In this paper, we solve the Duffing equation for given initial conditions. We introduce the concept of the discriminant for the Duffing equation and we solve it in three cases depending on sign of the discriminant. We also show the way the Duffing equation is applied in soliton theory.

1. Introduction

The nonlinear equation describing an oscillator with a cubic nonlinearity is called the Duffing equation. Duffing [1], a German engineer, wrote a comprehensive book about this in 1918. Since then there has been a tremendous amount of work done on this equation, including the development of solution methods (both analytical and numerical) and the use of these methods to investigate the dynamic behavior of physical systems that are described by the various forms of the Duffing equation. Because of its apparent and enigmatic simplicity, and because so much is now known about the Duffing equation, it is used by many researchers as an approximate model of many physical systems or as a convenient mathematical model to investigate new solution methods [2–7]. This equation exhibits an enormous range of well-known behavior in nonlinear dynamical systems and is used by many educators and researchers to illustrate such behavior. Since the 1970s, it has become really popular with researchers into chaos, as it is possibly one of the simplest equations that describes chaotic behavior of a system. This equation is also useful in the study of soliton solutions to important physics models such as KdV equation, mKdV equation, sine-Gordon equation, Klein–Gordon equation, nonlinear Schrodinger equation, and shallow water wave equation [8–18].

2. Undamped and Unforced Duffing Equation

Let $p$, $q$, $u_0$, and $u_0'$ be real numbers. The general solution to the undamped and unforced Duffing equation $u''(t) + pu(t) + qu^3(t) = 0$ may be expressed in terms of any of the twelve Jacobian elliptic functions, as shown in Table 1.

In this section, we will solve the initial value problem

$$u''(t) + pu(t) + qu^3(t) = 0, \quad u(0) = u_0, u'(0) = u'_0 \left( u_0^2 + u_0'^2 \right) q \neq 0. \quad (1)$$

The number

$$\Delta = \left( p + qu_0^2 \right)^2 + 2qu_0^2 \quad (2)$$
is called the discriminant for problem (1).

2.1. First Case: $\Delta > 0$. In the case, when $\dot{u}_0 = 0$, we get $\Delta = (p + q\dot{u}_0^2)^2$ so that $p + q\dot{u}_0^2 \neq 0$ and the problem reduces to

$$u''(t) + pu(t) + qu^3(t) = 0, \quad \text{subject to } u(0) = u_0, u'(0) = 0.$$  \hfill (3)

Its solution is given by

$$u(t) = u_0 \text{cn} \left( \sqrt{p + q\dot{u}_0^2} \ t, \frac{\dot{u}_0^2}{2(p + q\dot{u}_0^2)} \right).$$  \hfill (4)

Let $\dot{u}_0 \neq 0$. First of all, observe that if $u = u(t)$ is a solution to the ode $u''(t) + pu(t) + qu^3(t) = 0$, then $U(t) = u(t + C)$ is also a solution for any constant $C$. Secondly, let $y(t) = c_1 \text{cn} (\sqrt{\omega}, m) (c_1 = \text{a nonzero constant})$ be the Jacobi elliptic function cn with modulus $m$ and parameter $k$ defined by $k^2 = m$. We have

$$y''(t) + (1 - 2m)\omega y(t) + \frac{2m \omega}{c_1^2} y^3(t) = 0, \quad \text{for any } t, \text{any } c_1 \neq 0.$$  \hfill (5)

Therefore, comparing (1) and (5) gives

$$m = \frac{q_1^2}{2(p + q_1^2)},$$  \hfill (6)

$$\omega = p + q_1^2,$$

and we conclude that the analytic function

$$u = u(t) = c_1 \text{cn} \left( \sqrt{p + q_1^2} \ t + c_2, \frac{q_1^2}{2(p + q_1^2)} \right), \quad c_1 \neq 0,$$  \hfill (7)

is the general solution to the Duffing equation $u''(t) + pu(t) + qu^3(t) = 0$ for arbitrary constants $c_1$ and $c_2$. The values of these constants are determined from the initial conditions $u(0) = u_0$ and $u'(0) = \dot{u}_0$.

We have

$$\text{cn}(\text{cn}^{-1}(u_0, m), m) = u_0, \quad \text{so that } c_2 = \text{cn}^{-1}(u_0, m),$$

$$c_1 = \pm \sqrt{-\frac{p \pm \sqrt{(p + q_1^2)^2 + 2p\dot{u}_0^2}}{q}}, \quad \text{if } \dot{u}_0 < 0,$$

$$c_1 = -\sqrt{-\frac{p \pm \sqrt{(p + q_1^2)^2 + 2p\dot{u}_0^2} - p}{q}}, \quad \text{if } \dot{u}_0 \geq 0,$$  \hfill (13)

$$c_2 = \text{cn}^{-1} \left( \frac{u_0}{c_1} \left| \frac{q_1^2}{2(p + q_1^2)} \right\rangle, \quad \text{if } u_0 \neq 0,$$

$$c_2 = K \left( \frac{q_1^2}{2(p + q_1^2)} \right), \quad \text{if } u_0 = 0.$$  \hfill (14)

Making use of the addition formula

$$\text{cn}(u + v) = \frac{\text{cn}(u)\text{cn}(v) - \text{sn}(u)\text{dn}(u)\text{sn}(v)\text{dn}(v)}{1 - \text{sn}^2(u)\text{sn}^2(v)},$$

the solution may also be written in the form
2.2. Second Case: \( \Delta < 0 \). Define

\[
\delta = 2pu_0^2 + qu_0^4 + 2u_0^2.
\]

Since \( \Delta < 0 \), necessarily \( q < 0 \). From the equality

\[
\omega = \sqrt{\Delta},
\]

\[
m = \frac{1}{2} \left( 1 - \frac{p}{\sqrt{\Delta}} \right),
\]

\[
b = \frac{1}{2} \left( \frac{p + qu_0^2}{\sqrt{\Delta}} - 1 \right).
\]

The solution is periodic and its main period equals

\[
T = \frac{4K(m)}{\sqrt{\omega}}, \quad \text{for } 0 \leq m < 1.
\]

In the case, when \( m > 1 \), we make use if the identities

\[
\begin{align*}
\text{cn}(t, m) &= \text{dn} \left( \frac{\sqrt{m}t}{m}, \frac{1}{m} \right), \\
\text{sn}(t, m) &= \frac{1}{\sqrt{m}} \text{sn} \left( \frac{\sqrt{m}t}{m}, \frac{1}{m} \right), \\
\text{dn}(t, m) &= \text{cn} \left( \frac{\sqrt{m}t}{m}, \frac{1}{m} \right).
\end{align*}
\]

The main period will then be

\[
T = \frac{2K(1/m)}{\sqrt{m\omega}}, \quad m > 0.
\]

If \( m < 0 \), we transform the solution by means of the following identities:

\[
u(t) = -\sqrt{5 + 2\sqrt{6}\text{cn} \left( \frac{3}{4} \sqrt{4} \ 3t + \text{cn}^{-1} \left( \frac{1}{2} \left( \frac{5}{\sqrt{5 + 2\sqrt{6}}} \right) \right) \right) + \text{cn}^{-1} \left( \frac{1}{2} \left( \frac{5}{\sqrt{5 + 2\sqrt{6}}} \right) \right)},
\]

\[
= -3.14626\text{cn} \left( -2.21336t - (5.50011 - 3.13354\sqrt{-1}) \right) \left( 1.01031 \right).
\]

An equivalent expression without the imaginary unit is

\[
u(t) = \frac{\text{dn}(wt) + 2(\sqrt{6} - 2)\text{cn}(wt) \ | \ \text{sn}(wt)}{1 - 2(\sqrt{6} - 2)\text{sn}(wt)^2},
\]

See Figure 2 for a comparison with Runge–Kutta numerical solution (dashed curve).

Example 2. Let \( p = -5, q = 1, y_0 = 1, \) and \( \dot{y}_0 = 2. \) The solution to the i.v.p.

\[
u''(t) - 5\nu(t) + \nu^3(t) = 0, \quad \nu(0) = 0, \nu'(0) = 2,
\]

is given by

\[
u(t) = \frac{2\sqrt{3}\text{dn}(\sqrt{3}t \ | \ (1/3))\text{sn}(\sqrt{3}t \ | \ (1/3))}{\text{sn}(\sqrt{3}t \ | \ (1/3))^3 - 3}
\]

\[
= \frac{2\sqrt{3}}{3} \text{sd} \left( \sqrt{3}t \ | \ \frac{1}{3} \right).
\]

This solution is periodic with main period \( T = 4K(1/3)/\sqrt{3}. \) See Figure 1 for a comparison with Runge–Kutta numerical solution (dashed curve).

Example 1. Let \( p = 1, q = 1, y_0 = 0, \) and \( \dot{y}_0 = 2. \) The solution to the i.v.p.

\[
u''(t) + \nu(t) + \nu^3(t) = 0, \quad \nu(0) = 0, \nu'(0) = 2,
\]

is by

\[
u(t) = \text{cd} \left( \sqrt{1 - m^2}t \ | \ \frac{m}{m - 1} \right)
\]

\[
= \frac{1}{\sqrt{1 - m}} \text{sd} \left( \sqrt{1 - m^2}t \ | \ \frac{m}{m - 1} \right)
\]

\[
= \text{nd} \left( \sqrt{1 - m^2}t \ | \ \frac{m}{m - 1} \right).
\]

Remember that \( \text{cd} = \text{cn}/\text{dn}, \) \( \text{sd} = \text{sn}/\text{dn}, \) and \( \text{nd} = 1/\text{dn}. \) For reference, Tables 2–4 give useful conversion formulas.
| $t \mid m$ | $m > 1$ |
|---|---|
| $cn (t \mid m)$ | $= \sqrt{\beta t \mid 1/m}$ |
| $sn (t \mid m)$ | $= \sqrt{\beta t \mid 1/m}$ |
| $dn (t \mid m)$ | $= \sqrt{\beta t \mid 1/m}$ |
| $nc (t \mid m)$ | $= 1/\sqrt{\beta t \mid 1/m}$ |
| $ns (t \mid m)$ | $= \sqrt{\beta \ln (\sqrt{\beta t \mid 1/m})}$ |
| $nd (t \mid m)$ | $= 1/c\sqrt{\beta t \mid 1/m}$ |
| $sd (t \mid m)$ | $= \sqrt{\beta \ln (\sqrt{\beta t \mid 1/m})}$ |
| $ds (t \mid m)$ | $= \sqrt{\beta \ln (\sqrt{\beta t \mid 1/m})}$ |
| $sc (t \mid m)$ | $= \sqrt{\beta \ln (\sqrt{\beta t \mid 1/m})}$ |
| $cd (t \mid m)$ | $= \sqrt{\beta \ln (\sqrt{\beta t \mid 1/m})}$ |
| $dc (t \mid m)$ | $= \sqrt{\beta \ln (\sqrt{\beta t \mid 1/m})}$ |
Table 3: Conversion formulas for $m < 0$.

| Function | Formula |
|----------|---------|
| $cn(t | m) = cd(\sqrt{1 - mt} | (m/m - 1))$ | $sn(t | m) = sd(\sqrt{1 - mt} | (m/m - 1)/(\sqrt{1 - m})$ |
| $ss(t | m) = \sqrt{1 - m} / (\sqrt{1 - mt} | (m/m - 1))$ | $ds(t | m) = \sqrt{1 - m} / (\sqrt{1 - mt} | (m/m - 1))$ |
| $ns(t | m) = \sqrt{1 - m} / (\sqrt{1 - mt} | (m/m - 1))$ | $cs(t | m) = sd(\sqrt{1 - mt} | (m/m - 1)/(\sqrt{1 - m}$ |
| $nc(t | m) = \sqrt{1 - m} / (\sqrt{1 - mt} | (m/m - 1))$ | $cs(t | m) = sd(\sqrt{1 - mt} | (m/m - 1)/(\sqrt{1 - m}$ |
| $nc(t | m) = \sqrt{1 - m} / (\sqrt{1 - mt} | (m/m - 1))$ | $dc(t | m) = sd(\sqrt{1 - mt} | (m/m - 1)/(\sqrt{1 - m}$ |
| $dc(t | m) = sd(\sqrt{1 - mt} | (m/m - 1)/(\sqrt{1 - m}$ | $dn(t | m) = sd(\sqrt{1 - mt} | (m/m - 1)$ |
Table 4: Jacobi imaginary transformation.

| Function | Transformation |
|----------|----------------|
| \( cn(it, m) = nc(t, 1 - m) \) | \( sn(it, m) = i(sn(t, 1 - m)/cn(t, 1 - m)) \) |
| \( nc(it, m) = cn(t, 1 - m) \) | \( nd(it, m) = cn(t, 1 - m)/dn(t, 1 - m) \) |
| \( cd(it, m) = 1/dn(t, 1 - m) \) | \( dc(it, m) = dn(t, 1 - m) \) |
| \( dn(it, m) = dn(t, 1 - m)/cn(t, 1 - m) \) | \( sd(it, m) = i(sn(t, 1 - m)/dn(t, 1 - m)) \) |
| \( ds(it, m) = -i(dn(t, 1 - m)/sn(t, 1 - m)) \) | \( sc(it, m) = isn(t, 1 - m) \) |
| \( cs(it, m) = -i/sn(t, 1 - m) \) |
\[ v''(t) + \alpha v(t) + \beta v^3(t) = 0, \quad v(0) = v_0 := \frac{A + u_0}{A - u_0}, \quad v_0 := \frac{2Au_0}{(A - u_0)^2}. \]  

We have

\[ v''(t) = -\alpha v(t) - \beta v^3(t), \]  

\[ v'(t)^2 = D - \alpha v(t)^2 - \frac{1}{2} \beta v(t)^4, \]  

\[ D = \frac{2A^4 \alpha + A^3 \beta + 4A^3 u_0 \beta - 4A^2 u_0^2 \alpha + 6A^2 u_0^2 \beta + 8A^2 u_0^2 + 4Au_0^3 \beta + 2u_0^4 \alpha + u_0^4 \beta}{2(A - u_0)^4}. \]

Inserting the ansatz (30) into the ode \( u''(t) + pu(t) + qu^3(t) = 0 \) and taking into account (30) and (31), we get

\[ u''(t) + pu(t) + qu^3(t) = \frac{A}{(1 + v(t))^3}\left( A^2 q + p - 2\beta \right)v(t)^3 + (p + 2\alpha - 3A^2 q)v(t)^2 - \left( p + 2\alpha - 3A^2 q \right)v(t) - \left( A^2 q + 4D + p \right). \]
Equating to zero the coefficients of \( v^0(t), v^1(t), v^l(t), \) and \( v^f(t) \) in (32), we obtain an algebraic system. Solving it gives

\[
\alpha = \frac{1}{2}(3A^2q - p), \\
\beta = \frac{1}{2}(A^2q + p), \\
A = \sqrt{\frac{\delta}{-q}}
\]

where

\[
\bar{\omega} = \sqrt{\Delta}, \\
m = \frac{1}{2} - \frac{\alpha}{2\sqrt{\Delta}}, \\
\bar{b} = \frac{\alpha + \beta \nu_0^2}{2\sqrt{\Delta}} - \frac{1}{2}, \\
\nu_0 = \frac{A + u_0}{A - u_0}, \\
\nu_0 = \frac{2A\dot{u}_0}{(A - u_0)^2}, \\
\Delta = (\alpha + \nu_0^2 \beta^2) + 2\nu_0^2 \beta.
\]

The values of \( \alpha, \beta, \) and \( A \) are found from (33).

2.3. Third Case: \( \Delta = 0 \). When the discriminant vanishes, then \( q < 0 \) and the only solution to problem (1) with \( u'(0)^2 = \dot{u}_0^2 \) is

\[
u(t) = -\sqrt{\frac{p^2}{q}} \tanh \left( \sqrt{\frac{p}{2}} t - \tanh^{-1} \left( \sqrt{\frac{q}{p}} u_0 \right) \right),
\]

2.3. Third Case: \( \Delta = 0 \). When the discriminant vanishes, then \( q < 0 \) and the only solution to problem (1) with \( u'(0)^2 = \dot{u}_0^2 \) is

\[
u(t) = -\sqrt{\frac{p^2}{q}} \tanh \left( \sqrt{\frac{p}{2}} t - \tanh^{-1} \left( \sqrt{\frac{q}{p}} u_0 \right) \right),
\]

Observe that

\[
(a + \nu_0^2 \beta)^2 + 2\nu_0^2 \beta = 2\left( \sqrt{p^2(q\delta - (-q)\delta)} > 0. \right)
\]

Thus, the Duffing equation (29) has a positive discriminant. The solution to the i.v.p. (3) is then given by

\[
\frac{2\sqrt{\delta - q}}{1 + (\nu_0 \csc(\sqrt{\Delta} x | \bar{m}) + (\nu_0 / \sqrt{\omega}) \csc(\sqrt{\omega} x | \bar{m})) \sin(\sqrt{\omega} x | \bar{m})} \left[ \frac{1}{1 + 2\nu_0 \csc(\sqrt{\omega} x | \bar{m})} \right]
\]

which may be verified by direct computation.

3. New Trigonometric Jacobian Functions

Define the generalized cosine and sine functions as follows:

\[
\cos_\lambda(t) = \frac{-\sqrt{1 + \lambda} \cos(\sqrt{1 + \lambda} t)}{\sqrt{1 + \lambda} \cos^2(\sqrt{1 + \lambda} t)},
\]

\[
\sin_\lambda(t) = \frac{-\sqrt{1 + \lambda} \sin(\sqrt{1 + \lambda} t)}{\sqrt{1 + \lambda} \cos^2(\sqrt{1 + \lambda} t)}.
\]

Our aim is to find some \( \lambda \) so that

\[
\cn(t, m) = \cos_\lambda(t).
\]

Define

\[
R(t) = \frac{m x(t)^4}{4} + \frac{1}{2} (1 - 2m)x(t)^2 + \frac{1}{2} x'(t)^2 + \frac{1}{2} (m - 1).
\]

Observe that \( R(t) = 0 \) when \( x(t) = \cn(t, m) \). Let \( x(t) = \cos_\lambda(t) \). We have

\[
R(t) = \frac{1}{2(\lambda^2 \cos^2(\sqrt{\lambda + 1} t) + 1)^3}
\[
\cdot \left( \frac{1}{32} (14\lambda^2 + 2(m + 12)\lambda + 12m) + \frac{1}{32} (-17\lambda^2 - 32\lambda - \lambda m - 16m) \cos(2\sqrt{\lambda + 1} t)
\]

\[
+ \frac{1}{32} (2\lambda^2 + 8\lambda - 2\lambda m + 4m) \cos(4\sqrt{\lambda + 1} t) + \frac{1}{32} (\lambda^2 + \lambda m) \cos(6\sqrt{\lambda + 1} t) \right).
\]
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We will choose $\lambda$ so that

$$14\lambda^2 + 2(m + 12)\lambda + 12m = 0. \quad (42)$$

Define

$$\lambda = \frac{1}{14} \left( \sqrt{m^2 - 144m + 144 - (m + 12)} \right). \quad (43)$$

Define

$$cn_m(t) = \cos_{\lambda}(t), \quad \text{for} \quad -1 \leq m \leq 0.85,$$

$$cn_m(t) = \frac{1}{8} (m - 1) (\sinh(2t) - 2t) \tanh(t) \sec h(t) + \sec h(t), \quad \text{for} \quad 0.85 < m \leq 1,$$

$$\lambda = \frac{1}{14} \left( \sqrt{m^2 - 144m + 144 - (m + 12)} \right),$$

$$sn_m(t) = \sin_{\lambda}(t), \quad \text{for} \quad -1 \leq m \leq 0.85,$$

$$sn_m(t) = \frac{1}{4} (m - 1) \left( t \sec h^2(t) - \tanh(t) \right) + \tanh(t), \quad \text{for} \quad 0.85 < m \leq 1,$$

$$\lambda = \frac{1}{14} \left( \sqrt{m^2 - 144m + 144 - (m + 12)} \right).$$

Define

$$dn_m(t) = \sqrt{1 - msn_m^2(t)},$$

$$nc_m(t) = \frac{1}{cn_m(t)},$$

$$ns_m(t) = \frac{1}{sn_m(t)},$$

$$nd_m(t) = \frac{1}{dn_m(t)},$$

$$sc_m(t) = \frac{sn_m(t)}{cn_m(t)},$$

$$cs_m(t) = \frac{cn_m(t)}{sn_m(t)},$$

$$sd_m(t) = \frac{sn_m(t)}{dn_m(t)},$$

$$ds_m(t) = \frac{dn_m(t)}{sn_m(t)},$$

$$cd_m(t) = \frac{cn_m(t)}{dn_m(t)},$$

$$dc_m(t) = \frac{dn_m(t)}{cn_m(t)}.$$

The obtained approximations are good. This is seen from Tables 5 and 6.

We now will introduce new Jacobian “trigonometric functions” as follows:

We extend the new functions (44)–(47) $cn_m(t)$ and $sn_m(t)$ for $m > 1$ and $m < 0$ and imaginary argument it using Tables 1–3 replacing the $cn(t,m)$ with $cn_m(t)$ and $cn_m(t)$ with $cn_m(t)$ and so on.

4. Applications in Physics

Many partial differential equations arising in soliton theory may be reduced to odes or systems of odes by means of a traveling wave transformation. These odes are generally nonlinear and some of them are Duffing type equations. Let us consider some important models of soliton theory.

4.1. The Klein–Gordon–Zakharov (KGZ) Equation in Plasmas. The KGZ equation reads

$$q_{tt} - k^2 q_{xx} + a q + br q + c |q|^2 q = 0,$$

$$r_{tt} - k^2 r_{xx} = d (|q|^2)_{xx}. \quad (48)$$

We transform the KGZ by means of the traveling wave substitution

$$q(x,t) = u(x - \lambda t) \exp \left( \sqrt{-1} (-kx + \omega t + \theta) \right),$$

$$r(x,t) = v(x - \lambda t), \quad (49)$$

to obtain the system
\[
\begin{align*}
\frac{u''(\xi)}{(k - \lambda)(k + \lambda)} + \frac{u(\xi)(a + bv(\xi) + \kappa^2 k^2 - \omega^2) + cu(\xi)^3 + 2iu' (\xi)(kk^2 - \lambda \omega)}{(k - \lambda)(k + \lambda)} &= 0, \quad \xi = x - \lambda t, \\
\frac{v''(\xi)}{(k - \lambda)(k + \lambda)} + 2d\left(\frac{u(\xi)u''(\xi) + u'(\xi)^2}{(k - \lambda)(k + \lambda)}\right) &= 0, \quad \xi = x - \lambda t.
\end{align*}
\]

We choose \( \lambda \) so that \( \kappa = (\lambda \omega/k^2) \) and integrating the equation (51) twice taking null integration constants, we obtain
and the problem reduces to solve a Duffing equation.

4.2. The Sine-Gordon Equation. This is the equation

\[ \nu_{tt} = \alpha \nu_{xx} + \beta \sin(\nu). \]  

(53)

This important model appears in differential geometry and relativistic field theory. It is denominated following its similar form to the Klein–Gordon equation. The equation, as well as several solution techniques, was known in the 19th century, but the equation grew greatly in importance when it was realized that it led to solutions (“kink” and “antikink”) with the collision properties of solitons.

\[ 2(a - \lambda^2)(u(\xi)^2 - 1)u''(\xi) + 2u(\xi)\left(\lambda^2 - \alpha\right)u'(\xi)^2 - \beta\left(\lambda^2 - 1\right)^2 = 0. \]  

(55)

It may be easily verified that equation (55) holds for any solution \( u = u(\xi) \) of the Duffing equation

\[ u''(\xi) + \frac{\beta}{\alpha - \lambda^2} u(\xi) + \frac{2\beta}{\lambda^2 - \beta} u^3(\xi) = 0. \]  

(56)

4.3. The Pendulum Equation. This equation reads

\[ \theta''(t) + \omega^2 \sin(\theta(t)) = 0, \]  

\[ \theta(0) = \theta_0, \]  

\[ \theta'(0) = \theta_0. \]  

(57)

Let

\[ \theta(t) = 2 \arcsin(u(t)). \]  

(58)

Inserting ansatz (58) into (57) gives

\[ u(t)\left(u'(t)^2 + \omega^2(u(t)^2 - 1)^2\right) - (u(t)^2 - 1)u''(t) = 0. \]  

(59)

Equation (59) is satisfied for any solution \( u = u(t) \) to Duffing equation obeying

\[ u''(t) + \left(\rho u_0^2 + \frac{1}{2} q u_0^4 + \frac{1}{2} \omega^2 u_0^2\right) u(t) - 2 \omega^2 u(t)^3 = 0, \]  

\[ u(0) = u_0, u'(0) = u_0, \]  

where \( u_0 = u(0) = \sin\left(\frac{\theta_0}{2}\right), u_0 = u'(0) = \frac{\theta_0}{2} \cos\left(\frac{\theta_0}{2}\right) \).

(60)

4.4. The KdV Equation. This equation originated from soliton theory. It reads

\[ \partial_t u + \partial_x uu + a u \partial_x u = 0, a = \text{non zero constant}. \]  

(61)

Let \( u = \nu(x + \lambda t) = u(\xi) \). The traveling wave substitution gives \( \lambda \nu'(\xi) + a \nu(\xi) \nu'(\xi) + \nu''(\xi) = 0 \). Integrating once, we obtain

\[ \lambda \nu(\xi) + a \frac{\nu(\xi)^3}{3} + \nu''(\xi) = C, \]  

(62)

where \( C \) is the constant of integration. We seek a solution to the nonlinear ode (62) in the ansatz form

\[ \nu(\xi) = R + S y^2(\xi), \]  

where \( y''(\xi) + p y(\xi) + q y^3(\xi) = 0 \),
where the constants \( p, q, R, \) and \( S \) are to be determined. Since \( \int y' (ξ) (y'' (ξ) + p y (ξ) + q y^3 (ξ)) dξ = \int 0 \ dξ = D, \) it is clear that

\[
y'' (ξ) = D - p y (ξ)^2 - \frac{q}{2 y (ξ)^2} \quad (64)
\]

Inserting the ansatz (63) into (62) and taking into account (63), we obtain

\[
\left( \frac{a S^2}{2} - 3 q S \right) y (ξ)^4 + (a R S - 4 p S + \lambda S) y (ξ)^2 + \frac{a R^2}{2} + C + 2 D S + \lambda R = 0. \quad (65)
\]

4.5. The Nonlinear Schrödinger Equation. The nonlinear Schrödinger equation is among the most prominent equations in nonlinear science, applying to hydrodynamics, plasma physics, molecular biology, and optics. It has been studied for more than 40 years, and it is employed in numerous fields beyond plasma physics and nonlinear optics, where it originally appeared. The nonlinear Schrödinger equation (NLSE) is in the following form:

\[
\sqrt{-1} \frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^2} + \gamma |u|^2 u = 0, \quad (67)
\]

where \( \gamma \) is a non-zero real constant and \( u = u(x,t) \) is a complex valued function of two real variables \( x,t \). The Schrödinger equations occur in various areas of physics, including nonlinear optics, plasma physics, superconductivity, and quantum mechanics. The NLSE (67) exhibits soliton and periodic cnoidal wave solutions. Let

\[
u(t, x) = e^{i(a x + \beta t)} \eta(x - 2 \alpha t). \quad (68)
\]

Under this transformation, the NLSE (67) takes the form

\[
v'' (ξ) - \left( \alpha^2 + \beta \right) v(ξ) + 2 \gamma v^3 (ξ) = 0, \quad (69)
\]

which is a Duffing equation.

**Data Availability**

No data were used to support this study.
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