundary $\beta$, one may ask whether the corresponding liquidation strategy $Y^{\beta}$ exists and is unique. In order to answer this, we need to introduce the following functions related to $\beta$, which will bring benefits to our analysis:

\[ \gamma_{\beta}(y) = \beta(y) - y, \quad \text{for } y \in \mathbb{R}^+; \tag{4.7} \]
\[ \rho_{\beta}(z) = z - \beta^{-1}(z), \quad \text{for } z \in [\bar{z}, 0]; \tag{4.8} \]

\[ \beta^{-1}(z) = \inf \{ y \in \mathbb{R}^+ \mid \beta(y) \leq z \}, \quad \text{for } z \in [\bar{z}, 0]; \tag{4.9} \]
\[ \gamma_{\beta}^{-1}(x) = \inf \{ y \in \mathbb{R}^+ \mid \gamma_{\beta}(y) \leq x \}, \quad \text{for } x \in \mathbb{R}^-; \tag{4.10} \]
\[ \rho_{\beta}^{-1}(x) = \inf \{ z \in [\bar{z}, 0] \mid \rho_{\beta}(z) \geq x \}, \quad \text{for } x \in \mathbb{R}^- \tag{4.11} \]

Note that $\beta$ and $\gamma_{\beta}$ are càglàd, $\beta^{-1}$ and $\rho_{\beta}$ are càdlàg, and $\gamma_{\beta}^{-1}$ as well as $\rho_{\beta}^{-1}$ are both

Figure 3.4: An illustration of the strategy $Y^{\beta}$ corresponding to a boundary $\beta$. The intervention boundary, and prove that the strategy corresponding to such an optimal boundary exists, and is admissible and optimal.

Let us examine in more details the strategy corresponding to a given intervention boundary function $\beta$. We will consider any intervention boundary $\beta : \mathbb{R}^+ \to [\bar{z}, 0]$ which is decreasing, càdlàg, and satisfies $\beta(y) < 0$, for all $y > 0$. We also require that $\lim_{y \to \infty} \beta(y) = \bar{z}$ and $\beta(0) = 0$. Now, given an intervention boundary $\beta$, one may ask whether the corresponding liquidation strategy $Y^{\beta}$ exists and is unique. In order to answer this, we need to introduce the following functions related to $\beta$, which will bring benefits to our analysis:

\[ \gamma_{\beta}(y) = \beta(y) - y, \quad \text{for } y \in \mathbb{R}^+; \tag{4.7} \]
\[ \rho_{\beta}(z) = z - \beta^{-1}(z), \quad \text{for } z \in [\bar{z}, 0]; \tag{4.8} \]

\[ \beta^{-1}(z) = \inf \{ y \in \mathbb{R}^+ \mid \beta(y) \leq z \}, \quad \text{for } z \in [\bar{z}, 0]; \tag{4.9} \]
\[ \gamma_{\beta}^{-1}(x) = \inf \{ y \in \mathbb{R}^+ \mid \gamma_{\beta}(y) \leq x \}, \quad \text{for } x \in \mathbb{R}^-; \tag{4.10} \]
\[ \rho_{\beta}^{-1}(x) = \inf \{ z \in [\bar{z}, 0] \mid \rho_{\beta}(z) \geq x \}, \quad \text{for } x \in \mathbb{R}^- \tag{4.11} \]

Note that $\beta$ and $\gamma_{\beta}$ are càglàd, $\beta^{-1}$ and $\rho_{\beta}$ are càdlàg, and $\gamma_{\beta}^{-1}$ as well as $\rho_{\beta}^{-1}$ are both
continuous. Moreover, $\beta$, $\beta^{-1}$ and $\gamma_\beta^{-1}$ are decreasing, $\gamma_\beta$ is strictly decreasing, $\rho_\beta$ is strictly increasing, and $\rho_\beta^{-1}$ is increasing. Furthermore, it follows directly from the definitions of $\beta^{-1}$, $\gamma_\beta$, $\gamma_\beta^{-1}$, $\rho_\beta$ and $\rho_\beta^{-1}$ that the following three identities hold:

$$
\rho_\beta^{-1}(x) = x + \gamma_\beta^{-1}(x), \quad \text{for all } x \in \mathbb{R}^-; \quad (4.12)
$$

$$
\gamma_\beta^{-1}(\rho_\beta(z)) = \beta^{-1}(z), \quad \text{for all } z \in [\bar{z}, 0]; \quad (4.13)
$$

$$
\rho_\beta^{-1}(\gamma_\beta(y)) = \beta(y), \quad \text{for all } y \in \mathbb{R}^+. \quad (4.14)
$$

Also, by the definitions of $\mathcal{G}_\beta$, $\beta$ and $\beta^{-1}$, we see that the set $\mathcal{G}_\beta$ is the union of the graphs of functions $\beta$ and $\beta^{-1}$, restricted in $\mathcal{D}$.

Observe that if $z > \beta(y)$, then the strategy $Y^\beta$ corresponding to the intervention boundary described by $\beta$ consists of an initial sale of $\Delta$ number of shares so that $(y - \Delta, z - \Delta)$ is in $\mathcal{G}_\beta$ (see Figure 3.4). Let $Y_{0-}^\beta = y$ and $Y_0^\beta = y - \Delta$. Suppose $(y - \Delta, z - \Delta)$ is on the graph of $\beta$. Then $(y - \Delta, z - \Delta) = (y - \Delta, \beta(y - \Delta))$ and this equality is equivalent to

$$
\gamma_\beta(Y_0^\beta) = \beta(Y_0^\beta) - Y_0^\beta = z - y,
$$

from which it follows that $Y_0^\beta = \gamma_\beta^{-1}(z - y)$ and $\Delta = y - \gamma_\beta^{-1}(z - y)$. Now suppose $(y - \Delta, z - \Delta)$ is on the graph of $\beta^{-1}$, and let $Z_0^\gamma = z$ and $Z_0^\gamma = z - \Delta$. Then $(y - \Delta, z - \Delta) = (\beta^{-1}(z - \Delta), z - \Delta)$, which is equivalent to

$$
\rho_\beta(Z_0^\beta) = Z_0^\beta - \beta^{-1}(Z_0^\gamma) = z - y,
$$

and it follows that $Z_0^\beta = \rho_\beta^{-1}(z - y)$ and $\Delta = z - \rho_\beta^{-1}(z - y)$. According (4.12), the number $\Delta$ of shares in both of the aforementioned two cases can be expressed by

$$
\Delta = y - \gamma_\beta^{-1}(z - y) = z - \rho_\beta^{-1}(z - y).
$$

On the other hand, if $z \leq \beta(y)$, then the strategy $Y^\beta$ consists of an initial waiting period until $(Y_t^\beta, Z_t^\gamma)$ is on the graph of $\beta$ (see Figure 3.4). As long as no action is taken, we have $Y_t^\beta = y$, and with reference to (2.9) and (2.10), we obtain $Z_t^\gamma = H^{-1}(H(z) - t)$. The first

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3It can be checked that for any $x \in \mathbb{R}^-$, $\gamma_\beta^{-1}(x)$ and $\rho_\beta^{-1}(x)$ give out the $y$-coordinate and the $z$-coordinate of the intersection of the line $z = y + x$ and $\mathcal{G}_\beta$, respectively.
time $t_w$ that the state process is on the graph of $\beta$ is therefore given by

$$t_w = H(z) - H(\beta(y)).$$  \hfill (4.15)

Once the state process $(Y^\beta, Z^Y)$ is in the set $G^\beta$, the strategy $Y^\beta$ consists of taking minimal actions such that the state process remains in $G^\beta$ (see Figure 3.4). Therefore, $(Y^\beta_t, Z^Y_t) = (Y^\beta_t, \beta(Y^\beta_t))$ whenever $\beta(Y^\beta_t +) = \beta(Y^\beta_t)$. With reference to (2.6), this implies that $Y^\beta_t$ should solve

$$d\beta(Y^\beta_t) = -h(\beta(Y^\beta_t)) \, dt + dY^\beta_t,$$

which is equivalent to

$$d\gamma(\beta(Y^\beta_t)) = -h(\beta(Y^\beta_t)) \, dt.$$

If $\beta^{-1}(Z^Y_t) = \beta^{-1}(Z^Y_{t^-})$, then $(Y^\beta_t, Z^Y_t) = (\beta^{-1}(Z^Y_t), Z^Y_t)$. According to (2.6) and the definition of $\beta^{-1}$, $Z^Y_t$ should solve

$$dZ^Y_t = -h(Z^Y_t) \, dt.$$

Set

$$t_w = \begin{cases} 0, & \text{if } z > \beta(y), \\ H(z) - H(\beta(y)), & \text{if } z \leq \beta(y), \end{cases}$$

and

$$\bar{t} = \inf\{t \geq 0 \mid Y^\beta_t = 0\}. \hfill (4.16)$$

Denote by $\{y_n\}_{n \in \mathbb{I}}$ the set of discontinuity points of $\beta$. Then $\mathbb{I}$ is countable since $\beta$ is càglâd. Define $\{t_n\}_{n \in \mathbb{I}}$ by

$$t_n = \inf\{t \geq t_w \mid Y^\beta_t = y_n\}. \hfill (4.18)$$
and \( \{s_n\}_{n \in \mathbb{I}} \) by

\[
s_n = \inf \{ t \geq t_w \mid Y^\beta_t < y_n \}. \tag{4.19}
\]

If \( \{ t \geq t_w \mid Y^\beta_t = y_n \} = \emptyset \), set \( t_n = \infty \); and set \( s_n = \infty \), if \( \{ t \geq t_w \mid Y^\beta_t < y_n \} = \emptyset \). The following result establish existence and uniqueness of such a strategy \( Y^\beta \) corresponding to a given intervention boundary \( \beta \).

**Lemma 4.1.** Let \( (y, z) \in \mathcal{D} \) and \( \beta \) be an intervention boundary function. Suppose \( h \) is a resilience function satisfying Assumption 2.4 and \( H, \beta^{-1}, \gamma_\beta, \gamma_\beta^{-1}, t_w, \bar{t}, y_n, t_n \) and \( s_n \) are given by (2.8), (4.9), (4.7), (4.10) and (4.16)–(4.19) respectively. Let \( (Y^\beta_t)_{t \geq 0} = (Y^\beta_{t \wedge \bar{t}})_{t \geq 0} \), with \( Y^\beta_0 = y \), which denotes the decreasing càdlàg liquidation strategy corresponding to \( \beta \), and let \( (Z^\beta_Y)_{t \geq 0} \), with \( Z^\beta_0 = z \), be the state process of the bid order book associated with \( Y^\beta \). Suppose \( Y^\beta \) satisfies the following description:

(i) If \( y = 0 \), then liquidation is completed immediately; otherwise,

(ii) If \( z > \beta(y) \),

   (a) when \( y \in \bigcup_{n \in \mathbb{I}} (z - \beta(y_n) + y_n, z - \beta(y_n) + y_n] \), immediately sell \( y - \gamma_\beta^{-1}(z - y) \) number of shares. This block trade ensures \( Y^\beta_0 = \beta^{-1}(Z^\beta_0) \).

   (b) when \( y \in (z, \infty) \setminus \bigcup_{n \in \mathbb{I}} (z - \beta(y_n) + y_n, z - \beta(y_n) + y_n] \), immediately sell \( y - \gamma_\beta^{-1}(z - y) \) number of shares. This block trade ensures \( Z^\beta_0 = \beta(Y^\beta_0) \).

Then continuously sell shares so that \( (Y^\beta_t, Z^\beta_Y)_t \in \mathcal{G}^{\beta} \) for all \( t \in [t_w, \bar{t}] \).

(iii) If \( z \leq \beta(y) \), then wait until time \( t_w \). The time \( t_w \) has the property that \( Z^\beta_{t_w} = \beta(y) \). Continuously sell shares so that \( (Y^\beta_t, Z^\beta_Y)_t \in \mathcal{G}^{\beta} \) for all \( t \in [t_w, \bar{t}] \).

Such a strategy \( Y^\beta \) exists and is unique, and it is continuous for all \( t > 0 \). In particular,

\[
Y^\beta_t = y_n \quad \text{for} \quad t \in [t_w, \bar{t}] \cap \bigcup_{n \in \mathbb{I}} \{ t_n, s_n \}, \tag{4.20}
\]

with corresponding \( Z^\beta_Y_t \) being the unique solution to

\[
dZ^\beta_Y_t = -h(Z^\beta_Y_t) \, dt, \tag{4.21}
\]

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where

\[
Z_{tw}^\beta = \rho_\beta^{-1}(z - y) \text{ if } z > \beta(y), \quad \text{and} \quad Z_{tn}^\beta = \beta(Y_{tn}^\beta) \text{ for } t_n > t_w. \tag{4.22}
\]

Moreover,

\[
Z_t^\beta = \beta(Y_t^\beta), \quad \text{for } t \in [t_w, \bar{t}] \setminus \cup_{n \in \mathbb{I}}[t_n, s_n], \tag{4.23}
\]

where \(Y^\beta\) is the unique solution to

\[
d\gamma^\beta(Y_t^\beta) = -h(\beta(Y_t^\beta)) \, dt, \tag{4.24}
\]

with

\[
Y_{tw}^\beta = y \text{ if } z \leq \beta(y), \quad Y_{tw}^\beta = \gamma_\beta^{-1}(z - y) \text{ if } z > \beta(y), \quad \text{and} \quad Y_{sn}^\beta = y_n \text{ for } s_n > t_w. \tag{4.25}
\]

If \(t_w > 0\), then \(Y_{tw}^\beta = y\) and \(Z_{t}^\beta = H^{-1}(H(z) - t)\), for \(0 \leq t \leq t_w\).

We can also describe \(Z_{t}^\beta\) for \(t \in [\bar{t}, \infty)\) that it satisfies (4.21) with initial condition

\[
Z_{t}^\beta = \begin{cases} 
Z_{tw}^\beta, & \text{if } \bar{t} = t_w, \\
z, & \text{if } \bar{t} < t_w, \\
\beta(0+), & \text{if } \bar{t} > t_w. 
\end{cases} \tag{4.26}
\]

The value \(\beta(0+)\) can then be used to determine whether the liquidation period is finite or not. More specifically, we have that \(\beta(0+) < 0\) implies \(\bar{t} < \infty\). To see this, it is enough to consider

\[
\gamma_\beta(Y_{t}^\beta) - \gamma_\beta(Y_{\bar{t}}^\beta) = \int_{t}^{\bar{t}} -h(\beta(Y_u^\beta)) \, du
\]

which follows from (4.24) when there is no waiting period between the times \(t\) and \(\bar{t}\). To get a contradiction, suppose \(\bar{t} = \infty\). Then it is clear that \(\int_t^{\bar{t}} -h(\beta(Y_u^\beta)) \, du = \infty\), as \(\beta(Y_u^\beta)\) is bounded away from 0 on the interval \((t, \bar{t})\). However, \(\gamma_\beta(Y_{t}^\beta) - \gamma_\beta(Y_{\bar{t}}^\beta)\) is finite, so we get a contradiction.

It follows from the dynamics of \(Z_{t}^\beta\) that \(Z^\beta\) is càdlàg and increasing to 0. Moreover, the continuity of \(Y_{t}^\beta\) for \(t > 0\) implies that \(Z^\beta\) is also continuous for all \(t > 0\).
We now progress by deriving an explicit expression for the performance function associated with the strategy \( Y^\beta \) described by Lemma 4.1 for an arbitrary intervention boundary \( \beta \). This expression can then later be used to derive an explicit expression for the value function of our problem. For the strategy \( Y^\beta \) with associated state process \( Z^{Y^\beta} \), given an initial state \((y, z)\), and with reference to (3.7), we define the performance function \( J_\beta \) by

\[
J_\beta(y, z) = \int_0^\infty \left( \kappa_A(Y^\beta_t) + Ah(Z^{Y^\beta}_t)\psi(Z^{Y^\beta}_t) \right) dt, \tag{4.27}
\]

where \( Y^\beta_0 = y \), \( Z^{Y^\beta}_0 = z \) and \((y, z) \in \mathcal{D}\). Since \( \kappa_A(0) = 0 \), it follows that

\[
\int_{t_w}^{\infty} \left( \kappa_A(Y^\beta_t) + Ah(Z^{Y^\beta}_t)\psi(Z^{Y^\beta}_t) \right) dt = A \int_{\beta(0+)}^{Z^{Y^\beta}_{t_w}} \psi(u) du. \tag{4.28}
\]

Therefore, in cases (i) of Lemma 4.1,

\[
J_\beta(y, z) = A \int_0^z \psi(u) du. \tag{4.29}
\]

**Lemma 4.2.** Let \( \beta, Y^\beta, Z^{Y^\beta}, t_w \) and \( \bar{t} \) be defined as the same as in Lemma 4.1. If \( t_w < \bar{t} \), then

\[
\int_{t_w}^{\infty} \left( \kappa_A(Y^\beta_t) + Ah(Z^{Y^\beta}_t)\psi(Z^{Y^\beta}_t) \right) dt = \int_{\beta(0+)}^{Z^{Y^\beta}_{t_w}-Y^\beta_{t_w}} \left( \frac{\kappa_A(\gamma^{-1}_\beta(u))}{h(\rho^{-1}_\beta(u))} + A\psi(\rho^{-1}_\beta(u)) \right) du + A \int_{\beta(0+)}^{\rho^{-1}_\beta(z-w)} \psi(u) du,
\]

where \( \gamma^{-1}_\beta \) and \( \rho^{-1}_\beta \) are defined by (4.10) and (4.11), respectively.

In case (ii) (a) of Lemma 4.1 the strategy \( Y^\beta \) consists of an initial sale of \( y - \gamma^{-1}_\beta(z-y) = z - \rho^{-1}_\beta(z-y) \) number of shares. The state after the block sale is \((Y^\beta_0, Z^{Y^\beta}_0) = (\beta^{-1}(\rho^{-1}_\beta(z-y)),\left(\beta^{-1}(\rho^{-1}_\beta(z-y),z)\right)\).
\[ J_\beta(y, z) = J_\beta(\beta^{-1}(y), \beta^{-1}(z)) \]
\[ = \int_{\beta(0+)}^{\beta^{-1}(z-y)} \left( \frac{\kappa_A(\gamma^{-1}(u))}{h(\rho^{-1}(u))} + A\psi(\rho^{-1}(u)) \right) du + A \int_0^{\beta(0+)} \psi(u) du \]
\[ = \int_{\beta(0+)}^{\beta^{-1}(z-y)} \left( \frac{\kappa_A(\gamma^{-1}(u))}{h(\rho^{-1}(u))} + A\psi(\rho^{-1}(u)) \right) du + A \int_0^{\beta(0+)} \psi(u) du. \]

In case (ii) (b), we immediately sell \( y - \gamma^{-1}_\beta(z - y) \) number of shares at the beginning. The state after the block sale is \( (Y_0^\beta, Z_0^\gamma) = (\gamma^{-1}(z - y), \beta(\gamma^{-1}(z - y))) \). Hence, similar to the above calculation, we have

\[ J_\beta(y, z) = J_\beta(\gamma^{-1}_\beta(z - y), \beta(\gamma^{-1}_\beta(z - y))) \]
\[ = \int_{\beta(0+)}^{\gamma^{-1}_\beta(z - y)} \left( \frac{\kappa_A(\gamma^{-1}(u))}{h(\rho^{-1}(u))} + A\psi(\rho^{-1}(u)) \right) du + A \int_0^{\beta(0+)} \psi(u) du. \]

Therefore, we conclude that in case (ii) of Lemma 4.1

\[ J_\beta(y, z) = \int_{\beta(0+)}^{t_w} \left( \frac{\kappa_A(\gamma^{-1}(u))}{h(\rho^{-1}(u))} + A\psi(\rho^{-1}(u)) \right) du + A \int_0^{\beta(0+)} \psi(u) du. \] (4.30)

Moreover, in case (iii), \( z \leq \beta(y) \). So we need to wait until time \( t_w > 0 \) at which \( Z_{t_w}^\gamma = \beta(y) \). With reference to (2.8) and (4.15), we have

\[ t_w = H(z) - H(\beta(y)) = \int_{\beta(y)}^{z} \frac{1}{h(u)} du. \]

Also, observe that

\[ \int_0^{t_w} h(Z_t^\gamma) \psi(Z_t^\gamma) dt = - \int_0^{t_w} \psi(Z_t^\gamma) dZ_t^\gamma = - \int_z^{\beta(y)} \psi(u) du. \]
Hence in case (iii), the performance function is given by

\[
J_\beta(y, z) = \int_0^{t_w} \left( \kappa_A(y) + Ah(Z_t^{1,\beta}) \psi(Z_t^{1,\beta}) \right) dt + J_\beta(y, \beta(y)) \\
= \kappa_A(y) \int_0^z \frac{1}{h(u)} du - A \int_0^{\beta(y)} \psi(u) du \\
+ \int_0^{\gamma_\beta(y)} \left( \frac{\kappa_A(\gamma_\beta^{-1}(u))}{h(\rho_\beta^{-1}(u))} + A\psi(\rho_\beta^{-1}(u)) \right) du + A \int_0^{\beta(0+)} \psi(u) du. 
\]

(4.31)

Although this provides an explicit expression for \( J_\beta(y, z) \), it is not entirely straightforward to conclude about properties of continuity and differentiability for \( J_\beta(y, z) \) in \( y \) since \( \beta \) is only a càglàd function. However, we can calculate further that

\[
\int_0^{\gamma_\beta(y)} \left( \frac{\kappa_A(\gamma_\beta^{-1}(u))}{h(\rho_\beta^{-1}(u))} + A\psi(\rho_\beta^{-1}(u)) \right) du = \int_0^{\beta(0+)} \left( \kappa_A(u) + A\psi(u) \right) d\gamma_\beta(u) \\
+ \sum_{0 < u < y} \kappa_A(u) \int_0^{\beta(u+)} \frac{1}{h(x)} dx \\
+ A \sum_{0 < u < y} \int_\beta(u) \psi(s) ds,
\]

From this expression, as well as

\[
\int_0^{\beta(0+)} \psi(u) du = \int_0^{\beta(u+)} \psi(u) du + \sum_{0 < u < y} \int_\beta(u) \psi(s) ds,
\]

and

\[
\kappa_A(y)H(\beta(y)) = \kappa_A(0)H(\beta(0+)) + \int_0^{\beta(u+)} \kappa_A(u)H(\beta(u)) du + \int_0^{\beta(0+)} \frac{\kappa_A(u)}{h(\beta(u))} d\beta^c(u) \\
+ \sum_{0 < u < y} \kappa_A(u) \int_\beta(u) \frac{1}{h(x)} dx,
\]

it follows from (4.31) that the performance function \( J_\beta(y, z) \) in case (iii) of Lemma 4.1 admits
the expression

\[ J_\beta(y, z) = \kappa_A(y)H(z) + A \int_0^z \psi(u) \, du - \int_y^z \left( \frac{\kappa_A(u)}{h(\beta(u))} + A\psi(\beta(u)) + \kappa'_A(u)H(\beta(u)) \right) \, du. \]

(4.32)

In the above calculations, we have assumed the existence and finiteness of \( \lim_{u \to 0^+} \frac{\kappa_A(u)}{h(\beta(u))} \) and \( \lim_{u \to 0^+} \kappa'_A(u)H(\beta(u)) \). We have also used that \( \lim_{u \to y^-} \kappa_A(u) < \infty \) as well as \( \lim_{u \to y^-} \kappa'_A(u) < \infty \). The finiteness of \( \lim_{u \to 0^+} \kappa'_A(u)H(\beta(u)) \) together with (2.22) implies that \( \kappa_A(0)H(\beta(0^+)) = 0 \). For an optimal intervention boundary \( \beta \), all of these properties will be demonstrated below by Lemma 4.4.

Suppose \( \beta \) is an intervention boundary such that \( Y_\beta \) is optimal. Then according to the Hamilton-Jacobi-Bellman equation as well as (4.32), we have

\[ D_y^- v(y, z) + v_z(y, z) = \Gamma(z; y) - \Gamma(\beta(y); y) \leq 0, \quad \text{for all } (y, z) \in D, \]

where

\[ \Gamma(x; y) = A\psi(x) + \frac{\kappa_A(y)}{h(x)} + \kappa'_A(y)H(x). \]

Therefore, for any given \( y \), \( \beta(y) \) is a maximiser of \( \Gamma(x; y) \). The next lemma helps us characterise an intervention boundary \( \beta \) whose value maximises \( \Gamma(x; y) \) for any given \( y \), and it will be shown later that such a \( \beta \) is an optimal intervention boundary for our problem.

**Lemma 4.3.** For \( y \in (0, \bar{y}_A) \), define the function \( \Gamma(\cdot; y) : [\bar{z}, 0] \to \mathbb{R} \) by

\[ \Gamma(x; y) = A\psi(x) + \frac{\kappa_A(y)}{h(x)} + \kappa'_A(y)H(x), \quad \text{for } x \in (\bar{z}, 0), \]

(4.33)

and

\[ \Gamma(0; y) = \lim_{x \to 0^+} \Gamma(x; y), \quad \Gamma(\bar{z}; y) = \lim_{x \to \bar{z}} \Gamma(x; y). \]

Let \( \beta^* = \beta^*(y) \) and \( \beta_* = \beta_*(y) \) denote the functions defined as the largest and smallest
Let $\beta \in [\bar{z}, 0]$ satisfying

$$\max_{x \in [\bar{z}, 0]} \Gamma(x; y) = \Gamma(\beta; y),$$

respectively. Then for all $y \in (0, \bar{y}_A)$, we have $\bar{z} \leq \beta_*(y) \leq \beta^*(y) < 0$. Furthermore, if $\bar{y}_A < \infty$, write $\beta^*(y) = \beta_*(y) = \bar{z}$, for all $y > \bar{y}_A$. Set

$$\beta^*(0) = 0, \quad \beta_*(0) = \lim_{y \to 0^+} \beta_*(y),$$

and

$$\beta^*(\bar{y}_A) = \lim_{y \to \bar{y}_A^-} \beta^*(y), \quad \beta_*(\bar{y}_A) = \lim_{y \to \bar{y}_A^+} \beta_*(y).$$

This defines two unique decreasing functions $\beta^*, \beta_*$: $\mathbb{R}^+ \to [\bar{z}, 0]$ which are càglàd and càdlàg, respectively, and they are left and right-continuous versions of each other.

**Lemma 4.4.** Let $\beta^*$ be given by Lemma 4.3, it follows that if $\lim_{x \to y^-} \kappa_A(x) = \infty$ or $\lim_{x \to y^-} \kappa'_A(x) = \infty$, then $\lim_{x \to y^-} \beta^*(x) = \bar{z}$. Furthermore, we have

$$\lim_{y \to 0^+} \frac{\kappa_A(y)}{h(\beta^*(y))} = 0 \quad \text{and} \quad \lim_{y \to 0^+} \kappa'_A(y) H(\beta^*(y)) = 0. \quad (4.35)$$

Clearly, the function $\beta^*$ given in Lemma 4.3 satisfies the properties we require of an intervention boundary. With this intervention boundary, the proposition below provides an explicit expression for the value function that solves (4.1)–(4.4) with associated boundary condition $v(0, z) = A \int_{\bar{z}}^z \psi(u) \, du$, for all $z \in [\bar{z}, 0]$. As a consequence, the optimal liquidation strategy is characterised by this intervention boundary. Before proceeding, we make a few comments on the optimal intervention boundary and the associated optimal liquidation strategy. The property that the intervention boundary is non-increasing means that when the investor makes continuous sales, it is never optimal to implement a trading speed which decrease the current best bid price. In other words, the sell speed should be at most as large as the current speed of resilience. Therefore, the possible constant parts in the intervention boundary represent the situation that the current market risk is too large so that it is optimal to sell as quick as possible in order to reduce the stock position and hence the market risk. Moreover, discontinuity.
nuities in the intervention boundary correspond to an optimal liquidation strategy where it is optimal to wait and do no sales for a period of time. This can be interpreted as the current illiquidity cost\(^5\) is relatively large compared to the market risk, thus it is optimal to wait so that the best bid price increases to a level which is more preferable to the investor.

**Proposition 4.5.** Let \(\beta = \beta^*\) denote the largest solution to (4.34), and let \(\gamma^{-1}_\beta\) and \(\rho^{-1}_\beta\) be the corresponding functions defined by (4.10) and (4.11). Then the function \(v : \mathcal{D} \to \mathbb{R}\) given by that for \(z > \beta(y)\),

\[
v(y, z) = \int_{\beta(0+)}^{z-y} \left( \frac{\kappa_A(\gamma^{-1}_\beta(u))}{h(\rho^{-1}_\beta(u))} + A\psi(\rho^{-1}_\beta(u)) \right) du + A \int_{0}^{\beta(0+)} \psi(u) du, \tag{4.36}
\]

and for \(z \leq \beta(y)\),

\[
v(y, z) = \kappa_A(y)H(z) + A \int_{0}^{z} \psi(u) du - \int_{0}^{y} \left( \frac{\kappa_A(u)}{h(\beta(u))} + A\psi(\beta(u)) + \kappa'_A(u)H(\beta(u)) \right) du, \tag{4.37}
\]

is a \(C^{0,1}(\mathcal{D})\) solution to (4.1)-(4.4) with the boundary condition \(v(0, z) = A \int_{0}^{z} \psi(u) du\), for all \(z \in [\bar{z}, 0]\). Moreover, \(D^-v(y, z)\) is càglàd in \(y\) and continuous in \(z\).

Note that (4.36)-(4.37) agree with (4.29) when \(y = 0\). The following theorem verifies that the function \(v\) given by (4.36)-(4.37) is equal to the value function \(V\) given by (3.7), and that the strategy \(Y^\beta\) corresponding to \(\beta\) characterised by (4.34) is an optimal liquidation strategy. Hence, such a \(Y^\beta\) provides a solution to the utility maximization problem in (2.12).

**Theorem 4.6.** Denote the investor’s risk aversion by \(A\), the initial unaffected price by \(b\), and by \(c\) the initial cash position. We take \(\beta\) as the largest solution to (4.34) and \(v\) to be given by (4.36) and (4.37). Moreover, let \(V\) be given by (3.7). Then \(v = V\) on \(\mathcal{D}\) and

\[
\sup_{Y \in \mathcal{A}(y)} \mathbb{E}[U(C_\infty(Y))] = -\exp\left(-A(c + by) + A \int_{z}^{z-y} \psi(s) ds \right) \exp(v(y, z)),
\]

where \(z = Z_0^-\) is the initial state of the bid order book and \(y\) is the initial share position. The optimal strategy \(Y^*\) is equal to \(Y^\beta \in \mathcal{A}_D(y)\), where \(Y^\beta\) is the strategy described in Lemma 4.1 corresponding to \(\beta\) with \(Y^\beta_0 = y\).

\(\footnote{The illiquidity cost, or the price impact cost is described by \(F_\infty(Y)\) in (3.2), and it corresponds to the term \(\int_{0}^{\infty} h(Z_t^y) \psi(Z_t^y) dt\) in the simplified problem (3.2).} \)
5 Proofs

Proof of Lemma 3.2. With reference to the dynamic of $Z^Y$, we calculate that for $z \geq \bar{z}$,

\[
\int_0^Z \psi(u) \, du = \int_0^z \psi(u) \, du + \int_T^T \psi(Z_{t^-}^Y) \, dY_t^c
- \int_T^T h(Z_{t^-}^Y) \psi(Z_{t^-}^Y) \, dt + \sum_{0 \leq t \leq T} \int_{Z_{t^-}^Y + \Delta Y_t} \psi(u) \, du
\]

\[
= \int_0^z \psi(u) \, du + \int_T^T \psi(Z_{t^-}^Y) \, dY_t^c
- \int_T^T h(Z_{t^-}^Y) \psi(Z_{t^-}^Y) \, dt + \sum_{0 \leq t \leq T} \int_{Z_{t^-}^Y + \Delta Y_t} \psi(Z_{t^-}^Y + u) \, du.
\]

Then,

\[
F_T(Y) = \int_0^T \psi(Z_{t^-}^Y) \, dY_t^c + \sum_{0 \leq t \leq T} \int_{Z_{t^-}^Y} \psi(Z_{t^-}^Y + x) \, dx
\]

\[
= \int_0^Z \psi(u) \, du - \int_0^z \psi(u) \, du + \int_T^T h(Z_{t^-}^Y) \psi(Z_{t^-}^Y) \, dt
\]

\[
= \int_0^Z \psi(u) \, du + \int_0^T h(Z_{t^-}^Y) \psi(Z_{t^-}^Y) \, dt.
\]

Notice that for any admissible liquidation strategy $Y$, we have that either $Y$ and $Z^Y$ become 0 at the same time, or $Y$ becomes 0 at some time $s$ while $Z^Y_s < 0$. In the second case, for all $t > s$, $Z^Y$ satisfies

\[
dZ_t^Y = -h(Z_t^Y) \, dt.
\]

According to (2.9), we know that the solution to the above dynamic tends to 0, as $t \to \infty$. Therefore, $Z_t^Y \to 0$, as $t \to \infty$ in any case. Then it follows from the above expression of $F_T(Y)$ that

\[
F_\infty(Y) = \int_0^0 \psi(u) \, du + \int_0^\infty h(Z_{t^-}^Y) \psi(Z_{t^-}^Y) \, dt.
\]

\[
\square
\]

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**Proof of Lemma 4.1** We first prove that on any time interval $I$ contained in $[t_w, \bar{t}] \setminus \cup_{n \in \mathbb{I}} [t_n, s_n)$, there exists a unique solution to the dynamic (4.24). On such an interval $I$, the process $Y^\beta$ does not cross any jump of $\beta$. Thus, in terms of the function $\beta$, we shall only focus on those parts without jumps. Also, it is sufficient to consider $Y$ starting from time 0 (rather than starting at any time in $[t_w, \bar{t}] \setminus \cup_{n \in \mathbb{I}} [t_n, s_n)$). Write $Y^0_t = Y_0 > 0$ and

$$Y^{k+1}_t = \gamma_\beta^{-1}\left(\left\{\gamma_\beta(Y_0) - \int_0^t h(\beta(Y^k_u)) \, du\right\} \wedge \beta(0+)\right). \quad (5.1)$$

Let $T \in [0, \infty)$. Then

$$\sup_{0 \leq t \leq T} \left| \beta(Y^{k+1}_t) - \beta(Y^k_t) \right|$$

$$= \sup_{0 \leq t \leq T} \left| \left\{\gamma_\beta(Y_0) - \int_0^t h(\beta(Y^k_u)) \, du\right\} \wedge \beta(0+) - \left\{\gamma_\beta(Y_0) - \int_0^t h(\beta(Y^{k-1}_u)) \, du\right\} \wedge \beta(0+) + \gamma_\beta^{-1}\left(\left\{\gamma_\beta(Y_0) - \int_0^t h(\beta(Y^k_u)) \, du\right\} \wedge \beta(0+)\right) - \gamma_\beta^{-1}\left(\left\{\gamma_\beta(Y_0) - \int_0^t h(\beta(Y^{k-1}_u)) \, du\right\} \wedge \beta(0+)\right)\right|$$

$$\leq 2 \sup_{0 \leq t \leq T} \left| \left\{\gamma_\beta(Y_0) - \int_0^t h(\beta(Y^k_u)) \, du\right\} \wedge \beta(0+) - \left\{\gamma_\beta(Y_0) - \int_0^t h(\beta(Y^{k-1}_u)) \, du\right\} \wedge \beta(0+)\right|$$

$$\leq 2 \sup_{0 \leq t \leq T} \left| \int_0^t h(\beta(Y^k_u)) - h(\beta(Y^{k-1}_u)) \, du\right|$$

$$\leq 2L \int_0^T \left| \beta(Y^k_u) - \beta(Y^{k-1}_u) \right| \, du$$

$$\leq 2L \int_0^T \sup_{0 \leq u \leq t} \left| \beta(Y^k_u) - \beta(Y^{k-1}_u) \right| \, du,$$  \quad (5.2)

where the first equality holds because when $\beta$ has no jumps we have $\beta(\gamma_\beta^{-1}(x)) = x + \gamma_\beta^{-1}(x)$, the first inequality is due the triangle inequality and $|\gamma_\beta^{-1}(x) - \gamma_\beta^{-1}(y)| \leq |x - y|$, and the third inequality follows from the boundedness of the processes $\beta(Y^k)$ and $\beta(Y^{k-1})$ and the local Lipschitz continuity of $h$ with a Lipschitz constant $L$. By induction and with reference to (5.2), it can be shown that

$$\sup_{0 \leq t \leq T} \left| \beta(Y^{k+1}_t) - \beta(Y^k_t) \right| \leq \frac{(2LT)^k}{k!}2|\beta(Y_0)|.$$
Taking $k$ to infinity, we have that $\beta(Y^k_t)$ converges uniformly on $[0, T]$. Define $\beta_t = \lim_{k \to \infty} \beta(Y^k_t)$, for $t \in [0, T]$. Since $T \in [0, \infty)$ is arbitrary, it follows that $\beta_t = \lim_{k \to \infty} \beta(Y^k_t)$ for all $t \in [0, \infty)$. With reference to (5.1) and the dominated convergence theorem it follows that, for every $t \in [0, \infty)$, $(Y^k_t)_{t=0}^{\infty}$ is convergent. We define $Y^\beta_t = \lim_{k \to \infty} Y^k_t$. It can be checked that $Y^\beta$ decreases to 0. Then since $\beta$ is continuous, we obtain $\beta_t = \beta(Y^\beta_t)$, for all $t \in [0, \infty)$. Therefore, by sending $k$ to infinity in (5.1), since we only consider $Y^\beta_t$ before time $\bar{t}$, we have that

$$
Y^\beta_t = \gamma^{-1}_\beta \left( \gamma_\beta(Y^\beta_0) - \int_0^t h(\beta(Y^\beta_u)) \, du \right), \quad \text{for } t \leq \bar{t}.
$$

This proves the existence of a solution to the dynamic (4.24) on any time interval contained in $[t_w, \bar{t}] \setminus \cup_{n \in \mathbb{N}} [t_n, s_n)$. For uniqueness, let’s assume that $Y^{(1)}$ and $Y^{(2)}$ satisfy (4.24), where $Y_t^{(1)} = Y_t^{(2)}$ for $0 \leq t \leq t_1$, and $Y_t^{(1)} < Y_t^{(2)}$ for $t_1 < t < t_2$. Then for $t_1 < t < t_2$,

$$
Y_t^{(1)} = \gamma^{-1}_\beta \left( \gamma_\beta(Y_0^{(1)}) - \int_0^t h(\beta(Y_u^{(1)})) \, du \right) \\
\geq \gamma^{-1}_\beta \left( \gamma_\beta(Y_0^{(2)}) - \int_0^t h(\beta(Y_u^{(2)})) \, du \right) \\
= Y_t^{(2)},
$$

which contradicts the assumption that $Y_t^{(1)} < Y_t^{(2)}$ for $t_1 < t < t_2$. So the uniqueness holds. The existence and uniqueness of solution to the dynamic in (4.24) on any time interval contained in $[t_w, \bar{t}] \cap \bigcup_{n \in \mathbb{N}} [t_n, s_n)$ follow from the locally Lipschitz continuity of function $h$.

Now let $Y^\beta$ and $Z^Y^\beta$ be processes satisfying (4.20)–(4.26) with $(Y_0^\beta, Z_0^Y^\beta) = (y, z) \in \mathcal{D}$. Note that $(Y_t^\beta, Z_t^Y^\beta) \in \mathcal{G}$ for all $t \in [t_w, \bar{t})$. We need to show (2.6) is satisfied. We first focus on the case when $t \leq t_w$. Suppose $z > \beta(y)$, i.e. $t_w = 0$. Then in case (ii) (a),

$$
Y_0^\beta - Y_z^\beta = \beta^{-1}(\rho_0^{-1}(z - y)) - y \\
= \gamma^{-1}_\beta(z - y) - y \\
= (z - y + \gamma^{-1}_\beta(z - y)) - z \\
= Z_0^Y^\beta - Z_0^Y^\beta,
$$

where we have used the identity $\beta^{-1}(\rho_0^{-1}(z - y)) = \gamma^{-1}_\beta(z - y)$ which follows from (4.13) and
is valid under the condition of (ii) (a). In case (ii) (b), we obtain

\[ Z_Y^Y - Z_0^Y = \beta(\gamma_1^{-1}(z - y)) - z \]
\[ = z - y + \gamma_1^{-1}(z - y) - z \]
\[ = \gamma_1^{-1}(z - y) - y \]
\[ = Y^Y_0 - Y_{0-}^Y, \]

where \( \beta(\gamma_1^{-1}(z - y)) = \rho_1^{-1}(z - y) \) was used. Suppose \( z \leq \beta(y) \), i.e. \( t_w > 0 \). It can be checked that \( Z_t^Y = H^{-1}(H(z) - t) \) has dynamic (4.21). Because \( Y_t^Y \) is now constant, (2.6) is satisfied.

In the case when \( t > t_w \), \( Y_t^Y \) and \( Z_t^Y \) follow (4.20)–(4.26), which satisfy (2.6).

We next prove \( Y^Y_\bar{t} \) is càdlàg and decreasing. Note that by the definitions of \( t_n, s_n, t_w \) and \( t \) and (4.21), (4.24) and the first part of the proof, we have \( Y_t^Y \) and \( Z_t^Y \) are continuous when \( (Y_t^Y, Z_t^Y) \) is in each continuous part of the graph of \( \beta \) or \( \beta \), for \( t > 0 \). Also, each initial condition associated with the dynamics (4.21) and (4.24) is chosen to make \( Y_t^Y \) and \( Z_t^Y \) to be continuous at \( t_n, s_n \) and \( t_w \) when \( t_w > 0 \). It can also be seen that \( Y^Y_\bar{t} \) and \( Z^Y_\bar{t} \) are right continuous at \( t = 0 \). These together with the well-defined \( Y_{0-}^Y \) and \( Z_{0-}^Y \) imply that \( Y^Y \) and \( Z^Y \) are continuous for \( t > 0 \) and they are right-continuous with left-limit at \( t = 0 \). That \( Y^Y \) decreases to 0 follows from (4.20), (4.21), (4.24), and the first part of this proof. Finally, \( Z_t^Y = H^{-1}(H(z) - t) \), for \( 0 \leq t \leq t_w \) follows from (2.2).

Proof of Lemma 4.2. Let \( \{y_n\}_{n \in I} \) be the set of all points at which the intervention boundary \( \beta \) is discontinuous. Consider a time interval \([t, s] \subseteq [t_n, s_n]\) for some \( n \in I \), where \( t_n \) and \( s_n \) are given by (4.18) and (4.19). With reference to (2.6), we note that formally,

\[ dt = -\frac{d \rho_\beta(Z_t^Y)}{h(Z_t^Y)} \quad \forall t \in [t_n, s_n], \]
and hence,

$$
\int_t^s \left( \kappa_A(Y_t^\beta) + Ah(Z_r^Y) \psi(Z_r^Y) \right) \, dr
$$

$$
= \int_s^t \left( \kappa_A(\beta^{-1}(Z_r^Y)) + A\psi(Z_r^Y) \right) \, d\rho_{\beta}(Z_r^Y)
$$

$$
= \int_{\rho_{\beta}(Z_t^Y)} \left( \kappa_A(\gamma_{\beta}^{-1}(u)) + A\psi(\rho_{\beta}^{-1}(u)) \right) \, du
$$

$$
= \int_{Z_t^Y - Y_{t_n}^\beta} \left( \kappa_A(\gamma_{\beta}^{-1}(u)) + A\psi(\rho_{\beta}^{-1}(u)) \right) \, du,
$$

(5.3)

where we have used the identity in (4.13). Similarly, since

$$
dt = - \frac{d\gamma_{\beta}(Y_t^\beta)}{h(\beta(Y_t^\beta))} \quad \forall t \in [t_w, \bar{t}] \setminus \cup_{n \in \mathbb{I}} [t_n, s_n),
$$

applying (4.14), it can be calculated that on some time interval $[s, t] \subset [t_w, \bar{t}] \setminus \cup_{n \in \mathbb{I}} [t_n, s_n)$, for some $n \in \mathbb{I},$

$$
\int_s^t \left( \kappa_A(Y_t^\beta) + Ah(Z_r^Y) \psi(Z_r^Y) \right) \, dr
$$

$$
= \int_{Z_t^Y - Y_t^\beta} \left( \kappa_A(\gamma_{\beta}^{-1}(u)) + A\psi(\rho_{\beta}^{-1}(u)) \right) \, du,
$$

(5.4)

Let $t_w < \bar{t}$. Suppose the number of $t_n$ and $s_n$ in the interval $[t_w, \bar{t}]$ is equal to $m < \infty$ (possibly $m = 0$). Consider $r_0 \leq r_1 < \ldots < r_m < r_{m+1}$, where $r_0 = t_w$, $r_{m+1} = \bar{t}$ and for $k = 1, \ldots, m$, $r_k$ are equal to those $t_n, s_n \in [t_w, \bar{t}]$. We assume $r_1, \ldots, r_m$ are in an ascending order. Then it follows from (5.3), (5.4) and the continuity of $Y_t^\beta$ and $Z_t^Y$ when $t > 0$ that

$$
\int_{t_w}^{\bar{t}} \left( \kappa_A(Y_t^\beta) + Ah(Z_t^Y) \psi(Z_t^Y) \right) \, dt
$$

$$
= \sum_{k=0}^{m} \int_{r_k}^{r_{k+1}} \left( \kappa_A(Y_t^\beta) + Ah(Z_t^Y) \psi(Z_t^Y) \right) \, dt
$$

$$
= \int_{Z_{t_w}^Y - Y_{t_w}^\beta} \left( \kappa_A(\gamma_{\beta}^{-1}(u)) + A\psi(\rho_{\beta}^{-1}(u)) \right) \, du.
$$

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Suppose there are infinitely many \( t_n \) and \( s_n \) in the interval \([t_w, \bar{t}]\). Let \( r \in [t_w, \bar{t}] \) be an accumulation point of the sequence \( \{t_n\}_{n \in \mathbb{N}} \). Then without loss of generality, consider a subsequence \( \{t_{n_k}\}_{k=1}^\infty \subset [t_w, \bar{t}] \) increasing to \( r \). Consider some time interval \([t, s]\) in which \( r \) is the only accumulation point of \( \{t_n\}_{n \in \mathbb{I}} \). Then, it follows that

\[
\int_{t}^{s} \left( \kappa_A(Y_\beta^t) + Ah(Z_\beta^t) \psi(Z_\beta^t) \right) \, dt = \lim_{n \to \infty} \int_{t}^{t_{n_k}} \left( \kappa_A(Y_\beta^t) + Ah(Z_\beta^t) \psi(Z_\beta^t) \right) \, dt + \int_{r}^{s} \left( \kappa_A(Y_\beta^t) + Ah(Z_\beta^t) \psi(Z_\beta^t) \right) \, dt
\]

\[
= \lim_{n \to \infty} \int_{t}^{t_{n_k}} \left( \frac{\kappa_A(\gamma^{-1}_\beta(u))}{h(\rho^{-1}_\beta(u))} + A\psi(\rho^{-1}_\beta(u)) \right) \, du + \int_{r}^{s} \left( \frac{\kappa_A(\gamma^{-1}_\beta(u))}{h(\rho^{-1}_\beta(u))} + A\psi(\rho^{-1}_\beta(u)) \right) \, du,
\]

\[
= \int_{Z_\beta^t}^{Z_\beta^s} \left( \frac{\kappa_A(\gamma^{-1}_\beta(u))}{h(\rho^{-1}_\beta(u))} + A\psi(\rho^{-1}_\beta(u)) \right) \, du.
\]

This implies that

\[
\int_{t_{w}}^{\bar{t}} \left( \kappa_A(Y_\beta^t) + Ah(Z_\beta^t) \psi(Z_\beta^t) \right) \, dt = \int_{Z_\beta^{y_A}}^{Z_\beta^{y_A}} \left( \frac{\kappa_A(\gamma^{-1}_\beta(u))}{h(\rho^{-1}_\beta(u))} + A\psi(\rho^{-1}_\beta(u)) \right) \, du.
\]

Therefore the result follows from the above equality as well as (4.26) and (4.28).

**Proof of Lemma 4.3.** First notice that, for any \( y \in (0, \bar{y}_A) \), the function \( \Gamma(x, y) \) is concave in \( x \), but that this concavity may not be strict. Observe that for \( y \in (0, \bar{y}_A) \),

\[
\lim_{x \to 0^-} \Gamma(x; y) = -\infty.
\]

Also, \( \Gamma(x; y) \in \mathbb{R} \), for \( x \in [\bar{c}, 0) \). These imply that \( \bar{c} \leq \beta_s(y) \leq \beta^*(y) < 0 \), for all \( 0 < y < \bar{y}_A \). The largest and smallest solution to (4.34) uniquely define the functions \( \beta^* \) and \( \beta_s \). For
0 < y < y + \triangle < \bar{y}_A and x \in [\bar{x}, 0), we calculate that

\[
\frac{d}{dx} \left[ \Gamma(x; y + \triangle) - \Gamma(x; y) \right] = \frac{(\kappa_A(y + \triangle) - \kappa_A(y))h'(x)}{h^2(x)} + \frac{\kappa_A'(y + \triangle) - \kappa_A'(y)}{h(x)} < 0, \quad (5.5)
\]

since \( \kappa_A \) is convex and \( \kappa_A'(u) > 0 \), for \( u > 0 \). We want to show that \( \beta^* \) and \( \beta_* \) are decreasing functions. In order to get a contradiction, suppose that there exists \( y \in (0, \bar{y}_A) \) and \( \triangle > 0 \) such that \( \beta^*(y + \triangle) > \beta_*(y) \). With reference to (5.5), we obtain

\[
\Gamma(\beta^*(y + \triangle); y + \triangle) - \Gamma(\beta^*(y + \triangle); y) < \Gamma(\beta_*(y); y + \triangle) - \Gamma(\beta_*(y); y).
\]

However, this contradicts the definitions of \( \beta^* \) and \( \beta_* \) which imply that

\[
\Gamma(\beta^*(y + \triangle); y + \triangle) \geq \Gamma(\beta_*(y); y + \triangle) \quad \text{and} \quad \Gamma(\beta_*(y); y) \geq \Gamma(\beta^*(y + \triangle); y).
\]

Therefore, for all \( 0 < y < \bar{y}_A \),

\[
\beta_*(y + \triangle) \leq \beta^*(y + \triangle) \leq \beta_*(y) \leq \beta^*(y), \quad (5.6)
\]

and from which it follows that \( \beta^* \) and \( \beta_* \) are decreasing. By (4.33), we know that for \( \bar{x} \leq x < 0 \), \( \Gamma(x; y) \) is continuous in \( y \). Then for \( y \in (0, \bar{y}_A) \), we have

\[
\Gamma(\beta_*(y^+); y^+) = \Gamma(\beta_*(y^-); y^-) \leq \Gamma(\beta_*(y); y) = \Gamma(\beta^*(y); y^+) = \Gamma(\beta^*(y); y^-) = \Gamma(\beta^*(y); y).
\]

Since \( \beta^* \) and \( \beta_* \) are defined as the largest and smallest maximiser to (4.34) respectively, and because \( \beta^* \) and \( \beta_* \) are decreasing, it follows that \( \beta_*(y^+) = \beta_*(y) \) and \( \beta^*(y^-) = \beta^*(y) \). By monotonicity, the right limit of \( \beta^* \) and the left limit of \( \beta_* \) exist. Hence, we have proved that \( \beta^* \) is càglàd and \( \beta_* \) is càdlàg. The claim that \( \beta^* \) is the càglàd version of \( \beta_* \) and that \( \beta_* \) is the càdlàg version of \( \beta^* \) follows from (5.6). \( \square \)

**Proof of Lemma 4.4.** If \( y > \bar{y}_A \), then by the definition of \( \beta^* \), it holds that if \( \lim_{x \to y^-} \kappa_A(x) = \infty \) or \( \lim_{x \to y^-} \kappa'_A(x) = \infty \), then \( \lim_{x \to y^-} \beta^*(x) = \bar{x} \). The remaining case is when \( y = \bar{y}_A \).

We will prove this case by contradiction. Suppose \( \beta^*(\bar{y}_A) > \bar{x} \). For any \( x \in (\bar{x}, \beta^*(\bar{y}_A)) \) and
\( y \in (0, \bar{y}_A) \) such that \( \beta^*(y) \geq \beta^*(\bar{y}_A) \), we have

\[
A \psi(x) \leq A \left( \psi(x) - \psi(\beta^*(y)) \right) \\
\leq \kappa_A(y) \left( \frac{1}{h(\beta^*(y))} - \frac{1}{h(x)} \right) + \kappa'_A(y) \left( H(\beta^*(y)) - H(x) \right) \\
\leq \kappa_A(y) \left( \frac{1}{h(\beta^*(\bar{y}_A))} - \frac{1}{h(x)} \right) + \kappa'_A(y) \left( H(\beta^*(\bar{y}_A)) - H(x) \right).
\]

Taking \( y \) to be arbitrarily close to \( \bar{y}_A \) implies \( \psi(x) = -\infty \). This means \( x < \bar{z} \), which contradicts \( x > \bar{z} \). Hence, we conclude that \( \beta^*(\bar{y}_A) = -\infty \).

Next we prove (4.35). Observe that if \( \beta^*(0+) < 0 \), then (4.35) is true. However, if \( \beta^*(0+) = 0 \), then

\[
\frac{\kappa_A(y)}{h(\beta^*(y))} \geq \Gamma(x; y) - A \psi(\beta^*(y)) - \kappa'_A(y) H(\beta^*(y)) \geq \Gamma(x; y) - \kappa'_A(y) H(\beta^*(y)),
\]

from which it follows that for any \( x \in (\bar{z}, 0) \),

\[
0 \geq \liminf_{y \to 0+} \frac{\kappa_A(y)}{h(\beta^*(y))} \geq A \psi(x) - \limsup_{y \to 0+} \kappa'_A(y) H(\beta^*(y)),
\]

\[
0 \geq \limsup_{y \to 0+} \frac{\kappa_A(y)}{h(\beta^*(y))} \geq A \psi(x) - \liminf_{y \to 0+} \kappa'_A(y) H(\beta^*(y)).
\]

Therefore,

\[
0 \geq \limsup_{y \to 0+} \kappa'_A(y) H(\beta^*(y)) \geq A \psi(x),
\]

\[
0 \geq \liminf_{y \to 0+} \kappa'_A(y) H(\beta^*(y)) \geq A \psi(x).
\]

By letting \( x \) tend to 0, then with reference to (2.4), we get \( \lim_{y \to 0+} \kappa'_A(y) H(\beta^*(y)) = 0 \). Also, by letting \( x \) tend to 0 in (5.7) and (5.8), \( \lim_{y \to 0+} \frac{\kappa_A(y)}{h(\beta^*(y))} = 0 \) follows.

**Proof of Proposition 4.5.** To show that \( v \) is continuous, we first prove it is finite. With reference to (4.28)-(4.32), it is sufficient to show that the function \( J_\beta \) given by (4.27) is finite for \( \beta \) defined by Lemma 4.3. By the continuity of \( Y^\beta \) and \( Z^{Y^\beta} \) after time 0 and condition
we have that there exists some $s > 0$ such that
\[
\int_0^s \left( \kappa_A(Y_t^\beta) + Ah(Z_t^\beta)\psi(Z_t^\beta) \right) dt < \infty
\]  
(5.9)
and $Y_s^\beta < \bar{y}_A$. According to the condition in Lemma 4.4,
\[
\lim_{y \to 0^+} \frac{\kappa_A(y)}{h(\beta(y))} = 0,
\]
so it follows that there exists $C_1 > 0$ and $0 < \epsilon < \bar{y}_A$ such that
\[
\kappa_A(y) \leq -C_1 h(\beta(y)), \quad \text{for all } y \in [0, \epsilon].
\]
Since $\psi(Z_t^\beta)$ is bounded for all $t \geq s$ (it increases to 0), this together with the above inequality implies that
\[
\int_s^\infty \left( \kappa_A(Y_t^\beta) + Ah(Z_t^\beta)\psi(Z_t^\beta) \right) dt \leq \int_s^\infty \left( -C_1 h(\beta(Y_t^\beta)) - C_2 h(Z_t^\beta) \right) dt
\leq \int_s^\infty \left( -C_1 h(Z_t^\beta) - C_2 h(Z_t^\beta) \right) dt
\leq (C_1 + C_2) (Y_s^\beta - Z_s^\beta) < \infty,
\]  
(5.10)
where $C_2 > 0$ is some constant. Therefore, (5.9) and (5.10) together show that $v$ is finite.

Note that each expression given by (4.36) or (4.37) is continuous in $y$ and $z$. It is therefore sufficient to prove that $v$ is continuous across $G^\beta$. Write $J_u(y, z)$ to be the expression of $v(y, z)$ given by (4.36), and let $J_l(y, z)$ be the expression in (4.37). Suppose $(y, z)$ is a point on the graph of $\beta$, i.e., $z = \beta(y)$. Consider a sequence of points $(y_n, z_n)_{n=1}^\infty$ contained in $S^\beta \setminus G^\beta$, converging to $(y, z)$. With reference to (4.31) and (4.32), we calculate that
\[
\lim_{n \to \infty} v(y_n, z_n) = J_u(y, \beta(y)) = J_l(y, \beta(y)) = v(y, \beta(y)).
\]  
(5.11)
If $(y, z)$ lies on the graph of $\beta^{-1}$, i.e., $y = \beta^{-1}(z)$, then using the property that $\beta^{-1}(u) = \beta^{-1}(z)$, for $u \in (z, \beta(\beta^{-1}(z)))$, direct calculation results (5.11). We therefore conclude that $v$
is a continuous function. Differentiating $v$ gives

$$D^-_y v(y, z) = -\frac{\kappa_A(\gamma^{-1}_\beta(z - y))}{h(\rho^{-1}_\beta(z - y))} - A\psi(\rho^{-1}_\beta(z - y)), \quad z > \beta(y); \quad (5.12)$$

$$v_z(y, z) = \frac{\kappa_A(\gamma^{-1}_\beta(z - y))}{h(\rho^{-1}_\beta(z - y))} + A\psi(\rho^{-1}_\beta(z - y)), \quad z > \beta(y); \quad (5.13)$$

$$D^-_y v(y, z) = \kappa'_A(y)H(z) - \frac{\kappa_A(y)}{h(\beta(y))} - A\psi(\beta(y)) - \kappa'(y)H(\beta(y)), \quad z \leq \beta(y); \quad (5.14)$$

$$v_z(y, z) = \frac{\kappa_A(y)}{h(z)} + A\psi(z), \quad z \leq \beta(y). \quad (5.15)$$

These expressions are left-continuous with right limit in $y$ and continuous in $z$ (all of these expressions are continuous at $(0, 0)$, this is guaranteed by $(4.35)$). Also, it can be checked that for any $(y_n, z_n)_{n=1}^\infty \subseteq \mathcal{F}$, $(y, z) \in \mathcal{G}$ and $\lim_{n \to \infty}(y_n, z_n) = (y, z)$, we have $v_z(y_n, z_n) \to v_z(y, z)$, as $n \to \infty$. Further, $\lim_{z \to \beta(y)^+}D^-_y v(y, z) = D^-_y v(y, \beta(y))$. Therefore, we conclude that $v_z(y, z)$ is continuous, and $D^-_y v(y, z)$ is càglàd in $y$ and continuous in $z$.

Standard calculations show that $v$ satisfies $(4.1)$ and $(4.3)$. When $z = 0$, $(4.2)$ is clearly true. For $z \neq 0$, in order to verify $(4.2)$, we compute that when $z > \beta(y)$,

$$h(z)v_z(y, z) - \kappa_A(y) - Ah(z)\psi(z)$$

$$= h(z)\left\{\frac{\kappa_A(\gamma^{-1}_\beta(s))}{h(\rho^{-1}_\beta(s))} - \frac{\kappa_A(z - s)}{h(z)} + A\{\psi(\rho^{-1}_\beta(s)) - \psi(z)\}\right\}, \quad (5.16)$$

where $s = z - y$. Observe that $h(\rho^{-1}_\beta(s)) = 0$ implies $y = 0$, but $(4.1)$–$(4.4)$ are under the condition of $y > 0$. So $h(\rho^{-1}_\beta(s))$ is non-zero. By the definition of $\gamma^{-1}_\beta$, we must have $\gamma^{-1}_\beta(s) \in (0, \bar{y}_A)$ if $\beta(\bar{y}_A) = \bar{z}$, or $\gamma^{-1}_\beta(s) \in (0, \bar{y}_A)$ if $\beta(\bar{y}_A) > \bar{z}$. Then according to the limiting behaviour of $\beta$ in Lemma $(4.4)$, $\kappa_A(\gamma^{-1}_\beta(s))$ must be finite, so is $\kappa'_A(\gamma^{-1}_\beta(s))$. However, $\kappa_A(z - s)$ may be infinite, but then it follows that $(5.16)$ is negative. Otherwise, if $\kappa_A(y) < \infty$, write

$$G(s; z) = \frac{\kappa_A(\gamma^{-1}_\beta(s))}{h(\rho^{-1}_\beta(s))} - \frac{\kappa_A(z - s)}{h(z)} + A\{\psi(\rho^{-1}_\beta(s)) - \psi(z)\}.$$  

Then in order to verify $(4.2)$, it suffices to show $G(s; z) \geq 0$, for all $\rho^{-1}_\beta(s) < z < 0$. We
calculate that \( G(s; y) \) can be rewritten as
\[
G(s; z) = \left[ \Gamma(\rho^{-1}_\beta(s); \gamma^{-1}_\beta(s)) - \Gamma(z; \gamma^{-1}_\beta(s)) \right]
- \kappa'_A(\gamma^{-1}_\beta(s)) \left[ H(\rho^{-1}_\beta(s)) - H(z) \right]
+ \frac{1}{h(z)} \left[ \kappa_A(\gamma^{-1}_\beta(s)) - \kappa_A(z - s) \right],
\tag{5.17}
\]
where Lemma 4.3 verifies
\[
\Gamma(\rho^{-1}_\beta(s); \gamma^{-1}_\beta(s)) - \Gamma(z; \gamma^{-1}_\beta(s)) \geq 0.
\tag{5.18}
\]
We calculate that
\[
\frac{1}{h(z)} \left[ \kappa_A(\gamma^{-1}_\beta(s)) - \kappa_A(z - s) \right] - \kappa'_A(\gamma^{-1}_\beta(s)) \left[ H(\rho^{-1}_\beta(s)) - H(z) \right]
= \int_{\rho^{-1}_\beta(s)}^{z} \left( \frac{\kappa_A(u - s) - \kappa_A(\rho^{-1}_\beta(s) - s - u)}{h^2(u)} H'(u) + \frac{\kappa'_A(\rho^{-1}_\beta(s) - s - u)}{h(u)} \right) du
\geq 0.
\tag{5.19}
\]
Therefore, combining (5.17)–(5.19), (4.2) is verified. Furthermore, the definition of \( \beta \) yields
\[
D_y^- v(y, z) + v_z(y, z) = \kappa'_A(y) H(z) - \frac{\kappa_A(y)}{h(\beta(y))} + A \psi(z - y) - A \psi(\beta(y))
- \kappa'_A(y) H(\beta(y)) + \frac{\kappa_A(y)}{h(z)} + A \psi(z) - A \psi(\beta(y) - y)
= \Gamma(z; y) - \Gamma(\beta(y); y)
\leq 0.
\]
This verifies that (4.4) is true.

Finally, the boundary condition is satisfied by (4.36), because for any \( u \in [\beta(0+), z] \), we have \( \gamma^{-1}_\beta(u) = 0 \) and \( \rho^{-1}_\beta(u) = u \); and it is trivially satisfied by (4.37).

Proof of Theorem 4.6. Let \( \delta \) be a positive-valued \( C^\infty(\mathbb{R}) \) function with support on \([0, 1]\) satisfying \( \int_0^1 \delta(x) \, dx = 1 \), and define a sequence of functions \( \{\delta_n\}_{n=1}^\infty \) by
\[
\delta_n(s) = n \delta(ns), \quad s \geq 0.
\]
We mollify \( v \) to obtain a sequence of functions \( \{v^{(n)}\}_{n=1}^{\infty} \) which are given by

\[
v^{(n)}(y, z) = \int_{0}^{1} v(y - s, z) \delta_n(s) \, ds.
\]

(One may extend the lower bound of the domain of \( v(\cdot, z) \) properly so that \( v^{(n)} \) is well-defined at \( y = 0 \).) Then \( v^{(n)} \in C^{1,1}(\mathcal{D}) \), for all \( n \in \mathbb{N} \), and

\[
\begin{align*}
v(y, z) & = \lim_{n \to \infty} v^{(n)}(y, z), \\
v_z(y, z) & = \lim_{n \to \infty} v_z^{(n)}(y, z), \\
D_y v(y, z) & = \lim_{n \to \infty} v_{y}^{(n)}(y, z),
\end{align*}
\]

where the last equality is due to \( D_y v(y, z) \) being càdlàg in \( y \). Moreover, for every \((y_0, z_0) \in \mathcal{D}\) there exists a \( K > 0 \) such that on the set \( \{ (y, z) \in \mathcal{D} \, | \, z \geq y + z_0 - y_0 \} \),

\[
\begin{align*}
|v^{(n)}(y, z)| & \leq K, \quad n \in \mathbb{N}, \\
|v_y^{(n)}(y, z)| & \leq K, \quad n \in \mathbb{N}, \\
|v_z^{(n)}(y, z)| & \leq K, \quad n \in \mathbb{N}.
\end{align*}
\]

(If \( Y \) is admissible and \((Y_{0-}, Z_{0-}^Y) = (y_0, z_0)\), then \((Y_t, Z_t^Y) \in \{ (y, z) \in \mathcal{D} \, | \, z \geq y + z_0 - y_0 \} \), for all \( t \geq 0 \).) By Itô’s formula, we calculate that

\[
v^{(n)}(Y_T, Z_T^Y) + \int_{0}^{T} \left( \kappa_A(Y_{t-}) + Ah(Z_t^Y) \psi(Z_t^Y) \right) \, dt \\
= v^{(n)}(y, z) + \int_{0}^{T} \left( v_y^{(n)}(Y_{t-}, Z_t^Y) + v_z^{(n)}(Y_{t-}, Z_t^Y) \right) \, dY_t^c \\
\quad + \int_{0}^{T} \left( \kappa_A(Y_{t-}) + Ah(Z_{t-}^Y) \psi(Z_{t-}^Y) - v_z^{(n)}(Y_{t-}, Z_{t-}^Y) h(Z_{t-}^Y) \right) \, dt \\
\quad + \sum_{0 \leq t \leq T} \left\{ v^{(n)}(Y_{t-} + \Delta Y_t, Z_{t-}^Y + \Delta Z_{t-}^Y) - v^{(n)}(Y_{t-}, Z_{t-}^Y) \right\},
\]

(5.23)

for all \( Y \in \mathcal{A}_D(y) \). Observe that for \( t \geq 0 \),

\[
0 \leq -\int_{0}^{t} h(Z_u^Y) \, du = Z_t^Y - Y_t - Z_0^Y + Y_0 \leq y - z.
\]
Then, with reference to (5.20)–(5.22), we have
\[
\int_0^\infty \sup_{n \in \mathbb{N}} \left| v_n^{(n)}(Y_{t-}, Z_{t-}) h(Z_{t-}^Y) \right| dt \leq K(y-z).
\]
Similarly,
\[
\int_0^\infty \sup_{n \in \mathbb{N}} \left| v_n^{(n)}(Y_{t-}, Z_{t-}) + v_n^{(n)}(Y_{t-}, Z_{t-}^Y) \right| d(-Y_t^c) \leq 2Ky
\]
and
\[
\sum_{0 \leq t} \sup_{n \in \mathbb{N}} \left| v_n(Y_{t-} + \Delta Y_t, Z_{t-} + \Delta Y_t) - v_n(Y_{t-}, Z_{t-}) \right| \leq 2Ky.
\]
Hence, by (5.23) and the boundary condition \(v(0, z) = A \int_0^z \psi(u) du\), it follows from the dominated convergence theorem that for any \(Y \in \mathcal{A}_D(y)\),
\[
\int_0^\infty \left( \kappa_A(Y_{t-}) + Ah(Z_{t-}^Y) \psi(Z_{t-}^Y) \right) dt
\]
\[
= v(y, z) + \int_0^\infty \left( D_y v(Y_{t-}, Z_{t-}^Y) + v_z(Y_{t-}, Z_{t-}^Y) \right) dY_t^c
\]
\[
+ \int_0^\infty \left( \kappa_A(Y_{t-}) + Ah(Z_{t-}^Y) \psi(Z_{t-}^Y) - v_z(Y_{t-}, Z_{t-}^Y) h(Z_{t-}^Y) \right) dt
\]
\[
+ \sum_{t \geq 0} \left\{ v(Y_{t-} + \Delta Y_t, Z_{t-} + \Delta Y_t) - v(Y_{t-}, Z_{t-}) \right\},
\]
(5.24)
as \(n \to \infty\) and \(T \to \infty\). According to Proposition 4.5, \(v\) satisfies (4.1)–(4.4), and therefore,
\[
\int_0^\infty \left( \kappa_A(Y_{t-}) + Ah(Z_{t-}^Y) \psi(Z_{t-}^Y) \right) dt \geq v(y, z).
\]
(5.25)
Hence, \(V \geq v\).

From from (5.9)–(5.10), we know that with \(\beta\) being the largest solution to (4.34) and \(Y^\beta\) being the strategy described in Lemma 4.1 corresponding to \(\beta\), \(Y^\beta\) is admissible, in particular (2.3) is satisfied. Therefore, with reference to (5.25), in order to complete the proof, we need to show that (5.25) holds with equality for \(Y^\beta\). Observe that \(\Delta Y^\beta < 0\) only if \(t = 0\) and \(z > \beta(y)\). But by (4.1) and Proposition 4.5 we have that \(D_y v(y, z) + v_z(y, z) = 0\), for
$z > \beta(y)$. Therefore,

$$\sum_{t \geq 0} \left\{ v(Y_{t-}^\beta + \Delta Y_{t-}^\beta, Z_{t-}^\beta + \Delta Y_{t-}^\beta) - v(Y_{t-}^\beta, Z_{t-}^\beta) \right\} = 0.$$ 

For any $z \leq 0$, if $0 \leq t \leq t_w$, where $t_w$ is defined by (4.16), then $d(Y_t^\beta)^c = 0$, hence

$$\int_0^{t_w} \left( D_y v(Y_{t-}^\beta, Z_{t-}^\beta) + v_z(Y_{t-}^\beta, Z_{t-}^\beta) \right) d(Y_t^\beta)^c = 0;$$

if $t > t_w$, then $(Y_t^\beta, Z_t^\beta) \in G^\beta$, which implies

$$\int_{t_w}^\infty \left( D_y v(Y_{t-}^\beta, Z_{t-}^\beta) + v_z(Y_{t-}^\beta, Z_{t-}^\beta) \right) d(Y_t^\beta)^c = 0.$$

Finally we have

$$\int_0^\infty \left( \kappa_A(Y_{t-}^\beta) + Ah(Z_{t-}^\beta) \psi(Z_{t-}^{Y^\beta}) - v_z(Y_{t-}^\beta, Z_{t-}^\beta) h(Z_{t-}^{Y^\beta}) \right) dt = 0,$$

since the integrand is equal to 0, for all $(Y_t^\beta, Z_t^\beta) \in \overline{W}^\beta$, and the Lebesgue measure of the set of $t \geq 0$ for which $(Y_t^\beta, Z_t^\beta) \in \overline{S}^\beta \setminus G^\beta$ is 0. With reference to (5.24), we therefore conclude that $v = V$ and that $Y^* = Y^\beta \in \mathcal{A}_D(y)$ is an admissible optimal liquidation strategy for the optimization problem (3.7), and the result follows from (3.5).

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