UNIQUENESS OF A SMOOTH CONVEX BODY
WITH A UNIFORM CONE VOLUME MEASURE
IN THE NEIGHBORHOOD OF A BALL

ANDREA COLESANTI, GALYNA LIVSHYTS

ABSTRACT. The cone volume measure of a convex body $K$ is the measure $c_K$ on the sphere given by
$$c_K(\Omega) = \frac{1}{n} \int_{S^{n-1}} h_K(u) ds_K(u),$$
where $s_K$ is the surface area measure of $K$ and $h_K$ is the support function. Lutwak asked [31] if the cone volume measure determines a convex body uniquely; the answer is known to be affirmative in some partial cases. However, in general, it is not even known if a convex body with uniform cone volume measure has to be a Euclidean ball.

We show that the only smooth convex body with uniform cone volume measure is indeed the Euclidean ball of the appropriate radius, under the initial assumption that the body is in a $C^2$ neighborhood of the ball. The size of the neighborhood depends only on the dimension.

1. INTRODUCTION

We shall work in the Euclidean $n$-dimensional space $\mathbb{R}^n$. The unit ball shall be denoted by $B^n_2$ and the unit sphere by $S^{n-1}$. The Lebesgue volume of a measurable set $A \subset \mathbb{R}^n$ is denoted by $|A|$.

Let $s_K$ be the curvature measure of a convex set $K$ in $\mathbb{R}^n$. That is, $s_K$ is the push forward of the Hausdorff $(n-1)$-dimensional measure on $\partial K$ to the unit sphere under the Gauss map. The cone volume measure of a convex set $K$ is the measure on the sphere, defined as
$$c_K(\Omega) = \frac{1}{n} \int_{\Omega} h_K(u) ds_K(u),$$
where $h_K$ is the support function of $K$, that is
$$h_K(u) = \max_{x \in K} \langle x, u \rangle.$$

The name “cone volume measure” comes from the fact that
$$|K| = \int_{S^{n-1}} dc_K(u)$$
where $| \cdot |$ denotes the volume, i.e. the Lebesgue measure. One can check the validity of this equation in the case of polytopes, and the general case follows by approximation.

The following conjecture is due to Lutwak [31].

Conjecture 1.1. [Lutwak] If $K$ and $L$ are symmetric smooth strictly convex sets with $dc_K(u) = dc_L(u)$, then $K = L$.

In this manuscript, we verify this conjecture locally, in a $C^2$ neighborhood of Euclidean balls, in the case when the cone volume measure is uniform.
Theorem 1.2. Let \( n \geq 2 \) and let \( R > 0 \) be a constant. There exists \( \epsilon = \epsilon(n) > 0 \), which depends only on the dimension, such that, given a symmetric \( C^2 \)-smooth convex body \( K \) satisfying \( \| R - h_K \|_{C^2(S^{n-1})} \leq \epsilon(n)R \) and \( dc_K(u) = R^n du \), one has that \( K \) coincides with the Euclidean ball of radius \( R \).

Uniqueness questions such as Conjecture 1.1 are often related to Brunn-Minkowski type inequalities, and the equality cases in those inequalities. The classical Brunn-Minkowski inequality (in its multiplicative form) states that for every pair of Borel-measurable sets \( K \) and \( L \), and for a scalar \( \lambda \in [0, 1] \), one has

\[
|\lambda K + (1 - \lambda)L| \geq |K|^{\lambda}|L|^{1-\lambda}
\]

(in fact, we need to assume that \( \lambda K + (1 - \lambda)L \) is measurable as well). See, e.g., the extensive survey by Gardner [17] on the subject. Define the geometric average of convex bodies:

\[
\lambda K +_0 (1 - \lambda)L = \{ x \in \mathbb{R}^n : \langle x, u \rangle \leq h_K^\lambda(u)h_L^{1-\lambda}(u), \forall u \in S^{n-1} \}.
\]

The log-Brunn-Minkowski conjecture (see Boroczky, Lutwak, Yang, Zhang [7]) states that

\[
|\lambda K +_0 (1 - \lambda)L| \geq |K|^{\lambda}|L|^{1-\lambda}
\]

for every pair of symmetric convex sets \( K \) and \( L \). Important applications and motivations for this conjecture can be found in [8], [9]. In particular, it is equivalent to the famous B-conjecture (see [16]).

It is not difficult to see that the condition of symmetry in (2) is necessary. Böröczky, Lutwak, Yang and Zhang [7] showed that this conjecture holds for \( n = 2 \). Saroglou [46] and Cordero-Erasquin, Fradelizi, Maurey [16] proved that the conjecture is true when the sets \( K \) and \( L \) are unconditional (i.e. they are symmetric with respect to every coordinate hyperplane). Rotem [45] showed that log-Brunn-Minkowski conjecture holds for complex convex bodies. Note that the straightforward inclusion

\[
\lambda K +_0 (1 - \lambda)L \subseteq \lambda K + (1 - \lambda)L
\]

tells us that (2) is stronger than the classical Brunn-Minkowski inequality (1).

The structure of this paper is as follows: in Section 2 we discuss some preliminaries, in Section 3 we establish the stability of the log-Brunn-Minkowski inequality in the \( C^2 \) neighborhood of a ball, and in Section 4 we prove the Theorem 1.2.

Acknowledgement. The authors would like to thank Alina Stancu for pointing out Remark 4.3. The second author is supported by the AMS Simons travel grant. The work was partially supported by the National Science Foundation under Grant No. DMS-1440140 while the second author was in residence at the Mathematical Sciences Research Institute in Berkeley, California, during the Fall 2017 semester.

2. Preliminaries

We say that a convex body \( K \) is of class \( C^{2,+} \) if \( \partial K \) is of class \( C^2 \) and the Gauss curvature is strictly positive at every \( x \in \partial K \). In particular, if \( K \) is \( C^{2,+} \) then it admits unique outer unit normal \( \nu_K(x) \) at every boundary point \( x \). Recall that the Gauss map \( \nu_K : \partial K \to S^{n-1} \) is the map assigning the collection of unit normals to each point of \( \partial K \).

We recall that an orthonormal frame on the sphere is a map which associates a collection of \( n - 1 \) orthonormal vectors to every point of \( S^{n-1} \). Let \( \psi \in C^2(S^{n-1}) \); we denote by \( \psi_i(u) \) and \( \psi_{ij}(u) \), \( i, j \in \{1, \ldots, n - 1 \} \), the first and second covariant derivatives of \( \psi \) at
\( u \in S^{n-1} \), with respect to a fixed local orthonormal frame on an open subset of \( S^{n-1} \). We define the matrix

\[
Q(\psi; u) = (q_{ij})_{i,j=1,\ldots,n-1} = (\psi_{ij}(u) + \psi(u)\delta_{ij})_{i,j=1,\ldots,n-1},
\]

where the \( \delta_{ij} \)'s are the usual Kronecker symbols. On an occasion, instead of \( Q(\psi; u) \) we write \( Q(\psi) \). Note that \( Q(\psi; u) \) is symmetric by standard properties of covariant derivatives.

In what follows we shall often consider \( \psi \) to be a support function of a convex body \( K \). In this case \( Q(\psi) \) is called the curvature matrix of \( K \); this name comes from the fact that \( \det(Q(\psi)) \) is the density of the curvature measure \( s_K \), and therefore,

\[
|K| = \frac{1}{n} \int_{S^{n-1}} h_K(u) \det Q(h_K, u) du.
\]

(See, for instance, Koldobsky [22] for the proof.) We recall here a fact that will be frequently used in the paper (a proof can be deduced, for instance, from [48, Section 2.5]).

**Proposition 2.1.** Let \( K \in K^n \) and let \( h \) be its support function. Then \( K \) is of class \( C^{2,+} \) if and only if \( h \in C^{2}(S^{n-1}) \) and

\[
Q(h; u) > 0, \quad \forall u \in S^{n-1}.
\]

For \( g \in C^{2}(S^{n-1}) \), we set

\[
\|g\|_{C^{2}(S^{n-1})} = \|g\|_{L^\infty(S^{n-1})} + \|\nabla_s g\|_{L^\infty(S^{n-1})} + \sum_{i,j} \|g_{ij}\|_{L^\infty(S^{n-1})},
\]

where \( \nabla_s g \) denotes the spherical gradient of \( g \) (i.e. the vector having first covariant derivatives as components). We also set

\[
\|g\|_{L^2(S^{n-1})}^2 = \int_{S^{n-1}} g^2(u) du, \quad \|\nabla_s u\|_{L^2(S^{n-1})}^2 = \int_{S^{n-1}} \|\nabla_s(u)\|^2 du.
\]

### 2.1. Co-factor matrices

For a natural number \( N \), denote by \( \text{Sym}(N) \) the space of \( N \times N \) symmetric matrices. Given \( A \in \text{Sym}(N) \) we denote by \( a_{jk} \) its \( jk \)-th entry and write \( A = (a_{jk}) \). For \( j, k = 1, \ldots, N \) we set

\[
c_{jk}(A) = \frac{\partial \det}{\partial a_{jk}}(A).
\]

The matrix \( (c_{jk}(A)) \) is called the co-factor matrix of \( A \). We also set, for \( j, k, r, s = 1, \ldots, N \),

\[
c_{jk,rs}(A) = \frac{\partial^2 \det}{\partial a_{jk} \partial a_{rs}}(A).
\]

Recall that

\[
\det(A) = \frac{1}{N!} \sum \delta\left(\begin{array}{c} j_1, \ldots, j_N \\ k_1, \ldots, k_N \end{array}\right) a_{j_1k_1} \cdots a_{j_Nk_N},
\]

where the sum is taken over all possible indices \( j_s, k_s \in \{1, \ldots, N\} \) (for \( s = 1, \ldots, N \)) and the Kronecker symbol

\[
\delta\left(\begin{array}{c} j_1, \ldots, j_N \\ k_1, \ldots, k_N \end{array}\right).
\]
and (5), we derive for every \( j, k, r, s \):

\[
c_{jk}(A) = \frac{1}{(N-1)!} \sum \delta(j, j_1, \ldots, j_{N-1}) a_{jk} a_{j_{N-1}k_{N-1}},
\]

(7) \[
c_{jk,rs}(A) = \frac{1}{(N-2)!} \sum \delta(r, j, j_1, \ldots, j_{N-2}) a_{jk} a_{j_{N-2}k_{N-2}}.
\]

Remark 2.2. If \( A \in \text{Sym}(N) \) is invertible, then, by (7),

\[
(c_{jk}(A)) = \det(A) A^{-1}.
\]

In particular, if \( A = I_N \) (the identity matrix of order \( N \)), then \( (c_{jk}(I_N)) = I_N \).

Remark 2.3. Observe that, by (7), for every \( A = (a_{jk}) \in \text{Sym}(N) \),

\[
\sum_{j,k=1}^{N} c_{jk}(A) a_{jk} = N \det(A).
\]

Remark 2.4. Let \( A = (a_{ij}) \in \text{Sym}(N) \) and let \( M > 0 \) be such that

\[
|a_{jk}| \leq C, \quad \forall j, k = 1 \ldots, M.
\]

Then there exists some constant \( c = c(N) \) (i.e. depending only on \( N \)) such that, for every \( j, k, r, s = 1, \ldots, N \),

\[
|c_{jk}(A)| \leq c(N) M^{N-1}, \quad |c_{jk,rs}(A)| \leq c(N) M^{N-2}.
\]

Note that if \( g \equiv c \) on \( S^{n-1} \) then \( Q(g; u) = cI_{n-1} \) for every \( u \in S^{n-1} \).

2.2. The Cheng-Yau lemma and an extension. Let \( h \in C^{2,+}(S^{n-1}) \), and assume additionally that \( h \in C^3(S^{n-1}) \). Consider the co-factor matrix \( y \to C[Q(h; y)] \). This is a matrix of functions on \( S^{n-1} \). The lemma of Cheng and Yau (10) asserts that each column of this matrix is divergence-free.

Lemma 2.5 (Cheng-Yau.). Let \( h \in C^{2,+}(S^{n-1}) \cap C^3(S^{n-1}) \). Then, for every index \( j \in \{1, \ldots, n-1\} \) and for every \( y \in S^{n-1} \),

\[
\sum_{i=1}^{n-1} (c_{ij}[Q(h; y)])_i = 0,
\]

where the sub-script \( i \) denotes the derivative with respect to the \( i \)-th element of an orthonormal frame on \( S^{n-1} \).

For simplicity of notation we shall often write \( C(h), c_{ij}(h) \) and \( c_{ij,kl}(h) \) in place of \( C[Q(h)], c_{ij}[Q(h)] \) and \( c_{ij,kl}[Q(h)] \) respectively.

As a corollary of the previous result we have the following integration by parts formula. If \( h \in C^{2,+}(S^{n-1}) \cap C^3(S^{n-1}) \) and \( \psi, \phi \in C^2(S^{n-1}) \), then

(8) \[
\int_{S^{n-1}} \phi c_{ij}(h)(\psi_{ij} + \psi \delta_{ij})dy = \int_{S^{n-1}} \psi c_{ij}(h)(\phi_{ij} + \phi \delta_{ij})dy.
\]

The Lemma of Cheng and Yau admits the following extension (see Lemma 2.3 in the paper by the first-named author, Hug and Saorín-Gómez [14]). Note that we adopt the summation convention over repeated indices.
Lemma 2.6. Let $\psi \in C^2(S^{n-1})$ and $h \in C^2_+ (S^{n-1}) \cap C^3(S^{n-1})$. Then, for every $k \in \{1, \ldots, n-1\}$ and for every $y \in S^{n-1}$

$$
\sum_{l=1}^{n-1} (c_{ij,kl}(Q(h;y))(\psi_{ij} + \psi \delta_{ij}))_l = 0.
$$

Correspondingly we have, for every $h \in C^2_+ (S^{n-1}) \cap C^3(S^{n-1}), \psi, \varphi, \phi \in C^2(S^{n-1})$,

$$
\int_{S^{n-1}} \psi c_{ij,kl}(h)(\varphi_{ij} + \varphi \delta_{ij})(\phi_{kl} + \phi \delta_{kl})dy
$$

$$
= \int_{S^{n-1}} \phi c_{ij,kl}(h)(\varphi_{ij} + \varphi \delta_{ij})(\psi_{kl} + \psi \delta_{kl})dy.
$$

(9)

2.3. A Poincaré inequality for even functions on the sphere. Here we use some basic facts from the theory of spherical harmonics, which can be found, for instance in [48, Appendix]. We denote by $\Delta_\sigma$ the spherical Laplace operator (or Laplace-Beltrami operator), on $S^{n-1}$. The first eigenvalue of $\Delta_\sigma$ is 0, and the corresponding eigenspace is formed by constant functions. The second eigenvalue of $\Delta_\sigma$ is $n-1$, and the corresponding eigenspace is formed by the restrictions of linear functions of $\mathbb{R}^n$ to $S^{n-1}$. The third eigenvalue is $2n$, which implies, in particular, that for any even function $\psi \in C^2(S^{n-1})$ such that

$$
\int_{S^{n-1}} \psi du = 0,
$$

one has

(10)

$$
\int_{S^{n-1}} \psi^2 du \leq \frac{1}{2n} \int_{S^{n-1}} |\nabla_s \psi|^2 du.
$$

3. Computations of derivatives

Let $\psi \in C^2(S^{n-1})$, and let $s > 0$. We consider the function $h_s(u) = e^{s\psi(u)}$. We will denote derivatives with respect to the parameter $s$ by a dot, e.g.:

$$
\dot{h}_s(u) = \frac{d}{ds} h(u), \quad \ddot{h}_s(u) = \frac{d^2}{ds^2} h(u), \ldots
$$

Note that

(11)

$$
\dot{h}_s = \psi \dot{h}_s, \quad \ddot{h}_s = \psi^2 \dot{h}_s, \quad \dddot{h}_s = \psi^3 \dot{h}_s.
$$

Remark 3.1. As we may interchange the order of derivatives, for every $j, k = 1, \ldots, n-1$ we have

$$
q_{jk}(h) = \dot{q}_{jk}(h),
$$

and thus

$$
\dot{Q}(h) = Q(h).
$$

Similar equalities hold for successive derivatives in $s$.

Consider the volume function

(12)

$$
f(s) = \frac{1}{n} \int_{S^{n-1}} h_s(u) \det(Q(h_s; u))du.
$$

If $h_s$ is the support function of a convex body $K_s$ (as it will be in the sequel), $f$ represents the volume of $K_s$. 
Remark 3.2. The entries of $Q(h_s)$ are continuous functions of the second derivatives of $h_s$ and $Q(h_0) > 0$. Hence there exists $\eta_0 > 0$ such that if $\psi \in C^2(S^{n-1})$ is such that $\|\psi\|_{C^2(S^{n-1})} \leq \eta_0$, then

\begin{equation}
Q(e^{s\psi}; u) > 0 \quad \forall \ u \in S^{n-1}, \forall \ s \in [-2, 2].
\end{equation}

We shall use notation

\begin{equation}
\mathcal{U} = \{\psi \in C^2(S^{n-1}); \|\psi\|_{C^2(S^{n-1})} \leq \eta_0\}.
\end{equation}

Note that if $\psi \in \mathcal{U}$ then $f > 0$ in $[-2, 2]$. Moreover, in the case $h_0 \equiv 1$ we have $Q(h_0) = I_{n-1}$, and

\begin{equation}
f(0) = \frac{1}{n} |S^{n-1}|.
\end{equation}

Lemma 3.3. In the notations introduced above, we have, for every $s$:

\begin{equation}
f'(s) = \int_{S^{n-1}} \psi h_s \det(Q(h_s)) du;
\end{equation}

\begin{equation}
f''(s) = \int_{S^{n-1}} \left[\psi^2 h_s \det(Q(h_s)) + \psi h_s c_{jk}(h_s) q_{jk}(\psi h_s)\right] du;
\end{equation}

\begin{equation}
f'''(s) = \int_{S^{n-1}} h_s \left[\psi^3 \det(Q(h_s)) + 2\psi^2 c_{jk}(h_s) q_{jk}(\psi h_s)\right] du + \int_{S^{n-1}} h_s \left[\psi c_{jk,rs}(h_s) q_{jk}(\psi h_s) q_{rs}(\psi h_s) + c_{jk}(h_s) q_{jk}(\psi^2 h_s)\right] du.
\end{equation}

Proof. For brevity, we set

\[ c_{jk}(h) = c_{jk}(Q(h)). \]

We differentiate the function $f$ in $s$, and we adopt the summation convention over repeated indices.

\begin{align*}
f'(s) &= \frac{1}{n} \int_{S^{n-1}} \left[\dot{h} \det(Q(h)) + \dot{h} c_{jk}(h) \dot{q}_{jk}(h)\right] dy \\
&= \frac{1}{n} \int_{S^{n-1}} \left[\dot{h} \det(Q(h)) + \dot{h} c_{jk}(h) q_{jk}(\dot{h})\right] dy \\
&= \frac{1}{n} \int_{S^{n-1}} \left[\dot{h} \det(Q(h)) + \dot{h} c_{jk}(h) q_{jk}(h)\right] dy \\
&= \int_{S^{n-1}} \dot{h} \det(Q(h)) dy.
\end{align*}

Above we have used Remark 3.1 and the integration by parts formula (8).

Passing to the second derivative, we get:

\begin{align*}
f''(s) &= \int_{S^{n-1}} \left[\ddot{h} \det(Q(h)) + \ddot{h} c_{jk}(h) \dot{q}_{jk}(h)\right] du \\
&= \int_{S^{n-1}} \left[\ddot{h} \det(Q(h)) + \ddot{h} c_{jk}(h) q_{jk}(\dot{h})\right] du.
\end{align*}

Finally

\begin{align*}
f'''(s) &= \int_{S^{n-1}} \left[\dddot{h} \det(Q(h)) + 2\dddot{h} c_{jk}(h) q_{jk}(\dot{h})\right] du + \int_{S^{n-1}} \left\{\dddot{h} \left[c_{jk,rs}(h) q_{jk}(\dot{h}) q_{rs}(\dot{h}) + c_{jk} q_{jk}(\ddot{h})\right]\right\} du.
\end{align*}
Equalities (15), (16) and (17) follow from (11).

The next Corollary has appeared in [15].

Corollary 3.4. In the notations introduced before we have:

\[(18) \quad f'(0) = \int_{S^{n-1}} \psi du;\]
\[(19) \quad f''(0) = \int_{S^{n-1}} [n\psi^2 - |\nabla_s \psi|^2] du.\]

Proof. Equality (18) follows immediately from (15). Moreover, plugging \(s = 0\) in (16), and using the facts
\[c_{jk}(h_0) = \delta_{jk} \quad \text{and} \quad q_{jk}(\psi) = (\psi_{jk} + \psi \delta_{kj})\]
for every \(j, k = 1, \ldots, n - 1\), we get
\[f''(0) = \int_{S^{n-1}} [n\psi^2 + \psi \Delta_s \psi] du.\]

By the divergence theorem on \(S^{n-1}\) we deduce (19).

Lemma 3.5. For every \(\rho > 0\) there exists \(\eta > 0\), such that if \(\psi \in \mathcal{U}\) is an even function and it verifies:

- \(\int_{S^{n-1}} \psi du = 0;\)
- \(\|\psi\|_{C^2(S^{n-1})} \leq \eta;\)

then
\[|(\log f)'''(s)| \leq \rho \|\nabla_s \psi\|_{L^2(S^{n-1})}^2, \quad \forall \ s \in [-2, 2],\]
where \(f\) is defined as in (12) and \(h_s = e^{s\psi}\).

Proof. We have
\[(\log f)'''(s) = \frac{f'''(s)}{f(s)} - 3 \frac{f'(s)f''(s)}{f^2(s)} + 2 \frac{(f')^3(s)}{f^3(s)};\]
we first fix \(\eta_1 > 0\) such that \(\|\psi\|_{C^2(S^{n-1})} \leq \eta_1\) implies
\[f(s) \geq \frac{1}{4n}|S^{n-1}| = \frac{1}{4}f(0), \quad \forall \ s \in [-2, 2].\]
Hence
\[|(\log f)'''(s)| \leq C_0(|f'''(s)| + |f'(s)f''(s)| + |(f')^3(s)|) = C_0(T_1 + T_2 + T_3),\]
for some constant \(C_0 = C_0(n) = \frac{|S^{n-1}|}{n}\). Throughout this proof, we will denote by \(C\) a generic positive constant dependent on the dimension \(n\) and \(\eta_1\).

Bound of the term \(T_3\). There exists \(C\) such that
\[\|h_s\|_{C^2(S^{n-1})} = \|e^{s\psi}\|_{C^2(S^{n-1})} \leq C,\]
for every \(s \in [-2, 2]\) and every \(\psi \in \mathcal{U}\). Therefore
\[h_s(u) \det(Q(h_s; u)) \leq C, \quad \forall \ \psi \in \mathcal{U}.\]
Consequently, by Lemma 3.3 we may write two types of estimates
\[|f'(s)| \leq C\|\psi\|_{C^2(S^{n-1})}, \quad |f''(s)| \leq C\|\psi\|_{L^2(S^{n-1})}.\]
By (10), there exists \( \eta' > 0 \) such that

\[
|f''(s)| \leq C \| \psi \|^2_{L^2(\mathbb{R}^n)} + \int_{\mathbb{R}^n} \psi h_s c_{jk}(h_s)(\psi h_s \delta_{jk} + (\psi h_s)_{jk}) du
\]

if \( \psi \) verifies \( \| \psi \|^2_{C^2(\mathbb{R}^n)} \leq \eta' \).

**Bound of the term** \( T_2 \). By Lemma 3.3 (10) and the integration by parts formula (8), we have

\[
|f''(s)| \leq C \| \psi \|^2_{L^2(\mathbb{R}^n)} + \int_{\mathbb{R}^n} \psi h_s c_{jk}(h_s)(\psi h_s \delta_{jk} + (\psi h_s)_{jk}) du
\]

\[
\leq C \| \psi \|^2_{L^2(\mathbb{R}^n)} + \int_{\mathbb{R}^n} c_{jk}(h_s)(\psi h_s)_j(\psi h_s)_k du
\]

\[
\leq C \| \psi \|^2_{L^2(\mathbb{R}^n)} + C \| \nabla_s \psi \|^2_{L^2(\mathbb{R}^n)}
\]

(note that the first term was bounded using the argument as for the previous part of this proof). Hence we have the bound (20) for \( T_2 \) as well.

**Bound of the term** \( T_1 \). Equality (17) provides an expression of \( f'''(s) \) as the sum of four terms. Each of them can be treated as in the previous two cases, with the exception of \( \int_{\mathbb{R}^n} \psi h_s c_{jk,rs}(h_s) q_{rs}(\psi h_s) q_{jk}(\psi h_s) du \). We estimate it as follows:

\[
\left| \int_{\mathbb{R}^n} \psi h_s c_{jk,rs}(h_s) q_{rs}(\psi h_s) q_{jk}(\psi h_s) du \right|
\]

\[
\leq \int_{\mathbb{R}^n} \psi^2 h_s^2 c_{jk,rs}(h_s) q_{rs}(\psi h_s) \delta_{jk} du + \int_{\mathbb{R}^n} \psi h_s c_{jk,rs}(h_s) q_{rs}(\psi h_s)(\psi h_s)_{jk} du
\]

\[
\leq C \| \psi \|_{C^2(\mathbb{R}^n)} \| \psi \|^2_{L^2(\mathbb{R}^n)} + \int_{\mathbb{R}^n} c_{jk,rs}(h_s) q_{rs}(\psi h_s)(\psi h_s)_j(\psi h_s)_k du
\]

\[
\leq C \| \psi \|_{C^2(\mathbb{R}^n)} \| \psi \|^2_{L^2(\mathbb{R}^n)} + C \| \psi \|_{C^2(\mathbb{R}^n)} \| \nabla_s \psi \|^2_{L^2(\mathbb{R}^n)}
\]

We deduce that the upper bound (20) can be established for \( T_1 \). This concludes the proof. \( \square \)

**Lemma 3.6.** Let \( f \) be defined by (12). There exists \( \eta > 0 \) such that for every even \( \psi \in \mathcal{U} \) so that \( \| \psi \|_{C^2(\mathbb{R}^n)} \leq \eta \), the function \( \log(f(s)) \), is concave in \([-2, 2]\). Moreover it is strictly concave in this interval unless \( \psi \) is constant.

**Proof.** We first assume that

\[
(\log f)'''(s) = (\log f)'''(0) + s(\log f)''(0) = \frac{f(0)f''(0) - f'(0)^2}{f(0)^2} + s(\log f)''(0).
\]

It is shown in Lemma 3.3 that, for an arbitrary \( \rho > 0 \) there exists \( \eta > 0 \) such that if \( \| \psi \|_{C^2(\mathbb{R}^n)} \leq \eta \) then

\[
(\log f)'''(s) \leq \rho \| \nabla_s \psi \|^2_{L^2(\mathbb{R}^n)}, \quad \forall s \in [-2, 2].
\]
Using the last inequality along with Lemma 3.4 and (21) we have
\[(\log f)''(s) \leq \frac{1}{|S^{n-1}|} \int_{S^{n-1}} (n\psi^2(u) - |\nabla_s\psi(u)||^2)du + \rho||\nabla_s\psi||^2_{L^2}.
\]
By (10) we deduce
\[(\log f)''(s) \leq ||\nabla_s\psi||^2_{L^2(S^{n-1})} \left(2\rho - \frac{1}{2|S^{n-1}|}\right),
\]
which is negative as long as
\[\rho < \frac{1}{4|S^{n-1}|},
\]
and, with this choice, strictly negative unless \(\psi\) is a constant function.

Next we drop the assumption (21). For \(\psi \in C^2(S^{n-1})\), let
\[m_\psi = \frac{1}{|S^{n-1}|} \int_{S^{n-1}} \psi du, \quad \text{and} \quad \tilde{\psi} = \psi - m_\psi.
\]
Clearly \(\tilde{\psi} \in C^2(S^{n-1})\) and \(\tilde{\psi}\) verifies condition (21). Moreover,
\[||\tilde{\psi}||_{C^2(S^{n-1})} \leq ||\psi||_{C^2(S^{n-1})} + |m_\psi| \leq 2||\psi||_{C^2(S^{n-1})}.
\]
Consequently, \(\tilde{\psi} \in \mathcal{U}\) if \(||\psi||_{C^2(S^{n-1})} \leq \eta_0/2\). We also have:
\[\tilde{h}_s := e^{s\tilde{\psi}} = e^{s(\psi - m_\psi)} = e^{-sm_\psi} h_s.
\]
Hence
\[Q(\tilde{h}_s) = e^{-sm_\psi} Q(h_s).
\]
Consider
\[\tilde{f}(s) := \frac{1}{n} \int_{S^{n-1}} \tilde{h}_s \det(Q(\tilde{h}_s))du = e^{-nsm_\psi} f(s).
\]
We observe that \(\log(\tilde{f})\) and \(\log(f)\) differ by a linear term and convexity (resp. strict convexity) of \(f\) is equivalent to convexity (resp. strict convexity) of \(\tilde{f}\). On the other hand, by the first part of this proof \(\log(\tilde{f})\) is concave as long as \(||\tilde{\psi}||_{C^2(S^{n-1})}\) is sufficiently small, and this condition is verified when, in turn, \(||\psi||_{C^2(S^{n-1})}\) is sufficiently small. The proof is concluded.

\[\square\]

4. PROOF OF THE THEOREM.

We denote by \(B^n_2\) the unit ball in \(\mathbb{R}^n\), centered at the origin. The next Lemma significantly strengthens Theorem 1.4 from [15] in the case of Lebesgue measure, since we establish the existence of the neighborhood whose radius depends on dimension only. We remark, that it follows from a very recent Theorem 1.2 by Kolesnikov and Milman [25], which was unknown to us as of this writing.

**Lemma 4.1.** Let \(R > 0\) and \(n \geq 2\). There exists \(\varepsilon(n)\) such that for every symmetric convex \(C^2\)-smooth body \(K\) in \(\mathbb{R}^n\) such that \(\|h - R\|_{C^2(S^{n-1})} \leq \varepsilon(n)R\), where \(h\) is the support function of \(K\), we have
\[|\lambda K +_0 (1 - \lambda)(RB^n_2)| \geq |K|^\lambda |RB^n_2|^{1-\lambda} \quad \forall \lambda \in [0, 1].
\]
Moreover, equality holds if and only if \(K\) is a ball centered at the origin.
Proof. We assume $R = 1$; the general case can be deduced by a scaling argument.

We first suppose that $\|h - 1\|_{C^2(S^{n-1})} \leq 1/4$. This implies that $h > 0$ on $S^{n-1}$ and therefore we may write $h$ in the form $h = e^{\psi}$, where $\psi = \log(h) \in C^2(S^{n-1})$.

We select $\varepsilon_0 > 0$ such that $\|h - 1\|_{C^2(S^{n-1})} \leq \varepsilon_0$ implies $\|\psi\|_{C^2(S^{n-1})} \leq \eta_0$, i.e. $\psi \in U$ (see Remark 3.2). As a consequence of Proposition 2.1, $h_s = e^{s\psi}$ is the support function of a $C^{2,\beta}$ convex body, for every $s \in [-2, 2]$. In particular, for every $\lambda \in [0, 1]$, the function $e^{\lambda\psi}$ is the support function of $K^\lambda(B^n_2)^{1-\lambda}$.

There exists $\varepsilon > 0$ such that $\|h - 1\|_{C^2(S^{n-1})} \leq \varepsilon$ implies $\|\psi\|_{C^2(S^{n-1})} \leq \eta$, where $\eta > 0$ is the quantity indicated in Lemma 3.6. By the conclusion of Lemma 3.6 the function $f(\lambda) = |K^\lambda(B^n_2)|^{1-\lambda}$ is log-concave, and hence (22) follows. The equality case follows from the fact that the log-concavity of $f$ is strict unless $\psi$ is a constant function, which corresponds to the case when $K$ is a ball.

Lemma 4.1 implies:

**Lemma 4.2.** Let $R > 0$. Let $K$ be a symmetric strictly convex smooth body in $\mathbb{R}^n$, such that $\|h - R\|_{C^2(S^{n-1})} \leq \varepsilon(n)R$, where $h$ is the support function of $K$, and $\varepsilon(n)$ refers to Lemma 4.1. Then

\[
\int_{S^{n-1}} \log \frac{R}{h_K} dK(u) \geq |K| \log \frac{|K|}{|RB^n_2|},
\]

and

\[
\int_{S^{n-1}} \log \frac{h_K}{R} dRB^n_2(u) \geq |RB^n_2| \log \frac{|RB^n_2|}{|K|}.
\]

Moreover, the inequality in both inequalities holds if and only if $K$ is a Euclidean ball.

The implication Lemma 4.1 $\Rightarrow$ Lemma 4.2 follows from the fact that for every fixed pair of convex bodies $K$ and $L$, the validity of (22) implies the validity of (23) and (24), together with their equality cases, as was proved in [7]. We remark that for the reverse implication one requires the validity of log-Minkowski inequality for a larger class of sets, but that is not the direction we require.

**Remark 4.3.** Inequality (24), in fact, holds for symmetric convex sets $K$ not necessarily in a neighborhood of a ball; this is a simple consequence of Blaschke-Santaló inequality, and was proved by Guan and Ni in [20].

4.1. **Proof of the Theorem 1.2** Firstly, by integrating the condition $dK(u) = R^n du$ over the sphere, we get $|K| = |RB^n_2|$. Therefore, by (23)

\[
\int_{S^{n-1}} \log \frac{R}{h_K} dK(u) \geq 0,
\]

or, equivalently,

\[
\int_{S^{n-1}} \log RdK(u) \geq \int_{S^{n-1}} \log h_K dK(u).
\]

Using the fact that $dK(u) = R^n du$ once again, and then applying (24) we see that the right hand side of the above is equal to

\[
\int_{S^{n-1}} \log h_K dRB^n_2(u) \geq \int_{S^{n-1}} \log RD^n_2(u).
\]
Using the condition of the theorem once more, we see that the above is equal to
\[
\int_{S^{n-1}} \log Rdc_K(u).
\]
We have obtained a chain of inequalities starting and ending with the same expression, and hence equality must hold in all the inequalities. By Lemma 4.2 we conclude that \( K \) is a Euclidean ball. Since, in addition, \( |K| = |RB^n_2| \), we see that \( K = RB^n_2 \), which finishes the proof of the theorem. \( \square \)

REFERENCES

[1] K. Ball, An elementary introduction to modern convex geometry, Flavors of Geometry, MSRI Publications Volume 31, (1997), 58.
[2] K. Ball, Some remarks on the geometry of convex sets, Geometric aspects of functional analysis (1986/87), 224-231, Lecture Notes in Math., 1317, Springer, Berlin, 1988.
[3] S. Bobkov, M. Ledoux, From Brunn-Minkowski to Brascamp-Lieb and to logarithmic Sobolev inequalities, Geom. Funct. Anal. 10 (2000), no. 5, 1028-1052.
[4] S. Bobkov, M. Ledoux, From Brunn-Minkowski to sharp Sobolev inequalities, Ann. Mat. Pura Appl. (4) 187 (2008), no. 3, 369-384.
[5] T. Bonnesen, W. Fenchel Theory of convex bodies, BCS Associates, Moscow, Idaho, (1987), 173.
[6] C. Borell, Convex set functions in d-space, Period. Math. Hungar. 6 (1975), 111-136.
[7] K.J. Böröczky, E. Lutwak, D. Yang, G. Zhang, The log-Brunn-Minkowski inequality, Advances in Mathematics, 231 (2012), 1974-1997.
[8] K. J. Böröczky, E. Lutwak, D. Yang, G. Zhang, The logarithmic Minkowski Problem, Journal of the American Mathematical Society, 26 (2013), 831-852.
[9] K. J. Böröczky, E. Lutwak, D. Yang, G. Zhang, Affine images of isotropic measures, J. Diff. Geom., 99 (2015), 407-442.
[10] S. T. Cheng, S. T. Yau, On the regularity of solutions of the ν-dimensional Minkowski problem, Comm. Pure Appl. Math. 29 (1976), 495-516.
[11] A. Colesanti, From the Brunn-Minkowski inequality to a class of Poincaré type inequalities, Communications in Contemporary Mathematics, 10 n. 5 (2008), 765-772.
[12] A. Colesanti, D. Hug, E. Saorín-Gómez, A characterization of some mixed volumes via the Brunn-Minkowski inequality, Journal of Geometric Analysis (2012), 1-28.
[13] A. Colesanti, Log-concave functions, in Concentration, convexity and discrete structures, E. Carlen, M. Madiman, E. Werner eds. Springer, Berlin, 2017.
[14] A. Colesanti, D. Hug, E. Saorín-Gómez, Monotonicity and concavity of integral functionals, Communications in Contemporary Mathematics 19 (2017), 1-26.
[15] A. Colesanti, G. Livihsyts, A. Marsiglietti, On the stability of Brunn-Minkowski type inequalities, Journal of Functional Analysis, 273 (2017), 1120-1139.
[16] D. Cordero-Erausquin, M. Fradelizi, B. Maurey, The (B) conjecture for the Gaussian measure of dilates of symmetric convex sets and related problems, Journal of Functional Analysis 214 (2004) 410-427.
[17] R. Gardner, The Brunn-Minkowski inequality, Bull. Amer. Math. Soc. 39 (2002), 355-405.
[18] R. Gardner, A. Zvavitch, Gaussian Brunn-Minkowski-type inequalities, Trans. Amer. Math. Soc. 360 (2010), 5333-5353.
[19] H. Groemer, Geometric Applications of Fourier Series and Spherical Harmonics, Cambridge University Press, Cambridge, 1996.
[20] P. Guan, L. Ni, Entropy and a convergence theorem for Gauss curvature flow in high dimension, arXiv:1306.0625 (2013)
[21] M. Henk, E. Linke, Cone-volume measures of polytopes, http://arxiv.org/abs/1305.5335
[22] A. Koldobsky, Fourier Analysis in Convex Geometry, Math. Surveys and Monographs, AMS, Providence RI, 2005.
[23] A. V. Kolesnikov, E.Milman, Sharp Poincaré-type inequality for the Gaussian measure on the boundary of convex sets, to appear in GAFA Seminar Notes.
[24] A. V. Kolesnikov, E. Milman, Riemannian metrics on convex sets with applications to Poincaré and log-Sobolev inequalities, preprint.
[25] A. V. Kolesnikov, E. Milman, Local \( L_p \)-Brunn-Minkowski inequalities for \( p \neq 1 \), preprint.
[26] R. Latała, *On some inequalities for Gaussian measures*, Proceedings of the International Congress of Mathematicians, Beijing, Vol. II, Higher Ed. Press, Beijing, 2002, 813-822.

[27] L. Leindler, *On a certain converse of Hölder’s inequality. II*, Acta Sci. Math. (Szeged) 33 (1972), no. 3-4, 217-223.

[28] A. Livne Bar-on, *The B-conjecture for uniform measures in the plane*, in GAF A Seminar Notes, Lecture Notes in Math. 2116, Springer, Berlin, 2014, 10-?.

[29] G. V. Livshyts, *Maximal Surface Area of a convex set in \( \mathbb{R}^n \) with respect to log-concave rotation invariant measures*, in GAF A Seminar Notes, Lecture Notes in Math. 2116, Springer, Berlin, 2014, 355-384.

[30] G. Livshyts, A. Marsiglietti, P. Nayar, A. Zvavitch, *On the Brunn-Minkowski inequality for general measures with applications to new isoperimetric-type inequalities*, to appear in the Transactions of the AMS.

[31] E. Lutwak, *The Brunn-Minkowski-Firey theory I: Mixed volumes and the Minkowski problem*, J. Differential Geom. 38 (1993), no. 1, 131-150.

[32] E. Lutwak, *The Brunn-Minkowski-Firey theory II: Affine and geominimal surface areas*, Adv. Math. 118 (1996), 244-294.

[33] E. Lutwak and V. Oliker, *On the regularity of solutions to a generalization of the Minkowski problem*, J. Differential Geom. 41 (1995), no. 1, 227-246.

[34] E. Lutwak, D. Yang and G. Zhang, *\( L_p \) affine isoperimetric inequalities*, J. Diff. Geom. 56 (2000), 111-132.

[35] E. Lutwak, D. Yang and G. Zhang, *Sharp affine \( L_p \) Sobolev inequalities*, J. Diff. Geom. 62 (2002), 17-38.

[36] E. Lutwak, D. Yang and G. Zhang, *On the \( L_p \)-Minkowski problem*, Trans. Amer. Math. Soc. 356 (2004), 4359-4370.

[37] E. Lutwak, D. Yang and G. Zhang, *Optimal Sobolev norms and the \( L_p \) Minkowski problem*, IMRN, 2006, Art. ID 62987, 21 pp.

[38] A. Marsiglietti, *On the improvement of concavity of convex measures*, Proc. Amer. Math. Soc. 144 (2016), no. 2, 775-786.

[39] B. Maurey, *Some deviation inequalities*, Geom. Funct. Anal. 1 (1991), no. 2, 188-197.

[40] V. D. Milman, G. Schechtman, *Asymptotic Theory of finite-dimensional normed spaces*, Lecture notes in Math. 163, Springer, Berlin, 1980.

[41] A. Naor, *The Surface Measure and Cone Measure on the sphere of \( l_p \)*, Transactions of the AMS, Volume 359, Number 3 (March 2007), 1045-1079.

[42] P. Nayar, T. Tkocz, *A Note on a Brunn-Minkowski Inequality for the Gaussian Measure*, Proc. Amer. Math. Soc. 141 (2013), 11, 4027-4030.

[43] P. L. Nazarov, *On the maximal perimeter of a convex set in \( \mathbb{R}^n \) with respect to Gaussian measure*, Geometric Aspects of Func. Anal., 1807 (2003), 169-187.

[44] A. Prékopa, *logarithmic concave measures with applications to stochastic programming*, Acta Sci. Math. (Szeged) 32 (1971), 301-316.

[45] L. Rotem, *A letter: The log-Brunn-Minkowski inequality for complex bodies*, [http://www.tau.ac.il/~liranro1/papers/complexletter.pdf](http://www.tau.ac.il/~liranro1/papers/complexletter.pdf)

[46] C. Saroglou, *Remarks on the conjectured log-Brunn-Minkowski inequality*, Geom. Dedicata 177 (2015), 353-365.

[47] C. Saroglou, *More on logarithmic sums of convex bodies*, preprint, arXiv:1409.4346 [math.FA].

[48] R. Schneider, *Convex bodies: the Brunn-Minkowski theory, second expanded edition*, Encyclopedia of Mathematics and its Applications, Cambridge University Press, Cambridge, 2013.

[49] B. Uhrin, *Curvilinear extensions of the Brunn-Minkowski-Lusternik inequality*, Adv. Math. 109 (2) (1994), 288-312.

**DIPARTIMENTO DI MATematica E Informatica “U. Dini”, Università degli Studi di Firenze**

**E-mail address:** colesant@math.unifi.it

**School of Mathematics, Georgia Institute of Technology**

**E-mail address:** glivshyts6@math.gatech.edu