Asymptotically Good Quantum and Locally Testable Classical LDPC Codes

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Abstract

We study classical and quantum LDPC codes of constant rate obtained by the lifted product construction over non-abelian groups. We show that the obtained families of quantum LDPC codes are asymptotically good, which proves the qLDPC conjecture. Moreover, we show that the produced classical LDPC codes are also asymptotically good and locally testable with constant query and soundness parameters, which proves a well-known conjecture in the field of locally testable codes.

Introduction

Classical low-density parity-check (LDPC) codes [1], as well as their quantum counterparts [2], have many important applications in theory and practice. These codes are represented by sparse parity-check matrices, where the term sparse usually means that the corresponding Tanner graphs are of bounded degree. Besides numerous applications in data storage and transmission systems, such codes are often used to construct classical and quantum locally testable codes [3–5], where the sparseness of a code ensures the constant-query property, also known as the constant locality. Informally speaking, a classical locally testable code (LTC) is a code that comes with an efficient non-deterministic procedure that allows to test with high probability whether a given sequence is close to some codeword by looking at a very small, usually constant, number of randomly chosen bits from this sequence. There are several ways how one can formally define LTCs [6]. In this paper, we adopt a very simple combinatorial definition (see [7, Definition 11]) that implies a rather strong form of local testability. According to this definition, a linear code \( \mathcal{C} \subseteq \mathbb{F}_q^n \) is called \((w, s)\)-locally testable if it has a parity-check matrix \( H \) with rows of weight at most \( w \) such that for any vector \( x \in \mathbb{F}_q^n \) we have

\[
\frac{1}{m} |Hx| \geq \frac{s}{n} d(x, \mathcal{C}),
\]

where \( d(x, \mathcal{C}) := \min_{c \in \mathcal{C}} d(x, c) \), and \( d(\cdot, \cdot) \) is the Hamming distance. The parameters \( w \) and \( s \) are called the locality and soundness, respectively. As we already mentioned above, this definition implies a strong form of local testability. Indeed, if our test procedure picks a random row from \( H \) and finds the corresponding syndrome component, then the probability of rejection \( \text{rej}_H(x) = \)

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$\frac{1}{m}|Hx|$ grows at least linearly with the normalized minimum distance $\delta(x, \mathcal{C}) := \frac{1}{n}d(x, \mathcal{C})$ from the tested vector $x \in \mathbb{F}_q^n$ to the code $\mathcal{C}$. In fact, for any family of LDPC codes with $m = \Theta(n)$, it follows that $\frac{1}{m}|Hx|$ cannot grow more than linearly with $\delta(x, \mathcal{C})$ since for every $w$-limited parity-check matrix $H$ we get $|Hx| \leq w \cdot d(x, \mathcal{C})$, where by a $w$-limited matrix we mean a matrix with rows and columns of weight at most $w$.

In the case of quantum locally testable codes (qLTCs), introduced recently in [4], one can give a similar to the above definition if a sparse parity-check matrix $H$ is replaced by a local Hamiltonian $\mathcal{H}$ defining the quantum code. However, for a quantum CSS code $\mathcal{Q}$, obtained from a pair of classical codes $\mathcal{C}_X$ and $\mathcal{C}_Z$, it is possible [4, 7] to infer the local testability of $\mathcal{Q}$ from the local testability of $\mathcal{C}_X$ and $\mathcal{C}_Z$. Let us recall that a quantum CSS code $\mathcal{Q}$ of dimension $k$ is defined by a pair of classical linear codes $\mathcal{C}_X, \mathcal{C}_Z \subseteq \mathbb{F}_q^n$ such that $\mathcal{C}_Z^\perp \subseteq C_X$, and $\mathcal{C}_X^\perp \subseteq C_Z$. Its minimum distance $d$ is defined as $\min(d_X, d_Z)$, where $d_X$ and $d_Z$ are the minimal Hamming weights of the vectors from $\mathcal{C}_X \setminus \mathcal{C}_Z^\perp$ and $\mathcal{C}_Z \setminus \mathcal{C}_X^\perp$, respectively. In this case we often say that $\mathcal{Q}$ is an $[n, k, d]_q$ code. The codes $\mathcal{C}_X, \mathcal{C}_Z$ are usually represented respectively by parity-check matrices $H_X, H_Z$, and the condition $\mathcal{C}_Z^\perp \subseteq C_X$ is equivalent to $H_X H_Z^\perp = 0$, where $H_Z^\perp$ is the transpose of $H_Z$. It was shown in [7, Lemma 13] that if a CSS code $\mathcal{Q}$ is defined by two classical $(w, s)$-locally testable codes with parity-check matrices $H_X, H_Z$, then the quantum code $\mathcal{Q}$ is $(w, s')$-locally testable, where $s' := s \min\left(\frac{m_X}{m_X+m_Z}, \frac{m_Z}{m_X+m_Z}\right)$, and $m_X$ (resp. $m_Z$) is the number of rows in the matrix $H_X$ (resp. $H_Z$).

Classical and quantum LTCs have many interesting applications in theoretical computer science since they are intimately related to a number of important problems in complexity theory [4, 8]. A major open problem is whether there are such codes of constant locality $w$, constant rate, and constant normalized minimum distance, sometimes also known as the $c^3$-conjecture (in the context of classical codes [9]) and qLTC conjecture (in the quantum case [4]). In this respect, the situation for classical LTCs is much better than for their quantum counterparts since classical LTCs of almost constant rate have been known for a long time [10]. However, in the quantum case, even if the property of local testability is not required, it is still a widely open problem, known as the qLDCP conjecture [11], to obtain asymptotically good family of quantum LDPC (qLDPC) codes, i.e., with the constant rate and normalized minimum distance. Up until very recently [12–15], the best provable lower bounds on the distances of qLDPC codes were, up to polylogarithmic factors, at most of the order $\sqrt{n}$ as the number of qubits $n \to \infty$ [16–21]. At the same time, asymptotically good families of classical LDPC codes have been known since their introduction by Robert Gallager in the 1960s [1].

In the current work, we show the existence of classical LTCs of constant rate, constant locality, and constant normalized minimum distance. In particular, we prove the following theorem, which gives a positive answer to the $c^3$-conjecture. Let us recall that a classical linear code $\mathcal{C} \subseteq \mathbb{F}_q^n$ has the parameters $[n, k, d]_q$ if $k = \dim \mathcal{C}$ and $d = \min_{c \in \mathcal{C}\setminus\{0\}} |c|$.

**Theorem 1.** For every number $R \in (0,1/2)$ and finite field $\mathbb{F}_q$ it is possible to find universal constants $s$ and $w$ such that there exists an explicit family of $(w, s)$-locally testable classical LDPC codes with the parameters $[n, k \geq Rn, d = \Theta(n)]_q$ as $n \to \infty$.

In the quantum case, we obtained a somewhat weaker analog of the above theorem, given below, which shows the existence of asymptotically good families of qLDPC codes, not necessarily the locally testable ones. This gives an affirmative answer to the qLDCP conjecture.
Theorem 2. For every number $R \in (0, 1)$ and finite field $\mathbb{F}_q$ there exists an explicit family of quantum LDPC codes over $\mathbb{F}_q$ with the parameters $[n, k \geq Rn, d = \Theta(n)]_q$ as $n \to \infty$.

Remark 1. In the case of classical codes from Theorem 1, it is relatively easy to show that an algorithm, similar to the bit-flipping algorithm, corrects in linear time any error of weight up to the constant fraction of the code length $n$. As for the quantum codes from Theorem 2, we conjecture that it is also possible with a variant of the small-set-flip decoding algorithm from [22] (see also [20]).

The codes from the above two theorems are obtained using a tensor product of two chain complexes of free modules over some group algebra $\mathbb{F}_q G$. This abstract idea from homological algebra was used recently in [13] and also independently in [14] to construct quantum LDPC codes of very large distances. It corresponds to the lifted product over $\mathbb{F}_q G$ from [13] and is equivalent to the special case of the balanced product from [14]. In the current work, we also call this operation lifted product over $G$ or $G$-lifted product. A very important ingredient of the constructions from [13,14] is expander codes [23], which are the Tanner codes [24] obtained from spectral expander graphs. In [13] such graphs are obtained as $G$-lifts (i.e., regular $|G|$-fold covers) of some small base graph, where $G$ is a very large group. It is not hard to see that the obtained in this way expander codes are free modules over the group algebra $\mathbb{F}_q G$. Thus these codes can be used with the $G$-lifted product to obtain a 3-term chain complex $C$, which can also be considered as a quantum CSS code [25]. It is shown in [13, Example 3] that using a $G$-lifted product of two classical codes it is possible to obtain a qLDPC code of constant rate. Moreover, some particular examples of such codes [13, Example 4], indicate that these qLDPC codes may also have very large minimal distances, close to the distances of the classical codes used in the lifted product. However, if the group $G$ is abelian, then the upper bound on the minimum distance of such classical codes [13, Eq. 24] provides strong evidence that to obtain an asymptotically good family of qLDPC codes by a $G$-lifted product one has to use non-abelian groups (cf. the Conjecture from [14]).

In the current paper, we show that using a $G$-lifted product of two expander codes for a wide class of non-abelian groups $G$ it is possible to obtain qLDPC codes with the parameters as in Theorem 2. Moreover, we also show that, under some additional assumptions, if $H_X$ and $H_Z$ are the parity-check matrices of such qLDPC codes, then the classical codes $C_X^* := \ker H_X^*$ and $C_Z^* := \ker H_Z^*$ are locally testable with the parameters as in Theorem 1. The main technical tool in our proof of Theorems 1 and 2 is the notion of locally minimal (co)chain, often used in the context of high-dimensional expanders to show expansion properties in simplicial complexes [26]. It is known that such expansion properties can be used to show local testability of a classical code [27] and to give a lower bound on the minimum distance of a quantum code [20]. In the current work, we extend these ideas to a much more general context of (co)chain complexes with local system of coefficients.

\footnote{Note that the codes from [13] are CSS codes, while the codes from [14] are in general subsystem codes with sparse stabilizer generators.}

\footnote{In [13] this general idea was applied to cyclic groups.}

\footnote{A similar observation is also made in [14].}
1 Preliminaries

1.1 Chain complexes

In recent years, ideas from homological algebra\(^4\) found many interesting applications in the field of classical and quantum codes [25, 27, 29]. A common approach is to consider some based\(^5\) (co)chain complex of finite-dimensional vector spaces over a finite field \(\mathbb{F}_q\), and use it to define a code with the desired parameters. For example, a 2-term chain complex

\[
\mathbb{F}_q^n \xrightarrow{\partial_1} \mathbb{F}_q^m
\]

can be identified with the classical linear code \(\ker \partial_1\) defined by the \textit{parity-check matrix} \(H := \partial_1\). Here, the space \(\mathbb{F}_q^n\) of 1-chains corresponds to the \(n\) bits, while the space \(\mathbb{F}_q^m\) of 0-chains to the \(m\) checks. At the same time, a 3-term chain complex

\[
\mathcal{C} := \left( \mathbb{F}_q^{m_x} \xrightarrow{\partial_1} \mathbb{F}_q^n \xrightarrow{\partial_2} \mathbb{F}_q^{m_z} \right)
\]

can be identified with the quantum CSS \([n, k, d]_q\) code \(\mathcal{Q} = Q(H_X, H_Z)\) defined by the parity-check matrices \(H_X := \partial_1\) and \(H_Z := \partial_2^*\), where \(\partial_2^* : \mathbb{F}_q^m \to \mathbb{F}_q^n\) is the transpose of the map \(\partial_2 : \mathbb{F}_q^m \to \mathbb{F}_q^n\). In this case, the space \(\mathbb{F}_q^n\) of 1-chains corresponds to the \(n\) qubits, and the space \(\mathbb{F}_q^{m_z}\) of 0-chains (resp. the space \(\mathbb{F}_q^{m_x}\) of 2-cells) to the \(X\)-checks (resp. \(Z\)-checks). The length of \(\mathcal{Q}\) is equal to \(n = \dim \mathbb{F}_q^n\), while its dimension \(k\) is equal to the dimension of the first homology group \(H_1(\mathcal{C}) := \ker \partial_1 / \text{im} \partial_2 = \mathcal{C}_X / \mathcal{C}_Z^\perp\), where \(\mathcal{C}_X := \ker \partial_1\) and \(\mathcal{C}_Z := \ker \partial_2^*\). The minimum distance \(d = d(\mathcal{Q})\) can also be described in the language of homology groups if we consider the quotient vector space \(H_1(\mathcal{C})\) as a metric space, where the distance \(d(A, B)\) between homology classes \(A, B \in H_1(\mathcal{C})\) is defined as \(d(A, B) := |A - B|\) using the corresponding quotient Hamming norm \(|A| := \min_{a \in A} |a|\). It is easy to see that \(d = \min(d(H_1(\mathcal{C})), d(H_1(\mathcal{C}^*)))\), where

\[
\mathcal{C}^* := \left( \mathbb{F}_q^{m_x} \xrightarrow{\partial_1^*} \mathbb{F}_q^m \xrightarrow{\partial_2^*} \mathbb{F}_q^{m_z} \right)
\]

is the dual chain complex for \(\mathcal{C}\). The distances \(d(H_1(\mathcal{C}))\) and \(d(H_1(\mathcal{C}^*))\) are sometimes called the 1-systole and 1-cosystole distances of \(\mathcal{C}\).

1.2 Lifted product

In this work, we consider several new families of classical and quantum LDPC codes of constant rate based on the introduced recently lifted product construction [13], which generalizes many known constructions of quantum LDPC codes [2, 29-33]. This construction can be defined in terms of parity-check matrices (see Appendix B) and in the abstract language of homological algebra, which we prefer in the current work. Before we proceed, let us briefly remind some standard definitions

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\(^4\)In this text, we assume that the reader is familiar with the standard notions of homological algebra such as a (co)chain complex and the corresponding (co)homology groups. See Appendix A for some definitions and [28] for a short introduction into this subject.

\(^5\)The term \textit{based} means that the vector spaces of a (co)chain complex come with some distinguished bases. If in a vector space \(V\) we fix a basis \(V \subseteq V\), we can identify \(V\) and its dual space \(V^*\) with the coordinate space \(\mathbb{F}_q^{\dim V}\) in the standard way. This also allows us to identify linear maps between such spaces with the corresponding matrices.
from algebra. Consider some ring $R$. A left $R$-module $M$ is called free if there exists a set of elements $\{m_1, \ldots, m_r\} \subseteq M$ called basis such that every $m \in M$ is uniquely represented as:

$$m = a_1m_1 + \ldots + a_rm_r,$$

where $a_1, \ldots, a_r \in R$, and the parameter $r$ is called the rank$^6$ of $M$. Hence $M \cong R^r$, and if the ring $R$ is a field, then $M$ is just an $r$-dimensional vector space over $R$. A canonical example of a free $R$-module of rank $r$ is the module $RS$ of formal $R$-linear combinations of the elements of some set $S$, where $|S| = r$. One can also define free right $R$-modules in a similar way.

**Definition.** Suppose we have a finite-dimensional associative algebra $R$ over $F_q$ with some fixed basis $\mathcal{R} \subseteq R$. Consider two chain complexes $\mathcal{A} = \bigoplus_{i=0}^m \mathcal{A}_i$ and $\mathcal{B} = \bigoplus_{j=0}^n \mathcal{B}_j$ over $F_q$ such that the vector spaces $\mathcal{A}_i$ and $\mathcal{B}_j$ are also free $R$-modules with some distinguished bases (over $R$) $\mathcal{A}_R \subseteq \mathcal{A}$ and $\mathcal{B}_R \subseteq \mathcal{B}$, and the boundary maps $\partial_{\mathcal{A}}: \mathcal{A} \to \mathcal{A}$, $\partial_{\mathcal{B}}: \mathcal{B} \to \mathcal{B}$ are $R$-linear. If the algebra $R$ is not commutative, then we further assume that $R$ acts from the right on $\mathcal{A}$ and from the left on $\mathcal{B}$, i.e., $\mathcal{A}$ is a right free $R$-module, and $\mathcal{B}$ is a left free $R$-module. The lifted product of $\mathcal{A}$ and $\mathcal{B}$ over $R$ is their tensor product complex $\mathcal{A} \otimes_R \mathcal{B}$ (see [28, p. 7]), where for $k = 0, 1, \ldots, m+n$ the space of $k$-chains $(\mathcal{A} \otimes_R \mathcal{B})_k$ is equal to $\bigoplus_{i+j=k} \mathcal{A}_i \otimes_R \mathcal{B}_j$, while the boundary map $\partial: \mathcal{A} \otimes_R \mathcal{B} \to \mathcal{A} \otimes_R \mathcal{B}$ is defined for $a \in \mathcal{A}_i$, $b \in \mathcal{B}_j$ as$^7$

$$\partial(a \otimes_R b) := \partial_{\mathcal{A}}a \otimes_R b + (-1)^i a \otimes_R \partial_{\mathcal{B}}b,$$  

(1)

and extended by linearity. Furthermore, we always assume that the lifted product $\mathcal{C} = \mathcal{A} \otimes_R \mathcal{B}$ is a based chain complex of vector spaces over $F_q$. By definition its distinguished basis (over $F_q$) is given by

$$\mathcal{C} := \{a \cdot r \cdot b \mid a \in \mathcal{A}_R, b \in \mathcal{B}_R, r \in \mathcal{R}\},$$

where we used a short-hand notation:

$$a \cdot r \cdot b := ar \otimes_R b = a \otimes_R rb.$$  

(2)

From the properties of the tensor product $\otimes_R$ it follows that the map $(a, r, b) \mapsto a \cdot r \cdot b$ is $F_q$-multilinear, which means that for every $a, a' \in \mathcal{A}$, $b, b' \in \mathcal{B}$, and $r, r' \in \mathcal{R}$ we have:

$$(a + a') \cdot r \cdot b = a \cdot r \cdot b + a' \cdot r \cdot b,$$

$$a \cdot (r + r') \cdot b = a \cdot r \cdot b + a \cdot r' \cdot b,$$

$$a \cdot r \cdot (b + b') = a \cdot r \cdot b + a \cdot r \cdot b',$$

and for every $\lambda \in F_q$ we get:

$$(\lambda a) \cdot r \cdot b = a \cdot (\lambda r) \cdot b = a \cdot r \cdot (\lambda b) = \lambda(a \cdot r \cdot b).$$

We should note that if $R = F_q$, then the lifted product is equivalent to the product construction from [29], while if, in addition, we have $m = n = 1$, then it is the same as the hypergraph

$^6$Note that there are some infinite non-commutative rings $R$ such that $R^n \cong R^m$ when $m \neq n$. However, all the rings we consider here are either finite or commutative, and hence have the invariant basis number (IBN) property that implies that this never happens.

$^7$We should note that the sign $(-1)^i$ in this definition is only relevant in the case of finite fields of odd characteristic.
product [31]. Moreover, if \( m = n = 1 \) and \( R = \mathbb{F}_2[x]/(x^\ell - 1) \), it is essentially equivalent to the hyperbicycle codes construction [33]. It is also important to note that when \( m = n = 1 \) the complexes

\[
\mathcal{A} := (A_1 \xrightarrow{A} A_0) \quad \text{and} \quad \mathcal{B} := (B_1 \xrightarrow{B} B_0)
\]

are uniquely defined by the corresponding matrices \( A, B \) over \( R \). In this case, we denote the lifted product \( \mathcal{A} \otimes_R \mathcal{B} \) as \( \text{LP}(A, B) \) and usually identify it with the corresponding CSS code. Note that this code also has a concise description in terms of the parity-check matrices \( H_X \) and \( H_Z \) (see [13, Eq. 12] and Eq. 13 from Appendix B).

Though the lifted product can be defined over an arbitrary finite-dimensional associative algebra \( R \), the most interesting case \([13, 14]\) is when \( R \) is the group algebra \( \mathbb{F}_q G \) for some finite group \( G \). The elements of \( \mathbb{F}_q G \) are formal sums \( \sum_{g \in G} \alpha_g g \), where \( \alpha_g \in \mathbb{F}_q \). Consider elements \( a = \sum_{g \in G} \alpha_g g \) and \( b = \sum_{g \in G} \beta_g g \) from \( \mathbb{F}_q G \). Their sum \( a + b \) and product \( ab \) are defined as follows:

\[
a + b := \sum_{g \in G} (\alpha_g + \beta_g) g, \quad ab := \sum_{g \in G} \left( \sum_{h \in G} \alpha_h \beta_r \right) g.
\]

In this case, the condition that the vector spaces \( \mathcal{A} \) and \( \mathcal{B} \) over \( \mathbb{F}_q \) are free \( \mathbb{F}_q G \)-modules is equivalent to the condition that the group \( G \) has a free action\(^8\) on their bases over \( \mathbb{F}_q \) (from the right for \( \mathcal{A} \) and from the left for \( \mathcal{B} \)), which is extended by linearity to \( \mathcal{A} \) and \( \mathcal{B} \). Moreover, the boundary map \( \partial \) is \( \mathbb{F}_q G \)-linear if\(f\) it is an \( \mathbb{F}_q \)-linear map that commutes with the action of the group \( G \). Therefore in what follows, in tensor products over \( R = \mathbb{F}_q G \) instead of \( \otimes_R \) we write \( \otimes_G \), and assume that \( R := G \). Let \( \hat{\mathcal{A}}_G \) and \( \hat{\mathcal{B}}_G \) be respectively the distinguished bases (over \( \mathbb{F}_q G \)) of the the free \( \mathbb{F}_q G \)-modules \( \mathcal{A} \) and \( \mathcal{B} \). It is clear that the elements \( ag \) (resp. \( gb \)), where \( a \in \hat{\mathcal{A}}_G, g \in G, b \in \hat{\mathcal{B}}_G \), constitute the basis for \( \mathcal{A} \) (resp. \( \mathcal{B} \)), considered as a vector space over \( \mathbb{F}_q \). Moreover, we see, using short-hand notation (2), that the distinguished basis \( \hat{\mathcal{A}} \otimes_G \hat{\mathcal{B}} \) over \( \mathbb{F}_q \) consists of the elements \( a \cdot g \cdot b \), where \( a \in \hat{\mathcal{A}}_G, g \in G, b \in \hat{\mathcal{B}}_G \). Furthermore, we can express the boundary operator given in equation (1) as follows:

\[
\partial(a \cdot g \cdot b) := (\partial_A a) \cdot g \cdot b + (-1)^i a \cdot g \cdot (\partial_B b).
\]

(3)

We can also express the boundary operator \( \partial \) as

\[
\partial := \partial_A \otimes_G \text{id} + \text{id} \otimes_G \partial_B
\]

if, by definition, assume that \((\partial_A \otimes_G \text{id})(a \cdot g \cdot b) := (\partial_A a) \cdot g \cdot b \) and \((\text{id} \otimes_G \partial_B)(a \cdot g \cdot b) := (-1)^i a \cdot g \cdot (\partial_B b)\).

Remark 2. For any chain complex \( \mathcal{C} \) we can consider its dual chain complex \( \mathcal{C}^* \) obtained from \( \mathcal{C} \) if we replace the boundary map \( \partial \) of \( \mathcal{C} \) by its transpose map \( \partial^* \) (see Appendix A). It is not hard to see that if \( \mathcal{C} \) is a left (resp. right) \( G \)-module, then \( \mathcal{C}^* \) is a right (resp. left) \( G \)-module. Therefore if chain complexes \( \mathcal{A} \) and \( \mathcal{B} \) are right \( G \)-modules, we can consider the \( G \)-liftings \( \mathcal{A} \otimes \mathcal{B}^* \). In fact, for any set \( S \) with a left action \( (g, s) \mapsto g \cdot s \) (resp. a right action \( (s, g) \mapsto s \cdot g \)) of a group \( G \) we can also consider the corresponding right (resp. left) action of \( G \) defined as \((s, g) \mapsto g^{-1} \cdot s \) (resp. \((g, s) \mapsto s \cdot g^{-1}) \). Therefore if a group \( G \) has a right free action on a chain complex \( \mathcal{C} \), then

\footnote{A left (resp. right) action of a group \( G \) on a set \( S \) is called free if for every \( g \in G \) when we have \( g s = s \) (resp. \( s g = s \)) for some \( s \in S \), then \( g \) is the identity element of \( G \). Note that the sizes of all orbits of a free action are the same and equal to \(|G|\).}
Remark 3. Let us note that $G$-lifted product is a special case of balanced product from [14], where a non-free action of the group $G$ is also allowed. We should also emphasize that the first examples of the lifted products over $R = \mathbb{F}_2G$ for a non-abelian group $G$ were also considered in [14], while in [13] all the examples were only for the abelian case. In the current work, we also give new examples of non-abelian lifted products based on the double-cover of a Cayley graph, which are similar, though not equivalent, to the horizontal subsystem codes mentioned in the Conjecture from [11]. Generally speaking, the term $G$-lifted product, used in the current work, may seem redundant since it is just a special case of the balanced product. However, we think that this special case deserves its own name since the free action of $G$ implies that the obtained complex has a much more regular structure than in the general case. In some sense, the relation of $G$-lifted product to more general balanced product is similar to the relation of Cayley graphs to Schreier graphs. While the latter are more general, the former are usually much easier to describe and study.

The lifted product [13] and other similar product constructions [12], [14] were used recently to obtain quantum LDPC codes of almost linear minimum distance. In fact, all the codes of large distances from these papers were obtained from the codes equivalent to the codes $\text{LP}(A, 1 + x)$, where

$$R = \mathbb{F}_2[x]/(x^\ell - 1) \cong \mathbb{F}_{2^\ell},$$

and $C_\ell$ is the cyclic group of order $\ell$ generated by the element $x \in C_\ell$. We should also mention that the lifted product codes of the form $\text{LP}(A, b)$, where $b \in R$, were previously studied in [34] under the name GHP codes and shown to have surprisingly good error-correcting performance under the BP-OSD decoder.

### 1.3 Expander graphs and lifts

To produce linear maps $\varphi: \mathbb{F}_q^n \to \mathbb{F}_q^m$ with good expansion and coexpansion properties it was proposed in [13, 14] to use expander codes [23], i.e., the Tanner codes [24] defined on some spectral expander graph. Before we move on, let us recall some standard definitions related to expander graphs and Tanner codes.

Let $\Gamma$ be a graph$^9$ with the set of vertices $V(\Gamma)$ and the set of edges $E(\Gamma)$. If vertices $v, v' \in V(\Gamma)$ are connected by an edge $e \in E(\Gamma)$, we call $v, v'$ adjacent and denote this fact by $v \leftrightarrow v'$ or by $v \leftrightarrow_e v'$ when we want to emphasize the edge $e$. A graph $\Gamma$ is called $d$-regular if all its vertices have degree $d$. The adjacency matrix of a graph $\Gamma$ with $V(\Gamma) = \{v_1, \ldots, v_n\}$ is the matrix $A(\Gamma) = (a_{ij})_{n \times n}$, where $a_{ij}$ is the number of edges $e \in E(\Gamma)$ such that $v_i \leftrightarrow_e v_j$. Since $A(\Gamma)$ is a symmetric matrix, it has $n$ real-valued eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$. Let $\lambda_2(\Gamma) := \lambda_2$, and $\lambda(\Gamma) := \max(\lambda_2, \lambda_n)$. It is obvious that $\lambda_2(\Gamma) \leq \lambda(\Gamma)$. We call an $n$-vertex $d$-regular graph $\Gamma$ an $(n, d, \lambda)$-expander if $\lambda(\Gamma) \leq \lambda$. The term expander here means that the graph $\Gamma$ has a very good connectivity, which can be quantified by its Cheeger constant. Consider a subset of vertices $S \subseteq V(\Gamma)$ in the graph $\Gamma$. We call the set

$$\partial S := \{e \in E(\Gamma) \mid v \leftrightarrow_e v', v, v' \in S, v' \notin S\}$$

$^9$It may have loops and multiple edges.
the edge boundary, which is the set of all edges that go outside of $S$. The Cheeger constant $h(\Gamma)$ of the graph $\Gamma$ is defined as follows:

$$h(\Gamma) := \min_{0 < |S| \leq \frac{1}{2}|V(\Gamma)|} \frac{|\partial S|}{|S|}.$$  

Since for $d$-regular graphs it is known [35, Theorem 4.11] that $h(\Gamma) \geq \frac{1}{2}(d - \lambda_2(\Gamma))$, then the smaller the value of $\lambda_2(\Gamma)$, the higher the Cheeger constant. However, the Alon-Boppana bound [35, Theorem 5.3] implies that for $d$-regular graphs with $n$ vertices we have $\lambda_2(\Gamma) \geq 2\sqrt{d - 1} - o_n(1)$ as $n \to \infty$. There are a number of different constructions that almost attain this lower bound. In fact, it was shown in [36] that for any fixed $\varepsilon > 0$, a random $d$-regular graph with $n$ vertices has $\lambda_2(\Gamma) < 2\sqrt{d - 1} + \varepsilon$ with high probability as $n \to \infty$. A $d$-regular graphs $\Gamma$ that satisfy the condition $\lambda(\Gamma) \leq 2\sqrt{d - 1}$ is called Ramanujan. There are a number of explicit constructions of such graphs [37,38] that use Cayley graphs of some non-commutative groups (see [39] for a good survey).

We will see later that Tanner codes with such Ramanujan graphs (or their double-covers) can be used with the lifted product construction. The obtained chain complexes, which we can also consider as CSS codes, have very interesting expansion properties, similar to the ones studied in the theory of high-dimensional expanders (HDXs) [40]. We will show later that some of the standard definitions from this theory (e.g., the local minimality of (co)chains) can be naturally extended to a more broad context of based (co)chain complexes.

In [13], the graph $\hat{\Gamma}$ for the Tanner code was obtained as an $\ell$-lift of a small base graph $\Gamma$ using voltage assignments [41] with the cyclic group $C_\ell$ as the voltage group. Recall that an $\ell$-lift (also called an $\ell$-fold cover) of a base graph $\Gamma$ is a graph $\hat{\Gamma}$ obtained if we replace in the base graph each vertex $v \in V(\Gamma)$ with $\ell$ replicas $\hat{v}_1, \ldots, \hat{v}_\ell$, and replace each edge $e \in E(\Gamma)$ that connects vertices $v, v' \in V(\Gamma)$ with $\ell$ replicas $\hat{e}_1, \ldots, \hat{e}_\ell$ such that $\hat{e}_i$ connects in $\hat{\Gamma}$ the vertices $\hat{v}_i$ and $\hat{v}'_{\pi(i)}$, where $\pi \in S_\ell$ is some permutation on the set $\{1, \ldots, \ell\}$ (see Fig. 1). Note that the permutations for different edges may be different and are usually defined [41] by a voltage assignment using some group $G$, in which case we call the obtained graph a $G$-lift of $\Gamma$.

A voltage assignment for a graph $\Gamma$ with a voltage group $G$ is a map $\gamma : E(\Gamma) \to G$. Let us fix some orientation of the edges, i.e., a function $o : E(\Gamma) \to V(\Gamma) \times V(\Gamma)$, which tells us that the edge

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10 In this work we consider only non-bipartite Ramanujan graphs.

11 Multiple edges and loops are usually allowed in the base graph $\Gamma$.
$e$ is oriented from $v$ to $v'$ if $o(e) = (v, v')$. For any voltage assignment $\gamma$, we can obtain the $G$-lift $\hat{\Gamma}$ of the base graph $\Gamma$ called the \textit{(left) derived graph} for $\Gamma$ and $\gamma$, which we denote by $D(\Gamma, \gamma)$. To define $\hat{\Gamma} = D(\Gamma, \gamma)$ we first let $V(\hat{\Gamma}) := V(\Gamma) \times G$, $E(\hat{\Gamma}) := E(\Gamma) \times G$, and introduce the following short-hand notations: $\hat{v}_g := (v, g)$, $\hat{e}_g := (e, g)$, where $v \in V(\Gamma)$, $e \in E(\Gamma)$, $g \in G$. Now, if in the base graph $\Gamma$ an edge $e \in E(\Gamma)$ connects vertices $v, v'$ in $V(\Gamma)$, and $o(e) = (v, v')$, then in the derived graph $\hat{\Gamma}$, for every $g \in G$, the edge $\hat{e}_g$ connects the vertices $\hat{v}_g$ and $\hat{v}'_{\gamma(e)}$. One can also define the \textit{right derived graph} if the edge $\hat{e}_g$ connects the vertices $\hat{v}_g$ and $\hat{v}'_{\gamma(e)}$. We call the $G$-lifts obtained from the left and right derived graphs \textit{left} and \textit{right} respectively.

Note that a $G$-lift $\hat{\Gamma}$ obtained by a voltage assignment from a base graph $\Gamma$ is usually called a \textit{regular lift} or a \textit{regular cover} of $\Gamma$. If a group $G$ has a right (resp. left) free action on the vertices and edges of a graph, and the condition $v \leftrightarrow_e v'$ implies $vg \leftrightarrow_{eg} v'g$ (resp. $gv \leftrightarrow_{ge} gv'$) for every vertices $v, v'$, edge $e$, and $g \in G$, then we say that $G$ has a right (resp. left) \textit{free} action on this graph. One can easily check that for any left $G$-lift we can define a right action of $G$ if for every $\hat{v}_g \in V(\hat{\Gamma})$, $\hat{e}_g \in E(\hat{\Gamma})$, and $h \in G$ we put $\hat{v}_gh := \hat{v}_{gh}$, $\hat{e}_gh := e_{gh}$. In what follows, we consider only left $G$-lifts and omit the word “left”. Note that when the group $G$ is abelian, then there is no difference between left and right $G$-lifts.

When the voltage group is a cyclic group $C_\ell$, then the corresponding derived graphs are also called \textit{shift $\ell$-lifts} and the assigned voltages are called \textit{shifts}. In the special case when $\ell = 2$, and we assign to each edge $e$ of the base graph $\Gamma$ the non-identity shift from $C_2$, we obtain the bipartite graph $\hat{\Gamma}$ called the \textit{(bipartite) double-cover of $G$}. Since $\hat{\Gamma}$ is the tensor product of $\Gamma$ and the complete graph $K_2$, then it is not hard to show that $\lambda_2(\hat{\Gamma}) = \lambda(\Gamma)$. Hence this particular 2-lift almost preserves the spectral expansion properties. Note that if $G$ is a bipartite graph then $\hat{\Gamma}$ is a disconnected graph. Hence, it does not make a lot of sense to apply this simple construction more than once since on the second iteration one inevitably obtains a disconnected graph. However, the situation is not that bad if we apply a large shift $\ell$-lift only once. As it was shown in Theorem 1.2 from [42], if the base graph $\Gamma$ has good spectral expansion properties, then by using random shifts the obtained graph $\hat{\Gamma}$ also has good expansion properties, even when the lift size $\ell$ is very large. In [13], such graphs $\hat{\Gamma}$ were used to construct quasi-cyclic expander codes of very large lift size $\ell$ such that the corresponding parity-check matrix $H$ and its transpose $H^*$ have good expansion properties.

In the current work, we also obtain graphs $\hat{\Gamma}$ using voltage assignments. We start from a very small base graph $\Gamma$ such as the bouquet graph $B_w$ (one vertex, $w$ loops) or the graph $D_w$ (two vertices connected by $w$ multiple edges). Then we consider a finite group $G$ with some fixed $w$-element set of generators $S \subseteq G$ and assign each generator from $\hat{S} = \{s_1, \ldots, s_w\}$ to exactly one of the $w$ edges (see Fig. 2). It is not hard to see that the derived graphs for $B_w$ correspond to Cayley graphs $\text{Cay}(G, S)$ if the generating set $S$ is \textit{symmetric}, i.e. $S = \{s^{-1} \mid s \in S\}$, and there are no generators $s \in S$ such that $s = s^{-1}$. Let us remind that, given a finite group $G$ with some symmetric generating set $S$, the corresponding \textit{(left) Cayley graph} is the simple graph $\text{Cay}(G, S)$ with the set of vertices $V(\Gamma) := G$ and the set of edges $E(\Gamma) := \{\{g, sg\} \mid g \in G, s \in S\}$. Now if we assign the elements of a symmetric generating set $S$ of some finite group $G$ one-to-one to the $w$ edges of the graph $D_w$ (the orientation is shown in Fig. 2), then we obtain the graph $\text{Cay}_2(G, S)$, which is the double-cover of $\text{Cay}(G, S)$. The graph $\text{Cay}_2(G, S)$ has the set of vertices $V(\hat{\Gamma}) := G \times \{0, 1\}$ and the set of edges:

$$E(\hat{\Gamma}) := \{(g, 0), (sg, 1) \mid g \in G, s \in S\}.$$ 

Note that the free right action of the group $G$ on this graph is defined as $(g, a)h := (gh, a)$ and
The dual code where \( F \) is a \([n, k]_q \) code is usually defined either as the row space of a matrix \( G \) called the \textit{generator matrix}, or as the kernel of a matrix \( H \) called the \textit{parity-check matrix}. It is easy to see that \( GH^* = 0 \), \( \text{rk} \, G = k \), and \( \text{rk} \, H = n - k \). The code defined by a parity-check matrix \( H \) is denoted by \( \mathcal{C}(H) \). The vector space \( \mathbb{F}_q^n \) usually comes with the standard scalar product \( \langle x, y \rangle = x_1y_1 + \cdots + x_ny_n \). The \textit{dual code} \( \mathcal{C}^\perp \) for a linear \([n, k]_q \) code \( \mathcal{C} \) is the \([n, n-k]_q \) code
\[
\mathcal{C}^\perp = \{ x \in \mathbb{F}_q^n \mid \langle x, y \rangle = 0 \text{ for all } y \in \mathcal{C} \}.
\]

It is not hard to see that a generator matrix for \( \mathcal{C} \) is a parity-check matrix for \( \mathcal{C}^\perp \) and vice versa.

\footnote{The group \( \text{PSL}(\mathbb{F}_q^2) \) is the \textit{projective special linear} group for \( \mathbb{F}_q^2 \), i.e. the quotient of the group of matrices \( A \in \mathbb{F}_q^{2 \times 2} \) with \( \det A = 1 \) modulo its subgroup \( \{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \} \).}
Remark 4. Note that in the current work it is convenient to consider a slightly more general case, where instead of $\mathbb{F}_q^n$ we have an arbitrary based $n$-dimensional vector space $\mathcal{M}$ over $\mathbb{F}_q$ equipped with some distinguished basis $\mathcal{M} = \{m_1, \ldots, m_n\} \subseteq \mathcal{M}$. In this case, $\mathcal{M} \cong \mathbb{F}_q^n$, and we can consider subspaces $\mathcal{C} \subseteq \mathcal{M}$ as linear codes, and apply all the terminology we introduced above to this case as well.

In what follows, we will often use the following important definitions.

**Definition.** Consider two linear codes $\mathcal{C} \subseteq \mathcal{M}$ and $\mathcal{C}' \subseteq \mathcal{M}'$, where $\mathcal{M}$ and $\mathcal{M}'$ are two $n$-dimensional vector space over $\mathbb{F}_q$ with distinguished bases $\mathcal{M}$ and $\mathcal{M}'$ respectively. We say that $\mathcal{C}$ and $\mathcal{C}'$ are (permuation) equivalent and write $\mathcal{C} \sim \mathcal{C}'$ if there exists a linear map $\pi: \mathcal{M} \to \mathcal{M}'$ such that $\pi(\mathcal{M}) = \pi(\mathcal{M}')$ and $\pi(\mathcal{C}) = \pi(\mathcal{C}')$. We also say that two $m \times n$ matrices $A$ and $B$ are (permuation) equivalent and write $A \sim B$ if we can obtain one from another by some row/column permutations. It is clear that if $A \sim B$ then $\ker A \sim \ker B$.

### 1.5 Expander codes

In this subsection, we describe expander codes, which are Tanner codes obtained from expander graphs. We adopt a very convenient way, used in [43, 14] to represent these codes in the language of chain complexes and local systems. If $\mathcal{F}$ is some abelian group and $X$ is some $n$-element set, then we denote by $\mathcal{F}X$ the abelian group of all formal linear combinations $\sum_{x \in X} a_x x$ of the elements $x \in X$ with coefficients $a_x \in \mathcal{F}$. When $n = 1$, and $X = \{x\}$, we usually write $\mathbb{F}_q x$ instead of $\mathbb{F}_q \{x\}$. If $\mathcal{F} = \mathbb{F}_q$ then the group $\mathcal{F}X$ is isomorphic to the vector space $\mathbb{F}_q^X$. When $\mathcal{F} = \mathbb{F}_q^n$, the group $\mathcal{F}X$ can be identified with the vector space $\mathbb{F}_q^{|X|}$ of block vectors $(v_1, \ldots, v_n)$ with the blocks $v_i \in \mathbb{F}_q^n$, $i \in [n]$. If $S \subseteq X$ and $a = \sum_{x \in X} a_x x$, then $a|_S := \sum_{x \in S} a_x x$. Now let us introduce the following important definition.

**Definition.** Consider a graph $\Gamma = (V, E)$ and a collection $((\partial^{(v)})_{v \in V})$ of linear maps $\partial^{(v)}: \mathbb{F}_q E_v \to \mathbb{F}_q^V$ called local boundary maps, where $E_v$ is the set of edges incident to the vertex $v \in V$. A Tanner chain complex $\mathcal{T} = T_\bullet (\Gamma, (\partial^{(v)})_{v \in V})$ is a chain complex $\mathbb{F}_q E \xrightarrow{\partial} \mathbb{F}_q V$ such that for every $e \in E$ that connects $v$ and $v'$ we have:

$$\partial e := \partial^{(v)} e + \partial^{(v')} e.$$  (4)

Any Tanner complex $\mathcal{T}$ defines the global linear code $\mathcal{C} := \ker \partial$, also known as the Tanner code, and a number of local linear codes $\mathcal{C}_v := \ker \partial^{(v)}$, $v \in V$, also known as subcodes. We see that $c \in \mathcal{C}$ iff $c|_{E_v} \in \mathcal{C}_v$ for all $v \in V$. In what follows, we consider Tanner complexes where the matrices of all local boundary maps $\partial^{(v)}$ are equivalent to one matrix $h \in \mathbb{F}_q^{|E| \times |V|}$. Hence all local codes $\mathcal{C}_v$ are also equivalent to the same linear $[n, k, d]_q$ code $\mathcal{C}(h) = \ker h$. We denote the class of all such Tanner complexes on the graph $\Gamma$ as $\mathcal{T}(\Gamma, h)$.

We can lift Tanner complexes in a similar way as we lift graphs using voltage assignments. Consider a Tanner complex $\mathcal{T} = T_\bullet (\Gamma, (\partial^{(v)})_{v \in V})$ for the graph $\Gamma$. For any $G$-lift $\hat{\Gamma} = (\hat{V}, \hat{E})$, obtained from $\Gamma$ by a voltage assignment $\gamma: E(\Gamma) \to G$, we can define the $G$-lifted Tanner complex $\hat{\mathcal{T}} = D(\mathcal{T}, \gamma)$. It is convenient to represent $\hat{\mathcal{T}}$ as the complex

$$\mathbb{F}_q E \otimes \mathbb{F}_q G \xrightarrow{\hat{\partial}} \mathbb{F}_q^V \otimes \mathbb{F}_q G,$$

where by the tensor product $\otimes$ we mean the tensor product over $\mathbb{F}_q$. Since $\mathbb{F}_q^V \otimes \mathbb{F}_q G \cong \mathbb{F}_q \hat{V}$ and $\mathbb{F}_q E \otimes \mathbb{F}_q G \cong \mathbb{F}_q \hat{E}$, we can assume that $v \otimes g = (v, g)$ and $e \otimes g = (e, g)$ and still consider $\hat{\mathcal{T}}$ as
a Tanner complex

\[ \mathbb{F}_q \hat{E} \xrightarrow{\partial_1} \mathbb{F}_q \hat{V} \]

for the graph \( \hat{\Gamma} \). The boundary map \( \hat{\partial} \) of this complex is defined for every \( g \in G \) and \( e \in E \) with \( \alpha(e) = (v, v') \) as

\[ \hat{\partial}(e \otimes g) := \partial(v)e \otimes g + \partial(v')e \otimes \gamma(e)g, \]

and extended by linearity (cf. Equation (4)). Let \( \hat{\Gamma} = D(\Gamma, \gamma) \) is a G-lift of a graph \( \Gamma \). We denote by \( \mathcal{F}(\hat{\Gamma}, h) \) the class of all G-lifted Tanner complexes \( \hat{T} = D(\mathcal{T}, \gamma) \) where \( \mathcal{T} \in \mathcal{F}(\hat{\Gamma}, h) \).

Since the lifted Tanner complex \( \hat{T} \) is a right \( G \)-module\(^{13}\), we can use any such complex with the \( G \)-lifted product construction discussed earlier. Let us now consider a local \([w, k, d]_q \) code \( C(h) \) with the parity-check matrix \( h \in \mathbb{F}_q^{r \times w} \), and the Tanner complex \( \mathcal{T}(h) := \left( \mathbb{F}_q E(D_w) \xrightarrow{\partial_1} \mathbb{F}_q V(D_w) \right) \) with the boundary map defined as

\[ \hat{\partial} e_i := h_i v_1 + h_i v_2, \]

where \( E(D_w) = \{e_1, \ldots, e_w\} \), \( V(D_w) = \{v_1, v_2\} \), and \( h_i \) is the \( i \)-th column of the parity-check matrix \( h \). It is easy to see that the two local codes \( C_{v_1} \) and \( C_{v_2} \) of \( \mathcal{T}(h) \) are both equivalent to \( C(h) \). As it was already mentioned, we can obtain the double-cover \( \Gamma := \text{Cay}_2(G, S) \) of any Cayley graph \( \text{Cay}(G, S) \) as the \( G \)-lift of \( D_w \), where \( w = |S| \), by a one-to-one assignment of the \( w \) generators from \( S \) to the edges of \( D_w \) (see Fig. 2). Thus we can consider the lifted Tanner complex \( \mathcal{T}(\Gamma; h) := D(\mathcal{T}(h), \gamma) \), where \( \gamma \) is the corresponding voltage assignment map: \( \gamma(e_i) := s_i, i \in [w] \). It is not hard to check that the boundary map \( \hat{\partial} \) of this lifted complex acts on its bases as follows:

\[ \hat{\partial}(e_i \otimes g) = h_i v_1 \otimes g + h_i v_2 \otimes s_i g, \quad i \in [w]. \]

Let us remind that the chain complex \( \mathcal{T}(\Gamma; h) \) is a \( G \)-module.

Let us fix a graph \( \Gamma := \text{Cay}_2(G, S) \) and two parity-check matrices \( h \in \mathbb{F}_q^{r \times w} \), \( h' \in \mathbb{F}_q^{r' \times w} \). We can define the following 3-term chain complexes using the \( G \)-lifted product construction:

\[ C_\bullet(\Gamma; h, h') := \mathcal{T}(\Gamma; h) \otimes_G \mathcal{T}(\Gamma; h')^*, \]

\[ C'_\bullet(\Gamma; h, h') := \mathcal{T}(\Gamma; h) \otimes_G \mathcal{T}(\Gamma; h'). \]

**Remark 5.** Let \( X^{w-1, t} := \text{Cay}_2(G, S_{w-1, t}) \) be the \( w \)-regular graph from Example 1, where \( G = \text{PSL}(2, q) \). Consider the chain complexes \( C_\bullet(X^{w-1, t}; h, h') \) and \( C'_\bullet(X^{w-1, t}; h, h') \) respectively. In the current work, we use the first complex to show the existence of two asymptotically good families of codes: quantum LDPC codes and classical LTCs. However, as we can see from Theorem 1, the rate of the obtained LTCs is bounded above by 1/2. We conjecture that the complex \( C'_\bullet(X^{w-1, t}; h, h') \) can be used to obtain asymptotically good LTCs of rate arbitrary close to 1.

### 1.6 Posets and incidence chain complexes

In this subsection, we consider based chain complexes \( \mathcal{I} \) with integer coefficients\(^{14}\) and call the elements from the corresponding distinguished basis \( \mathcal{I} \) cells. We say that \( \mathcal{I} \) is an incidence chain complex if the matrix of its boundary map \( \partial \) contains only elements from \( \{-1, 0, 1\} \), and for every

\(^{13}\)We can multiply from the right on its basis as follows: \((e \otimes g)h := e \otimes gh\), \((v \otimes g)h := v \otimes gh\).

\(^{14}\)Chain complexes with integer coefficient are often used in algebraic topology to study the integral homology groups of CW-complexes.
such a complex we also define its cell poset, which can be viewed as a combinatorial structure that represents the incidence relation between the cells. In some sense, one can view the cell poset with the corresponding incidence chain complex as an abstract cell complex (see, e.g. [44, Section 2.12]), which generalizes the notion of an abstract simplicial complex and an abstract polytope [45].

Let $X$ be a poset, i.e., a set with a partial order $\leq$. We say that an element $a \in X$ covers an element $b \in X$ and write $a \prec b$ or $b \succ a$ if $a < b$, and there is no element $c \in X$ such that $a < c < b$. It is easy to see that any finite poset can be uniquely defined by its covering relation $\prec$ if we let $a \leq b$ iff there exists a sequence $c_0 \prec c_1 \prec \cdots \prec c_n$ of elements from $X$ such that $c_0 = a$, $c_n = b$, and $n \geq 0$. Let $\mathcal{C}$ be a based chain complex over some ring$^{15}$ $R$. We can define the partial order $\leq$ on the distinguished basis $\mathcal{C}$ if for every two cells $c, c' \in \mathcal{C}$ we put $c' \prec c$ iff $c' \in \text{supp} \partial c$. We call the poset $\mathcal{C}$ with the relation $\leq$ the cell poset of $\mathcal{C}$.

A graded poset is a poset $X$ equipped with a map $\rho: X \to \mathbb{Z}$ called a rank function such that for any $a, b \in X$ the following conditions hold:

1. if $a \leq b$ then $\rho(a) \leq \rho(b)$;
2. if $a \prec b$ then $\rho(b) = \rho(a) + 1$.

If $X$ is a finite graded poset, then it is not hard to see that it can be decomposed as

$$X = X(s) \sqcup X(s + 1) \sqcup \cdots \sqcup X(t),$$

where the subset $X(i) := \{a \in X \mid \rho(a) = i\}$ is called the $i$-th level of $X$, $i \in [s, t] \cap \mathbb{Z}$. It is clear that $X(s)$ and $X(t)$ correspond respectively to the sets of minimal and maximal elements in $X$. It is also trivial to check that the cell poset $\mathcal{C}$ of a based (co)chain complex $\mathcal{C}$ is a graded poset, where the levels correspond to the cells of the same dimension.

Another example of a graded poset, often studied in the context of HDXs, is an (abstract) simplicial complex on a finite non-empty set $V$, which is defined as a closed under taking subsets family $X \subseteq 2^V$. In this case, the partial order $\leq$ is just the set inclusion relation $\subseteq$, and $\rho(x) := |x| - 1$ for every $x \in X$. The elements $x \in X$ with $\rho(x) = i$ are called $i$-dimensional faces or just $i$-faces. The highest dimension of the faces from the simplicial complex $X$ is called its dimension. Let us note that a simple graph can be represented as a 1-dimensional simplicial complex, where the 0-faces and the 1-faces correspond respectively to the vertices and the edges of the graph. Hence we can also view an undirected graph $\Gamma$ as the graded poset with the levels $V(\Gamma)$ and $E(\Gamma)$, where for every $v \in V(\Gamma)$ and $e \in E(\Gamma)$ we have $v \prec e$ whenever $v$ is incident$^{16}$ to $e$. In fact, 2-level posets are equivalent to the incidence systems, and thus can be used to represent undirected multigraphs and hypergraphs as well.

In this work, it is convenient to define objects such as graphs, incidence systems, and simplicial complexes by the corresponding based chain complexes over $\mathbb{Z}$. In some way, we can view such complexes with integer coefficients as a vast generalization of these objects. For example, for any 2-level poset $X$ with the levels $V$ and $E$, we can define the based chain complex $\mathcal{C}_\bullet(X) := \left(\mathbb{Z}E \xrightarrow{\partial} \mathbb{Z}V\right)$ with the distinguished bases $\mathcal{C}_0 := V$, $\mathcal{C}_1 := E$, where

$$\partial e := \sum_{\substack{v \prec e \in E \atop v \in V}} v.$$  

---

$^{15}$In this section, we are interested in only two cases: $R = \mathbb{Z}$ and $R = \mathbb{F}_q$.

$^{16}$We assume here that the incident relation in a graph is asymmetric.
The matrix of $\partial_1$ is a zero–one matrix usually called the incidence matrix of $X$. For example, since we view an undirected graph $\Gamma$ as a 2-level poset, we can consider the corresponding chain complex $C_\ast(\Gamma)$. Now let $X$ be a simplicial complex with some fixed linear order $<_{V}$ on its set of vertices $V = X(0)$. Then we can define the chain complex $C_\ast(X)$ by the following diagram

$$
\begin{array}{c}
\mathbb{Z}X(n) \overset{\partial_n}{\longrightarrow} \cdots \overset{\partial_1}{\longrightarrow} \mathbb{Z}X(0) \overset{\partial_0}{\longrightarrow} \mathbb{Z}X(-1),
\end{array}
$$

where for every $k$-face $x = \{v_0, \ldots, v_k\} \in X$ such that $v_0 <_{V} \cdots <_{V} v_k$ the boundary map $\partial: \mathbb{Z}X \to \mathbb{Z}X$ is defined as $\partial x := \sum_{i=0}^{k} (-1)^i x \setminus \{v_i\}$, and then extended by linearity to all chains from $\mathbb{Z}X$. As we can see, the integer coefficients in the matrix of the boundary maps for $C_\ast(\Gamma)$ and $C_\ast(X)$ are from the set $\{-1, 0, 1\}$. Let us call any based chain complex $I$ with this property an incidence complex. Let $I$ be some incidence complex with a distinguished basis $X$. It is clear that its boundary map $\partial: \mathbb{Z}X \to \mathbb{Z}X$ acts on a cell $x \in X$ as

$$
\partial x = \sum_{x \succ x'} [x : x'] x',
$$

where the coefficient $[x : x'] \in \{-1, +1\}$ is called the incidence number for $x, x' \in X$. It is also convenient to assume that $[x : x'] = 0$ whenever $x \not\succ x'$. Let us note that since $\partial^2 = 0$, then for every $x, x'' \in X$ we obtain

$$
\sum_{x \succ x' \succ x''} [x : x'][x' : x''] = 0.
$$

### 1.7 Products of graphs

Suppose that $\hat{\Gamma}$ is a $G$-lift of some base graph $\Gamma$. As we know, the group $G$ has a free right action on $\hat{\Gamma}$. Thus we can consider (see Remark 2) the $G$-lifted product complex $C_\ast(\hat{\Gamma}) \otimes_G C_\ast(\hat{\Gamma})$, where $C_\ast(\Gamma)$ is the dual chain complex for $C_\ast(\Gamma)$. Let us denote its cell poset by $\hat{\Gamma} \times_G \hat{\Gamma}^*$. From the definition of $G$-lifted product and Eq. (3) it follows that

$$
\hat{\Gamma} \times_G \hat{\Gamma}^* = \{ x \cdot g \cdot y \mid x, y \in V(\Gamma) \cup E(\Gamma), g \in G \},
$$

and we have $x' \cdot g' \cdot y' \succ x \cdot g \cdot y$ iff one of the following conditions hold:

1. $\hat{x}'_g \succ\hat{\Gamma} \hat{x}_g$ and $y = y'$;
2. $x = x'$ and $\hat{y}'_g \preceq\hat{\Gamma} \hat{y}_g$;

where $\preceq\hat{\Gamma}$ is the covering relation in the graph $\hat{\Gamma}$, considered as a 2-level poset.

**Remark 6.** Note that the cell poset for the dual chain complex $C_\ast^*(\hat{\Gamma})$ is the dual poset $\hat{\Gamma}^*$ for $\hat{\Gamma}$, i.e., $x \prec\hat{\Gamma} y$ whenever $x \succ\hat{\Gamma} y$. This is the reason why in the second condition above the covering relation is in the opposite direction.

\[17\] In fact, sometimes it is also convenient to consider arbitrary integer coefficients. But this more general case is not covered here.
It is convenient to interpret the poset \( X = \hat{\Gamma} \times_G \hat{\Gamma}^* \) as a 2-dimensional geometric object\(^{18}\). An element \( x \cdot g \cdot y \in X \) is called:

- a **vertex** if \( x \in V(\Gamma), y \in V(\Gamma) \);
- a **horizontal edge** if \( x \in E(\Gamma), y \in V(\Gamma) \);
- a **vertical edge** if \( x \in V(\Gamma), y \in E(\Gamma) \);
- a **face** if \( x \in E(\Gamma), y \in V(\Gamma) \),

and the corresponding subsets of elements are denoted as \( V = V(X), E_{\rightarrow} = E_{\rightarrow}(X), E_{\uparrow} = E_{\uparrow}(X), \) and \( F = F(X) \). Let us note that the cell poset \( X \) is graded and has 3 levels: \( X(0) := E_{\uparrow} \) (the vertical edges), \( X(1) := F \cup V \) (the faces and vertices), and \( X(2) := E_{\rightarrow} \) (the horizontal edges).

These levels correspond to the terms of the chain complex \( C_{\bullet}(\hat{\Gamma}) \otimes_G C_{\bullet}(\hat{\Gamma}) \):

\[
\begin{align*}
Z E_{\rightarrow} &\xrightarrow{\partial_2} Z F \oplus Z V \xrightarrow{\partial_1} Z E_{\uparrow}.
\end{align*}
\]

**Remark 7.** Note that the levels in the poset \( X \) do not correspond to the natural geometrical dimension of the cells. However, for the proof of our main result it is helpful to consider \( X \) as a geometric object and with the corresponding incidence relation \( \text{inc}(\cdot, \cdot) \) in a standard geometrical sense. For example, every face from \( X \) can be represented geometrically as a square incident to two horizontal and two vertical edges, and to four vertices. We also use the following notations: if \( x \in X \) and \( S, T \subseteq X \), then

\[
S_x := \{ y \in S \mid \text{inc}(x, y) \},
\quad S_T := \{ y \in S \mid \text{inc}(x, y) \text{ for some } x \in S \},
\]

i.e. \( S_x \) is the subset of the elements from \( X \) incident to \( x \), \( S_T \) is the the subset of the elements from \( S \) incident to some element from \( T \). For example, \( X_v = \{ v \} \cup E_v \cup F_v \) is the set of all cells incident to \( v \), and called the **star** of \( v \), where \( E_v \) (resp. \( F_v \)) is the set of edges (resp. faces) incident to \( v \).

For the proof of our main result we will also need the 1-skeleton of \( X \) defined as the graph \( \Lambda = \Lambda(X) \) with \( V(\Lambda) := V, E(\Lambda) := E, \) and \( v \leftrightarrow_e v' \iff \text{inc}(v, e) \text{ and inc}(v', e) \). Denote by \( \Lambda^2 \) the graph with the same set of vertices \( V \) as \( \Lambda \), where two vertices \( v, v' \in V \) are connected whenever there exists a path of length 2 from \( v \) to \( v' \) in the graph \( \Lambda \).

### 1.8 Local systems

In this subsection, we consider a generalization of based chain complexes with coefficients from some field or ring to the complexes with **local system of coefficients**, where the chains are formal linear sums of cells with coefficients in arbitrary abelian groups. In fact, in this work, we are interested in the case when all these abelian groups are vector spaces over the same finite field \( \mathbb{F}_q \), and thus the corresponding chain complexes can be still considered as complexes of vector spaces over \( \mathbb{F}_q \).

\(^{18}\)In fact, it can be viewed as the 2-dimensional CW-complex obtained as the balanced product of a topological realization of the graph \( \hat{\Gamma} \) by itself. The **balanced product** of two topological spaces \( X \) and \( Y \) with a group \( G \) acting on the right on \( X \) and on the left on \( Y \) is the quotient space \( X \times_G Y := X \times Y / \sim \), where \((xg, y) \sim (x, gy)\) for \( x \in X, y \in Y, g \in G \).
some way, a complex with local coefficients gives us a high-level view of the corresponding complex over \( \mathbb{F}_q \).

Let \( X \) be some finite set, which we are going to use as an index set. If a vector space \( \mathcal{C} \) is the direct sum \( \bigoplus_{x \in X} \mathcal{F}_x \) of a collection of vector spaces \( \mathcal{F} = (\mathcal{F}_x)_{x \in X} \), then we can consider the elements of \( \mathcal{C} \) as formal sums \( \sum_{x \in X} a_x x \) of elements from \( X \), where for every \( x \in X \) the coefficient \( a_x \) is from the vector space \( \mathcal{F}_x \) called the local coefficient space of \( x \). In such cases, we also denote the vector space \( \mathcal{C} \) by \( F \mathcal{X} \) or by \( AX \) when all the local coefficient spaces are equal to the same space \( A \). If each local coefficient space \( \mathcal{F}_x \) comes with a distinguished basis \( \tilde{\mathcal{F}}_x \), then we assume that the distinguished basis for \( \mathcal{F} \mathcal{X} \) is the set \( \{ax \mid a \in \tilde{\mathcal{F}}_x, x \in X\} \), in which case we say that \( \mathcal{F} \mathcal{X} \) is based.

**Definition.** Given a poset \( X \) we say that \( \mathcal{F} \) is a local system of coefficients for \( X \) if to each \( x \in X \) we assign a vector space \( \mathcal{F}_x \), and to each \( x, x' \in X \) where \( x \geq x' \) we assign an \( \mathbb{F}_q \)-linear map \( \mathcal{F}_{x \to x'} : \mathcal{F}_x \to \mathcal{F}_{x'} \) such that whenever \( x \geq x' \geq x'' \) we have:

\[
\mathcal{F}_{x' \to x''} \circ \mathcal{F}_{x \to x'} = \mathcal{F}_{x \to x''}.
\]

**Remark 8.** Note that in the language of category theory we can view \( \mathcal{F} \) as a functor from a poset \( X \) to the category of vector spaces over \( \mathbb{F}_q \). Here we consider the poset \( X \) as a small category, where the object are the elements of \( X \), and we have an arrow \( x \to x' \) whenever \( x \geq x' \).

Given an incidence chain complex \( I \) with some local system \( \mathcal{F} \) on its cell poset \( X := \hat{I} \), we can consider the chain complex \( \mathcal{C}_\bullet(I; \mathcal{F}) \) as the vector space \( \mathcal{F} \mathcal{X} \) over \( \mathbb{F}_q \) with the boundary map \( \partial : \mathcal{F} \mathcal{X} \to \mathcal{F} \mathcal{X} \) defined on the elements \( ax \in \mathcal{F} \mathcal{X} \), where \( a \in \mathcal{F}_x, x \in X \), as follows:

\[
\partial(ax) := \sum_{x \geq x'} \sum_{x' \in X} [x : x'] \mathcal{F}_{x \to x'}(a)x',
\]

and extended to all formal sums \( \sum_{x \in X} a_x x \) by linearity. It is easy to prove that \( \partial^2 = 0 \). Indeed, it is enough to check that

\[
\partial^2(ax) := \partial \sum_{x \geq x'} [x : x'] \mathcal{F}_{x \to x'}(a)x' = \sum_{x \geq x'} [x : x'] \sum_{x'' \in X} [x' : x''] \mathcal{F}_{x \to x''}(a)x'' = 0,
\]

where the last step follows from (6). Note that if \( \mathcal{F} \mathcal{X} \) is based, then the chain complex \( \mathcal{C}_\bullet(I; \mathcal{F}) \) is also based.

**Remark 9.** With some small abuse of notation, we usually denote the complex \( \mathcal{C}_\bullet(I; \mathcal{F}) \) by \( \mathcal{C}_\bullet(X; \mathcal{F}) \), in which case we always assume that the cell poset \( X \) comes with the corresponding incidence complex \( I \), i.e., for every two elements \( x \geq x' \) from \( X \) their incidence number \( [x : x'] \in \{-1, +1\} \) is defined (cf. abstract cell complex from [44, Section 2.12]). In fact, in the case of complexes over the fields of characteristic 2, we can always assume that \( [x : x'] = 1 \) if \( x \geq x' \), and \( [x : x'] = 0 \) otherwise. Hence, in such cases, the poset \( X \) completely defines the corresponding incidence complex \( I \) by (5).

Consider a based chain complex \( \mathcal{C} = \mathcal{C}_\bullet(X; \mathcal{F}) \) over \( \mathbb{F}_q \). Let \( a = \sum_{x \in X} a_x x \in \mathcal{C} \), where each coefficient \( a_x \) is from the based vector space \( \mathcal{F}_x \) over \( \mathbb{F}_q \). We denote by \( \text{wt}(a) \) the standard Hamming weight of \( a \), considered as a vector over \( \mathbb{F}_q \). We also consider the block weight \( \text{wt}_X(a) \) defined as the number non-zero blocks in \( a \), viewed as a block vector \( (a_x)_{x \in X} \), i.e. we have

\[
\text{wt}_X(a) := \text{card}\{x \in X \mid a_x \neq 0\}.
\]
Sometimes we need to take into account only the blocks that correspond to some subset \( S \subseteq X \). In this case, we can define the block weight \( \text{wt}_S(a) := \operatorname{card}\{x \in S \mid a_x \neq 0\} \) relative to the subset \( S \subseteq X \). We also define \( \operatorname{supp}a := \{x \in X \mid a_x \neq 0\} \) and \( x|_S := \sum_{x \in S} a_x \), where \( a = \sum_{x \in X} a_x \).

Let \( \partial : \mathcal{F}X \to \mathcal{F}X \) be the boundary map of \( \mathcal{C} \). In some cases, we want to restrict its domain and codomain of \( \partial \). For every \( S, T \subseteq X \) we consider the map \( \partial_{S \to T} : \mathcal{F}S \to \mathcal{F}T \) defined as \( a \mapsto (\partial a)|_T \). From the definition it is clear that for every \( a \in \mathcal{F}X \) we have:

\[
(\partial(a|_S))|_T = \partial_{S \to T}(a|_S).
\]

As we already mentioned, local systems can be used to obtain a high-level view of a chain complex over \( \mathbb{F}_q \). For example, we can represent a Tanner complex

\[
\mathbb{T}(\Gamma, (\partial(v))_{v \in \mathcal{V}}) = \left( \mathbb{F}_q \mathcal{E} \xrightarrow{\partial_i} \mathbb{F}_q \mathcal{V} \right)
\]

for a graph \( \Gamma \) (considered as a 2-level poset) as the complex \( \mathcal{C}(\Gamma; \mathcal{F}) \), where for every \( v \in \mathcal{V} \) we have \( \mathcal{F}_v := \mathbb{F}_q^r \), for every \( e \in \mathcal{E} \) we have \( \mathcal{F}_e := \mathbb{F}_q^r \), and if \( e \) is incident to \( v \) then \( \mathcal{F}_{e \to v} := \partial(v)|_{\mathcal{F}_e} \). In the next subsection, we show that the \( G \)-lifted product of two \( G \)-lifted Tanner complexes can also be represented as a complex with a local system on the poset \( \hat{\Gamma} \times_G \hat{\Gamma}^* \) from Subsection 1.7.

### 2 Proof of the main results

#### 2.1 Local minimality

One of the key ideas used in the proof of our main result is the idea of local minimality. It was used previously in the context of cohomology of simplicial complexes with \( \mathbb{F}_2 \)-coefficients \([20, 26]\). In the current work, we extend this idea to a much more general context of (co)homology of abstract cell complexes with local systems of coefficients. As we mentioned before, by an abstract cell complex we mean a poset \( X \) with a map \( \partial : ZX \to ZX \) such that \( ZX \) is an incidence complex with the boundary map \( \partial \), and \( X \) is its cell poset.

Consider an abstract cell complex \( X \) and a based chain complex \( \mathcal{C} = \mathcal{C}(X; \mathcal{F}) \) of vector spaces

\[
\cdots \xrightarrow{\partial_{i+1}} \mathcal{C}_i \xrightarrow{\partial_i} \mathcal{C}_{i-1} \xrightarrow{\partial_{i-1}} \cdots
\]

over \( \mathbb{F}_q \), where \( \mathcal{F} \) is a local system on \( X \). Denote by \( | \cdot | \) the block weight \( \text{wt}_{X}(\cdot) \), which makes each term \( \mathcal{C}_i \) in this complex a normed abelian group (see Appendix C) with the norm \( | \cdot | \) and allows us to define the distance in the standard way: \( d(a, b) := |a - b| \), \( d(a, B) := \min_{b \in B} |a - b| \).

We also use \( | \cdot | \) to define for every \( i \in \mathbb{Z} \) the corresponding quotient norm on the \( i \)-th homology group \( H_i(\mathcal{C}) := Z_i(\mathcal{C})/B_i(\mathcal{C}) \) called the systolic norm by the formula \( |A| := \min_{a \in A} |a| \), where \( A \in H_i(\mathcal{C}) \), \( B_i(\mathcal{C}) = \text{im} \partial_{i+1} \), \( Z_i(\mathcal{C}) = \ker \partial_i \). This in turn allows us to define the distance on \( H_i(\mathcal{C}) \) as \( d(A, B) := |A - B| \) and consider the minimal distance of \( H_i(\mathcal{C}) \) given by the standard formulas:

\[
d(H_i(\mathcal{C})) := \min_{A \neq B} d(A, B) = \min_{A \in H_i(\mathcal{C}) \setminus \{B_i(\mathcal{C})\}} |A| = \min_{a \in Z_i(\mathcal{C}) \setminus B_i(\mathcal{C})} |a|.
\]

Note that the minimal distance of \( H_i(\mathcal{C}) \) is also called the \( i \)-systolic distance of \( \mathcal{C} \), while the distance \( d(H_i(\mathcal{C}^*)) \) of the dual chain complex \( \mathcal{C}^* \) is called its \( i \)-cosystolic distance. These distances are related
to the minimal distance \(d(Q)\) of the quantum CSS code \(Q = \mathcal{Q}(\partial_i, \partial_{i+1}^*)\) over \(\mathbb{F}_q\) defined by three consecutive terms of the complex
\[
\mathcal{C}_{i+1} \xrightarrow{\partial_{i+1}} \mathcal{C}_i \xrightarrow{\partial_i} \mathcal{C}_{i-1}.
\]
It is easy to see that \(d(Q) \geq \min(d(H_i(C), d(H_i(C^*)))\), where we have the equality if \(\mathcal{F}_x = \mathbb{F}_q\) for all \(x \in X\) since the block Hamming weight \(\text{wt}_X(\cdot)\) is less than or equal than the corresponding Hamming weight \(\text{wt}(\cdot)\).

**Definition.** We say that an \(i\)-chain \(c \in \mathcal{C}_i, i \in \mathbb{Z}\), is **locally minimal** (with respect to \(X\)) if \(|c + \partial ax| < |c|\) for all \(x \in X(i + 1)\) and \(a \in \mathcal{F}_x\). We also define the value
\[
d^{(i)}_{\text{LM}}(C) := \min\{|c| \mid c \in Z_i(C) \setminus \{0\}, c \text{ is locally minimal}\},
\]
which we call the \(i\)-th **locally-minimal distance** of \(C\). If we do not have non-zero locally minimal \(i\)-cycles, then we assume that \(d^{(i)}_{\text{LM}}(C) = \infty\).

The next lemma connects the locally minimal distance of the complex to the properties of the corresponding quantum and classical codes obtained from it. The first assertion can be used to obtain the lower bound on the minimal distance \(d(Q)\) of the corresponding quantum CSS code \(Q\), while the second one can be used to show that the space \(Z_{i+1}(C)\) is a locally testable code.

**Lemma 1.** Let \(C = C_\bullet(X; \mathcal{F})\) be a chain complex, where \(\mathcal{F}\) is a local system on \(X\). Then for every \(i \in \mathbb{Z}\) we have
\[
d(H_i(C)) \geq d^{(i)}_{\text{LM}}(C),
\]
and for every chain \(c \in \mathcal{C}_{i+1}\) such that \(|\partial c| < d^{(i)}_{\text{LM}}(C)\) we have
\[
|\partial c| \geq d(c, Z_{i+1}(C)). \tag{8}
\]

**Proof.** By definition
\[
d(H_i(C)) = |c_0| \quad \text{where} \quad c_0 := \arg\min_{c \in Z_i(C) \setminus B_i(C)} |c|.
\]
Since the element \(c_0\) has the minimal norm in the coset \(c_0 + B_i(C)\), it is also locally minimal. Hence we have \(d(H_i(C)) = |c_0| \geq d^{(i)}_{\text{LM}}(C)\).

We prove the second claim by induction on \(|\partial c|\). If \(|\partial c| = 0\) then \(d(c, Z_{i+1}(C)) = 0\), and (8) is true. Consider \(c \in \mathcal{C}_{i+1}\) such that \(0 < |\partial c| < d^{(i)}_{\text{LM}}(C)\). Since \(\partial c \in Z_i(C)\) and \(|\partial c| < d^{(i)}_{\text{LM}}(C)\), we see that \(\partial c\) cannot be locally minimal, and hence there exists \(a \in \mathcal{F}_x\) where \(x \in X(i + 1)\) such that \(|\partial(c + ax)| \leq |\partial c| - 1\). Therefore by the induction hypothesis we have \(|\partial(c + ax)| \geq d(c + ax, Z_{i+1}(C))\). Thus we obtain
\[
d(c, Z_{i+1}(C)) \leq d(c + ax, Z_{i+1}(C)) + |ax| \geq |\partial(c + ax)| + |ax| \leq |\partial c|,
\]
which completes the proof of the second claim. \(\square\)
2.2 Graph expansion

In this section, we prove several technical lemmas to establish expanding properties of the graphs \( \Lambda \) and \( \Lambda^2 \) from Subsection 1.7. If \( \Gamma = (V, E) \) is a simple graph, and \( S, T \subseteq E \), then by \( E_\Gamma(S, T) \), we denote the set of oriented edges connecting \( S \) and \( T \), i.e. \( E_\Gamma(S, T) := \{(s, t) \mid \{s, t\} \in E; s \in S, t \in T\} \). We also usually write \( E(S, T) \) and \( E(S) \) instead of \( E_\Gamma(S, T) \) and \( E_\Gamma(S) \) if the graph \( \Gamma \) is clear from the context.

**Definition.** We say that a graph \( \Gamma \) is an \((n, w, \lambda)\)-expander if it is a simple \( w \)-regular graph on \( n \) vertices such that \( \lambda = \lambda(G) \).

Let us now state without proof a well-known variant of the expander mixing lemma for \((n, w, \lambda)\)-regular graphs [35, Lemma 2.5].

**Lemma 2** (Expanding mixing lemma). If \( \Gamma = (V, E) \) is an \((n, w, \lambda)\)-expander graph, then for every \( S, T \subseteq V \) we have:

\[
\left| E(S, T) - w \frac{|S||T|}{n} \right| \leq \lambda \sqrt{|S||T|}.
\]

In what follows, it will be convenient to define a property of \((a, \lambda)\)-edge-expansion of a graph, which captures the edge expansion on small sets of vertices.

**Definition.** We say that a graph \( \Gamma \) is \((a, \lambda)\)-edge-expanding if for any \( S, T \subseteq V(\Gamma) \) such that \(|S|, |T| \leq a\) the following condition holds:

\[
|E(S, T)| \leq \lambda \sqrt{|S||T|}.
\]

**Lemma 3.** If \( \Gamma \) is an \((n, w, \lambda)\)-expander graph, then it is \((\lambda n/w, 2\lambda)\)-edge-expanding.

**Proof.** If \( \Gamma \) is a \( w \)-regular, then from Lemma 2 it follows that for any \( S, T \subseteq V(\Gamma) \) such that \(|S|, |T| \leq \lambda n/w \) we have \( \left| E(S, T) - w \frac{|S||T|}{n} \right| \leq \lambda \sqrt{|S||T|} \). Hence we have:

\[
|E(S, T)| \leq w \frac{|S||T|}{n} + \lambda \sqrt{|S||T|} \leq \left( \frac{\lambda n}{w} \cdot \frac{w}{n} + \lambda \right) \sqrt{|S||T|} = 2\lambda \sqrt{|S||T|},
\]

and the Lemma is proved.

**Lemma 4.** If \( \hat{\Gamma} \) is a \( G \)-lift of an \((a, \lambda)\)-edge-expanding base graph \( \Gamma \), then \( \hat{\Gamma} \) is \((a, |G|\cdot \lambda)\)-edge-expanding.

**Proof.** Consider subsets \( \hat{S}, \hat{T} \subseteq V(\hat{\Gamma}) \) such that \( |\hat{S}|, |\hat{T}| \leq a \), and let \( S, T \subseteq V(\Gamma) \) be their projections\(^{19}\) to the base graph \( \Gamma \). Since each edge of \( \Gamma \) is the projection of \( m = |G| \) edges from \( \hat{\Gamma} \), then using the edge-expansion of the base graph \( \Gamma \) we have:

\[
|E(\hat{S}, \hat{T})| \leq m |E(S, T)| \leq m \lambda \sqrt{|S||T|} \leq m \lambda \sqrt{|\hat{S}||\hat{T}|}.
\]

**Lemma 5.** Every graph \( \bar{X}^{w-1,t} \) from Example 1 is \((n/\sqrt{w}, 8\sqrt{w})\)-edge-expanding, where \( n = t(t^2 - 1) \) is the number of its vertices.

\(^{19}\)The projection of a vertex \((v, g) \in V(\hat{\Gamma}) \) is the vertex \( v \in V(\Gamma) \).
Proof. Since the Ramanujan graph $X^{w-1,t}$ is an $(n/2, w, 2\sqrt{w})$-graph, then by Lemma 3 it is $(n/\sqrt{w}, 4\sqrt{w})$-edge-expanding. Moreover, since the graph $X^{w-1,t}$ is a 2-lift of $X^{w-1,t}$, then by Lemma 4 it is $(n/\sqrt{w}, 8\sqrt{w})$-edge-expanding. □

Remark 10. In what follows, we are going to use the following properties of $(a, \lambda)$-edge-expansion, which are easy to prove.

1. If $a' \leq a$, $\lambda' \geq \lambda$, and the graph $\Gamma$ is $(a, \lambda)$-edge-expanding, then $\Gamma$ is $(a', \lambda')$-edge-expanding.

2. If a graph $\Gamma = (V, E)$ is $(a, \lambda)$-edge-expanding, and $\Gamma' = (V, E')$ is a subgraph of $\Gamma$ (i.e. $E' \subseteq E$), then $\Gamma'$ is also $(a, \lambda)$-edge-expanding.

3. If graphs $\Gamma_1, \ldots, \Gamma_m$ are $(a, \lambda)$-edge-expanding, then their disjoint union $\Gamma = \Gamma_1 \sqcup \cdots \sqcup \Gamma_m$ is also $(a, \lambda)$-edge-expanding.

4. If graphs $\Gamma_1, \ldots, \Gamma_m$ have the same set of vertices, and $\Gamma_i$ is $(a_i, \lambda_i)$-edge-expanding, then their union $\Gamma = \Gamma_1 \sqcup \cdots \sqcup \Gamma_m$ is $(\min_{i \in [m]} a_i, \sum_{i=1}^m \lambda_i)$-edge-expanding.

Lemma 6. Let $x_1, \ldots, x_n \in \mathbb{R}_+$, $y_1, \ldots, y_n \in \mathbb{R}_+$ be sequences of non-negative real numbers. Then

$$\min_{i \in [n]} x_i y_i \leq \bar{x} \bar{y},$$

where $\bar{x} = \frac{1}{n} \sum_{i=1}^n x_i$, $\bar{y} = \frac{1}{n} \sum_{i=1}^n y_i$.

Proof. By the Cauchy–Schwarz inequality for vectors $(\sqrt{x_i})_{i=1}^n$ and $(\sqrt{y_i})_{i=1}^n$ we have

$$\sum_{i=1}^n \sqrt{x_i} y_i \leq \left( \sum_{i=1}^n x_i \right)^{1/2} \cdot \left( \sum_{i=1}^n y_i \right)^{1/2} = n \sqrt{\bar{x} \bar{y}},$$

therefore $\min_{i \in [n]} \sqrt{x_i} y_i \leq \sqrt{\bar{x} \bar{y}}$, hence

$$\min_{i \in [n]} x_i y_i = \left( \min_{i \in [n]} \sqrt{x_i} y_i \right)^2 \leq \bar{x} \bar{y}.$$ □

Lemma 7. If a $w$-regular graph $\Gamma$ is $(a, \lambda)$-edge-expanding, then the graph $\Gamma^2$ is $(a/w, 2\lambda^2(1+\ln w))$-edge-expanding.

Proof. Let $S, T \subseteq V(\Gamma)$ and $|S|, |T| \leq a/w$. There are at most $w|S|$ vertices adjacent to the vertices from $S$. Let $v_1, v_2, \ldots$ be the sequence of the vertices incident to $S$ in the decreasing order of the number of length 2 paths from $S$ to $T$ that goes through each of these vertices. Consider the set $U_j = \{v_1, \ldots, v_j\}$ of size $j \leq a$. By the edge-expansion property of the graph $\Gamma$ we have

$$|E_\Gamma(U_j, S)| \leq \lambda \sqrt{\lambda |S|}, \quad |E_\Gamma(U_j, T)| \leq \lambda \sqrt{\lambda |T|}.$$

Hence, using Lemma 6 with $n = j$, $x_i = |E_\Gamma(\{v_i\}, S)|$, $y_i = |E_\Gamma(\{v_i\}, T)|$ the number of paths through the vertex $v_j$ is

$$x_j y_j = \min_{i \in [j]} x_i y_i \leq \frac{|E_\Gamma(U_j, S)|}{j} \cdot \frac{|E_\Gamma(U_j, T)|}{j} \leq \lambda^2 \sqrt{|S| |T|}. $$

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On the other hand, the degree of each vertex is $w$, and hence the total number of pairs of edges incident to each vertex is $w^2$. Hence, if we let $\mu := \lambda^2 \sqrt{|S||T|}$, then the total number of paths from $S$ to $T$ can be estimated as

$$|E_{\Gamma^2}(S, T)| \leq \sum_{j=1}^{|\mu|} \min\left( w^2, \mu/j \right) = w^2 \cdot \frac{\mu}{w^2} + \mu \sum_{j=\lceil \mu/w^2 \rceil}^{1} \frac{1}{j} < \mu(2 + \ln \mu - \ln(\mu/w^2)) = 2\mu(1 + \ln w).$$

Thus there are at most $2\lambda^2(1+\ln \mu)\sqrt{|S||T|}$ edges from $S$ to $T$ in $\Gamma^2$, and therefore $\Gamma^2$ is $(a/w, 2\lambda^2(1+\ln w))$-edge-expanding.

\[ \square \]

2.3 Proof outline

In this subsection, we give some definitions and an informal idea of the proof of our main results. Let $\hat{\Gamma} = (\hat{E}, \hat{V})$ be a $G$-lift of some base graph $\Gamma$. We assume that $\hat{\Gamma}$ is a $w$-regular $(a, \lambda)$-expanding graph with $n$ vertices. For example, we can use an infinite family of graphs $\bar{X}^{w-1,t}$ from Example 1, where by Lemma 5 we have $a = n/\sqrt{w}$, $\lambda = 8\sqrt{w}$.

Let $h \in \mathbb{F}_q^{r \times w}$, $h' \in \mathbb{F}_q^{r' \times w}$ be some full rank matrices, and $A \in \mathcal{S}_G(\hat{\Gamma}, h)$, $B \in \mathcal{S}_G(\hat{\Gamma}, h')$ be the corresponding $G$-lifted Tanner complexes:

$$A = \left( \mathbb{F}_q \hat{E} \otimes \mathbb{F}_q \hat{V} \right), \quad B = \left( \mathbb{F}_q \hat{E} \otimes \mathbb{F}_q \hat{V} \right).$$

Now since $G$ acts freely on $A$ and $B$, we can consider their $G$-lifted product complex $A \otimes_G B^*$ over $\mathbb{F}_q$ shown below:

\begin{equation*}
\begin{array}{c}
\mathbb{F}_q \hat{E} \otimes_G \mathbb{F}_q \hat{V} \quad \partial_2 \rightarrow \quad \mathbb{F}_q \hat{E} \otimes_G \mathbb{F}_q \hat{E} \oplus \mathbb{F}_q \hat{V} \otimes_G \mathbb{F}_q \hat{V} \quad \partial_1 \rightarrow \quad \mathbb{F}_q \hat{V} \otimes_G \mathbb{F}_2 \hat{E}.
\end{array}
\end{equation*}

It is convenient to represent $\mathcal{C}$ as the the chain complex $\mathcal{C} \bullet(X, \mathcal{F})$, where $\mathcal{F}$ is the local system on $X = \hat{\Gamma} \times_G \hat{\Gamma}^*$. Since the poset $X$ has three levels: $X(2) = E_\rightarrow$, $X(1) = F \cup V$, and $X(0) = E_\uparrow$, it is not hard to see that $\mathcal{C} \bullet(X, \mathcal{F})$ has the following form:

\begin{equation*}
\begin{array}{c}
\mathbb{F}_q \hat{E}_\rightarrow \quad \partial_2 \rightarrow \quad \mathbb{F}_q \hat{E} \oplus \mathbb{F}_q^{r \times r} \hat{V} \quad \partial_1 \rightarrow \quad \mathbb{F}_q \hat{E}_\uparrow,
\end{array}
\end{equation*}

where we identify $\mathbb{F}_q^r \otimes \mathbb{F}_q^{r'}$ with $\mathbb{F}_q^{r \times r'}$.

Remark 11. As we can see, $\mathcal{C}(X; \mathcal{F})$ gives us a high-level representation of the complex $\mathcal{C}$. For example, on the left of Fig. 3 you can find a graphical representation of the tensor product complex $A \otimes G B^*$, where $A \in \mathcal{S}(D_w, h)$, $B \in \mathcal{S}(D_w, h')$, and $w = 8$. For simplicity we consider in this example the tensor product instead of the $G$-lifted product. On the right of Fig. 3 you can see the “part” of this complex that corresponds to the faces and edges incident to one particular vertex $v \in V$.
Now we consider the classical code $Z_2(C) = \ker \partial_2$ and the quantum code $Q(C) := Q(\partial_1, \partial_2^*)$, and show that for some sufficiently large number $w$ we can choose the matrices $h, h'$ such that $Z_2(C)$ and $Q(C)$ satisfy the requirements of Theorems 1 and 2, respectively. The most difficult part of the proof is to show that $Z_2(C)$ is locally testable, and $Q(C)$ has linear minimum distance. However, from Lemma 1 it easily follows that if $C$ has the locally-minimal distance $d_{LM}^{(1)}(C) = \Theta(n)$ as $n \to \infty$, then both $Z_2(C)$ and $Q(C)$ have the desired properties. Therefore we need to show that for every 1-cycle $c \in Z_1(C)$ such that $|c| = o(n)$ as $n \to \infty$ we have $c = 0$, where $|c| = \text{wt}_X(c)$ is the block weight of $c$.

Let us fix some non-zero locally minimal 1-cycle $c = \sum_{x \in X(1)} c_x x \in Z_1(C)$. Hence we have $c \neq 0$ and $\partial c = 0$. Suppose $c = c_F + c_V$, where $c_F := c|_F$ and $c_V := c|_V$. Below we give a number of important definitions used in the rest of the paper. Note that some of them depend on the fixed 1-cycle $c$. However, for brevity, we usually do not mention $c$.

**Definition.** An element $x \in X(1)$ (a vertex or a face) is called active if $c_x \neq 0$. A vertical edge $e \in E_\uparrow$ is called active if it is incident to an active vertex or an active face. Furthermore, $e$ is called face-active if it is not incident to any active vertex (only to an active face).

We also need another type of vertices we call labeled that include active vertices as a special case. However, the number of labeled vertices is $O(|c|)$, and we can still use the expansion properties of the graphs involved in the proof. We define the set of labeled vertices as the minimal set of vertices such that:

1. every active vertex is labeled;
2. every vertex of a face-active edge adjacent to at least $m$ labeled vertices is labeled.

We also consider 2 types of labeled vertices:

1. a vertex is called $m$-edge-expanding if it is adjacent to at least $m$ labeled vertices in $\Lambda$;
2. a vertex is called \textit{s-face-expanding} if it is adjacent to at least \( s \) labeled vertices in \( \Lambda^2 \).

Sometimes, when the parameters \( m \) and \( s \) are clear from context we omit them and just say that a vertex is edge-expanding or face-expanding.

In the proof outlined below, we consider classical codes that are duals of the product codes. In Subsection 2.4 we define a special property of such codes called \((s, m, \beta)-expansion\). Informally speaking, this property corresponds to the local expansion in the complex \( \mathcal{C} \). In some sense, it plays a role similar to the role of the minimal distance of the local codes in the classical Sipser-Spielman proof from \cite{23} that expander codes have linear minimum distances.

Fix \( \varepsilon := 1/6 \), and put \( m := w^{1/2+\varepsilon} \), \( s := w^{1+\varepsilon} \). From Lemma 10 it follows that we can find a sufficiently large number \( w \) and choose matrices \( h \) and \( h' \) such that both pairs \((\text{im} h^*, \ker h')\) and \((\ker h, \text{im} h'^*)\) are \((s, 2m, \beta)\)-product-expanding.

In the proof, we often use expansion properties of the graphs \( \Lambda \) and \( \Lambda^2 \) defined in Subsection 1.7. Using the edge expansion of \( \hat{\Gamma} \) we show in Lemma 11 that \( \Lambda \) is \((\Theta(n), \lambda')\)-edge-expanding where \( \lambda' = \Theta(w^{1/2}) \). We also show in Lemma 12 that \( \Lambda^2 \) is \((\Theta(n), \lambda'')\)-edge-expanding, where \( \lambda'' = \Theta(w \ln w) \).

Suppose that \(|c| = o(n)\), i.e. the number of active vertices and faces is relatively small. Then the proof by contradiction is as follows:

1. since each labeled vertex is either active itself or incident to an active face, then the number of labeled vertices is \( O(|c|) = o(n) \), hence we can use the expansion properties of the graphs \( \Lambda \) and \( \Lambda^2 \) for subsets of labeled vertices;

2. using the expansion properties of the graph \( \hat{\Gamma} \) it is possible to show that each face-active edge is incident to a labeled vertex (Lemma 14);

3. note that, by definition, each labeled non-active vertex is \( m \)-edge-expanding;

4. using local minimality of \( c \) and \((s, 2m, \beta)\)-product-expansion of \((\text{im} h, \ker h')\) we can show that each active vertex is either \( m \)-edge-expanding or \( s \)-face-expanding (Lemma 15—the key lemma);

5. from the previous 2 items we have that each labeled vertex is either edge-expanding or face-expanding (Corollary 1);

6. thus using the expansion properties of \( \Lambda \) and \( \Lambda^2 \) we obtain a contradiction (Lemma 16):

   (a) from the \((\Theta(n), \lambda')\)-edge expansion of \( \Lambda \) we obtain that the ratio of the \( m \)-edge-expanding labeled vertices is \( \Theta(\lambda'/m) < 1/2 \) for a sufficiently large \( w \) since \( \lambda' = \Theta(w^{1/2}) \) and \( m = \Theta(w^{1/2+\varepsilon}) \);

   (b) from the \((\Theta(n), \lambda'')\)-edge expansion of \( \Lambda^2 \) we obtain that the ratio of the \( s \)-face-expanding labeled vertices is \( \Theta(\lambda''/s) < 1/2 \) for a sufficiently large \( w \) since \( \lambda'' = \Theta(w \ln w) \) and \( s = \Theta(w^{1+\varepsilon}) \);

   (c) the ratio of the labeled vertices that are either \( m \)-edge-expanding or \( s \)-face expanding is less than 1, which can be true only when the 1-cycle \( c \) is zero.

Since we obtained a contradiction, we have that \(|c| = \Theta(n)\), i.e., the locally minimal distance \( d_{LM}^{(1)}(C) = \Theta(n) \) as \( n \to \infty \), which is in turn of the same order as the length of the classical or quantum codes obtained from the chain complex \( \mathcal{C} \). Hence by Lemma 1 we get what we need.
2.4 Local expansion

In this section, we consider the dual code to the classical product code [46, 47] and study its expansion properties. Such codes are related to the local expansion properties of the $G$-lifted product of two Tanner complexes. Let $\mathcal{C}(h) \subseteq \mathbb{F}_q^w$ and $\mathcal{C}(h') \subseteq \mathbb{F}_q^w$ be linear codes with parity-check matrices $h$ and $h'$ respectively. Consider the code $\mathcal{C} = \ker(h \otimes h') \subseteq \mathbb{F}_q^w \otimes \mathbb{F}_q^w$. We will identify the elements $\sum_{i,j \in [w]} x_{ij} e_i \otimes e_j$ of $\mathbb{F}_q^w \otimes \mathbb{F}_q^w$ with the corresponding matrices $x = (x_{ij})_{i,j=1}^w \in \mathbb{F}_q^{w \times w}$. Note that the matrix $h \otimes h'$ is also a generator matrix for the product of the codes $\mathcal{C}^\perp(h)$ and $\mathcal{C}^\perp(h')$ with the generator matrices $h$, $h'$ respectively, which means $\mathcal{C}$ is the dual to this product code.

Remark 12. Using matrix representation, it is not hard to check that the codewords of $\mathcal{C}$ are precisely the matrices $x \in \mathbb{F}_q^{w \times w}$ such that $hxh^* = 0$. Therefore if $x \in \mathcal{C}$ then every row of the matrix $s_\rightarrow := hx$ is a codeword from $\mathcal{C}(h') = \ker h'$ and every column of the matrix $s_\uparrow := xh^*$ is a codeword from $\mathcal{C}(h) = \ker h$ (see Fig. 4).

Definition. A codeword $x \in \mathcal{C} = \ker(h \otimes h')$ is called $\Delta$-minimal if the following conditions hold:

1. $\text{wt}(x_i) \leq d(x_i, \mathcal{C}(h)) + \Delta$ for all $i \in [w]$,
2. $\text{wt}(x_j) \leq d(x_j, \mathcal{C}(h')) + \Delta$ for all $j \in [w]$,

which means we cannot decrease the weight of the matrix $x$ by more than $\Delta$ if we add any codeword from $\mathcal{C}(h)$ (resp. $\mathcal{C}(h')$) to some column (resp. row) of $x$. A pair of codes $(\mathcal{C}(h), \mathcal{C}(h'))$ is called $(s, m, \beta)$-product-expanding if for each $\beta$-minimal codeword $x \in \mathcal{C}$ and for each $A, B \subseteq [w]$ such that $|A|, |B| \geq w - m$ we have $\text{wt}_{A \times B}(x) \geq s$, where $\text{wt}_{A \times B}(x) := \text{wt}(x|_{A \times B})$.

In this section, we often use the following short-hand notations: $x^I := x|_{I \times [w]}$, $x^J := x|_{[w] \times J}$, and $x^I_J := x|_{I \times J}$, where $x \in \mathbb{F}_q^{w \times w}$, $I, J \subseteq [w]$.

Lemma 8. Let $h \in \mathbb{F}_q^{r \times w}$, $h' \in \mathbb{F}_q^{r' \times w}$ be parity-check matrices such that $d(\ker h), d(\ker h') \geq d$, and $x = (x_{ij})_{i,j=1}^w \in \mathbb{F}_q^{w \times w}$ be a $d/3$-minimal codeword of $\ker(h \otimes h')$. If there exist $A, B \subseteq [w]$ such that $|A| > w - d/3$, $|B| \geq w - d + 1$ and $x^B_A = 0$ or $x^A_B = 0$, then $x = 0$.

![Figure 4: Idea of the proof.](image-url)
Proof. Suppose that $x^B_A = 0$ for some $A, B \subseteq [w]$ such that $|A| > w - d/3, |B| \geq w - d + 1$ (shown on the left of Fig. 4). Since $|A| > w - d(\ker h'), |B| > w - d(\ker h)$, there exist information$^{20}$ sets $A' \subseteq A, B' \subseteq B$ of the codes $\ker h'$ and $\ker h$ respectively. Let $g'$ be the generator matrix in systematic form$^{21}$ for the information set $A'$. Consider matrices $\delta := x_A g'$ and $x' := x - \delta$. Since $\delta_A = x_{A'}$, we have $x'_{A'} = 0$. On the other hand, $\delta h' = x_A g' h' = 0$, and hence $h' x' h' = h (x - \delta) h' = h x h' - h \delta h' = 0$. Therefore $(h x') h' = 0$, and each row of $h x'$ is a codeword from $\ker h'$. The condition $x'_{A'} = 0$ implies that $(h x')_{A'} = 0$, and since $A'$ is an information set of $\ker h'$, we get $h x' = 0$, which means that every column of $x'$ is a codeword of $\ker h$ (shown on the right of Fig. 4). Note that we also have $x^B_{A'} = 0$, and thus $x^B_{A'} = 0$. Hence $\delta^B = x^B_{A'} = 0$, and therefore $x^B = 0$. Now since $B'$ is an information set of $\ker h$, $x'_{A'} = 0$, and $h x' = 0$, we have $x'_{A'} = 0$. Suppose $\delta \neq 0$. In this case, there exists $i \in [w]$ such that $\delta^i \neq 0$. Taking into account that $\delta^i \in \ker h'$, we obtain $\text{wt}(\delta^i) \geq d$. But since $x'_{A'} = 0$, and $|A| > w - d/3$, we have $\text{wt}(x^i) \leq w - |A| < d/3$, and thus $\text{wt}(x') \geq \text{wt}(\delta) - \text{wt}(x^i) > 2d/3 > \text{wt}(x^i) + d/3$, which contradicts the $d/3$-minimality of $x$. Thus $\delta = 0$, which implies that $x' = x$ and $h x = 0$, hence $d(x_j, \ker h) = 0$ for all $j \in [w]$. By $d/3$-minimality of $x$ we have $\text{wt}(x_j) \leq d/3 < d(\ker h)$, therefore $x_j = 0$ for all $j \in [w]$, i.e. $x = 0$. Hence we showed that $x^B = 0$ implies $x = 0$. Thus to prove the lemma it remains to show that $x^B = 0$ also implies $x = 0$, which can be shown in a similar way. \qed

Lemma 9. Let $h \in \mathbb{F}_q^{x \times w}$, $h' \in \mathbb{F}_q^{r' \times w}$, $d = \min(d(\ker h), d(\ker h'))$, $m \leq d/6$. Suppose $x \in \ker(h \otimes h')$ is a $d/3$-minimal non-zero codeword such that $\text{wt}_{A \times B}(x) < s$ for some $A, B \subseteq [w]$, $|A| = |B| = w - m$. Then $\text{rk}_{\mathbb{F}_q} h x \geq \frac{5}{3m} \cdot \frac{d^2}{s}$.

Proof. By Lemma 8 each submatrix of $w - d + 1$ columns of $x$ must have at least $d/3$ nonzero rows, and each submatrix of $w - d + 1$ rows of $x$ must have at least $d/3$ nonzero columns. In particular, $x$ has at least $d$ nonzero columns and at least $d$ nonzero rows. Indeed, otherwise we would have at least $w - d + 1$ zero rows or columns, which contradicts what we said earlier.

Let $k = \text{rk}_{\mathbb{F}_q} h x$, and $\{h \tilde{x}_1, \ldots, h \tilde{x}_k\}$ be a generating set for the column space of $h x$ with the minimal total weight $\text{wt}_{A}(\tilde{x}) := \text{wt}_{A}(\tilde{x}_1) + \cdots + \text{wt}_{A}(\tilde{x}_k)$, where $\tilde{x}$ is a matrix with the columns $\tilde{x}_1, \ldots, \tilde{x}_k$. Without loss of generality we assume that $\text{wt}_{A}(\tilde{x}_1) \leq \cdots \leq \text{wt}_{A}(\tilde{x}_k)$. By $\tilde{X}$ we denote the linear span $\langle \tilde{x}_1, \ldots, \tilde{x}_k\rangle$.

Let us show that $|\bigcup_{j=1}^k \text{supp}(\tilde{x}_j)| \geq d/3$. Denote $U = \bigcup_{j=1}^k \text{supp}(\tilde{x}_j)$. Suppose $|U| < d/3$. Since $|\bigcup_{i=1}^w \text{supp}(x_i)| \geq d$, there is a column $x_i$ such that $\text{supp}(x_i) \not\subseteq U$, hence $x_i \notin \tilde{X}$. However $h x_i \in h \tilde{X}$, and hence there exists some $y \in \ker h \setminus \{0\}$ such that $x_i + y \in \tilde{X}$. Since $\text{supp}(x_i + y) \subseteq U$, we have $\text{wt}(x_i + y) < d/3$ and $\text{wt}(x_i) \geq \text{wt}(y) - \text{wt}(x_i + y) > 2d/3 > \text{wt}(x_i + y) + d/3$, which contradicts the $d/3$-minimality of $x$, and hence our assumption is wrong, and $|U| \geq d/3$.

We have $\sum_{i=1}^k \text{wt}_{A}(\tilde{x}_i) \geq \left| \bigcup_{i=1}^k (\text{supp}(\tilde{x}_i) \cap A) \right| = |U \cap A| = |U \setminus ([w] \setminus A)| \geq |U| - (w - |A|) \geq \frac{d}{3} - m \geq \frac{d}{6}$.

Let $k'$ be the minimal number such that $\sum_{j=1}^{k'} \text{wt}_{A}(\tilde{x}_j) \geq d/6$, then $\text{wt}_{A}(\tilde{x}_{k'}) \geq \frac{d}{6k'} \geq \frac{d}{6k}$. Put $U_0 = \bigcup_{j=1}^{k'-1} \text{supp}(\tilde{x}_j)$. Each column $x_i$ is uniquely represented as $x_i = y_i + \tilde{x}_j x_i$ where $y_i \in \ker h$.

$^{20}$An information set for a linear code $C \subseteq \mathbb{F}_q^n$ is a smallest by inclusion index set $I \subseteq [n]$ such that for every $c \in C$ if $|c| = 0$ then $c = 0$. It is clear that for every $S \subseteq [n]$ such that $|S| > n - d(C)$ if for some codeword $c \in C$ we have $|c| = 0$ then $c = 0$. Hence there should exist an information set $I \subseteq S$.

$^{21}$A generator matrix $g$ is in systematic form for an information set $I$ if the submatrix $g_I$ is the identity matrix.
$a_i \in \mathbb{F}_q^k$. If $\text{supp} a_i \subseteq [k' - 1]$, then

$$\text{wt}(\tilde{x}a_i) \leq m + \text{wt}_A(\tilde{x}a_i) \leq m + \sum_{j=1}^{k'-1} \text{wt}_A(\tilde{x}j) < d/3,$$

and hence $y_i = 0$, otherwise $\text{wt}(x_i) \geq d - \text{wt}(\tilde{x}a_i) > 2d/3 \geq \text{wt}(x_i + y_i) + d/3$ which contradicts the $d/3$-minimality of $x$. Therefore $\text{supp} x_i \subseteq U_0$.

Since every $w - d + 1$ columns of $x$ have at least $d/3$ nonzero rows, there are at most $w - d$ columns $x_i$ such that $\text{supp} a_i \subseteq [k' - 1]$. Hence there exists a set $C \subseteq [w]$ of size $d$ such that $\text{max}(\text{supp} a_i) \geq k'$ for all $i \in C$. Note that if $j = \text{max}(\text{supp} a_i)$, then $\text{wt}_A(x_i) \geq \text{wt}_A(\tilde{x}j)$. Indeed, otherwise we can replace $\tilde{x}j$ by $x_i$ and reduce $\text{wt}_A(\tilde{x})$, which contradicts the minimality of $\text{wt}_A(\tilde{x})$. Hence $\text{wt}(x_i) \geq \text{wt}_A(\tilde{x}k_i) \geq \frac{d}{6k}$ for all $i \in C$, and therefore

$$\text{wt}_{A \times B}(x) \geq \text{wt}_{A \times (B \cap C)}(x) = \sum_{i \in B \cap C} \text{wt}_A(x_i) \geq \frac{|B \cap C|}{6k}.$$

Since $|B \cap C| = |C \setminus ([w] \setminus B)| \geq |C| - (w - |B|) = d - m$, we have

$$k \geq \frac{d|B \cap C|}{6\text{wt}_{A \times B}(x)} \geq \frac{d(d - m)}{6s} \geq \frac{5}{36} \cdot \frac{d^2}{s},$$

and the lemma is proved. \hfill \Box

**Lemma 10.** Let $\varepsilon \in (0, 1/4)$, $\alpha > 0$, $\gamma > 0$, $R_1 \in (0, 1)$, $R_2 \in (0, 1)$. Then there exist $\beta > 0$ and $\delta > 0$ such that for random matrices $g \in \mathbb{F}_q^{[R_1 w] \times w}$, $h' \in \mathbb{F}_q^{[R_2 w] \times w}$ the following three conditions hold with high probability as $w \to \infty$:

1. $\text{min}(d(\text{im} g^*), d(\text{ker} h')) \geq \delta w$;
2. the matrices $g$ and $h'$ have full rank;
3. the pair of codes $(\text{im} g^*, \text{ker} h')$ is $(\alpha w^{1+\varepsilon}, \gamma w^{1/2+\varepsilon}, \beta)$-product-expanding.

**Proof.** Let us start the proof by saying that the first two conditions follows from the probabilistic proof of the asymptotic Gilbert–Varshamov bound\textsuperscript{22}. Indeed, it is enough to choose $\delta \leq (q - 1)/q$ such that $H_q(\delta) = \min((1 - R_1)/2, R_2/2)$, where

$$H_q(x) := x \log_q(q - 1) - x \log_q x - (1 - x) \log_q(1 - x)$$

is the $q$-ary entropy function.

Now put $r_1 := [(1 - R_1)w]$, $r_2 := [R_2w]$, $d := \delta w$, $\beta := \delta/3$, and let us fix a full-rank matrix $g$ such that $d(\text{im} g^*) \geq d$. In the rest of the proof, we will consider all the probabilities conditioned on this choice of $g$.

Let $h \in \mathbb{F}_q^{[w-r_1] \times w}$ be a parity-check matrix of the code $\text{im} g^*$, and consider the code $C := \text{ker}(h \otimes h')$. The entries of the matrix $h'$ are independent uniformly distributed elements of $\mathbb{F}_q$.

\textsuperscript{22}We suppose that the entries of the both matrices are chosen uniformly and independently at random from $\mathbb{F}_q$.

\textsuperscript{23}Note that the probabilistic proof of the Gilbert–Varshamov bound can be used with a random code defined either by a random parity-check matrix or a random generator matrix. See [48] for a good review of this bound.
Now we estimate the probability that the code $C$ has a codeword of some particular form. It is convenient to interpret elements of $\mathbb{F}_q^w \otimes \mathbb{F}_q^w$ as $w \times w$ matrices over $\mathbb{F}_q$. In this interpretation every $x \in C$ satisfies the condition $h x h^t = 0$. Hence, for $x \in C$ we have
\begin{equation}
0 = h x h^t = s h^t s^t
\end{equation}
where $s = h x$. For matrix $u \in \mathbb{F}_q^{b \times b}$ by $u_i$ we denote the $i$-th column of $u$ and by $u^j$ we denote the $j$-th row of $u$.

When $h$ and $x$ are fixed, then (10) defines a system of linear equations on the elements of the matrix $h'$. To estimate the number of solutions we need to estimate the rank of this system. For all $j \in [r_2]$ we have $h'^{ij} \in \ker s$. Hence, the probability that the equation (10) satisfies is $q^{-r_2 \text{rk} s}$.

Put $\beta = d/3w = \delta/3$, $m = \gamma w^{1/2+\varepsilon}$ and suppose $w$ is sufficiently large such that $m \leq d/6$. By Lemma 9 for every $\beta$-w-minimal non-zero codewords $x \in C$ such that $\text{wt}(x) \leq \alpha w^{1+\varepsilon}$ we have $\text{rk} h x \geq \frac{5}{3w} \cdot \frac{d^2}{\alpha w^{1+\varepsilon}} = c_1 w^{1-\varepsilon}$ where $c_1 = \frac{5d^2}{3\alpha w}$. So, to summarize, we proved that if $(\ker h, \ker h')$ is not $(\alpha w^{1+\varepsilon}, m, \beta)$-product-expanding, and $m \leq \delta w/6$, then one of the following three cases is true:

1. $d(\ker h) < \delta w$;
2. $d(\ker h') < \delta w$;
3. there exist subsets $A, B \subseteq [w], |A| = |B| = w - m$ and a matrix $x \in \mathbb{F}_q^{w \times w}$ such that $\text{wt}(x|_{A \times B}) < \alpha w^{1+\varepsilon}$, $\text{rk} h x \geq c_1 w^{1-\varepsilon}$ and equation (10) is satisfied.

For every $i \in \{1, 2, 3\}$ let $p_i$ be the probability that the $i$-th case above holds if we choose the matrices $h$ and $h'$ uniformly at random. Recall that we have already chosen $\delta$ such that $p_1 \to 0$ and $p_2 \to 0$ as $w \to \infty$. Hence to complete the proof we also need to show that $p_3 \to 0$ as $w \to \infty$. To estimate the probability $p_3$ we need to estimate the number of ways one can choose the matrix $x$ such that the third case above holds. It is clear that we have

1. $\left(\frac{w^2}{m}\right)^2 < w^{2m}$ choices for the subsets $A$ and $B$;
2. less than $q^{2mw}$ choices for the elements of $x$ at the positions from $[w] \times [w] \setminus A \times B$;
3. less than $(\alpha w^{1+\varepsilon}) q^{aw^{1+\varepsilon}} < (qw)^{2aw^{1+\varepsilon}}$ choices for the elements of $x$ at the positions from $A \times B$.

Totally, we have $N$ choices of vector $x$, where
\[
\log_q N \leq \log_q q^{2mw} w^{2m} (qw)^{2aw^{1+\varepsilon}} = 2\gamma w^{3/2+\varepsilon} + 2\gamma w^{1/2+\varepsilon} \log_q w + 2\alpha w^{1+\varepsilon} (1 + \log_q w).
\]

For each choice of the vector $x$ the probability that (10) is satisfied equals to $q^{-r_2 \text{rk} h x} < q^{-c_1 w^{1-\varepsilon}} = q^{-c_2 w^{2-\varepsilon}}$ where $c_2 = c_1 (1 - R_2)$. Thus, by the union bound, the probability $p_3$ is bounded from above by $N q^{-c_2 w^{2-\varepsilon}}$, and we get
\[
\log_q p_3 \leq \log_q N - c_2 w^{2-\varepsilon} \leq \gamma w^{3/2+\varepsilon} + 2\gamma w^{1/2+\varepsilon} \log_q w + 2\alpha w^{1+\varepsilon} (\log_q w + 1) - c_2 w^{2-\varepsilon}.
\]

It is easy to see that $\log_q p_3 \to -\infty$ as $w \to \infty$ for any constants $\varepsilon < 1/4, \alpha > 0, \gamma > 0$. If $w$ is large enough then $m = \gamma w^{1/2+\varepsilon} < \delta w/6$. Hence the probability $p$ that $(\ker h, \ker h')$ is not $(\alpha w^{1+\varepsilon}, m, \beta)$-product-expanding is bounded from above by $p_1 + p_2 + p_3 \to 0$ as $w \to \infty$, and the lemma is proved. \qed

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2.5 Global expansion

In this subsection, the graphs \( \Lambda \) and \( \Lambda^2 \) are the graphs from Subsection 2.3.

Lemma 11. The graph \( \Lambda \) is \((a, 2\lambda)\)-edge-expanding.

Proof. Since \( E = E_{\rightarrow} \cup E_{\uparrow} \), we can split the graph \( \Lambda \) as \( \Lambda = \Lambda_{\rightarrow} \cup \Lambda_{\uparrow} \), where
\[
\Lambda_{\rightarrow} := V \cup E_{\rightarrow} = \{ x \cdot g \cdot y \mid x \in V(\Gamma) \cup E(\Gamma), y \in V(\Gamma), g \in G \},
\]
\[
\Lambda_{\uparrow} := V \cup E_{\uparrow} = \{ x \cdot g \cdot y \mid x \in V(\Gamma), y \in V(\Gamma) \cup E(\Gamma), g \in G \}.
\]
In terms of graphs, \( \Lambda_{\rightarrow} \) is the subgraph of \( \Lambda \) containing only horizontal edges, and \( \Lambda_{\uparrow} \) is the subgraph of \( \Lambda \) containing only vertical edges. It is easy to see that
\[
\Lambda_{\rightarrow} = \bigcup_{y \in V(\Gamma)} \Lambda_{\rightarrow}(y), \quad \Lambda_{\uparrow} = \bigcup_{x \in V(\Gamma)} \Lambda_{\uparrow}(x),
\]
where
\[
\Lambda_{\rightarrow}(y) = \{ x \cdot g \cdot y \mid x \in V(\Gamma) \cup E(\Gamma), g \in G \},
\]
\[
\Lambda_{\uparrow}(x) = \{ x \cdot g \cdot y \mid y \in V(\Gamma) \cup E(\Gamma), g \in G \}.
\]
Since the graphs \( \Lambda_{\rightarrow}(x) \) and \( \Lambda_{\uparrow}(y) \) are isomorphic to \( \hat{\Gamma} \), they are \((\alpha, \lambda)\)-edge-expanding. Hence, by property 3 of the edge expansion (see Remark 10), their disjoint unions \( \Lambda_{\rightarrow} \) and \( \Lambda_{\uparrow} \) have the same edge expansion. Therefore by property 4 of the edge expansion their union \( \Lambda \) is \((a, 2\lambda)\)-edge-expanding. \( \square \)

Lemma 12. The graph \( \Lambda^2 \) is \((a/2w, 8\lambda^2(\ln w + 2))\)-edge-expanding.

Proof. By Lemma 11 graph \( \Lambda \) is \((a, 2\lambda)\)-edge-expanding. From the definition of \( \Lambda \) it is easy to see that \( \Lambda \) is a \( 2w \)-regular graph. Hence by Lemma 7 the graph \( \Lambda^2 \) is \((a/2w, 8\lambda^2(1 + \ln(2w)))\)-edge-expanding. Since \( \ln(2w) < \ln w + 1 \), we obtain the assertion of the lemma. \( \square \)

In the rest of this subsection, we assume that \( c \) is some fixed locally minimal 1-cycle in the complex \( C_\bullet(X; \mathcal{F}) \) from Subsection 2.3.

Lemma 13. If \( \partial c = 0 \), then each face-active vertical edge is incident to at least \( d(\ker h) \) active faces.

Proof. Consider a face-active vertical edge \( e \). Then \( F_e \) is the set of faces incident to \( e \), and \( V_e \) is the set of (two) vertices incident to \( e \). Since \( e \) is face-active, \( c|_{V_e} = 0 \) but \( c|_{F_e} \neq 0 \). Since \( (\partial c)|_e \) depends only on \( c|_{F_e} \) and \( c|_{V_e} \), then using (7) we have
\[
0 = (\partial c)|_e = (\partial(c|_{F_e} + c|_{V_e}))|_e = \partial_{F_e \rightarrow e}(c|_{F_e})
\]
Since \( \partial_{F_e \rightarrow e} \sim h \), \( c|_{F_e} \neq 0 \), and \( \partial_{F_e \rightarrow e}(c|_{F_e}) = 0 \), we have that the number of active faces incident to the edge \( e \) is
\[
\text{wt}(c|_{F_e}) \geq d(\ker \partial_{F_e \rightarrow e}) = d(\ker h),
\]
and the lemma is proved. \( \square \)
Lemma 14. If \(d(\ker h) \geq 2m + \lambda\), \(\partial c = 0\) and \(\text{wt}(c) \leq a/w\), then every active edge is incident to a labeled vertex.

Proof. The number of active vertical edges is at most \(\text{wt}(c)w \leq a\). Let \(S \subseteq E_\uparrow\) be the set of active edges that are not incident to a labeled vertex, \(A \subseteq E_\uparrow\) be the set of active edges. By definition if an active edge is not incident to labeled vertices, then it is not incident to active vertices, then by definition it is face-active, hence \(S\) is a subset of face-active edges.

Consider the subposet \(\Lambda_\uparrow = E_\uparrow \cup F\) of the poset \(X\). Since each face from \(F\) is incident to exactly two vertical edges from \(E_\uparrow\), \(\Lambda_\uparrow\) can be interpreted as a graph with \(V(\Lambda_\uparrow) = E_\uparrow\) and \(E(\Lambda_\uparrow) = F\). We have

\[
\Lambda_\uparrow = \{ x \cdot g \cdot y \mid x \in V(\Gamma) \cup E(\Gamma), y \in E(\Gamma), g \in G \} = \bigcup_{y \in E(\Gamma)} \Lambda(y)
\]

where

\[
\Lambda(y) = \{ x \cdot g \cdot y \mid x \in V(\Gamma) \cup E(\Gamma), g \in G \} \simeq \hat{\Gamma}.
\]

By property 3 of edge expansion \(\Lambda_\uparrow\) has the same edge expansion as \(\hat{\Gamma}\), i.e. it is \((a, \lambda)\)-edge-expanding. The sets \(S\), \(A\) can be interpreted as a set of vertices of graph \(\Lambda_\uparrow\). From the edge expansion of \(\Lambda_\uparrow\) we have \(|E_{\Lambda_\uparrow}(S, S)| \leq \lambda|S|\).

On the other hand, by Lemma 13 since each edge \(e \in S\) is face-active, it is incident to at least \(d = d(\ker h)\) active faces, hence in the graph \(\Lambda_\uparrow\) it is adjacent to at least \(d \geq 2m + \lambda\) active edges, therefore \(|E_{\Lambda_\uparrow}(S, A)| \geq (\lambda + 2m)|S|\). Thus

\[
|E_{\Lambda_\uparrow}(S, A \setminus S)| = |E_{\Lambda_\uparrow}(S, A)| - |E_{\Lambda_\uparrow}(S, S)| \geq (\lambda + 2m)|S| - \lambda|S| = 2m|S|.
\]

Suppose, \(|S| \neq \emptyset\). Then there exists an edge \(e \in S\) adjacent to \(2m\) edges \(e_1, \ldots, e_{2m} \in A \setminus S\) in \(\Lambda_\uparrow\). By the definition of \(A\) and \(S\) each of the edges \(e_i\) is incident to some labeled vertex \(x_i\), which is adjacent to one of the two vertices of \(e\) in \(\Lambda\). Hence, there are \(2m\) different labeled vertices adjacent to one of the vertices of the edge \(e\), and therefore one of these vertices is adjacent to at least \(m\) labeled vertices, therefore it is labeled by definition. This contradicts the fact that the edge \(e\) is from \(S\) and cannot be incident to labeled vertices. Hence \(S = \emptyset\), and the lemma is proved.

In the next lemma, we need the following definition.

Definition. For a given vector \(s \in \mathbb{F}_q^r\) and a parity-check matrix \(h \in \mathbb{F}_q^{r \times w}\) we say that a vector \(x \in \mathbb{F}_q^w\) is an \((s, h)\)-coset leader if it has the minimal possible Hamming weight among the vectors from \(\{ x \in \mathbb{F}_q^w \mid hx = s \}\).

Lemma 15. Suppose the pair of codes \((\ker h, \mathrm{im} h^*)\) is \((s, 2m, \beta)\)-product-expanding, \(h^t\) has full rank, \(\beta w \geq 4m + 3\), \(d = \min(d(\ker h), d(\mathrm{im} h^*)) \geq 4m\), and \(m \geq \max(4s/d, \lambda)\). If \(c\) is a locally minimal 1-cycle, and \(\text{wt}(c) \leq a/w\), then for each active vertex \(v\) one of the following conditions holds:

1. \(v\) is \(m\)-edge-expanding (i.e. it is adjacent to at least \(m\) labeled vertices in \(\Lambda\));
2. \(v\) is \(s\)-face-expanding (i.e. it is adjacent to at least \(s\) labeled vertices in \(\Lambda^2\)).
Proof. Before we start, let us fix some active vertex \( v = v' \cdot g \cdot v''; \) \( v', v'' \in V(\Gamma), g \in G. \) Let \( y = c|_v \in \mathbb{F}_q^{r \times r'}, f = c|_{F_v} \in \mathbb{F}_q F_v. \) Then it is not hard to see that
\[
E_{\rightarrow v} = \{ e' \cdot g' \cdot v'' \in E \rightarrow | \hat{e}'_g^t \hat{v}_g' \}
\]
\[
E_{\uparrow v} = \{ v' \cdot g'' \cdot e'' \in E_{\uparrow} | \hat{e}'_g^{t} \hat{v}'_g \}
\]
\[
F_v = \{ e' \cdot g' g^{-1} g'' \cdot e'' \in F | \hat{e}'_g \hat{v}_g \}
\]
Since \(|E_{\rightarrow v}| = |E_{\uparrow v}| = w\) and each face from \( F_v \) is incident to one edge from \( E_{\uparrow} \) and one edge from \( E_{\rightarrow}, \) the set \( F_v \) is in natural one-to-one correspondence with the set \( E_{\rightarrow v} \times E_{\uparrow v} \) (see Fig. 5(a)) and we can represent the restriction \( f = c|_{F_v} \) as a \( w \times w \) matrix with the rows and columns indexed by the edges from \( E_{\uparrow v} \) and \( E_{\rightarrow v} \) respectively, i.e., \( f \in \mathbb{F}_q F_v \cong \mathbb{F}_q (E_{\rightarrow v} \times E_{\uparrow v}). \) Define the set
\[
N_t(v) := \{ v' \in V | v \leftrightarrow v', e \in E_{\uparrow}\},
\]
which consists of the vertices connected to \( v \) by vertical edges. Note that the set of elements from \( X(1) = V \cup F \) incident to the elements from \( E_{\uparrow v} \subseteq X(0) \) is equal to \( V_{E_{\uparrow v}} \cup F_{E_{\uparrow v}}, \) where \( V_{E_{\uparrow v}} = N_t(v) \cup \{ v \} \) and \( F_{E_{\uparrow v}} = F_v. \) Hence we obtain
\[
(\partial c)|_{E_{\uparrow v}} = (\partial(c|_v + c|_{F_v} + c|_{N_t(v)}))|_{E_{\uparrow v}} = (\partial_{E_{\rightarrow v}}(y) + \partial_{E_{\rightarrow v}}(f) + \partial_{N_t(v) \rightarrow E_{\uparrow v}}(c|_{N_t(v)})).
\]
Since \( A \in \mathcal{T}_G(\hat{\Gamma}, h), B \in \mathcal{T}_G(\hat{\Gamma}, h'), \) we have \( \partial_A^{(v')} \sim h \) and \( \partial_B^{(v'')} \sim h' \), therefore with a proper ordering of the edges in \( E_v \) we can identify \( \partial_{E_{\rightarrow v}} \) with \( I_r \otimes h^{r^*} \) and \( \partial_{E_{\rightarrow v}} \) with \( h \otimes I_w. \) Consider \( z_v := (I_r \otimes h^{r^*})y, z_F := (h \otimes I_w)f, \) and \( z_N := \partial_{N_t(v) \rightarrow E_{\uparrow v}}(c|_{N_t(v)}) \). Then we have
\[
0 = (\partial c)|_{E_{\uparrow v}} = z_v + z_F + z_N.
\]
There are two cases:

1. $|E_{tv} ∩ E_{tw}| = 1$, $\supp(\partial_{tv} - E_{tv}^\circ(c_i|_{tv})) \subset E_{tv} \cup E_{tw}$, and hence $\wtX(\partial_{tv} - E_{tv}^\circ(c_i|_{tv})) \leq \wtX(c_i|_{tv}) \leq 1$. Therefore we get

$$\wtX(z_N) = \wtX\left(\sum_{v' \in N_\Gamma(v)} \partial_{tv} - E_{tv}^\circ(c|_{tv})\right) \leq \sum_{v' \in N_\Gamma(v)} \wtX(\partial_{tv} - E_{tv}^\circ(c|_{tv})) \leq \sum_{v' \in N_\Gamma(v)} \wtX(c_i|_{tv}) = \wtX_{N_\Gamma}(c).$$

Note that $\wtX_{N_\Gamma}(c)$ is the number of active vertices adjacent to $v$ by vertical edges. If $\wtX(z_N) > m$, then the rest of the proof, we consider the most complex case when $\wtX(z_N) < m$. Let $A \subseteq E_{tv}$ (resp. $B \subseteq E_{tw}$) be the set of horizontal (resp. vertical) edges connecting $v$ with the unlabeled vertices. Each pair of edges in $A \times B$ determines a face incident to $v$ and not incident to the labeled vertices adjacent to $v$ in $\Lambda$. To prove the $s$-face expansion of $v$ first we need to show that $\wtX_{A \times B}(f) \geq s$. If $|A| \leq w - m$ or $|B| \leq w - m$, then there are at least $m$ labeled vertices adjacent to $v$ in $\Lambda$, hence $v$ is edge-expanding. In the rest of proof, we consider the case when $|A|, |B| > w - m$.

It this case, we have $\wtX(z_F + z_v) = \wtX(z_N) < m$. Let $z_v = (z_{w,1}^{1}, \ldots, z_{w,v}^{w}) = (I_r \otimes h^*)y$, $t = (t^1, \ldots, t^w) \in F_q^w \otimes F_q^w$, where for each $i \in [w]$ the vector $t^i$ is some $(z_{w,v}, h)$-coset leader. Then $(h \otimes I_w)t = z_v$ and

$$(h \otimes g)t = (I_r \otimes g)z_v = (I_r \otimes g'h^*)y = 0$$

where $g'$ is a parity-check matrix for the code $im h^*$. Hence $t \in \ker(h \otimes g')$. Consider $f' = f + t = (f^{t^1}, \ldots, f^{t^w})$. Component $f^{ti}$ we will call the $i$-th row of $f'$. We have $(h \otimes I_w)f' = z_F + z_v$. For each $i \in [w]$ we have one on the following cases:

1. $i \notin B$: the corresponding edge connects $v$ with active vertex;
2. $i \in B$ and $f^{ti} \neq 0$: in this case $h^{f^{ti}} = 0$, i.e. $f^{ti} \in \ker h \setminus \{0\}$, hence $|f^{ti}| \geq d$;
3. $i \in B$ and $f^{ti} = 0$: is this case $f^i = -t^i$, hence $\wtX(f^i|_A) = \wtX(t^i|_A)$.

Denote by $J_1$, $J_2$, and $J_3$ the sets of indices corresponding to these cases (see Fig. 5(b)). For these sets we have following relations:

$$[m] = J_1 \cup J_2 \cup J_3, \quad B = J_2 \cup J_3, \quad |J_1| < w.$$

There are two cases:

1. $|J_3| < w - 2m$. Then

$|J_2| = w - |J_1| - |J_3| > w - m - (w - 2m) = m \geq 4s/d.$

Each of the rows $f^i$ for $i \in J_2$ has weight at least $d$. On the other hand, for each $i \in J_2$ since $t^i$ is a $(z_{w,v}, h)$-coset leader and $f^{ti} \in \ker h$ we have $\wtX(t^i) \leq \wtX(t^i + f^{ti}) = \wtX(f^i)$, and hence $\wtX(f^{ti}) \leq \wtX(t^i) + \wtX(f^i) \leq 2\wtX(f^i)$. Therefore $\wtX(f^i) \geq \wtX(f^{ti})/2 \geq d/2$. Thus we obtain

$$\wtX(f|_{A \times B}) \geq \wtX(f|_{A \times J_2}) \geq |J_3| \left(\frac{d}{2} - (w - |A|) \right) \geq \frac{4s}{d} \left(\frac{d}{2} - m \right) \geq s.$$
satisfied, therefore each of these active edges is incident to a labeled vertex which is the opposite to $v$ in this face.

2. $|J_3| \geq w - 2m$. Each row $t^i$ has the minimal weight in the coset $t^i + \ker h$ since it is a $(z^i_v, h)$-coset leader. Suppose $t$ is not $\beta w$-minimal codeword. Then there is a row $t^j$ and a vector $\Delta t \in \im h^r$ such that $\wt(t^j + \Delta t) \leq \wt(t^j) - \beta w$. Since $t^j|_{J_3} = f^j|_{J_3}$, we have
\[
\wt(f^j + \Delta t) \leq \wt(f^j + t^j) + \wt(t^j + \Delta t) \leq 2m + \wt(t^j) - \beta w \leq \wt(f^j) + 4m - \beta w.
\]
Taking into account that $\beta w \geq 4m + 3$, we have $\wt(f^j + \Delta t) \leq \wt(f^j) - 3$. Since $\Delta t \in \im h^r$, there exists $u \in \mathbb{F}_q^r$ such that $\Delta t = h^r u$. Consider $ue_j \in C_0$, where $e_j \in E_{\rightarrow v}$ is the $j$-th horizontal edge such that $v \leftrightarrow v_j$, i.e., $e_j$ is incident to the faces corresponding to the $j$-th column of $t$. Then $\partial_{e_j \rightarrow F_{e_j}} \sim h^r$ and $|V_{e_j}| = 2$, therefore
\[
\wt_F(c + \partial(ue_j)) - \wt_F(c) = \wt(c|_{F_v} + \partial_{e_j \rightarrow F_{e_j}}(ue_j)) - \wt(c|_{F_v}) \leq -3,
\]
\[
\wt_V(c + \partial(ue_j)) - \wt_V(c) \leq |\text{supp}(ue_j) \cap V| \leq |V_{e_j}| = 2,
\]
thus
\[
\wt_X(c + \partial(ue_j)) - \wt_X(c) \leq -1
\]
which contradicts the local minimality of $c$. Hence our assumption is wrong, and $t$ is a $\beta w$-minimal codeword. Therefore from the $(s, 2m, \beta)$-product-expansion property of $(\ker h, \im h^r)$ we obtain $\wt(t|_{A \times J_3}) \geq s$, and it follows that
\[
\wt(f|_{A \times B}) \geq \wt(f|_{A \times J_3}) = \wt(t|_{A \times J_3}) \geq s.
\]
Thus in both cases $\wt(f|_{A \times B}) \geq s$. Each active face is incident to 2 active vertical edges. Since $d \geq 4m > 2m + \lambda$, conditions of Lemma 14 satisfied, therefore each of these active edges is incident
to a labeled vertex. For an active face \( x \in A \times B \) one of its vertical edges is incident to vertex \( v \); another vertical edge is incident to some other labeled vertex \( v_x \) which is not adjacent to \( v \) in \( \Lambda \), hence \( v_x \) is the opposite vertex to \( v \) in the face \( x \), i.e. it is connected to \( v \) by a path of length 2 consisting of one horizontal and one vertical edge in the graph \( \Lambda \) (See Fig. reffig:active). Therefore all the vertices \( v_x \) for \( x \in A \times B \) are different and adjacent to \( v \) in \( \Lambda^2 \). Thus \( v \) is face-expanding. \( \Box \)

From Lemma 15 and the definition of the labeled vertices we obtain the following result.

Corollary 1. Suppose the pair of codes \((\ker h, \im h^w)\) is \((s, 2m, \beta)-product-expanding, \beta w \geq 4m + 3, d = \min(d(\ker h), d(\im h^w)) \geq 4m, \) and \( m \geq \max(4s/d, \lambda) \). If \( c \) is a locally minimal 1-cycle, and \( \wt(c) \leq a/w \), then for each labeled vertex \( v \) one of the following conditions holds:

1. \( v \) is edge-expanding (i.e. it is adjacent to at least \( m \) labeled vertices in \( \Lambda \));
2. \( v \) is face-expanding (i.e. it is adjacent to at least \( s \) labeled vertices in \( \Lambda^2 \)).

Proof. If the vertex \( v \) is active, then the lemma assertion is true by Lemma 15. Otherwise, by definition, the vertex \( v \) is adjacent to at least \( m \) active vertices in \( \Lambda \), and hence it is \( m \)-edge-expanding. \( \Box \)

Lemma 16. Suppose the pair of codes \((\ker h, \im h^w)\) is \((s, 2m, \beta)-product-expanding, \beta w \geq 4m + 3, d = \min(d(\ker h), d(\im h^w)) \geq 4m, m \geq \max(4s/d, \lambda') \), and \( s \geq 2\lambda'' \) where \( \lambda' = 2\lambda \) and \( \lambda'' = 8\lambda^2(\ln w + 2) \) are respectively the edge expansion coefficients of the graphs \( \Lambda \) and \( \Lambda^2 \). If \( c \) is a locally minimal 1-cycle, and \( \wt(c) \leq a/w \), then \( c = 0 \).

Proof. Let \( L \) be the set of labeled vertices. Then by Corollary 1 each vertex \( v \in L \) is either \( m \)-edge-expanding or \( s \)-face-expanding, i.e. \( L = L_e \cup L_f \) where \( L_e \) is the set of \( m \)-edge-expanding vertices, \( L_f \) is the set of \( s \)-face-expanding vertices. By definition we have

\[
|E_\Lambda(L_e, L)| \geq m|L_e|, \quad |E_\Lambda^2(L_f, L)| \geq s|L_f|.
\]

(11)

Since each labeled vertex \( v \) is either active \((v \in \text{supp} v_V)\) or incident to a face-active edge, and hence adjacent to at least \( d \) active faces, we get

\[
|L| \leq \wt_X(v_V) + 4\wt_X(v_V)/d \leq \wt_X(v_V) + \wt_X(v_F) \leq \lambda_n/w^2.
\]

Hence by \((a, \lambda')\)-edge-expansion of \( \Lambda \) we have

\[
|E(L_e, L)| \leq \lambda' \sqrt{|L| |L_e|}.
\]

Similarly, from \((a/2w, \lambda'')\)-edge-expansion of \( \Lambda^2 \) we obtain

\[
|E(L_f, L)| \leq \lambda'' \sqrt{|L| |L_f|}.
\]

Taking into account (11), we obtain

\[
m|L_e| \leq \lambda' \sqrt{|L| |L_e|}, \quad s|L_f| \leq \lambda'' \sqrt{|L| |L_f|},
\]

and hence

\[
|L_e| \leq \left( \frac{\lambda'}{m} \right)^2 |L| \leq \frac{|L|}{4}, \quad |L_f| \leq \left( \frac{\lambda''}{s} \right)^2 |L| \leq \frac{|L|}{4}.
\]

Since \( |L| = |L_e \cup L_f| \leq |L|/2 \), we obtain \( |L| = 0 \). Since each active vertical edge by Lemma 14 contains labeled vertices, we have that the number of active vertical edges is 0, and hence \( c = 0 \). \( \Box \)
2.6 Proof of the theorems

Proposition 1. For every finite field $\mathbb{F}_q$, intervals $(\rho_0, \rho_1), (\rho_0', \rho_1') \subseteq (0, 1)$, constant $\mu > 0$, and infinite set $W \subseteq \mathbb{N}$, there exist matrices $h \in \mathbb{F}_q^{r \times w}$, $h' \in \mathbb{F}_q^{r' \times w}$ for sufficiently large $w \in W$ such that $r/w \in (\rho_0, \rho_1)$, $r'/w \in (\rho_0', \rho_1')$, and for every $G$-lifted $w$-regular $(a, \mu \sqrt{w})$-edge-expanding graph $\hat{\Gamma}$ and Tanner complexes $A \in \mathfrak{T}_G(\hat{\Gamma}, h)$, $B \in \mathfrak{T}_G(\hat{\Gamma}, h')$ with a free action of a group $G$ we have

$$d^{(1)}_{LM}(A \otimes_G B^*) \geq a/2w,$$

$$d^{(1)}_{LM}(B \otimes_G A^*) \geq a/2w.$$

Proof. Let $w$ be a parameter which we will fix later. Define $\varepsilon := 1/6$, $m := w^{1/2+\varepsilon}$, $s := w^{1+\varepsilon}$, $r := \left[\frac{1}{2}(\rho_0 + \rho_1)w\right]$, $r' := \left[\frac{1}{2}(\rho_0' + \rho_1')w\right]$. By Lemma 10 with $\alpha := 1$, $\gamma := 2$, there exist $\beta_1, \beta_2 > 0$ and $\delta_1, \delta_2 > 0$ such that for random matrices $h \in \mathbb{F}_q^{r \times w}$, $h' \in \mathbb{F}_q^{r' \times w}$ as $w \to \infty$ the following three conditions hold with high probability:

1. the matrices $h$ and $h'$ have maximal rank, i.e. $\text{rk} \ h = r$, $\text{rk} \ h' = r'$;
2. the pair $(\ker h, \text{im} h'^*)$ is $(s, 2m, \beta_1)$-product-expanding and $\min(d(\ker h), d(\text{im} h'^*)) \geq \delta_1 w$;
3. the pair $(\ker h', \text{im} h)$ is $(s, 2m, \beta_2)$-product-expanding and $\min(d(\ker h'), d(\text{im} h^*)) \geq \delta_2 w$.

Therefore by the union bound for a sufficiently large $w_0 \in \mathbb{N}$ for every $w \geq w_0$ there exists a pair $(h, h')$ that satisfies these three conditions. Let $\beta := \min(\beta_1, \beta_2)$, $d := \min(\delta_1, \delta_2)w$, $\lambda := \mu \sqrt{w}$, $\lambda' := 2\lambda$, $\lambda'' := 8\lambda^2 (\ln w + 2)$. We have

$$d = \Theta(w), \quad \lambda' = \Theta(w^{1/2}) = o(m), \quad \lambda'' = \Theta(w \ln w) = o(s), \quad m = \Theta(w^{1/2 + \varepsilon}) = o(w)$$

as $w \to \infty$. Hence there exists $w_1$ such that for every $w \geq w_1$ the following inequalities hold:

$$d > 4m, \quad \beta w \geq 4m + 3, \quad m > \max \left(\frac{4s}{d}, 2\lambda'\right), \quad s > 2\lambda''.$$

(12)

Since the set $W$ is infinite, we can take $w := \min\{w \in W \mid w \geq \max\{w_0, w_1\}\}$ and fix some pair $(h, h')$ that satisfy the conditions 1–3. Now consider a $G$-lifted $(a, \lambda)$-edge-expanding graph $\hat{\Gamma}$ and some $G$-lifted Tanner complexes $A \in \mathfrak{T}_G(\hat{\Gamma}, h)$, $B \in \mathfrak{T}_G(\hat{\Gamma}, h')$.

Since $\min(d(\ker h), d(\ker h'), d(\text{im} h^*), d(\text{im} h'^*)) \geq d$, and conditions (12) hold, we can apply Lemma 16 to the pair of codes $(h, h')$ and obtain that every non-zero locally-minimal 1-cycle of the chain complex $A \otimes_G B^*$ has the weight at least $a/2w$. Hence we have

$$d^{(1)}_{LM}(A \otimes_G B^*) \geq a/2w.$$

Since lemma 16 is also applicable to the pair $(h', h)$, we have

$$d^{(1)}_{LM}(B \otimes_G A^*) \geq a/2w,$$

which completes the proof of the proposition. \qed

Theorem 1. For every number $R \in (0, 1/2)$ and finite field $\mathbb{F}_q$ it is possible to find universal constants $s$ and $w$ such that there exists an explicit family of $(w, s)$-locally testable classical LDPC codes with the parameters $[n, k \geq Rn, d = \Theta(n)]_q$ as $n \to \infty$. 

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Proof. Fix some $R \in (0,1/2)$ and put $\varepsilon := (1-2R)/(6-2R)$. Consider the graphs $\bar{X}^{w-1,t}$ from Example 1. Note that for any $w$ from the infinite set $W := \{p+1 \in \mathbb{N} \mid p \equiv 1 \pmod{4} \text{ and } p \text{ is prime}\}$ there exist infinite family of graphs $\bar{X}^{w-1,t}$. By Lemma 5 every graph $\bar{X}^{w-1,t}$ is $(n_0(t)/\sqrt{w}, 8\sqrt{w})$-edge-expanding, where $n_0(t) = t(t^2 - 1) = |V(\bar{X}^{w-1,t})|$. Consider the chain complex

$$C := T(\bar{X}^{w-1,t}, h) \otimes_G T(\bar{X}^{w-1,t}, h'),$$

with the boundary operator $\partial$, where $G = \text{PSL}(F_2)$ and $\text{wt}_X(\cdot)$ defined on $C$, considered as a chain complex with a local system on the cell poset $X = \bar{X}^{w-1,t} \times_G (\bar{X}^{w-1,t})^*$. By Proposition 1 for the intervals $(1-\varepsilon, 1)$, $(0, \varepsilon)$ and the parameter $\mu = 8$ there exist $w \in W$ and matrices $h \in \mathbb{F}_q^w$, $h' \in \mathbb{F}_q^{w'}$ such that for every $\bar{X}^{w-1,t}$ we have

$$d_{\text{LM}}^{(1)}(C) \geq n_0(t)/2w\sqrt{w}$$

where $r/w > 1 - \varepsilon$, $r'/w < \varepsilon$. Let $n := \dim C_2$ and $m := \dim C_1$, then $n = 2n_0(t)rw$, $m = n_0(t)(w^2 + 4rr')$. Hence $d_{\text{LM}}^{(1)}(C) \geq \frac{n}{4w^2rw} > \frac{n}{4w^r}$. By Lemma 1 for all $c \in C_2$ we have

$$|\partial c| \geq \min(d_{\text{LM}}^{(1)}(C), |c + Z_2(C)|).$$

Since $|y| \leq \text{wt}(y)$ for $y \in C$ and $\text{wt}(c) \leq r|c| \leq w|c|$ for $c \in C_2$, taking into account that $n \geq \text{wt}(c + Z_2(C))$ finally we obtain

$$\text{wt}(\partial c) \geq \frac{1}{m} \min \left( \frac{n}{4w^r/2}, \frac{\text{wt}(c + Z_2(C))}{w} \right) \geq \frac{1}{4w^{r/2}} \text{wt}(c + Z_2(C)).$$

We have

$$\frac{ \frac{m}{n} = \frac{w^2 + 4rr'}{2rw} = \frac{1 + 4\cdot \frac{r'}{r}}{2w} \leq \frac{1 + 4\varepsilon}{2(1 - \varepsilon)} = 1 - R.}$$

In particular, we have $m < n$, hence

$$\frac{1}{m} \text{wt}(\partial c) \geq \frac{w^{-7/2}}{4m} \text{wt}(c + Z_2(C)) \geq \frac{w^{-7/2}}{4n} \text{wt}(c + Z_2(C)),$$

therefore $Z_2(C)$ is $(2w,s)$-locally testable code where $s = \frac{1}{4}w^{-7/2}$. The dimension $k = \dim Z_2(C)$ we have $k \geq n - m \geq Rn$.

To complete the proof we also need to show that the linear code $Z_2(C)$ has the minimal distance $\Theta(n)$ as $n \to \infty$. It is not hard to see that the minimal distance of $Z_2(C)$ is not less than the distance of the component Tanner code $T(\bar{X}^{w-1,t}, h)$, which is a classical expander code [23]. Thus if we fix a sufficiently large number $w$ such that $d(\ker h) > \lambda_2(\bar{X}^{w-1,t})$, then we obtain that $d(T(\bar{X}^{w-1,t}, h)) = \Theta(n)$ as $n \to \infty$. \hfill \square

Theorem 2. For every number $R \in (0,1)$ and finite field $\mathbb{F}_q$ there exists an explicit family of quantum LDPC codes over $\mathbb{F}_q$ with the parameters $[n,k \geq Rn, d = \Theta(n)]$ as $n \to \infty$.

Proof. Fix some $R \in (0,1)$. Consider the graphs $\bar{X}^{w-1,t}$ from Example 1. Note that for any $w$ from the infinite set $W := \{p+1 \in \mathbb{N} \mid p \equiv 1 \pmod{4} \text{ and } p \text{ is prime}\}$ there exist infinite family of graphs $\bar{X}^{w-1,t}$. By Lemma 5 graph $\bar{X}^{w-1,t}$ is $(n_0(t)/\sqrt{w}, 8\sqrt{w})$-edge-expanding where $n_0(t) = t(t^2 - 1) = |V(\bar{X}^{w-1,t})|$. Consider complex $C = T(\bar{X}^{w-1,t}, h) \otimes_G T(\bar{X}^{w-1,t}, h')^*$ with
boundary operator $\partial$ where $G = \text{PSL}(\mathbb{F}_q^2)$. Let $|\cdot|$ be block weight defined on $\mathcal{C}$. By Proposition 1 for $\rho_0 = \rho'_0 = 0$, $\rho_1 = \rho'_1 = (1 - R)/4$, and $\mu = 8$ there exist $w \in W$ and matrices $h \in \mathbb{F}_q^{r \times w}$, $h' \in \mathbb{F}_q^{r' \times w}$ such that for all $X^{w-1,t}$ we have
\[
d_{\text{LM}}^{(1)}(\mathcal{C}) \geq n_0(t)/2w\sqrt{w}, \quad d_{\text{LM}}^{(1)}(\mathcal{C}^*) \geq n_0(t)/2w\sqrt{w}
\]
where $r/w < (1 - R)/4$, $r'/w < (1 - R)/4$. Let $n := \dim \mathcal{C}_1$, then $n = n_0(t)(w^2 + 4rr') < 2w^2n_0(t)$. The chain complex $\mathcal{C}$ defines the quantum CSS code $\mathcal{Q} = \mathcal{Q}(H_X, H_Z)$ with the parity-check matrices $H_X := \partial_1$ and $H_Z := \partial_2$. By Lemma 1 for the complex $\mathcal{C}$ we have
\[
d_X(\mathcal{Q}) = d(H_1(\mathcal{C})) \geq d_{\text{LM}}^{(1)}(\mathcal{C}) \geq n_0(t)/2w\sqrt{w} > n/4w^{7/2}.
\]
Similarly, since the dual chain complex $\mathcal{C}^*$ is isomorphic to the chain complex $B \otimes_G A^*$, then by Lemma 1 we have
\[
d_Z(\mathcal{Q}) = d(H_1(\mathcal{C}^*)) \geq d_{\text{LM}}^{(1)}(\mathcal{C}^*) > n/4w^{7/2},
\]
and hence $d(\mathcal{Q}) = \min(d_X(\mathcal{Q}), d_Z(\mathcal{Q})) \geq 1/4w^{7/2}$. To complete the proof we also need to estimate the dimension $k = \dim(H_1(\mathcal{C}))$ of the quantum code $\mathcal{Q}$. We have
\[
dim \mathcal{C}_0 = 2n_0(t)rw = 2n \frac{rw}{w^2 + 4rr'} < n(1 - R)/2,
\]
\[
dim \mathcal{C}_2 = 2n_0(t)r'w = 2n \frac{r'w}{w^2 + 4rr'} < n(1 - R)/2,
\]
and therefore
\[
k = \dim(H_1(\mathcal{C})) \geq n - \dim \mathcal{C}_0 - \dim \mathcal{C}_2 > n - n(1 - R)/2 - n(1 - R)/2 = nR.
\]
Thus $\mathcal{Q}$ is a $w$-limited quantum CSS code with the parameters $[n, k \geq Rn, d \geq 1/4n/w^{7/2}]_q$. □

Conclusions

In this work, we showed that there exist asymptotically good families of quantum LDPC codes, which proves the well-known qLDPC conjecture. We also conjecture that a decoder, similar to the small-set-flip decoding algorithm from [22] (see also [20]), can be used to correct in linear time any adversarial errors up to the constant fraction of the code length.

The constructed qLDPC codes were obtained from the $G$-lifted product of two $G$-lifted Tanner complexes, and to obtain qLDPC codes of linear minimum distance a non-abelian group $G$ was used. In fact, it is not hard to see that Proposition 1 implies that using the $C_l$-lifted product of two $C_l$-lifted Tanner complexes from [13], where $C_l$ is the cyclic group of size $\ell = \Theta(n/\log n)$, one can obtain qLDPC codes with the parameters $[n, k = \Theta(n), d = \Theta(n/\log n)]_q$ as $n \to \infty$.

In addition, we show that the spaces $Z_2(\mathcal{C})$ of 2-cycles of the constructed in this work chain complexes $\mathcal{C}$ can be used to obtain asymptotically good families of classical LDPC codes, which are also locally testable with constant query and soundness parameters. This resolves an important conjecture in the field of locally-testable codes. We also hope that some of the methods developed in the current work can be used to show the existence of locally-testable qLDPC codes required to prove the qLTC and NLTS conjectures.

\[\text{We say that two based chain complexes } \mathcal{C} \text{ and } \mathcal{C}' \text{ over } \mathbb{F}_q \text{ are isomorphic if there exists a one-to-one } \mathbb{F}_q\text{-linear map } f: \mathcal{C} \to \mathcal{C}' \text{ such that } f(C_i) = C'_i \text{ for every } i \in \mathbb{Z}.\]
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A Chain complexes

Let $\mathbb{F}$ be a field. We say that an $n$-dimensional vector space $V$ over $\mathbb{F}$ is based if it comes with some distinguished basis $\tilde{V} := \{v_1, \ldots, v_n\} \subseteq V$. In this case we can naturally identify $V$ with the coordinate vector space $\mathbb{F}^n$. Moreover, we can consider the standard inner product $\langle v, v \rangle$ defined on the basis as $\langle v_i, v_j \rangle := \delta_{ij}$ and extend it by linearity. This also allows us to identify the dual vector space $V^* := \text{Hom}(V, \mathbb{F})$ with $V$ and hence with $\mathbb{F}^n$ if for every $v \in V$ we let $v(x) := \langle v, x \rangle$. Now consider an $\mathbb{F}$-linear map $\varphi: U \to V$ between based vector spaces $U \cong \mathbb{F}^m$ and $V \cong \mathbb{F}^n$. We usually identify such maps with the corresponding $m \times n$ matrix over $\mathbb{F}$. For every such map $\varphi: U \to V$, we can consider the corresponding transpose map $\varphi^*: V^* \to U^*$ that takes each linear
function \( f \in V^* \) to the function \( f \circ \varphi \in U^* \). It is easy to check that the \( n \times m \) matrix of the transposed map \( \varphi^* \) is the transpose of the matrix for \( \varphi \).

Consider a field \( \mathbb{F} \). A chain complex (over \( \mathbb{F} \)) is a collection of vector spaces\(^{25}\) \( (C_i)_{i \in \mathbb{Z}} \) over \( \mathbb{F} \), which is convenient to consider as one big vector space \( C = \bigoplus_{i \in \mathbb{Z}} C_i \), with some fixed linear operator \( \partial: C \to C \) called the boundary map such that \( \partial C_{i+1} \subseteq C_i \) and \( \partial^2 = 0 \) for all \( i \in \mathbb{Z} \). The condition \( \partial C_{i+1} \subseteq C_i \) says that one can define the maps \( \partial_i := \partial|_{C_i}: C_i \to C_{i-1}, \ i \in \mathbb{Z} \); while the condition \( \partial^2 = 0 \) implies that \( \partial_i \circ \partial_{i+1} = 0 \) for all \( i \in \mathbb{Z} \) or, equivalently, \( B_i(C) \subseteq Z_{i+1}(C) \), where \( B_i(C) := \ker \partial_i \). Therefore for every \( i \in \mathbb{Z} \) we can define the quotient group \( H_i(C) := Z_i(C)/B_i(C) \) called the \( i \)-th homology group of the complex \( C \). The elements from \( C_i, Z_i(C), \) and \( B_i(C) \) are called the \( i \)-chains, \( i \)-cycles, and \( i \)-boundaries of \( C \), respectively. We say that a complex \( C \) is based if every space \( C_i \) comes with a distinguished basis \( \mathcal{C}_i \subseteq C_i \), which elements are called \( i \)-cells. In this work we consider only bounded chain complexes, i.e., when \( C_i = 0 \) for all \( i \not\in [s, t] \). A bounded chain complex \( C \) is usually represented by the following diagram:

\[
\begin{array}{c}
C_s \xrightarrow{\partial_s} C_{s-1} \xrightarrow{\partial_{s-1}} \cdots \xrightarrow{\partial_{t+1}} C_t,
\end{array}
\]

where \( t-s+1 \) is called the length of \( C \). A complex of length \( n \) is also called an \( n \)-term complex.

The definition of a chain complex and the related terminology come from algebraic topology, where an \( i \)-cell \( c \in \mathcal{C}_i \) usually corresponds to some \( i \)-dimensional object, and \( \partial c \) is an algebraic representation of its \( (i-1) \)-dimensional boundary. For example, one can consider for any simple graph \( \Gamma = (V, E) \) its 2-term chain complex \( C_*(\Gamma; \mathbb{F}_2) \) over \( \mathbb{F}_2 \):

\[
\begin{array}{c}
\mathbb{F}_2 E \xrightarrow{\partial} \mathbb{F}_2 V, \\
\mathcal{C}_1 \xrightarrow{c} \mathcal{C}_0
\end{array}
\]

where \( \mathcal{C}_0 := V, \mathcal{C}_1 := E \), and the boundary map \( \partial \) is defined as \( \partial e := v+v' \), for every \( e = \{v, v'\} \in E \).

Sometimes it is also convenient to consider the dual notion of a chain complex called cochain complex. If we have a chain complex \( C \) we can obtain the corresponding cochain complex for \( C \) if we replace \( C \) by its dual vector space \( C^* := \text{Hom}(C, \mathbb{F}_q) \), and the boundary map \( \partial: C \to C \) by the corresponding coboundary map \( \delta: C^* \to C^* \) that takes each linear function \( x \mapsto f(x) \in C^* \) to \( x \mapsto f(\partial x) \in C^* \). Since \( \partial^2 = 0 \), it follows that \( \delta^2: x \mapsto f(\partial^2 x) \) is the zero map, and we also get \( \delta^2 = 0 \). Moreover, since \( C = \bigoplus_{i \in \mathbb{Z}} C_i \), we see that \( C^* = \bigoplus_{i \in \mathbb{Z}} C^i \) and \( \delta(C^i) \subseteq C^{i+1} \), where \( C^i := \text{Hom}(C_i, \mathbb{F}_q) \), \( i \in \mathbb{Z} \). Similar to the case of chain complexes, we can define the maps \( \delta_i := \delta|_{C^i}: C^i \to C^{i+1} \), and the condition \( \delta^2 = 0 \) implies that \( \delta_{i+1} \circ \delta_i = 0 \) for all \( i \in \mathbb{Z} \), or, equivalently, \( B^i(C) \subseteq Z^i(C) \), where \( B^i(C) := \ker \delta_{i-1}, Z^i(C) := \text{im} \delta_i \). Hence we have the spaces \( C^i, Z^i(C), \) and \( B^i(C) \) of cochains, i-cochains, and i-coboundaries, respectively. Since for every \( i \in \mathbb{Z} \) we have \( B^i(C) \subseteq Z^i(C) \), we can also define the quotient group \( H^i(C) := Z^i(C)/B^i(C) \) called the \( i \)-th cohomology group of \( C \).

Since in the current work we always assume that each \( C_i \) comes with some distinguished basis \( \mathcal{C}_i \), we can identify both \( C_i \) and \( C^i \) with the corresponding coordinate vector space \( \mathbb{F}_q^{n_i} \), where \( n_i := |\mathcal{C}_i| \). In this case, the maps \( \delta_i: \mathbb{F}_q^{n_i} \to \mathbb{F}_q^{n_{i-1}} \) and \( \delta_{i-1}: \mathbb{F}_q^{n_{i-1}} \to \mathbb{F}_q^{n_i} \) can be also identified with the corresponding matrices over \( \mathbb{F}_q \), and it is easy to verify that \( \delta_{i-1} \) is the transpose of \( \delta_i \).

Every chain (resp. cochain) complex can be also considered as a cochain (resp. chain) complex if we use the following convention \( C^i = C_{-i} \). Thus in what follows we are going to consider the

\(^{25}\)In fact, the definitions given below also can be generalized to the case when \( \mathbb{F} \) is an arbitrary commutative ring. In this case, instead of vector spaces over \( \mathbb{F} \) one should consider free \( \mathbb{F} \)-modules.
cochain complex $C^*$ also as the chain complex, in which case we call it the dual chain complex of $C$. For example, if we have a chain complex, corresponding to a quantum CSS code $Q$ with matrices $H_X$ and $H_Z$:

$$
C_\bullet(H_X, H_Z) := \left( \begin{array}{c c c c}
F_q \rightarrow & H_Z^* & F_q \rightarrow & H_X \\
C_1 & C_0 & C_{-1}
\end{array} \right),
$$

then its dual cochain complex for $C$ is

$$
C^\bullet(H_X, H_Z) := \left( \begin{array}{c c c c}
F_q \rightarrow & H_Z & F_q \rightarrow & H_X^* \\
C_1 & C_0 & C_{-1}
\end{array} \right),
$$

and its dual chain complex for $C$ is

$$
C^\bullet(H_X, H_Z) := \left( \begin{array}{c c c c}
F_q \rightarrow & H_X & F_q \rightarrow & H_Z^* \\
C_1 & C_0 & C_{-1}
\end{array} \right),
$$

and we see that $C^\bullet(H_X, H_Z) = C^\bullet(H_Z, H_X)$, i.e., the dual chain complex corresponds to the dual CSS code $Q^*$, where the roles of $H_X$ and $H_Z$ are reversed.

**B  Lifted product of two classical codes**

The lifted product was introduced in [13] as a way to generalize many known constructions [2,30–33] of qLDPC codes. The general idea was to lift the hypergraph product construction [31], which, for any two classical codes with parity-check matrices $A \in \mathbb{F}_q^{m_a \times n_a}$ and $B \in \mathbb{F}_q^{m_b \times n_b}$, gives the quantum CSS code $HP(A, B)$ with the following parity-check matrices:

$$
H_X := \left[ A \otimes I_{m_b}, -I_{m_a} \otimes B \right],
$$

$$
H_Z := \left[ I_{n_a} \otimes B^*, A^* \otimes I_{n_b} \right].
$$

If we replace the elements of the matrices $A := (a_{ij})_{m_a \times n_a}$ and $B := (b_{ij})_{m_b \times n_b}$ by some $\ell \times \ell$ matrices over $\mathbb{F}_q$, we obtain the $\ell$ times larger block matrices $\hat{A} := (\hat{a}_{ij})_{m_a \times n_a} \in \mathbb{F}_q^{\ell m_a \times \ell n_a}$ and $\hat{B} := (\hat{b}_{ij})_{m_b \times n_b} \in \mathbb{F}_q^{\ell m_b \times \ell n_b}$, which, in turn, are used to define the $\ell$ times larger analogs of $H_X$ and $H_Z$ in the following way:

$$
\hat{H}_X := \left[ \hat{A} \otimes I_{m_b}, -I_{m_a} \otimes \hat{B} \right],
$$

$$
\hat{H}_Z := \left[ I_{n_a} \otimes B^*, \hat{A}^* \otimes I_{n_b} \right],
$$

(13)

where in the transposed block matrices $\hat{A}^*$ and $\hat{B}^*$ we also transpose each $\ell \times \ell$ block. As it was shown in [13], if every block of $\hat{A}$ commutes with every block of $\hat{B}$, then this construction always gives a quantum CSS code with the parity-check matrices $\hat{H}_X$ and $\hat{H}_Z$, called the lifted product of $\hat{A}$, $\hat{B}$ and denoted by $LP(\hat{A}, \hat{B})$. Actually, it is easy to see that this commutativity condition is a necessary and sufficient condition to produce a well defined CSS code. Indeed, we have:

$$
\hat{H}_X \hat{H}_Z^* = 0 \iff (\hat{A} \otimes I_{m_b}) \left( I_{n_a} \otimes \hat{B} \right) = (I_{n_a} \otimes \hat{B}) (\hat{A} \otimes I_{m_b}),
$$

where the last equation is equivalent to $\hat{a}_{ij} \hat{b}_{st} = \delta_{st} \hat{a}_{ij}$ for all $i, j, s, t$.

---

26If the characteristics of $\mathbb{F}_q$ is 2, we can omit the sign in the definition of $H_X$. 

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The most straightforward way to make this general definition always work is to use $\ell \times \ell$ matrices from some commutative matrix ring $R \subseteq \mathbb{F}_q^{\ell \times \ell}$. However, it also works well with any $\ell$-dimensional associative algebra $R$ over $\mathbb{F}_q$, not necessary a commutative\(^{27}\) one, if we use the right (resp. left) regular matrix representation of its elements as the entries of $\hat{A}$ (resp. $\hat{B}$). Indeed, if we fix a basis in the algebra $R$, then the right (resp. left) regular matrix representation of an element $r \in R$ is defined as the $\ell \times \ell$ matrix of the linear operator $\rho_r := x \mapsto xr$ (resp. $\lambda_r := x \mapsto rx$). Since the multiplication in $R$ is associative, then for any $a, b \in R$ the operators $\rho_a$ and $\lambda_b$ always commute:

$$(\rho_a \lambda_b)(x) = (bx)a = b(xa) = (\lambda_b \rho_a)(x).$$

Hence, for any two matrices $A \in R^{m_0 \times n_0}$ and $B \in R^{m_0 \times n_0}$ we can replace their elements by the corresponding right and left matrix representations to obtain the block matrices $\hat{A}, \hat{B}$ and get the well-defined CSS code using Equation (13), which we denote by LP$(A, B)$.

Let us note that when the algebra $R$ is commutative, then $\rho_r = \lambda_r$ for each $r \in R$, and we do not need to distinguish the left and the right representations of $R$. A very simple example of a lifted product code in this case is Kitaev’s toric code \cite{Kitaev2003}, which can be obtained as LP$(1 + x, 1 + y)$ with the ring $R = \mathbb{F}_2[x, y]/(x^L - 1, y^L - 1)$. Another important example is Haah’s cubic code \cite{Haah2009}, which is equal to LP$(1 + x + y + z, 1 + xy + xz + yz)$, and $R = \mathbb{F}_2[x, y, z]/(x^L - 1, y^L - 1, z^L - 1)$. In these two examples the parameter $L$ is the lattice size. We see that in both these cases the ring $R$ is a group algebra $\mathbb{F}_qG$ for some finite group $G$. Indeed, $G = C_L^2$ for Kitaev’s code, and $G = C_L^3$ for Haah’s code, where $C_L$ is the cyclic group of order $L$.

**Remark 13.** Let us note that lifted products can also be used not only for group rings $R = \mathbb{F}_qG$. For example, if $R = \mathbb{F}_q[x]/(x^\ell - \alpha)$, where $\alpha \in \mathbb{F}_q^\times$, then any matrix $H \in R^{m \times n}$ defines the code $C(H)$, which is called quasi-twisted code, or constacyclic if $m = n = 1$. Such codes \cite{Negreiros2013, Negreiros2014, Negreiros2015} sometimes have better parameters than quasi-cyclic and cyclic codes, which are their special cases when $\alpha = 1$. Thus it is an interesting open problem whether lifted products of these classical codes can give quantum CSS codes with good parameters (cf. \cite{Negreiros2016}).

### C Normed abelian groups

Let $\mathcal{M}$ be a finite metric space with a distance function $d(x, y)$. For any non-empty subset $\mathcal{C} \subseteq \mathcal{M}$ we can define its minimal distance $d(\mathcal{C})$ as

$$d(\mathcal{C}) := \min\{d(x, y) \mid x \neq y; x, y \in \mathcal{C}\},$$

where we assume that $d(\mathcal{C}) := \infty$ if $|\mathcal{C}| = 1$.

We can also define $d(x, \mathcal{Y})$ and $d(\mathcal{X}, \mathcal{Y})$ for $x \in \mathcal{M}$ and $\mathcal{X}, \mathcal{Y} \subseteq \mathcal{M}$ in a straightforward way:

$$d(x, \mathcal{Y}) := \min_{y \in \mathcal{Y}} d(x, y),$$
$$d(\mathcal{X}, \mathcal{Y}) := \min_{x \in \mathcal{X}} \min_{y \in \mathcal{Y}} d(x, y).$$

In what follows, we always assume that the metric space $\mathcal{M}$ is an *abelian normed group*, which means that it has an abelian group structure $(\mathcal{M}, +, 0)$, and the distance $d(\cdot, \cdot)$ is invariant, i.e.,

\(^{27}\)Let us note that for all the examples of lifted products in \cite{Itagaki2017} the algebra $R$ is commutative, and the first examples of non-abelian lifted products first appeared in \cite{Itagaki2018} in the context of a very similar construction called *balanced product*.  

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\[ d(x + h, y + h) = d(x, y) \] for any \( x, y, h \in \mathcal{M} \). For example, if we have a based vector space \( \mathcal{M} \cong \mathbb{F}_q^2 \), then the standard Hamming distance \( d(x, y) := \text{wt}(x - y) \) is invariant. It is a well-known and easily verified fact that the invariant distances \( d(\cdot, \cdot) \) are in a one-to-one correspondence with the functions \( |\cdot| : \mathcal{M} \to \mathbb{R}_{\geq 0} \) called norms such that for all \( x, y \in \mathcal{M} \) we have:

\[
\begin{align*}
|x| = 0 & \iff x = 0, \quad (17) \\
||-x|| &= |x|, \quad (18) \\
|x + y| &\leq |x| + |y|; \quad (19)
\end{align*}
\]

where the correspondence is given by \( d(x, y) := |x - y| \) and \( |x| := d(x, 0) \). Such invariant distances on normed groups are sometimes also called group norm metrics \([53]\). One can easily check that if \( C \) is a subgroup of \( \mathcal{M} \), then the minimal distance \( d(C) \) can be also found by the formula:

\[
d(C) = \min_{x \in C \setminus \{0\}} |x|. \quad (20)
\]

In fact, a group norm metric on \( \mathcal{M} \) also induces the corresponding metric on the quotient group \( \mathfrak{M} = \mathcal{M}/\mathcal{N} \) called the quotient norm metric \([53]\), where \( \mathcal{N} \) is some subgroup of \( \mathcal{M} \). In this case, the norm \( |\mathcal{X}| \) for \( \mathcal{X} \in \mathfrak{M} \) is defined as

\[
|\mathcal{X}| := \min_{x \in \mathcal{X}} |x|. \quad (21)
\]

It is trivial to check that this norm satisfies (17)-(19), and the corresponding distance

\[
d(\mathcal{X}, \mathcal{Y}) := |\mathcal{X} - \mathcal{Y}|
\]

for \( \mathcal{X}, \mathcal{Y} \in \mathfrak{M} \) is equivalent to the distance defined by (16). Thus, the quotient group \( \mathfrak{M} \) is a metric space, and for any group \( \mathcal{C} \) such that \( \mathcal{N} \subseteq \mathcal{C} \subseteq \mathcal{M} \) we can define the minimal distance of the subgroup \( \mathcal{C} \) as in (14):

\[
d(\mathcal{C}) := \min\{d(\mathcal{X}, \mathcal{Y}) \mid \mathcal{X} \neq \mathcal{Y}; \mathcal{X}, \mathcal{Y} \in \mathcal{C}\}.
\]

In fact, using (20) and (21) we can get a much simpler formula:

\[
d(\mathcal{C}) = \min_{\mathcal{X} \in \mathcal{C} \setminus \{\mathcal{N}\}} |\mathcal{X}| = \min_{x \in \mathcal{C} \setminus \mathcal{N}} |x|. \quad (22)
\]

Moreover, if \([\cdot] : \mathcal{M} \to \mathfrak{M} \) is a canonical projection, giving by \( x \in \mathcal{M} \) its coset \([x] = x + \mathcal{N} \in \mathfrak{M}\), then we get: \( d([x], \mathcal{Y}) = d(x, \mathcal{Y}) \) and \( ||x|| = d(x, \mathcal{N}) \) for \( x \in \mathcal{M} \) and \( \mathcal{Y} \in \mathfrak{M} \). This allows us to define for any subgroup \( \mathcal{N} \subseteq \mathcal{M} \) a new norm on \( \mathcal{M} \) that we call a systolic norm as

\[
|x|_\mathcal{N} := ||x|| = d(x, \mathcal{N}).
\]
D List of symbols and standard notations

\[
\begin{align*}
[n] & \quad \text{set } \{1,2,\ldots,n\} \\
\mathbb{F}_q & \quad \text{finite field with } q \text{ elements} \\
R^{m \times n} & \quad \text{set of } m \times n \text{ matrices over } R \\
\mathcal{F} & \quad \text{abelian group of formal sums } \sum_{x \in X} a_x x \text{ with} \\
& \quad \text{coefficients } a_x \in \mathcal{F}_x \text{ in a local system } \mathcal{F} \\
\text{wt}(a) & \quad \text{Hamming weight of } a \in \mathbb{F}_q^n \\
\text{wt}_S(a) & \quad \text{block Hamming weight of } a \in \mathcal{F}X \text{ relative} \\
& \quad \text{to the subset } S \subseteq X \\
|a| & \quad \text{norm of } a \in A \text{ in a normed abelian group } A \\
supp a & \quad \text{support } \{x \in X \mid a_x \neq 0\} \text{ for } a \in \mathcal{F}X \\
a|_S & \quad \text{restriction } \sum_{x \in S} a_x x \text{ to the subset } S \subseteq X \\
\mathbb{K}G & \quad \text{group algebra over } \mathbb{K} \text{ for the group } G \\
v \leftrightarrow_{e} v' & \quad e \text{ connects } v \text{ and } v' \\
G \text{-lift} & \quad |G|\text{-fold regular cover} \\
E_T(S,T) & \quad \text{set of } (v,v') \in S \times T \text{ such that } v \leftrightarrow v' \text{ in } \Gamma \\
x \triangleright_P y & \quad x \text{ covers } y \text{ in a poset } P \\
\tilde{X} & \quad \text{double-cover of the Ramanujan graph } X^{p,q} \\
A \otimes_G B & \quad \text{G\text{-lifted product of complexes } } A \text{ and } B \\
X \times_G Y & \quad \text{G\text{-lifted product of abstract cell complexes } } X \text{ and } Y \\
[x : x'] & \quad \text{incidence number for } x \in X(i), x' \in X(i-1) \\
\mathcal{C}(H) & \quad \text{code with parity-check matrix } H \\
\mathcal{C}^\perp & \quad \text{dual code for } \mathcal{C} \\
A \sim B & \quad \text{permutation equivalent codes or matrices} \\
\ker A & \quad \text{kernel of the linear map } v \mapsto Av \\
\text{im } A & \quad \text{image of the linear map } v \mapsto Av \\
Z_i(\mathcal{C}), B_i(\mathcal{C}) & \quad \text{spaces of } i\text{-cycles and } i\text{-boundaries for } \mathcal{C} \\
H_i(\mathcal{C}) & \quad \text{i\text{-th homology group of } } \mathcal{C} \\
d_{LM}^{(i)}(\mathcal{C}) & \quad \text{i\text{-th locally minimal distance of } } \mathcal{C} \\
\partial_{S \rightarrow T} & \quad \text{restriction } \partial_{S \rightarrow T} : FS \rightarrow FT \text{ of a boundary} \\
& \quad \text{map } \partial : \mathcal{F}X \rightarrow \mathcal{F}X \text{ from } \mathcal{C}_c(X;F) \\
A^* & \quad \text{transpose map or transposed matrix for } A \\
\mathcal{C}^* & \quad \text{dual chain complex}
\end{align*}
\]