DIFFERENTIAL CALCULUS, TENSOR PRODUCTS AND THE IMPORTANCE OF NOTATION

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Abstract. An efficient coordinate-free notation is elucidated for differentiating matrix expressions and other functions between higher-dimensional vector spaces. This method of differentiation is known, but not explained well, in the literature. Teaching it early in the curriculum would avoid the tedium of element-wise differentiation and provide a better footing for understanding more advanced applications of calculus. Additionally, it is shown to lead naturally to tensor products, a topic previously considered too difficult to motivate quickly in elementary ways.

1. Introduction. The derivative of a function \( f : \mathbb{R} \to \mathbb{R} \), being a far-reaching concept, is taught early to students. Higher-dimensional functions \( f : \mathbb{R}^n \to \mathbb{R}^m \) can then be handled element-wise by computing the partial derivatives of the components of \( f = (f_1, \ldots, f_m) \). Yet an element-wise approach to calculus is uninformative and tedious. It does not provide an inherently systematic way of differentiating matrix expressions or functions between abstract vector spaces.

A known alternative exists. It uses the notation \( Df \) to denote the (Fréchet) derivative \([13, \text{Chapter 8}]\) of a function \( f : V \to W \) between normed vector spaces \( V \) and \( W \). The chain rule and product rule are typically written as

\[
D(f \circ g) = (Df \circ g) Dg,
\]

\[
D(fg) = (Df) g + f Dg.
\]

Books remain silent on how to apply these rules. Even just endeavouring to expand \( D^2(f \circ g) \) leads to confusion though:

\[
D^2(f \circ g) = D((Df \circ g) Dg)
\]

\[
= (D(Df \circ g)) Dg + (Df \circ g) D^2g
\]

\[
= ((D^2 f \circ g) Dg) Dg + (Df \circ g) D^2g.
\]

Taken literally, it makes no sense to multiply \( D^2 f \circ g \) with \( Dg \) twice. This confusion possibly caused an error in a well-regarded book \([2, \text{p. 3}]\) that was pointed out in \([9]\).

This article propounds a minor modification of the \( Df \) notation that avoids such confusion, and exemplifies that the notation makes differentiation easier, faster and more meaningful than working exclusively with gradients, Jacobians and Hessians \([15]\).

The modification involves the tensor product \( \otimes \). Importantly, \( \otimes \) can be introduced merely as a formal symbol separating the arguments of a function, and students can become familiar with manipulating \( \otimes \) as part of learning calculus. Later, it can be revealed that \( \otimes \) is actually a tensor product that reduces multi-linear maps to linear maps. This pedagogic approach might remove the difficulty students normally have with the concept of a tensor product.

The details of how to use tensor products to simplify working with derivatives are not readily found in the literature; no mention is made in the following textbooks on differential calculus \([1, \text{Chapter 2}], [13, \text{Chapter 8}], [25, \text{Chapter 4}], [26, \text{Chapter 5}]\), nor in the following textbooks on differential geometry \([2, \text{Chapter 1}], [4, \text{Chapter 1}]\).

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I.2], [13] Chapter I.3], [19]. Furthermore, Section 6 illustrates that a formal treatment actually requires some care.

Interestingly, although calculus is often considered elementary, many aspects of it are not elementary at all. A plethora of articles exist on the chain rule alone, including [6, 8–10, 18, 21, 23]. The existence of differentiable yet nowhere monotone functions [3], while true, is far from obvious. The history is not straightforward including [6, 8–10, 18, 21, 23]. The existence of differentiable yet now here monotone functions [6, 8–10, 18, 21, 23] is not elementary at all. A plethora of articles exist on the chain rule alone, actually requires some care.

The following approach is considerably simpler. Explanations follow in subsequent sections. Fix a matrix $Z$ of the same dimension as $X$. Then:

$$f(X + tZ) - f(X) = \text{tr} \left\{ (X + tZ)^T A (X + tZ) \right\} - \text{tr} \left\{ X^T AX \right\}$$

$$= (\text{tr} \left\{ Z^T AX \right\} + \text{tr} \left\{ X^T AZ \right\}) t + (\text{tr} \left\{ Z^T AZ \right\}) t^2.$$  

Since derivatives represent linear approximations, (2.2) shows the derivative of $f$ at $X$ in the direction $Z$ is $\text{tr} \left\{ Z^T AX \right\} + \text{tr} \left\{ X^T AZ \right\}$. The meaning may not be clear yet, but the calculation was simple!

The mapping $Z \rightarrow \text{tr} \left\{ Z^T AX \right\} + \text{tr} \left\{ X^T AZ \right\}$ is linear: if it sends $Z_1$ to $c_1$ and $Z_2$ to $c_2$ then it sends $\alpha Z_1 + \beta Z_2$ to $\alpha c_1 + \beta c_2$ for $\alpha, \beta \in \mathbb{R}$. This linear mapping is the (Fréchet) derivative of $f$.

$$Df(X) \cdot Z = \text{tr} \left\{ Z^T AX \right\} + \text{tr} \left\{ X^T AZ \right\}$$

If required, the Jacobian matrix can be read off as $(A + A^T)X$.

Treating $Z$ as a constant and differentiating (2.3) gives

$$(D^2 f(X) \cdot Z) \cdot T = \text{tr} \left\{ Z^T (A + A^T)T \right\}.$$  

The Hessian is $(A + A^T)$. The left-hand side of (2.5) is more commonly written as $D^2 f(X) \cdot (Z, T)$.

3. First-Order Derivatives and Gradients. The definition of the derivative $f'(x) = \lim_{t \to 0} t^{-1}[f(x + t) - f(x)]$ of a function $f: \mathbb{R} \to \mathbb{R}$ extends in several ways to functions $f: U \to V$ between normed vector spaces $U$ and $V$. The reader may take, for concreteness, $U$ and $V$ to be scalars $\mathbb{R}$, vectors $\mathbb{R}^n$ or matrices $\mathbb{R}^{n \times m}$.
One extension considers directional derivatives, reducing to the case $g: \mathbb{R} \to V$, $g(t) = f(x + tz)$ for fixed $x, z \in U$, for which the same formula can be used:

$$D_z f(x) = \lim_{t \to 0} \frac{f(x + tz) - f(x)}{t}. \quad (3.1)$$

If the limit exists for all $z$ then (3.1) is called the Gâteaux derivative of $f$ at $x$.

Another extension looks beyond (3.1) and focuses on the geometric meaning of $f'(x)$ as the rate of change in the direction $T$ directly with $Df$. Assume there exists a (continuous) linear function $A_x(z)$ such that

$$\lim_{z \to 0} \frac{\|f(x + z) - f(x) - A_x(z)\|}{\|z\|} = 0. \quad (3.2)$$

Then $A_x$ is unique and is called the Fréchet derivative of $f$ at $x$, denoted $Df(x)$. Sometimes, evaluation in a particular direction is denoted using a dot, as in (2.3).

Fréchet derivatives can be calculated by finding the directional derivatives $D_z f(x)$ then verifying $A_x(z) = D_z f(x)$ satisfies (3.2). Verification is unnecessary if the Fréchet derivative is known to exist by other means. The $f$ in Section 2 is a polynomial, hence its Fréchet derivative exists and can be found using directional derivatives, either explicitly as in (2.2) or, in more complicated situations, by using truncated Taylor series approximations. Of course, tables and rules could be used instead.

If $f: U \to \mathbb{R}$ is a scalar function then its gradient at $x$ is defined with respect to an inner product. This is often forgotten because the Euclidean inner product is chosen without mention in many textbooks. In matrix space, the Euclidean inner product is $\langle A, B \rangle = \text{tr} \{ B^T A \}$. For a fixed matrix $G$, $A(Z) = \langle G, Z \rangle$ is a linear functional, and every linear functional can be written this way. The gradient of $f$ at $X$ is the matrix $G_X$ such that $Df(X) \cdot Z = \langle G_X, Z \rangle$.

4. Second-order Derivatives and Hessians. The (Fréchet) derivative of a function $f: U \to V$ is $Df: U \to L(U; V)$ where $L(U; V)$ is the normed vector space of (continuous) linear maps from $U$ to $V$, with norm the operator norm. Applying $D$ to $Df$ yields the second-order derivative $D^2 f: U \to L(U; L(U; V))$. A second-order derivative requires not one, but two, directions: $(D^2 f(X) \cdot T) \cdot Z$. The right-hand side of (4.1) interprets this as the rate of change in the direction $T$ of the directional derivative $Df(X) \cdot Z$.

To the letter of the law, $D^2 f(X)$ is calculated from (2.4) as follows. Working directly with $Df(X) \cdot Z$ is not allowed because $Df(X)$ must be treated as an element of $L(U; V)$ when computing $D^2 f(X) \cdot T = D(Df)(X) \cdot T$. By assuming the Fréchet derivative exists, it suffices to work with directional derivatives:

$$D(Df)(X) \cdot T = \lim_{t \to 0} \frac{Df(X + tT) - Df(X)}{t}. \quad (4.1)$$
For clarity, let \( L_t = Df(X + tT) \in L(U; V) \). For fixed \( t \), both \( L_t - L_0 \) and \( (L_t - L_0)t^{-1} \) are linear operators in \( L(U; V) \). The vector space structure on \( L(U; V) \) is that induced by pointwise operations: \((L_t - L_0)t^{-1}\) evaluated at \( Z \) is \((L_t \cdot Z - L_0 \cdot Z)t^{-1}\) by definition. A sequence of linear operators converges if and only if it converges pointwise (throughout, all vector spaces are finite-dimensional for simplicity). Thus, the right-hand side of (4.1) can be determined pointwise:

\[
\left( \lim_{t \to 0} \frac{Df(X + tT) - Df(X)}{t} \right) \cdot Z = \lim_{t \to 0} \frac{Df(X + tT) \cdot Z - Df(X) \cdot Z}{t} = \operatorname{tr} \{ Z^T(A + A^T)T \}. \tag{4.2}
\]

In words, \( D(Df)(X) \cdot T \) is the linear operator \( Z \mapsto \operatorname{tr} \{ Z^T(A + A^T)T \} \).

A nominally different quantity is the derivative \( Dg(X) \cdot T \) where \( g(X) = Df(X) \cdot Z \) for a fixed \( Z \). Nevertheless, \( Dg(X) \cdot T = \operatorname{tr} \{ Z^T(A + A^T)T \} \), the same as (4.3). Indeed, the pointwise vector space structure on \( L(U; V) \) means

\[
(D^2f(X) \cdot T) \cdot Z = (D(Df)(X) \cdot T) \cdot Z = D(Df \cdot Z)(X) \cdot T. \tag{4.4}
\]

Therefore \( D^2f \) can be calculated from \( Df(X) \cdot Z \) by treating \( Z \) as a constant and differentiating with respect to \( X \). This is how (2.5) is obtained from (2.4).

The above notation is simple but cumbersome. Textbooks generally drop the variables, writing the chain rule and product rule as (4.1) and (4.2). Without variables though, deducing \((D^2(f \circ g))(X) \cdot T \cdot Z \) from (4.5) takes experience.

Including directions from the start reveals

\[
D(f \circ g) \cdot Z = (Df \circ g) \cdot (Dg \cdot Z), \tag{4.5}
\]

\[
D(f \cdot (g \cdot Z)) \cdot T = (Df \cdot T) \cdot (g \cdot Z) + f \cdot ((Dg \cdot T) \cdot Z), \tag{4.6}
\]

\[
(D^2(f \circ g) \cdot T) \cdot Z = ((D^2f \circ g) \cdot (Dg \cdot Z)) \cdot (Dg \cdot Z) + (Df \circ g) \cdot ((D^2g \cdot T) \cdot Z). \tag{4.7}
\]

Here, \( X \) is omitted because it is simple enough to feed it in to the terms requiring it. To be clear, \( Df \circ g \) means evaluate \( Df \) at \( g(X) \).

Neither approach is particularly friendly. Omitting variables omits important details while including variables is tedious; the reader is invited to derive (4.7) from either (4.5) and (4.6), or from (4.1) and

\[
D(f \cdot Z) = (Df \cdot Z) \cdot Z. \tag{4.8}
\]

For scalar fields \( f: U \to \mathbb{R} \), the unique linear operator \( H_X \) satisfying \((D^2f(X) \cdot T) \cdot Z = (H_X \cdot T, Z)\) is the Hessian of \( f \) at \( X \). The ordering is unimportant because \( D^2f(X) \) is symmetric: \((D^2f(X) \cdot T) \cdot Z = (D^2f(X) \cdot Z) \cdot T \) for all \( Z \) and \( T \). When the Euclidean inner product is used, \( H_X \) agrees with what is called the Hessian matrix (4.5).

5. A Tensor Product Notation for Derivatives. Technically, the product \((Df)g \) in (1.2) cannot be formed; given \( f: U \to L(V; W) \) and \( g: U \to L(Y; V) \), \( Df \) maps into \( L(U; L(V; W)) \) whereas \( g \) maps into \( L(Y; V) \). The tensor product allows replacing \( L(U; L(V; W)) \) by \( L(U \otimes V; W) \). Equation (1.2) should actually be written

\[
D(fg) = (Df)(I \otimes g) + f \cdot Dg \tag{5.1}
\]
where \( I \) is the identity map. This has the correctness of (4.8) and almost the same brevity as (4.2).
The tensor product can be understood simply as directing variables to their correct targets: the $g$ in $(Df)g$ blocks $Z$ from reaching $Df$ when applied on the right, while the $I$ in $Df(I \otimes g)$ allows the $Z$ through. Although the direct sum could also accomplish this, only the tensor product behaves correctly under differentiation:

$$D(f \otimes g) = (Df \otimes g) + (f \otimes Dg). \quad (5.2)$$

In particular, $I$ can be differentiated again by using $D(I \otimes g) = (DI \otimes g) + (I \otimes Dg) = (0 \otimes g) + (I \otimes Dg) = I \otimes Dg$. If $f$ is itself a derivative then $\frac{\partial}{\partial f}$ becomes

$$D(f \circ g) = (Df \circ g)(Dg \otimes I). \quad (5.3)$$

Following these rules gives

$$D^2(f \circ g) = (D^2f \circ g)(Dg \otimes I)(I \otimes Dg) + (Df \circ g)D^2g \quad (5.4)$$

$$= (D^2f \circ g)(Dg \otimes Dg) + (Df \circ g)D^2g, \quad (5.5)$$

$$D^3(f \circ g) = (D^3f \circ g)(Dg \otimes Dg \otimes Dg) + (D^2f \circ g)(Dg \otimes Dg \otimes Dg) \quad (5.6)$$

The remainder of this section gives the intermediate steps. Section 6 presents a formal description of the notation.

Start with $D(f \circ g) = (Df \circ g)Dg$. Differentiate to get $D(Df \circ g)(I \otimes Dg) + (Df \circ g)D^2g$. This time, $\frac{\partial}{\partial f}$ is required: $D(Df \circ g) = (D^2f \circ g)(Dg \otimes I)$. Tensor products of linear maps satisfy the rule $(A \otimes B)(C \otimes D) = (AC \otimes BD)$. Therefore, $(Dg \otimes I)(I \otimes Dg) = Dg \otimes Dg$.

To obtain $\frac{\partial}{\partial f}$, first apply the product rule $\frac{\partial}{\partial g}$ to the two additive terms in $\frac{\partial}{\partial f}$. Note $D(Dg \otimes Dg) = (D^2g \otimes Dg) + (Dg \otimes D^2g)$. At this point,

$$D^3(f \circ g) = (D^3f \circ g)(Dg \otimes I)(I \otimes (Dg \otimes Dg))$$

$$+ (D^2f \circ g)[(D^2g \otimes Dg) + (Dg \otimes D^2g)] \quad (5.7)$$

$$+ (D^2f \circ g)(Dg \otimes I)(I \otimes D^2g) + (Df \circ g)D^3g.$$

The first $I$ in $(5.7)$ acts on $U \otimes U$ whereas the second acts on $U$. Regardless, it is agreeable to equate $(Dg \otimes I)(I \otimes (Dg \otimes Dg))$ with $Dg \otimes (Dg \otimes Dg) = Dg \otimes Dg \otimes Dg$, and $(5.6)$ readily follows from $(5.7)$.

6. Formal Description. The notation used in Section 6 is derived below. Some intricacies appear but go unnoticed in practice. The notation simplifies the differentiation by hand of abstract expressions, such as when seeking bounds like the one in $(7.2)$. It is generally not needed for differentiating specific functions; see Section 2.

All spaces are finite-dimensional vector spaces. Basic properties of tensor products are used. The main principle is that canonical isomorphisms of vector spaces can be applied freely because they essentially commute with the Fréchet derivative.

Given $f : U \to V$, define $\bar{D}^k : U \to L(U \otimes \cdots \otimes U; V)$ by

$$\bar{D}^k f(X) \cdot (Z_1 \otimes \cdots \otimes Z_k) = ((D^k f(X) \cdot Z_1) \cdots) \cdot Z_k. \quad (6.1)$$

For $g : U \to L(V; W)$, define $\bar{D}^k_g : U \to L(U \otimes \cdots \otimes U \otimes V; W)$ by

$$\bar{D}^k_g (g) \cdot (Z_1 \otimes \cdots \otimes Z_k \otimes T) = (((D^k g(X) \cdot Z_1) \cdots) \cdot Z_k) \cdot T. \quad (6.2)$$
Although $D^2 f \neq D(D f)$, they agree up to a canonical linear isomorphism. In fact, $D^k f = D^{k-1}_f (D f)$. For all intents and purposes, $D^2_\nu (g)$ agrees with $D_{\nu \otimes V} (D_\nu (g))$ because applying the canonical identification $U \otimes (U \otimes V) \cong U \otimes U \otimes V$ in practice simply means omitting a pair of brackets.

Given $g: U \rightarrow L(V; W)$ and $h: U \rightarrow L(W; Y)$, the product rule is
\[
\bar{D}_V(h g) = \bar{D}_W(h) (I_U \otimes g) + h \bar{D}_V(g) \tag{6.3}
\]
where $I_U: U \rightarrow U$ is the identity map and $I_U \otimes g$ is a tensor field over $U$ whose value at $X \in U$ is $I_U \otimes g(x) \in L(U \otimes V; U \otimes W)$.

Related is the application of a linear map to a vector; given $f: U \rightarrow W$ then
\[
\bar{D}(h \cdot f) = \bar{D}_W(h) (I_U \boxtimes f) + h \bar{D}f \tag{6.4}
\]
where $(I_U \boxtimes f)(X)$ is the linear map $Z \mapsto (Z \otimes f(X))$. Later, by minor abuse of notation, $\otimes$ will replace $\boxtimes$. Since $L(U; V) \boxtimes W = L(U; V \otimes W) \cong L(U; V) \otimes W$, both $\boxtimes$ and $\otimes$ behave essentially the same way when differentiated.

For $d: V \rightarrow L(W; Y)$, $e: U \rightarrow V$ and $f: V \rightarrow W$, the two chain rules are
\[
\bar{D}(f \circ e) = (\bar{D}f \circ e) \bar{D}e, \tag{6.5}
\]
\[
\bar{D}_W(d \circ e) = (\bar{D}_W(d) \circ e) (\bar{D}e \otimes I_W) \tag{6.6}
\]
where $I_W: W \rightarrow W$ is the identity map.

For $e: U \rightarrow V$ and $f: U \rightarrow W$, the tensor product rule is
\[
\bar{D}(e \otimes f) = (\bar{D}e \boxtimes f) + (e \boxtimes \bar{D}f) \tag{6.7}
\]
where $\boxtimes$ combines a vector and a linear map to form a linear map, as in (6.4). For $g: U \rightarrow L(V; W)$ and $h: U \rightarrow L(C; Y)$, the tensor product rule is
\[
\bar{D}_V \otimes_C (g \otimes h) = (\bar{D}_V (g) \otimes h) + (g \otimes \bar{D}_C (h)). \tag{6.8}
\]
For $e: U \rightarrow V$ and $h: U \rightarrow L(C; Y)$,
\[
\bar{D}_C (e \boxtimes h) = (\bar{D}e \otimes h) + (e \boxtimes \bar{D}_C (h)). \tag{6.9}
\]

If the codomains of all functions are spaces of linear maps then the situation is particularly simple; (6.3), (6.4) and (6.8) suffice. This is the typical situation when computing higher-order derivatives because the codomain of the derivative of a function is a space of linear maps. It is possible to reduce to this situation by replacing $f: U \rightarrow W$ with $\bar{f}: U \rightarrow L(\mathbb{R}; W)$ where $f(X) = f(X) \cdot 1$. The $\cdot 1$ can be removed, the derivatives calculated, and the $\cdot 1$ applied at the very end. This explains the similarity of (6.3) and (6.7), and of (6.4), (6.8) and (6.9).

In practice, it is easier to replace $\boxtimes$ by $\otimes$ than replace $f$ by $\bar{f}$. No confusion arises because $\boxtimes$ and $\otimes$ behave the same way with respect to addition, multiplication and differentiation.

The subscripts on $\bar{D}$ used in (6.3)–(6.9) merely keep all derivatives in a consistent form and can be dropped. When computing higher-order derivatives recursively, to account for $\bar{D}^k$ differing from $\bar{D}^{k-1}$ by a linear isomorphism, it is only necessary to remove any remaining brackets in tensor products at the end of each step, e.g., replace $Dg \otimes (Dg \otimes Dg)$ by $Dg \otimes Dg \otimes Dg$ in (6.7).

Once $\boxtimes$ is replaced by $\otimes$ and the subscripts dropped on $\bar{D}$, the rules collapse to those in Section 5.
7. **Discussion.** Attention has been restricted to finite-dimensional vector spaces. In principle, the results remain valid in infinite dimensions, but a subtlety is that tensor products are not uniquely defined on Banach spaces; different choices of norms, and hence completions with respect to that norm, are possible [22].

Bounds on the (operator) norms of derivatives are important in a number of contexts, including for analysing the convergence of iterative algorithms in numerical analysis. The formal treatment in Section 6 justifies the calculation

\[
\|D^2(f \circ g)\| \leq \|D^2f \circ g\| \|Dg \otimes Dg\| + \|Df \circ g\| \|D^2g\| \tag{7.1}
\]

\[
\leq \|D^2f \circ g\| \|Dg\|^2 + \|Df \circ g\| \|D^2g\|. \tag{7.2}
\]

An alternative derivation to that of Section 6 could be based on treating $\otimes$ as a formal symbol used to direct variables to their correct targets and developing a mechanical calculus. This would follow the course of building a class $\Omega$ of allowable expressions, explaining how $D$ is applied to members of this class, and verifying the class is algorithmically closed under $D$.

8. **Relevance.** This section discusses several situations where coordinate-free differentiation simplifies matters.

The opening sentence of [10] asserts that formulae for differentiating composite functions are simple only in the case of first-order derivatives, where the chain rule applies. Yet requiring the first few derivatives of a composite function is a common occurrence, such as requiring for a Newton method the first two derivatives of a cost function $f: \mathbb{R}^{n \times m} \to \mathbb{R}$ given as the composition $f = g \circ h$.

The derivatives of $f$ may be calculated from the formula in [9] or [23], but unless one is already familiar with the notation in (3.1)–(3.2) of [23], the formula may be difficult to apply quickly with confidence. Perhaps less daunting is the formula labelled the chain rule for Hessian matrices, appearing in [12, p. 110]. Yet this involves the Kronecker product and may not yield a parsimonious description of the second-order derivative. Furthermore, it is inapplicable if the domain of $f$ is not a matrix space.

The author’s preferred choice is using (1.1), (5.1) and (5.3) to differentiate $f = g \circ h$ twice by hand. No complicated formulae are involved, and the same basic rules apply regardless of the actual domains of $g$ and $h$. Moreover, the standard rules for manipulating norms hold, hence bounds such as (7.2) are readily obtained.

The practical relevance of composite cost functions $f = g \circ h$ on interesting domains includes the theory and practice of optimisation on manifolds [7,16,17], where $g$ is a cost function on the manifold itself and $h$ is used to “pull” the cost function back locally to a function on the tangent space. Matrix manifolds such as the Stiefel manifold occur naturally in signal processing and the method exemplified in Section 2 is often the easiest way of differentiating functions on matrix manifolds.

A statistician wishing to estimate from data the entries of a symmetric matrix may be lead to studying cost functions whose domains are symmetric matrices. The coordinate-free framework handles this effortlessly: $Df(X) \cdot Z$ is defined exactly as before, where $X$ and $Z$ are symmetric matrices.

The function $f(X) = \log \det X$ of an invertible matrix $X$ is encountered in various situations, including in relation to maximum entropy methods [20]. (The article [20] itself implicitly advocates the coordinate-free approach to differentiation because the coordinate-free approach makes transparent the underlying geometry.) Differentiating $f(X) = \log \det X$ element-wise [15, pp. 149–151] is tedious and uninformative; as if by magic, a pleasing expression results. By comparison, a coordinate-free derivation
is easily written down and remembered. The starting point is \( \det(I + tZ) = I + \operatorname{tr}\{Z\} t + \cdots \) which can be derived by writing \( Z \) in Jordan canonical form \( Z = P^{-1}JP \) and expressing the determinant of an upper-triangular matrix as the product of the diagonal elements. Precisely,

\[
\det(I + tZ) = \det\left(P^{-1}(I + tJ)P\right) = \det(I + tJ) = \prod_i (1 + tJ_{ii}) = 1 + t \sum_i J_{ii} + \cdots = 1 + t \operatorname{tr}\{J\} + \cdots.
\]

Therefore,

\[
f(X + tZ) = \log \det\left(X(I + tX^{-1}Z)\right) = \log \det X + \log \det\left(I + tX^{-1}Z\right) = \log \det X + \log \left(1 + t \operatorname{tr}\{X^{-1}Z\} + \cdots\right) = \log \det X + t \operatorname{tr}\{X^{-1}Z\} + \cdots
\]

from which it follows immediately that \( Df(X) \cdot Z = \operatorname{tr}\{X^{-1}Z\} \).

9. Conclusion. Derivatives of matrix expressions arise frequently in the applied sciences \cite{15}. The traditional element-wise approach is tedious and uninformative compared with the \( Df \) notation (Section 2). It is incongruous with the ease with which \( Df \) can be taught that it is not as widely used as it profitably could be.

One could speculate the downfall of the \( Df \) notation is the difficulty encountered when repeatedly applying the chain and product rules (Section 1). This difficulty is eliminated by adopting the modified notation introduced in Section 5.

This modified notation is advocated to be taught to students early in the curriculum. The tensor product \( \otimes \) appearing in the notation can be treated merely as a formal symbol separating arguments to functions and which is differentiated analogously to the product rule, hence the \( \times \) in \( \otimes \). Furthermore, the notation is pedagogically interesting as an elementary yet genuine application of the tensor product.

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Appendix A. A Counterexample.

Even if the directional derivatives \( (3.1) \) exist and fit together linearly, the derivative \( (3.2) \) need not exist. Simple counterexamples are known. It is nevertheless insightful to derive a counterexample from first principles.

Consider the region in \( \mathbb{R}^2 \) between the parametrised curves \( t \mapsto (t, 0) \) and \( t \mapsto (t, 4t^2) \). If \( f: \mathbb{R}^2 \rightarrow \mathbb{R} \) is zero on and outside the boundary of the parametrised region, it will be initially zero for a short distance on every ray emanating from the origin, that is, \( D_z f(0, 0) = 0 \). If \( f \) additionally satisfies \( f(t, 2t^2) = t \) — a ridge of height \( t \) running along the curve \( t \mapsto (t, 2t^2) \) in the middle of the aforementioned region — then the derivative \( (3.2) \) of \( f \) at the origin would not exist.

Rigorously, let \( \alpha: \mathbb{R} \rightarrow [0, 1] \subset \mathbb{R} \) be a smooth bump function that is zero outside the open interval \( (1, 3) \subset \mathbb{R} \) and which satisfies \( \alpha(2) = 1 \). Let \( f(x, y) = x \alpha(yx^{-2}) \) if
$x \neq 0$ and $f(x, y) = 0$ otherwise. Away from the $y$-axis, $f$ is smooth and a fortiori continuous. If $(x_n, y_n) \to (0, y)$ then $|f(x_n, y_n)| \leq |x_n| \to 0$, proving $f$ is everywhere continuous. If $g(t) = f(at, bt)$ then there exists an $\epsilon > 0$ such that $g(t) = 0$ for $|t| < \epsilon$, that is, all directional derivatives at the origin are zero. This means that if $Df(0,0)$ exists it must be zero, yet the sequence $(x_n, y_n) = (n^{-1}, 2n^{-2})$ for $n = 1, 2, \cdots$ is such that $|f(x_n, y_n)| \|(x_n, y_n)\|^{-1} \to 1 \neq 0$. (The Euclidean norm has been used.)

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