On topological representation theory from quivers*

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December 1, 2020

Abstract

In this work, we introduce topological representations of a quiver as a system consisting of topological spaces and its relationships determined by the quiver. Such a setting gives a natural connection between topological representations of a quiver and diagrams of topological spaces. First, we investigate the relation between the category of topological representations and that of linear representations of a quiver via $P(Γ)-\mathcal{TOP}$ and $kΓ$-Mod, concerning (positively) graded or vertex (positively) graded modules. Second, we discuss the homological theory of topological representations of quivers via $Γ$-limit $\text{Lim}^Γ$ and using it, define the homology groups of topological representations of quivers via $H_n$. It is found that some properties of a quiver can be read from homology groups. Third, we investigate the homotopy theory of topological representations of quivers. We define the homotopy equivalence between two morphisms in Top–Repl$Γ$ and show that the parallel Homotopy Axiom also holds for top-representations based on the homotopy equivalence. Last, we mainly obtain the functor $AΓ$ from Top–Repl$Γ$ to Top and show that $AΓ$ preserves homotopy equivalence between morphisms. The relationship is established between the homotopy groups of a top-representation $(T,f)$ and the homotopy groups of $AΓ(T,f)$.

2010 Mathematics Subject Classifications: 16G20, 16E30, 20M30; 46H15, 54H10, 55Q05

Contents

1 Introduction and preliminaries 2
2 Top-representations of a quiver 4
3 Relations between top-representations and linear-representations of quivers 9
4 Homology groups of top-representations 12

*Project supported by the National Natural Science Foundation of China(No.12071422, No. 11971144), the Zhejiang Provincial Natural Science Foundation of China (No.LY19A010023) and High-level Scientific Research Foundation of Hebei Province.

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1 Introduction and preliminaries

Algebraic representation theory is a very important theory for us to understand and describe the structure of an algebra. It is also an important method to realize an abstract algebraic structure with concrete examples. The quiver theory is crucial to this theory.

- If the field, as a one-dimensional linear spaces over itself, placed on the vertices of a quiver are all the same with the base field, we then obtain the path algebra of this quiver.
- If one puts non-trivial linear spaces on the vertices of a quiver, we then obtain a representation of this quiver, which is equivalent to the representation of the path algebra of the quiver.
- If one places some algebraic bimodules on the vertices of a quiver, then we obtain the representation of a generalized path algebra (in fact, a tensor algebra), see [8].

Furthermore, one may ask what would happen if we put objects with more delicate structures on the vertices of a quiver.

In an early work [7], we considered an extreme case: replacing linear spaces at vertices with sets and linear maps with general maps, then the resulting structure is called the set-representation of the quiver $\Gamma$. It was proved that the category of set-representations of $\Gamma$ is equivalent to the $S$-system category of the multiplicative semigroup $P(\Gamma)$ of $\Gamma$. Its biggest difference is to not be longer an abelian category.

Why is the fact that representation categories are abelian important for us? The reason is that we consider it from the perspective of quivers (possibly, with relations). We hope to achieve the characterization of quivers or its path algebras (with corresponding relations) through the homological properties of the categories.

If we look at objects on vertices, and look at quivers only as the relationships among these objects on vertices, we do not need to pay too much attention to that whether the categories are abelian categories.

From this point of view, we can consider to put some objects on vertices that we want to care about, such as topological spaces. Therefore, we lead to the idea of topological representation theory of quivers.

We give a metaphor. A standard barbell consists of barbell bar and barbell pieces. When barbell pieces are very small, the focus of the barbell is the bar. When barbell pieces are replaced bigger and bigger, the focus of the barbell will become to the barbell pieces, no longer to the barbell bar.

The focus of the linear representations of a quiver is the quiver itself, while it is hoped that the quiver can be described by its representation. When we consider topological representations of a quiver on its vertices where topological spaces are put, the focus of topological representations of a quiver is no longer to depict the quiver, but to reflect the relationship among topological spaces placed on the vertices of the quiver. In this way, we can take topological representations of a quiver as a system consisting of topological spaces and its relationships determined by the quiver. Such a setting gives a natural connection between topological representations of a quiver and diagrams of topological spaces, although the emphasized views may be slightly different, where we concern more on the topological features of the quiver from the views of representation theory.
Now we are replacing the way of using representation category by topological space category to characterize a quiver. We call this phenomenon topologization, which is similar to the statement of geometrization. In this sense, this work is entitled as topological representation theory.

To begin, let us state some basic definitions and notations that will be used in this paper.

Let $X$ be a topological space, and let $G$ be a group. Suppose we have a map $\pi : G \times X \to X$ defined by sending $(g, x)$ to $gx$ for any $g \in G$ and $x \in X$. This leads to the following definition.

**Definition 1.1.** Let $X$ be a topological space $G$ be a group. For any $g \in G$, the left action $L_g$ on $X$ is a topological continuous map

$$L_g : X \to X$$

given by $x \mapsto gx$.

**Definition 1.2.** Let $S$ be a semigroup and $M$ a non-empty topological space. If the map $L_S : S \times M \to M$ satisfies $L_{s_2 s_1} (m) = L_{s_2} (L_{s_1} (m)) \ \forall s_1, s_2 \in S, \ \forall m \in M$, then $(M, L_S)$ is called a left $S$-topological system, or says, $S$ acts on the left of $M$.

For short, we denote $L_s(m)$ by $sm$, left $S$-topological system $(M, L_S)$ just as $M$. Similarly, we can define right $S$-topological systems.

**Definition 1.3.** Let $M, N$ are two $S$-Systems of topological spaces, a continuous map $f : M \to N$ is called an $S$-morphism from $M$ to $N$, if $f(sm) = sf(m), \ \forall s \in S \text{ and } \forall m \in M$. All left $S$-topological systems and all $S$-morphisms between them constitute a category, denoted by $S\text{-}\text{TOP}$.

Clearly, if the semigroup $S$ contains zero element, then any $S$-System $M$ must have an element $\theta$, such that $s\theta = \theta, \ \forall s \in S$. If moreover, $M$ contains a unique element $\theta_M$ satisfying $s\theta_M = \theta_M, \ 0m = 0_M, \ \forall s \in S \text{ and } \forall m \in M$, we call such $S$-System $M$ central. All central $S$-Systems and $S$-morphisms between them also constitute a category. Clearly it is a full sub-category of $S\text{-}\text{TOP}$.

An important object in our study is quiver, whose definition is given as follows.

A **quiver** $\Gamma = (\Gamma_0, \Gamma_1)$ is an oriented graph, where $\Gamma_0$ is the set of the vertices and $\Gamma_1$ is the set of arrows between vertices. A **sub-quiver** of $\Gamma$ is just its oriented sub-graph.

We say a quiver $\Gamma$ is a **finite quiver** if $\Gamma_0$ and $\Gamma_1$ are both finite sets. We denote by $s : \Gamma_1 \to \Gamma_0$ and $t : \Gamma_1 \to \Gamma_0$ the maps, where $s(\alpha) = i$ and $t(\alpha) = j$ when $\alpha : i \to j$ is an arrow from the vertex $i$ to the vertex $j$.

A **path** $p$ in the quiver $\Gamma$ is either an ordered sequence of arrows $p = \alpha_n \cdots \alpha_2 \alpha_1$ with $t(\alpha_l) = s(\alpha_{l+1})$ for $1 \leq l \leq n$, or the symbol $e_i$ for $i \in \Gamma_0$. We call the path $e_i$ trivial path and we define $s(e_i) = t(e_i) = i$. For a non-trivial path $p = \alpha_n \cdots \alpha_2 \alpha_1$, we define $s(p) = s(\alpha_1)$, and $t(p) = t(\alpha_n)$.

A vertex $i$ in $\Gamma_0$ is called a **sink** if there is no arrow $\alpha$ with $s(\alpha) = i$ and a source if there is no arrow $\alpha$ with $t(\alpha) = i$.

Now we sketch a outline of this paper.

In Section 2, we introduce the notion of topological representations of a quiver, as well as some basic categorical notions such as exact sequences, injective (projective) and indecomposable objects. As a result, we show the category of topological representations of a quiver is equivalent to that defined in Definition 1.3 while $S$ being some path semigroup of the quiver (see Theorem 2.4).

In Section 3, we investigate the relation between the category of topological representations and that of linear representations of a quiver via $P(\Gamma)\text{-}\text{TOP}^\circ$ and $k\Gamma\text{-}\text{Mod}$. By using the techincs developed in [9] concerning positively graded modules, the results in Section 2 allow us to show that
the property of the above two categories are actually equivalent when the property is (positively) graded or vertex (positively) graded (see Theorem 3.6, Corollary 3.9 and 3.10).

Section 4 is dedicated to discussing the homological theory of topological representations of quivers. We introduce the concept of Γ-limit \( \text{Lim}^\Gamma \) and use it to define the homology groups of topological representations of quivers, and moreover we discuss the compatibility between \( \text{Lim}^\Gamma \) and \( H_n \) (cf. Theorem 4.16). We show that the homology groups of topological representations can be well-defined by chain complexes (see Definition 4.13). Moreover, some properties of a quiver can be read from these homology groups (cf. Theorem 4.19, Theorems 4.22 and 4.24).

In Section 5, we discuss the homotopy theory of topological representations of quivers. We define the homotopy equivalence between two morphisms in \( \text{Top} \rightarrow \text{Rep}^\Gamma \) (see Definition 5.3), and show that the parallel Homotopy Axiom also holds for top-representations based on the homotopy equivalence defined here (see Theorem 5.5). Moreover, the parallel Excision theorem holds for top-representations (see Theorem 5.12).

In Section 6, we mainly study the functor \( A^\Gamma \text{Rep} \) (see Definition 6.2) from the category \( \text{Top} \rightarrow \text{Rep}^\Gamma \) to the category \( \text{Top} \). We show that \( A^\Gamma \text{Rep} \) preserves homotopy equivalence between morphisms (see Theorem 6.4). Moreover, we discuss some properties of the functor \( A^\Gamma \text{Rep} \) (see Theorem 6.5, 6.6 and 6.9). Finally, we establish the relationship between the homotopy groups of a top-representation \( (T,f) \) and the homotopy groups of \( A^\Gamma (T,f) \) (see Corollary 6.11).

2 Top-representations of a quiver

We start with the following definition.

**Definition 2.1.** Let \( \Gamma = (\Gamma_0, \Gamma_1) \) be a quiver with \( \Gamma_0 \) the set of vertices and \( \Gamma_1 \) the set of arrows between vertices. A top-representation \( (T,f) \) of a quiver \( \Gamma = (\Gamma_0, \Gamma_1) \) is a family of pairs of topological space \( \{T_i : i \in \Gamma_0\} \) together with continuous map \( f_\alpha : T_i \rightarrow T_j \) for each arrow \( \alpha : i \rightarrow j \).

We also call the top-representation of the quiver \( \Gamma \) defined in Definition 2.1 the \( \Gamma \)-representation via topological spaces.

Let \( (T,f) \) and \( (T',f') \) be two topological representations of \( \Gamma \). A morphism \( h : (T,f) \rightarrow (T',f') \) between two top-representations of \( \Gamma \) is a collection of continuous maps \( \{h_i : T_i \rightarrow T'_i\} \) such that for each arrow \( \alpha : i \rightarrow j \) in \( \Gamma_1 \) the following diagram:

\[
\begin{array}{ccc}
T_i & \xrightarrow{h_i} & T'_i \\
\downarrow{f_\alpha} & & \downarrow{f'_\alpha} \\
T_j & \xrightarrow{h_j} & T'_j
\end{array}
\]

commutes. If \( h : (T,f) \rightarrow (T',f') \) and \( g : (T',f') \rightarrow (T'',f'') \) are two morphisms between top-representations, then the composition \( gh \) is defined to be the collection of maps \( \{g_ih_i : T_i \rightarrow T''_i\} \) such that \( g_ih_i \) is defined. In this way, we get the category of topological representations of \( \Gamma \), which we denote by \( \text{Top} \rightarrow \text{Rep}^\Gamma \).

If we think from any set \( X \), there is a unique map \( \emptyset : X \rightarrow \emptyset \) and to any set \( Y \), there is a unique map \( \emptyset : \emptyset \rightarrow Y \). Then, we can allow \( T(i) = \emptyset \) in some special cases. For \( (T,f) \in \text{Top} \rightarrow \text{Rep}^\Gamma \), if \( T(j) = \emptyset \) and there exists an arrow \( \alpha : i \rightarrow j \), then \( T(i) \) must also be an emptyset. Then
we can define the zero object in $\text{Top-Rep} \Gamma$ as follows: $(T, f)$ is called the zero object, which we denote by $(\emptyset, 0)$, if $T(i) = \emptyset$ for all $i \in \Gamma_0$ and $f_\alpha = 1_\emptyset$ for each arrow $\alpha$ in $\Gamma_1$. For any two elements $(T, f), (T', f')$ in $\text{Top-Rep} \Gamma$, if $T(i) = \emptyset$ implies $T(i) = \emptyset$ (called condition $\gamma$), then $\text{Hom}_{\text{Top-Rep} \Gamma}((T, f), (T', f')) = \{(h_i)_{i \in \Gamma} | f'_j h_i = h_j f_\alpha, \forall \alpha : i \to j\}$. If the condition $\gamma$ doesn’t hold, then we set $\text{Hom}_{\text{Top-Rep} \Gamma}((T, f), (T', f')) = \{0\}$, where $0$ is the zero morphism from $(T, f)$ to $(T', f')$. For the object $(T, f) \in \text{Top-Rep} \Gamma$, $(T, f)$ is called a null object if there exists an $i_0 \in \Gamma_0$ such that $T(i_0) = \emptyset$, otherwise, it is called a non-null object.

If we think from any set $X$, there is a unique map $\emptyset : X \to \emptyset$ and to any set $Y$, there is a unique map $\emptyset : Y \to \emptyset$, then we can define the zero object in $\text{Top-Rep} \Gamma$ as follows: $(T, f)$ is called the zero object, which we denote by $(\emptyset, 0)$, if $T(i) = \emptyset$ for all $i \in \Gamma_0$ and $f_\alpha = 1_\emptyset$ for each arrow $\alpha$ in $\Gamma_1$.

An object $(S, f)$ is called a sub-object of an object $(T', f')$ in $\text{Top-Rep} \Gamma$, if $T(i) \subseteq T'(i)$ for all $i \in \Gamma_0$ and $f_\alpha = f'_\alpha$ for each arrow $\alpha$ starting from $i$.

A sum, or coproduct, of two objects $(T, f)$ and $(T', f')$ in $\text{Top-Rep} \Gamma$ is the object $(W, g)$, where $W(i) = T(i) \sqcup T'(i)$ for each $i \in \Gamma_0$ and $g_\alpha = f_\alpha \cup f'_\alpha$ for all $\alpha \in \Gamma_1$. And, we denote the the sum of $(T, f)$ and $(T', f')$ as $(T, f) \sqcup (T', f')$. An object $(T, f)$ is said to be indecomposable if it can not be written as the sum of any two nonzero top-representations. An object $(T, f)$ is simple if it has no proper nonzero sub-objects. Clearly, a simple object is indecomposable.

A product of two objects $(T, f)$ and $(T', f')$ in $\text{Top-Rep} \Gamma$ is the object $(H, h)$, where $H(i) = T(i) \times T'(i)$ for each $i \in \Gamma_0$ and $h_\alpha = f_\alpha \times f'_\alpha$ for all $\alpha \in \Gamma_1$. And, we denote the the product of $(T, f)$ and $(T', f')$ as $(T, f) \times (T', f')$.

Next, we illustrate with some examples.

**Example 2.2.** Let $(T, f)$ be an object in $\text{Top-Rep} \Gamma$ and $V(i) = \{(a, a) \mid a \in T(i)\}$, $g_\alpha = (f_\alpha \times f_\alpha) |_{V(i)}$ for each $i \in \Gamma_0$ and an arrow $\alpha$ starting from $i$, then $(V, g)$ is a sub-object of $(T, f) \times (T, f)$, which is denoted as $1_{(T, f)}$.

**Example 2.3.** Let $\Gamma$ be the quiver $1 \to 2$, $(T, f)$ and $(T', f')$ be two top-representations, where $T(1) = \{x_1, y_1\}$, $T(2) = \{x_2, y_2\}$, $T'(1) = \{x_1, y_1'\}$ and $T'(2) = \{x_2', y_2'\}$ are all spaces with discrete topology. Let $f_\alpha(x_1) = x_2$, $f_\alpha(y_1) = y_2$, $f'_\alpha(x_1) = f'_\alpha(y_1) = x_2$, and let $h_1 : T(1) \to T'(1)$ with $h_1(x_1) = y_1$ and $h_1(y_1) = x_1$, $h_2 : T(2) \to T'(2)$ with $h_2(x_2) = h_2(y_2) = x_2$, then $h = \{h_1, h_2\}$ is a morphism from $(T, f)$ to $(T', f')$.

Let $P(\Gamma)$ be the set consisting of 0 and all paths in the quiver $\Gamma$. Define a multiplication $\cdot$ on $P(\Gamma)$ as follows: $0 \cdot \rho = \rho \cdot 0 = 0$ for all $\rho \in P(\Gamma)$, for any two paths $\rho_1, \rho_2$ from $k$ to $t$, $\rho_{ji} \cdot \rho_{ik} = \left\{ \begin{array}{ll} 0, & \text{if } i \neq t \\ \rho_{ji} \rho_{ik}, & \text{if } i = t \end{array} \right.$ where $\rho_{ji} \rho_{ik}$ means the connection of $\rho_{ji}$ and $\rho_{ik}$ for $i = t$.

Then $P(\Gamma)$ becomes a semigroup with zero 0 under the multiplication $\cdot$. Omitting $\cdot$, we usually write $\rho_1 \rho_2$ instead of $\rho_1 \cdot \rho_2$.

Now, by Definition 1.3 we have a category $P(\Gamma)-\text{TOP}$, further we can define a subcategory $P(\Gamma)-\text{TOP}^0$ of it like this: the objects $M$ are $P(\Gamma)$-Systems satisfying (i) $P(\Gamma)M = M$; (ii) there is a unique element $\theta_M \in M$ such that $\{\theta_M\}$ is a component of $M$ as a isolated point and $0m = \theta_M$, for all $m \in M$. (Here $\theta_M$ acts as the "zero element" of $M$); (iii) if $e_i m \neq \theta_M$, then $am \neq \theta_M$, for all arrows $\alpha$ starting from $i$.

Let $M = \bigcup_{i \in \Gamma_0} M_i \backslash \{\theta_M\}$ for all $i \in \Gamma_0$. Then, it is easy to show that $M = \bigcup_{i \in \Gamma_0} M_i \backslash \{\theta_M\}$. Since $e_i : M \to M, x \mapsto e_i x$ is continuous and $\{\theta_M\}$ is a
close subset of \( M \), then \( e_i^{-1}\{\{\theta_M\}\} = M \setminus M_i \) is a close subset of \( M \). Thus, we have \( M_i \) is an open subset of \( M \) for each \( i \in \Gamma_0 \).

Note that for any \( \rho \in P(\Gamma) \), we always have \( \rho\theta_M = \rho(0m) = (\rho \cdot 0)m = 0m = \theta_M \). And clearly \( P(\Gamma) \) is an object of \( P(\Gamma)\text{-}\text{TOP}^\circ \), called the regular object, and \( \{\emptyset\} \) is the zero object of \( P(\Gamma)\text{-}\text{TOP}^\circ \), if we define the action as \( \rho\theta = \theta \) for all \( \rho \in P(\Gamma) \).

For two objects \( M \) and \( N \) in \( P(\Gamma)\text{-}\text{TOP}^\circ \), a morphism \( \varphi \colon M \to N \) is defined as a continuous map satisfying (i) \( \varphi(pm) = \rho\varphi(m) \) for any \( m \in M \) and \( \rho \in P(\Gamma) \); (ii) \( \varphi(m) \neq \theta_N \), if \( m \neq \theta_M \).

Note that (ii) is equivalent to saying \( \varphi(M \setminus \theta_M) \subseteq N \setminus \{\theta_N\} \), and when \( \rho = 0 \), from (i), it must hold that \( \varphi(\theta_M) = \theta_N \).

And the object \( M = \{\theta_M\} \in P(\Gamma)\text{-}\text{TOP}^\circ \) acts as a zero object, we denote it as \( \{0\} \).

For two objects \( M \) and \( N \) and a morphism \( \varphi \colon M \to N \) in \( P(\Gamma)\text{-}\text{TOP}^\circ \), it is easy to show that \( \varphi(e_iM \setminus \{\theta_M\}) \subseteq e_iN \setminus \{\theta_N\} \), for all \( i \in \Gamma_0 \). Thus, if \( e_{i_0}N \setminus \{\theta_N\} = \emptyset \) but \( e_{i_0}M \setminus \{\theta_M\} \) is nonempty for some \( i_0 \in \Gamma_0 \), which will lead to contradiction. In this case, we set \( \text{Hom}_{P(\Gamma)\text{-}\text{TOP}^\circ}(M,N) = \{0\} \), where the \( 0 \) is the zero morphism from \( M \) to \( N \). For any object \( M \in P(\Gamma)\text{-}\text{TOP}^\circ \), if \( e_{i_0}M \setminus \{\theta_M\} = \emptyset \) for some \( i_0 \in \Gamma_0 \), it is called the null element, otherwise, it is called the non-null element.

Then, \( P(\Gamma)\text{-}\text{TOP}^\circ \) is exactly a subcategory of the category \( P(\Gamma)\text{-}\text{TOP} \).

We have known from [1][2] that, for a field \( k \) and a finite quiver \( \Gamma \), there exists an equivalence between the two categories \( \text{Lin-Rep}_k \) and \( k\Gamma\text{-Mod} \), where \( \text{Lin-Rep}_k \) is the category of \( k \)-linear representations of \( \Gamma \) and \( k\Gamma\text{-Mod} \) the \( k\Gamma \)-module category. It is interesting for us to find that the similar result also holds between the two weaker categories \( \text{Top-Rep}_k \) and \( P(\Gamma)\text{-}\text{TOP}^\circ \), that is, we have :

**Theorem 2.4.** The two categories \( \text{Top-Rep}_k \) and \( P(\Gamma)\text{-}\text{TOP}^\circ \) are equivalent.

**Proof.** We start by defining two functors \( F \colon \text{Top-Rep}_k \to P(\Gamma)\text{-}\text{TOP}^\circ \) and \( H \colon P(\Gamma)\text{-}\text{TOP}^\circ \to \text{Top-Rep}_k \).

For an object \( (T,f) \) in \( \text{Top-Rep}_k \), set \( M = \bigcup_{i \in \Gamma_0} T(i) \cup \theta_M \), where \( \theta_M \) is an element which is not in \( T(i) \) for all \( i \in \Gamma_0 \). Define the action of \( P(\Gamma) \) on the set \( M \) as follows: for any \( m \in M \), \( \rho \in P(\Gamma) \),

(i) \( \rho m = \theta_M \), if \( \rho = 0 \);

(ii) \( \rho m = m \), if \( m \in T(i) \) and \( \rho = e_i \);

(iii) \( \rho m = f_{\alpha}(m) \), if \( m \in T(i) \) and \( \rho \) is an arrow \( \alpha : i \to j \);

(iv) \( \rho m = f_{\alpha_\cdot \cdot \cdot \alpha_1}(m) \), if \( m \in T(i) \), \( \rho = \alpha_\cdot \cdot \cdot \alpha_1 \) where \( \alpha_\cdot \cdot \cdot \alpha_1 \) are arrows and \( \alpha_1 \) starts from \( i \).

From this definition, it is easy to see that \( P(\Gamma)\theta_M = \theta_M \), and that if \( m \in T(i) \) but \( \rho \) does not start from \( i \), then \( \rho m = \rho(e_i m) = (\rho \cdot e_i)m = 0m = \theta_M \).

Clearly, \( M \) is a \( P(\Gamma) \)-System under the action defined above. Moreover, we can say \( M \) is an object of \( P(\Gamma)\text{-}\text{TOP}^\circ \). Firstly, the element \( \theta_M \) satisfies \( 0m = \theta_M \) for all \( m \in M \). And, obviously, \( P(\Gamma)\text{-}M \subseteq M \). Conversely, for all \( m \in M \), when \( m \neq \theta_M \), suppose \( m \in T(i) \) for some \( i \), then \( m = e_i m \); when \( m = \theta_M \), we have \( P(\Gamma)\theta_M = \theta_M \). Hence \( M \subseteq P(\Gamma)\text{-}M \). It follows that \( P(\Gamma)\text{-}M = M \). If \( e_i m \neq \theta_M \), which implies \( m \in T(i) \), then for all arrows as \( \alpha : i \to j \), \( \alpha m = f_{\alpha}(m) \in T(j) \), so \( \alpha m \neq \theta_M \). Then \( M \) is an object of \( P(\Gamma)\text{-}\text{TOP}^\circ \).

Now, we can start to define the functor \( F : \text{Top-Rep}_k \to P(\Gamma)\text{-}\text{TOP}^\circ \) by \( F(T,f) = M \).

Let \( h \) be a morphism from \( (T,f) \) to \( (T',f') \) in the category \( \text{Top-Rep}_k \). Then, for each \( i \in \Gamma_0 \), we have a map \( h_i : T(i) \to T'(i) \) satisfying the Figure (I), i.e. \( h_if_{\alpha} = f'_{\alpha}h_i \) for each arrow \( \alpha \) from \( i \) to \( j \). It has been known that \( M = F(T,f) = \bigcup_{i \in \Gamma_0} T(i) \cup \theta_M \) and \( M' = F(T',f') = \bigcup_{i \in \Gamma_0} T'(i) \cup \{\theta_M\} \).
Introducing a map \( \bar{h} : M \to M' \) satisfying that \( \bar{h}|_{T(i)} = h_i \) for all \( i \) and \( \bar{h}(\theta_M) = \theta_{M'} \). Thus, we can get \( h(\alpha m) = \alpha \bar{h}(m) \) for each \( m \in M \). Moreover, for each \( m \in M \), then \( \forall \rho = \alpha_1 \cdots \alpha_2 \alpha_1 \in P(\Gamma), m \in M \), we have

\[
\bar{h}(pm) = \bar{h}(f_{\alpha_1} \cdots f_{\alpha_2} f_{\alpha_1}(m)) = h(k(f_{\alpha_1} \cdots f_{\alpha_2} f_{\alpha_1}(m))) = f'_{\alpha_1} h_{k-1}(f_{\alpha_1} \cdots f_{\alpha_2} f_{\alpha_1}(m)) = \cdots = f'_{\alpha_1} \cdots f'_{\alpha_2} f'_{\alpha_1}(\bar{h}(m)) = \rho(\bar{h}(m)).
\]

This shows that \( F(h) = \bar{h} \), and so \( \bar{h} \) is a morphism in \( P(\Gamma)\text{-}\textsf{TOP} \). When \( m \neq \theta_M \), \( \bar{h}(m) \neq \theta_{M'} \) since \( \bar{h}(T(i)) = h_i(T(i)) \subseteq T'(i) \). Therefore \( \bar{h} \) is a morphism from \( M \) to \( M' \). This means one can set \( F(h) = \bar{h} \).

We next want to define a functor \( H : P(\Gamma)\text{-}\textsf{TOP}^o \to \textsf{Top}^{\text{Repl}} \). For an object \( M \) in category of \( P(\Gamma)\text{-}\textsf{TOP}^o \), let \( T(i) = e_i M \setminus \theta_M \). For all arrows \( \alpha : i \to j \), define \( f_{\alpha} : T(i) \to T(j) \) as follows: for all \( m \in T(i) \), suppose \( m = e_i m \), let \( f_{\alpha}(m) = \alpha m \), it is well-defined since \( \alpha m = \alpha(e_j n) = \alpha(n) \notin \theta_M \) and \( \alpha m = (e_j \alpha)m = e_j(\alpha m) \in T(j) \). Therefore let \( H(M) = (T,f) \), where \( S = \{ T(i) : i \in \Gamma_0 \} \), and \( f = \{ f_{\alpha} : \text{there is an arrow } \alpha \text{ from } i \text{ to } j \} \). Then \( H(M) \) is an object of category \( \textsf{Top}^{\text{Repl}} \).

If \( \varphi : M \to M' \) is a morphism in \( P(\Gamma)\text{-}\textsf{TOP}^o \), we have \( H(M) = (T,f) \), \( H(M') = (T',f') \), where \( T(i) = e_i M \setminus \theta_M \) and \( T'(i) = e_i M' \setminus \{ \theta_{M'} \} \). Since \( \varphi(e_i M) = e_i \varphi(M) \subseteq e_i M' \) and \( \varphi(m) \neq \theta_{M'} \) for all \( m \in M \) and \( m \neq \theta_M \), then we get \( \varphi_i : e_i M \setminus \theta_M \to e_i M' \setminus \{ \theta_{M'} \} \) by restriction, i.e. \( \varphi_i = \varphi|_{T(i)} : T(i) \to T'(i) \). For each arrow \( \alpha : i \to j \), we have \( \alpha \varphi_i(m) = \varphi(\alpha m) \), for all \( m \in M \). So \( \alpha \varphi_i(m) = \varphi_j(\alpha m) \), for all \( m \in T(i) \). Hence \( f'_{\alpha}\varphi_i = \varphi_j f_{\alpha} \) for any arrow \( \alpha : i \to j \). Therefore we can set \( H(\varphi) = \{ \varphi_i \}_{i \in \Gamma_0} \), which is a morphism in \( \textsf{Top}^{\text{Repl}} \).

Next, we will prove \( F \) and \( H \) are mutual-inverse equivalent functors. Let \( (T,f) \) be an object in \( \textsf{Top}^{\text{Repl}} \), then \( M = F(T,f) = \cup_{i \in \Gamma_0} T(j) \cup \{ \theta_M \} \) and \( e_i M \setminus \theta_M = e_i(\cup_{j \in \Gamma_0} T(j)) \setminus \{ \theta_M \} = e_i T(i) \setminus \theta_M = T(i) \). For an arrow \( \alpha : i \to j \) in \( \Gamma_1 \), the map \( f_{\alpha} : T(i) \to T(j) \) induces the map \( \bar{f}_{\alpha} : F(T,f) \to F(T,f) \) satisfying \( \bar{f}_{\alpha}(m) = \alpha m \) for all \( m \in F(T,f) \). The restriction of \( \bar{f}_{\alpha} \) on \( e_i F(T,f) \setminus \theta_M = T(i) \) is just \( f_{\alpha} \). So \( HF(T,f) = (T,f) \).

For a morphism \( h = \{ h_i \}_{i \in \Gamma_0} : (T,f) \to (T',f') \), we have \( F(h) = \bar{h} \) where \( \bar{h}|_{T(i)} = h_i, \bar{h}(\theta_M) = \theta_{M'} \). Due to the definition of \( H \), it follows \( HF(h) = \{ h_i \}_{i \in \Gamma_0} \). Thus, \( HF \) is \( \text{id} \) the identity functor in \( \textsf{Top}^{\text{Repl}} \).

Let \( M \) be an object in \( P(\Gamma)\text{-}\textsf{TOP}^o \), then \( H(M) = (T,f) \), where \( T(i) = e_i M \setminus \theta_M \) and

\[
f = \{ f_{\alpha} : T(i) \to T(j) \mid f_{\alpha}(m_i) = \alpha m_i \text{ for an arrow } \alpha : i \to j \text{ and } m_i \in T(i) \}.
\]

When \( i \neq j \), if there exists two elements \( m, m' \in M \), such that \( e_i m = e_j m' \neq \theta_M \), then for an arrow \( \alpha : i \to k \), \( \alpha(e_i m) = \alpha m \neq \theta_M \), but \( \alpha(e_j m') = (\alpha e_j)m' = 0 m' = \theta_M \), this is a contradiction. Hence \( T(i) \cap T(j) = \emptyset \) when \( i \neq j \). So if we can prove \( M = \cup_{i \in \Gamma_0} T(i) \cup \theta_M \), then \( FH(M) = M \). In fact, \( \cup_{i \in \Gamma_0} T(i) \cup \theta_M \subseteq P(\Gamma)M = M \). Conversely, for all \( m \in M \), if \( m = \theta_M \), it is clearly that \( m = \theta_M \). When \( m \neq \theta_M \), since \( m \in M = P(\Gamma)M \), there is \( \rho_{ji} \in P(\Gamma), m' \in M \), such that \( m = \rho_{ji} m' \). Clearly \( m' \neq \theta_M \), so \( m = \rho_{ji} m' = e_j(\rho_{ji} m') \in e_j M \setminus \theta_M = T(j) \). Therefore, \( M \subseteq \cup_{i \in \Gamma_0} T(i) \cup \theta_M \).
For a morphism $\varphi : M \to M'$, we have $H(\varphi) = \{ \varphi_i : T(i) \to T'(i) \mid \varphi_i = \varphi \mid_{T(i)} \}_{i \in \Gamma_0}$. Moreover, due to the definition of $F$, it follows $FH(\varphi) = \varphi$. Therefore, $FH = \text{id}$ the identity functor in $P(\Gamma)\text{-}\text{TOP}^\omega$. 

\[\square\]

**Remark 2.5.** According to Theorem 2.4 and some known results in the theory of $S$-Systems of semigroups, it is easy for us to obtain the following conclusions:

1. $P(\Gamma)$ is the coproduct of $P(\Gamma)e_i, i \in \Gamma_0$ in the category $P(\Gamma)\text{-}\text{TOP}^\omega$.
2. Let $M \in P(\Gamma)\text{-}\text{TOP}^\omega$, and $M_{i_0} = e_{i_0}M \setminus \{ \theta_M \} \neq \emptyset$ for some $i_0 \in \Gamma_0$. Then, for any $m_{i_0} \in M_{i_0}$, there exists a unique morphism $\varphi_{i_0}$ from $P(\Gamma)e_{i_0}$ to $M$ such that $\varphi_{i_0}(e_{i_0}) = m_{i_0}$.
3. Let $M \in P(\Gamma)\text{-}\text{TOP}^\omega$, and $M_i = e_iM \setminus \{ \theta_M \} \neq \emptyset$ for all $i \in \Gamma_0$. Then, for any $\{ m_i \}_{i \in \Gamma}$, there exists a unique morphism $\varphi$ from $P(\Gamma)$ to $M$ such that $\varphi(e_i) = m_i$ for all $i \in \Gamma$.
4. For each $i \in \Gamma_0$, $P(\Gamma)e_i$ is a projective object in category $P(\Gamma)\text{-}\text{TOP}^\omega$.
5. The regular object $P(\Gamma)$ is a projective object in category $P(\Gamma)\text{-}\text{TOP}^\omega$.

Due to Theorem 2.4 and the discussions of the product of two objects $(T, f)$ and $(T', f')$ in category $\text{Top-}\text{Rep}\Gamma$, for any two objects $M, N$ in category $P(\Gamma)\text{-}\text{TOP}^\omega$, the product of $M$ and $N$, denoted as $M \times N$, is $Q$, where $Q_i = M_i \times N_i$ and $\rho(x, y) = (px, py)$ for all $i \in \Gamma_0, (x, y) \in M_i \times N_i, \rho \in P(\Gamma)(M_0 = e_0M \setminus \{ \theta_M \}, N_0 = e_0N \setminus \{ \theta_N \}, Q_0 = e_0Q \setminus \{ \theta_Q \})$.

In the category $\text{Top-}\text{Rep}\Gamma$, for a morphism $h = \{ h_i \}_{i \in \Gamma_0} : (T, f) \to (T', f')$, we define the image $\text{Im}h$ to be the subobject $(U, g)$ of $(T', f')$, where $U(i) = \text{Im}h_i$ and $g_{\alpha} = f'_{\alpha}|_{\text{Im}h_i}$ for each arrow $\alpha : i \to j$. We define the kernel $\text{Ker}h = (V, f''')$ as the fibre product of $(T, f)$ and itself under the morphism $h$, i.e., a subobject of $(T, f) \times (T, f)$ with $V(i) = \{ (a, b) | a,b \in T(i) \text{with } h_i(a) = h_i(b) \}$ and $f''_{\alpha} = (f_{\alpha} \times f_{\alpha})|_{V(i)}$ for each arrow $\alpha : i \to j$. If each $h_i$ is injective (respectively surjective), we call $h$ a monomorphism (respectively an epimorphism), and $h$ is an isomorphism if and only if $h$ is both monomorphic and epimorphic. The morphism $h$ given in Example 2.2 is neither monomorphic nor epimorphic. We call the sequence $(T, f) \xrightarrow{h} (T', f') \xrightarrow{h'} (T'', f'')$ a related exact sequence if $(\text{Im}h \times \text{Im}h') \cup 1(T', f') = \text{Ker}h'$.

Similarly, for a morphism $h : M \to N$ in category $P(\Gamma)\text{-}\text{TOP}^\omega$ Ker$h$ is a subobject of $M \times M$ with $\text{Ker}h = \{ (x_1, x_2) | h(x_1) = h(x_2) \}$. We call the sequence in category $P(\Gamma)\text{-}\text{TOP}^\omega Q \xrightarrow{f} M \xrightarrow{g} N$ a related exact sequence if $(\text{Im}f \times \text{Im}f) \cup 1_M = \text{Kerg}$. Then, we have

**Proposition 2.6.** (i) The sequence $(\emptyset, 0) \to (T, f) \xrightarrow{h} (T', f')$ is related exact if and only if $h$ is a monomorphism.

(ii) Suppose $| T'(i) | \geq 2$ for all $i \in \Gamma_0$, then the sequence $(T, f) \xrightarrow{h} (T', f') \to (\emptyset, 0)$ is related exact if and only if $h$ is an epimorphism.

**Proof.** (i) $(\emptyset, 0) \to (T, f) \xrightarrow{h} (T', f')$ related exact

\[\iff \text{Ker}h = 1(T, f)\]

\[\iff (\text{Ker}h)(i) = \{ (a, a) | a \in T(i) \}, \forall i \in \Gamma_0 \]

\[\iff \{ (a, b) | a, b \in T(i), h_i(a) = h_i(b) \} = \{ (a, a) | a \in T(i) \}, \forall i \in \Gamma_0 \]

\[\iff h_i(a) = h_i(b) \text{ implies } a = b, \forall i \in \Gamma_0 \text{ and } a, b \in T(i) \]

\[\iff h_i \text{ is injective}, \forall i \in \Gamma_0 \]

\[\iff h \text{ is monomorphic.} \]
P and an arbitrary morphism called strongly graded is a graded semigroup, $S$ is called positively graded sub-semigroup, $S$. And a positively graded semigroup $S$ is essentially, simple, indecomposable) if and only if $a, b \in T'(i)$ and $a' \neq b$, then $(a, b) \in Imh_i \coprod Imh_i, \forall i \in \Gamma_0$

$\iff h_i$ is surjective, $\forall i \in \Gamma_0$
$\iff h$ is epimorphic.

An object $(T, f)$ is said to be projective if for an arbitrary epimorphism $h' : (T', f') \to (T'', f'')$, and an arbitrary morphism $h'' : (T, f) \to (T', f')$, there exists a morphism $h : (T, f) \to (T', f')$ such that $h'' = h'h$, i.e. we have the commutative diagram

$$
\begin{array}{c}
(T, f) \\
h \downarrow \\
(T', f') \\
\downarrow h' \\
(T'', f'')
\end{array}
$$

\textit{Figure(II)}

Dually, an object $(T, f)$ is said to be injective, if for an arbitrary monomorphism $h' : (T'', f'') \to (T', f')$ and an arbitrary morphism $h'' : (T', f') \to (T, f)$, there exists a morphism $h : (T', f') \to (T, f)$, such that $h'' = hh'$, i.e. we have the commutative diagram

$$
\begin{array}{c}
(T, f) \\
h' \downarrow \\
(T'', f'') \\
\downarrow h \\
(T', f')
\end{array}
$$

\textit{Figure(III)}

As a corollary, the following holds naturally:

**Corollary 2.7.** (i) An object $(V, f)$ in the category $\textit{Top-Rep}_\Gamma$ is projective (respectively injective, simple, indecomposable) if and only if $F(V, f)$ is projective (respectively injective, simple, indecomposable) in the category $P(\Gamma)-\textit{TOP}^\sigma$;

(ii) A sequence $(U, f) \to (V, g) \to (W, h)$ in the category $\textit{Top-Rep}_\Gamma$ is related exact if and only if the induced sequence $F(U, f) \to F(V, g) \to F(W, h)$ is related exact in the category $P(\Gamma)-\textit{TOP}^\sigma$.

### 3 Relations between top-representations and linear-representations of quivers

In this section, every semigroup mentioned contains a zero element and the quiver $\Gamma$ is finite.

A semigroup $S$ is graded if there exists a family of non-empty subsets $\{S_{ij}\}_{i \in \mathbb{Z}}$, where $S_{0j}$ is a sub-semigroup, $S_0 \subseteq S_{i+1} \subseteq S$, and $S_{ij} = \{0\}$ for $i \neq j$. When $S = \cup_{i > 0} S_{ij}$ is a graded semigroup, $S$ is called strongly graded. And a positively graded semigroup $S$ is called strongly graded if $S_{ij} = S_{i+1}$ for any $i, j > 0$.

Note that, the path semigroup $P(\Gamma)$ which consists of 0 and all paths in $\Gamma$ has a natural positive gradation: $P(\Gamma) = \cup_{i \geq 0}(P(\Gamma))_{(i)}$, where $(P(\Gamma))_{(i)}$ consists of 0 and all the paths whose length is $i$. This positive gradation of $P(\Gamma)$ is strongly graded obviously.
Let $S$ be a graded semigroup, $M$ be a $S$-System in $S\text{-TOP}^\alpha$, if there exists a family of nonempty subsets $\{M(i)\}_{i \in \mathbb{Z}}$, such that $M = \bigcup_{i \in \mathbb{Z}} M(i)$, $S(i)M(j) \subseteq M(i+j)$, and $M(i) \cap M(j) = \emptyset$ for $i \neq j$, then $M$ is said to be graded. Similarly, for a positively graded semigroup $S$, we can give the definition of positive gradation for $M$.

Let $M$ be a positively graded $P(\Gamma)$-system in $P(\Gamma)\text{-TOP}^\alpha$, where the gradation of $P(\Gamma)$ is natural. If every homogeneous component is the union of some $M_\upsilon = e_\upsilon M$, that is, for every vertex $\upsilon \in \Gamma_0$, $e_\upsilon M$ is contained in some a homogeneous component, then $M$ is said to be positively graded. Clearly, if $M$ is positively graded and for any vertex $\upsilon \in \Gamma_0$, $M_\upsilon$ contains at most one element except for $\emptyset_M$, then $M$ is vertex positively graded.

**Definition 3.1.** (7) (i) Function $F : \Gamma_0 \rightarrow \mathbb{Z}$ is called an arrow function on $\Gamma$, if $F(t(\alpha)) = F(s(\alpha)) + 1$ for any arrow $\alpha \in \Gamma_1$(see [4]).

(ii) If function $F : \Gamma_0 \rightarrow \mathbb{Z}^+ \cup \{0\}$ is an arrow function on $\Gamma$, we call $F$ an arrow positive function on $\Gamma$.

**Proposition 3.2.** (7) $F$ is an arrow positive function on a connected quiver $\Gamma$, $G : \Gamma_0 \rightarrow \mathbb{Z}^+ \cup \{0\}$ is another positive function, then $G$ is an arrow positive function on $\Gamma$ if and only if there exists an integer $k$, such that $F = G + k$.

**Definition 3.3.** (7) For a non-trivial path $\rho$ in a quiver $\Gamma$, if $s(\rho) = e(\rho)$, we say it is an oriented cycle. A sub-quiver $\Delta$ of a quiver $\Gamma$ is said to be a cycle, if when omitted the direction of all arrows, the graph, which we call the base graph, is closed. In a cycle, when the number of clockwise arrows equals to the number of anti-clockwise arrows, we say the cycle is symmetric.

By Definition 3.1 and 3.2, when a quiver has no cycle, we can always define an arrow positive function on it. And it is clearly that, an oriented cycle is not symmetric. Indeed, we have the following conclusions.

**Proposition 3.4.** (7) Assume $\Gamma$ is a finite connected quiver. Then there exists an arrow positive function on $\Gamma$ if and only if $\Gamma$ does not contain any non-symmetric cycle.

**Lemma 3.5.** For a finite quiver $\Gamma$, if all $P(\Gamma)$-Systems in $P(\Gamma)\text{-TOP}^\alpha$ are positively graded, then any cycle in $\Gamma$ is symmetric.

**Proof.** Suppose $\Gamma$ contains a cycle $\Delta$ with the base graph like Figure(IV), consider a special $P(\Gamma)$-System $M$ in $P(\Gamma)\text{-TOP}^\alpha$, its top-representation according to the equivalence in Theorem
2.3 is \((T, f)\), where all \(S(v)\) are equal and contain only one element, the maps between them are all identity maps.

Since \(M\) is positively graded, from its special construction it is also vertex positively graded. Define a function \(F: \Gamma_0 \rightarrow \mathbb{Z}^+ \cup \{0\}\) as follows: \(F(v) = i\), if \(e_i M \subseteq M_{(i)}\). It is easy to know \(F\) is an arrow positive function on \(\Gamma\), and so it is on \(\Delta\). Indeed, if for an arrow \(F\) \(F(v) = i\), then from the construction of \(M\), \(e_i M = \alpha(e_i M) \subseteq \alpha M_{(i)} \subseteq P(\Gamma)_{(i)} M_{(i)} \subseteq M_{(i+1)}\), i.e. \(F(\omega) = F(v) + 1\). By Lemma 3.2, \(\Delta\) is a symmetric cycle.

Thus, we get the main result of this section:

**Theorem 3.6.** Suppose \(\Gamma\) is a finite connected quiver. The following properties are equivalent:

(i) all \(P(\Gamma)\)-Systems in \(P(\Gamma)\)-\(\mathcal{TOP}^o\) are positively graded;

(ii) any cycle in \(\Gamma\) is symmetric;

(iii) there exists an arrow positive function on \(\Gamma\);

(iv) all \(P(\Gamma)\)-Systems in \(P(\Gamma)\)-\(\mathcal{TOP}^o\) are vertex positively graded.

**Proof.** (i)\(\Rightarrow\)(ii): By Lemma 3.5.

(ii)\(\Rightarrow\)(iii): By Lemma 3.3.

(iii)\(\Rightarrow\)(iv): Suppose \(F: \Gamma_0 \rightarrow \mathbb{Z}^+ \cup \{0\}\) is an arrow positive function on quiver \(\Gamma\), since for any \(P(\Gamma)\)-System \(M\) in \(P(\Gamma)\)-\(\mathcal{TOP}^o\), \(M = \cup_{i \in \Gamma_0} e_i M\), let \(M_i = \cup_{i \in \Gamma_0, F(i) = i} e_i M\), then \(M = \cup_{F(i) = i, i \in \Gamma_0} M_i\) is a positive gradation. Actually, for any arrow \(\alpha: v \rightarrow \omega\), \(F(\omega) = F(v) + 1\). Then when \(i \neq F(s(\alpha))\), \(\alpha M_i = \theta_M \subseteq M_{(i+1)}\), since \(\alpha M = \alpha e_i M\). And from the definition of \(M_i\) and \(\alpha M_{F(s(\alpha))} \subseteq e_{t(\alpha)} M \subseteq M_{(F(t(\alpha)))} = M_{(F(s(\alpha)) + 1)}\), we know \(M\) is vertex positively graded.

(iv)\(\Rightarrow\)(i): By the definition of vertex positively graded. \(\square\)

**Example 3.7.** Let \(\Gamma\) be a quiver as \(1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} 3 \xrightarrow{\alpha_3} \cdots \xrightarrow{\alpha_{n-1}} n\). Then all \(P(\Gamma)\)-Systems in \(P(\Gamma)\)-\(\mathcal{TOP}^o\) are positively graded.

Indeed, if \(M\) is a \(P(\Gamma)\)-System of \(P(\Gamma)\)-\(\mathcal{TOP}^o\), let \(M_i = e_i M, i = 1, 2, \cdots, n\). From the proof of Theorem 2.3, we know \(M = \cup_{i=1}^n (e_i M \setminus \theta_M) \cup \theta_M\), and \((e_i M \setminus \theta_M) \cap (e_j M \setminus \theta_M) = \emptyset\) when \(i \neq j\), so \(M = \cup_{i=1}^n M_i\) and \(M_i \cap M_{(j)} = \theta_M\) when \(i \neq j\). And the inclusion \((P(\Gamma))_{(i)} M_{(j)} \subseteq M_{(i+j)}\) is also easy to prove, here \(M_i = \theta_M\) for any \(i > n\).

In final, we discuss the relation between the category of top-representations and that of linear-representations.

**Theorem 3.8.** ([12]) Let \(\Gamma\) be a finite connected quiver with the corresponding path algebra \(k\Gamma\). Then the following statements are equivalent:

(i) all \(k\Gamma\)-modules are graded;

(ii) any cycle in \(\Gamma\) is symmetric;

(iii) there exists an arrow function on \(\Gamma\);

(iv) all \(k\Gamma\)-modules are vertex graded.

By Theorem 3.6 and Theorem 3.8, we know that for a finite connected quiver \(\Gamma\), there is an arrow function on it if and only if there is an arrow positive function on it. Since all the proofs in [4] about gradation were based on positive gradation, all conclusions about gradation in [4] can be equivalently replaced by the ones about positive gradation. Similarly, our results about positive
gradation in this paper can be equivalently replaced by the ones about gradation. Then through the common statement (ii) in Theorem 3.6 and Theorem 3.8 we have a collection of equivalent statements. In particular, we have the following corollaries:

**Corollary 3.9.** Let $\Gamma$ be a finite connected quiver and $k$ a field, then the following statements are equivalent:

1. all $k\Gamma$-modules are (positively) graded;
2. all $P(\Gamma)$-Systems in $P(\Gamma)\text{-TOP}^0$ are (positively) graded.

**Corollary 3.10.** Let $\Gamma$ be a finite connected quiver and $k$ a field, then the following statements are equivalent:

1. all $k\Gamma$-modules are vertex (positively) graded;
2. all $P(\Gamma)$-Systems in $P(\Gamma)\text{-TOP}^0$ are vertex (positively) graded.

From the two theorems above, we find that on a finite connected quiver $\Gamma$, there are some interesting relations between the categories $P(\Gamma)\text{-TOP}^0$ and $k\Gamma\text{-Mod}$, one of which is not abelian while the other is. Since the top-representation category $\text{Top-Rep}_\Gamma$ is equivalent to the category $P(\Gamma)\text{-TOP}^0$, and the linear-representation category $\text{Lin-Rep}_\Gamma$ is equivalent to the category $k\Gamma\text{-Mod}$, there are also some similar relations between the two representation categories $\text{Top-Rep}_\Gamma$ and $\text{Lin-Rep}_\Gamma$.

### 4 Homology groups of top-representations

In [6], for a quiver $\Gamma$, Henrik Holm and P. Jorgensen introduced a $\Gamma$-bundle associated to two chains. In [4], Bustamante J C, Dionne J and Smith D discussed homology theories for oriented algebras. In this section we hope to associate a quiver to chain complexes and use the homology groups of these chain complexes to reflect the quiver.

We recall from [10] some basic notations in algebraic topology. For any topological space and each integer $n \geq 0$, $S^n(X)$ denoted as the free abelian group with basis all continuous maps from $\Delta^n$ to $X$. For any positive integer $n$, there are $n+1$ continuous maps $\varepsilon^n_i (0 \leq i \leq n)$ from $\Delta^{n-1}$ to $\Delta^n(\Delta^n)$ is the standard n-simplex), defined as

$$\varepsilon^n_i(t_0, t_1, \ldots, t_{n-1}) = (t_0, t_1, \ldots, t_{i-1}, 0, t_{i+1}, \ldots, t_n) \quad (1)$$

Then for each $n \geq 0$, boundary operator $\partial_n$ from $S_n(X)$ to $S_{n-1}(X)$, defined as

$$\partial_n(f) = \sum_{i=1}^{n+1} (-1)^i f \varepsilon^n_i \quad (2)$$

where $f$ is a continuous map from $\Delta^n$ to $X$, $S_{-1}(X) = 0$, and $\partial \partial = 0$. $S(X)$ denotes the following chain complex:

$$\cdots \longrightarrow S_2(X) \xrightarrow{\partial_2} S_1(X) \xrightarrow{\partial_1} S_0(X) \longrightarrow 0 \quad (3)$$

If $f : X \rightarrow Y$ is continuous and if $\alpha : \Delta^n \rightarrow X$ is an n-simplex in $X$, then $f\alpha$ is an n-simplex in $Y$. Extending by linearity gives a homomorphism $S_n(f) : S_n(X) \rightarrow S_n(Y)$. Then, define $(S(f))_n = S_n(f)$ for all $n \geq -1$. Obviously, $S$ is a functor from $\text{Top}$ to $\mathbf{C}$, where $\text{Top}$ is the
category of topological spaces and continuous functions and \( \mathbf{C} \) is the category of chain complexes of abelian groups and chain maps.

For a continuous map between two topological spaces \( f : X \to Y \), \( S(f) \) denoted as \( f_* \). Then for each \( n \geq 0 \), \( H_n(X) = H_n(S(X)) \). And, \( H_n(f) = H_n(f_*) : H_n(X) \to H_n(Y) \). It is easy to show \( H_n \) is a functor from \( \mathbf{Top} \) to \( \mathbf{Ab} \) for each \( n \geq 0 \). For the emptyset \( \emptyset \), we set \( S(\emptyset) = 0 \). The coset \( z_n + B_n(X) \), where \( z_n \) is an \( n \)-cycle, is called the homology class of \( z_n \), and is denoted as \( \text{clsz}_n \) (see [10]).

We further introduce some notations. Let \( \mathbf{C} \text{– Rep}\Gamma \) be the category of \( \Gamma \)-representations via chain complexes of abelian groups, \((\mathcal{X}, f) \in \mathbf{C} \text{– Rep} \) \( \mathbf{C}(i) \) is a chain complex of abelian groups and \( f_{ij} \) is the chain morphism for any arrow \( i \to j \) in \( \Gamma \). Let \( \mathbf{Ab} \text{– Rep}\Gamma \) be the category of \( \Gamma \)-representations via abelian groups. That is, for any \( \Gamma \)-representation via abelian groups \((G, f) \in \mathbf{Ab} \text{– Rep}\Gamma \), \( G(i) \) is an abelian group and \( f_{ij} \) is a morphism between abelian groups for any \( i \to j \) in \( \Gamma \). Let \( \mathbf{C} \) denote the category of chain complexes of abelian groups, that is, for any \( A \in \mathbf{C} \), \( \dot{A} \) is a chain complex of abelian groups.

According to [3], for a complex chain of abelian groups:

\[
\lim_{\leftarrow} G_n = \{ (g_n)_{n \in \mathbb{Z}} \in \prod_{n \in \mathbb{Z}} G_n | f_n(g_n) = g_{n-1}, \forall n \in \mathbb{Z} \} \text{ is a a subgroup of } \prod_{n \in \mathbb{Z}} G_n.
\]

We can think the complex chain \( \lim_{\leftarrow} G \) to be on the infinite linear quiver \( A_\infty \). In follows, we think an (infinite) quiver to be a generalization of (infinite) linear quiver. So, as a generalization of \( \lim_{\leftarrow} \), we introduce the notation \( \lim^\Gamma \).

**Definition 4.1.** Let \( \Gamma \) be a quiver, and \((G, f) \) be a \( \Gamma \)-representation via abelian groups. Define

\[
\lim^\Gamma(G, f) = \{(x_i)_{i \in \Gamma} : x_i \in G(i) \text{ and } f_{ij}(x_i) = x_j \text{ if } f_{ij} \text{ exists} \} \subseteq \prod_{i \in \Gamma} G(i).
\]

We call \( \lim^\Gamma(G, f) \) the \( \Gamma \)-limit of \((G, f) \).

Note that \( \lim^\Gamma(G, f) \) is a subgroup of \( \prod_{i \in \Gamma} G(i) \), since for any \((x_i)_{i \in \Gamma}, (y_i)_{i \in \Gamma} \in \lim^\Gamma(G, f) \), we have \((x_i)_{i \in \Gamma} - (y_i)_{i \in \Gamma} = (x_i - y_i)_{i \in \Gamma} \) and \( f_{ij}(x_i - y_i) = f_{ij}(x_i) - f_{ij}(y_i) = x_j - y_j \) for all arrows \( i \to j \) in \( \Gamma_1 \), therefore \((x_i)_{i \in \Gamma} - (y_i)_{i \in \Gamma} \in \lim^\Gamma(G, f) \).

**Definition 4.2.** Let \( \Gamma \) be a quiver, and \((\mathcal{X}, f) \) be a \( \Gamma \)-representation via chain complexes of abelian groups. Define \( \lim^\Gamma(\mathcal{X}, f) \) as a subcomplex of \( \prod_{i \in \Gamma} \mathcal{X}(i) \), such that \( (\lim^\Gamma(\mathcal{X}, f))_n = \lim^\Gamma(C^n, f^n) \), where each \((C^n, f^n) \) is a \( \Gamma \)-representation via abelian groups and \( C^n(i) = \mathcal{X}(i)_n, (f^n)_{ij} = (f_{ij})_n \).

In the case that \( \Gamma \) is an acyclic quiver. Then \( \Gamma_0 \) is a partially ordered set with the partial order \( \geq \) defined as follows: for any two vertices \( i, j \in \Gamma_0 \), we define \( i \geq j \) if there is a path from \( i \) to \( j \). In particular, for any vertex \( i \in \Gamma_0 \), we say that there is a path of length zero from \( i \) to \( i \), i.e. \( i \geq i \).

For \((G, f) \in \mathbf{Ab} \text{– Rep}\Gamma \) and any \( i \geq j \in \Gamma_0 \), if \( i = j \) we set \( \theta_{ij} = \text{Id}_{G(i)} \); otherwise, there is at least a path \( i \to i_0 \to i_1 \to \cdots \to i_n \to j \) and then we set \( \theta_{ij} = f_{i_n,i_{n-1}} \cdots f_{i_1,i_0} \).

Note that for any \( i \geq j \in \Gamma_0 \), if there are more than one path from \( i \) to \( j \), then for any two paths

\[
i \to i_0 \to i_1 \to \cdots \to i_n \to j \quad \text{and} \quad i \to j_0 \to j_1 \to \cdots \to j_m \to j,
\]
it needs to assume that \( f_{j,m,j}f_{j,m-1,j,m} \cdots f_{j_0,j_1}f_{i,j_0} = f_{i,n,j}f_{i,n-1,i,n} \cdots f_{i_0,i_1}f_{i,i_0} \) so as to both be defined as \( \theta_{ij} \).

Thus, in the meaning of \( \prod \), \( (G(i), \theta_{ij})_{i \in \Gamma_0, i \geq j} \) is an inverse system in the category \( \text{Ab} \) over the partially ordered set \( \Gamma_0 \) and \( \lim_{\Gamma}(G, f) = \lim(G(i), \theta_{ij}) \).

Similarly, for any \((C, f) \in \text{C–Rep}\Gamma\), \((C(i), \theta_{ij})\) is an inverse system in the category \( \text{C} \) over the partially ordered set \( \Gamma_0 \) and \( \lim_{\Gamma}(C, f) = \lim(C(i), \theta_{ij}) \).

Note that we cannot define the partial order as above for a quiver \( \Gamma \) containing an oriented cycle: otherwise, we have \( i \neq j \) but \( i \geq j, j \geq i \), which is a contradiction.

Conversely, let \((M_i, \theta_{ij})_{i, j \in I, i \geq j}\) be an inverse system in the category \( \text{Ab} \) over the partially ordered set \( I \). Then we can define the quiver \( \Gamma_I = (\Gamma_0, \Gamma_1, s, t) \), where \( \Gamma_0 = I, \Gamma_1 = \{i \rightarrow j| i, j \in I \text{ and } i \geq j \} \). Let \((G, f) \in \text{Ab–Rep}\Gamma_I\), and \( G(i) = M_i, \forall i \in \Gamma_0, f_{ij} = \theta_{ij}, \forall i \rightarrow j \in \Gamma_1 \), then we have \( \lim_{\Gamma}(G, f) = \lim(M_i, \theta_{ij}) \) in the category \( \text{Ab–Rep}\Gamma_I \).

The converse discussion holds for an inverse system \((M_i, \theta_{ij})_{i, j \in I, i \geq j}\) from \( \text{C} \) to \( \text{C–Rep}\Gamma_I \).

**Definition 4.3.** Let \( \Gamma \) be a quiver. Define a functor \( S^\Gamma : \text{Top–Rep}\Gamma \rightarrow \text{C–Rep}\Gamma \). For all \((T, f) \in \text{Top–Rep}\Gamma\), \( S^\Gamma(T, f) = (A, h) \) where \( A(i) = S(T(i)) \) for all \( i \in \Gamma \) and \( h_{ij} = S(f_{ij}) \) for all \( i \rightarrow j \) in \( \Gamma \). For any \( \alpha : (T, f) \rightarrow (T', f') \), \( S^\Gamma(\alpha) : S^\Gamma(T, f) \rightarrow S^\Gamma(T', f') \) with \( (S^\Gamma(\alpha))(i) = S(\alpha_i) \).

Similarly, for each \( n \geq -1 \), we define a functor \( S_n^\Gamma : \text{Top–Rep}\Gamma \rightarrow \text{Ab–Rep}\Gamma \). For all \((T, f) \in \text{Top–Rep}\Gamma\), \( S_n^\Gamma(T, f) = (G, g) \) where \( G(i) = S_n(T(i)) \) for all \( i \in \Gamma \) and \( g_{ij} = S_n(f_{ij}) \) for all \( i \rightarrow j \) in \( \Gamma \). For any \( \alpha : (T, f) \rightarrow (T', f') \), \( S_n^\Gamma(\alpha) : S_n^\Gamma(T, f) \rightarrow S_n^\Gamma(T', f') \) with \( (S_n^\Gamma(\alpha))(i) = S_n(\alpha_i) \).

**Example 4.4.** Let \( \Gamma \) be a quiver: \( \cdots \rightarrow \cdot \leftarrow \cdot \). For any \((T, f) : X_1 \rightarrow X_2 \leftarrow X_3 \) with maps \( f_{12}, f_{32} \in \text{Top–Rep}\Gamma \),

\[
S^\Gamma(T, f) = S(X_1) \rightarrow S(X_2) \leftarrow S(X_3)
\]

with maps \( S(f_{12}), S(f_{32}) \) and \( S_n^\Gamma(T, f) = S_n(X_1) \rightarrow S_n(X_2) \leftarrow S_n(X_3) \) with maps \( S_n(f_{12}), S_n(f_{32}) \).

**Definition 4.5.** Let \( \Gamma \) be a quiver. For any \((T, f) \in \text{Top–Rep}\Gamma\), define \( \lim_{\Gamma}(T, f) = \lim\Gamma(S^\Gamma(T, f)) \).

From the definition, we know that \( (\lim\Gamma(T, f))_n = \lim\Gamma(S_n^\Gamma(T, f)) \) for all \( n \geq -1 \).

**Corollary 4.6.** Let \( \Gamma \) be a quiver. Then \( \lim\Gamma : \text{Top–Rep}\Gamma \rightarrow \text{C} \) is a functor.

**Example 4.7.** Let \( \Gamma \) be a finite quiver: \( \cdots \rightarrow \cdot \rightarrow \cdot \rightarrow \cdot \). For any \((T, f) \in \text{Top–Rep}\Gamma\), assume \((T, f) = X_1 \rightarrow X_2 \rightarrow \cdots \rightarrow X_n \). Obviously, \( \lim\Gamma(T, f) \simeq S(X_1) \).

**Lemma 4.8.** Let \( \Gamma \) be a connected quiver, and \((T, f) \in \text{Top–Rep}\Gamma\) with \( f_{ij} \) being isomorphisms for all \( i \rightarrow j \) in \( \Gamma \) and \( f_\alpha = f_\beta \) if both \( \alpha \) and \( \beta \) are arrows from \( i \) to \( j \). Then \( \lim\Gamma(T, f) \) is isomorphic to a subcomplex of \( S(T(i)) \) where \( i \in \Gamma \). Furthermore, \( \lim\Gamma(T, f) \simeq S(T(i)) \) for \( i \in \Gamma \) if \( \Gamma \) is acyclic.

**Proof.** Since \( \Gamma \) is connected and \( f_{ij} \) are isomorphisms for all \( i \rightarrow j \) in \( \Gamma \), we have \( T(i) \simeq T(j) \) and \( S(T(i)) \simeq S(T(j)) \) for all \( i, j \in \Gamma \). For each \( j \in \Gamma \), define \( L_j : \lim\Gamma(T, f) \rightarrow S(T(i)) \) such that for
all $n \geq 0$ $(L_j)_n : \lim^\Gamma(S_n^\Gamma(T, f)) \to S_n(T(i)), (\alpha_i)_{i \in \Gamma} \mapsto \alpha_j$. We will show $(L_j)_n$ are injective for all $j \in \Gamma$ and $n \geq 0$. If $\alpha_i = \beta_i$ for any $(\alpha_i)_{i \in \Gamma}, (\beta_i)_{i \in \Gamma} \in \lim^\Gamma(S_n^\Gamma(T, f))$. Since $\Gamma$ is connected and $f_{ij}$ are isomorphisms for all $i \to j$ in $\Gamma$, there exists an isomorphism $g_t : S_n(T(i)) \to S_n(T(t))$ for each $t \in \Gamma$ such that $\alpha_t = g_t(\alpha_i) = g_t(\beta_i) = \beta_t$. Therefore, $(\alpha_i)_{i \in \Gamma} = (\beta_i)_{i \in \Gamma}$ and $(L_j)_n$ is injective. This implies $L_j : \lim^\Gamma(T, f) \to S(T(j))$ is injective for each $j \in \Gamma$. There exist isomorphisms $g_{j_1,j_2}$ from $S(T(j_1))$ to $S(T(j_2))$ because $\Gamma$ is connected and $f_{ij}$ are isomorphisms for all $i \to j$ in $\Gamma$. We have the following diagram:

\[
\begin{array}{ccc}
\lim^\Gamma(T, f) & \xrightarrow{L_{j_1}} & S(T(j_1)) \\
\downarrow & & \downarrow \\
S(T(j_2)) & \xrightarrow{S_{j_1,j_2}} & S(T(j_2))
\end{array}
\]

If $\Gamma$ contains no circle, obviously $L_j$ is also surjective, that is $L_j$ is isomorphic, for each $j \in \Gamma$. Then, $\lim^\Gamma(T, f) \simeq S(T(j))$ for all $j \in \Gamma$.

\[\square\]

**Definition 4.9.** Let $\Gamma$ be a quiver, and $(T, f) \in \text{Top–Rep}\Gamma$. Define $H_n(T, f) = H_n(\lim^\Gamma(T, f))$ for all $n \geq 0$.

**Example 4.10.** Let $\Gamma$ be a connected quiver, $(T, f) \in \text{Top–Rep}\Gamma$ with $T(i) = \{x_i\}$ for all $i \in \Gamma$. Obviously, $\lim^\Gamma(T, f) \simeq S(T(i))$. Thus, $H_n(T, f) \simeq H_n(T(i))$, and then $H_n(T, f) = 0 \forall n \geq 1, H_0(T, f) = \mathbb{Z}$.

As in Definition 4.9, $H_n(T, f) = H_n(\lim^\Gamma(T, f))$ for all $n \geq 0$. For any morphism $(h_i)_{i \in \Gamma} : (T, f) \to (T', f')$ and $n \geq 0$, $H_n((h_i)_{i \in \Gamma}) = H_n(\lim^\Gamma(f)) : H_n(T, f) \to H_n(T', f')$. For each $n \geq 0$, $H_n(–)$ is a functor from $\text{Top–Rep}\Gamma$ to $\text{Ab}$.

Then, by Lemma 4.9, we can obtain the following corollaries.

**Corollary 4.11.** For each $n \geq 0$, $H_n(–)$ is a functor from $\text{Top–Rep}\Gamma$ to $\text{Ab}$.

**Corollary 4.12.** Let $\Gamma$ be a connected and acyclic quiver, and $(T, f) \in \text{Top–Rep}\Gamma$ with all $f_{ij}$ being isomorphisms and $f_\alpha = f_\beta$ if both $\alpha$ and $\beta$ are arrows from $i$ to $j$. Then $H_n(T, f) \simeq H_n(T(i)), \forall i \in \Gamma$ and $n \geq 0$.

**Definition 4.13.** Define functors $H_n^\Gamma$ from $\text{Top–Rep}\Gamma$ to $\text{Ab–Rep}\Gamma$, $n \geq 1$. For any object $(T, f)$ in $\text{Top–Rep}\Gamma$, $H_n^\Gamma(T, f) = (H_n, g_n)$ such that $H_n(i) = H_n(X_i)$ and $(g_n)_{ij} = H_n(f_{ij})$. For any morphism $(h_i)_{i \in \Gamma} : (T, f) \to (T', f')$ in $\text{Top–Rep}\Gamma$, $H_n^\Gamma((h_i)_{i \in \Gamma}) = (\lim H_n(h_i))_{i \in \Gamma}$.

Similarly, for each $n \in \mathbb{Z}$ we can define a functor from $\text{C–Rep}\Gamma$ to $\text{Ab–Rep}\Gamma$, also denoted as $H_n^\Gamma(–)$.

**Example 4.14.** Let $\Gamma$ be the quiver:

\[
\begin{array}{ccc}
\Gamma & \to & \Gamma \\
\end{array}
\]
For any \((T, f) : X_1 \rightarrow X_2 \leftarrow X_3\) with maps \(f_{12}, f_{32} \in \text{Top-Rep}_\Gamma\),
\[
H_n^\Gamma(T, f) = H_n(X_1) \rightarrow H_n(X_2) \leftarrow H_n(X_3)
\]
with maps \(H_n(f_{12}), H_n(f_{32})\).

**Example 4.15.** For any \((C, f) : C_1 \rightarrow C_2 \leftarrow C_3\) with chain morphisms \(f_{12}, f_{32}\) in \(\text{C-Rep}\),
\[
H_n^\Gamma(C, f) : H_n(C_1) \rightarrow H_n(C_2) \leftarrow H_n(C_3)
\]
with abelian group morphisms \(H_n(f_{12}), H_n(f_{32})\).

Let \(\Gamma\) be a quiver. We already know \(H_n, \lim^{\Gamma}, \lim^{\Gamma} H_n^\Gamma\) are two functors from \(\text{C-Rep}_\Gamma\) to \(\text{Ab}\) for each \(n \geq 0\).

**Theorem 4.16.** *(Commutativity of \(H_n\) and \(\lim^{\Gamma}\)) Let \(\Gamma\) be a quiver. For each \(n \geq 0\), there is a natural transformation \(\rho_n\) from functor \(H_n \lim^{\Gamma}\) to functor \(\lim^{\Gamma} H_n^\Gamma\).

**Proof.** We already know that \(H_n \lim^{\Gamma}, \lim^{\Gamma} H_n^\Gamma\) are functors from \(\text{C-Rep}_\Gamma\) to \(\text{Ab}\). For any \((A, f) \in \text{C-Rep}_\Gamma\), define \(\rho_{n, (A, f)} : H_n(\lim^{\Gamma}(A, f)) \rightarrow \lim^{\Gamma}(H_n^\Gamma(A, f)), \text{cls}((a_i)_{i \in \Gamma}) \mapsto (\text{cls}(a_i))_{i \in \Gamma}\). It is easy to show that \(\rho_{n, (A, f)}\) is a morphism between groups. And for any \(\alpha : (A, f) \rightarrow (B, g)\), we have
\[
\rho_{n, (B, g)} H_n(\lim^{\Gamma}(\alpha)) = \lim^{\Gamma}(H_n^\Gamma(\alpha)) \rho_{n, (A, f)}.
\]

Let \(\Gamma\) be a quiver. We already know \(H_n, \lim^{\Gamma}, H_n^\Gamma\) are two functors from \(\text{Top-Rep}_\Gamma\) to \(\text{Ab}\) for each \(n \geq 0\).

**Corollary 4.17.** There exists a natural transformation \(\psi : H_n \rightarrow \lim^{\Gamma} H_n^\Gamma\) for each \(n \geq 0\).

**Proof.** From the definition, we know that \(H_n = H_n \lim^{\Gamma} S^\Gamma, \lim^{\Gamma} H_n^\Gamma = \lim^{\Gamma} H_n^\Gamma S^\Gamma : \text{Top-Rep}_\Gamma \rightarrow \text{Ab}\). Applying Theorem 4.16, we can get the inclusion.

**Lemma 4.18.** Let \(\Gamma\) be a quiver, \(\Gamma = \Gamma_1 \cup \Gamma_2\) where \(\Gamma_1\) and \(\Gamma_2\) are sub-quivers of \(\Gamma\) and \(\Gamma_1 \cap \Gamma_2 = \emptyset\), \((G, g)\) be a \(\Gamma\)-representation via abelian groups, \((C, h)\) be a \(\Gamma\)-representation via chain complexes of abelian groups. Denote \((G_1, g_1)\) as a \(\Gamma_1\)-representation via abelian groups such that \(G_1(i) = G(i), (g_1)_{ij} = g_{ij}\) for all \(i \in \Gamma_0, i \rightarrow j \in \Gamma_1\). Similarly, define \((G_2, g_2), (C_1, h_1), (C_2, h_2)\). Then \(\lim^{\Gamma_1}(G, g) \simeq \lim^{\Gamma_1}(G_1, g_1) \prod \lim^{\Gamma_2}(G_2, g_2)\) and \(\lim^{\Gamma}(C, h) \simeq \lim^{\Gamma_1}(C_1, h_1) \prod \lim^{\Gamma_2}(C_2, h_2)\).

**Proof.** Define \(\phi\) from \(\lim^{\Gamma}(G, g)\) to \(\lim^{\Gamma_1}(G_1, g_1) \prod \lim^{\Gamma_2}(G_2, g_2)\), \((x_i)_{i \in \Gamma} \mapsto ((x_i)_{i \in \Gamma_1}, (x_i)_{i \in \Gamma_2})\).
Then it is easy to verify \(\phi\) is an abelian group isomorphism. Similarly, we can prove \(\lim^{\Gamma}(C, h) \simeq \lim^{\Gamma_1}(C_1, h_1) \prod \lim^{\Gamma_2}(C_2, h_2)\).

**Theorem 4.19.** Let \(\Gamma\) be a quiver, \(\Gamma = \Gamma_1 \cup \Gamma_2\) where \(\Gamma_1\) and \(\Gamma_2\) are sub-quivers of \(\Gamma\) and \(\Gamma_1 \cap \Gamma_2 = \emptyset\), and \((T, f) \in \text{Top-Rep}_\Gamma\). Denote \((T_1, f_1)\) as a top-representation of \(\Gamma_1\) such that \(T_1(i) = T(i), (f_1)_{ij} = f_{ij}\) for any \(i, j \in \Gamma_1\). Similarly, denote \((T_2, f_2)\) as a top-representation of \(\Gamma_2\). Then \(H_n(T, f) \simeq H_n(T_1, f_1) \prod H_n(T_2, f_2)\) for all \(n \geq 0\).
Proof. Since $\Gamma = \Gamma_1 \cup \Gamma_2, \Gamma_1 \cap \Gamma_2 = \emptyset$, we have

$$\lim^\Gamma(T, f) = \lim^\Gamma(S^\Gamma(T, f))$$

$$\simeq \lim^\Gamma_1(S^\Gamma_1(T_1, f_1)) \prod \lim^\Gamma_2(S^\Gamma_2(T_2, f_2)) \quad (4)$$

according to Lemma 4.18. Thus,

$$H_n(T, f) = H_n(\lim^\Gamma(T, f)) \simeq H_n(\lim^\Gamma_1(T_1, f_1)) \prod H_n(\lim^\Gamma_2(T_2, f_2)) = H_n(T_1, f_1) \prod H_n(T_2, f_2)$$

for all $n \geq 0$. \hfill \Box

**Corollary 4.20.** Let $\Gamma$ be a quiver and $(\Gamma_j)_{1 \leq j \leq m}$ be all components of $\Gamma$, $(T, f)$ be a top-representation of $\Gamma$. Then $H_n(T, f) = \prod_{1 \leq j \leq m} H_n(T_j, f_j)$ for all $n \geq 0$ where $(T_j, f_j)$ defined as above.

Proof. Note $\Gamma_{j_1} \cap \Gamma_{j_2} = \emptyset$ for $j_1 \neq j_2$ and $\Gamma = \cup_{1 \leq j \leq m} \Gamma_j$. Then the conclusion can be proved by induction on $m$. \hfill \Box

**Corollary 4.21.** Let $\Gamma$ be an acyclic quiver having $m$ components, $X$ be any topological space. Define $(T, f)$ as a top-representation of $\Gamma$ where $T(i) = X$ and $f_{ij} = Id_X$. Then $H_n(T, f) = (H_n(X))^m$ for all $n \geq 0$.

Proof. Using Corollary 4.12 and 4.20. \hfill \Box

Let $S^n = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} | x_1^2 + \ldots + x_{n+1}^2 = 1\}$ denote the n-dimensional unit sphere.

For $n = 1$, define a continuous map $F : S^1 \to S^1$ with $(\cos \theta, \sin \theta) \mapsto (\cos(\theta + t), \sin(\theta + t))$ where $ut \neq 2\pi$ for all $u \in \mathbb{Z}$. Apparently, $F$ is a homeomorphism and $F^k(x) \neq x$ for all $x \in S^1, k \geq 1$.

**Theorem 4.22.** Let $\Gamma$ be a connected quiver, and $(T, f)$ be a top-representation with $T(i) = S^1$, $f_{ij} = F$. Then $H_1(T, f) = \mathbb{Z}$ if $\Gamma$ is acyclic, and $H_1(T, f) = 0$ if $\Gamma$ contains oriented circle.

Proof. If $\Gamma$ is acyclic, then $H_1(T, f) = H_1(S^1)$ according to Corollary 4.12. If $\Gamma$ contains circle. Assume, the lengthen of this circle is $m(m \geq 1)$ and this circle is $i \to i+1 \to i+2 \to \cdots \to i+m-1 \to i$. Then we will show $lim^\Gamma(T, f) = 0$. For all $(\alpha_j)_{j \in \Gamma} \in (lim^\Gamma(T, f))_n$ where $n \geq 0$, we have

$$(f_{i,i+1})_*(\alpha_i) = F_*(\alpha_i) = \alpha_{i+1},$$

$$(f_{i+1,i+2})_*(\alpha_{i+1}) = F_*(\alpha_{i+1}) = \alpha_{i+2},$$

$$\ldots \ldots,$$

$$(f_{i+m-2,i+m-1})_*(\alpha_{i+m-2}) = F_*(\alpha_{i+m-2}) = \alpha_{m-1},$$

$$(f_{i+m-1,i})_*(\alpha_{i+m-1}) = F_*(\alpha_{i+m-1}) = \alpha_i.$$  \hfill (5)

Thus, we have $(F_*)^m(\alpha_i) = (F^m)_*(\alpha_i) = \alpha_i$. 

17
If \( \alpha_i \neq 0 \), assume \( \alpha_i = k_1f_1 + k_2f_2 + \cdots + k_vf_v \) where \( v \in \mathbb{N}^+, k_i \neq 0, f_i \neq f_j \) if \( i \neq j \), then we have
\[
(F^m)^* (\alpha_i) = k_1F^m f_1 + \cdots + k_vF^m f_v = k_1f_1 + \cdots + k_vf_v.
\]
(6)

Since \( f_1, \ldots, f_v, F^m f_1, \ldots, F^m f_v \) are basic elements in free abelian group \( S_n(S^1) \), then \( \{f_1, \ldots, f_v\} = \{F^m f_1, \ldots, F^m f_v\} \), that is, \( F^m f_i = f_s(i) \) for all \( 1 \leq i \leq v \) where \( s \in S_n \). Then \( (F^m)^m f_i = \text{Im}L_{s(i)} \) where \( s \neq f_s(i) \) for all \( i \in S^1 \). Thus for all \( x \in S^1 \), \( f_i(x) = F^m f_i (f_i(x)) \), which is a contradiction.

Now, we have \( \alpha_i = 0 \). Since \( \Gamma \) is connected and all \( (f_{ij})_{*} = F_s \) are isomorphisms, \( \alpha_j = 0 \) for all \( j \in \Gamma \). Thus, \( \lim^T(T, f) = 0 \) and \( H_1(T, f) = 0 \).

**Remark 4.23.** Define a continuous map \( \mathcal{T} : S^1 \to S^1, (x_1, x_2) \mapsto (-x_1, -x_2) \). Let \( X \) be a topological space, \( (g_i)_{1 \leq i \leq n} \) be a family of distinct continuous maps from \( X \) to \( S^1 \). We will show \( (g_1, \ldots, g_n) = \{h_1, \mathcal{T} h_1, \ldots, h_m, \mathcal{T} h_m\} \) where \( n = 2m, \{h_1, \ldots, h_m\} \subset \{g_1, \ldots, g_n\} = \{\mathcal{T} g_1, \ldots, \mathcal{T} g_n\} \). First, we have \( \mathcal{T} g_i = g_{\rho(i)} \) where \( \rho \in S_n \). Then \( g_i = \mathcal{T}^\alpha g_i = \mathcal{T} g_{\rho(i)} = g_{\rho^2(i)} \). Let \( \rho^2(i) = 1 \leq i \leq n \), thus \( \rho^2(i) = 1 \). Assume \( \rho \neq \alpha \). Then \( g_{\rho(i)} = g_{\rho^2(i)} \). If \( i \in \{i_1, i_2, \ldots, i_m\} \) then \( i \in \{i_2, i_3, \ldots, i_n \} \). Let \( h_1 = g_{i_{2-1}} \). Then \( \mathcal{T} h_1 = \mathcal{T} g_{i_{2-1}} = g_{\rho^2(i_{2-1})} = g_{i_{2-1}} \). Thus \( \{h_1, \ldots, g_n\} = \{h_1, \mathcal{T} h_1, \ldots, h_m, \mathcal{T} h_m\} \).

Let \( \Gamma \) be a connected quiver containing oriented circles, and \( (T, f) \in \text{Top-Rep} \) with \( T(i) = S^1, f_{ij} = f, \forall i, j \in \Gamma \). We select one point \( i_0 \in \Gamma \), from Lemma 4.21 we know that there exists an injective chain map \( L \) from \( \lim^T(T, f) \) to \( S(S^1) \). Then we have the following theorem:

**Theorem 4.24.** \( H_0 \frac{(S(S^1))}{(\text{Im}L)} \simeq \mathbb{Z}_2 \) if \( \Gamma \) contains at least one oriented circle whose length is odd; otherwise, \( H_0 \frac{(S(S^1))}{(\text{Im}L)} = 0 \).

**Proof.** If \( \Gamma \) contains no oriented circle whose length is odd, that is, \( \Gamma \) only contains oriented circles whose lengths are even. Since \( \mathcal{T}^n = I \) for any positive even number \( n \), then \( \text{Im}L = S(S^1) \) according to Lemma 4.21. Therefore, \( \frac{(S(S^1))}{(\text{Im}L)} = 0 \). If \( \Gamma \) contains at least one oriented circle whose length is odd. For any positive odd number \( n \), we have \( \mathcal{T}^n = \mathcal{T} \). For all \( n \geq 1 \), let \( B_n \) denote the free subgroup of \( S_n(S^1) \) with basis \( \{g \in \mathcal{T} \} \).

\[
\text{We first show } (\text{Im}L)_n = B_n \text{ for all } n \geq 0
\]

For all \( (\alpha_i)_{i \in \Gamma} \in (\text{Im}^T(T, f))_n \), then \( (f_{ij})_{*} (\alpha_i) = \mathcal{T} \alpha_i = \alpha_j \). Since \( \Gamma \) is connected and \( (\mathcal{T} \mathcal{T})^2 = I \), then \( \alpha_i = \alpha_{i_0} \) or \( \mathcal{T} \alpha_i \) for all \( i \in \Gamma \). We select at least one oriented circle whose length is odd. Then \( \mathcal{T} \alpha_i = \alpha_{i_0} \) or \( \mathcal{T} \alpha_i \) for all \( i \in \Gamma \). In both cases, we have \( \mathcal{T} \alpha_i = \alpha_{i_0} \). If \( \alpha_{i_0} = 0 \), then \( \mathcal{T} \mathcal{T} \alpha_i = \alpha_{i_0} = 0 \in B_n \). Assume \( \alpha_{i_0} \neq 0 \). Then \( \alpha_i = k_1(g_1 + \cdots + g_{i_1}) + \cdots + k_s(g_s + \cdots + g_{s_i}) \) where \( s \) is a positive integer, \( k_i(1 \leq i \leq s) \) are distinct nonzero integers, and \( g_{ij} \) are distinct continuous maps from \( \Delta^n \) to \( S^1 \). Then,
\[
\mathcal{T} \alpha_{i_0} = k_1(g_{11} + \cdots + g_{1i_1}) + \cdots + k_s(g_{s1} + \cdots + g_{s_i})
= k_1(g_{11} + \cdots + g_{1i_1}) + \cdots + k_s(g_{s1} + \cdots + g_{s_i}).
\]
Since \( k_i (1 \leq i \leq s) \) are distinct nonzero integers, and \( g_{ij} (1 \leq t \leq s, 1 \leq j \leq i_j) \) are distinct continuous maps from \( \Delta^n \) to \( S^1 \), then \( \overline{f} g_{1t} + \cdots + \overline{f} g_{n0} = g_{1t} + \cdots + g_{n0} \). Therefore, \( \{ g_{1t}, \ldots, \overline{f} g_{n0} \} = \{ g_{1t}, \ldots, g_{n0} \} \) for \( 1 \leq t \leq s \). According to Remark 123, there exists a subset \( \{ h_{t1}, \ldots, h_{tj} \} \) of \( \{ g_{1t}, \ldots, g_{n0} \} \). Then \( h_{t1} + \overline{f} h_{t1} + \cdots + h_{tj} + \overline{f} h_{tj} = g_{1t} + \cdots + g_{n0} \). Therefore

\[
\alpha_{i_0} = k_1 (g_{1t1} + \cdots + g_{1tj}) + \cdots + k_s (g_{1ts} + \cdots + g_{sjs}) = k_1 (h_{1t1} + \overline{f} h_{1t1} + \cdots + h_{1tj} + \overline{f} h_{1tj}) + \cdots + k_s (h_{st1} + \overline{f} h_{st1} + \cdots + h_{sjs} + \overline{f} h_{sjs}).
\] (9)

Then we have \( L_n (\alpha_{i_0})_{i \in \Gamma} = \alpha_{i_0} \in B_n \). From the above discussion, we know \( (Im L)_{n} \subset B_n, n \geq 0 \).

For any continuous map \( g \) from \( \Delta^n \) to \( S^1 \), let \( \alpha_{i_0} = g + \overline{f} g \). Then \( (\overline{f} g)^n (\alpha_{i_0}) = \alpha_{i_0} \) for any positive integer \( m \). Thus, there exists \( (\alpha_{i_0})_{i \in \Gamma} = \alpha_{i_0} \in (Im L)_{n} \) such that \( L_n (\alpha_{i_0})_{i \in \Gamma} = \alpha_{i_0} = g + \overline{f} g \in (Im L)_{n} \). It follows that \( g + \overline{f} g \) is the continuous map from \( \Delta^n \) to \( S^1 \) \( \subset (Im L)_{n} \) and \( B_n \subset (Im L)_{n}, n \geq 0 \). Then we have \( B_n = (Im L)_{n}, n \geq 0 \).

Particularly, \( (Im L)_{0} \) has basis \( \{ x + \overline{f} (x) | x \in S^1 \} \) and \( (Im L)_{1} \) has basis \( \{ \sigma + \overline{f} \sigma \sigma | \sigma \) is the path of \( S^1 \} \). Consider the subcomplex \( Im L \) of \( S(S^1) \). Since \( S^1 \) is path connected, it is easy to verify

\[
Im \partial_1 \{ \sum_{1 \leq t \leq n} k_t (x_t + \overline{f} (x_t)) | n \in \mathbb{N}^+, x_t \in S^1, \sum_{1 \leq t \leq n} k_t = 0 \}
\] (10)

Since \( (Im L)_{-1} = 0 \), we have

\[
H_0 (T, f) = H_0 (Lim^T (T, f)) \simeq H_0 (Im L) = \frac{B_0}{Im \partial_1}.
\]

Define \( \theta : B_0 \to \mathbb{Z}, \sum_{1 \leq t \leq n} k_t (x_t + \overline{f} (x_t)) \mapsto \sum_{1 \leq t \leq n} k_t \). It is easy to verify that \( \theta \) is surjective and \( Ker \theta = Im \partial_1 \). Then \( \overline{\theta} : H_0 (Im L) = \frac{B_0}{Im \partial_1} = \frac{B_0}{Ker \theta} \to \mathbb{Z}, cls \ (\sum_{1 \leq t \leq n} k_t (x_t + \overline{f} (x_t))) \mapsto \sum_{1 \leq t \leq n} k_t \)

is an isomorphism. Hence \( H_0 (T, f) \simeq H_0 (Im L) = \frac{B_0}{Im \partial_1} = \frac{B_0}{Ker \theta} \simeq \mathbb{Z} \). Let \( (\beta_{i_0})_{i \in \Gamma} \in (Lim^T (T, f))_{0} \) with \( \beta_{i_0} = x_0 + \overline{f} (x_0), x_0 \in S^1 \). Since \( H_0 (L) (cls (\beta_{i_0})_{i \in \Gamma}) = cls (x_0 + \overline{f} (x_0)), \theta (cls (x_0 + \overline{f} (x_0))) = 1, cls ((\beta_{i_0})_{i \in \Gamma}) \) is a generator of \( H_0 (T, f) \).

We have a short exact sequence:

\[
\xymatrix{ 0 \ar[r] & Lim^T (T, f) \ar[r]^L & S(S^1) \ar[r]^{\pi} & \frac{S(S^1)}{Im L} \ar[r] & 0 }
\]

Since \( (Lim^T (T, f))_{-1} = S_{-1} (S^1) = (\frac{S(S^1)}{Im L})_{-1} = 0 \), there exists an exact sequence:

\[
H_0 (T, f) \xrightarrow{H_0 (L)} H_0 (S^1) \xrightarrow{H_0 (\pi)} H_0 (\frac{S(S^1)}{Im L}) \to 0
\]

And \( H_0 (T, f) \simeq H_0 (S^1) \simeq \mathbb{Z}, cls ((\beta_{i_0})_{i \in \Gamma}) \) is a generator of \( H_0 (T, f) \), for any \( x, y \in S^1, cls (x) = cls (y) \) is a generator of \( H_0 (S^1) \). \( H_0 (L) (cls ((\beta_{i_0})_{i \in \Gamma})) = cls (L ((\beta_{i_0})_{i \in \Gamma})) = cls (\beta_{i_0}) = cls (x_0 + \overline{f} (x_0)) = cls (x_0) + cls (\overline{f} (x_0)) = 2cls (x_0) \). Thus, \( Im H_0 (L) = 2\mathbb{Z} cls (x_0) \) and \( H_0 (S^1) = \mathbb{Z} cls (x_0) \)

Then, \( H_0 (\frac{S(S^1)}{Im L}) \simeq \frac{H_0 (S^1)}{Ker H_0 (\pi)} = \frac{H_0 (S^1)}{Im H_0 (L)} \simeq \frac{\mathbb{Z}}{2\mathbb{Z}} = \mathbb{Z}/2\mathbb{Z} = \mathbb{Z}_2 \).
5 Homotopy equivalence in top-representations

**Definition 5.1.** Let $\Gamma$ be a quiver, $(\check{A}, f), (\check{B}, g) \in \mathbb{C} - \text{Rep} \Gamma$ and $\alpha, \beta : (\check{A}, f) \to (\check{B}, g)$. We say $\alpha$ is homotopic to $\beta$, denoted as $\alpha \simeq^t \beta$, if there exists a homotopy $F_i : \alpha_i \simeq^t \beta_i$ for each $i \in \Gamma$, and the following diagram commutes for all $n \in \mathbb{Z}$ if $f_{ij}$ exists:

\[
\begin{array}{c}
\begin{array}{ccc}
A(i)_{n-1} & \cdot & A(j)_{n-1} \\
(F_i)_{n-1} & \cdot & (F_j)_{n-1} \\
B(i)_n & \cdot & B(j)_n \\
\end{array}
\end{array}
\]

We also call $F : \alpha \simeq^t \beta$ a homotopy. It is easy to show that such “$\simeq^t$” is an equivalence in $\text{Hom}_{\mathbb{C} - \text{Rep}}((\check{A}, f), (\check{B}, g))$. And for any $(\check{A}, f), (\check{B}, g), (\check{C}, h) \in \mathbb{C} - \text{Rep} \Gamma$,

\[
\gamma \alpha \simeq^t \delta \beta : (\check{A}, f) \to (\check{C}, h)
\]

if $\gamma \simeq^t \delta : (\check{B}, g) \to (\check{C}, h)$ and $\alpha \simeq^t \beta : (\check{A}, f) \to (\check{B}, g)$.

Define $K((\check{A}, f), (\check{B}, g)) = \{ \alpha \in \text{Hom}_{\mathbb{C} - \text{Rep}}((\check{A}, f), (\check{B}, g)) | \alpha \simeq^t 0 \}$. Then $K((\check{A}, f), (\check{B}, g))$ is a subgroup of $\text{Hom}_{\mathbb{C} - \text{Rep}}((\check{A}, f), (\check{B}, g))$. Define the category $\textbf{K} - \text{Rep} \Gamma$ consisting of the same objects as $\mathbb{C} - \text{Rep} \Gamma$ with

\[
\text{Hom}_{\textbf{K} - \text{Rep}}((\check{A}, f), (\check{B}, g)) = \text{Hom}_{\mathbb{C} - \text{Rep}}((\check{A}, f), (\check{B}, g)) / K((\check{A}, f), (\check{B}, g)).
\] (11)

**Lemma 5.2.** The functor $\text{lim}^\Gamma : \mathbb{C} - \text{Rep} \Gamma \to \mathbb{C}$ induces a natural functor $\text{lim}^\Gamma : \textbf{K} - \text{Rep} \Gamma \to \textbf{K}$.

**Proof.** It suffices to show that for any $\alpha \simeq^t \beta : (\check{A}, f) \to (\check{B}, g)$ where $(\check{A}, f), (\check{B}, g) \in \mathbb{C} - \text{Rep} \Gamma$, we have $\text{lim}^\Gamma(\alpha) \simeq^t \text{lim}^\Gamma(\beta) : \text{lim}^\Gamma(\check{A}, f) \to \text{lim}^\Gamma(\check{B}, g)$. Assume $F : \alpha \simeq^t \beta$. Then for all $i \in \Gamma$, $F_i : \alpha_i \simeq^t \beta_i$, and we have the commutative diagram in Definition 5.1.

Furthermore, we have $\prod_{i \in \Gamma} F_i : \prod_{i \in \Gamma} \alpha_i \simeq^t \prod_{i \in \Gamma} \beta_i : \prod_{i \in \Gamma} A(i) \to \prod_{i \in \Gamma} B(i)$. Look at the following diagram:

\[
\begin{array}{c}
\begin{array}{ccc}
\prod_{i \in \Gamma} A(i)_{n+1} & \cdot & \prod_{i \in \Gamma} A(i)_n & \cdot & \prod_{i \in \Gamma} A(i)_{n-1} \\
\prod_{i \in \Gamma} B(i)_{n+1} & \cdot & \prod_{i \in \Gamma} B(i)_n & \cdot & \prod_{i \in \Gamma} B(i)_{n-1} \\
\end{array}
\end{array}
\]

We know that $\text{lim}^\Gamma(\check{A}, f)$ and $\text{lim}^\Gamma(\check{B}, g)$ are subcomplexes of $\prod_{i \in \Gamma} \check{A}(i)$ and $\prod_{i \in \Gamma} \check{B}(i)$ respectively. Thus $\text{lim}^\Gamma(\alpha) \simeq^t \text{lim}^\Gamma(\beta)$ holds if $\prod_{i \in \Gamma} (F_i)_n((\text{lim}^\Gamma(\check{A}, f))_n) \subset (\text{lim}^\Gamma(\check{B}, g))_{n+1}$. For all $\prod_{i \in \Gamma} (a_i) \in (\text{lim}^\Gamma(\check{A}, f))_n$, $\prod_{i \in \Gamma} (F_i)_n (\prod_{i \in \Gamma} (a_i)) = \prod_{i \in \Gamma} (F_i)_n (a_i)$, if $g_{ij}$ exists, according to the commutative diagram in Definition 5.1, we have

\[
(g_{ij})_{n+1}((F_i)_n(a_i)) = (F_j)_n((f_{ij})_n(a_i)) = (F_j)_n(a_j).
\] (12)

This completes the proof. \qed
Definition 5.3. Let $\Gamma$ be a quiver, $(T, f), (T', f') \in \text{Top–Rep}_\Gamma$ and $\mu, \nu : (T, f) \to (T', f')$. We say $\mu$ is homotopic to $\nu$, denoted as $\mu \simeq^t \nu$, if there exists a homotopy $H_i : \mu_i \simeq^t \nu_i$ for each $i \in \Gamma$, and the following diagram commutes if $f_{ij}$ exists:

\[
\begin{array}{ccc}
T(i) \times I \xrightarrow{H_i} T'(i) \\
\downarrow f_{ij} \times 1 \quad \downarrow f_{ij}' \\\nT(j) \times I \xrightarrow{H_j} T'(j)
\end{array}
\]

Meantime, we also call $H : \mu \simeq^t \nu$ a homotopy.

Remark that $H : \mu \simeq^t \nu$ in the category \textbf{Top–Rep}_$\Gamma$ has the same properties as the above $F : \mu \simeq^t \nu$ in the category \textbf{C–Rep}_$\Gamma$.

Remark 5.4. (10) We recall some well-known conclusions in algebraic topology. Let $X$ be any topological space, and define $\lambda^X : X \to X \times I, x \mapsto (x, i)$. Then there exist $P^X_n : S_n(X) \to S_{n+1}(X \times I)$ such that $P^X : S(\lambda^X_0) \simeq^t S(\lambda^X_1) : S(X) \to S(X \times I)$. For any continuous map $f : X \to Y$, there is a commutative diagram:

\[
\begin{array}{ccc}
S_n(X) \xrightarrow{P^X_n} S_{n+1}(X \times I) \\
\downarrow S_n(f) \quad \downarrow S_{n+1}(f \times 1) \\
S_n(Y) \xrightarrow{P^Y_n} S_{n+1}(Y \times I)
\end{array}
\]

From Definition 5.3 we claim that $H^\Gamma_n(\mu) = H^\Gamma_n(\nu)$ for all $n \geq 0$ if $\mu \simeq^t \nu : (T, f) \to (T', f')$. First, let us look at the parallel result for topological spaces.

Comparing with the fact that $H_n(f) = H_n(g)$ for all $n \geq 0$ if $f, g : X \to Y$ are homotopic (10, p75), we have the following theorem.

Theorem 5.5. (Homotopy Axiom) Let $(T, f), (T', f') \in \text{Top–Rep}_\Gamma$, and $\mu \simeq^t \nu : (T, f) \to (T', f')$. Then $S^I(\mu) \simeq^t S^I(\nu) : S^I(T, f) \to S^I(T', f')$, and therefore $H^\Gamma_n(\mu) = H^\Gamma_n(\nu)$ for all $n \geq 0$.

Proof. Assume $H : \mu \simeq^t \nu$. Then $H_i : \mu_i \simeq^t \nu_i$, that is, $H_i : T(i) \times I \to T'(i)$ and $H_i(-, 0) = \mu_i, H_i(-, 1) = \nu_i$. According to Remark 5.4, there exist $P^i_n : S_n(T(i)) \to S_{n+1}(T(i) \times I)$ such that $P^i : S(\lambda^T_0) \simeq^t S(\lambda^T_1)$, where $\lambda^T_k : T(i) \to T(i) \times I, x \mapsto (x, k)$ for $k = 0, 1$. Then

\[
S(H_i)P^i : S(H_i)S(\lambda^T_0) = S(\mu_i) \simeq^t S(H_i)S(\lambda^T_1) = S(\nu_i).
\]

We have the following commutative diagram if $f_{ij}$ exists:

\[
\begin{array}{ccc}
S_n(T(i)) \xrightarrow{P^i_n} S_{n+1}(T(i) \times I) \xrightarrow{S^{n+1}(H_i)} S_{n+1}(T'(i)) \\
\downarrow S_n(f_{ij}) \quad \downarrow S(f_{ij} \times 1) \quad \downarrow S_{n+1}(f_{ij}') \\
S_n(T(j)) \xrightarrow{P^j_n} S_{n+1}(T(j) \times I) \xrightarrow{S^{n+1}(H_j)} S_{n+1}(T'(j))
\end{array}
\]
The commutativity of the left square is based on Remark 5.4, and the commutativity of the right square is based on Definition 5.3. Therefore one gets $S^T(\mu) \simeq \alpha f T (T, f) \rightarrow S^T(T', f')$ based on equation (13) and commutative diagram (14). According to Lemma 5.2, $\lim_T(S^T(\mu)) \simeq \alpha f \lim^T(S^T(\nu)) : \lim^T(S^T(T, f)) \rightarrow \lim^T(S^T(T', f'))$. Thus for all $n \geq 0$, we have:

$$H_n(\mu) = H_n(\lim_T(S^T(\mu))) = H_n(\lim^T(S^T(\nu))) = H_n(\nu).$$

\[ \square \]

**Definition 5.6.** Let $\Gamma$ be a quiver, and $$(T, f), (T', f') \in \textbf{Top–Rep} \Gamma.$$ We say $(T, f)$ and $(T', f')$ have the same homotopy type, if there exists $\mu : (T, f) \rightarrow (T', f')$ and $\nu : (T', f') \rightarrow (T, f)$ such that $\mu \nu \simeq i \text{Id}_{(T', f')} \text{ and } \nu \mu \simeq i \text{Id}_{(T, f)}$.

For any two topological spaces $X, Y$, we already know their homotopy groups are isomorphic if they have the same homotopy type. Then it is natural to consider the parallel problem for two top-representations having the same homotopy type.

**Corollary 5.7.** (\cite{10}, p79) If $X$ and $Y$ have the same homotopy type, then $H_n(X) \simeq H_n(Y)$ for all $n \geq 0$, where the isomorphism is induced by any homotopy equivalence.

**Corollary 5.8.** If $(T, f), (T', f') \in \textbf{Top–Rep} \Gamma$ have the same homotopy type, then $H_n^T(T, f) \simeq H_n^T(T', f')$ for all $n \geq 0$.

**Corollary 5.9.** Let $(T, f)$ be a top-representation of a quiver $\Gamma$, with all $T(i)$ being convex linear topological spaces and all $f_{ij}$ being affine maps. If there exist $x_i \in T(i)$ for all $i \in \Gamma$ such that $f_{ij}(x_i) = x_j$, then $H_0(T, f) = \mathbb{Z}$ and $H_n(T, f) = 0$ for all $n \geq 1$.

**Proof.** Let $(T', f')$ be a top-representation of $\Gamma$ with $T'(i) = \{x_i\}$ for all $i \in \Gamma$. Define $\alpha : (T, f) \rightarrow (T', f')$ and $\beta : (T', f') \rightarrow (T, f)$, such that $\alpha_i : T(i) \rightarrow T'(i), x \mapsto x_i$ and $\beta_i : T'(i) \rightarrow T(i), x_i \mapsto x_i$. Then we have $\beta \alpha = 1_{(T', f')}$. Define $H_i : T(i) \times I \rightarrow T(i), (x, t) \mapsto tx + (1 − t)x_i$. Since $f_{ij}$ are affine maps, we have the following diagram:

$$
\begin{array}{ccc}
T(i) \times I & \xrightarrow{H_i} & T(i) \\
\downarrow f_{ij} \times 1 & & \downarrow f_{ij} \\
T(j) \times I & \xrightarrow{H_j} & T(j).
\end{array}
$$

Thus $H : \alpha \beta \simeq 1_{(T, f)}$ and $H_n(T, f) \simeq H_n(T', f')$ for all $n \geq 0$. This completes the proof according to Example 5.10. \[ \square \]

Let $(T, f)$ be a top-representation of a quiver $\Gamma$, and $T'(i)$ is a subspace of $T(i)$ for each $i \in \Gamma$ such that $f_{ij}(T'(i)) \subset T'(j)$. Then $(T', f)$ is a top-subrepresentation of top-representation $(T, f)$ and $S^T(T', f) \subset S^T(T, f)$. Thus we can define $H_n((T, f), (T', f)) = H_n(\lim^T((T, f), (T', f)))$ referring to the original correspondence in [5]. For any two top-representations $(T, f), (T', f')$ of the quiver $\Gamma$ and $(T'', f), (T'', f')$ being top-subrepresentations of $(T, f), (T', f')$ respectively and $\alpha : ((T, f), (T'', f)) \rightarrow ((T', f'), (T'', f'))$ meaning the restriction of $\alpha$ on $(T', f)$ is a morphism from $(T'', f)$ to $(T'', f')$, then $\alpha$ induces morphisms $H_n(\alpha) : H_n((T, f), (T'', f)) \rightarrow H_n((T', f'), (T'', f'))$ for all $n \geq 0$.
Theorem 5.10. Let $\Gamma$ be a quiver, and $(T, f)$ is a top-representation of $\Gamma$. And there exist $x_i \in T(i)$ for all $i \in \Gamma$ such that $f_{ij}(x_i) = x_j$ if $f_{ij} : T(i) \to T(j)$ exists. Then, $H_n(T(i), (X_0, f)) \simeq H_n(T(f))$ for all $n \geq 1$ and there exists a short exact sequence: $0 \longrightarrow \mathbb{Z} \longrightarrow H_0(T, f) \longrightarrow H_0((T, f), (X_0, f)) \longrightarrow 0$, where $(X_0, f)$ is a top-subrepresentation of $(T, f)$ with $X_0(i) = \{x_i\}$ for all $i \in \Gamma$.

Proof. We already know that $H_n(X_0, f) = 0$ for all $n \geq 1$ and $H_0(X_0, f) = \mathbb{Z}$. We have a short exact sequence: $0 \longrightarrow \lim^F(X_0, f) \longrightarrow \lim^F(T, f) \longrightarrow \frac{\lim^F(T, f)}{\lim^F(X_0, f)} \longrightarrow 0$. Apply $H_n(-)$ to this short exact sequence, we have a long exact sequence:

$$0 \longrightarrow H_0(X_0, f) \longrightarrow H_0(T, f) \longrightarrow H_0((T, f), (X_0, f)) \longrightarrow H_1(X_0, f) \longrightarrow H_1(T, f) \longrightarrow H_1((T, f), (X_0, f)) \longrightarrow \ldots \ldots$$

Since $H_n(X_0, f) = 0$ for all $n \geq 1$ and $H_0(X_0, f) = \mathbb{Z}$, we have $H_n((T, f), (X_0, f)) \simeq H_n(T, f)$ for all $n \geq 1$ and a short exact sequence: $0 \longrightarrow \mathbb{Z} \longrightarrow H_0(T, f) \longrightarrow H_0((T, f), (X_0, f)) \longrightarrow 0$. \qed

Remark 5.11. ([10]) Let $X$ be a topological space, and $X = X_1 \cup X_2$ where $X_1$ and $X_2$ are subspaces of $X$. Then $H_n(S(X)/(S(X_1) + S(X_2))) = 0$ for all $n \geq 0$.

Theorem 5.12. (Excision theorem) Let $(T, f)$ be a top-representation of a quiver $\Gamma$ such that $T(i) = (T''(i))^c \cup (T''(i))^c$ and $f_{ij}(T''(i)) \subset T''(j)$, $f_{ij}(T''(i)) \subset T''(j)$. Then for $n = 0, 1$, it holds that

$$H_n(\lim^F(S(T, f)/S(T''(f))) = 0).$$

Proof. We first prove that the functor $\lim^F : \text{Ab} \to \text{Rep}$ is left exact for any quiver $\Gamma$. For any short exact sequence in $\text{Ab} \to \text{Rep}$

$$0 \longrightarrow (A, f) \overset{\alpha}{\longrightarrow} (B, g) \overset{\beta}{\longrightarrow} (C, h) \longrightarrow 0,$$

we want show $0 \longrightarrow \lim^F(A, f) \longrightarrow \lim^F(B, g) \longrightarrow \lim^F(C, h) \longrightarrow 0$. Apparently, $\lim^F(\beta) \alpha$ is injective and $\lim^F(\beta) \alpha = 0$. It suffices to show $\ker \lim^F(\beta) \alpha \subset \text{Im} \lim^F(\alpha)$. For all $(b_i)_{i \in \Gamma} \in \ker \lim^F(\beta) \alpha$, we have $g_{ij}(b_i) = b_j$ and $\beta_i(b_i) = 0$. From the above exact sequence, we have $a_i \in A_i$ for each $i \in \Gamma$ such that $a_i(a_i) = b_i$. Then $b_j = \alpha_i(a_i) = g_{ij}(b_i) = g_{ij}(a_i(a_i)) = \alpha_j f_{ij}(a_i(a_i))$, and $a_j = f_{ij}(a_i)$ since $\alpha$ is injective. Therefore, $(a_i)_{i \in \Gamma} \in \lim^F(A, f)$ and $(b_i)_{i \in \Gamma} = \lim^F((a_i)_{i \in \Gamma}) \in \text{Im} \lim^F(\alpha)$

Let $(C, g) = \frac{\frac{S^F(T', f)}{S(T, f)}}{S(T, f)}$, then, $C(i) = \frac{S((T, f)))}{S(T, f)}$. From Remark, we know that $C(i)$ is exact. And let $(C^n, f^n)$ be in $\text{Ab} \to \text{Rep}$ with $C^n(i) = (C(i))_n$ and $(f^n)_{ij} = (f_{ij})_n$. From the above discussion, we know that there is an exact sequence in $\text{Ab} \to \text{Rep}$: $0 \longrightarrow (C_0, f^0) \longrightarrow (C_1, f^1) \longrightarrow (C_2, f^2) \longrightarrow \ldots \ldots$. Since $\lim^F$ is a left exact functor, we have an exact sequence: $0 \longrightarrow \lim^F(C_0, f^0) \longrightarrow \lim^F(C_1, f^1) \longrightarrow \lim^F(C_2, f^2) \longrightarrow \ldots \ldots$. From the definition, $(\lim^F(\frac{\frac{S^F(T', f)}{S(T, f)}}{S(T, f)}))_n = \lim^F(C^n, f^n)$ for all $n \geq 0$. Thus, $H_n(\lim^F(\frac{\frac{S^F(T', f)}{S(T, f)}}{S(T, f)})) = 0$ for $n = 0, 1$. \qed

Corollary 5.13. Let $\Gamma$ be a quiver, then the functor $\lim^F : \text{C} \to \text{Rep}$ is left exact.

Corollary 5.14. Let $\Gamma$ be a quiver, then the functor $\lim^F : \text{C} \to \text{C}$ is left exact.
6 On the functor $A\Gamma$ from $\text{Top}−\text{Rep}\Gamma$ to $\text{Top}$

Given a quiver $\Gamma$, it is natural to associate each top-representation $(T, f)$ of $\Gamma$ with a concrete topology, which is determined by both the quiver $\Gamma$ and the top-representation $(T, f)$. First, we define a functor $A\Gamma$ from $\text{Top}−\text{Rep}\Gamma$ to $\text{Top}$, which preserves homotopy equivalence property. Then by analyzing topological space $A\Gamma(T, f)$, we can obtain some properties of quivers, such as the connectivity of quivers and the numbers of components of quivers. Last, we established the relationship between the homotopy groups of top-representation and the homotopy groups of the corresponding topological space.

Throughout this section, we always assume the quiver $\Gamma$ is finite.

**Remark 6.1.** let $X$ be a topological space and $D$ a subset of $X \times X$, we know that $D$ can generate an equivalence relationship " $\simeq_D$ " on $X$. Let $f : X \to Y$ be a continuous map between two topological spaces, and $D, E$ be subsets of $X \times X, Y \times Y$ respectively. If $(f \times f)(D) \subset E$, that is, $\{(f(x_1), f(x_2))|(x_1, x_2) \in D\} \subset E$, then $f$ induces a continuous map $f \times f : X/ \simeq_D \to Y/ \simeq_E$, $\overline{x} \mapsto f(\overline{x})$. That is because:

Let $C^{-1} = \{(x_1, x_2)|(x_2, x_1) \in C\}$, $K^X = \{(x, x)|x \in X\}$, define $\tilde{C} = C \cup C^{-1} \cup K^X$. And define $\overline{C} = \{(x_1, x_2)|(x_2, x_1) \in C\}$ or there exist $v_1, \ldots, v_n \in X$ such that $(x_1, v_1), (v_1, v_2), \ldots, (v_n, x_2) \in \tilde{C}$. Then it is easy to show that $C$ generates $\overline{C}$. Similarly, we define $D, \overline{D}$. Then it suffices to show that $(f \times f)(\overline{C}) \subset \overline{D}$. Since $(f \times f)(C) \subset D$, $(f \times f)(C^{-1}) \subset D^{-1}$ and $(f \times f)(K^X) \subset K^Y$, that is, $(f \times f)(\overline{C}) \subset \overline{D}$. Then, obviously, $(f \times f)(\overline{C}) \subset \overline{D}$.

**Definition 6.2.** For a quiver $\Gamma$, $(T, f) \in \text{Top}−\text{Rep}\Gamma$, denote:

$$D = \{(x_i, x_j)|x_i \in T(i); x_j \in T(j); f_{ij}(x_i) = x_j; \forall i \to j \in \Gamma\} \subset \prod_{i \in \Gamma} T(i) \times \prod_{i \in \Gamma} T(i).$$

We define $A\Gamma$ from $\text{Top}−\text{Rep}\Gamma$ to $\text{Top}$ satisfying that

- $A\Gamma(T, f) = (\prod_{i \in \Gamma} T(i))/ \simeq_D$;
- for any morphism $\alpha : (T, f) \to (T', f')$, define

$$A\Gamma(\alpha) : A\Gamma(T, f) \to A\Gamma(T', f') \text{ such that } \overline{x} \mapsto (\prod_{i \in \Gamma} \alpha_i)(x).$$

**Theorem 6.3.** Let $\Gamma$ be a quiver. Then $A\Gamma$ is a functor from $\text{Top}−\text{Rep}\Gamma$ to $\text{Top}$.

**Proof.** For any two objects $(T, f), (T', f') \in \text{Top}−\text{Rep}\Gamma$, and any morphism $\alpha : (T, f) \to (T', f')$. We first show $A\Gamma(\alpha) : A\Gamma(T, f) \to A\Gamma(T', f')$ is a well-defined continuous map. From definition, we have

$$A\Gamma(T, f) = (\prod_{i \in \Gamma} T(i))/ \simeq_D,$$

$$A\Gamma(T', f') = (\prod_{i \in \Gamma} T'(i))/ \simeq_{D'},$$

where

$D = \{(x_i, x_j)|x_i \in T(i); x_j \in T(j); f_{ij}(x_i) = x_j; \forall i \to j \in \Gamma\}$,

$D' = \{(x'_i, x'_j)|x'_i \in T'(i); x'_j \in T'(j); f'_{ij}(x'_i) = x'_j; \forall i \to j \in \Gamma\}$. (15)
And $\prod_{i \in \Gamma} \alpha_i : \prod_{i \in \Gamma} T(i) \to \prod_{i \in \Gamma} T'(i)$ is a continuous map.

For all $(x_i, x_j) \in D$ and $i \to j \in \Gamma$, $(\prod_{i \in \Gamma} \alpha_i)(x_i), (\prod_{i \in \Gamma} \alpha_i)(x_j)) = (\alpha_i(x_i), \alpha_j(x_j))$, and $f_{ij}(\alpha_i(x_i)) = \alpha_j(f_{ij}(x_i)) = \alpha_j(x_j)$. Thus, $(\prod_{i \in \Gamma} \alpha_i)(x_i), (\prod_{i \in \Gamma} \alpha_i)(x_j)) \in D'$. According to Remark 6.1, $\prod_{i \in \Gamma} \alpha_i$ induces a continuous map $At^\Gamma(\alpha) : At^\Gamma(T, f) \to At^\Gamma(T', f')$, $x \to (\prod_{i \in \Gamma} \alpha_i)(x)$, that is, for all $x_i \in T(i), i \in \Gamma, At^\Gamma(\alpha)(x_i) = \alpha_i(x_i)$.

Obviously, we have $At^\Gamma(\beta \alpha) = At^\Gamma(\beta) At^\Gamma(\alpha), At^\Gamma(1_{(T,f)}) = 1_{At^\Gamma(T,f)}$, where $1_{(T,f)}$ is the identity morphism of $(T, f), \alpha \in Hom_{\text{Top} - \text{Repr}}((T, f), (T', f')) \text{, } \beta \in Hom_{\text{Top} - \text{Repr}}((T', f'), (T'', f''))$.

\begin{proof}
Assume that $H : \alpha \simeq^t \beta$, then $H_i : \alpha_i \simeq^t \beta_i$ for all $i \in \Gamma$ (that is $H_i : T(i) \times I \to T'(i), H_i(x,0) = \alpha_i(x), H_i(x,1) = \beta_i(x)$ for all $x \in T(i)$) and $H_j(f_{ij} \times 1) = f_{ij}H_i$ for all $i \to j \in \Gamma$. Thus $H : (T \times I, f \times 1) \to (T', f')$ is a morphism in $\text{Top} - \text{Repr}^\Gamma$ where $(T \times I)(i) = T(i) \times I, (f \times 1)_{ij} = f_{ij} \times 1 : T'(i) \times I \to T'(j) \times I$. Hence $At^\Gamma(H) : At^\Gamma(T \times I, f \times 1) \to At^\Gamma(T', f')$, $(x_i,t) \mapsto H_i(x_i,t)$ for $(x_i,t) \in T(i) \times I, i \in \Gamma$.

Next, we will show $At^\Gamma(T \times I, f \times 1) \simeq At^\Gamma(T, f) \times I$. We have

\begin{equation}
At^\Gamma(T \times I, f \times 1) = \left(\prod_{i \in \Gamma} T(i) \times I\right) / \simeq^C
\end{equation}

where

\begin{align*}
D &= \{(x_i,t), (x_j,t)) | (x_i,t) \in T(i) \times I; (x_j,t) \in T(j) \times I; (f_{ij}(x_i),t) = (x_j,t), \forall i \to j \in \Gamma}\ ,
C &= \{(x_i,x_j) | x_i \in T(i); x_j \in T(j); f_{ij}(x_i) = x_j; \forall i \to j \in \Gamma\}.
\end{align*}

Let $p$ be the natural map from $\prod_{i \in \Gamma} T(i) \times I$ to $(\prod_{i \in \Gamma} T(i)) / \simeq^C$, and $q$ be the natural map from $\prod_{i \in \Gamma} T(i) \times I$ to $(\prod_{i \in \Gamma} T(i)) / \simeq^C$, and $q$ be the natural map from $\prod_{i \in \Gamma} T(i) \times I$ to $(\prod_{i \in \Gamma} T(i)) / \simeq^C$. Then $p \times 1 : \prod_{i \in \Gamma} T(i) \times I = (\prod_{i \in \Gamma} T(i)) \times I \to (\prod_{i \in \Gamma} T(i)) / \simeq^C \times I$. For all $((x_i,t), (x_j,t)) \in D$, we have $(f_{ij}(x_i),t) = (x_j,t), \text{ that is, } f_{ij}(x_i) = x_j$ and $(x_i,x_j) \in C$. Thus $(p \times 1)(x_i,t) = (\bar{x}_i,t) = (\bar{x}_j,t) = (p \times 1)(x_j,t)$. Then there exists a map $\overline{p \times 1} : (\prod_{i \in \Gamma} T(i)) / \simeq^C \times I, (x_i,t) \mapsto (\bar{x}_i,t)$ such that the following diagram commutes:

\begin{equation}
\begin{array}{c}
\prod_{i \in \Gamma} T(i) \times I\ \searrow\ q \downarrow\ \nearrow\ \prod_{i \in \Gamma} T(i) / \simeq^C \times I \\
p \times 1 \searrow\ \nearrow\ p \times 1
\end{array}
\end{equation}

Since $q$ is an identification and $(p \times 1) q = p \times 1$ is continuous, it follows that $p \times 1 \Gamma$ is continuous.

Clearly, $p \times 1$ is surjective. For any $(x_1,t), (x_2,t) \in ((\prod_{i \in \Gamma} T(i)) / \simeq^C, (x_1,x_2) \in C$ then:

1. $x_1 = x_2$, thus $(x_1,t) = (x_2,t);
2. (x_1,x_2) \in C$, thus $(x_1,t), (x_2,t) \in D$ and $(x_1,t) = (x_2,t);\n
25
In a word, for any \((x_1,t), (x_2,t) \in (\prod_{i \in \Gamma} (T(i) \times I))/\sim^D\), \((x_1,t) \in C\) implies \((x_1,t) = (x_2,t)\). For any \((x_1,t), (x_2,t) \in (\prod_{i \in \Gamma} (T(i) \times I))/\sim^D\), we want to show that \((x_1,t) = (x_2,t)\). Since \((x_1,t), (x_2,t) \in C\), there exist \(v_1, \ldots, v_n \in \prod_{i \in \Gamma} T(i)\) such that \((x_1,v_1), (x_2,v_2), \ldots, (v_n,x_2) \in C\). Thus, \((x_1,t) = (v_1,t) = \cdots = (v_n,t) = (x_2,t)\). For any \((x_1,t), (x_2,t) \in (\prod_{i \in \Gamma} (T(i) \times I))/\sim^D\), if \(p \times 1((x_1,t)) = (\overline{x_1},t) = p \times 1((x_2,t)) = (\overline{x_2},t)\), then \(\overline{x_1} = \overline{x_2}\), which implies \((x_1,t) = (x_2,t)\). Therefore \(p \times 1\) is also injective. We have a commutative diagram:

\[
\begin{array}{ccc}
\prod_{i \in \Gamma} (T(i) \times I) & \xrightarrow{(\prod_{i \in \Gamma} (T(i) \times I))/\sim^D} & \prod_{i \in \Gamma} (T(i) \times I) / \sim^C \\
\downarrow{p \times 1} & & \downarrow{(p \times 1)^{-1}} \\
\end{array}
\]

Since \(I\) is locally compact and \(p\) is an identification, we have that \(p \times 1\) is an identification. Therefore, \((p \times I)^{-1}\) is continuous. One has

\[
(p \times I)^{-1}: At^\Gamma(T,f) \times I \simeq At^\Gamma(T \times I, f \times 1) \quad \text{via} \quad (\overline{x},t) \mapsto (\overline{x},t).
\]

Let

\[
F = At^\Gamma(H)(p \times I)^{-1}: At^\Gamma(T,f) \times I \to At^\Gamma(T', f') \quad \text{via} \quad (\overline{x},t) \mapsto H_i(\overline{x}_i)
\]

for \((x_1,t) \in T(i) \times I, i \in \Gamma\). Then

\[
F(\overline{x}_i, 0) = H_i(\overline{x}_i, 0) = \alpha_i(x_i) = At^\Gamma(\alpha)(\overline{x}), F(\overline{x}_i, 1) = H_i(x_1, 1) = \beta_i(x_i) = At^\Gamma(\beta)(\overline{x})
\]

for all \(x_i \in T(i), i \in \Gamma\). And, \(F: At^\Gamma(\alpha) \simeq At^\Gamma(\beta): At^\Gamma(t,f) \to At^\Gamma(T', f')\) in \(\text{Top}\).

\[
\square
\]

**Theorem 6.5. (Connectivity invariant)** Let \(\Gamma\) be a connected quiver, and \((T,f) \in \text{Top–Rep}^{\Gamma}\) with all \(T(i)\) being connected. Then, \(At^\Gamma(T,f)\) is connected.

**Proof.** Suppose on the contrary, then \(At^\Gamma(T,f) = X \cup Y, X \cap Y = \emptyset\) and \(X,Y\) are open close subsets of \(At^\Gamma(T,f)\). Let \(p\) be the natural map from \(\prod_{i \in \Gamma} T(i)\) to \(At^\Gamma(T,f)\).

Since \(p\) is surjective, \(P^{-1}(X), p^{-1}(Y)\) are nonempty and open close subsets of \(\prod_{i \in \Gamma} T(i)\), and \(\prod_{i \in \Gamma} T(i) = p^{-1}(X) \cup p^{-1}(Y), p^{-1}(X) \cap p^{-1}(Y) = \emptyset\).

Since \(T(i)\) are connected and \(P^{-1}(X), p^{-1}(Y)\) are open and close subsets of \(\prod_{i \in \Gamma} T(i)\), we have \(T(i) \subset P^{-1}(X)\) or \(T(i) \cap P^{-1}(X) = \emptyset, T(i) \subset P^{-1}(Y)\) or \(T(i) \cap P^{-1}(Y) = \emptyset\) for each \(i \in \Gamma\).

Therefore, \(p^{-1}(X) = \cup_{i \in \Gamma'} T(i), p^{-1}(Y) = \cup_{i \in \Gamma''} T(i)\) where \(\text{point}(\Gamma) = \Gamma' \cup \Gamma''\), \(\Gamma' \cap \Gamma'' = \emptyset\).

Because \(\Gamma\) is connected, there exist \(i_1 \in \Gamma', i_2 \in \Gamma''\) such that there is an arrow from \(i_1\) to \(i_2\) in \(\Gamma\). Select \(x \in T(i_1) \subset p^{-1}(X)\), and then \(f_{i_1,i_2}(x) \in T(i_2) \subset p^{-1}(Y)\). Thus \(p(x) = f_{i_1,i_2}(x) = p(\overline{f_{i_1,i_2}(x)}) \in X \cap Y\). This is a contradiction, and so \(At^\Gamma(T,f)\) is connected.

\[
\square
\]

**Theorem 6.6.** Let \(\Gamma\) be a quiver, \(\Gamma' = \Gamma' \cup \Gamma''\) where \(\Gamma'\) and \(\Gamma''\) are nontrivial sub-quivers of \(\Gamma\) and \(\Gamma' \cap \Gamma'' = \emptyset\), and \((T,f) \in \text{Top–Rep}^{\Gamma}\). Denote \((T', f')\) as a top-representation of \(\Gamma'\) such that \(T'(i) = T(i), (f')_{ij} = f_{ij}\) for any \(i,j \in \Gamma'\). Similarly, denote \((T'', f'')\) as a top-representation of \(\Gamma''\). Then, \(At^\Gamma(T,f) \simeq At^{\Gamma'}(T', f') \bigcup At^{\Gamma''}(T'', f'')\).
Proof. We know
\[
\text{At}^\Gamma(T,f) = \left( \bigcup_{i \in \Gamma} T(i) \right) / \sim^C, C = \{(x_i,x_j) | x_i \in T(i); x_j \in T(j); f_{ij}(x_i) = x_j; \forall i \to j \in \Gamma\}
\]
\[
\text{At}^{\Gamma'}(T',f') = \left( \bigcup_{i \in \Gamma'} T(i) \right) / \sim^{C'}, C' = \{(x_i,x_j) | x_i \in T(i); x_j \in T(j); f_{ij}(x_i) = x_j; \forall i \to j \in \Gamma'\}
\]
\[
\text{At}^{\Gamma''}(T'',f'') = \left( \bigcup_{i \in \Gamma''} T(i) \right) / \sim^{C''}, C'' = \{(x_i,x_j) | x_i \in T(i); x_j \in T(j); f_{ij}(x_i) = x_j; \forall i \to j \in \Gamma''\}.
\]
(19)

Since $\Gamma = \Gamma' \cup \Gamma''$, $\Gamma' \cap \Gamma'' = \emptyset$, we have $C = C' \cup C''$, $C' \cap C'' = \emptyset$. Denote the natural map $p: \bigcup_{i \in \Gamma} T(i) \to \text{At}^\Gamma(T,f), x \mapsto \overline{x}$. Let $X_1 = p(\bigcup_{i \in \Gamma'} T(i)), X_2 = p(\bigcup_{i \in \Gamma''} T(i))$. Clearly, $\text{At}^\Gamma(T,f) = X_1 \cup X_2$. We will show $X_1 \cap X_2 = \emptyset$. For any $x_1 \in \bigcup_{i \in \Gamma'} T(i), x_2 \in \bigcup_{i \in \Gamma''} T(i)$, $(x_1, x_2)$ is not in $\overline{C}$ since $\Gamma'$ and $\Gamma''$ are not connected. Then for any $x_1 \in \bigcup_{i \in \Gamma'} T(i), x_2 \in \bigcup_{i \in \Gamma''} T(i)$, $(x_1, x_2)$ is not in $\overline{C}$, which implies $X_1 \cap X_2 = \emptyset$.

Suppose $(x_1, x_2) \in \overline{C}$. There exist $v_1, v_2, \ldots, v_n \in \bigcup_{i \in \Gamma} T(i) = \bigcup_{i \in \Gamma'} T(i) \cup \bigcup_{i \in \Gamma''} T(i)$ such that $(x_1, v_1), (v_1, v_2), \ldots, (v_n, x_2) \in \overline{C}$. Since $x_1 \in \bigcup_{i \in \Gamma'} T(i), x_2 \in \bigcup_{i \in \Gamma''} T(i)$, we have $(x,y) \in \overline{C}$ where $x \in \bigcup_{i \in \Gamma'} T(i), y \in \bigcup_{i \in \Gamma''} T(i)$, which is a contradiction.

It is easy to show $p^{-1}(X_1) = \bigcup_{i \in \Gamma'} T(i), p^{-1}(X_2) = \bigcup_{i \in \Gamma''} T(i)$. Since $\bigcup_{i \in \Gamma'} T(i), \bigcup_{i \in \Gamma''} T(i)$ are open sets of $\bigcup_{i \in \Gamma} T(i)$, we have that $X_1, X_2$ are open sets of $\text{At}^\Gamma(T,f)$ and $\text{At}^\Gamma(T,f) = X_1 \cup X_2, X_1 \cap X_2 = \emptyset$. Thus, $\text{At}^\Gamma(T,f) \simeq X_1 \coprod X_2$.

It suffices to show $X_1 \simeq \text{At}^{\Gamma'}(T',f'), X_2 \simeq \text{At}^{\Gamma''}(T'',f'')$. Define
\[
f : \text{At}^{\Gamma'}(T',f') \to X_1, \text{ via } \overline{x_1} \mapsto \overline{x_1}, \text{ and } g : \text{At}^{\Gamma''}(T'',f'') \to X_2, \text{ via } \overline{x_2} \mapsto \overline{x_2}.
\]
It is easy to show $f, g$ are well-defined homeomorphisms. This completes the proof.

\[\square\]

Corollary 6.7. Let $\Gamma$ be a quiver, and $(\Gamma^k)_{1 \leq k \leq n}$ be all components of $\Gamma$. Denote $(T^k, f^k)$ as a top-representation of $\Gamma^k$ such that $T^k(i) = T(i), f^k_{ij} = f_{ij}$ for any $i, j \in \Gamma^k$. Then, $\text{At}^\Gamma(T,f) \simeq \prod_{1 \leq k \leq n} \text{At}^{\Gamma^k}(T^k,f^k)$.

Corollary 6.8. Let $\Gamma$ be a quiver, and $(T,f) \in \text{TopRep}_\Gamma$ with all $T(i)$ being connected. Then the number of components of $\Gamma$ is equal to the number of components of $\text{At}(T,f)$.

Proof. Theorem 6.5 and Corollary 6.7 \[\square\]

Theorem 6.9. Let $\Gamma$ be a quiver, $(T,f) \in \text{TopRep}_\Gamma$, and $(T',f), (T'',f)$ be sub-representations of $(T,f)$ such that $T(i) = T'(i) \cup T''(i), T'(i) \cap T''(i) = \emptyset$ for all $i \in \Gamma$. Let $L_1 : (T',f) \to (T,f)$ and $L_2 : (T'',f) \to (T,f)$ be inclusions. Then,

(i) $\text{At}^{\Gamma}_1(L_1)$ and $\text{At}^{\Gamma}_2(L_2)$ are injective and
\[
\text{At}^{\Gamma}(T,f) = \text{Im} \text{At}^{\Gamma}_1(L_1) \cup \text{Im} \text{At}^{\Gamma}_2(L_2), \text{ Im} \text{At}^{\Gamma}_1(L_1) \cap \text{Im} \text{At}^{\Gamma}_2(L_2) = \emptyset.
\]

(ii) Furthermore, $\text{At}^{\Gamma}(T',f) \simeq \text{Im} \text{At}^{\Gamma}_1(L_1)$ if $T'(i)$ is an open subset of $T(i)$ for each $i \in \Gamma$.

27
Proof. We know

\[At^f(T, f) = \prod_{i \in \Gamma} T(i)/\sim C, C = \{(x_i, x_j) | x_i \in T(i); x_j \in T(j); f_{ij}(x_i) = x_j; \forall i \rightarrow j \in \Gamma\}\]

\[At^f(T', f) = \prod_{i \in \Gamma} T'(i)/\sim C', C' = \{(x_i, x_j) | x_i \in T'(i); x_j \in T'(j); f_{ij}(x_i) = x_j; \forall i \rightarrow j \in \Gamma\}\]

\[At^f(T'', f) = \prod_{i \in \Gamma} T''(i)/\sim C'', C'' = \{(x_i, x_j) | x_i \in T''(i); x_j \in T''(j); f_{ij}(x_i) = x_j; \forall i \rightarrow j \in \Gamma\}.

(20)

We first show \(C = C' \cup C''\), \(C' \cap C'' = \emptyset\), \(\tilde{C} = \tilde{C}' \cup \tilde{C}'', \tilde{C}' \cap \tilde{C}'' = \emptyset\), \(\overline{C} = \overline{C}' \cup \overline{C}'', \overline{C}' \cap \overline{C}'' = \emptyset\).

Since \((\cup_{i \in \Gamma} T(i)) \cap (\cup_{i \in \Gamma} T''(i)) = \emptyset\), \(C' \cap C'' = \emptyset\). For any \((x_i, x_j) \in C\), we have \(x_i \in T(i) = T'(i) \cup T''(i), x_j \in T(j), f_{ij}(x_i) = x_j\). If \(x_i \in T'(i)\), then we have \(f_{ij}(x_i) = x_j \in T'(j)\). Therefore, \((x_i, x_j) \in C'\). Similarly, if \(x_i \in T''(i)\), we have \((x_i, x_j) \in C''\). Thus, \(C = C' \cup C''\).

Since \((\cup_{i \in \Gamma} T(i)) \cap (\cup_{i \in \Gamma} T''(i)) = \emptyset\), we have \(\overline{C}' \cap \overline{C}'' = \emptyset\). One then has \(C^{-1} = (C')^{-1} \cup (C'')^{-1}\) due to that \(C = C' \cup C''\). Because \(\prod_{i \in \Gamma} T'(i) = \prod_{i \in \Gamma} T''(i)\), we obtain \(K = K' \cup K''\), where \(K' = \{(x, x) | x \in \prod_{i \in \Gamma} T'(i)\}, K = \{(x, x) | x \in \prod_{i \in \Gamma} T''(i)\}\). Therefore, \(\tilde{C} = C \cup C^{-1} \cup K = C' \cup C'' \cup (C')^{-1} \cup (C'')^{-1} \cup K' \cup K'' = \tilde{C}' \cup \tilde{C}''\).

Before, \(\overline{C}' \cap \overline{C}'' = \emptyset\). For all \((x_1, x_2) \in C\), there exist \(v_1, v_2, \ldots, v_n \in \prod_{i \in \Gamma} T'(i)\) such that \((x_1, v_1), (v_1, v_2), \ldots, (v_n, x_2) \in \tilde{C}\). If \(x_1 \in \prod_{i \in \Gamma} T'(i)\), then \((x_1, v_1) \notin \overline{C}''\). Furthermore \((x_1, v_1) \in \tilde{C} = \tilde{C}' \cup \tilde{C}''\), thus \((x_1, v_1) \in \tilde{C}'\). We also have \(v_1 \in \prod_{i \in \Gamma} T'(i)\). Similarly, \((v_1, v_2) \notin \overline{C}''\), and \((v_1, v_2) \in \tilde{C} = \tilde{C}' \cup \tilde{C}''\), thus \((v_1, v_2) \in \tilde{C}'\). This process can be continued finitely, and eventually we have \((x_1, v_1), (v_1, v_2), \ldots, (v_n, x_2) \in \tilde{C}'\). It follows that \((x_1, x_2) \in \overline{C}'\). If \(x_1 \in \prod_{i \in \Gamma} T''(i)\), similarly, we can prove \((x_1, x_2) \in \overline{C}''\). Thus \(\overline{C} = \overline{C}' \cup \overline{C}''\).

The proof of (i):

For \(x \in \prod_{i \in \Gamma} T(i)\), let \(\overline{x}\) denote the equivalent class in \(At^f(T, f)\) represented by \(x\). For \(x \in \prod_{i \in \Gamma} T'(i)\), let \(\overline{x}'\) denote the equivalent class in \(At^f(T', f)\) represented by \(x\). Let \(p\) be the natural map from \(\prod_{i \in \Gamma} T'(i)\) to \(At^f(T', f)\), and \(p'\) be the natural map from \(\prod_{i \in \Gamma} T'(i)\) to \(At^f(T', f)\).

Look at the continuous map \(At^f(L_1) : At^f(T', f) \to At^f(T, f)\). For any \(\overline{x}_1, \overline{x}_2 \in At^f(T', f)\), if \(At^f(L_1)(\overline{x}_1) = \overline{x}_1 = At^f(L_1)(\overline{x}_2) = \overline{x}_2\), then \((x_1, x_2) \in \overline{C} = \overline{C}' \cup \overline{C}''\). Since \(x_1, x_2 \in \prod_{i \in \Gamma} T'(i)\), we have that \((x_1, x_2) \notin \overline{C}''\). Thus, \((x_1, x_2) \in \overline{C}'\), that is, \(\overline{x}_1 = \overline{x}_2\). Therefore, \(At^f(L_1)\) is injective. Similarly, we can prove \(At^f(L_2)\) is injective.

Since \(\prod_{i \in \Gamma} T(i) = \prod_{i \in \Gamma} T'(i) \cup \prod_{i \in \Gamma} T''(i)\), we obtain \(At^f(T, f) = ImAt^f(L_1) \cup ImAt^f(L_2)\). We want to show \(ImAt^f(L_1) \cap ImAt^f(L_2) = \emptyset\).

Suppose on the contrary, then there exist \(x_1 \in \prod_{i \in \Gamma} T'(i), x_2 \in \prod_{i \in \Gamma} T''(i)\) such that \(\overline{x}_1 = \overline{x}_2\), that is, \((x_1, x_2) \in \overline{C} = \overline{C}' \cup \overline{C}''\). However, \(x_1 \in \prod_{i \in \Gamma} T'(i), x_2 \in \prod_{i \in \Gamma} T''(i)\). Then we have \((x_1, x_2) \notin \overline{C}''\) and \((x_1, x_2) \notin \overline{C}'\), which is a contradiction.

Thus, we obtain \(ImAt^f(L_1) \cap ImAt^f(L_2) = \emptyset\).

The proof of (ii):
If $T'(i)$ is an open subset of $T(i)$ for each $i \in \Gamma$. In order to show $At^\Gamma(T', f) \simeq ImAt^\Gamma(L_1)$, it suffices to show $At^\Gamma(L_1) : At^\Gamma(T', f) \to ImAt^\Gamma(L_1)$ is an open map. For any open subset $\overline{X'}$ of $At^\Gamma(T', f)$, $(p')^{-1}(\overline{X'})$ is an open subset of $\bigsqcup_{i \in \Gamma} T'(i)$. We will prove $p^{-1}(At^\Gamma(L_1)(\overline{X'})) = (p')^{-1}(\overline{X'})$.

For all $x \in p^{-1}(At^\Gamma(L_1)(\overline{X'}))$, we have $p(x) \in At^\Gamma(L_1)(\overline{X'})$. Then, there exists $\overline{x}_1' \in \overline{X'}$ such that $\overline{x}_1' = At^\Gamma(L_1)(x) = \overline{x}_1$. Thus $(x, x_1) \in \overline{C} = \overline{C} \cup C''$. $x_1 \in \bigsqcup_{i \in \Gamma} T'(i)$, and thus $(x, x_1) \in \overline{C}', x \in \bigsqcup_{i \in \Gamma} T'(i)$. Hence we have $p'(x) = \overline{x}' = \overline{x}_1' \in \overline{X}'$. Therefore, $x \in (p')^{-1}(\overline{X'})$ and $p^{-1}(At^\Gamma(L_1)(\overline{X'})) \subset (p')^{-1}(\overline{X'})$.

For all $x \in (p')^{-1}(\overline{X'})$, we have $p'(x) = \overline{x}' \in \overline{X}'$. Thus, $At^\Gamma(L_1)(\overline{x}') = \overline{x}' = p(x) \in At^\Gamma(L_1)(\overline{X'})$. It implies that $x \in p^{-1}(At^\Gamma(L_1)(\overline{X'}))$ and $(p')^{-1}(\overline{X'}) \subset p^{-1}(At^\Gamma(L_1)(\overline{X'}))$.

Thus $p^{-1}(At^\Gamma(L_1)(\overline{X'})) = (p')^{-1}(\overline{X'})$ is an open subset of $\bigsqcup_{i \in \Gamma} T'(i)$. Since $\bigsqcup_{i \in \Gamma} T'(i)$ is an open subset of $\bigsqcup_{i \in \Gamma} T(i)$, it follows that $p^{-1}(At^\Gamma(L_1)(\overline{X'}))$ is an open subset of $\bigsqcup_{i \in \Gamma} T(i)$. Then $At^\Gamma(L_1)(\overline{X'})$ is an open subset of $At^\Gamma(T, f)$. This implies that $At^\Gamma(L_1)(\overline{X'}) = At^\Gamma(L_1)(\overline{X'}) \cap ImAt^\Gamma(L_1) = \bigsqcup_{i \in \Gamma} T(i)$ is an open map. This completes the proof.

Let $\Gamma$ be a quiver. We have already known that $SAT^\Gamma, lim\Gamma'$ are two functors from $\text{Top} \rightarrow \text{Rep}^\Gamma$ to $\text{C}$

**Theorem 6.10.** Let $\Gamma$ be a quiver with finite components. There exists a natural transformation $\sigma : lim\Gamma \rightarrow SAT^\Gamma$.

**Proof.** According to Corollary 6.7 and the proof of Theorem 4.19, we can assume the quiver is connected. For each $(T, f) \in \text{Top} \rightarrow \text{Rep}^\Gamma$, we try to define a chain morphism

$$\sigma_{(T, f)} : lim\Gamma(T, f) \rightarrow S(At^\Gamma(T, f)) \quad (21)$$

. From the definitions, we know that

$$(lim\Gamma(T, f))_n = lim\Gamma(S_n^\Gamma(T, f)), (S(At^\Gamma(T, f)))_n = S_n(At^\Gamma(T, f)). \quad (22)$$

Denote the natural map $p : \bigsqcup_{i \in \Gamma} T(i) \rightarrow At^\Gamma(T, f), x \mapsto \overline{x}$. For each $i \in \Gamma$, let $p_i$ denote the composition of the inclusion, from $T(i)$ to $\bigsqcup_{i \in \Gamma} T(i)$, and $p$.

Let $X$ be any topological space, and $F_1, F_2$ be continuous maps from $X$ to $T(i)$. $F_3$ be a continuous map from $X$ to $T(j)$ (There exists an arrow from $i$ to $j$ in $\Gamma$). We first show that $p_i F_1 = p_i F_2 = p_j F_3$ if $f_{ij} F_1 = f_{ij} F_2 = F_3$. For all $x \in X$, $f_{ij}(F_1(x)) = f_{ij}(F_2(x)) = F_3(x)$. Thus, we have $p_i(F_1(x)) = p_i(F_2(x)) = p_j(F_3(x))$.

For any $(\alpha_i)_{i \in \Gamma} \in lim^\Gamma(S_n^\Gamma(T, f))$. If there exists an arrow from $i$ to $j$, we know $(f_{ij})_*(\alpha_i) = \alpha_j$. We will show $(p_i)_*(\alpha_i) = (p_j)_*(\alpha_j)$ $p_i : T(i) \rightarrow At^\Gamma(T, f)$, and thus $(p_i)_*: S(T(i)) \rightarrow S(At^\Gamma(T, f))$. We will use $\overline{\alpha}$ to denote $(p_i)_*(\alpha_i)$.

$I$: $\alpha_j = 0$. If $\alpha_i = 0$, it is trivial $\overline{\alpha} = \overline{\alpha}$.
If \( \alpha_i \neq 0 \), suppose \( \alpha_i = k_1g_1 + \cdots + k_ng_n \) where \( n \in \mathbb{N}^+, k_i \in \mathbb{Z} \setminus \{0\} \) and \( g_u \neq g_v \) if \( u \neq v \). Since \((f_{ij})_*(\alpha_i) = k_1f_{ij}g_1 + \cdots + k_nf_{ij}g_n = 0\), we must have \( \alpha_i = l_1(g_u - g_v) + \cdots + l_ng_u - g_v \) where \( f_{ij}g_{at} = f_{ij}g_{vn}, 1 \leq t \leq s \). From the above discussion, we know \( \overline{g_u} = \overline{g_v}, 1 \leq t \leq s \). Thus

\[
\overline{\alpha_i} = l_1(\overline{g_u} - \overline{g_v}) + \cdots + l_n(\overline{g_u} - \overline{g_v}) = 0 = \overline{\alpha_j}
\]

(23)

\( II: \alpha_j \neq 0 \). Suppose

\[
\alpha_i = k_1g_1 + \cdots + k_ng_n, \alpha_j = c_1h_1 + \cdots + c_mh_m,
\]

where \( n, m \in \mathbb{N}^+, k_i, h_s \in \mathbb{Z} \setminus \{0\} \) and \( g_u \neq g_v \) if \( u \neq v, h_a \neq h_b \) if \( a \neq b \). Then \((f_{ij})_*(\alpha_i) = k_1f_{ij}g_1 + \cdots + k_nf_{ij}g_n \). Although \( g_u \neq g_v, u \neq v \), it is possible \( f_{ij}g_u = f_{ij}g_v, u \neq v \). Thus

\[
(f_{ij})_*(\alpha_i) = k_1f_{ij}g_1 + \cdots + k_nf_{ij}g_n \\
= l_1f_{ij}g_1 + \cdots + l_\lambda f_{ij}g_\lambda \\
= c_1h_1 + \cdots + c_mh_m,
\]

(25)

where \( f_{ij}g_1, \ldots, f_{ij}g_\lambda \) are distinct, \( i \neq 0 \) and \( \{t_1, \ldots, t_\lambda\} \) is a subset of \( \{1, 2, \ldots, n\} \). From above discussion, we know that \( \overline{g_u} = \overline{g_v} \). Then we have

\[
\overline{\alpha_i} = k_1\overline{g_1} + \cdots + k_n\overline{g_n} = l_1\overline{g_1} + \cdots + l_\lambda\overline{g_\lambda}.
\]

(26)

Since \( l_1f_{ij}g_1 + \cdots + l_\lambda f_{ij}g_\lambda = c_1h_1 + \cdots + c_mh_m, f_{ij}g_u \neq f_{ij}g_v \) if \( s \neq w, h_a \neq h_b \) if \( a \neq b \) \( i \neq 0, j \neq 0 \), then \( \lambda = m \) and \( \{f_{ij}g_1, \ldots, f_{ij}g_m\} = \{h_1 \ldots h_m\} \). Then, there exists \( \tau \in S_m \) such that \( f_{ij}g_u = h_{\tau(u)}, h_u = c_{\tau(u)} \) for all \( 1 \leq u \leq m \). From the above discussion, we know \( \overline{g_u} = \overline{h_{\tau(u)}} \) for all \( 1 \leq u \leq m \). Therefore, we have

\[
\overline{\alpha_i} = k_1\overline{g_1} + \cdots + k_n\overline{g_n} = l_1\overline{g_1} + \cdots + l_m\overline{g_m} = c_1h_1 + \cdots + c_mh_m,
\]

(27)

From the above discussion, we know that for any \( (\alpha_i)_{i \in \Gamma} \in \lim^F(S_n(T, f)) (n \geq 0) \), \((p_i)_*(\alpha_i) = (p_j)_*(\alpha_j) \) if there exists an arrow from \( i \) to \( j \). \( \Gamma \) is connected, and therefore \((p_i)_*(\alpha_i) = (p_j)_*(\alpha_j) \) in \( \mathcal{S}_n(\mathcal{A}^F(T, f)) \) for all \( i, j \in \Gamma \).

For each \( n \geq 0 \), we define

\[
(\sigma(T, f))_n : (\lim^F(T, f))_n = \lim^F(S_n(T, f)) \Rightarrow (S(\mathcal{A}^F(T, f)))_n = \mathcal{S}_n(\mathcal{A}^F(T, f))
\]

(28)

and we have \( \partial((\sigma(T, f))_n((\alpha_i)_{i \in \Gamma})) = \theta((p_i)_*(\alpha_i)) = (p_i)_*(\partial(\alpha_i)) = (\sigma(T, f))_{n-1}((\partial(\alpha_i))_{i \in \Gamma}) = (\sigma(T, f))_{n-1}(\partial((\alpha_i)_{i \in \Gamma})). \) Thus \( \sigma(T, f) : \lim^F(T, f) \Rightarrow S(\mathcal{A}^F(T, f)) \) is a chain map.

For any \( \theta : (T, f) \rightarrow (T', f') \), we will show the following diagram commutes:

\[
\begin{array}{ccc}
\lim^F(T, f) & \xrightarrow{\sigma(T, f)} & S(\mathcal{A}^F(T, f)) \\
\mid \quad \theta \downarrow & & \downarrow S(\mathcal{A}^F(\theta) = \mathcal{A}^F(\theta)_*) \\
\lim^F(T', f') & \xrightarrow{\sigma(T', f')} & S(\mathcal{A}^F(T', f'))
\end{array}
\]

30
For all $n \geq 0$, $(\alpha_i)_{i \in \Gamma} \in (\lim^{\Gamma}(T, f))_n = \lim^{\Gamma}(S_n^\Gamma(T, f))$,

\[
(\sigma_{(T, f)})_n((\alpha_i)_{i \in \Gamma}) = (p_i)_*(\alpha_i)
\]

\[
(At^{\Gamma}(\theta))_*((p_i)_*(\alpha_i)) = (At^{\Gamma}(\theta)p_i)_*(\alpha_i)
\]

\[
\lim^{\Gamma}(\theta)((\alpha_i)_{i \in \Gamma}) = ((\theta_i)_*(\alpha_i))_{i \in \Gamma}
\]

\[
(\sigma_{(T', f'))}_n(((\theta_i)_*(\alpha_i))_{i \in \Gamma}) = (p'_i)_*(((\theta_i)_*(\alpha_i)) = (p'_i\theta_i)_*(\alpha_i),
\]

where $p'_i : T'(i) \to At^{\Gamma}(T', f')$, defined similarly as $p_i$.

It suffices to show $At^{\Gamma}(\theta)p_i = p'_i\theta_i : T(i) \to At^{\Gamma}(T', f'), \forall i \in \Gamma$. For all $x_i \in T(i)$,

\[
At^{\Gamma}(\theta)(p_i(x_i)) = At^{\Gamma}(\theta)(\overline{x_i}) = \overline{\theta_i(x_i)} = p'_i(\theta_i(x_i)).
\]

This shows the above diagram is commutative, which completes the proof.

We know that $H_n(-), H_n At^{\Gamma}(-)$ are two functors from $\textbf{Top}-\text{Rep}_{\Gamma}$ to $\textbf{Ab}$ for each $n \geq 0$.

**Corollary 6.11.** For any quiver $\Gamma$ with finite components, each $n \geq 0$, we have a natural transformation from $H_n(-)$ to $H_n At^{\Gamma}(-)$.

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