ON A $q$-ANALOGUE OF THE $p$-ADIC GENERALIZED TWISTED $L$-FUNCTIONS AND $p$-ADIC $q$-INTEGRALS

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Abstract. The purpose of this paper is to define generalized twisted $q$-Bernoulli numbers by using $p$-adic $q$-integrals. Furthermore, we construct a $q$-analogue of the $p$-adic generalized twisted $L$-functions which interpolate generalized twisted $q$-Bernoulli numbers. This is the generalization of Kim's $h$-extension of $p$-adic $q$-$L$-function which was constructed in [5] and is a partial answer for the open question which was remained in [3].

§1. Introduction

Let us denote $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ sets of positive integer, integer, rational, real and complex numbers respectively. Let $p$ be prime and $x \in \mathbb{Q}$. Then $x = p^{v(x)} \frac{m}{n}$, where $m, n, v = v(x) \in \mathbb{Z}$, $m$ and $n$ are not divisible by $p$. Let $|x|_p = p^{-v(x)}$ and $|0|_p = 0$. Then $|x|_p$ is valuation on $\mathbb{Q}$ satisfying

$$|x + y|_p \leq \max\{|x|_p, |y|_p\}.$$ 

Completion of $\mathbb{Q}$ with respect to $|\cdot|_p$ is denoted by $\mathbb{Q}_p$ and called the field of $p$-adic rational numbers. $\mathbb{C}_p$ is the completion of algebraic closure of $\mathbb{Q}_p$ and $\mathbb{Z}_p = \{x \in \mathbb{Q}_p \mid |x|_p \leq 1\}$ is called the ring of $p$-adic rational integers (see [1, 2, 10, 12, 16]).

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Let $l$ be a fixed integer and let $p$ be a fixed prime number. We set
\[
X = \lim_{N \to \infty} (\mathbb{Z}/lp^N\mathbb{Z}), \\
X^* = \bigcup_{0 < a < lp, (a,p)=1} (a + lp\mathbb{Z}_p), \\
a + lp^N\mathbb{Z}_p = \{x \in X \mid x \equiv a \pmod{lp^N}\},
\]
where $N \in \mathbb{N}$ and $a \in \mathbb{Z}$ lies in $0 \leq a < lp^N$, cf. [3,7,8,9].

When one talks of $q$-extension, $q$ is considered in many ways such as an indeterminate, a complex number $q \in \mathbb{C}$, or a $p$-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, one normally assumes $|q| < 1$. If $q \in \mathbb{C}_p$, then we assume $|q - 1|_p < p^{-\frac{1}{p-1}}$, so that $q^x = \exp(x \log q)$ for each $x \in X$. We use the notation as $[x] = [x; q] = \frac{1 - q^x}{1 - q}$ for each $x \in X$. Hence $\lim_{q \to 1} [x] = x$, cf.[4, 16, 18, 19, 20]. For any positive integer $N$, we set
\[
\mu_q(a + lp^N\mathbb{Z}_p) = \frac{q^a}{lp^N}, \quad \text{cf.} \ [5, 6, 7, 8, 9, 10, 11, 12, 13, 14],
\]
and this can be extended to a distribution on $X$. This distribution yields an integral for each nonnegative integer $n$ (see [7]) :
\[
\int_{\mathbb{Z}_p} [x]^n \ d\mu_q(x) = \int_{X} [x]^n \ d\mu_q(x) = \beta_n(q),
\]
where $\beta_n(q)$ are the $n$-th Carlitz’s $q$-Bernoulli number, cf. [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14].

In the paper [17], Koblitz constructed $p$-adic $q - L$–function which interpolates Carlitz’s $q$-Bernoulli number at non-positive integers and suggested two questions. One of these two questions was solved by Kim (see [7]). In fact, Kim constructed $p$-adic $q$-integral and proved that Carlitz’s $q$-Bernoulli number can be represented as an $p$-adic $q$-integral by the $q$-analogue of the ordinary $p$-adic invariant measure. And also Kim is constructed a $h$-extension of $p$-adic $q - L$–function which interpolates the $h$-extension of $q$-Bernoulli numbers at non-positive integers (see [5, 6, 7, 6, 9, 10, 11, 12, 13, 14]). In [5, 12, 13], Kim constructed $p$-adic $q$-$L$-functions and he studied their properties . In [5], Kim introduced the $h$-extension of $p$-adic $q$-$L$-functions and investigated many interesting physical meaning. Also, In [15, 16], Koblitz defined $p$-adic twisted $L$-functions , and he constructed $p$-adic measures and integrations. And also Kim et al [3] constructed a $q$-analogue of the twisted Dirichlet’s $L$-function which interpolated the twisted Carlitz’s $q$-Bernoulli numbers, and they remained an open question in [3] as follows:

Find $q$-analogue of the $p$-adic twisted $L$-function which interpolates $q$-Bernoulli numbers $\beta^{(h)}_{m,w,\chi}(q)$, by means of a method provided by Kim, cf. [5].

In this paper, we will construct the ”twisted” $p$-adic generalized $q - L$–functions and generalized $q$-Bernoulli numbers to be a part of answer for the question which was remained by Kim el al in [3] by means of the same method provided by Kim in [5: p.98]. In section 2, we construct generalized twisted $q$-Bernoulli polynomials by using $p$-adic $q$-integrals by the same method of Kim, cf. [3, 5, 12, 13, 14, 20, 21, 22, 23]. We prove a formula between generalized twisted $q$-Bernoulli polynomials which is regarded as a generalization of Witt’s formula for Carlitz’s $q$-Bernoulli polynomials in [5, Eq (5.9)], [13] and [7, Theorem 2]. This means that the $q$-analogue of generalized twisted $q$-Bernoulli numbers
occur in the coefficients of some stirling type series. We also give construction of the distribution of the 
p-adic generalized twisted q-Bernoulli distribution. In section 3, we define the p-adic generalized twisted 
L-function and construct a q-analogue of the p-adic generalized twisted L-function which interpolate 
generalized twisted q-Bernoulli numbers on X. This result is related as a generalization of a q-analogue 
of the p-adic L-function which interpolate Carlitz’s q-bernoulli numbers in [5, 11, 12, 13], of p-adic 
generalized L-function which interpolates the h-extension of q-Bernoulli numbers at non-positive integers 
in [5, 6, 7].

§2. GENERALIZED TWISTED q-BERNOUlli POLYNOMIALS

In this section, we give generalized twisted q-Bernoulli polynomials by using p-adic q-integrals on 
X. Let UD(X) be the set of uniformly differentiable functions on X. For any \( f \in UD(X) \), T. Kim 
defined a q-analogue of an integral with respect to an p-adic invariant measure in [5] which is called 
p-adic q-integral. The p-adic q-integral was defined as follows:

\[
I_q(f) = \int_X f(x) \, d\mu_q(x)
= \lim_{N \to \infty} \frac{1}{[lp^N]} \sum_{0 \leq x < lp^N} f(x)q^x, \tag{1}
\]

cf. [4,5,6,7,8]. Note that

\[
I_1(f) = \lim_{q \to 1} I_q(f) = \int_X f(x) \, d\mu_1(x)
= \lim_{N \to \infty} \frac{1}{lp^N} \sum_{0 \leq x < lp^N} f(x), \tag{2}
\]

and that

\[
I_1(f_1) = I_1(f) + f'(x), \tag{3}
\]

where \( f_1(x) = f(x + 1) \).

Let \( T_p = \cup_{n \geq 1} C_{p^n} = \lim_{n \to \infty} \mathbb{Z}/p^n\mathbb{Z} \), where \( C_{p^n} = \{ \xi \in X \mid \xi p^n = 1 \} \) is the cyclic group of order 
p^n, see [9]. For \( \xi \in T_p \), we denote by \( \phi_\xi : \mathbb{Z}/p \to \mathbb{C}_p \) the locally constant function \( x \mapsto \xi^x \). If we take 
\( f(x) = \phi_\xi(x)e^{\xi tx} \), then we have that

\[
\int_X e^{tx} \phi_\xi(x) d\mu_1(x) = \frac{t}{we^t - 1}, \tag{4}
\]

cf. [5,8]. It is obvious from (3) that

\[
\int_X e^{tx} \chi(x) \phi_\xi(x) d\mu_1(x) = \frac{\sum_{a=1}^l \chi(a)\phi_\xi(a)e^{at}}{\xi^l e^t - 1}. \tag{5}
\]

Now we define the analogue of Bernoulli numbers as follows:

\[
e^{xt} \frac{t}{\xi^t e^t - 1} = \sum_{n=0}^\infty B_{n,\xi}(x) \frac{t^n}{n!}
\]

\[
\sum_{a=1}^l \chi(a)\phi_\xi(a)e^{at} = \frac{\sum_{n=0}^\infty B_{n,\xi,\chi} t^n}{\xi^l e^t - 1}, \tag{6}
\]
ON A \( q \)-ANALOGUE OF THE \( p \)-ADIC TWISTED L-FUNCTIONS \( \cdots \)

cf. [5,8]. By (4), (5) and (6), it is not difficult to see that
\[
\int_X x^n \phi_\xi(x) \, d\mu_1(x) = B_{n,\xi}
\] (7)
and
\[
\int_X \chi(x) x^n \phi_\xi(x) \, d\mu_1(x) = B_{n,\xi,\chi}.
\] (8)

From (7) and (8) we consider twisted \( q \)-Bernoulli numbers by using \( p \)-adic \( q \)-integral on \( \mathbb{Z}_p \). For \( \xi \in T_p \) and \( h \in \mathbb{Z} \), we define twisted \( q \)-Bernoulli polynomials as
\[
\beta_{m,\xi}(h, q)(x) = \int_{\mathbb{Z}_p^m} q^{(h-1)y} \xi^y [x+y]^m \, d\mu_q(y).
\] (9)

Observe that
\[
\lim_{q \to 1} \beta_{m,\xi}(h, q)(x) = B_{m,\xi}(x).
\]

When \( x = 0 \), we write \( \beta_{m,\xi}(0, q) = \beta_{m,\xi}(q) \), which are called twisted \( q \)-Bernoulli numbers. It follows from (9) that
\[
\beta_{m,\xi}(h, q)(x) = \frac{1}{(1-q)^{m-1}} \sum_{k=0}^{m} \binom{m}{k} q^{xk} (-1)^k \frac{k+h}{1-q^{h+k}\xi}.
\] (10)

The Eq.(10) is equivalent to
\[
\beta_{m,\xi}(h, q)(x) = -m \sum_{n=0}^{\infty} [n]^{m-1} q^{hn} \xi^n - (q-1)(m+h) \sum_{n=0}^{\infty} [n]^m q^{hn} \xi^n.
\] (11)

From (9), we obtain the below distribution relation for the twisted \( q \)-Bernoulli polynomials as follows. In fact, the proof of Lemma 1 is similar to the proof of Lemma 2 with \( \chi = 1 \).

**Lemma 1.** For \( n \geq 1 \), we have
\[
\beta_{n,\xi}(h, q)(x, q) = d^{n-1} \sum_{a=0}^{d-1} \xi^a q^{ha} \beta_{n,\xi}(h, \frac{a}{d}, q^d).
\]

For \( \xi \in T_p \) and \( h \in X \), we define generalized twisted \( q \)-Bernoulli polynomials as
\[
\beta_{n,\xi,\chi}(h, q)(x) = \int_X \chi(y) q^{(h-1)y} \xi^y [x+y]^n \, d\mu_q(y).
\] (12)

Observe that when \( \chi = 1 \),
\[
\beta_{n,\xi,1}(h, q)(x) = \int_X q^{(h-1)y} \xi^y [x+y]^n \, d\mu_q(y) = \beta_{n,\xi}(h, q)(x)
\] (13)
and
\[
\lim_{q \to 1} \beta_{n,\xi,\chi}(h, q)(x) = \int_X \chi(y) \xi^y [x+y]^n \, d\mu_1(y) = B_{n,\xi,\chi}(h, x),
\] (14)

where \( \beta_{n,\xi}(h, q)(x) \) is a twisted \( q \)-Bernoulli polynomial and \( B_{n,\xi,\chi}(h, x) \) is a generalized Bernoulli polynomial.
Lemma 2. For any \( n \geq 1 \), we have

\[
\beta_{n,\xi,\chi}^{(h)}(x, q) = [l]^{n-1} \sum_{a=0}^{l-1} \sum_{i=0}^{l-1} \frac{a+x}{l} \beta_{n,\xi,\chi}^{(h)}(\frac{a+x}{l}, q). 
\]

Proof. For each \( n \in \mathbb{N} \), we have

\[
\beta_{n,\xi,\chi}^{(h)}(x, q) = \int_X \chi(y) q^{(h-1)y} \xi^y [x + y]^n d\mu_q(y)
= \lim_{N \to \infty} \sum_{x_1=0}^{lp^N-1} \chi(x_1) \xi^x [x + x_1]^n \mu_q(x_1 + lp^N \mathbb{Z}_p)
= \lim_{N \to \infty} \frac{1}{[p^N]} \sum_{x_1=0}^{lp^N-1} \chi(x_1) \xi^x [x + x_1]^n q^x
= [l]^{n-1} \sum_{a=0}^{l-1} \chi(a) q^a \xi^a \lim_{N \to \infty} \frac{1}{[p^N : q^l]} \sum_{m=0}^{p^N-1} (q^l)^{(h-1)m} (\xi^l)^m \frac{x+a}{l} + m : q^l] \beta_{n,\xi,\chi}^{(h)}(\frac{x+a}{l}, q).
\]

We note that when \( x = 0 \), we have the distribution relation for the generalized twisted \( q \)-Bernoulli numbers as follows: for \( n \geq 1 \),

\[
\beta_{n,\xi,\chi}^{(h)}(0, q) = \beta_{n,\xi,\chi}^{(h)}(0, 1) = [l]^{n-1} \sum_{a=0}^{l-1} \chi(a) q^a \beta_{n,\xi,\chi}^{(h)}(\frac{a}{l}, q).
\]

and that when \( x = 0 \) and \( q = 1 \), we have the distribution relation for the generalized twisted Bernoulli numbers as follows: for \( n \geq 1 \),

\[
\beta_{n,\xi,\chi}^{(h)}(0, 1) = \beta_{n,\xi,\chi}^{(h)}(1) = [l]^{n-1} \sum_{a=0}^{l-1} \chi(a) q^a \beta_{n,\xi,\chi}^{(h)}(\frac{a}{l}).
\]

and that when \( x = 0 \) and \( \chi = 1 \), we have the distribution relation for the twisted \( q \)-Bernoulli polynomials as follows: for \( n \geq 1 \),

\[
\beta_{n,\xi}(0, q) = \beta_{n,\xi}(0, 1) = [l]^{n-1} \sum_{a=0}^{l-1} \xi^a q^a \beta_{n,\xi}^{(h)}(\frac{a}{l}, q).
\]

Lemma 1 and Lemma 2 are important for the construction of the \( p \)-adic generalized twisted \( q \)-Bernoulli distribution as follows.
Theorem 3. Let \( q \in \mathbb{C}_p \). For any positive integers \( N, n \) and \( l \), let \( \mu^{(h)}_{n, \xi} \) be defined by

\[
\mu^{(h)}_{n, \xi}(a + lp^N \mathbb{Z}_p) = [lp^N]^{n-1} q^{ha \xi a} \beta_{n, \xi}^{(h)}(\frac{a}{lp^N}, q lp^N).
\]

Then \( \mu^{(h)}_{n, \xi} \) extends uniquely to a distribution on \( X \).

Proof. It is suffices to show

\[
\sum_{i=1}^{p-1} \mu^{(h)}_{n, \xi}(a + ip^N + p^{N+1} \mathbb{Z}_p) = \mu^{(h)}_{n, \xi}(a + p^N \mathbb{Z}_p).
\]

Indeed, Lemma 1 and the definition of \( \mu^{(h)}_{n, \xi} \) imply that

\[
\sum_{i=1}^{p-1} \mu^{(h)}_{n, \xi}(a + ip^N + p^{N+1} \mathbb{Z}_p)
\]

\[
= \sum_{x=0}^{p-1} [p^{N+1}]^{n-1} q^{h(a+xp^N)} \xi^{a+xp^N} \beta_{n, xp^N+1}^{(h)}(\frac{a+xp^N}{p^{N+1}}, q^{xp^N+1})
\]

\[
= [p]^{n-1} q^{ha \xi a} [p^N : q^p]^{n-1} \sum_{x=0}^{p-1} (q^{xp^N})^x \beta_{n, (xp^N)^p}^{(h)}(\frac{a}{p^N}, q^{xp^N})
\]

\[
= [p]^{n-1} q^{ha \xi a} \beta_{n, p^N}^{(h)}(\frac{a}{p^N}, q^{p^N})
\]

\[
= \mu^{(h)}_{n, \xi}(a + p^N \mathbb{Z}_p).
\]

§3. A \( q \)-ANALOGUE OF THE \( p \)-ADIC TWISTED \( L \)-FUNCTIONS

Let \( \alpha \in X^*, \alpha \neq 1, n \geq 1 \). By the definition of \( \mu^{(h)}_{n, \xi, \chi} \), we easily see :

\[
\int_X \chi(x) d\mu^{(h)}_{n, \xi}(x) = \beta_{n, \xi, \chi}^{(h)}(q)
\]

\[
\int_{pX} \chi(x) d\mu^{(h)}_{n, \xi}(x) = [p]^{n-1} \chi(p) \beta_{n, p^N, \chi}^{(h)}(q^p)
\]

\[
\int_X \chi(x) d\mu^{(h)}_{n, q^\frac{1}{p}, \xi, \chi}(\alpha x) = \chi(\frac{1}{\alpha}) \beta_{n, q^{\frac{1}{p}}, \chi}^{(h)}(q^\frac{1}{p})
\]

\[
\int_{pX} \chi(x) d\mu^{(h)}_{n, q^\frac{1}{p}, \xi, \chi}(\alpha x) = [p; q^\frac{1}{p}]^{n-1} \chi(\frac{D}{\alpha}) \beta_{n, q^{\frac{1}{p}}, \chi}^{(h)}(q^\frac{1}{p}).
\]

For compact open set \( U \subset X \), we define

\[
\mu^{(h)}_{n, q, \alpha, \xi}(U) = \mu^{(h)}_{n, q, \xi}(U) - \alpha^{-1}[\alpha^{-1}; q]^{n-1} \mu^{(h)}_{n, q^{\frac{1}{p}}, \xi, \chi}(U).
\]
By the definition of $\mu_{n,q,\xi}^{(h)}$ and (19), we note that

$$\int_{X^*} \chi(x) d\mu_{n,q,\alpha,\xi}^{(h)}(x) = \beta_{n,\xi,\chi}^{(h)}(q) - [p]^{n-1} \chi(p) \beta_{n,\xi,\chi}^{(h)}(q^p)$$

$$- \frac{1}{\alpha} \left( \frac{1}{\alpha} \right)^{n-1} \chi(\frac{1}{\alpha}) \beta_{n,\xi,\chi}^{(h)}(q)$$

$$+ \frac{1}{\alpha} \left( \frac{p}{\alpha} \right)^{n-1} \chi(\frac{p}{\alpha}) \beta_{n,\xi,\chi}(q)$$

$$= (1 - \chi^p)(1 - \frac{1}{\alpha}) \beta_{n,\xi,\chi}^{(h)},$$

where the operator $\chi^y = \chi^{y,n,q,\xi}$ on $f(q, \xi)$ defined by

$$\chi^y f(q, \xi) = [y]^{n-1} \chi(y) f(q^y, \xi^y), \; \chi^x \chi^y = \chi^{x+y} \chi^y.$$

Let $x \in X$. We recall that $\{x\}_N$ denote the least nonnegative residue (mod $lp^N$) and that if $[x]_N = x - \{x\}_N$, then $[x]_N \in lp^N \mathbb{Z}_p$. Now we can define in [5] as follows:

$$\mu_{Mazur,1,\alpha}^{(h)}(a + lp^N \mathbb{Z}_p) = \frac{1}{h+1} + \frac{h}{\alpha} \cdot \frac{[\alpha]_N}{lp^N}.$$

By the same method of Kim in [5], we easily see:

$$\lim_{N \to \infty} \mu_{n,q,\alpha,\xi}^{(h)}(a + lp^N \mathbb{Z}_p)$$

$$= \lim_{N \to \infty} [l]^{n-1}((h + n)q^{(h+1)a} -hq^a)\xi^a(\frac{1}{h+1} + \frac{h}{\alpha} \cdot \frac{[\alpha]_N}{lp^N}).$$

Thus we have

$$\mu_{n,q,\alpha,\xi}^{(h)}(x) = [x]^{n-1}((h + n)q^{(h+1)x} -hq^{xh})\xi^x \mu_{Mazur,1,\alpha}^{(h)}(x).$$

**Theorem 4.** $\mu_{n,q,\alpha,\xi}^{(h)}$ are bounded $\mathbb{C}_p$-valued measure on $X$ for all $n \geq 1$ and $\alpha \in X^*, \alpha \neq 1$.

Now we define $< x > = [x; q] = [x; q]/w(x)$, where $w(x)$ is the Teichmüller character. For $|q - 1|_p < p^{-\frac{1}{n-1}}$, we note that $< x >^{pN} \equiv 1(mod p^N)$. By (21) and (23), we have the following:

$$\int_{X^*} \chi_n(x) d\mu_{n,q,\alpha,\xi}^{(h)}(x)$$

$$= \int_{X^*} \chi_n(x)[x]^{n-1}((h + n)q^{(h+1)x} -hq^{xh})\xi^x \mu_{Mazur,1,\alpha}^{(h)}(x)$$

$$= \int_{X^*} ((h + n)q^{(h+1)x} -hq^{xh}) < x >^{n-1} \xi^x \chi_1(x) \mu_{Mazur,1,\alpha}^{(h)}(x)$$

where $\chi_n(x) = \chi^{w^{-n}}(x)$. By using (24), we can construct a $q$-analogue of $p$-adic generalized twisted $L$-function.
Definition 5. For fixed $\alpha \in X^*$, $\alpha \neq 1$, we define a $h$-extension of $p$-adic generalized twisted $L$-function as follows:

$$L_{p,q,\xi}^{(h)}(s,\chi) = \frac{1}{1-s} \int_{X^*} \left( (h+1-s)q^{(h+1)x} -hq^{hx} \right) \xi^x < x >^{-s} \chi_1(x) d\mu_{Mazur,1,\alpha}^{(h)}(x),$$

for $s \in X$.

Theorem 6. For each $s \in \mathbb{Z}_p$ and $\alpha \in X^*, \alpha \neq 1$, we have

$$L_{p,q,\xi}^{(h)}(s,\chi) = \frac{1-s+h}{1-s} (q-1) \sum_{n=1}^{\infty} \frac{q^n \xi^n w^{s-1}(n)}{[n]^{s-1}} \chi(n) \left( \frac{1}{h} + \frac{h}{\alpha} \cdot \frac{[n\alpha]}{lp^N} \right)$$

$$\sum_{n=1}^{\infty} q^{hn} \xi^n [n]^{-s} w^{s-1}(n) \chi(n) \left( \frac{1}{h} + \frac{h}{\alpha} \cdot \frac{[n\alpha]}{lp^N} \right).$$

where $\sum_{n=1}^{\infty}^{\infty}$ means to sum over the rational integers prime to $p$ in the given range.

Proof. For each $s \in \mathbb{Z}_p$ and $x \in X^*$, we have

$$(h+1-s)q^{(h+1)x} -hq^{hx} =hq^{hx}(q^x-1) + (1-s)q^x q^{hx}$$

$$= (q-1)q^{hx}[x](h+1-s) + q^{hx}(1-s).$$

Thus

$$\frac{1}{1-s} \int_{X^*} \left( (h+1-s)q^{(h+1)x} -hq^{hx} \right) \xi^x < x >^{-s} \chi_1(x) d\mu_{Mazur,1,\alpha}^{(h)}(x)$$

$$= \frac{1}{1-s} \int_{X^*} [(q-1)q^{hx}[x](h+1-s) + q^{hx}(1-s)] \xi^x < x >^{-s} \chi_1(x) d\mu_{Mazur,1,\alpha}^{(h)}(x)$$

$$= \frac{1-s+h}{1-s} (q-1) \sum_{n=1}^{\infty} \frac{q^n \xi^n w^{s-1}(n)}{[n]^{s-1}} \chi(n) \left( \frac{1}{h} + \frac{h}{\alpha} \cdot \frac{[n\alpha]}{lp^N} \right)$$

$$\sum_{n=1}^{\infty} q^{hn} \xi^n [n]^{-s} w^{s-1}(n) \chi(n) \left( \frac{1}{h} + \frac{h}{\alpha} \cdot \frac{[n\alpha]}{lp^N} \right).$$

The equation (26) with $h = s - 1$ implies that

$$L_{p,q,\xi,\alpha}^{(s-1)}(s,\chi) = \sum_{n=1}^{\infty} q^{(s-1)n} \xi^n [n]^{-s} w^{s-1}(n) \chi(n) \left( \frac{1}{s} + \frac{s-1}{\alpha} \cdot \frac{[n\alpha]}{lp^N} \right).$$

Finally for each positive integer $m$, we can construct a $q$-analogue of the $p$-adic twisted $L$-function which interpolate a generalized $q$-Bernoulli number.
Theorem 7. For each $m \in \mathbb{N}$ and $\alpha \in X^*, \alpha \neq 1$, we have
\[
L^{(h)}_{p,q,\xi}(1-m, \chi) = -\frac{1}{m}(1 - \chi_m^p)(1 - \frac{1}{\alpha^m})w^{-m}\beta^{(h)}_{m,\xi,\chi}(q). \quad (28)
\]

Proof. For each $s \in \mathbb{Z}_p$, by using (21), we have
\[
\begin{align*}
L^{(h)}_{p,q,\xi,\alpha}(s, \chi)
&= \frac{1}{1-s} \int_{X^*} ((h + 1 - s)q^{(h+1)x} - hq^{hx})\xi^x < x >^{-s} \chi_1(x)d\mu^{(h)}_{Mazur,1,\alpha}(x) \\
&= \frac{1}{1-s} \int_{X^*} \chi_{1-s}(x)d\mu_{1-s;q,\alpha,\xi}(x) \\
&= \frac{1}{1-s}(1 - \chi_1^{p,s})(1 - \frac{1}{\alpha^s})\beta^{(h)}_{n,\xi,\chi}(q).
\end{align*}
\]

Thus
\[
L^{(h)}_{p,q,\xi}(1-m, \chi) = \frac{1}{m}(1 - \chi_m^p)(1 - \frac{1}{\alpha^m})\beta^{(h)}_{n,\xi,\chi}(q).
\]

Remark. In [5], Kim constructed the $h$-extension of $p$-adic $q$-$L$-functions. And the question to inquire the existence of the twisted $p$-adic $q$-$L$-functions was remained in [3]. This is still open. By means of the method provided by Kim in [5], we constructed the twisted $p$-adic $q$-$L$-function to be a part of an answer for the question which was remained in [3].

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