Intrinsic probability distributions for physical systems

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For a given metric \( g_{\mu\nu} \), which is identified as Fisher information metric, we generate new constraints for the probability distributions for physical systems. We postulate the existence of intrinsic probability distributions for physical systems, and calculate the probability distribution by optimizing the Fisher information metric under specified constraints. Accordingly, we get differential equations for the probability distributions.

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I. THE FISHER INFORMATION METRIC

Fisher information proposed by Fisher [1] is a way to estimate hidden parameters in a set of random variables. Since we want to retrieve information of certain parameters or coordinates in a statistical set for random variables \( X \). Accordingly, we define a probability distribution \( \rho(x|\xi) \), where \( \xi = (\xi_1, ..., \xi_n) \) is a set of parameters we want to estimate and \( x \) is a specific value picked from random variables \( X \). The Fisher information matrix is defined by [2]

\[
g_{ij}(\xi) = \int dx \rho(x|\xi) \frac{\partial \ln \rho(x|\xi)}{\partial \xi_i} \frac{\partial \ln \rho(x|\xi)}{\partial \xi_j} = \int dx \frac{1}{\rho(x|\xi)} \frac{\partial \rho(x|\xi)}{\partial \xi_i} \frac{\partial \rho(x|\xi)}{\partial \xi_j}.
\]

Let \( \rho(x|\xi) = \Psi^2(x|\xi) \), where \( \Psi \) is real. Then the Fisher information matrix can be rewritten as

\[
g_{ij}(\xi) = 4 \int dx \frac{\partial \Psi(x|\xi)}{\partial \xi_i} \frac{\partial \Psi(x|\xi)}{\partial \xi_j},
\]

which is symmetric, \( g_{ij}(\xi) = g_{ji}(\xi) \).

Fisher information matrix provides a natural distinguishability metric for probability distributions which can be expressed by [3, 4]

\[
ds^2_{PD} = \sum_j \frac{d\rho_j^2}{\rho_j} = 4 \sum_j d\Psi_j^2.
\]

Wootters called \( s_{PD} \) the statistical distance, which is defined by “maximum number of mutually distinguishable intermediate probabilities.” Hence, the Fisher information matrix can be generalized to Fisher information metric, which is a Riemannian metric on a smooth manifold.

A. Minkowski space in Cartesian coordinates

The flat space metric or Minkowski metric is \( \eta_{\mu\nu} = (-1, 1, 1, 1) \) [6]. The line element squared reads

\[
ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2.
\]

We define the Fisher information metric as follows [7]:

\[
\int_X d^4 x \rho(x|\xi) \left( \frac{1}{\rho(x|\xi)} \frac{\partial \rho(x|\xi)}{\partial \xi^\mu} \right) \left( \frac{1}{\rho(x|\xi)} \frac{\partial \rho(x|\xi)}{\partial \xi^\nu} \right) = \eta_{\mu\nu}.
\]

(1)

We shall confine our discussion to the spatial dimensions. Since the normalization condition on the distribution is

\[
\int d^4 x \rho(x|\xi) = 1,
\]

(2)

we should optimize the Fisher information subject to the constraint, Eq. (2). In terms of \( \Psi \), i.e., \( \rho(x|\xi) = \Psi^2(x|\xi) \) in one-dimension, the Fisher information reads

\[
\int_{-\infty}^{+\infty} dx \rho(x|\xi) \left( \frac{\partial \rho(x|\xi)}{\partial \xi^\mu} \right)^2 = 4 \int_{-\infty}^{+\infty} dx \left( \frac{\partial \Psi(x|\xi)}{\partial \xi^\mu} \right)^2.
\]

(3)

In order to obtain an objective function for minimization, we impose Eq. (2) on the Fisher information, Eq. (4),

\[
\int_X dx \left( \frac{\partial \Psi(x|\xi)}{\partial \xi^\mu} \right)^2 - \alpha^2 \int_X dx \Psi^2(x|\xi) = 0,
\]

giving the function

\[
\mathcal{L} = \left( \frac{\partial \Psi(x|\xi)}{\partial \xi^\mu} \right)^2 - \alpha^2 \Psi^2(x|\xi),
\]

where \( \alpha^2 \) is a Lagrange multiplier. The minimization of the objective function leads to

\[
\frac{\partial^2 \Psi(x|\xi)}{\partial y^2} + \alpha^2 \Psi(y) = 0. \tag{4}
\]

The superposition of \( \cos(\alpha \xi) \) and \( \sin(\alpha \xi) \) is a solution of Eq. (4). We get trivial and unlocalized solutions of Eq. (4).

Therefore, we need to seek localized measurable solutions. We observe that Eq. (4) has a trivial solution and
many others with the Gaussian distribution being a typical solution. To have a unique solution to Eq. (1), one must specify the domain and the local property of the solution.

We consider the Gaussian distribution in one dimension

\[ \rho(x|\xi) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(x-\xi)^2}{2\sigma^2}}, \quad (5) \]

where \( \xi = E[x] \) is the average value and \( \sigma \) is the deviation defined by \( \sigma^2 = E[(x-\xi)^2] = E[x^2] - (E[x])^2 \). Substituting Eq. (5) into Fisher information with a change of variable \( y = x - \xi \), we have

\[
\int_{-\infty}^{+\infty} dy \frac{1}{\rho(y)} \left( \frac{\partial \rho(y)}{\partial y} \right)^2 = 4 \int_{-\infty}^{+\infty} dy \left( \frac{\partial \Psi(y)}{\partial y} \right)^2.
\]

Then we optimize the Fisher information subject to the constraint, Eq. (5). In terms of \( \Psi \), i.e., \( \rho(x|\xi) = \Psi^2(x|\xi) \) in one dimension, the Fisher information reads

\[ \int dy \frac{1}{\rho(y)} \left( \frac{\partial \rho(y)}{\partial y} \right)^2 = 4 \int dy \left( \frac{\partial \Psi(y)}{\partial y} \right)^2. \]

In order to obtain an objective function for minimization, we impose Eq. (2) and Eq. (5) on the Fisher information, Eq. (3),

\[
\int d\tau \left( \frac{\partial \Psi(y)}{\partial \tau} \right)^2 + \kappa^2 \int dy y^2 \Psi^2(y) - \alpha^2 \int dy \Psi^2(y) = 0,
\]

giving the objective function

\[ \mathcal{L} = \left( \frac{\partial \Psi(y)}{\partial y} \right)^2 + \kappa^2 y^2 \Psi^2(y) - \alpha^2 \Psi^2(y), \]

where \( \kappa \) and \( \alpha \) are Lagrange multipliers. The minimization of the objective function leads to

\[ - \frac{\partial^2 \Psi(y)}{\partial y^2} + \kappa^2 y^2 \Psi(y) = \alpha^2 \Psi(y). \quad (8) \]

Eq. (8) has a solution of the Gaussian type.

B. Minkowski space in spherical coordinates

We now proceed to discuss the Minkowski metric in spherical coordinates with \( \eta_{\mu\nu} = (-c^2, 1, r^2, r^2 \sin^2 \theta) \). The line element squared reads

\[ ds^2 = -dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2, \]

where \( dt = \sqrt{-1}d\tau cdt \), and \( c \) is the speed of light. We rewrite the diagonal terms of Fisher information metric as

\[
\int d^4x \rho \left( \frac{\partial \rho}{\partial \tau} \right) \left( \frac{\partial \rho}{\partial \tau} \right) = -\int d^4x \eta^2 \rho,
\]

\[
\int d^4x \rho \left( \frac{\partial \rho}{\partial r} \right) \left( \frac{\partial \rho}{\partial r} \right) = \int d^4x \beta^2 \rho,
\]

\[
\int d^4x \rho \left( \frac{\partial \rho}{\partial \theta} \right) \left( \frac{\partial \rho}{\partial \theta} \right) = \int d^4x \gamma^2 \rho r^2,
\]

\[
\int d^4x \rho \left( \frac{\partial \rho}{\partial \phi} \right) \left( \frac{\partial \rho}{\partial \phi} \right) = \int d^4x \xi^2 \rho r^2 \sin^2 \theta.
\]

Above equations will satisfy if and only if the probability distribution \( \rho \) is separable. The volume integral \( \int_X d^4x \) is given by \( \int d\tau dr d\theta d\phi \). We define the probability distribution as \( \rho(x|\xi) = \Psi^2(x|\xi) \) with the normalization condition \( \int_X d^4x \Psi^2(x|\xi) = 1 \). Thus, \( \Psi(x|\xi) \) is a function of four parameters \( (r, \tau, \theta, \phi) \), i.e., \( \Psi = \Psi(r, \tau, \theta, \phi) \). We note that the unit vectors in the spherical polar coordinates are not constant; they depends on the position vector. In this system, the gradients in the spatial dimensions are:

\[ \nabla_r \Psi = \frac{\partial \Psi}{\partial r}, \quad \nabla_\theta \Psi = \frac{1}{r} \frac{\partial \Psi}{\partial \theta}, \quad \nabla_\phi \Psi = \frac{1}{r \sin \theta} \frac{\partial \Psi}{\partial \phi}. \]

The objective function for the Fisher information metric reads

\[ \mathcal{L} = (\nabla_r \Psi)^2 + (\nabla_\theta \Psi)^2 + (\nabla_\phi \Psi)^2 - \alpha^2 \Psi^2, \quad (9) \]

where we have imposed constraint Eq. (2) with Lagrange multiplier \( \alpha^2 \).
Assuming that $\Psi$ is separable, $\Psi(r, \theta, \phi) = R(r)\Theta(\theta)\Phi(\phi)$, we obtain

$$
\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{dR(r)}{dr} \right) + \left( \alpha^2 - \frac{\ell(\ell + 1)}{r^2} \right) R(r) = 0, \quad (11)
$$

$$
\frac{1}{\sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{d\Theta(\theta)}{d\theta} \right) + \left[ \ell(\ell + 1) - \frac{m^2}{\sin^2 \theta} \right] \Theta(\theta) = 0, \quad (12)
$$

$$
\frac{d^2\Phi(\phi)}{d\phi^2} + m^2\Phi(\phi) = 0. \quad (13)
$$

The solution which describes the universe’s wave function is

$$
\Psi(x| r, \theta, \phi) = A_j(\alpha r) \sqrt{\frac{(2\ell + 1)(\ell - |m|)!}{4\pi((\ell + |m|))!}} P^m_\ell(\cos \theta) e^{im\phi},
$$

where $\epsilon = (-1)^m$ for $m \geq 0$, and $\epsilon = 1$ for $m \leq 0$, $A$ is a normalization constant, $j_\ell(x)$ is the spherical Bessel function of order $\ell$, and $P^m_\ell$ the associated Legendre functions.

Now we consider the local property in radial estimation. For an optimal measurement, we consider the lower Cramér-Rao bound with

$$
\frac{1}{\sigma_r^2} \int dx (x - \xi^r)^2 \rho = \frac{1}{\sigma_r^2} \int dx r^2 \rho = 1.
$$

The objective function for the Fisher information metric

$$
\mathcal{L} = (\nabla_r \Psi)^2 + (\nabla_\theta \Psi)^2 + (\nabla_\phi \Psi)^2 + \kappa^2 r^2 \Psi^2 - \alpha^2 \Psi^2,
$$

where $\kappa^2$ is a multiplier. The Euler-Lagrange equation reads

$$
- \left[ \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \Psi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \Psi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \left( \frac{\partial^2 \Psi}{\partial \phi^2} \right) \right] + \kappa^2 r^2 \Psi = \alpha^2 \Psi. \quad (15)
$$

Again assuming that $\Psi$ is separable, $\Psi(r, \theta, \phi) = R(r)\Theta(\theta)\Phi(\phi)$, we obtain

$$
\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{dR(r)}{dr} \right) + \left( \alpha^2 - \kappa^2 r^2 - \frac{\ell(\ell + 1)}{r^2} \right) R(r) = 0, \quad (16)
$$

$$
\frac{1}{\sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{d\Theta(\theta)}{d\theta} \right) + \left[ \ell(\ell + 1) - \frac{m^2}{\sin^2 \theta} \right] \Theta(\theta) = 0, \quad (17)
$$

$$
\frac{d^2\Phi(\phi)}{d\phi^2} + m^2\Phi(\phi) = 0. \quad (18)
$$

The solution which describes the universe’s wave function reads

$$
\Psi(r, \theta, \phi) = B \frac{1}{2\ell + 1} \frac{1}{\sqrt{\pi}} \frac{1}{\sqrt{\pi}} L^{-\frac{\ell}{2} - \frac{1}{2}} \frac{1}{\sqrt{\pi}} (\kappa r^2) \times \epsilon \sqrt{\frac{(2\ell + 1)(\ell - |m|)!}{4\pi((\ell + |m|))!}} P^m_\ell(\cos \theta) e^{im\phi},
$$

where $\epsilon = (-1)^m$ for $m \geq 0$, and $\epsilon = 1$ for $m \leq 0$, $\ell = 0, 1, 2, ..., m = \pm \ell$, $B$ is a normalization constant and

$$
L^{-\frac{\ell}{2} - \frac{1}{2}} \frac{1}{\sqrt{\pi}} (\kappa r^2)
$$

are the generalized Laguerre polynomials.

C. Connection to quantum physics

From above derivations, we find that the universe is specified by configurations of probability distributions according to Fisher information metric. Although we have the differential equation, Eq. (10), to describe the probability distribution in the spherical Minkowski space, we still want to understand the phenomena when we put a test particle in the space. Therefore, we express the function $\Psi(x| r, \theta, \phi)$ in its momentum space by Fourier transform, as [5]
\[ \Psi(x|\tau, \theta, \phi) = \frac{1}{(2\pi\hbar)^{3/2}} \int dp_\tau dp_\theta dp_\phi \Phi(x|p_\tau, p_\theta, p_\phi)e^{-i(p_\tau \tau + p_\theta \theta + p_\phi \phi)/\hbar}, \] (19)

where we note that the dimension of Planck’s constant \( \hbar \) is same as angular momentum’s, \( \hbar = \hbar/2\pi \). We substitute Eq. (19) into Eq. (9) and get the Lagrangian as \( \mathcal{L} = \left( \frac{1}{2} \mu E_{kr} + \frac{L^2}{2r^2} - \alpha^2 \right) \Psi^2 \), where \( p_\theta \) is the canonical momentum to \( \theta \), \( L_\theta \) is the angular momentum for the variable \( \theta \), \( p_\phi \) is the canonical momentum to \( \phi \), and \( L_\phi \) is the angular momentum for the variable \( \phi \). From dimensional analysis, \( \alpha^2 \) has the dimension of energy so we choose \( \alpha^2 = E - V(r) \), where \( E \) is the total energy of the system and \( V(r) \) is the potential energy that depends on \( r \). By putting a test particle in the universe, we see that the behavior of the test particle obeys the Schrödinger equation.

### III. PROBABILITY DISTRIBUTION IN SCHWARZSCHILD GEOMETRY

The Schwarzschild metric \( g_{\mu\nu} = (-1, 1, 1, 1) \) for the four-dimensional coordinates \((\tau, r, \theta, \phi)\) with \( r_s = 2GM/c^2 \) being the Schwarzschild or gravitational radius, where \( G \) is the gravitational constant. The invariant line element

\[ ds^2 = -\left(1 - \frac{r_s}{r}\right) d\tau^2 + \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2, \]

where \( d\tau = c dt \), and \( c \) is the speed of light. We rewrite the diagonal terms of Fisher information metric

\[ \int_X d^4x \psi^2(1 - \frac{r_s}{r})^{-1/2} \frac{\partial \Psi}{\partial \tau^*}, \]
\[ \nabla_\tau \Psi = -i \left(1 - \frac{r_s}{r}\right)^{-1/2} \frac{\partial \Psi}{\partial \tau}, \]
\[ \nabla_r \Psi = \left(1 - \frac{r_s}{r}\right)^{1/2} \frac{\partial \Psi}{\partial r^*}, \]
\[ \frac{\partial \Psi}{r \partial \theta}, \nabla_\phi \Psi = \frac{1}{r \sin \theta} \frac{\partial \Psi}{\partial \phi^*}. \]

Then the objective function can be defined as

\[ \mathcal{L} = (\nabla_\tau \Psi)^2 + (\nabla_r \Psi)^2 + (\nabla_\theta \Psi)^2 + (\nabla_\phi \Psi)^2 - \alpha^2 \Psi^2. \] (20)

Eq. (20) describes a wave in the Schwarzschild space-time. Now, minimization of Eq. (20) gives

\[ \frac{1}{(1 - r_s/r)^2} \frac{\partial^2 \Psi}{\partial \tau^2} = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \left(1 - \frac{r_s}{r}\right) \frac{\partial \Psi}{\partial r} \right) - \left[ \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \Psi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \Psi}{\partial \phi^2} \right] = \alpha^2 \Psi. \] (21)

Assuming that \( \Psi(t, r, \theta, \phi) = T(\tau)R(r)\Theta(\theta)\Phi(\phi) \), we obtain

\[ \frac{d^2 T(\tau)}{d\tau^2} + \eta^2 T(\tau) = 0, \]
\[ \frac{1}{r^2} \frac{d}{dr} \left( r^2 \left(1 - \frac{r_s}{r}\right) \frac{dR(r)}{dr} + \left( \alpha^2 + \frac{\eta^2}{1 - r_s/r} - \frac{\ell(\ell + 1)}{r^2} \right) R(r) = 0, \]
\[ \frac{1}{\sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{d\Theta(\theta)}{d\theta} \right) + \left( \ell(\ell + 1) - \frac{m^2}{\sin^2 \theta} \right) \Theta(\theta) = 0, \]
\[ \frac{d^2 \Phi(\phi)}{d\phi^2} + m^2 \Phi(\phi) = 0. \] (25)
The temporal dependent solution to Eq. (22) is

\[ T(\tau) = e^{-\eta^r} \eta', \eta' = 0, \pm 1, \pm 2, \cdots. \]  (26)

The azimuthal and polar dependents solutions are

\[ \Phi(\phi) = e^{im\phi}, \quad m = 0, \pm 1, \pm 2, \cdots, \quad -\ell \leq m \leq +\ell, \]

\[ \Theta(\theta) = \epsilon \sqrt{\frac{(2\ell + 1)}{4\pi} \frac{(|m|)!}{(|\ell + |m|)!}} P^m_\ell (\cos \theta), \quad \ell = 0, 1, 2, \cdots. \]

Finally, the differential equation Eq. (23) describes the radial distribution, which does not have a simple solution. When \( 1 \gg r_s/r \), it can be approximated by the differential equation that describes the probability distribution in spherical Minkowski space, Eq. (11).

A. Is there any connection to quantum physics?

From the above derivations, we assign probability distributions to describe space-times according to Fisher information metric. Although we have the differential equation, Eq. (21), to describe the probability distribution in the Schwarzschild space-time, we should like to understand the phenomena when we put a test particle in the Schwarzschild space-time. We also like to generalize the concepts of Fisher information metric. When \( 1 \gg r_s/r \), it can be approximated by the differential equation that describes the probability distribution in spherical Minkowski space, Eq. (11).

IV. CONCLUSION

In this chapter, we have proposed a method to obtain the probability distributions for physical systems. From the relation between metric \( g_{\mu\nu} \) and Fisher information matrix, we can postulate that there exist distributions under certain constraints such that

\[ \int d^4x \left( \frac{\partial \Psi}{\partial \xi^\mu} \right)^* \left( \frac{\partial \Psi}{\partial \xi^\nu} \right) = g_{\mu\nu}(\xi), \]

where we need to consider the complex conjugate because wave function could be complex, but \( g_{\mu\nu}(\xi) \) are real.

Next, we have generated the wave function from optimizing the solution function that satisfies certain constraints. In other words, our purpose was to seek the maximum number of mutually distinguishable intermediate probabilities. From the derivation, we discover a differential equation for the wave function. For Minkowski metric, we obtained a differential equation in the Schrödinger equation form. As an example, a test particle in the Coulomb potential field has the hydrogen wave function solutions. The solutions of the hydrogen wave functions satisfy the general constraints of Fisher information metric.

After discussing the Minkowski metric, we considered the Schwarzschild metric which describes the space-time due to a spherical mass. We obtained a differential equation for the wave function in the Schwarzschild geometry. The differential equation, Eq. (21), can be solved by the method of separation of variables, with the radial differential equation, Eq. (23), remaining to be solved.

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Appendix A: Hydrogen Wave Function

With an appropriate assumption for the system, e.g., the total energy of the system is constant, we are able to get certain probability distributions, which satisfy the general constraints of Fisher information metric

\[ \int d^4x \left( \frac{\partial \Psi}{\partial \xi^\mu} \right)^* \left( \frac{\partial \Psi}{\partial \xi^\nu} \right) = g_{\mu\nu}(\xi). \]  (A1)

To prove that the solutions of Eq. (15) satisfy the constraints of Fisher information metric, we assume that there exists the potential field \( e/4\pi\epsilon_0 r \) in the space and put a negative charge test particle in the space with a total energy, \( E \). So we determine the multiplier \( \kappa^2 \) in Eq. (15) as \( \kappa^2 = E + e^2/4\pi\epsilon_0 r \), and, Eq. (15) becomes \([-\hbar^2 \nabla^2/2m - e^2/4\pi\epsilon_0 r] \Psi = E \Psi \), which is the hydrogenic Schrödinger equation. Hence we suppose the solution
of the hydrogenic Schrödinger equation should obey the constraints of Fisher information metric.

First, we rewrite the Fisher information metric elements

\[
\int_{X} d^{4}x \left( \frac{\partial \Psi}{\partial r} \right) \cdot \left( \frac{\partial \Psi}{\partial r} \right) = \int_{X} d^{3}x \beta^{2} |\Psi|^{2}, \tag{A2}
\]

\[
\int_{X} d^{4}x \left( \frac{\partial \Psi}{\partial \theta} \right) \cdot \left( \frac{\partial \Psi}{\partial \theta} \right) = \int_{X} d^{4}x \gamma^{2} |\Psi|^{2} r^{2}, \tag{A3}
\]

\[
\int_{X} d^{4}x \left( \frac{\partial \Psi}{\partial \phi} \right) \cdot \left( \frac{\partial \Psi}{\partial \phi} \right) = \int_{X} d^{4}x \zeta^{2} |\Psi|^{2} r^{2} \sin^{2} \theta, \tag{A4}
\]

where \( \beta, \gamma, \) and \( \zeta \) are multipliers. The volume integral \( \int_{X} d^{4}x \) is \( \sqrt{-g} d\tau d\theta d\phi. \) The above equations are satisfied if and only if the probability distribution \( \rho \) is separable. The objective functions for the Fisher information metric read

\[
\mathcal{L}_{r} = (\nabla_{r} \Psi)^{*}(\nabla_{r} \Psi) - \beta^{2} \Psi^{2},
\]

\[
\mathcal{L}_{\theta} = (\nabla_{\theta} \Psi)^{*}(\nabla_{\theta} \Psi) - \gamma^{2} \Psi^{2},
\]

\[
\mathcal{L}_{\phi} = (\nabla_{\phi} \Psi)^{*}(\nabla_{\phi} \Psi) - \zeta^{2} \Psi^{2}.
\]

The Euler-Lagrange equations read

\[
\frac{1}{r^{2}} \left[ \frac{\partial}{\partial r} \left( r^{2} \frac{\partial \Psi}{\partial \tau} \right) \right] + \beta^{2} \Psi = 0,
\]

\[
\frac{1}{r^{2} \sin \theta} \left[ \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \Psi}{\partial \theta} \right) \right] + \gamma^{2} \Psi = 0,
\]

\[
\frac{1}{r^{2} \sin^{2} \theta} \left[ \frac{\partial^{2} \Psi}{\partial \phi^{2}} \right] + \zeta^{2} \Psi = 0.
\]

We determine that \( \beta^{2} = \frac{\alpha^{2} - \ell(\ell + 1)}{r^{2} \sin^{2} \theta} \), \( \gamma^{2} = \frac{1}{r^{2}} [\ell(\ell + 1) - \frac{m^{2}}{\sin^{2} \theta}] \), and \( \zeta^{2} = \frac{m^{2}}{r^{2} \sin^{2} \theta} \) by comparing with Eqs. (11, 12, 13).

Let us now check the relations of Fisher information metric, Eqs. (A2, A3, A4) with the following wave function of the hydrogen atom,

\[
\Psi_{322} = \frac{1}{162\sqrt{a^{3}/2} a^{2}} e^{-r/3a} \sin^{2} \theta e^{i2\phi}.
\]

Then the elements of Fisher information metric are

\[
\int_{X} d\tau d\theta d\phi r^{2} \sin \theta \left( \frac{\partial \Psi_{322}}{\partial \tau} \right) \cdot \left( \frac{\partial \Psi_{322}}{\partial \tau} \right) = \frac{1}{45a^{2}},
\]

\[
\int_{X} d\tau d\theta d\phi r^{2} \sin \theta \left( \frac{\partial \Psi_{322}}{\partial \theta} \right) \cdot \left( \frac{\partial \Psi_{322}}{\partial \theta} \right) = 1,
\]

\[
\int_{X} d\tau d\theta d\phi r^{2} \sin \theta \left( \frac{\partial \Psi_{322}}{\partial \phi} \right) \cdot \left( \frac{\partial \Psi_{322}}{\partial \phi} \right) = 4.
\]

The \( \theta \) and \( \phi \) terms in the right-hand side of the elements of Fisher information metric are

\[
\int_{X} d\tau d\theta d\phi r^{2} \sin \theta \Psi_{322}^{*} \frac{r^{2}}{\sin^{2} \theta} \left( 2(2 + 1) - \frac{2^{2}}{\sin^{2} \theta} \right) \Psi_{322} = 1,
\]

\[
\int_{X} d\tau d\theta d\phi r^{2} \sin \theta \Psi_{322}^{*} r^{2} \sin^{2} \theta \left[ \frac{2^{2}}{r^{2} \sin^{2} \theta} \right] \Psi_{322} = 4.
\]

Consequently, we show that the wave functions of hydrogen atom satisfy the Fisher information metric, Eq. (A1).

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