Resonance near border-collision bifurcations in piecewise-smooth, continuous maps

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Abstract

Mode-locking regions (resonance tongues) formed by border-collision bifurcations of piecewise-smooth, continuous maps commonly exhibit a distinctive sausage-like geometry with pinch points called ‘shrinking points’. In this paper we extend our unfolding of the piecewise-linear case [Simpson and Meiss (2009 Nonlinearity 22 1123–44)] to show how shrinking points are destroyed by nonlinearity. We obtain a codimension-three unfolding of this shrinking point bifurcation for \(N\)-dimensional maps. We show that the destruction of the shrinking points generically occurs by the creation of a curve of saddle-node bifurcations that smooth one boundary of the sausage, leaving a kink in the other boundary.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

Piecewise-smooth systems are used to model a vast range of physical systems involving nonsmooth behaviour [1–4]. In this paper we study piecewise-smooth, continuous maps, i.e.

\[
x_{i+1} = F(x_i),
\]

where \(x_i \in \mathbb{R}^N\) and \(F\) is everywhere continuous but nondifferentiable on codimension-one surfaces in \(\mathbb{R}^N\) called switching manifolds. Such maps arise as Poincaré maps of Filippov systems near grazing-sliding and corner collisions [5, 6] or of some time-dependent piecewise-smooth flows [7]. They are often used as mathematical models of various discrete, nonsmooth systems, see, e.g. [8].

A fundamental and unique bifurcation of piecewise-smooth, continuous maps results from the collision of a fixed point with a switching manifold; it is known as a border-collision bifurcation. Except in degenerate cases, a border-collision bifurcation may be classified as...
either a border-collision fold at which two fixed points collide and annihilate or a border-collision persistence at which a single fixed point ‘crosses’ the switching manifold [1, 9]. Note that the collision of one point of a periodic solution of (1) with a switching manifold is also a border-collision bifurcation for the $n\text{th}$ iterate of (1) [9, 10]. Although complicated dynamics may be born in border-collision bifurcations, in this paper we study only the creation of periodic solutions.

We assume that the derivatives of the smooth components of (1), $D_x F$, are locally bounded (we exclude from consideration, for instance, square-root type maps [1, 11]). Then, generic border-collision bifurcations of (1) may be described by piecewise-linear maps. More precisely, structurally stable dynamics of a piecewise-smooth, continuous map near a border-collision bifurcation are described by the piecewise-linear, series expansion about the bifurcation. A consequence is that, to lowest order, the structurally stable invariant sets created at border-collision bifurcations grow linearly as the bifurcation parameter varies.

One example is the two-dimensional, piecewise-smooth, continuous map

$$f_\mu(x) = \begin{cases} \mu [1]_0 + \left[ \frac{2r_L \cos(2\pi \omega_L)}{-r_I^2} 1 \right] x + g^L(x), & s \leq 0, \\
\mu [1]_0 + \left[ \frac{2}{s_R} \cos(2\pi \omega_R) 1 \right] x + g^R(x), & s \geq 0, \end{cases}$$

(2)

where

$s = e_1^T x$ (the first component of the vector $x \in \mathbb{R}^2$).

$0 < r_L, s_R < 1, 0 < \omega_L, \omega_R < \frac{1}{2}, \mu \in \mathbb{R}$ is assumed to be small, and the functions $g^L(x)$ and $g^R(x)$ contain the terms of $f_\mu$ that are nonlinear in $x$. Map (2) is continuous only if $g^L(x) = g^R(x)$ whenever $s = 0$, i.e. on the switching manifold.

A border-collision bifurcation for (2) occurs at the origin when $\mu = 0$. If this bifurcation is nondegenerate, the local dynamics are independent of the nonlinear components, $g^L$ and $g^R$, see [12].

The piecewise-linear version of (2), i.e. with

$$g^L(x) = g^R(x) = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

(3)

is the canonical, piecewise-linear form for a border-collision analogue to a smooth Neimark–Sacker bifurcation. Indeed the fixed point for $\mu < 0$ has a pair of complex multipliers ($\lambda_{\pm} = r_i e^{\pm 2\pi i \omega_i}$) inside the unit circle that ‘jump’ at $\mu = 0$ outside the unit circle ($\lambda_{\pm} = \frac{1}{s_R} e^{\pm 2\pi i \omega_i}$) for $\mu > 0$. Depending upon the precise choice of parameter values, this border-collision bifurcation may generate periodic, quasiperiodic or chaotic solutions as well as combinations of these [13–16]. This class of border-collision bifurcations has been seen in models of dc/dc power converters [7] and optimization in economics [8, 17].

Figure 1(a) shows an example of a bifurcation diagram for (2) with (3) when a pair of period-seven orbits (7-cycles) is created at $\mu = 0$. These two orbits are shown in panel (b); one is attracting and the other is a saddle. We observe that the unstable manifold of the saddle forms an invariant circle that contains the attracting orbit. This border-collision bifurcation is nondegenerate; indeed, if we were to relax (3) then there is an $\varepsilon > 0$ such that whenever $0 < \mu < \varepsilon$ the dynamics are topologically equivalent to figure 1(b) near the origin. An example is shown in figure 2 where

$$g^L(x) = \begin{bmatrix} s^2 \\ 0 \end{bmatrix}, \quad g^R(x) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$  (4)
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Figure 1. Panel (a) shows a bifurcation diagram of (2) with (3) when \( r_L = 0.2, s_R = 0.95 \) and \( \omega_L = \omega_R = 0.287 \). When \( \mu < 0 \) the unique fixed point is attracting (the solid line) and when \( \mu > 0 \) it is repelling (the dotted line) and a pair of period-seven orbits exist. The solid lines for \( \mu > 0 \) in panel (a) show the attracting 7-cycle (two pairs of points have similar \( s \)-values). Panel (b) shows a phase portrait when \( \mu = 1 \). The dotted vertical line is the switching manifold. The attracting (saddle) 7-cycle is indicated by triangles (circles). The unstable manifold of the saddle forms an invariant circle (red); the curves forming its stable manifold (blue) intersect at the repelling fixed point (square).

Figure 2. A bifurcation diagram showing attracting orbits of (2) with nonlinearity (4) for the same parameter values as in figure 1. When \( \mu \) is small the dynamical behaviour near the origin is topologically equivalent to the linear case. However, at \( \mu \approx 1.202 \), the attracting 7-cycle undergoes border-collision beyond which there exists a complicated attracting set.

For this nonlinear system, there is also a saddle-node pair of period-seven orbits created at the border-collision bifurcation. The stable 7-cycle is attracting up to \( \mu \approx 1.202 \).

Note that piecewise-linear maps are particularly straightforward to analyse because any periodic orbit is the solution to a linear system. Furthermore, linearity implies that if \( \mathcal{I} \) is an invariant set of \( f_{\mu} \) then \( \lambda \mathcal{I} \) is an invariant set of \( f_{\lambda \mu} \) for any \( \lambda > 0 \). Consequently, it suffices to consider \( \mu = -1, 0, 1 \).
A two-parameter bifurcation diagram of the piecewise-linear case of (2) is shown in figure 3 for $\mu = 1$. The coloured regions are resonance (or Arnold) tongues within which there is an attracting periodic solution. Since the piecewise-linear case of the two-dimensional map (2) has a unique fixed point, we may define a rotation number for orbits as the average change in angle per iteration about the fixed point [13, 18–20]. Consequently, the resonance tongues can be labelled by the rotation number, $m/n$, of the corresponding periodic solution. The orbits shown in figure 1 lie in the $2/7$-tongue; this tongue intersects the $s_R = 1$ line at $\omega_R = 2/7 \approx 0.2857$.

The majority of the resonance tongues in figure 3 exhibit a structure that is often likened to a string of sausages. This structure was first observed in a one-dimensional sawtooth map [21, 22], and has since been described in higher dimensional maps such as (2), see for example [7, 8, 15, 17, 23]. As in [21], we refer to points where resonance tongues have zero width as shrinking points. An unfolding of shrinking points in piecewise-linear, continuous maps of arbitrary dimension was performed in [10]. There it was shown, upon imposing reasonable nondegeneracy assumptions, that any two-dimensional slice of parameter space in the neighbourhood of a shrinking point will resemble figure 4: shrinking points are codimension-two phenomena of piecewise-linear, continuous maps. In particular, near a shrinking point, the resonance tongue is locally a two-dimensional cone bounded by four curves. These boundary curves are pairwise tangent at the shrinking point so the cone boundaries are locally $C^1$. In the interior of the cone a primary $n$-cycle exists; it has some number of points, say $l$, located, ’left’, of the switching manifold ($l = 2$ for figure 4). This orbit collides and annihilates with another $n$-cycle in a border-collision fold bifurcation on the four boundary curves of the cone. This secondary periodic solution has $l - 1$ points to the left of the switching manifold within one half of the cone and $l + 1$ points to the left of the switching manifold within the other half of the cone. On each of the four boundary curves a different point of the primary orbit lies on the switching manifold.

3 In this paper we will only consider the case of ’non-terminating’ shrinking points defined in [10]. Thus we do not consider the ends of resonance tongues such as those at $s_R = 1$ in figure 3.
These results generalize to the nonlinear case in the following sense: regardless of the nonlinearites $g^L$ and $g^R$, the two-dimensional bifurcation structure shown in figure 3 will be essentially unchanged when $\mu$ is small enough and it will limit to the linear picture as $\mu \to 0^+$. However, when nonlinear terms are present the sausage structure is not preserved as $\mu$ increases; indeed, the shrinking points break apart as shown in figure 5. Consequently, when nonlinear terms are present the shrinking point phenomenon is codimension-three; we refer this scenario as a generalized shrinking point.

Figure 5 suggests that the break-up of shrinking points occurs in a regular fashion. The resonance tongues develop a nonzero width that increases with $\mu$. For $\mu > 0$ the right boundary of each resonance tongue appears to be smooth whereas each left boundary retains the kink that appeared at the shrinking point. It is interesting that, in contrast to the case for smooth maps [24], we observe no ‘strong’ resonance behaviour, i.e. special dynamics when $n < 5$.

The purpose of this paper is to determine the generic unfolding of generalized shrinking points for a piecewise-smooth, continuous map of arbitrary dimension. Here we summarize our results. Suppose that a border-collision bifurcation occurs when a parameter $\mu$ is zero, that resonance arises for $\mu > 0$ and that in the limit $\mu \to 0^+$, a two-parameter bifurcation diagram exhibits a shrinking point with its four boundary curves. For small $\mu > 0$ the four boundary curves maintain a common intersection point, $O$, but each curve is no longer necessarily tangent to the opposite curve at this point.

We will show that, under reasonable nondegeneracy assumptions, there exists a new bifurcation locus for each small enough fixed $\mu > 0$. This locus is a curve of saddle-node bifurcations of the primary $n$-cycle that is tangent to one boundary curve at a point $A$, and to an adjacent boundary curve on the other side of the shrinking point at $B$, see figure 6. This
Figure 5. Resonance tongues of (2) with (4) when \( r_L = 0.2 \) for three different values of \( \mu \). The uppermost plot is a magnification of figure 3. The lower two plots were computed numerically by estimating the eventual period of the forward orbit of the origin.

Figure 6. Bifurcation sets of (2) with (4) for \( r_L = 0.2 \) and three different \( \mu \) values. Panel (a) is identical to figure 4. In panels (b) and (c) the locus of classical saddle-node bifurcations of the primary orbit is indicated by a curve connecting A to B (red). Solid (dotted) curves correspond to border-collision fold bifurcations (border-collision persistence).

locus is smooth and collapses to the shrinking point as \( \mu \to 0^+ \). Two \( n \)-cycles that have the same itinerary as the primary \( n \)-cycle exist in a roughly triangular region bounded by \( O \), \( A \) and \( B \). If we let \( \theta_1 \) denote the angle made at \( O \) between the two border-collision curves across the region \( AOB \), see figure 6, and \( \theta_2 \) is the opposing angle made at \( O \) between the other two border-collision curves, then for small \( \mu > 0 \), \( \theta_1 < \theta_2 \).

A formal statement that includes these results is given in theorem 10 in section 6.

Here we summarize the rest of the paper. In the following section we present the \( N \)-dimensional map (6) that describes an arbitrary border-collision bifurcation. Concepts from
symbolic dynamics that are invaluable for describing periodic solutions of (6) are introduced in section 3. Key formulae for periodic solutions are obtained in section 4. Subsequently we impose the assumption that (6) is piecewise-$C^k$; this allows us to derive useful series expansions. Section 5 is devoted to defining generalized shrinking points as a codimension-three scenario. Finally in section 6 we unfold these points leading to theorem 10. The proofs of the theorem and lemma 9 are given in an appendix.

Throughout the paper we use $O(k) [o(k)]$ to denote terms that are order $k$ or larger (larger than order $k$) in all variables and parameters of a given expression.

2. Generic border-collision bifurcations

We restrict our attention to local dynamics of border-collision bifurcations, and thus study a piecewise-smooth series expansion of (1) about an arbitrary border-collision bifurcation. Throughout this paper we will assume that there is a single, smooth switching manifold in a neighbourhood of the border-collision bifurcation. This is typically the case in models since switching manifolds are usually defined by some simple physical constraint. For simplicity we assume that a coordinate transformation has been made so that the switching manifold corresponds to the vanishing of the first component of $x$ (see in particular [25] for details of such a transformation) and—to avoid the use of subscripts for components—we introduce the new variable

$$s = e_1^T x.$$  

(5)

A general piecewise-smooth map then takes the form

$$x_{i+1} = f(x_i; \xi) = \begin{cases} f^L(x_i; \xi), & s_i \leq 0, \\ f^R(x_i; \xi), & s_i \geq 0, \end{cases}$$  

(6)

where $\xi$ is a finite-dimensional vector of parameters. The switching manifold partitions phase space into two regions: the left half-space (where $s < 0$) and the right half-space (where $s > 0$).

We assume a border-collision bifurcation of a fixed point occurs at the origin when a parameter $\mu$ is zero and that the derivatives of the functions $f^L$ and $f^R$ exist here, so that $f^L$ and $f^R$ have series expansions

$$f^J(x_i; \xi) = \mu b(\xi) + A_J(\xi)x_i + g^J(x_i; \xi),$$  

(7)

where $J = L, R, \mu$ is the first component of the parameter vector $\xi$, and $A_J$ (an $N \times N$ matrix) and $b$ (an $N$-dimensional vector) depend continuously on $\xi$. The functions $g^J$ contain only terms that are nonlinear in $x$; that is, $g^J(x_i; \xi) = o(|x_i|)$ (this, for example, could include terms of order $|x_i|^2$ that arise naturally in Poincaré maps relating to sliding bifurcations [5, 26]). By continuity of (6), $A_L$ and $A_R$ are identical in their last $N-1$ columns and $g^L = g^R$ whenever $s = 0$.

If $A_J(\xi)$ does not have an eigenvalue 1, the half-map $f^J$ has a unique fixed point near the origin given explicitly by

$$x^{*J}(\xi) = (I - A_J(\xi))^{-1}b(\xi) \bigg|_{\mu = 0} + o(\mu).$$  

(8)

As seems to have been first noted by Feigin, see [9], a convenient expression for the first component of the vector, $x^{*J}$, is obtained with adjugate matrices [27, 28]:

$$\text{adj}(X)X = \det(X)I, \quad \text{for any } N \times N \text{ matrix } X.$$  

(9)
The point is that since \( A_L \) and \( A_R \) are identical in their last \( N - 1 \) columns, \( \text{adj}(I - A_L) \) and \( \text{adj}(I - A_R) \) share the same first row:

\[
\varrho^T(\xi) = e_1^T \text{adj}(I - A_L(\xi)) = e_1^T \text{adj}(I - A_R(\xi)).
\]

Consequently, multiplication of (8) by \( e_1^T \) on the left implies that the first component of the fixed point \( x^\star \) satisfies the useful formula

\[
s^\star(\xi) = \frac{\varrho^T(\xi)b(\xi)}{\det(I - A_J(\xi))} \bigg|_{\mu = 0} = \mu + o(\mu).
\]

In particular, we learn from (11) that both fixed points, if they exist, move away from the switching manifold linearly with respect to \( \mu \) if and only if \( \varrho^T(\xi)b(\xi) \big|_{\mu = 0} \neq 0 \). This condition is a nondegeneracy condition for the border-collision bifurcation. Under this assumption, the bifurcation is a border-collision fold bifurcation if \( \det(I - A_L(0)) \) and \( \det(I - A_R(0)) \) have opposite signs, and border-collision persistence if they have the same sign [9].

3. Symbolic dynamics

It is common in the study of piecewise-smooth systems for symbolic methods to be highly beneficial. In this paper we consider bi-infinite sequences, \( S \), constructed from the binary alphabet \( \{L, R\} \). In order to unfold shrinking points, we find it necessary to consider only what we have termed rotational symbol sequences. In [10] we defined these as particular finite collections. Instead of repeating this definition we provide here a definition that is more versatile in that it extends naturally to nonperiodic sequences and has been described elsewhere. Furthermore, to be consistent with combinatorics literature, sequences are always assumed to contain infinitely many elements (unlike in [10]).

**Definition 1.** For \( \alpha, \beta \in [0, 1) \), let \( S[\alpha, \beta] \) be the bi-infinite symbol sequence with \( i \)th element

\[
S[\alpha, \beta]_i = \begin{cases} L, & i\alpha \mod 1 \in [0, \beta) \\ R, & i\alpha \mod 1 \in (\beta, 1) \end{cases}
\]

for all \( i \in \mathbb{Z} \).

A visual representation of these sequences is provided by figure 7. These sequences seem to have first been studied by Slater [29, 30]. The sequences for which \( \alpha = \beta \neq \mathbb{Q} \) have been well studied and are known as rotation sequences; they are equivalent to Sturmian sequences [31], which are traditionally defined from a combinatorics viewpoint [32]. Some discussion of the case \( \alpha = \beta \in \mathbb{Q} \) is given in [33]. A related approach is to consider arithmetic sequences on \( \mathbb{Z}/n\mathbb{Z} \) for some \( n \in \mathbb{N} \) [34]. Similar sequences for rotational orbits of the Hénon map are described in [35].

**Definition 2 (Rotational symbol sequence).** For each \( l, m, n \in \mathbb{N} \) with \( l, m < n \) and \( \gcd(m, n) = 1 \), \( S[l/m, l/n] \) is a rotational symbol sequence.

Any periodic sequence, such as \( S[l/m, l/n] \), is generated by repeated copies of a finite collection from \( \{L, R\} \), termed a word. For instance for the word \( W = \text{LLRLRLR} \ldots \) generated sequence is \( S = \ldots \text{LLRLRLR} \ldots \). Here we let \( W[l/m, n] \) denote the word comprising \( i = 0, 1, \ldots, n - 1 \) elements of \( S[l/m, l/n] \).

As an example, let us compute \( S[3/7, 1/7] \). Here \( \alpha = \frac{3}{7} \), thus the numbers \( i\alpha \mod 1 \) of definition 1 for \( i = 0, 1, \ldots, n - 1 = 0, \frac{3}{7}, \frac{6}{7}, \frac{9}{7}, \frac{1}{7}, \frac{4}{7}, \frac{5}{7} \) and \( \beta = \frac{1}{7} \), therefore \( W[3, 2, 7] = \text{LLRLRLR} \).
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2πα
L
i=0
2πβ
L
i=1
R
i=2
R
i=3
R
i=4

the interval
[0, β)

Figure 7. A geometric portrayal of definition 1. First cut a circle with a vertical line that subtends an angle 2πβ as shown. Each real number, φ, is then represented by a point 2πφ radians clockwise from the lower intersection of the circle with the vertical line. Then S[α, β]i = L whenever φ = iα is located to the left of the vertical line, and S[α, β]i = R otherwise. In addition S[α, β]0 is always L.

Throughout this paper will we use the symbol d to denote the multiplicative inverse of m modulo n, for example d = 4 when m = 2 and n = 7 as above. Then

W[l, m, n]id = \begin{cases} L, & i = 0, \ldots, l - 1, \\ R, & i = l, \ldots, n - 1, \end{cases}

(12)

where id is taken modulo n. For clarity, throughout this paper we omit ‘mod n’ where it is clear modular arithmetic is being used.

Given a word W, we let W(l) denote the lth left cyclic permutation of W and W(l) denote the word that differs from W in only the lth element. For example if W = LRLRR then W(l) = LRLLR and W(2) = LRLRR (indexing of elements starts at i = 0). If S is the sequence generated by W we let S(l) and S(l) denote the sequences generated by W(l) and W(l), respectively.

A key property of rotational symbol sequences that is verified in [10] is

S[\frac{m}{n}, \frac{l}{n}]id = S[\frac{m}{n}, \frac{l}{n}]\mod(id).

(13)

4. Periodic solutions

For any symbol sequence S, the iterates of a point x0 ∈ R^N under the two half-maps of (6) in the order determined by S are

x_{i+1} = f^{S(i)}(x_i; \xi).

(14)

In general, this may be different from iterating x0 under map (6). However, if the sequence \{x_i\} satisfies the admissibility condition:

S_i = \begin{cases} L, & \text{whenever } s_i < 0, \\ R, & \text{whenever } s_i > 0 \end{cases}

(15)

for every i, then \{x_i\} coincides with the orbit of x0 under (6). Notice if s_i = 0 there is no restriction on S_i. When (15) holds for every i, \{x_i\} is admissible, otherwise it is virtual.
When \( S \) has period \( n \), the periodic orbits with this sequence are fixed points of the map
\[
S = f^{S_{n-1}} \circ \ldots \circ f^{S_0}.
\]
Some straightforward algebra leads to
\[
S(x; \xi) = P_S(\xi)b(\xi) \bigg|_{\mu=0} + o(\mu) + \left(M_S(\xi) \bigg|_{\mu=0} + o(\mu^0)\right)x + o(x),
\]
where
\[
M_S = A_{S_{n-1}} \ldots A_{S_0},
\]
\[
P_S = I + A_{S_{n-1}} + A_{S_{n-2}} + \cdots + A_{S_{n-1}} \ldots A_{S_1}.
\]
Notice \( P_S \) is independent of \( S_0 \), thus
\[
P_S = P_{S_0}.
\]
We now present six lemmas relating to periodic solutions that are useful for analysing shrinking points. A consequence of the following lemma is that \( M_S \) and \( M_{S_0} \) are identical in their last \( N-1 \) columns.

**Lemma 1.** For any \( N \times N \) matrix, \( X \), \( XA_R = XA_L + \zeta e_1^T \) for some \( \zeta \in \mathbb{R}^N \).

**Proof.** Since \( A_L \) and \( A_R \) are identical in all but possibly their first columns, we may write \( A_R = A_L + \hat{\zeta} e_1^T \) for some \( \hat{\zeta} \in \mathbb{R}^N \). This proves the result with \( \zeta = X \hat{\zeta} \).

If \( x_0 \) is a fixed point of \( S \) then the \( n \) points \( \{x_0, \ldots, x_{n-1}\} \) (where \( x_i \) is defined by (14)) describe a periodic solution (which may be virtual) that we will refer to as an \( S \)-cycle. The following two lemmas are generalizations of those given in [10].

**Lemma 2.** Suppose \( \{x_i\} \) is an \( S \)-cycle and \( x_j \) lies on the switching manifold, for some \( j \). Then \( \{x_i\} \) is also an \( S_j \)-cycle.

**Proof.** By continuity: \( f^L(x_j; \xi) = f^R(x_j; \xi) \), hence there is no restriction on the \( j \)th element of \( S \).

**Lemma 3.** Suppose \( \{x_i\} \) is an \( S \)-cycle. Then for any \( j \), \( \det \left(I - D_x f^{S(j)}(x_j; \xi)\right) = \det \left(I - D_x f^S(x_0; \xi)\right) \).

**Proof.** By the chain rule the Jacobian \( D_x f^S(x_0; \xi) \) may be written as a product of matrices:
\[
D_x f^S(x_0; \xi) = \prod_{i=0}^{n-1} D_x f^S(x_i; \xi).
\]
The spectrum of this product is unchanged if the \( N \) matrices are multiplied in an order that differs only cyclically from this one (refer to the proof of lemma 4 of [10]), which proves the result.

Whenever \( M_S(\xi) \) does not have an eigenvalue 1, the implicit function theorem implies that \( f^S \) has a unique fixed point near the origin, call it \( x^{sS} \). Using (17) we obtain
\[
x^{sS}(\xi) = (I - M_S(\xi))^{-1} P_S(\xi)b(\xi) \bigg|_{\mu=0} + o(\mu).
\]
We may derive a formula for the first component of \( x^{sS} \), denoted \( s^{sS} \), in the same spirit as (11) for fixed points of (6). Let
\[
\rho^T_S(\xi) = e_1^T \text{adj}(I - M_S(\xi)),
\]
\[
\rho^T_S(\xi) = e_1^T \text{adj}(I - M_S(\xi)),
\]
\[
\rho^T_S(\xi) = e_1^T \text{adj}(I - M_S(\xi)).
\]
then
\[
s^{sS}(\xi) = \left. \frac{q^T_S(\xi) P_S(\xi)b(\xi)}{\det(I - M_S(\xi))} \right|_{\mu=0} \mu + o(\mu). \tag{23}
\]

As in [12] we use two lemmas to derive a convenient formula for \(s^{sS}\).

**Lemma 4.** The matrices \(P_S(I - A_L)\) and \(I - M_S\) can differ in only their first columns.

**Proof.** Using (18) and (19),
\[
P_S(I - A_L) = (I + A_{S_{n-1}} + A_{S_{n-2}} \cdot A_{S_1} \cdot A_L)(I - A_L)
\]
expand and group terms differently:
\[
= I - A_{S_{n-1}} \cdot A_{S_{n-2}} \cdot A_{S_1} - A_L + A_{S_{n-1}}(A_{S_{n-2}} - A_L) + \cdots + A_{S_1} - A_L
\]
apply lemma 1:
\[
= I - M_S + \sum_{i=1}^n \xi_i e_i T
\]
where each \(\xi_i \in \mathbb{R}^N\).
\[
= I - M_S + (\sum_{i=1}^n \xi_i) e_1 T.
\]
\(\square\)

The choice of the symbol \(L\) in the statement of lemma 4 is arbitrary, i.e. the lemma remains true if \(L\) is replaced with \(R\).

**Lemma 5.** \(q_S^T P_S = \det(P_S)q^T\), where \(q_S^T\) is given by (22) and \(q^T\) is given by (10).

**Proof.** By lemma 4 we have
\[
e_1 T \text{adj}(P_S(I - A_L)) = e_1 T \text{adj}(I - M_S) = q_S^T
\]
\[
\Rightarrow e_1 T \text{adj}(I - A_L) \text{adj}(P_S) = q_S^T \quad \text{(since adj}(XY) = \text{adj}(Y)\text{adj}(X) \text{for any} \ X, Y)
\]
\[
\Rightarrow q^T \text{adj}(P_S) = q_S^T \quad \text{by (10)}
\]
\[
\Rightarrow \det(P_S)q^T = q_S^T P_S \quad \text{by (9)} \quad \square
\]

If \(I - M_S(\xi)\) is nonsingular, by lemma 5 and (23),
\[
s^{sS}(\xi) = \left. \frac{\det(P_S(\xi))}{\det(I - M_S(\xi))} q^T(\xi)b(\xi) \right|_{\mu=0} \mu + o(\mu). \tag{24}
\]

This expression relates the linear component of \(s^{sS}(\xi)\) simply in terms of \(q^ Tb\) (which appears in the fixed point equation (11)) and the determinants of \(P_S\) and \(I - M_S\). (Feigin’s result concerning the creation of 2-cycles at border-collision bifurcations (see [9]) follows from (24) by substituting \(S = LR\) and \(S = RL\).

We conclude this section with an important lemma that is most simply stated for (6) in the absence of nonlinear terms, for then all \(o(\mu)\) terms given above vanish. Although this result is proved in [10], with the use of lemma 5 we are now able to provide a pithier proof.

**Lemma 6.** Suppose the map (6) is piecewise-linear, that is \(g^L = g^R = 0\). Assume \(q^Tb \neq 0\) for some \(\xi\) with \(\mu \neq 0\).

(i) If \(I - M_S\) is nonsingular, then the unique fixed point of \(f^S, x^{sS}\), given by (21), lies on the switching manifold if and only if \(P_S\) is singular.

(ii) If \(P_S\) is nonsingular, then \(f^S\) has a fixed point if and only if \(I - M_S\) is nonsingular.
Proof. Since \( g^k = g^R = 0 \), (24) reduces to \( s^S = (\det(P_S)/\det(I-M_S))g^Tb\mu \), from which part (i) follows immediately. Similarly (21) reduces to \( x^S = (I-M_S)^{-1}P_Sb\mu \) which proves part (ii) when \( I-M_S \) is nonsingular. Suppose \( I-M_S \) is singular and suppose for a contradiction \( f^S \) has a fixed point \( x^S \). Then we have

\[
(I - M_S)x^S = P_Sb\mu,
\]

since \( g^L = g^R = 0 \), see (17). Multiplication of this by \( \varrho^T \) (22) on the left yields

\[
\det(I-M_S)x^S = \det(P_S)\varrho^Tb\mu,
\]

where we have also used (9) and lemma 5. This provides a contradiction because the left-hand side of the previous equation is zero, whereas by assumption the right-hand side is nonzero. □

5. Generalized shrinking points

At this stage we find it useful to impose the extra assumption that map (6) under investigation here is piecewise-\( C^K \), for some \( K \in \mathbb{N} \). This allows us to derive series expansions of smooth components of the map and iterates of the map. In particular, this assumption allows us to apply the centre manifold theorem necessary for proving the existence of saddle-node bifurcations in section 6.

In order to unfold a generalized shrinking point, we must first give a precise definition of such a point. We use the results of the previous section to write down assumptions that guarantee the existence of a periodic solution with two points on the switching manifold. As in [10], it is useful to assume that this periodic solution is admissible. To state this assumption we need to renormalize map (6).

Recall that map (7) depends upon an arbitrary vector of parameters \( \xi \), and that \( \mu \) denotes the first component of this vector. Scaling \( x \) by \( \mu \) gives a new map \( h \) defined through

\[
f^J(\mu z; \xi) = \mu h^J(z; \xi),
\]

where \( z \in \mathbb{R}^N \). Note that when \( f^J \) is \( C^K \) then \( h^J \) is \( C^{K-1} \), and using (7), it has the expansion

\[
h^J(z; \xi) = h(\xi) + A_J(\xi)z + O(\mu).
\]

For \( \mu \geq 0 \), the renormalized map for \( z \) is then

\[
z_{n+1} = h(z_n; \xi) = \begin{cases} h^L(z_n; \xi), & u_n \leq 0, \\ h^R(z_n; \xi), & u_n \geq 0, \end{cases}
\]

where

\[
u = e_1^Tz.
\]

Whenever \( \mu \geq 0 \), if \( x = \mu z \) then \( f(x; \xi) = \mu f(z; \xi) \). For any \( S \)-cycle (21), we can also let \( x^S(\xi) = \mu z^S(\xi) \) so that \( z^S(\xi) \) is \( C^{K-1} \) and is a fixed point of \( h^S = h^{S-1} \circ \cdots \circ h^S \). Since (25) is a ‘blow-up’ of phase space, points \( z \in \mathbb{R}^N \) are not necessarily near the origin when \( \mu \) is small. The renormalization effectively transfers the \( \mu \)-dependence of the piecewise-smooth map from the constant term to the nonlinear terms and this scaling is often helpful in the analysis. Note that (25) has nontrivial dynamics for \( \mu = 0 \); indeed in this case it is piecewise-linear and identical to the map studied in [10].

Recall that at shrinking points we observe a periodic solution with two points on the switching manifold. The existence of a solution with a point on the switching manifold can be related to the singularity of the matrices \( P_S(0) \) and \( P_{S^0-1,0}(0) \) by lemma 6. Moreover, if only one of these matrices is singular, the \( S \)-cycle undergoes normal border-collision for \( \mu \) near
zero. Thus the singularity of both matrices is needed to obtain a more general bifurcation, and this motivates the following definition.

**Definition 3 (Generalized shrinking point).** Consider map (6) with \( N \geq 2 \) and suppose \( \varrho^T(0)b(0) \neq 0 \). Let \( \mathcal{S} = S[m/n, 1/n] \) be a rotational symbol sequence with \( 2 \leq l \leq n - 2 \). Suppose \( P_{\mathcal{S}}(0) \) and \( P_{\mathcal{S}[m-1/n]}(0) \) are singular (the singularity condition).

Let
\[
\tilde{\mathcal{S}} = S^\varrho, \\
\hat{\mathcal{S}} = S^{\varrho^r},
\]
and assume \( I - M_{\tilde{\mathcal{S}}}(0) \) and \( I - M_{\hat{\mathcal{S}}}(0) \) are nonsingular. Let \( \{\hat{\xi}_i(\xi)\} \) be the unique \( \hat{\mathcal{S}} \)-cycle near the origin \( (\hat{\xi}_0(\xi) \text{ is given by (21))} \). Let
\[
y_i = \hat{z}_i(0),
\]
where \( \mu\hat{z}_i(\xi) = \hat{\xi}_i(\xi) \). If the orbit \( \{y_i\} \) is an admissible solution to (25) with \( \xi = 0 \), then we say that (6) is at a generalized shrinking point when \( \xi = 0 \).

The sequences \( \tilde{\mathcal{S}} \) (26) and \( \hat{\mathcal{S}} \) (27) are rotational with one less and one more \( L \) than \( \mathcal{S} = S[m/n, 1/n] \), respectively (specifically, \( \tilde{\mathcal{S}} = S[m/n, (l-1)/n] \) and \( \hat{\mathcal{S}} = S[m/n, (l+1)/n] \)). The periodic solution \( \{y_i\} \) is fundamental to the shrinking point. As stated in the following lemma, it has two points on the switching manifold. It follows that if \( \{\hat{\xi}_i(\xi)\} \) denotes the unique \( \hat{\mathcal{S}} \)-cycle near the origin, then \( \hat{z}_i(0) = \hat{\xi}_i(0) = y_i \). We let
\[
t_i = e^T_1 y_i.
\]

**Lemma 7.** Suppose (6) is at a generalized shrinking point when \( \xi = 0 \). Then,

(i) \( t_0 = t_{ld} = 0 \);
(ii) \( t_d, t_{(l-1)d} < 0 \), \( t_{-d}, t_{(l+1)d} > 0 \);
(iii) \( \{y_i\} \) has minimal period \( n \);

See [10] for a proof. Lemma 7 essentially states that the orbit \( \{y_i\} \) appears as in figure 8.

A consequence of lemma 7 is that several important matrices are singular. To see this, first note that the point \( y_0 \) is a fixed point of \( h^S(y; 0) \). By lemmas 2 and 7(i), \( y_0 \) is also a fixed point of \( h^{S^\varrho}(y; 0) \). Using (12), \( S^{\varrho^r} = S^{(-d)} \), and so \( y_d \) is a fixed point of \( h^\varrho(y; 0) \). The points \( y_0 \) and \( y_d \) are distinct (by lemma 7), in other words there are multiple \( S \)-cycles. Hence the matrix \( I - M_S(0) \) must be singular and consequently each \( P_{\mathcal{S}[l]}(0) \) is singular by lemma 6(ii) producing the following result (given also in [10]):

**Corollary 8.** Suppose (6) is at a generalized shrinking point when \( \xi = 0 \). Then,

(i) \( I - M_S(0) \) is singular;
(ii) \( P_{S[l]}(0) \) is singular, for all \( l \).

For reader convenience let us briefly summarize symbols used:

\[
x \in \mathbb{R}^N, \quad s = e^T_1 x, \\
y_i = \hat{z}_i(0) = \hat{\xi}_i(0), \quad t = e^T_1 y, \\
\mu z = x, \quad u = e^T_1 z.
\]
6. Unfolding generalized shrinking points

We begin by performing a change of coordinates, similar to that in [10], such that, locally, two boundaries of the associated resonance tongue lie on coordinate planes. We are given that $\tilde{u}_0(\xi)$ and $\tilde{u}_{ld}(\xi)$ are $C^{k-1}$ and $\tilde{u}_0(0) = \tilde{u}_{ld}(0) = 0$ (lemma 7(i)). Since a generalized shrinking point is a codimension-three phenomenon, we assume there are three bifurcation parameters

$$\xi = (\mu, \eta, \nu).$$

As long as the matrix

$$
\begin{bmatrix}
\frac{\partial \tilde{u}_0}{\partial \eta} & \frac{\partial \tilde{u}_0}{\partial \nu} \\
\frac{\partial \tilde{u}_{ld}}{\partial \eta} & \frac{\partial \tilde{u}_{ld}}{\partial \nu}
\end{bmatrix}_{\xi=0}
$$

is nonsingular, we may perform a nonlinear coordinate change such that

$$\tilde{u}_0(\xi) = \eta(1 + O(1)), \quad (30)$$

$$\tilde{u}_{ld}(\xi) = \nu(1 + O(1)). \quad (31)$$

Consequently, on the coordinate plane $\eta = 0$, the point $\tilde{x}_0$ of the $\tilde{S}$-cycle lies on the switching manifold. According to lemma 2, the $\tilde{S}$-cycle is also an $\hat{S}_0$-cycle here. Similarly on $\nu = 0$, $\tilde{x}_{ld}$ lies on the switching manifold. By lemma 2, here the $\tilde{S}$-cycle is also an $\hat{S}_{ld}$-cycle (this equality follows simply from (12)). Along the $\mu$-axis (in three-dimensional parameter space) both $\tilde{x}_0$ and $\tilde{x}_{ld}$ lie on the switching manifold so here the $\tilde{S}$-cycle is also an $\hat{S}$-cycle. Hence on the $\mu$-axis the orbits $\{\tilde{x}_i(\xi)\}$ and $\{\tilde{x}_i(\xi)\}$ are identical.

In order for $\xi$ to properly unfold a generalized shrinking point we need a nondegeneracy condition on the nonlinear terms of (6) that guarantees the shrinking point breaks apart in the usual manner as $\mu$ increases from 0. Along the $\mu$-axis, $\tilde{x}_0$ is a fixed point of $f^S$ and when $\mu = 0$ it has an associated multiplier of 1. We will assume that the algebraic multiplicity of this multiplier is one. An appropriate condition on the nonlinear terms is that this multiplier varies linearly (to lowest order) with respect to $\mu$, for small $\mu$. That is, we require

$$\left. \frac{\partial}{\partial \mu} \det \left( I - D_x f^{S}(\tilde{x}_0(\xi); \xi) \right) \right|_{\xi=0} \neq 0. \quad (32)$$
We are free to scale the parameter $\mu$ before performing the analysis. However, we find it is most instructive to merely fix the sign of $\mu$ in a way that ensures resonance arises for $\mu > 0$. It turns out that the following assumption achieves this effect:

$$\text{sgn} \left( \frac{\partial}{\partial \mu} \det \left( I - D_s f^S(\tilde{x}_0(\xi); \xi) \right) \bigg|_{\xi = 0} \right) = \text{sgn}(\det(I - M_0(0))).$$

(33)

Our unfolding theorem details the existence of admissible periodic solutions in regions of three-dimensional parameter space that are bounded by six different surfaces. As a consequence of the choice (30) and (31), three of these surfaces are simply the coordinate planes. The following lemma gives series expansions of the remaining three surfaces.

**Lemma 9.** Suppose the piecewise-$C^K$ map (6) is at a generalized shrinking point when $\xi = 0$ and that $K \geq 4$. Assume that the only eigenvalue of $M_0(0)$ on the unit circle is $1$ and that it has algebraic multiplicity one, and that (30), (31) and (33) hold. Let

$$\tilde{\delta} = \det(I - \tilde{M}_0(0)), \quad \hat{\delta} = \det(I - \hat{M}_0(0)),$$

(34)

$$k_0 = \left. \frac{\partial}{\partial \mu} \det(I - D_s f^S(\tilde{x}_0(\xi); \xi)) \right|_{\xi = 0} = \text{sgn}(\det(I - \tilde{M}_0(0))).$$

(35)

(36)

(37)

(38)

(39)

See appendix A for a proof. It is useful to consider the nonsingular, linear coordinate change:

$$\begin{bmatrix} \tilde{\delta} \\ \tilde{\eta} \\ \tilde{\nu} \end{bmatrix} = \begin{bmatrix} k_0 I_{(d-1)d} - \frac{1}{t_d} I_{d(d-1)} \\ \frac{t_d}{\delta t_{d+1}} I_{d(d-1)} \\ - \frac{1}{\delta t_{d+1}} I_{d(d-1)} \end{bmatrix},$$

(40)
The three-dimensional unfolding of a generalized shrinking point for the piecewise-smooth, continuous map (6). The surface \( \mu = 0 \) corresponds to the border-collision bifurcation of a fixed point. On each of the four surfaces, \( \eta = 0, \nu = 0, \eta = \phi_1(\mu, \nu) \) and \( \nu = \phi_2(\mu, \eta) \) (which share a mutual intersection with the \( \mu \)-axis), one point of an \( S \)-cycle lies on the switching manifold. The surface \( \Lambda(\xi) = 0 \) corresponds to classical saddle-node bifurcations of an \( S \)-cycle. This surface is approximately a cone, as made evident by transformation to the alternative \((\tilde{\mu}, \tilde{\eta}, \tilde{\nu})\)-coordinate system, (40), and tangentially intersects \( \eta = 0 \) and \( \nu = \phi_2(\mu, \eta) \) along the curves \((\mu, 0, \zeta_1(\mu))\) and \((\mu, \zeta_2(\mu), \phi_2(\mu, \zeta_2(\mu)))\), respectively. The plane \( \mu = \text{const} \), illustrates the cross-section shown in figure 10.

because substitution of (40) into (37) produces

\[
\Lambda(\xi) = \tilde{\mu}^2 + \tilde{\eta}^2 - \tilde{\nu}^2 + o(2). \tag{41}
\]

In this alternative coordinate system, saddle-node bifurcations occur approximately on a cone. All six surfaces are sketched for \( \mu \geq 0 \) in figure 9. The surface \( \Lambda(\xi) = 0 \) intersects the plane \( \eta = 0 \) tangentially along the curve \((\mu, 0, \zeta_1(\mu))\) and the surface \( \nu = \phi_2(\mu, \eta) \) tangentially along the curve \((\mu, \zeta_2(\mu), \phi_2(\mu, \zeta_2(\mu)))\).

Different two-dimensional slices of parameter space through figure 9 will produce vastly different bifurcation sets. Since slices defined by fixing the value of \( \mu \) have been shown earlier (see figure 6), in figure 9 we draw a plane at fixed \( \mu > 0 \) and show its curves of intersection with the nearby surfaces. This cross-section is shown again in figure 10. A close inspection of the formulae for \( \phi_1 \) and \( \phi_2 \) given lemma 9 reveals a specific geometrical arrangement near the generalized shrinking point as follows. Since \( \theta_d < 0, t(l+1) > 0 \) (lemma 7(ii)) and \( \text{sgn}(k_0) = \text{sgn}(\delta) \) (33), the coefficient for the \( \mu \nu \) term of \( \phi_1(\mu, \nu) \) is positive. Consequently, for small \( \mu > 0 \) the angle \( \theta_2 \) in figure 10 is greater than 90°. Similarly the coefficient for the \( \mu \eta \) term of \( \phi_2(\mu, \eta) \) is negative and so \( \theta_1 < 90° \). Any smooth distortion of figure 10 will preserve the property \( \theta_1 < \theta_2 \). The curves \( \eta = \phi_1(\mu, \nu) \) and \( \nu = \phi_2(\mu, \eta) \) bend left and down because the \( \nu^2 \) term of \( \phi_1(\mu, \nu) \) and the \( \eta^2 \) term of \( \phi_2(\mu, \eta) \) are both negative. The saddle-node locus is approximately a parabola.

Finally, we present the main result. See appendix A for a proof.

**Theorem 10.** Suppose (6) is at a generalized shrinking point when \( \xi = 0 \) and that \( K \geq 4 \). Assume that the only eigenvalue of \( M_3(0) \) on the unit circle is 1 and that it has algebraic multiplicity one. Assume \( y_0 \) and \( y_d \) are the only points of \( \{y_i\} \) that lie on the switching...
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\[ \theta_1 \]
\[ \theta_2 \]
\[ \nu = \phi_2(\mu, \eta) \]
\[ \eta = \phi_1(\mu, \nu) \]
\[ \nu = \zeta_1(\mu) \]
\[ \eta = 0 \]
\[ \delta_0 = 0 \]
\[ \nu = 0 \]
\[ \delta_{id} = 0 \]

\[ \Psi_1 \]
\[ \Psi_2 \]
\[ \Psi_3 \]

\[ \nu = \phi_2(\mu, \eta) \]
\[ \delta_0 = 0 \]
\[ \nu = \zeta_1(\mu) \]
\[ \eta = 0 \]

\[ \eta = \phi_1(\mu, \nu) \]
\[ \eta = \phi_2(\mu, \eta) \]
\[ \nu = \zeta_1(\mu) \]
\[ \Lambda(\xi) = 0 \]

**Figure 10.** A two-parameter bifurcation diagram for map (6) near a generalized shrinking point for fixed, small \( \mu > 0 \). As stated in theorem 10, an admissible \( S \)-cycle coexists with, (i) an admissible \( \hat{S} \)-cycle in \( \Psi_1 \); (ii) an admissible \( \check{S} \)-cycle in \( \Psi_2 \); (iii) a second admissible \( S \)-cycle in \( \Psi_3 \). Along the solid curves the two coexisting, admissible, periodic solutions undergo a border-collision fold bifurcation except on the curve connecting the points \( v = \zeta_1(\mu) \) and \( \eta = \zeta_2(\mu) \) which corresponds to a classical saddle-node bifurcation of the two periodic solutions. The boundary between \( \Psi_1 \) and \( \Psi_2 \) and the boundary between \( \Psi_2 \) and \( \Psi_3 \) correspond to border-collision persistence.

**7. Conclusions**

We have studied resonance arising from an arbitrary border-collision bifurcation of a piecewise-smooth, continuous map. When a periodic solution created in a border-collision bifurcation is rotational, in the sense described in section 3, the corresponding resonance tongue typically has a sausage-like geometry, see figure 3. Shrinking points break apart as parameters are varied to move away from the border-collision bifurcation, figure 6. We have proved theorem 10 which details the manner by which shrinking point destruction occurs. The results of the theorem are
in complete agreement with numerical results, figures 5 and 6. Theorem 10 does not provide an understanding of global properties of resonance tongues, for instance the observation that the majority of, or perhaps all of, the kinks in the resonance tongues of figure 5 appear on the left sides of the tongues. In addition, periodic solutions other than those described in theorem 10 may be created at $\mu = 0$ in a neighbourhood of a shrinking point. This corresponds to an overlapping of resonance tongues.

Some information regarding the stability of solutions may be gleaned from theorem 10. As in [12], let $a_i[S]$ denote the number of real stability multipliers of $S$ in $\Psi_i$ that are greater than one. Sufficiently near $\xi = 0$, since the coexisting periodic solutions in $\Psi_1$ and $\Psi_2$ collide in border-collision fold bifurcations, $a_i[S] + a_i[\hat{S}]$ and $a_i[S] + a_i[\tilde{S}]$ are odd [9]. Since the two $S$-cycles collide in a classical saddle-node bifurcation, $a_1[S] - a_2[S] = \pm 1$. For instance, for a two-dimensional map such as (2), if, $S$-cycles are attracting in $\Psi_1$, then, $\hat{S}$-cycles are saddles and the $S$-cycles are saddles in $\Psi_2$, matching figure 5.

If map (6) has an invariant circle that intersects the switching manifold at two points and the restriction of the map to the circle is a homeomorphism, then any periodic solution on the circle will have a corresponding symbol sequence that is rotational. Our results do not apply to periodic solutions born in border-collision bifurcations that are non-rotational. However such periodic solutions seem to be, in some sense, less common [13]. Note that we make no requirement that (6) is a homeomorphism or has an invariant circle, only that corresponding periodic solutions are rotational.

A limitation of theorem 10 is that it includes the assumption that map (6) is piecewise-$C^K$. Poincaré maps relating to sliding phenomena are generically piecewise-smooth, continuous with a $\frac{1}{2}, 2, \frac{5}{2}, \ldots$ type power expansion [5] and hence not apply to theorem 10. It seems reasonable that in this case a similar result with different scaling laws could apply, see [26]. A major hurdle in the analysis is that the centre manifold theorem (applied in the proof of theorem 10) is not immediately applicable to non-integer power expansions.

In our definition of the codimension-three generalized shrinking point (definition 3), we require that both matrices $I - M_{\hat{S}}$ and $I - M_{\tilde{S}}$ are nonsingular at $\xi = 0$. We then prove lemma 9 by computing parameter-dependent series expansions (see appendix A). However part (i) of lemma 9 is independent of the parameters and it seems there should be a more direct proof of this result. This problem remains for future work. Related problems that remain to be fully understood include the persistence of invariant topological circles created at border-collision bifurcations and the simultaneous occurrence of a border-collision bifurcation with a classical Neimark–Sacker bifurcation.

Appendix A. Proofs for section 6

Proof of lemma 9. We divide the proof into four major steps. In the first step we use the formulae for $\hat{u}_0(\xi)$ (30) and $\hat{u}_{id}(\xi)$ (31) to derive formulae for $\hat{u}_0$ and $\hat{u}_{id}$, enabling us to compute the border-collision boundaries $\eta = \phi_1(\mu, v)$ and $v = \phi_2(\mu, \eta)$. The method used is an extension of lemma 3 of [10] to account for the nonlinear terms in (6). In step 2 we continue this methodology to derive identities relating coefficients, verify (i) of the lemma and refine the formulae for $\phi_1$ and $\phi_2$. In the third step we compute the one-dimensional centre manifold of $f^S$ to identify saddle-node bifurcations on the surface $\Lambda = 0$. Lastly, in step 4 we study at the intersections of $\Lambda = 0$ with $\eta = 0$ and $v = \phi_2(\mu, \eta)$ to complete the proof. For convenience we write

$$\det(I - M_{\hat{S}}(0, \eta, v)) = k_1\eta + k_2v + O(2).$$  (42)
Step 1. Derive formulae for \( \hat{u}_0 \) and \( \hat{u}_{ld} \) and define \( \phi_1 \) and \( \phi_2 \).

Periodic solutions of (6) with associated symbol sequences that differ by a single symbol may be related algebraically (as shown below). However the sequences \( \hat{S} \) and \( \check{S} \) differ in two symbols, so in general we cannot effectively compare the \( \hat{S} \) and \( \check{S} \)-cycles. But if one of the four important quantities, \( \check{u}_0 \), \( \check{u}_{ld} \), \( \hat{u}_0 \) and \( \hat{u}_{ld} \) is zero, then one of the two \( n \)-cycles is also an \( S \)-cycle and the two \( n \)-cycles may indeed be effectively related. The four curves which make up the edges of a resonance tongue near a shrinking point correspond to where these four quantities are zero. We separate this step of the proof by where different assumptions on the parameters are made.

(a) Suppose \( n = 0 \). Then by (30), \( \check{u}_0 = 0 \). Thus the \( \hat{S} \)-cycle, \( \{ \check{x}_i \} \), is also an \( S \)-cycle. So each \( \check{x}_i \) is a fixed point of \( f^{S(0)} \). In particular, \( \check{x}_{ld} \) is a fixed point of \( f^{S(0)} \), which in the renormalized frame (25) says that \( \check{z}_{ld} \) is a fixed point of \( h^{S(0)} \):

\[
\check{z}_{ld}(\mu, 0, v) = h^{S(0)}(\check{z}_{ld}(\mu, 0, v); \mu, 0, v).
\]

Expressing \( h^{S(0)} \) as a Taylor series centred at \( y_{ld} \) (28) and evaluated at \( \check{z}_{ld} \) yields

\[
\check{z}_{ld}(\mu, 0, v) = h^{S(0)}(y_{ld}; \mu, 0, v) + D_{z_h}h^{S(0)}(y_{ld}; \mu, 0, v)(\check{z}_{ld}(\mu, 0, v) - y_{ld}) + O(3), \quad (43)
\]

where the next term in the expansion is \( O(\mu)O(\check{z}_{ld}(\mu, 0, v) - y_{ld}\xi) \) which is \( O(3) \) because \( \check{z}_{ld}(0, 0, 0) = y_{ld} \).

Now, \( \check{z}_{ld} \) is a fixed point of \( h^{S(0)} \) (for any sufficiently small \( \xi \)), therefore

\[
\check{z}_{ld}(\mu, 0, v) = h^{S(0)}(y_{ld}; \mu, 0, v) + D_{z_h}h^{S(0)}(y_{ld}; \mu, 0, v)(\check{z}_{ld}(\mu, 0, v) - y_{ld}) + O(3), \quad (44)
\]

where the right-hand side is the Taylor series of \( h^{S(0)} \) centred at \( y_{ld} \). But \( S(ld) = \check{S}(ld) \) (refer to section 3), thus whenever \( u = e_1^Tz = 0 \), we have \( h^{S(0)}(z; \xi) = h^{S(0)}(z; \xi) \) (by continuity of (6)). Since \( t_{ld} = e_1^Ty_{ld} = 0 \) (lemma 7(i)), we have

\[
h^{S(0)}(y_{ld}; \mu, 0, v) = h^{\check{S}(0)}(y_{ld}; \mu, 0, v). \quad (45)
\]

Furthermore, although in general the last \( N - 1 \) columns of \( D_zh^{S(0)} \) and \( D_zh^{\check{S}(0)} \) are not equal, since \( h^{S(0)} \equiv h^{\check{S}(0)} \) on the switching manifold, the last \( N - 1 \) columns of \( D_zh^{S(0)}(y_{ld}; \mu, 0, v) \) and \( D_zh^{\check{S}(0)}(y_{ld}; \mu, 0, v) \) are indeed equal. Consequently, the first row of their adjugates is the same:

\[
e_1^T\text{adj}(I - D_zh^{S(0)}(y_{ld}; \mu, 0, v)) = e_1^T\text{adj}(I - D_zh^{\check{S}(0)}(y_{ld}; \mu, 0, v)). \quad (46)
\]

The combination of (43), (44), (45) and \( t_{ld} = 0 \) produces

\[
(I - D_zh^{S(0)}(y_{ld}; \mu, 0, v))\check{z}_{ld}(\mu, 0, v) = (I - D_zh^{\check{S}(0)}(y_{ld}; \mu, 0, v))\check{z}_{ld}(\mu, 0, v) + O(3), \quad (47)
\]

and multiplication of both sides of (47) by (46) on the left (remembering (9) and \( u_i = e_1^Tz_i \)) leads to

\[
det(I - D_zh^{S(0)}(y_{ld}; \mu, 0, v))\check{u}_{ld}(\mu, 0, v) = det(I - D_zh^{\check{S}(0)}(y_{ld}; \mu, 0, v))\check{u}_{ld}(\mu, 0, v) + O(3).
\]
We now plug in known expansions (\(\hat{u}_{ld}\) is given by (31), det(\(I - D_x h^{\hat{S}^{(0)}}(y_{ld}; \mu, 0, v)) = \hat{\delta} + O(1)\) by (35) and lemma 3, det(\(I - D_x h^{\hat{S}^{(0)}}(y_{ld}; \mu, 0, v)) = k_0 \mu + k_2 v + O(2)\) by (36), (42) and lemma 3):

\[
(\hat{k}_0 \mu + k_2 v + O(2))v(1 + O(1)) = (\hat{\delta} + O(1))\hat{u}_{ld}(\mu, 0, v),
\]

which, upon rearranging, produces the following useful expression:

\[
\hat{u}_{ld}(\mu, 0, v) = \frac{k_0}{\delta} \mu v + \frac{k_2}{\delta} v^2 + O(3).
\]  

Below we use the same approach to derive further expressions for \(\hat{u}_0\) and \(\hat{u}_{ld}\) with different assumptions on the parameters. For brevity we will not provide the same level of detail.

(b) Suppose \(\eta = 0\) and \(\mu = 0\). As above, since \(\eta = 0\), the \(S\)-cycle is also an \(S\)-cycle. In particular, \(\hat{z}_{-d}\) is a fixed point of \(h^{S^{(-d)}}\). Here we express \(h^{S^{(-d)}}\) as a Taylor series centred at \(y_0\) (the reason for choosing \(y_0\) will soon become clear) and evaluated at \(\hat{z}_{-d}\):

\[
\hat{z}_{-d}(0, 0, v) = h^{S^{(-d)}}(y_0; 0, 0, v) + D_x h^{S^{(-d)}}(y_0; 0, 0, v)(\hat{z}_{-d}(0, 0, v) - y_0),
\]

where, unlike in (43), there is no next term in the expansion when \(\mu = 0\) the map is affine. Now, \(\hat{z}_0\) is a fixed point of \(h^{\hat{S}}\), therefore

\[
\hat{z}_0(0, 0, v) = h^{\hat{S}}(y_0; 0, 0, v) + D_x h^{\hat{S}}(y_0; 0, 0, v)(\hat{z}_0(0, 0, v) - y_0).
\]  

But \(S^{(-d)} = S^{\hat{\eta}}\) (by (12)), and since \(t_0 = 0\) (lemma 7(i)) we may perform the same simplification that we did above to combine (49) and (50) leaving

\[
(I - D_x h^{S^{(-d)}}(y_0; 0, 0, v))\hat{z}_{-d}(0, 0, v) = (I - D_x h^{\hat{S}}(y_0; 0, 0, v))\hat{z}_0(0, 0, v),
\]

\[
\Rightarrow \det(I - D_x h^{S^{(-d)}}(y_0; 0, 0, v))\hat{u}_{-d}(0, 0, v),
\]

\[
= \det(I - D_x h^{\hat{S}}(y_0; 0, 0, v))\hat{u}_0(0, 0, v),
\]

\[
\Rightarrow (k_2 v + O(v^2))(I_{-d} + O(v)) = (\hat{\delta} + O(v))\hat{u}_0(0, 0, v),
\]

\[
\Rightarrow \hat{u}_0(0, 0, v) = \frac{k_2 t_{-d}}{\delta} v + O(v^2).
\]  

(c) Suppose \(v = 0\). As discussed in section 6, when \(v = 0\) the \(S\)-cycle is also an \(S^{(-d)}\)-cycle. Therefore \(\hat{z}_0\) is a fixed point of \(h^{S^{(-d)}}\) and so

\[
\hat{z}_0(\mu, \eta, 0) = h^{S^{(-d)}}(y_0; \mu, \eta, 0) + D_x h^{S^{(-d)}}(y_0; \mu, \eta, 0)(\hat{z}_0(\mu, \eta, 0) - y_0) + O(3).
\]  

Also, \(\hat{z}_0\) is a fixed point of \(h^{\hat{S}}\), thus

\[
\hat{z}_0(\mu, \eta, 0) = h^{\hat{S}}(y_0; \mu, \eta, 0) + D_x h^{\hat{S}}(y_0; \mu, \eta, 0)(\hat{z}_0(\mu, \eta, 0) - y_0) + O(3).
\]

Combining (52) and (53) leads to (since \(S^{(-d)} = S^{\hat{\eta}}\))

\[
\det(I - D_x h^{S^{(-d)}}(y_0; \mu, \eta, 0))\hat{u}_0(\mu, \eta, 0) = \det(I - D_x h^{\hat{S}}(y_0; \mu, \eta, 0))\hat{u}_0(\mu, \eta, 0) + O(3).
\]

Take care to note that

\[
\frac{\hat{k}_0}{\mu} \equiv \left. \frac{\partial}{\partial \mu} \det(I - D_x f^{\hat{S}}(\hat{z}_0(\xi); \xi)) \right|_{\xi = 0}
\]  

is different from (32) due to the presence of nonlinear terms in (6) (below we will show that in fact \(\hat{k}_0 = -k_0\)). Consequently,

\[
(k_0 \mu + k_1 \eta + O(2))\eta(1 + O(1)) = (\hat{\delta} + O(1))\hat{u}_0(\mu, \eta, 0),
\]

\[
\Rightarrow \hat{u}_0(\mu, \eta, 0) = \frac{\hat{k}_0}{\delta} \mu \eta + \frac{k_1}{\delta} \eta^2 + O(3).
\]  

\[
\hat{u}_0(\mu, \eta, 0) = \frac{\hat{k}_0}{\delta} \mu \eta + \frac{k_1}{\delta} \eta^2 + O(3).
\]
(d) Suppose \( v = 0 \) and \( \mu = 0 \). Here \( \hat{z}_{(l+1)d} \) is a fixed point of \( h^{S(0)} \), so
\[
\hat{z}_{(l+1)d}(0, \eta, 0) = h^{S(0)}(y_{ld}; 0, \eta, 0) + D_h h^{S(0)}(y_{ld}; 0, \eta, 0) (\hat{z}_{(l+1)d}(0, \eta, 0) - y_{ld}).
\]
and \( \hat{z}_{ld} \) is a fixed point of \( h^{S(d)} \), so
\[
\hat{z}_{ld}(0, \eta, 0) = h^{S(d)}(y_{ld}; 0, \eta, 0) + D_h h^{S(d)}(y_{ld}; 0, \eta, 0) (\hat{z}_{ld}(0, \eta, 0) - y_{ld}),
\]
and since \( s^{(l)} = \hat{s}^{(ld)} \) we obtain
\[
det (I - D_h h^{S(d)}(y_{ld}; 0, \eta, 0)) \hat{u}_{(l+1)d}(0, \eta, 0) = det (I - D_h h^{S(d)}(y_{ld}; 0, \eta, 0)) \hat{u}_{ld}(0, \eta, 0),
\]
\[
\Rightarrow (k_1 \eta + O(\eta^2)) t_{(l+1)d} + O(\eta)) = (\hat{\delta} + O(\eta)) \hat{u}_{ld}(0, \eta, 0),
\]
\[
\Rightarrow \hat{u}_{ld}(0, \eta, 0) = \frac{k_{1t_{(l+1)d}}}{\hat{\delta}} \eta + O(\eta^2).
\]

We now apply the implicit function theorem to the \( C^{K-1} \) function \( \hat{u}_{ld}(\xi) \). By (48) and (58), there exists a unique \( C^{K-1} \) function \( \phi_1 \) such that for small \( \mu \) and \( v \), \( \hat{u}_{ld}(\mu, \phi_1(\mu, v), v) = 0 \) and
\[
\phi_1(\mu, v) = -\frac{k_0}{k_{1t_{(l+1)d}}} \mu v - \frac{k_2}{k_{1t_{(l+1)d}}} v^2 + O(3).
\]
Recall that \( \hat{u}_{ld} = 0 \) along the \( \mu \)-axis (a consequence of (30) and (31)), therefore \( \phi_1 = 0 \) whenever \( v = 0 \). Thus we may rewrite (59) as
\[
\phi_1(\mu, v) = v \left( -\frac{k_0}{k_{1t_{(l+1)d}}} \mu - \frac{k_2}{k_{1t_{(l+1)d}}} v + O(2) \right).
\]
Similarly by (51) and (55), there exists a unique \( C^{K-1} \) function \( \phi_2 \) such that for small \( \mu \) and \( \eta \), \( \hat{u}_{0}(\mu, \phi_2(\mu, \eta)) = 0 \) and
\[
\phi_2(\mu, \eta) = \eta \left( -\frac{\tilde{k}_0}{k_{2T-d}} \mu - \frac{k_{1}}{k_{2T-d}} \eta + O(2) \right),
\]
where the \( \eta \) may be factored in the same fashion as for (60).

**Step 2.** Verify (i) of the lemma and refine the formulae for \( \phi_1 \) and \( \phi_2 \), (60) and (61).

Here we continue to employ the methodology above to compare the \( \hat{s} \) and \( \hat{S} \)-cycles.

(a) Suppose \( \eta = \phi_1(\mu, v) \). Then \( \hat{u}_{ld} = 0 \), and so the \( \hat{S} \)-cycle is also an \( S \)-cycle. Thus, in particular, \( \tilde{z}_0 \) is a fixed point of \( h^{\hat{S}} \):
\[
\tilde{z}_0(\mu, \phi_1(\mu, v)) = h^{\hat{S}}(y_0; \mu, \phi_1(\mu, v)) + D_h h^{\hat{S}}(y_0; \mu, \phi_1(\mu, v)) (\tilde{z}_0(\mu, \phi_1(\mu, v)) - y_0) + O(3).
\]

Also \( \tilde{z}_0 \) is a fixed point of \( h^{\hat{S}} \):
\[
\tilde{z}_0(\mu, \phi_1(\mu, v)) = h^{\hat{S}}(y_0; \mu, \phi_1(\mu, v)) + D_h h^{\hat{S}}(y_0; \mu, \phi_1(\mu, v)) (\tilde{z}_0(\mu, \phi_1(\mu, v)) - y_0) + O(3).
\]

Then, since \( S = \hat{S} \),
\[
det (I - D_h h^{\hat{S}}(y_0; \mu, \phi_1(\mu, v))) \tilde{u}_0(\mu, \phi_1(\mu, v)) = det (I - D_h h^{\hat{S}}(y_0; \mu, \phi_1(\mu, v))) \tilde{u}_0(\mu, \phi_1(\mu, v)) + O(3).
\]
Unlike for similar expressions in step 1, we have already determined an expansion for each of the four components of (62) (in particular \( \tilde{u}_0(\mu, \phi_1(\mu, v), v) \) is given by (51), (55) and (60) and \( \tilde{u}_0(\mu, \phi_1(\mu, v), v) \) is given by (30) and (60), also recall lemma 3):
\[
(k_0\mu + k_2\nu + O(2)) \left( \frac{k_2t^d}{\delta} v + O(2) \right)
= (\tilde{\delta} + O(1)) \left( -\frac{k_0}{k_1(l+1)d} \mu v - \frac{k_2}{k_1(l+1)d} v^2 + O(3) \right) + O(3).
\]
Equating the second-order coefficients produces
\[
k_1k_2/(l+1)d/\delta = -\tilde{\delta} \quad (63).
\]
(b) Suppose \( \eta = \phi_1(\mu, v) \) and \( \mu = 0 \). In the same manner as above, equating the first components of the power series of \( h^{\delta(l-1)d} \) and \( h^{\delta(l+1)d} \) yields
\[
det(I - D_x h^{\delta(l-1)d} (y_{ld}; 0, \phi_1(0, v), v) u_{ld}(0, \phi_1(0, v), v))
= det(I - D_x h^{\delta(l+1)d} (y_{ld}; 0, \phi_1(0, v), v) u_{ld}(0, \phi_1(0, v), v)),
\]
and since \( \delta^{(l-1)d} = \delta^{(l+1)d} \),
\[
(k_2 v + O(v^2))(t_{l-1d} + O(v)) = (\tilde{\delta} + O(v))(v + O(v^2)),
\]
\[
\Rightarrow k_2 = \frac{\tilde{\delta}}{t_{l-1d}}. \quad (64)
\]
(c) Suppose \( v = \phi_2(\mu, \eta) \) and \( \mu = 0 \). Here, equating the first components of \( h^\delta \) and \( h^\tilde{\delta} \) produces
\[
det(I - D_x h^{\delta} (y_0; 0, \eta, \phi_2(0, \eta)) u_d(0, \eta, \phi_2(0, \eta))
= det(I - D_x h^\tilde{\delta} (y_0; 0, \eta, \phi_2(0, \eta)) u_0(0, \eta, \phi_2(0, \eta)),
\]
from which it follows that
\[
(k_1\eta + O(\eta^2))(t_d + O(\eta)) = (\tilde{\delta} + O(\eta))(\eta + O(\eta^2)),
\]
\[
\Rightarrow k_1 = \frac{\tilde{\delta}}{t_d}. \quad (65)
\]
We now combine above equations to demonstrate some parts of the lemma. By combining (63), (64) and (65) we obtain (i). Combining (60), (64) and (65) verifies (ii). Combining (61), (64) and (65) will verify (iii) once it is shown that \( k_0 = -k_0 \) (see below).

**Step 3.** Derive and analyse the one-dimensional centre manifold of \( f^\delta \) to obtain the function \( \Lambda(\xi) \) and identify saddle-node bifurcations of \( S \)-cycles.

When \( \xi \equiv (\mu, \eta, v) = 0, x = 0 \) is a fixed point of \( f^\delta(x; \xi) \) and the associated stability multipliers are the eigenvalues of \( D_x f^\delta(0; 0) = M^\delta(0) \). The matrix \( M^\delta(0) \) has an eigenvalue 1 of algebraic multiplicity one. Let \( v \in \mathbb{R}^N \) be the associated eigenvector, i.e. \( M^\delta(0)v = v \). Notice \( v \neq 0 \) implies \( e^1 v \neq 0 \) since if not then \( M^\delta(0)v = v \) which contradicts the assumption that \( I - M^\delta(0) \) is nonsingular (definition 3). In what follows we assume \( e^1 v = 1 \).

We now compute the restriction of \( f^\delta \) to the one-dimensional centre manifold. Let
\[
F(x; \xi) = \begin{bmatrix} f^\delta(x; \xi) \\ \xi \end{bmatrix},
\]
denote the \((N + 3)\)-dimensional, \(C^K\), extended map. The Jacobian,

\[
DF(0; 0) = \begin{bmatrix}
M_S(0) & P_S(0)b(0) & 0 & 0 \\
0 & 1
\end{bmatrix},
\]

(66)

has a four-dimensional centre space, \(E^c \in \mathbb{R}^{N+3}\), spanned by

\[
\begin{bmatrix}
\nu \\
0 \\
0 \\
0
\end{bmatrix}
\]

\[
\begin{bmatrix}
y_0 \\
1 \\
0 \\
0
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 \\
0 \\
0 \\
1
\end{bmatrix}
\]

since, in particular, \(y_0 = h^S(y_0; 0) = M_S(0)y_0 + P_S(0)b(0)\).

Since \(e_1^T \nu = 1\), we may use the centre manifold theorem to express the local centre manifold, \(W^c\), of \(f^S\), in terms of \(s = e_1^T x\) and \(\xi\). In particular, on \(W^c\),

\[
x = X(s; \xi) = sv + \mu y_0 + O(2),
\]

(67)

where \(X : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^N\) is \(C^{K-1}\). The first component of the restriction of \(f^S\) to \(W^c\) is given by

\[
s'(s; \xi) = e_1^T f^S(X(s; \xi); \xi)
\]

\[
= e_1^T \mu P_S(0)b(0) + e_1^T M_S(0)(sv + \mu y_0) + O(2)
\]

\[
= e_1^T (P_S(0)b(0) + M_S(0)y_0)\mu + e_1^T M_S(0)\nu s + O(2)
\]

\[
= s + O(2),
\]

since \(e_1^T (P_S(0)b(0) + M_S(0)y_0) = e_1^T y_0 = t_0 = 0\) and \(e_1^T M_S(0)\nu = e_1^T \nu = 1\). However, we require the knowledge of second-order terms of \(s'(s; \xi)\), so write

\[
s'(s; \xi) = s + c_1 s^2 + c_2 \mu s + c_3 \eta s + c_4 \nu s + c_5 \mu^2 + c_6 \mu \eta + c_7 \mu \nu + c_8 \eta^2 + c_9 \eta \nu + c_{10} \nu^2 + O(3).
\]

(68)

We now utilize known properties of \(f^S\) to determine an expression for the majority of the coefficients, \(c_i\).

1. When \(\mu = 0\), \(s = 0\) is a fixed point of \(f^S\), thus \(s'(0; 0, \eta, \nu) = 0\), hence \(c_0 = c_5 = c_{10} = 0\).

2. When \(\eta = 0\), \(s = \tilde{s}_0\) is a fixed point of \(f^S\) and since here \(s = \tilde{s}_0 = 0\), we have \(s'(0; \mu, 0, \nu) = 0\), hence \(c_5 = 0\).

3. Similarly when \(\nu = 0\), \(s = \tilde{s}_d\) is a fixed point of \(f^S\). Here \(s = \tilde{s}_d = \tilde{u}_d \mu = t_d \mu + O(2)\),

Thus we have reduced the centre manifold map (68) to

\[
s'(s; \xi) = s - \frac{c_2}{t_d} s^2 + c_2 \mu s + c_3 \eta s + c_4 \nu s - c_3 t_d \mu \eta + O(3).
\]

(69)

4. Let \(\lambda(\eta, \nu)\) be the eigenvalue of \(D_s f^S(0; \eta, \nu)\), \(C^{K-1}\) dependent on \(\eta\) and \(\nu\), for which \(\lambda(0, 0) = 1\). This eigenvalue is also the stability multiplier of the fixed point, \(s = 0\), of \(s'(s; 0, \eta, \nu)\), that is, by (69)

\[
\lambda(\eta, \nu) = \frac{\partial s'}{\partial s}(0; 0, \eta, \nu) = 1 + c_3 \eta + c_4 \nu + O(2).
\]

(70)
We have shown that the restriction of \( f^S(0; 0, \eta, \nu) \) is equal to the product of all \( N \) eigenvalues (counting algebraic multiplicity) of \( I - D_x f^S(0; 0, \eta, \nu) \). Thus
\[
\det(I - D_x f^S(0; 0, \eta, \nu)) = (1 - \lambda(\eta, \nu)) \mathcal{P}(\eta, \nu),
\]
where \( \mathcal{P}(\eta, \nu) \) denotes the product of the remaining \( N - 1 \) eigenvalues of \( I - D_x f^S(0; 0, \eta, \nu) \). \( \mathcal{P}(\eta, \nu) \) is \( C^{K-1} \) and \( \mathcal{P}(0, 0) \neq 0 \) since the algebraic multiplicity of the eigenvalue 1 of \( M_0(0) \) is one. Let
\[
\kappa = \mathcal{P}(0, 0),
\]
then, using (70),
\[
\det(I - D_x f^S(0; 0, \eta, \nu)) = (-c_1 \eta - c_4 \nu + O(2)) (\kappa + O(1))
\]
\[
= -c_1 \kappa \eta - c_4 \kappa \nu + O(2).
\]
Using (42), (64) and (65) we arrive at
\[
c_3 = \frac{-\delta}{\kappa l_d}, \quad c_4 = \frac{-\delta}{\kappa l_t(l-1)d}.
\]
(5) When \( \eta = \nu = 0 \), the \( \tilde{S} \)-cycle has two points on the switching manifold and coincides with the \( \tilde{S} \)-cycle. Both \( \tilde{x}_0(\mu, 0, 0) \) and \( \tilde{x}_d(\mu, 0, 0) \) are fixed points of \( f^S \).
Let \( \lambda_1(\mu) \) and \( \lambda_2(\mu) \) be the respective eigenvalues of \( D_x f^S(\tilde{x}_0(\mu, 0, 0); \mu, 0, 0) \) and \( D_x f^S(\tilde{x}_d(\mu, 0, 0); \mu, 0, 0) \), \( C^{K-1} \) dependent on \( \mu \) with \( \lambda_1(0) = \lambda_2(0) = 1 \). As above, since \( \tilde{x}_0(\mu, 0, 0) = 0 \), from (69)
\[
\lambda_1(\mu) = \frac{\partial s'}{\partial s}(\tilde{x}_0(\mu, 0, 0); \mu, 0, 0) = 1 + c_2 \mu + O(\mu^2),
\]
and then since \( \tilde{x}_d(\mu, 0, 0) = l_d \mu + O(\mu^2) \),
\[
\lambda_2(\mu) = \frac{\partial s'}{\partial s}(\tilde{x}_d(\mu, 0, 0); \mu, 0, 0) = 1 - \frac{2 c_2}{l_d} (l_d \mu + O(\mu^2)) + c_2 \mu + O(\mu^2)
\]
\[
= 1 - c_2 \mu + O(\mu^2).
\]
Again, as above,
\[
\det(I - D_x f^S(\tilde{x}_0(\mu, 0, 0); \mu, 0, 0) = -c_2 \kappa \mu + O(\mu^2),
\]
and \( \det(I - D_x f^S(\tilde{x}_d(\mu, 0, 0); \mu, 0, 0) = c_2 \kappa \mu + O(\mu^2). \)

But, recall (36), so
\[
c_2 = \frac{-k_0}{\kappa}
\]
and by (54)
\[
\tilde{k}_0 = -k_0. \tag{71}
\]
Substitution of (71) into (61) verifies (iii) of the lemma (using also (64) and (65)). It now only remains to demonstrate (iv) and (v) of the lemma.

We have shown that the restriction of \( f^S \) to \( W^C \) is
\[
s'(s; \xi) = s + \frac{k_0}{\kappa l_d} s^2 - \frac{k_0}{\kappa} \mu s - \frac{\delta}{\kappa l_d} \eta s - \frac{\delta}{\kappa l_t(l-1)d} \nu s + \frac{\delta}{\kappa} \mu \eta + O(3). \tag{72}
\]
We now look for saddle-node bifurcations of (72). Since \((\partial s'/\partial s)(0; 0) = 1\) and 
\((\partial^2 s'/\partial s^2)(0; 0) \neq 0\), the implicit function theorem implies that there exists a unique \(C^{K-2}\) function, \(\psi\) such that 
\[(\partial s'/\partial s)(\psi(\xi); \xi) = 1\] 
for small \(\xi\) and 
\[\psi(\xi) = \frac{t_d}{2}\mu + \frac{\delta}{2k_0} + \frac{\delta l_d}{2k_0(t_d-\delta)}v + O(2) . \tag{73}\]

Then saddle-node bifurcations occur when \(s'(\psi(\xi); \xi) = \psi(\xi)\). Let 
\[\Lambda(\xi) = -\frac{4k_0\kappa}{\delta t_d} (s'(\psi(\xi); \xi) - \psi(\xi)) . \tag{74}\]

Substitution of (72) and (73) into (74) yields (37). To complete verification of (iv) of the lemma we formally show that (72) has a saddle-node bifurcation at 
\[s = s(0) = 0\] 
and this is insufficient for a derivation of an associated multiplier of 1 as an 
\[S\] 
function 
\[\tilde{\Lambda}(\mu, \xi) = \lambda_1(\xi) \quad \text{by construction}, \quad (1)\] 
by construction, 
\[\frac{\partial s'}{\partial s}(\psi(\xi), \xi) = 1 \quad \text{when} \quad \Lambda(\xi) = 0 , \tag{2}\]

\[\frac{\partial s'}{\partial s} = \frac{\dot{\delta}l_d}{2k} \mu + O(2) \neq 0 \quad \text{when} \quad \mu > 0 \quad \text{verifying transversality (where} \quad \tilde{\nu} \quad \text{is given in (40)),} \tag{3}\]

\[\frac{\partial^2 s'}{\partial s^2} = \frac{-k_0}{\kappa t_d} + O(1) \neq 0 . \]

**Step 4.** Compute the intersections of \(\Lambda = 0\) with \(\eta = 0\) and \(v = \phi_2(\mu, \eta)\) to obtain the functions \(\xi_1\) and \(\xi_2\).

To determine where \(\Lambda(\xi) = 0\) intersects \(\eta = 0\) it is natural to look at \(\Lambda(\mu, 0, v) = 0\), but this is insufficient for a derivation of \(\xi_1(\mu)\) because we may not apply the implicit function theorem to \(\Lambda(\xi)\) since it contains no linear terms. Instead we use the fact that when \(\eta = 0\), the \(S\)-cycle is also an \(S\)-cycle. We have \(\Lambda = 0\) when in addition, this periodic solution has an associated multiplier of 1 as an \(S\)-cycle. This occurs when (using (36), (42) and (64))

\[\det(I - D_s f^\delta(\tilde{x}_0(\mu, 0, v); \mu, 0, v) = k_0\mu + \frac{\delta}{\delta t_d}v + O(2) = 0 . \]

Application of the implicit function theorem to the previous equation produces

\[v = \xi_1(\mu) = -\frac{k_0}{\delta} \mu + O(\mu^2) . \]

Moreover,

\[\Lambda(\mu, 0, v) = \frac{1}{t_d(t_d-\delta)}(v - \xi_1(\mu))^2 + o(|v - \xi_1(\mu)|^2), \]

and so for \(\mu > 0\), \(\Lambda \geq 0\) on the \(v\)-axis.

The curve \((\mu, \xi_2(\mu), \phi_2(\mu, \xi_2(\mu))\) along which \(\Lambda(\xi) = 0\) intersects the surface \(\phi_2(\mu, \eta)\), is easily computed in a similar fashion. When \(\mu > 0\), \(\Lambda \geq 0\) along \(v = \phi_2(\mu, \eta)\) and so \(\Lambda \leq 0\) only when \(\eta = 0\) and \(v \geq \phi_2(\mu, \eta)\) and stated in the final part of the lemma. \(\square\)

**Proof of theorem 10.** We begin by determining the region of admissibility of the \(S\)-cycle, \(\{\tilde{x}_i\}\).

From (28), \(\tilde{x}_i(\xi) = \mu(y_i + O(1))\), thus since by assumption \(n_d\) and \(y_0\) are the only points of \(y_i\) that lie on the switching manifold for small \(\xi\) with \(\mu > 0\), here \(\tilde{x}_i(\xi)\) lies on the same side of the switching manifold as \(y_i\) for each \(i \neq 0, l_d\). The \(n\)-cycle, \(\{y_i\}\), is admissible by assumption (for \(\mu > 0\), thus \(\{\tilde{x}_i(\xi)\}\) is admissible exactly when \(\tilde{x}_0, \tilde{x}_d \geq 0\) (since \(\tilde{x}_0 = \tilde{x}_d = R\)). By
Combining (67) and (75) yields (30) and (31). Similarly, for \( \mu > 0 \) the \( \hat{S} \)-cycle is admissible exactly when \( \hat{s}_0, \hat{s}_{td} \leq 0 \) (since \( \hat{S}_0 = \hat{S}_{td} = L \)). By lemma 9(ii), \( \hat{s}_{td}(\xi) = 0 \) when \( \eta = \phi_1(\mu, \nu) \). By (58), (65) and lemma 9(i), \( (\partial \hat{s}_{td}/\partial \eta)(\xi) = -(t_{d-1}d/t_{-d}) + O(\xi) \) which is positive for small \( \xi \), hence \( \hat{s}_{td} \leq 0 \) for \( \eta \leq \phi_1(\mu, \nu) \) when \( \mu > 0 \). Similarly by lemma 9(iii), \( \hat{s}_0(\xi) = 0 \) when \( \nu = \phi_2(\mu, \eta) \) and by (51), (64) and lemma 9(i), \( (\partial \hat{s}_0/\partial \nu)(\xi) = -(t_{d-1}d/t_{d+1}) + O(\xi) \) which is positive for small \( \xi \), hence \( \hat{s}_0 \leq 0 \) for \( \nu \leq \phi_2(\mu, \eta) \) when \( \mu > 0 \). Therefore the \( \hat{S} \)-cycle is admissible in \( \Psi_2 \).

It remains to verify admissibility of \( \hat{S} \)-cycles. For the theorem in [10] this was straightforward since, if it existed, the \( \hat{S} \)-cycle was unique. The situation here is more complicated because there may be two coexisting, admissible \( \hat{S} \)-cycles. In the proof of lemma 9 we determined the restriction of \( f^\hat{S} \) to the centre manifold through \( (s, \xi) = (0, 0) \). When \( \mu > 0 \) and \( \Lambda(\xi) > 0 \), locally map (72) has two distinct fixed points, say, \( s_{0,1} \) and \( s_{0,2} \). We will denote the corresponding \( \hat{S} \)-cycles of (6) by \( \{x_i,1\} \) and \( \{x_i,2\} \) and assume \( s_{0,1} \geq s_{0,2} \). For small \( \xi \) within \( \{\xi \mid \mu > 0, \Lambda(\xi) > 0\} \), \( s_{0,1} \) and \( s_{0,2} \) are \( C^k \) functions of \( \xi \).

On the surface \( \Lambda(\xi) = 0 \) the two solutions coincide (see (73)):

\[
s_{0,1}(\xi) = s_{0,2}(\xi) = \psi(\xi), \quad \text{when } \Lambda(\xi) = 0.
\]

At the intersection of \( \Lambda(\xi) = 0 \) and \( \eta = 0 \), namely \( (\mu, 0, \zeta_1(\mu)) \) (see lemma 9(v)), \( s_{0,1} = s_{0,2} = 0 \). Thus by (73), on \( \Lambda(\xi) = 0 \), \( s_{0,1} = s_{0,2} < 0 \) when \( \nu < \zeta_1(\mu) \) and \( s_{0,1} = s_{0,2} > 0 \) when \( \nu > \zeta_1(\mu) \).

Now, for \( \mu > 0 \), \( s_{0,1} \) and \( s_{0,2} \) can only be zero if \( \eta = 0 \) because if \( s = 0 \) is a fixed point of (72) then the corresponding \( \hat{S} \)-cycle would also be an \( \hat{S} \)-cycle which for \( \mu > 0 \) must be \( \{\xi\} \), so then \( \Lambda(\xi) = 0 \) and by (30) we would necessarily have \( \eta = 0 \). Consequently, one of \( s_{0,1} \) and \( s_{0,2} \) is zero when \( \eta = 0 \) and \( s_{0,1} \) and \( s_{0,2} \) are both nonzero when \( \eta \neq 0 \). Since we assume \( s_{0,1} > s_{0,2} \) for \( \Lambda(\xi) \neq 0 \), when \( \mu > 0 \) and \( \eta = 0 \) we must have \( s_{0,1} = 0 \) when \( \nu \leq \zeta_1(\mu) \) and \( s_{0,2} = 0 \) when \( \nu \geq \zeta_1(\mu) \). Consequently, \( s_{0,1} < 0 \) in \( \Psi_2 \cup \Psi_3 \) and \( s_{0,2} < 0 \) in \( \Psi_1 \cup \Psi_2 \cup \Psi_3 \).

Before we are able to perform a similar analysis of \( s_{i,j} \) for \( i \neq 0 \), we find it necessary to first derive an expression for \( s_{i,j} \) in terms of \( t_i \) and \( t_{d+i} \). Recall that the centre manifold, \( \Psi^c \), is given by (67) where \( M_S(0)v = v \) and \( e_1^Tv = 1 \). When \( \xi = 0 \), \( y_0 \) and \( y_d \) are both fixed points of \( h^\xi \), thus \( (I - M_S(0))y_0 = P_S(0)b(0) = (I - M_S(0))y_D \) and so

\[
(I - M_S(0))(y_D - y_0) = 0,
\]

and since \( y_0 \neq y_D \) (lemma 7) and the eigenvalue 1 of the matrix \( M_S(0) \) has algebraic multiplicity one, \( y_0 - y_D \) is a scalar multiple of \( v \). Due to the specified vector scaling we have

\[
v = \frac{1}{I_d}(y_D - y_0).
\]

Combining (67) and (75) yields

\[
x_{0,j}(\xi) = \left(\mu - \frac{s_{0,j}(\xi)}{I_d}\right)y_0 + \frac{s_{0,j}(\xi)}{I_d}y_D + O(2). \tag{76}
\]

This may be generalized to an expression for \( x_{i,j}(\xi) \) using \( x_{i+1,j} = \mu b + A_S x_{i,j} + O(2) \) and \( y_{i+1} = \mu b(0) + A_S(0)y_i \), from which we deduce

\[
s_{i,j}(\xi) = \left(\mu - \frac{s_{0,j}(\xi)}{I_d}\right)t_i + \frac{s_{0,j}(\xi)}{I_d}t_{d+i} + O(2), \tag{77}
\]

and hence

\[
s_{i,1}(\xi) - s_{i,2}(\xi) = -\frac{1}{I_d}(t_i - t_{d+i})(s_{0,1}(\xi) - s_{0,2}(\xi)) + O(2).
\]
Therefore for small \( \xi > 0 \) with \( \mu > 0 \) and \( \Lambda(\xi) > 0 \), \( s_{i,1} > s_{i,2} \) if \( t_i > t_{d+i} \) and \( s_{i,1} < s_{i,2} \) if \( t_i < t_{d+i} \) (because we assumed \( s_{0,1} > s_{0,2} \)).

Above we showed that when \( \eta = \nu = 0 \) and \( \mu > 0 \), \( s_{0,1} = 0 \) and \( s_{0,2} < 0 \). Thus here \( s_{d,1} = 0 \) and \( s_{d,2} > 0 \) and by lemma 9, along \( \eta = \phi_1(\mu, \nu) \), \( s_{d,1} = 0 \) and \( s_{d,2} > 0 \). From this it is easily follows that \( s_{d,1} > 0 \) in \( \Psi_2 \cup \Psi_3 \) and \( s_{d,2} > 0 \) in \( \Psi_1 \cup \Psi_2 \cup \Psi_3 \). Similarly when \( \nu = 0 \) and \( \mu > 0 \), \( s_{l-1,d,1} < 0 \) and \( s_{l-1,d,2} = 0 \) and consequently \( s_{l-1,d,1} < 0 \) in \( \Psi_1 \cup \Psi_2 \cup \Psi_3 \) and \( s_{l-1,d,2} < 0 \) in \( \Psi_1 \cup \Psi_3 \). By analogous arguments, \( s_{-d,1} > 0 \) in \( \Psi_1 \cup \Psi_2 \cup \Psi_3 \) and \( s_{-d,2} > 0 \) in \( \Psi_1 \cup \Psi_3 \). By (77), for \( i \neq 0 \), \( (l-1)d, ld, -d, s_{i,1} \) and \( s_{i,2} \) have the desired sign for admissibility for small \( \xi \) with \( \mu > 0 \). The above statements show that \( \{x_{i,1}\} \) is admissible in \( \Psi_2 \cup \Psi_3 \) and \( \{x_{i,2}\} \) is admissible in \( \Psi_1 \cup \Psi_3 \) which completes the proof.

\( \square \)

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