Weights, Weyl-Equivariant Maps and a Rank Conjecture

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ABSTRACT
In this note, given a pair \((g, \lambda)\), where \(g\) is a complex semisimple Lie algebra and \(\lambda \in \mathfrak{h}^*\) is a dominant integral weight of \(g\), where \(\mathfrak{h} \subseteq g\) is the real span of the coroots inside a fixed Cartan subalgebra, we associate an \(SU(2)\) and Weyl equivariant smooth map \(f: X \to (\mathbb{C}^n)_{\mathbb{R}}\), where \(X \subset \mathfrak{h} \oplus \mathbb{R}^3\) is the configuration space of regular triples in \(\mathfrak{h}\), and \(n, m\) depend on the initial data \((g, \lambda)\). We conjecture that, for any \(x \in X\), the rank of \(f(x)\) is at least the rank of a collinear configuration in \(X\) (collinear when viewed as an ordered \(r\)-tuple of points in \(\mathbb{R}^3\), with \(r\) being the rank of \(g\)). A stronger conjecture is also made using the singular values of a matrix representing \(f(x)\). This work is a generalization of the Atiyah-Sutcliffe problem to a Lie-theoretic setting.

1. Introduction
While its origin lies in Physics, more specifically in the work of Berry and Robbins [6] on a geometric explanation of the spin-statistics theorem, the Atiyah-Sutcliffe problem on configurations of points is a geometric problem. Consider

\[ C_n(\mathbb{R}^3) = \{ x = (x_1, \ldots, x_n) \in (\mathbb{R}^3)^n : x_a \neq x_b \text{ for all } a \neq b \} \]

where \(1 \leq a, b \leq n\). Also consider the flag manifold \(U(n)/T^n\). We note that the symmetric group \(\Sigma_n\) on \(n\) elements acts on \(C_n(\mathbb{R}^3)\) by permuting the \(n\) points \(x_a \in \mathbb{R}^3\), for \(1 \leq a \leq n\). Moreover, \(\Sigma_n\) also acts on the flag manifold \(U(n)/T^n\), thought of as a left coset space, by permuting the \(n\) columns of a matrix \(g \in U(n)\) representing a left coset \(gT^n \in U(n)/T^n\).

The Berry-Robbins problem asks whether there exists, for each \(n \geq 2\), a continuous map

\[ f_n : C_n(\mathbb{R}^3) \to U(n)/T^n \]

which is \(\Sigma_n\) equivariant.

The Berry-Robbins problem was first solved positively by M.F. Atiyah in [1]. However, the solution there had some unsatisfactory features. In that same article, and a subsequent article [2], other candidate maps \(f_n\) were presented, with more satisfactory features (for instance, these candidate maps are smooth), but they would be genuine solutions only provided some linear independence conjecture holds. Later in [5], Sir Michael Atiyah and Paul Sutcliffe found strong numerical evidence for linear independence, as well as for a stronger conjecture, which says that \(|D(x)| \geq 1\) for any \(x \in C_n(\mathbb{R}^3)\), where \(D : C_n(\mathbb{R}^3) \to \mathbb{C}\) is a (smooth) normalized determinant function whose non-vanishing is equivalent to the linear independence conjecture. These are the Atiyah-Sutcliffe conjectures 1 and 2 respectively. The authors of [5] also formulated a conjecture 3 which implies conjecture 2, but we will not explain it here.

In [4], M.F. Atiyah and R. Bielawski solved a Lie-theoretic generalization of the Berry-Robbins problem using Nahm’s equations. However, their solution was not explicit. In his Edinburgh Lectures on Geometry, Analysis and Physics [3], Sir Michael asked whether there exists a Lie-theoretic generalization of the Atiyah-Sutcliffe problem. In this article, we provide a positive answer to his question, and generalize the Atiyah-Sutcliffe problem to a Lie-theoretic setting.

2. A Lie-theoretic generalization of the Atiyah-Sutcliffe problem
Let \(g\) be a complex semisimple Lie algebra. Denote its Killing form by \((\cdot, \cdot)\). Let \(\mathfrak{h} \subset g\) be the real span of the coroots of \(g\) inside a fixed Cartan subalgebra (the latter being thus \(\mathfrak{h} \otimes \mathbb{C}\)). Let \(R \subset \mathfrak{h}^*\) be the set of all roots of \(g\) with respect to \(\mathfrak{h} \otimes \mathbb{C}\), denote by \(R^+ \subset \mathfrak{h}^*\) a choice of positive roots, and by \(\Phi \subset \mathfrak{h}^*\) the corresponding set of simple roots. Strictly speaking, a root \(\alpha\) of \(g\) is an element of \((\mathfrak{h} \otimes \mathbb{C})^*\) but, since each root \(\alpha\) is real-valued on \(\mathfrak{h}\), we consider each \(\alpha\) as an element of \(\mathfrak{h}^*\). It is known that the Killing form (up to a sign) on \(g\) restricts to an inner product on \(\mathfrak{h}\).

Denote by \(W\) the Weyl group of \((g, \mathfrak{h})\). Thus \(W\) is the group generated by reflections \(r_\alpha\) in \(\mathfrak{h}^*\) with respect to the hyperplane \(\mathfrak{h}^*\), as \(\alpha\) varies in the set of roots \(R\). We let \(\lambda \in \mathfrak{h}^*\) be a dominant integral weight of \(g \otimes \mathbb{C}\). What the latter condition amounts to is that
\[ \frac{2(\alpha, \lambda)}{2(\alpha, \alpha)} \]

is a nonnegative integer for any positive root \( \alpha \in R^+ \).

Denote by \( X \) the following configuration space

\[ X = \{ x \in \mathfrak{h} \otimes \mathbb{R}^3 : (\alpha \otimes 1)(x) \neq 0 \text{ for any } \alpha \in R^+ \} \]

where \((\alpha \otimes 1) : \mathfrak{h} \otimes \mathbb{R}^3 \to \mathbb{R}^3\) is the linear map obtained by tensoring \( \alpha \) with the identity map on \( \mathbb{R}^3 \).

From now on, we identify \( \mathfrak{h} \) with \( \mathfrak{h}^* \) via the (restricted) Killing form \((\cdot, \cdot)\), so that the Weyl group \( W \) acts naturally on \( \mathfrak{h} \). If we tensor this action with the trivial action of \( W \) on \( \mathbb{R}^3 \), we obtain an action of \( W \) on \( \mathfrak{h} \otimes \mathbb{R}^3 \), which preserves \( X \). On the other hand, \( SU(2) \) acts on \( \mathfrak{h} \otimes \mathbb{R}^3 \) via the tensor product of the trivial action on \( \mathfrak{h} \) and its natural action on \( \mathbb{R}^3 \) via its adjoint action, i.e. the 2-to-1 group homomorphism from \( SU(2) \) onto \( SO(3) \). This action of \( SU(2) \) preserves \( X \).

We let

\[ m = \sum_{\alpha \in R^+} \frac{2(\alpha, \lambda)}{2(\alpha, \alpha)} \]

and

\[ n = [W : W_\lambda], \]

where \( W_\lambda \) is the stabilizer of \( \lambda \) in \( W \).

Given the initial data \( (g, \mathfrak{h}, R^+, \lambda) \) as above, we will construct a smooth map \( f : X \to (P^m(\mathbb{C}))^n \). Let \( x \in X \). For any root \( \alpha \in R \), we define \( v_\alpha \in S^2 \subset \mathbb{R}^3 \) as the normalization (with respect to the Euclidean inner product on \( \mathbb{R}^3 \)) of

\[ (\alpha \otimes 1)(x) \in \mathbb{R}^3 \setminus \{0\}. \]

The Hopf map \( h : S^3 \to S^2 \) can be defined by

\[ h(u, v) = (2uv, |u|^2 - |v|^2) \]

where \( S^3 \subset \mathbb{C}^2 \) is the set of all \((u, v) \in \mathbb{C}^2\) such that \(|u|^2 + |v|^2 = 1\), and \( S^2 \subset \mathbb{C} \times \mathbb{R} \) is defined as the set of all \((z, x) \in \mathbb{C} \times \mathbb{R}\) such that \(|z|^2 + x^2 = 1\).

For every root \( \alpha \in R \), we choose a Hopf lift \((u_\alpha, v_\alpha) \in S^3\). Such a Hopf lift is unique up to a global factor in \( U(1) \). We then form the complex polynomial

\[ p_\alpha(t) = u_\alpha t - v_\alpha. \]

The elements of \( W/W_\lambda \) are in natural one-to-one correspondence with the Weyl orbit \( W.\lambda \) of \( \lambda \in \mathfrak{h}^* \). Let us say that

\[ W.\lambda = \{ \lambda_1, ..., \lambda_m \}. \]

Choose \( \lambda_1, ..., \lambda_m \in W \) so that \( g_k(\lambda_i) = \lambda_k \), for \( 1 \leq k \leq m \). For any \( 1 \leq k \leq n \), let

\[ p_k(t) = \prod_{\alpha \in R^+} (p_\alpha.\alpha(t))^{m_\alpha}, \]

where

\[ m_\alpha = \frac{2(\alpha, \lambda)}{2(\alpha, \alpha)}. \]

The latter is a nonnegative integer since \( \lambda \) is a dominant integral weight of \( g \otimes \mathbb{C} \) and \( \alpha \) is a positive root. We remark that the definition of \( p_k \) does not depend on the choice of representative \( g_k \) in its left coset \( g_kW_\lambda \), since another choice, say \( g_lw \), where \( w \in W_\lambda \), will only permute the factors of \( p_k \), since the map which maps \( \alpha \) to \( w.\alpha \) permutes the set of positive roots \( \alpha \) satisfying \((\alpha, \lambda) > 0\).

We define the map

\[ F : X \to (\mathbb{C}^{m+1} \setminus \{0\})^n / T^n \]

which maps each \( x \in X \) to the \( n \)-tuple of polynomials \((p_k(t)), \) for \( 1 \leq k \leq n \), where each polynomial is actually only defined up to multiplication by an element of \( U(1) \) (due to the \( U(1) \) ambiguity of each Hopf lift). Finally, the map

\[ f : X \to (P^m(\mathbb{C}))^n \]

is obtained by following the map \( F \) with the natural projection

\[ (\mathbb{C}^{m+1} \setminus \{0\})^n / T^n \to (P^m(\mathbb{C}))^n. \]

We note that an element \( x \in \mathfrak{h} \otimes \mathbb{R}^3 \) can be thought of as an \( r \)-tuple of points in \( \mathbb{R}^3 \), where \( r = \text{rank}(g) \cdot \text{dim}(\mathfrak{h}) \).

We then define the class of collinear configurations, as the set of all \( x \in X \) consisting of \( r \) collinear points in \( \mathbb{R}^3 \). Given \( x \in X \), \( f(x) \) can be represented by an \((m + 1) \times n \) complex matrix. Such a choice is not unique, as one can independently scale each column of such a matrix. It turns out that the rank of \( f(x) \), that is to say the rank of an \((m + 1) \times n \) complex matrix representing \( f(x) \), is the same for all collinear configurations \( x \). We denote this rank by \( r_{col} \).

Our first conjecture can now be phrased.

Conjecture (Generalized Conjecture 1). Given any \( x \in X \),

\[ \text{rank}(f(x)) \geq r_{col}. \]

Our second conjecture can be phrased using the singular values of a matrix representing \( F(x) \), and is a quantitative refinement of our first conjecture. More specifically, define the \((m + 1) \times (m + 1) \) matrix \( g \) by

\[ g = \text{diag}(\begin{pmatrix} m & -1 & \cdots & -1 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}), \]

and \( \Delta : X \to \mathbb{R} \) by

\[ \Delta(x) = s_{r_{col}}(\text{Sing}(\sqrt{g}F(x))) \]

where \( s_j \) denotes the \( j \)-th elementary symmetric polynomial of the diagonal entries of the matrix argument (which has 0s off the diagonal), and \( \text{Sing} \) denotes the middle matrix in the singular value decomposition, namely the matrix containing the singular values (and possibly zero(s)) as diagonal entries, with multiplicity taken into account. We note that the set of singular values does not depend on the choice of \((m + 1) \times n \) complex matrix representing \( F(x) \), since another such matrix is obtained from the first by multiplying by a diagonal unitary matrix from the right.

The matrix \( \sqrt{g} \) was used in the previous definition in order to make \( \Delta \) \( SU(2) \)-invariant.

We remark that \( \Delta \) is clearly non-negative. Moreover, its non-vanishing on \( X \) is equivalent to our Generalized Conjecture 1. We can now formulate our Generalized Conjecture 2. Define \( \delta : X \to \mathbb{R} \) by
Table 1. Numerical Tests for $\mathfrak{sl}(4)$. 

| Weight | Sample minimal $\delta_{\text{val}}$ | $r_{\text{coll}}$ | $(m + 1) \times n$ |
|--------|----------------------------------|-----------------|--------------------|
| [6 4 2 0] | 2467509.893687575 | 11 | 21 $\times$ 24 |
| [5 3 1 0] | 117.31.1059.42467832 | 16 | 18 $\times$ 24 |
| [4 2 0 0] | 601.134059559446 | 8 | 15 $\times$ 12 |
| [5 3 2 0] | 6374.063559302609 | 14 | 17 $\times$ 24 |
| [4 2 1 0] | 12.96192270429032 | 14 | 14 $\times$ 24 |
| [3 1 0 0] | 8.760098476990917 | 9 | 11 $\times$ 12 |
| [4 2 2 0] | 48.21104918942398 | 6 | 13 $\times$ 12 |
| [3 1 1 0] | 1.33576491104691 | 10 | 10 $\times$ 12 |
| [2 0 0 0] | 1.3720791603884346 | 4 | 7 $\times$ 4 |
| [5 4 2 0] | 844.997551976928 | 16 | 18 $\times$ 24 |
| [4 3 1 0] | 560.6200972121253 | 12 | 15 $\times$ 24 |
| [3 2 0 0] | 11.75361594621856 | 10 | 12 $\times$ 12 |
| [4 3 2 0] | 20.1067192041055 | 14 | 14 $\times$ 24 |
| [3 2 1 0] | 4.211242316907258 | 11 | 11 $\times$ 24 |
| [2 1 0 0] | 1.625397043159045 | 8 | 8 $\times$ 12 |
| [3 2 2 0] | 2.554250076222245 | 10 | 10 $\times$ 12 |
| [2 1 1 0] | 2.803319691109273 | 6 | 7 $\times$ 12 |
| [1 0 0 0] | 1.0477509152925308 | 4 | 4 $\times$ 4 |
| [4 4 2 0] | 664.2010890899873 | 8 | 15 $\times$ 12 |
| [3 3 1 0] | 25.47420339307807 | 10 | 12 $\times$ 12 |
| [2 2 0 0] | 1.78380735073747 | 5 | 9 $\times$ 6 |
| [3 3 2 0] | 22.79223230772062 | 9 | 11 $\times$ 12 |
| [2 2 1 0] | 1.53527494892929 | 8 | 8 $\times$ 12 |
| [1 1 0 0] | 1.1263470345826379 | 5 | 5 $\times$ 6 |
| [2 2 2 0] | 1.0106185196429186 | 4 | 7 $\times$ 4 |
| [1 1 1 0] | 1.0730733498926 | 4 | 4 $\times$ 4 |

$$
\delta(x) = \frac{\Delta(x)}{\Delta(x_{\text{coll}})}
$$

where $x_{\text{coll}} \in X$ is a collinear configuration (thinking of an element $x \in X$ as $r$ elements in $\mathbb{R}^3$, where $r$ is the rank of $g$). We note that the denominator is well defined.

Conjecture (Generalized Conjecture 2). Given any $x \in X$,

$$
\Delta(x) \geq \Delta(x_{\text{coll}}).
$$

Equivalently, this conjecture is equivalent to

$$
\delta(x) \geq 1
$$

for all $x \in X$.

These two conjectures are generalizations of the Atiyah-Sutcliffe conjectures 1 and 2 to a Lie theoretic setting. Indeed, if $g = \mathfrak{sl}(n)$, and

$$
\lambda = e^{1} + \cdots + e^{-1} - \frac{n-1}{n}(e^{1} + \cdots + e^{n})
$$

$$
= \frac{1}{n}(e^{1} + \cdots + e^{n-1} - (n-1)e^{n})
$$

where $e_k$ represents the diagonal $n \times n$ matrix having 1 at the $(k, k)$-entry, and 0 everywhere else, $\mathfrak{h}$ is given by

$$
\mathfrak{h} = \text{span}_{\mathbb{R}}(e_k - e_{k+1}; \text{for } 1 \leq k \leq n-1)
$$

and $(e^1, \ldots, e^n)$ represents the dual basis of the basis $(e_1, \ldots, e_n)$ of the space of diagonal matrices. Then for such choices, we claim that our map $f$ specializes to the Atiyah-Sutcliffe map, and that our two conjectures specialize to the Atiyah-Sutcliffe conjectures 1 and 2. Indeed, define, for $1 \leq k \leq n$,

$$
\lambda_k = e^1 + \cdots + e_k + \cdots + e^n - \frac{n-1}{n}(e^1 + \cdots + e^n)
$$

$$
= \frac{1}{n}(e^1 + \cdots + e_k + \cdots + e^n - (n-1)e^k)
$$

where the hat over a term means that this term is omitted. Note that $\lambda = \lambda_n$ and that the $\lambda_k$, $1 \leq k \leq n$, form the orbit of $\lambda$ under the Weyl group $W = \Sigma_n$ of $\mathfrak{sl}(n)$, where $\Sigma_n$ is the symmetric group on $n$ symbols. By calculating the multiplicities $m_{\lambda_k}$, one can see that the polynomial $p_k$ is nothing but the $k$-th Atiyah-Sutcliffe polynomial (which is also usually denoted by $p_k$ in the literature). Moreover, since the absolute value of the determinant of a complex square matrix is the product of the singular values, it follows that our $\Delta$ is nothing but $\|D\|$, where $D : C_{n}(\mathbb{R}^3) \to \mathbb{C}$ is the Atiyah-Sutcliffe determinant. Moreover, $D$ attains the value 1 at a collinear configuration. This is thus a generalization of the Atiyah-Sutcliffe problem to a Lie-theoretic setting.

Remark 2.1. The author’s $\text{Sp}(m)$ version of the Atiyah-Sutcliffe problem in [7] is also a special case of our Lie-theoretic generalization, and can be obtained by a suitable choice of weight $\lambda$ for the Lie algebra $\text{sp}(2m, \mathbb{C})$.

Remark 2.2. While Atiyah and Bielawski in [4] have found a Lie-theoretic solution to the so-called Berry-Robbins problem, it is not clear, as of now, how their non-elementary solution, which uses Nahm’s equations, is related to the more elementary Atiyah-Sutcliffe problem. It would be very interesting to try and relate the two approaches, if possible.

3. Numerical evidence

The author did some numerical testing of the Generalized Atiyah-Sutcliffe problem for $\mathfrak{sl}(4)$ for 26 different weights $\lambda$. For each such weight, the computer generated 1000 configurations pseudo-randomly for which it calculated their $\delta$’s, and then calculated the minimal $\delta$ among these 1000 configurations. Such a sample-minimum $\delta$ is shown in the Table 1 below, for these 26 weights, as well as the corresponding collinear rank $r_{\text{coll}}$ and the dimensions $(m + 1) \times n$ of the corresponding matrix representing $f(x)$. We can see that in all these cases, we found $\delta(x) \geq 1$, thus supporting our Generalized Conjecture 2.

We remark that the notation [3 3 2 0] corresponds to the orthogonal projection of $3e^1 + 3e^2 + e^3$ onto the orthogonal complement of $e^1 + e^2 + e^3$ corresponding to the condition of being trace free. The simulation above took about 30 minutes on a Macbook Pro 2015. We wish to run more numerical simulations in the future.

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Declaration of Interest

No potential conflict of interest was reported by the author.
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