A note on the resonant frequencies of rapidly rotating black holes

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I discuss the range of validity of Detweiler’s formula for the resonant frequencies of rapidly rotating Kerr black holes. While his formula is correct for extremal black holes, it has also been commonly accepted that it describes very well the resonant frequencies of near extremal black holes, and that therefore there is a large number of modes clustering on the real axis as the black hole becomes extremal. I will show that this last statement is not only incorrect, but that it also does not follow from Detweiler’s formula, provided it is handled with due care. It turns out that only the first \( n \ll -\log (r_+ - r_-)/r_+ \) modes are well described by that formula, which translates, for any astrophysical black hole, into one or two modes only. All existing numerical data gives further support to this claim. I also discuss some implications of this result for recent investigations on the late-time dynamics of rapidly rotating black holes.

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I. INTRODUCTION

Black hole spacetimes provide an interesting arena for classical wave propagation: they absorb, scatter and sometimes even amplify incident waves. The detailed response of a black hole to a given incident wave depends on the parameters characterizing the black hole, and thus detection of a black hole (by, for example, gravitational wave detectors) requires a deep knowledge of what precisely happens when a wave hits a black hole.

The first numerical studies on this \[1\] revealed that the response of a black hole to a given incident wavepacket is dominated by an exponentially decaying ringing phase. What’s more, the ringing frequency and decaying timescale of this signal depends only on the black hole parameters (mass, charge and angular momentum), and not on the particular details of the wave packet. Thus, a black hole, like so many other objects, has characteristic oscillation modes, called quasinormal modes (QNMs) (the associated characteristic frequencies are called quasinormal frequencies, QN frequencies or \( \omega_{\text{QN}} \)). It turns out that these modes are excited by almost any process taking place in the vicinities of a black hole, and thus a direct detection of black holes may well be done by identifying this characteristic ringing \[2\]. This possibility has stimulated a great deal of research on the computation of QNMs and QN frequencies of black holes \[3\] (see also \[4\] for another recent motivation for studying QNMs in black hole spacetimes). The situation for Schwarzschild black holes is well understood: in Schwarzschild coordinates, assume that the wave has the dependence

\[
\Psi(t, r, \theta, \phi) = e^{-i(\omega t + m \phi)} Y_{lm}(\theta) R(r),
\]

where \( Y_{lm}(\theta) \) are just the usual spherical harmonics. I assume \( \Psi \) is a scalar wave, otherwise the separation of the angular variables would have to be done with vectorial or tensorial spherical harmonics. Since the Schwarzschild geometry is spherically symmetric, the azimuthal number \( m \) won’t play any role. The QN frequencies are those frequencies \( \omega \) for which the field is outgoing at infinity and ingoing near the horizon \[2\]. Now, it turns out that for each angular number \( l \) there is a discrete, infinite number of frequencies satisfying these boundary conditions, and that they all have a negative imaginary part, i.e., if we write \( \omega_{\text{QN}} = \omega_{l} + i \omega_{l}' \) then \( \omega_{l}' < 0 \). According to the time dependence in \( \Psi \) the Schwarzschild geometry is then stable, since any perturbation will decay exponentially with time. The QN frequencies are arranged by growing magnitude of imaginary part. For instance, considering gravitational waves, the fundamental mode \( (n = 0) \) of quadrupole waves \( (l = 2) \) is \( \omega M = 0.37367 - 0.08896 \), while the first overtone \( (n = 1) \) is \( \omega M = 0.34671 - 0.27391 \). The fundamental mode will in general control the signal, because it is the least damped, and it’s quality factor \( Q \sim \omega_{l}/\omega_{l}' \) is of order unity. Thus, we see that Schwarzschild black holes are very poor resonators, with small \( Q \)’s.

The QNM spectrum for Kerr black holes is at the moment well understood, for small rotation at least \[5\]. Since this geometry is no longer spherically symmetric, the results depend on the azimuthal number \( m \). The numerical studies indicate that for \( m > 0 \) the quality factor increases dramatically as one increases the rotation because the magnitude of the imaginary part of the QN frequency decreases. Indeed, it looks like \( \omega_{l}' \rightarrow 0 \) as \( a \rightarrow M \), where \( a \) is the angular momentum per unit mass and I recall that \( a \) is bounded from above by \( M \); for \( a = M \) we say the black hole is extremal. Now, what can we gain from this? The first obvious answer is that if these modes get less damped, they should be easier to detect. However, this is so provided they are excited to a measurable amplitude (see \[6\] for a discussion on this point), which does not happen. Still, there is an analytical result by Detweiler \[5\] indicating that for very rapidly rotating black holes there seems to be an infinity of QN frequencies accumulating on the real axis. Thus, it could
be that even though a single QNM is hard to excite, all of them (which cluster on the real axis, and are therefore nearly undamped) could conspire to give a measurable effect. This was investigated by Glampedakis and Andersson recently, who showed that this effect is indeed there for extremal black holes. They also suggest that nearly extremal black holes could also display a similar behavior.

Here, I will discuss Detweiler’s formula and its range of validity, to see whether or not this or other phenomena involving rapidly rotating black holes could have any importance for astrophysical black holes. Contrary to common expectations, I find that this formula is of limited use for any astrophysical black hole. Indeed, it describes accurately only the first $n < -\log (r_+ - r_-)/r_+$ overtones of rapidly rotating black holes. For extremal black holes it should describe all the modes, and therefore there really is an infinite number of QN frequencies clustering on the real axis. For non-extremal black holes, say for example $(r_+ - r_-)/r_+ \sim 10^{-10}$ (which is still impossible to find in any astrophysical scenario) it describes well only a number $n << 20$ modes.

II. THE RESONANT FREQUENCIES OF NEAR EXTREMAL KERR BLACK HOLES

A Kerr black hole is characterized by two parameters, the mass $M$ and the angular momentum per unit mass $a$. It has an event horizon at $r_+ = M + \sqrt{M^2 - a^2}$ and a Cauchy horizon at $r_- = M - \sqrt{M^2 - a^2}$.

To analyze the QNMs and QN frequencies of black holes, one studies the wave equation that results from a linearization procedure. For Kerr black holes, the equations describing linearized waves were derived by Teukolsky. I shall not reproduce his equations here but instead refer the reader to the original works [3, 10]. The equations describing the evolution of scalar ($s = 0$), electromagnetic ($s = 1$) and gravitatonal ($s = 2$) perturbations are described by the following coupled equations:

\[
x(x + \sigma) \frac{d^2 \tilde{R}_{lm}}{dx^2} - \{2i\omega x^2 + 2x(2i\omega - s - 1) - (s + 1)\sigma + 4i\tau\} \frac{d\tilde{R}_{lm}}{dx} - \{2(2s + 1)i\omega(x + 1) + \lambda\} \tilde{R}_{lm} = 0 ,
\]

where 

\[
\sigma = \frac{r_+ - r_-}{r_+} ,
\]

\[
\omega_+ = \frac{a}{2Mr_+} ,
\]

\[
\tau = M(\omega - m\omega_+) ,
\]

\[
\tilde{\omega} = \omega r_+ ,
\]

\[
\delta^2 = 4\tilde{\omega}^2 - 1/4 - \lambda ,
\]

\[\lambda = E + a^2\omega^2 - 2m\omega_+ - s(s + 1).\]

These must be supplemented by “physically acceptable” boundary conditions. For a scattering problem, we allow in- and out-going waves at the asymptotic region ($r \to \infty$), but only in-going waves near the horizon, i.e., the field satisfies

\[\tilde{R}^\text{in} \sim \begin{cases} \frac{1}{Z^{\text{out}}}$e^{-2i\omega r} + Z^{\text{in}}$^{-1}$r \to r_+ , \\
\end{cases}\] as $r \to +\infty$.

For a study of resonant frequencies, we do not want waves incident from infinity, and thus it must be that $Z^{\text{in}} = 0$. I shall also be interested in the double limit $a \to M$, $\omega \to m\omega_+$, which corresponds to $\sigma \to 0$, $\tau \to 0$. A uniformly valid solution for $\tilde{R}_{lm}$ in this double limit was first found by Teukolsky and Press [10], following earlier work by Starobinsky and Starobinsky and Churilov [11]. Detweiler [3] then used that solution to find the QN frequencies of near extremal black holes. I shall now re-derive the condition for the QN frequencies of near extremal Kerr black holes, making explicit the assumptions that go with it.

Let us first consider equation (2) in the limit when $x >> \max(\sigma, \tau)$, i.e., for large radii. Then (2) is well approximated by

\[x^2 \frac{d^2 R_{lm}}{dx^2} - \{2i\omega x^2 + 2x(2i\omega - (s + 1))\} \frac{dR_{lm}}{dx} - \{2(2s + 1)i\omega(x + 1) + \lambda\} R_{lm} = 0 .\]

A solution to (11) that satisfies (10) can be written in terms of confluent hypergeometric functions [12]

\[R_{lm} = A x^{-s - 1/2 + 2i\omega + i\delta} M(1/2 + s + 2i\omega + i\delta, 1 + 2i\delta, 2i\omega x) + B(\delta \to -\delta)\]

where $A,B$ are constants and the notation $(\delta \to -\delta)$ means “replace $\delta$ by $-\delta$ in the preceding term”.

We next turn to the case when $x << 1$, i.e., try to find a solution that is valid close to the black hole’s horizon. In this regime, equation (2) can be written as

\[x(x + \sigma) \frac{d^2 \tilde{R}_{lm}}{dx^2} - \{2x(2i\omega - (s + 1)) - (s + 1)\sigma + 4i\tau\} \frac{d\tilde{R}_{lm}}{dx} - \{2(2s + 1)i\omega + \lambda\} \tilde{R}_{lm} = 0 .
\]

This is the hypergeometric equation, and one solution can be written as

\[\tilde{R}_{lm} = 2F_1(1/2 + s - 2i\omega + i\delta, 1/2 - 2i\omega + s - i\delta, 1 + s - 4i\tau/\sigma, -x/\sigma) .
\]
It is straightforward to verify that \( \frac{\text{d} F}{\text{d} t} \) → 1 as \( \mathbf{x} \to 0 \), which means that this solution has the desired “purely ingoing wave” behavior close to the event horizon.

The key point now is that there is an overlapping region, where both solutions are valid. Indeed, in the overlap region \( \max(\sigma, \tau) \ll x \ll 1 \) the solutions \([12]\) and \([13]\) both describe the solution, and can therefore be matched. To do the matching, take first the \( \mathbf{x} \to 0 \) limit of \([12]\) which yields,

\[
\tilde{R}_{lm} \to A x^{-s-1/2+2i\omega+i\delta} + B(\delta \to -\delta) \tag{15}
\]

Similarly, for \( \mathbf{x} \to \infty \), \([14]\) becomes,

\[
\tilde{R}_{lm} \to C(\delta) x^{-s-1/2+2i\omega+i\delta} + C(-\delta) x^{-s-1/2+2i\omega-i\delta}, \tag{16}
\]

where

\[
C(\delta) = \frac{\Gamma(1 + s - 4i\tau/\sigma)\Gamma(2i\delta)\Gamma(2s+1-2i\omega-i\delta)}{\Gamma(s + 1/2 - 2i\omega + i\delta)\Gamma(1/2 + 2i\omega + i\delta + 4i\tau/\sigma)} \tag{17}
\]

We can extract \( A \) and \( B \) by matching the solutions \([13]\) and \([15]\).

\[
A = C(\delta) \frac{\Gamma(1/2 + 2i\omega + i\delta + 4i\tau/\sigma)}{\Gamma(1/2 + 2i\omega + i\delta - 4i\tau/\sigma)} \tag{18}
\]

\[
B = C(-\delta) \frac{\Gamma(1/2 + 2i\omega + i\delta + 4i\tau/\sigma)}{\Gamma(1/2 + 2i\omega + i\delta - 4i\tau/\sigma)} \tag{19}
\]

On the other hand, approximating \([12]\) for \( \mathbf{x} \to \infty \) we get for the amplitudes \( Z_{\text{in}}, Z_{\text{out}} \),

\[
Z_{\text{in}} = A \frac{\Gamma(1 + 2i\delta)}{\Gamma(1/2 - s - 2i\omega + i\delta)} (-2i\omega)^{-1/2-s-2i\omega-i\delta} + B(\delta \to -\delta) \tag{20}
\]

\[
Z_{\text{out}} = A \frac{\Gamma(1 + 2i\delta)}{\Gamma(1/2 + s + 2i\omega + i\delta)} (2i\omega)^{-1/2+s+2i\omega-i\delta} + B(\delta \to -\delta) \tag{21}
\]

To find the resonant frequencies we now impose \( Z_{\text{in}} = 0 \). Using \([15]-[17]\) we find that for these frequencies the following condition must be satisfied

\[
-\frac{\Gamma(2i\delta)\Gamma(1 + 2i\delta)}{\Gamma(-2i\delta)\Gamma(1 - 2i\delta)} \left[ \frac{\Gamma(s + 1/2 - 2i\omega - i\delta)}{\Gamma(s + 1/2 - 2i\omega + i\delta)} \right]^2 = (-2i\omega)^{2i\delta} \frac{\Gamma(1/2 + 2i\omega + i\delta - 4i\tau/\sigma)}{\Gamma(1/2 + 2i\omega - i\delta - 4i\tau/\sigma)} \tag{22}
\]

### A. Detweiler’s formula

Up to this point, the only assumptions that entered the derivation of \([22]\) were that the black hole is very rapidly spinning and that \( \omega \sim \omega_+ \). Let us now proceed to derive Detweiler’s formula for the QN frequencies of rapidly rotating black holes.

**Table I:** Values of the constant \( \delta^2 \) for some \((l, m)\) pairs, for \( \omega = m/2 \), i.e., the extreme limit, and for scalar perturbations. Notice two important things: first, for \( l = m = 1 \) the quantity \( \delta^2 \) is negative, and thus \( \delta \) is purely imaginary. Second, for \( l = 6, m = 5 \), \( \delta \) is purely real. Thus, although the rule is that for \( l = m \) the quantity \( \delta \) is real, while for other \( l, m \) values it is imaginary, there are exceptions.

| \( l \) | \( m \) | \( \delta^2 \) |
|---|---|---|
| 1 | 0 | -0.4497 |
| 2 | 0 | -4.9144 |
| 2 | 2 | 0.8948 |
| 2 | 1 | -3.8792 |
| 3 | 3 | 3.7552 |

The quantity \( \delta \) is in general a complex quantity. However, the numerical studies show that for \( \omega \sim \omega_+ \), and for some \( l, m \) values it is almost purely real. I would like to emphasize that this fact is seen numerically, it hasn’t been analytically proved. Moreover, it seems to be true for most \( l = m \) modes but not only: \( \delta \) is real also for some other values of \( l, m \), and so there is nothing special about \( l = m \) modes, contrary to what is stated in \([6]\). To be more precise, I show in Table I the values of \( \delta^2 \) for some \( l, m \) pairs, assuming one is exactly in the extreme case, i.e., \( \omega = m/2 \). The Table refers to the scalar case \( s = 0 \), although the trend is similar for other \( s \) values. Notice that for \( l = m = 1 \) the quantity \( \delta^2 \) is negative, and notice also that for \( l = 6, m = 5 \) it is positive. So, there really is nothing special about \( l = m \) modes.

With this cleared, I shall go on assuming that \( \delta \) is a real quantity.

The left-hand side of equation \([22]\) has a well-defined limit as \( a \to M \) and \( \omega \to \omega_+ \). We represent that limit by

\[
\text{LHS} = q e^{iX}. \tag{23}
\]

Now, we cannot have a consistent solution unless \( \tau/\sigma \to \infty \) as \( a \to M \); this is because \( \tau/\sigma \) had some finite limit, then so would the Gamma functions in the RHS of equation \([22]\). And then we would have \( \omega \sim \omega_+ \), with \( X \) a well-defined quantity. But this would violate our assumptions that \( \omega \sim \omega_+ \). Therefore, let us proceed assuming that \( \tau/\sigma \to \infty \). Using Stirling’s formula \([12]\),

\[
\text{RHS} = (-8\omega_\tau)^{2i\delta}. \tag{24}
\]

In other words, a QNM must be a solution to

\[
f(\omega) = (-8\omega_\tau)^{2i\delta} - q e^{iX} = 0. \tag{25}
\]

Using \(-8\omega_\tau = \rho e^{i\xi}\) we see that solutions follow from (remembering that \( \delta \) is real and \( \omega \to \omega_+ \))

\[
\rho = \exp \left[ \frac{\chi - 2k\pi}{2\delta} \right], \tag{26}
\]
\[ \zeta = -\frac{1}{2\delta} \log q , \]  

where \( k \) is any integer number. This is Detweiler’s formula \(^7\). Since we are always assuming \( \omega \approx m\omega^* \), this can also be written as \(^\text{13} \) (just substitute \( -8\omega^* \tau \sim -4m(\omega - m/2) \))

\[ \omega M \approx \frac{m}{2} - \frac{1}{4m} \left[ \frac{e^{(x-2k\pi)/2\delta}}{\cos \zeta} - \frac{i}{4m} \frac{e^{(x-2k\pi)/2\delta}}{\sin \zeta} \right], \]  

(29)

It is possible to prove \(^\text{13} \) that \( \sin \zeta > 0 \), i.e. these QNMs are all damped.

**B. Discussion of Detweiler’s formula**

I now turn to discuss what is the range of validity of formula \(^\text{29} \). We have been assuming that \( \tau/\sigma \to \infty \), but there is a number \( k \) in \(^\text{29} \) at which this condition breaks down. In fact, \( \tau \gg \sigma \) implies, according to \(^\text{29} \), that in order of magnitude one should have

\[ e^{-k} \gg \frac{r_+ - r_-}{r_+} . \]  

(30)

Thus, there is an upper bound on \( k \) given by

\[ k < - \log \frac{r_+ - r_-}{r_+} . \]  

(31)

Notice that formula \(^\text{31} \) is basically a lower bound for the fundamental mode. In fact, with \(^\text{31} \) we get, from equation \(^\text{29} \) that \( \omega_1 M \gg (r_+ - r_-)/r_+ \).

This is a very restrictive bound. For instance, if we have \( (r_+ - r_-)/r_+ \approx 10^{-10} \), formula \(^\text{29} \) can only describe well the first \( n \ll 20 \) modes, where \( n \) now labels the overtone number. We know that accretion cannot spin a black hole beyond \( a = 0.998M \) \(^\text{3} \), and so, for any astrophysical black hole, Detweiler’s formula should only describe accurately one or two modes.

Extremal black holes are accurately described by \(^\text{29} \) for all \( n \). In fact, all one has to do to prove this is to put \( \sigma = 0 \) in Teukolsky’s equations. In this case equation \(^\text{13} \) would be replaced by a confluent hypergeometric, and the final outcome for the QN frequencies would still be equation \(^\text{29} \). Thus, for extremal black holes, there is indeed an infinity of modes clustering on the real axis.

**III. DISCUSSION AND IMPLICATIONS**

I have re-derived Detweiler’s \(^\text{3} \) result for the quasi-normal frequencies of rapidly rotating black holes, and I have shown to what extent it can be used. I have proved that for any astrophysical black hole, Detweiler’s approximation is likely to be accurate for just the fundamental mode. In \(^\text{3} \) it was shown that the signal from extremal black holes goes like \( t^{-4} \) at late times instead of the usual power-law \( t^{-2l-3} \) (for \( l = m \), but I refer the reader to \(^\text{14} \)) . Although they prove this for \( l = m \) modes there should be other modes decaying like this, according to the discussion in this paper. Now, the main ingredient in their result was the fact that for extremal black holes there is an infinity of long-lived modes near the real axis. What hope is there to have this same phenomenon for non-extremal, but rapidly rotating black holes? As I have shown, this effect will not be present for any astrophysical realistic black hole, not according to the analytical approximation, and not according to the numerical results in \(^\text{3} \). To conclude, I would like to draw attention to the fact that Detweiler’s approximation holds good for any \( (l, m) \) pair for which the quantity \( \delta \) is real. This includes most of the \( l = m \) modes, and some other modes, but not all. However, from the numerical work in \(^\text{3} \) it seems that all modes having positive \( m \) tend to cluster on the real axis, as the black hole approaches extremality. One is thus led to suspect that there should be some general argument, independent of \( \delta \), showing that all modes cluster on the real axis.

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