Optimal entanglement purifying via entanglement swapping

Bao-Sen Shi*, Yun-Kun Jiang and Guang-Can Guo†

Laboratory of Quantum Communication and Quantum Computation
Department of Physics
University of Science and Technology of China
Hefei, 230026, P. R. China

Abstract

It is known that entanglement swapping can be used to realize entanglement purifying. By this way, two particles belong to different non-maximally entangled pairs can be projected probabilistically to a maximally entangled state or to a less entangled state. In this report, we show, when the less entangled state is obtained, if a unitary transformation is introduced locally, then a maximally entangled state can be obtained probabilistically from this less entangled state. The total successful probability of our scheme is equal to the entanglement of a single pair purification (if two original pairs are in the same non-maximally entangled states) or to the smaller entanglement of a single pair purification of these two pairs (if two original pairs are not in the same non-maximally entangled states). The advantage of our scheme is no continuous indefinite iterative procedure is needed to achieve optimal purifying.

03.67.-a

*E-mail address: drshi@ustc.edu.cn
†E-mail address: gcguo@ustc.edu.cn
Entanglement is at the source of a number of pure quantum phenomena, such as the correlations violating Bell’s inequalities [1], quantum key distribution [2], quantum teleportation [3], Greanberger-Horne-Zeilinger [GHZ] correlations [4], and various other nonclassical interference phenomena [5]. Polarization entangled photons have been used to demonstrate both dense coding [6] and teleportation [7] in the laboratory. Teleportation has also been realized using path-entangled photons [8] and entangled electromagnetic field modes [9]. In order to realize these schemes, the entanglement between distant particles should be set up. One of possible way is entanglement swapping [10], which has been demonstrated experimentally [11]. Recently, Bose et al [12] showed that entanglement swapping can be used to realize entanglement purifying. In their scheme, if an ensemble of two photon pair is given, in which all pairs are in the same non-maximally entangled states, then two photons belong to different photon pairs can be projected probabilisticly into a maximally entangled Bell state or a less entangled state by their scheme. If one continues this process indefinitely, in the limit of an infinite sequence, the final ensemble generated will comprise of a certain fraction of Bell pairs and a certain fraction of completely disentangled pairs. This fraction of Bell pairs should be equal to twice the modulus square of the Schmidt coefficient of small magnitude of original pair. In this report, we show that if a unitary transformation is used followingly when a less entangled state is obtained by entanglement swapping, then, a maximally entangled Bell state can be obtained probabilisticly from this less entangled state. The maximum probability with which a Bell state can be obtained by our scheme is two the modulus square of the Schmidt coefficient of small magnitude, which means our scheme is optimal. One advantage of our scheme is no continuous indefinite iterative procedure is needed. Furthermore, if two particle pairs are not the same type of entangled state, two particles belong to different pairs can also be projected into a maximally entangled state with certain probability by the same way. This probability is equal to the smaller entanglement of single pair purification of these two pairs, which also means our scheme is optimal.

Let pairs of particles \((1, 2)\) and \((3, 4)\) be in the following entangled states:
\[ |\Phi\rangle_{12} = \alpha |00\rangle_{12} + \beta |11\rangle_{12}, \]  
\[ |\Phi\rangle_{34} = \alpha |00\rangle_{34} + \beta |11\rangle_{34}, \]

where, \(|\alpha| > |\beta|\), and \(|\alpha|^2 + |\beta|^2 = 1\). Suppose that the particle pair (1, 2) and the particle 3 belong to Alice and the particle 4 belongs to Bob. If a Bell state measurement on particles 2 and 3 is operated by Alice, then the particles 1 and 4 will be projected into one of the following states

\[ \langle \Phi^\pm \rangle_{23} |\Phi\rangle_{12} \otimes |\Phi\rangle_{34} = \frac{\alpha^2}{\sqrt{2}} |00\rangle_{14} \pm \frac{\beta^2}{\sqrt{2}} |11\rangle_{14}, \]  
\[ \langle \Psi^\pm \rangle_{23} |\Phi\rangle_{12} \otimes |\Phi\rangle_{34} = \alpha \beta \left[ \frac{1}{\sqrt{2}} (|01\rangle_{14} \pm |10\rangle_{14}) \right]. \]

Where \(|\Phi^\pm\rangle_{23} = \frac{1}{\sqrt{2}}(|00\rangle_{23} \pm |11\rangle_{23})\) and \(|\Psi^\pm\rangle_{23} = \frac{1}{\sqrt{2}}(|01\rangle_{23} \pm |10\rangle_{23})\). The particles 1 and 4 will be projected into a less entangled state with probability \(\frac{\alpha^4 + \beta^4}{2}\). In order to get optimal entanglement purifying, a unitary transformation in Alice’s side (or in Bob’s side, for example, in Alice’s side) follows when a less entangled state is obtained. To carry out this evolution, an auxiliary qubit with the original state \(|0\rangle_a\) is introduced by Alice. Under the basis \(\{|0\rangle_1 |0\rangle_a, |1\rangle_1 |0\rangle_a, |0\rangle_1 |1\rangle_a, |1\rangle_1 |1\rangle_a\}\), this unitary transformation can be written as

\[
\begin{bmatrix}
\frac{\beta^2}{\alpha^2} & 0 & \sqrt{1 - \frac{\beta^4}{\alpha^4}} & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & -1 \\
\sqrt{1 - \frac{\beta^4}{\alpha^4}} & 0 & -\frac{\beta^2}{\alpha^2} & 0
\end{bmatrix}.
\]

This transformation will transform Eq. (3) to the follow state

\[
\beta^2 \left[ \frac{1}{\sqrt{2}} (|00\rangle_{14} \pm |11\rangle_{14}) \right] |0\rangle_a + \frac{\alpha^2}{\sqrt{2}} \sqrt{1 - \frac{\beta^4}{\alpha^4}} |1\rangle_1 |0\rangle_4 |1\rangle_a. \]

A measurement to the auxiliary particle follows. If the result is \(|0\rangle_a\), then the particles 1 and 4 will be projected a maximally Bell state with probability \(\beta^4\). If the result of the measurement is \(|1\rangle_a\), the particles 1 and 4 are completely disentangled.
The maximally probability with which a Bell state can be obtained by purifying a single entangled pair is $2\beta^2$, which is equal to the maximum probability with which a Bell state can be obtained, so our scheme is optimal.

Next, we proceed to consider the case when particle pairs (1, 2) and (3, 4) are not in the same type of entangled state. Suppose the particles 1 and 2 are in the entangled state $|\Phi\rangle_{12}$ and the particles 3 and 4 are in another entangled state $|\Phi\rangle_{34}$ which are the following respectively

$$|\Phi\rangle_{12} = \alpha |00\rangle_{12} + \beta |11\rangle_{12} \quad (7)$$

and

$$|\Phi\rangle_{34} = a |00\rangle_{34} + b |11\rangle_{34}. \quad (8)$$

Where $|a| > |b|$, $|a|^2 + |b|^2 = 1$. Suppose that the particles 1, 2 and 3 belong to Alice and the particle 4 belongs to Bob. If Alice make a Bell state measurement on particles 2 and 3, then the particles 1 and 4 will be projected into one of the follow states

$$\langle \Phi^\pm |_{23} |\Phi\rangle_{12} \otimes |\Phi\rangle_{34} = \frac{\alpha a}{\sqrt{2}} |00\rangle_{14} \pm \frac{\beta b}{\sqrt{2}} |11\rangle_{14}, \quad (9)$$

$$\langle \Psi^\pm |_{23} |\Phi\rangle_{12} \otimes |\Phi\rangle_{34} = \frac{\alpha b}{\sqrt{2}} |01\rangle_{14} \pm \frac{\beta a}{\sqrt{2}} |10\rangle_{14}. \quad (10)$$

If Eq.(9) is obtained, a unitary transformation which is made on the particle 1 and an auxiliary qubit with the original state $|0\rangle_a$ is introduced by Alice (or on the particles 4 and auxiliary qubit $|0\rangle_a$ by Bob). Under the basis $\{|0\rangle_1 |0\rangle_a, |1\rangle_1 |0\rangle_a, |0\rangle_1 |1\rangle_a, |1\rangle_1 |1\rangle_a\}$, this unitary transformation is

$$\begin{bmatrix}
\frac{\beta b}{\alpha a} & 0 & \sqrt{1 - \beta^2\frac{b^2}{\alpha^2}} & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & -1 \\
\sqrt{1 - \beta^2\frac{b^2}{\alpha^2}} & 0 & -\frac{\beta b}{\alpha a} & 0
\end{bmatrix}. \quad (11)$$

Under this transformation, Eq. (9) will be transformed into the state
\[ \beta b \left[ \frac{1}{\sqrt{2}} (|00\rangle_{14} \pm |11\rangle_{14}) \right] |0\rangle_a + \frac{\alpha a}{\sqrt{2}} \sqrt{1 - \frac{\beta^2 b^2}{\alpha^2 a^2}} |1\rangle_1 |0\rangle_4 |1\rangle_a. \]  

(12)

A measurement on the auxiliary particle follows. If the result is $|0\rangle_a$, the particles 1 and 4 will be projected into a maximally entangled state with probability $\beta^2 b^2$. If the result is $|1\rangle_a$, the particles 1 and 4 are completely disentangled.

If Eq. (10) is obtained, two different cases should be considered:

1: $|\alpha b\rangle > |a\beta\rangle$

In this case, the unitary transformation on the particles 1 and auxiliary qubit is

\[
\begin{bmatrix}
\frac{\beta a}{ab} & 0 & \sqrt{1 - \frac{\beta^2 a^2}{\alpha^2 b^2}} & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & -1 \\
\sqrt{1 - \frac{\beta^2 a^2}{\alpha^2 b^2}} & 0 & -\frac{\beta a}{ab} & 0
\end{bmatrix}.
\]

(13)

By the same procedure, the particles 1 and 4 will be projected into a maximally entangled state with probability $a^2 \beta^2$.

2: $|\alpha b\rangle < |a\beta\rangle$

In this case, the probability of obtaining a maximally entangled is $\alpha^2 b^2$. The unitary transformation is on the particles 1 and auxiliary qubit is

\[
\begin{bmatrix}
\frac{\alpha b}{a^3} & 0 & \sqrt{1 - \frac{\alpha^2 b^2}{\alpha^2 \beta^2}} & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & -1 \\
\sqrt{1 - \frac{\alpha^2 b^2}{\alpha^2 \beta^2}} & 0 & -\frac{\alpha b}{a^3} & 0
\end{bmatrix}.
\]

(14)

The maximally probability of obtained a maximally entangled state is $2\beta^2$ or $2b^2$. The first means that entanglement of single pair purification of the $|\Phi\rangle_{12}$ is less than that of the state $|\Phi\rangle_{34}$. The second means that the entanglement of single pair purification of $|\Phi\rangle_{34}$ is less than that of the state $|\Phi\rangle_{12}$. This probability is equal to the smaller entanglement of single pair purification of these two pairs before entanglement swapping. Obviously, our scheme is optimal.
In conclusion. In the Ref. [12], an iterative procedure indefinitely is needed in order to achieve the optimal entanglement purifying. In our scheme, when a less entangled state is obtained during the entanglement purifying, if a unitary transformation is introduced locally, a maximally entangled state can be obtained with certain probability. The successful probability of our scheme is equal to the entanglement of a single pair purification if two original pairs are in the same non-maximally entangled states, or to the smaller entanglement of a single pair purification of these two pairs if they are not in the same non-maximally entangled states, which means our scheme is optimal. No continuous indefinite iterative procedure is needed, which makes our scheme implementable easily in practice.

This subject is supported by the National Natural Science Foundation of China (Grant No. 69907005).
REFERENCES

[1] J. S. Bell, Physics 1, 195 (1965)

[2] A. K. Ekert, Phys. Rev. Lett., 67, 661 (1991)

[3] C. H. Bennett, G. Brassard, C. Crepeau, R. Jozsa, A. Peres and W. K. Wootters, Phys. Rev. Lett, 70, 1895 (1993)

[4] M. Greenberger, M. A. Horne, A. Shimony and A. Zeilinger, Am. J. Phys, 58, 1131 (1990)

[5] Y. H. Shih and A. V. Sergienko, Phys. Rev. A., 50, 2564 (1994)

[6] K. Mattle, H. Weinfurter, P. G. Kwait and A. Zeilinger, Phys. Rev. Lett., 76, 4656 (1996)

[7] D. Bouweester, J-W. Pan, K. Mattle, M.Eibl, H. Weinfurter and A. Zeilinger Nature, 390, 575 (1997)

[8] D. Boschi, S. Branca, F. De Martini, L. Hardy and S. Popescu, Phys. Rev. Lett. 80, 1121 (1998)

[9] A. Furusawa, et. al, Science, 282, 706 (1998)

[10] M. Zukosawa, et. al, Phys. Rev. Lett. 71, 4287 (1993)

[11] J-W. Pan, et. al., Phys Rev. Lett., 80, 3891 (1998)

[12] S. Bose, et. al., Phys Rev. A., 60, 194 (1999)