Renormalization of quark propagator, vertex functions and twist-2 operators from
twisted-mass lattice QCD at $N_f=4$

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We present a precise non-perturbative determination of the renormalization constants in the mass independent RI'-MOM scheme. The lattice implementation uses the Iwasaki gauge action and four degenerate dynamical twisted mass fermions. The gauge configurations are provided by the ETM Collaboration. Renormalization constants for scalar, pseudo-scalar, vector and axial operators, as well as the quark propagator renormalization, are computed at three different values of the lattice spacing, two volumes and several twisted mass parameters. The method we developed allows for a precise cross-check of the running, thanks to the particular proper treatment of hypercubic artifacts. Results for the twist-2 operator $O_{44}$ are also presented.
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I. INTRODUCTION

Lattice QCD (LQCD) has proven to be a very powerful approach to study QCD and has become a precision technique for the ab initio computation of many QCD observables. In particular, the possibility to perform a rigorous non-perturbative renormalization is an essential feature of lattice calculations. QCD discretization on a space-time lattice provides indeed a well defined regularization of the theory, by introducing the lattice spacing as a natural cut-off. However, any comparison with physical results requires a precise control of the continuum limit. Renormalization allows, from bare quantities computed at finite lattice spacing, to obtain meaningful physical observables with the accuracy sought (typically of the percent level). Controlling as much as possible all statistical and systematic effects in the determination of the renormalization constants is crucial since the accuracy of the renormalization procedure directly affects the precision of the computed observables. For instance, the calculation of nucleon matrix elements, which remains an open challenge, involves a careful estimate of the corresponding renormalization constants, essential to compare lattice results to values deduced from experiments. A proper comparison of these matrix elements with experimental values represents both a challenge and an opportunity for lattice QCD.

The goal of this work is to present the computation of renormalization constants (RCs) for local and twist-2 fermionic bilinear operators using twisted mass fermion configurations with four dynamical quarks in the sea.

We use a modified version of the regularization invariant RI-MOM scheme known as RI’-MOM [1]. The renormalization conditions of an operator are imposed non-perturbatively on conveniently defined amputated projected Green functions computed between off-shell quark states, evaluated at a given momentum and in a fixed gauge. This scheme is mass-independent and renormalization constants are defined at zero quark mass. To carry out this renormalization study, the European Twisted Mass Collaboration (ETMC) has performed dedicated $N_f = 4$ simulations with four degenerate light quark masses. RCs for $N_f = 2 + 1 + 1$ ensembles are evaluated by extrapolating to the chiral limit the RCs computed with the $N_f = 4$ ensembles.

By using the lattice formalism, one is obliged to break some symmetries which are only recovered in the continuum limit, among which is the continuum rotation symmetry. In discrete Euclidean space, the $O(4)$ rotation symmetry is broken down to $H(4)$ or $H(3)$ hypercubic symmetry depending on whether the lattice setup is the same on spatial and temporal directions. As a consequence, there are lattice artifacts which are only invariant under $H(4)$ but not under $O(4)$. This is particularly an issue for the computation of quantities like the renormalization constants since the associated statistical errors are often quite small, and the uncertainties from lattice artifacts become visible, thus deserving a careful treatment. A popular solution is to use the “democratic cuts” to select data points with relatively small $H(4)$ lattice artifacts. Another approach, which is usually called the $H(4)$-extrapolation [2-4], is to include the lattice artifacts explicitly in the data analysis. This approach, applied in the present work, allows one to use a much wider range of data points and to extract information from the lattice simulations more efficiently.

A particularly interesting point of the $H(4)$-extrapolation procedure applied to the determination of renormalization constants, is to allow a precise study of their running. This key advantage provides the possibility to compare the evolution of the RCs obtained on the lattice with perturbative formulae and to perform an interesting estimate of the non-perturbative contributions.

This paper is organized as follows. After a brief description of the lattice set-up in section II, basic RI’-MOM formulae and our notation are defined in section III. The analysis procedure is explained in section IV, where Goldstone pole subtraction, $m_{PCAC}$ average and hypercubic corrections are detailed. A precise study of the running is presented in section V, both for local and twist-2 operators, with a special focus on lattice artifacts and higher order corrections. A comment is in order at this point. In the jargon of the renormalization community there is an abuse of the meaning of the word ”local”. All operators considered in this analysis such as densities, currents as well as twist-2 operators are of course local from the field theoretical viewpoint. However, it has prevailed within the renormalization community to refer to local operators in particular for the densities and the currents. Chiral extrapolations are performed in section VI and section VII presents in detail the way we convert our results to the $\overline{\text{MS}}$ scheme. In the penultimate section VIII we estimate the systematic errors and the final section contains our conclusions. It is noteworthy that our methods allow for the extraction of the $\langle A^2 \rangle$, the Landau gauge dimension-2 gluon condensate [5] that has rich phenomenological implications [6].

II. LATTICE SET-UP

The results presented here are based on the gauge field configurations generated by the ETMC using the Iwasaki gauge action and the twisted mass fermionic action. Since the RI’-MOM is a mass-independent scheme, where the renormalization conditions are imposed on the chiral limit, the ETMC has generated dedicated $N_f = 4$ ensembles with four light degenerate quarks [7], which would eventually allow for a more trustworthy chiral extrapolation. This is the reason we employ these configurations in our analysis, since the physical configurations with 2 light degenerate
u and d quarks and two heavier non degenerate s and c quark would introduce an extra source of rather uncontrolled systematic uncertainty. Of course the results obtained with our configurations are intended to renormalize bare matrix elements which are computed with the physical configurations. To achieve automatic $O(a)$ improvement, the twisted mass action is usually tuned to maximal twist, by tuning $m_{PCAC}$ quark mass to zero. However, in the case of four degenerate quarks, reaching the maximal twist is far from being a trivial task and an alternative option has been chosen. Ensembles are simulated in pairs, with opposite values of $m_{PCAC}$, and $O(a)$ artifacts are removed by averaging the quantities obtained from these two ensembles. Previous studies have indeed shown the feasibility of this approach [5,6]. We refer to Ref. [5] for more explicit details.

Two volumes, three values of the lattice spacing, and several values of the twisted mass have been considered in the analysis. The run parameters are summarized in Table I.

| ensemble | $\kappa$ | $am_{PCAC}$ | $a\mu$ (a$\mu_{sea}$ in bold) | confs |
|----------|---------|-------------|------------------|-------|
| 3p       | 0.156017 | +0.00559(14)| 0.0025, **0.0046**, 0.0090, 0.0152, 0.0201, 0.0249, 0.0297| 250   |
| 3m       | 0.156209 | -0.00585(08)| 0.0025, **0.0046**, 0.0090, 0.0152, 0.0201, 0.0249, 0.0297| 250   |
| 4p       | 0.155983 | +0.00685(12)| 0.0039, **0.0064**, 0.0112, 0.0184, 0.0240, 0.0295 | 210   |
| 4m       | 0.156250 | -0.00682(13)| 0.0039, **0.0064**, 0.0112, 0.0184, 0.0240, 0.0295 | 210   |
| 5p       | 0.156291 | -0.00821(11)| 0.0048, **0.0078**, 0.0119, 0.0190, 0.0242, 0.0293 | 220   |
| 5m       | 0.156291 | -0.00821(11)| 0.0048, **0.0078**, 0.0119, 0.0190, 0.0242, 0.0293 | 220   |

The lattice spacing values are respectively $a = 0.062$ fm for $\beta = 2.10$, $a = 0.078$ fm for $\beta = 1.95$ and $a = 0.086$ fm for $\beta = 1.90$ [7]. The fixing of the Landau gauge is achieved by the iterative minimization of a functional of the $SU(3)$ links with a combination of stochastic overrelaxation and Fourier acceleration.

### III. Renormalization Constants in the RI'-MOM Scheme

In this section we define the notation that we utilize, recall briefly the RI'-MOM scheme [1] and the explicit formulae that will be used in the computation. We refer the reader to Ref. [10] for a complete and pedagogical introduction to the RI'-MOM scheme. The RI'-MOM scheme is very widely used by many lattice collaborations [11–16].

#### A. Basics of RI'-MOM

We consider a generic bilinear fermion operator $O_\Gamma = \bar{q}_1 \Gamma q_2$ where $\Gamma$ is any Dirac structure, possibly multiplying a covariant derivative operator, and $q_1, q_2$ two fermionic fields. In the case of scalar, pseudo-scalar, vector and axial renormalisation constants, $\Gamma = 1, \gamma_5, \gamma_\mu, \gamma_5 \gamma_\mu$ respectively. To avoid contributions from disconnected diagrams, we focus mainly on the non-singlet quark operators, unless stated differently. The corresponding renormalized operator is defined as $O_R = Z_\OD O_\Gamma$. $Z_\OD$ is found in the RI' variant of the RI-MOM scheme by imposing at a scale $\mu$ large
enough (typically $\mu >> \Lambda_{QCD}$), that the amputated Green function in a fixed gauge (the Landau gauge in our case), equals its tree value, i.e. requiring that

$$Z_\Omega(\mu).Z_q^{-1}(\mu).\Gamma_\Omega(p)|_{p^2=\mu^2} = 1$$

$Z_q$ is the fermion field renormalization constant, determined through

$$Z_q(\mu^2 = p^2) = -i \frac{1}{12p^2} \text{Tr}[S^{-1}_{\text{bare}}(p)\hat{p}]$$

where $S_{\text{bare}}(p)$ is the bare quark propagator. At finite lattice spacing, the four-vector $p$ can be either the continuum momentum, or the lattice momentum $a p_\mu = \sin(a p_\mu)$. Both definitions differ only by $O(a^2)$ terms. Since RCs obtained using the continuum momentum already exhibit lattice spacing artifacts at tree-level, the lattice momentum definition is favored. $\Gamma_\Omega$ is defined in terms of the amputated Green function, or vertex, $\Lambda_\Omega$, by

$$\Gamma_\Omega(p) = \frac{1}{12} \text{Tr} \left( \Lambda_\Omega(p) \hat{P}_\Omega \right) \quad \text{where} \quad \Lambda_\Omega(p) = S_{q_1}^{-1}(p).G_\Omega(p).S_{q_2}^{-1}(p)$$

where $\hat{P}_\Omega$ is a suitable projector (see section below) and the Green function is defined in coordinate space by

$$G_\Omega(x, y) = <q_1(x) O_\Gamma \tilde{q}_2(y)>$$

On the lattice and in Fourier space, the Green function becomes

$$G_\Omega(p) = \int d^4x d^4y e^{-ip(x-y)} G_\Omega(x, y) = \frac{1}{N} \sum_i S_i^q(p|x, y) \Gamma_{\gamma_5} S_i^{q_1}(p|y) \gamma_{5},$$

where the sum runs over $N$ configurations, $S_i(p|x) = \int d^4y S_i(y, z) e^{-ipy}$ and $\Lambda_\Omega(p)$ in definition Eq. (3) reads

$$\Lambda_\Omega(p) = S_{q_1}^{-1}(p).G_\Omega(p).\gamma_{5} S_{q_2}^{-1}(p)\gamma_{5}.$$ (5)

It involves only one type of quark propagators since we have taken into account the properties of the twisted-mass formulation relating mass degenerate quarks.

We will study in particular in this work the twist-2 operator $O_{44}$. Twist-2 operators are of particular importance since they provide the leading contribution to the Operator Product Expansion (OPE) analysis of the deep inelastic scattering and this particular one is associated with the $\langle x|q\rangle = \int_0^1 dx x q(x)$ of the hadrons [17], where $x$ is the momentum fraction carried by the quark and $q(x)$ the associated longitudinal distribution.

For a general twist-2 operator, $O_{\Gamma}(z, z') = \tilde{q}_1(z) \Gamma(z, z') q_2(z')$, we take $\Gamma = \Gamma_\mu \hat{D}_\mu$ with $\hat{D}_\mu = \frac{i}{2}(\nabla_\mu + \nabla_\mu^*)$, where $\nabla$ and $\nabla^*$ are respectively the gauge covariant forward and backward derivatives, defined by

$$\nabla_\mu(x, y) = \delta_{x+y, y} U_\mu(x) - \delta_{x-y, y} U^*_\mu(y)$$

$$\nabla^*_\mu(x, y) = \delta_{x, y} - \delta_{y, x} U^*_\mu(y - \nu).$$

where $U$ are the gauge links. Inserting these definitions into the Green function and performing the Wick contractions lead to

$$G_\Omega(x, y) = -\frac{1}{2} \{ S_{q_1}(x, z) \Gamma_{\gamma_5} S_{q_2}(y, z) \gamma_{5} + S_{NL}(x, z) \Gamma_{\gamma_5} S_{q_1}(y, z) \gamma_{5} \}$$

where we have defined

$$S_{NL}(x, z) = S_{q_1}(x, z + \nu) U^*_\nu(z) \Gamma_\mu - S_{q_1}(x, z - \nu) U_\nu(z - \nu) \Gamma_\mu.$$ (10)

This "non-local propagator” $S_{NL}$, combining the neighboring propagators, is the solution of a Dirac equation with a modified source

$$\sum_y D(x, y) S_{NL}(y, z) = \delta_{x, z+x} U^*_\nu(z) \Gamma_\mu - \delta_{x, z-x} U_\nu(z - \nu) \Gamma_\mu$$

where $D(x, y)$ is the Dirac operator. Using these "non-local” propagators, from which we can construct all $\Gamma$ structures, we decrease the number of propagators to be computed, from nine to five (1 with a "local" source and 4 – one in each direction – with a "non-local” one). The advantage of our method is thus the reduced computational cost, since with only 5 inversions per configuration , we are able to extract all local and twist-2 renormalization constants for all momenta.

Finally, the Green function in momentum space becomes

$$G_\Omega(p) = -\frac{1}{2} \frac{1}{N} \sum_i \left\{ S_i^q(p|x) \gamma_5 S_{i,NL}^{q_1}(p|y) \gamma_5 + S_{i,NL}^{q_1}(p|y) \gamma_5 S_i^{q_1}(p|x) \gamma_5 \right\}.$$ (12)
B. Projectors

For scalar and pseudo-scalar operators, the projector $\hat{P}_O$ in (3) is simply $\gamma_0$ and $\gamma_5\gamma_0$ respectively. For vector and axial vertex functions however, "naive projectors" $\gamma_\mu$ and $\gamma_5\gamma_\mu$ do not project vertices onto different Lorentz structures. Indeed, the vertex function decomposes over the Dirac structures as

$$\Gamma V_\mu = \Sigma V_1 \gamma_\mu + \Sigma V_2 \frac{p_\mu}{p^2},$$  \hspace{1cm} (13)

$$\Gamma A_\mu = \Sigma A_1 \gamma_5 \gamma_\mu + \Sigma A_2 \frac{p_\mu}{p^2},$$  \hspace{1cm} (14)

with $\Sigma V_{1,2}$ and $\Sigma A_{1,2}$ being scalars multiplied by $3 \times 3$ identity matrices which we omit to simplify the notation. The correct projectors are actually given by

$$P V_\mu = \gamma_\mu - \frac{p_\mu}{p^2} \hat{p},$$

$$P A_\mu = \gamma_5 \gamma_\mu - \gamma_5 \frac{p_\mu}{p^2},$$  \hspace{1cm} (15)

and the corresponding form factors are then

$$\Sigma V^{A/V} = \frac{1}{12} \frac{p^2}{p^2 - p_\mu^2} \text{Tr} \left[ P^{A/V} \Gamma V^{A/V} \right].$$  \hspace{1cm} (16)

We have checked that the effect of using or not using these correct projectors has a rather small influence on $Z V$ and $Z A$. However, since statistical errors will turn out to be also small, we will use in what follows the correct projectors of Eq. (15).

In a similar way, twist-2 operators should be projected such that Lorentz structures are decoupled. Following the convention used in Ref. [18], we define a general symmetric and traceless twist-2 operator as

$$O_{\mu\nu} = \gamma_\mu D_{\nu} + \gamma_\nu D_{\mu} - \frac{1}{2} \delta_{\mu\nu} \gamma_\rho D^\rho.$$  \hspace{1cm} (17)

The corresponding Green function can be decomposed as

$$G_O = -\frac{1}{2} \Sigma_1(p) \left( \gamma_\mu p_\nu + \gamma_\nu p_\mu - \frac{1}{2} \delta_{\mu\nu} \hat{p} \right) - \Sigma_2(p) \hat{p} \left( p_\mu p_\nu - \frac{1}{4} p^2 \delta_{\mu\nu} \right).$$  \hspace{1cm} (18)

To project out the first form factor $\Sigma_1(p)$ we use (correcting minor typos in Ref. [18])

$$P_O = -p^2 \frac{\left( \frac{p_\mu p_\nu \hat{p}}{p^2} - \frac{\gamma_\mu p_\nu + \gamma_\nu p_\mu}{2} \right)}{4p_\mu^2 p_\nu^2 - p^2 (p_\mu^2 + p_\nu^2) - 2p_\mu p_\nu p^2 \delta_{\mu\nu}}.$$  \hspace{1cm} (19)

In particular, in the case of $O_{44}$ operator, we obtain

$$P_O = p^2 \frac{(\frac{p^2 \hat{p}}{p^2} - \gamma 4 p_4)}{4p_4^2 (p^2)}.$$  \hspace{1cm} (20)

IV. ANALYSIS PROCEDURE

For each value of the sea quark mass, two sets of gauge fields are produced, with opposite $m_{PCAC}$ values, corresponding to opposite values of the angle $\theta$, the latter being defined as the complementary to the twisted angle, see Ref. [1]. The first step consists in removing the Goldstone pole from vertex functions, for each ensemble, and in performing the valence quark mass extrapolation. The $\theta$ average is then done, before correcting for $H(4)$ artifacts. The different steps of the analysis are detailed below using a given set of ensembles, namely $3p/3m$ ensembles on a $32^3 \times 64$ lattice. The results presented in the next two sections concern charged currents and densities: $O = \bar{u} \Gamma d$ or $O = \bar{d} \Gamma u$. All plots in this section represent average of jackknife bins and error bars are also estimated by the jackknife method.
A. Pion mass and Goldstone pole subtraction

For each ensemble, the first step of the RCs analysis consists in subtracting the Goldstone pole contribution on vertex functions. This requires to compute the pion mass for each configurations set. Pion masses are determined before performing the $\theta$ average. The results are illustrated in Fig. 1 showing the pion mass as a function of the renormalized quark mass $M = \sqrt{(Z_A m_{PCAC})^2 + m_q^2}$ where an estimate of $Z_A = 0.78(0.73)$ has been taken for $\beta = 2.10$ from Ref. 7 (and $\beta = 1.95$ from Ref. 19).

![Figure 1](image)

**FIG. 1.** Pion mass for each $\beta = 2.10$ (LHS) and $\beta = 1.95$ (RHS) ensemble, before $\theta$ average. The $x$-axis is the renormalized quark mass $M = \sqrt{(Z_A m_{PCAC})^2 + m_q^2}$ and the $y$-axis is the pion mass squared, both in lattice units. The difference between $m/p$ ensembles illustrates the consequence of non maximal twist and $O(a)$ effects. The result of the straight line fit using pion mass values computed after $\theta$ average is shown in dashed blue curve.

After the jackknife bins of vertex functions (or $Z_q$, for the quark wavefunction renormalization constant) have been computed, an average over $H(3)$ invariants is performed. Then the Goldstone pole subtraction is done, bin by bin and before the $\theta$ average. The pole is taken as the pion mass squared at non maximal twist

$$\Gamma(p^2, \mu_{sea}) = A(p^2) + B(p^2)m^2 + \frac{C(p^2)}{m^2}$$

where $C(p^2)$ is the non perturbative Goldstone pole contribution 21. The value of the subtracted vertex functions extrapolated to zero mass for $u$ and $d$ quark also gives very similar results. This justifies the average over non-singlet operators that will be performed later in the analysis.

Only the pseudo-scalar vertex is expected to exhibit a Goldstone pole. We have however also inspected other vertices to check possible contamination or lattice artefacts. On Figure 2 for the $3p$ ensemble, the scalar vertex functions for the $u$ quark are plotted (filled symbols) versus the pion mass squared and compared with the subtracted values (empty circles). The extrapolated value is also indicated (star symbol). The difference between subtracted and non-subtracted values lies at the fourth digit for all valence quark masses for $a^2p^2 \geq \sim 0.5$, and it is not visible on the plot. For lower $a^2p^2$ values however, the subtraction effect is visible. Since no Goldstone pole contamination is expected for $Z_S$ 10, those effects are likely to be lattice artifacts. A similar conclusion holds for the quark renormalization constant and for the vector current. The axial vector current also has a coupling to the pion and this could be potentially a source of a problem but since this coupling is proportional to the momentum transfer of the process, which actually vanishes for our kinematical setup, it poses no problem either 21 22.

We stress that, since the low values of $a^2p^2$ will be excluded from the fit range considered when compensating for $H(4)$ artifacts (see next section), these lattice effects will not influence the final results.

Contrary to $\Gamma_S$, the pseudo-scalar vertex function shows, as expected, a strong pion mass dependence. Vertex functions for ensembles $3p$ are displayed in the RHS of Fig. 2 with the same legend conventions as for the plots of the scalar vertex. The Goldstone pole appears clearly and is thus subtracted according to Eq. 21.

The $p^2$ dependence of the chiral extrapolation coefficients are displayed in Fig. 3. The $1/m^2$ coefficient of the chiral extrapolation is, as expected, varying as $1/p^2$ for pseudo-scalar vertex, at large $p^2$. Over the large range of $a^2p^2$ values considered (typically for $a^2p^2 > 0.1$), this coefficient varies as $c_1/p^2 + (c_2/p^2)^2$. This is consistent with the expectation that the Goldstone pole can only appear in power suppressed non-perturbative contributions. For other vertices, it is globally compatible with zero.
FIG. 2. $u$ scalar (LHS) and pseudo-scalar (RHS) vertex functions versus pion mass squared (in lattice units) for ensemble 3p ($\beta = 2.10$) for several values of $a^2p^2$ ($ap^0 = \frac{\pi}{T}$ for all curves except the magenta one, for which $ap^0 = 21\frac{\pi}{T}$).

FIG. 3. Coefficient of the $1/m_q^2$ term (l.h.s.) and of the $m_q^2$ term (r.h.s.) in the chiral fit (in (21), $C(p^2)$ and $B(p^2)$ resp.) as a function of $1/p^2$ in lattice units, for ensemble 3p ($\beta = 2.10$). The green line serves as eye guidance mainly and represents a linear fit at large $p^2$.

B. $m_{PCAC}$ average and Hypercubic corrections

The $m_{PCAC}$ average is performed on the vertex function jackknife bins. Since they differ only by (small) lattice artifacts, non-singlet operators ($\bar{u}.O.d$ and $\bar{d}.O.u$) are also averaged at this stage, with an average weighted by jackknife errors. The scalar (RHS) and quark wave function (LHS) renormalization constants are represented for the representative pair ensembles $3m/3p$ (see Table I) in Fig. 4 as a function of $p^2$ in lattice units. $Z_q$ exhibits the usual strong half "fishbone" structure, typical of hypercubic artifacts, while other renormalization constants are also affected, although to a lesser extent.

The three vector components $\gamma_\mu$ (and similarly for the three axial ones $\gamma_\mu\gamma_5$) being very similar (not shown on these
The next step consists in correcting one of the two types of \( O(a^2) \) artifacts, namely the hypercubic artifacts which respect the \( H(4) \) symmetry group but not the \( O(4) \) one (the second type, i.e. \( O(a^2 p^2) \) artifacts, respecting the continuum \( O(4) \) rotation symmetry, will be treated non-perturbatively by introducing corrections in the running of the RCs, see section IV). A very powerful method has been developed \([2, 4]\), which does not rely on the selection of a small subset of momenta, thus keeping the maximum amount of information. A by-product of this procedure is the hypercubic correction can be applied either at the vertex level, or, as we did, directly on the propagators \([2, 29, 30]\), while its applications to renormalization have been presented in detail in \([3, 14, 31]\).

The effect of the hypercubic correction is clearly seen for \( Z_q \) in Fig. 6. Fig. 5 displays the renormalization constant for \( Z_{q4} \) before and after the hypercubic corrections. The "fishbone" structure is slightly less pronounced than for \( Z_{q} \) for instance.

\[
p^{[2]} = \sum_{\mu=1}^{4} p^2_{\mu}, \quad p^{[4]} = \sum_{\mu=1}^{4} p^4_{\mu}, \quad p^{[6]} = \sum_{\mu=1}^{4} p^6_{\mu}, \quad p^{[8]} = \sum_{\mu=1}^{4} p^8_{\mu},
\]

and denote the quantity \( Z(a p^4) \) (representing any renormalization constant) averaged over the cubic orbits as \( Z_{latt}(a^2 p^2, a^4 p^4, a^6 p^6, a p^4, a^2 \Lambda_{QCD}) \). We thus assume (and will check) that \( Z_{latt} \) can be Taylor-expanded around \( p^{[4]} = 0 \) up to values of \( \epsilon = a^2 p^{[4]} / p^2 \) significantly larger than 1 as

\[
Z_{latt}(a^2 p^2, a^4 p^4, a^6 p^6, a p^4, a^2 \Lambda_{QCD}) = Z_{hypcorrected}(a^2 p^2, a p^4, a^2 \Lambda_{QCD}) + R(a^2 p^2, a^2 \Lambda_{QCD}) \frac{a^2 p^{[4]}}{p^2} + \ldots,
\]

with

\[
R(a^2 p^2, a^2 \Lambda_{QCD}) = \frac{dZ_{latt}(a^2 p^2, 0, 0, 0, a^2 \Lambda_{QCD})}{d \epsilon} \big|_{\epsilon=0}.
\]
Finally, we note that there are ultraviolet artifacts which are functions of $a^2 p^2$ and are thus insensitive to hypercubic biases and not corrected by the above-mentioned method. They will be corrected simply by assuming $a^2 p^2$ terms in the final running fit.

As a summary, our analysis procedure to extract renormalization constants consists in the following steps:

- the $H(3)$ average of the vertex functions over $\vec{p}^2$ orbits,
- the $1/m^2_\pi$ term subtraction and valence chiral extrapolation, also done at the level of the vertex function, for each four-momentum,
- the $\theta$ average, performed on $Z_q$ or at the vertex level for other renormalization constants,
- the average over non-singlet $\bar{u}O_d$ and $\bar{d}O_u$ operators, weighted by jackknife errors,
• the average over equivalent $\mu$ ($= 1, 2, 3$) directions for vector and axial operators,
• the correction of hypercubic artifacts using an efficient and well-defined procedure.

The running of the RCs will be described in details in the next section.

V. RUNNING AND $O(a^2p^2)$ ARTIFACTS

The possibility to check the running of renormalization constants is an important feature of our analysis. This allows to study remaining lattice artifacts and non-perturbative contributions and to finally extract reliable values of the RCs.

A. Renormalization constants for quark and local operators

We consider the following expression for the running of the quark wave function RC

$$Z_q^{\text{hyp-corr}}(a^2p^2) = Z_q^{\text{pert}}(\mu^2) c_{0Z_q}(\frac{p^2}{\mu^2}, \alpha(\mu))$$

$$\times \left( 1 + \frac{\langle A^2 \rangle_{\mu^2} c_{2Z_q}(\frac{p^2}{\mu^2}, \alpha(\mu))}{32p^2 c_{0Z_q}(\frac{p^2}{\mu^2}, \alpha(\mu))} \right) c_{\overline{MS}}(\frac{p^2}{\mu^2}, \alpha(\mu)) + c_{a2p^2} a^2p^2 + c_{a4p^4} (a^2p^2)^2,$$  \( (25) \)

which was derived in Ref. [14] using an OPE analysis. The coefficients $c_{0Z_q}$ and $c_{\overline{MS}}$ are known from perturbation theory and the running formula contains lattice artifact terms $\propto a^2p^2$ and $\propto (a^2p^2)^2$, not yet removed. These additional terms are discussed below. We are left with four parameters to determine, namely the value of the RC $Z_q^{\text{pert}}(\mu^2)$ at a given renormalization scale $\mu$, the dimension-2 Landau gauge gluon condensate $\langle A^2 \rangle_{\mu^2}$ and the coefficients $c_{a2p^2}$ and $c_{a4p^4}$. In expression (25) appears explicitly the $1/p^2$ term generated by the gluonic condensate. For other renormalization constants, such a term can also appear (in $Z_S$ and $Z_P$ for instance) but we will not distinguish the different sources of possible artifacts (pion pole for $Z_P$, condensate, possible hadron contributions, lattice artifacts, . . .).

We illustrate below the results for $Z_q$ with the ensemble $3mp$, which is representative of the results we obtained with all other ensembles. We take $\Lambda_{QCD} = 316 \text{ MeV}$ from Ref. [26] and $\alpha_{\beta=2.10} = 0.062 \text{ fm}$ from Ref. [7]. The results for local renormalization constants are not sensitive to these values and changing $\alpha$ and $\Lambda_{QCD}$ over a wide range induces only a change in the local RCs values on the last digit. This however is not the case for twist-2 operators (see section V. B).

![FIG. 7. Running of $Z_q$ for ensemble $3mp$ ($\beta = 2.10$, $\mu = 0.0046$, volume $32^3 \times 64$) using different fitting formulae.](image-url)
Fig. 7 displays the running of $Z_q$ for the ensemble $3np$, fitted by different formulae, depending on whether or not lattice artifacts are included in the running. It can be shown that the standard OPE expression without any artifact correction (dot-dashed green curve) fails completely to describe the data. Adding only an $\alpha^2 p^2$ term decreases the $\chi^2/d.o.f.$ down to 2.85 but the running is still not correctly reproduced. To obtain a good fit, it is necessary to include both $\alpha^2 p^2$ and $(\alpha^2 p^2)^2$ terms. The running is then very well described ($\chi^2/d.o.f. = 0.26$) over the whole range of momenta and we get at 10 GeV $Z_q(\mu = 10 GeV) = 0.815(10)$. Errors quoted are at the moment only statistical. Precise estimations of the systematic errors will be performed in section VIII.

The same study is performed for scalar and pseudo-scalar RCs. $Z_S$ and $Z_P$ have the same running formula, namely

$$Z_{P/S}(\mu) = Z_{P/S}(\mu_0) \frac{c_{RI'MOM}(\mu)}{c_{RI'MOM}(\mu_0)} + c_{a2p2} \frac{p^2_{latt}}{p^2_{latt} + \frac{c_{p2m1}}{p^2_{latt}}}$$

(26)

where we have added $1/p^2_{latt}$ and $p^2_{latt}$ artifacts terms ($p_{latt}$ being the lattice momentum). We have 32

$$c_{RI'MOM}(\mu) = x_{\gamma_0} \left\{ 1 + (\gamma_1 - \bar{\beta}_1 \gamma_0) x + \frac{1}{2} \left[ (\gamma_1 - \bar{\beta}_1 \gamma_0)^3 + \gamma_2 + \bar{\beta}_1^2 \gamma_0 - \bar{\beta}_1 \gamma_1 - \bar{\beta}_2 \gamma_0 \right] x^2 
+ \left[ \frac{1}{6} (\gamma_1 - \bar{\beta}_1 \gamma_0)^3 + \frac{1}{2} (\gamma_1 - \bar{\beta}_1 \gamma_0)(\gamma_2 + \bar{\beta}_1 \gamma_0 - \bar{\beta}_2 \gamma_0) 
+ \frac{1}{3} (\gamma_3 - \bar{\beta}_1^2 \gamma_0 + 2 \bar{\beta}_1 \bar{\beta}_2 \gamma_0 - \bar{\beta}_3 \gamma_0 + \bar{\beta}_1^2 \gamma_1 - \bar{\beta}_2 \gamma_1 - \bar{\beta}_1 \gamma_2) \right] x^3 + O(x^4) \right\},$$

(27)

where $x = \alpha, \gamma_i = \gamma_i/\beta_0$ and $\bar{\beta}_i = \beta_i/\beta_0$. $\beta_i$ are the coefficients of the QCD beta-function and they are given at four-loop in 33.

Their expressions for scalar, pseudo-scalar operators and quark propagator can be written 11 33

$$\beta_0 = 11 - \frac{1}{3} N_f, \quad \beta_1 = 102 - \frac{44}{3} N_f, \quad \beta_2 = \frac{2857}{2} - \frac{5433}{16} N_f + \frac{433}{8} N_f^2,$$

and the anomalous dimensions $\gamma_i$ are given below

$$\gamma_{0/P}^S = -3 C_F, \quad \gamma_1^{S/P} = \frac{1}{2} \left( -404/3 + \frac{49}{9} N_f \right), \quad \gamma_2^{S/P} = \frac{1}{2} \left( -2498 + \left( \frac{4432}{27} + \frac{326}{3} \zeta(3) \right) N_f + \frac{280}{81} N_f^2 \right)$$

As for the quark renormalization constant, the standard running formula, i.e Eq. (26) without $1/p^2_{latt}$ and $p^2_{latt}$ terms, fails to describe the running of both $Z_S$ and $Z_P$, as illustrated in Fig. 8 (solid blue curves), though to a lesser extent than for $Z_q$. Additional terms are needed to take into account $O(\alpha^2 p^2)$ artifacts. The evolution of scalar and pseudo-scalar RCs can be perfectly reproduced with a $1/p^2_{latt}$ and $p^2_{latt}$ terms added to the standard running by fitting coefficients $c_{a2p2}$ and $c_{p2m1}$, leading to the dashed cyan curves in Fig. 8 ($\chi^2/d.o.f. = 1.14$ and $\chi^2/d.o.f. = 0.74$ for respectively $Z_S$ and $Z_P$). The scalar and pseudo-scalar RCs values obtained at 10 GeV for this ensemble are $Z_S(\mu = 10 GeV) = 0.869(4)$ and $Z_P(\mu = 10 GeV) = 0.623(2)$.

Scalar and pseudo-scalar RCs having the same anomalous dimension, they are expected to have similar running and their ratio should be constant. If $Z_P/Z_S$ is computed without properly taking into account lattice artifacts, the ratio varies by more than 20% on the momentum range considered (see Fig. 9 black circles). However, once $O(\alpha^2 p^2)$ artifacts have been separately removed from $Z_S$ and $Z_P$, the ratio becomes compatible with a constant within a very good accuracy, over the whole range of $p^2_{latt}$ values (see Fig. 9 red stars). This is an additional indication that lattice artifacts have been efficiently removed but also that the Goldstone pole has been correctly addressed.

Axial and vector renormalization constants do not run but it turns out that they exhibit a small $p^2_{latt}$ dependence, which is not surprising since all other local RCs also show this feature. Their variation does not reach more than 4% in total on the momentum range considered, but to extract reliable values of $Z_V$ and $Z_A$, we remove these artifacts by fitting this dependence, which turns out to be well described by a combination of $1/p^2_{latt}$ and $(p^2_{latt})^2$ terms. The results of the fit are shown in Fig. 10 and lead to values $Z_V = 0.688(5)$ and $Z_A = 0.761(4)$.

Table III summarizes the values obtained for local RCs, for all $N_f = 4$ ensembles considered.

In order to estimate the uncertainties on the RCs, coming from the lattice spacing determination, we vary $a$ by one $\sigma$ and check the influence of this variation on $Z_q$, $Z_S$, $Z_P$ and $Z_P/Z_S$ for a given ensemble, namely $2mp$, $\beta = 1.95$. Results are summarized in Table III. Scalar and pseudoscalar $Z$ factors are the most sensitive to the lattice spacing, whereas $Z_q$ varies less than one percent and as expected the ratio $Z_P/Z_S$ is remarkably constant.
FIG. 8. LHS: running of $Z_S$ for ensemble 3mp ($\beta = 2.10, \mu = 0.0046$, volume $32^3 \times 64$). The standard running formula is represented in solid blue line, the dashed cyan curve includes an $1/p^2$ and an $p^2$ term (in lattice units). This latter fit leads to $Z_S(10 \text{ GeV}) = 0.869(4)$. RHS.: Running of $Z_P$ with the standard running expression from (26) (solid blue curve), and adding an $1/p^2$ and an $p^2$ terms (in lattice units).

FIG. 9. $Z_P/Z_S$ for ensemble 3mp ($\beta = 2.10, \mu = 0.0046$, volume $32^3 \times 64$). Lattice artifacts have been removed separately from $Z_S$ and $Z_P$. The ratio of these two RCs is compatible with a constant over the whole $p^2$ interval considered and $Z_P/Z_S = 0.717(3)$.

B. Twist-2 operators

The running expression used for $Z_{44}$ is the same than for $Z_S$ (cf. Ref. [32], Eq. (70))

$$Z_{44}(\mu) = Z_{44}(\mu_0) \frac{c_{RI' MOM}(\mu)}{c_{RI' MOM}(\mu_0)} + c_{a2q2} \frac{p^2}{P_{latt}} + c_{p2m1} \frac{p^2_{latt}}{P_{latt}}$$

(28)
\[ \beta = 2.10 - 32^3 \times 64 \quad \beta = 1.95 - 24^3 \times 48 \quad \beta = 1.90 - 24^3 \times 48 \]

| \[Z_g^{\text{pert}} \ (\mu_R = a^{-1})\] | 3mp | 4mp | 5mp | 2mp | 3mp | 8mp | 1mp | 4mp |
|--------------------------------|------|------|------|------|------|------|------|------|
| \[Z_S \ (\mu_R = a^{-1})\] | 0.797(3) | 0.785(3) | 0.787(3) | 0.763(2) | 0.762(3) | 0.772(7) | 0.752(3) | 0.751(3) |
| \[Z_P \ (\mu_R = a^{-1})\] | 0.688(5) | 0.685(2) | 0.688(1) | 0.641(2) | 0.636(2) | 0.644(7) | 0.625(3) | 0.619(3) |
| \[Z_A\] | 0.761(4) | 0.753(2) | 0.756(1) | 0.727(2) | 0.725(2) | 0.733(6) | 0.721(2) | 0.713(2) |
| \[Z_P (I = 1)/Z_S (I = 1)\] | 0.717(3) | 0.724(3) | 0.720(3) | 0.645(5) | 0.634(5) | 0.632(5) | 0.597(7) | 0.602(5) |

**TABLE II.** Values of \([Z_q, Z_S, Z_P, Z_V, Z_A] \) and \([Z_P (I = 1)/Z_S (I = 1)]\) for all \(N_f = 4\) ensembles analyzed.

**TABLE III.** Dependence of local RCs on a lattice spacing variation, for ensemble 2mp.

\[ c^{RI'MOM}(\mu) = \exp \left\{ \int_x^x dx' \gamma(x') \right\} = x^{\bar{\gamma}_0} \left\{ 1 + (\bar{\gamma}_1 - \bar{\beta}_1 \bar{\gamma}_0) x \right. \]

\[ + \left. \frac{1}{2} \left[ (\bar{\gamma}_1 - \bar{\beta}_1 \bar{\gamma}_0)^2 + \bar{\gamma}_2 + \bar{\beta}_1^2 \bar{\gamma}_0 - \bar{\beta}_1 \bar{\gamma}_1 - \bar{\beta}_2 \bar{\gamma}_0 \right] x^2 + \mathcal{O}(x^3) \right\} \]
As for the local RCs, we have added artifacts to the standard running formula, and we fit the coefficients $c_2 p^2$ and $c_2 p^4$. The anomalous dimension for $O_{44}$ is taken from Ref. [34] and reminded here for completeness

$$\gamma_{O_{44}} = \frac{32}{9} a - \frac{4}{243} [378 N_f - 6005] a^2 + \frac{8}{6561} [10998 N_f^2 - 6318 \zeta(3) N_f - 467148 N_f - 524313 \zeta(3) + 3691019] a^3 + O(a^4)$$

(31)

with $a = \frac{g^2}{4 \pi^2}$.

As can be seen in Fig. 11, only small lattice artifacts are affecting $Z_{44}$, compared to the case of local RCs. When adding $p^2_{\text{latt}}$ and $1/(p^2_{\text{latt}})$ artifacts to the standard running expression Eq. (28), the $\chi^2$ of the fit is decreased and the $Z_{44}$ value changed by $3-5\%$.

![Figure 11](image-url)

**FIG. 11.** Running of $Z_{44}$ for ensemble $3mp$, $\beta = 2.10$, $\mu = 0.0046$, $L = 32$, $T = 64$. The black points are the data after hypercubic artifacts removal. The dashed blue curve is the standard running expression Eq. (28), and the solid red line includes $1/(p^2_{\text{latt}})$ and $p^2_{\text{latt}}$ artifacts.

The results are sensitive to the values of the lattice spacing $a$ and of $\Lambda_{QCD}$ at the percent level and the uncertainties on both $a$ and $\Lambda_{QCD}$ will be taken into account in the analysis of systematic errors (see section VIII).

VI. CHIRAL EXTRAPOLATION AND LATTICE SPACING DEPENDENCE

To get the final values of RCs at each $\beta$ value, we need to perform the chiral extrapolation. Left-hand side of Fig. 12 displays the pion mass dependence of local RCs for the three $\beta$ values under study. As can be shown, all renormalization constants only depend very weakly on the pion mass. We perform a linear extrapolation to the chiral limit (dashed lines). Also visible on Fig. 12 is the fact that, if RCs are constant with respect to the pion mass, they are, to various extent and with the noticeable exception of $Z_S$, dependent on $\beta$. This is particularly striking on $Z_P$, $Z_V$ and $Z_A$, and to a lesser extent on $Z_q$. To analyze further this variation, we plot in Fig. 12 (RHS), RCs in the chiral limit versus the lattice spacing squared in logarithmic scale. All RCs follow with a very high accuracy a $\log(a^2)$ variation. Although, since RCs are used in practice to renormalize matrix elements computed at a fixed $\beta$ value, it is not crucial to take this dependence into account in the analysis, it is still interesting to notice that the remaining lattice spacing dependence is in $\log(a^2)$.

Unlike local RCs, $Z_{44}$ exhibits a non negligible pion mass dependence, as shown on Figure 13. The values of $Z_{44}$ change by several percents (3-4 %) on the pion mass range considered (440-870 GeV) and we perform the chiral extrapolation by a constant fit. The pion mass dependence of $Z_{44}$ will be taken into account in evaluation of the systematics.
FIG. 12. LHS: \( N_f = 4 \) local RCs dependence with the pion mass. All RCs are given in the RI'-MOM scheme at 10 GeV. The straight dashed lines are constant fits for each \( \beta \) values. The red points correspond to \( \beta = 2.10 \), the black ones to \( \beta = 1.95 \), and the blue ones to \( \beta = 1.90 \). RHS: local RCs after chiral extrapolation, vs \( \log a^2 \). All RCs follow a linear dependence with \( \log a^2 \) to a very high accuracy.

FIG. 13. L.h.s.: \( N_f = 4 \) \( Z_{44} \) dependence with the pion mass. \( Z_{44} \) is given in the RI'-MOM scheme at 2 GeV. The straight dashed lines are constant fits for each \( \beta \) values. The red points correspond to \( \beta = 2.10 \), the black ones to \( \beta = 1.95 \), and the blue ones to \( \beta = 1.90 \).

VII. CONVERSION TO THE MS SCHEME AND EVOLUTION TO A REFERENCE SCALE

In order to make the connection with phenomenological calculations and experiments, which almost exclusively refer to the \( \overline{\text{MS}} \) scheme, we convert our renormalization factors from RI'-MOM to \( \overline{\text{MS}} \) using 3-loop perturbative conversion factors obtained from Ref. [18]. These latter are defined as \( Z_{q}^{\overline{\text{MS}}} = C_q^{-1} Z_{q}^{\text{RI}'-\text{MOM}} \) and \( Z_{O}^{\overline{\text{MS}}} = C_O^{-1} Z_{O}^{\text{RI}'-\text{MOM}} \). In
terms of the \( \overline{\text{MS}} \) coupling constant \( \alpha_{\overline{\text{MS}}} = \frac{\alpha^2}{16\pi^2} \), and in the Landau gauge, these functions read

\[
C_q = 1 + [5C_F - (82 - 24\zeta(3))] C_A + 28T_F N_f] \frac{C_F \alpha^2}{8} \\
+ \left[ (678024\zeta(3) + 22356\zeta(4) - 213840\zeta(5) - 1274056) C_A^2 \right. \\
- (228096\zeta(3) + 31104\zeta(4) - 103680\zeta(5) - 215352) C_A C_F + 31536 C_F^2 \\
\left. - (89856\zeta(3) - 760768) C_A T_F N_f + (68256 - 82944\zeta(3)) C_F T_F N_f - 100480T_F^2 N_f^2 \right] \frac{C_F \alpha^3}{5184} + \mathcal{O}(\alpha^4) \\
(32)
\]

\[
C_{S,P} = 1 - 4C_F \alpha \left[ (57 - 288\zeta(3)) C_F + 332T_F N_f + (432\zeta(3) - 1285) C_A \right] \frac{C_F \alpha^2}{24} \\
+ \left[ (-2493504\zeta(3) + 155520\zeta(5) + 2028348) C_A C_F - (-3368844\zeta(3) + 466560\zeta(5) + 6720046) C_A^2 \right. \\
+ (-532224\zeta(3) + 186624\zeta(4) + 3052384) C_A T_F N_f + (-331776\zeta(3) - 186624\zeta(4) + 958176) C_F T_F N_f \\
\left. - (-451008\zeta(3) - 933120\zeta(5) + 2091096) C_F^2 - (27648\zeta(3) + 244048) T_F^2 N_f^2 \right] \frac{C_F \alpha^3}{5184} + \mathcal{O}(\alpha^4) \\
(33)
\]

\[
C_{A,V} = 1 + \mathcal{O}(\alpha^4) \\
(34)
\]

\[
C_{44} = 1 + 31 \frac{C_F \alpha}{9} \left[ (-1782\zeta(3) + 6404) C_A + (1296\zeta(3) - 228) C_F - 2668T_F N_f \right] \frac{C_F \alpha^2}{162} \\
+ \left[ (-11944044\zeta(3) + 746496\zeta(4) + 524880\zeta(5) + 38226589) C_A^2 \right. \\
+ (-4914432\zeta(3) - 2239488\zeta(4) + 886460\zeta(5) + 3993332) C_A C_F \\
+ (369792\zeta(3) - 1492992\zeta(4) - 24752896) C_A T_F N_f \\
+ (10737792\zeta(3) + 1492992\zeta(4) - 9331200\zeta(5) - 3848760) C_F^2 \\
\left. - (-3234816\zeta(3) - 1492992\zeta(4) + 9980032) C_F T_F N_f \right] \frac{C_F \alpha^3}{69984} + \mathcal{O}(\alpha^4). \\
(35)
\]

where \( \zeta(n) \) is the Riemann zeta function and for the SU(3) color group, \( T_F = \frac{1}{2}, \ C_F = \frac{4}{3}, \ C_A = 3 \).

Using these expressions to convert our RI’-MOM results at a reference scale of 2 GeV to \( \overline{\text{MS}} \) values also at 2 GeV leads to the final RCs listed in Table IV.

| \( \beta \) | \( Z_A \) | \( Z_S \) | \( Z_P \) | \( Z_V \) | \( Z_A / Z_S \) | \( Z_{44} \) |
|----------|--------|--------|--------|--------|----------------|--------|
| 1.90     | 0.761(3) | 0.723(3) | 0.434(3) | 0.622(2) | 0.717(1) | 0.600(4) | 0.973(9) |
| 1.95     | 0.772(2) | 0.724(4) | 0.462(2) | 0.640(2) | 0.728(2) | 0.637(4) | 0.977(12) |
| 2.10     | 0.789(2) | 0.727(2) | 0.523(1) | 0.687(1) | 0.757(1) | 0.720(4) | 1.019(8) |

TABLE IV. Local \( N_f = 4 \) RCs in the \( \overline{\text{MS}} \) scheme at 2 GeV.

To estimate the effect of the truncation in the perturbative series, we have also converted our results to \( \overline{\text{MS}} \) at 2 GeV, but starting from RI’-MOM results at 10 GeV, converting them to \( \overline{\text{MS}} \) scheme at 10 GeV, and then evolving the \( \overline{\text{MS}} \) RCs from 10 GeV to 2 GeV using the scale dependence predicted by the renormalization group equation \[11\]

\[
R_{\mathcal{O}(\mu, \mu_0)} := \frac{Z_{\mathcal{O}(\mu)}}{Z_{\mathcal{O}(\mu_0)}} = \exp \left\{ - \int_{\mu_0^2}^{\mu^2} d\mu' \frac{\gamma(g)}{\beta(g)} \right\}. \\
(36)
\]

If the effect is negligible on \( Z_q \) (affecting only the last digit), it is of the order of 3.5% for \( Z_S \), 4% for \( Z_P \) and 2% for \( Z_{44} \). For a perturbative series, the effect of truncation is relatively small, but compared with the systematic errors, it is far from being negligible.

---

1 Setting the covariant gauge parameter \( \lambda_{RI'} \) to zero leads to \( \lambda_{\overline{\text{MS}}} = 0 \) and since in addition \( \alpha_{RI'} = \alpha_{\overline{\text{MS}}} + \mathcal{O}(\alpha^2) \) \[18\], these conversion functions have the same expression in the Landau gauge whether they are expressed in terms of \( \overline{\text{MS}} \) or RI’-MOM variables.
VIII. ESTIMATION OF THE SYSTEMATICS

The statistical uncertainties affecting the final results are rather small. Typically of the order of 1% for $Z_{44}$, and down to 2 – 5 per mil for local RCs. However, the analysis procedure leading to the final values of these RCs is quite complex and involves many non trivial steps and systematic errors turn out to be dominant compared to the tiny statistical ones. A very careful study of systematics is thus unavoidable to produce in fine reliable and meaningful results.

Sources of systematic uncertainties are manifold. They arise from the removal of hypercubic corrections, from the running fit, from the chiral extrapolation, and from the uncertainties on the lattice spacing and on $\Lambda_{QCD}$. We have carefully estimated each source of uncertainties and final results are given in Table V. The first parenthesis gives the statistical uncertainty. The second one comes from the systematics due to the hypercubic removal procedure, combined with the running fit range. We have varied both the range of $p^2$ used in the hypercubic removal procedure and in the running fit, separately, to estimate the maximal variation on the final RCs value. This leads to the systematics indicated in the second parenthesis. Finally, the last number indicates the systematics due to the chiral extrapolation.

| $\beta$ | $Z_q$ | $Z_S$ | $Z_P$ | $Z_V$ | $Z_A$ | $Z_P/Z_S$ | $Z_{44}$ |
|---------|-------|-------|-------|-------|-------|-----------|---------|
| 1.90    | 0.761(3)(5)(3) | 0.723(3)(5)(9) | 0.434(3)(3)(6) | 0.622(2)(1)(5) | 0.717(1)(2)(6) | 0.600(4)(2)(5) | 0.977(12)(11)(30) |
| 1.95    | 0.772(2)(6)(6) | 0.724(4)(5)(3) | 0.462(2)(4)(7) | 0.640(2)(1)(5) | 0.728(2)(2)(4) | 0.637(4)(4)(6) | 0.973(9)(7)(30) |
| 2.10    | 0.789(2)(6)(7) | 0.727(2)(5)(4) | 0.523(1)(4)(1) | 0.687(1)(2)(5) | 0.757(1)(2)(4) | 0.720(4)(2)(5) | 1.019(8)(6)(30) |

Table V. Final results for $N_f = 4$ local RCs in the $\overline{\text{MS}}$ scheme at 2 GeV, for each $\beta$ values considered. The first parenthesis gives the statistical uncertainty, the second one the systematics due to the hypercubic removal procedure, combined with the running fit range, and the last number indicates the systematics due to the chiral extrapolation.

The uncertainties on the lattice spacing $a$ and on the value of $\Lambda_{QCD}$ have also been propagated to the RCs. We vary $a$ by 10% and we take $\Lambda_{QCD} = 316(13)$ MeV from Ref. [26].

Systematics are estimated separately on local and on twist-2 renormalisation constants. They indeed behave quite differently, whether it concerns their pion mass dependence, or their sensitivity to lattice spacing and $\Lambda_{QCD}$. The hypercubic corrections and the running lead to an uncertainty which does not exceed 1%. The pion mass dependence of all local RCs is weak and the uncertainties associated with the chiral extrapolation small.

The uncertainty on the lattice spacing and on $\Lambda_{QCD}$, propagated to the local RCs, gives almost no sensitivity for $Z_q$ and $Z_P/Z_S$, and an effect of about 2% for $Z_S, Z_P$ (see Table III for example).

The situation is a bit different for $Z_{44}$. If uncertainties due to the $H(4)$ corrections and the running fit are of the order of the statistical errors, the dominant source of uncertainties comes clearly from the chiral extrapolation, which induces errors of the order of 3%. In addition, the errors on $a$ and $\Lambda_{QCD}$ produce an additional uncertainty on $Z_{44}$ of the order of 2%.

Finally we have compared our results with the values for local RCs given in Ref. [16]. These latter have been obtained using the “democratic” selection of momenta. Restricting our fitting interval to the one used in this reference (“method M1”) we find close results, a precise comparison being however difficult since only statistical errors are reported in [16]. In addition, taking into account statistical and systematic errors, our results are also compatible with those from [15] for $Z_{44}$.

IX. CONCLUSION

We have presented an original analysis of quark propagator, vertex functions and twist-2 operators renormalization constants for $N_f = 4$ twisted mass fermions. We have implemented a systematic and rigorous procedure to correct for hypercubic lattice artifacts. This non-perturbative method, avoiding the selection of momenta usually done in this kind of analysis and the use of perturbative formulae, allows to take advantage of all the data and to check the running over a wide range of momenta. We have applied our analysis procedure not only to local operators, but also to the twist-2 operator $O_{44}$. $O(a^2)$ lattice artefacts have also been efficiently subtracted. In order to compare with experimental values obtained for the corresponding matrix elements, all our results, obtained in the $\overline{\text{MS}}$ scheme, have been converted to the $\overline{\text{MS}}$ scheme at 2 GeV. A precise estimate of systematic errors have also been performed and these latter are shown to be dominant in the case of twist-2 operator $O_{44}$. Concerns could be raised
because of the fact that the RI-MOM scheme requires gauge fixing. There could in principle be fluctuations arising from the Gribov ambiguity. However, several studies explored this idea in the 90s \cite{33,38} as well as later on with Ginsparg-Wilson fermions \cite{22} and have shown that this effect is less than 1% and thus the dependence on gauge fixing is negligible. Of course one has to mention that all this previous work was in the quenched approximation.

The method developed here will be applied in the next future to the new gauge configurations, at the physical pion mass, generated by the ETM Collaboration.

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