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THE NON-SYMMETRIC OPERAD PRE-LIE IS FREE

NANTEL BERGERON AND MURIEL LIVERNET

Abstract. We prove that the pre-Lie operad is a free non-symmetric operad.

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Introduction

Operads are a specific tool for encoding type of algebras. For instance there are operads encoding associative algebras, commutative and associative algebras, Lie algebras, pre-Lie algebras, dendriform algebras, Poisson algebras and so on. A usual way of describing a type of algebras is by giving the generating operations and the relations among them. For instance a Lie algebra $L$ is a vector space together with a bilinear product, the bracket (the generating operation) satisfying the relations $[x, y] = -[y, x]$ and $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ for all $x, y, z \in L$. The vector space of all operations one can perform on $n$ distinct variables in a Lie algebra is $\mathcal{Lie}(n)$, the building block of the symmetric operad $\mathcal{Lie}$. Composition in the operad corresponds to composition of operations. The vector space $\mathcal{Lie}(n)$ has a natural action of the symmetric group, so it is a symmetric operad. The case of associative algebras can be considered in two different ways. An associative algebra $A$ is a vector space together with a product satisfying the relation $(xy)z = x(yz)$. The vector space of all operations one can perform on $n$ distinct variables in an associative algebra is $\mathcal{As}(n)$, the building block of the symmetric operad $\mathcal{As}$. The vector space $\mathcal{As}(n)$ has for basis the symmetric group $S_n$. But, in view of the relation, one can look also at the vector space of all order-preserving operations one can perform on $n$ distinct ordered variables in an associative algebra: this is a vector space of dimension 1 generated by the only operation $x_1 \cdots x_n$. So the non-symmetric operad $\tilde{\mathcal{As}}$ describing associative algebras is 1-dimensional for each $n$: this is the terminal object in the category of non-symmetric operads.

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Here is the connection between symmetric and non-symmetric operads. A symmetric operad $\mathcal{P}$ starts with a graded vector space $(\mathcal{P}(n))_{n \geq 0}$ together with an action of the symmetric group $S_n$ on $\mathcal{P}(n)$ for each $n$. This data is called a symmetric sequence or an $S$-module or a vector species. There is a forgetful functor from the category of vector species to the category of graded vector spaces, forgetting the action of the symmetric group. This functor has a left adjoint $S$ which corresponds to tensoring by the regular representation of the symmetric group. A symmetric (non-symmetric) operad is a monoid in the category of vector species (graded vector spaces). Again there is a forgetful functor from the category of symmetric operads to the category of non-symmetric operads admitting a left adjoint $S$. The symmetric operad $\mathcal{A}s$ is the image of the non-symmetric operad $\overset{\sim}{\mathcal{A}}s$ by $S$. It is clear that $\mathcal{L}ie$ is not in the image of $S$ since the Jacobi relation does not respect the order of the variables $x < y < z$ nor the anti-symmetry relation. Still one can regard $\mathcal{L}ie$ as a non-symmetric operad applying the forgetful functor. Salvatore and Tauraso proved in [5] that the operad $\mathcal{L}ie$ is a free non-symmetric operad.

A free non-symmetric operad describes type of algebras which have a set of generating operations and no relations between them. For instance, magmatic algebras are vector spaces together with a bilinear product. There is a well known free non-symmetric operad, the Stasheff operad, built on Stasheff polytopes, see e.g. [6]. An algebra over the Stasheff operad is a vector space $V$ together with an $n$-linear product: $V^\otimes n \to V$ for each $n$. From the point of view of homotopy theory, the category of operads is a Quillen category and free operads play an essential role in the homotopy category. One wants to replace an operad $\mathcal{P}$ by a quasi-free resolution, that is, a morphism of operads $\mathcal{Q} \to \mathcal{P}$ where $\mathcal{Q}$ is a free operad endowed with a differential realizing an isomorphism in homology. For instance, a quasi-free resolution of $\overset{\sim}{\mathcal{A}}s$, in the category of non-symmetric operads, is given by the Stasheff operad. Algebras over this operad are $A_\infty$-algebras (associative algebras up to homotopy). This gives us the motivation for studying whether a given symmetric operad is free as a non-symmetric operad or not.

In this paper we prove that the operad pre-Lie is a free non-symmetric operad. Pre-Lie algebras are vector spaces together with a bilinear product satisfying the relation $(x \ast y) \ast z - x \ast (y \ast z) = (x \ast z) \ast y - x \ast (z \ast y)$. The operad pre-Lie is based on labelled rooted trees which are of combinatorial interest. In the process of proving the main result, we describe another operad denoted $\mathcal{T}_{\text{Max}}$ also based on rooted trees and having the advantage of being the linearization of an operad in the category of sets. We prove that it is a free non-symmetric operad. The link between the two operads is made via a gradation on labelled rooted trees.

1. The pre-Lie operad and rooted trees

We first recall the definition of the pre-Lie operad based on labelled rooted trees as in [3]. For $n \in \mathbb{N}^+$, the set $\{1, \ldots, n\}$ is denoted by $[n]$ and $[0]$ denotes the empty set. The symmetric group on $k$ letters is denoted by $S_k$. There are many equivalent definitions of operads and we refer to [4] for basics on operads. We work over the
ground field $k$ and vector spaces are considered over $k$. Here are the definitions needed for the sequel.

**Definition 1.1.** A (reduced) non-symmetric operad is a graded vector space $(\mathcal{P}(n))_{n \geq 1}$, with a unit $1 \in \mathcal{P}(1) = k$, together with composition maps $a_i : \mathcal{P}(n) \otimes \mathcal{P}(m) \to \mathcal{P}(n + m - 1)$ for $1 \leq i \leq n$ satisfying the following relations: for $a \in \mathcal{P}(n)$, $b \in \mathcal{P}(m)$ and $c \in \mathcal{P}(\ell)$

\[
\begin{align*}
(a \circ_i b) \circ_{j+i-1} c &= a \circ_i (b \circ_j c), & \text{for } 1 \leq j \leq m, \\
(a \circ_i b) \circ_j c &= (a \circ_j c) \circ_{i+\ell-1} b, & \text{for } j < i, \\
1 \circ_i a &= a, \\
a \circ_i 1 &= a.
\end{align*}
\]

A non-trivial composition is a composition $a \circ_i b$ with $a \in \mathcal{P}(n), b \in \mathcal{P}(m)$ and $n, m > 1$.

If in addition each $\mathcal{P}(n)$ is acted on the right by the symmetric group $S_n$ and the composition maps are equivariant with respect to this action, then the collection $(\mathcal{P}(n))_n$ forms a symmetric operad. An algebra over an operad $\mathcal{P}$ is a vector space $X$ endowed with evaluation maps

\[
ev_n : \mathcal{P}(n) \otimes X^\otimes n \to X, \quad p \otimes x_1 \otimes \ldots \otimes x_n \mapsto p(x_1, \ldots, x_n)
\]

compatible with the composition maps $a_i$: for $p \in \mathcal{P}(n), q \in \mathcal{P}(m), x'_i s \in X$ one has

\[
(p \circ_i q)(x_1, \ldots, x_{n+m-1}) = p(x_1, \ldots, x_{i-1}, q(x_i, \ldots, x_{i+m-1}), x_{i+m}, \ldots, x_{n+m-1}).
\]

If the operad is symmetric the evaluation maps are required to be equivariant with respect to the action of the symmetric group as follows:

\[
(p \cdot \sigma)(x_1, \ldots, x_n) = p(x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(n)}).
\]

In the sequel an operad will always mean a reduced operad.

**Definition 1.2.** In this paper we will consider two type of trees: planar rooted trees will represent the composition maps in a non-symmetric operad (see [1,3]) and rooted trees will be the objects of our study (see [1,4]). Here are the definitions we will use in the sequel.

By a (planar) tree we mean a non empty finite connected contractible (planar) graph. All the trees considered are rooted.

In the planar case some edges (external edges or legs) will have only one adjacent vertex; the other edges are called internal edges. There is a distinguished leg called the root leg. The other legs are called the leaves. The choice of a root induces a natural orientation of the graph from the leaves to the root. Any vertex has incoming edges and only one outgoing edge. The arity of a vertex is the number of incoming edges. A tree with no vertices of arity one is called reduced. A planar rooted tree induces a structure of poset on the vertices, where $x < y$ if and only if there is an oriented path in the tree from $y$ to $x$. Let $x$ be a vertex of a planar rooted tree $T$. The full subtree $T^{(x)}$ of $T$ at $x$ is the subtree of $T$ containing all the vertices $y > x$ and all their adjacent edges. The root leg of $T^{(x)}$ is the half edge with adjacent vertex
$x$ induced by the unique outgoing edge of $x$. One represents a planar rooted tree like this:

![Planar rooted tree diagram]

In the abstract case (non-planar trees) every edge is an internal edge. The root vertex will be a distinguished vertex. The choice of a root induces a natural orientation of the graph towards the root. Any vertex has incoming edges and at most one outgoing edge. The other extremity of an incoming (outgoing) edge of the vertex $v$ is called an incoming (outgoing) vertex of the vertex $v$. The root vertex has no outgoing vertex. A rooted tree induces a structure of poset on the vertices, where $x < y$ if and only if there is an oriented path in the tree from $y$ to $x$. A leave is a maximal vertex for this order. The root is the only minimal vertex for this order.

Let $x$ be a vertex of a rooted tree $T$. The full subtree $T(x)$ of $T$ derived from the vertex $x$ is the subtree of $T$ containing all the vertices $y > x$. The root of $T(x)$ is $x$. One represents a rooted tree like this:

![Rooted tree diagram]

**Remark 1.3.** Reduced planar tree of operations: a convenient way to uniquely represent composition of operations in a non-symmetric operad $\mathcal{P}$ is to use a planar rooted tree as in Definition 1.2. An element $a \in \mathcal{P}(n)$ is represented by a planar rooted tree with a single vertex labelled by $a$ with $n$ incoming legs and a single outgoing leg:

![Reduced planar tree diagram]

The $n$ leaves are counted from left to right as $1, 2, \ldots, n$. Now if we have $a \in \mathcal{P}(n)$, $b \in \mathcal{P}(m)$ and $1 \leq i \leq n$ we represent the composition $a \circ_i b$ by the planar tree:

![Composition tree diagram]

The resulting tree has $n + m - 1$ leaves (counted from left to right) and represents an element of $\mathcal{P}(n + m - 1)$. The two first relations in Definition 1.1 corresponds to
the following two trees: for \( a \in P(n) \), \( b \in P(m) \) and \( c \in P(\ell) \) we can have

Each relation is obtained by writing down the two ways of interpreting the tree as a composition of operations. In general a planar tree \( T(a_1, a_2, \ldots, a_k) \) with \( k \) vertices labelled by elements \( a_i \in P(n_i) \) where \( n_i \) is the number of incoming edges at the \( i \)th vertex, corresponds to a unique composition of operations in \( P \) independent of any relations.

The two last relations in Definition 1.1 say that one can consider reduced trees (no vertices of arity 1) for reduced operads to represent non-trivial composition maps.

Any full subtree of \( T(a_1, a_2, \ldots, a_k) \) is completely determined by the position of its leaves; they form an interval \([p, q]\) where \( 1 \leq p \leq q \leq n_1 + n_2 + \cdots n_k - k + 1 \). A tree in position \([p, q]\) will mean the full subtree determined by the position \([p, q]\) of its leaves. If a full subtree in position \([p, q]\) has a single vertex labelled by \( a \in P(n) \), we identify this tree with the element \( a \in P(n) \). It is clear that \( n = q - p + 1 \).

Two trees of operations \( T(a_1, a_2, \ldots, a_k) \) and \( Y(b_1, b_2, \ldots, b_s) \) are distinct if and only if \( T \neq Y \) or there exists \( i \) such that \( a_i \neq b_i \).

**Definition 1.4.** Let \( S \) be a set. An \( S \)-labelled rooted tree is a non planar rooted tree as in Definition 1.2 whose vertices are in bijection with \( S \). If \( S = [n] \), then we talk about \( n \)-labelled rooted trees and denote by \( T(n) \) the set of those trees. It is acted on by the symmetric group by permuting the labels.

The set \( T(3) \) has for elements:

(1.1)

In general \( T(n) \) has \( n^{n-1} \) elements (see [1] for more details).

We denote by \( kT(n) \) the \( k \)-vector space spanned by \( T(n) \).

**Theorem 1.5.** [2, theorem 1.9] The collection \((kT(n))_{n \geq 1}\) forms a symmetric operad, the operad pre-Lie denoted by \( PL \). Algebras over this operad are pre-Lie algebras, that is, vector spaces \( L \) together with a product \( \ast \) satisfying the relation

\[
(x \ast y) \ast z - x \ast (y \ast z) = (x \ast z) \ast y - x \ast (z \ast y), \quad \forall x, y, z \in L.
\]

We recall the operad structure of \( PL \) as explained in [2]. A rooted tree is naturally oriented from the leaves to the root. The set \( \text{In}(T, i) \) of incoming vertices of a vertex \( i \) is the set of all vertices \( j \) such that \((j, i)\) is an edge oriented from \( j \) to \( i \). There is also at most one outgoing vertex of a vertex \( i \), i.e. a vertex \( r \) such that \((i, r)\) is an
oriented edge from $i$ to $r$, depending whether $i$ is the root of $T$ or not. For $T \in \mathcal{T}(n)$ and $S \in \mathcal{T}(m)$, we define

$$T \circ_i S = \sum_{f : \text{In}(T, i) \rightarrow [m]} T \circ_i^f S,$$

where $T \circ_i^f S$ is the rooted tree obtained by substituting the tree $S$ for the vertex $i$ in $T$. The outgoing vertex of $i$, if it exists, becomes the outgoing vertex of the root of $S$, whereas the incoming vertices of $i$ are grafted on the vertices of $S$ according to the map $f$. The root of $T \circ_i^f S$ is the root of $T$ if $i$ is not the root of $T$, and it is the root of $S$ if $i$ is the root of $T$. There is also a relabelling of the vertices of $T$ and $S$ in $T \circ_i^f S$: we add $i - 1$ to the labels of $S$ and $m - 1$ to the ones of $T$ which are greater than $i$. Here is an example:

$$(1.2) \quad \begin{array}{c}
\begin{array}{c}
1 \\
\circ_2 \\
3
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
1 \\
\circ_2 \\
4
\end{array} + \begin{array}{c}
\begin{array}{c}
1 \\
2
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
2 \\
3
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
2 \\
3
\end{array}
\end{array}
\end{array}$$

2. A gradation on labelled rooted trees

We introduce a gradation on labelled rooted trees. We prove that in the expansion of the composition of two rooted trees in the operad pre-Lie there is a unique rooted tree of maximal degree and a unique tree of minimal degree, yielding new non-symmetric operad structures on labelled rooted trees.

**Definition 2.1.** Let $T$ be an $n$-labelled rooted tree. Let $\{a, b\}$ denote a pair of two adjacent vertices labelled by $a$ and $b$. The degree of $\{a, b\}$ is $|a - b|$. The degree of $T$ denoted by $\deg(T)$ is the sum of the degrees of its pairs of adjacent vertices. For instance

$$\deg(\begin{array}{c}
1 \\
\circ_2 \\
3
\end{array}) = 2, \quad \deg(\begin{array}{c}
1 \\
\circ_3 \\
2
\end{array}) = 4, \quad \deg(\begin{array}{c}
2 \\
\circ_1 \\
3
\end{array}) = 5, \quad \deg(\begin{array}{c}
2 \\
\circ_3 \\
3
\end{array}) = 3$$

**Proposition 2.2.** In the expansion of $T \circ_i S$ in the operad pre-Lie, there is a unique tree of minimal degree and a unique tree of maximal degree.

For instance, in the equation (1.2) the rooted tree of minimal degree 3 is $\begin{array}{c}
\begin{array}{c}
3 \\
\circ_3 \\
2
\end{array}
\end{array}$ and the one of maximal degree 5 is $\begin{array}{c}
\begin{array}{c}
2 \\
\circ_3 \\
3
\end{array}
\end{array}$. The other ones are of degree 4.

**Proof—** Any tree in the expansion of $T \circ_i S$ writes $U_f := T \circ_i^f S$ for some $f : \text{In}(T, i) \rightarrow [m]$. To compute the degree of $U_f$, we compute the degree of a pair of two adjacent vertices $\{a, b\}$ in $U_f$. There are 4 cases to consider: a) the pair was previously in $S$ or b) it was previously in $T$ and each vertex was different from $i$, or c) it was in $T$ of the form $\{i, j\}$ for $j \in \text{In}(T, i)$ or d) if $i$ is not the root of $T$ it was of the form $\{i, k\}$ where $k$ is the outgoing vertex of $i$. 

\[\]
In case a) the degree of the pair in \( U_f \) is the same as it was in \( S \).

In case b), let \( \{a', b'\} \) be the corresponding pair in \( T \) before relabelling. The degree \( d \) of the pair \( \{a, b\} \) in \( U_f \) is the same as the degree \( d' \) of \( \{a', b'\} \) except if \( a' < i < b' \) or \( b' < i < a' \), where \( d = d' + m - 1 \). Let \( \text{gap}(T, i) \) be the number of adjacent pairs of vertices in \( T \) satisfying the latter condition.

In case c), let \( \{i, j\} \) be the pair in \( T \) which gives the pair \( \{a, b\} \) in \( U_f \). Let \( d' \) be the degree of \( \{i, j\} \). If \( j < i \) then \( \{a, b\} = \{f(j) + i - 1, j\} \). Its degree \( d \) is minimal and equals \( d' \) if \( f(j) = 1 \). It is maximal and equals \( d' + m - 1 \) if \( f(j) = m \). If \( j > i \) then \( \{a, b\} = \{f(j) + i - 1, j + m - 1\} \). Its degree \( d \) is minimal and equals \( d' \) if \( f(j) = m \). It is maximal and equals \( d' + m - 1 \) if \( f(j) = 1 \).

In case d), let \( d' \) be the degree of \( \{i, k\} \). If \( k < i \) then \( \{a, b\} = \{s + i - 1, k\} \) where \( s \) is the label of the root of \( S \). It has degree \( d' + s - 1 \). If \( k > i \), then \( \{a, b\} = \{s + i - 1, k + m - 1\} \) and has degree \( (m - s) + d' \). Let \( \epsilon(T, i, s) \) be \( 0, s - 1, m - s \) according to the different situations, 0 corresponding to the one where \( i \) is the root of \( T \).

As a conclusion

\[
(2.1) \quad \text{deg}(T) + \text{deg}(S) + \text{gap}(T, i)(m - 1) + \epsilon(T, i, s) \leq \text{deg}(U_f) \leq \text{deg}(T) + \text{deg}(S) + \text{gap}(T, i)(m - 1) + \epsilon(T, i, s) + |\text{In}(T, i)|(m - 1).
\]

There is a unique \( f_{\text{Min}} \) such that \( \text{deg}(U_{f_{\text{Min}}}) \) is minimal and there is a unique \( f_{\text{Max}} \) such that \( \text{deg}(U_{f_{\text{Max}}}) \) is maximal:

\[
(2.2) \quad f_{\text{Min}}(k) = \begin{cases} 1 & \text{if } k < i, \\ m & \text{if } k > i, \end{cases}
\]

\[
(2.3) \quad f_{\text{Max}}(k) = \begin{cases} m & \text{if } k < i, \\ 1 & \text{if } k > i, \end{cases}
\]

which ends the proof. \( \square \)

**Theorem 2.3.** There are two different non-symmetric operad structures on the collection \( (kT(n))_{n \geq 1} \) given by the composition maps \( T \circ_{\text{Min}} S \) on the one hand and \( T \circ_{\text{Max}} S \) on the other hand where \( f_{\text{Min}} \) and \( f_{\text{Max}} \) were defined in equations \((2.2)\) and \((2.3)\).

**Proof-** A rooted tree \( T \) is naturally oriented from its leaves to its root. Any edge is oriented and we denote by \( (a, b) \) an edge oriented from the vertex \( a \) to the vertex \( b \). Let \( E_T \) be the set of the oriented edges of the tree \( T \). For an integer \( a \neq i \) we denote by \( \tilde{a}^m_i \) the integer \( a \) if \( a < i \) or \( a + m - 1 \) if \( a > i \). Given a map \( f : \text{In}(T, i) \rightarrow [m] \), the set \( E_{T_S'}S \) has different type of elements:

- \( (a + i - 1, b + i - 1) \) for \( (a, b) \in E_S \);
- \( (\tilde{a}^m_i, \tilde{b}^m_i) \) for \( (a, b) \in E_T \) and \( a, b \neq i \);
- \( (\tilde{a}^m_i, f(a) + i - 1) \) for \( (a, i) \in E_T \);
- \( (i + s - 1, \tilde{b}^m_i) \) for \( (i, b) \in E_T \).
Let \( T \in \mathcal{T}(n), S \in \mathcal{T}(m) \) and \( U \in \mathcal{T}(p) \). In order to avoid confusion, we denote by \( f^{i,p}_{\text{Max}} \) the map sending \( k < i \) to \( p \) and \( l > i \) to 1. We would like to compare the trees

\[
V_1 = (T \circ f^{i,m}_{\text{Max}} S) \circ f^{i+1,p}_{\text{Max}} U \quad \text{and} \quad V_2 = T \circ f^{i,m+p-1}_{\text{Max}} (S \circ f^{i,p}_{\text{Max}} U):
\]

- In \( V_1 \) and \( V_2 \), any \((a, b) \in E_U\) converts to \((a + j + i - 2, b + j + i - 2)\).
- In \( V_1 \) and \( V_2 \), any \((a, b) \in E_S\) converts to \((\tilde{a}^p_j + i - 1, \tilde{b}^p_j + i - 1)\) if \( a, b \neq j \), or converts to \((\tilde{a}^p_j + i - 1, f^{i,p}_{\text{Max}}(a) + i + j - 2)\) if \( b = j \) or converts to \((j + i - 1 + u - 1, \tilde{b}^p_j + i - 1)\) if \( a = j \).
- In \( V_1 \) and \( V_2 \), any \((a, b) \in E_T\) with \( b \neq i \) converts to \((\tilde{a}^{p+m-1}_i, \tilde{b}^{p+m-1}_i)\).
- In \( V_1 \) and \( V_2 \), any \((a, i) \in E_T\) converts to \((\tilde{a}^{p+m-1}_i, f^{i,m+p-1}_{\text{Max}}(a) + i - 1)\).
- In \( V_1 \) and \( V_2 \), any \((i, b) \in E_T\) converts to \((i - 1 + \text{root}(S \circ_j U), \tilde{b}^{m+p-1}_i)\), where root\((S \circ_j U)\) is the root of \( S \circ_j U \). More precisely

\[
\text{root}(S \circ_j U) = \begin{cases} 
  s & \text{if } s < j \\
  u + j - 1 & \text{if } s = j \\
  s + p - 1 & \text{if } s > j.
\end{cases}
\]

The proof of

\[
(T \circ f^{i,m}_{\text{Max}} S) \circ f^{i,p}_{\text{Max}} U = (T \circ f^{i,p}_{\text{Max}} U) \circ f^{i+1,m}_{\text{Max}} S, \quad \text{for } j < i
\]

is similar and left to the reader. So is the proof with \( f_{\text{Min}} \) instead of \( f_{\text{Max}} \). \( \square \)

The two operads on labelled rooted trees defined by the theorem are denoted by \( \mathcal{T}_{\text{Max}} \) and \( \mathcal{T}_{\text{Min}} \). Note that they are linearization of operads in the category of sets. Actually the composition maps are defined at the level of the sets \( \mathcal{T}(n) \) and not only at the level of the vector spaces \( k\mathcal{T}(n) \). There is another operad built on rooted trees which has this property: the operad \( \mathcal{NAP} \) encoding non-associative permutative algebras in \( \mathcal{B} \), in which \( f_{\mathcal{NAP}} \) is the constant map with value the root of \( S \). This operad has the advantage of being a symmetric operad.

3. The Operad Pre-Lie is Free as a Non-Symmetric Operad

We show that \( \mathcal{T}_{\text{Max}} \) is a free non-symmetric operad. Using Proposition 2.2, we conclude that the operad Pre-Lie is free as a non-symmetric operad. To this end we need to introduce some notation on rooted trees.

**Definition 3.1.** Given two ordered sets \( S \) and \( T \), an order-preserving bijection \( \phi : S \to T \) induces a natural bijection between the set of \( S \)-labelled rooted trees and the set of \( T \)-labelled rooted trees also denoted by \( \phi \). A \( T \)-labelled rooted tree \( X \) is isomorphic to an \( S \)-labelled rooted tree \( Y \) if \( X = \phi(Y) \).

Given a rooted tree \( T \in \mathcal{T}(n) \) and a subset \( K \subseteq [n] \), we denote by \( T|_K \) the graph obtained from \( T \) by keeping only the vertices of \( T \) that are labelled by elements of \( K \) and only the edges of \( T \) that have two vertices labelled in \( K \). Remark that each connected component of \( T|_K \) is a rooted tree itself where the root is given by the unique vertex closest to the root of \( T \) in the component. Also, for \( c \in [n] \) we denote

\[ \text{root}(T|_c) \]
by \(T^{(c)}\) the full subtree of \(T\) derived from the vertex labelled by \(c\) (see Definition 1.2). For example if \(K = \{2, 3, 4, 5, 6\} \subset [7]\) and

\[
T = \begin{array}{c}
2 \\
7 \\
\hline \\
3 \\
\end{array}, \quad \text{we have } T|_K = \begin{array}{c}
2 \\
4 \\
\hline \\
3 \\
\end{array} \quad \text{and } T^{(1)} = \begin{array}{c}
1 \\
\end{array}.
\]

For \(1 \leq a < b \leq n\), \(T \in \mathcal{T}_{\text{Max}}(n - b + a)\) and \(S \in \mathcal{T}_{\text{Max}}(b - a + 1)\), let \(X = T \circ_a S\). Consider the interval \([a, b] = \{a, a + 1, \ldots, b\}\), clearly \(X|_{[a, b]}\) is isomorphic to \(S\) under the unique order-preserving bijection \([1, b - a + 1] \rightarrow [a, b]\). Let \(a \leq c \leq b\) be the label of the root of \(X|_{[a, b]}\). Remark that \(X^{(c)}\) is obtained from \(X|_{[a, b]}\) by grafting subtrees of \(X\) at the vertices \(a\) and \(b\) only. We can then characterize trees \(X\) that are obtained from a non-trivial composition \(T \circ_a S\) as follows:

**Definition 3.2.** A tree \(X \in \mathcal{T}_{\text{Max}}(n)\) is called decomposable if there exists \(1 \leq a < b \leq n\) with \((a, b) \neq (1, n)\) such that

(i) \(X|_{[a, b]}\) is a rooted tree. Let \(c\) be the label of its root. One has \(a \leq c \leq b\).

(ii) One has \(X^{(c)}|_{[a, b]} = X|_{[a, b]}\) and \(X^{(c)}\) is obtained from \(X|_{[a, b]}\) by grafting subtrees of \(X\) at the vertices \(a\) and \(b\) only.

(iii) All subtrees in \(X^{(c)} - X|_{[a, b]}\) attached at \(a\) have their root labelled in \([b + 1, n]\).

(iv) All subtrees in \(X^{(c)} - X|_{[a, b]}\) attached at \(b\) have their root labelled in \([1, a - 1]\).

It is clear from the discussion above and the definition of the operad \(\mathcal{T}_{\text{Max}}\) that \(X\) is decomposable if and only if it is the result of a non-trivial composition. Consequently, we say that \(X\) is indecomposable if it is not decomposable. That is there is no \(1 \leq a < b \leq n\) such that (i)–(iv) are satisfied. For example let

\[
X = \begin{array}{c}
7 \\
5 \\
\hline \\
4 \\
\end{array}, \quad X|_{[3, 5]} = \begin{array}{c}
3 \\
\hline \\
5 \\
\end{array} \quad \text{and } X^{(5)} = \begin{array}{c}
2 \\
\hline \\
4 \\
\end{array}.
\]

This tree \(X\) is decomposable since for \(1 \leq 3 < 5 \leq 8\) we have that \(X|_{[3, 5]}\) is a single tree and the subtrees of \(X^{(5)} - X|_{[3, 5]}\) are attached at 3 and 5 only. Moreover, the subtree attached at 3 has root labelled by 7 \(\in [6, 8]\) and the subtrees attached at 5 have roots labelled by 1, 2 \(\in [1, 2]\). Indeed, in \(\mathcal{T}_{\text{Max}}\) we have

\[
X = \begin{array}{c}
1 \\
\hline \\
4 \\
\end{array} \circ_3 \begin{array}{c}
2 \\
\hline \\
3 \\
\end{array}.
\]

The reader may check that the following are all the indecomposable trees of \(\mathcal{T}_{\text{Max}}\) up to arity 3:

\[
\begin{array}{c}
\circ \\
\end{array}^2, \quad \begin{array}{c}
\circ \\
\end{array}^1 \quad \text{and } \begin{array}{c}
\circ \\
\end{array}^0.
\]

**Theorem 3.3.** The non-symmetric operad \(\mathcal{T}_{\text{Max}}\) is a free non-symmetric operad.
Proof. If $T_{\text{Max}}$ is not free, then for some $n$ there is a tree $X \in T_{\text{Max}}(n)$ with two distinct constructions from indecomposables. In Remark 3.3 a non-trivial composition of operations is completely determined by a unique reduced planar rooted tree. We then have that $X = T(T_1, T_2, \ldots, T_r) = \mathcal{Y}(S_1, S_2, \ldots, S_k)$ where $T_1, \ldots, T_r, S_1, \ldots, S_k$ are indecomposables and $T(T_1, T_2, \ldots, T_r)$ and $\mathcal{Y}(S_1, S_2, \ldots, S_k)$ are two distinct trees of operations in $T_{\text{Max}}$ with $r, k > 1$.

The tree $X = T(T_1, T_2, \ldots, T_r)$ is decomposable (by assumption $r \geq 2$). We can find $1 \leq a < b \leq n$, such that $X_{[a,b]}$ is isomorphic to a single $T_i$ in position $[a,b]$ in $T(T_1, T_2, \ldots, T_r)$. Moreover $X_{[a,b]}$ satisfies (i)–(iv) of Definition 3.2.

If $X_{[a,b]}$ is also isomorphic to a tree $S_j$ in position $[a,b]$, then we replace $X$ by the smaller tree in $T_{\text{Max}}(n - b + a)$ that we obtain by removing $T_i$ in $T(T_1, T_2, \ldots, T_r)$ and removing $S_j$ in $\mathcal{Y}(S_1, S_2, \ldots, S_k)$. Clearly, this new smaller $X$ has two distinct constructions from indecomposables. We can thus assume that $X_{[a,b]}$ is not isomorphic to a single $S_j$ in position $[a,b]$ in $\mathcal{Y}(S_1, S_2, \ldots, S_k)$.

We now study how $X_{[a,b]}$ overlaps in the position $[a,b]$ of $\mathcal{Y}(S_1, S_2, \ldots, S_k)$. Remark first that since all $S_j$ are indecomposables, the interval $[a,b]$ cannot be part of a single $S_j$ of $\mathcal{Y}(S_1, S_2, \ldots, S_k)$. Indeed, that would imply that $S_j$ would contain a subtree satisfying Definition 3.2 which would be a contradiction.

We may assume that $a > 1$. To see this, assume that the only sub-interval $[a,b] \subset [1,n]$ such that $X_{[a,b]}$ is isomorphic to a single $T_i$ in position $[a,b]$ in $T(T_1, T_2, \ldots, T_r)$ is such that $a = 1$. Assume moreover that the only sub-interval $[a',b'] \subset [1,n]$ such that $X_{[a',b']} \in T_i$ in position $[a',b']$ in $\mathcal{Y}(S_1, S_2, \ldots, S_k)$ is such that $a' = 1$. Since $S_j$ is indecomposable, we must have $b > b'$. Similarly, since $T_i$ is indecomposable, we must have $b < b'$. This implies that $b = b'$ and $T_i = S_j$. This possibility was excluded above. So we must have $a > 1$ or $a' > 1$. In the case where $a = 1$ and $a' > 1$ we could just interchange the role of $T(T_1, T_2, \ldots, T_r)$ and $\mathcal{Y}(S_1, S_2, \ldots, S_k)$ and assume that we have $a > 1$.

Now, since $T_i$ is indecomposable, there is no subinterval $[c,d] \subset [a,b]$ such that $X_{[c,d]}$ is isomorphic to a full subtree of operations $\mathcal{Y}(S_{j_1}, S_{j_2}, \ldots, S_{j_k})$. Assume we can find $c < a \leq d < b$ such that $X_{[c,d]} \cong \mathcal{Y}(S_{j_1}, S_{j_2}, \ldots, S_{j_k})$ satisfies the Definition 3.2.

The graph $X_{[a,b]}$ is contained in the trees $X_{[a,b]}$ and $X_{[c,d]}$. Let $e$ be the label of the root $X_{[a,b]}$ and $f$ be the label of the root $X_{[c,d]}$. The two full subtrees $X^{(e)}$ and $X^{(f)}$ both contain $X_{[a,b]}$. This implies that either $X^{(f)}$ is fully contained in $X^{(e)}$, or $X^{(e)}$ is fully contained in $X^{(f)}$.

Let us assume that $X^{(f)}$ is fully contained in $X^{(e)}$, that means $X_{[a,b]}$ and $X_{[c,d]}$ are both subtrees of $X^{(e)}$. From Definition 3.2 we know that $X^{(e)}$ is obtained from $X_{[a,b]}$ by graphing subtrees of $X$ at the vertices $a$ and $b$ only. The vertex $c$ is in $X^{(e)}$ but not in $X_{[a,b]}$. It is part of a subtree attached to $a$ or $b$. Since $c$ is part of a subtree with root $f$ one has $f \not\in [a,b]$. The vertex $f$ is a (can not be $b$ since $f \leq d$) or is attached to $a$ or $b$. If $f$ is attached to $b$ then there is a path $c \to f \to b$. The tree $X_{[c,d]}$ has its root labelled by $f$ so there is a path $d \to f$. The tree $X_{[a,b]}$ contains

\[c < a \leq d < b \]
the vertices $b$ and $d$ and any path from $d$ to $b$ so there is a path $d \to f \to b$ in $X_{[a,b]}$. Hence $f = a$ for $f \not\in [a,b]$. As a conclusion $c$ is part of a subtree attached to $a$. By (iii) of Definition 3.2 applied to the tree $X_{[a,b]}$, the subtree must have a root $r \in [b+1,n]$. This is a contradiction, the root $r$ is part of any path joining $a$ and $c$ and $r \not\in [c,d]$, hence not in $X_{[c,d]}$. The case where $X^{(r)}$ is fully contained in $X^{(f)}$ is argued similarly, using condition (iv) of Definition 3.2, and leads to a contradiction as well.

The same argument holds in case we can find $a < c \leq b < d$.

The only case remaining is that the interval $[a,b]$ associated to any proper full subtree of $Y(S_1, \ldots, S_k)$ satisfies $[a,b] \cap [p,q] = \emptyset$ or $[a,b] \subset [p,q]$. There is at least one interval satisfying $[a,b] \subset [p,q]$ (take the full tree $Y(S_1, \ldots, S_k)$ and $[p,q] = [1,n]$). Let $[p,q]$ be the smallest interval such that $[a,b] \subset [p,q]$ and let $Y(S_1, \ldots, S_k)^{(S_j)} = Y(S_1, \ldots, S_k)$ be the full subtree it determines. Its root is labelled by $S_j$. The interval $[u,v]$ associated to any proper full subtree of $Y'(S_1, \ldots, S_u)$ satisfies $[a,b] \cap [u,v] = \emptyset$. Consequently $X_{[a,b]}$ is isomorphic to $S_j^{(a,b)}$ for some interval $[\alpha, \beta]$ isomorphic to $[a,b]$. This is impossible since $X$ satisfies the conditions of Definition 3.2 and $S_j$ is indecomposable.

We must conclude that $\mathcal{T}_{\text{Max}}$ is free. 

\begin{proof}

\begin{remark}

The non-symmetric operads $\mathcal{T}_{\text{Min}}$ and NAP are not free. Indeed, in the operad $\mathcal{T}_{\text{Min}}$ one has the following relation:

\[
\begin{array}{ccc}
2 & 1 & 2 \\
1 & 1 & 1 \\
\end{array} = \begin{array}{ccc}
2 & 1 & 2 \\
1 & 1 & 1 \\
\end{array} = \begin{array}{ccc}
2 & 1 & 1 \\
1 & 1 & 2 \\
\end{array} = \begin{array}{ccc}
2 & 1 & 1 \\
1 & 1 & 2 \\
\end{array} = 3
\end{array}
\]

And in the operad NAP one has the following relation

\[
\begin{array}{ccc}
2 & 1 & 2 \\
1 & 1 & 1 \\
\end{array} = \begin{array}{ccc}
2 & 1 & 2 \\
1 & 1 & 1 \\
\end{array} = \begin{array}{ccc}
2 & 1 & 1 \\
1 & 1 & 2 \\
\end{array} = \begin{array}{ccc}
2 & 1 & 1 \\
1 & 1 & 2 \\
\end{array} = 3
\end{array}
\]

\end{remark}

\begin{remark}

Let $k\mathcal{T}_{\text{Max}}^0(n)$ denote the $k$-vector space spanned by the indecomposables of $\mathcal{T}_{\text{Max}}(n)$ $(n > 1)$ and let $\beta_n$ be its dimension. Let $\alpha(x) = \sum_{n \geq 1} \alpha_n x^n$ be the Hilbert series associated to the free non-symmetric operad generated by the vector spaces $k\mathcal{T}_{\text{Max}}^0(n)$. It is well known (see e.g. [3]) that one has the identity

\[\beta(\alpha(x)) + x = \alpha(x),\]

where $\beta(x) = \sum_{n \geq 2} \beta_n x^n$. Theorem 3.3 implies that $\alpha_n = n^{n-1}$. As a consequence, we get that the Hilbert series for indecomposable of $\mathcal{T}_{\text{Max}}$ is

\[
\mathcal{H}_{\mathcal{T}_{\text{Max}}^0}(x) = \sum_{n \geq 2} \dim (k\mathcal{T}_{\text{Max}}^0(n)) x^n = 2x^2 + x^3 + 14x^4 + 146x^5 +
\]

\[+ 1994x^6 + 32853x^7 + 630320x^8 + 13759430x^9 + \cdots.
\]

\begin{corollary}

The non-symmetric operad pre-Lie is a free non-symmetric operad.
\end{corollary}

\end{proof}
Proof. Let \( \mathcal{F} \) be the free non-symmetric operad on indecomposable trees. By the universal property of \( \mathcal{F} \), there is a unique morphism of operads

\[
\phi: \mathcal{F} \to \mathcal{PL}
\]

extending the inclusion of indecomposable trees in \( \mathcal{PL} \). We prove that this map is surjective by induction on the degree of a tree. Trees of degree 1 are indecomposables (see Definition 3.2). Let \( t \in \mathcal{PL}(n) \) be a tree of degree \( k \geq n - 1 \). If \( t \) is indecomposable then \( t = \phi(t) \). If \( t \) is decomposable there are trees \( u \in \mathcal{PL}(r), v \in \mathcal{PL}(s) \), with \( r, s < n \) such that \( t = u \circ f_{\text{Max}}^i v \) in \( T_{\text{Max}} \). By Proposition 2.2 one has in \( \mathcal{PL} \)

\[
u \circ_i v = t + \sum_j t_j
\]

where \( t_j \in \mathcal{PL}(n) \) has degree \( k_j < k \). From equation (2.1) we deduce that the degrees of \( u \) and \( v \) are also lower than \( k \). By induction, the trees \( u, v \) and \( t' \)'s are in the image of \( \phi \), so is \( t \). Thus, the operad morphism \( \phi \) is surjective. Theorem 3.3 implies that the vector spaces \( \mathcal{F}(n) \) and \( \mathcal{PL}(n) \) have the same dimension, thus the operad morphism \( \phi \) is an isomorphism. \( \square \)

Remark 3.7. The Hilbert Series for the free non-symmetric operad on indecomposables and the operad \( \mathcal{PL} \) are the same as in Remark 3.3.

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