Quantum States Allowing Minimum Uncertainty Product of $\phi$ and $L_z$

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Abstract. We provide necessary and sufficient conditions for states to have an arbitrarily small uncertainty product of the azimuthal angle $\phi$ and its canonical moment $L_z$. We illustrate our results with analytical examples.

1. Introduction

The Newtonian determinism states that the present state of the universe determines its future precisely. At the beginning of the past century the advent of quantum mechanics exposed the determinism to great delusion. It turned out that in the quantum world the uncertainty prevails. Heisenberg, with his uncertainty principle, was the first to recognize the antagonism between classical and quantum mechanics [1]. He notice that for the position and its conjugate momentum the more concentrated the distribution of the position, the more uniform is the distribution of the momentum and vice-versa. The Heisenberg relation states that it is impossible to predict, with arbitrary certainty, the outcomes of measurements of two canonically conjugate observables.

The uncertainty relation was subsequently generalized by Robertson [2]. The variance of an observable $A$ for a given state $\psi$ is

\[ \sigma_A^2 = \langle A\psi, A\psi \rangle - |\langle \psi, A\psi \rangle|^2, \]

and the Heisenberg-Robertson (HR) uncertainty relation, in its most well known form, reads:

\[ \sigma_A \sigma_B \geq \frac{\hbar}{2} |\langle \psi, i[A,B] \psi \rangle|, \tag{1} \]

where $[A,B]$ is the commutator of observables $A$ and $B$.

The uncertainty principle has been one of the most intricate points in quantum mechanics [3, 4]. Besides its philosophical meaning it plays a major role in experimental
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physics of atomic scale as, for example, in the Bose-Einstein condensation [5], and electrons jump at random from one energy state which they could never reach except by fluctuations in their energy. Another manifestation of the uncertainty principle in the energy spectrum can be seen in the spectral linewidth that characterizes the width of a spectral line [6, 7].

An old problem concerning the uncertainty principle and whether the uncertainty relation (1) expresses it adequately appears if the quantum system is described in terms of angle variables [8]. When the Cartesian coordinates $(x, y, z)$ are changed to spherical ones $(r, \theta, \phi)$, equation (1) no longer provides a lower bound for the product of uncertainty of the azimuthal angle operator $\phi$ and its canonical conjugate momentum $L_z$ [8, 9]. The trouble arises since fluctuations on $\phi$ bigger than $2\pi$ do not have physical meaning. Consequently, if $\psi$ is sufficiently localized in the Fourier space, $\sigma_{L_z}$ is small $\sigma_{\phi}$ remains bounded and one may have uncertainty product $\sigma_{\phi}\sigma_{L_z}$ smaller than any given positive number. Recently this problem has attracted a great deal of attention [10, 11, 12, 13, 14].

The HR uncertainty relation for the angle and position has been criticized on several grounds and other mathematical formulations of the uncertainty principle have been proposed (see [10, 12, 15] for a contextualization). Examples of such attempts include the entropic relations relying on entropies instead of on the standard deviations of the observables [16, 17, 18, 19]; by introducing a unitary operator for phase $\phi$ [20, 21]; evaluating the commutator for functions that just belong to the domains of the angle and angular momentum operators [22, 23]; exchanging the angle with an absolutely continuous periodic function [24]; and expressing the lower bound as state dependent [15, 25].

Despite of these alternatives, expressing the uncertainty principle for angular operator by lower–bounding the product of the standard deviations is widely used. In particular, experimental confirmation of the uncertainty principle for the angular momentum and position has been carried out for intelligent states (states that saturates the uncertainty relation for $\phi$ and $L_z$ observables) [13]. Also recently, the relation between these intelligent states and the constrained minimum uncertainty product for the angular operator has shown to be important [14].

Motivated by the state–dependence of standard measures of uncertainty and the fact that some state features may be prepared or detected experimentally we shall investigate the class of states that allows for an arbitrarily small uncertainty product. For this, we introduce an one–parameter family of states $\{f_\alpha(\phi), \alpha > 0\}$, defined by the Fourier coefficients of $f_\alpha(\phi)/A_\alpha$ [27]

$$C_n(\alpha) = \frac{1}{2\pi A_\alpha} \int_{-\pi}^{\pi} e^{in\phi} f_\alpha(\phi) d\phi$$

, $n \in \mathbb{Z}$ (2)

with $A_\alpha$ fixed by the normalization $\int_{-\pi}^{\pi} |f_\alpha(\phi)|^2 d\phi = 1$ [see Eq. (9)].

In this paper, we provide necessary and sufficient conditions on these families that allow for an arbitrarily small uncertainty product. We demonstrate that arbitrarily small uncertainty product is attained if, and only if, a single nonvanishing Fourier coefficient
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$C_k(\alpha)$ decays, as a function of $\alpha$, slower than the others $C_n(\alpha)$ with $n \neq k$. Furthermore, we provide explicit examples of our result.

This paper is organized as follows: In Section 2, we discuss some problems associated with the HR relation. Our hypotheses on the states are given in Section 3. Our main result concerning the states which allow for an arbitrarily small uncertainty product is given in Section 4. In Section 5, we deduce the equations for $\sigma_\phi$ and $\sigma_{L_z}$. We provide examples of our result in Section 6 for the exponential decay and in Section 7 for the polynomial decay of the Fourier coefficients of the states. In Section 8, we show that replacing $\phi$ by $\sin \phi$ or $\cos \phi$ provides a good description of the HR relation. Section 9 contains a proof of our main result. Finally, in Section 10, we give our conclusions.

2. Pitfalls and Apparent Paradox

Let us start by introducing the operators $\phi$ and its canonical conjugate $L_z$. The phase is introduced as the angular displacement of the vector position:

$$\phi = \tan^{-1}\left(\frac{y}{x}\right).$$

The angle operator is usually defined as a multiplication operator either by the variable $\phi$ or by $25$

$$Y(\phi) = (\phi - \pi) \mod 2\pi + \pi.$$

When $\phi$ is defined on the lift, that is, without the mod $2\pi$, it is continuous but no longer periodic. Since $\phi$ and $\phi + 2\pi$ correspond to the same physical situation, the mod $2\pi$ operation in the range $[-\pi, \pi]$ is preferred. Here, we adopt $\phi$ as a multiplication operator by $\phi$ acting on the space of $2\pi$-periodic functions which is square integrable in the interval $[-\pi, \pi]$. For values in this range $\phi$ and $Y(\phi)$ do not differ from each other.

The canonical momentum associated with $\phi$ is given by

$$L_z = -i\hbar \left( x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right) = -i\hbar \frac{\partial}{\partial \phi}. \tag{3}$$

Under the (false) assumption that the commutation relation

$$[\phi, L_z] = i\hbar \tag{4}$$

holds on the domain in which $L_z$ and $\phi$ are self-adjoint operators, the HR uncertainty relation yields

$$\sigma_\phi \sigma_{L_z} \geq \frac{\hbar}{2}. \tag{5}$$

The product of uncertainty, however, can be made smaller than $\hbar/2$ for the majority of states $15, 13, 14$.

Another apparent paradox that appears by naïve assumptions on the domain of the operators involved is as follows. Let $|lm\rangle$ denote the spherical harmonic functions. From Eq. (4), we have

$$\langle lm'| [\phi, L_z] |lm\rangle = i\hbar \langle lm'|lm\rangle, \tag{6}$$
and this leads to the (wrong) conclusion
\[ \hbar (m - m') \langle lm' \mid \phi \mid lm \rangle = i \hbar \delta_{mm'}, \]
that \( 0 = 1 \) if \( m = m' \). See Examples 5 and 6 of [23].

Since the operator \( \phi \) multiplies the wave function by a bounded real number, it is Hermitian: \( \langle \psi_1, \phi \psi_2 \rangle = \langle \phi \psi_1, \psi_2 \rangle \), and self–adjoint operator in the Hilbert space \( \mathcal{H} \) of square integrable functions in \([-\pi, \pi]\). The operator \( L_z \), on the other hand, is defined in a closed domain \( D(L_z) \) of \( \mathcal{H} \). It may be extended as a self–adjoint operator if \( D(L_z) \) is the set of \( 2\pi \)–periodic absolutely continuous functions \( AC[-\pi, \pi] \) (see Section VIII.3 of [26]). Now, the domain \( D([\phi, L_z]) \) of the commutator \( [\phi, L_z] \) is given by the functions \( \psi \in AC[-\pi, \pi] \) such that \( \psi(-\pi) = \psi(\pi) = 0 \). As the eigenfunctions \( \psi_m(\phi) = e^{i m \phi} / \sqrt{2\pi} \) of \( L_z \) do not belong to \( D([\phi, L_z]) \), the commutator cannot acts over \( |lm\rangle \) and equation (6) doesn’t make sense. The apparent contradiction of (5) rests on the same problem: the domain \( D([\phi, L_z]) \) of functions in the r.h.s. of (1) is smaller than the domain \( D(L_z) \cap D(\phi) \) of the l.h.s. of (1) (see [23] for a detailed discussion).

An attempt to fix the domain problem in the uncertainty relation (5 ) is to abandon the commutator and introduce a sesquilinear form [15, 23] defined in \( D(L_z) \cap D(\phi) \). The uncertainty relation then reads
\[ \sigma_{\phi} \sigma_{L_z} \geq \left| i \langle \phi \psi, L_z \psi \rangle - i \langle L_z \psi, \phi \psi \rangle \right| = \frac{\hbar}{2} \left| 1 - 2\pi |\psi(\pi)|^2 \right| \]
which is now state–dependent (see [23, 10, 22], for details). Note that (7) and (5) agree if \( \psi \in D([\phi, L_z]) \), since a state \( \psi \) in the domain of the commutator satisfies \( \psi(\pi) = 0 \).

3. Set Up

The ground of our result is the Fourier expansions of \( f_\alpha(\phi) \):
\[ f_\alpha(\phi) = A_\alpha \sum_{n=-\infty}^{\infty} C_n(\alpha) e^{i n \phi}, \]
where \( C_n(\alpha) \) are the Fourier coefficients (frequency amplitudes) of \( f_\alpha(\phi)/A_\alpha \), given by Eq. (2), with \( A_\alpha \) fixed by the normalization:
\[ \langle f_\alpha(\phi), f_\alpha(\phi) \rangle = \int_{-\pi}^{\pi} |f_\alpha(\phi)|^2 d\phi = 2\pi |A_\alpha|^2 \sum_{n=-\infty}^{\infty} |C_n(\alpha)|^2 = 1. \]

For notational simplicity, whenever we do not specify the sum we understand the index running from \(-\infty\) to \( \infty \). Also, whenever there is no risk of confusion, we shall omit the index \( \alpha \) of the Fourier coefficients \( C_n \) and normalization constant \( |A|^2 \).

Admissible Family: Let \( \mathcal{F} \) be an one parameter family of periodic functions \( f_\alpha \) with
(i) nontrivial variance, that is, \( \sigma_\phi^2 \geq \inf_\alpha \sigma_\phi^2 = \kappa > 0 \); and Fourier coefficients such
that: (ii) \( \{ nC_n(\alpha) \} \in \ell_2 \) uniformly in \( \alpha \), that is, for every \( \epsilon > 0 \) there is \( N = N(\epsilon) \), independent of \( \alpha \), such that \( \sum_{n=j}^{m} n^2 |C_n(\alpha)|^2 < \epsilon \) for all \( m > j > N(\epsilon) \); (iii) there is an increasing sequence \( (\alpha_k)_{k\geq 1} \) such that \( C_n(\alpha_j) < C_n(\alpha_k) \) if \( j > k \). A family \( \mathcal{F} \) is said to be admissible if it satisfies (i), (ii) and (iii).

Condition (i) avoids a state \( f_\alpha \) to be in a neighborhood of the Dirac delta function \( \delta(\phi) \). One expects \( |f_\alpha(\pi)| \) to be small for such states, so the bound given by Eq. (7) already prevents the uncertainty product to be close to 0. Condition (ii) on uniformity is of technical nature and guarantees that the limit of a sum equals to sum of the limits of a given sequence. It will be used in Eqs. (14) and (27). The last condition (iii) is made here to give a relation of order inside the family, at least in terms of subsequences, as \( \alpha \) grows [28].

**Dominance Condition:** An admissible family \( \mathcal{F} \) satisfies the dominance condition if within its one-parameter family of Fourier Coefficients \( \{ C_n(\alpha) \} \) there is only one \( C_k(\alpha) \neq 0 \) \( \forall \alpha \) such that [29]

\[
\liminf_{\alpha \to \infty} \frac{C_n(\alpha)}{C_k(\alpha)} = 0, \quad \forall n \neq k. \tag{10}
\]

### 4. Theorem on Arbitrarily Small Uncertainty Product

Here we state our main results. We start by introducing the following

**Definition 1** Let the standard deviations \( \sigma_\phi(\alpha) \) and \( \sigma_{L_z}(\alpha) \) associated with a state \( f_\alpha \in \mathcal{F} \) be given by Eq. (11) and (13), respectively. An admissible family \( \mathcal{F} \) is said to allow an arbitrarily small uncertainty product if for every \( \epsilon > 0 \) there is an \( \alpha^* \in (0, \infty) \) such that

\[
\sigma_\phi(\alpha^*) \sigma_{L_z}(\alpha^*) < \epsilon.
\]

Our main result is then stated as follows:

**Theorem 1** An admissible family \( \mathcal{F} \) allows an arbitrarily small uncertainty product if, and only if, it satisfies the dominance condition.

From this theorem it follows:

**Corollary 1** Any state \( f_\alpha(\phi) \in \mathcal{F} \) whose Fourier coefficients are sufficiently localized in the Fourier space has uncertainty product smaller than the least value predicted by the HR relation (5).

It is worthy to note that our result does not depend on the decay of the coefficients, but only on the relative decay with respect to \( C_k \) as stated in Eq. (10). We illustrate our findings for two different decays. The proof of Theorem 1 is given in Section 9.

The consistency of Theorem 1 with the uncertainty relation (7) is as follows. The state \( f_\alpha \) whose Fourier coefficients satisfy the dominance condition (10) is such that \( |f_\alpha|^2 \) may be close to the uniform distribution for some large \( \alpha \) and this leads the r.h.s.
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of (7) to be close to 0. Theorem 1 goes, however, beyond what the uncertainty relation (7) can predict. It follows, in particular, from the prove of Theorem 1 that if the state $f_{\alpha}$ has two “dominant” Fourier coefficients, in the sense of (10), the uncertainty product cannot be smaller than the value predicted by relation (7). In Section 7, we give an examples of families of states of this type in which the uncertainty product differs from the lower bound (7) for all $\alpha$ (see Fig. 2).

5. Uncertainty Relations

In this section we give a formal derivation of the general formulas for the deviations $\sigma_\phi$ and $\sigma_{L_z}$, assuming that Eq. (8) holds. The deviation on the variable $\phi$ is given by:

$$\sigma_\phi^2 = \langle \phi^2 \rangle - |\langle \phi \rangle|^2,$$

and we start with the first term in the right-hand-side (r.h.s):

$$\langle \phi^2 \rangle = \int_{-\pi}^{\pi} \phi^2 |f(\phi)|^2 d\phi = |A|^2 \sum_{m,n} C_m^* C_n \int_{-\pi}^{\pi} e^{i(m-n)\phi} \phi^2 d\phi.$$

Splitting the sum into $m \neq n$ and $m = n$, evaluating the integrals, and using Eq. (9) we have:

$$\langle \phi^2 \rangle = \frac{\pi^2}{3} + 4\pi |A|^2 \xi$$

where

$$\xi = \sum_{m \neq n} C_m^* C_n \frac{(-1)^{(n-m)}}{(n-m)^2}.$$

For the second term in r.h.s. of (11), we have

$$\langle \phi \rangle = |A|^2 \sum_{m,n} C_m^* C_n \int_{-\pi}^{\pi} e^{i(n-m)\phi} \phi d\phi$$

$$= 2\pi |A|^2 \sum_{m \neq n} C_m^* C_n \frac{1}{i} \frac{(-1)^{n-m}}{n-m}.$$

Therefore the deviation is given by:

$$\sigma_\phi^2 = \frac{\pi^2}{3} + 4\pi |A|^2 \xi$$

$$- \left| 2\pi |A|^2 \sum_{m \neq n} C_m^* C_n \frac{(-1)^{n-m}}{n-m} \right|^2.$$

Next, we compute:

$$\sigma_{L_z}^2 = \langle L_z^2 \rangle - |\langle L_z \rangle|^2.$$
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Using condition (ii), we begin with

$$\langle L_z^2 \rangle = \langle L_z f, L_z f \rangle = |A|^2 \hbar^2 \sum_{m,n} C_m^* C_n \int_{-\pi}^{\pi} e^{i(n-m) \phi} d\phi. \quad (14)$$

The terms with $m \neq n$ vanish, while the terms with $n = m$ yield:

$$\langle L_z^2 \rangle = 2\pi |A|^2 \hbar^2 \sum_n |C_n|^2 n^2.$$

For the amount $\langle L_z \rangle = \langle f, L_z f \rangle$ we have analogously

$$|\langle f, L_z f \rangle|^2 = 4\pi^2 |A|^4 \hbar^2 \left( \sum_n |C_n|^2 n^2 \right)^2.$$

Thus, the deviation in $L_z$ is given by:

$$\sigma_{L_z}^2 = 2\pi \hbar^2 |A|^2 \sum_n |C_n|^2 n^2 - 4\pi^2 \hbar^2 |A|^4 \left( \sum_n |C_n|^2 n^2 \right)^2. \quad (15)$$

6. Fourier Coefficients with Exponential Decay

We restrict our attention to the case in which the frequency amplitudes $C_n$ decay exponentially fast in $|n|:

$$C_n = e^{-\alpha |n|}.$$

This and the next example capture most of the important features we wish to emphasize. Note that, $C_n$ is a real even function of $n$: $C_n = C_{-n}$ and $C_n^* = C_n$. The sequence $\{C_n(\alpha)\}$ satisfies hypotheses (ii) and (iii) but $f_\alpha(\phi)$ approaches the Dirac delta function $\delta(\phi)$ when $\alpha$ tends to 0: for any piecewise continuous periodic function $\psi$,

$$f_\alpha * \psi(\phi) = \int_{-\pi}^{\pi} f_\alpha(\phi - \zeta) \psi(\zeta) d\zeta \to (\psi(\phi + 0) + \psi(\phi - 0))/2$$

and converges uniformly in any closed interval of continuity.

The sequence $\{C_n(\alpha)\}$ satisfies, in addition, the dominance condition Eq. (10) with $k = 0$. As we shall see, the uncertainty product can be arbitrarily small despite of the noncompliance of (i).

From the properties of $C_n$ it follows that $\langle \phi \rangle = 0$. Note that the $1/(m-n)$ is odd, while the $C_m^* C_n(-1)^{n-m}$ is even. As a result the product is odd, and a symmetric sum over an odd function is zero. Therefore, we have

$$\sigma_\phi^2 = \frac{\pi^2}{3} + 2 \frac{e^{2\alpha} - 1}{e^{2\alpha} + 1} \xi(\alpha), \quad (16)$$

where

$$\xi(\alpha) = \sum_{m \neq n} e^{-\alpha |n|} e^{-\alpha |m|} \frac{(-1)^{n-m}}{(n-m)^2},$$
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therein we have explicitly written the dependence of $\xi$ on $\alpha$. It turns out that $(e^{2\alpha} - 1)\xi(\alpha)/(e^{2\alpha} + 1)$ is a monotone increasing function of $\alpha \in (0, \infty)$ and the limit as $\alpha \to 0$ and $\alpha \to \infty$ always exist. For the latter, we have

$$\lim_{\alpha \to \infty} \xi(\alpha) = 0,$$

and an explicit computation shows that $\sigma_\phi^2 = \frac{\pi^2}{3}(1 + O(e^{-\alpha}))$ holds for large $\alpha$ (see Appendix A). It thus follows that

$$\lim_{\alpha \to \infty} \sigma_\phi^2 = \frac{\pi^2}{3},$$  \hfill (17)

is an upper bound for $\sigma_\phi^2$. Since $\sigma_\phi^2$ remains bounded for all values of $\alpha$, its physical significance is assured. Note that $\sigma_\phi^2 = \pi^2/3$ is the deviation of a uniform state $\psi(\phi) = 1/\sqrt{2\pi}$, $\phi \in [-\pi, \pi]$.

The opposite situation yields:

$$2 \lim_{\alpha \to 0} \frac{e^{2\alpha} - 1}{e^{2\alpha} + 1} \xi(\alpha) = -\pi^2/3.$$

In Appendix A it is proved that, for $\alpha$ small enough,

$$\sigma_\phi^2 = \alpha^2 + O(\alpha^3),$$  \hfill (18)

Hence, it yields

$$\lim_{\alpha \to 0} \sigma_\phi^2 = 0.$$

For the deviation $\sigma_{L_z}$ (since $C_n$ is even it implies $\langle L_z \rangle = 0$) we have

$$\sigma_{L_z}^2 = 2\hbar^2 \frac{e^{2\alpha}}{(e^{2\alpha} - 1)^2}.$$

In the limit $\alpha \to 0$ we obtain

$$\sigma_{L_z}^2 = \frac{\hbar^2}{2\alpha^2}(1 + O(\alpha)),$$  \hfill (19)

and as $\alpha \to \infty$, we have

$$\sigma_{L_z}^2 = 2\hbar^2 \frac{1}{e^{2\alpha}}(1 + O(e^{-2\alpha})).$$  \hfill (20)

Hence, by Eq. (18,19), for $\alpha$ small enough the uncertainty product

$$\sigma_\phi^2 \sigma_{L_z}^2 = \frac{\hbar^2}{2}(1 + O(\alpha)),$$

asserts that the square of the uncertainty product reaches twice the smallest predicted values by the HR relation (recall $f_\alpha(\phi)$ approaches $\delta(\phi)$ in this limit and it is not affected by the boundary condition at $\pi$). For $\alpha$ large enough, by using Eq. (20,17), we have

$$\sigma_\phi^2 \sigma_{L_z}^2 = \frac{2\pi^2\hbar^2}{3} \frac{1}{e^{2\alpha}}(1 + O(e^{-\alpha})), $$

implying that the uncertainty product goes to zero exponentially fast with $\alpha$.

In Fig. 1 we depict the uncertainty product $\sigma_\phi \sigma_{L_z}/\hbar$ as a function of $\alpha$. One can see that the bound given by Eq. (5) holds only for $\alpha < 1.29639$ (see the dashed line).
7. Polynomial Decay of Fourier Coefficients

The fact that the Fourier coefficients with exponential decay have an arbitrarily small lower bound is not a privilege of this particular decay. Any other decay which fulfills the hypotheses will also do so.

In our next example we want to illustrate that if the hypothesis of a unique $C_k$ in Eq. (10) is not fulfilled, the uncertainty product is bounded from below as predicted by the HR uncertainty relation (5). We consider a symmetric family of Fourier coefficients but we set $C_0$ to zero. As a consequence, there are two coefficients with the same decay as a function of $\alpha$, and the dominance condition is no longer fulfilled by the family. So, according to Theorem 1 the uncertainty product cannot be made arbitrarily small.

In the following, we shall consider

$$C_n = |n|^{-\alpha}, \quad n \neq 0$$
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and $C_0 = 0$. If $\alpha \gg 1$ and $n \neq 0$ the polynomial decay gives an upper bound for the exponential decay. Note that in such limit $|n|^{-\alpha} > \alpha^{-|n|}$.

In this case, the normalization constant is given by

$$|A|^2 = \frac{1}{2\pi \sum_n |n|^{-2\alpha}}.$$  

The deviations now take the form

$$\sigma^2_\phi = \frac{\pi^2}{3} + \frac{1}{\sum_{n \geq 1} n^{-2\alpha}} \sum_{m \neq n} |n|^{-\alpha} |m|^{-\alpha} (\frac{-1}{n-m})^2$$

$$\sigma^2_{L_z} = \frac{\hbar^2}{\sum_{n \geq 1} n^{-2\alpha}} \sum_{n \geq 1} n^{-2(\alpha-1)}$$

In order to have $\sigma_{L_z}$ finite $\alpha$ must be bigger than $3/2$, which guarantees that $|A|^2$ is larger than 0. In the limit $\alpha \to 3/2$ the deviation $\sigma_{L_z}$ diverges, while $\sigma_\phi$ remains finite. The opposite situation yields:

$$\lim_{\alpha \to \infty} \sigma^2_\phi \sigma^2_{L_z} = \left( \frac{\pi^2}{3} + \frac{1}{2} \right) \hbar^2 \approx 3.78986 \hbar^2$$  \hspace{1cm} (21)$$

an uncertainty product larger than the least predicted value given by Eq. (5).

Similar results hold for the exponential decay if we set $C_0 = 0$. The profile of the uncertainty product for polynomial (solid line) and exponential (short dashed line) decays, as a function of $\alpha$, are shown in Figure 2.

8. Replacing $\phi$ by a Periodic Absolutely Continuous Function

As seen in Section 2, the trouble with HR uncertainty relation (5), with $A$ and $B$ replaced by angle operator $\phi$ and its canonical conjugate momentum $L_z$, is not with the commutation relation (4) but with the inequality

$$4 \langle \phi \psi, \phi \psi \rangle \langle L_z \psi, L_z \psi \rangle \geq \langle \psi, i [\phi, L_z] \psi \rangle^2$$

used to derive (5) from (4), which holds in a domain $D([\phi, L_z])$ much smaller than the domain $D(\phi) \cap D(L_z)$ of the left hand side. Among the possibilities to overcome this problem, see [16, 17, 18, 19, 21, 22, 23, 24, 25]. Here we illustrate the idea of replacing the operator $\phi$ by one periodic operator that is absolutely continuous [8, 24].

The basic idea is to introduce the operators $\sin \phi$ and $\cos \phi$ which satisfy the following commutation relation:

$$[\cos \phi, L_z] = -i\hbar \sin \phi$$

and

$$[\sin \phi, L_z] = i\hbar \cos \phi$$

now defined in the domain $D(\sin \phi) \cap D(L_z) = D(\cos \phi) \cap D(L_z)$. 


In this way, we can compute the new uncertainty relations

\[
\sigma_{L_z}^2 \sigma_{\sin \phi}^2 \geq \frac{\hbar^2}{4} (\cos \phi)^2 \tag{22}
\]

\[
\sigma_{L_z}^2 \sigma_{\cos \phi}^2 \geq \frac{\hbar^2}{4} (\sin \phi)^2 . \tag{23}
\]

Let us consider our previous example with the exponentially decaying frequency amplitudes, now applying the new operators. The deviation

\[
\sigma_{\cos \phi}^2 = (\cos^2 \phi) - (\cos \phi)^2 ,
\]

can be explicitly obtained. As a result we have

\[
\sigma_{\cos \phi}^2 = \frac{1}{2} \frac{e^{2\alpha} - e^{-2\alpha} + 4}{e^{2\alpha} + 1} - 4 \frac{e^{2\alpha}}{(e^{2\alpha} + 1)^2} .
\]
For \( \sin \phi \) the deviation is given by
\[
\sigma_{\sin \phi}^2 = \langle \sin^2 \phi \rangle - \langle \sin \phi \rangle^2,
\]
and \( \langle f, \sin \phi f \rangle = 0 \).
Thus after some manipulations we have
\[
\sigma_{\sin \phi}^2 = \frac{1}{2} \frac{e^{2\alpha} + e^{-2\alpha} - 2}{e^{2\alpha} + 1}.
\]

Note that for \( \cos \phi \) we have the relation
\[
\sigma_{L_z}\sigma_{\cos \phi} \geq 0,
\]
since \( \langle f, \sin \phi f \rangle = 0 \). This condition is always fulfilled. The next relation we have to analyze is:
\[
\sigma_{L_z}\sigma_{\sin \phi} \geq \frac{h^2}{4} \langle \cos \phi \rangle^2.
\]
(24)

Working the equations out, we have that Eq. (24) is equivalent to
\[
e^{2\alpha} + e^{-2\alpha} - 2 \geq 0,
\]
which is true for any \( \alpha \geq 0 \).

9. Proof of the Main Results

For convenience, and pedagogic purposes, we consider the case of symmetric Fourier coefficients \( |C_n(\alpha)| = |C_{-n}(\alpha)| \). Theorem I states that the uncertainty product is arbitrarily small if, and only if, there is only one coefficient \( C_k(\alpha) \) such that the rate \( C_n(\alpha)/C_k(\alpha) \) converges to zero as \( \alpha \) grows (dominance condition). For the symmetric case this coefficient must be
\[
C_0(\alpha) = \frac{1}{2\pi A_\alpha} \int_{-\pi}^{\pi} f_\alpha(\phi) d\phi
\]
which is proportional to the spatial average of \( f_\alpha \). \( C_0(\alpha) \) is the only possibility because otherwise it would always exist at least two terms which, as a function of \( \alpha \), decay slower than the other coefficients. Thus, if a family of Fourier coefficient is symmetric and the spatial average of the wave function is zero, our result implies in particular that it is impossible to make \( \sigma_{\phi}\sigma_{L_z} \) as small as one wishes.

We start by showing that if the assumptions in Theorem I are fulfilled then \( \sigma_{\phi}\sigma_{L_z} \) is arbitrarily small. The uncertainty of angular momentum is given by:
\[
\sigma_{L_z}^2 = 2\pi \hbar^2 |A|^2 \sum_n |C_n(\alpha)|^2 n^2
\]
(25)

with \( A \) defined by (9).

Given \( \varepsilon > 0 \), we show that
\[
2\pi^3 \hbar^2 |A|^2 \sum_n |C_n(\alpha)|^2 n^2 < \varepsilon
\]
holds for some \( \alpha = \alpha(\varepsilon) \). Introducing \( |d_n(\alpha)|^2 = |C_n(\alpha)|^2/|C_0(\alpha)|^2 \), Eq. (26) is equivalent to:
\[
\frac{\pi^2 \hbar^2}{\sum_n |d_n(\alpha)|^2} \sum_n |d_n(\alpha)|^2 n^2 < \varepsilon.
\]
Quantum States Allowing Minimum Uncertainty Product of $\phi$ and $L_z$

But since $\liminf_{\alpha \to \infty} |d_n(\alpha)| = 0$ for all $n \neq 0$, and the series $\sum_n |d_n(\alpha)|^2 n^2$ is uniformly convergent, by condition (ii), we have

$$\liminf_{\alpha \to \infty} \sum_n |d_n(\alpha)|^2 n^2 = \sum_n \liminf_{\alpha \to \infty} |d_n(\alpha)|^2 n^2 = 0. \quad (27)$$

Note that $\sum_n |d_n(\alpha)|^2 \geq 1$. Thus, by condition (iii) for any $\varepsilon > 0$ there is a $\alpha^*$ such that

$$\frac{\pi^2 \hbar^2}{\sum_n |d_n(\alpha^*)|^2} \sum_n |d_n(\alpha^*)|^2 n^2 < \varepsilon.$$ 

It follows from the definition of the deviation of $\phi$

$$\sigma^2 \phi = \int_{-\pi}^{\pi} \phi^2 |f_\alpha(\phi)|^2 d\phi \leq \pi^2 \int_{-\pi}^{\pi} |f_\alpha(\phi)|^2 d\phi.$$ 

This implies $\sigma^2 \phi \leq \pi^2$. Hence, it follows from (25) and (26) that

$$\sigma^2 \phi \sigma^2 L_z < \varepsilon,$$

and we finish the first part of the proof.

Next, we show the opposite implication. We want to show that outside our hypothesis there exists $\varepsilon > 0$ such that for all $\alpha \in (0, \infty)$

$$\sigma^2 \phi \sigma^2 L_z > \varepsilon \hbar^2$$

and the uncertainty product cannot be made arbitrarily small.

Let $k \neq 0$ be the smallest integer such that Eq. (10) holds, and introduce $d_n(\alpha) = C_n(\alpha)/|C_k(\alpha)|$. Here, for sake of simplicity, we assume that $k$ is unique, in the sense that only $|d_{-k}|$ and $|d_k|$ are different from zero as $\alpha \to \infty$.

By (i) we have $\sigma^2 \phi > \kappa$. Thus it suffices to demonstrate that $\sigma^2 L_z$ is bounded away from zero. To this end, we write

$$\sigma^2 L_z = \frac{\hbar^2}{\sum_n |d_n(\alpha)|^2} \sum_n |d_n(\alpha)|^2 n^2.$$ 

We split the sum in the numerator and denominator as

$$\sum_n |d_n|^2 = 2 + \sum_{|n|\neq k} |d_n|^2$$

and note that, by condition (ii), there is $K < \infty$ independent of $\alpha$ such that $\sum_{|n|\neq k} |d_n|^2 \leq 2K$. Hence,

$$\sigma^2 L_z \geq \frac{\hbar^2}{1 + K} \left( k^2 + \sum_{n \neq k} |d_n|^2 n^2 \right) \geq \frac{\hbar^2}{1 + K}$$

in view of $\sum_{n \neq k} |d_n|^2 n^2 \geq 0$ and $k \geq 1$. The uncertainty product can be bounded from below by

$$\sigma^2 \phi \sigma^2 L_z > \kappa \frac{\hbar^2}{1 + K}.$$
Since $K$ does not depend on $\alpha$ and $\kappa > 0$ is fixed, we can take $\varepsilon > 0$ so that $\kappa/(1+K) > \varepsilon$, concluding
\[
\sigma_\phi^2 \sigma_{L_z}^2 > \varepsilon \hbar^2.
\]
Our result also holds for asymmetric Fourier coefficients. We do not consider it here since the arguments are the same as for the symmetric case with further technicalities.

10. Conclusions

In conclusion, we have analyzed the uncertainty product for the azimuthal angle $\phi$ and its canonical conjugate moment $L_z$. We have provided necessary and sufficient conditions for a state to have an arbitrary small uncertainty product. These conditions are related to the existence of a Fourier coefficient of $f_\alpha$ which decays slower than the others Fourier modes. More precisely, a state allows for an arbitrary small uncertainty product if, and only if, there is only one coefficient $C_k(\alpha)$, such that $\liminf_{\alpha \to \infty} C_n(\alpha)/C_k(\alpha) = 0$ (the dominance condition).

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Appendix A. Estimation of $\sigma_\phi^2$ for $\lim \alpha \to 0$

Proceeding the variable change $k = n - m$ in $\xi(\alpha)$ we have
\[
\xi(\alpha) = \sum_{k \neq 0} \frac{(-1)^k}{k^2} \sum_n e^{-\alpha|n|} e^{-\alpha|n-k|}.
\]
Due to the modulo we must split the above equation as follows:
\[
\xi(\alpha) = \sum_{k \geq 1} \frac{(-1)^k}{k^2} \left[ \sum_{n \geq k} e^{-\alpha n} e^{-\alpha n} e^{ak} + \sum_{0 \leq n < k} e^{-\alpha n} e^{\alpha n} e^{-ak} + \sum_{n < 0} e^{\alpha n} e^{\alpha n} e^{-ak} \right] + \sum_{k \leq -1} \frac{(-1)^k}{k^2} \left[ \sum_{n > 0} e^{-\alpha n} e^{\alpha n} e^{ak} + \sum_{k < n \leq 0} e^{\alpha n} e^{\alpha n} e^{ak} + \sum_{n \leq k} e^{-\alpha n} e^{-\alpha n} e^{ak} \right].
\]
This can also be written as:
\[
\xi(\alpha) = 2 \sum_{k \geq 1} \frac{(-1)^k}{k^2} \left[ \left( \sum_{n > 0} e^{-2\alpha n} + k \right) e^{-ak} + \sum_{n \geq k} e^{-2\alpha n} e^{ak} \right] e^{-ak}
\]
Noting that $\sum_{n \geq k} e^{-2\alpha n} = e^{2\alpha - 2ak}/(e^{2\alpha} - 1)$, then
\[
\xi(\alpha) = 2 \sum_{k \geq 1} \frac{(-1)^k}{k^2} \left( \frac{e^{2\alpha} + 1}{e^{2\alpha} - 1 + k} \right) e^{-ak}
\]
Thus, we deviation takes the form:
\[
\sigma_\phi^2 = \frac{\pi^2}{3} + 4 \sum_{k \geq 1} \frac{(-1)^k}{k^2} e^{-\alpha k} + 4 \frac{e^{2\alpha} - 1}{e^{2\alpha} + 1} \sum_{k \geq 1} \frac{(-1)^k}{k} e^{-\alpha k}.
\]  

(A.1)

Introducing
\[
g(\alpha) = 4 \frac{e^{2\alpha} - 1}{e^{2\alpha} + 1} \sum_{k \geq 1} \frac{(-1)^k}{k} e^{-\alpha k},
\]  

(A.2)
in the limit of small \( \alpha \) we have
\[
\lim_{\alpha \to 0} \sigma_\phi^2 = \frac{\pi^2}{3} + 4 \sum_{k \geq 1} \frac{(-1)^k}{k^2} \lim_{\alpha \to 0} e^{-\alpha k} + \lim_{\alpha \to 0} f(\alpha),
\]

which equals
\[
\lim_{\alpha \to 0} \sigma_\phi^2 = \lim_{\alpha \to 0} g(\alpha),
\]

since \( \sum_{k \geq 1} \frac{(-1)^k}{k^2} = -\pi^2/12 \). To estimate \( g(\alpha) \), we note that
\[
\sum_{k \geq 1} \frac{(-1)^k}{k} e^{-\alpha k} = \int_\alpha ^\infty \sum_{k \geq 1} (-1)^k e^{-\zeta k} d\zeta
\]
\[
= - \int_\alpha ^\infty \frac{e^{-\zeta}}{1 + e^{-\zeta}} d\zeta
\]
\[
= - \ln \left(1 + e^{-\alpha}\right)
\]  

(A.3)
since the series converges absolutely for \( \alpha > 0 \) and the sum can be performed before the integral. Thus,
\[
g(\alpha) = -4 \frac{e^{2\alpha} - 1}{e^{2\alpha} + 1} \ln \left(1 + e^{-\alpha}\right)
\]
The expansion in power of \( \alpha \ll 1 \) up to third order gives
\[
g(\alpha) = -4\alpha \ln 2 + 2\alpha^2 + O(\alpha^3).  
\]  

(A.4)

Consequently, \( \lim_{\alpha \to 0} g(\alpha) = 0 \) and
\[
\lim_{\alpha \to 0} \sigma_\phi^2 = 0.
\]

Eq. (A.1) can be written in a closed form as
\[
\sigma_\phi^2 = \frac{\pi^2}{3} + 4 \text{Li}_2(-e^{-\alpha}) + g(\alpha),
\]  

(A.5)
where \( \text{Li}_2(z) = \sum_{k \geq 0} z^k/k^2 \) is the dilogarithm function, whose series in power of \( \alpha \) up to order 3 is given by
\[
\text{Li}_2(-e^{-\alpha}) = \sum_{k \geq 1} \frac{(-1)^k}{k^2} e^{-\alpha k} = \frac{-\pi^2}{12} + \alpha \ln 2 - \frac{\alpha^2}{4} + O(\alpha^3).
\]  

(A.6)
Replacing Eqs. (A.4) and (A.6) in Eq. (A.3) it yields
\[
\sigma_\phi^2 = \alpha^2 + O(\alpha^3),
\]
which dictates the behavior of the product \( \sigma_\phi \sigma_{L_z} \) as \( \alpha \to 0 \), as can be seen in Fig. 1.
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[26] Barry Simon and Michael Read, Methods of Modern Mathematical Physics II: Fourier Analysis, Self-Adjointness, Academic Press (1975).
[27] We write \( f_\alpha(\phi)/A_\alpha \) for convenience, so we can handle, for example, the Fourier coefficients and Eq. (3) easily.
[28] Alternatively, we could require the opposite situation \( C_n(\alpha_j) < C_n(\alpha_k) \) if \( k > j \) as \( \alpha \) goes to zero, or to a fixed \( \alpha_0 \).
[29] The \( \lim \inf \) in Eq. (10) means that there is at least one subsequence \( (\alpha_j)_{j \geq 1} \) such that \( \lim_{j \to \infty} C_n(\alpha_j)/C_k(\alpha_j) = 0 \) for all \( n \neq k \). No such subsequence exists if \( \lim \inf_{\alpha \to \infty} C_n(\alpha)/C_k(\alpha) > 0 \) for some \( n \neq k \).