CENTRO-AFFINE NORMAL FLOWS ON CURVES:
HARNACK ESTIMATES AND ANCIENT SOLUTIONS

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ABSTRACT. We prove that the only compact, origin-symmetric, strictly convex ancient solutions of the planar p centro-affine normal flows are contracting origin-centered ellipses.

1. Introduction

The setting of this paper is the two-dimensional Euclidean space, \( \mathbb{R}^2 \). A compact convex subset of \( \mathbb{R}^2 \) with non-empty interior is called a convex body. The set of smooth, strictly convex bodies in \( \mathbb{R}^2 \) is denoted by \( K \). Write \( K_0 \) for the set of smooth, strictly convex bodies whose interiors contain the origin of the plane.

Let \( K \) be a smooth, strictly convex body \( \mathbb{R}^2 \) and let \( X_K : \partial K \rightarrow \mathbb{R}^2 \) be a smooth embedding of \( \partial K \), the boundary of \( K \). Write \( S^1 \) for the unit circle and write \( \nu : \partial K \rightarrow S^1 \) for the Gauss map of \( \partial K \). That is, at each point \( x \in \partial K \), \( \nu(x) \) is the unit outwards normal at \( x \). The support function of \( K \in K_0 \) as a function on the unit circle is defined by \( s(z) := \langle X(\nu^{-1}(z)), z \rangle \), for each \( z \in S^1 \). We denote the curvature of \( \partial K \) by \( \kappa \) which as a function on \( \partial K \) is related to the support function by

\[
1 / \kappa(\nu^{-1}(z)) := \tau(z) = \frac{\partial^2}{\partial \theta^2} s(z) + s(z).
\]

Here and afterwards, we identify \( z = (\cos \theta, \sin \theta) \) with \( \theta \). The function \( \tau \) is called the radius of curvature. The affine support function of \( K \) is defined by \( \sigma : \partial K \rightarrow \mathbb{R} \) and \( \sigma(x) := s(\nu(x))^{1/3} (\nu(x)) \). The affine support function is invariant under the group of special linear transformations, \( SL(2) \), and it plays a basic role in our argument.

Let \( K \in K_0 \). A family of convex bodies \( \{K_t\}_t \subset K_0 \) given by the smooth map \( X : \partial K \times [0, T) \rightarrow \mathbb{R}^2 \) is said to be a solution to the \( p \) centro-affine normal flow, in short \( p \)-flow, with the initial data \( X_K \), if the following evolution equation is satisfied:

\[
\partial_t X(x,t) = - \left( \frac{\kappa(x,t)}{\langle X(x,t), \nu(x,t) \rangle^3} \right)^{\frac{p}{p-1}} \kappa^*(x,t) \nu(x,t), \quad X(\cdot,0) = X_K,
\]

for a fixed \( 1 < p < \infty \). In this equation, \( 0 < T < \infty \) is the maximal time that the solution exists, and \( \nu(x,t) \) is the unit normal to the curve \( X(\partial K, t) = \partial K_t \) at \( X(x,t) \). This family of flows for \( p > 1 \) was defined by Stancu [13]. The case \( p = 1 \) is the well-known affine normal flow whose asymptotic behavior was investigated by Sapiro and Tanembaum [12], and by Andrews in a more general
setting [2, 4]: Any convex solution to the affine normal flow, after appropriate rescaling converges to an ellipse in the $C^\infty$ norm. For $p > 1$, similar result was obtained with smooth, origin-symmetric, strictly convex initial data by the author and Stancu [7, 8]. Moreover, ancient solutions of the affine normal flow have been also classified: the only compact, convex ancient solutions of the affine normal flow are contracting ellipsoids. This result in $\mathbb{R}^n$, for $n \geq 3$, was proved by Loftin and Tsui [9] and in dimension two by S. Chen [5], and also by the author with a different method. We recall that a solution of flow is called an ancient solution if it exists on $(-\infty, T)$. Here we classify compact, origin-symmetric, strictly convex ancient solutions of the planar $p$ centro-affine normal flows:

**Theorem.** The only compact, origin-symmetric, strictly convex ancient solutions of the $p$-flows are contracting origin-centred ellipses.

Throughout this paper, we consider origin-symmetric solutions.

2. HARNACK ESTIMATE

In this section, we follow [1] to obtain the Harnack estimates for $p$-flows.

**Proposition.** Under the flow (1.1) we have $\partial_t \left( s^{1-\frac{2p}{p+2}} \frac{r^p}{\alpha} - \frac{r^p}{\alpha} + \frac{1}{\alpha^2} \right) \geq 0$.

**Proof.** For simplicity we set $\alpha = -\frac{p}{p+2}$. To prove the proposition, using the parabolic maximum principle we prove that the quantity defined by

\[
\mathcal{R} := t P - \frac{\alpha}{\alpha - 1} s^{1+3\alpha} \theta^\alpha
\]

remains negative as long as the flow exists. Here $P$ is defined as follows

\[
P := \partial_t \left( -s^{1+3\alpha} \theta^\alpha \right).
\]

**Lemma 2.1.** [7]

- $\partial_t s = -s^{1+3\alpha} \theta^\alpha$.
- $\partial_t \theta = -\left[ (s^{1+3\alpha} \theta^\alpha)_\theta + s^{1+3\alpha} \theta^\alpha \right]$.

Using the evolution equations of $s$ and $\theta$ we find

\[
P = (1+3\alpha)s^{1+6\alpha} \theta^2 + \alpha s^{1+3\alpha} \theta^{-1} \left[ (s^{1+3\alpha} \theta^\alpha)_\theta + s^{1+3\alpha} \theta^\alpha \right]
\]

\[
:= (1+3\alpha)s^{1+6\alpha} \theta^2 + \alpha s^{1+3\alpha} \theta^{-1} \mathcal{Q}.
\]

**Lemma 2.2.** We have the following evolution equation for $P$ as long as the flow exists:

\[
\partial_t P = -\alpha s^{1+3\alpha} \theta^{-1} \left[ P_{\theta\theta} + P \right] + \left[ (3\alpha + 1)(3\alpha + 2) - \frac{(\alpha - 1)(3\alpha + 1)^2}{\alpha} \right] s^{1+9\alpha} \theta^3
\]

\[
+ \left[ -3(3\alpha + 1) + \frac{2(\alpha - 1)(3\alpha + 1)}{\alpha} \right] s^{3\alpha} \theta^\alpha P - \frac{\alpha - 1}{\alpha} \frac{P^2}{s^{1+3\alpha} \theta^\alpha}.
\]
Proof. We repeatedly use the evolution equation of $s$ and $r$ given in Lemma 2.1.

\begin{equation}
\frac{\partial \rho}{\partial t} = -(1 + 3\alpha)(1 + 6\alpha)s^{1+3\alpha}r^{3\alpha} - 2\alpha(1 + 3\alpha)s^{1+6\alpha}r^{2\alpha - 1}\left[ (s^{1+3\alpha}r^\alpha)^{\theta\theta} + s^{1+3\alpha}r^\alpha \right] \\
- \alpha(1 + 3\alpha)s^{1+6\alpha}r^{2\alpha - 1}\left[ (s^{1+3\alpha}r^\alpha)^{\theta\theta} + s^{1+3\alpha}r^\alpha \right] \\
- \alpha(\alpha - 1)s^{1+3\alpha}r^{2\alpha - 2}\left[ (s^{1+3\alpha}r^\alpha)^{\theta\theta} + s^{1+3\alpha}r^\alpha \right]^2 - \alpha s^{1+3\alpha}r^{2\alpha - 1}[\rho_{\theta\theta} + \rho] \\
- \alpha(\alpha - 1)s^{1+3\alpha}r^{2\alpha - 2}Q^2 - \alpha s^{1+3\alpha}r^{2\alpha - 1}[\rho_{\theta\theta} + \rho].
\end{equation}

By the definition of $Q$, (2.2), we have

\begin{equation}
Q^2 = \frac{\rho^2}{\alpha^2s^{2+6\alpha}r^{2\alpha - 2}} - \frac{2(3\alpha + 1) \rho r^2}{\alpha^2s^{6\alpha}r^{2\alpha + 2}} + \frac{(3\alpha + 1)^2}{\alpha^2s^{1+3\alpha}r^{3\alpha}}.
\end{equation}

Substituting these expressions into the evolution equation of $\rho$ we find that

\begin{equation}
\frac{\partial \rho}{\partial t} = -\alpha s^{1+3\alpha}r^{2\alpha - 1}[\rho_{\theta\theta} + \rho] + \left[ (3\alpha + 1)(3\alpha + 2) - \frac{(\alpha - 1)(3\alpha + 1)^2}{\alpha} \right]s^{1+3\alpha}r^{3\alpha} \\
- 3(3\alpha + 1)s^{3\alpha}r^{2\alpha}\rho - \frac{\rho^2}{\alpha s^{1+3\alpha}r^{3\alpha}} + \frac{2(\alpha - 1)(3\alpha + 1)}{\alpha}s^{3\alpha}r^{2\alpha}\rho.
\end{equation}

This completes the proof of Lemma 2.2. \hfill \Box

We now proceed to find the evolution equation of $R$ which is defined by (2.1). First notice that

\begin{equation}
-\alpha s^{1+3\alpha}r^{2\alpha - 1}R_{\theta\theta} = -\alpha s^{1+3\alpha}r^{2\alpha - 1}\rho_{\theta\theta} + \frac{\rho^2}{\alpha - 1}s^{1+3\alpha}r^{2\alpha - 1}(s^{1+3\alpha}r^\alpha)^{\theta\theta}.
\end{equation}
Therefore, by Lemma 2.2 and identity (2.2) we get
\[ \partial_t \mathcal{R} \]
\[ = -t \alpha s^{1+3\alpha} \theta^{-1} \left[ \mathcal{P}_{\theta\theta} + \mathcal{P} \right] + t \left( 3\alpha + 1 \right) \left( 3\alpha + 2 \right) - \alpha - 1 \left( 3\alpha + 1 \right)^2 \right] s^{1+9\alpha} \theta^{3\alpha} + \mathcal{P} + \alpha \mathcal{P} \]
\[ = \alpha s^{1+3\alpha} \theta^{-1} \left( \mathcal{R}_{\theta\theta} + t \alpha s^{1+3\alpha} \theta^{-1} \mathcal{P}_{\theta\theta} \right) - \alpha s^{1+3\alpha} \theta^{-1} \left( 3\alpha + 1 \right) \left( 3\alpha + 2 \right) - \alpha - 1 \left( 3\alpha + 1 \right)^2 \right] s^{1+9\alpha} \theta^{3\alpha} + \mathcal{P} + \alpha \mathcal{P} \]
\[ = -\alpha s^{1+3\alpha} \theta^{-1} \mathcal{R}_{\theta\theta} + t \left( 3\alpha + 1 \right) \left( 3\alpha + 2 \right) - \alpha - 1 \left( 3\alpha + 1 \right)^2 \right] s^{1+9\alpha} \theta^{3\alpha} + \mathcal{P} + \alpha \mathcal{P} \]
\[ + \alpha \mathcal{P} - \alpha \mathcal{P} - \alpha s^{1+3\alpha} \theta^{-1} \mathcal{P} + \alpha \mathcal{P} s^{1+3\alpha} \theta^{-1} \mathcal{P} + \alpha s^{1+3\alpha} \theta^{-1} \left( 3\alpha + 1 \right) \left( 3\alpha + 2 \right) - \alpha - 1 \left( 3\alpha + 1 \right)^2 \right] s^{1+9\alpha} \theta^{3\alpha} + \mathcal{P} + \alpha \mathcal{P} \]
\[ = \alpha s^{1+3\alpha} \theta^{-1} \mathcal{R}_{\theta\theta} + t \left( 3\alpha + 1 \right) \left( 3\alpha + 2 \right) - \alpha - 1 \left( 3\alpha + 1 \right)^2 \right] s^{1+9\alpha} \theta^{3\alpha} + \mathcal{P} + \alpha \mathcal{P} \]
\[ - t \alpha s^{1+3\alpha} \theta^{-1} \mathcal{P} + \alpha \mathcal{P} s^{1+3\alpha} \theta^{-1} \mathcal{P} + \alpha s^{1+3\alpha} \theta^{-1} \left( 3\alpha + 1 \right) \left( 3\alpha + 2 \right) - \alpha - 1 \left( 3\alpha + 1 \right)^2 \right] s^{1+9\alpha} \theta^{3\alpha} + \mathcal{P} + \alpha \mathcal{P} \]

In the last expression, using the definition of \( \mathcal{R} \), identity (2.1), we replace \( t \mathcal{P} \) by \( \mathcal{R} + \frac{\alpha}{\alpha - 1} s^{1+3\alpha} \theta^\alpha \). Therefore, at the point where the maximum of \( \mathcal{R} \) is achieved we obtain
\[ \partial_t \mathcal{R} \]
\[ \leq \mathcal{R} \left[ -\alpha s^{1+3\alpha} \theta^{-1} - \frac{\alpha - 1}{\alpha} \mathcal{P} + \frac{2\left( \alpha - 1 \right)\left( 3\alpha + 1 \right)}{\alpha} s^{3\alpha} \theta^\alpha \right] \]
\[ + \frac{\alpha}{\alpha - 1} \left[ 2\left( \alpha - 1 \right)\left( 3\alpha + 1 \right) - 3\left( 3\alpha + 1 \right) \right] s^{2+6\alpha} \theta^{2\alpha} + \frac{\alpha\left( 3\alpha + 1 \right)}{\alpha - 1} s^{2+9\alpha} \theta^{3\alpha-1} \]
\[ + t \left[ 3\alpha + 1 \right)\left( 3\alpha + 2 \right) - \alpha - 1 \left( 3\alpha + 1 \right)^2 \right] s^{1+9\alpha} \theta^{3\alpha} \]
\[ \leq \mathcal{R} \left[ -\alpha s^{1+3\alpha} \theta^{-1} - \frac{\alpha - 1}{\alpha} \mathcal{P} + \frac{2\left( \alpha - 1 \right)\left( 3\alpha + 1 \right)}{\alpha} s^{3\alpha} \theta^\alpha \right] . \]

To get the last inequality, we used the fact that the terms on the second and third line are negative for \( p \geq 1 \). Hence, by the parabolic maximum principle and the fact that at the time zero we have \( \mathcal{R} \leq 0 \), we conclude \( \mathcal{R} = t \mathcal{P} - \frac{\alpha}{\alpha - 1} s^{1+3\alpha} \theta^\alpha \leq 0 \). Negativity of \( \mathcal{R} \) is equivalent to \( \partial_t \ln \left( s^{1+3\alpha} \theta^\alpha \right) \geq \frac{\alpha}{1+\frac{\alpha}{t}} \) for \( t > 0 \). From this we infer that \( \partial_t \left( s^{1+3\alpha} \theta^\alpha t^{\frac{\alpha}{1+\frac{\alpha}{t}}} \right) \geq 0 \) for \( t > 0 \). \( \square \)
Proposition 2.3. Ancient solutions of the flow (1.1) satisfy \( \partial_t \left( s \left( \frac{1}{r s^3} \right) \right) \geq 0 \).

Proof. By the Harnack estimate every solution of the flow (1.1) satisfies

\[
\partial_t \left( s \left( \frac{1}{r s^3} \right) \right) + \frac{p}{2t(p+1)} \left( s \left( \frac{1}{r s^3} \right) \right) \geq 0.
\]

We let the flow starts from a fixed time \( t_0 < 0 \). So the inequality (2.3) becomes

\[
\partial_t \left( s \left( \frac{1}{r s^3} \right) \right) + \frac{p}{2(t-t_0)(p+1)} \left( s \left( \frac{1}{r s^3} \right) \right) \geq 0.
\]

Now letting \( t_0 \) goes to \( -\infty \) proves the claim. \( \square \)

Corollary 2.4. Every ancient solution of the flow (1.1) satisfies \( \partial_t \left( s \left( \frac{1}{r s} \right) \right) \leq 0 \).

Proof. The \( s(\cdot, t) \) is decreasing on the time interval \( (-\infty, 0] \). The claim now follows from the previous proposition. \( \square \)

3. Affine differential setting

We will recall several definitions from affine differential geometry. Let \( \gamma : S^1 \to \mathbb{R}^2 \) be an embedded strictly convex curve with the curve parameter \( \theta \). Define \( g(\theta) := [\gamma_{\theta}, \gamma_{\theta\theta}]^{1/3} \), where for two vectors \( u, v \) in \( \mathbb{R}^2 \), \([u, v]\) denotes the determinant of the matrix with rows \( u \) and \( v \). The affine arc-length is defined as

\[
s(\theta) := \int_0^\theta g(\alpha) d\alpha.
\]

Furthermore, the affine normal vector \( n \) is given by \( n := \gamma_{ss} \). In the affine coordinate \( s \), there hold \([\gamma_s, \gamma_{ss}] = 1, \sigma = [\gamma, \gamma_s] \), and \( \sigma_{ss} + \sigma \mu = 1 \), where \( \mu = [\gamma_{ss}, \gamma_{sss}] \) is the affine curvature.

We can express the area of \( K \in \mathcal{K} \), denoted by \( A(K) \), in terms of affine invariant quantities:

\[
A(K) = \frac{1}{2} \int_{\partial K} \sigma ds.
\]

The \( p \)-affine perimeter of \( K \in \mathcal{K}_0 \) (for \( p = 1 \) the assumption \( K \in \mathcal{K}_0 \) is not necessary and we may take \( K \in \mathcal{K} \)), denoted by \( \Omega_p(K) \), is defined as

\[
\Omega_p(K) := \int_{\partial K} \sigma^{1 - \frac{3p}{p+2}} ds,
\]

\[10\]. We call the quantity \( \frac{\Omega_{p+2}^2(K)}{A^{2-p}(K)} \), the \( p \)-affine isoperimetric ratio and mention that it is invariant under \( GL(2) \). Moreover, for \( p > 1 \) the \( p \)-affine isoperimetric inequality states that if \( K \) has its centroid at the origin, then

\[
\frac{\Omega_{p+2}^2(K)}{A^{2-p}(K)} \leq 2^{p+2} \pi^{2p}
\]

and equality cases are obtained only for origin-centered ellipses. In the final section, we will use the 2-affine isoperimetric inequality.

Let \( K \in \mathcal{K}_0 \). The polar body of \( K \), denoted by \( K^* \), is a convex body in \( \mathcal{K}_0 \) defined by

\[
K^* = \{ y \in \mathbb{R}^2 \mid \langle x, y \rangle \leq 1, \forall x \in K \}.
\]
The area of $K^*$, denoted by $A^* = A(K^*)$, can be represented in terms of affine invariant quantities:

\[ A^* = \frac{1}{2} \int_{\partial K} \frac{1}{\sigma^2} ds = \frac{1}{2} \int_{S^1} \frac{1}{s^2} d\theta. \]

Let $K \in K_0$. We consider a family of convex bodies $\{K_t\}_t \subset K$, given by the smooth embeddings $X : \partial K \times [0, T) \rightarrow \mathbb{R}^2$, which are evolving according to (1.1). Then up to a time-dependant diffeomorphism, $\{K_t\}_t$ evolves according to

\[ \frac{\partial}{\partial t} X := \sigma^1 - \frac{3p}{p+2} n, \quad X(\cdot, 0) = X_K(\cdot). \] (3.2)

Therefore, classification of compact, origin-symmetric ancient solutions to (1.1) is equivalent to the classification of compact, origin-symmetric ancient solutions to (3.2). In what follows our reference flow is the evolution equation (3.2).

Notice that as a family of convex bodies evolve according to the evolution equation (3.2), in the Gauss parametrization their support functions and radii of curvature evolve according to Lemma 2.1. Assume $Q$ and $\bar{Q}$ are two smooth functions $Q : \partial K \times [0, T) \rightarrow \mathbb{R}, \bar{Q} : S^1 \times [0, T) \rightarrow \mathbb{R}$ that are related by $Q(x, t) = \bar{Q}(\nu(x, t), t)$. It can be easily verified that

\[ \frac{\partial}{\partial t} \bar{Q} = \frac{\partial}{\partial t} Q - Q_s \left( \sigma^1 - \frac{3p}{p+2} \right). \]

In particular, for ancient solutions of (3.2), in views of Corollary 2.4, $Q = \sigma$ must satisfy $0 \geq \frac{\partial}{\partial t} \sigma - \sigma_s \left( \sigma^1 - \frac{3p}{p+2} \right)$. The proceeding argument proves the next proposition.

**Proposition 3.1.** Every ancient solution satisfies $\frac{\partial}{\partial t} \sigma \leq - \left( \frac{3p}{p+2} - 1 \right) \sigma_s^2 \sigma - \frac{3p}{p+2} \sigma_s^2$.

The next two lemmas were proved in [7].

**Lemma 3.2.** [7, Lemma 3.1] The following evolution equations hold:

1. \( \frac{\partial}{\partial t} \sigma = \sigma^{1 - \frac{3p}{p+2}} \left( - \frac{4}{3} + \left( \frac{p}{p+2} + 1 \right) \left( 1 - \frac{3p}{p+2} \right) \frac{\sigma^2_s}{\sigma} + \frac{p}{p+2} \sigma_{ss} \right), \)

2. \( \frac{d}{dt} A = -\Omega_p \).

**Lemma 3.3.** [7, Section 6] The following evolution equation for $\Omega_l$ holds for every $l \geq 2$ and $p \geq 1$:

\[ \frac{d}{dt} \Omega_l(t) = \frac{2l(l-2)}{l+2} \int_{\gamma_l} \sigma^{1 - \frac{3p}{p+2} - \frac{2l}{l+2}} ds + \frac{18pl}{(l+2)^2(p+2)} \int_{\gamma_l} \sigma^{\frac{2l}{l+2} - \frac{2l}{l+2}} \sigma^2_{ss} ds, \]

where $\gamma_l := \partial K_l$ is the boundary of $K_l$.

**Lemma 3.4.** [13] The area product, $A(t)A^*(t)$, and the $p$-affine isoperimetric ratio are both non-decreasing along (3.2).

Write respectively $\max_{\gamma_l} \sigma$ and $\min_{\gamma_l} \sigma$ for $\sigma_M$ and $\sigma_m$.

**Lemma 3.5.** There is a constant $0 < c < \infty$ such that $\frac{\sigma_M}{\sigma_m^2} \leq c$ on $(-\infty, 0]$. 

Proof. By Corollary 3.1 and part (1) of Lemma 3.2 we have
\[- \left( \frac{3p}{p+2} - 1 \right) \frac{\sigma^2}{\sigma^3} \geq \frac{\partial_t \sigma}{\sigma^3}.\]

Integrating the inequality (3.4) against \(ds\) we obtain
\[
\frac{4}{3} \int_{\gamma_t} \frac{1}{\sigma^2} \, ds \geq \frac{p}{p+2} \left( 2 - \frac{3p}{p+2} \right) \int_{\gamma_t} \frac{\sigma^2}{\sigma^3} \, ds
= \frac{p}{p+2} \left( 3 - \frac{3p}{p+2} \right) \int_{\gamma_t} \frac{(\ln \sigma)^2}{\sigma} \, ds
\geq \frac{p}{p+2} \left( 3 - \frac{3p}{p+2} \right) \left( \int_{\gamma_t} \frac{|\ln \sigma|}{\sigma} \, ds \right)^2.
\]

Set \(d_p = \frac{p}{p+2} \left( 3 - \frac{3p}{p+2} \right) \). Applying the Hölder inequality to the left-hand side and the right-hand side of inequality (3.5) yields
\[
\left( \int_{\gamma_t} |(\ln \sigma)| \, ds \right)^2 \leq d_p A^*(t) A(t),
\]
for a new positive constant \(d_p\). Here we used the identities \(\int_{\gamma_t} \frac{1}{\pi} \, ds = 2A^*(t)\) and \(\int_{\gamma_t} \sigma \, ds = 2A(t)\). Now by Lemma 3.4 we have \(A(t)A^*(t) \leq A(0)A^*(0)\). This implies that
\[
\left( \frac{\ln \sigma}{\sigma_m} \right)^2 \leq d''_p,
\]
for a new positive constant \(d''_p\). Therefore, on \((-\infty, 0]\) we find that
\[
\frac{\sigma M}{\sigma m} \leq c
\]
for some positive constant \(c\). \(\square\)

Let \(\{K_t\}_t\) be a solution of (3.2). Then the family of convex bodies, \(\tilde{K}_t\), defined by
\[
\tilde{K}_t := \sqrt{\frac{\pi}{A(K_t)}} K_t
\]
is called a normalized solution to the \(p\)-flow, equivalently a solution that the area is fixed and is equal to \(\pi\).

Furnish every quantity associated with the normalized solution with a over-tile. For example, the support function, curvature, and the affine support function of \(\tilde{K}\) are denoted by \(\tilde{s}, \tilde{\kappa}, \text{and} \tilde{\sigma}\), respectively.

**Lemma 3.6.** There is a constant \(0 < c < \infty\) such that on the time interval \((-\infty, 0]\) we have
\[
\frac{\tilde{\sigma} M}{\tilde{\sigma} m} \leq c.
\]

**Proof.** The estimate (3.6) is scaling invariant, so the same estimate holds for the normalized solution. \(\square\)
Lemma 3.7. $\Omega_2(t)$ is non-decreasing along the $p$-flow. Moreover, we have

$$
\frac{d}{dt}\Omega_2(t) \geq \frac{9p}{4(p+2)} \int_{\gamma_t} \sigma^{-\frac{3p}{p+2}} \sigma_2^2 ds.
$$

Proof. Use the evolution equation (3.3) for $l = 2$. □

Corollary 3.8. There exists a constant $0 < b_p < \infty$ such that

$$
\frac{1}{\Omega_2^2(t)} < b_p
$$
on $(-\infty, 0]$.

Proof. Notice that $\Omega_2(t) = \left( \int_{\partial \gamma_t} \sigma^{-\frac{3}{2}} ds \right)$ is a $GL(2)$ invariant quantity. Therefore, we need only to prove the claim after applying appropriate $SL(2)$ transformations to the normalized solution of the flow. By the estimate (3.7) and the facts that $\Omega_2(K_t)$ is non-decreasing and $A(K_t) = \pi$ we have

$$
\frac{c^2}{2} \int_{M} \sigma_2^2 dv(t) \Omega_2(0) \geq \frac{1}{2} \int_{M} \sigma^2 dv(t) \Omega_2(0) \geq \frac{1}{2} \int_{M} \sigma^2 dv(t) \Omega_2(t) \geq \tilde{A}(t) = \pi.
$$

So we get $\left( \frac{\tilde{A}}{t} \right)^{3/2} \geq a > 0$ on $(-\infty, 0]$, for an $a$ independent of $t$. Moreover, as the affine support function is invariant under $SL(2)$ we may further assume, after applying a length minimizing special linear transformation at each time, that $\tilde{t}(t) < a' < \infty$, for an $a'$ independent of $t$. Therefore

$$
\frac{\tilde{A}^3(t)}{\tilde{A}(t)} = \left( \int_{t} \frac{\sigma^2 d\tilde{t}}{t} \right)^{3/2} \geq a'' > 0,
$$

for an $a''$ independent of $t$. Now the claim follows from the Hölder inequality:

$$
\left( \int_{t} \sigma^{-\frac{3}{2}} d\tilde{t} \right) \Omega_2^2(t)A(t)^{\frac{3}{2}} \geq \int_{t} \sigma^{-\frac{3}{2}} d\tilde{t} \int_{t} \sigma^{\frac{3}{2}} d\tilde{t} \geq \Omega_2^2(t),
$$

so

$$
\tilde{\Omega}_2(t) = \Omega_2(t) \geq \left( \frac{\Omega_2^3(t)}{A(t)} \right)^{\frac{3}{2}} = \left( \frac{\tilde{A}^3(t)}{\tilde{A}(t)} \right)^{\frac{3}{2}}.
$$

□

Corollary 3.9. As $K_t$ evolve by (3.2), then the following limit holds:

$$
\lim_{t \to -\infty} \left( \frac{A(t)}{\Omega_p(t)\Omega_2^2(t)} \right) \int_{\gamma_t} \left( \sigma^{\frac{3}{2}} \right)^2 ds = 0.
$$

Proof. Suppose on the contrary that there exists an $\varepsilon > 0$ small enough, such that

$$
\left( \frac{A(t)}{\Omega_p(t)\Omega_2^2(t)} \right) \int_{\gamma_t} \left( \sigma^{\frac{3}{2}} \right)^2 ds \geq \varepsilon \frac{\left( \frac{1}{4} - \frac{3p}{2(p+2)} \right)}{\frac{9p}{4(p+2)}}
$$
on $(-\infty, -N]$ for $N$ large enough. Then $\frac{d}{dt} \frac{1}{\Omega_2^2(t)} \leq \varepsilon \frac{d}{dt} \ln(A(t))$. So by integrating this last inequality against $dt$ and by Corollary 3.8 we get

$$
0 < \frac{1}{\Omega_2^2(t)} \leq \frac{1}{\Omega_2^2(0)} + \varepsilon \ln(A(0)) - \varepsilon \ln(A(t)) < b_p + \varepsilon \ln(A(0)) - \varepsilon \ln(A(t)).
$$
Letting $ t \to -\infty $ we reach to a contradiction: $ \lim_{ t \to -\infty } A(t) = +\infty $, that is, the right-hand side becomes negative for large values of $ t $. \qed

**Corollary 3.10.** For a sequence of times $ \{ t_k \} $ as $ t_k $ converge to $ -\infty $ we have $ \lim_{ t_k \to -\infty } \tilde{\sigma}(t_k) = 1 $. 

**Proof.** Notice that the quantity $ \left( \frac{A(t)}{\Omega_p(t)\Omega_2(t)} \right) \int_{\gamma_k} \left( \frac{1}{2} - \frac{3p}{p + 2} s \right)^2 d\tilde{s} $ is scaling invariant and $ \frac{\tilde{A}(t)}{\Omega_p(t)\Omega_2(t)} $ is bounded from below (By Lemmas 3.4 and 3.7, $ \Omega_p(t) \leq \tilde{\Omega}_p(0) $ and $ \tilde{\Omega}_2(t) \leq \tilde{\Omega}_2(0) $). Thus Corollary 3.9 implies that there exists a sequence of times $ \{ t_k \}_{k \in \mathbb{N}} $, such that $ \lim_{ k \to \infty } t_k = -\infty $ and 

$$ \lim_{ t_k \to -\infty } \int_{\gamma_k} \left( \frac{1}{2} - \frac{3p}{p + 2} s \right)^2 d\tilde{s} = 0. $$

On the other hand, by the Hölder inequality

$$ \frac{\left( \frac{1}{2} - \frac{3p}{p + 2} s \right)^2 (t_k) - \frac{1}{2} - \frac{3p}{p + 2} s \left( t_k \right)}{\Omega_1(t_k)} \leq \int_{\gamma_k} \left( \frac{1}{2} - \frac{3p}{p + 2} s \right)^2 d\tilde{s}. $$

Moreover, $ \tilde{\Omega}_1(t) $ is bounded from above: Indeed $ \left( \frac{\tilde{\Omega}_1(t)}{A(t)} \right)^{\frac{3}{p}} \leq \tilde{\Omega}_2(t) \leq \tilde{\Omega}_2(0) $. Therefore, we find that

$$ \lim_{ t_k \to -\infty } \left( \frac{1}{2} - \frac{3p}{p + 2} s \left( t_k \right) - \frac{1}{2} - \frac{3p}{p + 2} s \left( t_k \right) \right)^2 = 0. $$

Since $ \tilde{\sigma}_M \leq 1 $ and $ \tilde{\sigma}_m \geq 1 $ (see [3, Lemma 10]) the claim follows. \quad \Box

### 4. Proof of the Main Theorem

**Proof.** For each time $ t \in (-\infty, T) $, let $ T_1 \in SL(2) $ be a special linear transformation that the maximal ellipse contained in $ T_1 K_t $ is a disk. So by John’s ellipsoid lemma we have

$$ \frac{1}{\sqrt{2}} \leq s_{T_1 K_t} \leq \sqrt{2}. $$

Then by the Blaschke selection theorem, there is a subsequence of times, denoted again by $ \{ t_k \} $, such that $ \{ T_{ t_k } K_{ t_k } \} $ converges in the Hausdorff distance to an origin-symmetric convex body $ \tilde{K}_{-\infty} $, as $ t_k \to -\infty $. By Corollary 3.10, and by the weak convergence of the Monge-Ampère measures, the support function of $ \tilde{K}_{-\infty} $ is the generalized solution of the following Monge-Ampère equation on $ S^1 $:

$$ s^3(s_{\theta_0} + s) = 1. $$

Therefore, by Lemma 8.1 of Petty [6], $ \tilde{K}_{-\infty} $ is an origin-centered ellipse. This in turn implies that $ \lim_{ t \to -\infty } \tilde{\Omega}_2(t_k) = 2\pi $. On the other hand, by the 2-affine isoperimetric inequality, (3.1), and by Lemma 3.7, for $ t \in (\infty, 0] $ we have

$$ 2\pi \geq \tilde{\Omega}_2(t) \geq \lim_{ t_k \to -\infty } \tilde{\Omega}_2(t_k) = 2\pi. $$

Thus $ \frac{d}{dt} \tilde{\Omega}_2(t) \equiv 0 $ on $ (-\infty, 0) $. Hence, in view of Lemma 3.7, $ K_t $ is an origin-centred ellipse for every time $ t \in (-\infty, T) $. \quad \Box
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