Affine projection tensor geometry: Lie derivatives and isometries

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The generalized projection-tensor geometry introduced in an earlier paper is extended. A compact notation for families of projected objects is introduced and used to summarize the results of the previous paper and obtain fully projected decompositions of Lie derivatives of the projection tensor field, the metric and the projected parts of the metric. These results are applied to the analysis of spacetimes with isometries. The familiar cases of spacetimes with isotropic group orbits — cosmological models and spherical symmetry — are discussed as illustrations of the results.

I. INTRODUCTION

Many important problems in general relativity can be stated in terms of projections onto subspaces and the relationship of those projections to diffeomorphisms. From the earliest days of general relativity, exact solutions to Einstein’s equations have been sought by assuming isometry groups — metric preserving diffeomorphisms — and projecting the metric and other geometrical objects onto the orbits of those groups. In more recent times, the emphasis has shifted to organizing Einstein’s equations for efficient numerical evolution of spacetimes from initial data. However, the quantities which are needed for that endeavor are still the Lie derivatives — infinitesimal diffeomorphisms — of tensors which have been projected in various ways. Thus, the relationship between projections and diffeomorphisms remains at the center of our attempts to understand the dynamics of Einstein’s field equations. This paper, the second in a series on the geometry of projection-tensor fields, focuses on this relationship. The previous paper, Ref. 1, worked out all of the ways in which projection tensor fields can interact with a connection on a manifold. This paper performs that same task for Lie derivatives.

The point of this series of papers is the flexibility of a projection tensor geometry which is not restricted to normal projections or projections of codimension one. Normal projections of codimension one are familiar from the 3+1 approach to the initial value problems of spacetime field theories. Naturally, the conventional 3+1 results can be obtained by specializing the results in this series but there are more direct ways to obtain those results. This series of papers attempts to place those results in a wider context.

The definitions and results of projection tensor geometry are summarized in Sec. II. Although the content of Ref. 1 is summarized here, this treatment is not intended to be self-contained and the reader who wishes to understand this paper in detail should begin with Ref. 1. Section II introduces a notational convenience, assemblies of restricted tensors, or projection assemblies which make projection tensor expressions more compact, and identical in form to their counterparts in unprojected Riemannian geometry. This compact new notation is used to present the key results of Ref. 1. The main results of this paper are presented in Sec. IV which evaluates the Lie derivatives of projected geometrical structures and in Sec. V which considers the properties of projections onto the orbits of isometry groups.

As in Ref. 1, the paper closes with some familiar applications to provide a context for these results. In particular, Sec. VI revisits Birkhoff’s Theorem and shows how the results given here apply to spherically symmetric systems. Some natural generalizations of spherical symmetry lead to standard cosmological models as well as a five-dimensional Kaluza-Klein cosmological model which displays a spontaneous dimensional reduction. The interesting point about these examples is that they are all fundamentally the same, differing only by the “accidents” of dimension and metric signature.

II. SUMMARY OF PROJECTION TENSOR GEOMETRY

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A. Projection tensors and subspaces

A projection tensor field $H$ assigns to each point $P$ of a manifold, a map $H(P) : T_P \to T_P$ such that $H(P)^2 = H(P)$. The complement $V = I - H$ of a projection tensor field is also a projection tensor field. The pull-backs $H(P)^* : \tilde{T}_P \to \tilde{T}_P$ and $V(P)^* : \tilde{T}_P \to \tilde{T}_P$ are used to project one-forms. The tensor products of the subspaces $H(P)T_P$, $V(P)T_P$, $H(P)^* \tilde{T}_P$, $V(P)^* \tilde{T}_P$ are called fully projected tensor subspaces. Each fully projected subspace $S_P$ may be characterized by a projection operator $O(P)$ which takes any tensor $M$ of the appropriate rank into a tensor $O(P)M$ which is in $S_P$ and which acts as an identity operator on $S_P$. For example, $H$ itself belongs to the fully projected subspace $H(P)^* \tilde{T}_P \otimes H(P)T_P$ and the corresponding projection operator $O[H_H]$ takes any rank-$(1,1)$ tensor $M$ into a tensor in that subspace:

$$ (O[H_H]M)_{\alpha\beta} = H^\alpha_\rho H^\sigma_\beta M^{\rho\sigma}. $$

A tensor field which has values only in fully projected tensor subspaces will be referred to as a restricted tensor field. Projection tensor geometry seeks to express everything in terms of such restricted tensor fields.

The term ”restricted tensor field” is new in this paper. It replaces the term ”fully projected tensor field” which was used in Ref. 1. This change in terminology is needed because many restricted tensor fields have definitions which are restricted to a given fully projected subspace and are not simply projections of their counterparts in unprojected spacetime geometry.

Projections of higher-rank tensor fields can be cumbersome in standard index notation. I will use an abbreviated notation which places symbols for the projection tensors in a pattern to indicate where they would act in an index notation. The projection operator $O[H_H]$ shown above is one example. Another example is the projection

$$ (O[H_H] M)_{\alpha\beta\gamma} = M[H_H]_{\alpha\beta\gamma} = H^\rho_\alpha V^\beta_\beta H^\gamma_\gamma M^{\rho\beta\gamma}. $$

of a tensor $M$ into the tensor $M[H_H]$ which belongs to the fully projected subspace

$$ T[H_H]_{\rho} = H(P)^* \tilde{T}_P \otimes V(P)T_P \otimes H(P)^* \tilde{T}_P. $$

B. Projection curvatures

For each projection tensor field $H$, there is a projection curvature tensor

$$ h_H^{\gamma_{\alpha\beta}} = H^\rho_\alpha (H^\delta_\beta \nabla_\gamma H^\gamma_\rho) = \left( \nabla H \left[ H_H \right] \right)^{\gamma_{\alpha\beta}} \tag{1} $$

and a transpose projection curvature tensor

$$ h_{HT}^{\alpha_{\gamma\beta}} = H^\alpha_\rho \left( H^\delta_\beta \nabla_\gamma H^\rho_\gamma \right) = \left( \nabla H \left[ H_H \right] \right)^{\alpha_{\gamma\beta}}. \tag{2} $$

These tensor fields obey the projection identities

$$ h_H \left[ V_H H \right]^{\gamma_{\alpha\beta}} = h_H^{\gamma_{\alpha\beta}}, \quad h_H \left[ V_H \right]^{\alpha_{\gamma\beta}} = h_{HT}^{\alpha_{\gamma\beta}} \tag{3} $$

and are therefore restricted. Since the complement $V$ of $H$ is also a projection tensor field, it too has projection curvatures which obey identities which are the complements of the ones given above. The projection curvatures may be used to expand the covariant derivative of the projection tensor $H$ in terms of restricted tensor fields.

$$ \nabla_\delta H^\alpha_\beta = h_H^\alpha_{\beta\delta} - h_V^\alpha_{\beta\delta} + h_{HT}^\alpha_{\beta\delta} - h_V^{\alpha\beta\delta} \tag{4} $$

Some tensors derived from the projection curvatures are:

- Divergence $\theta_H^{\beta\rho} = h_H^{\beta\rho}$
- Twist $\omega_H^{\alpha_{\beta\delta}} = h_H^{\alpha_{[\beta\delta]}}$
- Expansion $\theta_H^{\alpha_{\beta\delta}} = h_H^{\alpha_{(\beta\delta)}}$
C. Restricted derivatives

For a restricted tensor field $M$, several projected derivatives can be defined. If $O$ is the projection operator which characterizes the subspace which $M$ belongs to, then the projected covariant derivative of $M$ is defined to be the tensor with components

$$D_\delta M = O \nabla_\delta M.$$  

Here, the indexes which are not associated with the derivative have been suppressed. Since $D_\delta M$ does not, itself, belong to a fully projected tensor subspace, it is useful to define derivatives which do. The restricted covariant derivatives of $M$ are defined to be

$$D_H \delta M = H^\sigma_\delta D_\sigma M, \quad D_V \delta M = V^\sigma_\delta D_\sigma M.$$  

The full decomposition of the covariant derivative of a fully projected tensor can be expressed in terms of its restricted derivatives and the projection curvatures. For a vector field $v$ with $v(P) \in HT_P$, the covariant derivative has the decomposition

$$\nabla_H \delta v^\alpha = H^\sigma_\delta \nabla_\sigma v^\alpha = D_H \delta v^\alpha + h^H_T \alpha_\rho \delta v^\rho$$  

$$\nabla_V \delta v^\alpha = V^\sigma_\delta \nabla_\sigma v^\alpha = D_V \delta v^\alpha - h^T \alpha_\rho \delta v^\rho.$$  

The covariant derivative of a vector field $v$ with $v(P) \in VT_P$ is given by the complement of the above decomposition. For a one-form field $\varphi$ with $\varphi(P) \in H^* T_P$ the covariant derivative has the decomposition

$$\nabla_H \delta \varphi_\alpha = D_H \delta \varphi_\alpha + h^H_T \alpha_\rho \delta \varphi_\rho$$  

$$\nabla_V \delta \varphi_\alpha = D_V \delta \varphi_\alpha - h^T \rho_\alpha \delta \varphi_\rho.$$  

and the complementary equations give the decomposition of a one-form field with the complementary projection property. In general, for an arbitrary rank fully projected tensor, $M$, the covariant derivative decomposition takes the form

$$\nabla_X \delta M = D_X \delta M + \text{corrections}.$$  

where $X$ can be either $H$ or $V$. There is a correction for each index on the tensor. Each correction consists of one of the two tensors $h_X$ or $h^T_X$ with either its first or its last index contracted with an index on $M$. Because of the projection curvature identities (Eq. (3)) there will always be just one way to form such a term. The sign of each term is positive when the tensor index being contracted is projected in the same way as the differentiating index and negative otherwise.

D. Restricted metric and metricity

The form-metric $g^{\mu \nu}$ is decomposed into the restricted tensors $g^{XY \mu \nu} = g[XY]^{\mu \nu}$ where the projection labels $X, Y$ stand for either $H$ or $V$. Similarly, the metricity tensor $Q^{\mu \nu_\rho} = -\nabla_\rho g^{\mu \nu}$ may be decomposed into restricted parts according to the following expressions

$$Q^{[HH]}_{H H} \mu \nu_\delta = Q^{HH \mu \nu_\delta} + g^{-1}_H \rho_\nu^\alpha h^T \rho_\delta \mu_\mu + g^H \rho_\nu \rho_\delta$$  

$$Q^{[HH]}_{H V} \mu \nu_\delta = Q^{HH \mu \nu_\delta} - g^{-1}_H \rho_\nu^\alpha h^V \rho_\delta - g^H \rho_\nu \rho_\delta$$  

$$Q^{[HV]}_{H H} \mu \nu_\delta = Q^{HV \mu \nu_\delta} - g^{-1}_H \rho_\nu^\alpha h^H \rho_\delta + g^V \rho_\nu \rho_\delta$$  

$$Q^{[HV]}_{H V} \mu \nu_\delta = Q^{HV \mu \nu_\delta} + g^{-1}_H \rho_\nu^\alpha h^V \rho_\delta - g^V \rho_\nu \rho_\delta$$  

$$Q^{[HH]}_{V V} \mu \nu_\delta = Q^{HH \mu \nu_\delta} + g^{-1}_H \rho_\nu^\alpha h^T \rho_\delta - g^H \rho_\nu \rho_\delta$$  

$$Q^{[HV]}_{V V} \mu \nu_\delta = Q^{HV \mu \nu_\delta} + g^{-1}_H \rho_\nu^\alpha h^V \rho_\delta - g^V \rho_\nu \rho_\delta.$$
and their complements. Here the restricted metricities
\[ Q^{XY \mu \nu}_Z = -D_Z g^{XY \mu \nu} \]
include the intrinsic metricity \( Q^{HH}_H \) associated with the subspaces \( HT_P \), the intrinsic metricity \( Q^{VV}_V \) associated with the subspaces \( VTP \) and a collection of unfamiliar objects such as \( Q^{HV}_H \) which I choose to call the cross-projected metricities.

Notice that the restricted metric tensors \( g^{XY \mu \nu} \) are just the projections \( g^{[XY]}_{\mu \nu} \) of the spacetime tensor field \( g^{\mu \nu} \) but the restricted metricities \( Q^{XY \mu \nu}_Z \) are defined by using restricted covariant derivatives within each fully projected tensor subspace and are not just projections of the full spacetime metricity.

E. Restricted torsion

The torsion tensor \( S^\rho_{\mu \nu} \) is defined through the commutator of covariant derivatives acting on a function \( f \) according to
\[ [\nabla_\nu, \nabla_\mu] f = S^\rho_{\mu \nu} \nabla_\rho f. \]  
(13)
It can be decomposed into restricted tensor fields by defining the restricted torsion tensors \( S^{Z\rho}_{XY, \mu \nu} \) according to
\[ [D_{X \nu}, D_{Y \mu}] f = S^{H\rho}_{XY, \mu \nu} D_{H \rho} f + S^{V\rho}_{XY, \mu \nu} D_{V \rho} f \]  
(14)
with the resulting decomposition
\[ S^{[H H \rho]}_{XY, \mu \nu} = S^{H \rho}_{H \mu \nu} \]  
(15)
\[ S^{[V H \rho]}_{XY, \mu \nu} = S^{V \rho}_{H \mu \nu} - 2h^H H_{\rho [\mu \nu]} \]  
(16)
\[ S^{[H V \rho]}_{XY, \mu \nu} = S^{H \rho}_{V \mu \nu} - h^T_{H \mu \nu \rho} \]  
(17)
as well as the complements of these expressions.

Notice once again, the critical distinction between restricted objects and the projections of the corresponding spacetime objects. In this case, the restricted torsions are clearly different from the projections of the full spacetime torsion.

III. PROJECTION ASSEMBLY NOTATION

A. Organizing collections of restricted tensors

1. Tensor index - projection label pairs

A restricted tensor such as the restricted torsion described above carries a projection label for each of its tensor indexes. Thus, \( S^{Z\rho}_{XY, \mu \nu} \) carries the label \( Z \) for the tensor index \( \rho \), the label \( X \) for the tensor index \( \mu \) and the label \( Y \) for the tensor index \( \nu \). The projection labels stand for projection tensors and indicate the projection identities which are associated with each tensor index. For example, the restricted torsion obeys the identity
\[ X^\sigma_{\mu} S^{Z \rho}_{XY, \sigma \nu} = S^{Z \rho}_{XY, \mu \nu} \]
as well as two more identities associated with its other indexes.

Represent each index-label pair by a single compound index so that the restricted torsion tensors become
\[ S^{[Z \rho]}_{XY, (X \rho) (Y \nu)} = S^{Z \rho}_{XY, \mu \nu}. \]
Abbreviate even more and use a single symbol \( (\rho) \) to stand for an index-label pair \( (X \rho) \) and interpret the summation convention on a repeated abbreviated symbol to imply a sum over both the visible index value \( \rho \) and the invisible projection-label \( X \). In this compact notation, the definition (Eq. 14) of the restricted torsion tensors becomes just
\[ [D_{(\nu)}, D_{(\mu)}] f = S^{(\rho) (\mu) (\nu)}_{(\rho) \mu \nu} D_{(\rho) \mu} f. \]
When an unrestricted tensor such as the full torsion tensor is projected, the result is also an object with corresponding projection labels and indexes and it too can be represented in terms of index-label pair indexes. Thus, the projections of the full torsion tensor can be represented as

\[ S \left[ Z^\rho \langle X^\mu \rangle \langle Y^\nu \rangle \right] = S \left[ Z^\rho X^\mu Y^\nu \right] \]

where the empty square bracket remains in order to distinguish these projections from the family of restricted torsion tensors.

2. Assemblies of restricted tensors

When a collection of restricted tensors is organized into a single object with projection labels, the result is a larger geometrical object which I choose to call an assembly. Thus, the index-label pair components \( S \langle \rho \rangle \langle \mu \rangle \langle \nu \rangle \) are regarded as specifying the restricted torsion assembly while the components \( S \left[ \langle \rho \rangle \langle \mu \rangle \langle \nu \rangle \right] \) are regarded as specifying the projected torsion assembly. Similarly, the array of restricted metricity components \( Q \langle \mu \rangle \langle \nu \rangle \langle \delta \rangle \) specify the restricted metricity assembly while the projections \( Q \left[ \langle \mu \rangle \langle \nu \rangle \langle \delta \rangle \right] \) specify the projected metricity assembly. The advantage of organizing restricted tensors into assemblies is that the index-label pair notation can then be used to make very compact expressions which are usually identical in structure to familiar unprojected tensor expressions.

3. Projection gradient assembly

It is particularly convenient to organize the projection curvatures into a single assembly, the projection gradient assembly. In terms of this object, the projection gradient decomposition (Eq. 4) becomes just

\[ \nabla H \left[ \langle \beta \rangle \langle \alpha \rangle \langle \delta \rangle \right] = \nabla H \langle \beta \rangle \langle \alpha \rangle \langle \delta \rangle \]

and the decomposition of the covariant derivative of a vector field \( v \in HT_P \) (compare to Eqs. 5 and 6) is

\[ \nabla \langle \delta \rangle v \langle \alpha \rangle = D \langle \delta \rangle v \langle \alpha \rangle + \nabla H \langle \alpha \rangle \langle \rho \rangle \langle \delta \rangle v \langle \rho \rangle \]  

while, for a form-field \( \varphi \in H\hat{T}_P \) (compare to Eqs. 7 and 8)

\[ \nabla \langle \delta \rangle \varphi \langle \beta \rangle = D \langle \delta \rangle \varphi \langle \beta \rangle + \varphi \langle \sigma \rangle C \langle \sigma \rangle \langle \rho \rangle \nabla H \langle \rho \rangle \langle \beta \rangle \langle \delta \rangle \]

4. The complementation tensor

From Eq. 18 the complement of the assembly \( \nabla H \) is \( \nabla V = -\nabla H \). so that the covariant derivative of a vector field \( v \in VT_P \) is given by Eq. 19 with the sign of \( \nabla H \) reversed. A similar reversal occurs for form-fields. In order to express this reversal in expressions for arbitrary vector or form fields, introduce the complementation tensor

\[ C^{\sigma \rho} = H^{\sigma \rho} - V^{\sigma \rho} \]

and the corresponding assembly so that, for any restricted vector field \( v \) and form-field \( \varphi \),

\[ \nabla \langle \delta \rangle v \langle \alpha \rangle = D \langle \delta \rangle v \langle \alpha \rangle + \nabla H \langle \alpha \rangle \langle \sigma \rangle C^{\langle \sigma \rangle \langle \rho \rangle} v \langle \rho \rangle \]  

\[ \nabla \langle \delta \rangle \varphi \langle \beta \rangle = D \langle \delta \rangle \varphi \langle \beta \rangle + \varphi \langle \sigma \rangle C^{\langle \sigma \rangle \langle \rho \rangle} \nabla H \langle \rho \rangle \langle \beta \rangle \langle \delta \rangle \]

It is no longer necessary to specify which subspace the vector \( v \) or the form \( \varphi \) is restricted to — only that it is restricted. When \( V \) is a timelike projection, the complementation tensor is just a representation of the time reversal operator. The covariant derivative of the complementation tensor follows from Eq. 18:

\[ \nabla C \left[ \langle \beta \rangle \langle \alpha \rangle \langle \delta \rangle \right] = 2 \nabla H \langle \beta \rangle \langle \alpha \rangle \langle \delta \rangle \]
The complementation and projection gradient assemblies obey the identity
\[ \nabla H^{(\alpha)}(\tau)(\delta)C^{(\tau)}(\sigma) = -C^{(\alpha)}(\rho)\nabla H^{(\rho)}(\sigma)(\delta) \]  
which suggests that Eqs. 21 and 22 can be simplified and made to appear more like the normal relationships between different types of derivative operators by defining the assembly
\[ K^{(\alpha)}(\sigma)(\delta) = \nabla H^{(\alpha)}(\tau)(\delta)C^{(\tau)}(\sigma) = -C^{(\alpha)}(\rho)\nabla H^{(\rho)}(\sigma)(\delta). \]  
The components of this object are
\[ K^{(H \beta)}(H \alpha)(H \delta) = 0, \quad K^{(H \beta)}(V \alpha)(H \delta) = -h^{T}_{V \alpha \beta \delta}, \quad K^{(V \beta)}(H \alpha)(V \delta) = 0, \quad K^{(H \beta)}(V \alpha)(V \delta) = 0 \]  
This assembly does not change sign under complementation and plays a satisfyingly familiar role as the generator of correction terms in the relationship between covariant and restricted derivatives. For a restricted tensor \( m^{a_{1}, a_{2}... , a_{n}}_{b_{1}, b_{2}... , b_{n}} \) of arbitrary rank, the decomposition of the gradient into restricted parts is just
\[ (\nabla m) []^{(a_{1})... (a_{n})} = D[\nabla g^{(a_{1})... (a_{n})}] = h^{T}_{a_{1}... a_{n}} \]  
where \( h^{T}_{a_{1}... a_{n}} \) is not.

The reason for emphasizing the distinction between restricted and projected objects should now be apparent. When a collection of restricted tensor fields is actually the set of all projections of a spacetime tensor, the corresponding assembly is just a representation of the \textit{original spacetime tensor field}. For example, the metric assembly \( g^{(\mu)(\nu)} \) is simply a representation of the spacetime metric tensor \( g^{\mu \nu} \). In an adapted frame, one would refer to this representation as "partitioning the matrix of coefficients". Similarly, the projected torsion assembly \( S^{(\mu)(\nu)} \) is just a representation of the spacetime torsion tensor \( S^{\mu}_{\nu \rho} \) but the restricted torsion assembly \( S^{(\mu)(\nu)} \) is not.

The concept of an assembly as a representation of a spacetime tensor can be made explicit by introducing a set of basis vectors \( e_{\mu} \) on the full tangent space \( T_{P} \) and noticing that the vectors
\[ e_{(H \mu)} = He_{\mu}, \quad e_{(V \mu)} = Ve_{\mu} \]
form an overcomplete basis for the subspaces \( HT_{P} \) and \( VT_{P} \) respectively. Similarly, the forms
\[ \omega^{(H \nu)} = H^{*}\omega^{\nu}, \quad \omega^{(V \nu)} = V^{*}\omega^{\nu} \]
are overcomplete bases for the subspaces \( H^{*}T_{P} \) and \( V^{*}T_{P} \). An assembly can be thought of as the expansion of a tensor in terms of these overcomplete sets of basis vectors. For example, the vector assembly \( v^{(a)} \) which has components \( v^{(H \alpha)} \) and \( v^{(V \alpha)} \) in \( HT_{P} \) and \( VT_{P} \) respectively may also be regarded as a representation of the vector
\[ v = v^{(\rho)}e_{(\rho)} = v^{(H \rho)}e_{(H \rho)} + v^{(V \rho)}e_{(V \rho)} = v^{H} + v^{V} \]
in the full tangent space \( T_{P} \).
B. Restricted curvatures

1. Definition in terms of assemblies

The restricted curvature tensors are defined by the action of restricted covariant derivatives on restricted vector fields according to

\[
\left( [D_{\beta}, D_{\alpha}] - S^{(\rho)}_{\alpha(\beta)} D_{(\rho)} \right) v^{(\gamma)} = v^{(\rho)} R^{(\gamma)}_{\rho(\gamma)\alpha\beta}.
\]

Because the restricted derivatives always project back into the same projected subspace, this definition implies that the first two index-label pairs of the restricted curvature must have the same label. In the notation of Ref. 1, the restricted curvature tensors are

\[
R_{(\rho)}^{(Z\gamma)}_{(X\alpha)(Y\beta)} = R^{Z}_{XY\rho} \gamma_{\alpha\beta}.
\]

For example, if the projection tensor \( H \) projects vectors into the space tangent to a spacelike hypersurface, then the restricted curvature \( R^{H}_{H\rho} \gamma_{\alpha\beta} \) is the familiar Riemannian curvature of the geometry intrinsic to the hypersurface. Similarly, when \( V \) is surface-forming, \( R^{V}_{V\rho} \gamma_{\alpha\beta} \) is the Riemannian curvature intrinsic to the surfaces which it forms. The remaining restricted curvature tensors are unfamiliar objects which I call cross-projection curvatures.

In order to take full advantage of the similarity of the restricted curvature definition to the full curvature definition, it is useful to let the projection labels on the first two indexes take on all values by stipulating that

\[
R_{(H\rho)}^{(V\gamma)}_{(\alpha)(\beta)} = R_{(V\rho)}^{(H\gamma)}_{(\alpha)(\beta)} = 0.
\]

This stipulation completes the definition of an assembly which I choose to call the restricted curvature assembly.

2. Identities of the restricted curvature assembly

Now convert the results of Ref. 1 into this compact notation. The restricted curvature assembly is found to obey the assembled torsion Bianchi identities.

\[
R_{[(\rho)]}^{(\gamma)}_{(\alpha)(\beta)} + D_{[(\rho)]} S^{(\gamma)}_{(\alpha)(\beta)} + S^{(\gamma)}_{(\sigma)[(\rho)]} S^{(\sigma)}_{(\alpha)(\beta)} = 0
\]

and the assembled curvature Bianchi identities

\[
D_{[(\rho)]} R_{[(\beta)]}^{(\gamma)}_{(\alpha)(\beta)} + R_{[(\beta)]}^{(\gamma)}_{(\sigma)[(\rho)]} S^{(\sigma)}_{(\alpha)(\beta)} = 0.
\]

Each of these identities expands out to a family of relationships connecting different restricted curvature tensors. The assembled torsion Bianchi identities in particular are useful for expressing the unfamiliar cross-projected curvature tensors in terms of more familiar objects.

In this same compact notation, the curvature-metricity identity which was discussed in Ref. 1 becomes

\[
2g^{(\rho)(\delta)} R_{(\rho)}^{(\gamma)}_{(\alpha)(\beta)} = 2D_{[(\alpha)]} Q^{(\gamma)(\delta)}_{(\beta)} + S^{(\rho)}_{(\alpha)(\beta)} Q^{(\gamma)(\delta)}_{(\rho)}
\]

Notice that the intrinsic curvature assembly, the restricted torsion assembly, and the restricted metricity assembly obey identities which have exactly the same structure as the usual identities of the unprojected curvature tensor.

3. Generalized Gauss-Codazzi curvature projections

A spectacular example of how the use of assemblies condenses complex expressions is the projection decomposition of the curvature tensor. A straightforward calculation from the definition of the full and restricted Riemann tensors and the covariant derivative decompositions implied by Eqs. \((22)\) and \((23)\) as well as an application of the identities given in Eq.\((24)\) and \((25)\) yields the expression

\[
R_{(\rho)}^{(\gamma)}_{(\alpha)(\beta)} = R_{(\rho)}^{(\gamma)}_{(\alpha)(\beta)} + 2D_{[(\beta)]} K^{(\gamma)}_{(\rho)(\alpha)} - S^{(\tau)}_{(\alpha)(\beta)} K^{(\gamma)}_{(\rho)(\tau)} - 2K^{(\gamma)}_{(\alpha)(\beta)} K^{(\gamma)}_{(\rho)(\beta)}.
\]
This one expression gives all of the projections of the curvature tensor in terms of components of the intrinsic and extrinsic curvature assemblies. A partial expansion of the full expression is useful because one finds two basic sub-expressions:

\[ R[ (H_{\rho}(H_{\gamma})_{(\alpha)_{(\beta)}} = R(H_{\rho})(H_{\gamma})_{(\alpha)_{(\beta)}} - 2K^{(\sigma)}(H_{\rho})[[\alpha]_{(\beta)} \right] \]

\[ R[ (H_{\rho})_{(\gamma)_{(\alpha)_{(\beta)}} = 2D_{(\beta)}K^{(\gamma)}(H_{\rho})_{[[\alpha]]} - S^{(\sigma)}(\alpha)_{(\beta)}K^{(\gamma)}(H_{\rho})_{(\gamma)} \right] \]

which clearly generalize the Gauss-Codazzi equations for surface embedding.

**IV. LIE DERIVATIVES**

**A. Basics**

1. **Standard Lie derivatives**

The Lie derivative of a vector field \( v \) with respect to a vector-field \( N \), may be obtained from the commutator

\[ (\mathcal{L}_N v) f = [N, v] f \]

for any function, \( f \). To relate the Lie derivative to the covariant derivative, write the commutator in the form

\[ (\mathcal{L}_N v) \delta \nabla f = [(N^\sigma \nabla_\sigma)(v^\rho \nabla_\rho) - (v^\rho \nabla_\rho)(N^\sigma \nabla_\sigma)] f. \]

The basic relationship between the Lie derivative and the covariant derivative then follows from the definition, Eq. 13, of the torsion tensor. In terms of the quantity

\[ \nabla' N^\delta = \nabla_\rho N^\delta - S^\delta_{\rho \sigma} N^\sigma \]

the relationship is found to be

\[ (\mathcal{L}_N v) \delta = N^\sigma \nabla_\sigma v^\rho - v^\rho \nabla' N^\delta. \]

The Lie derivative of an arbitrary rank tensors \( M \) is related to its covariant derivatives by

\[ (\mathcal{L}_N M)_{\delta_1 \ldots \delta_n}^{\alpha_1 \ldots \alpha_n} = N^\sigma \nabla_\sigma M_{\delta_1 \ldots \delta_n}^{\alpha_1 \ldots \alpha_n} \]

\[ + M^{\delta_1 \ldots \delta_n \rho \ldots \alpha_n} \nabla'_{\rho \alpha_n} N^\rho + \ldots + M^{\delta_1 \ldots \delta_n \rho \ldots \alpha_n} \nabla'_{\rho \alpha_n} N^\rho. \]

2. **Restricted Lie derivatives**

A restricted Lie derivative can be defined in the same way that the restricted covariant derivative was defined. For a restricted tensor-field \( M \) which is characterized by the projection

\[ O M = M \]

the restricted Lie derivative of \( M \) is defined to be

\[ L_N M = O \mathcal{L}_N M. \]

Just as the restricted covariant derivatives of the projection tensor field \( H \) are automatically zero, it is easy to see that

\[ L_N H = 0. \]

The restricted Lie derivative of an assembly of restricted tensor fields \( M \) is related to its restricted covariant derivatives by an expression with the same structure as Eq. 33

\[ (L_N M)_{\delta_1 \ldots \delta_n}^{\alpha_1 \ldots \alpha_n} = N^{\sigma} D^{\sigma}_{\sigma} M_{\delta_1 \ldots \delta_n}^{\alpha_1 \ldots \alpha_n} \]

\[ - M^{\delta_1 \ldots \delta_n}_{(\alpha_1) \ldots (\alpha_n)} D'_{(\rho)} N^{(\delta)} - \ldots - M^{\delta_1 \ldots \delta_n}_{(\alpha_1) \ldots (\alpha_n)} D'_{(\rho)} N^{(\delta)} \]

\[ + M^{\delta_1 \ldots \delta_n}_{(\rho) \ldots (\alpha_n)} D'_{(\alpha_n)} N^{(\rho)} + \ldots + M^{\delta_1 \ldots \delta_n}_{(\rho) \ldots (\sigma)} D'_{(\sigma)} N^{(\rho)}. \]

where

\[ D'_{(\rho)} N^{(\delta)} = D_{(\rho)} N^{(\delta)} - S^{(\delta)}_{(\rho) (\sigma)} N^{(\sigma)}. \]
B. Restricted Lie derivatives of geometrical structures

1. The projection tensor

Now work out the projection-tensor decomposition of $\mathcal{L}_N H$. Start with the assembly-notation version of the spacetime expression.

$$\mathcal{L}_N H^{(\alpha)}_{(\beta)} = N^{(\delta)} \nabla_{(\delta)} H^{(\alpha)}_{(\beta)} - H^{(\rho)}_{(\beta)} \nabla^{(\rho)} N^{(\alpha)} + H^{(\alpha)}_{(\rho)} \nabla^{(\rho)} N^{(\rho)}$$

Use Eq. 26 to represent the covariant derivatives in terms of assemblies of restricted objects and use Eq. 28 to eliminate the spacetime torsion tensor.

$$\mathcal{L}_N H^{(\alpha)}_{(\beta)} = H^{(\alpha)}_{(\rho)} D_{(\beta)} N^{(\rho)} - H^{(\rho)}_{(\beta)} D^{(\rho)} N^{(\alpha)} + N^{(\sigma)} \left( H^{(\rho)}_{(\beta)} S^{(\alpha)}_{(\rho)} - H^{(\alpha)}_{(\rho)} S^{(\rho)}_{(\beta)} \right)$$

This expression is actually antisymmetric under complementation, as the identity

$$\mathcal{L}_N (H + V) = \mathcal{L}_N H + \mathcal{L}_N V = 0$$

requires. It is useful to make this complementation antisymmetry manifest by writing the expression in the form

$$\mathcal{L}_N H^{(\alpha)}_{(\beta)} = \frac{1}{2} \left( C^{(\alpha)}_{(\tau)} D_{(\beta)} N^{(\tau)} - C^{(\tau)}_{(\sigma)} D^{(\tau)} N^{(\alpha)} \right) + \frac{1}{2} N^{(\sigma)} \left( C^{(\tau)}_{(\rho)} S^{(\alpha)}_{(\rho)} - C^{(\alpha)}_{(\rho)} S^{(\rho)}_{(\beta)} \right).$$

Simplify the result another way by finding its non-zero projections. Because the projections of the full spacetime torsion are usually set to zero, collect those terms together and write the result in the form

$$\mathcal{L}_N H^{[H]}_{[V]} \alpha \beta = -S_N \left[ [H] \right] \alpha \beta + D_{V \beta} N^{H \alpha} + h_{H \beta}^{\alpha} N^{H \sigma} - 2\omega^\alpha_{\beta \sigma} N^{V \sigma}$$

$$\mathcal{L}_N H \left[ V \right]^{H}_{H} \alpha \beta = S_N \left[ V \right]^{H}_{H} \alpha \beta - D_{H \beta} N^{V \alpha} - h_{V \beta}^{\alpha} N^{V \sigma} + 2\omega^\alpha_{H \beta \sigma} N^{H \sigma}$$

where the spacetime torsion terms are components of the assembly

$$S_N [\cdot]^{(\rho)}_{(\beta)} = S [\cdot]^{(\rho)}_{(\beta)} N^{(\beta)}.$$

2. The metric tensor

These techniques also yield an expression for the assembly of projections of the Lie derivative of the metric. Express the Lie derivative in terms of a covariant derivative and decompose the result into its restricted parts. Just as was the case for the Lie derivative of the projection tensor field, the result can be simplified in two ways. In terms of assemblies, the result is

$$\mathcal{L}_N g \left[ [\cdot]^{(\mu)}_{(\nu)} \right] = \left[ 2g^{(\rho)} \left( [\cdot]^{(\mu)}_{(\nu)} \right) - Q^{(\rho)}_{(\mu)} \right] N^{(\delta)}$$

$$\mathcal{L}_N g \left[ [H]^{\mu \nu} \right] = -Q_N \left[ [H]^{\mu \nu} \right] + g^{(\rho)} D^{(\rho)} N^{(\mu)} - g^{(\mu)} D^{(\rho)} N^{(\nu)}.$$

In terms of restricted objects, the projections of the Lie derivative are

$$\mathcal{L}_N g \left[ [H]^{\mu \nu} \right] = \left( g^{H \nu \rho} D^{H \rho} + g^{V \nu \rho} D^{V \rho} \right) N^{H \mu} + \left( g^{H \nu \rho} D^{H \rho} + g^{V \nu \rho} D^{V \rho} \right) N^{V \mu}$$

$$\mathcal{L}_N g \left[ [V]^{\mu \nu} \right] = \left( g^{H \nu \rho} D^{H \rho} + g^{V \nu \rho} D^{V \rho} \right) N^{V \mu} - \left( g^{H \nu \rho} D^{H \rho} + g^{V \nu \rho} D^{V \rho} \right) N^{H \mu}$$

and the complements of these expressions. Here, the terms which normally vanish in Riemannian geometries have been collected into the assembly

$$Q_N [\cdot]^{(\mu)}_{(\nu)} = \left( Q [\cdot]^{(\mu)}_{(\nu)} - g^{(\mu)} D^{(\rho)} N^{(\rho)} \right) N^{(\delta)}.$$
3. Restricted metric tensors

Consider each projection $g^{HH}, g^{HV}, g^{VH}, g^{VV}$ of the metric tensor as a separate restricted tensor field, compute the Lie derivatives of these tensor fields, and project the results in all possible ways. In each case, the result can be represented in the form of a projection assembly. The resulting assemblies can be found easily by applying the product rule and Eqs. (39) and (34) to expressions such as

$$g^{HH}(\alpha)(\beta) = H^{(\alpha)}(\rho)g^{(\rho)}(\sigma)H^{(\beta)}(\sigma)$$

in order to obtain

$$L_N g^{HH}(\alpha)(\beta) = -2g^{(H\rho)}(\beta)D_{(H\rho)}N^{(\alpha)} + (2g^{(H\rho)}(\beta)S^{(\alpha)}(H\rho)(\sigma) - Q(H\alpha)(H\beta)(\sigma))N^{(\alpha)}$$

$$L_N g^{HV}(\alpha)(\beta) = -g^{(H\rho)}(V\beta)D_{(H\rho)}N^{(\alpha)} + (g^{(H\rho)}(V\beta)S^{(\alpha)}(H\rho)(\sigma) + g^{(H\rho)}(V\beta)S^{(\beta)}(V\rho)(\sigma) - Q(H\alpha)(V\beta)(\sigma))N^{(\alpha)}$$

In terms of restricted objects, and the normally vanishing assemblies $Q_N$ and $S_N$ defined by Eqs. (42) and (38), the projections of these Lie derivatives are given by

$$L_N g^{HH}[HV]^{\alpha\beta} = -Q_N [HH]^{\alpha\beta} - 2g^{HH\rho\beta}D_{(H\rho)}N^{(\alpha)} + 2g^{HH\rho\beta}h^{\beta(\rho)\alpha}N_{H\sigma} + 2g^{HH\rho\beta}h^{\beta(\rho)\alpha}N_{V\sigma}$$

$$L_N g^{HV}[HV]^{\alpha\beta} = g^{HV\rho\sigma}S_N [V^H]^{\beta(\rho)} + g^{HV\rho\sigma}h^{\beta(\rho)\alpha}N_{H\sigma} - g^{HV\rho\sigma}h^{\beta(\rho)\alpha}N_{V\sigma}$$

$$L_N g^{HH}[VV]^{\alpha\beta} = 0$$

$$L_N g^{HV}[HV]^{\alpha\beta} = -Q_N [HV]^{\alpha\beta} - g^{HV\rho(\beta)D_{(H\rho)}N^{(\alpha)} + g^{HV\rho(\beta)D_{(H\rho)}N^{(\alpha)}} + g^{HV\rho(\beta)D_{(H\rho)}N^{(\alpha)}} + g^{HV\rho(\beta)D_{(H\rho)}N^{(\alpha)}}$$

$$L_N g^{HV}[HV]^{\alpha\beta} = g^{HV\rho\sigma}S_N [V^H]^{\beta(\rho)} + g^{HV\rho\sigma}h^{\beta(\rho)\alpha}N_{H\sigma} - g^{HV\rho\sigma}h^{\beta(\rho)\alpha}N_{V\sigma}$$

The remaining projections can be obtained by taking the complements of these and by using the symmetry of the metric tensor.

4. Generalized area change

An important consequence of Eq. (43) is a formula for the evolution of a generalized area element. Suppose that the dimension of $HT_P$ is $s$. The area element on surfaces tangent to the subspace $HT_P$ is just the unit $s$-form $\alpha$ in $H^*T_P$. Choose a vector field $N^\alpha$ in $VT_P$ and an $s$-form field $\sigma$ which is propagating along the integral curves of $N^\alpha$ by Lie-dragging so that it solves $L_{H^*}\sigma = 0$. So long as the chosen Lie-dragged $s$-form $\sigma$ obeys

$$H^* \neq 0, \quad g^{-1}(H^* \sigma, H^* \sigma) \neq 0,$$

the $H$-area element $\alpha$ can be constructed from $\sigma$. If the subspace $HT_P$ is spacelike ($k = 1$) or timelike ($k = -1$), then

$$\alpha = [kg^{-1}(H^* \sigma, H^* \sigma)]^{-1/2} H^* \sigma.$$
Because the restricted Lie derivative obeys $L_N H = 0$ and the field $\sigma$ has been defined by Lie-dragging, it is easy to see that $L_N (H^* \sigma) = 0$. The restricted Lie derivative of the $H$-area element is then

$$L_N \alpha = -\frac{1}{2} \left[ g^{-1} (H^* \sigma, H^* \sigma) \right]^{-1} \left[ L_N g^{-1} \right] (H^* \sigma, H^* \sigma) \alpha$$

Expand the forms in an adapted frame and find that $g^{-1} (H^* \sigma, H^* \sigma)$ is just the determinant of the matrix of coefficients $g^{H \rho \alpha}$ and, from the usual formula for the derivative of a determinant,

$$L_N \alpha = -\frac{1}{2} g^{H \rho \alpha} \frac{\partial}{\partial x^{\rho}} g^{H \rho \alpha} \alpha.$$

Eq. 45 now gives the result

$$g^{H \rho \alpha} \frac{\partial}{\partial x^{\rho}} g^{H \rho \alpha} (\sigma) = -L_N \alpha \quad (50)$$

Notice that, in general, $g^{H \rho \alpha}$ is not the matrix inverse of $g_{H \alpha \beta}$. By projecting the identity $g_{\alpha \beta} g^{\alpha \beta} = \delta_\beta^\rho$, one finds that the above equation takes the form

$$N^\sigma \frac{\partial}{\partial x^{\rho}} g^{H \rho \alpha} (\sigma) = -L_N \alpha \quad (51)$$

between the divergence and the rate of change of the projected area element is obtained.

C. Rule for differentiating restricted tensor fields

1. Restricted vector and form fields

Start with the expressions

$$v^{(\alpha)} = H^{(\alpha)} (\rho) v^{(\rho)} \quad \text{if} \quad v \in HT_P$$
$$v^{(\alpha)} = V^{(\alpha)} (\rho) v^{(\rho)} \quad \text{if} \quad v \in VT_P$$

Differentiate these expressions

$$\mathcal{L}_N v^{(\alpha)} = \left\{ \begin{array}{ll} \left( \mathcal{L}_N H^{(\alpha)} (\rho) \right) v^{(\rho)} + H^{(\alpha)} (\rho) \mathcal{L}_N v^{(\rho)} & \text{if} \quad v \in HT_P \\ -\left( \mathcal{L}_N H^{(\alpha)} (\rho) \right) v^{(\rho)} + V^{(\alpha)} (\rho) \mathcal{L}_N v^{(\rho)} & \text{if} \quad v \in VT_P \end{array} \right.$$ 

and notice that they are summarized by

$$\mathcal{L}_N v^{(\alpha)} = \left( \mathcal{L}_N H^{(\alpha)} (\sigma) \right) C^{(\sigma)} (\rho) v^{(\rho)} + L_N v^{(\alpha)}$$

Similarly, differentiate the projection identities satisfied by restricted forms and obtain

$$\mathcal{L}_N \varphi^{(\beta)} = \varphi^{(\rho)} C^{(\rho)} (\sigma) \left( \mathcal{L}_N H^{(\sigma)} (\beta) \right) + L_N \varphi^{(\beta)}$$

2. The Lie-derivative correction assembly

Evidently, the assembly

$$\mathcal{L}_N^{(\alpha)} (\beta) = C^{(\alpha)} (\sigma) \left( \mathcal{L}_N H^{(\sigma)} (\beta) \right) = -\left( \mathcal{L}_N H^{(\alpha)} (\sigma) \right) C^{(\sigma)} (\beta)$$
plays the same role in the projection decomposition of Lie derivatives as the quantity $\nabla^\rho_{\mu} N^\delta$ plays in their covariant derivative expressions. From Eq. [52] the general expression for this assembly is

$$\ell'_{N}^{(\alpha \rho \sigma)} = \frac{1}{2} \left( C^{\alpha \sigma \rho} D^{\sigma} N^{(\alpha \rho \sigma)} - C^{\alpha \sigma \rho} D^{\sigma} N^{(\alpha \rho \sigma)} \right) + \frac{1}{2} N^{(\alpha \rho \sigma)} \left( C^{\alpha \sigma \rho} S_{(\sigma)}^{(\rho \sigma)} - C^{\alpha \sigma \rho} S_{(\sigma)}^{(\rho \sigma)} \right) C^{(\sigma \rho \sigma)}.$$  

(52)

This expression is a good example of a case where the assembly notation is less compact and transparent than writing out all of its components. Define the Lie derivative correction assembly

$$\ell_{N}^{(\alpha \rho \sigma)} = \ell'_{N}^{(\alpha \rho \sigma)} + S_{N}^{(\alpha \rho \sigma)}$$

which plays a role analogous to the one played by $\nabla_{\rho} N_{\delta}$ in the rendering of Lie derivatives in terms of covariant derivatives. From Eqs. [52] and [57] this assembly can be expressed quite compactly by

$$\ell_{N}^{(H\alpha \beta \gamma \sigma)} = 0,$$

$$\ell_{N}^{(V\alpha \beta \gamma \sigma)} = D_{H \beta \gamma \sigma} N^{V \alpha} + \tilde{h}^{V \alpha \beta \sigma} N^{V \alpha} - 2\omega_{H \beta \gamma \sigma} N^{H \sigma},$$

and the complements of these expressions.

3. Restricted tensor fields of arbitrary rank

Consider an arbitrary-rank, restricted tensor-field $M^{\delta_{1} \delta_{2} \ldots \delta_{n} \alpha_{1} \alpha_{2} \ldots \alpha_{n}}$. Such a tensor field obeys a projection identity given by Eq. [52]. The Lie derivative of such a tensor field can be expressed in terms of restricted tensors by taking the Lie derivative of this identity with the result.

$$\left( \mathcal{L}_{N} M \right)^{(\delta_{1}) \ldots (\delta_{n})}_{(\alpha_{1}) \ldots (\alpha_{n})} = L_{N} M^{(\delta_{1}) \ldots (\delta_{n})}_{(\alpha_{1}) \ldots (\alpha_{n})} - M^{(\rho)}_{(\delta_{1}) \ldots (\delta_{n})}_{(\alpha_{1}) \ldots (\alpha_{n})} \ell'_{N}^{(\delta_{1})}_{(\rho)} - \ldots - M^{(\delta_{1}) \ldots (\rho)}_{(\alpha_{1}) \ldots (\alpha_{n})} \ell'_{N}^{(\delta_{n})}_{(\rho)} + M^{(\delta_{1}) \ldots (\delta_{n})}_{(\rho)} \ell_{N}^{(\rho)}_{(\alpha_{1}) \ldots (\alpha_{n})} + \ldots + M^{(\delta_{1}) \ldots (\delta_{n})}_{(\alpha_{1}) \ldots (\rho)} \ell'_{N}^{(\rho)}_{(\alpha_{1}) \ldots (\alpha_{n})}.$$  

(53)

4. Decomposing derivatives of unrestricted tensors

The preceding result may be used to express the projections of the Lie derivative of an unrestricted tensor $m$ in terms of its projections. Simply express the tensor $m$ as the sum of its projections, sum the preceding expression over all projection labels, and project the result on all free indexes. The resulting expression for $(\mathcal{L}_{N} m)^{(\delta_{1}) \ldots (\delta_{n})}_{(\alpha_{1}) \ldots (\alpha_{n})}$ looks exactly like Eq. [53] with $M$ replaced by $m$.

V. PROJECTIONS ONTO ISOMETRY GROUP ORBITS

A. Basics

1. Background

The study of metric-tensor symmetries is arguably the oldest topic in the field of general relativity. Most results in this area have been discovered and re-discovered, formulated, and re-formulated many times as fashions in notation have changed. This section is yet another chapter in this history of reformulation. The tilted projection tensor formulation is new and has some new insights to offer but, as far as I can tell, the particular properties which I will be discussing are not new. I apologize in advance to the very large number of colleagues whose work I have surely failed to cite.
2. Killing vectors

A vector field \( N \) on a manifold \( M \) generates an isometry or, equivalently, a motion of the metric tensor, if the corresponding Lie derivative of the metric tensor vanishes everywhere on \( M \). In the most general case, the resulting condition on \( N \) is

\[
\mathcal{L}_N g^{\alpha\beta} = -Q^{\alpha\beta} \delta N^\gamma - 2g^{\rho(\alpha} \nabla_{\rho'} N^{\beta)} = 0.
\]

A vector field which satisfies this condition everywhere on \( M \) is called a Killing vector field. Within this section, I will be assuming a metric-compatible, torsion-free connection so that Killing vectors obey the familiar form of Killing’s equation.

\[
g^{\rho(\alpha} \nabla_{\rho'} N^{\beta)} = 0
\]

3. Group orbits

Each Killing vector field generates a map of the manifold into itself. The set of all points \( O_P \) which can be reached by such maps, starting from a given point \( P \in M \) is called the isometry group orbit through the point \( P \). If there are \( n \) Killing vector fields then each orbit is a submanifold of dimension \( r \leq n \). When the dimension \( r \) of a group orbit is less than the dimensionality \( n \) of the isometry group, the Killing vectors cannot all be linearly independent everywhere on that orbit — At each point \( P \) of the orbit, some linear combination \( \Sigma P N \) of Killing vectors must vanish. The point \( P \) is then a fixed point of the motion generated by the combination \( \Sigma P N \). Spherical symmetry, with \( n = 3, r = 2 \) is the best known example of this situation.

4. Isotropy

Isotropies constrain the direction of invariant vector fields. If \( P \) is a fixed point of a subgroup \( I_P \) of a group of motions \( G \) then \( I_P \) induces mappings on the vector space \( HT_P \) which is tangent to the group orbit through \( P \). The group \( G \) is said to have an isotropic orbit at \( P \) if the subgroup \( I_P \) induces mappings which connect any two directions (i.e. rays) in \( HT_P \). In other words, there are no preferred directions on an isotropic orbit. The result that will be needed here is a very simple one: If \( G \) has an isotropic orbit, and a vector-field is tangent to the orbit and invariant under \( G \) then the vector-field vanishes everywhere on the orbit. This result can be used to obtain powerful constraints on an isotropic geometry by the technique of constructing group-invariant vector fields tangent to the group orbits and then setting those vector fields equal to zero.

B. The general case

1. Arbitrary orbit projections

Now consider a geometry with an isometry group and a projection tensor field \( H \) which projects vectors into the tangent spaces to group orbits. For any Killing vector field \( \xi \) one then has the relations \( H\xi = \xi \) or, equivalently, \( \xi^H = \xi^\alpha \) and \( \xi^V = 0 \) as well as Killing’s equation \( \mathcal{L}_\xi g^{\mu\nu} = 0 \). Equations \([40]\) and \([41]\) for the projections of the Lie derivative of the metric tensor then require

\[
2g^{H\rho(\nu} D_{H\rho} \xi^{\mu)} + 2g^{HV(\rho} D_{V\rho} \xi^{\nu)} = 0 \tag{54}
\]

and

\[
\left( g^{HV\rho} D_{H\rho} + g^{VV\rho} D_{V\rho} \right) \xi^{\mu} + \left[ g^{H\mu\rho} h_{H\nu}^{\nu} \sigma - g^{HV\mu\rho} h_{V\nu}^{\nu} \sigma \right] \xi^{\sigma} = 0 \tag{55}
\]

while the complement of Eq. \([40]\) requires

\[
\left[ g^{VV\mu\rho} h_{V\nu}^{\nu} \sigma + g^{VV\nu\rho} h_{V\sigma}^{\nu} \mu - g^{HV\mu\rho} h_{H\nu}^{\nu} \sigma - g^{HV\nu\rho} h_{H\mu}^{\mu} \sigma \right] \xi^{\sigma} = 0. \tag{56}
\]

The complement of Eq. \([41]\) simply requires Eq. \([55]\) again.
Because the Killing vectors span the tangent space $HT_P$ to the group orbit, Eq. 56 implies a restriction on the projection curvatures which can be written in the form

$$g^{VV\rho(\nu)h_T^{\mu\sigma}\rho}_{VV\sigma} = g^{HV\rho(\nu)h_H^{\mu}\rho}_{HV\sigma}.$$  \hspace{1cm} (57)

This restriction holds for any projection onto any group orbit no matter how the projection is tilted. In addition to this restriction, the fact that the projection is onto the tangent spaces to a surface requires (see Ref. 1)

$$h_H^{\mu}\sigma_{\rho} = h_H^{\mu}_{\rho\sigma}.$$  \hspace{1cm} (58)

Equations 54 and 55 imply restrictions on some of the connection coefficients. Because the Killing vectors $\{\xi_a\}$ span the tangent space to the orbit, the coefficients defined by

$$\Gamma_H^{H\mu}_{a\delta} = D_H\xi_a^\mu, \hspace{1cm} \Gamma_V^{H\mu}_{a\delta} = D_V\xi_a^\mu$$

contain all of the information needed to evaluate restricted covariant derivatives of tensor fields on an orbit. The restrictions on these coefficients are

$$2g^{HH\rho(\nu)\Gamma_H^{H\mu}_{a\rho}} + 2g^{HV\rho(\nu)\Gamma_V^{H\mu}_{a\rho}} = 0$$  \hspace{1cm} (59)

$$g^{HV\rho(\nu)\Gamma_H^{H\mu}_{a\rho}} + g^{VV\rho(\nu)\Gamma_V^{H\mu}_{a\rho}} + g^{HH\rho(\nu)h_H^{\mu}\sigma\rho} - g^{HV\rho(\nu)h_T^{\mu}\sigma\rho} - h_H^{\mu}\sigma_{\rho}\xi_a^\sigma = 0$$  \hspace{1cm} (60)

Equation 58 expresses the compatibility between the restricted connection induced on an orbit and the isometry group which is transitive on that orbit. Equation 60 ensures that the Fermi derivative ($D_V$ in this case) is compatible with the isometry group.

2. Group-invariant orbit projections

The projection tensor field can be specialized further. At least within an open set $\Omega$ which contains a non-degenerate group orbit one can find a reference surface $S_R$ which intersects each group orbit in $\Omega$ exactly once. At each point $P$ on $S_R$ let $HT_P$ be the tangent space to the group orbit through the point $P$ and choose $VT_P$ arbitrarily, thus specifying the tensor $H$ on $S_R$. Now use the Killing vector fields to Lie-drag this projection-tensor throughout $\Omega$. The result of this construction is a group-invariant projection onto the group orbits. Throughout the rest of this paper, it will be assumed that projections are group-invariant.

From Eq. 56 as well as the complement of Eq. 57, the vanishing Lie derivative of $H$ then requires each Killing vector to obey the restriction:

$$D_V\xi^\alpha = -h_H^{T\alpha}_{\beta}\sigma\xi^\sigma.$$  \hspace{1cm} (61)

For a set $\{\xi_a\}$ of Killing vectors which span the tangent space to the orbit, this restriction provides a direct formula for the mixed or Fermi-derivative connection coefficients

$$\Gamma_H^{H\alpha}_{a\beta} = -h_H^{T\alpha}_{\beta}\sigma\xi_a^\sigma$$  \hspace{1cm} (62)

and thus describes how to take the restricted derivative $D_V$ of orbit-tangent tensor fields. Equation 61, combined with the definition of the restricted curvature tensor, yields the result

$$R_V^{\tau\beta\alpha}_{\gamma\epsilon} = 2D_V(h_H^{T\beta}_{\gamma\epsilon})_\alpha + h_H^{T\alpha}_{\beta}[\gamma\epsilon]h_H^{T\beta}_{\gamma\epsilon}.$$  \hspace{1cm} (63)

3. Group-invariant evolutions

A group-invariant vector-field $N$ is a vector field which commutes with all of the Killing vectors. I will define a group-invariant evolution of the orbit $O$ to be a finite set of group-invariant vector fields $N[K]$ such that the vectors $VN[K](P)$ span the space $VT_P$ for every point $P$ on $O$. It is easy to see that non-null, orientable isometry orbits of codimension 1 always have group-invariant evolutions — the unit normal vector fields. In a generic spacetime with a group-invariant projection, one might construct a complete set of group-invariant vectors algebraically from the projected Riemann tensor $R_{VV\alpha_{\beta}\rho\nu}$ and its restricted derivatives. For example, in codimension 2, the vector $N^\alpha = g^{VV\alpha}_{\beta\rho}D_V\xi^\beta_{\rho\nu}$ and a unit vector orthogonal to it in $VT_P$ will provide a group-invariant evolution for those group orbits where $D_V\xi^\beta_{\rho\nu}$ is not null. In general, however, such constructions are difficult to carry out and not particularly instructive. In what follows, I will simply restrict my attention to group orbits which have group-invariant evolutions.
4. The divergence-orbit-area relation

For compact orbits, it is possible to extend Eq. 51 for the evolution of an area-element by integrating it over each orbit.

\[
\int_{\Omega} a \alpha N^\alpha \theta_{H \alpha} = - \int_{\Omega} N \alpha \quad (64)
\]

The assumption that this orbit has a group-invariant evolution lets us choose \( N^\alpha \) to be group-invariant. Since \( \theta_{H \alpha} \) is constructed from the projection tensor field \( H \) and the connection, it will be group-invariant if \( H \) is group-invariant. Thus, the scalar \( N^\alpha \theta_{H \alpha} \) is constant on the orbit and can be moved outside the integral. Because \( N \) commutes with the group generators, the Lie derivative on the right-hand side of the equation can be replaced by a derivative of the orbit-area function defined at each point \( P \) by

\[
a (P) = \int_{\Omega} \alpha.
\]

Eq. \((63)\) then becomes

\[
N^\alpha \theta_{H \alpha} a (P) = -L_N a (P) = -N \cdot da = -N \cdot V^* da = -N^\alpha D_{V \alpha} a.
\]

Because \( N \) is part of a group-invariant evolution which spans \( VT_P \), this result implies \( \theta_{H \alpha} a = -D_{V \alpha} a \) so that the divergence is the gradient of a potential.

\[
\theta_{H \alpha} = -D_{V \alpha} (\ln a) \quad (65)
\]

When the group orbits are not compact, this same argument can often be applied to a finite part \( A \) of an orbit which is mapped onto nearby orbits by the group-invariant evolution. The main complication which can occur is that the projection \( V \) and therefore the group-invariant evolution which spans it may not be surface-forming. In that case the vector fields in the evolution might not map \( A \) into the same subsets of neighboring orbits. When the \( V \)-twist tensor \( \omega_{V \rho \nu \gamma} \) is zero, \( V \) is surface-forming and Eq. \((65)\) still holds. Because the result depends on the logarithmic derivative of the area, it is not affected by the size of the chosen subset \( A \).

C. Normal projections onto group orbits

1. Normality

Although this sequence of papers removes the assumption that projection tensor fields are normal and discusses the geometry of arbitrarily tilted projections, it is important to understand just how powerful the normality condition is. At the level of the metric tensor, the assumption has a simple statement:

\[
g^{HV \alpha \beta} = 0.
\]

From this normality assumption with the further assumption

\[
Q^{\mu \nu \delta} = 0
\]

of metric compatibility for the full connection and the metricity decompositions (Eqs. 9, 10, 11, and 12) come three remarkable results: (1) From Eq. 11 and its complement comes the compatibility of the restricted derivatives \( D_{H \delta} \) and \( D_{V \delta} \) with the intrinsic metrics \( g^{HH \alpha \beta} \) and \( g^{VV \alpha \beta} \) on the corresponding subspaces

\[
Q^{HH \mu \nu \delta} = 0, \quad Q^{VV \mu \nu \delta} = 0.
\]

(2) From Eq. 10 and its complement come the vanishing Fermi derivatives of these intrinsic metrics

\[
Q^{HH \mu \nu \delta} = -D_{V \delta} g^{HH \mu \nu} = 0, \quad Q^{VV \mu \nu \delta} = -D_{H \delta} g^{VV \mu \nu} = 0
\]

(3) From Eq. 11 comes the relation

\[
g^{VV \mu \nu \delta} \theta_{H \alpha} - g^{HH \mu \nu \delta} \theta_{H \alpha} = 0
\]
while the complement of Eq. 11 yields the complementary relation.

With a normal projection tensor field, indexes can be raised and lowered on restricted tensors by using the intrinsic metrics so that the above relation becomes just

\[ h^{T\nu\rho}_{\delta\mu} = h_{H\nu\rho}^{\mu\delta} \]

and its complement. Thus, some of the possible restricted metric tensors have been set to zero by the normality conditions themselves (Eq. 66) half of the possible projection curvature tensors have been set equal to the other half and we have complete freedom to raise and lower the indexes of restricted tensors.

When \( H \) is normal, the subspace \( VT_P \) is completely determined once the subspace \( HT_P \) is given (provided that \( HT_P \) is not null). Furthermore, for a group with non-null orbits, the normal projection onto the orbits is unique and obviously group-invariant. The remainder of this paper assumes normal projection tensor fields unless stated otherwise.

2. Projection curvatures

The earlier results for arbitrary projection tensor fields may be specialized to the case of normal projections onto group orbits. Eq. 57 yields the condition \( h_{V\sigma}^{\mu\nu} = 0 \) so that only the twist part of the \( V \)-projection curvature remains.

\[ h^{T\sigma\nu}_{\nu\rho} = h_{V\sigmanu}^{\mu\rho} = \omega_{V\sigma\mu\nu} \]

Because the only non-zero projection tensors are \( h_H \) and \( \omega_V \) we have the luxury of omitting the projection subscripts and defining

\[ h^\nu_{\mu\delta} = h^\nu_{H\mu\delta}, \quad \omega^\nu_{\mu\delta} = \omega^\nu_{V\mu\delta}. \]

Since \( H \) is surface-forming, there is no \( H \)-twist part of \( h_H \). However it still has the decomposition

\[ h^\nu_{\mu\delta} = \sigma^\nu_{\mu\delta} + \frac{1}{8} \theta^\nu g_{HH\mu\delta} \]

where \( \sigma^\nu_{\mu\delta} \) is the shear tensor defined by

\[ \sigma^\nu_{\mu\delta} = h^\nu_{\mu\delta} - \frac{1}{8} g_{HH\mu\delta} h_{\nu\sigma}^\sigma \]

and \( \theta^\nu = h^\nu_{\sigma\sigma} \) is the divergence.

3. Torsion and Ricci curvatures

For this type of projection tensor field, the vanishing of the full torsion tensor leads to the restricted torsion tensors

\[ S_{HH\rho\mu\nu}^V = 0, \quad S_{HH\rho\mu\nu}^H = h_{\nu\rho}^\mu \]

\[ S_{VV\rho\mu\nu}^H = 2 \omega_{\mu\nu}^{\rho}, \quad S_{VV\rho\mu\nu}^V = \omega_{\nu\rho}^\mu \]

The Ricci curvature tensor decomposition then becomes

\[ R_{[HH]}_{\alpha\beta} = R_{HH}_{\alpha\beta}^H + D_{\nu\beta} h^\sigma_{\alpha\beta} - h^\sigma_{\alpha\beta} \theta_{\sigma} - \omega_{\alpha\sigma}^\rho \omega_{\beta\rho}^\sigma, \]

\[ R_{[VV]}_{\alpha\beta} = R_{VV}_{\alpha\beta}^V + D_{\nu\beta} \theta_{\alpha} + D_{\nu\sigma} \omega_{\alpha\beta} - h_{\alpha\rho}^\rho h_{\beta\sigma}^\sigma, \]

\[ R_{[HV]}_{\alpha\beta} = R_{HV}_{\alpha\beta}^H - D_{\nu\beta} \omega_{\alpha\rho}^\sigma - h^\rho_{\alpha\sigma} \omega_{\beta\rho}^\sigma + \theta_{\sigma} \omega_{\alpha\beta}^\sigma, \]

\[ R_{[VH]}_{\alpha\beta} = R_{HV}_{\alpha\beta}^V - D_{\nu\beta} h_{\alpha\beta}^\sigma + D_{\nu\beta} \theta_{\alpha} - \omega_{\alpha\sigma}^\rho h_{\beta\rho}^\sigma. \]
4. Torsion Bianchi identities

The cross-projected curvatures which appear in the last two equations are unfamiliar and the expression for $R_{VV}$ is not manifestly symmetric in its indexes. The projected Torsion Bianchi identities solve these problems. Specialize the identities given by Eq. 29 to the case of normal projections onto group orbits and contract them where possible to obtain the identities

$$R^V_{HV\nu\mu} = -D_{V\rho}\omega^\rho_{\nu\mu} + \omega^{\rho\nu}_{\mu\rho},$$

$$R^{H}_{VH\mu\nu} = D_{H\mu}\theta_{\gamma} - D_{H\rho}\h_{\gamma\rho\mu} + h_{\sigma\mu\rho}\omega^{\rho\sigma}_{\mu\gamma} - \theta_{\sigma}\omega^{\sigma\gamma}_{\rho\mu},$$

$$D_{V[\gamma\theta]} + D_{H\rho}\omega^\rho_{\nu\gamma} = 0.$$  

With the help of these identities, the cross-projections of the full Ricci tensor become fully explicit,

$$R_{VV\alpha\beta} = R_{V\alpha\beta} + D_{V\theta}(\beta_{\alpha}) - h_{\alpha\rho}\h_{\beta\rho}.$$  

(68)

while the normal projection becomes manifestly symmetric

$$R_{[V\varphi\mu\nu]} + \omega^{\sigma\mu}_{[\mu\nu]} = 0.$$  

(69)

The remaining nontrivial projected torsion Bianchi identity cannot be contracted but takes the form of an equation of motion for the $V$-twist tensor.

$$D_{V[\gamma\omega^\rho_{\mu\nu}]} + \omega^{\sigma\mu}_{[\mu\nu]} = 0.$$  

(70)

In Kaluza-Klein theories, the $V$-twist becomes the electromagnetic field tensor and this identity becomes the source-free Maxwell’s equation.

5. Normal orbit projections of Einstein’s equations

Equations 67, 68, and 69 may be used to obtain Einstein’s equations. It is convenient to decompose the resulting equations into trace and trace-free parts and to re-organize the trace parts so that Einstein’s equations become

$$(2 - s) R_{HH}^H - s R_{VV}^V + 2(1 - s) D \cdot \theta - (1 - s) \theta^2 + (2 - s) \omega^2 + s \sigma^2 = 16\pi\kappa T_{HH}$$

(71)

$$R_{HH}^H + D \cdot \theta - \theta^2 + \omega^2 = \frac{8\pi\kappa}{2 - d}(s T_{VV} + (2 - d + s) T_{HH})$$

(72)

$$TF R_{HH\alpha\beta}^H + D_{V\theta}\sigma^{\alpha\beta} - \theta_{\sigma}\sigma^{\alpha\beta} + TF\omega_{\alpha\rho}\omega^{\rho\beta} = 8\pi\kappa TF T_{HH\alpha\beta}$$

(73)

$$TF R_{VV\alpha\beta}^V + TF D_{V\theta}(\beta_{\alpha}) - TF\sigma^{\alpha\rho}_{\rho\beta} - \frac{1}{8} TF\theta_{\alpha}\theta_{\beta} = 8\pi\kappa TF T_{VV\alpha\beta}$$

(74)

$$-D_{V\rho}\omega^{\rho}_{\sigma\alpha} + \left(1 - \frac{1}{s}\right)D_{H\beta}\theta_{\alpha} - D_{H\sigma}\sigma^{\alpha}_{\beta} + 2\omega^{\sigma\beta}_{\rho\alpha}\theta_{\rho} = 8\pi\kappa T_{VH\alpha\beta}$$

(75)

with the abbreviations

$$\theta^2 = \theta_{\sigma}\theta^{\sigma}, \quad \omega^2 = -\omega_{\sigma\rho}\omega^{\sigma\rho}, \quad \sigma^2 = \sigma_{\sigma\rho}\sigma^{\sigma\rho}, \quad D \cdot \theta = D_{V\theta}\theta^{\sigma}$$

and the notation TF for the trace-free part of a second-rank tensor.
Equation 72 is the result of combining the trace equations and has two remarkable properties: (1) It does not involve the shear. (2) All of the dependence on dimensionality is associated with the stress-energy components. Because this one Einstein equation is the same for any spacetime with an isometry, it is worth examining more closely. In terms of the orbit-area \( a \), the remarkable Eq. 72 becomes even more remarkable:

\[
- \Delta V a + H_{HH} a = \frac{8\pi\kappa}{2-d} (sT V V + (2 - d + s) T_{HH}) a
\]

(76)

where

\[
\Delta V f = g^{\nu\alpha\beta} D_{\nu} a D_{\beta} f
\]

defines the harmonic operator \( \Delta V \) on the quotient space. In this form, I will refer to this combination of Einstein’s equations as the orbit-area equation. This equation determines the behavior of the orbit-area function which, in turn, determines the global topology of spacetime.

D. Isotropic orbits

1. Normality constraints

From the projection \( g_{HV} \) and a group-invariant vector field \( v \) construct the group-invariant form \( g_{HV \sigma\rho} v^\rho \) and then construct a forbidden orbit-tangent vector field by using \( g_{HH} \). The resulting constraint

\[
g_{HH \alpha\sigma} g_{HV \sigma\beta} = 0
\]

implies that the projection is normal whenever the group-orbits are non-null (so that \( g_{HH} \) can have an inverse). Similarly, construct the group-invariant form \( g_{V\nu\sigma\rho} v^\rho \) and then a forbidden orbit-tangent vector field \( g^{HV \alpha\sigma} g_{V\nu\sigma\rho} v^\rho \) to find the constraint

\[
g^{HV \alpha\sigma} g_{V\nu\sigma\rho} = 0
\]

which implies that the projection is normal whenever the quotient geometry is non-null.

2. Projection-curvature constraints

The simplest invariant vector field to construct is \( h^\nu_{\nu\rho\sigma} v^\rho u^\sigma \) where \( u, v \) are invariant vector fields. Because the resulting vector field lies in \( HTP \) and is tangent to the group-orbit, it can only be zero. The invariant group evolution assumption means that the vector fields \( u, v \) can be chosen arbitrarily from a set which spans \( VT_P \) at any one point \( P \) so the projection curvature obeys the constraint \( h^\nu_{\nu\rho\sigma} = 0 \). One consequence of this constraint is \( \omega^\nu_{\nu\rho\sigma} = 0 \) which implies that \( V \) is not only a normal projection but is surface-forming. Thus, isotropic orbits are always surface-orthogonal.

The projection curvature \( h^T_H \) requires a less direct approach. From \( h^T_H \) one can construct a symmetric tensor field

\[
t(v)_{\alpha\beta} = v^\tau h^T_H \tau^\rho (\alpha g_{HH \beta})_{\rho}
\]

and then seek solutions to the eigen-value equation \( t(v)_{\alpha\beta} \tau^\beta = \lambda g_{HH \alpha\beta} u^\rho \). If there are distinct eigenvalues, then there will be distinct eigenvectors which constitute group-invariant vector-fields which are tangent to the group orbits. To avoid this forbidden possibility, require \( t(v)_{\alpha\beta} = \alpha(v) g_{HH \alpha\beta} \) or

\[
h^T_H \tau^\rho (\alpha g_{HH \beta})_{\rho} = \alpha \tau g_{HH \alpha\beta}
\]

where \( \alpha \in V^* \hat{T}_P \). Similarly, the projection curvature \( h_H \) can be used to construct a second-rank group-invariant tensor field whose eigenvectors are the forbidden orbit-tangent invariant vector fields. The resulting constraint is

\[
g_{V\nu \tau \rho} h^T_H \rho(\alpha \beta) = \beta \tau g_{HH \alpha\beta}
\]

where \( \beta \in V^* \hat{T}_P \).
3. Summary of consequences of isotropic orbits

Let $V_d$ be a geometry which admits an isometry group $G_n$ and take $H$ to be a group-invariant projection tensor field which projects vectors tangent to $s$-dimensional orbits in the family $[V_d/G_n](s)$. From the previous section, this projection is necessarily normal and surface-orthogonal which means that cross-projected metric terms such as $g_{\nu H \alpha \beta}$ are all zero, the projected metrics $g_{HH}, g^{HH}, g_{VV}, g^{VV}$ may be used to raise and lower indexes (of the appropriate projection type) on tensors, and the projection curvatures $h$ and $h^T$ are the same except for index placement (which is now easy to change). The possible values of the projection curvatures are severely constrained and must have the form

$$h^T_{\nu \alpha \rho \sigma} = h_{\nu \alpha \rho \sigma} = 0$$

$$h^T_{H \rho \alpha \beta} = h_{H \rho \alpha \beta} = h_{\rho \alpha \beta} = \alpha^\rho g_{HH \alpha \beta}$$

where $\alpha$ is a vector in $VT_P$. From the definition of the divergence, $\theta_\rho$ it is easy to see that $\theta_\rho = s\alpha_\rho$ or

$$h^\rho_{\alpha \beta} = \frac{1}{s} \theta^\rho g_{HH \alpha \beta}$$

and

$$\sigma^\rho_{\alpha \beta} = 0.$$ (78)

4. Effective orbit-size parameter

Each orbit can be characterized by an effective size parameter $r$ which is defined by requiring the orbit area to be given by $a = Br^s$ for some constant $B$. For example, if the group is SO(2) then $s = 1$, the orbits are circles, $a$ can be chosen to be the total circumference of an orbit, and the choice $B = 2\pi$ makes $r$ the radius which an orbit would have in a flat embedding space. Similarly, for SO(3) the orbits are two-spheres and the choice $B = 4\pi^2$ makes $r$ again the flat embedding radius of an orbit — sometimes called the luminosity radius. When $s = 3$, the value of $r$ becomes a measure of the size of the universe in a cosmological model. Because $\theta_\alpha$ depends only on the logarithm of $a$, the choice of the constant $B$ is of no immediate consequence and the divergence takes the form

$$\theta_\alpha = -a^{-1}D_\alpha a = -sr^{-1}D_\alpha r.$$ (79)

The scalar curvature $R^H_{HH}$ of an orbit may also be expressed in terms of the effective size parameter $r$. When the Riemann curvature tensor of an orbit is expressed in a basis which is constructed from the group coordinates, its components are determined solely by the underlying group and are the same for all orbits. Since the Ricci tensor of an orbit is simply a contraction of the Riemann tensor without any use of the metric tensor, it too has group-basis components which are the same for all orbits. The scalar curvature $R^H_{HH} = g^H_{HH \alpha \beta} R^H_{HH \alpha \beta}$ therefore depends on the orbit only through the inverse metric components $g^H_{HH \alpha \beta} = \omega^{H \alpha} \cdot \omega^{H \beta}$ where the forms $\omega^{H \alpha}$ are defined in terms of the underlying group parameters (typically angles). Thus, one always has

$$R^H_{HH} = \frac{K_G}{r^2}.$$ (80)

The constant $K_G$ is determined by the group and by the choices which define the effective size function $r$. A simple way to calculate $K_G$ is to consider the orbits which are generated when the group acts on a flat spacetime manifold. For the groups SO(2), SO(3), and SO(4) the constants are found to be $K_G = 0, 2, 6$.

VI. APPLICATIONS

A. Einstein’s equations for isotropic spacetimes

With the simplifications which are afforded by Eqs. [5], [6], and [8], two of the trace-decomposed, normal orbit projections of Einstein’s equations (Eqs. [73] and [75]) become constraints on the stress-energy tensor while the remaining equations (Eqs. [71], [72], and [74]), become
\[(1 - s) \left[ (2 - s) r^{-2} (D_{V \sigma} r) (D_{V} r) - 2 r^{-1} D_{V \sigma} D_{V} r \right] + (2 - s) r^{-2} K_G = s R^V_{VV} + 16 \pi \kappa T_{HH} \]
\[(1 - s) r^{-2} (D_{V \sigma} r) (D_{V} r) - r^{-1} D_{V \sigma} D_{V} r + \frac{K_G}{4} = 8 \pi \kappa \left[ \frac{d - 2}{d - 2} T_{HH} - \frac{s}{d - 2} T_{VV} \right] \]
\[-s T_F \left[ r^{-1} D_{V \beta} D_{V \alpha} r - 2 r^{-2} (D_{V \beta} r) (D_{V \alpha} r) \right] = 8 \pi \kappa T F T_{V V \alpha \beta} - T F R^V_{V V \alpha \beta} \]

These three equations form a complete system for determining the orbit size function \( r \) and the quotient geometry. Notice that Eq. \( 83 \) does not depend on the metric tensor in any way except through the connection. It’s sole function is to determine the quotient space connection \( D_V \). Equations \( 82 \) and \( 81 \) determine the metric and matter variables including the orbit size function. However, as will be seen in these examples, the exact way that this system of equations functions is strongly affected by the "accidents" of dimension.

1. Codimension = 1: Friedman-Robertson-Walker cosmologies

For \( d - s = 1 \) the trace-free equation (Eq. \( 83 \)) is identically satisfied and the quotient-space connection is described by identifying a proper time function \( \tau \) and taking the single orthonormal-frame components of the restricted derivative to be just \( D_{V 0} f = df/d\tau \). The only remaining equations to be solved are Eq. \( 82 \) and Eq. \( 81 \) which become the familiar cosmological equations for the radius \( r \) of the universe.

As a specific and familiar example, consider the parameter values \( d = 4, s = 3 \) which correspond to isotropic, homogeneous cosmological models and take the stress-energy tensor to be that of a perfect fluid. One "accident" of this choice of dimensions is the simplicity of the perfect fluid stress-energy tensor:

\[ T^\mu_\nu = p H^\mu_\nu + p V^\mu_\nu = p H^\mu_\nu - \rho V^\mu_\nu. \]  

\[(84)\]

To obtain a definite example which will be needed later, take the equation of state to be that of incoherent radiation, \( p = \rho/3 \), so that the projected traces which enter into Eqs. \( 82 \) and \( 81 \) are

\[ T_{HH} = \rho, \quad T_{VV} = -\rho. \]

Choose a timelike unit vector \( e_0 = \partial/\partial \tau \) and denote proper-time derivatives by dots so that Equations \( 82 \) and Eq. \( 81 \) become

\[ 3 \frac{\ddot{r}}{r} + 6 \frac{\dot{r}^2}{r^2} + \frac{K_G}{r^2} = 8 \pi \kappa \rho \]

\[ -6 \frac{\ddot{r}}{r} - 3 \frac{\dot{r}^2}{r^2} - \frac{K_G}{2r^2} = 8 \pi \kappa \rho. \]

These equations can be used in one of two ways. Subtracting them gives the familiar second order equation for the radius of a radiation-dominated universe

\[ \frac{\ddot{r}}{r} + \frac{\dot{r}^2}{r^2} + \frac{K_G}{6r^2} = 0 \]  

\[(85)\]

while eliminating the second derivative between them gives the initial value constraint

\[ \frac{\dot{r}^2}{r^2} + \frac{K_G}{6r^2} = \frac{8}{3} \pi \kappa \rho. \]  

\[(86)\]

The constant \( K_G/6 \) is the usual topology parameter (often denoted \( k \)) which has values -1, 0, +1 for universes which are open, spatially flat, and closed respectively.
For $d - s = 2$ the Ricci curvature of the quotient space is proportional to the metric tensor. The trace-free part of the Ricci curvature then vanishes. Set the stress-energy tensor equal to zero in order to consider vacuum spacetimes and focus on spacelike group orbits. For $s = 2$, this case includes the exterior metrics of spherically symmetric systems. The trace-free part of Einstein’s equations then collapses to just

$$TF D_\beta D_\alpha r = 0 \quad (87)$$

This equation is supposed to determine the quotient geometry (or equivalently the geometry of the reference surface which is perpendicular to the group orbits) and it will be seen that it does that job admirably and simply.

Introduce orthonormal basis vectors $(e_0, e_1)$ on $V T_P$, aligned with the function $r$. Remember that these vector fields are directional derivatives and use the notation $(e_1 f) = f'$ where $f$ is any function on the reference surface.

Focus on a spacetime region where this function has a spacelike gradient so that, at each point the basis vectors can be chosen so that $(e_0 r) = 0$ and $(e_1 r) = r' > 0$. Within some local region, the function $r$ provides one coordinate on the two-dimensional reference surface and there will be another coordinate $t$ whose level-curves are the integral curves of the perpendicular vector field $e_1$. In terms of these coordinates, the orthonormal basis vectors are

$$e_0 = N \frac{\partial}{\partial t}, \quad e_1 = r' \frac{\partial}{\partial r}.$$  

In general the functions $r'$ and $N$ can depend on both coordinates.

There are only two non-zero independent orthonormal-frame connection coefficients:

$$\gamma_0 = \Gamma_V^V V^0_1 A = \Gamma_V^V V^1_0 A, \quad A = 0, 1.$$  

From the off-diagonal components of Eq. (87) come the two conditions

$$\gamma_1 r' = 0, \quad N \frac{\partial r'}{\partial t} = 0$$

which eliminate one connection component and ensure that $r'$ is independent of $t$ while the anti-trace combination of diagonal components of Eq. (87) gives $-\gamma_0 r'' + r'^2 = 0$ which can be solved for the other connection component.

Because the group-orbit projection tensor field is normal, the projected torsion vector $S^V V V^A_{01} e_A$ must vanish or

$$[e_0, e_1] - \gamma_0 e_0 + \gamma_1 e_1 = 0.$$  

This condition reduces to just one equation

$$-N' - \frac{r''}{r'} N = 0$$

which integrates to

$$N(r, t) = \frac{N_0(t)}{r'}.$$  

The remaining function $N_0$ can be absorbed by redefining $t$ so that $N(r, t) = 1/r'$.

To determine the rest of the geometry, specialize the orbit-area equation (Eq. 82) to this case and obtain

$$-4 \frac{r''}{r} - 2 \frac{r'^2}{r^2} + \frac{2}{r^2} = 0$$

which has the first integral $2 r'^2 - 2 r = -4 m$ where the integration constant $-4 m$ has been chosen with a certain amount of forethought. Solve this expression for $r'$ and find the expressions for the orthonormal basis vectors

$$e_0 = \frac{1}{\sqrt{1 - 2m/r}} \frac{\partial}{\partial t}, \quad e_1 = \sqrt{1 - 2m/r} \frac{\partial}{\partial r}.$$  

From the assumption that a vacuum spacetime has a group of motions with two dimensional isotropic orbits and the added restriction to a region where the orbit area function has a spacelike gradient, we have constructed the Schwarzschild solution. Along the way an additional isometry group has spontaneously appeared because this solution is static. This result is usually called Birkhoff’s Theorem. 

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2. Codimension = 2: Spherical symmetry and Birkhoff’s theorem
3. Codimension = 2: Five dimensional cosmology

Consider a five-dimensional manifold \( V_5 \) with a metric of signature \((-++++)\) and a group of motions \( G \) whose orbits are three-dimensional spacelike surfaces. In this case, the parameters introduced earlier are \( s = 3, \ d = 5 \). The quotient space \([V_5/G]\) (3) is two dimensional so the analysis which worked so well for spherically symmetric four-manifolds should work here as well. The group orbits are now three dimensional homogeneous and isotropic spaces. A family of these is a cosmology. Thus, we are looking for those solutions of Einstein’s vacuum equations in five dimensions which can describe cosmological models. This time, assume that the orbit size function \( r \) is timelike.

In \( V_{TP} \) (or equivalently, in the quotient space \( V_5/G \)) introduce the orthonormal basis \( \{ e_A \} \) with
\[
\begin{align*}
  e_0 \cdot e_0 &= -1, & e_1 \cdot e_1 &= 1 \\
  e_0 r &= e_0 (r) = \dot{r}, & e_1 r &= e_1 (r) = 0
\end{align*}
\]

and
\[
\begin{align*}
  e_0 &= \frac{\dot{r}}{r} \\
  e_1 &= \frac{\partial}{\partial \chi}
\end{align*}
\]

Notice that the unit vector \( e_0 \) can also be written in terms of the proper time \( \tau \) as \( e_0 = \partial/\partial \tau \). Thus, dots denote proper-time derivatives.

The geometry of the quotient space is determined, as before, by Eq. 87 which now consists of the conditions
\[
N \frac{\partial \dot{r}}{\partial \chi} = 0, \quad \gamma_0 = 0, \quad \ddot{r} - \gamma_1 \dot{r} = 0
\]

The condition that the projected torsion vanishes leads, as before, to the expression \( N = 1/\dot{r} \). All of these quantities are independent of the fifth dimensional coordinate \( \chi \). We have a surprise symmetry just as happened in the 4D case. It remains only to use the orbit-size equation (Eq. 82) to determine the behavior of \( r \). In this case, the equation takes a familiar form and is just Eq. 85 — the equation which governs a radiation-dominated universe. Working backward to the effective 4D stress-energy tensor can only yield the stress-energy of incoherent radiation — a result which has been noticed before in spherically symmetric solutions to the 5D Einstein equations.

In this curious parallel to Birkhoff’s theorem, the requirement that a five dimensional vacuum spacetime obey the Cosmological Principal by having three-dimensional space sections which are homogeneous and isotropic gives rise to a spontaneous additional symmetry which suppresses a dimension. The resulting solution can be regarded as a radiation dominated four-dimensional cosmological model with the radiation pressure supplied by a scalar field — a manifestation of the suppressed fifth dimension. Distances in that suppressed direction scale as the function \( N^{-1} \) which is proportional to the expansion rate \( \dot{r} \). Thus, as the universe evolves and the expansion rate slows, the fifth dimensional size of the universe collapses. This picture fits the usual scenario of a spontaneous dimensional reduction. The curious (and, as far as I can tell, new) aspect of this picture is that the dimensional reduction is not an independent contrivance of the initial conditions but a necessary accompaniment of those initial conditions which produce homogeneity and isotropy.

VII. DISCUSSION

The projection tensor framework which has been developed here offers two main advantages: (1) It is metric-independent and works for any connection. (2) It works for any combination of dimensions. The examples which have been developed in this paper have mostly exploited the second advantage, analyzing systems with arbitrary dimensionalities and then considering the accidents which happen when certain combinations of dimensions are chosen. The first advantage, metric-independence, has been exploited only in part. It is clearly useful to have metric-independent geometrical structures such as the divergence and twist of a projection-tensor field. The examples show how these structures can play such central roles in the Einstein equations that the spacetime metric tensor fades into the background. However, all of the examples rely upon normal projection tensor fields.

The normality condition, which depends directly upon the metric, is extremely powerful. It cuts the number of independent projection curvatures in half and makes the restricted covariant derivatives metric compatible on their respective subspaces and even between subspaces. The tilted projection-tensor formalism which has been developed here makes it possible to do without the normality condition and thus makes it possible to search for useful alternative conditions. A situation where an alternative condition would clearly be useful is projections onto null surfaces.
Another situation which calls for alternative conditions is the analysis of "non-projective" Kaluza-Klein unified field theories which are most naturally stated in terms of tilted projection tensor fields.

The applications have been restricted to isotropic spacetimes in order to provide simple and familiar examples. It is quite easy to extend the applications to include groups with anisotropic orbits and, in this way, revisit Kasner universes, Mixmaster universes, cylindrical waves, and the rotating star problem. The basic pattern for solving Einstein’s equations remains the same in all of these cases: (1) Solve the orbit-area equation (Eq. 74) for the orbit-area function, thus fixing the over-all topology of the spacetime and choosing a coordinate. (2) Solve the orbit-tangent, trace-free equation (Eq. 73) (a wave equation on the quotient space) for the shear which carries the gravitational wave degrees of freedom. (3) Solve the orbit-orthogonal, trace-free equation (Eq. 74) for the restricted connection on the space of orbits and in this way determine the geometry of the quotient space. In the solvable cases, these steps decouple cleanly from one another much as they do in the simple isotropic examples.

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