Multiplicative partition functions for reverse plane partitions derived from an integrable dynamical system

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January 25, 2017

Abstract

A close connection of reverse plane partitions with an integrable dynamical system called the discrete two-dimensional (2D) Toda molecule is clarified. It is shown that a multiplicative partition function for reverse plane partition of arbitrary shape with bounded parts can be obtained from each non-vanishing solution to the discrete 2D Toda molecule. As an example a partition function which generalizes MacMahon’s triple product formula as well as Gansner’s multi-trace generating function is derived from a specific solution to the dynamical system.

1 Introduction

Let \( \lambda \) be an (integer) partition or the corresponding Young diagram. A reverse plane partition of shape \( \lambda \) is a filling of cells in \( \lambda \) with nonnegative integers such that all rows and columns are weakly increasing. One of the most prominent results in the study of reverse (or ordinary) plane partitions is the discovery of multiplicative generating functions, namely those which can be nicely factored.

The first discovery is due to MacMahon [8] who proved the triple product formula

\[
\sum_{\pi} q^{\abs{\pi}} = \prod_{i=1}^{r} \prod_{j=1}^{c} \prod_{k=1}^{n} \frac{1 - q^{i+j+k-1}}{1 - q^{i+j+k-2}}
\]

for plane partitions \( \pi \) of \( r \times c \) rectangular shape with parts at most \( n \). MacMahon’s study on plane partitions was later revived by Stanley [9]. Among his vast amounts of results on plane partitions a multiplicative generating function involving the trace statistic is of great importance [10]. Stanley’s trace generating function was much refined by Gansner [11] who derived the multi-trace generating function

\[
\sum_{\pi \in \text{RPP}(\lambda)} \prod_{\ell=1}^{c-1} x_{\ell}^{\text{tr}_{\ell}(\pi)} = \prod_{(i,j) \in \lambda} \left( 1 - \prod_{\ell=j-\lambda'_{j}}^{\lambda_i-i} x_{\ell} \right)^{-1}
\]
where $\text{RPP}(\lambda)$ denotes the set of reverse plane partitions $\pi = (\pi_{i,j})$ of shape $\lambda$ with $r$ rows and $c$ columns, $\text{tr}_\ell(\pi) = \sum_{-i+j=\ell} \pi_{i,j}$ the $\ell$-trace, and $\lambda'$ the shape conjugate with $\lambda$. Note that plane partitions considered in (1) are those with bounded parts but reverse plane partitions in (2) are those with unbounded parts.

In this paper a close connection of reverse plane partitions with an integrable dynamical system, called the discrete two-dimensional (2D) Toda molecule [3], is clarified. Especially it is shown that a multiplicative partition functions for reverse plane partitions can be derived from each non-vanishing solution to the dynamical system (Theorem 6 in Section 4) where reverse plane partitions considered are those of arbitrary shape with bounded parts. As a concrete example a partition function which generalizes both MacMahon’s triple product formula (1) and Gansner’s multi-trace generating function (2) is derived from a specific solution (Theorem 7 in Section 5). The result is an extension of the author’s partition function for rectangular-shaped plane partitions [4, 5].

The key idea comes from a combinatorial interpretation of the discrete 2D Toda molecule in terms of non-intersecting lattice paths (Section 3). Note that Viennot [11] takes a similar approach to count non-intersecting Dyck paths by using the quotient-difference (qd) algorithm for Padé approximation.

2 Solutions to the discrete 2D Toda molecule

We see a brief review on the integrable dynamical system discussed throughout the paper. The discrete two-dimensional (2D) Toda molecule is one of the most typical integrable dynamical systems that was introduced as a discrete analogue of the Toda lattice [3]. The evolution of the discrete 2D Toda molecule is described by the difference equations

\[
\begin{align*}
a_n^{(s,t+1)} + b_n^{(s+1,t)} &= a_n^{(s,t)} + b_n^{(s,t)}, \quad (3a) \\
a_n^{(s+1,t)} b_{n+1}^{(s,t+1)} &= a_n b_{n+1}^{(s,t)}, \quad (3b) \\
(s, t) &\in \mathbb{Z}^2, \quad n \in \mathbb{Z}_{\geq 0}, \quad b_0^{(s,t)} = 0. \quad (3c)
\end{align*}
\]

Through a dependent variable transformation

\[
\begin{align*}
a_n^{(s,t)} &= \frac{r_n^{(s+1,t)}}{r_n^{(s+1,t)}} \frac{r_n^{(s,t)}}{r_n^{(s,t)}}, \\
b_n^{(s,t)} &= \frac{r_{n+1}^{(s,t+1)}}{r_n^{(s+1,t)}} \frac{r_n^{(s,t)}}{r_{n+1}^{(s,t+1)}},
\end{align*}
\]

with $\tau_0^{(s,t)} = 1$ the following equation is obtained from (3)

\[
\begin{align*}
\frac{r_{n-1}^{(s+1,t)}}{r_{n+1}^{(s+1,t)}} \frac{r_n^{(s,t)}}{r_n^{(s,t)} } - \frac{r_n^{(s+1,t+1)}}{r_n^{(s,t+1)}} \frac{r_n^{(s,t)}}{r_n^{(s,t)} } + \frac{r_n^{(s+1,t)}}{r_n^{(s,t+1)}} \frac{r_n^{(s,t+1)}}{r_n^{(s,t+1)}} &= 0, \quad (5a) \\
(s, t) &\in \mathbb{Z}^2, \quad n \in \mathbb{Z}_{\geq 1}, \quad \tau_0^{(s,t)} = 1, \quad (5b)
\end{align*}
\]

that is the so-called bilinear form of the discrete 2D Toda molecule.
The bilinear form \( \tau_n^{(s,t)} \) is solved by the determinant

\[
\tau_n^{(s,t)} = \det_{0 \leq i,j < n} (f_{s+i,t+j}) = \begin{vmatrix}
 f_{s,t} & \cdots & f_{s,t+n-1} \\
 \vdots & \ddots & \vdots \\
 f_{s+i,t} & \cdots & f_{s+i,t+n-1} \\
 \vdots & \ddots & \vdots \\
 f_{s+n-1,t} & \cdots & f_{s+n-1,t+n-1}
\end{vmatrix}
\]

where \( f = f_{i,j} \) is an arbitrary function defined over \( \mathbb{Z}^2 \). The discrete 2D Toda molecule \( \mathfrak{M} \) is therefore solved by \( a_n^{(s,t)}, b_n^{(s,t)} \) given by \( \mathfrak{M} \) with \( \mathfrak{M} \) if the determinant does not vanish. Conversely there exists a function \( f \) on \( \mathbb{Z}^2 \) which satisfies \( \mathfrak{M} \) with \( \mathfrak{M} \) for any solution \( a_n^{(s,t)} \neq 0, b_n^{(s,t)} \neq 0 \) to \( \mathfrak{M} \). It is not difficult to see the following correspondence between \( a_n^{(s,t)}, b_n^{(s,t)} \) and \( f \).

**Proposition 1.** For each solution \( a_n^{(s,t)} \neq 0, b_n^{(s,t)} \neq 0 \) to the discrete 2D Toda molecule \( \mathfrak{M} \) there exists a function \( f = f_{i,j} \) on \( \mathbb{Z}^2 \) which gives the same solution through \( \mathfrak{M} \) with \( \mathfrak{M} \). Moreover such an \( f \) is uniquely determined up to the transformation \( f_{i,j} \to \varphi_j f_{i,j} \) by any non-vanishing function \( \varphi = \varphi_j \) on \( \mathbb{Z} \).

Giving a non-vanishing solution \( a_n^{(s,t)}, b_n^{(s,t)} \) to the discrete 2D Toda molecule is thus essentially equivalent to giving a function \( f \) over \( \mathbb{Z}^2 \).

## 3 Lattice path combinatorics

In this paper we adopt a matrix-like coordinate to draw a square lattice \( \mathbb{Z}^2 \) where the nearest neighbors \((i + 1, j), (i - 1, j), (i, j + 1) \) and \((i, j - 1)\) of a lattice point \((i, j)\) are located on the south, north, east and west of \((i, j)\) respectively. We call a subset \( \mathbb{L} \) of \( \mathbb{Z}^2 \) regular such that (i) if \((i, j) \in \mathbb{L} \) then \((i+k, j+k) \in \mathbb{L} \) for all \( k \geq 1 \); (ii) if \((i, j) \in \mathbb{L} \) then \((i-k, j) \not\in \mathbb{L} \) and \((i, j-k) \not\in \mathbb{L} \) for some \( k \geq 1 \). We call a point \((i, j) \in \mathbb{L} \) a north boundary point if \((i-1, j) \not\in \mathbb{L} \); similarly a west boundary point if \((i, j-1) \not\in \mathbb{L} \). We call a point \((i, j) \in \mathbb{L} \) a convex corner if \((i, j)\) is a north and south boundary point. The interest is in lattice paths on a regular subset \( \mathbb{L} \) of \( \mathbb{Z}^2 \) consisting of north and east steps. See Figure 1 for example.

We think of a regular subset \( \mathbb{L} \) of \( \mathbb{Z}^2 \) as a graph with vertices \( \mathbb{L} \) and edges connecting nearest neighbors. We determine the weights of edges by using a solution \( a_n^{(s,t)}, b_n^{(s,t)} \) to the discrete 2D Toda molecule \( \mathfrak{M} \) as follows.

(a) The vertical edge with north endpoint at \((i, j)\) is weighted by \( a_n^{(i-n,j-n)} \) if \((i-n, j-n)\) is a west boundary point of \( \mathbb{L} \).

(b) The vertical edge with south endpoint at \((i, j)\) is weighted by \( b_n^{(i-n,j-n)} \) if \((i-n, j-n)\) is a north boundary point of \( \mathbb{L} \).
Figure 1: A regular subset $L$ of $\mathbb{Z}^2$ and a lattice path. North and west boundary points are marked blue and red respectively. Convex corners are those marked in both the colors.

(c) Every horizontal edge is weighted by 1.

See Figure 2 for example. We define the weight $w(L; a, b; P)$ of a lattice path $P$ on $L$ to be the product of the weights of all edges passed by $P$. We conventionally consider empty paths $P$ with no steps for which $w(L; a, b; P) = 1$. For $(i, j) \in L$ and $(k, \ell) \in L$ we further define

$$g(L; a, b; i, j; k, \ell) = \sum_P w(L; a, b; P)$$

where the sum ranges over all lattice paths on $L$ going from $(i, j)$ to $(k, \ell)$.

Let $x(j)$ denote the $x$-coordinate (or the vertical —) of the north boundary point of $L$ with $y$-coordinate (or horizontal —) equal to $j$; let $y(i)$ the $y$-coordinate of the west boundary point of $L$ with $x$-coordinate equal to $i$. The following theorem gives a combinatorial interpretation of the discrete 2D Toda molecule and refines Proposition 1.

**Theorem 2.** Let $a_n^{(s,t)} \neq 0$, $b_n^{(s,t)} \neq 0$ be a solution to the discrete 2D Toda molecule (3), and let a function $f = f_{i,j}$ on $\mathbb{Z}^2$ give the same solution through (4) with (6). Let $L$ be a regular subset of $\mathbb{Z}^2$. For $(i, j) \in L$ then

$$\frac{f_{i,j}}{f_{x(j),j}} = g(L; a, b; i, y(i); x(j), j) \quad (8)$$

In order to prove the theorem we use the following lemma.

**Lemma 3.** Let $L$ be a regular subset of $\mathbb{Z}^2$ with a convex corner $(s, t) \in L$. Let $L'$ denote the regular subset of $\mathbb{Z}^2$ obtained from $L$ by deleting $(s, t)$. For $(i, j)$ and $(k, \ell)$ in $L'$ with $i - j \neq s - t$ and $k - \ell \neq s - t$ then

$$g(L; a, b; i, j; k, \ell) = g(L'; a, b; i, j; k, \ell) \quad (9)$$
The difference between both \((i, j)\) enter the region between \(d\) that \((i, j)\) the identity (9) clearly holds since lattice paths going from \((i, j)\) to \((k, \ell)\) is outside the region between \(d_\) and \(d_+\). (Those may be on \(d_{+}\) ) If both \((i, j)\) and \((k, \ell)\) are either in the south of \(d_\) or in the north of \(d_+\) then the identity (9) clearly holds since lattice paths going from \((i, j)\) to \((k, \ell)\) never enter the region between \(d_\) and \(d_+\). In the rest of the proof we thus assume that \((i, j)\) is in the south of \(d_\) and \((k, \ell)\) in the north of \(d_+\).

Each lattice path \(P\) going from \((i, j)\) to \((k, \ell)\) is uniquely divided into three subpaths: \(P_\) from \((i, j)\) to \(d_\), \(Q\) of two steps between \(d_\) and \(d_+\) and \(P_\) from \(d_+\) to \((k, \ell)\). Obviously \(w(P_\) = \(w(P_\) where \(w\) and \(w\) are abbreviations of \(w(L; a, b; \cdot)\) and \(w(L'; a, b; \cdot)\) respectively. The proof of (9) thus amounts to showing that \(g(i, j; k, \ell) = g'(i, j; k, \ell)\) for each \((i, j)\) on \(d_\) and \((k, \ell)\) on \(d_\) where \(g\) and \(g'\) are abbreviations of \(g(L; a, b; \cdot)\) and \(g(L'; a, b; \cdot)\) respectively. Since \(Q\) is of two steps we have only three cases: (i) \((i, j) = (s + n, t + n + 1)\) and \((k, \ell) = (s + n, t + n + 1)\) for some \(n \geq 1\); (ii) \((i, j) = (s + n + 1, t + n)\) and \((k, \ell) = (s + n + 1, t + n)\) for some \(n \geq 0\); (iii) \((i, j) = (s + n + 1, t + n)\) and \((k, \ell) = (s + n - 1, t + n)\) for some \(n \geq 1\). See Figure 3.

Case (i): The unique lattice path going from \((i, j) = (s + n, t + n - 1)\) to \((k, \ell) = (s + n, t + n + 1)\) of two east steps is both on \(L\) and on \(L'\). Thus \(g(i, j; k, \ell) = g'(i, j; k, \ell) = 1\).

Case (ii): There are two lattice paths going from \((i, j) = (s + n+1, t+n)\) to \((k, \ell) = (s + n, t + n + 1)\) one of which is \(Q_\) going north and east, the other is \(Q_\) going east and north. If \(n \geq 1\) then \(Q_1\) and \(Q_2\) are both on \(L\) and on \(L'\), and
Figure 3: Proof of Lemma 8

\[ w(Q_1) = a_n(s, t), \quad w(Q_2) = b_n^{(s, t)}, \quad w'(Q_1) = a_n^{(s, t+1)} \quad \text{and} \quad w'(Q_2) = b_n^{(s+1, t)}. \]

Thus \( g(i, j; k, \ell) = a_n(s, t) + b_n(s, t) = a_n^{(s, t+1)} + b_n^{(s+1, t)} = g'(i, j; k, \ell) \) because of (3b).

If \( n = 0 \) then \( Q_1 \) and \( Q_2 \) are on \( L \) while only \( Q_2 \) on \( L' \), and \( w(Q_1) = a_0(s, t) \), \( w(Q_2) = b_0^{(s, t)} \) and \( w'(Q_1) = a_0^{(s, t+1)} \). \( Q_1 \) is not on \( L' \) since \( Q_1 \) passes through \((s, t) \notin L'\). Thus \( g(i, j; k, \ell) = a_0(s, t) + b_1^{(s, t)} = a_0^{(s, t+1)} = g'(i, j; k, \ell) \) because of (3n) with (3c).

Case (iii): The unique lattice path going from \((i, j) = (s + n + 1, t + n)\) to \((k, \ell) = (s + n - 1, t + n)\) of two north steps is both on \( L \) and on \( L' \). The weight of the lattice path is \( a_n(s, t)b_n^{(s, t)} \) on \( L \) and \( a_n^{(s, t+1)}b_n^{(s+1, t)} \) on \( L' \). Thus \( g(i, j; k, \ell) = a_n(s, t)b_n^{(s, t)} = a_n^{(s, t+1)}b_n^{(s+1, t)} = g'(i, j; k, \ell) \) because of (3n).

Proof of Theorem 2: Let \( L' \) denote the regular subset of \( \mathbb{Z}^2 \) defined by \( L' = \{ (s, t) \in L : s < i \text{ and } t < j \} \). From Lemma 8 then \( g(L'; a; b; i; y(i); x(j), j) = g(L'; a; b; i; y(i); x(j), j) \) because \( L' \) can be obtained from \( L \) by iterative deletion of convex corners. A lattice path going from \((i, y(i))\) to \((x(j), j)\) on \( L' \) is unique because such a lattice path cannot turn north until \((i, j)\) and cannot turn east from \((i, j)\). The weight of the unique lattice path on \( L' \) implies that
\[ g(L'; a; b; i; y(i); x(j), j) = \prod_{k=x(j)}^{i-1} a_0^{(k,j)}. \]

The last product is equal to \( f_{i,j}/f_{k(x),j} \) because \( a_0^{(k,j)} = f_{k+1,j}/f_{k,j} \) from (4) and (3).

Theorem 2 admits a combinatorial interpretation of the determinant \( \tau_n(s,t) \) by means of Gessel–Viennot–Lindström’s method [2, 7]. For \((s, t) \in L \) and \( n \geq 0 \) we define LP(L, s, t, n) to be the set of \( n \)-tuples \((P_0, \ldots, P_{n-1})\) of lattice paths on \( L \) such that (i) \( P_k \) goes from \((s + k, y(s) + k)\) to \((x(t) + k, t + k)\) for each \( 0 \leq k < n \), and (ii) \( P_0, \ldots, P_{n-1} \) are non-intersecting: \( P_j \cap P_k = \emptyset \) if \( j \neq k \). See Figure 4 the second figure shows an \( n \)-tuple of non-intersecting lattice paths.

Theorem 4. Let \( a_n(s,t) \neq 0, b_0^{(s,t)} \neq 0 \) be a solution to the discrete 2D Toda molecule [3], and let a function \( f = f_{i,j} \) on \( \mathbb{Z}^2 \) give the same solution through
Let $\mathbb{L}$ be a regular subset of $\mathbb{Z}^2$. For $(s, t) \in \mathbb{L}$ and $n \geq 0$ then
\begin{align}
\frac{\tau_n^{(s,t)}}{\tau_n^{(x(t),t)}} &= \sum_{(P_0', \ldots, P_{n-1}') \in \text{LP}(\mathbb{L}, s, t, n)} \prod_{k=0}^{n-1} w(\mathbb{L}; a; P_k') \tag{10}
\end{align}
where $\tau_n^{(s,t)} = \det_{0 \leq i,j < n} (f_{s+i,t+j})$.

**Proof.** Gessel–Viennot–Lindström’s method yields from Theorem 2 that
\begin{align}
\frac{\tau_n^{(s,t)}}{\tau_n^{(x(t),t)}} &= \sum_{(P_0', \ldots, P_{n-1}') \in \text{LP}(\mathbb{L}, s, t, n)} \prod_{k=0}^{n-1} w(\mathbb{L}; a; P_k') \tag{11}
\end{align}
where the sum ranges over all $n$-tuples $(P_0', \ldots, P_{n-1}')$ of non-intersecting lattice paths on $\mathbb{L}$ such that $P_k'$ goes from $(s + k, y(s + k))$ to $(x(t + k), t + k)$ for each $0 \leq k < n$. Eliminating the steps frozen due to the non-intersecting condition we obtain $(P_0', \ldots, P_{n-1}') \in \text{LP}(\mathbb{L}, s, t, n)$, see Figure 4 for example. The weight of the eliminated frozen steps is equal to the weight of a unique configuration of non-intersecting lattice paths from $(s, t)$ to $(x(t), t)$ for each $0 \leq k < n$. Thus $\tau_n^{(s,t)}$ is equal to the right-hand side of (10) multiplied by $\tau_n^{(x(t),t)}$.

Note that the left-hand side of (11) can be expressed as
\begin{align}
\frac{\tau_n^{(s,t)}}{\tau_n^{(x(t),t)}} &= \prod_{i=1}^{s-x(t)} \prod_{k=1}^{n} a_{(s-i,t)}^{(s-i,t)} \tag{12}
\end{align}
from (4). We can readily evaluate the sum in (10), a partition function for non-intersecting lattice paths, by using this formula.
Figure 5: The bijection between LP(\(L(\lambda), r, c, n\)) and RPP(\(\lambda, n\)) where \(\lambda = (5, 4, 4, 2, 1)\) with \(r = 5\) rows and \(c = 5\) columns, and \(n = 4\).

4 Multiplicative partition functions for reverse plane partitions

Let \(\lambda\) be a partition and let \(n \geq 0\). We write RPP(\(\lambda, n\)) for the set of reverse plane partitions of shape \(\lambda\) with parts at most \(n\). Let \(r\) and \(c\) denote the numbers of rows and columns in \(\lambda\) respectively. We then define a regular subset \(L(\lambda)\) of \(\mathbb{Z}^2\) by

\[
L(\lambda) = \{(i, j) \in \mathbb{Z}^2_{\geq 0}; j \geq c - \lambda_{r-i}\}
\]

where \(\lambda_i\) denotes the \(i\)-th part of \(\lambda\) for \(1 \leq i \leq r\) and \(\lambda_i = c\) for \(i \leq 0\). There is a bijection between LP(\(L(\lambda), r, c, n\)) and RPP(\(\lambda, n\)) which is described as follows.

(i) move the lattice path \(P_k\) northwest by \((-k, -k)\) for each \(0 \leq k < n\);

(ii) fill in the cells between \(P_{n-k-1}\) and \(P_{n-k}\) with \(k\) for each \(0 \leq k \leq n\) where \(P_{-1}\) is the lattice path going from \((r, 0)\) to \((0, c)\) along the border of \(L(\lambda)\) and \(P_n\) is that going east from \((r, 0)\), turning north at \((r, c)\) and going north to \((0, c)\);

(iii) rotate 180° to obtain a reverse plane partition in RPP(\(\lambda, n\)).

Figure 5 demonstrates the bijection by an example. It should be noted that this bijection is essentially the same as the classical interpretation of plane partitions by “zig-zag” non-intersecting paths [6].

We set up weight for reverse plane partitions which is equivalent to the weight for lattice paths defined in Section 3. Let \(\lambda' = (\lambda'_1, \ldots, \lambda'_c)\) denote the partition conjugate with \(\lambda\). We define \(\alpha_{i,j}\) by

\[
\alpha_{i+k,\lambda_i+k} = a_{n-k-1}^{(r-i,c-i)}, \quad \alpha_{\lambda'_j+k,j+k-1} = b_{n-k}^{(r-\lambda'_j,c-j)}
\]

for \(1 \leq i \leq r, 1 \leq j \leq c\) and \(k < n\) where \(a_n^{(s,t)} \neq 0, b_n^{(s,t)} \neq 0\) is a solution to the discrete 2D Toda molecule [3]. We then define the weight of a reverse plane
partition \( \pi \) by
\[
v(\lambda, n; a, b; \pi) = \prod_{(i,j) \in \lambda} v_{i,j}(\lambda, n; a, b; \pi) \quad \text{with} \quad (15a)
\]
\[
v_{i,j}(\lambda, n; a, b; \pi) = \prod_{k=1}^{\pi_{i,j}} \frac{\alpha_{i+k-1,j+k-2}}{\alpha_{i+k-1,j+k-1}}. \quad (15b)
\]

Lemma 5. Let \( \lambda \) be a partition with \( r \) rows and \( c \) columns, and let \( n \geq 0 \). Assume that \( \pi \in \text{RPP}(\lambda, n) \) and \( (P_0, \ldots, P_{n-1}) \in \text{LP}(\mathbb{L}(\lambda), r, c, n) \) corresponds to each other by the bijection. Then
\[
v(\lambda, n; a, b; \pi) = \frac{\prod_{k=0}^{n-1} w(\mathbb{L}(\lambda); a, b; P_k)}{\prod_{i=1}^{r} \prod_{k=1}^{n} a_k^{(r-i,c-\lambda)}}. \quad (16)
\]

Sketch of proof. Actually \( \alpha_{i,j} \) is defined so that \( v(\pi) = v(\lambda, n; a, b; \pi) \) is proportional to \( \prod_{k=0}^{n-1} w(P_k) \) with \( w(P) = w(\mathbb{L}(\lambda); a, b; P) \). That is, there exists a constant \( \kappa \) such that \( v(\pi) = \kappa \prod_{k=0}^{n-1} w(P_k) \). From (15), \( v(\lambda, n; a, b; \pi^0) = 1 \) for the empty reverse plane partition \( \pi^0 \in \text{RPP}(\lambda, n) \) whose parts are all 0. Thus \( \kappa^{-1} = \prod_{k=0}^{n-1} w(P_k^0) \) where \( (P_0^0, \ldots, P_{n-1}^0) \in \text{LP}(\mathbb{L}(\lambda), r, c, n) \) corresponds to \( \pi^0 \) by the bijection. We observe that \( P_k^0 \) goes from \( (r, 0) \) to \( (0, c) \) along the border of \( \mathbb{L}(\lambda) \), and \( P_0^0, \ldots, P_{n-1}^0 \) are copies of \( P_0^0 \). Especially \( w(P_k^0) = \prod_{i=1}^{r} a_k^{(r-i,c-\lambda)} \) and hence \( \kappa^{-1} \) is equal to the denominator of the right-hand side of (16).

The following is the main theorem of this paper.

Theorem 6. Let \( a_n^{(s,t)} \neq 0, b_n^{(s,t)} \neq 0 \) be a solution to the discrete 2D Toda molecule \( \mathfrak{3} \). Let \( \lambda \) be a partition with \( r \) rows and \( c \) columns, and let \( n \geq 0 \). Then
\[
\sum_{\pi \in \text{RPP}(\lambda, n)} v(\lambda, n; a, b; \pi) = \prod_{i=1}^{r} \prod_{k=1}^{n} a_k^{(r-i,c-\lambda)}. \quad (17)
\]

Proof. This theorem is a translation of Theorem \( \mathfrak{4} \) via the bijection with the help of (12) and Lemma 5.

Theorem 6 allows us to find a multiplicative partition function for reverse plane partitions of arbitrary shape with bounded parts from each non-vanishing solution to the discrete 2D Toda molecule \( \mathfrak{3} \).

5 An example

The discrete 2D Toda molecule \( \mathfrak{3} \) has the solution
\[
a_n^{(s,t)} = [p]^{s+n} [1 - a[p]_1^{*} [q]_1^{*}]^{s+n}, \quad (18a)
\]
\[
b_n^{(s,t)} = a[p]_1^{s+n-1} [q]_1^{*} (1 - [q]_t^{s+n}) \quad (18b)
\]
with the notation that \([z]^m_n = \prod_{\ell=m}^n z_\ell\) if \(m \leq n\), \([z]^n_n = 1\) if \(m = n + 1\) and \([z]^{n-1}_m = \prod_{\ell=m}^{n-1} z_\ell\) if \(m \geq n + 2\). The solution involves indeterminates \(a\) and \(p_\ell\), \(a\) for \(\ell \in \mathbb{Z}\) as parameters.

Let \(\lambda\) be a partition with \(r\) rows and \(c\) columns. Assume that

\[
a = [x]^{\lambda-n-r}_{c-x}, \quad p_i = [x]^{c-r+i\lambda_{i+1}}_{c-x}, \quad q_j = [x]^{\mu_j^i-j-r+c}_{m_{i+1}^j-j-r+c}. \tag{19}
\]

We then have a solution

\[
\phi_n^{(s,t)} = [x]^{c-r+s+n-\mu+n+1}_{c-r+s+1-\mu+1}(1 - \prod_{i=1}^{c-r+s+n-\mu+n+1} \mu_{i+1}^n - r - n - c), \tag{20a}
\]

\[
\psi_n^{(s,t)} = [x]^{c-r+s+n-1-\mu+n}(1 - \prod_{i=1}^{c-r+s+n-1-\mu+n} \mu_{i+1}^n - r - n + c). \tag{20b}
\]

Let \(n \geq 0\). The solution \((20)\) yields the weight of \((15)\) with \((14)\) given by

\[
v(\lambda, n; a, b; \pi) = \prod_{\ell=1}^{c-r} x_\ell^{tr(\pi)} \prod_{(i,j) \in \lambda} x_i^{1 - \prod_{k=1}^{j-i-1} [x]^{\lambda_{i-k}}_{j-k}} \prod_{(i,j) \in \lambda} x_j^{1 - \prod_{k=1}^{i-j-1} [x]^{\lambda_{j-k}}_{i-k}} \tag{21}
\]

As an instance of Theorem \(7\) we obtain the following multiplicative partition function for reverse plane partition.

**Theorem 7.** Let \(\lambda\) be a partition with \(r\) rows and \(c\) columns, and let \(n \geq 0\). Then

\[
\sum_{\pi \in RPP(\lambda, n)} v(\lambda, n; a, b; \pi) = \prod_{(i,j) \in \lambda} x_i^{1 - \prod_{k=1}^{j-i-1} [x]^{\lambda_{i-k}}_{j-k}} \prod_{(i,j) \in \lambda} x_j^{1 - \prod_{k=1}^{i-j-1} [x]^{\lambda_{j-k}}_{i-k}} \tag{22}
\]

where the weight \(v(\lambda, n; a, b; \pi)\) is given by \((21)\).

**Proof.** Substituting the solution \((20)\) for the right-hand side of \((17)\) we get

\[
\prod_{i=1}^{r} \prod_{k=1}^{n} 1 - [x]^{\lambda_{i-k+1}}_{k+1-\lambda_{k+1}} = \prod_{i=1}^{r} \prod_{j=1}^{n} 1 - [x]^{\lambda_{j-k}}_{j-k+1-\lambda_{j-k}} \tag{23a}
\]

\[
= \prod_{i=1}^{r} \prod_{j=1}^{n} 1 - [x]^{\lambda_{i-j}}_{i-j+1-\lambda_{i-j}}. \tag{23b}
\]

The last product is the same as the right-hand side of \((22)\).

The multiplicative partition function in Theorem \(7\) generalizes the multi-trace generating function \((2)\) by Gansner. Indeed \((22)\) reduces into \((2)\) as \(n \to \infty\) because \(\lim_{n \to \infty} [x]^{c-\text{const.}}_{c-\text{const.}} = 1\) as formal power series, \(\lim_{n \to \infty} \lambda_{r-n} = r\) and \(\lim_{n \to \infty} v(\lambda, n; a, b; \pi) = \prod_{\ell=1}^{c-r} x_\ell^{tr(\pi)}\).
Assuming \( x_\ell = q \) for \( \ell \in \mathbb{Z} \) we obtain the partition function

\[
\sum_{\pi \in \text{RPP}(\lambda, n)} v(\lambda, n; a, b; \pi) = \prod_{(i,j) \in \lambda} \frac{1 - q^{\lambda_i + \lambda'_j - i - j + n + 1}}{1 - q^{\lambda_i + \lambda'_j - i - j + 1}} \quad \text{with} \quad (24a)
\]

\[
v(\lambda, n; a, b; \pi) = q^{\mid \pi \mid} \prod_{(i,j) \in \lambda} \prod_{k=1}^{\pi_{i,j}} 1 - q^{n-i-k+1+\lambda_{n+j+k}} \prod_{(i,j) \in \lambda} 1 - q^{n-i-k+1+\lambda'_{n+j+k}} \quad (24b)
\]

from (21) and (22). If \( \lambda = (c^r) \), an \( r \times c \) rectangular shape, that becomes the triple product formula (1) by MacMahon. The partition function (24) is thus regarded as a generalization of (1) for reverse plane partitions of arbitrary shape.

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