Optimal performance of a quantum Otto refrigerator

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Abstract – We consider a quantum Otto refrigerator cycle of a time-dependent harmonic oscillator. We investigate the coefficient of performance at maximum figure of merit for adiabatic and nonadiabatic frequency modulations. We obtain analytical expressions for the optimal performance both in the high-temperature (classical) regime and in the low-temperature (quantum) limit. We moreover analyze the breakdown of the cooling cycle for strongly nonadiabatic driving protocols and derive analytical estimates for the minimal driving time allowed for cooling.

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Introduction. – Heat engines and refrigerators are two prominent examples of thermal machines. While heat engines produce work by absorbing heat from a hot reservoir, refrigerators consume work to extract heat from a cold reservoir [1]. In a sense, refrigerators thus appear as heat engines functioning in reverse. In equilibrium thermodynamics, this symmetry is reflected in their maximum efficiency (traditionally called “coefficient of performance” for refrigerators [1]): The maximum efficiency of any heat engine operating cyclically between two thermal reservoirs at temperatures $T_1$ and $T_2$ ($T_1 < T_2$) is given by the Carnot formula, $\eta = (T_2 - T_1)/T_2$. On the other hand, the corresponding maximum performance of any refrigerator is $\epsilon_c = T_1/(T_2 - T_1)$ [1]. However, this analogy between the two devices is only superficial and their finite-time behavior is radically different.

Thermodynamics has lately been extended into two different directions: finite-time thermodynamics [2,3] and quantum thermodynamics [4,5]. The equilibrium Carnot expressions $\eta$ and $\epsilon_c$ are only of limited practical importance since they may only be reached for machines that run infinitely slowly. In this quasistatic limit, the power output of a device vanishes. The efficiency at maximum power $\eta^*$ of a heat engine, and the corresponding coefficient of performance at maximum figure of merit $\epsilon^*$ of a refrigerator, are two quantities of greater significance for real machines that operate in finite time. Their investigation via the optimization of nonequilibrium processes beyond the quasistatic limit is one of the main goals of finite-time thermodynamics [2,3]. A central result for the efficiency at maximum power is the so-called Curzon-Ahlborn efficiency, $\eta_{\text{ca}} = 1 - \sqrt{1 - \eta}$, obtained by Reitlinger [6], Chambadal [7], Yvon [8], Novikov [9] and Curzon-Ahlborn [10] (see refs. [11,12] for a history of the formula).

The finite-time analysis of the performance of refrigerators is more involved than that of heat engines. It has for this reason only been completed much later [13–15]. One of the main difficulties is the identification of a proper optimization criterion: the maximization of the cooling power or the minimization of the power input, for instance, do not result in a temperature-dependent bound for the coefficient of performance, in contrast to what is known for heat engines [15]. The finite-time counterpart of the Curzon-Ahlborn efficiency for refrigerators was first obtained by Yan [13] and Yan and Chen [14] in the late 1980s and rediscovered by Velasco and coworkers in 1997 [15] (see also ref. [16]). It is given by $\epsilon_{\text{yan}} = \sqrt{1 + \epsilon_c} - 1$ with an optimization criterion (the figure of merit) that is taken as the product of the coefficient of performance $\epsilon$ and the cooling power of the refrigerator. It has recently been shown that this optimization criterion for refrigerators is indeed in true correspondence to the maximum power criterion for heat engines [17]. It also reflects the complementarity between the extracted heat and the coefficient of performance [16]. Additional results for the coefficient of performance at maximum figure of merit have been reported for quantum refrigerator models in [18,19].

Starting with the work of Scovil and Schulz-DuBois on maser heat engines [20], refrigerators [21] and heat pumps [22], the investigation of thermal machines has been extended to the low-temperature, quantum regime [23–29]. A standard model of a quantum heat engine consists of a single harmonic oscillator that is alternately connected to a hot and a cold reservoir [24,25]. Two important cases are usually
distinguished [29]: the adiabatic limit which corresponds to slow expansion/compression phases (compared to the free dynamics of the system) and the sudden limit of fast expansion/compression. In both instances, the finite-time performance of the machine can be studied analytically. A detailed investigation of the efficiency at maximum power of a quantum harmonic heat engine in these two limits may be found in ref. [29] (and references therein) in the high-temperature (classical) regime. As expected, the efficiency at maximum power $\eta^*$ coincides with the classical Curzon-Ahlborn expression for adiabatic driving. However, $\eta^*$ is much lower for sudden driving and is found to be bounded from above by one half instead of one. These results have been generalized to the low-temperature (quantum) domain in ref. [30] and were shown to depend explicitly on the Planck constant $\hbar$ (see also ref. [31]). However, to our knowledge, for refrigerators only the high-temperature (classical) expression of $\epsilon^*$ in the adiabatic limit has been determined so far. Analytical results are missing for the important low-temperature (quantum) domain, both for adiabatic and fast driving, as well as for fast driving in the classical regime.

In this paper, we consider a quantum Otto refrigerator cycle for a time-dependent harmonic oscillator, a paradigmatic model for a quantum thermal machine [32–36]. We first derive an exact formula for its coefficient of performance that is valid for any frequency driving and reservoir temperatures. We use this equation to evaluate the optimal coefficient of performance both in the high-temperature (classical) and low-temperature (quantum) regime, for slow and fast frequency modulations. We further show that the cooling cycle breaks down for strongly nonadiabatic driving and present an analytical estimate for the minimal driving time permitted for cooling.

Quantum Otto refrigerator. — The four branches of the cycle of a quantum Otto refrigerator for a harmonic oscillator are (see fig. 1): 1) Compression $A \rightarrow B$: the oscillator is isolated from the reservoirs and its frequency is unitarily increased from $\omega_1$ to $\omega_2$. Work is added to the system during this process whereas the von Neumann entropy is constant. 2) Hot isochore $B \rightarrow C$: the frequency is kept fixed while the oscillator thermalizes to state $C$ at the inverse temperature $\beta_2 = (k_B T_2)^{-1}$ of the hot reservoir. 3) Expansion $C \rightarrow D$: the frequency is unitarily decreased back to its initial value at constant von Neumann entropy. 4) Cold isochore $D \rightarrow A$: the system is brought back to the initial thermal state $A$ by coupling it, at fixed frequency, to the cold reservoir with inverse temperature $\beta_1 = (k_B T_1)^{-1}$. Expansion/compression need not be infinitely slow as assumed in equilibrium thermodynamics.

The analysis of the performance of the refrigerator requires the evaluation of the mean energies of the quantum oscillator at the four corners of the cycle:

$$\langle H \rangle_A = \frac{\hbar \omega_1}{2} \coth \left( \frac{\beta_1 \hbar \omega_1}{2} \right), \quad (1a)$$

$$\langle H \rangle_B = \frac{\hbar \omega_2}{2} Q_1^* \coth \left( \frac{\beta_1 \hbar \omega_1}{2} \right), \quad (1b)$$

$$\langle H \rangle_C = \frac{\hbar \omega_2}{2} \coth \left( \frac{\beta_2 \hbar \omega_2}{2} \right), \quad (1c)$$

$$\langle H \rangle_D = \frac{\hbar \omega_2}{2} Q_2^* \coth \left( \frac{\beta_2 \hbar \omega_2}{2} \right), \quad (1d)$$

The mean energies at points $A$ and $C$ are the standard equilibrium expressions for a thermal harmonic oscillator. The mean energies at points $B$ and $D$ are obtained by solving the Schrödinger equation for the time-dependent oscillator, following the method introduced by Husimi [37]. We have here introduced the dimensionless quantity $Q_{1,2}$ which depends on the speed of the frequency driving [37]. It is equal to $Q_{1,2}^* = 1$ for quasistatic frequency modulation and to $Q_{1,2}^* = (\omega_1^2 + \omega_2^2)/2(\omega_1 \omega_2)$ for a sudden frequency switch. The explicit expression of $Q_{1,2}$ for arbitrary frequency modulation can be found in refs. [38,39].

The mean work done on the system during the first and third strokes of the cycle is given by

$$\langle W_1 \rangle = \langle H \rangle_B - \langle H \rangle_A = \left( \frac{\hbar \omega_2}{2} Q_1^* - \frac{\hbar \omega_1}{2} \right) \coth \left( \frac{\beta_1 \hbar \omega_1}{2} \right), \quad (2)$$

and

$$\langle W_3 \rangle = \langle H \rangle_D - \langle H \rangle_C = \left( \frac{\hbar \omega_1}{2} Q_2^* - \frac{\hbar \omega_2}{2} \right) \coth \left( \frac{\beta_2 \hbar \omega_2}{2} \right). \quad (3)$$

On the other hand, the mean heat extracted during the fourth stroke, $\langle Q_4 \rangle = \langle H \rangle_A - \langle H \rangle_D$ (cooling part) reads

$$\langle Q_4 \rangle = \frac{\hbar \omega_1}{2} \left[ \coth \left( \frac{\beta_1 \hbar \omega_1}{2} \right) - Q_2^* \coth \left( \frac{\beta_2 \hbar \omega_2}{2} \right) \right]. \quad (4)$$

For a refrigerator, heat is absorbed from the cold reservoir, $\langle Q_4 \rangle \geq 0$ and flows into the hot reservoir, $\langle Q_2 \rangle \leq 0$. 60002-p2
The condition for cooling is thus that \( \omega_2/\omega_1 > \beta_1/\beta_2 \), since \( Q_2 \geq 1 \) for any driving protocol [37–39].

The coefficient of performance \( \epsilon \) of the Otto refrigerator, defined as the ratio of the heat removed from the cold reservoir \( \langle Q_4 \rangle \), to the total amount of work done per cycle, \( \langle W \rangle = (W_1 + W_3) \), follows as

\[
\epsilon = \frac{\langle Q_4 \rangle}{\langle W_1 \rangle + \langle W_3 \rangle} = \frac{\omega_1 [c(\beta_1 h\omega_1/2) - Q_2 c(\beta_2 h\omega_2/2)]}{(\omega_2 Q_1 - \omega_1) c(\beta_1 h\omega_1/2) - (\omega_2 - \omega_1) Q_2 c(\beta_2 h\omega_2/2)},
\]

(5)

where we have defined the function \( c(x) = \coth(x) \). This quantum expression is exact and valid for any frequency modulation. In the case of adiabatic processes, \( Q_1 = 1 \), the coefficient of performance reduces to [18,19,29],

\[
\epsilon_{ad} = \frac{\omega_1}{\omega_2 - \omega_1}.
\]

(6)

The above equation is positive provided that \( \omega_2 > \omega_1 \) and is always smaller than the Carnot expression, \( \epsilon < \epsilon_c \).

**Performance at maximum figure of merit.** – In order to optimize the performance of the quantum refrigerator, we introduce the figure of merit \( \chi \) defined as the product of the heat absorbed from the cold reservoir \( \langle Q_4 \rangle \), eq. (4), and the coefficient of performance \( \epsilon \), eq. (5), over the duration of a thermodynamic cycle \( t_{cycle} \) [13–17],

\[
\chi = \epsilon \langle Q_4 \rangle / t_{cycle}.
\]

(7)

The meaning of \( \chi \) may be understood by noting that the corresponding expression for a heat engine, \( \chi_{engine} = \eta \langle Q_2 \rangle / t_{cycle} = -\langle W \rangle / t_{cycle} \), is equal to its power output \( (\langle Q_2 \rangle \) being in this case the heat absorbed from the hot reservoir) [17]). Optimizing \( \chi \) is thus equivalent to optimizing the power for an engine. However, in contrast to heat engines, \( \chi \) depends quadratically on the heat, making the analysis more difficult. We next compute the optimal coefficient of performance \( \epsilon^* \) for slow and fast frequency transformations, both in the classical and quantum limits.

**Adiabatic frequency modulation.** We begin by considering quasistatic expansion/compression characterized by \( Q_{1,2} = 1 \). We assume that the hot reservoir obeys the (classical) high-temperature condition, \( \beta_2 h\omega_2 \ll 1 \), a situation which is relevant for many cooling scenarios. The figure of merit (7) then simplifies to

\[
\chi_{ad} = \left( \frac{\omega_1}{\omega_2 - \omega_1} \right) \left( \frac{h\omega_2}{2} \coth \left( \frac{\beta_1 h\omega_1}{2} \right) - \frac{\omega_1}{\beta_2 h\omega_2} \right) / t_{cycle}.
\]

(8)

This expression holds for arbitrary cold-reservoir temperatures. We optimize eq. (8) with respect to the final frequency \( \omega_2 \), assuming, as commonly done, that all other parameters such as temperatures, cycle time and initial frequency \( \omega_1 \) are fixed. By solving the equation \( \partial \chi_{ad} / \partial \omega_2 = 0 \), we find that the maximum figure of merit is obtained when the frequencies satisfy \( \omega_1 / \omega_2 = 1 - \sqrt{1 - \tau} \), where \( \tau = \beta_2 h\omega_1 \coth(\beta_1 h\omega_1/2)/2 \). The coefficient of performance at maximum \( \chi \) in the adiabatic limit follows as

\[
\epsilon_{ad}^* = \frac{1}{\sqrt{1 - \tau}} - 1.
\]

(9)

In the high-temperature regime, \( \beta_2 h\omega_1 \ll 1 \), we have \( \tau = \tau_q = \beta_2 / \beta_1 \) and we recover the known coefficient of performance at maximum figure of merit, \( \epsilon_{ad}^* = \sqrt{1 + \epsilon_c} = 1 \), of a classical refrigerator as derived in refs. [13–15].

The other hand, in the quantum limit where the cold reservoir is at low temperature, \( \beta_1 h\omega_1 \gg 1 \), we may use the expansion \( \coth(\beta_1 h\omega_1/2) \approx 1 + 2 \exp(-\beta_1 h\omega_1) \) in eq. (9) to write \( \tau = \tau_q \approx \beta_2 h\omega_1/2 + \beta_2 h\omega_1 \exp(-\beta_1 h\omega_1) \). The expression \( \epsilon_{ad}^* = 1/\sqrt{1 - \tau_q} = 1 \) is the semi-quantum generalization of the Yan-Chen optimal coefficient of performance. The parameter \( \tau \) has a simple physical interpretation as the ratio of the mean energies of the harmonic oscillator coupled, respectively, to the cold and hot reservoirs.

**Nonadiabatic frequency modulation.** For fast expansion/compression the parameter \( Q_{1,2} > 1 \). We assume as before that the hot reservoir is in the (classical) high-temperature regime \( \beta_2 h\omega_2 \ll 1 \). We first treat the case of weakly nonadiabatic driving \( (Q_{1,2}^* \ll 1) \) and defer the discussion of strongly nonadiabatic frequency modulation \( Q_{1,2}^* \gg 1 \) to the next section. In the limit of moderately fast evolution, we may linearize an arbitrary frequency driving protocol as

\[
\omega_1 = \omega_1 + (\omega_2 - \omega_1)\tau_0 = \omega_1 + \alpha \tau_0,
\]

(10)

with \( \alpha = (\omega_2 - \omega_1)/\tau_0 \ll 1 \). The driving time \( \tau_0 \) is here large, but finite, while it is infinitely large in the quasistatic limit. The parameter \( Q_{1,2}^* \) may be calculated using the formulas in [37–39] and expressed in terms of parabolic cylinder functions. We obtain, to lowest order in \( \alpha \),

\[
Q_{1,2}^* = 1 + y,
\]

(11)

with \( y = \alpha^2/(8\omega_2^2) \). The figure of merit (7) takes accordingly the form (for \( \beta_2 h\omega_2 \ll 1 \))

\[
\chi_{na} = \frac{h^2}{\omega_2^2} \left[ \frac{1}{\omega_2^2} (1 + y) - 1 \right] \left[ \frac{1}{\omega_2^2} (1 + y) + 1 \right] \left[ \omega_2^2 (1 + y) - 1 \right]^{-1} / t_{cycle}.
\]

(12)

Since the nonadiabatic correction \( y \) is small, its \( \omega_2 \)-dependence may be neglected for the optimization. Maximizing eq. (12) with respect to \( \omega_2 \) and keeping all other parameters constant as before, we obtain (with \( y \neq 0 \),

\[
\frac{(\omega_1/\omega_2)^3}{2 + \tau} - \frac{(\omega_1/\omega_2)^2}{1 + y} + 3\tau \frac{(\omega_1/\omega_2)}{1 + y} = \frac{\tau^2}{1 + y} = 0.
\]

(13)

Equation (13) is of third order in \( \omega_2^3 \) with two complex solutions (which are not relevant for the optimization
The Carnot coefficient of performance \( \epsilon \) is a function of the temperature ratio \( \tau_{cl} = T_1/T_2 \). The blue (solid) line shows the adiabatic case (9), while the red (dashed) line and the green (dash-dotted) lines show, respectively, the exact numerical result and the approximation (14) for the nonadiabatic frequency modulation for \( y = 0.01 \). The black (dotted) line is the Carnot coefficient of performance \( \epsilon_c = (\tau^{-1}_{cl} - 1)^{-1} \).

Equation (14) indicates that the nonadiabatic optimal coefficient of performance is always smaller than the adiabatic one, as expected. The respective classical and quantum limits of \( \epsilon_{na}^\ast \) are obtained replacing \( \tau \) by \( \tau_{cl} \) and \( \tau_q \).

Figure 2 shows the adiabatic optimal coefficient of performance \( \epsilon^{\ast cl} \) (blue solid line) in the classical regime, as well as the nonadiabatic result obtained from the exact solution of eq. (13) (red dashed line) and the approximation (14) to lowest order in \( y \) (green dot-dashed line). We observe that adiabatic and nonadiabatic coefficients of performance almost coincide for large temperature differences (small \( \tau_{cl} \)). However, they deviate significantly for small temperature differences (large \( \tau_{cl} \)). Both are always smaller than the maximum Carnot formula (black dotted line). Figure 3 displays the corresponding results for the quantum optimal coefficient of performance \( \epsilon^{\ast q} \) as a function of \( \tau_q \), with a similar behavior. Quite generically, the amount of heat \( \langle Q_4 \rangle \) that is extracted from the cold reservoir decreases with increasing temperature difference, or equivalently, decreasing cold temperature \( T_1 \) for fixed \( T_2 \). These results show that the optimum operation of a refrigerator is in the adiabatic regime, when the driving time is much longer than any characteristic times of the machine (here given by the oscillation period). Equation (14) allows to evaluate the optimal coefficient of performance when the adiabatic condition cannot be satisfied.

**Strongly nonadiabatic driving.** – The parameter \( Q_{1,2}^\ast \) increases with the degree of nonadiabaticity of the frequency protocols. Equation (4) reveals that the heat flow from the cold reservoir changes sign when \( Q_2^\ast \) is larger than a given threshold. In this regime, the refrigerator stops cooling: the device consumes work to pump heat into both reservoirs, as noted in ref. [40]. The physical origin of the breakdown of the cooling cycle is readily identified. Fast expansion/compression lead to nonadiabatic excitations of the harmonic oscillator that increase their mean energy. This extra energy is absorbed by the reservoirs during the thermalization phases. If the amount of energy induced by nonadiabatic transitions during the expansion step 3) exceeds the heat that is adiabatically extracted from the cold reservoir by the machine, the direction of the heat flow is reversed. This general mechanism explains why the cooling performance of the refrigerator is reduced by nonadiabatic frequency modulation, until it comes to an end. The maximal value of the parameter \( Q_2^\ast \) that is allowed before heat reversal occurs is given by

\[
Q_2^\ast = \frac{\coth(\beta_1\hbar \omega_1/2)}{\coth(\beta_2\hbar \omega_2/2)}. \tag{15}
\]

Determining the corresponding minimal driving time \( t_0^\ast \) is a difficult task as i) the latter depends on the particular frequency protocol and ii) \( Q_2^\ast \) is a nontrivial function of \( t_0^\ast \). In order to get an analytical estimate, we use the linear approximation (11) and find

\[
t_0^\ast = \sqrt{\frac{(\omega_2 - \omega_1)^2 \coth(\beta_2\hbar \omega_2/2)}{8\omega_2^4 (\coth(\beta_1\hbar \omega_1/2) - \coth(\beta_2\hbar \omega_2/2))}}. \tag{16}
\]

Strictly speaking, the use of the linear approximation (11) cannot be justified for strongly nonadiabatic driving. However, the estimate (16) is able to reproduce the main features of the minimal driving time. Figure 4 shows the
exact critical driving time \( t_0^* \) numerically obtained from eq. (15) (circles and dots) and the analytical estimate (16) (dot-dashed and dashed lines) as a function of the cold-reservoir temperature \( T_1 \), for the linear driving (10). We observe good qualitative agreement both for high and low temperatures\(^1\). The critical time \( t_0^* \) increases exponentially for decreasing \( T_1 \) and diverges when the condition \( \omega_1/\omega_2 = T_1/T_2 \) is verified. This equality corresponds to the maximum Carnot coefficient of performance. Consequently, strongly nonadiabatic expansion/compression are only possible for high \( T_1 \) (small \( t_0^* \)). For decreasing temperature \( T_1 \) (larger \( t_0^* \)), permitted cooling protocols get more and more adiabatic, confirming the behavior seen in figs. 2 and 3 at small \( \tau_{d,q} \). This phenomenon may be understood by noting that the heat extracted from the cold reservoir strongly decreases with decreasing temperature \( T_1 \), thus only allowing close-to-adiabatic processes to ensure positive heat extraction. It is the same mechanism that leads to strictly adiabatic processes in the limit of vanishing temperatures, as stated by the third law of thermodynamics [1].

\( T_1^* \geq \frac{\hbar\omega_1}{2k_B \coth \left[ (1 + y) \coth (\beta_2 \hbar \omega_2/2) \right]} \),

where eq. (11) has been used. Expression (17) is similar to that obtained in ref. [17] with a different approach for an adiabatically driven refrigerator in the presence of external noise. Thus, both nonadiabatic driving and external noise sources limit the coldest temperature achievable.

Our findings should be helpful for the design of harmonic quantum refrigerators, from nanomechanical to ion trap systems [30,31,41]. One could, for example, envision a three-ion configuration in a segmented linear Paul trap [42], with each ion in a different axial trapping potential. The ions could be weakly coupled via dipole-dipole [43] or Coulomb [44] interaction. The center ion may then be driven as an Otto refrigerator between the two outer ions by modulating its frequency. Assuming that the right ion is laser cooled at a Doppler temperature of \( T_2 = 0.7 \) mK (a value recently reached for \({}^{40}\text{Ca}^+\) [45]) and adiabatic driving between \( \omega_1 = 1 \) MHz and \( \omega_2 = 6 \) MHz, the left ion may be cooled down to \( T_1^* = 0.12 \) mK, eq. (17), well below the Doppler limit of \( T_D = 0.56 \) mK. The corresponding mean phonon number is \( \bar{n}_1 = [\exp(\hbar\beta_1 \omega_1) - 1]^{-1} = 2 \) and the optimal coefficient of performance \( \varepsilon_{ad}^* = 0.1 \), eq. (9), is close to the Carnot value of \( \varepsilon_c = 0.2 \).

**Conclusions.** We have studied the finite-time performance of a quantum Otto refrigerator for a time-dependent harmonic oscillator. We have derived analytical expressions for the coefficient of performance at maximum figure of merit for slow and fast expansion/compression, both in the high-temperature (classical) regime and in the low-temperature (quantum) limit. We have further discussed the breakdown of the cooling cycle for strongly nonadiabatic driving protocols and obtained analytical estimates for the minimal driving time allowed.

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Appendix

We here show that the first-order nonadiabatic expansion of the parameter $Q_{1,2}$ given in eq. (11) also yields a good estimate of the critical driving time $t_0^*$ for nonlinear driving. We consider the frequency protocol,

$$\omega_t^2 = \omega_0^2 + (\omega_0^2 - \omega_1^2)t/t_0 = \omega_0^2 + \alpha t \quad (A.1)$$

with $\alpha = (\omega_0^2 - \omega_1^2)/t_0$. The parameter $Q_{1,2}^*$ may be expressed in terms of Airy functions [38,39]. Equation (11) then reads $Q_{1,2}^* = 1 + \tilde{y}$ with $\tilde{y} = \alpha^2/(32\omega_0^2)$ and the critical driving time is found to be

$$t_0^* = \sqrt{\frac{(\omega_0^2 - \omega_1^2)^2}{32\omega_0^2(\coth(\beta_2\omega_0^2/2) - \coth(\beta_2\omega_1^2/2))}} \quad (A.2)$$

Figure 5 shows the exact critical driving time $t_0^*$ numerically obtained from eq. (15) (circles and dots) and the analytical estimate (A.1) (dot-dashed and dashed lines) as a function of $T_1$, for the nonlinear driving (A.1). We observe good qualitative agreement both for high and low temperatures as in fig. 4 for the linear driving (10).

REFERENCES

[1] Callen H. B., Thermodynamics and an Introduction to Thermostatistics (Wiley, New York) 1985.
[2] Andrensen B., Salamon P. and Berry R. S., Phys. Today, 37, issue No. 9 (1984) 62.
[3] Andrensen B., Angew. Chem. Int. Ed., 50 (2011) 2690.
[4] Gemmer J., Michel M. and Mahler G., Quantum Thermodynamics (Springer, Berlin) 2009.
[5] Kosloff R. and Levy A., Annu. Rev. Phys. Chem., 65 (2014) 365.
[6] Reitlinger H. B., Sur l’Utilisation de la Chaleur dans les Machines à Fuir (Vaillant-Carrmaone, Liège) 1929.
[7] Chamblad P., Thermodynamique de la Turbine à Gaz (Hermann, Paris) 1949.
[8] Yvon J., Proceedings of the International Conference on the Peaceful Uses of Atomic Energy, Vol. 2 (United Nations, Geneva) 1955, p. 337.
[9] Novikov I. I., At. Energy, 3 (1957) 409; J. Nucl. Energy II, 7 (1958) 125.
[10] Curzon F. L. and Ahlborn B., Am. J. Phys., 43 (1975) 22.
[11] Vaudrey A., Lanzetta F. and Feidt M., J. Non-Equilib. Thermodyn., 39 (2014) 199.
[12] Ouerdane H., Arpetit Y., Goupil C. and Lecocue P., Eur. Phys. J. ST, 224 (2015) 839.
[13] Yan Z., Cryogenics, 3 (1984) 17.
[14] Yan Z. and Chen J., J. Phys. D: Appl. Phys., 23 (1990) 136.
[15] Velasco S., Roco J. M. M., Medina A. and Hernández A. C., Phys. Rev. Lett., 78 (1997) 3241.
[16] Allahverdyan A. E., Hovhannisyan K. and Mahler G., Phys. Rev. E, 81 (2010) 051129.
[17] De Tomás C., Hernández A. C. and Roco J. M. M., Phys. Rev. E, 85 (2012) 010104.
[18] Biliukov J., Jahnke T. and Mahler G., Eur. Phys. J. B, 64 (2008) 105.
[19] Yuan Y., Wang R., He J., Ma Y. and Wang J., Phys. Rev. E, 90 (2014) 052151.
[20] Scovil H. E. D. and Schulz-DuBois E. O., Phys. Rev. Lett., 2 (1959) 262.
[21] Geusic J. E., Schulz-DuBois E. O., De Grasse R. W. and Scovil H. E. D., J. Appl. Phys., 30 (1959) 1113.
[22] Geusic J. E., Schulz-DuBois E. O. and Scovil H. E. D., Phys. Rev., 156 (1967) 343.
[23] Alicki R., J. Phys. A: Math. Gen., 12 (1979) L103.
[24] Kosloff R., J. Chem. Phys., 80 (1984) 1625.
[25] He J., Chen J. and Hua B., Phys. Rev. E, 65 (2002) 036145.
[26] Kiefer D., Phys. Rev. Lett., 93 (2004) 140403.
[27] Tonner F. and Mahler G., Phys. Rev. E, 72 (2005) 066118.
[28] Allahverdyan A. E., Ramandee S. J. and Mahler G., Phys. Rev. E, 77 (2008) 041118.
[29] Rezek Y. and Kosloff R., New J. Phys., 8 (2006) 83.
[30] Abah O., Rossnagel J., Jakob G., Deffner S., Schmidt-Kaler F., Singer K. and Lutz E., Phys. Rev. Lett., 109 (2012) 203006.
[31] Zhang K., Bariani F. and Meyste P., Phys. Rev. Lett., 112 (2014) 150602.
[32] Lin B., Chen J. and Hua B., J. Phys. D, 36 (2003) 406.
[33] Lin B. and Chen J., Phys. Rev. E, 68 (2003) 056117.
[34] Sánchez-Salas N. and Hernandez A. C., Phys. Rev. E, 70 (2004) 046134.
[35] Rezek Y., Salamon P., Hoffmann K. H. and Kosloff R., EPL, 85 (2009) 30008.
[36] Torrontegui E. and Kosloff R., Phys. Rev. E, 88 (2013) 032103.
[37] Hisimi K., Prog. Theor. Phys., 9 (1953) 381.
[38] Deffner S. and Lutz E., Phys. Rev. E, 77 (2008) 021128.
[39] Deffner S., Abah O. and Lutz E., Chem. Phys., 375 (2010) 290.
[40] Rezek Y., Heat Machines and Quantum Systems: Towards the Third Law, PhD Thesis, Hebrew University of Jerusalem, Israel (2011).
[41] Dechant A., Kiesel N. and Lutz E., Phys. Rev. Lett., 114 (2015) 183602.
[42] Walther A., Ziesel F., Rustner T., Dawkins S. T., Ott K., Hettrich M., Singer K., Schmidt-Kaler F. and Poschinger U. G., Phys. Rev. Lett., 109 (2012) 080501.
[43] Harlander M., Lechner R., Brownnutt M., Blatt R. and Hänsel W., Nature, 471 (2011) 200.
[44] Brown K. R., Ospelekaus C., Colombi Y., Wilson A. C., Leibfried D. and Wineland D. J., Nature, 471 (2011) 197.
[45] Roßnagel J., Tolazzi K. N., Schmidt-Kaler F. and Singer K., New J. Phys., 17 (2015) 045004.