Nonholonomic versus Vakonomic Dynamics

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Received November 5, 1998; revised June 7, 1999

The main aim of the present paper is to raise the doubt that vakonomic dynamics may not be satisfactory as a model for velocity dependent constraints.

Key Words: Lagrange equations with multipliers; nonholonomic dynamics; variational axiomatic kind dynamics.

1. INTRODUCTION

Mechanical systems with constraints on the velocities, called “nonholonomic constraints,” have paramount importance in engineering, in particular in robotics, vehicular dynamics and motion generation, so they are actively and deeply studied also from the theoretical point of view; see the recent paper by Bloch et al. [BK], the references therein, and Kupka and Oliva [KO]. We are not going to review the story of these studies, which started long ago with the work of Lagrange; let us just mention Chetaev among the most important contributors.

The mathematical model kept essentially unchanged until about 20 years ago when a new dynamics of velocity constrained mechanical systems was introduced by Kozlov [K], and was reported in the beautiful “Encyclopedia of Mathematical Sciences” edited by Arnol’d, see [A, Chap. 1, Sect. 4]. This new mechanics was called “vakonomic” being “variational axiomatic kind.”

The paper by Lewis and Murray [LM] studies the two mechanics from the theoretical point of view, and deals with a ball on a rotating table analytically, numerically, and experimentally. The experiment supports the nonholonomic framework, moreover the authors say on p. 808, “we were not able to produce any vakonomic simulations which resembled the experimental observations...” However, they also say on p. 809, “certainly,

1 Supported by the MURST and by the GNFM of the INDAM.
a more careful and exhaustive experimental effort on systems other than a ball on the rotating table would be valuable in providing data which would allow for a fair comparison on the nonholonomic and vakonomic methods.”

The previous paper by Kozlov [K, Part III, Sect. 5], relates the two different theories to the different ways the constraints can be realized, e.g., by large viscosity or additional masses, and it says “...vakonomic dynamics, which is an internally consistent model that can be applied to the description of the motion of any mechanical systems, is as “true” as traditional nonholonomic mechanics. The issue of the choice of model for each particular case is ultimately resolved by experiment.”

The aim of the present paper is to suggest the contrary: perhaps vakonomic mechanics is not satisfactory as a model for velocity dependent constraints. This opinion is based on the main example used by Kozlov to support his dynamics, namely the skate on an inclined plane, whose nonholonomic behaviour is rated “paradoxical” in [A, p. 19], and, on p. 36, to be compared with the vakonomic motion he studies.

The paradox seems to come from the fact that for the nonholonomic dynamics “...on the average the skate does not slide down the inclined plane...”

In Section 2 we recall the dynamics of natural systems with nonholonomic constraints, that is, Lagrange equation with multipliers, and we put it in normal form. It is a conservative dynamics. Moreover, we see that it is reversible, namely the set of solutions is invariant under time inversion. The example of the nonholonomic skate ends the section.

Section 3 deals with the same systems but with vakonomic dynamics. Now we have many more solutions since new “latent variables” \( \lambda \) appear ([A, p. 37], and the paper [K, III, p. 44] loosely relate this fact with the “unobservable quantities... in quantum mechanics”). This situation, quite strange for classical mechanics, is not overcome by passing to the Hamiltonian framework. Anyway, we prefer to keep our discussion in the Lagrangian framework, which is, perhaps, clearer for our purposes.

We show that also the vakonomic dynamics of natural systems is reversible, at least in a suitable sense, and we see no reasons why it should not be reversible, as nonholonomic dynamics is. The section ends with the study of some vakonomic solutions to the skate which seem paradoxical to me for all initial values of the “latent variable.”

Kharlamov [Kh] considers also the skate on a plane to criticize the vakonomic mechanics. His plane is horizontal, so he is able to obtain explicit expressions for the vakonomic solutions which correspond to the nonholonomic uniform motions along straight lines. The author concludes that the vakonomic skate has a “fanciful track.” However, the uniform motions along straight lines are vakonomic motions for the skate on the horizontal plane.
Finally, Section 4 deals with more general nonholonomic systems and revisits the nonholonomic skate by adding some dissipation on the rotational degree of freedom only. Now, the skate slides eventually down.

2. NONHOLONOMIC DYNAMICS OF NATURAL SYSTEMS

The Lagrange equation of Classical Mechanics is the celebrated

\[ \frac{d}{dt} \frac{\partial L}{\partial \dot{q}}(t, q, \dot{q}) - \frac{\partial L}{\partial q}(t, q, \dot{q}) = 0. \]  

(2.1)

The Lagrangian function \( L \) is assumed of class \( C^2 \) on some open connected subset of \( \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N \).

In particular we are interested in the natural Lagrangian functions,

\[ L(t, q, \dot{q}) = \frac{1}{2} \dot{q} \cdot A(q) \dot{q} - U(q), \quad A(q) = A(q)^T > 0, \]  

(2.2)

where \( A(q)^T \) is the transpose matrix of the \( N \times N \) positive definite matrix \( A(q) \), the central dot is the usual scalar product, \( \frac{1}{2} \dot{q} \cdot A(q) \dot{q} \) is called the kinetic energy, and \( U(q) \) the potential energy.

If one asks the mechanical system to obey the nonholonomic constraint equation

\[ B(q) \dot{q} = 0, \]  

(2.3)

where \( B(q) \) is an \( n \times N \) with \( n < N \), full rank matrix at each \( q \), then one has a nonholonomic system, which we briefly call a nonholonomic natural system, whose dynamics is ruled by the constraint (2.3) together with the following classic Lagrange equation with multipliers

\[ \frac{d}{dt} \frac{\partial L}{\partial \dot{q}}(t, q, \dot{q}) - \frac{\partial L}{\partial q}(t, q, \dot{q}) = B(q)^T \mu. \]  

(2.4)

The multipliers are the components of the new unknown function \( \mu(t) \) with values in \( \mathbb{R}^n \). In the sequel, we assume \( B \in C^2 \) as \( L \). For natural systems the dynamics is then ruled by equations of the form

\[ \begin{cases} [A(q)(\ddot{q} + \Gamma(q)[\dot{q}, \dot{q}])] + \nabla U(q) = B(q)^T \mu \\ B(q) \dot{q} = 0, \end{cases} \]  

(2.5)

where \( \nabla \) is the gradient operator, and \( \Gamma(q): \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}^N, (u, v) \mapsto \Gamma(q)[u, v] \), is a bilinear symmetric map whose components are called Christoffel symbols in the classical books.
We easily check that it is a conservative dynamics, namely the total energy \( \frac{1}{2} \dot{q} \cdot A(q) \dot{q} + U(q) \) is a first integral of (2.5).

For brevity, let us omit the functional dependences for a while. The \( N \times N \) matrix \( A \) is symmetric and positive definite, in particular then invertible, so we can define the new \( n \times n \) matrix \( C := BA^{-1}B^T \) which is non-singular, indeed \( C \xi = 0 \Rightarrow 0 = \xi \cdot C \xi = \xi \cdot BA^{-1}B^T \xi = B^T \xi \cdot A^{-1}B^T \xi \Rightarrow B^T \xi = 0 \) since \( A^{-1} \) is positive definite as \( A \); finally \( B^T \xi = 0 \Rightarrow \xi = 0 \) since \( B \) has full rank.

It is easy to see that the following \( (N+n) \times (N+n) \) matrices

\[
\begin{pmatrix}
A & -B^T \\
B & 0
\end{pmatrix},
\begin{pmatrix}
A^{-1} - A^{-1}B^T C^{-1} BA^{-1} & -A^{-1}B^T C^{-1} \\
-C^{-1} BA^{-1} & -C^{-1}
\end{pmatrix}
\] (2.6)

are inverse of each other. Thus the following system in the unknown functions \( q(t), \mu(t) \), which is obtained from (2.5) by differentiating the second equation with respect to \( t \),

\[
\begin{pmatrix}
A & -B^T \\
B & 0
\end{pmatrix}\begin{pmatrix}
\dot{q} \\
\mu
\end{pmatrix} + \begin{pmatrix}
A(q) \Gamma(q) [\dot{q}, \dot{q}] + \nabla U(q) \\
B'(q) [\dot{q}, \dot{q}]
\end{pmatrix} = 0
\] (2.7)

is equivalent to

\[
\begin{pmatrix}
\dot{q} \\
\mu
\end{pmatrix} = \begin{pmatrix}
A^{-1} - A^{-1}B^T C^{-1} BA^{-1} & -A^{-1}B^T C^{-1} \\
-C^{-1} BA^{-1} & -C^{-1}
\end{pmatrix}
\begin{pmatrix}
\dot{q} \\
\mu
\end{pmatrix} + \begin{pmatrix}
-A(q) \Gamma(q) [\dot{q}, \dot{q}] - \nabla U(q) \\
B'(q) [\dot{q}, \dot{q}]
\end{pmatrix}.
\] (2.8)

Reinstating the functional dependences and defining suitable new functions \( f, g \) we get the following “normal form” which is quadratic in \( \dot{q} \)

\[
\begin{pmatrix}
\dot{q} \\
\mu
\end{pmatrix} = \begin{pmatrix}
\dot{q} \\
\mu
\end{pmatrix} + \begin{pmatrix}
f(q) + g(q) [\dot{q}, \dot{q}] \\
h(q) + k(q) [\dot{q}, \dot{q}]
\end{pmatrix}.
\] (2.9)

We can get rid of the last equation and the unknown \( \mu \).

Since the second equation in (2.7) is obtained from (2.3) by differentiation, we have \( B(q) \dot{q} = \text{const} \) along the solutions of the first equation in (2.9). So, the constraint equation \( B(q) \dot{q} = 0 \) is satisfied provided it holds at some time \( t_0 \). Therefore, we arrive at the conclusion that all nonholonomic motions can be obtained from Cauchy problems of the following kind where an additional condition restricts the choice of the initial velocity

\[
\begin{pmatrix}
\dot{q} \\
\dot{q}(t_0)
\end{pmatrix} = \begin{pmatrix}
\dot{q}_0 \\
\dot{q}_0(t_0)
\end{pmatrix} \in \{(u, v) : B(u) v = 0\}
\] (2.10)
If \( q(t) \) is a solution to (2.10), then the function \( r(t) = q(-t) \) is also a non-holonomic motion which has initial data \( (r(-t_0), \ell(-t_0)) = (q_0, -q_0) \) at time \(-t_0\). So the set of nonholonomic motions is invariant under time reversal, and we can say that the nonholonomic dynamics of natural systems is reversible.

**Example. The Nonholonomic Skate.** Consider a skate (an homogeneous material segment) on an inclined plane with Cartesian coordinates \( x, y \). The \( x \)-axis points downward while the \( y \)-axis is horizontal. \( x \) and \( y \) will denote the coordinates of the center of the skate which has another Lagrangian coordinate: the rotation angle \( \phi \) it makes with the unit vector of the \( x \)-axis. We have a natural Lagrangian function as in formula (2.2)

\[
L(t, x, y, \phi, \dot{x}, \dot{y}, \dot{\phi}) = \frac{1}{2} (\dot{x}^2 + \dot{y}^2 + \dot{\phi}^2) + x. \tag{2.11}
\]

Suppose the center of the skate can only have velocities parallel to it, namely we consider the constraint equation

\[
\dot{x} \sin \phi - \dot{y} \cos \phi = 0. \tag{2.12}
\]

The nonholonomic equations (2.5) in the actual case are

\[
\begin{align*}
\dot{x} - 1 &= \mu \sin \phi \\
\dot{y} &= -\mu \cos \phi \\
\dot{\phi} &= 0 \\
\dot{x} \sin \phi - \dot{y} \cos \phi &= 0.
\end{align*} \tag{2.13}
\]

We easily get the following Cauchy problem which particularizes (2.10) to our system, to the initial time \( t_0 = 0 \), and to some special initial conditions (considered in [A, p. 19])

\[
\begin{align*}
\dot{x} &= \cos^2 \phi - \phi \sin \phi (\dot{x} \cos \phi + \dot{y} \sin \phi) \\
\dot{y} &= \cos \phi \sin \phi + \phi \cos \phi (\dot{x} \cos \phi + \dot{y} \sin \phi) \\
\dot{\phi} &= 0 \\
(\dot{x}(0), \dot{y}(0), \dot{\phi}(0)) &= (0, 0, 0, 0, w)
\end{align*} \tag{2.14}
\]

where \( w \neq 0 \). Notice that the initial data above are acceptable since they satisfy the constraint at \( t = 0 \) : \( \dot{x}(0) \sin \phi(0) - \dot{y}(0) \cos \phi(0) = 0 \). The solution is easily found

\[
x(t) = \frac{1}{2w^2} \sin^2 wt, \quad \dot{x}(t) = \frac{1}{2w^2} \left( wt - \frac{1}{2} \sin 2wt \right), \quad \phi(t) = wt. \tag{2.15}
\]
It is a cycloid. As remarked in [A, p. 19], on the average the skate does not slide down the inclined plane: \( 0 \leq x(t) \leq 1/2w^2 \).

We are also interested in the motion starting with the conditions \((0, 0, \pi/2, 0, \dot{y}_q, 0)\), which also satisfy the constraint

\[
x(t) = 0, \quad y(t) = \dot{y}_q t, \quad \phi(t) = \pi/2.
\]

(2.16)

3. VAKONOMIC DYNAMICS OF NATURAL SYSTEMS

As in Section 2, we consider the natural Lagrangian \( L(t, q, \dot{q}) \) in (2.2), and the constraint (2.3) with a full rank \( n \times N \) matrix \( B(q) \). As is well known, the nonholonomic system (2.5) is a consequence of d’Alembert’s principle which is not variational. If we adopt a variational approach by requiring the motion to be a stationary curve of the action functional \( q(\cdot) \mapsto \int_0^1 L(t, q(t), \dot{q}(t)) \, dt \) among all curves having the same end points and satisfying the nonholonomic constraints, then we get a vakonomic motion (see [A, pp. 32–34; [KO] for details). Moreover, the motion \( t \mapsto q(t) \) is vakonomic if and only if there exists a smooth curve \( t \mapsto \dot{\lambda}(t) \) in \( \mathbb{R}^n \), defined on the same time interval, such that the pair \((q(t), \dot{\lambda}(t))\) is a solution to the (unconstrained) variational problem associated to the Lagrangian function

\[
L(t, q, \dot{q}, \dot{\lambda}) = L(t, q, \dot{q}) - \dot{\lambda} \cdot B(q) \dot{q}
\]

(3.1)

(the scalar product being in \( \mathbb{R}^n \) as the values of the new unknown \( \dot{\lambda}(t) \)). Namely, the vakonomic motions can be obtained by the Lagrange equations

\[
\begin{cases}
\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0 \\
\frac{d}{dt} \frac{\partial L}{\partial \dot{\lambda}} - \frac{\partial L}{\partial \lambda} = 0,
\end{cases}
\]

(3.2)

As in the previous section we speak of natural systems, now vakonomic. Their dynamics is ruled by

\[
\begin{align*}
A(q) (\ddot{\dot{q}} + \Gamma(q)[\dot{q}, \dot{q}]) + \nabla U(q) - B(q)^T \dot{\lambda} - \dot{\lambda} \cdot B'(q) \dot{q} \\
+ \frac{\partial}{\partial q}(\dot{\lambda} \cdot B(q) \dot{q}) &= 0 \\
B(q) \dot{q} &= 0,
\end{align*}
\]

(3.3)
where, for each \(i \in \{1, ..., N\}\), the \(i\)-components of two of the previous expressions are

\[
(\lambda \cdot B(q) \dot{q})_i = \sum_{x=1}^{n} \lambda_x \sum_{j=1}^{N} \frac{\partial B_{ji}}{\partial q_j} q_j, \quad \frac{\partial}{\partial q}(\lambda \cdot B(q) \dot{q})_i = \sum_{x=1}^{n} \lambda_x \sum_{j=1}^{N} \frac{\partial B_{ji}}{\partial q_j} q_j.
\]

(3.4)

As in Section 2 we can differentiate the second equation in (3.3) with respect to \(t\) and write the system obtained from (3.3) in this way as

\[
\begin{pmatrix}
A(q) & -B(q)^T \\
-B(q) & 0
\end{pmatrix}
\begin{pmatrix}
\dot{q} \\
\dot{\lambda}
\end{pmatrix}
+ F(q, \dot{q}, \dot{\lambda}) = 0
\]

(3.5)

for a suitable \(F\). Again the \((N+n) \times (N+n)\) matrix is invertible since \(A(q) = A(q)^T > 0\) and \(B(q)\) has full rank at any point \(q\), and the inverse is as in (2.6). So we can solve (3.5) with respect to \((\dot{q}, \dot{\lambda})\) and get an equivalent system in normal form

\[
\begin{pmatrix}
\dot{q} \\
\dot{\lambda}
\end{pmatrix} = G(q, \dot{q}, \dot{\lambda}).
\]

(3.6)

The function \(B(q) \dot{q}\) is a first integral, i.e., constant along the solutions. If \(B(q(t_0)) \dot{q}(t_0) = 0\) at the initial time \(t_0\), then we have a vakonomic motion, moreover all vakonomic motions can be obtained in this way. However, now \(\lambda(t_0)\) is arbitrary in the Cauchy problem. It seems that we have too many vakonomic motions.

As in the nonholonomic dynamics of natural systems the total energy is conserved. However, now we cannot get rid of \(\lambda\).

In order to discuss the time reversibility, let us consider the new unknown function \(\sigma(t)\), with \(\dot{\sigma}(t) = \dot{\lambda}(t)\). Then Kozlov function (3.1) becomes quadratic in \((q^*, \sigma^*)\)

\[
L(t, q, \dot{q}, \sigma, \dot{\sigma}) = \frac{1}{2} \dot{q} \cdot A(q) \dot{q} - U(q) - \sigma \cdot B(q) \dot{q}
\]

(3.7)

and the related equations

\[
\begin{pmatrix}
\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} \\
\frac{d}{dt} \frac{\partial L}{\partial \dot{\sigma}} - \frac{\partial L}{\partial \sigma}
\end{pmatrix} = 0
\]

(3.8)

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give directly (3.5) with $\dot{\sigma}$ instead of $\dot{\lambda}$, $\dot{\phi}$ instead of $\dot{\tau}$, and no dependence on $\tau$

\[
\begin{pmatrix}
A(q) & -B(q)T \\
-B(q) & 0
\end{pmatrix}
\begin{pmatrix}
\dot{\eta} \\
\dot{\sigma}
\end{pmatrix}
+ F(q, \dot{q}, \dot{\phi}) = 0
\] (3.9)

This system can again be put in normal form

\[
\begin{pmatrix}
\dot{\tau} \\
\dot{\sigma}
\end{pmatrix}
= G(q, \dot{q}, \dot{\phi}).
\] (3.10)

with the first integral $B(q) \dot{\phi}$. Whenever this first integral vanishes, we get a vakonomic motion and all vakonomic motions can be obtained in this way. Finally, the solutions to (3.3) are the pairs $(q(t), \dot{\lambda}(t)) = (q(t), \dot{\sigma}(t))$.

Equation (3.10) is quadratic in the velocities, thus for any solution we have a corresponding reversed one: if $(q(t), \dot{\lambda}(t))$ is a vakonomic motion, then so is $(q(-t), -\dot{\lambda}(-t))$. Perhaps one can argue that these solutions are not necessarily physically acceptable at the same time, so accepting one and rejecting the other. I think that this point should be somehow justified: Without a good reason why should the vakonomic dynamics of natural systems miss the feature of time reversibility? Would not the very lack of time reversibility be a sufficient reason to reject the vakonomic model?

**Example. The Vakonomic Skate.** Consider again the skate as in Section 2, now vakonomic. The above Kozlov function (3.1) for this mechanical system is

\[
L(t, x, y, \phi, \dot{x}, \dot{y}, \dot{\phi}, \dot{\lambda}) = \frac{1}{2}(x^2 + y^2 + \phi^2) + x - \lambda (\dot{x} \sin \phi - \dot{y} \cos \phi).
\] (3.11)

The vakonomic equations are

\[
\begin{aligned}
\dot{x} - \dot{\lambda} \sin \phi - \dot{\phi} \cos \phi &= 1 \\
\dot{y} + \dot{\lambda} \cos \phi - \dot{\phi} \sin \phi &= 0 \\
\dot{\phi} + \lambda (\dot{x} \cos \phi + \dot{y} \sin \phi) &= 0 \\
\dot{x} \sin \phi - \dot{y} \cos \phi &= 0.
\end{aligned}
\] (3.12)

We easily get the following system

\[
\begin{aligned}
\dot{x} &= -\dot{\phi} \sin \phi (\dot{x} \cos \phi + \dot{y} \sin \phi) + \dot{\lambda} \phi \cos \phi + \phi^2 \\
\dot{y} &= +\dot{\phi} \cos \phi (\dot{x} \cos \phi + \dot{y} \sin \phi) + \dot{\lambda} \phi \sin \phi + \sin \phi \cos \phi \\
\dot{\phi} &= -\dot{\lambda}(\dot{x} \cos \phi + \dot{y} \sin \phi) \\
\dot{\lambda} &= -\sin \phi - \dot{\phi}(\dot{x} \cos \phi + \dot{y} \sin \phi).
\end{aligned}
\] (3.13)
which has the first integral
\[ \dot{x} \sin \phi - \dot{y} \cos \phi = a. \] (3.14)

The solutions to (3.13) with \( a = 0 \) are all the vakonomic motions.

The vakonomic motions to be compared to the nonholonomic motions (2.15) (we use the plural since there is a parameter \( w \)), according to Kozlov have \( \lambda(0) = 0 \) (this choice is probably justified by the results it gives), and is studied in [K, III, Sect. 3]. Their features are also reported in [A, pp. 35, 36]. Here, let us just quote from there that the skate slides monotonically down for \( t > 0 \), and almost all solutions tend to turn sideways: \( \phi(t) \) converges to one of the points \( \pi/2 + m\pi \) \((m \in \mathbb{Z})\), as \( t \to +\infty \). Let us add that for \( t < 0 \) it slides monotonically up.

Now, let us go to the nonholonomic motions (2.16). First of all, let us immediately remark that they have no corresponding vakonomic motions the extra variable \( \lambda \) notwithstanding, unless \( \dot{y}_0 = 0 \), the “vakonomic equilibrium,” for which the “latent variable” \( \lambda \) has the precise meaning of reversed time,
\[ x(t) = 0, \quad y(t) = 0, \quad \phi(t) = \pi/2, \quad \dot{\lambda}(t) = \dot{\lambda}(0) - t. \] (3.15)

Consider now (3.13) with the initial conditions
\[ (x(0), y(0), \phi(0), x(0), \dot{y}(0), \dot{\phi}(0), \dot{\lambda}(0)) = (0, 0, \pi/2, 0, \dot{y}_0, 0, \dot{\lambda}_0), \quad \dot{y}_0 \neq 0. \] (3.16)

For the moment assume that \( \dot{\lambda}_0 \neq 0 \) too. Then (3.13) give \( x(0) = 0 \), the second and the third derivative of the functions \( \phi(t) \) and \( x(t) \), respectively, at \( t = 0 \), are
\[ \ddot{\phi}(0) = -\dot{\lambda}_0 \dot{y}_0 \sin(\pi/2) = -\dot{\lambda}_0 \dot{y}_0, \quad x^{(3)}(0) = -\ddot{\phi}(0) \dot{y}(0) \sin^2(\pi/2) = \dot{\lambda}_0 \dot{y}_0^2. \] (3.17)

Therefore
\[ \dot{\lambda}_0 \neq 0 \Rightarrow \begin{cases} x(t) = \dot{\lambda}_0 \dot{y}_0^2 t^3/3! + o(t^3) \\ y(t) = \dot{y}_0 t + o(t) \\ \phi(t) = \pi/2 - \dot{\lambda}_0 \dot{y}_0^2 t^2/2 + o(t^2). \end{cases} \] (3.18)

If we accept the reversibility of vakonomic dynamics we arrive at the conclusion that either the solution we are analyzing, or the one obtained by time reversal, slides monotonically up in a full neighbourhood of \( t = 0 \). I guess we can rate this behaviour paradoxical given the initial conditions above.
Next, consider the other possibility $\lambda_0 = 0$. In this case (3.13) gives

$$
\begin{align*}
\dot{\phi}(0) &= 0, \quad \phi^{(3)}(0) = -\dot{\lambda}(0) \dot{y}(0) \sin(\pi/2) = \dot{y}_0 \sin^2(\pi/2) = \dot{y}_0, \\
x^{(2)}(0) &= 0, \quad x^{(3)}(0) = 0, \quad x^{(4)}(0) = -\phi^{(3)}(0) \dot{y}(0) \sin^2(\pi/2) = -\dot{y}_0^2.
\end{align*}
$$

(3.19)

So

$$
\begin{align*}
\lambda_0 = 0 \Rightarrow \begin{cases}
x(t) &= -\dot{y}_0^2 t^4/4! + o(t^4) \\
y(t) &= \dot{y}_0 t + o(t) \\
\phi(t) &= \pi/2 + \dot{y}_0 t^3/3! + o(t^3)
\end{cases}
\end{align*}
$$

(3.20)

which seems another paradox: the skate slides up for $t > 0$ (small enough) while both $x(0)$ and $\phi(0)$ vanish, and $\dot{\lambda}(0) = 0$ too.

Of course, also the “body-plus-fluid,” considered in [K2, p. 598] to answer the criticism in [Kh], has the same strange behaviour as well as any other physical system we may think should obey the same dynamics.

4. GENERAL NONHOLONOMIC DYNAMICS

For the final example, equations only slightly more general than those considered in Section 2 are needed. But full generality can be obtained with only little work from what described in Section 2, so we proceed to it.

The Lagrange equation with multipliers (2.4) is also studied for general Lagrangian functions $L(t, q, \dot{q})$, and has been also extended to constraints possibly non-linear in $\dot{q}$ by the work of Chetaev and others. Notice that justifying the extension was a difficult job, see the recent papers by Cardin and Favretti [CF] and Massa and Pagani [MP]; but concrete non-linear examples can be given, see Benenti [B].

The general equations for nonholonomic dynamics are

$$
\begin{align*}
\frac{d}{dt} \frac{\partial L}{\partial \dot{q}}(t, q, \dot{q}) - \frac{\partial L}{\partial q}(t, q, \dot{q}) &= \left( \frac{\partial b}{\partial \dot{q}}(t, q, \dot{q}) \right)^T \mu \\
b(t, q, \dot{q}) &= 0,
\end{align*}
$$

(4.1)

where, for each $t$, the values $q(t), \dot{q}(t) \in \mathbb{R}^N$, $\mu(t), b(t, q(t), \dot{q}(t)) \in \mathbb{R}^n$. The standard conditions to put (4.1) in normal form are

$$
\det S(t, q, \dot{q}) \neq 0, \quad \text{where} \quad S(t, q, \dot{q}) := \frac{\partial^2 L}{\partial \dot{q}^2}(t, q, \dot{q}) \quad \text{at all} \quad (t, q, \dot{q}).
$$

(4.2)
\[
\text{det } R(t, q, \dot{q}) \neq 0, \quad \text{at all } (t, q, \dot{q}),
\]

where
\[
R := DS^{-1}D^T, \quad \text{with } D(t, q, \dot{q}) := \frac{\partial b}{\partial \dot{q}}(t, q, \dot{q}). \quad (4.3)
\]

Indeed, differentiating the second equation (4.1) with respect to \(t\), and introducing suitable new functions \(r\) and \(s\), the system (4.1) gives
\[
\begin{align*}
S(t, q, \dot{q}) \ddot{q} - r(t, q, \dot{q}) &= D(t, q, \dot{q})^T \mu \\
- D(t, q, \dot{q}) \ddot{s} &= S(t, q, \dot{q}) \dot{q} - s(t, q, \dot{q}) = 0. \quad (4.4)
\end{align*}
\]

Next, we check at once that the following two \((N+n)\times(N+n)\) matrices are inverse of each other
\[
\begin{pmatrix} S & -D^T \\ -D & 0 \end{pmatrix}, \quad \begin{pmatrix} S^{-1} - S^{-1}D^TR^{-1}DS^{-1} & -S^{-1}D^TR^{-1} \\ -R^{-1}DS^{-1} & -R^{-1} \end{pmatrix}. \quad (4.5)
\]

Thus the system (4.4) is equivalent to the following one in normal form
\[
\begin{pmatrix} \ddot{q} \\ \mu \end{pmatrix} = \begin{pmatrix} S^{-1} - S^{-1}D^TR^{-1}DS^{-1} & -S^{-1}D^TR^{-1} \\ -R^{-1}DS^{-1} & -R^{-1} \end{pmatrix} \begin{pmatrix} r \\ s \end{pmatrix}. \quad (4.6)
\]

Since the second equation in (4.4) is obtained from the constraint equation by differentiating with respect to \(t\), we have that \(b(t, q, \dot{q}) = \text{const}\) along the solutions of the first equation in (4.6). So, \(b(t, q, \dot{q}) = 0\) is satisfied provided it holds at the initial time \(t_0\). Finally, we can get rid of the multipliers arriving at the following Cauchy problem which generalizes (2.10)
\[
\begin{cases}
\ddot{q} = F(t, q, \dot{q}) \\
(q(t_0), \dot{q}(t_0)) = (q_0, \dot{q}_0) \in \{(u, v); b(t_0, u, v) = 0\}
\end{cases} \quad (4.7)
\]

where the first equation is the first equation in (4.6), briefly written.

**Example.** The Nonholonomic Skate with Rotational Friction. We modify the Lagrangian function (2.11) so it has rotational dissipation
\[
L(t, x, y, \phi, \dot{x}, \dot{y}, \dot{\phi}) = \frac{1}{2} (\dot{x}^2 + \dot{y}^2 + e^{k} \dot{\phi}^2) + x, \quad (4.8)
\]

where \(k > 0\) is a new parameter.
The nonholonomic equations are
\[
\begin{align*}
\ddot{x} - 1 &= \mu \sin \phi \\
\dot{y} &= -\mu \cos \phi \\
\dot{\phi} + k\dot{\phi} &= 0 \\
\dot{x} \sin \phi - \dot{y} \cos \phi &= 0.
\end{align*}
\] (4.9)

Only the third equation is different from the one in (2.13). Instead of (2.14) we then have
\[
\begin{align*}
\ddot{x} &= \cos^2 \phi - \dot{\phi} \sin \phi (\dot{x} \cos \phi + \dot{y} \sin \phi) \\
\dot{y} &= \cos \phi \sin \phi + \dot{\phi} \cos \phi (\dot{x} \cos \phi + \dot{y} \sin \phi) \\
\dot{\phi} &= -k\dot{\phi}
\end{align*}
\] (4.10)

where \( w \neq 0 \). The solution is now
\[
\phi(t) = \frac{w}{k} (1 - e^{-kt}) , \quad x(t) = \frac{1}{2} \left( \int_0^t \cos(\phi(\tau)) \, d\tau \right)^2 , \\
y(t) = \int_0^t \sin(\phi(\xi)) \int_0^\xi \cos(\phi(\tau)) \, d\tau \, d\xi .
\] (4.11)

Notice from \( x(t) \) that the skate slides eventually down. Indeed, if
\[
\lim_{t \to +\infty} \phi(t) = \frac{w}{k} \left\{ \frac{\pi}{2} + m\pi : m \in \mathbb{Z} \right\}
\]
then
\[
\lim_{t \to +\infty} x(t) = +\infty ;
\] (4.12)

and otherwise \( x(t) \) converges to a strictly positive finite limit as \( t \to +\infty \).

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