BUSEMANN FUNCTIONS AND JULIA-WOLFF-CARATHÉODORY THEOREM FOR POLYDISCS

CHIARA FROSINI

Abstract. The classical Julia-Wolff-Carathéodory Theorem is one of the main tools to study the boundary behavior of holomorphic self-maps of the unit disc of $\mathbb{C}$. In this paper we prove a Julia-Wolff-Carathéodory’s type theorem in the case of the polydisc of $\mathbb{C}^n$. The Busemann functions are used to define a class of “generalized horospheres” for the polydisc and to extend the notion of non-tangential limit. With these new tools we give a generalization of the classical Julia’s Lemma and of the Lindelöf Theorem, which the new Julia-Wolff-Carathéodory Theorem relies upon.

1. Introduction

The Julia-Wolff-Carathéodory Theorem and its variants are powerful tools for investigating the boundary behavior of holomorphic self-maps of the unit disc $\Delta \subset \mathbb{C}$ (see, e.g. [2], [7], [9], [15], [21], [22]). The importance of this classical theorem (JWC’s Theorem for short) in different contexts such as the study of dynamics, extension of biholomorphisms, composition operators, semigroups of holomorphic maps, is well known and justifies several generalizations to higher and infinite dimensions due to various authors. We cite here Rudin [23] for the case of the unit ball in $\mathbb{C}^n$, Abate ([3], [4]) for strongly convex and strongly pseudoconvex domains, and Reich and Shoikhet [19] for the infinite dimensional case. The argument used to prove the classical JWC’s Theorem inspires its generalizations: let $f$ be a holomorphic self-map of a domain $D$ and $p \in \partial D$. If the distance of $f(z)$ from the boundary of the domain, $\text{dist}(f(z), \partial D)$, is “comparable” to $\text{dist}(z, \partial D)$ as $z \to p$ (no matter along which direction), then $f$ together with its normal derivatives has a limit at $p$ along some admissible directions. We point out that to find the class of these admissible directions is one of the main efforts to state and prove all JWC’s-type theorems.

In this paper we prove a JWC’s Theorem in the case of the polydisc $\Delta^n$ of $\mathbb{C}^n$ which generalizes the one obtained by Abate [1] (see also Jafari [14]). In order to achieve this result we first prove generalizations of the Julia’s Lemma and the Lindelöf Theorem. Our main issue is the use of Busemann sublevel sets [5] for the polydisc as “generalized horospheres” in the Julia’s Lemma. The Busemann sublevel sets are also used to define the analogous of the Koranyi regions. In contrast with what happens for the existing generalizations of the JWC’s Theorem, in our

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statement the class of admissible directions at a point, \( p \), of the boundary of the polydisc depends upon an entire family of complex geodesics \( [2] \) “passing through” \( p \).

In order to avoid technical complications and give a more geometric approach, we deal only with the bidisc \( \Delta^2 \) and use the following terminology. The symbol \( K_{\Delta^2} \) will denote the Kobayashi distance on \( \Delta^2 \). A map \( \Psi \in \text{Hol}(\Delta, \Delta^2) \) is called a complex geodesic passing through \( y \in \Delta^2 \) if it is an isometry between the Poincaré distance \( \omega \) of \( \Delta \) and the Kobayashi distance of \( \Delta^2 \) whose image closure contains the point \( y \). Let \( \Psi \) be a complex geodesic passing through a point \( y = (y_1, y_2) \in \partial\Delta^2 \) and suppose \( |y_1| = 1 \). It is well known that at least one component of \( \Psi \) is an automorphism of \( \Delta \) (see \( [2, \text{Proposition 2.6.10}] \)). Thus, up to re-parametrization, we can assume \( \Psi \) is given by \( \Delta \ni \phi \) with \( \phi \) an automorphism of \( \Delta \) (see \( [2, \text{pag.23}] \)). The Busemann sublevel set of center \( y \in \partial\Delta^2 \) and radius \( R > 0 \) of the function \( \lambda_g \) is not necessary a priori, we will assume that \( \phi \) is an automorphism of \( \Delta \) (see \( [2, \text{Proposition 2.6.10}] \)). Thus, up to re-parametrization, we can assume \( \phi \) is given by \( \Delta \ni \phi \) with \( \phi \) an automorphism of \( \Delta \) (see \( [2, \text{pag.23}] \)). The Busemann sublevel set of center \( y \in \partial\Delta^2 \) and radius \( R > 0 \) of the function \( \lambda_g \) is not necessary a priori, we will assume that \( \phi \) is an automorphism of \( \Delta \) (see \( [2, \text{Proposition 2.6.10}] \)). Thus, up to re-parametrization, we can assume \( \phi \) is given by \( \Delta \ni \phi \) with \( \phi \) an automorphism of \( \Delta \) (see \( [2, \text{pag.23}] \)).

Notice that every product of horocycles can be seen as a Busemann sublevel set in \( \Delta^2 \), boundary of \( \Delta \) and the Kobayashi distance of \( \Delta^2 \) depends upon an entire family of complex geodesics \( \Psi \) “pass through” \( y \).

There is only one Busemann sublevel set of a given radius \( R > 0 \) of the form \( \Delta \times E_{\lambda} \) if \( (y_1, y_2) \in \Delta \times \partial\Delta \) or \( E_{\lambda} \times \Delta \) if \( (y_1, y_2) \in \partial\Delta \times \Delta \).
In the sequel, we will denote by $B_{(\lambda_1,\lambda_2)}(y, R)$ (with $\lambda_1, \lambda_2 > 0$, possibly $+\infty$) the Busemann sublevel set given by $E_\Delta(y_1, \lambda_1 R) \times E_\Delta(y_2, \lambda_2 R)$, with the convention that $E_\Delta(y, \lambda R) = \Delta$ if either $y_i \in \Delta$ or $y_i \in \partial \Delta$ and $\lambda_i = +\infty$.

Our first result is the following version of Julia’s lemma:

**Theorem 1.** Let $f = (f_1, f_2) \in \text{Hol}(\Delta^2, \Delta^2)$. Let $x = (x_1, x_2) \in \partial(\Delta \times \Delta) = \partial \Delta^2$ and let (for example) $\varphi_g(x) = (\sigma, \pi)$ be a complex geodesic passing through $x$. Let

$$\frac{1}{2} \log \lambda_j := \lim_{t \to 1^-} [K_{\Delta^2}(0, \varphi_g(tx_1)) - \omega(0, f_j(\varphi_g(tx_1)))] \quad j = 1, 2$$

Suppose that either $\lambda_1 < \infty$ or $\lambda_2 < \infty$. Then there exists a point $y = (y_1, y_2) \in \partial \Delta^2$ such that for all $R > 0$

$$f(B_{(1,\lambda_2)}(y, R)) \subseteq B_{(\lambda_1,\lambda_2)}(y, R).$$

A second achievement is the proof of a generalization of the Lindelöf Theorem which is based on the definition of admissible limits. Let $x \in \partial \Delta^2$. A continuous curve $\sigma(t) \subset \Delta^2$ converging to $x$ as $t \to 1^-$ is called a $x-$curve. Let $\varphi_g : \Delta \to \Delta^2$ be a complex geodesic passing through $x$ and parameterized by $z \mapsto (g(z))$ with $g \in \text{Hol}(\Delta, \Delta)$. A holomorphic function $\tilde{\pi}_g : \Delta^2 \to \Delta$ such that: $\tilde{\pi}_g \circ \varphi_g = \text{id}_\Delta$ is called a $g-$left inverse of $\varphi_g$. The composition $\pi_g := \varphi_g \circ \tilde{\pi}_g : \Delta^2 \to \Delta^2$, (such that $\pi_g \circ \varphi_g = \varphi_g$, and $\tilde{\pi}_g \circ \tilde{\pi}_g = \tilde{\pi}_g$) is called a $g-$holomorphic retraction. The pair $(\varphi_g, \pi_g)$ is a $g$-projection device. Existence of $g$-projection devices, also known as Lempert’s projection devices, in convex domains is established in [20] (see also [13], [2]). In strongly convex domains the Lempert’s projection devices are essentially unique (see [9]) while in the bidisc various holomorphic retractions with different “fibers” may correspond to a given complex geodesic. The following definitions will be used in the statements of the generalizations of the Lindelöf Theorem and JWC’s Theorem.

**Definition 2.** Let $x \in \partial \Delta^2$ and $M > 1$. The $g$-Koranyi region $H_{\varphi_g}(x, M)$, of vertex $x$ and amplitude $M$ is:

$$H_{\varphi_g}(x, M) := \{z \in \Delta^2 : \lim_{r \to 1^-} K_{\Delta^2}(z, \varphi_g(r)) = K_{\Delta^2}(\varphi_g(0), \varphi_g(r)) + K_{\Delta^2}(\varphi_g(0), z) < 2 \log M\}.$$

A holomorphic function $f \in \text{Hol}(\Delta^2, \Delta)$ has $K_g-$limit equal to $L \in \mathbb{C}$ if $f$ approaches to $L$ inside any $g$-Koranyi region.

The function $f$ is $K_g-$bounded if $\forall M$ there exists a constant $C_M > 0$ such that $\|f(z)\| < C_M$ for all $z \in H_{\varphi_g}(x, M)$.

**Definition 3.** Let $\sigma(t) \subset \Delta^2$ be a $x-$curve.

- the curve $\sigma(t)$ is $g$-special if $K_{\Delta^2}(\sigma(t), \pi_g(\sigma(t))) \to 0$ as $t \to 1^-$.
- the curve $\sigma(t)$ is $g$-restricted if $\tilde{\pi}_g(\sigma(t)) \to \tilde{\pi}_g(x)$ non-tangentially as $t \to 1^-$, for $j = 1, 2$.

Moreover, if $h : \Delta^n \to \mathbb{C}$ is holomorphic we say that $h$ has restricted $K_g-$limit equal to $L \in \mathbb{C}$ if $h$ has limit $L$ along any curve which is $g$-special and $g$-restricted, and we write

$$\bar{K}_g \lim_{z \to x} h(z) = L.$$

The announced Lindelöf type Theorem, proved in this paper, has the following statement:
Theorem 4. Let \( f \in \text{Hol}(\Delta^2, \Delta) \) be a holomorphic function. Given \( x \in \partial \Delta^2 \) let \( \varphi_g \) be a complex geodesic passing through \( x \). Assume that \( f \) is \( K_g \)-bounded. If \( \sigma_0 \) is a \( g \)-special and \( g \)-restricted \( x \)-curve such that

\[
\lim_{t \to 1^-} f(\sigma_0(t)) = L
\]

then \( f \) admits restricted \( K_g \)-limit equal to \( L \) at \( x \).

The above result plays a key role in the proof of our main result:

Theorem 5. Let \( f \in \text{Hol}(\Delta^2, \Delta^2) \) and \( x \in \partial \Delta^2 \). Let \( \varphi_g \) be any complex geodesic passing through \( x \) and parameterized by \( \varphi_g(z) = (z, g(z)) \), with \( g \in \text{Hol}(\Delta, \Delta) \). Let \( \pi_g : \Delta^2 \to \Delta \) be the \( g \)-left inverse of \( \varphi_g \) given by \( \pi_g(z_1, z_2) = z_1 \). Suppose that for \( j = 1, 2 \)

\[
\frac{1}{2} \log \lambda_j = \lim_{t \to 1^-} [K_{\Delta^2}(0, \varphi_g(tx_1)) - \omega(0, f_j(\varphi_g(tx_1)))] < \infty.
\]

Then there exists a point \( y = (y_1, y_2) \in (\partial \Delta)^2 \) such that the restricted \( K_g \)-limit of \( f_j \) at \( x \) is \( y_j \) for \( j = 1, 2 \), and

\[
\begin{align*}
\tilde{K}_g \lim_{z \to x} \frac{y_j - f_j(z)}{1 - \pi_g(z)} &= \lambda_j \min\{1, \lambda_g\} \\
\tilde{K}_g \lim_{z \to x} \frac{y_j - f_j(z)}{1 - z_2} &= \lambda_j \max\{1, \lambda_g\}.
\end{align*}
\]

The paper is organized as follows: in Section 2 we study in detail the geometry the Busemann sublevel sets. In Section 3 we discuss of special and restricted curves. In Section 4 we introduce a new extension of the notion of non-tangential limits and in Section 5 we prove a new version of the Lindelöf Theorem. In Section 6 we give our extension of the classical Julia’s Lemma. In Section 7 we prove our generalization of the Julia-Wolff-Carathéodory Theorem. We end the paper in Section 8 with an application of our results to the study of the dynamics of fixed points free holomorphic self-maps of the bidisc. In fact in this section we give a geometrical interpretation of a result due to Hervé [13] in terms of the set of generalized Wolff points of a fixed point free \( f \in \text{Hol}(\Delta^2, \Delta^2) \).

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2. Busemann Functions and a Family of Horospheres

The aim of this section is to study in detail the Busemann sublevel sets and their relation with the horospheres in the polydisc. Let \( \varphi_g \in \text{Hol}(\Delta, \Delta^2) \) be a complex geodesic, passing through a point \( y = (y_1, y_2) \in \partial \Delta^2 \), parameterized as \( z \to (z, g(z)) \). Let denote by \( \lambda_g \) the boundary dilation coefficient of \( g \) at \( y_1 \), that is

\[
\lambda_g := \liminf_{z \to y_1} \frac{1 - |g(z)|}{1 - |z|}.
\]

In strongly convex domains of \( \mathbb{C}^n \) the definition of Busemann sublevel set is equivalent to the definition of horosphere, but in the bidisc this is no longer true. Let \( E(y, R) \) be the small horosphere of center \( y \in \Delta^n \) and radius \( R \) given by

\[
E(y, R) = \left\{ z \in D : \limsup_{w \to y} |K_{\Delta^n}(z, w) - K_{\Delta^n}(0, w)| < \frac{1}{2} \log R \right\},
\]
and let $F(y, R)$ be the big horosphere of center $y$ and radius $R$ given by

$$F(y, R) = \left\{ z \in D : \lim\inf_{w \to y}[K_{\Delta^n}(z, w) - K_{\Delta^n}(0, w)] < \frac{1}{2} \log R \right\}.$$ 

If $n = 1$ then $F(y, R) \equiv E(y, R) \equiv E_\Delta(y, R) \subset \Delta$, the horocycle centered in $y$ with radius $R$. If $n > 1$ then the following proposition holds:

**Proposition 6.** Let $\varphi_h(z) = (\theta(z), h(z))$ be a complex geodesic in $\Delta^2$ passing through a point $y \in \partial \Delta^2$, where $\theta \in \text{Aut}(\Delta)$ and $h \in \text{Hol}(\Delta, \Delta)$.

If $y = (e^{i\alpha_1}, e^{i\alpha_2}) \in (\partial \Delta)^2$, let $\lambda_\theta$ and $\lambda_h$ respectively be the boundary dilation coefficients of the maps $\theta$ and $h$, at $e^{i\alpha_1}$ then

$$\mathbb{B}^{\varphi_h}(y, R) = E_\Delta(e^{i\alpha_1}, \lambda_\theta R) \times E_\Delta(e^{i\alpha_2}, \lambda_h R)$$

If $y \in [\partial \Delta^2 \setminus (\partial \Delta)^2]$ then

$$\mathbb{B}^{\varphi_e}(y, R) \equiv E(y, R) \equiv F(y, R).$$

**Proof.** Up to conjugation with automorphisms, we can suppose $(e^{i\alpha_1}, e^{i\alpha_2}) = (1, 1)$. Then $h(1) = 1$, in the sense of non-tangential limit and $\theta(1) = 1$. Let first suppose that $x \in \mathbb{B}^{\varphi_h}((1, 1), R)$, then, by definition of Busemann sublevel sets, $\lim\sup_{r \to 1} \{\max\{\omega(x_1, \theta(r)), \omega(x_2, h(r))\} - \omega(0, r)\} \leq \frac{1}{2} \log R$. We consider the two following cases:

a) there exists a sequence $\{r_k\}_{k \in \mathbb{N}} \subseteq (0, 1)$ such that $r_k \to 1$ as $k \to \infty$ and $\max\{\omega(x_1, \theta(r_k)), \omega(x_2, h(r_k))\} = \omega(x_1, \theta(r_k))$, and

b) there exists a sequence $\{r_k\}_{k \in \mathbb{N}} \subseteq (0, 1)$ such that $r_k \to 1$ as $k \to \infty$ and $\max\{\omega(x_1, \theta(r_k)), \omega(x_2, h(r_k))\} = \omega(x_2, h(r_k)).$

In case a) we have that

$$\frac{1}{2} \log R \geq \lim_{r \to 1} \{\max\{\omega(x_1, \theta(r)), \omega(x_2, h(r))\} - \omega(0, r)\} = \lim_{k \to \infty} \{\omega(x_1, \theta(r_k)) - \omega(0, r_k)\} = \lim_{r \to 1} \{\omega(x_1, \theta(r_k)) - \omega(0, \theta(r_k)) + \omega(0, \theta(r_k)) - \omega(0, r_k)\} = \lim_{w \to 1^-} \{\omega(x_1, w) - \omega(0, w)\} + \frac{1}{2} \log \frac{1}{\lambda_\theta}.$$ 

It follows that $x_1 \in E_\Delta(1, \lambda_\theta R).$ Moreover

$$\frac{1}{2} \log R \geq \lim_{k \to \infty} \{\omega(x_2, h(r_k)) - \omega(0, r_k)\} = \lim_{k \to \infty} \{\omega(x_2, h(r_k)) - \omega(0, h(r_k)) + \omega(0, h(r_k)) - \omega(0, r_k)\} = \lim_{r \to 1} \{\omega(x_2, h(r)) - \omega(0, h(r))\} + \lim_{r \to 1} \{\omega(0, h(r)) - \omega(0, r)\} = \lim_{w \to 1^-} \{\omega(x_2, w) - \omega(0, w)\} + \frac{1}{2} \log \frac{1}{\lambda_h}$$

then $x_2 \in E_\Delta(1, \lambda_h R).$ Thus, in case a), the first inclusion is proved and $\mathbb{B}^{\varphi_h}((1, 1), R) \subseteq E_\Delta(1, \lambda_\theta R) \times E_\Delta(1, \lambda_h R).$
In case b) we notice that
\[
\frac{1}{2} \log R \geq \lim_{r \to 1} \left[ \max\{\omega(x_1, \theta(r)), \omega(x_2, h(r))\} - \omega(0, r) \right]
\]
\[
= \lim_{k \to \infty} \left[ \omega(x_2, h(r_k)) - \omega(0, r_k) \right]
= \lim_{r \to 1^-} \left[ \omega(x_2, h(r)) - \omega(0, r) \right]
\geq \lim_{k \to \infty} \left[ \omega(x_1, \theta(r_k)) - \omega(0, r_k) \right].
\]
then proceeding as in case a) it follows that \(x_1 \in E_\Delta(1, \lambda_\theta R)\) and \(x_2 \in E_\Delta(1, \lambda_\theta R)\).

We conclude that, also in this case, \(B^{\varphi_h}((1, 1), R) \subseteq E_\Delta(1, \lambda_\theta R) \times E_\Delta(1, \lambda_\theta R)\).

On the other hand, if a point \(x \in E_\Delta(1, \lambda_\theta R) \times E_\Delta(1, \lambda_\theta R)\) then by definition of horocycle
\[
\lim_{w \to 1^-} \left[ \omega(x_1, w) - \omega(0, w) \right] = \lim_{r \to 1^-} \left[ \omega(x_1, \theta(r)) - \omega(0, \theta(r)) \right] \leq \frac{1}{2} \log \lambda_\theta R
\]
and
\[
\lim_{r \to 1^-} \left[ \omega(x_2, h(r)) - \omega(0, h(r)) \right] \leq \frac{1}{2} \log \lambda_\theta R.
\]
Thus
\[
\lim_{r \to 1^-} \left[ \omega(x_2, h(r)) - \omega(0, r) \right] = \lim_{r \to 1^-} \left[ \omega(x_2, h(r)) - \omega(0, h(r)) + \omega(0, h(r)) - \omega(0, r) \right]
\leq \frac{1}{2} \log \lambda_\theta R \frac{1}{\lambda_\theta} = \frac{1}{2} \log R.
\]
Swapping \(h\) with \(\theta\), arguing as above, we have
\[
\lim_{r \to 1^-} \left[ \omega(x_1, \theta(r)) - \omega(0, r) \right] \leq \frac{1}{2} \log R.
\]
We conclude that
\[
(2.1) \quad \lim_{r \to 1^-} \left[ \max\{\omega(x_1, \theta(r)), \omega(x_2, h(r))\} - \omega(0, r) \right] \leq \frac{1}{2} \log R
\]
and \(\mathbb{B}^{\varphi_h}((1, 1), R) = E_\Delta(1, \lambda_\theta R) \times E_\Delta(1, \lambda_\theta R)\).

If \(y\) is a point of the flat component of \(\partial \Delta^2\), then \(E(y, R) \equiv F(y, R)\) and therefore
the limit that defines the small and big horospheres exists. Thus it follows immediately that \(\mathbb{B}^{\varphi}(y, R) \equiv E(y, R) \equiv F(y, R)\) for all geodesic \(\varphi\) and for all \(R > 0\).

Let us notice that if we consider the re-parametrization of \(\varphi_h(z) = (\theta(z), h(z))\)
given by \(\varphi_g(z) = (z, g(z))\) where \(g := h \circ \theta^{-1}\) we have
\[
\mathbb{B}^{\varphi_g}((1, 1), R) = E_\Delta(1, R) \times E_\Delta(1, \lambda_g R),
\]
where \(\lambda_g = \frac{\lambda_\theta}{\lambda_\theta}\). It follows by the same arguments used in Proposition 6.
For this reason, from now on, we consider only parametrization of the type \(z \to (z, g(z))\)
(respectively \(z \to (g(z), z)\) (see also the Introduction).

For later use we now compute explicitly the Busemann sublevel sets. We use the above notation. Let us first consider a point \(y = (y_1, y_2)\) contained in a flat component of the boundary of the bidisc, \(\partial \Delta^2\). As Proposition 6 states, the Busemann sublevel sets, centered in \(y\), coincide with the small and the big horosphere (see Abate [1] for an explicit description of small and big horospheres). On the other
hand let us consider a point $y = (y_1, y_2) \in (\partial \Delta)^2$. Without loss of generality we can suppose that $y = (1, 1)$. We claim that

\begin{equation}
E_\Delta(1, R) \times E_\Delta(1, \lambda_g R) = \left\{ z \in \Delta^2 : \max_{j=1,2} \frac{|1-z_j|^2}{1-|z_j|^2} \lim_{r \to 1} \frac{(1-r^2)}{1-|\varphi_g(r)|^2} \leq R \right\}.
\end{equation}

Indeed assume that

\[ \max_{j=1,2} \left\{ \frac{|1-z_1|^2}{1-|z_1|^2}, \frac{|1-z_2|^2}{1-|z_2|^2} \right\} \leq R. \]

Thus the two possibilities hold:

\begin{enumerate}
  \item $i) \max_{j=1,2} \left\{ \frac{|1-z_1|^2}{1-|z_1|^2}, \frac{|1-z_2|^2}{1-|z_2|^2} \right\} = \frac{|1-z_1|^2}{1-|z_1|^2}$ then $\frac{|1-z_1|^2}{1-|z_1|^2} \lambda_g \leq \frac{|1-z_1|^2}{1-|z_1|^2} \leq R$ and, by definition of horocycles, $z_1 \in E_\Delta(1, R)$ and $z_2 \in E_\Delta(1, \lambda_g R)$.
  \item $ii) \max_{j=1,2} \left\{ \frac{|1-z_1|^2}{1-|z_1|^2}, \frac{|1-z_2|^2}{1-|z_2|^2} \right\} = \frac{|1-z_2|^2}{1-|z_2|^2}$ then $\frac{|1-z_2|^2}{1-|z_2|^2} \lambda_g \leq \frac{|1-z_2|^2}{1-|z_2|^2} \leq R$ and, by definition of horocycles again $z_1 \in E_\Delta(1, R)$ and $z_2 \in E_\Delta(1, \lambda_g R)$. Namely $z \in E_\Delta(1, R) \times E_\Delta(1, \lambda_g R)$. Conversely let $z = (z_1, z_2) \in E_\Delta(1, R) \times E_\Delta(1, \lambda_g R) = \mathbb{B}^{\varphi_g}((1,1), R)$, with $\varphi_g(z) = (z, g(z))$. By definition of horocycles, it follows that either $\frac{|1-z_1|^2}{1-|z_1|^2} \leq R$ or $\frac{|1-z_2|^2}{1-|z_2|^2} \leq R$, hence

\[ \max_{j=1,2} \left\{ \frac{|1-z_1|^2}{1-|z_1|^2}, \frac{|1-z_2|^2}{1-|z_2|^2} \right\} \leq R. \]

By the very definition of sublevel sets of Busemann functions, $z = (z_1, z_2) \in \mathbb{B}^{\varphi_g}((1,1), R)$, proving the claim.

3. Special and restricted curves

Let be $x = (x_1, x_2) \in \partial \Delta^2$ and $\varphi_x : \Delta \to \Delta^2$ the complex geodesic passing through $x$, defined by $\varphi_x(z) = zx$. Let us denote by $d_x$ the Silov degree of $x$, that is the number of components of $x$ with absolute value $1$ and $\tilde{x} := (\tilde{x}_1, \tilde{x}_2)$ is the Silov part of $x$, defined by

\[ \tilde{x}_j = \begin{cases} x_j & \text{if } |x_j| = 1, \\ 0 & \text{if } |x_j| < 1. \end{cases} \]

In this setting Abate \cite{Abate} gave the following definition

\textbf{Definition 7.} We call the holomorphic function $\tilde{p}_x : \Delta^2 \to \Delta$ given by

\[ \tilde{p}_x(z) := \frac{1}{d_x}(z, \tilde{x}), \]

such that $\tilde{p}_x \circ \varphi_x = id_{\Delta^n}$, an Abate’s left inverse of $\varphi_x$.

We call the holomorphic function $p_x : \Delta^n \to \Delta^n$ given by $p_x(z) := \varphi_x \circ \tilde{p}_x$, such that $p_x \circ p_x = p_x$ and $p_x \circ \varphi_x = \varphi_x$ an Abate’s holomorphic retraction.

A $x$-curve $\sigma(t) \subset \Delta^n$ is $A$-special if $K_{\Delta^n}(\sigma(t), p_x(\sigma(t))) \to 0$, as $t \to 1^-$.

A $x$-curve $\sigma(t) \subset \Delta^n$ is $A$-restricted if $\tilde{p}_x(\sigma(t))$ converges to $1$ non-tangentially.

The pair $(p_x, \varphi_x)$ is called an $A$–projection device.
We notice that the $A$–projection device due to Abate is not unique, that is, given the complex geodesic $\varphi_x = xz$, the left inverse $\tilde{p}_x$, and the holomorphic retraction $p_x$ are not unique. Moreover $\varphi_x$ is not the unique complex geodesic passing through $x$. Thus we are led to give the following definitions:

**Definition 8.** Let $\varphi_g : \Delta \to \Delta^2$ be a complex geodesic passing through $x$ and parameterized by $z \mapsto (z, g(z))$; $g \in \text{Hol}(\Delta, \Delta)$.

A holomorphic function $\tilde{\pi}_g : \Delta^2 \to \Delta$ such that $\tilde{\pi}_g \circ \varphi_g = \text{id}_\Delta$ is called a $g$-left inverse function of $\varphi_g$.

A holomorphic function $\pi_g := \varphi_g \circ \tilde{\pi}_g : \Delta^2 \to \Delta^2$, such that $\pi_g \circ \varphi_g = \varphi_g$, and $\tilde{\pi}_g \circ \tilde{\pi}_g = \pi_g$ is called a $g$-holomorphic retraction.

The pair $(\varphi_g, \pi_g)$ is a $g$-projection device.

**Definition 9.** Let $\sigma(t) \subset \Delta^2$ be a $x$–curve.

- the curve $\sigma(t)$ is $g$-special if $K_{\Delta^2}(\sigma(t), \pi_g(\sigma(t))) \to 0$ as $t \to 1^-$.
- the curve $\sigma(t)$ is $g$-restricted if $\tilde{\pi}_g(\sigma(t)) \to \tilde{\pi}_g(x)$ non-tangentially as $t \to 1^-$, for $j = 1, 2$.

As a matter of notation, when we refer to the geodesic parameterized by $\varphi(z) = (z, z)$, we omit the index $g$, since $g = \text{id}_\Delta$. In addition we denote by $p$ the $A$–holomorphic retraction given in definition 4.

In this setting, an $x$–curve $\gamma$ is $A$–special and $A$–restricted if $K_{\Delta^2}(\gamma(t), p(\gamma(t))) \to 0$ as $t \to 1^-$ and $\tilde{p}(\gamma(t))$ approach to the point $\tilde{p}(x)$ non tangentially.

We notice that if $x$ is a point on a flat component of $\partial \Delta^2$, definitions 7 and 9 are equivalent, and we have that a $x$–curve $\gamma$ is $g$-special and $g$-restricted if and only if it is $A$-special and $A$-restricted. On the other hand if $x \in (\partial \Delta)^2$, we have the following characterization:

**Proposition 10.** Let denote by $\varphi_x(z)$ the complex geodesic passing through the point $x \in (\partial \Delta)^2$ parameterized by $z \to (xz)$ and let $\pi_g : \Delta^2 \to \varphi_x(\Delta)$ be any linear holomorphic retraction on the image of the complex geodesic $\varphi_x$. Let $\gamma(t) = (\gamma_1(t), \gamma_2(t))$ be a $g$–restricted, $x$–curve in $\Delta^2$. Then $\gamma$ is $g$–special if and only if

$$\lim_{t \to 1^-} \frac{\gamma_1(t) - a\gamma_1(t) - (1 - a)\gamma_2(t)}{1 - \gamma_1(t)[a\gamma_1(t) + (1 - a)\gamma_2(t)]} = 0$$

By an easy calculation we get:

$$\frac{\gamma_1(t) - a\gamma_1(t) - (1 - a)\gamma_2(t)}{1 - \gamma_1(t)[a\gamma_1(t) + (1 - a)\gamma_2(t)]} = \frac{(1 - a)(\frac{1 - \gamma_2(t)}{1 - \gamma_1(t)} - 1)}{\frac{1 - \gamma_2(t)}{1 - \gamma_1(t)} + (1 - a)\gamma_2(t)}$$

Thus if the curve $\gamma$ is $g$–special we necessarily have that $(1 - \gamma_2(t))/(1 - \gamma_1(t)) \to 1$ as $t \to 1^-$. On the other hand if

$$\lim_{t \to 1^-} \frac{1 - \gamma_2(t)}{1 - \gamma_1(t)} = 1,$$
taking into account that \( \gamma \) is \( g \)-restricted, then
\[
\left| \frac{1 - \gamma(t)}{1 - \gamma_1(t)} \right| \frac{a\gamma_1(t) + (1-a)\gamma_2(t)}{1 - \gamma_1(t)} + \frac{1 - |a\gamma_1(t) + (1-a)\gamma_2(t)|}{|1 - \gamma_1(t)|} \geq \frac{|1 - \gamma(t) - (1-a)\gamma_2(t)|}{M|1 - \gamma_1(t)|} = \frac{|a\frac{1-\gamma(t)}{1-\gamma_1(t)} + (1-a)\frac{1-\gamma_2(t)}{1-\gamma_1(t)}|}{M} \rightarrow \frac{1}{M}
\]
as \( t \to 1^- \), and condition (3.1) is also sufficient. \( \square \)

It is worth noticing that the Abate projection \( p \) is a special linear projection with \( a = b = \frac{1}{2} \):

**Proposition 11.** Let \( x \in (\partial \Delta)^2 \) and let \( \gamma(t) = (\gamma_1(t), \gamma_2(t)) \) be a \( x \)-curve in \( \Delta^2 \). Let \( \varphi_x(z) \) be the complex geodesic, parameterized by \( z \to (zx) \), passing through the point \( x \). Let \( \pi_g \) be any linear projection on \( \varphi_x \). Then \( \gamma \) is \( g \)-special and \( g \)-restricted if and only if \( \gamma \) is \( A \)-special and \( A \)-restricted.

**Proof.** Without loss of generality we suppose \( x = (1, 1) \) and thus \( \varphi_x(z) = \varphi(z) = (z, z) \). As in the proof of Proposition 11 we can write that \( \pi_g(z_1, z_2) = (a_2z_1 + (1 - a)z_2, a_2 + (1 - a)z_2) \), with \( a, b \in \mathbb{C} \).

We first prove the “\( \Longrightarrow \)” implication. We show that \( \gamma \) is \( g \)-special, and in particular that \( K_{\Delta^2}(\gamma(t), \pi_g(\gamma(t))) \to 0 \) as \( t \to 1^- \). By the triangular inequality
\[
K_{\Delta^2}(\gamma(t), \pi_g(\gamma(t))) \leq K_{\Delta^2}(\gamma(t), p(\gamma(t))) + K_{\Delta^2}(\pi_g(\gamma(t)), p(\gamma(t)))
\]
and, since \( \gamma \) is a \( A \)-special curve, \( K_{\Delta^2}(\gamma(t), p(\gamma(t))) \to 0 \) as \( t \to 1^- \). Moreover
\[
K_{\Delta^2}(\pi_g(\gamma(t)), p(\gamma(t))) = K_{\Delta^2}(\varphi(\pi_g(\gamma(t))), \varphi(\tilde{p}(\gamma(t)))) = \omega(\pi_g(\gamma(t)), \tilde{p}(\gamma(t))).
\]
We claim that \( \omega(\pi_g(\gamma(t)), \tilde{p}(\gamma(t))) \to 0 \) as \( t \to 1^- \). By the very definition of Poincaré metric, this is equivalent to
\[
\lim_{t \to 1^-} \frac{\left| \tilde{p}(\gamma(t)) - \pi_g(\gamma(t)) \right|}{\left| 1 - \tilde{p}(\gamma(t))\pi_g(\gamma(t)) \right|} = 0.
\]
First we notice:
\[
\frac{\left| \tilde{p}(\gamma(t)) - \pi_g(\gamma(t)) \right|}{\left| 1 - \tilde{p}(\gamma(t))\pi_g(\gamma(t)) \right|} = \frac{\left| \tilde{p}(\gamma(t)) - 1 + 1 - \pi_g(\gamma(t)) \right|}{\left| 1 - \tilde{p}(\gamma(t))\pi_g(\gamma(t)) \right|} = \left| \frac{1 - \pi_g(\gamma(t))}{1 - \tilde{p}(\gamma(t))} - 1 \right|.
\]
Moreover, by definition of \( A \)-projection device,
\[
\frac{1 - \pi_g(\gamma(t))}{1 - \tilde{p}(\gamma(t))} = \frac{1 - a\gamma_1 + (1-a)\gamma_2}{1 - \frac{1}{\gamma_1 + \gamma_2}} = \frac{1 - a\gamma_1 + (1-a)\gamma_2}{1 - \frac{1}{\gamma_1 + \gamma_2}} = \frac{2(1 - a\gamma_1 + (1-a)\gamma_2)}{1 - \gamma_1 + 1 - \gamma_2} = \frac{2a(1 - \gamma_1) + (1-a)(1 - \gamma_2)}{1 - \gamma_1 + 1 - \gamma_2}. \]
and by Proposition [10] we get
\[
\lim_{t \to 1^-} 2 \frac{a + (1 - a) \frac{1 - \gamma(t)}{\gamma(t)}(1 - \gamma(t))}{1 + (1 - \gamma(t))} = 1.
\]
Furthermore, since by hypothesis the curve \( \gamma \) is \( A \)-restricted, then there exists \( M > 1 \) such that
\[
\frac{|1 - \gamma(t)|}{1 - |\gamma(t)|} < M
\]
and in particular
\[
\left| \frac{1 - \bar{p}(\gamma(t))\bar{\pi}_g(\gamma(t))}{1 - \bar{p}(\gamma(t))} \right| = \frac{1 - |\bar{p}(\gamma(t))| |\bar{\pi}_g(\gamma(t))|}{|1 - \bar{p}(\gamma(t))|} \geq \frac{1 - |\bar{p}(\gamma(t))|}{|1 - \bar{p}(\gamma(t))|} > \frac{1}{M}.
\]
Then we conclude that
\[
\lim_{t \to 1^-} \omega(\bar{\pi}_g(\gamma(t)), \bar{p}(\gamma(t))) = 0
\]
and the curve \( \gamma \) is \( g \)-special. Let notice that by equation (3.3) and since the curve \( \gamma \) is \( A \)-restricted
\[
\frac{|1 - \bar{\pi}_g(\gamma(t))|}{1 - |\bar{\pi}_g(\gamma(t))|} = \frac{|1 - \bar{\pi}_g(\gamma(t))|}{|1 - \bar{p}(\gamma(t))|} \geq \frac{1 - |\bar{p}(\gamma(t))|}{|1 - \bar{p}(\gamma(t))|} < 4M
\]
and then the curve \( \gamma \) is \( g \)-restricted.

The last step consists in proving the “\( \Rightarrow \)” implication of the theorem. To do this it is sufficient to interchange the Abate’s projection \( p \) with the linear projection \( \pi_g \) in the proof above and the thesis easily follows.

**Remark 12.** By Proposition [11] it follows the Abate’s Julia-Wolff-Carathéodory theorem for linear projections.

The next question is what happens if we consider another geodesic passing through the point \( x \in (\partial \Delta)^2 \). Arguing as in Proposition [11] we have:

**Proposition 13.** Let \((\varphi_g, \pi_g)\) be a projection device. Let assume that the geodesic \( \varphi_g \) passes through a point \( x \in (\partial \Delta)^2 \) and set \( \lambda_g := \liminf_{z \to x} \frac{1 - |g(z)|}{1 - |z|} < \infty \). Let \( \gamma := (\gamma_1, \gamma_2) \) be a \( g \)-restricted \( x \)-curve in \( \Delta^2 \). Then \( \gamma \) is \( g \)-special if and only if
\[
\lim_{t \to 1^-} \frac{1 - \gamma_2(t)}{1 - \gamma_1(t)} = \lambda_g
\]

4. THE NON-TANGENTIAL LIMIT

The non-tangential limit in \( \Delta \) can be defined in two equivalent ways. We can say that a function \( f \in \text{Hol}(\Delta, \Delta) \) has non-tangential limit \( L \in \mathbb{C} \) at a point \( y \in \partial \Delta \) if \( f(z) \to L \) as \( z \to y \), inside any Stolz region, \( H(y, M) \) of vertex \( y \) and amplitude \( M > 1 \), where
\[
H(y, M) := \left\{ z \in \Delta : \frac{|y - z|}{1 - |z|} < M \right\}.
\]
We can equivalently say that \( f \in \text{Hol}(\Delta, \Delta) \) has non-tangential limit \( L \in \mathbb{C} \) at a point \( y \in \partial \Delta \) if \( f(\sigma(t)) \to L \) as \( t \to 1 \), along any curve \( \sigma : [0, 1) \to \Delta \) such that \( \sigma(t) \to y \) non-tangentially as \( t \to 1^- \). In [1] (see also [3]) Abate generalizes the Stolz
region giving the following definition of (small) Koranyi region (of vertex \( y \in \partial \Delta^2 \) and amplitude \( M \)),

\[
H(y, M) := \{ z \in \Delta^2 : \limsup_{w \to y} K_{\Delta^2}(z, w) - K_{\Delta^2}(0, w) + K_{\Delta^2}(0, z) < \log M \}.
\]

Thus an extension of the first definition of non-tangential limit becomes (see [1]):

**Definition 14.** A map \( f : \Delta^2 \to \mathbb{C} \) has \( K \)–limit \( L \in \mathbb{C} \) at \( y \in \partial \Delta^2 \) if \( f(z) \to L \) as \( z \to y \) inside any Koranyi region.

On the other hand, by means of \( A \)–special and \( A \)–restricted curves, Abate says that ([1]) a holomorphic function \( f : \Delta^2 \to \mathbb{C} \) has restricted \( K \)–limit \( L \) at \( x \) if \( f(\sigma(t)) \to L \) for any \( A \)–special and \( A \)–restricted \( x \)–curve \( \sigma(t) \subset \Delta^2 \) and we write \( \tilde{K} \)–limit as follows:

\[
\tilde{K}_{z \to x} f(z) = L.
\]

We notice that the definitions of \( K \)–limit and restricted \( K \)–limit are no more equivalent. More precisely if \( f \) has \( K \)–limit \( L \) at \( y \in \partial \Delta^2 \) then it has restricted \( K \)–limit too. The converse is false (see example in [1]). We extend these definitions by means of Busemann functions. The first step consists in giving the following extension of the notion of Stolz region:

**Definition 15.** Let \( x \in \partial \Delta^2 \) and \( M > 1 \), the \( g \)–Koranyi region \( H_{\phi_g}(x, M) \), of vertex \( x \) and amplitude \( M \) is:

\[
H_{\phi_g}(x, M) := \{ z \in \Delta^2 : \lim_{r \to 1^-} K_{\Delta^2}(z, \phi_g(r)) - K_{\Delta^2}(\phi_g(0), \phi_g(r)) + K_{\Delta^2}(\phi_g(0), z) < \log M \}.
\]

(4.1)

And then we, naturally, say that

**Definition 16.** A holomorphic function \( f \in \text{Hol}(\Delta^2, \Delta^2) \) has \( K_g \)–limit \( L \in \mathbb{C} \) if \( f \) approaches to \( L \) inside any \( g \)–Koranyi region.

If we consider the complex geodesic \( \phi(z) = (z, z) \) then the Koranyi region \( H((1, 1), M) \) coincide with the \( g \)–Koranyi region \( H_{\phi}(1, 1), M) \).

Moreover let \( (\phi_g, \pi_g) \) be a \( g \)–projection device as in Definition 8.

**Definition 17.** A holomorphic function \( h : \Delta^2 \to \mathbb{C} \) is said to have restricted \( K_g \)–limit \( L \) if \( h \) has limit \( L \) along any curve which is \( g \)–special and \( g \)–restricted, and we write \( \tilde{K}_{g} \)–limit as follows:

\[
\tilde{K}_{z \to x} h(z) = L.
\]

Obviously Definition 16 and Definition 17 are not equivalent but again the \( K_g \)–limit implies the restricted \( \tilde{K}_g \)–limit.

5. **Lindelöf Theorems**

The classical Lindelöf principle implies that if \( f \in \text{Hol}(\Delta, \Delta) \) has limit \( L \) along any given \( 1 \)–curve, then \( L \) is the non-tangential limit of \( f \) at 1. The first step to generalize this theorem to several complex variables consists in detecting a correct class of curves. Let \( (\phi_g, \pi_g) \) a \( g \)–projection device. The idea is to consider the \( g \)–special and \( g \)–restricted curves.

In this setting we prove the following first generalization of the Lindelöf principle:
**Theorem 18.** Let $f \in \text{Hol}(\Delta^2, \mathbb{C})$ be a bounded holomorphic function. Let $x \in \partial \Delta^2$. Assume there exists a $g$-special $x$-curve $\sigma$ such that
\[
\lim_{t \to 1^-} f(\sigma(t)) = L \in \mathbb{C}.
\]
Then $f$ has restricted $\tilde{K}_g$-limit $L$ at $x$.

**Proof.** This proof is similar to the one in [1] (see Theorem 2.1). We first observe that, given $\sigma$ a $g$-special $x$-curve,
\[
0 \leq \omega(f(\sigma(t)), f(\pi_g(\sigma(t)))) \leq K_{\Delta^2}(\sigma(t), \pi_g(\sigma(t))) \to 0
\]
as $t \to 1^-$. Therefore the limit of $f(\pi_g(\sigma(t)))$ exists, as $t \to 1^-$, if and only if the limit of $f(\sigma(t))$ as $t \to 1^-$ does, and the two limits are equal. In particular $f(\pi_g(\sigma(t))) \to L$ as $t \to 1^-$ and by classical Lindelöf principle $f(\pi_g(\sigma(t))) \to L$ for any $g$-restricted $x$-curve and by remark (5.1) it follows that $f(\sigma(t)) \to L$ for any $g$-restricted and $g$-special $x$-curve $\sigma$.

Since in the Julia-Wolff-Carathéodory theorem the functions we deal with are incremental ratios, a stronger result than Theorem [18] is needed. It is worthwhile to introduce some definitions and preliminary results.

**Definition 19.** Let $f \in \text{Hol}(\Delta^2, \mathbb{C})$. We say that $f$ is $K_g$-bounded if $\forall M$ there exists a constant $C_M > 0$ such that $||f(z)|| < C_M$ for all $z \in H_{\varphi_g}(x, M)$.

**Lemma 20.** Let $x \in \partial \Delta^2$ and let $(\varphi_g, \pi_g)$ a $g$-projection device. Suppose $\sigma$ an $x$-curve. Then $\sigma$ is $g$-restricted if and only if $\pi(\sigma(t)) \in H_{\varphi_g}(x, M)$ eventually.

**Proof.** The proof follows by definition of $g$-restricted curve. Indeed, since $\varphi_g$ is a geodesic and $\pi_g(\sigma(t)) = \varphi_g \circ \pi_g(\sigma(t))$
\[
\lim_{w \to x} \omega(\pi_g(\sigma(t)), w) - \omega(0, w) + \omega(0, \pi_g(\sigma(t))) < \log M
\]
if and only if
\[
\lim_{s \to 1^-} K_{\Delta^2}(\pi_g(\sigma(t)), \varphi_g(s)) - \omega(0, s) + K_{\Delta^2}(\varphi_g(0), \pi_g(\sigma(t))) = \lim_{w \to x} \omega(\pi_g(\sigma(t)), w) - \omega(0, w) + \omega(0, \pi_g(\sigma(t))) < \log M
\]

**Remark 21.** Consider $\sigma$ a $g$-special $x$-curve. We notice that it is possible to write $\sigma(t) := (\sigma_1(t), \sigma_2(t)) = \pi_g(\sigma(t)) + \alpha(t)$ with $\alpha(t) := (\alpha_1(t), \alpha_2(t)) \to (0, 0)$ as $t \to 1^-$. By definition of the projection $\pi_g$ we get that $\alpha_1(t) \equiv 0$ and $\alpha_2(t) \to 0$, as $t \to 1^-$. 

**Lemma 22.** Let $x \in \partial \Delta^2$ and $(\varphi_g, \pi_g)$ a $g$-projection device. Let $\sigma$ be an $x$-curve. Write $\sigma(t) = \pi_g(\sigma(t)) + \alpha(t)$ with $\alpha(t) \to 0$ as $t \to 1^-$. Then $\sigma$ is $g$-special if and only if
\[
\lim_{t \to 1^-} \frac{|\alpha_2(t)|}{1 - |g(\sigma_1(t))|} = 0.
\]
By the triangular inequality and by definition of Kobayashi distance in the bidisc, we have that
\[
K_{\Delta^2}(\sigma(t), \pi_g(\sigma(t))) = \max\{\omega(\sigma_1(t), \sigma_1(t)); \omega(\sigma_2(t), g(\sigma_1(t)))\}
\]
\[
= \frac{1}{2} \log \frac{1 + \frac{|\sigma_2(t)|}{1 - |g(\sigma_1(t))|}}{1 - \frac{|\sigma_2(t)|}{1 - |g(\sigma_1(t))|}} \leq \frac{1}{2} \log \frac{1 + \frac{|\alpha_2(t)|}{1 - |g(\sigma_1(t))|}}{1 - \frac{|\alpha_2(t)|}{1 - |g(\sigma_1(t))|}} \to 0
\]
as \( t \to 1^- \). Thus the curve \( \sigma \) is \( g \)-special and the first implication has been proved. On the other hand, let suppose that \( \sigma \) is \( g \)-special.

If, by contradiction, \( \lim_{t \to 1^-} \frac{|\alpha_2(t)|}{1 - |g(\sigma_1(t))|} \neq 0 \) then there exists \( \tilde{\epsilon} > 0 \) such that \( \frac{|\alpha_2(t)|}{1 - |g(\sigma_1(t))|} > \tilde{\epsilon} > 0 \). In particular there exists \( \varepsilon > 0 \) such that
\[
T := \frac{|\alpha_2(t)|}{1 - |g(\sigma_1(t))|^2} > \varepsilon > 0.
\]

Furthermore
\[
\frac{1 - g(\sigma_1(t))\sigma_2(t)}{1 - |g(\sigma_1(t))|^2} = \frac{1 - g(\sigma_1(t))(g(\sigma_1(t)) + \alpha_2(t))}{1 - |g(\sigma_1(t))|^2} = \frac{1 - \frac{g(\sigma_1(t))\alpha_2(t)}{1 - |g(\sigma_1(t))|^2}}{1 - |g(\sigma_1(t))|^2} \leq 1 + \left| \frac{g(\sigma_1(t))\alpha_2(t)}{1 - |g(\sigma_1(t))|^2} \right| = 1 + \frac{|g(\sigma_1(t))|}{1 - |g(\sigma_1(t))|^2} \left| \frac{\alpha_2(t)}{1 - |g(\sigma_1(t))|^2} \right| \leq (T + 1).
\]
and since \( T \to \frac{T}{1 + T} \) is a growing function, we have that
\[
\frac{|\alpha_2(t)|}{1 - g(\sigma_1(t))\sigma_2(t)} = \frac{|\alpha_2(t)|}{1 - |g(\sigma_1(t))|^2} \frac{1 - |g(\sigma_1(t))|^2}{1 - g(\sigma_1(t))\sigma_2(t)} \geq \frac{T}{1 + T} \to \frac{\varepsilon}{1 + \varepsilon} > 0
\]
which contradicts the hypothesis of \( g \)-speciality.

We have now the following result of Lindelöf type for Busemann functions:

**Theorem 23.** Let \( f \in \text{Hol}(\Delta^2, \Delta) \) be a holomorphic function. Given \( x \in \partial \Delta^2 \) let \( \varphi_g \) be a complex geodesic passing through \( x \) and \( (\varphi_g, \pi_g) \) a \( g \)-projection device. Assume that \( f \) is \( K_{\varphi_g} \)-bounded. If \( \sigma_0 \) is a \( g \)-special and \( g \)-restricted \( x \)-curve such that
\[
\lim_{t \to 1^-} f(\sigma_0(t)) = L
\]
then \( f \) admits restricted \( K_{\varphi_g} \)-limit equal to \( L \) at \( x \).

**Proof.** Let us consider a \( g \)-special and \( g \)-restricted \( x \)-curve \( \sigma \). By definition there exists a constant \( M > 1 \) such that \( \tilde{\pi}_g(\sigma(t)) \) approaches \( x_1 \) inside a Stolz region \( H(x_1, M) \). We claim that
\[
(5.4) \quad \forall M_1 > M, \; K_{H_{\varphi_g}(x, M_1)}(\sigma(t), \tilde{\pi}_g(\sigma(t))) \to 0 \text{ as } t \to 1^-.
\]
For any \( t \in [0, 1) \) let us consider the map \( \psi_t : \mathbb{C} \to \mathbb{C}^2 \) given by
\[
\phi_t(z) = \tilde{\pi}_g(\sigma(t)) + z[\sigma(t) - \tilde{\pi}_g(\sigma(t))].
\]
Let us notice that \( \phi_\ell(0) = \pi_g(\sigma(t)) \) and \( \phi_\ell(1) = \sigma(t) \). We claim that the following statement is true:

\[
\forall R > 0 \; \exists \; t_0 = t_0(R) \in [0, 1) \text{ such that } \forall \; t \in [0, 1) : t > t_0(R) \Rightarrow \phi_\ell(\Delta_R) \subset H_{\varphi_g}(x, M_1).
\]

(5.5)

Assuming (5.5) we get:

\[
R(t) := \sup\{r > 0 : \varphi(\Delta_r) \subset H_{\varphi_g}(x, M_1)\} \to \infty
\]

as \( t \to 1^- \), and since, by the very definition

\[
K_{H_{\varphi_g}(x, M_1)}(\sigma(t), \pi_g(\sigma(t))) \leq \inf\{\frac{1}{R} : \exists \varphi_g \in \text{Hol}(\Delta_R, H_{\varphi_g}(x, M_1)) : \varphi_g(0) = \pi_g(\sigma(t)) \text{ and } \varphi_g(1) = \sigma(t)\}
\]

then equation (5.6) follows from equation (5.6) and statement (5.5). Thus we are left to prove (5.5). Assume by contradiction that (5.5) is false. Then there exist \( M_1 > M \) and \( R_0 > 1 \) such that for any \( t_0 \in [0, 1) \) there are \( t' = t'(t_0) \in (t_0, 1) \) and \( z_0 = z_0(t_0) \in \Delta_{R_0} \) such that \( \psi_r(z_0) \notin H_{\varphi_g}(x, M_1) \). Moreover, by Proposition 20 \( \pi_g(\sigma(t)) \in H_{\varphi_g}(x, M_1) \) eventually, and in particular we can choose \( t_0' = t_0'(R_0) \in (0, 1) \) such that \( \pi_g(\sigma(t')) \in H_{\varphi_g}(x, M_1) \) for all \( t_0 > t_0' \). Being \( H_{\varphi_g}(x, M_1) \) open we can also assume that \( \psi_r(z_0) \in \partial H_{\varphi_g}(x, M_1) \) but \( \psi_r(z) \in H_{\varphi_g}(x, M_1) \) for all \( z \in \Delta_{\alpha z_0} \).

Remark 24. Let us notice that there exists \( t_0'' > 0 \) such that \( \psi_r(z) \notin \partial \Delta^2 \) and \( \psi_r(z) \in \Delta^2 \) for all \( t_0 > t_0'' \). Indeed, suppose by contradiction that, for any \( t_0 \in [0, 1) \) there are \( t' = t'(t_0) \in (t_0, 1) \) and \( z_0 = z_0(t_0) \in \Delta_{R_0} \) such that \( \psi_r(z_0) \in \partial H_{\varphi_g}(x, M_1) \cap \partial \Delta^2 \). This implies that it is possible to construct two sequences, say \( \{t_k\}_{k \in \mathbb{N}} = \{t_k(t_0)\}_{k \in \mathbb{N}} \subset (t_0, 1) \) and \( \{z_k\}_{k \in \mathbb{N}} = \{z_k(t_0)\}_{k \in \mathbb{N}} \subset \Delta_{R_0} \) such that \( \psi_r(t_k)(z_k) \in \partial H_{\varphi_g}(x, M_1) \cap \partial \Delta^2 \). Since

\[
\phi_\ell(t_k) = \pi_g(\sigma(t_k)) + z_k[\sigma(t_k) - \pi_g(\sigma(t_k))] = (\sigma_1(t_k), g(\sigma_1(t_k)) + z_k^0 \alpha_2(t_k))
\]

it follows that

\[
|g(\sigma_1(t_k)) + \alpha_2(t_k)| = 1.
\]

In particular

\[
0 = \frac{1 - |g(\sigma_1(t_k)) + z_k^0 \alpha_2(t_k)|}{1 - |g(\sigma_1(t_k))|} \geq \frac{1 - |g(\sigma_1(t_k))| - |z_k^0| |\alpha_2(t_k)|}{1 - |g(\sigma_1(t_k))|}
\]

\[
= 1 - \frac{|z_k^0| |\alpha_2(t_k)|}{1 - |g(\sigma_1(t_k))|} \geq 1 - \frac{R_0 |\alpha_2(t_k)|}{1 - |g(\sigma_1(t_k))|} \to 1^-
\]

as \( t \to 1^- \), a contradiction.

According to remark 24 and by definition of g-Koranyi region, we can write

\[
(5.7) \quad \log M_1 = \lim_{s \to 1^-} K_{\Delta^2}(\psi_r(z_0), \varphi(s)) - \omega(0, s) + K_{\Delta^2}(\psi_r(z_0), \varphi(0)).
\]
Furthermore for any \( z \in \Delta_{R_0} \)

\[
(5.8)\quad K_{\Delta z}(\psi_{t'}(z), \varphi(s)) = \omega(0, s) + K_{\Delta z}(\psi_{t'}(z), \varphi(0)) = \\
= \max\{\omega(\sigma_1(t'), s); \omega(g(\sigma_1(t')) + \alpha_2(t')z, g(s))\} - \omega(0, s) + \\
+ \max\{\omega(\sigma_1(t'), 0); \omega(g(\sigma_1(t')) + \alpha_2(t')z, g(0))\} \leq \\
\leq \omega(g(\sigma_1(t')) + \alpha_2(t')z, g(\sigma_1(t')) + \omega(\sigma_1(t'), s) - \omega(0, s) + \\
+ \omega(\sigma_1(t'), 0) + \omega(g(\sigma_1(t')) + \alpha_2(t')z, g(\sigma_1(t')))} = \\
= \omega(\sigma_1(t'), s) - \omega(0, s) + \omega(\sigma_1(t'), 0) + 2\omega(g(\sigma_1(t')) + \alpha_2(t')z, g(\sigma_1(t'))).
\]

Let us observe that

\[
(5.9)\quad \lim_{t' \to 1^-} \omega(g(\sigma_1(t')) + \alpha_2(t')z, g(\sigma_1(t'))) = 0
\]

uniformly for \( z \in \Delta_{R_0} \). Indeed by definition of Poincaré distance, we have

\[
\lim_{t' \to 1^-} \omega(g(\sigma_1(t')) + \alpha_2(t')z, g(\sigma_1(t'))) = \lim_{t' \to 1^-} \frac{1}{2} \log \frac{1 + \frac{\alpha_2(t')z}{1 - |g(\sigma_1(t'))|(|g(\sigma_1(t')) + \alpha_2(t')z)|}}{1 - \frac{\alpha_2(t')z}{1 - |g(\sigma_1(t'))|(|g(\sigma_1(t')) + \alpha_2(t')z)|}}
\]

and the argument of this logarithm tends to 1 since

\[
\frac{\alpha_2(t')z}{1 - |g(\sigma_1(t'))|(|g(\sigma_1(t')) + \alpha_2(t')z)|} \leq \frac{|\alpha_2(t')z|}{1 - |g(\sigma_1(t'))|(|g(\sigma_1(t')) + \alpha_2(t')z)|} \leq \frac{|\alpha_2(t')z|}{|\alpha_2(t')z|} \leq \frac{1}{1 - |g(\sigma_1(t'))|(|g(\sigma_1(t')) + \alpha_2(t')z)|} \to 0 \text{ as } t' \to 1^-.
\]

by Lemma 22. Thus equation (5.9) is proved. In particular it is true for \( z = z_0 \) and then by equations (5.7) and (5.8) we get

\[
\log M_1 \leq \lim_{s \to 1^-} \omega(\sigma_1(t'), s) - \omega(0, s) + \omega(\sigma_1(t'), 0) + 2\omega(g(\sigma_1(t')) + \alpha_2(t')z, g(\sigma_1(t')))
\]

and in particular, eventually,

\[
M < M_1 \leq \frac{|1 - \sigma_1(t')|}{1 - |\sigma_1(t')|}
\]

but it is a contradiction since \( \sigma \) is \( g \)-restricted. This concludes the proof of (5.5).

Now, since \( f \) is a \( K_g \)-bounded function, there exists \( c > 0 \) such that

\[
K_{\Delta z(M_1)}(f(\sigma_0(t)), f(\pi_g(\sigma_0(t)))) \leq K_{H_{\psi g}}(\sigma_0(t), \pi_g(\sigma_0(t))) \to 0 \text{ as } t \to 1^-.
\]

Now we can proceed as in Theorem 18 to complete the proof. \( \square \)

6. Julia’s Lemma

We want to give a new generalization to polydiscs of the Julia’s lemma, using the Busemann functions. The idea is to consider the rate of approach of \( f \) along particular directions given by geodesics passing through \( x \).
Definition 25. Let $f \in \text{Hol}(\Delta^2, \Delta)$ and $x \in \partial \Delta^2$. Let us consider a complex geodesic $\varphi_x \in \text{Hol}(\Delta, \Delta^2)$ passing through $x$. Let $\lambda_g$ be the boundary dilation coefficient of $g$ at $x_1$. The number $\lambda_{\varphi_x}(f)$ defined by

$$\frac{1}{2} \log \lambda_{\varphi_x}(f) := \lim_{t \to 1^-} K_{\Delta^2}(0, \varphi_x(t x_1)) - \omega(0, f(\varphi_x(t x_1))).$$

is the $\varphi_x$-boundary dilation coefficient of $f$ at $x$.

First we show that $\lambda_{\varphi_x}(f)$ is well defined. We prove this fact studying separately two cases:

1) $x \in (\partial \Delta)^2$ or

2) $x \in [(\partial \Delta^2) - (\partial \Delta)^2]$.

In the first case we can assume $x = (1, 1)$ and in the second one we suppose $x = (1, 0)$.

Remark 26. Suppose (as in case 1)) that $g \in \text{Hol}(\Delta, \Delta)$ has non-tangential limit 1 at the point 1. Let observe that if $\lambda_g < \infty$ then $\lambda_g = \lim_{t \to 1^-} \frac{1-|g(t)|}{1-t}$. Indeed we have

$$\lambda_g := \liminf_{z \to x} \frac{1-|g(z)|}{1-|z|} \leq \liminf_{t \to 1^-} \frac{1-|g(t)|}{1-t} \leq \limsup_{t \to 1^-} \frac{1-|g(t)|}{1-t}$$

and by the triangular inequality we get

$$\leq \limsup_{t \to 1^-} \frac{1-|g(t)|}{1-t} \leq \lim_{t \to 1^-} \frac{|1-g(t)|}{1-t} = \lambda_g$$

by the classical Julia-Wolff-Carathéodory theorem.

Let us consider the case 1) and let $\psi_g(z) = (\theta(z), g(\theta(z)))$ be another parametrization of the geodesic $\varphi_x$, with $\theta \in \text{Aut}(\Delta)$. We notice that $\theta(1) = 1$. As a matter of notation we call respectively $\lambda_{\varphi}$ and $\lambda_{\psi}$ the boundary dilation coefficient of $f$ at $x$ computed with respect to the parameterizations $z \to (z, g(z))$ and $z \to (\theta(z), g(\theta(z)))$. By the above definition we have that:

$$\frac{1}{2} \log \lambda_{\varphi} = \lim_{t \to 1^-} [K_{\Delta^2}(0, \varphi_g(t)) - \omega(0, f(\varphi_g(t)))]$$

$$= \lim_{t \to 1^-} [K_{\Delta^2}(0, \psi_g(\theta(t)))] - \omega(0, f(\psi_g(\theta(t))))]$$

$$= \lim_{t \to 1^-} [\max\{\omega(0, t), \omega(0, g(t))\} - \omega(0, f(\psi_g(\theta(t)))].$$

If $\lambda_g \geq 1$, then $\max\{\omega(0, t), \omega(0, g(t))\} = \omega(0, t)$ and the last member of equation (6.1) becomes

$$\lim_{t \to 1^-} [\omega(0, t) - \omega(0, \theta^{-1}(t)] + \omega(0, \theta^{-1}(t)) - \omega(0, f(\psi_g(\theta^{-1}(t))))]$$

$$\geq \liminf_{t \to 1^-} [\omega(0, t) - \omega(0, \theta^{-1}(t)] + \liminf_{z \to 1^-} [\omega(0, z) - \omega(0, f(\psi_g(z)))]$$

$$\geq \lim_{t \to 1^-} [\omega(0, t) - \omega(0, \theta^{-1}(t)] + \lim_{t \to 1^-} [\omega(0, t) - \omega(0, f(\psi_g(t)))]$$
Since \( f \circ \psi_g \) is a holomorphic self map of the unit disc, by remark\([26]\) the equation\([6.2]\) becomes
\[
\lim_{t \to 1^-} [\omega(0, t) - \omega(0, \theta^{-1}(t))] = \lim_{t \to 1^-} [\omega(0, t) - \omega(0, f(\psi_g(t)))] \\
= \lim_{t \to 1^-} [\omega(0, t) - \omega(0, \theta^{-1}(t))] + \lim_{t \to 1^-} [K_{\Delta z}(0, \psi_g(t))] \\
+ \lim_{t \to 1^-} [K_{\Delta z}(0, \psi_g(t)) - \omega(0, f(\psi_g(t)))]
\]
(6.3)
\[
= \lim_{t \to 1^-} [\omega(0, t) - \omega(0, \theta^{-1}(t))] + \lim_{t \to 1^-} [\omega(0, t) - \omega(0, \theta(t))] \\
+ \lim_{t \to 1^-} [K_{\Delta z}(0, \psi_g(t)) - \omega(0, f(\psi_g(t)))]
\]
\[
= \frac{1}{2} \log(\lambda_{\theta_1} \lambda_\theta \lambda_\psi) = \frac{1}{2} \log(\lambda_\psi)
\]

If \( \lambda_g \leq 1 \), then \( \max\{\omega(0, t), \omega(0, g(t))\} = \omega(0, g(t)) \) and the last member of equation\([6.1]\) becomes
\[
\lim_{t \to 1^-} \omega(0, g(t)) - \omega(0, t) = \omega(0, t) - \omega(0, \theta^{-1}(t)) + \omega(0, \theta^{-1}(t)) - \omega(0, f(\psi_g(\theta^{-1}(t))))
\]
\[
= \frac{1}{2} \log \frac{1}{\lambda_g \lambda_\theta} + \lim_{t \to 1^-} \omega(0, z) - \omega(0, f(\psi_g(z)))
\]
\[
= \frac{1}{2} \log \frac{1}{\lambda_g \lambda_\theta} + \lim_{t \to 1^-} \omega(0, t) - K_{\Delta z}(0, \psi_g(t)) + K_{\Delta z}(0, \psi_g(t)) - \omega(0, f(\psi_g(t)))
\]
\[
= \frac{1}{2} \log(\frac{1}{\lambda_\psi \lambda_g \lambda_\theta}) = \frac{1}{2} \log(\lambda_\psi)
\]

Then we have \( \lambda_\varphi \geq \lambda_\psi \). Swapping the roles of \( \lambda_\varphi \) and \( \lambda_\psi \) in the above inequalities, we also get that \( \lambda_\psi \geq \lambda_\varphi \) and thus \( \lambda_\varphi = \lambda_\psi \).

In case 2) we can suppose \( x = (1, 0) \), and we consider a complex geodesic \( \varphi_g \) passing through \((1, 0)\) parameterized by \( \varphi_g(z) = (z, g(z)) \). We also consider another parametrization \( \psi_g(z) = (\theta(z), g(\theta(z))) \), with \( \theta(z) \in \text{Aut}(\Delta) \). We notice that \( \theta(1) = 1 \). Since, in this case, \( \lambda_g = \infty \), we can repeat the calculation done in \((6.1)\), and in \((6.2)\) obtaining \( \lambda_\psi = \lambda_\varphi \). Thus \( \lambda_\varphi(f) \) is well defined. Furthermore we have an interesting property. Let be \( \frac{1}{2} \log \alpha_f := \lim_{w \to x} [K_{\Delta z}(0, w) - \omega(0, f(w))] \). Let notice that \( \alpha(f) \) is the boundary dilation coefficient of \( f \) at \( x \), defined by Abate in \([1]\) and the following property holds (see \([1]\) for the proof)
\[
\frac{1}{2} \log \alpha(f) = \lim_{t \to 1} [K_{\Delta z}(0, tx) - \omega(0, f(tx))].
\]

**Theorem 27.** Let \( f = (f_1, f_2) \in \text{Hol}(\Delta^2, \Delta^2) \). Let \( x = (x_1, x_2) \in \partial(\Delta \times \Delta) = \partial \Delta^2 \) and let (for example) \( \varphi_g = (z, g(z)) \) be a complex geodesic passing through \( x \). Let
\[
\frac{1}{2} \log \lambda_j := \lim_{t \to 2} [K_{\Delta z}(0, \varphi_g(tx_1)) - \omega(0, f_j(\varphi_g(tx_1)))] \quad j = 1, 2
\]
Suppose that either \( \lambda_1 < \infty \) or \( \lambda_2 < \infty \). Then there exists a point \( y = (y_1, y_2) \in \partial \Delta^2 \) such that for all \( R > 0 \)
\[
f(B(1, \lambda_\varphi)(x, R)) \subseteq B(\lambda_1, \lambda_\varphi)(y, R).
\]
**Proof.** Let us first suppose that \( \lambda_j < \infty \), for \( j = 1, 2 \) then
\[
\frac{1}{2} \log \lambda_j = \lim_{z \to x} K_{\Delta z}(0, z) - \omega(0, f_j(z)) < \infty \quad j = 1, 2.
\]
As shown by Abate in theorem 3.1 in [1], we can choose a sequence \( z_{\nu} \in \Delta^2 \), converging to \( x \), such that

\[
\lim_{\nu \to \infty} K_{\Delta^2}(0, z_{\nu}) - \omega(0, f_j(z_{\nu})) = \liminf_{z \to x} K_{\Delta^2}(0, z) - \omega(0, f_j(z)).
\]

Up to a subsequence, we can assume that \( f_j(z_{\nu}) \to y_j \in \Delta \). Since \( \Delta^2 \) is complete hyperbolic, we have that \( K_{\Delta^2}(0, z_{\nu}) \to +\infty \); therefore \( \omega(0, f_j(z_{\nu})) \to +\infty \) as well, and \( y_j \in \partial \Delta \). Thus there exists a point \( y = (y_1, y_2) \in (\partial \Delta)^2 \) such that \( f_j(z) \to y_j \) as \( z \to 1 \), \( j = 1, 2 \).

We claim that

\[
f(\mathbb{B}(1, \lambda_y)(x, R)) \subset \mathbb{B}(\lambda_1, \lambda_2)(y, R) \quad \forall R > 0.
\]

Without loss of generality let us suppose that \( x_1 = 1 \). Fix \( z \in \mathbb{B}(1, \lambda_y)(x, R) \). We have, for \( j = 1, 2 \), that

\[
\begin{align*}
\lim_{w \to y_j} \omega(f_j(z), w) - \omega(0, w) &= \lim_{s \to 1} \omega(f_j(z), f_j(\varphi_y(s))) - \omega(0, f_j(\varphi_y(s))) \\
&\leq \liminf_{s \to 1} K_{\Delta^2}(z, \varphi_y(s)) - \omega(0, f_j(\varphi_y(s))) \\
&= \liminf_{s \to 1} K_{\Delta^2}(z, \varphi_y(s)) - \omega(0, s) + \omega(0, s) - K_{\Delta^2}(0, \varphi_y(s)) + K_{\Delta^2}(0, \varphi_y(s)) - \omega(0, f_j(\varphi_y(s))) \\
&\leq \liminf_{s \to 1} K_{\Delta^2}(z, \varphi_y(s)) - \omega(0, s) + \lim_{s \to 1} K_{\Delta^2}(0, \varphi_y(s)) - \omega(0, f_j(\varphi_y(s))) \leq \frac{1}{2} \log \lambda_j R.
\end{align*}
\]

Then \( \forall R > 0 \)

\[
f(\mathbb{B}(1, \lambda_y)(x, R)) = f(E(x_1, R) \times E(x_2, \lambda_y R)) \subset E(y_1, \lambda_1 R) \times E(y_2, \lambda_2 R).
\]

Let suppose now that \( \lambda_1 = \infty \) and \( \lambda_2 < \infty \). By the above calculation we get that \( \forall R > 0 \)

\[
f(\mathbb{B}(1, \lambda_y)(x, R)) = f(E(x_1, R) \times E(x_2, \lambda_y R)) \subset \Delta \times E(y_2, \lambda_2 R).
\]

\[
\square
\]

Let us notice that the following proposition holds:

**Proposition 28.** For all complex geodesic \( \varphi_y \) passing through \( x \) such that the coefficient \( \lambda_y < \infty \) we have that \( \lambda_{\varphi_y}(f) \) is finite if and only if \( \alpha(f) \) is finite.

**Proof.** It is clear that \( \lambda_{\varphi_y}(f) \geq \alpha(f) \) and thus if \( \lambda_{\varphi_y}(f) \) is finite then also \( \alpha(f) \) does. On the other hand let us suppose that \( \alpha(f) \) is finite and let us denote by \( \varphi_x(z) \) the complex geodesic passing through the point \( x = (x_1, x_2) \in \partial \Delta^2 \) and parameterized by \( z \to zx \). Let us consider \( \varphi_y(z) = (z, g(z)) \) another complex geodesic passing through the point \( x \), and \( \pi_y \in \text{Hol}(\Delta^2, \Delta^2) \) the projection on the complex geodesic \( \varphi_y(z) \) given by \( \pi_y(z_1, z_2) = (z_1, g(z_1)) \). Note that \( \pi_y(\varphi_x(t)) = \pi_y(tx) = (tx_1, g(tx_1)) = \varphi_y(tx_1) \). Let us suppose, without loss of generality that
Then there exists a point $\tilde{j}$

$$x_1 = 1. \text{ Then }$$

$$\frac{1}{2} \log \lambda_{g}(f) = \lim_{t \to 1^-} K_{\Delta^2}(0, \varphi_{g}(t)) - \omega(0, f(\varphi_{g}(t)))$$

$$= \lim_{t \to 1^-} K_{\Delta^2}(0, \pi_{g}(\varphi_{x}(t))) - \omega(0, f(\pi_{g}(\varphi_{x}(t))))$$

$$= \lim_{t \to 1^-} K_{\Delta^2}(0, \pi_{g}(\varphi_{x}(t))) - K_{\Delta^2}(0, \varphi_{x}(t)) + K_{\Delta^2}(0, \varphi_{x}(t))$$

$$- \omega(0, f(\varphi_{x}(t))) + \omega(0, f(\varphi_{x}(t))) - \omega(0, f(\pi_{g}(\varphi_{x}(t))))$$

$$\leq \lim_{t \to 1^-} K_{\Delta^2}(0, \varphi_{x}(t)) - \omega(0, f(\varphi_{x}(t)))$$

$$+ 2K_{\Delta^2}(\varphi_{x}(t), \pi_{g}(\varphi_{x}(t)))$$

$$= \lim_{t \to 1^-} K_{\Delta^2}(0, \varphi_{x}(t)) - \omega(0, f(\varphi_{x}(t))) + 2\omega(t, g(t))$$

$$= \frac{1}{2} \log \alpha(f) + \log \lambda_{g}$$

and in our setting $\lambda_{g}$ is finite and then if $\alpha(f)$ is finite also $\lambda_{\varphi_{g}}(f)$ does. \qed

7. THE JULIA-WOLFF-CARATHÉODORY THEOREM

We are finally ready to state and prove our generalization of the Julia-Wolff-Carathéodory theorem obtained using Busemann functions.

**Theorem 29.** Let $f \in \text{Hol}(\Delta^2, \Delta^2)$ and $x \in \partial \Delta^2$. Let $\varphi_{g}$ be any complex geodesic passing through $x$ and parameterized by $\varphi_{g}(z) = (z, g(z))$, with $g \in \text{Hol}(\Delta, \Delta)$ such that

$$\frac{1}{2} \log \lambda_{j} = \lim_{t \to 1^-} K_{\Delta^2}(0, \varphi_{g}(tx)) - \omega(0, f_{j}(\varphi_{g}(tx))) < \infty.$$ 

for $j = 1, 2$. Let $\tilde{\pi}_{g}(z) : \Delta^{2} \to \Delta$ be the $g$-left-inverse of $\varphi_{g}$ given by $\tilde{\pi}_{g}(z_1, z_2) = z_1$. Then there exists a point $y = (y_1, y_2) \in \partial \Delta^2$ such that

$$\tilde{K}_{g} - \lim_{z \to x} \frac{y_j - f_j(z)}{1 - \tilde{\pi}(z)} = \lambda_{j} \min \{1, \lambda_{g}\}$$

$$\tilde{K}_{g} - \lim_{z \to x} \frac{y_j - f_j(z)}{1 - z_2} = \max \{1, \lambda_{g}\}$$

To prove this theorem we need first the following two lemmas:

**Lemma 30.** Let $f \in \text{Hol}(\Delta^2, \Delta^2)$ and $x \in \partial \Delta^2$. Suppose $|x_1| = 1$. Suppose there exists a complex geodesic $\varphi_{g}$ passing through $x$ and parameterized by $\varphi_{g}(z) = (z, g(z))$, with $g \in \text{Hol}(\Delta, \Delta)$ such that

$$\frac{1}{2} \log \lambda_{j} = \lim_{t \to 1^-} K_{\Delta^2}(0, \varphi_{g}(tx)) - \omega(0, f_{j}(\varphi_{g}(tx))) < \infty.$$ 

Let $\tilde{\pi}_{g}(z) : \Delta^{2} \to \Delta$ be the $g$-left-inverse of $\varphi_{g}$ given by $\tilde{\pi}_{g}(z_1, z_2) = z_1$. Then there exists a point $y = (y_1, y_2) \in \partial \Delta^2$ and a constant, say $c_{g} > 0$, depending on $g$, such that, given $M > 1$, for all $z \in H_{\varphi_{g}}(x, M)$

$$\left| \frac{y_j - f_j(z)}{1 - \tilde{\pi}(z)} \right| \leq 2\lambda_{1}M^2c_{g} \quad \text{and} \quad \left| \frac{y_j - f_j(z)}{1 - z_2} \right| \leq 2\lambda_{1}M^2c_{g}.$$
Proof. Without loss of generality let us suppose that $x_1 = 1$. Let $z \in H_{\varphi_y}(x, M)$ and set $\frac{1}{2}\log R := \log M - K_{\Delta^2}(\varphi_y(0), z)$. Thus

$$\lim_{s \to 1} K_{\Delta^2}(z, \varphi_y(s)) - K_{\Delta^2}(\varphi_y(0), \varphi_y(s)) < \frac{1}{2}\log R$$

which implies $z \in B_{(1, \lambda_1)}(x, R)$. By Lemma 27 there exists a point $y = (y_1, y_2) \in \partial\Delta^2$ and a complex geodesic $\varphi_{\tilde{y}}$ passing through $y$ and parameterized by $\varphi_{\tilde{y}}(z) = (z, \tilde{g}(z))$, with $\tilde{g} \in \text{Hol}(\Delta, \Delta)$ such that $\lambda_\tilde{y} = \frac{2}{\lambda_1}$ and $f(z) \in B_{(1, \lambda_\tilde{y})}(y, \lambda_1 R)$. Without loss of generality let us suppose that $y = (1, 1)$. In particular, by the very definition of Busemann sublevel sets, we have

$$\frac{1}{2}\log \lambda_1 R \geq \lim_{s \to 1} K_{\Delta^2}(f(z), \varphi_{\tilde{y}}(s)) - \omega(0, s)$$

for $j = 1, 2$. Moreover let us notice that $-\omega(0, f_j(z)) \leq \omega(f_j(z), s) - \omega(0, s) \forall s \in (0, 1)$.

For sake of clearness we argue for $j = 1$

$$\lim_{s \to 1} \omega(f_1(z), s) - \omega(0, s) - \omega(0, f_1(z)) \leq \log \lambda_1 R$$

then

$$\frac{|1 - f_1(z)|^2}{1 - |f_1(z)|^2} \frac{1 - |f_1(z)|}{1 + |f_1(z)|} = \left[\frac{|1 - f_1(z)|}{1 + |f_1(z)|}\right]^2 \leq (\lambda_1 R)^2.$$ 

Furthermore we know that

$$-2K_{\Delta^2}(\varphi_y(0), z) \leq 2K_{\Delta^2}(\varphi_y(0), 0) - 2K_{\Delta^2}(0, z)$$

$$= \log \frac{1 + ||\varphi_y(0)||}{1 - ||\varphi_y(0)||} \frac{1 - ||z||}{1 + ||z||} = \log \frac{1 + |g(0)|}{1 - |g(0)|} \frac{1 - ||z||}{1 + ||z||}.$$ 

and, by definition of $R$, set $c_g := 1 + |g(0)|$ then we get

$$\frac{|1 - f_1(z)|}{1 + |f_1(z)|} \leq \lambda_1 M^2 c_g \frac{1 - ||z||}{1 + ||z||} \leq \lambda_1 M^2 c_g \frac{1 - |z|}{1 + |z|} \text{ for } i = 1, 2$$

If $i = 1$, being $\hat{\pi}(z_1, z_2) = z_1$, then

$$\frac{|1 - f_1(z)|}{|1 - \hat{\pi}(z)|} \leq \lambda_1 M^2 c_g \frac{1 + |f_1(z)|}{1 + |z|} \leq 2\lambda_1 M^2 c_g$$

and if $i = 2$

$$\frac{|1 - f_1(z)|}{|1 - z_2|} \leq \lambda_1 M^2 c_g \frac{1 + |f_1(z)|}{1 + |z|} \leq 2\lambda_1 M^2 c_g.$$ 

With the same techniques we proved the statement for the second component $f_2$. \hfill \Box

Lemma 31. Let $f \in \text{Hol}(\Delta^2, \Delta^2)$ be a holomorphic function and $x \in \partial\Delta^2$. Suppose there exists a complex geodesic $\varphi_g$ passing through $x$ and parameterized by $\varphi_g(z) = (z, g(z))$, with $g \in \text{Hol}(\Delta, \Delta)$ such that

$$\frac{1}{2}\log \lambda_j = \lim_{t \to 1} K_{\Delta^2}(0, \varphi_g(tx_1)) - \omega(0, f_j(\varphi_g(tx_1))) < \infty.$$
Let \( \tilde{\pi}_g(z) : \Delta^2 \to \Delta \) be a \( g \)-left inverse of \( \varphi_g \) given by \( \tilde{\pi}_g(z_1, z_2) = z_1 \). Then there exists a point \( y = (y_1, y_2) \in \partial \Delta^2 \) such that, for \( j = 1, 2 \),

\[
(7.1) \lim_{s \to 1} \frac{1 - f_j(\varphi_g(sx_1))}{1 - sx_1} = \lambda_j \min \{1, \lambda_g\}
\]

\[
(7.2) \lim_{s \to 1} \frac{1 - f_j(\varphi_g(sx_1))}{1 - g(sx_1)} = \frac{\lambda_j}{\max \{1, \lambda_g\}}.
\]

**Proof.** Let us suppose that \( x_1 = 1 \). Let \( y \) the point given in Theorem 27 and without loss of generality let us suppose that \( y = (1, 1) \). By definition of \( \lambda_j \) we have that

\[
\lambda_j = \lim_{s \to 1} \frac{1 - |f_j(\varphi_g(s))|}{1 - |||\varphi_g(s)|||}.
\]

Moreover the limit (7.1) exists for the classic Julia-Wolff-Carathéodory theorem and also the limit (7.2) exists, since

\[
\frac{1 - f_j(\varphi_g(s))}{1 - g(s)} = \frac{1 - f_j(\varphi_g(s))}{1 - s} \frac{1 - s}{1 - g(s)}.
\]

Let proceed considering the following different cases (a) \( \lambda_g \geq 1 \) and (b) \( \lambda_g \leq 1 \). Let study the case (a). In this setting there exists a sequence \( \{s_k\}_{k \in \mathbb{N}} \in (0, 1) \) such that \( s_k \to 1 \) as \( k \to \infty \) and \( \lim_{k \to \infty} |\omega(0, s_k) - \omega(0, g(s_k))| \geq 0 \) which implies that \( |||\varphi_g(s_k)||| = s_k \) and then

\[
\lambda_j \min \{1, \lambda_g\} = \lim_{s \to 1} \frac{1 - |f_j(\varphi_g(s))|}{1 - |||\varphi_g(s)|||} = \lim_{k \to \infty} \frac{1 - |f_j(\varphi_g(s_k))|}{1 - |||\varphi_g(s_k)|||}
\]

\[
= \lim_{k \to \infty} \frac{1 - |f_j(\varphi_g(s_k))|}{1 - s_k} = \lim_{s \to 1} \frac{1 - |f_j(\varphi_g(s))|}{1 - s}
\]

and moreover,

\[
\frac{1 - f_j(\varphi_g(s))}{1 - g(s)} = \frac{1 - f_j(\varphi_g(s))}{1 - s} \frac{1 - s}{1 - g(s)} = \frac{\lambda_j}{\lambda_g} = \frac{\lambda_j}{\max \{1, \lambda_g\}}.
\]

Using the same techniques we proved that the above equalities hold also in case (b).

And now we are ready to prove Theorem 29.

**Proof of Theorem 29.** Let us suppose \( x_1 = 1 \). By Lemma 31 we have

\[
\lim_{s \to 1} \frac{1 - |f_j(\varphi_g(s))|}{1 - s} = \lambda_j \min \{1, \lambda_g\}
\]

and by Lemma 30 we know that the function \( \frac{1 - f_j(\varphi_g(s))}{1 - s} \) is \( K_g \)-bounded. Then, since the curve \( \varphi_g(s) \) is \( g \)-special and \( g \)-restricted, the conclusion of the proof follows by theorem 29. \( \square \)
8. Application to the dynamics

Let $f \in \text{Hol}(\Delta, \Delta)$ be without fixed points in $\Delta$. The classical Wolff lemma ensures the existence of a unique point $\tau \in \partial \Delta$ such that every horocycle centered in $\tau$ is sent in itself by $f$. The point $\tau$ is called the Wolff point of $f$.

Let $n \in \mathbb{N}$, and set $f^n = f \circ \cdots \circ f$ the composition of $f$ with itself $n-$times. We say that $\{f^n\}_{n \in \mathbb{N}}$ is the sequence of iterates of $f$. The Wolff-Denjoy lemma says that $\{f^n\}$ converges uniformly on compacta to the Wolff point $\tau$.

If we call target set, $T(f)$, the set of the limit points of the sequence of the iterates and we denote by $W(f)$ the set of the Wolff points of $f$, then in one complex variable, we have that $T(f) \equiv W(f) = \{\tau\}$.

In [12] we considered $f \in \text{Hol}(\Delta^2, \Delta^2)$ without fixed points in $\Delta^2$ and we defined the Wolff points of $f$ using the small and big horospheres.

**Definition 32.** [12] Let $f \in \text{Hol}(\Delta^2, \Delta^2)$ be without fixed points in $\Delta^2$. A point $\tau \in \partial \Delta^2$ is a Wolff point for $f$ if $f(E(\tau, R)) \subseteq E(\tau, R)$, for all $R > 0$.

In this setting, in [12] (see also [11]) we characterized the set of the Wolff points, $W(f)$, for a holomorphic self map $f$ of the bidisc without fixed points. As a spinning result (see also Hervé [13]), we get that $W(f) \subset T(f)$, where $T(f)$ is the target set of $f$ defined as follows ([11],[12]):

$$T(f) := \{x \in \overline{\Delta^2} : \exists \{k_n\} \subset \mathbb{N}, \exists z \in \Delta^2 \text{ such that } f^{k_n}(z) \to x \text{ as } n \to \infty\}.$$  

It turns out that this result can be improved using the Busemann functions.

**Definition 33.** Let $\tau \in \partial \Delta^2$. We say that $\tau$ is a generalized Wolff point for $f$ if there exists a geodesic $\varphi_\tau$, passing through the point $\tau$, such that every Busemann sublevel set, $B_{(1, \lambda_\tau)}(\tau, R)$, is sent in itself by $f$, that is $f(B_{(1, \lambda_\tau)}(\tau, R)) \subseteq B_{(1, \lambda_\tau)}(\tau, R)$ for every $R > 0$.

Let denote by $W_G(f)$ the set of the generalized Wolff points for $f$.

**Remark 34.** Let notice that if $\tau$ is contained in a flat component of the boundary then $\tau$ is a Wolff point if and only if $\tau$ is a generalized Wolff point. On the other hand, let consider a point $\tau$ of the Sílov boundary of the bidisc. If $\tau$ is a Wolff point for $f$ then $i$ is also a generalized Wolff point for $f$. The converse is, in general, false [12] (see also [11]).

**Remark 35.** Let observe that $W_G(f)$ is arcwise connected. The proof is the same of proposition 3.14 in [12] (see also [11]).

In order to state the result which characterizes the set $W_G(f)$, we need to introduce some definitions and results. Hervé proved the following useful theorem (see [13] Theorem 1):

**Theorem 36.** Let $f : \Delta^2 \to \Delta^2$ be a holomorphic map, without interior fixed points in $\Delta^2$, whose components are $f_1, f_2$. Then either

1. there exists a Wolff point, $e^{i\theta_1}$, of $f_1(\cdot, y)$, which does not depend on $y$ or
2. there exists a holomorphic function $F_1 : \Delta \to \Delta$, such that $f_1(F_1(y), y) = F_1(y), \forall y \in \Delta$. In this case $f_1(x, y) = x \Rightarrow x = F_1(y)$.

Let us remark that, if $f \neq id_{\Delta^2}$, then cases i) and ii) cannot hold at the same time. Motivated by the last mentioned result of Hervé we give the following definition ([11],[12]):
**Definition 37.** The holomorphic map $f : \Delta^2 \to \Delta^2$, whose components are $f_1, f_2$, is called of:

1. **first type** if:
   - there exists a holomorphic function $F_1 : \Delta \to \Delta$, such that $f_1(F_1(y), y) = F_1(y)$, $\forall \ y \in \Delta$ and
   - there exists a holomorphic function $F_2 : \Delta \to \Delta$, such that $f_2(x, F_2(x)) = F_2(x)$, $\forall \ x \in \Delta$.
2. **second type** if (up to switching $f_1$ with $f_2$):
   - there exists a Wolff point, $e^{i\alpha_1}$, of $f_1(\cdot, y)$, (necessarily independent of $y$) and
   - there exists a holomorphic function $F_2 : \Delta \to \Delta$, such that $f_2(x, F_2(x)) = F_2(x)$, $\forall \ x \in \Delta$.
3. **third type** if:
   - there exists a Wolff point, $e^{i\eta_1}$, of $f_1(\cdot, y)$, (independent of $y$) and
   - there exists a Wolff point, $e^{i\eta_2}$, of $f_2(x, \cdot)$, (independent of $x$).

In case $f$ is of **first type** and without interior fixed points in $\Delta^2$, then it turns out that $F_1 \circ F_2$ and $F_2 \circ F_1$ have a Wolff point (see Lemma 3.10 in [12] and also [11]). Let $e^{i\theta_1}$ (respectively $e^{i\theta_2}$) be the Wolff point of $F_1 \circ F_2$ (respectively $F_2 \circ F_1$). We also let $\lambda_1$ and $\lambda_2$ be, respectively, the boundary dilation coefficients of $F_1$ at $e^{i\theta_2}$ and of $F_2$ at $e^{i\theta_1}$ (see Lemma 3.10 in [12] and also [11]). In case $f$ is of **second type** we denote by $e^{i\alpha_1}$ the Wolff point of $f_1(\cdot, y)$, by $e^{i\alpha_2}$ the $K$-limit (or non-tangential limit) (see definition in [2]) of $F_2$ at $e^{i\alpha_1}$ (if it exists) and $k_2 := \lim_{x \to e^{i\alpha_1}} |F_2'(x)|$. In case $f$ is of **third type**, we set $e^{i\eta_1}$ and $e^{i\eta_2}$ to be, respectively, the Wolff points of $f_1(\cdot, y)$ and $f_2(x, \cdot)$. We let $\pi_j : \Delta^2 \to \Delta$ ($j = 1, 2$) be the projection on the $j$-th component. Finally without loss of generality we suppose that $e^{i\theta_1} = e^{i\theta_2} = e^{i\alpha_1} = e^{i\alpha_2} = e^{i\eta_1} = e^{i\eta_2} = 1$. With the above established notations we proved (see [12]) the following result:

**Theorem 38.** Let $f = (f_1, f_2)$ be a holomorphic map, without fixed points in the complex bidisc. If $f_1 \neq \pi_1$ and $f_2 \neq \pi_2$, then only the following five cases are possible:

1. $W(f) = \emptyset$ if and only if $f$ is of first type and $\lambda_i > 1$ for either $i = 1$ or $i = 2$;
2. $W(f) = (1, 1)$ if $f$ is of first type and $\lambda_i \leq 1$ for each $i = 1, 2$;
3. $W(f) = \{(1) \times \Delta\} \cup \{(1, 1)\}$ if $f$ is of second type and $k_2 \leq 1$;
4. $W(f) = \{(1) \times \Delta\}$ if $f$ is of second type and $k_2 > 1$;
5. $W(f) = \{(1) \times \Delta\} \cup \{(1, 1)\} \cup \{(\Delta \times \{1\}\}$ if $f$ is of third type.

On the other hand, if $f_1(x, y) = x$, $\forall \ y \in \Delta$, i.e if $f_1 = \pi_1$ (or respectively $f_2(x, y) = y$, $\forall \ x \in \Delta$, i.e $f_2 = \pi_2$ ) then:

1. $W(f) = (1 \times \Delta) \cup \{(1, 1)\} \cup \{(1) \times \Delta\}$ where $1$ is the Wolff point of $f_2(x, \cdot)$.
2. (or respectively $W(f) = (\Delta \times 1) \cup \{(1, 1)\} \cup \{(1) \times \Delta\}$ where $1$ is the Wolff point of $f_1(\cdot, y)$).

With the same techniques used in the proof of Theorem 48 ([11], [12]) we also get the following characterization of the generalized Wolff points of $f$:

**Theorem 39.** Let $f = (f_1, f_2)$ be a holomorphic map, without fixed points in the complex bidisc. If $f_1 \neq \pi_1$ and $f_2 \neq \pi_2$, then the following three cases are possible:
i) \( W_G(f) = (1, 1) \) if and only if \( f \) is of first type.

ii) \( W_G(f) = \{(1) \times \Delta\} \cup \{(1, 1)\} \) iff \( f \) is of second type;

iii) \( W_G(f) = \{(1) \times \Delta\} \cup \{(1, 1)\} \cup \{\Delta \times \{1\}\} \) iff \( f \) is of third type.

It is interesting to notice that, in this case, using the result of Hervé [13], about the target set of \( f \), we get that \( W_G(f) \equiv T(f) \).

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Dipartimento di Matematica “U. Dini”, Università di Firenze, Viale Morgagni 67/A, 50134 Firenze, Italy.
E-mail address: frosini@math.unifi.it