Research Article

Manseob Lee*

Asymptotic measure-expansiveness for generic diffeomorphisms

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Abstract: In this paper, we will assume $M$ to be a compact smooth manifold and $f : M \to M$ to be a diffeomorphism. We herein demonstrate that a $C^1$ generic diffeomorphism $f$ is Axiom A and has no cycles if $f$ is asymptotic measure expansive. Additionally, for a $C^1$ generic diffeomorphism $f$, if a homoclinic class $H(p, f)$ that contains a hyperbolic periodic point $p$ of $f$ is asymptotic measure-expansive, then $H(p, f)$ is hyperbolic of $f$.

Keywords: expansive, measure expansive, asymptotic measure expansive, generic, Axiom A, homoclinic class, hyperbolic

MSC 2020: 37C20, 37D20

1 Introduction

Throughout this paper, we will assume $M$ to be a compact smooth manifold and $d$ to be the distance on $M$ induced by a Riemannian metric $\| \cdot \|$. We also assume $f : M \to M$ to be a diffeomorphism and denote by $\text{Diff}(M)$ the set of diffeomorphisms of $M$ endowed with $C^1$ topology. It is to note that expansiveness has been earlier suggested in the study by Utz [1]. A diffeomorphism $f$ is said to be expansive if there exists a positive constant $\delta > 0$ such that for any two points $x, y \in M$ if $x \neq y$, and if there exists $k \in \mathbb{Z}$ such that $d(f^k(x), f^k(y)) > \delta$. Equivalently, if there is a positive constant $\delta > 0$ such that for any $x, y \in M$ if $d(f^i(x), f^i(y)) \leq \delta$, $\forall i \in \mathbb{Z}$, then $x = y$. Generally speaking, expansiveness means that if any two real orbits are separated by a small distance, the two orbits are identical, and therefore it is appropriate for studying smooth dynamic systems. Expansivities are hence a valuable notion in the investigation of hyperbolic structures (see [2–17], etc.).

Mañé [17] proved that a $C^1$ robustly expansive diffeomorphism $f$ is quasi Anosov, i.e., the set $\{ ||Df^n(v)|| : n \in \mathbb{Z} \}$ is unbounded for all $v \in TM \setminus \{0\}$.

Morales and Sirvent [18] introduced stochastic perspectives of expansiveness, called measure-expansiveness. Let us assume $M(M)$ to be the set of all Borel probability measures on $M$ endowed with the weak* topology and $M^*(M)$ to be the set of nonatomic measures $\mu \in M(M)$. It is known that $M^*(M) \subset M(M)$.

For any $\mu \in M^*(M)$, a closed $f$-invariant set $\Lambda \subset M$ is said to be $\mu$-expansive for $f$ if there is a positive constant $\delta > 0$ such that $\mu(\Gamma(\delta, x)) = 0 \ \forall x \in \Lambda$, where $\Gamma(\delta, x) = \{ y \in M : d(f^i(x), f^i(y)) \leq \delta \ \forall i \in \mathbb{Z} \}$.

Definition 1.1. A closed $f$-invariant set $\Lambda \subset M$ is said to be measure expansive for $f$ if $\Lambda$ is $\mu$-expansive for all $\mu \in M^*(M)$. If $\Lambda = M$, then we say that a diffeomorphism $f$ is measure expansive.

Here, $\delta$ is called a measure expansive constant of $f$. Now, we introduce a general notion of measure-expansiveness called the asymptotic measure expansive (see [19, Example 1.1]). The notion was suggested
in [19]. Let us assume that \( \mu \in M^*(M) \) is given. A closed \( f \)-invariant set \( \Lambda \subset M \) is said to be asymptotic \( \mu \)-expansive for \( f \) if there is \( \delta > 0 \) such that
\[
\lim_{n \to \infty} \mu(f^n(\Gamma(\delta, x))) = 0
\]
for any \( x \in \Lambda \).

**Definition 1.2.** Let us assume that \( f \in \text{Diff}(M) \), and \( \Lambda \subset M \) is a closed \( f \)-invariant set. We say that \( \Lambda \) is asymptotic measure expansive for \( f \) if there is a positive constant \( \delta > 0 \) such that \( \Lambda \) is asymptotic \( \mu \)-expansive for \( f \). Moreover, if \( \Lambda = M \), then we say that \( f \) is asymptotic measure expansive.

The following notion is suggested in [20]. A diffeomorphism \( f \) on \( M \) is said to be continuum-wise expansive if there is a positive constant \( \delta > 0 \) such that, for any nontrivial compact connected set \( \Lambda \), there is an integer \( n \in \mathbb{Z} \) such that \( \text{diam } f^n(\Lambda) \geq \varepsilon \), where \( \text{diam } \Lambda = \sup \{d(x, y) : x, y \in \Lambda \} \) for any subset \( \Lambda \subset M \) and \( \lambda \) is nontrivial, which means that \( \Lambda \) is neither one point nor one orbit.

Regarding the result of Artigue and Carrasco-Olivera [21], it is observed that a diffeomorphism \( f \) is measure-expansive if it is continuum-wise expansive. However, the converse is not true. We already know that a diffeomorphism \( f \) is measure-expansive if it is asymptotic measure-expansive. Here, the converse is also untrue. Therefore, we have a question:

**What is relation between asymptotic measure expansiveness and continuum-wise expansiveness?**

A closed \( f \)-invariant set \( \Lambda \subset M \) is called hyperbolic if a \( Df \)-invariant splitting \( T_{\Lambda}M = E^s \oplus E^u \), there exist constants \( C > 0 \) and \( 0 < \lambda < 1 \) such that \( \|Df^n(x)\| \leq C \lambda^n \) for \( x \in \Lambda \) and \( n \geq 0 \),
\[
(a) \quad \|Df^n(v)\| \leq \|\nabla f^n(v)\| \leq \|Df^n(w)\| \quad \text{for } v \in \mathbb{E}^s \setminus \{0\}, \text{ and}
\]
\[
(b) \quad \|Df^n(u)\| \leq \|\nabla f^n(u)\| \quad \text{for } u \in \mathbb{E}^u \setminus \{0\}.
\]
If \( \Lambda = M \), then a diffeomorphism \( f \) is said to be Anosov.

It is known that if \( \Lambda \) is hyperbolic for \( f \), then \( \Lambda \) is expansive, thus it is measure-expansive and asymptotic measure-expansive. A point \( x \in M \) is called periodic if there is \( n(x) > 0 \) such that \( f^{n(x)}(x) = x \), and a point \( x \in M \) is called non-wandering if \( k > 0 \) can be found such that \( f^k(U) \cap U \neq \emptyset \) for any neighborhood \( U \) of \( x \). We denote \( \text{Per}(f) \) as the set of all periodic points of \( f \) and \( \Omega(f) \) the set of all non wandering points of \( f \). It is known that \( \text{Per}(f) \subset \Omega(f) \). We say that \( f \) satisfies Axiom A if the nonwandering set \( \Omega(f) = \text{Per}(f) \) is hyperbolic. According to Aoki [22] and Hayashi [23], \( f \) satisfies Axiom A and has no-cycles if \( f \) is star.

In this paper, we consider sets of diffeomorphisms that are residual for the Baire category, i.e., sets that contain a countable intersection of dense and open subsets of \( \text{Diff}(M) \). Regarding \( C^1 \) generic diffeomorphisms, it is known that the periodic points are dense in \( \Omega(f) \) by Pugh’s closing lemma [24]. Using the \( C^1 \) generic property, Arbieto proved in [25] that \( f \) satisfies Axiom A and has no-cycles for a \( C^1 \) generic expansive diffeomorphism. Lee [26] proved that a \( C^1 \) generic measure expansive diffeomorphism \( f \) satisfies Axiom A and has no-cycle. Recently, Lee [27] proved that \( f \) satisfies Axiom A and has no-cycles for a \( C^1 \) generic continuum-wise expansive diffeomorphism. According to the abovementioned results, we consider general concepts of measure expansiveness. The following is the primary theorem of the paper.

**Theorem A.** For a \( C^1 \) generic \( f \in \text{Diff}(M) \), \( f \) satisfies Axiom A and has no-cycles if it is asymptotic measure-expansive.

For any hyperbolic periodic point \( p \), define the following sets \( W^s(p) = \{x \in M : f^i(x) \to p \text{ as } i \to \infty\} \) and \( W^u(p) = \{x \in M : f^i(x) \to p \text{ as } i \to -\infty\} \), where \( W^s(p) \) is called the stable manifold of \( p \) and \( W^u(p) \) is called the unstable manifold of \( p \). Denote by \( \text{dim } W^s(p) = \text{index}(p) \). We say that a hyperbolic \( p \in \text{Per}(f) \) is homoclinically related to \( q \in \text{Per}(f) \) if \( W^s(p) \cap W^u(p) \neq \emptyset \) and \( W^u(p) \cap W^s(q) \neq \emptyset \). We write \( p \sim q \). It is clear that \( \text{index}(p) = \text{index}(q) \) if \( p \sim q \).

A closed \( f \)-invariant set \( \Lambda \subset M \) is called transitive if we can take a point \( x \in \Lambda \) such that \( \overline{\text{Orb}(x)} = \Lambda \), where \( \overline{\Lambda} \) is the closure of \( \Lambda \). Denote \( H(p, f) = \{q \in \text{Per}(f) : q \sim p\} \), which is called the homoclinic class. It is
known that the set is a closed $f$-invariant and transitive set. Note that if a diffeomorphism $f$ satisfies Axiom A, then the nonwandering set $\Omega(f)$ is a disjoint union of transitive invariant closed subsets. In fact, these sets are homoclinic classes that each contain a hyperbolic periodic point. Several researchers are studying these sets and their hyperbolicity (see [4,28–34], etc.). We study whether the homoclinic class is hyperbolic using the asymptotic measure-expansiveness. Yang and Gan [34] proved that a homoclinic class $H(p, f)$ is hyperbolic if it is expansive for a $C^1$ generic diffeomorphism $f$. Koo et al. [35] proved that a locally maximal homoclinic class $H(p, f)$ is hyperbolic if it is measure-expansive for a $C^1$ generic $f$. Here, a closed $f$-invariant set $\Lambda \subset M$ is locally maximal if there exists a neighborhood $U$ of $\Lambda$ for which $\Lambda = \bigcap_{n \in \mathbb{Z}} f^n(U)$. Later, Lee proved in [32] that a homoclinic class $H(p, f)$ is hyperbolic if it is measure-expansive for a $C^1$ generic diffeomorphism $f$. The result is a general version of the proof in [35]. In [31], Lee proved that a homoclinic class $H(p, f)$ is hyperbolic if it is continuum-wise expansive for a $C^1$ generic $f$. According to the results, we prove the following:

**Theorem B.** A homoclinic class $H(p, f)$ is hyperbolic if it is asymptotic measure-expansive for a $C^1$ generic $f \in \text{Diff}(M)$.

## 2 Proof of Theorem A

Theorem A will be proven in this section, which requires some notions to be taken into account. A point $p \in \text{Per}(f)$ is weak hyperbolic if there is $g \in C^1$ close to $f$ such that the derivative map $D_p g\circ f^p$ has an eigenvalue $\lambda$ with $|\lambda| = 1$. For any $\varepsilon > 0$, we consider a closed curve $\eta$ to be $\varepsilon$ simply periodic if $\eta$ satisfies the following conditions:

(a) there is $k > 0$ such that $f^k(\eta) = \eta$,

(b) $0 < ||f^i(\eta)|| \leq \varepsilon$ for $0 \leq i \leq k$, and

(c) $\eta$ is normally hyperbolic (see [34]).

If a $p \in \text{Per}(f)$ is hyperbolic, then there are a $C^1$ neighborhood $U(f)$ of $f$ and a locally maximal neighborhood $U$ of $p$ such that there exists the hyperbolic periodic $p_\varepsilon = \bigcap_{n \in \mathbb{Z}} g^n(U)$ for any $g \in U(f)$. Here, $p_\varepsilon$ is called a continuation.

The following is called Franks’ lemma [36], which is a useful notion for a $C^1$ robust property.

**Lemma 2.1.** $U(f)$ be any given $C^1$ neighborhood of $f$. Then there exist $\varepsilon > 0$ and a $C^1$ neighborhood $U\varepsilon(f) \subset U(f)$ of $f$ such that, if a set $A = \{x_1, x_2, \ldots, x_k\}$, a neighborhood $U$ of $A$, and linear maps $L_i : T_x M \to T_{g^ix} M$ satisfy $||L_i - D_{g^ix}|| \leq \varepsilon$ for all $x_i \in A$, then for any $g \in U\varepsilon(f)$, there exists $g \in U(f)$ for which $g(x) = g(x)$ if $x \in A \cup (M\setminus U)$ and $D_{g^ix} \hat{g} = L_i$ for all $1 \leq i \leq k$.

**Lemma 2.2.** If a diffeomorphism $f$ has a weak hyperbolic periodic point, then for any neighborhood $U(f)$ of $f$ and any $\varepsilon > 0$, there are $g \in U(f)$ and a small curve $J$ with the following property:

(a) $J$ is $g$ periodic, i.e., there is $n \in \mathbb{Z}$ such that $g^n(J) = J$;

(b) the length of $g^i(J)$ is less than $\varepsilon$ for all $i \in \mathbb{Z}$;

(c) the endpoints of the curve $J$ are hyperbolic;

(d) $J$ is normally hyperbolic with respect to $g$ (see [37]).

**Proof.** Let us assume $p$ to be a weak hyperbolic periodic point of $f$ and $U(f)$ to be a $C^1$ neighborhood of $f$. For simplicity, we may assume that $f(p) = p$. According to Lemma 2.1, there is $g \in U(f)$ such that $g(p) = p$ and the derivative map $D_p g$ has an eigenvalue $\lambda$ with $|\lambda| = 1$, i.e., $g$ has a non hyperbolic periodic point $p$.

As given in the proof of [26, Lemma 2.2], $hC^1$ can be found close to $g$ (also, $h \in U(f)$) such that

(i) $h^k(J) = J$ for some $k \in \mathbb{Z}$, and

(ii) $h^k|_J : J \to J$ is the identity map.
In items (i) and (ii), \( k = 1 \) and \( k = 2 \) if the eigenvalue \( \lambda \) is a positive or negative real number, respectively. If the eigenvalue \( \lambda \) is a complex number, then one can take \( l > 0 \) such that \( k = l \). As in the proof of [26, Lemma 2.2], it is clear that \( \mathcal{J} \) is normally hyperbolic and the length of \( \mathcal{J} \) is less than \( \varepsilon \). Therefore, the small closed curve \( \mathcal{J} \) satisfies items (a), (b), and (d).

Finally, we show item (c). Let us assume that \( q \) and \( r \) are the endpoints of the closed curve \( \mathcal{J} \). For simplification, we assume that \( h^k = h \). It is observed that the eigenvalue of the derivative maps \( D_q h = 1 \) and \( D_r h = 1 \). Again, using Lemma 2.1, there is \( h \mathcal{C} \) close to \( h \) (also, \( h_1 \in \mathcal{U}(f) \)) such that \( h(q) = q \), \( h(r) = r \), and the norm of every eigenvalue of the derivative map \( D_q h_1 \) and \( D_r h_1 \) are not one. Therefore, we have a small curve \( \mathcal{J} \) that satisfies items (a), (b), (c), and (d). This completes the proof.

A diffeomorphism \( f \) is star if we can take a \( C^1 \) neighborhood \( \mathcal{U}(f) \) of \( f \) for which every periodic point of \( g \) is hyperbolic for \( g \in \mathcal{U}(f) \).

**Lemma 2.3.** There is a residual set \( \mathcal{R} \) subset in \( \text{Diff}(M) \) such that, for given \( f \in \mathcal{R} \), we have:

(a) either \( f \) is star or

(b) \( f \) has a simple periodic curve \( \mathcal{J} \) with hyperbolic endpoints.

**Proof.** The proof is similar to [26, Lemma 2.4].

**Proof of Theorem A.** For \( f \in \mathcal{R} \), we assume that \( f \) is asymptotic measure-expansive. According to Aoki [22] and Hayashi [23], it is sufficient to show that \( f \) is a star. Suppose, by contradiction, that \( f \) is not a star. If \( f \) is not a star, \( f \) has a simple periodic curve \( \mathcal{J} \) with hyperbolic endpoints according to Lemma 2.3. That is, there is \( k \in \mathbb{Z} \) such that \( f^k(\mathcal{J}) = \mathcal{J} \), the length of \( f^i(\mathcal{J}) \) is less than \( \varepsilon \) \((0 \leq i \leq k)\), the endpoints of the curve \( \mathcal{J} \) are hyperbolic, and \( \mathcal{J} \) is normally hyperbolic.

Let us now assume \( \nu_\mathcal{J} \) to be a normalized Lebesgue measure on \( \mathcal{J} \). \( \mu \in \mathcal{M}(M) \) is defined as

\[
\mu(B) = \frac{1}{k} \sum_{i=0}^{k-1} \nu_\mathcal{J}(f^{-i}(B) \cap f^i(\mathcal{J}))
\]

for any Borel set \( B \subset M \). It is clear that \( \mu \in \mathcal{M}^*(M) \) and \( \mu \) is invariant. For any small \( \delta > 0 \) and \( x \in \mathcal{J} \), we define \( \Phi(\delta, x) = \{ y \in \mathcal{J} : d(f^i(x), f^i(y)) \leq \delta \text{ for all } i \in \mathbb{Z} \} \subset \mathcal{J} \). It is also assumed that \( \Gamma(\delta, x) = \{ y \in M : d(f^i(x), f^i(y)) \leq \delta \text{ for all } i \in \mathbb{Z} \} \). It can therefore be seen that \( \Phi(\delta, x) \subset \Gamma(\delta, x) \). Because \( \Phi(\delta, x) \subset \mathcal{J} \), it is observed that \( f^n(\Phi(\delta, x)) = \Phi(\delta, x) \) for all \( n \in \mathbb{Z} \). Therefore, we know that

\[
\lim_{n \to \infty} \mu(f^n(\Phi(\delta, x))) \neq 0.
\]

\( \mu(f^n(\Gamma(\delta, x))) \to 0 \) as \( n \to \infty \) because \( f \) is asymptotic measure-expansive. This is a contradiction because \( \mu(f^n(\Phi(\delta, x))) \to 0(n \to \infty) \). Therefore, for \( C^1 \) generic \( f \), \( f \) satisfies both Axiom A and the no-cycle condition if \( f \) is asymptotic measure-expansive.

**3 Proof of Theorem B**

Theorem B will be proven in this section using various results of a \( C^1 \) generic property.

For any \( \delta > 0 \), we consider a point \( p \) to be a \( \delta \) weak hyperbolic periodic point if

\[
(1 - \lambda)^{\pi(p)} \leq |\lambda| \leq (1 + \delta)^{\pi(p)},
\]

where \( \lambda \) is the eigenvalue \( \lambda \) of \( D_p f \), and \( \pi(p) \) is the period of \( p \).

We consider \( f \) to be Kupka-Smale if every periodic point is hyperbolic and its stable and unstable manifolds are transversal intersections. It is well known that a diffeomorphism \( f \) is a residual subset of \( \text{Diff}(M) \) if it is Kupka-Smale (see [38]).
For any $\varepsilon > 0$, a sequence of points $\{x_i\}_{i=a}^b$ is a $\varepsilon$-pseudo orbit of $f$ if $d(f(x_i), x_{i+1}) < \varepsilon$ for all $-\infty \leq a \leq i < b \leq \infty$. A point $x \in M$ is called chain recurrent if there is a finite $\varepsilon$-pseudo orbit $\{x_i\}_{i=a}^b$ such that $x_a = x$ and $x_b = x$ for any $\varepsilon > 0$. Let us assume $\mathcal{C}(f)$ to be the set of all chain recurrence sets of $f$. We define a relation $\sim$ on $\mathcal{C}(f)$ by $x \sim y$ if there exists a finite $\varepsilon$-pseudo orbit $\{w_i\}_{i=0}^n$ for any $\varepsilon > 0$ such that $w_0 = x$, $w_n = y$, and another $\varepsilon$-pseudo orbit $\{z_i\}_{i=0}^n$ such that $z_0 = y$ and $z_n = x$. It is clear that $\sim$ is an equivalence relation on $\mathcal{C}(f)$. The equivalence classes are called the chain recurrent classes of $f$.

For any hyperbolic periodic point $p$, we denote $\mathcal{C}(p, f) = \{x \in M : x \sim p\}$. It is clear that $\mathcal{C}(p, f)$ is a closed $f$-invariant set and $H(p, f) \subset \mathcal{C}(p, f)$.

**Lemma 3.1.** There is a residual subset $\mathcal{G}_1$ in $\text{Diff}(M)$ such that, for given $f \in \mathcal{G}_1$, we have the following:
(a) $f$ is Kupka-Smale (see [38]);
(b) for any $\delta > 0$ and any $p \in \text{Per}(f)$, there exists $g \in \mathcal{U}(f)$ for any $\mathcal{C}^1$ neighborhood $\mathcal{U}(f)$ of $f$ such that $g$ has a $\delta$ simply periodic curve $I$, where the two endpoints of $I$ are homoclinically related to $p$. Therefore, $f$ has a $\delta$ simply periodic curve $J$ such that the two endpoints of $J$ are homoclinically related to $p$ (see [34]);
(c) $H(p, f) = \mathcal{C}(p, f)$ (see [39]).

**Proof.** See the proof of [33,34]. □

**Lemma 3.2.** Let us assume $q \in H(p, f) \cap \text{Per}(f)$ and $\delta > 0$ to be given. If $q$ is a $\delta$ weak hyperbolic periodic point for $f$, there exists $g \mathcal{C}^1$ close to $f$ such that $g$ has a $\delta$ simply periodic curve $I$ with endpoints that are homoclinically related to $p$.

**Proof.** For $f \in \mathcal{G}_1$, assume that $f$ is asymptotic measure-expansive. We shall derive a contradiction. Suppose that there is $q \in H(p, f) \cap \text{Per}(f)$ such that $q$ is weak hyperbolic. Therefore, there is $g \mathcal{C}^1$ close to $f$ such that $g$ has a $\delta$ simply periodic curve $I$ with endpoints that are homoclinically related to $p$ according to Lemma 3.2. Because $f \in \mathcal{G}_1$, $f$ has a $\delta$ simply periodic curve $J$ such that the two endpoints of $J$ are homoclinically related to $p$ by Lemma 3.1.

Assume that the period of $J$ is $L > 0$. Let us assume that $\mu_f$ be a normalized Lebesgue measure on $J$. $\chi \in M(M)$ is defined by

$$
\chi(B) = \frac{1}{L} \sum_{i=0}^{L-1} \mu_f(f^{-1}(B) \cap f^i(J))
$$

for any Borel set $B \subset M$. It is clear that $\chi$ is invariant and $\chi \in M^*(M)$. For any $\delta > 0$ and $x \in J$, we define $\Theta(\delta, x) = \{y \in J : d(f^i(x), f^i(y)) \leq \delta \text{ for all } i \in \mathbb{Z}\} \subset J$. Let us assume that $\Gamma(\delta, x) = \{y \in M : d(f^i(x), f^i(y)) \leq \delta \text{ for all } i \in \mathbb{Z}\}$. It is clear that $\Theta(\delta, x) \subset \Gamma(\delta, x)$. It can be observed that $f^{\mathbb{Z}}(\Theta(\delta, x)) = \Theta(\delta, x)$ for all $n \in \mathbb{Z}$ because $\Theta(\delta, x) \subset J$. Therefore, we know that

$$
\lim_{n \to \infty} \chi(f^n(\Theta(\delta, x))) \to 0.
$$

Because $H(p, f)$ is asymptotic measure-expansive, $\chi(f^n(\Gamma(\delta, x))) = 0$ as $n \to \infty$. This is a contradiction by (1) because $\Theta(\delta, x) \subset \Gamma(\delta, x)$. □

We assume $p$ to be a hyperbolic periodic point of $f$ having a period of $\pi(p)$. Therefore, if $\mu_1, \mu_2, \ldots, \mu_n$ are the eigenvalues of $D_pf$, then

$$
\lambda_i = \frac{1}{\pi(p)} \log |\mu_i|,
$$
for $i = 1, 2, \ldots, d$ are called the Lyapunov exponents of $p$. The following was proven by Wang [40]. In fact, Wang proved that there is a $q \in H(p, f) \cap \Per(f)$ such that $q$ has a Lyapunov exponent arbitrarily closed to 0 for a $C^1$ generic diffeomorphism $f$ if a homoclinic class $H(p, f)$ is not hyperbolic. It can be observed that for a $C^1$ generic $f$, if a periodic point $q \in H(p, f)$, then $q \sim p$. We modified the statement as follows:

**Lemma 3.4.** There is a residual subset $\mathcal{G}_2$ in $\Diff(M)$ such that there exists a weak hyperbolic periodic point $q \in H(p, f)$ for any $f \in \mathcal{G}_2$, if a homoclinic class $H(p, f)$ is not hyperbolic for $f$.

**Proof of Theorem B.** Let us assume that $f \in \mathcal{G}_3 \cap \mathcal{G}_2$ be asymptotic measure-expansive. We shall derive a contradiction. It is assumed that $H(p, f)$ is not hyperbolic. According to Lemma 3.4, there is $q \in H(p, f) \cap \Per(f)$ such that $q$ is a weak hyperbolic periodic point. This is a contradiction by Lemma 3.3 because $H(p, f)$ is asymptotic measure-expansive. Therefore, $H(p, f)$ is hyperbolic for $C^1$ generic diffeomorphism $f$ if a homoclinic class $H(p, f)$ is asymptotic measure-expansive for $f$. \hfill $\square$

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