Stability conditions on product threefolds of projective spaces and Abelian varieties

Naoki Koseki

ABSTRACT

In this paper, we prove the original Bogomolov–Gieseker type inequality conjecture for \( \mathbb{P}^1 \times S \), \( \mathbb{P}^2 \times C \) and \( \mathbb{P}^1 \times \mathbb{P}^1 \times C \), where \( S \) is an Abelian surface and \( C \) is an elliptic curve. In particular, there exist Bridgeland stability conditions on these threefolds.

Contents

1. Introduction ...............2 29
2. Preliminaries ...............2 31
3. Preparation for the main theorem ..........2 33
4 . P r o o f o f t h e m a i n theorem............2 35
Appendix. Counter-example for Conjecture 2.4 ........2 42
References ................2 43

1. Introduction

1.1. Motivation and results

The notion of stability conditions on a triangulated category was introduced by Bridgeland in his paper [10]. Bridgeland stability condition is a mathematical subject realizing Douglas’ II-stability in string theory [12–14]. It gives us new points of view in various scenes, such as birational geometry, counting invariants, mirror symmetry and so on (cf. [2–6, 8, 26–29]).

Constructing stability conditions on the derived category of coherent sheaves of a given smooth projective variety \( X \) is a starting problem for such applications. When \( \dim X \leq 2 \), the standard construction of stability conditions on \( D^b(X) \) was given in [1, 11]. In the case when \( \dim X = 3 \), the construction problem of stability conditions on \( D^b(X) \) is still open in general. In the paper [8], Bayer, Macrì and Toda proposed a conjectural approach for this problem. The problem was reduced to the conjectural Bogomolov–Gieseker (BG) type inequality for Chern characters (involving the third part of the Chern character) of certain semistable objects (called tilt-semistable objects) in the derived category. It is known that the original BG inequality conjecture holds for Abelian threefolds (cf. [7, 17, 18]), Fano threefolds of Picard rank one (cf. [8, 16, 19, 23]), some toric threefolds (cf. [9]) and their étale quotients (cf. [20]).

However, counter-examples for the original BG inequality conjecture were constructed in the case when \( X \) is the blow-up of a smooth projective threefold at a point (cf. [21, 24]). Furthermore, using the argument of [21], we can show that the BG inequality conjecture does not hold even when \( X \) is a Calabi–Yau threefold containing a plane. See Appendix of this

Received 24 March 2017; revised 31 October 2017; published online 29 December 2017.

2010 Mathematics Subject Classification 14F05 (primary), 14J50, 14K22, 14M25 (secondary).

This work was supported by the program for Leading Graduate Schools, MEXT, Japan, and also by grant-in-aid for JSPS Research Fellow 17J00664.
paper. Hence we need to modify the inequality in general. In this direction, it was shown that some modified versions of the BG inequality conjecture hold for every Fano threefolds (cf. [9, 22]). On the other hand, it seems still important to study for which variety the original BG inequality conjecture holds. In this paper, we give three new examples which satisfy the original BG inequality conjecture.

**Theorem 1.1.** Let $X$ be $\mathbb{P}^1 \times S$, $\mathbb{P}^2 \times C$, or $\mathbb{P}^1 \times \mathbb{P}^1 \times C$, where $S$ is an Abelian surface and $C$ is an elliptic curve. Then the original BG inequality conjecture holds for $X$.

See Theorem 2.6 for the precise statement. In particular, the above theorem implies.

**Theorem 1.2.** Let $X$ be as above. Then there exist Bridgeland stability conditions on $X$.

1.2. **Strategy of the proof of the main theorem**

The idea of proof is borrowed from that of [7, 9]. Roughly speaking, they considered the Euler characteristic $\chi(\mathcal{O}, m^*E)$ of the pullback of a given tilt-semistable object $E$ by the multiplication map (respectively, toric Frobenius morphism) $m: X \to X$ on an Abelian threefold (respectively, a toric threefold) $X$. Then by the Riemann–Roch theorem, we know that $\chi(\mathcal{O}, m^*E)$ is a polynomial of degree 6 (respectively, 3) with respect to $m$ and its leading coefficient is $\text{ch}_3(E)$.

On the other hand, they showed that $\text{ext}^i(\mathcal{O}, m^*E) = O(m^i)$ (respectively, $O(m^{i-1})$) for even $i$. In this way, they got an inequality for the third part of the Chern character, that is, $\text{ch}_3(E) \leq 0$.

To approximate $\text{ext}^i(\mathcal{O}, m^*E)$, it was important that $m$ is étale in the case when $X$ is an Abelian threefold, while the toric Frobenius splitting (Theorem 3.1) was essential when $X$ is a toric threefold.

In this paper, we consider the product of the multiplication map on an Abelian variety and the toric Frobenius morphisms on the projective spaces. Then we approximate $\text{ext}^i(\mathcal{O}, m^*E)$ combining the methods in [7, 9]. Note that our approach cannot apply to the product threefolds of an elliptic curve and other toric surfaces for a technical reason (see Remark 3.7).

1.3. **Plan of the paper**

The paper is organized as follows. In Section 2, we recall the notion of stability conditions. After that, we recall the work of [8] and state our main theorem. In Section 3, we collect key results which we will use in the proof of our main theorem. In Section 4, we prove our main theorem. In Appendix, we will show that the original BG inequality conjecture for a Calabi–Yau threefold containing a plane does not hold.

**Notation and Convention.** In this paper we always work over $\mathbb{C}$. We use the following notations:

- $\text{ch}^B = (\text{ch}_0^B, \ldots, \text{ch}_n^B) := e^{-B}.\text{ch}$, where ch denotes the Chern character and $B \in H^2(X; \mathbb{R})$;
- $\text{ch}^\beta := \text{ch}^{\beta H}$, where $H$ is an ample divisor and $\beta \in \mathbb{R}$;
- $H \cdot \text{ch}^B := (H^0.\text{ch}^B_0, \ldots, H\cdot\text{ch}^B_{n-1}, \text{ch}^B_n)$;
- $K(\mathcal{A})$ : the Grothendieck group of an Abelian category $\mathcal{A}$;
- $\text{hom}(E, F) := \dim \text{Hom}(E, F)$;
- $\text{ext}^i(E, F) := \dim \text{Ext}^i(E, F)$;
- $D^b(X) := D^b(\text{Coh}(X))$ : the bounded derived category of coherent sheaves on a smooth projective variety $X$. 
2. Preliminaries

2.1. Bridgeland stability condition

In this subsection, we recall the definition of stability conditions due to Bridgeland [10]. First we define the notion of stability functions.

**Definition 2.1.** (i) Let $\mathcal{A}$ be an Abelian category. A *stability function* on $\mathcal{A}$ is a group homomorphism $Z : K(\mathcal{A}) \to \mathbb{C}$ such that

$$Z(\mathcal{A} \setminus \{0\}) \subset \mathcal{H}.$$  

Here $\mathcal{H} := \mathbb{H} \cup \mathbb{R}_{<0}$ is the union of upper half plane and negative real line.

(ii) Let $Z$ be a stability function on an Abelian category $\mathcal{A}$. For $E \in \mathcal{A}$, define $\phi_Z(E) := \frac{1}{2\pi} \arg Z(E) \in (0, 1]$. Then $E$ is $Z$-semistable (respectively, stable) if for every proper non-zero subobject $F \subset E$,

$$\phi_Z(F) \leq (\text{respectively,} <) \phi_Z(E).$$

(iii) $Z$ satisfies the *Harder–Narashimhan property* (HN property) if the following property holds: For every non-zero object $E \in \mathcal{A}$, there exists a finite filtration $0 = E_0 \subset E_1 \subset \cdots \subset E_{n-1} \subset E_n = E$ such that $F_i := E_i/E_{i-1}$ is $Z$-semistable for every $i = 1, \ldots, n$ and

$$\phi_Z(E_1) > \cdots > \phi_Z(E_n).$$

Now we can define the notion of stability conditions on a triangulated category.

**Definition 2.2.** Let $\mathcal{D}$ be a triangulated category. A *stability condition* $\sigma = (Z, \mathcal{A})$ on $\mathcal{D}$ is a pair consisting of the heart of a bounded t-structure $\mathcal{A}$ on $\mathcal{D}$ and a stability function $Z$ on $\mathcal{A}$ (called central charge) satisfying the HN property.

2.2. Stability conditions on smooth projective varieties

In this subsection, we recall the works about the stability conditions on smooth projective varieties. Let $X$ be a smooth projective variety, $\omega$ an ample $\mathbb{R}$-divisor on $X$, and $B$ any $\mathbb{R}$-divisor on $X$. Conjecturally, a group homomorphism

$$Z_{\omega, B} := -\int_X e^{-i\omega \cdot \text{ch}^B} : K(X) \to \mathbb{C}$$

becomes the central charge of some stability condition on $D^b(X)$ (cf. [8, Conjecture 2.1.2]).

When $\dim X = 1$, the pair $(Z_{\omega, B}, \text{Coh}(X))$ is a stability condition on $X$ and this coincides with Mumford’s slope stability.

However in $\dim X \geq 2$, we need a more complicated construction of the wanted heart as follows. Let us define the slope function on $\text{Coh}(X)$ as

$$\mu_{\omega, B} : \text{Coh}(X) \to (-\infty, +\infty], \ E \mapsto \frac{\omega^{n-1} \cdot \text{ch}^B(E)}{\text{ch}^0(E)},$$

where $n = \dim X$. Define subcategories of $\text{Coh}(X)$ as follows:

$$\mathcal{T}_{\omega, B} := \{ T \in \text{Coh}(X) : T \text{ is } \mu_{\omega, B}\text{-semistable with } \mu_{\omega, B}(T) > 0 \},$$

$$\mathcal{F}_{\omega, B} := \{ F \in \text{Coh}(X) : F \text{ is } \mu_{\omega, B}\text{-semistable with } \mu_{\omega, B}(F) \leq 0 \}.$$
Here, we denote by \( \langle S \rangle \) the extension closure of a set of objects \( S \subset \text{Coh}(X) \). Due to the HN property of \( \mu \)-stability, the pair \( (\mathcal{T}_{\omega,B}, \mathcal{F}_{\omega,B}) \) is a torsion pair on \( \text{Coh}(X) \) in the sense of \cite{15}. Then we can construct a new heart, called the tilting heart of \( \text{Coh}(X) \) with respect to the torsion pair:

\[
\text{Coh}^{\omega,B}(X) := \langle \mathcal{F}_{\omega,B}[1], \mathcal{T}_{\omega,B} \rangle = \left\{ E \in \text{Coh}(X) : \begin{array}{l}
\mathcal{H}^{-1}(E) \in \mathcal{F}_{\omega,B}, \\
\mathcal{H}^0(E) \in \mathcal{T}_{\omega,B}, \\
\mathcal{H}^i(E) = 0 \text{ for } i \neq -1, 0
\end{array} \right\}.
\]

In \( \dim X = 2 \), \( \text{Coh}^{\omega,B}(X) \) is the required heart:

**Theorem 2.3** \cite{1, 11}. Let \( \dim X = 2 \). Then the pair \( (Z_{\omega,B}, \text{Coh}^{\omega,B}(X)) \) is a stability condition on \( X \).

In \( \dim X = 3 \), Bayer, Macrì and Toda provided the conjectural approach to construct the required heart \cite{8}. The idea is to tilt the heart \( \text{Coh}^{\omega,B}(X) \) once again using a new slope function. Let us recall the work \cite{8} of Bayer, Macrì and Toda. In the followings, assume that \( \dim X = 3 \). Let \( H \) be an ample divisor on \( X \) and let \( \omega := \alpha \sqrt{3}H, B := \beta H \) \((\alpha, \beta \in \mathbb{R}, \alpha > 0)\). Define a slope function on \( \text{Coh}^{\beta}(X) := \text{Coh}^{\omega,B}(X) \) as follows:

\[
\nu_{\alpha, \beta} = \nu_{\omega,B} : \text{Coh}^{\beta}(X) \to (\infty, +\infty), \quad E \mapsto \frac{H \cdot \text{ch}^2_{1}(E) - \frac{1}{2} \alpha^2 H^3 \cdot \text{ch}^3_{1}(E)}{H^2 \cdot \text{ch}^3_{1}(E)}.
\]

Then we can define the notion of \( \nu_{\alpha, \beta} \)-stability (or tilt-stability) as similar to the \( \mu_{\omega,B} \)-stability on \( \text{Coh}(X) \). Using the tilt-stability, the torsion pair \( (\mathcal{T}'_{\alpha,\beta}, \mathcal{F}'_{\alpha,\beta}) \) on \( \text{Coh}^{\beta}(X) \) is also defined similarly to \( (\mathcal{T}_{\omega,B}, \mathcal{F}_{\omega,B}) \) on \( \text{Coh}(X) \). Bayer, Macrì and Toda conjectured the following BG-type inequality for \( \nu_{\alpha, \beta} \)-semistable objects.

**Conjecture 2.4** \cite{8}. Let \( E \in \text{Coh}^{\beta}(X) \) be a \( \nu_{\alpha, \beta} \)-semistable object with \( \nu_{\alpha, \beta}(E) = 0 \). Then we have

\[
\text{ch}^3_{1}(E) \leq \frac{1}{6} \alpha^2 H^2 \cdot \text{ch}^3_{1}(E).
\]

Moreover, they showed that the above inequality implies the existence of a stability condition with the central charge \( Z_{\alpha, \beta} := Z_{\omega,B} \). Let \( \mathcal{A}_{\alpha, \beta} \) be a tilting heart of \( \text{Coh}^{\beta}(X) \) with respect to \( \nu_{\alpha, \beta} \)-stability, that is,

\[
\mathcal{A}_{\alpha, \beta} := \langle \mathcal{F}'_{\alpha,\beta}[1], \mathcal{T}'_{\alpha,\beta} \rangle.
\]

**Theorem 2.5** \cite{8}. Assume that Conjecture 2.4 holds. Then the pair \( (Z_{\alpha, \beta}, \mathcal{A}_{\alpha, \beta}) \) is a stability condition on \( X \).

Hence the construction problem of stability conditions on \( X \) is reduced to Conjecture 2.4. The main theorem of this paper is the following.

**Theorem 2.6.** Let \( X \) be \( \mathbb{P}^1 \times S, \mathbb{P}^2 \times C \) or \( \mathbb{P}^1 \times \mathbb{P}^1 \times C \), where \( S \) is an Abelian surface and \( C \) is an elliptic curve. Then for every ample divisor \( H \) on \( X \), \( \alpha > 0 \) and \( \beta \in \mathbb{R} \), Conjecture 2.4 holds.

**Remark 2.7.** In \cite{21, 24}, counter-examples for Conjecture 2.4 were obtained when \( X \) is the blow-up of a smooth projective threefold at a point. Furthermore, there exists a counter-example even when \( X \) is a Calabi–Yau threefold containing a plane. For the latter, see the Appendix of this paper.
2.3. Reduction theorem

In this subsection, we recall the further reduction of Conjecture 2.4 due to [7]. First we recall the notion of $\bar{\beta}$-stability.

**Definition 2.8.** Let $E \in \text{Coh}^3(X)$ be a $\nu_{\alpha,\beta}$-semistable object.

(i) We define

$$\bar{\beta}(E) := \begin{cases} \frac{H^2 \cdot \text{ch}_1(E) - \sqrt{\Delta_H(E)}}{H^3 \cdot \text{ch}_0(E)} & (\text{ch}_0(E) \neq 0) \\ \frac{H \cdot \text{ch}_2(E)}{H^2 \cdot \text{ch}_1(E)} & (\text{ch}_0(E) = 0), \end{cases}$$

where

$$\Delta_H(E) := (H^2 \cdot \text{ch}_1(E))^2 - 2(H^3 \cdot \text{ch}_0(E))(H \cdot \text{ch}_2(E)).$$

(ii) $E$ is $\bar{\beta}$-semistable (respectively, stable) if there exists an open neighborhood $V$ of $(0, \bar{\beta}(E))$ in $(\alpha, \beta)$-plane such that for every $(\alpha, \beta) \in V$ with $\alpha > 0$, $E$ is $\nu_{\alpha,\beta}$-semistable (respectively, stable).

**Remark 2.9.** In [8], it was shown that $\Delta_H(E)$ is non-negative for every $\nu_{\alpha,\beta}$-semistable object $E$.

Then Conjecture 2.4 is reduced as follows:

**Theorem 2.10 [7, Theorem 5.4].** Assume that for every $\bar{\beta}$-stable object $E$ with $\text{ch}_0(E) \geq 0$ and $\bar{\beta}(E) \in [0,1)$, we have

$$\text{ch}_3(\bar{\beta}(E)) \leq 0.$$  

Then Conjecture 2.4 holds for every $\alpha, \beta$.

3. Preparation for the main theorem

In this section, we collect key results which we will use in the proof of our main theorem. The first one is about the toric Frobenius push forward of line bundles:

**Theorem 3.1 [25].** Let $Y$ be a smooth projective toric variety, let $\{D_\rho\}_\rho$ be the torus invariant divisors. For $m \in \mathbb{Z}_{>0}$, denote the toric Frobenius morphism by $m_\#: Y \to Y$. Then for every divisor $D$ on $Y$, we have

$$m_\#(D) = \bigoplus_j L_j^{\ast \oplus \eta_j},$$

where

$$L_j := \mathcal{O}\left(\frac{1}{m}\left(-D + \sum_\rho a_\rho D_\rho\right)\right).$$

Here, integers $0 \leq a_\rho \leq m - 1$ move so that $L_j$ becomes an integral divisor and $\eta_j$ counts the multiplicity of $\{a_\rho\}$ which defines the same $L_j$. 

Remark 3.2. Let $Y = \mathbb{P}^n$ be a projective space. Let $a_\rho$ be as in Theorem 3.1. Then we have

$$-K_Y - \sum \frac{a_\rho D_\rho}{m} = \sum D_\rho - \sum \frac{a_\rho D_\rho}{m} \geq \frac{1}{m} \sum D_\rho$$

and hence $-K_Y - \sum a_\rho D_\rho/m$ is ample on $Y$. This fact will be used in Section 4.

The next one is about the preservation of tilt-stability under the pullback by finite étale morphisms.

Proposition 3.3 [7, Proposition 6.1]. Let $f: Y \to X$ be a finite étale surjective morphism between smooth projective threefolds. Let $\omega$ be an ample $\mathbb{R}$-divisor on $X$, $B$ an $\mathbb{R}$-divisor on $X$. Let $E \in D^b(X)$. Then

(i) $\nu_{f^*\omega, f^*B}(f^*E) = \nu_{\omega, B}(E);$  
(ii) $f^*E \in \text{Coh}_{f^*\omega, f^*B}(Y)$ if and only if $E \in \text{Coh}_{\omega, B}(X);$  
(iii) $f^*E$ is $\nu_{f^*\omega, f^*B}$-semistable (respectively, stable) if and only if $E$ is $\nu_{\omega, B}$-semistable (respectively, stable).

Example 3.4. Let $A$ be an Abelian variety of dim $\leq 2$, let $X = Y \times A$ be a product threefold. Let $m: A \to A$ be a multiplication map ($m \in \mathbb{Z}_{>0}$). Then $\text{id}_Y \times m: X \to X$ is a finite étale surjective morphism. Hence we can apply the above proposition to $\text{id}_Y \times m$.

The third one is about the tilt-stability of line bundles.

Lemma 3.5 [7, Corollary 3.11]. Let $X$ be a smooth projective threefold, $H$ an ample divisor on $X$. Assume that for every effective divisor $D$ on $X$, we have $H.D^2 \geq 0$. Then for every line bundle $L$ on $X$, $\alpha > 0$, and $\beta \in \mathbb{R}$, $L$ or $L[1]$ is $\nu_{\alpha, \beta}$-stable.

Example 3.6. (i) Let $C$ be an elliptic curve, $S$ an Abelian surface. Let $X$ be $\mathbb{P}^1 \times S, \mathbb{P}^2 \times C$ or $\langle \mathbb{P}^1 \times \mathbb{P}^1 \rangle \times C$. Then the assumption of the above lemma holds for every ample divisor on $X$, since there are no negative divisors on projective spaces or Abelian varieties.

(ii) Let $Y$ be any smooth projective toric surface other than $\mathbb{P}^2, \mathbb{P}^1 \times \mathbb{P}^1$. Let $X = Y \times C$. Since there exists a negative curve on $Y$, the assumption in the above lemma does not hold for any ample divisor on $X$.

Remark 3.7. The tilt-stability of line bundles is crucial in our proof of the main theorem. Hence our approach can not apply to threefolds in (ii) of Example 3.6.

The last one is about the approximation of dimensions of certain Ext-groups due to [7].

Proposition 3.8 [7]. Let $C$ be an elliptic curve, $S$ an Abelian surface. Let $X$ be $\mathbb{P}^1 \times S, \mathbb{P}^2 \times C$ or $\mathbb{P}^1 \times \mathbb{P}^1 \times C$. Let $f^{(m^2, m)} := m^2 \times m: X \to X$ be the product of the toric Frobenius morphism and the multiplication map. Let $E \in D^b(X)$ be a two-term complex concentrated in degrees $-1$ and $0$.

(i) If there exists an ample divisor $H'$ on $X$ such that

$$\text{hom} \left( \mathcal{O}(H'), f^{(m^2, m)*}E \right) = 0,$$
then
\[ \text{hom} \left( \mathcal{O}, f^{(m, m)^*} E \right) = O(m^4). \]

(ii) If there exists an ample divisor \( H' \) on \( X \) such that
\[ \text{ext}^2 \left( \mathcal{O}(-H'), f^{(m^2, m)^*} E \right) = 0, \]
then
\[ \text{ext}^2 \left( \mathcal{O}, f^{(m^2, m)^*} E \right) = O(m^4). \]

Proof. Summarizing the arguments of [7, Section 7], we get the result. \( \square \)

4. Proof of the main theorem

In this section, we prove our main theorem, Theorem 2.6. Let \( C \) be an elliptic curve and \( S \) an Abelian surface. Let \( X = Y \times Z \), where \( (Y, Z) = (\mathbb{P}^1, S), (\mathbb{P}^2, C), (\mathbb{P}^1 \times \mathbb{P}^1, C) \). Let \( H \) be an ample divisor on \( X \). Then \( H \) can be written as \( H = h + f \), where \( h, f \) are the pullback of some ample divisors on \( Y, Z \), respectively. For integers \( a, b \in \mathbb{Z} \geq 0 \), let \( f(a, b) := a \times b \) be the product of the toric Frobenius morphism \( a \) on \( Y \) and the multiplication map \( b \) on \( Z \). Furthermore, let us denote by \( D_\rho \in \text{NS}(X) \) the pullbacks of the torus invariant divisors on \( Y \).

Remark 4.1. Let \( m \in \mathbb{Z} > 0 \). Note that \( f^{(m^2, m)} \) acts on the even cohomology as follows:
\[ \bigoplus_{i=0}^{3} H^{2i}(X) \ni (x, y, z, w) \mapsto (x, m^2 y, m^4 z, m^6 w) \in \bigoplus_{i=0}^{3} H^{2i}(X). \]

We will use this property in the followings.

Let \( E \) be a \( \bar{\beta} \)-stable object with \( \text{ch}_0(E) \geq 0 \) and \( \bar{\beta} := \bar{\beta}(E) \in [0, 1) \). To prove Theorem 2.6, it is enough to show that \( \text{ch}^3_\bar{\beta}(E) \leq 0 \) by Theorem 2.10. We prove it in the following three subsections. We start with two easy lemmas which we will frequently use in the followings.

Lemma 4.2. For every \( E, F \in D^b(X) \), we have
\[ \text{Hom} \left( \omega_X^* \otimes f_{(a, b)}^*(E \otimes \omega_X), F \right) \cong \text{Hom} \left( E, f_{(a, b)}^* F \right). \]

Proof. Use Serre duality and the adjointness between \( f_{(a, b)}^* \) and \( f_{(a, b)}^* \). Note that we do not need to take derived functors since \( f_{(a, b)}^* \) is finite and flat. \( \square \)

Lemma 4.3. Let \( E \in \text{Coh} \bar{\beta}(X) \) be a \( \bar{\beta} \)-stable object with \( \text{ch}_0(E) \geq 0 \) and \( \bar{\beta} = \bar{\beta}(E) \in [0, 1), L \) a line bundle on \( X \).

(i) If \( \text{ch}_1^\beta(L) \) is ample, then
\[ \text{hom} (L, E) = 0. \]

(ii) If \( \text{ch}_1^\beta(L) \) is anti-ample, then
\[ \text{hom} (E, L[1]) = 0. \]
Proof. We only prove the first statement. The second one also follows from the similar computation. Since $\text{ch}_1^\beta(L)$ is ample, we have

$$H^2 \cdot \text{ch}_1^\beta(L) > 0,$$

By the first inequality, we have $L \in \text{Coh}^\beta(X)$. Moreover, by Proposition 3.5, $L$ is tilt-stable near $(0, \beta)$. On the other hand, the second inequality implies

$$\lim_{\alpha \to +0} \nu_{\alpha, \beta}(L) = \frac{H \cdot \text{ch}_1^\beta(L)^2}{H^2 \cdot \text{ch}_1^\beta(L)} > 0 = \lim_{\alpha \to 0} \nu_{\alpha, \beta}(E).$$

Hence by the tilt-stability of $L$ and $E$, we have $\text{hom}(L, E) = 0$. □

4.1. Integral case

Assume that $\beta = 0$. Let us consider $\chi(O, f^{(m^2, m)_*}E)$. By the Riemann–Roch theorem and Remark 4.1, we have

$$\chi(O, f^{(m^2, m)_*}E) = m^6 \text{ch}_3(E) + O(m^4).$$

On the other hand,

$$\chi(O, f^{(m^2, m)_*}E) \leq \text{hom}(O, f^{(m^2, m)_*}E) + \text{ext}^2(O, f^{(m^2, m)_*}E)$$

(4.1)
since $E$ is a two-term complex concentrated in degrees $-1$ and $0$.

By Proposition 3.8, the following two lemmas show that the RHS of the inequality (4.1) is of order $m^4$. Hence we must have $\text{ch}_3(E) \leq 0$ as required.

**Lemma 4.4.** We have

$$\text{hom}(O(-K_X + f), f^{(m^2, m)_*}E) = 0.$$

**Proof.** By Theorem 3.1 and Lemma 4.2, we have

$$\text{hom}(O(-K_X + f), f^{(m^2, m)_*}E) = \text{hom}(O(-K_X) \otimes f^*(m^2, 1)O(-K_X + f + K_X), f^{(1, m)_*}E)$$

$$= \text{hom}(O(-K_X + f) \otimes (\oplus L_j^{*\oplus \eta_j}), f^{(1, m)_*}E),$$

where

$$L_j = O \left( \frac{1}{m^2} \left( \sum_{\rho} a_{\rho} D_{\rho} \right) \right), \quad 0 \leq a_{\rho} \leq m^2 - 1.$$ 

As remarked in Remark 3.2, $O(-K_X) \otimes L_j^*$ is the pullback of an ample line bundle on $Y$. Hence $O(-K_X + f) \otimes L_j^*$ is ample on $X$. Note that $f^{(1, m)_*}E$ is $\beta$-stable with $\beta(f^{(1, m)_*}E) = 0$ (with respect to the polarization $f^{(1, m)_*}H$) by Proposition 3.3. Hence Lemma 4.3 implies that

$$\text{hom}(O(-K_X + f) \otimes L_j^*, f^{(1, m)_*}E) = 0.$$
Summing up, we conclude that 
\[
\text{hom}\left(\mathcal{O}(-K_X + f), f^{(m^2,m)*} E\right) = 0. 
\]

**Lemma 4.5.** We have
\[
\text{ext}^2 \left(\mathcal{O}(-h - f), f^{(m^2,m)*} E\right) = 0. 
\]

**Proof.** By Theorem 3.1, Serre duality, and the usual adjoint, we have
\[
\text{ext}^2 \left(\mathcal{O}(-h - f), f^{(m^2,m)*} E\right) = \text{hom}\left(f^{(m^2,m)*} E, \mathcal{O}(-h - f + K_X)\right) 
\]
\[
= \text{hom}\left(f^{(1,m)*} E, \mathcal{O}(-f) \otimes f^{(m^2,1)*} \mathcal{O}(-h + K_X)\right) 
\]
\[
= \text{hom}\left(f^{(1,m)*} E, \mathcal{O}(-f) \otimes (\oplus L_j^{(\eta_j)})\right), 
\]
where
\[
L_j = \mathcal{O}\left(\frac{1}{m^2} (h - K_X + \sum a_p D_p)\right), \quad 0 \leq a_p \leq m^2 - 1. 
\]
For all \(j\), \(\text{ch}_1(\mathcal{O}(-f) \otimes L_j)\) is anti-ample. Hence by Lemma 4.3,
\[
\text{ext}^2 \left(\mathcal{O}(-h - f), f^{(m^2,m)*} E\right) = 0. 
\]

### 4.2. Rational case

In this subsection, we assume that \(\beta = \frac{2}{q} \in \mathbb{Q}, \ q > 0, \) and \(q\) are coprime. We consider \(\chi(\mathcal{O}, f^{(m^2,m)*}(f^{(q^2,q)*} E \otimes \mathcal{O}(-pqH)))\). By Remark 4.1, we have
\[
\text{ch}_3\left(f^{(q^2,q)*} E \otimes \mathcal{O}(-pqH)\right) = q^6 \text{ch}_3^{\frac{2}{q}}(E) 
\]
and hence
\[
m^6 q^6 \text{ch}_3^{\frac{2}{q}}(E) + O(m^4) = \chi\left(\mathcal{O}, f^{(m^2,m)*}\left(f^{(q^2,q)*} E \otimes \mathcal{O}(-pqH)\right)\right) 
\]
\[
\leq \text{hom}\left(\mathcal{O}, f^{(m^2,m)*}\left(f^{(q^2,q)*} E \otimes \mathcal{O}(-pqH)\right)\right) 
\]
\[
+ \text{ext}^2 \left(\mathcal{O}, f^{(m^2,m)*}\left(f^{(q^2,q)*} E \otimes \mathcal{O}(-pqH)\right)\right). 
\]
As in the previous subsection, we will check the assumption in Proposition 3.8.

**Lemma 4.6.** We have
\[
\text{hom}\left(\mathcal{O}(-K_X + f), f^{(m^2,m)*}\left(f^{(q^2,q)*} E \otimes \mathcal{O}(-pqH)\right)\right) = 0. 
\]

**Proof.** Using Theorem 3.1 and Lemma 4.2, we have
\[
\text{hom}\left(\mathcal{O}(-K_X + f), f^{(m^2,m)*}\left(f^{(q^2,q)*} E \otimes \mathcal{O}(-pqH)\right)\right) 
\]
\[
= \text{hom}\left(\mathcal{O}(-K_X) \otimes \mathcal{O}(f) \otimes f^{(m^2,1)*} \mathcal{O}(-K_X + K_X), f^{(q^2,1)*} f^{(1,mq)*} E \otimes f^{(1,m)*} \mathcal{O}(-pqH)\right) 
\]
\[
= \text{hom}\left(\mathcal{O}(-K_X + f) \otimes (\oplus L_j^{(\eta_j)}), f^{(q^2,1)*} f^{(1,mq)*} E \otimes \mathcal{O}(-pqh - pqm^2 f)\right) 
\]
\[
\sum_j \eta_j \text{hom} \left( \mathcal{O}(pqmf^2 + f) \otimes \mathcal{O}(-fX + pqh) \otimes L_j^*, f(q^7,1)^* f^{(1,mq)^*} E \right)
\]

where

\[
L_j = \mathcal{O} \left( \frac{1}{m^2} \sum a_\rho D_\rho \right), \quad 0 \leq a_\rho \leq m^2 - 1 \quad \text{and}
\]

\[
R_{kj} = \mathcal{O} \left( \frac{1}{q^2} \left( -pqh + \frac{1}{m^2} \sum a_\rho D_\rho + \sum b_\rho D_\rho \right) \right), \quad 0 \leq b_\rho \leq q^2 - 1.
\]

By Lemma 4.3, it is enough to show that

\[
\text{ch}_1^{(h + m^2 q^2)} \mathcal{O}(pqmf^2 + f) \otimes \mathcal{O}(-fX) \otimes R_{kj}
\]
is ample. We can compute it as

\[
\text{ch}_1^{(h + m^2 q^2)} \mathcal{O}(pqmf^2 + f) \otimes \mathcal{O}(-fX) \otimes R_{kj}
= pqmf^2 + f - K_X + \frac{pqh}{q^2} - \frac{\sum \rho a_\rho D_\rho}{m^2 q^2}
- \frac{\sum \rho b_\rho D_\rho}{q^2} - \frac{p}{q} (h + m^2 q^2 f)
= f - K_X - \frac{\sum \rho a_\rho D_\rho}{m^2 q^2} - \frac{\sum \rho b_\rho D_\rho}{q^2}.
\]

Since \(-K_X = \sum \rho D_\rho, 0 \leq a_\rho \leq m^2 - 1, \text{ and } 0 \leq b_\rho \leq q^2 - 1, \text{ we have}

\[
f - K_X - \frac{\sum \rho a_\rho D_\rho}{m^2 q^2} - \frac{\sum \rho b_\rho D_\rho}{q^2}
\geq f + \frac{1}{m^2 q^2} (m^2 q^2 - (m^2 - 1) - m^2 (q^2 - 1)) \sum \rho D_\rho
\]

and it is ample on \(X\). We conclude that

\[
\text{hom} \left( \mathcal{O}(-K_X + f), f^{(m^2,m)*} \left( f(q^7,1)^* E \otimes \mathcal{O}(pqH) \right) \right) = 0.
\]

**Lemma 4.7.** We have

\[
\text{ext}^2 \left( \mathcal{O}(-h - f), f^{(m^2,m)*} \left( f(q^7,1)^* E \otimes \mathcal{O}(pqH) \right) \right) = 0.
\]
Proof. By Serre duality, adjunction and Theorem 3.1, we have
\[
\operatorname{ext}^2 \left( \mathcal{O} (-h - f), f^{(m^2, m)} \ast \left( f^{(q^2, q)} \ast E \otimes \mathcal{O} (-pqH) \right) \right)
\]
\[
= \operatorname{hom} \left( f^{(m^2, m)} \ast \left( f^{(q^2, q)} \ast E \otimes \mathcal{O} (-pqH) \right), \mathcal{O} (-f - h + K_X) [1] \right)
\]
\[
= \operatorname{hom} \left( f^{(1, m)} \ast \left( f^{(q^2, q)} \ast E \otimes \mathcal{O} (-pqH) \right), \mathcal{O} (-f) \otimes f^{(m^2, 1)}_* \mathcal{O} (-h + K_X) [1] \right)
\]
\[
= \sum_j \eta_j \operatorname{hom} \left( f^{(q^2, 1)} \ast f^{(1, mq)} \ast E, \mathcal{O} (pqh + pqm^2 f) \otimes \mathcal{O} (-f) \otimes (\oplus L_j^{\ast \oplus \eta_j}) [1] \right)
\]
\[
= \sum_j \eta_j \operatorname{hom} \left( f^{(1, mq)} \ast E, \mathcal{O} (pqm^2 f - f) \otimes \mathcal{O} (pqh) \otimes L_j [1] \right)
\]
\[
= \sum_j \eta_j \operatorname{hom} \left( f^{(mq)} \ast E, \mathcal{O} (pqm^2 f - f) \otimes f^{(q^2, 1)} \mathcal{O} (pqh) \otimes L_j [1] \right)
\]
\[
= \sum_j \eta_j \operatorname{hom} \left( f^{(mq)} \ast E, \mathcal{O} (pqm^2 f - f) \otimes (\oplus R^{\ast \oplus c}_{kj}) [1] \right).
\]

Here,
\[
L_j = \mathcal{O} \left( \frac{1}{m^2} \left( h - K_X + \sum a_\rho D_\rho \right) \right), \ 0 \leq a_\rho \leq m^2 - 1 \ \text{and}
\]
\[
R_{kj} = \mathcal{O} \left( \frac{1}{q^2} \left( -pqh + \frac{h - K_X + \sum a_\rho D_\rho}{m^2} + \sum b_\rho D_\rho \right) \right), \ 0 \leq b_\rho \leq q^2 - 1.
\]

As before, it is enough to show that
\[
\operatorname{ch}_1^{\frac{1}{2}(h + m^2 q^2 f)} \left( \mathcal{O} (pqm^2 f - f) \otimes R_{kj}^* \right)
\]
is anti-ample. Straightforward computation yields that
\[
\operatorname{ch}_1^{\frac{1}{2}(h + m^2 q^2 f)} \left( \mathcal{O} (pqm^2 f - f) \otimes R_{kj}^* \right) = pqm^2 f - f + \frac{p}{q} h - \frac{h - K_X + \sum a_\rho D_\rho}{m^2 q^2} - \frac{\sum b_\rho D_\rho}{q^2}
\]
\[
= -f - \frac{h - K_X + \sum a_\rho D_\rho}{m^2 q^2} - \frac{\sum b_\rho D_\rho}{q^2}.
\]

This is anti-ample on $X$. Hence we get the required result. \hfill \Box

4.3. Irrational case

Assume that $\bar{\beta}$ is irrational. Define
\[
V_\epsilon := \{ (\alpha, \beta) : 0 < \alpha < \epsilon, \bar{\beta} - \epsilon < \beta < \bar{\beta} + \epsilon \}.
\]
Take $\epsilon > 0$ small enough so that for every $(\alpha, \beta) \in V_\epsilon$, $E$ is $\nu_{\alpha, \beta}$-stable. By the Dirichlet approximation theorem, we can take a sequence $\{ \beta_n = \frac{p_n}{q_n} \}$ of rational numbers such that

$$|\bar{\beta} - \beta_n| < \frac{1}{q_n^2} < \epsilon$$

and $q_n \to +\infty$ as $n \to +\infty$. We compute $\chi(O, f^{(q_n^2, q_n)\ast} E \otimes O(-p_n q_n H))$. As before,

$$q_n^6 \chi_3^\beta(E) + O(q_n^4)$$

$$\leq q_n^6 \chi_3^\beta_n(E) + O(q_n^4)$$

$$\leq \chi \left( O, f^{(q_n^2, q_n)\ast} E \otimes O(-p_n q_n H) \right)$$

$$\leq \text{hom} \left( O, f^{(q_n^2, q_n)\ast} E \otimes O(-p_n q_n H) \right) + \text{ext}^2 \left( O, f^{(q_n^2, q_n)\ast} E \otimes O(-p_n q_n H) \right).$$

We will show that the last line of the above inequalities is of order $q_n^4$.

**Lemma 4.8.** Let $u, v \in \mathbb{Z}_{>0}$ such that $(u - 2)h + K_X$ is effective and $v > 2$. Then

$$\text{hom} \left( O(uh + vf), f^{(q_n^2, q_n)\ast} E \otimes O(-p_n q_n H) \right) = 0.$$

**Proof.** As in the rational case, we have

$$\text{hom} \left( O(uh + vf), f^{(q_n^2, q_n)\ast} E \otimes O(-p_n q_n H) \right)$$

$$= \text{hom} \left( O((p_n q_n + v) f) \otimes O(-K_X) \otimes \left( \bigoplus L_j^{\ast \oplus \eta_j} \right), f^{(1, q_n)\ast} E \right),$$

where

$$L_j = O \left( \frac{1}{q_n^2} (- (p_n q_n + u) h - K_X + \sum a_\rho D_\rho) \right), 0 \leq a_\rho \leq q_n^2 - 1.$$

Let $M_j := O((p_n q_n + v) f) \otimes O(-K_X) \otimes L_j^{\ast}, H^{(n)} := f^{(1, q_n)\ast} H$. For a while, assume that $\chi_1^{\beta_n H^{(n)}}(M_j) - 2 \frac{1}{q_n^2} H^{(n)}$ is ample. Then we can compute as

$$H^{(n)2} \cdot \chi_1^{\beta H^{(n)}}(M_j) = H^{(n)2} \cdot \left( \chi_1^{\beta_n H^{(n)}}(M_j) + (\beta_n - \bar{\beta}) H^{(n)} \right)$$

$$> H^{(n)2} \cdot \left( \chi_1^{\beta_n H^{(n)}}(M_j) - \frac{1}{q_n^2} H^{(n)} \right)$$

$$> 0$$

and

$$H^{(n)} \cdot \chi_1^{\beta H^{(n)}}(M_j)^2 = H^{(n)} \cdot \left( \chi_1^{\beta_n H^{(n)}}(M_j) + (\beta_n - \bar{\beta}) H^{(n)} \right)^2$$

$$> H^{(n)} \cdot \chi_1^{\beta_n H^{(n)}}(M_j) \cdot \left( \chi_1^{\beta_n H^{(n)}}(M_j) - 2 \frac{1}{q_n^2} H^{(n)} \right)$$

$$> 0,$$

which imply

$$\text{hom} \left( M_j, f^{(1, q_n)\ast} E \right) = 0.$$
by the proof of Lemma 4.3. Hence it is enough to show that $\text{ch}_1^{\beta_n H(n)}(M_j) - 2\frac{1}{q_n^2} H(n)$ is ample. As in the rational case, we have

$$\text{ch}_1^{\beta_n H(n)}(M_j) = vf - K_X + \frac{uh + K_X - \sum a_\rho D_\rho}{q_n^2}$$

and hence

$$\text{ch}_1^{\beta_n H(n)}(M_j) - 2\frac{1}{q_n^2} H(n) = vf - K_X + \frac{uh + K_X - \sum a_\rho D_\rho}{q_n^2} - 2\frac{h + q_n^2 f}{q_n^2} = (v - 2)f - K_X - \sum a_\rho D_\rho + \frac{(u - 2)h + K_X}{q_n^2}.$$ 

As observed in Remark 3.2,

$$-K_X - \sum a_\rho D_\rho$$

is the pullback of an ample divisor on $Y$. Hence if we take $u, v$ so that $v > 2$ and $(u - 2)h + K_X$ is effective, then

$$\text{ch}_1^{\beta_n H(n)}(M_j) - 2\frac{1}{q_n^2} H(n)$$

is ample on $X$. Note that these conditions does not depend on $n$. □

**Lemma 4.9.** Let $u, v \in \mathbb{Z}_{>0}$, $u, v > 2$. Then

$$\text{ext}^2 \left( \mathcal{O}(-uh - vf), f^{(q_n^2 q_n)} \mathcal{E} \otimes \mathcal{O}(-p_n q_n H) \right) = 0.$$ 

**Proof.** As in the rational case,

$$\text{ext}^2 \left( \mathcal{O}(-uh - vf), f^{(q_n^2 q_n)} \mathcal{E} \otimes \mathcal{O}(-p_n q_n H) \right) = \text{hom} \left( f^{(1, q_n)} \mathcal{E}, \mathcal{O}(p_n q_n f - vf) \otimes \left( \oplus L_j^{(\pm q_n)} \right) [1] \right),$$

where

$$L_j := \mathcal{O} \left( \frac{1}{q_n^2} \left( -p_n q_n h + uh - K_X + \sum a_\rho D_\rho \right) \right), 0 \leq a_\rho \leq q_n^2 - 1.$$ 

Let $M_j := \mathcal{O}(p_n q_n f - vf) \otimes L_j^{(\pm q_n)} H(n) := f^{(1, q_n)} H$. Assume that

$$\text{ch}_1^{\beta_n H(n)}(M_j) + 2\frac{1}{q_n^2} H(n)$$

is anti-ample. Then

$$H(n)^2 \cdot \text{ch}_1^{\beta H(n)}(M_j) = H(n)^2 \cdot \left( \text{ch}_1^{\beta_n H(n)}(M_j) + (\beta_n - \beta) H(n) \right) < H(n)^2 \cdot \left( \text{ch}_1^{\beta_n H(n)}(M_j) + \frac{1}{q_n^2} H(n) \right) < 0$$
and

\[ H^{(n)} \cdot \text{ch}_1 H^{(n)}(M_j)^2 = H^{(n)} \cdot \left( \text{ch}_1^{\beta_n H^{(n)}}(M_j) + (\beta_n - \bar{\beta}) H^{(n)} \right)^2 \]

\[ > H^{(n)} \cdot \text{ch}_1^{\beta_n H^{(n)}}(M_j) \left( \text{ch}_1^{\beta_n H^{(n)}}(M_j) + 2 \frac{1}{q_n} H^{(n)} \right) \]

\[ > 0. \]

Then the stability of \( M_j \) and \( f^{(1, q_n)} E \) shows that hom\((f^{(1, q_n)} E, M_j[1]) = 0 \) as required. Hence it is enough to show that

\[ \text{ch}_1^{\beta_n H^{(n)}}(M_j) + 2 \frac{1}{q_n} H^{(n)} \]

is anti-ample. We can compute it as

\[ \text{ch}_1^{\beta_n H^{(n)}}(M_j) + 2 \frac{1}{q_n} H^{(n)} = -vf - uh - K_X + \sum a_\rho D_\rho + 2 \frac{1}{q_n} (h + q_n f) \]

\[ = -(v - 2)f - \frac{(u - 2)h - K_X + \sum a_\rho D_\rho}{q_n^2}. \]

For \( u, v > 2 \), this is anti-ample. \( \square \)

**Appendix. Counter-example for Conjecture 2.4**

In this Appendix, we propose a counter-example for the original BG-type inequality conjecture. More precisely, we show the following proposition using the argument of [21]:

**Proposition A.1.** Let \( X \) be a Calabi–Yau threefold containing a plane \( \mathbb{P}^2 \cong D \subset X \). Then there exists an ample divisor \( H \) on \( X \), \( \alpha > 0 \), and \( \beta \in \mathbb{R} \) such that \( O_D[1] \in A_{\alpha, \beta} \) and

\[ Z_{\alpha, \beta} (O_D[1]) \in \mathbb{R}_{>0}. \]

This proves the pair \( (Z_{\alpha, \beta}, A_{\alpha, \beta}) \) is not a stability condition on \( X \). In particular, Conjecture 2.4 does not hold by Theorem 2.5.

First we explain how to take the ample divisor \( H \). Let \( l \subset D \) be a line. Then

\[ D^2 = -3l, D^3 = 9. \]

Let \( H' \) be any ample divisor on \( X \) and put \( L := 3H' + (H' \cdot l)D \). Then \( L \) is nef and big. Hence by the Kawamata–Shokurov basepoint-free theorem, some multiple of \( L \) defines a birational morphism \( f : X \to Y \), which only contracts \( D \). In particular, there exists an ample \( \mathbb{Q} \)-divisor \( A \) on \( Y \) such that \( L = f^* A \). On the other hand, \( -D \) is \( f \)-ample since \( O_D(-3) = \omega_D \cong O_D(D) \).

Hence \( H := mL - \frac{1}{3}D \) is ample on \( X \) for all \( m \gg 0 \).

The Chern character of \( O_D \) is computed as follows:

\[ H \cdot \text{ch}(O_D) = (0, \frac{9}{4}, \frac{9}{4}, \frac{3}{2}). \]

Similarly,

\[ H \cdot \text{ch}^H(O_D) = (0, \frac{9}{4}, 0, \frac{3}{8}). \]

Before beginning the proof of Proposition A.1, we recall the following aspect of the structure theorem of walls in tilt-stability.
Lemma A.2. Let $v \in \Lambda := \text{Im}(H \cdot \text{ch} : K(X) \to \mathbb{Q}^4)$. Let $E \in \text{Coh}^\beta_0(X)$ be an object with $H \cdot \text{ch}(E) = v$. Let $0 \to A \to E \to B \to 0$ be an exact sequence in $\text{Coh}^\beta_0(X)$ with $H \cdot \text{ch}(E) = v$ which defines a wall $W$ at $(\alpha_0, \beta_0) \in W$. Then for every $(\alpha, \beta) \in W$, we have $A, E, B \in \text{Coh}^\beta_0(X)$.

Proof. See, for example, [2, Lemma 6.3]. □

Now we can prove Proposition A.1.

Proof of Proposition A.1. The argument is exactly similar to [21]. Since $Z_{\alpha,1}(\mathcal{O}_D[1]) = \frac{3}{8}(1 - \alpha^2)$, $Z_{\alpha,1}(\mathcal{O}_D[1]) > 0$ if and only if $\alpha < 1$. On the other hand, since $\nu_{\alpha,1}(\mathcal{O}_D) = 0$, $\mathcal{O}_D[1] \in \mathcal{A}_{\alpha,1}$ if $\mathcal{O}_D$ is $\nu_{\alpha,1}$-semistable. Hence it is enough to show that there exists $\alpha \in (0, 1)$ such that $\mathcal{O}_D$ is $\nu_{\alpha,1}$-stable.

Let us consider the wall $W$ which is defined by a short exact sequence

$$0 \to A \to \mathcal{O}_D \to B \to 0.$$ 

Let us denote $(r(A), c(A), d(A), e(A)) := H \cdot \text{ch}(A)$, etc. The center of this semicircular wall $W$ is

$$\frac{d(\mathcal{O}_D)}{c(\mathcal{O}_D)} = 1.$$ 

Let $R$ be the radius of the wall $W$. We will bound $R$ from above. Since $\mathcal{H}^{-1}(B) \in \mathcal{F}_\beta$ and $A \in \mathcal{T}_\beta$ for all $(\alpha, \beta) \in W$, we have

$$\frac{c(\mathcal{H}^{-1}(B))}{r(\mathcal{H}^{-1}(B))} \leq 1 - R, \quad \frac{c(A)}{r(A)} \geq 1 + R.$$ 

Using the exact sequence

$$0 \to \mathcal{H}^{-1}(B) \to A \to \mathcal{O}_D \to \mathcal{H}^0(B) \to 0,$$ 

we get

$$r(A) = r(\mathcal{H}^{-1}(B)), \quad c(A) \leq c(\mathcal{H}^{-1}(B)) + \frac{9}{4}.$$ 

Using these inequalities, we have

$$R \leq \frac{9}{8r(A)} = \frac{9}{8m^3\text{ch}_0(A)L^3 - 9\text{ch}_0(A)} < 1$$ 

for $m > 1$. Since $\mathcal{O}_D$ is Gieseker stable, it is $\nu_{\alpha,\beta}$-stable for every $\alpha \gg 0, \beta \in \mathbb{R}$. By the bound of the radius of semicircular walls, we conclude that $\mathcal{O}_D$ is $\nu_{\alpha,1}$-stable for

$$\frac{9}{(8m^3L^3 - 9)} < \alpha < 1.$$ 

Acknowledgement. I would like to thank my supervisor Professor Yukinobu Toda. He suggested this problem to me and gave various comments and advices. I would also like to thank Genki Ouchi for useful discussions.

References
1. D. Arcara and A. Bertram, ‘Bridgeland-stable moduli spaces for $K$-trivial surfaces’, J. Eur. Math. Soc. 15 (2013) 1–38; with an appendix by Max Lieblich.
2. D. Arcara, A. Bertram, I. Coskun and J. Huizenga, ‘The minimal model program for the Hilbert scheme of points on $\mathbb{P}^2$ and Bridgeland stability’, Adv. Math. 235 (2013) 580–626.
1. A. Bayer, A. Bertram, E. Macrì and Y. Toda, ‘Bridgeland stability conditions of threefolds II: an application to Fujita’s conjecture’, J. Algebraic Geom. 23 (2014) 693–710.
2. A. Bayer and E. Macrì, ‘The space of stability conditions on the local projective plane’, Duke Math. J. 160 (2011) 263–322.
3. A. Bayer and E. Macrì, ‘MMP for moduli of sheaves on K3s via wall-crossing: nef and movable cones, Lagrangian fibrations’, Invent. Math. 198 (2014) 505–590.
4. A. Bayer and E. Macrì, ‘Projectivity and birational geometry of Bridgeland moduli spaces’, J. Amer. Math. Soc. 27 (2014) 707–752.
5. A. Bayer, E. Macrì and P. Stellari, ‘The space of stability conditions on Abelian threefolds, and on some Calabi–Yau threefolds’, Invent. Math. 206 (2016) 869–933.
6. A. Bayer, E. Macrì and Y. Toda, ‘Bridgeland stability conditions on threefolds I: Bogomolov–Gieseker type inequalities’, J. Algebraic Geom. 23 (2014) 117–163.
7. M. Bernardara, E. Macrì and P. Stellari, ‘The space of stability conditions on Abelian threefolds, and on some Calabi–Yau threefolds’, Invent. Math. 206 (2016) 869–933.
8. A. Bayer, E. Macrì and Y. Toda, ‘Bridgeland stability conditions on threefolds II: an application to Fujita’s conjecture’, J. Algebraic Geom. 23 (2014) 693–710.
9. A. Bayer and E. Macrì, ‘The space of stability conditions on the local projective plane’, Duke Math. J. 160 (2011) 263–322.
10. T. Bridgeland, ‘Stability conditions on triangulated categories’, Ann. of Math. (2) 166 (2007) 317–345.
11. T. Bridgeland, ‘Stability conditions on K3 surfaces’, Duke Math. J. 141 (2008) 241–291.
12. M. R. Douglas, ‘D-branes, categories and $\mathcal{N} = 1$ supersymmetry’, J. Math. Phys. 42 (2001) 2818–2843.
13. M. R. Douglas, ‘Dirichlet branes, homological mirror symmetry, and stability’, Proceedings of the International Congress of Mathematicians, Beijing, 2002, vol. III (Higher Ed. Press, Beijing, 2002) 395–408.
14. D. Happel, I. Reiten and S. O. Smalø, ‘Tilting in abelian categories and quasitilted algebras’, Mem. Amer. Math. Soc. 120 (1996) viii+ 88.
15. C. Li, ‘Stability conditions on Fano threefolds of Picard number one’, J. Eur. Math. Soc., to appear.
16. A. Maciocia and D. Piyaratne, ‘Fourier–Mukai transforms and Bridgeland stability conditions on abelian threefolds’, Algebr. Geom. 2 (2015) 270–297.
17. A. Maciocia and D. Piyaratne, ‘Fourier–Mukai transforms and Bridgeland stability conditions on abelian threefolds II’, Internat. J. Math. 27 (2016) 1650007, 27.
18. E. Macrì, ‘A generalized Bogomolov–Gieseker inequality for the three-dimensional projective space’, Algebra Number Theory 8 (2014) 173–190.
19. E. Macrì, S. Mehrotra and P. Stellari, ‘Inducing stability conditions’, J. Algebraic Geom. 18 (2009) 605–649.
20. E. Macrì, S. Mehrotra and P. Stellari, ‘Inducing stability conditions’, J. Algebraic Geom. 18 (2009) 605–649.
21. C. Martinez, ‘Failure of the generalized Bogomolov–Gieseker type inequality on blowups’, Preprint.
22. D. Piyaratne, ‘Generalized Bogomolov–Gieseker type inequalities on Fano 3-folds’, Preprint, 2016, arXiv:1607.07172v2.
23. B. Schmidt, ‘A generalized Bogomolov–Gieseker inequality for the smooth quadric threefold’, Bull. Lond. Math. Soc. 46 (2014) 915–923.
24. B. Schmidt, ‘Counterexample to the generalized Bogomolov–Gieseker inequality for threefolds’, Preprint, 2016, arXiv:1602.05055.
25. J. F. Thomsen, ‘Frobenius direct images of line bundles on toric varieties’, J. Algebra 226 (2000) 865–874.
26. Y. Toda, ‘Curve counting theories via stable objects I: DT/PT correspondence’, J. Amer. Math. Soc. 23 (2010) 1119–1157.
27. Y. Toda, ‘Curve counting theories via stable objects II: DT/ncDT flop formula’, J. reine angew. Math. 675 (2013) 1–51.
28. Y. Toda, ‘Stability conditions and extremal contractions’, Math. Ann. 357 (2013) 631–685.
29. Y. Toda, ‘Stability conditions and birational geometry of projective surfaces’, Compos. Math. 150 (2014) 1755–1788.