On the invariants of base changes of pencils of curves, II

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Introduction

Semistable reductions of pencils of curves have been studied by many authors in various ways. (cf. [AW, De, DM, Xi3]). In this part of the series, we shall investigate semistable reductions from the point of view of numerical invariants. As an application, we obtain two numerical criterions for a base change to be stabilizing, and for a fibration to be isotrivial. We also obtain a canonical class inequality for any fibration. Some other applications are presented.

Let \( f : S \rightarrow C \) be a fibration of a smooth complex projective surface \( S \) over a curve \( C \), and denote by \( g \) the genus of a general fiber of \( f \). We assume that \( g > 0 \) and \( S \) is relatively minimal with respect to \( f \), i.e., \( S \) has no \((-1)\)-curves contained in a fiber of \( f \). The basic relative numerical invariants of \( f \) are defined as follows,

\[
\begin{align*}
\chi_f &= \chi(\mathcal{O}_S) - (g - 1)(g(C) - 1), \\
K_f^2 &= K_S^2 - 8(g - 1)(g(C) - 1), \\
e_f &= \chi_{\text{top}}(S) - 4(g - 1)(g(C) - 1).
\end{align*}
\]

These invariants are nonnegative integers satisfying the Noether equality \( 12\chi_f = K_f^2 + e_f \). We denote by \( \omega_{S/C} = \omega_S \otimes f^*\omega_C \) the relative canonical sheaf of \( f \), and \( K_{S/C} \) the relative canonical divisor corresponding to \( \omega_{S/C} \). Then \( \chi_f = \deg f_*\omega_{S/C} \) and \( K_f^2 = K_{S/C}^2 \). If \( g > 1 \) and \( f \) is not locally trivial, then \( \chi_f \) and \( K_f^2 \) are positive (cf. [Ar, Be2, Pa] or [BPV, Theorem 18.2] and [Xi1, Theorem 2]), in this case, we define the slope of \( f \) as

\[
\lambda_f = K_f^2/\chi_f.
\]

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\[ e_f = \sum_F e_F = \sum_F (\chi_{\text{top}}(F) - (2 - 2g)) \text{ is zero iff } f \text{ is smooth.} \]

A fiber of \( f \) is called semistable if it consists of simple components meeting normally. \( f \) is said to be semistable if every fiber of it is semistable.

Let \( \pi : \tilde{C} \to C \) be a base change of degree \( d \). Then the pullback fibration \( \tilde{f} : \tilde{S} \to \tilde{C} \) of \( f \) with respect to \( \pi \) is defined as the relative minimal model of the desingularization of \( S \times_C \tilde{C} \to \tilde{C} \). (cf. Sect. 1.3). Since \( g > 0 \), the relative minimal model is unique, hence \( f \) is determined uniquely by \( f \) and \( \pi \). Due to Kodaira’s classification of singular fibers, the semistable reduction of an elliptic fibration is quite clear, so we always assume that \( g \geq 2 \).

We define \( \chi_{\pi} = \chi_f - \frac{1}{d} \chi_{\tilde{f}}, \quad K_{\pi}^2 = K_f^2 - \frac{1}{d} K_{\tilde{f}}^2, \quad e_{\pi} = e_f - \frac{1}{d} e_{\tilde{f}} \)

as the basic numerical invariants of \( \pi \) with respect to \( f \). Obviously, they are rational numbers satisfying \( 12 \chi_{\pi} = K_{\pi}^2 + e_{\pi} \). Xiao [Xi4] and I [Ta1] proved that these invariants are nonnegative, and one of them vanishes if and only if \( \pi \) is an invariant base change. (See Definition 1.7).

**Definition I.** We shall call \( \pi \) a stabilizing (resp. trivial) base change if all of the fibers of \( \tilde{f} \) (resp. \( f \)) over the branch locus \( B_\pi \) of \( \pi \) are semistable. We shall also call \( \pi \) the semistable reduction of the fibers over \( B_\pi \).

The well-known semistable reduction theorem says that for any fibration \( f \), there exists a base change \( \pi \) such that \( \tilde{f} \) is semistable. In particular, let \( \pi \) be a base change totally ramified over \( F \) (i.e., over \( f(F) \)) and some other semistable fibers, and let \( F' \) be the minimal embedded resolution of \( F \). If the degree of \( \pi \) is exactly the least common multiple of the multiplicities of the components in \( F' \), then it is well-known that \( \pi \) is stabilizing. We shall call \( \pi \) the canonical semistable reduction of \( F \), and denote it by \( \phi_F \).

**Definition II.** For any fiber \( F \) of \( f \), we define its basic invariants to be the basic invariants of \( \phi = \phi_F \), and denote them respectively by

\[
c_1^2(F) = K_{\phi}^2, \quad c_2(F) = e_{\phi}, \quad \chi_F = \chi_{\phi}.
\]

We shall show that these invariants are independent of the choice of the base changes (Lemma 2.3). They are nonnegative rational numbers satisfying the Noether equality

\[
12 \chi_F = c_1^2(F) + c_2(F).
\]

We can also see that one of them vanishes iff \( F \) is semistable (Lemma 1.8). In fact, these invariants can be computed directly from the embedded resolution of \( F \) (see Theorem 3.1 for the formulas). For simplicity, if \( B = F_1 + \cdots + F_s \), then we define \( c_1^2(B) = c_1^2(F_1) + \cdots + c_1^2(F_s) \). Similarly, we can define \( c_2(B) \) and \( \chi_B \).

**Definition III.** A fibration \( f : S \to C \) is trivial if \( S \) is isomorphic to \( F \times C \) over \( C \). It is isotrivial if it becomes trivial after a finite base change.

If \( \tilde{f} \) is a semistable model of \( f \) under a semistable reduction \( \pi \), then a natural problem is:
What is the effect of a non-semistable fiber on the invariants of \( \tilde{f} \)?

(See [Xi2, Problem 7]). In this paper the effect is completely determined.

In what follows, we denote by \( \mathcal{B}_\pi = f^*(B_\pi) \) the locus of branched fibers, and by \( \mathcal{R}_\pi = \tilde{f}^*(R_\pi) \) the pullback fibers of \( \mathcal{B}_\pi \), where \( R_\pi \) is the inverse image of \( B_\pi \) under \( \pi \).

The main results of this paper are the following.

**Theorem A.** Let \( f : S \to C \) be a fibration, and let \( \pi : \tilde{C} \to C \) be a base change of degree \( d \). Then

\[
K_\pi^2 = c_1^2(B_\pi) - \frac{1}{d} c_1^2(R_\pi), \quad e_\pi = c_2(B_\pi) - \frac{1}{d} c_2(R_\pi), \quad \chi_\pi = \chi_{B_\pi} - \frac{1}{d} \chi_{R_\pi}.
\]

**Corollary.** For any fibration \( f : S \to C \) and any base change \( \pi : \tilde{C} \to C \), we have

1) \( K_\pi^2 \leq c_1^2(B_\pi), \quad e_\pi \leq c_2(B_\pi), \quad \chi_\pi \leq \chi_{B_\pi} \),

and one of the equalities holds iff \( \pi \) is stabilizing.

2) \[
\sum_F c_1^2(F) \leq K_f^2, \quad \sum_F \chi_F \leq \chi_f, \quad \sum_F c_2(F) \leq e_f,
\]

where \( F \) runs over all of the non-semistable fibers of \( f \). Furthermore, one of the first two equalities holds iff \( f \) is isotrivial, and the last equality holds iff the semistable model of \( f \) is smooth.

3) If \( f \) is non-isotrivial and \( \pi \) is stabilizing, then we have

\[
\lambda_{\tilde{f}} = \frac{K_f^2 - c_1^2(B_\pi)}{\chi_f - \chi_{B_\pi}}.
\]

Hence the slope of \( \tilde{f} \) is completely determined by the branched non-semistable fibers.

From the point of view of the fibration itself, we can define

\[
I_K(f) = K_f^2 - \sum_F c_1^2(F), \quad I_\chi(f) = \chi_f - \sum_F \chi_F, \quad I_e(f) = e_f - \sum_F c_2(F),
\]

where \( F \) runs over all of the singular fibers of \( f \). Then we know that these numbers are nonnegative invariants of \( f \), and the first two invariants measure the isotriviality of \( f \). (Note that \( K_f^2 \) and \( \chi_f \) measure the local triviality of \( f \). If \( f \) is semistable, then these invariants are nothing but the standard relative invariants of the fibration. Furthermore, if we think of \( \mathcal{R}_\pi \) (resp. \( \mathcal{B}_\pi \)) as the set of singular fibers of \( \tilde{f} \) (resp. \( f \)), then it is easy to see that Theorem A holds too. Hence Theorem A can be restated as
**Theorem A'.** For any fibration $f$ and any base change $\pi$ of degree $d$, we have

\[ I_K(\tilde{f}) = dI_K(f), \quad I_{\chi}(\tilde{f}) = dI_{\chi}(f), \quad I_e(\tilde{f}) = dI_e(f). \]

Due to Theorem A, the study of the invariants of stabilizing base changes can be reduced to the local study of $c_1^2(F)$ and $c_2(F)$. First of all, from definition, it is trivial to see that

\[ c_2(F) \leq e_F (=: \chi_{\text{top}}(F) - (2 - 2g)), \]

with equality iff the semistable model of $F$ is a smooth fiber. In Sect. 3.3, we obtain

**Theorem B.**

\[ c_1^2(F) \leq 2c_2(F), \]

with equality iff $F = nF_{\text{red}}$ and $F_{\text{red}}$ has at worst ordinary double points as its singularities. Hence for any stabilizing base change $\pi$, we have

\[ K_\pi^2 \leq 8\chi_\pi. \]

In particular, if $S$ admits an isotrivial fibration, then

\[ K_S^2 \leq 8\chi(O_S), \]

with equality iff all of the singular fibers are some multiples of smooth curves.

We show that $c_1^2(F)$ is in fact bounded by the genus $g$, i.e.,

**Theorem C.**

\[ c_1^2(F) \leq 4g - 4. \]

As an application of this inequality, we obtain the following **canonical class inequality.**

**Theorem D.** If $f$ is a non-trivial fibration of genus $g \geq 2$ with $s$ singular fibers, then

\[ K_{S/C}^2 \leq (2g - 2)(2g(C) - 2 + 3s), \]

and if the equality holds, then $f$ is smooth, i.e., $s = 0$.

Note that other canonical class inequalities are already known for non-trivial semistable fibrations:

\[ K_{S/C}^2 \leq (2g - 2)(2g(C) - 2 + s); \]
\[ K_{S/C}^2 < 4g(g - 1)(2g(C) - 2 + s); \]
\[ K_{S/C}^2 \leq 8(g - 1)^2(2g(C) - 2 + s). \]

These inequalities are due respectively to Vojta [Vo], Szpiro [Sz], Esnault and Viehweg [EV]. Note that in [Ta2], we have shown that the equality in Vojta’s inequality implies the smoothness of $f$. 

As another application, we find some new phenomena for fibrations. (Sect. 4.1). For example, from the corollary above, we can see that every semistable model \( \tilde{f} \) of \( f \) has the same slope
\[
\lambda_{\tilde{f}} = \frac{I_K(f)}{I_\chi(f)}.
\]
From Theorem B we know that if \( \lambda_f > 8 \), then any non-trivial stabilizing base change \( \pi \) makes the slope increase. We have also found some relationships between non-semistable fibers and the slope of a fibration.

Finally, in Sect. 4.3, we consider the computation of the Horikawa number of a genus 3 non-semistable fiber \( F \) through semistable reductions. We reduce it to the computation for its semistable models \( \tilde{F} \).

Notations. If \( D \) is a local curve and \( p \in D \), then we denote by \( \nu_p \) the multiplicity of \( D \) at \( p \), and denote respectively by \( \mu_p, \delta_p, k_p \) the Milnor number, geometric genus and the number of local branches of \( (D_{\text{red}}, p) \). Hence \( \mu_p = 2\delta_p - k_p + 1 \). If \( F \) is a curve on a smooth surface, then we denote by \( \mu_F \) the total Milnor number of the singularities of \( F \).

If \( a, b \) are two natural numbers, then we denote by \( (a, b) \) the greatest common divisor of \( a \) and \( b \), and let \( [a, b] = \frac{(a,b)^2}{ab} \). \([x]\) is the greatest integer \( \leq x \).

1 Preliminaries and technical lemmas

1.1 Embedded resolution of curve singularities

Let \( (B, p) \subset \mathbb{C}^2 \) be a local curve (not necessarily reduced) in a neighborhood \( U_0 \) of \( p = (0,0) \). Assume that \( (B_{\text{red}}, p) \) is a singular point, we say also that \( p \) is a singular point of \( B \).

**Definition 1.1.** The embedded resolution of a curve singularity \( (B, p) = (B_0, p_0) \) is a sequence
\[
(U_0, B_0) \xrightarrow{\sigma_1} (U_1, B_1) \xrightarrow{\sigma_2} \cdots \xrightarrow{\sigma_r} (U_r, B_r)
\]
satisfying the following conditions.

1. \( \sigma_i \) is the blowing-up of \( U_{i-1} \) at a singular point \( p_{i-1} \in B_{i-1} \) with \( \mu_{p_{i-1}} > 1 \).
2. \( B_{r,\text{red}} \) has at worst ordinary double points as its singularities.
3. \( B_i \) is the total transformation of \( B_{i-1} \).

It is well-known that embedded resolution exists and is unique for any curve singularity \( (B, p) \subset \mathbb{C}^2 \).

We denote by \( m_i \) the multiplicity of \( (B_{i,\text{red}}, p_i) \). Let
\[
\alpha_p = \sum_{i=0}^{r-1} (m_i - 2)^2.
\] (1)

If \( q \in B_r \) is a double point, and \( a_q, b_q \) are the multiplicities of the two components of \( (B_r, q) \), then we let
\[
\beta_p = \sum_{q \in B_r} [a_q, b_q].
\] (2)
Lemma 1.2.

\[ \mu_p = \sum_{i=0}^{r-1} (m_i - 1)(m_i - 2) + k_p - 1, \quad (3) \]

\[ \delta_p = \frac{1}{2} \sum_{i=0}^{r-1} (m_i - 1)(m_i - 2) + k_p - 1. \quad (4) \]

**Proof.** In the embedded resolution, we let \( E_1 \cap (B_1 - E_1) = p_1, \ldots, p_s \). Then by [Ta1, Lemma 1.3] we have

\[ \mu_p = (m_p - 1)(m_p - 2) - 1 + \sum_{i=1}^{s} \mu_{p_i}. \quad (5) \]

On the other hand, it is obvious that

\[ k_p = \sum_{i=1}^{s} (k_{p_i} - 1), \quad (6) \]

hence (3) can be obtained easily by using induction on \( r \), and (4) follows from (3) and \( \mu_p = 2\delta_p - (k_p - 1) \). \( \text{Q.E.D.} \)

Lemma 1.3. For any singular point \((B, p)\), we have

\[ \alpha_p + \beta_p \leq \mu_p. \quad (7) \]

**Proof.** First we prove (7) for the case \( m_p = 2 \), i.e., \((B_{\text{red}}, p)\) is a double point. Assume that \((B, p)\) is defined by \( f(x, y) = 0 \) at 0.

If \( f = x^a(y + k)^{b} \) and \( k = 1 \), then \( \alpha_p = 0, \mu_p = 1 \) and \( \beta_p = [a, b] \), (7) is obvious. If \( k > 1 \), then by the computation of the embedded resolution, we have

\[ \alpha_p = k - 1, \mu_p = 2k - 1, \beta_p = 1 - \frac{1}{k} + [a, k(a + b)] + [b, k(a + b)] \leq 1, \]

hence (7) holds strictly.

If \( f = (x^2 + y^{2k+1})^n \), then

\[ \alpha_p = k, \mu_p = 2k, \beta_p = \frac{3}{2}(1 - \frac{1}{2k + 1}), \]

thus we can see that \( \alpha_p + \beta_p \leq \mu_p \).

Now we assume that \( m_p \geq 3 \). In this case, we shall prove (7) by using induction on \( \mu_p \). From (5) we know \( \mu_{p_i} < \mu_p \), by induction hypothesis, we have \( \alpha_{p_i} + \beta_{p_i} \leq \mu_{p_i} \). On the other hand, we know

\[ \beta_p = \sum_{i=1}^{s} \beta_{p_i}, \quad \alpha_p = (m_p - 2)^2 + \sum_{i=1}^{s} \alpha_{p_i}, \]
from (5), (7) follows immediately. \textbf{Q.E.D.}

1.2 On the resolution of the singularity of \( z^d = f(x, y) \)

Now we assume that \((B, p)\) is defined by \( f(x, y) = 0 \) at \( p = (0, 0) \). Let \( \Sigma \subset \mathbb{C}^3 \) be a local surface defined by \( z^d = f(x, y) \), and let \( V_0 \) be the normalization of \( \Sigma \). Then, \( V_0 \) is a \( d \)-cyclic cover \( \pi_0: V_0 \to U_0 \), the singular points of \( V_0 \) (lying over \( p \)) can be resolved by the embedded resolution of \((B, p)\), it goes as follows.

Let \( V_r \) be the normalization of \( U_r \times_{U_0} V_0 \), and let \( \eta: M \to V_r \) be the minimal resolution of the singularities of \( V_r \).

\( V_0 \xleftarrow{\pi} V_r \xleftarrow{\eta} M \)

\( \pi_0 \) \hspace{1cm} \( \pi_r \) \hspace{1cm} \( \pi_r \eta \)

\( U_0 \xleftarrow{\sigma} U_r \xrightarrow{=} U_r \)

Then \( \pi_r \) is a cyclic covering branched along \( B_r \). If near \( q \in B_r \), \( B_r \) is defined by \( x^a y^b = 0 \), then \( V_r \) is locally the normalization of \( z^d = x^a y^b \), which are cyclic quotient singularities, hence can be resolved by Jung-Hirzebruch method (cf. [BPV, p.83]). Hence \( \phi = \tau \eta: M \to V_0 \) is the resolution of \( V_0 \), we shall call \( \phi \) the \textit{embedded resolution} of \( V_0 \).

Denote by \( E_p = \sum_{i=1}^{s} E_i \) the exceptional curves of \( \phi \), and let \( K_\phi = \sum_{i=1}^{s} r_i E_i \) be the rational canonical divisor of \( E_p \), which is determined uniquely by the adjunction formula \( K_\phi E_i + E_i^2 = 2p_\phi(E_i) - 2 \). Then \( K_\phi^2 \) is an invariant of the resolution \( \phi \). If \( \phi \) is minimal, then \( K_\phi^2 = K_p^2 \leq 0 \) is an invariant of the singularities of \( V_0 \), which is independent of the resolution. \( K_p^2 = 0 \) iff \( V_0 \) has at worst rational double points as its singularities. At the end of this paper, we shall present Jung-Hirzebruch’s resolution and the computation of \( K^2 \) for the singularity defined by \( z^d = x^a y^b \). The following lemma can also be obtained from this resolution. (cf. Sect. 5, or [Xi3] and [BPV, p.83]).

\textbf{Lemma 1.4.} If \((B, p)\) is defined by \( x^a y^b = 0 \), and \( d \) is divided by \( a \) and \( b \), then \( E_p \) consists of \( d_p = (a, b) \) disjoint curves of type \( A_n \), where

\[ n = [a, b] \frac{d}{d_p} - 1. \]

\textbf{Lemma 1.5.} Assume that \( d \) is divided by all of the multiplicities of the components in the embedded resolution \( B_r \). Then

\[ -\frac{1}{d} K_\phi^2 = \alpha_p. \]

The proof of this lemma will be given in Sect. 5.

Now we recall the normalization of \( \Sigma \). (cf. [Ta1, Lemma 2.1]).
Lemma 1.6. For any point $p \in B$, $\pi_0^{-1}(p)$ consists of $d_p = \gcd(d, n_1, \ldots, n_s)$ points if there are exactly $s$ components $\Gamma_1, \ldots, \Gamma_s$ passing through $p$, where $n_i$ is the multiplicity of $\Gamma_i$ in $B$.

1.3 The construction of base changes

In this section, we recall the construction of the pullback fibration $\tilde{f}$ of $f : S \rightarrow C$ under a base change. Let $\pi : \tilde{C} \rightarrow C$ be a base change of degree $d$. Then the pullback fibration $\tilde{f} : \tilde{S} \rightarrow \tilde{C}$ of $f$ with respect to $\pi$ is defined as the relative minimal model of the desingularization of $S \times_C \tilde{C} \rightarrow \tilde{C}$. In fact, the pullback fibration $\tilde{f} : \tilde{S} \rightarrow \tilde{C}$ can be constructed as follows.

Let $\rho_1 : S_1 \rightarrow S \times_C \tilde{C}$ be the normalization of $S \times_C \tilde{C}$, let $\rho_2 : S_2 \rightarrow S_1$ be the minimal desingularization of $S_1$. Then we have a fibration $f_2 : S_2 \rightarrow \tilde{C}$. Let $\tilde{\rho} : S_2 \rightarrow \tilde{S}$ be the contraction of $(-1)$-curves such that $\tilde{f} : \tilde{S} \rightarrow \tilde{C}$ is a relative minimal model. Since we have assumed that $g > 1$, $\tilde{\rho}$ is unique. Hence $\tilde{f} : \tilde{S} \rightarrow \tilde{C}$ is determined uniquely by $f$ and $\pi$.

\[
\begin{array}{cccccc}
\tilde{S} & \xleftarrow{\tilde{\rho}} & S_2 & \xrightarrow{\rho_2} & S_1 & \xrightarrow{\rho_1} & S \\
\downarrow{\tilde{f}} & & \downarrow{f_2} & & \downarrow{f_1} & & \downarrow{f} \\
\tilde{C} & \equiv & \tilde{C} & \equiv & \tilde{C} & \equiv & C \\
\end{array}
\]

Let $\Pi_2 = \Pi' \circ \rho_1 \circ \rho_2 : S_2 \rightarrow S$.

**Definition 1.7.** If $\pi : \tilde{C} \rightarrow C$ is a base change satisfying

$$\tilde{\rho}^* K_{\tilde{S}/\tilde{C}} \equiv \Pi_2^* K_{S/C},$$

then we shall call it an *invariant base change*.

In fact, we have shown that if $g \geq 2$, then $\pi$ is invariant iff the fibers $F$ over the branch locus are reduced and $F$ has at worst $d_F$-simple singularities, where $d_F$ is the greatest ramification index of $\pi$ over $f(F)$. (cf. [Ta1, Lemma 2.2 and 2.3]). A *d-simple singularity* is a simple curve singularity $f(x, y) = 0$ such that $z^d = f(x, y)$ is a simple surface singularity. Hence 2-simple is $ADE$, 3-simple is $A_1, \cdots, A_4$, 4 and 5-simple are $A_1, A_2$, $d$-simple is $A_1$ if $d > 5$.

In particular, we can see that the canonical semistable reduction $\phi_F$ of $F$ is invariant if and only if $F$ is semistable. On the other hand, In [Xi4, Ta1] we have proved that the basic invariants of a base change are nonnegative, and one of them vanishes iff the base change is invariant. Hence we have

**Lemma 1.8.** If $g \geq 2$, then the invariants $c_1^2(F)$, $c_2(F)$ and $\chi_F$ are nonnegative, and one of them vanishes if and only if $F$ is semistable.

Let $F$ be a singular fiber. We always denote by $F'$ the embedded resolution of $F$, and denote by $M_F$ the least common multiple of the multiplicities of the components in $F'$. 
2 On the invariants of a base change

2.1 Local computations of $K^2_\pi$

In this section, we first consider the computation of the invariant $K^2_\pi$ for a base change $\pi : \widetilde{C} \rightarrow C$. Without loss of generality, we assume that $\pi$ is totally ramified over $p_1, \ldots, p_s$. Let $\rho_2$ be the embedded resolution of singularities, let $F_1, \ldots, F_s$ be the fibers of $f$ corresponding to $p_1, \ldots, p_s$, and let $B_\pi = \sum_{i=1}^s F_i = \sum_{\Gamma} n_{\Gamma}\Gamma$. From Lemma 1.6, it is easy to see that

$$K_{S_2} \equiv \Pi^2_2 \left( K_S + \sum_{\Gamma \subset B_\pi} \left( 1 - \frac{(d, n_{\Gamma})}{d} \right) \Gamma \right) + K_{\rho_2}, \tag{10}$$

where $K_{\rho_2}$ is the rational canonical divisor of the exceptional set of $\rho_2$. On the other hand, we have

$$K_{\widetilde{C}} = \pi^* \left( K_C + \sum_{i=1}^s \left( 1 - \frac{1}{d} \right) p_i \right). \tag{11}$$

Note that $f_2^* \pi^* = \Pi^2_2 f^*$, hence from (10) and (11) we can obtain

$$K_{S_2/\widetilde{C}} = \Pi^2_2 \left( K_{S/C} - \sum_{i=1}^s H_{F_i} \right) + K_{\rho_2},$$

where $H_i = \sum_{\Gamma \subset F_i} h_{\Gamma}\Gamma$, $h_{\Gamma} = n_{\Gamma} - 1 - \frac{1}{d}(n_{\Gamma} - (d, n_{\Gamma}))$. Hence

$$dK^2_f - K^2_{f_2} = d \sum_{i=1}^s (2H_{F_i} K_S - H^2_{F_i}) - K^2_{\rho_2}. \tag{12}$$

If we let $K^2_\pi(f_2) = K^2_f - \frac{1}{d} K^2_{f_2}$, then

$$K^2_\pi = K^2_\pi(f_2) - \frac{1}{d} \#\{ (-1)\text{-curves contracted by } \widetilde{\rho} \}.$$ 

**Proposition 2.1.** With the notations above, we have

$$K^2_\pi(f_2) = \sum_{i=1}^s (2H_{F_i} K_S - H^2_{F_i}) - \sum_{i=1}^s \sum_{p \in F_i} \frac{1}{d} K^2_p. \tag{12}$$

In the case when $\pi$ is the base change of $F_1, \ldots, F_s$, if $d$ is divided by $M_{F_i}$, $i = 1, \ldots s$, we can see that $H_{F_i} = F_i - F_{i,\text{red}}$. Note that we have (cf. [Ta1, (7)])

$$e_F = \chi_{\text{top}}(F) - (2 - 2g) = 2N_F + \mu_F,$$

where $N_F = g - p_a(F_{\text{red}}) = \frac{1}{2}((F - F_{\text{red}})K_S - F^2_{\text{red}})$ is an invariant of $F$. From Lemma 1.5 we have
Proposition 2.2. If $d$ is divided by $M_F$, for all $i$, then

$$K_\pi^2(f_2) = \sum_{i=1}^{s} (4N_{F_i} + F_{i, \text{red}}^2) + \sum_{i=1}^{s} \sum_{p \in F_i} \alpha_p. \quad (13)$$

Note that the right hand side of (13) is independent of $d$.

2.2 Proof of Theorem A

We consider first the composition of base changes.

Let $\pi_1 : C_1 \to C$ and $\pi_2 : \tilde{C} \to C_1$ be two base changes, let $f_1$ be the pullback fibration of $f$ under $\pi_1$, and let $f_2$ be that of $f_1$ under $\pi_2$. By the universal property of fiber product and the uniqueness of the relative canonical model (when $g > 0$), we know $f_2$ is nothing but the pullback fibration $\tilde{f}$ of $f$ under $\pi = \pi_1 \circ \pi_2$. Hence we have the basic equalities:

$$K_\pi^2 = K_{\pi_1}^2 + \frac{1}{\deg \pi_1} K_{\pi_2}^2,$$

$$e_\pi = e_{\pi_1} + \frac{1}{\deg \pi_1} e_{\pi_2},$$

$$\chi_\pi = \chi_{\pi_1} + \frac{1}{\deg \pi_1} \chi_{\pi_2}. \quad (14)$$

Lemma 2.3. Let $f : S \to C$ be a fibration, and let $F_1, \cdots, F_s$ be fibers of $f$. Considering all of the semistable reductions $\pi$ of $F_1, \cdots, F_s$, we have that $K_\pi^2$, $e_\pi$ and $\chi_\pi$ are independent of $\pi$.

Proof. Let $\pi_1 : C_1 \to C$ and $\pi_2 : C_2 \to C$ be two semistable reductions of $F_1, \cdots, F_s$, let $\deg \pi_i = d_i$, $i = 1, 2$, and let $f_i$ be the pullback fibration of $f$ under $\pi_i$. We shall prove that

$$K_{\pi_1}^2 = K_{\pi_2}^2.$$

For this, we consider the pullback of $\pi_1$ and $\pi_2$,

$$\pi = \pi_1 \times_C \pi_2 : \tilde{C} = C_1 \times_C C_2 \to C.$$

Note that if necessary, we can choose $\tilde{C}$ to be the normalization of a component of $C_1 \times_C C_2$. Let $p_i : \tilde{C} \to C_i$ be the $i$-th projection, it is obvious that $\deg p_1 = d_2$, $\deg p_2 = d_1$. Then we have $\pi = p_1 \circ \pi_1 = p_2 \circ \pi_2$ (composition of base changes). $\pi_1$ and $\pi_2$ are semistable reductions, so the fibers of $f_i$ over $F_1, \cdots, F_s$ are semistable, and thus $p_i$ is an invariant base change. It implies that $K_{p_i}^2 = 0$ for $i = 1, 2$. Then by using the basic equalities (14), we have

$$K_\pi^2 = K_{\pi_1}^2 = K_{\pi_2}^2.$$

The proofs for $\chi_\pi$ and $e_\pi$ are the same as above. Q.E.D.
Lemma 2.4. In the situation of Lemma 2.3, we have

\[ K_\pi^2 = \sum_{i=1}^{s} c_1^2(F_i), \quad e_\pi = \sum_{i=1}^{s} c_2(F_i), \quad \chi_\pi = \sum_{i=1}^{s} \chi_{F_i}. \]

Proof. By Lemma 2.3, we can assume that \( \pi \) is the pullback of the canonical semistable reductions \( \pi_i = \phi_{F_i} : C_i \to C, \ i = 1, \ldots, s \). We can assume that \( \pi_i \) is unramified over the fibers \( F_j \) for \( j \neq i \). Without loss of generality, we assume also that \( s = 2 \). As in the proof of Lemma 2.3, we have

\[ K_\pi^2 = K_{\pi_1}^2 + \frac{1}{d_1} K_{p_1}^2. \]

Since \( p_1 \) is a totally ramified semistable reduction, \( K_{p_1}^2 \) can be computed locally from the branched non-semistable fibers, which are the pullback of \( F_2 \) under \( \pi_1 \). Hence we know

\[ K_{p_1}^2 = d_1 K_{\pi_2}^2. \]

By definition, \( K_{\pi_i}^2 = c_1^2(F_i) \). Hence we have obtained the desired equality. Note that the local property used above holds for \( e_\pi \) and \( \chi_\pi \). Q.E.D.

Proof of Theorem A. Let \( \hat{\pi} : \hat{C} \to \hat{C} \) be a semistable reduction of the pullback fibers \( R_{\pi} \) of the branched fibers \( B_{\pi} \). Then we know that \( \pi \circ \hat{\pi} \) is also the semistable reduction of \( B_{\pi} \). By Lemma 2.4 and the basic equalities we can obtain the equalities in this theorem. Q.E.D.

3 On the invariants of non-semistable fibers

3.1 The computations of the invariants \( c_1^2, c_2 \) and \( \chi \)

In what follows, we shall consider the computation of the invariants \( c_1^2(F), c_2(F) \) and \( \chi_F \). By Noether equality, we only need to compute \( c_1^2 \) and \( c_2 \).

First note that if we use embedded resolution to resolve the singularities of \( F \), then the number

\[ c_{-1}(F) = \frac{1}{d} \# \{ \text{(-1)-curves over} \ F' \ \text{contracted by} \ \tilde{\rho} \} \]

is also independent of the stabilizing base change if \( d \) is divided by \( M_F \).

Theorem 3.1. \( c_1^2(F) = 4N_F + F_{\text{red}}^2 + \sum_{p \in F} \alpha_p - c_{-1}(F), \) \( c_2(F) = 2N_F + \mu_F - \sum_{p \in F} \beta_p + c_{-1}(F). \) \( (15) \)

Proof. The first formula has been proved in Proposition 2.2. In order to prove the second formula, we consider the stabilizing base change \( \pi \) of \( F \) whose degree
is divided by $M_F$. By definition, if $\bar{F}$ and $F_2$ are respectively the pullback fibers of $F$ in $\tilde{S}$ and $S_2$, (note that $S_2$ is the embedded resolution, not the minimal resolution), then we have

$$e_\pi = e_F - \frac{1}{d} e_{\bar{F}} = e_F - \frac{1}{d} e_{F_2} + c_{-1}(F).$$

Since $F_2$ is semistable, $e_{F_2}$ is the number of singular points of $F_2$, which is exactly the number $d \sum_{p \in F} \beta_p$ (Lemma 1.4). We have known that $e_F = 2N_F + \mu_F$, hence the second formula has been obtained.

**Q.E.D.**

**Remark.** From the formulas above and the Noether formula, we can see that $\chi_F$ is independent of $c_{-1}(F)$, hence it can be computed directly from embedded resolution. In fact, if we consider the canonical semistable reduction of $F$, then we can prove that

$$c_{-1}(F) = \sum_{q' \in F'} \beta_{q'},$$

where $F'$ is the embedded resolution of $F$, and $q'$ runs over the singular points of $F'$ such that the $(-2)$-curves coming from $p'$ are contracted to points of the semistable model of $F$.

**Example.** Note that the discussion above holds for elliptic fibrations. In this case, $K_F^2 = 0$ for all base changes, so we have $c_1^2(F) = 0$. By a direct computation we have

$$c_2(F) = 12 \chi_F = \begin{cases} 0, & \text{if } F \text{ is of type } mI_b, \\ 6, & \text{if } F \text{ is of type } I_b^\ast (b > 0), \\ e_F, & \text{otherwise.} \end{cases}$$

The result above shows the well-known fact that the semistable model of an elliptic fiber is smooth except for type $mI_b$ ($b > 0$) and type $I_b^\ast$ ($b > 0$).

### 3.2 Proof of Theorem C

**Lemma 3.2.**

$$\sum_{p \in F} \alpha_p \leq 2p_a(F_{\text{red}}), \quad (16)$$

the equality holds iff $p_a(F_{\text{red}}) = 0$, hence $F$ is a tree of nonsingular rational curves.

**Proof.** We shall use the notations of Sect. 1.1. By (1) and (4), we have

$$\alpha_p = 2\delta_p - \sum_{i=0}^{r-1} (m_i - 2) - 2(k_p - 1) \quad (17)$$

$$\leq 2\delta_p - 2(k_p - 1).$$
On the other hand, if the reduced part of $F$ consists of $l_F$ components $\Gamma_i$, $i = 1, \cdots, l_F$, then we have
\[
p_a(F_{\text{red}}) = \sum_{i=1}^{l_F} p_a(\widetilde{\Gamma}_i) + \sum_{p \in F} \delta_p - l_F + 1.
\]
where $\widetilde{\Gamma}_i$ is the normalization of $\Gamma_i$. Hence we only need to prove that
\[
\sum_{p \in F} (k_p - 1) \geq l_F - 1.
\]
But this inequality is an immediate consequence of the connectedness of $F$. So (16) holds.

If the equality in (16) holds, then from (17) we know that $\alpha_p = 0$ for any $p \in F$, hence $p_a(F_{\text{red}}) = 0$. Then from (18) and (19), we can see $F$ is a tree of smooth rational curves.

**Theorem 3.3.**
\[
c^2_1(F) \leq 4g - 4.
\]

**Proof.** From Lemma 3.2 we have
\[
c^2_1(F) \leq 4N_F + F^2_{\text{red}} + 2p_a(F_{\text{red}}) - c_{-1}(F),
\]
and the equality holds iff $p_a(F_{\text{red}}) = 0$. Hence it is easy to prove that
\[
c^2_1(F) + c_{-1}(F) \leq 4g - 3,
\]
and the equality holds iff $F$ satisfies
\[
p_a(F_{\text{red}}) = 0, \quad F_{\text{red}}K = 1.
\]
So it is enough to prove that for the fibers $F$ satisfying (21) we have $c_{-1}(F) \geq 1$.

Now we consider the canonical semistable reduction $\phi_F$ of $F$, we know that the degree of $\phi_F$ is $M_F$. We can see that the fiber $F$ satisfying (21) is a tree of a $(-3)$-curve $\Gamma$ and some $(-2)$-curves $E$. We note first that if $F$ contains a $(-2)$-curve $E$ such that $F_{\text{red}}$ has only one singular point $p$ on $E$, then $p$ is an ordinary double point of type $(n,2n)$, where $n$ is the multiplicity of $E$ in $F$. Since the pullback fiber $\widetilde{F}$ of $F$ is semistable, for any component $\Gamma$ in $\widetilde{F}$, $-\Gamma^2$ is the intersection number of $\Gamma$ with the other components. Thus we can see easily that the inverse image of $E$ in the minimal resolution surface consists of $n$ $(-1)$-curves, hence the exceptional curves of $p$ can be contracted to a point. That is to say we contracted $n + [n,2n]d - (n,2n) = d/2$ $(-1)$-curves (Lemma 1.4). Thus the contribution of $(E,p)$ to $c_{-1}(F)$ is $\frac{1}{2}$. On the other hand, we know easily that there are at least two such $(-2)$-curves in $F$, hence $c_{-1}(F) \geq 1$. This completes the proof. **Q.E.D.**
**Theorem 3.4.** For any singular fiber $F$, we have
\[ c_1^2(F) \leq 2c_2(F), \]  
with equality iff $F = nF_{\text{red}}$ and $F$ has at worst ordinary double points as its singularities.

*Proof.* From (15), we have
\[ 2c_2(F) - c_1^2(F) = 3c_{-1}(F) - F_{\text{red}}^2 + \sum_p (2\mu_p - 2\beta_p - \alpha_p). \]
Then by Lemma 1.3, $\mu_p - \beta_p \geq \alpha_p$, hence we have
\[ 2c_2(F) - c_1^2(F) \geq -F_{\text{red}}^2 + \sum_{p \in F} \alpha_p \geq 0. \]
If $c_1^2(F) = 2c_2(F)$, then $F_{\text{red}}^2 = 0$, and $\alpha_p = 0$ for all $p \in F$. By the well-known Zariski’s lemma [BPV, p.90], we have $F = nF_{\text{red}}$. Furthermore, $\alpha_p = 0$ implies that $p$ is an ordinary double point, so $F_{\text{red}}$ has at worst nodes as its singularities. The converse is obvious. \(\text{Q.E.D.}\)

**Proposition 3.5.** If all of the multiple components of $F$ are $(-2)$-curves, then
\[ c_1^2(F) \leq c_2(F). \]  
*Proof.* The proof is similar to that of Theorem 3.4. \(\text{Q.E.D.}\)

4 Applications

4.1 On the slopes of fibrations

From Theorem 3.4, Noether equality and the corollary to Theorem A, we have

**Theorem 4.1.** For any stabilizing base change $\pi$, we have
\[ K_\pi^2 \leq 8\chi_\pi. \]  
As in the case of fibrations, we have the following definition of slopes.

**Definition 4.2.** If $F$ is a non-semistable fiber, then $\chi_F \neq 0$, and so we can define the slope of $F$ as
\[ \lambda_F = \frac{c_1^2(F)}{\chi_F}. \]
From Theorem 3.4, we know $0 < \lambda_F \leq 8$.

If $\pi$ is a non-invariant base change, then we define the slope of $\pi$ as
\[ \lambda_\pi = \frac{K_\pi^2}{\chi_\pi}. \]

Note that a non-trivial stabilizing base change $\pi$ satisfies $\chi_\pi > 0$, so Theorem 4.1 says that its slope $\lambda_\pi \leq 8$.

We have known in the introduction that for a stabilizing base change $\pi$,
\[ (\lambda_f - \lambda_f)(\chi_f - \chi_{B_\pi}) = (\lambda_f - \lambda_\pi)\chi_{B_\pi}. \]  

(25)
Corollary 4.3. If \( f : S \to C \) is a non-semistable fibration with \( \lambda_f > 8 \), then through any non-trivial stabilizing base change, we have
\[
\lambda_\tilde{f} > \lambda_f.
\] (26)

In what follows, we shall consider a set of fibrations \( \Sigma \) which is invariant under base changes, i.e., if \( f \in \Sigma \), then \( \tilde{f} \in \Sigma \).

Corollary 4.4. Let \( f \) (resp. \( f' \)) be a fibration in \( \Sigma \) with maximal (resp. minimal) slope.

1) For any non-semistable fiber \( F \) of \( f \) (resp. \( f' \)), we have
\[
\lambda_F \geq \lambda_f, \quad (\text{resp. } \lambda_F \leq \lambda_{f'}).\n\]

2) If \( \lambda_f > 8 \), then \( f \) is semistable.

3) If \( \lambda_f > 6 \), then any non-semistable fiber of \( f \) has at least one multiple component which is not a \((-2)\)-curve.

Proof. Considering the canonical stabilizing base change of \( F \) and using (25), we can prove 1). 2) and 3) are immediate consequences of (22)–(25) and the assumption. Q.E.D.

Remark. This corollary can be used to classify singular fibers of a fibration with minimal slope in the sense above. For example,

I) Xiao [Xi1, Xi4] has proved that for any relatively minimal fibration \( f \) of genus \( g \),
\[
\lambda_f \geq 4 - 4/g.
\]
Furthermore, if \( f \) is a hyperelliptic fibration, then
\[
\lambda_f \leq 12 - \frac{4g + 2}{[g^2/2]}.
\]

II) If \( f \) is non-hyperelliptic, then the lower bounds \( \lambda_g \) of the slope are \( \lambda_3 = 3 \), \( \lambda_4 = 24/7 \), \( \lambda_5 = 40/11 \). (cf. [Ch, Ho, Ko, Re]).

If we consider fibrations over \( \mathbb{P}^1 \), and we only consider base changes with two ramification points, then the above results can also be used.

4.2 Canonical class inequality for general fibrations

First we recall Miyaoka’s inequality and refer to [Hi] for the details.

Lemma 4.5. [Mi] If \( S \) is a smooth minimal surface of general type, and \( E_1, \cdots, E_n \) are disjoint ADE curves on \( S \), then we have
\[
\sum_{i=1}^{n} m(E_i) \leq 3c_2(S) - c_1^2(S),
\]
where \( m(E) \) is defined as follows,

\[
m(A_r) = 3(r + 1) - \frac{3}{r + 1}; \quad m(D_r) = 3(r + 1) - \frac{3}{4(r - 2)}, \quad \text{for } r \geq 4;
\]

\[
m(E_6) = 21 - \frac{1}{8}; \quad m(E_7) = 24 - \frac{1}{16}; \quad m(E_8) = 27 - \frac{1}{40}.
\]

The condition “of general type” can be replaced by some other conditions. (cf. [Mi]).

**Theorem 4.6.** If \( f \) is a fibration of genus \( g > 1 \) over a curve \( C \) of genus \( b \), then

\[
K_{S/C}^2 \leq 3 \sum_{y \in C} \delta_y^\# + (2g - 2) \max(2b - 2, 0),
\]  

(27)

where \( \delta_y^\# = e_{F_y} - \frac{1}{3} \sum_{E \subset F_y} m(E) \leq 4g - 3 \), and the sum is taken over all of the disjoint ADE curves \( E \) in \( F_y \).

**Proof.** Inequality (27) is an immediate consequence of Miyaoka-Yau inequality (See Vojta’s proof, [Vo, Theorem 2.1]). So we only need to prove that \( \delta_y^\# \leq 4g - 3 \).

We denote by \( l_D \) the number of components of a curve \( D \), and by \( \tilde{D} \) the normalization of \( D \). Let \( F_{\text{red}} = D + \sum_{E \subset F} E \) be the reduced part of a fiber \( F \). Then

\[
\chi_{\text{top}}(F) = \chi_{\text{top}}(D) + \sum_{E \subset F} (\chi_{\text{top}}(E) - \#(D \cap E))
\]

\[
= \chi_{\text{top}}(\tilde{D}) - \sum_{p \in D} (k_p(D) - 1) + \sum_{E \subset F} (l_E + 1) - \#(D \cap \sum_{E \subset F} E)
\]

\[
\leq 2l_D + \sum_{E \subset F} (l_E + 1) - \sum_{p \in D} (k_p(D) - 1) - \sum_{E \subset F} \#(D \cap E).
\]

From the definition of \( m(E) \), we know

\[
m(E) \geq 3(l_E + 1) - 1,
\]

hence

\[
\delta_y^\# = 2g - 2 + \chi_{\text{top}}(F) - \sum_{E \subset F} m(E)
\]

\[
\leq 2g - 2 + 2l_D - \sum_{p \in D} (k_p(D) - 1) - \#(D \cap \sum_{E \subset F} E) + \sum_{E \subset F} 1.
\]

Since \( l_D \leq KD \leq 2g - 2 \), it is enough to prove that

\[
\sum_{p \in D} (k_p(D) - 1) + \sum_{E \subset F} (\#(D \cap E) - 1) \geq l_D - 1.
\]  

(28)

Indeed, if \( D \) is connected, then we have

\[
\sum_{p \in D} (k_p(D) - 1) \geq l_D - 1,
\]
hence (28) holds. If $D$ has $r$ connected components $D_1, \cdots, D_r$, then
\[
\sum_{p \in D} (k_p(D) - 1) \geq \sum_{i=1}^{r} (l_{D_i} - 1) = l_D - r.
\]
On the other hand, from the connectedness of $F$, we can prove easily that
\[
\sum_{E \subset F} (\#(D \cap E) - 1) \geq r - 1.
\]
Hence (28) holds too. \quad \text{Q.E.D.}

In [Vo], Vojta shows that if $f$ is a non-trivial semistable fibration with $s$ singular fibers, then
\[
K^2_{S/C} \leq (2g - 2)(2b - 2 + s).
\]
By using semistable reductions, we can obtain a similar inequality for general fibrations.

**Theorem 4.7.** If $f$ is a non-trivial fibration of genus $g \geq 2$ with $s$ singular fibers, then
\[
K^2_{S/C} \leq (2g - 2)(2b - 2 + 3s),
\]
(29)
and the equality holds only if $f$ is smooth.

**Proof.** We denote by $F_1, \cdots, F_s$ the $s$ singular fibers of $f$, and assume that $s > 0$. Note that $f$ is non-trivial, so $b = 0$ implies that $s \geq 2$. In fact, if $b = 0$ and $s = 2$, then $f$ is isotrivial (cf. [Be1]), thus $I_K(f) = 0$. In this case, the inequality (20) holds strictly, so does (29). Hence we can assume that $s \geq 3$ if $b = 0$.

First we claim that there exist some semistable reductions $\pi : \tilde{C} \rightarrow C$ of $f$ satisfying the following two conditions:

(i) $\pi$ is ramified uniformly over the $s$ critical points of $f$, and the ramification index of $\pi$ at any ramified point is exactly $e$.

(ii) $e$ is divided by $M_{F_i}$ for all $i$, and it can be arbitrarily large.

In fact, a base change satisfying the above two conditions must be a semistable reduction. If $b = g(C) > 0$, then the existence follows from Kodaira-Parshin’s construction. If $b = 0$, then $s \geq 3$ by assumption. Hence we can construct a base change totally ramified over the $s$ points. Then the existence is reduced to the case $b > 0$.

We let $\tilde{f} : \tilde{S} \rightarrow \tilde{C}$ be the semistable model of $f$ under the base change $\pi$ constructed above, $\tilde{s}$ the number of singular fibers of $\tilde{f}$, and $\tilde{b}$ the genus of $\tilde{C}$. Denote by $d$ the degree of $\pi$. Then we know that $\tilde{s} \leq ds/e$, and
\[
2\tilde{b} - 2 = d(2b - 2) + d \left(1 - \frac{1}{e}\right)s.
\]
From Theorem A, we obtain
\[
K^2_f = \frac{1}{d} K^2_{\tilde{f}} + \sum_{i=1}^{s} c_1^2(F_i),
\]
hence we have

\[
K_f^2 - (2g - 2)(2b - 2 + 3s) = \frac{1}{d} \left( K_f^2 - (2g - 2)(2\tilde{b} - 2 + \tilde{s}) \right) \\
+ \frac{2g - 2}{d} \left( \tilde{s} - \frac{d}{e}s \right) + \sum_{i=1}^{s} (c_1^2(F_i) - (4g - 4)).
\]

Consequently, combining Theorem 3.3 and Vojta’s canonical class inequality for semistable fibrations, we can obtain immediately (29). Since the equality in Vojta’s inequality implies the smoothness of the fibration (cf. [Ta2, Lemma 3.1]), we can see easily that if the equality in (29) holds, then \( f \) is smooth. \( \text{Q.E.D.} \)

4.3 On Horikawa number of a non-semistable fiber of genus 3

Let \( f : S \rightarrow C \) be a relatively minimal non-hyperelliptic fibration of genus 3, \( F \) a fiber of \( f \), and \( p = f(F) \). Then the Horikawa number of \( F \) is defined as (cf. [Re])

\[
H_F = \text{length coker} \left( S^2 f_* \omega_{S/C} \hookrightarrow f_* (\omega_{S/C}^\otimes 2) \right)_p.
\]

The global invariants of \( f \) depend on this number. In fact, Reid [Re] shows that

\[
K_f^2 - 3\chi_f = \sum_F H_F.
\] \hspace{1cm} (30)

In general, it is quite difficult to compute \( H_F \). The aim here is to try to reduce the computation for a non-semistable fiber to the computation for its semistable models, by using semistable reductions.

**Theorem 4.8.** Let \( \tilde{F} \) be the semistable model of \( F \) under a totally ramified stabilizing base change of degree \( d \). Then

\[
\frac{1}{d} H_{\tilde{F}} = H_F + \frac{1}{4} (c_2(F) - 3c_1^2(F)).
\] \hspace{1cm} (31)

**Proof.** We can assume that the branch locus of the base change consists of \( F \) and some generic smooth fibers, hence

\[
c_1^2(F) - 3\chi_F = K_\pi^2 - 3\chi_\pi \\
= K_f^2 - 3\chi_f - \frac{1}{d} (K_\tilde{f}^2 - 3\chi_\tilde{f}) \\
= H_F - \frac{1}{d} H_{\tilde{F}}.
\]

By using Noether formula, we can obtain (31). \( \text{Q.E.D.} \)
Examples. If $F$ is a genus 2 curve with an ordinary cusp, then we can take $d = 6$. Then the semistable model $\tilde{F}$ consists of a nonsingular elliptic curve $E$ and a nonsingular curve $C$ of genus 2, with $EC = 1$. Since $c_1^2(F) = \frac{1}{6}$ and $c_2(F) = \frac{11}{6}$, we have

$$H_{\tilde{F}} = 6H_F + 2.$$ 

If $F = 2C$, $C$ is a smooth curve of genus 2, then we can take $d = 2$, hence $\tilde{F}$ is a nonsingular hyperelliptic curve of genus 3. We have $c_1^2(F) = 4$, $c_2(F) = 2$. Hence

$$H_{\tilde{F}} = 2H_F - 5.$$ 

So we can compute directly the Horikawa numbers of some special singular fibers, e.g., if their semistable models are non-hyperelliptic curves of genus 3.

5 The proof of Lemma 1.5

In this section, we shall use freely the notations of Sect. 1.2. Note first that Lemma 1.5 is a special case of the following theorem.

Theorem 5.1. For the embedded resolution given in Sect. 1.2, we have

$$-K_{\phi}^2 = d \sum_{i=0}^{r-1} \left( m_i - 2 + \frac{1}{d} (m_i^*, d - m_i(d)) \right)^2 - \sum_{q \in B_i} K_q^2,$$

where $m_i, m_i^*, m_i(d)$ are respectively the multiplicities of $B_{i, \text{red}}$, $B_i$, $B_i(d)$ at $p_i$, and $B_i(d) = \sum_\Gamma (d, n_\Gamma) \Gamma$ if $B_i = \sum_\Gamma n_\Gamma \Gamma$.

Proof. Since we only need to find $K_{\phi}$, without loss of generality, we may assume that $U_0$ is a compact smooth surface, and the reduced curve of $B = B_0 = \sum_\Gamma n_\Gamma \Gamma$ has only one singular point $p$, (otherwise we can resolve in advance the other singularities of $B$ by using embedded resolution). So we have a formula similar to (11):

$$K_M = \phi^* \pi_0^* \left( K_{U_0} + \sum_{\Gamma \subset B} \left( 1 - \left( \frac{(d, n_\Gamma)}{d} \right) \Gamma \right) \right) + K_{\phi},$$

i.e.,

$$K_M = \eta^* \pi^* \sigma^* \left( K_{U_0} + B_{0, \text{red}} - \frac{1}{d} B_0(d) \right) + K_{\phi}. \quad (32)$$

On the other hand, $\pi_r$ is determined by $B_r = \sigma^*(B)$, hence we have also

$$K_M = \eta^* \pi_r^* \left( K_{U_r} + B_{r, \text{red}} - \frac{1}{d} B_r(d) \right) + K_{\eta}. \quad (33)$$

From Definition 1.1 it is easy to prove that

$$K_{U_i} + B_{i, \text{red}} - \frac{1}{d} B_i(d) = \sigma_i^* \left( K_{U_{i-1}} + B_{i-1, \text{red}} - \frac{1}{d} B_{i-1}(d) \right)$$

$$- \left( m_i - 2 + \frac{1}{d} (m_i^*, d - m_i(d)) \right) E_i. \quad (34)$$
If we denote by $E_i$ the total inverse image of $E_i$ in $U_r$, then from (32)–(34), we have

$$K_\phi = -\eta^*\pi^*_r \left( \sum_{i=1}^{r} \left( m_{i-1} - 2 + \frac{1}{d}(m_{i-1}^*, d) - m_{i-1}(d) \right) E_i \right) + K_\eta.$$  

Hence we obtain the desired equality. Q.E.D.

**Remark.** In order to resolve the singularities of $V_0$, we can also use the $d$-resolution of $(B, p)$, which is defined as in Definition 1.1 by replacing condition (3) with

$$(3')$$ If $E_i = \sigma_i^{-1}(p_{i-1})$ is the exceptional curve, then we have

$$B_i = \sigma_i^*(B_{i-1}) - d \left[ \frac{m_i^* - 1}{d} \right] E_i.$$  

Then we have also the formula in Theorem 5.1.

Finally, we consider the computation of $K^2_\eta$ by using Jung-Hirzebruch’s resolution.

Let $V$ be the normalization of the local surface defined by $z^d = x^a y^b$, and $\eta : M \rightarrow V$ the minimal resolution of the singularities. Then the resolution can be constructed as follows (cf. [BPV,p.83]).

I. If $(d, a, b) = (d, a') = (a', b) = 1$, then we let $q, q'$ be two integers with $0 < q, q' < d$, $aq + b \equiv 0 \pmod{d}$, $qq' \equiv 1 \pmod{d}$. If

$$\frac{d}{q} = e_1 - \frac{1}{e_2 - \frac{1}{\cdot \cdot \cdot - \frac{1}{e_r}}}$$

then the exceptional set of the resolution is a chain of $r$ rational curves

$$-e_1 \quad -e_2 \quad \cdots \quad -e_r$$

and we have

$$-K^2_\eta = \sum_{i=1}^{r} (e_i - 2) + \frac{q + q' + 2}{d} - 2.$$  

II. If $(d, a, b) = 1$, then the singularity of $V$ is isomorphic to the normalization of $z^{d'} = x^{a'} y^{b'}$, where $a = a'(d, a)$, $b = b'(d, b)$ and $d = d'(d, a)(d, b)$, hence the resolution and the computation are reduced to (I).

III. If $d_0 = (d, a, b) > 1$, then we have

$$z^d - x^a y^b = \prod_{i=1}^{d_0} \left( z^{d/d_0} - x^{a/d_0} y^{b/d_0} \exp \left( 2\pi i \sqrt{\frac{1}{d_0}} \right) \right).$$

Hence the singularity decomposes into $d_0$ singularities of type II.

In particular, if $d$ is divided by $a$ and $b$, by (III) and (II), we can see that $V$ consists of $d_0 = (a, b)$ singular points of type $A_n$, where

$$n = \frac{d_0 d}{ab} - 1.$$  

This is what we expected in Lemma 1.4.
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References

[Ar] Arakelov, S. Ju., Families of algebraic curves with fixed degeneracy, Math. USSR Izv. 5 (1971), 1277–1302.
[AW] Artin, M., Winters, G., Degenerate fibres and stable reduction of curves, Topology 10 (1971), 373–383.
[Be1] Beauville, A., Le nombre minimum de fibres singulières d’un courbe stable sur $\mathbb{P}^1$, in Séminaire sur les pinceaux de courbes de genre au moins deux, ed. L. Szpiro, Astérisque 86 (1981), 97–108.
[Be2] Beauville, A., L’inégalité $p_g \geq 2q - 4$ pour les surfaces de type général, Bull. Soc. Math. France 110 (1982), no. 3, 319–346.
[BPV] Barth, W., Peters, C., Van de Ven, A., Compact complex surfaces, Berlin, Heidelberg, New York: Springer, 1984.
[Ch] Chen, Z., On the lower bound of the slope of a non-hyperelliptic fibration of genus 4, Intern. J. Math. 4 (1993), no. 3, 367-378.
[De] Deschamps, M., Réduction semi-stable, in Séminaire sur les pinceaux de courbes de genre au moins deux, ed. L. Szpiro, Astérisque 86 (1981), 1–34.
[DM] Deligne, P., Mumford, D., The irreducibility of the space of curves of given genus, Publ. IHES 36 (1969), 75–109.
[EV] Esnault, H., Viehweg, E., Effective bounds for semipositive sheaves and the height of points on curves over complex functional fields, Compositio Mathematica 76 (1990), 69–85.
[Hi] Hirzebruch, F., Singularities of algebraic surfaces and characteristic numbers, The Lefschetz Centennial Conference, Part I (Mexico City), Contemp. Math., 58 (1986), Amer. Math. Soc. Providence, R.I., 141–155.
[Ho] Horikawa, E., Notes on canonical surfaces , Tohoku Math. J. 43 (1991), 141–148.
[Ko] Konno, K., Non-hyperelliptic fibrations of small genus and certain irregular canonical surfaces, Preprint (1992).
[Mi] Miyaoka, Y., The maximal number of quotient singularities on surfaces with given numerical invariants, Math. Ann. 268, 159–171.
[Pa] Parshin, A. N., Algebraic curves over function fields I, Math. USSR Izv. 2 (1968), 1145–1170.
[Re] Reid, M., Problems on pencils of small genus, Preprint.
[Sz] Szpiro, L., Propriété numériques de faisceau dualisant relatif, in Séminaire sur les pinceaux de courbes de genre au moins deux, ed. L. Szpiro, Astérisque 86 (1981), 44–78.
[Ta1] Tan, S.-L., On the invariants of base changes of pencils of curves, I, Manuscripta Math. 84 (1994), no. 3/4, 225–244.
[Ta2] Tan, S.-L., The minimal number of singular fibers of a semistable curve over $\mathbb{P}^1$, J. Algebraic Geometry (to appear).
[Vo] Vojta, P., Diophantine inequalities and Arakelov theory, in Lang, S., Introduction to Arakelov Theory (1988), Springer-Verlag, 155–178.
[Xi1] Xiao, G., Fibered algebraic surfaces with low slope, Math. Ann. 276 (1987), 449–466.
[Xi2] Xiao, G., Problem list, in: Birational geometry of algebraic varieties: open problems. The 23rd International Symposium of the Taniguchi Foundation, (1988), pp.36–41.
[Xi3] Xiao, G., On the stable reduction of pencils of curves, Math. Z. 203 (1990), 379–389.
[Xi4] Xiao, G., The fibrations of algebraic surfaces, Shanghai Scientific & Technical Publishers, 1992. (Chinese)

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