Errors and misconceptions in learning elementary linear algebra

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Abstract. Mostly student meet difficulty in learning elementary linear algebra (ELA). The aim of this research is to discuss errors, misconceptions and their causes in learning ELA. The subjects of this research were 41 chemistry students who took part in ELA course during the 2017/2018 academic year at Andalas University. The study use qualitative methods to explain how students reason to form errors and misconceptions. The participants were asked to answer four questions in system of linear equations as a part of the mid term examination questions. Students' solutions to the mid term examination questions were taken as the key source of data used to generate a list of errors and to extract the underlying misconceptions. The findings show that the average of student unable to: understanding the problem (E1), devising a plan (E2), carrying out the plan (E3.1 & E3.2), verified the solution (E4), respectively are E1 = 6.61%, E2 = 28.40%, E3.1 = 24.43%, E3.2 = 2.27%, E4 = 89.23% and one misconceptions revealed, that is Cramer's rule can be used to solve every kind of SLE.

1. Introduction

The system of linear equations (SLE) is one of the topics in ELA which have very important applications in the field of science and technology, so SLE is important to be included in any syllabus of undergraduate linear algebra [1,2]. In fact, some researcher (i.e. [3-10]), have found out that mostly student meet difficulty in ELA. According to Carlson, (1) the course is taught too early and the the students are to unsophisticated; (2) concept are introduced with less examples or applications [11]. According to Dubinsky, Study on the specific mental constructions that a student might make in order to understand these concepts is needed [12].

Misconceptions are one of the main reasons for students’ poor performance in mathematics [8]. Investigating misconceptions that a student might make in learning SLE, provides a solid foundation to consider pedagogical strategies for teaching and learning SLE.

Consider the SLE \( Ax = c \), where \( A = (a_{ij}) \) is an \( mxn \) matrix, \( y = (y_1, y_2, \ldots, y_n) \) is a vector in \( \mathbb{R}^n \), \( c = (c_1, c_2, \ldots, c_m) \) is a vector in \( \mathbb{R}^m \). A solution of a SLE in \( n \) unknowns, \( Ay = c \), is a sequence \( y_1 = s_1, y_2 = s_2, \ldots, y_n = s_n \) for which the substitution makes each equation a true statement, every SLE has no solution, one, or infinitely many solutions [13,14]. If SLE \( Ay = c \) has infinitely many solutions, then its solutions can be obtained by Gaussian elimination and Gauss-Jordan elimination, and if SLE \( Ay = c \) has only one solution, its solution also can be obtained by Cramer’s rule, and \( yx = A^{-1}c \), where \( A^{-1} \) is an inverse of \( A \[13\]. Learning outcome in SLE at department of chemistry Andalas University consists of: (1) solve SLE with one solution; (2) solve SLE with infinitele many solutions; (3) determine the condition
when the SLE has zero (no solution), one, or infinitely many solutions; (4) apply SLE to solve daily life problem. There were several ways to classify student conceptual understanding in mathematics, in abstract algebra, Hart was proposed four level of conceptual understanding [15]. The level of students’ achievement in SLE describe the capabilities in SLE, so can be used to classify student conceptual understanding in SLE.

In mathematics education, what is mean by errors and misconceptions? Errors and misconceptions are different, but they are related. According to Luneta & Makonye [16], an error is inaccuracies, but a misconception is the result of misunderstanding [17]. A misconception happen when a student believes a false concept as a true concept [18], and an error can be caused by the misconception [19]. Radatz [18] & Newman [20], classify students’errors in mathematical problem solving into certain categorizations. According to Polya, there are four steps in mathematical problem solving. These steps are: (1) understanding the problem, (2) devising a plan, (3) carrying out the plan, (4) verified solution [21], and then Schoenfeld expand these steps into five stages [22].

These study are devoted to answer “what are the student’s errors and misconceptions in learning elementary linear algebra and what are the possible causes of these errors and misconceptions”.

2. Materials and Methods
The study was undertaken through qualitative methods, qualitative methods is suitable to explain how students reason to form errors and misconceptions [23]. The subjects of this research were 41 student in ELA course offered to chemistry student during the 2017/2018 academic year at Andalas University students. Students’ solutions to the problem solving test (PST) were taken as the key source of data used to generate a list of errors and to extract the underlying misconceptions. Selected students were interviewed, the aim of the interviews was to provide additional information about common student errors on the test and to aid the process of understanding the underlying cause of these errors [24]. This determined the selection of questions for the interview. The initial analysis of the errors made on the test generated four broad categories of errors: Errors made in understanding the problem (E1); Errors which occurred in devising a plan (E2); Errors which appeared in carrying out the plan (E3); unverified solution (E4). Especially for E3 will be detail as: Technical error (e.g., basic skills in arithmetic) (E3.1); Wrong interpretation of symbols (E3.2) [21].

Criteria used in the development of the test were: it was designed to cover a range of learning outcome in SLE; as the aim of the research project was to develop an understanding of students’ errors, misconceptions and their causes in SLE, the test was designed to cover all methods in solve SLE. A first draft of the test was given to the two lecturers teaching first-year Mathematics Chemistry II courses at Andalas University. These lecturers were asked to comment on the standard of questions in the test and on the clarity of the questions. The final instrument of this study list in Table 1.

Each Students’ solutions to PST was graded on a scale of 0 to 4 in accordance with the level of completeness of student’s solutions. Based on the correctness (score 3 or 4) of their solution to PST, student were classified into one of four levels of conceptual understanding as follows: (1) level 0 (none of the four PST were correct); (2) level 1 (only PST1 was correct); level 2 (PST1 and PST2 were correct, but PST3 and PST4 were incorrect); level 3 (only PST4 was incorrect); level 4 ( all four PST were correct). Those student that not fit to one of the levels were classified as non fitters (NF) [15].
Table 1. Problem Solving Test (PST) for SLE

| Code | Question |
|------|----------|
| PST1 | Solve the following SLE |
|      | $5r + 3s + 2t = 17$ |
|      | $3r + 2s + t = 10$ |
|      | $r + s + t = 6$ |
| PST2 | Solve the following SLE |
|      | $3r + 2s + t = 5$ |
|      | $2r + s + t = -2$ |
|      | $5r + 3s + 2t = 3$ |
| PST3 | Determine $a$ and $b$ for which the system has no solutions, exactly one solution, or infinitely many solutions. |
|      | $r + s + t = 6$ |
|      | $r + 2s + 2t = 9$ |
|      | $2r + 2s + at = b$ |
| PST4 | An entrepreneur borrowed as much money as $6,000 from three banks with interest loans each bank is 6%, 7% and 8% every month, these entrepreneur each month pay interest loan of $440. Note that the number of loans in the bank’s third is three times the number of loans in the first bank. Determine the magnitude of the loans of the bank respectively. |

3. Result and Discussion

Based on 41 students’ solutions to problem solving test, the frequency and percentage of errors types related to forms: E1, E2, E3, and E4 are presented at Table 2.

| Errors types | PST1 | PST2 | PST3 | PST4 | Average |
|--------------|------|------|------|------|---------|
| E1           | 2 (4.88%) | 1 (2.44%) | 2 (4.88%) | 6 (14.63%) | 6.61% |
| E2           | 0 (0%) | 18 (45%) | 0 (0%) | 24 (68.57%) | 28.40% |
| E3.1         | 1 (2.56%) | 9 (40.90%) | 5 (12.82%) | 5 (45.45%) | 24.43% |
| E3.2         | 0 (0%) | 2 (9.09%) | 0 (0%) | 0 (0%) | 2.27% |
| E4           | 32 (84.21%) | 8 (72.72%) | 34 (100%) | 5 (100%) | 89.23% |

For PST1, Table 2 show that 2 of 41 students unable to understanding the problem (39 students can understanding the problem), all of 39 students that understanding the problem able in devising a plan, only one of 39 students (able in devising a plan) unable in carrying out the plan, causes basic skills in arithmetic, and 32 of 38 students (able in carrying out the plan) unverified its solution. For PST2, Table 2 show that 1 of 41 students unable to understanding the problem (40 students can understanding the problem), 18 of 40 students (understanding the problem) unable in devising a plan, 9 of 22 students (able in devising a plan) unable in carrying out the plan, causes wrong on basic skills in arithmetic, 2 of 22 students (able in carrying out the plan) unable in carrying out the plan, causes wrong on interpretation of symbols, 8 of 11 (able in carrying out the plan) unverified its solution. For PST 3, Table 2 show that 2 of 41 students unable to understanding the problem (39 students can understanding the problem), all students that understanding the problem, able in devising a plan, 5 of 39 students (able in devising a plan) unable in carrying out the plan, causes wrong on basic skills in arithmetic, and all students (34 of 34 student) that able in carrying out the plan, unverified its solution. For PST 4, Table 2 show that 6 of 41 students unable to understanding the problem (35 students can understanding the problem), 24 of 35 students that understanding the problem, unable in devising a plan, 5 of 11 students that able in devising a plan, unable in carrying out the plan, causes wrong on basic skills in arithmetic, and all students (5 of 5 students that able in carrying out the plan), unverified its solution.
On average, Table 2 shows that 6.61% of students unable to understand the problem, 28.40% of students able to understand the problem, 24.43% of students able to devise a plan, 24.43% of students able to carry out the plan, causes wrong on basic skills in arithmetic, 2.27% of students able to devise a plan unable to carry out the plan, causes wrong on interpretation of symbols, and 89.23% of students able to carry out the plan, unverified its solution.

Fachri is the only student unable to understand the problem of PST1 and PST2, to uncover the cause, researchers interviewed Fachri, quoting the results of the interview as follows.

Researchers: why not work on PST1 and PST2?
Fachri: I do not understand with “solve the system of linear equations.”
Researchers: If “solve the system of linear equations” is changed to “determine the solution of system of linear equations” are you understand?
Fachri: yes, I am understand,

Fatma is one of the students who are able to understand the problem of PST1, can choose the method for determining the solutions of PST1, i.e. Cramer’s rule, but fail to obtain a solution by the rules of mathematics, the cause is an error in the writing of determinant symbols in Cramer’s rule, quotation the results of the interview are as follows.

Researchers: Fatma, looking back your answered sheet of PST1, are sure of the truth of the answers?
Fatma: yes Sir, x = 1, y = 2, z = 3.
Researchers: looking back once again on what that you are wrote, it is mathematical symbol of determinat?
Fatma: hemm, oh yes Sir, I forgot to add the symbol of determinat, but the answer is correct Sir.

PST2, has infinitely many of solutions, so the methods that can be used is the method of Gaussian elimination and Gauss-Jordan Method. Riska is 1 of the 18 students who can not choose the proper method to solve PST2 quotation the results of the interview are as follows,

Researchers: why choose rules Cramer for PST2?
Riska: in my opinion, the Cramer’s rule is a simple methods.
Researchers: so, the solution is x = ∞, y = ∞, z = ∞? Is ∞ a real number?
Riska: yes Sir,
Researchers: Is Cramer’s rule can be used to find a solution of every system of linear equations?
Fachri encounter errors because the language factor, Fachri unable solve PST1 is caused by not familiar with the meaning of “Solve the system of linear equations” but if it is changed to “determine the solution of system of linear equations” Fachri can solve them. Fachri said that during the lecture of linear algebra, the phrase used in the search for solutions by the lecturer is” determine the solution of systems of linear equations “. So, in order to that similar things won’t happen again in the learning of linear algebra, the lecturer need to be consistent in the use of the sentence, and the sentence should be adjusted with the phrase used in the book reference, with the intent of enabling students helped in hooking the explanation lecturer in classrooms with the material in the book reference. These symptoms often occurred in the learning process, for example, Naseer in [25], stated that many students experience a lack of technical vocabulary, for example, students were unable to differentiate between “expand”, “factorize”, “solve”. Similar things are also expressed by Spooner in [17], that many students have problems in reading or interpreting the question in mathematics problem solving.

Fatma encounter errors due to the use of the mathematical symbol, Fatma failed in resolving system of linear equations caused by confusion in the use of the symbols of the determinant. In mathematics every symbol has the meaning of certain concepts which represent, so its use is not interchangeable, for example, the symbol ( ) or [ ] for the matrix and the symbol | | for determinant. Students errors in the use of mathematical symbols there have been many researched, for example, Elbrink found that many student in middle school or high school encounter errors in the use of the mathematics symbol [26].
4. Conclusions
Based on the results of the study it can be concluded that: (1) most of the students do not perform “looking back” in mathematical problem solving, (2) There are one misconception in learning system of linear equation that is Cramer’s rule can be used to find a solution of every system of linear equations, (3) Some errors in mathematical problem solving is caused by confusion in the use of the symbols and lack of technical vocabulary.

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