WELL-POSEDNESS AND PROPERTIES OF THE FLOW FOR SEMILINEAR BOUNDARY CONTROL SYSTEMS

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ABSTRACT. We derive conditions for well-posedness of semilinear evolution equations with unbounded input operators and the corresponding boundary control systems. Based on this, we provide sufficient conditions for Lipschitz continuity of the flow map, bounded-implies-continuation property, boundedness of reachability sets, etc. These properties represent a basic toolbox for stability and robustness analysis of semilinear boundary control systems.

We cover systems governed by general $C_0$-semigroups, and analytic semigroups that may have both boundary and distributed disturbances. We illustrate our findings on an example of a Burger’s equation with nonlinear local dynamics and both distributed and boundary disturbances.

1. INTRODUCTION

Semilinear evolution equations. In this work, we analyze the well-posedness and properties of the flow for semilinear evolution equations of the form

$$\dot{x}(t) = Ax(t) + B_2 f(x(t), u(t)) + B_1 u(t), \quad t > 0,$$

$$x(0) = x_0.$$  

Here $A$ generates a strongly continuous semigroup over a Banach space $X$, the operators $B$ and $B_2$ are admissible, and $f$ is a Lipschitz continuous in the first variable map. This class of systems is rather general:

- If $B$ and $B_2$ are bounded operators, (1) corresponds to the classic semilinear evolution equations covering broad classes of semilinear PDEs with distributed inputs. If $A$ is a bounded operator, such a theory was developed in [7]. In the case of unbounded generators $A$, we refer to [13], [15], [6, Chapter 11], [3], etc.

- If $B_2 = 0$, and $B$ is an admissible operator, then (1) reduces to the class of general linear control systems, that fully covers linear boundary control systems (see [2, 25], [58, 59], [11] for an overview). In particular, this class includes linear evolution PDEs with boundary inputs.

- Consider a linear system

$$\dot{x} = Ax + Bu,$$

with admissible $B$. Let us apply a feedback controller $u(x) = f(x, d_1) + d_2$ that is subject to additive actuator disturbance $d_2$ and further disturbance input $d_1$. Substituting this controller into (2), we arrive at systems (1), with $B_2 = B$. Therefore the systems (1) are sometimes called Lipschitz perturbations of linear systems.

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In [18], it was shown that the class of systems (1) includes 2D Navier–Stokes equations (under certain boundary conditions) with in-domain inputs and disturbances. Furthermore, in [18] the authors have designed an error feedback controller that guarantees approximate local velocity output tracking for a class of reference outputs. Viscous Burgers’ equation with nonlinear local terms and boundary inputs of Dirichlet or Neumann type falls into the class (1) as well.

In [53], it was shown that semilinear boundary control systems with linear boundary operators could be considered a special case of systems (1). In this case, it suffices to consider $B_2$ as the identity operator. Furthermore, in [53], the well-posedness and input-to-state stability of a class of analytic boundary control systems with nonlinear dynamics and a linear boundary operator were analyzed with the methods of operator theory.

ISS for infinite-dimensional systems. Our main motivation to analyze the systems (1) stems from the robust stability theory. During the last decade, we have witnessed tremendous progress in robust stability analysis of nonlinear infinite-dimensional systems subject to unknown unstructured disturbances. Input-to-state stability (ISS) framework admits a significant place in this development, striving to become a unifying paradigm for robust control and observation of PDEs and their interconnections, including ODE-PDE and PDE-PDE cascades [32, 40, 53].

Powerful techniques proposed to analyze the ISS property include: criteria of ISS and ISS-like properties in terms of weaker stability concepts [42, 22, 50], constructions of ISS Lyapunov functions for PDEs with in-domain and/or boundary controls [46, 57, 62, 10], efficient functional-analytic methods for the study of linear systems with unbounded input operators (including linear boundary control systems) [61, 22, 26, 24, 31, 35, 32], non-coercive ISS Lyapunov functions [42, 21], as well as small-gain stability analysis of finite [8, 28, 30, 37], and infinite networks, [9, 33, 39, 34], etc.

To make this powerful machinery work for any given system, one needs to verify its well-posedness, properties of reachability sets, and regularity of the flow induced by this system. In this paper, we develop sufficient conditions that help to derive these crucial properties.

State of the art. The systems (1) have been studied (up to the assumptions on $f$, and the choice of the space of admissible inputs) in [43] under the assumption that its linearization is an exponentially stable regular linear system in the sense of [58, 59, 55]. [43] ensures local well-posedness of regular nonlinear systems assuming the Lipschitz continuity of nonlinearity, and invoking regularity of the linearization. On this basis, the authors show in [43] that an error feedback controller designed for robust output regulation of a linearization of a regular nonlinear system achieves approximate local output regulation for the regular nonlinear systems.

Control of systems (1) has been studied recently in several papers. In particular, in [44], the exact controllability of a class of regular nonlinear systems was studied using back-and-forth iterations. A problem of robust observability was studied for a related class of systems in [27].

Stabilization of linear port-Hamiltonian systems by means of nonlinear boundary controllers was studied in [2, 48]. Bounded controls with saturations (a priori limitations of the input signal) have been employed for PDE control in [17, 56, 11]. Recently, several papers appeared that treat nonlinear boundary control systems within the input-to-state stability framework. Nonlinear boundary feedback was employed for the ISS stabilization of linear port-Hamiltonian systems in [52].
Several types of infinite-dimensional systems, distinct from (1), have been studied as well. One of such classes is time-variant infinite-dimensional semilinear systems that have been first studied (as far as the author is concerned) for systems without disturbances in [20]. Recently, in [51], sufficient conditions for well-posedness and uniform global stability have been obtained for scattering-passive semilinear systems (see [51, Theorem 3.8]).

Another important extension of (1) are semilinear systems with outputs. Such systems with globally Lipschitz nonlinearities have been analyzed in [59, Section 7], and it was shown that such systems are well-posed and forward complete provided that the Lipschitz constant is small enough. In [14] employing counterexamples, it was shown that semilinear systems with a nonlinear output boundary feedback might fail to be well-posed. Well-posedness of incrementally scattering-passive nonlinear systems with outputs has been analyzed in [54] by applying Crandall-Pazy theorem [5] on generation of nonlinear contraction semigroups to a Lax-Phillips nonlinear semigroup representing the system together with its inputs and outputs.

Contribution. Our first main result is the theorem guaranteeing (under proper conditions on \( f \) and input operators) the local existence and uniqueness of solutions for the system (1) with locally essentially bounded inputs \( u \) and \( d \).

There are several existence and uniqueness theorems in the literature. For example, [43, Proposition 3.2] covers semilinear systems with \( L^\infty \)-inputs; [59, Theorem 7.6], [17, Lemma 2.8] treat the case of bilinear systems of various type, and [53] considers the case of systems with linearly bounded nonlinearities.

In contrast to the usual formulations of such results (including a closely related result [43, Proposition 3.2]), we also provide a uniform existence time for solutions that controls the maximal deviation of the trajectory from the given set of initial conditions.

Next, we show that under natural conditions, the system (1) is a well-posed control system in the sense of [40]. Finally, we study the fundamental properties of the flow map, such as Lipschitz continuity with respect to initial states, boundedness of reachability sets, boundedness-implies-continuation property, etc.

These properties are important in their own right. But also, they are key components for the analysis of robust stability of systems (1) as we explained before.

The structure of semilinear boundary control systems allows combining the “linear” methods of admissibility theory with “nonlinear” methods, such as fixed point theorems and Lyapunov methods. We consider the case of general \( C_0 \)-semigroups and the special case of analytic semigroups, for which one can achieve stronger results. We consider semilinear parabolic systems with Dirichlet and Neumann boundary inputs as an example of our developments.

Outlook. Having developed conditions ensuring the well-posedness and “nice” properties of the flow map of systems (1), we can invoke for ISS analysis of (1) such powerful tools as coercive and non-coercive ISS Lyapunov functions [21], ISS superposition theorems [42], small-gain theorems for general systems [39], etc.

Notation. By \( \mathbb{N}, \mathbb{R}, \mathbb{R}_+ \), we denote the sets of natural, real, and nonnegative real numbers, respectively. \( \overline{S} \) denotes the closure of a set \( S \) (in a given topology). By \( V \subset U \) we mean that \( V \) is compactly contained in \( U \), that is, \( \overline{V} \subset U \).

Let \( S \) be a normed linear space. The distance from \( z \in S \) to the set \( Z \subset S \) we denote by \( \text{dist} (z, Z) := \inf \{ \| y - z \|_S : y \in Z \} \). We denote an open ball of radius \( r \) around \( Z \subset S \) by \( B_r, S (Z) := \{ y \in X : \text{dist} (y, Z) < r \} \), and we set also...
For normed linear spaces $X, U$, we denote by $L(X, U)$ the space of bounded linear operators from $X$ to $U$. We endow $L(X, U)$ with the standard operator norm $\|A\| := \sup_{\|x\|_X = 1} \|Ax\|_U$. We write for short $L(X) := L(X, X)$. By $\sigma(A)$, we denote the spectrum of a closed operator $A : D(A) \subset X \to X$, and by $\rho(A)$ its resolvent. By $\omega_0(T)$, we denote a growth bound of a $C_0$-semigroup $T$. The domain of definition, kernel, and image of an operator $A$ we denote by $D(A)$, $\text{Ker}(A)$, and $\text{Im}(A)$ respectively.

Let $X$ be a Banach space, and let $I$ be a closed subset of $\mathbb{R}$. We define the following spaces of vector-valued functions

$\begin{align*}
M(I, X) & := \{ f : I \to X : f \text{ is strongly measurable} \}, \\
L^p(I, X) & := \left\{ f \in M(I, X) : \|f\|_{L^p(I, X)} := \left( \int_I \|f(s)\|_{X}^p \, ds \right)^{1/p} < \infty \right\}, \\
L^p_{\text{loc}}(\mathbb{R}_+, X) & := \left\{ f \in L^p([0, t], X) \quad \forall t > 0 \right\}, \\
L^\infty(I, X) & := \left\{ f \in M(I, X) : \|f\|_{L^\infty(I, X)} := \sup_{s \in I} \|f(s)\|_X < \infty \right\}, \\
L^\infty_{\text{loc}}(I, X) & := \left\{ f \in L^\infty([0, t], X) \quad \forall t > 0 \right\}.
\end{align*}$

We denote also $L^p(a, b) := L^p([a, b], \mathbb{R})$, where $p \in [1, +\infty]$. The space $H^k(a, b)$, $k \in \mathbb{N}$, is a Sobolev space of functions $u \in L^2(a, b)$, such that for each natural $j \leq k$, the weak derivative $u^{(j)}$ exists and belongs to $L^2(a, b)$.

$H^k(a, b)$ is endowed with the norm

$\|u\|_{H^k(a, b)} := \left( \sum_{j=k}^{\infty} \int_a^b |u^{(j)}(x)|^2 \, dx \right)^{1/2}.$

$H^k_0(a, b)$ denotes the closure of smooth functions with compact support in $(a, b)$ in the norm of $H^k(a, b)$, $k \in \mathbb{N}$.

2. General class of systems

We start with a general definition of a control system.

**Definition 2.1.** Consider the triple $\Sigma = (X, U, \phi)$ consisting of

(i) A normed vector space $(X, \| \cdot \|_X)$, called the state space, endowed with the norm $\| \cdot \|_X$.

(ii) A normed vector space of inputs $U \subset \{ u : \mathbb{R}_+ \to U \}$ endowed with a norm $\| \cdot \|_U$, where $U$ is a normed vector space of input values. We assume that the following two axioms hold:

The axiom of shift invariance: for all $u \in U$ and all $\tau \geq 0$ the time shift $u(\cdot + \tau)$ belongs to $U$ with $\|u\|_U \geq \|u(\cdot + \tau)\|_U$.

The axiom of concatenation: for all $u_1, u_2 \in U$ and for all $t > 0$ the concatenation of $u_1$ and $u_2$ at time $t$, defined by

$u_1 \bigcirc_t u_2(\tau) := \begin{cases} u_1(\tau), & \text{if } \tau \in [0, t], \\
u_2(\tau - t), & \text{otherwise}, \end{cases}$

belongs to $U$. 

The triple $\Sigma$ is called a (control) system, if the following properties hold:

\(\Sigma 1\) The identity property: for every \((x, u) \in X \times U\) it holds that \(\phi(0, x, u) = x\).

\(\Sigma 2\) Causality: for every \((t, x, u) \in D_{\phi}\), for every \(\tilde{u} \in U\), such that \(u(s) = \tilde{u}(s)\) for all \(s \in [0, t]\) it holds that \([0, t] \times \{(x, \tilde{u})\} \subset D_{\phi}\) and \(\phi(t, x, \tilde{u}) = \phi(t, x, u)\).

\(\Sigma 3\) Continuity: for each \((x, u) \in X \times U\) the map \(t \mapsto \phi(t, x, u)\) is continuous on its maximal domain of definition.

\(\Sigma 4\) The cocycle property: for all \(x \in X\), \(u \in U\), for all \(t, h \geq 0\) so that \([0, t + h]\) \(\cap D_{\phi}\), we have

\[\phi(h, \phi(t, x, u), u(t + h)) = \phi(t + h, x, u)\].

Definition 2.2 can be viewed as a direct generalization, and a unification of the concepts of strongly continuous nonlinear semigroups [5] [4] with abstract linear control systems [60].

This class of systems encompasses control systems generated by ordinary differential equations (ODEs), switched systems, time-delay systems, evolution partial differential equations (PDEs), abstract differential equations in Banach spaces and many others [29] Chapter 1).

Definition 2.3. We say that a control system (as introduced in Definition 2.1) is forward complete (FC), if \(D_{\phi} = \mathbb{R}_+ \times X \times U\), that is for every \((x, u) \in X \times U\) and for all \(t \geq 0\) the value \(\phi(t, x, u) \in X\) is well-defined.

Forward completeness alone does not imply, in general, the existence of any uniform bounds on the trajectories emanating from bounded balls that are subject to uniformly bounded inputs. Systems exhibiting such bounds deserve a special name.

Definition 2.4. We say that a system \(\Sigma = (X, U, \phi)\) has bounded reachability sets (BRS), if for any \(C > 0\) and any \(\tau > 0\) it holds that

\[\sup \{ \|\phi(t, x, u)\|_X : \|x\|_X \leq C, \|u\|_U \leq C, \ t \in [0, \tau] \} < \infty\].

For a wide class of control systems, the boundedness of a solution implies the possibility of prolonging it to a larger interval, see [29] Chapter 1]. Next, we formulate this property for abstract systems:

Definition 2.4. We say that a system \(\Sigma\) satisfies the boundedness-implies-continuation (BIC) property if for each \((x, u) \in X \times U\) such that the maximal existence time \(t_m(x, u)\) is finite, and for all \(M > 0\), there exists \(t \in [0, t_m(x, u)]\) with \(\|\phi(t, x, u)\|_X > M\).

3. Semilinear evolution equations with unbounded input operators

Consider infinite-dimensional evolution equations of the form

\[(4a) \quad \dot{x}(t) = Ax(t) + B_2 f(x(t), u(t)) + Bu(t), \quad t > 0,
(4b) \quad x(0) = x_0,\]

where \(A : D(A) \subset X \to X\) generates a strongly continuous semigroup \(T = (T(t))_{t \geq 0}\) of bounded linear operators on a Banach space \(X\), \(U\) is a normed vector
space of input values, \( x_0 \in X \) is a given initial condition, \( f : X \times U \to V \) is a map defined on the whole \( X \times U \), and \( B : U \to X \), \( B_2 : V \to X \) is are possibly unbounded operators, that belong however to \( L(U, X_{-1}) \) and \( L(V, X_{-1}) \) respectively, where \( V \) is a Banach space. Here the extrapolation space \( X_{-1} \) is the closure of \( X \) in the norm \( x \mapsto \| (aI - A)^{-1} x \|_X \), \( x \in X \), where \( a \in \rho(A) \) (different choices of \( a \in \rho(A) \) induce equivalent norms on \( X \)).

3.1. Admissible input operators and mild solutions. First, consider the linear counterpart of the system \( 4 \).

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t), \quad t > 0, \\
x(0) &= x_0,
\end{align*}
\]

for the same \( A, B \) as above. As \( B \) is an unbounded operator and thus may not be defined on the whole space \( U \), one has to be careful when defining the concept of a solution for \( 5 \). Since \( B \in L(U, X_{-1}) \), it is natural to consider the system \( 5a \) on the space \( X_{-1} \). Note that the semigroup \( (T(t))_{t \geq 0} \) extends uniquely to a strongly continuous semigroup \( (T_{-1}(t))_{t \geq 0} \) on \( X_{-1} \) whose generator \( A_{-1} : X_{-1} \to X_{-1} \) is an extension of \( A \) with \( D(A_{-1}) = X \), see, e.g., \[12\] Section II.5. Recall the definitions of the spaces \( L^p, L^1_{\text{loc}} \) from Section \[ 1 \].

The mild solution of \( 5 \) for any \( x \in X \) and \( u \in L^1_{\text{loc}}(\mathbb{R}, X) \) is given by

\[
\phi_L(t,x,u) = T(t)x + \int_0^t T_{-1}(t-s)Bu(s)ds.
\]

The integral term here, however, belongs in general to \( X_{-1} \).

Thus, the existence and uniqueness of a mild solution depend on whether \( \int_0^t T_{-1}(t-s)Bu(s)ds \in X \). This leads to the following concept:

**Definition 3.1.** The operator \( B \in L(U, X_{-1}) \) is called \( q \)-admissible control operator for \( (T(t))_{t \geq 0} \), where \( 1 \leq q \leq \infty \), if there is \( t > 0 \) so that

\[
(6) \quad u \in L^q_{\text{loc}}(\mathbb{R}, U) \quad \Rightarrow \quad \int_0^t T_{-1}(t-s)Bu(s)ds \in X.
\]

Define for each \( t \geq 0 \) an operator \( \Phi(t) : L^q_{\text{loc}}(\mathbb{R}, U) \to X_{-1} \) by

\[
\Phi(t)u := \int_0^t T_{-1}(t-s)Bu(s)ds.
\]

Note that as \( B \in L(U, X_{-1}) \), the operators \( \Phi(t) \) are well-defined as maps from \( L^q_{\text{loc}}(\mathbb{R}, U) \) to \( X_{-1} \) for all \( t \). The next result (see \[ 20\] Proposition 4.2, \[ 35\] Proposition 4.2.2)) shows that \( q \)-admissibility of \( B \) ensures that the image of \( \Phi(t) \) is in \( X \) for all \( t \geq 0 \) and \( \Phi(t) \in L(L^q(\mathbb{R}, U), X) \) for all \( t > 0 \).

**Proposition 3.2.** Let \( X, U \) be Banach spaces and let \( q \in [1, \infty] \) be given. Then \( B \in L(U, X_{-1}) \) is \( q \)-admissible if and only if for all \( t > 0 \) there is \( h_t > 0 \) so that for all \( u \in L^q_{\text{loc}}(\mathbb{R}, U) \) it holds that \( \Phi(t)u \in X \) and

\[
(7) \quad \left\| \int_0^t T_{-1}(t-s)Bu(s)ds \right\|_X \leq h_t \|u\|_{L^q([0,t], U)}.
\]

The function \( t \mapsto h_t \) can be chosen to be nondecreasing in \( t \).

An important consequence of Proposition \[ 4.2\] is that well-posedness (and thus forward completeness) of the system \( 5 \) already implies boundedness of reachability sets property for \( 5a \), with a bound given by \( 7 \).

As \( t \mapsto h_t \) is nondecreasing in \( t \), there is a limit \( h_0 := \lim_{t \to 0+} h_t \geq 0 \), which is not necessarily zero. Operators for which \( h_0 = 0 \) deserve a special name.
Definition 3.3. Let $q \in [1, \infty]$. A $q$-admissible operator $B : U \rightarrow X$ is called zero-class $q$-admissible, if the constants $h_t$, $t > 0$ can be chosen such that $h_0 = 0$.

All $B \in L(U, X)$ are zero-class 1-admissible. If $X$ is reflexive, then 1-admissible operators are necessarily bounded. At the same time, there are unbounded zero-class admissible operators, see Proposition 5.5. Consider [23 Examples 3.8, 3.9] for unbounded admissible observation operators that are not zero-class admissible.

The above considerations motivate us to impose

Assumption 3.1. The operator $B \in L(U, X_{-1})$ is $\infty$-admissible, and the map $(t, u) \mapsto \Phi(t)u$ is continuous on $\mathbb{R}_+ \times L^0(\mathbb{R}_+, U)$.

In particular, this assumption holds if $B$ is a $q$-admissible operator with $q < \infty$, see [60 Proposition 2.3].

To define the concept of a mild solution, we also require the following:

Assumption 3.2. Let Assumption 3.1 hold. We assume that for all $u \in L^\infty(\mathbb{R}_+, U)$ and any $x \in C(\mathbb{R}_+, X)$ the map $s \mapsto f(x(s), u(s))$ is in $L^1_{loc}(\mathbb{R}_+, V)$, $B_2$ is zero-class $\infty$-admissible, and the map

$$t \mapsto \int_0^t T(t-s)B_2f(x(s), u(s))ds$$

is well-defined and continuous on $\mathbb{R}_+$.

Remark 3.4. Assumption 3.2 holds, in particular, if $B_2 \in L(V, X)$, and

(i) $f(x, u) = g(x) + Ru$, $x \in X$, $u \in U$, where $R \in L(U, V)$, and $g$ is continuous on $X$. Indeed, for a continuous $x$, the map $s \mapsto g(x(s))$ is continuous either, and thus Riemann integrable. The map $s \mapsto T(t-s)B_2Ru(s)$ is Bochner integrable for any $u \in L^1_{loc}(\mathbb{R}_+, U)$ by [11 Proposition 1.3.4], [25 Lemma 10.1.6]. This ensures that Assumption 3.2 holds.

(ii) If $f$ is continuous on $X \times U$, and $u$ is piecewise right-continuous, then the map $s \mapsto f(x(s), u(s))$ is also piecewise right-continuous, and thus it is Riemann integrable.

(iii) (ODE systems). Let $X = \mathbb{R}^n$, $U = \mathbb{R}^m$, $A = 0$ (and thus $T(t) = id$ for all $t$), $B_2 = id$, $B = 0$, and $f$ be continuous on $X \times U$. With these assumptions the equations (4) take form

$$\dot{x} = f(x, u).$$

Then for each $u \in L^\infty(\mathbb{R}_+, U)$ and each $x \in C(\mathbb{R}_+, X)$ the map $s \mapsto f(x(s), u(s))$ is Lebesgue integrable, and thus Assumption 3.2 holds.

Indeed, as $x$ is a solution of (4) on $[0, \tau)$, $x$ is continuous on $[0, \tau)$. By assumptions, $u$ is measurable on $[0, \tau)$, and $f$ is continuous on $\mathbb{R}^n \times \mathbb{R}^m$. Arguing similarly to [49 Proposition 7] (where it was shown that a composition of a continuous and measurable function defined on a measurable set $E$ is measurable on $E$), we see that the map $q : [0, \tau) \rightarrow \mathbb{R}^n$, $q(s) := f(x(s), u(s))$, is a measurable map. As $u$ is essentially bounded, and $x$ and $f$ map bounded sets into bounded sets, $q$ is essentially bounded on $[0, \tau)$. Thus, $q \in L^\infty(\mathbb{R}_+, \mathbb{R}^n)$, and thus $q$ is integrable on $[0, \tau)$.

Next we define mild solutions of (4).

Definition 3.5 (Mild solutions). Let Assumptions 3.1, 3.2 hold. Let also $\tau > 0$ be given. A function $x \in C([0, \tau], X)$ is called a mild solution of (4) on $[0, \tau]$...
corresponding to certain \( x_0 \in X \) and \( u \in L^\infty_\text{loc}(\mathbb{R}_+, U) \), if \( x \) solves the integral equation
\[
(9) \quad x(t) = T(t)x_0 + \int_0^t T(t-s)B_2f(x(s), u(s))ds + \int_0^t T_{-1}(t-s)Bu(s)ds.
\]
Here the integrals are Bochner integrals of \( X \)-valued maps, and \( T_{-1} \) is an extension of the semigroup \( T \) to the space \( X_{-1} \), see [12, p.126].

We say that \( x : \mathbb{R}_+ \to X \) is a mild solution of (4) on \( \mathbb{R}_+ \) corresponding to certain \( x_0 \in X \) and \( u \in L^\infty_\text{loc}(\mathbb{R}_+, U) \), if it is a mild solution of (4) (with \( x_0, u \)) on \([0, \tau]\) for all \( \tau > 0 \).

3.2. Local existence and uniqueness. Assumptions 3.1, 3.2 guarantee that the nonlinearities in (9) are well-defined. To ensure the existence and uniqueness of mild solutions, we impose further restrictions on \( f \).

**Definition 3.6.** We call \( f : X \times U \to V \)

(i) Lipschitz continuous (with respect to the first argument) on bounded subsets of \( X \) if for any \( C > 0 \) there is \( L(C) > 0 \), such that \( \forall x, y \in B_C \), \( \forall v \in B_{C,U} \) it holds that
\[
(10) \quad \| f(y, v) - f(x, v) \|_V \leq L(C)\|y - x\|_X.
\]

(ii) Lipschitz continuous (with respect to the first argument) on bounded subsets of \( X \), uniformly with respect to the other argument if for any \( C > 0 \) there is \( L(C) > 0 \), such that (10) holds for all \( x, y : \|x\|_X \leq C, \|y\|_X \leq C \), and all \( v \in U \).

(iii) uniformly globally Lipschitz continuous if (10) holds for all \( x, y \in X \), and all \( u \in U \) with \( L = \text{const} \) that does not depend on \( x, y, u \).

We omit the indication “with respect to the first argument” wherever this is clear from the context.

For the well-posedness analysis, we rely on the following assumption on the nonlinearity \( f \) in (4).

**Assumption 3.3.** The nonlinearity \( f \) satisfies the following properties:

(i) \( f : X \times U \to V \) is Lipschitz continuous on bounded subsets of \( X \).

(ii) \( f(x, \cdot) \) is continuous for all \( x \in X \).

(iii) There exist \( \sigma \in K_{\infty} \) and \( c > 0 \) so that for all \( u \in U \) the following holds:
\[
(11) \quad \|f(0, u)\|_V \leq \sigma(\|u\|_U) + c.
\]

Let \( S \) be a normed linear space. Define the distance from \( z \in S \) to the set \( Z \subset S \) by
\[
\text{dist} (z, Z) := \inf \{\|y - z\|_S : y \in Z\}.
\]
Further, denote an open ball of radius \( r \) around \( Z \subset S \) by
\[
B_{r,S}(Z) := \{y \in X : \text{dist} (y, Z) < r\}.
\]
We set also \( B_{r,S}(x) := B_{r,S}(\{x\}) \) for \( x \in X \), and \( B_r := B_{r,0}(0) \). If \( S = X \), we write for short \( B_r(Z) := B_{r,X}(Z), B_r(x) := B_{r,X}(x) \), etc.

Finally, for a set \( S \subset U \), denote the set of inputs with essential values in \( S \) as
\[
\mathcal{U}_S := \{u \in U : u(t) \in S, \text{ for a.e. } t \in \mathbb{R}_+\}.
\]

We start with the following sufficient condition for the existence and uniqueness of solutions of a system (4) with inputs in \( L^\infty(\mathbb{R}_+, U) \).
Theorem 3.7 (Picard-Lindelöf theorem). Let Assumptions 3.1, 3.2, 5.3 hold. Define \( h_0 \coloneqq \lim_{t \to +0} h_t \), where \( h_t \) are as in (7).

Assume that \((T(t))_{t \geq 0}\) is a strongly continuous semigroup, generated by \( A \) and satisfying for certain \( M \geq 1, \lambda > 0 \) the estimate

\begin{equation}
\|T(t)\| \leq Me^{Mt}, \quad t \geq 0.
\end{equation}

For any compact set \( Q \subset X \), any \( r > 0 \), any bounded set \( S \subset U \), and any \( \delta > 0 \), there is a time \( t_1 = t_1(Q, r, S, \delta) > 0 \), such that for any \( x_0 \in W := B_r(w) \) for some \( w \in Q \), and for any \( u \in U_S \) there is a unique mild solution of (11) on \([0, t_1]\), and it lies in the ball \( B_{M\delta + h_0\|u\|_{L^\infty([0, t_1], U)} + \delta}(w) \).

Proof. First, we show the claim for the case if \( Q \) is a single point in \( X \). Pick any \( C > 0 \) such that \( W := B_r(w) \subset B_C \), and \( U_S \subset B_{C, M\lambda} \). Pick any \( u \in U_S \). Take also any \( \delta > 0 \), and consider the following sets (depending on a parameter \( t > 0 \)):

\begin{equation}
\{ x \in C([0, t], X) : \sup_{s \in [0, t]} \|x(s) - w\|_X \leq Mr + h_0\|u\|_{L^\infty([0, t], U)} + \delta \},
\end{equation}

endowed with the metric \( \rho_t(x, y) := \sup_{s \in [0, t]} \|x(s) - y(s)\|_X \). As the sets \( Y_t \) are closed subsets of Banach spaces \( C([0, t], X) \), \( Y_t \) are complete metric spaces for all \( t > 0 \).

Pick any \( x_0 \in W \). We are going to prove that for small enough \( t \), the spaces \( Y_t \) are invariant under the operator \( \Phi_u \), defined for any \( x \in Y_t \) and all \( \tau \in [0, t] \) by

\[ \Phi_u(x)(\tau) = T(\tau)x_0 + \int_0^\tau T(\tau - s)B_2f(x(s), u(s))ds + \int_0^\tau T_{-1}(\tau - s)Bu(s)ds. \]

By Assumptions 3.1, 3.2 the function \( \Phi_u(x) \) is continuous.

Fix any \( t > 0 \) and pick any \( x \in Y_t \). As \( x_0 \in W = B_r(w) \), there is \( a \in B_r \) such that \( x_0 = w + a \).

Then for any \( \tau < t \), it holds that

\[
\|\Phi_u(x)(\tau) - w\|_X \\
\leq \|T(\tau)x_0 - w\|_X + \|\int_0^\tau T_{-1}(\tau - s)Bu(s)ds\|_X \\
+ \|\int_0^\tau T_{-1}(\tau - s)B_2f(x(s), u(s))ds\|_X \\
\leq \|T(\tau)(w + a) - w\|_X + h_\tau\|u\|_{L^\infty([0, \tau], U)} \\
+ c_\tau\|f(x(\cdot), u(\cdot))\|_{L^\infty([0, \tau], V)} \\
\leq \|T(\tau)w - w\|_X + \|T(\tau)a\|_X + h_\tau\|u\|_{L^\infty([0, \tau], U)} \\
+ c_\tau\|f(x(\cdot), u(\cdot)) - f(0, u(\cdot))\|_{L^\infty([0, \tau], V)} + c_\tau\|f(0, u(\cdot))\|_{L^\infty([0, \tau], V)}.\]

Now for all \( s \in [0, t] \)

\[
\|x(s)\|_X \leq \|w\|_X + Mr + h_0\|u\|_{L^\infty([0, t], U)} + \delta \\
\leq M(\|w\|_X + r) + h_0C + \delta \leq (M + h_0)C + \delta := K.
\]

In view of Assumption 5.3 (iii), it holds that

\[
\|f(0, u(s))\|_V \leq \sigma(\|u(s)\|_U) + c, \quad \text{for a.e. } s \in [0, t].
\]
As $M \geq 1$, it holds that $K > C$, and the Lipschitz continuity of $f$ on bounded balls ensures that there is $L(K) > 0$, such that for all $t \in [0, T]$

$$
\| \Phi_u(x)(\tau) - w \|_X \leq \| T(\tau)w - w \|_X + M e^{k \tau} r + h_t \| u \|_{L^\infty([0, t], U)} + c_t (L(K) \| x \|_{L^\infty([0, t], X)} + \sigma (\| u \|_{L^\infty([0, t], U)} + c))
$$

Since $T$ is a strongly continuous semigroup, $h_t \to h_0$ as $t \to +0$, and $c_t \to 0$ as $t \to +0$, from this estimate it is clear that there exists $t_1$, such that

$$
\| \Phi_u(x(t)) - w \|_X \leq M r + h_0 \| u \|_{L^\infty([0, t_1], U)} + \delta, \quad \text{for all } t \in [0, t_1].
$$

This means, that $Y_1$ is invariant with respect to $\Phi_u$ for all $t \in (0, t_1)$, and $t_1$ does not depend on the choice of $x_0 \in W$.

Now pick any $t > 0$, $\tau \in [0, t]$, and any $x, y \in Y_1$. Then it holds that

$$
\| \Phi_u(x)(\tau) - \Phi_u(y)(\tau) \|_X \leq \left\| \int_0^\tau T(\tau - s) B_2 (f(x(s), u(s)) - f(y(s), u(s))) ds \right\|_X
$$

$$
\leq c_t \| f(x(\cdot), u(\cdot)) - f(y(\cdot), u(\cdot)) \|_{L^\infty([0, \tau], V)}
$$

$$
\leq c_t K \rho_t(x, y)
$$

$$
\leq \frac{1}{2} \rho_t(x, y),
$$

for $t \leq t_2$, where $t_2 > 0$ is a small enough real number, that does not depend on the choice of $x_0 \in W$.

According to the Banach fixed point theorem, there exists a unique solution of $x(t) = \Phi_u(x(t))$ on $[0, \min\{t_1, t_2\}]$, which is a weak solution of (1).

**General compact $Q$.** Till now, we have shown that for any $w \in Q$, any $r > 0$, any bounded set $S \subset U$, and any $\delta > 0$, there is a time $t_1 = t_1(w, r, S, \delta) > 0$ (that we always take the maximal possible), such that for any $x_0 \in W := B_r(w)$, and for any $u \in U_S$ there is a unique solution of (4) on $[0, t_1]$, and it lies in the ball $B_{Mr + h_0 \| u \|_{L^\infty([0, t_1], U)} + \delta}(w)$.

It remains only to show that $t_1$ can be chosen uniformly in $w \in Q$, that is $\inf_{w \in Q} t_1(w, r, S, \delta) > 0$. Let this not be so, that is, $\inf_{w \in Q} t_1(w, r, S, \delta) = 0$. Then there is a sequence $(w_k) \subset Q$, such that the corresponding times $(t_1(w_k, r, S, \delta))_{k \in \mathbb{N}}$ monotonically decay to zero. As $Q$ is compact, there is a converging subsequence of $(w_k)$, converging to some $w^* \in Q$. However, $t_1(w^*, r, S, \delta) = 0$, which easily leads to a contradiction.

**Corollary 3.8** (Picard-Lindelöf theorem for zero-class admissible $B$ and quasi-contractive semigroups). Let Assumptions 1, 2, 3, 5 hold. Let also $B$ be zero-class admissible, and $T$ be a quasi-contractive strongly continuous semigroup, that is, there is $\lambda > 0$ such that

$$
(15) \quad \| T(t) \| \leq e^{\lambda t}, \quad t \geq 0.
$$

For any bounded ball $W = B_r(w) \subset X$, any bounded set $S \subset U$, and any $\delta > 0$, there is a time $t_1 = t_1(W, S, \delta) > 0$, such that for any $x_0 \in W$ and any $u \in U_S$ there is a unique solution of (4) on $[0, t_1]$, and it lies in the ball $B_{r + \delta}(w)$.

**Proof.** The claim follows directly from Theorem 3.7. \hfill $\Box$

**Remark 3.9.** Without an assumption of quasicontractivity, Corollary 3.8 does not hold. Consider the special case $f \equiv 0$ and $B \equiv 0$. Then the system (4) is linear,
and for a given $x_0 \in X$ the solution of (4) exists globally and equals $t \mapsto T(t)x_0$. 
Now take $w := 0$ and pick any $r > 0$ and $t_1 > 0$. Then
\[
\sup_{\tau \in [0,t_1]} \sup_{\|x\|_X \leq r} \|T(\tau)x\|_X = r \sup_{\tau \in [0,t_1]} \|T(\tau)\|.
\]
Since $T$ is merely strongly continuous, the map $t \mapsto \|T(t)\|$ does not have to be continuous at $t = 0$, and it may happen that $\lim_{t_1 \to 0} \sup_{\tau \in [0,t_1]} \|T(\tau)\| > 1$.
Hence, in general, it is not possible to prove that the solution starting at arbitrary $x_0 \in B_r(w)$, will stay in $B_{r+\delta}(w)$ during a sufficiently small and uniform in $x_0 \in B_r(w)$ time.

The following example shows that Theorem 3.7 does not hold in general if $W$ is a bounded set (and not only a bounded ball over a compact set), even for linear systems governed by contraction semigroups on a Hilbert space.

**Example 3.10.** Let $X = \ell_2$, and consider a diagonal semigroup, defined by $T(t)x := (e^{-kt} x_k)_k$, for all $x = (x_k)_k \in X$ and all $t \geq 0$. This semigroup is strongly continuous and contractive. Consider a bounded and closed set $W := \{x \in \ell_2 : \|x\|_X = 1\}$. Yet $\|T(t)e_k\|_X = e^{-kt}$, and thus for each $\delta > 0$ and for each time $t_1 > 0$, we can find $k \in \mathbb{N}$, such that $\|T(t_1)e_k\|_X < 1 - \delta$, which means that $T(t_1)e_k \notin B_\delta(W)$.

At the same time, a stronger Picard-Lindelöf-type theorem can be shown for uniformly continuous semigroups (this encompasses, in particular, the case of infinite ODE systems, also called “ensembles”), which fully extends the corresponding result for ODE systems, see [38, Chapter 1].

**Theorem 3.11 (Picard-Lindelöf theorem for uniformly continuous semigroups).**
Let Assumptions 3.1, 3.2, 3.3 hold, and let $B$ be zero-class $\infty$-admissible. Let further $T$ be a uniformly continuous semigroup (not necessarily quasincontractive). For any bounded set $W \subset X$, any bounded set $\mathcal{S} \subset \mathcal{U}$ and any $\delta > 0$, there is a time $\tau = \tau(W,\mathcal{S},\delta) > 0$, such that for any $x_0 \in W$, and $u \in \mathcal{U}_\mathcal{S}$ there is a unique solution of (4) on $[0,\tau]$, and it lies in $B_\delta(W)$.

**Proof.** Pick any $C > 0$ such that $W \subset B_C$, and $\mathcal{U}_\mathcal{S} \subset B_{C_H}$.
Take also any $\delta > 0$, and consider the following sets (depending on a parameter $t > 0$):
\[
Y_t := \{x \in C([0,t],X) : \text{dist}(x(t),W) \leq \delta \quad \forall t \in [0,t]\},
\]
endowed with the metric $\rho_t(x,y) := \sup_{s \in [0,t]} \|x(s) - y(s)\|_X$. As the sets $Y_t$ are closed subsets of Banach spaces $C([0,t],X)$, $Y_t$ are complete metric spaces for all $t > 0$.
Pick any $x_0 \in W$ and any $u \in \mathcal{U}_\mathcal{S}$. We are going to prove that for small enough $t$, the spaces $Y_t$ are invariant under the operator $\Phi_u$, defined for any $x \in Y_t$ and all $\tau \in [0,t]$ by
\[
\Phi_u(x)(\tau) = T(\tau)x_0 + \int_0^\tau T(\tau - s)B_2f(x(s),u(s))ds + \int_0^\tau T_{-1}(\tau - s)Bu(s)ds.
\]
By Assumptions 3.1, 3.2 the function $\Phi_u(x)$ is continuous.
Fix any \( t > 0 \) and pick any \( x \in Y_t \). Then for any \( \tau < t \), it holds that
\[
\text{dist} \left( \Phi_u(x)(\tau), W \right) \leq \| \Phi_u(x)(\tau) - x_0 \|_X
\]
\[
\leq \| T(\tau)x_0 - x_0 \|_X + \left\| \int_0^\tau T_{-1}(\tau - s)Bu(s)ds \right\|_X
\]
\[
\leq \| T(\tau) - I \|_X \| x_0 \|_X + h_\tau \| u \|_{L^\infty([0,\tau], U)}
\]
\[
+ c_\tau \| f(x(\cdot), u(\cdot)) \|_{L^\infty([0,\tau], V)}
\]
\[
\leq C\| T(\tau) - I \|_X + h_\tau \| u \|_{L^\infty([0,\tau], U)}
\]
\[
+ c_\tau \| f(x(\cdot), u(\cdot)) - f(0, u(\cdot)) \|_{L^\infty([0,\tau], V)} + c_\tau \| f(0, u(\cdot)) \|_{L^\infty([0,\tau], V)}.
\]
Now for all \( s \in [0, t] \)
\[
\| x(s) \|_X \leq C + \delta =: K.
\]
In view of Assumption 3.3(iii), it holds that
\[
\| f(0, u(s)) \|_V \leq \sigma(\| u(s) \|_U) + c, \quad \text{for a.e. } s \in [0, t].
\]
Now Lipschitz continuity of \( f \) on bounded balls ensures that there is \( L(K) > 0 \), such that for all \( \tau \in [0, t] \)
\[
\| \Phi_u(x)(\tau) - w \|_X \leq C\| T(\tau) - I \|_X \| x \|_{L^\infty([0,\tau], U)}
\]
\[
+ h_\tau \| u \|_{L^\infty([0,\tau], U)}
\]
\[
+ c_\tau \left( L(K)\| x \|_{L^\infty([0,\tau], U)} + \sigma(\| u \|_{L^\infty([0,\tau], U)}) + c \right)
\]
\[
\leq C\| T(\tau) - I \|_X \| x \|_{L^\infty([0,\tau], U)}
\]
\[
+ h_\tau \| u \|_{L^\infty([0,\tau], U)}
\]
\[
+ c_\tau \left( L(K)K + \sigma(C) + c \right).
\]
Since \( T \) is a uniformly continuous semigroup, \( h_\tau \to 0 \) as \( t \to +0 \), and \( c_\tau \to 0 \) as \( t \to +0 \), from this estimate it is clear that there exists \( t_1 > 0 \), depending solely on \( C \) and \( \delta \), such that
\[
\text{dist} \left( \Phi_u(x)(t), W \right) \leq \delta, \quad \text{for all } t \in [0, t_1].
\]
This means, that \( Y_t \) is invariant with respect to \( \Phi_u \) for all \( t \in (0, t_1] \), and \( t_1 \) does not depend on the choice of \( x_0 \in W \). The rest of the proof is analogous to the proof of Theorem 3.7. \( \square \)

**Remark 3.12.** In the above theorems, the existence time of solutions is uniform with respect to the initial values and inputs with norms bounded by \( C \). In general, the existence time of solutions cannot be chosen to be uniform if we do not restrict the norms of the inputs. For example, consider the system
\[
\dot{x}(t) = u(t)x^2(t).
\]
The maximal existence time of solutions with \( x(0) = 1 \) and \( u(t) \equiv k \) goes to zero as \( k \to \infty \).

### 3.3. Well-posedness.

Our next aim is to study the prolongations of solutions and their asymptotic properties. In this section, the input space is \( \mathcal{U} := L^\infty(\mathbb{R}_+, U) \).

**Definition 3.13.** Let \( x_1(\cdot), x_2(\cdot) \) be mild solutions of \( (1) \) defined on the intervals \([0, t_1)\) and \([0, t_2)\) respectively, \( t_1, t_2 > 0 \). We call \( x_2 \) an extension of \( x_1 \) if \( t_2 > t_1 \), and \( x_2(t) = x_1(t) \) for all \( t \in [0, t_1) \).

**Lemma 3.14.** Let Assumptions 3.3, 3.5, 3.8 hold. Take any \( x_0 \in X \) and \( u \in \mathcal{U} \). Any two solutions of \( (1) \) coincide in their common domain of existence.
Proof. Let $\phi_1, \phi_2$ be two mild solutions of (4) corresponding to $x_0 \in X$ and $u \in \mathcal{U}$, defined over $[0, t_1]$ and $[0, t_2]$ respectively. Assume that $t_1 \leq t_2$ (the other case is analogous).

By local existence and uniqueness theorem, there is some positive $t_3 \leq t_1$ (we take $t_3$ to be the maximal of such times), such that $\phi_1$ and $\phi_2$ coincide on $[0, t_3)$. If $t_3 = t_1$, then the claim is shown. If $t_3 < t_1$, then by continuity $\phi_1(t_3) = \phi_2(t_3)$. Now $\psi_1 : t \mapsto \phi_1(t_3 + t)$ and $\psi_2 : t \mapsto \phi_2(t_3 + t)$ are two mild solutions for the problem

$$\dot{x}(t) = Ax(t) + B_2 f(x(t), u(t + t_3)) + Bu(t_3 + \cdot), \quad x(0) = \phi_1(t_3),$$

By local existence and uniqueness, $\phi_1$ and $\phi_2$ coincide on $[0, t_3 + \varepsilon)$ for some $\varepsilon > 0$, which contradicts to the maximality of $t_3$. Hence $\phi_1$ and $\phi_2$ coincide on $[0, t_1)$.

\[\square\]

Definition 3.15. A solution $x(\cdot)$ of (4) is called

(i) maximal if there is no solution of (4) that extends $x(\cdot)$,

(ii) global if $x(\cdot)$ is defined on $\mathbb{R}_+$.

A central property of the system (4) is

Definition 3.16. We say that the system (4) is well-posed if for every initial value $x_0 \in X$ and every external input $u \in \mathcal{U}$, there exists a unique maximal solution that we denote by $\phi(\cdot, x_0, u) : [0, t_m(x_0, u)) \to X$, where $0 < t_m(x_0, u) \leq \infty$.

We call $t_m(x_0, u)$ the maximal existence time of a solution corresponding to $(x_0, u)$.

The map $\phi$, defined in Definition 3.16, and describing the evolution of the system (4), is called the flow map, or just flow. The domain of definition of the flow $\phi$ is

$$D_\phi := \cup_{x_0 \in X, u \in \mathcal{U}} [0, t_m(x_0, u)) \times \{(x_0, u)\}.$$ 

In the following pages, we will always deal with maximal solutions. We will usually denote the initial condition by $x \in X$.

We proceed with:

Theorem 3.17 (Well-posedness). Let Assumptions 3.1, 3.2, 3.3 hold. Then (4) is well-posed.

Proof. Pick any $x_0 \in X$ and $u \in \mathcal{U}$. Let $S$ be the set of all solutions $\xi$ of (4), defined on the corresponding domain of definition $I_\xi = [0, t_\xi)$.

Define $I := \cup_{\xi \in S} I_\xi = [0, t^*)$ for some $t^* > 0$, which can be either finite or infinite. For any $t \in I$ there is $\xi \in S$, such that $t \in I_\xi$. Define $\xi_{\max}(t) := \xi(t)$. Because of Lemma 3.14 the map $\xi_{\max}$ does not depend on the choice of $\xi$ above and thus is well-defined on $I$.

Furthermore, for each $t \in (0, t^*)$ there is $\xi \in S$, such that $\xi_{\max} = \xi$ on $[0, t + \varepsilon) \subset I$, for a certain $\varepsilon > 0$, and thus $\phi(\cdot, x_0, u) := \xi_{\max}$ is a solution of (4) corresponding to $x_0, u$. Finally, it is a maximal solution by construction.

Now we show that well-posed systems (4) are a special case of general control systems, introduced in Definition 2.1.

Theorem 3.18. Let (4) be well-posed. Then the triple $(X, \mathcal{U}, \phi)$, where $\phi$ is a flow map of (4), constitutes a control system in the sense of Definition 2.1.

Proof. By construction, all the axioms in the definition of a control system are fulfilled. In particular, continuity holds by the definition of a mild solution. We only need to check the cocycle property.
Take any initial condition \( x \in X \), any input \( u \in \mathcal{U} \), and any \( t, \tau \geq 0 \), such that \([0, t + \tau] \times \{(x, u)\} \subset D_\phi\). Define an input \( v \) by \( v(r) = u(r + \tau), \ r \geq 0 \).

Due to \( \mathbf{9} \), we have:
\[
\phi(t + \tau, x, u) = T(t + \tau)x + \int_0^{t+\tau} T_{-1}(t + \tau - s)B_2f(\phi(s, x, u), u(s))ds
+ \int_0^\tau T_{-1}(t + \tau - s)Bu(s)ds.
\]

As \( T(t) \) is a bounded operator, it can be taken out of the Bochner integral:
\[
\phi(t + \tau, x, u) = T(t)T(\tau)x + T(t)\int_0^\tau T_{-1}(\tau - s)B_2f(\phi(s, x, u), u(s))ds
+ T_{-1}(t)\int_0^\tau T_{-1}(\tau - s)Bu(s)ds
+ \int_0^\tau T_{-1}(\tau - s)B_2f(\phi(s, x, u), u(s))ds
+ \int_0^\tau T_{-1}(t + \tau - s)Bu(s)ds.
\]

As \( B \) is admissible, we have that \( \int_0^\tau T_{-1}(\tau - s)Bu(s)ds \in X \). Since \( T_{-1}(\cdot) \) coincides with \( T(\cdot) \) on \( X \), we infer
\[
T_{-1}(t)\int_0^\tau T_{-1}(\tau - s)Bu(s)ds = T(t)\int_0^\tau T_{-1}(\tau - s)Bu(s)ds.
\]

Finally,
\[
\phi(t + \tau, x, u) = T(t)\phi(\tau, x, u) + \int_0^t T_{-1}(t - s)B_2f(\phi(s + \tau, x, v), v(s))ds
+ \int_0^t T_{-1}(t - s)Bv(s)ds
= \phi(t, \phi(\tau, x, u), v),
\]
and the cocycle property holds.

We proceed with a general asymptotic property of the flow.

**Proposition 3.19.** Let Assumptions \( \mathbf{3.1, 3.2, 3.3} \) hold. Pick any \( x \in X \) and any \( u \in \mathcal{U} \). If \( t_m(x, u) \) is finite, then \( \| \phi(t, x, u) \|_X \to \infty \) as \( t \to t_m(x, u) - 0 \).

**Proof.** Pick any \( x \in X \) and any \( u \in \mathcal{U} \), and consider the corresponding maximal solution \( \phi(\cdot, x, u) \), defined on \([0, t_m(x, u))\). Assume that \( t_m(x, u) < +\infty \), but at the same time \( \lim_{t \to t_m(x, u)-0} \| \phi(t, x, u) \|_X < \infty \). Then there is a sequence \( (t_k) \), such that \( t_k \to t_m(x, u) \) as \( k \to \infty \) and \( \lim_{k \to \infty} \| \phi(t_k, x, u) \|_X < \infty \). Then also \( \sup_{k \in \mathbb{N}} \| \phi(t_k, x, u) \|_X =: C < \infty \).

Let \( \tau(C) > 0 \) be a uniform existence time for the solutions starting in the ball \( \overline{B_C} \) subject to inputs of a magnitude not exceeding \( \| u \| \), which exists and is positive in view of Theorem \( \mathbf{3.7} \). Then the solution of \( \mathbf{9} \) starting in \( \phi(t_k, x, u) \), corresponding to the input \( u(\cdot + t_k) \), exists and is unique on \([0, \tau(C)]\) by Theorem \( \mathbf{3.7} \), and by the cocycle property, we have that \( \phi(\cdot, x, u) \) can be prolonged to \([0, t_k + \tau(C)]\), which (since \( t_k \to t_m(x, u) \) as \( k \to \infty \)) contradicts to the maximality of the solution corresponding to \((x, u)\).

Hence \( \lim_{t \to t_m(x, u)-0} \| \phi(t, x, u) \|_X = \infty \), which implies the claim.

As a corollary of Proposition \( \mathbf{3.19} \) we obtain that any uniformly bounded maximal solution of \( \mathbf{9} \) is a global solution.
Corollary 3.20 (Boundedness-implies-continuation (BIC) property). Let Assumptions 3.1, 3.2, 3.3 hold. Let further the existence of solutions, stronger requirements on nonlinearity are needed. Lipschitz continuity guarantees the existence of local solutions. To ensure the global existence of solutions, stronger requirements on nonlinearity are needed.

3.4. Forward completeness and boundedness of reachability sets. Local Lipschitz continuity guarantees the existence of local solutions. To ensure the global existence of solutions, stronger requirements on nonlinearity are needed.

Proposition 3.21. Let Assumptions 3.1, 3.2, 3.3 hold. Let further $f$ be uniformly globally Lipschitz. Then (4)

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solution of (4), with a maximal existence time $t_1(x_0, u)$, which may be finite or infinite. Let $t_1(x_0, u)$ be finite.

Let $L > 0$ be a uniform global Lipschitz constant for $f$. As $\|T(t)\| \leq Me^{Lt}$ for some $M \geq 1$, $\lambda \geq 0$ and all $t \geq 0$, for any $t < t_1(x_0, u)$ we have according to the formula (19) the following estimates

\[
\|\phi(t, x_0, u)\| X \leq \|T(t)\| \|x_0\| X + \left\| \int_0^t T(t-s)Bu(s)ds \right\| X \\
\leq Me^{Lt} \|x_0\| X + h_1 \|u\| \|X\| + c_t \|T(t)\| \|x_0\| X + c_t \|f(0, u)\| \|X\| + c_t \|f(0, u)\| \|X\| + c_t \|f(0, u)\| \|X\| + c_t \|f(0, u)\| \|X\| + c_t \|f(0, u)\| \|X\| + c_t \|f(0, u)\| \|X\| + c_t \|f(0, u)\| \|X\|
\]

As $c_t \to 0$ as $t \to +0$, there is some $t_1 \in (0, t_1(x_0, u))$ such that $c_t L \leq \frac{1}{2}$. Then it holds that

\[ \sup_{t \in [0, t_1]} \|\phi(t, x_0, u)\| X \leq 2 \left( Me^{Lt_1} \|x_0\| X + h_1 \|u\| \|X\| + c_t \|f(0, u)\| \|X\| + c_t \|f(0, u)\| \|X\| + c_t \|f(0, u)\| \|X\| + c_t \|f(0, u)\| \|X\| + c_t \|f(0, u)\| \|X\| + c_t \|f(0, u)\| \|X\| + c_t \|f(0, u)\| \|X\| \right) \]

Note that $t_1$ does not depend on $x_0$ and $u$. Hence, using cocycle property and with $\phi(t_1, x_0, u)$ instead of $x_0$, we obtain a uniform bound for $\phi(\cdot, x_0, u)$ on $2t_1, 3t_1$, and so on. Thus, $\phi(\cdot, x_0, u)$ is uniformly bounded on $[0, t_1(x_0, u))$, and hence can be prolonged to a larger interval by BIC property, a contradiction to the definition of $t_1(x_0, u)$. Hence, $\Sigma$ is forward complete, and the estimate (17) iterated as above to larger intervals shows the BRS property.

3.5. Regularity of the flow map. We start with a basic result describing the exponential deviation between two trajectories.

Theorem 3.22. Let Assumptions 3.1, 3.2, 3.3 hold. Take $M \geq 1, \lambda \geq 0$ such that $\|T(t)\| \leq Me^{Lt}$ for all $t \geq 0$. Pick any $x_1, x_2 \in X$, any $u \in U$, and let $\phi(\cdot, x_1, u)$ and $\phi(\cdot, x_2, u)$ be defined on a certain common interval $[0, \tau]$.

Then there exists $R = R(x_1, x_2, \tau, u) > 0$, such that

\[ \|\phi(t, x_1, u) - \phi(t, x_2, u)\| X \leq 2M \|x_1 - x_2\| X e^{Rt}, \quad t \in [0, \tau]. \]

Proof. Pick any $x_1, x_2 \in X$, any $u \in U$, and let $\phi_i(t) := \phi(t, x_1, u)$, $i = 1, 2$ be the corresponding (unique) maximal solutions of (4) (guaranteed by Theorem 3.17), defined on $[0, \tau]$, for a certain $\tau > 0$.

Set

\[ K := \max \left\{ \sup_{0 \leq t \leq \tau} \|\phi_1(t)\| X, \sup_{0 \leq t \leq \tau} \|\phi_2(t)\| X, \|u\| \|X\| \right\} < \infty, \]

\[ \|\phi(t, x, u) - \phi(t, x, u)\| X \leq 2M \|x_1 - x_2\| X e^{Rt}, \quad t \in [0, \tau]. \]
where $K$ is finite due to continuity of trajectories.

Due to (10), and using Lipschitz continuity of $f$ (see (10)), we have for any $t \in [0, \tau]$:

$$
\|\phi_1(t) - \phi_2(t)\|_X \leq \|T(t)\|\|x_1 - x_2\|_X + \left\| \int_0^t T(t-s)B_2 \left( f(\phi_1(s), u(s)) - f(\phi_2(s), u(s)) \right) ds \right\|_X
$$

$$
\leq Me^{\lambda t}\|x_1 - x_2\|_X + c_1\|f(\phi_1(\cdot), u) - f(\phi_2(\cdot), u)\|_{L^\infty([0,t],X)}
$$

$$
\leq Me^{\lambda t}\|x_1 - x_2\|_X + c_1L(K)\|\phi_1(\cdot) - \phi_2(\cdot)\|_{L^\infty([0,t],X)}
$$

As $c_1 \to 0$ as $t \to +\infty$, there is some $t_1 \in (0, \tau)$ such that $c_1L(K) \leq \frac{1}{2}$. Note that $t_1$ depends on $\tau$ only (as $K$ does).

Then, taking the supremum of the previous expression over $[0, t]$, with $t < t_1$, we obtain that

$$
\|\phi_1(t) - \phi_2(t)\|_X \leq 2Me^{\lambda t}\|x_1 - x_2\|_X, \quad t \in [0, t_1].
$$

Take $k \in \mathbb{N}$ such that $kt_1 < \tau$ and $(k+1)t_1 > \tau$. Then, using the cocycle property, for any $l \in \mathbb{N}$, $l \leq k$ and all $t \in [0, t_1]$ s.t. $lt_1 + t < \tau$ we have

$$
\|\phi_1(lt_1 + t) - \phi_2(lt_1 + t)\|_X \leq (2M)^{l+1}e^{\lambda (lt_1+t)}\|x_1 - x_2\|_X
$$

$$
= 2M e^{\lambda l+\lambda t}\|x_1 - x_2\|_X
$$

$$
= 2M e^{R(t_1+t)}\|x_1 - x_2\|_X,
$$

where $R > \lambda$. Finally,

$$
\|\phi_1(s) - \phi_2(s)\|_X \leq 2Me^{Rs}\|x_1 - x_2\|_X, \quad s \in [0, \tau).
$$

\[\square\]

**Definition 3.23.** The flow of a forward complete control system $\Sigma = (X, U, \phi)$, is called Lipschitz continuous on compact intervals (for uniformly bounded inputs), if for any $\tau > 0$ and any $C > 0$ there exists $L > 0$ so that for any $x_1, x_2 \in \overline{B}_C$, for all $u \in B_{C, U}$, it holds that

$$
\|\phi(t, x_1, u) - \phi(t, x_2, u)\|_X \leq L\|x_1 - x_2\|_X, \quad t \in [0, \tau].
$$

**Theorem 3.22** estimates the deviation between two trajectories. To have a stronger result, showing the continuous dependence of the flow map $\phi$ on the initial conditions, we additionally assume the BRS property of (1).

**Theorem 3.24.** Let Assumptions 3.1, 3.2, 3.3. Let further (1) have BRS property. Then (1) has a flow that is Lipschitz continuous on compact intervals for uniformly bounded inputs.

**Proof.** Take any $C > 0$ and pick any $x_1, x_2 \in \overline{B}_C$, and any $u \in U$ with $\|u\|_U \leq C$. Let $\phi_i(t) := \phi(t, x_i, u)$, $i = 1, 2$ be the corresponding solutions of (1). These solutions are global since we assume that (1) is forward-complete.

As (1) is BRS, the following quantity is finite for any $\tau > 0$:

$$
K := \sup_{t \in [0, \tau], \ x \in \overline{B}_C, \ u \in B_{C, U}} \|\phi(t, x, u)\|_X < \infty.
$$

Following the lines of the proof of (3.22), we obtain the claim. \[\square\]
Definition 3.25. Let $\Sigma = (X, U, \phi)$ be a forward complete control system. We say that the flow $\phi$ depends continuously on inputs and on initial states, if for all $x \in X$, $u \in U$, $T > 0$, and all $\varepsilon > 0$ there exists $\delta > 0$, such that $\forall x' \in X : \|x - x'\|_X < \delta$ and $\forall u' \in U : \|u - u'\|_U < \delta$ it holds that 

$$\|\phi(t, x, u) - \phi(t, x', u')\|_X < \varepsilon, \ \forall t \in [0, T].$$

To obtain continuity of the flow map with respect to both states and inputs, which is important for the application of the density argument, we impose additional conditions on the nonlinearity $f$.

Theorem 3.26. Let Assumptions 3.1, 3.2, 3.3 hold. Let further there exists $q \in K_\infty$ such that for all $C > 0$ there is $L(C) > 0$: for all $x_1, x_2 \in \overline{B_C}$ and all $v_1, v_2 \in \overline{B_C U}$ it holds that

$$(20) \quad \|f(x_1, v_1) - f(x_2, v_2)\|_V \leq L(C)(\|x_1 - x_2\|_X + q(\|v_1 - v_2\|_U)).$$

Further let (4) have BRS property, and $B_2$ be a bounded operator. Then (4) depends continuously on initial states and inputs.

Proof. Take any $C > 0$ and pick any $x_1, x_2 \in \overline{B_C}$ and any $u_1, u_2 \in \overline{B_C U}$, $i = 1, 2$. Let $\phi_i(t) = \phi(t, x_i, u_i)$, $i = 1, 2$ be the corresponding global solutions. Due to (9), we have:

$$\|\phi_1(t) - \phi_2(t)\|_X \leq \|T(t)\|\|x_1 - x_2\|_X + h_t\|u_1 - u_2\|_U$$

$$+ \|B_2\| \int_0^t \|T(t - r)\|\|f(\phi_1(r), u_1(r)) - f(\phi_2(r), u_2(r))\|_V dr.$$

In view of the boundedness of reachability sets for the system (4), we have

$$K := \sup_{\|z\| \leq C, \|u\| \leq C, t \in [0,\tau]} \|\phi(t, z, u)\|_X < \infty.$$

As $\|T(t)\| \leq Me^{\lambda t}$ for some $M, \lambda \geq 0$ and all $t \geq 0$, and due to the property (20) with $L := L(K)$ (note that $K \geq C$), we can continue above estimates to obtain

$$\|\phi_1(t) - \phi_2(t)\|_X \leq Me^{\lambda t}\|x_1 - x_2\|_X + h_t\|u_1 - u_2\|_U$$

$$+ \|B_2\| \int_0^t Me^{\lambda(t-r)}L(K)\left(\|\phi_1(r) - \phi_2(r)\|_X + q(\|u_1(r) - u_2(r)\|_U)\right) dr.$$

Now define $z_i(r) := e^{-\lambda r}\phi_i(r)$, $i = 1, 2$, $r \geq 0$. Then

$$\|z_1(t) - z_2(t)\|_X \leq M\|x_1 - x_2\|_X + h_t\|u_1 - u_2\|_U + \|B_2\|ML(K)\int_0^t \|z_1(r) - z_2(r)\|_X dr$$

$$+ \|B_2\|ML(K) \int_0^t e^{-\lambda r}q(\|u_1(r) - u_2(r)\|_U) dr$$

$$\leq M\|x_1 - x_2\|_X + h_t\|u_1 - u_2\|_U + \|B_2\|ML(K)\int_0^t \|z_1(r) - z_2(r)\|_X dr$$

$$+ \|B_2\|ML(K) \int_0^t e^{-\lambda r}q(\|u_1 - u_2\|_U)$$

$$\leq M\|x_1 - x_2\|_X + h_t\|u_1 - u_2\|_U + \|B_2\|ML(K)\int_0^t \|z_1(r) - z_2(r)\|_X dr$$

$$+ \|B_2\|ML(K) \int_0^t e^{-\lambda r}q(\|u_1 - u_2\|_U).$$
In view of Gronwall’s inequality, we obtain
\[
\|z_1(t) - z_2(t)\|_X \leq \left( M\|x_1 - x_2\|_X + h_t\|u_1 - u_2\|_U \\
+ \frac{B_2\|ML(K)\|}{\lambda} q\left(\|u_1 - u_2\|_U\right)\right)e^{\frac{B_2\|ML(K)\|}{\lambda}t}
\]
or in the original variables
\[
\|\phi_1(t) - \phi_2(t)\|_X \leq \left( M\|x_1 - x_2\|_X \\
+ h_t\|u_1 - u_2\|_U + \frac{B_2\|ML(K)\|}{\lambda} q\left(\|u_1 - u_2\|_U\right)\right)e^{\frac{B_2\|ML(K)\|}{\lambda}t}.
\]
This implies that \((1)\) depends continuously on inputs and initial states. \(\square\)

3.6. Continuity at trivial equilibrium. Without loss of generality, we restrict our analysis to fixed points of the form \((0,0) \in X \times U\). Note that \((0,0)\) is in \(X \times U\) since both \(X\) and \(U\) are linear spaces.

For describing the behavior of solutions near the equilibrium, the following notion is of importance:

**Definition 3.27.** Consider a system \(\Sigma = (X, U, \phi)\) with equilibrium point \(0 \in X\). We say that \(\phi\) is continuous at the equilibrium if for every \(\varepsilon > 0\) and for any \(h > 0\) there exists a \(\delta = \delta(\varepsilon, h) > 0\), so that \([0, h] \times B_\delta \times B_\delta U \subset D_\varepsilon\), and

\[
(21) \quad t \in [0, h], \|x\|_X \leq \delta, \|u\|_U \leq \delta \Rightarrow \|\phi(t, x, u)\|_X \leq \varepsilon.
\]

In this case, we will also say that \(\Sigma\) has the CEP property.

CEP property is a “local version” of Lyapunov stability and is important, in particular, for the ISS superposition theorems [42] and for the applications of the non-coercive ISS Lyapunov theory [21].

**Lemma 3.28** (Continuity at equilibrium for \((1)\)). Let Assumptions Y.1, Y.2, Y.3 hold, and let \(f(0,0) = 0\). Then \(\phi\) is continuous at the equilibrium.

**Proof.** Consider the following auxiliary system
\[
(22a) \quad \dot{x}(t) = Ax(t) + B_2\tilde{f}(x(t), u(t)) + Bu(t), \quad t > 0,
(22b) \quad x(0) = x_0,
\]
where
\[
\tilde{f}(x, u) := f(\text{sat}(x), \text{sat}_2(u)), \quad x \in X, \ u \in U,
\]
and the saturation function is given for the vectors \(z\) in \(X\) and in \(U\) respectively by
\[
\text{sat}(z) := \begin{cases} z, & \|z\|_X \leq 1, \\ \frac{z}{\|z\|_X}, & \text{otherwise}, \end{cases}
\text{sat}_2(z) := \begin{cases} z, & \|z\|_U \leq 1, \\ \frac{z}{\|z\|_U}, & \text{otherwise}, \end{cases}
\]
As \(f\) satisfies Assumption 3.3 one can show that \(\tilde{f}\) is uniformly globally Lipschitz continuous. Hence, \((22)\) is forward complete and has BRS property by Proposition 3.21.

We denote the flow of \((22)\) by \(\tilde{\phi} = \tilde{\phi}(t, x, u)\). As \(f(x, u) = \tilde{f}(x, u)\) whenever \(\|x\|_X \leq 1\) and \(\|u\|_U \leq 1\), it holds also
\[
\phi(t, x, u) = \tilde{\phi}(t, x, u),
\]
provided that \(\|u\|_U \leq 1\), \(\phi(\cdot, x, u)\) exists on \([0, t]\), and \(\|\phi(s, x, u)\|_X \leq 1\) for all \(s \in [0, t]\).
Pick any $\varepsilon \in (0,1)$, $\tau \geq 0$, $\delta \in (0,\varepsilon)$, and choose any $x \in B_{\delta}$, as well as any $u \in B_{\delta}$. It holds that
$$ \| \tilde{\phi}(t, x, u) \|_{X} \leq \| \tilde{\phi}(t, x, u) - \tilde{\phi}(0, x, u) \|_{X} + \| \tilde{\phi}(0, x, u) \|_{X}. $$
Since (22) has BRS property, by Theorem 3.24, the flow of (22) is Lipschitz continuous on compact time intervals. Hence there exists a $L(\tau, \delta) > 0$ so that for $t \in [0, \tau]$
$$ (23) \quad \| \tilde{\phi}(t, x, u) - \tilde{\phi}(0, x, u) \|_{X} \leq L(\tau, \delta) \| x \|_{X} \leq L(\tau, \delta). $$

Let us estimate $\| \tilde{\phi}(t, 0, u) \|_{X}$. We have:
$$ \| \tilde{\phi}(t, 0, u) \|_{X} \leq \left\| \int_{0}^{t} T(t - s) B_{2} \tilde{f}(\tilde{\phi}(s, 0, u), u(s)) ds \right\|_{X} + \left\| \int_{0}^{t} T(t - s) B u(s) ds \right\|_{X} $$
$$ \leq c_{t} \text{ess sup}_{s \in [0, t]} \| \tilde{f}(\tilde{\phi}(s, 0, u), u(s)) \|_{X} + h_{t} \| u \|_{L^{\infty}([0, t], U)} $$
$$ \leq c_{t} \text{ess sup}_{s \in [0, t]} \| \tilde{f}(\tilde{\phi}(s, 0, u), u(s)) - \tilde{f}(0, u(s)) \|_{X} + c_{t} \text{ess sup}_{s \in [0, t]} \| \tilde{f}(0, u(s)) \|_{X} $$
$$ + h_{t} \| u \|_{L^{\infty}([0, t], U)}.$$
Since $\tilde{f}(0, \cdot)$ is continuous, for any $\varepsilon_{2} > 0$ there exists $\delta_{2} < \delta$ so that $u(s) \in B_{\delta_{2}}$ implies that $\| \tilde{f}(0, u(s)) - \tilde{f}(0, 0) \|_{X} \leq \varepsilon_{2}$. Since $\tilde{f}(0, 0) = 0$, for the above we have $\| \tilde{f}(0, u(s)) \|_{X} \leq \varepsilon_{2}$.

As $\tilde{f}$ is uniformly globally Lipschitz, there is $L > 0$ such that for the inputs satisfying $\| u \|_{U_{t}} \leq \delta_{2}$ we have
$$ \| \tilde{\phi}(t, 0, u) \|_{X} \leq c_{t} L \text{ess sup}_{s \in [0, t]} \| \tilde{\phi}(s, 0, u) \|_{X} + c_{t} \varepsilon_{2} + h_{t} \delta_{2}. $$
As $c_{t} \to 0$ for $t \to +0$, there is $t_{1} > 0$, such that $c_{t_{1}} L \leq \frac{1}{2}$.

Then we have that
$$ (24) \quad \| \tilde{\phi}(t, 0, u) \|_{X} \leq 2 c_{t_{1}} \varepsilon_{2} + 2 h_{t_{1}} \delta_{2}, \quad t \leq t_{1}. $$
Combining (23) with (24), we see that whenever $\| x \|_{X} \leq \delta_{2}$ and $\| u \|_{U_{t}} \leq \delta_{2}$, it holds that
$$ \| \tilde{\phi}(t, x, u) \|_{X} \leq L(\tau, \delta_{2}) \delta_{2} + 2 c_{t_{1}} \varepsilon_{2} + 2 h_{t_{1}} \delta_{2}, \quad t \leq t_{1}. $$
Now for any $\varepsilon < 1$ we can find $\delta_{2} < \varepsilon$, such that
$$ \| \tilde{\phi}(t, x, u) \|_{X} \leq \varepsilon, \quad t \leq t_{1}, \quad \| x \|_{X} \leq \delta_{2}, \quad \| u \|_{U_{t}} \leq \delta_{2}. $$
As $\tilde{\phi}(t, x, u) = \phi(t, x, u)$ whenever $\| \tilde{\phi}(t, x, u) \|_{X} < 1$, we obtain that
$$ \| \phi(t, x, u) \|_{X} \leq \varepsilon, \quad t \leq t_{1}, \quad \| x \|_{X} \leq \delta_{2}, \quad \| u \|_{U_{t}} \leq \delta_{2}. $$
Note that $t_{1}$ depends on $L$ only, and does not depend on $\delta_{2}$. Thus, one can find $\delta_{3} < \delta_{2}$, such that
$$ \| \phi(t, x, u) \|_{X} \leq \delta_{3}, \quad t \leq t_{1}, \quad \| x \|_{X} \leq \delta_{3}, \quad \| u \|_{U_{t}} \leq \delta_{3}. $$
By cocycle property, we obtain that
$$ \| \phi(t, x, u) \|_{X} \leq \varepsilon, \quad t \leq 2 t_{1}, \quad \| x \|_{X} \leq \delta_{3}, \quad \| u \|_{U_{t}} \leq \delta_{3}. $$
Iterating this procedure, we obtain that there is some $\omega > 0$, such that
$$ \| \phi(t, x, u) \|_{X} \leq \varepsilon, \quad t \in [0, \tau], \quad \| x \|_{X} \leq \omega, \quad \| u \|_{U_{t}} \leq \omega. $$
This shows the CEP property. $\Box$
4. Boundary control systems

Control systems governed by partial differential equations are defined by PDEs describing the dynamics inside of the spatial domain and boundary conditions, describing the dynamics of the system at the boundary of the domain. Such systems look (at first glance) quite differently from the evolution equations in Banach spaces, studied in Section 3. This motivated the development of a theory of abstract boundary control systems that allows for a more straightforward interpretation of PDEs in the language of semigroup theory.

4.1. Linear boundary control systems. Let $X$ and $U$ be Banach spaces. Consider a system

\begin{align}
\dot{x}(t) &= \hat{A}x(t), \quad x(0) = x_0, \\
\hat{R}x(t) &= u(t),
\end{align}

where the formal system operator $\hat{A} : D(\hat{A}) \subset X \to X$ is a linear operator, the control function $u$ takes values in $U$, and the boundary operator $\hat{R} : D(\hat{R}) \subset X \to U$ is linear and satisfies $D(\hat{A}) \subset D(\hat{R})$.

Definition 4.1. The system (25) is called a linear boundary control system (linear BCS) if the following conditions hold:

(i) The operator $A : D(A) \to X$ with $D(A) = D(\hat{A}) \cap \ker(\hat{R})$ defined by

\begin{equation}
Ax = \hat{A}x \quad \text{for} \quad x \in D(A)
\end{equation}

is the infinitesimal generator of a $C_0$-semigroup $(T(t))_{t \geq 0}$ on $X$;

(ii) There is an operator $R \in L(U, X)$ such that for all $u \in U$ we have $Ru \in D(\hat{A})$, $\hat{A}R \in L(U, X)$ and

\begin{equation}
\hat{R}Ru = u, \quad u \in U.
\end{equation}

The operator $R$ in this definition is sometimes called a lifting operator. Note that $R$ is not uniquely defined by the properties in the item (ii).

Item (i) of Definition 4.1 shows that for $u \equiv 0$ the equations (25) are well-posed. In particular, as $A$ is the generator of a certain strongly continuous semigroup $T(\cdot)$, for any $x \in D(A)$, it holds that $T(t)x \in D(\hat{A})$ and thus $T(t)x \in \ker(\hat{R})$ for all $t \geq 0$, which means that (25b) is satisfied.

Item (ii) of the definition implies, in particular, that the range of the operator $\hat{R}$ equals $U$, and thus the values of inputs are not restricted.

4.2. Semilinear boundary control systems. Let $(\hat{A}, \hat{R})$ be a linear BCS. We consider $D(\hat{A}) \subset X$ as a linear space equipped with the graph norm

\begin{equation}
\| \cdot \|_{D(\hat{A})} := \| \cdot \|_X + \| \hat{A} \cdot \|_X.
\end{equation}

Motivated by [53], we consider the following class of semilinear boundary control systems.

Definition 4.2. Consider a linear BCS $(\hat{A}, \hat{R})$. Consider the following system

\begin{align}
\dot{x}(t) &= \hat{A}x(t) + f(x(t), w(t)), \quad t > 0, \\
\hat{R}x(t) &= u(t), \quad t > 0, \\
x(0) &= x_0,
\end{align}

with a nonlinearity $f : X \times W \to X$, where $W$ is a Banach space.
The system (28) we call a semilinear boundary control system (semilinear BCS).

Following [53], we define classical solutions to the semilinear BCS (28).

**Definition 4.3.** Let \( x_0 \in D(\hat{\Delta}) \), \( T > 0 \) and \( u \in C([0, T], U) \). A function
\[
    x \in C([0, T], D(\hat{\Delta})) \cap C^1([0, T], X)
\]
is called a classical solution to the semilinear BCS (28) on \([0, T]\) if \( x(t) \in X \) for all \( t > 0 \) and the equations (28) are satisfied pointwise for \( t \in (0, T) \).

A function \( x : [0, \infty) \to X \) is called (global) classical solution to the semilinear BCS (28), if \( x|_{[0, T]} \) is a classical solution on \([0, T]\) for every \( T > 0 \).

If \( x \in C([0, T], D(\hat{\Delta})) \cap C^1([0, T], X) \) and \( x(t) \in X \) for all \( t > 0 \) and the equations (28) are satisfied pointwise for \( t \in (0, T) \), then we say that \( x \) is a classical solution on \([0, T]\).

Recall the following result (see [25, Corollary 10.1.4]):

**Proposition 4.4.** Let \( T \) be a strongly continuous semigroup over a Banach space \( X \) with the infinitesimal generator \( A \). Let also \( f \in C^1(\mathbb{R}_+, X) \). Then
\[
    \int_0^t T(t - r)f(r)dr \in D(A) \quad \text{for all} \quad t \geq 0,
\]
and the following holds:
\[
    A \int_0^t T(t - r)f(r)dr = \int_0^t T(t - r)f(r)dr + T(t)f(0) - f(t). \tag{29}
\]

The next theorem gives a representation for the (unique) solutions of (28) for smooth enough inputs.

**Theorem 4.5.** Consider the boundary control system (28) with \( f \in C(X \times W, X) \). Let \( u \in C^2([0, \tau], U) \), and \( w \in C([0, T], W) \) for some \( \tau > 0 \) and \( x_0 \in X \) be such that \( x_0 - Ru(0) \in D(A) \). Assume that the classical solution of semilinear BCS \( \phi(\cdot, x_0, u) \) exists on \([0, \tau]\). Then it can be represented as
\[
    \phi(t, x_0, u) = T(t)(x_0 - Ru(0))
\]
\[
    + \int_0^t T(t - r) \left( f(x(r), w(r)) + \hat{A}Ru(r) - \hat{R}u(r) \right) dr + Ru(t)
\]
\[
    = T(t)x_0 + \int_0^t T(t - r) \left( f(x(r), w(r)) + \hat{A}Ru(r) \right) dr
\]
\[
    - A \int_0^t T(t - r)Ru(r)dr
\]
\[
    = T(t)x_0 + \int_0^t T(-1)(t - r) \left( f(x(r), w(r)) + (\hat{A}R - A_{-1}R)u(r) \right) dr,
\]
where \( A_{-1} \) and \( T_{-1} \) are the extensions of the infinitesimal generator \( A \) and of the semigroup \( T \) to the extrapolation space \( X_{-1} \). Furthermore, \( A_{-1}R \in L(U, X_{-1}) \) (and thus \( \hat{A}R - A_{-1}R \in L(U, X_{-1}) \)).

The proof is similar to the proof of the linear case (see [10, Theorem 4.4], [53, pp. 93–94]), but we provide the detailed proof as we strive to be self-contained.

**Proof.** (a). Pick any \( \tau \geq 0 \), \( u \in C^2([0, \tau], U) \) and any \( x_0 \in X \): \( x_0 - Ru(0) \in D(A) \). Let \( x(\cdot) := \phi(\cdot, x_0, u) \) be a classical solution of the semilinear BCS. Define
\[
    v(t) := x(t) - Ru(t).
\]
We are going to show that this function is a classical solution of the following equation
\begin{equation}
\dot{v} = Av(t) + f(v(t) + Ru(t), w(t)) - R\dot{u}(t) + \hat{A}Ru(t)
\end{equation}
on $[0, \tau]$, in the sense that $v \in C([0, \tau], X)$, $v \in C^1([0, \tau], X)$, $v(t) \in D(A)$ for $t \in (0, \tau)$, and (31) holds with this $v$.

Firstly, it holds that
$\hat{R}v(t) = \hat{R}\phi(t, x_0, u) - u = 0$.

As $\text{Im}(R) \subset D(\hat{A})$ and $x$ is a classical solution of semilinear BCS, then $v(t) \in D(\hat{A}) \cap \text{Ker}(\hat{R}) = D(A)$.

Furthermore,
\begin{align*}
\dot{v}(t) &= \hat{A}x(t) + f(x(t), w(t)) - R\dot{u}(t) \\
&= A(v(t) + Ru(t)) + f(v(t) + Ru(t), w(t)) - R\dot{u}(t) \\
&= \dot{A}v(t) + f(v(t) + Ru(t), w(t)) - R\dot{u}(t) + \hat{A}Ru(t).
\end{align*}

Since $v(t) \in D(A)$, $A = \hat{A}$ on $D(A)$, and $v \in C^1([0, \tau], X)$, $v$ is a classical solution of (31).

As (31) is a classical semilinear evolution equation, $v$ can be represented (see [15] equation (1.2) on page 183) as
$\begin{equation}
\nu(t) = T(t)v(0) + \int_0^t T(t - r)\left( f(v(r) + Ru(r), w(r)) + \hat{A}Ru(r) - R\dot{u}(r) \right) dr,
\end{equation}$
and returning to the $x$-variable, we obtain (30a).

(b). By Proposition 14 it holds that
$\begin{equation}
\int_0^t T(t - r)\hat{R}\dot{u}(r) dr - Ru(t) = A \int_0^t T(t - r)Ru(r) dr - T(t)Ru(0).
\end{equation}$

Substituting this into (30a), we obtain (30b).

(c). Let $A_{-1}$ be the extension of $A$ to the extrapolation space $X_{-1}$, and let $T_{-1}$ be the extrapolated semigroup, generated by $A_{-1}$. Note that $R \in L(U, X)$, and $D(A_{-1}) = X$. Thus, the operator $A_{-1}R$ is well-defined as a linear operator from Banach space $U$ to Banach space $X_{-1}$ with $D(A_{-1}R) = U$. As $A_{-1}$ is the generator of a strongly continuous semigroup, it is closed. Thus, by [15] Proposition A.9 the operator $A_{-1}R$ is closed as a product of a closed and a bounded operator. By closed graph theorem (see, e.g., [15] Theorem A.3.52), $A_{-1}R \in L(U, X_{-1})$.

The map $r \mapsto T(t - r)Ru(r)$ is Bochner integrable in the space $X$ and thus also in $X_{-1}$ (even for any $u \in L^1_{\text{loc}}([0, \tau], U)$, see [15] Proposition 1.3.4). Furthermore, $T(t - r)Ru(r) \in X = D(A_{-1})$ for all $r \in [0, t]$.

Recall that $A_{-1}\nu_{-1}(s) = T_{-1}(s)A_{-1}$ for all $s \in \mathbb{R}_+$ on $D(A_{-1})$, see, e.g., [25] Theorem 5.2.2. Consider the map
$\begin{equation}
w: r \mapsto A_{-1}\nu_{-1}(t - r)Ru(r) = T_{-1}(t - r)A_{-1}Ru(r).
\end{equation}$

Since $A_{-1}R \in L(U, X_{-1})$ and $T_{-1}$ is a strongly continuous semigroup on $X_{-1}$, the function $w$ is Bochner integrable on $X_{-1}$, by [15] Proposition 1.3.4. Hence by Hille’s Lemma (see, e.g., [16] Theorem 3.7.12, [15] Proposition 1.1.7), we obtain for
all \( u \in C^2([0, \tau], U) \) that
\[
A \int_0^t T(t - r)Ru(dr) = A_{-1} \int_0^t T_{-1}(t - r)Ru(dr) \\
= \int_0^t A_{-1}T_{-1}(t - r)Ru(dr) = \int_0^t T_{-1}(t - r)A_{-1}Ru(dr).
\]
From this the formula \((30c)\) follows. \(\square\)

An advantage of the representation formula \((30a)\) is in the boundedness of the operators \(R\) and \(AR\) involved in the expression. Its disadvantage is that the derivative of \(u\) is employed. Still, the expression in the right-hand side of \((30a)\) makes sense for any \(x \in X\) and for any \(u \in H^1([0, \tau], U)\), and can be called a mild solution of BCS \((25)\), as is done, e.g., in \([25\text{, p. 146}]\).

The formula \((30c)\) does not involve any derivatives of inputs, and again is given in terms of a bounded operator \(AR - A_{-1}R \in L(U, X_{-1})\). Moreover, if we consider the expression in the right-hand side of \((30c)\) in the extrapolation spaces \(X_{-1}\), then it makes sense for all \(x \in X\) and all \(u \in L^1_{\text{loc}}(\mathbb{R}_+, U)\), and constitutes a mild solution of
\[
\dot{x}(t) = Ax(t) + f(x(t), w(t)) + Bu(t),
\]
with
\[
B := AR - A_{-1}R.
\]
This motivates us to define the mild solutions of semilinear BCS by means of the formula \((30c)\), as was proposed in \([33]\).

**Definition 4.6.** Let \((\dot{A}, \dot{R}, f)\) be a semilinear boundary control system with corresponding \(A, R\). Let \(x_0 \in X, T > 0, w \in L^1_{\text{loc}}([0, T], W)\), and \(u \in L^1_{\text{loc}}([0, T], U)\). A continuous function \(x : [0, T] \to X\) is called a mild solution to the semilinear BCS \((28)\) on \([0, T]\) if \(x(t) \in X\) for all \(t > 0\) and \(x\) solves
\[
x(t) = T(t)x_0 + \int_0^t T_{-1}(t - s)(f(x(s), w(s)) + Bu(s))ds,
\]
for all \(t \in [0, T]\) and where \(B = \dot{A}B_0 - A_{-1}B_0\). A function \(x : \mathbb{R}_+ \to X\) is called a global mild solution if \(x|_{[0, T]}\) is a mild solution on \([0, T]\) for all \(T > 0\).

In other words, \(x\) is a mild solution of a semilinear BCS \((4.2)\), if \(x\) is a mild solution of \((33)\) with \(B = \dot{A}R - A_{-1}\dot{R}\).

Thus, semilinear boundary control systems are a special case of semilinear evolution equations studied in Section \(3\), and we can use our well-posedness theory for semilinear evolution equations to analyze semilinear BCS.

### 5. Semilinear analytic systems

#### 5.1. Semilinear analytic systems and their mild solutions.
Consider again the system \((4)\) with \(B_2 = \text{id}\) that we restate next:
\[
\begin{align*}
\dot{x}(t) &= Ax(t) + f(x(t), u(t)) + Bu(t), \quad t > 0, \quad \text{(35a)} \\
x(0) &= x_0, \quad \text{(35b)}
\end{align*}
\]
In Section \(3\) we have assumed that \(f\) is a well-defined map from \(X \times U\) to \(X\). Although it sounds natural, it is, in fact, a quite restrictive assumption, as already basic nonlinearities, such as polynomial maps, do not satisfy it. Indeed, if \(f(x) = x^2\), where \(x \in X := L^2(0, 1)\), then \(f\) maps \(X\) to the space \(L^1(0, 1)\). However,
as $A$ generates an analytic semigroup, the requirements on $f$ can be considerably relaxed. Namely, we assume in this section that there is $\alpha \in [0, 1]$ such that $f$ is a well-defined map from $X_\alpha \times U$ to $X$.

Next, we define mild solutions of \ref{35}. Note that the nonlinearity $f$ is defined on $X_\alpha \times U$, and thus we must require that the mild solution lies in $X_\alpha$ for all positive times. We cannot expect such a nice behavior for general semigroups, but thanks to the smoothing effect of analytic semigroups, this is what we can expect in the analytic case.

**Definition 5.1 (Mild solutions).** Let $\tau > 0$ be given. A function $x \in C([0, \tau], X)$ is called a mild solution of \ref{35} on $[0, \tau]$ corresponding to certain $x_0 \in X$ and $u \in L^1_{\text{loc}}(\mathbb{R}^+; U)$, if $x(s) \in X_\alpha$ for $s \in [0, \tau]$, and $x$ solves the integral equation

$$(36) \quad x(t) = T(t)x_0 + \int_0^t T(t-s)f(x(s), u(s))\,ds + \int_0^t T_{-1}(t-s)Bu(s)\,ds.$$

We say that $x : \mathbb{R}^+ \to X$ is a mild solution of \ref{35} on $\mathbb{R}^+$ corresponding to certain $x_0 \in X$ and $u \in L^1_{\text{loc}}(\mathbb{R}^+, U)$, if it is a mild solution of \ref{35} (with $x_0, u$) on $[0, \tau]$ for all $\tau > 0$.

**Remark 5.2.** Note that if $\alpha = 0$, then $X_\alpha = X_0 = X$, and the concept of a mild solution introduced for general and analytic semigroups coincide.

**Assumption 5.1.** Let the following hold:

(i) $\alpha \in (0, 1)$,

(ii) $B \in L(U, X_{-1+\alpha+\varepsilon})$ for sufficiently small $\varepsilon > 0$,

(iii) $f \in C(X_\alpha \times U, X)$, and $f$ is Lipschitz continuous in the first argument in the following sense: for each $r > 0$ there is $L = L(r) > 0$ such that for each $x_1, x_2 \in B_{r, X_\alpha}$ and all $u \in B_{r, U}$ it holds that

$$(37) \quad \|f(x_1, u) - f(x_2, u)\|_X \leq L\|x_1 - x_2\|_{X_\alpha}.$$ 

(iv) For all $u \in L^{\infty}(\mathbb{R}^+, U)$ and any $x \in C(\mathbb{R}^+, X)$ with $x((0, +\infty)) \subset X_\alpha$, the map $s \mapsto f(x(s), u(s))$ is in $L^1_{\text{loc}}(\mathbb{R}^+, X)$ with $w > \frac{1}{1-\alpha}$.

(v) There is $\sigma \in K_\infty$ such that

$$(38) \quad \|f(0, u)\|_X \leq \sigma(\|u\|_{L^1}) + c, \quad u \in U.$$

**Remark 5.3.** By Proposition 5.3.8, the condition $B \in L(U, X_{-1+\alpha+\varepsilon})$ with $\alpha, \varepsilon > 0$ implies that $B$ is zero-class $q$-admissible for any $q \in (\frac{1}{\alpha+\varepsilon}, +\infty)$. This in turn implies that for such $q$ the map $t \mapsto \int_0^t T_{-1}(t-s)Bu(s)\,ds$ is continuous for any $u \in L^q(\mathbb{R}^+, U)$, by \ref{60} Proposition 2.3.1.

By Assumption 5.1(iv), and using Proposition 5.3, we see that for any $u \in L^{\infty}(\mathbb{R}^+, U)$ the map

$$(39) \quad t \mapsto \int_0^t T(t-s)f(x(s), u(s))\,ds$$

is well-defined and continuous.

Hence, if $x \in C(\mathbb{R}^+, X)$ with $x((0, +\infty)) \subset X_\alpha$, then for any $u \in L^{\infty}(\mathbb{R}^+, U)$ the right-hand side of \ref{35} is a continuous function of time.

### 5.2. Preliminaries for analytic semigroups and admissibility.

We start with the following “analytic” version of \ref{1} Proposition 1.3.4:

**Proposition 5.4.** Let $T$ be an analytic semigroup, $\alpha \in (0, 1)$, and $g \in L^w_{\text{loc}}(\mathbb{R}^+, X)$ with $w > \frac{1}{1-\alpha}$. Pick any $\omega > \omega_0(T)$. Then the map

$$(40) \quad \xi : t \mapsto \int_0^t (\omega I - A)^\alpha T(t-s)g(s)\,ds$$
is well-defined and continuous on \( \mathbb{R}_+ \).

Furthermore, for any \( \alpha > \omega_0(T) \) there is \( R = R(\kappa, \alpha) \) such that for any \( g \in L^\infty_{lo}([0, t], X) \) the following holds:

\[
\int_0^t \| (\omega I - A)^\alpha \int_0^s g(s) \|_X \, ds \leq Rt^{1-\alpha} e^{\kappa t} \| g \|_{L^\infty([0, t], X)}.
\]

Proof. We divide the proof into four parts.

(i). Well-definiteness. Since \( T \) is an analytic semigroup, \( T(t) \) maps \( X \) to \( D(A) \) for any \( t > 0 \). As \( X_\alpha \subset D(A) \) for all \( \alpha \in [0, 1] \), the integrand in (34) is in \( X \) for a.e. \( s \in [0, t] \). Let us show Bochner integrability of \( X \)-valued map \( s \mapsto (\omega I - A)^\alpha \int_0^s g(s) \) on \( [0, t] \).

As \( g \in L^\infty_{lo}(\mathbb{R}_+, X) \), by the criterion of Bochner integrability, \( g \) is strongly measurable and \( \int_I \| g(s) \|_X \, ds < \infty \) for any bounded interval \( I \subset \mathbb{R}_+ \).

Denote by \( \chi_\Omega \) the characteristic function of the set \( \Omega \subset \mathbb{R}_+ \). Recall that the map \( t \mapsto (\omega I - A)^\alpha T(t) \) is continuous outside of \( t = 0 \) in view of Proposition [A.10].

If \( g(s) = \chi_\Omega(s)x \) for some measurable \( \Omega \subset \mathbb{R}_+ \) and \( x \in X \), then the function

\[
s \mapsto (\omega I - A)^\alpha T(t-s)g(s) = (\omega I - A)^\alpha T(t-s)\chi_\Omega(s)x
\]

is measurable as a product of a measurable scalar function and a continuous (and thus measurable) vector-valued function. By linearity, \( s \mapsto (\omega I - A)^\alpha T(t-s)g(s) \) is strongly measurable if \( g \) is a simple function (see [1] Section 1.1 for definitions).

As \( g \) is strongly measurable, there is a sequence of simple functions \( (g_n)_{n \in \mathbb{N}} \), converging pointwise to \( g \) almost everywhere. Consider a sequence

\[
(s \mapsto (\omega I - A)^\alpha T(t-s)g_n(s))_{n \in \mathbb{N}}
\]

and take any \( s \in [0, t] \) such that \( g_n(s) \to g(s) \) as \( n \to \infty \). We have that

\[
\| (\omega I - A)^\alpha T(t-s)g_n(s) - (\omega I - A)^\alpha T(t-s)g(s) \|_X \leq \| (\omega I - A)^\alpha T(t-s) \|_X \| g_n(s) - g(s) \|_X \to 0, \quad n \to \infty.
\]

Hence a sequence of strongly measurable functions \( (g_n)_{n \in \mathbb{N}} \) converges a.e. to \( s \mapsto (\omega I - A)^\alpha T(t-s)g(s) \), and thus \( s \mapsto (\omega I - A)^\alpha T(t-s)g(s) \) is strongly measurable by [1] Corollary 1.1.2.

Furthermore, for any \( t > 0 \), using Proposition [A.10] we have that for any \( \kappa > \omega_0(T) \) there is \( M > 0 \) such that

\[
\int_0^t \| (\omega I - A)^\alpha T(t-s)g(s) \|_X \, ds \leq \int_0^t \| (\omega I - A)^\alpha T(t-s)g(s) \|_X \, ds \leq \frac{C_\alpha}{\kappa} e^{\kappa t} \| g(s) \|_X \, ds \leq MC_\alpha e^{\kappa t} \int_0^t \frac{1}{(t-s)^\alpha} \| g(s) \|_X \, ds.
\]

Using Hölder’s inequality with \( w > \frac{1}{1-\alpha} \) as in the assumptions of the proposition, we obtain

\[
\int_0^t \| (\omega I - A)^\alpha T(t-s)g(s) \|_X \, ds \leq MC_\alpha e^{\kappa t} \left( \int_0^t \frac{1}{(t-s)^\alpha} \, ds \right)^\frac{1}{1-\alpha} \left( \int_0^t \| g(s) \|_X^p \, ds \right)^{\frac{1}{p}} \leq \frac{MC_\alpha}{(1-\alpha)p} e^{\kappa t} \left( \int_0^t \| g(s) \|_X^p \, ds \right)^{\frac{1}{p}},
\]

where \( \frac{1}{p} + \frac{1}{w} = 1 \), and thus \( p \) satisfies \( p < \frac{1}{\alpha} \).
Finally, by [11] Theorem 1.1.4, the map $s \mapsto (\omega I - A)^\alpha T(t - s)g(s)$ is Bochner integrable on each $[0, t] \subset \mathbb{R}^+$. 

(ii). **Right-continuity.** We would like to show that $\xi$ defined by (38), is right-continuous on $\mathbb{R}^+$. To this end, we consider for $h > 0$ and $t \geq 0$ the expression

$$\xi(t + h) - \xi(t) = \int_0^{t+h} (\omega I - A)^\alpha T(t + h - s)g(s)ds - \int_0^t (\omega I - A)^\alpha T(t - s)g(s)ds$$

$$= \int_0^t (\omega I - A)^\alpha \left( T(t + h - s) - T(t - s) \right)g(s)ds$$

$$+ \int_0^{t+h} (\omega I - A)^\alpha T(t + h - s)g(s)ds$$

$$= (T(h) - I) \int_0^t (\omega I - A)^\alpha T(t - s)g(s)ds$$

$$+ \int_0^{t+h} (\omega I - A)^\alpha T(t + h - s)g(s)ds.$$ 

As shown in the first part of the proof, the integral $\int_0^t (\omega I - A)^\alpha T(t - s)g(s)ds$ converges in $X$. Since $T$ is strongly continuous, we have

$$(T(h) - I) \int_0^t (\omega I - A)^\alpha T(t - s)g(s)ds \to 0, \quad \text{whenever} \ h \to +0.$$ 

Furthermore, we estimate for a certain $K$ that does not depend on $h$:

$$\left\| \int_0^{t+h} (\omega I - A)^\alpha T(t + h - s)g(s)ds \right\|_X$$

$$\leq \int_0^{t+h} (\omega I - A)^\alpha T(t + h - s)\|g(s)\|_Xds$$

$$\leq K \int_0^{t+h} \frac{1}{(t + h - s)^\alpha} \|g(s)\|_Xds$$

and arguing similarly to (12), and using that $g \in L^w_{\text{loc}}(\mathbb{R}_+, X)$ with $w > \frac{1}{1-\alpha}$, the last term converges to 0 as $h \to +0$. Hence, $\xi(t + h) - \xi(t) \to 0$, as $h \to +0$.

(iii). **Left-continuity.** Now let $t > 0$ and $h > 0$. Consider

$$\xi(t) - \xi(t - h) = \int_0^t (\omega I - A)^\alpha T(t - s)g(s)ds - \int_0^{t-h} (\omega I - A)^\alpha T(t - h - s)g(s)ds$$

$$= \int_0^{t-h} (\omega I - A)^\alpha \left( T(t - s) - T(t - h - s) \right)g(s)ds$$

$$+ \int_{t-h}^t (\omega I - A)^\alpha T(t - s)g(s)ds.$$  

Then
\[
\|\xi(t) - \xi(t-h)\|_X \\
\leq \left\| \int_0^{t-h} (\omega I - A)^\alpha (T(t-s) - T(t-h-s)) g(s) \, ds \right\|_X \\
\quad + \left\| \int_{t-h}^t (\omega I - A)^\alpha T(t-s) g(s) \, ds \right\|_X \\
\leq \int_0^t \| (\omega I - A)^\alpha (T(t-s) - T(t-h-s)) g(s) \|_{X \mathcal{X}[0,t-h]} \, ds \\
\quad + \left\| \int_{t-h}^t (\omega I - A)^\alpha T(t-s) g(s) \, ds \right\|_X.
\]

It holds for all \( s \in [0, t-h) \) that
\[
\| (\omega I - A)^\alpha (T(t-s) - T(t-h-s)) g(s) \|_{X \mathcal{X}[0,t-h]} \\
\leq \left( \| (\omega I - A)^\alpha T(t-s) \| + \| (\omega I - A)^\alpha T(t-h-s) \| \right) \| g(s) \|_X \\
\leq K \left( \frac{1}{(t-s)^\alpha} + \frac{1}{(t-h-s)^\alpha} \right) \| g(s) \|_X,
\]
where \( K \) depends on \( t \), but does not depend on \( s, h \). As in the above arguments, the function in the rhs is integrable, and according to the (scalar) Lebesgue theorem on dominated convergence, it holds that
\[
\lim_{h \to +0} \int_0^{t-h} \| (\omega I - A)^\alpha (T(t-s) - T(t-h-s)) g(s) \|_X \, ds \\
= \int_0^t \lim_{h \to +0} \| (\omega I - A)^\alpha (T(t-s) - T(t-h-s)) g(s) \|_X \, ds = 0.
\]

Finally,
\[
\left\| \int_{t-h}^t (\omega I - A)^\alpha T(t-s) g(s) \, ds \right\|_X \to 0, \quad \text{as } h \to +0,
\]
as in the proof of the right-continuity. This shows that \( \| x(t+h) - x(t) \|_X \to 0 \) as \( h \to +0 \), and hence \( x \in C([0, \tau], X) \).

(iii). The estimate \((39)\). For the last claim we take \( g \in L^\infty_{\text{loc}}(\mathbb{R}^+, X) \) and continue the estimates in \((41)\) as follows:
\[
\int_0^t \| (\omega I - A)^\alpha T(t-s) g(s) \|_X \, ds \leq M C \alpha e^{\kappa t} \int_0^t \frac{1}{(t-s)^\alpha} \, ds \| g \|_{L^\infty([0,t], X)},
\]
and we obtain \((39)\) with \( R = \frac{MC}{1-\alpha} \). \( \Box \)

Next, we formulate a sufficient condition for the zero-class admissibility of input operators for analytic systems. Part (i) of the following proposition is contained in [33] Proposition 2.13. As we would like to show (ii), we give, however, the proof.

**Proposition 5.5.** Assume that \( A \) generates an analytic semigroup \( T \) and \( B \in L(U, X_{-\alpha}) \) for some \( \alpha \in (0, 1) \). Then:

(i) \( B \) is zero-class \( q \)-admissible for \( q \in \left( \frac{1}{\alpha}, +\infty \right] \).

(ii) For any \( \omega > \omega_0(T), \) any \( d \in [0, \alpha) \), and any \( \kappa > \omega_0(T) \) there is \( R > 0 \) such that for any \( g \in L^\infty_{\text{loc}}(\mathbb{R}^+, X) \) the map
\[
t \mapsto \int_0^t (\omega I - A)^\alpha T(t-s) B g(s) \, ds
\]
is continuous in $X$-norm, and the following holds:

\begin{equation}
\int_0^t \| (\omega I - A)^d T(t-s) B g(s) \|_X ds \leq R t^{\alpha-d} e^{\kappa t} \| g \|_{L^\infty([0,t],X)}.
\end{equation}

**Proof.** (i). Take any $\omega > \omega_0(T)$, and consider the corresponding norm on $X_{-1+\alpha}$ as in Definition A.9. We have

\begin{equation}
\| B \|_{L(U,X_{-1+\alpha})} = \sup_{u \in U : \| u \|_U = 1} \| Bu \|_{X_{-1+\alpha}} = \sup_{u \in U : \| u \|_U = 1} \| (\omega I - A)^{-1+\alpha} Bu \|_X = \| (\omega I - A)^{-1+\alpha} B \|_{L(U,X)},
\end{equation}

thus the condition $B \in L(U,X_{-1+\alpha})$ is equivalent to $(\omega I - A)^{-1+\alpha} B \in L(U,X)$. With this in mind, we have

\begin{equation}
T_{-1}(t) B = T_{-1}(t)(\omega I - A)^{-1+\alpha}(\omega I - A)^{-1+\alpha} B.
\end{equation}

Due to [45, Theorem 2.6.13, p. 74], on $X_{-1+\alpha} = D((\omega I - A)^{-1+\alpha})$ it holds that

\begin{equation}
T_{-1}(t)(\omega I - A)^{-1+\alpha} = (\omega I - A)^{-1+\alpha} T_{-1}(t).
\end{equation}

Now take any $f \in L^q_{\text{loc}}(\mathbb{R}^+,U)$ with $q > \frac{1}{\alpha}$. Representing

\begin{equation}
\int_0^t T_{-1}(t-s) B f(s) ds = \int_0^t (\omega I - A)^{-1+\alpha}_s T_{-1}(t-s)(\omega I - A)^{-1+\alpha} B f(s) ds
\end{equation}

and applying Proposition 5.4 and in particular the estimate (42) with $1 - \alpha$ instead of $\alpha$, $w := q$, and with $g := (\omega I - A)^{-1+\alpha} B f$ we see that $B$ is zero-class $q$-admissible for $q \in \left[\frac{1}{\alpha}, \infty\right)$.

(ii). Similarly, for $d < \alpha$ and $g \in L^\infty_{\text{loc}}(\mathbb{R}^+,U)$, consider the map

\begin{equation}
s \mapsto (\omega I - A)^d T_{-1}(t-s) B g(s) = (\omega I - A)^{1-\alpha+d} T_{-1}(t-s)(\omega I - A)^{-1+\alpha} B g(s).
\end{equation}

By Proposition 5.4, this map is Bochner integrable and in view of (49) with $1 - \alpha + d$ instead of $\alpha$ and $(\omega I - A)^{-1+\alpha} B g$ instead of $g$, we see that the map (43) is continuous and (44) holds. Setting $d := 0$ in (44), we obtain also zero-class $\infty$-admissibility of $B$. $\square$

5.3. **Local existence and uniqueness.** Our next result is the local existence and uniqueness theorem for analytic systems with initial states in $X_\alpha$ and the inputs in $U := L^\infty_{\text{loc}}(\mathbb{R}^+,U)$. Recall the notation $U_S$ from (12).

**Theorem 5.6** (Picard-Lindelöf theorem for analytic systems). Let Assumption 5.1 hold. Assume that $T$ is an analytic semigroup, satisfying for certain $M \geq 1$, $\lambda > 0$ the estimate

\begin{equation}
\| T(t) \| \leq M e^{\lambda t}, \quad t \geq 0.
\end{equation}

For any compact set $Q \subset X_\alpha$, any $r > 0$, any bounded set $S \subset U$, and any $\delta > 0$, there is a time $t_1 = t_1(Q,r,S,\delta) > 0$, such that for any $x_0 \in W := B_{r,X_\alpha}(w)$ for some $w \in Q$, and for any $u \in U_S$ there is a unique mild solution of (1) on $[0,t_1]$, and it lies in the ball $B_{M r + \delta, X_\alpha}(w)$.

**Proof.** First, we show the claim for the case if $Q = \{w\}$ is a single point in $X_\alpha$.

(i). Take any $\omega > \omega_0(T)$, and consider the corresponding space $X_\alpha$. Pick any $C > 0$ such that $W := B_{r,X_\alpha}(w) \subset B_{C,X_\alpha}$, and $U_S \subset B_{C,U}$. Pick any $u \in U_S$. Take also any $\delta > 0$, and consider the following sets (depending on a parameter $t > 0$):

\begin{equation}
Y_t := \{ y \in C([0,t],X) : \| y(s) - (\omega I - A)^{\alpha} w \|_X \leq M r + \delta \quad \forall s \in [0,t] \},
\end{equation}

[...continued...]
enforced with the metric \( p_t(y_1, y_2) := \sup_{s \in [0,t]} \| y_1(s) - y_2(s) \|_X \). As the sets \( Y_t \) are closed subsets of Banach spaces \( C([0,t], X) \), \( Y_t \) are complete metric spaces for all \( t > 0 \).

(ii). Pick any \( x_0 \in W \). We are going to prove that for small enough \( t \) the spaces \( Y_t \) are invariant under the operator \( \Phi_u \), defined for any \( y \in Y_t \) and all \( t \in [0, t] \) by

\[
\Phi_u(y)(\tau) = (\omega I - A)^\alpha T(\tau)x_0 + \int_0^\tau (\omega I - A)^\alpha T_{\tau-s}(\tau-s)Bu(s)ds
\]

(46) \[ + \int_0^\tau (\omega I - A)^\alpha T(\tau-s)f((\omega I - A)^{-\alpha}y(s), u(s))ds. \]

Since \( y \in C([0,t], X) \), the map \( s \mapsto (\omega I - A)^{-\alpha}y(s) \) is in \( C([0,t], X_\alpha) \), as for any \( s_1, s_2 \in [0,t] \) we have that

\[
\| (\omega I - A)^{-\alpha}y(s_1) - (\omega I - A)^{-\alpha}y(s_2) \|_{X_\alpha} = \| y(s_1) - y(s_2) \|_X.
\]

By Assumption 5.1, the map \( s \mapsto f((\omega I - A)^{-\alpha}y(s), u(s)) \) is in \( L^\alpha_{\text{loc}}(\mathbb{R}_+, X) \), with \( w > \frac{1}{1-\alpha} \). Proposition 5.4 ensures, that the map

\[
\tau \mapsto \int_0^\tau (\omega I - A)^\alpha T(\tau-s)f((\omega I - A)^{-\alpha}y(s), u(s))ds
\]

is continuous.

Since \( B \in L(U, X_{1+\alpha+t}) \), Proposition 5.5(ii) implies that

\[
\tau \mapsto \int_0^\tau (\omega I - A)^\alpha T_{\tau-s}(\tau-s)Bu(s)ds
\]

belongs to \( C([0,t], X) \).

Overall, the function \( \Phi_u(y) \) is continuous, and thus \( \Phi_u \) maps \( C([0,t], X) \) to \( C([0,t], X) \).

(iii). Fix any \( t > 0 \) and pick any \( y \in Y_t \). As \( x_0 \in W = B_{r,X_\alpha}(w) \), there is \( a \in X_\alpha \) with \( |a|_{X_\alpha} < r \) such that \( x_0 = w + a \).

Then for any \( \tau < t \), we obtain that

\[
\| \Phi_u(y)(\tau) - (\omega I - A)^\alpha w \|_X
\]

\[
\leq \left\| (\omega I - A)^\alpha T(\tau)x_0 - (\omega I - A)^\alpha w \right\|_X + \left\| \int_0^\tau (\omega I - A)^\alpha T_{\tau-s}(\tau-s)Bu(s)ds \right\|_X
\]

\[ + \int_0^\tau \left\| (\omega I - A)^\alpha T(\tau-s) \right\| \left\| f((\omega I - A)^{-\alpha}y(s), u(s)) \right\|_X ds. \]

We substitute \( x_0 := w + a \) into the first term. To estimate the second term, we use that \( (\omega I - A)^\alpha B \in L(U, X_{1+\alpha}) \). Thus, by Proposition 5.5 \( (\omega I - A)^\alpha B \) is zero-class \( \infty \)-admissible, and thus there is an increasing continuous function \( t \mapsto h_t \) satisfying \( h_0 = 0 \), such that:

\[
\Phi_t(y)(\tau) - (\omega I - A)^\alpha w \leq (\omega I - A)^\alpha T(\tau)w - (\omega I - A)^\alpha w \leq (\omega I - A)^\alpha T(\tau)w - (\omega I - A)^\alpha w \leq (\omega I - A)^\alpha T(\tau)w - (\omega I - A)^\alpha w \leq (\omega I - A)^\alpha T(\tau)w - (\omega I - A)^\alpha w
\]

(47) \[ + \left\| (\omega I - A)^\alpha T(\tau)a \right\|_X + h_\tau \| u \|_{L^\infty([0,\tau], U)} \]

\[ + \int_0^\tau C_\alpha e^{\lambda(\tau-s)} \left( \left\| f(0, u(s)) \right\|_X + \left\| f((\omega I - A)^{-\alpha}y(s), u(s)) - f(0, u(s)) \right\|_X \right) ds. \]

To estimate the latter expression, note that

- \( \| (\omega I - A)^\alpha a \|_X = \| a \|_{X_\alpha} < r \).
\(\bullet\) For all \(s \in [0, t]\) we have
\[
\| (\omega I - A)^{-\alpha} y(s) - 0 \|_{X_\alpha} = \| y(s) \|_X \leq \| (\omega I - A)^{\alpha} w \|_X + Mr + \delta
\leq M(\| w \|_{X_\alpha} + r) + \delta \leq MC + \delta := K.
\]

\(\bullet\) In view of Assumption \([5.1]\), it holds that
\[
\| f(0, u(s)) \|_X \leq \sigma(\| u(s) \|_{V'}) + c, \quad \text{for a.e. } s \in [0, t].
\]

\(\bullet\) \(h\) is a monotonically increasing continuous function.

As \(M \geq 1\), it holds that \(K > C\), and Lipschitz continuity of \(f\) on bounded balls ensures that there is \(L(K) > 0\), such that for all \(\tau \in [0, t]\)
\[
\| \Phi_\tau(y) - (\omega I - A)^{\alpha} w \|_X
\leq \| (\omega I - A)^{\alpha} T(\tau) w - (\omega I - A)^{\alpha} w \|_X + \| T(\tau) (\omega I - A)^{\alpha} w \|_X + h_\tau \| u \|_{L^\infty([0, \tau], U)}
+ \int_0^\tau \frac{C_\alpha}{(\tau - s)^{\alpha}} \| \Phi(s) \|_{U} \| (\omega I - A)^{-\alpha} y(s) \|_{X_\alpha} ds
\leq \sup_{\tau \in [0, t]} \| T(\tau) (\omega I - A)^{\alpha} w - (\omega I - A)^{\alpha} w \|_X + Me^{\lambda \tau} r + h_\tau \| u \|_{L^\infty([0, \tau], U)}
+ C_\alpha e^{\lambda t} (\sigma(C) + c + L(K)K) \int_0^t \frac{1}{s^{\alpha}} ds.
\]

Since \(T\) is a strongly continuous semigroup, and \(h_\tau \to 0\) as \(t \to +0\), from this estimate, it is clear that there exists \(t_1\), such that
\[
\| \Phi_\tau(y)(t) - w \|_X \leq Mr + \delta, \quad \text{for all } t \in [0, t_1].
\]

This means, that \(Y_t\) is invariant with respect to \(\Phi_\tau\) for all \(t \in (0, t_1]\), and \(t_1\) does not depend on the choice of \(x_0 \in W\).

(iv). Now pick any \(t > 0, \tau \in [0, t]\), and any \(y_1, y_2 \in Y_t\). Then it holds that
\[
\| \Phi_\tau(y_1)(\tau) - \Phi_\tau(y_2)(\tau) \|_X
\leq \int_0^\tau \| (\omega I - A)^{\alpha} T(\tau - s) \|
\cdot \| f((\omega I - A)^{-\alpha} y_1(s), u(s)) - f((\omega I - A)^{-\alpha} y_2(s), u(s)) \|_X ds
\leq L(K) \sup_{\tau \in [0, t]} \| (\omega I - A)^{\alpha} T(\tau) \| \int_0^t \| y_1(s) - y_2(s) \|_X ds
\leq L(K) \sup_{\tau \in [0, t]} \| (\omega I - A)^{\alpha} T(\tau) \| t_\rho(y_1, y_2)
\leq \frac{1}{2} \rho_\tau(y_1, y_2),
\]
for \(t \leq t_2\), where \(t_2 > 0\) is a small enough real number, that does not depend on the choice of \(x_0 \in W\).

According to Banach fixed point theorem, there exists a unique \(y \in Y_t\) that is a fixed point of \(\Phi_\tau\), that is
\[
y(\tau) = (\omega I - A)^{\alpha} T(\tau)x_0 + \int_0^\tau (\omega I - A)^{\alpha} T_{\tau - s} Bu(s) ds
+ \int_0^\tau (\omega I - A)^{\alpha} T(\tau - s) f((\omega I - A)^{-\alpha} y(s), u(s)) ds.
\]
(48)
on \([0, \min\{t_1, t_2\}]\).
As \((\omega I - A)^\alpha\) is invertible with a bounded inverse, \(y\) solves (49) if and only if \(y\) solves
\[
(\omega I - A)^{-\alpha} y(\tau) = T(\tau) x_0 + \int_0^\tau T_{\tau-s} B u(s) \, ds + \int_0^\tau T(\tau-s) f((\omega I - A)^{-\alpha} y(s), u(s)) \, ds.
\]
As \(y \in C([0, \min\{t_1, t_2\}], X),\) the map \(x := (\omega I - A)^{-\alpha} y\) is in \(C([0, \min\{t_1, t_2\}], X_\alpha)\), and is the unique mild solution of (35).

\[\text{(v). General compact } Q.\] Similar to the corresponding part of the proof of Theorem 3.7.

\[\square\]

\textbf{Remark 5.7.} We have proved our local existence result for initial conditions that are in \(X_\alpha\). To ensure local existence and uniqueness for the initial states outside of \(X_\alpha\), stronger requirements on \(f\) have to be imposed, see [15, Theorems 7.1.5, 7.1.6].

Introducing the concepts of maximal solutions and of well-posedness and arguing similar to Sections 3.2, 3.3 we obtain the following well-posedness theorem.

\textbf{Theorem 5.8.} Let \(A\) generate an analytic semigroup, Assumption 5.7 hold, and let \(U := L^\infty(\mathbb{R}_+, U)\). Then:

\(\text{(i) For each } x \in X_\alpha\) and each \(u \in U\), there is a unique maximal solution of (55) defined over \([0, t_m(x, u)]\). We denote this solution as \(\phi(\cdot, x, u)\).

\(\text{(ii) The triple } \Sigma := (X_\alpha, U, \phi)\text{ is a well-defined control system in the sense of Definition 2.1.}\)

\(\text{(iii) } \Sigma\text{ satisfies the BIC property, that is if for a certain } x \in X_\alpha\) and \(u \in U\) we have \(t_m(x, u) < \infty\), then \(\|\phi(t, x, u)\|_{X_\alpha} \to \infty\) as \(t \to t_m(x, u)\).

5.4. Global existence. Now we are going to derive sufficient conditions for forward completeness and BRS property.

\textbf{Proposition 5.9.} For any \(a, b \geq 0\), any \(\alpha, \beta \in [0, 1]\), and any \(T \in (0, +\infty)\) there is \(M = M(b, \alpha, \beta, T) > 0\) such that for any integrable function \(u : [0, T] \to \mathbb{R}\) satisfying for almost all \(t \in [0, T]\) the inequality
\[
0 \leq u(t) \leq a t^{-\alpha} + b \int_0^t (t-s)^{-\beta} u(s) \, ds
\]
it holds for a.e. \(t \in [0, T]\) that
\[
0 \leq u(t) \leq a M t^{-\alpha}.
\]

\textbf{Proof.} See [15] p. 6.

\[\square\]

Motivated by [15] Section 6.3, Theorem 3.3, we have the following result guaranteeing the forward completeness and BRS property for semilinear analytic systems.

\textbf{Theorem 5.10.} Let \(A\) generate an analytic semigroup, Assumption 5.7 hold, and let \(U := L^\infty(\mathbb{R}_+, U)\). Assume further that there are \(L, c > 0\) and \(\sigma \in K_{\infty}\) such that
\[
\|f(x, u)\|_X \leq L \|x\|_{X_\alpha} + \sigma(\|u\|_U) + c, \quad x \in X_\alpha, \quad u \in U.
\]

Then \(\Sigma := (X_\alpha, U, \phi)\) is a forward complete control system.
Proof. Take any positive \( \omega > \omega_0(T) \) and define \( X_\alpha \) as in (52).

We argue by a contradiction. Let \( \Sigma \) be not forward complete. Then there are \((x_0,u) \in X_\alpha \times \mathcal{U}\) such that \( t_m(x_0,u) < \infty \). By Theorem 5.8 we have that \( \| \phi(t,x_0,u) \|_{X_\alpha} \to \infty \) as \( t \to t_m(x_0,u) - 0 \).

For \( t < t_m(x_0,u) \) denote \( x(t) := \phi(t,x_0,u) \). As \( x(\cdot) \subset X_\alpha \), we can apply \((\omega I - A)^\alpha\) along the trajectory \( x(\cdot) \) to obtain

\[
(\omega I - A)^\alpha x(t) = (\omega I - A)^\alpha T(t)x_0 + \int_0^t (\omega I - A)^\alpha T(t-s)Bu(s)ds + \int_0^t (\omega I - A)^\alpha T(t-s)f(x(s),u(s))ds.
\]

Applying (63), we obtain

\[
\| x(t) \|_\alpha = \| (\omega I - A)^\alpha x(t) \|_X \\
\leq \| (\omega I - A)^\alpha T(t)x_0 \|_X + \int_0^t (\omega I - A)^\alpha T(t-s)Bu(s)ds \|_X \\
+ \int_0^t (\omega I - A)^\alpha T(t-s)\| f(x(s),u(s)) \|_X ds \\
\leq M e^{\omega t}(\| (\omega I - A)^\alpha x_0 \|_X + h_\tau \| u \|_{L^\infty([0,t],U)} \\
+ \int_0^t C_\alpha \| u \|_{L^\infty(\mathbb{R}+,U)} + \sigma(T,\omega_\alpha) + c \big) ds.
\]

Here the last term we have estimated as in (47), where \( h \) is a continuous increasing function with \( h_0 = 0 \). Defining \( z(t) := x(t)e^{-\omega t} \), we obtain from the previous estimate that

\[
\| z(t) \|_\alpha \leq M (\| (\omega I - A)^\alpha x_0 \|_X + \int_0^t \frac{C_\alpha}{s^\alpha} ds \big( \sigma(T,\omega_\alpha) + c \big) + h_\tau \| u \|_{L^\infty(\mathbb{R}+,U)} \\
+ \int_0^t \frac{LC_\alpha}{(t-s)^\alpha} \| z(s) \|_{X_\alpha} ds.
\]

Proposition 5.9 shows that \( z \), and hence \( x \), is uniformly bounded on \([0,t_m(x_0,u))\), and BIC property (Theorem 5.8(iii)) shows that \( t_m(x_0,u) \) is not the finite maximal existence time. A contradiction. \( \square \)

5.5. Example: well-posedness of a Burger’s equation with a distributed input. Motivated by [15] p. 57, we consider the following semilinear reaction-diffusion equation of Burgers’ type on a domain \([0,\pi]\), with distributed input \( u \), boundary input \( d \) at \( z = 0 \), and homogeneous Dirichlet boundary condition at \( \pi \).

\[
\begin{align*}
(51a) \quad x_t &= x_{zz} - xx_z + f(z,x(z,t)) + u(z,t), \quad z \in (0,\pi), \quad t > 0, \\
(51b) \quad x(0,t) &= d(t), \quad t > 0, \\
(51c) \quad x(\pi,t) &= 0.
\end{align*}
\]

Here \( f : [0,\pi] \times \mathbb{R} \to \mathbb{R} \) is measurable in \( z \), locally Lipschitz continuous in \( x \) uniformly in \( z \), and

\[
f(z,y) = h(z)g(|y|), \quad \text{for a.e.} \quad z \in [0,\pi], \quad \text{and all} \quad y \in \mathbb{R},
\]

where \( h \in L^2(0,\pi) \), and \( g \) is continuous, increasing, and both \( h \) and \( g \) are positive.

We denote \( X := L^2(0,\pi) \). The operator \( A := \frac{d^2}{dx^2} \) with the domain \( D(A) = H^2(0,\pi) \cap H_0^1(0,\pi) \) generates an analytic semigroup on \( X \).
We assume that the distributed input $u$ belongs to the space $U = L^\infty(\mathbb{R}_+, U)$, with $U := L^2(0, \pi)$, and the boundary input $d$ belongs to $D := L^\infty(\mathbb{R}_+, \mathbb{R})$.

The system (51) can be reformulated as a semilinear evolution equation
\begin{equation}
x_t = Ax + F(x) + u + Bd,
\end{equation}
where we slightly abuse the notation and use $x$ as an argument of the evolution equation.

The condition (ii) in Assumption 5.1 characterizing the admissibility properties of the boundary input operator $B$ holds in view of [53, Example 2.16].

The space $X_{1 \frac{1}{2}}$ corresponding to the operator $A$, is given by (see [58, Proposition 3.6.1])
\[ X_{1 \frac{1}{2}} = H^1_0(0, \pi), \]
which is a Banach space with the norm
\[ \|x\|_{1 \frac{1}{2}} := \left| \int_0^\pi |x'(z)|^2 dz \right|^{\frac{1}{2}}, \quad x \in X_{1 \frac{1}{2}}. \]

The nonlinearity $F : X_{1 \frac{1}{2}} \to X$ in (53) is given by
\[ F(x)(z) = -x(z)x'(z) + f(z, x(z)). \]

**Proposition 5.11.** For each $x_0 \in X_{1 \frac{1}{2}}$, each $u \in U = L^\infty_{loc}(\mathbb{R}_+, U)$, and each boundary input $d \in D = L^\infty_{loc}(\mathbb{R}_+, \mathbb{R})$ the system (51) possesses a unique maximal mild solution $\phi(\cdot, x_0, (u,d))$. The system $\Sigma = (X_{1 \frac{1}{2}}, U \times D, \phi)$ is a control system satisfying the BIC property.

**Proof.** We proceed in 3 steps:

**Step 1:** $F$ maps bounded sets of $X_{1 \frac{1}{2}}$ to bounded sets of $X$. Since the elements of $X_{1 \frac{1}{2}} = H^1_0(0, \pi)$ are absolutely continuous functions, using the Cauchy-Schwarz inequality, we obtain that for any $x \in X_{1 \frac{1}{2}}$ it holds that
\begin{equation}
\sup_{z \in (0, \pi)} |x(z)| = \sup_{z \in (0, \pi)} \left| \int_0^z x'(z)dz \right| \\
\leq \sup_{z \in (0, \pi)} \int_0^z |x'(z)|dz \\
= \int_0^\pi |x'(z)|dz \\
\leq \left| \int_0^\pi 1dz \right|^\frac{1}{2} \left| \int_0^\pi |x'(z)|^2 dz \right|^\frac{1}{2} \\
= \sqrt{\pi} \|x\|_{1 \frac{1}{2}}.
\end{equation}

For any $x \in X_{1 \frac{1}{2}}$ consider
\begin{equation}
\|F(x)\|_X^2 = \int_0^\pi |F(x)(z)|^2 dz \\
= \int_0^\pi |x(z)x'(z) + f(z, x(z))|^2 dz \\
\leq \int_0^\pi 2|x(z)x'(z)|^2 + 2|f(z, x(z))|^2 dz.
\end{equation}
Using (54) and (52), we continue the estimates as follows:
\[
\|F(x)\|_X^2 \leq \int_0^\pi 2|x'(z)|^2\pi \|x\|^2_2 + 2|h(z)|^2|g(|x(z)|)|^2\,dz \\
\leq 2\pi\|x\|^4_2 + 2\int_0^\pi |h(z)|^2|g(\sqrt{\pi}\|x\|_\frac{1}{2})|^2\,dz \\
= 2\pi\|x\|^4_2 + 2\|h\|_\frac{1}{2}^4\|g(\sqrt{\pi}\|x\|_\frac{1}{2})\|^2.
\]
Taking the square root and using that \(\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}\), for all \(a, b \geq 0\), we finally obtain
\[
\|F(x)\|_X \leq \sqrt{2\pi\|x\|^2_2 + 2\|h\|_\frac{1}{2}^4\|g(\sqrt{\pi}\|x\|_\frac{1}{2})\|^2}.
\]
This shows that \(F\) is well-defined as a map from \(X_\frac{1}{2}\) to \(X\) and \(F\) maps bounded sets of \(X_\frac{1}{2}\) to bounded sets of \(X\).

**Step 2: \(F\) is Lipschitz continuous in \(x\).** For \(x_0 \in X_\frac{1}{2}\), there is a neighborhood \(V\) of a compact set \(\{ (x, x_0(z)) : z \in [0, \pi]\} \) in \([0, \pi] \times \mathbb{R}\) and positive constants \(L, \theta\) so that for \((z, x_1) \in V\), \((z, x_2) \in V\) it holds that
\[
|f(z, x_1) - f(z, x_2)| \leq L|x_1 - x_2|.
\]
Thus, there is a neighborhood \(U\) of \(x_0\) in \(X_\frac{1}{2}\) such that \(x \in U\) implies that \((z, x(z)) \in V\) for a.e. \(z \in [0, \pi]\) and for \(x_1, x_2 \in U\) it holds that
\[
\left\| f(\cdot, x_1(\cdot)) - f(\cdot, x_2(\cdot)) \right\|_X = \int_0^\pi \left| f(z, x_1(z)) - f(z, x_2(z)) \right|^2\,dz \\
\leq L^2 \int_0^\pi |x_1(z) - x_2(z)|^2\,dz,
\]
and using (54) we have that
\[
\left\| f(\cdot, x_1(\cdot)) - f(\cdot, x_2(\cdot)) \right\|_X \leq L^2\pi^2\|x_1 - x_2\|_\frac{1}{2}^2.
\]
Taking the square root, we have that
\[
\left\| f(\cdot, x_1(\cdot)) - f(\cdot, x_2(\cdot)) \right\|_X \leq \pi L\|x_1 - x_2\|_\frac{1}{2}.
\]
Finally, for any \(x_1, x_2 \in X_\frac{1}{2}\) it holds that
\[
\|x_1x_1' - x_2x_2'\|_X \leq \|x_1(x_1' - x_2')\|_X + \|(x_1 - x_2)x_2'\|_X,
\]
and again using (54), we proceed to
\[
\|x_1x_1' - x_2x_2'\|_X \leq \sqrt{\pi}\|x_1\|_\frac{1}{2}\|x_1 - x_2\|_\frac{1}{2} + \sqrt{\pi}\|x_1 - x_2\|_\frac{1}{2}\|x_2\|_\frac{1}{2} \\
= \sqrt{\pi}(\|x_1\|_\frac{1}{2} + \|x_2\|_\frac{1}{2})\|x_1 - x_2\|_\frac{1}{2}.
\]
Combining (56) and (57), we obtain the required Lipschitz property for the function \(F\).

**Step 3: Application of general well-posedness theorems.** Finally, Theorem 5.9 shows that the system [51] possesses a unique mild solution for each \(x_0 \in X_\frac{1}{2}\), each \(u \in U = L^\infty_{loc}(\mathbb{R}_+, U)\), and each boundary input \(d \in D = L^\infty_{loc}(\mathbb{R}_+, \mathbb{R})\). Theorem 5.8 shows that \(\Sigma\) is a control system satisfying the BIC property.

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APPENDIX A. RECAP ON FRACTIONAL POWERS OF OPERATORS

In this section, we make a short recap of the fractional powers of sectorial operators, that we need in Section 5. A somewhat more detailed account of the properties of fractional powers of sectorial operators can be found, e.g., in [15, Section 1.4] and for a detailed treatment, we refer to [13].

Definition A.1. For $z \in \mathbb{C}$ with $\text{Re} \, z > 0$ the gamma function is defined as

\[ \Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \, dt. \]  

(58)

Definition A.2. Let $A$ be the generator of an exponentially stable analytic semigroup over a Banach space $X$. For $\alpha > 0$ define

\[ (-A)^{-\alpha} := \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} T(t) \, dt. \]  

(59)

As the semigroup $T$ is analytic, the map $t \to T(t)$ is differentiable on $(0, +\infty)$ (see, e.g., [12, Theorem 4.6, p. 101]), and hence continuous on this interval. As $T$ is also exponentially stable, the integral in (59) converges in the uniform operator topology. The following well-known property holds:

Proposition A.3. Let $A$ be the generator of an exponentially stable analytic semigroup over a Banach space $X$. For any $\alpha > 0$ the operator $(-A)^{-\alpha}$ belongs to $L(X)$, is injective and satisfies

\[ (-A)^{-\alpha}(-A)^{-\beta} = (-A)^{-(\alpha+\beta)}, \quad \alpha, \beta > 0. \]  

(60)

Definition A.4. Let $A$ be the generator of an exponentially stable analytic semigroup over a Banach space $X$. For $\alpha > 0$ define $(-A)^{\alpha}$ as the inverse of $(-A)^{-\alpha}$ with $\text{D}((-A)^{\alpha}) = \text{Im}((-A)^{-\alpha})$.

By definition, we set $A^0 := 1$.

Let us collect several basic properties of the fractional powers; see [45, Theorem 6.8, p. 72] for the first three items, and [15] Exercises, p. 26] for the last one.
Proposition A.5. Let $A$ be the generator of an exponentially stable analytic semigroup over a Banach space $X$.

(i) For $\alpha > 0$ the operators $(-A)^\alpha$ are closed and densely defined.

(ii) Whenever $\alpha > \beta > 0$, we have $D((-A)^\alpha) \subset D((-A)^\beta)$.

(iii) For all $\alpha, \beta \in \mathbb{R}$ it holds that

$$(-A)^{\alpha+\beta}x = (-A)^\alpha(-A)^\beta x,$$

for every $x \in D((-A)^\gamma)$, where $\gamma = \max\{\alpha, \beta, \alpha + \beta\}$.

(iv) $(-A)^\alpha T(t) = T(t)(-A)^\alpha$ on $D((-A)^\alpha)$, $t \geq 0$.

Using fractional powers of operators, we can define the spaces that will be important to deal with nonlinear equations governed by analytic semigroups.

Definition A.6. Let $A$ be the generator of an analytic semigroup over a Banach space $X$. Let $\alpha \geq 0$. Pick any $\omega > \omega_0(T)$ and define the space $X_\alpha$ and its norm as

$$X_\alpha := D((\omega I - A)^\alpha), \quad \|x\|_{X_\alpha} := \| (\omega I - A)^\alpha x \|_X, \quad x \in X_\alpha.$$  

In particular, by definition $X_0 = X$, and $X_1 = D(A)$ (with norms as in (62)).

Remark A.7. The choice of different $\omega > \omega_0(T)$ induces the same linear space $X_\alpha$ endowed with an equivalent norm.

The spaces $X_\alpha$ have a good structure:

Proposition A.8. Let $A$ be the generator of an exponentially stable analytic semigroup over a Banach space $X$. For any $\alpha \geq 0$, the space $X_\alpha$ is a Banach space. Furthermore, for all $\alpha \geq \beta \geq 0$ the space $X_\alpha$ is a dense subspace of $X_\beta$, with continuous inclusion.

Similarly to the space $X_{-1}$, we introduce

Definition A.9. For $\alpha > 0$ and $A$ as above, define the spaces $X_{-\alpha}$ as the completion of $X$ with respect to the norm $x \mapsto \| (\omega I - A)^{-\alpha} x \|_X$. Then we have for any $0 < \beta < \alpha < 1$ the following chain of continuous inclusions:

$$X_{-1} \supset X_{-\alpha} \supset X_{-\beta} \supset X = X_0 \supset X_\beta \supset X_\alpha \supset X_1.$$  

The following property holds:

Proposition A.10. Let $T$ be an analytic semigroup on a Banach space $X$ with a growth bound $\omega_0(T)$ and the generator $A$. Then for each $\omega, \kappa > \omega_0(T)$ and each $\alpha \in (0, 1)$ we have $\text{Im}(T(t)) \subset X_\alpha$, and there is $C_\alpha > 0$ such that

$$\| (\omega I - A)^\alpha T(t) \| \leq \frac{C_\alpha}{t^\alpha} e^{\kappa t}, \quad t > 0.$$  

Furthermore, the map $t \mapsto (\omega I - A)^\alpha T(t)$ is continuous on $(0, +\infty)$ in the uniform operator topology.