New Treatment of the Rotary Motion of a Rigid Body with Estimated Natural Frequency

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1. Introduction

Consider a rigid body of mass $M$ moves in an asymmetric field around a fixed point $O$ [1]. Let us assume that the surface of its ellipsoid of inertia is optimal, as well as the mass center. Let the frame $OX, OY,$ and $OZ$ be a fixed system in space, and the frame $Ox, Oy,$ and $Oz$ is the main axes frame for the surface of the ellipsoid of inertia of the body which moves with it. Initially, we consider the main axis $x$ for the surface of the ellipsoid of inertia that makes an angle $\xi_0 \neq (k\pi/2); k = 0, 1, 2$ with the fixed axis $Z$ in space. Let the body spins with small speed angular velocity $\omega$ about the axis $Z$. Suppose that $\rho, q,$ and $r$ represent the components of the angular velocity vector of the body about the main axes of the ellipsoid of the inertia surface; $\rho, \gamma',$ and $\gamma''$ are the directional cosines vector of the axis $Z$; $g$ is the acceleration of gravity; $A, B,$ and $C$ are the principal moments of inertia. The point $(x_0, y_0, z_0)$ is the center of mass in the moving coordinate system; $R$ is the position vector of the center of attraction $O_1$ on the fixed downward coordinate $Z$ axis, and $\rho$ is the position vector of the element $dm$. Let $\hat{1}, \hat{j}, \hat{k},$ and $\hat{Z}$ be the unit vectors in the shown directions (Figure 1). Consider $dF$ is the attraction force element due to the attracting center and acted on the element $dm$ at the point $p(x, y, z)$.

2. Formulation of the Problem

Without a loss of generality, we choose the positive direction of both the axis $z$ and the axis $x$ that do not make an obtuse angle $\xi_0$ with the direction of axis $Z$. Under the restriction on $\xi_0$ and the choice of the coordinate system, we get [2]

$$\gamma_0 \geq 0, \quad 0 < \gamma''_0 < 1. \quad (1)$$

The differential equations of motion can be reduced to an autonomous system of two degrees of freedom and one first integral as follows [3]:

$$4\ddot{p}_2 + p_2 = 4e^{-2}F(p_2, \dot{p}_2, \gamma_2, \dot{\gamma}_2, \epsilon), \quad (2)$$

$$\ddot{\gamma}_2 + \gamma_2 = e^{-2}\Phi(p_2, \dot{p}_2, \gamma_2, \dot{\gamma}_2, \epsilon), \quad (2)$$

$$\gamma_2^2 + \dot{\gamma}_2^2 + 2e^{-1}(\nu p_2 \gamma_2 + \nu_2 \dot{p}_2 \dot{\gamma}_2 + s_{21}) + e^{-2}(\ldots) = \gamma''_0 - 1, \quad (3)$$
where

\[ F = C_i A_i^{-1} p_2 \hat{p}_2^2 + x_0 \hat{p}_2 \hat{y}_2 - y_0 \alpha^{-1} p_2 \hat{y}_2 \\
- y_0 A_i^{-1} (A_i + \alpha^{-1}) \hat{y}_2 \hat{p}_2 - z_0 \alpha^{-1} p_2 \\
- 0.75 \epsilon a_1 p_2 - 0.25 p_2 s_{11} + A_i b^{-1} x_0 s_{21} + O(\epsilon^{-1}) + \cdots, \]

\[ \Phi = -(1 - C_i) A_i^{-1} p_2 \hat{p}_2 \hat{y}_2 + x_0 \hat{y}_2^2 - y_0 \epsilon \hat{y}_2 - z_0 b^{-1} \hat{y}_2 \\
+ x_0 b^{-1} - A_i^{-1} \hat{y}_2 \hat{p}_2 \\
+ 0.75 \epsilon (e + e_1) \hat{y}_2 - y_2 s_{11} \\
+ (1 + B_j) p_2 s_{21} + O(\epsilon^{-1}) + \cdots, \]

\[ \begin{align*}
    p_2 &= p_1 - \epsilon^{-1} (e + e_1) y_2, \\
    y_2 &= y_1 - \epsilon^{-1} \epsilon p_2, \\
    q_1 &= -A_i^{-1} \hat{p}_2 + \epsilon^{-1} A_i^{-1} (y_0 \alpha^{-1} - e_2 y_2) + \cdots, \\
    r_1 &= 1 + 0.5 \epsilon^{-2} s_{11} + \cdots, \\
    y_1' &= y_2 + \epsilon^{-1} p_2 + \cdots, \\
    y_2' &= 1 + \epsilon^{-1} s_{21} + \epsilon^{-2} (s_{22} - 0.5 s_{11}) + \cdots, \\
    p_1 &= \frac{p}{c(\gamma_0)}, \\
    r_1 &= \frac{r}{r_0}, \\
    y_1 &= \frac{y}{\gamma_0}, \\
    \tau &= r_0^{-1} \tau, \\
    \left( \epsilon = \frac{d}{d\tau} \right),
\end{align*} \]

\[ \begin{align*}
    s_{11} &= \frac{a(p_{20}^2 - p_{20}^2) + b(p_{20}^2 - p_{20}^2)}{A_i^{-2} - 2 [x_0^2 (y_2 - y_2) + y_0 (y_2 - y_2)]}, \\
    s_{21} &= a(p_{20} (y_2 - y_2) - b A_i^{-1} (p_{20} \hat{y}_2 \hat{p}_2), \\
    s_{22} &= a [\epsilon (p_{20}^2 - p_{20}^2) + \epsilon (y_2 - y_2) + e_1 (y_2 - y_2)] \\
    &+ b A_i^{-1} [-\gamma_2 (p_{20}^2 - p_{20}^2) + a^{-1} y_0 (y_2 - y_2) - e_2 (y_2 - y_2)],
\end{align*} \]

\[ \begin{align*}
    A_i &= C - B, \\
    (ABC), \\
    a &= \frac{A}{C}, \\
    (ab), \\
    c^2 &= \frac{M gl}{C}, \\
    \epsilon &= \frac{c \sqrt{\gamma_0}}{r_0}, \\
    x_0 &= l x_{00}, \\
    (xyz), \\
    \gamma^2 &= x_0^2 + y_0^2 + z_0^2, \\
    4 A_1 B_1 &= -1, \\
    e b &= 4 x_{00} A_1, \\
    3 \nu &= 4 (1 + B_1), \\
    3 e_1 &= 4 x_0 (A_i b^{-1} - a^{-1}), \\
    e_2 &= e_1 + a^{-1} z_0', \\
    \nu = \gamma - A_1^{-1}.
\end{align*} \]

The symbols like ABC are abbreviated equations.

### 3. Construction of Periodic Solutions with Zeros Basic Amplitudes

In this section, we use the suggested method for constructing the aimed solutions for the autonomous system (2). Consider the condition [4]

\[ p_2 (0,0) = \hat{p}_2 (0,0) = \hat{y}_2 (0, \epsilon) = 0. \]

The generating system for (2) is obtained when \( \epsilon \rightarrow \infty \) as follows:

\[ \begin{align*}
    4 \hat{p}_2 (0) + p_2 (0) &= 0, \\
    \hat{y}_2 (0) + y_2 (0) &= 0.
\end{align*} \]

The solutions for system (10) with a period \( T_0 = 4 \pi \) are

\[ \begin{align*}
    p_2 (0) &= a^*_b \cos (0.5 \tau), \\
    y_2 (0) &= b^*_0 \cos \tau,
\end{align*} \]

where \( a^*_b \) and \( b^*_0 \) are constants.
Let system (2) has periodic solutions with a period $T_0 + \alpha$ in the form [5]

$$
\begin{align*}
p_2 &= a^* \cos \psi + \sum_{n=1}^{N} \varepsilon^{-n} p_n^* (a^*, \psi) + O(\varepsilon^{-N-1}), \\
\gamma_2 &= b^* \cos \phi + \sum_{n=1}^{N} \varepsilon^{-n} \gamma_n^* (a^*, \phi) + O(\varepsilon^{-N-1}).
\end{align*}
$$

(12)

For these solutions, we let the initial conditions

$$
\begin{align*}
p_2(0, \varepsilon) &= a^* = a_0^* + a^*(\varepsilon), \\
\gamma_2(0, \varepsilon) &= b^* = b_0^* + b^*(\varepsilon),
\end{align*}
$$

(13)

Here, $a^*(\varepsilon), b^*(\varepsilon) \to 0$ as $\varepsilon \to \infty$. Considering first integral (3) with conditions (13), we get

$$
\begin{align*}
\dot{p}_2 &= -0.5a^* \sin \psi + O(\varepsilon^{-1}), \\
\dot{\gamma}_2 &= -b^* \sin \phi + O(\varepsilon^{-1}), \\
\ddot{p}_2 &= -0.25a^* \cos \psi + \varepsilon^{-1} \left[ 0.25 \frac{\partial^2 p_1^*}{\partial \psi^2} - a^* \psi_1 \cos \psi - A_1^* \sin \psi \right] \\
&\quad + \varepsilon^{-2} \left[ A_1^* \frac{\partial^2 p_1^*}{\partial a^* \partial \psi} - (A_1^* + 2A_1^* \psi_1) \sin \psi + A_1^* \frac{dA_1^*}{da^*} \cos \psi + 0.25 \frac{\partial^2 p_1^*}{\partial \psi^2} + \psi_1 \frac{\partial^2 p_1^*}{\partial \psi^2} - a^* (\psi_1 + 2\psi_2) \cos \psi - a^* A_1^* \sin \psi \frac{d\psi_1}{da^*} \right] + O(\varepsilon^{-3}). \\
\ddot{\gamma}_2 &= -b^* \cos \phi + \varepsilon^{-1} \left[ \frac{\partial^2 \gamma_1^*}{\partial \phi^2} - 2b^* \phi_1 \cos \phi \right] \\
&\quad + \varepsilon^{-2} \left[ \frac{\partial^2 \gamma_1^*}{\partial a^* \partial \phi} + 2\phi_1 \frac{\partial^2 \gamma_1^*}{\partial \phi^2} - b^* (\phi_1^2 + 2\phi_2) \cos \phi + 2A_1^* \frac{dA_1^*}{da^*} \cos \phi \right] + O(\varepsilon^{-3}).
\end{align*}
$$

(18)

Using equations (7), (12), and (18), we get

$$
\begin{align*}
s_{11}^{(0)} &= aa_0^2 \left( \cos^2 \psi_0 - \cos^2 \psi \right) - 0.25bA_1^{-1}a_0^2 \sin^2 \psi \\
&\quad - 2b^* \left[ x_1^*(\cos \phi_0 - \cos \phi) + y_0^* \sin \phi \right], \\
s_{21}^{(0)} &= a_0^* b_0^* \left[ a(\cos \psi_0 \cos \phi_0 - \cos \psi \cos \phi) + 0.5bA_1^{-1} \sin \psi \sin \phi \right], \\
s_{22}^{(0)} &= a \left[ a_0^* x_0^2 \left( \cos^2 \psi_0 - \cos^2 \psi \right) + eb_0^* (\cos \phi_0 - \cos \phi) + e_1 b_0^* (\cos \phi_0 - \cos \phi) \right] \\
&\quad + bA_1^{-1} \left[ 0.25y_2 a_0^* \sin^2 \psi + a^{-1} y_1^* b_0^* \sin \phi + e_2 b_0^* \sin^2 \phi \right],
\end{align*}
$$

(19)

where $\psi_0$ and $\phi_0$ are the initial values of the corresponding functions.
Canceling singular terms from (21) as in [6], we get

\[
F^{(0)} = 0.25 C_1 A_1^{-1} a_0^3 \cos \psi \sin^2 \psi + 0.5 a_0^2 b_0^* x_0' \sin \psi \sin \phi
\]

\[
+ a^{-1} a_0^* b_0^* y_0' \cos \psi \sin \phi + 0.5 A_1^{-1} (A_1 + a^{-1}) a_0^* b_0^* y_0' \sin \psi \cos \phi
\]

\[
- z_0' a_0^* \cos \psi - 0.75 v e_1 a_0^* \cos \psi
\]

\[
- 0.25 a_0^* \cos \psi \left[ a a_0^* (\cos^2 \psi - \cos^2 \phi) - 0.25 b A_1^{-2} a_0^* b_0^* \sin^2 \psi - 2 b_0^* \left[ x_0' (\cos \phi_0 - \cos \phi) + y_0' \sin \phi \right]\right]
\]

\[
+ A_1 b_0^{-1} x_0' a_0^* b_0^* \left[ a (\cos \psi \cos \phi_0 - \cos \psi \cos \phi) + 0.5 b A_1^{-1} \sin \psi \sin \phi \right],
\]

\[
\Phi^{(0)} = 0.25 (C_1 - 1) A_1^{-1} a_0^* b_0^* \sin 2 \psi \sin \phi + 0.5 x_0' b_0^* (1 - \cos 2 \phi)
\]

\[
+ 0.5 y_0' b_0^* \sin 2 \phi - z_0' b_0^{-1} \cos \phi + x_0' b_0^{-1}
\]

\[
- 0.125 A_1^{-2} a_0^* b_0^* (1 - \cos 2 \psi) \cos \phi + 0.75 v e_1 + 0.75 v e_1 b_0^* \cos \phi
\]

\[
- a a_0^* b_0^* \cos \phi + 0.5 a a_0^* b_0^* (1 + \cos 2 \psi) \cos \phi
\]

\[
+ 0.125 b A_1^{-2} a_0^* b_0^* (1 - \cos 2 \psi) \cos \phi + 2 x_0' b_0^* \cos \phi_0 \cos \phi
\]

\[
- x_0' b_0^* (1 + \cos 2 \phi) + y_0' b_0^* \sin 2 \phi
\]

\[
+ a_0^* b_0^* (1 + B_1) \left[ 0.5 b A_1^{-1} \sin \psi \sin \phi + a (\cos \psi_0 \cos \phi_0 - \cos \psi \cos \phi) \right] \cos \psi.
\]

Substituting from (12), (18), and (20) into (2) and equating coefficients of \(\epsilon^{-1}\) in both sides, we get

\[
\frac{\partial^2 p_1^*}{\partial \psi^2} + p_1^* = 4 a_0^* \psi_1 \cos \psi + 4 A_1^* \sin \psi,
\]

\[
\frac{\partial^2 y_1^*}{\partial \phi^2} + y_1^* = 2 b_0^* \phi_1 \cos \phi,
\]

\[
\frac{\partial^2 p_2^*}{\partial \psi^2} + p_2^* = 4 A_1^* \sin \psi + a_0^* \left[ 4 \psi_2 + 0.25 C_1 A_1^{-1} a_0^* - 3.25 a a_0^* b_0^* - 3 v e_1 + 0.125 b A_1^{-2} a_0^* + 2 x_0' b_0^* \cos \phi_0 \right] \cos \psi
\]

\[
+ 0.25 a_0^* \left( a - C_1 A_1^{-1} - 0.25 b A_1^{-2} \right) \cos 3 \psi + 4 a a_0^* x_0' A_1 b_0^* \cos \psi_0 \cos \phi_0
\]

\[
+ x_0' a_0^* b_0^* (1 - 2 a A_1 b_0^{-1}) \sin (\phi_0 - \phi) - x_0' a_0^* b_0^* (3 + 2 a A_1 b_0^{-1}) \sin (\phi + \phi)
\]

\[
+ y_0' a_0^* b_0^* (2 a_1^{-2} - A_1^{-1} a_1^{-1}) \cos (\phi_0 - \phi) + y_0' a_1 b_0^* (2 a_1^{-1} - A_1^{-1} a_1^{-1}) \sin (\phi + \phi),
\]

\[
\frac{\partial^2 y_2^*}{\partial \phi^2} + y_2^* = \left[ 2 \phi_2 - z_0' b_0^{-1} + 0.125 A_1^{-2} a_0^* (b - 1) + 0.75 v e_1 - a a_0^* b_0^* \cos \psi_0 - 0.5 a b_0 a_0^* + 2 x_0' b_0^* \cos \phi_0 \right] b_0^* \cos \phi
\]

\[
- 0.5 x_0' b_0^* + x_0' b_0^{-1} + 0.75 v e_1 (1 + B_1) a a_0^* b_0^* \cos \psi_0 \cos \phi_0 \cos \psi - 0.67 x_0' b_0^* \cos \phi_0 \cos \psi - 2 \phi + 1.5 y_0' b_0^* \sin 2 \phi
\]

\[
+ 0.5 a_0^* \left[ 0.25 A_1^{-2} (1 - b) - a B_1 + A_1^{-1} b_0^* (b - 1) \right] \cos (2 \psi - \phi)
\]

\[
+ \left[ 0.25 A_1^{-2} (1 - b) - a B_1 - A_1^{-1} b_0^* (b - 1) \right] \cos (2 \psi + \phi).
\]
\[ \psi_1 = A_1^* = \phi_1 = A_2^* = 0, \]
\[ \psi_2 = -0.06 C_1 A_1^{-1} a_0^2 + 0.81 a a_0^2 + z_{0a} - 1 + 0.75 v e_1 - 0.02 b A_1^{-1} a_0^2 - 0.5 x_j b_j^* \cos \phi_0, \]
\[ \phi_2 = 0.5 [z_{j} b_j^{-1} - 0.125 A_1^{-2} a_0^2 (b - 1) - 0.75 v e_1 + a a_0^2 (0.5 B_1 + \cos^2 \psi_0) - 2 x_j b_j^* \cos \phi_0]. \] (22)

Substituting from (22) into (15)–(17) and integrating, we obtain

\[ a^* = a_0^* \text{ (arbitrary const.),} \]
\[ \psi = 0.5 r + 0.5 \varepsilon^{-2} [-0.125 C_1 A_1^{-1} a_0^2 - 0.375 a a_0^2 + 2 a a_0^2 + 2 z_{0a} - 1 + 1.5 v e_1 - 0.31 b A_1^{-2} a_0^2 - x_j b_j^* \cos \phi_0] r, \]
\[ \phi = \tau + 0.5 \varepsilon^{-2} [z_{j} b_j^{-1} - 0.125 A_1^{-2} a_0^2 (b - 1) - 0.75 v e_1 + a a_0^2 (1 + 0.5 B_1) - 2 x_j b_j^*] r. \] (23)

From the previous results, we get
\[ \psi(0) = \psi_0 = 0, \]
\[ \phi(0) = \phi_0 = 0. \] (24)

From (13) and (23), we obtain \( a^* \) from the order greater than \( O(\varepsilon^{-2}) \).

The periodic solutions \( \rho_2 \) and \( \gamma_2 \) are obtained by substituting (22) and (23) into (21) and using (12) and (14). Finally, the periodic solutions \( \rho_1, q_1, r_1, \gamma_1, \) and \( \gamma''_1 \) are obtained from (5), (19), (23), and (24).

**4. Construction of Periodic Solutions with Nonzeros Basic Amplitudes**

We use the large parameter method [7] for constructing the periodic solutions with nonzeros basic amplitudes for system (2) when \( A < B < C \) or \( A > B > C \). Consider generating system (10) has periodic solutions with a period \( T_0 = 2\pi \) as follows:

\[ \rho_2^{(0)}(\tau) = E \cos (0.5 \tau - \mu), \]
\[ \gamma_2^{(0)}(\tau) = M_3 \cos \tau, \] (25)

where \( E = \sqrt{M_1^2 + M_2^2}, \mu = \tan^{-1} (M_2/M_1), \) and \( M_1, M_2, \) and \( M_3 \) are constants.

Let system (2) has periodic solutions with a period \( T_0 + \alpha \) that reduces to generating solutions (21) when \( \varepsilon \longrightarrow \infty \), where \( \alpha \) is a function of \( \varepsilon \) such that \( \alpha(\infty) = 0 \). Consider the following initial conditions:

\[ \rho_2(0, \varepsilon) = \tilde{M}_1, \]
\[ \dot{\rho}_2(0, \varepsilon) = 0.5 \tilde{M}_2, \]
\[ \gamma_2(0, \varepsilon) = \tilde{M}_3, \]
\[ \dot{\gamma}_2(0, \varepsilon) = 0. \] (26)

The notation \( \sim \) denotes the following substitution:
\[ M_i \longrightarrow \tilde{M}_i = M_i + \beta_i, \quad i = 1, 2, 3, \] (27)
where \( \beta_1, 0.5 \beta_2, \) and \( \beta_3 \) represent the deviations of the initial values of the required solutions from their values of the generating ones \( M_1, M_2, \) and \( M_3, \) respectively. These deviations are functions of \( \varepsilon \) and vanish when \( \varepsilon \longrightarrow \infty \). Now, we construct the required solutions in the following forms [8]:

\[ \rho_2 = \bar{E} \cos (\psi - \mu) + \sum_{n=1}^{\infty} \varepsilon^{-n} \rho_n^* (\bar{E}, \psi) + O(\varepsilon^{-N-1}), \]
\[ \gamma_2 = \tilde{M}_3 \cos \phi + \sum_{n=1}^{\infty} \varepsilon^{-n} \gamma_n^* (\bar{E}, \phi) + O(\varepsilon^{-N-1}), \] (28)

where \( \rho_n^* \) and \( \gamma_n^* \) are periodic functions in \( \psi \) and \( \phi \), respectively. The quantity \( \tilde{M}_3 \) is determined from the first integral (3). Let \( \bar{E}, \psi, \) and \( \phi \) be changed with time according to

\[ \frac{d\bar{E}}{d\tau} = \sum_{n=1}^{\infty} \varepsilon^{-n} E_n (\bar{E}) + O(\varepsilon^{-N-1}), \] (29)
\[ \frac{d\psi}{d\tau} = 0.5 + \sum_{n=1}^{\infty} \varepsilon^{-n} \psi_n (\bar{E}) + O(\varepsilon^{-N-1}), \] (30)
\[
\frac{d\phi}{dr} = 1 + \sum_{n=1}^{N} e^{-n} \phi_n(E) + O(e^{-N-1}).
\]  

Substituting initial conditions (26) into integral (3), when \( \tau = 0 \), we deduce that

\[
0 < M_3 = \frac{1 - y_0^5}{y_0} < \infty,
\]

\[
\beta_3 = e^{-1/2} M_1 + \ldots.
\]

The derivatives become

\[
\begin{align*}
\dot{p}_2 &= \frac{dE}{dr} \frac{\partial p_2}{\partial E} + \frac{d\psi}{dr} \frac{\partial p_2}{\partial \psi}, \\
\dot{\gamma}_2 &= \frac{dE}{dr} \frac{\partial \gamma_2}{\partial E} + \frac{d\phi}{dr} \frac{\partial \gamma_2}{\partial \phi}, \\
\dot{p}_3 &= \left( \frac{dE}{dr} \right)^2 \frac{\partial^2 p_3}{\partial E^2} + \frac{d^2E}{dr^2} \frac{\partial p_3}{\partial E} + 2 \frac{dE}{dr} \frac{d\psi}{dr} \frac{\partial^2 p_3}{\partial E \partial \psi} + \left( \frac{d\psi}{dr} \right)^2 \frac{\partial^2 p_3}{\partial \psi^2} + \frac{d^2E}{dr^2} \frac{\partial p_3}{\partial \psi^2}, \\
\dot{\gamma}_3 &= \left( \frac{dE}{dr} \right)^2 \frac{\partial^2 \gamma_3}{\partial E^2} + \frac{d^2E}{dr^2} \frac{\partial \gamma_3}{\partial E} + 2 \frac{dE}{dr} \frac{d\phi}{dr} \frac{\partial^2 \gamma_3}{\partial E \partial \phi} + \left( \frac{d\phi}{dr} \right)^2 \frac{\partial^2 \gamma_3}{\partial \phi^2} + \frac{d^2E}{dr^2} \frac{\partial \gamma_3}{\partial \phi^2}.
\end{align*}
\]

Using equations (7), (28), and (33), we get

\[
\begin{align*}
s_{11}^{(0)} &= E^2 \left[ (a \cos^2 \mu - 0.5) + 0.25bA_1^{-2}(\sin^2 \mu - 0.5) + 0.5(0.25bA_1^{-2} - a) \cos(\tau - 2\mu) \right] - 2M_3 \left[ x_0(1 - \cos \tau) + y_0 \sin \tau \right], \\
s_{21}^{(0)} &= M_3E \left[ a \cos \mu + 0.5(0.5bA_1^{-1} - a) \cos(0.5\tau + \mu) - 0.5(0.5bA_1^{-1} + a) \cos(1.5\tau - \mu) \right], \\
s_{22}^{(0)} &= E^2 \left[ va(\cos^2 \mu - 0.5) - 0.25bA_1^{-2}v_3(\sin^2 \mu - 0.5) - 0.5(2\cos \mu + 0.25bA_1^{-2}v_3) \cos(\tau - 2\mu) \right] \\
&\quad + 0.5M_3^2(\cos \tau) + M_3 \left[ a(1 - \cos \tau) + b y_0 \sin \tau \right].
\end{align*}
\]

Using (4), (28), (33), and (34), we obtain

\[
\begin{align*}
F^{(0)} &= 0.25C_1A_1^{-1}E^3 \cos(0.5\tau - \mu) \sin^2(0.5\tau - \mu) + EM_3 \left[ 0.5x_0^2 \sin(0.5\tau - \mu) + y_0^3 \right] + 0.5y_0^4 \left( A_1 + a^{-1} \right) M_3E \cos \tau \sin(0.5\tau - \mu) - E \left( \left( z_0^2 a^{-1} + 0.75ve_1 \right) \cos(0.5\tau - \mu) \right) - 0.25E \cos(0.5\tau - \mu) E^2 \left[ a \cos^2 \mu - 0.5 + 0.25bA_1^{-2}(\sin^2 \mu - 0.5) + 0.5(0.25bA_1^{-2} - a) \cos(\tau - 2\mu) \right] - 2M_3 \left[ x_0(1 - \cos \tau) + y_0 \sin \tau \right] \\
&\quad + A_1 \left( x_0^2 \tau \cos(0.5\tau - \mu) + y_0^4 \sin \tau \right) + 0.5M_3^2(\cos \tau) + 0.5M_3 \left[ (1 + a) \left( 1 + B_1 \right) \cos(0.5\tau - \mu) \right], \\
\Phi^{(0)} &= b^{-1}x_0^3 - 0.5M_3^2 \sin \mu - 0.75ve_1 \left\{ z_0^2 b^{-1} M_3 + 0.125A_1^{-1} M_3 E^3 - 0.75ve_1 M_3 \right\} + M_3E^2 \left[ a \cos^2 \mu - 0.5 + 0.25bA_1^{-2}(\sin^2 \mu - 0.5) \right] - 2M_3^2 \sin \tau + 0.5(1 - C_1)A_1^{-1} \cos \tau + 0.25A_1^{-2} \cos(\tau - 2\mu) - \cos(0.5\tau - \mu) + 0.25A_1^{-2} \cos \tau + 0.5A_1^{-2} \cos(\tau - 2\mu) - \cos(0.5\tau - \mu) + 0.25A_1^{-2} \cos \tau + 0.5A_1^{-2} \cos(\tau - 2\mu) - \cos(0.5\tau - \mu).
\end{align*}
\]
Substituting from (28), (33), and (35) into initial system (2) and equating coefficients of $\epsilon^{-1}$ and $\epsilon^{-2}$ in both sides, we obtain the following:

Coefficients of $\epsilon^{-1}$:

$$\frac{d^2 p_1^*}{d\psi^2} + p_1^* = 4 \left( E\psi_1 \cos \mu - E_1 \sin \mu \right) \cos \psi$$

$$+ 4 \left( E\psi_1 \sin \mu + E_1 \cos \mu \right) \sin \psi, \quad (36)$$

$$\frac{d^2 y_1^*}{d\phi^2} + y_1^* = 2\phi_1 M_3 \cos \phi.$$  

We neglect the singular terms [4] to get

$$\frac{d^2 p_1^*}{d\tau^2} + 0.25 p_2^* = \left[ E_1 \cos \mu + \psi_2 \sin \mu \right] \sin 0.5 \tau$$

$$- \left[ E_1 \sin \mu - \left( E\psi_2 + 0.06C_1 A_1^{-1} E^3 - z' \mu^{-1} E - 0.75v \tau \right) E - 0.25E^3 bA_1^{-2} \left( \sin^2 \mu - 0.5 \right) - 0.06E^3 bA_1^{-2} \phi_1 \right] \cos 0.5 \tau$$

$$+ A_1 b^{-1} x_0^2 M_3 E \cos \mu - 0.06E^3 \left( C_1 A_1^{-1} + 0.25bA_1^{-2} - a \right) \left( \cos 3 \mu \cos \tau + \sin \mu \sin 3 \mu \right)$$

$$+ M_3 E \left[ 0.5A_1 b^{-1} \left( 0.5bA_1^{-2} - a \right) x_0^2 \right] \cos \mu + 0.5 \left[ \frac{a^{-1} - 0.5A_1^{-1} \left( a_1 + a^{-1} \right) + 0.5 \phi_1 \mu \sin \mu \right] \cos 0.5 \tau$$

$$- 0.5M_3 E \left[ A_1 b^{-1} \left( 0.5bA_1^{-2} - a \right) x_0^2 \right] \cos \mu - \left[ a^{-1} - \left( 1 + A_1^{-1} a^{-1} \right) + 0.5 \phi_1 \mu \sin \mu \right] \cos 0.5 \tau$$

$$- 0.5M_3 E \left[ 0.25 + \left( 0.5 + A_1 b^{-1} a \right) x_0^2 \right] \cos \mu - \left[ a^{-1} + \left( 1 + A_1^{-1} a^{-1} \right) + 0.5 \phi_1 \mu \sin \mu \right] \cos 1.5 \tau$$

$$- 0.5M_3 E \left[ 1.5A_1 b^{-1} x_0^2 \mu \sin \mu - \left( a^{-1} + \left( 1 + A_1^{-1} a^{-1} \right) + 0.5 \phi_1 \mu \sin \mu \right] \cos 1.5 \tau, \quad (42)$$

Coefficients of $\epsilon^{-2}$:

$$\frac{d^2 y_2^*}{d\tau^2} + y_2^* = x_0^2 \left( b^{-1} - 0.5M_3^{-2} \right) + 0.75v$$

$$+ M_3 \left[ 2\phi_2 - z' \mu^{\phi_2} - 0.125bA_1^{-2} E^2 + 0.75v x_1^{-} - E^2 \left[ a \left( \cos^2 \mu - 0.5 \right) + \mu bA_1^{-2} \left( \sin^2 \mu - 0.5 \right) \right] \cos \tau$$

$$+ 2M_3 \left( x_1^{-} - 0.5aE^2 \right) \cos \mu$$

$$+ 1.5M_3 \left[ y_0^\phi \left( 0.5x_0^2 - \cos 2 \phi_1 \sin \mu \right) + \left( 1 + B_1 \right) x_0^2 \right] \cos \mu$$

$$+ 0.125M_3 E^2 \left[ \left( C_1 - 1 \right) A_1^{-1} + 0.5A_1^{-2} \left( 0.5bA_1^{-2} - 2a \right) \left( 1 + B_1 \right) bA_1^{-1} \cos 2 \mu \right.$$  

$$+ \left( 1 + B_1 \right) \left( bA_1^{-1} - 2a \right) \left( 0.5bA_1^{-2} - 2a \right) \left( 1 + B_1 \right) \left( bA_1^{-1} - 2a \right) \cos 2 \mu. \quad (43)$$

Neglecting singular terms from (42) and (43) yields [4]

$$E_2 = 0.125E \sin 2 \mu \left[ 0.25C_1 A_1^{-1} E^2 - 4z' \mu^{-1} - 3v x_1^{} - E^2 a \left( \cos^2 \mu - 0.5 \right) - 0.25E^2 bA_1^{-2} \left( \sin^2 \mu - 0.5 \right) - 0.25E^2 \left( 0.25bA_1^{-2} - a \right) + 2M_3 x_0^2 \right]$$

$$\psi_2 = 0.25 \cos \mu \left[ -0.25C_1 A_1^{-1} E^2 + 4z' \mu^{-1} + 3v x_1^{} + E^2 a \left( \cos^2 \mu - 0.5 \right) + 0.25bE^2 A_1^{-2} \left( \sin^2 \mu - 0.5 \right) + 0.25E^2 \left( 0.25bA_1^{-2} - a \right) - 2M_3 x_0^2 \right]$$

$$\phi_2 = 0.5 \left[ z' \mu^{-1} + 0.125A_1^{-2} E^2 - 0.75v x_1^{} + E^2 \left[ a \left( \cos^2 \mu - 0.5 \right) + 0.25bA_1^{-2} \left( \sin^2 \mu - 0.5 \right) \right] - 2M_3 x_0^2 + 0.5a(1 + B_1)E^2 \right]. \quad (44)$$
Substituting from (38), (40), and (44) into (29) and (30) and integrating, we get

\[
2E = 2E - \varepsilon^{-2} E \sin 2\mu \left[ -0.25C_1A_1^{-1}E^2 + 4z_0'a + 3ve_1 + E^2a \right] + 0.25bE^2 A_1^{-1} \left( \sin^2 \mu - 0.5 \right) + 0.25bE^2 \left( 0.25bA_1^{-2} - a \right) - 2M_3 x_0' \right] \tau + \cdots,
\]

\[
2\psi = \tau + 0.5\varepsilon^{-2} \left[ -0.25C_1A_1^{-1}E^2 + 4z_0'a + 3ve_1 + E^2a \right] + 0.25b E^2 \left( 0.25bA_1^{-2} - a \right) - 2M_3 x_0' \cos^2 \mu \tau + \cdots,
\]

\[
\phi = \tau + 0.25 \varepsilon^{-2} \left[ 2e_0' \tau \cos \theta + 0.25 \left( E^2 - 1.5ve_1 + E^2 \right) \left[ 2a \left( \cos^2 \mu - 0.5 \right) + 0.5b A_1^{-2} \left( \sin^2 \mu - 0.5 \right) \right] \right] - 4M_3 x_0' + a \left( 1 + B_1 \right) E^2 \right] \tau + \cdots.
\]

Substituting (44) into (42) and (43) and solving the resulted equations, we get \( p_2' \) and \( y_2' \). The periodic solutions \( p_2 \) and \( y_2 \) are constructed using (28), (32), (41), and (45).

Using (5) and (34), we get the first terms of the required solutions as follows:

\[
p_1 = M_1 \cos 0.5 \tau + M_2 \sin 0.5 \tau - \varepsilon^{-1} \left( \frac{x_0'}{bB_1} - e_1 M_3 \cos \tau \right) + \cdots,
\]

\[
q_1 = 0.5 A_1^{-1} \left( M_1 \sin 0.5 \tau - M_2 \cos 0.5 \tau \right) + \varepsilon^{-1} \left( \frac{y_0'}{aA_1} + e_2 A_1^{-1} M_3 \sin \tau \right) + \cdots,
\]

\[
r_1 = 1 + 0.25 \varepsilon^{-2} \left[ 2a M_2^2 - E^2 + 0.5b A_1^{-2} \left( M_2^2 - 0.5E^2 \right) \right] + \left( 0.25bA_1^{-2} - a \right) \left( M_1^2 - M_2^2 \right) \cos \tau + 2M_1 M_2 \sin \tau
\]

\[
- 4M_3 \left( x_0' \left( 1 - \cos \tau \right) + y_0' \sin \tau \right) \right] + \cdots,
\]

\[
\gamma_1 = M_3 \cos \tau + \varepsilon^{-1} \left( -M_1 \cos \tau + M_1 \cos 0.5 \tau + M_2 \sin 0.5 \tau \right) + \cdots,
\]

\[
\gamma_1' = -M_3 \sin \tau + \varepsilon^{-1} \left[ \gamma M_1 \sin \tau + 0.5 \gamma_2 \left( -M_1 \sin 0.5 \tau + M_2 \cos 0.5 \tau \right) \right] + \cdots,
\]

\[
\gamma_1'' = 1 + \varepsilon^{-1} \left[ M_1 \left( a \cos \mu + 0.5 \left( b A_1^{-1} - a \right) \cos \left( 0.5 \tau - \mu \right) - 0.25 \left( b A_1^{-1} + 2a \right) \cos \left( 1.5 \tau - \mu \right) \right)
\]

\[
+ \varepsilon^{-2} \left\{ \frac{0.5 M_3^2 z_0' \left( a - b \right)}{a + b - 1} + M_3 \left( 1 - a \right) \right\} \sin \tau - M_3 \left( 1 - a \right) \right\} x_0' \cos \tau - \frac{0.5 M_3^2 z_0' \left( a - b \right) \cos 2 \tau}{a + b - 1}
\]

\[
+ E^2 \left[ \left( a \cos^2 \mu - 0.5 \right) - 0.25b A_1^{-2} \left( \sin^2 \mu - 0.5 \right) \right] - 0.125 \left( 4a + b A_1^{-1} \right) \cos \left( 2 \tau - \mu \right)
\]

\[
- 0.125 \left( a \cos^2 \mu - 0.5 \right) + 0.25b A_1^{-2} \left( \sin^2 \mu - 0.5 \right) + 0.125 \left( b A_1^{-2} - 4 \right) \cos \left( 2 \tau - \mu \right) \right] \right] + \cdots.
\]

The correction of the period is

\[
a(\varepsilon) = \varepsilon^{-2} m \left[ 2M_3 x_0' - 2z_0' - 0.125 A_1^{-2} E^2 + E^2 \left[ a(\cos^2 \mu - 0.5) + 0.25b A_1^{-2} (\sin^2 \mu - 0.5) \right] - 0.5aE^2 \left( 1 + B_1 \right) \right] + \cdots.
\]

5. Geometric Interpretation of Motion

In this section, we describe the body motion using Euler’s angles \( \xi, \zeta, \) and \( \eta \) which come from the obtained solutions (Figure 2). Replacing the time \( t \) by \( t + t_0 \) where \( t_0 \) is an arbitrary interval, the periodic solutions remain periodic since the initial system is autonomous [9]. For this case, we obtain from (32),

\[
\eta_0 = 0.5 \pi + r_0^{-1} t_0 + \cdots,
\]

\[
\xi_0 = \tan^{-1} M_3,
\]

where \( \eta_0 \) and \( \xi_0 \) are arbitrary initial angles.

Making use of (46) and (49) when \( \tau = r_0^{-1} t \), we find Euler’s angles as follows:
where

\[ \begin{align*}
\xi(t) &= 0.5 \left( 0.5bA_1^{-1} - a \right) \cos \left( \frac{t}{2r_0} + \mu \right) - 0.5 \left( 0.5bA_1^{-1} + a \right) \cos \left( \frac{3t}{2r_0} - \mu \right), \\
\xi_2(t) &= y_0' a^{-1} A_1^{-1} \sin \frac{t}{r_0} + b^{-1} B_1^{-1} x_0' \cos \frac{t}{r_0} - 0.5 \tan \xi_0' \left( \frac{a - b}{a + b - 1} \right) \cos 2 \frac{t}{r_0} \\
&\quad - 0.5 \xi_0' \cos \xi_0 \left[ a (v - 0.5) + 0.25 b A_1^{-1} (v_2 + 0.5 A_1^{-1}) \right] \cos \left( \frac{t}{r_0} - 2 \mu \right), \\
\xi_1(t) &= \eta_1(t) = 0.67 \left( 1 + 0.5A_1^{-1} \right) \left( M_1 \sin \frac{3t}{2r_0} - M_2 \cos \frac{3t}{2r_0} \right) + (2 - A_1^{-1}) \left( M_2 \cos \frac{t}{2r_0} + M_1 \sin \frac{t}{2r_0} \right), \\
\xi_2(t) &= (Q_{11} + Q_{13} + Q_{16}) \sin \frac{t}{r_0} - (Q_{11}' + Q_{13}' - Q_{16}') \cos \frac{t}{r_0} \\
&\quad + 0.5 \left( Q_{12} \sin \frac{2t}{r_0} - Q_{12}' \cos \frac{2t}{r_0} \right) + 2 \left( Q_{14} \sin \frac{t}{2r_0} + Q_{14}' \cos \frac{t}{2r_0} \right) + 0.67 \left( Q_{15} \sin \frac{3t}{2r_0} - Q_{15}' \cos \frac{3t}{2r_0} \right), \\
\eta_2(t) &= h_{11} \sin \frac{t}{r_0} - h_{11}' \cos \frac{t}{r_0} + 0.5 \left( h_{12} \sin \frac{2t}{r_0} - h_{12}' \cos \frac{2t}{r_0} \right) + \left( h_{13} \sin \frac{t}{r_0} - h_{13}' \cos \frac{t}{r_0} \right) + 2 \left( h_{14} \sin \frac{t}{2r_0} + h_{14}' \cos \frac{t}{2r_0} \right) \\
&\quad + 0.67 \left( h_{15} \sin \frac{3t}{2r_0} - h_{15}' \cos \frac{3t}{2r_0} \right) + \left( h_{16} \sin \frac{t}{r_0} + h_{16}' \cos \frac{t}{r_0} \right) + 0.34 \left( h_{17} \sin \frac{3t}{2r_0} - h_{17}' \cos \frac{3t}{2r_0} \right). 
\end{align*} \]
Table 1: The analytical solutions $p_2$, $\gamma_2$, and their derivatives.

| t    | $p_2$  | $\gamma_2$ | $\dot{p}_2$ | $\dot{\gamma}_2$ |
|------|--------|-------------|--------------|-------------------|
| 0    | 1.5    | 11.06602    | 0.8164966    | 8.60977E-05       |
| 10   | 2.018443 | 8.279188    | 0.6022027    | -7.342405         |
| 20   | 2.361099 | 1.322297    | 0.3354627    | -10.98657         |
| 30   | 2.498126 | -6.300518   | 0.03950729   | -9.096887         |
| 40   | 2.417591 | -10.74971   | -0.259889    | -2.625185         |
| 50   | 2.126507 | -9.784271   | -0.5366514   | 5.168785          |
| 60   | 1.650225 | -3.890493   | -0.7666767   | 10.35926          |
| 70   | 1.030224 | 3.962964    | -0.9299318   | 10.33185          |
| 80   | 0.3205004 | 9.82036    | -1.012199    | 5.10033           |
| 90   | -0.4171353 | 10.73134   | -1.006313    | -2.700216         |
| 100  | -1.118443 | 6.237032    | -0.9127875   | -9.140733         |
| 110  | -1.722344 | -1.398863   | -0.7397668   | -10.97719         |
| 120  | -2.176246 | -8.330236   | -0.5023198   | -7.28457         |
| 130  | -2.440619 | -11.06585   | -0.2211253   | 0.07719428        |
| 140  | -2.492437 | -8.227805   | 0.07932674   | 7.400064          |
| 150  | -2.327188 | -1.245598   | 0.3728705    | 10.9956           |
| 160  | -1.959263 | 6.363904    | 0.6339409    | 9.052767          |
| 170  | -1.420706 | 10.76787    | 0.839801     | 2.550172          |
| 180  | -0.7584189 | 9.748108   | 0.9725226    | -5.236892         |
| 190  | -0.0300812 | 3.818282   | 1.020547     | -10.38613         |
| 200  | 0.7008771 | -4.034786   | 0.9796914    | -10.30393         |
| 210  | 1.370795  | -9.855532   | 0.8535145    | -5.031701         |
| 220  | 1.921331  | -10.71208   | 0.6530044    | 2.774997          |
| 230  | 2.304537  | -6.172976   | 0.3956243    | 9.183975          |
| 240  | 2.48704   | 1.475497    | 0.1037893    | 10.9671           |
| 250  | 2.452946  | 8.380901    | -0.1970856   | 7.226189          |
| 260  | 2.205224  | 11.06502    | -0.4807954   | -0.1544821        |
| 270  | 1.765449  | 8.175891    | -0.7226327   | -7.457364         |
| 280  | 1.171921  | 1.168729    | -0.9015357   | -11.00408         |
| 290  | 0.4763283 | -6.420705   | -1.001924    | -9.008176         |
| 300  | -0.2607465 | -10.78558  | -1.015054    | -2.474989         |

Table 2: The numerical solutions $p_2$, $\gamma_2$, and their derivatives.

| t    | $p_2$  | $\gamma_2$ | $\dot{p}_2$ | $\dot{\gamma}_2$ |
|------|--------|-------------|--------------|-------------------|
| 0    | 1.5    | 11.06602    | 0.8164966    | 8.60977E-05       |
| 10   | 2.018443 | 8.279188    | 0.6022027    | -7.342405         |
| 20   | 2.361099 | 1.322297    | 0.3354627    | -10.98657         |
| 30   | 2.498126 | -6.300518   | 0.03950729   | -9.096887         |
| 40   | 2.417591 | -10.74971   | -0.259889    | -2.625185         |
| 50   | 2.126507 | -9.784271   | -0.5366514   | 5.168785          |
| 60   | 1.650225 | -3.890493   | -0.7666767   | 10.35926          |
| 70   | 1.030224 | 3.962964    | -0.9299318   | 10.33185          |
| 80   | 0.3205004 | 9.82036    | -1.012199    | 5.10033           |
| 90   | -0.4171353 | 10.73134   | -1.006313    | -2.700216         |
| 100  | -1.118443 | 6.237032    | -0.9127875   | -9.140733         |
| 110  | -1.722344 | -1.398863   | -0.7397668   | -10.97719         |
| 120  | -2.176246 | -8.330236   | -0.5023198   | -7.28457         |
| 130  | -2.440619 | -11.06585   | -0.2211253   | 0.07719428        |
| 140  | -2.492437 | -8.227805   | 0.07932674   | 7.400064          |
| 150  | -2.327188 | -1.245598   | 0.3728705    | 10.9956           |
| 160  | -1.959263 | 6.363904    | 0.6339409    | 9.052767          |
| 170  | -1.420706 | 10.76787    | 0.839801     | 2.550172          |
| 180  | -0.7584189 | 9.748108   | 0.9725226    | -5.236892         |
| 190  | -0.0300812 | 3.818282   | 1.020547     | -10.38613         |
| 200  | 0.7008771 | -4.034786   | 0.9796914    | -10.30393         |
| 210  | 1.370795  | -9.855532   | 0.8535145    | -5.031701         |
6. The Numerical Solutions

In this section, we assume numerical values data for the parameters of a rigid body, and we achieve a computer program to solve the quasilinear system using the fourth order Runge–Kutta method [7]. We make another program to represent the analytical solutions numerically in a period $t$ between 0 and 300 (Table 1). We use the initial values from Table 1 for obtaining the numerical solutions represented in Table 2. The comparison between the obtained numerical solutions and analytical ones is presented to know the difference between them. The numerical and analytical solutions are in good agreement with others which proves the accuracy of used methods and obtained results.

| $t$  | $p_2$   | $y_2$   | $\dot{p}_2$ | $\dot{y}_2$ |
|------|---------|---------|-------------|-------------|
| 220  | 1.921283| -10.71119| 0.6530228   | 2.749476    |
| 230  | 2.304503| -6.19003  | 0.3956484   | 9.163264    |
| 240  | 2.487025| 1.44783   | 0.1038172   | 10.96256    |
| 250  | 2.452952| 8.356145  | -0.1970554  | 7.24163     |
| 260  | 2.205252| 11.05635  | -0.4807664  | -0.1255394  |
| 270  | 1.765499| 8.189053  | -0.7226074  | -7.429035   |
| 280  | 1.171988| 1.198392  | -0.9015169  | -10.99124   |
| 290  | 0.4764102| -6.395274 | -1.001913   | -9.018668   |
| 300  | -0.2606581| -10.76818 | -1.015054   | -2.504935   |

Figure 3: The stability of the analytical and numerical solutions $\dot{p}_2$ and $p_2$.

Figure 4: The stability of the analytical and numerical solutions $\dot{y}_2$ and $y_2$. 
7. Conclusion

The solutions (46) and the correction of the period (47) are obtained using the large parameter method, which had never been used for solving this kind of problem in the presence of the new assumptions for motion (the weak oscillations of the body about the minor or the major axis of the ellipsoid of inertia instead of the strong oscillations in the previous works). The advantage of this method is that the energy motion of the body is assumed to be sufficiently small instead of sufficiently large with other techniques [10–12]. Also, the obtained solutions treat a singular situation for the natural frequency which was excluded from previous works [13, 14].

Equations (50) and (51) describe the rotation of the body at any time and show that this motion depends on four arbitrary constants \( \xi_0, \zeta_0, \eta_0, \) and \( r_0 \), such that \( r_0 \) is sufficiently small. The obtained solutions give special cases of motions when \( M_1 = M_2 = 0 \) and when \( M_1 = 0, M_2 \neq 0, \) or \( M_2 = 0, M_1 \neq 0 \). Also, the obtained solutions give many gyroscopic motions, which depend on the values of the moments of inertia and the initial position of the body center of gravity. In the end, we obtain the case of regular precession [10] as a special case.

The analytical solutions (46) are represented indefinite intervals of time through computer programs (Table 1). The numerical solutions are obtained using the fourth order Runge–Kutta method in terms of another program (Table 2). Tables 1 and 2 give in detail the obtained results of both the analytical solutions and numerical ones. These results show that the analytical solutions are in full agreement with the numerical ones which proves the accuracy of the considered techniques and results. This case of study is considered as a general case of such ones studied in [5]. The stability phase diagrams of the solutions \( p_2 \) and \( \gamma_2 \) are given (Figures 3 and 4). From these diagrams, we note that the stability for both the analytical and the numerical solutions in full agreement. This gives the validity of the obtained solutions and the considered procedures. The considered procedures and results are very useful for the general reader’s concern with the new applications dealing with the use of functionally graded materials in such structures based on the recent works [15].

Data Availability

Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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