On cosmic no-hair in bimetric gravity and the Higuchi bound

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1. Introduction

It is well recognized that the large scale structure of the universe stems from primordial fluctuations generated quantum mechanically during inflation. Remarkably, the nature of the primordial fluctuations is independent of the initial conditions. This nice feature can be associated with the conjecture that the initial anisotropy and inhomogeneity rapidly disappear. This is called the cosmic no-hair conjecture. The cosmic no-hair is proved in an ideal situation [1]. Namely, a homogeneous expanding spacetime with a cosmological constant rapidly approaches de Sitter spacetime, i.e., the initial anisotropy decays in a Hubble time, when we assume that matter satisfies the strong and dominant energy conditions. In general, however, it is not clear whether the cosmic no-hair conjecture is correct or not. In fact, a counter-example to this conjecture has been found [2]. There, spin-1 gauge fields remain during inflation and the anisotropy does not necessarily vanish. Moreover, it turns out that anti-symmetric tensor fields can also generate anisotropy [3]. Hence, it is natural to explore the possibility that a symmetric spin-2 tensor as matter causes the violation of the cosmic no-hair conjecture. Historically, a model of massive spin-2 matter has been proposed as that of the meson [4], which can be regarded as bimetric gravity consisting of the physical metric and the other spin-2 tensor field. In order to treat the spin-2 matter, therefore, we need to construct a consistent ghost-free theory of bimetric gravity. Fortunately, this task has recently been accomplished [5–9].

Given a consistent model of spin-2 matter, we can study the cosmic no-hair conjecture. There are some reasons for us to expect that the conjecture can be violated. In the presence of spin-2 matter, it is inevitable that gravitons have mass as a consequence of mixing between the physical metric and the other spin-2 tensor field. When we consider massive gravitons in an expanding spacetime, the
decay time scale of the anisotropy is determined by comparing the Hubble scale with the effective mass of gravitons. For example, by taking the couplings of the physical metric and the spin-2 matter to be small, the Hubble friction term might be dominant compared with the effective mass term in the equation of motion, then the decay time scale becomes much longer than the Hubble time scale. Besides the above, there may be violation of the energy conditions in the presence of spin-2 matter [10]. Since the energy conditions are assumed in the proof of Ref. [1], it is not apparent whether the cosmic no-hair holds or not in bimetric gravity.

In this paper, we consider a cosmological constant in bimetric gravity as the limit of slow roll inflation. First, we concretely reveal the property of de Sitter solutions in bimetric gravity. Then, we investigate the fate of homogeneous anisotropic perturbations. If the effective mass is negative, the perturbations are unstable and the cosmic no-hair is, of course, broken. We stress that even if the effective mass is positive and the perturbations are stable, the cosmic no-hair may be broken. This is because, as mentioned above, if the effective mass is substantially smaller than the Hubble scale, the anisotropy may remain at the end of inflation. The effective mass can depend on the background geometry; therefore, it is important to study the background geometry in detail. Since, in known cases, the violation of the cosmic no-hair already appears at the linear level, we expect the linear analysis to reflect the feature at the nonlinear level.

When we consider massive gravitons in de Sitter spacetime, we also need to take into account the fact that the helicity-0 mode of massive gravitons becomes a ghost when the effective mass is below the Higuchi bound [11–14]. Note that this ghost is different from a Boulware–Deser type ghost [15], which is already removed by construction. Since there is no a priori reason to forbid the mass of gravitons violating the Higuchi bound, we also check if the effective mass satisfies the Higuchi bound.

The paper is organized as follows. In Sect. 2, we present ghost-free bimetric gravity and derive basic equations needed for the analysis. In Sect. 3, we study the cosmic no-hair in bimetric gravity in the presence of the cosmological constant in the physical sector. We find that de Sitter solutions are stable under homogeneous anisotropic perturbations and the small anisotropy rapidly decays. In Sect. 4, we introduce the other cosmological constant and investigate the fate of the anisotropy. We also study whether the Higuchi bound is satisfied or not. The final section is devoted to the conclusion.

In appendix A, we derive a set of equations used in the text.

### 2. Bimetric gravity

In this section, we introduce bimetric gravity [8,9] as a model of spin-2 matter and provide basic formulae. Historically, after the pioneering work of Ref. [4], bimetric gravity has been studied from time to time [16,17]. The model can be generalized to that of ghost-free multi-spin-2 matter [18,19].

Let us represent the physical metric and the other metric as $g_{\mu\nu}$ and $f_{\mu\nu}$, respectively. Note that we regard the other metric $f_{\mu\nu}$ as the spin-2 matter. We consider bimetric gravity with cosmological constants

$$S = \frac{M^2_g}{2} \int d^4x \sqrt{-g}(R[g_{\mu\nu}] - 2\Lambda_g) + \frac{M^2_f}{2} \int d^4x \sqrt{-f}(R[f_{\mu\nu}] - 2\Lambda_f) + m^2 \int d^4x \sqrt{-g} \sum_{n=1}^{3} \alpha_n F_n[L^{\mu}_\nu]. \quad (1)$$
where $M_g$ and $M_f$ are Planck constants of $g_{\mu\nu}$ and $f_{\mu\nu}$, and $R$ is the scalar curvature constructed from each metric. The interaction terms of the metrics are defined as

$$F_n[X^\mu_\nu] = \frac{1}{n!} \sum_{\sigma \in S_n} \text{sgn}(\sigma) X^{\mu_{\sigma(1)}}_{\mu_1} X^{\mu_{\sigma(2)}}_{\mu_2} \cdots X^{\mu_{\sigma(n)}}_{\mu_n},$$

$$L^\mu_\nu = \delta^\mu_\nu - (\sqrt{|g| - 1} f)^\mu_\nu.$$  

This combination of interaction terms gives no Boulware–Deser ghost [9]. Here, $m^2$ is a coupling constant of the metrics and $\{\alpha_n\}_{n=1,2,3}$ are arbitrary constants. We define the reduced Planck constant $M_e$ as

$$\frac{1}{M_e^2} = \frac{1}{M_g^2} + \frac{1}{M_f^2},$$

where $M_e$ is chosen so that $m$ coincides with the Fierz–Pauli mass [20] when we take the massive gravity limit. Note that we can regard $\Lambda_1$ as the potential energy of a scalar field in the slow roll approximation coupled to the physical metric $g_{\mu\nu}$, as in general relativity.

In this paper, we consider the simplest case $\alpha_2 = 1, \alpha_1 = \alpha_3 = 0$. Then, the action is written as

$$S = \frac{M_g^2}{2} \int d^4x \sqrt{-g} (R[g_{\mu\nu}] - 2\Lambda_g) + \frac{M_f^2}{2} \int d^4x \sqrt{-f} (R[f_{\mu\nu}] - 2\Lambda_f) + m^2 M_e^2 \int d^4x \sqrt{-g} F_2[L^\mu_\nu].$$  

where

$$F_2[L^\mu_\nu] = \frac{1}{2}([L]^2 - [L^2]), \quad [L] = L^\mu_\mu, \quad [L^2] = L^\nu_\nu L^\mu_\mu.$$  

We now present the basic equations and derive the formulae that will be used in the later analysis.

### 2.1. De Sitter solutions in bimetric gravity

In this subsection, we consider homogeneous and isotropic solutions in bimetric gravity [10,21–26]. We derive equations of motion and show that the solutions are de Sitter spacetimes.

We take the homogeneous and isotropic metric ansatz for $g_{\mu\nu}$ and $f_{\mu\nu}$,

$$ds^2 = -N^2(t)dt^2 + e^{2\alpha(t)}[dx^2 + dy^2 + dz^2],$$

and

$$ds^2 = -M^2(t)dt^2 + e^{2\beta(t)}[dx^2 + dy^2 + dz^2],$$

respectively. $M, N$ are lapse functions and $\alpha, \beta$ describe the isotropic expansion of each metric. Substituting the metric ansatz into the action, we obtain the Lagrangian

$$\mathcal{L} = M_g^2 e^{3\alpha} \left[ -\frac{3\alpha^2}{N} - N\Lambda_g \right] + M_f^2 e^{3\beta} \left[ -\frac{3\beta^2}{M} - M\Lambda_f \right] + m^2 M_e^2 Ne^{3\alpha} \left[ 6 - 9\epsilon + 3\epsilon^2 + \gamma(-3 + 3\epsilon) \right],$$  

where

$$\gamma = \frac{M}{N}, \quad \epsilon = e^{\beta - \alpha}. $$
Taking the variation with respect to each variable, we obtain the equations of motion for $\alpha$ and $\beta$:

\[
\left(\frac{\alpha'}{N}\right)' - \xi a_g (M - N\epsilon) \left(\frac{3}{2} - \epsilon\right) = 0, \tag{7}
\]
\[
\left(\frac{\beta'}{M}\right)' + \xi (1 - a_g) \epsilon^{-3} (M - N\epsilon) \left(\frac{3}{2} - \epsilon\right) = 0, \tag{8}
\]

and two constraints:

\[
\left(\frac{\alpha'}{N}\right)^2 = \lambda_g + \xi a_g (2 - \epsilon)(\epsilon - 1), \tag{9}
\]
\[
\left(\frac{\beta'}{M}\right)^2 = \lambda_f + \xi (1 - a_g) \epsilon^{-3} (1 - \epsilon), \tag{10}
\]

where we normalized parameters and time with $M_e$ as follows:

\[
a_g = \frac{M_e^2}{M^2}, \quad \xi = \frac{m^2}{M_e^2}, \quad \lambda_g = \frac{\Lambda_g}{3M_e^2}, \quad \lambda_f = \frac{\Lambda_f}{3M_e^2}, \quad \epsilon' = \frac{1}{M_e} \frac{d}{dt}. \tag{11}
\]

We notice that $a_g$ can take a value in the range $0 < a_g < 1$ from the definition of $M_e$. The detailed derivation can be found in Appendix A.

In bimetric gravity, the diagonal part of general coordinate invariance is preserved. Hence, the two constraints contain a first class constraint and a second class constraint. Thus, there exists a secondary constraint. Now, from (7) and (9) (or (8) and (10)), we can deduce the equation

\[
\xi \left(\frac{3}{2} - \epsilon\right) \left(\frac{\beta' e^\beta}{M} - \frac{\alpha' e^\alpha}{N}\right) = 0. \tag{12}
\]

The first factor can be taken to be zero. However, this is a special solution and it is known that this leads to a pathology [27–31]. Hence, we take the following branch:

\[
M = \frac{\beta'}{\alpha'} N\epsilon. \tag{13}
\]

This is nothing but the condition determining the Lagrange multiplier. From (9), (10), and (13), we obtain the secondary constraint

\[
g(\epsilon) = (\lambda_f + \xi a_g) \epsilon^3 - 3\xi a_g \epsilon^2 + [-\lambda_g + 2\xi a_g - \xi (1 - a_g)] \epsilon + \xi (1 - a_g) = 0. \tag{14}
\]

From the definition of $\epsilon$, $\epsilon$ should be positive and hence we should look for positive roots of the algebraic equation $g(\epsilon) = 0$. Since $\xi$, $a_g$, $\lambda_g$, and $\lambda_f$ are constants, a positive root of $g(\epsilon) = 0$ is also a constant that we represent as $\epsilon_0$. Then, taking the derivative of the definition of $\epsilon$, we derive $\alpha' = \beta'$ and hence $M = N\epsilon_0$. Now, we take a gauge $N = 1$ using the gauge degree of freedom. Then, we get $M = \epsilon_0$ constant. From (7) and (8), we can deduce $\alpha'' = \beta'' = 0$, which can be solved as $\alpha = H_0 M t$, $\beta = H_0 M t + \log (\epsilon_0)$, where $H_0$ is the Hubble scale, which is determined from the constraints as

\[
H_0^2 = \frac{\lambda_g + \xi a_g (2 - \epsilon_0)(\epsilon_0 - 1)}{\epsilon_0} = \frac{\lambda_f \epsilon_0^2 + \xi (1 - a_g) \frac{1 - \epsilon_0}{\epsilon_0}}{\epsilon_0}. \tag{15}
\]

Thus, we obtain two de Sitter spacetimes with the relation $f_{\mu\nu} = \epsilon_0^2 g_{\mu\nu}$, provided that $\epsilon_0$ is a positive root of $g(\epsilon) = 0$ and $H_0^2 > 0$ holds for $\epsilon_0$. 

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2.2. Fate of the anisotropy

In this subsection, we consider homogeneous anisotropic perturbations and examine how the anisotropy evolves. We also derive the effective mass of the massive graviton [14].

We take the homogeneous anisotropic metric ansatz

\[ ds^2 = -N^2(t)dt^2 + e^{2\alpha(t)}[e^{-4\sigma(t)}dx^2 + e^{2\sigma(t)}(dy^2 + dz^2)], \]

and

\[ ds'^2 = -M^2(t)dt^2 + e^{2\beta(t)}[e^{-4\lambda(t)}dx^2 + e^{2\lambda(t)}(dy^2 + dz^2)], \]

where \( \sigma \) and \( \lambda \) describe the anisotropic expansion of each metric. Here we assume that the anisotropy is small. Substituting the metric ansatz into the action and dropping the higher order terms, we can derive the quadratic Lagrangian:

\[ \delta^2 L = M^2 g e^{3\alpha} \dot{\sigma}^2 + M^2 f e^{3\beta} \dot{\lambda}^2 + m^2 M^2 \epsilon \left[ -9 \epsilon + 3 \epsilon^2 + 3 \gamma \epsilon \right] q^2, \]

where we define the new variable

\[ q = \lambda - \sigma. \]

Note that \( \sigma \) and \( \lambda \) can be regarded as zero modes of gravitons. From the above action, we can deduce the equations for \( \sigma \) as

\[ \sigma'' + 3H_0 \sigma' - \xi a_g \epsilon_0 (3 - 2 \epsilon_0) q = 0, \]

and for \( \lambda \) as

\[ \lambda'' + 3H_0 \lambda' + \xi (1 - a_g) \frac{1}{\epsilon_0} (3 - 2 \epsilon_0) q = 0. \]

By taking the difference of (21) and (20), it is easy to obtain

\[ q'' + 3H_0 q' + \xi \left[ a_g \epsilon_0 + (1 - a_g) \frac{1}{\epsilon_0} \right] (3 - 2 \epsilon_0) q = 0. \]

From this equation, we can read off the effective mass of the massive graviton as

\[ m^2_{\text{eff}} = \xi \left[ a_g \epsilon_0 + (1 - a_g) \frac{1}{\epsilon_0} \right] (3 - 2 \epsilon_0). \]

Since the effective mass is different from the bare mass \( \xi \), it is non-trivial if the effective mass is less than the Hubble scale even if the bare mass is so. By making the combination (20) \( \times \frac{1}{a_g} \) + (21) \( \times \frac{\epsilon_0^2}{1 - a_g} \), we have

\[ \left[ e^{3H_0 t} \left( \frac{\sigma'}{a_g} + \epsilon_0^2 \frac{\lambda'}{1 - a_g} \right) \right]' = 0. \]

This leads to a conserved quantity

\[ E = e^{3H_0 t} \left( \frac{\sigma'}{a_g} + \epsilon_0^2 \frac{\lambda'}{1 - a_g} \right), \]

which means that the mode

\[ \frac{\sigma'}{a_g} + \epsilon_0^2 \frac{\lambda'}{1 - a_g} \]

corresponds to the massless graviton. The existence of the massless mode is a reflection of the diagonal general coordinate invariance. From the conservation law (25), we see that this mode vanishes exponentially fast.
If we substitute $q = e^{i \omega t}$ into Eq. (22), we obtain

$$q = A \exp i \omega_+ t + B \exp i \omega_- t,$$

where

$$\omega_\pm = i \frac{3H_0}{2} \pm \sqrt{m_{\text{eff}}^2 - \frac{9H_0^2}{4}}$$

and $A, B$ are integral constants. If $m_{\text{eff}}^2$ is negative, $q$ exponentially grows like

$$q \sim B \exp \left( \sqrt{\left| m_{\text{eff}}^2 \right| + \frac{9H_0^2}{4}} - \frac{3H_0}{2} \right) t.$$  

Inversely, if $m_{\text{eff}}^2$ is positive, $q$ exponentially decays. When $m_{\text{eff}}^2 - \frac{9H_0^2}{4} > 0$, the decay time scale $\tau$ is $\tau = 2/3H_0$. On the other hand, if $m_{\text{eff}}^2 - \frac{9H_0^2}{4} < 0$, the time scale is evaluated as

$$\tau^{-1} = \left| \omega_- \right| = \frac{3H_0}{2} - \sqrt{\frac{H_0^2}{4} + (2H_0^2 - m_{\text{eff}}^2)}.$$  

Therefore, the decay time scale of the anisotropy $\tau$ is shorter than the Hubble time scale $1/H_0$ for $m_{\text{eff}}^2 > 2H_0^2$, and the opposite holds for $m_{\text{eff}}^2 < 2H_0^2$.

3. Decay of the anisotropy: cases $\lambda_f = 0$

First, we consider the situation $\lambda_f = 0$. The constant $\lambda_g$ can be regarded as the potential energy of a scalar field coupled to $g_{\mu\nu}$ in the slow roll approximation. We prove that there exists a de Sitter solution for $\lambda_g > 0$ and that the solution is stable under the anisotropic perturbations. We also see that the effective mass of the massive graviton is bounded from below $m_{\text{eff}}^2 > 3H_0^2$. This suggests that the anisotropy rapidly decays in a Hubble time.

When we take $\lambda_f = 0$, (14) and (15) become

$$g(\epsilon) = \xi a_g \epsilon^3 - 3\xi a_g \epsilon^2 + \left[ -\lambda_g + 2\xi a_g - \xi (1 - a_g) \right] \epsilon + \xi (1 - a_g) = 0$$

and

$$H_0^2 = \lambda_g + \xi a_g (2 - \epsilon_0)(\epsilon_0 - 1)$$

$$= \xi (1 - a_g) \frac{1 - \epsilon_0}{\epsilon_0}.$$  

From the second line of (32), we see that $\epsilon_0$ should be less than 1 so that $H_0$ is a real number. Then, from the first line of (32), $\lambda_g$ should have a positive lower bound. We assume that $\lambda_g$ is positive in the following. Then, we obtain

$$g(0) = \xi (1 - a_g) > 0, \quad g(1) = -\lambda_g < 0, \quad g(\epsilon) \to +\infty \text{ as } \epsilon \to +\infty.$$  

Thus, there is a positive root smaller than 1 and a positive root larger than 1. The root larger than 1 does not satisfy the condition $H_0^2 > 0$. It turns out that there is a single positive root $\epsilon_0$ in the range $0 < \epsilon_0 < 1$, where $H_0$ is a real number, in the case in which $\lambda_g$ is positive.

Next, we will see that the de Sitter solution derived above is always stable under anisotropic perturbations. Apparently, massless modes rapidly decay on the Hubble time scale. Then the stability
under the perturbations of $\sigma$ and $\lambda$ is determined by the sign of the mass term of the perturbation equation (22), as mentioned in Sect. 2.2, or the sign of

$$m_{\text{eff}}^2 = \xi \left[ a_g \epsilon_0 + \left( 1 - a_g \right) \frac{1}{\epsilon_0} \right] (3 - 2\epsilon_0).$$

Since the de Sitter solution satisfies $0 < \epsilon_0 < 1$, $m_{\text{eff}}^2$ is positive. Therefore, the de Sitter solution is stable under the perturbations of $\sigma$ and $\lambda$.

Furthermore, we can prove that $m_{\text{eff}}^2$ is bounded from below

$$m_{\text{eff}}^2 > 3H_0^2.$$  (35)

The effective mass of the massive graviton is bounded by the Hubble scale from below. Using the analysis in Sect. 2.2, we can see that the anisotropy rapidly decays in a Hubble time.

4. Decay of the anisotropy: cases $\lambda_f \neq 0$

In this section, we construct de Sitter solutions with $\lambda_f \neq 0$. Then, we check the perturbative stability of the de Sitter solutions. Finally, we examine whether the effective mass of the massive graviton can be smaller than the Hubble scale.

4.1. De Sitter solutions

We study de Sitter solutions and give a classification of them. What we should check is whether the roots of $g(\epsilon) = 0$ are positive and satisfy $H_0^2 > 0$.

4.1.1. When are the roots of $g(\epsilon) = 0$ positive? Since the behavior of $g(\epsilon)$ is largely determined by the leading term, $\lambda_f + \xi a_g$, we discuss the following three cases separately.

(1) In the case $\lambda_f > -\xi a_g$, the coefficient of the leading term in $g(\epsilon)$ is positive, which indicates that

$$g(\epsilon) \to -\infty \quad \text{as} \quad \epsilon \to -\infty, \quad g(\epsilon) \to +\infty \quad \text{as} \quad \epsilon \to +\infty.$$  

Combining the above with $g(0) = \xi (1 - a_g) > 0$, we see that there always exists a negative root. Since $g''(0) = -6\xi a_g < 0$, the inflection point must exist on the positive side of $\epsilon$. Therefore, the number of positive solutions can be characterized by the discriminant of $g(\epsilon) = 0$. If the discriminant is zero, a multiple positive root exists. On the other hand, if the discriminant
is positive, two positive roots exist. The discriminant of \( g(\epsilon) = 0 \) is given by

\[
D = -27(1 - a_g)^2 \left( \frac{\lambda_f}{\xi} + a_g \right)^2 + 2\bar{c}[2\bar{c}^2 + 27a_g(1 - a_g)] \left( \frac{\lambda_f}{\xi} + a_g \right) + 9a_g^2[\bar{c}^2 + 12a_g(1 - a_g)],
\]

where we define

\[
\bar{c} = \lambda_g/\xi - 2a_g + (1 - a_g).
\]

The condition that the discriminant is non-negative reads

\[
\lambda_- \leq \lambda_f \leq \lambda_+,
\]

where we define

\[
\frac{\lambda_+}{\xi} + a_g = \frac{1}{27(1 - a_g)^2} \left\{ \frac{\bar{c}[2\bar{c}^2 + 27a_g(1 - a_g)] \pm 2[\bar{c}^2 + 9a_g(1 - a_g)]^{3/2}}{\xi} \right\}.
\]

We can see that \( \lambda_- < -\xi a_g \) and \( \lambda_+ > -\xi a_g \) from (39), taking into account the inequality

\[
|2[\bar{c}^2 + 9a_g(1 - a_g)]^{3/2} - |\bar{c}[2\bar{c}^2 + 27a_g(1 - a_g)]| > 0.
\]

Thus, for \( \lambda_f = \lambda_+ \), there exists a single multiple positive root of \( g(\epsilon) = 0 \). Since we are considering the range \( \lambda_f > -\xi a_g \), there exist two positive roots for \( -\xi a_g < \lambda_f < \lambda_+ \).  

(2) In the case \( \lambda_f = -\xi a_g \), \( g(\epsilon) \) becomes a quadratic function of \( \epsilon \). Since the coefficient of the leading term \( -3\xi a_g \) is negative and \( g(0) = \xi(1 - a_g) > 0 \), there exists a single positive root.

(3) In the case \( \lambda_f < -\xi a_g \), the coefficient of the leading term in \( g(\epsilon) \) is negative, which leads to

\[
g(\epsilon) \to +\infty \text{ as } \epsilon \to -\infty, \quad g(\epsilon) \to -\infty \text{ as } \epsilon \to +\infty.
\]

Because of the fact that \( g(0) = \xi(1 - a_g) > 0 \), there always exists a positive root. Since \( g''(0) = -6\xi a_g < 0 \), the inflection point exists on the negative side of \( \epsilon \) in this case. Thus, other possible roots should be negative. Namely, there exists a single positive root for \( \lambda_f < -\xi a_g \).

We found that two positive roots exist for \( -\xi a_g < \lambda_f < \lambda_+ \) and a single positive root exists for \( \lambda_f \leq -\xi a_g \) and \( \lambda_f = \lambda_+ \).

Next, we check whether these roots satisfy the condition \( H_0^2 > 0 \).

4.1.2. Is \( H_0^2 > 0 \) satisfied?  Rewriting the first line of (15) as

\[
H_0^2 = \lambda_g + \xi a_g(2 - \epsilon_0)(\epsilon_0 - 1)
= \xi a_g \left[ -\left( \epsilon_0 - \frac{3}{2} \right)^2 + \frac{\lambda_g}{\xi a_g} + \frac{1}{4} \right],
\]

we see that \( \lambda_g > -\xi a_g/4 \) is at least needed for \( H_0^2 > 0 \). Therefore, we assume that \( \lambda_g > -\xi a_g/4 \) below. Then, we can factorize (40) as

\[
H_0^2 = -\xi a_g(\epsilon_0 - \epsilon_p)(\epsilon_0 - \epsilon_m),
\]

where we define

\[
\epsilon_p = \frac{3}{2} + \sqrt{\frac{\lambda_g}{\xi a_g} + \frac{1}{4}}, \quad \epsilon_m = \frac{3}{2} - \sqrt{\frac{\lambda_g}{\xi a_g} + \frac{1}{4}}.
\]
Note that \( \epsilon_p \) and \( \epsilon_m \) do not depend on \( \lambda_f \). Thus, in order to have \( H_0^2 > 0 \), we have to seek positive roots of \( g(\epsilon) = 0 \) in the range

\[
\epsilon_m < \epsilon_0 < \epsilon_p. \tag{43}
\]

As we discussed in the previous subsection, \( \lambda_f \leq \lambda_+ \) is needed for the existence of positive roots. We first consider the case \( \lambda_f = \lambda_+ \), for which there exists a single positive root. In this case, we have to solve \( g(\epsilon_*) = g'(\epsilon_*) = 0 \), which gives rise to the equation

\[
a_g \epsilon_*^2 + \frac{2}{3} \epsilon_* - (1 - a_g) = 0. \tag{44}
\]

The positive root of this equation is given by

\[
\epsilon_* = \frac{-c + \sqrt{c^2 + 9a_g(1 - a_g)}}{3a_g}. \tag{45}
\]

Thus, we see that

\[
H_0^2(\epsilon_*) = \frac{2\xi \left( \frac{\lambda_f}{\xi} + \frac{a_g}{4} \right) \sqrt{c^2 + 9a_g(1 - a_g)}}{\left( \frac{\lambda_f}{\xi} + \frac{a_g}{4} \right) + \frac{9}{4}a_g + (1 - a_g) + \sqrt{c^2 + 9a_g(1 - a_g)}} > 0. \tag{46}
\]

Therefore, the inequality \( \epsilon_m < \epsilon_* < \epsilon_p \) must hold.

As we decrease \( \lambda_f \) while fixing \( \lambda_g, a_g, \xi \), the discriminant of \( g(\epsilon) = 0 \) becomes positive. Thus, there will be two positive roots until \( \lambda_f \) reaches \(-\xi a_g\). We shall call the smaller one the inner root, \( \epsilon_{in} \), and the other one the outer root, \( \epsilon_{out} \). We note that \( \epsilon_{in} \) is always smaller than \( \epsilon_* \) and \( \epsilon_{out} \) is always larger than \( \epsilon_* \) because

\[
g(0) = \xi(1 - a_g) > 0, \quad g(\epsilon_*) = \epsilon_*^3(\lambda_f - \lambda_+) < 0, \quad g(\epsilon) \to +\infty \quad \text{as} \quad \epsilon \to +\infty. \]

We can regard the \( \lambda_f \leq -\xi a_g \) case as the inner root because the inner root is continuously connected to the positive root for \( \lambda_f < -\xi a_g \) when \( \lambda_f \) crosses \(-\xi a_g \) below.

We shall evaluate the first derivative of \( \epsilon_0 \) with respect to \( \lambda_f \) since we want to know the behavior of the roots when we decrease \( \lambda_f \). Differentiating \( g(\lambda_f, \epsilon_0(\lambda_f)) = 0 \) with respect to \( \lambda_f \):

\[
\frac{dg(\lambda_f, x(\lambda_f))}{d\lambda_f} \bigg|_{x=\epsilon_0} = 0, \tag{47}
\]

we obtain

\[
\frac{d\epsilon_0}{d\lambda_f} = -\frac{\epsilon_0^3}{\frac{dg(\epsilon)}{d\epsilon} \bigg|_{\epsilon=\epsilon_0}}. \tag{48}
\]

First, we discuss the outer root. Since \( g(\epsilon_*) < 0 \) and \( g(\epsilon) \to +\infty \) as \( \epsilon \to +\infty \), the outer root always satisfies

\[
\frac{dg(\epsilon)}{d\epsilon} \bigg|_{\epsilon=\epsilon_{out}} > 0. \tag{49}
\]

Then, from (48), we can see that

\[
\frac{d\epsilon_{out}}{d\lambda_f} < 0. \tag{50}
\]

Therefore, \( \epsilon_{out} \) starts from \( \epsilon_* \) at \( \lambda_f = \lambda_+ \) and \( \epsilon_{out} \) monotonically increases as \( \lambda_f \) decreases. We can expect that \( \epsilon_{out} \) will reach \( \epsilon_p \) at some point. Indeed, \( \epsilon_{out} \) reaches \( \epsilon_p \) when \( \lambda_f \) becomes small as

\[
\lambda_p = \xi(1 - a_g) \frac{\epsilon_p - 1}{\epsilon_3^3} > 0, \tag{51}
\]

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Fig. 1. We plot $g(\epsilon)$ for $\lambda_g \geq 2\xi a_g$. We set $a_g = 0.5$, $\lambda_g/\xi = 1.1$. Then $\lambda_+ / \xi \simeq 1.41$. As $\lambda_f$ decreases, the outer root increases and the inner root decreases. When $\lambda_f$ reaches $\lambda_p$, the outer root crosses $\epsilon_p$ above and $H^2_0(\epsilon_{\text{out}})$ becomes negative. But the inner root always satisfies $H^2_0(\epsilon_{\text{in}}) > 0$ since $\epsilon_m$ is non-positive.

where we used the fact that $H^2_0 = 0$ at $\epsilon_p$. Therefore, $\epsilon_{\text{out}}$ exists in the range $(\epsilon_m, \epsilon_p)$ if and only if $\lambda_f > \lambda_p$. We mention that $\lambda_p \to +\infty$ when $\lambda_g \to +\infty$ since $\epsilon_p \to +\infty$ (see (42)).

Next, we discuss the inner root. In turn, since $g(0) = \xi (1 - a_g) > 0$ and $g(\epsilon_*) < 0$, the inner root always satisfies

$$\left. \frac{dg(x)}{dx} \right|_{x=\epsilon_{\text{in}}} < 0.$$  

(52)

Then, from (48), we can see that

$$\frac{d\epsilon_{\text{in}}}{d\lambda_f} > 0.$$  

(53)

Therefore, $\epsilon_{\text{in}}$ starts from $\epsilon_*$ at $\lambda_f = \lambda_+$ and monotonically decreases as $\lambda_f$ decreases. Note that $\epsilon_{\text{in}} \to \left( \frac{\xi (1 - a_g)}{\lambda_f} \right)^{1/3} \to +0$ as $\lambda_f \to -\infty$. We can expect that $\epsilon_{\text{in}}$ will reach $\epsilon_m$ at some point. To see this, we need to notice that

$$\epsilon_m = \frac{3}{2} - \sqrt{\frac{\lambda_g}{\xi a_g} + \frac{1}{4}} = \frac{3}{2} - \sqrt{\frac{\lambda_g - 2\xi a_g}{\xi a_g} + 9}.$$  

(54)

changes the sign at $\lambda_g = 2\xi a_g$. Hence, we can consider the following two cases.

1. In the case $\lambda_g \geq 2\xi a_g$, $\epsilon_m$ is non-positive. Then $\epsilon_{\text{in}}$ cannot reach $\epsilon_m$ when we decrease $\lambda_f$. Therefore, $\epsilon_{\text{in}}$ always exists in the range $(\epsilon_m, \epsilon_p)$ and satisfies $H^2_0(\epsilon_{\text{in}}) > 0$ (see Fig. 1).

2. In the case $-\frac{1}{4}\xi a_g < \lambda_g < 2\xi a_g$, $\epsilon_m$ is positive. Then $\epsilon_{\text{in}}$ can reach $\epsilon_m$ when we decrease $\lambda_f$. Indeed, $\epsilon_{\text{in}}$ reaches $\epsilon_m$ when $\lambda_f$ becomes small as

$$\lambda_m = \xi (1 - a_g) \frac{\epsilon_m - 1}{\epsilon_m^3},$$  

(55)

where we used the fact that $H^2_0 = 0$ at $\epsilon_m$. Therefore, $\epsilon_{\text{in}}$ exists in the range $(\epsilon_m, \epsilon_p)$ and satisfies $H^2_0 > 0$ if and only if $\lambda_f > \lambda_m$. We mention that $\lambda_m \to -\infty$ when $\lambda_g \to 2\xi a_g - 0$ because $\epsilon_m \to +0$ (see (54)). In Fig. 2, we illustrate these features.
Fig. 2. We plot $g(\epsilon)$ for $-\frac{1}{4}\xi a_g < \lambda_g < 2\xi a_g$. We set $a_g = 0.5$, $\lambda_f/\xi = 0.5$. Then $\lambda_+/\xi = 0.5$. As $\lambda_f$ decreases, the outer root increases and the inner root decreases. When $\lambda_f$ reaches $\lambda_p$, the outer root crosses $\epsilon_p$ above and $H_0^2(\epsilon_{\text{out}})$ becomes negative. When $\lambda_f$ reaches $\lambda_m$, the inner root crosses $\epsilon_m$ below and $H_0^2(\epsilon_{\text{in}})$ becomes negative.

Table 1. For $\lambda_g \geq 2\xi a_g$.

| inner     | outer |
|-----------|-------|
| $\lambda_+ < \lambda_f$ | $\times$ | $\times$ |
| $\lambda_f = \lambda_+$ | $\circ$ | $\circ$ |
| $\lambda_p < \lambda_f < \lambda_+$ | $\circ$ | $\circ$ |
| $\lambda_f \leq \lambda_p$ | $\circ$ | $\times$ |

Table 2. For $-\frac{1}{4}\xi a_g < \lambda_g < 2\xi a_g$.

| inner     | outer |
|-----------|-------|
| $\lambda_+ < \lambda_f$ | $\times$ | $\times$ |
| $\lambda_f = \lambda_+$ | $\circ$ | $\circ$ |
| $\lambda_p < \lambda_f < \lambda_+$ | $\circ$ | $\circ$ |
| $\lambda_m < \lambda_f \leq \lambda_p$ | $\circ$ | $\times$ |
| $\lambda_f \leq \lambda_m$ | $\times$ | $\times$ |

We note that $\lambda_p > \lambda_m$ when $-\frac{1}{4}\xi a_g < \lambda_g < 2\xi a_g$. We can see this from the definitions of $\lambda_p$ and $\lambda_m$ as

$$\lambda_p - \lambda_m = \xi (1 - a_g) \frac{8 \left( \frac{\lambda_g}{\xi a_g} + \frac{1}{4} \right)^{\frac{3}{2}}}{\left( 2 - \frac{\lambda_g}{\xi a_g} \right)^{\frac{3}{2}}} > 0. \quad (56)$$

We summarize the results derived in this subsection in Table 1, Table 2, and Fig. 3.

In the tables, “$\circ$” means that there exists a positive root of $g(\epsilon) = 0$, which satisfies $H_0^2 > 0$, i.e., a de Sitter solution exists. Also, “$\times$” means that there exists no positive root or there exists a positive root for $g(\epsilon) = 0$ but $H_0^2 \leq 0$, i.e., no de Sitter solution exists. For $\lambda_g \leq -\frac{1}{4}\xi a_g$, there is no root satisfying $H_0^2 > 0$. Surprisingly, we have an upper bound for $\lambda_f$ and there exist de Sitter solutions even for arbitrary large negative $\lambda_f$ in the case $\lambda_g \geq 2\xi a_g$. We note that $\lambda_g - 2\xi a_g$ can be interpreted as an effective cosmological constant if we see the explicit constant term in the first line of (15). It is
remarkable that there also exists a de Sitter solution for the case in which the effective cosmological constant is zero.

4.2. Stability of de Sitter solutions

In this subsection, we examine the stability of de Sitter solutions. In Sect. 2.2, we saw that the sign of $m_{\text{eff}}^2$ determines the stability of de Sitter solutions, i.e., solutions are stable if $m_{\text{eff}}^2$ is positive and unstable if $m_{\text{eff}}^2$ is negative. Recalling the formula

$$m_{\text{eff}}^2 = \xi \left[ a_g \epsilon_0 + \left( 1 - a_g \right) \frac{1}{\epsilon_0} \right] (3 - 2 \epsilon_0),$$

we can see that $m_{\text{eff}}^2$ is positive when $\epsilon_0 < \frac{3}{2}$ and negative when $\epsilon_0 > \frac{3}{2}$. From now on, we suppose that $\lambda_g > -\frac{1}{4} \xi a_g$, so that $H_0^2 > 0$ is satisfied.

We know that $g(\epsilon) = 0$ has positive roots when $\lambda_f \leq \lambda_+$. We first consider the $\lambda_f = \lambda_+$ case, where there exists a multiple positive root $\epsilon_*$. Since we supposed that $\lambda_g > -\frac{1}{4} \xi a_g$, we can evaluate $\epsilon_*$ as

$$\frac{3}{2} - \epsilon_* = \frac{3 \left( \frac{\lambda_g}{\xi} + \frac{a_g}{4} \right) + \frac{9}{4} a_g + (1 - a_g) + \sqrt{c^2 + 9 a_g (1 - a_g)}}{\left( \frac{\lambda_g}{\xi} + \frac{a_g}{4} \right) + \frac{9}{4} a_g + (1 - a_g) + \sqrt{c^2 + 9 a_g (1 - a_g)}} > 0. \quad (57)$$

Thus, $m_{\text{eff}}^2$ is positive. Therefore, we find that the de Sitter solution corresponding to the multiple root is stable.

Next, we decrease $\lambda_f$ from $\lambda_+$. The inner root always satisfies $\epsilon_\text{in} < \epsilon_*$, as we mentioned in Sect. 4.1.2. We know that $\epsilon_*$ is smaller than $\frac{3}{2}$. Therefore, the inner root is always stable since $\epsilon_\text{in} < \epsilon_* < \frac{3}{2}$. On the other hand, the outer root always satisfies $\epsilon_\text{out} > \epsilon_*$. Since $\epsilon_*$ is smaller than $\frac{3}{2}$ and the outer root monotonically increases as $\lambda_f$ decreases, we can expect that $\epsilon_\text{out}$ will reach $\frac{3}{2}$ at
the value of $\epsilon$ from this expression, it is obvious that

$$\lambda_2 = \frac{4}{27} \left[ 3 \left( \lambda_g + \frac{\xi a_g}{4} \right) + \xi (1 - a_g) \right] > 0,$$

and the Hubble scale reads

$$H_0^2 \left( \frac{3}{2} \right) = \lambda_g + \frac{\xi a_g}{4} > 0.$$  

Note that $\lambda_\rho < \lambda_2 < \lambda_+$ because $\epsilon_\rho < \frac{3}{2} < \epsilon_2$ (see (42) and (57)). Therefore, the outer root is stable when $\lambda_f \geq \lambda_2$ and unstable when $\lambda_f < \lambda_2$.

### 4.3. Appearance of the Higuchi bound

In this subsection, we will evaluate the effective mass of the massive graviton corresponding to the anisotropy.

From the definition of $m_{\text{eff}}^2$ and the first line of (15), we can deduce the following expression:

$$m_{\text{eff}}^2(\epsilon_0) - 2H_0^2(\epsilon_0) = -\frac{3\xi}{\epsilon_0} \left[ a_g \epsilon_0^2 + \frac{2}{3} \bar{c} \epsilon_0 - (1 - a_g) \right]$$

$$= \frac{3\xi a_g}{\epsilon_0} (\epsilon_* - \epsilon_0)(\epsilon_0 - \epsilon_2),$$

where $\epsilon_*$ is given in (45) and we define

$$\epsilon_2 = -\bar{c} - \sqrt{\bar{c}^2 + 9a_g(1 - a_g)} < 0.$$  

Since $\epsilon_2$ is negative, the sign of $m_{\text{eff}}^2 - 2H_0^2$ depends on that of $(\epsilon_* - \epsilon_0)$. Namely, $\epsilon_0 = \epsilon_*$ is equivalent to $m_{\text{eff}}^2 = 2H_0^2$, $\epsilon_0 < \epsilon_* \leq \epsilon_\rho$ leads to $m_{\text{eff}}^2 > 2H_0^2$, and $\epsilon_* < \epsilon_0$ leads to $m_{\text{eff}}^2 < 2H_0^2$. When $\lambda_f = \lambda_+$, the multiple root $\epsilon_*$ obviously satisfies $m_{\text{eff}}^2 = 2H_0^2$. When $\lambda_f < \lambda_+$, there are two positive roots for $g(\epsilon) = 0$. The inner root always satisfies $\epsilon_{\text{in}} < \epsilon_*$, as we mentioned in Sect. 4.1.2. Hence, the inner root always satisfies $m_{\text{eff}}^2 > 2H_0^2$. On the other hand, the outer root always satisfies $\epsilon_{\text{out}} > \epsilon_*$. Therefore, the outer root always satisfies $m_{\text{eff}}^2 < 2H_0^2$.

Remarkably, the equation $m_{\text{eff}}^2 - 2H_0^2 = 0$ coincides with the equation determining the multiple root $\epsilon_*$ (see (44) and (60)). This is the reason why the bifurcation point of de Sitter solutions is exactly the same as the Higuchi bound.

Note that the anisotropy decays more rapidly than the Hubble time scale $1/H_0$ for the inner root and it decays more slowly than $1/H_0$ or exponentially grows for the outer root if we use the analysis of Sect. 2.2.

Finally, we shall see that the ratio of the effective mass to the Hubble scale monotonically varies along the line that the value of $\epsilon_0$ is constant on the $\lambda_g$-$\lambda_f$ plane. We define $\zeta$ as the ratio of the effective mass to the Hubble scale,

$$\zeta = \frac{m_{\text{eff}}^2}{H_0^2} = \frac{\xi [a_g \epsilon_0 + (1 - a_g) \frac{1}{\epsilon_0}] (3 - 2\epsilon_0)}{\lambda_g + \xi a_g (2 - \epsilon_0)(\epsilon_0 - 1)}.$$  

From this expression, it is obvious that $\partial \zeta / \partial \lambda_g |_{\epsilon_0 = \text{const.}} < 0$ for $\epsilon_0 > \frac{3}{2}$ where $\zeta$ is positive, and $\partial \zeta / \partial \lambda_g |_{\epsilon_0 = \text{const.}} > 0$ for $\epsilon_0 < \frac{3}{2}$ where $\zeta$ is negative.
Fig. 4. We plot the $\zeta = \text{const.}$ curves of the inner root on the $\lambda_g - \lambda_f$ plane. We set $a_g = 0.5$. The inner root is always stable since $m_{\text{eff}}^2$ is positive. In this figure, the $\zeta = 2$ and $\zeta = +\infty$ curves coincide with the $\lambda_f = \lambda_+$ and $\lambda_f = \lambda_m$ curves, respectively. Note that, if we start from a point on $\lambda_f = \lambda_+$, $\zeta$ monotonically increases along the $\epsilon_{\text{in}} = \text{const.}$ line.

Fig. 5. We plot the $\zeta = \text{const.}$ curves of the outer root on the $\lambda_g - \lambda_f$ plane. We set $a_g = 0.5$. The outer root is stable above $\lambda_\frac{1}{2}$ where $m_{\text{eff}}^2$ is positive and unstable below $\lambda_\frac{1}{2}$ where $m_{\text{eff}}^2$ is negative. In this figure, the $\zeta = 2$, $\zeta = 0$, and $\zeta = -\infty$ curves coincide with the $\lambda_f = \lambda_+$, $\lambda_f = \lambda_\frac{1}{2}$, and $\lambda_f = \lambda_p$ curves, respectively. We see that, if we start from a point on $\lambda_f = \lambda_+$, $\zeta$ monotonically decreases along the $\epsilon_{\text{out}} = \text{const.}$ line in the stable region and monotonically increases in the unstable region.

We will check how the line where $\epsilon_0$ is constant can be drawn on the $\lambda_g - \lambda_f$ plane. When we fix the value of $\epsilon_0$, $g(\epsilon_0) = 0$ gives the relation between $\lambda_g$ and $\lambda_f$ as

$$\lambda_f = \frac{1}{\epsilon_0^2} \lambda_g - \xi a_g + \frac{3a_g \xi \epsilon_0^2 + (1 - 3a_g) \xi \epsilon_0 - \xi (1 - a_g)}{\epsilon_0^3}. \quad (62)$$
On the $\lambda_g - \lambda_f$ plane, each point in the $\lambda_f < \lambda_+$ region determines two lines: one for the inner root and the other for the outer root. From the fact that

$$
\frac{d\lambda_+}{d\lambda_g} = \left( \frac{-c + \sqrt{c^2 + 9a_g(1 - a_g)}}{3(1 - a_g)} \right)^2 = \frac{1}{\epsilon^2},
$$

(63)

each line is tangential to the $\lambda_f = \lambda_+$ curve. We also know that $\lambda_f = \lambda_+$ is a convex function since, from

$$
\frac{de_0}{d\lambda_g} = \frac{-\epsilon_0}{\xi_0 \sqrt{c^2 + 9a_g(1 - a_g)}} < 0,
$$

(64)

we can obtain

$$
\frac{d^2\lambda_+}{d\lambda_g^2} = -\frac{2d\epsilon_0}{\epsilon_0^2 d\lambda_g} > 0.
$$

(65)

Therefore, the series of lines cover the whole region satisfying $\lambda_f < \lambda_+$.

Using these formulae, we plotted Figs. 4 and 5.

5. Conclusion

We investigated the cosmic no-hair conjecture in the presence of spin-2 matter. More precisely, we studied the cosmic no-hair conjecture in bimetric gravity using the perturbative method. First, we analyzed de Sitter solutions and found that there are two branches of de Sitter solutions. The ratio of the scale factors of the metrics $\epsilon$ has a different range in each branch; one has $\epsilon < \epsilon_*$ and the other has $\epsilon > \epsilon_*$. Next, we examined the fate of homogeneous anisotropic perturbations. Since the effective mass depends on the background quantity $\epsilon$, the fate of the anisotropy is determined by $\epsilon$. Indeed, the branch satisfying $\epsilon < \epsilon_*$ is stable, but the other is stable in the range $\epsilon_* < \epsilon < \frac{3}{2}$ and unstable in the range $\frac{3}{2} < \epsilon$. Furthermore, we found that the stable branch satisfies the Higuchi bound and the other does not, since the bifurcation point of the two branches exactly coincides with the Higuchi bound, i.e., $\epsilon_*$ realizes $m_\text{eff}^2 = 2H_0^2$. It is non-trivial that the effective mass coincides with the Higuchi bound, on the boundary of the region where de Sitter solutions exist. The fact that the stable branch is bounded by the Higuchi bound, which has the order of the Hubble scale from below, means that the cosmic no-hair holds for the branch. Thus, we concluded that there exists a de Sitter solution for which the cosmic no-hair holds and the effective mass satisfies the Higuchi bound under the homogeneous anisotropic perturbations. Since the cosmic no-hair conjecture is already violated at the linear level in known cases, our result indicates that the cosmic no-hair conjecture is correct in bimetric gravity, even though we have not given the nonlinear analysis.

For future work, it would be interesting to explore the meaning behind the curious fact that the bifurcation point of the two branches of de Sitter solutions coincides with the Higuchi bound. Moreover, since there is no violation of the cosmic no-hair for at least one branch of de Sitter solutions in our analysis, we can consider inflation in bimetric gravity without pathologies. It would be important to clarify what kind of signatures peculiar to bimetric gravity appear, e.g., in the cosmic microwave background radiation.

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Appendix A. Derivation of basic equations

In this appendix, we derive the basic equations.

**Ansatz and Lagrangian**

We start with the anisotropic metric ansatz for $g_{\mu\nu}$ and $f_{\mu\nu}$:

\[
  ds^2 = -N^2(t)dt^2 + e^{2\alpha(t)}[e^{-4\sigma(t)}dx^2 + e^{2\sigma(t)}(dy^2 + dz^2)],
\]

and

\[
  ds'^2 = -M^2(t)dt^2 + e^{2\beta(t)}[e^{-4\lambda(t)}dx^2 + e^{2\lambda(t)}(dy^2 + dz^2)],
\]

respectively. From these metrics, scalar curvatures are calculated as

\[
  R[g_{\mu\nu}] = \frac{1}{N^2}(-6\dot{\alpha}^2 + 6\dot{\sigma}^2), \quad R[f_{\mu\nu}] = \frac{1}{M^2}(-6\dot{\beta}^2 + 6\dot{\lambda}^2).
\]

(A1)

Moreover, $g^{-1}f$ is given by

\[
  g^{-1}f = \begin{pmatrix}
    (M/N)^2 & e^{2\beta - 2\alpha - 4\lambda + 4\sigma} \\
    e^{2\beta - 2\alpha + 2\lambda - 2\sigma} & e^{2\beta - 2\alpha + 2\lambda - 2\sigma} \\
    e^{2\beta - 2\alpha + 2\lambda - 2\sigma} & e^{2\beta - 2\alpha + 2\lambda - 2\sigma}
  \end{pmatrix} = \begin{pmatrix}
    A^2 & B^2 \\
    B & B
  \end{pmatrix},
\]

where we have defined the variables as

\[
  \gamma = \frac{M}{N}, \quad \epsilon = e^{\beta - \alpha}, \quad \eta = e^{\lambda - \sigma}, \quad A = \epsilon\eta^{-2} = e^{\beta - \alpha - 2\lambda + 2\sigma}, \quad B = \epsilon\eta = e^{\beta - \alpha + \lambda - \sigma}.
\]

Thus, we obtain

\[
  L = 1 - \sqrt{g^{-1}f} = \begin{pmatrix}
    1 - \gamma \\
    1 - A \\
    1 - B
  \end{pmatrix},
\]

(A2)

Then, we can calculate the interaction term as

\[
  F_2 = \frac{1}{2}(L^2 - L^2) = \frac{1}{2}(4A - 2B - \gamma^2 - (4 - 2A - 4B + A^2 + 2B^2 - 2\gamma + \gamma^2)) = [6 - 3A - 6B + B(2A + B) + \gamma(-3 + A + 2B)].
\]

(A3)

Therefore, the Lagrangian reads

\[
  \mathcal{L} = M_S^2 e^{3\alpha} \left[ \frac{3}{N}(-\dot{\alpha}^2 + \dot{\sigma}^2) - N\Lambda_S \right] + M_f^2 e^{3\beta} \left[ \frac{3}{M}(\dot{\beta}^2 + \dot{\lambda}^2) - M\Lambda_f \right] + m^2 M_S^2 N e^{3\alpha} [6 - 3A - 6B + B(2A + B) + \gamma(-3 + A + 2B)].
\]

(A4)
Equations of motion and constraints

We normalize parameters and time with $M_e$ as follows:

\[ a_g = \frac{M_e^2}{M^2}, \quad \xi = \frac{m^2}{M_e^2}, \quad \lambda_g = \frac{\Lambda_g}{3M_e^2}, \quad \lambda_f = \frac{\Lambda_f}{3M_e^2}, \quad \gamma = \cdot / M_e. \]

Note that $0 < a_g < 1$ from the definition of $M_e$.

From the Lagrangian, we obtain the equations of motion

\[ \left( \frac{\alpha'}{N} \right)' + 3\frac{\sigma'^2}{N} + \frac{1}{6}\xi a_g[N(3A + 6B - 2B(2A + B)) - M(9 - 2A - 4B)] = 0, \quad (A5) \]

\[ \left( \frac{\beta'}{M} \right)' + 3\frac{\lambda'^2}{M} - \frac{1}{6}\xi(1 - a_g)\frac{1}{\epsilon^3}[N(3A + 6B - 2B(2A + B)) - M(9 - 2A - 4B)] = 0, \quad (A6) \]

\[ \left( \frac{\sigma'}{N} \right)' = \lambda_g + 3\frac{\alpha' \sigma'}{N} + \frac{1}{3}\xi a_g(A - B)[N(3 - B) - M] = 0, \quad (A7) \]

\[ \left( \frac{\lambda'}{M} \right)' = \frac{3\beta' \lambda'}{M} - \frac{1}{3}\xi(1 - a_g)\frac{1}{\epsilon^3}(A - B)[N(3 - B) - M] = 0, \quad (A8) \]

and the constraints

\[ \left( \frac{\alpha'}{N} \right)^2 - \left( \frac{\sigma'}{N} \right)^2 = \lambda_g + \frac{1}{3}\xi a_g[-6 + 3A + 6B - B(2A + B)], \quad (A9) \]

\[ \left( \frac{\beta'}{M} \right)^2 - \left( \frac{\lambda'}{M} \right)^2 = \lambda_f + \frac{1}{3}\xi(1 - a_g)\frac{1}{\epsilon^3}(3 - A - 2B). \quad (A10) \]

It is easy to find the consistency relation

\[ \frac{M}{N} = \frac{\beta'(3A + 6B - 2B(2A + B)) - \lambda'(2A - 2B)(3 - B)}{\alpha'(9 - 2A - 4B) - \sigma'(2A - 2B)}. \quad (A11) \]

From the linear combination of (A7) and (A8), we can also obtain a conserved quantity:

\[ E = \frac{1}{a_g} \frac{e^{3\alpha' \sigma'}}{N} + \frac{1}{1 - a_g} \frac{e^{3\beta' \lambda'}}{M}. \quad (A12) \]

References

[1] R. M. Wald, Phys. Rev. D 28, 2118 (1983).
[2] M.-a. Watanabe, S. Kanno, and J. Soda, Phys. Rev. Lett. 102, 191302 (2009).
[3] M.-a. Watanabe, S. Kanno, and J. Soda, Prog. Theor. Phys. 123, 1041 (2010).
[4] C. J. Isham, A. Salam, and J. A. Strathdee, Phys. Rev. D 3, 867 (1971).
[5] C. de Rham and G. Gabadadze, Phys. Rev. D 82, 044020 (2010).
[6] C. de Rham, G. Gabadadze, and A. J. Tolley, Phys. Rev. Lett. 106, 231101 (2011).
[7] S. F. Hassan and R. A. Rosen, Phys. Rev. Lett. 108, 041101 (2012).
[8] S. F. Hassan and R. A. Rosen, J. High Energy Phys. 1202, 126 (2012).
[9] S. F. Hassan and R. A. Rosen, J. High Energy Phys. 1204, 123 (2012).
[10] M. S. Volkov, J. High Energy Phys. 1201, 035 (2012).
[11] A. Higuchi, Nucl. Phys. B 282, 397 (1987).
[12] C. de Rham and S. Renaux-Petel, arXiv:1206.3482 [hep-th].
[13] M. Fasiello and A. J. Tolley, arXiv:1206.3852 [hep-th].
[14] S. F. Hassan, A. Schmidt-May, and M. von Strauss, arXiv:1208.1797 [hep-th].
[15] D. G. Boulware and S. Deser, Phys. Rev. D 6, 3368 (1972).
[16] T. Damour, I. I. Kogan, and A. Papazoglou, Phys. Rev. D 66, 104025 (2002).
[17] T. Damour and I. I. Kogan, Phys. Rev. D 66, 104024 (2002).
[18] N. Khosravi, N. Rahmanpour, H. R. Sepangi, and S. Shahidi, Phys. Rev. D 85, 024049 (2012).
[19] K. Nomura and J. Soda, arXiv:1207.3637 [hep-th].
[20] M. Fierz and W. Pauli, Proc. R. Soc. Lond. A 173, 211 (1939).
[21] M. von Strauss, A. Schmidt-May, J. Enander, E. Mortsell, and S. F. Hassan, J. Cosmol. Astropart. Phys. 1203, 042 (2012).
[22] D. Comelli, M. Crisostomi, F. Nesti, and L. Pilo, J. High Energy Phys. 1203, 067 (2012); 1206, 020 (2012) [erratum].
[23] M. S. Volkov, Phys. Rev. D 86, 061502 (2012).
[24] M. Berg, I. Buchberger, J. Enander, E. Mortsell, and S. Sjors, arXiv:1206.3496 [gr-qc].
[25] S. F. Hassan, A. Schmidt-May, and M. von Strauss, arXiv:1208.1515 [hep-th].
[26] Y. Akrami, T. S. Koivisto, and M. Sandstad, arXiv:1209.0457 [astro-ph.CO].
[27] A. De Felice, A. E. Gumrukcuoglu, and S. Mukohyama, Phys. Rev. Lett. 109, 171101 (2012).
[28] A. E. Gumrukcuoglu, C. Lin, and S. Mukohyama, Phys. Lett. B 717, 295 (2012).
[29] G. Tasinato, K. Koyama, and G. Niz, arXiv:1210.3627 [hep-th].
[30] A. E. Gumrukcuoglu, C. Lin, and S. Mukohyama, J. Cosmology Astropart. Phys. 1203, 006 (2012).
[31] D. Comelli, M. Crisostomi, and L. Pilo, J. High Energy Phys. 1206, 085 (2012).