Abstract—Coded caching is a technique that promises huge reductions in network traffic in content-delivery networks. However, the original formulation and several subsequent contributions in the area, assume that the file requests from the users are synchronized, i.e., they arrive at the server at the same time. In this work we formulate and study the coded caching problem when the file requests from the users arrive at different times. We assume that each user also has a prescribed deadline by which they want their request to be completed. In the offline case, we assume that the server knows the arrival times before starting transmission and in the online case, the user requests are revealed to the server over time. We present a linear programming formulation for the offline case that minimizes the overall rate subject to constraint that each user meets his/her deadline. While the online case is much harder, we introduce a novel heuristic for the online case and show that under certain conditions, with high probability the request of each user can be satisfied with her/his deadline. Our simulation results indicate that in the presence of mild asynchronism, much of the benefit of coded caching can still be leveraged.

Index Terms—coded caching, asynchronous, deadlines, linear programming

I. INTRODUCTION

Caching is a core component of solving the problem of large scale content delivery over the Internet. Conventional caching typically relies on placing popular content closer to the end users. Statistically, popular content is requested more frequently and the cache can be used to serve the user requests in this case. Contacting the central server that has all the content is not needed. This serves to reduce the induced network traffic.

In their pioneering work [1], Maddah-Ali and Niesen considered the usage of coding in the caching problem. In this so-called “coded caching” setting, there is a server containing \( N \) files, \( K \) users each with a cache that can store up to \( M \) files. The users are connected to the server via an error-free shared link (see Fig. 1). The system operates in two distinct phases. In the placement phase the content of the caches is populated by server. This phase does not depend on the future requests of the users which are assumed to be arbitrary. In the delivery phase each user makes a request and the server transmits potentially coded signals to satisfy the requests of the users. The work of [1] demonstrated that significant reductions in the network traffic were possible as compared to conventional caching. Crucially, these gains continue to hold even if the popularity of the files is not taken into account.

While this is a significant result, the original formulation of the coded caching problem assumes that the user requests are synchronized, i.e., all file requests from the users arrive at the server at the same time. Henceforth, we refer to this as the synchronous setting. From a practical perspective, it is important to consider the asynchronous setting where user requests arrive at different times. In this case, a simple strategy would be to wait for the last request to arrive and then apply the scheme of [1]. Such a strategy will be quite good in terms of the overall rate of transmission from the server. However, this may be quite bad for an end user’s experience, e.g., the delay experienced by the users will essentially be dominated by the arrival time of the last request.

In this work we formulate and study the coded caching problem when the user requests arrive at different times. Each user has a specific deadline by which his/her demand needs to be satisfied. The goal is to schedule transmission of packets so that each user is able to recover the requested file from the transmitted packets and his/her cache content within the prescribed deadline. We present algorithms for both the offline and online versions of this problem.

This paper is organized as follows. In Section II we discuss the background and related work and overview our main contributions. The problem formulation appears in Section III. Sections IV and V discuss our work on the offline and the online versions of the problem, respectively. We conclude the paper with a discussion of opportunities for future work in Section VII.

II. BACKGROUND, RELATED WORK AND SUMMARY OF CONTRIBUTIONS

A coded caching system contains a server with \( N \) files, denoted \( W_n, n = 1, \ldots, N \), each of size \( F \) subfiles, where a subfile is a basic unit of storage. The system also contains \( K \) users each connected to the server through an error free, broadcast shared link. Each of the users is equipped with a local cache. The \( i \)-th cache is of size \( M_i F \) subfiles. We denote the cache content of user \( i \) by \( Z_i \), where \( Z_i \) is a function of \( W_1, \ldots, W_N \). Our formulation supports users with different cache sizes. A block diagram of a coded caching system is depicted in Fig. 1.

In this work, we assume that an uncoded placement scheme is being used by the coded caching system, i.e., user \( i \in [K] \) caches at most an \( M_i F \)-sized subset of the total number of subfiles in the server. It is well recognized that the delivery phase in this case corresponds to an index coding problem [2]. While the optimal solution for an arbitrary index coding problem is known to be hard, techniques such as clique cover on the side information graph are well-recognized to have good performance [2]. In this case each transmitted equation from the server is such that a certain number of users “benefit”
from it simultaneously. Under this assumption, we formulate and study the asynchronous coded caching problem when the file requests arrive at the server at different times. Each user specifies a deadline by which he/she expects the request to be satisfied. We assume that
- the delivery phase proceeds via a clique cover and
- transmitting a single packet over the shared link takes a certain number of time slots.

We study the rate gains of coded caching under this setup, i.e., among the class of strategies that allow the users to meet their deadlines, we attempt to determine those where the server transmits the fewest number of packets. Both the offline and the online versions of the problem are studied. In the offline scenario, we assume that information about all request arrival times and deadlines are known to the server before transmission, whereas in the online scenario, the arrival times and deadlines are revealed to the server as time progresses.

A. Main contributions
- Linear programming (LP) formulation in the offline case. We propose a LP in the offline scenario that determines a schedule for the equations transmitted from the server. If feasible, the schedule is such that each user meets its deadlines and the rate of transmission from the server is minimized. This allows us to study the effect of asynchronism on the coded caching rate. We demonstrate that a feasible point of the LP can be interpreted as a coding solution that can be used by the server.

The computational complexity of solving this LP can be quite high for a large number of users. Accordingly, we develop a dual decomposition technique where the dual problem decouples into a set of independent minimum cost network flow problems which can be solved in an efficient manner.

- A novel online algorithm. For the online problem we demonstrate that in general coding within subfiles of the same file is essential. Interestingly, in the synchronous case this is not needed. We propose an novel online algorithm that is inspired by recursively solving the offline LP and interpreting the corresponding output appropriately. Under certain condition, we also show that the algorithm will result in a solution that satisfies the deadline constraints with high probability.

For both scenarios we present exhaustive simulation results that corroborate our findings and demonstrate the superiority of our algorithm with respect to prior work. Overall, our work indicates that under mild asynchronism, much of the benefits of coded caching can still be leveraged.

B. Related Work

The area of coded caching has seen a flurry of research activity along several dimensions in recent years. From a theoretical perspective, significant work has attempted to understand the fundamental rate limits of a coded caching system. Extensions of the basic model to general networks have been examined in [7]–[9]. Issues related to subpacketization (i.e., the number of subfiles) have been considered in [10]–[12]. A high subpacketization level can cause several issues in practical implementations.

In [13], [14], we presented a preliminary version of the offline formulation and the dual decomposition method. The current paper includes a more compact representation of the associated optimization problems, exhaustive simulation results and all the proofs. Moreover, the part dealing with online algorithms has not appeared in prior work.

There are relatively few prior works that have considered asynchronism within the context of coded caching. To our best knowledge, it was first studied in [15]. They considered the decentralized coded caching model [16], and considered a situation where each subfile has a specific deadline. Only the online case was considered and heuristics for transmission from the server were proposed. The heuristics are found to have good performance. However, the transmission time for a packet was not considered in their formulation. Reference [17] also considers the asynchronous setting; again, they do not consider the transmission time of a transmitted packet. In that sense, their setting is closer to the work of [15] and can be viewed as a set of rules that the server should follow in the online case. [17] (Section III.C) also considers an offline setting for the centralized placement scheme of [16]. In contrast, our LP formulation can be viewed as a bound on the possible performance of any online scheme. Our proposed online algorithm has significantly better performance than the ones presented in [15].

Reference [16] (Section V.C) also discusses the issue of asynchronism within the context of decentralized coded caching, without considering deadlines or packet transmission times. They advocate a further subpacketization of each subfile (referred to as a segment in [16]). It is important to note that any system will need to commit to a certain subpacketization scheme before deployment. Given this subpacketization and with user specified deadlines, the formalism of our work and our algorithms can be used to arrive at schemes that address asynchronous requests.

The work of [18] proposes an algorithm for the online scenario under the assumption of decentralized coded caching for reducing the worst-case load of fronthaul links in fog radio access networks (F-RANs); this is a different model than ours. Their work does not take transmission time into account and considers the scenario where each user has the same deadline.
The asynchronous setting has also been considered in [19] for video delivery by taking into account an appropriately defined audience retention rate. Their work considers a probabilistic arrival model and presents a decentralized coded caching scheme for it.

III. PROBLEM FORMULATION AND PRELIMINARIES

We assume that time $\tau \geq 0$ is slotted. Let $[n]$ denote the set $\{1, \ldots, n\}$ and the symbol $\oplus$ represent the XOR operation. We assume that the server contains $N \geq K$ files denoted by $W_n, n = 1, \ldots, N$. The subfiles are denoted by $W_{n,f}$ so that $W_n = \{W_{n,f} : f \in [F]\}$ and the cache of user $i$ by $Z_i \subseteq \{W_{n,f} : n \in [N], f \in [F]\}$. $Z_i$ contains at most $M_iF$ subfiles. In the delivery phase, user $i$ requests file $W_{d_i}$, where $d_i \in [N]$, from the server. We let $\Omega^{(i)}$ denote the indices of the subfiles that are not present in the $i$-th user’s cache, i.e.,

$$\Omega^{(i)} = \{f : f \in [F], W_{d_i,f} \notin Z_i\}.$$ 

The equations in the delivery phase are assumed to be of the all-but-one type.

Definition 1: All-but-one equation. Consider an equation $E$ such that

$$E = \oplus_{l=1}^{\ell} W_{d_i,l}.$$ 

We say that $E$ is of the all-but-one type if for each $l \in [\ell]$, we have $W_{d_i,l} \notin Z_i$ and $W_{d_i,l} \in Z_{ik}$ for all $k \in [\ell] \setminus \{l\}$. It is evident that an all-but-one equation transmitted from the server allows each of the users participating in the equation to recover a missing subfile that they need. The asynchronous coded caching problem can be formulated as follows.

Inputs.

- User requests. User $i$ requests file $W_{d_i}$, with $d_i \in [N]$ at time $T_i$.
- Deadlines. The $i$-th user needs to be satisfied by time $T_i + \Delta_i$, where $\Delta_i$ is a positive integer.
- Transmission delay. Each subfile needs $r$ time-slots to be transmitted over the shared link, i.e., each subfile can be treated as equivalent to $r$ packets, where each packet can be transmitted in one time slot.

As the problem is symmetric with respect to users, w.l.o.g. we assume that $T_1 \leq T_2 \leq \ldots \leq T_K$. Let $T_{\text{max}} = \max_{1 \leq i \leq K}(T_i + \Delta_i)$. Note that upon sorting the set of arrival times and deadlines, i.e., $\cup_{\ell=1}^{K} \{T_i, T_i + \Delta_i\}$, we can divide the interval $[T_1, T_{\text{max}}]$ into at most $2K - 1$ non-overlapping intervals. Let the integer $\beta$, where $1 \leq \beta \leq 2K - 1$ denote the number of intervals. Let $\Pi_1, \ldots, \Pi_{\beta}$ represent the intervals where $\Pi_i$ appears before $\Pi_j$ if $i < j$; $|\Pi_i|$ denotes the length of interval $\Pi_i$. The intervals are left-closed and right-open. An easy to see but very useful property of the intervals that we have defined is that for a given $i$, either $[T_i, T_i + \Delta_i] \cap \Pi_\ell = \Pi_\ell$ or $[T_i, T_i + \Delta_i] \cap \Pi_\ell = \emptyset$. Fig. 2 shows an example when $K = 3$. We define $U_i = \{i \in [K] : [T_i, T_i + \Delta_i] \cap \Pi_\ell = \Pi_\ell\}$, and $D_i = \{d_i \in [N] : i \in U_i\}$. Thus, $U_i$ is the set of active users in time interval $\Pi_\ell$ and $D_i$ is the corresponding set of active file requests.

Fig. 2: Offline solution corresponding to the Example 1. The double-headed arrows show the active time slots for each user. The transmitted equations are shown above the timeline.

Outputs.

- Transmissions at each time slot. If the problem is feasible, the schedule specifies which equations (of the all-but-one type) need to be transmitted at each time. The schedule is such that each user can recover all its missing subfiles within its deadline. The equations transmitted at time $\tau \in \Pi_\ell$ only depend on $D_i$.

We consider two versions of the above problem.

- Offline version. In the offline version, we assume the server is aware of $\{T_i, \Delta_i, d_i\}_{i=1}^{K}$ at $\tau = 0$. However, at time $\tau \in \Pi_\ell$ the transmitted equation(s) will only depend on $D_i$, i.e., the server cannot start sending missing subfiles for a given user until its request arrives.
- Online version. In the online version, information about the file requests are revealed to the server as time progresses. At each time $\tau$, the server only has information about $\{T_i, \Delta_i, d_i\}$ if $T_i \leq \tau$, i.e., the requests that have arrived by time $\tau$.

We begin by defining some relevant sets; for convenience, a tabulated list of most of the items needed in the subsequent sections can be found in Table I. Consider a subset of users $U \subseteq [K]$. For each user $i \in U$ we let $F_{(i,U)}$ denote the indices of all missing subfiles of the $i$-th user that have been stored in the cache of the other users in $U$, i.e.,

$$F_{(i,U)} = \{f \in \Omega^{(i)} : W_{d_i,f} \in Z_j \text{ for all } j \in U \setminus \{i\}\}.$$ 

Definition 2: User Group. A subset $U \subseteq [K]$ is said to be a user group if $F_{(i,U)} \neq \emptyset$ for all users $i \in U$ so that there is at least one all-but-one type equation associated with $U$.

We note that for a user group $U$ there are $\prod_{i \in U} |F_{(i,U)}|$ different all-but-one equations. Recall that $U_\ell$ is the set of active users in time interval $\Pi_\ell$ and $D_\ell$ represents their file requests. Let $U_\ell$ be a subset of the power set of $U_\ell$ (i.e. the set of all subsets of $U_\ell$) such that each element in $U_\ell$ is an user group (cf. Definition 2). For any $U \subseteq [K]$, let $\mathcal{I}_U$ be the set of indices of all time intervals where the users in $U$ are simultaneously active, i.e.,

$$\mathcal{I}_U = \{\ell : [T_\ell, T_\ell + \Delta_\ell] \cap \Pi_\ell = \Pi_\ell, \forall i \in U\}.$$ 

For each missing subfile $W_{(d_i,f)}$ (where $f \in \Omega^{(i)}$) we let $U_{(i,f)}$ denote the set of user groups where it can be transmitted, i.e.,

$$U_{(i,f)} = \{U \in \bigcup_{\ell=1}^{\beta} U_\ell : i \in U, f \in F_{(i,U)}\}.$$ 

We note here that for a fixed $i$, there are potentially multiple indices $f_1, f_2, \ldots, f_\ell \in \Omega^{(i)}$ such that $U \in U_{(i,f_j)}$ for $j = 1, \ldots, \ell.$
Example 1: Consider a system (shown in Fig. 2) with $N = 3$ files, $W_1$, $W_2$, and $W_3$ where each file is divided into three subfiles, so that $F = 3$. There are $K = 3$ users with the following cache content, $Z_1 = \{W_{2,1}, W_{2,2}, W_{3,3}\}$, $Z_2 = \{W_{1,2}, W_{2,3}, W_{3,1}\}$, and $Z_3 = \{W_{2,2}, W_{3,2}, W_{2,1}\}$. Thus, $M_i = 1$ for $i \in [K]$. The arrival times are $T_1 = 0$, $T_2 = 1$, $T_3 = 3$, and deadlines are $\Delta_1 = 5$, $\Delta_2 = 4$, and $\Delta_3 = 2$. The $i$-th user requests file $W_i$, for $i = 1, \ldots, 3$. Therefore, $\Omega^{(1)} = \{1, 2, 3\}$, $\Omega^{(2)} = \{1, 2\}$, and $\Omega^{(3)} = \{1, 3\}$.

In this system we have $F_{(1,1,2)} = \{1\}$ as $W_{1,1} \in Z_2$, and $F_{(2,1,2)} = \{1, 2\}$ as $W_{2,2} \in Z_1$. Therefore, $\{1, 2\}$ is an user group and the corresponding all-but-one equations are $W_{1,1} \oplus W_{2,1}$ and $W_{1,1} \oplus W_{2,2}$. However, $F_{(1,1,3)} = \emptyset$ thus $\{1, 3\}$ is not an user group.

As $U = \{1, 2\}$ is an user group, we have $U_2 = \{\{1\}, \{2\}, \{1, 2\}\}$. The set of time intervals where user group $\{1, 2\}$ is active is $I_{\{1,2\}} = \{2, 3\}$. Finally, note that user group $U = \{2, 3\}$ is a member of $U_{(2,1)}$ since $2 \in U$ and $1 \in F_{(2,1)} = \{1, 2\}$. Similarly, $U \in U_{(2,2)}$ as well since $2 \in U$ and $2 \in F_{(2,2)}$.

IV. OFFLINE ASYNCHRONOUS CODED CACHING

In this section, we discuss the offline version of the problem where the server has the knowledge of the arrival times/deadlines of all the requests at $\tau = 0$. The offline solution of the system in Example 1 is depicted in Fig. 2 where the transmitted equation in each time slot appears above the timeline. It can be verified that each user can recover the missing subfiles that they need. In what follows we argue that the offline setting can be cast as a linear programming problem.

A. Linear programming formulation

For each time interval $\Pi_\ell$ with $\ell = 1, \ldots, \beta$ and for each $U \in U_\ell$ we define variable $x_{U}(\ell) \in [0, |\Pi_\ell|]$ that represents the portion of time interval $\Pi_\ell$ that is allocated to an equation that benefits user group $U$. The actual equation will be determined shortly. For each missing subfile $W_{(d,f)}$ and each $U \in U_{(i,f)}$ we define variable $y_{(i,f)}(U) \in [0, r]$ that represents the portion of the missing subfile $W_{(d,f)}$ transmitted within some or all of the equations associated with $x_{U}(\ell)$ for $\ell \in I_\ell$. As pointed out before, for a fixed $i$, $U$ can be used to transmit different missing subfiles needed by user $i$. However, a single equation can only help recover one missing subfile needed by $i$. Thus, $\sum_{\ell \in I_\ell} x_{U}(\ell)$ must be shared between the appropriate $y_{(i,f)}(U)$’s. Accordingly, we need the following constraint for user $i$ and a user group $U$ which contains $i$.

$$\sum_{f \in F_{(i,U)}} y_{(i,f)}(U) \leq \sum_{\ell \in I_\ell} x_{U}(\ell).$$

In addition, at time interval $\Pi_\ell$ at most $|\Pi_\ell|$ packets can be transmitted, so that $\sum_{U \in U_\ell} x_{U}(\ell) \leq |\Pi_\ell|$. To ensure that each missing subfile $W_{(d,f)}$ is transmitted in exactly $r$ time slots we have $\sum_{U \in U_{(i,f)}} y_{(i,f)}(U) = r$.

The following LP minimizes the overall rate of transmission from the server while respecting all the deadline constraints of the users.

$$\min \sum_{\ell = 1}^{\beta} \sum_{U \in U_\ell} x_{U}(\ell)$$

s.t. $\sum_{U \in U_\ell} x_{U}(\ell) \leq |\Pi_\ell|$, for $\ell = 1, \ldots, \beta,$

$\sum_{f \in F_{(i,U)}} y_{(i,f)}(U) \leq \sum_{\ell \in I_\ell} x_{U}(\ell)$, for $i \in U$, $U \in \cup_{\ell = 1}^{\beta} U_\ell,$

$\sum_{U \in U_{(i,f)}} y_{(i,f)}(U) = r$, for $f \in \Omega^{(i)}$, $i \in [K],$

$y_{(i,f)}(U) \geq 0$, for all $i \in [K], \ell \in [\beta], U \in \cup_{\ell = 1}^{\beta} U_\ell.$

Note that [15] considers the case when each missing subfile has a prescribed deadline. Our LP above can be modified in a straightforward manner to incorporate this aspect.

B. Interpretation of feasible point of $\Omega$ as a coding solution

We start by assigning time intervals to user groups. The time interval $\Pi_\ell$, $\ell \in [\beta]$, will be arbitrarily assigned to user groups $U \in U_\ell$ so that the time assigned to one user group does not overlap with another. The constraint $\sum_{U \in U_\ell} x_{U}(\ell) \leq |\Pi_\ell|$ implies that such an assignment exists. For each user group $U$ and each $i \in U$, suppose that $f_1, \ldots, f_l \in F_{(i,U)}$ are such that $y_{(i,f_j)}(U) \neq 0$ for $j = 1, \ldots, l$. We assign $y_{(i,f_j)}(U)$ part of the total time allocated to user group $U$, i.e., $\sum_{\ell \in I_\ell} x_{U}(\ell)$, to the missing subfile $W_{d,f_j}$ for $j = 1, \ldots, l$. The constraint $\sum_{f \in F_{(i,U)}} y_{(i,f)}(U) \leq \sum_{\ell \in I_\ell} x_{U}(\ell)$ ensures that such an assignment always exists, i.e., it is possible to assign $y_{(i,f)}(U)$’s (for fixed $i$) to the available (strictly) positive $x_{U}(\ell)$’s, such that there is no overlap between them. This assignment is not unique in general. However, this is not a problem as any assignment can be used to determine the equations. This process is repeated for all users $i \in U$.

The equation transmitted on a particular interval is simply the XOR of the subfile indices that map to that interval. This equation is valid since the missing subfile $W_{d,f}$ with $f \in F_{(i,U)}$ is in the cache of all the users in $U \setminus \{i\}$.

Finally, according to the constraint $\sum_{U \in U_{(i,f)}} y_{(i,f)}(U) = r$, each missing subfile $W_{d,f}$ is transmitted in its entirety in some equation. The following example serves to illustrate the arguments above.

### Table I: List of variables used in the description

| Variable | Description |
|----------|-------------|
| $T_i$    | arrival time of user $i$ |
| $T_i + \Delta_i$ | deadline of user $i$ |
| $\beta$ | number of time intervals |
| $\Pi_\ell$ | time interval $\ell$ |
| $U_\ell$ | set of the active users in time interval $\ell$ |
| $\Omega^{(i)}$ | set of the indices of missing subfiles of user $i$ |
| $U_{(i,f)}$ | set of all subsets of $U_\ell$ that are user groups |
| $\Omega^{(i)}(U)$ | set of the indices $\ell \in [\beta]$ so that $U \in U_\ell$ |
| $x_{U}(\ell)$ | portion of $\Pi_\ell$ allocated to user group $U$ |
| $y_{(i,f)}(U)$ | portion of $W_{d,f}$ transmitted within user group $U$ |
| $F_{(i,U)}$ | set of the indices of $f \in \Omega^{(i)}$ that can be transmitted within $U$ |

Note that [15] considers the case when each missing subfile has a prescribed deadline. Our LP above can be modified in a straightforward manner to incorporate this aspect.
time assigned to \(\{1, 2\}\)  
blue: time assigned to \(\{2, 3\}\)  
red: time assigned to \(\{2, 3\}\)  
green: time assigned to \(\{3, 1\}\)  
blue: time assigned to \(\{y_{1,2}\}\)  
red: time assigned to \(\{y_{1,2}\}\)  
green: time assigned to \(\{y_{1,2}\}\)

Fig. 3: Interpretation of feasible point in (1) for Example 1. For readability, only equations corresponding to user groups \(\{1, 2\}\) and \(\{2, 3\}\) are depicted.

**Example 2:** Consider again the system in Example 1. Part of a feasible solution to the LP in (1), corresponding to user groups \(U = \{2, 3\}\) and \(U' = \{1, 2\}\), is presented below:

\[
\begin{align*}
{x}_{1,2}(2) &= 0.5, & {x}_{1,2}(3) &= 0.5, & {x}_{(2,3)}(3) &= 1, \\
y_{1,1}(\{1, 2\}) &= 1, & y_{2,1}(\{1, 2\}) &= 0.5, & y_{2,2}(\{1, 2\}) &= 0.5, \\
y_{2,1}(\{2, 3\}) &= 0.5, & y_{2,2}(\{2, 3\}) &= 0.5, & y_{3,1}(\{2, 3\}) &= 1.
\end{align*}
\]

According to the solution, \(x_{(2,3)}(3) = 1\). Therefore, only one unit of \(\Pi_3\) is assigned to \(U\) (though \(\Pi_3 = 2\)). This is denoted by the light blue color line in Fig. 3. For user 3 \(\in U\), there is only one missing subfile in \(F_{\{2,3\}}\), namely \(W_{3,1}\). As \(y_{2,1}(\{2, 3\}) = 1\) it is assigned to \(x_{(2,3)}(3)\) in its entirety. This is depicted by the gray line in Fig. 3. For user 2 in \(U\) we have \(F_{\{2,3\}} = \{1, 2\}\). The solution specifies \(y_{2,1}(\{2, 3\}) = y_{2,2}(\{2, 3\}) = 0.5\). Thus, we assign the first half of \(x_{(2,3)}(3)\) to missing subfile \(W_{2,1}\) and the second half to \(W_{2,2}\) (see the dark blue and dotted dark blue lines in Fig. 3). Accordingly, the server transmits equations such that the first half of the time interval assigned to user group \(U\) corresponds to the \(E_2 = W_{2,1} \oplus W_{3,1}\) whereas the second half corresponds to \(E_3 = W_{2,2} \oplus W_{3,1}\). The interpretation of the user group \(U'\) is similar (see Fig. 3).

**Remark 1:** The output of the above LP will typically result in a fractional solution for the variables. A fractional solution can be interpreted by assuming that each packet that is transmitted over the shared edge can be subdivided as finely as needed. Thus, in each time slot we can transmit multiple equations that may serve potentially different subsets of users. This assumption is reasonable if the underlying subfiles and hence the packets are quite large. In any case the above LP provides a lower bound on the performance of a solution where integrality constraints are enforced.

**Remark 2:** We note that for the offline solution, within a given time interval, the user groups can be assigned in any order according to the \(x_{U}(f)\)'s as long as they don’t overlap. Moreover the assignment of \(y_{i,f}(U)\)'s is also arbitrary as long the constraints of the LP are respected. However for the online case (cf. Section 5), ordering does matter since we make a best effort decision on each individual slot as we have no knowledge of future arrivals.

**Remark 3:** The complexity of our solution does not have any dependence on the arrival times \(T_i\)'s and the deadlines \(\Delta_i\)'s. Our formulation of the LP in terms of the intervals allows us to circumvent this potential dependence. A straightforward formulation of the above problem would assign variables for each time slot which would be very expensive.

Nevertheless, the complexity of the solving the LP does grow quite quickly (cubic) in the problem parameters. Next, we discuss a solution based on dual decomposition.

### C. Dual Decomposition based LP solution

As it stands, the LP in (1) cannot be interpreted as a network flow. Yet, intuitively one can view the missing subfiles from each user as flowing through the user groups and getting absorbed in sinks that correspond to their valid time intervals. However, the flows corresponding to different users can be shared as the all-but-one equations allow different users to benefit from the same equation. We note here that a similar sharing of flows also occurs in the problem of minimum cost multicast with network coding [20]. The LP in (1) can however be modified slightly so that the corresponding dual function is such that it can be evaluated by solving a set of *decoupled minimum cost network flow optimizations.*

1. **Decoupling procedure:** For each user \(i \in U\) the variable \(x_{U}(f)\) represents the amount of flow corresponding to user \(i\) outgoing from user group \(U\) to time interval \(\Pi_i\). Evidently, this amount can’t be more than \(x_{U}(f)\). Therefore, we have

\[
x_{U}(f) \leq x_{U}(f),
\]

which holds for all \(i \in U\) and all \(U \in U_i, f \in [\beta]\). We define \(\Omega_\ell^{i} \subseteq U_\ell\) to be the subset of possible user groups at time interval \(\Pi_\ell\) that include user \(i\), i.e., \(i \in U\) for all \(U \in \Omega_\ell^{i}\).

By the flow interpretation of \(x_{U}(f)\), we have

\[
f \in F_{\{f, U\}} \sum_{\ell \in U_{\ell}^{i}} y_{i, f}(U) = \sum_{\ell \in U_{\ell}^{i}} x(f)_{U}(f),
\]

for all \(U \in \cup_{\ell=1}^{\beta} U_\ell^{i}\). For \(i = 1, \ldots, K\), let \(C_i\) denote the following set of constraints.

\[
f \in F_{\{f, U\}} \sum_{\ell \in U_{\ell}^{i}} y_{i, f}(U) = \sum_{\ell \in U_{\ell}^{i}} x(f)_{U}(f), \quad f \in \Omega_\ell^{i},
\]

\[
U \in U_\ell^{i}, \quad \beta \leq \sum_{\ell \in U_{\ell}^{i}} x(f)_{U}(f) \geq 0, \quad U \in U_\ell^{i}, i \in [\beta],
\]

Then, the original LP can be compactly rewritten as

\[
\begin{align*}
\min \sum_{\ell=1}^{\beta} \sum_{U \subseteq U_\ell^{i}} x(f)_{U}(f) & \quad (2) \\
\text{s.t.} \quad x(f)_{U}(f) \leq x(f)_{U}(f) & \quad f \in U_\ell^{i}, i \in [\beta], \ell \in [\beta],
\end{align*}
\]

It is not too hard to see that the LPs in (1) and (2) are equivalent. The only difference with respect to (1) is the introduction of variables \(x(f)_{U}(f)\) (for appropriate ranges of \(i, U\) and \(f\)) such that the second set of inequality constraints in (1) are replaced by equality constraints. Moreover, the original constraints are maintained by setting \(x(f)_{U}(f) \leq x(f)_{U}(f)\).

We proceed by considering the dual of the LP in (2) with respect to the constraints that
involve the variables \( x_U(\ell) \). The Lagrangian
\[
\mathcal{L}(\{x_U(\ell), x^{(i)}_U(\ell), \lambda^{(i)}_U(\ell)\}_{i \in U, U \in \mathcal{U}, \ell \in [\beta]}, \{\zeta_\ell\}_{\ell \in [\beta]})
\]
can be expressed as
\[
\mathcal{L} = \sum_{\ell \in U} \sum_{s \in U_{\ell}} x_U(\ell) + \sum_{\ell \in U} \sum_{s \in U} \lambda^{(i)}_U(\ell) \left( x^{(i)}_U(\ell) - x_U(\ell) \right)
+ \sum_{\ell \in U} \zeta_\ell \left( \sum_{s \in U_{\ell}} x_U(\ell) - |\Pi_\ell| \right)
\]
where \( \lambda^{(i)}_U(\ell) \)'s and \( \zeta_\ell \)'s are nonnegative dual variables. It turns out that minimizing the Lagrangian for fixed dual variables can be simplified by defining \( \gamma^{(i)}_U(\ell) = \lambda^{(i)}_U(\ell)/(1 + \zeta_\ell) \) for \( i \in U, U \in \mathcal{U}, \) and \( \ell \in [\beta] \). We define \( \Gamma^{(i)} = \{\gamma^{(i)}_U(\ell), \ell \in \mathcal{I}_U, U \in \mathcal{U}_i, \ell \in [\beta]\} \), \( x = \{x_U(\ell), U \in \mathcal{U}, \ell \in [\beta]\} \), and \( x^{(i)} = \{x^{(i)}_U(\ell), \ell \in \mathcal{I}_U, U \in \mathcal{U}_i\} \). The dual function \( g(\Gamma^{(1)}, \ldots, \Gamma^{(K)}, \{\zeta_\ell\}_{\ell \in [\beta]}) \) is obtained by solving for
\[
\min_{x, x^{(1)}, \ldots, x^{(K)}} \mathcal{L} \quad \text{s.t.} \quad C_1, C_2, \ldots, C_K.
\]
It is evident that the dual function \( g(\Gamma^{(1)}, \ldots, \Gamma^{(K)}, \{\zeta_\ell\}_{\ell \in [\beta]}) \) takes a nontrivial value only if
\[
\sum_{i \in U} \gamma^{(i)}_U(\ell) = 1, \quad \forall U \in \mathcal{U}_i, \ell \in [\beta].
\]
The evaluation of \( g(\Gamma^{(1)}, \ldots, \Gamma^{(K)}, \{\zeta_\ell\}_{\ell \in [\beta]}) \) at a fixed set of dual variables \( \Gamma^{(i)} \)'s and \( \zeta_\ell \)'s can therefore be written as
\[
\min_{x^{(1)}, \ldots, x^{(K)}} \sum_{i \in U} \sum_{s \in U} (1 + \zeta_\ell) \gamma^{(i)}_U(\ell) x^{(i)}_U(\ell) - \sum_{i \in U} \zeta_\ell |\Pi_\ell| \quad \text{s.t.} \quad C_i, C_2, \ldots, C_K.
\]
We emphasize that (3) is still a convex problem and that \( \gamma^{(i)}_U(\ell), \zeta_\ell \geq 0 \). Let \( h_i(\Gamma_i, \{\zeta_\ell\}_{\ell \in [\beta]}), i \in [K] \) be
\[
h_i(\Gamma_i, \{\zeta_\ell\}_{\ell \in [\beta]} = \min_{x^{(i)}} \sum_{i \in U} \sum_{s \in U} (1 + \zeta_\ell) \gamma^{(i)}_U(\ell) x^{(i)}_U(\ell),
\]
with the constraint \( C_i \).
Then, the dual function becomes
\[
g(\Gamma^{(1)}, \ldots, \Gamma^{(K)}, \{\zeta_\ell\}_{\ell \in [\beta]}) =
\sum_{i = 1}^{K} h_i(\Gamma_i, \{\zeta_\ell\}_{\ell \in [\beta]} - \sum_{\ell \in [\beta]} \zeta_\ell |\Pi_\ell|,
\]
if \( \sum_{i \in U} \gamma^{(i)}_U(\ell) = 1 \) for all \( U \in \mathcal{U}_i, \ell \in [\beta] \). We present an approach to maximize the dual function in (5) shortly.

The sub-problem in (4) for fixed \( \Gamma_i \) and \( \{\zeta_\ell\}_{\ell \in [\beta]} \) is a standard minimum-cost flow problem. The associated flow network corresponding to user \( i, i \in [K] \), depends on \( \Gamma_i \) and \( \{\zeta_\ell\}_{\ell \in [\beta]} \) and we denote it by \( \mathcal{N}_i(\Gamma_i, \{\zeta_\ell\}_{\ell \in [\beta]} \). It contains a source node \( s \) and three intermediate layers followed by a terminal node \( t \) (see Fig. 4 for an example). The nodes in the first, second, and third layer correspond to missing subfiles in \( \Omega^{(i)} \), user groups in \( U \in \mathcal{U}^{(i)} \), and time intervals \( \{\Pi_\ell : \ell \in [\beta] \) and \( i \in U_i\} \) respectively. The edges in \( \mathcal{N}_i(\Gamma_i, \{\zeta_\ell\}_{\ell \in [\beta]} \) can be expressed as follows. There are \( |\Omega^{(i)}| \) edges going from source node \( s \) to each of missing subfiles in \( \Omega^{(i)} \). Also, for each \( f \in \mathcal{F}_{i,U} \) there is an edge going from missing subfile node \( f \) to user group node \( U \). Furthermore, there is an edge going from user group \( U \in \cup_{\ell \in [\beta]} |\Pi_\ell| \) to time interval \( \Pi_\ell \) for each \( \ell \in \mathcal{I}_U \). Finally, corresponding to each time interval in \( \{\Pi_\ell : \ell \in [\beta] \) and \( i \in U_i\} \) there is an edge going from this time interval to the terminal node \( t \).

In flow network \( \mathcal{N}_i(\Gamma_i, \{\zeta_\ell\}_{\ell \in [\beta]} \), \( i \in [K] \), a zero cost is assigned to all edges except those from the user group nodes to the time intervals. The cost of the edge between user group \( U \) and time interval \( \Pi_\ell \) is \((1 + \zeta_\ell) \gamma^{(i)}_U(\ell) \). The edge between time interval \( \Pi_\ell \) and the terminal node has a capacity constraint of \( |\Pi_\ell| \) and the edge between the source node and a missing subfile has a capacity constraint of \( r \); the other edges have no capacity constraint. The variable \( x^{(i)}_U(\ell) \) is the amount of flow carried by the edge from user group \( U \) to time interval \( \Pi_\ell \). The source injects a flow of value \( |\Omega^{(i)}| r \) which needs to be absorbed in the terminal.

We emphasize that minimum cost network flow algorithms have been subject of much investigation recent within the optimization literature and large scale instances can be solved very quickly. For our work we leverage Capacity Scaling algorithms within the open-source LEMON package [21].

2) Maximizing the dual function: The dual function in (5) is concave (as it can be expressed as the pointwise infimum of a family of affine functions of the dual variables (22)). We exploit the projected subgradient method to maximize the dual function iteratively. Let \( x_U^{(i)}(\ell, n - 1) \) for all \( i \in [K], U \in \mathcal{U}_i \) denote the optimal point of (4) when solved for \( i \in [K] \) at the \( n - 1 \)-th iteration. Let \( \{\gamma^{(i)}_U(\ell, n - 1), \zeta^{(i)}(n - 1), \forall U \in \mathcal{U}_i, \ell \in [\beta], i \in [K] \} \) denote a feasible dual point of (5) at the \( (n - 1) \)-th iteration.

According to the subgradient method for the \( n \)-th iteration we first determine for \( i \in [K] \)
\[
\delta^{(i)}_U(\ell, n) = \gamma^{(i)}_U(\ell, n - 1) + \theta_n x^{(i)}_U(\ell, n)(1 + \zeta^{(i)}(n - 1)),
\]
where \( \theta_n \) is the step size. These intermediate variables are projected onto the feasible set and primal recovery is per-
formed by the method of \cite{23}. The details can be found in the Appendix \[A\]. Numerical results appear in Section \[VI\].

V. ONLINE ASYNCHRONOUS CODED CACHING

In the online scenario, at time $\tau$ only information about the already arrived requests are known to the server, i.e., it only knows $T_i$, $d_i$, and $\Delta_i$ for $i \in [K]$ such that $T_i \leq \tau$. Ideally, one would want to design an online algorithm that is guaranteed to be feasible whenever the corresponding offline version is feasible. However, this appears to be a hard problem. Specifically, routinely used algorithms such as earliest-deadline-first (EDF) do not have this property \cite{24}. In this section we discuss certain characteristics of the online solution that distinguish it from the offline solution (Section \[V-A\]) and our proposed online algorithm and its properties (Section \[V-B\]).

A. Necessity of coding across missing subfiles of a user

Example 3: Consider a system with $N = K = 3$ and $M_i = 1$ with $Z_i = \{W_{n,i} : n \in [N]\}$ for $i \in [K]$. The arrival times and deadlines of the users are $T_i = i$, and $\Delta_i = 2$ for $i \in [K]$ (as shown in Fig. 5). We assume that user $i$ is interested in files $W_i$ for $i \in [K]$ and that transmitting a subfile takes a single time slot, i.e., $r = 1$.

Suppose that there is an adversary that makes decisions on when a particular user request arrives. We assume that the adversary can see the decisions made by the server. Suppose that the server does not code across any user’s missing subfiles. At $\tau = 1$, it has the choice to transmit either $W_{1,2}$ or $W_{1,3}$. We emphasize that it has to transmit either of these as the deadline for user 1 is $T_1 + \Delta_1 = 3$. If the server transmits $W_{1,3}$, then the adversary can force the arrival of the third user with $(T_3, \Delta_3) = (2, 2)$ and subsequently the arrival of the second user with $(T_2, \Delta_2) = (3, 2)$. In this case, the server is forced to transmit $W_{1,2}$ at $\tau = 2$, which implies that user 3 misses its deadline. In a similar manner, if the server transmits $W_{1,2}$, the adversary can easily generate an arrival pattern so that user 2 misses its deadline.

This issue can be circumvented if we transmit a linear combination of both $W_{1,2}$ and $W_{1,3}$ in the first time slot as shown in Fig. 5. Intuitively, this is the correct strategy since transmitting $W_{1,3} \oplus W_{1,2}$ allows the server to hedge its bets against the identity of the next request arrival. This example demonstrates that coding across missing subfiles of user 1 is strictly better than the alternative. We emphasize that the synchronized model of \cite{1} and the offline scenario do not require this.

Accordingly, for the online scenario we treat each missing subfile $W_{d_i,f}$ as an element of a large enough finite field $\mathbb{F}$. This allows us to consider linear combinations of the missing subfiles over $\mathbb{F}$. Note that any equation of the form

\[ \bigoplus_{i \in U} \bigoplus_{f \in \mathcal{F}(i,U)} \alpha_{i,f} W_{d_i,f}, \]

where the coefficients $\alpha_{i,f}$ belong to the field $\mathbb{F}$ is also an all-but-one equation from which user $i$ can recover $\bigoplus_{f \in \mathcal{F}(i,U)} \alpha_{i,f} W_{d_i,f}$.

B. Recursive LP based algorithm

The online scenario differs significantly from the offline one. At time $\tau$ our only decision is to transmit an equation in the time slot $[\tau, \tau + 1)$. In particular, it is possible that a request arrives at $\tau + 1$ and that can change the situation drastically. It makes intuitive sense to transmit equations that benefit a large number of users. However, we also need to take into account the deadline constraints of each user. These requirements need to be balanced.

Our proposed online algorithm leverages the offline LP for enforcing the deadline constraints. We solve an LP which is similar to \cite{1} each time a new user request comes into the system. This specifies a set of $x_{U}(\ell)$ and $y_{i,f}(U)$ variables. However, in the offline case, the ordering of the $x_{U}(\ell)$ within an interval does not matter (cf. Remark \[2\]). In the online case, this is no longer true. As we have no knowledge of future arrivals, it becomes important to choose the “best” user group for the time slot in which transmission needs to take place. Furthermore in each time slot, exactly one equation is transmitted, i.e., for each time slot only one user group is chosen for transmission. In contrast, in the offline LP, the fractional nature of the solution may require sharing of a time slot between multiple user groups.

Accordingly, on the $x_{U}(\ell)$ variables we first decide a candidate list of feasible user groups that can be chosen for transmission at each time slot. We calculate a metric for each feasible user group $U$ depending upon (i) the stringency of the deadlines of the users in $U$, and (ii) the benefit of this equation to the participating users. If this metric is above a system-defined threshold, we transmit an equation corresponding to this user group in the time slot following the user’s request. Following this we update certain variables and the process continues for each time slot thereafter. When the next user request arrives into the system, the history of the variable assignments is used to solve a new LP (similar to \cite{1}), and the process continues recursively.

Consider a time $\tau = T_k$ when the request of the $k$-th user arrives at the server. We let $\mathcal{U}_{\text{act}}(\tau)$ be the set of user groups associated with the previously transmitted equations. We also let $z_{U}(\tau)$ be the total time allocated to equations corresponding to user group $U$ prior to time $\tau$. Thus, if in time interval $[\tau, \tau + 1)$ the server transmits an equation that exclusively benefit users in $U$ then $z_{U}(\tau + 1) = z_{U}(\tau) + 1$ otherwise $z_{U}(\tau + 1) = z_{U}(\tau)$. Time intervals $\Pi_{1,k}, \ldots, \Pi_{\beta_k,k}$ are formed by the set of times in

\[ \{T_k\} \cup \{T_i + \Delta_i : i \in [k], T_i + \Delta_i > T_k\}. \]

As in the offline case in \cite{1}, the sets of active users $U_{\ell,k}$, user groups $\mathcal{U}_{\ell,k}$ and $\mathcal{I}^{(k)}_{\ell}$ are defined corresponding to these
time intervals, e.g., $U_{ℓ, k}$ is the set of active users in $Π_{ℓ, k}$. Moreover, $V_k$ is a set of user groups that either already have been transmitted or might be transmitted after $τ = T_k$. That is $V_k = U_{\text{seen}}(τ) \cup \{U_{ℓ, k} : ℓ \in [β_k]\}$. The variables $x_U(ℓ)$’s and $y_{i, f}(U)$’s have the same interpretation as the offline case. With these variables, the server solves the following LP.

$$\min \sum_{ℓ=1}^{β_k} \sum_{U ∈ U_{ℓ, k}} x_U(ℓ) \quad (6)$$

subject to

$$s.t. \sum_{U ∈ U_{ℓ, k}} x_U(ℓ) ≤ [Π_{ℓ, k}], \quad \text{for } ℓ = 1, \ldots, β_k$$

$$\sum_{f ∈ F_{i, U}} y_{i, f}(U) ≤ \sum_{ℓ ∈ Z_U^{(k)}} x_U(ℓ) + z_U(Τ_k) \quad \text{for } i ∈ U, U ∈ V_k$$

$$\sum_{U ∈ U_{i, f}} y_{i, f}(U) = r, \quad \text{for } f ∈ Ω_i(ℓ), \quad \forall i ∈ U_{ℓ, k}, ℓ ≤ β_k$$

$$x_U(ℓ), y_{i, f}(U) ≥ 0, \quad \text{for } i ∈ [K], ℓ ∈ [β], U ∈ V_k.$$

An important feature of time intervals $Π_{1, k}, \ldots, Π_{β_k, k}$ is that these time intervals end at a deadline and except the first time interval $Π_{1, k}$ that starts with arrival time $T_k$, the other time intervals start with a deadline. Thus, we have $U_{t+1, k} ⊆ U_{t, k}$, i.e., the set of active users in interval $Π_{t+1, k}$ is a subset of the active users in interval $Π_{t, k}$ for the range of $ℓ$.

Upon solving the LP in (6), the server makes a decision on the equation that will be transmitted in time slot $[T_k, T_k + 1]$. Towards this end, it creates a list of candidate user groups. Let $\{x_U^*, ℓ \in U_{t, k}, ℓ = 1, \ldots, β_k\}$ be the solution of (6) and let $X^* = \{x_U^*(ℓ) : x_U^*(ℓ) ≥ 1\}$. The elements of $X^*$ are first ordered based on time intervals. Then, among the elements with the same time interval, they are ordered based on length of user group. Therefore, for two elements $x_U^*(ℓ), x_U^*(ℓ’)$ in $X^*$ we say $x_U^*(ℓ) ≤ x_U^*(ℓ’)$ if $ℓ < ℓ’$, or if $ℓ = ℓ’$ and $|U| ≥ |U’|$. We let $X^*_{\text{sorted}}$ denote the sorted version of $X^*$ using this procedure. Let $v_τ(ℓ)$ be the number of missing packets (subfiles when $r = 1$) that have been transmitted for user $i$ until time $τ$; this value is tracked in Algorithm [III].

Next, we compute a metric $η_U(τ)$ for each $U ∈ X_{\text{sorted}}$ that measures the overall benefit of transmitting an equation corresponding to user group $U$. If a user group $U$ is chosen for transmission, it may in general benefit different users differently. For instance, if $U$ has been used for transmission in the past, then the current transmission may be less beneficial to some of the users or of no benefit. We demonstrate this by means of the following example.

Example 4: Consider a system $N = K = 5$, $M_i = 2$ for all users $i ∈ [K]$, and $r = 1$. The placement scheme is the same as [III] so that each file is divided to $F = 10$ subfiles and each user misses 6 subfiles. The cache content and missing subfiles are specified below.

$$Z_1 = \{W_{(n, f)} : n ∈ [5], f = 1, 2, 3, 4\}, \quad Ω^{(1)} = \{5, 6, 7, 8, 9, 10\}$$

$$Z_2 = \{W_{(n, f)} : n ∈ [5], f = 1, 3, 6, 7\}, \quad Ω^{(2)} = \{2, 3, 4, 8, 9, 10\}$$

$$Z_3 = \{W_{(n, f)} : n ∈ [5], f = 2, 5, 8, 9\}, \quad Ω^{(3)} = \{1, 3, 4, 6, 7, 10\}$$

$$Z_4 = \{W_{(n, f)} : n ∈ [5], f = 3, 6, 8, 10\}, \quad Ω^{(4)} = \{1, 2, 4, 5, 7, 9\}$$

$$Z_5 = \{W_{(n, f)} : n ∈ [5], f = 4, 7, 9, 10\}, \quad Ω^{(5)} = \{1, 2, 3, 5, 6, 8\}.$$

We assume that the current time is $τ = 8$ and that the request of users 1, 2, 3 have arrived to the server. More specifically, we have $T_1 = 0, T_2 = 2, T_3 = 3, T_4 = 6$ with deadlines $Δ_i = 15$ for all users $i ∈ [K]$. The server has already transmitted eight equations so that

$$U_{\text{seen}}(τ) = \{\{1\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}, \{1, 2, 4\}, \{2, 3, 4\}\}.$$  

At $τ = 8$ the server solves problem (6) with earlier parameters set to $z_{\{1\}}(τ) = 2, z_{\{2\}}(τ) = 1$ for all user groups $U ∈ U_{\text{seen}}(τ) \setminus \{\{1\}\}$, and $z_{\{1\}}(τ) = 0$ otherwise (see Fig. 6). Then, solving (6) for $k = 5$ yields the following $x_U(ℓ)$ variables.

$$x_{(2,3,5)}(1) = 1, \quad x_{(2,4,5)}(1) = 1, \quad x_{(3,4,5)}(1) = 1, \quad x_{(3,4)}(1) = 1, \quad x_{(4,5)}(1) = 1, \quad x_{(2,3,4)}(5) = 2. \quad (7)$$

Suppose that the server uses this solution as follows. It schedules user groups $\{2, 3, 5\}, \{2, 4, 5\}$, and $\{3, 4, 5\}$ at time slots beginning on $τ = 8$, $τ = 9$, and $τ = 10$ respectively. Before $τ = 8$, the third user has benefitted from user groups $\{1, 3\}, \{2, 3\}, \{1, 2, 3\}$, and $\{2, 3, 4\}$.

The third user can recover missing subfiles $W_{(d_3,1)}$ and $W_{(d_3,6)}$ from transmitted equations associated with user groups $\{1, 2, 3\}$ and $\{2, 3, 4\}$ respectively. Moreover, it can obtain a linear combination of subfiles $W_{(d_3, f)}$, $f = 1, 6, 7$ and $W_{(d_3, f)}, f = 1, 3, 4$ from the equations associated with the user groups $\{2, 3\}$ and $\{1, 3\}$ respectively. It is easy to see that subfiles $W_{(d_3,1)}, W_{(d_3,6)}$, and $W_{(d_3,7)}$, along with a linear combination of subfiles $W_{(d_3, f)}, f = 3, 4$ can be recovered from these equations. Now, at time slot $τ = 8$ the server transmits an equation associated with user group $\{2, 3, 5\}$. Clearly, the fifth user benefits from this equation since this user can recover missing subfile $W_{(d_5,5)}$ as $F_{5,2,3,5} = \{5\}$. Similarly, since $F_{3,2,3,5} = \{7\}$ the third user can recover only the missing subfile $W_{(d_5,7)}$ from this equation. However this subfile has been recovered from the earlier equations. Therefore, this equation and user group is not beneficial for the third user. Similarly, one can show that the second user also benefits nothing from this equation and that this equation is only useful for the fifth user. Therefore, the users in a user group might benefit partially or not benefit from the transmitted equation associated with the user group.

Thus, we need a measure of how useful a given $U$ is to a user $i$ when $U$ is used for transmission at a given time slot. This can naturally be described in terms of a related LP that we now describe. For each element $x_U(ℓ)$ in $X_{\text{sorted}}$ let $w_{(i, U)}(τ)$ denote the maximum number of missing packets that are recovered by user $i$ if user group $U$ was chosen.
for transmission at time $\tau$. Let us assume that the server chooses $\tilde{U}$ to transmit an equation at the next time slot and let $\tilde{U}_{\text{sent}}(\tau) = U_{\text{sent}}(\tau) \cup \{ \tilde{U} \}$. Under this assumption we let $\tilde{z}_U(\tau)$, for $U \in \tilde{U}_{\text{sent}}(\tau)$, be such that $\tilde{z}_U(\tau) = z(U)(\tau)$ for $U \in \tilde{U}_{\text{sent}}(\tau) \setminus \{ \tilde{U} \}$ and if $\tilde{U} \in \tilde{U}_{\text{sent}}(\tau)$ then $\tilde{z}_U(\tau) = z(\tilde{U})(\tau) + 1$; otherwise $\tilde{z}_U(\tau) = 1$. Consider the set of all user groups in $\tilde{U}_{\text{sent}}(\tau)$. For each user group $U$ in this set there are $\tilde{z}_U(\tau)$ time slots available. To compute $w(i, \tilde{U})(\tau)$, we need to find an assignment of the missing subfiles in $\Omega^{(i)}$ to each of these time slots so that number of the recovered missing subfiles of the $i$-th user is minimized. We let $\tilde{y}(i, f)(U)$, for $U \in \tilde{U}_{\text{sent}}(\tau)$ with $U \ni i$, to have the same interpretation as $y(i, f)(U)$ in (1). This is equivalent to finding $\tilde{y}(i, f)(U)$’s that maximize $\sum_{U \in \tilde{U}_{\text{sent}}(\tau)} w(i, f)(U)$ under the following constraints. Since for each user group $U \in \tilde{U}_{\text{sent}}(\tau)$ there are $\tilde{z}_U(\tau)$ time slots available, therefore we have $\sum_{f \in F(i)} \tilde{y}(i, f)(U) \leq \tilde{z}_U(\tau)$. Each missing subfile in $\Omega^{(i)}$ needs $r$ time slots but not all of them can be recoverable. Therefore, we have $\sum_{U \in \tilde{U}(i)} (r \tilde{y}(i, f)(U)) \leq \tilde{z}_U(\tau)$. Thus, $w(i, \tilde{U})(\tau)$ can be obtained as the objective function of the following LP:

$$\begin{align*}
\max \sum_{U \in \tilde{U}_{\text{sent}}(\tau)} \sum_{f \in F(i)} \tilde{y}(i, f)(U) \\
\text{s.t.} \sum_{f \in F(i)} \tilde{y}(i, f)(U) \leq \tilde{z}_U(\tau) \quad \forall U \in \tilde{U}_{\text{sent}}(\tau), U \ni i \\
\sum_{f \in F(i)} \tilde{y}(i, f)(U) \leq r \quad \forall f \in \Omega^{(i)}, U \in \tilde{U}(i), \tilde{U}_{\text{sent}}(\tau) \\
\tilde{y}(i, f)(U) \geq 0.
\end{align*}$$

Remark 4: The LP in (8) can also be expressed as a maximum flow problem. The associated flow network consists of a source node $s$, a node for each $f \in \Omega^{(i)}$, a node for each user group $U \in \tilde{U}_{\text{sent}}(\tau)$, and a terminal node $t$. There are edges with capacity $r$ going from $s$ to each $f \in \Omega^{(i)}$ and edges from $f \in \Omega^{(i)}$ to node $U \in \tilde{U}_{\text{sent}}(\tau)$ if $f \in F(i, U)$. The flow on such an edge is $\tilde{y}(i, f)(U)$. Moreover, from each node $U \in \tilde{U}_{\text{sent}}(\tau)$ to $t$ there exist an edge of capacity $\tilde{z}_U(\tau)$. These capacity constraints model the first two inequality constraints in (8). Fig. 7 illustrates an example of this network. It is well-known that if all capacities in a flow network are integers, there exists an integral maximum flow (Micali, Chapter 7). Therefore, there exists an integral solution for $\tilde{y}(i, f)(U)$’s in (8) if $\tilde{z}_U(\tau)$’s are integers.

**Example 5:** In this example we show how the LP in (8) addresses the issue highlighted in Example 4. We consider the setting of this example and solve (8) for the user groups in (7). For simplicity, we only discuss the results for user groups $\{2, 3, 5\}$ and $\{3, 4, 5\}$. The other user groups follow the same rule. Before proceeding, we note that $v_2(\tau) = 5$, $v_3(\tau) = 4$, $v_4(\tau) = 2$, and $v_5(\tau) = 0$. Our goal is to see how these numbers change if the server decides to transmit an equation associated with either of user groups $\{2, 3, 5\}$ or $\{3, 4, 5\}$. If the server chooses $\tilde{U} = \{2, 3, 5\}$ then a nonzero solution for the $\tilde{y}(i, f)(U)$’s for the corresponding users is

$$\tilde{y}_{\{2, 2\}}(\{2, 3\}) = 1, \quad \tilde{y}_{\{2, 2\}}(\{2, 3\}) = 1, \quad \tilde{y}_{\{2, 6\}}(\{2, 3\}) = 1, \quad \tilde{y}_{\{2, 4\}}(\{1, 2\}) = 1, \quad \tilde{y}_{\{2, 3\}}(\{2, 3\}) = 1, \quad \tilde{y}_{\{3, 3\}}(\{1, 3\}) = 0.0, \quad \tilde{y}_{\{3, 4\}}(\{1, 3\}) = 1.0, \quad \tilde{y}_{\{3, 7\}}(\{2, 3\}) = 1, \quad \tilde{y}_{\{3, 1\}}(\{1, 2, 3\}) = 1, \quad \tilde{y}_{\{2, 3\}}(\{2, 3, 4\}) = 1, \quad \tilde{y}_{\{5, 3\}}(\{2, 3, 5\}) = 1.$$

This solution results in $w_{\{2, 3, 5\}}(\tau) = 5, w_{\{3, 3, 5\}}(\tau) = 4, w_{\{5, 2, 3, 5\}}(\tau) = 1$. Therefore, $w_{\{2, 3, 5\}}(\tau) - v_i(\tau)$ is zero for $i = 2, 3$ and one for $i = 5$. This implies that $w_{\{2, 3, 5\}}(\tau)$ correctly captures the benefits of transmitting an equation corresponding to each user in $\{2, 3, 5\}$.

Now, we repeat the same analysis for user group $\{3, 4, 5\}$. After solving (8) for this user group and user 3, the only change comparing to the solution of this user for the user group $\{2, 3, 5\}$ is that $\tilde{y}_{\{3, 3\}}(\{3, 4, 5\}) = 1$ and thus $w_{\{3, 3, 4, 5\}}(\tau) = 5$. For the other users we have $w_{\{3, 3, 4, 5\}}(\tau) = 3$ and $w_{\{5, 3, 4, 5\}}(\tau) = 1$. Therefore, $w_{\{3, 3, 4, 5\}}(\tau) - v_i(\tau)$ for all $i \in \{3, 4, 5\}$ and more users benefit from this user group than $\{2, 3, 5\}$.

Note that user $i$ needs to recover $r \Omega^{(i)} - v_i(\tau)$ missing packets and it has $T_i + \Delta_i - \tau$ time slots to obtain them. Thus, the ratio of these quantities is a measure of the stringency of the deadline of user $i$. Furthermore, based on the above discussion $w_{\{i, \tilde{U}\}}(\tau) - v_i(\tau)$ indicates the number of packets that are useful to user $i$ if $\tilde{U}$ was chosen for transmission. Therefore the metric $\eta_U(\tau)$ is obtained by the following weighted sum:

$$\eta_U(\tau) = \sum_{i \in U} \frac{r \Omega^{(i)} - v_i(\tau)}{T_i + \Delta_i - \tau} (w_{\{i, \tilde{U}\}}(\tau) - v_i(\tau)).$$

At time $\tau = T_k$, the server picks the first element $x_k^*(\ell) \in X_{\text{sorted}}^*$ such that $\eta_U(\tau) \geq \eta_0$ for some threshold $\eta_0$ and transmits an equation corresponding to it. Unlike the synchronous case, we choose a random linear combination of all missing subfiles of user $i$ that can be transmitted by user group $U$.

When $r > 1$, we subdivide a missing subfile into $r$ packets that are denoted $W_{\{d, f\}}$ for $j = 1, \ldots, r$. Thus, the server transmits

$$\bigoplus_{i \in U} \bigoplus_{f \in F(i)} \bigoplus_{j=1}^r \alpha_{\{i, f, j, m\}} W_{\{d, f\}},$$

at time interval $[\tau, \tau + 1)$ where $m$ denotes the $m$-th equation transmitted by the server and $\alpha_{\{i, f, j, m\}}$ are chosen independently and uniformly at random from the finite field $F$. If none of the elements in $x_k^*(\ell) \in X_{\text{sorted}}^*$ satisfy $\eta_U(\tau) \geq \eta_0$ then nothing will be transmitted at this time interval.

If a new user request does not come at time $\tau + 1$, then the server updates the user group values and then solves (8).
Algorithm 1 Recursive LP Algorithm

Input: Caches $Z_i$ for $i \in [K]$, $\eta_i$, $\{T_i, \Delta_i\}$, for $i \in [K]$.

1: Initialization:
2: set $\mathcal{U}_{sent}(0) \leftarrow \emptyset$, $\mathcal{X}_{off} \leftarrow \emptyset$, $\ell_{off} \leftarrow 0$, $m \leftarrow 1$ and $k \leftarrow 1$.
3: set $\mathcal{M}_i = \emptyset$, and $v_i(0) = 0$ for $i = 1, \ldots, K$.
4: for $\tau = 0, 1, 2, \ldots, T_{\text{max}}$ do
5: if $\tau = T_i + \Delta_i$ and $v_i(\tau) < r_1|\Omega^{(i)}|$ for some $i$ then
6: return INFEASIBLE.
7: end if
8: if $\tau = T_i$ (a new user makes request) for some $i$ then
9: $k = \arg \max_{i \in [K]} \tau = T_i$.
10: Solve LP (6). Form $X^*$ and then $X_{\text{sorted}}^*$.
11: end if
12: If $\tau = T_i$ or $\tau = T_i + \Delta_i$ for some $i$ then $\ell_{off} \leftarrow \ell_{off} + 1$.
13: if $X_{\text{sorted}}^* \neq \emptyset$ then
14: Pick first in order $x^*_{\tau, \ell}(\ell) \in X^*_{\text{sorted}}$ with $\eta_{\tau, \ell}(\tau) \geq \eta_0$.
15: Randomly select $\alpha_{i, f, j, m}$’s from $\mathcal{F}$ and send $x^*_{\tau, \ell}(\ell) \in \mathcal{X}_{\text{off}}(\tau)$, otherwise $x^*_{\tau, \ell}(\ell) \in \mathcal{X}_{\text{off}}(\tau) \cup \{\mathcal{U}^*\}$.
16: if $x^*_{\tau, \ell}(\ell) \in \mathcal{X}_{\text{off}}(\tau)$ then $z^*_{\tau, \ell}(\tau + 1) \leftarrow z^*_{\tau, \ell}(\tau) + 1$, otherwise $z^*_{\tau, \ell}(\tau + 1) = 1$ and $\mathcal{U}_{sent}(\tau + 1) \leftarrow \mathcal{X}_{\text{off}}(\tau) \cup \{\mathcal{U}^*\}$.
17: if $\tilde{x}_{\tau, \ell}(\ell) \in \mathcal{X}_{\text{off}}$ then $\tilde{x}_{\tau, \ell}(\ell) \leftarrow \tilde{x}_{\tau, \ell}(\ell_{off}) + 1$, otherwise $\tilde{x}_{\tau, \ell}(\ell) = 1$ and $X_{\tau, \ell}(\ell) \leftarrow \mathcal{X}_{\text{off}} \cup \{\mathcal{U}^*\}$.
18: set $x^*_{\tau, \ell}(\ell) \leftarrow x^*_{\tau, \ell}(\ell - 1)$, if $x^*_{\tau, \ell}(\ell) < 1$ remove it from $X_{\text{sorted}}^*$.
19: For all $i \in U^*$, set $v_i(\tau + 1) \leftarrow w_i(\ell, \ell)$, set $\mathcal{M}_i \leftarrow \mathcal{M}_i \cup \{\ell\}$, then $m \leftarrow m + 1$.
20: for all $i \in [K] \setminus U^*$ set $v_i(\tau + 1) \leftarrow v_i(\tau)$.
21: for all $i \in [K] \setminus U^*$ set $v_i(\tau + 1) \leftarrow v_i(\tau)$.
22: end if
23: end for

It is not difficult to verify that for any $x_{\tau, \ell}(\ell) \in \mathcal{X}_{\text{off}}$ user group $U$ is a member of $\mathcal{U}_{\ell}$. Moreover, the algorithm assigns integer values to $x_{\tau, \ell}(\ell)$. Now, for any $U \in \mathcal{U}_{\ell}$ in (1), we set $x_{\tau, \ell}(\ell) \in \mathcal{X}_{\text{off}}(\tau)$ if $x_{\tau, \ell}(\ell) \in \mathcal{X}_{\text{off}}(\ell) = 0$. Therefore, $x_{\tau, \ell}(\ell)$’s take integer values. Since at each time only one equation is transmitted in Algorithm 1 the first condition $\sum_{U \in \mathcal{U}_{\ell}} x_{\tau, \ell}(\ell) \leq |\mathcal{U}_{\ell}|$ holds for all $\ell \in [\beta]$.

For each $i \in [K]$ we define $\tau_i = \tau_i + 1$ to be the last time slot that user $i$ benefits from the equation transmitted by the server. Clearly we have that $\tau_i(\tau_i + 1) \geq |\Omega^{(i)}|$ otherwise Algorithm 1 will be infeasible at $\tau = T_i + \Delta_i$. We let $U_{i, \text{last}}$ to be the user group associated with this equation where $i \in U_{i, \text{last}}$.

Note that Algorithm 1 tracks a set $\mathcal{U}_{sent}(\tau)$ that contains all the user groups that have been used by the algorithm before the time $\tau$. We let $y_{(i, f)}(U, f) \in \mathcal{F}(i, U)$ and $U \in \mathcal{U}_{sent}(\tau_i)$ with $U \ni i$ be the solution of (8) when solving it for $w_{i, U_{i, \text{last}}}(\tau_i)$. Then, for each $U \in \mathcal{U}_{\ell}$ with $U \ni i$ and for each $f \in \mathcal{F}(i, U)$ we assign $y_{(i, f)}(U) = y_{(i, f)}(U_f)$ if $U \in \mathcal{U}_{sent}(\tau_i)$ and $y_{(i, f)}(U) = 0$ otherwise. We apply this assignment for all $\tau_i \in [1, K]$. Algorithm 1 assigns integer values to $z_{(\tau)}$’s. From Remark 3, it follows that there exists an integral solution for $y_{(i, f)}(U)$’s and consequently the $y_{(i, f)}(U)$’s as well. With these assignments, we now demonstrate that the second and third conditions in (1) hold.

For the second condition we note that if $U \notin \mathcal{U}_{sent}(\tau_i)$ then $y_{(i, f)}(U, f) = 0$ and we have nothing to show. For $U \in \mathcal{U}_{sent}(\tau_i)$ we have that $y_{(i, f)}(U) = y_{(i, f)}(U_f)$. Recall that $y_{(i, f)}(U)$ is the solution of (8) at time $\tau = \tau_i$. By the way that $z_{(\tau)}$ has been updated in Algorithm 1, we have $z_{(\tau)}(\tau_i) \leq z_{(\tau)}(T_{\text{max}})$. Therefore, we have $z_{(\tau)}(\tau_i + 1) \leq z_{(\tau)}(T_{\text{max}}) = \sum_{\ell \in \mathcal{U}_{\ell}} \tilde{x}_{\tau, \ell}(\ell)$ and from (8) for $w_{i, U_{i, \text{last}}}(\tau_i)$,

$$\sum_{f \in \mathcal{F}(i, U)} y_{(i, f)}(U) = \sum_{f \in \mathcal{F}(i, U)} \tilde{y}_{(i, f)}(U) \leq \tilde{x}_{\tau, \ell}(\ell) \leq x_{\tau, \ell}(\ell + 1)$$

For the third condition, consider any user $i \in [K]$ and any $f \in \Omega^{(i)}$. Recalling the definition of $\tau_i$ and $w_{i, U_{i, \text{last}}}(\tau_i)$, we know that $w_{i, U_{i, \text{last}}}(\tau_i) = v_i(\tau_i + 1) \geq |\Omega^{(i)}|$ which implies that in (8), we have

$$|\Omega^{(i)}| \leq \sum_{U \in \mathcal{U}_{sent}(\tau_i)} \sum_{U \in \mathcal{U}_{sent}(\tau_i)} \sum_{f \in \Omega^{(i)}} y_{(i, f)}(U) \leq \sum_{f \in \Omega^{(i)}} 1 = |\Omega^{(i)}|,$$

where the last inequality comes from the second constraint in (8). The middle equality holds by counting arguments for missing subfiles $f \in \Omega^{(i)}$ and user groups in $U \in \mathcal{U}_{sent}(\tau_i)$. To verify this, consider a bipartite graph in which the left and right nodes correspond to $f \in \Omega^{(i)}$ and $U \in \mathcal{U}_{sent}(\tau_i)$ with $U \ni i$ respectively. There is an edge between nodes corresponding to $f$ and $U$ if and only if $f \in \mathcal{F}(i, U)$. We let $y_{(i, f)}(U)$ to be the label of this edge. By the definition of $\mathcal{U}_{i, f}$ we know that $f \in \mathcal{F}(i, U)$ implies $U \in \mathcal{U}(i, f)$. Therefore, outgoing edges from the node corresponding to $f$ are the edges between $f$ and the nodes $U \in \mathcal{U}(i, f) \cap \mathcal{U}_{sent}(\tau_i)$. Similarly, the outgoing edges between node $U \in \mathcal{U}_{sent}(\tau_i)$ and
with \( U \ni i \) are the edges between \( U \) and \( f \in F_{i,U} \). By counting \( \hat{y}_{i,f}(U) \) in two ways, from the left and right nodes, we have the required equality. Therefore, we have that \( \sum_{U \in U_{i,f}} y_{i,U}(\hat{y}_{i,f}(U)) = 1 \) for any \( f \in \Omega^{(i)} \). This further implies that \( \sum_{U \in U_{i,f}} y_{i,U}(1) = 1 \) for all \( f \in \Omega^{(i)} \) and ends the proof.

The following lemma shows that if Algorithm 1 does not return \textquoteleft INFEASIBLE \textquoteright; then with high probability each user recovers its missing subfiles from the transmitted equations.

\textbf{Lemma 1:} If Algorithm 1 does not return \textquoteleft INFEASIBLE \textquoteright; then with probability at least \( \left( 1 - \frac{1}{|F|} \right)^{rKF} \) all requests will be satisfied within their deadline.

\textbf{Proof:} For simplicity, in the discussion below we assume that \( r = 1 \). The proof for \( r > 1 \) follows in straightforward manner. By the way that \( M_i \) and \( v_i(\tau) \) are updated in Algorithm 1 we have \( |M_i| = v_i(\tau) \) at each time \( \tau \). Furthermore, \( v_i(\tau_{\text{max}}) = |\Omega^{(i)}| \) for all \( i \in [K] \). Therefore, each user \( i \in [K] \) benefits from \( |\Omega^{(i)}| \) equations. For a \( m \in M_i \), let \( \bigoplus_{f \in F_{i,U}} \alpha_{i,f,m}^{(i)} W_{d,f} \) represent the \( m \)-th equation (the dependence on index \( j \) is suppressed since we assume that \( r = 1 \)). User \( i \in U \) can recover \( \bigoplus_{f \in F_{i,U}} \alpha_{i,f,m}^{(i)} W_{d,f} \) from this equation since the missing subfiles \( W_{d',f'} \), for \( f' \in F_{i,U} \) and \( j \in U \setminus \{ i \} \), exist in the cache of user \( i \).

For each user \( i \in [K] \) we define matrix \( B_i \in \mathbb{F}[\Omega^{(i)}] \times [\Omega^{(i)}] \) whose rows and columns correspond to equation numbers in \( M_i \) and missing subfiles in \( \Omega^{(i)} \) respectively. For \( m \in M_i \), assume that \( m \)-th equation is associated with user group \( U \), where \( i \in U \). Then, the entry of \( B_i \) for the row and column corresponding to \( m \in M_i \) and \( f \in \Omega^{(i)} \) is \( \alpha_{i,f,m} \) if \( f \in F_{i,U} \) and zero otherwise. Therefore, if matrix \( B_i \) is invertible then user \( i \) can recover all the missing subfiles \( W_{d,f} \), for \( f \in \Omega^{(i)} \), from equations \( \sum_{f \in F_{i,U}} \alpha_{i,f,m} W_{d,f} \) for \( m \in M_i \). Thus, we need to show that the determinant of \( B_i \) is nonzero for all \( i \in [K] \) with high probability.

Towards this end, let \( h_i(\{ \alpha_{i,f,m} \}, f \in \Omega^{(i)}, m \in M_i) \) denote the determinant of \( B_i \); we treat the \( \left\{ \alpha_{i,f,m} \right\} \) as indeterminates at this point. Note that since Algorithm 1 did not return \textquoteleft INFEASIBLE \textquoteright; we have a feasible integral solution for the corresponding offline LP (cf. Claim 1). Thus, there exists an interpretation of this solution (cf. Section IV-B) such that in each time slot, only one equation is transmitted, i.e., unlike a fractional solution, we do not need to potentially transmit multiple equations in the same time slot. This in turn implies that there is a setting for coefficients \( \alpha_{i,f,m} \) with \( \alpha_{i,f,m} \in \{0, 1\} \) such that the multivariate polynomial \( h_i \) evaluates to a non-zero value over \( \mathbb{F} \), i.e., \( h_i \) is not identically zero. This further implies that \( h = \prod_{i \in [K]} h_i \) is not identically zero. Now, since each \( \alpha_{i,f,m} \) appears only once in \( B_i \), thus its degree in polynomial \( h_i \) is one. Also, \( h_i \) is a polynomial of degree \( |\Omega^{(i)}| \leq F \) thus \( h \) is a polynomial of degree at most \( KF \). Therefore, we can use Lemma 4 in [24] to show that by choosing \( \alpha_{i,f,m} \)'s independently and uniformly at random from \( \mathbb{F} \), the determinants of \( B_i \)'s, \( i \in [K] \), are nonzero with probability at least \( \left( 1 - \frac{1}{|F|} \right)^{KF} \).

When \( r > 1 \) we will need to split a missing subfile

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|}
\hline
\( (K,t) \) & No. of nodes & No. of edges & Exec. time (min) \tabularnewline \hline
\( (10, 2) \) & 980, 101 & 17, 684, 980 & 8.026 \tabularnewline \hline
\( (20, 4) \) & 118, 542 & 27, 785, 203 & 10.68 \tabularnewline \hline
\( (40, 2) \) & 61, 959 & 567, 780 & 64 \tabularnewline \hline
\( (20, 2) \) & 7, 542 & 43, 507 & 1.9 \tabularnewline \hline
\( (10, 4) \) & 3, 917 & 29, 369 & 0.8 \tabularnewline \hline
\( (10, 2) \) & 915 & 3, 866 & 0.08 \tabularnewline \hline
\end{tabular}
\caption{Execution time for solving the LP using our approach; we run 1000 iterations of subgradient ascent. Columns 2 & 3 indicate the size of the associated flow network. The table is ordered by the number of nodes in the flow network.}
\end{table}

\section{Simulation Results and Comparisons with Prior Work}

In this section we present simulation results for both the proposed offline and the online algorithms. Prior work in this area is primarily the work of [15] that presents heuristics for the online scenario. However, we note that [15] works with deadlines for subfiles and does not take into account the time taken to transmit a packet. It uses intuitively plausible rules to decide the equations transmitted by the server depending on the deadlines of the users.

For both scenarios, the request arrival times \( \{ T_i, i \in [K] \} \) are generated according to a Poisson process with parameter \( \lambda F \). The arrival time is quantized to the nearest time slot. The deadlines \( \Delta_i, i \in [K] \) are generated uniformly at random from the range \( \Delta_{\min}, \Delta_{\max} \) (these values will be specified for each setting below).

\subsection{Offline scenario simulation}

In the first set of simulations we examine the execution time of our approach for various values of \( (K,t) \) where \( t = KM/N \) is an integer; the placement scheme in [11] was used. In these simulations we set \( r = 1, \lambda = 0.4, F = \frac{(K_f)}{K_f} \), \( \Delta_{\min} = (K_f - 1), \) and \( \Delta_{\max} = \left( \frac{K_f}{K_f} + 1 \right) \). Table II shows the details of the overall execution time and the size of the corresponding flow networks for the various instances. The last column of the table corresponds to the execution time (in MATLAB) of the LP in [11], while the second-last column corresponds to the execution time of the proposed approach above. It is evident that the proposed approach is significantly faster. In fact, memory requirements make it infeasible to even formulate the problems corresponding to the first three rows in MATLAB. Fig. 8 shows the convergence of the primal recovery procedure to the actual rate for a system with \( N = K = 20, t = 2, \) and \( r = 1 \). It can be observed that there is a clear convergence of the solution to the optimal value.

\subsection{Online scenario simulation}

For the online scenario we consider both centralized [11] and decentralized [16] placement schemes for a system with \( N = K = 6 \) and \( M = 2 \) with \( \Delta_{\min} = (KM/N)F \) and \( \Delta_{\max} = \).
For each experiment we run 200 trials for generating the arrivals. For the centralized case, we use the placement scheme of [1] and the placement is fixed during each experiment. In the decentralized scheme, at each trial the cache content of each user is independently and uniformly chosen as well.

For each set of generated arrivals, we first run the offline LP to check whether it is feasible. The online algorithm is run only if the offline LP is feasible. The online algorithm requires a threshold \( \eta_0 \) (see Section V-B). We run simulations with a low threshold (case I) and a high threshold (case II). The coding gain is defined as the ratio of the uncoded rate to the rate achieved by the system. Fig. 9 (a) and Fig. 10 (a) depict plots of the coding gain vs. \( 1/(F \lambda) \) in centralized and decentralized cases, respectively. As \( \lambda \) decreases, the arrivals are spaced further apart on average, and the coding gain of any scheme is expected to reduce. The coding gain is computed by taking an average over all instances where a given scheme is feasible. For the offline scheme this means that we take the average of all instances where it is feasible. For the case II of the online algorithm, some of arrival patterns may result in infeasibility; these instances were not taken into account when computing the average coding gain. This explains why the coding gain of case II sometimes appears to be higher than the offline algorithm. However, the coding gain of case I is significantly lower, because of its low threshold.

The feasibility probability of a scheme vs. the arrival rate is plotted in Fig. 9 (b) and Fig. 10 (b) for the centralized and decentralized placement schemes respectively. As expected the low threshold online algorithm has a very high feasibility probability \( \approx 1 \) for a range of arrival parameters, while the high threshold algorithm has a lower feasibility probability.

For both plots, we also include the results of [15]. In this scheme feasibility and coding gain can be traded off by setting a threshold for the defined misfit function (Section III in [15]). We use this scheme by setting the threshold to zero; this is the so-called First-Fit Rule in [15]. The First-Fit rule prefers

\[ \frac{1}{(F \lambda)}, \text{Poisson Process parameter} \]

feasibility over coding gain. The setting in [15] considers a scenario where each subfile has a deadline. We have adapted their algorithm for our case. It can be observed that the feasibility probability of [15] is quite poor. Accordingly we also plot the fraction of subfiles that meet the deadline; this is somewhat better. The coding gain numbers for [15] are also quite unreliable as the algorithm is infeasible in most cases. Thus, we do not plot it.

C. Scenario where individual subfiles have deadlines

The work of [15] considers a situation where each subfile has its own deadline. This is inspired by applications such as video delivery over the Internet. We emphasize that this setting can be captured by our techniques. In particular, suppose that each user requests a set of subfiles from the server where the subfile requests arrive at different times and each subfile has a different deadline. In this case we can treat each subfile request of user \( i \in [K] \) as corresponding to a distinct virtual user whose cache content is the same as user \( i \). However, the requests of the users are different. In this situation, each virtual user has precisely one missing subfile. Thus, the issue of coding over the corresponding subfiles does not arise.
schemes with cache of size 200 trials and at each trial, the cache content of each user is populated randomly and uniformly. In this simulation, we set \( \eta_0 = 0.4 - \frac{\eta}{2} \) and \( \eta_0 = 0.8 - \frac{\eta}{2} \) in Case I and Case II respectively.

Our setting is again one where \( K = N = 6, M = 2 \). Each file is subdivided into \( F = 20 \) subfiles. Arrival times and deadlines are generated similar to the previous simulations with Poisson parameters \( 1/(\lambda F) \) and the deadlines are randomly chosen uniformly from \([\Delta_{\min}, \Delta_{\max}]\) with \( \Delta_{\min} = KMF/N \) and \( \Delta_{\max} = KF \). Similar to the previous experiments we run 200 trials and at each trial, the cache content of each user is populated randomly and uniformly among all placement schemes with cache of size \( MF \) subfiles. Thus, different users might request different number of chunks from the server. The only difference is that here each requested chunk has its own arrival time and deadline. The results are illustrated in Fig. 10. It can be observed that our proposed approach provides significantly superior coding gain and feasibility probability as compared to the work of \[15\].

VII. CONCLUSIONS AND FUTURE WORK

In this work we considered the asynchronous coded caching problem where user requests (with deadlines) arrive at the main server at different times. We considered both offline and online versions of this problem. We demonstrated that under the assumption of all but one equations, the offline scenario can be solved by a linear program (LP). Moreover, we presented a low-complexity solution to this LP based on dual decomposition. In contrast to the synchronous case and the offline scenario, we show that the online scenario requires coding across missing subfiles of a given user. Furthermore, we present an online algorithm that leverages the offline LP in a recursive fashion. Extensive simulation results indicate that our proposed algorithm significantly outperforms prior algorithms.

Our online algorithm considers the situation where there is no knowledge about future request arrival times and file identities; this corresponds to a worst-case scenario. It would be interesting to consider cases where there is statistical information available on the arrival times and file popularity and investigate how this knowledge can be used to further improve the performance of the algorithm. Variants of the
problem, with soft deadline constraints may also be of interest.

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APPENDIX

A. Quadratic Projection and Primal Recovery in Dual Decomposition

For the projection of $\tilde{\gamma}_{U}^{(i)}(\ell, n)$ and $\tilde{\gamma}_{U}^{(i)}(n)$ to the constraint space we simply set $\tilde{\gamma}_{U}^{(i)}(n) = \max (\tilde{\gamma}_{U}^{(i)}(n), 0)$ and $\{\tilde{\gamma}_{U}^{(i)}(\ell, n), \forall i \in U\}$ is obtained via the following quadratic optimization.

$$\min_{\{\tilde{v}_{U}^{(i)}; \forall i \in U\}} \sum_{i \in U} (\tilde{v}_{U}^{(i)} - \tilde{z}_{U}^{(i)}(\ell, n))^{2} \quad \text{s.t.} \quad \sum_{i \in U} \tilde{v}_{U}^{(i)}(i) = 1. \tag{9}$$

In [20] Appendix I an algorithm has been proposed to solve (9). This solution can be explained as follows. For fixed $\ell \in [\beta]$ and for each $U \in \mathcal{U}_{\ell}$, we sort $\tilde{\gamma}_{U}^{(i)}(\ell, n)$ so that $\tilde{\gamma}_{U}^{(i)}(\ell, n) \geq \ldots \geq \tilde{\gamma}_{U}^{(i)}(\ell, n)$. We take $\hat{k}$ to be the minimum $k$ such that

$$\frac{1}{k} \left( 1 - \sum_{j=1}^{k} \tilde{z}_{U}^{(i)}(\ell, n) \right) \leq -\tilde{z}_{U}^{(i+1)}(\ell, n)$$

or let $\hat{k} = |U|$ if such a $k$ doesn’t exist. Then $\tilde{\gamma}_{U}^{(i)}(\ell, n) = \tilde{\gamma}_{U}^{(i)}(\ell, n) + 1 - \sum_{k=1}^{\hat{k}} \tilde{z}_{U}^{(i)}(\ell, n)$ if $j \in [\hat{k}]$ and zero otherwise.

The initial setting for the dual variables is chosen as $\gamma_{U}^{(i)}(\ell, 0) = 1/|U|$, for $i \in U$, $U \in \mathcal{U}_{\ell}$, $\ell \in [\beta]$, and $\gamma_{U}^{(i)}(\ell) = 0$ for $\ell \notin [\beta]$.

### Primal Recovery:

After solving the dual problem, the primal variables, i.e., $x_{U}(\ell, n)'s$, are recovered by the method of [20]

whereby

$$x_{U}(\ell, n) = \sum_{i=1}^{n} \mu_{U}(n) \left( \max_{\ell \in [\beta]} x_{U}^{(i)}(\ell, l) \right) \tag{10}$$

where $\mu_{U}(n)'s$ are sequence of convex combination weights for each non-negative integer $n$, i.e. $\sum_{l=1}^{\gamma_{1,n}} \mu_{U}(n) = 1$ and $\mu_{U}(n) \geq 0$ for all $l = 1, \ldots, n$. In [20], it has been shown that the step size $\eta_{n}$ and convex combination weights $\mu_{U}(n)$ are chosen so that

- $\eta_{n} \geq \eta_{n-1,n}$ for all $l = 2, \ldots, n$ and $n = 0, 1, \ldots$
- $\Delta_{\eta_{n}} \rightarrow 0$ as $n \rightarrow \infty$, and
- $\eta_{1,n} \rightarrow 0$ as $n \rightarrow \infty$ and $\eta_{n,n} \leq \delta$ for all $n = 0, 1, \ldots$ for some $\delta > 0$.

Then $\{x_{U}(\ell, n)'s, \ell \in [\beta], U \in \mathcal{U}_{\ell}\}$ is an optimal primal solution. Here $\eta_{n} = \frac{\mu_{U}(n)}{\eta_{n}}$ and $\Delta_{\eta_{n}} = \max_{\ell=2,\ldots,n}(\eta_{n,n} - \eta_{n-1,n})$. Some sequences for $\eta_{n}$ and $\mu_{U}(n)$ that satisfy the above conditions has been proposed by [20]. Among them we choose $\mu_{U}(n) = \frac{1}{n}$ and $\eta_{n} = n^{-\alpha}$ where $0 < \alpha < 1$. Then, the primal solution will be updated as,

$$x_{U}(\ell, n + 1) = \frac{n}{n + 1} x_{U}(\ell, n) + \frac{\max_{\ell \in [\beta]} x_{U}^{(i)}(\ell, n)}{n + 1}. \tag{11}$$