A Quarter-symmetric Metric Connection on Almost Contact $B$–metric Manifolds

Şenay Bulut

Eskişehir Technical University, Science Faculty, Department of Mathematics, Eskişehir, Turkey

Abstract.

The aim of this paper is to study the notion of a quarter-symmetric metric connection on an almost contact $B$–metric manifold $(M, \varphi, \xi, \eta, g)$. We obtain the relation between the Levi-Civita connection and the quarter-symmetric metric connection on $(M, \varphi, \xi, \eta, g)$. We investigate the curvature tensor, Ricci tensor and scalar curvature tensor with respect to the quarter-symmetric metric connection. In case the manifold $(M, \varphi, \xi, \eta, g)$ is a Sasaki-like almost contact $B$–metric manifold, we get some formulas. Finally, we give some examples of a quarter-symmetric metric connection.

1. Introduction

The investigations of a quarter-symmetric metric connection in a differentiable manifold with affine connection take a central place in the study of the differential geometry. In 1975, it was defined and studied by Golab[11]. The systematic study of the quarter-symmetric metric connection was continued by [2, 3]. The quarter-symmetric metric connection in Riemannian, Kaehlerian and Sasakian manifolds was studied by [1, 4, 5, 13]. The quarter-symmetric metric connection on Riemannian manifold with an almost contact structure and pseudo-Riemannian manifolds was studied by [12, 14].

A classification of the space of the torsion tensors on almost contact $B$–metric manifolds is made in [8]. According to the classification we determine the class of the torsion tensor of the quarter-symmetric metric connection.

Sasaki-like almost contact $B$–metric manifolds was studied in [7]. We investigate the quarter-symmetric metric connection on Sasaki-like almost contact $B$–metric manifolds.

We organize the present paper as follows: Section 2 contains the basic known results of almost contact $B$–metric manifolds and Sasaki-like almost contact $B$–metric manifolds. The brief results of the quarter-symmetric metric connection on an almost contact $B$–metric manifold are given in Section 3. In Section 4, the properties of the curvature tensors corresponding to the quarter-symmetric metric connection on Sasaki-like almost contact $B$–metric manifolds are investigated. In the last section, we construct some examples of almost contact $B$–metric manifolds equipped with the quarter-symmetric metric connection and verify our results.

2010 Mathematics Subject Classification. Primary 53C05; Secondary 53C15, 53D10

Keywords. A quarter-symmetric metric connection; Almost contact $B$–metric manifold; Sasaki-like almost contact $B$–metric manifold.

Received: 19 July 2019; Revised: 24 September 2019; Accepted: 08 October 2019

Communicated by Ljubica Velimirović

Email address: skarapazar@eskisehir.edu.tr (Şenay Bulut)
Convention: Let $\mathcal{M} = (\mathcal{M}, \varphi, \xi, \eta, g)$ be a $(2n + 1)$-dimensional almost contact $B$-metric manifold. Let $\mathcal{M}$ be $(2n + 1)$-dimensional almost contact structure $(\varphi, \eta, \xi)$ consisting of an endomorphism $\varphi$ of the tangent bundle, a Reeb vector field $\xi$, its dual 1-form $\eta$ such that the following relations are satisfied:

$$\varphi^2 = -I + \eta \otimes \xi,$$

$$\eta(\xi) = 1, \quad \eta \circ \varphi = 0, \quad \varphi \xi = 0.$$  \hspace{1cm} (1)

Then, $(\mathcal{M}, \varphi, \xi, \eta)$ is called almost contact manifold. Moreover, if the almost contact manifold $(\mathcal{M}, \varphi, \xi, \eta)$ is endowed with a pseudo-Riemannian metric $g$ of signature $(n + 1, n)$ compatible with the almost contact structure in the following way

$$g(\varphi x, \varphi y) = -g(x, y) + \eta(x)\eta(y),$$

then $(\mathcal{M}, \varphi, \xi, \eta, g)$ is called almost contact $B$-metric manifold.

2. Almost Contact $B$–metric Manifolds

Let $\mathcal{M}$ be $(2n + 1)$–dimensional almost contact manifold with an almost contact structure $(\varphi, \eta, \xi)$ consisting of an endomorphism $\varphi$ of the tangent bundle, a Reeb vector field $\xi$, its dual 1–form $\eta$ such that the following relations are satisfied:

$$\varphi^2 = -I + \eta \otimes \xi,$$

$$\eta(\xi) = 1, \quad \eta \circ \varphi = 0, \quad \varphi \xi = 0.$$  \hspace{1cm} (1)

Then, $(\mathcal{M}, \varphi, \xi, \eta)$ is called almost contact manifold. Moreover, if the almost contact manifold $(\mathcal{M}, \varphi, \xi, \eta)$ is endowed with a pseudo-Riemannian metric $g$ of signature $(n + 1, n)$ compatible with the almost contact structure in the following way

$$g(\varphi x, \varphi y) = -g(x, y) + \eta(x)\eta(y),$$

then $(\mathcal{M}, \varphi, \xi, \eta, g)$ is called almost contact $B$–metric manifold.

2n–dimensional contact distribution $H = \ker \eta$, induced by the contact 1–form $\eta$, can be considered as the horizontal distribution. The restriction of $\varphi$ to $H$ is an almost complex structure, and the restriction of $g$ to $H$ is a Norden metric, i.e.,

$$g|_H(\varphi|_H(x), \varphi|_H(y)) = -g(x, y)$$

for any $x, y \in \chi(H)$. Thus, $(H, \varphi|_H)$ can be considered as $2n$–dimensional almost complex manifold with Norden metric.

The structure group of the almost contact $B$–metric manifolds is $O(n, \mathbb{C}) \times 1$, that is, $O(n, \mathbb{C}) \times 1$ consists of $(2n + 1) \times (2n + 1)$ matrices of the following type

$$\begin{pmatrix}
A & B & 0_{n \times 1} \\
-B & A & 0_{n \times 1} \\
0_{1 \times n} & 0_{1 \times n} & 1
\end{pmatrix}, \quad AA^t - BB^t = I_n, \quad AB^t + BA^t = 0_{n \times n},$$

where $A, B \in GL(n, \mathbb{R})$ and $I_n$ and $0_n$ are the unit matrix and zero matrix, respectively.

The fundamental tensor $F$ of type $(0, 3)$ on $(\mathcal{M}, \varphi, \xi, \eta, g)$ is determined by

$$F(x, y, z) = g((\nabla_x \varphi)y, z),$$

where $\nabla$ is the Levi-Civita connection of $g$. Moreover, the tensor $F$ has the following properties:

$$F(x, y, z) = F(x, z, y) = F(x, \varphi y, \varphi z) + \eta(y)F(x, \xi, z) + \eta(z)F(x, \xi, \xi),$$

$$(\nabla_x \eta)y = g(\nabla_x \xi, y) = F(x, \varphi y, \xi).$$  \hspace{1cm} (3)

2.1. Sasaki-like Almost Contact $B$–metric Manifolds

An almost contact $B$–metric manifold $(\mathcal{M}, \varphi, \xi, \eta, g)$ is called a Sasaki-like almost contact $B$–metric manifold if the tensor $F$ satisfies the following conditions:

$$F(X, Y, Z) = F(\xi, Y, Z) = F(\xi, \xi, Z) = 0,$$

$$F(X, Y, \xi) = -g(X, Y).$$

If $(\mathcal{M}, \varphi, \xi, \eta, g)$ is a Sasaki-like almost contact $B$–metric manifold, then the following conditions are given in [7]:
3. A Quarter-symmetric Metric Connection on Almost Contact B–Metric Manifolds

If $T$ is the torsion tensor of a linear connection $D$ given by

$$T(x, y) = D_x y - D_y x - [x, y],$$

then the corresponding tensor of type $(0, 3)$ is determined by

$$T(x, y, z) = g(T(x, y), z).$$

It is well-known that any metric connection $D$ is completely determined by its torsion tensor with

$$2g(D_x y - \nabla_x y, z) = T(x, y, z) - T(y, z, x) + T(z, x, y).$$

**Definition 3.1.** A linear connection $\tilde{\nabla}$ on an almost contact $B$–metric manifold is called a quarter-symmetric connection if its torsion tensor $T$ of the connection $\tilde{\nabla}$ satisfies the condition

$$T(x, y) = \eta(y)\phi x - \eta(x)\phi y.$$  

If moreover, the connection $\tilde{\nabla}$ satisfies the condition

$$(\tilde{\nabla}_x g)(y, z) = 0,$$

for all $x, y, z \in \chi(M)$, then $\tilde{\nabla}$ is called a quarter-symmetric metric connection, otherwise it is called a quarter-symmetric non-metric connection.

Let us define a connection $\tilde{\nabla}_x y$ by the following equation:

$$2g(\tilde{\nabla}_x y, z) = xg(y, z) + yg(z, x) - zg(x, y) + g([x, y], z) - g([y, z], x) + g(z, x), y) + g(\eta(y)\phi x - \eta(x)\phi y, z) + g(\eta(x)\phi z - \eta(z)\phi x, y),$$

where $x, y, z \in \chi(M)$. This connection $\tilde{\nabla}$ satisfies the following conditions:

$$\tilde{\nabla}_x (y + z) = \tilde{\nabla}_x y + \tilde{\nabla}_x z,$$
$$\tilde{\nabla}_{x+y} z = \tilde{\nabla}_x z + \tilde{\nabla}_y z,$$
$$\tilde{\nabla}_x f y = f\tilde{\nabla}_x y,$$
$$\tilde{\nabla}_x (f y) = f\tilde{\nabla}_x y + x(f)y,$$

a. The covariant derivative $\nabla\varphi$ satisfies the equality

$$(\nabla_z \varphi)y = -g(x, y)\xi - \eta(y)x + 2\eta(x)\eta(y)\xi.$$ (4)

b. The manifold $M$ is normal, i.e., $N = 0$, the fundamental 1–form $\eta$ is closed, i.e., $d\eta = 0$ and the integral curves of $\xi$ are geodesics, i.e., $\nabla_z \xi = 0$.

c. The covariant derivative $\nabla\eta$ satisfies the equality

$$(\nabla_z \eta)Y = -g(X, \varphi Y).$$

d. The 1–forms $\theta$ and $\theta^*$ satisfy the equalities $\theta = -2n \eta$ and $\theta^* = 0$.

e. $\nabla_z X = -\varphi X - [X, \xi]$.

f. $\nabla_z \xi = -\varphi x$. 

3. A Quarter-symmetric Metric Connection on Almost Contact B–Metric Manifolds

If $T$ is the torsion tensor of a linear connection $D$ given by

$$T(x, y) = D_x y - D_y x - [x, y],$$

then the corresponding tensor of type $(0, 3)$ is determined by

$$T(x, y, z) = g(T(x, y), z).$$

It is well-known that any metric connection $D$ is completely determined by its torsion tensor with

$$2g(D_x y - \nabla_x y, z) = T(x, y, z) - T(y, z, x) + T(z, x, y).$$

**Definition 3.1.** A linear connection $\tilde{\nabla}$ on an almost contact $B$–metric manifold is called a quarter-symmetric connection if its torsion tensor $T$ of the connection $\tilde{\nabla}$ satisfies the condition

$$T(x, y) = \eta(y)\phi x - \eta(x)\phi y.$$ (8)

If moreover, the connection $\tilde{\nabla}$ satisfies the condition

$$(\tilde{\nabla}_x g)(y, z) = 0,$$ (9)

for all $x, y, z \in \chi(M)$, then $\tilde{\nabla}$ is called a quarter-symmetric metric connection, otherwise it is called a quarter-symmetric non-metric connection.

Let us define a connection $\tilde{\nabla}_x y$ by the following equation:

$$2g(\tilde{\nabla}_x y, z) = xg(y, z) + yg(z, x) - zg(x, y) + g([x, y], z) - g([y, z], x) + g(z, x), y) + g(\eta(y)\phi x - \eta(x)\phi y, z) + g(\eta(x)\phi z - \eta(z)\phi x, y),$$ (10)

where $x, y, z \in \chi(M)$. This connection $\tilde{\nabla}$ satisfies the following conditions:

$$\tilde{\nabla}_x (y + z) = \tilde{\nabla}_x y + \tilde{\nabla}_x z,$$
$$\tilde{\nabla}_{x+y} z = \tilde{\nabla}_x z + \tilde{\nabla}_y z,$$
$$\tilde{\nabla}_x f y = f\tilde{\nabla}_x y,$$
$$\tilde{\nabla}_x (f y) = f\tilde{\nabla}_x y + x(f)y,$$ (11)
for all $x, y, z \in \chi(M)$ and $f \in C^\infty(M)$. Therefore, the connection $\nabla$ determines a linear connection on $(M, g)$. According to (10), we have the following relation:

$$g(\nabla_x y, z) - g(\nabla_y x, z) = g([x, y], z) + g(\eta(y)\varphi x - \eta(x)\varphi y, z).$$

(12)

Then, we get

$$T(x, y) = \nabla_x y - \nabla_y x - [x, y] = \eta(y)\varphi x - \eta(x)\varphi y.$$  

(13)

Moreover, it can be easily verified that $\nabla$ is compatible with the metric $g$ on $M$, i.e.,

$$\nabla g = 0.$$  

(14)

$\nabla$ determines metric connection on $(M, \varphi, \xi, \eta, g)$.

**Theorem 3.2.** If $(M, \varphi, \xi, \eta, g)$ is an almost contact $B$–metric manifold, then there exists a unique linear connection $\nabla$ satisfying the conditions (13) and (14).

Now we give a relation between the Levi-Civita connection $\nabla$ and the quarter-symmetric metric connection $\nabla$ on $(M, \varphi, \xi, \eta, g)$. Let

$$\nabla_x y = \nabla_x y + U(x, y),$$

where $U(x, y)$ is a tensor of type $(1, 2)$. It can be seen that

$$U(x, y) = \frac{1}{2}[T(x, y) + S(x, y) + S(y, x)],$$

where

$$g(S(x, y), z) = g(T(z, x), y).$$

From (13) we get

$$U(x, y) = \eta(y)\varphi x - \eta(qx, y)\xi.$$  

(15)

Hence, a quarter-symmetric metric connection $\nabla$ on $(M, \varphi, \xi, \eta, g)$ is given by

$$\nabla_x y = \nabla_x y + \eta(y)\varphi x - \eta(qx, y)\xi.$$  

(16)

Conversely, it is easy to show that a linear connection $\nabla$ on $(M, \varphi, \xi, \eta, g)$ defined by (16) determines a quarter-symmetric metric connection.

If $T$ is the torsion tensor of a quarter-symmetric metric connection $\nabla$, then the corresponding tensor of type $(0, 3)$ is given by

$$T(x, y, z) = \eta(y)g(\varphi x, z) - \eta(x)g(\varphi y, z).$$

(17)

In particular, we have

$$T(\varphi x, \varphi y, z) = 0, \text{ and}$$

$$T(x, y, \xi) = 0.$$  

(18)

(19)

The classification of the space of the torsion tensors with respect to almost contact $B$–metric structure is made in [8]. The class of the torsion tensor corresponding to the quarter-symmetric metric connection is determined in the following.

**Proposition 3.3.** The torsion $T$ of the quarter-symmetric metric connection $\nabla$ on $(M, \varphi, \xi, \eta, g)$ belongs to $T_{10}$. 


Proof. Let $\mathcal{T}$ be a vector space of all tensors $T$ of type $(0,3)$ over $T_p(M)$ having skew-symmetry by the first two arguments, i.e.,

$$\mathcal{T} = \{T(x, y, z) \in \mathbb{R}|T(x, y, z) = -T(y, x, z), x, y, z \in T_pM\}.$$ 

Firstly, we have the operator $p_1: \mathcal{T} \rightarrow \mathcal{T}$ by

$$p_1(T)(x, y, z) = -T(q^2x, q^2y, q^2z).$$

We have the following orthogonal decomposition of $\mathcal{T}$ by the image and the kernel of $p_1$:

$$W_1 = \text{im}(p_1) = \{T \in \mathcal{T} \mid p_1(T) = T\}, \quad W_2 = \ker(p_1) = \{T \in \mathcal{T} \mid p_1(T) = 0\}.$$ 

From (18) we obtain $p_1(T) = 0$, namely, $T \in \ker(p_1) = W_1^\perp$. Now consider the operator $p_2: W_1^\perp \rightarrow W_1^\perp$ defined by

$$p_2(T)(x, y, z) = \eta(z)T(q^2x, q^2y, \xi) = \eta(z)g(T(q^2x, q^2y), \xi).$$

(20)

Since $p_2 \circ p_2 = p_2$, we have the following decomposition of $W_1^\perp$:

$$W_2 = \text{im}(p_2) = \{T \in W_1^\perp \mid p_2(T) = T\}, \quad W_2^\perp = \ker(p_2) = \{T \in W_1^\perp \mid p_2(T) = 0\}.$$ 

According to (19) we get $p_2(T) = 0$, that is, $T \in \ker(p_2) = W_2^\perp$. We consider the operator $p_3: W_2^\perp \rightarrow W_2^\perp$ defined by

$$p_3(T)(x, y, z) = \eta(x)T(\xi, q^2y, q^2z) + \eta(y)T(q^2x, \xi, q^2z).$$

By using the equalities given in (2), (6) and (13), the above equality is written in the form

$$p_3(T)(x, y, z) = \eta(x)T(\xi, q^2y, q^2z) + \eta(y)T(q^2x, \xi, q^2z) = \eta(x)g(\eta y, q^2z) + \eta(y)g(\xi, q^2z) = -\eta(x)g(\eta y, q^2z) + \eta(y)g(\xi, q^2z)$$

(21)

Then, $p_3(T) = T$, that is, $T \in \text{im}(p_3) = W_3$. The following operators $L_{3,0}$ and $L_{3,1}$ are involutive isometries on $W_3$:

$$L_{3,0}(T)(x, y, z) = \eta(x)T(\xi, q^2y, q^2z) - \eta(y)T(\xi, q^2x, q^2z),$$

$$L_{3,1}(T)(x, y, z) = \eta(x)T(\xi, q^2x, q^2z) - \eta(y)T(\xi, q^2x, q^2z).$$

(22)

From (17) we get $L_{3,0}(T) = -T$, namely, $T \in W_3^\perp$ and $L_{3,1}(T) = T$, namely, $T \in W_3$. Where

$$W_3 = \{T \in W_3 \mid L_{3,0}(T) = -T\}, \quad W_3 = \{T \in W_3 \mid L_{3,1}(T) = T\}.$$ 

The torsion forms $t$ and $t'$ of $T$ are defined by

$$t(x) = g^{ij}T(x, e_i, e_j),$$

$$t'(x) = g^{ij}T(x, e_i, qe_j),$$

with respect to the basis $\{\xi, e_1, \ldots, e_2n\}$, respectively. By using the torsion tensor $T$ in (17) the torsion forms $t$ and $t'$ can be easily calculated as $t = 0, t' \neq 0$. Hence, $T \in W_{3,1} = T_{10}$ where

$$W_{3,1} = \{T \in W_3 \mid t = 0, t' \neq 0\}.$$ 

□
Note that for almost contact B–metric manifold \((M, \varphi, \xi, \eta, g)\) with respect to the basis \([\xi, e_1, \ldots, e_{2n}]\) we have the following relation:
\[
\tilde{\nabla}_x y = \nabla_x y, \quad \tilde{\nabla}_x \xi = \nabla_x \xi + \varphi x.
\]

(23)

Let \((M, \varphi, \xi, \eta, g)\) be an almost contact B–metric manifold. The curvature tensor of type \((1, 3)\) is defined by
\[
R(x, y)z = \nabla_x \nabla_y z - \nabla_y \nabla_x z - \nabla_{[x,y]} z.
\]

If \(R(x, y, z, w) = g(R(x, y)z, w)\), then the Ricci tensor \(Ric\), the scalar curvature \(Scal\) and \(\ast\) scalar curvature \(Scal^*\) are, respectively, defined by
\[
Ric(x, y) = \sum_{i=0}^{2n} \varepsilon_i Ric(e_i, x, y, e_i),
\]
\[
Scal = \sum_{i=0}^{2n} \varepsilon_i Ric(e_i, e_i),
\]
\[
Scal^* = \sum_{i=0}^{2n} \varepsilon_i Ric(e_i, \varphi e_i).
\]

(24)

For more details see [6, 7].

It is well-known that the manifold \((M, \varphi, \xi, \eta, g)\) is called Einstein if the Ricci tensor \(Ric\) is proportional to the metric tensor \(g\), i.e. \(Ric = \lambda g\), \(\lambda \in \mathbb{R}\). Moreover, the manifold \(M\) is called an \(\eta\)–complex-Einstein manifold if the Ricci tensor \(Ric\) satisfies the condition
\[
Ric = \lambda g + \mu \varphi g + \nu \eta \otimes \eta,
\]

(25)

where \(\lambda, \mu, \nu \in \mathbb{R}\) and \(\varphi(x, y) = g(x, \varphi y) + \eta(x)\eta(y)\). If \(\mu = 0\), we call \(M\) an \(\eta\)–Einstein manifold.

The relation between curvature tensors with respect to the Levi-Civita connection and the quarter-symmetric metric connection on almost contact B–metric manifold \((M, \varphi, \xi, \eta, g)\) is given by
\[
\tilde{R}(x, y)z = R(x, y)z + \eta(z)\nabla_x (\varphi y) - \nabla_y (\varphi x) - \varphi [x, y]
- g(z, \varphi y + \nabla_x \xi)\varphi x + g(z, \varphi x + \nabla_x \xi)\varphi y
- g(\nabla_x (\varphi y) - \nabla_y (\varphi x) - \varphi [x, y], z)\xi - g(\varphi y, z)\nabla_x \xi + g(\varphi x, z)\nabla_y \xi,
\]

(26)

where \(\nabla_x (\varphi y) - \nabla_y (\varphi x) - \varphi [x, y] = (\nabla_x \varphi) y - (\nabla_y \varphi) x\).

When the structures are \(\nabla\)–parallel, i.e. \(\nabla \varphi = \nabla \xi = \nabla \eta = \nabla g = 0\), the almost contact B–metric manifold belongs to the class \(\mathcal{F}_0\). If the almost contact B–metric manifold is in the special class \(\mathcal{F}_0\), then the relation between the curvature tensors \(\tilde{R}\) and \(R\) is given by
\[
\tilde{R}(x, y)z = R(x, y)z - g(z, \varphi y)\varphi x + g(z, \varphi x)\varphi y.
\]

(27)

4. A Quarter-symmetric Metric Connection on Sasaki-like Almost Contact B–Metric Manifolds

There is considerable interest in natural connections having some additional geometric or algebraic properties about their torsion[9]. In this section we show that the quarter-symmetric metric connection \(\nabla\) on Sasaki-like almost contact B–metric manifolds is a natural connection and investigate curvature properties on these manifolds.

Theorem 4.1. The quarter-symmetric metric connection \(\nabla\) on Sasaki-like almost contact B–metric manifolds is a natural connection, i.e. \(\nabla \varphi = \nabla \xi = \nabla \eta = \nabla g = 0\).
Proof. By using (4) we obtain the following:

\[
\begin{align*}
(\tilde{\nabla}_x \varphi)y &= \tilde{\nabla}_x (\varphi y) - \varphi(\tilde{\nabla}_x y) \\
&= \tilde{\nabla}_x (\varphi y) - \varphi(\varphi x, \varphi y)\xi - \varphi(\tilde{\nabla}_x y) - \eta(y)\varphi^2 x \\
&= (\tilde{\nabla}_x \varphi)y + \varphi(x, y)\xi - \eta(x)\eta(y)\xi + \eta(y)x - \eta(x)\eta(y)\xi \\
&= 0.
\end{align*}
\]

The equality \( \tilde{\nabla}_\xi = \tilde{\nabla}_\eta = \tilde{\nabla}_1 = 0 \) follow immediately from (14), (28) and second part of (3).

The following Proposition given in [7] gives some properties of Sasaki-like almost contact \( B^- \) metric manifolds with Levi-Civita connection.

**Proposition 4.2.** On a Sasaki-like almost contact \( B^- \) metric manifold \( (M, \varphi, \xi, \eta, 1) \) the following formulas hold:

a. \( R(x, y)\xi = \eta(y)x - \eta(x)y \).

b. \([X, \xi] \in H\).

c. \( \nabla_\xi X = -\varphi X - [X, \xi] \in H\).

d. \( R(\xi, X)\xi = -X \).

e. \( \text{Ric}(y, \xi) = 2\eta(y) \).

f. \( \text{Ric}(\xi, \xi) = 2n \).

If \((M, \varphi, \xi, \eta, g)\) is a Sasaki-like almost contact \( B^- \) metric manifold, then we have the following relation:

\[
\tilde{R}(x, y)z = R(x, y)z - \eta(y)\eta(z)x + \eta(x)\eta(z)y + \eta(y)\varphi(x, z)\xi - \eta(x)\varphi(y, z)\xi + g(\varphi y, z)\varphi x - g(\varphi x, z)\varphi y.
\]

(29)

Set \( z = \xi \) into (29) and use (1) to obtain

\[
\tilde{R}(x, y)\xi = R(x, y)\xi - \eta(y)x + \eta(x)y.
\]

(30)

Set \( x = \xi \) and \( y \to x \) into (30) to get

\[
\tilde{R}(\xi, x)\xi = R(\xi, x)\xi - \eta(x)\xi + x.
\]

Moreover, for any \( X \in \chi(H) \) the above equality implies by Proposition (4.2)(d)

\[
\tilde{R}(\xi, X)\xi = 0.
\]

By virtue of the identity in the Proposition (4.2)(f) we have

\[
\tilde{\text{Ric}}(\xi, \xi) = \text{Ric}(\xi, \xi) - 2n = 0.
\]

(31)

5. Examples

In this section we construct a number of examples of almost contact \( B^- \) metric manifolds with the quarter-symmetric metric connection.
5.1. Example 1

Consider the real connected 3-dimensional Lie group \( L \) with the left invariant vector fields \( \{\xi = e_0, e_1, e_2\} \). Let the non-zero commutators of the corresponding Lie algebra be
\[
[\xi, e_1] = ae_2, \quad [\xi, e_2] = -ae_1,
\]
where \( a \in \mathbb{R} \). Let an almost contact \( B \)-metric structure be defined by
\[
g(e_0, e_0) = g(e_1, e_1) = -g(e_2, e_2) = 1, \quad \varphi(e_1) = e_2, \quad \varphi(e_2) = -e_1, \quad \varphi(\xi) = 0, \quad \eta(\xi) = 1. \tag{32}
\]
It can be easily shown that \( L \) is an almost contact \( B \)-metric manifold. The non-zero components of the Levi-Civita connection \( \nabla \) and the quarter-symmetric metric connection \( \tilde{\nabla} \) are respectively given by
\[
\begin{align*}
\nabla_{e_1}\xi &= -ae_2, \quad \nabla_{e_2}\xi = ae_1, \quad \nabla_{e_2}e_1 = -ae_1, \\
\tilde{\nabla}_{e_1}\xi &= (1-a)e_2, \quad \tilde{\nabla}_{e_2}\xi = (a-1)e_1, \\
\tilde{\nabla}_{e_2}e_1 &= (1-a)e_2.
\end{align*}
\]

The non-zero components of the curvature tensors \( R \) and \( \tilde{R} \) corresponding to the connections \( \nabla \) and \( \tilde{\nabla} \) are given by
\[
\begin{align*}
R_{010} &= -a^2e_1, \quad R_{020} = -a^2e_2, \quad R_{011} = a^2\xi, \\
R_{022} &= -a^2\xi, \quad R_{121} = a^2e_2, \quad R_{122} = a^2e_1, \\
\tilde{R}_{010} &= (a-a^2)e_1, \quad \tilde{R}_{020} = (a-a^2)e_2, \quad \tilde{R}_{011} = (a^2-a)\xi, \\
\tilde{R}_{022} &= (a-a^2)\xi, \quad \tilde{R}_{121} = (a-1)^2e_2, \quad \tilde{R}_{122} = (a-1)^2e_1.
\end{align*}
\]
Moreover, the non-zero-components of Ricci tensors \( \text{Ric} \) and \( \tilde{\text{Ric}} \) can be easily calculated as
\[
\begin{align*}
\text{Ric}_{00} &= 2a^2, \quad \text{Ric}_{11} = \text{Ric}_{22} = 0, \\
\tilde{\text{Ric}}_{00} &= 2(a^2-a), \quad \tilde{\text{Ric}}_{11} = a-1, \quad \tilde{\text{Ric}}_{22} = 1-a.
\end{align*}
\]

By using above components of \( R \) and \( \tilde{R} \) the scalar curvatures \( \text{Scal}, \tilde{\text{Scal}} \) and \( \tilde{\text{Scal}}^c \) can be easily calculated as
\[
\text{Scal} = 2a^2, \quad \tilde{\text{Scal}} = 2a^2 - 2, \quad \tilde{\text{Scal}}^c = 0.
\]
The Ricci tensor \( \tilde{\text{Ric}} \) satisfies the condition
\[
\text{Ric} = \tilde{\text{Ric}} = (a-1)g + (2a^2 - 3a + 1)\eta \otimes \eta.
\]
Then, the manifold \( M \) is an \( \eta \)-Einstein manifold.

In particular, in case of \( a = 1 \) we get \( \tilde{R} = 0 \). Namely, \( L \) has a flat quarter-symmetric metric connection.

5.2. Example 2

In \cite{10} a real connected Lie group \( L \) as a manifold from the class \( \mathcal{F}_5 \) is introduced. Now we consider this example. In this case, \( L \) is a 3–dimensional real connected Lie group and its associated Lie algebra with a global basis \( \{\xi = e_0, e_1, e_2\} \) of the left invariant vector fields on \( L \) is defined by
\[
[\xi, e_1] = ae_2, \quad [\xi, e_2] = ae_1, \quad [e_1, e_2] = 0,
\]
where \( \lambda \in \mathbb{R} \). Let an almost \( B \)-metric contact structure be defined by (32). Then, \( (L, \varphi, \xi, \eta, g) \) is a 3–dimensional almost contact \( B \)-metric manifold.
By using the Koszul formula, we get the non-zero covariant derivatives of \( e_i \) with respect to the Levi-Civita connection \( \nabla \) and the quarter-symmetric metric connection \( \bar{\nabla} \) as follows:

\[
\nabla \alpha e_1 = -\nabla \alpha e_2 = \alpha \xi, \quad \nabla \alpha \xi = -\alpha e_1, \quad \nabla \xi \xi = -\alpha e_2,
\]

\[
\bar{\nabla} \alpha e_1 = -\bar{\nabla} \alpha e_2 = \alpha \xi, \quad \bar{\nabla} \alpha \xi = -\alpha e_1 + e_2, \quad \bar{\nabla} e_2 \xi = -\alpha e_2 - e_1,
\]

The non-zero components of the curvature tensor \( R \) and \( \tilde{R} \) are respectively given by

\[
R_{010} = R_{122} = a^2 e_1, \quad R_{020} = R_{121} = a^2 e_2, \quad R_{022} = -R_{011} = a^2 \xi,
\]

\[
\tilde{R}_{010} = \alpha^2 e_1 - \alpha e_2, \quad \tilde{R}_{020} = \alpha^2 e_2 + \alpha e_1, \quad \tilde{R}_{121} = (a^2 + 1)e_2, \quad \tilde{R}_{021} = \tilde{R}_{012} = -\alpha \xi, \quad \tilde{R}_{011} = -\tilde{R}_{022} = -\alpha^2 \xi, \quad \tilde{R}_{122} = (a^2 + 1)e_1.
\]

The non-zero components of the Ricci tensors \( \text{Ric} \) and \( \tilde{\text{Ric}} \) with respect to the connections \( \nabla \) and \( \bar{\nabla} \) can be easily calculated as follows:

\[
\text{Ric}_{00} = \text{Ric}_{11} = -\text{Ric}_{22} = -2a^2, \quad \text{Ric}_{00} = -2a^2, \quad \text{Ric}_{22} = 2a^2 + 1, \quad \text{Ric}_{11} = -a.
\]

The scalar curvatures \( \text{Scal} \) and \( \tilde{\text{Scal}} \) with respect to \( \nabla \) and \( \bar{\nabla} \) are computed by \( \text{Scal} = -6a^2 \) and \( \tilde{\text{Scal}} = -6a^2 - 2 \), respectively. \( \text{Scal} \) and \( \tilde{\text{Scal}} \) are negative for all \( a \in \mathbb{R} \). While *scalar curvature* \( \text{Scal}^* \) with respect to \( \nabla \) is zero, *scalar curvature* \( \tilde{\text{Scal}}^* \) with respect to \( \bar{\nabla} \) is \(-2a\). The Ricci tensor \( \tilde{\text{Ric}} \) with respect to the Levi-Civita connection \( \nabla \) satisfies the condition

\[
\tilde{\text{Ric}} = -2a^2 g.
\]

Then, the Lie group \( L \) is an Einstein manifold. The Ricci tensor \( \tilde{\text{Ric}} \) with respect to the quarter-symmetric metric connection \( \bar{\nabla} \) satisfies the condition

\[
\tilde{\text{Ric}} = (-2a^2 - 1)g + a \tilde{g} + (1 - a) \eta \otimes \eta.
\]

Then, the Lie group \( L \) with respect to the quarter-symmetric metric connection \( \bar{\nabla} \) is an \( \eta \) complex-Einstein manifold.

### 5.3. Example 3

Let us consider the Lie group \( G \) of dimension 5 with a basis of left-invariant vector fields \( \{ \xi = e_0, e_1, e_2, e_3, e_4 \} \) defined by the commutators

\[
[\xi, e_1] = \lambda e_2 + \mu e_4 + e_3, \quad [\xi, e_2] = -\lambda e_1 - \mu e_3 + e_4,
\]

\[
[\xi, e_3] = -e_1 - \mu e_2 + \lambda e_4, \quad [\xi, e_4] = \mu e_1 - e_2 - \lambda e_3,
\]

where \( \lambda, \mu \in \mathbb{R} \). Define an invariant almost contact \( B \)-metric structure on \( G \) by

\[
g(e_1, e_1) = g(e_2, e_2) = -g(e_3, e_3) = -g(e_4, e_4) = g(\xi, \xi) = 1, \quad g(e_5) = e_5, \quad g(\xi) = 0, \quad \eta (\xi) = 1.
\]

By using the Koszul formula the non-zero connection 1-forms of the Levi-Civita connection \( \nabla \) are calculated in ([7]) as follows:

\[
\nabla_1 e_1 = \lambda e_2 + \mu e_4, \quad \nabla_2 e_2 = -\lambda e_1 - \mu e_3,
\]

\[
\nabla_3 e_3 = -\mu e_2 + \lambda e_4, \quad \nabla_4 e_4 = \mu e_1 - \lambda e_3,
\]

\[
\nabla_5 e_5 = -e_2, \quad \nabla_5 \xi = -e_4, \quad \nabla_5 e_1 = e_1, \quad \nabla_5 e_3 = e_2.
\]

\[
\nabla_5 e_1 = \nabla_5 e_2 = \nabla_5 e_3 = \nabla_5 e_4 = \nabla_5 e_5 = -\xi.
\]
It can be easily checked that the constructed manifold $(G, \varphi, \xi, \eta, g)$ is a Sasaki-like almost contact $B$-metric manifold. A quarter-symmetric metric connection $\tilde{\nabla}$ on $(G, \varphi, \xi, \eta, g)$ is given by (20). The non-zero connection 1-forms of the quarter-symmetric metric connection $\tilde{\nabla}$ can be calculated as follows:

\[
\begin{align*}
\tilde{\nabla}_\xi e_1 &= \lambda e_2 + \mu e_4, \\
\tilde{\nabla}_\xi e_2 &= -\lambda e_1 - \mu e_3, \\
\tilde{\nabla}_\xi e_3 &= -\mu e_2 + \lambda e_4, \\
\tilde{\nabla}_\xi e_4 &= \mu e_1 - \lambda e_3.
\end{align*}
\] (45)

Then, we get $\tilde{R}(e_i, e_j)e_k = 0$ for $i, j, k = 0, \ldots, 4$. That is, $\tilde{R} = 0$. Hence, the manifold $(G, \varphi, \xi, \eta, g)$ has a flat quarter-symmetric metric connection. In particular, if we take $\lambda = 0$ and $\mu = 0$, then it can be verified that all covariant derivatives $\tilde{\nabla}_e e_j$ are zero, that is, $\tilde{\nabla} = 0$. The curvature tensor $R$ with respect to the Levi-Civita connection $\nabla$ is not zero but, $\tilde{R} = 0$.

Kaynaklar

[1] K. Yano, T. Imai, Quarter-symmetric metric connections and their curvature tensors, Tensor, N.S. 38 (1982) 13–18.
[2] S. C. Rastogi, On quarter-symmetric metric connection, C. R. Acad. Sci. Bulgar 31 (1978) 811–814.
[3] S. C. Rastogi, On quarter-symmetric metric connection, Tensor, 44(2) (1987) 133–141.
[4] A. K. Mondal, U. C. De, Some properties of a quarter symmetric connection on a Sasakian manifold, Bull. Math. Anal. Appl. 3 (2009) 99–108.
[5] U. C. De, J. Sengupta, Quarter-symmetric metric connection on a Sasakian manifold, Commun. Fac. Sci. Univ. Ank. Series A1 49 (2000) 7–13.
[6] B. O’Neill, Semi-Riemannian Geometry, Academic Press, New York, 1983.
[7] S. Ivanov, H. Manev, M. Manev, Sasaki-like almost contact complex Riemannian manifolds, J. Geo. and Phys 107(2016) 136–148.
[8] M. Manev, M. Ivanova, A classification of the torsion tensors on almost contact manifolds with $B$-metric, Central European Journal of Mathematics, 12(10) (2014) 1416–1432.
[9] M. Manev, M. Ivanova, Canonical-type connection on almost contact manifolds with $B$–metric, Ann. Glob. Anal. Geom. 43 (2013) 397–408.
[10] H. Manev, D. Mekerov, Lie groups as 3-dimensional almost contact $B$-metric manifolds, J. Geo. 106 (2015) 229–242.
[11] S. Golab, On semi-symmetric and quarter-symmetric linear connections, Tensor N.S., 29 (1975) 249–254.
[12] I. E. Hirica, L. Nicolescu, On quarter-symmetric metric connections on pseudo-Riemannian manifolds, Balkan Journal of Geo. 16(1) (2011) 56–65.
[13] R. S. Mishra, S. N. Pandey, On quasi-symmetric metric F-connections, Tensor, N.S. 34 (1980) 1–7.
[14] S. Mukhopadhyay, A. K. Roy, B. Barua, Some properties of a quarter-symmetric metric connection on a Riemannian manifold, Soochow J. of Math. 17(2) (1992) 205–211.