The Gap Function $\phi(k, \omega)$ for a Two-leg t-J Ladder and the Pairing Interaction

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Abstract

The gap function $\phi(k, \omega)$, determined from a Lanczos calculation for a doped 2-leg t-J ladder, is used to provide insight into the spatial and temporal structure of the pairing interaction. It implies that this interaction is a local near-neighbor coupling which is retarded. The onset frequency of the interaction is set by the energy of an $S = 1$ magnon-hole-pair and it is spread out over a frequency region of order the bandwidth.

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The momentum and frequency dependence of the superconducting gap function $\phi(k, \omega)$ provides information on the spatial and dynamic structure of the pairing interaction. Traditionally, electron tunneling has been used to explore the frequency dependence of the gap \[1\]. Presently, angular resolved photoemission spectroscopy \[2\] (ARPES) and scanning tunneling microscopy \[3, 4\] (STM) open the possibility of obtaining both $k$ and $\omega$ information about the superconducting gap. However, approaches to extract this information on $\phi(k, \omega)$ from such experiments are still being explored. As has been previously discussed, a $k$ and $\omega$ dependent gap function for finite t-J lattices can be obtained using Lanczos exact diagonalization \[5, 6\]. Here, using results for $\phi(k, \omega)$ obtained for a doped 2-leg t-J ladder, we explore what can be learned once data for $\phi(k, \omega)$ becomes available.

The Hamiltonian for a 2-leg t-J ladder can be written as

$$H = J_{\text{rung}} \sum_i \left( S_{i1} \cdot S_{i2} - \frac{1}{4} n_{i1} n_{i2} \right) + J_{\text{leg}} \sum_{i,\alpha} \left( S_{i+1,\alpha} \cdot S_{i,\alpha} - \frac{1}{4} n_{i+1,\alpha} n_{i\alpha} \right)$$

$$- t_{\text{rung}} \sum_i \left( c_{i1}^\dagger c_{i2} + \text{h.c.} \right) - t_{\text{leg}} \sum_{i,\alpha} \left( c_{i+1,\alpha}^\dagger c_{i\alpha} + \text{h.c.} \right)$$

(1)

where $c_{i\alpha}$ are projected hole operators (spin indices are omitted) and $\alpha = (1, 2)$ labels the two legs of the ladder. We will consider the isotropic case in which $J_{\text{rung}} = J_{\text{leg}} = J$ and $t_{\text{rung}} = t_{\text{leg}} = t$ and our calculations will be carried out for a periodic $2 \times 12$ ladder at 1/8 and 1/6 hole doping.

As previously discussed, a gap function can be extracted by combining Lanczos results for the usual one-electron Green’s function $G(k, \omega)$ with the Fourier transform of Gorkov’s off-diagonal Green’s function

$$F(k; t) = i \langle T c_{-k,-\sigma}(t/2)c_{k,\sigma}(-t/2) \rangle$$

(2)

Here, for a finite system, this expectation value is taken between the ground states for $N$ and $N - 2$ particles and we choose the phase of $F$ to be zero. We will take $N = 22$, $N - 2 = 20$ corresponding to an average filling $n \simeq \frac{21}{24} = 0.875$. The Dyson equations \[7\] relating $G$ and $F$ are

$$[Z(k, \omega) \omega - (\epsilon_k + X(k, \omega))] G(k, \omega) = 1 - \phi(k, \omega) F(k, \omega)$$

(3)

$$[Z(-k, -\omega) \omega - (\epsilon_k + X(-k, -\omega))] F(k, \omega) = -\phi(k, \omega) G(k, \omega)$$

(4)
with $Z$ and $X$ the usual Nambu self energies and $\phi(k, \omega)$ the gap function. Then, using the even ($k \to -k$) parity of $Z(k, \omega)$, $X(k, \omega)$, $\epsilon_k$ and $\phi(k, \omega)$ along with the even $\omega$ dependence of $Z(k, \omega)$, $X(k, \omega)$ and $\phi(k, \omega)$, we have

$$\phi(k, \omega) = \frac{F(k, \omega)}{F^2(k, \omega) + G(k, \omega)G(k, -\omega)}.$$  \hfill (5)

Alternatively, $Z(k, \omega)$ can be eliminated to obtain an expression for the superconducting gap

$$\Delta(k, \omega) = \frac{\phi(k, \omega)}{Z(k, \omega)} = \frac{2\omega F(k, \omega)}{G(k, \omega) - G(k, -\omega)}. \hfill (6)$$

From a numerical calculation of $G(k, \omega)$ and $F(k, \omega)$, $\Delta(k, \omega)$ has been obtained for a 32-site, t-J cluster \cite{5}. This extended previous work by Ohta et. al \cite{8} who fit the spectral weight $\text{Im} F(k, \omega)$ to a $d_{x^2-y^2}$ BCS-Bogoliubov quasiparticle form in which the frequency dependence of the gap was neglected.

In contrast to the long range order of the superconducting ground state of a 2D lattice, a 2-leg ladder exhibits power law pair field correlations \cite{10, 11} which decay as $x^{-1/\kappa_\rho}$. Here, $\kappa_\rho$ is the Luttinger liquid parameter associated with the massless charge mode. This implies that for a ladder of length $L$, the off-diagonal Green’s function $F(k, \omega)$ decays \cite{12, 13, 14} as $(\xi/L)^{1/\kappa_\rho}$. Here, the coherence length $\xi$ is proportional to the inverse of the gap. For our doped ladder $L = 12$ is of order this coherence length so that we can probe the internal $k - \omega$ structure of a pair.

For $J/t = 0.4$ and 1/8 doping, the $k$-dependence of the zero frequency gap $\phi(k_x, k_y, \omega = 0)$ is plotted in Fig. 1 for the bonding ($k_y = 0$) and antibonding ($k_y = \pi$) bands. For comparison, the solid curves correspond to $\phi_0 (a \cos k_x - \cos k_y)$ with $\phi_0 = 0.3$ and $a = 0.8$. This d-wave-like $k$-dependence of the gap function is similar to the behavior found in previous studies of both the t-J \cite{8} and Hubbard ladders \cite{9}. It implies that the spatial structure of the pairing interaction is dominantly a near-neighbor interaction.

The frequency dependence of the real and imaginary parts of $\phi(k, \omega)$ are shown in Fig. 2 for $k$ values which are near the fermi surface of the bonding (red) and antibonding (green) bands respectively. In general, the gap function $\phi(k, \omega)$ is a complex frequency-dependent function $\phi_1 + i\phi_2$ with the imaginary part associated with dynamic decay processes. For our doped 2-leg ladder with $J/t = 0.4$, the magnitudes of the zero frequency gaps $\Delta_0 = \Delta(k_F, 0)$ obtained from eq. (6), for both the bonding and antibonding fermi points, are of order 0.15$t$. The gap, $\Delta_0$, is reduced from $\phi(k_F, 0)$ due to the renormalization factor $Z$. The onset of
the imaginary part of $\phi_2(k, \omega)$ seen in Fig. 2(b) appears to occur somewhat below $3\Delta_0$. In addition, the peaking in $\phi_2(k, \omega)$ and the rapid rise in $\phi_1(k, \omega)$ as this onset frequency is approached suggest that a particular excitation mode occurs at a frequency $\Omega$ such that $\omega = \Delta_0 + \Omega$ determines the onset of $\phi_2(k, \omega)$.

The occurrence of an onset peak in $\phi_2(k, \omega)$ at threshold and the short-range nature of the interaction involving scattering of pairs from the bonding to antibonding band ($q_y \sim \pi$), imply that the pairing in this energy regime is mediated by an $S = 1$ channel. This follows from the form of the coherence factor which varies as $\frac{1}{2} \left( 1 \pm \frac{\Delta(p+q,\omega)\Delta(p,\omega)}{E(p+q)E(p)} \right)$, with the plus sign associated with the charge channel and the minus sign with the $S = 1$ spin channel. For a “d-wave-like” gap with $q = (k_F(\text{bonding}) - k_F(\text{antibonding}), \pi)$ and $\omega$ near threshold, the coherence factor goes to 1 for the spin channel and vanishes for the charge channel.

A plot of the low-energy $S = 1$ excitations for a doped ladder is shown in Fig. 3(a). The solid diamonds show a collective $S = 1$ bound magnon-hole-pair [15, 16] mode and the open symbols the $S = 1$ particle-hole continuum. A measure of the spectral weight of the $S = 1$ channel which couples to $\phi(k, \omega)$ is given by the d-wave projection of the spin fluctuation spectral weight

$$V_d(\omega) = \frac{1}{N^2} \sum_{k, k'} (\cos k_x - \cos k_y) (\cos k'_x - \cos k'_y) S(k - k', \omega)$$

(7)

with

$$S(q, \omega) = -\frac{1}{\pi} \text{Im} \left\langle S^\alpha_{-q} \frac{1}{\omega + E_0 + i\eta - H} S^\alpha_q \right\rangle.$$  

(8)

Here $S^\alpha_q$ is the Fourier transform of the $\alpha$ spin component. A plot of $V_d(\omega)$ versus $\omega$ is shown in Fig. 3(b). The peak at $\Omega \approx 0.15t$ arises from the bound magnon-hole-pair mode and the high frequency weight comes from the particle-hole continuum. We believe that the onset of $\phi_2(k, \omega)$ seen in Fig. 2 is due to a process in which a single particle excitation with energy $\omega = \Omega_m + \Delta_0$ emits a bound magnon-hole-pair and drops down to the gap edge while the remaining $\phi_2(k, \omega) \neq 0$ region arises from a coupling to the $S = 1$ continuum.

In order to further characterize the dynamic nature of the pairing interaction, we introduce

$$I(k, \Omega) = \frac{1}{\pi} \int_0^\Omega d\omega' \frac{\phi_2(k, \omega')}{\omega'}.$$  

(9)

Then, since the gap function satisfies a dispersion relation, one has

$$\phi_1(k, 0) = I(k, \Omega \rightarrow \infty) + \phi_{\text{static}}(k).$$

(10)
Here, $\phi_{\text{static}}(k)$ represents a non-retarded contribution. For example, if one were to make a mean-field approximation in which

$$J \left( \mathbf{S}_i \cdot \mathbf{S}_j - \frac{1}{4} n_i n_j \right) \rightarrow -J \left( \langle \Delta_{i\delta} \rangle \Delta_{i\delta}^\dagger + h.c. \right)$$

one would have an effective attractive pairing interaction which is independent of frequency which would contribute to $\phi_{\text{static}}$. In Fig. 4(a), we have plotted $I(k, \Omega) / \phi_1(k, 0)$ versus $\Omega$ and one sees that for $J/t = 0.4$, $I(k, \Omega)$ saturates at over 80% of $\phi_1(k, 0)$. This implies that the dominant part of the pairing interaction comes from dynamic processes. Also shown, Fig. 4(b), are similar results for $J/t=0.8$. In this, unphysically large, $J/t$ regime only 60% of the $\omega = 0$ gap function is associated with a dynamic pairing interaction. This is similar to earlier results for the spatial structure of a pair \cite{17, 18} in which near-neighbor sites for the two holes making up a pair were favored for large values of $J/t$. For physical values of $J/t$, next-near-neighbor (diagonal) hole-hole occupation was favored. The diagonal structure arises dynamically and reflects the retarded nature of the pairing interaction for physical values of $J/t$.

In summary, knowledge of the $k$- and $\omega$-dependence of the gap function provides information on the spatial and temporal structure of the pairing interaction. Numerical solutions for the two-leg ladder show that the pairing interaction has a short-range, near-neighbor form, leading to momentum transfer processes which scatter pairs between the bonding and antibonding states. The d-wave-like momentum dependence of the gap function and the $\omega$ onset of $\phi_2(k, \omega)$ imply that the pairing interaction in this energy regime is mediated by the $S = 1$ channel. This channel contains both a bound magnon-pair state and a continuum of particle-hole excitations. We believe that the magnon-pair mode is responsible for the onset behavior seen in $\phi_2(k, \omega)$. The dispersion relation for $\phi_1(k, 0)$ shows that for physical values of $J/t$, the dominant part of the interaction is dynamic with contributions coming from both the magnon-hole-pair mode and the particle-hole continuum.
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FIG. 1: The $k_x$ dependence of $\phi_1(k_x,k_y,\omega = 0)$ for the bonding $k_y = 0$ (red) and antibonding $k_y = \pi$ (green) bands. For comparison, the solid curves show $\phi_0(a \cos k_x - \cos k_y)$ with $\phi_0 = 0.3$ and $a = 0.8$.

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FIG. 2: Real (a) and imaginary (b) parts of the gap function $\phi(k, \omega)$ versus $\omega$ for a $2 \times 12$ ladder at values of $k$ near the bonding (red) and antibonding (green) fermi momenta [$\sim (\frac{2\pi}{3}, 0)$ and $\sim (\frac{\pi}{3}, \pi)$ respectively] for $J/t = 0.4$. Here, $\Delta_0 \sim \Delta(k_F, \Delta_0)$ and $\Omega_m$ is the minimum energy for a bound magnon-hole-pair excitation.
FIG. 3: (a) Low-energy electron-hole $S = 1$ excitations for a $1/8$-doped $J/t = 0.4$ ladder for $q_y = 0$ (circles) and $q_y = \pi$ (diamonds) momenta. The solid diamonds denote a bound magnon-hole-pair collective mode \[6\]. (b) “d-wave” projection of the spin fluctuation spectral weight with $\Omega_m$ the minimum energy of the bound magnon-hole-pair collective mode (1/6 doping).

FIG. 4: The integrated contribution of $\phi_2(k, \omega)/\omega$ to $\phi_1(k, 0)$ for the bonding (red), antibonding (green) $k$-values shown in Fig. 2. (a) for $J/t = 0.4$ and (b) for $J/t = 0.8$. The non-retarded contributions are shown by arrows.