Determinantal hypersurfaces

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Introduction

(0.1) We discuss in this paper which homogeneous form on $\mathbb{P}^n$ can be written as the determinant of a matrix with homogeneous entries (possibly symmetric), or the pfaffian of a skew-symmetric matrix. This question has been considered in various particular cases (see the historical comments below), and we believe that the general result is well-known from the experts; but we have been unable to find it in the literature. The aim of this paper is to fill this gap.

We will discuss at the outset the general structure theorems; roughly, they show that expressing a homogeneous form $F$ as a determinant (resp. a pfaffian) is equivalent to produce a line bundle (resp. a rank 2 vector bundle) of a certain type on the hypersurface $F = 0$. The rest of the paper consists of applications. We have restricted our attention to smooth hypersurfaces; in fact we are particularly interested in the case when the generic form of degree $d$ in $\mathbb{P}^n$ can be written in one of the above forms. When this is the case, the moduli space of pairs $(X, E)$, where $X$ is a smooth hypersurface of degree $d$ in $\mathbb{P}^n$ and $E$ a rank 1 or 2 vector bundle satisfying appropriate conditions, appears as a quotient of an open subset of a certain vector space of matrices; in particular, this moduli space is unirational. This is the case for instance of the universal family of Jacobians of plane curves (Cor. 3.6), or of intermediate Jacobians of cubic threefolds (Cor. 8.8).

Unfortunately this situation does not occur too frequently: we will show that only curves and cubic surfaces admit generically a determinantal equation. The situation is slightly better for pfaffians: plane curves of any degree, surfaces of degree $\leq 15$ and threefolds of degree $\leq 5$ can be generically defined by a linear pfaffian.

(0.2) Historical comments

The representation of curves and surfaces of small degree as linear determinants is a classical subject. The case of cubic surfaces was already known in the middle of the last century [G]; other examples of curves and surfaces are treated in [S]. The general homogeneous forms which can be expressed as linear determinants are determined in [D]. A modern treatment for plane curves appears in [C-T]; the result has been rediscovered a number of times since then.

The representation of the plane quartic as a symmetric determinant goes back again to 1855 [H]; plane curves of any degree are treated in [Di]. Cubic and quartic
surfaces defined by linear symmetric determinants ("symmetroids") have been also studied early [Ca]. Surfaces of arbitrary degree are thoroughly treated in [C1]; an overview of the use of symmetric resolutions can be found in [C2].

Finally, the only reference we know about pfaffians is Adler’s proof that a generic cubic threefold can be written as a linear pfaffian ([A-R], App. V).

(0.3) Conventions
We work over an arbitrary field \( k \), not necessarily algebraically closed. Unless explicitly stated, all geometric objects are defined over \( k \).

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1. General results: determinants

(1.1) Let \( \mathcal{F} \) be a coherent sheaf on \( \mathbb{P}^n \). We will say that \( \mathcal{F} \) is arithmetically Cohen-Macaulay (ACM for short) if:

a) \( \mathcal{F} \) is Cohen-Macaulay, that is, the \( \mathcal{O}_x \)-module \( \mathcal{F}_x \) is Cohen-Macaulay for every \( x \) in \( \mathbb{P}^n \);

b) \( H^i(\mathbb{P}^n, \mathcal{F}(j)) = 0 \) for \( 1 \leq i \leq \dim(\text{Supp} \mathcal{F}) - 1 \) and \( j \in \mathbb{Z} \).

Put \( S^n = k[X_0, \ldots, X_n] = \bigoplus_{j \in \mathbb{Z}} H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(j)) \) (we will often drop the superscript \( n \) if there is no danger of confusion). Following EGA, we denote by \( \Gamma_*(\mathcal{F}) \) the \( S \)-module \( \bigoplus_{j \in \mathbb{Z}} H^0(\mathbb{P}^n, \mathcal{F}(j)) \). The following well-known remark explains the terminology:

**Proposition 1.2.**— The sheaf \( \mathcal{F} \) is ACM if and only if the \( S \)-module \( \Gamma_*(\mathcal{F}) \) is Cohen-Macaulay.

**Proof:** Let \( U := \mathbb{A}^{n+1} - \{0\} \). The projection \( p : U \to \mathbb{P}^n \) is affine, and satisfies \( p_\ast \mathcal{O}_U = \bigoplus_{j \in \mathbb{Z}} \mathcal{O}_{\mathbb{P}^n}(j) \). The \( S \)-module \( \Gamma_*(\mathcal{F}) \) defines a coherent sheaf \( \tilde{\mathcal{F}} \) on \( \mathbb{A}^{n+1} \), whose restriction to \( U \) is isomorphic to \( p^\ast \mathcal{F} \). Therefore \( H^i(U, \tilde{\mathcal{F}}) \) is isomorphic to \( \bigoplus_{j \in \mathbb{Z}} H^i(\mathbb{P}^n, \mathcal{F}(j)) \). The long exact sequence of local cohomology

\[
\ldots \to H^i_{\{0\}}(\mathbb{A}^{n+1}, \tilde{\mathcal{F}}) \to H^i(\mathbb{A}^{n+1}, \tilde{\mathcal{F}}) \to H^i(U, \tilde{\mathcal{F}}) \to \ldots
\]

implies \( H^0_{\{0\}}(\mathbb{A}^{n+1}, \tilde{\mathcal{F}}) = H^1_{\{0\}}(\mathbb{A}^{n+1}, \tilde{\mathcal{F}}) = 0 \), and give isomorphisms

\[
\bigoplus_{j \in \mathbb{Z}} H^i(\mathbb{P}^n, \mathcal{F}(j)) \cong H^i_{\{0\}}(\mathbb{A}^{n+1}, \tilde{\mathcal{F}})
\]

for \( i \geq 1 \).

Thus condition b) of (1.1) is equivalent to \( H^i_{\{0\}}(\tilde{\mathcal{F}}) = 0 \) for \( i < \dim(\tilde{\mathcal{F}}) \), that is to \( \tilde{\mathcal{F}}_0 \) being Cohen-Macaulay. On the other hand, since \( p \) is smooth, condition a) is equivalent to \( \tilde{\mathcal{F}}_v \) being Cohen-Macaulay for all \( v \in U \), hence the Proposition.

Let us mention incidentally the following corollary, due to Horrocks:
Corollary 1.3.— A locally free sheaf $F$ on $\mathbb{P}^n$ with $H^i(\mathbb{P}^n, F(j)) = 0$ for $1 \leq i \leq n - 1$ and $j \in \mathbb{Z}$ splits as a direct sum of line bundles.

Proof: The $S$-module $\Gamma_*(F)$ is Cohen-Macaulay of maximal dimension, hence projective; it is therefore free as a $S$-graded module, that is isomorphic to a direct sum $S(d_1) \oplus \cdots \oplus S(d_r)$ ([Bo], §8, Prop. 8). Since $F$ is the sheaf on Proj($S$) associated to $\Gamma_*(F)$, it is isomorphic to $O_{\mathbb{P}^n}(d_1) \oplus \cdots \oplus O_{\mathbb{P}^n}(d_r)$.

Theorem A.— Let $F$ be an ACM sheaf on $\mathbb{P}^n$, of dimension $n - 1$. There exists an exact sequence

$$0 \to \bigoplus_{i=1}^\ell O_{\mathbb{P}^n}(e_i) \to \bigoplus_{i=1}^\ell O_{\mathbb{P}^n}(d_i) \to F \to 0 . \quad \text{(A1)}$$

Conversely, let $M : \bigoplus_{i=1}^\ell O_{\mathbb{P}^n}(e_i) \to \bigoplus_{i=1}^\ell O_{\mathbb{P}^n}(d_i)$ be an injective homomorphism; the cokernel of $M$ is ACM and its support is the hypersurface $\det M = 0$.

Proof: Suppose that $F$ is ACM of dimension $n - 1$. The Cohen-Macaulay $S$-module $\Gamma_*(F)$ has projective dimension 1; by Hilbert’s theorem ([Bo], §8, Cor. 3 of Prop. 8), it admits a free graded resolution of the form

$$0 \to \bigoplus_{i=1}^\ell S(e_i) \to \bigoplus_{i=1}^\ell S(d_i) \to \Gamma_*(F) \to 0 , \quad \text{(A2)}$$

which gives (A1) by taking the associated sheaves on $\mathbb{P}^n$.

Conversely, suppose given the exact sequence (A1). The support of $F$ consists of the points $x$ of $\mathbb{P}^n$ where $M(x)$ is not injective, that is where $\det M(x) = 0$. Since $M$ is generically injective this is a hypersurface in $\mathbb{P}^n$.

For every $x \in \mathbb{P}^n$, the $O_{\mathbb{P}^n,x}$-module $F_x$ has projective dimension $\leq 1$, hence $\depth \geq \dim O_{\mathbb{P}^n,x} - 1 = \dim F_x$; thus it is Cohen-Macaulay. From (A1) we deduce $H^i(\mathbb{P}^n, F(j)) = 0$ for $1 \leq i \leq n - 2$, hence $F$ is ACM. •

(1.4) The homomorphism $M$ is given by a matrix $(m_{ij}) \in M_\ell(S)$, with $m_{ij}$ homogeneous of degree $(d_i - e_j)$; we will use the same letter $M$ to denote this matrix.

(1.5) Let $F$ be an ACM sheaf on $\mathbb{P}^n$ of dimension $n - 1$. We will always take for (A2) a minimal graded free resolution of $\Gamma_*(F)$: this means that the images in $\Gamma_*(F)$ of the generators of $S(d_i)$ ($1 \leq i \leq \ell$) form a minimal system of generators of the $S$-module $\Gamma_*(F)$ . Such a resolution is unique up to isomorphism. The resolution (A2) is minimal if and only if the matrix $(m_{ij})$ is zero modulo $(X_0, \ldots, X_n)$, that is, if and only if $m_{ij} = 0$ whenever $d_i = e_j$.

We will refer to the corresponding exact sequence (A1), slightly abusively, as the minimal resolution of the sheaf $F$. 
(1.6) The minimal resolution $0 \rightarrow L_1 \rightarrow L_0 \rightarrow \mathcal{F} \rightarrow 0$, with $L_1 = \bigoplus_{i=1}^{\ell} \mathcal{O}_{\mathbb{P}^n}(e_i)$ and $L_0 = \bigoplus_{i=1}^{\ell} \mathcal{O}_{\mathbb{P}^n}(d_i)$, is unique up to isomorphism, but this isomorphism is not unique in general; it is unique if $\max(e_j) < \min(d_i)$ (in particular in the linear case). Indeed this condition implies $\text{Hom}(L_0, L_1) = 0$, and therefore the map $\text{End}(L_0) \rightarrow \text{Hom}(L_0, \mathcal{F})$ is injective; thus the only automorphism of $L_0$ which induces the identity on $\mathcal{F}$ is the identity. If moreover every automorphism of $\mathcal{F}$ is scalar, we see that the only pairs of automorphisms $P \in \text{Aut}(L_0)$, $Q \in \text{Aut}(L_1)$ such that $PM = MQ$ are the pairs $(\lambda, \lambda)$ for $\lambda \in k^*$. 

(1.7) In this paper we will mainly use Theorem A in the following way: we will start from an integral (usually smooth) hypersurface $X$ and a vector bundle $E$ of rank $r$ on $X$; we will still say that $E$ is ACM if it is so as an $\mathcal{O}_{\mathbb{P}^n}$-module, that is, $H^i(X, \mathcal{F}(j)) = 0$ for $1 \leq i \leq n-2$ and $j \in \mathbb{Z}$. For such a sheaf Theorem A provides a minimal resolution (A1); localizing at the generic point of $X$ and using the structure theorem for matrices over a principal ring we get $\det M = F^r$, where $F = 0$ is an equation of $X$. This gives the following corollary:

**Corollary 1.8.** Let $X$ be a smooth hypersurface in $\mathbb{P}^n$, given by an equation $F = 0$.

a) Let $L$ be a line bundle on $X$ with $H^i(X, L(j)) = 0$ for $1 \leq i \leq n-2$ and all $j \in \mathbb{Z}$. Then $L$ admits a minimal resolution 

$$0 \rightarrow \bigoplus_{i=1}^{\ell} \mathcal{O}_{\mathbb{P}^n}(e_i) \xrightarrow{M} \bigoplus_{i=1}^{\ell} \mathcal{O}_{\mathbb{P}^n}(d_i) \rightarrow L \rightarrow 0$$

with $F = \det M$.

b) Conversely, let $M = (m_{ij}) \in \mathcal{M}_\ell(S)$, with $m_{ij}$ homogeneous of degree $(d_i - e_j)$ and $F = \det M$. Then the cokernel of $M : \bigoplus_{i=1}^{\ell} \mathcal{O}_{\mathbb{P}^n}(e_i) \rightarrow \bigoplus_{i=1}^{\ell} \mathcal{O}_{\mathbb{P}^n}(d_i)$ is a line bundle $L$ on $X$ with the above properties. 

(1.9) The apparent generality of this Corollary is somewhat misleading: taking for $L$ the line bundle $\mathcal{O}_X(j)$ gives rise to the trivial case $\ell = 1$, $M = (F)$. Thus if Pic$(X)$ is generated by $\mathcal{O}_X(1)$ the hypersurface can not be defined by a $\ell \times \ell$ determinant with $\ell > 1$. So interesting situations occur only for curves and surfaces. In particular, we infer from the Noether-Lefschetz theorem that the generic hypersurface of degree $d$ in $\mathbb{P}^n$ can be expressed in a non-trivial way as a determinant only if $n = 2$ or $n = 3$ and $d \leq 3$. On the other hand we will see in (3.1) and (6.4) that any smooth plane curve or cubic surface can be defined by a linear determinant.

(1.10) Conversely, given integers $e_i, d_j$, one may ask whether a general matrix $(m_{ij}) \in \mathcal{M}_\ell(S)$ with $\deg m_{ij} = d_i - e_j$ defines a smooth curve or surface. If we order
the numbers $e_i, d_j$ so that $e_1 \leq \ldots \leq e_\ell$ and $d_1 \leq \ldots \leq d_\ell$, a sufficient condition is the inequality $d_i > e_{i+1}$ for $1 \leq i < \ell$. Indeed we can consider the matrix

$$
M = \begin{pmatrix}
F_1 & G_1 & 0 & \cdots & 0 \\
0 & F_2 & G_2 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & & & F_{\ell-1} & G_{\ell-1} \\
G_\ell & 0 & \cdots & 0 & F_\ell
\end{pmatrix}
$$

where the entries are product of linear forms. Then $\det M$ can be written in the form $\prod L_i \prod P_j$, where $L_i, P_j$ are arbitrary linear forms. We obtain in this way, for instance, the Fermat hypersurface $^1 \sum X_i^d = 0$ in $\mathbf{P}^2$ or $\mathbf{P}^3$.

The integers $e_i, d_j$ which occur in the minimal resolution are determined by the $\mathcal{S}$-module $\Gamma_*(\mathcal{F})$; we will see some examples in the next sections. We will be particularly interested by the case when the entries $(m_{ij})$ are linear forms; in this case we will say for short that the matrix $M$ is linear. There is a handy characterization of the sheaves which give rise to linear matrices:

**Proposition 1.11.** Let $\mathcal{F}$ be a coherent sheaf on $\mathbf{P}^n$. The following conditions are equivalent:

(i) There exists an exact sequence

$$
0 \to \mathcal{O}_{\mathbf{P}^n}(-1)^\ell \to \mathcal{O}_{\mathbf{P}^n}^n \to \mathcal{F} \to 0 ;
$$

(ii) $\mathcal{F}$ is ACM of dimension $n - 1$, and

$$
H^0(\mathbf{P}^n, \mathcal{F}(-1)) = H^{n-1}(\mathbf{P}^n, \mathcal{F}(1 - n)) = 0 .
$$

**Proof:** In view of Theorem A the implication (i) $\Rightarrow$ (ii) is clear. Assume that (ii) holds; then $H^i(\mathbf{P}^n, \mathcal{F}(-i)) = 0$ for $i \geq 1$, that is, $\mathcal{F}$ is 0-regular in the sense of Mumford ([Md], lect. 14). By loc. cit., this implies that $\mathcal{F}$ is spanned by its global sections and that the natural map

$$
H^0(\mathbf{P}^n, \mathcal{F}(j)) \otimes H^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(1)) \to H^0(\mathbf{P}^n, \mathcal{F}(j + 1))
$$

is surjective for $j \geq 0$. Since $H^0(\mathbf{P}^n, \mathcal{F}(-1)) = 0$, this means that the multiplication map $\mathcal{S} \otimes_k H^0(\mathbf{P}^n, \mathcal{F}) \to \Gamma_*(\mathcal{F})$ is surjective, and therefore the minimal resolution of $\mathcal{F}$ takes the form:

$$
0 \to \bigoplus_{i=1}^\ell \mathcal{O}_{\mathbf{P}^n}(e_i) \xrightarrow{M} \mathcal{O}_{\mathbf{P}^n}^n \xrightarrow{p} \mathcal{F} \to 0
$$

with $\ell = \dim H^0(\mathbf{P}^n, \mathcal{F})$. Since $H^0(p)$ is bijective and $H^{n-1}(\mathbf{P}^n, \mathcal{F}(1 - n)) = 0$, we must have $e_i = -1$ for all $i$. ■

We can again reformulate this result as:

$^1$ If $\text{char}(k) \mid d$ consider the surface $X_0(X_0^{d-1} + X_1^{d-1}) + (X_1 + X_2)(X_2^{d-1} + X_3^{d-1}) = 0$. 
Corollary 1.12.— Let $X$ be a smooth hypersurface of degree $d$ in $\mathbb{P}^n$, given by an equation $F = 0$.

a) Let $L$ be a line bundle on $X$ with $H^i(X, L(j)) = 0$ for $1 \leq i \leq n-2$ and all $j \in \mathbb{Z}$, and $H^0(X, L(-1)) = H^{n-1}(X, L(1-n)) = 0$. There exists a $d \times d$ linear matrix $M$ such that $F = \det M$, and an exact sequence

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^n}(-1)^d \xrightarrow{M} \mathcal{O}_{\mathbb{P}^n}^d \rightarrow L \rightarrow 0.$$

b) Conversely, let $M$ be a $d \times d$ linear matrix such that $F = \det M$. Then the cokernel of $M : \mathcal{O}_{\mathbb{P}^n}(-1)^d \rightarrow \mathcal{O}_{\mathbb{P}^n}^d$ is a line bundle $L$ on $X$ with the above properties. ■

2. General results: symmetric determinants and pfaffians

(2.1) We will now put an extra data on our ACM sheaf. Let $F$ be a torsion-free sheaf on an integral variety $X$, and $L$ a line bundle on $X$; a bilinear form $\varphi : F \otimes \mathcal{O}_X F \rightarrow L$ is said to be invertible if the associated homomorphism $\kappa : F \rightarrow \text{Hom}_{\mathcal{O}_X}(F, L)$ is an isomorphism. We will consider forms which are $\varepsilon$-symmetric ($\varepsilon = \pm 1$), that is, such that $^t\kappa = \varepsilon \kappa$.

**Theorem B.**— Assume $\text{char}(k) \neq 2$. Let $X$ be an integral hypersurface of degree $d$ in $\mathbb{P}^n$, and $F$ a torsion-free ACM sheaf on $X$, equipped with an $\varepsilon$-symmetric invertible form $F \otimes F \rightarrow \mathcal{O}_X(d+t)$ ($t \in \mathbb{Z}$). Then $F$ admits a resolution

$$0 \rightarrow L_0^*(t) \xrightarrow{M} L_0 \rightarrow F \rightarrow 0,$$

(B1)

where $L_0 = \oplus \mathcal{O}_{\mathbb{P}^n}(d_i)$ and $M$ is $\varepsilon$-symmetric, that is, $^tM = \varepsilon M$.

Conversely, if a sheaf $F$ on $X$ fits into the exact sequence (B1), it is ACM, torsion-free, and admits an $\varepsilon$-symmetric invertible form $F \otimes F \rightarrow \mathcal{O}_X(d+t)$.

**Proof:** Consider a minimal resolution

$$0 \rightarrow L_1 \xrightarrow{M} L_0 \xrightarrow{p} F \rightarrow 0$$

of $F$. Applying the functor $\text{Hom}_{\mathcal{O}_{\mathbb{P}^n}}(\ast, \mathcal{O}_{\mathbb{P}^n}(t))$ gives an exact sequence

$$0 \rightarrow L_0^*(t) \xrightarrow{^tM} L_1^*(t) \rightarrow \text{Ext}_{\mathcal{O}_{\mathbb{P}^n}}^1(F, \mathcal{O}_X(t)) \rightarrow 0$$

and the vanishing of $\text{Ext}_{\mathcal{O}_{\mathbb{P}^n}}^i(F, \mathcal{O}_X(t))$ for $i \neq 1$.

Let $i$ be the embedding of $X$ into $\mathbb{P}^n$; put $F' = \text{Hom}_{\mathcal{O}_X}(F, \mathcal{O}_X(d+t))$. Grothendieck duality provides a canonical isomorphism $\text{Ext}_{\mathcal{O}_{\mathbb{P}^n}}^1(F, \mathcal{O}_X(t)) \cong i_! F'$.
Thus the above exact sequence gives a minimal resolution of the $\mathcal{O}_{\mathbb{P}^n}$-module $F'$; the isomorphism $\kappa : \mathcal{F} \to \mathcal{F}'$ extends to an isomorphism of resolutions:

$$
0 \to L_1 \xrightarrow{M} L_0 \xrightarrow{p} \mathcal{F} \to 0
$$

Applying the functor $\text{Hom}_{\mathcal{O}_{\mathbb{P}^n}}(\ast, \mathcal{O}_{\mathbb{P}^n}(t))$ leads to another commutative diagram:

$$
0 \to L_1 \xrightarrow{M} L_0 \xrightarrow{p} \mathcal{F} \to 0
$$

Since $t\kappa = \varepsilon \kappa$, we have $q \circ tB = t\kappa \circ p = \varepsilon q \circ A$, which means that there exists a map $C : L_0 \to L_0(t)$ such that $tB - \varepsilon A = tMC$.

Since $tBM = tM^iA$, we have

$$
tMC = (tB - \varepsilon A)M = t(AM) - \varepsilon(AM) = -\varepsilon tM^iCM
$$

and therefore the map $A' := A + \frac{\varepsilon}{2}tMC$ satisfies $t(A'M) = \varepsilon A'M$. Moreover we still have $q \circ A' = \kappa \circ p$, so $A'$ is an isomorphism. We have an exact sequence

$$
0 \to L_0(t) \xrightarrow{M'} L_0 \xrightarrow{p} \mathcal{F} \to 0
$$

where $M' := A'^{-1} tM$ satisfies $tM' = \varepsilon M'$.

Conversely, starting from the exact sequence (B1), Grothendieck duality implies as above an isomorphism $\kappa : \mathcal{F} \to \text{Hom}(\mathcal{F}, \mathcal{O}_X(d + t))$; applying again the functor $\text{Hom}_{\mathcal{O}_{\mathbb{P}^n}}(\ast, \mathcal{O}_{\mathbb{P}^n}(t))$ we obtain $t\kappa = \varepsilon \kappa$.

Remark 2.2.-- The result remains valid in characteristic 2 under the extra hypothesis $\max(e_j) < \min(d_i)$: indeed, with the above notation, the relation $q \circ A = q \circ tB$ implies then directly $A = tB$ (1.6), and we can take $M' = A^{-1} tM$.

F. Catanese pointed out that his proof in [C1] for symmetric surfaces extends readily to the case considered here; it has the advantage of working equally well in characteristic 2, without the above restriction on the degrees.
Assume again $\max(e_j) < \min(d_i)$. Let $0 \to \mathcal{P}_0^* (t') \xrightarrow{M'} \mathcal{P}_0 \xrightarrow{p'} \mathcal{F} \to 0$ be another resolution (B1) of $\mathcal{F}$; then we have $t = t'$ and a commutative diagram

\[
\begin{array}{ccccccccc}
0 & \to & L_0^* (t) & \xrightarrow{M} & L_0 & \xrightarrow{p} & \mathcal{F} & \to & 0 \\
\downarrow {B} & & \downarrow {A} & & \downarrow {\cong} & & \downarrow {\cong} & & \\
0 & \to & \mathcal{P}_0^* (t) & \xrightarrow{M'} & \mathcal{P}_0 & \xrightarrow{q} & \mathcal{F} & \to & 0
\end{array}
\]

where the vertical arrows are isomorphisms.

We have $AM = M'B$, hence, since $M$ and $M'$ are $\varepsilon$-symmetric, $M'A = 'BM'$, and therefore $'BAM = M'AB$. By 1.6 this implies $'AB = \lambda I$ for some $\lambda \in k^*$. Multiplying $A$ by a scalar we get $M' = AM'A$. Thus all $\varepsilon$-symmetric matrices providing a minimal resolution of $\mathcal{F}$ are conjugate under the action of $\text{Aut}(L_0)$.

In the same way we see that every automorphism of $\mathcal{F}$ is induced by a matrix $A \in \text{Aut}(L_0)$ such that $AM'A = \lambda M$ for some $\lambda \in k^*$.

As above let us rephrase Theorem B in the way we will mostly use it:

**Corollary 2.4.** - Assume $\text{char}(k) \neq 2$. Let $X$ be an integral hypersurface of degree $d$ in $\mathbb{P}^n$, and $E$ an ACM line bundle on $X$ with $E^2 \cong \mathcal{O}_X(d+t)$ (resp. an ACM rank 2 vector bundle on $X$ with determinant $\mathcal{O}_X(d+t)$). There exists a symmetric (resp. skew-symmetric) matrix $M = (m_{ij}) \in M_\ell(S)$, with $m_{ij}$ homogeneous of degree $d_i + d_j - t$, and an exact sequence

\[
0 \to \bigoplus_{i=1}^\ell \mathcal{O}_{\mathbb{P}^n}(t-d_i) \xrightarrow{M} \bigoplus_{i=1}^\ell \mathcal{O}_{\mathbb{P}^n}(d_i) \to E \to 0 ;
\]

$X$ is defined by the equation $\det M = 0$ (resp. $\text{pf } M = 0$). If $H^0(X, E(1)) = 0$ and $t = -1$, the matrix $M$ is linear, and the exact sequence takes the form

\[
0 \to \mathcal{O}_{\mathbb{P}^n}(-1)^r \xrightarrow{M} \mathcal{O}_{\mathbb{P}^n}^{r^d} \to E \to 0
\]

with $r = \text{rk } E$.

**Proof:** By assumption $E$ carries an $\varepsilon$-symmetric form $E \otimes E \to \mathcal{O}_X(d+t)$, with $\varepsilon = (-1)^{r-1}$. Then Theorem B provides the above minimal resolution; by (1.7) we have $F = \det M$ if $r = 1$ and $F^2 = \det M = (\text{pf } M)^2$ if $r = 2$. Moreover if $t = -1$ we have $H^{n-1}(X, E(1-n)) \cong H^0(X, E(1))^*$ by Serre duality, so the last assertion follows from Prop. 1.11. 

3. Plane curves as determinants

Let \( C \) be a smooth plane curve of degree \( d \), defined by an equation \( F = 0 \). We denote by \( g = \frac{1}{2}(d-1)(d-2) \) the genus of \( C \). Any line bundle \( L \) on \( C \) is ACM, hence admits a minimal resolution (A1), with \( \det M = F \).

The case of line bundles of degree \( g - 1 \) follows directly from Cor. 1.12 (applied to \( L(1) \)):

**Proposition 3.1.**— a) Let \( L \) be a line bundle of degree \( g - 1 \) on \( C \) with \( H^0(X, L) = 0 \). There exists a \( d \times d \) linear matrix \( M \) such that \( F = \det M \), and an exact sequence

\[
0 \to \mathcal{O}_{\mathbb{P}^2}(-2)^d \xrightarrow{M} \mathcal{O}_{\mathbb{P}^2}(-1)^d \to L \to 0 .
\]

b) Conversely, let \( M \) be a \( d \times d \) linear matrix such that \( F = \det M \). Then the cokernel of \( M : \mathcal{O}_{\mathbb{P}^2}(-2)^d \to \mathcal{O}_{\mathbb{P}^2}(-1)^d \) is a line bundle \( L \) on \( C \) of degree \( g - 1 \) with \( H^0(X, L) = 0 \).

(3.2) Let \( |O_{\mathbb{P}^2}(d)|_{\text{sm}} \) be the open subset of the projective space \( |O_{\mathbb{P}^2}(d)| \) parametrizing smooth plane curves of degree \( d \). For \( \delta \in \mathbb{Z} \), let \( J^\delta_d \to |O_{\mathbb{P}^2}(d)|_{\text{sm}} \) be the family of degree \( \delta \) Jacobians: \( J^\delta_d \) parametrizes pairs \((C, L)\) of a smooth plane curve of degree \( d \) and a line bundle of degree \( \delta \) on \( C \). Finally we denote by \( \Theta_d \) the divisor in \( J^{g-1}_d \) consisting of pairs \((C, L)\) with \( H^0(C, L) \neq 0 \). It is an ample divisor, so its complement in \( J^{g-1}_d \) is affine.

Let \( \mathcal{L}_d \) the open subset of the vector space of linear matrices \( M \in M_d(S^2) \) such that the equation \( \det M = 0 \) defines a smooth plane curve \( C_M \) in \( \mathbb{P}^2 \). By associating to \( M \) the curve \( C_M \) and the line bundle \( L_M := \text{Coker}[O_{\mathbb{P}^2}(-2)^d \xrightarrow{M} O_{\mathbb{P}^2}(-1)^d] \) on \( C_M \) we define a morphism \( \pi : \mathcal{L}_d \to J^{g-1}_d - \Theta_d \). The group \( \text{GL}(d) \times \text{GL}(d) \) acts on \( \mathcal{L}_d \) by \( (P, Q) \cdot M = PMQ^{-1} \); this action factors through the quotient \( G_d \) of \( \text{GL}(d) \times \text{GL}(d) \) by \( \mathbb{G}_m \) embedded diagonally.

**Proposition 3.3.**— The group \( G_d \) acts freely and properly on \( \mathcal{L}_d \); the morphism \( \pi \) induces an isomorphism \( \mathcal{L}_d / G_d \to J^{g-1}_d - \Theta_d \).

**Proof:** This is proved for instance in [B3], §3; let us give a proof based on our present methods. Let \( M \in \mathcal{L}_d \), \( (P, Q) \in \text{GL}(d) \times \text{GL}(d) \), and \( M' = PMQ^{-1} \). Then \( \det M' = \det M \) up to a scalar, and we have a commutative diagram

\[
\begin{array}{ccccccccc}
0 & \to & O_{\mathbb{P}^2}(-1)^d & \xrightarrow{M} & O_{\mathbb{P}^2}^d & \to & L_M & \to & 0 \\
\downarrow Q & & \downarrow P & & \downarrow \iota & & & & \\
0 & \to & O_{\mathbb{P}^2}^d & \xrightarrow{M'} & O_{\mathbb{P}^2}(-1)^d & \to & L_{M'} & \to & 0 \\
\end{array}
\]  

(3.3.a)
thus π factors through a morphism \( \mathcal{L}_d/G_d \to \mathcal{J}_d^{g-1} \rightarrow \Theta_d \). Conversely, if two matrices \( M \) and \( M' \) give rise to isomorphic pairs, the minimal resolution of \( L_M \) and \( L'_M \) are isomorphic, so we have a diagram (3.3.a), which shows that \( M \) and \( M' \) are conjugate under \( G_d \). Thus the orbits of \( G_d \) in \( \mathcal{M}_d \) are isomorphic to the fibres of \( \pi \), hence are closed. Moreover by (1.6) the stabilizer of \( M \) in \( GL(d) \times GL(d) \) reduces to \( G_m \) embedded diagonally, hence \( G_d \) acts freely on \( \mathcal{L}_d \). This proves our assertions.

Remark 3.4.— A simpler birational presentation of the quotient \( GL(d)\mathcal{L}_d/GL(d) \) (and therefore of \( \mathcal{J}_d^{g-1} \)) is obtained as follows. Let \( D_d \) be the closed subset of \( \mathcal{L}_d \) consisting of matrices of the form \( X_0I_d + X_1M_1 + X_2M_2 \); it is isomorphic to an affine open subset of \( M_d \times M_d \), where \( M_d \) denotes the \( k \)-variety of \((d \times d)\)-matrices. Then \( G_dD_d \) is an open affine subset of \( \mathcal{L}_d \), and the stabilizer of \( D_d \) in \( G_d \) is \( PGL(d) \) acting on \( M_d \times M_d \) by conjugation. We thus have an open embedding \( D_d/PGL(d) \hookrightarrow GL(d)\mathcal{L}_d/GL(d) \).

These quotients are of course unirational. It is a classical question to decide whether they are rational: this would have interesting applications in algebra (where the function field of \( D_d/PGL(d) \) is known as the “center of the generic division algebra”) and in geometry (\( D_d/PGL(d) \) is birationally equivalent to the moduli space of stable rank \( d \) vector bundles on \( P^2 \) with \( c_1 = 0 \), \( c_2 = d \)). The rationality is known only for \( d \leq 4 \). We refer to [L] for an excellent survey of these questions.

It is amusing to observe that the universal Jacobian \( \mathcal{J}_d^g \) is rational ([B3], 3.4): using the rational map \( \mathcal{J}_d^g \dashrightarrow Sym^g(P^2) \) which maps a general pair \((C,L)\) to the unique element of \( |L| \), we see that \( \mathcal{J}_d^g \) is birational to a projective fibre bundle over the rational variety \( Sym^g(P^2) \). Unfortunately this does not seem to have any implication on the more interesting question of the rationality of \( \mathcal{J}_d^{g-1} \).

We will now determine the minimal resolution of a generic line bundle \( L \) of arbitrary degree on a generic plane curve. Replacing \( L \) by \( L(t) \) for some \( t \in \mathbb{Z} \) we can assume \( g - 1 \leq \deg L \leq g - 1 + d \).

Proposition 3.5.— Let \( L \) be a line bundle of degree \( g - 1 + p \) on \( C \), with \( 0 \leq p \leq d \). The following conditions are equivalent:
(i) \( H^0(C,L(-1)) = H^1(C,L) = 0 \), and the natural map
\[ \mu_0 : H^0(C,L) \otimes H^0(C,O_C(1)) \to H^0(C,L(1)) \]
is of maximal rank (that is, injective for \( p \leq \frac{d}{2} \) and surjective for \( p \geq \frac{d}{2} \));
(ii) There is an exact sequence
\[ 0 \to \mathcal{O}_{P^2}(-2)^{d-p} \xrightarrow{M} \mathcal{O}_{P^2}(-1)^{d-2p} \oplus \mathcal{O}_{P^2}^p \to L \to 0 \quad \text{if} \quad p \leq \frac{d}{2}, \]
\[ 0 \to \mathcal{O}_{P^2}(-2)^{d-p} \oplus \mathcal{O}_{P^2}(-1)^{2p-d} \xrightarrow{M} \mathcal{O}_{P^2}^p \to L \to 0 \quad \text{if} \quad p \geq \frac{d}{2}. \]
with \( \det M = F \).

The set of pairs \((C, L)\) satisfying these conditions is Zariski dense in \( J_d^{g-1+p} \) (and open if \( k = \bar{k} \)).

**Proof**: Assume that (i) holds. The natural maps

\[
\mu_j : H^0(C, L(j)) \otimes H^0(C, \mathcal{O}_C(1)) \to H^0(C, L(j + 1))
\]

are surjective for \( j \geq 1 \) because \( H^1(C, L) = 0 \) [Md]; since \( H^0(C, L(-1)) = 0 \), this means that the \( S^2 \)-module \( \Gamma_*(L) \) is generated by homogeneous elements of degree 0 and 1. In other words, the minimal resolution of \( L \) takes the form

\[
0 \to \bigoplus_{i=1}^{p+q} \mathcal{O}_{\mathbb{P}^2}(e_i) \xrightarrow{M} \mathcal{O}_{\mathbb{P}^2}(-1)^q \oplus \mathcal{O}_{\mathbb{P}^2}^p \to L \to 0
\]

for some integer \( q \geq 0 \) (observe that \( \dim H^0(C, L) = p \) by Riemann-Roch). The vanishing of \( H^1(C, L) \) and the minimality of the resolution imply \( e_i \in \{-2, -1\} \), so we have

\[
0 \to \mathcal{O}_{\mathbb{P}^2}(-2)^{d-p} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)^r \xrightarrow{M} \mathcal{O}_{\mathbb{P}^2}(-1)^q \oplus \mathcal{O}_{\mathbb{P}^2}^p \to L \to 0
\]

(3.5a)

with \( r = 2p - d + q \). After tensor product with \( \mathcal{O}_{\mathbb{P}^2}(1) \) the cohomology exact sequence gives

\[
q = \dim \text{Coker } \mu_0 , \quad r = \dim \text{Ker } \mu_0 ,
\]

(3.5b)

from which (ii) follows.

If (ii) holds, we have the exact sequence (3.5a) with \( r = 0 \) (if \( p \leq \frac{d}{2} \)) or \( q = 0 \) (if \( p \geq \frac{d}{2} \)). By (3.5b) \( \mu_0 \) is of maximal rank; the vanishing of \( H^0(C, L(-1)) \) and \( H^1(C, L) \) is clear.

Let \( V \) be the vector space of matrices \( M \) appearing in (ii), and \( V_0 \) the open subset of matrices whose determinant defines a smooth curve; observe that \( V_0 \) is non-empty by (1.10). As in (3.3) we get a morphism \( \pi : V_0 \to J_d^{g-1+p} \); since property (i) is open in \( J_d^{g-1+p} \), \( \pi \) is dominant. The last assertion of the Proposition follows.

We just proved:

**Corollary 3.6.** — The variety \( J_d^\delta \) is unirational for all \( \delta \in \mathbb{Z} \).

**Example 3.7.** — Let us consider the relative Jacobian \( J_d^0 \). We have \( g - 1 = \frac{1}{2}d(d - 3) \), so if \( d \) is odd, \( J_d^0 \) is canonically isomorphic to \( J_d^{g-1} \). Assume \( d = 2e \), so that \( J_d^0 \) is canonically isomorphic to \( J_d^{g-1+e} \). For \((C, L)\) generic in \( J_d^{g-1+e} \) the minimal resolution of \( L \) takes the form

\[
0 \to \mathcal{O}_{\mathbb{P}^2}(-2)^e \xrightarrow{M} \mathcal{O}_{\mathbb{P}^2}^e \to L \to 0
\]
so the equation of \( C \) can be written as the determinant of a matrix \( M \in M_5(e(S^2)) \) with quadratic entries. Writing such a matrix as \( M = \sum X_iX_jM_{ij} \), we see as in (3.4) that \( J^0_d \) is birationally equivalent to the quotient of \( M_5^5 \) by \( GL(e) \) acting by conjugation. This quotient is birationally equivalent to a vector bundle over \( M_5^5/GL(e) \) \([L]\); in particular, we see that the variety \( J^0_d \) is rational for \( d = 4, 6 \) or 8.

4. Plane curves as symmetric determinants

By Corollary 2.4, any line bundle \( L \) on \( C \) with \( L \otimes 2 \cong O_C(s) \) admits a symmetric minimal resolution. There are (at least) two interesting applications.

(4.1) Theta-characteristics

Recall that a theta-characteristic on a smooth curve \( C \) is a line bundle \( \kappa \) such that \( \kappa \otimes 2 \cong K_C \). We write \( h^0(\kappa) := \dim H^0(C, \kappa) \).

**Proposition 4.2.** – Let \( C \) be a smooth plane curve, defined by an equation \( F = 0 \), and \( \kappa \) a theta-characteristic on \( C \).

a) If \( h^0(\kappa) = 0 \), \( \kappa \) admits a minimal resolution

\[
0 \rightarrow O_{P^2}(-2)^d \xrightarrow{M} O_{P^2}(-1)^d \rightarrow \kappa \rightarrow 0,
\]

where the matrix \( M \in M_d(S^2) \) is symmetric (linear) and \( \det M = F \).

b) If \( h^0(\kappa) = 1 \), \( \kappa \) admits a minimal resolution

\[
0 \rightarrow O_{P^2}(-2)^{d-3} \oplus O_{P^2}(-3) \xrightarrow{M} O_{P^2}(-1)^{d-3} \oplus O_{P^2} \rightarrow \kappa \rightarrow 0
\]

with a symmetric matrix \( M \in M_{d-2}(S^2) \) satisfying \( \det M = F \), and of the form

\[
M = \begin{pmatrix}
L_{1,1} & \cdots & L_{1,d-3} & Q_1 \\
\vdots & \ddots & \vdots & \vdots \\
L_{1,d-3} & \cdots & L_{d-3,d-3} & Q_{d-3} \\
Q_1 & \cdots & Q_{d-3} & H
\end{pmatrix}
\]

where the forms \( L_{ij}, Q_i, H \) are linear, quadratic and cubic respectively.

Conversely, the cokernel of a symmetric matrix \( M \) as in a) (resp. b)) is a theta-characteristic \( \kappa \) on \( C \) with \( h^0(\kappa) = 0 \) (resp. \( h^0(\kappa) = 1 \)).

Part a) is well-known, and goes back essentially to Dixon [Di]. Part b) is stated for instance (without proof) in [B1], 6.27. Geometrically, when \( \text{char}(k) \neq 2 \), a) means that \( C \) is the discriminant curve of a net of quadrics in \( P^{d-1} \); b) means that \( C \) is the discriminant curve of the quadric bundle obtained by projecting the cubic hypersurface \( \sum U_iU_jL_{ij} + \sum U_iQ_i + H = 0 \) in the projective space \( P^{d-1} \) with coordinates \( U_1, \ldots, U_{d-3}, X_0, X_1, X_2 \) from the subspace \( X_0 = X_1 = X_2 = 0 \).
Proof: Part a) follows directly from Cor. 2.4 (applied to $E = \kappa(1)$).

Let $\kappa$ be a theta-characteristic on $C$, with $h^0(\kappa) = 1$. Then $H^1(C, \kappa(1)) = H^0(C, \kappa(-1))^* = 0$, so $\Gamma_{\ast}(\kappa)$ is generated by its elements of degree 0, 1 and 2. In view of 2.4, the minimal resolution of $\kappa$ is of the form

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-1)^q \oplus \mathcal{O}_{\mathbb{P}^2}(-2)^p \oplus \mathcal{O}_{\mathbb{P}^2}(-3) \overset{M}{\longrightarrow} \mathcal{O}_{\mathbb{P}^2}(-2)^q \oplus \mathcal{O}_{\mathbb{P}^2}(-1)^p \oplus \mathcal{O}_{\mathbb{P}^2} \longrightarrow \kappa \to 0$$

for some non-negative integers $p, q$. Since the resolution is minimal, the summand $\mathcal{O}_{\mathbb{P}^2}(-1)^q$ in the first term is mapped into $\mathcal{O}_{\mathbb{P}^2}$; this implies $q \leq 1$, and in fact $q = 0$ because otherwise the non-zero section of $\kappa$ would be annihilated by some linear form. This gives the form of the resolution (and of the matrix $M$) in part b).

Assume now $\text{char}(k) = 0$. The moduli space of pairs $(C, \kappa)$, where $C$ is a smooth plane curve of degree $d$ and $\kappa$ a theta-characteristic on $C$, has two components, corresponding to the parity of $h^0(\kappa)$, plus one special component when $d$ is odd consisting of the pairs $(C, \mathcal{O}_C((d - 3)/2))$ ([B2], Prop. 3); a general element $(C, \kappa)$ in a non-special component satisfies $h^0(\kappa) \leq 1$.

**Corollary 4.3.** Each component of the moduli space of smooth plane curves with a theta-characteristic is unirational.

**Remark 4.4.** If $k$ is algebraically closed, any smooth plane curve admits a theta-characteristic $L$ with $H^0(L) = 0$: this follows (via the Riemann singularity theorem) from the classical fact that the theta divisor of a principally polarized Abelian variety cannot contain all points of order 2 (see for instance [I], Ch. IV, lemma 11). Thus every smooth plane curve can be defined by a symmetric linear determinant. Actually every plane curve $C$ admits such a representation: one reduces readily to the case when $C$ is integral; then one applies Theorem B to the sheaf $\pi_* L$, where $\pi : C' \to C$ is the normalization of $C$ and $L$ is a theta-characteristic on $C'$ with $H^0(C', L) = 0$.  

(4.5) **Half-periods**

We assume again $\text{char}(k) = 0$. Let us consider now the moduli space $\mathcal{R}_d$ of pairs $(C, \alpha)$, where $C$ is a smooth plane curve of degree $d$ and $\alpha$ a “half-period”, that is, a non-trivial line bundle on $C$ with $\alpha \otimes 2 \cong \mathcal{O}_C$. If $d$ is odd the map $(C, \alpha) \mapsto (C, \alpha((d - 3)/2))$ gives an isomorphism of $\mathcal{R}_d$ onto the above moduli space; we thus restrict to case $d$ even, say $d = 2e$. It follows then from [B2], Prop. 2 that $\mathcal{R}_d$ is irreducible.

---

2 This works equally well in all characteristics $\neq 2$, but references are lacking.

3 This remark answers a question of F. Catanese.
Proposition 4.6. — For \((C, \alpha)\) general in \(\mathcal{R}_d\), the line bundle \(\alpha\) admits a minimal resolution
\[
0 \rightarrow \mathcal{O}_{\mathbb{P}^2}(-e - 1)^e \xrightarrow{M} \mathcal{O}_{\mathbb{P}^2}(-e + 1)^e \rightarrow \alpha \rightarrow 0,
\]
where the matrix \(M \in M_e(S^2)\) is symmetric (with quadratic entries) and \(\det M = F\).

Proof: In view of Cor. 2.4, this amounts to say that the line bundle \(\alpha(e - 1)\) satisfies the equivalent conditions of Prop. 3.5. As in 3.5, it suffices to exhibit a symmetric matrix \(M \in M_e(S^2)\) with quadratic entries such that the equation \(\det M = 0\) defines a smooth plane curve.

Start with a symmetric linear matrix \((L_{ij}) \in M_e(S)\) such that the curve \(\Gamma\) defined by \(\det(L_{ij}) = 0\) is smooth (such a matrix exists by Prop. 4.2). Changing coordinates if necessary we can assume that \(\Gamma\) is transverse to the coordinate axes and does not pass through the intersection point of any two axes. Consider the covering \(\pi: \mathbb{P}^2 \rightarrow \mathbb{P}^2\) given by \(\pi(X_0, X_1, X_2) = (X_0^2, X_1^2, X_2^2)\). The pull-back of \(\Gamma\) by \(\pi\) is smooth by our hypotheses; it is defined by the determinant of the symmetric matrix \(M = (L_{ij}(X_0^2, X_1^2, X_2^2))\) with quadratic entries. ■

Corollary 4.7. — The moduli space \(\mathcal{R}_d\) is unirational. ■

5. Plane curves as pfaffians

Again any rank 2 vector bundle \(E\) on the plane curve \(C\) with determinant \(\mathcal{O}_C(s)\) for some integer \(s\) admits a skew-symmetric resolution. Let us restrict our attention to the linear case. Cor. 2.4 applied to \(E(1)\) gives:

Proposition 5.1. — Let \(C\) be a smooth plane curve of degree \(d\), \(E\) a rank 2 vector bundle on \(C\) with \(\det E \cong K_C\) and \(H^0(C, E) = 0\). Then \(E\) admits a minimal resolution
\[
0 \rightarrow \mathcal{O}_{\mathbb{P}^2}(-2)2d \xrightarrow{M} \mathcal{O}_{\mathbb{P}^2}(-1)^{2d} \rightarrow E \rightarrow 0
\]
where the matrix \(M \in M_{2d}(S^2)\) is linear skew-symmetric and \(\text{pf}\, M = F\). ■

Note that the condition \(H^0(C, E) = 0\) implies that \(E\) is semi-stable.

Corollary 5.2. — The moduli space of pairs \((C, E)\), where \(C\) is a smooth plane curve of degree \(d\) and \(E\) a semi-stable rank 2 vector bundle on \(C\) with determinant \(K_C\), is unirational. ■

This is not surprising in this case, since the fibres of the projection to \(|\mathcal{O}_{\mathbb{P}^2}(d)|\) are already unirational.

(5.3) Another consequence of the Proposition is that if \(d \geq 4\) and \(M\) is general enough, the corresponding vector bundle \(E_M = \text{Coker}\, M\) is stable and therefore simple, that is, \(\text{End}(M) = k\). In view of (2.3) this means that given 3 generic
skew-symmetric matrices $M_0, M_1, M_2 \in M_{2d}(k)$, the equations $^tA M_i A = M_i$ for $i = 0, 1, 2$ imply $A = \pm I$.

6. Surfaces as determinants

(6.1) Let $S$ be a smooth surface of degree $d$ in $\mathbb{P}^3$, defined by an equation $F = 0$. Let $C$ be a curve in $S$, and $L = \mathcal{O}_S(C)$. Using the exact sequence $0 \to L^{-1} \to \mathcal{O}_S \to \mathcal{O}_C \to 0$ and Serre duality, we see that $L$ is ACM if and only if $C$ is projectively normal in $\mathbb{P}^3$, that is, the restriction map $H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(j)) \to H^0(C, \mathcal{O}_C(j))$ is surjective for every $j \in \mathbb{Z}$. Since any line bundle is of the form $\mathcal{O}_S(C)$ after some twist, this characterizes the ACM line bundles on $S$. Thus any projectively normal curve contained in $S$ gives rise to an expression of $F$ as the determinant of a matrix $M \in M_{2d}(\mathbb{P}^3)$. Recall however that a hypersurface section of $S$ gives the trivial case $M = (F)$; a curve $C$ defined in $\mathbb{P}^3$ by two equations $A = B = 0$ produces a $2 \times 2$-matrix $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$.

We will now restrict our study to linear determinants.

**Proposition 6.2.** Let $C$ be a projectively normal curve on $S$, of degree $\frac{1}{2}d(d-1)$ and genus $\frac{1}{6}(d-2)(d-3)(2d+1)$. The line bundle $\mathcal{O}_S(C)$ admits a minimal resolution

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-d) \xrightarrow{M} \mathcal{O}_{\mathbb{P}^3}^d \to \mathcal{O}_S(C) \to 0$$

with $\det M = F$.

Conversely, let $M \in M_d(\mathbb{P}^3)$ be a linear matrix such that $\det M = F$; the cokernel of $M : \mathcal{O}_{\mathbb{P}^3}(-d) \to \mathcal{O}_{\mathbb{P}^3}^d$ is isomorphic to $\mathcal{O}_S(C)$, where $C$ is a smooth projectively normal curve on $S$ with the above degree and genus.

**Proof:** Let $C$ be a curve on $S$; put $L = \mathcal{O}_S(C)$. A straightforward Riemann-Roch computation shows that the given condition on the degree and genus of $C$ is equivalent to $\chi(L(-1)) = \chi(L(-2)) = 0$. If $C$ is projectively normal the spaces $H^1(S, L(j))$ vanish (6.1), therefore the above condition is also equivalent to $H^0(S, L(-1)) = H^2(S, L(-2)) = 0$; this is exactly what we need to apply Cor. 1.12.

Conversely, given a matrix $M$, let $L = \text{Coker} M$; in view of the above all we have to prove is that the linear system $|L|$ contains a smooth curve. This is obvious in characteristic $0$ since $L$ is spanned by its global sections. In the general case, we first observe that the restriction of $L$ to any smooth hyperplane section $H$ of $S$ is very ample: indeed from the resolution $0 \to \mathcal{O}_{\mathbb{P}^2}(-d) \to \mathcal{O}_{\mathbb{P}^2}^d \to L|_H \to 0$ we get $H^1(H, L|_H(-1)) = 0$, hence $H^1(H, L|_H(-x-y)) = 0$ for all $x, y \in H$. It follows that the linear system $|L|$ on $S$ separates two points $x, y \in S$ (possibly infinitely close) unless the line $(x, y)$ lies in $S$; in other words, the morphism $\varphi_L : S \to \mathbb{P}^{d-1}$
defined by $|L|$ contracts finitely many lines, and embeds the complement of these lines. Then a general hyperplane in $\mathbb{P}^{d-1}$ cuts down a smooth curve $C \in |L|$.

(6.3) Under the hypotheses of the Proposition, Grothendieck duality provides a dual exact sequence (see the proof of Theorem B):

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-1)^d \xrightarrow{i^* M} \mathcal{O}_{\mathbb{P}^3}^d \to L^{-1}(d-1) \to 0;$$

in other words, the involution $M \mapsto i^* M$ on the space of linear matrices corresponds to the involution $L \mapsto L^{-1}(d-1)$ on $\text{Pic}(S)$.

As we already pointed out, a general form of degree $d$ on $\mathbb{P}^3$ can be represented as a linear determinant only for $d \leq 3$, the only non-trivial case being $d = 3$. There we find the following classical result [G]:

**Corollary 6.4.**– Assume $k$ is algebraically closed. A smooth cubic surface can be defined by an equation $\det M = 0$, where $M$ is a $3 \times 3$-linear matrix. There are 72 such representations (up to the action of $\text{GL}(3) \times \text{GL}(3)$ by left and right multiplication), corresponding in a one-to-one way to the linear systems of twisted cubics on $S$.

There are various ways of describing the set of linear systems of twisted cubics on $S$: they also correspond to the birational morphisms $S \to \mathbb{P}^2$, or to the sets of 6 lines on $S$ which do not intersect each other. In terms of these, the involution $M \mapsto i^* M$ corresponds to the Schäfli involution which associates to such a set $\{\ell_1, \ldots, \ell_6\}$ the unique set $\{\ell'_1, \ldots, \ell'_6\}$ such that the 12 lines $\ell_i, \ell'_j$ form a double-six, that is satisfy $\ell_i \cap \ell'_i = \emptyset$ and $\ell_i \cdot \ell'_j = 1$ for $i \neq j$.

As a consequence, the space of pairs $(S, \lambda)$, where $S$ is a smooth cubic surface and $\lambda$ a set of 6 non-intersecting lines, is rational: as in (3.4) it is birational to the quotient of $(\mathbb{P}^3)^3$ by the group $\text{GL}(3)$ acting by conjugation, and we know that this quotient is rational.

In the case of a non-necessarily algebraically closed field, we find the following result of B. Segre [Se]:

**Corollary 6.5.**– Let $S$ be a smooth cubic surface. The following conditions are equivalent:

(i) $S$ can be defined by an equation $\det M = 0$, where $M$ is a $3 \times 3$-linear matrix;
(ii) $S$ contains a twisted cubic;
(iii) $S$ admits a birational morphism to $\mathbb{P}^2$;
(iv) $S$ contains a rational point and a set (defined over $k$) of 6 non-intersecting lines.

**Proof:** The equivalence of (i), (ii) and (iii) follows from Prop. 6.2. The implication (iii) $\Rightarrow$ (iv) is clear. If (iv) holds, the surface obtained from $S$ by blowing down the
set of 6 non-intersecting lines is isomorphic to $\mathbb{P}^2$ over $\overline{k}$ and contains a rational point, hence is $k$-isomorphic to $\mathbb{P}^2$.

**Corollary 6.6.** A smooth quartic surface is determinantal if and only if it contains a non-hyperelliptic curve of genus 3, embedded in $\mathbb{P}^3$ by a linear system of degree 6.

**Proof:** The only point to check is that a curve $C$ of genus 3, embedded in $\mathbb{P}^3$ by a linear system $|L|$ of degree 6, is projectively normal if and only if it is not hyperelliptic. Since $H^1(C,L) = 0$, the projective normality reduces using the base-point free pencil trick to the surjectivity of the restriction map $H^0(\mathbb{P}^3,\mathcal{O}_{\mathbb{P}^3}(2)) \to H^0(C,L^2)$; or equivalently, since both spaces have the same dimension, to its injectivity. One checks that $C$ is contained in a quadric if and only if it is hyperelliptic.

(6.7) There is another approach to Prop. 6.2, which we will now sketch. Given the linear matrix $M$, let $C$ be the divisor of the section of $L = \text{Coker} M$ corresponding to the first basis vector of $\mathcal{O}_\mathbb{P}^d$. Using (6.3) we see easily that the curve $C$ is defined in $\mathbb{P}^3$ by the maximal minors of the matrix $N$ obtained from $M$ by deleting the first row. Conversely, since $C$ is projectively normal, it admits a resolution

$$0 \to \bigoplus_{j=1}^{\ell-1} \mathcal{O}_{\mathbb{P}^3}(e_j) \xrightarrow{N} \bigoplus_{i=1}^{\ell} \mathcal{O}_{\mathbb{P}^3}(d_i) \xrightarrow{\Delta} \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_C \to 0,$$

where $\Delta$ is given by the maximal minors of $N$; with some work one finds $\ell = d$, $e_1 = \ldots = e_{d-1} = -d$ and $d_1 = \ldots = d_d = -(d-1)$. It follows easily that any surface of degree $d$ containing $C$ is defined by the determinant of a linear matrix obtained by adding one row to $N$.

(6.8) As indicated in the introduction, we will not consider surfaces defined by symmetric determinants, though this is again a classical and rich story; we refer to [C1] or [C2] for a modern treatment.

7. Surfaces as Pfaffians

*From now on we assume char($k$) = 0 (see 7.3).*

(7.1) Again we will restrict ourselves to the linear case, that is to surfaces $S \subset \mathbb{P}^3$ defined by an equation $\text{pf} M = 0$, where $M$ is a $(2d) \times (2d)$ skew-symmetric linear matrix.

Let $Z$ be a finite reduced subscheme of $\mathbb{P}^n$, of degree $c$, and $I_Z$ its homogeneous ideal in $S^n$. $Z$ is said to be *arithmetically Gorenstein* if the algebra $R := S/I_Z$ is Gorenstein. This implies that there exists an integer $N$ such that:

$^4$ The degree of $Z$ is by definition $\dim_k H^0(Z, \mathcal{O}_Z)$.
a) $\dim R_p + \dim R_{N-p} = c$ for all $p \in \mathbb{Z}$.

The integer $N$ is uniquely determined: it is the largest integer such that $\dim R_N < c$. By lack of a better name we will call it the index of $Z$.

Assume $k = \bar{k}$. By [D-G-O], thm. 5, the subscheme $Z$ is arithmetically Gorenstein if and only if it satisfies a) and:

b) $Z$ has the Cayley-Bacharach property w.r.t. the linear system $|O_{P^n}(N)|$; that is, for each point $z \in Z$, every element of $|O_{P^n}(N)|$ containing $Z \setminus z$ contains $Z$.

In general, $Z$ is arithmetically Gorenstein if and only if $Z \otimes_k \bar{k}$ has the same property.

Let $Z \subset P^3$ be a finite arithmetically Gorenstein subscheme, contained in a surface $S$ of degree $d$. Let $\mathcal{I}_Z$ be the sheaf of ideals of $Z$ in $\mathcal{O}_S$. Using the exact sequence $0 \to \mathcal{I}_Z \to \mathcal{O}_S \to \mathcal{O}_Z \to 0$, property a) for $p = N$ gives $\dim H^1(S, \mathcal{I}_Z(N)) = 1$. Thus there exists a unique non-trivial extension (up to automorphism)

$$0 \to \mathcal{O}_S \to E \to \mathcal{I}_Z(N-d+4) \to 0.$$ 

We claim that $E$ is locally free. To check this we can assume that $k$ is algebraically closed; then b) is equivalent to $H^1(S, \mathcal{I}_Z'(N)) = 0$ for each proper subset $Z' \subset Z$, which implies our assertion by [G-H]. We will say that $E$ is the vector bundle associated to $Z$.

**Proposition 7.2.** Let $S$ be a smooth surface of degree $d$ in $P^3$. The following conditions are equivalent:

(i) $S$ can be defined by an equation $pf M = 0$, where $M$ is a skew-symmetric linear $(2d) \times (2d)$ matrix;

(ii) $S$ contains a finite arithmetically Gorenstein reduced subscheme $Z$ of index $2d-5$, not contained in any surface of degree $d-2$.

More precisely, under hypothesis (ii), the rank 2 vector bundle $E$ associated to $Z$ admits a minimal resolution

$$0 \to \mathcal{O}_{P^n}(-1)^{2d} \xrightarrow{M} \mathcal{O}_{P^n}^{2d} \to E \to 0;$$

the degree of $Z$ is $\frac{1}{6}d(d-1)(2d-1)$.

**Proof:** If (i) holds the vector bundle $E := \text{Coker } M$ is spanned by its global sections; let $Z$ be the zero locus of a general section of $E$. Under (i) or (ii) we have an exact sequence

$$0 \to \mathcal{O}_S \to E \to \mathcal{I}_Z(d-1) \to 0.$$ (7.2.a)

In view of Prop. 2.4, we have to prove the equivalence of:

- $E$ is ACM and $H^0(S, E(-1)) = 0$;
- $Z$ is arithmetically Gorenstein and $H^0(S, \mathcal{I}_Z(d-2)) = 0$. 

To do that we can assume $k = \bar{k}$. The fact that $E$ is locally free implies that $Z$ has the Cayley-Bacharach property w.r.t. $|\mathcal{O}_{\mathbb{P}^3}(2d-5)|$ [G-H]. The sequence (7.2.a) provides an isomorphism

$$H^0(S, E(-1)) \sim \rightarrow H^0(S, \mathcal{I}_Z(d-2)),$$

and gives rise for each $j \in Z$ to an exact sequence

$$0 \rightarrow H^1(S, E(j)) \rightarrow H^1(S, \mathcal{I}_Z(d-1+j)) \overset{\partial}{\rightarrow} H^2(S, \mathcal{O}_S(j)).$$

Using the exact sequence $0 \rightarrow \mathcal{I}_Z \rightarrow \mathcal{O}_S \rightarrow \mathcal{O}_Z \rightarrow 0$, we can identify $H^1(S, \mathcal{I}_Z(k))$ with the cokernel of the restriction map $r_k : H^0(S, \mathcal{O}_S(k)) \rightarrow H^0(Z, \mathcal{O}_Z(k))$; the map $H^0(Z, \mathcal{O}_Z(d-1+j)) \rightarrow H^2(S, \mathcal{O}_S(j))$ deduced from $\partial$ is identified by Serre duality to the transpose of $r_{d-4-j}$. Therefore the vanishing of $H^1(S, E(j))$ is equivalent to $\text{Im} r_{d-1+j} = \text{Ker} t r_{d-4-j} = (\text{Im} r_{d-4-j})^\perp$, that is to $\dim R_{d-1+j} = c - \dim R_{d-4-j}$.

This proves the equivalence of (i) and (ii).

Under these equivalent conditions, we have $\text{Card} Z = c_2(E)$; this number can be computed for instance using Riemann-Roch and $\chi(E) = 2d$.

Remarks 7.3. a) We have to restrict to the characteristic 0 case because we do not know how to prove that the zero locus of a general section of $E$ is smooth in characteristic $p$. The same problem occurs in higher dimension as well.

b) As in (6.7) we could use another approach: using the Buchsbaum-Eisenbud theorem [B-E] one shows that $\mathcal{I}_Z$ is generated by the $(2d-2) \times (2d-2)$ pfaffians extracted from a skew-symmetric linear $(2d-1) \times (2d-1)$-matrix $N$; then $X$ is defined by the pfaffian of the matrix $\begin{pmatrix} N & C \\ -tC & 0 \end{pmatrix}$, where $C$ is a column of linear forms.

Examples 7.4. For a cubic surface we have $\deg Z = 5$, and $N = 1$. If $k = \bar{k}$ a subset $Z$ is arithmetically Gorenstein if and only if any 4 points in $Z$ are linearly independent, that is, $Z$ is in general position.

For a quartic the subset $Z$ should have 14 points, not be contained in a quadric, and satisfy the Cayley-Bacharach property w.r.t. $|\mathcal{O}_S(3)|$.

(7.5) Let us observe that for each $d$ there exists smooth surfaces defined by the pfaffian of a $(2d) \times (2d)$ skew-symmetric linear matrix, and therefore containing a subset $Z$ with the properties of the Proposition; we can take for instance $M = \begin{pmatrix} 0 & N \\ -tN & 0 \end{pmatrix}$, where $N$ is a linear $d \times d$ matrix: we have $\text{pf} M = \det N$, and we can choose $N$ so that the surface $\det N = 0$ is smooth (1.10). The corresponding vector bundle $E$ is $L \oplus L^{-1}(d-1)$, where $L$ is the line bundle $\text{Coker} N$; the zero set $Z$ of a general section of $E$ is the intersection of two curves on $S$ of the type described in Prop. 6.2 (see also 8.3).
We will now investigate when a generic surface of degree \( d \) can be written as a linear pfaffian.

**Proposition 7.6.**— Assume that \( k \) is algebraically closed.

a) Every cubic surface can be defined by a linear pfaffian.

b) A general surface of degree \( d \) in \( \mathbb{P}^3 \) can be defined by a linear pfaffian if and only if \( d \leq 15 \).

**Proof:** a) follows from Proposition 7.2 and Example 7.4. Let \( S_d \) be the variety of linear skew-symmetric matrices \( M \in M_{2d}(S^3) \) such that the equation \( \text{pf} M = 0 \) defines a smooth surface in \( \mathbb{P}^3 \). Consider the map \( \text{pf} : S_d \to |\mathcal{O}_{\mathbb{P}^3}(d)| \). We have \( \dim S_d/\text{GL}(2d) = 4d(2d-1) - 4d^2 = 4d(d-1) \); an easy computation gives \( 4d(d-1) < \dim |\mathcal{O}_{\mathbb{P}^3}(d)| \) for \( d \geq 16 \), which gives the “only if” part of b).

To prove the remaining part we use Adler’s method ([A-R], App. V), namely we prove that the differential of \( \text{pf} \) is surjective at a general matrix \( M \in S_d \). As in loc. cit., a standard computation shows that this is equivalent to the fact that the vector space \( H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(d)) \) is spanned by the forms \( X_k M_{ij} \), where \( M_{ij} \) is the pfaffian of the skew-symmetric matrix obtained from \( M \) by deleting the rows and columns of index \( i \) and \( j \). This has been checked by F. Schreyer using the computer algebra system Macaulay 2: a script is provided in the Appendix.

We do not consider the proof of b) as completely satisfactory, since it relies on a computer checking which does not provide any clue as why the result holds. The following lemma explains better the meaning of the result. Recall that we associate to a matrix \( M \in S_d \) the smooth surface \( S_M \) defined by \( \text{pf} M = 0 \) and the vector bundle \( E_M := \text{Coker}[\mathcal{O}_{\mathbb{P}^n}(-1)^d \xrightarrow{M} \mathcal{O}_{\mathbb{P}^n}^d] \) on \( S_M \).

**Lemma 7.7.**— The pfaffian map \( \text{pf} : S_d \to |\mathcal{O}_{\mathbb{P}^3}(d)| \) is dominant if and only if \( H^2(S_M, \mathcal{E}nd_0(E_M)) \) vanishes for a general \( M \) in \( S_d \).

(As usual \( \mathcal{E}nd_0(E) \) denotes the bundle of traceless endomorphisms of \( E \).)

**Proof:** We will restrict our attention to matrices \( M \) such that \( E_M \) is simple, that is, has only scalar endomorphisms. According to 2.3, this means that the only matrices \( A \in M_d(k) \) such that \( AM^4A = M \) are \( \pm I \). The matrices \( M \) with this property form an open subset \( S_d^s \) of \( S_d \), which is non-empty by 5.3.

We consider the map \( \text{pf} : S_d^s \to |\mathcal{O}_{\mathbb{P}^3}(d)| \); its fibre at a point \( S \in |\mathcal{O}_{\mathbb{P}^3}(d)| \) is the moduli space of simple ACM rank 2 vector bundles on \( S \) with \( \text{det} E = \mathcal{O}_S(d-1) \) and \( H^0(S, E(-1)) = 0 \). A straightforward computation gives

\[
\dim S_d/\text{GL}(2d) = \dim |\mathcal{O}_{\mathbb{P}^3}(d)| - \chi(\mathcal{E}nd_0(E_M)) = \dim |\mathcal{O}_{\mathbb{P}^3}(d)| + \dim H^1(S_M, \mathcal{E}nd_0(E_M)) - \dim H^2(S_M, \mathcal{E}nd_0(E_M))
\]

(7.7.a)

for any matrix \( M \in S_d^s \).
If $H^2(S_M, \mathcal{E}nd_0(E_M)) = 0$, the moduli space of simple vector bundles on $S_M$ is smooth of dimension $\dim H^1(S_M, \mathcal{E}nd_0(E_M))$ at $[E_M]$. It then follows from (7.7.a) that $pf$ is dominant.

Conversely, assume that $pf$ is dominant. Let $S$ be a generic surface of degree $d$; the fibre $pf^{-1}(S)$ can be identified with an open subset of the moduli space of simple rank 2 bundles $E$ on $S$ with $\det E = O_S(d - 1)$ and $c_2(E) = \frac{1}{6}d(d - 1)(2d - 1)$. Being smooth, this open subset is of dimension $\dim H^1(S, \mathcal{E}nd_0(E))$. Comparing with (7.7.a) gives $H^2(S, \mathcal{E}nd_0(E)) = 0$.

(7.8) Thus assertion b) of Prop. 7.6 is equivalent to the fact that on a general surface $S$ of degree $d$, the moduli space of ACM rank 2 vector bundles with $\det E = O_S(d - 1)$ and $H^0(S, E(-1)) = 0$ is smooth of the expected dimension $-\chi(\mathcal{E}nd_0(E))$ for $d \leq 15$. We were not able to prove this directly, except for the obvious case $d = 4$ where the vanishing of $H^2(S, \mathcal{E}nd_0(E))$ follows from Serre duality.

8. Threefolds as linear pfaffians

(8.1) Let us first briefly recall Serre’s construction. Let $X$ be a projective manifold of dimension $\geq 3$ and $E$ a rank 2 vector bundle on $X$, spanned by its global sections; put $L = \det E$. Then the zero locus of a general section $s$ of $E$ is a submanifold $V$ of codimension 2 in $X$, and there is an exact sequence

$$0 \rightarrow \mathcal{O}_X \xrightarrow{s} E \rightarrow \mathcal{I}_V L \rightarrow 0;$$

it follows that $K_V$ is isomorphic to $(K_X \otimes L)|_V$. Conversely, given a codimension 2 submanifold $V \subset X$ and a line bundle $L$ on $X$ such that $K_V \cong (K_X \otimes L)|_V$, there exists a rank 2 vector bundle $E$ and a section $s \in H^0(X, E)$ such that $Z(s) = V$; if moreover $V$ is connected, the pair $(E, s)$ is uniquely determined up to isomorphism. We will refer to $E$ as the vector bundle associated to $V$.

Recall that a submanifold $V$ of $\mathbb{P}^n$ is said to be arithmetically Cohen-Macaulay if the sheaf $\mathcal{O}_V$ is ACM and $V$ is projectively normal. This implies in particular $H^0(V, \mathcal{O}_V) = k$, so $V$ is connected.

**Proposition 8.2.** Let $X$ be a smooth hypersurface of degree $d$ in $\mathbb{P}^n$ ($n = 4$ or 5). The following conditions are equivalent:

(i) $X$ can be defined by an equation $pf M = 0$, where $M$ is a skew-symmetric linear $(2d) \times (2d)$ matrix;

(ii) $X$ contains a codimension 2 submanifold $V$ which is arithmetically Cohen-Macaulay, not contained in any hypersurface of degree $d - 2$, and such that $K_V \cong \mathcal{O}_V(2d - 2 - n)$. 

More precisely, under hypothesis (ii), the rank 2 vector bundle $E$ associated to $V$ admits a minimal resolution

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-1)^{2d} \xrightarrow{\mathcal{M}} \mathcal{O}_{\mathbb{P}^n}^{2d} \to E \to 0;$$

the variety $V$ has degree $\frac{1}{6}d(d-1)(2d-1)$.

**Proof:** If (i) holds the vector bundle $E := \text{Coker } M$ is spanned by its global sections; let $V$ be the zero locus of a general section of $E$. Under (i) or (ii) we have an exact sequence

$$0 \to \mathcal{O}_X \to E \to \mathcal{I}_V(d-1) \to 0.$$

By Serre duality, $E$ is ACM if and only if $H^i(X, E(j)) = 0$ for $1 \leq i \leq n-3$; in view of the above exact sequence this is equivalent to $V$ being arithmetically Cohen-Macaulay. Similarly the condition $H^0(X, E(-1)) = 0$ translates as $H^0(X, \mathcal{I}(d-2)) = 0$; we conclude by cor. 2.4.

The degree of $V$ can be deduced for instance from (7.2) by restriction to a general 3-dimensional linear subspace. ■

(8.3) Note that there exist indeed smooth threefolds and fourfolds satisfying the equivalent conditions of the Proposition. One way to see this is to consider the vector space $M_{2d}^{ss}$ of skew-symmetric $(2d) \times (2d)$ matrices, and the universal pfaffian hypersurface $X_d \subset \mathbb{P}(M_{2d}^{ss})$ consisting of singular matrices. The singular locus of $X_d$ consists of those matrices which are of rank $\leq 2d-4$, and has codimension 6. Therefore for $n \leq 5$ a generic $\mathbb{P}^n \subset \mathbb{P}(M_{2d}^{ss})$ intersects $X_d$ along a smooth hypersurface in $\mathbb{P}^n$, defined by the vanishing of a linear pfaffian.

(8.4) The cubic threefold

**Proposition 8.5.**— If $k = \bar{k}$, every smooth cubic threefold can be defined by an equation $\text{pf } M = 0$, where $M$ is a skew-symmetric linear $6 \times 6$ matrix.

As mentioned in the introduction, this result is due to Adler ([A-R], App. V) in the case of a generic cubic threefold.

**Proof:** Let $X$ be a smooth cubic threefold. In view of Prop. 8.2, we have to prove that $X$ contains a normal elliptic quintic curve. This is essentially in [M-T], Remark 4.9; we sketch the argument since the result we need is not explicitly stated there. We first observe that $X$ contains a non-normal elliptic quintic curve $C$ (that is, contained in a hyperplane); in fact any smooth hyperplane section $S$ of $X$ contains finitely many 5-dimensional families of such curves (represent $S$ as $\mathbb{P}^2$ blown up at 6 points and consider the linear system of plane cubics passing through 4 of the 6 points). Varying the hyperplane section gives a 8-dimensional family of non-normal elliptic quintic curves in $S$. 
Let $C$ be one of these curves; the normal bundle $N_{C/V}$ fits into an exact sequence

$$0 \to \mathcal{O}_C(1) \to N_{C/V} \to N_{C/S} \to 0,$$

from which one deduces $H^1(C, N_{C/V}) = 0$ and $\dim H^0(C, N_{C/V}) = 10$. Therefore the Hilbert scheme of curves of degree 5 and arithmetic genus 0 in $V$ is smooth of dimension 10 at $C$. The general member of the component containing $C$ is a smooth elliptic quintic curve not contained in any hyperplane, and therefore projectively normal.

(8.6) By Prop. 2.4, a rank 2 vector bundle $E$ on $X$ is associated to a normal elliptic quintic if and only if $F = E(-1)$ satisfies $\det F = \mathcal{O}_X$ and $H^0(X, F) = 0$; since $\text{Pic}(X) = \mathbb{Z}$, this last condition means that $F$ is stable (with respect to $\mathcal{O}_X(1)$). Let $\mathcal{M}_X$ be the moduli space of stable ACM rank 2 vector bundles on $X$ with trivial determinant; it is smooth of dimension 5 [M-T]. By a theorem of Druel [Dr], this is also the moduli space of stable rank 2 vector bundles on $X$ with $c_1 = 0$ and $c_2 = 2\ell$, where $\ell$ denotes the class of a line in $H^4(X, \mathbb{Z})$; we will not need this result here.

Let us now vary $X$ and consider the space $\mathcal{M}$ of pairs $(X, F)$, where $X$ is a smooth element of $|\mathcal{O}_{P^4}(3)|$ and $F \in \mathcal{M}_X$. By the Proposition we have a dominant rational map from the space of linear skew-symmetric matrices $M \in M_6(S^4)$ onto the space $\mathcal{M}$, which is therefore unirational.

(8.7) Thanks to [M-T], this has the following nice consequence. We now assume $k = \mathbb{C}$. Let $|\mathcal{O}_{P^4}(3)|_{sm}$ be the open subset of $|\mathcal{O}_{P^4}(3)|$ parametrizing smooth cubic threefolds. The intermediate Jacobians of cubic threefolds fit into a universal family $\mathcal{J} \to |\mathcal{O}_{P^4}(3)|_{sm}$. More generally, for each integer $k$ we can define a twisted intermediate Jacobian $J^k(X)$, which parametrizes one-dimensional cycles on $X$ with cohomology class $k\ell$; this is a principal homogeneous space under the usual intermediate Jacobian $J^0(X)$. These spaces fit into a family $\mathcal{J}^k$ over $|\mathcal{O}_{P^4}(3)|_{sm}$; while each $J^k(X)$ is isomorphic to $J^0(X)$, it is not clear that $\mathcal{J}^k$ is isomorphic to $\mathcal{J}$. However the class of a plane section is a canonical element in each $J^3(X)$, giving a section of the fibration $\mathcal{J}^3 \to |\mathcal{O}_{P^4}(3)|_{sm}$; this provides canonical isomorphisms $\mathcal{J}^k \cong \mathcal{J}^{k+3}$ above $|\mathcal{O}_{P^4}(3)|_{sm}$. Note also that for $p \in \mathbb{Z}$ the multiplication map $\mathcal{J}^k \times_{\mathcal{J}} \mathcal{J}^{pk}$ is a finite étale covering, since it is so on each fibre.

**Corollary 8.8.** The intermediate Jacobian $\mathcal{J}$ of the universal family of cubic threefolds is unirational.

**Proof:** Associating to a pair $(X, F)$ in $\mathcal{M}$ the class of $c_2(F)$ defines a morphism $\mathcal{M} \to \mathcal{J}^2$ above $|\mathcal{O}_{P^4}(3)|_{sm}$. By [M-T] this morphism is étale, hence dominant; thus $\mathcal{J}^2$ is unirational. Using the maps $\mathcal{J}^2 \times \mathcal{J}^6 \cong \mathcal{J}$, we conclude that $\mathcal{J}$ is unirational. ■
Let us discuss the case of higher degree threefolds.

**Proposition 8.9.** Assume that $k$ is algebraically closed. A general threefold of degree $d$ in $\mathbb{P}^4$ can be defined by a linear pfaffian if and only if $d \leq 5$.

**Proof:** Let us denote again by $S_d$ the space of linear skew-symmetric matrices $M \in M_{2d}(S^4)$ such that the equation $\text{pf} M = 0$ defines a smooth hypersurface $X_M \subset \mathbb{P}^4$. As before the group $\text{GL}(2d)$ acts freely and properly on $S_d$, and the map $\text{pf} : S_d \to |\mathcal{O}_{\mathbb{P}^4}(d)|$ factors through $S_d/\text{GL}(2d)$.

An easy computation gives $\dim S_d/\text{GL}(2d) < \dim |\mathcal{O}_{\mathbb{P}^4}(d)|$ for $d \geq 6$, so a general threefold of degree $\geq 6$ is not pfaffian. For $d = 4$ and $5$ one checks as in 7.6 that the differential of $\text{pf}$ at a generic matrix is surjective (Appendix; for $d = 4$ this was also observed in [I-M]).

(8.10) Exactly as in lemma 7.7 we find that the map $\text{pf} : S_d \to |\mathcal{O}_{\mathbb{P}^4}(d)|$ is dominant if and only if $H^2(X_M, \text{End}_0(E_M)) = 0$ for $M$ general in $S_d$ – that is, if the moduli space of the vector bundles we are considering on a general quartic or quintic threefold has the expected dimension. We see in particular that there is a finite number of ways of representing a general quintic as a pfaffian; this number is an instance of the generalized Casson invariant considered by Thomas [T]. It would be of course quite interesting to determine it.

**9. Fourfolds as linear pfaffians**

(9.1) Let us keep the notation of (8.9) for fourfolds in $\mathbb{P}^5$. We find in this case $\dim S_d/\text{GL}(2d) < \dim |\mathcal{O}_{\mathbb{P}^5}(d)|$ for $d \geq 3$, so a general hypersurface of degree $\geq 3$ in $\mathbb{P}^5$ cannot be defined by the vanishing of a linear pfaffian (a smooth hyperquadric can of course, since it is isomorphic to the Grassmannian of lines in $\mathbb{P}^3$ in the Plücker embedding). For $d = 3$ one finds $\dim S_3/\text{GL}(6) = \dim |\mathcal{O}_{\mathbb{P}^5}(3)| - 1$.

**Proposition 9.2.** a) A (smooth) cubic fourfold $X \subset \mathbb{P}^5$ is pfaffian if and only if it contains a Del Pezzo surface of degree 5.

b) Assume $k = \mathbb{C}$. The map $\text{pf} : S_3/\text{GL}(6) \to |\mathcal{O}_{\mathbb{P}^5}(3)|$ is generically injective. In particular, pfaffian cubic fourfolds form a hypersurface in the space of all smooth cubic fourfolds.

The pfaffian cubics play a key role in the proof that the variety of lines contained in a cubic fourfold is irreducible symplectic [B-D]. Cubic fourfolds containing a Del Pezzo surface of degree 5 were already considered by Fano [F].

**Proof:** Part a) follows at once from Prop. 8.2; let us prove part b).

Let us introduce a 6-dimensional vector space $V$ and the space $\text{Alt}(V)$ of bilinear alternate forms on $V$; we will view $S_3$ as an open subset of $\text{Alt}(V)^6 = \ldots$
Alt(V) \otimes_k k^6. The map \( pf : S_3 \to |O_{P^5}(3)| \) associates to a sextuple \((\varphi_0, \ldots, \varphi_5)\) the hypersurface \( pf(\sum_i X_i \varphi_i) = 0 \). The group \( GL(6) \) acts on \( S_3 \) through its action on \( k^6 \); this action commutes with the action of \( GL(V) \), and the map \( pf : S_3/ GL(V) \to |O_{P^5}(3)| \) is \( GL(6) \)-equivariant. The orbits of \( GL(6) \) in \( S_3 \) correspond to 6-dimensional subspaces \( L \subset Alt(V) \); to such a subspace is associated the isomorphism class of the cubic hypersurface \( X_L \) of degenerate forms in \( P(L) \).

Since the action of \( GL(6) \) is generically free on \( |O_{P^5}(3)| \), it is sufficient to prove that the isomorphism class of \( X_L \) determines \( L \) (up to the action of \( GL(V) \)).

The orthogonal \( L^\perp \) of \( L \) in \( \Lambda^2 V \) is 9-dimensional; the locus of rank 2 bivectors in \( P(L^\perp) \) is a K3 surface \( S \) of genus 8 [B-D]. By [M], a general K3 surface of genus 8 is obtained in this way, and this representation is unique: the surface \( S \) determines the space \( L^\perp \subset \Lambda^2 V \) (and therefore also the space \( L \subset Alt(V) \)) up to the action of \( GL(V) \). So what we need to prove is that the cubic \( X_L \) determines the K3 surface \( S \) up to projective isomorphism.

We proved in [B-D] that the variety \( F \) of lines contained in \( X_L \) is a (complex) symplectic manifold, isomorphic to the Hilbert scheme \( S[2] \); in particular, the group \( H^2(F, \mathbb{Z}) \) carries a canonical quadratic form, and there is a Hodge isometry

\[
H^2(F, \mathbb{Z}) \longrightarrow H^2(S, \mathbb{Z}) \oplus \mathbb{Z} \delta ,
\]

where \( H^2(S, \mathbb{Z}) \) is endowed with the intersection form and \( \delta \) is a class of type \((1, 1)\) and square \(-2\). The polarization of \( F \) given by the embedding in the Grassmannian \( G(2, 6) \) corresponds to the class \( 2l - 5\delta \), where \( l \) is the polarization on \( S \) deduced from the embedding \( S \subset P(L^\perp) \).

Let \( L \) and \( L' \) be two subspaces of \( Alt(V) \) which produce isomorphic cubics; let \((S, l)\) and \((S', l')\) be the corresponding polarized K3 surfaces. We then have a Hodge isometry

\[
\varphi : H^2(S, \mathbb{Z}) \oplus \mathbb{Z} \delta \longrightarrow H^2(S', \mathbb{Z}) \oplus \mathbb{Z} \delta'
\]

which maps the class \( 2l - 5\delta \) to the corresponding class \( 2l' - 5\delta' \). Assume \( \text{Pic}(S) = \mathbb{Z}l \). Then we have \( \text{Pic}(S') = \mathbb{Z}l' \), and \( \varphi \) induces an isometry \( \mathbb{Z}l \oplus \mathbb{Z} \delta \longrightarrow \mathbb{Z}l' \oplus \mathbb{Z} \delta' \) which maps \( 2l - 5\delta \) onto \( 2l' - 5\delta' \); an easy computation shows that this implies \( \varphi(\delta) = \varphi(\delta') \). Thus \( \varphi \) induces a Hodge isometry of \( H^2(S, \mathbb{Z}) \) onto \( H^2(S', \mathbb{Z}) \) mapping \( l \) to \( l' \); by the Torelli theorem for K3 surfaces this implies that \((S, l)\) and \((S', l')\) are isomorphic. ■
Appendix

Hypersurfaces are generically pfaffian in the expected range

Frank-Olaf Schreyer

In this appendix we prove by a Macaulay 2 computation that a generic surface of degree \( d \leq 15 \) in \( \mathbb{P}^3 \), and a general threefold of degree \( d \leq 5 \) in \( \mathbb{P}^4 \), can be defined by the pfaffian of a skew-symmetric \( 2d \times 2d \) matrix with linear entries (Propositions 7.6 and 8.9 in the text). As explained in the paper, it is sufficient to prove that for some matrix \( M \) of this type the space of homogeneous forms of degree \( d \) is equal to \( \mathfrak{m} \cdot \text{pfaffians}(2d - 2, M) \), where \( \mathfrak{m} \) is the ideal spanned by the coordinates and \( \text{pfaffians}(2d - 2, M) \) the ideal of submaximal pfaffians of \( M \). We compute the dimension of the latter space at randomly chosen skew symmetric matrices over a finite field using Macaulay 2 \([G-S]\). The computation is within the range of nowadays computers. On the computer “alice” of the Mathematical Science Research Institute at Berkeley the following code was executed in about 2 hours of cpu. The output verifies the result.

```plaintext
isPrime(31991)
kk=ZZ/31991 -- this is a field

randomSkewMatrix = (e,S) -> (  
  -- returns a random e x e skew symmetric matrix  
  -- with linear entries in the ring S  
  N:=binomial(e,2);  
  R:=kk[t_0..t_(N-1)];  
  G:=genericSkewMatrix(R,t_0,e);  
  substitute(G,random(S^0),S^N{-1}))  
) -- end randomSkewMatrix

subPfaffiansViaSyzygies = (M) -> (  
  -- This is an alternative to the command pfaffians(2d-2,M).  
  -- It returns the generators of the ideal of the 2d-2 pfaffians  
  -- of the linear 2d x 2d skew symmetric matrix M computed  
  -- using the structure theorem of [B-E]:  
  -- Under a mild genericity condition on the submatrix M1  
  -- the syzygies of the 2d-1 x 2d-1 skew matrix M1 are its 2d-1  
  -- principal pfaffians.  
  -- If the computation fails, then the standard way is used.  
  d:=lift((rank source M)/2,ZZ);  
  syzygiesGivePfaffians=true; i:=0; S:=ring M;
```

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J:=generators ideal0_S;
while syzygiesGivePfaffians==true and (i<(2*d)) do (  
    -- take i-th 2d-1 x 2d-1 skew submatrix
    M1:=transpose((transpose(M_{0..(i-1),(i+1)..(2*d-1)}))
                _{0..(i-1),(i+1)..(2*d-1)});
    N1:=syz(M1,DegreeLimit=>d);
    syzygiesGivePfaffians=((degrees source N1) == {{d}});
    if syzygiesGivePfaffians==true then
        J=(J|flatten(N1));
        i=i+1;
    );
    if syzygiesGivePfaffians then (mingens image J)
    else (mingens image pfaffians(2*d-1,M))
) -- end subPfaffiansViaSyzygies

isDominant=(r,d) -> (  
    S:=kk[x_0..x_r]; M:=randomSkewMatrix(2*d,S);
    J:=subPfaffiansViaSyzygies(M);
    N=syz(J,DegreeLimit=>d);
    -- DegreeLimit=> d is carefully choosen to compute only
    -- linear syzygies. From this the number of kk-linear
    ---independent elements of degree d in the ideal
    -- with generated by J can be computed:
    cd=binomial(d+r,r)-(r+1)*rank(target N)+(rank source N);
    cd==0) -- end isDominant

lowerBoundForDominantDegree = (r) -> (  
    dominant:=true; d:=2;
    while dominant do
        (d=d+1;dominant=isDominant(r,d));
    d-1)

isDominant(5,3)
    cd
    time d4=lowerBoundForDominantDegree(4)
    time d3=lowerBoundForDominantDegree(3)

Note that we used the method to compute pfaffians via syzygies, since this  
is faster than the command pfaffians(2*d-2,M). The reason is that syzygy  
computations are fast, while the pfaffian command does not utilize much special  
structure. For comments on the commands and the Macaulay 2 language we refer  
to the on-line help.

Notice that the computation also shows that the closure of the scheme of  
pfaffian cubic 4-folds form a hypersurface in |O_{P^5}(3)|.
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