Some remarks on relations between the $\mu$-parameters of regular graphs

N.N. Davtyan$^1$, R.R. Kamalian$^2$

$^1$Ijevan Branch of Yerevan State University, e-mail: nndavtyan@gmail.com,
$^2$The Institute for Informatics and Automation Problems of NAS RA, e-mail: rrkamalian@yahoo.com

Abstract

For an undirected, simple, finite, connected graph $G$, we denote by $V(G)$ and $E(G)$ the sets of its vertices and edges, respectively. A function $\varphi : E(G) \to \{1, \ldots, t\}$ is called a proper edge $t$-coloring of a graph $G$, if adjacent edges are colored differently and each of $t$ colors is used. The least value of $t$ for which there exists a proper edge $t$-coloring of a graph $G$ is denoted by $\chi'(G)$. For any graph $G$, and for any integer $t$ satisfying the inequality $\chi'(G) \leq t \leq |E(G)|$, we denote by $\alpha(G,t)$ the set of all proper edge $t$-colorings of $G$. Let us also define a set $\alpha(G)$ of all proper edge colorings of a graph $G$:

$$\alpha(G) \equiv \bigcup_{t=\chi'(G)}^{|E(G)|} \alpha(G,t).$$

An arbitrary nonempty finite subset of consecutive integers is called an interval. If $\varphi \in \alpha(G)$ and $x \in V(G)$, then the set of colors of edges of $G$ which are incident with $x$ is denoted by $S_G(x, \varphi)$ and is called a spectrum of the vertex $x$ of the graph $G$ at the proper edge coloring $\varphi$. If $G$ is a graph and $\varphi \in \alpha(G)$, then define $f_G(\varphi) \equiv |\{x \in V(G) : S_G(x, \varphi) \text{ is an interval}\}|$.

For a graph $G$ and any integer $t$, satisfying the inequality $\chi'(G) \leq t \leq |E(G)|$, we define:

$$\mu_1(G,t) \equiv \min_{\varphi \in \alpha(G,t)} f_G(\varphi), \quad \mu_2(G,t) \equiv \max_{\varphi \in \alpha(G,t)} f_G(\varphi).$$

For any graph $G$, we set:

$$\mu_{11}(G) \equiv \min_{\chi'(G) \leq t \leq |E(G)|} \mu_1(G,t), \quad \mu_{12}(G) \equiv \max_{\chi'(G) \leq t \leq |E(G)|} \mu_1(G,t),$$

$$\mu_{21}(G) \equiv \min_{\chi'(G) \leq t \leq |E(G)|} \mu_2(G,t), \quad \mu_{22}(G) \equiv \max_{\chi'(G) \leq t \leq |E(G)|} \mu_2(G,t).$$

For regular graphs, some relations between the $\mu$-parameters are obtained.

Keywords: regular graph, proper edge coloring, interval spectrum, $\mu$-parameters, game.

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We consider finite, undirected, connected graphs without loops and multiple edges containing at least one edge. For any graph $G$, we denote by $V(G)$ and $E(G)$ the sets of vertices and edges of $G$, respectively. For any $x \in V(G)$, $d_G(x)$ denotes the degree of the vertex $x$ in $G$. For a graph $G$, $\delta(G)$ and $\Delta(G)$ denote the minimum and maximum degrees of vertices in $G$, respectively.

An arbitrary nonempty finite subset of consecutive integers is called an interval. An interval $\{a, b\}$ for any integer $a$ with the minimum element $a$ and the maximum element $b$ is denoted by $[a, b]$.

A function $\varphi : E(G) \to [1, t]$ is called a proper edge $t$-coloring of a graph $G$, if each of $t$ colors is used, and adjacent edges are colored differently.

The minimum value of $t$ for which there exists a proper edge $t$-coloring of a graph $G$ is denoted by $\chi'(G)$ \[1\].

For any graph $G$, and for any $t \in [\chi'(G), |E(G)|]$, we denote by $\alpha(G,t)$ the set of all proper edge $t$-colorings of $G$.

Let us also define a set $\alpha(G)$ of all proper edge colorings of a graph $G$:

$$\alpha(G) \equiv \bigcup_{t=\chi'(G)}^{\left|E(G)\right|} \alpha(G,t).$$

If $\varphi \in \alpha(G)$ and $x \in V(G)$, then the set $\{\varphi(e)/e \in E(G), e$ is incident with $x\}$ is called a spectrum of the vertex $x$ of the graph $G$ at the proper edge coloring $\varphi$ and is denoted by $S_G(x, \varphi)$.

If $G$ is a graph, $\varphi \in \alpha(G)$, then set $V_{\text{int}}(G, \varphi) \equiv \{x \in V(G)/S_G(x, \varphi) \text{ is an interval}\}$ and $f_G(\varphi) \equiv |V_{\text{int}}(G, \varphi)|$. A proper edge coloring $\varphi \in \alpha(G)$ is called an interval edge coloring \[2\] of the graph $G$ iff $f_G(\varphi) = |V(G)|$. The set of all graphs having an interval edge coloring is denoted by $\mathcal{I}$. The terms and concepts which are not defined can be found in \[5\].

For a graph $G$, and for any $t \in [\chi'(G), |E(G)|]$, we set \[6\]:

$$\mu_1(G,t) \equiv \min_{\varphi \in \alpha(G,t)} f_G(\varphi), \quad \mu_2(G,t) \equiv \max_{\varphi \in \alpha(G,t)} f_G(\varphi).$$

For any graph $G$, we set \[6\]:

$$\mu_{11}(G) \equiv \min_{\chi'(G) \leq t \leq \left|E(G)\right|} \mu_1(G,t), \quad \mu_{12}(G) \equiv \max_{\chi'(G) \leq t \leq \left|E(G)\right|} \mu_1(G,t),$$

$$\mu_{21}(G) \equiv \min_{\chi'(G) \leq t \leq \left|E(G)\right|} \mu_2(G,t), \quad \mu_{22}(G) \equiv \max_{\chi'(G) \leq t \leq \left|E(G)\right|} \mu_2(G,t).$$

Clearly, the $\mu$-parameters are correctly defined for an arbitrary graph. Some remarks on their interpretations in games are given in \[7\,8\].

The exact values of the parameters $\mu_{11}$, $\mu_{12}$, $\mu_{21}$ and $\mu_{22}$ are found for simple paths, simple cycles and simple cycles with a chord \[9\,10\], "Möbius ladders" \[6\,11\], complete graphs \[12\], complete bipartite graphs \[13\,14\], prisms \[11\,15\], $n$-dimensional cubes \[7\,15\,16\] and the Petersen graph \[8\]. The exact values of $\mu_{11}$ and $\mu_{22}$ for trees are found in \[17\]. The exact value of $\mu_{12}$ for an arbitrary tree is found in \[18\] (see also \[19\,20\]).

In this paper some relations between the $\mu$-parameters of regular graphs are obtained.

In the rest part of this paper we admit an additional condition: an arbitrary graph $G$ satisfies the inequality $\delta(G) \geq 2$.

**Theorem 1.** \[9\,10\] For any integer $k \geq 2$, the following equalities hold:

1) $\mu_{12}(C_{2k}) = \mu_{22}(C_{2k}) = 2k$, 

2) $\mu_{21}(C_{2k}) = \mu_{11}(C_{2k}) = 2k - 1$.

3) $\mu_{11}(C_{2k+1}) = \mu_{22}(C_{2k+1}) = k$.

4) $\mu_{21}(C_{2k+1}) = \mu_{11}(C_{2k+1}) = k$.
2) \( \mu_{21}(C_{2k}) = 2k - 1 \),

3) \( \mu_{11}(C_{2k}) = \begin{cases} 1, & \text{if } k = 2 \\ 0, & \text{if } k \geq 3 \end{cases} \)

Theorem 2. [9,10] For any positive integer \( k \), the following equalities hold:

1) \( \mu_{12}(C_{2k+1}) = 2 \),

2) \( \mu_{21}(C_{2k+1}) = \mu_{22}(C_{2k+1}) = 2k \),

3) \( \mu_{11}(C_{2k+1}) = \begin{cases} 2, & \text{if } k = 1 \\ 0, & \text{if } k \geq 2 \end{cases} \)

Corollary 1. [9,10] For any integer \( k \geq 2 \), the inequalities \( \mu_{21}(C_{2k}) < \mu_{12}(C_{2k}) \) and \( \mu_{12}(C_{2k}) < \mu_{21}(C_{2k+1}) \) hold.

Theorem 3. [9,10] For any graph \( G \), the inequalities \( \mu_{11}(G) \leq \mu_{12}(G) \leq \mu_{22}(G) \), \( \mu_{11}(G) \leq \mu_{21}(G) \leq \mu_{22}(G) \) hold.

Remark 1. [9,10] Corollary 1 means that there are graphs \( G \) for which \( \mu_{21}(G) < \mu_{12}(G) \) and there are also graphs \( G \) for which \( \mu_{12}(G) < \mu_{21}(G) \).

Theorem 4. [9] If \( G \) is a regular graph with \( \chi'(G) = \Delta(G) \), then \( \mu_{12}(G) = |V(G)| \).

Theorem 5. [27] If \( G \) is an \( r \)-regular graph, and \( \varphi \in \alpha(G, |E(G)|) \), then

\[ |V_{int}(G, \varphi)| \leq \left\lfloor \frac{r \cdot |V(G)| - 2}{2 \cdot (r - 1)} \right\rfloor. \]

Corollary 2. If \( G \) is an \( r \)-regular graph, then

\[ \mu_{2}(G, |E(G)|) \leq \left\lfloor \frac{r \cdot |V(G)| - 2}{2 \cdot (r - 1)} \right\rfloor. \]

Corollary 3. If \( G \) is an \( r \)-regular graph, then

\[ \mu_{21}(G) \leq \left\lfloor \frac{r \cdot |V(G)| - 2}{2 \cdot (r - 1)} \right\rfloor. \]

Proposition 1. For arbitrary integers \( r \geq 2 \) and \( n \geq 1 \), the inequality

\[ \left\lfloor \frac{r \cdot n - 2}{2 \cdot (r - 1)} \right\rfloor \leq n - 1 \]

holds.

Proof.

\[ \left\lfloor \frac{r n - 2}{2 \cdot (r - 1)} \right\rfloor = \left\lfloor \frac{n}{2} + \frac{n - 2}{2 \cdot (r - 1)} \right\rfloor \leq \left\lfloor \frac{n}{2} + \frac{n - 2}{2} \right\rfloor = n - 1. \]

The Proposition is proved.

Corollary 4. If \( G \) is a regular graph, then \( \mu_{21}(G) \leq |V(G)| - 1 \).
From corollary 4 and theorem 4 we obtain

**Corollary 5.** For an arbitrary regular graph $G$ with $\chi'(G) = \Delta(G)$, the inequality $\mu_{21}(G) < \mu_{12}(G)$ holds.

**Theorem 6.** For an arbitrary regular graph $G$, the following four statements are equivalent:

1) $\chi'(G) = \Delta(G)$,
2) $G \in \mathcal{N}$,
3) $\mu_{22}(G) = |V(G)|$,
4) $\mu_{12}(G) = |V(G)|$.

**Proof.** The equivalence between 1) and 2) was proved in [2, 4]. The equivalence between 2) and 3) is evident.

Let us show the equivalence between 1) and 4).

If $\chi'(G) = \Delta(G)$, then, by theorem 4, we have the equality $\mu_{12}(G) = |V(G)|$. It means that 1) $\Rightarrow$ 4).

Now suppose that $\mu_{12}(G) = |V(G)|$. By theorem 3, we have also the equality $\mu_{22}(G) = |V(G)|$. Consequently, using the equivalence between 2) and 3), we have also the relation $G \in \mathcal{N}$. Finally, using the equivalence between 1) and 2), we have also the equality $\chi'(G) = \Delta(G)$. Thus, 4) $\Rightarrow$ 1).

The Theorem is proved.

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