The inverse problem of an impenetrable sound-hard body in acoustic scattering

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ABSTRACT
We study the inverse problem of recovering the scatterer shape from the far-field pattern (FFP) of the scattered wave in presence of noise. This problem is ill-posed and is usually addressed via regularization. Instead, a direct approach to denoise the FFP using wavelet technique is proposed by us. We are interested in methods that deal with the scatterer of the general shape which may be described by a finite number of parameters. To study the effectiveness of the technique we concentrate on simple bodies such as ellipses, where the analytic solution to the forward scattering problem is known. The shape parameters are found based on a least-square error estimator. Two cases with the FFP corrupted by Gaussian noise and/or computational error from a finite element method are considered. We also consider the case where only partial data is known in the far field.

1. Introduction
This paper deals with the inverse problem of recovering the scatterer shape from the far-field pattern of the scattered wave. Inverse problems of this type occur in various application such as remote sensing, ultrasound tomography and seismic imaging. They are difficult to solve since they are ill-posed and nonlinear. The ill-posedness is usually addressed via regularization. Most of the reconstruction methods, such as linear sampling, factorization method [3], incorporate different types of regularization. For example, in [4] the regularization consists of using a level set representation of the obstacle shape while in [5] Newton iterations with a Tikhonov regularization is applied.

Instead, we propose to directly smooth the far field pattern before the reconstruction. After smoothing, any known traditional reconstruction methods can be used with or without additional regularization. We use the wavelet denoising technique to efficiently remove noise from the data. Our technique is less data demanding than the usual regularization methods and may be interpreted as an implicit regularization imposed on the far field pattern rather than on the solution. The efficiency of the proposed technique is demonstrated on the problem of recovering the semi-axis of ellipse from the far-field pattern.

2. General description of the problem
The problem is formulated as recovering a body shape \( \Gamma \) from the measured far-field pattern. Usually, the body shape \( \Gamma \) is parameterized by a few parameters that should be recovered. The noise source may be either experimental or computational. The problem is complicated by the fact that in practice only a fraction of the data can be obtained. The given far-field pattern
is only measured at discrete angles and one cannot usually obtain the data in a complete circle/sphere in the far field but only in some portion of the farfield.

We consider the problem for the total wave $u = u^{inc} + u^s$ in terms of the incoming wave, $u^{inc} = e^{ikx\cdot d}$, $|d| = 1$, where $d$ is the direction of the incident plane wave. We solve the reduced wave equation in 2 or 3 dimensions exterior to a given body, so

$$\Delta^2 u + k^2 u = 0 \text{ in } \mathbb{R}^n \setminus \Omega,$$

where $\Omega$ is the scattering object.

Along the boundary one can impose either a Dirichlet condition corresponding to a sound-soft body or a Neumann condition corresponding to a sound-hard body. In this study we consider an impenetrable sound-hard body and so $\frac{\partial u}{\partial n} = 0$ on $\Gamma$ the boundary of the obstacle. The far field asymptotic solution is given by

$$u_s(x) = \frac{e^{ikr}}{r^{n-1}} \left[ u_\infty \left( \frac{x}{r} \right) + O \left( \frac{1}{r} \right) \right] \text{ as } r \to \infty, u_\infty \left( \frac{x}{r} \right)$$

is the far field pattern (FFP). For the forward problem to be well posed we impose the Sommerfeld radiation condition in the farfield. In $n$ dimensions this is given by:

$$r^{n-1} \left( \frac{\partial u_s}{\partial r} - iku_s \right) \to 0 \text{ as } r \to \infty$$

The inverse problem is to reconstruct $\Gamma$ given the FFP. We express this as $F(\Gamma) \left( \frac{x}{r} \right) = u_\infty \left( \frac{x}{r} \right)$. The operator $F$ assigns to every suitable boundary $\Gamma$ the corresponding far field pattern. The inverse problem is not a well-posed problem, i.e. it is well known that adding a small perturbation to the far-field pattern can cause an exponentially large change to the shape of the scatterer [2]. Furthermore, since the scatterer is described by finite number of parameters, in general there will be no solution that exactly matches the given far-field pattern. Hence, we shall only consider a least-squares solution to the problem. If $\tilde{u}_\infty$ is a perturbation of the exact far-field pattern then we shall attempt to find $\Gamma$ which minimizes

$$\| u_\infty - \tilde{u}_\infty \|^2$$

The conjugate gradient method (CG) is used to minimize $\| u_\infty - \tilde{u}_\infty \|^2$ [10].

We shall consider two cases. The first one is a model based on the exact solution of the Helmholtz equation for the ellipse [6]-[9]. The second one uses a numerical solution of the Helmholtz equation for the ellipse based on the finite element method [11], [12] with additional Gaussian noise. The noisy far field pattern is filtered using the wavelet denoising technique [14]-[18].

3. Model based on the exact solution of the Helmholtz equation about an ellipse

We present the exact solution in the elliptical coordinates:

$$x = a \cos \theta = f \cosh \xi \cos \theta$$
$$y = b \sin \theta = f \sinh \xi \sin \theta$$
$$f = \sqrt{a^2 - b^2}, \cosh \xi_1 = \frac{a}{f}$$

and is given by [6]:

$$P = -2\pi \sum_{m=0}^\infty (-1)^m \frac{1}{N_m^{(o)}} \frac{R_m^{(1)}(c, \xi_1)}{R_m^{(3)}(c, \xi_1)} S_\infty(c, \cos \varphi_0) S_\infty(c, \cos \theta)$$

$$- 2\pi \sum_{m=0}^\infty (-1)^m \frac{1}{N_m^{(o)}} \frac{R_m^{(1)}(c, \xi_1)}{R_m^{(3)}(c, \xi_1)} S_\infty(c, \cos \varphi_0) S_\infty(c, \cos \theta),$$

(2)
The notation in solution (2) is due to Stratton[7] and Bowman et al.[6]. The Stratton normalization that is different from the modern notation for Mathieu functions by M. Abramowitz and I. Stegun [1] and S. Zhang [8]. The relation between the recent and modern notation is given by:

\[
Re_m(j) = \sqrt{\pi 2 M^{(1)}} \left( \frac{(kf)^2}{4}, \xi \right)
\]

\[
Ro_m(j) = \sqrt{\pi 2 M^{(1)}} \left( \frac{(kf)^2}{4}, \xi \right)
\]

\[
\frac{1}{\sqrt{N_m^{(e)}}} Se_m(kf, \cos \theta) = \frac{1}{\sqrt{\pi}} ce_m \left( \frac{(kf)^2}{4}, \theta \right)
\]

\[
\frac{1}{\sqrt{N_m^{(o)}}} So_m(kf, \cos \theta) = \frac{1}{\sqrt{\pi}} co_m \left( \frac{(kf)^2}{4}, \theta \right)
\]

where \( c = kf = k\sqrt{a^2 - b^2} \), \( \xi = \ln \left( \frac{a}{b} + \sqrt{\left( \frac{a}{b} \right)^2 - 1} \right) \) and \( a, b \) are major and minor axes of the obstacle ellipse. \( Re_m(j), Ro_m(j), j = 1, 2, 3 \) are respectively the even and odd Mathieu radial function while \( Se_m, So_m \) are the even and odd angular Mathieu functions. The quantities \( N^{(e)}_m, N^{(o)}_m \) are normalization coefficients defined by Stratton [7]. In our simulations we use the software developed by D. Erricolo [9] that uses normalization of Ince. 1

4. Exact Theoretical Solution
The first case is without noise in the far field pattern. We seek for the ellipse \( \xi = \xi_1 \), i.e. its semi-axes \( a \) and \( b \), that minimize the least-square error between the resultant discrete FFP and the original given FFP:

\[
F(a, b) = \sum_{n=0}^{m} |P(c, \xi, \theta_n) - P(c, \xi_1, \theta_n)|^2
\]  

The conjugate gradient method (CG) [10] is used to minimize \( F(a, b) \). The gradient of \( F(c, \xi) \) is approximated using the numerical formulae:

\[
F'_c(c, \xi) = \frac{F(c + h, \xi) - F(c - h, \xi)}{2h}
\]

\[
F'_\xi(c, \xi) = \frac{F(c, \xi + h) - F(c, \xi - h)}{2h}
\]

5. Numerical solution based on the finite element method
This section presents a solution of the exterior Helmholtz equation using a finite element method (FEM) with linear elements. An absorbing boundary condition is imposed at the outer circular surface using the formula of Kriegsmann et. al. [12]. The FFP is calculated by integrating along the ellipse with the specified \( u \) and the calculated \( \frac{\partial u}{\partial n} \). This is given by

\[
P(a, b) = u_\infty(\vec{x}) = \frac{e^{i\frac{\pi}{4}}}{\sqrt{8\pi k}} \int_{\Gamma} \left( \frac{\partial u^s(y)}{\partial n} + ik\vec{x} \cdot yu^s(y) \right) e^{-ik\vec{x} \cdot y} ds
\]

1 The code by [9] is written in Fortran and was converted to Matlab by us.
where \( \tilde{x} \in S = \{ x \in \mathbb{R}^2 | ||x||_2 = 1 \} \) and \( n \) is the outward normal to the ellipse \( \Gamma \).

We consider a more realistic case where the FFP is corrupted by additive Gaussian noise and only a partial aperture is available in the far field. To simulate the corruption with Gaussian noise the Matlab pseudorandom generator for the normal distribution \(^2\) was used.

### 5.1. Wavelet Denoising

In this section we propose to use the wavelet denoising technique \([15]-[18]\) to smooth the noisy far field before recovering the shape parameters. Let us present the FFP by using a wavelet expansion

\[
f(x) = \sum_{k=-\infty}^{\infty} c_{Jk} \varphi_{Jk} + \sum_{j=J}^{\infty} \sum_{k=-\infty}^{\infty} d_{jk} \psi_{jk}
\]

where

\[
\begin{align*}
  c_{jk} &= (f, \varphi_{jk}) \\
  d_{jk} &= (f, \psi_{jk})
\end{align*}
\]

and \( J \) is the starting index (usually, \( J = 0 \)). It is well known \([13]\) that random noise will show up mostly in the \( d \)-coefficients. So by setting the smaller coefficients to zero, much of the noise will be eliminated. The denoising technique was proposed and analyzed in great detail by Donoho \([15]-[18]\) and by Antoniadis \([14]\); it is often referred to as wavelet shrinkage. A thresholding function \( T_\varepsilon \) is applied to \( d \)-coefficients: \( d_{nk} \rightarrow T_\varepsilon(d_{nk}) \) The most commonly used thresholding function are hard thresholding

\[
T_\varepsilon = \begin{cases} 0, & |x| \leq \varepsilon \\ x, & |x| > \varepsilon \end{cases}
\]

or soft thresholding

\[
T_\varepsilon = \begin{cases} x - \varepsilon, & x > \varepsilon \\ 0, & |x| \leq \varepsilon \\ -x + \varepsilon, & x > -\varepsilon \end{cases}
\]

where \( \varepsilon \in [0, \infty) \) is the threshold parameter. The general de-noising procedure involves three steps:

(i) **Decomposition:** Select a wavelet type mother function and the decompose FFP to some level \( N \).

(ii) **Thresholding:** Apply soft thresholding to the \( d \)-coefficients with the threshold parameter selected adaptively per level.

(iii) **Synthesis:** Reconstruct the signal using the modified \( d \)-coefficients.

The main decision in using the wavelet technique are the choice of the mother wavelet and the threshold parameter selection technique. We found the symmlet8 \([19]\) mother function to be useful in our application. Instead of sticking to a single threshold parameter, we used several different parameter setting techniques (‘universal’, ‘minimax’ and adaptive ‘minimax’) and averaged the obtained signals.

\(^2\) The pseudorandom generators are computing programs that generate the samples from the predefined distribution. They depend on the variable called current state. For the same current state the same sampling is guaranteed, changing of the state leads to new sampling.
5.2. Unsupervised Robust Estimation

We propose an estimation process that enables us to obtain robust estimation results. The idea is to filter out unreasonable estimations \( a_i, b_i \) obtained for some states of the pseudo-generator.

For this goal, we remove results \( a_i, b_i \) that satisfy the inequalities:

\[
|M_a - a_i| > 2.5 s_a
\]

and

\[
|M_b - b_i| > 2.5 s_b
\]

where \( M_a, M_b \) is a median over shape results corresponding the all random state and \( s_a, s_b \) is a kind of standard deviation, but calculated using the median instead of a simple average. The final shape sizes per states are calculated as the mean of the remaining data. This procedure, efficiently removes outlier calculations from consideration.

6. Numerical Simulation

This section presents the reconstruction results for the cases described in Sections 4-5. In our simulations, we consider several wave numbers and aspect ratios of the scattering ellipse. The reconstruction quality is measured as a reconstruction error between the estimated and the real true value:

\[
E = \sqrt{(\hat{a} - a)^2 + (\hat{b} - b)^2},
\]

where \( a \) and \( b \) are the exact ellipse values and \( \hat{a}, \hat{b} \) are the values estimated by the CG method.

For the cases with the noise in the FFP the reported error is obtained by averaging the estimated values \( \hat{a}, \hat{b} \) over multiple runs for the chosen level of noise, but using different noise realization.

6.1. Results for the Exact Theoretical Solution

This section presents results of reconstruction without noise in the FFP (Section 4). The results are presented in Figures 1-2 for ellipses with an aspect ratio 2 : 1 and 5 : 1, respectively. The number of directions (i.e. the number of discrete FFP samples) that are used is equal 2 and the incident angle is equal \( = 45^\circ \). We conducted experiments for three wave numbers \( k = 1, 2, 3 \) and starting from different initial guesses for the parameters \( a, b \).

As can be seen from Fig. 1 for the ellipse with an aspect ratio 2 : 1 and wave number \( k = 1 \), conjugate gradient converges for a large range of initial guesses. For \( k = 2 \) and 3 the initial guess is more important. The simultaneous maximal radius of convergence for all wave numbers \( k = 1, 2, 3 \) is equal to 0.4. We will use this fact later when considering models with noise.

A similar analysis for the ellipse with an aspect ratio 5 : 1 (Fig. 2) shows that the simultaneous radius of convergence for the wave numbers \( k = 1, 2, 3 \) is equal to 0.4. We again observe that when the wave number grows the convergence is more sensitive to the initial condition.

6.2. Results for Numerical Solution

This section presents the same results as in the previous section, but with noise added to the FFP which was calculated using a finite element method. As was discussed in Section 1 the inverse reconstruction problem is ill-posed and it becomes more difficult in the presence of noise. To alleviate this problem the noisy FFP was filtered using the wavelet denoising technique before reconstruction. It is also clear that to recover the shape in the presence of noise more data is required; we use 5 directions (5 FFP samples) for reconstruction.

The reconstruction results are presented in Figures 3-4 for the ellipses of aspect ratio 2 : 1 and 5 : 1, respectively. Based on our experience for the case without noise we start from the same
initial guess with a radius of convergence 0.3. We also run multiple experiments for different noise realization\(^3\).

In comparison with the case without noise, in the presence of noise the reconstruction error for wave number \( k = 1 \) is larger than for wave numbers \( k = 2, 3 \). Thus, in the presence of noise the ill-posedness of the reconstruction problem is more severe for wave number \( k = 1 \).

\(^3\) Noise realization are controlled by the seed stage parameter of the random pseudo-generator of Matlab.
As was discussed in section 5 the final estimation values are obtained by averaging $\hat{a}$, $\hat{b}$ over different stage values (see Table 1). There are some noise realizations for which the reconstruction error is quite large (Figures 3-4). As was discussed in section 5.2 we can improve the estimation by excluding the unreasonable values. The estimation results for this technique are presented in Table 1, last row. It is seen that this method leads to an improvement in the reconstruction of the scattering ellipse.
Table 1. The averaged error for two methods: standard and robust. In the robust method the unreasonable estimations are excluded from averaging.

| method | aspect ratio 2 : 1 | aspect ratio 5 : 1 |
|--------|-------------------|-------------------|
|        | $K = 1$ | $K = 2$ | $K = 3$ | $K = 1$ | $K = 2$ | $K = 3$ |
| standard | 0.3194 | 0.0892 | 0.1287 | 0.1862 | 0.0758 | 0.1427 |
| robust | 0.0665 | 0.0844 | 0.0532 | 0.1338 | 0.0846 | 0.0789 |

7. Summary
We have considered the scattering of a plane wave about a ellipses with various aspect ratios and with wave number $k = 1, 2, 3$. The shape reconstruction problem is converted to a least squares problem and then solved by conjugate gradient without any regularization. We have considered several cases without and with noise.

For noisy data we reduce the noise by using a Wavelet denoising technique prior to reconstruction. To improve the robustness of the method we develop technique to eliminate unreasonable estimations from consideration.

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