Stochastic Bilevel Distributed Optimization over a Network

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Abstract

Bilevel optimization has been applied to a wide variety of machine learning models. Numerous stochastic bilevel optimization algorithms have been developed in recent years. However, most of them restrict their focus on the single-machine setting so that they are incapable of handling the distributed data. To address this issue, under the setting where all participants compose a network and perform the peer-to-peer communication in this network, we developed two novel distributed stochastic bilevel optimization algorithms based on the gradient tracking communication mechanism and two different gradient estimators. Additionally, we show that they can achieve $O\left(\frac{1}{\epsilon^2(1 - \lambda)^2}\right)$ and $O\left(\frac{1}{\epsilon^{3/2}(1 - \lambda)^2}\right)$ convergence rate respectively to obtain the $\epsilon$-accuracy solution, where $1 - \lambda$ denotes the spectral gap of the communication network. To our knowledge, this is the first work achieving these theoretical results. Finally, we applied our algorithms to practical machine learning models, and the experimental results confirmed the efficacy of our algorithms.

1 Introduction

Bilevel optimization is an important learning paradigm in machine learning. It consists of an upper-level optimization problem and a lower-level optimization problem, where the objective function of the upper-level optimization problem depends on the solution of the lower-level one. This kind of learning paradigm covers numerous machine learning models, such as hyperparameter optimization [4, 5], meta-learning [6, 22], neural architecture search [19], to name a few. Thus, it is of importance and necessity to develop efficient optimization algorithms for solving the bilevel optimization problem.

In fact, the bilevel structure makes it difficult to compute the gradient of the outer-level optimization problem since it involves the computation of Hessian and Jacobian matrices. In the past few years, numerous optimization algorithms have been proposed to address this challenge. For instance, [7, 11, 12, 1] developed stochastic-gradient-based optimization algorithms, which are able to efficiently estimate Hessian and Jacobian matrices. Recently, to accelerate the convergence speed, [9] developed a momentum-based optimization algorithm, and [27, 14, 10] proposed the variance-reduced optimization algorithms. Both categories are able to improve the estimation of the full gradient. Thus, they can achieve faster convergence speed than [7, 11, 12, 1]. However, all these bilevel optimization algorithms restrict their focus on the non-parallel setting. As a result, they are not applicable to the distributed data.

In this work, we aim to develop decentralized bilevel optimization algorithms to solve the following bilevel distributed optimization problem, which resides over a network:

$$\min_{x \in \mathbb{R}^d_x} \frac{1}{K} \sum_{k=1}^{K} f^{(k)}(x, y^*(x)), \quad s.t. \quad y^*(x) = \arg \min_{y \in \mathbb{R}^d_y} \frac{1}{K} \sum_{k=1}^{K} g^{(k)}(x, y), \quad (1)$$

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where \( x \in \mathbb{R}^{d_x} \) and \( y \in \mathbb{R}^{d_y} \) are the model parameters, \( g^{(k)}(x, y) = \mathbb{E}_{\zeta \sim S_y^{(k)}} [g^{(k)}(x, y; \zeta)] \) denotes the objective function of the lower-level subproblem in the \( k \)-th participant and \( S_y^{(k)} \) is the data distribution of the \( k \)-th participant. \( f^{(k)}(x, y) = \mathbb{E}_{\xi \sim S_x^{(k)}} [f^{(k)}(x, y; \xi)] \) is the objective function of the upper-level subproblem in the \( k \)-th participant and correspondingly \( S_x^{(k)} \) is the data distribution of the \( k \)-th participant, \( K \) is the total number of the participants. From Eq. (1), it is easy to know that each participant possesses its own data, which will be used to learn the model parameters \((x, y)\) via the collaboration among all participants. In this work, it is assumed that all participants compose a network where the participant performs peer-to-peer communication. Thus, it is a decentralized bilevel optimization problem.

Decentralized optimization has been extensively studied in recent years due to its great potential in real-world machine learning tasks, such as data analysis on Internet-of-Things (IoT) devices. To address the challenges under various settings, several decentralized optimization algorithms have been proposed. For example, \([18]\) developed the decentralized stochastic gradient descent (DSGD) algorithm based on the gossip communication mechanism and established its convergence rate for nonconvex problems. \([23, 26, 30, 29]\) proposed the decentralized stochastic variance-reduced gradient descent algorithms based on the gradient tracking communication mechanism, which improve the convergence rate of DSGD. Additionally, some efforts \([15, 24, 25, 16]\) were made to improve the communication complexity of decentralized optimization algorithms by compressing the communicated variables or skipping the communication round.

However, all aforementioned decentralized optimization algorithms restrict their focus on the single-level minimization problem, which are not applicable to the bilevel optimization problem. In particular, bilevel optimization involves the computation of Hessian and Jacobian matrices. If communicating these two matrices, it will incur a large communication complexity. Thus, it is not clear whether these matrices should be communicated like the gradient. Moreover, considering the interaction between the bilevel structure and the communication mechanism, how will the decentralized bilevel optimization algorithm converge? Particularly, the stochastic hypergradient regarding \( x \) of the outer-level objective function is a biased estimation of the full gradient. How does this biased estimator affect the consensus error? All these problems regarding the algorithmic design and theoretical analysis for decentralized bilevel optimization are still unexplored.

To address the aforementioned problems, we proposed two novel decentralized bilevel optimization algorithms. Specifically, on the algorithmic design side, we developed a momentum-based decentralized stochastic bilevel optimization (MDO) algorithm, which takes advantage of the momentum to update model parameters, and a variance-reduction-based decentralized stochastic bilevel optimization (VRDO) algorithm, which leverages the variance-reduced gradient to update model parameters. Both of them employ the gradient tracking communication mechanism. In other words, only model parameters and stochastic gradients are communicated among participants. In this way, the computation of Hessian and Jacobian matrices is restricted in each participant. On the theoretical analysis side, we established the convergence rate of our two algorithms. Specifically, we investigated how the biased gradient estimator affects the consensus error in the presence of the momentum and variance-reduced gradient. Our theoretical analysis shows that MDO achieves the \( O(\frac{1}{\epsilon^2(1-\lambda)^2}) \) convergence rate and VRDO enjoys the \( O(\frac{1}{\epsilon^2\tau(1-\lambda^2)}) \) convergence rate to obtain the \( \epsilon \)-accuracy solution. Our contributions are summarized as follows:

- We developed two novel decentralized bilevel optimization algorithms for solving Eq. (1), which demonstrated how to update model parameters locally and communicate them across different participants.
- We established the convergence rate of our proposed algorithms, which demonstrated how the bilevel structure, the gradient estimator, and the communication mechanism affect the convergence rate.
- We applied our algorithms to the practical machine learning task. The empirical results confirm the superiority of our algorithms.

2 Related Works

Bilevel optimization has been widely applied to numerous machine learning applications. For instance, in the hyperparameter optimization task, the upper-level problem optimizes the hyperparameter and
the lower-level problem optimizes the model parameter of the machine learning model. In the meta-learning task, the upper-level problem learns the task-shared model parameters while the lower-level problem learns the task-specific model parameters\cite{12}. When optimizing these kinds of bilevel machine learning models, the challenge lies in the computation of the inverse Hessian matrix \((\nabla^2_{yy}g^{(k)})^{-1}\). To address this issue,\cite{7} developed a Hessian inverse approximation strategy, which employs stochastic samples to compute an approximation for \((\nabla^2_{yy}g^{(k)})^{-1}\). Meanwhile, it employs the double-loop mechanism, where \(y\) is updated for multiple times before updating \(x\), to obtain a good approximation for \(y^*(x)\).\cite{12} further employs a large batch size to improve the approximation for \(y^*(x)\). On the contrary,\cite{11} developed a single-loop method, which employs different step sizes for the model parameters \(x\) and \(y\) such that each update \(y\) is a good approximation for the optimal solution \(y^*(x)\). It is worth noting that these single-loop and double-loop algorithms employ stochastic gradients to update model parameters, which suffer from a large estimation variance. To address this problem,\cite{9} developed a single-loop algorithm, which leverages a variance-reduced gradient estimator to update the model parameter \(x\). However, they fail to achieve an improved theoretical convergence rate. Recently,\cite{27,14,10} resort to more advanced variance-reduced gradient estimators to accelerate the convergence rate. Specifically,\cite{27,14,10} combines the STORM \cite{2} gradient estimator and the single-loop mechanism so that they can achieve a better theoretical convergence rate than the stochastic-gradient-based algorithm and the momentum-based algorithm. Moreover,\cite{27} combines the SPIDER \cite{3} gradient estimator and the double-loop mechanism, which actually achieve the same theoretical convergence rate with that based on STORM. However, all these algorithms only investigate the non-parallel situation. Thus, their theoretical analysis does not hold anymore for the distributed scenario.

Decentralized optimization has also been applied to a wide variety of machine learning applications in recent years. Compared to the parameter-server setting, the decentralized communication is robust to the single-node failure since the participant conducts peer-to-peer communication. Recently,\cite{13} investigated the convergence rate of the standard decentralized stochastic gradient descent (SGD) algorithm for nonconvex problems.\cite{28} developed a decentralized stochastic gradient descent with momentum algorithm, which has the same theoretical convergence rate as\cite{13}.\cite{21,20} developed a decentralized SGD based on the gradient tracking communication mechanism. Later, some variance-reduced algorithms were proposed to accelerate the convergence rate. For instance,\cite{23} utilizes the SPIDER \cite{3} gradient estimator,\cite{26,30} employ the STORM \cite{2} gradient estimator, and\cite{29} resorts to the ZeroSARAH \cite{17} gradient estimator for improving the sample and communication complexities. However, all these decentralized optimization algorithms are not application to the decentralized bilevel optimization problem. On the one hand, they focus on the single-level problem. Thus, their theoretical analysis is incapable of handling the interaction between two levels of functions. On the other hand, those algorithms are based on the standard stochastic gradient, which is an unbiased estimation for the full gradient. On the contrary, the stochastic hypergradient of the upper-level objective function is biased, which incurs new challenges when bounding the consensus error. Thus, it is necessary to develop new theoretical analysis strategies to investigate the convergence rate of decentralized bilevel optimization algorithms.

3 Preliminaries

Stochastic Hypergradient. Throughout this paper, we denote \(F^{(k)}(x) = f^{(k)}(x, y^*(x))\) and \(F(x) = \frac{1}{K}\sum_{k=1}^{K} F^{(k)}(x)\). Then, the full gradient of \(F^{(k)}(x)\) is defined as follows:

\[
\nabla F^{(k)}(x) = \nabla_x f^{(k)}(x, y^*(x)) - \nabla^2_{xy} g^{(k)}(x, y^*(x)) (\nabla^2_{yy} g^{(k)}(x, y^*(x)))^{-1} \nabla_y f^{(k)}(x, y^*(x)),
\]

where \(\nabla^2_{xy} g^{(k)}(x, y^*(x))\) is the Jacobian matrix and \(\nabla^2_{yy} g^{(k)}(x, y^*(x))\) is the Hessian matrix. Note that \(\nabla F^{(k)}(x)\) is also called hypergradient. Since \(y^*(x)\) is typically not easy to obtain in each iteration, following\cite{7}, we can use the following gradient to approximate it:

\[
\nabla F^{(k)}(x, y) = \nabla_x f^{(k)}(x, y) - \nabla^2_{xy} g^{(k)}(x, y) (\nabla^2_{yy} g^{(k)}(x, y))^{-1} \nabla_y f^{(k)}(x, y),
\]

Moreover, because the inverse of Hessian matrix is difficult to compute, following the Hessian inverse approximation strategy proposed in\cite{7}, we can use the following stochastic hypergradient to
approximate it:
\[
\nabla \tilde{F}^{(k)}(x, y; \xi) = \nabla_x f^{(k)}(x, y; \xi) \\
- \nabla^2_{xy} g^{(k)}(x, y; \zeta_0) \frac{J}{L_{gy}} \prod_{j=1}^{\bar{J}} \left( I - \frac{1}{L_{gy}} \nabla^2_{xy} g^{(k)}(x, y; \zeta_j) \right) \nabla_y f^{(k)}(x, y; \xi),
\]
where \( L_{gy} \) the Lipschitz-continuous constant, which is defined in Assumption \( \xi = \{ \xi, \zeta_0, \zeta_1, \ldots, \zeta_J \} \) and \( \bar{J} \) is randomly selected from \( \{0, 1, 2, \ldots, J\} \) where \( J \) is a positive integer. Note that we let \( \prod_{j=1}^{\bar{J}} \left( I - \frac{1}{L_{gy}} \nabla^2_{xy} g^{(k)}(x, y; \zeta_j) \right) = I \) when \( \bar{J} = 0 \). Moreover, we denote the expectation of the stochastic hypergradient as follows:
\[
\nabla \tilde{F}^{(k)}(x, y) \triangleq \mathbb{E}[\nabla \tilde{F}^{(k)}(x, y; \xi)] = \nabla_x f^{(k)}(x, y) \\
- \nabla^2_{xy} g^{(k)}(x, y) \mathbb{E} \left[ \frac{J}{L_{gy}} \prod_{j=1}^{\bar{J}} \left( I - \frac{1}{L_{gy}} \nabla^2_{xy} g^{(k)}(x, y; \zeta_j) \right) \right] \nabla_y f^{(k)}(x, y),
\]
(5)

Since \( \nabla \tilde{F}^{(k)}(x, y) \neq \nabla F^{(k)}(x, y) \), the stochastic hypergradient \( \nabla \tilde{F}^{(k)}(x, y; \xi) \) is a biased estimator for \( \nabla F^{(k)}(x, y) \). The detailed bias is shown in Lemma 3.

**Notations.** Throughout this paper, \( x_t^{(k)} \) and \( y_t^{(k)} \) denote the model parameters of the \( k \)-th participant in the \( t \)-th iteration. Additionally, we denote \( X_t = [x_t^{(1)}, \ldots, x_t^{(K)}], Y_t = [y_t^{(1)}, \ldots, y_t^{(K)}], \Delta_t^{\xi} = [\nabla \tilde{F}^{(1)}(x_t^{(1)}, y_t^{(1)}; \xi_t^{(1)}), \ldots, \nabla \tilde{F}^{(K)}(x_t^{(K)}, y_t^{(K)}; \xi_t^{(K)})], \) and \( g_t^{(k)} = [\nabla g(x_t^{(1)}, y_t^{(1)}; \zeta_t^{(1)}), \ldots, \nabla g(x_t^{(K)}, y_t^{(K)}; \zeta_t^{(K)})]. \) Moreover, we denote \( \bar{x}_t = \frac{1}{K} \sum_{k=1}^{K} x_t^{(k)} \) and \( \bar{y}_t = \frac{1}{K} \sum_{k=1}^{K} y_t^{(k)} \). Furthermore, the adjacency matrix of the communication network is denoted by \( W = [w_{ij}] \in \mathbb{R}^{K \times K} \), where \( w_{ij} > 0 \) indicates the \( i \)-th participant is connected with the \( j \)-th participant and otherwise \( w_{ij} = 0 \). The adjacency matrix satisfies the following assumption.

**Assumption 1.** \( W \) satisfies \( W^T = W, W1 = 1, \) and \( 1^T W = 1^T \). Its eigenvalues satisfy \( |\lambda_1| \leq \cdots \leq |\lambda_2| < |\lambda_1| = 1. \)

Then, the spectral gap of \( W \) can be represented by \( 1 - \lambda \) where \( \lambda = \frac{1}{2} |\lambda_2| \). Note that \( 1 - \lambda \in (0, 1] \), where the upper bound is achieved when the communication network is fully connected.

## 4 Decentralized Stochastic Bilevel Optimization Algorithms

**Algorithm 1 MDBO**

**Input:** \( z_0^{(k)} = x_0, y_0^{(k)} = y_0, \eta > 0, \alpha_1 > 0, \alpha_2 > 0, \beta_1 > 0, \beta_2 > 0. \)

1. for \( t = 0, \ldots, T - 1 \) do
2. if \( t == 0 \) then
3. \( U_t = \Delta_t^{\xi}, V_t = \Delta_t^{\zeta}, Z_t^{\xi} = \Delta_t^{\xi}, Z_t^{\zeta} = \Delta_t^{\zeta}, \)
4. else
5. \( U_t = (1 - \alpha_1 \eta)U_{t-1} + \alpha_1 \eta \Delta_t^{\xi}, V_t = (1 - \alpha_2 \eta)V_{t-1} + \alpha_2 \eta \Delta_t^{\zeta}, \)
6. \( Z_t^{\xi} = Z_{t-1}^{\xi} W + U_t - U_{t-1}, Z_t^{\zeta} = Z_{t-1}^{\zeta} W + V_t - V_{t-1}, \)
7. end if
8. \( X_{t+1} = X_t - \eta X_t (I - W) - \beta_1 \eta Z_t^{\xi}, Y_{t+1} = Y_t - \eta Y_t (I - W) - \beta_2 \eta Z_t^{\zeta}, \)
9. end for

**Momentum-based Decentralized Stochastic Bilevel Optimization Algorithm.** In Algorithm 1 we developed a momentum-based decentralized stochastic bilevel optimization (MDBO) algorithm. The main idea is to use the momentum to update the model parameters \( x \) and \( y \) at each participant and then perform communication. Specifically, the momentum in the first iteration is initialized as
where \( \alpha > 0 \), \( \eta > 0 \), and \( \alpha \eta^2 \in (0, 1) \). Note that \( \Delta t^{-1} \) denotes the stochastic hypergradient which is computed based on the model parameters \( X_{t-1} \) and \( Y_{t-1} \) in the \( t-1 \) iteration, as well as the selected samples in the \( t \)-th iteration. \( V_t \) is updated in the same way. Then, based on this variance-reduced gradient estimator, each participant leverages the gradient tracking communication mechanism to exchange the momentum and model parameter updates across participants. In detail, \( Z^F_t \in \mathbb{R}^{d_1 \times K} \) and \( Z^g_t \in \mathbb{R}^{d_2 \times K} \) are the tracked momentum for \( U_t \) and \( V_t \), respectively. In the first iteration, they are initialized as the stochastic gradient as shown in Line 3 of Algorithm 1. In other iterations, the momentum is updated as follows:

\[
Z^F_t = Z^F_{t-1} W + U_t - U_{t-1}, \quad Z^g_t = Z^g_{t-1} W + V_t - V_{t-1},
\]

where \( Z^F_{t-1} W \) denotes the communication operation. Based on the tracked momentum, the model parameters \( x \) and \( y \) are updated as follows:

\[
X_{t+1} = X_t - \eta X_t (I - W) - \beta_1 \eta Z^F_t, \quad Y_{t+1} = Y_t - \eta Y_t (I - W) - \beta_2 \eta Z^g_t,
\]

where \( \eta \in (0, 1) \), \( \beta_1 \) and \( \beta_2 \) are positive, \( X_t W \) and \( Y_t W \) indicate the communication of model parameters across participants. In fact, by reformulating this updating rule, it is easy to know that \( X_{t+1} \) is the combination of the local model parameter \( X_t \) and the update \( X_t W - \beta_1 Z^F_t \) that is based on the neighboring participants’ information. In summary, the computation of stochastic gradients/hypergradients, Hessian matrix, and Jacobian matrix is restricted in each participant. Only the momentum and model parameters are communicated across participants in our algorithm.

Algorithm 2 VRDBO

**Input:** \( x_0^{(k)} = x_0, \ y_0^{(k)} = y_0, \ \eta > 0, \ \alpha_1 > 0, \ \alpha_2 > 0, \ \beta_1 > 0, \ \beta_2 > 0 \).

1: for \( t = 0, \cdots, T - 1 \) do
2: \hspace{1em} if \( t == 0 \) then
3: \hspace{2em} \( U_t = \Delta^F_t, \ V_t = \Delta^g_t, \ Z^F_t = \Delta^F_t, \ Z^g_t = \Delta^g_t \),
4: \hspace{1em} else
5: \hspace{2em} \( U_t = (1 - \alpha_1 \eta^2) (U_{t-1} + \Delta^F_{t-1} - \Delta^F_{t-1}) + \alpha_1 \eta^2 \Delta^F_{t-1} \),
6: \hspace{2em} \( V_t = (1 - \alpha_2 \eta^2) (V_{t-1} + \Delta^g_{t-1} - \Delta^g_{t-1}) + \alpha_2 \eta^2 \Delta^g_{t-1} \),
7: \hspace{2em} \( Z^F_t = Z^F_{t-1} W + U_t - U_{t-1}, \ Z^g_t = Z^g_{t-1} W + V_t - V_{t-1} \),
8: \hspace{1em} end if
9: \hspace{2em} \( X_{t+1} = X_t - \eta X_t (I - W) - \beta_1 \eta Z^F_t, \ Y_{t+1} = Y_t - \eta Y_t (I - W) - \beta_2 \eta Z^g_t \),
10: end for

Decentralized Stochastic Bilevel Optimization Algorithm Based on Variance-Reduced Gradient. Existing non-parallel algorithms have shown that the momentum-based approach does not achieve a better theoretical convergence rate even though it demonstrates better empirical convergence performance [9][13]. Thus, we further developed a new algorithm: variance-reduction-based decentralized stochastic bilevel optimization (VRDBO) algorithm, which takes advantage of the variance-reduced gradient estimator to accelerate the convergence rate. The details are shown in Algorithm 2. Specifically, VRDBO utilizes the STORM [2] gradient estimator to control the variance of stochastic gradients/hypergradients, which is defined as follows:

\[
U_t = (1 - \alpha_1 \eta^2) (U_{t-1} + \Delta^F_{t-1} - \Delta^F_{t-1}) + \alpha_1 \eta^2 \Delta^F_{t-1},
\]

where \( \alpha > 0, \ \eta > 0 \), and \( \alpha \eta^2 \in (0, 1) \).
exchange the tracked gradients and model parameters to update local model parameters, which is shown in Lines 7 and 9 of Algorithm 2. In summary, compared to Algorithm 1, VRDBO utilizes a variance-reduced gradient estimator to control the variance of stochastic gradients. Thus, it is supposed to enjoy a better convergence rate than MDBO, which will be shown in next section.

5 Convergence Analysis

To investigate the convergence rate of our two algorithms, we first introduce two common assumptions for both algorithms and then introduce the algorithm-specific assumptions.

Assumption 2. For any $x \in \mathbb{R}^d_x$ and $k \in \{1, 2, \ldots, K\}$, the lower-level function $g^{(k)}(x, y)$ is $\mu$-strongly convex with respect to $y$.

Assumption 3. For any $k \in \{1, 2, \ldots, K\}$, the first and second order stochastic gradients of the upper-level and lower-level loss functions have bounded variance $\sigma > 0$.

5.1 Convergence Rate of Algorithm 1

Similar to the non-parallel algorithms [7, 11, 12], our Algorithm 1 requires a weaker assumption regarding the smoothness of the loss function compared with Algorithm 2 which is shown as follows.

Assumption 4. For any $k \in \{1, 2, \ldots, K\}$, $\nabla_x f^{(k)}(x, y)$ is Lipschitz continuous with the constant $L_f > 0$, $\nabla_y f^{(k)}(x, y)$ is Lipschitz continuous with the constant $L_f > 0$, i.e.,

\[
\begin{align*}
\|\nabla_x f^{(k)}(x_1, y_1) - \nabla_x f^{(k)}(x_2, y_2)\| &\leq L_f \| (x_1, y_1) - (x_2, y_2) \|, \\
\|\nabla_y f^{(k)}(x_1, y_1) - \nabla_y f^{(k)}(x_2, y_2)\| &\leq L_f \| (x_1, y_1) - (x_2, y_2) \|
\end{align*}
\]

hold for any $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^d_x \times \mathbb{R}^d_y$. Moreover, $\|\nabla_y f^{(k)}(x, y)\| \leq C_f$ with the constant $C_f > 0$ for $(x, y) \in \mathbb{R}^d_x \times \mathbb{R}^d_y$.

Assumption 5. For any $k \in \{1, 2, \ldots, K\}$, $\nabla_y g^{(k)}(x, y)$ is Lipschitz continuous with the constant $L_{g_y} > 0$, $\nabla_y^2 g^{(k)}(x, y)$ is Lipschitz continuous with the constant $L_{g_{yy}} > 0$, i.e.,

\[
\begin{align*}
\|\nabla_y g^{(k)}(x_1, y_1) - \nabla_y g^{(k)}(x_2, y_2)\| &\leq L_{g_y} \| (x_1, y_1) - (x_2, y_2) \|, \\
\|\nabla_y^2 g^{(k)}(x_1, y_1) - \nabla_y^2 g^{(k)}(x_2, y_2)\| &\leq L_{g_{yy}} \| (x_1, y_1) - (x_2, y_2) \|
\end{align*}
\]

hold for any $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^d_x \times \mathbb{R}^d_y$. Moreover, $\|\nabla_y g^{(k)}(x, y)\| \leq C_{g_y}$ with the constant $C_{g_y} > 0$ and $\mu \leq \nabla_y^2 g^{(k)}(x, y; \zeta) \leq L_{g_{yy}}$ for any $(x, y) \in \mathbb{R}^d_x \times \mathbb{R}^d_y$.

Based on these assumptions, we are able to establish the convergence rate of Algorithm 1 as follows.

Theorem 1. Given Assumptions 2-5 if $\alpha_1 > 0$, $\alpha_2 > 0$, $\eta < \min\{1, \frac{\beta_1}{2\bar{L}_F}, \frac{1}{\alpha_1}, \frac{1}{\alpha_2}\}$, $\beta_1 \leq \min\{\frac{\sigma_2^2\mu^2}{16L_yU_y}, \frac{\mu}{16L_yU_y\sqrt{2L_y^2/\alpha_1^2+50L_y^2/\alpha_2^2}}, \frac{2\bar{L}_F}{\sqrt{(3+36/\alpha_1^2)L_{g_y}^2+(50+900/\alpha_2^2)L_{g_y}^2}}\}$ and $\beta_2 \leq \min\{\frac{\mu}{12L_yU_y\sqrt{2L_y^2/\alpha_1^2+50L_y^2/\alpha_2^2}L_{g_y}^2}, \frac{5(1-\lambda)^2L_{g_y}^2}{2L_yU_y\sqrt{(3+36/\alpha_1^2)L_{g_y}^2+(50+900/\alpha_2^2)L_{g_y}^2}}\}$, the convergence rate of Algorithm 1 is

\[
\frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}[\|\nabla F(\bar{x}_t)\|^2 + L_F^2 \tilde{y}_t - y^*(\bar{x}_t)] \leq \frac{2(F(x_0) - F(x_*)}{\eta\beta_1 T} + \frac{12L_F^2}{\beta_2 T j} \tilde{y}_t - y^*(\bar{x}_t) + \frac{6C_{g_{yy}}^2}{\mu^2} (1 - \frac{\mu}{L_{g_y}})^{2J} \frac{\mu}{\alpha_1^2} + \frac{12\sigma_2^2}{\mu^2} \frac{\alpha_2^2}{\mu^2} + \frac{10\alpha_1\eta^2}{\mu^2} + \frac{300\alpha_2\eta L_{g_y}^2\sigma_2^2}{\mu^2}
\]

where the definition of $L_y$, $L_F$, $L_{g_y}$, $\sigma_\tilde{y}$ is shown in Lemmas 1, 4, 5.

Corollary 1. Given the same condition with Theorem 1 by choosing $T = O\left(\frac{1}{\epsilon(1-\lambda)^2}\right)$, $\eta = O(\epsilon)$, $J = O(\log \frac{1}{\epsilon})$, $\beta_1 = O((1-\lambda)^2)$, $\beta_2 = O((1-\lambda)^2)$, $\alpha_1 = O(1)$, and $\alpha_2 = O(1)$, Algorithm 1
can achieve the ε-accuracy solution: $\frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}[\|\nabla F(x_t)\|^2 + L_F^2 \|y_t - y^*(x_t)\|^2] \leq O(\epsilon)$. Then, the communication complexity is $O\left(\frac{1}{\epsilon^{2(1-\lambda)^2}}\right)$, the gradient complexity and Jacobian-vector product complexity is $O\left(\frac{1}{\epsilon^{2(1-\lambda)^2}}\right)$, and the Hessian-vector product complexity is $O\left(\frac{1}{\epsilon^{2(1-\lambda)^2}}\right)$.

5.2 Convergence Rate of Algorithm 2

Since Algorithm 2 employs the variance-reduced gradient estimator, we introduce the following mean-square Lipschitz smoothness assumption for the upper-level and lower-level objective functions, which is also used by existing variance-reduced bilevel optimization algorithms [27, 10]. Please note that all variance-reduced gradient descent algorithms [24, 3] require the mean-square Lipschitz smoothness assumption to establish the convergence rate.

Assumption 6. For any $k \in \{1, 2, \ldots, K\}$, $\nabla_x f^k(x, y)$ is Lipschitz continuous with the constant $\ell_{f_x} > 0$, $\nabla_y f^k(x, y)$ is Lipschitz continuous with the constant $\ell_{f_y} > 0$, i.e.,

$$
\mathbb{E}[\|\nabla_x f^k(x_1, y_1; \xi) - \nabla_x f^k(x_2, y_2; \xi)\|^2] \leq \ell_{f_x} \|x_1 - x_2\|^2,
\mathbb{E}[\|\nabla_y f^k(x_1, y_1; \xi) - \nabla_y f^k(x_2, y_2; \xi)\|^2] \leq \ell_{f_y} \|y_1 - y_2\|^2,
$$

(13)

hold for any $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^d \times \mathbb{R}^d$. Moreover, $\mathbb{E}[\|\nabla_y f^k(x, y; \xi)\|^2] \leq c_{f_y}$ with the constant $c_{f_y} > 0$ for any $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$.

Assumption 7. For any $k \in \{1, 2, \ldots, K\}$, $\nabla_y g^k(x, y)$ is Lipschitz continuous with the constant $\ell_{g_y} > 0$, $\nabla_y g^k(x, y)$ is Lipschitz continuous with the constant $\ell_{g_y} > 0$, $\nabla_y g^k(x, y)$ is Lipschitz continuous with the constant $\ell_{g_y} > 0$, i.e.,

$$
\mathbb{E}[\|\nabla_y g^k(x_1, y_1; \xi) - \nabla_y g^k(x_2, y_2; \xi)\|^2] \leq \ell_{g_y} \|x_1 - x_2\|^2,
\mathbb{E}[\|\nabla_y g^k(x_1, y_1; \xi) - \nabla_y g^k(x_2, y_2; \xi)\|^2] \leq \ell_{g_y} \|y_1 - y_2\|^2,
$$

(14)

hold for any $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^d \times \mathbb{R}^d$. Moreover, $\mathbb{E}[\|\nabla_y g^k(x, y; \xi)\|^2] \leq c_{g_y}$ with the constant $c_{g_y} > 0$ and $\mu \leq \|\nabla_y g^k(x, y; \xi)\| \leq c_{g_y}$ for any $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$.

Theorem 2. Given Assumptions 3, 4, 6, 7 if $\alpha_1 > 0$, $\alpha_2 > 0$, $\eta < \min\{1, \frac{1}{5\lambda}, \frac{1}{(1-\lambda)}\}$, $\beta_1 \leq \min\{\frac{1}{4\mu}, \frac{1}{\mu L_{\hat{F}}}, \frac{5(1-\lambda)^2}{\mu L_{\hat{F}}}\}$, and $\beta_2 \leq \min\{\frac{2(1-\lambda)^2}{\mu L_{\hat{F}}}, \frac{1}{\mu \eta^T}, \frac{1}{\mu L_{\hat{F}}}, \frac{1}{\mu \eta^T}, \frac{1}{\mu \eta^T}, \frac{1}{\mu \eta^T}\}$, the convergence rate of Algorithm 2 is

$$
\frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}[\|\nabla F(x_t)\|^2 + L_F^2 \|y_t - y^*(x_t)\|^2] \leq \frac{2(F(x_0) - F(x_*))}{\eta T} + \frac{12L_F^2}{\mu \eta^T} \|y_0 - y^*(x_0)\|^2 + \frac{6C^2_g \|\nabla_y g^k(x, y; \xi)\|^2}{\mu^2} (1 - \frac{\mu L_{g_y}}{\mu^2}) + \frac{17\alpha_1 \eta^2 \sigma^2 \mu}{\mu^2} + \frac{450\alpha_2 \eta^2 \sigma^2 \mu^2}{\mu^2},
$$

(15)

where the definition of $L_{\mu}, L_F, L_{\hat{F}}, \sigma_{\hat{F}}$ is shown in Lemmas 14, 17, 18.

Corollary 2. Given the same condition with Theorem 2 by choosing $T = O\left(\frac{1}{\epsilon^{2(1-\lambda)^2}}\right)$, $J = O(\log \frac{1}{\epsilon})$, $\beta_1 = O((1 - \lambda)^2)$, $\beta_2 = O((1 - \lambda)^2)$, $\alpha_1 = O(1)$, and $\alpha_2 = O(1)$, Algorithm 2 can achieve the ε-accuracy solution: $\frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}[\|\nabla F(x_t)\|^2 + L_F^2 \|y_t - y^*(x_t)\|^2] \leq O(\epsilon)$. Then, the communication complexity is $O\left(\frac{1}{\epsilon^{2(1-\lambda)^2}}\right)$, which is better than $O\left(\frac{1}{\epsilon^{2(1-\lambda)^2}}\right)$ of Algorithm 7. Additionally, the gradient complexity and Jacobian-vector product complexity of Algorithm 2 are $O\left(\frac{1}{\epsilon^{2(1-\lambda)^2}}\right)$ and the Hessian-vector product complexity is $O\left(\frac{1}{\epsilon^{2(1-\lambda)^2}}\right)$, all of which are better than Algorithm 7.

6 Experiments

In this section, we conduct experiments to verify the performance of our proposed algorithms. In particular, we apply our algorithms to the hyperparameter optimization of the logistic regression
model \cite{13}, which is defined as follows:
\[
\min_{x \in \mathbb{R}^d} \frac{1}{K} \sum_{k=1}^{K} \frac{1}{n_{val}^{(k)}} \sum_{i=1}^{n_{val}^{(k)}} \ell_{CE}(y^*(x)^{T} a_{val,i}^{(k)}, b_{val,i}^{(k)})
\]
\[
s.t. \quad y^*(x) = \arg \min_{y \in \mathbb{R}^{d \times c}} \frac{1}{K} \sum_{k=1}^{K} \frac{1}{n_{tr}^{(k)}} \sum_{i=1}^{n_{tr}^{(k)}} \ell_{CE}(y^{T} a_{tr,i}^{(k)}, b_{tr,i}^{(k)}) + \frac{1}{c d} \sum_{p=1}^{c} \sum_{q=1}^{d} \exp(x_q) y_{pq}^2,
\]
where \((a_{val,i}^{(k)}, b_{val,i}^{(k)}) \in \mathbb{R}^d \times \mathbb{R}^c\) denotes the \(i\)-th validation sample’s feature and label of the \(k\)-th participant, \((a_{tr,i}^{(k)}, b_{tr,i}^{(k)})\) represents the training sample, \(n_{val}^{(k)}\) is the number of validation samples in the \(k\)-th participant, \(n_{tr}^{(k)}\) is the number of training samples, \(\ell_{CE}\) is the cross-entropy loss function, \(x \in \mathbb{R}^d\) represents the hyperparameter, \(y \in \mathbb{R}^{d \times c}\) denotes the model parameter.

In our experiments, we use three binary benchmark classification datasets\cite{14} a9a, ijcnn1, and covtype. In particular, a9a has 32,561 samples, ijcnn1 has 49,990 samples, and covtype has 581,012 samples. We randomly select 30\% samples as the validation set and the remaining samples as the training set. Then, they are randomly and evenly put to each participant. To demonstrate the performance of our algorithms, we compare them with three state-of-the-art algorithms\cite{13} SEMA \cite{9}, MSTSA \cite{11}, MRBO \cite{27}. Specifically, SEMA also employs the momentum technique as our MDBO algorithm, MSTSA applies the STORM gradient estimator to the upper-level problem, and MRBO utilizes the STORM gradient estimator to both upper-level and lower-level problems. To make a fair comparison, we use the same batch size for all algorithms. In particular, the batch size is set to 400 for baseline methods. As for our algorithms, the batch size of each participant is 400/K where \(K\) is the total number of participants. Moreover, we set \(\eta = 0.1\) and \(\beta_1 = \beta_2 = 1.0\) for both algorithms, \(\alpha_1 = \alpha_2 = 1.0\) for MDBO, and \(\alpha_1 = \alpha_2 = 7.0\) for VRDBO. Correspondingly, the learning rate of baseline algorithms is set to 0.1, while the coefficient for momentum or STORM of baseline algorithms is equivalently set as our methods. Furthermore, when estimating the stochastic hypergradient, \(J\) is set to 10 for all algorithms.

In Figure\cite{1} we employ 4 workers and the network is a ring network. Here, we show the training loss function value with respect to the number of iterations. There are two observations. First, our algorithms can converge to the same value as the non-parallel algorithm. In particular, our MDBO can converge to the same value as SEMA, which uses the same momentum technique as ours. Additionally, our VRDBO can also converge to the same value as the non-parallel counterpart MRBO. This observation confirms the correctness of our algorithms. Second, VRDBO converges faster than MRBO. The reason is that VRDBO employs a variance-reduced gradient estimator. Thus, it converges faster, which is consistent with the non-parallel counterparts. This observation further confirms the correctness of our algorithms. In Figure\cite{2} we show the training loss function value with respect to the used samples in each participant for demonstrating the acceleration effect. We can find...
that each participant in our algorithms needs much less samples to converge than their non-parallel counterparts. Thus, our algorithms for each participant converge faster than baseline algorithms.

In Figure 3 we demonstrate how the communication network affects the convergence of our algorithms. In particular, we generate three Erdos-Renyi random networks, where the number nodes is 16 and the probability for generating edges is set to 0.2, 0.4, and 0.6, respectively. The covtype dataset is used and the batch size is set to 1,600. From Figure 3 we have similar observations as Figures 1, 2. Specifically, our decentralized algorithms converge to the same value as the non-parallel counterparts and they are more sample-efficient than baseline methods for all configurations. This further confirms the correctness and effectiveness of our algorithms.

Figure 3: The convergence performance when using different random communication networks, which are generated with different edge probabilities. The first row shows the training loss with respect to iterations. The second row shows the training loss with respect to the consumed samples in each participant.

7 Conclusions

In this paper, we studied how to facilitate bilevel optimization to a decentralized communication network. Our proposed algorithms demonstrated how to address this problem. In addition, we also established the convergence rate, demonstrating how the communication network affects the convergence rate. To our knowledge, this is the first work achieving these favorable results. Moreover, extensive experimental results confirm the correctness and effectiveness of our algorithms.
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