Unitary evolution for anisotropic quantum cosmologies: models with variable spatial curvature

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Abstract
Contrary to the general belief, there has recently been quite a few examples of unitary evolution of quantum cosmological models. The present work gives more examples, namely Bianchi type VI and type II. These examples are important as they involve varying spatial curvature unlike the most talked about homogeneous but anisotropic cosmological models like Bianchi I, V and IX. We exhibit either an explicit example of the unitary solutions of the Wheeler–DeWitt equation, or at least show that a self-adjoint extension is possible.

Keywords: quantum cosmology, unitary evolution, Bianchi models

1. Introduction
A quantum description of the Universe should emerge from a quantum theory of gravity which still eludes the reach in a generally accepted form. Quantum cosmology is a moderately ambitious programme where quantum mechanical principles are employed in a gravitational system in the absence of a more general quantum theory of gravity. Of course quantum cosmology has its own motivation, such as looking for a resolution to the problem of singularity at the birth of the Universe. The basic framework for quantum cosmology is provided by the Wheeler–DeWitt equation [1–3]. Amongst the infinite possible metrics, only a particular form is normally chosen by hand from the consideration of symmetry. This is the usual minisuperspace which reduces the degrees of freedom to a finite number and thus makes the problem tractable. There are quite a few reviews which discuss the development of the subject and some of its conceptual problems [4–6].

Although quantum cosmology is treated as a typical problem in gravity theory, the Wheeler–DeWitt formulation actually has a very general appeal, the approach is very similar to the usual practice in standard quantum physics. From the classical Lagrangian, the momenta corresponding to the identified coordinates are discovered so as to write the Hamiltonian, the variables are then promoted to operators (usually in the coordinate representation), and the relevant Schrödinger-like equations are explored which govern the system. In some cases, the Wheeler–DeWitt equation has a straightforward analogue with some situations in other branches of physics. For instance, one anisotropic quantum cosmological model, namely the Bianchi-I model eventually reduces to the standard quantum mechanical problem with an inverse square potential [7] which has many applications in physics [8].

One major problem of quantum cosmology is that the quantization of anisotropic models are believed to give rise to a non-unitary evolution of the wave function resulting in a nonconservation of probability. It is interesting to note that this non-unitarity is often apt to be invisible in the absence of a properly oriented scalar time parameter in the scheme of quantization [9, 10]. In a relativistic theory, time itself is a coordinate and fails to be the scalar parameter against which the evolution should be studied. In fact the problem of the proper identification of time in quantum cosmology is a subject by itself and dealt with by many researchers [11–14].

A novel idea about the identification of time through the evolution of a fluid present in the model appeared to work very well. The method, where the fluid variables are endowed with dynamical degrees of freedom through certain
thermodynamic potentials [15, 16], was suggested by Lapchinskii and Rubakov [17]. It has been shown that the time parameter that emerges out of the fluid evolution has the required monotonicity as well as the correct orientation [7]. This Schutz formalism is now very widely used in quantizing cosmological models [7, 18–24].

Until very recently, the non-conservation of probability in anisotropic models had almost been generally accepted as a pathology, and had been ascribed to the hyperbolicity of the Hamiltonian [20]. Not that the anisotropic models are of utmost importance so far as the observed Universe is concerned, but this feature of non-unitarity renders the quantization scheme vulnerable. It should also be noted that observations do indicate anisotropy in the Cosmic Microwave Background, which however is quite compatible with an isotropic Universe statistically. These fluctuations are essentially local effects, consistent with the requirements for the structure formation.

There has now been a new development in this picture. Majumder and Banerjee [21] showed that a suitable ordering of operators can lead to a alleviation of the problem, meaning that the probability is conserved except for a small period of time. Later it was clearly shown by Pal and Banerjee [7, 23] that the said non-unitarity can actually be attributed to either an ordering of operators or to a bad choice of variables. With a suitable ordering, examples of unitary evolution were exhibited in Bianchi I, V and IX models. The degree of difficulty in integration allowed only a few cases of choice of $\alpha$ which determine the equation of state ($P = \alpha \rho$) for which the desired unitarity was established. However, even a few examples are good enough to indicate that the problem is not actually pathological and can be cured. Very recently an example of a unitary evolution for a Kantowski–Sachs model has been given by Pal and Banerjee [24]. It was also shown by Pal [25] that this unitarity is achieved not at the cost of anisotropy itself. It deserves mention that the conservation of probability for a Kantowski–Sachs model prevails in non-commutative quantum cosmology as well [26].

Except for the Kantowski–Sachs cosmology, all other examples of the anisotropic Bianchi models stated have one unifying feature, they are all of constant spatial curvature. The motivation for the present work is to show that the possibility of a self adjoint extension and hence a unitary evolution is not a characteristic of models with a constant spatial curvature, this is in fact more general and can be extended to models with variable curvature of spatial hypersurfaces as well. Two specific examples, namely Bianchi II and VI are dealt with in the following sections. We should mention that non-conservation of probability, if it is there, is a pathology for a quantum mechanical description of any system, so the motivation of the present work is actually of a very general interest.

Section 2 deals with the formalism and takes up the example of the Bianchi VI model. Section 3 deals with the Bianchi II model. In the fourth section a note on the spatial curvature is given. The last section includes a summary and a discussion of the results obtained.

2. The formalism and Bianchi VI models

We start with the standard Einstein–Hilbert action for gravity along with a perfect fluid given by

$$A = \int_M d^4x \sqrt{-g} R + \int_M d^4x \sqrt{-g} P,$$

where $R$ is the Ricci Scalar, $g$ is the determinant of the metric and $P$ is the pressure of the ideal fluid. The first term corresponds to the gravity sector and the second term is due to the matter sector. Here we have ignored the contributions from the boundary as it would not contribute to the variation. The units are so chosen that $16\pi G = 1$.

A Bianchi VI model is given by the metric

$$ds^2 = n^2(t)dt^2 - a^2(t)dx^2 - e^{mb^2(t)d^2y^2 - e^c(t)dz^2},$$

where the lapse function $n$ and $a, b, c$ are functions of time $t$ and $m$ is a constant.

From the metric given above, we can write the Ricci Scalar as

$$\sqrt{-g} R = e^{\alpha - \frac{\alpha}{n}} \left[ \frac{2}{n} (abc + bca + acb) \right]$$

$$- \frac{2}{n} [abc + bca + cab]$$

$$+ \frac{n^2bc}{4a} (m^2 - m + 1).$$

Using this, we can find the action for the gravity sector from equation (1) which is given as

$$A_g = \int dt \left[ - \frac{2}{n} [abc + bca + cab]$$

$$+ \frac{n^2bc}{4a} (m^2 - m + 1) \right],$$

where an overhead dot indicates a derivative with respect to time.

Now we make a set of transformation of variables as

$$a(t) = e^{\beta_1},$$

$$b(t) = e^{\beta_2 + \sqrt{3}(\beta_1 - \beta_2)},$$

$$c(t) = e^{\beta_3 - \sqrt{3}(\beta_1 - \beta_2)}.$$

This introduces a constraint $a^2 = bc$, but the model is still remains Bianchi Type VI without any loss of the typical characteristics of the model. Such type of transformation of variables has been extensively used in the literature [7, 20, 21]. One can now write the Lagrangian density of the gravity sector as

$$\mathcal{L}_g = - \frac{6\beta_1}{n} \left[ \beta_1^2 - (\beta_1 - \beta_2)^2 \right]$$

$$+ e^{-2\beta_1} n^2 (m^2 - m + 1).$$

(8)
Here $\beta_0, \beta_1$, and $\beta_2$ have been treated as coordinates. So corresponding Canonical momentum will be $p_0, p_1$, and $p_2$ where $p_i = \frac{\partial E}{\partial \beta_i}$. It is easy to check that one has $p_i = -\dot{p}_i$. Hence we can write the corresponding Hamiltonian as

$$H_{\varepsilon} = -ne^{-3\beta_0} \left[ \frac{1}{24}(p_0^2 - p_1^2 - 12(m^2 - m + 1)e^{4\beta_0}) \right].$$

(9)

With the widely used technique, developed by Lapchinskii and Rubakov [17] by using the Schutz formalism of writing the fluid parameters in terms of thermodynamic variables [15, 16], the action the fluid sector can be written as

$$A_f = \int dt L_f$$

$$= \int dt \left[ n^{-\frac{1}{2}}e^{3\beta_0} \frac{\alpha}{(1 + \alpha)^{1+\nu}} (\dot{\varepsilon} + \theta \dot{S}) + \frac{1}{2} \varepsilon \right].$$

(10)

Here $\varepsilon, \theta, S$ are thermodynamic potentials. A constant volume factor $V$ comes out of the integral in both (4) and (10). This $V$ is inconsequential as it can be absorbed into the subsequent variational principle. With a canonically transformed set of variables $T, \varepsilon'$ in place of $S, \varepsilon$, one can finally write down the Hamiltonian for the fluid sector as

$$H_f = ne^{-3\beta_0}e^{3(1-\alpha)\beta_0} p_T.$$

(11)

The canonical transformation is given by the set of equations

$$T = -p_S \exp(-S)p^\alpha,$$

$$p_T = p_T^{\nu+1} \exp(S),$$

$$
\varepsilon' = \varepsilon + (\alpha + 1) \frac{p_S}{p_T},
$$

$$p' = p_T.$$

(12-15)

This method and the canonical nature of the transformation are comprehensively discussed in [7].

The net or the super Hamiltonian is

$$H = H_{\varepsilon} + H_f = -ne^{-3\beta_0}\left[ p_0^2 - p_1^2 - 12(m^2 - m + 1)e^{3(1-\alpha)\beta_0}p_T \right].$$

(16)

Using the Hamiltonian constraint $H = 0$, which can be obtained by varying the action $A_{\varepsilon} + A_f$ with respect to the lapse function $n$, one can write the Wheeler–DeWitt equation as

$$\left[ e^{3(1-\alpha)\beta_0} \frac{\partial^2}{\partial \beta_0} - e^{3(1-\alpha)\beta_0} \frac{\partial^2}{\partial \beta_1^2} + 12(m^2 - m + 1)e^{3(1+\alpha)\beta_0} \right] \psi = 24i \frac{\partial}{\partial T} \psi.$$

(17)

This equation is obtained after we promote the momenta to the corresponding operators given by $p_i = -i\frac{\partial}{\partial \beta_i}$ in the units of $\hbar = 1$.

It is interesting to note that for a particular value of $m = m_0$ where $m_0$ is a root of equation $m^2 - m + 1 = 0$, the spatial curvature vanishes and the equation (17) reduces to the corresponding equation for a Bianchi Type I model [7]. We shall discuss the solution of the Wheeler–DeWitt equation in two different cases, namely $\alpha = 1$ and $\alpha \neq 1$.

### 2.1. Stiff fluid: $\alpha = 1$

For a stiff fluid ($P = \rho$), the equation (17) becomes simple and can be easily separated. It looks like

$$\left[ \frac{\partial^2}{\partial \beta_0^2} - \frac{\partial^2}{\partial \beta_1^2} + 12(m^2 - m + 1)e^{4\beta_0} \right] \psi = 24i \frac{\partial}{\partial T} \psi.$$  

(18)

With the separation ansatz

$$\psi = e^{2k_i\beta_i}(\beta_0) e^{-i\nu T},$$

one can write

$$\frac{\partial^2 \phi}{\partial \beta_0^2} + (4k_0^2 - 24E + 4N^2 e^{4\beta_0})\phi = 0,$$

(20)

where $N^2 = 3(m^2 - m + 1)$. After making the change in variable as $q = Ne^{2\beta_0}$, above equation can be written as

$$q^2 \frac{\partial^2 \phi}{\partial q^2} + q \frac{\partial \phi}{\partial q} + [q^2 - (6E - k_0^2)]\phi = 0.$$  

(21)

Solution of this equation can be written in terms of Bessel’s functions as

$$\phi(q) = J_\nu(q),$$

(22)

where $\nu = \sqrt{6E - k_0^2}$. Now for the construction of the wave packet, we need to fix $\nu$. If we take $\varepsilon = -\nu^2 = k_0^2 - 6E$ then wave packet can have following expression

$$\Psi = \Phi(q) \zeta(\beta_+ e^{i\nu T/6}).$$

(23)

where

$$\zeta(\beta_+) = \int dk_+ e^{-i(k_+ - k_0)q} e^{i\left(2k_+\beta_+ - \frac{4k_+^2}{3}\right)}.$$  

(24)

The norm indeed comes out to be positive and finite (for the details of the calculations, we refer to work of Pal and Banerjee [24]). Thus one indeed has a unitary time evolution.

### 2.2. General perfect fluid: $\alpha \neq 1$

Now we shall take the more complicated case of $\alpha \neq 1$ and try to solve the Wheeler–DeWitt equation (17). We use a specific type of operator ordering with which equation (17)
takes the form
\[
\left[ e^{2(\alpha-1)\beta_0} \frac{\partial}{\partial \beta_0} e^{2(\alpha-1)\beta_0} \frac{\partial}{\partial \beta_0} - e^{2(\alpha-1)\beta_0} \frac{\partial^2}{\partial \beta^2} + 12(m^2 - m + 1)e^{3\alpha + 1}\beta_0 \right] \Psi = 24e \frac{\partial}{\partial \tau} \Psi. \tag{25}
\]

Now with the standard separation of variable as,
\[
\Psi(\beta_0, \beta_+, T) = \phi(\beta_0)e^{i\beta_+}e^{-iET},
\]
the equation for \( \phi \) becomes
\[
\left[ e^{2(\alpha-1)\beta_0} \frac{\partial}{\partial \beta_0} e^{2(\alpha-1)\beta_0} \frac{\partial}{\partial \beta_0} + e^{2(\alpha-1)\beta_0}k^2_+ \right.
\]
\[
+ 12(m^2 - m + 1)e^{3\alpha + 1}\beta_0 - 24E] \phi = 0. \tag{27}
\]

For \( \alpha = 1 \) we make a transformation of variable as
\[
\chi = e^{-2(\alpha-1)\beta_0}, \tag{28}
\]
and write equation (27) as
\[
\frac{9}{4}(1 - \alpha)^2 \frac{\partial^2 \phi}{\partial \chi^2} + \frac{k^2_+}{\chi^2} \phi
\]
\[
+ 12(m^2 - m + 1)\chi^{2(\alpha-1)}\phi - 24E\phi = 0. \tag{29}
\]

We define some parameters as
\[
\sigma = \frac{4k^2_+}{9(1 - \alpha)^2}, \quad E' = \frac{32}{3(1 - \alpha)^2}E, \quad M^2 = \frac{16(m^2 - m + 1)}{3(1 - \alpha)^2}. \tag{30, 31}
\]
Equation (29) can now be written as
\[
-\frac{\partial^2 \phi}{\partial \chi^2} - \frac{\sigma^2}{\chi^2} \phi - M^2 \chi^{2(\alpha-1)}\phi = -E'\phi. \tag{33}
\]

The above equation can be compared to 
\[
-H_\chi = -\frac{\partial^2}{\partial \chi^2} + V(\chi) \quad \text{with} \quad V(\chi) = -\frac{\sigma^2}{\chi^2} - M^2 \chi^{2(\alpha-1)/m}\]
which is a continuous and real valued function on the half line, and one can show that the Hamiltonian \( H_\chi \) admits self-adjoint extension as \( H_\chi \) has equal deficiency indices. For a systematic and detailed description of the self-adjoint extension we can refer to the text by Reed and Simon [27]. See also [28,29].

So it can be said that for perfect fluid with \( \alpha = 1 \) Bianchi VI quantum models do admit a unitarity evolution.

2.3. \( \alpha = -\frac{1}{3} \)

We take a specific choice, where \( \rho + 3P = 0 \), as an example. This equation of state will make equation (33) much simpler. With \( \alpha = -1/3 \), the term \(-M^2\chi^{2(1/3)/m}\) becomes a constant \((M^2)\). Equation (33) becomes
\[
-\frac{\partial^2 \phi}{\partial \chi^2} - \frac{\sigma^2}{\chi^2} \phi = -(E' - M^2)\phi, \tag{34}
\]
which is in fact a well known Schrodinger equation of a particle with mass \( m = 1/2 \) in an attractive inverse square potential. Solution to above can be given as,
\[
\phi_\chi(\chi) = \sqrt{\frac{\chi}{X}}[AH_\chi^{(1)}(\chi) + BH_\chi^{(2)}(\chi)], \tag{35}
\]
\[
\phi_\phi(\chi) = \sqrt{\frac{\chi}{X}}[AH_\phi^{(1)}(\chi) + BH_\phi^{(2)}(\chi)]. \tag{36}
\]

for \( \sigma > 1/4 \) and \( \sigma < 1/4 \) and \( \beta = \sqrt{\sigma - 1/4} \) and \( \beta = \sqrt{1/4 - \sigma} \) respectively. Here both \( \alpha \) and \( \beta \) are real numbers and in both cases the energy spectra is given as
\[
E' = M^2 - \chi^2. \tag{37}
\]

Self-adjoint extension guarantees that \([\beta/A] \) takes a value so as to conserve probability and make the model unitarity. The details of the calculations are omitted, as the analysis is similar to that described in reference [23].

3. Bianchi II models

Bianchi Type II model is given the line element
\[
ds^2 = dt^2 - a^2(t)[dr^2 - b^2(t)d\theta^2 - 2a^2(t)\theta dr d\phi], \tag{38}
\]
and the process is a bit more involved for the presence of the non-diagonal terms in the metric.

The Ricci scalar \( R \) in this case is given by
\[
R = -\frac{a^2}{2b^4} - \frac{4ab}{ab} - \frac{2b^2}{b^2} - \frac{2a}{a} - \frac{4b}{b}. \tag{39}
\]

If we define a new variable \( \beta = ab \) as prescribed in [20], then Lagrangian density for gravity sector looks like
\[
\mathcal{L}_g = \frac{2\beta^2\alpha^2}{a^4} - \frac{2\beta^2}{a} - \frac{a^4}{2\beta^2}, \tag{40}
\]
and the corresponding Hamiltonian density for gravity sector can be written as
\[
H_g = \frac{a^4\beta^2}{8\beta^2} - \frac{a^4}{8\beta^2} + \frac{a^4}{2\beta^2}. \tag{41}
\]

Using Schutz’s formalism and proper identification of time as we did before, the Hamiltonian density for fluid sector can be written as
\[
H_f = a^4\beta^{-2}\rho_f. \tag{42}
\]

The super Hamiltonian can now be written in following form
\[
H = H_g + H_f = \frac{a^4\rho_f^2}{8\beta^2} - \frac{a^4}{8\beta^2} + \frac{a^4}{2\beta^2} + a^4\beta^{-2}\rho_f. \tag{43}
\]

As an example we take up the case of a stiff fluid given by \( \alpha = 1 \).
After promoting the momenta by operators as usual, the Wheeler–DeWitt equation $H \Psi = 0$ takes following form
\[
-a^2 \frac{\partial^2 \Psi}{\partial a^2} + \beta^2 \frac{\partial^2 \Psi}{\partial \beta^2} + \frac{a^4}{2} \frac{\partial \Psi}{\partial T} = 0.
\] (44)

Using a separation of variables
\[
\Psi = e^{-\beta T} \phi(\alpha) \psi(\beta),
\] (45)
we get following equations for $\psi$ and $\phi$ respectively
\[
-\frac{d^2 \psi}{d \beta^2} + \frac{8k}{\beta^2} \psi = 0,
\] (46)
\[
a^2 \frac{d^2 \phi}{da^2} - 4a^4 \phi - 8(k - \epsilon) \phi = 0.
\] (47)

With $\phi = \frac{\psi}{\sqrt{\gamma}}$ and $\chi = a^2$, last equation can be written as
\[
-\frac{d^2 \phi_0}{d \chi^2} - \frac{\sigma}{\chi^2} \phi_0 = -\phi_0,
\] (48)
where $\sigma = \left[ \frac{1}{4} - 2(k - E) \right]$. Equations (46) and (48) are the governing equations for Bianchi Type II with a stiff fluid.

Equations for both $\psi$ and $\phi$ can be mapped to a Schroedinger equation for a particle in an inverse square potential. In order to get a solution we actually have ensure an attractive regime, which requires $k \leq 0$, $E \leq k - \frac{3}{32}$. We see that both the equations are that for inverse square potentials, and thus a self-adjoint extension is possible. This case is actually very similar to the Bianchi IX model as discussed in [23]. So we do not discuss this in detail.

4. A note on spatial curvature

It has already been mentioned that the difference between the present examples of Bianchi VI and Bianchi II models on one hand, and most of the models discussed earlier on the other, is the fact that the present models have variable spatial curvature as opposed to most the models discussed in connection with the quantization of cosmological models according to the Wheeler–DeWitt scheme. In a $(3 + 1)$ decomposition of the spacetime metric, one can calculate the Ricci curvature $\mathcal{R}$ of the three dimensional space section, embedded in a four dimensional spacetime. Both Bianchi VI and II will have $\mathcal{R}$ which vary with time. For example, if we take the Bianchi II metric as an example in a more general form, than that used here (equation (2)), given by
\[
d s^2 = n^2(t) dt^2 - a^2(t) d\mathbf{x}^2 - e^{-m(b^2(t))} dy^2 - e^{n(c^2(t))} dz^2,
\] (49)
the 3-space Ricci curvature looks like
\[
\mathcal{R} = \frac{m^2 - m + l^2}{2a^2(t)},
\] (50)
which indeed is a function of the cosmic time $t$ through $a$. In the case of the metric (2), this becomes
\[
\mathcal{R} = \frac{m^2 - m + l^2}{2a^2(t)}.\] (51)

The most talked about anisotropic models, like Bianchi I, V and IX all have constant $\mathcal{R}$. For example, if we put $m = l = 0$ in the metric (49), we get a Bianchi I metric, and for this choice, equation (50) clearly shows that $\mathcal{R} = 0$. For some more information regarding the spatial curvature, we refer to the recent work by Akarsu and Kilinc [30].

5. Discussion and conclusion

The present work deals with two examples of anisotropic quantum cosmological models with varying spatial curvature, namely Bianchi VI and II. We show that there is indeed a possibility of finding unitary evolution of the system. The earlier work on anisotropic models with constant spatial curvature [7, 23] disproved the belief that anisotropic quantum cosmologies generically suffer from a pathology of non-unitarity. The present work now strongly drives home the fact that this feature is not at all a characteristic of models with constant spatial curvature. It was also shown before that the unitarity is not achieved at the cost of anisotropy itself [25]. One can now indeed work with quantum cosmologies far more confidently, as there is actually no built-in generic non-conservation of probability in the models.

Very recently it has been shown that in fact all homogeneous models, isotropic or anisotropic, quite generally have a self-adjoint extension [31], although the extension is not unique. The present work gives two more examples, and consolidates the result proved in [31]. The examples chosen indeed have physical implications. It has been shown very recently that a Bianchi VI model plays an important role in producing anisotropic inflation [32]. We also refer to the work of Barrow [33] for various cosmological implications of Bianchi type VI models. Bianchi II models, on the other hand, are instrumental in understanding the Belinskii, Khalatnikov, Lifshitz conjecture in the discussion of spacelike singularities [34, 35].

Thus the standard canonical quantization of cosmological models via the Wheeler–DeWitt equation still proves to be useful in the absence of a more general quantum theory of gravity. The more challenging work will now be the quantization of inhomogeneous cosmological models.

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References

[1] DeWitt B S 1967 Phys. Rev. 160 1113
[2] Wheeler J A Superspace and the nature of quantum geometrodynamics Batelle Recontres (New York: Benjamin)
[3] Misner C W 1969 Phys. Rev. 186 1319
[4] Wiltshire D L arXiv:gr-qc/0101003
[5] Halliwell J J 1991 Quantum Cosmology and Baby Universes ed S Coleman, J B Hartle, T Piran and S Weinberg (Singapore: World Scientific)
[6] Pinto-Neto N and Fabris J C 2013 Class. Quant. Grav. 30 143001
[7] Pal S and Banerjee N 2014 Phys. Rev. D 90 104001
[8] Gopalakrishnan S 2006 Self-adjointness and the renormalization of singular potentials BA (Hons) Thesis Amherst College
[9] Lidsey J E 1995 Phys. Lett. B 352 207
[10] Pinto-Neto N, Velasco A F and Collistete R Jr 2000 Phys. Lett. A 277 194
[11] Kuchar K V 1991 Conceptual Problems in Quantum Gravity ed A Ashtekar and J Stachel (Boston: Birkhause)
[12] Isham C J 1993 Integrable Systems, Quantum Groups and Quantum Field Theory ed L A Ibort and M A Rodriguez (Dordrecht: Kluwer)
[13] Rovelli C arXiv:gr-qc/9903.3832
[14] Anderson E arXiv:gr-qc/1009.2157
[15] Schutz B F 1970 Phys. Rev. D 2 2762
[16] Schutz B F 1971 Phys. Rev. D 4 3559
[17] Lachinskii V G and Rubakov V A 1977 Theor. Math. Phys. 33 1076
[18] Alvarenga F G and Lemos N A 1998 Gen. Relativ. Gravit. 30 681
[19] Alvarenga F G, Fabris J C, Lemos N A and Monerat G A 2002 Gen. Relativ. Gravit. 34 651
[20] Alvarenga F G, Batista A B, Fabris J C, Lemos N A and Gonzales S V B 2003 Gen. Relativ. Gravit. 35 1639
[21] Majumder B and Banerjee N 2013 Gen. Relativ. Gravit. 45 1
[22] Almeida C R, Batista A B, Fabris J C and Moniz P R L V arXiv:1501.04170
[23] Pal S and Banerjee N 2015 Phys. Rev. D 91 044042
[24] Pal S and Banerjee N 2015 Class. Quant. Grav. 32 205005
[25] Pal S 2016 Class. Quant. Grav. 33 045007
[26] Bastos C, Bertolami O, Dias N C and Prata J N 2008 Phys. Rev. D 78 023516
[27] Reed M and Simon B 1975 Methods of Modern Mathematical Physics vol 2 2nd edn (INC: Academic)
[28] Essin A M and Griffiths D J 2005 Am. J. Phys. 74 109
[29] Gupta K S and Rajeev S G 1993 Phys. Rev. D 48 5940
[30] Akarsu O and Kilinc C B 2011 Int. J. Theor. Phys. 50 1967
[31] Pal S and Banerjee N arXiv:1601.00460
[32] Kao W F and Lin I-C 2011 Phys. Rev. D 83 0630004
[33] Barrow J D 1984 Mon. Not. R. Astron. Soc. 211 221
[34] Belinskii V A, Khalatnikov I M and Lifshitz E M 1970 Adv. Phys. 19 525
[35] Belinskii V A, Khalatnikov I M and Lifshitz E M 1982 Adv. Phys. 31 639