Caching in Combination Networks: Novel Multicast Message Generation and Delivery by Leveraging the Network Topology

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Abstract—Maddah-Ali and Niesen’s original coded caching scheme for shared-link broadcast networks is now known to be optimal to within a factor two, and has been applied to other types of networks. For practical reasons, this paper considers that a server communicates to cache-aided users through \(H\) intermediate relays. In particular, it focuses on combination networks where each of the \(K = \binom{N}{r}\) users is connected to a distinct \(r\)-subset of relays. By leveraging the symmetric topology of the network, this paper proposes a novel method to general multicast messages and to deliver them to the users. By numerical evaluations, the proposed scheme is shown to reduce the download time compared to the schemes available in the literature. The idea is then extended to decentralized combination networks, more general relay networks, and combination networks with cache-aided relays and users. Also in these cases the proposed scheme outperforms known ones.

I. INTRODUCTION

Caching locally popular content is known to reduce network load from servers to users during peak traffic times. A caching scheme comprises two phases: (i) Placement phase: a server places parts of its library into the users’ caches without knowledge of later demands. If each user directly stores some bits of the files, the placement is called uncoded. Centralized caching systems allow for coordination among users in the placement phase, while decentralized ones do not. (ii) Delivery phase: each user requests one file. According to the users’ demands and cache contents, the server transmits packets to satisfy all the user demands.

In [1] and [2], Maddah-Ali and Niesen proposed caching schemes for centralized (cMAN) and decentralized (dMAN) shared-link networks, respectively, where \(K\) users with cache size MB bits are connected to a central server with N files of B bits through a single error-free shared link. The placement phase is uncoded, while the delivery phase uses linear network coding to deliver carefully designed multicast messages to the users. The optimality of cMAN, in terms of the number of broadcast bits (referred to as load) for the worst-case set of demands, under the constraint of uncoded placement and \(N \geq K\) was shown in [3], and later extended to \(N < K\) and to dMAN with uniform demands in [4]. Uncoded placement is optimal to within a factor 2 for shared-link networks [5].

In practice users and servers may communicate through intermediate relays. The caching problem for general relay networks was firstly considered in [6], where a scheme based on cMAN placement and an interference alignment scheme to transmit cMAN multicast messages was proposed. Several worked followed, such as [7], [8], but since it is hard to characterize the fundamental limits of general relay networks, focus has recently shifted to a symmetric network referred to as combination network [9]. As illustrated in Fig. 1 in a combination network there are \(H\) relays and \(K = \binom{N}{r}\) users with cache size of MB bits, where each user is connected to a different \(r\)-subset of relays, and all links are error-free and orthogonal. The objective is to minimize the download time for the worst-case demands. The available literature mainly follows the two-step separation principle formalized in [8]: (a) cMAN-type uncoded cache placement and multicast message generation, and (b) message delivery that aims to match the network multicast capacity region (i.e., the generation of the multicast messages is independent to the network topology). Examples of such a separation approach are [10], [11], [6]. In [10] two schemes were proposed, one based on routing and the other based on a combination network coding. In [11] we proposed a delivery scheme that leverages the structure of the network and then improved upon it for the case \(M = N/K\) by interference elimination. Another approach for this problem was proposed in [12], [13], where the combination network was split into \(H\) shared-link networks and then cMAN delivery was used in each one. With a coded cache placement based on MDS (maximum separable distance) codes, the scheme in [13] achieves the same performance of [12] but without the constraint in [12] that \(r\) divides \(H\). This work departs from these two lines of work and proposes a novel way to generate multicast messages by leveraging the network topology.

Contributions and Paper Organization: We start by considering centralized combination networks. In Section III we propose a novel delivery scheme which generates multicast messages by leveraging the network topology. Numerical results show that the proposed scheme outperforms existing schemes. We then extend this idea to other classes of relay networks. In Section IV we extend our novel delivery scheme to decentralized combination networks, to general relay networks, and finally to combination networks with both cache-aided relays and users.
B. MAN Caching Schemes in Shared-Link Networks

Since we will use MAN placement in our proposed delivery scheme, we summarize here some known results for the shared-link model (where all the K users are directly connected to the server through an error-free shared-link). For sake of space, we consider here only the case K ≤ N.

Centralized Cache-aided Systems: Let M = \( t \frac{B}{K} \) for some positive integer \( t \in [0 : K] \). In the cMAN placement phase, each file is divided into \( \binom{K}{t} \) non-overlapping sub-files of length \( \frac{B}{\binom{K}{t}} \) bits. The sub-files of \( F_i \) are denoted by \( F_{i,W} \) for \( W \subseteq [K] \) where \( |W| = t \). User \( k \in [K] \) fills its cache as

\[
Z_k = \left( F_{i,W} : k \in W, \; |W| = t, \; i \in [N] \right).
\]

In the delivery phase with demand vector \( d \), for each \( J \subseteq [K] \) where \( |J| = t + 1 \), the multicast messages

\[
W_J := \oplus_{k \in J} F_{d_k,J \setminus \{k\}},
\]

are generated—since user \( k \in J \) wants \( F_{d_k,J \setminus \{k\}} \) and knows \( F_{d_j,J \setminus \{j\}} \) for all \( j \in J \setminus \{k\} \). Each of these blocks can be successfully recovered from \( W_J \). The load is

\[
\left( \frac{M}{N} \right)^{|W|} \left( 1 - \frac{M}{N} \right)^{K-|W|}.
\]

Decentralized Cache-aided Systems: In the dMAN placement phase, each user caches a subset of MB/N bits of each file, chosen uniformly and independently at random. Given the cache content of users, the bits of the files are naturally grouped into sub-files \( F_{i,W} \), where \( F_{i,W} \) is the set of bits of file \( i \in [N] \) that are only cached only by the users in \( W \subseteq [K] \). When \( B \to \infty \) it can be shown that

\[
\frac{|F_{i,W}|}{B} \to \left( \frac{M}{N} \right)^{|W|} \left( 1 - \frac{M}{N} \right)^{K-|W|} \quad \text{in probability.}
\]

In the delivery phase, for each \( t' \in [0 : K - 1] \), all the \( \left( \binom{K}{t'} \right) \) sub-files \( F_{i,W} : |W| = t', \; i \in [N] \) are gathered together; since they all have approximately the same normalized length as in (3), the cMAN delivery phase is used for \( M = t'N/K \) to deliver them. The achieved load is \( \left( \frac{N}{K} - 1 \right) \left[ 1 - \left( 1 - \frac{M}{N} \right)^{K} \right] \). The load can be further reduced when \( K > N \) [14].

C. Bit-Borrowing

To conclude this section, we introduce the bit-borrowing idea proposed in [14, 15], which we will also use in our novel proposed scheme. For decentralized shared-link caching problems with non-uniform demands or finite file size B, the sub-files in (3) may have different lengths, as [14] may not hold. If this is the case, instead of zero-paddings the sub-files to meet the length of the longest one, which leads to inefficient transmissions, we can borrow bits from some sub-files to ‘lengthen’ short sub-files in such a way that the borrowed bits need not to be transmitted at a later stage. More precisely, if \( |F_{d_k,J \setminus \{k\}}| \leq \max_{k \in J \setminus \{k\}} |F_{d_k,J \setminus \{k\}}| \), we take bits from some \( F_{d\ell,W} \), where \( \ell \notin W \) and \( J \setminus \{\ell\} \subseteq W \) (because \( F_{d\ell,W} \) is also demanded by user \( \ell \) and known by the users in \( J \setminus \{\ell\} \)) and add those bits to \( F_{d_k,J \setminus \{k\}} \).

III. NOVEL ACHIEVABLE DELIVERY SCHEME

For combination networks with cMAN placement, the schemes in [10, 6, 11] first create multicast messages as in (3) and then deliver them to the users by various methods;
for example, in \([11]\) one approach for \(t = 1\) is to use network coding to achieve interference elimination. In this section we propose a delivery scheme, referred to as Separate Relay Decoding Delivery Scheme (SRDS), based on a novel way to create multicast messages: by leveraging the symmetries in the topology of combination networks, each multicast message sent to relay \(h \in \mathcal{H}\) is such that it is useful for the largest possible subset of \(\mathcal{H}_h\) (i.e., users connected to relay \(h\)). We highlight key novelties by way of an example.

A. Example for \(r = 2\)

Consider the network in Fig. \([11]\) with \(N = K = 6\) and \(M = t = 2\). With cMAN, each file \(F_1\) is partitioned into \(\binom{6}{3} = 15\) sub-files of length \(B = 15\) bits. Let \(d = (1 : 6)\).

Step 1 [Subfile partition]: For each \(F_{d_k \cup W}, k \in [K]\) where \(k \notin W\), we seek to find the set of relays in \(\mathcal{H}_h\) each of which is connected to the largest number of users in \(W\). Consider the following examples.

For sub-file \(F_{1, (2, 3)}\), which is demanded by user \(k = 1\) and cached by the users in \(W = \{2, 3\}\), we have that \(\mathcal{U}_1 = \{1, 2, 3\}\) (relay 1 is connected to users in \(W\)) and \(\mathcal{U}_2 = \{1, 4, 5\}\) (relay 2 is not connected to any user in \(W\)). So we solve \(S = \{\max_{h \in \mathcal{H}_k} |\mathcal{U}_h \cap\ W| = 1\}\), (i.e., relay \(h = 1\)). Since \(|S| = 1\) we simply add \(F_{1, (2, 3)}\) to the set \(\mathcal{T}^1_{k, W \cup U}: k = 1\) from relay \(h \in S\) and already known by the users in \(W \cap \mathcal{U}_h = \{2, 3\}\) who are also connected to relay \(h = 1\).

Consider now \(F_{1, (2, 3)}\) where \(k = 1\) and \(W = \{2, 5\}\). Sub-file \(F_{1, (2, 3)}\) is also demanded by user 1, who is connected to relays \(H_1 = \{1, 2\}\). Relay 1 is connected to users \(W \cap \mathcal{U}_1 = \{2, 5\} \cap \{1, 2, 3\} = \{2\}\), while relay 2 is not connected to any user in \(W\). So we solve \(S = \{\max_{h \in \mathcal{H}_k} |\mathcal{U}_h \cap W| = 1\}\), (i.e., relay \(h = 1\)), and thus we have \(S = \{1\}\) we simply add \(F_{1, (2, 3)}\) to the set \(\mathcal{T}^1_{k, W \cup U}: k = 1\) from relay \(h \in S\) and already known by the users in \(W \cap \mathcal{U}_h = \{2, 3\}\) who are also connected to relay \(h = 1\).

Step 2 [Multicast Message Generation]: In this example, for each relay \(k \in H\) and each \(J \subset \mathcal{U}_k\), we have \(\min_{h \in J} |\mathcal{T}^h_{k, J \setminus \{k\}}| = \max_{h \in J} |\mathcal{T}^h_{k, J \setminus \{k\}}|\), i.e., only a function of \(|J|\). We thus create the multicast messages \(W^h_j := \bigoplus_{k \in J} \mathcal{T}^h_{k, J \setminus \{k\}}\). For example, the server transmits to relay 1 the following messages:

\[
\begin{align*}
W_{1, (2, 3)} &= \mathcal{T}^1_{1, (2)} + \mathcal{T}^1_{2, (1, 3)} + \mathcal{T}^1_{3, (1, 2)} \text{ of length } B = 15 \text{ bits,} \\
W_{1, (2)} &= \mathcal{T}^1_{1, (2)} + \mathcal{T}^1_{2, (1)} \text{ of length } 2B = 30 \text{ bits,} \\
W_{1, (3)} &= \mathcal{T}^1_{1, (3)} + \mathcal{T}^1_{3, (1)} \text{ of length } 2B = 30 \text{ bits, and} \\
W_{1, (2, 3)} &= \mathcal{T}^1_{2, (3)} + \mathcal{T}^1_{3, (2)} \text{ of length } 2B = 30 \text{ bits.}
\end{align*}
\]

Step 3 [Multicast Message Delivery]: Finally, for each relay \(h \in \mathcal{H}\) and each set \(J \subset \mathcal{U}_h\) where \(W^h_j \neq 0\), relay \(h\) forwards \(W^h_j\) to the users in \(k \in J\).

The normalized (by the file length) number of bits sent from the server to each relay is the same, thus the achieved max link-load is \(7/15 = 14/30\). The max link-loads of the schemes in \([11, 10, 6, 13]\) are \(17/30, 20/30, 20/30\) and \(15/30\), respectively. So our proposed scheme performs the best.

B. General Scheme of SRDS

The key in above example is that for each relay \(h \in \mathcal{H}\) and each set \(J \subset \mathcal{U}_h\), the length of the message \(\mathcal{T}^h_{k, J \setminus \{k\}}\) only depends on \(|J|\). However, if \(r > 2\), we may have \(\min_{k \in J} |\mathcal{T}^h_{k, J \setminus \{k\}}| < \max_{k \in J} |\mathcal{T}^h_{k, J \setminus \{k\}}|\). In order to ‘equalize’ the lengths of the various parts involved in the linear combinations for the multicast messages, we propose to use the bit-borrowing idea described in Section II-C. The pseudo code of the proposed SRDS delivery scheme is in Appendix B.

Step 1 [Subfile partition]: For each user \(k \in \mathcal{K}\) and each set \(W \subset \mathcal{K} \setminus \{k\}\) where \(|W| = t\), we search for the set of relays \(S \subset \mathcal{H}_k\), each relay in which is connected to the largest number of users in \(W\), i.e., \(\max_{h \in \mathcal{H}_k} |\mathcal{U}_h \cap W|\). We partition \(F_{d_k \cup W} \subset |S|\) equal-length pieces \(F_{d_k \cup W} = \left(\mathcal{T}^{[S]}_{k, W \cup U}, h \in S\right)\). For each relay \(h \in S\), we add \(F_{d_k \cup W} \subset T_{k, W \cup U}, h \in S\), where \(T_{k, W \cup U}\) represents the set of bits needed to be recovered by user \(k\) from relay \(h\) and already known by the users in \(W \cup \mathcal{U}_h\) who are also connected to relay \(h\).

Step 2 [Multicast Message Generation]: Focus on each relay \(h \in \mathcal{H}\) and each set \(J \subset \mathcal{U}_h\) where \(W^h_j \neq 0\). For each user \(k \in \mathcal{J}\), if \(|\mathcal{T}^h_{k, J \setminus \{k\}}| < \max_{h \in \mathcal{J}} |\mathcal{T}^h_{k, J \setminus \{k\}}|\), we use the bit-borrowing idea described in Section II-C. We take bits from \(T_{k, W}\) where \(J \setminus \{k\} \subset W\) and \(k \notin W\) and add them to \(T^h_{k, J \setminus \{k\}}\) so that these borrowed bits need not to be transmitted to \(k\) later. Since \(J \setminus \{k\} \subset W\), the users in \(J \setminus \{k\}\) also knows \(T^h_{k, W}\). After considering all the users in \(J\), the server forms the multicast messages

\[
W^h_j := \bigoplus_{k \in J} \mathcal{T}^h_{k, J \setminus \{k\}}.
\]  

Step 3 [Multicast Message Delivery]: For each relay \(h \in \mathcal{H}\) and each set \(J \subset \mathcal{U}_h\) where \(W^h_j \neq 0\), the server sends \(W^h_j\) to relay \(h\), who then forwards it to each user \(k \in J\).

C. Achievable max link-load for \(r = 2\)

In Appendix A we show that, when \(r = 2\) the bit-borrowing step is not needed (as in the example in Section III-A). In this case the achieved max link-load is as follows.
For combination network with end-user caches with \( r = 2 \) and \( t = \frac{KM}{N} \in [0 : K] \), the max link-load is
\[
R_u^* \leq R_{SRDS} := \frac{K(1 - M/N)}{H(\frac{K-1}{2})} X_{K,H}
\]
(6)
\[
X_{K,H} := \sum_{b_1=0}^{\min(t,H-2)} \left( \frac{1}{b_1 + 1} \left( \frac{H-2}{b_1} \right)^2 \left( \frac{H-2}{b_2} \right)^{t-2b_1} \right) + \sum_{b_1=0}^{\min(t,H-2)} \sum_{b_2=b_1}^{b_1-1} \frac{2}{b_1 + 1} \left( \frac{H-2}{b_1} \right) \left( \frac{H-2}{b_2} \right) \left( \frac{H-2}{b_2} \right)^{t-b_1-b_2}.
\]
(7)

D. Numerical Evaluations

In this section, we compare the performance of our proposed delivery scheme with that of existing schemes for centralized combination network for \( H = 6, r = 3, N = K = 20 \). Notice that when \( M = N/K = 1 \), for each illustrated scheme in Fig. 2, we use the interference elimination scheme proposed in [11]. Fig. 2 shows that our proposed scheme outperforms the schemes in [11] and in [13], which are better than the schemes in [6].

IV. Extensions

In this section, we discuss applications and extensions of our proposed SRDS scheme to models other than centralized combination networks where only end-users have caches.

A. Decentralized Combination Networks

Following steps similar to [2], we extend our proposed delivery scheme to decentralized combination networks. The cache placement phase is the same as dMAN. Then for each \( t' \in [0 : K - 1] \), we gather the sub-files which are only known by \( t' \) users and use our proposed delivery scheme to encode those sub-files. More precisely, to encode the sub-files only known by one user, we use the interference elimination scheme proposed in [11], while to encode the sub-files known by more than one users we use SRDS. In Fig. 3 we compare the performance of the proposed decentralized scheme to those proposed in [10], [6], [11] for \( H = 6, r = 2, N = K = 15 \). Notice that for each illustrated scheme in Fig. 3 we use the interference elimination scheme proposed in [11] to encode the sub-files known by only one user. Notice that the placement of the schemes in [13], [12] are designed with the knowledge of the position (the connected relays) of each user in the delivery phase. Hence, it is not possible to extend these schemes to the decentralized case. Fig. 3 shows the inferiority of our proposed scheme over known schemes.

B. General Relay Networks

Since SRDS does not rely on the symmetric topology of combination networks, it can be used in general relay networks with end-user-caches—as opposed to the schemes in [13], [12] that work for the relay networks where each user is connected to the same number of relays and each relay is connected to the same number of users (we refer to such networks as symmetric networks). In symmetric network, we can directly use SRDS as shown by the next example.

Example 1 (Symmetric Network). Consider a relay network with end-user-caches where \( N = K = 5, H = 5, M = 2 \) and
\[
U_1 = \{1, 2, 3\}, \quad U_2 = \{1, 3, 4\}, \quad U_3 = \{1, 4, 5\},
\]
\[
U_4 = \{2, 4, 5\}, \quad U_5 = \{2, 3, 5\}.
\]
Each user is connected to three relays and each relay is connected to three users. Let \( d = (1 : 5) \). With SRDS, the load to each relay is 1/4, outperforming the schemes in [11], in [13], in [10] and in [6] whose loads equal 4/15, 13/45, 1/3 and 3/5, respectively.

SRDS is designed to minimize the total link-loads to all the relays. For symmetric networks, minimizing the total link-loads to all the relays is equivalent to minimize the max link-load. However, for asymmetric networks, we may need to further ‘balance’ the link-load to each relay, which will be shown in the following example.

Example 2 (Asymmetric Network). Consider a relay network with end-user-caches where \( N = K = 5, H = 5, M = 2 \) and
\[
U_1 = \{1, 2, 3\}, \quad U_2 = \{1, 3, 4\}, \quad U_3 = \{1, 4, 5\},
\]
\[
U_4 = \{2, 4, 5\}, \quad U_5 = \{2, 3, 5\}.
\]
i.e., we changed $U_4$ from $\{2, 4, 5\}$ to Example 1 to $\{3, 4, 5\}$ so that the number of connected relays to each user is not the same. Let $d = (1:5)$. We use SRDS and indicate the multicast messages as $W_{J,L} = \bigoplus_{k \in \mathbb{K}} h_k \mathcal{J}(k)$, where in this example we added the subscript $L$ to indicate that the message has length $L$ bits. Then, we have the server transmit

$$
W_{1, \{1,3\}, B/30} \rightarrow 1, \\
W_{1, \{1,3\}, B/30} \rightarrow 2, \\
W_{1, \{1,4\}, B/30} \rightarrow 3, \\
W_{1, \{1,5\}, B/30} \rightarrow 4, \\
W_{1, \{2,5\}, B/30} \rightarrow 5.
$$

It can be seen that the link-loads to relay 1 to 5 are $1/3, 7/30, 17/60, 7/30$ and $1/3$, respectively. So the achieved max link-load is $1/3$; and the achieved max link-loads by the schemes in [7], [10] and in [13] are $1/3, 1/2$ and $3/5$, respectively.

One can further improve on SRDS by observing that the link-load to relay 1 (or relay 5) is the largest. Thus, instead of transmitting $W_{1, \{1,3\}, B/30}$ to relay 1, we can transmit it to relay 2 which is also connected to users in $\{1,3\}$. Similarly, instead of transmitting $W_{3, \{1,5\}, B/30}$ to relay 5, we can transmit it to relay 4 which is also connected to users in $\{3,5\}$. With this modification, the achieved max link-load is reduced to $3/10$, which is equal to the cut-set outer bound under the constraint of uncoded placement proposed in [11].

The observation at the end of Example 2 can be translated in an improvement of the Algorithm in Appendix B as described in Appendix C.

C. Networks with Cache-Aided Relays and Users

Combination networks with both cache-aided relays and users were considered in [13], where each relay store $M_1 B$ bits and each user can store $M_2 B$ bits, for $(M_1, M_2) \in [0, N]^2$. The objective is to determine the lower convex envelop of the load (number of transmitted bits in the delivery phase) pairs

$$
(\max_{h \in \mathbb{H}} R_h(d, Z), \max_{h \in \mathbb{H}, i \in \mathcal{U}_k} R_{h,i}(d, Z))
$$

for the worst case demands $d$ for a given placement $Z$.

We propose a phase change combining the ideas of the placement in [13] and cMAN. We divide each file $F_i$ into two non-overlapping parts, $F_{i1}$ and $F_{i2}$ where $|F_{i1}| = B \min\{rM_1/N, 1\}$ and $|F_{i2}| = B \max\{1 - rM_1/N, 0\}$. In the placement phase, each relay caches $|F_{i1}|/r$ random linear combinations of $F_{i1}$ for each file $i \in [N]$. Fix two integers $t_3 \in [0 : K]$ and $t_4 \in [0 : K]$. Each user firstly randomly and independently caches $t_3|F_{i1}|/K$ bits of $F_i$ for each $i \in [N]$. We then divide each $F_{i2}$ into $t_4|F_{i2}|$ non-overlapping equal-length parts, each of which is denoted by $F_{i2,w}$ where $W \subseteq [K]$ and $|W| = t_4$. Each user $k$ then caches $F_{i2,w}$ for each $i \in [N]$ if $k \in W$. Hence, $M_2 = \frac{t_3 \min\{rM_1/N, 1\} + \max\{1 - rM_1/N, 0\}t_4}{K}$. In the delivery phase, each relay transmits $F_{i2,w}$ randomly linear combinations of $F_{i1}$ to each user $k \in \mathcal{U}_k$, and uses SRDS to let each user $k$ recover $F_{i2,w}$. The following example compares this proposed scheme and the one in [13].

Example 3. Consider the network in Fig. 7 with $N = 6$, $M_1 = 1$, $M_2 = 2$ and $d = (1:6)$. We divide each file into two parts $F_{i1}$ and $F_{i2}$, where $|F_{i1}| = B/3$ and $|F_{i2}| = 2B/3$. From our proposed caching scheme with $t_3 = 1$ and $t_4 = 2$, the achieved link-load pair is $(14/45, 1/3)$ while the scheme in [13] leads to $(1/3, 1/3)$. We thus see that SRDS is able to lower the link-load from the source to the relays.

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APPENDIX A

PROOF OF THEOREM 1

When $r = 2$, each user $k \in [K]$ is connected to two relays and each relay is connected to $H - 1$ users, say $\mathcal{H}_k = \{h, h'\}$ for which it holds that $\mathcal{U}_h \cap \mathcal{U}_{h'} = \emptyset$. We want to compute $|T_{k,\mathcal{J}}^h|$ for one relay $h \in \mathcal{H}$, one user $k \in \mathcal{U}_h$ and one set $\mathcal{J} \subseteq \mathcal{U}_k$. We consider two cases.

Case 1. The number of $W \subseteq [K] \setminus \{k\}$, where $|W| = t$, $\mathcal{U}_h \cap W \neq \emptyset$ and $|U| = |U \cap W| = |W| - 1$, is $K_r(H - 1)\left(\begin{bmatrix} t-1 \\ r-1 \end{bmatrix}\right)$. Here, $Kr/H - 1 - H - 2$ is the number of users connected to relay $h'$ besides $k$. $\binom{r}{t}$ is the number of users which are connected neither to $h$ nor to $h'$, and $t - 2(|J| - 1)$ represents the number of users in $W$ which are not connected to the relays in $\mathcal{H}_k$. For each of this type of $W$, we divide $F_{d,k,W}$ into two non-overlapping equal-length parts and put one part in $T_{k,\mathcal{J}}^h$ and the other part in $T_{k,\mathcal{J}}^{W}$. Case 2. The number of $W \subseteq [K] \setminus \{k\}$, where $|W| = t$, $\mathcal{U}_h \cap W \neq \emptyset$ and $|U| < |U \cap W| = |W| - 1$, is $\sum_{b_2 = 0}^{\min\{t-1, Kr/H - 1\}} \binom{r-b_2}{t-1} \binom{K_r(H - 1) - b_2}{t - 1 - (|J| - 1) - b_2}$. For this type of $W$, we put $F_{d,k,W}$ in $T_{k,\mathcal{J}}^h$. Hence, we showed that for each user $k \in \mathcal{J}$, $|T_{k,\mathcal{J}}^h|$ is identical and we need not to use the bit-borrowing step. So we encode each $F_{d,k,W}$ (or each partitioned piece of it) by a sum including $\max_{h \in \mathcal{H}}|\mathcal{U}_h \cap \mathcal{W}|$ sub-files (or partitioned pieces with the same length). Therefore, link-load to all the relays for transmitting $F_{d,k,W}$ is $|F_{d,k,W}|/\max_{h \in \mathcal{H}}|\mathcal{U}_h \cap \mathcal{W}|$. By considering each integer $b_1 = |J| - 1 \in [0 : \min\{t, Kr/H - 1\}]$ and that $Kr/H - 1 = H - 2$, the max link-load achieved by SRDS is as in [6].

APPENDIX B

PSEUDO CODE OF ALGORITHM 1: SRDS

1) input: $F_{i,W}$ where $i \in [N]$, $W \subseteq [K]$ and $|W| = t$; initialization: $t_1 = 1$; $T_{k,J} = 0$ for each $h \in [H], k \in \mathcal{U}_h$ and $J \subset \mathcal{U}_h \setminus \{k\}$.

2) for each $k \in [K]$ and each $W \subseteq [K]$ where $k \notin W$;

a) $S = \arg \max_{h \in \mathcal{H}} |W \cap \mathcal{U}_h|$; divide $F_{d,k,W}$ into $|S|$ non-overlapping parts with equal length, $F_{d,k,W} = \{F_{d,k,W,h} : h \in S\}$;

b) for each $h \in S$, pad $F_{d,k,W,h}$ at the end of $T_{k,W \cup \mathcal{U}_h}$.

3) for each $h \in [H]$ and each $J \subseteq \mathcal{U}_h$ where $|J| = t_1$. 

a) $m_1 = \max_{k \in \mathcal{J}} |T_{k, \mathcal{J}}^h(k)|$.
b) For each $k \in \mathcal{J}$, if $|T_{k, \mathcal{J}}^h(k)| < m_1$, then
   i) $R_e = m_1 - |T_{k, \mathcal{J}}^h(k)|$; $t_2 = |\mathcal{J}|$; (The $R_e$ represents the number of bits to be borrowed.)
   ii) $D = \{W \subseteq U_h : k \notin W, \mathcal{J} \setminus \{k\} : W, |W| = t_2, T_{k, W}^h \neq \emptyset\}$; (D represents the set of bits which can be borrowed.)
   iii) If $R_e \geq \sum_{W \in D} |T_{k, W}^h|$, then $C = \cup_{W \in D} T_{k, W}^h$; else then
      A) $C = \emptyset$; sort all the sets $W \in D$ by the length of $T_{k, W}^h$, such that $|T_{k, W}^h|$ represents the set where $|T_{k, W}^h| = \max_{W \subseteq D} T_{k, W}^h$, while $D(|D|)$ represents the set where $|T_{k, W}^h| = \min_{W \subseteq D} T_{k, W}^h$; assume $T_{k, W}^h |D| = \emptyset$;
      B) $a$ is the minimum number in \{ $\forall i \in \{2 : |D| : \sum_{j\in [1 : a] - 1} T_{k, D(j)}^h - (i - 1) |T_{k, D(i)}^h| \geq R_e \}$ $\cup \{ |D| + 1\}$;
      C) For each $i \in [a - 1]$, pad the first \{$T_{k, D(i)}^h - \sum_{h=1}^{a-1} T_{k, D(i)}^h - R_e$\} bits of $T_{k, D(i)}^h$ at the end of $C$;
   iv) Pad the bits in $C$ at the end of $T_{k, \mathcal{J}}^h(k)$;
   v) For each $W \in D$, update $T_{k, W}^h = T_{k, W}^h \setminus C$;
   vi) $R_e = R_e - |C|$; if $R_e > 0$ and $t_2 < |U_h| - 1$, then $t_2 = t_2 + 1$ and go to step 3-b-i);
   c) Let $W_{\mathcal{J}} = \oplus_{k \in \mathcal{J}} T_{k, \mathcal{J}}^h(k)$;
4) If $t_1 < |U_h|$, $t_1 = t_1 + 1$ and go to step 3);
5) For each relay $h$ and each $\mathcal{J} \subseteq U_h$, where $W_{\mathcal{J}} \neq \emptyset$, transmit $W_{\mathcal{J}}^h$ to relay $h$ and relay $h$ transmits $W_{\mathcal{J}}^h$ to each user in $\mathcal{J}$;

Remark. If we need to pad $R_e$ bits to the end $T_{k, \mathcal{J}}^h(k)$, firstly we find the set \{$T_{k, W}^h : k \notin W, W \supseteq W_1, |W| = |W_1| + 1, T_{k, W}^h \neq \emptyset$\}. If the number of bits in this set is not larger than $R_e$, we pad them at the end of $T_{k, W}^h$ and focus on the set \{$T_{k, W}^h : k \notin W, W \supseteq W_1, |W| = |W_1| + 2, T_{k, W}^h \neq \emptyset$\}. Otherwise, we need to choose which $R_e$ bits in the found set to be padded. We propose a novel way different to [14] and [15], which is described in Step 3-b-iii) of Algorithm 1. We sort all the sets in $D = \{ W : k \notin W, W \supseteq W_1, |W| = |W_1| + 1, T_{k, W}^h \neq \emptyset $\} by the length of $T_{k, W}^h$ with a decreasing order, where $D(1)$ represents the set with the largest length and $|T_{k, D(1)}^h| \geq |T_{k, D(2)}^h| \geq \cdots \cdots$. We firstly move the first $\min\{R_e, |T_{k, D(1)}^h| - |T_{k, D(2)}^h|\}$ bits of $T_{k, D(1)}^h$ to $C$. We update $R_e$ representing the number of remaining bits. If $R_e > 0$, we move the first $\min\{R_e/2, |T_{k, D(2)}^h| - |T_{k, D(3)}^h|\}$ bits of the remaining bits in $T_{k, D(1)}^h$ and the first $\min\{R_e/2, |T_{k, D(2)}^h| - |T_{k, D(3)}^h|\}$ bits of $T_{k, D(2)}^h$ to $C$. We repeat this procedure until $R_e$ becomes 0. Notice that we can compute that the chosen bits are from $T_{k, D(j)}^h$ where $j \in [a-1]$ after $a$ is defined in Step 3-b-iii-B) of Algorithm 1. After removing these bits from their original sets, it can be seen that $|T_{k, D(1)}^h| = \cdots = |T_{k, D(a-1)}^h|$.

APPENDIX C

PSEUDO CODE OF ALGORITHM 2: NEW STEP 5) FOR ALGORITHM 1

1) initialization: $i = 1$; Define that $L_h = \sum_{\mathcal{J} \subseteq U_h} |W_{\mathcal{J}}^h|$ for each $h \in [H]$;
2) sort all the relays $h \in [H]$ by $L_h$, i.e., $C(1)$ represents the relay whose $L_h$ is maximal and $C(\mathcal{H})$ represents the relay whose $L_h$ is minimal;
3) if there exists a set $\mathcal{J} \subseteq U_{C(i)}$ such that $|W_{\mathcal{J}}^C(i)| > 0$ and there exists a set of relays (denoted by $Q$) where $|Q| \subseteq \{C(i + 1), \ldots, C(\mathcal{H})\}$ and each relay $h \in Q$ is connected to all the users in $\mathcal{J}$, then
   a) choose one relay $h \in Q$ where $L(h) = \max_{h \in Q} L(h)$, and move $\min\{|W_{\mathcal{J}}^C(i)|, (L_C(i) - L_h)/2\}$ bits from $|W_{\mathcal{J}}^C(i)|$ to $W_h^R$;
   b) update $L(h)$ for relay $h$ and update $L(C(i))$ for relay $C(i)$;
else then, $i = i + 1$;
4) If $i < H$, go to Step 2) of Algorithm 2;
5) For each relay $h$ and each $\mathcal{J} \subseteq U_h$, where $W_{\mathcal{J}} \neq \emptyset$, transmit $W_{\mathcal{J}}^h$ to relay $h$ and relay $h$ transmits $W_{\mathcal{J}}^h$ to each user in $\mathcal{J}$;

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