OPERATOR INTERPRETATION OF RESONANCES ARISING IN SPECTRAL PROBLEMS FOR $2 \times 2$ MATRIX HAMILTONIANS$^{*\dagger}$

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We consider the analytic continuation of the transfer function for a $2 \times 2$ matrix Hamiltonian into the unphysical sheets of the energy Riemann surface. We construct non-selfadjoint operators representing operator roots of the transfer function which reproduce certain parts of its spectrum including resonances situated in the unphysical sheets neighboring the physical sheet. On this basis, completeness and basis properties for the root vectors of the transfer function (including those for the resonances) are proved.

1. Introduction. In this work we deal with Hamiltonian of the $2 \times 2$ matrix form

$$\mathbf{H} = \begin{pmatrix} A_0 & B_{01} \\ B_{10} & A_1 \end{pmatrix}$$

It is assumed that the operator $\mathbf{H}$ acts in an orthogonal sum $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ of the Hilbert spaces $\mathcal{H}_0$ and $\mathcal{H}_1$ while the entries $A_0 : \mathcal{H}_0 \to \mathcal{H}_0$, and $A_1 : \mathcal{H}_1 \to \mathcal{H}_1$, are selfadjoint operators. The couplings $B_{ij} : \mathcal{H}_j \to \mathcal{H}_i$, $i \neq j$, $B_{01} = B_{10}^*$, are assumed to be bounded operators. We are especially interested in the physically typical case where the spectrum of, say, $A_1$ is partly or totally embedded into the absolutely continuous spectrum of $A_0$ and the transfer function $M_1(z) = A_1 - z + V_1(z)$, where $V_1(z) = B_{10}(z - A_0)^{-1}B_{01}$, admits analytic continuation (as an operator-valued function) through the absolutely continuous spectrum of the entry $A_0$ into the

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unphysical sheet(s) of the energy $z$ plane. Notice that the resolvent $(H - z)^{-1}$ of the operator $H$ can be expressed explicitly in terms of the inverse transfer function $M_1^{-1}(z)$. Therefore, in studying the spectral properties of the transfer function $M_1$ one studies at the same time the spectral properties of the operator matrix $H$.

We construct an operator-valued function $V(X)$ (see [9]) on the space of operators in $H_1$ possessing the property: $V_1(X)\psi_1 = V_1(z)\psi_1$ for any eigenvector $\psi_1$ of $X$ corresponding to an eigenvalue $z$ and then study the equation

$$H_1 = A_1 + V_1(H_1).$$

(1)

In the case where the spectra of $A_0$ and $A_1$ have no intersection, Eq. (1) is reduced (cf. [9]) to an operator Riccati equation for a similarity transform which allows to find invariant subspaces of the matrix $H$ admitting a graph representation [1, 2, 8] (also see [6]). Now we prove the solvability of Eq. (1) in the case where the spectra of $A_1$ and $A_0$ overlap, cf. also [8]. However in this case the solutions of (1) already represent non-selfadjoint operators.

The problem considered is closely related to the resonances generated by the matrix $H$. Regarding a definition of the resonance and history of the subject see, e.g., the book [11]. A recent survey of the literature on resonances can be found in [10]. Throughout the paper we treat resonances as the discrete spectrum of the transfer function $M_1$ situated in the unphysical sheets of its Riemann surface since points of this spectrum automatically correspond to the poles of the analytic continuation of the resolvent $(H - z)^{-1}$ (understood in the form sense). Using the fact that the root vectors of the solutions $H_1$ of the equation (1) are at the same time such vectors for $M_1$, we prove completeness and even basis properties for the root vectors including those for the resonances. The results obtained allow immediate applications in particular to the scattering problems for multichannel Schrödinger equation.

A detailed exposition of the material presented including proofs in the case of essentially more general spectral situation will be given in the extended paper [7].

2. Analytic continuation of the transfer function. For the sake of simplicity we assume in the report that all the spectrum $\sigma(A_0)$ of the entry $A_0$ is absolutely continuous consisting of an only interval $J^0 = (\mu_0^{(1)}, \mu_0^{(2)})$ with $-\infty \leq \mu_0^{(1)} < \mu_0^{(2)} \leq \infty$ while all the spectrum $\sigma(A_1)$ of the entry $A_1$ is totally embedded into the interval $J^0$. Therefore, we assume $\sigma(A_1) \subset J^0$.

Let $E_0$ be the spectral measure for the entry $A_0$, $A_0 = \int_{\sigma(A_0)} \lambda dE_0(\lambda)$. Then the function $V_1(z)$ can be written

$$V_1(z) = \int_{\sigma(A_0)} dK_B(\mu)(z - \mu)^{-1}$$

with

$$K_B(\mu) = B_{10}E_0(\mu)B_{01}$$
Lemma 1 The analytic continuation of the transfer function $M_1(z)$, $z \in \mathbb{C} \setminus \sigma(A_0)$, through the spectral interval $J^0$ into the subdomain $D(\Gamma^l) \subset D^l$, $l = \pm 1$, bounded by this interval and a $K_B$-bounded contour $\Gamma^l$ is given by

$$M_1(z, \Gamma^l) = A_1 - z + V_1(z, \Gamma^l) \quad \text{with} \quad V_1(z, \Gamma^l) = \int_{\Gamma^l} d\mu K_B'(\mu) (z - \mu)^{-1}. \quad (2)$$

For $z \in D^l \cap D(\Gamma^l)$ one has $M_1(z, \Gamma^l) = M_1(z) + 2\pi i \mu K_B(z)$.

Proof is reduced to the observation that the function $M_1(z, \Gamma^l)$ is holomorphic for $z \in \mathbb{C} \setminus \Gamma^l$ and coincides with $M_1(z)$ for $z \in \mathbb{C} \setminus D(\Gamma^l)$. The last equation representing $M_1(z, \Gamma^l)$ via $M_1(z)$ is obtained from (2) using the Residue Theorem.

The latest statement of the lemma shows that the transfer function $M_1$ has at least two-sheeted Riemann surface. The sheet of the complex plane $\mathbb{C}$ where the transfer function $M_1(z)$ is considered together with the resolvent $(\mathbf{H} - z)^{-1}$ initially is said to be the physical sheet. The remaining sheets of the Riemann surface of $M_1$ are said to be unphysical sheets. In the present work we only deal with the unphysical sheets connected through the interval $J^0$ immediately to the physical sheet.

3. Example. To explain our reasons to introduce the function $K_B'(\mu)$ we will briefly consider an example closely related to the multichannel Schrödinger operator. Let $\mathcal{H}_0 = L_2(\mathbb{R}^n)$, $n \geq 1$, and $A_0$ be the Laplacian, $A_0 = -\Delta$, defined on the

where $E^0(\mu)$ is the spectral function of $A_0$, $E^0(\mu) = E_0\left( (-\infty, \mu) \right)$. We suppose that the function $K_B(\mu)$ is differentiable in $\mu \in J^0$ in the operator norm topology. Further, we suppose that the derivative $K_B'(\mu)$ is continuous within the closed interval $J^0$ and, moreover, that it admits analytic continuation from this interval to a simply connected domain situated, say, in $\mathbb{C}^-$. Let this domain be called $D^-$. We assume that the boundary of $D^-$ includes the entire spectral interval $J^0$. Since $K_B'(\mu)$ represents a selfadjoint operator for $\mu \in J^0$ and $J^0 \subset \mathbb{R}$, the function $K_B'(\mu)$ also automatically admits analytic continuation from $J^0$ into the domain $D^+$, symmetric to $D^-$ with respect to the real axis and $|K_B'(\mu)|^* = K_B'(\mu)$, $\mu \in D^\pm$.

Let $\Gamma^l$, $l = \pm 1$, be a rectifiable Jordan curve in $D^l$ resulting from continuous deformation of the interval $J^0$, the (finite) end points of this interval being fixed. The quantity

$$\nu_0(B, \Gamma^l) = \int_{\Gamma^l} |d\mu| \|K_B'(\mu)\|$$

where $|d\mu|$ stands for the Lebesgue measure on $\Gamma^l$, is called variation of the function $K_B(\mu)$ along the contour $\Gamma^l$. We suppose that there exists a contour (contours) $\Gamma^l$ where the value $\nu_0(B, \Gamma^l)$ is finite including also the case of the unbounded interval $J^0$. The contours $\Gamma^l$ satisfying the condition $\nu_0(B, \Gamma^l) < \infty$ are said to be $K_B$-bounded contours.
Sobolev space $W^2_2(\mathbb{R}^n)$. We impose no restrictions on the Hilbert space $\mathcal{H}_1$ and on the selfadjoint entry $A_1$ except for the assumption that $\sigma(A_1) \subseteq \sigma(A_0) = [0, +\infty)$.

Let $\tilde{b} \in L^2(\mathbb{R}^n, \mathcal{H}_1)$ be an $\mathcal{H}_1$-valued function of the variable $x \in \mathbb{R}^n$ whose Fourier transform

$$b(p) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} dx \, \tilde{b}(x) \exp(i(p, x)), \quad p \in \mathbb{R}^n,$$

represents a continuous function of $p$ with respect to the norm topology in $\mathcal{H}_1$. We define the coupling $B_{10}$ as

$$B_{10}u_0 = \int_{\mathbb{R}^n} dx \, \tilde{b}(x)u_0(x), \quad u_0 \in \mathcal{H}_0.$$

Obviously, the operator $B_{10}$ is bounded and $\|B_{10}\| \leq \|\tilde{b}\|_{L^2(\mathbb{R}^n, \mathcal{H}_1)}$. The adjoint operator $B_{01}$ reads

$$(B_{01}u_1)(x) = \langle u_1, \tilde{b}(x) \rangle_{\mathcal{H}_1}, \quad u_1 \in \mathcal{H}_1.$$

Denote by $B_r$ the open ball in $\mathbb{R}^n$ centered at the origin and having radius $r$, $B_r = \{ p \in \mathbb{R}^n : |p| < r \}$. For $\mu > 0$ the value of the spectral function $E^0(\mu)$ of the Laplacian $-\Delta$ represents the integral operator in $L^2(\mathbb{R}^n)$ whose kernel reads as the Fourier transform of the characteristic function of the ball $B_{\sqrt{\mu}}$ (see, e.g., Ref. [3], §4.2 of Ch. 8),

$$E^0(\mu; x, x') = \frac{1}{(2\pi)^n} \int_{B_{\sqrt{\mu}}} dp \, \exp(i(p, x - x')).$$

If $\mu \leq 0$ then $E^0(\mu; x, x') = 0$.

A simple computation in the example considered shows that, for $\mu > 0$,

$$K_B'(\mu) = \frac{d}{d\mu} \left( B_{10} E^0(\mu) B_{01} \right) = \frac{1}{2} \mu^{(n-2)/2} \int_{S^{n-1}} d\hat{p} \, b(\mu^{1/2}\hat{p}) \langle \cdot, b(\mu^{1/2}\hat{p}) \rangle_{\mathcal{H}_1},$$

where $S^{n-1}$ stands for the unit sphere in $\mathbb{R}^n$. Thus, for the analytic continuability of $K'_B(\mu)$ into a domain $D \subset \mathbb{C}$ surrounding the spectrum of the entry $A_1$ it suffices to require the analytic continuability of the function $b(p)$ into an appropriate domain of $\mathbb{C}^n$. In particular, if $b(p)$ admits an analytic continuation into a “strip” $|\text{Im} \, p| < a$ for some $a > 0$, then the function $K'_B(\mu)$ is holomorphic in $\mu$ in the parabolic domain $\text{Re} \, \mu > -a^2 + \frac{1}{4a^2}(\text{Im} \, \mu)^2$ cut along the interval $(-a^2, 0]$. Notice that the holomorphy of $b(p)$ in a strip corresponds to the case of an exponentially decreasing $\tilde{b}(x)$ as $x \to \infty$ which is often encountered in applications.

**4. Factorization theorem.** Let the spectrum of a linear operator $Y : \mathcal{H}_1 \to \mathcal{H}_1$ is separated from a $K_B$-bounded contour $\Gamma$. Then one can define the operator

$$V_1(Y, \Gamma) = \int_{\Gamma} d\mu \, K_B'(\mu)(Y - \mu)^{-1}.$$
Obviously, this operator is bounded,
\[ \|V_1(Y, \Gamma)\| \leq \mathcal{V}_0(B, \Gamma) \sup_{\mu \in \Gamma} \|Y - \mu\|^{-1}. \]

In what follows we consider the equation (cf. [9])
\[ X = V_1(A_1 + X, \Gamma). \tag{3} \]

This equation possesses the following important characteristic property: If \( X \) is a solution of (3) and \( u_1 \) is an eigenvector of \( H_1 = A_1 + X \), \( H_1u_1 = zu_1 \), then \( zu_1 = A_1u_1 + V_1(H_1, \Gamma)u_1 = A_1u_1 + V_1(z, \Gamma)u_1 \). This implies that any eigenvalue \( z \) of \( H_1 \) is automatically an eigenvalue for the continued transfer function \( M_1(z, \Gamma^l) \) and \( u_1 \) is its eigenvector. Thus, having found the solution(s) of the equation (3) one obtains an effective means of studying the spectral properties of the transfer function \( M_1(z, \Gamma) \) itself, referring to well known facts of Operator Theory [4], [3].

**Theorem 1** Let a \( K_B \)-bounded contour \( \Gamma \) satisfy the condition
\[ \mathcal{V}_0(B, \Gamma) < \frac{1}{4} d_0^2(\Gamma) \tag{4} \]
where \( d_0(\Gamma) = \text{dist}\{\sigma(A_1), \Gamma\} \). Then Eq. (3) is uniquely solvable in any ball including operators \( X : \mathcal{H}_1 \to \mathcal{H}_1 \) the norms of which are bounded as \( \|X\| \leq r \) with \( r_{\text{min}}(\Gamma) \leq r < r_{\text{max}}(\Gamma) \) where
\[ r_{\text{min}}(\Gamma) = d_0(\Gamma)/2 - \sqrt{d_0^2(\Gamma)/4 - \mathcal{V}_0(B, \Gamma)}, \quad r_{\text{max}}(\Gamma) = d_0(\Gamma) - \sqrt{\mathcal{V}_0(B, \Gamma)}. \tag{5} \]

In fact, the solution \( X \) belongs to the smallest ball \( \|X\| \leq r_{\text{min}}(\Gamma) \).

One can prove this statement making use of the Banach’s Fixed Point Theorem (see [3]). One can even prove that if the index \( l = \pm 1 \) is fixed then, under the condition (4), the solution \( X \) does not depend on a concrete contour \( \Gamma \subset D^l \). Moreover, this solution satisfies the inequality \( \|X\| \leq r_0(B) \) with \( r_0(B) = \inf_{\Gamma^l : \omega(B, \Gamma^l) > 0} r_{\text{min}}(\Gamma^l) \) where \( \omega(B, \Gamma^l) = d_0^2(\Gamma^l) - 4\mathcal{V}_0(B, \Gamma^l) \). The value of \( r_0(B) \) does not depend on \( l \). But when \( l \) changes, the solution \( X \) can also change. For this reason we supply it in the following with the index \( l \) writing \( X^{(l)} \). As a matter of fact, the operators \( H_1^{(l)} = A_1 + X^{(l)}, l = \pm 1 \), represent operator roots of the transfer function \( M_1 \).

**Theorem 2** Let \( \Gamma^l \) be a contour satisfying the condition (4) and \( H_1^{(l)} = A_1 + X^{(l)} \) where \( X^{(l)} \) is the above solution of the basic equation (3). Then, for \( z \in \mathbb{C} \setminus \Gamma^l \), the transfer function \( M_1(z, \Gamma^l) \) admits the factorization
\[ M_1(z, \Gamma^l) = W_1(z, \Gamma^l)(H_1^{(l)} - z) \tag{6} \]
where \( W_1(z, \Gamma^l) \) is a bounded operator in \( \mathcal{H}_1 \),
\[ W_1(z, \Gamma^l) = I_1 - \int_{\Gamma^l} d\mu K_B^l(\mu)(H_1^{(l)} - \mu)^{-1}(\mu - z)^{-1}. \]
Here, $I_1$ denotes the identity operator in $\mathcal{H}_1$. For $\text{dist}\{z, \sigma(A_1)\} \leq d_0(\Gamma^i)/2$ the operator $W_1(z,\Gamma^i)$ is boundedly invertible.

**Theorem 3** The spectrum $\sigma(H_1^{(l)})$ of the operator $H_1^{(l)} = A_1 + X^{(l)}$ belongs to the closed $r_0(B)$-vicinity

$$\mathcal{O}_{r_0}(A_1) = \{ z \in \mathbb{C} : \text{dist}\{z, \sigma(A_1)\} \leq r_0(B) \}$$

of the spectrum of $A_1$. Moreover, the spectrum of $M_1(\cdot, \Gamma^i)$ in

$$\mathcal{O}_{d_0/2}(A_1) = \{ z : z \in \mathbb{C}, \text{dist}\{z, \sigma(A_1)\} \leq d_0(\Gamma^i)/2 \}$$

is only represented by the spectrum of $H_1^{(l)}$ including the complex spectrum (in particular the resonances).

Let

$$\Omega^{(l)} = \int_{\Gamma^i} d\mu (H_1^{(l)})^* - \mu)^{-1} K_B(\mu) (H_1^{(l)} - \mu)^{-1}$$

where $\Gamma^i$ stands for a contour satisfying the condition (4).

**Theorem 4** The operators $\Omega^{(l)}$, $l = \pm 1$, possess the following properties (cf. 5):

$$
\|\Omega^{(l)}\| < 1, \quad \Omega^{(-l)} = \Omega^{(l)^*}, \\
- \frac{1}{2\pi i} \int_{\gamma} dz [M_1(z, \Gamma^i)]^{-1} = (I_1 + \Omega^{(l)})^{-1}, \\
- \frac{1}{2\pi i} \int_{\gamma} dz z [M_1(z, \Gamma^i)]^{-1} = (I_1 + \Omega^{(l)})^{-1} H_1^{(-l)^*} = H_1^{(l)} (I_1 + \Omega^{(l)})^{-1}
$$

where $\gamma$ stands for an arbitrary rectifiable closed contour going in the positive direction around the spectrum of $H_1^{(l)}$ inside the set $\mathcal{O}_{d_0(\Gamma)/2}(A_1)$. The integration over $\gamma$ is understood in the strong sense.

The formulas (7) and (8) allow one, in principle, to find the operators $H_1^{(l)}$ and, thus, to resolve the equation (3) only using the contour integration of the inverse transfer function $[M_1(z, \Gamma^i)]^{-1}$. Also, Eq. (8) implies that the spectrum of $H_1^{(-l)^*}$ coincides with the spectrum of $H_1^{(l)}$.

**5. Completeness and basis properties.** In the following we restrict ourselves to the case where the resolvent $R_1(z) = (A_1 - z)^{-1}$ of the entry $A_1$ is a compact operator in $\mathcal{H}_1$ for $z \in \rho(A_1)$. Then, according to Theorem IV.3.17 of 6, the operators $H_1^{(l)}$ also have compact resolvents since $X^{(l)}$ are bounded operators.

Denote by $\mathcal{H}_{1,\lambda}^{(l)}$ the algebraic eigenspace of $H_1^{(l)}$ corresponding to an eigenvalue $\lambda$. Let $m_\lambda$ be the algebraic multiplicity, $m_\lambda = \dim \mathcal{H}_{1,\lambda}^{(l)}$, $m_\lambda < \infty$, and $\psi_{\lambda,i}^{(l)}$, $i = 1, 2, \ldots, m_\lambda$, be the root vectors of $H_1^{(l)}$ forming a basis of the subspace $\mathcal{H}_{1,\lambda}^{(l)}$. Regarding these vectors we already have the following assertion representing a particular case of Theorem V.10.1 from 6.
Theorem 5 The closure of the linear span of the system \( \{ \psi_{\lambda,i}^{(l)}, \lambda \in \sigma(H_1^{(l)}), i = 1, 2, \ldots, m_\lambda \} \) coincides with \( H_1 \).

We shall consider the case where the spectrum \( \sigma(A_1) \) includes infinitely many points and the entry \( A_1 \) is semibounded from below. The eigenvalues \( \lambda_i^{(A_1)} \), \( i = 1, 2, \ldots, \) of the operator \( A_1 \) are assumed to be enumerated in increasing order; \( \lim_{i \to \infty} \lambda_i^{(A_1)} = +\infty \). Since we assume \( \sigma(A_1) \subset J^0 \) this assumption means that the interval \( J^0 \) is infinite, \( \mu_0^{(2)} = \infty \). Also, we suppose that there is a number \( i_0 \) such that for any \( i \geq i_0 \)
\[
\lambda_i^{(A_1)} - \lambda_{i-1}^{(A_1)} > 2r.
\]
with some \( r > r_0(B) \). Let \( \gamma_0 \) be a circle centered at \( z = (\lambda_1^{(A_1)} + \lambda_{i_0}^{(A_1)})/2 \) and having the radius \((\lambda_{i_0}^{(A_1)} - \lambda_1^{(A_1)})/2 + r\) while \( \gamma_i \) for \( i \geq i_0 \) be the circles centered at \( \lambda_i^{(A_1)} \) and having the radius \( r \). Let us introduce the projections
\[
Q_i^{(l)} = -\frac{1}{2\pi i} \int_{\gamma_i} dz (H_1^{(l)} - z)^{-1}, \quad i = 0, i_0, i_0 + 1, \ldots.
\]
The subspaces \( N_i^{(l)} = Q_i^{(l)} H_1 \) are invariant under \( H_1^{(l)} \); \( \dim N_i^{(l)} \) coincides with a sum of algebraic multiplicities of the eigenvalues \( \lambda \in \sigma(H_1^{(l)}) \) lying inside \( \gamma_i \).

Lemma 2 Under the condition (9) the sequence \( N_i^{(l)}, \quad i = 0, i_0, i_0 + 1, \ldots, \) is \( \omega \)-linearly independent and complete in \( H_1 \).

The next theorem represents a slightly extended statement of Theorems V.4.15 and V.4.16 of [5] (the extension only concerns a possible degeneracy of the eigenvalues of \( A_1 \)).

Theorem 6 Assume \( \lambda_{i+1}^{(A_1)} - \lambda_i^{(A_1)} \to \infty \) as \( i \to \infty \). Let \( i_0 \) be a number starting from which the inequality \( \lambda_i^{(A_1)} - \lambda_{i-1}^{(A_1)} > 4r_0 \) holds. Then the following limit exists
\[
s - \lim_{n \to \infty} \sum_{i=0, i \geq i_0}^{n} Q_i^{(l)} = I_1.
\]
Additionally, assume that \( \sum_{i=1}^{\infty} (\lambda_{i+1}^{(A_1)} - \lambda_i^{(A_1)})^{-2} < \infty \). Then (10) is true for any renumbering of \( Q_i^{(l)} \) and there exists a constant \( C \) such that \( \left\| \sum_{i \in I} Q_i^{(l)} \right\| \leq C \) for any finite set \( I \) of integers \( i = 0, i \geq i_0 \).
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