Inequalities Involving Berezin Norm and Berezin Number

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Abstract
We obtain new inequalities involving Berezin norm and Berezin number of bounded linear operators defined on a reproducing kernel Hilbert space \( \mathcal{H} \). Among many inequalities obtained here, it is shown that if \( A \) is a positive bounded linear operator on \( \mathcal{H} \), then \( \|A\|_{ber} = \text{ber}(A) \), where \( \|A\|_{ber} \) and \( \text{ber}(A) \) are the Berezin norm and Berezin number of \( A \), respectively. In contrast to the numerical radius, this equality does not hold for selfadjoint operators, which highlights the necessity of studying Berezin number inequalities independently.

Keywords Berezin norm · Berezin number · Reproducing kernel Hilbert space

Mathematics Subject Classification 47A30 · 15A60 · 47A12

1 Introduction

Let \( \mathbb{B}(\mathbb{H}) \) denote the \( \mathbb{C}^\ast \)-algebra of all bounded linear operators on a complex Hilbert space \( \mathbb{H} \) with the usual inner product \( \langle \cdot, \cdot \rangle \), and \( \| \cdot \| \) is the norm induced by the inner product \( \langle \cdot, \cdot \rangle \). The alphabet \( I \) stands for the identity operator on \( \mathbb{H} \). An operator \( A \in \mathbb{B}(\mathbb{H}) \) is positive if and only if \( \langle Ax, x \rangle \geq 0 \) for all \( x \in \mathbb{H} \), and we write \( A \geq 0 \). Let \( A \in \mathbb{B}(\mathbb{H}) \). The adjoint of \( A \) is denoted by \( A^\ast \) and \( |A| \) denotes the positive operator.
(A* A)^{1/2}. The Cartesian decomposition of A is given by A = \Re(A) + i \Im(A), where \Re(A) and \Im(A) denote the real part and the imaginary part of A, respectively, i.e., \Re(A) = \frac{A + A^*}{2} and \Im(A) = \frac{A - A^*}{2i}. For a ∈ B(H), the numerical range of A, denoted as W(A), is the collection of complex scalars \langle Ax, x \rangle for x ∈ H with ||x|| = 1. More precisely,

W(A) = \{ \langle Ax, x \rangle : x ∈ H, ||x|| = 1 \}.

The numerical radius and the usual operator norm of A are denoted by w(A) and ||A||, respectively. Recall that

w(A) = \sup \{ ||Ax|| : x ∈ H, ||x|| = 1 \}

and

||A|| = \sup \{ ||Ax, y|| : x, y ∈ H, ||x|| = ||y|| = 1 \}.

It is easy to verify that w(.) defines a norm on B(H) and is equivalent to the usual operator norm. In particular, for all A ∈ B(H), the following inequality holds

\frac{1}{2} ||A|| ≤ w(A) ≤ ||A||. \tag{1.1}

For the latest and recent improvements of the above inequalities in (1.1) one can see [6–9] and references therein.

A reproducing kernel Hilbert space (RKHS in short) \mathcal{H} = \mathcal{H}(\Omega) is a Hilbert space of all complex valued functions on a non-empty set \Omega, which has the property that for every \lambda ∈ \Omega the map E_{\lambda} : \mathcal{H} → \mathbb{C} defined by E_{\lambda}(f) = f(\lambda), is continuous linear functional on \mathcal{H}. Throughout the article, a reproducing kernel Hilbert space on the set \Omega is denoted by \mathcal{H}. By the Riesz representation theorem, for each \lambda ∈ \Omega there exists a unique function k_{\lambda} ∈ \mathcal{H} such that f(\lambda) = \langle f, k_{\lambda} \rangle for all f ∈ \mathcal{H}. The collection of functions \{k_{\lambda} : \lambda ∈ \Omega\} is called the reproducing kernel of \mathcal{H}. The normalized reproducing kernel of \mathcal{H} is the collection of functions \{\tilde{k}_{\lambda} = k_{\lambda}/||k_{\lambda}|| : \lambda ∈ \Omega\}. Let A ∈ B(\mathcal{H}). The function \tilde{A} defined on \Omega by \tilde{A}(\lambda) = \langle \tilde{A} k_{\lambda}, \tilde{A} k_{\lambda} \rangle, is called the Berezin symbol of A. The Berezin set of A, denoted as \text{Ber}(A), is defined as \text{Ber}(A) = \{ \tilde{A}(\lambda) : \lambda ∈ \Omega \}. The Berezin number and Berezin norm of A denoted as ber(A) and ||A||_{ber}, respectively are defined as

ber(A) = \sup \{ ||\tilde{A}(\lambda)|| : \lambda ∈ \Omega \} = \sup \{ ||\langle \tilde{A} k_{\lambda}, \tilde{A} k_{\lambda} \rangle|| : \lambda ∈ \Omega \}

and

||A||_{ber} = \sup \{ ||\langle \tilde{A} k_{\lambda}, \tilde{A} k_{\mu} \rangle|| : \lambda, \mu ∈ \Omega \}.

For A, B ∈ B(\mathcal{H}) it is clear from the definition of the Berezin number and the Berezin norm that the following properties hold:
(i) $\text{ber}(\alpha A) = |\alpha|\text{ber}(A)$ for all $\alpha \in \mathbb{C}$,
(ii) $\text{ber}(A + B) \leq \text{ber}(A) + \text{ber}(B)$,
(iii) $\text{ber}(A) \leq \|A\|_{\text{ber}}$,
(iv) $\|\alpha A\|_{\text{ber}} = |\alpha|\|A\|_{\text{ber}}$ for all $\alpha \in \mathbb{C}$,
(v) $\|A + B\|_{\text{ber}} \leq \|A\|_{\text{ber}} + \|B\|_{\text{ber}}$,
(vi) $\|A\|_{\text{ber}} = \|A^*\|_{\text{ber}}$ and $\text{ber}(A) = \text{ber}(A^*)$.

Also, it is clear that for $A \in \mathbb{B}(\mathcal{H})$,

$$\text{Ber}(A) \subseteq W(A), \text{ber}(A) \leq w(A) \quad \text{and} \quad \|A\|_{\text{ber}} \leq \|A\|.$$ 

The Berezin symbol has been studied in details for Toeplitz and Hankel operators on Hardy and Bergman spaces. Recall that the Hardy–Hilbert space of the unit disk $\mathbb{D} = \{\lambda \in \mathbb{C} : |\lambda| < 1\}$, is denoted by $H^2(\mathbb{D})$, is a RKHS of analytic functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ defined on $\mathbb{D}$ such that $\sum_{n=0}^{\infty} |a_n|^2 < \infty$, with reproducing kernel $k_{\lambda}(z) = \sum_{n=0}^{\infty} \lambda^n z^n = 1/(1 - \lambda z)$, see [29, pp. 11–12].

Considering the operator $A$ on $H^2(\mathbb{D})$ defined as $Af(w) = (f(w), w)w^2 (f \in H^2(\mathbb{D}), w \in \mathbb{D})$, we have $A(\lambda) = \langle Ak_{\lambda}(w), k_{\lambda}(w) \rangle = |\lambda|^4(1 - |\lambda|^2)$. Therefore, $\text{Ber}(A) = \{|\lambda|^4(1 - |\lambda|^2) : \lambda \in \mathbb{D}\} = [0, 4/27] \subset [0, 1] = W(A)$, and so $\text{ber}(A) = 4/27 < 1 = w(A)$. On the other hand, considering the operator $M_z$ on $H^2(\mathbb{D})$ defined as $M_z f(w) = w f(w)$ ($f \in H^2(\mathbb{D}), w \in \mathbb{D}$), we have $\text{ber}(M_z) = 1 = w(M_z)$. Since, the collection of normalized reproducing kernel of $\mathcal{H}$ is a subset of the unit sphere of $\mathcal{H}$, so the numerical radius and the Berezin number of an operator on $\mathcal{H}$ may not be equal. However, the previous example provides the equality. Thus it is natural to ask, when the numerical radius is equal to the Berezin number? Note that if there exists a sequence $\{\lambda_n\}$ in $\Omega$ such that $\lim_{n \to \infty} |A(\lambda_n)| = w(A)$, then $\text{ber}(A) = w(A)$.

The Berezin number inequalities have been studied by many mathematicians over the years, the interested readers can see [1–3, 16, 17, 19, 21, 22, 33, 34, 37].

In this paper, we obtain generalized inequalities involving Berezin norm and Berezin number of reproducing kernel Hilbert space operators. As special cases, we derive several inequalities that refine the existing ones. Further, we also give some usual operator norm inequalities of complex Hilbert space operators.

2 The Berezin Norm Inequalities

We start with the following lemmas that will be used to develop new results in this article.

Lemma 2.1 [31] Let $A \in \mathbb{B}(\mathcal{H})$ be positive, and let $x \in \mathcal{H}$ with $\|x\| = 1$. Then

$$\langle Ax, x \rangle^r \leq \langle A^r x, x \rangle \quad \text{for all} \quad r \geq 1.$$ 

Lemma 2.2 [15] Let $A \in \mathbb{B}(\mathcal{H})$, and let $x, y \in \mathcal{H}$. Then

$$|\langle Ax, y \rangle|^2 \leq |\langle A^{2\alpha} x, x \rangle| |\langle A^{2(1-\alpha)} y, y \rangle| \quad \text{for all} \quad \alpha \in [0, 1].$$
Lemma 2.3 [23] For $a, b \geq 0$, $0 < \alpha < 1$ and $r \neq 0$, let $M_r(a, b, \alpha) = (a^r \alpha + (1 - \alpha)b^r)^{1/r}$ and $M_0(a, b, \alpha) = a^\alpha b^{1-\alpha}$. Then

$$M_r(a, b, \alpha) \leq M_s(a, b, \alpha)$$ for $r \leq s$.

Now, we are in a position to prove a general inequality involving the Berezin norm and Berezin number, which leads to several inequalities as special cases.

Theorem 2.4 Let $A, B, C, D, X, Y \in \mathcal{B}(\mathcal{H})$, and let $\alpha \in [0, 1]$. Then

$$\left\| \frac{A^*XB + C^*YD}{2} \right\|^2_{ber} \leq ber^{1/r} \left( \frac{(B^*|X|^{2\alpha}B)^r + (D^*|Y|^{2\alpha}D)^r}{2} \right) \leq ber^{1/s} \left( \frac{(A^*|X|^{2(1-\alpha)}A)^s + (C^*|Y|^{2(1-\alpha)}C)^s}{2} \right),$$

for all $r, s \geq 1$.

**Proof** Let $\hat{k}_\lambda$ and $\hat{k}_\mu$ be two normalized reproducing kernel of $\mathcal{H}$. Then,

$$\frac{1}{4} \left| \langle (A^*XB + C^*YD)\hat{k}_\lambda, \hat{k}_\mu \rangle \right|^2 \leq \frac{1}{4} \left( |\langle A^*XB\hat{k}_\lambda, \hat{k}_\mu \rangle| + |\langle C^*YD\hat{k}_\lambda, \hat{k}_\mu \rangle| \right)^2$$

$$= \frac{1}{4} \left( |\langle XB\hat{k}_\lambda, A\hat{k}_\mu \rangle| + |\langle YD\hat{k}_\lambda, C\hat{k}_\mu \rangle| \right)^2 \leq \frac{1}{4} \left( \langle |X|^{2\alpha}B\hat{k}_\lambda, \hat{k}_\mu \rangle \right)^{1/2} \left( \langle |X|^{2(1-\alpha)}A\hat{k}_\lambda, \hat{k}_\mu \rangle \right)^{1/2}$$

$$+ \langle |Y|^{2\alpha}D\hat{k}_\lambda, \hat{k}_\mu \rangle \left( \langle |Y|^{2(1-\alpha)}C\hat{k}_\lambda, \hat{k}_\mu \rangle \right)^{1/2} \right)^2 \left( \text{by Lemma 2.2} \right)$$

$$\leq \frac{1}{4} \left( \langle B^*|X|^{2\alpha}B\hat{k}_\lambda, \hat{k}_\mu \rangle \right)^{1/2} \left( \langle A^*|X|^{2(1-\alpha)}A\hat{k}_\lambda, \hat{k}_\mu \rangle \right)^{1/2}$$

$$+ \langle D^*|Y|^{2\alpha}D\hat{k}_\lambda, \hat{k}_\mu \rangle \left( \langle C^*|Y|^{2(1-\alpha)}C\hat{k}_\lambda, \hat{k}_\mu \rangle \right)^{1/2} \right)^2 \left( \text{using the inequality } (ab + cd)^2 \leq (a^2 + c^2)(b^2 + d^2) \text{ for all } a, b, c, d \in \mathbb{R} \right)$$

$$= \frac{1}{4} \left( \langle B^*|X|^{2\alpha}B\hat{k}_\lambda, \hat{k}_\mu \rangle + \langle D^*|Y|^{2\alpha}D\hat{k}_\lambda, \hat{k}_\mu \rangle \right)$$

$$\times \left( \langle A^*|X|^{2(1-\alpha)}A\hat{k}_\lambda, \hat{k}_\mu \rangle + \langle C^*|Y|^{2(1-\alpha)}C\hat{k}_\lambda, \hat{k}_\mu \rangle \right).$$
We have, let $A$ be as in Corollary 2.6. Therefore, taking the supremum over all $\lambda, \mu \in \Omega$, we get the desired inequality. 

**Example 2.5** Consider $\mathbb{C}^3$ as a RKHS on the set $\Omega = \{1, 2, 3\}$, see in [29, pp. 4–5]. Then the standard orthonormal basis $\{e_1, e_2, e_3\}$ of $\mathbb{C}^3$ is precisely the set of normalized reproducing kernel of $\mathbb{C}^3$. If we take $A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $C = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 4 \end{pmatrix}$ and $X = Y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, then from Theorem 2.4 (for $r = 1, s = 2$) we have, $\|A^*XB + C^*YD\|^2_{ber} \leq 27\sqrt{10}/2$.

The following corollary follows easily from Theorem 2.4 by taking $A = B = C = D = I$, $X = A$ and $Y = B$.

**Corollary 2.6** Let $A, B \in \mathbb{B}(\mathcal{H})$, and let $\alpha \in [0, 1]$. Then

$$\left\| \frac{A + B}{2} \right\|^2_{ber} \leq \text{ber}^{1/r} \left( \frac{|A|^{2\alpha r} + |B|^{2\alpha r}}{2} \right) \left( \frac{|A^*|^{2(1-\alpha)s} + |B^*|^{2(1-\alpha)s}}{2} \right),$$

for all $r, s \geq 1$.

**In particular, for $s = r$**
\[ \|A + B\|_{ber}^{r} \leq 2^{r-1} \text{ber}^{1/2} \left( |A|^{2\alpha r} + |B|^{2\alpha r} \right) \text{ber}^{1/2} \left( |A^*|^{2(1-\alpha)r} + |B^*|^{2(1-\alpha)r} \right), \]

(2.1)

for all \( r \geq 1 \).

**Remark 2.7** It was proved in [30, Theorem 2.17] that if \( A, B \in \mathbb{B}(\mathcal{H}) \), then

\[ \|A + B\|_{ber}^{r} \leq 2^{r-2} \left( \text{ber}(|A|^{2\alpha r} + |B|^{2\alpha r}) + \text{ber}(|A^*|^{2(1-\alpha)r} + |B^*|^{2(1-\alpha)r}) \right), \]

(2.2)

for \( 0 < \alpha < 1 \) and for all \( r \geq 1 \). Clearly, by applying AM-GM inequality, we infer that

\[ \text{ber}^{1/2} \left( |A|^{2\alpha r} + |B|^{2\alpha r} \right) \text{ber}^{1/2} \left( |A^*|^{2(1-\alpha)r} + |B^*|^{2(1-\alpha)r} \right) \leq \frac{1}{2} \left( \text{ber}(|A|^{2\alpha r} + |B|^{2\alpha r}) + \text{ber}(|A^*|^{2(1-\alpha)r} + |B^*|^{2(1-\alpha)r}) \right). \]

Thus, the inequality (2.1) is sharper than the inequality (2.2).

Again, by considering \( X = Y = I \) in Theorem 2.4, we get the following inequality.

**Corollary 2.8** Let \( A, B, C, D \in \mathbb{B}(\mathcal{H}) \). Then

\[ \left\| \frac{A^*B + C^*D}{2} \right\|_{ber}^{2} \leq \text{ber}^{1/r} \left( \frac{|A|^2 + |C|^2}{2} \right) \text{ber}^{1/s} \left( \frac{|B|^2 + |D|^2}{2} \right), \]

(2.3)

for all \( r, s \geq 1 \).

**Remark 2.9** It was proved in [18, Theorem 3.5.] that if \( A, B, C, D \in \mathbb{B}(\mathcal{H}) \), then

\[ \text{ber}^2 \left( \frac{A^*B + C^*D}{2} \right) \leq \text{ber}^{1/r} \left( \frac{|A|^2 + |C|^2}{2} \right) \text{ber}^{1/s} \left( \frac{|B|^2 + |D|^2}{2} \right), \]

(2.4)

for all \( r, s \geq 1 \).

Since \( \text{ber}(A^*B + C^*D) \leq \|A^*B + C^*D\|_{ber} \), the inequality (2.4) follows from (2.3).

Next inequality follows from Corollary 2.8 by taking \( s = r \).

**Corollary 2.10** Let \( A, B, C, D \in \mathbb{B}(\mathcal{H}) \). Then

\[ \left\| \frac{A^*B + C^*D}{2} \right\|_{ber}^{2r} \leq \text{ber} \left( \frac{|A|^2 + |C|^2}{2} \right) \text{ber} \left( \frac{|B|^2 + |D|^2}{2} \right), \]

for all \( r \geq 1 \).

Now, we prove an interesting equality for positive operators.
Proposition 2.11 If $A \in \mathcal{B}(\mathcal{H})$ is positive (i.e., $A \geq 0$), then

$$\|A\|_{ber} = \text{ber}(A).$$

Proof Putting $A = B = X, C = D = 0$ and $r = 1$ in Corollary 2.10, we have

$$\|X^*X\|_{ber} \leq \text{ber}(X^*X) \text{ for every } X \in \mathcal{B}(\mathcal{H}).$$

Therefore, for every $X \in \mathcal{B}(\mathcal{H})$

$$\|X^*X\|_{ber} = \text{ber}(X^*X). \quad (2.5)$$

Since $A$ is positive, there exists a $Y \in \mathcal{B}(\mathcal{H})$ such that $A = Y^*Y$. This argument together with (2.5) gives that $\|A\|_{ber} = \text{ber}(A).$ $\square$

In the following example we show that the above proposition may not be true for general selfadjoint operators.

Example 2.12 Consider $\mathbb{C}^{2n}$ as a RKHS on the set $\Omega = \{1, 2, \ldots, 2n\}$ (see in [29, pp. 4-5]). Let $\{e_1, e_2, \ldots, e_{2n}\}$ be the standard orthonormal basis for $\mathbb{C}^{2n}$, i.e., $e_i$ be the function defined by

$$e_i(j) = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$$

for $i, j \in \{1, 2, \ldots, 2n\}$. Then $\{e_1, e_2, \ldots, e_{2n}\}$ is the set of all normalized reproducing kernel functions for $\mathbb{C}^{2n}$. Consider a selfadjoint (not positive) operator $A$ on the RKHS $\mathbb{C}^{2n}$ defined as the matrix

$$A = \begin{pmatrix} 0 & 0 \cdots 0 & 1 \\ 0 & 0 \vdots 1 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 1 \cdots 0 \\ 1 & 0 & \cdots 0 \end{pmatrix}_{2n \times 2n}.$$

Then, we have

$$\text{ber}(A) = \sup \{ |\langle Ae_i, e_i \rangle| : i \in \{1, 2, \ldots, 2n\} \} = 0$$

and

$$\|A\|_{ber} = \sup \{ |\langle Ae_i, e_j \rangle| : i, j \in \{1, 2, \ldots, 2n\} \} = 1.$$
Corollary 2.13 Let \( A, B \in \mathcal{B}(\mathcal{H}) \). Then

(i) \[
\left\| \frac{A^*B + B^*A}{2} \right\|_{ber}^2 \leq \ber^{1/r} \left( \frac{|A|^{2r} + |B|^{2r}}{2} \right) \ber^{1/s} \left( \frac{|A|^{2s} + |B|^{2s}}{2} \right), \quad \text{for all } r, s \geq 1.
\]

(ii) \[
\left\| \frac{A^*B + B^*A}{2} \right\|_{ber}^r \leq \ber \left( \frac{|A|^{2r} + |B|^{2r}}{2} \right), \quad \text{for all } r \geq 1.
\]

In particular, for \( r = 1 \)

\[
\|A^* B + B^* A\|_{ber} \leq \ber (A^* A + B^* B) = \|A^* A + B^* B\|_{ber}.
\] (2.6)

Now, taking \( A = C = I, B = A \) and \( D = B \) in Corollary 2.8 we get the following corollary.

Corollary 2.14 Let \( A, B \in \mathcal{B}(\mathcal{H}) \). Then

\[
\left\| \frac{A + B}{2} \right\|_{ber}^{2r} \leq \ber \left( \frac{|A|^{2r} + |B|^{2r}}{2} \right),
\] (2.7)

for all \( r \geq 1 \).

In particular, for \( r = 1 \)

\[
\|A + B\|_{ber}^2 \leq 2 \ber (A^* A + B^* B) = 2 \|A^* A + B^* B\|_{ber}
\] (2.8)

If we take \( A = \Re(A) \) and \( B = i \Im(A) \) in (2.7), then we get

\[
\|A\|_{ber}^{2r} \leq 2^{2r-1} \ber \left( \Re(A)^{2r} + \Im(A)^{2r} \right),
\] (2.9)

for all \( r \geq 1 \).

Also, if we take \( A = A \) and \( B = A^* \) in (2.7), then we get

\[
\|\Re(A)\|_{ber}^{2r} \leq \ber \left( \frac{|A|^{2r} + |A^*|^{2r}}{2} \right) \quad \text{for all } r \geq 1
\] (2.10)

and if we take \( A = A \) and \( B = -A^* \) in (2.7), then we get

\[
\|\Im(A)\|_{ber}^{2r} \leq \ber \left( \frac{|A|^{2r} + |A^*|^{2r}}{2} \right) \quad \text{for all } r \geq 1.
\] (2.11)

The following result follows from Corollary 2.8 by considering \( A = A^* \), \( B = A \), \( C = B^* \) and \( D = B \).

Corollary 2.15 Let \( A, B \in \mathcal{B}(\mathcal{H}) \). Then

(i) \[
\left\| \frac{A^2 + B^2}{2} \right\|_{ber}^2 \leq \ber^{1/r} \left( \frac{|A|^{2r} + |B|^{2r}}{2} \right) \ber^{1/s} \left( \frac{|A|^{2s} + |B|^{2s}}{2} \right), \quad \text{for all } r, s \geq 1.
\]

(ii) \[
\left\| \frac{A^2 + B^2}{2} \right\|_{ber}^{2r} \leq \ber \left( \frac{|A|^{2r} + |B|^{2r}}{2} \right) \ber \left( \frac{|A^*|^{2r} + |B^*|^{2r}}{2} \right), \quad \text{for all } r \geq 1.
\]
In particular, for \( r = 1 \)

\[
\left\| A^2 + B^2 \right\|_{ber}^2 \leq ber(A^*A + B^*B)ber(AA^* + BB^*) = \| A^*A + B^*B \|_{ber} \| AA^* + BB^* \|_{ber}.
\] (2.12)

Also, the following inequality follows from Corollary 2.8 by choosing \( A = I, D = I, C = B^* \) and \( B = A \).

**Corollary 2.16** Let \( A, B \in \mathcal{B}(\mathcal{H}) \). Then

\[
\left\| \frac{A + B}{2} \right\|_{ber}^2 \leq ber^{1/r} \left( \frac{|A|^{2r} + I}{2} \right) ber^{1/s} \left( \frac{|B|^{2s} + I}{2} \right),
\] (2.13)

for all \( r, s \geq 1 \).

In particular, for \( B = A \)

\[
\| A \|_{ber}^2 \leq ber^{1/r} \left( \frac{|A|^{2r} + I}{2} \right) ber^{1/s} \left( \frac{|A^*|^{2s} + I}{2} \right),
\] (2.14)

for all \( r, s \geq 1 \).

Moreover, for \( s = r \)

\[
\| A \|_{ber}^{2r} \leq ber \left( \frac{|A|^{2r} + I}{2} \right) ber \left( \frac{|A^*|^{2r} + I}{2} \right),
\] (2.15)

for all \( r \geq 1 \).

Now, taking \( A = A^* \) and \( C = D = 0 \) in Corollary 2.8 we get the following result.

**Corollary 2.17** Let \( A, B \in \mathcal{B}(\mathcal{H}) \). Then

(i) \( \| AB \|_{ber}^2 \leq 2^{2-1/r-1/s} ber^{1/r} \left( |A|^{2r} \right) ber^{1/s} \left( |B|^{2s} \right) \) for all \( r, s \geq 1 \).

(ii) \( \| AB \|_{ber}^{2r} \leq 2^{2r-2} ber \left( |A|^{2r} \right) ber \left( |B|^{2r} \right) \) for all \( r \geq 1 \).

In particular, for \( r = 1 \)

\[
\| AB \|_{ber} \leq ber^{1/2} \left( AA^* \right) ber^{1/2} \left( B^*B \right) = \| AA^* \|_{ber}^{1/2} \| B^*B \|_{ber}^{1/2}.
\] (2.16)

Also, the next result follows from Corollary 2.8 by taking \( A = A^*, B = B, C = \pm B^* \) and \( D = A \).

**Corollary 2.18** Let \( A, B \in \mathcal{B}(\mathcal{H}) \). Then

(i) \( \frac{AB + BA}{2} \|_{ber}^2 \leq ber^{1/r} \left( \frac{|A|^{2r} + |B|^{2r}}{2} \right) ber^{1/s} \left( \frac{|A^*|^{2r} + |B^*|^{2r}}{2} \right) \), for all \( r, s \geq 1 \).

(ii) \( \frac{AB + BA}{2} \|_{ber}^{2r} \leq ber \left( \frac{|A|^{2r} + |B|^{2r}}{2} \right) ber \left( \frac{|A^*|^{2r} + |B^*|^{2r}}{2} \right) \), for all \( r \geq 1 \).
In particular, for $B = A^*$ in Corollary 2.18 (ii), we have
\[
\|AA^* \pm A^*A\|_{ber}^r \leq 2^{r-1}\text{ber} \left((A^*A)^r + (AA^*)^r\right) \text{ for all } r \geq 1. \tag{2.17}
\]

Moreover, for $r = 1$ in (2.17), we have
\[
\|AA^* - A^*A\|_{ber} \leq \text{ber} \left(A^*A + AA^*\right) = \|AA^* + A^*A\|_{ber}. \tag{2.18}
\]

In the following theorem we obtain an upper bound for the Berezin norm of the sum of the product of two positive operators.

**Theorem 2.19** Let $A, B \in \mathcal{B}(\mathcal{H})$ be positive. Then
\[
\left\|\frac{A^\alpha B^{1-\alpha} + A^{1-\alpha} B^\alpha}{2}\right\|_{ber}^2 \leq \text{ber}^{1/r} \left(\frac{A^{2\alpha r} + A^{2(1-\alpha)r}}{2}\right) \text{ber}^{1/s} \left(\frac{B^{2\alpha s} + B^{2(1-\alpha)s}}{2}\right),
\]
for all $r, s \geq 1$ and for all $\alpha \in [0, 1]$.

**Proof** Suppose that $\hat{k}_\lambda$ and $\hat{k}_\mu$ are two normalized reproducing kernel of $\mathcal{H}$. Then
\[
|\langle (A^\alpha B^{1-\alpha} + A^{1-\alpha} B^\alpha)\hat{k}_\lambda, \hat{k}_\mu\rangle|^2 \leq |\langle (A^\alpha B^{1-\alpha}\hat{k}_\lambda, \hat{k}_\mu)\rangle| + |\langle A^{1-\alpha} B^\alpha\hat{k}_\lambda, \hat{k}_\mu\rangle|.
\]
\[
= |\langle B^{1-\alpha}\hat{k}_\lambda, A^\alpha\hat{k}_\mu\rangle| + |\langle B^\alpha\hat{k}_\lambda, A^{1-\alpha}\hat{k}_\mu\rangle|.
\]
\[
\leq \left\|B^{1-\alpha}\hat{k}_\lambda\right\|_A^2 \left\|A^\alpha\hat{k}_\mu\right\| + \left\|B^\alpha\hat{k}_\lambda\right\| \left\|A^{1-\alpha}\hat{k}_\mu\right\|^2
\]
\[
= \left\langle A^{2\alpha}\hat{k}_\mu, \hat{k}_\mu\right\rangle^{1/2} \left\langle B^{2(1-\alpha)}\hat{k}_\lambda, \hat{k}_\lambda\right\rangle^{1/2} + \left\langle A^{2(1-\alpha)}\hat{k}_\mu, \hat{k}_\mu\right\rangle^{1/2} \left\langle B^{2\alpha}\hat{k}_\lambda, \hat{k}_\lambda\right\rangle^{1/2}
\]
\[
\leq 4 \left\langle \frac{A^{2\alpha}\hat{k}_\mu, \hat{k}_\mu}{2} + \frac{A^{2(1-\alpha)}\hat{k}_\mu, \hat{k}_\mu}{2} \right\rangle^{1/2} \left\langle \frac{B^{2\alpha}\hat{k}_\lambda, \hat{k}_\lambda}{2} + \frac{B^{2(1-\alpha)}\hat{k}_\lambda, \hat{k}_\lambda}{2} \right\rangle^{1/2}
\]
(using the inequality $(ab + cd)^2 \leq (a^2 + c^2)(b^2 + d^2)$ for all $a, b, c, d \in \mathbb{R}$)
\[
\leq 4 \left\langle \frac{A^{2\alpha}\hat{k}_\mu, \hat{k}_\mu}{2} + \frac{A^{2(1-\alpha)}\hat{k}_\mu, \hat{k}_\mu}{2} \right\rangle^{1/2} \left\langle \frac{B^{2\alpha}\hat{k}_\lambda, \hat{k}_\lambda}/s + \frac{B^{2(1-\alpha)}\hat{k}_\lambda, \hat{k}_\lambda}/s}{2} \right\rangle^{1/2}
\]
(by Lemma 2.3)
\[
\leq 4 \left\langle \frac{A^{2\alpha}(\hat{k}_\mu, \hat{k}_\mu)\alpha r + A^{2(1-\alpha)}(\hat{k}_\mu, \hat{k}_\mu)}{2} \right\rangle^{1/2} \left\langle \frac{B^{2\alpha}(\hat{k}_\lambda, \hat{k}_\lambda)/s + B^{2(1-\alpha)}(\hat{k}_\lambda, \hat{k}_\lambda)/s}{2} \right\rangle^{1/2}
\]
(by Lemma 2.1)
\[
= 4 \left\langle \frac{A^{2\alpha r} + A^{2(1-\alpha)r}}{2}\hat{k}_\mu, \hat{k}_\mu\right\rangle^{1/2} \left\langle \frac{B^{2\alpha s} + B^{2(1-\alpha)s}}{2}\hat{k}_\lambda, \hat{k}_\lambda\right\rangle^{1/2}
\]
\[
\leq 4 \text{ber}^{1/r} \left(\frac{A^{2\alpha r} + A^{2(1-\alpha)r}}{2}\right) \text{ber}^{1/s} \left(\frac{B^{2\alpha s} + B^{2(1-\alpha)s}}{2}\right).
Therefore, taking the supremum over all \( \lambda, \mu \in \Omega \), we get the desired inequality. \( \square \)

**Example 2.20** Consider \( \mathbb{C}^3 \) as a RKHS on the set \( \Omega = \{1, 2, 3\} \). Then the standard orthonormal basis \( \{e_1, e_2, e_3\} \) of \( \mathbb{C}^3 \) is precisely the set of all normalized reproducing kernel of \( \mathbb{C}^3 \). If we consider \( A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \) and \( B = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 6 \end{pmatrix} \), then from Theorem 2.19 (for \( \alpha = 1/3, r = 3, s = 3/2 \)) we have,

\[
\| A^{\alpha} B^{1-\alpha} + A^{1-\alpha} B^\alpha \|_{ber}^2 \leq \frac{34}{35} \frac{1}{37} \frac{2}{3}.
\]

Considering \( s = r \) in Theorem 2.19, we infer the following corollary.

**Corollary 2.21** Let \( A, B \in \mathbb{B} (H) \) be positive. Then

\[
\| A^{\alpha} B^{1-\alpha} + A^{1-\alpha} B^\alpha \|_{ber}^2 \leq 2^{2r-2} \ber \left( A^{2\alpha r} + A^{2(1-\alpha)r} \right) \ber \left( B^{2\alpha r} + B^{2(1-\alpha)r} \right),
\]

(2.19)

for all \( r \geq 1 \) and for all \( \alpha \in [0, 1] \).

In particular, for \( \alpha = \frac{1}{2} \) and \( r = 1 \), we have

\[
\| A^{1/2} B^{1/2} \|_{ber} \leq \ber^{1/2} (A) \ber^{1/2} (B).
\]

(2.20)

Moreover, if \( AB = BA \), then

\[
\| \sqrt{AB} \|_{ber} \leq \sqrt{\ber (A) \ber (B)}.
\]

(2.21)

Following Proposition 2.11, since \( \ber (\sqrt{AB}) = \| \sqrt{AB} \|_{ber} \), the inequality (2.21) is same as the existing inequality [32, Corollary 2.10], namely, \( \ber (\sqrt{AB}) \leq \sqrt{\ber (A) \ber (B)} \), where \( A, B \in \mathbb{B} (H) \) with \( A, B \geq 0 \) and \( AB = BA \).

For \( A, B \in \mathbb{B} (H) \) and \( \alpha \in [0, 1] \), the \( \alpha \)-weighted arithmetic mean of \( A \) and \( B \) is given by \( \alpha A + (1 - \alpha)B \). Now, in the following theorem we obtain an upper bound for the Berezin norm for \( \alpha \)-weighted arithmetic mean of two operators.

**Theorem 2.22** Let \( A, B \in \mathbb{B} (H) \). Then

\[
\| \alpha A + (1 - \alpha)B \|_{ber}^2 \leq \ber \left( \alpha^2 |A|^2 + (1 - \alpha)^2 |B|^2 \right) + 2\alpha(1 - \alpha) \ber (B^* A),
\]

for all \( \alpha \in [0, 1] \).

**Proof** Let \( \hat{k}_\lambda \) and \( \hat{k}_\mu \) be two normalized reproducing kernel of \( H \). Then

\[
|\langle \alpha A + (1 - \alpha)B \hat{k}_\lambda, \hat{k}_\mu \rangle|^2 \\
\leq \| \alpha A + (1 - \alpha)B \hat{k}_\lambda \|^2
\]
Let $A$.

**Corollary 2.24**

Upper bound for the Berezin norm of the sum of two operators.

\[
\|\alpha A + (1 - \alpha)B\|_{ber}^2 \leq \text{ber} \left( \alpha^2|A|^2 + (1 - \alpha)^2|B|^2 \right) + 2\alpha(1 - \alpha)\text{ber}(B^*A). 
\]

Therefore, taking the supremum over all $\lambda, \mu \in \Omega$, we get

\[
\|\alpha A + (1 - \alpha)B\|_{ber}^2 \leq \text{ber} \left( \alpha^2|A|^2 + (1 - \alpha)^2|B|^2 \right) + 2\alpha(1 - \alpha)\text{ber}(B^*A).
\]

\[\square\]

**Example 2.23** Consider $\mathbb{C}^3$ as a RKHS on the set $\Omega = \{1, 2, 3\}$. Then the standard orthonormal basis $\{e_1, e_2, e_3\}$ of $\mathbb{C}^3$ is precisely the set of all normalized reproducing kernel of $\mathbb{C}^3$. If we consider the matrices $A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, then from Theorem 2.22 (for $\alpha = 1/3$) we have, $\|\alpha A + (1 - \alpha)B\|_{ber}^2 \leq 13/9$.

Putting $\alpha = \frac{1}{2}$ in Theorem 2.22, we get the following corollary which presents upper bound for the Berezin norm of the sum of two operators.

**Corollary 2.24** Let $A, B \in \mathbb{B}(\mathcal{H})$. Then

\[
\|A + B\|_{ber}^2 \leq \text{ber} \left( |A|^2 + |B|^2 \right) + 2\text{ber}(B^*A).
\]

**Remark 2.25** The following inequalities for the sum of the product of operators defined on a complex Hilbert space can be obtained using analogous argument as described in Theorems 2.4, 2.19 and 2.22.

(i) Let $A, B, C, D, X, Y \in \mathbb{B}(\mathbb{H})$, and let $\alpha \in [0, 1]$. Then

\[
\left\| \frac{A^*XB + C^*YD}{2} \right\|^{2r} \leq \left\| \frac{B^*X^{2\alpha}B^r + (D^*Y^{2\alpha}D^r)^r}{2} \right\|^{1/s} \left\| \frac{(A^*X^{2(1-\alpha)}A)^s + (C^*Y^{2(1-\alpha)}C^s)^s}{2} \right\|^{1/s},
\]

for all $r, s \geq 1$.

(ii) Let $A, B, C, D \in \mathbb{B}(\mathbb{H})$. Then

\[
\left\| \frac{A^*B + C^*D}{2} \right\|^{2r} \leq \left\| \frac{(B^*B)^r + (D^*D)^r}{2} \right\|^{1/s} \left\| \frac{(A^*A)^s + (C^*C)^s}{2} \right\|^{1/s},
\]

for all $r, s \geq 1$. 
(iii) Let $A, B \in \mathcal{B}(\mathbb{H})$ be positive. Then
\[
\left\| \frac{A^\alpha B^{1-\alpha} + A^{1-\alpha} B^\alpha}{2} \right\|_2^2 \leq \left\| \frac{A^{2\alpha r} + A^{2(1-\alpha)s}}{2} \right\|^{1/r}_r \left\| \frac{B^{2\alpha r} + B^{2(1-\alpha)s}}{2} \right\|^{1/s}_s,
\]
for all $r, s \geq 1$ and for all $\alpha \in [0, 1]$.

(iv) Let $A, B \in \mathcal{B}(\mathbb{H})$. Then
\[
\|\alpha A + (1 - \alpha)B\|^2 \leq \left\| \alpha^2 |A|^2 + (1 - \alpha)^2 |B|^2 \right\| + 2\alpha(1 - \alpha)w(B^* A),
\]
for all $\alpha \in [0, 1]$.

Note that the inequality obtained in (ii) was developed independently in [12, Th. 3.]

3 Applications

Reproducing kernel Hilbert spaces have become an important tool in approximation theory, machine learning theory, theory of integral operators and they play a valuable role in complex analysis. The Berezin symbol and Berezin number has large application in the study of various questions of operator theory in the functional Hilbert space, quantum physics and non-commutative geometry, see [4, 5, 13, 14, 24, 25, 27]. These are the important tools to study operators on Hardy and Bergman spaces, especially for Toeplitz and Hankel operators. The properties of Berezin symbol of an operator provides important information about that operator. In the most familiar functional Hilbert spaces, including Hardy and Bergman spaces, it uniquely determines the operators, i.e., for all $\lambda \in \Omega$, $\tilde{A}(\lambda) = \tilde{B}(\lambda)$ implies $A = B$. Also the invertibility of an operator on functional Hilbert space is related with the Berezin symbol of the operator, see [26]. The results obtained here may help to study the behavior of bounded linear operators defined on a RKHS. In particular, in Proposition 2.11, we show that for any positive operator $A$, the equality $\text{ber}(A) = \|A\|_{\text{ber}}$ holds. This equality may be helpful to study for the positive operators on RKHS.

Reproducing kernel Hilbert spaces are fundamental in Kernel methods in machine learning. This is an active area of research where kernel functions show up repeatedly. Kernels have been used in a lot of real world applications including microarray data analysis and image denoising. The performance of kernel methods intensely depends on the choice of the kernel in practical applications. It is not an easy task to find the appropriate kernel for a particular application. For relevant research in kernel construction, kernel selection, and multiple kernel learning, see [10, 11, 20, 28, 35, 36] and references therein. The equality in Proposition 2.11 may be helpful in kernel methods for positive kernel matrices. Moreover, the inequalities involving Berezin number and Berezin norm may help to study the same.

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**Data Availability** Authors declare that data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

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