A Bayesian analysis of inflationary primordial spectrum models using Planck data

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Abstract. The current available Cosmic Microwave Background (CMB) data show an anomalously low value of the CMB temperature fluctuations at large angular scales (ℓ < 40). This lack of power is not explained by the minimal ΛCDM model, and one of the possible mechanisms explored in the literature to address this problem is the presence of features in the primordial power spectrum (PPS) motivated by the early universe physics. In this paper, we analyse a set of cutoff inflationary PPS models using a Bayesian model comparison approach in light of the latest CMB data from the Planck Collaboration. Our results show that the standard power-law parameterisation is preferred over all models considered in the analysis, which motivates the search for alternative explanations for the observed lack of power in the CMB anisotropy spectrum.

Keywords: cosmological parameters from CMBR, inflation, physics of the early universe

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1 Introduction

The predictions of the minimal cosmological constant ($\Lambda$) + cold dark matter (CDM) model with a primordial power spectrum of the power-law type shows a good agreement with the current CMB observations [1, 2]. However, despite of its consistency with CMB observations there are some tensions which emerge when different data sets at intermediate scales ($z \lesssim 1$) are analysed in the context of this model. Some examples are the present-day value of the Hubble parameter, estimates of the power spectrum amplitude on scales of $8h^{-1}$ Mpc, and measurements of the matter density parameter (see, e.g., [3] for a general discussion and the references therein for details).

Another intriguing aspect of the current data are the features on the CMB temperature power spectrum which are not fully explained by the standard cosmological model, even if the $\Lambda$CDM model was found in accordance with the Planck data at $\sim 2\sigma$ confidence level [4]. In particular, the lack of angular power at large scale (see [5–8] for an exhaustive discussion) was firstly noticed by the Cosmic Background Explorer (COBE) [9] satellite and later confirmed by the Wilkinson Microwave Anisotropy Probe (WMAP) experiment [10] and by the Planck satellite [11]. Although the deviation from the $\Lambda$CDM best-fit prediction lies in the cosmic variance uncertainty, the possibility that it is due to a physical mechanism in the early universe cannot be excluded. Indeed, if one admits that CMB anisotropies are sourced by quantum fluctuations generated during inflation, thus this lack of power could be explained by some mechanisms, such as a negative running of the spectral index [12–16] or a feature at large wavelengths of the primordial power spectrum able to produce a depletion of power. Such features can be obtained, for example, from a brief violation of the slow-roll condition [17–19], or assuming an inflationary epoch preceded by matter or radiation domination [21].
Since features in the PPS has been the most common mechanism to address the problem of lack of power at low multipoles in the CMB anisotropy spectrum, in what follows we present a Bayesian model selection analysis of single-field inflationary models able to explain this problem, the so-called “cutoff PPS models”. Such models ranges from the simplest empirical models [24–28] to more complex ones, where the modulations in the PPS are obtained by a given physical mechanism [21, 23, 26, 27, 29, 30]. This is the case if one assumes an inflationary epoch preceded by matter or radiation domination [21, 29, 30], or that the onset of a slow-roll phase coincides with the time when the largest observable scales exited the Hubble radius during inflation [23], or even consider a fast rolling stage in the evolution of the inflaton field at the beginning of the inflationary phase [26, 27]. Our study differs from previous investigations (see, e.g., ref. [26]) in two aspects. First, we use the most recent data from the Planck Collaboration. Second, we perform an accurate Bayesian analysis of such PPS models in order to investigate their compatibility with the high accuracy of the Planck data. Differently from the statistical methods used in ref. [26], the Bayesian model comparison selects the best-fit model by achieving the best compromise between quality of fit and predictivity and by evaluating whether the extra complexity of a model is required by the data, preferring the model that describes the data well over a large fraction of their prior volume.

This paper is organised as follows. Section 2 briefly introduces the inflationary model and reviews the class of inflationary models considered in this work. In section 3 we discuss the observational data sets and priors used in the analysis as well as the Bayesian model selection method adopted. In section 4 we discuss the results and present a comparison with previous analysis. We end the paper by summarising the main results in section 5.

2 Inflationary scenarios

The standard inflationary dynamics is governed by the action,

$$S = \int d^4x \sqrt{-g} \left[ \frac{M^2_{Pl}}{2} R - \frac{1}{2} g_{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right],$$

where $R$ is the four-dimensional Ricci scalar derived using the metric $g_{\mu\nu}$ and $V(\phi)$ is the potential energy of the inflaton field. The dynamics of the inflation field is governed by the Friedman and Klein-Gordon equations

$$H^2 = \frac{1}{3M^2_{Pl}} \left( \frac{1}{2} \dot{\phi}^2 + V(\phi) \right),$$

$$\ddot{\phi} + 3H \dot{\phi} + V'(\phi) = 0,$$

where $H$ is the Hubble parameter and the derivatives with respect to the cosmic time and scalar field are denoted, respectively, by dots and primes. In the slow-roll regime inflation is realised by the single scalar field $\phi$ slowly rolling down its potential $V(\phi)$. It is characterised by the slow-roll parameters:

$$\epsilon \equiv -\frac{\dot{H}}{H^2} = \frac{M^2_{Pl}}{2} \left( \frac{V'(\phi)}{V(\phi)} \right)^2,$$

$$\eta \equiv \epsilon + \delta = M^2_{Pl} \left( \frac{V''(\phi)}{V(\phi)} \right),$$

$$\xi \equiv M^2_{Pl} \frac{V'(\phi)V'''(\phi)}{V(\phi)^2},$$
where $\delta = -\ddot{\phi}/H\dot{\phi}$. In terms of these parameters, inflation happens when $\epsilon \ll 1$ and lasts for a sufficiently long time for $\eta \ll 1$. One should note that the acceleration condition ($\epsilon \ll 1$) also implies that the comoving Hubble radius $(aH)^{-1}$ is a decreasing function of time.

In order to study the curvature perturbation $\mathcal{R}$ produced due to fluctuations in the scalar field $\phi$, one can use the Mukhanov-Sasaki equation [31]

$$u'' + \left(k^2 - \frac{z''}{z}\right)u_{k} = 0,$$

where $u = -z\mathcal{R}$ and $z \equiv a\dot{\phi}/H$. The solution of this equation can be obtained by considering a Bunch-Davies vacuum, in which all modes of cosmological interest are well inside the horizon at sufficiently early times $(k/aH \gg 1)$, such that [32]

$$u_{k}(\tau) \rightarrow \frac{1}{\sqrt{2k}}e^{-ik\tau}.$$  \hfill (2.8)

We can define the primordial power spectrum of curvature perturbations $P_{\mathcal{R}}(k)$ in terms of the vacuum expectation value of $\mathcal{R}$

$$< \mathcal{R}^{*}(k)\mathcal{R}(k') >= \frac{2\pi^2}{k^3}\delta^{3}(k-k')P_{\mathcal{R}}(k),$$

where $\delta$ is the Dirac delta function and the factor $2\pi^2/k^3$ is chosen to obey the usual Fourier conventions. On the other hand, $P_{\mathcal{R}}(k)$ is related to $u_{k}$ and $z$ via:

$$P_{\mathcal{R}}(k) = \frac{k^3}{2\pi^2} \left|\frac{u_{k}}{z}\right|^2.$$  \hfill (2.10)

The simplest shape of the primordial power spectrum in the standard $\Lambda$CDM cosmology is the power law parameterization, which can be obtained considering the slow-roll approximation of the single-inflaton field [12]:

$$P_{PL}(k) = A_{s} \left(\frac{k}{k_{0}}\right)^{n_{s}-1},$$

where $n_{s}$ is the spectral index, which is constant for power law models ($n_{s} = 1$ corresponds to a scale-invariant, Harrison-Zel’doovich-Peebles power spectrum), $A_{s}$ is the spectral amplitude and $k_{0}$ is the pivot scale set equal to 0.05 Mpc$^{-1}$. In terms of the slow-roll parameters we can evaluate the primordial power spectrum parameters as

$$A_{s} \simeq \frac{V(\phi)}{24\pi^2\epsilon M_{Pl}^2} \quad \text{and} \quad n_{s} \simeq 1 + 2\eta - 6\epsilon.$$  \hfill (2.12)

Similarly, inflation also predicts tensor perturbations (gravity waves) which produces a tensor spectrum $P_{t}(k)$ written as

$$P_{t}(k) = A_{t} \left(\frac{k}{k_{0}}\right)^{n_{t}},$$

where $A_{t}$ and $n_{t}$ are, respectively, the tensor amplitude and tensor spectral index. Again, in terms of the slow-roll parameters we can rewritten them as

$$A_{t} \simeq \frac{3V(\phi)}{2\pi^2 M_{Pl}^2} \quad \text{and} \quad n_{t} \simeq -\frac{r}{8},$$  \hfill (2.14)
where $r \equiv \mathcal{P}_t(k)/\mathcal{P}_R(k) \simeq 16\epsilon$ is the tensor-to-scalar ratio (the relative amplitude of the tensor to scalar modes). However, considering the recent BICEP2 results [33] we set $r = 0$ or $\mathcal{P}_t(k) = 0$.

In the next section, we describe the reference model used in our analysis, the power-law potential, and the inflationary primordial power spectra with infrared cutoff explored in this work. The latter use in general the additional cutoff parameter, $k_c$, which denotes the mode where the model diverges from the reference one. These cutoff potentials can be divided into two main categories, namely, the empirical parameterisations, able to produce the low power at high scales, and the physical motivated models, that can modulate the primordial potential with an observed feature. For illustration purposes, in figure 1 we show the features introduced in the primordial spectrum by such inflationary models using the best fit parameter values of table 2.

2.1 Power law (PL)

The Power Law potential is given by eq. (2.11) and can be considered part of the standard cosmology, as the scalar index $n_s$ and primordial amplitude $A_s$ are included in the minimal set of six cosmological parameters of the $\Lambda$CDM model. The most recent temperature data of the Planck Collaboration [1] exclude the exact scale invariance, $n_s = 1$, at more than $5\sigma$, constraining the spectral index to $n_s = 0.9655 \pm 0.0062$ and the primordial amplitude to $\ln(10^{10} A_s) = 3.089 \pm 0.036$ (we refer the reader to ref. [34] for a discussion on the $n_s = 1$ case). In our work we choose to use this PL parameterization (dashed black line in figure 1) as the reference model. It is worth noticing that all the models discussed in this paper can be written as modulation over the power law model, i.e.,

$$P(k) = \mathcal{P}_{PL}(k) \times F(k, \Theta)$$  \hspace{1cm} (2.15)

where $F(k, \Theta)$ is the modulation part and $\Theta$ is a vector which characterizes the extra parameters.
2.2 Running spectral index (RN)

Possibly, the slightest deviation from the PL power spectrum is obtained by considering the
dependence of the spectral index with the scale through the parameter $\alpha_s = dn_s/d \ln k$. The
“running of the spectral index” [12–16] is the second order deviation from the scale invariance,
which is the one extra parameter considered in this model, and can be expressed in terms of
slow roll parameters as

$$\alpha_s \simeq -2\xi + 16\epsilon\eta - 24\epsilon^2.$$  \hspace{1cm} (2.16)

Although the variation of the spectral index is expected to be small (of the order of $10^{-3}$ in
the slow-roll approximation), this correction leads to a suppression of power at large scale
in the power spectrum, as showed by the solid red line (RN) in figure 1. The running
parameter is constraint to $\alpha_s = -0.0084 \pm 0.0082$ at 68% CL by the Planck Collaboration
using temperature data (Planck TT (2015)) [1], and the joint constraint including high-$\ell$
polarization data is $\alpha_s = -0.0057 \pm 0.0071$ at 68% CL. Even with the low statistical
significance with which is presently measured its behaviour could point to a deviation from
the scale-invariant power law model. It is worth mentioning that even though a sizable
value of $\alpha_s$ can violate the slow roll approximation, there are models in which the running
parameter can be large while still respecting the slow-roll approximation [13–15, 35–37].

In the present work we use the standard parameterisation for the PL model with running
of the spectral index:

$$\ln P(k) = \ln A_s + (n_s - 1) \ln \left(\frac{k}{k_0}\right) + \frac{\alpha_s}{2} \ln^2 \left(\frac{k}{k_0}\right),$$  \hspace{1cm} (2.17)

that using the formalism of eq. (2.15) corresponds to $F(k, \Theta) \equiv F(k, \alpha_s) = \left(\frac{k}{k_0}\right)^{\alpha_s}$.

2.3 Sharp cutoff (SC)

The simplest empirical model able to describe the observations at large scales is given by the
functional form:

$$P(k) = \begin{cases} 
A_s \left(\frac{k}{k_c}\right)^{n_s-1}, & \text{for } k > k_c \\
0, & \text{otherwise}
\end{cases}$$

where $F(k, \Theta)$ is reduced to $F(\Theta) \equiv F(k_c) = (k_0/k_c)^{n_s-1}$ for $k > k_c$ and zero otherwise,
being the extra parameter $k_c$ the scale at which the power drops to zero. This amounts
to saying that the constraints on the cosmological parameters remain unchanged only for
small values of $k_c$. Indeed, this cutoff parameter states the scale at which the primordial
potential departs from the power law $P_{PL}(k)$: for small values of $k_c$ only the low-
multipoles are affected, since for high $k_c$ also small scales (higher multipoles) would be modified. Notice
that we are not concerned about the form of the spectrum near the cutoff, instead we are
here interested in parameterizing the $k_c$ scale. Previous work have considered this model and
found constraints on the cutoff scale by using WMAP 9 year + Planck TT (2013) data [20].

2.4 Exponential cutoff (EC)

Another phenomenological parameterization of cutoff can be expressed in the simple exponential
form [27, 28]:

$$P(k) = P_{PL}(k) \left[1 - e^{-(k/k_c)^\gamma}\right],$$  \hspace{1cm} (2.18)
where \( F(k, \Theta) \equiv F(k, k_c, \alpha) = \left[ 1 - e^{-\left(\frac{k}{k_c}\right)^\alpha} \right] \). This model introduces two extra parameters: the first one, \( k_c \), is the scale of the cutoff and \( \alpha \) is a measure of its steepness. We can see its behavior as the yellow curve in figure 1 (EC). Like the previous one, it also recovers the simple power law form at small angular scales, such that the constraints on the cosmological parameters are not affected for small values of \( k_c \). This model was already constrained in literature by using Planck TT (2015) data [1, 20].

2.5 Pre-inflationary radiation domination (PIR)

This model has been proposed in the context of spontaneous symmetry breaking phase transitions [21, 38], arising in gauge theories of elementary-particle interactions [39]. It considers a Universe containing two components during the phase transition, namely, the radiation and the vacuum energy, and assumes that the pre-inflationary Universe was in a radiation-dominated phase which eventually evolved to a vacuum-energy-dominated (or de Sitter) phase [21]. As shown in [21, 29, 30], this can lead to modulations in the PPS, such as an infrared cutoff with a “bump”.

We consider the following functional form [23]:

\[
P(k) = A_s k^{1-n_s} \left( \frac{k_0}{k} \right)^{n_s-1} \left| e^{-2iy(1+2iy)} - 1 - 2y^2 \right|^2,
\]

where \( y = k/k_c \) and the modulation part \( F(k, \Theta) \) is given by:

\[
F(k, \Theta) \equiv F(k, k_c) = \left( \frac{k_0}{k^2} \right)^{n_s-1} \left| e^{-2iy(1+2iy)} - 1 - 2y^2 \right|^2.
\]

The extra parameter \( k_c \) is the cutoff scale set by the Hubble parameter at the onset of inflation and the current horizon crosses the Hubble radius around the onset of inflation. The behaviour of this potential is shown by the magenta line (PIR) in figure 1. A recent work have constrained the cutoff scale using WMAP 9 year + Planck TT (2013) data [26].

2.6 Pre-inflationary kinetic domination (PIK)

A power spectrum amplitude suppression is also obtained assuming an inflationary stage where the velocity of the scalar field is not negligible, without necessarily meaning the interruption of inflation [27]. However, in order to affect the low-\( \ell \) multipoles, this stage should occur very close to the beginning of the inflation, e.g., assuming a pre-inflationary phase with domination of the kinetic term. Thus, the difference of the vacuum in the inflationary kinetic domination phase (relative to the fast-rolling inflationary phase) would imprint a feature in the power spectrum at large scales, corresponding to first modes that crossed out of the Hubble radius at the onset of inflation (see [23] and references therein).

The PIK model is given by [23, 26, 27]

\[
P(k) = \frac{H_{\text{inf}}^2}{2\pi^2} k |A - B|^2,
\]

with

\[
A = \frac{e^{-ik/H_{\text{inf}}}}{\sqrt{32H_{\text{inf}}^3/\pi}} \left[ \mathcal{H}_0^{(2)} \left( \frac{k}{2H_{\text{inf}}} \right) - \left( \frac{H_{\text{inf}}}{k} + i \right) \mathcal{H}_1^{(2)} \left( \frac{k}{2H_{\text{inf}}} \right) \right],
\]

\[
B = \frac{e^{ik/H_{\text{inf}}}}{\sqrt{32H_{\text{inf}}^3/\pi}} \left[ \mathcal{H}_0^{(2)} \left( \frac{k}{2H_{\text{inf}}} \right) - \left( \frac{H_{\text{inf}}}{k} - i \right) \mathcal{H}_1^{(2)} \left( \frac{k}{2H_{\text{inf}}} \right) \right],
\]
where $H_{\text{inf}}$ characterises the Hubble parameter during inflation. $\mathcal{H}_0^{(2)}$ and $\mathcal{H}_1^{(2)}$ stand for the Hankel function of the second kind with order 0 and 1, respectively.

Under the assumption that the standard inflationary conditions of quantum fluctuations originate in the Bunch-Davies vacuum [26, 29], the primordial power spectrum can be rewritten as:

$$P(k) = A_s^* \left( \frac{k}{k_0} \right)^{n_s-1} \frac{H_{\text{inf}}^2 k |A - B|^2}{2\pi^2},$$

with

$$A_s^* = \frac{2\pi^2}{H_{\text{inf}}^2 k_0 |A(k_0) - B(k_0)|^2}.$$

In this case, the modulation part can be written as:

$$F(k, \Theta) \equiv F(k, H_{\text{inf}}) = \left( \frac{k}{k_0} \right) \frac{|A - B|^2}{|A(k_0) - B(k_0)|^2},$$

and for our analysis purpose, we will treat the $H_{\text{inf}}$ as the extra free parameter, as done in ref. [26]. This potential is shown in figure 1 with light blue line (PIK). The parameter $H_{\text{inf}}$ was already constrained in a previous work using a joint WMAP 9 year + Planck TT (2013) analysis [26].

### 2.7 Starobinsky (SB)

This model was proposed by Starobinsky [40] and assumes that the potential of the effective scalar field, which controls the inflationary phase, has a singularity in the form of a sharp change in its slope. This feature would be able to produce an infrared cutoff followed by the bump that arises naturally as the first peak of a damped ringing [23]. One can choose the value of the scalar field where the slope changes abruptly to be $\phi_0$, and let the slope of the potential above and below $\phi_0$ be $A_+$ and $A_-$, respectively, in such way that the general form of the scalar field potential can be expressed as:

$$V(\phi) = \begin{cases} V_0 + A_+ (\phi - \phi_0), & \text{for } \phi > \phi_0 \\ V_0 + A_- (\phi - \phi_0), & \text{for } \phi < \phi_0 \end{cases},$$

where $V_0$ is the value of the potential at $\phi = \phi_0$, $A_+$ and $A_-$ are model parameters greater than 0.

The resulting power spectrum for this model is [40]

$$P(k) = P_{PL}(k) D^2(y, \Delta),$$

with the modulation part being $F(k, \Theta) \equiv F(k, k_c, \Delta) = D^2(y, \Delta)$, where we consider $y = k/k_c$ and $\Delta = \frac{A_+ - A_-}{A_+}$ as the two extra parameters of this model. The transfer function $D^2(y, \Delta)$ is responsible for making the underlying power spectrum non-flat around the point $k_c$ [40–42], when $\Delta \phi \approx (\phi - \phi_0)$ is small,

$$D^2(y, \Delta) = \left[ 1 + \frac{9\Delta^2}{2} \left( \frac{1}{y} + \frac{1}{y^2} \right)^2 + \frac{3\Delta}{2} \left( 4 + 3\Delta - \frac{3\Delta}{y^2} \right) \frac{1}{y^2} \cos 2y + \right.$$  
$$\left. + 3\Delta \left( 1 - (1 + 3\Delta) \frac{1}{y^2} - \frac{3\Delta}{y^4} \right) \frac{1}{y} \sin 2y \right].$$

(2.26)
We stress that \( k_c \) denotes the location of the step but without effect on the shape of the spectrum. It is important to notice that the power spectrum \( P(k) \) has a sharp decrease followed by a bump at small \( k \) with large oscillations and a flat upper plateau on small scales, for \( R = A_+/A_- < 1 \) (see [40, 42] for more details). For \( R = A_+/A_- > 1 \) the power spectrum \( P(k) \) has a step-down like feature (toward large \( k \)).

The behavior of this potential is shown with blue line (SB) in figure 1. This model was also previous constrained in literature using the Planck TT (2015) data [20].

2.8 Starobinsky cutoff (SBC)

The last model we consider is the Starobinsky exponential cutoff [23, 26]:

\[
P(k) = P_{PL}(k) \left[ 1 - e^{-(ck/k_c)^\alpha} \right] D^2(y, \Delta), \tag{2.27}
\]

where \( D^2(y, \Delta) \) is the transfer function of the Starobinsky feature described in section 2.7 (SB model), \( \epsilon \) sets the ratio of the two cutoff scales involved and \( \alpha \) is a measure of the steepness of the cutoff. In this model, the modulation part englobes the transfer function multiplied by the exponential contribution,

\[
F(k, k_c, \alpha, \Delta) = D^2(y, \Delta) \left[ 1 - e^{-(ck/k_c)^\alpha} \right]. \tag{2.28}
\]

We follow [26] and fix \( \epsilon = 1 \), which reduces the number of degrees of freedom and the degeneracy problem without affecting the final results. The extra parameters are the same of the SB model plus the steepness parameter \( \alpha \). Its behavior is shown by the solid black line (SBC) in figure 1. We notice that this model showed the best agreement with previous CMB data when compared with other cutoff potentials [23, 26]. It is worth pointing out that a previous work using a joint WMAP 9 + Planck (2013) analysis have constrained the extra parameters \( k_c, \alpha \) and \( \Delta \) [26].

3 Methodology

We adopt a Bayesian approach to model selection, since we are interested in knowing whether the latest cosmological data support the inclusion of extra parameters to explain the features in the primordial power spectrum. In our analysis, we choose to use the CMB data set from the latest release of the Planck Collaboration [11], considering the high-\( l \) Planck temperature data from the 100-,143-, and 217-GHz full-mission T maps in the range of 30 < \( l < 2508 \), and the low-\( P \) data in the range of 2 < \( l < 29 \) by the joint TT,EE,BB and TE likelihood.

The minimal ΛCDM model with the PL primordial power spectrum is assumed as reference and is parameterized with the usual set of cosmological parameters: the baryon density, \( \Omega_b h^2 \), the cold dark matter density, \( \Omega_c h^2 \), the ratio between the sound horizon and the angular diameter distance at decoupling, \( \theta \), the optical depth, \( \tau \), the primordial scalar amplitude, \( A_s \), and the primordial spectral index \( n_s \). We also choose to work with flat priors on these parameters, as listed in the first six lines of table 1. We consider purely adiabatic initial conditions and fix the sum of neutrino masses to 0.06 eV, setting the pivot scale at \( k_* = 0.05 \) Mpc\(^{-1} \). In addition to the parameters above we also vary the nuisance foregrounds parameters [11].

The key quantity for Bayesian model comparison is the Bayesian evidence, or marginal likelihood, and is calculated here by implementing the nested sampling algorithm of MultiNest [43, 44] in the current release of the package COSMO MC [46]. In order to computing
| Parameter Name                        | Symbol | Prior Ranges |
|--------------------------------------|--------|--------------|
| Baryon Density                       | $\Omega_b h^2$ | [0.005 : 0.1] |
| Cold Dark Matter Density             | $\Omega_c h^2$ | [0.001 : 0.99] |
| Angular size of Acoustic Horizon     | $\theta$ | [0.5 : 10.0] |
| Optical Depth                        | $\tau$ | [0.01 : 0.8] |
| Scalar Spectral Index                | $n_s$ | [0.8 : 1.2] |
| Scalar Amplitude                     | $\log 10^{10} A_s \,^1$ | [2.0 : 4.0] |
| Hubble Parameter at Inflation        | $H_{\text{inf}}$ (Mpc$^{-1}$) | $[10^{-7} : 10^{-2}]$ |
| Running Index                        | $\alpha_s$ | $[-1.0 : 1.0]$ |
| Cutoff Parameter                     | $k_c$ (Mpc$^{-1}$) | [0.0 : 0.01] |
| Cutoff Steepness Parameter           | $\alpha$ | [1.0 : 15.0] |
| Starobinsky Parameter                | $\Delta$ | [0.0 : 1.0] |

Table 1. Priors on the model parameters.

| Parameter | Mean | Best-fit |
|-----------|------|----------|
| $\alpha_s$ | $-0.007 \pm 0.007$ | $-0.007$ |
| $10^4 k_c$ | $2.478 \pm 0.894$ | $3.035$ |
| $H_{\text{inf}}$ | $3.476 \pm 0.343$ | $8.175$ |
| $\Delta$ | $0.0859 \pm 0.0856$ | $0.0841$ |

Table 2. 68% confidence limits for the primordial parameters using Planck TT+lowP data. The $\Delta \chi^2_{\text{best}}$ and the $\ln B_{ij}$ refers to the difference with respect to the minimal $\Lambda$CDM model, i.e. $\Delta \chi^2_{\text{best}} = \chi^2_{\Lambda\text{CDM}} - \chi^2_{\text{model}}$.

the angular power spectrum of CMB anisotropies for each model considered here, we modify the CAMB code [47], included in CosmoMC. Finally, we use the Bound Optimization BY Quadratic Approximation (BOBYQA) algorithm [48] through the Powell’s routines as implemented in CosmoMC, to obtain the results for the best-fit values of the parameters and to maximize the likelihood itself.

In dealing with model comparison, the evidence is the prime tool to evaluation of a model’s performance in the light of the data [49]. This quantity is based on Bayes’ theorem, given by:

$$p(\theta \mid d, M) = \frac{p(d \mid \theta, M)\pi(\theta \mid M)}{p(d \mid M)}, \quad (3.1)$$

which relates the posterior probability for the parameters $\theta$ given the data $d$ under a model $M$,

$^{1}k_0 = 0.05 \text{ Mpc}^{-1}$. 

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\( p(\theta \mid d, M) \), with the likelihood, \( p(d \mid \theta, M) \), and the prior probability distribution function (which encodes our state of knowledge before seeing the data), \( \pi(\theta \mid M) \). The evidence, the denominator of the Bayes’ theorem, is given by

\[
\begin{align*}
p(d \mid M) &= \int_{\Omega} p(d \mid \theta, M) \pi(\theta \mid M) d\theta,
\end{align*}
\]

where \( \Omega \) extends over the region(s) of parameter space contained within the iso-likelihood contour [44]. We use the evidence to discriminate two competing models by taking the ratio

\[
B_{ij} = \frac{p(d \mid M_i)}{p(d \mid M_j)},
\]

that is the Bayes factor of the model \( i \) relative to the model \( j \) (that hereafter we assume to be the \( \Lambda \)CDM one). It is worth pointing out that the evidence rewards models that achieve the best compromise between quality of fit and complexity [50]. Indeed, models with a large number of free parameters, not required by the data, are penalised for the wasted parameter space (see refs. [51–59] for recent work with Bayesian model selection in cosmology). The most usual way to rank the models of interest is adopting a scale to interpret the values of \( \ln B_{ij} \) in terms of the strength of the evidence of a chosen reference model \( M_j \): \( \ln B_{ij} = 0 - 1 \), \( \ln B_{ij} = 1 - 2.5 \), \( \ln B_{ij} = 2.5 - 5 \) and \( \ln B_{ij} > 5 \) indicate, respectively, an inconclusive, weak, moderate and strong preference of the model \( M_i \) relative to the model \( M_j \) (negative values stands for the opposite, which means, an inconclusive, weak, moderate and strong preference of the model \( M_j \) with respect to the model \( M_i \)). We refer the reader to ref. [49] for a more complete discussion about this scale, which is a revisited and more conservative version of the Jeffreys’ scale [60]. Finally, we choose to use the most accurate Importance Nested Sampling (INS) [45, 61] to calculate the Bayesian evidence, requiring a INS Global Log-Evidence error of \( \leq 0.02. \)

4 Results

The main results of our analysis are showed in tables 3 and 2. There, we list the constraints on the usual cosmological parameters and on the extra primordial parameters for each model considered in this work. In the last lines of the table 2, we present the values of \( \Delta \chi^2_{\text{best}} \) and \( \ln B_{ij} \) (Bayes factor), which are obtained considering the \( \Lambda \)CDM cosmology as the reference model. Figure 2 (left) shows the 2D contours and the marginal one-dimensional posterior distributions for the parameters describing the RN model at 68% and 95% confidence level. For brevity, we do not show similar plots for the other models discussed in section 2. Additionally, we show in figure 2 (right) the angular power spectra for all models we have examined, which are obtained using the best fit values of table 2.

We mainly compare our results to the recent work of Iqbal et al. (2015) [26] (IQ15) where a Markov-Chain Monte Carlo analysis was performed using the WMAP-9yr data [62] jointly with the first data release of the Planck collaboration (2013) [63]. It is worth emphasizing that both the Akaike Information Criteria (AIC) and Bayesian Information Criteria (BIC) adopted by IQ15 differ from the Bayesian model selection approach discussed in section 3, as they consider only the point that maximizes the posterior probability distribution to compare the models, taking into account both the number of data points and the number of extra parameters of the models under consideration.
Table 3. 68% confidence limits for the cosmological parameters using Planck TT+lowP data.

| Model | $100\Omega_b h^2$ | $\Omega_c h^2$ | $\theta$ | $\tau$ | $\ln 10^{10} A_s$ | $n_s$ | $H_0$ |
|-------|-----------------|----------------|----------|--------|-----------------|------|-------|
| AC1DM | 2.22 ± 0.022    | 0.1197 ± 0.0021 | 1.04085 ± 0.00045 | 0.077 ± 0.018 | 3.088 ± 0.034 | 0.9654 ± 0.0059 | 67.32 ± 0.95 |
| RN    | 2.237 ± 0.026   | 0.1196 ± 0.0021 | 1.04093 ± 0.00047 | 0.088 ± 0.021 | 3.112 ± 0.041 | 0.9652 ± 0.0062 | 67.51 ± 0.97 |
| SC    | 2.221 ± 0.023   | 0.1197 ± 0.0022 | 1.04086 ± 0.00048 | 0.082 ± 0.019 | 3.284 ± 0.039 | 0.9652 ± 0.0061 | 67.32 ± 0.97 |
| EC    | 2.224 ± 0.024   | 0.1197 ± 0.0022 | 1.04086 ± 0.00048 | 0.085 ± 0.022 | 3.104 ± 0.042 | 0.9651 ± 0.0066 | 67.34 ± 1.01 |
| PIR   | 2.222 ± 0.023   | 0.1197 ± 0.0021 | 1.04086 ± 0.00046 | 0.076 ± 0.019 | 2.983 ± 0.046 | 1.035 ± 0.0061 | 67.33 ± 0.94 |
| PIK   | 2.225 ± 0.024   | 0.1194 ± 0.0023 | 1.04089 ± 0.00049 | 0.084 ± 0.021 | 3.100 ± 0.040 | 0.9662 ± 0.0064 | 67.45 ± 1.02 |
| SB    | 2.222 ± 0.023   | 0.120 ± 0.0022  | 1.04084 ± 0.00047 | 0.085 ± 0.020 | 3.104 ± 0.039 | 0.9641 ± 0.0065 | 67.18 ± 0.99 |
| SBC   | 2.224 ± 0.024   | 0.120 ± 0.0022  | 1.04089 ± 0.00048 | 0.084 ± 0.021 | 3.100 ± 0.040 | 0.9659 ± 0.0061 | 67.44 ± 1.00 |

Figure 2. Left: two-dimensional probability distribution and one-dimensional probability distribution for the parameters $A_s$, $n_s$, and $\alpha_s$ of the primordial power spectrum for the model RN. Right: the best-fit angular power spectra for all models considered in the analysis. The data points correspond to the latest release of Planck data [2] and the lower panel show the residuals with respect to the reference model ($\Lambda$CDM).

In what concerns the constraints on the cosmological parameters, we note that the introduction of primordial features does not produce significant changes. The only exception is for the PIR model, whose spectral index $n_s$ mean value grows up until a red spectral tilt ($n_s > 1$). Our results for the RN model are fully consistent with that found by IQ15 and the most recent Plank Collaboration analysis [1], i.e., confirming that the zero value for the running of the spectral index $\alpha_s$ is off by 68% (C.L.). Even with a better $\chi^2$ with respect to the PL model, the deviation from the standard cosmological model is too low to be supported from the data and we find that the model is *moderately* discarded, also confirming the recent results of Heavens et al. [59].

For the empirical parameterization SC, the constraints for the cutoff scale $k_c$ show good agreement with those found by IQ15. However, the Bayesian model comparison performed here shows that the SC model is *strongly* disfavored with respect to the PL model, while the AIC value found by IQ15 for this model indicates that it is as good as the PL model.

$^2k_0 = 0.05\ Mpc^{-1}.$
The high point of the EC and SBC models is that the posterior probability density distribution for the $\alpha$ parameter never goes to zero, i.e. it doesn’t have a closed prior volume. This implies that we can’t calculate a proper Bayesian evidence and, at the best, we can only calculate the BIC or AIC value for these models in order to compare them with others works. This would continue to happen even if we increase the prior range for $\alpha$, since the observable predictions do not change for higher value of such a parameter, i.e. the probability density distribution remains unchanged. Hereupon, we compute the BIC value given by

$$BIC = -2\ln \mathcal{L}(d|\theta) + k \ln N,$$

where the number of data points $N$ is 2499 (Planck CamSpec+commander $\ell$’s) and the number of extra parameters are $k = 2$ and $k = 3$, for EC and SBC models, respectively. In this way, the $\Delta BIC \equiv BIC_i - BIC_{\Lambda CDM}$ represents the preference of the reference model over model $i$, and $\Delta BIC \leq 2$, $2 < \Delta BIC \leq 6$, $6 < \Delta BIC \leq 10$ and $\Delta BIC \geq 10$ means weak, positive, strong and very strong support for the reference model, respectively [65]. We compare EC and $\Lambda CDM$ models and find that $\Delta BIC = 17.01$, which means that the PL model is strongly favoured over the EC model, which is in accordance with what IQ15 found. At the same time, SBC model shows $\Delta BIC = 23.68$ that is found strongly disfavoured with respect to the standard model.

In what concerns the physical motivated models, the cutoff scale of PIR parameterization is in concordance with the constraints of IQ15. As we can see in the right panel figure 2 (magenta line) its prediction is very close to the $\Lambda CDM$ curve ($\Delta \chi^2_{\text{best}} \sim 0$). For the PIK model, we find a value of $k_c$ bigger than was found by IQ15. This implies an extended oscillation (until $\ell \sim 20$) and a lower power at large scales. The Bayes factor shows that these models are strongly disfavoured compared to the PL model, which is in agreement with the IQ15 results. For the SB model, we find that, although our constraints for the primordial parameters ($k_c$ and $\Delta$) are fully consistent with those found by IQ15, the result of the model selection disagree. The results of IQ15 (AIC) favour such model with respect to the reference model while our Bayesian analysis indicate that it is strongly disfavoured.

5 Conclusion

An intriguing aspect of the current available CMB data is the anomalously low value of the CMB temperature fluctuations up to multipole $\ell < 40$. In the present literature, the most common mechanisms used to address this problem is to consider features in the primordial power spectrum motivated by the early universe physics.

In this work we have used the most recent data release of the Planck Collaboration and performed a Bayesian model selection analysis to test the observational viability of a set of PPS models. The constraints on cosmological parameters of our results were found in agreement with previous analysis [1, 20, 26, 59] while bounds on the extra PPS parameters were discussed in the Results section. Using a revised version of the Jeffreys’ scale given in ref. [49], our results indicate a moderate evidence in favor of the power-law model, adopted in the standard $\Lambda CDM$ cosmology, with respect to the Running spectral index model (RN), which confirms the recent results of ref. [59]. Moreover, in accordance with BIC analysis obtained in ref. [26], we have found that the standard power-law parameterisation is always

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3See, for example, the recent work [64], which study the cases when improper priors are combined with a likelihood function resulting in an improper posterior distribution that is not finitely integrable, leading to a resulting inference that may or may not be based on a probability distribution.
favoured with strong evidence when compared to all the other PPS models considered in our analysis (see table 2). We could not find the evidence for the EC and SBC models since it is not possible to have a closed prior volume. However the $\Delta BIC$ values we found are in accordance with IQ15, pointing that the PL model is strongly supported over these models. We stress that these are the first analyses in literature that consider a Bayesian model comparison for almost all the selected models in this paper. We believe that the results of the present analysis rule out these models as a possible explanation for the lack of power observed at large angular scales of the CMB power spectrum and motivate the search for alternative solutions of this open problem.

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