Almost-zero-energy Eigenvalues of Some Broken Supersymmetric Systems

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Abstract

For a quantum mechanical system with broken supersymmetry, we present a simple method of determining the ground state when the corresponding energy eigenvalue is sufficiently small. A concise formula is derived for the approximate ground state energy in an associated, well-separated, asymmetric double-well-type potential. Our discussion is also relevant for the analysis of the fermion bound state in the kink-antikink scalar background.

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Supersymmetry (SUSY) and its breaking are fundamental issues in theoretical particle physics. There have also been numerous applications of SUSY to quantum-mechanical potential problems [1, 2], based on the observation that the spectrum of the Hamiltonian

\[ H_+ = -\frac{d^2}{dx^2} + V_+(x) , \quad V_+(x) = W^2(x) + W'(x) \]  

\[ (W(x) \text{ is the superpotential, and we set } \hbar = 2m = 1) \text{ is related through SUSY to that of the partner Hamiltonian} \]

\[ H_- = -\frac{d^2}{dx^2} + V_-(x) , \quad V_-(x) = W^2(x) - W'(x) . \]  

This formalism has provided us with a number of exactly solvable quantum mechanical systems for which energy eigenvalues and eigenfunctions can be found in closed forms. The key properties that made such feat possible are unbroken SUSY, manifested by the vanishing energy for the ground state of \( H_- \) (or \( H_+ \)), and shape invariance under the change of parameters for the given potentials [3]-[5]. This approach can sometimes be extended to parameter ranges corresponding to (spontaneously) broken SUSY [6, 7]. But, with SUSY broken, the ground state energy is no longer equal to zero and this jeopardizes the possibility of obtaining exact analytic results by the SUSY-based method in a crucial way.

In this work we will show that, in some broken SUSY case for which the lowest energy \( \bar{E}(> 0) \) for the Hamiltonian \( H_+ \) or \( H_- \) is sufficiently small, a simple perturbative scheme leading to an easy evaluation of \( \bar{E} \) can be developed. Our discussion finds useful application in studying the almost-zero-energy fermion modes in the background of a soliton-antisoliton pair.

The superpotential relevant for our discussion is given as follows. Let \( \sigma_R(x) \) be a generic function with the properties

\[ \sigma_R(x) > 0 \quad , \quad \text{for } x > 0 \text{ and } |x| \text{ not very small} , \]

\[ \sigma_R(x) \rightarrow -v \quad , \quad \text{for } x < 0 \text{ and } |x| \text{ not very small} \]  

and \( \sigma_L(x) \) the one with the properties

\[ \sigma_L(x) > 0 \quad , \quad \text{for } x < 0 \text{ and } |x| \text{ not very small} , \]

\[ \sigma_L(x) \rightarrow -v \quad , \quad \text{for } x > 0 \text{ and } |x| \text{ not very small} \]  

so that the related potentials \( V_{R\pm}(x) \equiv \sigma_R^2 \pm \sigma_R'(x) \) and \( V_{L\pm}(x) \equiv \sigma_L^2 \pm \sigma_L'(x) \) may have the general shapes shown in Figs.1 and 2, respectively. Then the superpotential appropriate to our case is obtained by combining these two types of functions as

\[ W(x) = \sigma_R(x - l_1) + \sigma_L(x - l_2) + v \]  

with \( L \equiv |l_1 - l_2| \) taken to be reasonably large (so that \( W(x) \) may have a flat basin between the points \( x = l_2 \) and \( x = l_1 \)).\(^1\) See Fig.3 for the schematic plots of \( W(x) \) and the related

\(^1\) At the later stage of our discussion we will use the fact if \( l^* \) denotes a certain point in the flat basin, the approximation \( \sigma_R(x - l_1) + v \simeq 0 \) for \( x < l^* \) or \( \sigma_L(x - l_2) + v \simeq 0 \) for \( x > l^* \) is valid.
potentials $V_{\pm}(x) \equiv W^2(x) \pm W'(x)$. Both $W(\infty)$ and $W(-\infty)$ being positive, this corresponds to the case of broken SUSY [1,2]; but, for the posited superpotential (with $L$ large), the ground state energy $\bar{E}$ is expected to be rather small. [Our superpotential will be an even function of $x$ if $\sigma_L(x)$ happens to be the mirror image of $\sigma_R(x)$, i.e., $\sigma_L(x) = \sigma_R(-x)$, and $(l_1, l_2)$ are equal to $(\frac{L}{2}, -\frac{L}{2})].$

For $W(x)$ specified as above, the corresponding Hamiltonians $H_{\pm}$ involve the potentials which can approximately be described by the sum of two well-separated potentials (aside from a constant term), i.e.,

$$V_{\pm}(x) \sim V_{R\pm}(x - l_1) + V_{L\pm}(x - l_2) - v^2.$$  \hfill (6)

These correspond to asymmetric double wells even when $W(x)$ is an even function, and hence the well-known approximation schemes used for symmetric double wells (e.g., instanton methods, tight-binding approximations) would not be much useful. [Note that, for the ground state of our Hamiltonian with the potential $V_+ \sim V_{R+} + V_{L+} - v^2$ (or $V_- \sim V_{R-} + V_{L-} - v^2$), the tight-binding approximation is plainly not available — while the local Hamiltonian involving $V_{L+}$ (or $V_{R-}$) allows a zero-energy bound state, no zero-energy state exists for the local Hamiltonian involving $V_{R+}$ (or $V_{L-}$)]. But, for a supersymmetric system, one can always consider a pair of coupled first-order differential equations instead of the second-order Schrödinger equations. Our perturbative approach for the ground state is based on the analysis of these first-order equations, and as a result we obtain a remarkably simple formula for the lowest eigenvalue $\bar{E}$. It is simply the square of the product of the two zero-energy eigenfunctions (allowed with the potentials $V_{L+}(x - l_2)$ and $V_{R-}(x - l_1)$ separately), evaluated at an arbitrary chosen point $l^*$ in the flat middle region of the superpotential $W(x)$. See the expression (21) below.

Consider a matrix Hamiltonian

$$\mathcal{H} = \begin{pmatrix} 0 & A^\dagger \\ A & 0 \end{pmatrix}$$  \hfill (7)

with

$$A = \partial_x + W(x) \quad , \quad A^\dagger = -\partial_x + W(x).$$  \hfill (8)

The corresponding eigenvalue equation, $\mathcal{H}\Psi(x) = \omega\Psi(x)$ with $\Psi(x) = \begin{pmatrix} \Psi_1(x) \\ \Psi_2(x) \end{pmatrix}$, reduces to a pair of first-order differential equations

$$A \Psi_1(x) = \omega \Psi_2(x),$$
$$A^\dagger \Psi_2(x) = \omega \Psi_1(x).$$ \hfill (9a)

(9b)

Then, based on the relations

$$A^\dagger A = -\partial^2_x + W^2(x) - W'(x) \equiv H_- \quad , \quad AA^\dagger = -\partial^2_x + W^2(x) + W'(x) \equiv H_+,$$  \hfill (10)

one finds that the functions $\Psi_1(x)$ and $\Psi_2(x)$ are eigenfunctions of the Schrödinger Hamiltonians $H_-$ and $H_+$ (with the same energy $E = \omega^2$), respectively. Further, it is not difficult to show that if $\Psi(x) = \begin{pmatrix} \Psi_1(x) \\ \Psi_2(x) \end{pmatrix}$ is an eigenvector of $\mathcal{H}$ with eigenvalue $\omega$, $\tilde{\Psi}(x) = \begin{pmatrix} \Psi_1(x) \\ -\Psi_2(x) \end{pmatrix}$
corresponds to an eigenvector of $\mathcal{H}$ with eigenvalue $-\omega$. For our later application, we also remark that the above matrix Hamiltonian can be written as

$$\mathcal{H} = -i\gamma^0 \gamma^1 \partial_x + \gamma^0 W(x)$$  \hspace{1cm} (11)

with $\gamma^0 = \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\gamma^1 = i\sigma^3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$. That is, it describes the Dirac Hamiltonian (defined on a line) in the presence of an external scalar field $W(x)$ [8].

Let us now specialize to the case with the superpotential given by the form (5) with $L$ large. If $\varphi_1(x)$ is the ground-state eigenfunction of the Hamiltonian $\mathcal{H}_-$ with a small energy $E \equiv \omega^2$ and $\varphi_2(x)$ that of the isospectral partner $\mathcal{H}_+$ with the same energy, they will have to satisfy the integral equations (derived from Eqs. (9a) and (9b))

$$\varphi_1(x) = e^{-\int_{l_1}^{x} W(y) dy} \left\{ D_1 + \bar{\omega} \int_{a_1}^{x} dy \left[ e^{\int_{a_1}^{y} W(z) dz} \varphi_2(y) \right] \right\}, \hspace{1cm} (12a)$$

$$\varphi_2(x) = e^{\int_{l_2}^{x} W(y) dy} \left\{ D_2 - \bar{\omega} \int_{a_2}^{x} dy \left[ e^{-\int_{a_2}^{y} W(z) dz} \varphi_1(y) \right] \right\}, \hspace{1cm} (12b)$$

for suitable constants $a_1$, $a_2$, $D_1$ and $D_2$. Clearly, for small enough $\bar{\omega}$, we may iterate these equations to learn about the corresponding eigenfunctions. Then, let $\varphi_1^0(x)$ and $\varphi_2^0(x)$ denote the limiting expressions of $\varphi_1(x)$ and $\varphi_2(x)$ for very large $L$ (and therefore very small $\bar{\omega}$). Here, considering the general shapes of the potentials $V_{\pm}$ (see also Eq.(6)) shown in Fig.3, $\varphi_1^0(x)$ ($\varphi_2^0(x)$) can be taken as the normalized zero-energy eigenfunction of $-\partial_x^2 + V_{R-}(x-l_1)$ ($-\partial_x^2 + V_{L+}(x-l_2)$). This in turn implies that $\varphi_1^0(x)$ and $\varphi_2^0(x)$ satisfy the first-order differential equations

$$[\partial_x + \sigma_R(x-l_1)]\varphi_1^0(x) = 0 \hspace{1cm} [\partial_x + \sigma_L(x-l_2)]\varphi_2^0(x) = 0. \hspace{1cm} (13)$$

Hence we have the explicit expressions

$$\varphi_1^0(x) = C_1 e^{-\int_{l_1}^{x} \sigma_R(y-l_1) dy} \hspace{1cm} \varphi_2^0(x) = C_2 e^{\int_{l_2}^{x} \sigma_L(y-l_2) dy} \hspace{1cm} (14)$$

with appropriate normalization constants $C_1$ and $C_2$, which are taken to be positive. [In view of Eqs. (9a) and (9b), both $\varphi_1^0(x)$ and $\varphi_2^0(x)$ are chosen to be real]. As these informations are used with our integral equations (12a) and (12b), we are led to the following approximate expressions for $\varphi_1(x)$ and $\varphi_2(x)$ (valid for large $L$):

$$\varphi_1(x) \simeq e^{-\int_{l_1}^{x} W(y) dy} \left\{ C_1 + \bar{\omega} \int_{l_1}^{x} dy \left[ e^{\int_{l_1}^{y} W(z) dz} \varphi_2^0(y) \right] \right\} \hspace{1cm} (15a)$$

$$\varphi_2(x) \simeq e^{\int_{l_2}^{x} W(y) dy} \left\{ C_2 - \bar{\omega} \int_{l_2}^{x} dy \left[ e^{-\int_{l_2}^{y} W(z) dz} \varphi_1^0(y) \right] \right\} \hspace{1cm} (15b)$$

with the value $l^*$ chosen conveniently at some point in the flat middle region of the superpotential so that both $|l_1 - l^*|$ and $|l^* - l_2|$ may become $\mathcal{O}(L)$. We then note that the expression (15a) as $x \to -\infty$ (and similarly that in Eq.(15b) as $x \to \infty$) would blow up unless the value of $\bar{\omega}$ were chosen such that

$$C_1 + \bar{\omega} \int_{l_1}^{-\infty} dy \left[ e^{\int_{l_1}^{y} W(z) dz} \varphi_2^0(y) \right] = 0. \hspace{1cm} (16)$$
Here, with the explicit expressions for $W(x)$ and $\varphi_2^0(x)$ given in Eqs. (5) and (14), we observe that
\[
e^{\int_1^x W(z) dz} \simeq e^{\int_1^{l^*} \sigma_R(z-l_1) dz} e^{\int_0^{l^*} \sigma_L(z-l_2) dz} e^{\int_0^x \sigma_L(z-l_2) dz}
= C_1 [\varphi_1^0(l^*) \varphi_2^0(l^*)]^{-1} \varphi_2^0(y) \quad \text{for} \ y \in [-\infty, l^*],
\] (17)
\[
\int_{l^*}^{-\infty} dy [\varphi_2^0(y)]^2 \simeq - \int_{-\infty}^{\infty} dy [\varphi_2^0(y)]^2 = -1,
\] (18)
and hence the ‘eigenvalue condition’ (16) reduces to the form
\[
C_1 - \tilde{\omega} C_1 [\varphi_1^0(l^*) \varphi_2^0(l^*)]^{-1} = 0.
\] (19)
We thus find
\[
\tilde{\omega} = \varphi_2^0(l^*) \varphi_2^0(l^*) = C_1 C_2 e^{-\int_{l_1}^{l^*} \sigma_R(y-l_1) dy + \int_{l_2}^{l^*} \sigma_L(y-l_2) dy}
\] (20)
and so, for the desired ground-state energy, the formula
\[
\tilde{E} = [\varphi_1^0(l^*) \varphi_2^0(l^*)]^2.
\] (21)

[Notice that, for an arbitrary choice of $l^*$ in the flat middle region of the superpotential, the same value for $\tilde{E}$ results]. The same expression for $\tilde{E}$ is obtained by analogous analyses with Eq.(15b). With the eigenvalue $\tilde{\omega}$ determined in this manner, the corresponding wave functions $\varphi_1(x)$ and $\varphi_2(x)$ (up to normalization) are now expressed as
\[
\varphi_1(x) \simeq \begin{cases} \varphi_1^0(x) \{1 + \tilde{\omega} \int_{l_x}^x dy [\varphi_2^0(y)]^{-1} \varphi_2^0(y)\}, & x > l^* \\
\tilde{\omega} [\varphi_2^0(x)]^{-1} \int_{-\infty}^x dy [\varphi_2^0(y)]^2, & x < l^*
\end{cases}
\] (22)
\[
\varphi_2(x) \simeq \begin{cases} \tilde{\omega} [\varphi_2^0(x)]^{-1} \int_{-\infty}^x dy [\varphi_2^0(y)]^2, & x > l^* \\
\varphi_2^0(x) \{1 + \tilde{\omega} \int_{l_x}^{l^*} dy [\varphi_2^0(y)]^{-1} \varphi_2^0(y)\}, & x < l^*.
\end{cases}
\] (23)

We expect that a judicious use of the tight-binding approximation with the matrix Hamiltonian (7), taking $\begin{pmatrix} \varphi_1^0(x) \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ \varphi_2^0(x) \end{pmatrix}$ as the degenerate (i.e., zero energy) eigenstates of the corresponding local Hamiltonians, lead to the same conclusion as above. This is supported by the observation that, for the eigenvalue $\tilde{\omega}$, the same result (i.e., Eq.(20)) follows from the calculation based on the formula
\[
\pm \tilde{\omega} = \int_{-\infty}^{\infty} dx \, \varphi_\pm^0(x) \mathcal{H} \varphi_\pm^0(x),
\] (24)
taking $\varphi_\pm^0(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} \varphi_1^0(x) \\ \pm \varphi_2^0(x) \end{pmatrix}$, the usual zeroth-order states in the tight-binding approximation. Indeed, one can verify that the given integral
\[
\int_{-\infty}^{\infty} dx \, \varphi_\pm^0(x) \mathcal{H} \varphi_\pm^0(x) = \int_{-\infty}^{\infty} dx \left[ \sigma_R(x-l_1) + v \right] \varphi_\pm^0(x) \varphi_\pm^0(x)
\] (25)

\[But, since our matrix Hamiltonian (which is the Dirac Hamiltonian) is unbounded from below, some care must be exercised.
is well approximated by the product \( \varphi_0^1(l^*) \varphi_0^2(l^*) \).

As an explicit example, consider the superpotential of the form

\[
W(x) = v \, \text{sgn}(x - \frac{L}{2}) - v \, \text{sgn}(x + \frac{L}{2}) + v ,
\]

i.e., in our notation, \((l_1, l_2) = (\frac{L}{2}, -\frac{L}{2}), \sigma_R(x - l_1) = v \, \text{sgn}(x - \frac{L}{2}), \) and \(\sigma_L(x - l_2) = -v \, \text{sgn}(x + \frac{L}{2})\). Given this, the potentials of the Schrödinger Hamiltonians \(H_\pm\) will be

\[
V_\pm(x) = v^2 \pm 2v \, \delta(x - \frac{L}{2}) \pm 2v \, \delta(x + \frac{L}{2}).
\]

For these systems, one can of course find the exact ground state energy \(\bar{E}\) by solving the appropriate Schrödinger equations. This exercise shows that \(\bar{E}\) is the root of the equation

\[
\bar{E} = v^2 e^{-2\frac{v}{\mu} \sqrt{\frac{v^2}{\mu^2} - \bar{E}}},
\]

and hence, for large \(L\), we have

\[
\bar{E} \simeq v^2 e^{-2\mu L}.
\]

Let us see whether our formula (21) yields the same. The normalized solutions of Eq.(13) are trivially found here:

\[
\varphi_0^1(x) = \sqrt{v \, e^{-v|x - \frac{L}{2}|}}, \quad \varphi_0^2(x) = \sqrt{v \, e^{-v|x + \frac{L}{2}|}}.
\]

Then, from Eq.(21), we have that \(\bar{E}\) (for large \(L\)) should equal \([\varphi_0^1(0) \varphi_0^2(0)]^2 = v^2 e^{-2\mu L}\). Hence a complete agreement.

More physically relevant example is provided by the Dirac Hamiltonian (11) with the scalar field taken to represent the kink-antikink pair,

\[
W(x) = v \, \tanh \left( \frac{\mu}{2}(x - \frac{L}{2}) \right) - v \, \tanh \left( \frac{\mu}{2}(x + \frac{L}{2}) \right) + v .
\]

Here the scalar field \(\sigma_R(x - \frac{L}{2}) = v \, \tanh \left( \frac{\mu}{2}(x - \frac{L}{2}) \right)\) represents a kink located at \(x = \frac{L}{2}\), and \(\sigma_L(x + \frac{L}{2}) = -v \, \tanh \left( \frac{\mu}{2}(x + \frac{L}{2}) \right)\) an antikink at \(x = -\frac{L}{2}\) [9]. The widely-separated kink-antikink configuration, described by the form (30), has received attention in Refs. [10–12]. Especially interesting is the almost zero-energy mode of the Dirac Hamiltonian, in connection with the role of the so-called Jackiw-Rebbi mode [10] (which refers to the zero-energy fermion mode [13] in the kink or antikink background) when a kink and an antikink are simultaneously present. In the kink or antikink background the Jackiw-Rebbi mode is represented by \(\begin{pmatrix} \varphi_0^1(x) \\ 0 \end{pmatrix}\) or \(\begin{pmatrix} 0 \\ \varphi_0^2(x) \end{pmatrix}\), if \(\varphi_0^1(x)\) and \(\varphi_0^2(x)\) denote the normalized solutions of Eq.(13):

\[
\varphi_0^1(x) = C \left[ \cosh \frac{\mu}{2}(x - \frac{L}{2}) \right]^{-\frac{2\mu}{\rho}}, \quad \varphi_0^2(x) = C \left[ \cosh \frac{\mu}{2}(x + \frac{L}{2}) \right]^{-\frac{2\mu}{\rho}},
\]

where \(C = \left\{ \frac{\mu}{2} \Gamma(-\frac{\mu}{\rho}) \right\}^{1/2}\). Then, in the above kink-antikink scalar background, one can immediately find the energy of the almost-zero-energy fermion eigenmode by using our
formula (20) — it is equal to $\pm \bar{\omega}$, with\(^3\)

\[
\bar{\omega} = \varphi_1^0(0)\varphi_2^0(0) = C^2[\cosh \left( \frac{\mu L}{4} \right)]^{-\frac{4\nu}{\mu}} \approx \frac{\mu}{2} e^{\frac{4\nu}{\mu}} \Gamma\left( \frac{2\nu}{\mu} + \frac{1}{2} \right) \Gamma\left( \frac{2\mu}{\nu} \right) \Gamma\left( \frac{1}{2} \right) e^{-\nu L}.
\]

(32)

The exponential dependence of $\bar{\omega}$ on the distance $L$ was previously noted in Ref. [10]. This $L$-dependent fermion energy shift will contribute to the effective potential between the kink and the antikink. For instance, in the vacuum sector where all negative energy fermion modes are to be occupied, the contribution from this mode, i.e., that with energy $-\bar{\omega}$ will become more negative as $L$ decreases, thus producing an attractive interaction between the kink and the antikink (if only this mode is taken into account).

In this work we investigated on some special properties pertaining to the ground state of a quantum mechanical Hamiltonian with broken supersymmetry, when the corresponding eigenvalue is small. A direct perturbative analysis based on the first-order coupled equations has been used to obtain a very simple (approximate) expression for the ground state energy. Our formula (21) should be useful in finding the ground state energy of a Schrödinger Hamiltonian the potential in which can be approximated by the form (6).

ACKNOWLEDGMENT

We would like to thank Seok Kim for very useful discussions. This work was supported in part by the BK21 project of the Ministry of Education, Korea, and also by Korea Research Foundation Grant 2001-015-DP0085 (C.L.).

\(^3\)Based on some heuristic arguments the author of Ref. [12] also identified $\bar{\omega}$ with the expectation value in Eq.(24). But his final expression for $\bar{\omega}$ is apparently not consistent with our result in Eq.(32)—this is due to the calculational error made in Ref. [12] (e.g., wrong fermion zero-energy wave functions used).
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FIG. 1. Schematic plots of $\sigma_R(x)$ and $V_{R\pm}(x) \equiv \sigma_R^2(x) \pm \sigma_R'(x)$

FIG. 2. Schematic plots of $\sigma_L(x)$ and $V_{L\pm}(x) \equiv \sigma_L^2(x) \pm \sigma_L'(x)$
FIG. 3. Schematic plots of $W(x)$ and $V_{\pm}(x) \equiv W^2(x) \pm W'(x)$