ON INDEX DIVISORS AND MONOGENITY OF CERTAIN NUMBER FIELDS DEFINED BY TRINOMIALS $x^6 + ax + b$

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Abstract. For a number field $K$ defined by a trinomial $F(x) = x^6 + ax + b \in \mathbb{Z}[x]$, Jakhar and Kumar gave some necessary conditions on $a$ and $b$, which guarantee the non-monogenity of $K$ [25]. In this paper, for every prime integer $p$, we characterize when $p$ is a common index divisor of $K$. In particular, if any one of these conditions holds, then $K$ is not monogenic. In such a way our proposed results extend those of Jakhar and Kumar.

1. Introduction

Let $K = \mathbb{Q}(\alpha)$ be a number field generated by a complex root $\alpha$ of a monic irreducible polynomial $F(x)$ of degree $n$ over $\mathbb{Q}$ and $\mathbb{Z}_K$ its ring of integers. The field $K$ is called monogenic if $\mathbb{Z}_K$ has a power integral basis $(1, \theta, \ldots, \theta^{n-1})$ for some $\theta \in \mathbb{Z}_K$ and $K$ is not monogenic otherwise. Monogenity of number fields is a classical problem of algebraic number theory, going back to Dedekind, Hasse and Hensel [17, 21, 22]. There are extensive computational results in the literature of testing the monogenity of number fields and constructing power integral bases, and it was treated by different approaches, mainly by Gaál, Győry, Pohst, and Pethö with their research teams. These methods are based on the arithmetic of the index form equations. They studied monogenity of several algebraic number fields (see [2, 17, 14, 15, 16, 34]). Recently many authors have been interested on monogenity of number fields defined by trinomials. In [27, 28], Khanduja et al. studied the integral closedness of some number fields defined by trinomials. Their results are refined by Ibarra et al. with computation of the densities (see [26]). Also in [29, 30, 31, 32], Jones et al. studied monogenity of some irreducible trinomials. According to Jones’s definition, a monic irreducible polynomial $F(x) \in \mathbb{Z}[x]$ is monogenic if $\mathbb{Z}[\alpha]$ is integrally closed, where $\alpha$ is a complex root of $F(x)$. Remark that the results given in [27, 28, 29, 30, 31, 32], can only decide on the integral closedness of $\mathbb{Z}[\alpha]$, but cannot test whether the field is monogenic or not. The monogenity of polynomials can be used partially in the study of monogenity of number fields, but the converse is not true; because it is possible that a number field generated by a complex root $\alpha$ of a non monogenic polynomial can be monogenic. Therefore Jones’s and Khanduja’s results cover partially the study of monogenity of number fields defined by trinomials. In [1], Ben Yakkou and El Fadil gave sufficient conditions on coefficients of a trinomial
which guarantee the non-monogenity of the number field defined by such a trinomial. Also in [13], Gaál studied the multi-monogenity of number fields defined by some sextic irreducible trinomials. In [10], for \( p = 2, 3 \), we characterized when \( p \) is a common index divisor of any number field generated by a complex root of an irreducible trinomial \( x^5 + ax^2 + b \). In [25], for any number \( K \) defined by a trinomial \( F(x) = x^6 + ax + b \in \mathbb{Z}[x] \), Jakhar and Kumar gave some necessary conditions on \( a \) and \( b \), which guarantee the non-monogenity of \( K \). Based on the form of the factorization of \( p\mathbb{Z}_K \), Engstrom calculated \( v_p(i(K)) \) for any number field \( K \) of degree less than 8 and any prime integer less than 7 [12]. He showed that for sextic number fields, he proved that \( i(K) = 2^k \times 3^l \times 5^h \) for some integers \( k, l, h \) such that \( 0 \leq k \leq 8, 0 \leq l \leq 3 \), and \( h = 0, 1 \). In this paper, for every prime integer \( p \) and any number field \( K \) defined by an irreducible trinomial \( F(x) = x^6 + ax + b \in \mathbb{Z}[x] \), we characterize when does \( p \) is a common index divisor of \( K \). In particular, under any of the mentioned conditions \( K \) is not monogenic. In such a way our proposed results extend those of Jakhar and Kumar given in [25].

2. Main Results

Let \( K \) be a number field generated by a complex root of a trinomial \( F(x) = x^6 + ax + b \in \mathbb{Z}[x] \). Without loss of generality, we can assume that for every rational prime integer \( p, v_p(a) < 5 \) or \( v_p(b) < 6 \). Along this paper, for every integer \( a \in \mathbb{Z} \) and a prime integer \( p \), let \( a_p = \frac{a}{p^{v_p(a)}} \). For every \((a, b, c, d) \in \mathbb{Z}^4 \) and \( n \in \mathbb{N} \), the notation \((a, b) = (c, d) \pmod{n}\) means that \( a \equiv c \pmod{n} \) and \( b \equiv d \pmod{n} \). Also \((a, b) \in I \pmod{n}\) means that \((a, b) = (c, d) \pmod{n}\) for some \((b, c) \in I\).

Recall that [26] Theorem 3.2 characterizes, when the ring \( \mathbb{Z}[\alpha] \) is integrally closed, which means when does the polynomial \( x^6 + ax + b \) is monogenic. Proposition 2.1 gives an infinite family of number fields generated by a root \( \alpha \) of non-monogenic trinomials which are monogenic. In such a way, the results given in [28, 29, 30, 31, 32, 26], cannot decide on monogenity of \( K \).

**Proposition 2.1.** Let \( p \) be a prime integer and \( F(x) = x^6 + p^v x + p^5 \in \mathbb{Z}[x] \) a polynomial with \( v \geq 5 \) an integer. Then \( F(x) \) is irreducible over \( \mathbb{Q} \). Let \( K \) be the number field generated by a root \( \alpha \) of \( F(x) \). If \( 5^5 p^{v_5 - 25} - 6^6 \) is square free, then \( \mathbb{Z}[\alpha] \) is not integrally closed, \( K \) is monogenic, and \( \theta = \frac{\alpha^5}{p} \) generates a power integral basis of \( K \).

In the remainder of this section, for every prime \( p \), we give sufficient and necessary conditions on \( a \) and \( b \) so that \( p \) is a common index divisor of \( K \). If any one of these conditions holds, then \( K \) is not monogenic. In particular, our proposed results extend those of Jakhar and Kumar given in [25].

**Theorem 2.2.** The prime integer 2 is a common index divisor of \( K \) if and only if one of the following conditions holds:

1. \((\overline{a}, \overline{b}) \in \{(0, 7), (0, 3)\} \pmod{8}\).
(2) \((\overline{a}, \overline{b}) \in \{(6,9), (14,1)\}\) (mod 16).
(3) \((\overline{a}, \overline{b}) \in \{(2,17), (18,1)\}\) (mod 32).
(4) \((\overline{a}, \overline{b}) \in \{(10,9), (10,5)(14,13), (2,13), (6,5)\}\) (mod 16) and \(v_3(b + as + s^6) = 2v_3(a + 6s^5)\) for some integer \(s\) for which \(F(x)\) is \(x - s\)-regular with respect to \(p = 2\).

In particular, if any one of these conditions holds, then \(K\) is not monogenic.

Section 5 gives a method to calculate an integer \(s\) such that \(F(x)\) is \(x - s\)-regular with respect to \(p = 2\).

**Theorem 2.3.** The prime integer 3 is a common index divisor of \(K\) if and only if one of the following conditions holds:

1. \(a \equiv 0 \pmod{9}\) and \(b \equiv -1 \pmod{9}\).
2. \(a \equiv 21 \pmod{27}\), \(b \equiv -(1 + a) \pmod{81}\), and \(2v_3(a + 6) < v_3(a + b + 1) + 1\).
3. \(a \equiv 21 \pmod{27}\) and \(b \equiv -(1 + a) \pmod{81}\), \(2v_3(a + 6) > v_3(a + b + 1) + 1\), \(v_3(a + b + 1)\) is odd, and \((a + b + 1)_3 = 1\).
4. \(a \equiv 21 \pmod{27}\), \(b \equiv -(1 + a) \pmod{81}\), \(v_3(a + b + 1) = 2v_3(a + 6) - 1\), \((a + 6)_3 \equiv (b + a + 1)_3 \equiv -1 \pmod{3}\), and \(v_3(b + as + s^6) > 2v_3(a + 6s^5) - 1\) or \(v_3(b + as + s^6) < 2v_3(a + 6s^5) - 1\) and \((b + as + s^6)_3 \equiv 1 \pmod{3}\) for some integer \(s\) such that \(F(x)\) is \(x - s\)-regular with respect to \(p = 3\).
5. \(a \equiv 6 \pmod{27}\), \(b \equiv -1 + a \pmod{81}\), and \(2v_3(a - 6) < 2v_3(-a + b + 1) + 1\).
6. \(a \equiv 6 \pmod{27}\), \(b \equiv -1 + a \pmod{81}\), \(2v_3(a - 6) > v_3(-a + b + 1) + 1\), \(v_3(-a + b + 1)\) is odd, and \((b + 1 - a)_3 \equiv 1 \pmod{3}\).
7. \(a \equiv 6 \pmod{27}\), \(b \equiv -1 + a \pmod{81}\), \(2v_3(a - 6) = v_3(b + 1 - a) + 1\), \((a - 6)_3 \equiv (1 + b - a)_3 \equiv -1 \pmod{3}\), and \(v_3(b + as + s^6) > 2v_3(a + 6s^5) - 1\) or \(v_3(b + as + s^6) < 2v_3(a + 6s^5) - 1\) and \((b + as + s^6)_3 \equiv 1 \pmod{3}\) for some integer \(s\) such that \(F(x)\) is \(x - s\)-regular with respect to \(p = 3\).

In particular, if one of these conditions holds, then \(K\) is not monogenic.

Section 5 gives how to calculate an integer \(s\) for which \(F(x)\) is \(x - s\)-regular with respect to \(p = 3\).

**Remark 1.**

1. For \(a = 0\), if \(b \equiv 1 \pmod{4}\), then by Theorem 2.2, we conclude that the pure sextic number field generated by \(\sqrt[n]{b}\) is not monogenic.
2. For \(a = 30\) and \(b = 5\), we have \(F(x)\) is irreducible over \(\mathbb{Q}\) as it is 5-Eisenstein. Let \(K\) be the number field defined by \(F(x)\). Since \((a, b) \equiv (6, 5) \pmod{8}\), we conclude by Theorem 2.2(1) that 2 is a common index divisor of \(K\), and so \(K\) is not monogenic. As \(\Delta(F) = 2^9 \cdot 3^8 \cdot 5^3 \cdot 7 \cdot 31\), \(v_2(\Delta(F))\) is odd. Thus neither Theorem 1.3 nor Theorem 1.1 of [25] does cover this case.

Also let \((a, b) = (21, 2165)\). Since \(x^6 + ax + b = x^6 + x + 1\) is irreducible over \(\mathbb{F}_2\), we conclude that \(F(x)\) is irreducible over \(\mathbb{Q}\). Let \(K\) be the number field defined by \(F(x)\). Remark that [25] Theorem 1.1 does not cover this case. Moreover since \(a \equiv 21 \pmod{81}\) and \(1 + b + a = 3^7\), we have \(v_3(a + 6) = 3\), \(v_3(a + b + 1) = 7\), and \((a + b + 1)_3 \equiv 1 \pmod{3}\). Thus by Theorem 2.3(2), we conclude that 3 is a common index divisor \(K\), and so \(K\) is not monogenic.

In particular, Jakhar’s and Kumar’s results given in [25] does not characterize...
the monogenity of number fields defined by $x^6 + ax + b$ and our results extend those of Jakhar and Kumar.

(3) The field $K$ can be non monogenic even if the index $i(K) = 1$. It suffices to consider the number field generated by a root of the polynomial $F(x) = x^6 - 17$, which is irreducible as it is 17-Eisenstein. Since $17 \equiv -1 \pmod{9}$, by [16, sec: 5.4, case: 6.5, p 141], $K$ is not monogenic. But as $(a, b) = (0, -17)$, then neither Theorem 2.2 nor Theorem 2.3 covers this case.

(4) The unique method which allows to test whether $K$ is monogenic is to calculate the solutions of the index form equation of $K$, which is very complicated in the case of sextic number fields (see for instance [16, 17]).

**Proposition 2.4.** For every prime integer $p \geq 5$, for every values of $a$ and $b$, $p$ is not a common index divisor of $K$.

3. A short introduction to Newton polygons applied in prime ideal factorization

Newton polygons techniques play a key role in number theory, namely in prime ideal factorization and calculation of index. This is a standard method which is rather technical but very efficient to apply. We have introduced the corresponding concepts in several former papers. Here we only give a brief introduction which makes our proofs understandable. For a detailed description, we refer to [18, 35, 9].

Let $F(x) \in \mathbb{Z}[x]$ be a monic irreducible polynomial with a root $\alpha$ and set $K = \mathbb{Q}(\alpha)$. For a prime integer $p$, a well-known theorem of Dedekind [20, Chapter I, Proposition 8.3] says that if $p$ does not divide the index $(\mathbb{Z}_K : \mathbb{Z}[\alpha])$, then the prime ideal factorization $p\mathbb{Z}_K$ can be derived directly from the factorization of $F(x)$ modulo $p$. Also, we shall need Dedekind’s criterion to test if $p$ does not divide the index $(\mathbb{Z}_K : \mathbb{Z}[\alpha])$ [4, Theorem 6.1.4]. When Dedekind’s criterion fails, that is, $p$ divides the index $(\mathbb{Z}_K : \mathbb{Z}[\alpha])$ for every primitive element $\alpha \in K$, then for such primes and number fields, it is not possible to obtain the prime ideal factorization of $p\mathbb{Z}_K$ by Dedekind’s theorem. In 1928, Ore developed an alternative approach for obtaining the index $(\mathbb{Z}_K : \mathbb{Z}[\alpha])$, the absolute discriminant of $K$, and the prime ideal factorization of the rational primes in a number field $K$ by using Newton polygons (see for instance [11, 19, 9]). For any prime integer $p$ and for any monic polynomial $\phi \in \mathbb{Z}[x]$ whose reduction is irreducible in $\mathbb{F}_p[x]$, let $\mathbb{F}_\phi$ be the finite field $\mathbb{F}_p[x]/(\phi)$. For any monic polynomial $F(x) \in \mathbb{Z}[x]$, upon to the Euclidean division by successive powers of $\phi$, we expand $F(x)$ as $F(x) = a_0(x) + a_1(x)\phi(x) + \cdots + a_l(x)\phi(x)^l$ with $\deg(a_i(x)) < \deg(\phi)$ for every $i = 0, \ldots, l$. Such a development is unique and called the $\phi$-expansion of $F(x)$. To any coefficient $a_i(x)$ we attach $u_i = v_p(a_i(x)) \in \mathbb{Z} \cup \{\infty\}$. The $\phi$-Newton polygon of $F(x)$ with respect to $p$, is the lower boundary of the convex envelope of the set of points $\{(i, u_i), a_i(x) \neq 0\}$ in the Euclidean plane, which we denote by $N_\phi(F)$. The $\phi$-Newton polygon of $F$, is the process of joining the obtained edges $S_1, \ldots, S_t$ ordered by increasing slopes, which can be expressed as $N_\phi(F) = S_1 + \cdots + S_t$. The principal $\phi$-Newton polygon of $F$, denoted $N^\phi_\phi(F)$, is the part of the polygon $N_\phi(F)$,
which is determined by joining all sides of negative slopes. For every side $S$ of $N^+_\phi(F)$, the length of $S$, denoted $l(S)$, is the length of its projection to the $x$-axis and its height, denoted $h(S)$, is the length of its projection to the $y$-axis. Let $d = \text{GCD}(l(S), h(S))$ be the degree of $S$. For every side $S$ of $N_\phi(F)$, with initial point $(s, u_0)$, length $l$, and for every $i = 0, \ldots, l$, we attach the following residue coefficient $c_i \in \mathbb{F}_\phi$ as follows:

$$c_i = \begin{cases} 
0, & \text{if } (s + i, u_{s+i}) \text{ lies strictly above } S, \\
\left( \frac{a_{s+i}(x)}{p^n_{s+i}} \right) \mod (p, \phi(x)), & \text{if } (s + i, u_{s+i}) \text{ lies on } S.
\end{cases}$$

where $(p, \phi(x))$ is the maximal ideal of $\mathbb{Z}[x]$ generated by $p$ and $\phi$. Let $\lambda = -h/e$ be the slope of $S$, where $h$ and $e$ are two positive coprime integers. Then $d = l/e$ is the degree of $S$. Since the points with integer coordinates lying on $S$ are exactly

$$(s, u_s), (s + e, u_s - h), \ldots, (s + de, u_s - dh),$$

if $i$ is not a multiple of $e$, then $(s + i, u_{s+i})$ does not lie on $S$, and so $c_i = 0$. Let

$$R_\lambda(F)(y) = t_d y^d + t_{d-1} y^{d-1} + \cdots + t_1 y + t_0 \in \mathbb{F}_\phi[y],$$

called the residual polynomial of $F(x)$ associated to the side $S$, where for every $i = 0, \ldots, d$, $t_i = c_i e$. Let $N^+_\phi(F) = S_1 + \cdots + S_t$ be the principal $\phi$-Newton polygon of $F$ with respect to $p$.

We say that $F$ is a $\phi$-regular polynomial with respect to $p$, if $R_\lambda(F)(y)$ is square free in $\mathbb{F}_\phi[y]$ for every $i = 1, \ldots, t$. The polynomial $F$ is said to be $p$-regular if

$$\overline{F(x)} = \prod_{i=1}^r \overline{\phi_i(x)}$$

for some monic polynomials $\phi_1, \ldots, \phi_r$ of $\mathbb{Z}[x]$ such that $\overline{\phi_1}, \ldots, \overline{\phi_r}$ are irreducible coprime polynomials over $\mathbb{F}_p$ and $F$ is a $\phi_i$-regular polynomial with respect to $p$ for every $i = 1, \ldots, r$.

The theorem of Ore plays a key role for proving our main Theorems. Let $\phi \in \mathbb{Z}[x]$ be a monic polynomial, with $\overline{\phi(x)}$ is irreducible in $\mathbb{F}_p[x]$. As defined in \cite{11} Def. 1.3, the $\phi$-index of $F(x)$, denoted by $\text{ind}_\phi(F)$, is $\text{deg}(\phi)$ times the number of points with natural integer coordinates that lie below or on the polygon $N^+_\phi(F)$, strictly above the horizontal axis, and strictly beyond the vertical axis. Now, assume that $\overline{F(x)} = \prod_{i=1}^r \overline{\phi_i(x)}$ is the factorization of $\overline{F(x)}$ in $\mathbb{F}_p[x]$, where every $\phi_i \in \mathbb{Z}[x]$ is a monic polynomial, with $\overline{\phi_i(x)}$ is irreducible in $\mathbb{F}_p[x]$, $\overline{\phi_1(x)}, \ldots, \overline{\phi_r(x)}$ are pairwise coprime. For every $i = 1, \ldots, r$, let $N^+_{\phi_i}(F) = S_{i1} + \cdots + S_{it_i}$ be the principal $\phi_i$-Newton polygon of $F$ with respect to $p$. For every $j = 1, \ldots, t_i$, let $R_{\lambda_{ij}}(F) = \prod_{s=1}^{s_{i_j}} y_{ij}^{a_{ij}}(y)$ be the factorization of $R_{\lambda_{ij}}(F)(y)$ in $\mathbb{F}_\phi[y]$. Then we have the following theorem of Ore:
**Theorem 3.1. (Theorem of Ore)**

Under the above hypothesis, if \( F(x) \) is \( p \)-regular, then

\[
p\mathbb{Z}_K = \prod_{i=1}^{r} \prod_{j=1}^{t_i} \prod_{s=1}^{s_{ij}} p_{ij}^{e_{ij}}
\]

is the factorization of \( p\mathbb{Z}_K \) into powers of prime ideals of \( \mathbb{Z}_K \), where \( e_{ij} \) is the smallest positive integer satisfying \( e_{ij} > 6 \mathbb{Z} \) and \( f_{ij} = \deg(\phi_i) \times \deg(\psi_{ij}) \) is the residue degree of \( \psi_{ij} \) over \( p \) for every \( (i, j, s) \).

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4. Proofs of our main results

**Proof of Theorem 2.1**

Since for \( \phi = x, F(x) \equiv \phi^6 \pmod{p} \) and \( N_{\mathbb{Q}}(F) = S \) has a single side joining \((0, 5)\) and \((6, 0)\) with respect to \( p \), then the side \( S \) is of degree 1 (because \( \gcd(5, 6) = 1 \)), and so \( R_1(F)(y) \) is irreducible over \( \mathbb{F}_p \). Therefore \( F(x) \) is irreducible over \( \mathbb{Q}_p \). Let \( L = \mathbb{Q}_p(\alpha) \) and \( K = \mathbb{Q}(\alpha) \). Since \( \mathbb{Q}_p \) is a Henselian field, there is a unique valuation \( \omega \) of \( L \) extending \( v_p \). Let \( \theta = \frac{\alpha^5}{p^4} \) and let us show that \( \theta \in \mathbb{Z}_K \) and \( \mathbb{Z}_K = \mathbb{Z}[\theta] \). First, by [12, Corollary 3.1.4], in order to show that \( \theta \in \mathbb{Z}_K \), we need to show that \( \omega(\theta) \geq 0 \). Since \( N_{\mathbb{Q}}(F) = S \) has a single side of slope \(-5/6\), we conclude that \( \omega(\alpha) = 5/6 \), and so \( \omega(\theta) = 5 \times \frac{5}{6} - 4 = \frac{1}{6} \). Since 5 and 6 are coprime, we conclude that \( K = \mathbb{Q}(\theta) \).

Let \( g(x) \) be the minimal polynomial of \( \theta \) over \( \mathbb{Q} \). By the formula relating roots and coefficients of a monic polynomial, we conclude that \( g(x) = x^6 + \sum_{i=1}^{6} (-1)^i s_{ij} x^{6-i} \), where \( s_i = \sum_{k_1 < \cdots < k_i} \theta_{k_1} \cdots \theta_{k_i} \) and \( \theta_{k_1}, \ldots, \theta_6 \) are the \( \mathbb{Q}_p \)-conjugates of \( \theta \). Since there is a unique valuation extending \( v_p \) to any algebraic extension of \( \mathbb{Q}_p \), we conclude that \( \omega(\theta_i) = 1/6 \) for every \( i = 1, \ldots, 6 \). Thus \( v_p(s_i) = \omega(\theta_1 \cdots \theta_6) = 6 \times 1/6 = 1 \) and \( v_p(s_i) \geq 1/6 \) for every \( i = 1, \ldots, 5 \), which means that \( g(x) \) is a \( p \)-Eisenstein polynomial. Hence \( p \) does not divide the index \( (\mathbb{Z}_K : \mathbb{Z}[\theta]) \). As by hypothesis \( p \) is the unique positive prime integer such that \( p^2 \) divides \( \Delta(F(x)) \) and by definition of \( \theta, p \) is the unique positive prime integer candidate to divide \( (\mathbb{Z}[\alpha] : \mathbb{Z}[\theta]) \), we conclude that for every prime integer \( q, q \) does not divide \( (\mathbb{Z}_K : \mathbb{Z}[\theta]) \), which means that \( \mathbb{Z}_K = \mathbb{Z}[\theta] \). \( \square \)

The index of a field \( K \) is defined by \( i(K) = \gcd((\mathbb{Z}_K : \mathbb{Z}[\alpha]) \mid K = \mathbb{Q}(\alpha) \text{ and } \alpha \in \mathbb{Z}_K) \).

A rational prime \( p \) dividing \( i(K) \) is called a prime common index divisor of \( K \). If \( \mathbb{Z}_K \) has a power integral basis, then \( i(K) = 1 \). Therefore a field having a prime common index divisor is not monogenic. The existence of prime common index divisors was first established in 1871 by Dedekind who exhibited examples in cubic and quartic number fields. For example, he considered the cubic field \( K \) generated by a complex root of \( x^3 - x^2 - 2x - 8 \) and showed that the prime 2 splits completely in \( K \). So, if we suppose that \( K \) is monogenic, then we would be able to find a cubic polynomial generating \( K \), that splits completely into distinct polynomials of degree 1 in \( \mathbb{F}_2[x] \).
Since there are only 2 distinct polynomials of degree 1 in $\mathbb{F}_2[x]$, this is impossible. Based on these ideas and using Kronecker’s theory of algebraic number fields, Hensel gave necessary and sufficient conditions on the so-called “index divisors of $K$” for any prime integer $p$ to be a prime common index divisor [24].

**Remark 2.** It is well known that for any number field $K$ and a prime integer $p$, if $p$ is a common index divisor of $K$, then $p \leq n$, where $n$ is the degree of $K$ (see [3]). It follows that the unique prime candidates to be a prime common index divisor of $K$ are 2, 3, and 5.

For the proof of Theorems 2.2 and 2.3, we need the following lemma, which characterizes the prime common index divisors of $K$.

**Lemma 4.1.** Let $p$ be a rational prime integer and $K$ be a number field. For every positive integer $f$, let $P_f$ be the number of distinct prime ideals of $\mathbb{Z}_K$ lying above $p$ with residue degree $f$ and $N_f$ the number of monic irreducible polynomials of $\mathbb{F}_p[x]$ of degree $f$. Then $p$ is a prime common index divisor of $K$ if and only if $P_f > N_f$ for some positive integer $f$.

**Remark 3.** In order to prove Theorems 2.2 and 2.3, we don’t need to determine the factorization of $p\mathbb{Z}_K$ explicitly. But according to Lemma 4.1, we need only to show that $P_f > N_f$ for some adequate positive integer $f$. So in practice the second point of Theorem 5.1 could replaced by the following: If $l_i = 1$ or $d_{ij} = 1$ or $a_{ijk} = 1$ for some $(i, j, k)$ according to notation of Theorem 5.1, then $\psi_{ijk}$ provides a prime ideal $\pi_{ijk}$ of $\mathbb{Z}_K$ lying above $p$ with residue degree $f_{ijk} = m_i \cdot t_{ijk}$, where $m_i = \deg(\phi_i)$ and $t_{ijk} = \deg(\psi_{ijk})$ and $p\mathbb{Z}_K = \pi_{ijk}^t I$, where the factorization of the ideal $I$ can be derived from the other factors of each residual factors of $F(x)$.

**Proof of Theorem 2.2**

Since $\Delta(F) = 6^6b^5 - 5^5a^5$, we conclude that 2 is a candidate divisor ($\mathbb{Z}_K : \mathbb{Z}[a]$) if and only if 2 divides $a$. Assume that 2 divides $a$. Then

1. If 2 does not divide $b$, then $F(x) = (\phi_1 \cdot \phi_2)^2$ in $\mathbb{F}_2[x]$, where $\phi_1 = x - 1$ and $\phi_2 = x^2 + x + 1$. Consider the following expansions of $F(x)$:

   (1) \[ F(x) = \phi_1^6 + 6\phi_1^5 + 15\phi_1^4 + 20\phi_1^3 + 15\phi_1^2 + (6 + a)\phi_1 + 1 + a + b, \]

   (2) \[ = \phi_2^3 - 3x\phi_2^2 + (2x - 2)\phi_2 + ax + 1 + b. \]

   (a) If $a \equiv 0 \pmod{8}$ and $b \equiv 7 \pmod{8}$, then $N_{\phi_1}(F) = S_{21} + S_{22}$ has two sides joining the points $(0, v), (1, 1), \text{and } (2, 0)$ with $v \geq 3$, and so $S_{2i}$ is of degree 1 for every $i = 1, 2$. Thus $\phi_2$ provides two distinct prime ideals of $\mathbb{Z}_K$ lying above 2 with residue degree 2 each. As there is a unique monic irreducible polynomial in $\mathbb{F}_2[x]$, by Remark 3, 2 is a common index divisor of $K$.

   (b) If $a \equiv 0 \pmod{8}$ and $b \equiv 3 \pmod{8}$, then $N_{\phi_2}(F) = S_{21}$ has a single side joining $(0, 2), (1, 1), \text{and } (2, 0)$ with $R_1(F)(y) = xy^2 + (x - 1)y + 1$ its attached residual polynomial of $F(x)$. Since $R_1(y) = xy^2 + (x - 1)y + 1 = x(y - 1)(y - x^2)$,
\( \phi_2 \) provides two distinct prime ideals of \( \mathbb{Z}_K \) lying above 2 with residue degree 2 each. Again by Remark \([3] \) 2 is a common index divisor of \( K \).

(c) If \( a \equiv 0 \pmod{4} \) and \( b \equiv 1 \pmod{4} \), then \( N_{\phi_1}^+(F) = S_{11} \) has a single side of height 1 for every \( i = 1, 2 \). Thus 2 does not divide the index \( \langle \mathbb{Z}_K : \mathbb{Z}[\alpha] \rangle \).

(d) If \( a \equiv 4 \pmod{8} \) and \( b \equiv 7 \pmod{8} \), then \( N_{\phi_1}(F) = S_{11} \) has a single side joining the points \((0, 2), (1, 1)\), and \((2, 0)\). Thus the residual polynomial of \( F(x) \) associated to \( S_{11} \) is \( R_1(F)(y) = y^2 + y + 1 \), which is irreducible over \( \mathbb{F}_1 \cong \mathbb{F}_2 \). Similarly, \( N_{\phi_2}(F) = S_{21} \), with \( R_1(F)(y) = xy^2 + (x - 1)y + x \) the attached residual polynomial of \( F(x) \). Since \( xy^2 + (x - 1)y + x \) is irreducible over \( \mathbb{F}_1 \), we conclude by Theorem \([3] \) \( 2Z_K = p_{11}p_{21} \), with \( f_{11} = 2 \) and \( f_{21} = 4 \). Thus 2 is not a common index divisor of \( K \).

(e) If \( a \equiv 4 \pmod{8} \) and \( b \equiv 3 \pmod{8} \), then \( N_{\phi_1}^+(F) = S_{11} + S_{12} \) has two sides joining the points \((0, v), (1, 1)\), and \((2, 0)\) with \( v \geq 3 \), and so \( S_{11} \) is of degree 1 for every \( i = 1, 2 \). Similarly, \( N_{\phi_2}(F) = S_{21} \), with \( R_1(F)(y) = xy^2 + (x - 1)y + x + 1 \) the attached residual polynomial of \( F(x) \). Since \( xy^2 + (x - 1)y + x + 1 \) is irreducible over \( \mathbb{F}_1 \), we conclude by Theorem \([5] \) \( 2Z_K = p_{11}p_{12}p_{21} \), with \( f_{11} = f_{12} = 1 \) and \( f_{21} = 4 \). Thus 2 is not a common index divisor of \( K \).

(f) If \( a \equiv 2 \pmod{4} \) and \( b \equiv 3 \pmod{4} \), then \( N_{\phi_1}^+(F) = S_{11} \) has a single side of height 1 for every \( i = 1, 2 \). Thus 2 does not divide \( \langle \mathbb{Z}_K : \mathbb{Z}[\alpha] \rangle \).

(g) For \((a, b) \equiv (6, 1) \pmod{8}\), let \( v = \nu_2(1 + b + a) \). Then \( v \geq 3 \).
   If \( v = 3; (a, b) \in \{(6, 1), (14, 9) \pmod{16}\} \), then \( N_{\phi_1}^+(F) = S_{11} \) has a single side of degree 1, and so \( 2Z_K = p_{11}^2p_{21}^2 \), with \( f_{11} = 1 \) and \( f_{21} = 2 \).
   If \( v = 4; (a, b) \in \{(6, 9), (14, 1), (22, 25), (30, 17) \pmod{32}\} \), then \( N_{\phi_2}^+(F) = S_{21} \) has a single side joining \((0, 4), (1, 2)\), and \((2, 0)\) with \( y^2 + y + 1 \) its attached residual polynomial of \( F(x) \). Thus \( 2Z_K = p_{11}p_{12}p_{21}^2 \), with \( f_{11} = f_{12} = 2 \), and so 2 is a common index divisor of \( K \).
   If \( v \geq 5; (a, b) \in \{(6, 25), (14, 17), (22, 9), (30, 1) \pmod{32}\} \), then \( N_{\phi_2}^+(F) = S_{21} \) has a single side joining \((0, 1)\) and \((2, 0)\) with \( y^2 + y + 1 \) its attached residual polynomial of \( F(x) \). Thus \( 2Z_K = p_{11}p_{12}p_{21}^2 \), with \( f_{11} = f_{12} = 2 \), and so 2 is a common index divisor of \( K \).

(h) For \((a, b) \equiv (6, 5) \pmod{8}\), let \( \phi = x - 3 \). Then \( F(x) = \cdots + 1215\phi^2 + (a + 1458)\phi + (1 + b + 3a + 728) \). If \( a \equiv 6 \pmod{16} \), then \( \nu_2(6 + 1458) = 3 \) and \( \nu_2(728) = 3 \), we have the following:
   If \( (a, b) \in \{(14, 5), (6, 13) \pmod{16}\} \), then \( N_{\phi}^+(F) = S_{11} \) has a single side joining \((0, 3)\) and \((2, 0)\). Thus \( 2Z_K = p_{11}^2p_{21}^2 \), with \( f_{11} = 1 \) and \( f_{21} = 2 \).
   If \( (a, b) \in \{(14, 13), (6, 5) \pmod{16}\} \), then go to section \([5]\) to calculate an integer \( s \) satisfying \( F(x) \) is \( x - s \)-regular with respect to \( p = 2 \).

(i) For \((a, b) \equiv (2, 5) \pmod{8}\), If \((a, b) \in \{(2, 5), (10, 13) \pmod{16}\} \), then \( N_{\phi_1}^+(F) = S_{11} \) has a single side joining \((0, 3)\) and \((2, 0)\). Thus \( 2Z_K = p_{11}^2p_{21}^2 \), with \( f_{11} = 1 \) and \( f_{21} = 2 \).
If \((a, b) \in \{(2, 13), (10, 5)\} \pmod{16}\), then go to section 5 to calculate an integer \(s\) satisfying \(F(x) = x - s\)-regular with respect to \(p = 2\).

(j) For \((a, b) \equiv (2, 1) \pmod{8}\), if \((a, b) \in \{(2, 9), (10, 1)\} \pmod{16}\), then for \(\phi = x - 3\), we have \(F(x) = \cdots + 1215\phi^2 + (a + 1458)\phi + (1 + b + 3a + 728)\) and \(N^+_\phi(F) = S_{11}\) has a single side joining \((0, 3)\) and \((2, 0)\). Thus, \(2\mathbb{Z}_K = v_1^2\mathfrak{p}_{21}\), with \(f_1 = 1\) and \(f_{21} = 2\).

If \((a, b) \equiv (2, 17) \pmod{16}\) or \((a, b) \equiv (18, 1) \pmod{16}\), then for \(\phi = x - 3\), we have \(F(x) = \cdots + 1215\phi^2 + (a + 1458)\phi + (1 + b + 3a + 728)\). In this case since \(N^+_\phi(F) = S_{21}\) has a single side of height 1, we conclude that 2 is a common index divisor of \(K\) if and only if \(\phi\) provides a prime ideal of \(\mathbb{Z}_K\) lying above 2 with residue degree 2. That is if and only if \(1 + b + 3a + 728 \equiv 16 \pmod{32}\), i.e., \((a, b) \equiv (2, 17), (18, 1) \pmod{32}\).

If \((a, b) \equiv (10, 9) \pmod{16}\), then go to section 5 to calculate an integer \(s\) satisfying \(F(x) = x - s\)-regular with respect to \(p = 2\).

(2) If 2 divides \(b\), then \(\overline{F(x)} = \phi^6\) in \(\mathbb{F}_2[x]\), with \(\phi = x\). It follows that:

(a) If \(6v_2(a) < 5v_2(b)\), then \(N^+_\phi(F) = S_1 + S_2\) has two sides joining the points \((0, v_2(b)), (0, v_2(a))\), and \((6, 0)\). Since by assumption, \(v_2(b) \leq 5\) or \(v_2(a) \leq 4\), we conclude that \(v_2(a) \leq 4\), and each side of \(N^+_\phi(F)\) is of degree 1. Thus \(2\mathbb{Z}_K = v_1^5\mathfrak{p}_{21}\) with \(f_{11} = f_{21} = 1\).

(b) If \(6v_2(a) \geq 5v_2(b)\), then \(N^+_\phi(F) = S_1\) has a single side joining the points \((0, v_2(b))\) and \((6, 0)\). Let \(d_1\) be the degree of \(S\). Then \(d_1 \in \{1, 2, 3\}\).

If \(d_1 = 1\), then \(2\mathbb{Z}_K = v_6\) with residue degree \(f_1 = 1\). If \(d_1 = 2; v_2(b) \in \{2, 4\}\), then \(e_1 = 3\) is the ramification degree of \(S_1\). Since \(R_1(F)(y) = (y + 1)^2\) is the residual polynomial of \(F(x)\) attached to \(S_1\), the Newton polygon of \(F(x)\) of second order \(N_2(F)\) has length 2. Thus \(N_2(F)\) has two sides of degree 1 each or \(N_2(F)\) has a single side and its attached residual polynomial has at most degree 2, and so there at most two prime ideals of \(\mathbb{Z}_K\) lying above 2.

If \(d_1 = 3\), then \(e_1 = 2\) and \(R_1(F)(y) = y^3 - 1 = (y - 1)(y^2 + y + 1)\) is the residual polynomial of \(F(x)\) attached to \(S_1\). Thus \(2\mathbb{Z}_K = v_2^2\mathfrak{p}_{22}\) with \(f_{21} = 1\) and \(f_{22} = 2\).

Proof of Theorem 2.3

For \(p = 3\), since \(\Delta(F) = 6^6b^5 - 5^5a^6\), we conclude that 3 is a candidate to divide \((\mathbb{Z}_K : \mathbb{Z}[\alpha])\) if and only if 3 divides \(a\). Assume that 3 divides \(a\). Then we have the following cases:

(1) If \(b \equiv 1 \pmod{3}\), then \(\overline{F(x)} = \phi^3\) in \(\mathbb{F}_3[x]\), with \(\phi = x^2 + 1\). Let \(F(x) = \phi^3 - 3\phi^2 + 3\phi + (ax + b - 1)\) and \(N^+_\phi(F)\) the principal \(\phi\)-Newton polygon of \(F(x)\).

Since the length of \(N^+_\phi(F)\) is three, we conclude that \(\phi\) can provide at most 3 prime ideals of \(\mathbb{Z}_K\) lying above 3. Thus 3 is not a common index divisor of \(K\).
(2) If \( b \equiv -1 \pmod{3} \), then \( F(x) = (\phi_1 \phi_2)^3 \) in \( \mathbb{F}_3[x] \), with \( \phi_1 = x - 1 \) and \( \phi_1 = x + 1 \). Let

\[
F(x) = \phi_1^6 + 6\phi_1^5 + 15\phi_1^4 + 20\phi_1^3 + 15\phi_1^2 + (6 + a)\phi_1 + 1 + a + b,
\]

\[
(4) \quad \phi_2^6 - 6\phi_2^5 + 15\phi_2^4 - 20\phi_2^3 + 15\phi_2^2 + (6 - a)\phi_2 + 1 + b - a.
\]

It follows that:

(a) If \( a \equiv 0 \pmod{9} \) and \( b \equiv -1 \pmod{9} \), then \( N_{\phi_1}^+(F) = S_{11} + S_{12} \) has two sides joining \((0, v), (1, 1), \text{ and } (3, 0)\), where \( v \geq 2 \). Thus \( 3\mathbb{Z}_K = p_{11} p_{21}^2 p_{22} \), with residue degree 1 each. As there is exactly three monic polynomials of degree 1 modulo 3, we conclude that 3 is a common index divisor of \( K \).

(b) If \( a \equiv 0 \pmod{9} \) and \( b \not\equiv -1 \pmod{9} \) or \( a \not\equiv 0 \pmod{9} \) and \( b \equiv -1 \pmod{9} \), then \( N_{\phi_1}^+(F) = S_{11} \) has a single side of height 1 for every \( i = 1, 2 \). Thus 3 does not divide \( (\mathbb{Z}_K : \mathbb{Z}[a]) \).

(c) If \( a \equiv 3 \pmod{9} \) and \( b \equiv -1 + a \pmod{9} \), then \( N_{\phi_1}^+(F) = S_{11} \) has a single side of height 1 and \( N_{\phi_2}^+(F) = S_{21} + S_{22} \) has two sides joining \((0, v), (1, 1), \text{ and } (3, 0)\) with \( v \geq 2 \). Thus \( 3\mathbb{Z}_K = p_{11}^3 p_{21}^2 p_{22} \), with residue degree 1 each, and so 3 is not a common index divisor of \( K \).

(d) If \( a \equiv 3 \pmod{9} \) and \( b \equiv -(1 + a) \pmod{9} \), then \( N_{\phi_2}^+(F) = S_{21} \) has a single side of height 1. For \( N_{\phi_1}^+(F) = S_{11} \), let \( \mu = \nu_3(a + 6) \) and \( \tau = \nu_3(b + 1 + a) \).

If \( \tau = 2 \), then \( N_{\phi_1}(F) = S_{11} \) has a single side of height 2, and so \( 3\mathbb{Z}_K = p_{11}^3 p_{21}^3 \), with residue degree 1 each, and so 3 is not a common index divisor of \( K \).

The following table summarizes the case \( \mu = 2 \) and \( \tau = 3 \):

Table A
In all cases, $\phi_1$ provides at most two prime ideals of $\mathbb{Z}_k$ lying above 3 and $\phi_2$ provides a unique prime ideal of $\mathbb{Z}_k$ lying above 3. Thus 3 is not a common index divisor of $K$.

If $\mu = 2$ and $\tau \geq 4$, then $N_{\phi_1}^+(F) = S_{11} + S_{12}$ has two sides such that $S_{12}$ is of degree 1 and $S_{11}$ is of degree 2 and its attached residual polynomial of $F(x)$ is $R_1(F)(y) = -y^2 - y + 1$. More precisely, if $a \equiv 3 \pmod{27}$ and $b \equiv -(1 + a) \pmod{81}$, then $R_1(F)(y) = -y^2 - y + 1$, which is irreducible over $\mathbb{F}_{31} = \mathbb{F}_3$. If $a \equiv 12 \pmod{27}$ and $b \equiv -(1 + a) \pmod{81}$, then let us replace $\phi_1$ by $\phi = x + 2$. Then $F(x) = \phi^6 - 12\phi^5 + 60\phi^4 - 160\phi^3 + 240\phi^2 + (6a - 192)\phi + (1 + b - 2a + 63)$. Since $a - 192 \equiv 9 \pmod{27}$ and $1 + b - 2a + 63 \equiv 0 \pmod{27}$, if $1 + b - 2a + 63 \equiv 27 \pmod{81}$, then $N_{\phi}^+(F) = S$ has a single side joining $(0, 3)$ and $(3, 0)$ with $R_1(F)(y) = -y^3 - y^2 + y + 1 = -(y^2 + 1)(y - 1)$. Hence $\phi$ provides two distinct prime ideals of $\mathbb{Z}_k$ lying above 3. Therefore, 3 is not a common index divisor of $K$. If $1 + b - 2a + 63 \equiv -27 \pmod{81}$, then $N_{\phi}^+(F) = S$ has a single side joining $(0, 3)$ and $(3, 0)$ with $R_1(F)(y) = -y^3 - y^2 + y - 1$, which is irreducible over $\mathbb{F}_3$. Hence $\phi$ provides a unique prime ideal of $\mathbb{Z}_k$ lying above 3. Therefore, 3 is not a common index divisor of $K$. If $S + T$ has two sides joining $(0, v), (1, 2)$ and $(3, 0)$ with $R_1(F)(y) = -y^2 - y + 1$ the residual polynomial of $F(x)$ attached to $T$. Hence $\phi$ provides a unique prime ideal of $\mathbb{Z}_k$ lying above 3. Therefore, 3 is not a common index divisor of $K$.

If $\mu \geq 3$ and $\tau = 3$, then $N_{\phi_1}^+(F) = S_{11}$ has a single side with $R_1(F)(y) = $
\(-y^3 - y^2 \pm 1\) the residual polynomial of \(F(x)\) attached to \(S_{11}\). Since 
\[-y^3 - y^2 - 1 = -(y^2 + 2y + 2)(y + 1)\] and 
\[-y^3 - y^2 + 1\] is irreducible over \(\mathbb{F}_3\), we conclude that 3 is not a common index divisor of \(K\).

Assume that \(\mu \geq 3\) and \(\tau \geq 4\), which means \((a, b) \in \{(21, 59), (48, 32), (75, 5)\} \mod 81\).

In this case we have the following:

If \(2\mu > \tau + 1\) and \(\tau = 2k + 1\) for some positive integer \(k \geq 2\), then 
\(N_{\phi_1}^+(F) = S_{11} + S_{12}\) has two sides such that the residual polynomial of \(F(x)\) attached to \(S_{11}\) is 
\(-y^2 + (b + 1 + a)_3\). Therefore, 3 is a common index divisor of \(K\) if and only if 
\((b + 1 + a)_3 \equiv 1 \mod 3\).

If \(2\mu > \tau + 1\) and \(\tau = 2k\) for some positive integer \(k \geq 2\), then 
\(N_{\phi_1}^+(F) = S_{11} + S_{12}\) has two sides of degree one each. Therefore, 3 is not a common index divisor of \(K\).

If \(2\mu < \tau + 1\), then 
\(N_{\phi_1}^+(F) = S_{11} + S_{12} + S_{13}\), and so there are 4 prime ideals of \(\mathbb{Z}_K\) lying above 3 with residue degree 1 each. Hence 3 is a common index divisor of \(K\).

If \(2\mu = \tau + 1\), then 
\(N_{\phi_1}^+(F) = S_{11} + S_{12}\) has two sides such that the residual polynomial of \(F(x)\) attached to \(S_{11}\) is 
\(-y^2 + (a + 6)_3y + (b + 1 + a)_3\).

(e) If \(a \equiv 6 \mod 9\) and \(b \equiv -(1 + a) \mod 9\), then 
\(N_{\phi_2}^+(F) = S_{21}\) has a single side of height 1 and 
\(N_{\phi_2}^+(F) = S_{11} + S_{12}\) has two sides joining the points 
\((0, v), (1, 1)\), and \((3, 0)\) with \(v \geq 2\). Thus each side is of degree 1. Thus there are exactly three prime ideals of \(\mathbb{Z}_K\) lying above 3, and so 3 is not a common index divisor of \(K\).

(f) If \(a \equiv 6 \mod 9\) and \(b \equiv -1 + a \mod 9\), then 
\(N_{\phi_1}^+(F) = S_{11}\) has a single side of height 1. For 
\(N_{\phi_2}^+(F)\), let \(\mu = v_3(a - 6)\) and \(\tau = v_3(b + 1 - a)\).

If \(\tau = 2\), then 
\(N_{\phi_2}^+(F) = S_{21}\) has a single side of height 2, and so 
\(3\mathbb{Z}_K = p_1^3v_3^3\), with residue degree 1 each prime factor.

The following table summarizes the case \(\mu = 2\) and \(\tau = 3\):

| \(a \mod 81\) | \(b \mod 81\) | \(\frac{a - 6}{9} \mod 3\) | \(\frac{b + 1 - a}{27} \mod 3\) | \(R_1(F)\) |
|---|---|---|---|---|
| 15 | 41 | 1 | 1 | \(y^2 - y^2 + y + 1\) |
| 15 | 68 | 1 | -1 | \((y^2 + 1)(y - 1)\) |
| 24 | 50 | -1 | 1 | \((y + 1)(y - 1)^2\) |
| 24 | 77 | -1 | -1 | \(y^2 - y^2 - y - 1\) |
| 42 | 68 | 1 | -1 | \(y^2 - y^2 + y + 1\) |
| 42 | 14 | 1 | -1 | \((y^2 + 1)(y - 1)\) |
| 51 | 23 | -1 | -1 | \(y^2 - y^2 - y - 1\) |
| 51 | 77 | -1 | 1 | \((y + 1)(y - 1)^2\) |
| 69 | 41 | 1 | -1 | \((y^2 + 1)(y - 1)\) |
| 69 | 14 | 1 | 1 | \(y^2 - y^2 + y + 1\) |
| 78 | 23 | -1 | 1 | \((y + 1)(y - 1)^2\) |
| 78 | 50 | -1 | -1 | \(y^2 - y^2 - y - 1\) |
If \((a, b) \in \{(15, 14), (24, 23), (42, 41), (51, 50), (69, 68), (78, 77)\}\) (mod 81), then \(N_{\phi_2}^+(F) = S_{21} + S_{22}\) has two sides joining \((0, v), (1, 2), (2, 1),\) and \((3, 0)\) with \(v \geq 4, R_1(F)(y) = y^2 - y + 1\) the residual polynomial of \(F(x)\) associated to \(S_{22}\) and \(S_{21}\) is of degree \(1\). Since \(R_1(F)(y)\) is irreducible over \(\mathbb{F}_3\), there are exactly three prime ideals of \(\mathbb{Z}_K\) lying above \(3\).

For \((a, b) \in \{(6, 32), (6, 59), (33, 5), (33, 59), (60, 5), (60, 32)\}\) (mod 81), we have \(N_{\phi_2}^+(F) = S_{21}\) has a single side joining \((0, 3), (2, 1),\) and \((3, 0)\) with \(R_1(F)(y) = y^3 - y^2 + 1\) the residual polynomial of \(F(x)\) associated to \(S_{22}\). Since \(R_1(F)(y)\) is irreducible over \(\mathbb{F}_3\), we conclude that there are exactly two prime ideals of \(\mathbb{Z}_K\) lying above \(3\).

For \((a, b) \in \{(24, 50), (78, 23), (51, 77)\}\) (mod 81), let \(\phi = x - 2\). Then \(F(x) = \cdots + 160\phi^3 + 240\phi^2(a + 192)\phi + (1 + b - a + 3a + 63)\). It follows that \(N_{\phi}^+(F) = S\) has a single side joining the points \((0, 3)\) and \((3, 0)\) such that for \((a, b) \equiv (24, 50)\) (mod 81), \(R_1(F)(y) = y^3 - y^2 - y - 1\), which is irreducible and For \((a, b) \in \{(78, 23), (51, 77)\}\) (mod 81), \(R_1(F)(y) = y^3 - y^2 - 1 = (y + 1)(y^2 + y - 1)\). Thus there are exactly three prime ideals of \(\mathbb{Z}_K\) lying above \(3\).

For \(\mu \geq 3\) and \(\tau \geq 4; a \equiv 6\) (mod 27) and \(b \equiv -1 + a\) (mod 81), we have the following case:

If \(2\mu > \tau + 1\), then \(N_{\phi_2}^+(F) = S_{21} + S_{22}\) has two sides joining \((0, v), (1, 2),\) and \((3, 0)\) with \(v \geq 4\) such that \(R_1(F)(y) = y^2 - y + 1\) is the residual polynomial of \(F(x)\) associated to \(S_{22}\) and \(S_{21}\) is of degree \(1\). Thus there are exactly three prime ideals of \(\mathbb{Z}_K\) lying above \(3\).

If \(\mu \geq 3, \tau \geq 4, 2\mu < \tau + 1,\) and \(\tau\) is even, then there are exactly three prime ideals of \(\mathbb{Z}_K\) lying above \(3\).

If \(\mu \geq 3, \tau \geq 4, 2\mu < \tau + 1,\) and \(\tau\) is even, then \(N_{\phi_2}^+(F) = S_{21} + S_{22} + S_{23}\), and so \(3\) is a common index divisor of \(K\).

If \(2\mu = \tau + 1\), then \(N_{\phi_2}^+(F) = S_{21} + S_{22}\) has a two sides such that the residual polynomial of \(F(x)\) attached to \(S_{21}\) is \(-y^2 + (a - 6)y + (b + 1 - a)\). Thus if \((a - 6) \not\equiv -1\) (mod 3) or \((b + 1 - a) \not\equiv -1\) (mod 3), then \(3\) is not a common index divisor of \(K\). If \((a - 6) \equiv -1\) (mod 3) and \((b + 1 - a) \equiv -1\) (mod 3), then go to the section \(5\) to calculate an integer \(s\) satisfying \(F(x)\) is \(x - s\)-regular with respect to \(p = 3\).

**Proof of Proposition 2.4**

Since the candidate prime integers to divide \(i(K)\) are \(2, 3,\) and \(5\), we have to show that \(5\) is not a common index divisor of \(K\). As \(\Delta(F) = 5^5a^6 - 6^6b^5\), if \(5\) divides \(i(K)\), then \(5\) divides \(b\).
(1) If \( a \not\equiv 0 \pmod{5} \), \( \overline{F(x)} = (x + a)^5 x \) in \( F_5[x] \). Let \( \phi = x + a \) and \( F(x) = \phi^6 - 6a\phi^5 + 15a^2\phi^4 - 20a^3\phi^3 + 15a^4\phi^2 + a(1 - 6a^5)\phi + (b - a^2(1 - a^4)) \) be the \( \phi \)-expansion of \( F(x) \). Since \( v_5(15a^4) = 1 \), \( N_\phi^0(F) \) has at most 3 sides and \( \phi \) can provide at most 3 prime ideals of \( \mathbb{Z}_K \) lying above 5. Thus there are at most 4 prime ideals of \( \mathbb{Z}_K \) lying above 5, and so 5 is not a common index divisor of \( K \).

(2) If 5 divides \( a \) and \( v_5(a) < v_5(b) \), then there are exactly 2 prime ideals of \( \mathbb{Z}_K \) lying above 5.

(3) If 5 divides \( a \) and \( v_5(a) \geq v_5(b) \), then \( v_5(b) \leq 5 \), and so \( N_\phi^+(F) \) has a single side of degree 1. Thus there is a unique prime ideal of \( \mathbb{Z}_K \) lying above 5.

5. Special cases and examples

Let \( p \) be a prime integer and \( K \) a number field defined by a monic irreducible polynomial \( F(x) \in \mathbb{Z}[x] \) with \( F(x) = \phi^6 + a_5\phi^5 + a_4\phi^4 + a_3\phi^3 + a_2\phi^2 + a_1\phi + a_0 \) for some \( \phi = x - a \in \mathbb{Z}[x] \) and \( (a_0, \ldots, a_5) \in \mathbb{Z}^6 \). Assume that \( N_\phi^0(F) \) has a non-trivial \( k \)-component for some \( k \in \mathbb{N} \); a side of slope \(-k\) and length \( l \neq 0 \). Assume also that \( l \geq 2 \) and \( R_1(F)(y) = \pm(y - u)^2 \) \( u \in \mathbb{Z} \) is the residual polynomial of \( F(x) \) associated to this side for some integer \( u \). Then we can construct an element \( s \in \mathbb{Z}_p \) such that \( F(x) \) is \( x - s \)-regular. Such an element \( s \) is called a regular element of \( F(x) \) with respect to \( \phi \). How to construct such a regular element \( s \) by theorem of the polygon, \( F(x) = F_1(x)F_2(x) \) in \( \mathbb{Z}_p[x] \) such that \( F_2 \) is monic, \( N_\phi^0(F_2) \) has a single side of slope \(-k\), and \( R_1(F_2)(y) = \pm(y - u)^2 \) is the residual polynomial of \( F_2 \) associated to this side. Let \( u_0 = u, s_1 = s_0 + p^ku_0, \) and \( \phi = x - s_1 \). Then \( F_2(x) = a_1\phi^2 + b_1\phi + c_1 \) for some \( (a_1, b_1, c_1) \in \mathbb{Z}_p^3 \) such that \( v_p(a_1) = 0, v_p(b_1) \geq k+1 \) and \( v_p(c_1) \geq 2k+1 \). If \( 2v_p(b_1) \geq v_p(c_1) \), then we can repeat the same process. In this case \( ind_{\phi_1}(F) \geq ind_{\phi_0}(F) + 1 \). Thus \( (\mathbb{Z}_K : \mathbb{Z}[\alpha]) \geq ind_{\phi_1}(F) \geq ind_{\phi_0}(F) + 1 \). Since \( (\mathbb{Z}_K : \mathbb{Z}[\alpha]) \) is finite, this process cannot continue infinitely. Thus after a finite number of iterations, this process will provide a regular element.

For \( (a, b) \in \{(14, 13), (6, 5), (2, 13), (10, 5), (10, 9)\} \pmod{16} \), we have \( N_\phi^+(F) \) has a single side of height 1. By Remark 3.3 \( \phi_2 \) provides a unique prime ideal \( \mathfrak{p}_{21} \) of \( \mathbb{Z}_K \) lying above 2 with residue degree 2. Also \( N_\phi^0(F) = S \) has a single side of height \( 2^{2k} \) for some positive integer \( k \) with \( R_1(F)(y) = (y - 1)^2 \) its residual attached polynomial of \( F(x) \), where \( \phi = x - 1 \). Let us replace \( \phi \) by \( x - s \) with s a regular element of \( F(x) = \cdots + a_1\phi^2 + b_1\phi + c_1 \). If \( v_2(b_1) \geq h + 1 \) and \( v_2(c_1) = 2h + 1 \), then \( N_\phi^+(F) = S_1 \) has a single side of degree 1, and so \( \phi \) provides a unique prime ideal of \( \mathbb{Z}_K \) lying above 2 with residue degree 1. If \( v_2(b_1) = h \) and \( v_2(c_1) = 2h \), then \( N_\phi^+(F) = S_1 \) has a single side joining \( (0, 2h), (1, h) \), and \( (2, 0) \). Thus \( \phi \) provides a unique prime ideal of \( \mathbb{Z}_K \) lying above 2 with residue degree 2. If \( 2v_2(b_1) < v_2(c_1) \), then \( \phi \) provides two prime ideals of \( \mathbb{Z}_K \) lying above 2 with residue degree 1 each. Recall that in this case, 2 is a common index divisor of \( K \) if and only if \( v_2(b_1) = h \) and \( v_2(c_1) = 2h \) for some positive integer \( h \).
Corollary 5.1.  
(1) If \((a, b) \equiv (14, 13) \pmod{16}\), then 2 divides \(i(K)\) if and only if 
\((a, b) \equiv (14, 29) \pmod{32}\) or \((a, b) \equiv (30, 13) \pmod{32}\).  

(2) If \((a, b) \equiv (2, 13) \pmod{16}\), then 2 divides \(i(K)\) if and only if \(b \equiv -(1 + a) + 
64 \pmod{128}\) or \(b \equiv 5a + 73 \pmod{128}\).  

Proof. Recall that \(F(x) = \phi_6^c + 6\phi_5^2 + 15\phi_4^2 + 20\phi_3^2 + 15\phi_2^2 + (6 + a)\phi_1 + 1 + a + b\) and 
\(F(x) = \phi_3^3 - 3x\phi_2^2 + (2x - 2)\phi_2 + ax + 1 + b\) for \(\phi_1 = x - 1\) and \(\phi_2 = x^2 + x + 1\). Since 
\(\nu_2(a) = 1\), then \(\phi_2\) provides a unique prime ideal of \(\mathbb{Z}_K\) lying above 2 with residue 
degree 2 and ramification index 2. It follows that 2 is a common divisor of \(K\) if and only if \(\phi_1\) provides a unique prime ideal of \(\mathbb{Z}_K\) lying above 2 with residue degree 2.

(1) If \((a, b) \equiv (14, 13) \pmod{16}\), then by a refinement of the expansion; we replace 
\(\phi_1\) by \(x + 1\) and we consider 
\(F(x) = \phi_6^c - 6\phi_5^2 + 15\phi_4^2 - 20\phi_3^2 + 15\phi_2^2 + (-6 + 
a)\phi_1 + 1 - a + b\). Since \(a + 6 \equiv 4 \pmod{8}\) and \(1 - a + b \equiv 0 \pmod{8}\), then \(\phi_1\) provides a unique prime ideal of \(\mathbb{Z}_K\) lying above 2 with residue degree 2 if and only if \(b \equiv a + 15 \pmod{32}\), which means that \((a, b) \equiv (14, 29) \pmod{32}\) or \((a, b) \equiv (30, 13) \pmod{32}\). When this occurs, then \(2\mathbb{Z}_K = \nu_1\nu_2\) with residue degree 2 each prime factor.

(2) For \((a, b) \equiv (2, 13) \pmod{16}\), since \(\nu_2(a + 6) = 3\) and \(a + b + 1 \equiv 0 \pmod{16}\), then 
2 divides \(i(K)\) if and only if \(N_{\phi_1}(F)\) has a single side of height 6 for \(\phi_1 = x - 1\) 
or \(N_{\phi_1}(F)\) has a single side of height 4 and \(N_{\phi_1}(F)\) has a single side of height 6 
for \(\phi_1 = x + 5\).

\(\square\)

Similarly, if \(a \equiv 21 \pmod{27}\) and \(b \equiv -(1 + a) \pmod{81}\), then we have to repeat 
the previous process until to get a regular element of \(F(x)\) with respect to \(\phi = x - 1\). 
Again if \(a \equiv 6 \pmod{27}\) and \(b \equiv a - 1 \pmod{81}\), then we have to repeat the previous 
process until to get a regular element of \(F(x)\) with respect to \(\phi = x + 1\). 
Now assume that \(a \equiv 21 \pmod{27}\) and \(b \equiv -(1 + a) \pmod{81}\).

(1) If \(a \equiv 21\) or \(48 \pmod{81}\) and \(b \equiv -(1 + a) + 81 \pmod{3^5}\), then for \(\phi = x - 1\), 
\(N_{\phi}(F) = S_{11} + S_{12}\) has two sides such that \(R_1(F(y)) = -y^2 + 1 = -(y - 1)(y + 1)\). 
Thus there are 4 prime ideals of \(\mathbb{Z}_K\) lying above 3 with residue degree 1 each. 
Therefore 3 is a common index divisor of \(K\).

(2) If \(a \equiv 21\) or \(48 \pmod{81}\) and \(b \equiv -(1 + a) + 162 \pmod{3^5}\), then for \(\phi = x - 1\), 
\(N_{\phi}(F) = S_{11} + S_{12}\) has two sides such that \(R_1(F(y)) = -(y^2 + 1)\), which is 
irreducible over \(\mathbb{F}_3\). Thus there are exactly 3 prime ideals of \(\mathbb{Z}_K\) lying above 3. 
Therefore 3 is not a common index divisor of \(K\).

(3) If \(a \equiv 21\) or \(48 \pmod{81}\) and \(b \equiv -(1 + a) \pmod{3^6}\), then for \(\phi = x - 1\), 
\(N_{\phi}(F) = S_{11} + S_{12} + S_{13}\) has three sides such with degree 1 each. Thus there are 
4 prime ideals of \(\mathbb{Z}_K\) lying above 3 with residue degree 1 each. Therefore 3 is a 
common index divisor of \(K\).

(4) If \(a \equiv 21 \pmod{81}\) and \(b \equiv -(1 + a) \pm 3^5 \pmod{3^6}\), then for \(\phi = x - 1\), 
\(N_{\phi}(F) = S_{11} + S_{12}\) has two sides such that \(R_1(F(y)) = -y^2 + y \pm 1\), which is irreducible 
over \(\mathbb{F}_3\). Thus 3 is not a common index divisor of \(K\).
(5) If \(a \equiv 48 \pmod{81}\) and \(b \equiv -(1+a) + 3^5 \pmod{3^6}\), then for \(\phi = x - 1\), \(N_\phi^+(F) = S_{11} + S_{12}\) has two sides such that \(R_1(F)(y) = -y^2 - y + 1\), which is irreducible over \(\mathbb{F}_\phi = \mathbb{F}_3\). Thus 3 is not a common index divisor of \(K\).

(6) If \(a \equiv 48 \pmod{81}\) and \(b \equiv -(1+a) + 2 \cdot 3^5 \pmod{3^6}\), then for \(\phi = x - 1\), \(N_\phi^+(F) = S_{11} + S_{12}\) has two sides such that \(R_1(F)(y) = -y^2 - y - 1 = -(y - 1)^2\) in \(\mathbb{F}_3[y]\). In this case, we have to replace \(\phi_1\) by \(\phi = x - 10\). For example for \(a \equiv 48 \pmod{3^6}\) and \(b \equiv -(1+a) + 2 \cdot 3^5 + 3^6 \pmod{3^6}\), we have \(N_\phi^+(F) = S_{11} + S_{12}\) has two sides joining \((0,7), (1,4), (2,1),\) and \((3,0)\) and \(R_1(F)(y) = -y^2 + y - 1\) in \(\mathbb{F}_3[y]\). Thus 3 is not a common index divisor of \(K\).

Examples. Let \(K\) be a number field generated by a complex root \(\alpha\) of an irreducible trinomial \(F(x) \in \mathbb{Z}[x]\). Based on prime ideal factorization of \(p\mathbb{Z}_k\) and Engström’s result in [12], we evaluate \(i(K)\). Let \(x_p = v_p(i(K))\).

1. For \(F(x) = x^6 + 18x^5 + 33\), we have \(F(x)\) is 3-Eisenstein. Thus \(F(x)\) is irreducible over \(\mathbb{Q}\) and 3 does not divide \((\mathbb{Z}_k : \mathbb{Z}[\alpha])\), and so \(v_3(i(K)) = 0\). By Theorem 2.2(2) and its proof \(2\mathbb{Z}_k = p_1^v p_2^w\) with residue degree 2 each prime ideal. By [12], we have \(v_2(i(K)) = 1\), and so \(i(K) = 2\). Therefore \(K\) is not monogenic.

2. For \(F(x) = x^6 + 72x - 1\), by Theorems 2.2(1) and 2.3(1), 6 divides \(i(K)\). Since \(2\mathbb{Z}_k = p_1 p_2 p_3 p_4\) with \(f_1 = f_2 = 1\) and \(f_3 = f_4 = 2\), we conclude by [12] that \(v_2(i(K)) = 2\). Again since \(3\mathbb{Z}_k = p_1^v p_2^w p_3^z\) with \(f_1 = f_2 = f_3 = 1\), we conclude that \(v_3(i(K)) = 1\). Thus \(i(K) = 12\), and so \(K\) is not monogenic.

3. For \(F(x) = x^6 + 10x + 25\), since \(2\mathbb{Z}_k = p_1^v p_2^w\) with \(f_1 = f_2 = 2\), we conclude by [12] that \(v_2(i(K)) = 1\). Since 3 does not divide \(i(K)\), we get \(i(K) = 2\), and so \(K\) is not monogenic.

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