A Pseudo-Formal Linearization Using Chebyshev Expansion and Its Application to Nonlinear Observer for Nonlinear Scalar-Measurement Systems

Kazuo Komatsu\(^1\) and Hitoshi Takata\(^2\)

\(^1\)National Institute of Technology, Kumamoto College, 2659-2 Suya, Koshi, Kumamoto 861-1102, Japan
\(^2\)Kagoshima University, 1-21-40 Korimoto, Kagoshima 890-0065, Japan
E-mail: \(^1\)kaz@kumamoto-nct.ac.jp

Abstract This paper is concerned with a pseudo-formal linearization method using Chebyshev expansion and its application to a nonlinear observer for nonlinear scalar-measurement systems. The given nonlinear autonomous dynamic system is linearized into an augmented linear system with respect to a linearization function that consists of polynomials of state variables by a pseudo-formal linearization method using Chebyshev expansion. As an application of this method, a nonlinear observer is discussed. An augmented measurement vector that consists of polynomials of measurement data is introduced and is transformed into an augmented linear one by the pseudo-formal linearization technique using Chebyshev expansion. Thereby, a linear system theory is applied to both the linearized dynamic and measurement systems in order to design a new nonlinear observer. Numerical experiments indicate that the performance of the presented method is superior to that of the previous method.

Keywords: nonlinear system, pseudo-formal linearization, nonlinear observer, Chebyshev expansion, linearization function

1. Introduction

When treating nonlinear problems for systems described by nonlinear differential equations, it might be difficult to implant design techniques in practical systems because of their nonlinearities. One of the solutions is to exploit linearization techniques so that the theories for linear systems can be applied to them. Such solutions \([1]-[7]\), mainly using geometric methods have been proposed, including pseudo-formal linearization. In our previous works \([8]-[9]\), we presented a pseudo-formal linearization by means of a piecewise Taylor expansion considering up to higher-order terms and an automatic choosing function \([10]\), which can improve the accuracy of a formal linearization method.

In this paper, we develop pseudo-formal linearization using Chebyshev expansion. In the previous works, we applied the Taylor expansion for linearization, which has the property that an approximation error grows rapidly when the state value departs from the nominal operating point, whereas Chebyshev expansion is one of the best approximations under the maximum norm. Thus, the accuracy of the linearization is expected to be improved with the introduction of a formal linearization function that consists of polynomials, a nonlinear autonomous dynamic system is linearized by pseudo-formal linearization with respect to the formal linearization function based on Chebyshev expansion. As an application of this method, we introduce a nonlinear observer for nonlinear scalar-measurement systems. To improve the accuracy of the nonlinear observer, an augmented measurement vector \([8]\) is introduced and then the given nonlinear measurement equation is also linearized with respect to the formal linearization function by pseudo-formal linearization using Chebyshev expansion. Thus, the linear system theories can be applied to these linearized systems to design a nonlinear observer. This approach might be highly useful in, for example, augmented automatic choosing control \([10]\).

Numerical experiments show that the presented method is superior to the previous method \([8]\).
2. Statement of Problems

Nonlinear autonomous dynamic and measurement equations are described by
\[
\begin{align*}
\Sigma_1: \dot{x}(t) &= f(x(t)), \quad x(0) = x_0 \in D \subset \mathbb{R}^n \quad (1) \\
\eta(t) &= h(x(t)) \subset R \quad (2)
\end{align*}
\]
where \( t \) denotes time, \( \cdot = d/dt \), \( x = [x_1, \ldots, x_n]^T \) is an \( n \)-dimensional state vector, and \( f \) is a sufficiently smooth nonlinear vector-valued function. \( D \) is a domain denoted by the Cartesian product \( D = \prod_{i=1}^n [m_i - p_i, m_i + p_i] \ (m_i \in R, p_i > 0) \). \( \eta \) is a scalar measurement and \( h \) is a nonlinear function and continuously differentiable. The problems are to translate these nonlinear systems into linear systems more accurately, and to determine the state of the nonlinear dynamic system from the given measurement data \( \eta \).

3. Pseudo-Formal Linearization

First, we introduce a vector-valued separable function in order to linearize the given nonlinear state differential equation in Eq.(1) as
\[
C : D \rightarrow \mathbb{R}^L \quad (3)
\]
We set \( C = [I : 0] \) (\( I : L \times L \) unit matrix) for simplicity. For example, if \( x_1 \) is the element of \( x \) that gives the highest nonlinearity of Eq.(1) (see Eq.(30)), the function is set to
\[
C(x) = x_1 \in D \subset R \quad (L = 1)
\]
Considering the nonlinearity of the given nonlinear dynamic system, the domain \( D \) is divided into \((M + 1)\) subdomains (see Fig.1):
\[
D = \bigcup_{k=0}^M D_k \quad (4)
\]
where \( D_M = D - \bigcup_{k=0}^{M-1} D_k \) and \( C^{-1}(D_0) \ni 0 \). \( D_k (0 \leq k \leq M) \) endowed with a lexicographic order is the Cartesian product \( D_k = \prod_{j=1}^L (a_{kj}, b_{kj}) \), \( (a_{kj} < b_{kj}) \).

We here introduce an automatic choosing function of the sigmoid type [10],
\[
I_k(\zeta) = \prod_{j=1}^L \left[ 1 - \frac{1}{1 + \exp(2\mu(\zeta_j - a_{kj}))} \right] - \frac{1}{1 + \exp(-2\mu(\zeta_j - b_{kj}))} \right], \quad (0 \leq k \leq M - 1) \quad (5)
\]

Fig. 1 Pseudo-formal linearization using Chebyshev expansion

\[
I_M(\zeta) = 1 - \sum_{k=0}^{M-1} I_k(\zeta)
\]
so that
\[
\sum_{k=0}^M I_k(\zeta) = 1 \quad (6)
\]
where \( \zeta = [\zeta_1, \ldots, \zeta_L]^T = C(x) \) and \( \mu \) is a positive real value. \( I_k(\zeta) \) is analytic and almost unity on \( D_k \), otherwise it is almost zero (see Fig. 1).

Secondly, the state vector \( x \) is changed into \( y \) to apply Chebyshev expansion, so that \( y \) has the basic domain of the Chebyshev polynomials \( D_0 = \prod_{i=1}^L [-1, 1] \) and \( y \) is rewritten as
\[
y = \varphi^{(k)^{-1}}(x - M^{(k)}) \in D_0 \quad (7)
\]
where
\[
\varphi^{(k)} = \begin{bmatrix}
y_1 \\
\vdots \\
y_L \\
\end{bmatrix}, \quad M^{(k)} = \begin{bmatrix}
m_1^{(k)} \\
\vdots \\
m_L^{(k)} \\
\end{bmatrix}
\]
\[
m_i^{(k)} = \frac{1}{2} (a_{kj} + b_{kj}), \quad p_i^{(k)} = \frac{1}{2} (b_{kj} - a_{kj})
\]
The given dynamic system (Eq. (1)) becomes
\begin{equation}
\dot{y}(t) = P^{(k)}(\mathcal{O}(\dot{y}(t) + \mathcal{M}^{(k)}))
\end{equation}

Thirdly, we define an $N$th-order formal linearization function that consists of polynomials defined by
\[
\phi(x) = [x_1, x_2, \ldots, x_n, \frac{x_1^2}{2}, \frac{x_1^2}{2}, \ldots, \frac{x_n^2}{2}, \ldots]
\]
\[
\cdots, \frac{x_1^n}{r_1!r_2!\cdots r_n!}, \cdots, \frac{x_n^n}{r_1!r_2!\cdots r_n!}]^T
\]
\[
= [\phi_{(10\ldots0)}(x), \cdots, \phi_{(1\ldots0)}(x), \cdots, \phi_{(N\ldotsN)}(x)]^T
\]
\end{equation}
Deriving the derivative of each element of $\phi$ along with the solution of the nonlinear system (Eq. (1)), we obtain
\[
\dot{\phi}_{(1\ldots0)}(x) = \frac{\partial}{\partial x^T} \phi_{(1\ldots0)}(x) \cdot \dot{x}
\]
\[
= \frac{\partial}{\partial x^T} \phi_{(1\ldots0)}(x) \cdot f(x) + \frac{\partial}{\partial y^T} P^{(k)}^{-1}
\times \phi_{(1\ldots0)}(P^{(k)} y + \mathcal{M}^{(k)}) \cdot f(P^{(k)} y + \mathcal{M}^{(k)})
\equiv G_{(1\ldots0)}^{(k)}(y)
\end{equation}
Fourthly, we apply Chebyshev expansion to the nonlinear terms $G_{(1\ldots0)}^{(k)}(y)$ in Eq.(10). The Chebyshev polynomials $\{T_r(\cdot)\}$ are defined as
\[
T_r(y) = \cos^{(r)} \cdot \cos^{(1)} y_i, \ (r = 0, 1, 2, \cdots)
\]
or
\[
T_0(y_i) = 1, \ T_1(y_i) = y_i, \ T_2(y_i) = 2y_i^2 - 1,
\]
\[
T_3(y_i) = 4y_i^3 - 3y_i, \ T_4(y_i) = 8y_i^4 - 8y_i^2 + 1, \cdots
\]
Applying Chebyshev expansion up to the $N$th order on each subdomain $D_k$, $G_{(1\ldots0)}^{(k)}(y)$ is approximated by
\[
G_{(1\ldots0)}^{(k)}(y) \approx \sum_{q_1=0}^N \cdots \sum_{q_n=0}^N C_{(q_1\ldotsq_n)}^{(k)(1\ldots0)} \cdot T_{(q_1\ldotsq_n)}(y)
\]
\[
\equiv G_{(1\ldots0)}^{(k)}(y)
\end{equation}
Substituting this $G_{(1\ldots0)}^{(k)}(y)$ into Eq.(10),
\[
\dot{\phi}_{(1\ldots0)}(x) \approx \sum_{j_1=0}^N \cdots \sum_{j_n=0}^N A_{(j_1\ldotsj_n)}^{(k)(1\ldots0)}(x) \phi_{(j_1\ldotsj_n)}(x)
\]
where
\[
A_{(j_1\ldotsj_n)}^{(k)(1\ldots0)} = \frac{\partial}{\partial x^{(j_1\ldotsj_n)}} P^{(k)} - 1 (y - \mathcal{M}(k)) \big|_{x=0}
\]
Namely,
\[
\dot{\phi}_{(1\ldots0)}(x) \approx [A_{(1\ldots0)}^{(k)(1\ldots0)}, A_{(1\ldots0)}^{(k)(0\ldots0)}, \cdots, A_{(1\ldots0)}^{(k)(0\ldots0)}],
\]
\[
\cdots, A_{(1\ldots0)}^{(k)(0\ldots0)} \phi_{(x)} + A_{(1\ldots0)}^{(k)(0\ldots0)}
\end{equation}
Thus, it follows that on a subdomain $D_k$,\begin{equation}
\dot{\phi}(x) \approx A^{(k)}(x) + b^{(k)}\end{equation}
where
\[
A^{(k)} = \left[ A_{(1\ldots0)}^{(k)(1\ldots0)} \right], \ b^{(k)} = \left[ A_{(1\ldots0)}^{(k)(0\ldots0)} \right]
\]
We unite $(M+1)$ linearized systems (Eq.(15)) on subdomains into a single linear system on the whole domain by using Eq.(5), as
\[
\dot{\varphi}(x) = \sum_{k=0}^M \hat{A}(\zeta) \varphi(x) I_k(\zeta) \approx \sum_{k=0}^M (A^{(k)}(x) + b^{(k)}) I_k(\zeta)
\]
\[
\approx \bar{A}(\zeta) \varphi(x) + \bar{b}(\zeta)
\]
where
\[
\bar{A}(\zeta) = \sum_{k=0}^M A^{(k)} I_k(\zeta), \ \bar{b}(\zeta) = \sum_{k=0}^M b^{(k)} I_k(\zeta)
\]
Finally a pseudo-formal linearization system is defined as
\[
\Sigma_2: \dot{z}(t) = \bar{A}(\zeta) z(t) + \bar{b}(\zeta), \ z(0) = \varphi(x(0))
\end{equation}
From Eq.(9), its inversion is carried out using
\[
\dot{\hat{x}}(t) = [I, 0, \cdots, 0] z(t)
\]
as the approximated value of $x(t)$, where $I$ is the $n \times n$ unit matrix.

4. Nonlinear Observer

Next we linearize the measurement equation (Eq. (2)) by pseudo-formal linearization using Chebyshev expansion. To improve the accuracy of estimation, an $\ell$th-order measurement vector $Y$ [8] is introduced as
\[
Y = [y_1^\ell, y_2^\ell, \cdots, y_r^\ell, \cdots, y_t^\ell]^T
\end{equation}
We apply Chebyshev expansion up to the $N$th order by the formal linearization function as

$$\hat{G}^{(k)}(r) \approx \frac{2^{n-\gamma}}{\pi n} \int_{-1}^{1} \cdots \int_{-1}^{1} G^{(k)}(r)(y)$$

\begin{align*}
\times \frac{1}{\sqrt{1-y_1^2} \cdots \sqrt{1-y_n^2}} dy_1 dy_2 \cdots dy_n
\end{align*}

$\gamma = \{ \text{the number of } q_i = 0 : 1 \leq i \leq n \}$

Substituting this $\hat{G}^{(k)}(r)(y)$, $Y_r$ is

$$Y_r \approx \sum_{j_0=0}^{N} \cdots \sum_{j_n=0}^{N} H^{(j_1 \cdots j_n)}(r) \phi(j_1 \cdots j_n)(x)$$

Thus, $\hat{h}^{(k)}(x)$ on a subdomain $D_k$ is approximated by the formal linearization function as

$$Y_r = \frac{\hat{h}^{(r)}(x)}{r!} \approx \left[ H^{(k)(10 \cdots 0)}(r), H^{(k)(01 \cdots 0)}(r), \cdots, H^{(k)(j_1 \cdots j_n)}(r), \cdots, H^{(k)(N \cdots N)}(r) \right] \phi(x) + H^{(k)(00 \cdots 0)}(r)$$

and a linear measurement equation with respect to $\phi$ is obtained as

$$Y \approx \left[ H^{(k)(j_1 \cdots j_n)}(r) \phi(x) + H^{(k)(00 \cdots 0)}(r) \right] = H^{(k)}(r) \phi(x) + d^{(k)}$$

Therefore, a pseudo-formal linearization system for the measurement equation is approximately derived as

$$Y(t) \approx \tilde{H}(\zeta)z(t) + \tilde{d}(\zeta)$$

where

$$\tilde{H}(\zeta) = \sum_{k=0}^{M} H^{(k)}(\zeta), \quad \tilde{d}(\zeta) = \sum_{k=0}^{M} d^{(k)}I_{k}(\zeta)$$

To the linearized systems in Eqs.(17) and (25), the linear observer theory [11] is applied and an identity observer is obtained as

$$\dot{\tilde{z}}(t) = \tilde{A}(\zeta)\tilde{z}(t) + \tilde{b}(\zeta) + K(t)\left(Y(t) - \tilde{H}(\zeta)\tilde{z}(t) - \tilde{d}(\zeta)\right)$$

$$= \sum_{k=0}^{M} \left\{(A^{(k)} \tilde{z}(t) + b^{(k)}) + K^{(k)}(t)(Y(t) - H^{(k)}(t)\tilde{z}(t) - d^{(k)})\right\}I_{k}(\zeta)$$

where $\zeta = C(z)$ and $K^{(k)}(t)$ is the observer gain on a subdomain $D_k$ given by

$$K^{(k)}(t) = \frac{1}{2} P^{(k)}(t)H^{(k)T}S^{(k)}(t)$$

$P^{(k)}(t)$ satisfies the matrix Riccati differential equation

$$\dot{P}^{(k)}(t) = A^{(k)}P^{(k)}(t) + P^{(k)}(t)A^{(k)T} + Q^{(k)}(t)$$

$$- P^{(k)}(t)H^{(k)T}S^{(k)}(t)H^{(k)}P^{(k)}(t)$$

where $Q^{(k)}(t)$, $S^{(k)}(t)$, and $P^{(k)}(0)$ are arbitrary real symmetric positive definite matrices.

From Eq.(18), the estimate $\hat{x}(t)$ of a nonlinear observer becomes

$$\dot{\hat{x}}(t) = [I, 0, \cdots, 0] \hat{z}(t)$$

5. Numerical Examples

To show the effectiveness of the approach, numerical experiments on the presented pseudo-formal linearization and simulations of a nonlinear observer as its application are illustrated.

5.1 Pseudo-formal linearization

Consider a pendulum system [8]. Let $\theta$ denote the angle subtended by the rod and the vertical axis through the pivot point. This system can be written as

$$\frac{d^2}{dt^2} \theta(t) + \frac{d}{dt} \theta(t) + \sin(\theta(t)) = 0$$

Setting the state variables as

$$x_1 = \theta, \quad x_2 = \dot{\theta}$$
the given system is rewritten as
\[
\dot{x}(t) = \begin{bmatrix}
x_3(t) \\
\sin(x_1(t)) - x_2(t)
\end{bmatrix} = f(x(t)) \quad (30)
\]

In order to apply the linearization, we set \( C(x) = x_1 \) because of its highest nonlinearity \( \sin(x_1) \) and consider the domain of \( x_1 \) as
\[
D = [-\frac{3}{4}\pi, \frac{3}{4}\pi]
\]

We divide this whole domain into three subdomains \( (M = 2) \) (see Fig. 1) as
\[
D_0 = [-\frac{3}{4}\pi, -\frac{1}{4}\pi), \quad D_1 = [-\frac{1}{4}\pi, \frac{1}{4}\pi), \quad D_2 = [\frac{1}{4}\pi, \frac{3}{4}\pi]
\]
The system parameters are set as
\[
x(0) = [1.8, -2.5]^T
\]
\[
\mathcal{M}^{(0)} = \begin{bmatrix}
-\pi \\
\frac{\pi}{2}
\end{bmatrix}, \quad \mathcal{M}^{(1)} = \begin{bmatrix} 0 \\ -1.1 \end{bmatrix}, \quad \mathcal{M}^{(2)} = \begin{bmatrix} \frac{\pi}{2} \\ -1.1 \end{bmatrix}
\]
\[
\mathcal{P}^{(k)} = \begin{bmatrix}
1.1 \pi \\ 0 \\ 1.5
\end{bmatrix} \quad (k = 0, 1, 2)
\]

Figure 2 shows the true value \( x(t) \), which is a solution of the given system (Eq.(29)), and the approximated values \( \hat{x}(t) \) given by Eq.(18) when the order of the linearization function \( N \) is varied from 1 to 3 and the parameter of the automatic choosing function is set at \( \mu = 10 \).

To clarify the difference in the approximation errors in Fig. 2, Fig. 3 shows the integral square errors of the estimation
\[
J(t) = \int_0^t (x(\tau) - \hat{x}(\tau))^T (x(\tau) - \hat{x}(\tau)) d\tau \quad (31)
\]
for the various orders in these cases. \( J(t) (N = 3, \text{old}) \) refers to a result obtained by the previous method [8] when the order of the linearization function \( N \) is 3 and nominal operating points are set at the center point of each subdomain as
\[
\{ \hat{x}_{10}, \hat{x}_{11}, \hat{x}_{12} \} = \{-\frac{1}{2}\pi, 0, \frac{1}{2}\pi\}
\]
for comparison.

### 5.2 Nonlinear observer

A nonlinear dynamic system is considered to be given by Eq.(30) and a measurement equation is assumed to be
\[
\eta = \frac{1}{5} \cos(x_1(t)) + \frac{1}{5} x_2(t) \equiv h(x(t)) \quad (32)
\]
The subdomains and system parameters are the same as in the previous subsection. The initial point of the system is
\[
x(0) = [1.8, -2.5]^T
\]
and the parameters for the nonlinear observer are set as
\[
\hat{x}(0) = [0.8, -2]^T, \quad Q^{(k)}(t) = 20I, \quad S^{(k)}(t) = 50
\]
\[
P^{(k)}(0) = 20I \quad (k = 0, 1, 2), \quad \mu = 10
\]

Figure 4 shows the true value \( x(t) \) and the estimates \( \hat{x}(t) \) in Eq.(28) obtained by the presented method when the order \( N \) is 2 and \( m \) is varied from 1 to 2. Figure 5 shows the integral square errors of the estimation
\[
J(t) = \int_0^t (x(\tau) - \hat{x}(\tau))^T (x(\tau) - \hat{x}(\tau)) d\tau
\]
in this case. \( N = 2, m = 1, 2 \text{(old)} \) refer to results obtained by the previous method [8].
6. Conclusions

We have considered a pseudo-formal linearization method using Chebyshev expansion and synthesized a nonlinear observer as an application of this method for both multidimensional nonlinear autonomous systems and nonlinear scalar-measurement systems. Numerical experiments show that the accuracy of this method is improved as the order of the augmented measurement vector increases and is better than that of the previous method. It is left for future studies to extend this method to multidimensional measurement systems and nonlinear control problems.

References

[1] A. J. Krener: Approximate linearization by state feedback and coordinate change, Syst. Control Lett., Vol.5, pp.181-185, 1984.
[2] W. T. Baumann and W. J. Rugh: Feedback control of nonlinear systems by extended linearization, IEEE Trans. Autom. Control, Vol.31, No.1, pp.40-46, 1986.
[3] R. Marino: On the largest feedback linearizable subsystem, Syst. Control Lett., Vol.6, pp.345-351, 1986.
[4] R. R. Kadiyala: A tool box for approximate linearization on nonlinear systems, IEEE Control Syst. Mag., Vol.13, No.2, pp.47-57, 1993.
[5] A. Ishidori: Nonlinear Control Systems II, Springer-Verlag, 1999.
[6] J. Lei and H. K. Khalil: Feedback linearization for nonlinear systems with time-varying input and output delays by using high-gain predictors, IEEE Trans. Autom. Control, Vol.61, No.8, pp.2262-2268, 2016.
[7] S. Yang, P. Wang and Y. Tang: Feedback linearization-based current control strategy for modular multilevel converters, IEEE Trans. Power Electron., Vol.33, No.1, pp.161-174, 2018.
[8] K. Komatsu and H. Takata: A nonlinear observer via pseudo-formal linearization for both state and measurement equations of nonlinear scalar-measurement systems, Journal of Signal Processing, Vol.21, No.6, pp.291-296, 2017.
[9] H. Takata and K. Komatsu: A pseudo-formal linearization of polynomial type for nonlinear systems and its applications, Journal of Signal Processing, Vol.22, No.1, pp.9-16, 2018.
[10] H. Takata, T. Hachino, Y. Hino, K. Yumokuchi, H. Miyajima and K. Komatsu: Augmented automatic choosing control of modified filter type for nonlinear noisy measurement systems, Journal of Signal Processing, Vol.16, No.6, pp.563-569, 2012.
[11] G. W. Johnson: A deterministic theory of estimation and control, IEEE Trans. Autom. Control, Vol.14, pp.380-384, 1974.

Kazuo Komatsu received his B.S. degree in computer science and Dr. Eng. degree in electrical engineering from Kyushu Institute of Technology in 1985 and 1995, respectively. He is currently a Professor at the Department of Human-Oriented Information Systems Engineering in National Institute of Technology, Kumamoto College. His research interests include formal linearization for nonlinear systems and its applications. He is a member of the RISP.

Hitoshi Takata received his B.S. degree in electrical engineering from Kyushu Institute of Technology in 1968 and his M.S. and Dr. Eng. degrees in electrical engineering from Kyushu University in 1970 and 1974, respectively. He is currently a Professor Emeritus at Kagoshima University. His research interests include the control, linearization, and identification of nonlinear systems.

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