3+1d QHE, elasticity tetrads and mixed axial-gravitational anomalies

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Elasticity tetrads with torsion describe the hydrodynamic elasticity theory of crystals with dislocations. These tetrads have the canonical dimensions of inverse length and are the proper variables allowing the description of the integer quantum Hall effect — as well as the intrinsic, anomalous quantum Hall effect in topological insulators — in odd spatial dimensions. In all even space dimensions the momentum-space topological invariants – the Chern numbers — are exact differentials. They can be expressed in general form in terms of a system of three deformed crystallographic coordinate planes, surfaces of constant phase \(X^a(x) = 2\pi n^a, n^a \in \mathbb{Z}\) with \(a = 1, 2, 3\) in three dimensions. The intersections of the three constant surfaces

\[ X^1(r, t) = 2\pi n^1, \quad X^2(r, t) = 2\pi n^2, \quad X^3(r, t) = 2\pi n^3, \]

are points of the (possibly deformed) crystal lattice

\[ L = \{ r = R(n_1, n_2, n_3) \mid r \in \mathbb{R}^3, n^a \in \mathbb{Z}^3 \}. \]

The elasticity tetrads are gradients of the phase function

\[ E^a_{\mu}(x) = \partial_{\mu}X^a(x) \]

and have units of crystal momentum. In the simplest undeformed case, \(X^a(r, t) = K^a \cdot r\), where \(E^{(0)\mu} = K^\mu\) are the (primitive) reciprocal lattice vectors \(K^a\). In the general case, they depend on space and time and are quantized in terms of the lattice \(L\) in Eq. (1).

In the absence of dislocations, the tensor \(E^a_{\mu}(x)\) satisfies the integrability condition:

\[ dE^a = \frac{1}{2} (\partial_\mu E^a_\nu(x) - \partial_\nu E^a_\mu(x)) dx^\mu \wedge dx^\nu = 0. \]

The scalar phase fields \(N_a X^a\), where \(N_a\) is a set of integer (weak) topological invariants for a 3+1d quantum Hall system related to a QH lattice system will play an important role in the following. Before introducing the effective response, we review the simpler 2+1d case in the next section.
III. 2+1D TOPOLOGICAL ACTION FOR QHE

The topological Chern-Simons action for the IQHE and for the anomalous, intrinsic (i.e. without external magnetic field/flux) AQHE in the $D = 2 + 1$-dimensional crystalline insulator is

$$S_{4D}[A_{\mu}] = \frac{1}{4\pi} N \int d^2 x dt \, \epsilon^{\nu \alpha \beta} A_{\nu} \partial_{\alpha} A_{\beta}. \quad (5)$$

Here $|\epsilon| = \hbar = 1$ and the electromagnetic U(1) gauge field $A_{\mu}$ has dimensions of momentum. The prefactors in the topological response are expressed in terms of the integer topological momentum space invariant $23,30,32,33$. In a given case

$$N = \frac{1}{8\pi^2} \epsilon_{ij} \int_{-\infty}^{\infty} d\omega \int_{BZ} dS \times \text{Tr}[(G\partial_{\omega}G^{-1})(G\partial_{k_i}G^{-1})(G\partial_{k_j}G^{-1})]. \quad (6)$$

The momentum integral is over the 2d torus of the two-dimensional Brillouin zone (BZ).

The physics of the quantum Hall effect can be seen to arise due to the Callan-Harvey anomaly inflow of the Chern-Simons action from the bulk to the boundary $23,30,32,33$. In fact, the 1+1d boundary current $J^{\mu}_{\text{bdry}}$ realizes the (consistent) 1+1d chiral anomaly

$$\partial_{\nu} J^{\mu}_{\text{bdry}} = \frac{N}{8\pi} \epsilon^{\mu
u} F^{\text{bdry}}_{\mu\nu}, \quad (7)$$

where $F^{\text{bdry}}_{\mu\nu} = \partial_{\nu} A_{\mu} - \partial_{\mu} A_{\nu}$ is along the boundary. In this way the protected edge modes arise from the cancellation of bulk and boundary gauge anomalies $23,30,32,33$. The integer $N$ is a topological invariant of the system and in particular remains locally well-defined under smooth deformations of the lattice. Under sufficiently strong deformations or disorder one can have regions of different $N(x)$ with the associated chiral edge modes. In that case, the global invariant, if any, is defined by the topological charge of the dominating cluster which percolates through the system $23,30,32,33$.

IV. 3+1D TOPOLOGICAL ACTION FOR QHE

Using the elasticity tetrads, the generalization of the 2+1d QH response to 3+1d is in principle straightforward and is discussed in detail in Appendix A. The extension of the 2+1-dimensional electromagnetic response to 3+1-dimensions may contain the following topological terms with elasticity tetrads $E^{\mu\nu}_{\alpha}$.

$$S_{4D}^a[E^{\alpha\mu a}, A_{\mu}] = \frac{N_a}{8\pi^2} \int d^4 x \, E^{\mu\nu\alpha\beta}_{\alpha a} A_{\nu} \partial_{\alpha} A_{\beta}. \quad (8)$$

where $a = 1, 2, 3$ labels the spatial lattice directions in three space dimensions. These can for instance label the lattice planes in a layered system of IQH states, chiral superfluids or crystalline topological insulators with AQHE. The new elastic field $N_a E^{a}_{\nu} = N_a \partial_{\nu} X^{a} = \partial_{\nu}(N_a X^{a})$ is due to crystalline QH order in the system.

In general, the $E^{a\nu}_{\mu}(x)$ are slowly varying non-trivial dynamic fields in spacetime. Note that the tetrads $E^{a\nu}_{\mu}(x)$ in Eq. (5) have the dimension of the momentum, and thus the $S_{4D}^a$ in Eq. (5) are dimensionless (in the units $\hbar = 1$). It follows that the prefactors in the topological response Eq. (5) can be combined and the total response expressed in terms of the dimensionless topological momentum space invariants is given by:

$$S_{4D}[A_{\mu}] = \frac{1}{8\pi^2} \sum_{a=1}^{3} N_a \int d^4 x \, E^{a\mu\alpha\beta}_{\alpha} A_{\nu} \partial_{\alpha} A_{\beta}, \quad (9)$$

The integer coefficients $N_a$ are the integrals of the Green's functions:

$$N_a = \frac{1}{8\pi^2} \epsilon_{ijk} \int_{-\infty}^{\infty} d\omega \int_{BZ} dS^{ij}_a \times \text{Tr}[(G\partial_{\omega}G^{-1})(G\partial_{k_i}G^{-1})(G\partial_{k_j}G^{-1})], \quad (10)$$

where now the momentum integral is over the 2D BZ torus, determined by the normal $E^a = E^a_{\nu}$. A similar expression for the 3+1d QH was proposed in Refs. 14.

A. Gauge invariance of the 3+1d action

In the deformed crystalline systems in 3+1d, the tetrads $E^{\mu\nu}_{\alpha}$, which enter the 3+1d Chern-Simons action, depend slowly on space and time. For arbitrary background fields, the spacetime dependence violates the gauge invariance of the action. However, in the absence of dislocations the latter does not happen for the elastic-tetrad parameters $E^{\mu\nu}_{\alpha}(x)$ due to condition $dE^a = 0$ in Eq. (4). The variation $\delta_{\phi} S$ under $\delta A_{\mu} = \partial_{\mu} \phi$ is identically zero modulo the bulk/boundary QH currents of the sample.

Here is the main difference between the topological insulator and a gapless system in 3+1d, e.g. a Weyl semimetal. In that case, integrating out gapless fermions produced non-local anomalous terms in the effective action that lead to axial anomalies in the bulk. While the chiral current is not a well-defined concept (except at low energy close to a Weyl point) and does not need to be conserved, overall gauge invariance must hold. The gauge anomalies of physical currents have to cancel in the bulk. In this sense, the case of a gapless semimetal with axial fields with the so-called consistent bulk axial anomaly is different from the 3+1d QH anomaly inflow discussed here.
B. Hall conductivity

The Chern-Simons action Eq. (9) gives the conductivity tensor that depends on space and time:

\[ \sigma_{ij}(x) = \frac{e^2}{2\pi h} \epsilon_{ijk} \sum_{a=1}^{3} N_a E_k^{(0)a}(x). \] (11)

In the non-deformed crystal, where \( E_k^{(0)a} \) are primitive reciprocal lattice vectors, this equation transforms to the well-known equation with quantized conductivity: \[ \sigma_{ij} = \frac{e^2}{2\pi h} \epsilon_{ijk} G_k, \] (12)

where \( G_k \) is a reciprocal lattice vector, which is expressed in terms of the topological invariants \( N_a \) and the primitive reciprocal lattice vectors \( E_k^{(0)a} = K^a \):

\[ G_k = \sum_{a=1}^{3} N_a E_k^{(0)a}. \] (13)

In the deformed crystal, the conductivity tensor is space-time dependent, and thus is not universally quantized. However, the response of the conductivity to deformation is quantized:

\[ \frac{d\sigma_{ij}}{dE_k} = \frac{e^2}{2\pi h} \epsilon_{ijk} N_a. \] (14)

The Hall current is

\[ J^\mu = \frac{-1}{4\pi^2} \sum_{a=1}^{3} N_a \epsilon^{\mu\nu\alpha\beta} E_\nu^{(0)a} \partial_\alpha A_\beta + \frac{1}{8\pi^2} \sum_{a=1}^{3} N_a \epsilon^{\mu\alpha\beta\gamma} \partial_\alpha E_\gamma^{(0)a} A_\nu \] (15)

and has a bulk and a topological defect/boundary component. In the absence of dislocations, \( dE^\alpha = 0 \) in Eq. (14), and the current represents a dissipationless, fully reversible current, which is conserved due to \( U(1) \) gauge invariance

\[ \partial_\mu J^\mu = \partial_\mu J^\mu_{\text{bulk}} = 0. \] (16)

C. Chiral magnetic effect

The action Eq. (9) and current Eq. (15) also describe the chiral magnetic effect (CME). That is in the presence of periodic directions varying in time, time-dependence \( X^a(r_t) \) appears. In the CME an electric current along an applied magnetic field is induced:

\[ J = \frac{1}{4\pi^2} \sum_{a=1}^{3} N_a E_t^{(0)a} B. \] (17)

This current contains \( E_t^{(0)a} = \partial X^a/\partial t \) and thus it vanishes in equilibrium in agreement with Bloch theorem, according to which in the ground state of the system or in the equilibrium state in general, the total current is absent, see e.g. Ref. [1]. Here we restrict to spatial lattices under deformations. The CME for a time-periodic insulator with timelike \( E_t^{(0)a} = \omega_t \delta^{(0)}_t \) (with Floquet drive \( \omega_F \) and the temporal invariant \( N_t \neq 0 \) was pointed out in Ref. [15]. This can be extended to spatial deformations as well.

V. CALLAN-HARVEY EFFECT ON DISLOCATIONS AND MIXED ANOMALY

The constraint Eq. (1) is violated in the presence of topological defects – dislocations. The density of dislocations equals the torsion for the elasticity tetrads (with vanishing spin connection):

\[ T_{kl} = (\partial_k E_t^{(0)a} - \partial_t E_k^{(0)a}), \] (18)

similar to the role of spacetime torsion in gravitational theories. The nonzero dislocation density or torsion violates the conservation of the Hall current:

\[ \partial_\mu J^\mu = \frac{-1}{8\pi^2} \sum_{a=1}^{3} N_a T^a_{\mu\nu}. \] (19)

This mixed anomaly represents the Callan-Harvey mechanism of anomaly cancellation which is provided here by the fermion zero modes living on dislocations. As shown in for Dirac fermions in the presence of complex vortex-like axionic mass, which can be taken as a topological model for the 3+1d QH system, the action remains well-defined and gauge invariant in the presence of dislocation, i.e. 2π ambiguities in the phase field \( X^a \), due to zero modes with 1+1D covariant anomaly along the dislocation string,

\[ \partial_\mu J^\mu = \partial_\mu J^\mu_{\text{bulk}} + \partial_\mu J^\mu_{\text{dislocation}} = 0. \] (20)

VI. EXTENSION TO OTHER DIMENSIONS

The anomaly equation Eq. (19) can straightforwardly extended to \( D = 2 + 2n \) spacetime dimensions:

\[ \partial_\mu J^\mu \propto \frac{1}{8\pi^2} \sum_{a=1}^{2n+1} N_a T^a \wedge F \wedge ... \wedge F. \] (21)

In addition to the torsional field strength \( T^a = dE^a \), it contains the product of \( n \ U(1) \) gauge field strengths \( F = dA \), while the integer valued topological invariants \( N_a \) are expressed in terms of \( 2n+1 \)-dimensional integrals in frequency-momentum space. The Eq. (21) is valid even in case of \( n = 0 \), i.e. in one spatial dimension. Consider gapped systems of electrons
in the one dimensional chain with the action (see also\cite{42})
\[ S_{1+1D}[A] = \frac{N_1}{2\pi} \int dx dt \, \epsilon^{\mu\nu} E^{1}_\mu A^\nu. \] (22)
The index \( N_1 \) is defined via the Green’s function
\[ G^{-1}(\omega, k_x), \]
\[ N_1 = \frac{1}{2\pi i} \int d\omega \, \text{Tr} \, G(k_x, \omega) \partial_\omega G^{-1}(k_x, \omega). \] (23)
The index \( N_1 \) is the same for any \( k_x \). Note that in the
3+1d case the index \( N_a \) in Eq. (10) is the same for any
cross section \( S_a \) of the three-dimensional Brillouin zone.
In the 1+1 case the cross section corresponds to one point
\( k_x \) in the one-dimensional Brillouin zone.
The Eq. (22) gives the electric current
\[ J^\mu = \frac{N_1}{2\pi} \epsilon^{\mu\nu} E^{1}_\nu. \] (24)
The conservation of this current
\[ \partial_\mu J^\mu = N_a dE^a = N_1 dE^1 = 0, \] (25)
has simple interpretation. The condition \( dE^a = 0 \) is
equivalent to the conservation of the sites of the 1d lattice,
whereas the index \( N_1 \) corresponds to the number of
the electrons per site, which is integer for band insulators.
As a result the number of the electrons is trivially
conserved under adiabatic deformations.
Since any one-dimensional insulator is described by the
topological invariant \( N_1 \), which can only change when the
gap closes, we may call any 1D insulator topological,
although the topology of the filled states can only be
detected by higher invariants – the Chern numbers. In
fact, this is very similar to ordinary metals with Fermi
surfaces\cite{56}. The gapless Fermi surface represents a topol-
ogical object in momentum space protected by an invar-
iant similar to \( N_1 \). Topology provides the stability of the
Fermi surface with respect to interactions and explains
why metals can be described by Landau Fermi-liquid the-
dory. In this sense metals can be considered as a topolo-
gical materials, making the zeroth-order invariant \( N_1 \) one
of the most important topological invariants in the hier-
archy of the topological invariants for fermionic systems.
In particular it gives rise to the Luttinger theorem: the
number of states in the region between two Fermi points
in 1+1d does not depend on interaction. This can be
generalized for any closed Fermi surface or insulator in
higher dimensions\cite{56}.

**VII. DISCUSSION AND CONCLUSION**

In conclusion, the dimensionful elasticity tetrads
are the proper hydrodynamic variables related to the
weakly protected IQHE/AQHE/CME on general even-
dimensional spacetime backgrounds, and can be utilized
to describe the topological response in systems in odd
spatial dimensions, including elastic and geometric deformations. The Eq. (19) and its extension to higher
dimensions \cite{44–50,56} describe the mixed anomaly in terms of
the gauge fields and elasticity torsion.
On the other hand the contribution of the gravita-
tional space-time torsion \( \tilde{T}^a \), where \( a = 0, 1, 2, 3 \) is a local
Lorentz index, to the chiral current \( J^5_\mu \) – the Nieh-Yan
(NY) term\cite{44–47,56–58}
\[ \partial_\mu J^{5\mu} \propto \Lambda^2 \eta_{ab} \tilde{T}^a \wedge \tilde{T}^b, \] (26)
contains the non-universal dimensional ultraviolet cut-
off parameter \( \Lambda = 1/l \) for Lorentz invariant systems with the
local Minkowski metric \( \eta_{ab} \). This, however, does not allow to apply this anomaly to condensed matter sys-
tems, where the Lorentz invariance is absent, and the
ultraviolet regularization may depend on the direction in
momentum space.
In contrast to the torsional contribution of the NY
term, the mixed anomaly in Eqs. (19) and (21) ex-
pressed in terms of the elasticity tetrads and torsion
do not contain any parameters, except for topological quan-
tum numbers. This is because the elastic tetrads (tor-
sion) has the canonical dimensions of \([\ell]^{-1}\) (resp. \([\ell]^{-2}\)
instead of the conventional \([l]^{0}\) (resp. \([l]^{-1}\)) for the gravita-
tional spacetime tetrads (torsion). In summary, we have
shown here that this allows one to write many mixed
“quasi-topological” Chern-Simons terms, analogous to
mixed axial-gravitational anomalies, in 3+1d quantum
Hall systems with weak crystalline symmetries. Simi-
larly, the dimensional elasticity tetrads allow us to write
a mixed elasticity gravitational anomaly in 3+1d, as the
dimensional extension of the 2+1d gravitational framing
anomaly\cite{23,51–53}:
\[ S_{\text{eff},g} = \sum_{a=1}^{3} \frac{N_a}{192\pi^2} \int d^3x dt \, E^a \wedge \left( \Gamma^{\mu}_\nu \wedge d\Gamma^\nu_\rho \right.
\left. + \frac{2}{3} \Gamma^{\mu}_\nu \wedge \Gamma^\nu_\rho \wedge \Gamma^\rho_\mu \right), \] (27)
where \( \Gamma^{\mu}_\nu dx^\lambda \) are the Christoffel symbol one-forms of the
spacetime metric. This mixed elastic-gravitational form
of the Chern-Simons term, although third order in deriva-
tives and therefore beyond linear response, imply the gen-
eralization of the 2+1d thermal Hall effect\cite{44–50,51–53} to
3+1d quantum Hall systems and topological insulators
with intrinsic Hall effect.
We have considered the effective 3+1d QH re-
sponse and elasticity tetrad background fields due to
a lattice with topological charges, dislocations and its
Chern-Simons -like form, reminiscent of mixed axial-
gravitational response, and anomaly inflow. Finally let
us speculate on the hypothetical connection of the elas-
ticity tetrads \( E^a_\mu \) to real spacetime gravity. While the
microscopic structure of the deep relativistic quantum
spacetime vacuum is not known, phenomenological ap-
proaches and effective field theory can be used to describe
the effects of the vacuum degrees of freedom. In one of
these scenarios, it is assumed that the spacetime vacuum has the properties of a 3 + 1d super-plastic crystalline medium constructed from $E^a$,\(^{57,57}\). As in the condensed matter elasticity theory, dislocations and disclinations in this spacetime crystal describe torsion and curvature of general relativity.\(^{11,28,60,61}\). More specifically, in this superplastic model of gravity, the size of the elementary cell in the vacuum spacetime crystal is not fixed but in principle can vary arbitrarily with no elementary Planck scale spacetime lattice. As a result, the action for the gravitational field\(^{57}\)

$$ S[E^a_\mu] = \int_{\mathbb{R}^3} \mathcal{D}[x] \, (K R + \Lambda) \, , $$

(28)

contains only dimensionless quantities. Here the metric $g_{\mu\nu} = \eta_{ab} E^a_\mu E^b_\nu$ and the Ricci curvature scalar $R[E^a_\mu]$ are expressed in terms of the elasticity constants in the standard way, thus making the gravitational constant $K$ (the inverse Newton constant $K \sim 1/G$) and the cosmological constant $\Lambda$ dimensionless, $[K] = [R] = [\Lambda] = 1$. The same is for the higher order gravitational terms ($[R^2] = [R^{\mu\nu} R_{\mu\nu}] = 1$) and for the other (non-gravitational) physical quantities, such as mass: $[M] = 1$. All this means that if gravity is related to the elasticity tensors, all the measurable physical quantities are dimensionless.

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**Appendix A: 3+1d QH and semiclassical expansion**

The above Hall conductivity can be obtained via the semi-classical Green function formalism.\(^{5,29,36}\) The assumptions are gauge invariance and semiclassical expansion to lowest order. The effective action for a slowly varying background field $A_\mu(x,t)$ is simply

$$ \log \frac{Z[A]}{Z_0} = i S_{\text{eff}}[A] = \text{itr} \ln \frac{G[A]}{G_0} \quad (A1) $$

Now we want to obtain the effective action to first order in gradients of $A_\mu(x)$. We use the fact that the coordinate dependence of $A_\mu(x)$ is semiclassical, i.e. a field slowly varying on the scale of the lattice momenta. In the semiclassical phase-space analysis, with $G \equiv G[A] \equiv G(p, \omega; A_\mu(x,t))$, we obtain to the lowest order in the gradients of $A_\mu(x)$

$$ S_{\text{eff}}[A] = - \frac{i}{2} \int \frac{d^3 p d\omega}{(2\pi)^3} \int d^3 x \, dt \quad (A2) $$

$$ \text{tr} [G \partial_{x^\mu} G^{-1} G \partial_{x^\rho} G^{-1} G \partial_{x^\nu} G^{-1} \times (G \partial_{x^\iota} G^{-1})]_{[A=0]} A_\nu $$

where $x^\mu = (t,x)$ and $k_\mu = (\omega, -p)$ and

$$ G \partial_{x^\mu} G^{-1} = G \partial_{k_\mu} G^{-1} \big|_{A=0} \partial_{x^\mu} A_\nu. \quad (A3) $$

The trace in the matrix product of Green functions is antisymmetric in the indices of $k_\mu$-derivatives. With the convention $\epsilon^{xyz} = -1$, we arrive to

$$ S_{\text{eff}}[A] = \frac{i}{12} \int d^3 x dt \, \epsilon^{\mu\nu\rho} A_\mu \partial_\rho A_\gamma \times (A4) $$

$$ \int \frac{d^3 \mathbf{p} d\omega}{(2\pi)^4} \epsilon_{\mu\nu\lambda\rho} \text{tr} [(G \partial_{k_\rho} G^{-1})(G \partial_{k_\iota} G^{-1})]_{A=0} \quad (A5) $$

The momentum space factor can be separated to be of the form

$$ \int \frac{d^3 \mathbf{p} d\omega}{24\pi^2} \times \text{tr} [(G \partial_{k_\rho} G^{-1})(G \partial_{k_\iota} G^{-1})]_{A=0} \quad (A6) $$

evaluated in imaginary time over the cross-sectional reciprocal space perpendicular to the reciprocal lattice direction $a$ Topologically a (pinched) 3-torus and $N_a(p^a)$ is an element of $\pi_3(GL(n,\mathbb{C})) = \mathbb{Z}$.

We conclude that

$$ S_{\text{eff}}[A] = \frac{1}{8\pi^2} N_a \int d^3 x dt \, \epsilon^{\mu\nu\lambda\rho} E^{(0)a}_\mu A_\nu \partial_\lambda A_\rho. \quad (A7) $$

where $E^{(0)a}_\mu = (K^a)_\mu \delta^a_\mu = \int dp^a, i = x, y, z$ are the reciprocal lattice vectors normal to the different lattice planes with invariants $N_a$. This is the statement that the 3+1d is a described by the weak vector invariant $N_a E^{(0)a}_\mu$ protected by the crystalline symmetry.

### 1. Generalization to deformations

It is well-known that the topological winding number $N_a$ is stable against small variations $\delta G(k_\mu, k_\rho, k_\iota)$ of the Green’s function that do not close the gap.\(^{28,29}\) We consider small deformations $x^\mu \rightarrow x^\mu + \xi^\mu(x), \xi \ll 1$. Under these, the reciprocal vectors change as

$$ E^{(0)a}_\mu = \int dp^a \rightarrow E^{a}_\mu(x) \quad (A8) $$

$$ \approx \int dp^a(x) \approx \frac{2\pi}{d} (\delta^a_\mu - \partial_\mu \xi^a) \quad (A9) $$

where $E^{a}_\mu(x)$ is a semiclassical, slowly varying field at the lattice scale $d$ in reciprocal space. Its properties are

$$ E^{a}_\mu(x) E^{a}_\mu(x) = 1 + O(\xi^2). \quad (A10) $$

which defines the local normal direction of a set of lattice planes. Note that the invariants $N_a$ are assumed to be...
constant and independent of deformations throughout. The final result is Eq. (11) in the main text:

\[ S_{\text{eff}}[A] = \frac{1}{8\pi^2} N_a \int d^3x dt \epsilon^{\mu\nu\lambda\rho} E_{\mu}^a A_{\nu} \partial_\lambda A_\rho. \]  

(A11)

In this way, the elasticity tetrads, i.e. elementary deformed reciprocal and direct lattice vectors, \( E_{\mu}^a = \partial_\mu X^a \) and the inverse, \( E^a_\mu(x) E^\beta_{\mu}(x) = \delta^a_\beta, \quad E^a_\mu(x) E^\alpha_\mu(x) = \delta^a_\alpha, \quad a = 1, 2, 3. \) (A12)

appear descending from the lattice field \( N_a X^a. \) As discussed in the text, overall gauge invariance is maintained even in the presence of deformations and topological singularities in \( X^a \) (dislocations). The Hall conductance is

\[ \sigma_{ij} = \frac{\epsilon_{ijk} \sum N_a E^a_j(x)}{4\pi^2} \]  

(A13)

i.e. the conductance is quantized as \( N = \text{gcd}(N_1, N_2, N_3) \) in planes perpendicular to the primitive layer normal \( G_i = \sum_a N_a E^a_i(x) \). The reciprocal vector \( G = G_i \) is the weak Hall index of a weak 3+1d Chern insulator.

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