ERRATUM

PERIODS OF DRINFELD MODULES AND LOCAL SHTUKAS WITH COMPLEX MULTIPLICATION – ERRATA

URS HARTL \textsuperscript{1} AND RAJNEESH KUMAR SINGH \textsuperscript{2}

\textsuperscript{1}Universität Münster, Mathematisches Institut, Einsteinstr. 62, D-48149 Münster, Germany (https://www.uni-muenster.de/Arithm/hartl/)

\textsuperscript{2}Ramakrishna Mission, Vivekananda University, PO Belur Math, Dist Howrah 711202, West Bengal, India

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B. errata

B.1. First Error

In [2, formulas (1.13) and (1.12) and Definition 5.21] it is claimed that $v(\omega_\psi)$ and $v_\psi(u)$ are integers. However, in general they only lie in the rational numbers $\mathbb{Q}$, because the valuation $v$ is normalized to be an isomorphism $v: Q_v^\times/A_v^\times \to \mathbb{Z}$, but the arguments of $v$ in both formulas lie in $Q_{v, \text{alg}}$ instead of $Q_v$.

This error is harmless, as the integrality of $v_\psi(u)$ and $v(\omega_\psi)$ is nowhere used.

B.2. Second Error

In [2, formula (1.13) and Definition 4.10] there is an error in the definition of $v(\omega_\psi)$.

As in most of [2], we fix a finite separable semi-simple $Q$-algebra $E$. That is, $E$ is a product of finite separable field extensions of $Q$. We fix a finite place $v$ of $Q$ and consider the decomposition of the separable $Q_v$-algebra $E_v := E \otimes_Q Q_v = E_{v,1} \times \cdots \times E_{v,s}$ into a product of finite field extensions $E_{v,i}$ of $Q_v$ as after [2, Definition 4.1]. We fix a finite Galois extension $K \subset Q_{v, \text{alg}}$ of $Q$ and we let $L := K_v \subset Q_{v, \text{alg}}$ be the closure of $K$. It is a finite Galois extension of $Q_v$. We fix a $\psi \in H_E$. The canonical extension $\psi \otimes \text{id}_{Q_v}: E_v \to L$ will be denoted again by $\psi$ and factors through the quotient $E_{v,i}(\psi)$ of $E_v$; see [2, Definition 4.5].

Let $\hat{M}$ be a local shtuka over $R := O_L$ with complex multiplication by $O_{E_v}$ as in [2, Definition 4.3]. It may arise from a good model $\mathcal{M}$ of an $A$-motive over $R$ as in [2, Both authors acknowledge support by the Deutsche Forschungsgemeinschaft (DFG) in the form of SFB 878 and Germany’s Excellence Strategy EXC 2044–390685587 “Mathematics Münster: Dynamics–Geometry–Structure”. The first author was also supported by the DFG in form of Project-ID 427320536 – SFB 1442.
Example 3.2. We consider the one-dimensional $L$-vector space

$$H^\psi(\hat{M},L) := \{ \omega \in H^1_{\text{dR}}(\hat{M},L) : [a]^* \omega = \psi(a) \cdot \omega \ \forall a \in \mathcal{O}_{E_v} \}$$

$$\sim \to H^1_{\text{dR}}(\hat{M},L)/([a]^* - \psi(a) : a \in \mathcal{O}_{E_v}) \cdot H^1_{\text{dR}}(\hat{M},L), \quad (B.1)$$

where the isomorphism comes from [2, Proposition 4.9] using the fact that $E$ is separable over $Q$.

In [2, formula (1.13) and Definition 4.10] there is an error in the definition of $v(\omega_\psi)$ for $L[y_{i(\psi)} - \psi(y_{i(\psi)})]$-generators $\omega_\psi$ of $H^\psi(\hat{M},L[y_{i(\psi)} - \psi(y_{i(\psi)})])$. Namely, there, as a reference integral structure on the $L$-vector space

$$H^\psi(\hat{M},L) = H^\psi(\hat{M},L[y_{i(\psi)} - \psi(y_{i(\psi)})])/\langle y_{i(\psi)} - \psi(y_{i(\psi)}) \rangle,$$

the $R$-module

$$\tilde{H}^\psi(\hat{M},R) := \{ \omega \in H^1_{\text{dR}}(\hat{M},R) : [a]^* \omega = \psi(a) \cdot \omega \ \forall a \in \mathcal{O}_{E_v} \}$$

was used (which was denoted without the ~ on $H$) and in [2, Formula (1.13) and Definition 4.10]. Then $v(\omega_\psi)$ was defined to be

$$v^\sim(\omega_\psi) := v(\tilde{x}) \in Q, \quad (B.2)$$

where $\tilde{x} \in L^\times$ is such that $\tilde{x}^{-1}(\omega_\psi \mod y_{i(\psi)} - \psi(y_{i(\psi)}))$ is an $R$-generator of $\tilde{H}^\psi(\hat{M},R)$.

(To clarify the error we write $v^\sim(\omega_\psi)$ instead of $v(\omega_\psi)$ in this erratum.)

However, in the rest of [2] the $R$-submodule

$$H^\psi(\hat{M},R) := H^1_{\text{dR}}(\hat{M},R)/\langle [a]^* - \psi(a) : a \in \mathcal{O}_{E_v} \rangle \cdot H^1_{\text{dR}}(\hat{M},R) \subset H^\psi(\hat{M},L)$$

is used as a reference integral structure on $H^\psi(\hat{M},L)$. (See Lemma B.1 for why the latter is an inclusion and how the two integral structures can be compared.) Correspondingly, the following definition for $v(\omega_\psi)$ is used in [2]

$$v(\omega_\psi) := v(x) \in Q, \quad (B.3)$$

where $x \in L^\times$ is such that $x^{-1}(\omega_\psi \mod y_{i(\psi)} - \psi(y_{i(\psi)}))$ is an $R$-generator of $H^\psi(\hat{M},R)$. Indeed, in [2, Section 5.12] the generator $\omega^0_\psi := 1$ of $H^\psi(\hat{M},R)$ is used, which might not lie in $\tilde{H}^\psi(\hat{M},R)$. Afterwards, any other generator $\omega_\psi$ is compared to the generator $\omega^0_\psi$.

This error occurs in [2, Theorems 1.3 and 5.24 and Corollaries 5.22 and 5.25]. In terms of the valuation $v^\sim(\omega_\psi)$ from formula (B.2), all these theorems and corollaries have to be reformulated, as explained later. However, with the definition of $v(\omega_\psi)$ in formula (B.3), all these theorems and corollaries are correct.

Note that if $\hat{M} = \hat{M}_\psi(\mathcal{M})$ arises from a good model $\mathcal{M}$ of an $A$-motive over $R$ as in [2, Example 3.2] then $H^1_{\text{dR}}(\hat{M},R) = H^1_{\text{dR}}(\mathcal{M},R) := \sigma^* \mathcal{M} \otimes_{\mathcal{O}_R} \gamma \otimes \text{id}_R$, and hence

$$\tilde{H}^\psi(\hat{M},R) = \tilde{H}^\psi(\mathcal{M},R) := \{ \omega \in H^1_{\text{dR}}(\mathcal{M},R) : [a]^* \omega = \psi(a) \cdot \omega \ \forall a \in \mathcal{O}_E \},$$

$$H^\psi(\hat{M},R) = H^\psi(\mathcal{M},R) := H^1_{\text{dR}}(\mathcal{M},R)/\langle [a]^* - \psi(a) : a \in \mathcal{O}_E \rangle \cdot H^1_{\text{dR}}(\mathcal{M},R).$$
inside \( \hat{H}_\psi^j(\hat{M}, L) = \hat{H}_\psi^j(\hat{M}, L) = \tilde{H}_\psi^j(\hat{M}, R) \otimes_R L = \hat{H}_\psi^j(\hat{M}, R) \otimes_R L \).

We next show how the two integral structures can be compared.

**Lemma B.1.** The integral structures \( \hat{H}_\psi^j(\hat{M}, R) \) and \( \hat{H}_\psi^j(\hat{M}, R) \) are free \( R \)-modules of rank one and contained in the \( L \)-vector space \( \hat{H}_\psi^j(\hat{M}, L) \). The natural \( R \)-morphism

\[
\hat{H}_\psi^j(\hat{M}, R) \hookrightarrow H^1_{\text{dR}}(\hat{M}, R) \twoheadrightarrow \hat{H}_\psi^j(\hat{M}, R)
\]

is injective with cokernel isomorphic to \( R/R_\psi \) still has to be proved. Note that the argument will not use the specific situation of de Rham cohomology of local shtukas. It will only use the isomorphism (B.1) coming from [2, Proposition 4.9] and the freeness of the \( R \)-module \( H^1_{\text{dR}}(\hat{M}, R) \) over \( O_{E_v} \otimes A_v R \) (see later).

The \( Q_v \)-algebra \( E_v \) acts on \( \hat{H}_\psi^j(\hat{M}, L) \) through the character \( \psi : E_v \rightarrow E_v, (\psi) \hookrightarrow L \). By [3, §III.6, Proposition 12], there exists an element \( y \in O_{E_v, i(\psi)} \) such that \( O_{E_v, i(\psi)} = A_v[y] = A_v[Y]/(m) \), where \( m \in A_v[Y] \) is the minimal polynomial of \( y \) over \( A_v \). The image \( \gamma(m) \) under the map \( \gamma : A_v[Y] \rightarrow R[Y] \) has \( \psi(y) \) as a zero and correspondingly factors as

\[
\gamma(m) = (Y - \psi(y)) \cdot g(Y)
\]

for a monic polynomial \( g(Y) \in R[Y] \). The derivative \( m' := \frac{d m}{d y} \in A_v[Y] \) satisfies

\[
\psi(m'(y)) = \gamma(m)'(\psi(y)) = g(\psi(y)).
\]

Recall that \( A_{v, R} \) is the \( v \)-adic completion of \( A_v \). By [2, Proposition 4.8] we can decompose \( \hat{M} = \bigoplus_{i=1}^s \hat{M}_i \) into local shtukas \( \hat{M}_i \) over \( R \) with complex multiplication by \( O_{E_v, i} \). In particular,

\[
\hat{H}_\psi^j(\hat{M}, R) := \{ \omega \in H^1_{\text{dR}}(\hat{M}_i(\psi), R) : [a]^* \omega = \psi(a) \cdot \omega \ \forall a \in O_{E_v, i} \}
\]

and

\[
\hat{H}_\psi^j(\hat{M}, R) := H^1_{\text{dR}}(\hat{M}_i(\psi), R)/([a]^* - \psi(a) : a \in O_{E_v, i}) \cdot H^1_{\text{dR}}(\hat{M}_i(\psi), R)
\]

can be computed from

\[
H^1_{\text{dR}}(\hat{M}_i(\psi), R) := \sigma^* \hat{M}_i(\psi) \otimes_{A_{v, R}, \gamma \otimes \text{id}_R} R
\]

instead of \( H^1_{\text{dR}}(\hat{M}, R) \). Moreover, by the same proposition \( \hat{M} \) is free over \( O_{E_v, R} := O_{E_v} \otimes A_{v, R} R[z]/2 \) = \( O_{E_v} \otimes R \) of rank one, and we may choose a generator of \( \hat{M} \). This generator
provides an isomorphism

\[ H_{dR}^{1}(\hat{M}, R) \cong (O_{E_{\psi}} \otimes_{\mathbb{F}} R)_{A_{\psi}, R} \otimes_{\mathbb{F}} R = O_{E_{\psi}} \otimes_{\mathbb{F}} R = R[Y]/(\gamma(m)). \]

Since \([a]^* - \psi(a) = [a]^* - \gamma(a)\) for \(a \in A_{\psi}\) already annihilates \(H_{dR}^{1}(\hat{M}, R)\), this yields the upper vertical isomorphisms in the following diagram:

\[
\begin{array}{c}
\tilde{H}^{\psi}(\hat{M}, R) \\
\cong
\end{array}
\begin{array}{c}
\rightarrow
\end{array}
\begin{array}{c}
H^{\psi}(\hat{M}, R)
\end{array}
\]

\[
\begin{array}{c}
\{ f \in O_{E_{\psi}} \otimes_{A_{\psi}, R} R : (y \otimes 1 - 1 \otimes \psi(y)) \cdot f = 0 \} \\
\cong
\end{array}
\begin{array}{c}
\rightarrow
\end{array}
\begin{array}{c}
(O_{E_{\psi}} \otimes_{A_{\psi}, R} R)/(y \otimes 1 - 1 \otimes \psi(y))
\end{array}
\]

\[
\begin{array}{c}
\{ f \in R[Y]/(\gamma(m)) : (Y - \psi(y)) \cdot f = 0 \} \\
\cong
\end{array}
\begin{array}{c}
\rightarrow
\end{array}
\begin{array}{c}
R[Y]/(\gamma(m), Y - \psi(y))
\end{array}
\]

\[
\begin{array}{c}
g(Y) \cdot R[Y]/(\gamma(m)) \\
\cong
\end{array}
\begin{array}{c}
\rightarrow
\end{array}
\begin{array}{c}
R[Y]/(Y - \psi(y)).
\end{array}
\]

The injectivity of the horizontal maps follows from diagram (B.4). The lower left equality holds because \(R[Y]\) has no \((Y - \psi(y))-\)torsion. Next, \(H^{\psi}(\hat{M}, R) \cong R[Y]/(Y - \psi(y)) \cong R\) is free, and hence contained in \(H^{\psi}(\hat{M}, R) \otimes_{R} L = H^{\psi}(\hat{M}, L)\). Finally, the image of the lower horizontal map is the ideal

\[ R \cdot g(\psi(y)) = R \cdot \psi(m'(y)) = R \cdot \psi(D_{E_{\psi}}/Q) = R \cdot \psi(D_{E}/Q) \]

(see [3, §III.4, Proposition 10 and §III.6, Corollary 2]).

**Corollary B.2.** For an \(L\)-generator \(\omega_{\psi}\) of \(H^{\psi}(\hat{M}, L)\), the two valuations in formulas (B.2) and (B.3) satisfy

\[ v(\omega_{\psi}) - v^\sim(\omega_{\psi}) = v(D_{\psi}/Q) = v(\psi(D_{E}/Q)) = v(\psi(D_{E}/Q)). \]

**Proof.** Let \(x, \tilde{x} \in L^\times\) be elements such that \(x^{-1}\omega_{\psi}\) is an \(R\)-generator of \(H^{\psi}(\hat{M}, R)\) and \(\tilde{x}^{-1}\omega_{\psi}\) is an \(R\)-generator of \(H^{\psi}(\hat{M}, R)\). Then \(x/\tilde{x}\) is an \(R\)-generator of \(\psi(D_{E}/Q)\) by Lemma B.1, and the corollary follows. \(\square\)

Now let \(M\) be an \(A\)-motive over a finite Galois extension \(K \subset Q_{\text{alg}}\) of \(Q\) with complex multiplication by a finite separable semi-simple \(Q\)-algebra \(E\). Assume that \(\psi(E) \subset K\) for all \(\psi \in H_{E}\). Fix a \(\psi \in H_{E}\) and let \(\omega_{\psi}\) be a generator of the \(K[y_{\psi} - \psi(y_{\psi})]\)-module \(H^{\psi}(\hat{M}, K[y_{\psi} - \psi(y_{\psi})])\). For every \(\eta \in H_{K}\), let \(\omega_{\psi}^{\eta} \in H^{\psi}(\hat{M}, K[y_{\psi} - \eta \psi(y_{\psi})])\) be obtained by extension of scalars via \(\eta\). With the corollary and the computation

\[
\sum_{\eta \in H_{K}} v(\omega_{\psi}^{\eta}) - v^\sim(\omega_{\psi}^{\eta}) = \sum_{\eta \in H_{K}} v(\eta \psi(D_{E}/Q)) = v\left(\prod_{\eta \in H_{K}} \eta \psi(D_{E}/Q)\right) = v\left(N_{K/Q}(D_{E}/Q)\right) = \left[K : \psi(E)\right] \cdot v(D_{E}/A),
\]

\[ v\left(N_{\psi(E)/Q}(N_{K/\psi(E)}(D_{\psi(E)}/Q))\right) = [K : \psi(E)] \cdot v(D_{\psi(O_{E})}/A), \]
we obtain a reformulation of [2, Theorems 1.3 and 5.24 and Corollaries 5.22 and 5.25] in terms of $v^\sim(\omega_\psi)$, which is even more analogous to [1, Theorem II.1.1(i)].

**Theorem 1.3’.** Let $\omega_\psi$ be a generator of the $K[y_\psi - \psi(y_\psi)]$-module $H^\psi(\hat{M}, K[y_\psi - \psi(y_\psi)])$. For every $\eta \in H_K$, let $M^\eta$ and $\omega_\psi^\eta \in H^\psi(\hat{M}, K[y_\eta - \eta \psi(y_\eta)])$ be obtained by extension of scalars via $\eta$, and choose an $E$-generator $u_\eta \in H_{1, \text{Betti}}(M^\eta, Q)$. Then for every place $v \neq \infty$ of $C$, we have

$$\frac{1}{#H_K} \sum_{\eta \in H_K} v(\int_{u_\eta} \omega_\psi^\eta) = Z_v(a^0_{E, \psi, \Phi}, 1) - \mu_{\text{Art}}, v(a^0_{E, \psi, \Phi}) + \frac{1}{#H_K} \sum_{\eta \in H_K} (v^\sim(\omega_\psi^\eta) + v_{\eta \psi}(u_\eta)). \tag*{□}$$

**Corollary 5.22’.** Let $\varphi, \psi \in H_E$, with $i(\varphi) = i(\psi) = i$ and assume that $E_{\varphi, i}$ is separable over $Q_v$. Let $u \in H_{1, v}(\hat{M}_{E, \varphi}, Q_v)$ be a generator as an $E_v$-module and let $\omega_\psi$ be an $L[y_i - \psi(y_i)]$-generator of $H^\psi(\hat{M}_{E, \varphi}, L[y_i - \psi(y_i)])$. Then $\int_u \omega_\psi := u \otimes \id_{C_v((z - \zeta))} (h_{v, d\text{R}}(\omega_\psi))$ has valuation

$$v(\int_u \omega_\psi) = Z_v(a_{E, \psi, \Phi}, 1) - \mu_{\text{Art}}, v(a_{E, \psi, \Phi}) + v^\sim(\omega_\psi) + v_\psi(u),$$

where $v^\sim(\omega_\psi)$ and $v_\psi(u)$ are defined in formula (B.2) and [2, Definition 5.21] respectively. \tag*{□}

**Theorem 5.24’.** Let $\hat{M}$ be a local shtuka over $R$ with complex multiplication by the ring of integers $O_{E_v}$ in a commutative, semi-simple, separable $Q_v$-algebra $E_v$ with local CM-type $\Phi$, and assume that $\psi(E_v) \subset L$ for all $\psi \in H_{E_v}$ and that $L$ is separable over $Q_v$. Let $u \in H_{1, v}(\hat{M}, Q_v)$ be an $E_v$-generator and let $\omega_\psi$ be an $L[y_i(\psi) - \psi(y_i(\psi)))]$-generator of $H^\psi(\hat{M}, L[y_i(\psi) - \psi(y_i(\psi))])$. Then the period $\int_u \omega_\psi := u \otimes \id_{C_v((z - \zeta))} (h_{v, d\text{R}}(\omega_\psi))$ has valuation

$$v(\int_u \omega_\psi) = Z_v(a_{E, \psi, \Phi, 1} - \mu_{\text{Art}}, v(a_{E, \psi, \Phi}) + v^\sim(\omega_\psi) + v_\psi(u),$$

where $v^\sim(\omega_\psi)$ and $v_\psi(u)$ are defined in formula (B.2) and [2, Definition 5.21] respectively. \tag*{□}

**Corollary 5.25’.** Keep the situation of Theorem 5.24’. For every $\eta \in H_L$, note that $i(\eta \psi) = i(\psi)$, let $\eta \hat{M}$ and $\omega_\psi^\eta \in H^\eta(\hat{M}, L[y_i(\psi) - \eta \psi(y_i(\psi))])$ be obtained by extension of scalars via $\eta$ and choose an $E_v$-generator $u_\eta \in H_{1, v}(\hat{M}^\eta, Q_v)$. Then

$$\frac{1}{#H_L} \sum_{\eta \in H_L} v(\int_{u_\eta} \omega_\psi^\eta) = Z_v(a^0_{E, \psi, \Phi, 1} - \mu_{\text{Art}}, v(a^0_{E, \psi, \Phi}) + \frac{1}{#H_L} \sum_{\eta \in H_L} (v^\sim(\omega_\psi^\eta) + v_{\eta \psi}(u_\eta)). \tag*{□}$$

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