GROUP APPROXIMATION IN CAYLEY TOPOLOGY AND
COARSE GEOMETRY,
PART II: FIBERED COARSE EMBEDDINGS.

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Abstract. The objective of this series is to study metric geometric properties
of disjoint unions of amenable Cayley graphs by group properties of the Cayley
accumulation points in the space of marked groups. In this Part II, we prove
that a disjoint union admits a fibred coarse embedding into a Hilbert space (in
a generalized sense) if and only if the Cayley boundary of the sequence in the
space of marked groups is uniformly a-T-menable. We furthermore extend this
result to ones with other target spaces. By combining our main results with
constructions of Arzhantseva–Osajda and Osajda, we construct two systems of
markings of a certain sequence of finite groups with two opposite extreme behaviors
of the resulting two disjoint unions: With respect to one marking, the space has
property A. On the other hand, with respect to the other, the space does not admit
fibred coarse embeddings into Banach spaces with non-trivial type (for instance,
uniformly convex Banach spaces) or Hadamard manifolds; the Cayley limit group
is, furthermore, non-exact.

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1. Introduction

The main topics of this paper are the fibred coarse embeddings of disjoint unions of Cayley graphs and equivariant coarse embeddings of groups. Before proceeding to these two concepts, we first recall the definition of (genuine) coarse embeddings. By generalized metrics, we mean metrics that possibly take the value $+\infty$. A basic example of generalized metric spaces is constructed as follows. For a sequence of metric spaces $(X_m, d_m)_{m \in \mathbb{N}}$, we define a generalized metric $d$ on $\bigsqcup_{m \in \mathbb{N}} X_m$ by $d(x, y) = d_m(x, y)$ if $x, y \in X_m$ for some $m$ and $d(x, y) = +\infty$ otherwise. We call the resulting generalized metric space $(\bigsqcup_{m \in \mathbb{N}} X_m, d)$ the disjoint union, and simply write it as $\bigsqcup_{m \in \mathbb{N}} X_m$.

**Definition 1.1.** Let $(X, d_X)$ be a generalized metric space and $\mathcal{M}$ be a non-empty class of (genuine) metric spaces.

1. Let $(M, d_M)$ be a (generalized) metric space. A (possibly discontinuous and possibly non-injective) map $f: X \to M$ is said to be a coarse embedding, if there exist two non-decreasing functions $\rho, \omega: [0, \infty) \to [0, \infty)$ that are proper (namely, $\lim_{r \to +\infty} \rho(r) = \lim_{r \to +\infty} \omega(r) = +\infty$) such that for all $x_1, x_2 \in X$ such that $d_X(x_1, x_2) < +\infty,$

   $$\rho(d_X(x_1, x_2)) \leq d_M(f(x_1), f(x_2)) \leq \omega(d_X(x_1, x_2))$$

holds true. The $\rho$, $\omega$, $(\rho, \omega)$ are, respectively, called a compression function, an expansion function and a control pair for $f$. 


(2) We say that \( X \) admits a coarse embedding into \( \mathcal{M} \) if there exist \( M \in \mathcal{M} \) and a coarse embedding \( f: X \to M \).

(3) We say the pair \((\rho, \omega)\) of two non-decreasing proper functions \([0, \infty) \to [0, \infty)\) is a control pair for \( X \) into \( \mathcal{M} \) if there exist \( M \in \mathcal{M} \) and a coarse embedding \( f: X \to M \) such that \((\rho, \omega)\) is a control pair for \( f \). Denote by \( \mathcal{CP}_\mathcal{M}(X) \) the set of all control pairs for \( X \) into \( \mathcal{M} \).

(4) If \( X = G \) is a marked group (with the metric \( d_G \); see Subsection 2.1), we write \( \mathcal{CP}_\mathcal{M}(X) \) as \( \mathcal{CP}_\mathcal{M}^\ast(G) \) in order to distinguish it from the set \( \mathcal{CP}_\mathcal{M}^\sharp(G) \) of equivariant control pairs; compare with Definition 2.11.

The notion of fibred coarse embeddings was introduced by Chen–Wang–Yu \( \text{[CWW14a]} \). This is a weakening of the (genuine) coarse embeddability; see Remark 2.6. In this paper, since we consider the disjoint union of possibly infinite graphs, we relax the condition on exceptional sets, and call the modified notion that of fibred coarse embeddings in a generalized sense; see Definition 2.4. This new notion coincides with the original notion of \( \text{[CWW14a]} \) for a coarse disjoint union of finite graphs; see Remark 2.5. In \( \text{[CWW14a]} \), they proved that if a coarse disjoint union \( X \) of finite graphs of uniformly bounded degree admits a fibred coarse embedding into a Hilbert space, then the maximal Baum–Connes conjecture holds for \( X \). Furthermore, Chen–Wang–Wang \( \text{[CWW14b]} \) proved that if \( X \) above admits a fibred coarse embedding into a complete, connected and simply connected Riemannian manifold with non-positive sectional curvature, then the coarse Novikov conjecture holds for \( X \).

The concept of equivariant coarse embedding is defined for finitely generated groups in terms of isometric actions. It relates to Gromov’s \( a\)-T-menability if the target space is a Hilbert space, and to \( a\)-\( \mathcal{M}\)-menability in general cases; see Definition 2.11.

We employ the space of \((k)\)-marked groups \( \mathcal{G}(k) \) to study a relationship between these two notions. This space was introduced by R. I. Grigorchuk \( \text{[Grig84, Section 6]} \), and it is the space of (equivalence classes of) all pairs of a group and a \( k \)-generating ordered set. The space \( \mathcal{G}(k) \) is equipped with the topology of local convergence as rooted diagrams. This topology is sometimes called the Cayley topology, and it is compact and metrizable. We will briefly recall \( \mathcal{G}(k) \) in Subsection 2.1. For a sequence \( (G_m)_{m \in \mathbb{N}} \), we consider the following two objects:

- The disjoint union \( \bigsqcup_{m \in \mathbb{N}} \text{Cay}(G_m) \) of Cayley graphs, which is a generalized metric space without group structure.

- The Cayley boundary \( \partial_{\text{Cay}}(G_m)_{m \in \mathbb{N}}(\subseteq \mathcal{G}(k)) \), defined as follows.

**Definition 1.2.** The Cayley boundary \( \partial_{\text{Cay}}(G_m)_{m \in \mathbb{N}} \) is defined as the set of all accumulation points of \( (G_m)_{m \in \mathbb{N}} \) in \( \mathcal{G}(k) \) in the Cayley topology.

It forms a non-empty compact set, consisting of marked groups \( G_\infty \in \partial_{\text{Cay}}(G_m)_{m \in \mathbb{N}} \).

The following, Proposition 1.4 and Theorem A are our main results; they provide quantitative relationships between control pairs for each side of embeddings, as well as qualitative aspects. Our results employ several operations on classes of metric
spaces:
\[ \mathcal{M} \ni UP(\mathcal{M}), \mathcal{F}_q(\mathcal{M}), \mathcal{FS}_q(\mathcal{M}), \ell_q(\mathcal{M}), \quad \text{for } q \in [1, \infty). \]

See Subsections 2.4 and 2.5, respectively, for the definitions and examples.

**Definition 1.3.** Let \( K \) be a non-empty subset of \( \mathcal{G}(k) \) \((k \in \mathbb{N}_{\geq 1})\). For a non-empty class of metric spaces \( \mathcal{M} \), we say that \( K \) is uniformly \( a-\mathcal{M}\)-menable if it admits equivariant equi-coarse embeddings into \( \mathcal{M} \). That means, there exists a common pair \((\rho, \omega)\) of non-decreasing proper functions \([0, \infty) \to [0, \infty)\) such that for every \( G \in K \), \((\rho, \omega)\) is an equivariant control pair from \( G \) into \( \mathcal{M} \). In short, it holds that
\[ \bigcap_{G \in \mathcal{K}} CP^\sharp_{\mathcal{M}}(G) \neq \emptyset; \]
see Definition 2.11 for related definitions.

**Proposition 1.4.** Let \( \mathcal{M} \) be a non-empty class of metric spaces. Let \((G_m)_{m \in \mathbb{N}}\) be a sequence in \( \mathcal{G}(k) \) \((k \in \mathbb{N}_{\geq 1})\). If \( \bigsqcup_{m \in \mathbb{N}} Cay(G_m) \) admits a fibred coarse embedding into \( \mathcal{M} \) in a generalized sense, then \( \partial_{\text{Cay}}(G_m)_{m \in \mathbb{N}} \) admits equi-coarse embeddings into \( UP(\mathcal{M}) \); that means,
\[ \bigcap_{G, \in \partial_{\text{Cay}}(G_m)_{m}} CP^*_{UP(\mathcal{M})}(G_{\infty}) \neq \emptyset. \]

If \( \mathcal{M} \) consists only of Banach spaces, then the following holds true: If \( \bigsqcup_{m \in \mathbb{N}} Cay(G_m) \) admits a fibred coarse embedding into \( \mathcal{M} \) in a generalized sense, then \( \partial_{\text{Cay}}(G_m)_{m \in \mathbb{N}} \) admits equi-coarse embeddings into the original class \( \mathcal{M} \).

See Proposition 3.3 for a further generalization to disjoint unions of connected graphs with uniformly bounded degree, not necessarily those of Cayley graphs.

**Theorem A.** Let \( \mathcal{M} \) be a non-empty class of metric spaces. Let \((G_m)_{m \in \mathbb{N}}\) be a sequence in \( \mathcal{G}(k) \) \((k \in \mathbb{N}_{\geq 1})\).

(i) (1) Assume that all \( G_m, m \in \mathbb{N} \), are finite. Then, for every \( q \in [1, \infty) \), the following holds true: If \( \bigsqcup_{m \in \mathbb{N}} Cay(G_m) \) admits a fibred coarse embedding into \( \mathcal{M} \) in a generalized sense, then \( \partial_{\text{Cay}}(G_m)_{m \in \mathbb{N}} \) is uniformly \( a-F_q(M) \)-menable.

Moreover, it holds that for every \( q \in [1, \infty) \),
\[ CP^\text{fib}_\mathcal{M} \left( \bigsqcup_m Cay(G_m) \right) \subseteq \bigcap_{G_{\infty} \in \partial_{\text{Cay}}(G_m)_{m}} CP^\sharp_{F_q(\mathcal{M})}(G_{\infty}). \]

(2) Assume that all \( G_m, m \in \mathbb{N} \), are amenable. Assume moreover that
- either there exists \( q \in (1, \infty) \) such that for every \( M \in \mathcal{M} \), there exists an element \( L \) in \( F_q(M) \) that is uniquely geodesic, namely, for every \( z, w \in L \), there exists a unique geodesic \( c: [0, d(z, w)] \to L \) from \( z \) to \( w \); see our notation at the end of Introduction, or
- the class \( \mathcal{M} \) consists only of Banach spaces (with no restriction on \( q \in [1, \infty) \)).
Then for every such $q$ in the first case (respectively, for every $q \in [1, \infty)$ in the second case) the following holds true: If $\bigsqcup_{m \in \mathbb{N}} \text{Cay}(G_m)$ admits a fibred coarse embedding into $\mathcal{M}$ in a generalized sense, then $\partial_{\text{Cay}}(G_m)_{m \in \mathbb{N}}$ is uniformly $a$-$\mathcal{FS}_q(\mathcal{M})$-menable.

Moreover, it holds that for every such $q$ above,

$$
\mathcal{C}P^{\text{fib}}_{\mathcal{M}} \left( \bigsqcup_m \text{Cay}(G_m) \right) \subseteq \bigcap_{G_m \in \partial_{\text{Cay}}(G_m)_{m \in \mathbb{N}}} \mathcal{C}P^4_{\mathcal{FS}_q(\mathcal{M})}(G_\infty).
$$

(ii) For every $q \in [1, \infty)$, the following holds true: If the Cayley boundary $\partial_{\text{Cay}}(G_m)_{m \in \mathbb{N}}$ is uniformly $a$-$\mathcal{M}$-menable, then the disjoint union $\bigsqcup_{m \in \mathbb{N}} \text{Cay}(G_m)$ admits a fibred coarse embedding into $\ell_q(\mathcal{M})$ in a generalized sense.

Moreover, it holds that for every $q \in [1, \infty)$,

$$
\bigcap_{G_m \in \partial_{\text{Cay}}(G_m)_{m \in \mathbb{N}}} \mathcal{C}P^4_{\mathcal{M}}(G_\infty) \subseteq \mathcal{C}P^{\text{fib}}_{\ell_q(\mathcal{M})} \left( \bigsqcup_m \text{Cay}(G_m) \right).
$$

If $(G_m)_{m \in \mathbb{N}}$ is a convergent sequence, then we may replace $\ell_q(\mathcal{M})$ with the original class $\mathcal{M}$ in the assertions above; in that case, it holds that

$$
\mathcal{C}P^4_{\mathcal{M}}(G_\infty) \subseteq \mathcal{C}P^{\text{fib}}_{\mathcal{M}} \left( \bigsqcup_m \text{Cay}(G_m) \right),
$$

where $G_\infty$ is the Cayley limit group of $(G_m)_m$.

As a corollary, we obtain the following.

**Corollary B.** Let $(G_m)_{m \in \mathbb{N}}$ a sequence of amenable marked groups in $\mathcal{G}(k)$.

1. The disjoint union $\bigsqcup_{m \in \mathbb{N}} \text{Cay}(G_m)$ admits a fibred coarse embedding into a Hilbert space in a generalized sense if and only if $\partial_{\text{Cay}}(G_m)_{m \in \mathbb{N}}$ is uniformly $a$-$\mathcal{T}$-menable.

2. Let $\mathcal{M}$ be either of the following classes. Then, $\bigsqcup_{m \in \mathbb{N}} \text{Cay}(G_m)$ admits a fibred coarse embedding into $\mathcal{M}$ in a generalized sense if and only if $\partial_{\text{Cay}}(G_m)_{m \in \mathbb{N}}$ is uniformly $a$-$\mathcal{M}$-menable.

   2.1 For $q \in [1, \infty)$, $L_q$ denoting $L_q([0,1], \mathbb{R})$.

   2.2 For $r \in (1, 2]$ and for $C > 0$, $B^\text{type}_r,C$ being the class defined in (4) of Example 2.18.

   2.3 For $\delta_0 \in [0, 1]$, $\text{CAT}(0)_{\leq \delta_0}$ denoting the class of all complete CAT(0) spaces with Izeki–Nayatani invariant at most $\delta_0$; see (1) of Example 2.19 for the definitions.

Furthermore, for $\mathcal{M}$ being a class Hilbert, the class of all Hilbert spaces, or being as in (2), we have that

$$
\mathcal{C}P^{\text{fib}}_{\mathcal{M}} \left( \bigsqcup_m \text{Cay}(G_m) \right) = \bigcap_{G_m \in \partial_{\text{Cay}}(G_m)_{m \in \mathbb{N}}} \mathcal{C}P^4_{\mathcal{M}}(G_\infty).
$$
Note that (1) in Corollary 3 is essentially a special case of (2-1) with \( q = 2 \).

Some work has been done by other researchers before our results in the context of box spaces for RF (Residually Finite) groups. If a finitely generated infinite group \( G \) with a finite generating set \( S \) admits a chain \( (N_m)_{m \in \mathbb{N}}, N_{m+1} \leq N_m, \) of normal subgroups of finite index in \( G \) such that \( \bigcap_{m \in \mathbb{N}} N_m = \{e\} \), then the box space of \( G \) is defined by

\[
\Box G = \bigcup_{m \in \mathbb{N}} \text{Cay}(G/N_m; S \mod N_m),
\]

where \( \bigcup_m \) denotes a coarse disjoint union (see [MS13, Definition 2.15.(2)] and Subsection 2.2). Chen–Wang–Wang [CWW13] showed that \( \Box G \) admits a fibred coarse embedding into a Hilbert space if and only if \( G \) is a-T-menable. They also showed that for a metric space \( M \), if \( G \) is a-M-menable, then \( \Box G \) admits a fibred coarse embedding into \( M \). It supplies several examples that admit fibred coarse embeddings into Hilbert spaces, but that do not admit genuine coarse embeddings; compare with Example 7.12.

Here we stress that the following points are visible only after extending our framework from the class of box spaces to our general class; see the definitions of RF/LEF/LEA groups in Definition 2.2.

(a) The Cayley boundary \( \partial_{\text{Cay}}(G_m) \) may consist of infinitely many points.

(b) Even when \( \partial_{\text{Cay}}(G_m) \) is a singleton \( \{G_\infty\} \), the Cayley limit group \( G_\infty = \lim_m G_m \) is in the class of LEA (Locally Embeddable into Amenable groups) group when \( G_m, m \in \mathbb{N}, \) is amenable; it is in the class of LEF (Locally Embeddable into Finite groups) group when \( G_m, m \in \mathbb{N}, \) is furthermore finite. In general, the implications

\[
\text{RF} \implies \text{LEF} \implies \text{LEA}
\]

hold, and none of the implications can be reversed; see [MS13, Subsection 2.2].

(c) Coarse properties of \( \bigcup_{m \in \mathbb{N}} \text{Cay}(G_m, S_m) \) may be considerably affected by the choice of the system \( (S_m)_m \) of generators of \( G_m \), even when it might look a slight change.

To illustrate point (c) above, we study the following example.

Example 1.5. (1) Fix a prime \( p \). For \( n \in \mathbb{N}_{\geq 1}, \) denote by \( \mathbb{F}_{p^n} \) the finite field of order \( p^n \). It is well-known that the multiplicative group \( \mathbb{F}_{p^n}^\times \) is cyclic; for each \( p \) and each \( n \), we fix a generator \( t_n = t_{p^n} \in \mathbb{F}_{p^n} \) of it. Fix a sequence \( (n_m)_{m \in 2\mathbb{N}+1 \geq 3} \) of positive integers such that \( \lim_{m \to \infty} n_m = +\infty \).

Let \( G_m = \text{SL}(m, \mathbb{F}_{p^{n_m}}) \). Then for (odd) \( m \in 2\mathbb{N} + 1 \geq 3, \) we consider the following two systems \( (S_m)_{m \in 2\mathbb{N}+1 \geq 3}, (T_m)_{m \in 2\mathbb{N}+1 \geq 3} \) of generators of \( G_m \).
• For \( m \in 2\mathbb{N} + 1_{\geq 3} \), \( S_m = (\sigma^{(m)}, v^{(m)}, \tau^{(m)}) \). Here

\[
\sigma^{(m)} = \begin{pmatrix} 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \vdots & \\ \vdots & 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix}, \quad v^{(m)} = \begin{pmatrix} 1 & t_m & 0 & \cdots & 0 \\ 0 & 1 & 0 & \vdots & \\ \vdots & 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix},
\]

and

\[
\tau^{(m)} = \begin{pmatrix} 0 & \cdots & \cdots & 0 & 1 \\ 1 & 0 & 0 & \vdots & \\ \vdots & 1 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix}.
\]

Define \( X' = X'_{p,(l_m)} = \bigsqcup_{m \in 2\mathbb{N} + 1_{\geq 3}} \text{Cay}(G_m; S_m). \)

• For \( m \in 2\mathbb{N} + 1_{\geq 3} \), \( T_m = (\sigma^{(m)}, \sigma^{(m)}, v^{(m)}, \tau^{(m)}) \). Here \( \sigma^{(m)}, v^{(m)} \) and \( \tau^{(m)} \)

are the same as above, and \( \sigma^{(m)} = t\sigma^{(m)} \) is the transpose of \( \sigma^{(m)} \).

Define \( Y' = Y'_{p,(l_m)} = \bigsqcup_{m \in 2\mathbb{N} + 1_{\geq 3}} \text{Cay}(G_m; T_m). \)

(2) Let \( (l_m)_{m \in 2\mathbb{N} + 1_{\geq 3}} \) be a sequence of integers at least 2 such that \( \lim_{m \to \infty} l_m = \infty \).

For (odd) \( m \in 2\mathbb{N} + 1_{\geq 3} \), set \( H_m = 0.1(Z/l_mZ) \) and take two markings \( P_m, Q_m \) as follows:

• Set \( P_m = (\sigma^{(m)}, \tau^{(m)}) \), where \( \sigma^{(m)} \) and \( \tau^{(m)} \)

are the matrices with exactly the same entries of 0 and 1 as in, respectively, \( \sigma^{(m)} \) and \( \tau^{(m)} \) as in (1) above.

Define \( V' = V'_{l_m} = \bigsqcup_{m \in 2\mathbb{N} + 1_{\geq 3}} \text{Cay}(H_m; P_m). \)

• Set \( Q_m = (\sigma^{(m)}, \sigma^{(m)}, \tau^{(m)}) \), where \( \sigma^{(m)} \) and \( \tau^{(m)} \)

are the same as \( P_m \), and \( \sigma^{(m)} = t\sigma^{(m)} \).

Define \( W' = W'_{l_m} = \bigsqcup_{m \in 2\mathbb{N} + 1_{\geq 3}} \text{Cay}(H_m; Q_m). \)

In our Part I paper [MS13, Remark 5.9] (see also Corollary B and Proposition 2.11 therein), we proved that \( X' \) and \( V' \)

above have property A of G. Yu [Yu00]. We do not recall the definition of property A here; see [Yu00] or [MS13, Definition 2.17]. Yu [Yu00] showed that property A implies the coarse embeddability into a Hilbert space.

By the Dvoretzky theorem [BL00, Chapter 12] and a theorem of M. I. Ostrovskii [Ost12], it then follows that a locally finite generalized metric space with property A admits a coarse embedding into every infinite dimensional Banach space. At the other end of the spectrum, we obtain the following. For every prime \( p \), set

\[
\delta(p) = 1 - \frac{1}{2 \left( 1 - \frac{\sqrt{p}}{p+1} \right)} \quad (\in (0, \frac{1}{2}));
\]

see Remark 7.4 for the background of this constant. For \( \delta_0 \in (0, 1] \), let \( \mathcal{CAT}(0)_{<\delta_0} \)

denote the class of all complete \( \text{CAT}(0) \) spaces whose Izeki–Nayatani invariants are strictly less than \( \delta_0 \); see also (1) of Example 2.19.

**Corollary 1.6.** Let \( Y', V', W' \) as in Example 1.5.
(1) The spaces $Y', V', W'$ do not admit a fibred coarse embedding in a generalized sense into $(\prod_{\aleph_0} \mathcal{Q}T)_{\ell_1}$ or into $(\prod_{\aleph_0} \mathcal{M})_{\ell_2}$, where $\mathcal{Q}T$ denotes the class of all quasi-trees and $\mathcal{M}$ is the class of all finite dimensional, complete, connected and simply connected Riemannian manifolds with strictly negative sectional curvature that are cocompact; see Remark 2.20 for the definitions.

(2) The space $Y' = Y_{p,(n_m)_{m=1}}$ does not admit a fibred coarse embedding in a generalized sense into $\mathcal{B}_{\text{type} > 1}$, the class of all Banach spaces with (linear, also known as Rademacher) type $> 1$; see (4) of Example 2.18. Neither does $Y'$ admit a fibred coarse embedding in a generalized sense into $\text{CAT}(0)_{<\delta(p)}$.

(3) The space $W'$ does not admit a fibred coarse embedding in a generalized sense into $\mathcal{B}_{\beta<1/2}$; see (5) of Example 2.18.

For every prime $p$, the class $\text{CAT}(0)_{<\delta(p)}$ as in (2) above includes $\text{CAT}(0)_{\leq 0}$, and the subclass $\text{CAT}(0)_{\leq 0}$ contains all (possibly infinite dimensional) complete, connected and simply connected Riemannian manifolds with non-positive sectional curvature. Hence, the space $Y'$ above does not admit a fibred coarse embedding in a generalized sense into such spaces. Neither does it admit a fibred coarse embedding (in a generalized sense) into uniformly convex Banach spaces (see (7) of Example 2.18 for the definition) because every uniformly convex Banach space is contained in $\mathcal{B}_{\text{type} > 1}$. See [LJ89] and [BL00] on geometry of Banach spaces. After work [BBF15] of Bestvina–Bromberg–Fujiwara, study of actions on finite products of quasi-trees has been paid an intensive attention.

**Remark 1.7.** If we hope to allow even integers $m$ as well as odd ones, then it will be more natural to consider unimodular linear groups $\text{SL}^\pm$ rather than special linear groups $\text{SL}$; see [MS13, Subsection 5.3] for the discussion in that framework and compare with examples [MS13, Example 1.5 and Example 2.10] in our Part I paper.

Directions concerning point (c) above are closely related to the following two questions:

- For a fixed (infinite) group $G$ and a fixed $k \in \mathbb{N}_{\geq 1}$, to which (marked) groups can a sequence of $(G; S_m)_{m \in \mathbb{N}}$ converge, where $(S_m)_{m \in \mathbb{N}}$ is a system of $k$-markings of $G$? This problem is studied in the paper of L. Bartholdi and A. Erschler [BE15] in terms of a pre-order which they introduced there.

- For a compact (metrizable) infinite group $K$ and two finitely generated dense subgroups $\Lambda_1$ and $\Lambda_2$ in $K$, how different can the group properties of $\Lambda_1$ and $\Lambda_2$ be? This question is related to a conjecture by A. Lubotzky and B. Weiss [LW93, Conjecture 1.2], which asserted that it would be impossible that $\Lambda_1$ and $\Lambda_2$ are, respectively, amenable and Kazhdan (namely, with property (T)). This conjecture was resolved in the negative by M. Ershov and A. Jaikin-Zapirain [EJZ10, Subsection 6.3]; M. Kassabov [Kas07] also constructed a counterexample to the Lubotzky–Weiss conjecture by using a result of Y. Shalom [Sha06].

In Section 6, we explain several gadgets which can be used to study the former question. One simple but striking tool is an absorption lemma (Lemma 6.14), which we learned from a paper of Bartholdi and Erschler [BE15, Lemma 6.13]. A standard
wreath product \( G \wr H \) is a key tool for this; see Subsection 6.1 for the definition. We will deal with the latter question in a distinct paper [Mim18] from the present one, because the focus may be distinct from coarse geometry of disjoint unions.

We apply constructions described in Section 6 to an example of D. Osajda [Osa16] of an RF group, or that of Arzhantseva–Osajda [AO14] of a LEF group, without property A. It provides us with the following extreme examples; see also Remark 7.5.

**Theorem C.** (i) (See Proposition 7.6 for the detailed statement.) There exist a sequence of finite groups \((G_n)_{n \in \mathbb{N}}\) and \(d \in \mathbb{N}\) such that the following holds true: For every prime \(p\) and for every sequence \((l_n)_{n \in \mathbb{N}}\) of integers at least 2 such that \(\lim_{n \to \infty} l_n = \infty\), there exist three systems \((S_n)_n\), \((T_n)_n\) and \((U_n)_n\) of \(d\)-markings of \((H_n,p, (l_n)_n) = G_n \wr \text{SL}(2n+3, \mathbb{F}_{p^d})\)) such that the following hold true:

- The sequence \((H_n,p; S_n))_{n \in \mathbb{N}}\) converges in the Cayley topology to a group without property A (in other words, it is a non-exact group), but the Cayley limit group is \(a\)-\(T\)-menable.
- The sequence \((H_n,p; T_n))_{n \in \mathbb{N}}\) converges in the Cayley topology to a group without property A; in addition, the Cayley limit group is not \(a\)-\(\mathcal{M}\)-menable for \(\mathcal{M} = \mathcal{B}_{\text{type} > 1}\) or \(\mathcal{M} = \mathcal{CAT}(0, \delta(p))\), where \(\delta(p)\) is described as above Corollary 7.6.

(ii) (See Proposition 7.7 for the detailed statement.) In (i), for every prime \(p\), we may replace \((H_n,p)_n\) with \((S([h_n,p]))_n\), a sequence of finite symmetric groups of specified degrees. Here \((h_n,p)_{n \geq 1}\) is a sequence of integers at least 2 that satisfies that \(\lim_{n \to \infty} h_n,p = \infty\), which depends on \(p\). The symbol \([h_n,p]\) denotes the set \(\{1, 2, \ldots, h_n, p\}\); see our notation of symmetric groups at the end of Introduction.

By Theorem A of our Part I paper [MS13], the disjoint union \(\bigsqcup_{n \in \mathbb{N}} \text{Cay}(H_n,p; S_n)\) has property A. By (ii) of Theorem A \(\bigsqcup_{n \in \mathbb{N}} \text{Cay}(H_n,p; T_n)\) admits a fibred coarse embedding into a Hilbert space. At the other end of the spectrum, by (i) of Theorem A \(\bigsqcup_{n \in \mathbb{N}} \text{Cay}(H_n,p; U_n)\) does not admit a fibred coarse embedding into \(\mathcal{B}_{\text{type} > 1}\) or \(\mathcal{CAT}(0, \delta(p))\). We also are able to take \(S_n = (s_i^{(n)})_{1 \leq i \leq d}, T_n = (t_i^{(n)})_{1 \leq i \leq d}, U_n = (u_i^{(n)})_{1 \leq i \leq d}\) such that for a fixed \(i \in [d]\), the elements \(s_i^{(n)}, t_i^{(n)}\) and \(u_i^{(n)}\) are all conjugate (but group elements in \(H_n,p\) which conjugate them depend on \(i\)).

T. Pillon introduced a notion of fibred property A and showed that a box space of a group has this property if and only if the group has property A. In this aspect, it is furthermore plausible that \(\bigsqcup_{n \in \mathbb{N}} \text{Cay}(H_n,p; T_n)\) and \(\bigsqcup_{n \in \mathbb{N}} \text{Cay}(H_n,p; U_n)\) both fail to enjoy fibred property A. These examples may indicate that, beyond the world of box spaces, it is no longer reasonable to write disjoint unions as \(\bigsqcup_n G_n\) without expressing markings.

We, moreover, observe that point (a) above is striking in the study of fibred coarse embeddings: Unlike amenability and property (T), uniformity is not automatic for
with respect to fibred coarse embeddings; see also point (c). Metric spaces and fibred coarse embeddings may be taken in the original sense.

The disjoint union \( X \times X \) admits a fibred coarse embedding in a generalized sense into the Hilbert space.

Let \( X \) be a Banach space that is sphere equivalent to \( CAT(0) \) space. Then \( X \) admits a fibred coarse embedding into \( CAT(0)_{\leq 1} \).

Finally, we construct some generalized metric spaces that have very exotic features with respect to fibred coarse embeddings; see also point (c) above for (ii).

**Theorem D.** (i) There exists a sequence \( (\Gamma_l)_{l \in \mathbb{N}} \) of finite graphs of uniformly bounded degree such that all of the following hold true.

- The sequence \( (\Gamma_l)_{l} \) forms an expander family; see Definition 7.8.
- The disjoint union \( \bigsqcup_{l \in \mathbb{N}} \Gamma_l \) does not admit a fibred coarse embedding in a generalized sense into \( CAT(0)_{<1} \) or the class of Banach spaces that are sphere equivalent (see below) to Hilbert spaces.

- There exists a complete \( CAT(0) \) space \( M \) such that \( \bigsqcup_{l \in \mathbb{N}} \Gamma_l \) admits a biLipschitz embedding into \( M \), namely, it admits a coarse embedding with control pair \((\rho, \omega)\), where \( \rho \) and \( \omega \) are both linear functions.

(ii) There exist \( (k_l)_{l \in \mathbb{N}} \) of a sequence of natural numbers at least 2 with \( \lim_{l \to \infty} k_l = \infty \) and two (ordered) systems of generators \( (\Xi_l)_{l \in \mathbb{N}}, (\Omega_l)_{l \in \mathbb{N}} \) of symmetric groups \( \mathcal{G}([k_l]) \) over \( [k_l] = \{1, 2, \ldots, k_l\} \) that satisfy all of the following.

1. For all \( l \in \mathbb{N} \), \( \sharp(\Xi_l) = 8 \) and \( \sharp(\Omega_l) = 9 \). For each \( l \in \mathbb{N} \), \( \Omega_l \) is constructed by adding one extra element to \( \Xi_l \).
2. The disjoint union \( \bigsqcup_{l \in \mathbb{N}} \text{Cay}(\mathcal{G}([k_l]); \Xi_l) \) has property A.
3. The disjoint union \( \bigsqcup_{l \in \mathbb{N}} \text{Cay}(\mathcal{G}([k_l]); \Omega_l) \) does not admit a fibred coarse embedding in a generalized sense into any of these spaces:
   - Banach spaces of non-trivial type, and Banach spaces that are sphere equivalent to Banach spaces of non-trivial type.
   - Elements in \( CAT(0)_{<1} \).

Here two Banach spaces are said to be sphere equivalent if there exists a bijection \( \Phi \) between the unit spheres such that \( \Phi \) and \( \Phi^{-1} \) are both uniformly continuous;
see [Mim15] for details. Note that many reasonable CAT(0) spaces, including all Euclidean buildings associated with simple algebraic groups, belong to the class \( \text{CAT}(0)_{<1} \); see a paper of T. Toyoda [Toy10] for other examples of elements in \( \text{CAT}(0)_{<1} \). For the proof of Theorem D, we utilize the notion of embedded expanders; see Definition 7.8 and Proposition 7.14 for details.

Our proof of Theorem A is inspired by a trick by Gromov, [dCTV07, Proposition 4.4] for Hilbert spaces and [NP11, Section 9] for general Banach spaces, as we will explain in Sections 3 and 4. Independently to our results, S. Arnt [Arn16] applied this trick in a special situation where the coarse disjoint union is a box space (in particular, all \( G_m, m \in \mathbb{N} \), are finite) and the target class consists only of Banach spaces. For the case where \( \mathcal{M} = \text{Hilbert} \), V. Alekseev and M. Finn-Sell [AFS16] extended the framework of Theorem A for the case where \( (G_m)_m \) is a LEF approximation of \( G_\infty \), see Definition 2.2, to a sofic approximation of a sofic group. However, in that generality, only one direction (the direction of (i) in Theorem A) can be deduced; see the construction of a counterexample to the other direction by T. Kaiser [Kai17], which is explained below Theorem 5.3 in the concerning reference [Kai17]. Compare also with our points (a), (b) with LEA approximations, and the case where \( \mathcal{M} \) is general.

**Notation and Conventions:** We use \( G \) for a (non-marked) group and \( G \) for a marked group. We write the group unit of \( G \) as \( e_G \). A finite generating set \( S \) of \( G \) is regarded as an ordered set (sometimes an ordered multi-set) \( S = (s_1, s_2, \ldots, s_k) \) so that \((G; S)\) is seen as a marked group. A marked group \( G = (G; S) \) is said to be finite (respectively, amenable, and a-T-menable) if so is \( G \). For \( k \in \mathbb{N}_{\geq 1} \), we denote by \( F_k \) the free \( k \)-marked group, namely, \( F_k = (F_k; a_1, \ldots, a_k) \). Here \((a_1, \ldots, a_k)\) denotes a free basis of \( F_k \). For a non-empty set \( B \), denote by \( \mathcal{S}(B) \) the full symmetric group, and by \( \mathcal{S}_{<\aleph_0}(B) \) the symmetric group with finite support, namely, the group of all permutations on \( B \) that fix all but finitely many elements in \( B \). For \( R \in \mathbb{R}_{\geq 0} \), let \([R]\) denote the integer part of \( R \). For \( m \in \mathbb{N}_{\geq 1} \), let \([m]\) = \( \{1, 2, \ldots, m\} \). We use the terminology *isometries* for surjective ones; we use *geodesics* for minimal ones; namely, a *geodesic* from \( y \in \mathcal{M} \) to \( z \in \mathcal{M} \) is an isometric embedding \( c: [0, d(y, z)] \to \mathcal{M} \). We always exclude the empty-set from metric spaces. For a metric space \( \mathcal{M} \), we write the class \( \{\mathcal{M}\} \) consisting only of \( \mathcal{M} \) as \( \mathcal{M} \) for short.

**Organization of the paper:** In Section 2, we briefly recall the space of marked groups and the Cayley topology, and the definition of fibred coarse embeddings (in a generalized sense). In Subsections 2.4 and 2.5 we respectively, define several operations to classes of metric spaces and provide examples. In section 3 we explain the key idea to non-linear version of Gromov’s trick in relation to (pointed) metric ultraproducts. There we prove Proposition 1.4. Section 4 is devoted to the proof of (i) of Theorem A. It is done by the non-linear version of Gromov’s trick. In Section 5 we prove (ii) of Theorem A and Corollary B. Section 6 is for description of several gadgets that may be used in the space of marked groups, such as
the absorption lemma (Lemma 6.4). In Section 7, we discuss various examples to apply Theorem A (and Proposition 3.3), including the proofs of Corollary 1.6 and Corollary 1.8. Theorem C (Propositions 7.6 and 7.7) is proved in Subsection 7.2. Theorem D is verified in Subsections 7.5 and 7.6.

2. Preliminaries

2.1. Space of $k$-marked groups and Cayley topology. We recall basic facts of the Cayley topology from our Part I paper [MS13]; see Subsection 2.1 there for more details. Fix $k \in \mathbb{N}_{\geq 1}$. A $k$-marked group $G = (G; S) = (G; s_1, s_2, \ldots, s_k)$ is a pair of a finitely generated group $G$ and an ordered $k$-tuple $S = (s_1, \ldots, s_k)$ of generators of $G$ (as a group). From a $k$-marked group $G$, we construct two combinatorial objects, the Cayley diagram $\text{CayD}(G)$ and the Cayley graph $\text{Cay}(G)$ of $G$ as follows. The former is defined as a diagram (edge-colored and edge-oriented graph), with the edge coloring set $[k]$, by setting the vertex set as $G$ and by putting edges of the form $(g, s_j g)$ with orientation from $g$ to $s_j g$ in color $j(\in [k])$ for every $j \in [k]$ and every $g \in G$. The latter is the graph (with no edge colorings or no edge orientations) constructed by forgetting the edge-colorings/orientations of $\text{CayD}(G)$. Both of them are endowed with the shortest path metric $d_G$ (in $\text{CayD}(G)$, we ignore the edge-orientation to consider $d_G$) on the vertex set $G$. In this way, we regard $\text{CayD}(G)$ and $\text{Cay}(G)$ as geometric objects. We also consider $G$ itself as a metric space with this metric $d_G$; in other words, $d_G$ on $G$ is the right-invariant word metric with respect to $S$.

For $\emptyset \neq Y \subseteq G$ and for $R \in \mathbb{N}_{\geq 1}$, denote by $\partial_G(Y, R)$ the $R$-neighborhood of $Y$ in $d_G$, namely, the set of all $h \in G$ such that there exists $g \in Y$ such that $d_G(g, h) \leq R$. If $Y = \{g\}$, then we simply write $\partial_G(\{g\}, R)$ as $B_G(g, R)$ (closed ball of radius $R$ centered at $g$). In this setting, we define $B_{\text{CayD}(G)}(g, R)$ by restricting the vertex set of $\text{CayD}(G)$ to $B_G(g, R)$ and by taking the induced sub-diagram (more precisely, we collect all edges connecting vertices in $B_G(g, R)$ with remembering its edge-colorings/orientations). By declaring $g$ to be the root, $B_{\text{CayD}(G)}(g, R)$ has the structure of a rooted diagram. Note that $B_{\text{CayD}(G)}(e_G, R)$ completely remembers the multiplication table of $G$ up to word length $\lfloor R/2 \rfloor$.

Denote by $\mathcal{G}(k)$ the set of all $k$-marked groups (up to marked group isomorphisms). This space is equipped with a natural topology, the Cayley topology, which is metrizable and compact. One definition of that topology is the induced topology of the product topology on $\{0, 1\}^{F_k}$ to the set of all normal subgroups in $F_k$; there is a natural one-to-one correspondence between that subset of $\{0, 1\}^{F_k}$ and $\mathcal{G}(k)$ by the standard marked quotient map $F_k \twoheadrightarrow G$. Another characterization of this topology is the topology of local convergence (also known as the Gromov–Hausdorff convergence in this setting) among rooted diagrams, as stated in the following lemma (Lemma 2.4 in [MS13]). Here for two groups $G, H$ and for two subsets $e_G \in K_1 \subseteq G$ and $e_H \in K_2 \subseteq H$, a map $\beta : K_1 \to K_2$ is called a partial homomorphism if

$$\text{for all } g_1, g_2 \in K_1 \text{ such that } g_1 g_2 \in K_1, \quad \beta(g_1 g_2) = \beta(g_1)\beta(g_2)$$

holds true. The map $\beta$ is called a partial isomorphism if it is furthermore bijective.
Lemma 2.1. In \( \mathcal{G}(k) \), \((G_m)_{m \in \mathbb{N}}\) converges to \( G_\infty \) if and only if the following holds true:

\((\star)\) “For every \( m \in \mathbb{N} \), there exists \( R_m \in \mathbb{N} \) such that \( \lim_{m \to \infty} R_m = +\infty \) and

\[ B_{\text{CayD}(G_m)}(e_{G_m}, R_m) \cong B_{\text{CayD}(G_\infty)}(e_{G_\infty}, R_m) \]

as rooted diagrams.”

Here an isomorphism of rooted diagrams means a graph automorphism that preserves edge-colorings (in \([k]\)) and edge-orientations and that sends the root of the former diagram to the root of the latter.

In other words, for \( G = (G; s_1, \ldots, s_k) \in \mathcal{G}(k) \), if we define for each \( R \in \mathbb{N} \),

\[ N(G, R) = \{ H = (H; t_1, \ldots, t_k) \in \mathcal{G}(k) : \text{the map } t_j \mapsto s_j \text{ induces}
\]

a partial isomorphism \( \beta_{H,G,R} : B_H(e_H, R) \to B_G(e_G, R) \}, \)

then \( \{N(G, R)\}_{R \in \mathbb{N}} \) forms an (open) neighborhood system of \( G \).

Indeed, for every \( m \in \mathbb{N} \), set \( R_m \) to be the largest \( R \) such that \( m \geq m_R \), where \( m_R \) is as in [MS13] Lemma 2.4.

We write the convergence in the Cayley topology as \( \lim_{m \to \infty} G_m = G_\infty \) or \( G_m \xrightarrow{\text{Cay}} G_\infty \). The readers who are not familiar with the Cayley topology may consult Section 5 in our Part I paper [MS13], specially Lemma 5.1 therein, for pedagogical examples of the Cayley convergence.

We also recall the definitions of RF/LEF/LEA groups; recall these abbreviations from Introduction.

Definition 2.2. Let \( G \) be a finitely generated group.

(1) The group \( G \) is said to be RF if there exists a sequence \((N_m)_{m \in \mathbb{N}}\) of finite index normal subgroups of \( G \) such that

\[ \liminf_{m \to \infty} N_m = \bigcup_{m \in \mathbb{N}} \bigcap_{n \in \mathbb{N}, n \geq m} N_n \]

holds true.

(2) The group \( G \) is said to be LEF if for some (equivalently, every) marking \( G \) of \( G \), there exists a Cayley convergent sequence consisting of finite marked groups that converges to \( G \).

(3) The group \( G \) is said to be LEA if for some (equivalently, every) marking \( G \) of \( G \), there exists a Cayley convergent sequence consisting of amenable marked groups that converges to \( G \).

We say a sequence \((G_m)_{m}\) is a LEF (respectively, LEA) approximation of \( G \) if it consists of finite (respectively, amenable) marked groups converging to \( G \) in the Cayley topology. A LEF approximation is moreover called an RF approximation if it consists of marked group quotients; namely, for every \( m \), there exists a group quotient map \( \varphi_m : G \twoheadrightarrow G_m \) that sends the marking \( S = (s_1, \ldots, s_k) \) of \( G \) to that \( S_m = (s_1^{(m)}, \ldots, s_k^{(m)}) \) of \( G_m \) with preserving the orders on them: \( \varphi_m(s_j) = s_j^{(m)} \) for every \( j \in [k] \). An RF approximation of \((G; S)\) is of the form \(((G/N_m; S \mod N_m))_{m \in \mathbb{N}}\), where \((N_m)_{m \in \mathbb{N}}\) satisfies the conditions of Definition 2.2 (1).
In (1) in the definition above, we may relax the condition of \( \liminf_{m \to \infty} N_m = \{ e_G \} \) to \( \cap_{m} N_m = \{ e_G \} \); indeed, if we set new \( (N'_m)_m \) as \( N'_m = \cap_{n \in \mathbb{N} \leq m} N_n \), then \( \liminf_{m \to \infty} N'_m = \{ e_G \} \) is equivalent to \( \cap_m N'_m = \{ e_G \} \). However, if we hope to have an RF approximation out of \( (N_m)_m \) by taking marked group quotients, then the right condition is the former one, not the latter.

**Remark 2.3.** If a marked group \( G \) is finitely presented (this is independent of the choice of markings), then the set of all marked group quotients of \( G \) forms an open set. Hence, in that case, every LEF approximation eventually is an RF approximation; see [MS13, Subsection 2.1].

2.2. **Fibred coarse embeddings.** Recall from Introduction the construction of the disjoint union \( \bigsqcup_{m \in \mathbb{N}} X_m \) out of a sequence of metric spaces \( (X_m, d_m)_{m \in \mathbb{N}} \). If every \( X_m \) has finite diameter (the diameter is defined as the supremum of the distances between two points in the metric space), then we may construct a coarse disjoint union \( \bigsqcup_{m \in \mathbb{N}} X_m \), which is a (genuine) metric space. However, we do not go into details in this paper; instead, we refer the readers to [MS13, Definition 2.15.(2)] on this notion.

In this paper, we study fibred coarse embeddings from the disjoint union constructed above. For this purpose, we relax the definition of the fibred coarse embeddings as follows. For a generalized metric space \( X \), we say that \( X \) is uniformly locally finite if for every \( R \in \mathbb{R}_{\geq 0} \), there exists \( C \in \mathbb{N} \) such that every closed \( R \)-ball (for every center \( x \in X \)) has cardinality at most \( C \). For a sequence of metric spaces \( (X_m)_{m \in \mathbb{N}} \), we say that it is equi-uniformly locally finite if every \( X_m \) is uniformly locally finite and if moreover \( C = C(R) \) is taken uniformly on \( m \in \mathbb{N} \) for every \( R \in \mathbb{R}_{> 0} \). If \( (X_m)_{m \in \mathbb{N}} \) is equi-uniformly locally finite, then the disjoint union \( X = \bigsqcup_{m \in \mathbb{N}} X_m \) is uniformly locally finite.

**Definition 2.4.** Let \( \mathcal{M} \) be a non-empty class of metric spaces. Let \( (X, d) = \bigsqcup_{m \in \mathbb{N}} X_m \) be the disjoint union of a sequence of metric spaces \( (X_m)_{m \in \mathbb{N}} \) that is equi-uniformly locally finite. Let \( \rho, \omega : [0, \infty) \to [0, \infty) \) be two non-decreasing proper functions.

1. We say that \( X \) admits a \((\rho, \omega)\)-fibred coarse embedding into \( \mathcal{M} \) in a generalized sense if there exists \( M \in \mathcal{M} \) such that the following holds true: There exist
   - a field of metric spaces \((M_x)_{x \in X}\) over \( X \) such that each \( M_x \) is isometric to \( M \),
   - a section \( s : X \to \bigsqcup_{x \in X} M_x \), (namely, \( s(x) \in M_x \) for every \( x \in X \)), such that for every \( R \in \mathbb{R}_{> 0} \) there exists \( m(R) \in \mathbb{N}_{\geq 1} \) such that for each non-empty subset \( C \subseteq X \setminus \left( \bigsqcup_{n < m(R)} X_m \right) \) of diameter at most \( R \), there exists a “trivialization” \( t_{C,R} : (M_x)_{x \in C} \to C \times M \) such that the following holds. The restriction of \( t_{C,R} \) to the fibre \( M_x, x \in C \), is an isometry \( t_{C,R}(x) : M_x \to M \) that satisfies
     1. for every \( x_1, x_2 \in C \),
        \[ \rho(d(x_1, x_2)) \leq d_M(t_{C,R}(x_1)(s(x_1)), t_{C,R}(x_2)(s(x_2))) \leq \omega(d(x_1, x_2)) \];
(2) for every two subsets \( C_1, C_2 \subseteq X \setminus \left( \bigcup_{m \in \mathbb{N} < m(R)} X_m \right) \) of diameter at most \( R \) with \( C_1 \cap C_2 \neq \emptyset \), there exists an isometry \( t_{C_1,C_2,R} : M \to Y \) such that \( t_{C_1,R}(x) \circ t_{C_2,R}(x)^{-1} = t_{C_1,C_2,R} \) for all \( x \in C_1 \cap C_2 \).

We say that \( X \) admits a fibred coarse embedding into \( M \) if it admits a \((\rho, \omega)\)-fibred coarse embedding into \( M \) for some (non-decreasing, proper) pair \((\rho, \omega)\).

(ii) We say \((\rho, \omega)\) is a control pair for fibred coarse embeddings in a generalized sense for \( X \) into \( M \) if there exists a \((\rho, \omega)\)-fibred coarse embedding from \( X \) to \( M \) in a generalized sense. Denote by \( \mathcal{CP}_M^\text{fib}(X) \) the set of all control pairs above. The functions \( \rho \) and \( \omega \) are, respectively, called a compression function and an expansion function in the setting above.

Note that if a non-empty set \( C \) of \( X = \bigcup_{m \in \mathbb{N}} X_m \) is of bounded diameter, then there exists a unique \( m \in \mathbb{N} \) such that \( C \subseteq X_m \).

**Remark 2.5.** In the original formulation in [CWW14a, Definition 2.1] (for the case \( M \) being the class of all Hilbert spaces), for each \( R \in \mathbb{N} \), we are allowed to choose a bounded exceptional set \( K \), and consider \( C \) of diameter at most \( R \) from \( X \setminus K \).

In our definition in generalized sense, we relax this process and allow to take \( K = \bigcup_{m \in \mathbb{N} < m(R)} X_m \), the disjoint union of finitely many components in \( X = \bigcup_{m \in \mathbb{N}} X_m \).

Therefore, in Definition 2.6, if all \( X_m, m \in \mathbb{N} \), are finite, then our notion of the fibred coarse embeddability in a generalized sense coincides with that of the fibred coarse embeddability in the original sense from a coarse disjoint union \( \coprod_{m \in \mathbb{N}} X_m \).

In this paper, we discuss quantitative aspects (control pairs) for fibred coarse embeddings (in a generalized sense) as well as qualitative aspects (the property itself). For this purpose, disjoint unions are more suited than coarse ones.

**Remark 2.6.** The fibred coarse embeddability into \( M \) (in a general sense) is weaker than the the (genuine) coarse embeddability. Indeed, if \( f : X = \bigcup_{m} X_m \to M \) is a coarse embedding with control pair \((\rho, \omega)\), then set \( m(R) = 0 \) for all \( R \) and \( M_x = M \) for all \( x \in X \). Let \( s : X \to \bigsqcup_{x \in X} M \) be \( s(x) = f(x) \), and \( t_{C,R} = \text{id}_M \) for all \( R \) and for all \( C \) of diameter at most \( R \). This gives rise to a \((\rho, \omega)\)-fibred coarse embedding in a generalized sense into \( M \).

**Remark 2.7.** Strictly speaking, if \( M = \mathcal{E} \) consists of Banach spaces, then we furthermore assume that all isometries in the definition above are affine. However, by the Mazur–Ulam theorem [BL00, Chapter 14.1], the restriction on isometries in the case \( M = \mathcal{E} \) is automatic if \( \mathcal{E} \) consists of real Banach spaces. If \( \mathcal{E} \) consists of complex Banach spaces, then we need to care the affine property. However, in some case, this problem is automatically fixed in the following way: Consider a fibred coarse embedding into some complex Banach space \( E \) with respect to which \( t_{C,R}(x) \) above are isometries, not necessarily affine. Then, regard \( E \) as a real Banach space and write it as \( E_\mathbb{R} \); by the Mazur–Ulam theorem, \( t_{C,R}(x) \) are real affine. Take the Taylor complexification \( \widetilde{E}_\mathbb{R} \) of \( E_\mathbb{R} \) as follows: As a real vector space, \( \widetilde{E}_\mathbb{R} = E_\mathbb{R} \oplus E_\mathbb{R} \), and complex multiplication is defined by

\[
(a + b\sqrt{-1})(\xi, \eta) = (a\xi - b\eta, b\xi + a\eta), \quad a, b \in \mathbb{R}, \ \xi, \eta \in E_\mathbb{R}.
\]
(We may regard \((\xi, \eta) \in \tilde{E}_R\) as \(\xi + \sqrt{-1}\eta\).) The norm is defined as
\[
\|((\xi, \eta))\|_T = \sup_{\theta \in [0, 2\pi]} \|\cos t\xi - (\sin t)\eta\|_E.
\]
The resulting \((\tilde{E}_R, \| \cdot \|_T)\) is a complex Banach space. Finally, we define \(\tilde{E}_R\) from the original \(\tilde{t}_{C,R}(x)\) by
\[
\tilde{t}_{C,R}(x)(y) = \frac{1}{\sqrt{2}}(t_{C,R}(x)(y), t_{C,R}(x)(y)) \in \tilde{E}_R, \quad y \in (\tilde{E}_R)_x.
\]
This gives rise to a fibred coarse embedding into \(\tilde{E}_R\) with the same control pair as the original one. By construction, now these \(\tilde{t}_{C,R}(x)\) are complex affine.

Thus if the class \(\mathcal{E}\) is closed under the procedure \(E \mapsto (\tilde{E}_R, \| \cdot \|_T)\), then the issue of the affine property is essentially automatic. This closeness holds for some classes of our concern in this paper, for instance \(\mathcal{B}_{\text{type} > 1}\) and \(\mathcal{B}_{\beta < 1/2}\): see Example 2.18.

For this reason, in what follows, we do not go into details of the issue between isometries and affine isometries.

Remark 2.8. Though it was implicit in the original formulation [CWW14a, Definition 2.1], the “trivialization” \(t_C = t_{C,R}\) in Definition 2.4 is allowed to be inconsistent on changing \(R\). More precisely, for \(0 \leq R_1 < R_2\) and for \(C \subseteq X \setminus \left(\bigsqcup_{m \leq m_1} X_m\right)\) of diameter at most \(R_1\), we do not require that \(t_{C,R_1} = t_{C,R_2}\). This observation is important in our proof of (ii) of Theorem A.

We observe the following two lemmata. Here for a metric space \(X, x \in X\) and \(R \in \mathbb{R}_{>0}\), denote by \(B_X(x, R)\) the closed ball of radius \(R\) centered at \(x\).

Lemma 2.9. Let \((X_m)_{m \in \mathbb{N}}\) be a sequence of metric spaces that is equi-uniformly locally finite. Let \(X = \bigsqcup_{m \in \mathbb{N}} X_m\). Let \(\mathcal{M}\) be a non-empty class of metric spaces. Let \(\rho, \omega: [0, \infty) \to [0, \infty)\) be two non-decreasing proper functions. Then, \(X\) admits a \((\rho, \omega)\)-fibred coarse embedding into \(\mathcal{M}\) in a generalized sense if and only if there exists \(M \in \mathcal{M}\) such that the following holds true: There exist
- a field of metric spaces \((M_x)_{x \in X}\) which are all isometric to \(M\),
- there exists a section \(s: X \to \bigsqcup_{x \in X} M_x\),
- a sequence of non-negative real numbers \((R'_m)_{m \in \mathbb{N}}\) such that \(\lim_{m \to \infty} R'_m = +\infty\),
- a local trivialization \(t_{g,R'_m}: \bigsqcup_{x \in B_{X_m}(g,R'_m)} M_x \to B_{X_m}(g,R'_m) \times M\), for each \(m \in \mathbb{N}\) and each \(g \in X_m\),

such that the following hold:

1. For every \(n \in \mathbb{N}\), for every \(g \in X_m\) and every \(x \in B_{X_m}(g,R'_m)\), the restriction \(t_{g,R'_m}(x): M_x \to M\) of \(t_{g,R'_m}\) is isometry;
2. for every \(x_1, x_2 \in B_{X_m}(g,R'_m)\),
   \[
   \rho(d(x_1, x_2)) \leq d_M(t_{g,R'_m}(x_1)(s(x_1)), t_{g,R'_m}(x_2)(s(x_2))) \leq \omega(d(x_1, x_2));
   \]
3. if \(B_{X_m}(g_1,R'_m) \cap B_{X_m}(g_2,R'_m) \neq \emptyset\), there exists an isometry \(t_{g_1,R'_m}: M \to M\) such that \(t_{g_1,R'_m}(x) \circ t_{g_2,R'_m}(x)^{-1} = t_{g_1,g_2,R}\) for all \(x \in B_{X_m}(g_1,R'_m) \cap B_{X_m}(g_2,R'_m)\).
Lemma 2.10. In the setting of Lemma 2.9, let $Y = \bigsqcup_{n \in \mathbb{N}} Y_{m_n}$ be such that $(m_n)_{n \in \mathbb{N}}$ is a subsequence of $(m)_{m \in \mathbb{N}}$ and for each $n \in \mathbb{N}$, $Y_{m_n}$ is a non-empty subset of $X_{m_n}$ equipped with the induced metric. Then, if $X$ admits a fibred coarse embedding into $\mathcal{M}$ in a generalized sense with control pair $(\rho, \omega)$, then so does $Y$.

Proofs of Lemma 2.9 and Lemma 2.10. Lemma 2.10 is obvious.

To show the conditions as in Lemma 2.9, take $R \mapsto m^{(R)}$ as in Definition 2.4 set $R'_m = \min \{ \sup \{ r \in \mathbb{R}_{\geq 0} : m \geq m^{(r)} \}, m \}$ for every $m \in \mathbb{N}$, where we set $m_0 = 0$. By construction, $\lim_{m \to \infty} R'_m = +\infty$. For $g \in X_m$, set $t_{g,R'_m}$ as $t_{B_{X_m}(g,R'_m),R'_m}$.

To show the converse direction, for each $R \in \mathbb{R}_{\geq 0}$, set $m^{(R)} = \max \{ m \in \mathbb{N} : R'_m < R \} + 1$. Since $\lim_{m \to \infty} R'_m = +\infty$, it holds that $m_R \in \mathbb{N}$. For each $C \subseteq X \setminus \bigcup_{m \leq m^{(R)}} X_m$ of diameter at most $R$, there exist (unique) $m \in \mathbb{N}$ and (non-unique) $g \in X_m$ such that $C \subseteq B_{X_m}(g,R)$. Since $R'_m \geq R$ for all $m \leq m^{(R)}$, we may define $t_{g,R,R'_m}$ as the restriction of $t_{g,R_R'_m}$ on $C$. There is an ambiguity on the choice of $g$; however, if we fix the choices for all $C$, then condition (3) as in Lemma 2.9 ensures condition (2) as in Definition 2.4. Recall also Remark 2.8.

2.3. Equivariant coarse embeddings and a-$\mathcal{M}$-menability. In Section 4 in our Part I paper [MSB13], we recall the definition of a-$\mathcal{T}$-menability for finitely generated groups. Here we generalize this concept in terms of other target spaces. The following property should be stated as a-$F_{\mathcal{M}}$-menability in the strict sense. However, through communications with Arnt, we have agreed to use the terminology of a-$\mathcal{M}$-menability to avoid messes on notation. In the following definition, recall that a marked group $G$ is naturally equipped the metric $d_G$; see Subsection 2.1.

Definition 2.11. Let $G$ be a marked group and $\mathcal{M}$ be a non-empty class of metric spaces.

1. The marked group $G$ is said to be $a$-$\mathcal{M}$-menable if there exist $M \in \mathcal{M}$ and a coarse embedding $f : (G,d_G) \to M$ such that the following condition is satisfied: The map $f$ is of the form

\[ f(g) = y \cdot \alpha(g), \]

where $\alpha : M \act G$ is a right action by isometries and $y \in M$. We say that a coarse embedding $f$ is $G$-equivariant if it satisfies the condition above.

2. We say a finitely generated group $G$ is $a$-$\mathcal{M}$-menable if for some (equivalently, all) marking $G = (G;S)$ of $G$, $G$ is $a$-$\mathcal{M}$-menable.

3. The pair $(\rho, \omega)$ of two non-decreasing proper functions $[0, \infty) \to [0, \infty)$ is called an equivariant control pair for $G$ into $\mathcal{M}$ if there exist $M \in \mathcal{M}$ and an $G$-equivariant coarse embedding $f : G \to M$ such that $(\rho, \omega)$ is a control pair for $f$. In this case, we call $\rho$ and $\omega$, respectively, an equivariant compression function and an equivariant expansion function from $G$ into $\mathcal{M}$.

4. We denote by $\mathcal{CP}^d_{\mathcal{M}}(G)$ be the set of all equivariant control pairs for $G$ into $\mathcal{M}$.

In the definition above, we take a right action, not a left action, because we equip marked groups with right-invariant metrics.
Remark 2.12. Similarly to Remark 2.7, strictly speaking, if \( \mathcal{M} = \mathcal{E} \) consists of Banach spaces, then we require that the action \( \alpha \) above is affine. However, for the same reason as one in Remark 2.7, we do not go into details of this issue in this paper.

Let \( \text{Hilbert} \) denote the class of all Hilbert spaces. Then the notion of a-\( \text{Hilbert} \)-menability coincides with that of a-T-menability.

Remark 2.13. We warn that, unlike some other literature, the control pair \((\rho, \omega)\) is regarded as the pair of concrete functions, not only as growth orders. In particular, if \((\rho, \omega) \in \mathcal{CP}^\sharp_M(G)\) and if \(C_1, C_2 > 0\), it does not necessarily hold that \((C_1\rho, C_2\omega) \in \mathcal{CP}^\sharp_M(G)\). If we consider a class \( \mathcal{M} \) that is not necessarily closed under rescaling, this remark applies even when \(C_1 = C_2\). A similar issue to above applies to \(\mathcal{CP}_M(X)\) and \(\mathcal{CP}^\sharp_M(X)\).

For a fixed finitely generated group \(G\) and for a fixed equivariant coarse embedding \(f: G \to M\), equivariant compression functions for \(f\) depends on markings of \(G\) up to constant multiplication. Therefore, the set \(\mathcal{CP}^\sharp_M(G)\) \(\) does depend on the choice of markings; recall our discussion above. This observation is important because the Cayley boundary \(\partial_{\text{Cay}}(G_m)_m\) of a sequence \((G_m)_m\) may possibly consist of infinitely many marked groups.

2.4. Several operations on (pointed) metric spaces: \(\ell_q\)-products and metric ultraproducts. A pointed metric space \((M, y)\) is a (genuine) metric space \(M = (M, d_M)\) with a base point \(y \in M\). We define certain operations on a class of metric spaces, which will be employed in what follows.

Definition 2.14. Let \(q \in [1, \infty)\). Let \(B\) be a non-empty set that is at most countable. Let \((r_j)_{j \in B}\) be such that \(r_j \in (0, \infty)\) for all \(j \in B\). For a sequence \((M_j, d_j, y_j)_{j \in B}\) of pointed metric spaces, define the (pointed) \(\ell_p\)-product with scaling \((r_j)_j\), denoted by \((\prod_{j \in B} (M_j, y_j, r_j))_{\ell_p}\), by

\[
\left(\prod_{j \in B} (M_j, y_j, r_j)\right)_{\ell_q} = \left\{ (z_j)_{j \in B} : \left(\sum_{j \in B} (r_j d_j(z_j, y_j))^q\right)^{1/q} < \infty \right\}
\]

with the metric

\[
d_q(r_j)_j((z_j)_j, (w_j)_j) = \left(\sum_{j \in B} (r_j d_j(z_j, w_j))^q\right)^{1/q}, \quad (z_j)_j, (w_j)_j \in \left(\prod_{j \in B} (M_j, y_j, r_j)\right)_{\ell_q}
\]

and with the base point \((y_j)_j\).

If the scaling factor \((r_j)_j\) is all \(1\) \((r_j = 1\) for all \(j \in B\)), then we simply write \((\prod_{j \in B} (M_j, y_j, 1))_{\ell_q}\) as \((\prod_{j \in B} (M_j, y_j))_{\ell_q}\). This space is called the (pointed) \(\ell_q\)-product of \((M_j, y_j)_j\). If \(M_j\) are Banach spaces, then it is usually called the pointed \(\ell_q\)-sum.

If \(\#(B) < \infty\), then (the isometry type of) the resulting space \((\prod_{j \in B} (M_j, y_j, r_j))_{\ell_q}\) does not depend on the choice \((y_j)_j\) of base points. In that case, we write it as \((\prod_{j \in B} (M_j, r_j))_{\ell_q}\) for short.
We now switch our subject to \((\text{pointed})\) metric ultraproducts. A ultrafilter \(\mathcal{U}\) over \(\mathbb{N}\) has a one-to-one correspondence to a probability mean \(\nu\) (finitely additive measure with \(\nu(\mathbb{N}) = 1\)) on \(\mathbb{N}\) that is \(\{0, 1\}\)-valued and is defined over all subsets of \(\mathbb{N}\). The correspondence is given by setting that \(A \in \mathcal{U}\) if and only if \(\nu(A) = 1\). The cofinite filter \(\mathcal{U}_{\text{cofin}} = \{A \subseteq \mathbb{N} : |(\mathbb{N} \setminus A)\} < \infty\}\) is a filter, but not an ultrafilter. A non-principal ultrafilter \(\mathcal{U}\) is an ultrafilter that includes \(\mathcal{U}_{\text{cofin}}\) (as a subfilter). In what follows, fix a non-principal ultrafilter \(\mathcal{U}\) over \(\mathbb{N}\).

For a sequence \((r_m)_{m \in \mathbb{N}}\) in \(\mathbb{R}\) and for \(r_\infty \in \mathbb{R}\), we say that \(\lim\_\mathcal{U} r_m = r_\infty\) if it holds that

for every \(\varepsilon > 0\), \(\{m \in \mathbb{N} : |r_\infty - r_m| < \varepsilon\} \in \mathcal{U}\).

By local compactness and Hausdorff property of \(\mathbb{R}\), it is standard to show that every bounded real sequence \((r_m)_{m \in \mathbb{N}}\) has a unique \(\mathcal{U}\)-limit. The limit in general depends on the choice of a non-principal ultrafilter \(\mathcal{U}\). However, if \(\lim_{m \to \infty} r_m \) exists, then \(\lim\_\mathcal{U} r_m\) coincides with the limit above.

We now consider a sequence \(((M_m, d_m, y_m))_{m \in \mathbb{N}}\) of pointed metric spaces. Set

\[
\left(\prod_{m \in \mathbb{N}} (M_m, y_m)\right)_{\ell_\infty} = \{(z_m)_{m \in \mathbb{N}} : \sup_{m \in \mathbb{N}} d_m(z_m, y_m) < \infty\}
\]

and define \(d_\mathcal{U}\) by setting for \((z_m)_{m}, (w_m)_{m} \in \prod_{m \in \mathbb{N}} (M_m, y_m)_{\ell_\infty}\),

\[d_\mathcal{U}((z_m)_{m}, (w_m)_{m}) = \lim_{\mathcal{U}} d_m(z_m, w_m).
\]

This is a \(\text{pseudo}\)-metric, namely, \(d_\mathcal{U}\) does not separate points in general. To obtain a genuine metric space, introduce an equivalence relation \(\sim_{d_\mathcal{U} = 0}\) on \(\prod_{m \in \mathbb{N}} (M_m, y_m)_{\ell_\infty}\) by defining \((z_m)_{m} \sim_{d_\mathcal{U} = 0} (w_m)_{m}\) by \(d_\mathcal{U}((z_m)_{m}, (w_m)_{m}) = 0\). Finally, the quotient space

\[
\lim_{\mathcal{U}} (M_m, y_m) = \left(\prod_{m \in \mathbb{N}} (M_m, y_m)\right)_{\ell_\infty} / \sim_{d_\mathcal{U} = 0}
\]

is equipped with a genuine metric \(d_\mathcal{U}\). We call the resulting space the (\text{pointed}) metric ultraproduct of \((M_m, y_m)\) with respect to \(\mathcal{U}\). We write the equivalence class with respect to \(\sim_{d_\mathcal{U} = 0}\) of \((z_m)_{m}\) as \([((z_m)_{m})]_\mathcal{U}\).

**Definition 2.15.** Let \(\mathcal{M}\) be a non-empty class of metric spaces. We define \(\mathcal{UP}(\mathcal{M})\) to be the class of all pointed metric ultraproducts (after forgetting metric ultraproducts) of a single space in \(\mathcal{M}\). More precisely, it is the class of all spaces (isometric to those) of the form \(\lim_{\mathcal{U}} (M, y_m)\). Here \(M \in \mathcal{M}\) and for every \(m \in \mathbb{N}\), \(y_m \in M_m\); \(\mathcal{U}\) runs over all non-principal ultrafilters on \(\mathbb{N}\).

**Definition 2.16.** Let \(\mathcal{M}\) be a non-empty class of metric spaces. Fix \(q \in [1, \infty)\). We define the following three new classes, \(\ell_q(\mathcal{M})\), \(\mathcal{F}_q(\mathcal{M})\) and \(\mathcal{FS}_q(\mathcal{M})\) of metric spaces constructed from \(\mathcal{M}\).

1. We define \(\ell_q(\mathcal{M})\) as the class of all metric spaces (that is isometric to ones) of the form \((\prod_{j \in B} (M_j, y_j))_{\ell_q}\) (after forgetting the base point) for a non-empty at most countable sets \(B\) and for \(M_j \in \mathcal{M}\) and \(y_j \in M_j\) for \(j \in B\).
(2) We define $\mathcal{F}_q(\mathcal{M})$ as the class of all metric spaces (that is isometric to ones) that are constructed by the following three steps.

- **(Step 1.)** Take $M \in \mathcal{M}$.
- **(Step 2.)** Consider all metric spaces of the form $\left(\prod_{f \in F} (M, \frac{1}{\ell(f)})\right)_{\ell_q}$ for non-empty finite sets $F$. Here $\left(\frac{1}{\ell(f)}\right)_{f \in F}$ means that we take the constant scaling factor $\frac{1}{\ell(f)}$.
- **(Step 3.)** Take an arbitrary sequence $(\ell_m)$ as the class of all metric spaces constructed in Step 2.

(3) The new class $\mathcal{FS}_q(\mathcal{M})$ is defined if every element $L$ in $\mathcal{F}_q(\mathcal{M})$ is a geodesic space. Namely, for every $z, w \in L$, there exists a geodesic $c: [0, d(z, w)] \to L$ connecting $z$ and $w$; recall our notation from Introduction. If this is the case, then we construct $\mathcal{FS}_q(\mathcal{M})$ in the following way.

- If $\mathcal{M}$ consists only of Banach spaces, then every element $L$ in $\mathcal{F}_q(\mathcal{M})$ has a structure of affine Banach spaces. Then set $\mathcal{FS}_q(\mathcal{M})$ as the class of all Banach spaces isometrically affinely isomorphic to non-empty closed affine subspaces of $L$ for all $L \in \mathcal{F}_q(\mathcal{M})$.
- Otherwise, define $\mathcal{FS}_q(\mathcal{M})$ to be the class of all metric spaces isometric to non-empty closed convex subsets $L_0$ of $L$ (equipped with the induced metric from $L$) for all $L \in \mathcal{F}_q(\mathcal{M})$. Here a non-empty subset $L_0 \subseteq L$ is said to be **convex** if for every $z, w \in L_0$ and for every geodesic $c: [0, d(z, w)] \to L$ from $z$ to $w$ in $L$, $c$ (more precisely, the image $c([0, d(z, w)])$) is included in $L_0$.

Note that unlike the construction of $\ell_q(\mathcal{M})$, in Step 1 of the construction of $\mathcal{F}_q(\mathcal{M})$, we use a single $M \in \mathcal{M}$ to take the $\ell_q$-product with scaling. (Similarly for $\mathcal{UP}(\mathcal{M})$.) The symbol $\mathcal{F}$ in (2) stands for **finite** and Følner. The symbol $\mathcal{FS}$ in (3) stands for Følner and **subspaces** (or **subsets**).

**Remark 2.17.** The scaling factor $\frac{1}{\ell(f)}_{f \in F}$ in Step 2 above is chosen exactly in order to ensure that the diagonal embedding $M \hookrightarrow \left(\prod_{f \in F} (M, \frac{1}{\ell(f)})\right)_{\ell_q}$; $z \mapsto (z, z, \ldots, z)$ is isometric.

**2.5. Examples of classes of metric spaces.** We here exhibit several examples of a class $\mathcal{M}$ for which some of $\ell_q(\mathcal{M}) \subseteq \mathcal{M}$, $\mathcal{F}_q(\mathcal{M}) \subseteq \mathcal{M}$ and $\mathcal{FS}_q(\mathcal{M}) \subseteq \mathcal{M}$ holds for certain $q \in [1, \infty)$, or $\mathcal{UP}(\mathcal{M}) \subseteq \mathcal{M}$.

**Example 2.18.** First we consider classes of Banach spaces.

1. Let $r \in [1, \infty)$. Then $\mathcal{M} = \ell_r(= \{\ell_r\})$ satisfies $\ell_q(\mathcal{M}) \subseteq \mathcal{M}$ for $q = r$.
2. More generally to (1), let $\mathcal{M} = B_{L_r}$ denote the class of all $L_r$-spaces (over all measure spaces). (We fix $\mathbb{R}$ or $\mathbb{C}$, and construct the class above over the fixed coefficient field.) Then, $\ell_q(\mathcal{M}) \subseteq \mathcal{M}$ for $q = r$. Furthermore, Krivine showed that $\mathcal{UP}(\mathcal{M}) \subseteq \mathcal{M}$; see the survey [Hei80]. It implies that $\mathcal{F}_r(\mathcal{M}) \subseteq \mathcal{M}$. 
In particular, by letting $r = 2$, we observe that for $\mathcal{M} = \mathcal{H}$-Hilbert (the class of all Hilbert spaces), $\ell_2(\mathcal{M}) \subseteq \mathcal{M}$ and $\mathcal{F}_2(\mathcal{M}) \subseteq \mathcal{M}$; the proof of the latter item is much easier than that for the general $L_r$-case. In that case, moreover, $\mathcal{F}_2(\mathcal{M}) \subseteq \mathcal{M}$ holds.

(3) More generally to (2), let $\mathcal{M} = B_{NCL}$ denote the class of all non-commutative $L_r$-spaces (associated with all von-Neumann algebras). Then, $\ell_r(\mathcal{M}) \subseteq \mathcal{M}$ and $\mathcal{U}(\mathcal{M}) \subseteq \mathcal{M}$; the latter follows from work of Raynaud [Ray02]. It also holds that $\mathcal{F}_r(\mathcal{M}) \subseteq \mathcal{M}$.

(4) A Banach space $E$ is said to be of non-trivial \textit{(linear or Rademacher) type} if there exists $r \in (1, 2]$ and a constant $C > 0$ such that the following holds true: For every $m \in \mathbb{N}_{\geq 1}$ and for every $(\xi_i)_{i \in [m]}$ in $E$,

$$\mathbb{E}_{(\epsilon_i)_{i \in \mathbb{N}}}[\sum_{i \in [m]} \epsilon_i \xi_i^r] \leq C^r \sum_{i \in [m]} \|\xi_i\|^r.$$ 

If the inequality above is satisfied for fixed $r$ and $C$, we say that $E$ has a \textit{type $r$ constant $C$}. Here $\mathbb{E}_{(\epsilon_i)_{i \in \mathbb{N}}}[\cdot]$ means the expected value (average) over the uniform distribution of $(\epsilon_i)_{i \in \mathbb{N}}$ over \{-1, 1\}$^m$. Let $\mathcal{M} = B_{\text{type} > 1}$ denote the class of all complex Banach spaces of non-trivial type. Then, $\mathcal{U}(\mathcal{M}) \subseteq \mathcal{M}$ and $\mathcal{F}_2(\mathcal{M}) \subseteq \mathcal{F}_2(\mathcal{M}) \subseteq \mathcal{M}$ hold true. Moreover, for a fixed $r \in (1, 2]$ and $C > 0$, if we consider the subclass $B^{\text{type}}_{r,C}$ of all complex Banach spaces that have type $r$ constants $C$, then $\ell_r(B^{\text{type}}_{r,C}) \subseteq B^{\text{type}}_{r,C}$. See [TJ89] for details of types of Banach spaces.

Celebrated work of V. Lafforgue [Laf08, Laf09] yield fixed point properties with respect to $B_{\text{type} > 1}$; see (1) of Theorem 7.3.

(5) N. Tomczak-Jaegermann [TJ89, Chapter 6], and T. de Laat and M. de la Salle [dLdS18] studied quantities $(d_k(E))_{k \in \mathbb{N}_{\geq 1}}$ and the class $B_{\beta < 1/2}$ (see also [dLMdS16, Formula (1.1)]), which are defined as follows: For two isomorphic (but not necessarily isometrically) Banach spaces $E_1$ and $E_2$, the Banach–Mazur distance $d_{BM}(E_1, E_2)$ is defined to be the infimum of $\|T\|\|T^{-1}\|$, where $T$: $E_1 \xrightarrow{\approx} E_2$ runs over all isomorphisms between $E_1$ and $E_2$, and $\|\cdot\|$ denotes the operator norm. For a complex Banach space $E$, for each $k \in \mathbb{N}_{\geq 1}$, we define $d_k(E)$ by

$$d_k(E) = \sup \left\{ d_{BM}(E', \ell_{2,C}^{\dim_{C}(E')}): \dim_{C}(E') \leq k \right\},$$

where $E'$ runs over all (complex) linear subspaces of $E$ with the condition above. Here $\ell_{2,C}^m$ denotes the $m$-dimensional complex $\ell_2$-space for $m \in \mathbb{N}$. The class $B_{\beta < 1/2}$ is defined as the class of all complex Banach spaces $E$ for which there exist $0 < \beta < 1/2$ and $C > 0$ such that the condition

$$d_k(E) \leq C k^\beta$$

is satisfied. Then, it follows that $\mathcal{U}(B_{\beta < 1/2}) \subseteq B_{\beta < 1/2}$ and that $\mathcal{F}_2(B_{\beta < 1/2}) \subseteq \mathcal{F}_2(\mathcal{M}) \subseteq B_{\beta < 1/2}$. Moreover, for fixed $\beta \in (0, 1/2)$ and $C > 0$, if we denote by
\( \mathcal{B}_{\beta,C} \) the class of all complex Banach spaces such that the condition above holds for that pair \((\beta, C)\), then \( \ell_2(\mathcal{B}_{\beta,C}) \subseteq \mathcal{B}_{\beta,C} \) holds.

A fact states that a complex Banach space \( E \) is of non-trivial type if and only if \( \lim_{k \to \infty} k^{-1/2} d_k(E) = 0 \); see [TJ89]. In particular, \( \mathcal{B}_{\beta<1/2} \subseteq \mathcal{B}_{\text{type}>1} \). It is not known whether the inclusion above is strict.

de Laat–Mimura–de la Salle [dLMdlS16] studied fixed point properties with respect to \( \mathcal{B}_{\beta<1/2} \); see (3) of Theorem 7.3

(6) Similarly to (4), for each \( r \in [2, \infty) \) and each \( C > 0 \), we define the class \( \mathcal{B}_{r,C}^{\text{cotype}} \) as that of all Banach spaces that satisfy the cotype \( r \) inequality with constant \( C \):

\[
\begin{align*}
\mathbb{E}_{(\epsilon_i)} \left[ \sum_{i \in [m]} \epsilon_i \xi_i \right] & \geq C^{-r} \sum_{i \in [m]} \| \xi_i \|^r.
\end{align*}
\]

Here the expected value in the left-hand side is defined as one in (4). Then for \( \mathcal{M} = \mathcal{B}_{r,C}^{\text{cotype}} \), it holds that \( \ell_r(\mathcal{M}) \subseteq \mathcal{M} \), \( \mathcal{UP}(\mathcal{M}) \subseteq \mathcal{M} \), and \( \mathcal{FS}_r(\mathcal{M}) \subseteq \mathcal{M} \). The union \( \mathcal{B}_{\text{cotype}<\infty} = \bigcup_{r,C} \mathcal{B}_{r,C}^{\text{cotype}} \) equals the class of all Banach spaces of non-trivial cotype.

(7) A Banach space \( E \) is said to be uniformly convex if there exists a strictly positive real-valued function \( \Delta : (0, 2] \to [0, \infty) \) such that the following holds: For every \( \xi, \eta \in S(E) \) with \( \xi \neq \eta \),

\[
1 - \frac{\|\xi + \eta\|}{2} \geq \Delta(\|\xi - \eta\|).
\]

Here \( S(E) \) denotes the unit sphere of \( E \). For a fixed \( r \in [2, \infty) \), if there exists \( C > 0 \) such that \( \Delta \) above satisfies that \( \Delta(\epsilon) \geq C \epsilon^r \) for all \( \epsilon \in (0, 2] \), then we say that \( E \) is uniformly convex with modulus of convexity of \( r \)-type. Ball–Carlen–Lieb [BCL94, Proposition 7] showed that the condition above is equivalent to saying that there exists \( C' > 0 \) such that for all \( \xi, \eta \in X \) and for all \( t \in [0, 1] \), the following inequality holds true:

\[
\|(1 - t)\xi + t\eta\| \leq (1 - t)\|\xi\|^r + t\|\eta\|^r - (C')^r t(1 - t)\|\xi - \eta\|^r.
\]

They also made estimate between \( C \) and \( C' \) above. In this paper, we say a Banach space \( E \) is \( r \)-uniformly convex with constant \( C' \) if the inequality above is satisfied; this terminology is consistent with that of \( r \)-uniformly convex metric spaces in more general framework; see (2) of Example 2.19.

A Banach space \( E \) is said to be superreflexive if every (equivalently, some) metric ultrapower \( \lim_{\mathcal{U}}(E, 0) \) is reflexive. Enflo’s characterization states that \( E \) is superreflexive if and only if \( E \) is isomorphic to a uniformly convex Banach space. A theorem of G. Pisier [Pis75] shows that, moreover, for every superreflexive Banach \( E \), there exists \( r \in [2, \infty) \) such that \( E \) is isomorphic to a uniformly convex Banach space with modulus of convexity of power type \( r \). For \( r \in [2, \infty) \), for \( C' > 0 \) and for \( D \geq 1 \), we define the class \( \mathcal{B}_{r,C',D}^{\text{sr}} \) as that of all Banach spaces whose Banach–Mazur distance at most \( D \) to \( r \)-uniformly convex Banach spaces with constant \( C' \). Then, for \( \mathcal{M} = \mathcal{B}_{r,C',D}^{\text{sr}} \), it holds that \( \ell_r(\mathcal{M}) \subseteq \mathcal{M} \), \( \mathcal{UP}(\mathcal{M}) \subseteq \mathcal{M} \), and \( \mathcal{FS}_r(\mathcal{M}) \subseteq \mathcal{M} \). By aforementioned theorems in
Example 2.19. We furthermore provide classes of non-linear metric spaces.

(1) Let $M$ be a geodesic space. Then, $M$ is CAT(0) if and only if for every $x \in M$ and for every geodesic $c: [0, d(y, z)] \to M$ with $c(0) = y$ and $c(d(y, z)) = z$ and for every $0 \leq t \leq 1$, the following inequality

$$d(x, c_t)^2 \leq (1-t)d(x, y)^2 + td(x, z)^2 - t(1-t)d(y, z)^2$$

holds true, where $c_t$ denotes $c(td(y, z))$; see [BH99 Chapter II.1] for details.

H. Izeki and S. Nayatani [IN05] defined an invariant $\delta = \delta(M)$ for a complete CAT(0) space $(M, d_M)$ as follows, which is now called the Izeki–Nayatani invariant: Let $\mathcal{P}_{<0}(M)$ denote the set of all finitely supported probability measures on $M$ supported on more than one point. In other words, each $\mu \in \mathcal{P}_{<0}(M)$ is of the form $\sum_{i=1}^{k} t_i \text{Dirac}_{p_i}$ with $t_i > 0$ for $i \in [k]$, $\sum_{i=1}^{k} t_i = 1$ and $k \in \mathbb{N}_{\geq 2}$. Here $\text{Dirac}_p$ means the Dirac mass at $p$. Note that for such $\mu$, there exists a unique point $\overline{\mu} \in M$ that minimizes the function $M \ni x \mapsto \sum_{i=1}^{k} t_i d_M(p_i, x)^2 \in \mathbb{R}_{\geq 0}$;

this point $\overline{\mu}$ is called the barycenter of $\mu$. For such $\mu$, define

$$\delta(\mu) = \inf \left\{ \frac{\left\| \sum_{i=1}^{k} t_i f(p_i) \right\|^2}{\sum_{i=1}^{k} t_i \left\| f(p_i) \right\|^2} : \left\| f(p_i) \right\| = d_M(p_i, \overline{\mu}), \left\| f(p_i) - f(p_j) \right\| \leq d_M(p_i, p_j) \right\}.$$  

Here $f$ runs over all maps from $\text{supp}(\mu)$ to $L_2 = L_2([0, 1])$ that satisfies the two conditions indicated above, and $i$ and $j$ there vary all indices in $[k]$. Finally, the Izeki–Nayatani invariant $\delta(M)$ is defined as

$$\delta(M) = \sup_{\mu \in \mathcal{P}_{<0}(M)} \delta(\mu).$$

This invariant takes values in $[0, 1]$. For instance, if $M$ is a tree (or $\mathbb{R}$-tree), a Hilbert space or a (possibly infinite dimensional) Hadamard manifold (that is, a complete, connected and simply-connected Riemannian manifold with non-positive sectional curvature), then $\delta(M)$ is computed to be 0.

Fix $\delta_0 \in [0, 1]$. We define a class

$$\mathcal{M} = \mathcal{CAT}(0)_{\leq \delta_0} = \{ M : M \text{ is complete and CAT}(0), \delta(M) \leq \delta_0 \}.$$  

Then for each $\delta_0$, the class $\mathcal{M} = \mathcal{CAT}(0)_{\leq \delta_0}$ satisfies

$$\ell_2(\mathcal{M}) \subseteq \mathcal{M}, \mathcal{F}_2(\mathcal{M}) \subseteq \mathcal{F}_2(\mathcal{M}) \subseteq \mathcal{M}, \text{ and } \mathcal{UP}(\mathcal{M}) \subseteq \mathcal{M}.$$  

We refer the readers to [IN05] and [Toy09] for discussions on the Izeki–Nayatani invariants.
For $\delta_0 \in (0, 1]$, if we set $M$ as

$$\text{CAT}(0)_{<\delta_0} = \bigcup_{\delta' < \delta_0} \text{CAT}(0)_{\leq \delta'},$$

then it holds that

$$F_2(M) \subseteq F_{\delta} \subseteq M, \quad \text{and} \quad UP(M) \subseteq M.$$ 

Note that all CAT(0) spaces are uniquely geodesic; indeed, by setting $t = 1/2$, we observe that for every pair $(y, z)$, a geodesic midpoint of it is unique.

Izeki and Nayatani [IN05] studied fixed point properties with respect to $\text{CAT}(0)_{<\delta_0}$ for certain $\delta_0$; see (2) of Theorem 18.3

(2) Fix $r \in [2, \infty)$. Then, some $\ell_r$-analogue of item (1) may be defined as follows:

Let $C \in (0, 1]$. A geodesic space $M$ is said to be $r$-uniformly convex with constant $C$ if for every $x \in M$ and for every geodesic $c: [0, d(y, z)] \to M$ with $c(0) = y$ and $c(d(y, z)) = z$ and for every $0 \leq t \leq 1$, the following inequality

$$d(x, c_t)^r \leq (1-t)d(x, y)^r + t d(x, z)^r - C' t(1-t)d(y, z)^r$$

holds true, where $c_t$ denotes $c(td(y, z))$; see also [NS14]; compare with the inequality of $r$-uniformly convex Banach spaces in (7) of Example 2.18. The Clarkson inequality (see for instance [BL00]) shows that $L_r$ is $r$-uniformly convex with a certain constant $C_r$. For fixed $C \in (0, C_r]$, we write the class of all $r$-uniformly convex (geodesic) spaces with constant $C$ as $UC_{r,C}$. Then, for every $C \in (0, C_r]$, $M = UC_{r,C}$ satisfies that

$$\ell_r(M) \subseteq M, \quad F_r(M) \subseteq F_{\delta} \subseteq M \quad \text{and} \quad UP(M) \subseteq M.$$ 

We now claim that all $r$-uniformly convex metric spaces $M$ are uniquely geodesic. To see this, let $y, z \in M$ and let $c^{(1)}, c^{(2)}: [0, d(y, z)] \to M$ be two geodesic from $y$ to $z$. Fix $t \in [0, 1]$. Take a geodesic $c': [0, d(c^{(1)}, c^{(2)})] \to M$ from $c^{(1)}$ to $c^{(2)}$. Apply the inequality of $r$-convexity with $t = 1/2$ and $c = c'$ respectively for $(x, y, z) = (y, c^{(1)}, c^{(2)})$ and for $(x, y, z) = (z, c^{(1)}, c^{(2)})$. Then we have that

$$d(y, c^{(1)}_{1/2})^r \leq t^r d(y, z)^r - (C'' / 4) d(c^{(1)}, c^{(2)})^r,$$

and

$$d(z, c^{(1)}_{1/2})^r \leq (1-t)^r d(y, z)^r - (C'' / 4) d(c^{(1)}, c^{(2)})^r,$$

where $C'' > 0$ is the constant associated with the $r$-uniform convexity inequality for $M$. If $d(c^{(1)}, c^{(2)}) > 0$, then it would imply that

$$d(y, z) \leq d(y, c^{(1)}_{1/2}) + d(z, c^{(1)}_{1/2})$$

$$< (t^r d(y, z)^r)^{1/r} + ((1-t)^r d(y, z)^r)^{1/r}$$

$$= d(y, z);$$

a contradiction. Therefore, $c^{(1)} \equiv c^{(2)}$, and we are done.

However, the class $M = UC_{r,C}$ itself seems too enormous. We need to restrict it to a reasonably smaller subclass, and some variant of the Izeki–Nayatani invariant would be demanded in the current framework.
(3) Since we choose our scaling factor in Step 2 in $\mathcal{F}_q(M)$ in a specific manner, a certain quasification of classes in (1) and (2) satisfies some closeness property. For instance, for a given $\epsilon > 0$, we may define that a geodesic space $M$ is $\epsilon$-coarse CAT(0) if the following holds true: For every $x \in M$ and for every geodesic $c : [0, d(y, z)] \to M$ with $c(0) = y$ and $c(d(y, z)) = z$ and for every $0 \leq t \leq 1$, the following inequality holds:

$$d(x, c_t)^2 \leq (1 - t)d(x, y)^2 + td(x, z)^2 - t(1 - t)d(y, z)^2 + \epsilon,$$

where $c_t = c(td(y, z))$. Then, for every fixed $\epsilon$, the class $\mathcal{M}$ of all $\epsilon$-coarse CAT(0)-spaces satisfies that $\mathcal{F}_2(\mathcal{M}) \subseteq \mathcal{M}$. It also holds that $\mathcal{F}S_2(\mathcal{M}) \subseteq \mathcal{M}$, but in general elements in $\mathcal{F}S_2(\mathcal{M})$ may not be uniquely geodesic; compare with statements of Theorem A.(i),(2) and Proposition 3.7.

Again, this class $\mathcal{M}$ itself is too huge, and we need to bound wildness in a reasonable way.

(4) For a metric space $M$, concepts of the Enflo type [Enf78], the metric cotype (in the sense of A. Naor and M. Mendel [MN08]) and the Markov type [Bal92] are defined in terms of certain inequalities. For fixed values of parameters (if we consider the metric cotype, then we also need to fix a function $f : \mathbb{N}_{\geq 1} \to 2^{\mathbb{N}_{\geq 2}}$ such that $m \leq f(n)$ holds in the definition; compare with [MN08, Definition 1.1]), we define a class $\mathcal{M}$ of metric spaces that satisfy the concerning inequality associated with these fixed values of parameters. Then this $\mathcal{M}$ satisfies that $\ell_r(\mathcal{M})$, $\mathcal{UP}(\mathcal{M}) \subseteq \mathcal{M}$, and $\mathcal{F}_r(\mathcal{M}) \subseteq \mathcal{M}$, where $r \in [1, \infty)$ is the value of the Enflo type/metric cotype/Markov type. Moreover, if we switch $\mathcal{M}$ to the subclass $\mathcal{M}'$ of complete and geodesic ones in $\mathcal{M}$, then it also holds that $\mathcal{F}S_r(\mathcal{M}') \subseteq \mathcal{M}'$. However, similarly to (2) and (3), these classes are big and we may need to refine them to certain subclasses.

Note that there is a notion of the Markov cotype of a metric space, but that it may not be suited for the study of $\mathcal{F}S_r(\mathcal{M})$. This is because in the definition, we need to take extra points according to given points in the metric space, and the existence of such extra points may fail if we consider a convex subset, rather than the original space. For treatises on these concepts, we refer to [MN08] and [MNT13].

Remark 2.20. The main difference between $\mathcal{F}_q(\mathcal{M})$ and $\mathcal{UP}(\mathcal{M})$ is that the latter does not take (finite) $\ell_q$-products (or rescaling) before taking metric ultraproducts. Therefore, the latter procedure may preserve some “dimension” under certain conditions. First we consider the class $\mathbb{R}\mathcal{T}$ of all $\mathbb{R}$-trees (namely, geodesic 0-hyperbolic metric spaces). By the four-point condition of Gromov-hyperbolicity [BH99, Chapter III, Remark 1.21], it follows that $\mathcal{UP}(\mathbb{R}\mathcal{T}) \subseteq \mathbb{R}\mathcal{T}$. Even if we consider a smaller class $\mathcal{T}$ of all simplicial trees (considered as geodesic spaces, possibly with uncountably many vertices), then $\mathcal{UP}(\mathcal{T}) \subseteq \mathcal{T}$. This is because we may endow a metric ultraproduct with a simplicial structure by declaring vertices to be (equivalence classes of) bounded sequence of vertices; we draw edges between those with the limit distance 1.
We consider the class $\mathcal{QT}$ of quasi-trees, namely, graphs (considered as geodesic spaces, possibly with uncountably many vertices) that are quasi-isometric ([BH99 Chapter I. Definition 8.14]) to simplicial trees. By the argument above, we see that $\mathcal{UP}(\mathcal{QT}) \subseteq \mathcal{QT}$; recall that we fix a single element of $\mathcal{M}$ and take pointed metric ultraproducts of it to construct $\mathcal{UP}(\mathcal{M})$.

**Definition 2.21.** Let $\mathcal{M}$ be a non-empty class of metric spaces and $q \in [1, \infty)$. Denote $(\prod_{<\aleph_0} \mathcal{M})_{\ell_q}$ by the class of all metric spaces (isometric to) finite $\ell_q$-products $(\prod_{j \in F} M_j)_{\ell_q}$, where $1 \leq \sharp(F) < \infty$ and $M_j \in \mathcal{M}$ for all $j \in F$.

Since for a fixed $m \in \mathbb{N}_{\geq 2}$, taking an $m$-fold product is compatible with taking a metric ultraproduct, we conclude that $\mathcal{UP}((\prod_{<\aleph_0} \mathcal{QT})_{\ell_1}) \subseteq (\prod_{<\aleph_0} \mathcal{QT})_{\ell_1}$. (We may replace $\ell_1$ simultaneously with $\ell_q$ for each $q \in (1, \infty)$.)

Another construction is the following. Let $\mathcal{M} = \mathcal{M}$ be a proper metric space that is cocompact. Here the properness means that all closed bounded balls are compact; $\mathcal{M}$ is said to be cocompact if the full isometry group of $\mathcal{M}$ acts on $\mathcal{M}$ cocompactly. Then, $\mathcal{UP}(\mathcal{M}) = \mathcal{M}$. Here the cocompactness assumption is needed in order to make control on choices of base points $(y_m)_m$ to take a pointed metric ultraproduct.

**Remark 2.22.** Here we make more precise on what “dimension” means in examples in Remark 2.20. Gromov [Gro93] introduce an analogue of covering dimension in coarse geometry for a generalized metric space $\mathcal{M}$. This concept is called the asymptotic dimension, written as $\text{asdim}$; see [NY12 Chapter 2.2] for the definition. This is an invariant under coarse equivalence. Moreover, it is showed that if $f : X \to M$ is a coarse embedding, then it holds that

$$\text{asdim}(X) \leq \text{asdim}(M).$$

Also, a finite product of spaces with finite asymptotic dimension has finite asymptotic dimension. See [NY12 Proposition 2.2.4, Theorem 2.2.5 and Example 2.4.1] for details of these facts. Every tree has asymptotic dimension at most 1 ([NY12 Proposition 2.3.1]; see also [BS07 Proposition 10.2.1] for $\mathbb{R}$-trees). Hence, every element in $(\prod_{<\aleph_0} \mathcal{QT})_{\ell_1}$ has finite asymptotic dimension. For $M$ a complete, connected and simply connected finite dimensional Riemannian manifold with sectional curvature strictly negative, results in 1.E′ in [Gro93] implies the following: If such $M$ is cocompact, then $\text{asdim}(M) < \infty$.

### 3. Idea of the proof: Metric ultraproducts and the key to non-linear version of Gromov’s trick

We explain how metric ultraproducts play a role in the proofs of Proposition 1.4 and (i) of Theorem A.

**3.1. Metric ultraproducts and proof of Proposition 1.4.** To illustrate the ideas, we first prove the following result, which is a weakening of Proposition 1.4.

**Lemma 3.1.** Let $\mathcal{M}$ be a non-empty class of metric spaces. Let $(G_m)_{m \in \mathbb{N}}$ be a convergent sequence in $\mathcal{G}(k)$ ($k \in \mathbb{N}_{\geq 1}$) and let $G_\infty$ be the limit. Let $\rho, \omega : [0, \infty) \to [0, \infty)$ be two non-decreasing proper functions. Then, the following holds true: If...
\[ \bigcup_{m \in \mathbb{N}} \text{Cay}(G_m) \text{ admits a (genuine) coarse embedding into } M \text{ with control pair } (\rho, \omega), \]
then \( G_\infty \) admits a coarse embeddings into \( \mathcal{UP}(M) \) with the same control pair \( (\rho, \omega) \).

If \( M \) consists only of Banach spaces, then the following holds true: If \( \bigcup_{m \in \mathbb{N}} \text{Cay}(G_m) \) admits a coarse embedding into \( M \), then \( G_\infty \) admits a coarse embedding into the original class \( M \).

One key to the proof of Lemma 3.1 is \((*)\) in Lemma 2.1. For each \( m \in \mathbb{N} \), take \( R_m \in \mathbb{N} \) as in there. Hence, \( \beta_{G_m,G_\infty,R_m} \) gives a complete identification between \( R_m \)-balls \( B_{\text{CayD}(G_m)}(e_{G_m}, R_m) \) and \( B_{\text{CayD}(G_\infty)}(e_{G_\infty}, R_m) \); also, \( R_m \to +\infty \) as \( m \to \infty \). By employing this identification and by taking a pointed metric ultraproduct associated with a well-chosen sequence of base points \((y_m)_{m \in \mathbb{N}}\), we construct a coarse embedding \( \text{Cay}(G_\infty) \to \lim_U(M, y_m) \) out of the original coarse embedding \( \bigcup_{m \in \mathbb{N}} \text{Cay}(G_m) \to M \). The precise argument goes as follows.

Proof of Lemma 3.1. Let \( M \in \mathcal{M} \). Suppose there exists a coarse embedding
\[ f: \bigcup_{m \in \mathbb{N}} \text{Cay}(G_m) \to M \]
with control pair \((\rho, \omega)\). For every \( m \in \mathbb{N} \), take \( R_m \) as above.

Now, for each \( g \in G_\infty \), we associate the following sequence \((y(g)_m)_{m \in \mathbb{N}}\) of points in \( M \):
\[ y(g)_m = \begin{cases} f((\beta_{G_m,G_\infty,R_m})^{-1}(g)), & \text{if } g \in B_{\text{CayD}(G_\infty)}(e_{G_\infty}, R_m), \\ f(e_{G_m}), & \text{otherwise}. \end{cases} \]

By \((*)\), we observe the following:
- For every \( g \in G_\infty \),
  \[ \sup_{m \in \mathbb{N}} d_M(y(g)_m, f(e_{G_m})) \leq \omega(d_{G_\infty}(e_{G_\infty}, g)) < \infty. \]
- For every \( g_1, g_2 \in G_\infty \), let \( m_{g_1,g_2} \) be the smallest \( m \) such that for every \( n \geq m \), it holds that \( g_1, g_2 \in B_{\text{CayD}(G_\infty)}(e_{G_\infty}, R_n) \). (Since \( \lim_{m \to \infty} R_m = +\infty \), such \( m \) exists.) Then, for all \( m \geq m_{g_1,g_2} \), it holds that
  \[ \rho(d_{G_\infty}(g_1, g_2)) \leq d_M(y(g_1)_m, y(g_2)_m) \leq \omega(d_{G_\infty}(g_1, g_2)). \]

Finally, fix a non-principal ultrafilter \( U \) over \( \mathbb{N} \) and take the pointed metric ultraproduct \( M_U = \lim_U(M, d_M, f(e_{G_m})) \); we define the following map
\[ f_\infty: (\text{Cay}(G_\infty), d_{G_\infty}) \to M_U; \quad g \mapsto [(y(g)_m)_{m \in \mathbb{N}}]_U. \]
By the two observations above, we conclude that this \( f_\infty \) is well-defined, and that it is a coarse embedding with the same control pair \((\rho, \omega)\) as one for the original \( f \).

If \( M = E \) is a Banach space, then the arguments in the paper of Ostrovskii [Ost12] indicate a way to construct a coarse embedding from \( \text{Cay}(G_\infty) \) to the original \( E \) out of the metric ultraproduct construction above; this procedure will affect the control pair by some multiplicative and additive factors. \( \square \)

Proof of Proposition 2.10. Let \( M \in \mathcal{M} \). Suppose there exists a fibred coarse embedding from \( \bigcup_{m \in \mathbb{N}} \text{Cay}(G_m) \) into \( M \) with control pair \((\rho, \omega)\). Let \( G_\infty \in \partial_{\text{Cay}(G_m)} \). By definition, there exists a subsequence \((G_{m_n})_n \subset (G_m) \) that converges to \( G_\infty \) in \( \mathcal{G}(k) \). By Lemma 2.10, there exists a fibred coarse embedding from \( \bigcup_n \text{Cay}(G_{m_n}) \)
into $M$ with control pair $(\rho, \omega)$. Thus, we may assume that $(G_m)_{m \in \mathbb{N}}$ itself converges to $G_\infty$.

For every $m \in \mathbb{N}_{\geq 1}$, take $R_m$ as in (\star) and take $R'_m$ as in Lemma 2.9. Let $R''_m$ be the minimum of $R_m$ and $R'_m$. By construction, $\lim_{m \to \infty} R''_m = +\infty$. Take the local trivialization

$$t_{e_{G_m}, R'_m} : \bigcup_{x \in B_{\text{Cay}}(G_m)(e_{G_m}, R'_m)} M_x \to B_{\text{Cay}}(G_m)(e_{G_m}, R'_m) \times M.$$  

Define a map

$$f_m : B_{\text{Cay}}(G_m)(e_{G_m}, R'_m) \to M; \quad x \mapsto t_{e_{G_m}, R'_m}(x)(s(x)).$$

By (2) in Lemma 2.9 this $f_m$ is a coarse embedding with compression pair $(\rho, \omega)$.

Then, we may modify the construction of $(y(g)_m)$ as in the proof of Lemma 3.1 by setting for every $m \in \mathbb{N}$,

$$y(g)_m = \begin{cases}  
  f_m((\beta_{G_m, G_\infty, R''_m})^{-1}(g)), & \text{if } g \in B_{\text{Cay}}(G_\infty)(e_{G_\infty}, R''_m), \\
  f_m(e_{G_m}), & \text{otherwise.} 
\end{cases}$$

Then it will complete our proof of Proposition 1.4. \qed

**Remark 3.2.** To prove these lemma and proposition, we do not use the property that $\beta_{G_m, G_\infty}$ is an isomorphism as rooted diagrams; what we needed above is this map is an isomorphism as rooted (non-labelled, non-oriented) graphs. From this point of view, we consider the space of rooted graphs with bounded degree and generalize Proposition 1.4 in the following manner; see Proposition 3.3 for the conclusion.

Fix $k \in \mathbb{N}_{\geq 2}$. We set $\mathcal{R}(k)$ as the space of all connected graphs (without labellings/orientations) $(\Gamma, r_\Gamma)$ with roots $r_\Gamma(\in V(\Gamma))$ such that the degrees of all vertices are at most $k$. We say $\phi : (\Gamma_1, r_{\Gamma_1}) \cong (\Gamma_2, r_{\Gamma_2})$ is an isomorphism as rooted graphs if $\phi(r_{\Gamma_1}) = r_{\Gamma_2}$ and if $\phi$ is a graph isomorphism. In $\mathcal{R}(k)$, we identify two rooted graphs that are isomorphic in the sense above. We endow $\mathcal{R}(k)$ with the topology of local convergence as rooted graphs. This means, $((\Gamma_m, r_{\Gamma_m}))_{m \in \mathbb{N}}$ converges to $(\Gamma_\infty, r_{\Gamma_\infty})$ if for every $R \in \mathbb{N}_{\geq 1}$, there exists $m_R \in \mathbb{N}$ such that for every $m \geq m_R$, the $R$-balls $B_{\Gamma_m}(r_{\Gamma_m}, R)$ and $B_{\Gamma_\infty}(r_{\Gamma_\infty}, R)$, centered at roots, are isomorphic as rooted graphs. The space $\mathcal{R}(k)$, equipped with this topology, is a compact metrizable space.

Consider a sequence $(\Gamma_m)_{m \in \mathbb{N}}$ of connected graphs with all degrees at most $k$. Then, each $\Gamma_m$ forms a (possibly, non-singleton) subset $\widetilde{\Gamma_m} = \{(\Gamma_m, v) : v \in V(\Gamma_m)\}$ of $\mathcal{R}(k)$; we define the rooted graph boundary of $(\Gamma_m)$ by the set of all possible accumulation points of $\bigcup_{m \in \mathbb{N}} \widetilde{\Gamma_m}$ in $\mathcal{R}(k)$ as $m \to \infty$. We write it as $\partial_r(\Gamma_m)_{m \in \mathbb{N}}$.

**Proposition 3.3.** Let $k \in \mathbb{N}_{\geq 2}$. Let $\mathcal{M}$ be a non-empty class of metric spaces. Let $(\Gamma_m)_{m \in \mathbb{N}}$ be a sequence of connected graphs with all degrees at most $k$. If $\bigcup_{m \in \mathbb{N}} \Gamma_m$ admits a fibred coarse embedding into $\mathcal{M}$ in a generalized sense, then the rooted graph boundary $\partial_r(\Gamma_m)_{m \in \mathbb{N}}$ admits equi-coarse embeddings into $\mathcal{UP}(\mathcal{M})$; that means,

$$\bigcap_{(\Gamma_\infty, r_\infty) \in \partial_r(\Gamma_m)} \mathcal{CP}_{\mathcal{UP}(\mathcal{M})}(\Gamma_\infty) \neq \emptyset.$$
If $\mathcal{M}$ consists only of Banach spaces, then the following holds true: If $\bigsqcup_{m \in \mathbb{N}} \Gamma_m$ admits a fibred coarse embedding into $\mathcal{M}$ in a generalized sense, then $\partial_r(\Gamma_m)_{m \in \mathbb{N}}$ admits equi-coarse embeddings into $\mathcal{M}$.

3.2. Metric ultraproducts of fragmentary actions. In Subsection 3.1, we see how to recover (non-equivariant) coarse embeddings from Cayley limit groups out of a (fibred) coarse embeddings of the disjoint union. See for instance a survey of Y. Stalder [Sta09, Theorem 3.12] for the standard argument.

On this recovery procedure, what we need is not the global actions of the whole groups $G_m$, but local actions of balls; compare with the proof of Lemma 3.1. Here we give the definition of a fragmentary action of a subset of a group, which is a local version of the action of the whole group.

**Definition 3.4.** Let $M$ be a metric space. Let $G$ be a group and $e_G \in K \subseteq G$ be a subset. A partial homomorphism from $K$ to the isometry group $\text{Isom}(M)$ is called a fragmentary action of $K$ on $M$. In other words, a right fragmentary action $\alpha: M \curvearrowright K$ (where for all $g \in K$, $\alpha(g)$ is an isometry on $M$) satisfies the following property: For every $g_1, g_2 \in K$ such that $g_1g_2 \in K$,

$$z \cdot \alpha(g_1g_2) = (z \cdot \alpha(g_1)) \cdot \alpha(g_2)$$

for all $z \in M$.

We use the word “fragmentary” because the terminology “partial action” is referred to a quite different concept in the literature.

**Proposition 3.5.** Let $G_m \overset{\text{Cay}}{\rightarrow} G_\infty$. Let $\rho, \omega: [0, \infty) \to [0, \infty)$ be two non-decreasing proper functions. Assume that for every $m \in \mathbb{N}$, there exists $r_m \in \mathbb{N}_{\geq 1}$ with $\lim_{m \to \infty} r_m = +\infty$ such that the following holds: For every $m \in \mathbb{N}$, there exists an pointed isometric right fragmentary action $(\alpha_m, M_m, y_m)$ of $B_{G_m}(e_{G_m}, r_m)$

$$\alpha: M_m \curvearrowright B_{G_m}(e_{G_m}, r_m), \quad y_m \in M_m$$

such that the orbit map of $y_m$ is an (equivariant) coarse embedding of $(B_{G_m}(e_{G_m}, r_m), d_{G_m})$ with (equivariant) control pair $(\rho, \omega)$.

Then, for every non-principal ultrafilter $\mathcal{U}$ over $\mathbb{N}$, there exists a pointed isometric right action $(\alpha_\mathcal{U}, M_\mathcal{U}, y_\mathcal{U})$ of $G_\infty$ such that the orbit map of $y_\mathcal{U}$ is an (equivariant) coarse embedding of $(G_\infty, d_{G_\infty})$ with equivariant control pair $(\rho, \omega)$. Here $M_\mathcal{U} = \lim_\mathcal{U}(M_m, y_m)$ and $y_\mathcal{U} = [(y_m)_{m \in \mathbb{N}}]_\mathcal{U}$.

**Proof.** For every $m \in \mathbb{N}$, take $R_m \in \mathbb{N}$ and $\beta_{G_m, G_\infty, R_m}$ as in $(\ast)$. Set $R''_m = \min\{R_m, r_m\}$. For each $g \in G_\infty$, define $\alpha'_m(g): M_m \to M'_m$; $z \mapsto z \cdot \alpha'_m(g)$ by

$$z \cdot \alpha'_m(g) = \begin{cases} z \cdot \alpha_m((\beta_{G_m, G_\infty, R_m})^{-1}(g)), & \text{if } g \in B_{G_\infty}(e_{G_\infty}, R''_m), \\ z, & \text{otherwise}. \end{cases}$$

By construction, the restriction of $\alpha'_m$ on $B_{G_\infty}(e_{G_\infty}, R''_m)$ is a fragmentary action.

Finally, for every $g \in G_\infty$, define $\alpha_\mathcal{U}(g): M_\mathcal{U} \to M_\mathcal{U}$ by

$$[(z_m)_m]_\mathcal{U} \cdot \alpha_\mathcal{U}(g) = [(z_m \cdot \alpha'_m(g))_m]_\mathcal{U} \quad \text{for every } [(z_m)_m]_\mathcal{U} \in M_\mathcal{U}.$$
It is straightforward to check that this is well-defined. Since \( \lim_{m \to \infty} R_m'' = +\infty \), this \( \alpha_U \) is a (global) action of \( G_\infty \) on \( M_U \) (by isometries). By assumption, it furthermore holds that for every \( g_1, g_2 \in G_\infty \),

\[
\rho(d_{G_\infty}(g_1, g_2)) \leq d_{M_U}(\alpha_U(g_1), \alpha_U(g_2)) \leq \omega(d_{G_\infty}(g_1, g_2)),
\]

as desired; compare with the proofs of Lemma 3.1 and Proposition 1.4.

\[\square\]

3.3. Key to the non-linear version of Gromov’s trick. Proposition 3.5 will be used for the proof of (i). (1) of Theorem A. To deal with (i). (2), we employ the following definition.

**Definition 3.6.** Let \( G \) be a group and \( e_G \in K \subseteq G \) be a subset. Let \( M \) be a metric space and \( y \in M \). Let \( \epsilon \geq 0 \). We say that a map \( \alpha : K \to \text{Isom}(M) \) is an \( \epsilon \)-almost fragmentary (right) action at \( y \) if the following condition is fulfilled: For every \( g_1, g_2 \in K \) such that \( g_1 g_2 \in K \),

\[
d(y \cdot \alpha(g_1 g_2), (y \cdot \alpha(g_1)) \cdot \alpha(g_2)) \leq \epsilon.
\]

If \( K = G \) and \( \alpha \) is a \( 0 \)-almost fragmentary at \( y \), then \( \alpha : G \to \text{Isom}(M) \) gives rise to a genuine action on the \( G \)-orbit \( \{ y \cdot \alpha(g) : g \in G \} \) of \( y \).

On the proposition above, recall from Theorem A the definitions of convex subsets of geodesic spaces and of the unique geodesic condition.

**Proposition 3.7.** Let \( G_m \overset{\text{Cay}}{\to} G_\infty \). Let \( \rho, \omega : [0, \infty) \to [0, \infty) \) be two non-decreasing proper functions. Assume that for every \( m \in \mathbb{N} \), there exist \( r_m \in \mathbb{N}_{\geq 1} \) with \( \lim_{m \to \infty} r_m = +\infty \), a sequence \( (\rho_m, \omega_m)_m \) of two non-decreasing proper functions \( [0, \infty) \to [0, \infty) \), \( \epsilon_m \geq 0 \) with \( \lim_{m \to \infty} \epsilon_m = 0 \) and \( y_m \in M_m \) such that the following conditions hold:

- For every \( m \in \mathbb{N} \), there exists an \( \epsilon_m \)-almost (right) fragmentary action \( \alpha_m \) at \( y_m \) of \( B_{G_m}(\epsilon_m, r_m) \) (by isometries) on \( M_m \).
- Two sequences \( (\rho_m)_m \) and \( (\omega_m)_m \), respectively, converge to \( \rho \) and \( \omega \) pointwise.
- For every \( m \in \mathbb{N} \) and for every \( g_1, g_2 \in B_{G_m}(\epsilon_m, r_m) \), it holds that

\[
\rho_m(d_{G_m}(g_1, g_2)) \leq d_{M_m}(y_m \cdot \alpha_m(g_1), y_m \cdot \alpha_m(g_2)) \leq \omega_m(d_{G_m}(g_1, g_2)).
\]

Assume that there exists a non-principal ultrafilter \( U \) over \( \mathbb{N} \) such that \( M_U = \lim_U(M_m, y_m) \) is uniquely geodesic.

Then, for every such \( U \) over \( \mathbb{N} \), there exist a closed convex subset \( L_0 \) of \( M_U \) and an isometric right (genuine) action \( (\alpha_U, L_0) \) of \( G_\infty \) that satisfy all of the following conditions:

- For \( y_U = [(y_m)_m]_U \), it holds that \( \{ y_U \cdot \alpha_U(g) : g \in G_\infty \} \subseteq L_0 \).
- The orbit map of \( y_U \) by \( \alpha_U \) is an (equivariant) coarse embedding of \( (G_\infty, d_{G_\infty}) \) (into \( L_0 \)) with equivariant control pair \( (\rho, \omega) \).

Here we equip \( L_0 \) with the induced metric from that of \( M_U \).

**Proof.** For each \( g \in G_\infty \), the construction of \( \alpha_U(g) : M_U \to M_U \) is exactly the same as one in the proof of Proposition 3.5. Indeed, since each \( \alpha_m(h) \), for \( h \in B_{G_m}(\epsilon_m, r_m) \),
is isometric, \( \alpha_m \) is \( \varepsilon \)-almost fragmentary action at \( y \) and the “orbit map” of \( y_m \) by \( \alpha_m \) is a coarse embedding with control pair \((\rho_m, \omega_m)\), it follows that for each \( g \in G_\infty \),

\[
\sup_{m \in \mathbb{N}} d_{M_m}(z_m \cdot \alpha'_m(g), y_m) < \infty \quad \text{for every } (z_m)_m \in \left( \prod_{m}(M_m, y_m) \right)_{l_\infty};
\]

recall that \( \rho_m \) and \( \omega_m \) are non-decreasing. The construction of \( \alpha_U(g) \) above is well-defined, and \( \alpha_U(g) \) is an isometry. We, however, warn that in general, \( \alpha_U(gh) \) may not coincide with \( \alpha_U(g) \circ \alpha_U(h) \) (the composition is from left to right) as maps \( M_U \to M_U \).

Nevertheless, we observe that \( \alpha_U: G_\infty \to \text{Isom}(M_U) \) is 0-almost fragmentary action at \( y_U \) because \( \lim_{m \to \infty} \epsilon_m = 0 \). Therefore, it is a genuine action on \( L' = \{ y_U \cdot \alpha_U(g) : g \in G_\infty \} \). For every \( g_1, g_2 \in G_\infty \), define

\[
L_{g_1, g_2} = \{ z \in M_U : z \cdot (\alpha(g_1) \circ \alpha(g_2) \circ \alpha(g_1 g_2)^{-1}) = z \}.
\]

Because \( \alpha(g_1) \circ \alpha(g_2) \circ \alpha(g_1 g_2)^{-1} \) is an isometry and we assume that \( M_U \) is uniquely geodesic, each \( L_{g_1, g_2} \) is a closed convex subset of \( M_U \) with \( L' \subseteq L_{g_1, g_2} \). (Observe that every isometry sends geodesics to geodesics.) Finally, take

\[
L_0 = \bigcap_{g_1, g_2 \in G_\infty} L_{g_1, g_2} \quad (\supseteq L').
\]

Then \( L_0 = L_0 \cdot \alpha_U(G_\infty) \) holds, and \( \alpha_U \) gives rise to a genuine action on \( L_0 \). We rewrite the restriction of \( \alpha_U \) on \( L_0 \) as \( \alpha_U : L_0 \rhd G_\infty \); it satisfies the required two conditions.

Remark 3.8. We may remove the assumption of the unique geodesic property on \( M_U = E_U \) if \( M = \mathcal{E} \) consists only of Banach spaces. Indeed, if we assume that all \( \alpha_m \) are complex affine, then take \( L_0 \) as the closure of the algebraic complex affine span of \( L' \); this \( L_0 \) is a non-empty complex affine subspace of \( E_U \). Even if we do not assume it, the Mazur–Ulam theorem states that all \( \alpha_m \) are real affine. Then we can take a desired \( L_0 \) as a non-empty real affine subspace of \( E_U \).

4. FROM FIBRED COARSE EMBEDDINGS TO EQUIVARIANT EMBEDDINGS OF GROUPS IN THE CAYLEY BOUNDARY

In this section, we prove item (i) of Theorem A. As mentioned in Introduction, our idea of the proof(s) is based on a trick of Gromov. We first demonstrate the proof of (i).(1) in Subsection 4.1. Then we proceed to the proof of (i).(2) in Subsection 4.2.

4.1. Proof for finite marked group sequences. We already know from Proposition 1.4 the way to recover (non-equivariant) coarse embeddings of groups in the Cayley boundary from local information from the fibred coarse embedding. The point in our proof is how to recover moreover equivariant coarse embeddings. The key tool here is Proposition 3.5.

Proof of (i).(1) of Theorem A. Similarly to the proof of Proposition 1.4, we may assume that \( (G_m)_{m \in \mathbb{N}} \) is a convergent sequence. Let \( G_\infty \) be the Cayley limit of it. Assume that \( \bigsqcup_{m \in \mathbb{N}} \text{Cay}(G_m) \) admits a fibred coarse embedding into \( M, M \in M, \)
Lemma 4.1. We claim the following.

Recall Definition 2.14.

Let \( q \in [1, \infty) \).

For each \( m \in \mathbb{N} \), set

\[
M_{m,q} = \left( \prod_{x \in G_m} \left( M, \left( \frac{1}{e(G_m)} \right)^{1/q} \right) \right)_{t_q}
\]

We claim the following.

**Lemma 4.1.** (1) This \( \alpha_m \) is a fragmentary action (by isometries)

\[
M_{m,q} \sim B_{G_m}(e_{G_m}, R_m).
\]

(2) Let \( y_m = (y_{m,x})_{x \in G_m} \in M_{m,q} \), where \( y_{m,x} = t_{x,R_m}(x)(s(x)) \) for every \( x \in G_m \).

Then the orbit map of \( y_m \) by the fragmentary action \( \alpha_m \) is an (equivariant) coarse embedding from \( B_{G_m}(e_{G_m}, R_m) \) into \( M_{m,q} \) with control pair \( (\rho, \omega) \).

**Proof of Lemma 4.1.** Since all \( t_{x,gx,R_m} \) are isometries, \( \alpha_m(g) \) is an isometry for all \( g \in G_m \). Assume that \( g_1, g_2, g_1g_2 \in B_{G_m}(e_{G_m}, R_m) \). Then for each \( x \in G_m \),

\[
g_1g_2x \in B_{G_m}(g_1g_2x, R_m) \cap B_{G_m}(g_2x, R_m) \cap B_{G_m}(x, R_m),
\]

it holds that

\[
t_{x,g_2x,R_m} \circ t_{g_2x,g_1g_2x,R_m} = t_{x,R_m}(g_1g_2x) \circ (t_{g_2x,R_m}(g_1g_2x))^{-1} \circ t_{g_2x,R_m}(g_1g_2x) \circ (t_{g_1g_2x,R_m}(g_1g_2x))^{-1}
\]

\[
= t_{x,R_m}(g_1g_2x) \circ (t_{g_1g_2x,R_m}(g_1g_2x))^{-1}
\]

\[
= t_{x,g_1g_2x,R_m}.
\]

Therefore, we have that by setting \( w_x = t_{x,g_1x,R_m}(z_{g_1x}) \),

\[
((z_x)_x \cdot \alpha(g_1)) \cdot \alpha(g_2) = (t_{x,g_2x,R_m}(w_{g_2x}))_x
\]

\[
= (t_{x,g_2x,R_m} \circ t_{g_2x,g_1g_2x,R_m}(z_{g_1g_2x}))_x
\]

\[
= (t_{x,g_1g_2x,R_m}(z_{g_1g_2x}))_x
\]

\[
= (z_x)_x \cdot \alpha(g_1g_2).
\]

This proves (1).

For (2), observe that for every \( g \in B_{G_m}(x, R'_m) \) and every \( x \in G_m \),

\[
d_M(y_{m,x}, t_{x,gx,R'_m}(y_{m,gx})) = d_M(t_{x,R'_m}(x)(s(x)), t_{x,R'_m}(gx)(s(gx))).
\]

By assumption and by recalling Remark 2.17, we verify (2).

By applying Proposition 3.5 with \( r_m = R_m' \), we obtain from Lemma 4.1 an equivariant coarse embedding from \( G_\infty \) into \( \lim_t(M_{m,q}, y_m) \) with equivariant control pair \( (\rho, \omega) \). Since \( \lim_t(M_{m,q}, y_m) \in F_q(M) \), it proves the desired assertions.
4.2. Non-linear version of Gromov’s trick and proof for amenable group sequences. In order to extend the argument as in Subsection 4.1 to the case of amenable marked group sequences, we employ a Følner set of \(G_m\) instead of \(G_m\) itself and utilize Proposition 3.7. This idea dates back to Gromov, and well-known if \(M = \text{Hilb}\). We extend this framework to possibly non-linear settings.

For \(\epsilon > 0\) and for \(R \in \mathbb{N}\), an \((\epsilon, R)\)-Følner set \(F\) of a marked group \(G\) is a non-empty finite subset of \(G\) that satisfies
\[
\frac{\#(\partial_G(F, R))}{\#(F)} < \epsilon.
\]

Amenability of \(G\) is characterized by the existence of \((\epsilon, R)\)-Følner sets for all \(\epsilon(>0)\) and for all \(R\) (this property does not depend on the choices of markings of \(G\)).

**Proof of (i),(2) of Theorem A.** We describe the modifications needed from the proof (i),(1) of Theorem A. Fix \(q \in [1, \infty)\). For each \(m \in \mathbb{N}\), choose \(\delta_m > 0\) appropriately (we will specify later) and take an \((\delta_m, R'_m)\)-Følner set \(F(m)\) of \(G_m\). Set
\[
M_{m,q} = \left( \prod_{x \in F(m)} (M, \left( \frac{1}{\#(F(m))} \right)^{1/q}) \right)^{\ell_q}.
\]

For every \(g \in B_{G_m}(\epsilon g_m, R'_m)\), let \(\alpha_m(g) : M_{m,q} \to M_{m,q}\) be \((z_{x})_{x \in F(m)} \cdot \alpha_m(g) = (w_{x})_{x}\) for \((z_{x})_{x} \in M_{m,q}\). Here \(w_{x} \in M\) is defined by
\[
w_{x} = \begin{cases} t_{x,gx,R'_m}(z_{gx}), & \text{if } gx \in F(m), \\ z_{x}, & \text{otherwise}. \end{cases}
\]

We claim the following.

**Lemma 4.2.** Let \(y_m = (y_{m,x})_{x \in G_m} \in M_{m,q}\), where \(y_{m,x} = t_{x,R'_m}(x)(s(x))\) for every \(x \in G_m\). Let \(\delta'_{m,q} = \delta_{m}^{1/q} \omega(R'_m)\).

1. For every \(g \in B_{G_m}(\epsilon g_m, R'_m)\), \(\alpha_m(g)\) is an isometry.
2. This \(\alpha_m\) is a \(3\delta'_{m,q}\)-almost fragmentary action of \(B_{G_m}(\epsilon g_m, [R'_m/2])\) at \(y_m\); recall Definition 3.6.
3. For every \(g_1, g_2 \in B_{G_m}(\epsilon g_m, [R'_m/2])\),
\[
(1 - 2\delta_m) \rho(d_{G_m}(g_1, g_2)) \leq d_{M_m}(y_m \cdot \alpha_m(g_1), y_m \cdot \alpha_m(g_2)) \leq \omega(d_{G_m}(g_1, g_2)) + 2\delta'_{m,q}.
\]

**Proof of Lemma 4.2.** Item (1) is by construction. For (2), let \(g_1, g_2 \in B_{G_m}(\epsilon g_m, [R'_m/2])\) such that \(g_1g_2 \in B_{G_m}(\epsilon g_m, [R'_m/2])\). Let \(F_{\text{good}}^{(m)} = F^{(m)} \cap (g_1^{-1}F^{(m)}) \cap (g_2^{-1}F^{(m)}) \cap ((g_1g_2)^{-1}F^{(m)})\) and \(F_{\text{bad}}^{(m)} = F^{(m)} \setminus F_{\text{good}}^{(m)}\). Then, by the Følner property for \(F\),
\[
\frac{\#(F_{\text{bad}}^{(m)})}{\#(F^{(m)})} \leq 3\delta_m \frac{\#(F^{(m)})}{\#(F^{(m)})}.
\]
Note that by the proof of Lemma 4.1 for all \(x \in F^{(m)}\good\),
\[
((y_m \cdot \alpha(g_1)) \cdot \alpha(g_2))(x) = (y_m \cdot \alpha(g_1g_2))(x),
\]
where \((\cdot)(x)\) indicates the \(x\)-th coordinate.
Now let \( x \in F_{\text{bad}}^{(m)} \). Then, similarly to one above, we have that
\[
\begin{align*}
d_M((y_m \cdot \alpha(g_1)) \cdot \alpha(g_2))(x), (y_m \cdot \alpha(g_1 g_2))(x) & \leq \max\{\omega(d_{G_m}(\gamma, \gamma')) : \gamma, \gamma' \in \{e_{G_m}, g_1, g_2, g_1 g_2\}\} \\
& = \omega(\max\{d_{G_m}(\gamma, \gamma') : \gamma, \gamma' \in \{e_{G_m}, g_1, g_2, g_1 g_2\}\}) \\
& \leq \omega(R'_m).
\end{align*}
\]

By recalling that we take the scaling factor \((1/\sharp(F^{(m)}))^{1/q}\) to construct \(M_{m,q}\) from \(M\), we conclude that
\[
\begin{align*}
d_{M,m,q}((y_m \cdot \alpha(g_1)) \cdot \alpha(g_2), y_m \cdot \alpha(g_1 g_2)) & \leq \left(3\delta_m\sharp(F^{(m)}) \cdot \frac{\omega(R'_m)^q}{\sharp(F^{(m)})}\right)^{\frac{1}{q}} \\
& \leq 3\delta_m^{1/q}\omega(R'_m),
\end{align*}
\]
as desired. Item (3) will be proved in a manner quite similar to one above. \(\Box\)

For given \( q \in [1, \infty) \), \( \omega : [0, \infty) \to [0, \infty) \) and \( (R'_m)_m \), we can choose \( (\delta_m)_m, \delta_m > 0 \), such that \( \lim_{m \to \infty} \delta_m = 0 \) and \( \lim_{m \to \infty} \delta'_m = 0 \). Finally, apply Proposition 3.7 with \( r_m = [R'_m/2], \ v_m = \max\{2\delta_m, 3\delta'_m\} \) and \( (\rho_m, \omega_m) = ((1 - \varepsilon_m)\rho, \omega + \varepsilon_m) \), and we thus obtain the conclusion. If \( \mathcal{M} \) only consists of Banach spaces, then consult also Remarks 2.7, 2.12 and 3.8. \(\Box\)

By setting \( G_m \equiv G \) for a fixed amenable group and by restricting embeddings to genuine coarse embeddings (recall Remark 2.6), we in particular have the following corollary. It may be regarded as a non-linear version of Gromov’s trick. Although this may have been previously observed by other researchers, we include it for the sake of convenience of the readers; compare with [NP11, Theorem 9.1] for the case of Banach spaces.

**Corollary 4.3.** Let \( \mathcal{M} \) be a non-empty class of metric spaces that satisfies the conditions as in Theorem A (i). Assume that for some of such \( q \), it holds that \( \mathcal{FS}_q(\mathcal{M}) \subset \mathcal{M} \). Then for every amenable marked group \( G \),
\[
\mathcal{CP}^*_{\mathcal{M}}(G) = \mathcal{CP}^*_{\mathcal{M}}(G).
\]

On the other hand, for non-amenable marked groups, \( \mathcal{CP}^*_{\mathcal{M}}(G) \) is much restrictive than \( \mathcal{CP}^*_{\mathcal{M}}(G) \). For instance, E. Guentner and J. Kaminker [GK04, proposition 4.2] showed that for every \( a \in (0, 1) \), there exists \( C > 0 \) such that \( (C r^a, r) \in \mathcal{CP}^*_{\text{Hilbert}}(F_2) \). However, they [GK04, Theorem 5.3] also proved that if there exist \( a \in (1/2, 1] \) and \( C > 0 \) such that \( (C r^a, r) \in \mathcal{CP}^*_{\text{Hilbert}}(G) \), then \( G \) must be amenable.

5. FROM EQUIVARIANT EQUI-COARSE EMBEDDINGS OF THE CAYLEY BOUNDARY TO FIBRED COARSE EMBEDDINGS

Here we prove (ii) of Theorem A. Unlike the proofs in Section 4, we do not need to impose conditions on \( G_m, m \in \mathbb{N} \). First, we provide the proof where \( (G_m)_m \) is a convergent sequence.
Proof of (ii) of Theorem A for the case where $\partial_{\text{Cay}}(G_m)_m = 1$. Let $G_\infty$ be the Cayley limit of $(G_m)_m$. For each $m \in \mathbb{N}$, take $R_m$ and $\beta_{G_m, G_\infty, R_m}$ as in Lemma 2.1. Assume that there exist $M \in \mathcal{M}$ and an equivariant coarse embedding from $G_\infty$ into $M$ with equivariant control pair $(\rho, \omega)$. Let $\alpha : M \to G_\infty$ be an action by isometries and $y \in M$ such that the orbit map $G_\infty \ni g_\infty \mapsto y \cdot \alpha(g) \in M$ gives the (equivariant) coarse embedding above. Write as $X = \bigsqcup_{m \in \mathbb{N}} \text{Cay}(G_m)$.

Let $R'_m = [R_m/2]$ for every $m \in \mathbb{N}$ and $M_x = M$ for every $x \in X$. Define a section $s : X \to \bigsqcup_{x \in X} M$ by $s(x) = y(\in M = M_x)$ for every $x \in X$. Now for $m \in \mathbb{N}$, $g \in G_m$, define a local trivialization $t_{g, R'_m} : \bigsqcup_{x \in B_{G_m}(g, R'_m)} M \to B_{G_m}(g, R'_m) \times M$ by

$$(t_{g, R'_m}(x))(z) = z \cdot \alpha(\beta_{G_m, G_\infty, R_m}(xg^{-1})), \quad \text{for } x \in B_{G_m}(g, R'_m) \text{ and } z \in M.$$ 

Here note that since $G_m$ acts on $\text{Cay}(G_m)$ by right, $B_{G_m}(g, R'_m)g^{-1} = B_{G_m}(e_G, R'_m)$.

In what follows, we will verify conditions (1)–(3) of Lemma 2.9. For (1), for each $x \in B_{G_m}(g, R'_m)$, the map $t_{g, R'_m}(x) : M \to M$ is an isometry. For $x_1, x_2 \in B_{G_m}(g, R'_m)$,

$$d_M((t_{g, R'_m}(x_1))(s(x_1)), (t_{g, R'_m}(x_2))(s(x_2))) = d_M(y \cdot \alpha(\beta_{G_m, G_\infty, R_m}(x_1g^{-1})), y \cdot \alpha(\beta_{G_m, G_\infty, R_m}(x_2g^{-1})))$$

Since

$$d_{G_\infty}(\beta_{G_m, G_\infty, R_m}(x_1g^{-1}), \beta_{G_m, G_\infty, R_m}(x_2g^{-1})) = d_{G_m}(x_1g^{-1}, x_2g^{-1}) = d_{G_m}(x_1, x_2),$$

it follows (2). Finally, we check (3). Let $B_{G_m}(g, R'_m) \cap B_{G_m}(g', R'_m) \neq \emptyset$. For each $x \in B_{G_m}(g_1, R'_m) \cap B_{G_m}(g_2, R'_m)$,

$$((t_{g_1, R'_m}(x))(z) \circ (t_{g_2, R'_m}(x)^{-1}))(z) = (z \cdot \alpha((\beta_{G_m, G_\infty, R_m}(xg_2^{-1}))^{-1}) \cdot \alpha(\beta_{G_m, G_\infty, R_m}(xg_1^{-1})))) = (z \cdot \alpha((\beta_{G_m, G_\infty, R_m}(g_2x^{-1}))^{-1}) \cdot \alpha(\beta_{G_m, G_\infty, R_m}(xg_1^{-1})))) = z \cdot \alpha((\beta_{G_m, G_\infty, R_m}(g_2x^{-1})\beta_{G_m, G_\infty, R_m}(xg_1^{-1})))) = z \cdot \alpha((\beta_{G_m, G_\infty, R_m}(g_2x^{-1}xg_1^{-1})) = z \cdot \alpha((\beta_{G_m, G_\infty, R_m}(g_2g_1^{-1}))).$$

Indeed, here we observe that $\beta_{G_m, G_\infty, R_m}$ is a partial isomorphism from $B_{G_m}(e_G, R_m)$ to $B_{G_\infty}(e_G, R_m)$ and that $g_2g_1^{-1} \in B_{G_m}(e_G, 2R'_m) \subseteq B_{G_m}(e_G, R_m)$. The expression in the very below side of the equalities above is independent of $x \in B_{G_m}(g_1, R'_m) \cap B_{G_m}(g_2, R'_m)$.

We proceed to the proof of the general case; we here employ the class $\ell_q(M)$. Recall the definition of an open neighborhood $N(G, R)$ of $G$ from Lemma 2.1.
Proof of (ii) of Theorem A in full generality. For each $R \in \mathbb{N}$, \{\!\{N(H, R) : H \in \partial_{\text{Cay}}(G_m)\}_m\} is an open cover of \partial_{\text{Cay}}(G_m)_m. By compactness of \partial_{\text{Cay}}(G_m)_m, there exist $i(R) \in \mathbb{N}$ and $H_0^{(R)}, \ldots, H_{i(R)}^{(R)}$ such that

$$
\bigcup_{i=0}^{i(R)} N(H_i^{(R)}, R) \supseteq \partial_{\text{Cay}}(G_m)_m.
$$

Let $H_0^{(R)}, \ldots, H_{i(R)}^{(R)}$ be, respectively, the underlying groups of $H_0^{(R)}, \ldots, H_{i(R)}^{(R)}$. By definition of \partial_{\text{Cay}}(G)_m, for each $R$, there exists $n_R \in \mathbb{N}_{\geq 1}$ such that,

$$
\bigcup_{i=0}^{i(R)} N(H_i^{(R)}, R) \supseteq (G_m)_m \setminus \{G_0, \ldots, G_{n_R-1}\}
$$

holds, where $\overline{\{\}}_{\text{Cay}}$ denotes the closure in the Cayley topology. Note that for $R = 0$, then the left-hand side above, in fact, includes $\overline{(G_m)_m}$. For each $R \in \mathbb{N}$ and for every $m \in \mathbb{N}_{\geq n_R}$, choose $0 \leq i \leq i(R)$ such that $G_m \in N(H_i^{(R)}, R)$ (if there exist at least two such $i$, choose the smallest $i$). We write this $i$ as $i_m^{(R)}$.

Set a new disjoint union as

$$
X' = \bigcup_{R \in \mathbb{N}} \left( \bigcup_{0 \leq i \leq i(R)} \left( \bigcup_{n_R \leq m \leq n_R+1+R : i_m^{(R)} = i} \text{Cay}(G_m) \right) \right).
$$

Now assume that $\partial_{\text{Cay}}(G_m)_m$ is uniformly $\alpha$-$\mathcal{M}$-menable; there exists a pair $(\rho, \omega)$ of non-decreasing proper functions $[0, \infty) \to [0, \infty)$ such that

$$(\rho, \omega) \in \bigcap_{H \in \partial_{\text{Cay}}(G_m)_m} \mathcal{C}P_{\mathcal{M}}^q(H).$$

In particular, for every $R \in \mathbb{N}$ and for every $0 \leq i \leq i(R)$, there exist $M_i^{(R)} \in \mathcal{M}$, $y_i^{(R)} \in M_i^{(R)}$ and an action $\alpha_i^{(R)} : M_i^{(R)} \acts H_i^{(R)}$ such that the orbit map $H_i^{(R)} \ni h \mapsto y_i^{(R)} \cdot \alpha_i^{(R)}(h) \in M_i^{(R)}$ is an (equivariant) coarse embedding with equivariant control pair $(\rho, \omega)$. Fix $q \in [1, \infty)$ and define

$$
M_q = \left( \prod_{R \in \mathbb{N}} \left( \prod_{0 \leq i \leq i(R)} (M_i^{(R)}, y_i^{(R)}) \right) \right)_{\ell_q}.
$$

Note that this is an (at most) countable $\ell_q$-product; hence $M_q \in \ell_q(\mathcal{M})$.

By Lemma 2.10 it suffices to construct a fibred coarse embedding in a generalized sense from $X'$ into $M_q$. Let $(M_q)_x = M_q$ for all $x \in X'$ and $s : X' \to \bigsqcup_{x \in X'} M_q$ be $s(x) = (y_j^{(r)})_{r,j}$. For each $n_R \leq m \leq n_R+1+R$ with $i_m^{(R)} = i$, consider the component $\text{Cay}(G_m)$ in $X'$ associated with these $R$ and $i$. Set $R_m' = [R/2]$ and construct $t_{g, R_m'}$ by

$$
(t_{g, R_m'}(x))(z)_{r,j} = (w_{r,j})_{r,j}.
$$
for $x \in B_{G_m}(g, R_m')$ and for $(z)_{r,j} \in M_q$, where,
\[
(w)_{r,j} = \begin{cases} 
2_r \cdot \alpha_i^{(R_m)(B_m H_m^{(R_m)}(xg^{-1}))}, & \text{if } (r, j) = (R, i), \\
2_r, & \text{otherwise.}
\end{cases}
\]

Then in a similar argument to one in the previous proof for the case where $\sharp(\partial_{Cay}(G_m)) = 1$, we may verify conditions (1)–(3) in Lemma 2.9; recall also Remark 2.8. Furthermore, we obtain that
\[
(\rho, \omega) \in CP_{M_q} \left( \bigsqcup_{m \in \mathbb{N}} \text{Cay}(G_m) \right).
\]

Proof of Corollary 2.13 For the case where $M = \mathcal{H}$, $\mathcal{B}_{r,C}$ or $\mathcal{CAT}(0) \leq \delta_0$, the assertions immediate follow from Theorem A; see also arguments in Examples 2.18 and 2.19. For $M = L_q$, Naor and Y. Peres employed the classification of separable closed subspaces of $L_q$-spaces and indicated a way to coming back to $L_q$ from $FS_q(L_q)$; see the last assertion of [NP11, Theorem 9.1].

\[\square\]

6. Gadgets in the space of marked groups

In this section, we explain some gadgets to produce different markings of a group (or a sequence of finite groups) whose accumulation points in the space of marked groups have different nature.

6.1. Standard (restricted) wreath products and two conditions on group properties. First, we recall the definition of standard (restricted) wreath products; see also [MS13, Proposition 2.9]. For two groups $G$ and $H$, $G \wr H$ is defined to be $(\bigoplus_{h \in H} G) \rtimes H$, where $H$ acts by permutation of coordinates by right. For $g \in G$ and $h \in H$, by $g \delta_h$ we denote the element in $\bigoplus_{h \in H} G$ whose $h$-entry is $g$ and all of the other entries are $e_G$. We use $e$ for the group unit of $\bigoplus_{h \in H} G$. If $G = (G; s_1, \ldots, s_k)$ and $H = (H; t_1, \ldots, t_l)$ are two marked groups, then we endow $G \wr H$ with the standard $(k + l)$-marking as follows:
\[
((s_1 \delta_{e_H}, e_H), (s_2 \delta_{e_H}, e_H), \ldots, (s_k \delta_{e_H}, e_H), (e, t_1), (e, t_2), \ldots, (e, t_l)).
\]

We write the marked group of $G \wr H$ with the standard marking above as $G \wr H$. Then, for $G_m \to G_\infty$ and $H_n \to H_\infty$ (respectively in $G(k)$ and $G(l)$) in the Cayley topology, we have that as $\min\{m, n\} \to \infty$,
\[
G_m \wr H_n \xrightarrow{\text{Cay}} G_\infty \wr H_\infty \text{ in } G(k + l);
\]
see §2.4. Theorem in [VG97]. This may be clear to the readers who are familiar with a relationship between wreath products and random walks.

We consider a group property $P$ for finitely generated groups (or more generally, for countable discrete groups) that satisfies the following two conditions:

(Conditions on the property $P$.)
• If $G$ has $\mathcal{P}$, then so does $G \wr H$ for every finitely generated infinite amenable group $H$.
• If an infinite $G$ has $\mathcal{P}$, then so does $\mathfrak{S}_{<\aleph_0}(G) \rtimes G$; recall our notation of symmetric groups from Introduction. Here $G$ acts on $\mathfrak{S}_{<\aleph_0}(G)$ by the permutations induced by right action $G \curvearrowright G$.

Example 6.1. We exhibit some examples of $\mathcal{P}$ that satisfy the two conditions above.

(1) Amenability satisfies the two conditions above. Indeed, it is defined also for countable discrete group, and it is a classical fact that amenability is closed under taking group extensions. For the second condition, observe that $\mathfrak{S}_{<\aleph_0}(G)$ above is locally finite (an increasing union of finite groups) and hence amenable.

(2) A-T-menability fulfills both of the two conditions. Indeed, it is also defined for countable discrete groups, and [CCJ+01, Example 6.1.6] states that the extension of an a-T-menable group (normal subgroup side) by an amenable group (quotient side) is a-T-menable. The validity of the second condition is explicitly written in a celebrated paper of Cornulier–Stalder–Valette [CSV12, Example 5.4], as follows.

Theorem 6.2 (Cornulier–Stalder–Valette). Let $H$ be a countable infinite group. If $H$ is a-T-menable, then so is $\mathfrak{S}_{<\aleph_0}(H) \rtimes H$.

To see this, more precisely, apply Corollary 5.3 to Example 3.4 with $X = G$ in the concerning reference [CSV12]. In this work, they showed that a-T-menability is closed under taking standard wreath products.

(3) Both of property A and the coarse embeddability into a Hilbert space, respectively, satisfy the two conditions: The proof for property A is similar to one in (1); see [ADR00] for the fact that property A is closed under taking group extensions. For the coarse embeddability, the second condition is fulfilled due to [CSV12, Theorem 5.10]; compare with (2).

(4) The failure of property A (equivalently, the non-exactness in the context of $C^*$-algebras; see [Oza00]) satisfies the two conditions. More generally, if a group property $\mathcal{Q}$ passes to (finitely generated) subgroups, then the property $\mathcal{P}$, defined as the negation of $\mathcal{Q}$, fulfills both of the conditions. For instance, we may set $\mathcal{P}$ as the failure of a-T-menability as well.

(5) A non-example is Kazhdan’s property (T); see also Remark 6.3 below. We refer the readers to [BdlHV08] for comprehensive study on this property.

Remark 6.3. The first condition on $\mathcal{P}$ is used in the context of the absorption lemma, Lemma 6.4. For applications, we may relax the condition to saying that $G \wr H$ remains to have $\mathcal{P}$ for certain infinite $H$, according to the way we utilize the absorption lemma. We note that even in this setting, property (T) does not fit in our framework. Indeed, if $\sharp(H) = \infty$, then $G \wr H$ never has property (T) unless $G$ is trivial; see [BdlHV08, Proposition 2.8.2] (note that notation of $\Gamma \wr H$ in [BdlHV08] corresponds to that of $H \wr \Gamma$ in the present paper).
6.2. Absorption Lemma. The following lemma, which may be called an absorption lemma, enables us to absorb a group into some abelian group by taking the wreath product by an infinite group. The original form in the paper of Bartholdi and Erschler [BE15] stated it in terms of permutational (restricted) wreath products; here we formulate it for a simpler case.

**Lemma 6.4** (Special case of Lemma 6.1 in Bartholdi–Erschler [BE13].) Let $G$ be a finitely generated group and fix $(g_1, \ldots, g_k)$ a marking of $G$. For each $j \in [k]$, let $C_j$ be the cyclic group of the same order as for $g_j$. Then, for every infinite and finitely generated group $P$, there exists a system of marking $(S_m)_{m \in \mathbb{N}}$ of $G \wr P$ with fixed size such that

$$(G \wr P; S_m) \xrightarrow{\text{Cay}} (C_1 \times C_2 \times \cdots \times C_k) \wr P,$$

with a suitable marking of the Cayley limit group.

For the sake of completeness, we include (idea of) the proof.

**Proof.** Fix a marking $T = (t_1, \ldots, t_l)$ of $P$. Since $P$ is infinite, for every $m \in \mathbb{N}$, there exists $e \in P$ such that $B_{Cay(D(P;T)}(x_j^{(m)}, m), j \in [k]$, are mutually disjoint. Now, define a system $(S_m)_{m \in \mathbb{N}}$ of markings of $G \wr P$ by

$$S_m = ((g_1 \delta_{eP}, eP), (g_2 \delta_{x_2^{(m)}}, eP), \ldots, (g_k \delta_{x_k^{(m)}}, eP), (e, t_1), \ldots, (e, t_l)),$$

where $e$ means the group unit of $\bigoplus P$. Let $(S_m)_1$ be the set of the first $k$ elements in the marking $S_m$. Then the following holds true: For $\gamma_1, \gamma_2$ elements in $G \wr P$ of the form $\tau^{-1} \sigma \tau$, $\sigma \in (S_m)_1$, $\tau \in P$, if $\gamma_1, \gamma_2$ and $\gamma_1 \gamma_2$ are all contained in the ball $B_{Cay(D(P;S_m)}(e_{G\wr P}, m)$ of radius $m$, then $\gamma_1$ and $\gamma_2$ commute. By a similar reasoning to one in the proof of [MS13] Lemma 5.1), we conclude that as $m \to \infty$,

$$(G \wr P; S_m) \xrightarrow{\text{Cay}} (C_1 \times C_2 \times \cdots \times C_k) \wr P,$$

with a suitable marking of the Cayley limit group. \hfill \square

Since the constant sequence of $G \wr P$ with a fixed standard marking converge to itself, Lemma 6.4 can be utilized as a source of producing two systems of different markings of a group that produce Cayley limit groups of quite different nature.

Note that the construction above will be moreover used to produce two systems of different markings of a sequence of finite groups with the same feature as one above, in the following way: Let $G_\infty$ and $P_\infty$ be (infinite) LEF marked groups and take $(G_m)_{m \in \mathbb{N}}$ and $(P_m)_{m \in \mathbb{N}}$ LEF approximations; recall Definition 2.2, respectively, of them.

For $m \in \mathbb{N} \cup \{\infty\}$, let $G_m$ and $P_m$, respectively, be the underlying group of $G_\infty$ and $P_\infty$. Then in the terminology as in Subsection 6.1, the standard marking $G_m \wr P_m$ of $G_m \wr P_m$ converges to $G_\infty \wr P_\infty$ in the Cayley topology. On the other hand, by Lemma 6.4, $G_\infty \wr P_\infty$ admits a system of markings with respect to which the resulting marked groups converge to one with underlying group $(C_1 \times \cdots \times C_k) \wr P_\infty$.

Set this sequence of marked group as $((G_m \wr P_m; S_m))_{m \in \mathbb{N}}$. Then, for each $m \in \mathbb{N}$, there exists a marking $S_m$ of $G_m \wr P_m$ such that $(G_m \wr P_m; S_m)$ is sufficiently close to $(G_\infty \wr P_\infty; S_m^{(\infty)})$, more precisely, $\lim_{m \to \infty} R_m = +\infty$ holds for $R_m$ that is taken for each $m \in \mathbb{N}$ as in (\star) in Lemma 2.1. Then, with respect to this system of
corresponding marking of Proof. Fix (Lemma 6.5. Let $G$ be a LEF group. Let $(G_m)_{m \in \mathbb{N}}$ be a sequence of finite groups that is obtained from the underlying groups of a LEF approximation of $G$ (with a fixed marking). Then, there exist two different systems of markings $(S_m)_m$ and $(T_m)_m$ of $(G_m \wr (\mathbb{Z}/m\mathbb{Z}))_{m \in \mathbb{N} \geq 3}$ such that

- The sequence $(G_m \wr (\mathbb{Z}/m\mathbb{Z}); S_m)_{m \in \mathbb{N} \geq 3}$ converges in the Cayley topology to a solvable marked group.
- The sequence $(G_m \wr (\mathbb{Z}/m\mathbb{Z}); T_m)_{m \in \mathbb{N} \geq 4}$ converges in the Cayley topology to $G \wr \mathbb{Z}$ with a suitable marking.

Proof. Fix $(g_1, \ldots, g_k)$ a marking of $G$. For every $m \in \mathbb{N} \geq 3$, let $(g_1^{(m)}), \ldots, g_k^{(m)})$ be the corresponding marking of $G_m$ in the LEF approximation. Note that $((\mathbb{Z}/m\mathbb{Z}; 1))_{m \in \mathbb{N} \geq 3}$ converges in the Cayley topology; here we can take $R_m = \lfloor (m - 1)/2 \rfloor$ as in (•) in Lemma 2.1. For every $m \in \mathbb{N} \geq 3$, set

$$r_m = \min \{ R_m, \left\lceil \frac{\text{diam}(\text{CayD}(\mathbb{Z}/m\mathbb{Z}); 1)}{4k} \right\rceil \}.$$ 

Then, $\lim_{m \to \infty} r_m = +\infty$. There exist $0 = x_1^{(m)}, x_2^{(m)}, \ldots, x_k^{(m)} \in \mathbb{Z}/m\mathbb{Z}$ such that $B_{\text{CayD}(\mathbb{Z}/m\mathbb{Z}; 1)}(x_j^{(m)}, r_m), j \in [k]$, are mutually disjoint. Finally, define two systems $(S_m)_m$ and $(T_m)_m$ of markings of $(G_m \wr (\mathbb{Z}/m\mathbb{Z}))_{m \in \mathbb{N} \geq 3}$ by

$$S_m = ((g_1^{(m)} \delta_{x_1^{(m)}}, e_{\mathbb{Z}/m\mathbb{Z}}), (g_2^{(m)} \delta_{x_2^{(m)}}, e_{\mathbb{Z}/m\mathbb{Z}}), \ldots, (g_k^{(m)} \delta_{x_k^{(m)}}, e_{\mathbb{Z}/m\mathbb{Z}}), (e, 1)),$$

$$T_m = ((g_1^{(m)} \delta_{x_1^{(m)}}, e_{\mathbb{Z}/m\mathbb{Z}}), (g_2^{(m)} \delta_{x_2^{(m)}}, e_{\mathbb{Z}/m\mathbb{Z}}), \ldots, (g_k^{(m)} \delta_{x_k^{(m)}}, e_{\mathbb{Z}/m\mathbb{Z}}), (e, 1)),$$

where $e$ means the group unit of $\bigoplus_{\mathbb{Z}/m\mathbb{Z}} G_m$. (Hence $T_m$ is the standard marking of $G_m \wr (\mathbb{Z}/m\mathbb{Z})$.) Then, we have that

$$(G_m \wr (\mathbb{Z}/m\mathbb{Z}); S_m) \xrightarrow{\text{Cay}} (C_1 \times C_2 \times \cdots \times C_k) \wr \mathbb{Z},$$

$$(G_m \wr (\mathbb{Z}/m\mathbb{Z}); T_m) \xrightarrow{\text{Cay}} G \wr \mathbb{Z},$$

where for every $j \in [k]$, $C_j$ is the cyclic group of the same order as for $g_j$.

Remark 6.6. In Lemma 6.5, if $G$ above is generated by torsion elements, then we may have the Cayley limit group in the former assertion as a solvable group with asymptotic dimension 1.

For convenience of the readers, we summarize our arguments above.
Corollary 6.7. Let $G$ be an infinite LEF group and $P$ be an infinite amenable LEF group. Let $(G_m)_{m \in \mathbb{N}}$ and $(P_m)_{m \in \mathbb{N}}$ be sequences of underlying groups of LEF approximations, respectively, of $G$ and $P$ (with some markings). Let $\mathcal{P}$ be a group property that satisfies the first conditions of ones in Subsection 6.1. Assume that $G$ satisfies $\mathcal{P}$.

Then there exist two systems of markings $(S_m)_{m \in \mathbb{N}}$ and $(T_m)_{m \in \mathbb{N}}$,

\[
S_m = (s_1^{(m)}, s_2^{(m)}, \ldots, s_d^{(m)}),
T_m = (t_1^{(m)}, t_2^{(m)}, \ldots, t_d^{(m)})
\]

of $(G_m \wr P_m)_{m \in \mathbb{N}}$ of fixed size $d$ such that

- The sequence $((G_m \wr P_m; S_m))_{m \in \mathbb{N}}$ converges in the Cayley topology to a marked group whose underlying group is of the form $(C_1 \times \cdots \times C_k) \wr P$, where $C_j$, $j \in [k]$, are cyclic. In particular, this group is amenable.
- The sequence $((G_m \wr P_m; T_m))_{m \in \mathbb{N}}$ converges in the Cayley topology to a marked group whose underlying group is $G \wr P$. In particular, this group has $\mathcal{P}$.

Moreover, for every $m \in \mathbb{N}$ and for each $i \in [d]$, there exists $h_i(= h_{m,i}) \in G_m \wr P_m$ such that $t_i^{(m)} = h_i^{-1}s_i^{(m)}h_i$ holds true.

Proof. All but the last assertion follow from the arguments above. The last assertion is by our construction; compare with it for the case $P = \mathbb{Z}$ and $P_m = \mathbb{Z}/m\mathbb{Z}$ in Lemma 6.5. \hfill \Box

Remark 6.8. K. W. Gruenberg [Gru57] showed that a wreath product $G \wr H$ with an infinite $H$ is never RF unless $G$ is abelian. Hence, our construction as in Corollary 6.7 may be available only after we extend our framework from RF approximations to LEF ones. In addition, if $G$ is not abelian, then the Cayley convergence of $(G_m \wr P_m; S_m)$ to the amenable marked group $(C_1 \times \cdots \times C_k) \wr P$ above is a LEF approximation, but not an RF one. This is because $C_1 \times \cdots \times C_k$ is abelian but $G_m$ for large $m$ is not.

6.3. A gadget to encode information into symmetric groups. In this subsection, we recall our construction in [MS13, Remark 5.3] in the Part I paper, which allows us to encode all information into symmetric groups. Recall our notation of symmetric groups from Introduction. For simplicity, we only consider the case of LEF approximations: Suppose that $(G_m = (G_m, t_1^{(m)}, \ldots, t_k^{(m)}))_{m \in \mathbb{N}}$ is a LEF approximation of $G_\infty = (G_\infty, t_1^{(\infty)}, \ldots, t_k^{(\infty)})$. Without loss of generality, we may assume that $t_j^{(m)} \neq e_{G_m}$ for every $m \in \mathbb{N} \cup \{\infty\}$ and every $j \in [k]$.

For a (at most countable) group $G$ and for $\gamma \in G \setminus \{e_G\}$, define two elements $\chi_\gamma$ and $\theta_\gamma$ in $\mathfrak{S}(G)$ as

\[
\chi_\gamma = (\text{the transposition on } \{e_G, \gamma\}),
\theta_\gamma = (\text{the permutation on } G \text{ produced by the right-multiplication of } \gamma).
\]

It is then straightforward to see that for every $m \in \mathbb{N}$, $(\chi_1^{(m)}, \ldots, \chi_k^{(m)}, \theta_1^{(m)}, \ldots, \theta_k^{(m)})$ is a $2k$-marking of $\mathfrak{S}(G_m)$. Indeed, for every $\gamma \in G_m \setminus \{e_{G_m}\}$, the transposition on
\( \{e_{G_m}, \gamma\} \) may be written as some product of these \( 2k \) elements. Thus, we obtain a Cayley convergent sequence of groups in \( G(2k) \),

\[
(\mathcal{S}(G_m); \chi_{i_1}^{(m)}, \ldots, \chi_{i_k}^{(m)}, \theta_{i_1}^{(m)}, \ldots, \theta_{i_k}^{(m)})
\]

\[
\xrightarrow{\text{Cay}} (\mathcal{S}_{<\aleph_0}(G_{\infty}) \rtimes G_{\infty}; \chi_{i_1}^{(\infty)}, \ldots, \chi_{i_k}^{(\infty)}, \theta_{i_1}^{(\infty)}, \ldots, \theta_{i_k}^{(\infty)}).
\]

Recall from Subsection 6.1 that \( G_{\infty} \) acts on \( \mathcal{S}_{<\aleph_0}(G_{\infty}) \) by permutations given by right multiplications.

This construction yields the following lemma.

**Lemma 6.9.** Let \( G \) be an infinite LEF group, and \( (G_m)_{m \in \mathbb{N}} \) be a sequence of underlying groups of a LEF approximation of \( G \) (with some markings). Let \( \mathcal{P} \) be a group property. Assume that \( \mathcal{P} \) satisfies the second condition of the two conditions as in Subsection 6.1.

Then, there exists a sequence \( (l_m)_{m \in \mathbb{N}} \) of integers at least 2 with \( \lim_{m \to \infty} l_m = \infty \) and a sequence of markings \( (\tilde{S}_m)_{m \in \mathbb{N}} \) of \( (\mathcal{S}(l_m))_{m \in \mathbb{N}} \) such that \( (\mathcal{S}(l_m)); \tilde{S}_m)_{m \in \mathbb{N}} \) converges in the Cayley topology to a marked group whose underlying group is isomorphic to \( \mathcal{S}_{<\aleph_0}(G) \rtimes G \). In particular, the Cayley limit group has \( \mathcal{P} \).

**Proof.** For every \( m \in \mathbb{N} \), set \( l_m = \sharp(G_m) \) and identify \( \mathcal{S}(l_m) \) with \( \mathcal{S}(G_m) \). Then the construction of markings of symmetric groups indicated in this subsection ends our proof. \( \square \)

**Corollary 6.10.** Let \( G \) and \( (G_m)_{m \in \mathbb{N}} \) be as in Corollary 6.7. Let \( \mathcal{P} \) be a group property that satisfies the two conditions as in Subsection 6.1. Assume that \( G \) satisfies \( \mathcal{P} \).

Then there exist a sequence \( (l_m)_{m \in \mathbb{N}} \) of integers at least 2 such that \( \lim_{m \to \infty} l_m = \infty \) and two systems of markings \( (\tilde{S}_m)_{m \in \mathbb{N}} \) and \( (\tilde{T}_m)_{m \in \mathbb{N}} \),

\[
\tilde{S}_m = (\tilde{s}_1^{(m)}, \tilde{s}_2^{(m)}, \ldots, \tilde{s}_{d'}^{(m)}),
\tilde{T}_m = (\tilde{t}_1^{(m)}, \tilde{t}_2^{(m)}, \ldots, \tilde{t}_{d'}^{(m)})
\]

of \( (\mathcal{S}(l_m))_{m \in \mathbb{N}} \) of fixed size \( d' \) such that

- The sequence \( ((\mathcal{S}(l_m)); \tilde{S}_m)_{m \in \mathbb{N}} \) converges in the Cayley topology to an amenable marked group,
- The sequence \( ((\mathcal{S}(l_m)); \tilde{T}_m)_{m \in \mathbb{N}} \) converges in the Cayley topology to a marked group whose underlying group has \( \mathcal{P} \).

Moreover, for every \( m \in \mathbb{N} \) and for each \( i \in [d'] \), there exists \( \tilde{h}_i (= \tilde{h}_{m,i}) \in \mathcal{S}(l_m) \) such that \( \tilde{t}_i^{(m)} = \tilde{h}_i^{-1} \tilde{s}_i^{(m)} \tilde{h}_i \) holds true.

**Proof.** Take an infinite amenable LEF group \( P \). Combine Corollary 6.7 and Lemma 6.9. \( \square \)

7. Examples
7.1. **Special linear groups.** Here we discuss coarse properties of $X', Y', V', W'$ as in Example 1.5. In our Part I paper [MS13, Remark 5.9], we observed that

$$(G_m; S_m) \xrightarrow{\text{Cay}} N_>(\mathbb{Z}, \mathbb{F}_p[t]) \rtimes \mathbb{Z}, \quad (G_m; T_m) \xrightarrow{\text{Cay}} \text{SL}(\mathbb{Z}, \mathbb{F}_p[t]) \rtimes \mathbb{Z},$$

with respectively suitable markings of the Cayley limit groups. Here for a unital commutative ring $A$ (associative), the group $\text{SL}(\mathbb{Z}, A)$ denotes the union of $\text{SL}(K, A) = \{ g \in \text{Mat}_{K \times K}(A) : \det(g) = 1 \}$ over all finite non-empty sets $K \subseteq \mathbb{Z}$ (via the natural inclusion $\text{SL}(K, A) \hookrightarrow \text{SL}(\mathbb{Z}, A)$). Similarly, $N_>(\mathbb{Z}, A)$ denotes the union of $N_>(K, A) = \{ g \in \text{Mat}_{K \times K}(A) : (g)_{i,i} = 1 \text{ for all } i \in K, \ (g)_{i,j} = 0 \text{ for all } i > j, i, j \in K \}$ over all finite non-empty sets $K \subseteq \mathbb{Z}$. Here $>$ is the standard total order on $\mathbb{Z}$. The actions of $\mathbb{Z}$ in the semi-direct products above are given by the right translation of $\mathbb{Z}$ on the coordinate set $\mathbb{Z}$. In the Part I paper [MS13, Remark 5.9], we deduced property $A$ for $\mathcal{X}'$ and $\mathcal{V}'$ by amenability of the Cayley limit groups $N_>(\mathbb{Z}, \mathbb{F}_p[t]) \rtimes \mathbb{Z}$ and $N_>(\mathbb{Z}, \mathbb{Z}) \rtimes \mathbb{Z}$.

**Definition 7.1.** Let $\mathcal{M}$ be a non-empty class of metric spaces. We say that a group $G$ has property $(F, \mathcal{M})$ if for every $M \in \mathcal{M}$, every action $\alpha : M \curvearrowright G$ by isometries admits a global fixed point.

**Remark 7.2.** If $\mathcal{M} = \mathcal{E}$ only consists of Banach spaces, then we assume $\alpha$ to be moreover affine. However, we do not go in details on this issue; recall Remark 2.7.

The following are showed by several researchers.

**Theorem 7.3.** (1) (V. Lafforgue [Laf08, Laf09]) For every prime $p$ and for every $n \in \mathbb{N}_{\geq 3}$, the group $\text{SL}(n, \mathbb{F}_p[t])$ has property $(F_{\text{type} > 1})$.

(2) (Izeki–Nayatani [IN05]) For every prime $p$ and for every $n \in \mathbb{N}_{\geq 3}$, the group $\text{SL}(n, \mathbb{F}_p[t])$ has property $(F_{\text{CAT}(0) < 1})$.

(3) (de Laat–Mimura–de la Salle [LMS16]) For every $E \in \mathcal{B}_{3 < 1/2}$, there exists $N_E \in \mathbb{N}_{\geq 3}$ such that for every $n \in \mathbb{N}_{\geq N_E}$, the group $\text{SL}(n, \mathbb{Z})$ has property $(F_E)$.

Indeed, for (2), exactly the same estimate of the first positive Laplace eigenvalue $\lambda_1$ for a link graph as one for a uniform lattices in $\text{SL}(n, \mathbb{Q}_p)$, which is given in [IN05 Section 6, Example 1], applies to that for a uniform lattices in $\text{SL}(n, \mathbb{F}_p((t^{-1})))$; the estimate gives that $\lambda_1 = 1 - (\sqrt{p}/(p + 1))$, where $\mathbb{F}_p((t^{-1}))$ denotes the (local) field of formal Laurent series with indeterminate $t^{-1}$ over $\mathbb{F}_p$. This is because local information is the same for buildings associated with $\text{PGL}(n, \mathbb{Q}_p)$ and for those associated with $\text{PGL}(n, \mathbb{F}_p((t^{-1})))$. For every prime $p$, the estimate above of $\lambda_1$ is strictly bigger than $1/2$. Then, by [IN05 Theorem 1.1], every uniform lattice in $\text{SL}(n, \mathbb{F}_p((t^{-1})))$ has property $(F_{\text{CAT}(0) < 0})$. Even though $\text{SL}(n, \mathbb{F}_p[t])$ is a non-uniform lattice in $\text{SL}(n, \mathbb{F}_p((t^{-1})))$, we obtain the same conclusion as in (2) through $L_2$-induction process; see [BFGM07 Section 8].

**Remark 7.4.** On (2) of Theorem 7.3, with the aid of [IN05 Proposition 6.3], the following strengthening holds true: For every prime $p$ and for every $n \in \mathbb{N}_{\geq 3}$, the
group $\text{SL}(n, \mathbb{F}_p[t])$ has property $(F_M)$, where $M = \text{CAT}(0, \delta(p))$ and $\delta(p)$ is described above Corollary 1.6. The key here is for every $\delta_0 < \delta(p)$, it holds that

$$(1 - \delta_0) \left(1 - \frac{\sqrt{p}}{p + 1}\right) > \frac{1}{2}.$$ 

Proof of Corollary 1.6. To show (1), observe that $\text{SL}(\mathbb{Z}, \mathbb{F}_p[t]) \rtimes \mathbb{Z}, \mathbb{N} > (\mathbb{Z}, \mathbb{Z}) \rtimes \mathbb{Z}, \text{SL}(\mathbb{Z}, \mathbb{Z}) \rtimes \mathbb{Z}$ have asymptotic dimensions $\infty$. Then combine it with Proposition 1.4 and Remark 2.22. Items (2) and (3) follow from (i),(1) of Theorem A (and Corollary B) and Theorem 7.3, together with Remark 7.4; compare with discussions in Examples 2.18 and 2.19.

7.2. Three markings one of whose limit is amenable but the others do not have property A (are non-exact). We prove Theorem C. The main ingredient is a remarkable result by Osajda [Osa16] of the existence of a (finitely generated) RF group without property A. Before the aforementioned paper of Osajda, it follows from the argument of G. Arzhantseva and Osajda [AO14] that a LEF group without property A exists. What is needed in what follows is the LEF property.

Remark 7.5. Osajda pointed out to the authors that although it is implicit in his paper, the resulting group (RF but without property A) in [Osa16] is furthermore a-T-menable. To see this, he used a method developed in [Osa14] to transfer wall structures on the finite presented graphical small cancellation groups in his construction at all finitary stages to that on the infinitely presented limit group. See also [AO14]. We employ this a-T-menability in our proof of Theorem C.

Here we state the precise assertion of (i) of Theorem C.

Proposition 7.6. There exist a sequence of finite groups $(G_n)_{n \in \mathbb{N}}$ and $d \in \mathbb{N}$ such that the following holds true: For every prime $p$ and for every sequence $(l_n)_{n \in \mathbb{N}}$ of integers at least 2 such that $\lim_{n \to \infty} l_n = \infty$, there exist three systems $(S_n)_n$, $(T_n)_n$ and $(U_n)_n$ of $d$-markings

$$S_n = (s_1^{(n)}, s_2^{(n)}, \ldots, s_d^{(n)}),$$

$$T_n = (t_1^{(n)}, t_2^{(n)}, \ldots, t_d^{(n)}),$$

$$U_n = (u_1^{(n)}, u_2^{(n)}, \ldots, u_d^{(n)}),$$

of $(H_{n,p}(= H_{n,p,(l_n)_{n}}) = G_n \rtimes \text{SL}(2n + 3, \mathbb{F}_p))_{n \in \mathbb{N}}$, such that the following hold true:

- For every $n \in \mathbb{N}$ and for every $i \in [d]$, there exist $h_i = h_{n,p,i} \in H_{n,p}$ and $k_i = k_{n,p,i} \in H_{n,p}$ such that

  $$h_i^{-1} s_i^{(n)} h_i = t_i^{(n)} \quad \text{and} \quad k_i^{-1} s_i^{(n)} k_i = u_i^{(n)}.$$  

- The sequence $((H_{n,p}; S_n))_{n \in \mathbb{N}}$ converges in the Cayley topology to an amenable group.

- The sequence $((H_{n,p}; T_n))_{n \in \mathbb{N}}$ converges in the Cayley topology to a group without property A, but the Cayley limit group is a-T-menable.
• The sequence \(((\mathcal{H}_{n,p} U_n))_{n \in \mathbb{N}}\) converges in the Cayley topology to a group without property A. Moreover, the Cayley limit group is not a-\(\mathcal{M}\)-menable for \(\mathcal{M} = \mathcal{B}_{\text{type} > 1}\) or \(\mathcal{M} = \mathcal{CAT}(0)_{< \delta(p)}\).

An outline of our construction in Proposition 7.6 goes as follows: We combine the construction in Subsection 6.1 (Corollary 6.7) with our examples as in Subsection 7.1 (Example 7.5),

\[
(\text{SL}(m, \mathbb{F}_{p^n}); \sigma^{(m)}, \nu^{(m)}, \tau^{(m)}) \xrightarrow{\text{Cay}} N_{>}(\mathbb{Z}, \mathbb{F}_{p^n}) \rtimes \mathbb{Z},
\]

\[
(\text{SL}(m, \mathbb{F}_{p^n}); \sigma^{(m)}, \sigma^{(m)}_2, \nu^{(m)}, \tau^{(m)}) \xrightarrow{\text{Cay}} \text{SL}(\mathbb{Z}, \mathbb{F}_{p^n}) \rtimes \mathbb{Z},
\]

with respect to suitable markings of the Cayley limit groups. Here recall that the former limit group \(N_{>}(\mathbb{Z}, \mathbb{F}_{p^n}) \rtimes \mathbb{Z}\) is amenable, whereas the latter \(\text{SL}(\mathbb{Z}, \mathbb{F}_{p^n}) \rtimes \mathbb{Z}\) contains a copy of \(\text{SL}(3, \mathbb{F}_{p^n})\), which has property (F\(_{\mathcal{M}}\)) for \(\mathcal{M} = \mathcal{B}_{\text{type} > 1}\) and \(\mathcal{M} = \mathcal{CAT}(0)_{< \delta(p)}\).

For the sake of completeness, we exhibit our proof in a more detailed way; compare with arguments in Lemma 6.5 and Corollary 6.7.

Proof of Proposition 7.6 Let \(G\) be the (finitely generated) RF group without property A constructed in \([\text{Os16}])\, and \(S = (g_1, \ldots, g_k)\) be a \(k\)-marking of \(G\). Take \((G_n)_{n \in \mathbb{N}}\) an RF approximation of \((G; S)\) (in fact, what we need here in principle are a LEF group without property A and a LEF approximation of it; see \([\text{AO14}])\). For every \(n \in \mathbb{N}\), write \(G_n = (G_n; g_1, \ldots, g_k)\). Recall from Remark 7.5 that this \(G\) is a-T-menable.

Recall two systems \((\sigma^{(m)}_n, \nu^{(m)}_n, \tau^{(m)}_n)_{m \in \mathbb{Z}, n \geq 1}\) and \((\sigma^{(m)}_n, \sigma^{(m)}_2, \nu^{(m)}_n, \tau^{(m)}_n)_{m \in \mathbb{Z}, n \geq 1}\) of markings of \((\text{SL}(m, \mathbb{F}_{p^n}))_{n \in \mathbb{N}}\) from (1) of Example 7.5. Set \(m = 2n + 3\), and rewrite \(n_m\) and \(\sigma^{(m)}_n, \sigma^{(m)}_2, \nu^{(m)}_n, \tau^{(m)}_n\), respectively, as \(l_n\) and \(\sigma_n, \sigma'_n, \nu_n, \tau_n\). Hence, we have two markings \((\sigma_n, \nu_n, \tau_n)\) and \((\sigma_n, \sigma'_n, \nu_n, \tau_n)\) of \(\text{SL}(2n + 3, \mathbb{F}_{p^n})\).

Let \(H_{n,p} = G_n \rtimes \text{SL}(2n + 3, \mathbb{F}_{p^n})\). Let \(d = k + 4\). Then,

\[
(\text{SL}(2n + 3, \mathbb{F}_{p^n}); \sigma_n, \nu_n, \tau_n) \xrightarrow{\text{Cay}} N_{>}(\mathbb{Z}, \mathbb{F}_{p^n}) \rtimes \mathbb{Z},
\]

with the suitable marking of \(N_{>}(\mathbb{Z}, \mathbb{F}_{p^n}) \rtimes \mathbb{Z}\). For each \(n \in \mathbb{N}\), take \(R_n \in \mathbb{N}\) as in (\(\ast\)) in Lemma 2.1 associated with the convergence above. Let

\[
r_n = \min\{R_n, \left[\frac{\text{diam} (\text{CayD}(\text{SL}(2n + 3, \mathbb{F}_{p^n}); \sigma_n, \nu_n, \tau_n))}{4k}\right]\}.
\]

Then \(\lim_{m \to \infty} r_n = +\infty\). By definition of \(r_n\), for each \(n \in \mathbb{N}\), there exists \(x_1^{(n)} = \epsilon_{\text{SL}(2n + 3, \mathbb{F}_{p^n})}, x_2^{(n)} = \epsilon_{\text{SL}(2n + 3, \mathbb{F}_{p^n})}, \ldots, x_k^{(n)} = \epsilon_{\text{SL}(2n + 3, \mathbb{F}_{p^n})}\) such that the \(r_n\)-balls in the Cayley diagram \(\text{CayD}(\text{SL}(2n + 3, \mathbb{F}_{p^n}); \sigma_n, \nu_n, \tau_n)\) centered at \(x_j^{(n)}\), \(j \in [k]\), are mutually disjoint. Finally, for every \(n \in \mathbb{N}\), set a marking \(S_n\) of \(H_{n,p}\) as

\[
S_n = ((g_1^{(n)}_1 \delta_{\epsilon_{\text{SL}(2n + 3, \mathbb{F}_{p^n})}}, e_{\text{SL}(2n + 3, \mathbb{F}_{p^n})}), (g_2^{(n)} \delta_{x_2^{(n)}}, e_{\text{SL}(2n + 3, \mathbb{F}_{p^n})}), \ldots, (g_k^{(n)} \delta_{x_k^{(n)}}, e_{\text{SL}(2n + 3, \mathbb{F}_{p^n})})),
\]

\[
(e, \sigma_n), (e, (\sigma_n)^{-1}), (e, \nu_n), (e, \tau_n)),
\]
where \( e \) is the group unit of \( \bigoplus_{\SL(2n+3,F_{p^n})} G_n \). (The \((e, (\sigma_n)^{-1})\) above is redundant as a marking; this element is added only in order to meet the first condition of the proposition.) Compare with the proof of Lemma 6.3.

For the other two markings \((T_n)_n\) and \((U_n)_n\), without employing \( x_1^{(n)}, \ldots, x_k^{(n)} \), we simply set

\[
T_n = ( (g_1^{(n)} \delta_{\SL(2n+1,F_{p^n})}, e_{\SL(2n+3,F_{p^n})}), \ldots, (g_k^{(n)} \delta_{\SL(2n+3,F_{p^n})}, e_{\SL(2n+3,F_{p^n})}), (e, (\sigma_n)^{-1}), (e, v_n), (e, \tau_n) ),
\]

\[
U_n = ( (g_1^{(n)} \delta_{\SL(2n+1,F_{p^n})}, e_{\SL(2n+3,F_{p^n})}), \ldots, (g_k^{(n)} \delta_{\SL(2n+3,F_{p^n})}, e_{\SL(2n+3,F_{p^n})}), (e, (\sigma'_n)^{-1}), (e, v_n), (e, \tau_n) ).
\]

Then as \( n \to \infty \), respectively with suitable markings of the Cayley limit groups,

\[
(H_{n,p}; S_n) \xrightarrow{\text{Cay}} (C_1 \times C_2 \times \cdots \times C_k) \wr (N_>(\Z, F_p[i]) \rtimes \Z),
\]

\[
(H_{n,p}; T_n) \xrightarrow{\text{Cay}} G \wr (N_>(\Z, F_p[i]) \rtimes \Z),
\]

\[
(H_{n,p}; U_n) \xrightarrow{\text{Cay}} G \wr (\SL(\Z, F_p[i]) \rtimes \Z),
\]

where \( C_1, \ldots, C_k \) are as in Lemma 6.4. By Remark 7.5 Theorem 7.3 and Remark 7.4, we have the second, third and fourth assertions; see also arguments in Example 6.1. The first assertion is by construction; observe that \((\sigma_n)^{-1}\) and \((\sigma'_n)\) are conjugate in \( \SL(2n+3,F_{p^n}) \). It ends our proof. \( \square \)

Secondly, we transfer this construction to the framework of symmetric groups. The precise statement of \((ii)\) of Theorem C is as follows.

**Proposition 7.7.** There exists \( d' \in \N \) with the following condition. For every prime \( p \), there exist a sequence \((h_{n,p})_{n \in \N}\) of integers at least 2 such that \( \lim_{n \to \infty} h_{n,p} = \infty \) that satisfies the following: There exist three systems \((\tilde{S}_n)_n\), \((\tilde{T}_n)_n\) and \((\tilde{U}_n)_n\) of \( d' \)-markings

\[
\tilde{S}_n = (\tilde{s}_1^{(n)}, \ldots, \tilde{s}_{d'}^{(n)}), \quad \tilde{T}_n = (\tilde{t}_1^{(n)}, \ldots, \tilde{t}_{d'}^{(n)}), \quad \tilde{U}_n = (\tilde{u}_1^{(n)}, \ldots, \tilde{u}_{d'}^{(n)}),
\]

of \((\mathcal{G}([h_{n,p}]))_{n \in \N}\) such that the following hold true:

- For every \( n \in \N \) and for every \( i \in [d'] \), there exist \( \tilde{h}_i = \tilde{h}_{n,p,i} \in \mathcal{G}([h_{n,p}]) \) and \( \tilde{k}_i = \tilde{k}_{n,p,i} \in H_{n,p} \) such that

  \[
  \tilde{h}_i^{-1} \tilde{s}_i^{(n)} \tilde{h}_i = \tilde{t}_i^{(n)} \quad \text{and} \quad \tilde{k}_i^{-1} \tilde{s}_i^{(n)} \tilde{k}_i = \tilde{u}_i^{(n)}.
  \]

- The sequence \((\mathcal{G}([h_{n,p}]); \tilde{S}_n)_{n \in \N}\) converges in the Cayley topology to an amenable group.

- The sequence \((\mathcal{G}([h_{n,p}]); \tilde{T}_n)_{n \in \N}\) converges in the Cayley topology to a group without property A, but the Cayley limit group is \( a-T \)-menable.

- The sequence \((\mathcal{G}([h_{n,p}]); \tilde{U}_n)_{n \in \N}\) converges in the Cayley topology to a group without property A. Moreover, the Cayley limit group is not \( a-M \)-menable for \( M = B_{\text{type} > 1} \) or \( M = \mathcal{CAT}(0, <\delta(p)) \).
Proof. Our contraction is built upon that in the proof of Proposition 7.6. Set \( d' = 2d \), and take \( l_n = n \) for \( n \in \mathbb{N} \) in that proof. For every \( n \in \mathbb{N} \), set \( h_{n,p} = \sharp(H_{n,p}) \) and identify \( \mathcal{G}(\{h_{n,p}\}) \) with \( \mathcal{G}(H_{n,p}) \). Transfer all constructions of \((S_n)_n, (T_n)_n \) and \((U_n)_n \) into three systems \((\tilde{S}_n)_n, (\tilde{T}_n)_n \) and \((\tilde{U}_n)_n \) of \( d' \)-markings of \((\mathcal{G}(\{h_{n,p}\}))_n \) in the way indicated in Subsection 6.3. This together with arguments in Example 6.1 will complete our proof. Indeed, employ Theorem 6.2 for the proof of \( a \)-\( T \)-menability of the Cayley limit group of \((\langle \mathcal{G}(\{h_{n,p}\}); \hat{T}_n \rangle)_n \). More precisely, the concerning limit group is of the form \( \mathcal{G}_{<\mathbb{N}_0}(H) \rtimes H \), where

\[
H = G \wr (\mathbb{N}_0, \mathbb{F}_p[t]) \rtimes \mathbb{Z}
\]

and \( G \) is as in the proof of Proposition 7.6. \( \square \)

The proofs of Propisitions 7.6 and 7.7 complete the proof of Theorem C.

7.3. Embedded Banach expanders. In this subsection, we give a definition of embedded Banach expanders.

Definition 7.8. Let \( \mathcal{E} \) be a non-empty class of Banach spaces and fix \( q \in [1, \infty) \). A sequence of finite connected graphs \((\Gamma_m)_m \in \mathbb{N} \) of uniformly bounded degree is said to admit embedded Banach \((\mathcal{E}, q)\)-expanders if there exist a subsequence \((m_n)_n \in \mathbb{N} \) of \((m)_m \) and a sequence of finite connected graphs \((\Lambda_{m_n})_n \in \mathbb{N} \) such that all of the following hold true:

- There exists \( D > 0 \) such that for each \( n \in \mathbb{N} \), there exists an injective map \( \iota_{m_n} : V(\Lambda_{m_n}) \to V(\Gamma_{m_n}) \) between the vertex sets such that the map \( \iota_{m_n} : (V(\Lambda_{m_n}), d_{\Lambda_{m_n}}) \to (V(\Gamma_{m_n}), d_{\Gamma_{m_n}}) \) is \( D \)-Lipschitz.
- There exists \( d \in \mathbb{N}_{\geq 2} \) such that for every \( n \), each vertex of \( \Lambda_{m_n} \) has degree at most \( d \).
- The number \( \sharp(V(\Lambda_{m_n})) \) tends to \( \infty \) as \( n \to \infty \).
- (Poincaré-type inequality) For every \( E \in \mathcal{E} \), there exists \( C_E > 0 \) such that the following holds true: For every \( n \in \mathbb{N} \) and for every map \( f_{m_n} : V(\Lambda_{m_n}) \to E \), it holds that

\[
\frac{1}{\#(V(\Lambda_{m_n}))} \sum_{v \in V(\Lambda_{m_n})} \|f_{m_n}(v) - m(f_{m_n})\|^q \leq C_E \left( \frac{1}{\#(V(\Lambda_{m_n}))} \sum_{e = (v,w) \in E(\Lambda_{m_n})} \|f_{m_n}(v) - f_{m_n}(w)\|^q \right),
\]

where \( m(f_{m_n})_n \) denotes the mean of \( f_{m_n} \):

\[
m(f_{m_n}) = \frac{1}{\#(V(\Lambda_{m_n}))} \left( \sum_{v \in V(\Lambda_{m_n})} f_{m_n}(v) \right) \ (\in E).
\]

The sum on the right-hand side of the inequality above runs over all edges \( e \in E(\Lambda_{m_n}) \) in \( \Lambda_{m_n} \), and for each \( e \in E(\Lambda_{m_n}) \), by writing \( e = (v,w) \) we express that \( e \) connects the vertices \( v \) and \( w \).

We say that \((\Gamma_m)_m \in \mathbb{N} \) is a family of Banach \((\mathcal{E}, q)\)-expanders if we can take \( m_n = m \) and \( \Lambda_{m_n} = \Gamma_m \) (that also means that \( \iota_m = \text{id}_{V(\Gamma_m)} \)) for every \( n \in \mathbb{N} \).
The concept of ordinary expanders is one with \((\mathcal{E}, q) = (\text{Hilbert}, 2)\). It is known from work of Q. Cheng [Che16] that the condition of being Banach \((\mathcal{E}, q)\)-expanders does not depend on the choice of the exponent \(q \in [1, \infty)\).

The following is a variant of the well-known fact asserting that expanders do not admit a coarse embedding into a Hilbert space. For the sake of completeness, we provide a proof; compare with the proof of [NY12, Theorem 5.6.5].

**Proposition 7.9.** Let \(\mathcal{E}\) be a non-empty class of Banach spaces and let \(q \in [1, \infty)\). If a sequence of finite connected graphs \((\Gamma_m)_{m \in \mathbb{N}}\) of uniformly bounded degree admits embedded Banach \((\mathcal{E}, q)\)-expanders \((\Lambda_{mn})_{n \in \mathbb{N}}\), then for the disjoint union \(\bigsqcup_{m \in \mathbb{N}} (\Gamma_m, d_{\Gamma_m})\) does not admit a coarse embedding into \(\mathcal{E}\).

In particular, if \((\Gamma_m)_{m \in \mathbb{N}}\) admits embedded (ordinary) expanders, then \(\bigsqcup_{m \in \mathbb{N}} (\Gamma_m, d_{\Gamma_m})\) does not admit a coarse embedding into a Hilbert space.

**Proof.** Suppose that there exists a coarse embedding \(f : \bigsqcup_{m \in \mathbb{N}} (\Gamma_m, d_{\Gamma_m}) \to E\) with control pair \((\rho, \omega)\). Then for every \(n \in \mathbb{N}\) and for every \(v, w \in V(\Lambda_{mn})\) adjacent in \(\Lambda_{mn}\), it holds that \(\|f(\nu(v)) - f(\nu(w))\| \leq \omega(D)\). By the Poincaré-type inequality in the conditions above, we therefore have that

\[
\frac{1}{\sharp(V(\Lambda_{mn}))} \sum_{v \in V(\Lambda_{mn})} \|f(t_{mn}(v)) - m((f \circ t_{mn})|_{V(\Lambda_{mn})})\|^q \leq C_E d \cdot \omega(D)^q.
\]

Since the right-hand side of the inequality above is independent of \(n\), the images \(f(t_{mn}(V(\Lambda_{mn})))\) must be concentrated around its mean \(m((f \circ t_{mn})|_{V(\Lambda_{mn})})\). It contradicts the properness of \(\rho\) as \(n \to \infty\), because \(t_{mn}\) is injective, \(\sharp(V(\Lambda_{mn})) \to \infty\), and \((\Gamma_m)_m\) is of uniformly bounded degree. \(\Box\)

The proof above works for a more general setting of graphs that admit weakly embedded expanders; see [AT15].

The following is deduced from [Mim15, Theorem A]; compare with [Nao14, Theorem 1.10] and [Oza04, Appendix A].

**Proposition 7.10.** Let \(F\) be a Banach space and let \(q \in [1, \infty)\). Let \(E\) be a Banach space that is sphere equivalent to \(F\), namely, there exists a bijection \(\Phi : S(F) \to S(E)\) between unit spheres such that \(\Phi\) and \(\Phi^{-1}\) are both uniformly continuous. If a sequence of finite connected graphs \((\Gamma_m)_{m \in \mathbb{N}}\) admits embedded Banach \((F, q)\)-expanders, then it admits embedded Banach \((E, q)\)-expanders.

There exists a notion of expanders with target in non-linear metric spaces; see [MN15] and [Nao14]. By combining this with [IN05, Proposition 6.3], we have the following.

**Proposition 7.11.** If a sequence of finite connected graphs of uniformly bounded degree \((\Gamma_m)_{m \in \mathbb{N}}\) admits embedded expanders, then \(\bigsqcup_{m \in \mathbb{N}} \Gamma_m\) does not admit a coarse embedding into \(\text{CAT}(0)_{<1}\).

Mendel and Naor [MN15] constructed a complete \(\text{CAT}(0)\) space \(M\) and a sequence of graphs \((\Gamma_m)_m\) such that \((\Gamma_m)_m\) forms an expander family with respect to \(M\), but that expanders coming from random graphs are not expanders with respect to \(M\). This \(M\) must have the Izeki–Nayatani invariant 1.
7.4. **Uniformity is not automatic for a-M-menability.** For a non-empty class of metric spaces, we say that a non-empty set $K \subseteq G(k)$ is pointwise $a$-$M$-menable if every $G \in K$ is $a$-$M$-menable. Concerning amenability and property (T), uniformity is automatic for Cayley-compact subsets, namely, the pointwise property automatically implies the uniform one; see [MSI13 Proposition 3.4] and [MOSS15 Proposition 5.1]. In contrast, concerning $a$-$M$-menability, uniformity is not automatic, as the example below indicates.

**Example 7.12.** The classical ping-pong argument shows that $F_2 \cong (G_0; \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix})$ as marked groups, where $G_0(\simeq F_2)$ is the group generated by these two elements. For each odd prime $p$, consider mod $p$ reduction. Then $G_0$ maps onto $\text{SL}(2, \mathbb{F}_p)$ and

$$(\text{SL}(2, \mathbb{F}_p); \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \mod p, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \mod p) \xrightarrow{\text{Cay}} F_2,$$

as $p \to \infty$. We write the marked group in the left-hand side as $G_p$. Then $K = \{G_p : p \text{ odd prime}\} \cup \{F_2\}$ is a compact subset in $G(2)$. This set $K$ is pointwise $a$-T-menable, but not uniformly $a$-T-menable. Indeed, for the latter assertion, by work of A. Selberg [Sel65], it follows that $(\text{Cay}(G_p))_p$ forms an expander family; see also [Lub10]. By Proposition 7.9 there does not exist a common pair $(p, \omega)$ that serves as a control pair of all of the $G_p$, $p$ odd primes.

In this example, the obstruction to uniformity is the coarse non-embeddability, not the equivariant one of the sequence. Hence, we are able to utilize this observation to prove Corollary 7.8 as follows.

**Proof of Corollary 7.8** By the way of contradiction. Assume that $\bigcup_{m \in \mathbb{N}} \Gamma_m$ does not admit a coarse embedding into $\mathcal{M}$. Choose an element $\Lambda_\infty$ in the rooted graph boundary $\partial_r(\Lambda_n)_{n \in \mathbb{N}}$; recall the definition from Remark 3.2. Then, for every $m \in \mathbb{N}$, the subsequence $(\Gamma_m \times \Lambda_n)_{n \in \mathbb{N}}$, with changing roots, has $\Gamma_m \times \Lambda_\infty$ as an accumulation point in the space of rooted graphs. Hence by Proposition 3.3 in particular, $(\Gamma_m \times \Lambda_\infty)_{m \in \mathbb{N}}$, must admit equi-coarse embeddings into $\mathcal{M}$. This contradicts the coarse non-embeddability of $\bigcup_{m \in \mathbb{N}} \Gamma_m$ into $\mathcal{M}$. \hfill $\square$  

7.5. **Upper triangular products.** We saw in the previous subsection that by taking the disjoint union $\bigcup_{m,n \in \mathbb{N}} (\Gamma_m \times \Lambda_n)$, we can embed a copy of each $\Gamma_m$ and $\Lambda_n$ (as an isometrically embedded subgraph) in the rooted graph boundary. In what follows, we slightly modify this construction and call the resulting object the *upper triangular product*. We exhibit it in the context of the space of marked groups.

Let $(G_m)_{m \in \mathbb{N}} \subseteq G(k_1)$ and $(H_n)_{n \in \mathbb{N}} \subseteq G(k_2)$. For each $G = (G; s_1, \ldots, s_{k_1})$ and $G = (H; t_1, \ldots, t_{k_2})$, define the *direct product marked group* $G \times H$ by

$$G \times H = (G \times H; (s_1, e_H), \ldots, (s_{k_1}, e_H), (e_G, t_1), \ldots, (e_G, t_{k_2})) \in G(k_1 + k_2)).$$

**Definition 7.13.** The upper triangular product of $\bigcup_{m \in \mathbb{N}} \text{Cay}(G_m)$ and $\bigcup_{n \in \mathbb{N}} \text{Cay}(H_n)$ is defined by

$$\bigcup_{(m,n) \in \mathbb{N} \times \mathbb{N}, m \leq n} \text{Cay}(G_m \times H_n).$$
equipped with the total order given by comparison firstly on \( n \) and secondly on \( m \)
on the index set \( \{(m, n)\} \); namely, \( (0, 0) < (0, 1) < (1, 1) < (0, 2) < (1, 2) < (2, 2) < (0, 3) < \cdots \). If \((G_m)_m\) and \((H_n)_n\) are indexed by sets that are respectively order isomorphic to \((\mathbb{N}, >)\), then we modify the order accordingly.

We write the sequence \((G_m \times H_n)_{(m, n) \in \mathbb{N}^2}, m \leq n\) in \(\mathcal{G}(k_1 + k_2)\), with the enumeration with respect to the order above, identified with that by \(l \in \mathbb{N}\), as \((G_m)_m \sqcup (H_n)_n\).

Note that for the upper triangular product,
\[
\partial_{\text{Cay}}((G_m)_m \sqcup (H_n)_n) = \left( \bigcup_{m \in \mathbb{N}} (G_m) \times \partial_{\text{Cay}}(H_n)_n \right) \cup \left( \bigcup_{G_\infty \in \partial_{\text{Cay}}(G_m)_m} (G_\infty \times \partial_{\text{Cay}}(H_n)_n) \right)
\]
In this way, we can embed (isomorphic and isometric copies of) \((G_m)_m\) in the Cayley boundary (as subgroups of respectively suitable Cayley boundary groups).

**Proof of (i) of Theorem [D].** Take the sequence of marked groups \((G_p)_p\) over odd primes \(p\) as in Example 7.12 and construct the upper triangular product \((H_l)_l \in \mathbb{N} = (G_p)_p \sqcup (G_p)_p\). Set \(\Gamma_l\) as \(\text{Cay}(H_l)\). Since \((\text{Cay}(G_p))_p\) forms an expander family, so does \((\Gamma_l)_{l \in \mathbb{N}}\). The Cayley boundary of that sequence contains an isometric copy of \((\text{Cay}(G_p))_p\); hence by Proposition 1.4 together with Propositions 7.9, 7.10 and 7.11, we confirm the second assertion. To see the third assertion, G. A. Margulis showed that there exists \(c > 0\) such that for all odd prime \(p\),
\[
girth(\text{Cay}(G_p)) \geq c \cdot \text{diam}(\text{Cay}(G_p))
\]
holds, where the *girth* of a connected graph is the length of shortest cycle; see [DSV03, Appendix A]. For such sequence of finite graphs \((\text{Cay}(G_p))_p\), T. Kondo [Kon12] constructed a complete CAT(0) space \(M_0 = M_0((\text{Cay}(G_p))_p)\) such that the disjoint union \(\bigsqcup_p \text{Cay}(G_p)\) embeds biLipschitzly into \(M_0\). Therefore, the disjoint union of \((\Gamma_l)_l\) admits a biLipschitz embedding into \(M = (M_0 \times M_0)_\ell_2\).

### 7.6. Embedded expanders from fixed point property, and exotic examples from symmetric groups
Here we prove (ii) of Theorem [D]. First we prove the following proposition, which may be of its own interest. It may be regarded as a generalization of [MOSS15, Corollary 1.2] of our Part III paper.

**Proposition 7.14.** Let \((G_m = (G_m; s_1^{(m)}, \ldots, s_k^{(m)}))_{m \in \mathbb{N}}\) be a Cayley convergent sequence consisting of finite marked groups and \(G = (G; s_1, \ldots, s_k)\) be the limit. Let \(\mathcal{E}\) be a non-empty class of Banach spaces that satisfies both of the following two conditions:

1. There exists \(q \in [1, \infty)\) such that for every \(E \in \mathcal{E}\), it holds that \(\ell_q(\mathbb{N}, E) \in \mathcal{E}\).
2. The class \(\mathcal{E}\) can be written as a union of subclasses
\[
\mathcal{E} = \bigcup_{\lambda} \mathcal{E}_\lambda
\]
such that each such subclass \(\mathcal{E}_\lambda\) satisfies the following: For every \((E_m)_{m \in \mathbb{N}}\) with \(E_m \in \mathcal{E}_\lambda\) for every \(m\), there exists a non-principal ultrafilter \(\mathcal{U}\) over \(\mathbb{N}\) such that \(\lim_{\mathcal{U}}(E_m, 0) \in \mathcal{E}_\lambda\).
Assume that $G$ contains an infinite subgroup $H$ with property $(F_\mathcal{E})$. Then the sequence of Cayley graphs $(\text{Cay}(G_m))_{m \in \mathbb{N}}$ admits embedded Banach $(\mathcal{E}, q)$-expanders.

By combining this with Proposition 7.13, we deduce that the disjoint union $\bigsqcup_{m \in \mathbb{N}} \text{Cay}(G_m)$ of such a sequence does not admit a coarse embedding into $\mathcal{E}$.

To prove Proposition 7.14, we employ the following three results.

**Lemma 7.15.** Assume that a non-empty class of Banach spaces $\mathcal{E}$ satisfies condition (1) as in Proposition 7.14. Then, if a countable discrete group $H$ satisfies property $(F_\mathcal{E})$, then $H$ is finitely generated.

**Proof.** Generalize the proof of [BdlHV08, Proposition 2.4.1 and Corollary 2.4.2]. □

**Proposition 7.16.** Assume that a non-empty class of Banach spaces $\mathcal{E}$ satisfies condition (1) as in Proposition 7.14. Let $H = (H; T)$ be an infinite marked group such that $H$ has property $(F_\mathcal{E})$. Let $(H_n = (H_n; T_n))_{n \in \mathbb{N}}$ be a sequence of finite marked group quotients (recall the definition from Definition 2.2) such that $\lim_{n \to \infty} \sharp(H_n) = \infty$.

Then, the sequence $(\text{Cay}(H_n))_{n \in \mathbb{N}}$ forms a family of Banach $(\mathcal{E}, q)$-expanders.

**Proof.** By [BFGM07, 3.a], $H$ has property $(T_\mathcal{E})$ in the sense of Bader–Furman–Gelander–Monod. For each $E \in \mathcal{E}$, in particular, $H$ has property $(T_{\ell_q(\mathbb{N}, E)})$. This implies that the $(\tau)$-type constant associated with $(H, \ell_q(\mathbb{N}, E))$, defined in our Part I paper [MS13, Definition 6.6.(2)], is strictly positive. Then, in a similar argument to one in the proof of [MS13, Lemma 6.8] (by replacing the square sums there with $q$-sums), we deduce that $(\text{Cay}(H_n))_{n \in \mathbb{N}}$ satisfies the Poincaré-type inequality as in Definition 7.8. By construction, degrees are bounded by $2k$, and $(\infty >) \sharp(H_n) \to \infty$. □

**Proposition 7.17.** Assume that a non-empty class of Banach spaces $\mathcal{E}$ satisfies condition (2) as in Proposition 7.14 with $\mathcal{E}_\lambda = \mathcal{E}$. Let $H = (H; T)$ be an infinite marked group such that $H$ has property $(F_\mathcal{E})$. Then there exists a finitely presented marked group $\tilde{H}$ such that it has property $(F_\mathcal{E})$ and there exists a marked quotient map $\tilde{H} \twoheadrightarrow H$.

**Proof.** This follows from a well-known Gromov–Schoen argument; see the survey [Sta09] of Stalder. More precisely, [Sta09, Theorem 1.5] implies that the subset of all marked groups in $G(k)$ with property $(F_\mathcal{E})$ forms an open subset in the Cayley topology. Here $k = \sharp(T)$. If $H$ itself is finitely presented, then we are done. Otherwise, there exists a Cayley convergent sequence $(\tilde{H}_m)_{m \in \mathbb{N}}$ to $H$ consisting of finitely presented marked groups, constructed by putting relations of $H$ one by one. By the openness property above, there must exist $m \in \mathbb{N}$ such that $\tilde{H}_m$ has property $(F_\mathcal{E})$. This $\tilde{H}_m$ is a desired $\tilde{H}$. □

On Proposition 7.17, the case where $\mathcal{E} = \mathcal{H}$ Hilbert was proved by Shalom [Sha00]; see also [KS97]. In this case, property $(F_{\mathcal{H}})$ (for countable discrete groups) is
equivalent to the celebrated property (T) of D. Kazhdan; see [BilHV08] on property (T), including this equivalence (the Delorme–Guichardet theorem).

Proof of Proposition 7.14. By Lemma 7.13, $H$ is finitely generated. Fix a finite generating set $T = (t_1, \ldots, t_l)$ of $H$. Then, each $t_j$, $j \in [l]$, may be written as a product of elements in $S = (s_1, \ldots, s_k)$; fix such an expressions for each $j \in [l]$. For each $m \in \mathbb{N}$, $t_j^{(m)}$, $j \in [l]$, be the element in $G_m$ constructed by replacing $s_i$ with $s_i^{(m)}$ in that expression for all $i \in [k]$. Let $H_m(\leq G_m)$ be the group generated by these $t_1^{(m)}, \ldots, t_k^{(m)}$. Then for every $m \in \mathbb{N}$, $H_m$ is finite, and

$$(H_m; t_1^{(m)}, \ldots, t_k^{(m)}) \xrightarrow{\text{Cay}} H.$$ 

Now fix $E \in \mathcal{E}$. Then by condition (1) and (2), there exists a subclass $\mathcal{E}_\lambda$ as in (2) of $\mathcal{E}$ that contains $\ell_q(\mathbb{N}, E)$. We apply Proposition 7.17 to $\mathcal{E}_\lambda$ and take finitely presented marked lift $\tilde{H}$ of $H$ with property $(F_\mathcal{E}_\lambda)$. Then by finite presentation of $\tilde{H}$, the set of all marked group quotients of $\tilde{H}$ is an open neighborhood of $H$; recall Remark 2.3. In particular, the sequence $((H_m; t_1^{(m)}, \ldots, t_k^{(m)}))_m$ eventually consists of marked group quotient of $\tilde{H}$. Therefore, Proposition 7.16 applies and $(\Lambda_m)_m = (\text{Cay}(H_m; t_1^{(m)}, \ldots, t_k^{(m)}))_m$ forms a Banach $(E, q)$-expander family. (Strictly speaking, for small $m$, the marked group might not be a marked group quotients of $\tilde{H}$. However, since these are only finitely many, they do not affect the Banach $(E, q)$-expander property.) Because this holds for each $E \in \mathcal{E}$, $(\Lambda_m)_m$ forms a Banach $(\mathcal{E}, q)$-expander family.

Finally we go back to the original graphs $(\Gamma_m)_{m \in \mathbb{N}} = (\text{Cay}(G_m))_{m \in \mathbb{N}}$. First, the vertex set $V(\Lambda_m) = H_m$ injects into $V(\Gamma_m) = G_m$ via $\iota_m : H_m \hookrightarrow G_m$ (as a subgroup $H_m \leq G_m$). Moreover, by construction of $(t_1^{(m)}, \ldots, t_k^{(m)})$, there exists $D > 0$ such that for every $m \in \mathbb{N}$, the map $(\Lambda_m, d_{\Lambda_m}) \to (\Gamma_m, d_{\Gamma_m})$ induced by $\iota_m$ is $D$-Lipschitz. This ends our proof. \hfill \square

Proof of (ii) of Theorem 1.2. We take two sequences of marked groups $((G_m; S_m))_{m \in 2\mathbb{N} + 1 \geq 3}$ and $((G_m; T_m))_{m \in 2\mathbb{N} + 1 \geq 3}$ as in Example 1.5. More precisely, $G_m = \text{SL}(m, \mathbb{F}_{p^m})$, $S_m = (\sigma(m), \nu(m), \tau(m))$ and $T_m = (\sigma^{(m)}, \sigma^{(m)}, \nu^{(m)})$. Let $(H_n)_{n \in \mathbb{N} \geq 3} = ((\mathbb{Z}/n\mathbb{Z}; 1))_n$. Then, take upper triangular products

$$(I_t)_{t \in \mathbb{N}} = ((G_m; S_m))_m \lor (H_n)_n \text{ in } G(4),$$

$$(J_t)_{t \in \mathbb{N}} = ((G_m; T_m))_m \lor (H_n)_n \text{ in } G(5).$$

By construction, concerning Cayley boundaries, we have that

$$\partial_{\text{Cay}}(I_t) = \{(G_m; \sigma^{(m)}, \nu^{(m)}, \tau^{(m)}) : m \in 2\mathbb{N} + 1 \geq 3\} \times \mathbb{Z}$$

$$\cup \{(N, \mathbb{Z}[\mathbb{F}_p[t]]) \times \mathbb{Z}; \sigma^{(\infty)}, \nu^{(\infty)}, \tau^{(\infty)} \times \mathbb{Z}\},$$

$$\partial_{\text{Cay}}(J_t) = \{(G_m; \sigma^{(m)}, \sigma^{(m)}, \nu^{(m)}, \tau^{(m)}) : m \in 2\mathbb{N} + 1 \geq 3\} \times \mathbb{Z}$$

$$\cup \{(\mathbb{S}L(\mathbb{Z}, \mathbb{F}_p[t])) \times \mathbb{Z}; \sigma^{(\infty)}, \sigma^{(\infty)}, \nu^{(\infty)}, \tau^{(\infty)} \times \mathbb{Z}\},$$

for some markings $(\sigma^{(\infty)}, \nu^{(\infty)}, \tau^{(\infty)})$ and $(\sigma^{(\infty)}, \sigma^{(\infty)}, \nu^{(\infty)}, \tau^{(\infty)})$. Here $\mathbb{Z} = (\mathbb{Z}; 1)$. 


Note that for each \(l \in \mathbb{N}\), the underlying groups of \(I_l\) and \(J_l\) are the same; we write it as \(K_l\). The marking of \(I_l\) is of the form \((b_1^{(l)}, b_2^{(l)}, b_3^{(l)}, c^{(l)})\) and the one of \(J_l\) is of the form \((b_1^{(l)}, b_1^{(l)}, b_2^{(l)}, b_3^{(l)}, c^{(l)})\). Here \(b_1, b_1, b_2, b_3\) are associated, respectively, with \(\sigma, \sigma', \nu, \tau, \) and \(c\) corresponds to the generator 1 of \(H_n\).

Finally, we employ the construction as in Subsection 6.3 and transfer these into symmetric groups. More precisely, consider two systems of markings \((\Xi_l)_{l \in \mathbb{N}}\) and \((\Omega_l)_{l \in \mathbb{N}}\) of \((\mathcal{G}(K_l))_{l \in \mathbb{N}}\) by

\[
\Xi_l = (\chi_{b_1^{(l)}}, \chi_{b_2^{(l)}}, \chi_{b_3^{(l)}}, \chi_{c^{(l)}}, \theta_{b_1^{(l)}}, \theta_{b_2^{(l)}}, \theta_{b_3^{(l)}}, \theta_{c^{(l)}}),
\]

\[
\Omega_l = (\chi_{b_1^{(l)}}, \chi_{b_1^{(l)}}, \chi_{b_2^{(l)}}, \chi_{b_3^{(l)}}, \chi_{c^{(l)}}, \theta_{b_1^{(l)}}, \theta_{b_2^{(l)}}, \theta_{b_3^{(l)}}, \theta_{c^{(l)}}), \theta_{b_4^{(l)}});
\]

recall our construction of \(\gamma\) and \(\theta\) from Subsection 6.3.

In what follows, we check assertions of (ii) in Theorem D. Item (1) is by construction. To see (2), all underlying groups appearing in \(\partial_{\text{Cay}}(I_l)\) are

\[
\mathcal{G}_{<N_0}(G_m \times \mathbb{Z}) \rtimes (G_m \times \mathbb{Z}), \quad m \in 2\mathbb{N} + 1_{\geq 3}, \quad \text{and} \quad \mathcal{G}_{<N_0}(\hat{G}_\infty) \rtimes (\hat{G}_\infty),
\]

where \(\hat{G}_\infty = (N_{>0}\mathbb{Z}, \mathbb{F}_p[t]) \rtimes \mathbb{Z}\). Since all of them are amenable, [MS13 Theorem A] implies that the disjoint union \(\bigsqcup_l \partial_{\text{Cay}}(I_l)\) has property A.

Finally, we deal with (3). In a similar argument to one above, we see that the Cayley boundary \(\partial_{\text{Cay}}(J_l)\) contains an isomorphic and isometric copies of \(((G_m; T_m))_{m \in 2\mathbb{N} + 1_{\geq 3}}\) (as subgroups of respectively suitable Cayley boundary groups). Now recall that

\[
(G_m; T_m) \overset{\text{Cay}}{\longrightarrow} \text{SL}(\mathbb{Z}, \mathbb{F}_p[t]) \rtimes \mathbb{Z}
\]

with respect to a suitable marking of the limit, and that the Cayley limit group contains \(\text{SL}(3, \mathbb{F}_p[t])\), which has property \((\mathcal{F}_{\mathcal{B}_{\text{type} \geq 1}})\). Note that the class \(\mathcal{B}_{\text{type} \geq 1}\) fulfills the two conditions in Proposition 7.14. Indeed, to see (2), decompose

\[
\mathcal{B}_{\text{type} \geq 1} = \bigcup_{r \in (1, 2), \ C > 0} \mathcal{B}_{r, C}^{\text{type}}.
\]

Hence by Proposition 7.14 we conclude that \((\text{Cay}(G_m; T_m))_{m \in 2\mathbb{N} + 1_{\geq 3}}\) admits embedded Banach \((\mathcal{B}_{\text{type} \geq 1}, 2)\)-expanders. This with Propositions 7.9, 7.10 and 7.11 imply that \(\partial_{\text{Cay}}(J_l)\) does not admit equi-coarse embeddings into \(\mathcal{M}\), where \(\mathcal{M}\) is either of the two classes as in the assertion of (3). Thus by Proposition 1.4 we complete the proof. Here for every \(l \in \mathbb{N}\), we set \(k_l = \sharp(K_l)\) and identify \(\mathcal{G}([k_l])\) with \(\mathcal{G}(K_l)\). \(\square\)

Remark 7.18. In this specific example above, we do not need to appeal to Proposition 7.17 to obtain a finitely presented lift with property \((\mathcal{F}_{\mathcal{B}_{\text{type} \geq 1}})\). Indeed, it follows from work of H. Behr [Beh98] that \(\text{SL}(n, \mathbb{F}_p[t])\) is finitely presented for every prime \(p\) and for every \(r \in \mathbb{N}_{\geq 1}\), provided that \(n \geq 4\). Thus the Cayley limit group \(\text{SL}(\mathbb{Z}, \mathbb{F}_p[t]) \rtimes \mathbb{Z}\) of our concern in the example above contains a copy of a finitely presented group \(\text{SL}(4, \mathbb{F}_p[t])\) with property \((\mathcal{F}_{\mathcal{B}_{\text{type} \geq 1}})\) as a subgroup.

We make a final remark, which is similar to one in the Part I paper [MS13]: The construction above is “semi-explicit” because in general, there is an issue to have an explicit generator of \(\mathbb{F}_p^\times\). To obtain a fully explicit construction, replace coefficient rings \((\mathbb{F}_p^n)^m\) with explicit other quotient rings of \(\mathbb{F}_p[t]\); for instance take \((\mathbb{F}_p[t]/(t^m - t))^m\), and replace \((t_m \in \mathbb{F}_p^n)^m\) with \((t \in \mathbb{F}_p[t]/(t^m - t))^m\).
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