Equilibria and Systemic Risk in Saturated Networks

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Abstract

We undertake a fundamental study of network equilibria modeled as solutions of fixed point of monotone linear functions with saturation nonlinearities. The considered model extends one originally proposed to study systemic risk in networks of financial institutions interconnected by mutual obligations and is one of the simplest continuous models accounting for shock propagation phenomena and cascading failure effects. We first derive explicit expressions for network equilibria and prove necessary and sufficient conditions for their uniqueness encompassing and generalizing several results in the literature. Then, we study jump discontinuities of the network equilibria when the exogenous flows cross a certain critical region consisting of the union of finitely many linear submanifolds of co-dimension 1. This is of particular interest in the financial systems context, as it shows that even small shocks affecting the values of the assets of few nodes, can trigger catastrophic aggregated loss to the system and cause the default of several agents.

1 Introduction

The ability to predict and optimize the behavior of complex socio-technical systems is one of this century’s main scientific themes. It is becoming clearer and clearer how the behavior of large-scale infrastructural, social, economic, and financial networks can have a huge societal impact, e.g., enabling or limiting access to essential services, such as mobility and energy, influencing the outcome of elections, or destabilizing entire economies. A central aspect of such network systems is the role played by interconnections in amplifying and propagating shocks through cascading mechanisms that may increase the fragility of a system [1, 2, 3, 4]. The term systemic risk refers to the possibility that even small shocks localized in a limited part of the network can propagate and amplify through cascading mechanisms, thus possibly achieving a significant global impact [5, 6, 7, 8].

In this realm, a central problem is to find adequate models for networks, that are sufficiently elaborate to incorporate such propagation phenomena, yet simple enough to allow for mathematical tractability. Simple contagion models such as epidemic contact models prove inadequate as based on purely pairwise interactions. A more complex model taking into account cumulative neighborhood effects is the linear threshold model [9, 10, 11] whose applicability is however limited by the fact that states of the nodes are described by pure binary variables simply expressing whether the node has been affected by the shock. In most of the applicable contexts where the network represents a physical infrastructure or an interconnected financial system, this simplification is way too strong. Indeed, in these cases, the cascading mechanism is rather triggered by a process naturally working with continuous variables such as, e.g., power flows in electric grids, traffic volumes in transportation systems, or assets values and payments in financial networks.

In this paper, we undertake a fundamental study of a model, originally proposed in the seminal work [12] to study systemic risk in networks of financial institutions interconnected by mutual obligations. It is one of the simplest continuous models where the shock propagation phenomena are caused by a saturation nonlinearity of a positive linear static system. Precisely, we consider the following fixed point equation

\[
x_i = \min \left\{ \max \left\{ \sum_{j=1}^{n} x_j P_{ji} + c_i, 0 \right\}, w_i \right\}, \quad i = 1, \ldots, n,
\]
where $P$ is a row-stochastic or sub-stochastic $n \times n$ matrix and $w \in \mathbb{R}^n$ is a nonnegative vector that jointly describe a network and where $c \in \mathbb{R}^n$ is an exogenous flow vector. Equation (1) can be more compactly rewritten as

$$x = S_0^w(P'x + c).$$

(2)

where $S_0^w$ denotes the vector saturation function

$$(S_0^w(x))_i = \min\{\max\{x_i, 0\}, w_i\}, \quad i = 1, \ldots, n,$$

(3)

We shall refer to vectors $x$ that are solutions of (2) as equilibria of the network $(P, w)$ with exogenous flow $c$. While standard arguments guarantee existence of such network equilibria, their uniqueness and dependence on the exogenous flows prove to be more delicate issues that will be the object of this paper.

In financial networks [12], the entries of the vector $w$ represent the obligations of the various institutions (say banks), those of the exogenous flow $c$ represent the balance between assets possessed by the entities and their obligations towards institutions outside the network, while $P$ describes the way obligations of an entity are split among the others and describe the way the financial system is interconnected. An equilibrium $x$ represents, in this context, a set of payments that clear the network in a consistent way. Previous works including [13, 7, 14, 15] have analyzed conditions for uniqueness of the clearing payment $x$ and studied its dependence on the exogenous flow vector $c$. Their main focus is to understand the extent to which a shock hitting the value of the assets of a single node $i$ (perturbation of $c_i$) reflects on the entire network and leads to possible cascade effects. More precisely, they define the concept of a default node as one for which the quantity $\sum_j P_{ji} x_j + c_i$ (representing the liquidity of the entity $i$) is below the value of its obligation $w_i$ and distinguishing in partial or total default if, respectively, $\sum_j P_{ji} x_j + c_i > 0$ or not. Although simple, this framework turns out to be very useful for analyzing how losses propagate through the financial system.

Considering the importance of this application, we will discuss the financial model more accurately in Section 2.1. Network equilibria defined as fixed points of equation (1) can also be seen as the Nash equilibria of a properly defined quadratic games — as illustrated in Section 2.2 — as well as the equilibrium points of a dynamical flow network with exogenous flows and capacity constraints that is briefly discussed in Section 2.3 and whose dynamical behavior has been analyzed in detail in [10].

The present paper presents a systematic study of the equilibria described by equation (2) and gives four fundamental contributions:

(i) We prove that the set of nodes in partial or full default (in the financial jargon discussed above) is invariant through the set of all equilibria.

(ii) We analyze the structure of the solutions with respect to the topological properties of the graph. In particular we show how to effectively construct all equilibria starting from anyone of them.

(iii) We prove a general result that provides necessary and sufficient conditions for uniqueness of the equilibrium, which subsumes the other theorems assessing uniqueness present in literature. In particular, in the case of a strongly connected network, conditions for uniqueness can be easily checked a priori without doing any preliminary computation.

(iv) We show that the equilibrium exhibits a jump discontinuity with respect to the exogenous flow vector $c$ when this is crossing certain linear sub-manifolds where uniqueness of equilibrium is lost. This is of particular interest in the financial context, as it shows that even small shocks affecting the values of the assets of few nodes, can trigger catastrophic aggregated loss to the system and cause the default of several agents. We provide an analytical description of the discontinuity set and we quantify the size of the jump for an aggregated loss function.

The rest of this paper is organized as follows. The remainder of this Introduction is devoted to a discussion of some related literature as well as a brief explanation of the notational conventions to be followed throughout the paper. Section 2 is devoted to present three motivating examples for the model considered in the paper. Section 3 is devoted to establishing a number of preliminary results on the structure of the equilibria. Uniqueness results as well a general expression describing all solutions in non uniqueness cases is presented in Section 4. Section 5 is devoted to the analysis of jump discontinuities in the equilibrium with respect to variation of the exogenous flow vector. The papers ends with Section 6 dedicated to draw some conclusions and open problems.
1.1 Related Literature

The seminal work by Eisenberg and Noe [12] triggered the development of a large literature on similar models. Ren et al. [15] explore several conditions on the uniqueness of the clearing payment vector. It is shown that the clearing payment vector is, in general, not unique, but rather a certain kind of net value of a bank is. Liu and Statum [13] use linear programming to provide a sensitivity analysis of Eisenberg and Noe model with respect to certain parameters. Elsinger et al. [17] use the Eisenberg and Noe model together with standard tools from modern risk management to assess systemic financial stability of the Austrian banking system; they find that financial contagion is rare but can nonetheless wipe out a major part of the banking system. Cifuentes et al. [18] extend the original Eisenberg and Noe model by considering fire sales; Rogers et al [19] include bankruptcy costs using a recovery function that drops discontinuously at the default boundary and then decreases linearly with the amount of assets available. Elsinger and Helmut [20] enrich the original model by considering cross-holdings that lead to additional spillover effects when a shock hits the network. Glasserman and Young [14] estimate the extent to which interconnections increase expected losses and defaults under a wide range of shock distributions and they provide bounds on the potential magnitude of network effects on contagion and loss amplification; they also provide an extensive survey on the financial networks topic in [21]. Acemoglu et al. [8] develop a unified framework for the study of how network interactions can function as a mechanism for propagation and amplification of microeconomic shocks while in [7] they provide an analysis of a particular case of the Eisenberg and Noe model and they prove some rigorous results about the resilience of different network topologies depending on the shock magnitude.

1.2 Notation

We briefly explain the notation to be used throughout this paper. We denote vectors with lower case and matrices with upper case. Sets are denoted with calligraphic letters. A subscript associated to vectors, for instance \(v_A\), represents the sub-vector that is the restriction of \(v \in \mathbb{R}^n\) over the indexes contained in the set \(A \subseteq \{1, 2, \ldots, n\}\). The same notation holds for matrices: \(P_{AB}\) represents the sub-matrix of \(P\) obtained by considering rows and columns associated to the indexes contained in sets \(A\) and \(B\), respectively. We indicate with \(\mathbb{1}\) the all-1 vector (independently from its dimension). We consider on \(\mathbb{R}^n\) the partial order given by entry-wise ordering: if \(x, y \in \mathbb{R}^n\), \(x \leq y\) if \(x_i \leq y_i\) for every \(i\) and we use the strict inequality symbol < if moreover the two vectors are different. Throughout the paper the ordering of vectors will be meant in this sense.

A non-negative matrix \(P \in \mathbb{R}^{n \times n}_+\) is called (row) sub-stochastic if the sum of the elements in each row never exceeds 1, namely \(P\mathbb{1} \leq \mathbb{1}\). Notice that in the literature it is often assumed that sub-stochastic matrices have the additional property that for at least one row there is strict inequality. Here we prefer not to follow this convention and in this way our class of sub-stochastic matrices contains also the stochastic matrices that are those for which \(P\mathbb{1} = \mathbb{1}\).

2 Three applications

In this section, we describe three motivating applications. We start in Section 2.1 by presenting a model of financial networks generalizing the one first considered in [12]. We then provide an interpretation in terms of Nash equilibria of a certain quadratic game, as explained in Section 2.2. Finally, in Section 2.3 we discuss a dynamical flow network model with fixed routing and capacity constraints whose rest points coincide with the equilibria.

2.1 Payment equilibria in financial networks

We consider a set \(V = \{1, \ldots, n\}\) of financial entities (e.g., banks, broke dealers,...) interconnected by internal and external obligations that are specified by a non-negative matrix \(W \in \mathbb{R}^{n \times n}\) and three vectors \(a, b, u \in \mathbb{R}^n_+\) that have the following interpretations:

- \(W_{ij} \geq 0\) is the liability of node \(i\) to node \(j\);
- \(a_i\) is the total value of assets and credits of \(i\) from external entities;
- \(b_i\) is the total liability of node \(i\) to external non-financial entities;
- \(u_i\) is the total liability of node \(i\) to external financial entities.
The quantity \( v_i = \sum_j W_{ji} - \sum_j W_{ij} + a_i - b_i - u_i \) is the net worth of node \( i \). If the condition \( v_i \geq 0 \) is verified for every \( i \in \mathcal{V} \), it means that each node is fully liable and in principle capable to pay back all its liabilities to the nodes in the network as well the external ones. In case when instead some nodes do not satisfy the condition \( v_i \geq 0 \), namely they are not fully liable, it is necessary to determine a consistent set of payments among the various nodes.

Put \( w_i = \sum_j W_{ij} + u_i \) and

\[
P_{ij} = \begin{cases} 
W_{ij}/w_i & \text{if } w_i > 0 \\
0 & \text{otherwise}
\end{cases}
\]

We define by \( X_{ij} \) the payment from node \( i \) to node \( j \) and by \( X_{io} \) the payment from node \( i \) to external financial entities. Assuming that liabilities to non-financial entities have a higher seniority and that all other payments (including those to external financial entities) should be proportional to the corresponding liabilities, a consistent set of payments among the nodes has to satisfy the relations

\[
X_{ij} = \min \{ P_{ij} \max \{ \sum_k X_{ki} + a_i - b_i, 0 \}, W_{ij} \} \\
X_{io} = \min \{ u_i/w_i \max \{ \sum_k X_{ki} + a_i - b_i, 0 \}, u_i \}
\]

(4)

Let \( x_i = \sum_j X_{ij} + X_{io} \) be the total payment of node \( i \) to the financial entities. Summing the relations in (4) and using the fact that \( W_{ij} = w_i P_{ij} \), we obtain

\[
x_i = \min \left\{ \max \left\{ \sum_k X_{ki} + a_i - b_i, 0 \right\}, w_i \right\}
\]

(5)

so that, \( X_{ij} = x_i P_{ij} \). Relation (5) can thus be rewritten as

\[
x_i = \min \left\{ \max \left\{ \sum_k x_k P_{ki} + a_i - b_i, 0 \right\}, w_i \right\}
\]

(6)

This set of relations is equivalent to (4). Indeed, if the vector \( x \) solves (6), then \( X_{ij} = x_i P_{ij} \) solves (4). This coincides with (1) with exogenous flow \( c = a - b \).

Notice that the matrix \( P \) is sub-stochastic in its strict sense (i.e., at least one row does not sum to 1) when either there exist nodes with a positive liability towards external financial entities, or nodes with no financial liabilities.

In this financial setting, it is often considered the case when we start from a fully liable configuration, that is \( v_i \geq 0 \) for all \( i \), leading to a solution \( x \) of (6) such that \( x_i \geq v_i \) for all \( i \). We then assume that the outside assets suffer a shock \( \epsilon \in \mathbb{R}_+^n \) so that their values reduce to \( a - \epsilon \) possibly making some of the \( v_i \)'s negative. The study of the amount of nodes in default \( x_i < v_i \) has a function of the shock \( \epsilon \) one of the key issues.

### 2.2 A quadratic network game

Consider a quadratic network game with player set \( \mathcal{V} = \{1, 2, \ldots, n\} \) where each player \( i \) chooses her action \( X_i = (X_{i0}, X_{i1}, \ldots, X_{in}) \in \mathbb{R}_+^{n+1} \) so as to maximize her utility

\[
u_i(X) = - \sum_{j \in \{0, \ldots, n\} : W_{ij} > 0} \frac{W_{ij}}{2} \left( \frac{X_{ij}}{W_{ij}} - 1 \right)^2,
\]

(7)

under the following constraints:

\[
\sum_{j=0}^n X_{ij} \leq \sum_{j=1}^n X_{ji} + c_i.
\]

(8)

Above, \( W \in \mathbb{R}_+^{n \times n} \) is a nonnegative zero-diagonal matrix such that \( w_i = \sum_{j=0}^n W_{ij} > 0 \) for \( 1 \leq i \leq n \), \( c \in \mathbb{R}^n \) is a given vector, and \( X \in \mathbb{R}_+^{(n-1) \times n} \) denotes the configuration of all players' actions.

Note that the best response of player \( i \) necessarily assigns \( X_{ij} = 0 \) for every \( j = 0, \ldots, n \) whenever \( \sum_{j=1}^n X_{ji} + c_i \leq 0 \) and \( X_{ij} = W_{ij} \) for every \( j = 0, \ldots, n \) whenever \( \sum_{j=1}^n X_{ji} + c_i \geq w_i \). On the other hand, when \( 0 < \sum_{j=1}^n X_{ji} + c_i < w_i \),

\[
\sum_{j=0}^n X_{ij} \leq \sum_{j=1}^n X_{ji} + c_i.
\]
then the constraint (8) is met with equality by the best response of a player $i$ and this can then be determined by setting to 0 the partial derivatives of the Lagrangian

$$
\sum_{j \in \{0, \ldots, n\}; W_{ij} > 0} \frac{W_{ij}}{2} \left( \frac{X_{ij}}{W_{ij}} - 1 \right)^2 + \lambda \left( \sum_{j=0}^{n} X_{ij} - \sum_{j=1}^{n} X_{ji} - c_i \right).
$$

We thus get

$$X_{ij} = P_{ij} \left( \sum_{k=1}^{n} X_{ki} + c_i \right), \quad 0 \leq j \leq n,$$

where

$$P_{ij} = \frac{W_{ij}}{w_i}, \quad 0 \leq i \leq n, \quad 1 \leq j \leq n.$$

Putting together our findings in the three cases, the best response of a player $i = 1, \ldots, n$ reads

$$X_{ij} = P_{ij} S_{w_i} \left( \sum_{k=1}^{n} X_{ki} + c_i \right), \quad 0 \leq j \leq n.$$

The Nash equilibria of this game are those $X^* \in \mathbb{R}_{++}^{n(n+1)}$ with entries $X_{ij}^* = x_i P_{ij}$, where $x = X^* \mathbf{1}$ is a solution of (2).

### 2.3 Dynamical flow networks with finite capacities and losses

Consider a dynamical flow network consisting of finitely many cells $i = 1, 2, \ldots, n$, each containing a quantity $x_i(t)$ of the same commodity at time $t \geq 0$. The state of the system is described by the vector $x(t) = (x_i(t))_{1 \leq i \leq n}$ and evolves in continuous time as the cells can exchange flow both among themselves and with the external environment. The network flow dynamics is then described by the system of ordinary differential equations

$$
\dot{x}_i = f_i(x), \quad i = 1, \ldots, n,
$$

where $f_i(x)$ represents the total instantaneous net flow (inflow minus outflow) on the cell $i$. The form of the instantaneous flow $f_i(x)$ follows from the considerations below:

- The instantaneous outflow from cell $i$ is exactly $x_i$. A constant fraction $P_{ij} \geq 0$ of the quantity of commodity $x_i$ flows directly towards another cell $j \neq i$ in the network, while the remaining part $(1 - \sum_j P_{ij})x_i$ leaves the network directly.
- The instantaneous inflow to cell $i$ is a quantity between 0 and $w_i$.
- Besides the inflow from other cells, each cell $i$ possibly receives a constant instantaneous exogenous inflow $a_i \geq 0$ from outside the network while a constant instantaneous outflow $b_i \geq 0$ is drained, if available, directly from cell $i$ towards the external environment. The part possibly exceeding $w_i$ is lost.

This leads to the following form for the total instantaneous net flow:

$$f_i(x) = S_{w_i} ^{w_i} \left( \sum_j P_{ji} x_j + c_i \right) - x_i, \quad 1 \leq i \leq n,$$

where $c_i = a_i - b_i$. It is then clear that the rest points of the dynamical flow network (9)–(10) coincide with the equilibria of the network $(P, w)$ with exogenous flow $c$. In fact, as proven in [16], every solution of the dynamical flow network (9)–(10) with initial state $x(0)$ converges to some network equilibrium in the limit as $t$ grows large.
3 Structural properties of the equilibria

In this section, we assume we have fixed a sub-stochastic matrix \( P \in \mathbb{R}^{n \times n}_+ \), a nonnegative vector \( w \in \mathbb{R}^n_+ \), and a vector \( c \in \mathbb{R}^n \), and we study the equilibria of the network \((P, w)\) with exogenous flow \( c \), namely the solutions of the fixed-point equation \( \text{2} \). We denote by 
\[
\mathcal{X} = \{ x \in \mathbb{R}^n : \text{2} \}
\]
the set of such equilibria. It is useful to introduce the complete lattice 
\[
\mathcal{L}_0^w = \{ x \in \mathbb{R}^n : 0 \leq x \leq w \},
\]
and note that the vector saturation function \( S_0^w : \mathbb{R}^n \to \mathcal{L}_0^w \). This implies that necessarily \( \mathcal{X} \subseteq \mathcal{L}_0^w \).

Existence of such equilibria directly follows from Brower’s fixed point theorem, since the lattice \( \mathcal{L}_0^w \) is a nonempty, convex, and compact set, and \( x \mapsto S_0^w(P'x + c) \) maps \( \mathcal{L}_0^w \) in itself with continuity. Hence, we always have that the equilibrium set \( \mathcal{X} \) is nonempty. On the other hand, equilibria are not unique in general: e.g., in the simple network with \( n = 2 \), \( P_{ij} = |i - j| \) for \( i, j = 1, 2 \), \( w = 1 \), and \( c = 0 \), the equilibria are all the vectors \( x = (t, t) \) for \( 0 \leq t \leq 1 \).

In the analysis of the structure of \( \mathcal{X} \), an important role will be played by the graph underlying the matrix \( P \), namely the directed graph \( G_P = (\mathcal{V}, \mathcal{E}) \) with node set \( \mathcal{V} = \{1, 2, \ldots, n\} \), and link set \( \mathcal{E} = \{(i, j) \in \mathcal{V} \times \mathcal{V} : P_{ij} > 0\} \).

The contribution of this rest of this section is threefold. First, we prove that among the equilibria there is always a minimal one and a maximal one. Moreover, we show that such minimal and maximal equilibria both depend monotonically on the exogenous flow vector \( c \) and that they can be computed by a natural iterative distributed algorithm. Finally, we get to the main result of this section introducing a fundamental partition of the node set into three subsets and prove that such partition is invariant with respect to the entire set of equilibria. Such results will be instrumental to all derivations in the next sections.

3.1 Minimal and maximal equilibria

We start with the following simple result.

**Lemma 3.1.** Let \( w \in \mathbb{R}^n_+ \) be a nonnegative vector and let \( P \in \mathbb{R}^{n \times n}_+ \) be a nonnegative matrix. Then, the map
\[
(x, c) \mapsto S_0^w(P'x + c)
\]
is continuous and monotone nondecreasing from \( \mathbb{R}^n \times \mathbb{R}^n \) to the complete lattice \( \mathcal{L}_0^w \subseteq \mathbb{R}^n \).

**Proof.** Continuity follows immediately from the fact that \( S_0^w \) is the composition of the saturation function \( S_0^w \) with the linear map \( x \mapsto P'x + c \). Similarly, monotonicity is implied by the fact that both those functions preserve the natural partial ordering of \( \mathbb{R}^n \) (this is true for the linear map because \( P \) is a nonnegative matrix).

We now consider the iterative algorithm
\[
x(t + 1) = S_0^w(P'x(t) + c), \quad t \geq 0,
\]
whose fixed points coincide with the solutions of \( \text{2} \).

**Proposition 3.1.1.** Let \( \mathcal{X} \) be the set of equilibria of the network \((P, w)\) with exogenous flow vector \( c \in \mathbb{R}^n \). Let \( x(t) \) be the sequence determined by \( \text{13} \) with initial condition \( x(0) = x_0 \). Then,

(i) The set \( \mathcal{X} \) admits a minimal element \( \underline{x} \) and a maximal element \( \overline{x} \) (in the sense of the usual entry-wise ordering in \( \mathbb{R}^n \));

(ii) If \( x_0 = 0 \), then \( x(t) \to \underline{x} \) for \( t \to +\infty \);

(iii) If \( x_0 = 1 \), then \( x(t) \to \overline{x} \) for \( t \to +\infty \).

Indicating with \( \underline{x}(c) \) and \( \overline{x}(c) \) the minimal and maximal equilibrium as a function of the exogenous flow vector \( c \), we have that

(iv) \( c \mapsto \underline{x}(c) \) and \( c \mapsto \overline{x}(c) \) are monotone nondecreasing maps from \( \mathbb{R}^n \) to \( \mathcal{L}_0^w \).
Proof. For simplicity of notation, throughout the proof, we put \( \Phi(x) = S^0_w(P'x + c) \).

Assume first that \( x_0 = 0 \). In this case the sequence \( x(t) \) determined by (14) is non-decreasing and this can be shown with an induction argument on \( t \geq 0 \). Indeed, on the one hand, since \( \Phi(0) \in L^w_0 \), we have that \( x(0) = 0 \leq \Phi(0) = x(1) \).

On the other hand, since \( \Phi(x) \) is monotone by Lemma 3.1 we have that, if \( x(t) \leq x(t + 1) \) for some \( t \geq 0 \), then \( x(t + 2) = \Phi(x(t + 1)) \geq \Phi(x(t)) = x(t + 1) \). Being non-decreasing and bounded, the sequence \( x(t) \) admits limit \( x \).

Since \( \Phi \) is continuous, \( x = \Phi(x) \) is necessarily a solution of (14).

We now prove by contradiction that \( x \) is minimal in \( \mathcal{X} \). If not, let \( x^* \in \mathcal{X} \) be such that \( x^* < x \) and let \( t^* = \sup\{t \geq 0 : x(t) \leq x^*\} \). Notice that, since \( x(0) = 0 \leq x^* \) and \( \lim x(t) = x \), \( t^* \) is necessarily finite and such that \( x(t^*) \leq x^* \). Monotonicity of \( \Phi \) implies that \( x(t^* + 1) = \Phi(x(t^*)) \leq \Phi(x^*) = x^* \), thus contradicting the way \( t^* \) was defined. Therefore, \( x \) is minimal in \( \mathcal{X} \). This proves (ii) and the first part of (i).

A completely parallel argument shows that, with initial condition \( x(0) = w \) the iterations (13) converge to a maximal solution \( \mathfrak{U} \) of (2). This proves (iii) and the second part of (i).

We now prove (iv). Consider two vectors \( c_1, c_2 \in \mathbb{R}^n \) such that \( c_1 \leq c_2 \) and let \( x_1(t) \) and \( x_2(t) \) the two sequences determined by the iterative algorithm (13) with, respectively, \( c = c_1 \) and \( c = c_2 \), and with initial condition \( x_1(0) = x_2(0) = 0 \). Using (ii), we obtain that the two sequences converge, respectively, to the minimal solutions \( \mathfrak{U}(c_1) \) and \( \mathfrak{U}(c_2) \). Using Lemma 3.1 we can see that

\[
x_1(t) \leq x_2(t) \Rightarrow x_1(t + 1) = S^0_w(P'x_1(t) + c_1) \leq S^0_w(P'x_2(t) + c_2) = x_2(t + 1).
\]

Since \( x_1(0) = x_2(0) \), by induction we have that \( x_1(t) \leq x_2(t) \) for all \( t \). This yields \( \mathfrak{U}(c_1) \leq \mathfrak{U}(c_2) \). We have proven that \( \mathfrak{U}(c) \) is monotone nondecreasing. The same property for the maximal solution \( \mathfrak{U}(c) \) follows by an equivalent argument.

Remark. Observe that the iteration (13) can be implemented as a distributed algorithm, whereby at each time \( t \geq 0 \) each node \( i \in \mathcal{V} \) using only the current states \( x_j(t) \) of its in-neighbors \( \{j \in \mathcal{V} : P_{ji} > 0\} \) update its state according to \( x_i(t + 1) = S^0_w(\sum_j P_{ji}x_j(t) + c_i) \). The complexity of each iteration of (13) is therefore of the order of the number of links in the network.

3.2 A fundamental partition

We now show that, regardless of whether the system (2) admits a unique solution or multiple ones, the node set \( \mathcal{V} \) can be partitioned in three subsets such that every equilibrium \( x \) has entries saturated to \( w \), unsaturated, and, respectively, saturated to \( 0 \) on these subsets. Remarkably, such partition can be proven to be equilibrium-invariant, a fact that will play a key role in the future analysis.

For every equilibrium \( x \in \mathcal{X} \), we proceed by partitioning the node set \( \mathcal{V} \) as

\[
\mathcal{V} = \mathcal{V}_x^+ \cup \mathcal{V}_x^- \cup \mathcal{V}_0^-,
\]

where:

- \( \mathcal{V}_x^+ = \{i \in \mathcal{V} : c_i + \sum_{k \neq i} P_{ki}x_k > w_i\} \) is the set of surplus nodes;
- \( \mathcal{V}_0^- = \{i \in \mathcal{V} : 0 \leq c_i + \sum_{k \neq i} P_{ki}x_k \leq w_i\} \) is the set of exposed nodes;
- \( \mathcal{V}_x^- = \{i \in \mathcal{V} : c_i + \sum_{k \neq i} P_{ki}x_k < 0\} \) is the set of deficit nodes.

By the way these sets have been defined, it follows that

\[
\begin{align*}
x_i &= 0, & \text{if } i \in \mathcal{V}_x^+ \\
x_i &= w_i, & \text{if } i \in \mathcal{V}_x^- \\
x_i &= c_i + \sum_{j \neq i} P_{ji}x_j, & \text{if } i \in \mathcal{V}_0^-.
\end{align*}
\]

The following is the key result of this section and will be instrumental to all our future derivations.

**Proposition 3.1.2.** The partition (14) is invariant over all equilibria \( x \in \mathcal{X} \).
Proof: We consider the maximal equilibrium \( \mathbf{x} \) and any another equilibrium \( x \) and we show that they share the same node partition \([14]\). To begin with, notice that, since \( \mathbf{x} \geq x \), we have \( V_+^x \supseteq V_+^x \) and \( V_-^x \subseteq V_-^x \). Let us split nodes in five different classes, \( C_1, C_2, C_3, C_4, C_5 \), and write any vector \( y \in \mathbb{R}^V \) in a block form accordingly \( y = (y_1, y_2, y_3, y_4, y_5) \).

For simplicity of notation indicate \( Q_{ij} := (P')_{C_j} \).

The five classes of nodes correspond to the four possible cases in which \( \mathbf{x} \) and \( x \) can differ and are precisely defined below:

- \( C_1 = V_+^x \) is the set of nodes that are surplus for both equilibria; it holds \( w_1 = x_1^+ = x_1 \);
- \( C_2 = V_+^x \setminus V_+^x \) is the set of nodes that are surplus for \( \mathbf{x} \) but not for \( x \); it holds \( w_2 = x_2 = \sum_{k=1}^4 Q_{2k}x_k + c_2 \) and \( x_2 \geq \sum_{k=1}^4 Q_{2k}x_k + c_2 \);
- \( C_3 = V_+^x \cap \mathbb{R}^0 \) is the set of nodes that are exposed for both equilibria; it holds \( x_3 = \sum_{k=1}^4 Q_{3k}x_k + c_3 \);
- \( C_4 = V_+^x \setminus V_+^x \) is the set of nodes that are exposed for \( \mathbf{x} \) and deficit for \( x \); it holds \( x_4 = \sum_{k=1}^4 Q_{4k}x_k + c_4 \) and \( x_4 = 0 \geq \sum_{k=1}^4 Q_{4k}x_k + c_4 \);
- \( C_5 = V_+^x \) is the set of nodes that are deficit for both equilibria; it holds \( x_5 = x_5 = 0 \).

For our claim to hold, we need to prove that \( C_2 = C_4 = 0 \). We put \( \Delta x = \mathbf{x} - x \geq 0 \) and we notice that, for classes \( C_1 \) and \( C_5 \) we have that \( \Delta x_1 = \Delta x_5 = 0 \). For the remaining blocks we can write:

\[
\begin{align*}
\Delta x_2 &= x_2 - x_2 < \sum_{k=1}^4 Q_{2k}x_k + c_2 - \sum_{k=1}^4 Q_{2k}x_k - c_2 = \sum_{k=2}^4 Q_{2k}\Delta x_k \\
\Delta x_3 &= x_3 - x_3 = \sum_{k=1}^4 Q_{3k}x_k + c_3 - \sum_{k=1}^4 Q_{3k}x_k - c_3 = \sum_{k=2}^4 Q_{3k}\Delta x_k \\
\Delta x_4 &= x_4 - x_4 < \sum_{k=1}^4 Q_{4k}x_k + c_4 - \sum_{k=1}^4 Q_{4k}x_k - c_4 = \sum_{k=2}^4 Q_{4k}\Delta x_k
\end{align*}
\]

Since the matrix

\[
Q = \begin{bmatrix}
Q_{22} & Q_{23} & Q_{24} \\
Q_{32} & Q_{33} & Q_{34} \\
Q_{42} & Q_{43} & Q_{44}
\end{bmatrix}
\]

is a sub-matrix of \( P' \), necessarily its transpose \( Q' \) is sub-stochastic. By contradiction, if \( C_2 \cup C_4 \neq \emptyset \), using \([10]\), the fact that \( \Delta x_2 + \Delta x_4 > 0 \), and the fact that \( 1'Q \leq 1' \), we obtain

\[
1'\Delta x_2 + 1'\Delta x_3 + 1'\Delta x_4 < 1'Q' \Delta x_2 + \Delta x_3 + \Delta x_4 \leq 1'\Delta x_2 + 1'\Delta x_3 + 1'\Delta x_4 .
\]

This contradiction implies that, necessarily, \( C_2 = C_4 = 0 \), thus proving the result.

We gather some immediate consequences of Proposition \( 3.1.2 \) in the following result.

Corollary 3.1.1. For every network \( (P, w) \) and exogenous flow \( c \), there exists a partition of the node set

\[
\mathcal{V} = \mathcal{V}_+ \cup \mathcal{V}_0 \cup \mathcal{V}_- ,
\]

such that

(i) for every equilibrium \( x \in \mathcal{X} \)

\[
\begin{align*}
x_{\mathcal{V}_-} &= 0 , & x_{\mathcal{V}_0} &= P_{\mathcal{V}_0}x_{\mathcal{V}_0} + c_{\mathcal{V}_0}, & x_{\mathcal{V}_+} &= w_{\mathcal{V}_+} ; \\
x_{\mathcal{V}_-}^{(1)} &= x_{\mathcal{V}_-}^{(2)} , & x_{\mathcal{V}_+}^{(1)} &= x_{\mathcal{V}_+}^{(2)} , & x_{\mathcal{V}_+}^{(1)} &= x_{\mathcal{V}_+}^{(2)} .
\end{align*}
\]

(ii) for every two equilibria \( x^{(1)}, x^{(2)} \in \mathcal{X} \),

Corollary 3.1.1 implies that the check for uniqueness can always be restricted to those entries of the equilibria that are in \( \mathcal{V}_0 \) and that such entries of the equilibria solve a linear system of equations. However, the outstanding difficulty in the analysis of the equilibrium set \( \mathcal{X} \) is that the partition \([13]\) is not known a priori. This will be dealt with in the following section.
4 Uniqueness of the equilibria

In this section, we derive necessary and sufficient conditions for uniqueness of equilibria in a network \((P, w)\) with exogenous flow \(c\). We shall first consider two fundamental special cases — when the matrix \(P\) is sub-stochastic and out-connected (Section 4.1) and, respectively, when \(P\) is stochastic and irreducible (Section 4.2) — and then build on them in order to state and prove the general result (Section 4.3).

4.1 The out-connected case

Consider a sub-stochastic matrix \(P \in \mathbb{R}^{n \times n}\) and let \(O = \{i \in V: \sum_j P_{ij} < 1\}\) be the set of the rows of \(P\) that sum up to less than 1. The matrix \(P\) is said to be out-connected if the set \(O\) is nonempty and globally reachable in the graph \(G_P\) (i.e., from every node there is a path to a node in \(O\)). The spectrum of any out-connected sub-stochastic matrix is contained in the open unitary disk centered in 0, as stated in the following result, whose proof can be found, e.g., in [22].

**Lemma 4.1.** Let \(P \in \mathbb{R}^{n \times n}\) be an out-connected sub-stochastic matrix. Then, \(P\) has spectral radius strictly smaller than 1.

By combining Proposition 3.1.2 and Lemma 4.1 we can prove the following first uniqueness result.

**Proposition 4.1.1.** For a network \((P, w)\) with \(P\) sub-stochastic out-connected and for any exogenous flows \(c\), the equilibrium is unique.

**Proof.** Let \(x^{(1)}, x^{(2)} \in X\) and put \(y = x^{(1)} - x^{(2)}\). We know from point (ii) of Corollary 3.1.1 that \(y_i = 0\) for every \(i \in V_0 \cup V_+\). The proof is finished if \(V_0 = \emptyset\). Otherwise, let \(z \in \mathbb{R}^{V_0}\) and \(Q \in \mathbb{R}^{V_0 \times V_0}\) be the restrictions of \(y\) to \(V_0\) and of \(P\) to \(V_0 \times V_0\), respectively. It then follows from point (i) of Corollary 3.1.1 that \(z\) satisfies the equation \(z = Qz\).

By Lemma 4.1, the matrix \((I - Q)\) is invertible and thus \(z = 0\). We thus have that \(x^{(1)} = x^{(2)}\) and this completes the proof.

4.2 The irreducible stochastic case

We now study the case when \(P\) is an irreducible stochastic matrix (i.e., the associated graph \(G_P\) is strongly connected). The following result gathers standard algebraic properties of irreducible stochastic matrices that will be instrumental in the following derivations.

**Lemma 4.2.** Let \(P \in \mathbb{R}^{n \times n}\) be an irreducible stochastic matrix. Then:

(i) the matrix \(I - P\) has rank \(n - 1\) and there exists a unique invariant probability vector \(\pi = P'\pi\). Moreover, such invariant probability vector has positive entries \(\pi_i > 0\) for every \(i \in V\);

(ii) for every zero-sum vector \(c \in \mathbb{R}^n\), the serie

\[
\nu = \frac{1}{2} \sum_{k \geq 0} \left( I + \frac{P'}{2} \right)^k c
\]

is convergent and its limit \(\nu\) satisfies \(\nu = P'\nu + c\);

(iii) any square matrix \(Q\) obtained as the restriction of \(P\) to a proper subset of nodes \(U \subseteq V\) has spectral radius smaller than 1.

**Proof.** (i) This is quite standard, c.f., e.g., [23].

(ii) Let \(\overline{P} = (I + P)/2\). Then, \(\overline{P}\) is stochastic irreducible and aperiodic so that the matrix series

\[
H = \frac{1}{2} \sum_{k \geq 0} \left( \overline{P}^k - \overline{1}\pi' \right)
\]
is convergent. Observe that, for any zero-sum vector \( c \), we have

\[
H'c = \lim_{t \to +\infty} \frac{1}{2} \sum_{k=0}^{t} \left( (P')^k - \pi' \right) c = \lim_{t \to +\infty} \frac{1}{2} \sum_{k=0}^{t} (P')^k c = \nu.
\]

Observe that

\[
\frac{1}{2} H + \frac{1}{2} H P = HP = \frac{1}{2} \sum_{k \geq 0} (P')^{k+1} - \pi' P = \frac{1}{2} \sum_{k \geq 1} (P')^k - \pi' = H - \frac{1}{2} I + \frac{1}{2} \pi',
\]

so that \( H = HP + I - \pi' \), which in turn implies that

\[
\nu = H'c = P'H'c + c - \pi'c = \nu + c,
\]

where the last equality follows since \( \pi'c = 0 \). This completes the proof.

(iii) It is sufficient to apply Lemma 4.2.1 upon noticing that the square matrix \( Q \) obtained as the restriction of an irreducible stochastic matrix \( P \) to any proper subset of nodes \( U \subseteq V \) is an out-connected sub-stochastic matrix. \( \Box \)

We can now state and prove the following result giving, for irreducible networks, an explicit characterization of the condition of non-uniqueness as well a representation of the set of equilibria.

**Proposition 4.2.1.** For a network \((P, w)\) with \( P \) stochastic irreducible, and for any exogenous flow \( c \), the set of equilibria \( \mathcal{X} \) contains more than one elements if and only if \( \sum_i c_i = 0 \) and

\[
\min_i \left\{ \frac{\nu_i}{\pi_i} \right\} + \min_i \left\{ \frac{w_i - \nu_i}{\pi_i} \right\} > 0,
\]

where \( \nu \) and \( \pi \) are defined as in Lemma 4.2. Moreover, in this case, the set of equilibria is given by

\[
\mathcal{X} = \left\{ x = \nu + \alpha \pi : -\min_i \left\{ \frac{\nu_i}{\pi_i} \right\} \leq \alpha \leq \min_i \left\{ \frac{w_i - \nu_i}{\pi_i} \right\} \right\}.
\]

**Proof.** We first analyze the solution on \( \mathbb{R}^n \) of the non-saturated linear system

\[
x = P'x + c.
\]

Since \( P \) is in this case row-stochastic, summing up all the entries of both sides of (24) gives

\[
\mathbb{1}'x = \mathbb{1}'P'x + \mathbb{1}'c = \mathbb{1}'x + \mathbb{1}'c
\]

so that, for solutions to exist, it must hold that \( \sum_i c_i = \mathbb{1}'c = 0 \). On the other hand, if condition \( \sum_i c_i = 0 \) is satisfied, since \( P \) is irreducible, Lemma 4.2 ensures that \( 1 - P \) has rank \( n - 1 \) and that the set of solutions of (24) coincides with the line

\[
\mathcal{H} = \left\{ x = \nu + \alpha \pi : \alpha \in \mathbb{R} \right\},
\]

with the vectors \( \nu \) and \( \pi \) defined as in Lemma 4.2 (ii) and (i), respectively.

Notice that solutions of the linear system (24) that stay in the complete lattice are automatically equilibria. In other words,

\[
\mathcal{H} \cap L^w_v \subseteq \mathcal{X}
\]

Observe, moreover, that \( \mathcal{H} \cap L^w_v \) coincides with the right-hand side of (23) and that condition (22) is equivalent to saying that \( \mathcal{H} \cap L^w_v \) is a segment of strictly positive length.

We are now ready to prove the statements of the theorem.

(i) Suppose first that there are multiple equilibria, namely \( |\mathcal{X}| > 1 \). Consider the node set partition (18) that is common to all solutions of system (2) by Proposition 4.1.1. If \( V_0 \cup V_+ \neq \emptyset \), since \( V_0 \) is a proper subset of \( V \), Lemma 4.2 (iii) guarantees that the restriction \( Q \) of \( P \) to \( V_0 \times V_0 \) has spectral radius smaller than 1. Arguing exactly as in the proof of Proposition 4.1.1 we then deduce that \( |\mathcal{X}| = 1 \). This contradicts the assumption made. Therefore, necessarily \( V = V_0 \). In this case, it follows from point (i) of Corollary 4.1.1 that all equilibria are solutions of (24), namely \( \mathcal{H} \cap L^w_v = \mathcal{X} \). By previous considerations, since this set is not empty, the condition \( \mathbb{1}'c = 0 \) must hold. Moreover, \( |\mathcal{X}| > 1 \) implies that \( \mathcal{H} \cap L^w_v \) must be a segment of positive length that, as previously observed, is equivalent to condition (22).
(ii) Suppose now that \( \sum c_i = 0 \) and that condition (22) holds true. Then previous considerations imply that \( H \cap L^w_0 = \mathcal{X} \) is a segment of positive length. Non-uniqueness of equilibria is thus proven.

(iii) Finally notice that if any of the two equivalent conditions hold, then, from proof (i) we have that \( H \cap L^w_0 = \mathcal{X} \) and this is equivalent to representation (23).

This completes the proof.

The result above has a simple geometric interpretation in part already exploited in the proof. Assuming that \( \sum c_i = 0 \), the line \( H \) defined in (25) is the set of solutions of the non-saturated linear system (24). The non-uniqueness condition (22) is simply the condition that this line intersects the interior part of the lattice \( L^w_0 \) and the set of equilibria in this case is the segment obtained by this intersection. The minimal and maximal equilibria are the boundary points of this interval. We notice that the arguments used in the proof also show that in the case of non-uniqueness, necessarily all nodes must be exposed nodes, namely \( \mathcal{V} = \mathcal{V}_0 \).

**Example:**

Consider the network \((P, w)\) where

\[
P = \begin{bmatrix}
0 & 0.75 & 0.25 \\
0 & 0 & 1 \\
0.3 & 0.7 & 0
\end{bmatrix}, \quad w = \begin{bmatrix}
5 \\
3 \\
2
\end{bmatrix}.
\]

Notice that the matrix \( P \) is stochastic and irreducible. The associated graph \( G_P \) is depicted in Figure 1.

![Figure 1: The network of Example 4.2](image)

We analyze uniqueness for two possible exogenous flows

\[
c^{(1)} = [-1, 1, 0]' \quad c^{(2)} = [-2, 2, 0]'.
\]

Notice first that \( \sum c_i^{(1)} = \sum c_i^{(2)} = 0 \). We now check condition (22).

Indicating with \( \nu^i \), for \( i = 1, 2 \), the vector defined in (21) for \( c = c^c \) and with \( \pi \) the invariant probability of \( P \), a direct computation shows that

\[
\min_i \left\{ \frac{\nu^1_i}{\pi_i} \right\} + \min_i \left\{ \frac{w_i - \nu^1_i}{\pi_i} \right\} \approx 1.60 > 0
\]

\[
\min_i \left\{ \frac{\nu^2_i}{\pi_i} \right\} + \min_i \left\{ \frac{w_i - \nu^2_i}{\pi_i} \right\} \approx -6.41 < 0 \tag{26}
\]

By Proposition 4.2.1 we deduce that for the flow \( c^{(1)} \) there are multiple equilibria, while for the flow \( c^{(2)} \) the equilibrium is unique.

The structure of the equilibria is shown in Figure 2. Notice how in the first case the line \( H \) intersects the open part of the lattice, while in the second is external. The unique equilibrium in this last case is a single point necessarily on the boundary of the lattice as some of its entries are saturated either to 0 or to \( w_i \).
Equilibria and Systemic Risk in Saturated Networks

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(a) The network \((P, w)\) with exogenous flow \(c^{(1)}\) admits multiple equilibria (the red dots and the red segment).

(b) The network \((P, w)\) with exogenous flow \(c^{(2)}\) admits a unique equilibrium (the red dot).

Figure 2: Sets of equilibria for the network.

4.3 Uniqueness in the general case

We now study uniqueness of equilibria for general networks \((P, w)\), where \(P\) is an arbitrary sub-stochastic matrix and \(w\) an arbitrary nonnegative vector.

In order to proceed, we need to introduce some further graph-theoretic notions. Consider the graph \(G_P = (\mathcal{V}, \mathcal{E})\) associated with \(P\) that, we recall, has node set \(\mathcal{V} = \{1, 2, \ldots, n\}\) and link set \(\mathcal{E} = \{(i, j) \in \mathcal{V} \times \mathcal{V} : P_{ij} > 0\}\). A trapping set is any nonempty subset of nodes \(S \subseteq \mathcal{V}\) with no links pointing out of it, i.e., such that \(P_{ij} = 0\) for every \(i \in S\) and \(j \in \mathcal{V} \setminus S\). A trapping set is referred to as irreducible if it does not strictly contain any other trapping set. A sink node \(j\) is a node such that \(\{j\}\) is a trapping set (and, as such, it is necessarily irreducible). The transient part \(T \subseteq \mathcal{V}\) of a graph is the subset of nodes that do not belong to any irreducible trapping set. The node set \(\mathcal{V}\) can be partitioned as the disjoint union

\[\mathcal{V} = T \cup S_1 \cup S_2 \cup \ldots \cup S_m,\]

of the transient part \(T\) and of the distinct irreducible trapping sets \(S_l\) for \(1 \leq l \leq m\), with \(0 \leq m \leq n\). Observe that partition (27) is determined by the matrix \(P\) only. In the special case when \(G_P\) is strongly connected, the whole node set \(\mathcal{V}\) is itself an irreducible trapping set, so that in this case \(T = \emptyset, m = 1,\) and \(S_1 = \mathcal{V}\). See Figure 3 for an example of a graph with a nonempty transient part \(T\) and a single trapping set \(S_1\).

Figure 3: A graph \(G\) containing an irreducible trapping set \(S_1\) and a nonempty transient part \(T\).

The following result gives a complete analysis of the structure of the equilibria for generic networks. We recall the standing notation assumption: if \(x\) is a vector and \(P\) is a matrix, \(x_A, P_{AB}\) denote restrictions to, respectively, the index sets \(A\) or \(A \times B\).

**Theorem 4.3.** Consider a network \((P, w)\), where \(P\) is any sub-stochastic matrix, and an exogenous flow \(c\). Consider the partition (27) of the node set into its transient part and its irreducible trapping sets of the graph \(G_P\). Then, the equilibria \(x \in X\) split according to such partition

\[x = (x_T, x_{S_1}, \ldots, x_{S_m})\]

in such a way that
(i) their projection $x_T$ on the transient part $T$ is unique; 
(ii) their projection $x_{S_l}$ on a trapping set $S_l$ that is out-connected or such that

$$
\sum_{j \in S_l} \left( c_j + \sum_{i \in T} P_{ij} x_i \right) \neq 0,
$$

is unique;

(iii) their projection $x_{S_l}$ on a trapping set $S_l$ that is not out-connected and such that

$$
\sum_{j \in S_l} \left( c_j + \sum_{i \in T} P_{ij} x_i \right) = 0,
$$

is non-unique if and only if

$$
\min_i \left\{ \frac{\nu_i^{(l)}}{\pi_i^{(l)}} \right\} + \min_i \left\{ \frac{w_i - \nu_i^{(l)}}{\pi_i^{(l)}} \right\} > 0,
$$

where

- $\pi^{(l)} = (P_{S_lS_l})' \pi^{(l)}$ is the unique invariant probability vector of the block $P_{S_lS_l}$;
- $\nu_i^{(l)} = \frac{1}{2} \sum_{k \geq 0} \left( \frac{1}{2} + (P_{S_l})' \right)^k (c_{S_l} + (P_{T}S_l)'x_T)$

Moreover, in this case, the projection $x_{S_l}$ of any equilibrium satisfies

$$
x_{S_l} = \nu^{(l)} + \alpha \pi^{(l)}, \quad -\min_i \left\{ \frac{\nu_i^{(l)}}{\pi_i^{(l)}} \right\} \leq \alpha \leq \min_i \left\{ \frac{w_i - \nu_i^{(l)}}{\pi_i^{(l)}} \right\}.
$$

Proof. First, we observe that, by the way the transient part $T$ is defined, the projection of any equilibrium $x$ on $T$ satisfies

$$
x_T = S_0^{w_T} \left( (P_{TT})' x_T + c_T \right),
$$

The fact that $T$ is the transient part implies that the matrix $P_{TT}$ is sub-stochastic and out-connected, so that we can readily apply Proposition 4.1.1 to prove claim (i).

We now study separately the behavior on each trapping set $S_l$, for $l = 1, \ldots, m$ observing that the relative projection of the equilibrium $x$ satisfies

$$
x_{S_l} = S_0^{w_{l_S}} \left( (P_{S_lS_l})' x_{S_l} + (P_{T}S_l)'x_T + c_{S_l} \right),
$$

Claims (ii) and (iii) then follow by applying Proposition 4.2.1.

5 Continuity and the lack thereof

In this section, we study how, for a given network $(P, w)$, the set of equilibria depends on the exogenous flow vector $c$. This analysis is crucial to study the way exogenous shocks affect the network equilibrium. Indeed, recall that in considered financial network model, we have that $c = a - b - \epsilon$ where $a$ and $b$ represent the vector of assets values and liabilities towards external entities and $\epsilon$ is the vector of shock affecting the value of the assets. Thus, the resilience of the system with respect to shocks is in the end determined by the way solutions depend on the parameter vector $c$.

In the following we will consider a fixed network $(P, w)$ and a varying flow vector $c$ and use the notation

$$
\mathcal{X}(c), \quad \varphi(c), \quad \underline{c}(c)
$$

to denote, respectively, the set of equilibria, the maximal and the minimal equilibria of $(P, w)$ with exogenous flow $c$.

Moreover, let

$$
\mathcal{U} = \{ c \in \mathbb{R}^n : |\mathcal{X}(c)| = 1 \}, \quad \mathcal{M} = \mathbb{R}^n \setminus \mathcal{U},
$$

where

- $\mathcal{X}(c)$ is the set of equilibria of the system with exogenous flow $c$;
- $\varphi(c)$ is the maximal equilibrium of the system with exogenous flow $c$;
- $\underline{c}(c)$ is the minimal equilibrium of the system with exogenous flow $c$.


be the subsets of exogenous flow vectors for which there is a unique equilibrium and, respectively, there are multiple equilibria. For exogenous flow vectors \( c \in \mathcal{U} \), we shall also use the notation

\[
x(c) = \underline{x}(c) = \overline{x}(c)
\]

for the unique equilibrium.

The following result gives a complete picture of the behavior of the set of equilibria \( \mathcal{X}(c) \) as a function of \( c \). It shows that the set \( \mathcal{M} \) where the equilibrium is not unique is contained in the union of a finite number of linear manifolds of co-dimension 1 in \( \mathbb{R}^n \), that outside of it the equilibrium \( x(c) \) is a piecewise continuous function of the exogenous flow \( c \) that undergoes jump discontinuities when \( c \) crosses the non-uniqueness set \( \mathcal{M} \).

**Theorem 5.1.** Let \((P,w)\) be a network. Let \( \mathcal{U} \) and \( \mathcal{M} \) be defined as in (32) and let \( m \) be number of irreducible trapping sets in \( \mathcal{G}_P \). Then,

(i) the non-uniqueness set \( \mathcal{M} \) is the union of at most \( m \) linear sub-manifolds of co-dimension 1 in \( \mathbb{R}^n \);

(ii) the map \( c \rightarrow x(c) \) is continuous on the uniqueness set \( \mathcal{U} \);

(iii) for every exogenous flow \( c^* \in \mathcal{M} \),

\[
\lim\inf_{c \in \mathcal{U}} x(c) = \underline{x}(c^*), \quad \lim\sup_{c \in \mathcal{U}} x(c) = \overline{x}(c^*).
\]

**Proof.** (i) Consider the partition (27) of the node set of the graph \( \mathcal{G}_P \) into its transient part and its irreducible trapping sets. It then follows from Theorem 4.3(iii) that an exogenous flow \( c \in \mathcal{M} \) must satisfy equation (28) for some irreducible trapping set \( S_0 \). Each such condition determines a linear sub-manifold of co-dimension 1 in \( \mathbb{R}^n \). Therefore, we can conclude that the set \( \mathcal{M} \) of those exogenous flow vectors \( c \) for which the equilibrium are not unique is the union of at most \( m \) linear sub-manifolds of co-dimension 1 in \( \mathbb{R}^n \).

(ii) and (iii): Consider first any exogenous flow \( c^* \in \mathbb{R}^n \) and a sequence \((c(k))_{k \geq 1}\) in \( \mathbb{R}^n \) such that

\[
c(k) \xrightarrow{k \to +\infty} c^*, \quad \underline{x}(c(k)) \xrightarrow{k \to +\infty} x^*.
\]

Since

\[
\underline{x}(c(k)) = S_0^w (P' \underline{x}(c(k)) + c(k)),
\]

for all \( k \geq 1 \), passing to the limit in both sides of the above we obtain by continuity that

\[
x^* = S_0^w (P' x^* + c^*),
\]

i.e., \( x^* \in \mathcal{X}(c^*) \), so that in particular

\[
\underline{x}(c^*) \leq x^* \leq \overline{x}(c^*). \tag{34}
\]

The arbitrariness of the sequence \((c(k))_k\) satisfying (33) and (34) imply that

\[
\lim\inf_{c \in \mathcal{U}} x(c) \geq \underline{x}(c^*) \tag{35}
\]

Consider now \( c(k) = c^* - 1/k \mathbb{1} \). A continuity argument shows that for those trapping sets for which condition (28) holds true with \( c = c^* \), it continues to hold true with \( c = c(k) \) for sufficiently large \( k \). On the other hand, for those trapping sets for which condition (29) holds true for \( c = c^* \), condition (29) holds true instead for \( c = c(k) \) and sufficiently large \( k \). This says that, independently from \( c^* \), \( c(k) \in \mathcal{U} \) for sufficiently large \( k \). It then follows from Proposition 4.1.4 (iv) that, since \( c(k) \xrightarrow{k \to +\infty} c^* \) and \( c(k) \leq c^* \) for every \( k \geq 1 \),

\[
x(c(k)) = \overline{x}(c(k)) c(k) \xrightarrow{k \to +\infty} x^* \leq \overline{x}(c^*).
\]

By combining the above with (35) we deduce that

\[
\lim\inf_{c \in \mathcal{U}} x(c) = \underline{x}(c^*) \tag{36}
\]
A completely analogous argument allows one to prove that
\[
\limsup_{c \in \mathcal{U}} x(c) = \overline{x}(c^*)
\] (37)

Now, in case that \(c^* \in \mathcal{U}\), we have that \(\underline{x}(c^*) = \overline{x}(c^*)\) and then relations (36) and (37) together yield (ii). In case instead that \(c^* \in \mathcal{M}\), relations (36) and (37) together prove (iii).

We finally characterize the size of the maximum jump \(\overline{x}(c^*) - \underline{x}(c^*)\).

**Corollary 5.1.1.** Let \((P, w)\) be a network. Consider the partition \([27]\) of the node set into its transient part \(T\) and its irreducible trapping sets \(S_l\) for \(l = 1, \ldots, m\). For each \(l\), let \(\pi^{(l)}\) be the invariant probability vector of the block \(P_{S_l}S_l\). Then, the maximal norm of a jump discontinuity in the equilibrium is given by
\[
\max_{c \in \mathbb{R}^n} \|\overline{x}(c) - \underline{x}(c)\|_p = \sum_{1 \leq l \leq m} \left( \min_{i \in S_l} \frac{w_i}{\pi_i^{(l)}} \right)^p \|\pi^{(l)}\|_p^p,
\] (38)

**Proof.** First observe that, for every \(c \in \mathbb{R}^n\), Theorem 4.3 guarantees that there exist scalars \(\alpha_l\) such that
\[
(\overline{x}(c))_T - (\underline{x}(c))_T = 0, \quad (\overline{x}(c))_{S_l} - (\underline{x}(c))_{S_l} = \alpha_l \pi^{(l)},
\] (39)

for \(1 \leq l \leq m\). Now, observe that \(0 \leq \overline{x}(c) - \underline{x}(c) \leq w_i\) for every \(i \in V\), so that \(0 \leq \alpha_l \pi_i^{(l)} \leq w_i\) for every \(i \in S_l\), which in turn implies that
\[
0 \leq \alpha_l \leq \min_{i \in S_l} \frac{w_i}{\pi_i^{(l)}},
\] (40)

for \(1 \leq l \leq m\). It then follows from (36) and (40) that
\[
\|\overline{x}(c) - \underline{x}(c)\|_p^p \leq \sum_{1 \leq l \leq m} \left( \min_{i \in S_l} \frac{w_i}{\pi_i^{(l)}} \right)^p \|\pi^{(l)}\|_p^p,
\]

for every \(c \in \mathbb{R}^n\).

It remains to prove equality in (38). For this, consider \(c = 0\) and notice that, since \(0 \in X(0)\), then \(\underline{x}(0) = 0\). Notice now that on each irreducible trapping set \(S_l\), the non-uniqueness conditions (29) and (30) of Theorem 4.3 hold true (since \(\nu^{(l)} = 0\)) and it follows from expression (31) that
\[
(\overline{x}(0))_{S_l} = \min_{i \in S_l} \frac{w_i}{\pi_i^{(l)}} \pi^{(l)}
\]

Therefore,
\[
\|\overline{x}(0) - \underline{x}(0)\|_p^p = \sum_{1 \leq l \leq m} \left( \min_{i \in S_l} \frac{w_i}{\pi_i^{(l)}} \right)^p \|\pi^{(l)}\|_p^p,
\]

thus completing the proof.

We conclude this section with the following example.

**Example:**

In this example we will show how the jump discontinuity that occurs when the exogenous flow \(c\) crosses the critical set \(\mathcal{M}\) can be interpreted in the context of financial networks. For sake of simplicity, we will consider a strongly connected graph, but the analysis can be carried on in the general case.

To measure the aggregated effect of a shock on a network, it is useful to introduce a risk measure known as *systemic loss* as defined in [21]. Let \(c^0\) an arbitrary exogenous flow for which all nodes in the financial network are fully liable, i.e., such that \(x(c^0) = w\), and let \(c \leq c^0\) be the exogenous flow after a shock. Then, define the systemic loss as the aggregated difference between the net worth of the nodes before the shock, i.e., \(\nu^0 = P^0w + c^0 - w\), and
their net worth after the shock, i.e. \( v = P'x(c) + c - w \). Observe that
\[
I(c^*, c) = I'(v^0 - v) = I'(P'w + e^0 - w - (P'x(c) + c - w)) = I'c^0 - I'c + I'w - I'x(c).
\]
In the rightmost side of the above, the term \((I'c^0 - I'(c))\) represents the direct loss inflicted by the shock, while \((I'w - I'x(c))\), called the shortfall term, represents the indirect loss triggered by reduced payments.

Let us now compute the size of the jump of the loss function when the exogenous flow \( c \) crosses the critical region \( \mathcal{M} \). By using the representation of solutions derived in Proposition 4.2.1, we obtain that for every \( c^* \in \mathcal{M} \), the jump in the loss is given by
\[
\Delta l(c^*) = \liminf_{c \in U^-} l(c^0, c) - \limsup_{c \in U^+} l(c^0, c) = I'(x^*) - x(c^*)
\]
\[
= I'(\nu + \min_i \left\{ \frac{w_i - \nu_i}{\pi_i} \right\} \pi - \nu + \min_i \left\{ \frac{\nu_i}{\pi_i} \right\} \pi)
\]
\[
= \min_i \left\{ \frac{\nu_i}{\pi_i} \right\} + \min_i \left\{ \frac{w_i - \nu_i}{\pi_i} \right\}.
\]

Let us consider again the network of Example 4.2 (Figure 1) and compute how the loss function varies when an exogenous shock of magnitude \( \epsilon \in [0, 14] \) hits the network. Specifically, let us consider an initial vector \( c^0 = [5, 2, 2]' \) and an exogenous shock \( c \) such that:
\[
c = c^0 - \epsilon q, \quad q = \begin{bmatrix} 0.07 \\ 0.59 \\ 0.34 \end{bmatrix}, \quad \epsilon \in [0, 14].
\]
Notice that we have decided to perturb the vector \( c^0 \) along the particular direction \( q \); such \( q \) can be interpreted as a normalized vector collecting the sensibilities of nodes with respect to the shock magnitude \( \epsilon \).

The jump discontinuity must happen when uniqueness conditions fail to hold, which, by Proposition 4.2.1 means that first of all it must be \( I'c = 0 \implies \epsilon = 9 \), which gives us a candidate critical vector \( c^* = [4.4, -3.3, -1.1]' \). For such \( c^* \), a direct computation shows that \( \min_i \left\{ \frac{\nu_i}{\pi_i} \right\} + \min_i \left\{ \frac{w_i - \nu_i}{\pi_i} \right\} \approx 14.6 > 0 \), which implies that \( c^* \in \mathcal{M} \). Let us plot the loss function and the solution vector \( x \) as functions of \( \epsilon \) in Fig. 4.

Let us compute the size of the jump of the loss function when the exogenous flow \( c \) crosses the critical region \( \mathcal{M} \). By using the representation of solutions derived in Proposition 4.2.1, we obtain that for every \( c^* \in \mathcal{M} \), the jump in the loss is given by
\[
\Delta l(c^*) = \min_i \left\{ \frac{\nu_i}{\pi_i} \right\} + \min_i \left\{ \frac{w_i - \nu_i}{\pi_i} \right\} \approx 14.6.
\]
It is particularly useful also to plot the solution vector $x$ as a function of $\epsilon$ (Figure 4(b)); we can notice that all nodes are solvent for $\epsilon < 6.5$ while for $\epsilon \approx 6.5$ node 2 goes bankrupt as its outflow falls below $w_2 = 3$. As the shock magnitude increases, we reach the discontinuity point at $\epsilon = 9$ where the network suffers a dramatic crisis as node 1 and 3 suddenly default. Notice in particular how node 3 goes from fully solvent ($x_3 = w_3$) to completely insolvent ($x_3 = 0$) around $\epsilon = 9$.

6 Conclusions

This paper has analyzed network equilibria modeled as the solutions of a linear fixed point equation with a saturation inequality. Necessary and sufficient conditions for uniqueness and a general expression describing all such equilibria for a general network have been proved. Finally, the dependence of the network equilibria on the exogenous flows in the network has been studied highlighting the existence of jump discontinuities. This model was first considered to determining clearing payments in the context of networked financial institutions interconnected by obligations and it is one of the simplest continuous model where shock propagation phenomena and cascading failure effects may occur. Our results contribute to an in-depth analysis of such phenomena.

The understanding of the extent to which the network topology determines the structure of the solutions as well the possibility of these cascading effects to occur is still not sufficiently understood. As a future project, we aim at studying this for random networks with prescribed degree distributions.

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