CYLINDERS AS INVARIANT CMC SURFACES IN SIMPLY CONNECTED HOMOGENEOUS 3-MANIFOLDS

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ABSTRACT. We give a geometric proof for the existence of cylinders with constant mean curvature $H > H(E)$ in a non-compact simply connected homogeneous 3-manifold $E$. Our cylinders are invariant under a 1-parameter group of isometries and generated by a simple closed curve. The value $H(E)$ denotes the critical mean curvature for which constant mean curvature spheres exist. For the spaces $\text{Sol}_3$ and $\text{PSL}_2(\mathbb{R})$ existence of these surfaces has not been established yet. We include interesting computed examples of cylinders in $\text{Sol}_3$ generated by non-embedded simple closed curves.

INTRODUCTION

Recently constant mean curvature surfaces have been studied in various simply connected 3-dimensional homogeneous manifolds which are possibly non-isotropic, for example in [AR05], [Tor10], [MP12] and [Mee13]. Invariant surfaces with constant mean curvature are a classical topic in differential geometry. The mean curvature equation reduces to an ordinary differential equation. When this ODE has a simple closed curve as solution we call the invariant surface generated by this curve an $\text{MC}_H$-cylinder. Our approach to this problem is as follows: We consider invariant CMC surfaces generated by graphical curves. A comparison with CMC spheres yields properties of the graph which let us extend it to a simple closed solution curve.

In the first part of this paper we work in $\text{Sol}_3$. This homogeneous space with a 3-dimensional isometry group can be considered a Riemannian fibration $\text{Sol}_3 \to \mathbb{R}$ with $\mathbb{R}^2$-fibers and base $\mathbb{R}$. At each point of $\text{Sol}_3$ there are three distinguished geodesics which admit rotations of angle $\pi$: The base and two orthogonal lines in a $\mathbb{R}^2$-fiber. Since $\text{Sol}_3$ is also a metric Lie group left-translations along any of these three geodesics define a one-parameter family of isometries. For each of these geodesics we construct surfaces which are invariant under the corresponding family of left-translations. Without the need to state the ODEs explicitly we can prove, in Theorem 5 and Theorem 9, that embedded $\text{MC}_H$-cylinders with $H > 0$ exist. For surfaces invariant by left-translations along the base the existence of such surfaces was conjectured by [Lop14], on the grounds of computed examples. We believe our families are the first examples of embedded annuli with constant mean curvature $H > 0$ in $\text{Sol}_3$.

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We also include images of computed examples of $\text{mC}_H$-cylinders in $\text{Sol}_3$ which have the same invariance, but are only immersed (Figure 8 and Figure 9). This class seems large. In fact we conjecture that there are infinitely many simple closed solution curves with self-intersections which generate these examples. This surprising phenomenon cannot occur in ambient spaces with higher dimensional isometry groups, for instance $\mathbb{R}^3$, where translations along and rotations about the same axis commute and thus imply rotational invariance of translationally invariant surfaces.

In the second part we consider Riemannian fibrations $E \to B$ with geodesic fibers. They are parametrized as $E(\kappa, \tau)$-spaces with base curvature $\kappa$ and bundle curvature $\tau$. We exclude the compact case of the Berger spheres, which admit CMC spheres which are possibly self-intersecting; comparison spheres for maximum principle are not available.

The $E(\kappa, \tau)$-spaces have 4- or 6-dimensional isometry groups. In case of a 4-dimensional isometry groups rotations about non-vertical geodesics need not be isometries, and their respective geodesic tubes need not have constant mean curvature. However, translations along geodesics are still isometries.

A reasoning similar to the first part proves existence of $\text{mC}_H$-cylinders with $H > H(E)$, invariant under translation along those geodesic axes which have a geodesic projection into the base space $B$, see Theorem 17. For $\tau = 0$ this includes tilded $\text{mC}_H$-cylinders; we also get horizontal $\text{mC}_H$-cylinders in $\text{PSL}_2(\mathbb{R})$osa. Again, we do not need to refer to the explicit form of the ODE. In Theorem 20 we calculate the horizontal diameter of these surfaces. The argument is based on a weight formula for CMC surfaces.

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Part 1. $\text{mC}_H$-cylinders in $\text{Sol}_3$

In Section 1 of this part we describe the metric Lie group $\text{Sol}_3$ as a semi-direct product $\mathbb{R}^2 \ltimes_A \mathbb{R}$. Section 2 is devoted to constant mean curvature surfaces invariant under translations along the base: One problem concerns the ODE satisfied by a graph generating such a surface. The other problem is the geometric discussion of the ODE and the extension of the graphical solution to a simple closed embedded curve. For this class of surfaces we also include images of computed examples. In Secion 3 we proceed analogously and construct $\text{mC}_H$-cylinders invariant under translations along a diagonal in a $\mathbb{R}^2$-fiber of $\text{Sol}_3$.

1. Preliminaries on $\text{Sol}_3$

The space $\text{Sol}_3$ is a simply connected homogeneous 3-manifold diffeomorphic to $\mathbb{R}^3$ and as such a metric Lie group. We describe a model for this space and some properties.

Model. We endow $\mathbb{R}^3$ with the Riemannian metric

$$\langle \cdot, \cdot \rangle_{(x,y,z)} = e^{2z}dx^2 + e^{-2z}dy^2 + dz^2,$$

(1)
and set $\text{Sol}_3 := (\mathbb{R}^3, \langle \cdot, \cdot \rangle)$. The multiplication

$$(x_1, y_1, z_1) \ast (x_2, y_2, z_2) := (x_1 + e^{-z_1}x_2, y_1 + e^{z_1}y_2, z_1 + z_2)$$

turns $\text{Sol}_3$ into a metric Lie group, i.e. for $a \in \text{Sol}_3$ the left-multiplication

$$\ell_a : \text{Sol}_3 \to \text{Sol}_3, \quad \ell_a(g) := a \ast g$$

is an isometry of $\text{Sol}_3$.

We remark that $\text{Sol}_3$ can be considered a Riemannian fibration $\text{Sol}_3 \to \mathbb{R}$, $(x, y, z) \mapsto z$ with $\mathbb{R}^2$-fibers over the $z$-axis.

**Canonical frame and Riemannian connection.** At the origin let $(\partial_x, \partial_y, \partial_z)$ be the standard Euclidean frame. A left-translation from the origin to $p = (x, y, z)$ gives the orthonormal frame

$$E_1 = e^{-z} \partial_x, \quad E_2 = e^z \partial_y, \quad E_3 = \partial_z.$$

The Riemannian connection with respect to this frame has the following representation:

$$\nabla_{E_1} E_1 = -E_3, \quad \nabla_{E_1} E_2 = 0, \quad \nabla_{E_1} E_3 = E_1,$$

$$\nabla_{E_2} E_1 = 0, \quad \nabla_{E_2} E_2 = E_3, \quad \nabla_{E_2} E_3 = -E_2,$$

$$\nabla_{E_3} E_1 = 0, \quad \nabla_{E_3} E_2 = 0, \quad \nabla_{E_3} E_3 = 0.$$

**Special geodesics and induced isometries.** We consider the unit-speed geodesics

$$c : \mathbb{R} \to \text{Sol}_3, \quad c(s) := (0, 0, s) \quad \text{and} \quad c_{\pm} : \mathbb{R} \to \text{Sol}_3, \quad c_{\pm}(s) := \frac{1}{\sqrt{2}}(s, \pm s, 0).$$

Since $\text{Sol}_3$ is a metric Lie group we obtain a one-parameter family of isometries $(\Phi_s)_{s \in \mathbb{R}}$ by setting

$$\Phi_s : \text{Sol}_3 \to \text{Sol}_3, \quad \Phi_s(x, y, z) := \ell_{c(s)}(x, y, z) = (e^{-s}x, e^s y, z + s), \quad s \in \mathbb{R}.$$

We call the family $\Gamma := (\Phi_s)_{s \in \mathbb{R}}$ *translations along c*.

Another one-parameter family of isometries $\Gamma_{\pm} := (\Phi_{\pm s})_{s \in \mathbb{R}}$, *translations along $c_{\pm}$*, is defined by

$$\Phi_{\pm s} : \text{Sol}_3 \to \text{Sol}_3, \quad \Phi_{\pm s}(x, y, z) := \ell_{c_{\pm}(s)}(x, y, z) = \left(x + \frac{s}{\sqrt{2}}, y \pm \frac{s}{\sqrt{2}}, z\right), \quad s \in \mathbb{R}.$$

Each $x$-$z$ plane is a totally geodesic submanifold of $\text{Sol}_3$: Indeed, $E_2$ is normal and $\langle \nabla_{E_i} E_j, E_2 \rangle = 0$ for $i, j \in \{1, 3\}$. Similarly the $y$-$z$ planes are totally geodesic, too, with normal $E_1$ and $\langle \nabla_{E_i} E_j, E_1 \rangle = 0$ for $i, j \in \{2, 3\}$. In particular, reflections in all these planes are isometries of $\text{Sol}_3$. We denote the reflection in $\{x = 0\}$ by $\sigma_{yz}$ and reflection in $\{y = 0\}$ by $\sigma_{xz}$.

The following two properties are less obvious and can be verified directly from the form of the metric (1):

Rotations by an angle $\pi$ about $c$ and $c_{\pm}$ are isometries of $\text{Sol}_3$. We denote them by

$$\psi : \text{Sol}_3 \to \text{Sol}_3, \quad \Psi(x, y, z) := (-x, -y, z)$$

and

$$\psi_{\pm} : \text{Sol}_3 \to \text{Sol}_3, \quad \Psi_{\pm}(x, y, z) := (\pm y, \pm x, -z).$$
Killing fields. Left-translation along each of the coordinate axes defines a one-parameter group of isometries, generated by the following three Killing fields:

$$K_1 = e^z E_1 = \partial_x, \quad K_2 = e^{-z} E_2 = \partial_y, \quad K_3 = -xK_1 + yK_2 + E_3.$$ 

This is useful to describe Killing graphs defined on \{y = 0\} as well as for translations in the x-y plane \{z = 0\}. We will use this later.

Constant mean curvature spheres. In [Mee13] CMC spheres in $\text{Sol}_3$ are studied. We need the following property of $H$-spheres in $\text{Sol}_3$:

**Proposition 1.** Let $H > 0$ and $S_H$ be a sphere of constant mean curvature $H$ in $\text{Sol}_3$, centered at $(0,0,0)$. Then \{x = 0\} and \{y = 0\} are mirror planes of $S_H$ and $S_H$ is a bi-graph with respect to each mirror plane. The minimal and maximal values of the x, y and z coordinates arise on the respective coordinate axes.

**Proof.** The first part is stated in [Mee13]. The last claim is a consequence of the Gauss map being a diffeomorphism: If minimum and maximum were attained elsewhere the Gauss map could not be injective because $S_H$ is invariant by rotations of angle $\pi$ about each coordinate axis. \(\square\)

### 2. Surfaces invariant under translations along $c$

In this section we study constant mean curvature surfaces invariant under translation along the base $c$ of $\text{Sol}_3$. First we describe properties of the differential equation for constant mean curvature surfaces invariant by $\Gamma$. These are natural implications by the geometry of $\text{Sol}_3$. Then we discuss the solution of this ODE geometrically. We use the maximum principle to derive properties, which let us extend the respective graph by reflections to an embedded closed solution curve. We also discuss further solutions obtained numerically.

#### 2.1. ODE for surfaces invariant under translations along $c$

The foliation by x-y planes of $\text{Sol}_3$ stays invariant under translations along $c$. Therefore it is sufficient to consider a curve in the fiber

$$S_0 := \{(x, y, z) \in \text{Sol}_3 : z = 0\}$$

as generating curve of a surface invariant by translation along $c$.

Explicitly, for $C^2$-functions $x : J \to \mathbb{R}$ and $y : J \to \mathbb{R}$, defined on an open interval $J \subset \mathbb{R}$, the curve

$$\gamma : J \to \text{Sol}_3, \quad \gamma(t) := (x(t), y(t), 0)$$

is in $S_0$ and the invariant surface generated by translation of $\gamma$ along $c$ is parametrized by

$$f : \mathbb{R} \times J \to \text{Sol}_3, \quad f(s, t) := \Phi_s(\gamma(t)) = (e^{-s}x(t), e^s y(t), s). \quad (3)$$

The mean curvature of $f$ is independent of $s$, i.e. $H = H(t)$. Requiring $H$ to be constant leads to an ordinary differential equation for $\gamma$. Such surfaces were studied in [LM14] and [Lop14], too, but for $H > 0$ the mean curvature equation appears too complicated for explicit solutions or qualitative discussions involving first integrals.

We will consider graphical solutions, for which the ODE can be described as follows:
Proposition 2. Let $H \in \mathbb{R}$.

(a) There is a smooth function $F : \mathbb{R}^3 \to \mathbb{R}$ such that the invariant surface

$$f : \mathbb{R} \times J \to \text{Sol}_3, \quad f(s, t) := \Phi_s(t, h(t), 0), \quad \text{where } h \in C^2(J, \mathbb{R}),$$

has constant mean curvature $H$ with respect to the upper normal if and only if

$$h''(t) = F(t, h(t), h'(t)) \quad \text{for all } t \in J. \quad (4)$$

(b) The invariant surface $\tilde{f} : \mathbb{R} \times J \to \text{Sol}_3, \quad \tilde{f}(s, t) := \Phi_{-s}(h(t), h(t), 0)$ has constant mean curvature $H$ if and only if $h \in C^2(J, \mathbb{R})$ satisfies (4), i.e. $x$-graphs and $y$-graphs as generating curves of invariant surfaces with constant mean curvature $H$ satisfy the same ODE.

Proof. (a) Let $v_1 := \partial_s f$ and $v_2 := \partial_t f$. We denote the upper normal to $f$ by $N$, so that $g_{ij} := \langle v_i, v_j \rangle$ and $b_{ij} := \langle \nabla v_i, v_j, N \rangle$ for $i, j \in \{1, 2\}$ are the coefficients of the first and second fundamental form. Then the mean curvature of $f$ is given by

$$H = \frac{b_{11}g^{11} + 2b_{12}g^{12} + b_{22}g^{22}}{2}.$$ 

We have

$$v_2 = E_1 + h'E_2 \quad \text{and} \quad \nabla_{v_2}v_2 = \nabla_{v_2}E_1 + h'\nabla_{v_2}E_2 + h''E_2$$

Here we note that $H$ depends on $t, h(t), h'(t)$ and $h''(t)$.

We assume $H$ to be constant and therefore get an implicit differential equation depending on $h'$ and $h''$. Now we want to show that we can solve this implicit equation for $h''$.

Obviously $w$ is independent of $h''$ and the only term containing $h''$ is

$$\frac{b_{22}g^{22}}{2} = \frac{b_{22}g_{11}}{2} = \frac{\langle \nabla_{v_2}v_2, N \rangle g_{11}}{2\det(g)} = \frac{\langle h''(N, E_2) + \langle w, N \rangle g_{11} \rangle}{2\det(g)}.$$ 

The surface $f$ is a Killing graph with respect to the Killing field $K_2 = \partial y = e^{-s}E_2$, so that $\langle N, E_2 \rangle$ is positive, because $N$ is chosen as upper normal. We also have $g_{11} > 0$ because the Killing field generated by translation along $c$ is non-trivial.

Therefore we can solve the implicit equation for $h''$ and get a function $F : \mathbb{R}^3 \to \mathbb{R}$ with $h''(t) = F(t, h(t), h'(t))$. The function $F$ is smooth because each $\Phi_s$ is smooth and so are $g$ and $b$. It is defined on all of $\mathbb{R}^3$ because we can prescribe any kind of function $h : J \to \mathbb{R}$.

(b) The equation $\Phi_{-s} \circ \psi_+ = \psi_+ \circ \Phi_s$ implies $\tilde{f} = \psi_+ \circ f$, i.e. $\tilde{f}$ and $f$ are isometric.

Thus the claim about the ODE follows from (a).

2.2. Half-cylinder solution and its extension to an \textit{MC}H-cylinder with axis $c$. We consider the ODE for surfaces invariant by translations along the base $c$ first. We can apply the Picard-Lindelöf Theorem to (4) because $F$ is smooth. We obtain a maximal solution $h$. For constant mean curvature $H > 0$ the maximum principle yields some properties by comparing the surface $f$ with spheres of constant mean curvature $H$, which will justify the name “half-cylinder”:
Lemma 3. Given $a, b \in \mathbb{R}$ and $H > 0$, there is a unique maximal solution $h : I_{\max} \to \mathbb{R}$ with $h(0) = a$ and $h'(0) = b$ satisfying (4). For each $a, b \in \mathbb{R}$ it has the following properties:

(a) [x-boundedness]: There are real numbers $R_- = R_-(a, b) < R_+ = R_+(a, b)$ such that $I_{\max} = (R_-, R_+)$.

(b) [y-boundedness]: There is $K = K(a, b) > 0$ such that $\lim_{t \to R_\pm} |h(t)| \leq K$.

(c) [Asymptotic behaviour]: We have $\lim_{t \to R_\pm} h'(t) = \pm \infty$.

(d) [Monotonicity]: There is $t_0 \in (R_-, R_+)$ with $h'(t_0) = 0$. On $(R_-, t_0)$ the function $h$ is monotonically decreasing and on $(t_0, R_+)$ it is monotonically increasing.

(e) [Symmetry]: For $b = 0$ we have $R = R(a) := R_+(a, 0) = -R_-(a, 0)$ for the maximal solution with $h(0) = a$ and $h'(0) = 0$.

Proof. Let $h : I_{\max} \to \mathbb{R}$ be the unique maximal solution of $h''(t) = F(t, h(t), h'(t))$ with $h(0) = a$ and $h'(0) = b$ and $\Sigma$ be the surface generated by $(t, h(t), 0)$. We will use frequently that translations along the $x$-axis or $y$-axis are isometries of $\text{Sol}_3$ and that $f$ is an invariant surface. Here we also use Proposition 1.

(a): Assume $\sup I_{\max} = \infty$ or $\inf I_{\max} = -\infty$; say, without loss of generality, $\sup I_{\max} = \infty$. Consider a constant mean curvature $H$ sphere $S_H$ centered at $(0, 0, 0)$ in $\text{Sol}_3$ and let $\Pi_y : \text{Sol}_3 \to \mathbb{R}^2$ be defined by $\Pi_y(x, y, z) := (x, y)$.

Due to our assumption and the compactness of $S_H$ we can translate $S_H$ to a sphere $S_H(p)$ centered at a point $p$ such that $\Pi_y(S_H(p))$ is contained in $\Pi_y(\Sigma)$. Moving spheres along some $y$-axis $\Gamma$ through $p$ inside the mean convex side of $\Sigma$ towards the surface leads to a first tangential contact point in the interior of $\Sigma$. The maximum principle then shows $\Sigma = S_H$, which is a contradiction. See Figure 1 on page 7.

(b): If this were false, then $h$ could only be unbounded for $t \to R_\pm$. Without loss of generality, $\lim_{t \to R_-} h(t) = -\infty$. Both cases are ruled out by moving spheres towards $f|_{\mathbb{R}^\times(0, R_+)}$; compare with Figure 2 on page 8.

(c): Let $\tilde{F} : \mathbb{R}^3 \to \mathbb{R}^2$, $\tilde{F}(\tau, \xi, \eta) := (\eta, F(\tau, \xi, \eta))$, so that the maximal solution of $x'(t) = \tilde{F}(t, x(t))$ with $x(0) = (a, b)$ is given by $y(t) := (h(t), h'(t))$.

We know the phase space of $\tilde{F}$ is $\mathbb{R}^3$. General ODE theory implies that $I_{\max} \to \mathbb{R}^2$, $t \mapsto (t, y(t)) = (t, h(t), h'(t))$ leaves every compact subset in $\mathbb{R}^3$, in particular $[R_-, R_+] \times [-K, K] \times [-C, C]$ for every $C > 0$. In view of (a) and (b) this implies $\lim_{t \to R_\pm} |h'(t)| = \infty$.

Let us now confirm the sign of $\lim_{t \to R_+} h'(t)$. On the contrary, suppose $\lim_{t \to R_+} h'(t) = -\infty$ and consider

$$\alpha := \inf \{ t \in (0, R_+) : h'|_{(t, R_+)} < 0 \}.$$ 

For $t_0 \in (\alpha, R_+)$ let $\Gamma$ be defined by

$$\Gamma : [t_0, +\infty) \to \text{Sol}_3, \quad s \mapsto (s, h(t_0), 0).$$
Figure 1. Lemma 3 (a): Comparison argument indicating that $\sup I_{\text{max}} = \infty$ and $\inf I_{\text{max}} = -\infty$ are impossible

Moving spheres along $\Gamma$ we get a first tangential point of contact in the interior of the surface $f|_{\mathbb{R} \times (\alpha, R_+)}$. The normals of $S_H$ and $f|_{\mathbb{R} \times (\alpha, R_+)}$ coincide at this point because of $\lim_{t \to R_+} h'(t) = -\infty$. Thus the maximum principle yields a contradiction; see Figure 3 on page 8. For $\lim_{t \to R_-} h'(t) = -\infty$ we argue similarly.

(d): The existence of $t_0 \in I_{\text{max}}$ with $h'(t_0) = 0$ is clear since $h'$ is continuous and changes sign at least once by (c). Assume $h$ were not strictly monotonically increasing on $(t_0, R_+)$. Then $\beta := \sup \{ t \in (t_0, R_+): h'(t) \leq 0 \}$ is strictly larger than $t_0$ and $h$ defined on $[\beta, t_0]$. To rule out this case we apply the maximum principle to the surface $f|_{\mathbb{R} \times (t_0, \beta)}$, and move spheres to this surface having no boundary contact; see Figure 4 on page 9. We reason similarly for $h$ on $(R_-, t_0)$.

(e): It is easy to check $\sigma_{yz} \circ \Phi_s = \Phi_s \circ \sigma_{yz}$, so that a reflection of the solution through $\{x = 0\}$ gives another solution of the same ODE. If we assume $b = 0$, then the initial
values \( h(0) = a \) and \( h'(0) = 0 \) are invariant under \( \sigma_{y_2} \), so that we obtain the same solution. This proves \( R_+ = -R_- \). □

We are interested in a particular solution of the ODE, see Figure 5:

**Proposition 4** (0-height solution). There is \( a_0 \in \mathbb{R} \) such that \( h(\pm R(a_0)) = 0 \) for the maximal solution with \( h(0) = a_0 \) and \( h'(0) = 0 \). Furthermore we have \( R(a_0) = -a_0 \).

**Proof.** The function \( \varphi : \mathbb{R} \to \mathbb{R}, \varphi(a) := h(R(a)) \) is continuous. For \( a = 0 \) monotonicity implies \( \varphi(a) > 0 \). If we had \( \varphi(a) > 0 \) for all \( a \leq 0 \), then we could find \( \tilde{a} < 0 \) such that it were possible to move a sphere to the surface \( f|_{\mathbb{R} \times (0,R(\tilde{a}))} \) without touching its boundary.
a contradiction. So there is some $\tilde{a} < 0$ with $\varphi(\tilde{a}) < 0$, and by the intermediate value theorem we get $a_0 \in (-\infty, 0)$ with $\varphi(a_0) = 0$.

To show $R(a_0) = -a_0$ let us consider $\tilde{f} := \psi_+ \circ f$. Then Proposition 2 (b) implies that $h$ is also a 0-height solution to the initial values $h(0) = -R(a_0)$ and $h'(0) = 0$ with $I_{\text{max}} = (a_0, -a_0)$. This can only hold for $R(a_0) = -a_0$. $\square$

We use one 0-height solution to obtain a smoothly embedded closed curve $\gamma$ generating an invariant cylinder $f$ with constant mean curvature $H > 0$.

**Theorem 5.** Consider the metric Lie group $\text{Sol}_3$ as $\mathbb{R}^3$ with left-invariant Riemannian metric $\langle \cdot, \cdot \rangle = e^{2z} dx^2 + e^{-2z} dy^2 + dz^2$ and let $\Gamma$ be the family of left-translations along the $z$-axis $c : \mathbb{R} \to \text{Sol}_3$, $c(s) = (0, 0, s)$. Then for each $H > 0$ there is a smooth embedded
simple closed curve $\gamma$ in $S_0 = \{ z = 0 \}$ which generates a $\Gamma$-invariant embedded surface $f$ with constant mean curvature $H$. The surface is invariant by the dihedral subgroup of order 8, generated by $\{ \sigma_{xz}, \sigma_{yz}, \psi_{\pm} \}$.

In the following we refer to these surfaces as $\text{mCH}$-cylinders with axis $c$.

**Proof.** Let $h: (-R, R) \to \mathbb{R}$ be a 0-height solution. We have $\sigma_{xz} \circ \Phi_s = \Phi_s \circ \sigma_{xz}$, so that we can extend the surface by reflecting through $\{ y = 0 \}$. This extension gives rise to a closed curve $\gamma$. The curve $\gamma$ is smooth since $h$ is asymptotic to a $y$-axis. Monotonicity of $h$ implies embeddedness of $\gamma$. This proves the claim about the generating curve.

For the isometry group we note that invariance by $\sigma_{xz}$ is obvious by construction of $\gamma$. The invariance by $\sigma_{yz}$ follows from Lemma 3 (e). Similarly we argue for the invariance by $\sigma_{\pm}$: Due to $R(a_0) = -a_0$ the initial values of the half-cylinder solutions remain invariant, hence we get the same solution. \hfill $\square$

**Remark 6.** We conjecture there is exactly one 0-height solution, but we do not have a proof at hand. If there were 0-height solutions $h_0$ and $h_1$ to initial values $h_0(0) = a_0$ and $h_1(0) = a_1$ respectively, then both would satisfy $R(a_0) = -a_0 \neq -a_1 = R(a_1)$. Then one solution would be above the other one, so one cylinder would be on the mean convex side of the other one. However we cannot get a point of tangential contact by moving one solution up along the $y$-axis because translations along the $y$-axis and translations along $c$ do not commute. It seems we need a halfspace theorem. The general halfspace theorem by Mazet [Maz13] has two crucial assumptions: First, it requires parabolicity of our cylinders, that is, they must be conformal to a punctured plane, an assumption which is satisfied in our case due to translational invariance. Second, there is an assumption on the mean curvature of equidistant surfaces to the given $\text{mCH}$-cylinder. It appears difficult to verify and we do not know whether the second assumption holds.

**Remark 7.** We used Mathematica to calculate the $\text{mCH}$-cylinders with axis $c$. We have computed the ODE in Proposition 21. [Lop14] also has a numerical example, but we believe it is less precise due to a different approach of exhibiting the initial value $h(0) = a$ numerically.

We set $H = 1$. Upon iteration we calculated for $a := -0.642176$ that

$$h(R(a)) < 10^{-7} \quad \text{and} \quad R(a) = -a = 0.642176,$$

as expected by Proposition 4. Finally we extended the solution curve by a reflection through $\{ y = 0 \}$. See Figure 6.

We note that $h(0) \approx -0.6425$ in [Lop14], which we consider less precise. For instance, it does not satisfy $R(a) = -a$ numerically and for this value we get $h(R(a)) \approx 2 \cdot 10^{-4}$.

It is natural to look at the family of $\text{mCH}$-cylinders with $H \in (0, \infty)$. Computations with Mathematica, illustrated by Figure 7, are evidence for the following:

**Conjecture.** The $\text{mCH}$-cylinders with axis $c$ form an analytical family in $H \in (0, \infty)$. For $H \to 0$ the surfaces are unbounded and for $H \to \infty$ they shrink to $c$. 
2.3. **Conjecture on non-embedded solutions with axis** $c$. A shooting method leads to computed examples of non-embedded $\text{MC}_1$-cylinders with axis $c$ in $\text{Sol}_3$. We shoot orthogonally from the diagonal $c_+$ and aim at the $y$-axis. Assume the solution curve meets the $y$-axis at $T > 0$. Then $y'(T)$ is the angle between $\gamma$ and the $y$-axis. We extend this portion by a rotation of angle $\pi$ about $c_+$ and reflections through $\{x = 0\}$ and $\{y = 0\}$ to a closed curve $\gamma = (x, y, 0)$. The resulting curve is built up from 8 such portions, possibly non-smooth at multiples of $T$.

Recall that the turning number $\text{turn}(\gamma)$ satisfies

$$2\pi \text{turn}(\gamma) = \int_0^{8T} \kappa_{\text{eucl}}(\gamma) \, dt + \text{ext}(\gamma),$$

where the second term $\text{ext}(\gamma) = 8y'(T)$ denotes the sum of the exterior angles. If $\gamma$ meets the $y$-axis orthogonally at $T$ then $\gamma$ is smooth and $\text{ext}(\gamma) = 0$.

To compute examples we fix $H = 1$ and proceed as follows:
Figure 7. Generating curve of an \( MCH \)-cylinder with axis \( c \): From outer to inner contour the mean curvature \( H \) takes the values 0.5, 0.6, 0.65, 0.7 and 1.

- Take \( \gamma(0) = (d, d, 0) \) for some \( d \in \mathbb{R} \) and \( \gamma'(0) = \frac{1}{\sqrt{2}}(-1, 1, 0) \) as initial values.
- Suppose the resulting curve meets the \( y \)-axis at time \( T = T(d) > 0 \).
- Vary \( d \) while maintaining the same turning number of closed extension curve.
- Exhibit \( d_1 \) and \( d_2 \) with \( y'(T(d_1)) < 0 \) and \( y'(T(d_2)) > 0 \). An intermediate value argument gives some \( d_0 \) between \( d_1 \) and \( d_2 \) with \( y'(T(d_0)) = 0 \).

With this ansatz we computed solutions with turning number 9 and 17, shown in Figure 8a and Figure 8b.

Aiming at the other diagonal \( c_- \) instead of the \( y \)-axis we find solutions with further turning numbers. See Figure 8c and Figure 8d for solutions with turning number 13 and 21.

It is straightforward to compute more examples with turning number \( 5 + 4k \) where \( k \in \mathbb{N} \). The particular value \( d = 0.429474 \) corresponds to the solution generating the embedded cylinder.
Moreover, we computed an example with turning number 5 for $H = \frac{1}{2}$ and $d = -0.965$. Increasing $H$ as well as $d$, we computed examples with turning number 5 up to $H = 0.759$. It appears that these solutions with turning number 5 degenerate to the fivefold cover of a cylinder solution for some $H_0 \in (0.759, 1)$; see Figure 9.

**Conjecture.** For each $H > 0$ there is $m = m(H) \in \mathbb{N}$ such that for every natural number $k \geq 1$ there exists a non-embedded closed curve with turning number $m + 4k$ as generating curve of a $\Gamma$-invariant surface with constant mean curvature $H$.

A proof of this conjecture seems beyond the techniques used in the present paper.
3. **Surfaces invariant under translations along** $c_{\pm}$

For constant mean curvature surfaces invariant under $\Gamma_{\pm}$ we will proceed as for those invariant under $\Gamma$.

3.1. **ODE for surfaces invariant by translations along** $c_{\pm}$. For our second surface family we can consider the foliation $(S_{\pm,s})_{s \in \mathbb{R}}$ of planes above diagonals in the $x$-$y$-plane. We have

$$S_{\pm,s} = \left\{ \left( \frac{x}{\sqrt{2}} \mp \frac{x}{\sqrt{2}}, z \right) : x, z \in \mathbb{R} \right\}, \quad s \in \mathbb{R}. $$
Obviously, this foliation is invariant by translations along $c_{\pm}$. For surfaces invariant by $\Gamma_{\pm}$ a discussion as in the previous subsection gives the following result for the ODE of graphical solutions:

**Proposition 8.** Let $H \in \mathbb{R}$. There is a smooth function $F : \mathbb{R}^3 \rightarrow \mathbb{R}$ such that the invariant surface

$$f : \mathbb{R} \times J \rightarrow \text{Sol}_3, \quad f(s, t) := \Phi_{\pm, s} \left( \frac{t}{\sqrt{2}}, \mp \frac{t}{\sqrt{2}}, h(t) \right)$$

where $h \in C^2(J, \mathbb{R})$,

has constant mean curvature $H$ with respect to the upper normal if and only if

$$h''(t) = F(t, h(t), h'(t)) \quad \text{for all } t \in J. \quad (5)$$

### 3.2. $\text{MCH}$-cylinders with axis $c_{\pm}$

The discussion from above is also applicable for $\Gamma_{\pm}$-invariant surfaces, so that we only state the result obtained in this case and indicate the differences in the proof:

**Theorem 9.** Consider the metric Lie group $\text{Sol}_3$ as $\mathbb{R}^3$ with left-invariant Riemannian metric $\langle \cdot, \cdot \rangle = e^{2z} dx^2 + e^{-2z} dy^2 + dz^2$ and let $\Gamma_{\pm}$ be the family of left-translations along $c_{\pm} : \mathbb{R} \rightarrow \text{Sol}_3$, $c(s) = \left( \frac{s}{\sqrt{2}}, \pm \frac{s}{\sqrt{2}}, 0 \right)$. Then for each $H > 0$ there is a smooth embedded simple closed curve $\gamma$ in $S_{\pm,0} = \left\{ \left( \frac{s}{\sqrt{2}}, \pm \frac{s}{\sqrt{2}}, z \right) : x, z \in \mathbb{R} \right\}$ which generates a $\Gamma_{\pm}$-invariant embedded surface $f$ with constant mean curvature $H$. It is invariant by $\psi_+$ and $\psi_-$. We call this surface $\text{MCH}$-cylinder with axis $c_{\pm}$.

**Proof.** First, (5) has a maximal solution. In order to obtain a symmetric solution as in Lemma 3 (e) we fix the initial value at $h'(0) = 0$ and argue as follows: The rotations of angle $\pi$ about $c_+$ and $c_-$ commute with translation along $c_{\pm}$. This shows the symmetry in this case, the other items are proved in the same way.

For the 0-height solution in this case we argue exactly as in Proposition 4. Since $\psi_{\pm}$ commutes with translation along $c_{\pm}$ we can extend a 0-height solution to a smooth embedded closed curve. This finishes the proof. \(\Box\)

**Remark 10.** The open problems and conjectures for $\text{MCH}$-cylinders with axis $c_{\pm}$ can be stated in the same way.

### Part 2. $\text{MCH}$-cylinders in non-compact $E(\kappa, \tau)$-spaces

The $E(\kappa, \tau)$-spaces are Riemannian fibrations $E \rightarrow B$ with geodesic fibers, bundle curvature $\tau \in \mathbb{R}$ and base curvature $\kappa \in \mathbb{R}$. In Section 4 we describe these spaces. Most results concerning constant mean curvature surfaces then become “horizontal” or “vertical” generalizations of results in $\mathbb{R}^3$. It turns out that the arguments given in Section 2 and 3 carry over to prove existence of tilded $\text{MCH}$-cylinders in $E(\kappa, 0)$ and of horizontal $\text{MCH}$-cylinders in $E(\kappa, \tau)$ for $\tau \neq 0$. In the final Section we compute the horizontal diameter of a horizontal $\text{MCH}$-cylinder in $E(\kappa, \tau)$-spaces with $\kappa \leq 0$. 


4. Non-compact $E(\kappa, \tau)$-spaces

The $E(\kappa, \tau)$-spaces are simply connected homogeneous 3-manifolds diffeomorphic to $\mathbb{R}^3$ or $\mathbb{S}^2 \times \mathbb{R}$ and arise as Riemannian fibrations $E \rightarrow B$ with geodesic fibers, where $B$ has curvature $\kappa \in \mathbb{R}$ and the bundle curvature is $\tau \in \mathbb{R}$. Because we exclude the Berger spheres, that is $\kappa > 0$ and $\tau \neq 0$ arbitrary, we may assume $E = B \times \mathbb{R}$.

4.1. General properties. The $E(\kappa, \tau)$-spaces have some geometric properties, which can be stated without an explicit model.

Geodesics. In $E = B \times \mathbb{R}$ the vertical translations $T_s : E \rightarrow E$, $T_s(p, t) = (p, t + s)$ for $s \in \mathbb{R}$ are isometries, giving rise to a Killing field $\xi$. As a consequence of Clairaut’s Theorem geodesics have the following property (see, e.g., [Eng06, Lemma 3.7]):

**Proposition 11.** Let $c : \mathbb{R} \rightarrow E(\kappa, \tau)$ be a unit-speed geodesic. Then there is $\alpha \in [0, \pi]$ with $\langle c', \xi \circ c \rangle \equiv \cos(\alpha)$. We call $\alpha$ slope of $c$ with respect to $\xi$. Furthermore the projection $\tilde{c} := \Pi \circ c$ is a curve of constant geodesic curvature $-\tau \cot(\alpha)$ in $B$. We call the geodesic fibers, corresponding to $\alpha = 0$, vertical geodesics. They admit arbitrary rotations as isometries. On the other hand, the case of $\alpha = \frac{\pi}{2}$ corresponds to horizontal geodesics. They admit rotations by an angle $\pi$.

For $\tau = 0$ geodesics project to geodesics of the base space $B$, while for $\tau \neq 0$ only horizontal geodesics project to geodesics of $B$. In the following we are only considering geodesics $c$ in $E$ which project onto geodesics in $B$ and have slope $\alpha \in (0, \frac{\pi}{2}]$.

Foliation by vertical planes and induced translations. Let $c$ be a geodesic whose projection $\tilde{c} := \Pi \circ c$ is also geodesic. Then there is a foliation of $B$ or of $B = \mathbb{S}^2$ minus two points by geodesics $(\tilde{\gamma}_s)_{s \in \mathbb{R}}$ perpendicular to $\tilde{c}$ such that $\tilde{\gamma}_s(0) = \tilde{c}(s)$ for all $s \in \mathbb{R}$. The vertical planes $P_s := (\gamma_s \times \mathbb{R})_{s \in \mathbb{R}}$ are therefore a foliation of $E$ respectively $\mathbb{S}^2 \times \mathbb{R}$ minus two vertical lines. As $\alpha \neq 0$ the geodesic $c$ meets each $P_s$ transversally.

Geodesics in $E(\kappa, \tau)$ spaces are orbits of one-parameter families of isometries, for a proof see [Eng06, Theorem 2.5]. Such a one-parameter family can be chosen as follows: In the base $B$, let $(\tilde{\psi}_s)_{s \in \mathbb{R}}$ be the family of translations along $\tilde{c}$ with $\tilde{\psi}_s(\tilde{c}(0)) = \tilde{c}(s)$. By [Man14, Corollary 2.11] we can lift each $\tilde{\psi}_s$ horizontally and obtain an orientation-preserving isometry $\psi_s : E \rightarrow E$. Vertical translations $T_\alpha$ commute with $\psi_s$ so that we can consider

$$\Phi_s := \psi_s \sin(\alpha) \circ T_\alpha \cos(\alpha)$$

This defines a one-parameter family of isometries $\Gamma := (\Phi_s)_{s \in \mathbb{R}}$ in $E$, which by construction satisfies $\Phi_s(c(0)) = c(s)$. We refer to the isometries as translations along $c$, and list some straightforward properties:

**Proposition 12.** Let $c$ be a geodesic in $E(\kappa, \tau)$ with geodesic projection $\tilde{c}$ and let $\Gamma$ be the family of translations along $c$.

(a) For $\tau = 0$ reflection through $\tilde{c} \times \mathbb{R}$ is an isometry commuting with $\Gamma$.

(b) For $\tau \neq 0$ the geodesic $c$ horizontal and rotation of angle $\pi$ about $c$ commutes with $\Gamma$.

(c) For any $\tau \in \mathbb{R}$ we have the following:
The horizontal lift $\gamma$ of $\tilde{\gamma}_0$ with $\gamma(0) = c(0)$ is a horizontal geodesic and the rotation of angle $\pi$ about $\gamma$, denoted by $\sigma$, satisfies $\sigma \circ \Phi_s = \Phi_{-s} \circ \sigma$.

Vertical translations and $\Gamma$ commute.

Constant mean curvature spheres. We quote various results on constant mean curvature spheres in non-compact $E(\kappa, \tau)$-spaces. They are

- invariant by rotation about a fiber,
- unique up to isometries and
- embedded.

By [AR05, Theorem 6], any immersed constant mean curvature sphere in a non-compact $E(\kappa, \tau)$-space is a rotational sphere, which also implies uniqueness up to isometries. Explicit examples of rotationally-invariant CMC spheres in $H_2 \times \mathbb{R}$, $S^2 \times \mathbb{R}$, $Nil_3$ and $PSL_2(\mathbb{R})$ can be found in the following papers: [Onn08, ST05, HH89, FMP99, Pen10].

The assumption on $E$ being non-compact is crucial for embeddedness. In [Tor10] there are examples of non-embedded rotationally-invariant CMC spheres in Berger spheres.

4.2. $E(\kappa, \tau)$ spaces with $\kappa \leq 0$. The $E(\kappa, \tau)$-spaces with $\kappa \leq 0$ are simply connected homogeneous 3-manifolds diffeomorphic to $\mathbb{R}^3$ and arise as metric Lie groups. We describe a model and some geometric properties. The advantage of this model is that the limits $\kappa \to 0$ and $\tau \to 0$ are well-defined, also on the level of orthonormal frames.

**Model.** For our purpose the classification of [MP12] provides a convenient description of these spaces. For $\kappa \leq 0$ and $\tau \in \mathbb{R}$ let

$$A(\kappa, \tau) := \begin{pmatrix} \sqrt{-\kappa} & 0 \\ 2\tau & 0 \end{pmatrix}.$$ 

We want to compute

$$\begin{pmatrix} a_{11}(z) & a_{12}(z) \\ a_{21}(z) & a_{22}(z) \end{pmatrix} := e^{zA(\kappa, \tau)}.$$ 

For $\kappa < 0$ we have

$$e^{zA(\kappa, \tau)} = \begin{pmatrix} 2\tau^{-1} & \sqrt{-\kappa} \\ e^{z\sqrt{-\kappa}} - 1 & 1 \end{pmatrix}$$ 

and for $\kappa = 0$ we get

$$e^{zA(0, \tau)} = \begin{pmatrix} 1 & 0 \\ 2\tau z & 0 \end{pmatrix}.$$ 

We observe $\lim_{\kappa \to 0} e^{zA(\kappa, \tau)} = e^{zA(0, \tau)}$ for all $z, \tau \in \mathbb{R}$ so that the first expression also makes sense for $\kappa = 0$.

The space $\mathbb{R}^2 \times A(\kappa, \tau) \mathbb{R}$ is a metric Lie group with group structure

$$(x_1, y_1, z_1) * (x_2, y_2, z_2) := \left((x_1, y_1) + e^{zA(\kappa, \tau)}(x_2, y_2), z_1 + z_2\right)$$

and Riemannian metric

$$\langle \cdot, \cdot \rangle_{(x, y, z)} = \left(e^{-2z\sqrt{-\kappa}} - \frac{4\tau}{\kappa}(e^{-z\sqrt{-\kappa}} - 1)^2 \right) dx^2 + dy^2 + dz^2 + \frac{2\tau}{\sqrt{-\kappa}}(e^{-z\sqrt{-\kappa}} - 1)(dx \otimes dy + dy \otimes dx).$$
The canonical orthonormal frame, obtained by left-translation of the Euclidean frame from the origin \((0, 0, 0)\), is
\[
E_1(x, y, z) = e^{\sqrt{-\kappa}} \partial_x + \frac{2\tau}{\sqrt{-\kappa}} \left(e^{\sqrt{-\kappa}} - 1\right) \partial_y, \quad E_2(x, y, z) = \partial_y, \quad E_3(x, y, z) = \partial_z.
\]

The Riemannian connection with respect to this frame has the following representation:
\[
\nabla_{E_1} E_1 = \sqrt{-\kappa} E_1, \quad \nabla_{E_1} E_2 = \tau E_3, \quad \nabla_{E_1} E_3 = -\sqrt{-\kappa} E_1 - \tau E_2,
\]
\[
\nabla_{E_2} E_1 = \tau E_3, \quad \nabla_{E_2} E_2 = 0, \quad \nabla_{E_2} E_3 = -\tau E_1,
\]
\[
\nabla_{E_3} E_1 = \tau E_2, \quad \nabla_{E_3} E_2 = -\tau E_1, \quad \nabla_{E_3} E_3 = 0.
\]

**Proposition 13.** Let \(\kappa \leq 0\) and \(\tau \in \mathbb{R}\). On \(\mathbb{R}^2\) we consider the Riemannian metric
\[
\tilde{g}_{(x,z)} = e^{-2\sqrt{-\kappa}} dx^2 + dz^2.
\]
Then \(\Pi: (\mathbb{R}^3, \langle \cdot, \cdot \rangle) \to (\mathbb{R}^2, \tilde{g}), (x, y, z) \mapsto (x, z)\) is a Riemannian submersion with geodesic fibers over the simply connected surface \((\mathbb{R}^2, \tilde{g})\) with constant curvature \(\kappa\). This submersion has bundle curvature \(\tau\), so that \((\mathbb{R}^3, \langle \cdot, \cdot \rangle)\) is isometric to \(E(\kappa, \tau)\).

**Sketch of proof.** We can refer to various Theorems in [MP12], but let us give the explicit argument:

- The vertical space is spanned by \(E_2\) while the horizontal space is spanned by \(E_1\) and \(E_3\).
- For a horizontal vector \(v = \lambda E_1 + \mu E_3\) we have \(\tilde{g}_{(x,z)}(d\Pi v, d\Pi v) = \lambda^2 + \mu^2\), so that \(\Pi\) is indeed a Riemannian submersion.
- In view of the Riemannian connection we have \(\langle R(E_1, E_3)E_3, E_1 \rangle = \kappa - 3\tau^2\), so that \((\mathbb{R}^2, \tilde{g})\) is a simply connected surface with constant curvature \(\kappa\).
- We also have \(\langle R(E_2, E_3)E_3, E_2 \rangle = \tau^2\), which proves the claim about the bundle curvature. \(\square\)

**A horizontal geodesic and the induced translations.** The unit-speed curve
\[
c: E \to E, \quad c(s) := (0, 0, s)
\]
is a horizontal geodesic.

Our model is a metric Lie group and so a one-parameter family of isometries is
\[
\Phi_s: E \to E, \quad \Phi_s(x, y, z) := \left(e^{\sqrt{-\kappa}} x, \frac{2\tau}{\sqrt{-\kappa}} \left(e^{\sqrt{-\kappa}} - 1\right) x + y, z + s\right),
\]
which preserves \(c\). We refer to \(\Gamma := (\Phi_s)_{s \in \mathbb{R}}\) as translations along \(c\). The infinitesimal generator or Killing field of \(\Gamma\) at \((x, y, z) \in E\) is given by
\[
K_{(x,y,z)} := \frac{d}{ds}\Phi_s(x, y, z) = x \sqrt{-\kappa} e^{-z \sqrt{-\kappa}} E_1 + 2\tau x e^{-z \sqrt{-\kappa}} E_2 + E_3. \tag{7}
\]
We observe \(K\) is independent of \(y\).
**Foliation by vertical planes.** We want to exhibit vertical planes \((P_s)_{s \in \mathbb{R}}\) as in subsection 4.1. In fact, for the vertical planes \((P_s)_{s \in \mathbb{R}}\) with \(P_s = \Pi^{-1}(\tilde{\gamma}_s)\) we will only need the curve \(\tilde{\gamma}_0\) explicitly.

**Proposition 14.** Consider

\[
\tilde{\gamma}_0 : \mathbb{R} \to \mathbb{R}^2, \quad \tilde{\gamma}_0(t) := \begin{cases} 
\left( \frac{\tanh(t \sqrt{-\kappa})}{\sqrt{-\kappa}}, \frac{\log(\text{sech}(t \sqrt{-\kappa}))}{\sqrt{-\kappa}} \right) & \text{for } \kappa < 0, \\
(t, 0) & \text{for } \kappa = 0.
\end{cases}
\] (8)

Then \(\tilde{\gamma}_0(t)\) is also a continuous function of \(\kappa\): For each \(t \in \mathbb{R}\) the limit of \(\tilde{\gamma}_0(t)\) for \(\kappa < 0\) and \(\kappa \to 0\) exists and equals \((t, 0)\). Moreover \(\tilde{\gamma}_0\) is a unit-speed geodesic in \(\mathbb{R}^2\) with respect to the metric induced by the Riemannian submersion \(\Pi : \mathbb{R}^2 \ltimes A(\kappa, \tau) \to \mathbb{R}^2\).

Each horizontal lift \(\gamma\) of \(\tilde{\gamma}_0\) satisfies

\[
\gamma'(t) = \text{sech}(t \sqrt{-\kappa}) E_1 - \tanh(t \sqrt{-\kappa}) E_3.
\] (9)

**Sketch of proof.** The claim about the continuity of \(\tilde{\gamma}_0(t)\) is clear.

For \(\kappa = 0\) we have \(\tilde{\gamma}_0(t) = (t, 0)\) and the metric induced on \(\mathbb{R}^2\) is the Euclidean one, so that \(\tilde{\gamma}_0\) is geodesic.

For \(\kappa < 0\) we consider the upper half-plane \(\mathbb{H} := \{(u, v) : v > 0\}\) and note that

\[
g(u, v) := \frac{1}{-\kappa} \text{sech}(\sqrt{-\kappa})
\]
defines a metric of constant sectional curvature \(\kappa\) on \(\mathbb{H}\). Then

\[
\mathbb{R} \to \mathbb{H}, \quad t \mapsto (\tanh(t \sqrt{-\kappa}), \text{sech}(t \sqrt{-\kappa}))
\]

parametrizes a unit-speed geodesic semi-circle through \((0, 1)\). One can check that

\[
\varphi : \mathbb{R}^2 \to \mathbb{H}, \quad \varphi(x, z) := \left( x \sqrt{-\kappa}, e^{z \sqrt{-\kappa}} \right)
\]
is an isometry with

\[
\varphi^{-1} : \mathbb{H} \to \mathbb{R}^2, \quad \varphi^{-1}(u, v) = \left( \frac{u}{\sqrt{-\kappa}}, \frac{\log(v)}{\sqrt{-\kappa}} \right).
\]

Applying \(\varphi^{-1}\) to the geodesic in \(\mathbb{H}\) proves the claim about \(\tilde{\gamma}_0\). Regarding the horizontal lift \(\gamma\) we observe the following for \(v := \text{sech}(t \sqrt{-\kappa}) E_1 - \tanh(t \sqrt{-\kappa}) E_3\):

- \(v\) is horizontal,
- \(\nabla_n v \equiv 0\) and
- \(d\Pi v \equiv \tilde{\gamma}_0'\).

This completes the proof. \(\square\)

### 5. TRANSLATIONALLY-INvariant CYLINDERS as ODE SOLUTIONS

In this section we carry over the arguments used in the first part of the paper:

- As in case of Sol\(_3\) we consider translationally-invariant surfaces whose generating curves are graphical.
- The geometric discussion of the ODE for the graphical solution and its extension to a simple closed embedded curve carry over from Sol\(_3\) almost literally, so that we only state what is different.
5.1. ODE for translationally-invariant surfaces of constant mean curvature. The foliation by vertical planes $(P_s)_{s \in \mathbb{R}}$ is preserved by $\Gamma$. For $C^2$-functions $x, y: J \to \mathbb{R}$ consider the unit-speed curve $\beta: J \to E, \beta(t) := T_{y(t)}(\gamma(x(t)))$, which is contained in the vertical plane $P_0$. A surface invariant by translation along $c$ is parametrized by

$$f: \mathbb{R} \times J \to E, \quad f(s, t) := \Phi_s(\beta(t)).$$

(10)

We specialize to $x(t) = t$ and $h(t) = y(t)$, i.e. we are considering vertical graphs over $\gamma$. For these vertical graphs over $\gamma$ we study the ODE for constant mean curvature:

**Proposition 15.** Let $H$ be in $\mathbb{R}$. There exists a smooth function $F: \mathbb{R}^2 \to \mathbb{R}$ such that the invariant surface

$$f: \mathbb{R} \times J \to E, \quad f(s, t) := \Phi_s(T_{h(t)}(\gamma(t))), \quad \text{where } h \in C^2(J, \mathbb{R}),$$

has constant mean curvature $H$ with respect to the upper normal if and only if

$$h''(t) = F(t, h'(t)) \quad \text{for all } t \in J.$$  

(11)

**Proof.** Let $v_1 := \partial_s f$ and $v_2 := \partial_t f$. We denote the upper normal to $f$ by $N$, so that $g_{ij} := \langle v_i, v_j \rangle$ and $b_{ij} := \langle \nabla v_i, v_j, N \rangle$ for $i, j \in \{1, 2\}$ are the coefficients of the first and second fundamental form. Then the mean curvature of $f$ is given by

$$H = \frac{b_{11}g_{11}^2 + 2b_{12}g_{12} + b_{22}g_{22}^2}{2}.$$  

Here we note that $H$ depends on $t$, $h'(t)$ and $h''(t)$, but not on $h(t)$ itself. This is due to the existence of vertical translations commuting with $\Gamma$.

We assume $H$ to be constant and therefore get an implicit differential equation depending on $h'(t)$ and $h''(t)$. Now we want to show we can solve this implicit equation for $h''(t)$. We have

$$v_2 = \gamma' + h'\xi \quad \text{and} \quad \nabla_{v_2}v_2 = \sum_{\omega = 1}^{\omega = \Omega} \langle \nabla_{v_2}v_2, N \rangle g_{11} = \langle h''(N, \xi) + \langle w, N \rangle \rangle g_{11} \frac{g_{11}}{2\det(g)}.$$  

We obviously have $w = w(t, h'(t))$ and so the only term containing $h''(t)$ is

$$\frac{b_{22}g_{11}}{2\det(g)} = \frac{\langle v_2, v_2, N \rangle g_{11}}{2\det(g)} = \frac{\langle h''(N, \xi) + \langle w, N \rangle \rangle g_{11}}{2\det(g)}.$$  

The surface $f$ is a Killing graph with respect to the Killing field $\xi$, so that $\langle N, \xi \rangle$ is positive for $N$ is the upper normal. We also have $g_{11} > 0$ since $\Gamma$ does never act trivially. Hence we can solve the implicit equation for $h''(t)$ and obtain a function $F: \mathbb{R}^2 \to \mathbb{R}$ with $h''(t) = F(t, h'(t))$. This function $F$ is smooth because each $\Phi_s$ is smooth and thus are $g$ and $b$. It is defined on whole $\mathbb{R}^2$ because we can prescribe any kind of function $h: J \to \mathbb{R}$.  

\[ \square \]

5.2. Geometric discussion of the ODE: Half-cylinder solution and its extension to an embedded cylinder. The following lemma corresponds to Lemma 3:

**Lemma 16.** Given $a, b \in \mathbb{R}$ and $H > H(E)$, the Picard-Lindelöf Theorem gives a unique maximal solution $h: I_{\text{max}} \to \mathbb{R}$ with $h(0) = a$ and $h'(0) = b$ satisfying (11). For each $a, b \in \mathbb{R}$ it has the following properties:
(a) [Horizontal boundedness]: There are real numbers $R_\pm = R_\pm (a, b) < R_\mp = R_\mp (a, b)$ such that $I_{\text{max}} = (R_-, R_+).

(b) [Vertical boundedness]: There is $K = K(a, b) > 0$ such that $\lim_{t \to R_\pm} |h(t)| \leq K$.

(c) [Asymptotic behaviour]: We have $\lim_{t \to R_\pm} h'(t) = \pm \infty$.

(d) [Monotonicity]: There is $t_0 \in (R_-, R_+) \text{ with } h'(t_0) = 0$. On $(R_-, t_0)$ the function $h$ is monotonically decreasing and on $(t_0, R_+)$ it is monotonically increasing.

(e) [Symmetry]: For $b = 0$ we have $R := R_+ = -R_-$ for the maximal solution with $h(0) = a$ and $h'(0) = 0$.

Proof. Items (a) to (d) can be proven in the same way as in the case of $\text{Sol}_3$, see Lemma 3.

(e): For $b = 0$ the tangent $\beta'(0)$ is horizontal. If $\tau = 0$ we can reflect through the vertical plane $\tilde{c} \times \mathbb{R}$. This reflection and $\Gamma$ commute and so the reflected graph satisfies the same ODE. Moreover the initial values are invariant. Hence the reflection leaves the solution invariant.

In case $\tau \neq 0$ let us translate such that $a = 0$. Applying a rotation of angle $\pi$ about $c$ and $\gamma$ yields a graphical solution satisfying the same ODE (by Proposition 12 (b) and (c)) and the initial values remain invariant.

In both cases this implies $R_+ = -R_-$.

The symmetric solution from (e) generates a horizontal cylinder for all $\tau$ and a cylinder with sloped axis for $\tau = 0$, that is, in $\mathbb{H}^2 \times \mathbb{R}$ and $\mathbb{S}^2 \times \mathbb{R}$.

Theorem 17. Let $c$ be a geodesic in a non-compact $E(\kappa, \tau)$-space with geodesic projection, that is $c$ has slope $\alpha \in (0, \frac{\pi}{2})$ for $\tau = 0$ and $\alpha = \frac{\pi}{2}$ for $\tau \neq 0$; let $\Gamma$ be the family of translations along $c$. For each $H > H(E)$ there is a smoothly embedded simple closed curve $\beta$ which generates a $\Gamma$-invariant surface $f$ with constant mean curvature $H$. The surface $f$ is embedded, except for $\mathbb{S}^2 \times \mathbb{R}$.

For arbitrary $\tau$ the surface is invariant by a rotation of angle $\pi$ about $\gamma$. For $\tau = 0$ the surface has a vertical mirror plane containing the axis. If the axis is horizontal the surface is invariant by a rotation of angle $\pi$ about its axis.

Proof. Let $h: (-R, R) \to \mathbb{R}$ be the symmetric solution from Lemma 16, i.e. $h'(0) = 0$. After a vertical translation we may assume $h(R) = h(-R) = 0$: in view of Lemma 16 (c) the graph meets $\gamma$ orthogonally at $t = \pm R$.

We extend the graph $h$ by $\sigma$ to a closed curve $\tilde{\beta}$, where $\sigma$ denotes the rotation of angle $\pi$ about $\gamma$. The curve $\tilde{\beta}$ is smooth because of the graph’s asymptotic behaviour and monotonicity of $h$ implies embeddedness of $\tilde{\beta}$. Due to $\sigma \circ \Phi_s = \Phi_{-s} \circ \sigma$ from Proposition 12 the curve $\tilde{\beta}$ is generating a translationally-invariant surface with constant mean curvature. The surface is embedded except for $\mathbb{S}^2 \times \mathbb{R}$ where translations can also be screw-motions.

The claimed symmetries follow from Proposition 12 (a) and (b).

Remark 18. In [Pen10] invariant surfaces in $E(-1, \tau)$ were studied: In the upper-half plane model surfaces invariant by hyperbolic and parabolic isometries in $E(-1, \tau)$ were discussed by considering graphs $h$ as above and choosing special coordinates. Penafiel obtained an integral representation for the graphs $h$ which lead to surfaces with constant
mean curvature \( H \). For some values of \( H \) and \( \tau \) the integral can be computed explicitly, but for surfaces invariant by hyperbolic isometries with \( H > \frac{1}{2} \) and \( \tau \neq 0 \) this has not been the case.

6. **Horizontal diameter of an \( \text{MCH} \)-cylinder with horizontal axis**

We conclude this paper with an application of the weight formula, i.e. by a flux computation: Without calculating the ODE for translationally-invariant surfaces it is possible to determine \( I_{\max} \) and thus the horizontal diameter of a horizontal \( \text{MCH} \)-cylinder. We exclude \( \Sigma^2 \times \mathbb{R} \) from the present discussion, i.e. we are only considering \( \widehat{E}(\kappa, \tau) \)-spaces with \( \kappa \leq 0 \).

In Section 5 we considered graphs above the horizontal lift \( \gamma \) of \( \bar{\gamma}_0 \) with \( \gamma(0) = c(0) \) as generating graphs of the half-cylinder solution. To calculate \( I_{\max} = (\tau, R) \) of these graphs let us reparametrize the graph on \([0, R]\) by arc-length:

**Lemma 19.** Let \( H > H(E) \) and let \( h: (-R, R) \to \mathbb{R} \) be the symmetric maximal solution with \( h(R) = 0 \) and \( h(0) = a \), established in Lemma 16. An orientation-preserving reparametrization of the graph on \([0, R]\) by arc-length can be realized by a parametrized curve \( \beta: [0, L] \to E \), \( \beta = \gamma \circ x + (0, y, 0) \) for some \( x, y \in C^2([0, L], \mathbb{R}) \) with the following properties:

- \( L \) is the arc-length of the graph on \([0, R]\), that is \( L = \int_0^R \sqrt{1 + h'^2(t)} \, dt \),
- \( \beta \) respects the initial values of the graph, i.e. \( \beta(0) = (0, a, 0) \) and \( \beta'(0) = \gamma'(0) \),
- \( \beta(L) = \gamma(R) \) and \( \beta'(L) = E_2 \).

For the invariant surface \( f: \mathbb{R} \times [0, L] \to E \), \( f(s, t) := \Phi_s(\beta(t)) \) the tangent vectors are

\[
\begin{align*}
v_1 := \frac{\partial f}{\partial s} &= \sinh \left( x \sqrt{-\kappa} \right) E_1 + 2\tau \sinh \left( x \sqrt{-\kappa} \right) E_2 + E_3, \\
v_2 := \frac{\partial f}{\partial t} &= x' \sech \left( x \sqrt{-\kappa} \right) E_1 + y'(t) E_2 - x' \tanh \left( x \sqrt{-\kappa} \right) E_3.
\end{align*}
\]

**Proof.** The claim about the reparametrization is clear.

For the tangent vector \( v_1 \) we have

\[ v_1 = K_{\gamma(x(t)) + (0, y(t), 0)}. \]

Since the Killing field \( K \) is independent of \( y \) and \( \gamma \) is the horizontal lift of \( \bar{\gamma}_0 \) it suffices to insert \( x = \frac{\tanh(x(t) \sqrt{-\kappa})}{\sqrt{-\kappa}} \) and \( z = \frac{\log(\sech(x(t) \sqrt{-\kappa}))}{\sqrt{-\kappa}} \) into \( K_{(x, y, z)} \), given by (7), to show (12). For (13) we note \( v_2 = x' \gamma' \circ x + y' E_2 \) and refer to (9). \( \square \)

The horizontal diameter of a horizontal \( \text{MCH} \)-cylinder can be computed using the weight formula; it is independent of \( \tau \).

**Theorem 20.** For the symmetric solution from Lemma 16 we have

\[ R = \frac{1}{\sqrt{-\kappa}} \arctanh \left( \frac{\sqrt{-\kappa}}{2H} \right). \]

Therefore the horizontal diameter of a horizontal \( \text{MCH} \)-cylinder is \( 2R \). The \( \text{MCH} \)-cylinders with axis \( c \), considered as a one-parameter family depending on \( H \in (H(E), \infty) \), are unbounded for \( H \to H(E) = \frac{\sqrt{-\kappa}}{2} \) and converge to the horizontal geodesic \( c \) for \( H \to \infty \).
Proof. Let \( h: (-R, R) \to \mathbb{R} \) be the maximal solution from Lemma 16 (e) and \( \beta \) the reparametrization of \( h|_{[0,R]} \) by arc-length as in Lemma 19. We use the weight formula to determine the explicit value of \( R \). We consider the invariant surface

\[
f: \mathbb{R} \times [0, L] \to \mathbb{R}^3, \quad f(s,t) = \Phi_s(\gamma(x(t)) + (0, y(t), 0)).
\]

For a bounded domain \( \Omega \subseteq \mathbb{R} \times [0, L] \) with \( \partial \Omega \) a closed Jordan curve we let \( \eta \) be the outer unit conormal along \( f(\partial \Omega) \) and \( N \) the inner normal of the surface. The weight formula (see [HdLR05, Proposition 3] for a proof in a general Riemannian 3-manifold) yields

\[
2H \int_{f(\Omega)} \langle N, Y \rangle = \int_{f(\partial \Omega)} \langle \eta, Y \rangle, \quad Y \text{ Killing field.} \quad (14)
\]

We apply (14) to the Killing field \( Y = \xi = E_2 \) and set \( \Omega := [0, 1] \times [0, L] \).

![Figure 10. Application of weight formula](image)

We need some geometric data of the invariant surface \( f \), which are easily computed with Lemma 19:

The entries of the induced metric \( g = (\langle v_j, v_k \rangle)_{1 \leq j, k \leq 2} \) on \( \mathbb{R} \times J \) are

\[
\begin{align*}
g_{11} &= \cosh^2(x\sqrt{-\kappa}) + 4\tau^2 \sinh^2(x\sqrt{-\kappa}), \\
g_{12} &= 2\tau \sinh(x\sqrt{-\kappa})y'(t), \\
g_{22} &= x'^2 + y'^2 \quad (15)
\end{align*}
\]

with

\[
\det(g) = \cosh^2(x\sqrt{-\kappa})(x'^2 + y'^2 + 4\tau^2 \tanh^2(x\sqrt{-\kappa})x'^2). \quad (16)
\]

The inner normal \( N \) to \( f \) satisfies

\[
\sqrt{\det(g)}N = \cosh(x\sqrt{-\kappa})y'\left(-\sech(x\sqrt{-\kappa})E_1 + \tanh(x\sqrt{-\kappa})E_3\right) \\
+ \cosh(x\sqrt{-\kappa})x'E_2 \\
- 2\tau \sinh(x\sqrt{-\kappa})x'\left(\tanh(x\sqrt{-\kappa})E_1 + \sech(x\sqrt{-\kappa})E_3\right). \quad (17)
\]
First we compute the left-hand side of (14). In view of (17) we get
\[ 2H \int_{f(t)} (N, E_2) = 2H \int_{[0,1] \times [0,L]} x'(t) \cosh(x(t)\sqrt{-\kappa}) \, ds \, dt = \frac{2H}{\sqrt{-\kappa}} \sinh(R\sqrt{-\kappa}). \]
To compute the right-hand side of (14) we decompose the boundary parametrization as
\[ f(\partial \Omega) = \beta_1 \oplus \beta_2 \oplus \beta_3 \oplus \beta_4, \]
where
\[ \beta_1(t) = f(0, t), \quad \beta_2(s) = f(s, L), \quad \beta_3(t) = f(1, L-t), \quad \beta_4(s) = f(1-s, 0). \]
See Figure 10 on page 23. We denote by \( \eta_1 \) to \( \eta_4 \) the respective unit conormals along \( \beta_1 \) to \( \beta_4 \). Due to \( \beta_3(t) = \Phi_4(\beta_1(L-t)) \) we have \( \beta_3'(t) = -\beta_1'(L-t) \) and thus \( \eta_3(t) = -\eta_1(L-t) \). Since \( E_2 \) is a constant Killing field this implies
\[ \int_{\beta_1} \langle \eta_1, E_2 \rangle + \int_{\beta_3} \langle \eta_3, E_2 \rangle = 0. \]
To determine \( \int_{\beta_4} \langle \eta_4, E_2 \rangle \) note that \( \beta_4'(s) = -\frac{\partial f}{\partial s}(1-s, 0) = E_3 \) and \( \frac{\partial f}{\partial t}(1-s, 0) = E_1 \), i.e. \( \eta_4 = E_1 \). This shows
\[ \int_{\beta_4} \langle \eta_4, E_2 \rangle = 0. \]
Finally we consider \( \int_{\beta_2} \langle \eta_2, E_3 \rangle \). We note \( \beta_2'(s) = v_1 \) and for the conormal we get
\[ \eta_2 = \frac{1}{\sqrt{g_{11}} \sqrt{\det(g)}} (-g_{12}v_1 + g_{11}v_2). \]
At \( L \) we have
\[ x(L) = R, \quad x'(L) = 0, \quad y(L) = 0 \quad \text{and} \quad y'(L) = 1, \]
so that in view of (12), (13) and (15) we get
\[ \langle \eta_4, E_2 \rangle \sqrt{g_{11}} = -4\tau^2 \sinh^2(R\sqrt{-\kappa}) + \cosh^2(R\sqrt{-\kappa}) + 4\tau^2 \sinh^2(R\sqrt{-\kappa}) \cosh(R\sqrt{-\kappa}) \]
\[ = \cosh(R\sqrt{-\kappa}). \]
Noting that \( \sqrt{\langle \beta_4', \beta_4' \rangle} = \sqrt{\langle v_1, v_1 \rangle} = \sqrt{g_{11}} \) we get
\[ \int_{\beta_4} \langle \eta_4, E_2 \rangle = \int_{[0,1]} \left[ \langle \eta_4, E_2 \rangle \cdot \sqrt{\langle \beta_4', \beta_4' \rangle} \right] \, ds = \cosh(\sqrt{-\kappa}R). \]
Combining these results yields
\[ \frac{2H}{\sqrt{-\kappa}} \sinh(R\sqrt{-\kappa}) = \cosh(R\sqrt{-\kappa}). \]
Because of \( 2H > \sqrt{-\kappa} \) we can solve this equation for \( R \) and get
\[ R = \frac{1}{\sqrt{-\kappa}} \arctanh \left( \frac{\sqrt{-\kappa}}{2H} \right). \]
The unboundedness for \( H \to H(E) \) is clear since \( \arctanh(u) \) is unbounded for \( u \to 1 \).

The convergence to \( c \) for \( H \to \infty \) follows by comparison with CMC spheres; here we use embeddedness of \( \text{mCH} \)-cylinders in \( E(\kappa, \tau) \)-spaces with \( \kappa \leq 0 \).
APPENDIX A. COMPUTATION OF ODE FOR CYLINDERS IN Sol3

The mean curvature of an surface \( f \) invariant by translations along the base in Sol3 is easy to compute in terms of the orthonormal frame from Section 1:

**Proposition 21.** Let \( f \) be as in (3), i.e., \( f \) parametrizes a surface invariant by translations along the base \( c \) in Sol3. Then we have

\[
C := \sqrt{\det(g)} = \sqrt{x'^2 + y'^2 + (x'y + xy')^2}
\]

for the induced Riemannian metric on \( \mathbb{R} \times J \). Moreover the mean curvature \( H \) of \( f \) in terms of \( \gamma = (x, y, 0) \) with respect to the inner normal satisfies the equation

\[
2HC^3 = \left[ xy'-x'y + (x^2-y^2)(xy'+x'y) \right] \cdot \left[ x'^2 + y'^2 \right] \\
+ 2(y'y + xx')(yy' - xx')(xy' + x'y) \\
+ (x^2+y^2+1) \left( x'y'' - x''y' + (x^2-y^2)(xy' + x'y) \right).
\]  

(18)

**Sketch of proof.** We have

\[
v_1 := \frac{\partial f}{\partial s}(s, t) = \begin{pmatrix} -e^{-s}x(t) \\ e^{s}y(t) \\ 1 \end{pmatrix} = -xE_1 + yE_2 + E_3
\]

and

\[
v_2 := \frac{\partial f}{\partial t}(s, t) = \begin{pmatrix} e^{-s}x'(t) \\ e^{s}y'(t) \\ 0 \end{pmatrix} = x'E_1 + y'E_2.
\]

Thus the upper normal \( N \) to \( f \) is

\[
N = \frac{1}{\sqrt{x'^2 + y'^2 + (x'y + xy')^2}} \left[ -y'E_1 + x'E_2 - (xy' + x'y)E_3 \right].
\]

(19)

The entries of the induced metric \( g = \langle v_j, v_k \rangle_{1 \leq j, k \leq 2} \) on \( \mathbb{R} \times J \) are

\[
g_{11} = x^2 + y^2 + 1, \\
g_{12} = -xx' + yy', \\
g_{22} = x'^2 + y'^2.
\]

(20)

Furthermore let us compute \( \nabla_{v_j} v_k \) for \( j, k \in \{1, 2\} \):

\[
\nabla_{v_j} v_1 = -xE_1 - yE_2 + (y^2 - x^2)E_4, \\
\nabla_{v_j} v_2 = (xx' + yy')E_3, \\
\nabla_{v_2} v_2 = x''E_1 + y''E_2 + \left( y^2 - x^2 \right)E_3.
\]

It can be checked that \( C := \sqrt{\det(g)} \) agrees with the denominator of the coefficients in (19), i.e. we have

\[
C \cdot N = -y'E_1 + x'E_2 - (xy' + x'y)E_3.
\]
Thus the second fundamental form $b = (\langle \nabla v_j v_k, N \rangle)_{1 \leq j, k \leq 2}$ satisfies:

$$
Cb_{11} = xy' - x'y + (x^2 - y^2)(xy' + x'y),
$$
$$
Cb_{12} = -(xx' + yy')(xy' + x'y),
$$
$$
Cb_{22} = -x''y' + x'y'' + (x'^2 - y'^2)(xy' + x'y).
$$

In order to verify (18), the previous expressions must be plugged into

$$
2H = \frac{g_{12}}{C^2}b_{11} - 2\frac{g_{12}}{C^2}b_{12} + \frac{g_{11}}{C^2}b_{22} = \frac{g_{22}Cb_{11} - 2g_{12}Cb_{12} + g_{11}Cb_{22}}{C^3}.
$$

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