On The Moduli of Surfaces Admitting Genus Two Fibrations Over Elliptic Curves

Gülay Kaya

Abstract

In this paper, we study the structure, deformations and the moduli spaces of complex projective surfaces admitting genus two fibrations over elliptic curves. We observe that, a surface admitting a smooth fibration as above is elliptic and we employ results on the moduli of polarized elliptic surfaces, to construct moduli spaces of these smooth fibrations. In the case of nonsmooth fibrations, we relate the moduli spaces to the Hurwitz schemes \( \mathcal{H}(1,X(d),n) \) of morphisms of degree \( n \) from elliptic curves to the modular curve \( X(d) \), \( d \geq 3 \). Ultimately, we show that the moduli spaces in the nonsmooth case are fiber spaces over the affine line \( \mathbb{A}^1 \) with fibers determined by the components of \( \mathcal{H}(1,X(d),n) \).

1 Introduction

The aim of this paper is to work out the structure, deformations and the moduli spaces of complex projective surfaces admitting genus two fibrations over elliptic curves.

In the literature, the cases of albanese fibrations with fiber genus two over arbitrary base curves and nonalbanese fibrations over curves of genus \( g \geq 2 \) have been studied extensively ([12], [13] for the former type and [5], [9], [8] for the latter). We aim at complementing these results by examining the case of fibrations with irregularity \( q(S) = 2 \) over elliptic curves. These fibrations are of nonalbanese type and have Kodaira dimension \( \kappa(S) = 1 \) (respectively 2) in case the given fibration is smooth (respectively non-smooth).

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In the smooth case, we fix the type of our surfaces as \( \tau = (1,0;2,2;2,1) \), according to the generalized definition of polarized elliptic surfaces ([I], p.210). Then we consider the moduli problem for surfaces \( S \) of type \( \tau \) as given. To get a more natural description of this moduli problem we observe the following. Roughly, forgetting \( \tau \), the moduli of the surfaces we consider is closely related to the moduli of isogenies of degree two of elliptic curves (the base curves) and the moduli of smooth genus two curves \( C \) admitting an elliptic subcover \( C \to E \) of degree two. The functor corresponding to the first moduli is coarsely represented by affine modular curve \( Y_0(2) \). The functor corresponding to the latter one is coarsely represented by an open subscheme \( \mathcal{H} \) of \( \mathcal{A}_{2,1}(X(2) \times X(2))/SL_2(\mathbb{Z}) \) ([II], p.210). We obtain

**Proposition 1.1.** (i) The corresponding functor is coarsely represented by an irreducible scheme \( \mathcal{M} \) of dimension 3.

(ii) There exists a natural surjective morphism \( \phi : \mathcal{M} \to \mathcal{H} \).

In the nonsmooth case, even though our main interest is in fibrations over elliptic curves, we will also prove the algebraic version of the main result in [II] describing the structure of the moduli spaces \( \mathcal{M}(g,K^2,\chi) \) of surfaces fibered over curves of genus \( g \geq 2 \). We note that the result for base curves of genus \( g \geq 2 \) is stronger than the result in the case of \( g = 1 \). This is due to the fact that we can not prove Lemma 2.11 in full strength when \( g = 1 \).

**Theorem 1.2.** Let \( K^2, \chi \) and \( g \geq 2 \) be given and let \( \mathcal{H}(g,X(d),n) \) be the Hurwitz scheme of morphisms of degree \( n \) from curves of genus \( g \) onto \( X(d) \). Then we have morphisms \( \Phi : \mathcal{M}(g,K^2,\chi) \to \mathbb{A}^1 \) and \( \Psi_{E'} : \mathcal{H}(g,X(d),n) \to \mathcal{M}(g,K^2,\chi) \) for any fixed elliptic curve \( E' \) such that

(i) \( \Psi_{E'} \) establishes a one-to-one correspondence between the components \( \mathcal{H}_i \) of \( \mathcal{H}(g,X(d),n) \) and the components \( \mathcal{M}_i \) of \( \mathcal{M}(g,K^2,\chi) \),

(ii) \( \Phi : \mathcal{M}_i \to \mathbb{A}^1 \) is a fibration with \( \Psi_{E'}(\mathcal{H}_i) \) as the fiber over \( [E'] \in \mathbb{A}^1 \).

**Theorem 1.3.** Let \( \mathcal{M}_i \) be a connected component of \( \mathcal{M}(1,K^2,\chi) \). Then we have a morphism \( \Phi : \mathcal{M}_i \to \mathbb{A}^1 \) (given on closed points by \( [X] \mapsto [E'] \) if \( X \) is of type \( (E',d) \) such that the fiber over \( [E'] \in \mathbb{A}^1 \) is a disjoint union

\[
\bigsqcup_j \Psi_{E'}(\mathcal{H}(1,X(d),n)_j).
\]

We work over the complex numbers \( \mathbb{C} \) and use the following standard notation:

- \( S \) is a smooth projective surface.
- \( c_1(S), c_2(S) \) denote the first and the second Chern classes of \( S \), respectively.
\(\kappa(S), \; q(S)\) are the Kodaira dimension and the irregularity of \(S\), respectively.

\(K(S), \; \chi(S)\) are the canonical class and the holomorphic Euler characteristic of \(S\).

For fixed \(K^2\) and \(\chi\), \(\mathcal{M}(g, K^2, \chi)\) is the moduli space of surfaces of general type admitting genus two fibration with irregularity \(q = g + 1\) and slope \(\lambda\) which satisfies the slope formula \(K^2 = \lambda \chi + (8 - \lambda)(g - 1)\).

2 Structure and Deformations of Genus Two Fibrations

First we discuss the case of smooth fibrations.

**Lemma 2.1.** Let \(\pi : S \to E\) be a smooth genus two fibration over an elliptic curve \(E\) with \(q(S) = 2\). Then \(S\) admits an elliptic fibration with two double fibers of the form \(2.E\). All other fibers are smooth and are isomorphic to \(E'\) (the double cover of \(E\) corresponding to the monodromy representation arising from \(\pi\)).

*Proof.* Let \(F\) be the general fiber of the fibration \(\pi : S \to E\). Since \(\pi : S \to E\) is a smooth genus two fibration, \(\pi\) is iso with monodromy group \(G \subset \text{Aut}(F)\) which is cyclic of order two ([14], Proposition 2.12, p.30) and \(F \to E'' = F/G\) ramifies precisely over two points \(p_1, p_2 \in E''\) (by Riemann-Hurwitz formula). Hence, the composite map \(E' \times F \to F \to E''\) induces a natural fibration \(S \to E''\) with generic fiber \(E'\) and with two double fibers of the form \(2.(E'/G) = 2.E\) over \(p_1, p_2\). This proves the Lemma.

**Lemma 2.2.** Let \(\psi : S \to T\) be a deformation of \(S\). Then there exists an elliptic curve \(E'' \to T\) and two sections \(s_1, s_2 : T \to E''\) such that \(S \to T\) factors through \(E''\). Furthermore, \(S \to E''\) is smooth outside \(s_1(T) \cup s_2(T)\) and for each \(t \in T\) the restriction of \(S \to E''\) induces an elliptic fibration \(\mathcal{S}_t \to E''_t\) with precisely two double fibers (over \(s_1(t), s_2(t)\)).

*Proof.* By standard results in deformation theory, we know that for all \(t \in T\), \(S_t\) is a minimal surface and \(\kappa(S_t) = 1\). Furthermore, each \(S_t\) admits an elliptic fibration exactly of the same type as \(S\) ([11], Proposition 7.1, p.111) and it follows from ([11], Proposition 7.11(iii), p.128) that there exists an elliptic curve \(E'' \to T\) through which \(S \to T\) factors. Since each surface \(S_t\) is an “elliptic surface of general type” in the terminology of [2] (i.e., \(P(n) \geq 2\) for some \(n\)), Proposition 10 in [2] applies to prove the existence of two sections \(s_1, s_2 : T \to E''\) ([11], Lemma 1.9) with the properties stated in the Lemma.
Remarks 2.3. 1) The existence of $E''$ over $T$ can be proved simply by observing that the fibrations in the family $S$ are induced from $m$-th canonical map for $m$ sufficiently large ([11], p.194). More precisely, we take $\mathbb{P}(\psi^*\omega_S^{\otimes m})$ over $T$ and the morphism $S \to \mathbb{P}(\psi^*\omega_S^{\otimes m})$ induced by the homomorphism $\psi^*(\psi^*\omega_S^{\otimes m}) \to \omega_S^{\otimes m} \to 0$. $E''$ is the image of this $m$-th canonical map.

2) Lemma 2.2 indicates a relation between the moduli of smooth genus two fibrations over elliptic curves with irregularity $q = 2$ and the moduli of elliptic surfaces. In order to apply the results of ([10], [11]), we need the following observation:

Lemma 2.4. An elliptic surface with exactly two double fibers as the singular fibers, admits a smooth genus two fibration.

Proof. Let $\pi : S \to E''$ be an elliptic fibration over an elliptic curve $E''$ with two double fibers over $p_1, p_2 \in E''$ and generic fiber $E'$. Let $\pi_j : B = J(S) \to E''$ be the Jacobian fibration and $B' = E' \times E'' \to E''$ be the trivial elliptic fibration. Both, $B$ and $B'$, are elliptic fibrations with sections. Since the associated $j$-invariants are equal and constant, we can find isomorphic compatible lifts $\rho$ and $\rho'$ defined by $B$ and $B'$, respectively ([1], p.41). Hence, the Jacobian surface $J(S)$ of $S$ is trivial, being isomorphic to $B' = E' \times E''$ ([14], Theorem 3.14(ii), p.45). Moreover, $R^1\pi_*\mathcal{O}_S \cong R^1(\pi_j)_*\mathcal{O}_{J(S)}$ is trivial and so $L = (R^1\pi_*\mathcal{O}_S)^\vee$ is trivial.

Therefore, $\chi(\mathcal{O}_S) = \deg L = 0$ ([1], Proposition 3.18, p.48), $p_g(S) = g(E'') = 1$ ([11], Proposition 3.22(i), p.49) and so $q(S) = 1 + p_g(S) - \chi(\mathcal{O}_S) = 2$. By the universal property of albanese varieties there is a morphism $Alb(S) \to E''$. Hence, $Alb(S)$ is a reducible abelian variety and by the complete reducibility property of abelian varieties ([1], Theorem 1, p.173) it has a projection to an elliptic curve $E$, $\pi_1 : Alb(S) \to E$, which restricts to a nonconstant necessarily étale morphism from $E'$. The pullback $S \times_E E'$ is the product $F \times E'$ where $F$ is the general fiber of the composite map $S \to Alb(S) \to E$ ([6], E.8.6, p.151). $g(F) = 2$ since $F$ is a double cover of $E''$ which is ramified at two points.

Remark 2.5. Combining the results in Lemma 2.2 and Lemma 2.4, we see that a surface $S$ admitting a smooth genus two fibration deforms only to surfaces of the same type.

Next, we consider the case of nonsmooth fibrations. Let $\pi : S \to C$ be a nonsmooth genus two fibration with $q(S) = g(C) + 1$. Then there is a unique rational number $\lambda = \lambda(\pi)$, which is called the slope of $\pi$, such that $K^2 = \lambda \chi + (8 - \lambda)(g(C) - 1)$, and one has $2 \leq \lambda \leq 7$ for nontrivial nonsmooth fibrations ([14], p.22). Let $F$ be a smooth fiber of $\pi$ and $J(F)$ its Jacobian. We have a projection $p_F : J(F) \to E$ onto the fixed part of the associated relative Jacobian. Let $d$ be the degree of the composite map $F \to J(F) \to E$. Then $d$ is called the degree associated to $\pi$ and $\pi : S \to C$ is said to be of type $(E, d)$. We have $\lambda = 7 - \frac{6}{d}$ ([14], Corollaire 2, p.50). Since we will consider only semistable fibrations, we
have \( d \geq 3 \) ([14], Corollaire, p.47). As an immediate result of ([14], Théorème 3.10, p.44) we obtain

**Theorem 2.6.** Let \( E \) be an elliptic curve, \( d \) an integer \( \geq 3 \). There exists a genus two fibration of type \((E, d)\)

\[ \Phi : S(E, d) \to X(d) \]

over the modular curve \( X(d) \) which is universal in the following sense: any genus two fibration \( \pi : S \to C \) with slope \( \lambda = 7 - \frac{6}{d} \) and with \( E \) as the fixed part of the Jacobian fibration corresponding to \( \pi \) (i.e., \( \pi \) is of type \((E, d)\)) is the minimal desingularization of the pullback \( f^*(S(E, d)) \) via a surjective holomorphic map \( f : C \to X(d) \).

**Remark 2.7.** Since \( g(X(d)) \geq 3 \) for \( d \geq 7 \), it follows that fibrations over elliptic curves have \( d \in \{3, 4, 5, 6\} \). We recall that \( X(d) \sim \mathbb{P}^1 \) for \( d = 3, 4, 5 \) and \( X(6) \) is the elliptic curve with \( j(X(6)) = 0 \).

Given \( f : C \to X(d) \), the surface \( f^*(S(E, d)) \) has singularities only if \( f \) ramifies over some points in the singular locus of \( \Phi : S(E, d) \to X(d) \). A singular fiber of \( \Phi \) is either an elliptic curve with a single node or two smooth elliptic curves intersecting transversally at a single point ([14], Lemme 3.11, Théorème 3.16). Hence, singularities of \( f^*(S(E, d)) \) are all type \( A_k \) for some \( k \) depending on the singular point.

As a consequence of this observation we see that we can apply simultaneous desingularization to a family of surfaces obtained via a family of surjective morphisms onto \( X(d) \).

**Lemma 2.8.** For a fibration \( \pi : S \to C \) over a curve \( C \) of genus \( \geq 1 \) arising from a map \( f : C \to X(d) \) of degree \( n \) we have \( c_2(S) > 0 \) and \( K^2 = c_1^2(S) > 0 \). In particular, since \( S \) is minimal, it is a surface of general type.

**Proof.** Let \( \phi : S(E', d) \to X(d) \) be the corresponding fibration. Then \( c_2(S) = -n \deg(R^1\phi_*\mathcal{O}_{S(E', d)}) > 0 \). Using the relations \( c_1^2(S) = \lambda \chi(S) + (8 - \lambda)(g(C) - 1) \) and \( 12\chi(S) = c_1^2(S) + c_2(S) \), we have \( c_1^2(S) > 0 \).

**Lemma 2.9.** Let \( S_i \to C_i, i = 1, 2 \) be two fibrations of the same type \((E, d)\), corresponding to morphisms \( f_i : C_i \to X(d) \). Then

\( (i) \) \( S_i \) have the same invariants \( K^2, \chi \) if and only if \( \deg(f_1) = \deg(f_2) \),

\( (ii) \) \( S_1 \) and \( S_2 \) are isomorphic as surfaces if and only if \( C_1 = C_2 \) and there exist automorphisms \( \alpha \in \text{Aut}(C_1), \beta \in \text{Aut}(X(d)) \) such that \( f_1 \circ \alpha = \beta \circ f_2 \).

**Proof.** (i) This is Lemma 1 in [19].
(ii) That $C_1 = C_2$ follows from the uniqueness of such a fibration on a given surface ([9, Lemma 2(i)]). Then the rest of the statement is a consequence of the minimality of the surfaces $S_1$ and $S_2$, since the given condition is necessary and sufficient for the surfaces $f_i^*(S(E,d))$ to be birationally equivalent. □

**Lemma 2.10.** $S$ admitting a fibration as described over an elliptic curve, exists if and only if $K^2$ and $\chi$ have the following values:

| $\lambda$ | $K^2$ | $\chi$ |
|-----------|-------|-------|
| 5         | $5n$  | $n$   |
| $11/2$    | $11n$ | $2n$  |
| $29/5$    | $29n$ | $5n$  |
| 6         | $36n$ | $6n$  |

where $n \geq 2$ in the first three rows and $n \geq 1$ in the last row.

*Proof.* For $n \geq 2$ and for any elliptic curve $E$ we have morphisms $E \to \mathbb{P}^1$ of degree $n$ and for any such a map, using the formulae in the proof of Lemma 2.8 and observing that $-\deg(R^1\phi_*O_{S(E',d)}) = 7, 13, 31$ for $d = 3, 4, 5$, respectively, ([14], p.52), we find the values of $K^2$ and $\chi$ given in the first three rows of the table.

Since $X(6)$ is an elliptic curve, by the same computation, this time using the existence of isogenies of any order and the fact that $-\deg(R^1\phi_*O_{S(E',6)}) = 6$ we obtain the last row. □

We will need the following Lemma ([9], Lemma 2).

**Lemma 2.11.** Let $\psi : S \to T$ be a deformation over a connected base, of a surface $S$ admitting a genus 2 fibration with slope $\lambda$ over a curve $C$ of genus $g \geq 2$. Then

(i) each fiber $S_t$ of $\psi$ admits such a fibration $S_t \to C_t$ which is unique,

(ii) the slope $\lambda$ is constant on $T$,

(iii) the degree of the map $C_t \to X(d)$ inducing the fiber space $S_t \to C_t$ is constant.

In case of elliptic base curves (ii) and (iii) of Lemma 2.11 remain unchanged when we consider a family $S \to T$ of surfaces having a fibration of the given form. Moreover, the fibration over any such curve is also unique. However, we do not know if (i) holds, too.
3 Moduli Problem of Genus Two Fibrations

Let \( \pi : S \to E \) be a smooth genus two fibration with fiber \( F \) and \( \pi_1 : S \to E'' \) be the corresponding elliptic fibration with two double fibers. In fact, these double fibers are sections of \( \pi \), say \( s_1, s_2 \). Consider the divisor \( s_1(E) + F \) on \( S \). We have

\[
(s_1(E) + F)^2 = 2s_1(E).F = 2 > 0
\]

and for any irreducible curve \( C \) in \( S \)

\[
(s_1(E) + F).C = s_1(E).C + F.C > 0.
\]

Hence, \( s_1(E) + F \) is an ample divisor on \( S \), by Nakai’s ampleness criterion. Let \( \eta \) be the numerical equivalence class of the line bundle corresponding to this ample divisor in \( \text{Num}(S) \) (the group of numerical equivalence classes of line bundles on \( S \)). Then \( d := \eta^2 = (s_1(E) + F)^2 = 2 \) and \( e := \eta.f = (s_1(E) + F).E' = F.E' = 1 \) where \( f \) is the class of the general fiber \( E' \) of \( \pi_1 \). Hence, our surfaces are of type \( \tau = (1, 0; 2, 2, 2; 2) \) according to the generalized definition of polarized elliptic surfaces given in \([11], \text{p.} 210\). Moreover, any surface of type \( \tau \) is one of our surfaces.

We consider the functor \( G_\tau : \text{Sch} \to \text{Sets} \) defined by \( G_\tau(T) = \text{set of all isomorphism classes of families of polarized elliptic surfaces of type } \tau \text{ over } T \). We have

**Proposition 3.1.** \( G_\tau \) is coarsely represented by an irreducible scheme \( \mathcal{M} \) of dimension 3.

**Proof.** Existence of \( \mathcal{M} \) follows from \([11], \text{Theorem} \ 2.15, \text{p.} 211\). The proof of this theorem shows that there is a finite map \( \mathcal{M} \to Y'' \), where \( Y'' \) is an open subscheme of \( Y' = E_{1,0} \times_{\mathbb{A}^1} M_{1,2} \). Here \( E_{1,0} \) denotes the moduli scheme for Weierstrass surfaces with base genus \( g = 1 \) and \( \chi = 0 \) which exists by \([10]\), and \( M_{1,2} \) is the moduli scheme for elliptic curves with two distinguished points. \( E_{1,0} \) splits into a disjoint union of irreducible subschemes \( E_{1,0}^n \) for \( n = 1, 2, 3, 4, 6 \) \([10], \text{p.} 182\) where each \( E_{1,0}^n \) represents the subfunctor corresponding to Weierstrass surfaces for which the order of the module \( L = (R^1p_*\mathcal{O})^\vee \) is \( n \).

Let \( S \) be an elliptic surface of type \( \tau \). Since all fibers of the elliptic fibration on \( S \) are irreducible, the Weierstrass fibration associated to \( S \) is the Jacobian fibration of \( S \) \([11], \text{p.} 191\). In the proof of Lemma \([2, \text{p.} 183]\) we have seen that the Jacobian of such a surface is a trivial product of two elliptic curves. So the relevant part of \( E_{1,0} \) is \( E_{1,0}^1 \) which corresponds to trivial \( L \). Hence, \( Y'' \) is an open subscheme of \( E_{1,0}^1 \times_{\mathbb{A}^1} M_{1,2} \). By \([10], \text{Lemma} \ 10, \text{p.} 182\) \( E_{1,0}^1 \cong \mathbb{A}^2 \). Therefore, \( \dim(\mathcal{M}) = \dim(E_{1,0}^1 \times_{\mathbb{A}^1} M_{1,2}) = 3 \).

In the preceding section we have observed that the moduli of the surfaces we consider is closely related to the moduli of isogenies of degree two of elliptic
curves (the base curves) and the moduli of smooth genus two curves $C$ admitting an elliptic subcover $C \to E$ of degree two. The functor $Y_0$ the first moduli is coarsely represented by affine modular curve $Y_0(2)$. As for the latter, we have the affine surface $A_{2,1} = (X(2) \times X(2))/SL_2(\mathbb{Z})$ (8, p.210) which coarsely represents the functor associated to the triplets $\{(A, \Theta, E)\}$ where $A$ is an abelian surface, $\Theta$ is a principal polarization, $E$ is an elliptic subgroup of $A$ and $\deg(\Theta|_E) = 2$. Let $C$ be a curve of genus two with Jacobian $J_C$ and canonical polarization $\Theta$. Then there is a bijective correspondence between the set of isomorphism classes of (minimal) elliptic subcovers $f : C \to E$ of degree $\deg(f) = 2$ and the set of elliptic subgroups $E \leq J_C$ of $J_C$ of degree $\deg_\Theta(E) = 2$ (8, Theorem 1.9, p.202). Therefore, the functor $M'_2$ of isomorphism classes of pairs $(C, \mathcal{E})$ of (relative) smooth curves of genus two and elliptic subcovers $(C \to \mathcal{E})$ of degree two is coarsely represented by the open subscheme $H = \Phi^{-1}(t(M_2))$ of $A_{2,1}$, where $t : M_2 \to A_2$ is the Torelli map which associates to a curve its canonically polarized Jacobian and $\Phi : A_{2,1} \to A_2$ is the map which forgets $E$ in the triplets.

**Proposition 3.2.** There exists a natural surjective morphism $\phi : M \to H$.

**Proof.** Consider an object in $G_\tau(T)$ for some $T$; i.e. a family $S \to T$ which factors over $\mathcal{E}''$ (Lemma 2.2). Let $f_1, f_2 \in \mathcal{O}_{\mathcal{E}''/T}$ such that $(f_i) = s_i(T), i = 1, 2.$ Then the natural injection $f : \mathcal{O}_{\mathcal{E}''/T} \to \mathcal{O}_{\mathcal{E}''/T}[\sqrt{f_1 f_2}]$ gives a double cover $(f) : F/T \to \mathcal{E}''/T$ over $T$ ramified along $(f_1) \cup (f_2) = s_1(T) \cup s_2(T)$, where $F = \text{Spec}(\mathcal{O}_{\mathcal{E}''}[t]/(t^2 - f_1 f_2))$. Moreover, since for any $t \in T$, $F_t \to \mathcal{E}''$ is a double cover ramified at two points, we have $g(F_t) = 2$ by Riemann-Hurwitz formula. Hence, $F_t$ is a smooth genus two curve. Hence, the pair $(F, \mathcal{E}'')$ corresponds to a point in $H(T)$. By functoriality of this construction, we obtain natural morphism $\phi : M \to H$. 

Next we consider moduli of surfaces with nonsmooth genus two fibrations of nonalbanese type. In Section 2 we have seen that a surface of type $(E', d)$ is the desingularization of $f^*(S(E', d))$ for some morphism $f : C \to X(d)$. Hence, such a surface $S$ can be deformed in two ways; we can deform $E'$ to other elliptic curves and we can deform the map $f$. Therefore, in describing the moduli spaces of such surfaces under consideration, we need to clarify the relation of these spaces to the Hurwitz spaces $H(g, X(d), n)$ of morphisms of degree $n$ from curves of genus $g$ to the modular curve $X(d)$.

**Theorem 3.3.** Let $K^2$, $\chi$ and $g \geq 2$ be given and let $H(g, X(d), n)$ be the Hurwitz scheme of morphisms of degree $n$ from curves of genus $g$ onto $X(d)$. Then we have morphisms $\Phi : M(g, K^2, \chi) \to \mathbb{A}^1$ and $\Psi_{E'} : H(g, X(d), n) \to M(g, K^2, \chi)$ for any elliptic curve $E'$ such that 

(i) $\Psi_{E'}$ establishes a one-to-one correspondence between the components $H_i$ of $H(g, X(d), n)$ and the components $M_i$ of $M(g, K^2, \chi)$,
(ii) $\Phi : \mathcal{M}_{t} \to \mathbb{A}^1$ is a fibration with $\Psi_{E'}(\mathcal{H}_{i})$ as the fiber over $[E'] \in \mathbb{A}^1$.

This result is a consequence of Lemma 3.4 and Lemma 3.5.

**Lemma 3.4.** There exists a morphism $\Phi : \mathcal{M}(g, K^2, \chi) \to \mathbb{A}^1$ which maps the class $[S] \in \mathcal{M}(g, K^2, \chi)$ to the class $[E] \in \mathbb{A}^1$ of the elliptic curve associated to the fibration on $S$. $\Phi$ is surjective on each component of $\mathcal{M}(g, K^2, \chi)$.

**Proof.** Let $M(g, \lambda) : \text{Sch}/\mathbb{C} \to \text{Sets}$ be the functor defined by $M(g, \lambda)(T) = \text{isomorphism classes of families of surfaces over } T$ admitting genus 2 fibrations over curves of genus $g$, with slope $\lambda$. To prove the lemma, it suffices to construct a morphism of functors $M(g, \lambda) \to h_{\mathbb{A}^1}$ as described in the lemma. This, on the other hand, follows once we prove that for any $T \in \text{Sch}/\mathbb{C}$ and for a given family of surfaces $\mathcal{S} \to T$, the map $T \to \mathbb{A}^1$ defined by $t \mapsto [E_t]$ where $[E_t]$ is the fixed part of the jacobian fibration on $\mathcal{S}_t$, is a morphism.

This last claim being local over the base, we assume that $\mathcal{S} \to T$ is projective and we consider the relative albanese morphism $\alpha : \mathcal{S} \to \text{Alb}_{\mathcal{S}/T}$; the image $\mathcal{E} = \alpha(S)$ is a family of smooth isotrivial elliptic surfaces over $T$ and the base of the fibration on $\mathcal{E}_t$ is $C_t = \text{the base of the fibration on } \mathcal{S}_t$. It is well known that for such a family of elliptic surfaces, the base curves glue to give a relative curve $\mathcal{C}$ and the map $\mathcal{E} \to T$ factors over $\mathcal{C}$. Since the fibres of $\mathcal{E}_t \to C_t$ are constant, the morphism $\mathcal{C} \to \mathbb{A}^1$ corresponding to the elliptic curves $\mathcal{E}/\mathcal{C}$ coincides with the map $T \to \mathbb{A}^1$ defined above, which completes the proof of the claim.

To prove the surjectivity of $\Phi$, we take any connected component of $\mathcal{M}(g, K^2, \chi)$ and a surface $S$ of type $(E, d)$ corresponding to a point in this component. We let $f : C \to X(d)$ be the map inducing the fibration on $S$. For any family of elliptic curves $\mathcal{E} \to T$, we have a genus two curve $\mathcal{F} \to H_{\mathcal{E}/T, d, -1}$, where $H_{\mathcal{E}/T, d, -1}$ is an open subscheme of $X(d) \times_{\mathbb{C}} T$, universal for normalized genus two covers ([1], Definition on p.13) of degree $d$ of $\mathcal{E}/T$ ([1], Thm. 1.1). From $(f, id) : C \times T \to X(d) \times_{\mathbb{C}} T$ we obtain a $T$-morphism $F : U \to H_{\mathcal{E}/T, d, -1}$ where $U$ is an open subscheme of $C \times T$. Completing the family of genus two curves $F^*(\mathcal{F})/U$ to a family over $C \times T$, and then applying simultaneous desingularization we get a family of smooth surfaces $\mathcal{S} \to T'$ where $T' \to T$ is a finite Galois base extension. Since $\mathcal{S}$ contains $S$ as one of the fibers, its moduli lies in the same component of $\mathcal{M}(g, K^2, \chi)$ as the modulus of $S$. For an arbitrary elliptic curve $E'$, choosing $\mathcal{E} \to T$ as a deformation of $E$ to $E'$, we see that $\Phi$ restricted to this moduli has $[E'] \in \mathbb{A}^1$ in its image. This completes the proof of the lemma. \qed

Let $C \to T$ be a family of smooth curves of genus $g$ and let $F : C \to X(d) \times T$ be a family of morphisms of degree $n$. For a fixed elliptic curve $E'$, applying simultaneous desingularization to the family of surfaces $F^*(S(E', d))$ we obtain a family of fibered surfaces $\mathcal{S} \to T'$ over a Galois extension $T' \to T$.
with group $G$, which defines a morphism $\alpha : T' \to \mathcal{M}(g, K^2, \chi)$. $\alpha$, being $G$-invariant, descends to a morphism $T \to \mathcal{M}(g, K^2, \chi)$. Clearly, this construction is functorial and by the defining property of coarse moduli spaces we get a morphism $\Psi_{E'} : \mathcal{H}(g, X(d), n) \to \mathcal{M}(g, K^2, \chi)$. To a given connected component $\mathcal{H}_i$ of $\mathcal{H}(g, X(d), n)$ we assign the component $\mathcal{M}_i$ of $\mathcal{M}(g, K^2, \chi)$ which contains $\Psi_{E'}(\mathcal{H}_i)$.

**Lemma 3.5.** The above assignment induces a one-to-one correspondence between the connected components of $\mathcal{M}(g, K^2, \chi)$ and those of $\mathcal{H}(g, X(d), n)$. Moreover, we have $\Psi_{E'}(\mathcal{H}_i) = \Phi^{-1}_{M_i}([E'])$.

**Proof.** Since by Lemma 3.4 each component $\mathcal{M}_i$ of $\mathcal{M}(g, K^2, \chi)$ contains the moduli of a surface of type $(E', \lambda)$, it suffices to check that in each $\mathcal{M}_i$ we have the image under $\Psi_{E'}$ of a unique component of $\mathcal{H}(g, X(d), n)$.

Let $\mathcal{M}_i$ be the component of $\mathcal{M}(g, K^2, \chi)$ which contains $\Psi_{E'}(\mathcal{H}_i)$. Fix $[S_1] \in \Psi_{E'}(\mathcal{H}_i)$ and let $[S_2] \in \mathcal{M}_i$ be an arbitrary point and let $C_i, j = 1, 2$ be the base curves of the corresponding fibrations. Then, the surfaces $S_1$ and $S_2$ deform to each other. Since deformations of the surfaces under consideration are induced from deformations of the fibrations (proof of Lemma 3.4), it follows that $f_1 : C_1 \to X(d)$ deforms to a morphism $\overline{f}_2 : C_2 \to X(d)$. Therefore, $f_1, \overline{f}_2$ belong to $\mathcal{H}(g, X(d), n)$, $\mathcal{H}(g, X(d), n)$. On the other hand, by (Lemma 2.9 (ii)), $f_2$ and $\overline{f}_2$ satisfy a relation of the form $f_2 \circ \alpha = \beta \circ \overline{f}_2$ for some $\alpha \in Aut(C_1), \beta \in Aut(X(d))$. Therefore, $f_2$ and $\overline{f}_2$, hence, $f_1$ and $f_2$ lie in $\mathcal{H}_i$. This proves the first part of the lemma. The second statement is obvious. \hfill $\square$

In the case of elliptic base curves we can not prove that a given deformation of our surfaces $S \to T$ arises from the deformation of the associated maps $f_i : C_t \to X(d), t \in T$. Therefore, by exactly the same proof we obtain the following weaker result:

**Theorem 3.6.** Let $\mathcal{M}_i$ be a connected component of $\mathcal{M}(1, K^2, \chi)$. Then we have a morphism $\Phi : \mathcal{M}_i \to \mathbb{A}^1$ (given on closed points by $[X] \to [E']$ if $X$ is of type $(E', d)$) such that the fiber over $[E'] \in \mathbb{A}^1$ is a disjoint union

$$\bigsqcup_j \Psi_{E'}(\mathcal{H}(1, X(d), n)_j).$$

**Remarks 3.7.** 1) When $\lambda = 6$, one can prove that $\mathcal{M}(g, K^2, \chi) = \bigsqcup_{i=1}^N \mathbb{A}^1_i$ where $N$ is the number of distinct étale covers of degree $n$ of the elliptic curve $X(6)$ (\cite{H}, Theorem 2.3).

2) Another shortcoming of the result in case of base genus $g = 1$ is that we do not know if each $\mathcal{M}_i$ is a connected component of the corresponding moduli space $\mathcal{M}_{K^2, \chi}$ of surfaces of general type.
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Gülay Kaya
Department of Mathematics
Galatasaray University
Ciragan Cad. No:36 34357 Ortakoy, Istanbul, Turkey