Some evidence for the Coleman–Oort conjecture

Diego Conti¹ · Alessandro Ghigi² · Roberto Pignatelli³

Received: 7 June 2021 / Accepted: 17 November 2021 / Published online: 16 December 2021
© The Author(s) under exclusive licence to The Royal Academy of Sciences, Madrid 2021

Abstract
The Coleman–Oort conjecture says that for large $g$ there are no positive-dimensional Shimura subvarieties of $A_g$ generically contained in the Jacobian locus. Counterexamples are known for $g \leq 7$. They can all be constructed using families of Galois coverings of curves satisfying a numerical condition. These families are already classified in cases where: (a) the Galois group is cyclic, (b) it is abelian and the family is 1-dimensional, or c) $g \leq 9$. By means of carefully designed computations and theoretical arguments excluding a large number of cases we are able to prove that for $g \leq 100$ there are no other families than those already known.

Keywords Shimura varieties · Group actions on algebraic curves · Computational group theory

Mathematics Subject Classification Primary 14G35 · 14J10 · 14Q05; Secondary 20F99

1 Introduction

1.1 Denote by $A_g$ the moduli space of principally polarized complex abelian varieties of dimension $g$, by $M_g$ the moduli space of smooth complex algebraic curves of genus $g$ and
by \( j : M_g \to A_g \) the period mapping (or Torelli mapping), which associates to \([C] \in M_g\) the moduli point of the Jacobian variety \(JC\) provided with the theta polarization. The Jacobian locus is the image \( j(M_g) \).

By \( j(M_g) \) we denote the closure of \( j(M_g) \) in \( A_g \).

On \( A_g \) there is a tautological \( \mathbb{Q} \)-variation of the Hodge structure (in the orbifold sense): if \( A \) is a principally polarized abelian variety, the fibre over its moduli point \([A] \in A_g\) is \( H^1(A, \mathbb{Q}) \) with its Hodge structure of weight 1. In general, given a variation of the Hodge structure \( H \to B \), it is interesting to consider the points \( b \in B \) where the Hodge structure is “more symmetric” than over the general point. Making precise the meaning of “more symmetric” requires some effort. In the simplest case this means that the Hodge structure has more automorphism than usual. For example for the variation over \( A_1 \), the general point has no automorphisms beyond \( \{ \pm 1 \} \), while the points with more automorphisms represent the well-known elliptic curves with automorphisms \( \mathbb{Z}/4\mathbb{Z} \) or \( \mathbb{Z}/6\mathbb{Z} \). The general case is more complicated since the symmetry is not at the level of automorphisms but is detected by Hodge classes in general tensor spaces. The loci obtained in this way are called the Hodge loci of the variation of the Hodge structure. In the case of \( A_g \) they are also called special subvarieties or Shimura subvarieties. (See \([27, \S 3.3]\) and \([18]\).) A subvariety \( Z \subset A_g \) is said to be generically contained in \( j(M_g) \) if \( Z \subset j(M_g) \) and \( Z \cap j(M_g) \neq \emptyset \). Arithmetical considerations led first Coleman and later Oort \([28]\) to the following

**Conjecture 1.2** (Coleman–Oort) For large \( g \) there are no special subvarieties of positive dimension generically contained in \( j(M_g) \).

(See \([27, \S 4]\) for more details.) This expectation is also motivated by another stronger expectation originating from the point of view of differential geometry: special subvarieties are totally geodesic with respect to the locally symmetric (orbifold) metric on \( A_g \) (the one coming from the Siegel space). If one believes that \( j(M_g) \) bears no strong relation to the ambient geometry of \( A_g \), in particular that it is very curved inside \( A_g \), then it is natural to expect that \( j(M_g) \) contains generically no totally geodesic subvarieties, and in particular no Shimura subvarieties (see \([9,17,19]\) for results in this direction).

What makes the problem more interesting is that for low genus examples of such Shimura varieties generically contained in \( j(M_g) \) do exist! All the examples known so far are in genus \( g \leq 7 \) and arise from one of the following two constructions.

**1.3 First construction.** Let \( G \) be a finite group acting on a curve \( C \). Consider the family of curves \( \mathcal{C} \to B \) with a \( G \)-action of the same topological type (see below for the precise definition). For every \( m, H^0(C_b, mK_{C_b}) \) is a representation of \( G \) and its equivalence class is independent of \( b \in B \). Denote by \( B' \subset M_g \) the moduli image of \( B \) and by \( Z \) the closure of \( j(B') \) in \( A_g \). In \([14,16]\) it is proven that if

\[
\dim(S^2(H^0(K_{C_b})))^G = \dim H^0(2K_{C_b})^G, \quad (*)
\]

then \( Z \) is a Shimura variety generically contained in \( j(M_g) \). We also say that the family of \( G \)-covers \( \mathcal{C} \to B \) yields a Shimura variety to mean that \( Z \) is Shimura. We refer to such a Shimura variety as a counter-example to Coleman–Oort conjecture. Several counter-examples are known, see Theorem 1.5 below.

**1.4 Second construction.** Consider a Shimura variety \( Z \) generically contained in \( j(M_g) \) obtained as in 1.3 from a family of \( G \)-curves \( \mathcal{C} \to B \). Denote by \( g' \) the genus of \( C_b/G \). Let \( \text{Nm} : J_{C_b} \to J(C_b/G) \) be the norm map of the covering \( f_b : C_b \to C_b/G \), defined by \( \text{Nm}(\sum_i p_i) := \sum_i f_b(p_i) \). Then (\( \ker \text{Nm} \))\(^0 \subset JC_b \) is an abelian subvariety, the generalized
Prym variety of the covering $f_b$. The theta polarization of $JC_b$ restricts to a polarization of some type $\delta$ on the Prym variety. We get maps
\[
\varphi : B \rightarrow M_g, \quad \varphi(b) := [C_b/G],
\]
\[
\mathcal{P} : B \rightarrow A^g_{g'-g}, \quad \mathcal{P}(b) := [(\ker Nm)^0].
\]
$\mathcal{P}$ is the generalized Prym map. If $g' = 0$ the map $\varphi$ is of course constant, $A^g_{g'-g} = A_g$ and $\mathcal{P}$ is just the Torelli map, so we get nothing new. If instead $g' > 0$, the irreducible components of the fibres of $\mathcal{P}$ and $\varphi$ are totally geodesic subvarieties and countably many of them are in fact Shimura, see [22] and [15, Thm. 3.9, Thm. 3.11]. Thus for $g' > 0$ this construction gives uncountably many totally geodesic non-Shimura varieties generically contained in $j(M_g)$ and countably many Shimura varieties generically contained in $j(M_g)$.

Let us summarize what is known about the counter-examples obtained via these constructions.

**Theorem 1.5**  
a) There are 38 families of Galois coverings of the projective line satisfying (*) with $2 \leq g \leq 7$. For $g \leq 9$ there are no other counter-examples. See [14,26,27,31].

b) There are 6 families of Galois coverings of elliptic curves satisfying (*) with $2 \leq g \leq 4$. For $g \leq 9$ there are no other counter-examples. See [16].

c) If a family satisfies (*) and $g' > 0$, then necessarily $g' = 1$ and the family is one of those in (b). See [15].

**1.6** Note that we focus on $g \geq 2$, since for $g = 1$ there are infinitely many 1-dimensional families satisfying (*).

In fact, for every elliptic curve $C$ the involution $p \mapsto -p$ acts trivially on both $S^2H^0(K_C)$ and $H^0(2K_C)$. Let $G$ be the group of the biholomorphisms of $C$ generated by it and by a finite group of translations. Then $S^2H^0(K_C)^G = S^2H^0(K_C) \cong \mathbb{C} \cong H^0(2K_C) = H^0(2K_C)^G$, so giving examples of (*) with $G$ of order arbitrarily high. Two of these families are listed in Table 2 in [14].

However all these families are irrelevant for the Coleman–Oort conjecture, since in all cases $B' = M_1$. Note also that some of the families of Theorem 1.5 yield the same Shimura variety, i.e. have the same image in moduli, see [14,16].

**1.7** It follows from Theorem 1.5 (c) that all the cases where (*) holds and $g' > 0$ are already known and also that no new examples can be found using the second construction 1.4. Therefore, in order to construct new examples using the two methods above (or to exclude the existence of such examples) we can restrict to the first construction with $g' = 0$, i.e. $C_b/G = \mathbb{P}^1$.

The purpose of this paper is to provide further evidence for the Coleman–Oort conjecture, employing a computational approach complemented by theoretical arguments. Our result is the following improvement of Theorem 1.5.

**Theorem 1.8** The positive-dimensional families of Galois covers satisfying (*) with $2 \leq g \leq 100$ are only those of Theorem 1.5.

**1.9** The fact that we found no new families at all is strong evidence that there are no more families satisfying (*). Since all known counter-examples to the Coleman–Oort conjecture can be constructed using these families, this also suggests that either further counter-examples do not exist or they are of a completely different nature.
1.10 An important point to stress is the following. Condition (*) is sufficient for a family to yield a Shimura variety. In general it is unknown if it is also necessary. In this paper we only check whether condition (*) holds. So we cannot exclude that these families give rise to counter-examples to Coleman–Oort conjecture.

1.11 Families of $G$-covers are identified by data of combinatorial and group-theoretical nature. We explain this in Sect. 2. So the basic strategy is obviously to list all these data and check condition (*) for each datum in the list. Since the list of these data is extremely long, one needs to avoid unnecessary computations. The first observation is that many data give rise to the same family. More precisely call two data $\Delta$ and $\Delta'$ Hurwitz equivalent if they have the same group $G$ and if the families corresponding to them are isomorphic as families of algebraic curves with $G$-action. It turns out that Hurwitz equivalence classes can be huge. To check condition (*) for all the families of some genus, one would start by choosing a representative out of any Hurwitz equivalence class, and proceed by checking (*) for all the representatives. However, the identification of a single representative inside each class is a daunting task, since the classes are huge and Hurwitz equivalence is rather complicated.

(An algorithm dealing with Hurwitz equivalence appears in [2]. It was used in [14,16]. An improvement of this algorithm is given in [3]. We hope to address the problem of algorithmic computation of Hurwitz equivalence in future work.)

Luckily there is another equivalence relation on data, much coarser than the Hurwitz equivalence, which is appropriate to our problem: if $\Delta = (G, g_1, \ldots, g_r)$, then the number $N = N(\Delta)$ only depends on the conjugacy classes $C_1 = [g_1], \ldots, C_r = [g_r]$. Also the order of these is completely irrelevant. The unordered sequence $(C_1, \ldots, C_r)$ is called a refined passport. (See Definition 3.7.) So our problem depends only on refined passports, more precisely on their $\text{Aut}(G)$-orbits, which are considerably less in number than Hurwitz equivalence classes, leading to much shorter execution times. Notice that in some cases refined passports (even if taken up to the action of $\text{Aut}(G)$) are still too many to be stored simultaneously into memory, but this is not a problem, since we only need to perform an iteration to check (*) on each individually.

Even after this great simplification the computation remains quite formidable, at least for the computers at our disposal. We use a number of tricks to reduce the data that must be considered. Several exclusions (e.g. cyclic groups) follow from previous results (see Theorem 3.3). We complement them with Corollary 3.13, which effectively eliminates more than 90% of the data, including some of the hardest cases, thus allowing us to complete the computation.

1.12 For the implementation of the algorithm we used MAGMA [5], which is quite suited to the task at hand since it allows working with groups, group actions and representations, in particular computing characters, orbits and stabilizers; furthermore, it contains a database of groups of small order. Our code is available at [11].

The problem lends itself easily to parallelization, since each group and signature is treated independently; however, MAGMA does not support parallelization natively. The first part of the computation (Algorithm 1) was parallelized using the standard tool [33]. On the other hand, the rest of the computation can become quite memory-intensive; this leads to technical difficulties, mainly concerning situations in which one of the processes is terminated for lack of memory, which were addressed by writing the ad hoc external program [10] to run the MAGMA script.

Using a computer with 56 Intel Xeon 2.60GHz CPU and 128 GB of RAM we were able to finish the computations in less than 3 days.
1.13 The plan of the paper is as follows. In Sect. 2 we recall the description of the families of $G$-curves and some basic facts concerning the multiplication map on sections of the canonical bundle, which is related with condition (\(^*\)). At the end we prove Lemma 2.17, which deals with the behaviour of condition (\(^*\)) when passing from a given family to a quotient by a normal subgroup. In Sect. 3 we gather several facts of quite different nature, some well-known, some new, which we have found useful to rule out several cases. This has been essential in order to complete the computation. Finally Sect. 4 contains a thorough explanation of the algorithm.

2 Families of $G$-curves

2.1 The purpose of this section is to describe some group-theoretic and combinatorial data from which one can construct algebraic families of curves with prescribed symmetry. We will denote by $\Delta$ the datum and by $\mathcal{C}_\Delta \to B_\Delta$ the corresponding family of curves. The image of $B_\Delta$ in $M_g$ will be denoted by $M_\Delta$. We are interested in the closure of $M_\Delta$ in $A_g$. As explained in 1.3, when (\(^*\)) holds this closure is a Shimura variety generically contained in the Jacobian locus. This is explained in more detail at the end of this section, together with some related remarks on the multiplication map.

In the following, unless otherwise stated, we assume that the genus is at least 2. For $r \geq 3$, set

$$\Gamma_r:=\left\{\gamma_1, \ldots, \gamma_r \mid \prod_{i=1}^r \gamma_i = 1\right\}.$$ 

**Definition 2.2** If $G$ is a finite group an epimorphism $\theta : \Gamma_r \to G$ is called *admissible* if $\theta(\gamma_i) \neq 1$ for $i = 1, \ldots, r$. An $r$-datum is a pair $\Delta = (G, \theta)$ where $G$ is a finite group and $\theta : \Gamma_r \to G$ is an admissible epimorphism. The *signature* of $\Delta$ is the vector $m:=(m_1, \ldots, m_r)$ where $m_i:=\text{ord}(\theta(\gamma_i))$. The *genus* of $\Delta$, denoted by $g(\Delta)$, is defined by the Riemann-Hurwitz formula:

$$2(g(\Delta) - 1) = |G| \left(-2 + \sum_{i=1}^r \left(1 - \frac{1}{m_i}\right)\right) \tag{2.1}$$

We let $\mathscr{D}$ or simply $\mathcal{D}$ denote the set of all $r$-data.

2.3 Orient $S^2$ by the outer normal. Consider smooth regular arcs $\tilde{a}_i$ in $S^2$ joining $p_0$ to $p_1$ such that for $i \neq j$ $\tilde{a}_i$ and $\tilde{a}_j$ intersect only at $p_0$. Assume also that the tangent vectors at $p_0$ are all distinct and follow each other in counterclockwise order. Next consider loops $\alpha_i$ based at $p_0$ constructed as follows: $\alpha_i$ starts at $p_0$, travels along $\tilde{a}_i$ until near $p_i$, there travels counterclockwise along a small circle around $p_i$, finally goes back to $p_i$ again along $\tilde{a}_i$. The circles have to be pairwise disjoint. We call the resulting set of generators $\{[\alpha_1], \ldots, [\alpha_r]\}$ a *geometric basis* of $\pi_1(S^2 - P, p_0)$. Once a geometric basis is fixed, there is a well-defined isomorphism

$$\chi : \Gamma_r \to \pi_1(S^2 - P, p_0)$$

such that $\chi(\gamma_i) = [\alpha_i]$.

2.4 The following geometric setting gives rise to data (and it is the main motivation for them). Let $X$ be a compact (connected) Riemann surface. Assume that a finite group $G$ acts effectively and holomorphically on $X$ in such a way that $X/G = \mathbb{P}^1$. Let $P:=\{p_1, \ldots, p_r\}$ be the critical values of $\pi : X \to \mathbb{P}^1 \cong S^2$. Fix $p_0 \in S^2 - P$ and a geometric basis

\[ \text{Springer} \]
The group embedding depends on family uniquely completed to a proper holomorphic map \( \pi \) by the Riemann-Hurwitz formula and \( n_t \) is the cardinality of the stabilizer of points in \( \pi^{-1}(p_t) \). We are going to show that each datum arises from a covering \( X \rightarrow \mathbb{P}^1 = X/G \).

2.5 Assume from now on that \( r \geq 3 \) and denote by \( T_{0,r} \) the Teichmüller space in genus 0 and with \( r \) marked points. The definition of \( T_{0,r} \) is as follows. Fix \( r+1 \) distinct points \( p_0, \ldots, p_r \) on \( S^2 \). For simplicity set \( P = (p_1, \ldots, p_r) \). Consider triples of the form \((\mathbb{P}^1, x, [f])\) where \( x = (x_1, \ldots, x_r) \) is an \( r \)-tuple of distinct points in \( \mathbb{P}^1 \) and \([f] \) is an isotopy class of orientation preserving homeomorphisms \( f : (\mathbb{P}^1, x) \rightarrow (S^2, P) \). Two such triples \((\mathbb{P}^1, x, [f])\) and \((\mathbb{P}^1, x', [f'])\) are equivalent if there is a biholomorphism \( \phi : \mathbb{P}^1 \rightarrow \mathbb{P}^1 \) such that \( \phi(x_i) = x_i' \) for any \( i \) and \([f] = [f' \circ \phi] \). The Teichmüller space \( T_{0,r} \) is the set of all equivalence classes, see e.g. [1, Chap. 15] for more details.

2.6 Fix a geometric basis \( \mathcal{B} = \{[\alpha_i]\}_{i=1}^r \) of \( \pi_1(S^2 - P, p_0) \) with corresponding isomorphism \( \chi : \Gamma_r \cong \pi_1(S^2 - P, p_0) \). Finally fix a point \( p_0 \in \pi^{-1}(p_0) \). As is well-known there is a morphism \( \theta : \pi_1(S^2 - P, p_0) \rightarrow G \) such that for \( [\alpha] \in \pi_1(S^2 - P, p_0) \) the lifting of \( \alpha \) starting at \( p_0 \) ends at \( g \cdot p_0 \) where \( g = \theta(\alpha) \). Since \( X \) is connected \( \theta \) is surjective. Therefore \( \Delta := (G, \theta := \theta \circ \chi) \) is an \( r \)-datum, \( g(\Delta) = g(X) \) by the Riemann-Hurwitz formula and \( n_t \) is the cardinality of the stabilizer of points in \( \pi^{-1}(p_t) \).

We get an induced isomorphism \( \pi_0 : C_t \rightarrow \mathbb{P}^1 \). Moreover there is an isotopy class of homeomorphisms \( \chi) = \chi_0 : \mathbb{P}^1 \rightarrow C_t \) giving rise to a covering \( \pi : \Sigma_0 \rightarrow S^2 \). By the topological part of Riemann’s Existence Theorem this can be completed to a branched cover \( \pi : \Sigma \rightarrow S^2 \). Given a point \( t = ([\mathbb{P}^1, x, [f]]) \in T_{0,r}, \) the homeomorphism \( f \) restricts to a homeomorphism of \( \mathbb{P}^1 - x \) onto \( S^2 - P \). We get an induced isomorphism \( f_* : \pi_1(\mathbb{P}^1 - x, f^{-1}(p_0)) \cong \pi_1(S^2 - P, p_0) \).

Thus \( \theta \circ \chi^{-1} \circ f_* : \pi_1(\mathbb{P}^1 - x, f^{-1}(p_0)) \rightarrow G \) is an epimorphism and this gives rise to a topological covering \( \pi_0 : C_t \rightarrow \mathbb{P}^1 - x \). Here \( C_t \) is an open differentiable surface. Since \( \pi_0 \) is a local diffeomorphism, there is a unique complex structure on \( C_t \) making \( \pi_0 \) holomorphic. By the holomorphic part of Riemann’s Existence Theorem \( C_t \) and \( \pi_0 \) may be uniquely completed to a proper holomorphic map \( \pi_t : C_t \rightarrow \mathbb{P}^1 \) and the \( G \)-action extends to \( C_t \). Moreover there is an isotopy class of homeomorphisms \( f_t : C_t \rightarrow \Sigma \) that cover \( f_t \). As \( t \) varies in \( T_{0,r} \) this construction yields a holomorphic map to the Teichmüller space of \( \Sigma \):

\[
\Phi_\Delta : T_{0,r} \rightarrow T_g \cong T(\Sigma), \quad t \mapsto [C_t, [f_t]].
\]

The group \( G \) embeds in the mapping class group of \( \Sigma \), which we denote by \( \text{Mod}_g \). This embedding depends on \( \theta \) and we denote by \( G_\theta \subset \text{Mod}_g \) its image. The image of \( \Phi_\Delta \) coincides with \( T_{G_\theta}^{\Delta} \), the set of fixed points of \( G_\theta \) on \( T_g \). As such it is a complex submanifold. We denote it by \( T_\Delta \).

The image of \( T_\Delta \) in the moduli space \( \text{Mod}_g \) is an irreducible algebraic subvariety of dimension \( (r-3) \) that we denote by \( M_\Delta \). (See e.g. [2,7,8,21] for more details.) As explained in [21, p. 79] the map \( T_\Delta \rightarrow M_\Delta \) factors through an intermediate variety \( \tilde{M}_\Delta \):

\[
T_\Delta \rightarrow \tilde{M}_\Delta \rightarrow M_\Delta.
\]

Thus \( \pi_\Delta : \tilde{M}_\Delta \rightarrow B_\Delta \) is the normalization of \( M_\Delta \). There is a finite cover \( B_\Delta \rightarrow \tilde{M}_\Delta \) and a universal family

\[
\pi_\Delta : \tilde{E}_\Delta \rightarrow B_\Delta.
\]

We call it the family of \( G \)-curves associated to \( \Delta \). The proofs of these assertions can be found in [21] (where \( T_g(H_0) \) corresponds in our notation to \( T_\Delta, \tilde{M}(H_0) \) to \( \tilde{M}_\Delta \), \( M(H_0) \) to \( M_\Delta \) and \( \tilde{M}^{\text{pure}}(H_0) \) to \( B_\Delta \)). Note that

\[
\dim M_\Delta = \dim B_\Delta = r-3.
\]
2.7 In this construction the choice of the base point \( p_0 \) is irrelevant. In fact (up to isomorphism) the ramified covering \( \Sigma \to S^2 \) only depends on \( N := \ker \theta \circ \chi^{-1} \triangleleft \pi_1(S^2 - P, p_0) \). Two isomorphism \( \pi(S^2 - P, p_0) \to \pi_1(S^2 - P, p'_0) \) differ by an inner automorphism, so the map from normal subgroups of \( \pi(S^2 - P, p_0) \) to those of \( \pi(S^2 - P, p'_0) \) is well defined. This proves that \( T_\Delta \) and hence also \( M_\Delta, \tilde{M}_\Delta \) and the family \( \pi_\Delta : \mathcal{C}_\Delta \to B_\Delta \) do not depend on the choice of the base point \( p_0 \).

2.8 On the other hand the construction of \( T_\Delta, M_\Delta, \tilde{M}_\Delta, \pi_\Delta \) does depend on the choice of the geometric basis. Let \( \mathcal{B} = \{ [\alpha_i] \}_{i=1}^r \) be another geometric basis. and let \( \tilde{\chi} : \Gamma_r \to \pi_1(S^2 - P, p_0) \) be the corresponding isomorphism. Then \( \mu := \tilde{\chi} \circ \chi^{-1} \in \text{Aut} \pi(S^2 - P, p_0) \) has two special properties: (1) for every \( i = 1, \ldots, r, \mu([\alpha_i]) = [\tilde{\alpha}_i] \) is conjugate to \([\alpha_j]\) for some \( j \); (2) the induced homomorphism on the cohomology group \( H_2(\pi_1(S^2 - P, p_0), \mathbb{Z}) \) is the identity. By a variant of the Dehn–Nielsen Theorem (see e.g. [12, §8.2.7 p. 233] or [34, Thm. 5.7.1 p. 197]) there is an orientation-preserving diffeomorphism \( \varphi : (S^2 - P, p_0) \to (S^2 - P, p_0) \) such that \( \mu = \varphi_* \). Let \( \Sigma \) and \( \tilde{\Sigma} \) be the coverings of \( S^2 \) obtained from \( \chi \) and \( \tilde{\chi} \). If \( N = \ker \theta \circ \chi^{-1} \) and \( \tilde{N} = \ker \theta \circ (\tilde{\chi})^{-1} \), then \( \varphi_*(N) = \tilde{N} \). By the Lifting Theorem there is an orientation-preserving diffeomorphism \( \tilde{\varphi} : \Sigma \to \tilde{\Sigma} \) that covers \( \varphi \). This gives rise to a biholomorphism \( T(\Sigma) \to T(\Sigma') \) which maps \( T_\Delta \) constructed using \( \chi \) to \( T_\Delta \) constructed using \( \tilde{\chi} \). The identification \( T_{\tilde{\varphi}} = T(\Sigma) \) is defined up to the action of \( \text{Mod}_g \) and the discussion above shows that also \( T_\Delta \) is well defined up to this action. In particular \( M_\Delta, \tilde{M}_\Delta, B_\Delta \) and \( \pi_\Delta \) are completely independent of the choice of the geometric basis.

2.9 There is a representation
\[
\rho : G \to \text{GL} H^0(C, K_C)^G, \quad \rho(g) := (g^{-1})^*. \tag{2.3}
\]
The equivalence class of this representation is independent of \( f \in B_\Delta \).

For later use we recall the following observation, already used in the proof of [15, Thm. 2.3].

**Proposition 2.10** Let \( G \) be a finite group of automorphisms of a curve \( C \), and consider the subspace of invariants \( H^0(C, 2K_C)^G \). Then the multiplication map
\[
m_C^G : S^2 H^0(C, K_C)^G \to H^0(C, 2K_C)^G
\]
is surjective unless \( C \) is hyperelliptic (so of genus at least 2) and there is a small deformation \( C_\xi \) of the complex structure of \( C \) such that all elements of \( G \) remain holomorphic and the general curve \( C_\xi \) is not hyperelliptic.

In particular, for a fixed \( r \)-datum \( \Delta = (G, \theta) \), the map \( m_C^\Delta \) is surjective for the general \( C \in B_\Delta \).

**Proof** Let \( g \) be the genus of \( C \). The statement is obvious for \( g \leq 1 \) since the \( G \)-equivariant map \( S^2(H^0(C, K_C)) \to H^0(C, 2K_C) \) is an isomorphism (among spaces of dimension \( g \)). If \( C \) is not hyperelliptic, then the statement follows similarly since the map \( S^2(H^0(C, K_C)) \to H^0(C, 2K_C) \) is surjective by M. Noether’s Theorem.

We can then assume that \( C \) is hyperelliptic. Let \( \sigma \) be the hyperelliptic involution. It is well-known that \( \sigma \) acts as the multiplication by \(-1\) on \( H^0(C, K_C) \), so trivially on \( S^2(H^0(C, K_C)) \), and that the multiplication map \( S^2(H^0(C, K_C)) \to H^0(C, 2K_C)(\sigma) \) is surjective.

We distinguish two cases.

1. If \( \sigma \in G \) then the surjectivity of \( m_C^G \) follows by the surjectivity of the map \( S^2(H^0(C, K_C)) \to H^0(C, 2K_C)(\sigma) \).
(2) If $\sigma \notin G$ we denote by $\tilde{G}$ the group of automorphisms of $G$ generated by $G$ and $\sigma$. Then $m^G_C$ is surjective. Moreover $S^2(H^0(C, K_C))^{\tilde{G}} = S^2(H^0(C, K_C))^{\tilde{G}}$ so we need $H^0(C, 2K_C)^{\tilde{G}} \cong H^0(C, 2K_C)^{\tilde{G}}$, that is equivalent to $H^0(C, 2K_C)^{\tilde{G}} \subset H^0(C, 2K_C)^{\sigma}$. Dualizing, this is equivalent to $H^1(C, T_C)^{\tilde{G}} \subset H^1(C, T_C)^{\sigma}$, which amounts to asking that every small deformation of the pair $(C, G)$ remain hyperelliptic.

$\square$

2.11 We notice that the exceptional case in Proposition 2.10 occurs. Consider for example family (27) in [14, Table 2]. A direct computation shows that this 3-dimensional family of curves of genus 3 with an action of $(\mathbb{Z}/2\mathbb{Z})^2$ intersects the hyperelliptic locus in the 2-dimensional family of curves with an action of $(\mathbb{Z}/2\mathbb{Z})^3$ considered in [30, Table 2-Five critical values-(b)]. If $C$ belongs to this latter family, then $3 = h^0(C, 2K_C)^{\tilde{G}} \neq h^0(C, 2K_C)^{\tilde{G}} = 2$ and therefore $m^G_C$ has corank 1.

2.12 Consider now a datum $\Delta$ and the family $\pi_\Delta : \mathcal{C}_\Delta \to \mathcal{B}_\Delta$. As $t$ varies in $\mathcal{B}_\Delta$, the domain and codomain of $m^G_{C_t}$ do not change in dimension. Set

$$N(\Delta) := \dim (S^2H^0(C_t, K_{C_t}))^G$$

(2.4)

Theorem 2.13 If $g = g(\Delta) \geq 2$ and

$$N(\Delta) = r - 3, \quad (*)$$

then $j(M_\Delta)$ (closure in $A_g$) is a special subvariety of PEL type of $A_g$ that is generically contained in the Jacobian locus.

(See [14, Thm. 3.9] and [16, Thm. 3.7].)

2.14 The idea of Theorem 2.13 is that from $\Delta$ one can construct both $M_\Delta$ and a Shimura subvariety $Z_\Delta \subset A_g$ with $N(\Delta) = \dim Z_\Delta$. By construction $j(M_\Delta) \subset Z_\Delta$ and both $M_\Delta$ and $Z_\Delta$ are irreducible algebraic subvarieties. By (2.2) $\dim M_\Delta = r - 3$. Since $j$ is an injective morphism of algebraic varieties, when $g \geq 2$ we always have $N \geq r - 3$. If (*) holds, then $j(M_\Delta)$ is dense in $Z_\Delta$.

2.15 Note also that (when $g \geq 2$) for any $t \in \mathcal{B}_\Delta$ we have $\dim H^0(2K_{C_t})^G = \dim H^1(T_{C_t})^G = \dim \mathcal{B}_\Delta = r - 3$. Hence condition (*) in Theorem 2.13 coincides with condition (*) of the Introduction. It amounts to asking that domain and codomain of $m^G_{C_t}$ have the same dimension. By Proposition 2.10 this is then equivalent to asking that, for general $t$, $m^G_{C_t}$ is injective.

2.16 We now wish to prove a lemma that is helpful to rule out a priori some groups.

Let $\Delta = (G, \theta)$ be a datum and let $H$ be a normal subgroup of $G$. Set $K := G/H$ and let $\pi : G \to K$ be the canonical projection. The composition $\pi \circ \theta : \Gamma_r \to G \to K$ is an epimorphism, but it is not necessarily admissible, since some of the $\gamma_i \in \Gamma_r$ might map to 1. We can throw them away obtaining an admissible epimorphism $\bar{\theta} : \Gamma_s \to K$ for some $s \leq r$. In terms of spherical generators this means the following: if $\theta(\gamma_i) = g_i$ and $k_i = \pi(g_i)$, then $\bar{\theta} = (k_1, \ldots, k_s)$ where we omit all the $k_i$ that equal 1. So we get a new datum $\Delta = (K, \bar{\theta})$. This corresponds to the following geometric situation. $\Delta$ gives rise to
the family $\pi_\Delta : \mathcal{C}_\Delta \to B_\Delta$. We can quotient each fibre $C_t$ by $H$ getting a curve $F_t := C_t/H$ on which $K$ acts:

$$C_t \xrightarrow{\pi} F_t = C_t/H$$

The curves $F_t$ form a family $\mathcal{F} \to B_\Delta$. If $g(F_t) \geq 2$, out of the datum $\tilde{\mathcal{F}}$ we can form the family $\mathcal{C}_\Delta \to B_\Delta$ as explained in 2.6. Then $\mathcal{F}$ is a pull-back of this family, i.e. $f^*\mathcal{C}_\Delta = \mathcal{F}$ for some holomorphic map $f : B_\Delta \to B_\Delta$.

**Lemma 2.17** In the above situation, assume that $g(F) \geq 2$. If (*) holds for $\tilde{\Delta}$, then it holds also for $\Delta$.

**Proof** Write for simplicity $C = C_t$ and $F = F_t$. We have two pull-back maps:

$$p^* : H^0(K_F) \leftarrow H^0(K_C), \quad p^* : H^0(2K_F) \leftarrow H^0(2K_C).$$

From the first one we obtain also an injection

$$f := S^2p^* : S^2H^0(K_F) \leftarrow S^2H^0(K_C).$$

Since $p^*H^0(K_F) = H^0(K_C)^H$ then

$$f((S^2H^0(K_F))^K) \subset (S^2H^0(K_C))^G.$$

Thus, we get a commutative diagram

\[
\begin{array}{ccc}
(S^2H^0(K_F))^K & \xrightarrow{m^K_F} & H^0(2K_F)^K \\
\downarrow f & & \downarrow p^* \\
(S^2H^0(K_C))^G & \xrightarrow{m^G_C} & H^0(2K_C)^G.
\end{array}
\]

from which

$m^G_C$ injective $\Rightarrow m^K_F$ injective.

As explained in 2.15, if (*) holds for $\Delta$, then $m^G_C$ is injective for general $C$ and therefore $m^K_F$ is injective for general $F$, so $N(\tilde{\Delta}) = S^2H^0(K_F)^K \leq H^0(2K_F)^K$. But since $g(F) \geq 2$, the discussion in 2.14 shows that $N(\tilde{\Delta}) \geq s - 3 = H^0(2K_F)^K$. Thus $N(\tilde{\Delta}) = s - 3$, i.e. $\Delta$ satisfies (*). $\square$

3 Avoiding unnecessary computations

This section collects several results that allow to rule out a priori various cases avoiding some parts, sometimes really substantial, of the computation. We briefly explain its contents.

Lemmata 3.1 and 3.2 use the same ideas underlying the proof of the Hurwitz theorem to ensure that signatures exist only in some ranges. Theorem 3.3 summarizes results of Moonen and Mohajer-Zuo, saying that no new counter-examples exist in certain cases.

In 3.4 we introduce spherical systems of generators, recall the Chevalley-Weil formula, define refined passports and show that $N(\Delta)$ only depends on the refined passport of the
generators. We then recall Eichler’s formula. It is used in the proof of Theorem 3.12, which says that no counter-example exists with $G = (Z/2Z)^k$ for $g \geq 4$. Its Corollary 3.13 is the main tool to cut down the number of computations to be done. Other such tools are Frobenius’ test (Corollary 3.15) and an elementary observation on the abelianization of a group admitting a spherical system of generators (Sect. 3.16).

**Lemma 3.1** If $(G, \theta)$ is an r-datum of genus $g$ and $G$ contains an element of order $> 4(g - 1)$, then either $r = 3$, i.e. the family is 0-dimensional, or it coincides with family (5) in [14, Table 2].

If $x \in G$ has order $> 4(g - 1)$, then by definition $H := (x)$ is a large automorphism group of $C$. So the Lemma follows immediately from Proposition 4.5 in [23]. The idea of using upper bounds for the order of single elements of $G$ comes from Corollary 5.10 in [4], where the classical bound of Wiman was used. The theorem of Kulkarni that we use here is more precise.

**Lemma 3.2** Let $\Delta = (G, \theta)$ be an r-datum with genus $g \geq 2$ and $r \geq 4$. If the datum corresponds to an action of $G$ on a smooth curve $X$ with $X/G = \mathbb{P}^1$, then (a) $r \leq 2g + 2$ with equality only for $X$ hyperelliptic and $G$ generated by the hyperelliptic involution, (b) $r \leq 4 + \frac{s(g - 1)}{d}$ and (c) $|G| \leq 12(g - 1)$.

**Proof** The arguments are extremely classical, but for the reader’s convenience we give the proof. Set $d := |G|, \delta := \sum_{i=1}^r \frac{1}{m_i}$ and $\mu := r - 2 - \delta$. By the Riemann-Hurwitz formula,

$$2(g - 1) = d \cdot \mu$$  \hfill (3.1)

Assume $2 \leq m_1 \leq m_2 \leq \ldots \leq m_r$. Since $g \geq 2, \mu > 0$. For $x > 0$ set $f(x) := 1 - 1/x$. Then $\mu = \sum_{i=1}^r f(m_i) - 2$. Since $f$ is increasing $\mu \geq r \cdot f(2) - 2 = (r - 4)/2$. Using $d \geq 2$ and (3.1), this gives $g - 1 \geq (r - 4)/2$, i.e. the inequality in (a). If equality holds $|G| = 2$, so the curves are hyperelliptic. By a dimensional count the family coincides with that of hyperelliptic curves. This proves (a).

Set $A(r) := \{x = (x_1, \ldots, x_r) \in \mathbb{Z}^r : x_i \geq 2, \sum x_i > 2, f(x_i) > 2\}$ and

$$\bar{\mu}(r) := \min_{x \in A(r)} \left\{ \sum_{i=1}^r f(x_i) - 2 \right\}$$

Using the fact that $f$ is strictly increasing one verifies that for $r = 4$ the minimum is achieved at $x = (2, 2, 2, 3)$ and $\bar{\mu}(4) = 1/6$, while for $r \geq 5$ the minimum is achieved at $x = (2, \ldots, 2)$ and $\bar{\mu}(r) = r/2 - 2$. So for any $r \geq 4$ we have $\bar{\mu}(r) \geq (r - 4)/2$. Let now $m$ be the signature of the datum $(G, \theta)$. Then $m \in A(r)$, so $\mu \geq \bar{\mu}(r)$. Thus (3.1) gives $2(g - 1)/d \geq \bar{\mu}(r) \geq (r - 4)/2$, which is the inequality in (b).

If $r = 4$, (3.1) gives $2(g - 1)/d \geq \bar{\mu}(4) = 1/6$, which is equivalent to the inequality in (c). If $r > 4$ in the same way we get $d \leq 4(g - 1)/(r - 4) \leq 4(g - 1)$. But $4(g - 1)/(r - 4) \leq 4(g - 1)/12(g - 1)$ for every value of $r$. \hfill $\Box$

**Theorem 3.3** The data $\Delta = (G, \theta)$ satisfying (*) with $G$ cyclic or with $G$ abelian and $r = 4$ are Hurwitz equivalent to those mentioned in Theorem 1.5. Moreover for such data (*) is necessary for $Z_\Delta$ to be a Shimura subvariety.

These results are due to Moonen [26] and Mohajer-Zuo [25, Thms. 3.1 and 6.2].
3.4 If \( G \) is a finite group, giving an \( r \)-datum \( \Delta = (G, \theta) \) is equivalent to giving a list of generators \( g_1, \ldots, g_r \) of \( G \) such that \( g_i \neq 1 \) for any \( i \) and subject to the constraint \( g_1 \cdots g_r = 1 \). Indeed, this defines an epimorphism \( \theta : \Gamma_r \to G \) by \( \theta(\gamma_i) = g_i \). From now on we will write \( \Delta \in \mathcal{D}^r \) as \( \Delta = (G, g_1, \ldots, g_r) \), and we will call \((g_1, \ldots, g_r)\) a \textit{spherical system of generators} of the group \( G \).

Let \( \chi_\rho \) denote the character of the representation \( \rho \) defined in (2.3). As explained in [14, §§2.9ff] the number \( N(\Delta) \) in (2.4) can be computed from \( \chi_\rho \):

\[
N(\Delta) = \frac{1}{2|G|} \sum_{a \in G} (\chi_\rho(a^2) + \chi_\rho(a)^2).
\] (3.2)

So to test (*) one needs to compute \( \chi_\rho \) for a datum \( \Delta \). There are two ways to do that: using Eichler’s trace formula or the Chevalley–Weil formula. We need both and we start from the Chevalley–Weil formula.

3.5 Next, fix a datum \( \Delta = (G, g_1, \ldots, g_r) \) and let \( m_j := \text{ord}(g_j) \) as usual. Denote by \( \text{Irr} \ G \) the set of irreducible characters of \( G \). For each \( \chi \in \text{Irr} \ G \) fix a representation \( \sigma_\chi \) with character \( \chi \). For \( n \in \mathbb{N}, n > 0 \) set \( \zeta_n := \exp(2\pi i/n) \). If \( \chi \in \text{Irr} \ G, 1 \leq j \leq r \) and \( 0 \leq \alpha < m_j \), denote by \( N_{j, \alpha} \) the multiplicity of \( \zeta_{m_j}^\alpha \) as an eigenvalue of \( \sigma_\chi(g_j) \).

**Theorem 3.6 (Chevalley–Weil)** If \( \Delta = (G, g_1, \ldots, g_r) \) is a datum for the Galois covering \( C \to \mathbb{P}^1 \), then the multiplicity \( \mu_\chi \) of \( \chi \in \text{Irr} \ G \) in \( \rho \) is

\[
\mu_\chi = -\deg \chi + \sum_{j=1}^{r} \sum_{\alpha=0}^{m_j-1} N_{j, \alpha} \frac{\alpha}{m_j} + \varepsilon,
\] (3.3)

where \( \varepsilon = 1 \) if \( \chi \) is the trivial character and \( \varepsilon = 0 \) otherwise.

A nice reference for the Chevalley-Weil formula is [20, Ch. 1]. Our implementation uses this formula to compute \( \chi_\rho \) and hence \( N(\Delta) \). In fact we use the same algorithm as Gleißner, which is based in turn on [32], but with code optimized for our setting (see 4.6).

We will denote the symmetric group on \( r \) letters by \( \Sigma_r \); given \( \sigma \in \Sigma_r \), we will write \( \sigma_i \) for \( \sigma(i) \).

**Definition 3.7** Given a finite group \( G \) let \( C_G \) or simply \( C \) denote the set of conjugacy classes of \( G \). The symmetric group \( \Sigma_r \) acts on

\[
C_G^r := C_G \times \cdots \times C_G.
\]

A \textit{refined passport} with \( r \) branch points for the group \( G \) is an element of \( C_G^r / \Sigma_r \). Thus a refined passport is an unordered sequence of conjugacy classes of \( G \). Given \( \Delta = (G, g_1, \ldots, g_r) \), the \textit{refined passport} of \( \Delta \) is the class of \((g_1], \ldots, [g_r])\) in \( C_G^r / \Sigma_r \).

Note that this definition is slightly different from those of [24,29]: we do not assume that a refined passport comes from a datum.

3.8 It is clear that the numbers \( N_{j, \alpha} \) defined in 3.5 do not change if \( g_j \) is replaced by another element \( g_j' \in G \) which is conjugate to \( g_j \). Another observation is that obviously the sum in (3.3) is independent of the order. Thus \( N(\Delta) \) depends only on the refined passport of \( \Delta \). This elementary observation is at the basis of our approach to the computation.
Lemma 3.9 Let $G$ be a finite group and let $C_i \in C_G$ for $i = 1, \ldots, r$. Assume that there is a datum $\Delta = (G, g_1, \ldots, g_r)$ with $g_i \in C_i$ for $i = 1, \ldots, r$. Then for any $\sigma \in \Sigma_r$ there is a datum $(G, \gamma_1, \ldots, \gamma_r)$ such that $\gamma_i \in C_{\sigma_i}$ for $i = 1, \ldots, r$.

Proof Since $\Sigma_r$ is generated by simple transpositions, it is enough to prove the result for $\sigma = (j, j+1), 1 \leq j < r$. Set

$$\gamma_i = g_i, \quad \text{for } i \neq [j, j+1], \quad \gamma_j = g_j g_{j+1} g_j^{-1}, \quad \gamma_{j+1} = g_j.$$

Then $(G, \gamma_1, \ldots, \gamma_r)$ is still a datum and $\gamma_i \in C_{\sigma_i}$ for any $i$. □

3.10 Now we turn to Eichler’s formula, which is important to rule out a class of groups. Recall that if $a \in G, p \in C$ and $a \cdot p = p$, then $da(p) \in \text{End} T_p C = \mathbb{C}$ is a root of unity, see e.g. [13, p. 106].

Theorem 3.11 (Eichler Trace Formula) If $a \in G, a \neq 1$ then

$$\chi_\rho(a) = 1 - \sum_{p \in \text{Fix}(a)} \frac{1}{1 - da(p)}. \quad (3.4)$$

See e.g. [13, Thm. V.2.9, p. 281].

Theorem 3.12 Let $\Delta = (G, g_1, \ldots, g_r)$ be a datum corresponding to a covering $C \to \mathbb{P}^1$ with $G \cong (\mathbb{Z}/2\mathbb{Z})^k$. If $g(C) \geq 4$, then (*) does not hold for $\Delta$.

Proof The families fulfilling condition (*) with genus up to 7 have been classified in [14, Theorems 5.4 and 5.5] and are all listed in [14, Table 2]: inspecting the table we see that we may assume $g(C) \geq 8$.

Since all elements $a$ in $G, a \neq 1$, have order 2, by the Hurwitz formula

$$\chi_\rho(1) = g(C) = 1 + \frac{|G|}{4}(r - 4) = 1 + 2^{k-2}(r - 4).$$

Moreover for all $p \in \text{Fix}(a), da(p) = -1 \in \mathbb{R}$ and then, by (3.4) for all $a \in G, \chi_\rho(a) \in \mathbb{R}$. In particular all summands in the expression of $N$ in (3.2) are real numbers and

$$N(\Delta) = \frac{1}{2|G|} \sum_{a \in G} (\chi_\rho(a^2) + \chi_\rho(a)^2) \geq \frac{1}{2^{k+1}} \sum_{a \in G} (\chi_\rho(1) + \chi_\rho(a)^2) \geq \frac{\left(\sum_{a \in G} \chi_\rho(1)\right)^2 + \chi_\rho(1)^2}{2^{k+1}} = g(C) \left(\frac{1}{2} + \frac{g(C)}{2^{k+1}}\right) = g(C) \left(\frac{1}{2} + \frac{g(C)}{2^{k+1}}\right) + \frac{r - 4}{8} = g(C) \left(\frac{1}{2} + \frac{r}{8}\right) > g(C) \left(\frac{r}{8}\right) \geq r$$

contradicting (*). □

Considering Lemma 2.17 we deduce the following stronger result:

Corollary 3.13 Let $\Delta = (G, g_1, \ldots, g_r)$ be a datum corresponding to a covering $C \to \mathbb{P}^1$. If there is a surjective map $G \to (\mathbb{Z}/2\mathbb{Z})^4$, then (*) does not hold for $\Delta$.

Proof Assume by contradiction that (*) holds.

Let $H$ be the kernel of the surjection $G \to (\mathbb{Z}/2\mathbb{Z})^4$ and consider the family of the curves $F_i = C_i/H \to \mathbb{P}^1$ as in 2.16. They are Galois covers with datum $\Delta = ((\mathbb{Z}/2\mathbb{Z})^4, h_1, \ldots, h_\Delta)$.
Since each set of generators of \((\mathbb{Z}/2\mathbb{Z})^4\) has cardinality at least 4, then \(s \geq 5\). This implies \(g(F) \geq 2\) by the Hurwitz formula and \(g(F) \leq 4\) by Lemma 2.17 and Theorem 3.12.

The Galois covers of \(\mathbb{P}^1\) with genus among 2 and 4 having 4 or more branch points are listed in [14, Table 2]: we see that the group \((\mathbb{Z}/2\mathbb{Z})^4\) does not occur, reaching an absurd. □

The Galois group \(G\) of family (34) in [14, Table 2] admits \((\mathbb{Z}/2\mathbb{Z})^3\) as a quotient. Thus one cannot improve the above Corollary by substituting \((\mathbb{Z}/2\mathbb{Z})^4\) with one of its proper quotients. In fact applying Lemma 2.17 to this case yields one of the families of elliptic curves mentioned after Theorem 1.5.

There is another useful criterion, already used by Breuer [6] and Paulhus [29]. Indeed, for some elements \(c\), one can ascertain \textit{a priori} that \(\pi^{-1}(c) = p^{-1}(\tilde{c})\) does not contain any system of generators at all. This is based on a theorem of Frobenius. (See [24, p. 406] for a proof.)

**Theorem 3.14** (Frobenius’ formula) Given a finite group \(G\) and conjugacy classes \(C_1, \ldots, C_r\), the number of \(r\)-ples \((g_1, \ldots, g_r) \in C_1 \times \cdots \times C_r\) such that \(\prod g_i = 1\) is

\[
\frac{|C_1| \cdots |C_r|}{|G|} \sum_{\chi \in \text{Irr}(G)} \chi(C_1) \cdots \chi(C_r) \chi(1)^{r-2}.
\]

Notice that this condition is independent of the order.

**Corollary 3.15** Let \(G\) be a group and \((C_1, \ldots, C_r)\) a refined passport. If

\[
\sum_{\chi \in \text{Irr}(G)} \frac{\chi(C_1) \cdots \chi(C_r)}{\chi(1)^{r-2}} = 0,
\]

then there is no datum \((G, g_1, \ldots, g_r)\) with refined passport \((C_1, \ldots, C_r)\).

3.16 We conclude with a useful elementary observation. Assume that a group \(G\) admits a system of spherical generators \((g_1, \ldots, g_r)\) with signature \((m_1, \ldots, m_r)\). Decompose its abelianization \(\text{Ab} G = \mathbb{Z}/k_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/k_p\mathbb{Z}\) with \(k_1 \cdots k_p\) (i.e. the \(k_i\)’s are the invariant factors). Since for any \(j\), \(\text{Ab} G\) is generated by the images of \(g_1, \ldots, \hat{g}_j, \ldots, g_r\), it follows that \(p \leq r - 1\) and that \(k_p\) divides \(\text{lcm}(m_1, \ldots, \hat{m}_j, \ldots, m_r)\) for any \(j\).

**4 The algorithm**

4.1 Given a group \(G\), let \(C_G\) be the set of its conjugacy classes. Recall from Definition 3.7 that a refined passport on \(G\) with \(r\) branch points is an unordered sequence of \(r\) conjugacy classes of \(G\), i.e. an element of \(C_G^r/\Sigma_r\). If a refined passport contains a spherical system of generators \(\Delta = (g_1, \ldots, g_r), \ g(\Delta)\) and \(N(\Delta)\) only depend on the refined passport of \(\Delta\). We will say that a refined passport is a \textit{counter-example of genus} \(g\) if it contains a spherical system of generators \(\Delta\) with \(g(\Delta) = g\) such that \((*)\) holds. Notice that refined passports that satisfy \((*)\) formally but do not contain a spherical system of generators are excluded by this definition. The group \(\text{Aut} G\) acts both on \(C_G\) and on the set of refined passports.

4.2 We illustrate an algorithm to attack the following:

**Problem 4.1** For fixed \(g \geq 2\), list groups \(G\) and counter-examples of genus \(g\) on \(G\) with \(r \geq 4\) branch points, one for each orbit of \(\text{Aut}(G)\), leaving aside those with \(G\) cyclic and those with \(G\) abelian and \(r = 4\).
Our basic strategy is to fix \( r \), and then choose one refined passport of genus \( g \) with \( r \) branch points in each \( \text{Aut}(G) \)-orbit. If (*) holds, it then suffices to determine whether the refined passport contains a system of spherical systems of generators.

### 4.3
As in [2,14], we use signature as an invariant. Using the notation of Definition 2.2 signature defines a map

\[
\mathcal{D}^r \rightarrow \mathbb{N}^r, \quad (g_1, \ldots, g_r) \mapsto (\text{ord}(g_1), \ldots, \text{ord}(g_r)).
\]

Since the order of an element only depends on its conjugacy class, the signature of a spherical system of generators \((g_1, \ldots, g_r)\) only depends on the conjugacy classes \(([g_1], \ldots, [g_r])\).

Corresponding to the fact that refined passports are taken up to reordering (Lemma 3.9), signatures can be considered up to permutation, i.e. we can restrict to signatures satisfying

\[
m_1 \leq \cdots \leq m_r.
\]

We iterate over the order \( d = |G| \). For fixed \( d \), let \( \mathcal{S}_{d,g} \) be the set of finite sequences \( m = (m_1, \ldots, m_r) \) such that

\[
\begin{align*}
&S1) \quad 4 \leq r \leq \frac{4(g-1)}{d} + 4 \text{ and } d \leq 12(g-1); \\
&S2) \quad \text{each } m_i \text{ is a divisor of } d; \\
&S3) \quad 1 < m_i < d; \\
&S4) \quad g \text{ and } m \text{ satisfy (2.1);} \\
&S5) \quad m_1 \leq \cdots \leq m_r;
\end{align*}
\]

By Lemma 3.2, the signature of a spherical system of generators \( \Delta \) with \( r \geq 4 \) and \( g(\Delta) = g \) must satisfy (S1); the restriction \( r \geq 4 \) ensures that the family is positive-dimensional, see (2.2); the restriction \( m_i < d \) in (S3) is motivated by the fact that we are only interested in noncyclic groups \( G \).

The set of “admissible” signatures \( \mathcal{S}_{d,g} \) is computed by Algorithm 1. In the implementation, we found it convenient to compute each \( \mathcal{S}_{d,g} \) for \( 2 \leq g \leq g_{\text{max}} \) simultaneously, and then store the result on disk for later retrieval, rather than iterate over \( g \); this prevents repeating some computations.

### 4.4
Elements of \( C_G^r / \Sigma_r \) (i.e. refined passports) can be viewed as multisets. Given a set \( X \), a multiset of elements of \( X \) can be defined as a set \( \{(x_1, n_1), \ldots, (x_k, n_k)\} \) where the \( x_i \) are pairwise disjoint elements of \( X \) and the \( n_i \) are nonnegative integers representing the multiplicity of \( x_i \). In fact, it is customary to require the \( n_i \) to be positive, but it will be convenient for our purposes to allow them to be zero as well. We will write a multiset as \( \{x_1^{n_1}, \ldots, x_k^{n_k}\} \). A set \( \{x_1, \ldots, x_k\} \) can be identified with the multiset \( \{x_1^1, \ldots, x_k^1\} \), and the union of two multisets is defined in the obvious way by adding multiplicities.

It will also be convenient to represent elements of \( \mathcal{S}_{d,g} \) as multisets of integers \( \{m_1^{n_1}, \ldots, m_k^{n_k}\} \); for instance, the signature \((2,2,3,3,3)\) will be represented by the multiset \((2^2, 3^3)\).

### 4.5
Problem 4.1 can then be addressed by iterating through the signatures \( m \in \mathcal{S}_{d,g} \) computed in Algorithm 1 and groups \( G \) of order \( d \). A refined passport with signature \( m \) only exists on a group \( G \) if there is at least one element of order \( m_j \) for every \( m_j \in m \); we therefore discard groups and signatures that do not satisfy this condition. More groups and signatures can be eliminated by taking advantage of Lemma 3.1, Corollary 3.13 and the observation in 3.16. This procedure is displayed in Algorithm 2, which reduces the problem to identifying counter-examples for fixed group and signature. Notice that on line 25 the signature \( \{m_1^{n_1}, \ldots, m_k^{n_k}\} \) is converted into a multiset

\[
\{A_1^{n_1}, \ldots, A_k^{n_k}\} \subset \mathcal{P}(C_G), \quad (4.1)
\]
where each $A_i$ is the subset of $C_G$ of conjugacy classes of order $m_i$. This is the basis for the recursion of Algorithm 4.

4.6 At this point we need to determine the counter-examples with a given signature $\mathbf{m}$ and group $G$. This is achieved by picking one refined passport with signature $\mathbf{m}$ in each Aut($G$)-orbit, then verifying whether (*) holds and the refined passport contains a spherical system of generators.

The iteration through one refined passport in each Aut($G$)-orbit is performed in Algorithm 4. A refined passport with signature $\{m_1^{n_1}, \ldots, m_k^{n_k}\}$ is obtained by choosing $n_i$ conjugacy classes with order $m_i$ for each $1 \leq i \leq k$; in terms of (4.1), for each $i$ we must choose a multiset $S_i$ of $n_i$ elements of $A_i$, counted with multiplicities. We can write $S_i$ in a unique way as a union of sets $\bigcup_j B_{ij}$, where $B_{i1} \supset B_{i2} \supset \ldots$ is a definitely empty sequence of subsets of $A_i$; this means that the multiplicity of $C$ in $S_i$ is the number of indices $j$ such that $C$ is in $B_{ij}$. Thus, iterating through the possible multisets $S_i$ is equivalent to iterating through sequences $A_i \supset B_{i1} \supset B_{i2} \supset \ldots$, $\sum |B_{ij}| = n_i$.

This must be repeated for each $i = 1, \ldots, k$.

Our goal is to perform a similar iteration by choosing a single element in each Aut($G$)-orbit. To begin with, our algorithm picks a subset $B$ of $A_k$ with $1 \leq h \leq n_k$ elements, representing $B_{k1}$ in the notation above. For each choice of $B$, the function recursively iterates through refined passports obtained by taking the union of $B$ and a refined passport with $n_i$ elements in each $A_i$, $i < k$ and $n_k - h$ elements in $B$. The recursive call iterates through one refined passport for each $H$-orbit, where $H$ is the stabilizer of $B$ in Aut($G$). Top-level iteration over one subset $B$ for each Aut($G$)-orbit completes the algorithm.

This approach requires a much lower amount of memory than determining all possible refined passports first and then picking one in each Aut($G$)-orbit. Notice also that the refined passports produced by the algorithm are elaborated sequentially, and not stored simultaneously into memory. Nevertheless, the algorithm must iterate through one subset of $A_k$ for each Aut($G$)-orbit, and we are not aware of any efficient way of doing this without storing all subsets of fixed cardinality in memory. This is the one point in the whole algorithm where memory consumption can be significant.

Algorithm 3 determines whether a refined passport is a counter-example; first, the condition of Theorem 3.14 is verified, i.e. whether $\sum \chi(C_1) \cdots \chi(C_r) / \chi(1)^{r-2}$ is nonzero; if so, we will say that $(C_1, \ldots, C_r)$ passes Frobenius’ test. Then, condition (*) is tested by selecting random elements inside each $C_i$ and computing $N(\Delta)$ by (3.3). Notice that each term $\sum_{\alpha} N_j,\alpha / m_j$ appearing in (3.3) only depends on the corresponding $g_j$, and the characters $\chi$ only depend on the group $G$. Thus, it suffices to compute these data at the beginning of the computation, when $G$ is fixed, making the computation of (3.3) in the iteration quite fast. Only when both Frobenius’ test and (*) hold does the algorithm perform the most computationally expensive step, namely checking whether $C_1 \times \cdots \times C_r$ contain a spherical system of generators, by straightforward iteration.
4.7 For abelian groups \( G \), conjugacy classes contain a single element, and the algorithm can be improved.

First, observe that Frobenius’ test is useless in this case: the product \( C_1 \times \cdots \times C_r \) contains a single element \((g_1, \ldots, g_r)\), so the condition \( \prod g_i = 1 \) is best verified directly.

Second, since refined passports contain a single element of \( G' \), we effectively iterate through elements of \( G' \). However, in a spherical system of generators \((g_1, \ldots, g_r)\) any element is determined by the others, so we can iterate through “short sequences” \((g_1, \ldots, \hat{g}_j, \ldots, g_r)\). Thus, we proceed as follows.

We fix \( m_j \) in \( m = (m_1, \ldots, m_r) \) such that the number of elements of \( G \) with order \( m_j \) is largest; then, we use a scheme analogous to Algorithm 4 to iterate through \((r - 1)\)-ples \((g_1, \ldots, \hat{g}_j, \ldots, g_r)\) with signature \((m_1, \ldots, \hat{m}_j, \ldots, m_r)\), one for each \( \text{Aut}(G) \)-orbit. We then define \( g_j \) as the inverse of \( g_1 \cdots \hat{g}_j \cdots g_r \); if \( g_j \) has order \( m_j \), the \((g_1, \ldots, g_r)\) is a candidate for a spherical system of generators with signature \( m \). At this point, we test condition (*) and, if it holds, whether the elements \( g_1, \ldots, g_r \) generate the group \( G \).

This is clearly faster than a plain application of Algorithm 4, because an \( r \)-fold iteration is replaced by an \((r - 1)\)-fold iteration. Notice, however, that the same counter-example can appear more than once in the output, if this method is applied to cases where \( m_j \) has multiplicity greater than one, say \( m_j = m_{j+1} \). Indeed, a counter-example \((g_1, \ldots, \hat{g}_j, \ldots, g_r)\) can be obtained by completing two different short sequences, namely \((g_1, \ldots, \hat{g}_j, \ldots, g_r)\) and \((g_1, \ldots, \hat{g}_{j+1}, \ldots, g_r)\). If the two short sequences lie in different \( \text{Aut}(G) \)-orbits, the output will contain two counter-examples in the \( \text{Aut}(G) \)-orbit of \((g_1, \ldots, g_r)\).

---

**Algorithm 1: Computing the signatures**

```
input : integers \( g \geq 2, d \geq 2 \)
output: the set of signatures \( \mathcal{S}_{d,g} \)

Function \( \mathcal{S}_{d,g}(d, g) \)

1. if \( d \) prime then
2. return \( \emptyset \) // (S3) cannot be satisfied
3. \( \mathcal{S}_{d,g} \leftarrow \emptyset \)
4. for \( r \) satisfying \((S1)\) do
5. \( D \leftarrow \{ n \in \mathbb{N} \mid 2 \leq n < d, \ n \text{ divides } d \} \);
6. for \( m_1, \ldots, m_r \in D, m_1 \leq \cdots \leq m_r \) do
7. \( \text{if } (m_1, \ldots, m_r) \text{ satisfies } (S4) \text{ and } (S5) \text{ then} \)
8. \( \text{insert } (m_1, \ldots, m_r) \text{ in } \mathcal{S}_{d,g} \)
9. return \( \mathcal{S}_{d,g} \)
```
Algorithm 2: Find counter-examples of genus $g$ with $r \geq 4$ branch points

input: an integer $g \geq 2$
output: counter-examples of genus $g$ with $r \geq 4$ branch points, one for each $\text{Aut}(G)$-orbit

Function $\text{admissible}(G, m)$

1. $O \leftarrow \{\text{ord}(g) | g \in G\}$;
2. if $r = 4$ and $G$ abelian then
   return false;
3. else if $r > 4$ and $G$ cyclic then
   return false;
4. else if some $m_i$ is not in $O$ then
   return false
5. else if $g > 2$ and some $o \in O$ is greater than $4(g - 1)$ then
   return false // Lemma 3.1
6. else
decompose the abelianization of $G$ as $\mathbb{Z}/k_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/k_p\mathbb{Z}$, with each $k_i$ dividing $k_{i+1}$;
7. if at least 4 elements in $(k_1, \ldots, k_p)$ are even then
   return false // $G$ surjects over $(\mathbb{Z}/2\mathbb{Z})^4$ (Corollary 3.13)
8. if $p \geq r$ then
   return false // $r - 1$ elements cannot generate $G$
9. else if exists $j$ such that $k_p \nmid \text{lcm}(m_1, \ldots, m_j, \ldots, m_r)$ then
   return false // § 3.16
10. return true // passed all tests

21. for $2 \leq d \leq 12(g - 1)$ do
22.   determine $\mathcal{S}_{d,g}$ by Algorithm 1;
23.   for $m = \{m_1, \ldots, m_r\}$ in $\mathcal{S}_{d,g}$ do
24.     for $G$ group of order $d$ do
25.       if $\text{admissible}(G, m)$ then
26.         for $1 \leq i \leq k$ do
27.             $A_i \leftarrow \{C \in C_G | \text{ord}(C) = m_i\}$
28.             CounterExamplesIn$(G, \{A_1^{m_1}, \ldots, A_k^{m_k}\})$ // Find counter-examples for group $G$ and signature $m$ using Algorithm 4

Algorithm 3: Determine whether a refined passport is a counter-example

input: a group $G$, a refined passport $(C_1, \ldots, C_r)$
output: true if the refined passport is a counter-example, false otherwise

Function $\text{IsCounterExample}(G, C_1, \ldots, C_r)$

1. if $(C_1, \ldots, C_r)$ passes Frobenius' test and $N = r - 3$ then
2.   for $(g_1, \ldots, g_{r-1})$ in $C_1 \times \cdots \times C_{r-1}$ do
3.     $gr \leftarrow (g_1 \cdots g_{r-1})^{-1}$;
4.     if $gr \in C_r$ and $(g_1, \ldots, g_{r-1}) = G$ then
5.       return true
6. return false
Algorithm 4: Find counter-examples for fixed group and signature

\textbf{input}: A group \( G \) and a nonempty multiset \( \{A_1^{n_1}, \ldots, A_k^{n_k}\} \) where each \( A_i \) is a nonempty set of conjugacy classes of \( G \) and each \( n_i \) is a nonnegative integer

\textbf{output}: Counter-examples obtained by choosing \( n_i \) elements in each \( A_i \), one for each \( \text{Aut}(G) \)-orbit

1 Function \( \text{CounterExamplesIn}(G, \{A_1^{n_1}, \ldots, A_k^{n_k}\}, S=\emptyset, H=\text{Aut}(G)) \)
\hspace{1cm} // This is a recursive function using two arguments with default values: \( S \) is a multiset of conjugacy classes of \( G \); \( H \) is a subgroup of \( \text{Aut}(G) \) acting on each \( A_i \)
2 if \( k = 0 \) then
3\hspace{1cm} if \( \text{IsCounterExample}(G, S) \) then
4\hspace{2cm} print \( G, S \)
5\hspace{1cm} else if \( n_k = 0 \) then
6\hspace{2cm} \text{CounterExamplesIn}(G, \{A_1^{n_1}, \ldots, A_{k-1}^{n_{k-1}}\}, S, H)
7\hspace{1cm} else if \( A_k \) contains a single element \( a \) then
8\hspace{2cm} \text{CounterExamplesIn}(G, \{A_1^{n_1}, \ldots, A_{k-1}^{n_{k-1}}\}, S \cup \{a^{n_k}\}, H)
9\hspace{1cm} else
10\hspace{2cm} \text{CounterExamplesWith}(G, \{A_1^{n_1}, \ldots, A_{k-1}^{n_{k-1}}\}, S, H, A_k, n_k)

11 Function \( \text{CounterExamplesWith}(G, \{A_1^{n_1}, \ldots, A_k^{n_k}\}, S, H, A, n) \)
\hspace{1cm} // Helper function that iterates through subsets of \( A \)
12 for \( 1 \leq h \leq n \) do
13\hspace{1cm} \( X \leftarrow \{B \subset A \mid |B| = h\} \)
14\hspace{1cm} for one \( B \) in each \( H \)-orbit of \( X \) do
15\hspace{2cm} \( K \leftarrow \text{stabilizer of } B \text{ for action of } H \text{ on } X \)
16\hspace{2cm} \text{CounterExamplesIn}(G, \{A_1^{n_1}, \ldots, A_k^{n_k}, B^{n-h}\}, S \cup B, K)

Acknowledgements The authors would like to thank Paola Frediani for help with Lemma 2.17 and Matteo Penegini and Fabio Perroni for interesting discussions related to the subject of this work. The second author would like to thank Matteo Garofano and Gabriele Merli for technical help with the installation and the maintenance of the server used for the computations.

References

1. Arbarello, E., Cornalba, M., Griffiths, P.A.: Geometry of Algebraic Curves, vol. II. Springer, New York (2011)
2. Bauer, I., Catanese, F., Grunewald, F., Pignatelli, R.: Quotients of products of curves, new surfaces with \( p_g = 0 \) and their fundamental groups. Am. J. Math. 134, 993–1049 (2012)
3. Bauer, I., Pignatelli, R.: The classification of minimal product-quotient surfaces with \( p_g = 0 \). Math. Comput. 81(280), 2389–2418 (2012)
4. Bauer, I., Pignatelli, R.: Product-quotient surfaces: new invariants and algorithms. Groups Geom. Dyn. 10(1), 319–363 (2016)
5. Bosma, W., Cannon, J., Playoust, C.: The Magma algebra system. I. The user language. J. Symb. Comput. 24, 235–265 (1997)
6. Breuer, T.: Characters and Automorphism Groups of Compact Riemann Surfaces. Cambridge University Press, Cambridge (2000)
7. Catanese, F., Lönne, M., Perroni, F.: Irreducibility of the space of dihedral covers of the projective line of a given numerical type. Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl. 22(3), 291–309 (2011)
8. Catanese, F., Lönne, M., Perroni, F.: The irreducible components of the moduli space of dihedral covers of algebraic curves. Groups Geom. Dyn. 9(4), 1185–1229 (2015)
9. Colombo, E., Frediani, P., Ghigi, A.: On totally geodesic submanifolds in the Jacobian locus. Int. J. Math. 26(1), 1550005 (2015)
Some evidence for the Coleman–Oort conjecture

10. Conti, D.: https://github.com/diego-conti/hlidskjalf
11. Conti, D., Ghigi, A., Pignatelli, R.: https://github.com/diego-conti/centone
12. Farb, B., Margalit, D.: A Primer on Mapping Class Groups. Princeton University Press, Princeton (2012)
13. Farkas, H.M., Kra, I.: Riemann Surfaces, Volume 71 of Graduate Texts in Mathematics, 2nd edn. Springer, New York (1992)
14. Frediani, P., Ghigi, A., Penegini, M.: Shimura varieties in the Torelli locus via Galois coverings. Int. Math. Res. Not. 20, 10595–10623 (2015)
15. Frediani, P., Ghigi, A.I.: Spelta Infinitely many Shimura varieties in the Jacobian locus for \( g \leq 4 \). Preprint. arXiv:1910.13245 To appear on Ann. Sc. Norm. Super, Pisa (2019)
16. Frediani, P., Penegini, M., Porru, P.: Shimura varieties in the Torelli locus via Galois coverings of elliptic curves. Geom. Dedic. 181, 177–192 (2016)
17. Frediani, P., Pirola, G.P.: On the geometry of the second fundamental form of the Torelli map. Proc. Am. Math. Soc. 149(3), 1011–1024 (2021)
18. Ghigi, A.: On some differential-geometric aspects of the Torelli map. Boll. Unione Mat. Ital. 12(1–2), 133–144 (2019)
19. Ghigi, A., Pirola, G.P., Torelli, S.: Totally geodesic subvarities in the moduli space of curves. Commun. Contemp. Math. 23(3), 2050020 (2021)
20. Gleißner, C.: Threefolds Isogenous to a Product and Product quotient Threefolds with Canonical Singularities. Ph.D. Dissertation. Bayreuth (2016). https://epub.uni-bayreuth.de/2981
21. González Díez, G., Harvey, W.J.: Moduli of Riemann surfaces with symmetry. In: Harvey, W.J., Maclachlan, C. (eds.) Discrete groups and geometry (Birmingham, 1991), pp. 75–93. Cambridge University Press, Cambridge (1992)
22. Grushevsky, S., Möller, M.: Explicit formulas for infinitely many Shimura curves in genus 4. Asian J. Math. 22(2), 381–390 (2018)
23. Kulkarni, R.S.: Riemann surfaces admitting large automorphism groups, In: Extremal Riemann surfaces (San Francisco, CA, 1995), Contemporary Mathematics, 201, pp. 63–79. AMS (1997)
24. Lando, S.K., Zvonkin, A.K.: Graphs on Surfaces and Their Applications, Volume 141 of Encyclopaedia of Mathematical Sciences. Springer, Berlin (2004)
25. Mohajer, A., Zuo, K.: On Shimura subvarieties generated by families of abelian covers of \( \mathbb{P}^1 \). J. Pure Appl. Algebra 222(4), 931–949 (2018)
26. Moonen, B.: Special subvarieties arising from families of cyclic covers of the projective line. Doc. Math. 15, 793–819 (2010)
27. Moonen, B., Oort, F.: The Torelli locus and special subvarieties. In: Farkas, G., Morrison, I. (eds.) Handbook of Moduli, vol. II, pp. 549–94. International Press, Boston (2013)
28. Oort, F.: Canonical liftings and dense sets of CM-points. In: Catanese, F. (ed.) Arithmetic geometry (Cortona, 1994), Sympos. Math., XXXVII, pp. 228–234. Cambridge University Press, Cambridge (1997)
29. Paulhus, J.: A database of group actions on Riemann surfaces. Preprint (2017) https://paulhus.math.grinnell.edu/Paulhus-lmfdb.pdf
30. Pignatelli, R., Raso, C.: Riemann surfaces with a quasi large abelian group of automorphisms. Matematiche (Catania) 66(2), 77–90 (2011)
31. Rohde, J.C.: Cyclic Coverings, Calabi-Yau Manifolds and Complex Multiplication. Lecture Notes in Mathematics, vol. 1975. Springer, Berlin (2009)
32. Swinarski, D.: https://faculty.fordham.edu/dswinarski/ivrg/
33. Tange, O.: GNU parallel: the command-line power tool. Login USENIX Mag. 36(1), 42–47 (2011)
34. Zieschang, H., Vogt, E., Coldewey, H.-D.: Surfaces and Planar Discontinuous Groups. LNM Volume 835 of Lecture Notes in Mathematics. Springer, Berlin (1980)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.