Existence and classification of overtwisted contact structures in all dimensions

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To Misha Gromov with admiration

Abstract

We establish a parametric extension $h$-principle for overtwisted contact structures on manifolds of all dimensions, which is the direct generalization of the 3-dimensional result from [12]. It implies, in particular, that any closed manifold admits a contact structure in any given homotopy class of almost contact structures.

1 Introduction

A contact structure on a $(2n + 1)$-dimensional manifold $M$ is a completely non-integrable hyperplane field $\xi \subset TM$. Defining $\xi$ by a Pfaffian equation $\{\alpha = 0\}$ where $\alpha$ is a 1-form, possibly with coefficients in a local system for a non-coorientable $\xi$, then the complete non-integrability is equivalent to $\alpha \wedge d\alpha^n$ being non-vanishing on $M$. An equivalent definition of the contact condition is that the complement of the 0-section of the total space of the conormal bundle $L_\xi \subset T^*M$ is a symplectic submanifold of $T^*M$ with its canonical symplectic structure $d(pdq)$.

The corresponding formal homotopy counterpart of a contact structure is an almost contact structure, which is a defined up to a scalar factor pair $(\alpha, \omega)$ where $\alpha$ is non-vanishing 1-form on $M$, possibly with local coefficients in a non-trivial 1-bundle, and

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$\omega$ is a non-degenerate two-form on the hyperplane field $\xi = \{\alpha = 0\}$ with coefficients in the same local system. Thus, both in the contact and almost contact cases, the hyperplane field is endowed with a conformal class of symplectic structures. In the co-orientable case, i.e. when $TM/\xi$ is trivialized by $\alpha$, the existence of an almost contact structure is equivalent to the existence of a \textit{stable almost complex structure} on $M$, i.e. a complex structure on the bundle $TM \oplus \varepsilon^1$ where $\varepsilon^1$ is the trivial line bundle over $M$.

The current paper concerns with basic topological questions about contact structures: existence, extension and homotopy. This problem has a long history. It was first explicitly formulated, probably, in S.S. Chern’s paper [3]. In 1969 M. Gromov [28] proved a parametric $h$-principle for contact structures on an open manifold $M$: any almost contact structure is homotopic to a genuine one, and two contact structures are homotopic if they are homotopic as almost contact structures, see Theorem 7.1 below for a more precise formulation of Gromov's theorem.

For closed manifolds a lot of progress was achieved in the 3-dimensional case beginning from the work of J. Martinet [36] and R. Lutz [34] who solved the non-parametric existence problem for 3-manifolds. D. Bennequin [2] showed that the 1-parametric $h$-principle fails for contact structures on $S^3$ and Y. Eliashberg in [12] introduced a dichotomy of 3-dimensional contact manifolds into \textit{tight} and \textit{overtwisted} and established a parametric $h$-principle for overtwisted ones: any almost contact homotopy class on a closed 3-manifold contains a unique up to isotopy overtwisted contact structure. Tight contact structures were also classified on several classes of 3-manifolds, see e.g. [14, 25, 32, 33]. V. Colin, E. Giroux and K. Honda proved in [10] that any atoroidal contact 3-manifold admits at most finitely many non-isotopic tight contact structures.

Significant progress in the problem of construction of contact structures on closed manifolds was achieved in the 5-dimensional case beginning from the work of H. Geiges [19, 20] and H. Geiges and C.B. Thomas [22, 23], and followed by the work of R. Casals, D.M. Pancholi and F. Presas [7] and J. Etnyre [18], where there was established existence of contact structures on any 5-manifolds in any homotopy class of almost contact structures. For manifolds of dimension $> 5$ the results are more scarce. The work [13] implied existence of contact structures on all closed $(2n + 1)$-dimensional manifolds that bound almost complex manifolds with the homotopy type of $(n + 1)$-dimensional cell complexes, provided $n \geq 2$. F. Bourgeois [5] proved that for any closed contact manifold $M$ and any surface $\Sigma$ with genus at least one, the product $M \times \Sigma$ admits a contact structure, using work of E. Giroux [24]. This positively answered a long standing problem about existence of contact structures...
on tori of dimension \(2n + 1 > 5\) (a contact structure on \(T^5\) was first constructed by R. Lutz in [35]).

Non-homotopic, but formally homotopic contact structures were constructed on higher dimensional manifolds as well, see e.g. [40]. As far as we know, before the current paper there were no known general results concerning extension of contact structures in dimension greater than three.

**Theorem 1.1.** Let \(M\) be a \((2n + 1)\)-manifold, \(A \subset M\) be a closed set, and \(\xi\) be an almost contact structure on \(M\). If \(\xi\) is genuine on \(O^p A \subset M\) then \(\xi\) is homotopic relative to \(A\) to a genuine contact structure. In particular, any almost contact structure on a closed manifold is homotopic to a genuine contact structure.

Here we are using Gromov’s notation \(O^p A\) for any unspecified open neighborhood of a closed subset \(A \subset M\).

In Section 3 we will define the notion of an overtwisted contact structure for any odd dimensional manifold. Deferring the definition until Section 3.2 we will say here that a contact manifold \((M^{2n+1}, \xi)\) is called overtwisted if it admits a contact embedding of a piecewise smooth \(2n\)-disc \(D_{\text{ot}}\) with a certain model germ \(\zeta_{\text{ot}}\) of a contact structure. In the 3-dimensional case this notion is equivalent to the standard notion introduced in [12]. See Section 10 for further discussion of the overtwisting property.

Given a \((2n + 1)\)-dimensional manifold \(M\), let \(A\) be a closed subset such that \(M \setminus A\) is connected, and let \(\xi_0\) be an almost contact structure \(M\) that is a genuine contact structure on \(O^p A\). Define \(\text{Cont}_{\text{ot}}(M; A, \xi_0)\) to be the space of contact structures on \(M\) that are overtwisted on \(M \setminus A\) and coincide with \(\xi_0\) on \(O^p A\). The notation \(\text{cont}(M; A, \xi_0)\) stands for the space of almost contact structures that agree with \(\xi_0\) on \(O^p A\). Let

\[
 j : \text{Cont}_{\text{ot}}(M; A, \xi_0) \to \text{cont}(M; A, \xi_0)
\]

be the inclusion map. For an embedding \(\phi : D_{\text{ot}} \to M \setminus A\), let \(\text{Cont}_{\text{ot}}(M; A, \xi_0, \phi)\) and \(\text{cont}_{\text{ot}}(M; A, \xi_0, \phi)\) be the subspaces of \(\text{Cont}_{\text{ot}}(M; A, \xi_0)\) and \(\text{cont}_{\text{ot}}(M; A, \xi_0)\) of contact and almost contact structures for which \(\phi : (D_{\text{ot}}, \zeta_{\text{ot}}) \to (M, \xi)\) is a contact embedding.

**Theorem 1.2.** The inclusion map induces an isomorphism

\[
 j_* : \pi_0(\text{Cont}_{\text{ot}}(M; A, \xi_0)) \to \pi_0(\text{cont}(M; A, \xi_0))
\]

and moreover the map

\[
 j : \text{Cont}_{\text{ot}}(M; A, \xi_0, \phi) \to \text{cont}_{\text{ot}}(M; A, \xi_0, \phi)
\]
is a (weak) homotopy equivalence.

As an immediate corollary we have the following

**Corollary 1.3.** On any closed manifold $M$ any almost contact structure is homotopic to an overtwisted contact structure which is unique up to isotopy.

We also have the following corollary (see Section 3.6 for the proof) concerning isomcontact embeddings into an overtwisted contact manifold.

**Corollary 1.4.** Let $(M^{2n+1}, \xi)$ be a connected overtwisted contact manifold and let $(N^{2n+1}, \zeta)$ be an open contact manifold of the same dimension. Let $f : N \to M$ be a smooth embedding covered by a contact bundle homomorphism $\Phi : TN \to TM$, that is $\Phi(\zeta_x) = \xi|_{f(x)}$ and $\Phi$ preserves the conformal symplectic structures on $\zeta$ and $\xi$. If $df$ and $\Phi$ are homotopic as injective bundle homomorphisms $TN \to TM$, then $f$ is isotopic to a contact embedding $\tilde{f} : (N, \zeta) \to (M, \xi)$. In particular, an open ball with any contact structure embeds into any overtwisted contact manifold of the same dimension.

We note that there were many proposals for defining the overtwisting phenomenon in dimension greater than three. We claim that our notion is stronger than any other possible notions, in the sense that any exotic phenomenon, e.g. a plastikstufe \cite{38}, can be found in any overtwisted contact manifold. Indeed suppose we are given some exotic model $(A, \zeta)$, which is an open contact manifold, and assume it formally embeds into an equidimensional $(M, \xi_{\text{ot}})$, then by Corollary 1.4 $(A, \zeta)$ admits a genuine contact embedding into $(M, \xi_{\text{ot}})$.

In particular, the known results about contact manifolds with a plastikstufe apply to overtwisted manifolds as well:

- **Overtwisted contact manifolds are not (semi-positively) symplectically fillable** \cite{38};
- **The Weinstein conjecture holds for any contact form defining an overtwisted contact structure on a closed manifold** \cite{1};
- **Any Legendrian submanifold whose complement is overtwisted is loose** \cite{37}. Conversely, any loose Legendrian in an overtwisted ambient manifold has an overtwisted complement.
As it is customary in the $h$-principle type framework, a parametric $h$-principle yields results about leafwise structures on foliations, see e.g. [28]. In particular, in [11] the parametric $h$-principle [12] for overtwisted contact structures on a 3-manifold was used for the construction of leafwise contact structures on codimension one foliations on 4-manifolds.

Let $\mathcal{F}$ be a smooth $(2n + 1)$-dimensional foliation on a manifold $V$ of dimension $m = 2n + 1 + q$.

**Theorem 1.5.** Any leafwise almost contact structure on $\mathcal{F}$ is homotopic to a genuine leafwise contact structure.

A leafwise contact structure $\xi$ on a codimension $q$ foliation $\mathcal{F}$ on a manifold $V$ of dimension $2n + 1 + q$ is called overtwisted if there exist disjoint embeddings

$$h_i : T_i \times B \to V$$

for $i = 1, \ldots, N$,

where $(B, \zeta)$ is a $(2n + 1)$-dimensional overtwisted contact ball and each $T_i$ is a compact $q$-dimensional manifold with boundary, such that

- each leaf of $\mathcal{F}$ is intersected by one of these embeddings, and
- for each $i = 1, \ldots, N$ and $\tau \in T_i$ the restriction $h_i|_{\tau \times B}$ is a contact embedding of $(B, \zeta)$ into some leaf of $\mathcal{F}$ with its contact structure.

The set of embeddings $h_1, \ldots, h_N$ is called an **overtwisted basis** of the overtwisted leafwise contact structure $\xi$ on $\mathcal{F}$.

For a closed subset $A \subset V$, let $\xi_0$ be a leafwise contact structure on $\mathcal{F}|_{O p A}$, and let $h_i : T_i \times B \to V \setminus A$ for $i = 1, \ldots, N$ be a collection of disjoint embeddings. Define

$$\text{Cont}_{\text{ot}}(\mathcal{F}; A, \xi_0, h_1, \ldots, h_N)$$

to be the space of leafwise contact structures $\mathcal{F}$ that coincide with $\xi_0$ over $O p A$ and such that $\{h_i\}_{i \in \{1, \ldots, N\}}$ is an overtwisted basis for $\mathcal{F}|_{V \setminus A}$.

Define

$$\text{cont}_{\text{ot}}(\mathcal{F}; A, \xi_0, h_1, \ldots, h_n)$$

to be the analogous space of leafwise almost contact structures on $\mathcal{F}$.

**Theorem 1.6.** The inclusion map

$$\text{Cont}_{\text{ot}}(\mathcal{F}; A, \xi_0, h_1, \ldots, h_N) \to \text{cont}_{\text{ot}}(\mathcal{F}; A, \xi_0, h_1, \ldots, h_N)$$

is a (weak) homotopy equivalence.
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**Remark 1.7.** If $V$ is closed then an analog of Gray-Moser’s theorem still holds even though the leaves could be non-compact. Indeed, the leafwise vector field produced by Moser’s argument is integrable because $V$ is compact, and hence it generates the flow realizing the prescribed deformation of the leafwise contact structure. Therefore, a homotopical classification of leafwise contact structures coincides with their isotopical classification.

**Plan of the paper.** Because of Gromov’s $h$-principle for contact structures on open manifolds, the entire problem can be reduced to a local extension problem of when a germ of a contact structure on the $2n$-sphere $\partial B^{2n+1}$ can be extended to a contact structure on $B^{2n+1}$. Our proof is based on the two main results: Proposition 3.1, which reduces the extension problem to a unique model in every dimension, and Proposition 3.9, which provides an extension of the connected sum of this universal model with a neighborhood of an overtwisted $2n$-disc $D_{ot}$ defined in Section 3.2. We formulate Propositions 3.1 and 3.9 in Section 3 and then deduce from them Theorem 1.1. We then continue Section 3 with Propositions 3.10 and 3.11 which are parametric analogs of the preceding propositions, and then prove Theorem 1.2 and Corollary 1.4. The proofs of Theorems 1.5 and 1.6 concerning leafwise contact structures on a foliation, are postponed till Section 9.

In Section 4 we study the notion of domination between contact shells and prove Proposition 4.8 and its corollary Proposition 4.9, which can be thought of as certain disorderability results for the group of contactomorphisms of a contact ball. These results are used in an essential way in the proof of Propositions 3.1 and 3.10 in Section 8. We prove the main extension results, Propositions 3.9 and 3.11, in Section 5.

Propositions 3.1 and 3.10 are proved in Section 8. This is done by gradually standardizing the extension problem in Sections 6–7. First, in Section 6 we reduce it to extension of germs of contact structures induced by a certain family of immersions of $S^{2n}$ into the standard contact $\mathbb{R}^{2n+1}$. This part is fairly standard, and the proof uses the traditional $h$-principle type techniques going back to Gromov’s papers [28, 29] and Eliashberg-Mishachev’s paper [15]. In Section 6 we show how the extension problem of Section 6 can be reduced to the extension of some special models determined by contact Hamiltonians. Finally, to complete the proof of Propositions 3.1 and 3.10 we introduce in Section 8 equivariant coverings and use them to further reduce the problem to just one universal extension model in any given dimension.

The final Section 10 is devoted to further comments regarding the overtwisting property. We also provide an explicit classification of overtwisted contact structures on spheres.
The above diagram outlines the logical dependency of the major propositions in the paper. Notice that the left three columns together give the proof of Theorem 1.1, whereas the right three columns together prove Theorem 1.2. The double arrow between Propositions 6.12 and 3.1 indicates that 6.12 is used in the proof of 3.1 twice in an essential way. The diagram is symmetrical about the central column, in the sense that any two propositions which are opposite of each other are parametric/non-parametric versions of the same result.

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2 Basic notions

2.1 Notation and conventions

Throughout the paper, we will often refer to discs of dimension 2n − 1, 2n, and 2n + 1. For the sake of clarity, we will always use the convention dim B = 2n + 1, dim D = 2n, and dim Δ = 2n − 1. When we occasionally refer to discs of other dimensions we will explicitly write their dimension as a superscript, e.g. D^m. All
discs will be assumed diffeomorphic to closed balls, with possibly piecewise smooth boundary.

Functions, contact structures, etc, on a subset $A$ of a manifold $M$ will always be assumed given on a neighborhood $O p A \subset M$. Throughout the paper, the notation $I$ stands for the interval $I = [0, 1]$ and $S^1$ for the circle $S^1 = \mathbb{R}/\mathbb{Z}$. The notation $A \Subset B$ stands for compact inclusion, meaning $\overline{A} \subset \text{Int} B$.

As the standard model of contact structures in $\mathbb{R}^{2n-1} = \mathbb{R} \times (\mathbb{R}^2)^{n-1}$ we choose

$$\xi_{st} := \{\lambda_{st}^{2n-1} := dz + \sum_{i=1}^{n-1} u_i d\varphi_i = 0\}$$

where $(r_i, \varphi_i)$ are polar coordinates in copies of $\mathbb{R}^2$ with $\varphi_i \in S^1$ and $u_i := r_i^2$. We always use the contact form $\lambda_{st}^{2n-1}$ throughout the paper. On $\mathbb{R}^{2n+1}$ we will use two equivalent contact structures, both defined by

$$\xi_{st} := \{\lambda_{st}^{2n-1} + vdt = 0\}$$

where the coordinates $(v, t)$ have two possible meanings. For $\mathbb{R}^{2n-1} \times \mathbb{R}^2$ we will take $v := r^2$ and $t \in S^1$ where $(r, t)$ are polar coordinates on $\mathbb{R}^2$, while for $\mathbb{R}^{2n-1} \times T^*\mathbb{R}$ we will take $v := -y_n$ and $t := x_n$. In each case it will be explicitly clarified which model contact structure is considered.

A compact domain in $(\mathbb{R}^{2n-1}, \xi_{std})$ will be called star-shaped if its boundary is transverse to the contact vector field $Z = z \frac{\partial}{\partial z} + \sum_{i=1}^{n-1} u_i \frac{\partial}{\partial u_i}$. An abstract contact $(2n-1)$-dimensional closed ball will be called star-shaped if it is contactomorphic to a star-shaped domain in $(\mathbb{R}^{2n-1}, \xi_{st})$.

A hypersurface $\Sigma \subset (M, \xi = \ker \lambda)$ in a contact manifold has a singular 1-dimensional characteristic distribution $\ell \subset T\Sigma \cap \xi$, defined to be the kernel of the 2-form $d\lambda|_{T\Sigma \cap \xi}$, with singularities where $\xi = T\Sigma$. The distribution $\ell$ integrates to a singular characteristic foliation $\mathcal{F}$ with a transverse contact structure, that is contact structures on hypersurfaces $Y \subset \Sigma$ transverse to $\mathcal{F}$, which is invariant with respect to monodromy along the leaves of $\mathcal{F}$. The characteristic foliation $\mathcal{F}$ and its transverse contact structure determines the germ of $\xi$ along $\Sigma$ up to a diffeomorphism fixed on $\Sigma$.

### 2.2 Shells

We will need below some specific models for germs of contact structures along the boundary sphere of a $(2n+1)$-dimensional ball $B$ with piecewise smooth (i.e. stratified
A contact shell will be an almost contact structure \(\xi\) on a ball \(B\) such that \(\xi\) is genuine near \(\partial B\). A contact shell \((B, \xi)\) is called solid if \(\xi\) is a genuine contact structure. An equivalence between two contact shells \((B, \xi)\) and \((B', \xi')\) is a diffeomorphism \(g : B \to B'\) such that \(g_* \xi\) coincides with \(\xi'\) on \(O_p \partial B'\) and \(g_* \xi\) is homotopic to \(\xi'\) through almost contact structures fixed on \(O_p \partial B'\).

Given two shells \(\zeta_+ = (B_+, \xi_+)\) and \(\zeta_- = (B_-, \xi_-)\) we say that \(\zeta_+\) dominates \(\zeta_-\) if there exists both

- a shell \(\tilde{\zeta} = (B, \xi)\) with an equivalence \(g : (B, \xi) \to (B_+, \xi_+)\) of contact shells,
- an embedding \(h : B_- \to B\) such that \(h^* \xi = \xi_-\) and \(\xi\) is a genuine contact structure on \(B \setminus \text{Int } h(B_-)\).

We will refer to the composition \(g \circ h : (B_-, \xi_-) \to (B_+ , \xi_+)\) as a subordination map. Notice that, if \((B_+, \xi_+)\) dominates \((B_-, \xi_-)\) and \((B_-, \xi_-)\) is solid, then \((B_+, \xi_+)\) is equivalent to a solid shell. If both shells \((B_-, \xi_-)\) and \((B_+, \xi_+)\) are solid, then the subordination map is called solid if it is a contact embedding.

A gluing place on a contact shell \((B, \xi)\) is a smooth point \(p \in \partial B\) where \(T_p \partial B = \xi|_p\). Given two gluing places \(p_i \in (B_i, \xi_i)\) on contact shells, the standard topological boundary connected sum construction can be performed in a straightforward way.

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\[\text{Fig. 2.1: Domination of contact shells, where } \xi \text{ is genuine in a neighborhood of the gray region and } \xi|_{h(B_-)} \cong \xi_- \text{ as almost contact structures.}\]
at the points \( p_i \) to produce a contact shell \((B_0\#B_1, \xi_0\#\xi_1)\), which we will call the \textit{boundary connected sum} of the shells \((B_i, \xi_i)\) at the boundary points \( p_i \). We refer the reader to Section 5.1 for precise definitions, and only say here that we can make the shells \((B_i, \xi_i)\) isomorphic near \( p_i \) via an orientation reversing diffeomorphism by a \( C^1 \)-perturbation of the shells that fix the contact planes \( \xi_i |_{p_i} \).

### 2.3 Circular model shells

Here we will describe a contact shell model associated to contact Hamiltonians, which will play a key role in this paper for it is these models that we will use to define overtwisted discs.

Let \( \Delta \subset \mathbb{R}^{2n-1} \) be a compact star-shaped domain and consider a smooth function

\[
K : \Delta \times S^1 \to \mathbb{R} \quad \text{with} \quad K|_{\partial \Delta \times S^1} > 0.
\]

Throughout the paper we will use the notation \((K, \Delta)\) to refer to a such a contact Hamiltonian on a star-shaped domain. For a constant \( C \in \mathbb{R} \), we can define a piecewise smooth \((2n + 1)\)-dimensional ball associated to \((K, \Delta)\) by

\[
B_{K,C} := \{(x, v, t) \in \Delta \times \mathbb{R}^2 : v \leq K(x, t) + C\} \subset \mathbb{R}^{2n-1} \times \mathbb{R}^2
\]

provided \( C + \min(K) > 0 \). Now consider a smooth family of functions

\[
\rho(z) : \mathbb{R} \to \mathbb{R}
\]

for \((x, t) \in \Delta \times S^1\) such that

\[
\rho_{(x,t)} : \mathbb{R}_{\geq 0} \to \mathbb{R} \quad \text{for} \quad (x, t) \in \Delta \times S^1
\]

such that

\[
\rho_{(x,t)} : \mathbb{R}_{\geq 0} \to \mathbb{R} \quad \text{for} \quad (x, t) \in \Delta \times S^1
\]
(i) \( \rho_{(x,t)}(0) = 0 \) for all \((x, t) \in \Delta \times S^1 \),

(ii) \( \rho_{(x,t)}(v) = v - C \) when \((x, v, t) \in \mathcal{O}_p \{ v = K(x, t) + C \} \), and

(iii) \( \partial_v \rho_{(x,t)}(v) > 0 \) when \((x, v, t) \in \mathcal{O}_p \{ v \leq K(x, t) + C, x \in \partial \Delta \} \).

See Figure 2.2 for a schematic picture of such a family of functions. Such a family, which exists by (1), defines an almost contact structure on \( B_{K,C} \) given by

\[
\eta_{K,\rho} := (\alpha_\rho, \omega) \quad \text{with} \quad \alpha_\rho := \lambda_{st} + \rho dt \quad \text{and} \quad \omega := d\lambda_{st} + dv \wedge dt
\]

where smoothness and condition (i) ensures \( \alpha_\rho \) is a well defined 1-form.

**Lemma 2.1.** The pair \( (B_{K,C}, \eta_{K,\rho}) \) is a contact shell, which up to equivalence is independent of the choice of \( \rho \) and \( C \).

**Proof.** One checks \( \alpha_\rho \) satisfies the contact condition

\[
\alpha_\rho \wedge (d\alpha_\rho)^n > 0 \quad \text{whenever} \quad \partial_v \rho > 0
\]

and hence \( (B_{K,C}, \eta_{K,\rho}) \) is a contact shell by conditions (ii) and (iii).

Consider the special case of two choices \( \rho_0 \) and \( \rho_1 \) as in (3) for the same \( C \). We can pick a family of diffeomorphisms \( \phi_{(x,t)} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \) such that

\[
\phi_{(x,t)}(v) = (\rho_1^{-1} \circ \rho_0)_{(x,t)}(v) \quad \text{on} \quad \mathcal{O}_p \{ v = K(x, t) + C \} \cup \mathcal{O}_p \partial \Delta
\]
and this family induces a diffeomorphism \( \Phi : B_{K,C} \to B_{K,C} \) such that \( \Phi^* \alpha_{\rho_1} = \alpha_{\rho_1 \circ \phi} \) and \( \alpha_{\rho_1 \circ \phi} = \alpha_{\rho_0} \) on \( \mathcal{O} p \, \partial B_{K,C} \). It follows that \( \Phi \) is an equivalence between \( \eta_{K,\rho_0} \) and \( \eta_{K,\rho_1} \) since we have the straight line homotopy from \( \alpha_{\rho_1 \circ \phi} \) to \( \alpha_{\rho_0} \).

Given two choices \((C_0, \rho_0)\) and \((C_1, \rho_1)\), we can pick a family of diffeomorphisms \( \psi_{(x,t)} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \) such that

\[
\psi_{(x,t)}(v) = v + (C_1 - C_0) \quad \text{on} \quad \mathcal{O} p \{ v = C_0 + K(x,t) \}.
\]

Now to see \((B_{K,C_0}, \eta_{K,\rho_0})\) and \((B_{K,C_1}, \eta_{K,\rho_1})\) are equivalent, just note that \((B_{K,C_0}, \eta_{K,\rho_0})\) and \((B_{K,C_0}, \eta_{K,\rho_1 \circ \phi})\) are equivalent via the previous special case and \( \psi \) induces an isomorphism \( \Psi : (B_{K,C_0}, \alpha_{\rho_1 \circ \phi}) \to (B_{K,C_1}, \alpha_{\rho_1}) \).

We will use the notation \((B_{K,C}, \eta_{K,\rho})\) throughout the paper for this specific construction, though we will usually drop \( C \) and \( \rho \) from the notation and write \((B_K, \eta_K)\) when the particular choice will be irrelevant. We will refer to this contact shell as the circle model associated to \((\Delta, K)\).

**Remark 2.2.** This construction can produce a genuine contact structure on \(B_K\) if and only if \( K > 0 \). If \( K > 0 \), then from \([5]\) it follows \((B_{K,0}, \eta_{K,\rho})\) for \( \rho(v) = v \) is a contact ball. Conversely if \( K(x_0, t_0) \leq 0 \) for some \((x_0, t_0)\), then \((B_{K,C}, \eta_{K,\rho})\) is never contact by \([5]\) since since conditions (i) and (ii) on \( \rho \) force \( \partial_c \rho(x_0, t_0)(v_0) = 0 \) for some \( v_0 \).

The contact germ \((\partial B_K, \eta_K)\) without its almost contact extension can be described more directly in the following way. Consider the contact germs on the hypersurfaces

\[
\tilde{\Sigma}_{1,K} = \{ (x,v,t) : v = K(x,t) \} \subset (\Delta \times T^* S^1, \ker(\lambda_{st} + v \, dt)) \quad \text{and} \quad \tilde{\Sigma}_{2,K} = \{ (x,v,t) : 0 \leq v \leq K(x,t) , \ x \in \partial \Delta \} \subset (\Delta \times \mathbb{R}^2, \ker(\lambda_{st} + v \, dt)).
\]

These germs can be glued together via the natural identification between neighborhoods of their boundaries, to form a contact germ \( \tilde{\eta}_K \) on \( \tilde{\Sigma}_K := \tilde{\Sigma}_{1,K} \cup \tilde{\Sigma}_{2,K} \).

**Lemma 2.3.** The contact germs \((\partial B_K, \eta_K)\) and \((\tilde{\Sigma}_K, \tilde{\eta}_K)\) are contactomorphic.

**Proof.** We have that the boundary \( \partial B_{K,C} = \Sigma_{1,K,C} \cup \Sigma_{2,K,C} \) where

\[
\Sigma_{1,K,C} := \{ (x,v,t) \in \Delta \times \mathbb{R}^2 : v = K(x,t) + C \} \quad \text{and} \quad \Sigma_{2,K,C} := \{ (x,v,t) \in \Delta \times \mathbb{R}^2 : 0 \leq v \leq K(x,t) + C , \ x \in \partial \Delta \}.
\]

Recalling the 1-form \( \alpha_{\rho} = \lambda_{st} + \rho \, dt \) is a contact form near \( \partial B_{K,C} \subset \Delta \times \mathbb{R}^2 \), just note that \( \rho \) induces contactomorphisms of neighborhoods

\[
(\mathcal{O} p \, \Sigma_{j,K,C} , \ker(\alpha_{\rho})) \to (\mathcal{O} p \, \tilde{\Sigma}_{j,K} , \ker(\lambda_{st} + v \, dt))
\]

for \( j = 0, 1 \) by construction. \( \square \)
2.4 The cylindrical domain

Throughout the paper we will often use the following star-shaped cylindrical domain

\[ \Delta_{\text{cyl}} := D^{2n-2} \times [-1,1] = \{|z| \leq 1, \ u \leq 1\} \subset (\mathbb{R}^{2n-1}, \xi_{\text{st}}). \]

where \( D^{2n-2} := \{u = \sum_{i}^{n-1} u_i \leq 1\} \subset \mathbb{R}^{2n-2} \) is the unit ball.

Also observe for any contact Hamiltonian \((K, \Delta_{\text{cyl}})\) the north pole and south pole

\[ P_{\pm 1} := (0, \pm 1, 0) \in (\partial B_K, \eta_K) \]

in the coordinates \((u, z, v) \in \mathbb{R}^{2n-1} \times \mathbb{R}^2\) are gluing places in the sense of Section 2.2. When performing a boundary connected sum of such models \((B_K \# B_{K'}, \eta_K \# \eta_{K'})\) we will always use the north pole of \(B_K\) and the south pole of \(B_{K'}\). See Section 5.1 for more details on the gluing construction.

3 Proof of Theorems 1.1 and 1.2

3.1 Construction of contact structures with universal holes

Proposition 3.1, which we prove in Section 8.1, and which represents one half of the proof of Theorem 1.1, constructs from an almost contact structure a contact structure in the complement of a finite number of disjoint \((2n+1)\)-balls where the germ of the contact structure on the boundaries of the balls has a unique universal form:

Proposition 3.1. For fixed dimension \(2n+1\) there exists a contact Hamiltonian \((K_{\text{univ}}, \Delta_{\text{cyl}})\), specified in Lemma 8.7, such that the following holds. For any almost contact manifold \((M, \xi)\) as in Theorem 1.1 there exists an almost contact structure \(\xi'\) on \(M\), which is homotopic to \(\xi\) relative \(A\) through almost contact structures, and a finite collection of disjoint balls \(B_i \subset M \setminus A\) for \(i = 1, \ldots, L\), with piecewise smooth boundary such that

- \(\xi'\) is a genuine contact structure on \(M \setminus \bigcup_1^L \text{Int} B_i\)
- the contact shells \(\xi'|_{B_i}\) are equivalent to \((B_{K_{\text{univ}}}, \eta_{K_{\text{univ}}})\) for \(i = 1, \ldots, L\).

Remark 3.2. If \((B_K, \eta_K)\) is dominated by \((B_{K_{\text{univ}}}, \eta_{K_{\text{univ}}})\), then we can \(K\) in the statement of Proposition 3.1 in lieu of \(K_{\text{univ}}\). In particular by Lemma 4.7 in the
3-dimensional case we can take $K_{\text{univ}} : [-1, 1] \to \mathbb{R}$ to be any somewhere negative function. Our proof in higher dimension is not constructive, and we do not know an effective criterion which would allow one to verify whether a particular function $K_{\text{univ}}$ satisfies 3.1. Of course, it is easy to construct a 1-parameter family of Hamiltonians $K^\varepsilon$ so that any Hamiltonian $K$ is less than $K^\varepsilon$ for sufficiently small $\varepsilon > 0$ (see Example 3.4). We can then take $K_{\text{univ}} = K^\varepsilon$ for sufficiently small $\varepsilon$. It would be interesting to find such a general criterion for which Hamiltonians can be taken as $K_{\text{univ}}$.

3.2 Overtwisted discs and filling of universal holes

Proposition 3.9, which we formulate in this section and prove in Section 5.2.1, will combine with Proposition 3.1 to prove Theorem 1.1 in Section 3.3.

A smooth function $k : \mathbb{R} \geq 0 \to \mathbb{R}$ is called special if $k(1) > 0$ and

$$ak\left(\frac{1}{a}\right) < k(1) \text{ for all } a > 1 \text{ and } u \geq 0.$$ (6)

This implies that $k(0) < 0$, and hence $k(u)$ has a zero in $(0, 1)$. By differentiating (6) with respect to $a$ we conclude that

$$k(u) - uk'(u) < 0 \text{ for all } u \geq 0$$ (7)

which means that the $y$-intercept of all tangent lines to the graph of $k$ is negative.

We call a function $K : \Delta_{\text{cyl}} \to \mathbb{R}$ spherically symmetric if it depends only on the coordinates $(u, z)$ where $u = \sum_{i=1}^{n-1} u_i$, in this case by a slight abuse of notation will write $K(u, z)$, rather than $\tilde{K}(u, z)$ for some function $\tilde{K} : [0, 1] \times [-1, 1] \to \mathbb{R}$.

Definition 3.3. A spherically symmetric piecewise smooth contact Hamiltonian $K : \Delta_{\text{cyl}} \to \mathbb{R}$ is called special if

(i) $K > 0$ on $\partial \Delta_{\text{cyl}}$;
(ii) $K(u, -\delta) = K(u, \delta)$ for $\delta \in \mathcal{O}_p \{1\}$ and $u \in [0, 1]$;
(iii) If $\delta \in \mathcal{O}_p \{1\}$, then $K(u, z) \leq K(u, \delta)$ for $u \in [0, 1]$ and $|z| \leq \delta$;

and there exist $z_D \in (-1, 1)$ and a special function $k : \mathbb{R} \geq 0 \to \mathbb{R}$ such that

(iv) $K$ is non-increasing in the coordinate $z$ for $z \in \mathcal{O}_p [-1, z_D] \subset [-1, 1]$;
(v) $K(u, z) \geq k(u)$ with equality if $z \in \mathcal{O}_p \{z_D\}$.

When $n = 1$, where $\Delta_{\text{cyl}} = [-1, 1]$, condition (v) can be replaced with $K(z_D) < 0$.

As the following example shows special contact Hamiltonians exist and furthermore for any particular contact Hamiltonian $(K', \Delta_{\text{cyl}})$ positive on $\partial \Delta_{\text{cyl}} \times S^1$ there is a special contact Hamiltonian $K : \Delta_{\text{cyl}} \to \mathbb{R}$ such that $K < K'$.

**Example 3.4.** For positive constants $a, b, \lambda$ with $b < 1$ and $\lambda > \frac{a}{1-b}$, define the special piecewise smooth function

$$k(u) = \begin{cases} 
\lambda(u - b) - a & \text{if } u \geq b \\
-a & \text{if } u \leq b
\end{cases}$$

and the special piecewise smooth contact Hamiltonian

$$K(u, z) = \max\{k(u), k(|z|)\}.$$  

By a perturbation of $K$ near its singular set, we may construct a smooth special contact Hamiltonian $\tilde{K}$ that is $C^0$-close to $K$, though smoothness of $K$ will not be needed in the proof.

Let $K : \Delta_{\text{cyl}} \to \mathbb{R}$ be a special contact Hamiltonian and define $(D_K, \eta_K)$ to be the contact germ on the 2n-dimensional disc

$$D_K := \{(x, v, t) \in \partial B_K : z(x) \in [-1, z_D]\} \subset (B_K, \eta_K)$$  

where $z_D$ is the constant in Definition 3.3. Notice that $D_K$ inherits the south pole of the corresponding circular model and the coorientation of $\partial B_K$ as a boundary.

**Definition 3.5.** Let $K_{\text{univ}}$ be as in Proposition 3.1. An overtwisted disc $(D_{\text{ot}}, \eta_{\text{ot}})$ is a 2n-dimensional disc with a germ of a contact structure such that there is a contactomorphism

$$(D_{\text{ot}}, \eta_{\text{ot}}) \cong (D_K, \eta_K)$$

where $K$ is some special contact Hamiltonian such that $K < K_{\text{univ}}$. A contact manifold $(M^{2n+1}, \xi)$ is overtwisted if it admits a contact embedding $(D_{\text{ot}}, \eta_{\text{ot}}) \to (M, \xi)$ of some overtwisted disc.

**Example 3.6.** In the 3-dimensional case, it follows from Lemma 4.7 that the disc

$$D_K := \{(z, v, t) \in \partial B_K : z \in [-1, z_D]\} \subset (B_K, \eta_K)$$

is overtwisted in the sense of Definition 3.5 for any special contact Hamiltonian, i.e. a somewhere negative function on the interval $[-1, 1]$, positive near the end-points $\pm 1$. 

Proof of Theorems 1.1 and 1.2

Fig. 3.1: A 2-dimensional overtwisted disc \((D_{ot}, \eta_{ot})\) with its characteristic foliation.

**Remark 3.7.** The definition of the overtwisted disc \((D_K, \eta_K)\) depends on the choice of a special Hamiltonian \(K < K_{univ}\), and the germs \(\eta_K\) need not to be contactomorphic when we vary \(K\). However, as Corollary 1.4 shows, for any two special Hamiltonians \(K, K' < K_{univ}\) any neighborhood of \((D_K, \eta_K)\) contains \((D_{K'}, \eta_{K'})\).

As the following proposition shows any overtwisted contact manifold contains infinitely many disjoint overtwisted discs.

**Proposition 3.8.** Every neighborhood of an overtwisted disc in a contact manifold contains a foliation by overtwisted discs.

We prove Proposition 3.8 at the end of Section 4.2.

Given a special contact Hamiltonian \(K : \Delta_{cyl} \to \mathbb{R}\), the contact germ \((D_K, \eta_K)\) has the following remarkable property, which we will prove in Section 5.2.2. Suppose one has a \((2n + 1)\)-dimensional contact ball \((B, \xi)\) with piecewise smooth boundary such that \((D_K, \eta_K) \subset (\partial B, \xi)\) where the co-orientation of \(D_K\) coincides with the outward co-orientation of \(\partial B\).

**Proposition 3.9.** Let \(K_0, K, K_0 \geq K\), be two contact Hamiltonians and \(K\) is special. Then the contact shell \((B_{K_0} \# B, \eta_{K_0} \# \xi)\) given by performing a boundary connected sum at the north pole of \(B_{K_0}\) and the south pole of \(D_K \subset \partial B\) is equivalent to a genuine contact structure.

### 3.3 Proof of Theorem 1.1

Choose a ball \(B \subset M \setminus A\) with piecewise smooth boundary and deform the almost contact structure \(\xi\) to make it a contact structure on \(B\) with an overtwisted disc.
Proof of Theorems 1.1 and 1.2

\((D_{\text{ot}}, \eta_{\text{ot}}) \subset (\partial B, \xi)\) on its boundary. This can be done since any two almost contact structures on the ball are homotopic if we do not require the homotopy to be fixed on \(\partial B\).

Using Proposition 3.1 we deform the almost contact structure \(\xi\) relative to \(A \cup B\) to an almost contact structure \(\xi\) on \(M\), which is genuine in the complement of finitely many disjoint balls \(B_1, \ldots, B_N \subset M \setminus (A \cup B)\) where each \((B_i, \xi|_{B_i})\) is isomorphic to \((B_{\text{Kuniv}}, \eta_{\text{Kuniv}})\) as almost contact structures.

According to Proposition 3.8 we can pick disjoint balls \(B'_i \subset \text{Int}(B), i = 1, \ldots, N\), each with an overtwisted disc on their boundary \((D_{\text{ot}}', \eta_{\text{ot}}') \subset (\partial B'_i, \xi)\). As we will describe in Section 5.1 we can perform an ambient boundary connected sum \(B_i \# B'_i \subset M \setminus A\) such that the sets \(B_i \# B'_i\) are disjoint for \(i = 1, \ldots, N\) and there are isomorphisms of almost contact structures

\[(B_i \# B'_i, \xi|_{B_i \# B'_i}) \cong (B_i \# B'_i, \eta_{\text{Kuniv}} |_{B_i \# B'_i}).\]

Now for \(i = 1, \ldots, L\) by definition we have \((D_{\text{ot}}', \eta_{\text{ot}}') = (D_{K_i}, \eta_{K_i})\) for special contact Hamiltonians \(K_i\) such that \(K_i < K_{\text{univ}}\). Therefore we can apply Proposition 3.9 to homotope \(\xi|_{B_i \# B'_i}\) relative to the boundary to a genuine contact structure on \(B_i \# B'_i\) for each \(i = 1, \ldots, N\). The result will be a contact structure on \(M\) that is homotopic relative to \(A\) to the original almost contact structure.

### 3.4 Fibered structures

To prove the parametric version of the Theorem 1.1 we need to discuss the parametric form of the introduced above notions. The parameter space, always denoted by \(T\), will be assumed to be a compact manifold of dimension \(q\), possibly with boundary, and we will use the letter \(\tau\) for points in \(T\).

A family of (almost) contact structures \(\{\xi_{\tau}\}_{\tau \in T}\) on a manifold \(M\) can be equivalently viewed as a fiberwise, or as we also say fibered (almost) contact structure \(T\xi\) on the total space of the trivial fibration \(T^\tau M := T \times M \to T\), which coincides with \(\xi_{\tau}\) on the fiber \(M^\tau := \tau \times M, \tau \in T\).

A fibered contact shell \((T^\tau M, T\xi)\) is a fibered almost contact structure that is genuine on \(\text{Op}(\partial(T^\tau M))\), by which we mean \((M^\tau, \xi^\tau)\) is genuine for all \(\tau \in \text{Op}(\partial T)\) and \((\text{Op}(M^\tau), \xi^\tau)\) is genuine for all \(\tau \in T\). An equivalence between fibered contact shells

\[G : (T_1 B_1, T_1 \xi_1) \to (T_2 B_2, T_2 \xi_2)\]

is a diffeomorphism covering a diffeomorphism \(g : T_1 \to T_2\) such that \(G^*(T_2 \xi_2)\) and \(T_1 \xi_1\) are homotopic relative to \(\text{Op}(\partial(T_1 B_1))\) through fibered almost contact structures.
on $T_1 B_1$. In particular this requires $G : (B^1_{\tau}, \xi^1_{\tau}) \rightarrow (B^2_{\tau}, \xi^2_{\tau})$ to be an equivalence of contact shells for all $\tau \in T_1$ and to be a contactomorphism when $\tau \in \mathcal{O} p \partial T_1$.

Given fibered contact shells $T^\pm \zeta^\pm = (T^\pm B^\pm, T^\pm \xi^\pm)$ we say $T^+ \zeta^+ \text{ dominates } T^- \zeta^-$ if there is a third fibered contact shell $\zeta = (T B, T \xi)$ such that

- there is a fibered equivalence $G : T \zeta \rightarrow T^+ \zeta^+$ and
- a fiberwise embedding $H : T^- B^- \rightarrow T B$ covering an embedding $h : T_- \rightarrow T$ such that $H^*(T \zeta) = T^- \zeta^-$ and $T^\pm \xi$ is genuine on $T B \setminus H(\text{Int} T^- B^-)$.

We will refer to the embedding $G \circ H : (T^- B^- , T^- \xi^-) \rightarrow (T^+ B^+, T^+ \xi^+)$ as a subordination map.

Finally we note that the boundary connected sum construction can be performed in the fibered set-up to define a fibered connected sum

$$(T B_1 \#^T B_2, T \xi_1 \#^T \xi_2)$$

with fibers $(B^1_{\tau} \# B^2_{\tau}, \xi^1_{\tau} \# \xi^2_{\tau})$ provided that we are given a family of boundary points $p^1_{\tau} \in \partial B^1_{\tau}$ and $p^2_{\tau} \in \partial B^2_{\tau}$ as in the non-parametric case.

### 3.5 Parametric contact structures with universal holes

Given a special contact Hamiltonian $K : \Delta_{cyl} \rightarrow \mathbb{R}$ we define a function $E : \mathbb{R} \rightarrow \mathbb{R}$ by the formula $E(u, z) := K(u, 1)$. By assumption, we have $K \leq E$ on $\Delta_{cyl}$. We further define a family of contact Hamiltonians $K^{(s)} : \Delta_{cyl} \rightarrow \mathbb{R}$ by

$$K^{(s)} := sK + (1 - s)E \quad \text{for } s \in [0, 1].$$

Given a disc $T := D^q \subset \mathbb{R}^q$ pick a bump function $\delta : T \rightarrow [0, 1]$ with support in the interior of $T$ and consider the family of contact Hamiltonians $K^{(\delta(\tau))} : \Delta_{cyl} \rightarrow \mathbb{R}$ parametrized by $\tau \in T$ and the fibered circular model shell over $T$

$$\left( T B_K, \tau \eta_K \right) \quad \text{where } T B_K = \bigcup_{\tau \in T} \{ \tau \} \times B_{K^{(\delta(\tau))}}$$

and the fiber over $\tau \in T$ is given by $(B_{K^{(\delta(\tau))}}, \eta_{K^{(\delta(\tau))}})$.

Recall Proposition [3.1] and its contact Hamiltonian $K_{\text{univ}} : \Delta_{cyl} \rightarrow \mathbb{R}$. The next proposition, which we prove in Section [8.2], is the parametric generalization of Proposition [3.1] and says that any fibered almost contact structure is equivalent to a fibered almost contact structure that is genuine away from holes modeled on $(T B_{K_{\text{univ}}}, \tau \eta_{K_{\text{univ}}})$. 

Proposition 3.10. Let $T = D^q$ and let $A \subset M$ be a closed subset. Every fibered almost contact structure $T^\xi_0$ on $TM = T \times M$ that is genuine on $(T \times \mathcal{O}p A) \cup \mathcal{O}p \partial T \times M$ is homotopic relative to $(T \times A) \cup (\partial T \times M)$ through fibered almost contact structures on $TM$ to some structure $T^\xi$ with the following property:

There is a collection of disjoint embedded fibered shells $T^i B_i \subset TM$ over (not necessarily disjoint) $q$-dimensional discs $T_i \subset T$ for $i = 1, \ldots, L$ such that

(i) the fibers of $T^\xi$ are genuine contact structures away from $\bigcup_{i=1}^L \text{Int}(T^i B_i)$ and

(ii) the fibered contact shells $(T^i B_i, T^i \xi)$ and $(T^i B_{K_{\text{univ}}}, T^i \eta_{K_{\text{univ}}})$ are equivalent.

Furthermore for every $C \subset \{1, \ldots, L\}$ the intersection $\bigcap_{i \in C} T_i$ either empty or a disc.

Recall the setting of Proposition 3.9, where $(B, \xi)$ is a $(2n+1)$-dimensional contact ball for which there is a special contact Hamiltonian $K : \Delta_{cyl} \to \mathbb{R}$ such that $(D_K, \eta_K) \subset (\partial B, \xi)$ where the co-orientation of $D_K$ coincides with the outward co-orientation of $\partial B$. The following proposition, which we prove in Section 5.2.2, is the parametric generalization of Proposition 3.9 where $(T^B, T^\xi)$ is the fibered contact structure $T \times (B, \xi)$.

Proposition 3.11. Let $(K_0, \Delta_{cyl})$ be a contact Hamiltonian and consider the fibered contact shell

$$(T^B_{K_0} \# T^B, T^\eta_{K_0} \# T^\xi)$$

given by performing a boundary connected sum on each fiber over $\tau \in T$ at the north pole of $B_{K_0}^{(\delta(\tau))}$ and the south pole of $D_K \subset \partial B$. If $K \leq K_0$ is special, then $(T^B_{K_0} \# T^B, T^\eta_{K_0} \# T^\xi)$ is fibered equivalent to a genuine fibered contact structure.

3.6 Proof of Theorem 1.2 and Corollary 1.4

Theorem 1.2 is an immediate corollary of the following theorem, which is a fibered version of Theorem 1.1. In particular for each $q \geq 0$, we see that

$$j_* : \pi_q(\text{Cont}_{\text{tot}}(M; A, \xi_0, \phi)) \to \pi_q(\text{Cont}_{\text{tot}}(M; A, \xi_0, \phi))$$

is an isomorphism by applying Theorem 3.12 in the cases of $D^q$ and $D^{q+1}$.

Theorem 3.12. Let $T = D^q$ and $A \subset M$ be a closed subset such that $M \setminus A$ is connected, and let $T^\xi$ be a fibered almost contact structure on $TM$ which is genuine on $(T \times \mathcal{O}p A) \cup (\partial T \times M)$. If there exists a fixed overtwisted disc $(D_{\text{ot}}, \eta_{\text{ot}}) \subset M \setminus A$
such that for all $\tau \in T$ the inclusion $(D_{\text{ot}}, \eta_{\text{ot}}) \subset (M \setminus A, \xi^\tau)$ is a contact embedding, then $^T \xi$ is homotopic to a fibered genuine contact structure through fibered almost contact structures fixed on $(T \times (A \cup D_{\text{ot}})) \cup (\partial T \times M)$

Proof of Theorem 3.12: By assumption there is a piecewise smooth disc $D_{\text{ot}} \subset M \setminus A$ such that all almost contact structures $\xi^\tau$ for $\tau \in T$ are genuine on $Op \; D_{\text{ot}}$ and restrict to $D_{\text{ot}}$ as $\eta_{\text{ot}}$. Since $(D_{\text{ot}}, \eta_{\text{ot}})$ determines the germ of the contact structure we may pick a ball $B \subset Op \; D_{\text{ot}}$ with $D_{\text{ot}} \subset \partial B$ and assume $(^TB, ^T \xi) = T \times (B, \xi)$.

By applying Proposition 3.10 we may assume there is a collection of disjoint fibered balls $T_i B_i \subset M \setminus (A \cup B)$ over a collection of discs $T_i \subset T$ for $i = 1, \ldots, L$ such that

(i) $^T \xi$ is genuine away from $\bigcup_{i=1}^{L} \text{Int} (T_i B_i)$ and

(ii) the fibered shells $(T_i B_i, ^T \xi)$ and $(T_i B_{\text{Kuniv}}, ^T \eta_{\text{Kuniv}})$ are equivalent.

Apply Proposition 3.8 to get $L$ disjoint balls $B_i' \subset \text{Int} (B \setminus (D_{\text{ot}} \cup A))$ with an overtwisted disc $(D_{\text{ot}}', \eta_{\text{ot}}) \subset (\partial B_i', \xi)$ in each of them.

It follows from Lemma 9.1, proven in Section 9 below, that for each $j$ we can find a parametric family of embedded paths $T_j \gamma_j$ connecting $T_j B_j$ to $T_j B_j'$ in $T \times (M \setminus A \cup D_{\text{ot}})$. Moreover, using Gromov’s parametric $h$-principle for transverse paths, see [30], we can assume the constructed paths are transverse.

As we explain in Section 5.1, with these parametric paths we can form disjoint parametric ambient boundary connected sums $T_i C_j \subset T_i (M \setminus (A \cup D_{\text{ot}}))$ for each $j = 1, \ldots, L$, between the fibered shells $(T_j B_j, ^T \xi)$ and $(T_j B_{\text{Kuniv}}, ^T \eta_{\text{Kuniv}})$. Furthermore, by Section 5.1 and property (ii) above we have isomorphisms of fibered almost contact structures

$$(^T C_j, ^T \xi) \cong (T_i B_{\text{Kuniv}} \# T_j B_j', ^T \eta_{\text{Kuniv}} \# ^T \xi).$$

Applying Proposition 3.11 inductively for $j = 1, \ldots, L$ we deform $^T \xi$ on these connected sums relative to their boundary to get a fibered genuine contact structure on $^T M$.

Proof of Corollary 1.4: By an isotopy of $f$ we can arrange that the complement $M \setminus f(N)$ is overtwisted and the closure $f(N)$ is compact. Then, slightly reducing if necessary the manifold $M$, we can assume it non-compact and overtwisted at infinity. Let us exhaust $N$ by compact subsets: $N = \bigcup_{j} C_j$, such that $C_j \subset \text{Int} C_{j+1}$.

\end{proof}
and $V \setminus C_j$ is connected, $j = 1, \ldots$. Set $C_0 := \emptyset$. The result follows by induction from the following claim:

Suppose we are given an embedding $f^{j-1} : N \to M$ which is contact on $O_p C_{j-1}$ and a homotopy of bundle isomorphisms $\Phi_t^{-1} : TN \to TM$ covering $f^{j-1}$ such that the following property $P^{j-1}$ is satisfied:

$$P^{j-1} : \text{The homotopy } \Phi_t^{-1} \text{ is contact on } T(N)|_{O_p C_{j-1}} \text{ for all } t \in [0,1], \Phi_0^{-1} \text{ is contact everywhere, and } \Phi_t^{-1} = df^{j-1}. $$

Then there exists a pair $(f^j, \Phi_t^j)$ which satisfies $P^j$ and such that $f^{j-1}$ and $f^j$ are isotopic via an isotopy fixed on $C_{j-1}$.

Let $\xi_t$ be a family of almost contact structures on $M$ such that $\xi_t = (\Phi_t^{-1})_* \zeta$ on $f^{j-1}(C_j)$ and $\xi_t = \xi$ outside $f^{j-1}(C_{j+1})$. We note that $\xi_0 = \xi$ on $f^{j-1}(C_j)$, and $\xi_t = \xi$ on $f^{j-1}(C_{j-1})$ for all $t \in [0,1]$. Theorem 1.2 allows us to construct a compactly supported homotopy $\tilde{\xi}_t$ of genuine contact structures on $M$, $t \in [0,1]$, connecting $\tilde{\xi}_0 = \xi$ and a contact structure $\tilde{\xi}_1$ which coincides with $\xi_1$ on $f^{j-1}(C_j)$. Moreover, this can be done to ensure existence of a homotopy $\Psi_t : TM \to TM$ of bundle isomorphisms such that $\Psi_0 = \text{Id}$, $\Psi_0^* \tilde{\xi}_t = \xi_t$, and $\Psi_t|_{f^{j-1}(C_{j+1})} = \text{Id}$, $t \in [0,1]$. Then Gray’s theorem [27] provides us with a compactly supported diffeotopy $\phi_t : M \to M$, $t \in [0,1]$, such that $\phi_0 = \text{Id}$, $\phi_t^* \xi_t = \tilde{\xi}_t$ and $\phi_t|_{f^{j-1}(C_{j+1})} = \text{Id}$. Set $f^j := \phi_1 \circ f^{j-1}$ and $\Phi_t^j := df^j \circ \Psi_t^j \circ \Phi_t^{j-1}$, $t \in [0,1]$. Then $\Phi_t^j = df^j$, $(\Phi_t^j)*\xi = (\Phi_t^{j-1})* \circ (\Psi_t) \circ (df_t)*\xi = (\Phi_t^{j-1})* \circ \Psi_t^* \xi_t = (\Phi_t^{j-1})* \xi_t$. Hence, $(\Phi_t^j)*\xi|_{C_j} = \zeta$ for all $t \in [0,1]$. We also have $(\Psi_0^*\xi)t = \zeta$ everywhere. Thus, the pair $(f^j, \Phi_t^j)$ satisfies $P^j$, and the claim follows by induction.

\[\Box\]

4 Domination and conjugation for Hamiltonian contact shells

Recall the notation $(K, \Delta)$ to refer to a contact Hamiltonian $K$ on a star-shaped domain $\Delta \subset (\mathbb{R}^{2n-1}, \xi_\star)$ such that $K|_{\partial \Delta \times S^1} > 0$ as in [1].

In this section we will develop two properties of Hamiltonian contact shells that make them well-suited for the purposes of this paper. Namely in Section 4.1 we show that a natural partial order $(K, \Delta) \leq (K', \Delta')$ is compatible with the partial order on contact shells given by domination, and in Section 4.2 we show the action of $\text{Cont}(\Delta)$ on a contact Hamiltonian $(K, \Delta)$ by conjugation preserves the equivalence class of the associated contact shell.

A simple, but very important observation is then made in Section 4.3 where we
show how conjugation can be used to make some contact Hamiltonians $(K, \Delta)$ much smaller with respect to the partial order. For instance in the 3-dimensional case where $\Delta \subset \mathbb{R}$ is an interval, we prove that up to conjugation $K : \Delta \to \mathbb{R}$ is a minimal element for the partial order if $K$ is somewhere negative. In higher dimensions, the existence of a minimal element up to conjugation seems to no longer be true, but the weaker Propositions 4.8 and 4.9 hold in general and they suffice for our purposes.

4.1 A partial order on contact Hamiltonians with domains

Let us introduce a partial order on contact Hamiltonians with domains where

$$(K, \Delta) \leq (K', \Delta')$$

is defined to mean $\Delta \subset \Delta'$ together with

$$K(x, t) \leq K'(x, t) \quad \text{for all } x \in \Delta \quad \text{and} \quad (11)$$

$$0 < K'(x, t) \quad \text{for all } x \in \Delta' \setminus \Delta. \quad (12)$$

This partial order is compatible with domination of contact shells.

**Lemma 4.1.** If $(K, \Delta) \leq (K', \Delta')$, then $(B_K, \eta_K)$ is dominated by $(B_{K'}, \eta_{K'})$. More specifically, given a contact shell $(B_{K,C}, \eta_{K,\rho})$ there is a shell $(B_{K',C'}, \eta_{K',\rho'})$ such that

$$(B_{K,C}, \eta_{K,\rho}) \subset (B_{K',C'}, \eta_{K',\rho'})$$

the inclusion is a subordination map.

**Proof.** If $C' \geq C$, then by (11) we have $(B_{K,C}, \eta_{K,\rho}) \subset (B_{K',C'}, \eta_{K',\rho'})$ and it will be an embedding of almost contact structures whenever

$$\rho' = \rho \quad \text{on} \quad \mathcal{O}_p B_{K,C} \subset B_{K',C'}.$$

If we pick the extension so that

$$\partial_v \rho'_{(x,t)}(v) > 0 \quad \text{on} \quad \mathcal{O}_p \{x \in \Delta, v \geq K(x, t) + C\} \cup \mathcal{O}_p \{x \in \Delta' \setminus \text{Int } \Delta\},$$

which is possible on the latter region by (12), it follows that $\eta_{K',\rho'}$ is contact on $\mathcal{O}_p (B_{K',C'} \setminus \text{Int } B_{K,C})$ and hence the inclusion is a subordination map. \qed
4 Domination and conjugation for Hamiltonian contact shells

4.2 Conjugation of contact Hamiltonians

Given a contact manifold \((M,\alpha)\) and a contact Hamiltonian \(K : M \times S^1 \to \mathbb{R}\) let \(\{\phi_K^t\}_{t \in [0,1]}\) be the unique contact isotopy with \(\phi_K^0 = 1\) and

\[
\alpha(\partial_t \phi_K^t(x)) = K(\phi_K^t(x), t).
\]

For a contactomorphism \(\Phi : (M,\alpha) \to (M',\alpha')\) define the push-forward Hamiltonian

\[
\Phi_* K : M' \times S^1 \to \mathbb{R} \quad \text{by} \quad (\Phi_* K)(\Phi(x), t) = c_\Phi(x) K(x, t).
\]

where \(c_\Phi : M \to \mathbb{R}_{>0}\) satisfies \(\Phi^* \alpha' = c_\Phi \alpha\). One can verify

\[
\{\Phi \phi_K^t \Phi^{-1}\}_{t \in [0,1]} = \{\phi_{\Phi_* K}^t\}_{t \in [0,1]}
\]

so \(\Phi_*\) corresponds with conjugating by \(\Phi\).

In this paper we will primarily be concerned with contactomorphisms \(\Phi : \Delta \to \Delta'\) between star-shaped domains in \((\mathbb{R}^{2n-1}, \xi_{st})\) where \(c_\Phi : \Delta \to \mathbb{R}_{>0}\) is defined by

\[
\Phi^* \lambda_{st} = c_\Phi \lambda_{st}.
\]

It is clear that if \((K, \Delta)\) satisfies \([1]\), then \((\Phi_* K, \Delta')\) does as well. As the next lemma shows the push-forward operation induces an equivalence of contact shells.

**Lemma 4.2.** A contactomorphism between star-shaped domains \(\Phi : \Delta \to \Delta'\) in \((\mathbb{R}^{2n-1}, \xi_{st})\) induces an equivalence of the contact shells

\[
\hat{\Phi} : (B_K, \eta_K) \to (B_{\Phi_* K}, \eta_{\Phi_* K})
\]

defined by \((K, \Delta)\) and \((\Phi_* K, \Delta')\).

**Proof.** For a given model \((B_{K,C}, \eta_{K,C})\) we will build a model \((B_{\Phi_* K,C}, \eta_{\Phi_* K,C})\) such that the two models are isomorphic as almost contact structures.

For \(\tilde{C} + \min(\Phi_* K) > 0\), pick a family of diffeomorphisms for \((x, t) \in \Delta \times S^1\)

\[
\phi_{(x,t)} : [0, K(x, t) + C] \to [0, c_\Phi(x) K(x, t) + \tilde{C}]
\]

and define a smooth family of functions for \((x, t) \in \Delta \times S^1\)

\[
\tilde{\rho}_{(\phi_{(x,t)}, t)} : [0, c_\Phi(x) K(x, t) + \tilde{C}] \to \mathbb{R} \quad \text{by} \quad \tilde{\rho}_{(\phi_{(x,t)}, t)}(v) = c_\Phi(x) \rho_{(x,t)}(\phi_{(x,t)}^{-1}(v)).
\]
One see one \( \tilde{\rho} \) satisfies the conditions in (3) to define \((B_{\Phi,K,\tilde{C}},\eta_{\Phi,K,\tilde{\rho}})\) provided
\[
\phi_{(x,t)}(v) = c_{\Phi}(x)(v - C) + \tilde{C} \quad \text{for} \quad (x,t,v) \in \mathcal{O}_p \{v = K(x,t) + C\}.
\]

It follows by construction that the diffeomorphism
\[
\hat{\Phi} : (B_{K,C,\eta_{K,C}}) \to (B_{\Phi,K,\tilde{C}},\eta_{\Phi,K,\tilde{\rho}}) \quad \text{defined by} \quad \hat{\Phi}(x,v,t) = (\Phi(x),\phi_{(x,t)}(v),t)
\]
is an isomorphism of almost contact structures.

### 4.2.1 Foliations of overtwisted discs

For a first example of this push-forward procedure, we will prove Proposition 3.8 as a corollary of Lemma 4.2 above and Lemma 4.3 below. For \( \delta \in \mathcal{O}_p \{1\} \) observe the contactomorphism \( C_{\delta} : \mathbb{R}^{2n-1} \to \mathbb{R}^{2n-1} \) given by
\[
C_{\delta}(u_1, \ldots, u_{n-1}, \phi_1, \ldots, \phi_{n-1}, z) = \left(\frac{u_1}{\delta}, \ldots, \frac{u_{n-1}}{\delta}, \phi_1, \ldots, \phi_{n-1}, \frac{z}{\delta}\right)
\]
maps \( C_{\delta}(\Delta_{\delta}) = \Delta_{\text{cyl}} \) where \( \Delta_{\delta} := \{|z| \leq \delta, u \leq \delta\} \).

**Lemma 4.3.** Let \( K : \Delta_{\text{cyl}} \to \mathbb{R} \) a special contact Hamiltonian and define
\[
K_{\delta} : \Delta_{\delta} \to \mathbb{R} \quad \text{by} \quad K_{\delta} := K + (\delta - 1).
\]
If \( \delta < 1 \) is sufficiently close to 1, then \( \tilde{K}_{\delta} := (C_{\delta})_* K_{\delta} : \Delta_{\text{cyl}} \to \mathbb{R} \) is also a special contact Hamiltonian.

**Proof.** Provided \( \delta \) is sufficiently close to 1, it is clear that
\[
\tilde{K}_{\delta} := (C_{\delta})_* K_{\delta} = \frac{K \circ C_{\delta}^{-1}}{\delta} + \frac{\delta - 1}{\delta}
\]
satisfies items (i)-(iv) in Definition 3.3 with \( \tilde{z}_D = z_D/\delta \). If \( k : \mathbb{R}_{\geq 0} \to \mathbb{R} \) is the special function for \( K \), then let \( \tilde{k}_\delta(u) := \frac{k(\delta u)}{\delta} + \frac{\delta - 1}{\delta} \). Using that \( k \) is special, computing for \( a > 1 \) and recalling \( \delta < 1 \), we see
\[
a \tilde{k}_\delta(u/a) - \tilde{k}_\delta(u) < (a - 1)\frac{\delta - 1}{\delta} < 0
\]
so therefore \( \tilde{k}_\delta \) is special provided \( \delta \) is close enough to 1 so that \( \tilde{k}_\delta(1) > 0 \). Therefore item (v) in Definition 3.3 holds for \( \tilde{K}_{\delta} \) with respect to \( \tilde{k}_\delta \). \( \square \)
Proof of Proposition 3.8. Consider an overtwisted disc \((D_K, \eta_K)\) defined by a special contact Hamiltonian \(K : \Delta_{cyl} \to \mathbb{R}\). For \(\delta \in [1 - \varepsilon, 1]\), let \(\Delta_\delta = \{|z| \leq \delta, u \leq \delta\}\) and consider the family of contact Hamiltonians
\[
K_\delta : \Delta_\delta \to \mathbb{R} \quad \text{where} \quad K_\delta := K + (\delta - 1).
\]
Observe any neighborhood of \((\partial B_K, \eta_K)\) contains a foliation \(\{(\partial B_{K_\delta}, \eta_{K_\delta})\}_{\delta \in [1 - \varepsilon, 1]}\) provided \(\varepsilon > 0\) is small enough.

Furthermore, when \(\varepsilon > 0\) is sufficiently small, Lemmas 4.2 and 4.3 give us a family of special contact Hamiltonians \(\tilde{K}_\delta : \Delta_{cyl} \to \mathbb{R}\) such that \(\tilde{K}_\delta < K_{\text{univ}}\) together with contactomorphisms \((\partial B_{K_\delta}, \eta_{K_\delta}) \cong (\partial B_{\tilde{K}_\delta}, \eta_{\tilde{K}_\delta})\).

Therefore every neighborhood of \((D_K, \eta_K)\) contains a foliation \(\{(D_{K_\delta}, \eta_{K_\delta})\}_{\delta \in [1 - \varepsilon, 1]}\) of overtwisted discs. \(\square\)

4.2.2 Embeddings of contact Hamiltonian shells

As a second application of the push-forward procedure, we have the following lemma about embeddings of contact Hamiltonian shells.

Lemma 4.4. Let \((B_{K,C}, \eta_{K,\rho})\) be a contact shell structure for \((K, \Delta)\). For any other \((K', \Delta')\) there exists a contact shell structure \((B_{K',C'}, \eta_{K',\rho'})\) together with an embedding of almost contact structures
\[
(B_{K,C}, \eta_{K,\rho}) \to (B_{K',C'}, \eta_{K',\rho'}).
\]
If \(\Delta \subset \text{Int} \Delta'\), then the embedding can be taken to be an inclusion map.

Proof. Since \(\Delta'\) is star-shaped, there is a contactomorphism \(\Phi \in \text{Cont}_0^c(\mathbb{R}^{2n-1})\) such that \(\Delta \subset \text{Int} \Phi(\Delta')\) and therefore by Lemma 4.2 without loss of generality we may assume \(\Delta \subset \text{Int} \Delta'\).

Given the contact shell structure \((B_{K,C}, \eta_{K,\rho})\), pick any contact shell \((B_{K',C'}, \eta_{K',\rho'})\) subject to the additional conditions that
\[
K'(x, t) + C' > K(x, t) + C \quad \text{for all} \quad (x, t) \in \Delta \times S^1
\]
and the smooth family of functions \(\rho_{(x,t)} : \mathbb{R}_{\geq 0} \to \mathbb{R}\) for \((x, t) \in \Delta' \times S^1\) satisfy
\[
\rho' = \rho \quad \text{on} \quad \text{Op} B_{K,C} \subset B_{K',C'},
\]
where the latter is always possible since $\Delta \subset \text{Int}(\Delta')$. By (14) we have an inclusion
\[(B_{K,C},\eta_{K,C}) \subset (B_{K',C'},\eta_{K',C'})\] (16)
and by (15) it is an embedding of almost contact structures.

\textbf{Remark 4.5.} If the inclusion (16) was a subordination map, then
\[\partial_v \rho'_{(x,t)}(v) > 0 \quad \text{for all } (x,v,t) \in \mathcal{O}_p \{ x \in \Delta, v \geq K(x,t) + C \}\]
which together with (14) and (15) imply $K'(x,t) > K(x,t)$ for all $x \in \Delta$ since
\[K'(x,t) - K(x,t) = \rho'_{(x,t)}(K'(x,t) + C') - \rho'_{(x,t)}(K(x,t) + C) > 0.\]
A similar argument shows why assuming $\Delta \subset \Delta'$ is not sufficient, since the conditions (14), (15), and $\partial_v \rho' > 0$ on $\partial \Delta'$ imply $K'(x,t) > K(x,t)$ for all $x \in \partial \Delta \cap \partial \Delta'$.

\subsection{Changing the contactomorphism type of the domain}

Recall that star-shaped domains $\Delta \subset (\mathbb{R}^{2n-1},\xi_{st})$ are ones for which the contact vector field $Z = z \frac{\partial}{\partial z} + u \frac{\partial}{\partial u}$ is transverse to $\partial \Delta$ and we denote the flow of a vector field $X$ by $X^t$. While not all star-shaped domains are contactomorphic, up to mutual domination of contact shells $(B_K,\eta_K)$ the choice of domain does not matter.

\textbf{Lemma 4.6.} For any contact Hamiltonian $(K,\Delta)$ and star-shaped domain $\Delta'$ there is a contact Hamiltonian $(K',\Delta')$ such that $(B_K,\eta_K)$ dominates $(B_{K'},\eta_{K'}).$

\textbf{Proof.} For any neighborhood $U \supset \partial \Delta$ there is a contactomorphism $\Phi \in \text{Cont}_0^{c}(\mathbb{R}^{2n-1})$ such that $\Phi(\Delta') \subset \Delta$ and $\Phi(\partial \Delta') \subset U$. To see this first note without loss of generality we may assume $\Delta' \subset \Delta$ by replacing $\Delta'$ with $Z^{-N}(\Delta')$ for $N$ sufficiently large. After this reduction, the required contactomorphism is given by $\tilde{Z}^T$ for $T$ sufficiently large, where $\tilde{Z}$ is a contact vector field with $\text{supp}(\tilde{Z}) \subset \text{Int} \Delta$ and $\tilde{Z} = Z$ on $\mathcal{O}_p Z^{-\epsilon}(\Delta)$ where $\Delta \setminus Z^{-\epsilon}(\Delta) \subset U$.

Now pick $U \supset \partial \Delta$ to be such that $K|_{U \times S^1} > 0$, take the constructed contactomorphism $\Phi$ above, and consider the contact Hamiltonian $K' = \Phi^{-1}(K|_{\Phi(\Delta')})$ on $\Delta'$. It follows from Lemmas \ref{lem:4.1} and \ref{lem:4.2} that $(B_K,\eta_K)$ dominates $(B_{K'},\eta_{K'})$. \hfill \Box

\subsection{Domination up to conjugation}

If we want to prove the contact shell $(B_K,\eta_K)$ is dominated by the shell $(B_{K'},\eta_{K'})$ then Lemmas \ref{lem:4.1} and \ref{lem:4.2} instruct us to care about the partial order from Section 4.1 up to conjugation. In particular it is enough to find a contact embedding $\Phi : \Delta \to \Delta'$ such that $(\Phi_*K,\Phi(\Delta)) \leq (K',\Delta')$ to prove $(B_K,\eta_K)$ is dominated $(B_{K'},\eta_{K'})$. 
4 Domination and conjugation for Hamiltonian contact shells

4.3.1 Minimal elements up to conjugation in the 3-dimensional case

In the 3-dimensional case where $\Delta \subset \mathbb{R}$ is always a closed interval, up to conjugation any somewhere negative Hamiltonian $(K, \Delta)$ is minimal with respect to the partial order from Section 4.1.

**Lemma 4.7.** Let $(K, \Delta)$ be somewhere negative where $\Delta = [-1, 1]$. For any other contact Hamiltonian $(\tilde{K}, \Delta)$ there is a contactomorphism $\Phi \in \text{Cont}_0(\Delta)$ such that $(\Phi^* K, \Delta) \leq (\tilde{K}, \Delta)$, and hence $(B_K, \eta_K)$ is dominated by $(B_{\tilde{K}}, \eta_{\tilde{K}})$.

**Proof.** Without loss of generality assume $K(0) < 0$ and pick a $\delta > 0$ such that

$$K(z) < -\delta \text{ if } |z| \in [0, a] \text{ for some } a \in (0, 1),$$

$$\tilde{K}(z) > \delta \text{ if } |z| \in [1 - \varepsilon, 1] \text{ for some } \varepsilon \in (0, 1).$$

For $0 < \sigma \ll 1$, pick a diffeomorphism $\Phi : [-1, 1] \to [-1, 1]$ such that linearly

$$[-\sigma, \sigma] \text{ maps onto } [-1 + 2\sigma, 1 - 2\sigma] \text{ and } \pm [2\sigma, 1] \text{ maps onto } \pm [1 - \sigma, 1].$$

Since $(\Phi^* K)(\Phi(z)) = \Phi'(z)K(z)$ we can pick $\sigma$ sufficiently small so that

$$(\Phi^* K)(z) < -\frac{1-2\sigma}{\sigma} \delta < \tilde{K}(z)$$

if $|z| \in [0, 1 - 2\sigma]$

$$(\Phi^* K)(z) < 0 < \tilde{K}(z)$$

if $|z| \in [1 - 2\sigma, 1 - \sigma]$

$$(\Phi^* K)(z) \leq \frac{\sigma}{1-2\sigma} \max(K) < \delta < \tilde{K}(z)$$

if $|z| \in [1 - \sigma, 1]$

and hence get that $\Phi^* K < \tilde{K}$. $\square$

As a consequence of this lemma, the 3-dimensional case simplifies by making Section 8 unnecessary and allows us to give an effective description of an overtwisted disc. It seems unlikely to us (though we do not have a proof) disc that the generalization of Lemma 4.7 holds when $\dim(\Delta_{cyl}) \geq 3$. The immediate obstacle to adapting the proof is essentially that $O_D \partial \Delta_{cyl}$ is not a star-shaped domain in higher dimensions, while for $\Delta_{cyl} = [-1, 1]$ we get two intervals which are star-shaped.

4.3.2 Remnants of the 3-dimensional case

Proposition 4.8 and its corollary Proposition 4.9 below, represent the remnants of the 3-dimensional Lemma 4.7 that survive to higher dimensions.
Proposition 4.9 essentially says that up to conjugation the only part of \((K, \Delta)\) that is relevant for the partial order is \(K|_{\{K \geq 0\}}\) whereas for instance \(\min(K)\) is irrelevant if \(K < 0\) somewhere. It will play a key role in Section 8 where we prove the existence universal contact shells.

Given a domain \(\Delta \subset \mathbb{R}^{2n-1}\) let

\[ F_+ (\Delta) := \{ K \in C^0(\Delta) : \text{supp}(K) \subset \text{Int} \Delta, K \geq 0, \text{ and } K \neq 0 \} \]

and consider the action of \(\mathfrak{D}_0(\Delta) := \text{Cont}_c^0(\text{Int} \Delta)\) on \(F_+ (\Delta)\) given by

\[ \Phi_* K := (c_\Phi \cdot K) \circ \Phi^{-1} \text{ for } K \in F_+ (\Delta) \text{ and } \Phi \in \mathfrak{D}_0(\Delta) \]

i.e. the push-forward operation from (13).

**Proposition 4.8.** If \(\Delta \subset (\mathbb{R}^{2n-1}, \xi_{st})\) is star-shaped, then for any two \(K, H \in F_+ (\Delta)\) there is a contactomorphism \(\Phi \in \mathfrak{D}_0(\Delta)\) such that \(\Phi_* K \geq H\).

**Proof.** Without loss of generalize assume \(\Delta\) is star-shaped with respect to the radial vector field \(Z\) and that \(K(0) > 0\). Pick a sufficiently small neighborhood \(U \ni 0\) so that for some \(T > 0\):

\[ \inf_U (K) > 0, \text{ supp}(H) \subset Z^T(U) \subset \text{Int} \Delta, \text{ and } e^T \inf_U (K) > \max(H) \]

where \(Z^t : \mathbb{R}^{2n-1} \to \mathbb{R}^{2n-1}\) is the flow of \(Z\) and satisfies \((Z^t)^* \lambda_{st} = e^t \lambda_{st}\). Let \(\tilde{Z}\) be another contact vector field supported in \(\text{Int} (\Delta)\) and equal to \(Z\) on \(Z^T(U)\). It follows the contactomorphism \(\Phi := \tilde{Z}^T \in \mathfrak{D}_0(\Delta)\) satisfies \(\Phi_* K \geq H\) since

\[ (\Phi_* K)(x) = (c_\Phi \cdot K)(\Phi^{-1}(x)) \geq e^T \inf_U (K) \geq H(x) \text{ if } x \in \text{supp}(H) \]

and \(\Phi_* K \geq 0\) otherwise. \(\square\)

Note that Proposition 4.8 shows that on the conjugacy classes of elements of the positive cone \(\mathcal{C} := \{ f \in \mathfrak{D}_0; f \geq \text{Id}, f \neq \text{Id} \}\) partial order from [10] is trivial and it would be to understand for which contact manifolds the analog of Proposition 4.8 holds. As pointed out to us by L. Polterovich, a non-trivial bi-invariant metric on \(\text{Cont}_c^0\) compatible with the notion of order on \(\text{Cont}_c^0\) from [10] provides an obstruction to Proposition 4.8. For instance Sandon’s metric [39] shows Proposition 4.8 does not hold for \(\mathbb{D}_R^{2n} \times S^1\) with contact form \(dz + \sum_{i=1}^n u_i d\phi_i\) where \(\mathbb{D}_R^{2n}\) is a \(2n\)-disc of a sufficiently large radius \(R\).

As an application of Proposition 4.8 we show in this next proposition that condition (11) in the definition of the partial order \((K, \Delta) \leq (K', \Delta')\) from Section 4.1 can be weaken so that there is still domination of the contact shells.
Proposition 4.9. Consider contact Hamiltonians $K_i : \Delta \to \mathbb{R}$ defining contact shells $(B_{K_i}, \eta_{K_i})$ for $i = 1, 2$. If there is a star-shaped domain $\tilde{\Delta} \subset \text{Int } \Delta$ such that

$$K_0 \leq K_1 \text{ on } \partial (\Delta \setminus \text{Int } \tilde{\Delta}), \quad 0 \leq K_1 \text{ on } \partial \tilde{\Delta}, \quad \text{and } K_0 \leq 0 \text{ on } \partial \tilde{\Delta}$$

with $K_0|_{\text{Int } \Delta} \neq 0$, then the contact shell $(B_{K_0}, \eta_{K_0})$ is dominated by $(B_{K_1}, \eta_{K_1})$.

Proof. The assumptions ensure we can pick contact Hamiltonians $\tilde{K}_i : \Delta \to \mathbb{R}$ defining contact shells $(B_{\tilde{K}_i}, \eta_{\tilde{K}_i})$ so that

(i) $K_0 \leq \tilde{K}_0$ and $\tilde{K}_1 \leq K_1$,

(ii) $\tilde{K}_0 \leq \tilde{K}_1$ on $\Delta \setminus \tilde{\Delta}$, and

(iii) $-\tilde{K}_i|_{\tilde{\Delta}} \in F_+(\tilde{\Delta})$ for $i = 1, 2$.

By item (i) and Lemma 4.1 it suffices to show $(B_{\tilde{K}_0}, \eta_{\tilde{K}_0})$ is dominated by $(B_{\tilde{K}_1}, \eta_{\tilde{K}_1})$. Applying Proposition 4.8 to item (iii) gives a $\Phi \in \text{Cont}_0^c(\text{Int } \tilde{\Delta})$ such that

$$\Phi_*(\tilde{K}_0|_{\tilde{\Delta}}) \leq \tilde{K}_1|_{\tilde{\Delta}}.$$

Together with item (ii) this means $\Phi_* \tilde{K}_0 \leq \tilde{K}_1$ where we think of $\Phi \in \text{Cont}_0^c(\text{Int } \Delta)$ and therefore $(B_{\tilde{K}_0}, \eta_{\tilde{K}_0})$ is dominated by $(B_{\tilde{K}_1}, \eta_{\tilde{K}_1})$ by Lemmas 4.2 and 4.1.

We also have the following parametric version of Proposition 4.9.
Proposition 4.10. Assume that $\Delta \subset \mathbb{R}^{2n-1}$ is a star-shaped domain. Let $\Delta' \subset \Delta$ be a smooth star-shaped subdomain and let $K^\tau : \Delta \to \mathbb{R}$, $\tau \in T$, be a family time-independent functions satisfying $K^\tau|_{\Delta \setminus \text{Int} \Delta'} > 0$. Suppose that $K^\tau > 0$ for $\tau$ in a closed subset $A \subset T$. Then for any $\delta > 0$, there exists a family $\tilde{K}^\tau$ such that

- $\tilde{K}^\tau = K^\tau$ on $\Delta \setminus \text{Int} \Delta'$ and $\tilde{K}^\tau > -\delta$, $\tau \in T$;
- $\tilde{K}^\tau = K^\tau$ for $\tau \in A$;
- there exists a family of subordination maps $h^\tau : \eta_{\tilde{K}^\tau} \to \eta_{K^\tau}$ which are identity maps for $\tau \in A$.

5 Filling of the universal circular models

We prove in this section Propositions 3.9 and 3.11. In this section, we will always take

$$\Delta = \Delta_{cyl} = \{ u \leq 1, |z| \leq 1 \} \subset (\mathbb{R}^{2n-1}, \xi_{st}) \text{ where } u = u_1 + \cdots + u_{n-1}.$$ 

All contact Hamiltonians $(K, \Delta)$ will be assumed time independent and spherically symmetric, i.e. functions $K(u, z)$ of only the $u$ and $z$ variables.

The contactomorphism of $(\mathbb{R}^{2n-1}, \xi_{st})$ that is translation in the $z$-coordinate will be

$$Z^\tau : \mathbb{R}^{2n-1} \to \mathbb{R}^{2n-1} \text{ where } Z^\tau(q, z) = (q, z + \tau)$$

using coordinates $(q, z) \in \mathbb{R}^{2n-2} \times \mathbb{R}$.

5.1 Boundary connected sum

5.1.1 Abstract boundary connected sum

Consider the $\mathbb{R}^{2n}$ with polar coordinates $(u_1, \varphi_1, \ldots, u_{n-1}, \varphi_{n-1}, v, t)$ equipped with the radial Liouville form and vector field

$$\theta := \sum_{i=1}^{n-1} u_i d\varphi_i + vt$$

and $L := \sum_{i=1}^{n-1} u_i \frac{\partial}{\partial u_i} + v \frac{\partial}{\partial v}$.

and denote by $L^t : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ the Liouville flow.
A gluing disc for a contact shell \((W, \zeta)\) is a smooth embedding \(\iota: D \rightarrow \partial W\), where \(D \subset \mathbb{R}^{2n}\) is compact domain, star-shaped with respect to \(L\), and with piecewise smooth boundary such that \(\iota^*\alpha = \theta\) for a choice of a contact form \(\alpha\) for \(\zeta\) in \(\mathcal{O}_P \partial W\). Note this implies \(\iota(0) \in \partial W\) is a gluing place in the sense of Section 2.2 and that the Reeb vector field \(R_\alpha\) is transverse to \(\iota(D)\).

Given contact shells \((W_{\pm}^{2n+1}, \zeta_{\pm})\) with gluing discs \(\iota_{\pm}: D \rightarrow \partial W_{\pm}\) such that \(\iota_+\) preserves and \(\iota_-\) reverses orientation, the Reeb flows define contact embeddings 
\[
\Phi_+: D \times (-\epsilon, 0] \rightarrow \mathcal{O} \quad \text{with} \quad \Phi_+^*\alpha_+ = dz + \theta, \\
\Phi_-: D \times [0, \epsilon) \rightarrow \mathcal{O} \quad \text{with} \quad \Phi_-^*\alpha_- = dz + \theta, 
\]
(17) such that \(\Phi_{\pm}|_{D \times 0} = \iota_{\pm}\). For \(\ell > 0\) pick a smooth function \(\beta: [-\ell, \ell] \rightarrow \mathbb{R}_{\geq 0}\) such that \(\beta(z) = 0\) near \(\pm \ell\) and denote \(D(z) := L^{-\beta(z)}(D)\). Define the abstract boundary connected sum to be the almost contact manifold
\[
(W_+ \#_T W_-, \zeta_+ \#_T \zeta_-) := \left((W_+, \zeta_+) \cup (T, \ker(dz + \theta)) \cup (W_-, \zeta_-)\right) / \sim
\]
(18) where
\[
T = \{(p, z) \in \mathbb{R}^{2n} \times [-\ell, \ell] : p \in D(z)\} \subset \mathbb{R}^{2n+1}
\]
(19) and one identifies 
\[
\Phi_+(p, 0) \sim (p, -\ell) \in T \quad \text{and} \quad \Phi_-(p, 0) \sim (p, \ell) \in T.
\]

### 5.1.2 Abstract connected sum of \(S^1\)-model contact shells

Consider a Hamiltonian contact shell \((B_{K,C}, \eta_{K,\rho})\) associated to a contact Hamiltonian \((K, \Delta)\). There are canonical gluing discs
\[
D_{\pm} = \{u \leq 1, v \leq K(u, \pm 1)\} \subset \mathbb{R}^{2n}
\]
with maps \(\iota_{\pm}: D_{\pm} \rightarrow (\partial B_{K,C}, \eta_{K,\rho})\)
\[
\iota_{\pm}(q, v, t) = (q, \pm 1, \rho_{(q, \pm 1)}^{-1}(v), t) \in \mathbb{R}^{2n-1} \times \mathbb{R}^2,
\]
where \(\iota_{\pm}(0, 0) = (0, \pm 1, 0)\) are the north and south poles of \(B_K\).

For two contact Hamiltonians \(K_\pm: \Delta \rightarrow \mathbb{R}\) assume \(E(u) = K_\pm(u, \pm 1)\) is well defined. For any \(\ell > 0\) and a smooth function \(\beta: [-\ell, \ell] \rightarrow \mathbb{R}_{\geq 0}\) such that \(\beta = 0\) near \(z = \pm \ell\), define the domain 
\[
\Delta_{\beta, \ell} \Delta := Z_{1+\ell}(\Delta) \cup T_{\beta, \ell} \cup Z_{1+\ell}(\Delta) \subset \mathbb{R}^{2n-1}
\]
(20)
where

\[ T_{\beta,\ell} := \{ u \leq e^{-\beta(z)}, |z| \leq 1 \} \subset \mathbb{R}^{2n-1} \]

and define the contact Hamiltonian \( K_+ \#_\beta K_- : \Delta \#_{\beta,\ell} \Delta \to \mathbb{R} \) by

\[
(K_+ \#_\beta K_-)(u, z) = \begin{cases} 
(K_+ \circ Z_{1+\ell})(u, z) & \text{on } Z_{1+\ell}^{-1}(\Delta) \\
e^{-\beta(z)}E(u) & \text{for } (z, q) \in T_{\beta,\ell} \\
(K_- \circ Z_{1+\ell}^{-1})(u, z) & \text{on } Z_{1+\ell}(\Delta)
\end{cases}
\]

Going forward we will drop \( \beta \) from the notation when \( \beta \equiv 0 \).

It follows from Example 5.8 below that \( \Delta \#_{\beta,\ell} \Delta \) is star-shaped since it is contactomorphism to \( \Delta \#_{\ell} \Delta \), which is star-shaped with respect to \( Z = \frac{\partial}{\partial z} + L \), and hence \( (K_+ \#_\beta K_-, \Delta \#_{\beta,\ell} \Delta) \) defines an \( S^1 \)-model contact shell

\[
(B_{K_+ \#_\beta K_-}, \eta_{K_+ \#_\beta K_-})
\]

as in Section 2.3. It is straightforward to check that we have the following lemma.

**Lemma 5.1.** The contact shell \( (B_{K_+ \#_\beta K_-}, \eta_{K_+ \#_\beta K_-}) \) is equivalent to the abstract connected sum \( (B_{K_+ \# T K_-}, \eta_{K_+ \# T K_-}) \) with tube

\[
T = \{ u \leq e^{-\beta(z)}, v \leq e^{-\beta(z)}E(u) \} \subset \mathbb{R}^{2n+1}
\]

where the connected sum is done at the north pole of \( B_{K_+} \) and the south pole of \( B_{K_-} \).

### 5.1.3 Ambient boundary connected sum

Suppose in an almost contact manifold \( (W^{2n+1}, \xi) \) there are disjoint codimension 0 submanifolds \( W_\pm \subset \text{Int } W \) with piecewise smooth boundary such that \( \xi \) is a genuine
contact structure in \( \mathcal{O} \partial W \pm \). Assume the contact shells \((W \pm, \xi)\) are equipped with gluing discs \( \iota_\pm : D \to \partial W \pm \) where \( \iota_\pm^* \alpha = \theta \) for a contact form \( \alpha \) for \( \xi \) such that \( \iota_+ \) preserves and \( \iota_- \) reverses orientation.

For a smooth embedding \( \gamma : [0, 1] \to \text{Int } W \) such that

- \( \gamma(0) = \iota_+(0), \gamma(1) = \iota_-(0) \), and \( \gamma(t) \notin W_+ \cup W_- \) otherwise;
- \( \xi \) is a genuine contact structure on \( \mathcal{O} \partial \Gamma \) where \( \Gamma := \gamma([0, 1]) \);
- \( \gamma \) is transverse to \( \xi \),

we can think of \((W_+ \cup \mathcal{O} \partial \Gamma \cup W_-, \xi)\) as an ambient boundary connected sum of the shells \((W_\pm, \xi)\). This is made precise with the following lemma.

**Lemma 5.2.** Every neighborhood \((W_+ \cup \mathcal{O} \partial \Gamma \cup W_-, \xi)\) contains the image of an almost contact embedding of an abstract connected sum \((W_+ \# T W_-, \xi \# T \xi)\).

**Proof.** The gluing discs \( \iota_\pm : D \to \partial W_\pm \) extend to Darboux embeddings

\[
\Phi_\pm : D \times (\mp \ell - \varepsilon, \mp \ell + \varepsilon) \to \mathcal{O} \partial \iota_\pm(D) \quad \text{with} \quad \Phi_\pm^* \alpha = dz + \theta \quad \text{and} \quad \Phi_\pm|_{D \times \mp \ell} = \iota_\pm
\]

and moreover one can ensure \( \Phi_+^{-1}(\Gamma) = 0 \times [-\ell, -\ell + \varepsilon] \) and \( \Phi_-^{-1}(\Gamma) = 0 \times (\ell - \varepsilon, \ell] \).

By the neighborhood theorem for transverse curves in a contact manifold, for \( N > 0 \) sufficiently large the embeddings \( \Phi_\pm \) can be extended (after possibly decreasing \( \varepsilon \)) to a contact embedding

\[
\Phi : \left(D \times (-\ell - \varepsilon, \ell + \varepsilon)\right) \cup \left(L^{-N}(D) \times [-\ell, \ell]\right) \cup \left(D \times (\ell - \varepsilon, \ell + \varepsilon)\right) \to \text{Int } W,
\]

whose image is contained in \( \mathcal{O} \partial (\iota_+(D) \cup \Gamma \cup \iota_-(D)) \) and such that \( \Phi(0 \times [-\ell, \ell]) = \Gamma \).

Picking \( \beta : [-\ell, \ell] \to \mathbb{R}_{\geq 0} \) such that the tube

\[
T = \{(p, z) \in [-\ell, \ell] \times \mathbb{R}^{2n} : p \in L^{-\beta(z)} D\}
\]

is contained in the domain of \( \Phi \), we can now use \( \Phi \) to define the required contact embedding \((W_+ \# T W_-, \xi \# T \xi) \to (W, \xi)\). \(\square\)
5.2 Filling a connected sum of a shell with a neighborhood of an overtwisted disc

For the rest of this section we will let \((K, \Delta)\) be a special Hamiltonian associated to a special function \(k: \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}\) such that

\[
K(u, z) = k(u) \quad \text{when} \quad z \in \mathcal{O}_p[z_D', z_D] \tag{22}
\]

where \(z_D' \in (-1, z_D)\) is sufficiently close to \(z_D\) and recall that \(E(u) := K(u, \pm 1)\) is well-defined and satisfies \(K \leq E\). For \(\varepsilon' > 0\), define

\[
K' = K - \varepsilon' \quad \text{and} \quad \Delta' = \{u \leq 1 - \varepsilon', |z| \leq 1 - \varepsilon'\}
\]

and assume \(\varepsilon' > 0\) is small enough so that \(K'|_{\partial \Delta'} > 0\).

The goal of this subsection is the proof of Proposition 3.9 and its parametric version Proposition 3.11. All the connected sums as in Sections 5.1.1 and 5.1.2 will be done with a fixed choice of function \(\beta: [-\ell, \ell] \rightarrow \mathbb{R}_{\geq 0}\), which we will suppress from the notation. In particular we will be considering abstract connected sums such as

\[
(B_K \# B_K, \eta_K \# \eta_K) \quad \text{and} \quad (B_K \# K, \eta_K \# K)
\]

where we will always use the north pole gluing place on the first factor and the south pole gluing place on the second factor. We will also freely use Lemma 5.1 to identify such connect sums.

By Lemma 4.1 we can arrange the inclusion

\[
(B_{K', C}, \eta_{K', \rho}) \hookrightarrow (B_{K, C}, \eta_{K, \rho})
\]

to be a subordination map, so that we have a \((2n + 1)\)-dimensional contact annulus

\[
(A, \xi_A) := (B_{K, C} \setminus \text{Int } B_{K', C}, \ker \eta_{K, \rho}|_A).
\]

Define the contact ball \((B, \xi_B) \subset (A, \xi_A)\) given by

\[
B := \{(x, v, t) \in A : z(x) \in [-1, z_D]\} \tag{23}
\]

and by design the \(2n\)-dimensional disc \((D_K, \eta_K) \subset (\partial B, \xi_B)\) appears with the correct coorientation.
5 Filling of the universal circular models

5.2.1 Non-parametric version

To prove Proposition 3.9 it will suffice to show the contact shell \((B_K \# B, \eta_K \# \xi_B)\), defined as a subset of \((B_K \# B_K, \eta_K \# \eta_K)\), is equivalent to a genuine contact structure. Denoting by \(\iota: \Delta \to \Delta \# \Delta\) the inclusion into the right hand factor, we will prove in Lemma 5.4(i) below that there is a family of contact embeddings \(\Theta_\sigma: \Delta \to \text{Int}(\Delta \# \Delta)\) for \(\sigma \in [0,1]\) with \(\Theta_0 = \iota\) such that \(\Theta_\sigma = \iota\) in \(O_p\{z \in [z_D, 1]\}\) for all \(\sigma \in [0,1]\) and \(\Theta := \Theta_1\) satisfies

\[
\Theta^* K', \Theta(\Delta') < (K \# K, \Delta \# \Delta).
\]

Proof of Proposition 3.9. It suffices to prove \((B_K \# B, \eta_K \# \xi_B)\) is equivalent to a genuine contact structure, since it is dominated by \((B_K \# B_K, \eta_K \# \eta_K)\) if we pick \(\varepsilon' > 0\) sufficiently small in the definition of \((B, \xi_B)\).

By Lemmas 4.1, 4.2, and 4.4 we can pick a family of contact shell structures on \((B_K \# B_K, \eta_K \# \eta_K, \hat{\rho})\) such that there is a family of contact shell embeddings

\[
\hat{\Theta}_\sigma: (B_K', \eta_K') \to (B_K \# B_K, \eta_K \# \eta_K, \hat{\rho}_\sigma)\]

with \(\hat{\Theta}_1\) a subordination map. We can arrange that \(\eta_K \# \hat{\rho}_0 = \eta_K \# \eta_K\) and for all \(\sigma \in [0,1]\) to have

\[
\eta_K \# \hat{\rho}_\sigma = \eta_K \# \eta_K \quad \text{on} \quad O_p i\{z \in [z_D, 1]\}
\]

where \(i: B_K \to B_K \# B_K\) is the inclusion into the right hand factor.

We can pick an isotopy \(\{\Psi_\sigma\}_{\sigma \in [0,1]}\) of \(B_K \# B_K\) based at the identity and supported away from the boundary such that

Fig. 5.2: On the left: The union of the grey regions is \(B_K\), the dark grey region is \(B_K'\), and the light grey region is \((A, \xi_A)\). On the right: The contact ball \((B, \xi_B) \subset B_K\) obtained from \((A, \xi_A)\).
Fig. 5.3: Images of the almost contact embeddings $\hat{\Theta}_\sigma : B_{K'} \to B_K \# K$ in dark grey. The white regions denote where outside of $\hat{\Theta}_\sigma (B_{K'})$ the almost contact structure $\eta_{K \# K, \hat{\rho}^\sigma}$ is not genuine.

(i) $\Psi_\sigma \circ \hat{\iota} = \hat{\Theta}_\sigma : B_{K'} \to B_K \# B_K$,

(ii) $\Psi_\sigma = \text{Id}$ on $\mathcal{O} \hat{\iota} (\{ z \in [z_D, 1] \})$, and

(iii) $\Psi_1 (B_K \# A) = (B_K \# B_K) \setminus \text{Int} \hat{\Theta}(B_{K'})$.

Observe that a point in $\mathcal{O} \partial (B_K \# B)$ is one of the following regions

(i) $\mathcal{O} \partial (B_K \# B_K)$ where $\Psi_\sigma = \text{Id}$ and $\eta_{K \# B} = \eta_{K \# K} = \eta_{K \# K, \hat{\rho}^\sigma}$

(ii) $\mathcal{O} \hat{\iota} (\{ z = z_D \})$ where $\Psi_\sigma = \text{Id}$ and $\eta_{K \# B} = \eta_{K \# K} = \eta_{K \# K, \hat{\rho}^\sigma}$

(iii) $\mathcal{O} \hat{\iota} (\partial B_{K'})$ where $\eta_{K \# B} = \hat{\iota}_* \eta_{K'} = \hat{\iota}_* \hat{\Theta}_\sigma^* (\eta_{K \# K, \hat{\rho}^\sigma}) = \Psi_\sigma^* (\eta_{K \# K, \hat{\rho}^\sigma})$
This shows $\xi_\sigma := \Psi_\sigma^*(\eta_{K\#K,\hat{\rho}_0})$ is a family of equivalent contact shells on $B_K\#B$ with $\xi_0 = \eta_{K\#B}$. We know $\eta_{K\#K,\hat{\rho}_1}$ is a contact structure away from $\text{Int} \hat{\Theta}(B_{K'})$, since $\hat{\Theta}_1$ is a subordination map, and therefore $\xi_1$ is a genuine contact structure on $B_K\#B$. 

\[ \hat{\Theta}_s^s : (B_{K'}, \eta_{K'}) \to (B_K\#B, \eta_{K\#K,\hat{\rho}_2}) \]

and associated isotopies $\{\Psi_\sigma^s\}_{\sigma \in [0,1]}$ of $\eta_{K\#K} \# B_K$, which lead to contact shell structures

$\hat{\xi}_s^s := (\Psi_\sigma^s)^*(\eta_{K\#K,\hat{\rho}_2})$ on $B_K\#B$

that define a family of fibered contact shells $I^s \hat{\xi}_\sigma$ on $I^s B_K\#I^s B$. It follows from the second part of Lemma 5.4 that

\[ \left( (\Theta_\sigma)_*, K', \Theta_\sigma(\Delta') \right) < (K(\#K, \Delta\#\Delta) \text{ if } s \in \mathcal{O} \{0\}, \]

and therefore we can arrange for $\hat{\Theta}_s^s$ to be a subordination map when $s \in \mathcal{O} \{0\}$.

With this set-up the proof of Proposition 3.9 shows we can ensure the family of fibered contact shells $I^s \hat{\xi}_\sigma$ is such that $I^s \hat{\xi}_0 = I^s \hat{\xi}$ as well as

\[ \mathcal{O} \{s = 0\} \cup \bigcup_{s \in [0,1]} \mathcal{O} \partial(B_K\#B) \subset I^s B_K\#I^s B \]

through fibered families of contact shells on $I^s B_K\#I^s B$ to a fibered family of genuine contact structures.

\textbf{Proposition 5.3.} The fibered family of contact shells $I^s \xi$ is homotopic relative to

\[ \mathcal{O} \{s = 0\} \cup \bigcup_{s \in [0,1]} \mathcal{O} \partial(B_K\#B) \subset I^s B_K\#I^s B \]

through fibered families of contact shells on $I^s B_K\#I^s B$ to a fibered family of genuine contact structures.

\textbf{Proof.} Inspecting the proof of Proposition 3.9 shows that it can be done parametrically. In particular we can get a family of contact shell embeddings

\[ \hat{\Theta}_s^s : (B_{K'}, \eta_{K'}) \to (B_K\#B, \eta_{K\#K,\hat{\rho}_2}) \]

and associated isotopies $\{\Psi_\sigma^s\}_{\sigma \in [0,1]}$ of $B_K\#B$, which lead to contact shell structures

$\hat{\xi}_s^s := (\Psi_\sigma^s)^*(\eta_{K\#K,\hat{\rho}_2})$ on $B_K\#B$

that define a family of fibered contact shells $I^s \hat{\xi}_\sigma$ on $I^s B_K\#I^s B$. It follows from the second part of Lemma 5.4 that

\[ \left( (\Theta_\sigma)_*, K', \Theta_\sigma(\Delta') \right) < (K(\#K, \Delta\#\Delta) \text{ if } s \in \mathcal{O} \{0\}, \]

and therefore we can arrange for $\hat{\Theta}_s^s$ to be a subordination map when $s \in \mathcal{O} \{0\}$.

With this set-up the proof of Proposition 3.9 shows we can ensure the family of fibered contact shells $I^s \hat{\xi}_\sigma$ is such that $I^s \hat{\xi}_0 = I^s \hat{\xi}$ as well as

\[ \mathcal{O} \{s = 0\} \cup \bigcup_{s \in [0,1]} \mathcal{O} \partial(B_K\#B) \subset I^s B_K\#I^s B \]

through fibered families of contact shells on $I^s B_K\#I^s B$ to a fibered family of genuine contact structures.

\textbf{Proof.} Inspecting the proof of Proposition 3.9 shows that it can be done parametrically. In particular we can get a family of contact shell embeddings

\[ \hat{\Theta}_s^s : (B_{K'}, \eta_{K'}) \to (B_K\#B, \eta_{K\#K,\hat{\rho}_2}) \]

and associated isotopies $\{\Psi_\sigma^s\}_{\sigma \in [0,1]}$ of $B_K\#B$, which lead to contact shell structures

$\hat{\xi}_s^s := (\Psi_\sigma^s)^*(\eta_{K\#K,\hat{\rho}_2})$ on $B_K\#B$

that define a family of fibered contact shells $I^s \hat{\xi}_\sigma$ on $I^s B_K\#I^s B$. It follows from the second part of Lemma 5.4 that

\[ \left( (\Theta_\sigma)_*, K', \Theta_\sigma(\Delta') \right) < (K(\#K, \Delta\#\Delta) \text{ if } s \in \mathcal{O} \{0\}, \]

and therefore we can arrange for $\hat{\Theta}_s^s$ to be a subordination map when $s \in \mathcal{O} \{0\}$.

With this set-up the proof of Proposition 3.9 shows we can ensure the family of fibered contact shells $I^s \hat{\xi}_\sigma$ is such that $I^s \hat{\xi}_0 = I^s \hat{\xi}$ as well as

\[ \mathcal{O} \{s = 0\} \cup \bigcup_{s \in [0,1]} \mathcal{O} \partial(B_K\#B) \subset I^s B_K\#I^s B \]

through fibered families of contact shells on $I^s B_K\#I^s B$ to a fibered family of genuine contact structures.
(i) \( \hat{\zeta}_s^\sigma = \zeta_s^\sigma \) on \( \mathcal{O}_p \partial (B_{K(s)} \# B) \) for all \( s \) and \( \sigma \),

(ii) \( \hat{\zeta}_1^s \) is a genuine contact structure for all \( s \), and

(iii) \( \hat{\zeta}_s^\sigma \) is a genuine contact structure for all \( (\sigma, s) \in [0, 1] \times [0, 3a] \) for some \( a > 0 \).

Pick any smooth function

\[
 f : [0, 1] \times [0, 1] \to [0, 1] \quad \text{with} \quad f(\sigma, s) = \begin{cases} 
 0 & \text{if } \sigma = 0 \\
 0 & \text{if } s \in [0, a] \\
 1 & \text{if } s \in [2a, 1] \text{ and } \sigma = 1
\end{cases}
\]

and define the family of contact shells \( \zeta_s^\sigma := \hat{\zeta}_s^f(\sigma, s) \) on \( B_{K(s)} \# B \), which represents a homotopy of fibered families of contact shells

\[ \{ I_{\zeta_\sigma} \}_{\sigma \in [0, 1]} \quad \text{on} \quad \mathcal{I} B_{K \#} B. \]

It follows from item (i) and the fact \( f(\sigma, s) = 0 \) if \( s \in [0, a] \), that this homotopy is relative is the appropriate set. Observe that \( \xi_1^s \) is a genuine contact structure for all \( s \in [0, 1] \), since either \( s \leq 3a \) and \( \zeta_1^s := \hat{\zeta}_f(1, s) \) is genuine by item (iii), or \( s \geq 2a \) and \( \zeta_1^s := \hat{\zeta}_f(1, s) = \hat{\zeta}_1^s \) is genuine by item (ii). Therefore we have the desired homotopy between \( \mathcal{I}_\zeta = \mathcal{I}_0 \) and a fibered family of genuine contact structures \( \mathcal{I}_1 \).

**Proof of Proposition 3.11.** Recall that \( (T B_{K_0} \# T B, T \eta_{K_0} \# T \xi) \) is the fibered contact shell, which at the point \( \tau \in T = D^g \) is given by

\[
 (B_{K(\delta(\tau))} \# B, \eta_{K(\delta(\tau))} \# \xi)
\]

where \( \delta : T \to [0, 1] \) is a bump function that vanishes near the boundary. It suffices to prove \( (T B_{K} \# T B, T \eta_{K} \# T \xi) \) is fibered equivalent to a fibered contact structure over \( T \), since it is dominated by \( (T B_{K_0} \# T B, T \eta_{K_0} \# T \xi) \) if we pick \( \varepsilon' \) sufficiently small in the definition of \( (B, \xi_B) \).

In the notation of Proposition 5.3 we have the identification

\[
 \zeta^{\delta(\tau)} = \eta_{K(\delta(\tau))} \# \xi_B \quad \text{as contact shell structures on} \quad B_{K(\delta(\tau))} \# B
\]

and a fibered contact structure on \( (T B_{K} \# T B, T \xi_1) \) with contact structure

\[
 \zeta^{\delta(\tau)}_1 \quad \text{on the fiber} \quad B_{K(\delta(\tau))} \# B.
\]

Since \( \delta(\tau) = 0 \) if \( \tau \in \mathcal{O}_p \partial T \), the homotopy constructed in Proposition 5.3 when used fiberwise gives a homotopy between \( T \eta_{K} \# T \xi_B \) and \( T \xi_1 \) that shows they are fibered equivalent. \( \square \)
5.3 Main lemma

Consider the connected sums \((K\# K, \Delta\# \Delta)\) and \((E\# K, \Delta\# \Delta)\) as in Section 5.1.2. The main goal of this section will be to prove the following lemma, which we will break up into two sublemmas below.

**Lemma 5.4.** There is a family of contact embeddings for \(\sigma \in [0, 1]\)

\[
\Theta_\sigma : \Delta \to \Delta' \quad \text{with } \Theta_\sigma = Z_{1+\ell} \text{ on } \mathcal{O} \{ z \in [z_D, 1] \}
\]

based at \(\Theta_0 := Z_{1+\ell}\) such that

(i) \(((\Theta_1), K', \Theta_1(\Delta')) < (K\# K, \Delta\# \Delta)\) and

(ii) \(((\Theta_\sigma), K', \Theta_\sigma(\Delta')) < (E\# K, \Delta\# \Delta)\) for all \(\sigma \in [0, 1]\).

**Proof.** It follows from Lemma 5.7 that it suffices to prove this lemma when \(\beta \equiv 0\) and this special case is proved in Lemma 5.9. \(\square\)
Let us remark that Section 5.2.1 only used the first part of Lemma 5.4 while Section 5.2.2 used the both parts.

**Remark 5.5.** In the 3-dimensional case, where \( \Delta = [-1, 1] \), this lemma essentially follows from Lemma 4.7.

### 5.3.1 Transverse scaling and simplifying the neck region

**Transverse scaling.** A orientation preserving diffeomorphism \( h : \mathbb{R} \to \mathbb{R} \) defines a contactomorphism \( \Phi_h \) of \((\mathbb{R}^{2n-1}, \xi_{st})\) by

\[
\Phi_h(u_i, \varphi_i, z) = (h'(z)u_i, \varphi_i, h(z))
\]

where \( \Phi_h^{-1} = \Phi_{h^{-1}} \). By (13) we have

\[
(\Phi_h)_* H(u, z) = h'(h^{-1}(z)) H\left(\frac{u}{h'(h^{-1}(z))}, h^{-1}(z)\right)
\]

for a contact Hamiltonian \( H(u, z) : \mathbb{R}^{2n-1} \to \mathbb{R} \).

**Example 5.6.** For our purposes \( \Phi \) should be thought of as a way to manipulate the \( z \)-variable at the cost of a scaling factor on the \( u \)-variable, in particular we have a contactomorphism between domains in \((\mathbb{R}^{2n-1}, \xi_{st})\)

\[
\Phi_h : \{ u \leq f(z), z \in [a, b] \} \to \{ u \leq (h' \cdot f)(h^{-1}(z)), z \in [h(a), h(b)] \}
\]

where \( f : \mathbb{R} \to \mathbb{R}_{>0} \).

This contactomorphism allows us to reduce the proof of Lemma 5.4 to when \( \beta \equiv 0 \).

**Lemma 5.7.** For every connected sum \((K \#_\beta K, \Delta \#_{\beta, \ell} \Delta)\), if \( \ell' > \ell \) is sufficiently large, then there is a contact embedding

\[
\Phi : \Delta \#_{\ell'} \Delta \to \Delta \#_{\beta, \ell} \Delta \quad \text{with} \quad \Phi = Z_{\pm (\ell - \ell')} \text{ on } \mathcal{O} \{ \pm z \geq \ell' \}
\]

such that \((\Phi_*(K \# K), \Phi(\Delta \#_{\ell'} \Delta)) \leq (K \#_\beta K, \Delta \#_{\beta, \ell} \Delta)\).

**Proof.** Recall \( E(u) := K(u, \pm 1) > 0 \) with \( K \leq E \) and pick a constant

\[
0 < C < \frac{\min(E)}{\max(E)} \leq 1.
\]
Pick a diffeomorphism $h : [-\ell', \ell'] \to [-\ell, \ell]$ with $h'(z) = 1$ on $z \in O_p \{ \pm \ell \}$ and
\[ h'(h^{-1}(z)) \leq Ce^{-\beta(z)} \tag{29} \]
which is possible provided
\[ \ell' > \frac{1}{2C} \int_{-\ell}^{\ell} e^{\beta(z)} dz \]
Extend $h$ by translation to get a diffeomorphism $h : \mathbb{R} \to \mathbb{R}$ and consider the associated contactomorphism $\Phi_h : (\mathbb{R}^{2n-1}, \xi) \to (\mathbb{R}^{2n-1}, \xi)$ from \cite{27}. This is the desired contact embedding for by (29) we have
\[ \Phi_h(\Delta \# \xi, \Delta) = \{ u \leq h'(h^{-1}(z)), z \in [-2 - \ell, 2 + \ell] \} \subset \Delta \# \xi, \Delta. \]
To check the order on the Hamiltonians, it suffices to check on $\Phi_h(T\ell')$ where we have
\[ (\Phi_h)_* E(u, z) = h'(h^{-1}(z)) E\left(\frac{u}{h'(h^{-1}(z))}\right) < e^{-\beta(z)} E(u) = (K \# K)(u, z) \]
by (28) and (29).

### 5.3.2 The twist contactomorphism and a special case of Lemma 5.4

We will use the transverse scaling contactomorphisms $\Phi_h$ together with the following contactomorphism.

**Twist contactomorphism.** For $g \in C^\infty(\mathbb{R})$ and $z_0 \in \mathbb{R}$, define
\[ \Psi_{g, z_0}(u_i, \varphi_i, z) := \left( \frac{u_i}{1 + g(z)u_i}, \varphi_i - \int_{z_0}^{z} g(s) ds, z \right) \]
which is a contactomorphism between the subsets of $(\mathbb{R}^{2n-1}, \xi_{st})$
\[ \Psi_{g, z_0} : \{ 1 + g(z)u > 0 \} \to \{ 1 - g(z)u > 0 \} \]
where $\Psi_{g, z_0}^{-1} = \Psi_{-g, z_0}$. By (13) we have
\[ (\Psi_{g, z_0}_*) H(u, z) = (1 - g(z)u) H\left(\frac{u}{1 - g(z)u}, z \right) \tag{30} \]
for a contact Hamiltonian $H(u, z) : \mathbb{R}^{2n-1} \to \mathbb{R}$.

**Example 5.8.** For our purposes $\Psi$ should be thought of as a way to manipulate the $u$-variable at the cost of a rotation in the angular coordinates, in particular we have a contactomorphism between domains in $(\mathbb{R}^{2n-1}, \xi_{st})$
\[ \Psi_g : \{ u \leq f_2(z) \} \to \{ u \leq f_1(z) \} \]
where $f_j : \mathbb{R} \to \mathbb{R}_{>0}$ and $g(z) = \frac{1}{f_1(z)} - \frac{1}{f_2(z)}$. 


Composing twist and scaling. Fix an orientation preserving diffeomorphism \( h : \mathbb{R} \to \mathbb{R} \) and define \( g(z) := 1 - \frac{1}{h'(h^{-1}(z))} \). It follows from Examples 5.6 and 5.8

\[
\Gamma_{h,z_0} := \Psi_{g,z_0} \circ \Phi_h : \{ u \leq 1, \ z \in [a, b] \} \to \{ u \leq 1, \ z \in [h(a), h(b)] \}
\]

is a contactomorphism of these domains in \((\mathbb{R}^{2n-1}, \xi_{st})\). So \( \Gamma_{h,z_0} \) lets us change the \( z \)-length of a region without changing the \( u \)-width, albeit still at the cost of a rotation in the angular coordinates.

Computing shows that

\[
\Gamma_{h,z_0}(u_i, \varphi_i, z) = \left( \frac{h'(z)u_i}{1 + (h'(z) - 1)u} ; \varphi_i - \int_{z_0}^z \left( 1 - \frac{1}{h'(h^{-1}(s))} \right) ds , \ h(z) \right)
\]

so if \( h(z) = z + \tau \) on \( z \in A \subset \mathbb{R} \) and \( z_0 \in h(A) \), then \( \Gamma_{h,z_0} \) is just translation

\[
\Gamma_{h,z_0} = Z_{\tau} \quad \text{on} \quad \{ z \in A \} \subset \mathbb{R}^{2n-1}.
\]

If we define

\[
\tilde{h}(u, z) := h'(h^{-1}(z)) - (h'(h^{-1}(z)) - 1)u
\]

then for a contact Hamiltonian \( H(u, z) : \mathbb{R}^{2n-1} \to \mathbb{R} \) we have that

\[
(\Gamma_{h,z_0})_* H(u, z) = \tilde{h}(u, z) \ H\left( \frac{u}{\tilde{h}(u, z)}, h^{-1}(z) \right).
\]

Proving Lemma 5.4 when \( \beta \equiv 0 \)

Pick a family of diffeomorphisms \( h_\sigma : \mathbb{R} \to \mathbb{R} \) for \( \sigma \in [0, 1] \) such that

\[
h_\sigma(z) = \begin{cases} 
  z + (1 - 2\sigma)(1 + \ell) & \text{for } z \in \mathcal{O}_p(-\infty, z_D') \\
  h_\sigma'(z) \geq 1 & \text{for } z \in [z_D', z_D] \\
  z + 1 + \ell & \text{for } z \in \mathcal{O}_p(z_D, \infty) 
\end{cases}
\]

Recall the contactomorphism \( \Gamma_{h,z_0} \) from (31) and define the contact embeddings

\[
\Gamma_\sigma := \Gamma_{h_\sigma, z + \ell} : \Delta \to \Delta \#_\ell \Delta \quad \text{for } s \in [0, 1].
\]

By (32), we see \( \Gamma_0 = Z_{1+\ell} \) and on \( \mathcal{O}_p \{ z \in [z_D, 1] \} \) we have \( \Gamma_\sigma = Z_{1+\ell} \) for all \( \sigma \in [0, 1] \). With this family of contactomorphisms we can prove Lemma 5.4 with the simplifying assumption that \( \beta = 0 \).
Filling of the universal circular models

\[ u z^2 + \ell - (z_1 - 1) - \ell h_0(h_1(z_D)) = \Gamma_0 \]

\[ h_0(z_D) \]

\[ -2 - \ell \quad h_0(z_D) \quad 2 + \ell \]

\[ h_1(z_D) \]

\[ -1 \quad -2 - \ell \quad 1 \]

\[ 2 + \ell \]

\[ \Gamma_{\sigma} \]

\[ \Gamma_0 \]

\[ \Gamma_1 \]

\[ \text{Fig. 5.5: The family of diffeomorphisms } h_{\sigma} \text{ and embeddings } \Gamma_\sigma : \Delta \to \Delta \# \ell \Delta. \text{ The union of the grey regions denote the image } \Gamma_\sigma(\Delta) \text{ while the dark grey regions denote the image } \Gamma_\sigma(\Delta') \text{ for } \Delta' = \{|z| \leq 1 - \varepsilon', u \leq 1 - \varepsilon\}. \]

**Lemma 5.9.** The family of contact embeddings \( \Gamma_\sigma : \Delta \to \Delta \# \ell \Delta \) for \( \sigma \in [0, 1] \) satisfy:

- (i) \( (\Gamma_\sigma)_* K \leq \varepsilon # K \) on \( \Gamma_\sigma(\Delta) \) for all \( \sigma \in [0, 1] \), and
- (ii) \( (\Gamma_1)_* K \leq K # K \) on \( \Gamma_1(\Delta) \).

**Proof of Lemma 5.9.** By (34) we have

\[ (\Gamma_\sigma)_* K(u, z) = \tilde{h}_\sigma(u, z) K\left(\frac{u}{\tilde{h}_\sigma(u, z)}, h_\sigma^{-1}(z)\right), \]

where recall from (33) that

\[ \tilde{h}_\sigma(u, z) := h_\sigma'(h_\sigma^{-1}(z)) - (h_\sigma'(h_\sigma^{-1}(z)) - 1) u \geq 1 \]
where the inequality follows from $h'_\sigma(h^{-1}_\sigma(z)) \geq 1$ and $u \leq 1$. Bringing these two together, we have

$$(\Gamma_\sigma)_*K(u, z) = \begin{cases}
K(u, h^{-1}_\sigma(z)) & \text{if } z \in \mathcal{O}_p h_\sigma([-1, z'_D]) \\
\leq k(u) & \text{if } z \in h_\sigma([z'_D, z_D]) \\
K(u, z - (1 + \ell)) & \text{if } z \in \mathcal{O}_p [z_D + 1 + \ell, 2 + \ell]
\end{cases}$$

using that $h_\sigma$ is translations on the ends, while on the middle we have

$$(\Gamma_\sigma)_*K(u, z) = (\Gamma_\sigma)_*k(u, z) = \tilde{h}_\sigma(u, z) k\left(\frac{u}{\tilde{h}_\sigma(u, z)}\right) \leq k(u)$$

using (22) and (6).

To verify $(i)$, since $$(E\#K)(u, z) = \begin{cases}
E(u) & \text{if } z \in [-2 - \ell, \ell] \\
K(u, z - (1 + \ell)) & \text{if } z \in [\ell, 2 + \ell]
\end{cases}$$

it follows from (37) and the inequalities $k(u) \leq K(u, z) \leq E(u)$.

that it suffices to check $$K(u, h^{-1}_\sigma(z)) \leq K(u, z - (1 + \ell)) \text{ when } z \in [\ell, h_\sigma(z_D)].$$

Since $h^{-1}_\sigma(z) = z - (1 - 2\sigma)(1 + \ell)$ here, this is equivalent to $$K(u, z + 2\sigma(1 + \ell)) \leq K(u, z) \text{ when } z \in [-1, z'_D - 2\sigma(1 + \ell)]$$

and this latter condition follows from Definition 3.3.

To verify $(ii)$, using (35) we see (37) at $\sigma = 1$ becomes

$$(\Gamma_1)_*K(u, z) = \begin{cases}
K(u, z + (1 + \ell)) & \text{if } z \in \mathcal{O}_p [-2 - \ell, z'_D - 1 - \ell] \\
\leq k(u) & \text{if } z \in [z'_D - 1 - \ell, z_D + 1 + \ell] \\
K(u, z - (1 + \ell)) & \text{if } z \in \mathcal{O}_p [z_D + 1 + \ell, 2 + \ell]
\end{cases}$$

while by definition

$$(K\#K)(u, z) = \begin{cases}
K(u, z + (1 + \ell)) & \text{if } z \in [-2 - \ell, -\ell] \\
E(u) & \text{if } z \in [-\ell, \ell] \\
K(u, z - (1 + \ell)) & \text{if } z \in [\ell, 2 + \ell]
\end{cases}$$

so $(ii)$ follows from (38).
6 Contact structures with holes

The goal of this section is Proposition 6.2 and its parametric version Proposition 7.6, which are the first steps in proving Propositions 3.1 and Proposition 3.10.

6.1 Semi-contact structures

Let $\Sigma$ be a closed $2n$-dimensional manifold. A semi-contact structure on an annulus $C = \Sigma \times [a,b]$ is a smooth family $\{\zeta_s\}_{s \in [a,b]}$ such that $\zeta_s$ is a germ of a contact structure along the slice $\Sigma_s := \Sigma \times s$. If $\{\alpha_s\}_{s \in [a,b]}$ is a smooth family of 1-forms with $\zeta_s = \ker \alpha_s$ on $O^p \Sigma_s$, then one gets an almost contact structure $(\lambda, \omega)$ on $C$ where

$$\lambda(x, s) = \alpha_s(x, s) \text{ and } \omega(x, s) = d\alpha_s(x, s).$$

It follows that every semi-contact structure on $C$ defines an almost contact structure on $C$ that equals $\zeta_s$ on $TC|\Sigma_s$.

Given a contact structure $\xi$ on $\Sigma \times \mathbb{R}$ and a smooth family of functions $\psi_s : \Sigma \to \mathbb{R}$ for $s \in [a, b]$, if we pick $\Psi_s : O^p \Sigma_s \to O^p (\text{graph } \psi_s) \subset \Sigma \times \mathbb{R}$ to be a smooth family of diffeomorphisms such that $\Psi_s|_{\Sigma_s} = \text{Id} \times \psi_s$, then we can define a semi-contact structure on $\Sigma \times [a, b]$ by $\zeta_s := \Psi_s^* \xi$. Any semi-contact structure of this form will be said to be of immersion type.

**Remark 6.1.** The term is motivated by the fact that on the boundary of each domain $\Sigma[a', b'] := \Sigma \times [a', b']$ for $a \leq a' < b' < b$ the structure $\zeta|_{\partial \Sigma[a', b']} := \zeta$ is induced from the genuine contact structure $\xi$ by an immersion $\partial \Sigma[a', b'] \to \Sigma \times \mathbb{R}$. Of course, this is an immersion of a very special type, which maps the boundary components $\Sigma \times a'$ and $\Sigma \times b'$ onto intersecting graphical hypersurfaces.

6.2 Saucers

A saucer is a domain $B \subset D \times \mathbb{R}$, where $D$ is a $2n$-disc possibly with a piecewise smooth boundary, of the form

$$B = \{(w, v) \in D \times \mathbb{R} : f_-(w) \leq v \leq f_+(w)\}$$

where $f_\pm : D \to \mathbb{R}$ are smooth functions such that $f_- < f_+$ on $\text{Int } D$ and whose $\infty$-jets coincide along $\partial D$. Observe that every saucer comes with a family of discs

$$D_s = \{(w, v) \in D \times \mathbb{R} : v = (1-s)f_-(w) + sf_+(w)\} \quad \text{for } s \in [0, 1]$$
such that the interiors $\text{Int} \, D_s$ foliate $\text{Int} \, B$ and the family of discs $D_s$ coincide with their $\infty$-jets along their common boundary $S = \partial D_s$, which is called the border of the saucer $B$.

A semi-contact structure on a saucer $B$ is a family $\{\zeta_s\}_{s \in [0,1]}$ of germs of contact structures along the discs $D_s$ for $s \in [0,1]$, which coincide as germs along the border $S$. As in Section 6.1 a semi-contact structure on a saucer $B$ defines an almost contact structure $\xi$ on $B$. Furthermore $(B, \xi)$ is a contact shell since $\zeta_0$ and $\zeta_1$ are germs of contact structures on $D_0$ and $D_1$ and the family $\zeta_s$ coincide along the border of $B$.

### 6.3 Regular semi-contact saucers

In $(\mathbb{R}^{2n+1}, \xi_{\text{std}}^{2n+1}) = \{\lambda_{\text{std}}^{2n-1} + v dt = 0\}$ where $v := -y_n$ and $t := x_n$, define the hyperplane $\Pi := \{v = 0\}$. Observe the characteristic foliation on $\Pi \subset (\mathbb{R}^{2n+1}, \xi_{\text{std}}^{2n+1})$ is formed by the fibers of the projection $\pi : \Pi \to \mathbb{R}^{2n-1}$ given by $\pi(x, t) = x$ for $x \in \mathbb{R}^{2n-1}$.

![Fig. 6.1: A typical regular contact saucer.](image)

Let $D \subset \Pi$ be a $2n$-disc and let $\phi : D \to \mathbb{R}$ be a smooth function such that $\phi > 0$ on $\text{Int} \, D \cap \mathcal{O}_p (\partial D)$ and whose $\infty$-jet vanishes on $\partial D$. Let $F : D \to \mathbb{R}$ be a function, compactly supported in $\text{Int} \, D$, so that $\phi + F$ is positive on $\text{Int} \, D$. Define the saucer $B := \{(w, v) \in D \times \mathbb{R} : w \in D, \ 0 \leq v \leq \phi(w) + F(w)\}$. Note that up to a canonical diffeomorphism the saucer $B$ is independent of a choice of the function $F$. There is a natural family of diffeomorphisms between $D_s \subset B$ and the graphs

$$\Gamma_{s\phi} := \{v = s\phi(w), \ w \in D\} \subset \mathbb{R}^{2n+1}$$

whose $\infty$-jets coincide along the border.

Define $\sigma_{\phi} = \{\zeta_s\}$ to be the semi-contact structure on $B$ where $\zeta_s$ is the pullback of the germ of the contact structure on $\Gamma_{s\phi} \subset (\mathbb{R}^{2n+1}, \xi_{\text{std}}^{2n+1})$. We see that $\phi$ defines the contact shell $(B, \sigma_{\phi})$ up to diffeomorphism of the domain.

Parametrize $B$ with coordinates $(w, s) \in D \times [0,1]$, so that $D_{s_0} = \{s = s_0\}$ and consider the map

$$\Phi : B \to \mathbb{R}^{2n+1} \quad \text{where} \quad \Phi(w, s) = (w, s\phi(w)).$$
If $\phi$ is positive everywhere on $\text{Int } D$, then $\Phi$ is an embedding, and hence $\sigma_\phi$ is a genuine contact structure since it can be identified with $\Phi^* \xi_{2n+1}$. Similarly for $2n$-discs $D' \subset D$ and associated semi-contact structures $\sigma_{\phi'}$ and $\sigma_\phi$ a contact shell $\sigma_{\phi'}$ is dominated by a shell $\sigma_\phi$ if $\phi' \leq \phi|_{D'}$ and $\phi|_{\text{Int } D \setminus D'} > 0$.

An embedded $2n$-disc $D \subset \Pi$ is called regular if

- the characteristic foliation $\mathcal{F}$ on $D \subset (\mathbb{R}^{2n+1}, \xi_{2n+1})$ is diffeomorphic to the characteristic foliation on the standard round disc in $\Pi$;
- the ball $\Delta := D/\mathcal{F}$ with its induced contact structure is star-shaped.

An embedded $2n$-disc $D \subset (M^{2n+1}, \xi)$ in a contact manifold is regular if the contact germ of $\xi$ on $D$ is contactomorphic to the contact germ of a regular disc in $\Pi$. A semi-contact saucer is regular if it is equivalent to a semi-contact saucer of the form $(B, \sigma_\phi)$ defined over a regular $2n$-disc $D \subset \Pi$.

In Section 7 we will prove the following proposition.

**Proposition 6.2.** Let $M$ be a $(2n+1)$-manifold, $A \subset M$ a closed subset, and $\xi_0$ an almost contact structure on $M$ that is genuine on $\partial p A \subset M$. There exists a finite number of embedded saucers $B_i \subset M$ for $i = 1, \ldots, N$ such that $\xi_0$ is homotopic relative to $A$ to an almost contact structure $\xi_1$ which is genuine on $M \setminus \bigcup_{i=1}^N B_i$ and whose restriction to each saucer $B_i$ is semi-contact and regular.
6.4 Fibered saucers

Slightly stretching the definition of a fibered shell we will allow \((2n+1)\)-dimensional discs \(B^\tau\) for \(\tau \in \partial T\) to degenerate into \(2n\)-dimensional discs, as in the following definition of fibered saucers. A domain \(T B \subset T \times D \times \mathbb{R}\) is called a fibered saucer if \(T = D^q\) and it has a form

\[ T B = \{(\tau, x, v) \in T \times D \times \mathbb{R} : f_-(\tau, x) \leq v \leq f_+ (\tau, x)\}, \]

where \(f_{\pm} : T \times D \to \mathbb{R}\) are two \(C^\infty\)-functions such that \(f_-(\tau, x) < f_+ (\tau, x)\) for all \((\tau, x) \in \text{Int} (T \times D)\) and \(f_{\pm}\) coincide along \(\partial (T \times D)\) together with their \(\infty\)-jet. Every fibered saucer comes with a family of discs

\[ D^\tau_s = \{(\tau, x, v) : x \in D, v = (1-s)f_-(\tau, x) + sf_+ (\tau, x)\} \]

where for fixed \(\tau \in T\) the discs \(\{D^\tau_s\}_{s \in [0,1]}\) coincide with their \(\infty\)-jets along their common boundary \(S^\tau = \partial D^\tau_s\). We call the union \(T S := \bigcup_{\tau \in T} S^\tau\) the border of the fibered saucer \(T B\).

A fibered semi-contact structure \(T \xi\) on a fibered saucer \(B\) is a family \(\zeta^\tau_s\) of germs of contact structures along discs \(D^\tau_s\) for \(s \in [0,1]\) and \(\tau \in T\), which coincide along the border \(T S\). A fibered semi-contact structure defines a fibered almost contact structure on \(T B\). In particular, any fibered semi-contact structure on a fibered saucer \(T B\) defines a fibered contact shell.

A fibered semi-contact structure on a fibered saucer \(B\) is called regular if the saucer \((B^\tau, \xi^\tau)\) is regular for each \(\tau \in \text{Int} T\). More precisely a fibered semi-contact saucer \(T \zeta = (T B, T \xi)\) is regular if there exists a regular \(2n\)-ball \(D \subset \Pi\) and a \(C^\infty\)-function \(\phi : T D = \bigcup_{\tau \in T} \{\tau \times D\} \to \mathbb{R}\) such that

- \(\phi\) vanishes with its \(\infty\)-jet along \(\partial (T D)\), and \(\phi > 0\) on \(\partial \phi \cap \text{Int} T D\);

- for each \(s \in [0,1]\) the contact structure \(\zeta_s\) is induced by an embedding onto a neighborhood of the graph \(\{y_n = s\phi (\tau, x), \tau \in T, x \in D^\tau\} \subset \mathbb{R}^{2n+1}\);

- the disc \(D\) is regular.

Thus a fibered regular semi-contact saucer is determined by the function \(\phi\), and we will denote it by \(T \sigma^\phi\).
6.5 Interval model

Proposition 6.2 says any contact shell dominates a collection of regular semi-contact saucers. So the next step towards proving Proposition 3.1 will be to relate regular semi-contact structures and circle model contact shells and this will be the goal of the remainder of the section.

We will start by introducing one more model contact shell, which we call an interval model, and it will help us interpolate between regular semi-contact saucers and circle models shells.

Recall that the standard contact \((\mathbb{R}^{2n-1}, \xi_{st})\) with \(\xi_{st}\) is given by the contact form

\[
\lambda_{st} = dz + \sum_{i=1}^{n-1} u_i d\varphi_i .
\]

In this section the notation \((v,t)\) stands for canonical coordinates on the cotangent bundle \(T^*I\).

For \(\Delta \subset \mathbb{R}^{2n-1}\) a compact star-shaped domain and a contact Hamiltonian

\[
K : \Delta \times S^1 \to \mathbb{R}\ 	ext{ such that } K|_{\partial \Delta \times S^1} > 0 \text{ and } K|_{\Delta \times \{0\}} > 0
\]

we will build a contact shell structure, similar to the circle model, on a piecewise smooth \((2n+1)\)-dimensional ball

\[
(B^I_{K,C}, \eta^I_{K,C}) \subset \Delta \times T^*I
\]

which we will refer to as the interval model contact shell for \(K\).

For any constant \(C > -\min(K)\), define the domain

\[
B^I_{K,C} := \{(x,v,t) \in \Delta \times T^*I : 0 \leq v \leq K(x,t) + C\}
\]

which is a piecewise smooth \((2n+1)\)-dimensional ball in \(\mathbb{R}^{2n-1} \times T^*I\) whose diffeomorphism type is independent of the choice of \(C\). Denote the boundary by

\[
\Sigma^I_{K,C} = \partial B^I_{K,C} = \Sigma^I_{0,K,C} \cup \Sigma^I_{1,K,C} \cup \Sigma^I_{2,K,C}
\]

where

\[
\Sigma^I_{0,K,C} = \{(x,v,t) : v = 0\} \subset \Delta \times T^*I
\]

\[
\Sigma^I_{1,K,C} = \{(x,v,t) : v = K(x,t) + C\} \subset \Delta \times T^*I
\]

\[
\Sigma^I_{2,K,C} = \{(x,v,t) : 0 \leq v \leq K(x,t) + C, (x,t) \in \partial(\Delta \times I)\} \subset \Delta \times T^*I.
\]

Now pick a smooth family of functions

\[
\rho_{(x,t)} : \mathbb{R}_{\geq 0} \to \mathbb{R} \ 	ext{ for } (x,t) \in \Delta \times I \text{ such that }
\]

(40)
(i) \( \rho_{(x,t)}(v) = v \) when \( v \in O \{0\} \),
(ii) \( \rho_{(x,t)}(v) = v - C \) for \( (x,t,v) \in O \{v \geq K(x,t) + C\} \), and
(iii) \( \partial_v \rho_{(x,t)}(v) > 0 \) for \( (x,t) \in O \partial(\Delta \times I) \)
which is possible by \( [39] \), and consider the distribution on \( \Delta \times T^*I \)
\[
\ker \alpha_\rho \text{ for the 1-form } \alpha_\rho = \lambda_{st} + \rho dt.
\]

We now have the following lemma, whose proof is analogous to Lemma 2.1.

**Lemma 6.3.** The almost contact structure given by \( \alpha_\rho \) defines a contact shell \((B_I,\eta_I)\) that is independent of the choice of \( \rho \) and \( C \), up to equivalence. If \( K > 0 \), then the contact germ \((\Sigma^I_I,\eta^I_I)\) extends canonically to a contact structure on \( B_I \).

Similarly, we also have a direct description of the contact germ \((\Sigma^I_I,\eta^I_I)\) without the shell given by gluing together the contact germs on the hypersurfaces
\[
\tilde{\Sigma}^I_{0,K} = \{ (x,v,t) : v = 0 \} \subset \Delta \times T^*I \\
\tilde{\Sigma}^I_{1,K} = \{ (x,v,t) : v = K(x,t) \} \subset \Delta \times T^*I \\
\tilde{\Sigma}^I_{2,K} = \{ (x,v,t) : 0 \leq v \leq K(x,t), (x,t) \in \partial(\Delta \times I) \} \subset \Delta \times T^*I
\]
to form a contact germ on \( \tilde{\Sigma}^I_K := \tilde{\Sigma}^I_{0,K} \cup \tilde{\Sigma}^I_{1,K} \cup \tilde{\Sigma}^I_{2,K} \).

**Lemma 6.4.** The contact germs on \( \Sigma^I_K \) and \( \tilde{\Sigma}^I_K \) are contactomorphic.

The proof is completely analogous to Lemma 2.3. Notice one important distinction compared to the circle model: the contact germ on \( \tilde{\Sigma}^I_K \) is defined by a global immersion of the sphere into \( \Delta \times T^*I \) (piecewise smooth, topologically embedded at the non-smooth points). This property allows us to use the interval model as a bridge between regular contact saucers and the circle model.

### 6.6 Relations between the model contact shells

We will now establish some domination relations between our three models.

**Proposition 6.5.** For star-shaped domains \( \Delta' \subset \text{Int} \Delta \), let \( K : \Delta \times S^1 \to \mathbb{R} \) be such that \( K|_{\Delta \times 0} > 0 \) and \( K|_{\Delta \setminus \Delta' \times S^1} > 0 \). For \( \Delta' := K|_{\Delta' \times S^1} \), the interval model contact shell \((B_I,\eta_I)\) dominates the circle model contact shell \((B_{K'},\eta_{K'})\).
Proof. Fix a $C > -\min K$ and a $\rho$ as in (40) that defines contact shell models

$$(B_K, \eta_{K,\rho}), (B_{K'}, \eta_{K',\rho}), \text{ and } (B^I_K, \eta_{K,\rho}^I).$$

Take any $\varepsilon > 0$ such that $K|_{\Delta \times [-\varepsilon, \varepsilon]} > \varepsilon$ and consider the domain

$$B^\varepsilon_K := B^I_K \setminus (\{v \leq \varepsilon\} \cup \{t \in [-\varepsilon, \varepsilon]\}) = \{(x, v, t) \in \Delta \times T^* I : \varepsilon \leq v \leq K(x, t) + C, \varepsilon \leq t \leq 1 - \varepsilon\}. \quad (41)$$

Note that $\eta_{K,\rho}^I$ restricted to $B^\varepsilon_K$ defines a contact shell $(B^\varepsilon_K, \eta_{K,\varepsilon}^I)$ that we will call the keyhole model, and it follows from (41) that $(B^\varepsilon_K, \eta_{K,\varepsilon}^I)$ dominates $(B_K, \eta_{K,\rho})$. It remains to show for sufficiently small $\varepsilon$ the shell $(B^\varepsilon_K, \eta_{K,\varepsilon}^I)$ dominates $(B_{K'}, \eta_{K',\rho})$.

Note that $(B^\varepsilon_K, \eta_{K,\varepsilon}^I)$ can be cut out of $(B_K, \eta_{K,\rho})$ by the same inequalities (41) where $(v, t)$ are viewed as coordinates $v = r^2$ and $t = \frac{\phi}{2\pi}$ on $\mathbb{R}^2$, rather than on $T^* I$. This embedding is shown on Figure 6.4 and explains the term ‘keyhole’.

For standard coordinates $(q, p) \in \mathbb{R}^2$ where $q = \sqrt{v} \cos(2\pi t)$ and $p = \sqrt{v} \sin(2\pi t)$, by the assumptions on $\rho$ in (40), the 1-form on $\Delta \times \mathbb{R}^2$ defining $\eta_{K,\rho}$ can be written as

$$\alpha_{\rho} = \lambda_{\text{st}} + \frac{\rho(v)}{2\pi v} (gdp - pdq)$$

and on $\Delta \times \mathcal{O} p \{ (q, 0) \in \mathbb{R}^2 : q \geq -2\delta \}$ is a genuine contact form for some $\delta > 0$.

Pick a smooth function $k : \Delta \to [-\delta, \infty)$ such that both $k(x) = -\delta$ on $\mathcal{O} p \partial \Delta$ and $k(x) = K(x, 0)$ on $\mathcal{O} p \Delta'$, and

$$\Gamma_k := \{(x, q, 0) \in \Delta \times \mathbb{R}^2 : -2\delta \leq q \leq k(x)\} \subset B_K.$$

Consider a smooth isotopy $\{\psi_s\}_{s \in [0, 1]}$ of $\Delta \times \{q \geq -2\delta, p = 0\}$ of the form $\psi_s(x, q) = (x, g_s(x, q))$, supported away from $\partial B_K$, and such that

$$\psi_1(\Gamma_k) = \{(x, q, 0) \in \Delta \times \mathbb{R}^2 : -2\delta \leq q \leq -\delta\} \subset B_K.$$
Since this isotopy preserves $\alpha_{\rho}|_{\Delta \times \{p=0\}} = \lambda_{st}$, it follows from a Moser-method argument (c.f. [21, Theorem 2.6.13]) that $\psi_s$ can be extended to a contact isotopy $\Psi_s$ of $B_K$ supported in $\Delta \times \mathcal{O}_{\rho} \{(q, 0) \in \mathbb{R}^2 : q \geq -2\delta\}$.

If $\varepsilon$ is small enough, then the contactomorphism $\Psi_1$ satisfies $\Psi_1(B_{K'}) \subset B_{\varepsilon}^\varepsilon$ and hence the keyhole model shell $(B_{K'}, \eta_{K'})$ dominates the circular model shell $(B_{K''}, \eta_{K''})$. □

We also have the following parametric version of Proposition 6.5 whose proof is analogous.

**Proposition 6.6.** Let $K^\tau : \Delta \times S^1 \to \mathbb{R}$ be a family of contact Hamiltonians such that $K^\tau|_{\Delta \times 0} > 0$, and $K^\tau|_{\partial \Delta \times S^1} > 0$. If $\Delta' \subset \text{Int} \Delta$ is a star-shaped domain and $K'' := K^\tau|_{\Delta' \times S^1}$, then the fibered shell $^T_\eta_{K'}$ dominates $^T_\eta_{K''}$.

The next proposition relates our saucer models from the previous section with the interval models discussed here.

**Proposition 6.7.** Let $\zeta = (B, \xi)$ be a regular semi-contact saucer viewed as a shell, then $\zeta$ dominates an interval model $\eta_K^I$ for some $K : \Delta \times I \to \mathbb{R}$.

To prove this we will need the following two lemmas 6.8 and 6.9.

**Lemma 6.8.** Let $\Delta$ be a compact contact manifold with boundary with a fixed contact form. Suppose $h, g : \Delta \to \Delta$ are contactomorphisms that are the time 1 maps of
isotopies generated by contact Hamiltonians $H, G : \Delta \times I \to \mathbb{R}$ that vanish with their $\infty$-jet on $\partial \Delta$. If $h = g$ on $\mathcal{O}p \partial \Delta$, then $h$ can be generated as the time 1 map of a Hamiltonian $\tilde{H} : \Delta \times I \to \mathbb{R}$ where $\tilde{H} = G$ on $\mathcal{O}p \partial (\Delta \times I)$.

\textbf{Proof.} Denote by $h_t$ and $g_t$ the contact diffeotopies generated by $H$ and $G$. Pick a contact diffeotopy such that $\tilde{h}_t = h_t$ on $\partial \Delta$ as $\infty$-jets and

$$
\tilde{h}_t = \begin{cases} 
  g_t & \text{when } t \in [0, \varepsilon] \\
  h_t & \text{when } t \in [2\varepsilon, 1 - 2\varepsilon] \\
  g_t \circ g_1^{-1} \circ h_1 & \text{when } t \in [1 - \varepsilon, 1]
\end{cases}
$$

Observe $\tilde{H} = G$ when $t \in [0, \varepsilon] \cup [1 - \varepsilon, 1]$ if $\tilde{h}_t$ is generated by $\tilde{H} : \Delta \times I \to \mathbb{R}$. Hence without loss of generality we may assume $H = G$ when $t \in \mathcal{O}p \partial I$.

Since $h_t$ and $g_t$ are $C^\infty$-small on $\mathcal{O}p \partial \Delta$, we can pick an isotopy

$$
\psi^s : \Delta \times I \to \Delta \times I \quad \text{for } s \in [0, 1] \text{ with } \psi^0 = \text{Id}
$$

supported in $\Delta \times \text{Int} I$ and such that for $x \in \mathcal{O}p \partial \Delta$ we have

$$
\psi^s(\cdot, t) \text{ is a contactomorphism and } \psi^1(h_t(x), t) = (g_t(x), t).
$$

Applying the Gray–Moser argument parametrically in $t$ builds a contact isotopy $\tilde{\psi}_t$ such that $\tilde{\psi}_t = \text{Id}$ when $t \in \mathcal{O}p \partial I$ and $\tilde{\psi}_1(h_t(x)) = g_t(x)$ when $x \in \mathcal{O}p \partial \Delta$. Defining $\tilde{h}_t := \tilde{\psi}_t \circ h_t$ and $\tilde{H}$ to be its generating contact Hamiltonian gives the result. \hfill \square

For the following lemma let $\Pi := \{(w, t, v) \in \mathbb{R}^{2n+1} \times T^*\mathbb{R} : v = 0\} \subset (\mathbb{R}^{2n+1}, \xi_{st}^{2n+1})$.

\textbf{Lemma 6.9.} For a star-shaped domain $\Delta \subset (\mathbb{R}^{2n-1}, \xi_{st})$ consider the disc

$$
D = \{(w, t) \in \Delta \times \mathbb{R} : h_-(w) \leq t \leq h_+(w)\} \subset \Pi
$$

where $h_\pm : \Delta \to \mathbb{R}$ are $C^\infty$-functions such that $h_- < h_+$. If $(B, \sigma_\phi)$ is a contact saucer defined over $D$, then it is equivalent to a contact saucer $(\tilde{B}, \sigma_{\tilde{\phi}})$ defined over $\tilde{D} = \Delta \times [0, 1]$.

\textbf{Proof.} We can assume that $B = \{0 \leq v \leq \Phi(w, t) : (w, t) \in D\}$, where the function $\Phi$ is positive on $\text{Int} D$ and coincides with $\phi$ on $\mathcal{O}p \partial D$, and hence $\partial B = D \cup \Gamma$ where $\Gamma := \{v = \Phi(w, t) : (w, t) \in D\}$.

Given a point $w \in \Delta$, consider a leaf $\ell_w$ through the point $(w, h_-(w)) \in D$ of the characteristic foliation $\mathcal{F}_\phi$ on the graph $\Gamma_\phi := \{v = \phi(w, t) : (w, t) \in D\}$ and let
the graph of the function \( \phi \) characteristic foliation \( F \) whose \( \infty \)
extends to a contactomorphism holonomy contactomorphism \( f \)
\( \Gamma \)-isotopy of \( \Delta \) defined by \( G \).

Consider the diffeomorphism \( G : D \to \tilde{D} = \Delta \times [0, 1] \)
defined by the formula

\[
G(w, t) = \left( w, \frac{t - h_-(w)}{h_+(w) - h_-(w)} \right).
\]

Since \( \lambda_{st} + v dt \) restricted to \( D \) and \( \tilde{D} \) is \( \lambda_{st} \) and is preserved by \( G \), it follows that \( G \)
extends to a contactomorphism \( G : \mathcal{O}_p D \to \mathcal{O}_p \tilde{D} \).

The characteristic foliation \( \mathcal{F}_{\tilde{\phi}} \) on the graph \( \Gamma_{\tilde{\phi}} = \{ v = \tilde{\phi}(w, t) : (w, t) \in \tilde{D} \} \)
is represented by \( \frac{\partial}{\partial t} \) for \( \tilde{\phi} \) is the contact vector field on \( \Delta \) for \( \tilde{\phi} : \Delta \to \mathbb{R} \)
thought of as a contact Hamiltonian. It follows that the holonomy contactomorphism \( \tilde{f} : \Delta \to \Delta \)
declared similarly to \( f_{\phi} \), coincides with the time one map of the contact
isotopy of \( \Delta \) defined by \( \tilde{\phi} : \Delta \times [0, 1] \to \mathbb{R} \)
thought of as a time-dependent contact Hamiltonian. According to Lemma 6.8 we can modify \( \tilde{\phi} \)
keeping it fixed over \( \mathcal{O}_p \partial D \), to make the holonomy contactomorphism \( \tilde{f}_{\phi} \) equal to \( f_{\phi} \).

Since the holonomy maps \( \tilde{f}_{\phi} \) and \( f_{\phi} \) are equal, it follows that there is a diffeomorphism \( F : \Gamma_{\phi} \to \Gamma_{\tilde{\phi}} \)
equal to \( G \) on \( \mathcal{O}_p \partial \Gamma_{\phi} \), mapping the characteristic foliation \( \mathcal{F}_{\phi} \)
to the characteristic foliation \( \mathcal{F}_{\tilde{\phi}} \), and with the form

\[
F(w, t, \phi(v(t))) = (f(w, t), \tilde{\phi}(f(w(t)))) \quad \text{for } (v, t) \in D
\]

for some diffeomorphism \( f : D \to \tilde{D} \). It follows that \( F \) extends to a contactomorphism
of neighborhoods \( F : \mathcal{O}_p \Gamma_{\phi} \to \mathcal{O}_p \Gamma_{\tilde{\phi}} \).

Let \((\tilde{B}, \sigma_{\tilde{\phi}})\) be a contact saucer over \( \tilde{D} \),
where \( \tilde{B} = \{ 0 \leq v \leq \tilde{\Phi}(w, t), (w, t) \in \tilde{D} \} \)
for some function \( \Phi : \tilde{B} \to \mathbb{R} \) that coincides with \( \tilde{\phi} \) on \( \mathcal{O}_p \partial \tilde{D} \) and is positive on \( \text{Int} \tilde{D} \).

Note \( \partial \tilde{B} = \tilde{D} \cup \tilde{\Gamma} \) where \( \tilde{\Gamma} := \{ v = \tilde{\Phi}(w, t), (w, t) \in \tilde{D} \} \).
Let us define a diffeomorphism \( H : \partial \tilde{B} \to \partial \tilde{B} \) so that

\[
H|_{\mathcal{O}_p \partial D} = G \quad \text{and} \quad H|_{\Gamma_{\phi}}(w, t, \Phi(v(t))) = (f(w, t), \Phi(f(v(t)))).
\]

This diffeomorphism matches the traces of contact structures on the boundaries \( \partial B \)
and \( \partial \tilde{B} \) of the saucers, and hence extends to a contactomorphism between \( \mathcal{O}_p \partial B \)
and \( \mathcal{O}_p \partial \tilde{B} \) which can be further extended to an equivalence between the saucers
\((B, \sigma_{\phi})\) and \((\tilde{B}, \sigma_{\tilde{\phi}})\). \( \square \)
Proof of Proposition 6.7. By the definition of regular semi-contact saucer from Section 6.2 we may assume that \( \zeta = (B, \sigma_\phi) \) where \( B \) is defined over a regular domain \( D \subset \Pi := \{ v = 0 \} \subset \mathbb{R}^{2n+1} \) and \( \sigma_\phi \) is given by a family of contact structures \( \zeta_s \) on neighborhoods of graphs \( D_s := \{ y_n = s \phi(w), \ w \in D \} \) where \( \phi : D \to \mathbb{R} \) is a \( C^\infty \)-function supported in \( D \) which is positive on \( \mathcal{O}_{p \partial D} \cap \text{Int} \ D \).

By the regularity assumption of \( D \) the projection \( \pi : D \to \mathbb{R}^{2n-1} \) is equivalent to the linear projection of the round ball and the image \( \Delta = \pi(D) \) is contactomorphic to a star-shaped domain in \( \mathbb{R}^{2n-1} \). Choose a slightly smaller star shaped ball \( \Delta' \subset \text{Int} \Delta \) such that \( \phi|_{\text{Int} \ D \cap \text{Int} \ D'} > 0 \) where \( D' := \pi^{-1}(\Delta') \cap D \). Note that the characteristic foliation on \( D' \) is not a regular foliation, rather it is diffeomorphic to the product foliation of the 2n − 1 disc and the interval. There exist functions \( h_+ : \Delta' \to \mathbb{R} \), \( h_- < h_+ \), such that \( D' = \{(w,t); \ h_-(w) \leq t \leq h_+(w), \ w \in \Delta' \} \subset \Pi \). Choose a function \( \phi' : D' \to \mathbb{R} \) that defines an immersion type semi-contact saucer \( (B', \sigma_{\phi'}) \) over \( D' \) such that \( \phi' \leq \phi|_{D'} \) and hence \( (B', \sigma_{\phi'}) \) is dominated by \( (B, \sigma_\phi) \).

Hence, we can apply Lemma 6.9 and find a function \( \tilde{\phi} : \tilde{D} := \Delta' \times [0, 1] \to \mathbb{R} \) which is positive near \( \partial \tilde{D} \) and such that the corresponding saucer \( (\tilde{B}, \sigma_{\tilde{\phi}}) \) over \( \tilde{D} \) is equivalent to \( (B', \sigma_{\phi'}) \).

Let us rescale the saucer \( (\tilde{B}, \sigma_{\tilde{\phi}}) \) by an affine contactomorphism of \( \mathbb{R}^{2n+1}_{st} \)

\[
(z, u, \varphi, t, v) \mapsto \left( (1 + \delta)^2 z, (1 + \delta)^2 u, \varphi, (1 + \delta)t - \frac{\delta}{2}, (1 + \delta)v \right)
\]

for \( \delta > 0 \), to an equivalent saucer \( (\hat{B}, \sigma_{\hat{\phi}}) \) over the domain \( \hat{D} := \hat{\Delta} \times [-\frac{\delta}{2}, 1 + \frac{\delta}{2}] \).

The notation \( \varphi \) stands for the tuple \( (\varphi_1, \ldots, \varphi_{n-1}) \) of angular coordinates. Note that \( \Delta' \subset \text{Int} \hat{\Delta} \) and we may choose \( \delta \) sufficiently small so that \( \hat{\phi}|_{\text{Int} \hat{D} \cap \text{Int} \hat{D}} > 0 \).

Then the restriction \( K := \hat{\phi}|_{\hat{D}} \) of the function \( \hat{\phi} \) to the domain \( \hat{D} = \Delta' \times [0, 1] \) defines an interval model shell \( (B^I_K, \eta^I_K) \). It is dominated by the saucer \( (\hat{B}, \sigma_{\hat{\phi}}) \), which is equivalent to \( (B', \sigma_{\phi'}) \), which in turn dominated by \( (B, \sigma_\phi) \).

Similarly, one can prove the following parametric version of Proposition 6.7.

**Proposition 6.10.** Let \( T\zeta = (T\, B, T\, \xi) \) be a fibered regular semi-contact saucer. Then \( T\zeta \) dominates a fibered interval model \( T\eta^K_I \) for some \( K : (T\, \Delta) \times I \to \mathbb{R} \).

**Remark 6.11.** Let us point out that the shells \( (B'^T, \xi'^T) \) degenerate when \( T \) approaches \( \partial T' \). Hence, the subordination map \( (T\, B^I_K, T\, \eta^I_K) \) to \( (T\, B, T\, \xi) \) has to cover an embedding \( T \to \text{Int} T \).
The next proposition is the main result in this section.

**Proposition 6.12.** If \((B, \xi) = \sigma_\phi\) is a regular semi-contact saucer viewed as a shell, then there is a time-independent contact Hamiltonian \(K : \Delta \to \mathbb{R}\) such that \((B, \xi)\) dominates the circle model contact shell \((B_K, \eta_K)\).

**Proof.** We first use Proposition 6.7 to find a dominated by \(\zeta\) interval model \(\eta_{I\tilde{K}}\) for some \(\tilde{K} : D \times I \to \mathbb{R}\). Then, we apply Proposition 6.5 to get a circular model contact shell \((B_{K'}, \eta_{K'})\) dominated by \((B_{I\tilde{K}}, \eta_{I\tilde{K}})\). Finally, choosing a time-independent contact Hamiltonian \(K < K'\) and applying Lemma 4.1 we get the required circle model contact shell \((B_K, \eta_K)\) dominated by \((B, \xi)\).

7 Reduction to saucers

7.1 Construction of contact structures in the complement of saucers

The goal of this section is to prove the Proposition 6.2. The starting point of the proof is Gromov’s \(h\)-principle for contact structures on open manifolds, which we will now formulate. Given a \((2n+1)\)-dimensional manifold \(M\), possibly with boundary, a closed subset \(A \subset M\) and a contact structure \(\xi_0\) on \(O_p A \subset M\) define \(\text{Cont}(M; A, \xi_0)\) to be the space of contact structures on \(M\) that coincide with \(\xi_0\) on \(O_p A\) and \(\text{cont}(M; A, \xi_0)\) to be the space of almost contact structures that agree with \(\xi_0\) on \(O_p A\). Let \(j : \text{Cont}(M; A, \xi_0) \to \text{cont}(M; A, \xi_0)\) be the inclusion map. We say that the pair \((M, A)\) is relatively open if for any point \(x \in M \setminus A\) either there exists a path in \(M \setminus A\) connecting \(x\) with a boundary point of \(M\), or a proper path \(\gamma : [0, \infty) \to M \setminus A\) with \(\gamma(0) = x\).

**Theorem 7.1.** [M. Gromov, 28, 30] Let \(M\) be a \((2n+1)\)-manifold, \(A \subset M\) a closed subset, and \(\xi_0\) a contact structure on \(O_p A \subset M\). Suppose that \((M, A)\) is relatively
open. Then the inclusion

\[ j : \text{Cont}(M; A, \xi_0) \rightarrow \text{cont}(M; A, \xi_0) \]

is a homotopy equivalence.

As we will see, Proposition 6.2 follows from the following special case.

**Lemma 7.2.** For a closed manifold \( \Sigma \), any semi-contact structure \( \xi = \{ \xi_s \} \) on the annulus \( C = \Sigma \times [0,1] \) is homotopic relative \( \partial \Sigma \times 0 \) to an almost contact structure \( \tilde{\xi} \) which is a genuine contact structure in the complement of finitely many saucers \( B_1, \ldots, B_k \subset C \) and such that the restriction \( \tilde{\xi}|_{B_j} \) for \( j = 1, \ldots, k \), is semi-contact and regular.

**Proof of Proposition 6.2.** Choose an embedded annulus \( C = S^{2n} \times [0,1] \subset M \setminus A \) and first use the existence part of Gromov’s h-principle 7.1 to deform \( \xi_0 \) relative to \( A \) to an almost contact structure which is genuine on \( M \setminus C \). Next with use the 1-parametric part of Theorem 7.1 applied to the family of neighborhoods of spheres \( S^{2n} \times t \) for \( t \in [0,1] \) to make the almost contact structure semi-contact on \( C \). Finally we use Lemma 7.2 to complete the proof.

We will need two lemmas in order to prove Lemma 7.2.
Observe we are free to partition $[0, 1] = \bigcup_{i=0}^{N} [a_i, a_{i+1}]$ where $a_i < a_j$ for $i < j$ and prove the lemma for the restriction of the semi-contact structure to each $\Sigma \times [a_i, a_{i+1}]$, which we will do multiple times through the proof.

**Lemma 7.3.** For a closed manifold $\Sigma$ and a semi-contact structure $\{\zeta_s\}$ on $\Sigma \times [0, 1]$ there exists $N > 0$ such that the restriction of $\{\zeta_s\}$ to $\Sigma \times [a_i, a_{i+1}]$ for $a_i := a + \frac{i}{N}$ is of immersion type for each $i = 0, \ldots, N - 1$.

**Proof.** Choose $\varepsilon > 0$ such that the contact structure $\zeta_s$ is defined on $\Sigma \times [s - \varepsilon, s + \varepsilon]$ for each $s \in [a, b]$. We will view $\zeta_s$ as a family of contact structures on $\Sigma \times [-\varepsilon, \varepsilon]$. For each $s_0$ and $\sigma > 0$ sufficiently small, a Darboux-Moser type argument implies there is an isotopy $\phi_{s_0}^s : \Sigma \times [-\sigma, \sigma] \to \Sigma \times [-\varepsilon, \varepsilon]$ such that $\phi_{s_0}^s = \text{Id}$ and $(\phi_{s_0}^s)^* \zeta_s = \zeta_{s_0}$ for $s \in [s_0, s_0 + \sigma]$. Moreover by shrinking $\sigma$ if necessary we can ensure that the hypersurfaces $\phi_{s_0}^s(\Sigma \times 0)$ are graphical in $\Sigma \times [-\varepsilon, \varepsilon]$. Hence for any $s_0 \in [a, b]$ the restriction of $\zeta_s$ to $[s_0, s_0 + \sigma]$ is of immersion type and therefore choosing $N > \frac{1}{\sigma}$ we get the required partition of $\Sigma \times [a, b]$ into the annuli of immersion type. \qed

**Lemma 7.4.** Let $\{\xi_s\}$ be a semi-contact structure on $\Sigma \times [0, 1]$. Then, after partitioning, $\{\xi_s\}$ is equivalent to a semi-contact structure $\{\zeta_s\}$ which is immersion type satisfying the following properties. There exists a smooth function $\psi : \Sigma \to [-\frac{R}{2}, \frac{R}{2}]$ and a contact structure $\mu$ on $\Sigma \times [-R, R]$ such that

- the germ of contact structure $\zeta_s = \Psi_s^* \mu$ on $\Sigma \times [s - \delta, s + \delta]$ where

  $$\Psi_s : \Sigma \times [s - \delta, s + \delta] \to \Sigma \times [-R, R]$$

  by the embedding $(x, s + t) \mapsto (x, s\psi(x) + t) \in \Sigma \times [-R, R]$ where $x \in \Sigma$ and $t \in [-\delta, \delta]$;

- there are closed domains $V \subset \Sigma$ and $\hat{V} \subset \text{Int} V$ such that $\psi|_V > 0$, and the contact structure $\mu$ is transverse to graphs of functions $s\psi$ over $\hat{W} = \Sigma \setminus \text{Int} \hat{V}$ for all $s \in [0, 1]$.

**Proof.** Let us endow $\Sigma \times [-R, R]$ with the product metric. Using Lemma 7.3 we may assume, by passing to a partition, that the semi-contact structure $\{\zeta_s\}$ on the annulus $C$ is of immersion type. So there is a contact structure $\mu$ on $\Sigma \times [-R, R]$ and a smooth family of embeddings

$$\Psi_s : \hat{C} := \Sigma \times [-\delta, \delta] \to \Sigma \times [-R, R] \quad \text{for } s \in [0, 1]$$
such that $\Psi_s(x, u) = (x, \psi_s(x) + u)$ for $u \in [-\delta, \delta]$, and $\Psi_s^* \mu$ is identified with $\zeta_s$. Furthermore, the partition argument also allows us to assume that $\psi_0 = 0$ and that the $C^1$-norm of $\psi_1$ is arbitrary small. We will impose the appropriate bound on its $C^1$-norm further down in the proof.

Define a subset

$$\widehat{V} := \{ x \in \Sigma : \text{the angle between } \mu_{x,0} \text{ and } T_{x,0} \Sigma \text{ in } T_{x,0} \widehat{C} \text{ is at most } \frac{\pi}{4} \}.$$ 

Assuming $\mu$ and $\Sigma$ cooriented, we present $\widehat{V}$ as a disjoint union $\widehat{V} = \widehat{V}_+ \cup \widehat{V}_-$ defined by whether the vector field $\frac{\partial}{\partial u}$ defines the same or opposite coorientation of $\mu|_\Sigma$ and $T \Sigma$.

We can choose a contact form $\alpha$ for $\mu$ such that its Reeb vector field $R$ satisfies $R(x, u) = \pm \frac{\partial}{\partial u}$ for $x \in \widehat{V}_\pm$ and $u \leq \sigma$ for a sufficiently small $\sigma > 0$. We will assume $\sigma > 0$ small enough to also ensure that

$$|\angle(\mu_{x,u}, \mu_{x,0})| \leq \frac{\pi}{16}$$

for $x \in \Sigma$ and $|u| \leq \sigma$.

Consider any smooth function $H : \Sigma \rightarrow [-1, 1]$ equal to $\pm 1$ on $\widehat{V}_\pm$. We further extend this function to $\Sigma \times [-R, R]$ as independent of the coordinate $u$ and then multiply by the cut-off function $\beta(u)$ equal to 1 on $[-\frac{R}{3}, \frac{R}{3}]$ and equal to 0 outside $[-\frac{R}{2}, \frac{R}{2}]$.

We will keep the notation $H$ for the resulting function on $\Sigma \times [-R, R]$.

Let $h : \Sigma \times [-R, R] \rightarrow \Sigma \times [-R, R]$, $t \in [0, 1]$, be the contact isotopy generated by $H$ as its contact Hamiltonian. Choose $\varepsilon \in (0, \frac{\sigma}{2})$ so small that $dh_\varepsilon$ rotates any hyperplane by an angle $< \frac{\pi}{16}$ and $||h_\varepsilon||_{C^0} < \sigma$. Then $\Gamma_\varepsilon := h_\varepsilon(\Sigma \times 0)$ is graphical, $\Gamma_\varepsilon = \{ u = \phi_\varepsilon(x) \}$ for a function $\phi_\varepsilon : \Sigma \rightarrow \mathbb{R}$ which satisfies the following conditions:

1. $\phi_\varepsilon|_{\widehat{V}} = \varepsilon$;
2. for every $x \in \widehat{W} := \Sigma \setminus \text{Int } \widehat{V}$ we have $\angle(T_z \Sigma, \mu_z) > \frac{\pi}{8}$, $z = (x, \phi_\varepsilon(x)) \in \Gamma_\varepsilon$.

As it was explained above, the function $\psi_1 : \Sigma \rightarrow \mathbb{R}$ entering the definition of the semi-contact annulus can be chosen arbitrarily $C^1$-small. Hence, by requiring it to satisfy the conditions $|\psi_1| < \frac{\pi}{8}$ and $||\psi_1||_{C^1} < \tan \frac{\pi}{16}$ we can conclude that $\widetilde{\Gamma} := h_\varepsilon(\{ u = \psi_1(x), x \in \Sigma \})$ is graphical, $\widetilde{\Gamma} = \{ u = \widetilde{\psi}_1(x) \}$ for a function $\widetilde{\psi}_1 : \Sigma \rightarrow \mathbb{R}$ which satisfies the following conditions.
\begin{itemize}
  \item \( \tilde{\psi}_1|_{\tilde{V}} \geq \frac{\alpha}{2} \);
  \item for every \( x \in \tilde{W} = \Sigma \setminus \text{Int} \tilde{V} \) and \( s \in [0, 1] \) we have
    \[
    \angle(T_z \tilde{\Sigma}_s, \mu_z) > \frac{\pi}{16}, \quad z = (x, s\tilde{\psi}_1(x)) \in \tilde{\Gamma}_s = \{ u = s\tilde{\psi}_1(x) \}.
    \]
\end{itemize}

Let us denote \( V := \{ \tilde{\psi}_1 \geq \frac{\alpha}{4} \} \), so that \( \tilde{V} \subset \text{Int} V \). Note that the family of contact structures \( \tilde{\zeta}_s \) on \( \Sigma \times s \subset \Sigma \times [0, 1] \) induced by \( \mu \) on the neighborhoods of graphs \( \Gamma_s \) of functions \( s\tilde{\psi} \) defines a semi-contact structure \( \{ \zeta_s \} \) which is equivalent to \( \{ \xi_s \} \). This concludes the proof of Lemma 7.4.

**Proof of Lemma 7.2.** According to Lemma 7.4 we may assume that the semi-contact structure \( \{ \zeta_s \} \) on \( \Sigma \times [0, 1] \) is of immersion type with the following properties. There exist a contact structure \( \mu \) on \( \Sigma \times [-R, R] \) and a function \( \psi : \Sigma \to [-R^2, R^2] \) such that

\begin{enumerate}
  \item the germ of contact structure \( \zeta_s \) is induced from \( \mu \) on a neighborhood of the \( \Sigma \times s \) by an embedding \( (x, s) \mapsto (x, s\psi(x) + t) \in \Sigma \times [-R, R], x \in \Sigma, t \in [-\delta, \delta] \);
  \item there are closed domains \( V \subset \Sigma \) and \( \tilde{V} \subset \text{Int} V \) such that \( \psi|_V > 0 \), and the contact structure \( \mu \) is transverse to graphs of functions \( s\psi \) over \( \tilde{W} := \Sigma \setminus \text{Int} \tilde{V} \) for all \( s \in [0, 1] \).
\end{enumerate}

We will keep the notation \( \psi \) for the restriction of \( \psi \) to \( \tilde{W} \). Note that \( \psi \) can be presented as the difference \( \psi = \psi_+ - \psi_- \) of two positive functions \( \psi_\pm \in C^\infty(\tilde{W}) \) such that the graphs of the functions \( s\psi_\pm \) are transverse to \( \mu \).

Let \( \{ U_i \}_{i=1}^N \) be a finite covering of \( W := \Sigma \setminus \text{Int} V \) by interiors of balls with smooth boundaries and such that \( \bigcup_{i=1}^N U_i \subset \tilde{W} \).

Let \( \{ \lambda_\pm^i : \Sigma \to [0, 1] \} \) be two partitions of unity on \( W \) subordinate to the covering \( \{ U_i \}_{i=1}^N \), \( \sum_{i=1}^N \lambda_\pm^i|_W = 1 \), such that \( \text{Support}(\lambda_\pm^i) \subset \text{Support}(\lambda_\pm^i), i = 1, \ldots, N \), and \( \sum_{i=1}^N \lambda_\pm^i|_{\tilde{W}} \leq 1 \).

For \( 0 \leq k \leq N \) define
\[
L_k := \sum_{i=1}^k \lambda_\pm^i
\]
noting \( L_N|_W = 1 \) and \( L_N|_{\tilde{V}} = 0 \) shows \( \tilde{V} \subset U \subset \Sigma \setminus W \), where \( U := \{ L_N < 1 \} \).
For $1 \leq i \leq N$ define the functions
\[
\psi_i^\pm := \psi_i^\pm \lambda_i^\pm : \Sigma \to \mathbb{R} \quad \text{and} \quad \psi_i := \psi_i^+ - \psi_i^- : \Sigma \to \mathbb{R}
\]
and for $0 \leq k \leq N$ the functions
\[
\Psi_k^\pm := \sum_{l=1}^{k} \psi_l^\pm \quad \text{and} \quad \Psi_k := \Psi_k^+ - \Psi_k^- = \sum_{l=1}^{k} \psi_l.
\]
One can further ensure that the graphs of the functions $\Psi^\pm$ are transverse to $\mu$. Let $\Gamma(\Psi_k) := \{u = \Psi_k(x), \ x \in \Sigma\} \subset \hat{C} = \Sigma \times [-R, R]$ be the graph of $\Psi_k$ and likewise $\Gamma(L_k) := \{u = L_k(x), \ x \in \Sigma\} \subset C = \Sigma \times [-1, 1]$ be the graph of $L_k$. Set $\Gamma_L := \bigcup_{k=0}^{N} \Gamma(L_k)$ and consider the map $p : \Gamma_L \to \Sigma \times [-R, R]$ to be well defined because if $(x, u) \in \Gamma(L_i) \cap \Gamma(L_j)$ for $0 \leq i < j \leq N$ then $\Psi_i^-(x) = \Psi_j^-(x)$. Indeed, $L_i(x) = L_j(x)$ implies that $\psi_i^+(x) = 0$ for all $i < l \leq j$, and hence $\psi_i^-(x) = 0$ for $i < l \leq j$ because $\text{Support}(\psi_i^-) \subset \text{Support}(\psi_i^+)$. Note that $p(\Gamma_L) = \bigcup_{i=0}^{N} \Gamma(\Psi_i)$. The map $p$ extends to an immersion $P : \mathcal{O} \Gamma_L \to \Sigma \times [-R, R]$. See Figure 7.2.

The complement $C \setminus \Gamma_L$ is a union of the domain $\Omega := \{(x, u); \ \Phi_N(x) \leq u \leq 1, \ x \in U\}$ and the interiors of saucers $B_i$ bounded by the graphs $\Gamma(L_{i-1})$ and $\Gamma(L_i)$ over the balls $U_i$, $i = 1, \ldots, N$.

We extend the immersion $P$ to $\Omega$ as a diffeomorphism $\Omega \to \{(x, u); \ \Psi_N(x) \leq u \leq \psi_1(x), \ x \in U\}$ which is fiberwise linear with respect to the projection to $\Sigma$. 

**Fig. 7.2:** The set $\Gamma_L$ used in the proof of the lemma. The region below a bump function is a saucer, so a partition of unity decomposes a general region into small saucers. Each saucer comes equipped with a map to $\hat{C}$ making it a contact shell.
It remains to extend the induced contact structure $P^*\mu$ on $\mathcal{O}_p \Gamma \cup \Omega$ as a regular semi-contact structure on the saucers. The following lemma is obvious.

**Lemma 7.5.** Let $\Sigma$ be a hypersurface in a $(2n+1)$-dimensional contact manifold transversal to the contact structure, and $f : D^{2n} \to \Sigma$ a smooth embedding of the unit $2n$-ball. Then there exists $\varepsilon > 0$ such that the disc $f(D^{2n}_\varepsilon)$ is regular.

It follows that the covering by balls $U^+_i$ can be chosen to ensure that the discs $(U^+_i, \zeta_0)$ are regular. If the function $\psi$ is sufficiently $C^1$-small then for each $i = 1, \ldots, N$ the graphs $\Gamma^-_i \subset \Gamma(\Psi_{i-1})$ and $\Gamma^+_i \subset \Gamma(\Psi_i)$ of the functions $\Psi_{i-1|\mathcal{U}_i}$ and $\Psi_i|\mathcal{U}_i$, respectively, are regular as well. Hence there is a contactomorphism $g_i$ between a neighborhood $O_i \supset \Gamma^-_i$ and a neighborhood of a disc in $\Pi = \{y_\alpha = 0\} \subset \mathbb{R}^{n+1}$. Again, if $\psi$ is sufficiently small then the neighborhood $O_i$ contains the disc $\Gamma^+_i$ and, moreover, $g_i(\Gamma^+_i)$ is transverse to the vector field $\frac{\partial}{\partial y_\alpha}$, which ensures the regularity of the contact saucer $B_i, i = 1, \ldots, N$.

Finally, it remains to observe that the required $C^1$-smallness of the function $\psi$ can be achieved by passing to a partition. This concludes the proof of Lemma 7.2.

### 7.2 Contact structures in the complement of saucers.
#### Parametric version

We prove in this section the following parametric version of Proposition 6.2.

**Proposition 7.6.** Let $M$ be a $(2n+1)$-manifold and $A \subset M$ a closed set, so that $M \setminus A$ is connected. Let $T^\xi_0$ be a fibered almost contact structure on $TM$ which is genuine on $(T \times \mathcal{O}_p A) \cup (\partial T \times M) \subset T \times M$. Then there exist a finite number of (possibly overlapping) discs $T_i \subset T$ and disjoint embedded fibered saucers $T_i B_i \subset TM, i = 1, \ldots, N$, such that $T^\xi_0$ is homotopic rel. $(T \times A) \cup \partial T \times M$ to a fibered almost contact structure $T^\xi_1$ which is genuine on $TM \setminus \bigcup B_i$ and whose restriction to each fibered saucer $T_i B_i$ is semi-contact and regular. Moreover, we can choose the discs in such a way that any non-empty intersection $T_{i_1} \cap \cdots \cap T_{i_k}, 1 \leq i_1 < \cdots < i_k \leq L$, is again a disc with piecewise smooth boundary.

As in the non-parametric case the following lemma is the main part of the proof.

**Lemma 7.7.** Any fibered semi-contact structure $T^\xi = \{\xi^s\}_{s \in [0,1], \tau \in T}$ on the fibered annulus $T^C := T \times \Sigma \times [0,1]$ is homotopic relative $T \times \mathcal{O}_p \partial C \cup \partial T \times C$ to a fibered almost contact structure $T^\zeta$ which is a genuine contact structure in the complement of finitely many fibered saucers $T_i B_1, \ldots, T_k B_k \subset T^C$, and such that the restriction
Lemma 7.8. Given a fibered semi-contact structure $T\xi$ on $T \times \Sigma \times [0,1]$ there exists $N > 0$ such that the restriction of $T\xi$ to $T \times \Sigma \times [a_i,a_{i+1}]$, $a_i := a + \frac{b-a}{N}$ is of immersion type for each $i = 0,\ldots,N-1$.

Proof of Lemma 7.8 Using Lemma 7.4 we can assume that the fibered semi-contact structure $T\xi$ on the annulus $T\Sigma \times C$ is of immersion type, i.e. there exists a fibered contact structure $T\mu = \{\mu_{\tau} \in T\}$ on $T\hat{\Sigma} := T \times \Sigma \times [-R,R]$ and a family of functions $\psi^\tau_s : \Sigma \rightarrow [-r,r]$, $r < R$, $s \in [0,1], \tau \in T$, such that the contact structure $\zeta^\tau$ on a neighborhood of $\Sigma_{s}^\tau$ is induced from $\mu^\tau$ by an embedding of this neighborhood onto a neighborhood of the graph of the function $\psi^\tau_s$.

Let $T' \subset \text{Int} T''$, $T'' \subset \text{Int} T$ be two slightly smaller compact parameter spaces such that the semi-contact structure $\mu^\tau$ is contact for $\tau \in T \setminus \text{Int} T'$, i.e. $\psi^\tau_s(x) > \psi^\tau_{s'}(x)$ for any $x \in \Sigma$, $\tau \in T \setminus \text{Int} T'$ and $s > s'$.

Similarly to the non-parametric case (see Lemma 7.4) we can reduce to the case when the following property holds: $\psi^\tau_0 = 0, \psi^\tau_s = s\psi^\tau$ and there exist domains $\hat{W}, W \subset T \times \Sigma$, $\text{Int} \hat{W} \ni W$ such that $\psi^\tau(x) > 0$ for $(x,\tau) \in T' \times \Sigma \setminus \text{Int} W$, and the contact structures $\mu^\tau$ transverse to the graphs $\Gamma^\tau_s$ of the functions $s\psi^\tau$ over $V^\tau := (\Sigma \setminus \text{Int} W) \cap \tau \times \Sigma$.

Denote by $\psi$ the function $\psi(\tau,x,s) := \psi^\tau(x,s)$ for $(\tau,x,s) \in W$, and present $\psi$ as the difference of two positive functions, $\psi = \psi^+ - \psi^-$.

Let us choose a finite covering of $W$ denoted $\{U_i\}$, such that $U_i = \text{Int} T_i \times \text{Int} \Delta^\pm_i$, where $\Delta_i \subset \Sigma$, $T_i \subset T''$, are balls with smooth boundaries, $i = 1,\ldots,N$, and $\bigcup_{i=1}^{N} U_i \subset \hat{W}$. Here we choose the discs $T_i \subset T$ in such a way that any non-empty intersection $T_{i_1} \cap \cdots \cap T_{i_k}$, $1 \leq i_1 < \cdots < i_k \leq L$, is again a disc with piecewise smooth boundary. More geometric constraints on the coverings will be imposed below.

Let $\lambda^\pm_i$ be two partitions of unity over $\hat{W}$ subordinated to $U_i$ so that $\text{Support}(\lambda^-_i) \in \text{Support}(\lambda^+_i)$,

$$\sum_{i=1}^{N} \lambda^\pm_i|_W = 1, \text{ and } \sum_{i=1}^{L} \lambda^\pm_i|_{\hat{W}} \leq 1.$$
Let us denote
\[ \psi_i^\pm := \psi_i^\pm \chi_i^\pm, \psi_i := \psi_i^+ - \psi_i^- \text{, } i = 1, \ldots, N. \]
Set
\[ \Psi_k^\pm := \sum_{i=1}^k \psi_i^\pm, \Psi_k := \Psi_k^+ - \Psi_k^- = \sum_{i=1}^k \psi_i, \Phi_k := \sum_{i=1}^k \chi_i^+. \]
Note that \( L_N|_W = 1 \) and \( L_N|_{\hat{\psi}} = 0 \), so \( V \subset U := \{L_N(x) < 1\} \subset T \times \Sigma \setminus W. \)
In \( T\hat{C} = T \times \Sigma \times [0,1] \) we let \( \Gamma(L_k) \) be the graph of the function \( L_k \), and in \( T\hat{C} = T \times \Sigma \times [-R, R] \) we let \( \Gamma(\Psi_k) \) be the graph of the function \( \Psi_k \). Set \( \Gamma_L = \bigcup_0^N \Gamma(L_i) \subset T\hat{C}. \)
Consider the map \( p : \Gamma_L \to T \times \Sigma \times [-R, R] \) given by the formula
\[ p(\tau, x, s) = (\tau, x, \psi^+ \psi^+(\tau, x)s - \Psi^- (\tau, x)) \text{ for } (\tau, x, s) \in \Gamma(L_i). \]
This map is well defined because if \( (\tau, x, s) \in \Gamma(L_i) \cap \Gamma(L_j) \) for \( 0 \leq i < j \leq N \) then \( \Psi_-^\pm (\tau, x) = \Psi_-^j (\tau, x) \). Indeed, \( \Phi_i(\tau, x) = \Phi_j(\tau, x) \) implies that \( \psi^+_i (\tau, x) = 0 \) for all \( i \leq l < j \), and hence \( \psi^-_i (\tau, x) = 0 \) for \( i < l \leq j \) because \( \text{Support}(\psi^-_i) \subset \text{Support}(\psi^+_i) \). Note that \( p(\Gamma_L) = \bigcup_0^N \Gamma(\Psi_i) \). The map \( p \) extends to an immersion \( P : \mathcal{O} p \Gamma_L \to T \times \Sigma \times [-R, R] \).
The complement \( T\hat{C} \setminus \Gamma_L \) is a union of the domain \( \Omega := \{(x, s); \Phi_N(\tau, x) \leq s \leq 1, \tau \in T, x \in U\} \) and interiors of fibered saucers \( T_i B_i \) bounded by the graphs \( \Gamma(L_i-1) \) and \( \Gamma(L_i) \) over the balls \( \overline{U_i}, i = 1, \ldots, N \).
We extend the immersion \( P \) to \( \Omega \) as a fiberwise linear, with respect to the projection to \( T \times \Sigma \), diffeomorphism
\[ \Omega \to \{(\tau, x, u); \Psi_N(\tau, x) \leq u \leq \psi(\tau, x), (\tau, x) \in U\}. \]
It remains to extend the induced contact structure \( P^*(T\mu) \) on \( \mathcal{O} p \Gamma_L \cup \Omega \) as a fibered regular semi-contact structure to the saucers.
It follows from Lemma \( 7.5 \) that the covering by balls \( U_i \) can be chosen to ensure that for each \( i = 1, \ldots, N \) and \( \tau \in T_i^+ \) the disc \( (\tau \times \Delta^+, \zeta^+_0) \) is regular. If the functions \( \psi_1 \) and \( \psi_0 \) are sufficiently \( C^1 \)-close than for each \( i = 1, \ldots, N \) the graphs \( \Gamma^-_i \subset \Gamma(\Psi_i-1) \) and \( \Gamma^+_i \subset \Gamma(\Psi_i) \) of the functions \( \Psi_i-1|_{\overline{T^+_i}} \) and \( \Psi_i|_{\overline{T^+_i}} \), respectively, are fibered over \( T_i^+ \) by regular discs as well. Hence, there is a fibered over \( T_i^+ \) contactomorphism \( g_i \) between a neighborhood \( O_i \supset \Gamma^-_i \) and a neighborhood of a fibered disc in \( T_i \times \Pi = \{y_n = 0\} \subset \mathbb{R}^{n+1} \). Again, if \( \psi \) is sufficiently close then
the neighborhood $O_i$ contains the disc $\Gamma_i^+$ and, moreover, $g_i(\Gamma_i^+)$ is transverse to the vector field $\frac{\partial}{\partial y_i}$, which ensures the regularity of the fibered contact saucer $T_i B_i$, $i = 1, \ldots, N$.

Finally, it remains to observe that the required smallness of the function $\psi$ can be achieved by passing to a partition.

**Proof of Proposition 7.6.** First assume $T = D^q$. Choose an embedded annulus $C = S^{2n} \times [0, 1] \subset M \setminus A$ and first use the $q$-parametric part of Gromov’s h-principle to deform $T \xi_0$ rel. $(T \times A) \cup (\partial T \times M)$ to a fibered almost contact structure which is genuine fibered contact structure on $T \times M \setminus C$. Next, with use of the $(q + 1)$-parametric part of Theorem 7.1 applied to the family of neighborhoods of spheres $\tau \times S^{2n} \times t$, $\tau \in T, t \in [0, 1]$ we make the fibered almost contact structure semi-contact on $T C = T \times C$. Finally we use Lemma 7.7 to complete the proof. For general $T$ we triangulate it and inductively over skeleta apply the previous proof to each simplex.

8 Reduction to a universal model

We prove in this section Proposition 3.1.

8.1 Equivariant coverings

The key step in the proof of Proposition 3.1 is the following

**Proposition 8.1.** For a fixed dimension, there is a finite list of saucers $\{(B_p, \zeta_p)\}$ for $p = 1, \ldots, L$, with the following property. For any circle model contact shell $(B_K, \eta_K)$ defined by a time-independent contact Hamiltonian $K : \Delta \rightarrow \mathbb{R}$, there exist finitely many disjoint balls $B_i \subset B_K$ for $i = 1, \ldots, q$ so that the contact shell $(B_K, \eta_K)$ is homotopic relative to $O p \partial B_K$ to an almost contact structure $\xi$ that is genuinely contact on $B_K \setminus \bigcup_{i=1}^q B_i$ and each contact shell $\xi|_{B_i}$ is equivalent to one of the saucers $(B_p, \zeta_p)$ for $p = 1, \ldots, L$.

**Remark 8.2.** The proof of Proposition 8.1 follows roughly the same scheme as the proof of Proposition 6.2 but uses the idea of equivariant coverings in a crucial way. The basic idea can be seen in the following trivial observation about real functions. Consider the piecewise constant function $\phi : \mathbb{R} \rightarrow \mathbb{R}$ which is equal to 1
on \([0, 1) \cup [2, 3)\), equal to \(-3\) on \([1, 2)\), and 0 elsewhere. Let the group \(\mathbb{Z}\) act on \(\mathbb{R}\) by translation: \(j \in \mathbb{Z}\) being identified with the map \(x \mapsto x + j\). Then the function \(\sum_{j=1}^{k} \phi \circ j\) is equal to 1 on \([0, 1) \cup [k + 2, k + 3)\) and it is strictly negative on \([1, k + 2)\).

The key point of this example is two-fold: firstly, that a function which is negative on an arbitrarily large portion of its support can be written as a sum of functions which are negative on a small subset of their support. And secondly, that in fact these functions can be taken to be translations of a single function by a group action.

Consider \(\mathbb{R}^{2n+1}\) with the contact structure \(\xi_{st}\) given by the form

\[
\alpha = dz + \sum_{i=1}^{n-1} (x_i dy_i - y_i dx_i) - y_n dx_n = dz + \sum_{i=1}^{n-1} u_i d\varphi_i - y_n dx_n.
\]

Denote \(\Pi = \{y_n = 0\}\). In the group of contactomorphisms \(\text{Diff}(\mathbb{R}^{2n+1}, \xi_{st})\) consider the 2\(n\)-dimensional lattice \(\Theta\) generated by the following transformations:

- translations
  \[
  T_z : (x, y, z) \mapsto (x, y, z + 1) \\
  T_{x_n} : (x_1, \ldots, x_n, y, z) \mapsto (x_1, \ldots, x_n + \frac{1}{2}, y, z),
  \]
- sheers
  \[
  S_{y_j} : (x_1, \ldots, y_j, \ldots, y_n, z, t) \mapsto (x_1, \ldots, y_j + 1, \ldots, y_n, z + x_j), \quad j = 1, \ldots, n-1 \\
  S_{x_j} : (x_1, \ldots, x_j, \ldots, x_n, y, z) \mapsto (x_1, \ldots, x_j + 1, \ldots, x_n, y, z - y_j), \quad j = 1, \ldots, n-1.
  \]

Notice that \(\Theta\) preserves \(\Pi\), we have \(S_{y_j} S_{x_j} S_{y_j}^{-1} S_{x_j}^{-1} = T_z^2\), and all other transformations commute. Hence every element of \(\Theta\) may be written as

\[
S_{x_1}^{k_1} \cdots S_{x_{n-1}}^{k_{n-1}} S_{y_1}^{l_1} \cdots S_{y_{n-1}}^{l_{n-1}} T_{x_n}^{k_n} T_{z}^{l_n}
\]

and from this it follows that \(\Theta\) acts properly discontinuously on \(\Pi\), that is for any compact set \(Q \subseteq \Pi\), the set

\[
S(Q) := \{g \in \Theta : g(Q) \cap Q \neq \emptyset\} \subset \Theta
\]

is finite.
For any positive number $N$, let $C_N$ be the scaling contactomorphism

$$(x, y, z) \mapsto (Nx, Ny, N^2 z).$$

Let $\Theta_N = C_N^{-1} \Theta C_N$, that is $\Theta_N$ is the group generated by translations $T_{j,N} := C_N^{-1} \circ T_j \circ C_N$, $T_{z,N} := C_N^{-1} \circ T_z \circ C_N$ and sheers $S_{j,N} := C_N^{-1} \circ S_j \circ C_N$. Say that a compact set $Q$ generates a $\Theta_N$-equivariant cover of $\Pi$ if $\Theta_N \cdot \text{Int}(Q) = \Pi$. Below we will always assume that $N$ is a positive integer, in this case the element $T_{x_n} := T_{N}x_n$ belongs to $\Theta_N$. We denote by $\Upsilon$ the normal subgroup in $\Theta_N$ generated by $T_{2x_n}$, and by $\hat{\Theta}_N$ the quotient group $\Theta_N/\Upsilon$. We call a compact set $Q \subset \Pi$ sufficiently small if $T_{2x_n}(Q) \cap Q = \emptyset$.

Notice that the quotient of $\mathbb{R}^{2+n}$ by the contactomorphism $T_{2x_n}$ is the contact manifold

$$\left(\mathbb{R}^{2n-1} \times T^*S^1, \ker(dz - \sum_{i=1}^{n-1} (x_idy_i - y_idx_i) + vdt)\right)$$

where $v = -y_n$ is identified with the fiber coordinate of $T^*S^1$ and the base coordinate $t \in \mathbb{R}/\mathbb{Z}$ is given by the quotient by translation $T_{2x_n}^2$. Denote this quotient by $\pi : \mathbb{R}^{2n+1} \to \mathbb{R}^{2n-1} \times T^*S^1$. The group $\hat{\Theta}_N$ can be viewed as a subgroup of the group of contactomorphisms of $\mathbb{R}^{2n-1} \times T^*S^1$ preserving $\Pi = \pi(\Pi) \simeq \mathbb{R}^{2n-1} \times \mathbb{R}/\mathbb{Z}$. Any compactly supported function $\Phi : \Pi \to \mathbb{R}$ defines a function $\sum_{h \in \Upsilon} \Phi \circ h^{-1}$ which is 1-periodic in the $x_n$-variable, and therefore defines a function $\hat{\Phi} : \hat{\Pi} \to \mathbb{R}$.

**Remark 8.3.**

a) If $Q$ generates a $\Theta$-equivariant covering of $\Pi$ then $Q_N := C_N^{-1}(Q)$ generates a $\Theta_N$-equivariant covering. For a sufficiently large $N$ the set $Q_N$ is sufficiently small.

b) Suppose that $a > 1/2$. Then the parallelepiped

$$P = \{|z| \leq a, |x_j|, |y_j| \leq a, 1 \leq j \leq n-1, 0 < x_n \leq a, y_n = 0\} \subset \Pi$$

generates a $\Theta$-equivariant covering of $\Pi$. If $a < 1$ then $P$ is sufficiently small. In particular, there are sufficiently small sets generating equivariant coverings.

c) If $Q' \subset Q \subset \Pi$ are two compact sets, and $Q'$ generates a $\Theta_N$-equivariant covering of $\Pi$, then so does $Q$.

Let us fix for the rest of the paper a regular sufficiently small disc $Q \subset \Pi$ and a smaller disc $Q' \subset \text{Int} Q$ which generates a $\Theta$-equivariant covering of $\Pi$. We Denote by $m$ the cardinality $|S(Q)|$ of the set $S(Q)$. 

We also fix two non-negative $C^\infty$-functions $\phi_+, \phi_- : \Pi \to \mathbb{R}$ which are supported in $Q$ which satisfy the following conditions:

(i) $\phi_+|_{\text{Int } Q} > 0$, $\phi_-|_{Q'} > 0$, and $\phi_-|_{\partial_p(\partial Q)} = 0$;

(ii) $\max(\phi|_{Q'}) < -(m + 1)\mu$, where $\phi := \phi_+ - \phi_-$, $\mu := \max(\phi)$ and $m$ is the cardinality of the set $S(Q)$ defined in (42);

(iii) denote $\phi^s = \phi_+ - s\phi_-$, $s \in [0, 1]$, (so that $\phi^s \geq \phi^1 = \phi$). For any finite subset $F \subset \Theta$ denote

$$\Phi^s_F := \mu + \sum_{g \in F} \phi^s \circ g^{-1}|_Q, \quad s \in [0, 1].$$

Then the graph $y_n = \Phi^+_F(q)$, $q \in Q$, with the induced contact structure is regular.

**Remark 8.4.** In condition (iii) elements $g \in F$ with $g(Q) \cap Q = \emptyset$ are irrelevant, so it suffices to verify (iii) only for subsets $F$ of the finite set $S(Q)$. Hence the condition can always be satisfied by taking $\phi_+$ and $\phi_-$ sufficiently small (e.g. replacing the pair $(\phi_+, \phi_-)$ which satisfy (i) and (ii) by $(\varepsilon \phi_+, \varepsilon \phi_-)$ for a sufficiently small $\varepsilon > 0$.)

Let us linearly order in any way all the elements of $\Theta$: $g_1, g_2, \ldots$ and order accordingly $\hat{\Theta}_N$, we fix this ordering during the paper. Define functions $\Pi \to \mathbb{R}$ by the formulas

$$\Phi_k := \mu + \sum_{j=1}^k \phi \circ g_j^{-1} \quad \text{and} \quad \Psi_k = \mu + \sum_{j=1}^k \phi_+ \circ g_j^{-1}$$

for $k = 0, 1, \ldots$, and note $\Phi_0 = \Psi_0 \equiv \mu$.

Let

$$\Phi_{k-1}^- := \{y_n = \Phi_{k-1}(x_1, \ldots, x_n, y_1, \ldots, y_{n-1}, z); (x_1, \ldots, x_n, y_1, \ldots, y_{n-1}, z) \in g_k(Q)\}$$

be the graph of $\Phi_{k-1}$ over the set $g_k(Q)$, and similarly denote by $\Phi^+_{k-1}$ the graph of $\Phi_k$ over $g_k(Q)$. Denote by $\Psi_{k-1}^+$ the graph of $\Psi_k$ over $g_k(Q)$ and by $\Psi^-_{k-1}$ the graph of $\Psi_{k-1}$ over $g_k(Q)$. Define $B_k$ to be the saucer

$$B_k := \{\Psi_{k-1}(x_1, \ldots, x_n, y_1, \ldots, y_{n-1}, z) \leq y_n \leq \Psi_k(x_1, \ldots, x_n, y_1, \ldots, y_{n-1}, z), (x_1, \ldots, x_n, y_1, \ldots, y_{n-1}, z) \in g_k(Q)\}$$

bounded by $\Psi^-_{k-1}$ and $\Psi^+_{k-1}$. Similar to the proof of Proposition 6.2 we observe that there is an immersion $\partial_p \partial B_k \to \mathbb{R}^{2n+1}$ which maps $\Psi^-_{k-1} \to \Phi^-_{k-1}$ diffeomorphically.
The induced contact structure $\zeta_k$ with its canonical semi-contact extension to $B_k$ defines a shell structure on the saucer $B_k$.

More generally for $s \in [0, 1]$ set

$$\Phi^s_k := \mu + \sum_{j=1}^{k} \phi^u \circ g_j^{-1}$$

so that $\Phi^1_k = \Phi_k$ and $\Phi^0_k = \Psi_k$. Define the regular contact saucer $(B_k, \zeta^s_k)$, $k = 1, \ldots$, as induced by an immersion of $O \rightarrow \mathbb{R}^{2n+1}$ that maps $\partial B_k = \Psi \Gamma_k^+ \cup \Psi \Gamma_k^-$ diffeomorphically onto the graphs of $\Phi^s_k$ and $\Phi^s_{k-1}$ over $g_k(Q)$. Regularity is ensured by the above condition (iii).

Therefore, the above construction builds 1-parameter families for $s \in [0, 1]$ of regular contact saucers $(B_k, \zeta^s_k)$, $k = 0, 1, \ldots$. However as the next lemma shows, up to equivalence the number of these is always bounded by $L = 2^m$, where $m := |S(Q)|$ is the cardinality of the set from [42].

**Lemma 8.5.** Up to equivalence the above construction builds at most $L = 2^m$, $m = |S(Q)|$, one-parametric families of regular contact saucers $(B_k, \zeta^u_k)$.

**Proof.** By the contactomorphism $g_k^{-1}$ we know $\zeta^s_k$ is equivalent to a saucer whose boundary contact germ is defined by the two graphs over $Q$:

$$\{y_n = (\Phi^s_{k-1} \circ g_k)|_Q\} \text{ and } \{y_n = (\Phi^u_k \circ g_k)|_Q\}.$$ 

However $\Phi^s_k|_Q \circ g_k = \mu + \sum_{j=1}^{k} (\phi^u \circ (g_j^{-1} g_k))|_Q$ and the number of different sums of this type is bounded above the number $L = 2^m$ of finite subsets of the set $S(Q)$.

Given a positive $N$ and an element $g \in \Theta_N$ we denote $\phi_{g,N} := \frac{1}{N} \phi \circ C_N \circ g^{-1}$. Notice that by the contactomorphism $C_N^{-1}$ the regular semi-contact saucer which is defined over the domain $Q$ by the functions $\Phi^s_{k-1}$ and $\Phi^s_k$ is equivalent to the saucer over the domain $C_N^{-1}(Q)$ defined by the functions $\Phi^s_{k-1,N} := \frac{\mu}{N} + \sum_{j=1}^{k-1} \phi^u_{g_j,N}$ and

$$\Phi^s_{k,N} := \frac{\mu}{N} + \sum_{j=1}^{k} \phi^s_{g_j,N}, \text{ where } \phi^s_{g_j,N} := \frac{1}{N} \phi^s \circ C_N \circ g^{-1}.$$ 

Consider the function $\widehat{\phi}_{g,N} : \widehat{\Pi} \rightarrow \mathbb{R}$ We note that $\widehat{\phi}_{g,N} = \widehat{\phi}_{g',N}$ if $g, g'$ are in the same conjugacy class from $\hat{\Theta}_N = \Theta_N/\Upsilon$, so that in the notation for $\widehat{\phi}_{g,N}$ we can use $g \in \hat{\Theta}_N$. 


Lemma 8.6. With the above choices of \(Q, Q', \phi_+, \phi_-\), for any bounded open domains \(U'\) and \(U \ni U'\) in \(\mathbb{R}^{2n-1}\) and any \(C^\infty\)-function \(K: U \to \mathbb{R}\) which is positive on \((U \setminus U')\), there exist \(N > 0\) and a finite subset \(\Lambda \subset \hat{\Theta}_N\) such that

\[
\begin{align*}
&\bullet \quad U' \times S^1 \subseteq \bigcup_{g \in \Lambda} g(\text{Int} \pi(Q'_N)) \subset \bigcup_{g \in \Lambda} g(\text{Int} \pi(Q_N)) \subseteq U \times S^1; \\
&\bullet \quad \sum_{g \in \Lambda} \hat{\phi}_{g,N} < \begin{cases} 
-\frac{2\mu}{N} & \text{on } U' \times S^1; \\
K - \frac{\mu}{N} & \text{on } (U \setminus U') \times S^1.
\end{cases}
\end{align*}
\]

Proof. Suppose \(K: U \to \mathbb{R}\) is given. Since \(K\) is positive on \(U \setminus U'\), we may fix some \(\varepsilon > 0\) with the property that the set \(P := \{(x, y, z) \in U \setminus U': K(x, y, z) > \varepsilon\}\) disconnects \(U'\) from \(\partial U\). Notice that the conclusion of the lemma only becomes stronger if we enlarge the set \(U' \subseteq U\). With this in mind we redefine \(U'\) to be the interior of the union of all components of \(U \setminus P\) which are disjoint from \(\partial U\).

Set \(\hat{Q}_N := \pi(Q_N), \hat{Q}'_N := \pi(Q'_N)\). For a sufficiently large \(N\) there exists a finite set \(\Lambda \subset \hat{\Theta}\) such that \((U' \cup P) \times S^1 \ni \bigcup_{g \in \Lambda} g(\hat{Q}_N) \supset \bigcup_{g \in \Lambda} g(\hat{Q}'_N) \ni U' \times S^1\). Furthermore, suppose that

\[
N > (m + 1)\mu\varepsilon^{-1}.
\]

Then, using (43) we get on \(P \times S^1\)

\[
\sum_{g \in \Lambda} \hat{\phi}_{g,N} < \frac{m\mu}{N} = \frac{(m+1)\mu}{N} - \frac{\mu}{N} < \varepsilon - \frac{\mu}{N} < K - \frac{\mu}{N}.
\]

On the other hand, on \(U' \times S^1\) we have

\[
\sum_{g \in \Lambda} \hat{\phi}_{g,N} < -\frac{(m+1)\mu}{N} + \frac{(m-1)\mu}{N} < -\frac{2\mu}{N}.
\]

Indeed this holds, for given \((x, y, z) \in g(Q'_N)\) because according to inequality (ii) a single negative term \(\hat{\phi}_{g,N}(x, y, z)\) is larger in absolute value by at least \(\frac{2\mu}{N}\) than the sum of all positive terms (the denominator \(N\) appears because of the scaling factor of the function in the definition of \(\hat{\phi}_{g,N}\)).

Proof of Proposition 8.1. Let \(U = \text{Int} \Delta\) and \(U' \subseteq U\) be a star-shaped subset such that \(K|_{U \setminus U'} > 0\). Apply Lemma 8.6 providing an integer \(N > 0\), and a finite set...
Λ ⊂ ̃ΘN. Then the corresponding function

$$\Phi = \Phi^{S1,N} = \frac{\mu}{N} + \sum_{g \in \Lambda} \phi_{g,N} : \Delta \times S^1 \to \mathbb{R}$$

satisfies $\Phi(w,t) < K(v)$ for $w \in \Delta \setminus U'$ and $\Phi|_{U' \times S^1} < -\frac{\mu}{N}$. According to Proposition 4.9 there exists a contact Hamiltonian $\tilde{K}$ so that $\eta_{\tilde{K}}$ is dominated by $\eta_K$, where $K|_{\Delta \setminus U'} = \tilde{K}|_{\Delta \setminus U'}$ and $\tilde{K}|_{U'} > -\frac{\mu}{N}$. Therefore $\Phi(w,t) < \tilde{K}(w)$ for all $(w,t) \in \Delta \times S^1$. The function $\Phi$ is equal $\frac{\mu}{N} > 0$ near $\partial \Delta \times S^1$ and hence defines a circular shell model $\eta_{\Phi}$ which is dominated by $\eta_{\tilde{K}}$. Hence it sufficient to prove the required extension result for $\eta_{\Phi}$. We order $\Lambda$ using the chosen ordering of $\hat{\Theta}$ and define functions

$$\Phi_k = \frac{\mu}{N} + \sum_{j=1}^{k} \phi_{g_j,N} : \Delta \times S^1 \to \mathbb{R}, \ k = 0, \ldots, |\Lambda|,$$

where $|\Lambda|$ is the cardinality of $\Lambda$. We have $\Phi_0 = \frac{\mu}{N}$ and $\Phi|_{\Lambda} = \Phi$. The shells $\eta_{\Phi_k}$ and $\eta_{\Phi_{k-1}}$ differ by one of the regular saucers $(B_p, \zeta_p)$, from the finite list provided by Lemma 8.5, while the shell $\eta_{\Phi_0}$ is solid, since $\Phi_0 > 0$ everywhere.

Now we are ready to prove Proposition 3.1.

**Proof of Proposition 3.1.** Proposition 6.2 allows us to assume that $\xi$ is contact outside of a finite collection of disjoint saucers $\{B_i\}_{1 \leq i \leq N}$, so that the restriction $\xi|_{B_i}$ for each $i = 1, \ldots, N$, is a regular semi-contact saucer. Using Proposition 6.12 we replace saucers by circular model shells defined by time-independent contact Hamiltonians. Applying Proposition 8.1 we can further reduce to the case of a contact structure in the complement of saucers from the finite list $(B_p, \zeta_p)$, $p = 1, \ldots, L$. Using again Proposition 6.12 we replace saucers by circular model shells $(B_{K_p}, \eta_{K_p})$ defined by time-independent contact Hamiltonians. We may then choose any special Hamiltonian $K_{univ}$ satisfying $K_{univ}(x) < \min_p K_p(x)$.

### 8.2 The standardization of the holes in the parametric case

In this section we prove Proposition 3.10.

Given a special Hamiltonian $K : \Delta_{cyl} \to \mathbb{R}$ we recall the notation from Section 3.1

$$K^{(s)} := sK + (1 - s)E, \ s \in [0,1], \text{ where } E(u,z) := K(u,1),$$
Lemma 8.7. There exist a special Hamiltonian $K_{\text{univ}} : \Delta_{\text{cyl}} \to \mathbb{R}$ and a non-increasing function $\theta : [0, 1] \to [0, 1]$ with $\theta(0) = 0$ and $\theta(1) = 1$, which depend only on the choice of $Q, Q', \phi_+, \phi_-$, and such that for each $p = 1, \ldots, 2^m$ there exists a family of subordination maps

$$\eta_{K_{\text{univ}}^{(\theta(s))}} \to \eta_{p} := (B_p, \zeta_p^s), \ s \in [0, 1],$$

which are solid for $s = 0$.

Proof. We note that there exists $\delta > 0$ such that the regular saucer $(B_p, \zeta_p^s)$ is solid for $s \in [0, \delta]$, i.e. the contact structure on its boundary is extended inside as a genuine contact structure. Proposition 6.13 implies that the family of saucers $(B_p, \zeta_p^s)$ dominates a family of circular models $\tilde{K}_p^s$, where $\tilde{K}_p^s > 0$ for $s \in [0, \delta']$, $p = 1, \ldots, 2^m$, for some $\delta' < \delta$. We also note that Lemma 4.6 allows us to assume that the domain $\Delta$ in the definition of the Hamiltonians $\tilde{K}_p^s$ coincides with $\Delta_{\text{cyl}}$. Choose as $K_{\text{univ}}$ any special contact Hamiltonian which satisfies

$$K_{\text{univ}} < \min_{s \in [0, 1], p=1,\ldots,2^m} \tilde{K}_p^s.$$  

(See Example 3.4.) There exists $\delta'' \in (0, \delta')$ such that $\tilde{K}_p^s > K_{\text{univ}}^{(0)}$ for all $s \in [0, 1], p = 1, \ldots, 2^m$. Choose a non-decreasing function $\theta : [0, 1] \to [0, 1]$ such that $\theta(s) = 0$ for $s \in [0, \frac{\delta'}{2}]$ and $\theta(s) = 1$ for $s \in [\delta'', 1)$. Then $K_{\text{univ}}^{(\theta(s))} < \tilde{K}_p^s$ for all $s \in [0, 1], p = 1, \ldots, 2^n$. Hence, by Lemma 4.1 one can arrange the inclusion maps $\eta_{K_{\text{univ}}^{(\theta(s))}} \to \eta_{p}^{s}$ be subordinations.

Remark 8.8. It is not clear that any Hamiltonian $K_{\text{univ}}$ satisfying Proposition 3.1 also satisfies Lemma 8.7, or conversely. But once we know that there are two Hamiltonians separately satisfying Proposition 3.1 and Lemma 8.7 we can simply choose $K_{\text{univ}}$ to be less than both of them, and this Hamiltonian will suffice for both.

Let $T = D^q \subset \mathbb{R}^q$ be the unit disc. Choose any decreasing $C^\infty$-function $\theta : [0, 1] \to [0, 1]$ which is equal to 1 on $[0, \frac{1}{4}]$ and to 0 on $[\frac{3}{4}, 1]$.

**Proposition 8.9.** There is a universal finite list of families saucers $(B_p^s, \zeta_p^s), \ p = 1, \ldots, L, s \in [0, 1]$, where $L$ depends only on dimension $n$, with the following property. Let $K^\tau : \Delta \to \mathbb{R}, \ \tau \in T$, be a family of time-independent contact Hamiltonians parameterized by the unit disc $T = D^q \subset \mathbb{R}^q$, and such that $K^\tau(x) > 0$ for $(\tau, x) \in \partial(T \times \Delta)$. Let $(T B = T \times B, T \eta)$ be the fibered circular shell defined by this family. Denote by $T \eta_p$ the shell corresponding to the family of saucers $\eta_p^{(\theta(||\tau||))}$. Then there
exist finitely many balls $B_i \subset B$, $i = 1, \ldots, N$, with piecewise smooth boundary, so that the fibered contact shell $T \eta$ (viewed as a fibered almost contact structure on $TB$) is homotopic rel. $\partial B \times T$ to a fibered almost contact structure $T \xi$ which is genuinely contact on $T \times (B \setminus \bigcup B_i)$, and such that each fibered contact shell $T \xi|_{B_i}$ is equivalent to one of the fibered saucer shells $T \eta_p$, $p = 1, \ldots, L$.

**Proof.** First, we can choose $K^s : \Delta \rightarrow \mathbb{R}$, $s \in [0, 1]$, so that $K^s|\tau| \leq K^\tau$ everywhere, $K^\tau > 0$ and $K^s|\partial \Delta > 0$ for all $s \in [0, 1]$. It, therefore, suffices to prove the proposition for the family $K^s|\tau|$. We can also assume that $K^0(x) \leq K^s(x)$ for any $x \in \Delta$, $s \in [0, 1]$.

Let $U = \text{Int} \Delta$ and $U' \Subset U$ be a star-shaped subset such that $K^s|\tau|_{U \setminus U'} > 0$ for all $\tau \in T$. We also choose $\delta > 0$ so that $K^s > 0$ for all $s \in [1 - \delta, 1]$. Apply Lemma 8.6 to $K^0$, providing an integer $N > 0$, and a finite set $\Lambda = \{g_1, \ldots, g_k\} \subset \hat{\Theta}_N$, so that the function

$$\Phi = \Phi^{S^1, N} = \frac{\mu}{N} + \sum_{g \in \Lambda} \phi_{g, N} : \Delta \times S^1 \rightarrow \mathbb{R}$$

satisfies $\Phi(w, t) < K^s(w)$ for $w \in \Delta \setminus U'$ and $\Phi|_{U' \times S^1} < -\frac{\mu}{N}$. Choosing $N$ large enough we can also arrange that $\min_{||r|| \geq 1 - \delta} K^\tau > \Psi = \Psi^{S^1, N} = \frac{\mu}{N} + \sum_{g \in \Lambda} \phi_{+, g, N} = \frac{\mu}{N} + \frac{1}{N} \sum_{g \in \Lambda} (\phi_{+} \circ C_N \circ g^{-1})$.

According to Proposition 4.10 there exists a family of functions $\tilde{K}^s$, $s \in [0, 1]$, such that

- $\tilde{K}^s = K^s$ on $\Delta \setminus U'$, $\tilde{K}^s > -\frac{\mu}{N}$, $s \in [0, 1]$;

- $\tilde{K}^s = K^s$ for $s \in [1 - \delta, 1]$;

- there exists a family of subordination maps $h^s : \eta_{\tilde{K}^s} \rightarrow \eta_{\tilde{K}^s}$ which are identity maps for $s \in [1 - \delta, 1]$.

In particular, $\Phi(w, t) < \tilde{K}^s(w)$ for all $w \in \Delta$, $t \in S^1$, $s \in [0, 1]$.

Recall the notation $\phi^s = \phi_+ - s\phi_-$, $\phi_{g, N} = \frac{1}{N} \phi^s \circ C_N \circ g^{-1}$ from Section 8.1. Choose a diffeomorphism $f : [0, 1] \rightarrow [0, 1]$ such $f(1 - \frac{\delta}{2}) = \frac{2}{3}$, $f(1 - \delta) = \frac{1}{3}$. Then the function $\tilde{\theta} := \theta \circ f$ satisfies $\tilde{\theta}|_{[0, 1 - \delta]} = 1$ and $\tilde{\theta}|_{[1 - \frac{2}{3}, 1]} = 0$. 

We define the families of functions $\Phi^s := \frac{1}{N} + \sum_{j=1}^{i} \left( \phi_{g_j,N}^{\tilde{\theta}(s)} \right): \Delta \times S^1 \to \mathbb{R}$, so we have $\Phi^s_0 = \frac{1}{N}$, $\Phi^s_{|\Lambda|} = \Phi$ for $s \leq 1 - \delta$, and $\Phi^1_{|\Lambda|} = \Psi$. Here $|\Lambda|$ is the cardinality of $\Lambda$.

The function $\Phi^s := \Phi^s_k$ for each $s \in [0, 1]$ is equal near $\partial \Delta \times S^1$, and it satisfies the inequality $\Phi^s < \tilde{K}^s$. Indeed, for $s \in [0, 1 - \delta]$ we have $\Phi^s = \Phi < \tilde{K}^s$, and for $s \in [1 - \delta, 1]$ we have $\Phi^s < \Psi < K^u = \tilde{K}^s$. Therefore, the family of circular shell models $\eta_{\Phi^s_{|\tau|}}$ is dominated by $\eta_{\tilde{K}^s_{|\tau|}}$, and hence it is sufficient to prove the required extension result for the family $\eta_{\Phi^s_{|\tau|}}$.

The family of model shells $\eta_{\Phi^s_{|\tau|}}$ and $\eta_{\Phi^s_{|\tau| - 1}}$, $\tau \in T$, differ by one of the regular saucer families $(B_p, \zeta_p^{\tilde{\theta}(|\tau|)})$, $p = 1, \ldots, L = 2^m$, from the finite list provided by Lemma 8.5. The shell $\eta_{\Phi^s_{|\tau|}}$ is solid for all $\tau \in T$, since $\Phi^s_{|\tau|} > 0$ everywhere. Similarly the saucers $(B_p, \zeta_p^{a(\kappa(|\tau|))})$ for $\tau \in \mathcal{O}p \partial T$ are solid for $\tau \in \mathcal{O}p \partial T$, because we have $\Phi^s_{|\tau|} = \Phi^s_{|\tau| - 1}$ for all $j = 1, \ldots, |\Lambda|$. But the fibered saucer corresponding to the family $(B_p, \zeta_p^{\tilde{\theta}(|\tau|)})$ is equivalent to $\eta_{\Phi^s_{|\tau|}}$. 

**Proof of Proposition 3.10.** Proposition 7.6 allows us to assume that $T\xi_0$ is fibered contact outside of a finite collection of disjoint $\{B_i\}_{1 \leq i \leq N}$, so that the restriction for each $i = 1, \ldots, N$, $\xi_0|_{B_i}$ is a fibered regular semi-contact saucer. Applying Proposition 8.9 we further reduce the holes to a finite list of fibered saucers $\tau \eta_p$, $p = 1, \ldots, L$. Proposition 6.13 allows us to replace each saucer $\tau \eta_p$, $p = 1, \ldots, L$, by a fibered circular model shell $\tau \eta_{K_p} \tau \in T$. But then, using Lemma 8.7 we conclude that each circular model shell $\tau \eta_{K_p}$ dominates the fibered circular model $\tau \eta_{\eta_{K_{univ}}} := \{\eta_{\Phi^{|\kappa(\tilde{\theta}(|\tau|))|}}\}_{\tau \in T}$. 

**9 Leafwise contact structures**

Theorem 1.5 follows from Theorem 1.6 because any leafwise almost contact structure is homotopic to a structure from $\text{cont}_0(F; h_1, \ldots, h_n)$ for an appropriate choice of embeddings $h_1, \ldots, h_N$. Hence, it is sufficient to prove Theorem 1.6.

We begin with the following lemma, which we already used in Section 3.6 in the proof of Theorem 3.12.

**Lemma 9.1.** Let $U$ be a connected manifold of dimension $m > 1$, $T$ a compact contractible set, $T_1, \ldots, T_k \subset T$ its compact subsets such that

\begin{equation}
9 \text{ Leafwise contact structures}
\end{equation}
Let $B$ be a closed $m$-dimensional ball with a given point $p \in \partial B$, $S_j : T_j \times B \to T_j \times U$, $S_j(\tau, x) = (\tau, s_j(\tau, x))$, and $S_\pm : T \times B \to T \times U$, $S_\pm(\tau, x) = (\tau, s_\pm(\tau, x))$, be pairwise disjoint fiberwise smooth embeddings. Then there exists a fiberwise embedding $S : T \times [-1, 1] \to T \times U$ such that

(i) $S(\tau, \pm 1) = S_\pm(\tau, p)$, $\tau \in T$;

(ii) $S(T \times [-1, 1]) \cap k \bigcup_{j=1}^k S_j(T_j \times B) = \emptyset$;

(iii) $S(T \times (-1, 1)) \cap (S_-(T \times B) \cup S_+(T \times B)) = \emptyset$.

Proof. We will prove the statement by induction in $k$. When $k = 0$ the statement follows from the fact that the space of maps of the contractible set $T$ into the space of pairs of disjoint embeddings of $B$ into $U$ is connected, and hence by a fiberwise isotopy we can assume that the embeddings $s_\pm(\tau, \cdot) : B \to U$ are independent of $\tau$, i.e. $s_\pm(\tau, x) = \tilde{s}_\pm(x)$ for all $(\tau, x) \in T \times B$. Then to construct the required embedding it is sufficient to connect the points $\tilde{s}_\pm(p)$ by an embedded arc in $U$ which does not intersect the ball $\tilde{s}_\pm(B)$ in its interior points.

Suppose that the statement is already proven for $k = j$ (and any $U$). Suppose first that one of the $k = j + 1$ sets $T_1, \ldots, T_k$, say $T_k$, coincides with $T$. By a fiberwise isotopy we can make the embedding $s_k(\tau, \cdot) : B \to U$ independent of $\tau$, i.e. $s_k(\tau, x) = \tilde{s}_k(x)$ for all $(\tau, x) \in T \times B$. Therefore, the statement reduces to the case of $k - 1 = j$ sets $T_1, \ldots, T_j$ and their embeddings into $\tilde{U} = U \setminus \tilde{s}_k(B)$, which is connected as well.

Consider now the general case. By an argument as above, we can assume that the embeddings $s_\pm(\tau, \cdot)$ are independent of $\tau$, i.e. $s_\pm(\tau, x) = \tilde{s}_\pm(x)$ for all $(\tau, x) \in T \times B$. Denote $\tilde{U} := U \setminus \tilde{s}_+(B)$. Suppose that $T_k$ is a proper subset of $T$. Set $\tilde{T} := T_k$, $\tilde{T}_i := T_i \cap T_k$, $\tilde{S}_i := S_i|_{\tilde{T}_i \times B}$, $i = 1, \ldots, k - 1$, $\tilde{S}_- := S_-|_{T_k \times B}$, $\tilde{S}_+ := S_+|_{T_k \times B}$. Note that the sets $\tilde{T}_i$, $i = 1, \ldots, k - 1$, and $\tilde{T}$ satisfy the condition $(\ast)$.

Considering $\tilde{S}_i$ as embeddings into $\tilde{T}_i \times \tilde{U}$, and $\tilde{S}_\pm$ as embeddings into $\tilde{T} \times \tilde{U}$ we can apply the inductive hypothesis to construct a fiberwise embedding $\tilde{S} : \tilde{T} \times [-1, 1] \to \tilde{T} \times \tilde{U}$ such that

$-\tilde{S}(\tau, \pm 1) = S_\pm(\tau, \tilde{p})$, $\tau \in \tilde{T}$, where $\tilde{p} \in \partial B$, is a point different from $p$;
\[ S(T \times [-1, 1]) \cap \bigcup_{1}^{k-1} S_j(T_j \times B) = \emptyset; \]

\[ S(T \times (-1, 1)) \cap \left( S_- (T \times B) \cup S_+ (T \times B) \right) = \emptyset. \]

Using the embedding \( S \) we can make a fiberwise connected sum of the embeddings \( S \pm \) to construct a fiberwise embedding \( \tilde{S} : T \times B \to T \times \hat{U} \) with the following properties:

- \( \tilde{S}_- (T \times B) \cap \bigcup_{1}^{k-1} S_j(T_j \times B) = \emptyset; \)
- \( \tilde{S}_- (T \times B) \supset S_- (T \times B) \cup S_k(T_k \times B); \)
- the embeddings \( \tilde{S}_- \) and \( S_- \) coincides near \( T \times p \in T \times \partial B \).

Hence, by applying again the inductional hypothesis to the embeddings \( \tilde{S}_- , S \) and \( S_j , j = 1, \ldots, k - 1 \), we can construct a fiberwise embedding \( S : T \times [-1, 1] \to T \times U \) with the required properties.

**Proof of Theorem 1.6.** Let \( T \) be an \( m \)-ball. We need to prove that any map

\[ (T, \partial T) \to (\text{cont}_{\text{at}}(\bar{F}; h_1, \ldots, h_N), \text{Cont}_{\text{at}}(\bar{F}; h_1, \ldots, h_N)) \]

is homotopic rel. \( \partial T \) to a map into \( \text{Cont}_{\text{at}}(\bar{F}; h_1, \ldots, h_N) \). In other words, let \( \xi_\tau \in \text{cont}_{\text{at}}(\bar{F}; h_1, \ldots, h_n), \tau \in T \), be a family of leafwise almost contact structures which are genuine leafwise contact structures for \( \tau \in \partial T \). We will construct a homotopy rel. \( \partial T \) to a family of genuine leafwise contact structures \( \xi_\tau, \tau \in T \).

Consider a foliation \( \hat{F} \) on \( T \times V \) with leaves \( \tau \times L \) where \( \tau \in T \) and \( L \) a leaf of \( F \). Let \( \hat{h}_j : T \times T_j \times B \to T \times V \) be the embeddings, given by

\[ \hat{h}_j(\tau, \tau', x) = (\tau, h_j(\tau', x), (\tau, \tau', x) \in T \times T_j \times B, j = 1, \ldots, N. \]

The family \( \xi_\tau, \tau \in T \), can be viewed as a leafwise almost contact structure \( \Xi \) from \( \text{cont}(\hat{F}; \hat{h}_1, \ldots, \hat{h}_N) \) which is genuine on leaves \( \tau \times L \) for \( \tau \in \partial T \). Moreover, we can assume that \( \Xi \) is a genuine leafwise contact structure on a neighborhood \( U \supset \partial T \times V \) and neighborhoods \( U_j \supset \hat{h}_j(T \times T_j \times B), j = 1, \ldots, N. \)

There exists a triangulation \( T \) of \( T \times V \) with the following properties:
• there are compact subcomplexes \( \hat{U}, \hat{U}_j, j = 1, \ldots, N \) of the triangulation \( T \) such that \( \partial T \times V \subset \hat{U} \subset U, \hat{h}_j(T \times T_j \times B) \subset \hat{U}_j \subset U_j, j = 1, \ldots, N; \)

• the restriction \( T_0 \) of the triangulation \( T \) to \( T \times V \setminus \text{Int}(\hat{U} \cup \bigcup_1^N \hat{U}_j) \) is transverse to the foliation \( \hat{F} \);

• For every top-dimensional simplex \( \sigma \) of \( T_0 \) there exists a submersion \( \pi_\sigma : \text{Int} \sigma \to B^{q+m} \) which is a fibration over an open \((q + m)\)-ball with the ball fibers, and such that the pre-images \( \pi_\sigma^{-1}(s), s \in B^{q+m} \), are intersections of the leaves of \( \hat{F} \) with \( \text{Int} \sigma \).

Applying Gromov’s parametric h-principle for contact structures on open manifolds (see [28] and Theorem 7.1) inductively over skeleta of the triangulation we can deform \( \Xi \), keeping it fixed on \( \hat{U} \cup \bigcup_1^N \hat{U}_j \), to make it leafwise genuinely contact in a neighborhood of the codimension 1 skeleton of the triangulation \( T_0 \).

Our next goal is to further deform \( \Xi \) on each top-dimensional simplex \( \sigma \) of \( T_0 \), keeping it fixed on \( \partial \sigma \), to make in leafwise genuine contact structure on \( \sigma \). Let us choose one of such simplices. There exists a compact subset \( \sigma \subset \text{Int} \sigma \) such that the leafwise almost contact structure \( \Xi \) is genuine on \( \sigma \setminus \text{Int} \sigma \) and \( \pi_\sigma|_\sigma \) is a fibration over a closed \((m + q)\)-ball \( X \) with fibers diffeomorphic to a closed \((2n + 1)\)-ball.

Hence, \( \Xi|_\sigma \) can be viewed as a fibered over \( X \) almost contact structure on \( \sigma \), and applying Proposition 7.6 we can further deform \( \Xi \) keeping it fixed on \( \partial \sigma \), to make it genuine away from a finite number of disjoint domains \( Z_i \) fibered over \( X_i \subset X \) with piecewise smooth boundary, \( i = 1, \ldots, K \). These domains are not necessarily disjoint but could be chosen arbitrary small and in such a way that all non-empty intersections \( X_{i_1} \cap \cdots \cap X_{i_k}, 1 \leq i_1 < \cdots < i_k \leq K \), are again balls with piecewise smooth boundaries. Let \( Y, Y \subset \sigma \), and \( Y_i, Y_i \subset Z_i \), be subfibrations of the fibrations \( \sigma \to X \) and \( Z_i \to X_i, i = 1, \ldots, K \), formed by boundaries of the corresponding ball-fibers.

Next, we use Lemma 9.1 to construct for each \( X_j \) a fiberwise embedding \( S_j : X_j \times [0, 1] \to \bigcup_i Z_i \cup Z_j \) with \( S_j(\tau, 0) \in Y_j \) and \( S_j(\tau, 1) \in Y \). Recall that by assumption every point \( (\tau, x) \in Y \) can be connected to a point on the boundary of one of the overtwisted balls \( B_{i, \tau, \tau'} : = h_i(\tau \times B_{\tau'}), i = 1, \ldots, N, \tau \in T_i \) by an embedded path in the corresponding leaf. This path can be chosen inside an arbitrarily small
neighborhood of the codimension 1 skeleton of the triangulation $T_0$. Hence, if the sets $X_j$ are chosen sufficiently small we can extend each of the embeddings $S_j$ to a leafwise embedding $\tilde{S}_j : X_j \times [0, 2] \to V$ such that

- $S_j(\tau, 0) \in Y_j$;
- $S_j(\tau, 2) \in h_i(T_i \times \partial B)$ for some $i = i(\sigma, j)$.

Moreover, using Proposition 3.8 to increase the number of embeddings $h_i$ we can additionally arrange that the map $(\sigma, j) \mapsto i(\sigma, j)$ is injective. Then, successively applying Theorem 3.12 to neighborhoods of $Z_j \cup S_j(X_j \times [0, 2]) \cup \bigcup_{\tau \in X_j} h_i(\sigma, j)(S(\tau, 2) \times B)$ for all top-dimensional simplices $\sigma$ of the triangulation we deform $\Xi$ to make it leafwise genuinely contact on these neighborhoods.

10 The overtwisted contact structures. Discussion

Recall the definition of an overtwisted contact structure from Section 3.2: a contact structure $\xi$ on a manifold $M$ is called overtwisted if there is a contact embedding $(D_{ot}, \xi_{ot}) \to (M, \xi)$, see Section 3.2. We note that the disc $D_{ot}$ is only piecewise smooth. We do not know if it is possible to characterize overtwisted structures in dimension $> 3$ by existence of a smooth overtwisted disc.

In the 3-dimensional case a contact structure which is overtwisted in our sense is also overtwisted in the sense of [12]. This should be clear from the picture of the characteristic foliation on the disc $D_{ot}$, see Figure 3.1. The converse is also true. This can be seen directly by finding a copy of $(D_{ot}, \xi_{ot})$ in a neighborhood of the traditional overtwisted disc, or indirectly, from the classification theorem from [12]. Indeed, one can first find a contact structure on the ball with standard boundary which contains $(D_{ot}, \xi_{ot})$ and which is in the standard almost contact class. Then, implanting this ball in an overtwisted contact manifold does not change the isotopy class of this structure.

The overtwisting property can be characterized in many other equivalent ways. We believe that the best characterization is yet to be found. Note, however, that all definitions for which Theorem 1.2 holds are equivalent. We describe below some possible variations of the definition of the overtwisting property.
The definition of an overtwisted disc depends on a choice of the special contact Hamiltonian $K_\varepsilon : \Delta_{\text{cyl}} \to \mathbb{R}$, where $\Delta_{\text{cyl}} = D \times [-1, 1]$, where $D$ is the unit ball in $\mathbb{R}^{2n-2}$. Suppose $\tilde{D} \subset \mathbb{R}^{2n-2}$ be any other star-shaped domain with a piecewise smooth boundary. Denote

$$\tilde{\Delta}_{\text{cyl}} := \{(x, z) \in \mathbb{R}^{2n-2} \times \mathbb{R} ; \ x \in \tilde{D}, \ |z| \leq 1\}, \ \Delta_{\text{cyl}}^- := \Delta_{\text{cyl}} \cap \{z \leq 0\}$$

and $\tilde{\Delta}_{\text{cyl}}^- := \tilde{\Delta}_{\text{cyl}} \cap \{z \leq 0\}$.

Let $C_+$ be the space of continuous piecewise smooth functions $\tilde{\Delta}_{\text{cyl}}^- \to \mathbb{R}$ which are positive on $O \partial \tilde{\Delta}_{\text{cyl}} \cap \tilde{\Delta}_{\text{cyl}}^-$. Given two functions $K_\pm \in C_+$ such that $K_- < K_+$ we denote

$$U_{K_-,K_+} = \{(x, v, t) : K_-(x, t) \leq v \leq K_+(x, t), \ z(x) \leq 0\} \subset \tilde{\Delta}_{\text{cyl}}^- \times T^*S^1, \ker(\lambda_{\text{st}} + v dt)$$

and $\Sigma_{K_+} = \{(x, v, t) : 0 \leq v \leq K_+(x, t), \ x \in \partial \Delta, \ z(x) \leq 0\} \subset \Delta \times \mathbb{R}^2, \ker(\lambda_{\text{st}} + v dt)$.

Gluing these pieces together via the natural identification between their common parts we define $\hat{U}_{K_-,K_+} := U_{K_-,K_+} \cup \Sigma_{K_+}$.

**Lemma 10.1.** For any $K_+ \in C_+(\tilde{\Delta})$ there exists $K_- \in C_+(\tilde{\Delta})$, such that $K_- < K_+$ such that $\hat{U}_{K_-,K_+}$ is overtwisted.

**Proof.** Choosing a representative $\eta$ of the contact germ along $\Sigma_{K_+}$ let $U$ be a neighborhood of $\partial \tilde{\Delta}_{\text{cyl}}$ such that $K|_{U \cap \tilde{\Delta}_{\text{cyl}}} > 0$ and the contact structure $\eta$ is defined on $\{(x, v, t); x \in U \cap \{z \geq 0\}, \ v \leq K_+(x)\}$. There is a contact embedding $\Phi : \Delta_{\text{cyl}}^- := \tilde{\Delta}_{\text{cyl}}^-$ such that $\Phi(\partial \Delta_{\text{cyl}} \cap \Delta_{\text{cyl}}^-) \subset U$ and $\Phi(\Delta_{\text{cyl}}^- \cap \{z = 0\}) \subset \{z = 0\}$. Indeed, the contact vector field $Z = L + z \frac{\partial}{\partial z}$, $L = \sum_{i=1}^{n-1} u_i \frac{\partial}{\partial u_i}$ is given by the contact Hamiltonian $z$ with respect to the standard contact form $\lambda_{\text{st}} = dz + \sum_{i=1}^{n-1} u_i d\phi_i$. Consider a cut-off function $\sigma : \mathbb{R}^{2n-1} \to \mathbb{R}_+$ which is equal to 1 on $\tilde{\Delta}_{\text{cyl}} \setminus U$ and supported in $\text{Int} \tilde{\Delta}_{\text{cyl}}$, and let $\tilde{Z}$ denote the contact vector field defined by the contact Hamiltonian $K := z \sigma$. Let us observe that $\tilde{Z}$ is tangent to the hyperplane $\{z = 0\}$ because $K$ vanishes on this
hyperplane. Let $Z^t$ and $\tilde{Z}^t$ be the contact flows generated by $Z$ and $\tilde{Z}$. Then the formula $\Phi := \tilde{Z}^t \circ Z^{-t}$ is the required contact embedding for appropriately chosen positive constants $C, \tilde{C}$.

For an appropriate choice of a special Hamiltonian $K < K_{\text{univ}}$ we have $\Phi_* K < K_{\text{univ}}$. On the other hand, there exists $K_+ \in C_+$ such that $\Phi_* K > K_{\text{univ}}$. Hence, the overtwisted disc $D_{\text{ot}} = D_K$ embeds into an arbitrarily small neighborhood of $\hat{U}_{K_{\text{univ}}}$, i.e. $\hat{U}_{K_{\text{univ}}}$ is overtwisted.

**Wrinkles and overtwisting**

Consider the standard contact $\left( \mathbb{R}^{2n+1}, \xi_{\text{st}} = \{ dz + \sum_{i=1}^{n-1} u_i d\varphi_i - y_n dx_n = 0 \} \right)$. Let $B$ denote the unit ball in $\mathbb{R}^{2n+1}$ and $w : B \to \mathbb{R}^{2n+1}$ be the standard wrinkle (see [15]), i.e. a map given by the formula

$$(v, y_n) \mapsto (v, y_n^3 - 3\alpha(r)y_n), \; v \in \mathbb{R}^{2n},$$

where $r := ||v||^2$, and $\alpha : [0, 1] \to \mathbb{R}$ is a $C^\infty$-function which is positive on $\left( \frac{1}{4}, \frac{3}{4} \right)$, negative on $\left( \frac{3}{4}, 1 \right]$, constant near 0, has a negative derivative at $\frac{3}{4}$, and satisfies the inequality $\alpha(r) \leq 1 - r^2$. Denote $W : \{ y_n^2 \leq \alpha(r) \}$.

Note that Corollary 1.4 allows us to construct a contact embedding of $(U \setminus W, (w^*\zeta)|_{U\setminus W})$ into any overtwisted contact manifold of the same dimension. One can also show, though we do not know a simple proof of this fact, that $(U \setminus W, (w^*\zeta)|_{U\setminus W})$ contains an overtwisted disc. Hence, a contact structure $\xi$ on a manifold $M$ is overtwisted if and only if there exists a neighborhood $U \supset W$ in $B$, and a contact embedding $(U \setminus W, (w^*\zeta)|_{U\setminus W}) \to (M, \xi)$.

**Stabilization of overtwisted contact manifolds**

Given a contact manifold $(Y, \xi)$ with a fixed contact form $\lambda$ its $k$-stabilization is the contact manifold $Y^\text{stab}_k := Y \times \mathbb{R}^{2k}$ endowed with the contact structure $\xi^\text{stab}_k := \{ \lambda + \sum_{i=1}^{k} v_i d\phi_i = 0 \}$. It is straightforward to check that up to a canonical contactomorphism the contact manifold $(Y^\text{stab}_k, \xi^\text{stab}_k)$ is independent of the choice of the contact form $\lambda$.

After the first version of the current paper was posted on arXiv, R. Casals and F. Presas observed that the $k$-stabilization for any $k \geq 0$ preserves the overtwisting property, i.e.
Theorem 10.2 ([4]). The $k$-stabilization $(Y_k^{\text{stab}}, \xi_k^{\text{stab}})$ of any overtwisted contact manifold $(Y, \xi)$ is overtwisted for any $k \geq 0$.

In particular, this implies that an overtwisted contact manifold of dimension $2n + 1$ can be equivalently defined as a contact manifold containing the $(2n-2)$-stabilization of a neighborhood of the standard 3-dimensional overtwisted disc.

Note that Theorem 10.2 also implies Corollary 10.3. For any overtwisted contact manifold $(M, \xi = \{ \lambda = 0 \})$ the contact manifold $(M \times T^*S^1, \{ \lambda + vdt = 0 \})$ is overtwisted. Moreover, $M \times T^*_+S^1 := (M \times T^*S^1) \cap \{ v > 0 \}$ is overtwisted as well.

Proof. $(M \times T^*_+S^1, \lambda + vdt)$ is contactomorphic to $(M \times \mathbb{R}^2 \setminus 0, \lambda + xdy - ydx)$. On the other hand, there exists a contact embedding $(M \times D^2_R, \lambda + xdy - ydx) \to (M \times D^2_R, \lambda + xdy - ydx)$. It can be defined, for instance, by the formula $(w, x, y) \mapsto (\mathfrak{R}^{-2Ry}(w), x + 2R, y)$, where $\mathfrak{R}$ is the Reeb flow of the contact form $\lambda$. But according to Theorem 10.2 the product $(M \times D^2_R, \{ \lambda + xdy - ydx = 0 \})$ is overtwisted if the radius of the 2-disc $D^2_R$ is sufficiently large, and the claim follows.

Overtwisting and (non)-orderability

In [16] there was introduced a relation $\leq$ on the universal cover $\widetilde{\text{Cont}}(Y, \xi)$ of the identity component of the group of contactomorphisms of $(Y, \xi)$. Namely, $f \leq g$ for $f, g \in \widetilde{\text{Cont}}(Y, \xi)$ if there is a path in $\widetilde{\text{Cont}}(Y, \xi)$ connecting $f$ to $g$ which is generated by a non-negative contact Hamiltonian. This relation is either trivial (e.g. in the case of the standard contact sphere of dimension $> 1$, see [17], and in this case the contact manifold $(Y, \xi)$ is called non-orderable, or it is a genuine partial order, e.g. in the case of $\mathbb{R}P^{2n-1}$ (see [26]) or the unit cotangent bundle $UT^*(M)$ of a closed manifold $M$, see [17, 9]. It is an open longstanding question whether a 3-dimensional closed overtwisted contact manifold is orderable or not (see [6, 8] for partial results in this direction).

We define below a possibly weaker notion of the above order relation, for which all contact manifolds which are known to be orderable in the sense of $\leq$ are orderable as well, while overtwisted contact manifolds are unorderable.

Let us recall a property of the relation $\leq$ from [16]. Given a contact manifold $(Y, \xi)$ with a fixed contact for $\lambda$ consider the contact manifold $(Y \times T^*S^1, \{ \lambda + vdt = 0 \})$, where $S^1 = \mathbb{R}/\mathbb{Z}$. Let $f \in \text{Cont}(Y, \xi)$ be generated by a time dependent contact
Hamiltonian $K_t : Y \to \mathbb{R}$ which can be assumed 1-periodic in $t$. We consider the domain $V^+(f) = \{ v + K_t(x) \geq 0, x \in Y \} \subset Y \times T^*S^1$. If $f \leq g$ then there exists a contact diffeotopy

$$h_t : Y \times T^*S^1 \to Y \times T^*S^1,$$

such that $h_0 = \text{Id}$ and $h_1(V^+(f)) \subset V^+(g)$. (44)

However it is not known whether the converse is true. Thus, it seems natural to introduce a weaker relation: we say that $f \preceq g$ if there exists an isotopy $h_t$ as in (44). If $f \leq g$ then $f \preceq g$ but we do not know whether these relations are equivalent or not. However, as we already stated above, in all known to us cases of contact orderability in the sense of $\leq$ one can also prove weak orderability in the sense of $\preceq$. On the other hand,

**Theorem 10.4.** Any closed overtwisted contact manifold is not weakly orderable.

**Proof.** According to Corollary 10.3 for a sufficiently large contact Hamiltonian $K > 0$ the domain $V^+(K)$ is overtwisted. Hence, Corollary 1.4 from allows us to construct an isotopy of $V^+(2K)$ into $V^+(K)$ inside $Y \times T^*S^1$. This isotopy extends to a global diffeotopy, and hence $g_{2K} \preceq g_K$, where $g_H$ denotes a time 1 map of the contact Hamiltonian $H$. On the other hand, we clearly have $g_K \preceq g_{2K}$, i.e. the order $\preceq$ is trivial.

\[\square\]

**Classification of overtwisted contact structures on spheres**

We will finish the paper by discussing the classification of overtwisted contact structures on $S^{2n+1}$ explicitly. Almost contact structures on the sphere $S^{2n+1}$ are classified by the homotopy group $\pi_{2n+1}(SO(2n+2)/U(n+1))$. The following lemma computes this group.

**Lemma 10.5** (Bruno Harris, [31]).

$$\pi_{2n+1}(SO(2n+2)/U(n+1)) = \begin{cases} 
\mathbb{Z}/n!\mathbb{Z}, & n = 4k; \\
\mathbb{Z}, & n = 4k + 1; \\
\mathbb{Z}/n\mathbb{Z}, & n = 4k + 2; \\
\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}, & n = 4k + 3; 
\end{cases}$$

\[\text{We thank Soren Galatius for providing this reference.}\]
Thus, Corollary 1.3 implies that on spheres $S^{8k+1}$, $k > 0$, there are exactly $(4k)!$ different overtwisted contact structures, on spheres $S^{8k+5}$, $k \geq 0$, there are $\frac{(4k+2)!}{2}$ different overtwisted contact structures, while on all other spheres there are infinitely many. In particular, there is a unique overtwisted contact structure on $S^5$.

It is interesting to note that $S^5$ has infinitely many tight, i.e. non-overtwisted contact structures. Besides the standard contact structure, these are examples given by Brieskorn spheres (see [40]). The full classification of tight contact structures on any manifold of dimension $> 3$ is an open problem.

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