Doxastic logic: a new approach

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ABSTRACT
In this paper, I develop a new set of doxastic logical systems and I show how they can be used to solve several well-known problems in doxastic logic, for example the so-called problem of logical omniscience. According to this puzzle, the notions of knowledge and belief that are used in ordinary epistemic and doxastic symbolic systems are too idealised. Hence, those systems cannot be used to model ordinary human or human-like agents’ beliefs. At best, they can describe idealised individuals. The systems in this paper can be used to symbolise not only the doxastic states of perfectly rational individuals, but also the beliefs of finite humans (and human-like agents). Proof-theoretically, I will use a tableau technique. Every system is combined with predicate logic with necessary identity and ‘possibilist’ quantifiers and modal logic with two kinds of modal operators for relative and absolute necessity. The semantics is a possible world semantics. Finally, I prove that every tableau system in the paper is sound and complete with respect to its semantics.

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1. Introduction
In this paper, I develop a new set of doxastic logical systems that include two doxastic operators B and C, two kinds of modal operators for relative and absolute necessity, ‘possibilist’ quantifiers and an identity sign for necessary identity. B and C are sentential operators that take individual terms and sentences as arguments and give sentences as values. The formal sentence $B_dA$ reads ‘individual $d$ believes that $A$’ and the formal sentence $C_dA$ reads ‘it is doxastically conceivable for $d$ that $A$’, or ‘it is doxastically imaginable to $d$ that $A$’ or ‘it is doxastically possible for $d$ that $A$’. Proof-theoretically, I will develop a set of indexed tableau systems. The meaning of the various symbolic expressions in our systems is described by a kind of possible world semantics. Finally, I prove that every tableau system in the paper is sound and complete with respect to its semantics.

Epistemic and doxastic logic has often been developed as a kind of modal logic (Fagin, Halpern, Moses, & Vardi, 1995; Hintikka, 1962; Meyer & van der Hoek, 1995). There are, however, several well-known problems with various standard epistemic and
doxastic systems, for instance, the so-called problem of logical omniscience. It is, therefore, plausible to explore some alternatives to this approach. The systems in this paper avoid many of the problems with the standard systems without totally abandoning the basic modal paradigm.\(^1\)

There are many good philosophical reasons to be attracted to the systems developed in this paper. I cannot discuss every possible argument, but I will briefly go through three of the most obvious ones. First, they can be used to solve the problem of logical omniscience. Second, they can be used to explain the validity of many intuitively valid arguments that cannot be proved in standard doxastic logic. Third, they solve the puzzles in a rather parsimonious way: we do not have to introduce any new entities, such as impossible worlds.

Reason 1 suggests that standard doxastic systems are too strong, and Reason 2 suggests that they are too weak. This indicates that the standard systems are in deep trouble. One can avoid problem 1 by making the standard systems weaker. However, then they might become too weak. One can avoid problem 2 by making the standard systems stronger, but then they might become too strong. The solution in this paper seems more promising.

**Reason 1: The problem of logical omniscience.** The problem of logical omniscience is a well-known puzzle in epistemic and doxastic logic (see Meyer & van der Hoek, 1995, pp. 71–89, for an introduction). According to this problem, the notions of knowledge and belief that are used in ordinary epistemic and doxastic symbolic systems are too idealised. All the following rules of inference hold, for example, in most standard systems (\(B_iA\) reads ‘individual \(i\) believes that \(A\)’):

- If \(A \leftrightarrow C\) is a theorem, then \(B_iA \leftrightarrow B_iC\) is a theorem (for every \(i\) and \(A\) and \(C\)) (Belief of equivalent formulas).
- If \(A \rightarrow C\) is a theorem, then \(B_iA \rightarrow B_iC\) is a theorem (for every \(i\) and \(A\) and \(C\)) (Closure under valid implication).
- If \(A\) is a theorem, then \(B_iA\) is a theorem (for every \(i\) and \(A\)) (Belief of valid formulas).

If doxastic logic is supposed to be modelling human or human-like agents (non-human animals, computers, robots, etc.), then all these inference patterns are unreasonable. It is unlikely that an organism such as a human could ever believe every valid sentence no matter how complex it is. It is also implausible to assume that a human-like agent’s beliefs are closed under valid implication; that is, it is unreasonable to believe that she believes every logical consequence of what she believes no matter how complicated it is. Furthermore, it does not seem to be the case that for every \(A\) and \(B\) that are logically equivalent, if the agent believes \(A\), she also believes \(B\), and vice versa, no matter how complicated these assertions are. That the rules of inference above hold in standard doxastic systems is a consequence of the fact that doxastic (and epistemic) logic traditionally has been developed as a form of normal modal logic. The first rule (Belief of equivalent formulas) is a problem also for many non-normal systems. Several theorems that can be proved in (most) standard systems might also be problematic. Here are some examples:

\[(B_iA \land B_i(A \rightarrow C)) \rightarrow B_iC.\]
\[\neg(B_iA \land B_i\neg A).\]
\[B_i(A \land C) \leftrightarrow (B_iA \land B_iC).\]
In our systems, none of the rules of inference and none of the formulas above hold. So, we can solve the problem of logical omniscience. From the fact that someone believes something, almost nothing of interest follows. Nevertheless, the inference rules and the formulas above might seem reasonable if we consider a perfectly rational (reasonable or wise) individual. We will, for example, see that a sentence of the following kind $Rc \rightarrow \neg((BcA \land Bc\neg A))$ (if $c$ is perfectly rational, then it is not the case that $c$ believes that $A$ and $c$ believes that not-$A$) holds in certain systems. In other words, in those systems we can prove that no perfectly rational (wise, reasonable) individual has inconsistent beliefs. It might perhaps also be the case that no one, in fact, has any inconsistent beliefs at some time. But even if this were true, it seems unreasonable to assume that it is a logical truth that no one has inconsistent beliefs. Something can be contingently true without being logically true, and it does not seem to be a task for a doxastic logician to decide whether there are any contingent truths of this kind. In any case, the fact that we can use the systems in this paper to solve the problem of logical omniscience is a good reason to study them.

There are many other possible solutions to the problem of logical omniscience. One can use classical modal logic and not normal modal logic as a model for doxastic logic, one can treat the belief-operator as a possibility-operator and not as a necessity-operator, one can make a distinction between implicit and explicit beliefs or between awareness and explicit beliefs, one can introduce the notions of local reasoning and opaque knowledge and beliefs, and principles of implicit beliefs, one can use fusion models or impossible world semantics and one can use various non-modal approaches to doxastic logic to try to solve this problem.\(^2\) It is beyond the scope of this paper to discuss all solutions that have been suggested in the literature in detail. However, every solution that I am aware of seems to me to be either intuitively too strong or intuitively too weak or simultaneously both too strong and too weak. A system is too strong if we can prove too much in it, that is, if we can prove things in this system that are counterintuitive; and it is too weak, if we can prove too little in it, that is, if we cannot prove everything that we want to be able to prove. For example, classical systems seem to be too strong, since Belief of equivalent formulas (see above) still holds in such systems, and systems that use impossible worlds often appear to be too weak. It seems to be intuitively plausible that a perfectly rational individual will not believe any contradiction. However, in systems based on some kind of impossible world semantics we cannot show this. Of course, our intuitions are not infallible, but if we can construct a system that is consistent with our intuitions such a system appears to be preferable to systems that have counterintuitive consequences. Some of the solutions also postulate various new kinds of entities that might be ontologically problematic, for example impossible objects and impossible worlds. The systems in this paper are both partly weaker and partly stronger than many standard systems. Several intuitively problematic sentences and arguments that are valid in standard systems are not valid in our systems, and several intuitively plausible sentences and arguments that are not valid in standard systems are valid in our systems. Consequently, we can avoid many problems with classical doxastic logic and with many other solutions to the problem of logical omniscience. This is a good reason to be interested in the results in this paper.
Reason 2: The problem of intuitively valid arguments. There are many arguments that are intuitively valid which cannot be proved in standard doxastic logic. Here is one example:

**The unmarried teacher argument**

Every student believes that if the teacher is a bachelor then the teacher is an unmarried man.

Every student believes that the teacher is a bachelor.

Susan is a student.

Hence,

If Susan is perfectly rational (wise), she believes that the teacher is an unmarried man.

This argument is intuitively valid. It seems to be impossible that the premises are true and the conclusion false. Informally, we can reason as follows. From the first and the third premise, it follows that Susan believes that if the teacher is a bachelor, then the teacher is an unmarried man. From the second and the third premise, it follows that Susan believes that the teacher is a bachelor. Hence, Susan believes both that if the teacher is a bachelor then the teacher is an unmarried man and that the teacher is a bachelor. If the contents of Susan's beliefs are true, it follows that the teacher is an unmarried man. Hence, if she is perfectly rational, she believes that the teacher is an unmarried man. Yet, in standard doxastic systems, we cannot prove that the unmarried teacher argument is valid. In standard doxastic logic, we cannot even quantify over believers in any natural way. In general, we cannot adequately symbolise the expressions ‘everyone who is such and such believes that’ and ‘someone who is such and such believes that’ in such systems. However, in every system in this paper, we can symbolise these phrases. In Section 7, I will prove that the conclusion follows from the premises in the unmarried teacher argument. Since there are countless other arguments of this kind, this is a good reason to be attracted to the systems in this paper.

Let us now consider an argument that is intuitively invalid.

**The conscientious student argument**

(1) Every student believes that if she studies hard she deserves a good grade.

(2) Every student believes that she studies hard.

Hence,

(3) Every student believes that she deserves a good grade.

If it is true that it is true that if \(x\) studies hard \(x\) deserves a good grade and it is true that \(x\) studies hard, then it is true that \(x\) deserves a good grade (for every \(x\)). However, if someone is not perfectly rational, she might not have thought about the matter and believe that if she studies hard then she deserves a good grade and also believe that she studies hard even though she does not believe that she deserves a good grade. Someone might be filled with ‘irrational’ self-doubt. So, the conscientious student argument is intuitively invalid. It seems to be possible that the premises are true and the conclusion false. In Section 7, I will show that the argument is invalid in the class of all models and I will verify this claim by constructing a countermodel. This example will
illustrate how we can use the semantic tableau method to generate countermodels and to prove that an argument (or sentence) is invalid.

Reason 3: The problem of economy. Finally, the systems in this paper solve the puzzles above in a rather parsimonious and conservative way. We do not have to introduce any new entities such as impossible worlds, and we do not have to abandon the modal paradigm completely. The results in this paper are conservative in the sense that every system is an extension of classical propositional logic: there are no truth-value gaps and there are no truth-value gluts; every (closed) sentence is either true or false (in a world) and there are no true contradictions. Moreover, the tableau rules for the possibilist quantifiers and the modal operators are classical. The primitive quantifiers are, in effect, used to quantify over absolutely everything, including merely possible objects (if there are any). So, in every possible world they vary over all possible objects, not only over all the things that happen to exist at this world. However, we also define a pair of actualist quantifiers that have existential import (see Definition 2.1). Furthermore, when it comes to the doxastic part of the systems, we do not have to abandon the modal paradigm completely. When the denotation of ‘a’ is not perfectly rational in a world, ‘B_a’ and ‘C_a’ behave as if they were ordinary predicates in this world; but when the denotation of ‘a’ is perfectly rational in a world, ‘B_a’ and ‘C_a’ behave as if they were ordinary modal operators in this world at this time. Consequently, if a is not perfectly rational, almost nothing of interest follows from the proposition that a believes something. However, if we assume that a is perfectly rational, we can derive all sorts of interesting consequences from this proposition. Exactly what it means to be perfectly rational and exactly what follows from the claim that a perfectly rational individual believes something seems to be a matter of choice, and in different systems we can derive different consequences.4

All in all, I conclude that we have very good reasons to be interested in the systems presented in this paper.

The paper is divided into seven main sections. Section 2 deals with the syntax and Section 3 with the semantics of our systems. In Section 4, I describe the proof theory of our logics and Section 5 includes some examples of theorems. Section 6 contains soundness and completeness proofs for every system and Section 7 includes two examples of derivations in doxastic logic.

2. Syntax

2.1. Alphabet

Terms

(i) A set of variables x_1, x_2, x_3 . . .
(ii) A set of constants (rigid designators) c_1, c_2, c_3 . . .

Predicates

(iii) For every natural number n > 0, n-place predicate symbols p_{1n}, p_{2n}, p_{3n} . . .
(iv) The monadic existence predicate $E$ and the monadic rationality predicate $R$. 
I will use $x$, $y$ and $z$ ... for arbitrary variables, $a$, $b$, $c$ ... for arbitrary constants, and $s$ and $t$ for arbitrary terms (with or without primes or subscripts). $F_n$, $G_n$, $H_n$ ... stand for arbitrary $n$-place predicates. The subscript will be omitted if it can be read off from the context.

### 2.2. Language

I will use the following language, $\mathcal{L}$, in this paper:

(i) Any constant or variable is a term.
(ii) If $t_1, \ldots, t_n$ are any terms and $P$ is any $n$-place predicate, $Pt_1 \ldots t_n$ is an atomic formula.
(iii) If $t$ is a term, $Et$ (‘$t$ exists’) is an atomic formula and $Rt$ (‘$t$ is perfectly rational [reasonable, wise]’) is an atomic formula.
(iv) If $s$ and $t$ are terms, then $s = t$ (‘$s$ is identical with $t$’) is an atomic formula.
(v) If $A$ and $B$ are formulas, so are $\neg A$, $(A \land B)$, $(A \lor B)$, $(A \rightarrow B)$ and $(A \leftrightarrow B)$.
(vi) If $A$ is a formula, so are $\forall A$ (‘it is universally [or absolutely] necessary that $A$’), $\exists A$ (‘it is universally [or absolutely] possible that $A$’), $\Box A$ (‘it is [relatively] necessary that $A$’) and $\Diamond A$ (‘it is [relatively] possible that $A$’).
(vii) If $A$ is any formula and $t$ is any term, then $B_t A$ (‘$t$ believes that [it is the case that] $A$’) and $C_t A$ (‘it is doxastically conceivable for $t$ that $A$’) are formulas.
(viii) If $A$ is any formula and $x$ is any variable, then $\Pi x A$ (‘for every [possible] $x$: $A$’) and $\Sigma x A$ (‘for some [possible] $x$: $A$’) are formulas.
(ix) Nothing else is a formula.
A, B, C, D . . . stand for arbitrary formulas, and Γ, Φ . . . for finite sets of closed formulas. The concepts of bound and free variables, and open and closed formulas, are defined in the usual way. \( (A)[t/x] \) is the formula obtained by substituting \( t \) for every free occurrence of \( x \) in \( A \). The definition is standard. Note that substitutions are performed also within the scope of the doxastic operators. Brackets around formulas are usually dropped if the result is not ambiguous.

**Definition 2.1 (ACTUALIST QUANTIFIERS):** \( \forall x A \) (‘for every existing \( x \) \( A \)) = \( \Pi x (Ex \rightarrow A) \) and \( \exists x A \) (‘for some existing \( x A \)) = \( \Sigma x (Ex \land A) \).

### 3. Semantics

#### 3.1. Models

**Definition 3.1 (MODEL):** A model \( \mathcal{M} \) is a relational structure \( \langle D, W, \mathcal{R}, \mathcal{D}, \nu \rangle \), where \( D \) is a non-empty set of individuals (the domain), \( W \) is a non-empty set of possible worlds, \( \mathcal{R} \) is a binary alethic accessibility relation (\( \mathcal{R} \) is a subset of \( W \times W \)), \( \mathcal{D} \) is a ternary doxastic accessibility relation (\( \mathcal{D} \) is a subset of \( D \times W \times W \)), and \( \nu \) is an interpretation function.

\( \mathcal{R} \) is used to define the truth conditions for sentences that begin with the alethic operators \( \Box \) and \( \Diamond \), and \( \mathcal{D} \) is used to define the truth conditions for sentences that begin with the doxastic operators \( B \) and \( C \). Informally, \( \mathcal{R}_{\omega \omega'} \) says that the possible world \( \omega' \) is alethically (relatively) accessible from the possible world \( \omega \), and \( \mathcal{D}_{\delta \omega \omega'} \) that the possible world \( \omega' \) is doxastically accessible to the individual \( \delta \) from the possible world \( \omega \), or that \( \delta \) can see \( \omega' \) from \( \omega \).

Every constant in our language is a kind of rigid designator, it refers to the same individual in every possible world. In other words, the valuation function \( \nu \) assigns every constant \( c \) an element \( \nu(c) \) of \( D \). The extension of a predicate, however, may change from world to world and it may be empty in a world, that is, \( \nu \) assigns every possible world \( \omega \) in \( W \) and \( n \)-place predicate \( P \) a subset \( \nu_\omega(P) \) (the extension of \( P \) in \( \omega \)) of \( D^n \). We shall say that \( \nu_\omega(P) \) is the set of \( n \)-tuples that satisfy \( P \) in the world \( \omega \).

The predicate \( R \) has a special meaning in our systems. ‘\( Rc \)’ says that \( c \) is **perfectly rational**, **perfectly reasonable** or **perfectly wise**. Exactly what these expressions mean can, of course, be debated. By imposing various conditions on the doxastic accessibility relation \( \mathcal{D} \) (Section 3.3), we obtain several different interpretations of the predicate \( R \). **Consistency** is one prima facie plausible condition, but we might also want to include some very strong properties in our concept of perfect wisdom (rationality), for example **infallibility** or **doxastic omniscience**. If \( \nu(c) \) is in the extension of \( R \) at the possible world \( \omega \), this means that \( \nu(c) \) is perfectly rational, reasonable or wise in \( \omega \). \( R \) functions as an ordinary predicate. Hence, an individual \( \delta \) may be in \( R \)'s extension in one possible world even though \( \delta \) is not in \( R \)'s extension in every possible world. Consequently, the fact that an individual \( \delta \) is perfectly rational, reasonable or wise in a possible world does not entail that \( \delta \) is perfectly rational, reasonable or wise in every possible world. We can, if we want, add the extra assumption that every perfectly rational individual is necessarily perfectly rational to any system in this paper (see the semantic condition \( C-UR \) (Table 4) in Section 3.3.4). \( R \) plays an important role in our systems, as should be
obvious from the definitions of the truth conditions for sentences of the forms $B_o A$ and $C_o A$ (see Section 3.2 below). In Section 4.2, we will see that $R$ also plays an important role in our various tableau rules.\footnote{Let $M$ be a model. Then the language of $M$, $L(M)$, is obtained by adding a constant $k_d$ such that $v(k_d) = d$ to the language for every member $d \in D$. Thus, every object in the domain of a model has at least one name in our language while several different constants may refer to one and the same object.}

The valuation function assigns extensions to so-called matrices. Given any closed doxastic formula of the form $B_i A$ or $C_i A$, we shall construct its matrix as follows. Let $m$ be the least number greater than every $n$ such that $x_n$ occurs bound in $A$. From left to right, replace every occurrence of an individual constant with $x_m, x_{m+1}, \text{etc.}$ The result is the formula’s matrix. The matrix of $B_c Pd$ is $B_0 x_1 Px_2$; the matrix of $B_c Pdd$ is $B_0 x_1 x_2 x_3$; the matrix of $C_d (F a \land G b c)$ is $C_0 x_1 (F x_2 \land G x_3 x_4)$; the matrix of $B_c \Pi x_1 (F x_1 \to G c)$ is $B_0 \Pi x_1 (F x_1 \to G x_3)$; the matrix of $B_c B_d \Sigma x_2 P x_2$ is $B_0 x_3 B_4 \Sigma x_2 x_3 x_2 P x_2$, etc. $A[a_1, \ldots, a_n/x_1, \ldots, x_n]$ is the result of replacing $x_1$ by $a_1$, and $\ldots$, and $x_n$ by $a_n$ in $A$. $A[a_1, \ldots, a_n/x_1, \ldots, x_n]$ will be abbreviated as $A[a_1, \ldots, a_n/x_1, \ldots, x_n]$. If $M$ is any matrix of the form $B_i A$ or $C_i A$ with free variables $x_1, \ldots, x_n$, then $v_\omega(M) \subseteq D^n$. Note that $M$ always includes at least one free variable.\footnote{Let $M$ be a matrix where $x_m$ is the first free variable in $M$ and $a_m$ is the constant in $M[a_1, \ldots, a_n/\overrightarrow{x}]$ that replaces $x_m$. Then the truth conditions for closed doxastic formulas of the form $M[a_1, \ldots, a_n/\overrightarrow{x}]$, when $v_\omega(R a_m) = 0$, are defined in terms of the extension of $M$ in $\omega$ (see condition (ii) in Section 3.2 below). $v_\omega(=) = \{(d, d) : d \in D\}$, i.e. the extension of the identity predicate is the same in every possible world (in a model). This means that all identities (and non-identities) are both absolutely and relatively necessary. The existence predicate $E$ functions as an ordinary predicate. The extension of this predicate may vary from one world to another. $Ec$ is true in a possible world intuitively means that $v(c)$ exists in this world.}

Let $M$ be a matrix where $x_m$ is the first free variable in $M$ and $a_m$ is the constant in $M[a_1, \ldots, a_n/\overrightarrow{x}]$ that replaces $x_m$. Then the truth conditions for closed doxastic formulas of the form $M[a_1, \ldots, a_n/\overrightarrow{x}]$, when $v_\omega(R a_m) = 0$, are given in (ii) below.

(i) $v_\omega(P d_1 \ldots a_n) = 1$ iff $\langle v(a_1), \ldots, v(a_n) \rangle \in v_\omega(P)$.

Let $M$ be a matrix where $x_m$ is the first free variable in $M$ and $a_m$ is the constant in $M[a_1, \ldots, a_n/\overrightarrow{x}]$ that replaces $x_m$. Then the truth conditions for closed doxastic formulas of the form $M[a_1, \ldots, a_n/\overrightarrow{x}]$, when $v_\omega(R a_m) = 0$, are given in (ii) below.

(ii) $v_\omega(M[a_1, \ldots, a_n/\overrightarrow{x}]) = 1$ iff $\langle v(a_1), \ldots, v(a_n) \rangle \in v_\omega(M)$.
(iii) $v_\omega(\neg A) = 1$ iff $v_\omega(A) = 0$.
(iv) $v_\omega(A \land B) = 1$ iff $v_\omega(A) = 1$ and $v_\omega(B) = 1$.
(v) $v_\omega(\cup A) = 1$ iff $\forall \omega' \in W: v_\omega'(A) = 1$. 

3.2. Truth conditions

Let us consider the truth conditions for some sentences in our language. ($1 = \text{True}$ and $0 = \text{False}; \forall \omega' \in W'$ is read as ‘for all possible worlds $\omega'$ in $W'$; and $\exists \omega' \in W'$ is read as ‘for some possible world $\omega'$ in $W'$. The truth conditions for the omitted sentences are standard.) We extend the interpretation function so that every closed formula $A$ is assigned exactly one truth-value $v_\omega(A)$ in each world $\omega$.

(i) $v_\omega(P d_1 \ldots a_n) = 1$ iff $\langle v(a_1), \ldots, v(a_n) \rangle \in v_\omega(P)$.
(vi) \(v_\omega([M]A) = 1 \text{ iff } \exists \omega' \in W: v_{\omega'}(A) = 1\).
(vii) \(v_\omega(\square A) = 1 \text{ iff } \forall \omega' \in W \text{ s.t. } R\omega\omega': v_{\omega'}(A) = 1\).
(viii) \(v_\omega(\langle A) = 1 \text{ iff } \exists \omega' \in W \text{ s.t. } R\omega\omega': v_{\omega'}(A) = 1\).
(ix) \(v_\omega(\Pi x A) = 1 \text{ iff for all } k_d \in L(M), v_\omega(A[k_d/x]) = 1\).
(x) \(v_\omega(\Sigma x A) = 1 \text{ iff for some } k_d \in L(M), v_\omega(A[k_d/x]) = 1\).
(xi) \(v_\omega(B_c A) = 1 \text{ iff for all } \omega' \text{ such that } D\omega(A)\omega': v_{\omega'}(A) = 1\), given that \(v(a)\) is an element in \(v_\omega(R)\), if \(v(a)\) is not an element in \(v_\omega(R)\), then \(B_c A\) is assigned a truth-value in \(\omega\) in a way that does not depend on the value of \(A\) (see condition (ii) above).
(xii) \(v_\omega(C_{a} A) = 1 \text{ iff for at least one } \omega' \text{ such that } D\omega(A)\omega': v_{\omega'}(A) = 1\), given that \(v(a)\) is an element in \(v_\omega(R)\), if \(v(a)\) is not an element in \(v_\omega(R)\), then \(C_{a} A\) is assigned a truth-value in \(\omega\) in a way that does not depend on the value of \(A\) (see condition (ii) above).

\(\Pi\) and \(\Sigma\) are substitutional quantifiers. Nevertheless, we can also call them ‘possibilist’ because they, in effect, vary over every object in the domain and the domain is the same in every possible world. Intuitively, conditions (xi) and (xii) mean the following. If \(v(a)\) is not perfectly rational in a possible world, \(B_c A\) and \(C_{a} A\) behave as if they are ordinary predicates in this world; and if \(v(a)\) is perfectly rational in a possible world, \(B_c\) and \(C_{a}\) behave as ordinary modal operators in this world.

Here is an example to help explain condition (ii) above. Consider the closed doxastic formula \(B_c P_{x^3}\). The matrix of this formula is \(B_{x_1} P_{x_2 x_3}\) note that the first occurrence of \(c\) is replaced by \(x_1\) and the second by \(x_2\). \(B_{x_1} P_{x_2 x_3}[c, c, d/x_1, x_2, x_3] = B_c P_{x_3}\). \(x_1\) is the first free variable in the matrix \(B_{x_1} P_{x_2 x_3}\) and \(c\) is the constant in \(B_{x_1} P_{x_2 x_3}[c, c, d/x_1, x_2, x_3]\) that replaces \(x_1\). Hence, by condition (ii) above, if \(v_\omega(Rc) = 0\), then \(v_\omega(B_{x_1} P_{x_2 x_3}[c, c, d/x_1, x_2, x_3]) = 1\) just in case \(\langle v(c), v(c), v(d) \rangle \in v_\omega(B_{x_1} P_{x_2 x_3})\). Suppose that \(v_\omega(Rc) = 0\). Then, \(v_\omega(B_{x_1} P_{x_2 x_3}[c, c, d/x_1, x_2, x_3]) = 1\) iff \(\langle v(c), v(c), v(d) \rangle \in v_\omega(B_{x_1} P_{x_2 x_3})\). The interpretation function \(v\) assigns extensions to matrices in possible worlds. So, \(\langle v(c), v(c), v(d) \rangle\) is either in the extension of \(B_{x_1} P_{x_2 x_3}\) in \(\omega\) or not. Suppose that \(\langle v(c), v(c), v(d) \rangle \in v_\omega(B_{x_1} P_{x_2 x_3})\). Then, \(v_\omega(B_{x_1} P_{x_2 x_3}[c, c, d/x_1, x_2, x_3]) = 1\). Since, \(B_{x_1} P_{x_2 x_3}[c, c, d/x_1, x_2, x_3] = B_c P_{x_3}\), this means that \(v_\omega(B_c P_{x_3}) = 1\), that is, that \(B_c P_{x_3}\) is true in \(\omega\). Suppose, instead that it is not the case that \(\langle v(c), v(c), v(d) \rangle \in v_\omega(B_{x_1} P_{x_2 x_3})\). Then, it is not the case that \(v_\omega(B_{x_1} P_{x_2 x_3}[c, c, d/x_1, x_2, x_3]) = 1\). Since, \(B_{x_1} P_{x_2 x_3}[c, c, d/x_1, x_2, x_3] = B_c P_{x_3}\), it is not the case that \(v_\omega(B_c P_{x_3}) = 1\), that is, \(B_c P_{x_3}\) is false in \(\omega\).

We are now in a position to define some important semantic concepts.

**Definition 3.2**: 
(i) (SATISFIABILITY IN A MODEL). A set of sentences is satisfiable in a model \(M\) iff there is a world in \(M\) in which every element of this set is true.
(ii) (VALIDITY IN A CLASS OF MODELS). A sentence \(A\) is valid in a class of models \(M\) iff \(A\) is true in every world in every model in \(M\).
(iii) (LOGICAL CONSEQUENCE IN A CLASS OF MODELS). A sentence \(B\) is a logical consequence of a set of sentences \(\Gamma\) in a class of models \(M\) \((M, \Gamma \models B)\) iff for every model \(M\) in \(M\) and world \(\omega\) in \(M\) if all elements of \(\Gamma\) are true in \(\omega\) in \(M\), then \(B\) is true in \(\omega\) in \(M\). If \(M, \Gamma \models B\), we also say that \(\Gamma\) entails \(B\) in \(M\) and that
3.3. Conditions on models

In this section, I will introduce some conditions that can be used to classify different kinds of models. The conditions concern the formal properties of the accessibility relations, the relationships between the various accessibility relations and the relationships between the accessibility relations and the valuation function. Table 1 includes information about the alethic accessibility relation. The conditions in this table are mentioned in most introductions to modal logic (see, for example, Blackburn, de Rijke, & Venema, 2001; Chellas, 1980; Fitting & Mendelsohn, 1998; Garson, 2006; Priest, 2008). Given almost any concept of necessity, it is reasonable to assume that the alethic accessibility relation is an equivalence relation and, hence, that it satisfies all conditions in Table 1. Some clauses (C—dT, C—dD and C—d4) in Table 2 have been mentioned in the literature (Fitting, Thalmann, & Voronkov, 2001). All other conditions are new.

By combining the clauses in this section in various ways, we can generate many different doxastic systems. Exactly which conditions we should accept will depend on what we mean or should mean by ‘perfectly rational’ (reasonable, wise). There might be good reasons to accept all (or almost all) conditions in this section. Having said that, it might also be interesting to see what follows if we accept some smaller class. The more conditions we accept, the more content we pack into the concept of rationality (wisdom). Perhaps we should distinguish between different concepts of rationality and talk about ‘rationality1’, ‘rationality2’, etc., and not just about ‘rationality’. Then, we can use different systems to explicate these different notions. Even though the conditions in this section should be more or less self-explanatory, I have added a few comments about some of the new clauses. There are many interesting relationships between the various conditions that I do not have space to discuss in this paper. I will, however, mention a few connections. Some combinations may be more philosophically interesting than others. Some combinations might be philosophically implausible.

| Table 1. Conditions on the relation R. |
|--------------------------------------|
| Condition | Formalisation of condition |
| C—dT | ∀ωRωω |
| C—aD | ∀ω∃ω′Rωω′ |
| C—aB | ∀ω∃ω′(Rωω′ → Rωω′) |
| C—dA | ∀ω∀ω′∀ω′′(Rωω′ ∧ Rωω′′ → Rωω′′) |
| C—a5 | ∀ω∀ω′∀ω′′(Rωω′ ∧ Rωω′′ → Rωω′′) |

| Table 2. Conditions on the relation D. |
|--------------------------------------|
| Condition | Formalisation of condition |
| C—dO | ∀ψ∀ωω′(Dωω′ → ω = ω′) |
| C—dT | ∀ψ∀ωDωω |
| C—dD | ∀ψ∀ωω′(Dωω′) |
| C—dB | ∀ψ∀ωω′(Dωω′ → Dωω′) |
| C—d4 | ∀ψ∀ωω′ω′′(Dωω′ ∧ Dωω′′ → Dωω′′) |
| C—d5 | ∀ψ∀ωω′ω′′(Dωω′ ∧ Dωω′′ → Dωω′′) |
3.3.1. Conditions on the relation \( R \)

3.3.2. Conditions on the relation \( D \)

Some of the conditions in Table 2 are similar to the conditions in Table 1, and to some well-known clauses that are often used in epistemic and doxastic logic. Nevertheless, there are also some important differences. \( R \) is a binary relation, while \( D \) is a ternary relation. ‘C’ in ‘\( C\rightarrow D \)’ stands for ‘condition’ and ‘\( d \)’, for ‘doxastic’. \( C\rightarrow D \) is called ‘\( C\rightarrow D \)’ because it is similar to the well-known condition \( D \) (as in ‘Deontic’) in ordinary alethic (modal) logic. Similar remarks apply to the other clauses in this section. If it is clear from the context that we are talking about a semantic condition, I will often omit the initial \( C \).

The (semantic) condition of (doxastic) omniscience. According to condition \( C\rightarrow O \), it holds that for every (individual) \( \delta \) and for all (possible worlds) \( \omega \) and \( \omega' \): if \( \omega' \) is doxastically accessible from \( \omega \) to \( \delta \), then \( \omega \) is identical to \( \omega' \). In other words, in every world there is at most one possible world that is doxastically accessible from this world (to an individual), namely this world itself. This condition corresponds to the tableau rule \( T\rightarrow O \) in Table 10. In any model that satisfies this condition, the principle of (doxastic) omniscience, \( \Pi x(Rx \rightarrow (A \rightarrow BxA)) \) (Table 17), which says that every perfectly rational (wise) individual believes everything that is true, is valid. It is not plausible to assume that actual human beings have access to every fact whatsoever, but it might be reasonable to assume that perfectly wise individuals are different. ‘\( O \)’ in ‘\( C\rightarrow O \)’ stands for ‘doxastic omniscience’ since it is reasonable to call an individual who believes everything that is true doxastically omniscient.

The (semantic) condition of infallibility. \( C\rightarrow T \) says that the doxastic accessibility relation is ‘reflexive’: every possible world is doxastically accessible from itself (for every individual). This condition corresponds to the tableau rule \( T\rightarrow T \) in Table 10. In any model that satisfies this condition, the principle of infallibility, \( \Pi x(Rx \rightarrow (BxA \rightarrow A)) \) (Table 17), is valid. This principle says that everything a perfectly wise individual believes is true. We call \( C\rightarrow T \) the ‘infallibility condition’ since it is reasonable to say that an individual who does not believe anything that is false is infallible. Again, this condition does not entail that no one has any false beliefs. We can only conclude that no one who is perfectly rational has any false beliefs.

Together \( C\rightarrow O \) and \( C\rightarrow T \) say that every agent has doxastic access to the world where she is and to no other world. If we accept both of these conditions, we can show that every perfectly rational individual is both omniscient and infallible, that is, that the following schema is valid \( \Pi x(Rx \rightarrow (BxA \leftrightarrow A)) \). This formula says that every perfectly rational being believes that \( A \) if and only if \( A \). Note that the following sentences are not valid even if we assume both \( C\rightarrow O \) and \( C\rightarrow T \): \( \Pi x(A \rightarrow BxA) \) (everyone believes everything that is true), \( \Pi x(BxA \rightarrow A) \) (everyone is infallible). \( \Pi x(A \rightarrow BxA) \) and \( \Pi x(BxA \rightarrow A) \) are usually not accepted in doxastic logic. Nonetheless, it might be interesting to note that something like \( \Pi x(Rx \rightarrow (BxA \rightarrow A)) \) has a long tradition in philosophy. The Stoics seem to have accepted this proposition, or something very similar. Thus, according to Diogenes Laertius, the Stoics thought that

\[ \] the wise man will never form mere opinion, that is to say, he will never give assent to anything false\ldots. the wise are infallible, not being liable to error. (See Diogenes Laertius (1925), Book VII, Zeno, p. 227)
The concept of doxastic omniscience is a neglected topic in the literature on epistemic and doxastic logic. It is an interesting concept that deserves further investigation. Some might think it is obvious that we should include the notion in the concept of perfect wisdom (rationality); otherwise, how could we call this wisdom perfect? Others might think that it is obvious that perfect rationality does not entail doxastic omniscience since rationality only has to do with internal consistency among beliefs, or because they believe that this condition (together with some other plausible thesis) has unreasonable consequences. ‘Facts’ do not entail anything about beliefs (not even the beliefs about a perfectly rational or wise individual), and beliefs do not entail anything about ‘facts’. If this is true, we should neither accept the condition of doxastic omniscience nor the condition of infallibility. On the other hand, is it not reasonable to call a being that is infallible and omniscient wiser than a being who is not? If it is, how can the latter be perfectly wise (rational)? In the next section, I will consider to call a being that is infallible and omniscient wiser than a being who is not? If it is, how can the latter be perfectly wise (rational)? In the next section, I will consider a ‘weaker’ condition of omniscience, the so-called semantic condition of necessity-omniscience (Section 3.3.3). I will also consider a weaker form of ‘infallibility’, so-called possibility-infallibility.8

3.3.3. Conditions concerning the relation between \( \mathbb{R} \) and \( \mathcal{D} \)

The conditions in Table 3 concern the relationship between the doxastic accessibility relation and the alethic accessibility relation. They correspond to the tableau rules in Table 13.

‘MB’ in ‘C – MB’ stands for ‘Must Belief’, and ‘BP’ in ‘C – BP’ for ‘Belief Possibility’. C – ad4 (as in ‘alethic doxastic 4’) is called ‘C – ad4’ because it is similar to the well-known alethic (modal) condition C – 4, and similarly for C – ad5. ‘CMP’ in ‘C – CMP’ is an abbreviation of ‘Conceivability Must Permutation’, and ‘BMP’ in ‘C – BMP’ is an abbreviation of ‘Belief Must Permutation’.

The (semantic) condition of (doxastic) necessity-omniscience. C – MB says that for every (individual) \( \delta \), for every (possible world) \( \omega \) and for every (possible world) \( \omega' \), \( \omega' \) is doxastically accessible from \( \omega \) (to \( \delta \)) only if \( \omega' \) is alethically accessible from \( \omega \). In other words, if C – MB holds, then it is not the case that \( \delta \) can see \( \omega' \) from \( \omega \) if \( \omega' \) is not alethically accessible from \( \omega \). C – MB corresponds to the tableau rule T – MB in Table 13.

In every class of models that satisfies this condition, the principle of (doxastic) necessity-omniscience, \( \Pi x(\mathbb{R}x \rightarrow (\Box A \rightarrow Bx A)) \), which says that every perfectly wise individual believes every necessary truth, is valid. If we assume C – MB, we can also establish the validity of the principle of consequence-consistency, \( \Pi x(\mathbb{R}x \rightarrow ((Bx A \land \Box (A \rightarrow B)) \rightarrow Bx B)) \) (Table 20), which says that every perfectly wise person believes every necessary implication of the things she believes. So, this condition is philosophically

| Condition | Formalisation of condition |
|-----------|-----------------------------|
| C – MB    | \( \forall \delta \forall \omega' (\Box \omega' \rightarrow \omega') \) |
| C – BP    | \( \forall \delta \forall \omega' (\Box \omega \land \omega') \) |
| C – ad4   | \( \forall \delta \forall \omega' (\Box \omega \land \Box \omega') \rightarrow \omega' \) |
| C – ad5   | \( \forall \delta \forall \omega' (\Box \omega \land \Box \omega') \rightarrow \omega' \) |
| C – CMP   | \( \forall \delta \forall \omega' (\Box \omega \land \Box \omega') \rightarrow \omega' \) |
| C – BMP   | \( \forall \delta \forall \omega' (\Box \omega \land \Box \omega') \rightarrow \omega' \) |

Table 3. Conditions concerning the relation between \( \mathbb{R} \) and \( \mathcal{D} \).
The concept of necessity-omniscience is ‘weaker’ than the concept of omniscience. If someone believes everything that is true, she believes everything that is necessarily true, but the converse does not hold. It is possible to believe everything that is necessarily true without believing everything that is (in fact) true (given that there are things that are true but not necessarily true).

The (semantic) condition of possibility-infallibility. According to $C - BP$, for every (individual) $\delta$, there is for every (possible world) $\omega$ a (possible world) $\omega'$ such that $\delta$ can see $\omega'$ from $\omega$ and $\omega'$ is alethically accessible from $\omega$. In other words, in every possible world, there is at least one possible world that is alethically and doxastically accessible (to $\delta$). $C - BP$ corresponds to the tableau rule $T - BP$ in Table 13. This condition is similar to condition $C - dD$ (Table 2). $C - BP$ entails $C - dD$, but $C - dD$ (in itself) does not entail $C - BP$. In every class of models that satisfy this condition, the principle of possibility-infallibility is valid: $\Pi x (Rx \rightarrow (B_x A \rightarrow \Diamond A))$ ('For every $x$: if $x$ is perfectly rational, then $x$ believes that it is the case that $A$ only if $A$ is possible'). In other words, according to this condition, a perfectly wise individual does not believe anything impossible. This is an intuitively interesting principle. If $c$ believes something that is impossible, $c$’s belief will inevitably be false. $C - BP$ is also similar to the semantic condition of infallibility. If everything a person believes is true, then everything she believes is possible (given that everything that is true is possible). However, the converse does not hold. From the fact that everything a person believes is possible, it does not follow that everything she believes is (in fact) true (given that there are things that are possible but that are not true). So, the concept of possibility-infallibility is ‘weaker’ than the concept of infallibility.

The (semantic) condition of the necessity of beliefs. $C - ad4$ says that for every (individual) $\delta$, for every (possible worlds) $\omega$, $\omega'$ and $\omega''$, if $\omega'$ is alethically accessible from $\omega$ and $\omega''$ is doxastically accessible from $\omega'$ to $\delta$, then $\omega''$ is doxastically accessible from $\omega$ to $\delta$. $C - ad4$ corresponds to the tableau rule $T - ad4$ in Table 13. In the class of models that satisfy this condition (and $C - UR$ or $C - FTR$ in Table 4), the principle of the necessity of beliefs holds, $\Pi x (Rx \rightarrow (B_x A \rightarrow \Box B_x A))$, which says that if a perfectly wise individual believes something, it is (relatively) necessary that she believes it. Note that the following formula is not valid, even if we assume $C - ad4$, $\Pi x (Rx \rightarrow (B_x A \rightarrow \bigcup B_x A))$. It is still possible that a perfectly rational being believes something without it being the case that it is absolutely necessary that she believes it.

The (semantic) condition of the necessity of non-beliefs. According to $C - ad5$, for every (individual) $\delta$, for every (possible worlds) $\omega$, $\omega'$ and $\omega''$, if $\omega'$ is alethically accessible from $\omega$ and $\omega''$ is doxastically accessible from $\omega$ to $\delta$, then $\omega''$ is doxastically accessible from $\omega'$ to $\delta$. $C - ad5$ corresponds to the tableau rule $T - ad5$ in Table 13. In the class of models that satisfy this condition (and $C - UR$ or $C - FTR$ in Table 4), the principle of the necessity of non-beliefs holds, $\Pi x (Rx \rightarrow (\neg B_x A \rightarrow \Box \neg B_x A))$, which says that if a perfectly wise individual does not believe something, it is (relatively) necessary that she

| Condition | Formalisation of condition |
|-----------|---------------------------|
| $C - FTR$ | $\forall \omega_1 \forall \omega_2 \forall c ((\Box \omega_1 \omega_2 \land (v_{\omega_1} ( Rc ) = 1)) \Rightarrow v_{\omega_2} ( Rc ) = 1)$. |
| $C - UR$  | $\forall \omega_1 \forall \omega_2 \forall c ((v_{\omega_1} ( Rc ) = 1) \Rightarrow (v_{\omega_2} ( Rc ) = 1))$. |
does not believe it. We can also show that the principle of the necessity of conceivable
is valid in this class of models, \( \Pi x(Rx \rightarrow (CxA \rightarrow \Box CxA)) \), which says that if something
is (doxastically) conceivable to a perfectly wise individual, then it (relatively) necessary
that it is conceivable to her. Note that the following formula is not valid, even if
we assume \( C \rightarrow \text{ad5} \), \( \Pi x(Rx \rightarrow (\neg BxA \rightarrow \cup BxA)) \). It is still possible that a perfectly
rational being does not believe something without it being the case that it is absolutely
necessary that she does not believe it. Nor is the following formula a theorem
\( \Pi x(Rx \rightarrow (CxA \rightarrow \cup CxA)) \).

Other (semantic) conditions. \( C \rightarrow \text{CMP} \) corresponds to the tableau rule \( T \rightarrow \text{CMP} \) in
Table 13 and \( C \rightarrow \text{BMP} \) to the tableau rule \( T \rightarrow \text{BMP} \) in the same table. In the class of
models that satisfy \( C \rightarrow \text{CMP} \) (and \( C \rightarrow \text{UR} \) or \( C \rightarrow \text{FTR} \) in Table 4), we can show that the
following sentence is valid \( \Pi x(Rx \rightarrow (CxA \rightarrow \Box CxA)) \); and in the class of models that
satisfy \( C \rightarrow \text{BMP} \) (and \( C \rightarrow \text{UR} \) or \( C \rightarrow \text{FTR} \) in Table 4) we can show that the following formula
is valid \( \Pi x(Rx \rightarrow (BxA \rightarrow \neg BxA)) \). These sentences are kinds of permutation
principles.

According to the condition of omniscience as well as the condition of necessity-
omniscience (in combination with infallibility), wisdom includes more than just inner
consistency or perfect reasoning skills; it also involves correct ‘factual’ beliefs or at
least correct beliefs about ‘facts’ that are (relatively) necessary. For example, you cannot
be truly wise without having true beliefs about necessary conditions for various
important ends. If we want to pack such ‘factual’ beliefs (knowledge) into the con-
cept of wisdom, it might be plausible to accept those conditions. If we want a weaker
notion of wisdom or rationality, we can omit them and concentrate on conditions
that have to do with the consistency of an individual’s beliefs; we can, for example,
accept condition \( C \rightarrow \text{dD} \). As already pointed out, it might be the case that we
want to use different analyses of the concept of wisdom (rationality) for different
purposes.

3.3.4. Conditions on the valuation function \( v \) in a model
The semantic conditions \( C \rightarrow \text{FTR} \) and \( C \rightarrow \text{UR} \) (Table 4) correspond to the tableau rules
\( T \rightarrow \text{FTR} \) and \( T \rightarrow \text{UR} \) (Table 12), respectively. It follows from \( C \rightarrow \text{UR} \) that every perfectly
rational individual is necessarily perfectly rational (See Section 4.2.8, Table 12, for more
on this).

Finally, to show how one can justify the claims in this section, I will verify the proposi-
tion that the principle of (doxastic) necessity-omniscience, \( \Pi x(Rx \rightarrow (\Box A \rightarrow BxA)) \),
is valid in the class of all models that satisfy \( C \rightarrow MB \). Suppose that \( \Pi x(Rx \rightarrow (\Box A \rightarrow BxA)) \) is
not valid in the class of all models that satisfy \( C \rightarrow MB \). Then \( \Pi x(Rx \rightarrow (\Box A \rightarrow BxA)) \)
is false in some possible world \( \omega \) in some model \( M \) in this class. Hence, for some
c: \( Rc \rightarrow (\Box A \rightarrow BxA) \) is false in \( \omega \) in \( M \). Accordingly, \( Rc \) is true in \( \omega \) in \( M \), \( \Box A \) is true in
\( \omega \) in \( M \) and \( BxA \) is false in \( \omega \) in \( M \). Since \( BxA \) is false in \( \omega \) in \( M \) and \( Rc \) is true in \( \omega \) in \( M \),
there is a possible world \( \omega ' \) in \( M \) that is doxastically accessible from \( \omega \) to \( c \) in \( M \) in
which \( A \) is false. It follows that \( \omega ' \) is alethically accessible from \( \omega \) in \( M \), for \( \omega ' \) is dox-
astically accessible from \( \omega \) to \( c \) in \( M \) and \( M \) satisfies \( C \rightarrow MB \). Therefore, \( A \) is true in \( \omega ' \) in
\( M \), since \( \Box A \) is true in \( \omega \) in \( M \) and \( \omega ' \) is alethically accessible from \( \omega \) in \( M \). But this is
absurd. Hence, our assumption cannot be true. It follows that \( \Pi x(Rx \rightarrow (\Box A \rightarrow BxA)) \)
is valid in the class of all models that satisfy \( C \rightarrow MB \). Q.E.D.
3.4. Model classes and the logic of a class of models

We can use the conditions mentioned in Section 3.3 to obtain a classification of the set of all models into various kinds. Let $M(C_1, \ldots, C_n)$ be the class of (all) models that satisfy the conditions $C_1, \ldots, C_n$. Then, $M(C - dT, C - d4, C - d5)$ is the class of (all) models that satisfy the conditions $C - dT, C - d4$ and $C - d5$, etc.

We can say that the set of all sentences in a language that are valid in a class of models $M$ is the (logical) system of $M$, or the logic of $M$, in symbols $S(M)$. For example, $S(M(C - dT, C - d4, C - d5))$ (the system of $M(C - dT, C - d4, C - d5)$) is the class of sentences (in our language) that are valid in the class of (all) models that satisfy the conditions $C - dT, C - d4$ and $C - d5$.

We can define a large set of systems by using this classification of model classes. In the next section, I will introduce a set of semantic tableau systems that exactly correspond to these semantically defined logics.

4. Proof theory

4.1. Semantic tableaux

In Section 4, I will introduce a set of tableau rules. Then, I will show how these rules can be used to construct a large set of non-equivalent tableau systems. All systems are extensions of ordinary propositional logic. All systems also include rules for a pair of ‘possibilist’ quantifiers, a pair of ‘absolute’ necessity operators and a pair of ‘relative’ necessity operators. The propositional part of the systems is similar to systems introduced by Jeffrey (1967) and Smullyan (1968), and the modal part is similar to systems discussed by, among others, Priest (2008). For more information about the tableau method and various kinds of tableau systems, see, for example, D’Agostino, Gabbay, Hähnle, and Posegga (1999) and Fitting and Mendelsohn (1998).

4.2. Tableau rules

In this section, I will introduce a set of tableau rules that can be used to construct a large set of tableau systems (Section 4.3). They should be more or less self-explanatory. Nevertheless, I will briefly discuss some of the new rules.

4.2.1. Propositional rules

see Table 5

4.2.2. Basic alethic rules (ba-rules)

see Table 6

4.2.3. Basic doxastic rules (bd-rules)

see Table 7

Intuitively, ‘$Rc, i$’ in the doxastic rules says that the individual denoted by ‘$c$’ is perfectly rational, reasonable or wise in the possible world denoted by ‘$i$’, and ‘$iDcj$’ says that the possible world denoted by ‘$j$’ is doxastically accessible to the individual
Table 5. Propositional rules.

| ¬¬ | ∧  | ¬∧  |
|----|----|-----|
| ¬¬A, i | (A ∧ B), i | ¬(A ∧ B), i |
| ↓   | ↓     | ¬A, i |
| A, i | A, i  | ¬A, i |
|   | B, i  | ¬B, i |
| ∨   | ¬∨    | →   |
| (A ∨ B), i | ¬(A ∨ B), i | (A → B), i |
| ↓   | ↓     | ¬A, i |
| A, i B, i | ¬A, i  | ¬A, i B, i |
|   | ¬B, i | ¬B, i |
| ¬ → | ↔    | ¬↔  |
| ¬(A → B), i | (A ↔ B), i | ¬(A ↔ B), i |
| ↓   | ↑     | ¬A, i |
| A, i A, i | ¬A, i  | ¬A, i |

Table 6. Basic alethic rules.

| U | M | □ | ◊ |
|---|---|---|---|
| ¬U | ¬M | ¬□ | ¬◊ |
| ¬¬U, i | ¬¬M, i | ¬¬□, i | ¬¬◊, i |
| ↓   | ↓     | iRj   | ↓     |
| A, i A, i | ¬A, i  | ¬A, i |
| for any j where j is new A, j A, j |
| ¬U   | ¬M    | ¬□    | ¬◊   |

Table 7. Basic doxastic rules.

| B  | C  | ¬B  | ¬C  |
|----|----|-----|-----|
| Rc, i | Rc, i | Rc, i | Rc, i |
| BcA, i | CcA, i | ¬BcA, i | ¬CcA, i |
| iDcj | iDcj | Cc¬A, i | Bc¬A, i |
| ↓   | ↓     | ↓     | ↓     |
| A, j | A, j  | A, j  | A, j  |
| where j is new |

denoted by ‘c’ in the possible world denoted by ‘i’. The basic doxastic rules hold for every constant c (i.e. c can be replaced by any constant in these rules).

4.2.4. Possibilist quantifiers

see Table 8

Note that a and c in the quantifier rules are rigid constants—we never instantiate with variables; a is any constant on the branch and c is a constant new to the branch.

4.2.5. Alethic accessibility rules (a-rules)

The alethic accessibility rules in Table 9 correspond to the semantic conditions in Table 1.
Table 8. Quantifier rules.

| $\Pi xA, i$ | $\Sigma xA, i$ | $\neg \Pi xA, i$ | $\neg \Sigma xA, i$ |
|-------------|----------------|-----------------|-----------------|
| $\downarrow$ | $\downarrow$ | $\downarrow$ | $\downarrow$ |
| $\downarrow$ | $\downarrow$ | $\downarrow$ | $\downarrow$ |

for every constant $a$ where $c$ is new on the branch, a new if there are no constants on the branch.

4.2.6. **Doxastic accessibility rules (d-rules)**

The doxastic accessibility rules in Table 10 correspond to the semantic conditions in Table 2.

4.2.7. **The CUT-rule (CUT), (CUTR) and the (world) identity rules**

We could use a more restricted CUT rule, CUTR, where ‘A’ in CUT is replaced by ‘$Rc$’ where $c$ is a constant (that occurs as an index to some doxastic operator) on the branch. In fact, in the completeness proofs we will use CUTR and not CUT. Yet, CUT is often more useful in proving many theorems and derived rules.

There are two identity rules: $T \rightarrow ldI$ and $T \rightarrow ldII$ (both abbreviated ld). $\alpha(i)$ is a line in a tableau that includes ‘i’, and $\alpha(j)$ is like $\alpha(i)$ except that ‘i’ is replaced by ‘j’. That is, if $\alpha(i)$ is A, i, then $\alpha(j)$ is A, j; if $\alpha(i)$ is $kRi$, then $\alpha(j)$ is $kRj$; if $\alpha(i)$ is $i = k$, then $\alpha(j)$ is $j = k$, etc. If $\alpha(i)$ is A, i we only apply the rule when A is atomic or of the form $B_i A$ or $C_i A$ given that $\neg R$, $i$ is on the branch.

Table 9. Alethic accessibility rules.

| $T \rightarrow dD$ | $T \rightarrow dT$ | $T \rightarrow dB$ | $T \rightarrow d4$ | $T \rightarrow d5$ |
|-------------------|-------------------|-------------------|-------------------|-------------------|
| $i$               | $i$               | $iRj$             | $iRj$             | $iRj$             |
| $\downarrow$      | $\downarrow$      | $\downarrow$      | $\downarrow$      | $\downarrow$      |
| $iRj$             | $iRi$             | $jRk$             | $iRk$             | $jRk$             |
| where $j$ is new  |                   |                   |                   |                   |

Table 10. Doxastic accessibility rules.

| $T \rightarrow dO$ | $T \rightarrow dT$ | $T \rightarrow dB$ | $T \rightarrow dD$ | $T \rightarrow d4$ | $T \rightarrow d5$ |
|-------------------|-------------------|-------------------|-------------------|-------------------|-------------------|
| $iDcj$            | $iDcj$            | $iDcj$            | $iDcj$            | $iDcj$            | $iDcj$            |
| $\downarrow$      | $\downarrow$      | $\downarrow$      | $\downarrow$      | $\downarrow$      | $\downarrow$      |
| $iDci$            | $jDci$            | $iDcj$            | $jDck$            | $iDck$            | $jDck$            |
| where $j$ is new  |                   |                   |                   |                   |                   |

Table 11. CUT and (world) identity rules.

| CUT               | $T \rightarrow ldI$ | $T \rightarrow ldII$ |
|-------------------|----------------------|-----------------------|
| $\alpha(i)$       | $\alpha(i)$          | $\alpha(i)$          |
| $\neg R$, $i$     | $\alpha(j)$          | $\alpha(j)$          |
| for every $A$ and $i$ | $\alpha(j)$          | $\alpha(j)$          |
\textit{CUTR} (or \textit{CUT}), \textit{T−IdI} and \textit{T−IdII} are included in every system in this paper. Still, in every system that does not include \textit{T−dO}, the identity rules are redundant and can, in principle, be omitted.

4.2.8. Transfer rules, etc.

‘FT’ in ‘\textit{T−FTR}’ is an abbreviation of ‘Forward Transfer’, and ‘R’ in ‘\textit{T−FTR}’ and ‘\textit{T−UR}’ of ‘Rationality’. The tableau rules in Table 12 correspond to the semantic conditions in Table 4.

In every system that includes \textit{T−FTR} and \textit{T−MB} (Table 13), we can prove that the following sentence is a theorem \( \Pi x (Rx \rightarrow B_x Rx) \), which says that everyone who is perfectly rational (wise) believes that she is perfectly rational (wise).

If a system includes \textit{T−FTR}, \textit{T−MB} and \textit{T−dD} (Table 10), we can prove that \( \Pi x (Rx \rightarrow C_x Rx) \) is a theorem in this system. This formula says that everyone who is perfectly rational is such that it is conceivable to her that she is perfectly rational.

In every system that includes \textit{T−UR}, we can prove the following sentence \( \Pi x (Rx \rightarrow \Box Rx) \), which says that every perfectly rational (wise) individual is necessarily perfectly rational (wise).

The transfer rules (the rules in Table 12) are not included in every system. Whether they should be added seems to be a matter of choice. In standard doxastic systems, that every believer is perfectly rational is usually built into the logic. The systems in the present paper are, therefore, more flexible.

4.2.9. Alethic-doxastic accessibility rules (ad-rules)

The alethic-doxastic rules in Table 13 correspond to the semantic conditions introduced in Table 3.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|c|c|}
\hline
\textbf{Table 12. Transfer rules.} \\
\textit{T−FTR} & \textit{T−UR} \\
\hline
\textit{Rc} & \textit{Rc} & \textit{i} & \textit{i} & \\
\hline
\textit{idR} & \textit{idR} & \textit{iR} & \textit{iR} & \textit{iDc} & \textit{iDc} & \textit{iDc} & \textit{iDc} \\
\hline
\textit{Rc}, \textit{j} & \textit{Rc}, \textit{j} & \textit{Rc}, \textit{j} & \textit{Rc}, \textit{j} & \textit{Rc}, \textit{j} & \textit{Rc}, \textit{j} & \textit{Rc}, \textit{j} & \textit{Rc}, \textit{j}  \\
\hline
\end{tabular}
\end{table}

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|}
\hline
\textbf{Table 13. Alethic-doxastic accessibility rules.} \\
\textit{T−MB} & \textit{T−BP} \\
\hline
\textit{iDcj} & \textit{i} & \textit{i} & \textit{i} & \textit{i} & \textit{i}  \\
\textit{idR} & \textit{idR} & \textit{idR} & \textit{idR} & \textit{idR} & \textit{idR} & \textit{idR} & \textit{idR} & \textit{idR} \\
\hline
\textit{Rc} & \textit{Rc} & \textit{Rc} & \textit{Rc} & \textit{Rc} & \textit{Rc} & \textit{Rc} & \textit{Rc} & \textit{Rc} \\
\hline
\textit{Rc}, \textit{i} & \textit{Rc}, \textit{i} & \textit{Rc}, \textit{j} & \textit{Rc}, \textit{j} & \textit{Rc}, \textit{j} & \textit{Rc}, \textit{j} & \textit{Rc}, \textit{j} & \textit{Rc}, \textit{j} & \textit{Rc}, \textit{j}  \\
\hline
\textit{Rc}, \textit{j} & \textit{Rc}, \textit{j} & \textit{Rc}, \textit{j} & \textit{Rc}, \textit{j} & \textit{Rc}, \textit{j} & \textit{Rc}, \textit{j} & \textit{Rc}, \textit{j} & \textit{Rc}, \textit{j} & \textit{Rc}, \textit{j}  \\
\hline
\textit{Rc}, \textit{j} & \textit{Rc}, \textit{j} & \textit{Rc}, \textit{j} & \textit{Rc}, \textit{j} & \textit{Rc}, \textit{j} & \textit{Rc}, \textit{j} & \textit{Rc}, \textit{j} & \textit{Rc}, \textit{j} & \textit{Rc}, \textit{j}  \\
\hline
\end{tabular}
\end{table}
4.2.10. Identity rules

see Table 14

In the identity rules $R$ stands for ‘reflexive’, $S$ for ‘substitution (of identities)’, $N$ for ‘necessary identity’ and $D$ for ‘(doxastic) accessibility’. The star in $(T - R =)$ indicates that $(T - R =)$ is a rule without premises; we may add $t = t, i$ to any open branch in a tree. $(T - S =)$ is applied only ‘within worlds’, and we usually only apply the rule when $A$ is atomic. However, we shall also allow applications of the following kind. Let $M$ be a matrix where $x_m$ is the first free variable in $M$ and $a_m$ is the constant in $M[a_1, \ldots, a, \ldots, a_n/ \bar{x}]$ that replaces $x_m$. Furthermore, suppose we have $a = b, i, M[a_1, \ldots, a, \ldots, a_n/ \bar{x}], i$ and $\neg Ra_m, i$ on the branch. Then we may apply $(T - S =)$ to obtain an extension of the branch that includes $M[a_1, \ldots, b, \ldots, a_n/ \bar{x}], i$.

In every system that contains $(T - S =)$ and $(T - D =)$, we can prove the following theorems: $(B_xA \land c = d) \rightarrow B_dA, (C_xA \land c = d) \rightarrow C_dA, \Pi x\Pi y((B_xA \land x = y) \rightarrow B_yA)$ and $\Pi x\Pi y((C_xA \land x = y) \rightarrow C_yA)$. Intuitively, these formulas are reasonable. It seems to be necessarily true that if Samuel Clemens believes that the evening star is the morning star and Samuel Clemens is Mark Twain, then Mark Twain believes that the evening star is the morning star. In every system that includes $(T - N =)$, we can prove that all identities and non-identities are (absolutely and relatively) necessary, that is, we can prove all of the following theorems: $\Pi x\Pi y(x = y \rightarrow \exists x = y), \Pi x\Pi y(x = y \rightarrow \exists x = y), \Pi x\Pi y(\exists x = y \rightarrow \exists x = y)$ and $\Pi x\Pi y(\exists x = y \rightarrow \exists x = y)$. Recall that every constant is treated as a rigid designator in this paper.

4.3. Tableau systems and some basic proof-theoretical concepts

A tableau system is a set of tableau rules. A doxastic tableau system (or logic) is a tableau system that includes all propositional rules, all basic alethic rules, all basic doxastic rules, the rules for the possibilist quantifiers, CUTR, $T-\text{IdI}$ and $T-\text{IdII}$, and all identity rules. The smallest doxastic logic will be called $D$. By adding various additional rules, we obtain a large class of stronger doxastic systems.

Let $aA_1, \ldots, A_n, dB_1, \ldots, B_n, aC_1, \ldots, C_n, TrD_1, \ldots, D_n$ be the doxastic logic that includes the alethic accessibility rules $A_1, \ldots, A_n$, the doxastic accessibility rules $B_1, \ldots, B_n$, the alethic-doxastic accessibility rules $C_1, \ldots, C_n$, and the transfer rules $D_1, \ldots, D_n$. Then $aTAdD4AdD5adPMBiB45$ (redundant letters are omitted) is the doxastic system that includes the rules $aT, aB, a4, dD, d4, d5, BP, MB, ad4$ and $ad5$. 

### Table 14. Identity rules.

| $T - R =$ | $T - S =$ | $T - N =$ | $T - D =$ |
|----------|----------|----------|----------|
| $=$      | $s = t, i$ | $a = b, i$ | $a = b, i$ |
| $\downarrow$ | $A[s/x], i$ | $\downarrow$ | $jDak$ |
| $t = t, i$ | $\downarrow$ | $a = b, j$ | $\downarrow$ |
| for every $t$ | $A[t/x], i$ | for any $j$ | $jDbk$ |
| on the branch | where $A$ is of a certain form (see below, Section 4.2.10) | | |
A tree is a structure that looks, something, like this (for more information on the concept of a tree, see, for example, Smullyan, 1966, 1968, pp. 3–4):

```
•
•
•
•
•
```

The dots are called nodes and the top node is called the root. Nodes without successors are called tips or leaves. Any path from the root down a series of arrows to a tip is called a branch.

A (semantic) tableau is a tree like this where the nodes have the following form: $A, i$, where $A$ is a formula in $\mathcal{L}$ and $i \in \{0, 1, 2, 3, \ldots\}$, or something of the form $iRj$, $iDcj$ or $i = j$ where $i, j \in \{0, 1, 2, 3, \ldots\}$ and $c$ is a constant in $\mathcal{L}$. The arrows in a tree indicate relations among the nodes. Arrows may be omitted if the structure of the tree can be seen without them.

Different tableau systems contain different tableau rules which, intuitively, tell us how to ‘extend branches’ from given nodes.

A branch in a tableau is closed iff there is a formula $A$ and a number $i$, such that both $A, i$ and $\neg A, i$ occur on the branch; it is open just in case it is not closed. A tableau itself is closed iff every branch in it is closed; it is open iff it is not closed.

Semantic tableaux can be used to check validity of sentences and logical consequence between sets of sentences and sentences. It can be seen as a systematic search for a model in which the class of every sentence on a branch is satisfiable. If the tableau is closed, there is no such model.

We are now in a position to define some important proof-theoretical concepts.

Let $S$ be any system in this paper in the following definitions and let an $S$-tableau be a tableau generated in accordance with the rules in $S$.

**Definition 4.1:**

(i) (PROOF IN A SYSTEM). A proof of $A$ in $S$ is a closed $S$-tableau for $\neg A, 0$, that is, a closed $S$-tableau whose root consists of $\neg A, 0$.

(ii) (THEOREM IN A SYSTEM). $A$ is a theorem in $S$ or provable in $S$ iff there is a proof of $A$ in $S$, that is, iff there is a closed $S$-tableau for $\neg A, 0$.

(iii) (DERIVATION IN A SYSTEM). A derivation in the system $S$ of $B$ from the set of formulas $\Gamma$, is a closed $S$-tableau whose initial list comprises $A, 0$ for every $A \in \Gamma$ and $\neg B, 0$. The sentences in $\Gamma$ are called the premises of the derivation and $B$ is called the conclusion of the derivation. The initial list of a tableau consists of the first nodes in this tableau whose satisfiability we are testing.

(iv) (PROOF-THEORETIC CONSEQUENCE IN A SYSTEM). $B$ is a proof-theoretic consequence of the set of formulas $\Gamma$ in $S$ or $B$ is derivable from a set of formulas $\Gamma$ in $S$ $(\Gamma \vdash S B)$ iff there is a derivation of $B$ in $S$ from $\Gamma$, that is, iff there is a closed $S$-tableau whose initial list comprises $A, 0$ for every $A \in \Gamma$ and $\neg B, 0$. 

5. Examples of theorems

In this section, I will mention some sentences that can be proved in various systems. The informal reading of the formulas should be obvious. Every sentence in Table 15 is a theorem in every system in this paper, every sentence in Table 16 is a theorem in every system that includes the tableau rule $T - dD$, etc.

### Table 15. Some theorems in every system.

| Theorem | System |
|---------|--------|
| $\Pi(x(Rx \rightarrow (B_x(A \wedge B) \rightarrow (B_xA \wedge B)))$ | Every |
| $\Pi(x(Rx \rightarrow ((B_xA \vee B_xB) \rightarrow B_x(A \vee B)))$ | Every |
| $\Pi(x(Rx \rightarrow (C_xA \wedge B) \rightarrow (C_xA \wedge C_xB)))$ | Every |
| $\Pi(x(Rx \rightarrow (C_x(A \vee B) \rightarrow (C_xA \wedge C_xB)))$ | Every |
| $\Pi(x(Rx \rightarrow ((B_xA \rightarrow B) \rightarrow (B_xA \rightarrow B_x)))$ | Every |
| $\Pi(x(Rx \rightarrow (B_x(A \rightarrow B) \rightarrow (C_xA \rightarrow C_xB)))$ | Every |
| $\Pi(x(Rx \rightarrow ((C_xA \wedge B_x(A \rightarrow B)) \rightarrow C_xB))$ | Every |
| $\Pi(x(Rx \rightarrow (B_x(A \rightarrow B) \rightarrow (B_x \neg B \rightarrow B_x \neg A)))$ | Every |
| $\Pi(x(Rx \rightarrow ((B_x \neg B \wedge B_x(A \rightarrow B)) \rightarrow B_x \neg A))$ | Every |
| $\Pi(x(Rx \rightarrow (B_x(A \leftrightarrow B) \rightarrow (B_xB \leftrightarrow B_x)))$ | Every |
| $\Pi(x(Rx \rightarrow (B_x(A \leftrightarrow B) \rightarrow (C_xA \leftrightarrow C_xB)))$ | Every |
| $\Pi(x(Rx \rightarrow (B_x(A \leftrightarrow B) \rightarrow (\neg B_xA \leftrightarrow \neg B_x)))$ | Every |

### Table 16. Some theorems in $dD$.

| Theorem | System |
|---------|--------|
| $\Pi(x(Rx \rightarrow (B_xA \rightarrow C_xA))$ | $dD$ |
| $\Pi(x(Rx \rightarrow (C_x(A \vee C_x \neg A)))$ | $dD$ |
| $\Pi(x(Rx \rightarrow (\neg (B_x(A \vee B) \wedge (B_x \neg A \wedge B_x \neg B))))$ | $dD$ |
| $\Pi(x(Rx \rightarrow (B_x(A \rightarrow B) \rightarrow (B_xA \rightarrow C_xB)))$ | $dD$ |
| $\Pi(x(Rx \rightarrow ((B_xA \wedge B_x(A \rightarrow B)) \rightarrow C_xB))$ | $dD$ |
| $\Pi(x(Rx \rightarrow (B_x(A \rightarrow B) \rightarrow (B_x \neg B \rightarrow \neg B_xA)))$ | $dD$ |
| $\Pi(x(Rx \rightarrow ((B_x \neg B \wedge B_x(A \rightarrow B)) \rightarrow \neg B_xA))$ | $dD$ |

### Table 17. Some basic doxastic theorems.

| Theorem | Systems |
|---------|---------|
| The principle of doxastic omniscience $\Pi(x(Rx \rightarrow (A \rightarrow B_xA))$ | $dO$ |
| The principle of infallibility $\Pi(x(Rx \rightarrow (B_xA \rightarrow A))$ | $dT$ |
| The principle of doxastic consistency $\Pi(x(Rx \rightarrow (\neg (B_xA \wedge B_x \neg A)))$ | $dD$ |
| The principle of positive introspection $\Pi(x(Rx \rightarrow (B_xA \rightarrow B_xB_xA))$ | $d4UR$ |
| The principle of negative introspection $\Pi(x(Rx \rightarrow (\neg B_xA \rightarrow B_x \neg B_xA))$ | $d5UR$ |
| Some other theorems $\Pi(x((B_xA \wedge B_xR_x) \rightarrow (B_xA \rightarrow B_xB_xA))$ | $d4$ |
| $\Pi(x((B_xA \wedge B_xR_x) \rightarrow (C_xA \rightarrow B_xC_xA))$ | $d5$ |
| $\Pi(x((B_xA \wedge B_xR_x) \rightarrow B_x(C_xB_xA \rightarrow A))$ | $d5d4$ |
| $\Pi(x(Rx \rightarrow (C_xA \rightarrow B_xC_xA))$ | $d5UR$ |
| $\Pi(x(Rx \rightarrow B_x(C_xB_xA \rightarrow A))$ | $d5UR$ |
| $\Pi(x(Rx \rightarrow B_x(C_xB_xA \rightarrow A))$ | $d5UR$ |
Section 4.3). Then we shall say that the class of models, Definition 6.1 (SOUNDNESS AND COMPLETENESS): Let $M$, corresponds to $S$ just in case $M = M(C - A_1, \ldots, C - A_n, C - B_1, \ldots, C - B_n, C - C_1, \ldots, C - C_n, C - D_1, \ldots, C - D_n)$. See Tables 17–20.

6. Soundness and completeness theorems

In this section, I will prove that every system in this paper is sound and complete with respect to its semantics. The concepts of soundness and completeness are defined as usual.

Definition 6.1 (SOUNDNESS AND COMPLETENESS): Let $S = aA_1, \ldots, A_n dB_1, \ldots, B_n adC_1, \ldots, C_n TrD_1, \ldots, D_n$ be a doxastic tableau system as defined above (see Section 4.3). Then we shall say that the class of models, $M$, corresponds to $S$ just in case $M = M(C - A_1, \ldots, C - A_n, C - B_1, \ldots, C - B_n, C - C_1, \ldots, C - C_n, C - D_1, \ldots, C - D_n)$.
S is sound with respect to $M$ iff $\Gamma \vdash S$ entails $M$, $\Gamma \vDash A$. $S$ is complete with respect to $M$ just in case $M$, $\Gamma \vDash A$ entails $\Gamma \vdash S$.

**Lemma 6.2 (Locality):** Let $\mathcal{M}_1 = \langle D, W, \mathcal{R}, \mathcal{D}, v_1 \rangle$ and $\mathcal{M}_2 = \langle D, W, \mathcal{R}, \mathcal{D}, v_2 \rangle$ be two models. Since they have the same domain the language of the two is the same: $\mathcal{L}(\mathcal{M}_1) = \mathcal{L}(\mathcal{M}_2)$. Let us call this language $\mathcal{L}$. Furthermore, let $A$ be any closed formula of $\mathcal{L}$ such that $v_1$ and $v_2$ agree on the denotations of all the predicates, constants and matrices in it. Then for all $\omega \in W$: $v_{1,\omega}(A) = v_{2,\omega}(A)$.

**Proof:** The proof is by recursion on formulas; ‘the IH’ refers to the induction hypothesis.

Atomic formulas. $v_{1,\omega}(Pa_1 \ldots a_\omega) = 1$ iff $(v_1(a_1), \ldots, v_1(a_\omega)) \in v_{1,\omega}(P)$ iff $(v_2(a_1), \ldots, v_2(a_\omega)) \in v_{2,\omega}(P)$ iff $v_{2,\omega}(Pa_1 \ldots a_\omega) = 1$.

Suppose that $v_{1,\omega}(Ra_m) = 0$, that $M$ is a matrix where $x_m$ is the first free variable in $M$ and that $a_m$ is the constant in $M[a_1, \ldots, a_\omega / x]$ that replaces $x_m$. Then: $v_{2,\omega}(Ra_m) = 0$ and $v_{1,\omega}(M[a_1, \ldots, a_\omega / x]) = 1$ iff $(v_1(a_1), \ldots, v_1(a_\omega)) \in v_{1,\omega}(M)$ iff $(v_2(a_1), \ldots, v_2(a_\omega)) \in v_{2,\omega}(M)$ iff $v_{2,\omega}(M[a_1, \ldots, a_\omega / x]) = 1$.

Truth-functional connectives. Straightforward.

$(\Box)$. $v_{1,\omega}(\Box B) = 1$ iff for all $\omega'$ such that $\mathcal{R}_{\omega'}$, $v_{1,\omega'}(B) = 1$ iff for all $\omega'$ such that $\mathcal{R}_{\omega'}$, $v_{2,\omega'}(B) = 1$. Hence, $v_{2,\omega}(\Box B) = 1$.

The case for the other alethic operators is similar.

$(B_{c}C)$. $A$ is of the form $B_{c}C$. Assume that $v_{1,\omega}(B_{c}C) = 1$. We have two cases: $v_{1,\omega}(RC) = 0$ or $v_{1,\omega}(RC) = 1$. Suppose $v_{1,\omega}(RC) = 0$. Then $v_{2,\omega}(RC) = 0$. Hence, $v_{2,\omega}(B_{c}C) = 1$. And vice versa. Suppose $v_{1,\omega}(RC) = 1$. Then for all $\omega'$ such that $\mathcal{D}v_{1}(c)_{\omega'}$: $v_{1,\omega'}(C) = 1$. Accordingly, for all $\omega'$ such that $\mathcal{D}v_{2}(c)_{\omega'}$: $v_{2,\omega'}(C) = 1$ [by assumption and the IH]. Furthermore, $v_{2,\omega}(RC) = 1$. Hence, $v_{2,\omega}(B_{c}C) = 1$. And vice versa. Consequently, $v_{1,\omega}(B_{c}C) = 1$ iff $v_{2,\omega}(B_{c}C) = 1$.

The case for $C_{c}B$ is similar.

$(\Pi)$. $v_{1,\omega}(\Pi x B) = 1$ iff for all $k_d \in L$, $v_{1,\omega}(B[k_d / x]) = 1$ iff for all $k_d \in L$, $v_{2,\omega}(B[k_d / x]) = 1$ [by the IH, and the fact that $v_{1,\omega}(k_d) = v_{2,\omega}(k_d) = d$] iff $v_{2,\omega}(\Pi x B) = 1$.

The case for the particular quantifier is similar.

**Lemma 6.3 (Denotation):** Let $\mathcal{M} = \langle D, W, \mathcal{R}, \mathcal{D}, v \rangle$ be any model. Let $A$ be any formula of $\mathcal{L}(\mathcal{M})$ with at most one free variable, $x$, and $a$ and $b$ be any two constants such that $v(a) = v(b)$. Then for any $\omega \in W$: $v_{\omega}(A[a/x]) = v_{\omega}(A[b/x])$.

**Proof:** The proof is by induction on formulas.

Atomic formulas. (To illustrate, we assume that the formula has one occurrence of ‘$a$’ distinct from each $a_i$) $v_{\omega}(Pa_1 \ldots a_\omega) = 1$ iff $(v(a_1), \ldots, v(a_\omega)) \in v_{\omega}(P)$ iff $(v(a_1), \ldots, v(b), \ldots, v(a_\omega)) \in v_{\omega}(P)$ iff $v_{\omega}(Pa_1 \ldots a_\omega) = 1$.

Suppose $v_{\omega}(Ra_m) = 0$, that $M$ is a matrix where $x_m$ is the first free variable in $M$ and that $a_m$ is the constant in $M[a_1, \ldots, a_\omega / x]$ such that $v_{\omega}(M[a_1, \ldots, a_\omega / x]) = 1$ iff $(v(a_1), \ldots, v(a_\omega)) \in v_{\omega}(M)$ iff $(v(a_1), \ldots, v(b), \ldots, v(a_\omega)) \in v_{\omega}(M)$ iff $v_{\omega}(M[a_1, \ldots, b, \ldots, a_\omega / x]) = 1$.
Truth-functional connectives. Straightforward.

\[ \square, v_\omega(\square B[a/x]) = 1 \text{ iff for all } \omega' \text{ such that } \mathcal{M} \omega \omega', v_\omega(B[a/x]) = 1 \text{ iff for all } \omega' \text{ such that } \mathcal{M} \omega \omega', v_\omega'(B(b/x)) = 1 \text{ [the IH] iff } v_\omega(\square B[b/x]) = 1. \]

The arguments for the other primitive alethic operators are similar.

\[ (B_j). A \text{ is of the form } B_j C. \text{ Either } v_\omega(\mathcal{R}_j) = 1 \text{ or } v_\omega(\mathcal{R}_j) = 0. \text{ We have already shown that the result holds if } v_\omega(\mathcal{R}_j) = 0. \text{ Accordingly, suppose that } v_\omega(\mathcal{R}_j) = 1. \text{ Since } x \text{ is the only free variable, } t \text{ cannot be a variable distinct from } x. \text{ So, } t \text{ is either } x \text{ or a constant. Suppose } t = x. \text{ Then } v_\omega(B_j C[a/x]) = 1 \text{ iff } v_\omega(B_j C[b/x]) = 1 \text{ iff for all } \omega' \text{ such that } \mathcal{D} v(a) \omega \omega', v_\omega(C[a/x]) = 1 \text{ iff for all } \omega' \text{ such that } \mathcal{D} v(b) \omega \omega', v_\omega(C[b/x]) = 1 \text{ [by the fact that } v(a) = v(b) \text{ and the IH] iff } v_\omega(B_j C[b/x]) = 1. \text{ Suppose } t \text{ is a constant, say } c. \text{ Then } v_\omega(B_j C[a/x]) = 1 \text{ iff for all } \omega' \text{ such that } \mathcal{D} v(c) \omega \omega', v_\omega(C[a/x]) = 1 \text{ iff for all } \omega' \text{ such that } \mathcal{D} v(c) \omega \omega', v_\omega(C[b/x]) = 1 \text{ [by the IH] iff } v_\omega(B_j C[b/x]) = 1. \]

The case for \( C_i \) is similar.

\[ (\Pi). \text{ Let } A \text{ be of the form } \Pi y B. \text{ If } x = y, \text{ then } A[a/x] = A[b/x] = A, \text{ so the result is trivial. Accordingly, suppose that } x \text{ and } y \text{ are distinct. Then, } (\Pi y B)[b/x] = \Pi y(B[b/x]) \text{ and } (B[b/x])[a/y] = (B[a/y])[b/x], v_\omega((\Pi y B)[a/x]) = 1 \text{ iff } v_\omega((\Pi y B)[b/x]) = 1 \text{ iff for all } k_d \in \mathcal{L}(\mathcal{M}), v_\omega((B[a/x])[k_d/y]) = 1 \text{ iff for all } k_d \in \mathcal{L}(\mathcal{M}), v_\omega((B[b/x])[k_d/y]) = 1 \text{ [the IH] iff for all } k_d \in \mathcal{L}(\mathcal{M}), v_\omega((B[b/x])[k_d/y]) = 1 \text{ iff } v_\omega((\Pi y B)[b/x]) = 1. \]

The case for the particular quantifier (\( \Sigma \)) is similar.

### 6.1. Soundness theorem

Let \( \mathcal{M} = \langle D, W, \mathcal{R}, \mathcal{D}, v \rangle \) be any model and \( B \) any branch of a tableau. Then \( B \) is satisfiable in \( \mathcal{M} \) iff there is a function \( f \) from 0, 1, 2, \ldots to \( W \) such that

1. A is true in \( f(i) \) in \( \mathcal{M} \), for every node \( A, i \) on \( B \),
2. if \( i \mathcal{R} j \) is on \( B \), then \( \mathcal{M} f(i) f(j) \) in \( \mathcal{M} \),
3. if \( i \mathcal{D} c j \) is on \( B \), then \( \mathcal{D} v(c) f(i) f(j) \) in \( \mathcal{M} \),
4. if \( i = j \) is on \( B \), then \( f(i) = f(j) \).

If these conditions are fulfilled, we say that \( f \) shows that \( B \) is satisfiable in \( \mathcal{M} \).

**Lemma 6.4 (Soundness Lemma):** Let \( B \) be any branch of a tableau and \( \mathcal{M} \) be any model. If \( B \) is satisfiable in \( \mathcal{M} \) and a tableau rule is applied to it, then there is a model \( \mathcal{M}' \) and an extension of \( B, B' \), such that \( B' \) is satisfiable in \( \mathcal{M}' \).

**Proof:** The proof is by induction on the height of the derivation. Let \( f \) be a function that shows that the branch \( B \) is satisfiable in \( \mathcal{M} \).

Connectives and the modal operators. Straightforward.

\( (B) \) Suppose that \( R c, i, B_c C, i, \) and \( i \mathcal{D} c j \) are on \( B \), and that we apply the \( B \)-rule. Then we get an extension of \( B \) that includes \( C, j \). Since \( B \) is satisfiable in \( \mathcal{M}, B_c C \) is true in \( f(i) \) and \( R c \) is true in \( f(i) \). Furthermore, for any \( i \) and \( j \) such that \( i \mathcal{D} c j \) is on \( B, \mathcal{D} v(c) f(i) f(j) \). Thus by the truth conditions for \( B_c C, C \) is true in \( f(j) \).

\( (C) \) Suppose that \( R c, i, C_c B, i \) are on \( B \) and that we apply the \( C \)-rule to get an extension of \( B \) that includes nodes of the form \( i \mathcal{D} c j \) and \( B, j \). Since \( B \) is satisfiable in \( \mathcal{M}, C_c B \) is
true in \( f(i) \) and \( Rc \) is true in \( f(i) \). Hence, for some \( \omega \) in \( W \), \( \mathcal{D} v(c)f(i)\omega \) and \( B \) is true in \( \omega \) [by the truth conditions for \( CcB \) and the fact that \( Rc \) is true in \( f(i) \)]. Let \( f' \) be the same as \( f \) except that \( f'(j) = \omega \). Since \( f \) and \( f' \) differ only at \( j \), \( f' \) shows that \( B \) is satisfiable in \( \mathcal{M} \). Furthermore, by definition \( \mathcal{D} v(c)f'(i)f'(j) \), and \( B \) is true in \( f'(j) \).

\((-B) \) and \((-C) \). Similar.

\((\Pi) \). Suppose that \( \Pi x A, i \) is on \( B \) and that we apply the \( \Pi \)-rule to get an extension of \( B \) that includes a node of the form \( A[a/x], i \). \( \mathcal{M} \) makes \( \Pi x A \) true in \( f(i) \). For \( B \) is satisfiable in \( \mathcal{M} \). Hence, \( A[kd/x] \) is true in \( f(i) \) in \( \mathcal{M} \), for all \( kd \in \mathcal{L}(\mathcal{M}) \). Let \( d \) be such that \( v(a) = v(kd) \). By the Denotation Lemma, \( A[a/x] \) is true in \( f(i) \) in \( \mathcal{M} \). Accordingly, we can take \( \mathcal{M}' \) to be \( \mathcal{M} \).

\((\Sigma) \). Suppose that \( \Sigma x A, i \) is on \( B \) and that we apply the \( \Sigma \)-rule to get an extension of \( B \) that includes a node of the form \( A[c/x], i \) (where \( c \) is new). Since \( B \) is satisfiable in \( \mathcal{M} \), \( \Sigma x A \) is true in \( f(i) \) in \( \mathcal{M} \). Consequently, there is some \( kd \in \mathcal{L}(\mathcal{M}) \) such that \( \mathcal{M} \) makes \( A[kd/x] \) true in \( f(i) \). Let \( \mathcal{M}' = \langle D, W', R', \mathcal{D}, v' \rangle \) be the same as \( \mathcal{M} \) except that \( v'(c) = d \). Since \( c \) does not occur in \( A[kd/x] \), \( A[kd/x] \) is true in \( f(i) \) in \( \mathcal{M}' \), by the Locality Lemma. By the Denotation Lemma and the fact that \( v'(c) = d = v'(kd) \), \( A[c/x] \) is true in \( f(i) \) in \( \mathcal{M}' \). Moreover, \( \mathcal{M}' \) makes all other formulas on the branch true at their respective worlds as well, by the Locality Lemma. For \( c \) does not occur in any other formula on the branch.

\((-\Pi) \) and \((-\Sigma) \). Straightforward.

Accessibility rules. I will consider two examples to illustrate the method.

\((T-\text{ad} 4) \). Suppose that \( iRj \) and \( jDck \) are on \( B \), and that we apply \((T-\text{ad} 4) \) to give an extended branch containing \( iDck \). Since \( B \) is satisfiable in \( \mathcal{M} \), \( iRf(i)f(j) \) and \( \mathcal{D} v(c)f(i)f(k) \). Hence, \( \mathcal{D} v(c)f(i)f(k) \) since \( \mathcal{M} \) satisfies condition \( C-\text{ad} 4 \). Hence, the extension of \( B \) is satisfiable in \( \mathcal{M} \).

\((T-S =) \). Suppose we have \( s=t \), and \( A[s/x], i \) on the branch, and that we apply \((T-S =) \) to obtain an extension of the branch that includes \( A[t/x], i \). Since \( f \) shows that the branch is satisfiable in \( \mathcal{M} \), \( v(s) = v(t) \). Suppose that \( A \) is atomic and has the following form: \( Pa_1 \ldots s \ldots a_n \). Then \( Pa_1 \ldots s \ldots a_n \) is true in \( f(i) \), i.e. \( \langle v(a_1), \ldots, v(s), \ldots, v(a_n) \rangle \) is an element in \( P \)’s extension in \( f(i) \). Accordingly, \( \langle v(a_1), \ldots, v(t), \ldots, v(a_n) \rangle \) is an element in \( P \)’s extension in \( f(i) \), i.e. \( Pa_1 \ldots t \ldots a_n \) is true in \( f(i) \) in \( \mathcal{M} \). So, we may take \( \mathcal{M}' \) to be \( \mathcal{M} \).

Let \( \mathcal{M} \) be a matrix where \( x_m \) is the first free variable in \( M \) and \( a_m \) is the constant in \( M[a_1, \ldots, a, \ldots, a_n/ \overrightarrow{x}] \) \( (M[a_1, \ldots, b, \ldots, a_n/ \overrightarrow{x}] \) that replaces \( x_m \). Furthermore, to illustrate, suppose we have \( a = b, i, M[a_1, \ldots, a, \ldots, a_n/ \overrightarrow{x}], i \) and \( \neg Ra_m, i \) on the branch (and that \( a_m \) is not \( a (b)) \), and that we apply \((T-S =) \) to obtain an extension of the branch that includes \( M[a_1, \ldots, b, \ldots, a_n/ \overrightarrow{x}], i \). Since \( f \) shows that the branch is satisfiable in \( \mathcal{M} \), \( v_{f(i)}(Ra_m) = 0, v(a) = v(b) \) and \( \langle v(a_1), \ldots, v(a), \ldots, v(a_n) \rangle \in v_{f(i)}(M) \). Hence, \( \langle v(a_1), \ldots, v(b), \ldots, v(a_n) \rangle \in v_{f(i)}(M) \). It follows that \( M[a_1, \ldots, b, \ldots, a_n/ \overrightarrow{x}] \) is true in \( f(i) \). In conclusion, the extension of \( B \) is satisfiable in \( \mathcal{M} \). (All other cases are proved similarly.)

**Theorem 6.5 (Soundness Theorem):** Every system \( S \) in this paper is sound with respect to its semantics.
Proof: Suppose that $B$ does not follow from $\Gamma$ in $M$, where $M$ is the class of models that corresponds to $S$. Then every premise in $\Gamma$ is true and the conclusion $B$ false in some world $\omega$ in some model $M'$ in $M$. Consider an $S$-tableau whose initial list consists of $A$, $0$ for every $A \in \Gamma$ and $\neg B$, $0$, where ‘0’ refers to $\omega$. Then the initial list is satisfiable in $M$. Every time we apply a rule to this list it produces at least one extension that is satisfiable in a model $M'$ in $M$ (by the Soundness Lemma). So, we can find a whole branch such that every initial section of this branch is satisfiable in some model $M''$ in $M$. If this branch is closed, then some sentence is both true and false in some possible world in $M''$. Yet, this is impossible. Hence, the tableau is open. Accordingly, $B$ is not derivable from $\Gamma$ in $S$. In conclusion, if $B$ is derivable from $\Gamma$ in $M$, then $B$ follows from $\Gamma$ in $M$. ■

6.2. Completeness theorem

In this section, I will prove that every system in this paper is complete with respect to its semantics. However, first I will introduce some important concepts.

Intuitively, a complete tableau is a tableau where every rule that can be applied has been applied. In this sense, there may be several different (complete) tableaux for the same sentence or set of sentences, some more complex than others, and the tableau rules may be applied in different orders. To produce a complete tableau, we can use the following method (which is usually not the simplest one). (1) For every open branch on the tree, one at a time, start from its root and move towards its tip. Apply any rule that produces something new to the branch (if the application of a rule would result in just repeating lines already on the same branch of the tableau, it should not be applied). For example, $\Sigma$ is applied at most once to a node of the form $\Sigma xA_i$. We do not apply any rules to a branch that is closed. Some rules may have several possible applications, e.g. $\square$ and $\Pi$. Then make all applications at once. (2) When we have done this for all open branches on the tree, we repeat the procedure. Some rules introduce new possible worlds, for example $T \neg aD$ and $T \neg BP$. If a rule introduces a new possible world, it is applied once at the tip of every open branch at the end of every cycle when we have moved through all nodes. If a system includes several different rules that introduce new possible worlds ($R1$, $R2 \ldots$), we alternate between them. The first time we use $R1$ once; the second time we use $R2$ once, etc. Before we conclude a cycle and begin to move through all nodes again we split the end of every open branch in the tree and add $Rc$, $i$ to the left node and $\neg Rc$, $i$ to the right node, for every constant $c$ (that occurs as an index to some doxastic operator on the tree) and $i$ on the branch. If there is still something to do according to this method, the tableau is incomplete; if not, it is complete.

Definition 6.6 (INDUCED MODEL): Let $B$ be an open complete branch of a tableau, let $i$, $j$, $k$, etc. be numbers on $B$, and let $l$ be the set of numbers on $B$. We shall say that $i \equiv j$ just in case $i = j$, or ‘$i = j$’ or ‘$j = i$’ occurs on $B$. $\equiv$ is an equivalence relation and $[i]$ is the equivalence class of $i$. Moreover, let $C$ be the set of all constants on $B$. Define $a \sim b$ to mean that $a = b$, $0$ is on the branch. $a \sim b$ is obviously an equivalence relation. Let $[a]$ be the equivalence class of $a$ under $\sim$. The model $M = (D, W, R, \square, \Pi, \omega)$ induced by $B$ is defined as follows. $D = \{[a] : a \in C\}$ (or, if $C = \emptyset$, $D = \{0\}$ for an arbitrary $0$). ($0$ is not in the extension of anything.) $W = \{\omega_i : i \text{ occurs on } B\}$, $R\omega_i\omega_j$ iff $iRj$ occurs
on $B$, and $\Delta v(a)\omega_{ij}\omega_{ij}$ iff $iDaj$ occurs on $B$. $v(a) = [a]$, and $\langle [a_1], \ldots, [a_n] \rangle \in v_{\omega_{ij}}(P)$ iff $Pa_1 \ldots a_n, i$ is on $B$, given that $P$ is any $n$-place predicate other than identity. If $\neg Ra_m, i$ occurs on $B$ and $M$ is an $n$-place matrix with instantiations on the branch (where $x_m$ is the first free variable in $M$ and $a_m$ is the constant in $M[a_1, \ldots, a_n/\stackrel{\rightarrow}{x}]$ that replaces $x_m$), then $\langle [a_1], \ldots, [a_n] \rangle \in v_{\omega_{ij}}(M) \iff M[a_1, \ldots, a_n/\stackrel{\rightarrow}{x}], i$ occurs on $B$. (Due to the identity rules this is well defined. For example, if we have $iDcj$ and $j = k$ on $B$, we also have $iDck$ on $B$ by the identity rules and the fact that the branch is complete, etc.) When we have $a = b, 0, b = c, 0$, etc. we choose one single object for all constants to denote.

If our tableau system does not include $T-dO, \equiv$ is reduced to identity and $[i] = i$. Hence, in such systems, we may take $W$ to be $\{\omega : i \text{ occurs on } B\}$ and dispense with the equivalence classes.

**Lemma 6.7 (Completeness Lemma):** Let $B$ be an open branch in a complete tableau and let $M$ be a model induced by $B$. Then, for every formula $A$:

(i) if $A, i$ is on $B$, then $v_{\omega_{ij}}(A) = 1$, and
(ii) if $\neg A, i$ is on $B$, then $v_{\omega_{ij}}(A) = 0$.

**Proof:** The proof is by induction on the complexity of $A$.

(i) Atomic formulas. $Pa_1 \ldots a_n, i$ is on $B \Rightarrow \langle [a_1], \ldots, [a_n] \rangle \in v_{\omega_{ij}}(P) \Rightarrow \langle v(a_1), \ldots, v(a_n) \rangle \in v_{\omega_{ij}}(Pa_1 \ldots a_n) = 1$.

$a = b, i$ is on $B \Rightarrow a \sim b (T - N \Rightarrow) [a] = [b] \Rightarrow v(a) = v(b) \Rightarrow v_{\omega_{ij}}(a = b) = 1$.

Suppose that $M$ is a matrix where $x_m$ is the first free variable and $a_m$ is the constant in $M[a_1, \ldots, a_n/\stackrel{\rightarrow}{x}]$ that replaces $x_m$ and that $v_{\omega_{ij}}(Ra_m) = 0$. Then: $M[a_1, \ldots, a_n/\stackrel{\rightarrow}{x}], i$ occurs on $B \Rightarrow \langle [a_1], \ldots, [a_n] \rangle \in v_{\omega_{ij}}(M) \Rightarrow \langle v(a_1), \ldots, v(a_n) \rangle \in v_{\omega_{ij}}(M) = v_{\omega_{ij}}(M[a_1, \ldots, a_n/\stackrel{\rightarrow}{x}]) = 1$.

Other truth-functional connectives and modal operators. Straightforward.

Doxastic operators. ($B$). Suppose $B, D, i$ is on $B$. Moreover, suppose that $Rc, i$ is not on $B$. Then $\neg Rc, i$ is on $B$ by $\text{CUTR}$. Consequently, $B, D$ is true in $\omega_{ij}$ by definition and previous steps. Suppose $Rc, i$ is on $B$. Then since the branch is complete, the $B$-rule has been applied and for every $j$ such that $iDcj$ is on $B$, $D, j$ is on $B$. By the induction hypothesis, $D$ is true in every $\omega_{ij}$ such that $\Delta v(c)\omega_{ij}\omega_{ij}$. Since $Rc, i$ is on $B$, $v(c)$ is perfectly rational in $\omega_{ij}$. It follows that $B, D$ is true in $\omega_{ij}$, as required.

(C). Similar as for ($B$).

Quantifiers. ($\Sigma$). Suppose that $\Sigma xD, i$ is on the branch. Since the tableau is complete ($\Sigma$) has been applied. Accordingly, for some $c$, $D[c/x], i$ is on the branch. Hence, $v_{\omega_{ij}}(D[c/x]) = 1$, by (IH). For some $k_d \in L(M)$, $v(c) = d$ and $v(k_d) = d$. Consequently, $v_{\omega_{ij}}(D[k_d/x]) = 1$, by the Denotation Lemma. It follows that $v_{\omega_{ij}}(\Sigma xD) = 1$.

The case for $\Pi$ is similar.

(ii) Atomic formulas.

$\neg Pa_1 \ldots a_n, i$ is on $B \Rightarrow Pa_1 \ldots a_n, i$ is not on $B$ ($B$ open) $\Rightarrow \langle [a_1], \ldots, [a_n] \rangle \notin v_{\omega_{ij}}(P) \Rightarrow \langle v(a_1), \ldots, v(a_n) \rangle \notin v_{\omega_{ij}}(P) \Rightarrow v_{\omega_{ij}}(Pa_1 \ldots a_n) = 0$. 
\(\neg a = b, i\) is on \(B\) \(\Rightarrow a = b, 0\) is not on \(B\) (\(B\) open) \(\Rightarrow\) it is not the case that \(a \sim b\) 

\(\Rightarrow [a] \neq [b] \Rightarrow v(a) \neq v(b) \Rightarrow v_{\omega[]} (a = b) = 0.\)

Suppose that \(M\) is a matrix and \(x_m^i\) is the first free variable and \(a_m\) is the constant in \(\text{in} \{a_1, \ldots, a_n/ \overrightarrow{x}\}\) that replaces \(x_m\) and that \(v_{\omega[]} (Ra_m) = 0.\) Then: \(\neg M[a_1, \ldots, a_n/ \overrightarrow{x}], i\) occurs on \(B\) \(\Rightarrow M[a_1, \ldots, a_n/ \overrightarrow{x}], i\) is not on \(B\) (\(B\) open) \(\Rightarrow \langle [a_1], \ldots, [a_n] \rangle \neq v_{\omega[]} (M) \Rightarrow \langle v(a_1), \ldots, v(a_n) \rangle \neq v_{\omega[]} (M) \Rightarrow v_{\omega[]} (M) \Rightarrow v_{\omega[]} (M[a_1, \ldots, a_n/ \overrightarrow{x}]) = 0.\)

Other truth-functional connectives and modal operators. Straightforward.

Doxastic operators. (\(\neg B\)). Suppose \(\neg B, c, i\) is on \(B\). Furthermore, suppose that \(Rc, i\) is on \(B\) [by CUTR]. Hence, \(B, c\) is false in \(\omega[i]\) by definition and previous steps. Suppose \(Rc, i\) is on \(B\). Then the \(\neg B\)-rule has been applied to \(\neg B, c, i\) and we have \(C, \neg D, i\) on \(B\). For the branch is complete. Then the \(C\)-rule has been applied to \(C, \neg D, i\), since the branch is complete. Hence, for some new \(j, iDc, j\) and \(\neg D, j\) occur on \(B\). By the induction hypothesis, \(D, v(c) \omega[i]\omega[j], and D is false in \(\omega[i]\). Since \(Rc, i\) is on \(B\), \(v(c)\) is perfectly rational in \(\omega[i]\). Consequently, \(B, c\) is false in \(\omega[i]\), as required.

(\(\neg C\)). Similar as for (\(\neg B\)).

Quantifiers. (\(\neg \Sigma\)). Suppose that \(\neg \Sigma x D, i\) is on the branch. Since the tableau is complete (\(\neg \Sigma\) has been applied. So, \(\Pi x \neg D, i\) on the branch. Again, since the tableau is complete (\(\Pi\) has been applied. Thus, for all \(c \in C, \neg D, c/ \overrightarrow{x}, i\) is on the branch. Consequently, \(v_{\omega[]} (D, c/ \overrightarrow{x}) = 0\) for all \(c \in C\) [by the induction hypothesis]. If \(k_d \in L(\mathcal{M})\), then for some \(c \in C, v(c) = v(k_d)\). By the Denotation Lemma, for all \(k_d \in L(\mathcal{M}), v_{\omega[]} (D, k_d/ \overrightarrow{x}) = 0\). Consequently, \(v_{\omega[]} (D x D) = 0.\)

(\(\neg \Pi\)). Straightforward.

Theorem 6.8 (Completeness Theorem): Every system in this paper is complete with respect to its semantics.

Proof: First we prove that the theorem holds for our weakest system \(D\). Then we extend the theorem to all extensions of this system. Let \(M\) be the class of models that corresponds to \(D\).

Suppose that \(B\) is not derivable from \(\Gamma\) in \(D\): then it is not the case that there is a closed \(D\)-tableau whose initial list comprises \(A, 0\) for every \(A\) in \(\Gamma\) and \(\neg B, 0\). Let \(t\) be a complete \(D\)-tableau whose initial list comprises \(A, 0\) for every \(A\) in \(\Gamma\) and \(\neg B, 0\). Then \(t\) is not closed—in other words, it is open. Hence, there is at least one open branch in \(t\). Let \(B\) be an open branch in \(t\). The model induced by \(B\) makes all the premises in \(\Gamma\) true and \(B\) false in \(\omega[i]\). Accordingly, it is not the case that \(B\) follows from \(\Gamma\) in \(M\). In conclusion, if \(B\) follows from \(\Gamma\) in \(M\), then \(B\) is derivable from \(\Gamma\) in \(D\).

To prove that all extensions of \(D\) are complete with respect to their semantics, we have to check that the model induced by the open branch \(B\) is of the right kind. To do this we first check that this is true for every single semantic condition. Then we combine each of the individual arguments. I will go through some steps to illustrate the method.

\(C-d5\). Suppose that \(D, v(c) \omega[i] \omega[j]\) and \(D, v(c) \omega[i] \omega[k]\). Then \(iDc, j\) and \(iDc, k\) occur on \(B\) [by the definition of an induced model]. Since \(B\) is complete, \((T-d5)\) has been applied and \(jDc, k\) occurs on \(B\). It follows that \(D, v(c) \omega[i] \omega[k]\), as required [by the definition of an induced model].
C = BC. Suppose that \( \omega_{[i]} \) is in \( W \). Then \( i \) occurs on \( B \) [by the definition of an induced model]. Since \( B \) is complete (\( T - BC \) has been applied. Accordingly, for some \( j \), \( iDcj \) and \( iRj \) are on \( B \). Thus, for some \( \omega_{[i]} \), \( \mathcal{D}v(c)\omega_{[i]}\omega_{[j]} \) and \( \mathcal{R}\omega_{[i]}\omega_{[j]} \), as required [by the definition of an induced model].

C = CMP. Suppose that \( \mathcal{D}v(c)\omega_{[i]}\omega_{[j]} \) and \( \mathcal{R}\omega_{[i]}\omega_{[j]} \). Then \( iDcj \) and \( iRk \) occur on \( B \) [by the definition of an induced model]. Since \( B \) is complete (\( T - CMP \) has been applied. Hence, for some \( l, jRl \) and \( kDcl \) are on \( B \). Accordingly, for some \( \omega_{[l]} \), \( R\omega_{[j]}\omega_{[l]} \) and \( \mathcal{D}v(c)\omega_{[k]}\omega_{[l]} \), as required [by the definition of an induced model].

7. Examples of derivations in our systems, the unmarried teacher argument and the conscientious student argument

In this section, I will show that the conclusion in the unmarried teacher argument is derivable from the premises in every system in this paper and that the conscientious student argument is invalid in the class of all models (Section 1). It follows that the unmarried teacher argument is valid (in the class of all models) [by the Soundness Theorem]. The argument can be symbolised in the following way. Premises: \( \Pi x(Sx \rightarrow Bx(Bt \rightarrow Ut)) \), \( \Pi x(Sx \rightarrow BxBt) \), \( Ss \). Conclusion: \( Rs \rightarrow BsUt \), where ‘\( Sx \)’ says that \( x \) is a student, ‘\( Bt \)’ says that the teacher is a bachelor, ‘\( Ut \)’ says that the teacher is an unmarried man, and ‘\( s \)’ refers to Susan. This argument is intuitively valid, but it seems impossible to prove this in any standard doxastic systems. As we saw in the introduction, this is a good reason to be attracted to the systems developed in the present paper.

To prove that the conclusion is derivable from the premises, we construct a semantic tableau that begins with all premises and the negation of the conclusion. Since this tableau is closed and we have only used the rules of the basic system \( D \), it constitutes a derivation of the conclusion from the premises in this system. Consequently, the conclusion follows from the premises in the class of all models. Here is our proof. (‘\( MP \)’ stands for the derived rule ‘Modus Ponens’.)

(1) \( \Pi x(Sx \rightarrow Bx(Bt \rightarrow Ut)) \), 0
(2) \( \Pi x(Sx \rightarrow BxBt) \), 0
(3) \( Ss \), 0
(4) \( \neg(Rs \rightarrow BsUt) \), 0
(5) \( Rs \), 0 [4, \( \neg \rightarrow \)]
(6) \( \neg BsUt \), 0 [4, \( \neg \rightarrow \)]
(7) \( Ss \rightarrow Bs(Bt \rightarrow Ut) \), 0 [1, \( \Pi \)]
(8) \( Ss \rightarrow BsBt \), 0 [2, \( \Pi \)]
(9) \( Bs(Bt \rightarrow Ut) \), 0 [3, 7, \( MP \)]
(10) \( BsBt \), 0 [3, 8, \( MP \)]
(11) \( Cs\neg Ut \), 0 [5, 6, \( \neg B \)]
(12) \( 0Ds1 \) [5, 11, \( C \)]
(13) \( \neg Ut \), 1 [5, 11, \( C \)]
(14) \( Bs \rightarrow Ut \), 1 [5, 9, 12, \( B \)]
(15) \( Bs \), 1 [5, 10, 12, \( B \)]
(16) \( Ut \), 1 [14, 15, \( MP \)]
(17) \( * \) [13, 16]
Let us now turn to the conscientious student argument. Let $Sx$ stand for ‘$x$ is a student’, $Hx$ for ‘$x$ studies hard’, and $Dx$ for ‘$x$ deserves a good grade’. Then the argument can be symbolised in the following way: (1) $\Pi x(Sx \rightarrow B_x(Hx \rightarrow Dx))$ (Every student believes that if she studies hard she deserves a good grade), (2) $\Pi x(Sx \rightarrow B_x Hx)$ (Every student believes that she studies hard), Hence, (3) $\Pi x(Sx \rightarrow B_x Dx)$ (Every student believes that she deserves a good grade).

To prove that an argument is not valid we construct an open complete tableau that begins with the premises and the negation of the conclusion. Then we use an open branch in the tree to read off a countermodel. More precisely, if $A_1, \ldots, A_n$ are the premises in the argument and $B$ is the conclusion, then we construct a semantic tableau that begins with $A_1, 0, \ldots, A_n, 0$ and $\neg B, 0$. Here is a tableau for the conscientious student argument.

(1) $\Pi x(Sx \rightarrow B_x(Hx \rightarrow Dx)), 0$
(2) $\Pi x(Sx \rightarrow B_xHx), 0$
(3) $\neg \Pi x(Sx \rightarrow B_x Dx), 0$
(4) $\Sigma x \neg(Sx \rightarrow B_x Dx), 0 [3, \neg \Pi]$
(5) $\neg(Sc \rightarrow B_c Dc), 0 [4, \Sigma]$
(6) $Sc, 0 [5, \neg \rightarrow]$
(7) $\neg B_c Dc, 0 [5, \neg \rightarrow]$
(8) $Sc \rightarrow B_c(Hc \rightarrow Dc), 0 [1, \Pi]$
(9) $Sc \rightarrow B_c Hc, 0 [2, \Pi]$

\hspace{1cm}

| (10) $\neg Sc, 0$ | (11) $B_c(Hc \rightarrow Dc), 0 [8, \rightarrow]$ |
|-----------------|------------------------------------------|
| (12) * [6, 10]  | (13) $\neg Sc, 0$                       |
|                 | (14) $B_c Hc, 0 [9, \rightarrow]$       |
|                 | (15) * [6, 13]                          |
|                 | (16) $c = c, 0 [T - R =]$                |
|                 | (17) $Rc, 0$                            |
|                 | (18) $\neg Rc, 0$ [CUTR]                |

The open branch to the left in this tree closes in a few more steps (it is left to the reader to verify this claim). However, the right branch is open and complete. Accordingly, the whole tableau is open (and complete). Consequently, the conclusion in the conscientious student argument is not derivable from the premises. It follows that the argument is invalid (by the completeness results in Section 6).
branch. $x_1$ is the first free variable in $B_{x_1} Dx_2$ and $c$ is the constant in $B_{x_1} Dx_2[c, c/x_1, x_2]$ that replaces $x_1$. So, $[c, [c]]$ is not an element in $v_{ω_0}(B_{x_1} Dx_2)(v_{ω_0}(B_{x_1} Dx_2)$ is empty). Since $¬Rc, 0$ occurs on our open branch, $Rc$ is false in $ω_0$. If $Rc$ is false in $ω_0$ in $M$, then $B_{x_1} Dx_2[c, c/x_1, x_2]$ is true in $ω_0$ in $M$ iff $⟨v(c), v(c)⟩$ is in $v_{ω_0}(B_{x_1} Dx_2)$. Hence, $B_{x_1} Dx_2[c, c/x_1, x_2]$ is true in $ω_0$ in $M$ iff $⟨v(c), v(c)⟩$ is in $v_{ω_0}(B_{x_1} Dx_2)$. $⟨v(c), v(c)⟩$ is not in $v_{ω_0}(B_{x_1} Dx_2)$. Consequently, it is not the case that $B_{x_1} Dx_2[c, c/x_1, x_2]$ is true in $ω_0$ in $M$. $B_{x_1} Dx_2[c, c/x_1, x_2] = B_c DC$. It follows that it is not the case that $B_c DC$ is true in $ω_0$ in $M$, that is, $B_c DC$ is false in $ω_0$ in $M$.

$B_{x_1}(Hx_2 → Dx_3)[c, c, c/x_1, x_2, x_3], 0$ (that is, $B_c (Hc → DC), 0$) occurs on the branch. $x_1$ is the first free variable in $B_{x_1}(Hx_2 → Dx_3)$ and $c$ is the constant in $B_{x_1}(Hx_2 → Dx_3)[c, c, c/x_1, x_2, x_3]$ that replaces $x_1$. Consequently, $[c, [c], [c]]$ is an element in $v_{ω_0}(B_{x_1}(Hx_2 → Dx_3))$. If $Rc$ is false in $ω_0$ in $M$, then $B_{x_1}(Hx_2 → Dx_3)[c, c, c/x_1, x_2, x_3]$ is true in $ω_0$ in $M$ iff $⟨v(c), v(c), v(c)⟩$ is in $v_{ω_0}(B_{x_1}(Hx_2 → Dx_3))$. So, $B_{x_1}(Hx_2 → Dx_3)[c, c, c/x_1, x_2, x_3]$ is true in $ω_0$ in $M$ iff $⟨v(c), v(c), v(c)⟩$ is in $v_{ω_0}(B_{x_1}(Hx_2 → Dx_3))$. $⟨v(c), v(c), v(c)⟩$ is in $v_{ω_0}(B_{x_1}(Hx_2 → Dx_3))$. Consequently, $B_{x_1}(Hx_2 → Dx_3)[c, c, c/x_1, x_2, x_3]$ is true in $ω_0$ in $M$. Note that $B_{x_1}(Hx_2 → Dx_3)[c, c, c/x_1, x_2, x_3] = B_c (Hc → DC)$. It follows that $B_c (Hc → DC)$ is true in $ω_0$ in $M$.

Similarly, we can show that $B_c Hc$ is true in $ω_0$ in $M$, since $B_c Hc, 0$ is on the branch. $[c]$ is in the extension of $S$ in $ω_0$. Hence, $Sc$ is true in $ω_0$. Therefore, $Sc → B_c (Hc → DC)$ is true in $ω_0$. Since $[c]$ is the only object in our domain, it follows that $Πx(Sx → B_c (Hx → Dx))$ is true in $ω_0$. Similarly, we can show that $Πx(Sx → B_c Hx)$ is true in $ω_0$. This establishes that both premises in the conscientious student argument are true in $ω_0$ in $M$. On the other hand, since $Sc$ is true in $ω_0$ and $B_c DC$ is false in $ω_0$, $Sc → B_c DC$ is false in $ω_0$. It follows that $Πx(Sx → B_c Dx)$ is false in $ω_0$. This establishes that the conclusion in the conscientious student argument is false in $ω_0$ in $M$. In other words, the premises in the conscientious student argument are true in $ω_0$ in $M$ while the conclusion is false in $ω_0$ in $M$. Hence, the argument is invalid. Q.E.D.

Notes

1. For more on epistemic and doxastic logic and many relevant references, see, for example, Alechina and Logan (2010), Bacharach, Gérard-Varet, Mongin, and Shin (1997), Galeazzi and Lorini (2016), Goeth and Gribomont (2006), Hendricks (2005, Chapter 6), Hendricks, Jörgensen, and Pedersen (2003), Lenzen (1978, 1980, 2004), Rescher (2005), van Benthem (2006), van Ditmarsch, van der Hoek, and Kooi (2008), von Kutschera (1976, 1981, Chapter 1). Some information about the history of the subject can be found in Boh (1993) and Knuuttila (2008). For some views on the relationship between knowledge and belief, see, for example, Askounis, Koutras, and Zikos (2016), Goeth and Gribomont (2006), Kraus and Lehmann (1988) and Stalnaker (2006). I focus on the concept of belief in this paper. Short introductions to first-order epistemic logic can be found in Fagin et al. (1995, pp. 80–91) and Meyer and van der Hoek (1995, pp. 225–229). For more on how to combine modal logic and predicate logic, see, for example, Barcan (Marcus) (1946), Carnap (1946), Fitting and Mendelsohn (1998), Garson (1984, 2006), Hintikka (1961), Hughes and Cresswell (1968), Parks (1976), Priest (2008), Thomason (1970).

2. For more on the problem of logical omniscience and some possible solutions, see, for example, Fagin and Halpern (1988), Girle (1998), Goeth and Gillet (1991), Goeth and Gribomont (2006), Hocutt (1972), Jaspars (1991), Levesque (1984), McLane (1979), Rantala (1982), Sim (1997, 2000), Thijssen (1992), van der Hoek and Meyer (1989), Yap (2014).
3. However, see Fitting et al. (2001) and Corsi and Orlandelli (2013).

4. Some might think that the systems that are developed in this paper are too weak. Almost nothing follows from the proposition that an agent that is not perfectly rational believes something. But is it not reasonable to assume that, for example, the following principles hold also for individuals that are not perfectly rational: \( B_c(A \land B) \rightarrow B_cA \) and \( B_cA \leftrightarrow \neg C_c \neg A \). Two anonymous reviewers independently raised this worry. Should there not be some connections between the truth-values of different beliefs (or propositions about beliefs), for example between \( B_c(A \land B) \) and \( B_cA \), even for individuals that are not perfectly rational? As one reviewer pointed out, being not perfectly rational does not mean being ignorant. However, this does not strike me as a particularly serious problem. The point is that (for example) \( B_c(A \land B) \) does not entail \( B_cA \) (for every c). This does not necessarily mean that there are no interesting relations between the truth-values of \( B_c(A \land B) \) and \( B_cA \) (for example). Many people who believe that A-and-B probably also believe that A; but, according to our systems, it is not logically necessary that someone who believes that A-and-B also believes that A. Obviously, \( B_c(A \land B) \) does not entail that it is not the case that \( B_cA \) in any system in this paper. All our systems are consistent with the proposition that everyone who believes that A-and-B also, in fact, believes that A. It is, in principle, possible to add \( \Pi(x)(B_c(A \land B) \rightarrow B_cA) \) as an ‘axiom’ to any system in this paper or to use this formula as a premise in various derivations. Yet, it is obviously conceivable that someone believes that A-and-B without believing that A, and we can think and reason about individuals that are not perfectly rational. In classical doxastic logic this is impossible, we cannot speak about and reason about an agent that, for example, believes that A-and-B without believing that A (at least we cannot do this in any natural way). This is clearly implausible. So even if it were true that everyone who believes that A-and-B also, in fact, believes that A, it would not be a logical truth. I think that one of the reviewers is correct when he or she says that ‘being not perfectly rational does not mean being ignorant’. But the systems that I discuss do not entail that an agent is ignorant if she is not perfectly rational. A person that is not perfectly rational can still be rational in many ways.

5. For more on these concepts, see, for example Bostock (1997, p. 79), Church (1996, p. 170) and Epstein (2006, p. 65). The first occurrence of x in a formula of the form \( B_xA \) (and \( C_xA \)) is free.

6. For more on the concept of rationality, see Mele (2004).

7. The idea of using matrices is borrowed from Priest (2005, Ch. 1–2).

8. In the philosophy of religion, the concept of epistemic omniscience has been discussed for a long time (see, for example, Taliaferro, 1998, Chapter 5). We can say that an individual is epistemically omniscient iff she knows everything (i.e. everything that is true). If knowledge implies belief, it follows that everyone who is epistemically omniscient is doxastically omniscient. Theistic philosophers usually think that God is epistemically omniscient. If this is correct (and God exists), then God is doxastically omniscient. It also follows that if an individual is epistemically omniscient and doxastically consistent, she is infallible. And if someone is infallible and epistemically omniscient, she believes something iff she knows it (given that knowledge implies belief). If knowledge implies truth, it follows that every (epistemically) omniscient individual knows something iff it is true and, hence, believes something iff it is true. For more on epistemology in general and the concept of knowledge in particular, see, for example, Niiniluoto, Sintonen, and Wolenski (2004) and Sosa and Kim (2000). I will not say anything more about epistemic omniscience or about the philosophy of religion in this paper.

9. \( C-UR \) is a theoretically interesting condition that is philosophically problematic. If all perfectly rational individuals necessarily are perfectly rational, a being that is in fact not perfectly rational cannot be perfectly rational in some other possible world. Yet, no actual human being is, in fact, perfectly rational (or at least so it seems). If this is true,
no actual human being could have been perfectly rational. Nevertheless, it seems perfectly possible and interesting to consider what would have been the case if some actual human being would have been perfectly rational. So, if we want to use counterfactuals with such ‘antecedents’, we should probably reject $C-UR$. Of course, our systems do not include any symbols for counterfactuals. But they could, in principle, be augmented with operators of this kind.

10. Epistemic and doxastic logic have usually been studied axiomatically. Ajspur, Goranko, and Shkatov (2013) introduces a tableau-based decision procedure for a multiagent epistemic logic. See also Halpern and Moses (1992).

11. We have used the weakest system $D$ to construct our tableau. Hence, when I say that the conclusion is ‘not derivable’, I mean that it is ‘not derivable in the system $D’$. Yet, it is possible to show that the conclusion is not derivable from the premises in any system in this paper.

12. Since we use $D$, ‘invalid’ here means ‘invalid in the class of all models’. However, it is also invalid in all other classes of models we consider in this paper. It is left to the reader to verify this.

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