Ising Model Scaling Behaviour on $z$-Preserving Small-World Networks

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Abstract

We have investigated the anomalous scaling behaviour of the Ising model on small-world networks based on 2- and 3-dimensional lattices using Monte Carlo simulations. Our main result is that even at low $p$, the shift in the critical temperature $\Delta T_c$ scales as $p^s$, with $s \approx 0.50$ for 2-D systems, $s \approx 0.698$ for 3-D and $s \approx 0.75$ for 4-D. We have also verified that a $z$-preserving rewiring algorithm still exhibits small-world effects and yet is more directly comparable with the conventional Ising model; the small-world effect is due to enhanced long-range correlations and not the change in effective dimension. We find the critical exponents $\beta$ and $\nu$ exhibit a monotonic change between an Ising-like transition and mean-field behaviour in 2- and 3-dimensional systems.

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I. INTRODUCTION

Small-world networks \cite{1} are now an important class of complex network and have been used for many purposes ranging from social network models \cite{2} to quantum gravity \cite{3}. Complex networks have many unusual properties \cite{4} but one of the most valuable aspects of the small-world model is that it allows a parameterized interpolation between those properties exhibited by a random graph \cite{5} and those by regular lattices. Understanding the process that lead to small-world networks is non-trivial \cite{6,7} and one important approach involves comparing the critical phase transitional behaviour of small-world systems with well-studied phase transition models. The Ising model provides a valuable test model for studying phase transitions on various networks. Study of the 1-D Ising model \cite{8} has led to some insights and relatively recent work by Herrero \cite{9,10} and others \cite{11} raises some interesting questions about the behaviour of the Ising model on small-world rewired lattices.

The Ising model can be adapted in a number of ways such as adding long-range weak interactions \cite{12}, however we are interested in preserving the underpinning model structure and consider solely the effect on spatial distortions and in particular shortcuts. It is still unclear whether real physical materials exhibit small-world magnetic properties \cite{13} but one can speculate about small-world “shortcut” effects that might arise when effectively one-dimensional structures such as protein chains are folded. Two-dimensional crumpled sheet systems embedded in a three-dimensional space could also exhibit real-space shortcuts. We might further speculate that higher dimensional systems such as quantum gravity models, that are not restricted to three Euclidean dimensions, could also exhibit physically relevant small-world transitional behaviour.

The Ising model is usually formulated on a regular $d$-dimensional hyper-cubic lattice where each magnetic spin variable is connected to $2 \times d$ nearest neighbouring sites. The model Hamiltonian is usually written in the form:

$$ H = - \sum_{i \neq j} J_{ij} S_i S_j \quad (1) $$

where $S_i = \pm 1$, $i = 1, 2, \ldots, N$ sites, and $J_{ij}$ is $|J| = 1/k_B T$ is the ferromagnetic coupling over neighbouring sites $i,j$ on the network.

The model has a critical temperature of $T_c = \frac{1}{J k_B} = 0$ in one dimension, but displays finite transition temperatures due to the spontaneous magnetization effects in higher dimensions.
Specifically in two- and three-dimensions the Ising-type phase transitions have been very well studied and the critical temperatures are known exactly in two dimensions \[14\] and approximately from computer simulations such as \[15\] in three dimensions. In systems of four dimensions and higher a finite temperature phase transition still occurs, but the nature of the transition is well described by mean-field theory. Values for the transition temperature in these higher dimensional systems are also known.

The regular lattice Ising model can be perturbed in a number of controlled ways: by removing bonds or “damaging” the system; by rewiring the bonds; or by adding extra bonds. Herrero and others have employed the Watts-Strogatz rewiring model which preserves the number of total bonds, but does not preserve the number of bonds connecting a given site. Svenson and Johnson explored damage spreading in small-world Ising systems \[16\], again using the Watt-Strogatz approach.

Generally, the Ising model phase transitional temperature is systematically shifted by these damaged, rewired or added links. In the case of the small-world rewiring, individual sites can become connected to sites very far away physically. These long-range bonds encourage and help the long-range correlations that manifest the spontaneous magnetization and hence give rise to the peculiarly Ising-like critical phenomena. The Ising model critical temperature rises as long-range order is encouraged by the rewiring. Physically, it is easier for the system to maintain long-range order against the thermal disordering effects of higher temperatures than it would be without the rewired long-range bonds. This can be measured as a monotonic dependence of the critical temperature \(T_c(p)\) on the small-wiring rewiring probability \(p\).

It has been speculated that under the Watts-Strogatz small-world rewiring this shift in \(T_c\) is partially due to the effective change in the dimensionality of the system as individual sites can have more or less than \(z = 2 \times d\) bond-connected neighbours. In this paper we explore an alternative small-world rewiring model that adjusts pairs of bonds and is able to preserve exactly the coordination number \(z\) for all spin sites.

This means that as far as individual spin-sites are concerned their environment is locally identical to the conventional \(z = 2 \times d\) lattice-based Ising model. We explore the behaviour of the Ising model under such a rewiring and compare it with the Watts-Strogatz rewiring studied by Herrero and others. We are able to show that the small-world effect is not dependent on an effective dimension change but solely on the long-range nature of the
interactions themselves.

In our model, spins still have precisely \( z \) “nearest neighbour” sites to which they are directly connected and with which they are topologically directly coupled via the Ising model coupling parameter \( J \). However in terms of a physical space interpretation, the rewired links mean that nearest neighbouring sites can now be at arbitrary physical distances apart. This can be interpreted as filling physical space with “worm-holes” or applying an elaborate manifold folding of physical space.

Like that of [9], our study is based on Monte Carlo simulations of the Ising system on rewired lattices. We have been able to study larger systems than prior work for both two- and three-dimensions and have also made some preliminary studies of four- and five dimensional systems. Of particular importance is the need to study systems that are large enough to support the so called small-\( p \) regime. It transpires that the power law dependence on important properties of the model on the rewiring probability parameter \( p \) requires an analysis across logarithmic system sizes and length scales. Consequently we have needed to simulate large Ising systems of up to \( 1024^2 \) and \( 384^3 \).

Equilibration requirements for large system sizes over a large parameter space set of \( p \) and \( T \) values needed a faster Monte Carlo simulation algorithm than the local Metropolis method used by prior small-world Ising work. We therefore have explored non-local cluster-based updating methods such as that of Wolff [17] for our simulations. We certainly expected the Wolff cluster method to perform entirely correctly on a \( z \)-preserving rewiring model, and indeed it does. We have also, however, verified that it produces results on the non-\( z \)-preserving Watts-Strogatz rewired system that are consistent with those using the localised Metropolis updating method.

The main question of interest is how the critical phenomenological behaviour [18] of the model changes due to the small-world links. Approximate theoretical models for the \( p \)-concentration dependence of \( T_c \), such as those given in [19], depend on arguments based on domain wall formation energies in the Ising model. In the work reported here, we show that the critical temperature is monotonically shifted upwards from its regular lattice value in \( d = 2, 3, 4, 5 \) and that it is very-well characterised by a power law in \( p \). We also consider the critical exponents \( \beta \) and \( \nu \) and our data shows a gradual but monotonic change with increasing \( p \) from an Ising-like phase transition to a mean-field-like transition in both two- and three-dimensions.
Although a great deal of practical lore exists in the literature concerning the operational methods of obtaining critical values for the regular Ising model, it is more difficult to obtain corresponding values for small-world systems due to less well studied $p$-regimes and the need for appropriate averaging of rewired lattice configurations. We therefore describe our computational method in detail in section II. Our main results concern the nature of the rewiring models and the interpretation of the parameter $p$ and in section III we describe our rewiring algorithm. In section IV we present some results of the Monte Carlo simulations and for some static network properties such as the maximum path length in a rewired lattice. Finally we suggest some conclusions and areas for further simulation in section V.

II. COMPUTATIONAL METHOD

Herrero noted the uncertainty as to whether his simulated system sizes were sufficiently large to truly be able to explore the “small-$p$” regime. We have experimented with a variety of system sizes in both two- and three-dimensions. We believe a useful rule of thumb is that at least 100 of the system’s bonds need to be rewired to have a reliable and measurable effect. This places a practical lower limit on the $p$-values that can be explored, given the practical upper limits on system sizes that can be reliably simulated.

In view of the need to simulate large systems sizes, for long measurement periods and for many different $p$-values near criticality, we felt it was important to investigate fast algorithms such as cluster updating methods. The Wolff cluster updating algorithm [17] carries the Ising spin configuration to a new point in phase space by constructing a cluster of like spins with an appropriate probability, based on the coupling (and hence the simulated temperature). This algorithm is particularly effective at temperatures near the critical point as it can flip very large spin clusters and effectively overcomes the critical slowing down of a local update method such as Metropolis. In this work we are interested in equilibrium properties, not dynamical ones and therefore the time evolution properties of the update algorithm need not resemble any physical process. We verified our implementation of the Wolff algorithm against implementations of both Metropolis and Glauber [20] Monte Carlo updates. For the work reported here, we note no discernible difference in quality of equilibrium properties estimated.
Computational Resourcing Issues

Although the work was carried out on a mix of 32-bit and 64-bit microprocessors, we found that it is still only tractable to simulate systems that comfortably fit within approximately 2 GBytes of memory. This is the addressable limit attainable using a 32-bit signed integer, and although a bit-based model such as the Ising model can be implemented using a compact storage scheme, for implementations of the Wolff cluster algorithm we require a full integer for each of the $N$ spin sites. Furthermore, when investigating non-regular networks we need to explicitly store the neighbour addresses for each site. For our code, we optimise for speed using a storage scheme that records the neighbour arcs for every site – thus storing a bond’s source and destination site twice.

In the case of a rewiring model that allows different sites to have different coordination numbers, this scheme gives rise to a storage budget of one spin-bit; one mark-bit; one coordination number (integer) and on average $2 \times d$ site addresses per spin site. The practical upshot of this is that we can, in principal, simulate Ising systems of up $384^3$ sites. In practice however, even with the Wolff algorithm to assist with equilibration and decorrelation near the critical temperature, we found that the processors available to us for this work would take over 1 week of wall-clock time to attain a useful measurement for one $T - p$ pair. Consequently we found that size limits of around $N = 256^3$ (around 16 million sites) were more practical. We anticipate that larger sizes (approaching $512^3$) will be practical with 64-bit address spaces and the next generation of processor speeds.

In applying the Binder cumulant method we found it useful to run triples of systems size, such as $N=224^3, 240^3, 256^3$ in three dimensions, or $992^2, 1024^2$ and $1056^2$ in two dimensions. Although only two cumulant curves are needed to obtain an intersection, three combine to also give a measure of uncertainty.

Monte Carlo Algorithmic Issues

The Wolff cluster algorithm is not efficient when performing an initial quench from a hot or random spin configuration, since the clusters it builds tend to be very small. Given the unknown dependencies on the parameter $p$ we erred on the side of caution and generated a completely independent start configuration for each experimental $T - p$ pair investigated.
Typically we quenched a hot starting spin configuration to finite $T$ using on average 1000 Metropolis hits per site. Although the debate in the literature over the effect of sweeping artifacts seems now resolved, we avoided any bias accidentally introduced by sweeps by performing the Metropolis equilibration site hits in a random order. So in effect we apply $N \times 1000$ randomly chosen local Metropolis hits following the quench. Since we are typically interested in behaviour near $T^*_p$, we typically used the Wolff cluster method predominantly during measurement, but again to err on the side of caution, we used a hybrid update step consisting of 100 Wolff cluster hits, followed by $N$ local Metropolis hits for each measurement.

In determining the Binder cumulant and other statistics, we typically made 12 million measurement steps, dividing them into blocks of 1 million, and discarding the first two blocks as additional equilibration. For the smaller system sizes, we found that 100-150 thousand measurement steps were sufficient. The data blocking approach gave us a measure of uncertainty in each cumulant value.

For the large systems sizes used it appears that good self-averaging properties hold and the results are largely independent of the random rewiring pattern chosen for a particular run at a particular $p$ value. For small-$p$ values and also for smaller system sizes we found a perceptible experimental spread of results and it was necessary to repeat runs at the same $T-p$ value. In the work reported here we typically combined measured cumulant values from 16 or 25 completely independent runs. Study of the statistics indicated a satisfactory central-limiting behaviour and a ready calculation of uncertainties in the cumulants so obtained.

In most of the work reported here, we used the lagged-Fibonacci random number generator of Marsaglia [21]. For some simulations on 64-bit platforms we used an implementation of the Mersenne-Twistor generator algorithm [22]. We are not aware of any concerns regarding periodicity or correlations due to either of these generators.

III. REWIRING ALGORITHMS

The value of a small-world rewiring procedure is that the amount of space folding can be specified statistically by a single parameter $p$, with the extreme value $p = 0$ corresponding to a normal periodic hyper-cubic geometry lattice. The case $p = 1$ corresponds to a random network of spins.

Herrero and other authors report using a network rewiring algorithm that maintains an
average coordination number $z$ for each spin site, but that does allow individual spins to have a lesser or greater $z$ value. This small-world rewiring algorithm is essentially that described by Watts and Strogatz and works by randomly selecting $p \times N \times d$ of the $N \times d$ original regular-lattice bonds and reassigning them to link random spin sites.

This algorithm can be implemented as:

1. choose a spin site $A$ at random
2. choose one of its $2 \times d$ existing neighbour $B$ at random
3. choose another (distinct) spin site $C$
4. re-wire $A$ to $C$, disconnecting $A$ from $B$

In this rewiring model $p$ is the probability that any of the $z \times N$ regular bonds has been rewired. It can be verified experimentally that a requested $p$ value from the rewiring algorithm has been implemented by performing a bond-by-bond comparison with each bond’s regular lattice endpoints.

One difficulty with this model is that the system is also influenced by percolation transition effects. Even at small-$p$ values individual spin sites have a finite probability of becoming completely disconnected from the rest of the system. In practice the system will have a finite number of monomer and dimer spin sites, separate from the one giant component. At large $p$ values (greater than 0.1 for example) this is a serious effect and the spin system is typically fragmented into more than one major component. Although this can be compensated for by only making measurements on the largest component, it is an operational annoyance as well as seriously worsening finite-size effects that impinge on the calculation of the critical temperature and exponents. Above the bond percolation threshold of $p = 0.5$ the system is nearly always very fragmented and consists of a number of similar-sized components.

A major aim of our work was to investigate the effect of a $z$-preserving network rewiring algorithm both in terms of how it compared to the Watts-Strogatz edge-rewiring algorithm and also how to interpret its probabilistic rewiring parameter $p$. To preserve coordination number $z$ exactly and for all spin sites it is necessary to re-wire the regular lattice in terms of pairs of edges.

Our network is constructed from the starting lattice using a similar rewiring procedure. We modify the bonds of each site so that it still has degree $z = 2 \times d$ bonds per site in
dimension $d$, but that they may link, with probability $p$, to a randomly chosen site elsewhere in the original lattice. We ensure each site links to other sites at most once, and there are no self-bonds in our system. This is feasible below the percolation threshold. Specifically, our procedure is:

1. choose a spin site $A$ at random
2. choose one of its $2 \times d$ existing neighbours $B$ at random
3. choose another (distinct) spin site $C$
4. choose one of its neighbours $D$ at random
5. ensure $A$, $B$, $C$ and $D$ are all distinct to avoid self bonds and multiple bonds
6. re-wire $A$ to $C$ and $B$ to $D$, thus exactly preserving $z$ for all of $A$, $B$, $C$ and $D$

Repeating this procedure $p \times N \times d$ times achieves a rewiring of the regular lattice to have an effective rewiring probability $p^*$ which can be subsequently measured by comparing spin-site neighbour-lists to that of the regular lattice.

This algorithm has the additional desirable property that it is guaranteed not to fragment the lattice into multiple components since each site still connects to $z$ distinct other sites.

![Diagram](image)

**FIG. 1:** Different rewiring models: a) Watts-Strogatz non-$z$ preserving model, b) Our $z$-preserving rewiring model.

Figure 1 shows an example of the two rewiring models for a regular square lattice substrate.

In practice, the achieved or effective rewiring probability $p^* < p$ by an amount that is of order $p^2$ since our algorithm allows a site to be “re-rewired”. There is therefore an
exclusion effect that gives rise to a correction term in $p$. Since, $p^*$ is readily measured, this is only an operational inconvenience and a chosen effective $p^*$ can be set with an appropriate choice of $p$ given the solvable quadratic relationship between them and experimentally fitted coefficients.

Over and above this exclusion correction however, it is important to note that pairs of bonds have been rewired. This pairing leads to a factor of two in the effective value of $p$ for our model when we compare it with the Watts-Strogatz rewiring model. This is explained when we consider the actual effect of a pair of rewired links connecting sites $A$, $B$, $C$, $D$ as shown in figure 1. If $A$ and $B$ were originally neighbours, and so were $C$ and $D$, and now $A$, $B$ are connected via a long-range pair of links to $C$, $D$ then the net effect is really only as if there were one long range link. This manifests itself in the role of $p$ for our $z$-preserving rewiring model. In comparing the two models, we need to measure the actual $p$ implemented by comparing with the regular lattice bond end-points. Then we should treat the effective $p$ for the $z$-preserving model as half the effective $p$ achieved by the Watts-Strogatz model. This is further evidence that it is the long-range nature of the rewired bonds that is behind the small-world critical shifts in the Ising model – the local topological details (such as which particular neighbour of $A$, $B$, etc. was rewired) are much less important.

IV. RESULTS AND DISCUSSION

The effect of the small-world network rewiring is to add long-range interactions across the lattice. This enables the formation of correlations across the spin sites and consequently makes it easier for the Ising system to maintain spontaneous magnetization at a higher critical temperature than it otherwise would. This effect is not dissimilar to the finite-size effects of simulating periodic lattice models. The periodicity means that the system is able to support spin correlations at a higher temperature that it otherwise could and consequently the critical temperature is shifted to a higher value than that of a real system in the thermodynamic limit. A key step to understanding the small-world effect is to quantify the resulting shift in the critical temperature for the different network rewiring models.
Binder Cumulant Analysis

We compute the temperature shift $\Delta T_c = T^p_{c} - T^p_{c=0}$ based on the critical temperature measured for a small-world rewired Ising model system, compared with that of an unperturbed system on a regular lattice.

![Graph showing Binder Cumulants](image)

**FIG. 2: Binder Cumulants**

The critical temperature of a Monte Carlo simulated system can be calculated using the fourth-order Binder cumulant method [23, 24]. The cumulant:

$$U_N = 1 - \frac{\langle M^4 \rangle_N}{3\langle M^2 \rangle^2_N}$$  \hspace{1cm} (2)

is defined at different network sizes $N$, where $N = L^d$ based on the edge length $L$ of our substrate lattice. The cumulant shows a transition edge at the critical temperature. The cumulant curves for different $N$ (or $L$) values coincide at the critical temperature and this gives a way of extrapolating to the thermodynamic limit from relatively small sized simulations. Figure 2 shows the form of the cumulant and how it has a sharper transition edge at larger system sizes. A practical method of obtaining a critical temperature is to simulate at least three different system sizes and by fitting straight lines to the linear region of the cumulant curve around the critical temperature, calculate the intercept and an uncertainty estimate. For all the work we report in this paper we used at least three system sizes. For small systems multiple independent rewiring configurations can be sampled. For very large systems needed for estimating low-$p$ behaviour it is impractical to run more than a few independent samples and consequently the measurement uncertainties are very much
greater. Generally we have been able to estimate the critical temperatures for two- and three-dimensional systems to around four significant figures, to three significant figures for four-dimensional systems. Our five-dimensional system simulations are limited to qualitatively showing that a small-world effect takes place.

![Graph showing variation of critical temperature](image)

**FIG. 3:** Variation of the Critical Temperature against small-world (bond pair) rewiring probability for Ising model in 2, 3, 4 and 5 dimensions, scaled by the underpinning regular lattice coordination number $z$. Error-bars are comparable to symbol sizes and lines joining points are guides to the eye only.

Generally, obtaining the shifted critical temperatures for different $p$-values was an iterative procedure. We employed small lattice sizes to home in on approximate locations then used progressively larger system sizes to refine precision and uncertainties. Figure 3 shows the qualitative behaviour of the critical temperature as it varies with $p$ for different dimensionalities.

Analysis of the shift $\Delta T_c = T_c^p - T_c^{p=0}$ in the critical temperature from the regular lattice value of shows a power law of the form $\Delta T_c^{(p)} \approx p^s$. This is shown on a logarithmic scale for three dimensions in figure 4 where we have included all our three-dimensional data for both the Watts-Strogatz rewiring model and our $z$-preserving model (after suitable corrections to the meaning of $p$), as well as data taken from [9]. We believe that within the bounds of experimental uncertainty these all agree.

This confirms that well below the percolation limit, where the system is essentially one single component, the behaviour is independent of the rewiring model. We have been able
to extend the simulation results of Herrero to much larger system sizes and hence much smaller values of $p$.

![Graph](image_url)

**FIG. 4:** Collapse of Wiring model 1 data onto Model 3 (actual $p$) after factor of two scaling

We studied both the Watts-Strogatz and our $z$-preserving rewiring models in detail on small-world systems constructed from 2-D, 3-D and 4-D hyper-cubic lattices. For each dimension we studied at least three decades of $p$ values and in the case of 3-D systems we studied six decades of $p$ down to $10^{-6}$ on systems of up to $N \approx 384^3$. A preliminary study of a 5-D system also indicates that the small-world shift in $T_c$ does take place, although our 5-D data is as yet insufficient to determine a useful value of $s$.

For $p \lesssim 0.1$ we are below the percolation regime and the system consists of one large component for the W-S model. In this regime our rewiring model and the W-S rewiring model are in agreement within the limits of experimental error, and furthermore agree closely with the 3-D data reported for a W-S rewiring model [9]. Combining all data we find $s \approx 0.698 \pm 0.002$. Although we have less data for 2-D systems, we investigated larger system sizes (up to $1024^2$) than Herrero, and do not find the tail-off he reports for small-$p$. We find a very good fit for $s \approx 0.50 \pm 0.001$ for 2-D systems.

We conclude that the $\Delta T_c^{(p)} \approx p^s$ power-law is a very good description of all the data and any deviation from it is indeed due to finite-size limitations that do not properly achieve the small-$p$ regime as Herrero correctly suggests for his data.
Critical Exponent $\beta$

It remains to explore the critical exponents of the simulated small-world system to determine in what manner the system transitions between the Ising-type transitions at $p = 0$ in two- and three-dimensions, and the mean-field like transitions at high-$p$ values.

The critical exponent $\beta$ describes how the Ising model order parameter – its magnetization $M$ – diverges close to the critical temperature. It is usually defined by:

$$\langle M \rangle \sim |T_c - T|^\beta, T < T_c$$

(3)

and we define the reduced temperature $t \equiv T_c - T$ so that $\langle M \rangle \sim t^\beta, t > 0$.

We can follow Herrero and study the logarithmic derivative:

$$\mu(t) = \frac{d\log\langle M \rangle}{d\log t}$$

(4)

This gives a qualitative indication of how the behaviour is changing as $p$ is varied.

The logarithmic derivative $\mu$ curves for different $p$ values appear to intercept the $t = 0$ axis at values between a 3-D Ising-type transition $\beta$ value of $\approx 0.325^{25}$ and the mean-field value of 0.5. Our 3-D data has quite high uncertainties in it due to small numbers of samples with large system sizes, but it suggests a monotonic increase in the limiting value of the $t = 0$ intercept for $\mu$ with rising $p$. Our 2-D data also shows a steady change between the 2-D Ising value for $\beta$ of 0.125 and the mean-field value.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig5}
\caption{Deriving $\beta$ from the Magnetization}
\end{figure}

Figure 5 shows the magnetization averaged over 25 sample sizes for a three-dimensional system at a finite value of $p$ as we approach the critical temperature from below. We can
estimate the limiting value of the slope at low $t$ (as shown in the inset) and use this to estimate values of $\beta$.

Applying this procedure we find a good straight line fit of $\log \beta$ vs $\log p$ characterised by $\beta \approx p^{0.10\pm0.005}$ in 2-D and $\beta \approx p^{0.08\pm0.002}$ in 3-D.

We conclude that the model transitions continuously from Ising-like behaviour to mean-field behaviour as $p$ is increased. We find no difference in these values for the two different rewiring models. This again emphasises the power of the small-world parameter $p$ in interpolating between two different behaviour regimes.

**Critical Exponent $\nu$**

The correlation length near the critical temperature is known to scale as:

$$\xi \sim |T - T_c|^{-\nu} \quad (5)$$

and the correlation length exponent $\nu$ is known to have values of $\nu = 0.632$ for 3-D systems; $\nu = 1$ for a 2-D system and a value of $\nu = 0.5$ from mean-field theory.

At the linearised region near the critical temperature, an expansion indicates that the Binder cumulant depends upon approximately on temperature as:

$$U_N(T) \approx U^* + U_1 \left(1 - \frac{T}{T_C}\right) N^{\frac{1}{\nu}} \quad (6)$$

so that

$$\frac{\Delta U_N}{\Delta T} \propto -N^{\frac{1}{\nu}} \quad (7)$$

On the work we report here on small-world systems based upon regular lattices with a fixed $z$, we can also write:

$$\frac{\Delta U_N}{\Delta T} \propto -L^{\frac{4}{\nu}} \quad (8)$$

Straight line fits to the experimental Binder cumulants around the critical temperature yields values of $\nu$ that are between the requisite Ising value and mean-field values. The experimental uncertainties in our data for $\nu$ are quite high however, and although the values suggest a monotonic change between Ising and mean-field behaviour as $p$ is increased, we are unable to identify a meaningful functional form to characterise the variation with $p$. 
**p-Dependence of the Transition**

It seems that the small-world transitional behaviour of the Ising model is intimately tied with enhancements to the systems ability to support long-range correlations. It is therefore useful to consider the length scales present in the rewired lattice. A useful metric is the maximum path length connecting two spin sites, as counted in terms of number of traversed bonds or “hops” along edges of the graph.

![Graph showing the dependence of the transition on p](image)

**FIG. 6:** Mean All-pairs Dijkstra distance $\langle L \rangle$ averaged over 25 samples of $32 \times 32$ lattices perturbed with the Watts-Strogatz (W-S) and our $z$-preserving (z-p) rewiring models.

Figure 6 shows the Mean All-pairs distance $\langle L \rangle$ calculated using the Dijkstra algorithm \[28\] for a 2-D lattice when perturbed with the Watts-Strogatz (W-S) and our $z$-preserving (z-p) rewiring models. The calculation is shown for a $32 \times 32$ system with the two different rewiring algorithms applied. The underpinning regular lattice is periodic so the furthest apart any two spin sites can be is $L/2 = 16$ edge units. This is shown on the inset plot on a linear scale, where the curves tend towards a maximum value of 16 at $p = 0$.

There are three distinct $p$-regimes shown in the figure. At small values of $p$, for which the lattice size is big enough to support a reasonable number of rewired bonds, the log – log plot shows that $\langle L \rangle \approx p^m$. We find $m \approx 0.20$ for this data set. At $p$-values that are too small for the lattice to support, the straight line tails off. At high values of $p$ the value of $\langle L \rangle$ falls off to the value for a random lattice.

Note that at high $p$ values, our $z$-preserving rewiring algorithm gives a different $\langle L \rangle$ value from the W-S rewiring algorithm. This was discussed in section III and is due to
exclusion effects of our algorithm in avoiding self-bonds and multiple bonds. At high-$p$ the Watts-Strogatz rewiring algorithm breaks up the system into multiple components. This is reflected in the averaged Dijkstra distance calculations, and for the W-S rewiring $\langle L \rangle$ falls to zero in the limit of a completely random lattice.

Herrero hypothesized that the order-disorder transition temperature for small $p$ becomes

$$T_c - T_c^{p=0} \sim p^{\frac{1}{\nu d}}$$

(9)

where $\nu$ is the critical exponent of the regular (unperturbed) lattice. His data supports this in a 2-D system, as does ours. However for a 3-D system his data disagree with this hypothesis and he speculated this was due to too small system sizes and insufficiently small $p$. This conclusion was also reached by [27] for the XY model. Nevertheless our value of $s \approx 0.698$ for a 3-D system, for which we have quite a high degree of confidence, and which is derived from much larger simulated system sizes, is still in disagreement with equation [9] Furthermore, in the case of our 4-D system, which has mean-field behaviour in the regular unperturbed lattice case as well as in the rewired case, our measured value of $s = 0.75$ also clearly disagrees with this hypothesis.

V. CONCLUSIONS

Generally the effect of the rewiring is to shift the the critical temperature $T_c$ upwards. We have determined this remains the case in dimensions 2, 3, 4 and 5. We have also found that the precise local rewiring details are not important to the nature of the small-world phase transition, providing the appropriate interpretation of parameter $p$ is made. We find that the shift in $T_c$ seems to go as a power law in $p$ in all dimensions.

Since our re-wire model preserves $z$ and hence the effective dimensionality of the underpinning lattice, we believe it is this effect that principally gives rise to the $p$-concentration dependences of $T_c$ and not the change in effective dimensionality of the system that arises when links are added. The small-world shortcuts support long-range spin-spin correlations above the normal $T_c$ value. It appears that it is the presence of long-range correlations that essentially constitute the nature of the phase transition and the critical behaviour; it is not unreasonable that the small-worlding of the lattice has this effect.

Operationally we have found that a cluster-updating method such as that of Wolff is
entirely satisfactory and its use may assist in investigating more closely the small-world behaviour in large systems sizes at higher dimensionality. The exact nature of the small-world transition and in particular its interplay with the Ising transition are still not entirely clear.

We believe that the shift from Ising-like behaviour to mean-field behaviour is a gradual one, but it remains to ascertain this with better statistical sampling in higher dimensional systems. A particular area worthy of further attention is the nature of cluster break-up in the system under a Watts-Strogatz rewiring, and the associated path and correlation lengths that arise for different cluster components.

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