The conformal factor in the parameter-free construction of spin-gauge variables for gravity

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Abstract

The newly found conformal decomposition in canonical general relativity is applied to drastically simplify the recently formulated parameter-free construction of spin-gauge variables for gravity. The resulting framework preserves many of the main structures of the existing canonical framework for loop quantum gravity related to the spin network and Thiemann’s regularization. However, the Barbero-Immirzi parameter is now converted into the conformal factor as a canonical variable. It behaves like a scalar field but is somehow non-dynamical since the effective Hamiltonian constraint does not depend on its momentum. The essential steps of the mathematical derivation of this parameter-free framework for the spin-gauge variables of gravity are spelled out. The implications for the loop quantum gravity programme are briefly discussed.

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I. THE ROLE OF THE CONFORMAL FACTOR IN CANONICAL GRAVITY

In recent series of works [1, 2, 3, 4], the canonical theory of general relativity has been constructed in terms of the conformal equivalence classes of spatial metrics. The work was motivated by the following two ideas. First, the conformal 3-geometry, rather than the 3-geometry, may well carry the true dynamics of general relativity (GR) [5, 6, 7]. Secondly, the free parameter responsible for the Barbero-Immirzi ambiguity of the present loop quantum gravity is of conformal nature, and may be removed by an extension of the phase space for GR using the conformal symmetry. This led to the canonical framework in [1, 2] where the conformal 3-metric, the mean extrinsic curvature and their respective momenta act as canonical variables. A new first class constraint, the conformal constraint, was introduced to offset the conformal redundancy. However, its quantum implementation appears to be impeded by the complexity of the formalism where the conformal factor as a key subexpression is a highly non-polynomial function of other variables. This will cause the required regularization for quantization hard to resolve.

Fortunately, it was then found in [4] that the conformal factor need not be used this way and can receive the canonical variable status if a suitable canonical transformation is performed. A quick way of demonstrating this is as follows: Starting with the Arnowitt-Deser-Misner (ADM) variables $g_{ab}$ and $p^{ab}$, we introduce the conformally related quantities $\tilde{g}_{ab}$ and $\tilde{p}^{ab}$ by

\begin{align}
  g_{ab} &= \phi^4 \tilde{g}_{ab} \quad (1) \\
  p^{ab} &= \phi^{-4} \tilde{p}^{ab} \quad (2)
\end{align}

using the conformal factor $\phi = \phi(x)$. Throughout this work, we shall use a bar over a quantity to indicate that the quantity has been obtained with a rescaling using a power of $\phi$. The strategy is to turn $\tilde{g}_{ab}$ and $\phi$ into new configuration variables. If the momentum of $\tilde{g}_{ab}$ is $\tilde{p}^{ab}$, then the momentum of $\phi$ must be identified. In addition, there should be an additional constraint to compensate the conformal redundancy in [1] and [2]. To this end, let us calculate the following:

\begin{align}
  p^{ab} \dot{g}_{ab} &= \phi^{-4} \tilde{p}^{ab} \left[ \phi^4 \dot{\tilde{g}}_{ab} + 4 \phi^3 \dot{\phi} \tilde{g}_{ab} \right] \\
  &= \tilde{p}^{ab} \dot{\tilde{g}}_{ab} + \pi \dot{\phi} \quad (3)
\end{align}
where
\[ \pi := 4 \phi^{-1} \bar{g}_{ab} \bar{p}^{ab} = -8 \phi^{-1} \mu K \] (4)
using the mean extrinsic curvature \( K \) and the scale factor \( \mu := \sqrt{\text{det} g_{ab}} \). This implies that the variables \((\bar{g}_{ab}, \bar{p}^{ab}, \phi, \pi)\) may be treated as a canonical set if the constraint \( C \) given by
\[ C := \bar{g}_{ab} \bar{p}^{ab} - \frac{1}{4} \phi \pi \] (5)
vanishes weakly. We have therefore identified \((\bar{g}_{ab}, \bar{p}^{ab}; \phi, \pi)\) as the new canonical variables together with \( C \) as the new (conformal) constraint for GR. In terms of these variables, the ADM diffeomorphism and Hamiltonian constraints become
\[ \mathcal{H}_a = -2 \bar{\nabla}_b \bar{p}^b_a + \pi \phi,a + 4(ln \phi,a) C \] (6)
\[ \mathcal{H} = \phi^{-6} \bar{\mu} \bar{g}_{abcd} \bar{p}^{ab} \bar{p}^{cd} - \phi^2 \bar{\mu} \bar{R} + 8 \bar{\mu} \phi \bar{\Delta} \phi \] (7)
respectively. Here we have used the scale factor \( \bar{\mu} := \sqrt{\text{det} \bar{g}_{ab}} \), Levi-Civita connection \( \bar{\nabla} \), Ricci scalar curvature \( \bar{R} \) and Laplacian \( \bar{\Delta} := \bar{g}^{ab} \bar{\nabla}_a \bar{\nabla}_b \), associated with the conformal metric \( \bar{g}_{ab} \). In (6), the last term \( 4(ln \phi,a) C \) can be dropped to define an effective diffeomorphism constraint \([4]\).

II. TRIAD VARIABLES FOR GRAVITY

The canonical framework in the preceding section will be applied to obtain a triad formulation of GR with extended conformal symmetry. Before we carry that out, however, it is useful to briefly review the standard triad formulation of GR with the canonical variables \((E^a_i, K^i_a)\). Here \( E^a_i \) is the densitized triad and \( K^i_a \) the extrinsic curvature. In terms of these variables, the ADM variables take the following forms:
\[ g_{ab} = \mu^2 E^i_a E^i_b \] (8)
\[ p^{ab} = \frac{1}{2} \mu^{-2} K^c_i \left[ E^c_j E^i_a E^b_j - E^c_j E^a_i E^b_j \right] \] (9)
where \( E^i_a \) is the inverse of \( E^a_i \). It follows that
\[ \bar{p}^{ab} g_{ab} = -E^a_i K^i_a \] (10)
and
\[ p^{ab} \dot{g}_{ab} = -\dot{E}_a^a K_a^i - \frac{1}{\mu^2} E_i^a \dot{E}_a^b K_{ab} \] (11)
with the constraint
\[ K_{ab} := \mu K_{[a}^i e_{b]}^i = \mu^2 K_{[a}^i E_{b]}^i \] (12)
to vanish weakly. This justifies \((K_a^i, E_i^a)\) as canonical variables. Instead of working with the constraint \(K_{ab}\) it is more convenient to adopt the equivalent constraint
\[ C_k := \epsilon_{kij} K_{ai} E_{aj} = -\frac{1}{\mu^2} \epsilon_{kij} K_{ab} E_i^a E_j^b \] (13)
since it generates the rotation of the triad. We shall therefore refer to \(C_k\) as the “spin constraint”.

In terms of the canonical variables \((K_a^i, E_i^a)\), the ADM diffeomorphism and Hamiltonian constraints then become
\[ H_a = 2 E_k^b \nabla_a K^k_{[b]} - \frac{1}{2} \epsilon_{ijk} E_a^i E_j^b \nabla_b C_k \] (14)
and
\[ H = -\frac{1}{2\mu} K_{[a}^i K_{b]}^j E_i^a E_j^b - \mu R + \frac{1}{8\mu} C_k C_k \] (15)
respectively.

### III. CONFORMAL TREATMENT OF THE TRIAD VARIABLES

The rescaling relations (1) and (2) give rise to the conformal triad variables \((\bar{K}_a^i, \bar{E}_i^a)\) satisfying
\[ E_i^a = \phi^4 \bar{E}_i^a \] (16)
\[ K_a^i = \phi^{-4} \bar{K}_a^i \] (17)
Using (4) and and the identities
\[ K_a^i E_i^a = \bar{K}_a^i \bar{E}_i^a = 2\mu K \] (18)
we can calculate that
\[ E_i^a \dot{K}_a^i = \phi^4 \bar{E}_i^a [\phi^{-4} \dot{\bar{K}}_a^i - 4 \phi^{-5} \dot{\phi} \dot{\bar{K}}_a^i] \]
\[ = \bar{E}_i^a \dot{\bar{K}}_a^i - 4 \phi^{-1} \bar{E}_i^a \dot{\bar{K}}_a^i \dot{\phi} \]
\[ = \bar{E}_i^a \dot{\bar{K}}_a^i + \pi \dot{\phi}. \] (19)
This establishes the variables \((\bar{K}_a^i, \bar{E}_a^i; \phi, \pi)\) as canonical variables. By using (10), we see that the conformal constraint \(C\) defined in (5) now takes the form

\[
C = -\bar{K}_a^i \bar{E}_a^i - \frac{1}{4} \phi \pi
\]  

(20)
The spin constraint defined in (13) then becomes

\[
C_k = \epsilon_{kij} \bar{K}_{a[i} \bar{E}_{j]}^a.
\]  

(21)

In terms of the variables \((\bar{K}_a^i, \bar{E}_a^i; \phi, \pi)\) the conformal metric and its momentum are given by

\[
\bar{g}_{ab} = \bar{\mu}^2 \bar{E}_a^i \bar{E}_b^i
\]  

(22)
\[
\bar{p}^{ab} = \bar{\mu}^{-2} \bar{K}_{c}^i \left[ \bar{E}_j^a \bar{E}_j^b - \bar{E}_j^a \bar{E}_j^b \right].
\]  

(23)

By substituting (22), (23) into (6), (7) and making use of the expressions in (14) and (15) with the replacement \((E_a^i, K_a^i) \rightarrow (\bar{E}_a^i, \bar{K}_a^i)\) we get the diffeomorphism constraint and Hamiltonian constraint respectively in the following forms

\[
H_a = 2 \bar{E}_b^b \nabla_a [\bar{K}_b^k] + \pi \phi_a \pi - \frac{1}{2} \epsilon_{ijk} \bar{E}_a^i \bar{E}_j^b \nabla_b C_k + 4 (\ln \phi)_a C
\]  

(24)
\[
H = -\frac{1}{2} \phi^{-6} \bar{\mu}^{-1} \bar{K}_{[a}^i \bar{K}_b^j] \bar{E}_a^i \bar{E}_j^b - \phi^2 \bar{\mu} \bar{R} + 8 \bar{\mu} \phi \Delta \phi + \frac{1}{8} \phi^{-6} \bar{\mu}^{-1} C_k C_k.
\]  

(25)

IV. STANDARD SPIN-GAUZE FORMALISM

We now briefly review the existing real spin-gauge formalism for gravity [8, 9, 10, 11] before moving on to our final form of the spin-gauge formalism by assimilating the the conformal treatment in the preceding section. For any positive constant \(\beta\), introduce the spin connection

\[
\tilde{A}_a^i := \Gamma_a^i + \beta K_a^i = \Gamma_a^i + \bar{K}_a^i
\]  

(26)
where \(\bar{K}_a^i := \beta K_a^i\). Further, introduce the scaled triad

\[
\tilde{E}_a^i := \beta^{-1} E_a^i.
\]  

(27)

Here and below, we use the tilde to emphasize a quantity’s dependence on the parameter \(\beta\). This parameter is called the Barbero-Immirzi parameter. It can be show that the variables
\( (\tilde{A}_a, \tilde{E}^a_i) \) are canonical. They are used for the existing spin-gauge formalism for gravity. The curvature of the spin connection \( \tilde{A}_a \) is given by

\[
\tilde{F}^{ab}_k := 2 \partial_{[a} \tilde{A}^k_{b]} + \epsilon_{kij} \tilde{A}^i_a \tilde{A}^j_b.
\] (28)

Denoting by \( \tilde{D}_a \) the covariant derivative associated with \( \tilde{A}^i_a \), we can express the spin constraint in the form of the “Gauss law” as:

\[
\tilde{C}_k := \tilde{D}_a \tilde{E}^a_k = \tilde{E}^a_{k,a} + \epsilon_{kij} \tilde{A}^i_a \tilde{E}^j_b = \nabla_a \tilde{E}^a_k + \epsilon_{kij} K^i_a E^j_b = \mathcal{C}_k.
\] (29)

In the above the torsion-free condition \( \nabla_a \tilde{E}^a_k = 0 \) for the Levi-Civita spin connection \( \nabla_a \) associated with the metric \( g_{ab} \) has been used. Using the relations

\[
\tilde{F}^{ab}_k \tilde{E}^b_k = 2 \nabla_{[a} \tilde{K}^b_{b]} \tilde{E}^b_k \tilde{K}^k_a \tilde{C}_k
\] (30)

and

\[
\left[ \beta^2 \epsilon_{kij} \tilde{F}^{ab}_k - 2 \beta^2 \tilde{K}^i_{[a} \tilde{K}^j_{b]} \right] \tilde{E}^a_i \tilde{E}^b_j = -\mu^2 R - 2 \beta^2 \tilde{E}^a_k \nabla_c \tilde{C}_k
\] (31)

we see that (14) and (15) become

\[
\mathcal{H}_a = \tilde{F}^{ab}_k \tilde{E}^b_k - \tilde{A}^k_a \tilde{C}_k + \Gamma^k_a \tilde{C}_k - \frac{1}{2} \epsilon_{ijk} \tilde{E}^i_a \tilde{E}^b_j \nabla_b \tilde{C}_k
\] (32)

and

\[
\mathcal{H} = \beta^{1/2} \tilde{\mu}^{-1} \left[ \epsilon_{ijk} \tilde{F}^{ab}_k - \frac{4 \beta^2 + 1}{2 \beta^2} \tilde{K}^i_{[a} \tilde{K}^j_{b]} \right] \tilde{E}^a_i \tilde{E}^b_j + 2 \beta^{1/2} \tilde{\mu}^{-1} \tilde{E}^a_k \nabla_c \tilde{C}_k + \frac{1}{8} \beta^{-3/2} \tilde{\mu}^{-1} \tilde{C}_k \tilde{C}_k.
\] (33)

Here the scale factor of the metric defined using the triad \( \tilde{E}^a_i \) is denoted by \( \tilde{\mu} = \beta^{-3/2} \mu \). As per [12], we summarize the standard spin-gauge formalism of gravity by listing its effective Hamiltonian constraint:

\[
\tilde{\mathcal{C}}_\perp = \beta^{1/2} \tilde{\mu}^{-1} \left[ \epsilon_{ijk} \tilde{F}^{ab}_k - \frac{4 \beta^2 + 1}{2 \beta^2} \tilde{K}^i_{[a} \tilde{K}^j_{b]} \right] \tilde{E}^a_i \tilde{E}^b_j
\] (34)

and diffeomorphism constraint

\[
\tilde{\mathcal{C}}_a = \tilde{F}^{ab}_k \tilde{E}^b_k - \tilde{A}^k_a \tilde{C}_k
\] (35)

together with the spin constraint

\[
\tilde{\mathcal{C}}_k := \tilde{D}_a \tilde{E}^a_k.
\] (36)
V. PARAMETER-FREE APPROACH TO THE SPIN-GAUGE FORMALISM USING THE CONFORMAL METHOD

The parameter dependence of the standard spin-gauge formalism discussed above is due to the fact that the following inequality:

$$\mathcal{H}[K^i_a, E^a_i] \neq \mathcal{H}[\beta K^i_a, \beta^{-1} E^a_i]$$

holds for any constant $\beta \neq 1$. This can be traced back to Kuchař’s observation that the kinetic and potential terms are rescaled differently under a constant conformal transformation [13]. When the phase space of GR is extended by conformal symmetry, the situation is completely different. From the expression of $\mathcal{H}$ in (25) it is clear that we do have the following equation:

$$\mathcal{H}[\bar{K}^i_a, \bar{E}^a_i; \phi, \pi] = \mathcal{H}[eta \bar{K}^i_a, \beta^{-1} \bar{E}^a_i; \beta^{1/4} \phi, \beta^{-1/4} \pi].$$

(38)

Therefore, if we introduce the alternative spin connection variable

$$\bar{A}^i_a := \bar{\Gamma}^i_a + \bar{K}^i_a = \bar{\Gamma}^i_a + \phi^4 K^i_a$$

(39)

and consider

$$\bar{E}^a_i = \phi^{-4} E^a_i$$

(40)

as its conjugate momentum, then unlike the construction of $\tilde{A}^i_a$ and $\tilde{E}^a_i$ in (26) and (27), any similar multiplicative constant can be absorbed into the conformal factor $\phi$ together with an inverse rescaling of $\pi$. As with the spin gauge treatment in the preceding section, the curvature of the spin connection $\bar{A}^i_a$ is given by

$$\bar{F}^k_{ab} := 2 \partial_{[a} \bar{A}^k_b] + \epsilon_{kij} \bar{A}^i_a \bar{A}^j_b.$$  

(41)

Denoting by $\bar{D}_a$ the covariant derivative associated with $\bar{A}^i_a$ we can also express the spin constraint in the form of the Gauss law as:

$$\bar{C}_k := \bar{D}_a \bar{E}^a_k = \bar{E}^a_{k,a} + \epsilon_{kij} \bar{A}^i_a \bar{E}^a_j = \bar{\nabla}_a \bar{E}^a_k + \epsilon_{kij} \bar{K}^i_a \bar{E}^a_j = C_k.$$  

(42)

Here the torsion-free condition $\bar{\nabla}_a \bar{E}^a_k = 0$ for the Levi-Civita spin connection $\bar{\nabla}_a$ associated with the conformal metric $\bar{g}_{ab}$ [22] has been used. Using the relations

$$\bar{F}^k_{ab} \bar{E}^b_k = 2 \bar{\nabla}_{[a} \bar{K}^k_b] \bar{E}^b_k + \bar{K}^k_a \bar{C}_k$$

(43)
and

\[
\left[ \epsilon_{kij} F^k_{ab} - 2 K^i_{[a} K^j_{b]} \right] E^a_i E^b_j = -\mu^2 R - 2 E^c_i \nabla_c C_k
\] (44)

analogous to (30) and (31), we see that (24) and (25) become

\[
\mathcal{H}_a = \vec{F}^k_{ab} \vec{E}^b_k - \vec{A}_a \vec{C}_k + \pi \phi_{,a} + \vec{\Gamma}_i^k \vec{C}_k - \frac{1}{2} \epsilon_{ijk} \vec{E}^i_a \vec{E}^b_j \nabla_b \vec{C}_k
\] (45)

and

\[
\mathcal{H} = \phi^2 \bar{\mu}^{-1} \left[ \epsilon_{ijk} \vec{F}^k_{ab} - \frac{4 \phi^8 + 1}{2 \phi^8} \vec{K}^i_{[a} \vec{K}^j_{b]} \right] \vec{E}^a_i \vec{E}^b_j + 8 \bar{\mu} \phi \Delta \phi + 2 \phi^2 \bar{\mu}^{-1} \vec{E}^c_i \nabla_c \vec{C}_k + \frac{1}{8} \phi^{-6} \bar{\mu}^{-1} \vec{C}_k \vec{C}_k.
\] (46)

This leads us to the final expressions for the effective Hamiltonian constraint

\[
\tilde{C}_\perp = \phi^2 \bar{\mu}^{-1} \left[ \epsilon_{ijk} \vec{F}^k_{ab} - \frac{4 \phi^8 + 1}{2 \phi^8} \vec{K}^i_{[a} \vec{K}^j_{b]} \right] \vec{E}^a_i \vec{E}^b_j + 8 \bar{\mu} \phi \Delta \phi
\] (47)

effective diffeomorphism constraint

\[
\tilde{C}_a = \vec{F}^k_{ab} \vec{E}^b_k - \vec{A}_a \vec{C}_k + \pi \phi_{,a}
\] (48)

spin constraint

\[
\tilde{C}_k = \vec{D}_a \vec{E}^a_k
\] (49)

and conformal constraint

\[
\tilde{C} = -\vec{K}^i_a \vec{E}^a_i - \frac{1}{4} \phi \pi.
\] (50)

VI. CONCLUDING REMARKS

It is readily seen that, apart from “small” extra terms involving \( \phi \) and \( \pi \), the structures of the first three constraints in (47), (48) and (49) above resemble very closely that in (34), (35) and (36). In fact, the constraints \( \tilde{C}_\perp, \tilde{C}_a \) and \( \tilde{C}_k \) reduce to \( \tilde{C}_\perp, \tilde{C}_a \) and \( \tilde{C}_k \) on substituting \( \phi \rightarrow \beta^{1/4} \). However, it is the introduction of the canonical variable \( \phi \) that makes our spin-gauge formalism parameter-free. It is interesting to observe that the effective Hamiltonian \( \tilde{C}_\perp \) is independent of \( \pi \). Consequently, the evolution of \( \phi \) is like a gauge effect and is dictated by the effective diffeomorphism constraint \( \tilde{C}_a \) and the conformal constraint \( \tilde{C} \). This is consistent with one of our motivating ideas that the true dynamics of GR is in the conformal 3-geometry, rather than the conformal factor. An important technical implication of this is related to the
regularization of the Hamiltonian operator. It is envisaged that the \((\bar{A}_a^k, \phi)\)-representation is to be used for quantization. While the spin connection \(\bar{A}_a^k\) can be treated with spin-networks, the conformal factor \(\phi\) will be treated similar to a coupled scalar field. The appearance of \(\phi\) in the first term in (47) should not spoil the regularization schemes developed by Thiemann \[14, 15\], since \(\phi\) commutes with all operators there just like the Barbero-Immirzi parameter in the existing loop quantum gravity. It remains to solve the quantum conformal constraint equation

\[
\bar{C} \Psi[\bar{A}_a^k, \phi] = 0. \tag{51}
\]

Addressing this problem will require the construction of “conformally related spin networks” which will form a subject for future investigation.

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