DUAL PROPERTIES AND JOINT SPECTRA
FOR SOLVABLE LIE ALGEBRAS OF OPERATORS

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ABSTRACT. Given $L$ a solvable Lie Algebra of operators acting on a Banach space $E$, we study the action of the opposite algebra of $L$, $L'$, on $E^*$. Moreover, we extend Slodkowski joint spectra, $\sigma_{\delta,k}$, $\sigma_{\pi,k}$, and we study its usual spectral properties.

1. Introduction

In [1] we defined a joint spectrum for a finite dimensional complex solvable Lie algebras of operators $L$ acting on a Banach space $E$ and we denoted it by $Sp(L, E)$. We also proved that $Sp(L, E)$ is a compact non void subset of $L^2_\perp = \{ f \in L : f(L^2) = 0 \}$. Besides, if $I$ is an ideal of $L$, the projection property holds. Furthermore, if $L$ is a commutative algebra, this spectrum reduces to the Taylor joint spectrum ([5]).

Let $a$ be an $n$-tuple of commuting operators acting on $E$, $a = (a_1, ..., a_n)$. Let $a^*$ be the adjoint $n$-tuple of $a$, i.e., $a^* = (a_1^*, ..., a_n^*)$, where $a_i^*$ is the adjoint operator of $a_i$. Then $a^*$ is an $n$-tuple of commuting operators acting on $E^*$, the dual space of $E$. If $\sigma(a)$ (respectively $\sigma(a^*)$) denotes the Taylor joint spectrum of $a$ (respectively $a^*$), it is well known that $\sigma(a) = \sigma(a^*)$. If we consider a solvable non commutative Lie algebra of operators $L$ contained in $\mathcal{L}(E)$, the space of bounded linear maps on $E$, its dual, $L^* = \{ x^* : x \in L \}$ defines a solvable Lie subalgebra of $\mathcal{L}(E^*)$ with the opposite bracket of $L$. One may ask if the joint spectra of $L$ and $L^*$ in the sense of [1] coincide. In the solvable non commutative case, in general, the answer is no.

We study this problem and prove that $Sp(L, E)$ and $Sp(L^*, E^*)$ are related: one is obtained from the other by a translation, i.e., $Sp(L, E) = Sp(L^*, E^*) + c$, where $c$ is a constant. Moreover, we characterize this constant in terms of the algebra and prove that in the nilpotent case $c = 0$.

In the second part of our work, we study $\sigma_{\delta,k}$ and $\sigma_{\pi,k}$, the Slodkowski spectra of [4]. We extend then to the case of solvable Lie algebras of operators and verify the usual spectral properties: they are compact, non void sets and the projection property for ideals still holds.

The paper is organized as follows. In section 2 we review several definitions and results of [1]. In section 3 we study the relation between $Sp(L, E)$ and $Sp(L^*, E^*)$, the dual property. In section 4 we extend Slodkowski spectrum and prove its spectral properties.
2. Preliminaries

We shall briefly recall several definitions and results related to the spectrum of a solvable Lie algebra of operators ([1]).

From now on, $L$ denotes a complex finite dimensional solvable Lie algebra. $E$ denotes a Banach space on which $L$ acts as right continuous operators, i.e., $L$ is a Lie subalgebra of $L(E)$.

Let $f$ be a character of $L$ and suppose that $n = \text{dim } L$. Let us consider the following complex, $(E \otimes \wedge L, d(f))$, where $\wedge L$ denotes the exterior algebra of $L$ and $d(f)$:

$$d_p(f) : E \otimes \wedge^p L \to E \otimes \wedge^{p-1} L,$$

$$d_p(f)e(x_1 \wedge \ldots \wedge x_p) = \sum_{k=1}^{k=p} (-1)^{k+1} e(x_k - f(x_k))\langle x_1 \wedge \ldots \wedge \hat{x}_1 \wedge \ldots \wedge x_p \rangle$$

$$+ \sum_{1 \leq k < l \leq p} (-1)^{k+l} e([x_k,x_l] \wedge x_1 \wedge \ldots \wedge \hat{x}_k \wedge \ldots \wedge \hat{x}_l \wedge \ldots \wedge x_p),$$

where $\hat{}$ means deletion. If $p \leq 0$ or $p \geq n + 1$, we also define $d_p(f) \equiv 0$.

Let $H_*(E \otimes \wedge L, d(f))$ denotes the homology of the complex $(E \otimes \wedge L, d(f))$.

**Definition 2.1.** Let $L$ and $E$ be as above. The set $\{f \in L^* : f(L^2) = 0$ and $H_*(E \otimes \wedge L, d(f)) \neq 0 \}$ is the spectrum of $L$ acting on $E$, and it is denoted by $Sp(L,E)$.

**Theorem 2.2.** If $L$ is a commutative Lie algebra, $Sp(L,E)$ reduces to the Taylor joint spectrum.

**Theorem 2.3.** $Sp(L,E)$ is a compact non void subset of $L^*$.

**Theorem 2.4.** (Projection property) Let $I$ be an ideal of $L$ and $\pi$ the projection map from $L^*$ onto $I^*$, then

$$Sp(I, E) = \pi(Sp(L,E)).$$

As in [1], we consider an $n-1$ dimensional ideal of $L$, $L_{n-1}$, and we decompose $E \otimes \wedge^p L$ in the following way

$$E \otimes \wedge^p L = (E \otimes \wedge^p L_{n-1}) \oplus (E \otimes \wedge^{p-1} L_{n-1}) \wedge \langle x_n \rangle,$$

where $x_n \in L$ and it is such that $L_{n-1} \oplus \langle x_n \rangle = L$.

If $\tilde{f}$ denotes the restriction of $f$ to $L_{n-1}$, we may consider the complex $(E \otimes \wedge^p L_{n-1}, d(\tilde{f}))$.

As $L_{n-1}$ is an ideal of codimension 1 in $L$, we may decompose the operator $d_p(f)$ as follows

$$d_p(f) : E \otimes \wedge^p L_{n-1} \to E \otimes \wedge^{p-1} L_{n-1},$$
\[ d_p(f) = \tilde{d}_p(\tilde{f}), \]  
\[ d_p(f): E \otimes \wedge^{p-1} L_{n-1} \wedge \langle x_n \rangle \rightarrow E \otimes \wedge^{p-1} L_{n-1} \oplus E \otimes \wedge^{p-2} L_{n-1} \wedge \langle x_n \rangle, \]

where \( \wedge \) means deletion and \( x_i (1 \leq i \leq p - 1) \) belongs to \( L_{n-1} \).

Now we consider the following morphism defined in [3,2].

Let \( \theta(x_n) = \) the derivation of \( \wedge L \) that extends the map \( ad(x_n) \), \( ad(x_n)(y) = [x_n, y] \ (y \in L) \),

\[ \theta(x_n)\langle x_1 \wedge \ldots \wedge x_p \rangle = \sum_{i=1}^{p} \langle x_1 \wedge \ldots \wedge ad(x_n)(x_i) \wedge \ldots \wedge x_p \rangle. \]

\( \theta(x_n) \) satisfies:

\[ \theta(x_n)(ab) = (\theta(x_n)a)b + a(\theta(x_n)b), \]

\[ \theta(x_n)w = w\theta(x_n), \]

where \( w: \wedge L \rightarrow \wedge L \) is the map

\[ w(\langle x_1 \wedge \ldots \wedge x_p \rangle) = (-1)^p \langle x_1 \wedge \ldots \wedge x_p \rangle. \]

Let \( u \) belong to \( \wedge L \) and \( \epsilon(u) \) be the following endomorphism defined of \( \wedge L \):

\[ \epsilon(u)v = u \wedge v \ (v \in \wedge L). \]

As \( (\wedge L)^* \) may be identified with \( \wedge L^* \), let \( \iota(u) \) be the dual map of \( \epsilon(u) \), \( \iota(u): \wedge L^* \rightarrow \wedge L^* \).

Besides, we consider \( \theta^*(x_n) \), the dual map of \( -\theta(x_n) \).

As \( \epsilon(u \wedge v) = \epsilon(u)\epsilon(v) \), \( \iota(w \wedge z) = \iota(w)\iota(z) \ (u, v \in \wedge L, w, z \in \wedge L^*) \).

As in [3,7] we define an isomorphism \( \rho \)

\[ \rho: \wedge L^* \rightarrow \wedge L, \]

\[ \rho(a) = \iota(a).w, \]

where \( a \in \wedge L^* \), \( w = \langle x_1 \wedge \ldots \wedge x_{n-1} \wedge x_n \rangle \) and \( \{x_1 \ldots x_n\} \) is a basis of \( L_{n-1} \). Note that \( \rho \) applies \( \wedge^p L^* \) isomorphically onto \( \wedge^{n-p} L \).
3. The Dual Property

Let $L$ and $E$ be as in section 2. Let $E^*$ be the space of continuous functionals on $E$. Let $L'$ be the solvable Lie algebra defined as follows: as vector space, $L = L'$, and the bracket of $L'$ is the opposite of the one of $L$; that is: $[x, y]' = -[y, x] = [y, x]$. $L'$ is a complex finite dimensional solvable Lie algebra and $L'^2 = L'^2$.

As $L$ acts as right continuous operators on $E$, the space $L^* = \{ x^*: x \in L \}$ has the Lie structure of the algebra $L'$ and acts as right continuous operators on $E^*$.

Observe that in the definition of $\epsilon$, $\iota$ and $\rho$, we only consider the structure of $L$ as vector space. As $L$ and $L'$ coincide as vector spaces, then $\wedge L = \wedge L'$ and we may consider

$$\rho: \wedge L^* \to \wedge L'.$$

If $L_{n-1}$ is an ideal of codimension 1 of $L$, $L_{n-1} = L_{n-1}$ is an ideal of codimension 1 of $L'$. Moreover, if $x_n \in L$ is such that $L_{n-1} \oplus \langle x_n \rangle = L$, then $L_{n-1} \oplus \langle x_n \rangle = L'$.

Let $\theta(x_n)$ be the derivation of $\wedge L'$ that extends the map $ad(x_n)$, $ad(x_n)(y) = [x_n, y]'$. By [2, Chapter V, Section 3], there exist a basis of $L$, $\{ x_i \}_{1 \leq i \leq n}$, such that

$$[x_j, x_i] = \sum_{h=1}^{i} c_{ij}^h x_h \ (i < j) \quad (10)$$

and $L_{n-1}$ has the basis $\{ x_i \}_{1 \leq i \leq n-1}$. If $w = \langle x_1 \wedge \cdots \wedge x_n \rangle$, then

$$\theta(x_n)w = \theta(x_n)\langle x_1 \wedge \cdots \wedge x_n \rangle$$

$$= \sum_{i=1}^{n} \langle x_1 \wedge \cdots \wedge x_{i-1} \wedge [x_n, x_i] \wedge \cdots \wedge x_n \rangle$$

$$= \sum_{i=1}^{n} c_{in}^i \langle x_1 \wedge \cdots \wedge x_n \rangle$$

$$= (\text{trace} \ \theta(x_n)(x_n))w.$$

As in [3,7], if $a \in \wedge L^*$, we have

$$\rho \theta^*(x_n) = \theta(x_n) \rho(a) - \iota(a) \theta(x_n)w. \quad (11)$$

Then

$$\rho \theta^*(x_n) = \rho \theta - (\text{trace} \ ad(x_n)) \rho. \quad (12)$$

As $L'$ has the opposite bracket of $L$,

$$\rho \theta^*(x_n) = - (\theta'(x_n) + \text{trace} \ ad(x_n)) \rho. \quad (13)$$

Let us consider the maps $1_{E^*} \otimes \theta(x_n)$, $1_{E^*} \otimes \rho$, $1_{E^*} \otimes \theta(x_n)$, $1_{E^*} \otimes \theta'(x_n)$ and let us still denote then by $\theta^*(x_n)$, $\rho$, $\theta(x_n)$, $\theta'(x_n)$, respectively. We observe that formula (11), (12), (13) remain true.

Let us decompose, as in section 2, $E^* \otimes \wedge^p L^* \ (E^* \otimes \wedge^{n-p} L')$, respectively) as the sum

$$E^* \otimes \wedge^p L_{n-1}^* \oplus E^* \otimes \wedge^{p-1} L_{n-1}^* \wedge \langle x_n \rangle,$$

$$E^* \otimes \wedge^{n-p} L_{n-1}^* \oplus E^* \otimes \wedge^{n-p-1} L_{n-1}^* \wedge \langle x_n \rangle,$$

respectively,
where $L_{n-1}^*$ is the subspace of $L^*$ generated by $\{y_j\}_{1 \leq j \leq n-1}$ ($\{y_j\}_{1 \leq j \leq n}$ is the dual basis of $\{x_j\}_{1 \leq j \leq n}$).

A standard calculation shows the following facts:

$$\rho(E^* \otimes \wedge^p L_{n-1}^*) = (E^* \otimes \wedge^{n-p-1} L_{n-1}) \wedge \langle x_n \rangle, \quad (14)$$
$$\rho(E^* \otimes \wedge^{p-1} L_{n-1} \wedge \langle y_n \rangle) = E^* \otimes \wedge^{n-p} L_{n-1}'$$
$$\rho|E^* \otimes \wedge^p L_{n-1}^* = \rho_{n-1} \wedge \langle x_n \rangle, \quad (16)$$
$$\rho|E^* \otimes \wedge^{p-1} L_{n-1}^* \wedge \langle y_n \rangle = \rho_{n-1}(-1)^{n-p} \rho, \quad (17)$$

where $\rho_{n-1}$ is the isomorphism associated to the algebra $L_{n-1}$.

**Proposition 3.1.** Let $L$, $L'$, $E$, $L_{n-1}$ and $\rho$ be as above and let $f$ belong to $L_2$ and $g$ to $L_2'$ such that $f(x_n) + \text{trace } \theta(x_n) = g(x_n)$. Then, the following diagram commutes:

$$\begin{array}{ccc}
E^* \otimes \wedge^p L_{n-1}^* & \xrightarrow{L_{p+1}'} & E^* \otimes \wedge^p L_{n-1}^* \wedge \langle y_n \rangle \\
\rho \downarrow & & \downarrow \rho \\
E^* \otimes \wedge^{n-p-1} L_{n-1}^* \wedge \langle x_n \rangle & \xrightarrow{L_{n-p}'} & E^* \otimes \wedge^{n-p-1} L_{n-1}^* \wedge \langle x_n \rangle,
\end{array}$$

where $L_{n-p}'$ is the operator involved in the definition of $d_{n-p}(g)$ and $L_{n-1}'$ is the adjoint operator of $L_{p+1}$, which is involved in the definition of $d_{p+1}(f)$ (see (3), (4)).

**Proof.** By (4), (5)

$$L_{p+1} = -(x_n - f(x_n)) + \theta(x_n).$$

Then,

$$L_{p+1}' = -(x_n^* - f(x_n)) - \theta^*(x_n).$$

As the bracket of $L'$ is the opposite of the one of $L$, by (4) and (5),

$$L_{n-p}' = -(x_n^* - g(x_n)) + \theta(x_n).$$

Then

$$L_{n-p}' \rho = -(x_n^* - g(x_n)) \rho + \theta'(x_n) \rho.$$

On the other hand, by (13)

$$\rho L_{p+1}' = -(x_n^* - f(x_n)) \rho - \rho \theta^*(x_n)$$

$$= -(x_n^* - f(x_n)) \rho + \theta'(x_n) \rho + \text{trace } f(x_n) \rho$$

$$= -(x_n^* - (f(x_n) + \text{trace}(x_n))) \rho + \theta' \rho$$

$$= -(x_n^* - g(x_n)) \rho + \theta'(x_n) \rho$$

$$= L_{n-p}' \rho.$$  

\qed
Theorem 3.2. Let \( L, L', E, \) and \( \rho \) be as above. Let \( \{x_i\}_{1 \leq i \leq n} \) be the basis of \( L \) defined in (10). Let \( f \) belong to \( L^2 \) and \( g \) to \( L'^2 \) such that
\[
g = f + (\text{trace } \tilde{\theta}(x_1), \ldots, \text{trace } \tilde{\theta}(x_n)),
\]
where \( \tilde{\theta}(x_i) \) is the restriction of \( \theta(x_i) \) to \( L_i \), the ideal generated by \( \{x_l\}_{1 \leq l \leq i} \) (\( 1 \leq i \leq n \)). Then, if we consider the adjoint complex of \((E \otimes \land L, d(f))\) and the complex \((E \otimes \land L', d(g))\), for each \( p \),
\[
d'_{n-p}(g) \rho w = \rho d^*_p(f).
\]
That is, the following diagram commutes:
\[
\begin{array}{ccc}
E^* \otimes \land^p L & \xrightarrow{d^*_{p+1}} & E^* \otimes \land^{p+1} L \\
\rho w & \downarrow & \downarrow \rho \\
E^* \otimes \land^{n-p} L' & \xrightarrow{d'_{n-p}} & E^* \otimes \land^{n-p-1} L',
\end{array}
\]
where \( w \) is the map of (8).

Proof. By means of an induction argument the proof may be derived from Proposition 3.1. \( \square \)

Theorem 3.3. Let \( L, L' \) and \( E \) be as above. If we consider the basis of \( L \) \( \{x_i\}_{1 \leq i \leq n} \) defined in (10), then in terms of the dual basis in \( L^* (= L') \) we have
\[
Sp(L, E) + (\text{trace } \tilde{\theta}(x_1), \ldots, \text{trace } \tilde{\theta}(x_n)) = Sp(L', E^*),
\]
where \( \tilde{\theta}(x_i) \) is as in Proposition 3.1.

Proof. Is a consequence of Theorem 3.2 and [4, Lemma 2.1]. \( \square \)

Theorem 3.4. If \( L \) is a nilpotent Lie Algebra, then
\[
Sp(L, E) = Sp(L', E^*).
\]

Proof. By [2, Chapter V, Section 1], the basis \( \{x_i\}_{1 \leq i \leq n} \) of (10) may be chosen such that
\[
[x_j, x_i] = \sum_{h=1}^{i-1} c^h_{ji} x_h \quad (j > i).
\]
Then \( \tilde{\theta}(x_i) = 0 \). \( \square \)

Remark 3.5. Let \( L \) be a solvable Lie Algebra. Let \( n = \text{dim}(L) \) and \( k = \text{dim}(L^2) \). By [2, Chapter V, Section 3], the basis of (10) may be chosen such that \( \{x_j\}_{1 \leq j \leq k} \) generates \( L^2 \). As \( L^2 \) is a nilpotent ideal of \( L \), \( \text{trace } \theta(x_i) = 0 \), \( 1 \leq i \leq k \). Then, we have the following proposition.

Proposition 3.6. Let \( L, L', \) and \( E \) be as usual. Let us consider a basis of \( L \) as in the previous remark. Then
\[
Sp(L, E) + (0, \ldots, 0, \text{trace } \tilde{\theta}(x_{k+1}), \ldots, \text{trace } \tilde{\theta}(x_n)) = Sp(L', E^*)
\]
Then, as we have seen, if \( L \) is a nilpotent Lie algebra, the dual property is essentially the one of the commutative case. However, if \( L \) is a solvable non nilpotent Lie algebra, this property fails. For example, if \( L \) is the algebra
\[
L = \langle x_1 \rangle \oplus \langle x_2 \rangle, \quad [x_2, x_1] = x_1,
\]
then,
\[
\text{trace } \theta(x_2) = 1, \quad \text{trace } \theta(x_1) = 0
\]
and
\[
Sp(L, E) + (0, 1) = Sp(L', E^*)
\]

4. The extension of Slodkowski joint spectra

Let \( L \) and \( E \) be as in section 2. We give an homological version of Slodkowski spectra instead of the cohomological one of \([4]\).

Let \( \Sigma_p(L, E) \) be the set \( \{ f \in L^{2^\pm} : H_p(E \otimes \Lambda L, d(f)) \neq 0 \} = \{ f \in L^{2^\pm} : R(d_{p+1}(f)) \neq \text{Ker}(d_p(f)) \} \).

**Definition 4.1.** Let \( L \) and \( E \) be as in section 2.
\[
\sigma_{\delta, k}(L, E) = \bigcup_{0 \leq p \leq n} \Sigma_p(L, E),
\]
\[
\sigma_{\pi, k} = \bigcup_{k \leq p \leq n} \Sigma_p(L, E) \bigcup \{ f \in L^{2^\pm} : R(d_k(f)) \text{ is closed} \},
\]
where \( 0 \leq k \leq n \).

We shall see that \( \sigma_{\delta, k}(L, E) \) and \( \sigma_{\pi, k}(L, E) \) are compact non void subsets of \( L^* \) and that they verify the projection property for ideals.

Observe that \( \sigma_{\delta, n}(L, E) = \sigma_{\pi, 0}(L, E) = Sp(L, E) \).

Let us show that \( \sigma_{\delta, k}(L, E) \) has the usual properties of a spectrum.

**Theorem 4.2.** Let \( L \) and \( E \) be as usual. Then \( \sigma_{\delta, k}(L, E) \) is a compact subset of \( L^* \).

**Proof.** As \( \sigma_{\delta, k}(L, E) \) is contained in \( Sp(L, E) \), it is enough to prove that \( \sigma_{\delta, k}(L, E) \) is closed in \( L^* \). Let us consider the complex \( (E \otimes \Lambda^p L, d_k(f)) \) \((0 \leq k \leq p + 1)\).

\[
E \otimes \Lambda^{p+1} L \xrightarrow{d_{p+1}(f)} E \otimes \Lambda^p L \to \cdots \to E \otimes L \xrightarrow{d_1(f)} E \to 0.
\]

This complex is a parameterized chain complex of Banach spaces on \( L^{2^\pm} \) in the sense of \([5, \text{Definition 2.1}]\). By \([5, \text{Theorem 2.1}]\), \( \{ f \in L^{2^\pm} : (E \otimes \Lambda^k L, d_k(f)) \) \((0 \leq k \leq p + 1) \) is not exact \} is a closed set of \( L^{2^\pm} \), and then in \( L^* \). However, this set is exactly \( \sigma_{\delta, k}(L, E) \).

Let \( L_{n-1} \) be an ideal of codimension 1 of \( L \) and let \( x_n \) in \( L \) be such that \( L_{n-1} \oplus \langle x_n \rangle = L \). As in \([5]\) and \([1]\), we consider a short exact sequence of complex. Let \( f \) belong to \( L^* \) and denote \( \tilde{f} \) its restriction to \( L_{n-1} \). Then
\[
0 \to (E \otimes \Lambda L_{n-1}, \tilde{d}(\tilde{f})) \xrightarrow{i} (E \otimes \Lambda L, d(f)) \xrightarrow{p} (E \otimes \Lambda L_{n-1}, \tilde{d}(\tilde{f})) \to 0,
\]
where \( p \) is the following map. As in section 2 we decompose \( E \otimes \Lambda^k L \),
\[
E \otimes \Lambda^k L = E \otimes \Lambda^k L_{n-1} \oplus E \otimes \Lambda^{k-1} L_{n-1} \wedge \langle x_n \rangle,
\]
\[ p(E \otimes \wedge^k L_{n-1}) = 0, \]
\[ p(e(x_1 \wedge \cdots \wedge x_{k-1} \wedge x_n)) = (-1)^{k-1} e(x_1 \wedge \cdots \wedge x_{k-1}), \]
where \( x_i \in L_{n-1}, 1 \leq i \leq k - 1. \)

**Remark 4.3.** (i) \( d(f)|E \otimes \wedge L_{n-1} = \tilde{d}(\tilde{f}). \)
(ii) As in [1], \( H_p(E \otimes \wedge L, d(f)) = \text{Tor}_p^{(U)}(E, C(f)). \)
(iii) As in [5] and [1], we have a long exact sequence of \( U(L) \) modules, where \( U(L) \) is the universal algebra of \( L \)
\[ \rightarrow H_p(E \otimes \wedge L_{n-1}, \tilde{d}(\tilde{f})) \xrightarrow{i} H_p(E \otimes \wedge L, d(f)) \xrightarrow{p} \]
\[ \rightarrow H_{p-1}(E \otimes \wedge L_{n-1}, \tilde{d}(\tilde{f})) \xrightarrow{\delta_{sp-1}} H_{p-1}(E \otimes \wedge L_{n-1}, \tilde{d}(\tilde{f})) \rightarrow, \]
where \( \delta, p \) is the connecting operator.
(iv) As in [1], we observe that if we regard the \( U(L) \) module \( H_p(E \otimes \wedge L_{n-1}, \tilde{d}(\tilde{f})) \) as \( U(L_{n-1}) \) module, then we obtain \( \text{Tor}_p^{(U)}(E, C(\tilde{f})). \) Then, as \( U(L_{n-1}) \) is a subalgebra with unit of \( U(L), \)
\[ \tilde{f} \in \bigoplus_p (L_{n-1}, E) \text{ if and only if } H_p(E \otimes \wedge L_{n-1}, \tilde{d}(\tilde{f})) \neq 0 \text{ as } U(L) \text{ module.} \]

**Proposition 4.4.** Let \( L, L_{n-1}, E, f, \) and \( \tilde{f} \) be as above. Then, if \( f \) belongs to \( \Sigma_p(L, E), \) \( \tilde{f} \) belongs to \( \bigcup \Sigma_p(L_{n-1}, E). \)

**Proof.** By Remark 4.3 (iv), \( \tilde{f} \not\in \Sigma_p(L_{n-1}, E) \bigcup \Sigma_p(L_{n-1}, E) \) if and only if \( H_i(E \otimes \wedge L_{n-1}, \tilde{d}(\tilde{f})) = 0 \) as \( U(L) \) module \( (i = p, p - 1) \). By Remark 4.3 (iii), \( H_p(E \otimes \wedge L, d(f)) = 0, \) i.e., \( f \not\in \Sigma_p(L, E). \)

**Proposition 4.5.** Let \( L, L_{n-1} \) and \( E \) be as usual. Let \( \Pi : L^* \rightarrow L_{n-1}^* \) be the projection map. Then
\[ \Pi(\sigma_{\delta, k}(L, E)) \subseteq \sigma_{\delta, k}(L_{n-1}, E). \]

**Proof.** Is a consequence of Proposition 4.4. \( \Box \)

**Proposition 4.6.** Let \( L, L_{n-1} \) and \( E \) be as usual and consider \( \Pi \) as in Proposition 4.5. Then, \( \sigma_{\delta, k}(L_{n-1}, E) = \Pi(\sigma_{\delta, k}(L, E)). \)

**Proof.** By Proposition 4.5 it is enough to show that
\[ \sigma_{\delta, k}(L_{n-1}, E) \subseteq \Pi(\sigma_{\delta, k}(L, E)). \]

By refining an argument of [1] and [5], we shall see that if \( \tilde{f} \) belongs to \( \Sigma_p(L_{n-1}, E), \) then there is an extension of \( \tilde{f} \) to \( L^*, f, \) such that \( f \in \Sigma_p(L, E). \)

First of all, as \( \tilde{f} \in \Sigma_p(L_{n-1}, E) \subseteq Sp(L, E), \) by [1, Theorem 3], if \( g \) is an extension of \( \tilde{f} \) to \( L^*, g(L^2) = 0, \) i.e., \( g \in L^{2+}. \)

Let us suppose that our claim is false, equivalently, \( H_p(E \otimes \wedge L, d(f)) = 0, \) \( \forall f \in L^{2+}, \Pi(f) = \tilde{f}. \)

Let us consider the connecting map associated to the long exact sequence of Remark 4.3 (iii),
\[ \delta_{sp} : H_p(E \otimes \wedge L_{n-1}, \tilde{d}(\tilde{f})) \rightarrow H_p(E \otimes \wedge L_{n-1}, \tilde{d}(\tilde{f})). \]
As $H_p(E \otimes \Lambda L, d(f)) = 0$, $\forall f \in L^2$, $\Pi(f) = \tilde{f}$, $\delta_{sp}(f)$ is a surjective map.

Let $k \in E \otimes \Lambda^{p+1}L_{n-1}$ be such that $d_p(\tilde{f})(k) = 0$. Then, it is well known that if $m \in E \otimes \Lambda^{p+1}L$ is such that $p(m) = k$, $\delta_{sp}([k]) = [d_{p+1}(f)(m)]$, where $p$ is the map defined in (18). Let us consider $m = k \wedge x_n \in E \otimes \Lambda^p L_{n-1} \langle x_n \rangle$. Then, $p((-1)^p m) = k$ and

$$\delta_{sp}(f)([k]) = (-1)^p[d_{p+1}(f)(m)].$$

Since,

$$d_{p+1}(f)(m) = d_{p+1}(f)(k \wedge \langle x_n \rangle)$$

$$= (d\tilde{f})(k) \wedge \langle x_n \rangle + (-1)^{p+1}L_{p+1}(k)$$

$$= (-1)^{p+1}L_{p+1}(k) \in (E \otimes \Lambda^p L_{n-1}),$$

$$\delta_{sp}(f)[k] = -[L_{p+1}(k)].$$

Moreover, by Equations (4) and (5), $L_{p+1}(k) = -k(x_n - f(x_n) + \theta(x_n)(k)$. Then

$$0 = d_p(f)(d_{p+1}(f)(m))$$

$$= d_p(f)(k(x_n - f(x_n)) + d_p(f)\theta(x_n)(k)$$

$$= (-d_p(f)(k))(x_n - f(x_n)) + d_p(f)\theta(x_n)(k)$$

$$= d_p(f)\theta(x_n)(k),$$

which implies that,

$$\delta_{sp}(f) = [k](x_n - f(x_n)) - [\theta(x_n)k].$$

Let us consider the complex of Banach spaces and maps

$$E \otimes \Lambda^{p+1}L \xrightarrow{d_{p+1}(f)} E \otimes \Lambda^p L \xrightarrow{d_p(f)} E \otimes \Lambda^{p-1}L. \quad (19)$$

Then, this is an analytically parameterized complex of Banach spaces on $\mathbb{C}$, which is exact $\forall f \in L^2$, $\Pi(f) = \tilde{f}$, and exact at $\infty$ ([5, Section 2]).

As $\delta_{sp}$ differs by a constant term of the connecting map of [5, Lemma 1.3], the argument of [5, Lemma 3.1] still applies to the complex (19). Then $H_p(E \otimes \Lambda^p L_{n-1}, d\tilde{f})) = 0$ as $U(L)$ module. By Remark 4.3 (iv) we finish the proof. □

**Theorem 4.7.** Let $L$ and $E$ be as usual. Let $I$ be an ideal of $L$. Then

$$\sigma_{\delta k}(I, E) = \Pi(\sigma_{\delta k}(L, E)),$$

where $\Pi$ denotes the projection map.

**Proof.** By [2, Chapter V, Section 3], Proposition 5 and an inductive argument, we conclude the proof of the theorem. □

**Theorem 4.8.** Let $L$ and $E$ be as usual. Then $\sigma_{\delta k}(L, E)$ is a non void set of $L^*$. **Proof.** It is a consequence of [2, Chapter V, Section 3], Theorem 5 and the one dimensional case. □
Theorem 4.9. Let $L$ and $E$ be as usual. Let $L'$, $\{x_i\}_{1 \leq i \leq n}$ and $\tilde{\vartheta}(x_i)$ ($1 \leq i \leq n$) be as in Theorem 3.2. Then, in terms of the dual basis of $\{x_i\}_{1 \leq i \leq n}$, $L'$, $\{x_i\}_{1 \leq i \leq n}$ and $\tilde{\vartheta}(x_i)$ ($1 \leq i \leq n$).

(i) $\sigma_{\delta k}(L, E) + \text{trace} \tilde{\vartheta}(x_1, \ldots, \text{trace} \tilde{\vartheta}(x_n)) = \sigma_{\pi k}(L', E^*)$.

(ii) $\sigma_{\pi k}(L, E) = \sigma_{\delta n-k}(L', E) - \text{trace} \tilde{\vartheta}(x_1, \ldots, \text{trace} \tilde{\vartheta}(x_n))$.

Proof. It is a consequence of [4, Lemma 2.1] and Theorem 3.2. □

Theorem 4.10. Let $L$ and $E$ be as usual. Then $\sigma_{\pi k}(L, E)$ is a compact subset of $L^*$ and if $I$ is an ideal of $L$ and $\Pi$ is the projection map from $L^*$ onto $I$, then $\sigma_{\pi k}(L, E) = \Pi(\sigma_{\pi k}(L, E))$.

Proof. It is a consequence of Theorems 4.2, 4.7, 4.8 and 4.9. □

Remark 4.11. In [1] it was proved that the projection property for subspaces that are not ideals fails for $Sp(L, E)$. As $\sigma_{\pi 0}(L, E) = \sigma_{\delta n}(L, E) = Sp(L, E)$, the same result remains true, in general, for $\sigma_{\delta k}(L, E)$ and $\sigma_{\pi k}(L, E)$.

Theorem 4.12. Let $L$ be a nilpotent Lie algebra and let $E$, $L'$, $\{x_i\}_{1 \leq i \leq n}$ and $\tilde{\vartheta}(x_i)$ ($1 \leq i \leq n$) be as in Theorem 3.2. Then,

(i) $\sigma_{\delta k}(L, E) = \sigma_{\pi k}(L', E^*)$.

(ii) $\sigma_{\pi k}(L, E) = \sigma_{\delta n-k}(L', E^*)$.

Proof. It is a consequence of Theorem 4.9 and the proof of Theorem 3.4. □

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