NONPARAMETRIC REGRESSION FOR LOCALLY STATIONARY RANDOM FIELDS UNDER STOCHASTIC SAMPLING DESIGN

DAISUKE KURISU

ABSTRACT. In this study, we develop an asymptotic theory of nonparametric regression for locally stationary random fields (LSRFs) \( \{ X_{s,A_n} : s \in R_n \} \) in \( \mathbb{R}^p \) observed at irregularly spaced locations in \( R_n = [0, A_n]^d \subset \mathbb{R}^d \). We first derive the uniform convergence rate of general kernel estimators, followed by the asymptotic normality of an estimator for the mean function of the model. Moreover, we consider additive models to avoid the curse of dimensionality arising from the dependence of the convergence rate of estimators on the number of covariates. Subsequently, we derive the uniform convergence rate and joint asymptotic normality of the estimators for additive functions. We also introduce approximately \( m_n \)-dependent RFs to provide examples of LSRFs. We find that these RFs include a wide class of Lévy-driven moving average RFs.

Keywords: nonparametric regression, locally stationary random field, irregularly spaced data, additive model, Lévy-driven moving average random field.

1. Introduction

In this study, we consider the following model:

\[
Y_{s_j,A_n} = m \left( \frac{s_j}{A_n}, X_{s_j,A_n} \right) + \epsilon_{s_j,A_n}, \quad s_j \in R_n, \ j = 1, \ldots, n, \quad (1.1)
\]

where \( E[\epsilon_{s,A_n} | X_{s,A_n}] = 0 \) and \( R_n = [0, A_n]^d \subset \mathbb{R}^d \) is a sampling region with \( A_n \to \infty \) as \( n \to \infty \). Here, \( Y_{s_j,A_n} \) and \( X_{s_j,A_n} \) are random variables of dimensions 1 and \( p \), respectively. We assume that \( \{ X_{s,A_n} : s \in R_n \} \) is a locally stationary random field on \( R_n \subset \mathbb{R}^d \) \( (d \geq 2) \). Locally stationary processes, as proposed by Dahlhaus (1997), are nonstationary time series that allow parameters of the time series to be time-dependent. They can be approximated by a stationary time series locally in time, which enables asymptotic theories to be established for the estimation of time-dependent characteristics. In time series analysis, locally stationary models are mainly considered in a parametric framework with time-varying coefficients. For example, we refer to Dahlhaus et al. (1997), Dahlhaus and Subba Rao (2000), Fryzlewicz et al. (2008), Hafner and Linton (2010), Koo and Linton (2012), Zhou (2014) and Truquet (2017). Nonparametric methods for stationary and nonstationary time series models have also been developed. We refer to Masry (1996), Fan and Yao (2003) and Hansen (2008) for stationary time series and Kristensen (2009), Zhou and Wu (2009), Kristensen (2011), Vogt (2012), Zhang and Wu (2015), Truquet (2019), Kurisu (2021a), and Kurisu (2021b) for recent contributions on locally stationary (functional) time series. Furthermore, we refer to Dahlhaus et al. (2013) for general theory in the literature of locally stationary processes.

Date: First version: April 16, 2020. This version: July 7, 2022.

D. Kurisu is partially supported by JSPS KAKENHI Grant Numbers 17H02513 and 20K13468. The author would like to thank Paul Doukhan, Kengo Kato, Yasumasa Matsuda, Taisuke Otsu and Robert Stelzer for their helpful comments and suggestions.
Recently, statistical inference of spatial regression models for geostatistical data has drawn considerable attention in certain economic and scientific fields, such as spatial econometrics, ecology, and seismology. Nonparametric methods for spatial data have also been the focus of attention. Recent contributions include those by Hallin et al. (2004), Machkouri and Stoica (2008), Hallin et al. (2009), Robinson (2011), Jenish (2012), Lu and Tjostheim (2014), Li (2016), Machkouri et al. (2017), and Kurisu (2019), who studied nonparametric inference and estimation of mean functions of spatial regression models. Compared with the abovementioned studies regarding time series analysis, studies pertaining to nonparametric methods for locally stationary random fields are scarce even for nonparametric density estimation as well as nonparametric regression despite empirical interest in modeling spatially varying dependence structures. Moreover, for the analysis of spatial data on $\mathbb{R}^d$, it is generally assumed that the underlying model is a parametric Gaussian process. We refer to Fuglstad et al. (2015) for the empirical motivation and discussion regarding the modeling of nonstationary random fields and Steel and Fuentes (2010) for discussion regarding the importance of nonparametric, nonstationary, and non-Gaussian spatial models. To bridge the gap between theory and practice, we first extend the concept of a locally stationary time series by Dahlhaus (1997) to random fields and consider a locally stationary random field $\{X_{s,A_n} : s \in \mathbb{R}_n\} (\mathbb{R}_n \subset \mathbb{R}^d)$ as a nonstationary random field that can be approximated by a stationary random field locally at each spatial point $s \in \mathbb{R}_n$. A detailed definition of locally stationary random fields is provided in Section 2. Next, we provide a complete asymptotic theory for the nonparametric estimators of the model (1.1). The model (1.1) can be regarded as a natural extension of (i) nonparametric regression to the locally stationary time series considered in Vogt (2012) and Zhang and Wu (2015), and (ii) nonparametric spatial regression investigated in Hallin et al. (2004), Robinson (2011), Jenish (2012), Li (2016), and Kurisu (2019) to locally stationary random fields, and (iii) linear regression models with spatially varying coefficients to a nonlinear framework. Therefore, the model includes a wide range of key models for nonstationary random fields.

The objectives of this study are to (i) derive the uniform convergence rate of kernel estimators for the density function of $X_{s,A_n}$ and the mean function $m$ in the model (1.1) over compact sets; (ii) derive the asymptotic normality of the estimators at a specified point; and (iii) provide examples of locally stationary random fields on $\mathbb{R}^d$ with a detailed discussion of their properties. To attain the first and second objectives, we first derive the uniform convergence rate of the important general kernel estimators; the result is crucial for demonstrating our main results. As general estimators include a wide range of kernel-based estimators such as the Nadaraya-Watson estimators, the general results are of independent interest. Although these results are general, the estimators are affected by dimensionality because their convergence rate depends on the number of covariates. Hence, we consider additive models and derive the uniform convergence rate and joint asymptotic normality of kernel estimators for additive functions based on the backfitting method developed by Mammen et al. (1999) and Vogt (2012). Our results are extensions of the results for time series in Vogt (2012) to random fields with irregularly spaced observations, which include irregularly spaced time series as a special case. To the best of our knowledge, this is the first study where a complete asymptotic theory of density estimation, nonparametric regression and additive models for locally stationary random fields on $\mathbb{R}^d$ are developed with a detailed discussion on concrete examples of locally stationary random fields.

Technically, the proofs of our results are significantly different from those for equally distant time series (mixing sequence) or spatial data observed on lattice points, as our focus is on spatial
dependence and irregularly spaced sampling points. In many scientific fields, such as ecology, meteorology, seismology, and spatial econometrics, sampling points are naturally irregular. In fact, measurement stations cannot be placed on a regular grid owing to physical constraints. The stochastic sampling design assumed in this study allows the sampling sites to have a possibly non-uniform density across the sampling region, thereby enabling the number of sampling sites to increase at different rates with respect to the volume of the region \(O(A_n^d)\). In this study, we work with both the pure increasing domain case \((\lim_{n \to \infty} nA_n^{-d} = \kappa \in (0, \infty))\) and the mixed increasing domain case \((\lim_{n \to \infty} nA_n^{-d} = \infty)\). As recent contributions pertaining to the design of stochastic spatial sampling, we refer to Shao (2010) for time series and Lahiri (2003b), Lahiri and Zhu (2006), and Bandypadhyay et al. (2015) for random fields. In Lu and Tjøstheim (2014), they show asymptotic normality of a kernel density estimator of a strictly stationary random field on \(\mathbb{R}^2\) under non-stochastic irregularly spaced observations based on the technique developed in Bolthausen (1982). Our approach is quite different from theirs and their sampling framework seems not compatible to develop asymptotic theory for locally stationary random fields. More precisely, in the proofs of our main results, we combine the big and small block techniques of Bernstein (1926) and the coupling technique for mixing sequences of Yu (1994) to construct a sequence of independent blocks in a finite sample size. However, their applications are nontrivial for addressing irregularly spaced data and spatial dependence simultaneously. Discussion on the mixing condition is provided in Section 2.

To attain the third objective of our study, we discuss examples of locally stationary random fields on \(\mathbb{R}^d\) that satisfy our mixing conditions and the other regularity conditions mentioned in Section 3. For this, we introduce the concept of *approximately \(m_n\)-dependent* locally stationary random fields \((m_n \to \infty\) as \(n \to \infty)\) and we extend continuous autoregressive and moving average (CARMA)-type random fields developed in Brockwell and Matsuda (2017) and Matsuda and Yuan (2020) to locally stationary CARMA-type random fields. CARMA random fields are characterized by solutions of (fractional) stochastic partial differential equations (cf. Berger (2020)) and are known as a rich class of models for spatial data (cf. Brockwell and Matsuda (2017), Matsuda and Yajima (2018) and Matsuda and Yuan (2020)). However, their mixing properties have not been investigated. We can show that a wide class of Lévy-driven moving average random fields, which includes the locally stationary CARMA-type random fields, comprise (approximately \(m_n\)-dependent) locally stationary random fields. One of the key features of CARMA random fields is that they can represent non-Gaussian random fields as well as Gaussian random fields if the driving Lévy random measures are purely non-Gaussian. On the other hand, the statistical models in most of existing papers for spatial data on \(\mathbb{R}^2\) are based on Gaussian processes and for non-Gaussian processes, it is often difficult to check mixing conditions as considered in Lahiri and Zhu (2006) and Bandypadhyay et al. (2015), for example. Moreover, in many empirical applications, the assumption of Gaussianity in spatial models is not necessarily adequate (see Steel and Fuentes (2010) for example). As a result, this study also contributes to the flexible modeling of nonparametric, nonstationary and possibly non-Gaussian random fields on \(\mathbb{R}^d\). Therefore, this paper not only extends the scope of empirical analysis for spatial and spatio-temporal data (discussed in Section 6) but also addresses open questions on the dependence structure of statistical models built on CARMA random fields.

The rest of this paper is organized as follows. In Section 2, we define locally stationary random fields and describe our sampling scheme to consider irregularly spaced data and mixing conditions.
for random fields. In Section 3, we present the uniform convergence rate of general kernel estimators as well as the estimators of the mean function \( m \) in the model \( \{X_t\} \) over compact sets. The asymptotic normality of the estimator of the mean function is also provided. In Section 4, we consider the additive models and provide the uniform convergence rate and joint asymptotic normality of the kernel estimators for the additive functions. In Section 5, we discuss examples of locally stationary random fields by introducing the concept of \( m_n \)-dependent random fields; we also provide examples of locally stationary Lévy-driven moving average random fields. We also discuss some extensions of our results and applications to spatio-temporal data in Section 6. All the proofs are deferred to Appendix.

1.1. Notations. For \( \mathbf{x} = (x_1, \ldots, x_d)' \in \mathbb{R}^d \), let \( |\mathbf{x}| = |x_1| + \cdots + |x_d| \) and \( \|\mathbf{x}\| = (x_1^2 + \cdots + x_d^2)^{1/2} \) denote the \( \ell^1 \) and \( \ell^2 \) norm on \( \mathbb{R}^d \). For \( \mathbf{x} = (x_1, \ldots, x_d)', \mathbf{y} = (y_1, \ldots, y_d) \in \mathbb{R}^d \), the notation \( \mathbf{x} \leq \mathbf{y} \) means that \( x_j \leq y_j \) for all \( j = 1, \ldots, d \). For any set \( A \subseteq \mathbb{R}^d \), let \( |A| \) denote the Lebesgue measure of \( A \) and \( \llbracket A \rrbracket \) denote the number of elements in \( A \). For any positive sequences \( a_n \) and \( b_n \), we write \( a_n \lesssim b_n \) if a constant \( C > 0 \) independent of \( n \) exists such that \( a_n \leq C b_n \) for all \( n \), \( a_n \sim b_n \) if \( a_n \lesssim b_n \) and \( b_n \lesssim a_n \), and \( a_n \ll b_n \) if \( a_n/b_n \to 0 \) as \( n \to \infty \). For \( a \in \mathbb{R} \) and \( b > 0 \), we use the shorthand notation \( [a \pm b] = [a - b, a + b] \). We use the notation \( \xrightarrow{d} \) to denote convergence in distribution. For random variables \( X \) and \( Y \), we write \( X \xrightarrow{d} Y \) if they have the same distribution. \( N(\mu, \Sigma) \) denotes a (multivariate) normal distribution with mean \( \mu \) and a covariance matrix \( \Sigma \). Let \( P_S \) denote the joint probability distribution of the sequence of independent and identically distributed (i.i.d.) random vectors \( \{S_{0,j}\}_{j \geq 1} \), and let \( P_{|S} \) denote the conditional probability distribution given \( \{S_{0,j}\}_{j \geq 1} \). Let \( E_{|S} \) and \( \text{Var}_{|S} \) denote the conditional expectation and variance given \( \{S_{0,j}\}_{j \geq 1} \), respectively. \( B(\mathbb{R}^d) \) denotes the Borel \( \sigma \)-field on \( \mathbb{R}^d \).

2. Settings

In this section, we introduce locally stationary random fields on \( \mathbb{R}^d \) that extend the concept of the locally stationary process on \( \mathbb{R} \). Furthermore, we discuss the sampling design of irregularly spaced locations and the dependence structures for random fields.

2.1. Local stationarity. Intuitively, a random field \( \{X_{s,j,A_n} : s \in R_n\} \) is locally stationary if it behaves approximately stationary in local space. To ensure that it is locally stationary around each rescaled space point \( u \), a process \( \{X_{s,A_n}\} \) can be approximated by a stationary random field \( \{X_u(s) : s \in \mathbb{R}^d\} \) stochastically. See [Dahlhaus and Subba Rao (2006)] for an example. This concept can be defined as follows.

**Definition 2.1.** The process \( \{X_{s,A_n} = (X_{s,A_n}^1, \ldots, X_{s,A_n}^p)' : s \in R_n\} \) is locally stationary if for each rescaled space point \( u \in [0,1]^d \), there exists an associated random field \( \{X_u(s) : s \in \mathbb{R}^d\} \) with the following properties:

(i) \( \{X_u(s) : s \in \mathbb{R}^d\} \) is strictly stationary with density \( f_X(u) \).

(ii) It holds that
\[
\|X_{s,A_n} - X_u(s)\|_1 \leq \left( \frac{\|s - u\|}{A_n} + \frac{1}{A_n^2} \right) U_{s,A_n}^{(u)} \text{ a.s.},
\]

where \( \{U_{s,A_n}^{(u)}\} \) is a process of positive variables satisfying \( E[(U_{s,A_n}^{(u)})^\rho] < C \) for some \( \rho > 0, C < \infty \) that is independent of \( u, s, \) and \( A_n \), and where \( \| \cdot \|_1 \) and \( \| \cdot \|_2 \) denote arbitrary norms on \( \mathbb{R}^p \) and \( \mathbb{R}^d \), respectively.
Definition 2.1 is a natural extension of the concept of local stationarity for time series introduced in [Dahlhaus (1997)]. We discuss examples of locally stationary random fields in Section 5. In particular, we demonstrate that a wide class of random fields, which includes locally stationary versions of Lévy-driven moving average random fields, satisfy the Condition (2.1). We also refer to Pezo (2018) and Matsuda and Yajima (2018) for other definitions of locally stationary spatial and spatio-temporal processes by using their spectral representations, respectively.

2.2. Sampling design. To account for irregularly spaced data, we consider the stochastic sampling design. First, we define the sampling region $R_n$. Let $\{A_n\}_{n \geq 1}$ be a sequence of positive numbers such that $A_n \to \infty$ as $n \to \infty$. We consider the following set as the sampling region:

\[ R_n = [0, A_n]^d. \]  

(2.2)

Next, we introduce our (stochastic) sampling designs. Let $f_S(s_0)$ be a continuous, everywhere positive probability density function on $R_0 = [0, 1]^d$, and let $\{S_{0,j}\}_{j \geq 1}$ be a sequence of i.i.d. random vectors with probability density $f_S(s_0)$ such that $\{S_{0,j}\}_{j \geq 1}$ and $\{X_{s,A} : s \in R_n\}$ are defined on a common probability space $(\Omega, \mathcal{F}, P)$ and are independent. We assume that the sampling sites $s_1, \ldots, s_n$ are obtained from realizations $s_{0,1}, \ldots, s_{0,n}$ of random vectors $S_{0,1}, \ldots, S_{0,n}$ by the following relation:

\[ s_j = A_n s_{0,j}, \quad j = 1, \ldots, n. \]

Herein, we assume that $nA_n^{-d} \to \infty$ as $n \to \infty$. We also assume the following conditions on the sampling scheme:

**Assumption 2.1.** (S1) For any $\alpha \in \mathbb{N}^d$ with $|\alpha| = 1, 2$, $\partial^\alpha f_S(s)$ exists and is continuous on $(0,1)^d$.

(S2) $C_0 \leq nA_n^{-d} \leq C_1 n^{\eta_1}$ for some $0 < C_0 < C_1 < \infty$ and small $\eta_1 \in (0,1)$.

(S3) Let $A_{1,n}$ and $A_{2,n}$ be two positive numbers such that as $n \to \infty$, $A_{1,n}, A_{2,n} \to \infty$ and $\frac{A_{1,n}}{A_n} + \frac{A_{2,n}}{A_n} \leq C_0^{-1} n^{-\eta} \to 0$ for some $C_0 > 0$ and $\eta > 0$.

Condition (S2) implies that our sampling design allows both the pure increasing domain case ($\lim_{n \to \infty} nA_n^{-d} = \kappa \in (0, \infty)$) and the mixed increasing domain case ($\lim_{n \to \infty} nA_n^{-d} = \infty$). This implies that our study addresses the infill sampling criteria in the stochastic design case (cf. Cressie (1993) and Lahiri (2003b)), which is of interest in geostatistical and environmental monitoring applications (cf. Lahiri (2003b) and Lahiri et al. (1999)). Moreover, in stochastic sampling design, the sampling density can be nonuniform. There is another approach for irregularly spaced sampling sites based on a homogeneous Poisson point process (cf. Chapter 8 in Cressie (1993)). For a sample size $n$, the approach allows the sampling sites to have only a uniform distribution over the sampling region. Therefore, the stochastic sampling design in this study is flexible compared with that based on homogeneous Poisson point processes and is therefore useful for practical applications. It may be worthwhile to consider sampling designs based on point processes that allow nonuniform sampling sites such as Cox or Hawkes point processes; however, we did not consider them because we believe that our sampling design is sufficiently practical for numerous applications. Condition (S3) is considered to decompose the sampling region $R_n$ into big and small blocks. See also the proof of Proposition 3.1 in Appendix A for details.

**Remark 2.1.** In practice, $A_n$ can be determined using the diameter of the sampling region. See Hall and Patil (1994) and Matsuda and Yajima (2009) for examples. We can relax the assumption
on $R_n$ to a more general situation, i.e.,

$$R_n = \prod_{j=1}^d [0, A_{j,n}],$$

where $A_{j,n}$ are sequences of positive constants with $A_{j,n} \to \infty$ as $n \to \infty$. To avoid complicated results, we assumed (2.2). See also Section 6.1 for further discussion on the general sampling region and extensions of our results to the general cases.

2.3. Mixing condition. Now we define $\beta$-mixing coefficients for a random field $\widetilde{X}$. Let $\sigma_{\widetilde{X}}(T) = \sigma(\{\widetilde{X}(s) : s \in T\})$ be the $\sigma$-field generated by variables $\{\widetilde{X}(s) : s \in T\}$, $T \subset \mathbb{R}^d$. For subsets $T_1$ and $T_2$ of $\mathbb{R}^d$, let

$$\tilde{\beta}(T_1, T_2) = \sup \frac{1}{2} \sum_{j=1}^J \sum_{k=1}^K |P(A_j \cap B_k) - P(A_j)P(B_k)|,$$

where the supremum is taken over all pairs of (finite) partitions $\{A_1, \ldots, A_J\}$ and $\{B_1, \ldots, B_K\}$ of $\mathbb{R}^d$ such that $A_j \in \sigma_{\widetilde{X}}(T_1)$ and $B_k \in \sigma_{\widetilde{X}}(T_2)$. Furthermore, let $d(T_1, T_2) = \inf\{|x-y| : x \in T_1, y \in T_2\}$, where $|x| = \sum_{j=1}^d |x_j|$ for $x \in \mathbb{R}^d$, and let $\mathcal{R}(b)$ be the collection of all finite disjoint unions of cubes in $\mathbb{R}^d$ with a total volume not exceeding $b$. Subsequently, the $\beta$-mixing coefficients for the random field $\widetilde{X}$ can be defined as

$$\beta(a; b) = \sup \{\tilde{\beta}(T_1, T_2) : d(T_1, T_2) \geq a, T_1, T_2 \in \mathcal{R}(b)\}.$$

(2.3)

We assume that a non-increasing function $\beta_1$ with $\lim_{a \to \infty} \beta_1(a) = 0$ and a non-decreasing function $g_1$ exist such that the $\beta$-mixing coefficient $\beta(a; b)$ satisfies the following inequality:

$$\beta(a; b) \leq \beta_1(a)g_1(b), \quad a > 0, b > 0,$$

where $g_1$ may be unbounded for $d \geq 2$.

In Section 5 we can show that our results also hold for a locally stationary random field $\{X_{s,A_n} : s \in R_n\}$ which is approximated by a $\beta$-mixing random field, i.e.,

$$X_{s,A_n} = \widetilde{X}_{s,A_n} + \epsilon_{s,A_n},$$

(2.4)

where $\widetilde{X}_{s,A_n}$ is a $\beta$-mixing random field with a $\beta$-mixing coefficient such that $\beta(a; b) = 0$ for $a \geq A_{2,n}$ (i.e. $A_{2,n}$-dependent) and $\epsilon_{s,A_n}$ is a “residual” random field which is asymptotically negligible. See also Definition 5.1 in Section 5 for the meaning of asymptotic negligibility.

We can also define $\alpha$-mixing coefficients $\alpha(a; b)$ for a random field $X$ similar to the way we defined $\beta(a; b)$ and it is known that $\beta$-mixing implies $\alpha$-mixing in general. However, it is difficult to compute $\beta$- (and even for $\alpha$-)mixing coefficients of a given random field on $\mathbb{R}^d$, which is a key different point from time series data. Moreover, almost all existing papers assume technical conditions on mixing coefficients which are difficult to check except for some Gaussian random fields, $m$-dependent random fields on $\mathbb{R}^2$ ($m > 0$: fix) and linear random fields on $\mathbb{Z}^2$ (cf. [Hallin et al. 2004], [Lahiri and Zhu 2006], [Lu and Tjøstheim 2014], [Bandypadhyay et al. 2015] and references therein). In Section 5.3 we introduce a concept of approximately $m_n$-dependent random fields. We can check that a wide class of random fields on $\mathbb{R}^d$ which includes non-Gaussian random fields as well as Gaussian random fields satisfies (2.4) and we can show that our results hold for such a class of random fields. See Section 5.3 for details.
Remark 2.2. It is important to restrict the size of index sets $T_1$ and $T_2$ in the definition of $\beta(a; b)$. To see this, we define the $\beta$-mixing coefficients of a random field $X$ as a natural extension of the $\beta$-mixing coefficients for the time series as follows: Let $O_1$ and $O_2$ be half-planes with boundaries $L_1$ and $L_2$, respectively. For each real number $a > 0$, we define the following quantity
\[ \beta(a) = \sup \{ \beta(O_1, O_2) : d(O_1, O_2) \geq a \}, \]
where $\sup$ is taken over all pairs of parallel lines $L_1$ and $L_2$ such that $d(L_1, L_2) \geq a$. Subsequently, we obtain the following result:

**Theorem 2.1** (Theorem 1 in Bradley (1989)). Suppose $\{X(s) : s \in \mathbb{R}^2\}$ is a strictly stationary mixing random field, and $a > 0$ is a real number. Then $\beta(a) = 1$ or 0.

This implies that if a random field $X$ is $\beta$-mixing with respect to the $\beta$-mixing coefficient $\beta(\cdot)$ ($\lim_{a \to \infty} \beta(a) = 0$), then the random field $X$ is “$m$”-dependent, i.e., $\beta(a) = 0$ for some $a > m$, where $m$ is a fixed positive constant. For practical purposes, this is extremely restrictive. Therefore, we adopted the definition (2.3) for $\beta$-mixing random fields. We refer to Bradley (1993), Doukhan (1994), and Dedecker et al. (2007) for details regarding the mixing coefficients for random fields.

2.4. Discussion on mixing conditions. It is often assumed that $X$ is $\alpha$-mixing and blocking techniques are applied to construct asymptotically independent blocks of observations and to show asymptotic normality of estimators based on the convergence of characteristic functions (cf. Proposition 2.6 in Fan and Yao (2003)). See also Lahiri (2003b), Lahiri and Zhu (2006) and Bandyopadhyay et al. (2015) for example. Vogt (2012) used an exponential inequality for an equidistant $\alpha$-mixing sequence (Theorem 2.1 in Liebscher (1996)) to derive uniform convergence rates of kernel estimators for locally stationary time series. We can show pointwise convergence results under $\alpha$-mixing conditions without changing proofs. Precisely, the central limit theorems (Theorems 3.2 and 4.2) below hold under the same Assumptions by replacing conditions on $\beta$-mixing coefficient $\beta(a; b)$ with conditions on $\alpha$-mixing coefficient $\alpha(a; b)$ that is defined as follows: Let $\vec{X}$ be a random field on $\mathbb{R}^d$. For any two subsets $T_1$ and $T_2$ of $\mathbb{R}^d$, let $\vec{\alpha}(T_1, T_2) = \sup \{|P(A \cap B) - P(A)P(B)| : A \in \sigma_{\vec{X}}(T_1), B \in \sigma_{\vec{X}}(T_2)\}$. Then the $\alpha$-mixing coefficient of the random field $\vec{X}$ is defined as
\[ \alpha(a; b) = \sup \{ \vec{\alpha}(T_1, T_2) : d(T_1, T_2) \geq a, T_2, T_2 \in \mathcal{R}(b) \}. \]

As with the $\beta$-mixing coefficients, we also assume that a non-increasing function $\alpha_1$ with $\lim_{a \to \infty} \alpha_1(a) = 0$ and a non-decreasing function $g_1'$ exist such that the $\alpha$-mixing coefficient $\alpha(a; b)$ satisfies the following inequality:
\[ \alpha(a; b) \leq \alpha_1(a)g_1'(b), \quad a > 0, b > 0, \]
where $g_1'$ may be unbounded for $d \geq 2$. We refer to Lahiri and Zhu (2006) and Bandyopadhyay et al. (2015) for the definition and discussion of $\alpha$-mixing coefficients for random fields on $\mathbb{R}^d$.

On the other hand, the proofs of main results in this paper are based on a general blocking technique designed for irregularly spaced sampling sites. Moreover, to derive the uniform convergence rates of the kernel estimators, we need to care about the effect of non-equidistant sampling sites when applying a maximal inequality and it requires additional work compared with the case that sampling sites are equidistant. Indeed, in place of using results for (regularly spaced) stationary sequence, which cannot be applied to the analysis of irregularly spaced non-stationary data, we construct “exactly” independent blocks of observations and apply results for independent data to
the independent blocks since there is no practical guidance for introducing an order to spatial points as opposed to time series. Precisely, we first reduce the dependent data to not asymptotically but exactly independent blocks in finite sample by using the key result in [Yu (1994)](Corollary 2.7) which does not require any dependence structure and regularly spaced sampling sites. Then we apply the Bernstein’s inequality for independent and possibly not identically distributed random variables to the independent blocks. It also should be noted that according to Remarks (ii) after the proof Lemma 4.1 in Yu (1994), the result on construction of independent blocks for β-mixing sequence would not hold for α-mixing sequences. Therefore, we work with β-mixing sequence for uniform estimation of regression functions.

As a recent contribution in econometrics, Chernozhukov et al. (2019) extended high-dimensional central limit theorems for independent data developed by Chernozhukov et al. (2013, 2017) to possibly nonstationary β-mixing time series under a nonstochastic sampling design using the technique in [Yu (1994)](see). See also Section 6.2 for further discussion on the mixing conditions.

3. Main results

In this section, we consider general kernel estimators and derive their uniform convergence rates. Based on the result, we derive the uniform convergence rate and asymptotic normality of an estimator for the mean function in the model (1.1).

3.1. Kernel estimation for regression functions. We consider the following kernel estimator for \( m(u, x) \) in the model (1.1):

\[
\hat{m}(u, x) = \frac{\sum_{j=1}^{n} K_h(u - s_j/A_n) \prod_{\ell=1}^{p} K_h(x_\ell - X_{s_j,A_n}) Y_{s_j,A_n}}{\sum_{j=1}^{n} K_h(u - s_j/A_n) \prod_{\ell=1}^{p} K_h(x_\ell - X_{s_j,A_n})}.
\]

(3.1)

Here, \( X_{s,A_n} = (X_{s,A_n}^1, \ldots, X_{s,A_n}^p)' \), \( x = (x_1, \ldots, x_p)' \) and \( u = (u_1, \ldots, u_p)' \in \mathbb{R}^p \). \( K \) denotes a one-dimensional kernel function, and we used the notations \( K_h(v) = K(v/h) \), \( \hat{K}_h(u) = K(u/h) \) where \( \hat{K}(u) = \prod_{j=1}^{d} K(u_j) \) and \( u/h = (u_1/h, \ldots, u_d/h)' \).

Before we state the main results, we summarize the assumptions made for the model (1.1) and kernel functions. These assumptions are standard; similar assumptions are made in [Vogt (2012)] and [Zhang and Wu (2015)].

Assumption 3.1. (M1) The process \( \{X_{s,A_n}\} \) is locally stationary. Hence, for each space point \( u \in [0,1]^d \), a strictly stationary random field \( \{X_u(s)\} \) exists such that

\[
\|X_{s,A_n} - X_u(s)\| \leq \left( \frac{|s - A_n|}{A_n} + \frac{1}{A_n^d} \right) U_{s,A_n}(u) \text{ a.s.}
\]

with \( E[(U_{s,A_n}(u))^\rho] \leq C \) for some \( \rho > 0 \).

(M2) The density \( f(u, x) = f_{X_u(s)}(x) \) of the variable \( X_u(s) \) is smooth in \( u \). In particular, \( f(u, x) \) is partially differentiable with respect to (w.r.t.) \( u \in (0,1)^d \) for each \( x \in \mathbb{R}^p \), and the derivatives \( \partial_u f(u, x) = \frac{\partial}{\partial u_i} f(u, x) \), \( 1 \leq i \leq d \) are continuous.

(M3) The β-mixing coefficients of the array \( \{X_{s,A_n}, \epsilon_{s,A_n}\} \) satisfy \( \beta(a;b) \leq \beta_1(a)g_1(b) \) with \( \beta_1(a) \to 0 \) as \( a \to \infty \).

(M4) \( f(u, x) \) is partially differentiable w.r.t. \( x \) for each \( u \in [0,1]^d \). The derivatives \( \partial_{x_i} f(u, x) := \frac{\partial}{\partial x_i} f(u, x), 1 \leq i \leq p \) are continuous.
(M5) \( m(u, x) \) is twice continuously partially differentiable with first derivatives \( \partial_u m(u, x) = \frac{\partial}{\partial u_i} m(u, x), \partial_x m(u, x) = \frac{\partial}{\partial x_i} m(u, x) \) and second derivatives \( \partial^2 m(u, x) = \frac{\partial^2}{\partial u_i \partial u_j} m(u, x), \partial^2 m(u, x) = \frac{\partial^2}{\partial x_i \partial x_j} m(u, x) \).

**Remark 3.1.** We can verify that a wide class of random fields satisfies Condition (M1). Indeed, our results can be applied to a class of locally stationary Lévy-driven moving average random fields that include non-stationary CARMA random fields and CARMA random fields are known as a rich class of models for spatial data (cf. Brockwell and Matsuda (2017), Matsuda and Yajima (2018) and Matsuda and Yuan (2020)). See also Section 5 for detailed discussion on the properties of locally stationary Lévy-driven moving average random fields.

**Assumption 3.2.** (KB1) The kernel \( K \) is symmetric around zero, bounded, and has a compact support, i.e., \( K(v) = 0 \) for all \( |v| > C_1 \) for some \( C_1 < \infty \). Moreover, \( K \) is Lipschitz continuous, i.e., \( |K(v_1) - K(v_2)| \leq C_2 |v_1 - v_2| \) for some \( C_2 < \infty \) and all \( v_1, v_2 \in \mathbb{R} \).

(KB2) The bandwidth \( h \) is assumed to converge to zero at least at a polynomial rate, that is, there exists a small \( \xi_1 > 0 \) such that \( h \leq Cn^{-\xi_1} \) for some constant \( 0 < C < \infty \).

3.2. **Uniform convergence rates for general kernel estimators.** As a first step to study the asymptotic properties of estimators (3.1), we analyze the following general kernel estimator:

\[
\hat{\psi}(u, x) = \frac{1}{nh^{d+1}} \sum_{j=1}^{n} \hat{K}_h \left( \frac{u - s_j}{A_n} \right) \prod_{\ell=1}^{p} K_h \left( x_\ell - X_{s_j, A_n} \right) W_{s_j, A_n},
\]

where \( \hat{K}(u) = \prod_{j=1}^{d} K(u_j) \) and \( \{W_{s_j, A_n}\} \) is an array of one-dimensional random variables. Many kernel estimators, such as Nadaraya–Watson estimators, can be represented by (3.2). In this study, we use the results with \( W_{s,A_n} = 1 \) and \( W_{s,A_n} = \epsilon_{s,A_n} \).

Next, we derived the uniform convergence rate of \( \hat{\psi}(u, x) - E[\hat{\psi}(u, x)] \). We assumed the following for the components in (3.2). Similar assumptions are made in Hansen (2008), Kristensen (2009), and Vogt (2012).

**Assumption 3.3.** (U1) It holds that \( E[|W_{s,A_n}|^\zeta] \leq C \) for some \( \zeta > 2 \) and \( C < \infty \).

(U2) The \( \beta \)-mixing coefficients of the array \( \{X_{s,A_n}, W_{s,A_n}\} \) satisfy \( \beta(a; b) \leq \beta_1(a) g_1(b) \) with \( \beta_1(a) \to 0 \) as \( a \to \infty \).

(U3) Let \( f_{X_{s,A_n}} \) be the density of \( X_{s,A_n} \). For any compact set \( S_c \subset \mathbb{R}^p \), a constant \( C = C(S_c) \) exists such that

\[
\sup_{s,A_n} \sup_{x \in S_c} f_{X_{s,A_n}}(x) \leq C \quad \text{and} \quad \sup_{s,A_n} E[|W_{s,A_n}|^\zeta |X_{s,A_n} = x] \leq C.
\]

Moreover, for all distinct \( s_1, s_2 \in R_n \),

\[
\sup_{s_1, s_2, A_n} \sup_{x_1, x_2 \in S_c} E[|W_{s_1,A_n}|W_{s_2,A_n}|X_{s_1,A_n} = x_1, X_{s_2,A_n} = x_2] \leq C.
\]

Furthermore, we assume the following regularity conditions.

**Assumption 3.4.** Let \( a_n = \sqrt{\frac{\log n}{nh^{d+1}}} \). As \( n \to \infty \),

(1) \( h^{-(d+p)/2} a_n h^{d+1} A_n^{d+1} A_n^{-d} (A_2 n^{-1}) \to 0 \) and \( A_2 n^{d} A_n^{-d} n h^{d+1} (\log n) \to 0 \),

(2) \( n^{1/2} h^{(d+p)/2} A_n^{1/2} n^{1/2} \geq C_0 n^\eta \) for some \( 0 < C_0 < \infty \) and \( \eta > 0 \),

(3) \( A_n^{d/p} h^n \to \infty \),
where \( \zeta \) is a positive constant that appears in Assumption 3.3.

Discussions on the assumptions regarding the mixing condition in Assumptions 3.4 and 3.6 (introduced in Section 3.3) are given in Section 5 and Appendix A.

**Proposition 3.1.** Let \( S_c \) be a compact subset of \( \mathbb{R}^p \). Then, under Assumptions 2.1, 3.1, 3.2, 3.3 and 3.4 the following results hold for \( P_S \) almost surely:

\[
\sup_{u \in [0,1]^d, x \in S_c} |\hat{\psi}(u, x) - m(u, x)| = O_P(\sqrt{\log n}/nh^{d+p})
\]

3.3. **Asymptotic properties of \( \hat{m} \).** Next, we present our main results for the estimation of mean function \( m \) for the model (1.1).

**Theorem 3.1.** Let \( I_h = [C_1 h, 1 - C_1 h]^d \) and let \( S_c \) be a compact subset of \( \mathbb{R}^p \). Suppose that \( \inf_{u \in [0,1]^d, x \in S_c} f(u, x) > 0 \). Then, under Assumptions 2.1, 3.1, 3.2, 3.3 (with \( W_{s_j, A_n} = 1 \) and \( \epsilon_{s_j, A_n} \)) and 3.4 the following result holds for \( P_S \) almost surely:

\[
\sup_{u \in I_h, x \in S_c} |\hat{m}(u, x) - m(u, x)| = O_P(\sqrt{\log n}/nh^{d+p} + h^2 + 1/A_n h^p),
\]

where \( r = \min\{1, \rho\} \).

The term \( A_n^{-\rho} h^{-p} \) in the convergence rate arises from the local stationarity of \( X_{s,A_n} \), i.e., the approximation error of \( X_{s,A_n} \) by a stationary random field \( X_u(s) \).

To demonstrate the asymptotic normality of the estimator, we additionally assume the following conditions:

**Assumption 3.5.** (Ua1) (U1), (U2) and (U3) in Assumption 3.3 hold. (Ua2) for all distinct \( s_1, s_2, s_3 \in R_n \), there exists a constant \( C < \infty \) such that

\[
\sup_{s_1, s_2, s_3, A_n, x_1, x_2, x_3 \in S_c} E[||W_{s_1, A_n}||W_{s_2, A_n}||W_{s_3, A_n}||X_{s_1, A_n} = x_1, X_{s_2, A_n} = x_2, X_{s_3, A_n} = x_3|] \leq C.
\]

**Assumption 3.6.** As \( n \to \infty \),

(Ra1) (R1) and (R2) in Assumption 3.4 hold.

(Ra2) \( A_n^{d/3}h^{p+2} \to \infty \).

(Ra3)

\[
\left( \frac{1}{nh^{p+2}} \right)^{1/3} \left( \frac{A_{1,n}}{A_n} \right)^{2d/3} \left( \frac{A_{2,n}}{A_{1,n}} \right)^{2/3} \left( \frac{1}{A_{1,n}} \right)^{1/3} \left( \frac{A_{4,n}}{A_{1,n}} \right)^{1/3} \sum_{k=1}^{A_{4,n}/A_{1,n}} k^{d-1} \beta_1^{1/3} (kA_{1,n} + A_{2,n}) \to 0.
\]

The asymptotic normality of the kernel estimators can be established under \( \alpha \)-mixing that is weaker than \( \beta \)-mixing. See also (2.5) for the definition of \( \alpha \)-mixing coefficients for random fields.

**Theorem 3.2.** Suppose that \( f(u, x) > 0, f_S(u) > 0 \) and \( \epsilon_{s_j, A_n} = \sigma \left( \frac{s_j}{A_n}, x \right) \epsilon_j \), where \( \sigma(\cdot, \cdot) \) is continuous and \( \{\epsilon_j\}_{j=1}^n \) is a sequence of i.i.d. random variables with mean zero and variance 1. Moreover, suppose \( nh^{d+p+4} \to c_0 \) for a constant \( c_0 \). Then under Assumptions 2.1, 3.1, 3.2, 3.5 and 3.6 with \( W_{s_j, A_n} = 1 \) and \( \epsilon_{s_j, A_n} \) in Assumption 3.5, the following result holds for \( P_S \) almost surely:

\[
\sqrt{nh^{d+p}}(\hat{m}(u, x) - m(u, x)) \overset{d}{\to} N(B_{u,x}, V_{u,x}),(u,x).
\]
where
\[
B_{u,x} = \sqrt{c_0 \frac{\kappa_2}{2}} \left\{ \sum_{i=1}^{d} (2\partial_{u_i} m(u,x)\partial_{u_i} f(u,x) + \partial_{u_i u_i}^2 m(u,x)f(u,x)) + \sum_{k=1}^{p} (2\partial_{x_k} m(u,x)\partial_{x_k} f(u,x) + \partial_{x_k x_k}^2 m(u,x)f(u,x)) \right\}
\]
and
\[
V_{u,x} = \kappa_0^{d+p} \sigma^2(u,x)/(f_S(u)f(u,x)) \quad \text{with} \quad \kappa_0 = \int_{\mathbb{R}} K^2(x)dx \quad \text{and} \quad \kappa_2 = \int_{\mathbb{R}} x^2 K(x)dx.
\]
The same result holds true even if we replace the conditions on \(\beta\)-mixing coefficients in Assumptions 3.1, 3.5 and 3.6 with those on \(\alpha\)-mixing coefficients defined by (2.5).

The asymptotic variance \(V_{u,x}\) depends on the sampling density \(f_S(u)\) through \(1/f_S(u)\). This dependence of variance on sampling density differs in stochastic and nonstochastic (or equidistant) sampling designs. Intuitively, the result implies that around “hot-spots,” where more sampling sites tend to be selected, the asymptotic variance of the estimator is smaller than that of the other spatial points. The same applies to Theorem 4.2.

Remark 3.2. As shown in Theorem 3.2, the expressions of asymptotic bias and variance of the kernel estimator are very similar in structure to those from a standard random design for stationary time series and random fields. Therefore, we conjecture that the methods to choose the bandwidth in such a design can be adapted to our setting. In particular, using the formulas for the asymptotic bias and variance from Theorem 3.2 it should be possible to select the bandwidth via plug-in methods.

Remark 3.3. Set \(A_n^d = O(n^{1-\bar{\eta}_1})\) for some \(\bar{\eta}_1 \in [0,1)\), \(A_{1,n} = O(A_n^{\gamma A_1})\), \(A_{2,n} = O(A_n^{\gamma A_2})\) with \(0 < \gamma A_2 < \gamma A_1 < 1/3\) and \(r = \min\{1,\rho\} = 1\). Assume that we can take a sufficiently large \(\zeta > 2\) such that \(\frac{2}{\zeta} < (1 - \bar{\eta}_1)(1 - 3\gamma A_1)\). Then, Assumption 3.6 is satisfied for \(d \geq 1\) and \(p \geq 1\). See Remarks A.4 and B.1 for details.

4. Additive models

In the previous section, we considered general kernel estimators and discussed their asymptotic properties; however, the estimators are adversely affected by dimensionality. In particular, the convergence rate \(O_{\text{Pr}} \left( \sqrt{\log n/nh^{d+p}} \right)\) of the estimators deteriorated as the dimension of the covariates \(p\) increases. Hence, we consider additive models inspired by the idea presented in Vogt (2012), which is based on the smooth backfitting method developed by Mammen et al. (1999), and studied the asymptotic properties of the estimators of additive functions.

4.1. Construction of estimators. We place the following structural constraint on \(m(u,x)\):
\[
E[Y_s,A_n | X_{s,A_n} = x] = m \left( \frac{s}{A_n}, x \right) = m_0 \left( \frac{s}{A_n} \right) + \sum_{\ell=1}^{p} m_\ell \left( \frac{s}{A_n}, x_\ell \right).
\]
Model (4.1) is a natural extension of the following linear regression models with spatially varying coefficients (cf. Gelfand et al. (2003)):

$$Y_{s,A_n} = \beta_0 \left( \frac{s}{A_n} \right) + \sum_{\ell=1}^{p} \beta_{\ell} \left( \frac{s}{A_n} \right) X_{s,A_n}^{\ell} + \epsilon_{s,A_n}.$$ 

**Remark 4.1.** In this paper, we focus on the circumvention of the curse of dimensionality that comes from the number of covariates since in some applications it seems not suitable to consider additive components over the spatial coordinates. For example, many environmental and climate data such as temperature and precipitation would have not spatial coordinate-wise but location specific features. Moreover, one of our motivation is extending the spatially varying linear regression models in Gelfand et al. (2003), which is one of the influential papers in spatial statistics, to nonparametric, non-stationary and non-Gaussian settings. They apply their methods to the analysis of log selling price of single family homes and their regression model does not include additive components over spatial coordinates. Brockwell and Matsuda (2017) also considers regression models that do not include additive components over spatial coordinates and estimating parameters of a CARMA random field with an application of their methods to land price data. Therefore, we believe our modeling would not be restrictive in many empirical applications.

To identify the additive function of the model (4.1) within a unit cube $[0,1]^p$, we impose the condition that

$$\int m_{\ell}(u,x_{\ell}) p_{\ell}(u,x_{\ell}) dx_{\ell} = 0, \; \ell = 1, \ldots, p$$

and all rescaled space points $u \in [0,1]^d$. Here, the functions $p_{\ell}(u,x_{\ell}) = \int_{\mathbb{R}^p} p(u,x) dx - \ell$ are the marginals of the density

$$p(u,x) = \frac{I(x \in [0,1]^p)f(u,x)}{P(X_u(0) \in [0,1]^p)},$$

where $f(u,\cdot)$ is the density of the strictly stationary random field $\{X_u(s)\}$.

To estimate the functions $m_0, \ldots, m_p$, we apply the strategy used in Vogt (2012), which is based on the smooth backfitting technique developed in Mammen et al. (1999). First, we introduce the auxiliary estimates.

$$\hat{\tilde{p}}(u,x) = \frac{1}{n_{[0,1]^p}} \sum_{j=1}^{n} I(X_{s_j,A_n} \in [0,1]^p) \tilde{K}_h \left( u, \frac{s_j}{A_n} \right) \prod_{\ell=1}^{p} K_h \left( x_{\ell}, X_{s_j,A_n}^{\ell} \right),$$

$$\hat{\tilde{m}}(u,x) = \frac{1}{n_{[0,1]^p}} \sum_{j=1}^{n} I(X_{s_j,A_n} \in [0,1]^p) \tilde{K}_h \left( u, \frac{s_j}{A_n} \right) \prod_{\ell=1}^{p} K_h \left( x_{\ell}, X_{s_j,A_n}^{\ell} \right) Y_{s_j,A_n}/\hat{\tilde{p}}(u,x),$$

where $\tilde{K}_h \left( u, \frac{s_j}{A_n} \right) = \prod_{\ell=1}^{d} K_h \left( u_{\ell}, \frac{s_{j,\ell}}{A_n} \right)$, $\hat{\tilde{p}}(u,x)$ is a kernel estimator for the function $f_S(u)p(u,x)$, and $\hat{\tilde{m}}(u,x)$ is a $(p+d)$-dimensional kernel estimator that estimates $m(u,x)$ for $x \in [0,1]^p$. In the aforementioned definitions,

$$n_{[0,1]^p} = \sum_{j=1}^{n} \tilde{K}_h \left( u, \frac{s_j}{A_n} \right) I(X_{s_j,A_n} \in [0,1]^p)/\tilde{f}_S(u).$$
is the number of observations in the unit cube $[0,1]^p$, where only space points close to \( u \) are considered. Furthermore, \( \tilde{f}_{S}(u) = \frac{1}{n} \sum_{j=1}^{n} \tilde{K}_{h} \left( u, \frac{S_{j}}{A_{n}} \right) \), and

\[
K_{h}(v, w) = I(v, w \in [0,1]) \frac{K_{h}(v-w)}{\int_{[0,1]} K_{h}(s-w)ds}
\]
is a modified kernel weight. This weight possesses the property that \( \int_{[0,1]} K_{h}(v,w)dw = 1 \) for all \( w \in [0,1] \), which is required to derive the asymptotic properties of the backfitting estimates.

Given the estimators \( \hat{p} \) and \( \hat{m} \), we define the smooth backfitting estimates \( \hat{m}_0(u), \hat{m}_1(u, \cdot), \ldots, \hat{m}_d(u, \cdot) \) of the functions \( m_0(u), m_1(u, \cdot), \ldots, m_d(u, \cdot) \) at the space point \( u \in [0,1]^d \) as the minimizer of the criterion

\[
\int_{[0,1]^p} \left\{ \hat{m}(u, w) - \left( g_0 + \sum_{\ell=1}^{p} g_{\ell}(w_{\ell}) \right) \right\}^2 \hat{p}(u, w)dw, \tag{4.2}
\]

where the minimization runs over all additive functions \( g(x) = g_0 + \sum_{\ell=1}^{p} g_{\ell}(x_{\ell}) \), whose components are normalized to satisfy

\[
\int g_{\ell}(w_{\ell})\hat{p}_{\ell}(u, w_{\ell})dw_{\ell} = 0, \ \ell = 1, \ldots, p.
\]

Here, \( \hat{p}_{\ell}(u, x_{\ell}) = \int_{[0,1]^{p-1}} \hat{p}(u, x)dx_{\cdot-\ell} \) is the marginal of the kernel density \( \hat{p}(u, \cdot) \) at point \( x_{\ell} \). According to (4.2), the estimate \( \hat{m}(u, \cdot) = \hat{m}_0(u) + \sum_{\ell=1}^{p} \hat{m}_{\ell}(u, \cdot) \) is an \( L^2 \)-projection of the full dimensional kernel estimate \( \hat{m}(u, \cdot) \) on to the subspace of additive functions, where the projection is performed with respect to the density estimate \( \hat{p}(u, \cdot) \).

By differentiation, we can demonstrate that the solution of (4.2) is characterized by the system of equations

\[
\hat{m}_{\ell}(u, x_{\ell}) = \hat{m}_{\ell}(u, x_{\ell}) - \sum_{k \neq \ell} \int_{[0,1]} \hat{m}_{k}(u, x_{k})\frac{\hat{p}_{\ell,k}(u, x_{\ell}, x_{k})}{\hat{p}_{\ell}(u, x_{\ell})}dx_{k} - \hat{m}_0(u)
\]
together with

\[
\int \hat{m}_{\ell}(u, w_{\ell})\hat{p}_{\ell}(u, w_{\ell})dw_{\ell} = 0, \ \ell = 1, \ldots, p,
\]

where \( \hat{p}_{\ell} \) and \( \hat{p}_{\ell,k} \) are kernel density estimates, and \( \hat{m}_{\ell} \) is a kernel estimator defined as

\[
\hat{p}_{\ell}(u, x_{\ell}) = \frac{1}{n_{[0,1]^p}} \sum_{j=1}^{n} I \left( X_{s_{j},A_{n}} \in [0,1]^p \right) \hat{K}_{h} \left( u, \frac{S_{j}}{A_{n}} \right) \hat{K}_{h} \left( x_{\ell}, X_{s_{j},A_{n}} \right),
\]

\[
\hat{p}_{\ell,k}(u, x_{\ell}, x_{k}) = \frac{1}{n_{[0,1]^p}} \sum_{j=1}^{n} I \left( X_{s_{j},A_{n}} \in [0,1]^p \right) \hat{K}_{h} \left( u, \frac{S_{j}}{A_{n}} \right) \hat{K}_{h} \left( x_{\ell}, X_{s_{j},A_{n}} \right) \hat{K}_{h} \left( x_{k}, X_{s_{j},A_{n}} \right),
\]

\[
\hat{m}_{\ell}(u, x_{\ell}) = \frac{1}{n_{[0,1]^p}} \sum_{j=1}^{n} I \left( X_{s_{j},A_{n}} \in [0,1]^p \right) \hat{K}_{h} \left( u, \frac{S_{j}}{A_{n}} \right) \hat{K}_{h} \left( x_{\ell}, X_{s_{j},A_{n}} \right) Y_{s_{j},A_{n}}/\hat{p}_{\ell}(u, x_{\ell}).
\]

Moreover, the estimate \( \hat{m}_0(u) \) of the model constant at space point \( u \) is expressed as

\[
\hat{m}_0(u) = \frac{1}{n_{[0,1]^p}} \sum_{j=1}^{n} I \left( X_{s_{j},A_{n}} \in [0,1]^p \right) \hat{K}_{h} \left( u, \frac{S_{j}}{A_{n}} \right) Y_{s_{j},A_{n}}/\tilde{f}_{S}(u),
\]

where \( \tilde{f}_{S}(u) = \frac{1}{n_{[0,1]^p}} \sum_{j=1}^{n} I \left( X_{s_{j},A_{n}} \in [0,1]^p \right) \tilde{K}_{h} \left( u, \frac{S_{j}}{A_{n}} \right) \).
4.2. Asymptotic properties of estimators. Now we present the uniform convergence rate and joint asymptotic normality of the estimators. Before describing the results, we summarize the set of assumptions made.

**Assumption 4.1.** Let 
\[ a_n = \sqrt{\frac{\log n}{nh^{d+1}}} \] 
and \( r = \min\{\rho, 1\} \), where \( \rho \) is a positive constant that appears in Assumption 3.1. As \( n \to \infty \),

(Rb1) \( h^{-(d+1)}a_n^{d+1}A_n^{d^2}A_1\beta(A_{2n}; A_n^d) \to 0 \) and \( A_n^{d^2}A_1\beta(A_{2n}; A_n^d) \to 0 \),

(Rb2) \( \frac{n^{1/2}h^{(d+1)/2}}{A_1^{r}n^{3/2}} \geq C_0n^\eta \) for some \( C_0 > 0 \) and \( \eta > 0 \),

(Rb3) \( A_n^{dr}h^3 \to \infty \) and \( A_n^{dr/(1+r)}h^2 \to \infty \),

(Rb4) \( nh^{d+4} \to \infty \).

(Rb5) (Ra3) in Assumption 3.6 holds with \( \zeta \) a positive constant that appears in Assumption 3.5.

The following results are extension of Theorems 5.1 and 5.2 in Vogt (2012) to locally stationary random fields with irregularly spaced observations.

**Theorem 4.1.** Let \( I_{h,0} = [2C_1h, 1-2C_1h] \) and \( I_h = [2C_1h, 1-2C_1h]^d \). Suppose that \( \inf_{u \in [0,1]^d, x \in [0,1]^p} f(u, x) > 0 \) and \( \inf_{u \in [0,1]^d} f_S(u) > 0 \). Then, under Assumptions 2.1, 3.1, 3.2, 3.5 and 4.1 with \( \epsilon_{s_j, A_n} = 1 \) and \( \epsilon_{s_j, A_n} \) and 3.1, the following result holds for \( P_S \) almost surely:

\[
\sup_{u \in I_{h,0}, x \in I_h} |\tilde{m}_\ell(u, x) - m_\ell(u, x)| = O_{P_S} \left( \sqrt{\frac{\log n}{nh^{d+1}}} + h^2 \right), \ \ell = 1, \ldots, p.
\]

As is the case with general regression function, the asymptotic normality of the kernel estimators for additive functions can be established under \( \alpha \)-mixing.

**Theorem 4.2.** Suppose that \( \inf_{u \in [0,1]^d, x \in [0,1]^p} f(u, x) > 0 \) and \( \inf_{u \in [0,1]^d} f_S(u) > 0 \). Moreover, suppose that \( \epsilon_{s_j, A_n} \) given \( X_{s_j, A_n} \) are i.i.d. random variables and continuous functions \( \sigma_\ell(u, x) \), \( 1 \leq \ell \leq p \) exist such that \( \sigma_\ell^2\epsilon_0^2 \to 0 \) and \( n_{[0,1]^p}h^{d+5} - c_0 = o_{P_S}(1) \) for a constant \( c_0 \), \( P_S \) almost surely. Then, under Assumptions 2.1, 3.1, 3.2, 3.5 and 3.1 with \( \epsilon_{s_j, A_n} = 1 \) and \( \epsilon_{s_j, A_n} \) in Assumption 3.5, for any \( u \in (0,1)^d, x_1, \ldots, x_p \in (0,1), \) the following result holds for \( P_S \) almost surely:

\[
\sqrt{n_{[0,1]^p}h^{d+1}} \begin{pmatrix}
\tilde{m}_1(u, x) - m_1(u, x) \\
\vdots \\
\tilde{m}_p(u, x) - m_p(u, x)
\end{pmatrix} \xrightarrow{d} N(B_{u,x}, V_{u,x}).
\]

Here, \( V_{u,x} = \text{diag}(v_1(u, x), \ldots, v_p(u, x)) \) is a \( p \times p \) diagonal matrix with

\[
v_\ell(u, x) = \kappa_0^{d+1} \sigma_\ell^2(u, x)/f_S(u)p_\ell(u, x),
\]

where \( \kappa_0 = \int_{\mathbb{R}} K^2(x)dx \). \( B_{u,x} \) has the form

\[
B_{u,x} = \sqrt{\epsilon_0^2(\beta_1(u, x) - \gamma_1(u), \ldots, \beta_p(u, x) - \gamma_p(u))}.
\]

The functions \( \beta_\ell(u, \cdot) \) are defined as the minimizer of the problem

\[
\int_{\mathbb{R}^p} \left\{ \beta(u, x) - \left( b_0 + \sum_{\ell=1}^p b_\ell(x) \right) \right\}^2 p(u, x)dx,
\]
where the minimization runs over all additive functions \( b(x) = b_0 + \sum_{\ell=1}^p b_{\ell}(x) \) with \( \int_{\mathbb{R}} b_{\ell}(x) p_{\ell}(u, x) dx = 0 \); the function \( \beta(u, x) \) is provided in Lemma \[B.7\] in Appendix. \( \gamma(u) \) can be characterized by the equation \( \int_{\mathbb{R}} \alpha_{\ell}(u, x) \tilde{p}_{\ell}(u, x) dx = h^2 \gamma_{\ell}(u) + o_{P, \ell}(h^2) \), where \( \alpha_{\ell}(u, x) \) are defined in Lemma \[B.6\].

The same result holds true even if we replace the conditions on \( \alpha \)-mixing coefficients in Assumptions \[3.1\], \[3.2\] and \[4.1\] with those on \( \alpha \)-mixing coefficients defined by \[2.5\].

**Remark 4.2.** Set \( A_n^d = n^{1-q_1} \) for some \( q_1 \in [0, 1) \) and \( q_2 \geq 1 \). \( A_{1,n} = O(A_{n}^{\gamma_A}) \) with \( \gamma_{A_1} \in (0, 1/3) \) and \( r = 1 \). Assume that we can take a sufficiently large \( \zeta > 2 \). In this case, Assumption \[4.4\] is satisfied for \( d \geq 1 \) and \( p \geq 1 \).

5. **Examples of locally stationary random fields on \( \mathbb{R}^d \)**

In this section, we present examples of locally stationary random fields. In particular, we discuss Lévy-driven moving average (MA) random fields, which represent a wide class of random fields. One of the important features of this class is that it includes non-Gaussian random fields in addition to Gaussian random fields. Moreover, we provide sufficient conditions to demonstrate that a random field is locally stationary and can be approximated by a stationary Lévy-driven MA random field. We focus on the case of \( p = 1 \) to simplify our discussion. The results in this section can be extended to the multivariate case \( (p \geq 2) \). See Appendix \[D\] herein for the definition of multivariate Lévy-driven MA random fields on \( \mathbb{R}^d \) and for a discussion on their properties.

### 5.1. Lévy-driven MA random fields.

Brockwell and Matsuda (2017) generalized CARMA\((p, q)\) processes on \( \mathbb{R} \) to CARMA\((p, q)\) random fields on \( \mathbb{R}^d \), which is a special class of Lévy-driven MA random fields. See also Brockwell (2000), Brockwell (2001), Marquardt and Stelzer (2007) and Schlemm and Stelzer (2012) for examples of the CARMA process on \( \mathbb{R} \) and Matsuda and Yuan (2020) for multivariate extension of CARMA random fields on \( \mathbb{R}^2 \) and their parametric inference. Let \( L = \{L(A) : A \in \mathcal{B}(\mathbb{R}^d)\} \) be an infinitely divisible random measure on some probability space \( (\Omega, \mathcal{A}, P) \), i.e., a random measure such that

1. for each sequence \( (E_m)_{m \in \mathbb{N}} \) of disjoint sets in \( \mathcal{B}(\mathbb{R}^d) \),
   a. \( L(\bigcup_{m=1}^{\infty} E_m) = \sum_{m=1}^{\infty} L(E_m) \) a.s. whenever \( \bigcup_{m=1}^{\infty} E_m \in \mathcal{B}(\mathbb{R}^d) \),
   b. \( (L(E_m))_{m \in \mathbb{N}} \) is a sequence of independent random variables.

2. the random variable \( L(A) \) has an infinitely divisible distribution for any \( A \in \mathcal{B}(\mathbb{R}^d) \).

The characteristic function of \( L(A) \) which will be denoted by \( \varphi_{L(A)}(t) \), has a Lévy–Khintchine representation of the form \( \varphi_{L(A)}(t) = \exp(\langle A | \psi(t) \rangle) \) with

\[
\psi(t) = it\gamma_0 + \frac{1}{2}t^2\sigma_0 + \int_{\mathbb{R}} \left\{ e^{itx} - 1 - itxI(x \in [-1, 1]) \right\} \nu_0(x) dx
\]

where \( i = \sqrt{-1}, \gamma_0 \in \mathbb{R}, 0 \leq \sigma_0 < \infty, \nu_0 \) is a Lévy density with \( \int_{\mathbb{R}} \min\{1, x^2\} \nu_0(x) dx < \infty \), and \( |A| \) is the Lebesgue measure of \( A \). The triplet \( (\gamma_0, \sigma_0, \nu_0) \) is called the Lévy characteristic of \( L \); it uniquely determines the distribution of random measure \( L \). We refer to Sato (1999) and Bertoin (1996) as standard references on Lévy processes and Rajput and Rosinski (1989) for details on the theory of infinitely divisible measures and fields. Let \( a(z) = z^p + a_1 z^{p-1} + \cdots + a_p \) be a polynomial of degree \( p \) with real coefficients and distinct negative zeros \( \lambda_1, \ldots, \lambda_p \), and let \( b(z) = b_0 + b_1 z + \cdots + b_q z^q = \prod_{i=1}^{q} (z - \xi_i) \) be a polynomial with real coefficients and real zeros \( \xi_1, \ldots, \xi_q \) such that \( b_q = 1 \) and \( 0 \leq q < p \) and \( \lambda_i^2 \neq \xi_j^2 \) for all \( i \) and \( j \). Define \( a(z) = \prod_{i=1}^{p} (z^2 - \lambda_i^2) \).
and \( b(z) = \prod_{i=1}^{q}(z^2 - \xi_i^2) \). A Lévy-driven MA random field driven by an infinitely divisible random measure \( L \), which we call Lévy random measure, is defined by
\[
X(s) = \int_{\mathbb{R}^d} g(s-v)L(dv)
\]
for every \( s \in \mathbb{R}^d \). In particular, when \( g(\cdot) \) is a kernel function of the form
\[
g(s) = \sum_{i=1}^{p} \frac{b(\lambda_i)}{a'(\lambda_i)} e^{\lambda_i\|s\|},
\]
where \( a' \) denotes the derivative of the polynomial \( a \), \( X(s) \) is a univariate (isotropic) CARMA\((p, q)\) random field.

**Remark 5.1** (Connections to SPDEs). Berger [2020] characterizes (5.1) as a solution of a (fractional) stochastic partial differential equation (SPDE). The author also extends the concept of CARMA\((p, q)\) random fields as a strictly stationary solution of an SPDE. The uniqueness of the solution is discussed in Berger [2011].

### 5.2. Stationary distribution of Lévy-driven MA random fields

When the Lévy-driven MA random field (5.1) is strictly stationary, the characteristic function of the stationary distribution of \( X \) is expressed as
\[
\varphi_X(0)(t) = \mathbb{E}[e^{itX(0)}] = \exp \left( \int_{\mathbb{R}^d} H(tg(s))ds \right),
\]
where
\[
\int_{\mathbb{R}^d} H(tg(s))ds = it\gamma_1 - \frac{1}{2}t^2\sigma_1 + \int_{\mathbb{R}} \{e^{itx} - 1 - itxI(x \in [-1,1])\} \nu_1(x)dx,
\]
with
\[
\gamma_1 = \int_{\mathbb{R}^d} U(g(s))ds, \quad \sigma_1 = \sigma_0 \int_{\mathbb{R}^d} g^2(s)ds, \quad \nu_1(x) = \int_{S_g} \frac{1}{g(s)}\nu_0 \left( \frac{x}{g(s)} \right) ds.
\]
Here, \( S_g = \text{supp}(g) = \{s \in \mathbb{R}^d : g(s) \neq 0 \} \) denotes the support of \( g \); the function \( U \) is defined as follows:
\[
U(u) = u \left( \gamma_0 + \int_{\mathbb{R}} x \{I(ux \in [-1,1]) - I(x \in [-1,1])\} \nu_0(x)dx \right).
\]
The triplet \((\gamma_1, \sigma_1, \nu_1)\) is again referred to as the Lévy characteristic of \( X(0) \) and determines the distribution of \( X(0) \) uniquely. See Karcher et al. [2019] for details. The representation of (5.3) implies that the stationary distribution of \( X(s) \) has a density function when the Lévy random measure \( L \) is Gaussian, i.e., \((\gamma_0, \sigma_0, \nu_0) = (\gamma_0, \sigma_0, 0)\). When \( L \) is purely non-Gaussian, i.e., \((\gamma_0, \sigma_0, \nu_0) = (\gamma_0, 0, \nu_0)\), we can demonstrate that \( X(s) \) has a bounded continuous, infinitely differentiable density if
\[
\nu_0(x) = \frac{c_\beta}{|x|^{1+\beta}}g_0(x), \quad x \neq 0
\]
where \( c_\beta > 0, g_0 \) is a positive, continuous, and bounded function on \( \mathbb{R} \) with \( \lim_{|x| \to 0}g_0(x) = 1 \) and \( \beta \in (0, 1) \). In fact, we can verify that a constant \( \tilde{C} > 0 \) exists such that
\[
u_0(x)\geq\tilde{C}|u|^{2-\alpha}
\]
for any $|u| \geq c_0 > 1$, where $\alpha = 2 - \beta \in (0, 2)$. This implies the existence of the density function of $X(s)$. See Lemma 4.3.9 for the proof of (5.3) and Lemma 4.3.1 for the sufficient condition for the existence of the density of $X(s)$ with a purely non-Gaussian Lévy random measure in a multivariate case. A Lévy process with a Lévy density of the form (5.3) is called a locally stable Lévy process and it can represent important Lévy processes such as tempered stable and normal inverse Gaussian processes (cf. Section 2 in Masuda (2012), Section 5 in Kato and Kurisu (2020) and Kurisu et al. (2021)).

5.3. **Locally stationary Lévy-driven MA random fields.** Next, we present examples of locally stationary Lévy-driven MA random fields. Consider the following processes

$$X_{s,A_n} = \int_{\mathbb{R}^d} g \left( \frac{s}{A_n}, \|s - v\| \right) L(dv)$$

and

$$X_u(s) = \int_{\mathbb{R}^d} g \left( u, \|s - v\| \right) L(dv),$$

where $g : [0, 1]^d \times [0, \infty) \to \mathbb{R}$ is a bounded function such that $|g(u, \cdot) - g(v, \cdot)| \leq C \|u - v\| g(\cdot)$ with $C < \infty$ and for any $u \in [0, 1]^d$, $\int_{\mathbb{R}^d} |g(u, \|s\|)| + |\bar{g}(s)| ds < \infty$ and $\int_{\mathbb{R}^d} (g^2(u, \|s\|) + \bar{g}^2(s)) ds < \infty$. Notably, $X_u(s)$ is a strictly stationary random field for each $u$. If $E[|L(A)|^{q'}] < \infty$ for any $A \in \mathcal{B}(\mathbb{R}^d)$ with a bounded Lebesgue measure $|A|$ and for $q' = 1, 2$, we have that

$$E[X_u(s)] = \mu_0 \int_{\mathbb{R}^d} g(u, \|s\|) ds$$

and

$$E[(X_u(s))^2] = \sigma_0^2 \int_{\mathbb{R}^d} g^2(u, \|s\|) ds,$$

where $\mu_0 = -i\psi'(0)$ and $\sigma_0^2 = -\psi''(0)$. Let $m > 0$. Define the function $\iota(\cdot : m) : [0, \infty) \to [0, 1]$ as

$$\iota(x : m) = \begin{cases} 1 & \text{if } 0 \leq x \leq \frac{m}{2}, \\ -\frac{2}{m} x + 2 & \text{if } \frac{m}{2} < x \leq m, \\ 0 & \text{if } m < x \end{cases}$$

and consider the process $X_u(s : A_{2,n}) = \int_{\mathbb{R}^d} g(u, \|s - v\|) \iota(\|s - v\| : A_{2,n}) L(dv)$. Notably, $X_u(s : A_{2,n})$ is also an $A_{2,n}$-dependent strictly stationary random field, i.e., the $\beta$-mixing coefficients $\beta(a; b) = \beta_1(a) g_1(b)$ of $X_u(s : A_{2,n})$ satisfy $\beta_1(a) = 0$ for $a \geq 2A_{2,n}$. Observe that

$$|X_{s,A_n} - X_u(s : A_{2,n})| \leq |X_{s,A_n} - X_u(s)| + |X_u(s) - X_u(s : A_{2,n})|$$

where

$$\begin{align*}
&= \int_{\mathbb{R}^d} \left| g \left( \frac{s}{A_n}, \|s - v\| \right) - g \left( u, \|s - v\| \right) \right| L(dv) \\
&\quad + \frac{1}{A_n^d} \int_{\mathbb{R}^d} A_n^d |g(u, \|s - v\|) | (1 - \iota(\|s - v\| : A_{2,n}) ) L(dv) \\
&\leq \left( \left| \frac{s}{A_n} - u \right| \int_{\mathbb{R}^d} Cg(\|s - v\|) |(1 - \iota(\|s - v\| : A_{2,n}) ) L(dv) \\
&\quad + \frac{1}{A_n^d} \int_{\mathbb{R}^d} A_n^d |g(u, \|s - v\|) |(1 - \iota(\|s - v\| : A_{2,n}) ) L(dv) \\
&\quad =: g_{u, A_n, A_{2,n}}(\|s - v\|) \right) \\
&\quad \leq \left( \left| \frac{s}{A_n} - u \right| + \frac{1}{A_n^d} \right) \int_{\mathbb{R}^d} \left( Cg(\|s - v\|) + A_n^d g_{u, A_n, A_{2,n}}(\|s - v\|) \right) L(dv)\\n&\quad =: \left( \left| \frac{s}{A_n} - u \right| + \frac{1}{A_n^d} \right) U_{s,A_n}(u).
\end{align*}$$

(5.6)
Here, we define $|L(A)|$ as the absolute value of the random variable $L(A)$ for any $A \in B(\mathbb{R}^d)$. If we assume that

$$\sup_{n \geq 1} \sup_{u \in [0,1]^d} \int_{\mathbb{R}^d} (A_n^d g_u(\|s\| : A_{2,n}) + A_{2,n}^d g_u^2(\|s\| : A_{2,n})) ds < \infty,$$  

(5.7)

we obtain

$$E[U^2_{s,A_n}(u)] \leq C \sup_{n \geq 1} \sup_{u \in [0,1]^d} \left( \int_{\mathbb{R}^d} g_{u,A_n,A_{2,n}}^2(\|s\|) ds + \left( \int_{\mathbb{R}^d} g_{u,A_n,A_{2,n}}(\|s\|) ds \right)^2 \right) < \infty.$$  

(5.8)

See Brockwell and Matsuda (2017) for details regarding the computation of the moments of Lévy-driven MA random fields. Note that (5.6) and (5.8) imply that $X_{s,A_n}$ is a locally stationary random field.

Remark 5.2. We also note that (5.6) and (5.8) imply that $X_{s,A_n}$ can be approximated by an $A_{2,n}$-dependent locally stationary random field under the condition (5.7). In this case, the random field $X_{s,A_n}$ is an approximately $A_{2,n}$-dependent locally stationary random field. Let $c_1, \ldots, c_{p_1}$ be positive constants and $r_1(\cdot), \ldots, r_{p_1}(\cdot)$ be continuous functions on $[0,1]^d$ such that $|r_k(u) - r_k(v)| \leq C\|u - v\|$, $1 \leq k \leq p_1$ with $C < \infty$. Consider the function

$$g(u, \|s\|) = \sum_{k=1}^{p_1} r_k(u) e^{-c_k\|s\|}.$$  

(5.9)

We can demonstrate that a Lévy-driven MA random field with the kernel function (5.9) satisfies (5.6); furthermore, locally stationary random fields include a wide class of CARMA-type random fields. For a discussion on general multivariate cases and examples that satisfy our regularity conditions, see also Appendix D herein.

Next, we introduce the concept of approximately $m_n$-dependence for locally stationary random fields.

Definition 5.1 (Approximately $m_n$-dependence for locally stationary random field). Let $m_n$ be a sequence of positive constants with $m_n \rightarrow \infty$ as $n \rightarrow \infty$. We define a locally stationary random field $X_{s,A_n} = \{X_{s,A_n} = (X_{s,A_n}^1, \ldots, X_{s,A_n}^p) : s \in R_n\}$ in $\mathbb{R}^p$ to be an approximately $m_n$-dependent random field if $X_{s,A_n}$ can be represented as a sum of $m_n$-dependent random field $\{X_{s,A_n;m_n}\}$ and the “residual” random field $\{e_{s,A_n;m_n} = (e_{s,A_n;m_n}^1, \ldots, e_{s,A_n;m_n}^p)\}$, which satisfies the following condition:

(Ma0) For some $q > 1$ such that $\frac{q}{\zeta - 1}$ is an even integer,

$$\max_{1 \leq j \leq n} \max_{1 \leq \ell \leq p} \sup_{s \in R_n} E|s|^{\frac{q}{\zeta - 1}} \left( \sum_{m_n}^{\frac{q}{\zeta - 1}} \right) \leq \gamma_c(m_n), \ P_s\text{-a.s.}$$

where $\zeta > 2$ is the constant in Assumption 3.3 and $\gamma_c(\cdot)$ is a decreasing function such that for some $\eta_2 \in (0,1)$

$$\gamma_c(m_n) (n + \frac{A_{1,n}^d \log n}{n^{\frac{1-\eta_2}{2}}})^{\frac{d+2}{d+2}} \rightarrow 0 \ as \ n \rightarrow \infty.$$  

(5.10)

We can show that a univariate locally stationary CARMA-type random field with an exponential decay kernel such as (5.9) and a Lévy random measure that has finite moments up to the $q\zeta/(\zeta - 1)$th
moment, where \( q \zeta / (\zeta - 1) \) is an even integer, i.e., \( E[L([0,1]^d)] < \infty \) for \( 1 \leq q' \leq q \zeta / (\zeta - 1) \), satisfies Condition (Ma0) with
\[
\gamma_t(x) \leq C x^{(d-1)(\zeta-1) / 2} e^{-c_0 x / 2}
\]
for a constant \( C < \infty \), where \( c_0 = \min_{1 \leq p \leq 1} c_p \). This implies (5.10). In this case, the CARMA-type process is approximately \( D(\log n) \)-dependent for a sufficiently large \( D > 0 \). See Lemma [B.8] for the proof of (5.11). See also Appendix [D] for a discussion on multivariate cases.

**Remark 5.3.** We introduced the notion of approximately \( m_n \)-dependent random fields to give some examples of random fields that satisfy our assumptions. For this, it could be possible to adopt the \( m \)-dependence approximation technique in Machkouri et al. (2013) and Machkouri (2014), for example. On the other hand, the proofs of our results are based on a general blocking technique designed for irregularly spaced sampling sites. Moreover, to derive the uniform convergence rate of the kernel estimator, we need to care about the effect of non-equidistant sampling sites when applying a maximal inequality, which requires additional work compared with the case that sampling sites are equidistant. See also Section [2.4] for a discussion of mixing conditions.

Now we summarize our discussion in this subsection.

**Assumption 5.1.** (Ma1) The process \( \{X_{s,A_n}\} \) is an approximately \( A_{2,n} \)-dependent random field. Hence, \( X_{s,A_n} \) can be decomposed into a sum of \( A_{2,n} \)-dependent random fields \( \{X_{s,A_n;A_{2,n}}\} \) and residual random fields \( \{e_{s,A_n;A_{2,n}}\} \), which satisfies Condition (Ma0).

(Ma2) The process \( \{X_{s,A_n}\} \) is an approximately \( A_{2,n} \)-dependent locally stationary random field. Therefore, for each space point \( u \in [0,1]^d \), an \( A_{2,n} \)-dependent strictly stationary random field \( \{X_u(s : A_{2,n})\} \) exists such that
\[
\|X_{s,A_n} - X_u(s : A_{2,n})\| \leq \left( \left\| \frac{s}{A_n} - u \right\| + \frac{1}{A_n^2} \right) U_{s,A_n;A_{2,n}}(u) \text{ a.s.}
\]
with \( E[(U_{s,A_n;A_{2,n}}(u))^{\rho}] \leq C \) for some \( \rho > 0 \).

(Ma3) (M2) and (M4) in Assumption [3.1] hold by replacing \( X_u(s) \) with \( X_u(s : A_{2,n}) \).

(Ma4) (M3) and (M5) in Assumption [3.1] hold.

**Corollary 5.1.** Proposition [3.1] Theorems [3.1] [3.2] holds even if we replace Assumption [3.1] with Assumption [5.1].

**Corollary 5.2.** Theorems [4.1] and [4.2] hold even if we replace Assumption [3.1] with Assumption [5.1] (with \( p = 1 \) in [5.10]).

**Remark 5.4.** Let \( \tilde{\beta}(a;b) = \tilde{\beta}_1(a)\tilde{g}_1(b) \) denote the \( \beta \)-mixing coefficients of \( X_{s,A_n;A_{2,n}} \). When the process \( \{X_{s,A_n}\} \) is an approximately \( A_{2,n} \)-dependent random field and an approximately \( A_{2,n} \)-dependent locally stationary random field, we can replace \( X_{s,A_n} \) with \( X_{s,A_n;A_{2,n}} \) in our analysis. In this case, because the \( \beta \)-mixing coefficients of \( X_{s,A_n;A_{2,n}} \) satisfy \( \tilde{\beta}(A_{2,n};A_n^d) = \tilde{\beta}_1(A_{2,n})\tilde{g}_1(A_n^d) = 0 \), conditions on \( \beta \)-mixing coefficients (R1), (Ra3), (Rb1) are satisfied automatically. See also Remark [A.3] in the Appendix.

Brockwell and Matsuda (2017) developed CARMA random fields and they discuss in the introduction of their paper that even for CARMA random fields, which is a special class of Lévy-driven moving average random fields, the random fields are already generates a much rich class of random fields. Therefore, Lévy-driven moving average random fields are one of the most flexible and
concrete class of random fields from both theoretical and practical point of view. Moreover, it is difficult to compute $\alpha$- or $\beta$-mixing coefficients for non-Gaussian random fields on $\mathbb{R}^d$ and there would be no general tools for the computation. The dependence structure of CARMA random fields on $\mathbb{R}^d$ have not been studied although there are some results on $\beta$-mixing properties of multivariate CARMA processes on $\mathbb{R}$ in [20] for example. Indeed, existing papers on statistical analysis of CARMA random fields on $\mathbb{R}^d$ (cf. [17] and [20]) consider parametric estimation of second-order stationary CARMA random fields but their $(\alpha$- or $\beta$-) mixing properties are not addressed. Hence it is still an open question that CARMA random fields are $\beta$- (or $\alpha$-) mixing. Then, our results in Section 5 are important contributions in the sense that our results in Sections 3 and 4 can be applied to Lévy-driven moving average random fields that is a more flexible class of random fields than CARMA random fields by providing sufficient conditions for $m_n$-dependent approximation of locally stationary random fields and showing that locally stationary CARMA-type random fields can be approximated by $m_n$-dependent random fields.

6. Discussion

In this section we discuss some directions of possible extensions of our results and the model (1.1) for applications to spatio-temporal data.

6.1. Extensions.

6.1.1. On sampling region. It is possible to extend the definition of the sampling region $R_n$ to a more general case that includes nonstandard shapes. For example, we can adopt the definition of sampling regions in [20] as follows: First, we define the sampling region $R_n$. Let $R_0^*$ be an open connected subset of $(-2,2]^d$ containing $[-1,1]^d$ and let $R_0$ be a Borel set satisfying $R_0^* \subset R_0 \subset \overline{R_0}$, where for any set $S \subset \mathbb{R}^d$, $\overline{S}$ denotes its closure. Let $\{A_n\}_{n \geq 1}$ be a sequence of positive numbers such that $A_n \to \infty$ as $n \to \infty$ and consider $R_n = A_n R_0$ as a sampling region. We also assume that for any sequence of positive numbers $\{a_n\}_{n \geq 1}$ with $a_n \to 0$ as $n \to \infty$, the number of cubes of the form $a_n (\ell + [0,1]^d)$, $\ell \in \mathbb{Z}^d$ with their lower left corner $a_n \ell$ on the lattice $a_n \mathbb{Z}^d$ that intersect both $R_0$ and $R_0^*$ is $O(a_n^{-d+1})$ as $n \to \infty$. Moreover, let $f$ be a continuous, everywhere positive probability density function on $R_0$, and let $\{S_{0,i}\}_{i \geq 1}$ be a sequence of i.i.d. random vectors with density $f$. Assume that $\{S_{0,i}\}_{i \geq 1}$ and $X_{s,A_n}$ are independent.

The boundary condition on the prototype set $R_0$ holds in many practical situations, including many convex subsets in $\mathbb{R}^d$, such as spheres, ellipsoids, polyhedrons, as well as many nonconvex sets in $\mathbb{R}^d$. See also [20] and Chapter 12 in [20] for further discussion on the boundary condition.

Under this setting on the sampling region, our results hold under the same assumptions and the proofs are the same. Furthermore, for any set $S \subset \mathbb{R}^d$, let $\text{int}(S) = S \setminus \overline{S}$ be the internal part of $S$ and for any $\delta > 0$, let $S^\delta = \{s \in \mathbb{R}^d : d(s,S) \leq \delta\}$ where $d(s,S) = \inf_{x \in S} \|s - x\|$. Then by replacing $[0,1]^d$ and $(0,1)^d$ that appear in Assumptions 2.1 and 3.1 with $R_0$ and $\text{int}(R_0)$, it is also possible to show uniform convergence results that correspond to Proposition 3.1, Theorem 3.1 and Theorem 4.1 over $(u, x) \in V_0 \times S_c$, $V_{0,h} \times S_c$ and $V_{0,2h} \times I_{h,0}$, respectively. Here, $S_c$ is a compact subset of $\mathbb{R}^d$, and $V_{0,h}, V_{0,2h}$ and $V_0$ are compact subsets of $R_0$ such that $V_{0,2h} \subset V_{0,h}$ and $V_{0,2h}^c = V_{0,h}^c = V_0$. We omit the proofs since they are almost the same as those of Proposition 3.1, Theorem 3.1 and Theorem 4.1.
6.1.2. Constructing confidence bands for regression functions. As an extension of Theorem 3.2 it is straightforward to show joint asymptotic normality of \( \hat{m} \) over finite number of design points \( \{(u_j, x_j)\}_{j=1}^L \) and verify that \( \hat{m}(u_j, x_j) \) are asymptotically independent. Building on the result, it should be possible to construct simple confidence bands by plug-in methods and linear interpolations of the following joint confidence intervals: Let \( \xi_1, \ldots, \xi_L \) be i.i.d. standard normal random variables, and let \( q_\tau \) satisfy \( P(\max_{1 \leq j \leq L} |\xi_j| > q_\tau) = \tau \) for \( \tau \in (0, 1) \) and take a bandwidth \( h \) such that \( h^2 \ll 1/(nh^{d+\tilde{p}}) \). Then,

\[
C_j(1-\tau) = \left[ \hat{m}(u_j, x_j) \pm \sqrt{\frac{V_{u_j,x_j}}{nh^{d+\tilde{p}} q_\tau}} \right], \quad j = 1, \ldots, L
\]

are joint asymptotic 100(1-\( \tau \))% confidence intervals of \( m \). Here, \( [a \pm b] = [a-b, a+b] \) for \( a \in \mathbb{R} \) and \( b > 0 \). More generally, there could be two possible ways to construct confidence bands of the regression function. The first way is based on a Gumbel approximation as considered in Zhao and Wu (2008). The second way is based on intermediate (high-dimensional) Gaussian approximations as considered in Horowitz and Lee (2012). However, we believe that both approaches require additional substantial work and there would be no previous studies on the construction of uniform confidence bands for locally stationary time series or random fields. Therefore, we only explain the idea of both approaches formally to keep the tight focus of the present paper and leave the extension as a future research topic. The idea of both approaches is as follows. Instead of constructing uniform confidence bands of the regression function over \( I_h \) and \( S_c \), we first consider a discretized version of \( I_h \) and \( S_c \) in Theorem 3.1 such that \( \{u_{n,j}\}_{j=1}^{N_u}, \{x_{n,k}\}_{k=1}^{N_x} \) where \( u_{n,j} \in I_h \) and \( x_{n,k} \in S_c \) and the design points \( \{u_{n,j}\}_{j=1}^{N_u}, \{x_{n,k}\}_{k=1}^{N_x} \) are asymptotically dense in \( I_h \) and \( S_c \), i.e. \( N_u, N_x \to \infty, \max_{1 \leq j \neq j' \leq N_u} \|u_{n,j} - u_{n,j'}\| \to 0 \) and \( \max_{1 \leq k_1 \neq k_2 \leq N_x} \|x_{n,k_1} - x_{n,k_2}\| \to 0 \) as \( n \to \infty \). Then we could construct “asymptotically” uniform confidence bands of the regression function by linear interpolations of joint confidence intervals over the design points combining Gumbel or intermediate Gaussian approximations.

6.2. Further discussion on mixing conditions. Here we discuss further on the mixing conditions in this paper. See also Section 2.4 for discussion on mixing conditions from another technical aspects. Consider the following decomposition of a random field \( X_{s,A_n} \) :

\[
X_{s,A_n} = X_{s,A_n} + \xi_{s,A_n}, \quad (6.1)
\]

where \( X_{s,A_n} \) is a locally stationary \( \beta \)-mixing (for uniform estimation) or \( \alpha \)-mixing (for asymptotic normality) random field that satisfies Assumptions 3.1, 3.5 and 3.6 (for general case) or Assumptions 3.1, 3.5 and 4.1 (for additive models), and \( \xi_{s,A_n} = (\xi_{s,A_n}^1, \ldots, \xi_{s,A_n}^p)' \) is a residual random field which is asymptotically negligible. The decomposition (6.1) implies that \( X_{s,A_n} \) is not necessarily \( m_n \)-dependent. A similar argument in Remark A.3 in Appendix A yields that to show the asymptotic negligibility of \( \xi_{s,A_n} \) for our results to hold, it is sufficient to verify the following condition: For some \( q > 1 \) such that \( \frac{q^q}{\xi^q} \) is an even integer,

\[
\max_{1 \leq j \leq n} \max_{1 \leq \ell \leq p} E_{S}[\xi_{s,A_n}^{\xi^{q/\ell}}] \leq c_{\xi}(n), \quad P_{S\text{-a.s.}}
\]
where $\zeta > 2$ is the constant in Assumption 3.3 and $\nu_2(\cdot)$ is a decreasing function such that for some $\eta_2 \in (0,1)$

$$\frac{\nu_2(n + A_n^d \log n)}{n^{(1-\eta_2)/2} h^{d+p+1}} \to 0 \text{ as } n \to \infty. \quad (6.2)$$

As far as (6.2) is satisfied, the random field $X_{s,A_n}$ itself is not necessarily $\beta$-mixing (or $\alpha$-mixing) in general and $\bar{\epsilon}_{s,A_n}$ can accommodate complex dependence structures of $X_{s,A_n}$ that cannot be captured by $\beta$-mixing (or $\alpha$-mixing) random fields. As a special case, $\bar{X}_{s,A_n}$ can be an approximately $m_n$-dependent locally stationary random field and in this case $\bar{X}_{s,A_n}$ also can be decomposed into $m_n$-dependent random field $\bar{X}_{s,A_n,m_n}$ and a residual random field $\bar{\epsilon}_{s,A_n,m_n}$ that corresponds to the approximation error. Then $X_{s,A_n}$ can be represented as follows:

$$X_{s,A_n} = \bar{X}_{s,A_n,m_n} + \bar{\epsilon}_{s,A_n,m_n} + \bar{\epsilon}_{s,A_n}.$$

The Lévy-driven moving average random fields considered in Section 5 of the revised manuscript are examples when $\bar{\epsilon}_{s,A_n} = 0$.

6.3. Spatio-temporal modeling. Consider a spatially locally stationary spatio-temporal data $\{X_{s_j,A_n} : s_j \in R_n\}$ where $X_{s_j,A_n} = (X_{s_j,A_n}(t_1), \ldots, X_{s_j,A_n}(t_{p+1})) : s \in R_n$, that is, at each location $s_j$, $X_{s_j,A_n}$ is a time series observed at time points $t_1 < \ldots < t_{p+1}$ with $p \geq 1$ fixed. Our results can be applied to the following spatio-temporal model:

$$X_{s_j,A_n}(t_{p+1}) = m\left(\frac{s_j}{A_n}, X_{s_j,A_n}(t_1), \ldots, X_{s_j,A_n}(t_p)\right) + \epsilon_{s_j,A_n}, \quad j = 1, \ldots, n.$$

In our framework, the time points can be non-equidistant and the process can be temporally nonstationary since we consider finite number of time points. Additionally, our sampling design is related to that of Altmeyer and Reiß (2019). They study nonparametric estimation of a linear SPDE with a spatially-varying coefficient under the asymptotic regime with fixed time horizon and with the spatial resolution of the observations tending to zero.

6.4. More discussion on locally stationary Lévy-driven MA random fields. Our argument to show (5.6)-(5.8) is limited to non-Gaussian (or pure jump) MA random fields. However, the argument can be extended to the cases that allow the existence of non-trivial Gaussian part of the random measure $L$ as follows: Consider the following Lévy-driven MA random fields with possibly non-vanishing Gaussian part.

$$X_{s,A_n} = \int_{\mathbb{R}^d} g\left(\frac{s}{A_n}, ||s - v||\right) L(dv) \text{ and } X_u(s) = \int_{\mathbb{R}^d} g(u, ||s - v||) L(dv),$$

where $g : [0,1]^d \times [0,\infty) \to \mathbb{R}$ is a bounded function such that $|g(u, \cdot) - g(v, \cdot)| \leq C ||u - v|| \tilde{g}(\cdot)$ with $C < \infty$ and for any $u \in [0,1]^d$,

$$\int_{\mathbb{R}^d} (|g(u, ||s||)| + \tilde{g}(s)) ds < \infty \text{ and } \int_{\mathbb{R}^d} (g^2(u, ||s||) + \tilde{g}^2(s)) ds < \infty.$$

If $E[|L(A)| q'] < \infty$ for any $A \in \mathcal{B}(\mathbb{R}^d)$ with a bounded Lebesgue measure $|A|$ and for $q' = 1, 2$, we have that

$$E[X_u(s)] = \mu_0 \int_{\mathbb{R}^d} g(u, ||s||) ds \text{ and } E\left[(X_u(s))^2\right] = \sigma_0^2 \int_{\mathbb{R}^d} g^2(u, ||s||) ds,$$
where $\mu_0 \in \mathbb{R}$ and $\sigma^2_0 > 0$. Let $m > 0$. Define the function $\iota(\cdot : m) : [0, \infty) \to [0, 1]$ as

$$
\iota(x : m) = \begin{cases} 
1 & \text{if } 0 \leq x \leq \frac{m}{2}, \\
-\frac{2}{m}x + 2 & \text{if } \frac{m}{2} < x \leq m, \\
0 & \text{if } m < x 
\end{cases}
$$

and consider the process $X_u(s : A_{2,n}) = \int_{\mathbb{R}^d} g(u, \|s - v\|) \iota(\|s - v\| : A_{2,n}) L(dv)$.

Assume that $\|s/A_n - u\| > 0$. Observe that

$$
|X_{s,A_n} - X_u(s : A_{2,n})| \leq |X_{s,A_n} - X_u(s)| + |X_u(s) - X_u(s : A_{2,n})|
$$

\[= \left| \int_{\mathbb{R}^d} g \left( \frac{s}{A_n}, \|s - v\| \right) - g(u, \|s - v\|) \right| L(dv) \]

\[+ \frac{1}{A_n^d} \int_{\mathbb{R}^d} A_n^d g \left( u, \|s - v\| \right) \left( 1 - \iota(\|s - v\| : A_{2,n}) \right) L(dv) \]

\[= U_{s,A_n}(u) \]

\[\leq \left( \left| \frac{s}{A_n} - u \right| + \frac{1}{A_n^d} \right) \{ |U_{1,s,A_n}(u)| + |U_{2,s,A_n}(u)| \} \]

\[= \left( \left| \frac{s}{A_n} - u \right| + \frac{1}{A_n^d} \right) U_{s,A_n}(u). \]

Note that

$$
E \left[ U_{1,s,A_n}(u) \right]^2 = \sigma^2_0 \int_{\mathbb{R}^d} \left( g \left( \frac{s}{A_n}, \|z\| \right) - g(u, \|z\|) \right)^2 \frac{\|s/A_n - u\|}{\|z\|} \, dz \leq C^2 \sigma^2_0 \int_{\mathbb{R}^d} g^2(\|z\|) \, dz,
$$

$$
E \left[ U_{2,s,A_n}(u) \right]^2 = \int_{\mathbb{R}^d} A_n^{2d} g_u^2(\|z\| : A_{2,n}) \, dz
$$

If we assume that

$$
\sup_{n \geq 1} \sup_{u \in [0,1]^d} \int_{\mathbb{R}^d} (A_n^d g_u(\|s\| : A_{2,n}) + A_n^{2d} g_u^2(\|s\| : A_{2,n})) \, ds < \infty,
$$

then we obtain

$$
E[U_{s,A_n}(u)] \leq C \sup_{n \geq 1} \sup_{u \in [0,1]^d} \left( \int_{\mathbb{R}^d} g_{u,A_n,A_{2,n}}^2(\|s\|) \, ds + \left( \int_{\mathbb{R}^d} g_{u,A_n,A_{2,n}}(\|s\|) \, ds \right)^2 \right) < \infty.
$$


APPENDIX A. PROOFS

A.1. Proofs for Section 3

Proof of Proposition 3.1. Define $B = \{(u, x) \in \mathbb{R}^{d+p} : u \in [0,1]^d, x \in S_c\}$, $a_n = \sqrt{\log n/nh^{d+p}}$ and $\tau_n = \rho_n n^{1/\zeta}$ with $\rho_n = (\log n)^{\zeta_0}$ for some $\zeta_0 > 0$. Define

$$\hat{\psi}_1(u, x) = \frac{1}{nh^{d+p}} \sum_{j=1}^{n} \bar{K}_h\left(u - \frac{s_j}{A_n}\right) \prod_{\ell_2} K_h\left(x_{\ell_2} - X_{s_j, A_n}^{\ell_2}\right) W_{s_j, A_n} I(|W_{s_j, A_n}| \leq \tau_n),$$

$$\hat{\psi}_2(u, x) = \frac{1}{nh^{d+p}} \sum_{j=1}^{n} \bar{K}_h\left(u - \frac{s_j}{A_n}\right) \prod_{\ell_2} K_h\left(x_{\ell_2} - X_{s_j, A_n}^{\ell_2}\right) W_{s_j, A_n} I(|W_{s_j, A_n}| > \tau_n).$$

Note that $\hat{\psi}(u, x) - E_{|S}|\hat{\psi}(u, x)| = \hat{\psi}_1(u, x) - E_{|S}|\hat{\psi}_1(u, x)| + \hat{\psi}_2(u, x) - E_{|S}|\hat{\psi}_2(u, x)|$.

(Step 1) First we consider the term $P_{|S}\left(\sup_{(u, x) \in B} \left|\hat{\psi}_2(u, x)\right| > C a_n\right) \leq P_{|S}\left(|W_{s_j, A_n}| > \tau_n\right.$ for some $j = 1, \ldots, n$)

$$\leq \tau_n^{-\zeta} \sum_{j=1}^{n} E_{|S}|W_{s_j, A_n}|^\zeta \leq C n \tau_n^{-\zeta} = \rho_n^{-\zeta} \to 0 P_S - a.s.$$

$$E_{|S}\left[|\hat{\psi}_2(u, x)|\right] \leq \frac{1}{nh^{d+p}} \sum_{j=1}^{n} \bar{K}_h\left(u - \frac{s_j}{A_n}\right) \int_{\mathbb{R}^d} \prod_{\ell=1}^{p} K_h(x_{\ell} - w_{\ell})$$

$$\times E_{|S}|W_{s_j, A_n}| I(|W_{s_j, A_n}| > \tau_n) |X_{s_j, A_n} = w| f_{X_{s_j, A_n}}(w) dw$$

$$= \frac{1}{nh^{d}} \sum_{j=1}^{n} \bar{K}_h\left(u - \frac{s_j}{A_n}\right) \int_{\mathbb{R}^d} \prod_{\ell=1}^{p} K(\varphi_{\ell})$$

$$\times E_{|S}|W_{s_j, A_n}| I(|W_{s_j, A_n}| > \tau_n) |X_{s_j, A_n} = x - h\varphi| f_{X_{s_j, A_n}}(x - h\varphi) d\varphi$$

$$= \frac{1}{nh^{d}} \sum_{j=1}^{n} \bar{K}_h\left(u - \frac{s_j}{A_n}\right) \frac{1}{\tau_n^{-\zeta-1}} \int_{\mathbb{R}^d} \prod_{\ell=1}^{p} K(\varphi_{\ell})$$

$$\times E_{|S}|W_{s_j, A_n}|^\zeta I(|W_{s_j, A_n}| > \tau_n) |X_{s_j, A_n} = x - h\varphi| f_{X_{s_j, A_n}}(x - h\varphi) d\varphi$$

$$\leq \frac{C}{\tau_n^{-\zeta-1}} \sum_{j=1}^{n} \bar{K}_h\left(u - \frac{s_j}{A_n}\right)$$

$$= \frac{C}{\tau_n^{-\zeta-1}} \left(f_S(u) + O\left(\sqrt{\frac{\log n}{nh^{d+p}}} + h^2\right)\right) \leq \frac{C}{\tau_n^{-\zeta-1}} = C \rho_n^{-(\zeta-1)} n^{-(\zeta-1)/\zeta} \leq C a_n P_S - a.s.$$

In the last equation, we used Lemma C.3. As a result,

$$\sup_{(u, x) \in B} \left|\hat{\psi}_2(u, x) - E_{|S}|\hat{\psi}_2(u, x)|\right| = O_{P_{|S}}(a_n).$$

(Step 2) Now we consider the term $\hat{\psi}_1(u, x) - E_{|S}|\hat{\psi}_1(u, x)|$. First we introduce some notations.

For $\ell = (\ell_1, \ldots, \ell_d)' \in \mathbb{Z}^d$, let $\Gamma_n(\ell; 0) = (\ell + (0,1)^d) A_{3,n}$ where $A_{3,n} = A_{1,n} + A_{2,n}$ and divide
\(\Gamma_n(\ell; 0)\) into \(2^d\) hypercubes as follows:

\[
\Gamma_n(\ell; \epsilon) = \prod_{j=1}^{d} I_j(\epsilon_j), \quad \epsilon = (\epsilon_1, \ldots, \epsilon_d)' \in \{1, 2\}^d,
\]

where, for \(1 \leq j \leq d\),

\[
I_j(\epsilon_j) = \begin{cases} 
(\ell_j \lambda_{3,n} + \lambda_{1,n}) & \text{if } \epsilon_1 = 1, \\
(\ell_j \lambda_{3,n} + \lambda_{1,n}, (\ell_j + 1) \lambda_{3,n}) & \text{if } \epsilon_1 = 2.
\end{cases}
\]

Note that

\[|\Gamma_n(\ell; \epsilon)| = \lambda_{1,n}^{q(\epsilon)} \lambda_{2,n}^{d-q(\epsilon)} \tag{A.1}\]

for any \(\ell \in \mathbb{Z}^d\) and \(\epsilon \in \{1, 2\}^d\), where \(q(\epsilon) = \{1 \leq j \leq d : \epsilon_j = 1\}\).

Let \(L_n = \{\ell \in \mathbb{Z}^d : \Gamma_n(\ell, 0) \cap R_n \neq \emptyset\}\) denote the index set of all hypercubes \(\Gamma_n(\ell, 0)\) that are contained in or boundary of \(R_n\). Moreover, let \(L_{1,n} = \{\ell \in \mathbb{Z}^d : \Gamma_n(\ell, 0) \subset R_n\}\) denote the index set of all hypercubes \(\Gamma_n(\ell, 0)\) that are contained in \(R_n\), and let \(L_{2,n} = \{\ell \in \mathbb{Z}^d : \Gamma_n(\ell, 0) \cap R_n \neq 0, \Gamma_n(\ell, 0) \cap R_n^c \neq \emptyset\}\) be the index set of boundary hypercubes.

**Remark A.1.** Let \(\epsilon_0 = (1, \ldots, 1)'\). The partitions \(\Gamma_n(\ell; \epsilon_0)\) correspond to “big blocks” and the partitions \(\Gamma(\ell; \epsilon)\) for \(\epsilon \neq \epsilon_0\) correspond to “small blocks”.

Define

\[
Z'_{s,A_n}(u, x) = K_h \left( u - \frac{s}{A_n} \right) \prod_{\ell=1}^p K_h \left( x_{\ell} - X_{s,A_n}^\ell \right) W_{s,A_n} I(|W_{s,A_n}| \leq \tau_n)
\]

\[= E_i \left[ \prod_{\ell=1}^p K_h \left( x_{\ell} - X_{s,A_n}^\ell \right) W_{s,A_n} I(|W_{s,A_n}| \leq \tau_n) \right]. \tag{A.2}\]

Observe that

\[
\sum_{j=1}^n Z'_{s,j,A_n}(u, x) = \sum_{\ell \in L_{1,n} \cup L_{2,n}} \underbrace{Z'_{s,A_n}(u, x)}_{\text{big blocks}} + \sum_{\ell \in L_{1,n}} \sum_{\epsilon \neq \epsilon_0} \underbrace{Z'_{s,A_n}(u, x)}_{\text{small blocks}} + \sum_{\ell \in L_{2,n}} \sum_{\epsilon \neq \epsilon_0} \underbrace{Z'_{s,A_n}(u, x)}_{\text{small blocks}}
\]

where \(Z'_{s,A_n}(u, x) = \sum_{j:s,j \in \Gamma_n(\ell, \epsilon)} Z'_{s,A_n}(u, x)\). Let \(\{Z'_{s,A_n}(u, x)\}_{\ell \in L_{1,n} \cup L_{2,n}}\) be independent random variables such that \(Z'_{s,A_n}(u, x) \overset{d}{=} Z'_{s,A_n}(u, x)\). Note that for distinct \(\ell_1\) and \(\ell_2\), \(d(\Gamma_n(\ell_1; \epsilon), \Gamma_n(\ell_2; \epsilon)) \geq A_{2,n}\). Applying Corollary 2.7 in \cite{Yin1994} (Lemma 2.2 in this paper) with \(m = \left( \frac{A_{-1,n}}{A_{1,n}} \right)^d\) and \(\beta(Q) = \beta(A_{2,n}; A_{n}^d)\), we have that

\[
\sup_{\ell > 0} \left| P_{\epsilon} \left( \left| \sum_{\ell \in L_{1,n} \cup L_{2,n}} Z'_{s,A_n}(u, x) \right| > t \right) - P_{\epsilon} \left( \left| \sum_{\ell \in L_{1,n} \cup L_{2,n}} \tilde{Z}'_{s,A_n}(u, x) \right| > t \right) \right| \leq C \left( \frac{A_{n}}{A_{1,n}} \right)^d \beta(A_{2,n}; A_{n}^d) \cdot P_{\epsilon} - a.s.
\]
\[
\sup_{t > 0} \left| P_{|S} \left( \left| \sum_{\ell \in L_{1,n}} Z_{A_n}^{(\ell, \varepsilon)}(u, x) \right| > t \right) - P_{|S} \left( \left| \sum_{\ell \in L_{2,n}} \tilde{Z}_{A_n}^{(\ell, \varepsilon)}(u, x) \right| > t \right) \right| \\
\leq C \left( \frac{A_n}{A_{1,n}} \right)^d \beta(A_{2,n}; A_n^d) P_S - a.s.
\]

\[
\sup_{t > 0} \left| P_{|S} \left( \left| \sum_{\ell \in L_{1,n}} Z_{A_n}^{(\ell, \varepsilon)}(u, x) \right| > t \right) - P_{|S} \left( \left| \sum_{\ell \in L_{2,n}} \tilde{Z}_{A_n}^{(\ell, \varepsilon)}(u, x) \right| > t \right) \right| \\
\leq C \left( \frac{A_n}{A_{1,n}} \right)^d \beta(A_{2,n}; A_n^d) P_S - a.s.
\]

Since \( A_n^d A_{1,n}^d \beta(A_{2,n}; A_n^d) \to 0 \) as \( n \to \infty \), these results imply that

\[
\sum_{\ell \in L_{1,n} \cup L_{2,n}} Z_{A_n}^{(\ell, \varepsilon)}(u, x) = O \left( \sum_{\ell \in L_{1,n} \cup L_{2,n}} \tilde{Z}_{A_n}^{(\ell, \varepsilon)}(u, x) \right) P_S - a.s.
\]

\[
\sum_{\ell \in L_{1,n}} Z_{A_n}^{(\ell, \varepsilon)}(u, x) = O \left( \sum_{\ell \in L_{1,n}} \tilde{Z}_{A_n}^{(\ell, \varepsilon)}(u, x) \right) P_S - a.s.
\]

\[
\sum_{\ell \in L_{2,n}} Z_{A_n}^{(\ell, \varepsilon)}(u, x) = O \left( \sum_{\ell \in L_{2,n}} \tilde{Z}_{A_n}^{(\ell, \varepsilon)}(u, x) \right) P_S - a.s.
\]

Now we show \( \sup_{(u, x) \in B} \left| \hat{v}_1(u, x) - E_{|S}[\hat{v}_1(u, x)] \right| = O_{P_{|S}}(a_n) \). Cover the region \( B \) with \( N \leq C h^{-(d+p)} a_n^{-(d+p)} \) balls \( B_n = \{(u, x) \in \mathbb{R}^{d+p} : \| (u, x) - (u_n, x_n) \|_\infty \leq a_n h \} \) and use \( (u_n, x_n) \) to denote the mid point of \( B_n \), where \( \| x_1 - x_2 \|_\infty := \max_{1 \leq j \leq d} |x_{1,j} - x_{2,j}| \). In addition, let \( K^*(w, v) = C \prod_{k=1}^d K(w_k) \prod_{j=1}^p I(|v_j| \leq 2C_1) \) for \( (w, v) \in \mathbb{R}^{d+p} \), where \( K \) satisfies Assumption 3.2 (KB1) with \( \tilde{K}^*(w, v) \geq C_0 \prod_{k=1}^d I(|w_k| \leq 2C_1) \prod_{j=1}^p I(|v_j| \leq 2C_1) \). Note that for \( (u, x) \in B_n \) and sufficiently large \( n \),

\[
\left| \tilde{K}_h \left( u - \frac{s}{A_n} \right) \prod_{\ell=1}^p K_h \left( x_{\ell} - X_{s,A_n}^{\ell} \right) - K_h \left( u - \frac{s}{A_n} \right) \prod_{\ell=1}^p K_h \left( x_{\ell,n} - X_{s,A_n}^{\ell} \right) \right| \\
\leq a_n K_h^*(u - \frac{s}{A_n}, x_n - X_{s,A_n})
\]

with \( K_h^*(v) = K^*(v/h) \). For \( \ell \in L_{1,n} \cup L_{2,n} \) and \( \varepsilon \in \{1, 2\}^d \), define \( \tilde{Z}_{A_n}^{(\ell, \varepsilon)}(u, x) \) by replacing

\[
\tilde{K}_h \left( u - \frac{s}{A_n} \right) \prod_{\ell=2}^p K_h \left( x_{\ell,2} - X_{s,A_n}^{\ell} \right) W_{s,A_n} I(|W_{s,A_n}| \leq \tau_n)
\]

with

\[
K_h^* \left( u - \frac{s_j}{A_n}, x_n - X_{s,A_n} \right) W_{s,A_n} I(|W_{s,A_n}| \leq \tau_n)
\]

26
Moreover, let \( \tilde{\psi}_1(u, x) \) in the definition of \( Z_{s_j,A_n}^j(u, x) \) and define

\[
\tilde{\psi}_1(u, x) = \frac{1}{nh^{p+d}} \sum_{j=1}^n K_n \left( u_n - \frac{S_j}{A_n}, x_n - X_{s_j,A_n} \right) |W_{s_j,A_n}| I(|W_{s_j,A_n}| \leq \tau_n).
\]

Note that \( E_{|S|} [ |\tilde{\psi}_1(u, x)|] \leq M < \infty \) for some sufficiently large \( M \), \( P_{S} \)-a.s. Then we obtain

\[
\sup_{(u,x) \in B} |\tilde{\psi}_1(u, x) - E_{|S|}[\tilde{\psi}_1(u, x)]| \\
\leq |\tilde{\psi}_1(u_n, x_n) - E_{|S|}[\tilde{\psi}_1(u_n, x_n)]| + a_n \left( |\tilde{\psi}_1(u_n, x_n)| + E_{|S|}[|\tilde{\psi}_1(u_n, x_n)|] \right) \\
\leq |\tilde{\psi}_1(u_n, x_n) - E_{|S|}[\tilde{\psi}_1(u_n, x_n)]| + |\tilde{\psi}_1(u_n, x_n) - E_{|S|}[\tilde{\psi}_1(u_n, x_n)]| + 2Ma_n
\]

\[
\leq \sum_{\ell \in L_1 \cup L_2} Z_{A_n}^\prime(\ell, e_0)(u_n, x_n) + \sum_{e \neq e_0} \sum_{\ell \in L_1} Z_{A_n}^\prime(\ell, e)(u_n, x_n) + \sum_{e \neq e_0} \sum_{\ell \in L_2} Z_{A_n}^\prime(\ell, e)(u_n, x_n) \\
+ 2Ma_n.
\]

Moreover, let \( \{\tilde{Z}_{A_n}^{\prime}(\ell, e)(u, x)\}_{\ell \in L_1 \cup L_2} \) be independent random variables where \( \tilde{Z}_{A_n}^{\prime}(\ell, e)(u, x) \) \( \overset{\text{d}}{=} \) \( Z_{A_n}^{\prime}(\ell, e)(u, x) \). Then applying Corollary 2.7 in [Y1994] to \( \{\tilde{Z}_{A_n}^{\prime}(\ell, e)(u, x)\}_{\ell \in L_1 \cup L_2} \) and \( \{\tilde{Z}_{A_n}^{\prime}(\ell, e)(u, x)\}_{\ell \in L_1 \cup L_2} \), we have that

\[
P_{|S|} \left( \sup_{(u,x) \in B} |\tilde{\psi}_1(u, x) - E_{|S|}[\tilde{\psi}_1(u, x)]| > 2^{d+1}Ma_n \right)
\]

\[
\leq N \max_{1 \leq k \leq N} P_{|S|} \left( \sup_{(u,x) \in B_k} |\tilde{\psi}_1(u, x) - E_{|S|}[\tilde{\psi}_1(u, x)]| > 2^{d+1}Ma_n \right)
\]

\[
\leq \sum_{e \in \{1,2\}^d} \tilde{Q}_n(e) + \sum_{e \in \{1,2\}^d} \tilde{Q}_n(e) + 2^{d+1}N \left( \frac{A_n}{A_{1,n}} \right)^d \beta(A_{2,n}; A_n^d),
\]

where

\[
\tilde{Q}_n(e_0) = N \max_{1 \leq k \leq N} P_{|S|} \left( \sum_{\ell \in L_1 \cup L_2} \tilde{Z}_{A_n}^{\prime}(\ell, e_0)(u_k, x_k) > Ma_n nh^{d+p} \right),
\]

\[
\tilde{Q}_n(e_0) = N \max_{1 \leq k \leq N} P_{|S|} \left( \sum_{\ell \in L_1 \cup L_2} \tilde{Z}_{A_n}^{\prime}(\ell, e_0)(u_k, x_k) > Ma_n nh^{d+p} \right),
\]

\[27\]
and for $\epsilon \neq \epsilon_0$,

$$\tilde{Q}_n(\epsilon) = N \max_{1 \leq k \leq N} P_{1|S} \left( \left| \sum_{\ell \in L_{1,n} \cup L_{2,n}} \tilde{Z}'_{A_n}^{(\epsilon)}(u_k, x_k) \right| > M_n n h^{d+p} \right),$$

$$\bar{Q}_n(\epsilon) = N \max_{1 \leq k \leq N} P_{1|S} \left( \left| \sum_{\ell \in L_{1,n} \cup L_{2,n}} \tilde{Z}''_{A_n}^{(\epsilon)}(u_k, x_k) \right| > M_n n h^{d+p} \right).$$

Since the proof is similar, we restrict our attention to $\tilde{Q}_n(\epsilon)$, $\epsilon \neq \epsilon_0$. Note that

$$P_{1|S} \left( \left| \sum_{\ell \in L_{1,n} \cup L_{2,n}} \tilde{Z}'_{A_n}^{(\epsilon)}(u_k, x_k) \right| > M_n n h^{d+p} \right) \leq 2 P_{1|S} \left( \left| \sum_{\ell \in L_{1,n} \cup L_{2,n}} \tilde{Z}'_{A_n}^{(\epsilon)}(u_k, x_k) \right| > M_n n h^{d+p} \right)$$

Since $\tilde{Z}'_{A_n}^{(\epsilon)}(u_k, x_k)$ are zero-mean random variables with

$$\left| \tilde{Z}'_{A_n}^{(\epsilon)}(u_k, x_k) \right| \leq CA_{1,n}^{d-1} A_{2,n} (\log n) \tau_n, \quad P_S - a.s. \text{ (from Lemma C.1)}$$

$$E_{1|S} \left[ \left( \tilde{Z}'_{A_n}^{(\epsilon)}(u_k, x_k) \right)^2 \right] \leq C h^{d+p} A_{1,n}^{d-1} A_{2,n} (\log n), \quad P_S - a.s., \quad (A.3)$$

Lemma C.6 yields that

$$P_{1|S} \left( \sum_{\ell \in L_{1,n} \cup L_{2,n}} \tilde{Z}'_{A_n}^{(\epsilon)}(u_k, x_k) > M_n n h^{d+p} \right)$$

$$\leq \exp \left( - \frac{M_n h^{d+p} \log n}{\left( \frac{A_{1,n}}{A_{1,n}} \right)^d A_{1,n}^{d-1} A_{2,n} h^{d+p} (\log n) + \frac{M_1^{1/2} n^{1/2} h^{(d+p)/2} (\log n)^{1/2} A_{1,n}^{d-1} A_{2,n} \tau_n}{3}} \right).$$

For (A.3), see Lemma B.1. Observe that

$$\frac{n h^{d+p} \log n}{\left( \frac{A_{1,n}}{A_{1,n}} \right)^d A_{1,n}^{d-1} A_{2,n} h^{d+p} (\log n)} = n A_n^{-d} \left( \frac{A_{1,n}}{A_{2,n}} \right) \geq \frac{A_{1,n}}{A_{2,n}} \geq n^{\eta},$$

$$\frac{n^{1/2} h^{(d+p)/2} (\log n)^{1/2} A_{1,n}^{d-1} A_{2,n} \tau_n}{A_{1,n}^{d} \left( \frac{A_{2,n}}{A_{1,n}} \right) \rho_n n^{1/\zeta}} \geq C_0 n^{\eta/2}.$$  

Taking $M > 0$ sufficiently large, this shows the desired result. \hfill \square

**Remark A.2.** Let $\eta_1 \in (0,1)$, $\gamma_2$, $\gamma_A$ with $\gamma_A > \gamma_2$. Define

$$A_n^d = n^{1-\eta_1}, \quad n h^{p+d} = n^{\gamma_2}, \quad A_{1,n} = A_{1,n}^{\gamma_A}, \quad A_{2,n} = A_{2,n}^{\gamma_A}.$$  

Note that

$$\frac{n^{1/2} h^{(d+p)/2} (\log n)^{1/2}}{A_{1,n}^{d} \left( \frac{A_{2,n}}{A_{1,n}} \right) \rho_n n^{1/\zeta}} \geq \frac{n^{\gamma_2/2} (\log n)^{1/2}}{A_{1,n}^{\gamma_2} \rho_n n^{1/\zeta}} = \frac{n^{\gamma_2/2} (\log n)^{1/2}}{\rho_n n^{(1-\eta_1)\gamma_A + 1/\zeta}}$$

$$= \frac{(\log n)^{1/2}}{\rho_n} \frac{\tau_2}{n^{(1-\eta_1)\gamma_A + 1/\zeta}} = 28$$
For \( n^{\frac{22}{9}}-(1-m)\gamma_A - \frac{1}{\xi} \gtrsim n^\eta \) for some \( \eta > 0 \), we need
\[
1 - \eta_1 \gamma_A + \frac{1}{\zeta} \lesssim \frac{\gamma_2}{2}.
\] (A.4)

**Remark A.3.** Assume that \( X_{s,A_n} \) satisfies Conditions (Ma1) and (Ma2) in Assumption [5.1]. Observe that
\[
Z^{(\ell,e)}_{A_n}(u, x) = Z^{(\ell,e)}_{1,A_n}(u, x : A_{2,n}) + Z^{(\ell,e)}_{2,A_n}(u, x : A_{2,n})
\]
\[
= \sum_{j:s_j \in \Gamma_n(\ell,e) \cap R_n} Z_{1,s_j,A_n}(u, x : A_{2,n}) + \sum_{j:s_j \in \Gamma_n(\ell,e) \cap R_n} Z_{2,s_j,A_n}(u, x : A_{2,n})
\]
where
\[
Z_{1,s_j,A_n}(u, x : A_{2,n}) = K_h \left( u - \frac{s_j}{A_n} \right) \left\{ \prod_{\ell=1}^{p} K_h \left( x_{\ell} - X_{s_j,A_n;A_{2,n}}^\ell \right) W_{s,A_n} \right\}
\]
\[
- E_{|S} \left[ \prod_{\ell=1}^{p} K_h \left( x_{\ell} - X_{s_j,A_n;A_{2,n}}^\ell \right) W_{s,A_n} \right]
\]
and \( Z_{2,s_j,A_n}(u, x : A_{2,n}) = \tilde{Z}_{s_j,A_n}(u, x) - Z_{1,s_j,A_n}(u, x : A_{2,n}) \). Note that
\[
|E_{|S} \left[ Z_{2,s_j,A_n}(u, x : A_{2,n}) \right] |
\]
\[
\leq 2K_h \left( u - \frac{s_j}{A_n} \right) E_{|S} \left[ \prod_{\ell=1}^{p} K_h \left( x_{\ell} - X_{s_j,A_n;A_{2,n}}^\ell \right) - \prod_{\ell=1}^{p} K_h \left( x_{\ell} - X_{s_j,A_n}^\ell \right) \right] \left| W_{s,A_n} \right|
\]
\[
\leq 2C \sum_{\ell=1}^{p} E_{|S} \left[ K_h \left( x_{\ell} - X_{s_j,A_n;A_{2,n}}^\ell \right) - K_h \left( x_{\ell} - X_{s_j,A_n}^\ell \right) \right] \left| W_{s,A_n} \right|
\]
\[
\leq 2CpE_{|S} \left[ \max_{1 \leq \ell \leq p} \left| X_{s,A_n}^\ell - X_{s_j,A_n;A_{2,n}}^\ell \right| \right] \left| W_{s,A_n} \right|
\]
\[
\leq 2Cp E_{|S} \left[ \max_{1 \leq \ell \leq p} \left| \epsilon_{s_j,A_n;A_{2,n}} \right| \right] \left| W_{s,A_n} \right|
\]
\[
\leq \frac{2Cp}{h} E_{|S} \left[ \max_{1 \leq \ell \leq p} \left| \epsilon_{s_j,A_n;A_{2,n}} \right| \right] \left| W_{s,A_n} \right|
\]
\[
\leq \frac{2Cp^{1+1/q}}{h} E_{|S} \left[ \max_{1 \leq \ell \leq p} \left| \epsilon_{s_j,A_n;A_{2,n}} \right| \right] \left| W_{s,A_n} \right|
\]
\[
\leq 2Cp^{1+1/q}h^{-1} \gamma_\ell(A_{2,n}).
\]
Here, we used Lipschitz continuity of \( K \) in the third inequality and Lemma 2.2.2 in van der Vaart and Wellner (1996) in the fifth inequality. Applying Markov’s inequality and Lemma [C.1] this yields that
\[
P_{|S} \left( \sum_{\ell \in E_{1,n} \cup E_{2,n}} \left| Z^{(\ell,e)}_{2,A_n}(u, x : A_{2,n}) \right| > \sqrt{n^{1-\eta_2} h^{d+p}} \right)
\]
\[
\leq \frac{\sum_{\ell \in E_{1,n} \cup E_{2,n}} \sum_{j:s_j \in \Gamma_n(\ell,e) \cap R_n} E_{|S} \left[ \left| Z_{2,s_j,A_n}(u, x : A_{2,n}) \right| \right]}{\sqrt{n^{1-\eta_2} h^{d+p}}}
\]
\[
\leq C \left( \frac{A_n/A_{1,n}}{A_{1,n}} \right)^d A_{1,n}^d \left( n A_{n}^{-d} + \log n \right) \gamma_\ell(A_{2,n})
\]
\[
\to 0 \text{ as } n \to \infty.
\] (A.5)
Define $Z_{1,A_n}^{(\ell, e)}(u, x)$ by replacing $X_{s_j, A_n}$ in definition of $Z_{1,A_n}^{(\ell, e)}(u, x)$ with $X_{s_j, A_n; A_2, n}$ and define $Z_{2,A_n}^{(\ell, e)}(u, x) = Z_{1,A_n}^{(\ell, e)}(u, x) - Z_{1,A_n}^{(\ell, e)}(u, x)$. Observe that

\[
P_{|s} \left( \sum_{\ell \in L_1, n \cup L_2, n} Z_{A_n}^{(\ell, e)}(u, x) > t \right)
= P_{|s} \left( \left\{ \sum_{\ell \in L_1, n \cup L_2, n} Z_{A_n}^{(\ell, e)}(u, x) > t \right\} \cap \left\{ \sum_{\ell \in L_1, n \cup L_2, n} Z_{2,A_n}^{(\ell, e)}(u, x) > t_0 \right\} \right)
+ P_{|s} \left( \left\{ \sum_{\ell \in L_1, n \cup L_2, n} Z_{1,A_n}^{(\ell, e)}(u, x) > t \right\} \cap \left\{ \sum_{\ell \in L_1, n \cup L_2, n} Z_{2,A_n}^{(\ell, e)}(u, x) \leq t_0 \right\} \right)
\leq P_{|s} \left( \sum_{\ell \in L_1, n \cup L_2, n} Z_{2,A_n}^{(\ell, e)}(u, x) > t_0 \right) + P_{|s} \left( \sum_{\ell \in L_1, n \cup L_2, n} Z_{1,A_n}^{(\ell, e)}(u, x) > t - t_0 \right)
\]

Set $t = M n_{A,n} n_{A}d^{+}p$ and $t_0 = \sqrt{n_{A,n} n_{A}d^{+}p} \leq \sqrt{n_{A,n} n_{A}d^{+}p}$. Let $\tilde{\beta}(a; b)$ be the $\beta$-mixing coefficients of $X_{s,A_n; A_2, n}$. Applying \textbf{[A.5]} and Lemma \textbf{[C.2]} (Corollary 2.7 in \textbf{[1994]}), we have that

\[
P_{|s} \left( \sum_{\ell \in L_1, n \cup L_2, n} Z_{A_n}^{(\ell, e)}(u, x) > M n_{A,n} n_{A}d^{+}p \right) \leq P_{|s} \left( \sum_{\ell \in L_1, n \cup L_2, n} Z_{1,A_n}^{(\ell, e)}(u, x) > 2 M n_{A,n} n_{A}d^{+}p \right) + o(1)
\leq P_{|s} \left( \sum_{\ell \in L_1, n \cup L_2, n} \tilde{Z}_{1,A_n}^{(\ell, e)}(u, x) > 2 M n_{A,n} n_{A}d^{+}p \right) + \left( \frac{A_{n}}{A_{1,n}} \right)^{d} \tilde{\beta}(A_{2,n}; A_{1,n}^{d}) \quad o(1),
\]

where \{\tilde{Z}_{1,A_n}^{(\ell, e)}(u, x)\} are independent random variables with \tilde{Z}_{1,A_n}^{(\ell, e)}(u, x) \overset{d}{=} Z_{1,A_n}^{(\ell, e)}(u, x)$. This implies that under (Ma1) and (Ma2), we can replace $X_{s,A_n}$ with $X_{s,A_n; A_2, n}$ in our analysis.

**Proof of Theorem \textbf{[3.1]}** Observe that

\[
\tilde{m}(u, x) - m(u, x) = \frac{1}{f(u, x)} \left( \tilde{g}_1(u, x) + \tilde{g}_2(u, x) - m(u, x) \tilde{f}(u, x) \right),
\]

where

\[
\tilde{f}(u, x) = \frac{1}{n_{h_{d^{+}p}}} \sum_{j=1}^{n} \hat{K}_{h} \left( u - \frac{s_j}{A_{n}} \right) \prod_{\ell=1}^{p} K_{h} \left( x_{\ell} - X_{s_j, A_{n}}^{\ell} \right),
\]

\[
\tilde{g}_1(u, x) = \frac{1}{n_{h_{d^{+}p}}} \sum_{j=1}^{n} \hat{K}_{h} \left( u - \frac{s_j}{A_{n}} \right) \prod_{\ell=1}^{p} K_{h} \left( x_{\ell} - X_{s_j, A_{n}}^{\ell} \right) \epsilon_{s_j, A_{n}},
\]

\[
\tilde{g}_2(u, x) = \frac{1}{n_{h_{d^{+}p}}} \sum_{j=1}^{n} \hat{K}_{h} \left( u - \frac{s_j}{A_{n}} \right) \prod_{\ell=1}^{p} K_{h} \left( x_{\ell} - X_{s_j, A_{n}}^{\ell} \right) m \left( \frac{s_j}{A_{n}}, X_{s_j, A_{n}} \right).
\]

(Step1) First we give a sketch of the proof. In Steps 2 and 3, we show the following four results:

(i) $\sup_{u \in [0,1]^d, x \in S_\epsilon} |\tilde{g}_1(u, x)| = O_{P_{|s}} \left( \sqrt{\left( \log n \right) / n h_{d^{+}p}} \right)$, \textit{P}-a.s.
(i) Let \( K_2 = \int_{\mathbb{R}} x^2 K(x) dx \).

\[
\sup_{u \in h, x \in S_c} E_{|S} \left[ \hat{g}_2(u, x) - m(u, x) \hat{f}(u, x) \right] = \frac{1}{h^2} K_2 - O_P(1), \quad P_S - a.s.
\]

(Step2) In this step, we show (iii). Let \( K_0 : \mathbb{R} \to \mathbb{R} \) be a Lipschitz continuous function with support \([-qC_1, qC_1] \) for some \( q > 1 \). Assume that \( K_0(x) = 1 \) for all \( x \in [-C_1, C_1] \) and write \( K_0(x) = K_0(x/h) \). Observe that

\[
E_{|S} \left[ \hat{g}_2(u, x) - m(u, x) \hat{f}(u, x) \right] = \sum_{i=1}^{4} Q_i(u, x),
\]

where

\[
Q_i(u, x) = \frac{1}{nh^{d+p}} \sum_{j=1}^{n} \bar{K}_h \left( u - \frac{s_j}{A_n} \right) q_i(u, x)
\]
and
\[
q_1(u, x) = E_{|S} \left[ \prod_{\ell=1}^{p} K_{0,h}(x_\ell - X_{s_j,A_n}) \prod_{\ell=1}^{p} K_h(x_\ell - X_{\frac{s_j}{A_n}}(s_j)) \left\{ m \left( \frac{s_j}{A_n}, X_{s_j,A_n} \right) - m(u, x) \right\} \right],
\]
\[
q_2(u, x) = E_{|S} \left[ \prod_{\ell=1}^{p} K_{0,h}(x_\ell - X_{s_j,A_n}) \prod_{\ell=1}^{p} K_h(x_\ell - X_{\frac{s_j}{A_n}}(s_j)) \left\{ m \left( \frac{s_j}{A_n}, X_{s_j,A_n} \right) - m(u, x) \right\} \right],
\]
\[
q_3(u, x) = E_{|S} \left[ \prod_{\ell=1}^{p} K_{0,h}(x_\ell - X_{s_j,A_n}) \prod_{\ell=1}^{p} K_h(x_\ell - X_{\frac{s_j}{A_n}}(s_j)) \left\{ m \left( \frac{s_j}{A_n}, X_{s_j,A_n} \right) - m(u, x) \right\} \right],
\]
\[
q_4(u, x) = E_{|S} \left[ \prod_{\ell=1}^{p} K_h \left( x_\ell - X_{\frac{s_j}{A_n}}(s_j) \right) \left\{ m \left( \frac{s_j}{A_n}, X_{s_j,A_n} \right) - m(u, x) \right\} \right].
\]

We first consider \( Q_1(u, x) \). Since the kernel \( K \) is bounded, we can use the telescoping argument to get that
\[
\left| \prod_{\ell=1}^{p} K_h(x_\ell - X_{s_j,A_n}) - \prod_{\ell=1}^{p} K_h \left( x_\ell - X_{\frac{s_j}{A_n}}(s_j) \right) \right| \leq C \sum_{\ell=1}^{p} \left| K_h(x_\ell - X_{s_j,A_n}) - K_h(x_\ell - X_{\frac{s_j}{A_n}}(s_j)) \right|.
\]

Once again using the boundedness of \( K \), we can find a constant \( C < \infty \) such that
\[
\left| K_h(x_\ell - X_{s_j,A_n}) - K_h \left( x_\ell - X_{\frac{s_j}{A_n}}(s_j) \right) \right| \leq \left| K_h(x_\ell - X_{s_j,A_n}) - K_h \left( x_\ell - X_{\frac{s_j}{A_n}}(s_j) \right) \right|^r,
\]
where \( r = \min\{\rho, 1\} \). Therefore,
\[
\left| \prod_{\ell=1}^{p} K_h(x_\ell - X_{s_j,A_n}) - \prod_{\ell=1}^{p} K_h \left( x_\ell - X_{\frac{s_j}{A_n}}(s_j) \right) \right| \leq C \sum_{\ell=1}^{p} \left| K_h(x_\ell - X_{s_j,A_n}) - K_h \left( x_\ell - X_{\frac{s_j}{A_n}}(s_j) \right) \right|^r.
\]

Applying this inequality, we have that
\[
Q_1(u, x) \leq \frac{C}{n h^{d+p}} \sum_{j=1}^{n} K_{0,h} \left( u - \frac{s_j}{A_n} \right) E_{|S} \left[ \sum_{\ell=1}^{p} \left| K_h(x_\ell - X_{s_j,A_n}) - K_h \left( x_\ell - X_{\frac{s_j}{A_n}}(s_j) \right) \right|^r \right]
\]
\[
\times \left| \prod_{\ell=1}^{p} K_{0,h}(x_\ell - X_{s_j,A_n}) \right| m \left( \frac{s_j}{A_n}, X_{s_j,A_n} \right) - m(u, x) \right].
\]
Note that \( \prod_{\ell=1}^{p} K_{i,h}(x_{\ell} - X_{s_{j}, \bar{A}_{n}}) \) \[ m \left( \frac{s_{j}}{A_{n}}, X_{s_{j}, \bar{A}_{n}} \right) - m(u, x) \leq C h. \] Since \( K \) is Lipschitz, \( X_{s_{j}, \bar{A}_{n}} - X_{s_{j}, \bar{A}_{n}}(s_{j}/A_{n}) \) and the variable \( U_{s_{j}, \bar{A}_{n}}(s_{j}/A_{n}) \) have finite \( r \)-th moment, we have that

\[
Q_{1}(u, x) \leq \frac{C}{nh^{d+p-1}} \sum_{j=1}^{n} \tilde{K}_{h} \left( u - \frac{s_{j}}{A_{n}} \right) \mathbb{E} |s| \left[ \sum_{\ell=1}^{p} \left| K_{h}(x_{\ell} - X_{s_{j}, \bar{A}_{n}}) - K_{h} \left( x_{\ell} - X_{s_{j}, \bar{A}_{n}} \left( \frac{s_{j}}{A_{n}} \right) \right) \right|^{r} \right]
\]

\[
\leq \frac{C}{nh^{d+p-1}} \sum_{j=1}^{n} \tilde{K}_{h} \left( u - \frac{s_{j}}{A_{n}} \right) \mathbb{E} |s| \left[ \sum_{k=1}^{p} \left| \frac{1}{A_{n}^{d}} U_{s_{j}, \bar{A}_{n}} \left( \frac{s_{j}}{A_{n}} \right) \right|^{r} \right] \leq \frac{C}{A_{n}^{d} h^{p-1+r}} \quad \text{(from Lemma C.3)}
\]

uniformly in \( u \) and \( x \). Using similar arguments, we can also show that

\[
\sup_{u \in l_{h}, x \in S_{c}} |Q_{2}(u, x)| \leq \frac{C}{A_{n}^{d} h^{p}}, \quad \sup_{u \in l_{h}, x \in S_{c}} |Q_{3}(u, x)| \leq \frac{C}{A_{n}^{d} h^{p-1+r}}.
\]

Finally, applying Lemmas C.4 and C.5 and using the assumptions on the smoothness of \( m \) and \( f \), we have that

\[
Q_{4}(u, x) = h^{2} K_{2} f_{s}(u) \left\{ \sum_{i=1}^{d} \left( 2 \partial_{u_{i}} m(u, x) \partial_{u_{i}} f(u, x) + \partial_{u_{i} u_{i}}^{2} m(u, x) f(u, x) \right) + \sum_{k=1}^{p} \left( 2 \partial_{x_{i}} m(u, x) \partial_{x_{i}} f(u, x) + \partial_{x_{i} x_{i}}^{2} m(u, x) f(u, x) \right) \right\} + o(h^{2})
\]

uniformly in \( u \) and \( x \). Combining the results on \( Q_{i}(u, x), 1 \leq i \leq 4 \) yields (iii). \( \square \)

**Remark A.4.** Let \( h \sim n^{1/(d+p+4)} \) and \( A_{n}^{d} = n^{1-\eta} \) for some \( \eta \in [0, 1) \).

\[
\frac{1}{A_{n}^{d} h^{p+1}} \leq h^{2} \Leftrightarrow 1 \leq A_{n}^{d} h^{p+2} = n^{r(1-\eta)} \left( \frac{p+2}{r(d+p+4)} \right) \leq 1 - \eta \geq \frac{p+2}{r(d+p+4)}.
\]

If \( r = 1 \),

\[
1 - \eta \geq \frac{p+2}{d+p+4} \Leftrightarrow \frac{d+2}{d+p+4} \geq \eta.
\] (A.6)

(A.6) is satisfied for \( d \geq 1 \) and \( p \geq 1 \).

**Proof of Theorem 3.2.** Observe that

\[
\sqrt{nh^{d+p}}(\hat{m}(u, x) - m(u, x)) = \sqrt{nh^{d+p}} \left( \hat{g}_{1}(u, x) + \hat{g}_{2}(u, x) - m(u, x) \hat{f}(u, x) \right).
\]

Define

\[
B(u, x) = \sqrt{nh^{d+p}} \left( \hat{g}_{2}(u, x) - m(u, x) \hat{f}(u, x) \right), \quad V(u, x) = \sqrt{nh^{d+p}} \hat{g}_{1}(u, x).
\]

\( B(u, x) \) converge in \( P_{S}-\)probability to \( B_{u,x} \) defined in Theorem 3.2. This follows from (iii) in the proof of Theorem 3.1 and the fact that \( B(u, x) - E_{S}|B(u, x)| = o_{P_{S}}(1) \). To prove the latter, it is sufficient to prove \( \text{Var}_{S}(B(u, x)) = o(1) \), \( P_{S}\) a.s., which can be shown by similar arguments used
in the proof Lemma B.2. Using the i.i.d. assumption on \( \{\epsilon_j\} \), \( K(u) = \prod_{\ell=1}^{d} K(u_{\ell}) \) and Lemma C.4 the asymptotic variance in (A.7) can be computed as follows:

\[
\text{Var}_{|S}(V(u, x)) = \text{Var}_{|S} \left( \frac{1}{nh^{d+p}} \sum_{j=1}^{n} \tilde{K}_h \left( u - \frac{s_j}{A_n} \right) \prod_{\ell=1}^{p} K_h \left( x_{\ell} - X_{s_j, A_n} \epsilon_{s_j, A_n} \right) \right) \\
= \frac{1}{nh^{d+p}} \sum_{j=1}^{n} \tilde{K}_h^2 \left( u - \frac{s_j}{A_n} \right) \int_{\mathbb{R}^p} \prod_{\ell=1}^{p} K_h \left( x_{\ell} - w_{\ell} \right) E_{|S} \left[ \epsilon_{s_j, A_n}^2 | X_{s_j, A_n} = w \right] f_{X_{s_j, A_n}}(w) dw \\
= \frac{1}{nh^d} \sum_{j=1}^{n} \tilde{K}_h^2 \left( u - \frac{s_j}{A_n} \right) \int_{\mathbb{R}^p} \prod_{\ell=1}^{p} K \left( \varphi_{\ell} \right) \sigma^2 \left( \frac{s_j}{A_n}, x - h\varphi \right) f_{X_{s_j, A_n}}(x - h\varphi) d\varphi \\
= \frac{1}{nh^d} \sum_{j=1}^{n} \tilde{K}_h^2 \left( u - \frac{s_j}{A_n} \right) \left( \kappa_0^p \sigma^2 \left( \frac{s_j}{A_n}, x \right) f \left( \frac{s_j}{A_n}, x \right) \right) + o(1) \\
= \kappa_0^d + f_S(u) \sigma^2(u, x) f(u, x) + o(1) \quad P_S - a.s.
\]

Moreover, \( V(u, x) \) is asymptotically normal. In particular,

\[
V(u, x) \xrightarrow{d} N(0, \kappa_0^d + f_S(u) \sigma^2(u, x) f(u, x)). \tag{A.7}
\]

We can show (A.7) by applying blocking arguments. Decompose \( V(u, x) \) into some big-blocks and small-blocks as in (A.2). We can neglect the small blocks by applying Lemma B.2 and use mixing conditions to replace the big blocks by independent random variables. This allows us to apply a Lyapunov’s condition for the central limit theorem for sum of independent random variables to get the result. We omit the details as the proof is similar to that of Theorem 3.1 in Lahiri (2003b) under the standard strictly stationary \( \alpha \)-mixing settings for random fields. \( \Box \)

### A.2. Proofs for Section 4.

**Proof of Theorems 4.1 and 4.2.** Since we can prove the desired result by applying almost the same strategy in the proof of Theorems 5.1 and 5.2 in Vogt (2012), we omit the proof. We note that it suffices to check conditions (A1)-(A6), (A8) and (A9) in Mammen et al. (1999) to obtain the desired results. We can check those conditions by applying almost the same argument in the proof of Theorems 5.1 and 5.2 in Vogt (2012) and applying Proposition 3.1, Theorems 3.1 and Lemmas B.4-B.6 in this paper, which correspond to Lemmas C.2-C.4 in Vogt (2012). \( \Box \)

### Appendix B. Auxiliary lemmas

Define

\[
\tilde{Z}_{s, A_n}(u, x) = \tilde{K}_h \left( u - \frac{s}{A_n} \right) \left\{ \prod_{\ell=2}^{p} K_h \left( x_{\ell} - X_{s, A_n} \right) W_{s, A_n} - E_{|S} \left[ \prod_{\ell=2}^{p} K_h \left( x_{\ell} - X_{s, A_n} \right) W_{s, A_n} \right] \right\}, \tag{B.1}
\]
Lemma B.1. Under Assumptions 2.4, 3.2, 3.3, and 3.4,

\[ E_{|S} \left[ \left( Z_{A_n}^{(\ell,\epsilon)}(u, x) \right)^2 \right] \leq CA_{1,n}^{-1} A_{2,n} (nA_n^{-d} + \log n) h^{p+d} P_S - a.s. \]

Proof. Observe that

\[ E_{|S} \left[ \left( Z_{A_n}^{(\ell,\epsilon)}(u, x) \right)^2 \right] = \sum_{j: s_j \in \Gamma_n(\epsilon) \cap R_n} E_{|S} \left[ Z_{s_j, A_n}^2(u, x) \right] + \sum_{j_1 \neq j_2: s_{j_1}, s_{j_2} \in \Gamma_n(\epsilon) \cap R_n} E_{|S} \left[ Z_{s_{j_1}, A_n}(u, x) Z_{s_{j_2}, A_n}(u, x) \right]. \]

Note that

\[ E_{|S} \left[ Z_{s_j, A_n}^2(u, x) \right] = K_h^2 \left( u - \frac{s_j}{A_n} \right) \left( E_{|S} \left[ \prod_{\ell_2=1}^{p} K_h^2 \left( x_{\ell_2} - X_{s_{j_2}, A_n}^{\ell_2} \right) W_{s_{j_2}, A_n}^2 \right] - \left( E_{|S} \left[ \prod_{\ell_2=1}^{p} K_h \left( x_{\ell_2} - X_{s_{j_2}, A_n}^{\ell_2} \right) W_{s_{j_2}, A_n} \right] \right)^2 \right). \]

Observe that

\[ E_{|S} \left[ \prod_{\ell_2=1}^{p} K_h \left( x_{\ell_2} - X_{s_{j_2}, A_n}^{\ell_2} \right) W_{s_{j_2}, A_n} \right] \]

\[ = \int_{\mathbb{R}^p} \prod_{\ell_2=1}^{p} K_h \left( x_{\ell_2} - X_{s_{j_2}, A_n}^{\ell_2} \right) E_{|S} \left[ W_{s_{j_2}, A_n} \mid X_{s_{j_2}, A_n} = w \right] f_{X_{s_{j_2}, A_n}}(w) \, dw \]

\[ = h^p \int_{\mathbb{R}^p} \prod_{\ell_2=1}^{p} K_h \left( \varphi_{\ell_2} \right) E_{|S} \left[ W_{s_{j_2}, A_n} \mid X_{s_{j_2}, A_n} = x - h \varphi_{\ell_2} \right] f_{X_{s_{j_2}, A_n}}(x - h \varphi_{\ell_2}) \, d\varphi \leq C \|K\|_\infty^p h^p, \]

where \( \|K\|_\infty = \sup_{x \in \mathbb{R}} |K(x)| \). Likewise,

\[ E_{|S} \left[ \prod_{\ell_2=1}^{p} K_h^2 \left( x_{\ell_2} - X_{s_{j_2}, A_n}^{\ell_2} \right) W_{s_{j_2}, A_n}^2 \right] \leq C \|K\|_\infty^{2p} h^p. \]

Then

\[ E_{|S} \left[ Z_{s_j, A_n}^2(u, x) \right] \leq C (h^p + h^{2p}) \|K\|_\infty^{2p} K_h^2 \left( u - \frac{s_j}{A_n} \right) \leq Ch \|K\|_\infty^{2p} K_h^2 \left( u - \frac{s_j}{A_n} \right) P_S - a.s. \]

(B.2)
Likewise,

\[
\left| E_{\cdot|S} \left[ Z_{s_{j_1},A_n}(u, x) Z_{s_{j_2},A_n}(u, x) \right] \right| \leq C h^{2p} \left\| K \right\|_{\infty}^{2p} \bar{K}_h \left( u - \frac{s_{j_1}}{A_n} \right) \bar{K}_h \left( u - \frac{s_{j_2}}{A_n} \right) P_S - a.s. \quad (B.3)
\]

Then Lemmas C.3 and C.1 imply that

\[
\sum_{j: s_j \in \Gamma_n(\ell; \varepsilon) \cap R_n} \bar{K}_h \left( u - \frac{s_j}{A_n} \right) \leq C \sum_{j: s_j \in \Gamma_n(\ell; \varepsilon) \cap R_n} \bar{K}_h \left( u - \frac{s_j}{A_n} \right) \leq C h^d \left[ \left\{ j : s_j \in \Gamma_n(\ell; \varepsilon) \cap R_n \right\} \right]^{2}
\]

\[
\leq C h^d A_{1,n}^{d-1} A_{2,n}(nA^{-d} + \log n), \quad P_S - a.s.,
\]

Since \( A_{1,n} A_{2,n}(nA^{-d} + \log n)h^{p+d} \leq A_{1,n}^d (nA^{-d} + \log n)h^{p+d} = o(1) \), (B.2) and (B.3) yield that

\[
E_{\cdot|S} \left[ \left( Z_{\bar{A}_n}(u, x) \right)^2 \right] \leq C \left\{ A_{1,n}^{d-1} A_{2,n}(nA^{-d} + \log n)h^{p+d} + A_{2,n}^{2(d-1)} A_{2,n}^2 (n^2 A^{-2d} + \log^2 n)h^{2(d+p)} \right\}
\]

\[
\leq CA_{1,n}^{d-1} A_{2,n}(nA^{-d} + \log n)h^{d+p}, \quad P_S - a.s.
\]

**Remark B.1.** As we defined in Remark A.2 consider

\[
A_n^d = n^{1-\eta_1}, nh^{p+d} = n^{\gamma_2}, A_{1,n} = A_n^{\gamma A_1}, A_{2,n} = A_n^{\gamma A_2}
\]

Note that \( A_{1,n}^{d-1} A_{2,n} A_n^{-d} nh^{d+p} \leq A_{1,n}^d A_n^{-d} nh^{d+p} = n^{-(1-\eta)(1-\gamma A_1)+\gamma_2} \). For \( A_{1,n}^d A_n^{-d} nh^{d+p} \leq n^{-c} \) for some \( c > 0 \), we need

\[
\gamma_2 < (1 - \eta_1)(1 - \gamma A_1). \quad (B.4)
\]

Note that \( A_{1,n}^{d-1} A_{2,n} h^{d+p} = n^{-1} A_{1,n}^{d-1} A_{2,n} nh^{p+d} \leq n^{-1} A_{1,n}^d nh^{p+d} = n^{(1-\eta_1)(1-\gamma A_1)+\gamma_2-1} \). For \( A_{1,n}^d h^{d+p} = o((\log n)^{-1}) \), we need

\[
(1 - \eta_1)\gamma A_1 + \gamma_2 < 1. \quad (B.5)
\]

(B.5) and (B.4) implies that

\[
\gamma_2 < \min \{1 - (1 - \eta_1)\gamma A_1, (1 - \eta_1)(1 - \gamma A_1) \} = (1 - \eta_1)(1 - \gamma A_1). \quad (B.6)
\]

(A.4) and (B.6) imply that \( 2(1 - \eta_1)\gamma A_1 + \frac{2}{\zeta} < \gamma_2 < (1 - \eta_1)(1 - \gamma A_1) \). For this, we need

\[
2(1 - \eta_1)\gamma A_1 + \frac{2}{\zeta} < (1 - \eta_1)(1 - \gamma A_1) \iff \frac{2}{\zeta} < (1 - \eta_1)(1 - 3\gamma A_1).
\]
Lemma B.2. Under Assumptions 2.1, 3.2, 3.5 and 3.6

\[
\frac{1}{nh^{d+p}} \text{Var}_l \left( \sum_{\ell \in L_{1,n}} Z_{A_n}^{(\ell, \epsilon)}(u, x) \right) = o(1), \quad P_S - \text{a.s.} \tag{B.7}
\]

\[
\frac{1}{nh^{d+p}} \text{Var}_l \left( \sum_{\ell \in L_{2,n}} Z_{A_n}^{(\ell, \epsilon)}(u, x) \right) = o(1), \quad P_S - \text{a.s.} \tag{B.8}
\]

Proof. Since the proof is similar, we only show (B.7). Note that

\[
\frac{1}{nh^{d+p}} \text{Var}_l \left( \sum_{\ell \in L_{1,n}} Z_{A_n}^{(\ell, \epsilon)}(u, x) \right) = \frac{1}{nh^{d+p}} \sum_{\ell \in L_{1,n}} E_{\mid S} \left[ \left( Z_{A_n}^{(\ell, \epsilon)}(u, x) \right)^2 \right]
\]

\[
+ \frac{1}{nh^{d+p}} \sum_{\ell_1, \ell_2 \in L_{1,n}, \ell_1 \neq \ell_2} E_{\mid S} \left[ Z_{A_n}^{(\ell_1, \epsilon)}(u, x) Z_{A_n}^{(\ell_2, \epsilon)}(u, x) \right] =: I_1 + I_2.
\]

As a result of Lemma B.1

\[
I_1 \leq Cn^{-1}h^{-(d+p)} \left( \frac{A_n}{A_{1,n}} \right)^d A_{1,n}^{d-1} A_{2,n}(nA_n^{-d} + \log n)h^{d+p} = C \frac{A_{2,n}}{A_{1,n}}(\log n) = o(1).
\]

Applying Theorem 1.1 in [Rio (2013)], we have that

\[
E_{\mid S} \left[ Z_{A_n}^{(\ell_1, \epsilon)}(u, x) Z_{A_n}^{(\ell_2, \epsilon)}(u, x) \right] \leq E_{\mid S} \left[ \left( Z_{A_n}^{(\ell_1, \epsilon)}(u, x) \right)^3 \right]^{1/3} \beta_1^{1/3} (d(\ell_1, \ell_2) A_{2,n}) g_1^{1/3} (A_{1,n})
\]

where \( d(\ell_1, \ell_2) = \min_{1 \leq j \leq d} |\ell_1^j - \ell_2^j| \). A similar argument to show (B.2) and (B.3) yield that

\[
E_{\mid S} \left[ \left( Z_{A_n}^{(\ell_1, \epsilon)}(u, x) \right)^3 \right] \leq CA_{1,n}^{d-1} A_{2,n}(nA_n^{-d} + \log n)h^{d+p}.
\]

Therefore, similar arguments in the proof of Theorem 3.1 in [Lahiri (2003b)] yield

\[
I_2 \leq C \left( \frac{A_{1,n}^{d-1} A_{2,n}(nA_n^{-d} + \log n)h^{p+d}}{nh^{d+p}} \right)^{2/3} \sum_{\ell_1, \ell_2 \in L_{1,n}, \ell_1 \neq \ell_2} \beta_1^{1/3} \left( \left| \ell_1 - \ell_2 \right| - d \right) A_{3,n} + A_{2,n} g_1^{1/3} (A_{1,n})
\]

\[
\leq C \left\{ \left( \frac{1}{nh^{d+p}} \right)^{1/3} \left( \frac{A_{1,n}}{A_n} \right)^{2d/3} \left( \frac{A_{2,n}}{A_{1,n}} \right)^{2/3} + \frac{A_{1,n}^{d-1} A_{2,n}^{1/3}}{nh^{(d+p)/3}} \right\}
\]

\[
\times g_1^{1/3} (A_{1,n}) \left\{ \beta_1^{1/3} (A_{2,n}) + \sum_{k=1}^{A_{n}/A_{1,n}} k^{d-1} \beta_1^{1/3} (kA_{3,n} + A_{2,n}) \right\} = o(1)
\]

where \(|\ell_1 - \ell_2| = \sum_{j=1}^{d} |\ell_{1,j} - \ell_{2,j}|\).

\[\square\]

Lemma B.3. Define \( \tilde{n}_0 = E_{\mid S} \left[ \tilde{\eta}_{[0,1]} \right] \) where

\[
\tilde{\eta}_{[0,1]} = \sum_{j=1}^{n} \tilde{K}_h \left( u, \frac{s_j}{A_n} \right) I(X_{s_j}A_n \in [0,1])
\]

37
Suppose Assumptions 2.1, 3.1, 3.2, 3.5 (with $W_{s,A_n} = 1$ and $\epsilon_{s,A_n}$) and 4.1 hold. Then uniformly for $u \in I_h$,

$$\frac{\tilde{n}_0}{n} = f_s(u)P(X_u(0) \in [0,1]^p) + O\left(A_n^{-\frac{\rho}{1+\rho}}\right) + o(h) \quad P_S - a.s.$$ (B.9)

and

$$\frac{\tilde{n}[0,1]^p - \tilde{n}_0}{n_0} = OP_\mid_S \left(\sqrt{\log n/n^d}\right) \quad P_S - a.s.$$ (B.10)

**Proof.** (Step 1) In this step, we show (B.9). Define $U_{s,A_n} = U_{s,A_n}(s/A_n)$. Recall that $\|X_{s,A_n} - X_{s,A_n}(s)\| \leq \frac{1}{A_n}U_{s,A_n}$ almost surely with $E[U_{s,A_n}^p] < C \leq \infty$ for some $\rho > 0$. Observe that for sufficiently large $C < \infty$,

$$E[I(X_{s,A_n} \in [0,1]^p)] = E[I(X_{s,A_n} \in [0,1]^p, \|X_{s,A_n} - X_{s,A_n}(s/A_n)\| \leq A_n^{-d}U_{s,A_n})]
\geq E[I(X_{s,A_n}(s) \in [CA_n^{-d}U_{s,A_n}, 1 - CA_n^{-d}U_{s,A_n}]^p)]
\leq E[I(X_{s,A_n}(s) \in [-CA_n^{-d}U_{s,A_n}, 1 + CA_n^{-d}U_{s,A_n}]^p)].$$

Define

$$B_L = \frac{1}{n} \sum_{j=1}^n \bar{K}_h \left( u, \frac{s_j}{A_n} \right) E_1 \mid S \left[ I(X_{s_j/A_n}(s_j) \in [CA_n^{-d}U_{s_j,A_n}, 1 - CA_n^{-d}U_{s_j,A_n}]^p) \right],$$

$$B_U = \frac{1}{n} \sum_{j=1}^n \bar{K}_h \left( u, \frac{s_j}{A_n} \right) E_1 \mid S \left[ I(X_{s_j/A_n}(s_j) \in [-CA_n^{-d}U_{s_j,A_n}, 1 + CA_n^{-d}U_{s_j,A_n}]^p) \right].$$

From the definitions of $B_L$ and $B_U$, $B_L \leq \frac{\tilde{n}_0}{n} \leq B_U$. Let $q \in (0,1)$ and write $B_U = B_U, + B_U,$. where

$$B_{U,1} = \frac{1}{n} \sum_{j=1}^n \bar{K}_h \left( u, \frac{s_j}{A_n} \right) E_1 \mid S \left[ I(X_{s_j/A_n}(s_j) \in [-CA_n^{-d}U_{s_j,A_n}, 1 + CA_n^{-d}U_{s_j,A_n}]^p, U_{s_j,A_n} \leq A_n^{dq}) \right]$$

and $B_{U,2} = B_U - B_{U,1}$. Applying Lemma C.5, we have that

$$B_{U,1} \leq \frac{1}{n} \sum_{j=1}^n \bar{K}_h \left( u, \frac{s_j}{A_n} \right) E_1 \mid S \left[ I(X_{s_j/A_n}(s) \in [-CA_n^{-d(q-1)}, 1 + CA_n^{-d(q-1)}]^p) \right]
= \int_{\mathbb{R}^p} I(x \in [-CA_n^{-d(q-1)}, 1 + CA_n^{-d(q-1)}]^p) \frac{1}{n} \sum_{j=1}^n \bar{K}_h \left( u, \frac{s_j}{A_n} \right) f \left( \frac{s_j}{A_n}, x \right) \, dx
= f_s(u) \int_{\mathbb{R}^p} I(x \in [-CA_n^{-d(q-1)}, 1 + CA_n^{-d(q-1)}]^p) f(u, x) \, dx + o(h)
= f_s(u) \int_{\mathbb{R}^p} I(x \in [0,1]^p) f(u, x) \, dx + O(A_n^{-d(1-q)}) + o(h), \quad P_S - a.s.
uniformly over $I_h$. Moreover, applying Lemma C.3 we also have that

$$B_{U,2} \leq \frac{1}{n} \sum_{j=1}^{n} K_h \left( u, \frac{s_i}{A_n} \right) E_{|S|} \left[ I(U_{s_j,A_n} > A_n^{dq}) \right]$$

$$\leq \frac{1}{n} \sum_{j=1}^{n} K_h \left( u, \frac{s_i}{A_n} \right) E_{|S|} \left[ (U_{s_j,A_n}/A_n^{dq})^B \right] \leq \frac{C}{A_n^{dp}} P_S - a.s.$$ 

Set $q = \frac{1}{1+\rho}$. Then we have that

$$B_U \leq f_S(u)P(X_u(0) \in [0,1]^P) + O(A_n^{-\frac{dp}{1+\rho}}) + o(h)$$  \hspace{1cm} \text{(B.11)}$$

uniformly over $I_h$, $P_S$-a.s. Likewise,

$$B_U \geq f_S(u)P(X_u(0) \in [0,1]^P) - O(A_n^{-\frac{dp}{1+\rho}}) - o(h)$$  \hspace{1cm} \text{(B.12)}$$

(Step2) Now we show (B.10). Applying Proposition 3.1 and (B.9), we have that

$$\frac{n}{n_0} \times \frac{n_0}{n} \times \frac{n}{n_0} \times \frac{n_0}{n} = O(1) \times O_P(u) \times \sqrt{\frac{\log n}{nh^d}} = O_P(u) \times \sqrt{\frac{\log n}{nh^d}}.$$  \hspace{1cm} \text{Proposition 3.1}$$

\hfill \Box$$

Define $n_0 = E[|n_0|]$. Lemmas B.3 and C.3 imply that

$$\frac{n}{n_0} \times \frac{n_0}{n} \times \frac{n}{n_0} \times \frac{n_0}{n} = O(1) \times O_P(u) \times \sqrt{\frac{\log n}{nh^d}} = O_P(u) \times \sqrt{\frac{\log n}{nh^d}}.$$  \hspace{1cm} \text{Proposition 3.1}$$

\hfill \Box$$

\textbf{Lemma B.4.} Let $\kappa_0(w) = \int_{\mathbb{R}} K_h(w,v)dv$. Suppose Assumptions 2.1 3.1 3.2 3.3 (with $W_{s,A_n} = 1$ and $\epsilon_{s,A_n}$) and \textit{4.1} hold. Then

$$\sup_{u \in I_h, x_1 \in [0,1]} |\hat{p}_t(u, x_1) - p_t(u, x_1)| = O_P(u) \left( \frac{\log n}{nh^{d+1}} \right) + O \left( \frac{1}{A_n^{dp+h^{p+r}}} \right) + o(h),$$

$$\sup_{u \in I_h, x_1 \in [0,1]} |\hat{p}_t(u, x_1) - \kappa_0(x_1)p_t(u, x_1)| = O_P(u) \left( \frac{\log n}{nh^{d+1}} \right) + O \left( \frac{1}{A_n^{dp+h^{p+r}}} \right) + o(h),$$

$$\sup_{u \in I_h, x_1, x_2 \in [0,1]} |\hat{p}_t(u, x_1, x_2) - p_t(u, x_1, x_2)| = O_P(u) \left( \frac{\log n}{nh^{d+2}} \right) + O \left( \frac{1}{A_n^{dp+h^{p+r}}} \right) + o(h),$$

and

$$\sup_{u \in I_h, x_1, x_2 \in [0,1]} |\hat{p}_t(u, x_1, x_2) - \kappa_0(x_1)\kappa_0(x_2)p_t(u, x_1, x_2)| = O_P(u) \left( \frac{\log n}{nh^{d+2}} \right) + O \left( \frac{1}{A_n^{dp+h^{p+r}}} \right) + o(h)$$

39
Proof. Since the proof is similar, we only give the proof for $\hat{\mu}_\ell(u, x_\ell)$. Define
\[
\hat{\mu}_\ell(u, x_\ell) = \frac{1}{n_0} \sum_{j=1}^n I(X_{s_j, A_n} \in [0, 1]^p) \bar{K}_h \left( u, \frac{s_j}{A_n} \right) K_h(x_\ell, X_{s_j, A_n}^\ell).
\]
Applying Lemma B.3, we have that
\[
\hat{\mu}_\ell(u, x_\ell) = \left( 1 + \frac{n[0,1]^p - n_0}{n_0} \right)^{-1} \hat{\mu}_\ell(u, x_\ell) = \left( 1 - \frac{n[0,1]^p - n_0}{n_0} + O_{P\mid S} \left( \left( \frac{n[0,1]^p - n_0}{n_0} \right)^2 \right) \right) \hat{\mu}_\ell(u, x_\ell)
\]
\[
= \hat{\mu}_\ell(u, x_\ell) + O_{P\mid S} \left( \sqrt{\frac{\log n}{nh^d}} \right)
\]
uniformly for $u \in I_h$ and $x_\ell \in [0,1]$. Applying similar arguments in the proof of Theorem 3.1 to $\hat{\mu}_\ell(u, x_\ell)$, we obtain the desired result.

Decompose $\hat{m}_\ell(u, x_\ell) = \hat{m}_{1,\ell}(u, x_\ell) + \hat{m}_{2,\ell}(u, x_\ell)$, where
\[
\hat{m}_{1,\ell}(u, x_\ell) = \frac{1}{\hat{\mu}_\ell(u, x_\ell)n[0,1]^p} \sum_{j=1}^n I(X_{s_j, A_n} \in [0, 1]^p) \bar{K}_h \left( u, \frac{s_j}{A_n} \right) K_h(x_\ell, X_{s_j, A_n}^\ell) \epsilon_{s_j, A_n},
\]
\[
\hat{m}_{2,\ell}(u, x_\ell) = \frac{1}{\hat{\mu}_\ell(u, x_\ell)n[0,1]^p} \sum_{j=1}^n I(X_{s_j, A_n} \in [0, 1]^p) \bar{K}_h \left( u, \frac{s_j}{A_n} \right) K_h(x_\ell, X_{s_j, A_n}^\ell)
\]
\[
\times \left( m_0 \left( \frac{s_j}{A_n} \right) + \sum_{k=1}^p m_k \left( \frac{s_j}{A_n}, X_{s_j, A_n}^k \right) \right).
\]

Lemma B.5. Suppose Assumptions 2.1, 3.1, 3.2, 3.5 (with $W_{s,A_n} = 1$ and $\epsilon_{s,A_n}$) and 4.1 hold. Then
\[
\sup_{u \in [0,1]^d, x_\ell \in [0,1]} |\hat{m}_{1,\ell}(u, x_\ell)| = O_{P\mid S} \left( \sqrt{\frac{\log n}{nh^d + 1}} \right).
\]

Proof. Replacing $n[0,1]^p$ in the definition of $\hat{m}_{1,\ell}$ by $n_0$ and applying Proposition 3.1 gives the desired result.

Lemma B.6. Let $I_{h,0} = [0,1] \setminus I_{0,h}$ and $I_{h}^c = [0,1]^d \setminus I_h$. Suppose Assumptions 2.1, 3.1, 3.2, 3.5 (with $W_{s,A_n} = 1$ and $\epsilon_{s,A_n}$), and 4.1 hold. Then
\[
\sup_{u \in I_{h,0}, x_\ell \in I_{h,0}} |\tilde{m}_{2,j}(u, x_\ell) - \hat{\mu}_\ell(u, x_\ell)| = o_{P\mid S}(h^2),
\]
\[
\sup_{u \in I_{h,0}, x_\ell \in I_{h,0}^c} |\tilde{m}_{2,j}(u, x_\ell) - \hat{\mu}_\ell(u, x_\ell)| = O_{P\mid S}(h^2),
\]
where
\[
\hat{\mu}_\ell(u, x_\ell) = \alpha_0(u) + \alpha_\ell(u, x_\ell) + \sum_{k \neq \ell} \int_{\mathbb{R}} \alpha_k(u, x_k) \frac{\hat{\mu}_k(u, x_k, x_\ell)}{\hat{\mu}_\ell(u, x_\ell)} dx_k + h^2 \int_{\mathbb{R}^{p-1}} \beta(u, x) \frac{p(u, x)}{\hat{\mu}_\ell(u, x_\ell)} dx_{-\ell},
\]
40
\[ \alpha_0(u) = m_0(u) + h \sum_{i=1}^{d} \kappa_1(u_i) \partial_u m_0(u) \]
\[ + \frac{h^2}{2} \left( \sum_{i=1}^{d} \kappa_2(u_i) \partial_u m_0(u) + \sum_{1 \leq i_1 \leq i_2 \leq d, i_1 \neq i_2} \kappa_1(u_{i_1}) \kappa_1(u_{i_2}) \partial_{u_{i_1} u_{i_2}} m_0(u) \right), \]
\[ \alpha_k(u, x_k) = m_k(u, x_k) + h \left\{ \sum_{i=1}^{d} \kappa_1(u_i) \partial_u m_k(u, x_k) + \prod_{i=1}^{d} \kappa_0(u_i) \kappa_1(x_k) \partial_{x_k} m_k(u, x_k) \right\}, \]
\[ \beta(u, x) = \kappa_2 \sum_{i=1}^{d} \partial_u m_0(u) \partial_u \log p(u, x) + \kappa_2 \sum_{k=1}^{p} \left\{ \sum_{i=1}^{d} \partial_u m_k(u, x_k) \partial_{u_i} \log p(u, x) \right\} \]
\[ + \frac{1}{2} \sum_{i=1}^{d} \partial_{u_i u_i} m_k(u, x_k) + \partial_{x_k} m_k(u, x_k) \partial_{x_k} \log p(u, x) + \frac{1}{2} \partial_{x_k x_k} m_k(u, x_k) \right\}. \]

Here, \( \kappa_2 = \int_{\mathbb{R}} x^2 K(x) dx \) and \( \kappa_j(v) = \int_{\mathbb{R}} w^j K_h(v, w) dw \) for \( j = 0, 1, 2, \).

**Proof.** Although the detailed proof is lengthy and involved, we can obtain the desired result by applying almost the same strategy in the proof of Lemma C.4 in [Vogt (2012)]. Therefore, we omit the proof. We note that Lemmas B.3, C.4 and C.5 in this paper, which correspond to Lemmas C.1, B.1 and B.2 in [Vogt (2012)], and the similar argument in the proof of Proposition 3.1 and Theorem 6.1 in this paper are applied for the proof. See also [Mammen et al. (1999)] for the original idea of the proof. \( \square \)

**Lemma B.7.** Suppose Assumptions 2.1, 3.1, 3.2, 3.5 (with \( W_{s,A_n} = 1 \) and \( \epsilon_{s,A_n} \)) and 4.1 hold. Then
\[ \sup_{u \in \mathcal{I}_h} |\tilde{m}_0(u) - m_0(u)| = O_{\| \cdot \|} \left( \sqrt{\frac{\log n}{nh^d}} + h^2 \right). \]

**Proof.** Replacing \( n_{[0,1]^d} \) by \( n_0 \) in the definition of \( \tilde{m}_0 \) and applying similar arguments in the proof of Theorem 3.1 we obtain the desired result. \( \square \)

**Lemma B.8.** Suppose that \( E[|L([0,1]^d)|^q] < \infty \) for \( 1 \leq q' \leq q \) where \( q \) is some even integer. Then a univariate CARMA-type random field with a kernel function \( g \) of the form (5.9) satisfies (5.11).

**Proof.** Observe that
\[ X_{s,A_n} = \int_{\mathbb{R}^d} g \left( \frac{s}{A_n}, \|s - v\| \right) L(dv) \]
\[ = \int_{\mathbb{R}^d} g \left( \frac{s}{A_n}, \|s - v\| \right) \iota(\|s - v\|: A_{2,n}) L(dv) + \int_{\mathbb{R}^d} g \left( \frac{s}{A_n}, \|s - v\| \right) (1 - \iota(\|s - v\|: A_{2,n})) L(dv) \]
\[ =: X_{s,A_n; A_{2,n}} + \epsilon_{s,A_n; A_{2,n}}. \]
Note that $X_{s,A_n;A_{2n}}$ is $A_{2n}$-dependent from the definition of the function $\iota$. Then we have that
\[
E[|\epsilon_{s,A_n;A_{2n}}|^q] \lesssim \int_{\mathbb{R}^d} e^{-q\|u\|} (1 - \iota(\|u\| : A_{2n}))^q \, du
\]
\[
\lesssim \int_{\|u\| \geq A_{2n}/2} e^{-q\|u\|} \left( 1 + \frac{\|u\|}{A_{2n}} \right)^q \|u\|^q \, du
\]
\[
\lesssim 2^{q-1} \int_{\|u\| \geq A_{2n}/2} e^{-q\|u\|} \left( 1 + \frac{2\|u\|^q}{A_{2n}^q} \right) \, du
\]
\[
\lesssim e^{-\frac{q\|u\|_{A_{2n}}}{2}} \left( 1 + \frac{(A_{2n}/2)^q}{A_{2n}} \right) (A_{2n}/2)^{-1} \lesssim A_{2n}^{q-1} e^{-q\|u\|_{A_{2n}}^2}.
\]
For the first inequality, we used the finiteness of $E[|L([0,1])|^q]$ (see also Brockwell and Matsuda (2017) for computation of moments of a Lévy-driven MA process). For the second inequality, we used the definition of $\iota$. Hence we obtain the desired result.

Lemma B.9. Let $c$ be a positive constant and $g$ be a bounded function such that $\int_{\mathbb{R}^d} |g(v)| \, dv < \infty$ and $\lim_{\|v\| \to \infty} g(v) = 0$. Suppose that the Lévy random measure $L$ of a random field $X(s) = \int_{\mathbb{R}} g(s - v)L(ds)$ is purely non-Gaussian with Lévy density
\[
\nu_0(x) = \frac{c_3}{|x|^{1+\beta}} g_0(x), \quad x \neq 0
\]
where $\beta \in (0,1)$, $c_3 > 0$ and $g_0$ is a positive, continuous and bounded function on $\mathbb{R}$ with $\lim_{|x| \to 0} g_0(x) = 1$. Then there exists a constant $\tilde{C} > 0$ such that
\[
u_0(x) = \frac{c_3}{|x|^{1+\beta}} g_0(x), \quad x \neq 0
\]
\[
|u|^2 \int_{\mathbb{R}^d} g^2(v) \int_{|x| \leq \frac{1}{|g(v)| \|v\|}} x^2 \nu_0(x) \, dx \, dv \geq \tilde{C}|u|^{2-\alpha}
\]
for any $|u| \geq c_0 > 1$, where $\alpha = 2 - \beta \in (0,2)$.

Proof. From the assumption on $g$, there exists a bounded set $S_{c_0} \subset \mathbb{R}^d$ such that $S_{c_0} \subset \{v \in \mathbb{R}^d : \frac{1}{c_0} \leq |g(v)| \}$. Then we have that
\[
|u|^2 \int_{\mathbb{R}^d} g^2(v) \int_{|x| \leq \frac{1}{|g(v)| \|v\|}} x^2 \nu_0(x) \, dx \, dv
\]
\[
= |u|^2 \int_{\frac{1}{|g(v)| \|v\|} \leq 1} g^2(v) \int_{|x| \leq \frac{1}{|g(v)| \|v\|}} x^2 \nu_0(x) \, dx \, dv + |u|^2 \int_{\frac{1}{|g(v)| \|v\|} > 1} g^2(v) \int_{|x| \leq \frac{1}{|g(v)| \|v\|}} x^2 \nu_0(x) \, dx \, dv
\]
\[
\geq |u|^2 \int_{\frac{1}{|g(v)| \|v\|} \leq 1} g^2(v) \int_{|x| \leq \frac{1}{|g(v)| \|v\|}} |x|^{1-\beta} g_0(x) \, dx \, dv + |u|^2 \int_{\frac{1}{|g(v)| \|v\|} > 1} g^2(v) \int_{|x| \leq 1} x^2 \nu_0(x) \, dx \, dv
\]
\[
\geq |u|^2 \int_{\frac{1}{|g(v)| \|v\|} \leq 1} g^2(v) |u|^{\beta-2} |g(v)|^{\beta-2} \, dv + |u|^2 \int_{\frac{1}{|g(v)| \|v\|} > 1} g^2(v) \, dv
\]
\[
\geq |u|^\beta \int_{\frac{1}{|g(v)| \|v\|} \leq 1} |g(v)|^\beta \, dv \geq |u|^\beta \int_{\frac{1}{c_0} \leq |g(v)|} |g(v)|^\beta \, dv \geq |u|^\beta \int_{S_{c_0}} |g(v)|^\beta \, dv \geq |u|^\beta.
\]
Then we obtain the desired result. \qed
APPENDIX C. TECHNICAL LEMMAS

We refer to the following lemmas without those proofs.

**Lemma C.1** (Lemmas A.1 and 5.1 in [Lahiri (2003b)]). Let \( \mathcal{I}_n = \{ i \in \mathbb{Z}^d : (i + (0, 1]^d) \cap R_n \neq \emptyset \} \). Then we have that

\[
P_S \left( \sum_{j=1}^{n} 1\{ A_n S_{0,j} \in (i + (0, 1]^d) \cap R_n > 2(\log n + nA_n^{d}) \text{ for some } i \in \mathcal{I}_n, \text{ i.o.} \} = 0 \right)
\]

and

\[
P_S \left( \sum_{j=1}^{n} 1\{ A_n S_{0,j} \in \Gamma_n(\ell; \epsilon) \} > CA_{1,n}^{d-q} A_{2,n}^{d} nA_n^{d} \text{ for some } \ell \in L_{1,n}, \text{ i.o.} \} = 0 \right)
\]

for any \( \epsilon \in \{1, 2\}^d \), where \( C > 0 \) is a sufficiently large constant.

**Remark C.1.** Lemma [C.1] implies that each \( \Gamma_n(\ell; \epsilon) \) contains at most \( CA_{1,n}^{d-q} A_{2,n}^{d} nA_n^{d} \) samples \( P_S \)-almost surely.

We may define the \( \beta \)-mixing coefficients for any probability measure \( Q \) on a product measure space \( (\Omega_1 \times \Omega_2, \Sigma_1 \times \Sigma_2) \) as follows:

**Definition C.1** (Definition 2.5 in [Yu (1994)]). Suppose that \( Q_1 \) and \( Q_2 \) are the marginal probability measures of \( Q \) on \( (\Omega_1, \Sigma_1) \) and \( (\Omega_2, \Sigma_2) \), respectively. Then we define

\[
\beta(\Sigma_1, \Sigma_2, Q) = P \sup \{ |Q(B|\Sigma_1) - Q_2(B)| : B \in \Sigma_2 \}
\]

**Lemma C.2** (Corollary 2.7 in [Yu (1994)]). Let \( m \geq 1 \) and let \( Q \) be a probability measure on a product space \((\prod_{i=1}^{m} \Omega_i, \prod_{i=1}^{m} \Sigma_i)\) with marginal measures \( Q_i \) on \((\Omega_i, \Sigma_i)\). Suppose that \( h \) is a bounded measurable function on the product probability space such that \( |h| \leq M_h < \infty \). Let \( Q_a^b \) (with \( 1 \leq a \leq b \)) be the marginal measure on \((\prod_{i=a}^{b} \Omega_i, \prod_{i=a}^{b} \Sigma_i)\). Write

\[
\beta(Q) = \sup_{1 \leq i \leq m-1} \beta \left( \prod_{j=1}^{i} \Sigma_j, \Sigma_{i+1}, Q_1^{i+1} \right).
\]

Suppose that, for all \( 1 \leq k \leq m-1 \),

\[
\| Q - Q_1^k \times Q_{k+1}^m \| \leq \beta(Q),
\]

where \( Q_1^k \times Q_{k+1}^m \) is a product measure and \( \| \cdot \| \) is \( 1/2 \) of the total variation norm. Then

\[
|Qh - Ph| \leq M_h (m-1) \beta(Q).
\]

where \( P = \prod_{i=1}^{m} Q_i \), \( Qh = \int \! h dQ \), and \( Ph = \int \! h dP \).

**Remark C.2.** Lemma [C.2] is a key tool to construct independent blocks for \( \beta \)-mixing sequence. Note that Lemma [C.2] holds for each finite \( n \).

**Assumption C.1.**

(KD1) The kernel \( \bar{K} : \mathbb{R}^d \to [0, \infty) \) is bounded and has compact support \([-C, C]^d\). Moreover,

\[
\int_{[-C,C]^d} \bar{K}(x) dx = 1, \quad \int_{[-C,C]^d} x^\alpha \bar{K}(x) dx = 0, \text{ for any } \alpha \in \mathbb{Z}^d \text{ with } |\alpha| = 1,
\]

and \( |\bar{K}(u) - \bar{K}(v)| \leq C \| u - v \| \).
(KD2) For any $\alpha \in \mathbb{Z}^d$ with $|\alpha| = 1, 2$, $\partial^\alpha f_S(s)$ exist and continuous on $(0, 1)^d$.

Define $\tilde{f}_S(u) = \frac{1}{nh^d} \sum_{j=1}^n \tilde{K}_h (u - S_{0,j})$.

Lemma C.3 (Theorem 2 in Masry (1996)). Under Assumption C.1 and $h \to 0$ such that $nh^d/(\log n) \to \infty$ as $n \to \infty$, we have that

$$\sup_{u \in [0,1]^d} \left| f_S(u) - f_S(u) \right| = O \left( \sqrt{\frac{\log n}{nh^d} + h^2} \right) P_S - a.s.$$

Similar arguments of the proof of Lemma C.3 yields the following Lemmas C.4 and C.5.

Lemma C.4. Under Assumption C.1 and $h \to 0$ such that $nh^d/(\log n) \to \infty$ as $n \to \infty$, we have that

$$\sup_{u \in I_h} \left| \frac{1}{nh^d} \sum_{j=1}^n \tilde{K}_h (u - S_{0,j}) \left( \frac{u - S_{0,j}}{h} \right)^k - \frac{1}{h^d} \int_{\mathbb{R}^d} \tilde{K}_h (u - w) \left( \frac{u - w}{h} \right)^k f_S(w) dw \right|$$

$$= O \left( \sqrt{\frac{\log n}{nh^d}} \right) P_S - a.s.$$

for any $k \in \mathbb{Z}^d$ with $|k| = 0, 1, 2$, where $x^k = \prod_{\ell=1}^d x^k_{\ell}$.

Lemma C.5. Let $g : [0,1]^d \times \mathbb{R}^p \to \mathbb{R}$, $(u, x) \to g(u, x)$ be continuously partially differentiable w.r.t. $u$. Under Assumption C.1 and $h \to 0$ such that $nh^d/(\log n) \to \infty$ as $n \to \infty$, we have that

$$\sup_{u \in I_h, x \in S_c} \left| \frac{1}{nh^d} \sum_{j=1}^n \tilde{K}_h^m (u - S_{0,j}) g(S_{0,j}, x) - \tilde{\kappa}_m f_S(u) g(u, x) \right| = O \left( \sqrt{\frac{\log n}{nh^d}} \right) + o(h)$$

$P_S$-a.s. for $m = 1, 2$, where $\tilde{\kappa}_m = \int_{\mathbb{R}^d} \tilde{K}_h^m (x) dx$.

Lemma C.6 (Bernstein’s inequality). Let $X_1, \ldots, X_n$ be independent zero-mean random variables. Suppose that $\max_{1 \leq i \leq n} |X_i| \leq M < \infty$ a.s. Then, for all $t > 0$,

$$P \left( \sum_{i=1}^n X_i \geq t \right) \leq \exp \left( -\frac{t^2}{\sum_{j=1}^n \mathbb{E}[X_j^2]} + \frac{Mt}{3} \right).$$

Appendix D. Multivariate Lévy-driven moving average random fields

In this section, we discuss multivariate extension of univariate Lévy-driven moving average random fields and give examples of multivariate locally stationary random fields. Let $M_p(\mathbb{R})$ denote the space of all real $p \times p$ matrices and let $\text{tr}(A)$ denote the trace of the matrix $A$.

Let $L = \{L(A) = (L_1(A), \ldots, L_p(A))' : A \in \mathcal{B}(\mathbb{R}^d)\}$ be an $\mathbb{R}^p$-valued infinitely divisible random measure on some probability space $(\Omega, \mathcal{A}, P)$, i.e., a random measure such that

1. for each sequence $(E_m)_{m \in \mathbb{N}}$ of disjoint sets in $\mathcal{B}(\mathbb{R}^d)$ it holds
   (a) $L(\bigcup_{m=1}^\infty E_m) = \sum_{m=1}^\infty L(E_m)$ a.s., whenever $\bigcup_{m=1}^\infty E_m \in \mathcal{B}(\mathbb{R}^d)$,
   (b) $(L(E_m))_{m \in \mathbb{N}}$ is a sequence of independent random vectors.
2. the random variable $L(A)$ has an infinitely divisible distribution for any $A \in \mathcal{B}(\mathbb{R}^d)$. 

44
The characteristic function of $L(A)$ which will be denoted by $\varphi_{L(A)}(t)$ ($t = (t_1, \ldots, t_p)' \in \mathbb{R}^p$), has a Lévy-Khintchine representation of the form $\varphi_{L(A)}(t) = \exp \left( |A| \psi(t) \right)$ with
\[
\psi(t) = i \langle t, \gamma_0 \rangle - \frac{1}{2} t' \Sigma_0 t + \int_{\mathbb{R}^p} \left\{ e^{i \langle t, x \rangle} - 1 - i \langle t, x \rangle I(\|x\| \leq 1) \right\} \nu_0(x) dx
\]
where $i = \sqrt{-1}$, $\gamma_0 \in \mathbb{R}^p$, $\Sigma_0$ is a $p \times p$ positive definite matrix, $\nu_0$ is a Lévy density with $\int_{\mathbb{R}^p} \min\{1, \|x\|^2\} \nu_0(x) dx < \infty$ and $|A|$ is the Lebesgue measure of $A$. The triplet $(\gamma_0, \Sigma_0, \nu_0)$ is called the Lévy characteristic of $L$ and it uniquely determines the distribution of the random measure $L$.

Consider the stochastic process $X = \{X(s) = (X_1(s), \ldots, X_p(s))' : s \in \mathbb{R}^d\}$ given by
\[
X(s) = \int_{\mathbb{R}^d} g(s, v)L(dv), \tag{D.1}
\]
where $g : \mathbb{R}^d \times \mathbb{R}^d \rightarrow M_p(\mathbb{R})$ is a measurable function and $L$ is a $p$-dimensional Lévy random measure. Assume that
\[
\int_{\mathbb{R}^d} \text{tr} \left( g(s, v) \Sigma_0 g(s, v)' \right) dv < \infty, \tag{D.2}
\]
\[
\int_{\mathbb{R}^d} \min\{\|g(s, v)x\|^2, 1\} \nu_0(x) dx dv < \infty, \quad \text{and} \tag{D.3}
\]
\[
\int_{\mathbb{R}^d} \left\| g(s, v) \gamma_0 + \int_{\mathbb{R}^p} g(s, v)x \left( I(\|g(s, v)x\| \leq 1) - I(\|x\| \leq 1) \right) \nu_0(x) dx \right\| dv < \infty. \tag{D.4}
\]
Then the law of $X(s)$ for all $s \in \mathbb{R}^d$ is infinitely divisible with characteristic function
\[
E[e^{i \langle t, X(s) \rangle}] = \exp \left\{ i \langle t, \int_{\mathbb{R}^d} g(s, v) \gamma_0 dv \rangle - \frac{1}{2} t' \left( \int_{\mathbb{R}^d} g(s, v) \Sigma_0 g(s, v)' dv \right) t \right. \\
+ \left. \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^p} (e^{i \langle t, g(s, v)x \rangle} - 1 - i \langle t, g(s, v)x \rangle I(\|g(s, v)x\| \leq 1) \nu_0(x) dx \right) dv \right\}.
\]
Therefore, the Lévy characteristics $(\gamma_{X(s)}, \Sigma_{X(s)}, \nu_{X(s)})$ of $X(s)$ are given by
\[
\gamma_{X(s)} = \int_{\mathbb{R}^d} g(s, v) \gamma_0 dv
\]
\[
+ \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^p} g(s, v)x \left( I(\|g(s, v)x\| \leq 1) - I(\|x\| \leq 1) \right) \nu_0(x) dx \right) dv,
\]
\[
\Sigma_{X(s)} = \int_{\mathbb{R}^d} g(s, v) \Sigma_0 g(s, v)' dv,
\]
\[
\nu_{X(s)}(B) = \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^p} I(g(s, v)x \in B) \nu_0(x) dx \right) dv, \quad B \in \mathcal{B}(\mathbb{R}^p).
\]
These results follow from Theorem 3.1, Proposition 2.17 and Corollary 2.19 in Sato (2006).

**Remark D.1.** Assumptions (D.2)–(D.4) are necessary and sufficient conditions for the existence of the stochastic integral (D.1). We refer to Rajput and Rosinski (1989) and Sato (2006) for details.
If \( g(s, v) = g(\|s - v\|) \), that is, \( X \) is a strictly stationary isotropic random field, then we have that
\[
E[e^{i \langle t, X(s) \rangle}] = \exp \left\{ i \langle t, \int_{\mathbb{R}^d} g(\|v\|) \gamma_0 dv \rangle - \frac{1}{2} t^2 \left( \int_{\mathbb{R}^d} g(\|v\|) \Sigma_0 g(\|v\|)' dv \right) \right\} \\
+ \int_{\mathbb{R}^d} \left\{ \left( \int_{\mathbb{R}^p} e^{i \langle t, g(\|v\|) \rangle} - 1 - i \langle t, g(\|v\|) \rangle I(\|g(\|v\|)\| \leq 1) v_0(x) dx \right) dv \right\}.
\]

Then \( X \) has a density function when \( L \) is Gaussian, that is, \((\gamma_0, \Sigma_0, \nu_0) = (\gamma_0, \Sigma_0, 0)\). We can also give a sufficient condition for the existence of the density function of \( X \) when \( L \) is purely non-Gaussian, that is, \((\gamma_0, \Sigma_0, \nu_0) = (\gamma_0, 0, \nu_0)\).

**Lemma D.1.** Suppose that there exist an \( \alpha \in (0, 2) \) and a constant \( C > 0 \) such that
\[
\int_{\mathbb{R}^d} \int_{\mathbb{R}^p} |\langle u, g(\|v\|) \rangle|^2 I(\|\langle u, g(\|v\|) \rangle\| \leq 1) \nu_0(x) dx dv \geq C \|u\|^{2-\alpha}
\]
for any vector \( u \) with \( \|u\| \geq c_0 > 1 \). Then \( X \) with a purely non-Gaussian Lévy random field \( L \) has a bounded continuous, infinitely often differentiable functions whose derivatives are bounded.

**Proof.** From Proposition 0.2 in [Picard (1996)](#), it is sufficient to show that \( \int \|u\|^k |\varphi(u)| du < \infty \) for any non-negative integer \( k \), where \( \varphi(u) \) denotes the characteristic function of \( X(s) \). Observe that
\[
|\varphi(t)| = \left| \exp \left\{ \int_{\mathbb{R}^d} \int_{\mathbb{R}^p} \left( e^{i \langle t, g(\|v\|) \rangle} - 1 - i \langle t, g(\|v\|) \rangle I(\|g(\|v\|)\| \leq 1) v_0(x) dx \right) dv \right\} \right|^{1/2}
\]
\[
= \exp \left\{ \int_{\mathbb{R}^d} \int_{\mathbb{R}^p} \left( \cos(\langle t, g(\|v\|) \rangle) - 1 \right) I(\|\langle u, g(\|v\|) \rangle\| \leq 1) \nu_0(x) dx dv \right\}
\]
\[
\leq \exp \left\{ \int_{\mathbb{R}^d} \int_{\mathbb{R}^p} \left( \cos(\langle t, g(\|v\|) \rangle) - 1 \right) I (\|\langle u, g(\|v\|) \rangle\| \leq 1) \nu_0(x) dx dv \right\}
\]

Then using the inequality \( 1 - \cos(z) \geq 2(z/\pi)^2 \) for \( |z| \leq \pi \), we have that
\[
|\varphi(u)| \leq \exp \left\{ -C \int_{\mathbb{R}^d} \int_{\mathbb{R}^p} |\langle u, g(\|v\|) \rangle|^2 I (\|\langle u, g(\|v\|) \rangle\| \leq 1) \nu_0(x) dx dv \right\}
\]
\[
\leq \exp(-\tilde{C}\|u\|^{2-\alpha})
\]
for some constants \( C, \tilde{C} > 0 \). Therefore, we complete the proof. \( \square \)

For \( g(u, s) = g(u, \|s\|) : [0, 1]^d \times \mathbb{R}^d \to M_p(\mathbb{R}) \) with bounded components, assume that \( |g_{j,k}(u, \cdot) - g_{j,k}(v, \cdot)| \leq C \|s - v\| \bar{g}_{j,k}(\cdot) \) with \( C < \infty \) and for any \( u \in [0, 1]^d \),
\[
\max_{1 \leq j, k \leq p} \int_{\mathbb{R}^d} \left( |g_{j,k}(u, \|s\|)| + |\bar{g}_{j,k}(s)| + g_{j,k}^2(u, \|s\|) + \bar{g}_{j,k}^2(s) \right) ds < \infty.
\]

Consider the processes
\[
X_{s, A_n} = \int_{\mathbb{R}^d} g \left( \frac{s}{A_n}, \|s - v\| \right) L(dv), \quad X_u(s) = \int_{\mathbb{R}^d} g(u, \|s - v\|) L(dv)
\]
\[
X_u(s : A_{2,n}) = \int_{\mathbb{R}^d} g(u, \|s - v\|) \iota(\|s - v\| : A_{2,n}) L(dv).
\]
Note that \(X_u(s)\) and \(X_u(s : A_{2,n})\) are strictly stationary random fields for each \(u\). In particular, \(X_u(s : A_{2,n})\) is \(A_{2,n}\)-dependent. Assume that \(\max_{1 \leq k \leq p} E[|L_k(A)|^q] < \infty\) for \(1 \leq q \leq q_0\) where \(q_0\) is an integer such that \(q_0 \geq 2\). Observe that

\[
\|X_{s,A\nu} - X_u(s)\|_p \leq \|X_{s,A\nu} - X_u(s)\| + \|X_u(s) - X_u(s : A_{2,n})\|
\]

\[
\leq C \left( \frac{s}{A_n} - u \right) + \sum_{j=1}^p \int_{\mathbb{R}^d} \sum_{k=1}^{p} |g_{j,k} (\|s - v\|)| |L_j(dv)|
\]

\[
+ \frac{1}{A_n} \sum_{j=1}^p \int_{\mathbb{R}^d} \sum_{k=1}^{p} A_n^d |g_{j,k} (u, \|s - v\|) |(1 - \nu (\|s - v\| : A_{2,n}))|L_j(dv)|
\]

\[
\leq \left( \left\| \frac{s}{A_n} - u \right\| + \frac{1}{A_n} \right) U_{s,A\nu}(u),
\]

where

\[
U_{s,A\nu}(u) = \sum_{j=1}^p \int_{\mathbb{R}^d} \sum_{k=1}^{p} \left( C |g_{j,k} (\|s - v\|)| + A_n^d |g_{j,k} (u, \|s - v\|) |(1 - \nu (\|s - v\| : A_{2,n}))| \right) |L_j(dv)|.
\]

Assume that

\[
\sup_{n \geq 1} \sup_{\|u\| \leq 1} \max_{1 \leq j,k \leq p} \int_{\mathbb{R}^d} (A_n^d |g_{u,j,k}(\|s\| : A_{2,n}) + A_n^{2d} g_{u,j,k}^2 (\|s\| : A_{2,n})) ds < \infty.
\]

Then we can show that \(E[|U_{s,A\nu}(u)|^2] < \infty\) and Condition (Ma2) in Assumption 5.1 is satisfied. Let \(\{c_{j,k}\}_{1 \leq j,k \leq p}\) be positive constants and \(\{r_{j,k,\ell}(\cdot)\}_{1 \leq \ell \leq p, 1 \leq j,k \leq p}\) are continuous functions on \([0, 1]^d\) such that \(|r_{j,k,\ell}(u) - r_{j,k,\ell}(v)| \leq C \|u - v\|\) for \(1 \leq \ell \leq p, 1 \leq j,k \leq p_1\) with \(C < \infty\). Consider the function

\[
g(u, \|s\|) = \left( \sum_{\ell=1}^{p_{j,k}} r_{j,k,\ell}(u) e^{-c_{j,k,\ell} \|s\|} \right)_{1 \leq j,k \leq p}.
\]

Then we can show that the process \(X_{s,A\nu} = \int_{\mathbb{R}^d} g \left( \frac{s}{A_n}, \|s - v\| \right) L(dv)\) is an approximately \(D(\log n)\)-dependent random field for sufficiently large \(D > 0\) by applying a similar argument in the proof of Lemma B.8.

References

Altmeier, R. and Reiβ, M. (2019). Nonparametric estimation for linear SPDEs from local measurements. forthcoming in Ann. Appl. Probab.

Bandyopadhyay, S., Lahiri, S. N. and Nordman, D. J. (2015). A frequency domain empirical likelihood method for irregularly spaced spatial data. Ann. Statist. 43 519-545.

Berger, D. (2019). Lévy driven linear and semilinear stochastic partial differential equations. Stochastic Process. Appl. 130, 5865-5887.

Berger, D. (2020). Lévy driven CARMA generalized processes and stochastic partial differential equations. Stochastic Process. Appl. 130, 5865-5887.

Bernstein, S. N. (1926). Sur l’extension du théorème limite du calcul des probabilités aux sommes de quantités dépendantes. Ann. Math. 21 1-59.

Bertoin, J. (1996). Lévy Processes. Cambridge University Press.
Bolthausen, E. (1982). On the central limit theorem for stationary mixing random fields. *Ann. Probab.* **10**, 1047-1050.

Bosq, D. (1998). *Nonparametric Statistics for Stochastic Processes: Estimation and Prediction*, 2nd ed. Lecture Notes in Statistics **110** Springer, New York.

Bradley, R. C. (1989). A caution on mixing conditions for random fields. *Statist. Probab. Lett.* **8**, 489-491.

Bradley, R. C. (1993). Some examples of mixing random fields. *Rocky Mountain J. Math.* **23**, 495-519.

Brockwell, P.J. (2000). *Continuous-time ARMA processes*, Handbook of Statistics: Stochastic Processes, Theory and Methods (eds. C.R. Rao and D. N. Shanbhag), Elsevier, Amsterdam.

Brockwell, P.J. (2001). Lévy-driven CARMA processes. *Ann. Inst. Statist. Math.* **53**, 113-124.

Brockwell, P.J. and Matsuda, Y. (2017). Continuous auto-regressive moving average random fields on $\mathbb{R}^n$. *J. Roy. Statist. Soc. Ser. B Stat. Methodol.* **79**, 833-857.

Chernozhukov, V., Chetverikov, D. and Kato, K. (2013). Gaussian approximations and multiplier bootstrap for maxima of high-dimensional random vectors. *Ann. Statist.* **41**, 2786-2819.

Chernozhukov, V., Chetverikov, D. and Kato, K. (2017). Central limit theorems and bootstrap in high dimensions. *Ann. Probab.* **45**, 2309-2352.

Chernozhukov, V., Chetverikov, D. and Kato, K. (2019). Inference on causal and structural parameters using many moment inequalities. *Rev. Econ. Stud.* **86**, 1867-1900.

Cressie, N. (1993). *Statistics for Spatial Data*. Rev. ed. Wiley, New York.

Dahlhaus, R. (1997). Fitting time series models to nonstationary processes. *Ann. Statist.* **25**, 1-37.

Dahlhaus, R., Neumann, M.H. and von Sachs, R. (1999). Nonlinear wavelet estimation of time-varying autoregressive processes. *Bernoulli* **5**, 873-906.

Dahlhaus, R., Richter, S. and Wu, W.B. (2000). A likelihood approximation for locally stationary processes. *Ann. Statist.* **28**, 1762-1794.

Dahlhaus, R., Richter, S. and Wu, W.B. (2019). Towards a general theory for nonlinear locally stationary processes. *Bernoulli* **25**, 1013-1044.

Dahlhaus, R. and Subba Rao, S. (2006). Statistical inference for time-varying ARCH processes. *Ann. Statist.* **34**, 1075-1114.

Dedecker, J., Doukhan, P., Lang, G., Leon, J.R., Louhichi, S. and Prieur, C. (2007). *Weak Dependence: With Examples and Applications*, vol 190. Lecture Notes in Statistics. Springer, New York.

Doukhan, P. (1994). *Mixing: Properties and Examples*. Springer.

Fan, J. and Yao, Q. (2003). *Nonlinear time series: Nonparametric and Parametric Methods*, Springer, New York.

Fryzlewicz, P., Sapatinas, T. and Subba Rao, S. (2008). Normalized least-squares estimation in time-varying ARCH models. *Ann. Statist.* **36**, 742-786.

Fuglstad, G.-A., Simpson, D., Lindgren, F. and Rue, H. (2015). Does non-stationary spatial data always require non-stationary random fields? *Spatial Statistics* **14**, 505-531.

Gelfand, A.E., Kim, H.-J., Sirmans, C.F. and Banerjee, S. (2003). Spatial modeling with spatially varying coefficient processes. *J. Amer. Statist. Assoc.* **98**, 387-396.

Hafner, C. M. and Linton, O. (2010). Efficient estimation of a multivariate multiplicative volatility model. *J. Econometrics* **159**, 55-73.

Hallin, M., Lu, Z. and Tran, L. (2004). Local linear spatial regression. *Ann. Statist.* **32**, 2469-2500.

48
Hallin, M., Lu, Z. and Yu, K. (2009). Local linear spatial quantile regression. *Bernoulli* 15 659-686.

Hansen, B.E. (2008). Uniform convergence rates for kernel estimation with dependent data. *Econometric Theory* 24 726-748.

Hall, P. and Patil, P. (1994). Properties of nonparametric estimators of autocovariance for stationary random fields. *Probab. Theory Related Fields* 99 399-424.

Horowitz, J. L. and Lee, S. (2012) Uniform confidence bands for functions estimated nonparametrically with instrumental variables. *J. Econometrics* 168, 175-188.

Jenish, N. (2012). Nonparametric spatial regression under near-epoch dependence. *J. Econometrics* 167 224-239.

Karcher, W., Roth, S., Spodarev, E. and Walk, C. (2019). An inverse problem for infinitely divisible moving average random fields. *Stat. Inference Stoch. Process.* 22 263-306.

Kato, K. and Kurisu, D. (2020). Bootstrap confidence bands for spectral estimation of Lévy densities under high-frequency observations. *Stochastic Process. Appl.* 130 1159-1205.

Kristensen, D. (2009). Uniform convergence rates of kernel estimators with heterogeneous dependent data. *Econometric Theory* 25 1433-1445.

Kristensen, D. (2011). Stationary approximations of time-inhomogenous Markov chains with applications. Mimeo.

Kurisu, D. (2019). On nonparametric inference for spatial regression models under domain expanding and infill asymptotics. *Statist. Probab. Lett.* 154 108543.

Kurisu, D. (2021a). Nonparametric regression for locally stationary functional time series. [arXiv:2105.07613](https://arxiv.org/abs/2105.07613).

Kurisu, D. (2021b). On the estimation of locally stationary functional time series. [arXiv:2105.11873](https://arxiv.org/abs/2105.11873).

Kurisu, D., Kato, K. and Shao, X. (2021). Gaussian approximation and spatially dependent wild bootstrap for high-dimensional spatial data. [arXiv:2103.10720](https://arxiv.org/abs/2103.10720).

Koo, B. and Linton, O. (2012). Semiparametric estimation of locally stationary diffusion models. *J. Econometrics* 170 210-233.

Lahiri, S.N. (2003a). *Resampling Methods for Dependent Data*. Springer.

Lahiri, S.N. (2003b). Central limit theorems for weighted sum of a spatial process under a class of stochastic and fixed design. *Sankhya Ser. A* 65 356-388.

Lahiri, S. N., Kaiser, M. S., Cressie, N. and Hsu, N.-J. (1999). Prediction of spatial cumulative distribution functions using subsampling (with discussion). *J. Amer. Statist. Assoc.* 94 86-110.

Lahiri, S.N. and Zhu, J. (2006). Resampling methods for spatial regression models under a class of stochastic designs. *Ann. Statist.* 34 1774-1813.

Li, L. (2016). Nonparametric regression on random fields with random design using wavelet method. *Stat. Inference Stoch. Process.* 19 51-69.

Liebscher, E. (1996). Strong convergence of sums of α-mixing random variables with applications to density estimation. *Stochastic Process. Appl.* 65 69-80.

Lu, Z. and Tjostheim, D. (2014). Nonparametric estimation of probability density functions for irregularly observed spatial data. *J. Amer. Statist. Soc.* 109 1546-1564.

Machkouri, M.E. and Stoica, R. (2008). Asymptotic normality of kernel estimates in a regression model for random fields. *J. Nonparametric Statist.* 22 955-971.

Machkouri, M. El. (2014). Kernel density estimation for stationary random fields. *ALEA, Lat. Am. J. Probab. Math. Stat.* 11 259-279.
Machkouri, M. El., Volny, D. and Wu, W. B. (2013). A central limit theorem for stationary random fields. *Stochastic Process. Appl.* **123** 1-14.

Machkouri, M.E., Es-Sebaiy, K. and Ouassou, I. (2017). On local linear regression for strong mixing random fields. *J. Multivariate Anal.* **156** 103-115.

Mammen, E., Linton, O. and Nielsen, J. (1999). The existence and asymptotic properties of a backfitting projection algorithm under weak conditions. *Ann. Statist.* **27** 1443-1490.

Marquardt, T. and Stelzer, R. (2007). Multivariate CARMA processes. *Stochastic Process. Appl.* **117** 96-120.

Masry, E. (1996). Multivariate local polynomial regression for time series: uniform strong consistency and rates. *J. Time Ser. Anal.* **17** 571-599.

Masuda, H. (2019). Non-Gaussian quasi-likelihood estimation of SDE driven by locally stable Lévy processes. *Stochastic Process. Appl.* **129** 1013-1059.

Matsuda, Y. and Yajima, Y. (2009). Fourier analysis of irregularly spaced data on $\mathbb{R}^d$. *J. Roy. Stat. Soc. Ser. B Stat. Methodol.* **71** 191-217.

Matsuda, Y. and Yajima, Y. (2018). Locally stationary spatio-temporal processes. *Jpn. J. Data Sci.* **4** 41-57.

Matsuda, Y. and Yuan, X. (2020). Multivariate CARMA random fields. DSSR Discussion Papers No.113.

Pezo, D. (2018). Local stationarity for spatial data. PhD thesis, Technische Universität Kaiserslautern.

Picard, J. (1996). On the existence of smooth densities for jump processes. *Probab. Theory Related Fields* **105** 481-511.

Rajput, B.S. and Rosinski, J. (1989). Spectral representations of infinitely divisible processes. *Probab. Theory Relat. Fields* **82** 451-487.

Rio, E. (2013). Inequalities and limit theorems for weakly dependent sequences. cel-00867106v2.

Robinson, P. (2011). Asymptotic theory for nonparametric regression with spatial data. *J. Econometrics* **165** 5-19.

Sato, K. (1999). *Lévy Processes and Infinitely Divisible Distributions*. Cambridge University Press.

Sato, K. (2006). Additive processes and stochastic integrals. *Illinois Journal of Mathematics* **50** 825-851.

Schlemm, E. and Stelzer, R. (2012). Multivariate CARMA processes, continuous-time state space models and complete regularity of the innovations of the sampled processes. *Bernoulli* **18** 46-63.

Shao, X. (2010). The dependent wild bootstrap. *J. Amer. Statist. Assoc.* **105** 218-235.

Steel, M.F.J. and Fuentes, M. (2010). Non-Gaussian and nonparametric models for continuous spatial data. in Gelfand et al.(Eds.), *Handbook of Spatial Statistics*, Chapman & Hall/CRC, pp149-170.

Truquet, L. (2017). Parameter stability and semiparametric inference in time varying autoregressive conditional heteroscedasticity models. *J. Roy. Statist. Soc. Ser. B Stat. Methodol.* **79** 1391-1414.

Truquet, L. (2019). Local stationarity and time-inhomogeneous Markov chains. *Ann. Statist.* **47** 2023-2050.

van der Vaart, A.W. and Wellner, J.A.(1996). *Weak Convergence and Empirical Processes with Applications to Statistics*. Springer.
Vogt, M. (2012). Nonparametric regression for locally stationary time series. *Ann. Statist.* **40** 2601-2633.

Yu, B. (1994). Rates of convergence for empirical processes of stationary mixing sequences. *Ann. Probab.* **22** 94-116.

Zhang, T. and Wu, W. B. (2015). Time-varying nonlinear regression models: Nonparametric estimation and model selection. *Ann. Statist.* **43** 741-768.

Zhou, Z. (2014). Nonparametric specification for non-stationary time series regression. *Bernoulli* **20** 78-108.

Zhao, Z. and Wu, W. B. (2008). Confidence bands in nonparametric time series regression. *Ann. Statist.* **36** 1854-1878.

Zhou, Z. and Wu, W. B. (2009). Local linear quantile estimation for nonstationary time series. *Ann. Statist.* **37** 2696-2729.

(D. Kurisu) Graduate School of International Social Sciences, Yokohama National University, 79-4 Tokiwadai, Hodogaya-ku, Yokohama 240-8501, Japan.

*Email address: kurisu-daisuke-jr@ynu.ac.jp*