An Application of Renormalization Group Techniques to Classical Information Theory

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Abstract

We apply Renormalization Group (RG) techniques to Classical Information Theory, in the limit of large codeword size $n$. In particular, we apply RG techniques to (1) noiseless coding (i.e., a coding used for compression) and (2) noisy coding (i.e., a coding used for channel transmission). Shannon’s “first” and “second” theorems refer to (1) and (2), respectively. Our RG technique uses composition class (CC) ideas, so we call our technique Composition Class Renormalization Group (CCRG). Often, CC’s are called “types” instead of CC’s, and their theory is referred to as the “Method of Types”. For (1) and (2), we find that the probability of error can be expressed as an Error Function whose argument contains variables that obey renormalization group equations. We describe a computer program called WimpyRG-C1.0 that implements the ideas of this paper. C++ source code for WimpyRG-C1.0 is publicly available.
1 Introduction

Renormalization Group (RG) techniques are a panoply of techniques that serve to obtain asymptotic limits. RG techniques usually apply to a system with a very large number of degrees of freedom that is described by a partition function $Z$. Most RG techniques comprise an iterative step (i.e., a step which is performed repeatedly) consisting of a decimation followed by a rescaling. Decimation involves reducing the number of degrees of freedom. Rescaling involves rescaling the variables of $Z$ so as to bring $Z$ to the same form it had before the previous decimation. (Curiously, in Roman times, the word “decimate” meant to kill 1 out of every 10 prisoners. The modern meaning of the word is more like killing 9 out of every 10).

In this paper, we apply RG techniques to Classical Information Theory in the limit of large codeword size $n$. In particular, we apply RG techniques to (1) noiseless coding (i.e., a coding used for compression) and (2) noisy coding (i.e., a coding used for channel transmission). Shannon’s “first” and “second” theorems refer to (1) and (2), respectively. For (1), we consider the special case of Csiszár-Körner (CK) universal code. For (2), we consider the special case of random encoding and maximum-likelihood (ML) decoding. For these special cases of (1) and (2), we find that the probability of error can be expressed as an Error Function (see Appendix A) whose argument contains variables that obey RG equations.

Of course, there is no unique way of applying RG techniques to Classical Information Theory. The way shown in this paper is new, to our knowledge. Our RG technique uses composition class (CC) ideas, so we call our technique Composition Class Renormalization Group (CCRG). Often, CC’s are called “types” instead of CC’s, and their theory is referred to as the “Method of Types”.

We end this paper by describing the internal algorithms and typical input and output of a computer program called WimpyRG-C1.0 that implements the ideas of this paper. (The 1.0 is the version number. The C before the 1.0 stands for “Classical”, to distinguish this program from a Q (Quantum) version of WimpyRG that we expect to deliver in the future.) C++ source code for WimpyRG-C1.0 is publicly available, at www.ar-tiste.com/WimpyRG.html.

This paper straddles two fields (RG and Classical Information Theory) which are seldom used together within previous literature. It is therefore most likely that the reader is not closely acquainted with both of these fields. To help readers acquainted with only one of these two fields, the author has strived to make this paper as self-contained as reasonably possible.

Before embarking on long, complicated calculations, let us discuss a simple example that illustrates the manner in which we will apply RG ideas to Information Theory in this paper.

We show in this paper that the probability of error for both noiseless and noisy coding can be expressed as an integral of the following type:
\begin{equation}
I = \int_{\xi}^{+\infty} dx \, e^{-nf(x)}, \tag{1}
\end{equation}

where \( n \gg 1 \). Suppose \( f : \text{Reals} \rightarrow \text{Reals} \) is a convex (i.e., shaped like a cup \( \bigcup \)) function with a minimum at \( x_0 \). Let \( \Delta x = x - x_0 \), \( \Delta \xi = \xi - x_0 \), and \( F(\Delta x) = f(x) \). Then \( I \) can be rewritten as

\begin{equation}
I = \int_{\Delta \xi}^{+\infty} d\Delta x \, e^{-nF(\Delta x)} . \tag{2}
\end{equation}

\( I \) can be approximated as follows

\begin{equation}
I \approx e^{-nF(\Delta \xi)} . \tag{3}
\end{equation}

This approximation for \( I \) is the leading term of an asymptotic expansion. This method of obtaining asymptotic expansions of integrals is usually called Laplace’s Method \cite{laplace}, named after the inventor of the closely related Laplace Transform. Unfortunately, the \( I \)-approximation given by Eq.(3) is poor for those \( \Delta \xi_0 \) for which \( F(\Delta \xi_0) = 0 \). Indeed, \( e^{-nF(\Delta \xi_0)} \) is indeterminate because \( nF(\Delta \xi_0) = \infty \cdot 0 \). Our goal is to devise an \( I \)-approximation that overcomes this limitation.

Suppose, for example, that \( F \) is quadratic in \( \Delta x \):

\begin{equation}
F(\Delta x) = \frac{a}{2}(\Delta x)^2 , \tag{4}
\end{equation}

for some \( a > 0 \). Then we can do the integration in Eq.(2) exactly in terms of Error Functions (see Appendix \[A\])

\begin{align}
I &= \int_{\Delta \xi}^{+\infty} d\Delta x \, e^{-n\frac{a}{2}(\Delta x)^2} \\
&= \sqrt{\frac{\pi}{2na}} \text{erfc} \left( \Delta \xi \sqrt{\frac{na}{2}} \right) . \tag{5a}
\end{align}

Using RG ideas, we can generalize this result, valid only for a quadratic \( F \), to more general types of \( F \). In Eq.(2), let us rescale the parameters \( \Delta \xi, n \) and the integration variable \( \Delta x \), but keep the value of \( I \) fixed. Then

\begin{equation}
I = \int_{\Delta \xi^\wedge}^{+\infty} d\Delta x^\wedge \, J \, e^{-n^\wedge F^\wedge(\Delta x)} , \tag{6}
\end{equation}

where \( J \) is a Jacobian, and where, for some parameter \( s > 0 \), we define

\begin{equation}
n^\wedge = e^s n , \tag{7}
\end{equation}

and

\begin{equation}
F^\wedge(\Delta x) = F(\Delta x^\wedge) = e^{-s}F(\Delta x) . \tag{8}
\end{equation}
For \( s = \delta s \) where \( 0 < \delta s << 1 \), we get:
\[
\delta s = \frac{-\delta F}{F}. \tag{9}
\]
From Eq.(9), we get the following “RG Equation”:
\[
\frac{d\Delta \xi^{(s)}}{ds} = \frac{-F(\Delta \xi^{(s)})}{F_1(\Delta \xi^{(s)})}, \tag{10}
\]
where \( F_n \) is the \( n \)th derivative of \( F \), and we have replaced the symbol \( \wedge \) by \( (s) \). Of course, this RG equation is trivial and can be solved immediately:
\[
\Delta \xi^{(s)} = F^{-1}(e^{-sF(\Delta \xi)}). \tag{11}
\]
In the more complicated examples presented later in this paper, one gets a system of coupled RG equations with complicated boundary conditions. Such systems of RG equations usually cannot be solved exactly, but they can be solved numerically with a computer.

We can calculate the Jacobian \( J \) as follows:
\[
\Delta x^{(\delta s)} = \Delta x + \delta \Delta x = \Delta x - \delta s \frac{F}{F_1}, \tag{12}
\]
so
\[
J^{-1} = \left| \frac{\partial \Delta x^{(\delta s)}}{\partial \Delta x} \right| = \left| 1 - \delta s \left( 1 - \frac{FF_2}{(F_1)^2} \right) \right|. \tag{13}
\]
Note that we are justified in setting \( J \approx 1 \) if we are only interested in finding \( I \) to leading order in \( n \).

Suppose \( \Delta \xi > 0 \). Since \( F(\Delta \xi) \) is a convex function with minimum at the origin, as \( s \) increases (and therefore also \( n \) increases), then, according to Eq.(10), \( \Delta \xi \) decreases. Likewise, if \( \Delta \xi < 0 \), then as \( s \) increases, \( \Delta \xi \) increases. In both cases, \( \Delta \xi \) is attracted to zero as \( s \) increases. By making \( s \) large enough, we can make \( \Delta \xi \) small enough so that \( F \) is well approximated by its quadratic approximation:
\[
I = \int_{\Delta \xi^{(s)}}^{+\infty} d\Delta x^{(s)} \ J \ e^{-n(s)F^{(s)}(\Delta x)} \tag{14a}
\]
\[
\approx e^{-n(s)F(0)} \int_{\Delta \xi^{(s)}}^{+\infty} d\Delta x^{(s)} \ e^{-n(s)F_2(0)}\frac{1}{2}(\Delta x)^2 \tag{14b}
\]
\[
\approx e^{-n(s)F(0)} \sqrt{\frac{\pi}{2n(s)F_2(0)}} \text{erfc}\left( \frac{\Delta \xi^{(s)}}{\sqrt{\frac{n(s)F_2(0)}{2}}} \right). \tag{14c}
\]
In Eq.(14), to go from line (a) to (b), we replaced \( F \) by its Taylor expansion up to second order (this is valid for very large \( s \)) and we approximated \( J \) by one (this is
valid to leading order in $n$). Eq. (14c) is typical of the type of approximations that we propose in this paper.

Before leaving our toy example, it is instructive to compare the $I$-approximation Eq. (14c) to the exact answer in case $F$ is quadratic. So assume $F(0) = 0$ and $F_2(0) = a$ as in Eq. (11). For such an $F$, one can show from Eq. (11) that

$$
\Delta \xi^{(s)} = e^{-\frac{s^2}{2}} \Delta \xi .
$$

(15)

Furthermore, one can show from Eq. (13) that

$$
J^{-1} = e^{-\frac{s^2}{2}} .
$$

(16)

By definition,

$$
n^{(s)} = e^{s} n .
$$

(17)

Thus,

$$
I = \int_{\Delta \xi^{(s)}}^{+\infty} d\Delta x^{(s)} J e^{-n^{(s)} F^{(s)}(\Delta x)}
$$

(18a)

$$
= e^{\frac{s^2}{2}} \sqrt{\frac{\pi}{2n^{(s)} a}} \text{erfc} \left( \Delta \xi^{(s)} \sqrt{\frac{n^{(s)} a}{2}} \right)
$$

(18b)

$$
= \sqrt{\frac{\pi}{2na}} \text{erfc} \left( \Delta \xi \sqrt{\frac{na}{2}} \right)
$$

(18c)

Hence, we see that for a quadratic $F$, $I$-approximation Eq. (14c) differs from the exact answer by a factor of $e^{\frac{s^2}{2}}$. This discrepancy is due to the fact that we neglected the Jacobian in deriving $I$-approximation Eq. (14c).
2 Notation

In this section, we will present some notation that will be used throughout the paper.

RHS and LHS will stand for “right hand side” and “left hand side”, respectively. When we say “x (ditto, y) is A (ditto, B)” we will mean that x is A and y is B.

The number of elements in a set S will be denoted by |S|. Let \( Z_{a,b} = \{a, a+1, a+2, \ldots, b\} \) for any integers \( a \leq b \). Let \( x^{\#n} \) represent an n-letter codeword. We reserve the upper index location for the label of a letter in the codeword. Thus, we will denote the codeword \( i \) and its \( \delta \)-function. Indeed, there is. Suppose \( K > 0 \). The following equation is easy to

\[
\delta(x) = \int_{-\infty}^{+\infty} \frac{dk}{2\pi} e^{ikx} e^{-\epsilon k^2}, \tag{19}
\]

for some infinitesimal \( \epsilon > 0 \). The Dirac function \( \delta(x) \) has unit area: \( \int_{-\infty}^{+\infty} \delta(x) = 1 \), and is sharply peaked at \( x = 0 \). The identity \( \delta(x) = \frac{d}{dx} \theta(x > 0) \) is easily proven using the sharply peaked and unit area properties of \( \delta(x) \). This identity connecting the Dirac delta function and the unit step function leads us to suspect that there is an integral representation for the unit step function, analogous to Eq. (19) for the Dirac delta function. Indeed, there is. Suppose \( K > 0 \). The following equation is easy to
prove using contour integration in the complex plane:

\[ \theta(x > 0) = \frac{1}{2\pi i} \int_{K-i\infty}^{K+i\infty} \frac{dk}{k} e^{kx}. \]  (20)

See Fig. 1. For \( x > 0 \), the integration contour can be deformed so that it wraps around the point \( k = 0 \). By integrating around this pole, it is easy to show that for \( x > 0 \), the RHS of Eq. (20) equals 1. For \( x < 0 \), the integration contour can be deformed so that it wraps around the point \( k = +\infty \). Thus, for \( x < 0 \), the RHS of Eq. (20) equals 0.

![Figure 1: For complex integral Eq. (20), one can deform the contour of integration differently for \( x < 0 \) and \( x > 0 \).](image)

The Shannon entropy associated with the random variable \( x \) will be represented by any of the following:

\[ H_{P_x}(x) = H(P_x) = H(\bar{P}) = H(P(x))_{\forall x} = -\sum_x P(x) \ln P(x). \]  (21)

Likewise, the relative entropy (also called the Kullback Liebler distance) between two probability distributions \( P(x) \) and \( Q(x) \) will be represented by any of the following:

\[ D(P_x/Q_x) = D(\bar{P}/\bar{Q}) = D(P(x)/Q(x))_{\forall x} = \sum_x P(x) \ln \frac{P(x)}{Q(x)}. \]  (22)

We will also use the conditional entropy
\[ H(x|y) = -\sum_{x,y} P(x, y) \ln P(x|y) , \] (23)

and the mutual entropy:

\[ H(x : y) = \sum_{x,y} P(x, y) \ln \frac{P(x, y)}{P(x)P(y)} . \] (24)

Note that we have defined our entropies in terms of base \( e \) rather than base 2 logs. Of course, \( \log_a(x) = \frac{\log_b x}{\log_b a} \) so \( \log_2 x = \frac{\ln x}{\ln 2} \).

Let \( DP = \Pi \{dP(x)\}_{\forall x} \). For any function \( f : \text{Reals}^N \rightarrow \text{Reals} \), define

\[ \int DP f(P) = \prod \left\{ \int_{-\infty}^{\infty} dP(x) \right\}_{\forall x} f(P) , \] (25)

and

\[ \int_{pd(S_x)} DP f(P) = \int DP \theta(P \geq 0)\delta(\sum_x P(x) - 1)f(P) . \] (26)

It is easy to prove by induction that

\[ \int_{pd(S_x)} DP 1 = \frac{1}{(N_x - 1)!} . \] (27)

3 Composition Classes

In this section we will discuss composition classes (CC’s). Often, CC’s are called “types” instead of CC’s, and their theory is referred to as the “Method of Types”. The term “type” is very vague, so we will shun it, and use the more specific term CC. This section reviews and extends standard material on CC’s as found in, for example, the books by Cover and Thomas [2] and the one by Blahut [3].

In the mathematical theory of Statistics, one often considers a sequence of \( n \) random variables \((x^1, x^2, \ldots, x^n) = x^n = \bar{x} \in S^n_x\). Information Theory also deals with such sequences, where they are called a word (or codeword or block) of letters (or symbols) \( x \) from the alphabet \( S_x \). We will assume the simplest case, wherein the \( n \) random variables are independent, identically distributed (i.i.d.), and each \( x^i \) is distributed (“drawn”) according to a probability distribution \( Q : S_x \rightarrow [0, 1] \). In what follows, we will often refer to \( Q \) as the Center of Mass (CM) probability distribution, (The reason for this name will be explained later.)

Let \( n(x|\bar{x}) \) represent the number of times that the letter \( x \) occurs in the word \( \bar{x} \). A composition class \( C(\bar{x}) \) (also called a “type” or “empirical distribution” or “relative frequency”) is defined by

\[ C(\bar{x}) = \{ \bar{y} : \forall x \in S_x, n(x|\bar{x}) = n(x|\bar{y}) \} . \] (28)
Clearly, this defines an equivalence relation on (and a disjoint partition for) the set $S_n^x$. To each CC, there corresponds a probability distribution given by

$$P_{C(x)}(x) = \frac{n(x|\bar{x})}{n}$$  \hspace{1cm} (29)

for all $x \in S_n^x$. In the notation $C(\bar{x})$, the CC is specified by giving one of its elements $\bar{x}$. Alternatively, one can specify a CC by giving its probability distribution:

$$C(P) = \{ \bar{x} : P_{C(\bar{x})} = P \}.$$  \hspace{1cm} (30)

Hence $C(P_{C(\bar{y})}) = C(\bar{y})$.

Define $\left( \begin{array}{c} S_n^a \\ S_n^b \end{array} \right)$ to be the set of all $2 \times n$ matrices $\left( \begin{array}{c} \bar{a} \\ \bar{b} \end{array} \right)$, where $\bar{a} \in S_n^a$ and $\bar{b} \in S_n^b$ are $n$-dimensional row vectors. For some $x^n \in \left( \begin{array}{c} S_n^a \\ S_n^b \end{array} \right)$, the CC denoted by $C(x^n) = C(\left( \begin{array}{c} \bar{a} \\ \bar{b} \end{array} \right))$ is defined as before, as the set of all $2 \times n$ matrices $y^n \in \left( \begin{array}{c} S_n^a \\ S_n^b \end{array} \right)$ such that, for all column vectors $x = \left( \begin{array}{c} a \\ b \end{array} \right)$ with $a \in S_n^a$ and $b \in S_n^b$, one has $n(x|y^n) = n(x|x^n)$.

For any $A \subset S_n^a$, it is convenient to define the following two sets:

$$C(A) = \{ C(\bar{x}) : \forall \bar{x} \in A \}.$$  \hspace{1cm} (31)

$$P(A) = \{ P_{C(\bar{x})} : \forall \bar{x} \in A \} \subset pd(S_n^x).$$  \hspace{1cm} (32)

Note that these two sets are in 1-1 correspondence. For $A = S_n^a$, they become $C(S_n^a)$ and $P(S_n^a)$.

For large $n$, we can easily estimate the number of elements in a CC and the number of $C(\bar{x})$ for all $\bar{x} \in S_n^a$.

**Claim 3.1** As $n \to \infty$,

$$|C(\bar{x})| \approx \frac{\exp \left[ nH(P_{C(\bar{x})}) \right]}{(2\pi n)^{\frac{1}{2}(N_2-1)}} \sqrt{\prod P_{C(\bar{x})}},$$  \hspace{1cm} (33)

and

$$|C(S_n^a)| \approx \frac{n^{N_2-1}}{(N_2 - 1)!}.$$  \hspace{1cm} (34)
proof:

The exact number of elements in $C(\vec{x})$ is given by

$$|C(\vec{x})| = \frac{n!}{\prod \{n(x|\vec{x})!\}_{\forall x}} .$$  \hspace{1cm} (35)

Recall the first term of Stirling’s asymptotic expansion, for large $n$, of the factorial $n!$:

$$n! \approx \sqrt{2\pi n} \ e^{-n} n^n .$$  \hspace{1cm} (36)

Applying this approximation to the factorials in Eq. (35) immediately yields Eq. (33). Ref. [2] proves that $|C(\vec{x})|$ is bounded below and above as follows:

$$\frac{1}{(n+1)^{N_{\vec{x}}}} \leq |C(\vec{x})| \leq \exp \left[ nH(P_C(\vec{x})) \right] .$$  \hspace{1cm} (37)

Since $P_{C(\vec{x})} = \frac{1}{n}(n_1, n_2, \ldots, n_{N_{\vec{x}}})$, where $n_1, n_2, \ldots, n_{N_{\vec{x}}} \in \mathbb{Z}_{0,n}$, it follows that the exact number of CC’s in $pd(S_{\vec{x}})$ is given by

$$|C(S_{\vec{x}}^n)| = \sum_{n_1=0}^{n} \sum_{n_2=0}^{n} \cdots \sum_{n_{N_{\vec{x}}}=0}^{n} \delta(\sum_{j=1}^{N_{\vec{x}}} n_j, n) .$$  \hspace{1cm} (38)

The previous equation immediately implies that

$$|C(S_{\vec{x}}^n)| \leq (n + 1)^{N_{\vec{x}}} .$$  \hspace{1cm} (39)

Suppose $f : \text{Reals} \to \text{Reals}$. For large $n$:

$$\sum_{k=0}^{n} f(k) \approx \int_{0}^{n+1} dk \ f(k) .$$  \hspace{1cm} (40)

For any $n$:

$$\sum_{k=0}^{n} \delta(k, k_0) = \theta(0 \leq k_0 \leq n) = \int_{0}^{n} dk \ \delta(k - k_0) .$$  \hspace{1cm} (41)

We can use the previous two equations to approximate all sums in Eq. (38) by integrals. This yields:

$$|C(S_{\vec{x}}^n)| \approx \int_{0}^{n+1} dn_1 \int_{0}^{n+1} dn_2 \cdots \int_{0}^{n+1} dn_{N_{\vec{x}}} \delta(\sum_{j=1}^{N_{\vec{x}}} n_j - n)$$  \hspace{1cm} (42a)

$$\approx n^{N_{\vec{x}}-1} \int_{0}^{1} dP_1 \int_{0}^{1} dP_2 \cdots \int_{0}^{1} dP_{N_{\vec{x}}} \delta(\sum_{j=1}^{N_{\vec{x}}} P_j - 1)$$  \hspace{1cm} (42b)

$$\approx \frac{n^{N_{\vec{x}}-1}}{(N_{\vec{x}}-1)!} .$$  \hspace{1cm} (42c)
QED

Let $Q(\vec{x})$ stand for the joint probability of the components of $\vec{x}$. Since we will assume that these components are i.i.d.,

$$Q(\vec{x}) = \prod Q(x^i)_{\forall i}. \hspace{1cm} (43)$$

For $A \subset S_x^n$, let

$$Q(A) = \sum_{\vec{x} \in A} Q(\vec{x}). \hspace{1cm} (44)$$

$Q(\vec{x})$ can be expressed in terms of relative entropy as follows:

$$Q(\vec{x}) = \prod Q(x)_{\forall x,} \hspace{1cm} (45a)$$

$$= \exp \left[ n \sum_x P_C(\vec{x})(x) \ln Q(x) \right] \hspace{1cm} (45b)$$

$$= \exp \left[ -n H(P_C(\vec{x})) - n D(P_C(\vec{x})/Q) \right]. \hspace{1cm} (45c)$$

Combining this expression for $Q(\vec{x})$ with the approximation Eq. (33) for $|C(\vec{x})|$ yields

$$Q(C(\vec{x})) = |C(\vec{x})|Q(\vec{x}) \approx \frac{\exp \left[ -n D(P_C(\vec{x})/Q) \right]}{(2\pi n)^{\frac{1}{2} (N_x - 1)}} \sqrt{\prod P_C(\vec{x})}. \hspace{1cm} (46b)$$

Figure 2: Probability simplex $pd(S_x)$ for $N_x = 3$. Two especially important points of the simplex are its geometric center $\Omega$ and its center of mass (CM) $Q$. The graph on the right illustrates how the entropy $H(P)$ decreases monotonically as point $P$ moves from the geometric center to the edges.
The set \( pd(S_n) \) is in 1-1 correspondence with a simplex in space \( \text{Reals}^N \). For example, for \( N = 3 \), this probability simplex is the region of \( \text{Reals}^3 \) that connects the corners \((1, 0, 0), (0, 1, 0) \) and \((0, 0, 1)\). Fig 2 shows \( pd(S_n) \) for \( N = 3 \). The probability distributions \( P_{C(\vec{x})} \) form a finite subset of this simplex. In Fig 2, the \( P_{C(\vec{x})} \) are represented by dots inside \( pd(S_n) \). Other notable points of \( pd(S_n) \) are its geometric center \( \Omega = \left( \frac{1}{N} \right)^n \) and the CM distribution \( Q(x) \). From Eq. (43), it follows that the closer a CC is to the geometric center \( \Omega \), the more elements the CC has. If we represent CC’s by points of \( pd(S^n) \) with varying diameters, where fatter points represent CC’s with more elements, then the diameter of the points decreases as we travel away from \( \Omega \). From Eq. (16), it follows that the closer a CC is to the CM distribution \( Q(x) \), the more probable the CC is. As in Fig 2, if we show only the most probable CC’s, then most of the CC’s shown cluster around the point \( Q(x) \) (This is why we call \( Q(x) \) and \( Q(\vec{x}) \) the CM distribution.)

As mentioned in the introduction, most RG methods comprise an iterative step, (i.e., a step that is performed repeatedly) consisting of a decimation followed by a rescaling. CCRG is slightly different from this. In CCRG, we perform a preliminary reduction that reduces a very large (i.e., infinite as \( n \to \infty \)) number of degrees of freedom to a small, fixed (i.e., \( n \) independent) number. This is accomplished by replacing sums like \( \sum_{\vec{x}} \), that run over \( n \) discrete degrees of freedom, by integrals like \( \int D\vec{x} \), that run over the far fewer \( N \) continuous degrees of freedom that specify a point of \( pd(S_n) \). After this preliminary reduction, we perform an iterative step consisting of an infinitesimal rescaling of \( n \) followed by a rescaling of all other parameters in such a way that the form of the partition function is not changed by the iterative step.

The following two claims embody the preliminary reduction step of CCRG.

**Claim 3.2 (Reduction Formula 1)** Suppose \( f : pd(S_n) \to \text{Reals} \). Define

\[
I(f) = r(n, N) \int_{pd(S_n)} D\vec{x} \frac{e^{nH(P)}}{\prod \{P\}} f(P) , \tag{47}
\]

where

\[
r(n, N) = \left( \frac{n}{2\pi} \right)^{(N-1)/2} . \tag{48}
\]

Then

\[
\sum_{\vec{x}} f(P_{C(\vec{x})}) \approx I(f) , \tag{49}
\]

and

\[
I(1) \approx N^m . \tag{50}
\]
proof:

\[
\sum_{\vec{x}} f(P_{C(\vec{x})}) = \sum_{C(\vec{x}) \in \mathcal{C}(S_n^\nu)} |C(\vec{x})| f(P_{C(\vec{x})}) 
\]

(51a)

\[
= |\mathcal{C}(S_n^\nu)| \frac{\sum_{C(\vec{x}) \in \mathcal{C}(S_n^\nu)} |C(\vec{x})| f(P_{C(\vec{x})})}{\sum_{C(\vec{x}) \in \mathcal{C}(S_n^\nu)}} 
\]

(51b)

\[
= |\mathcal{C}(S_n^\nu)| \frac{\sum_{P_{C(\vec{x})} \in \mathcal{P}(S_n^\nu)} |C(\vec{x})| f(P_{C(\vec{x})})}{\sum_{P_{C(\vec{x})} \in \mathcal{P}(S_n^\nu)}} 
\]

(51c)

\[
\approx |\mathcal{C}(S_n^\nu)| \frac{\int_{pd(S_n^\nu)} \mathcal{D}P |C(\vec{x})| f(P)}{\int_{pd(S_n^\nu)} \mathcal{D}P} 
\]

(51d)

\[
\approx I(f) 
\]

(51e)

In Eq.(51), we went from line (d) to (e) by substituting previously derived values for |\mathcal{C}(S_n^\nu)|, |C(\vec{x})| and \int_{pd(S_n^\nu)} \mathcal{D}P. This proves Eq.(49).

If we substitute \( f = 1 \) into the LHS of Eq.(49), we get \( N_n^\nu \). But what if we substitute \( f = 1 \) into the RHS of Eq.(49)? Does this also yield \( N_n^\nu \)? Yes. Let’s see how.

Define \( \Delta P(x) = P(x) - \frac{1}{N_x^n} \). If we expand \( H(P) \) about the point \( \Omega \), we get: (See Appendix B for a compendium of Taylor expansions related to Information Theory)

\[
H(P) \approx \ln N_x^n - \frac{N_x^n}{2} \sum_x [\Delta P(x)]^2 + \mathcal{O}((\Delta P)^3). 
\]

(52)

For large \( n \), most of \( I(1) \) comes from the vicinity of \( \Omega \). Since \( \Omega \) is far away from the boundary of the probability simplex, the constraint \( \theta(P \geq 0) \) can be ignored in \( I(1) \). Thus, \( I(1) \) can be approximated by:

\[
I(1) \approx r(n, N_x^n) N_x^n \frac{n}{2} \int \mathcal{D}P \delta(\sum_x \Delta P(x)) \exp \left(-\frac{nN_x^n}{2} \sum_x [\Delta P(x)]^2 \right) 
\]

(53a)

\[
\approx N_x^n. 
\]

(53b)

In Eq.(53), to go from line (a) to (b), we performed the integration using the Gaussian integration formulae of Appendix C. QED

Claim 3.3 (Reduction Formula 2) Suppose \( f : pd(S_n^\nu_y) \rightarrow \text{Reals} \). Define

\[
J(f) = \frac{r(n, N_x^n - N_y^n)}{\prod \{P_y^n\}^{N_x^n - 1}} \int \mathcal{D}P_y^n \theta(P_y^n \geq 0) \prod \{\delta(P_y^n(y) - P_C(\vec{y})(y))\}_{y^n_y} 
\]

\[
\exp[nH_{P_x^n}(x^n|y^n)] \frac{\sqrt{\prod \{P(x|y^n)\}_{y^n_x,y^n}} f(P_x^n)}{\prod \{P(x|y^n)\}_{y^n_x,y^n}}. 
\]

(54)
Then

\[ \sum_{\vec{x}} f(P_{\vec{C}}(\vec{x}, \vec{y})) \approx J(f), \quad (55) \]

and

\[ J(1) \approx N_\Lambda^n. \quad (56) \]

**proof:** Clearly,

\[ \sum_{\vec{x}} f(P_{\vec{C}}(\vec{x}, \vec{y})) = \sum_{\vec{x}_1, \vec{y}_1} \delta(\vec{y}_1, \vec{y}) f(P_{\vec{C}}(\vec{x}_1, \vec{y}_1)). \quad (57) \]

We would like to transform the sum over the words \( \vec{x}_1 \) and \( \vec{y}_1 \) into a sum over “coarser” items: namely, a sum over CC’s like \( C(\vec{x}_1, \vec{y}_1) \). These CC’s are in 1-1 correspondence with their probability distributions \( P_{\vec{C}}(\vec{x}_1, \vec{y}_1) \), and a sum over these distributions can be approximated by an integral over the probability simplex \( pd(S_{\vec{y}}) \). All this can be accomplished if we approximate the Kronecker delta for points \( \vec{y} \) by a suitably normalized Dirac delta function for distributions \( P_{\vec{C}}(\vec{y}) \). So let us do the following replacement:

\[ \delta(\vec{y}_1, \vec{y}) \rightarrow K \prod \{\delta(P_{\vec{C}(\vec{y})}(y) - P_{\vec{C}(\vec{y})}(y))\}_{\vec{y}}. \quad (58) \]

We choose the value of the normalization constant \( K \) to be

\[ K = \frac{\sqrt{\prod\{P_{\vec{C}(\vec{y})}\}}} {r(n, N_y) \delta(\sum_y P_{\vec{C}(\vec{y})}(y) - 1)}. \quad (59) \]

(Division by a Dirac delta function is allowed as an intermediate step, before taking the \( \epsilon \) parameter of Eq.(19) to zero.) The reason for choosing this value for \( K \) is as follows. Using Reduction Formula 1 and Eq.(58), one gets

\[ 1 = \sum_{\vec{y}} \delta(\vec{y}_1, \vec{y}) \approx r(n, N_y) \int \mathcal{D}P_y \theta(P_y \geq 0) \delta(\sum_y P_y(y) - 1) \frac{\exp[-nH(P_y)]} {\sqrt{\prod\{P_y\}}} K \prod \{\delta(P_{\vec{C}(\vec{y})}(y) - P_{\vec{y}}(y))\}_{\vec{y}}. \quad (60) \]

The previous equation is satisfied for the value of \( K \) given by Eq.(59).
To show Eq. (55), one replaces the Kronecker delta $\delta(y_1, y)$ in the RHS of Eq. (57) by a coarser delta, in accordance with the prescription Eq. (58). Then one applies Reduction Formula 1 to the result. This proves Eq. (55).

If we substitute $f = 1$ into the LHS of Eq. (55), we get $N^n_x$. But what if we substitute $f = 1$ into the RHS of Eq. (55)? Does this also yield $\tilde{N}^n_x$? Yes. Here is a sketch of the proof. The proof comprises two main steps: First, use the results of Appendix D to convert $J(1)$ from an integral of the form $\int \prod \{dP(x, y)\} \forall x, y (\cdot)$ to a product over all $y$ of integrals of the form $\int \prod \{dP(x|y)\} \forall x (\cdot)$. Second, apply the Gaussian integration formulae of Appendix C. QED

4 Noiseless Coding

In this section we will discuss Noiseless Coding (i.e., a coding used in compression). In particular, we will calculate the probability of error, in the limit of large word size $n$, for compression using the Csiszár-Körner (CK) universal code.

4.1 Error Model

This section reviews the usual error model for compression using CK universal code. Subsequent sections will apply CCRG to it.

![Diagram of encoding and decoding](image)

**Figure 3:** Encoding and Decoding maps for Noiseless Coding.

A block source emits a stream of $n$-letter words $(x^1, x^2, \ldots, x^n) = x^n = \vec{x} \in S^n_Z$. Each word is modelled as a sequence of $n$ i.i.d. random variables $x^i$ distributed according to $Q(x^i)$, where $i \in Z_{1,n}$. 

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Suppose that, as shown in Fig. 3: (1) Each word $\vec{x} \in S^n$ is mapped by an encoder function $E$ into a message $E(\vec{x}) = m \in \mathbb{Z}_0^M$. (2) Each message $m$ is in turn mapped by a decoder function $D$ into a word $D(m) = \vec{x}' \in S^n$. A block code is characterized by: the probability distribution $Q$ of its source, its encoder function $E$ and its decoder function $D$. The block code is said to be universal if $E$ and $D$ do not depend on $Q$.

Assume that $|\text{Image}(E)| = |E(S^n)| \approx M$. The compression factor or code rate $R$ of the encoder is defined by

$$R = \frac{\ln M}{n} = (\ln 2) \frac{\log_2 M}{n}.$$ (61)

Note that if $N \geq 2$, then $\frac{\log_2 M}{n} = \frac{n_{\text{out}}}{n_{\text{in}}}$ where $n_{\text{out}}$ (ditto, $n_{\text{in}}$) is the encoder output (ditto, input) measured in bits. Note that $R \leq \ln N$ because $M \leq N^n$. For a fixed rate block code, $R$ is fixed as $n \to \infty$.

The probability of error for the code is given by

$$p_{\text{err}} = \sum_{\vec{x}} Q(\vec{x}) \theta(D \circ E(\vec{x}) \neq \vec{x}).$$ (62)

Assume a fixed rate block code and let

$$\tilde{R} = R - N \frac{\ln(n + 1)}{n}.$$ (63)

Of course, for large $n$, $\tilde{R} \approx R$. Let

$$A_{\text{pass}} = \{\vec{x} \in S^n \mid H(P_{C(\vec{x})}) \leq \tilde{R}\}, \quad A_{\text{stop}} = S^n - A_{\text{pass}}.$$ (64)

$|A_{\text{pass}}| \leq M$ because

$$|A_{\text{pass}}| = \sum_{C(\vec{x})} |C(\vec{x})| \theta(H(P_{C(\vec{x})}) \leq \tilde{R})$$ (65a)

$$\leq \sum_{C(\vec{x})} e^{nH(P_{C(\vec{x})})} \theta(H(P_{C(\vec{x})}) \leq \tilde{R})$$ (65b)

$$\leq e^{n\tilde{R}} |C(S^n)|$$ (65c)

$$\leq e^{n\tilde{R}} (n + 1)^N = e^{n\tilde{R}} = M.$$ (65d)

If $|A_{\text{pass}}| \approx M$, then $|A_{\text{stop}}| \approx N^n - M = e^{n\ln N} - e^{n\tilde{R}}$. Since $R \leq \ln N$, $|A_{\text{stop}}| \gg |A_{\text{pass}}|$ for large $n$.

We can number the elements of $A_{\text{pass}}$ from 1 to $|A_{\text{pass}}|$. Call $m(\vec{x})$ the number assigned to $\vec{x} \in A_{\text{pass}}$. The CK universal code is a fixed rate block code with encoding and decoding functions defined by:

$$E(\vec{x}) = \begin{cases} m(\vec{x}) \text{ if } \vec{x} \in A_{\text{pass}} \\ 0 \text{ if } \vec{x} \notin A_{\text{pass}} \end{cases}.$$ (66)
\[ D(m) = \begin{cases} E^{-1}(m) & \text{if } m \in Z_{\leq |A_{\text{pass}}|} \\ \text{any } \bar{x} \notin A_{\text{pass}} & \text{if } m = 0 \end{cases} \] (67)

Note that low entropy words (i.e., those \( \bar{x} \) with \( H(P_C(\bar{x})) < R \)) belong to \( A_{\text{pass}} \) and are coded, whereas the high entropy words (i.e., those \( \bar{x} \) with \( H(P_C(\bar{x})) > R \)) belong to \( A_{\text{stop}} \) and are not coded. Thus, the CK universal code can be described as a low pass filter of word entropy. Why are low entropy words preferable to high entropy ones for coding? Because for \( R = H(Q) \), \( Q(A_{\text{pass}}) \) and \( Q(A_{\text{stop}}) \) are comparable even though \( |A_{\text{pass}}| < |A_{\text{stop}}| \). Note that

\[ \theta(D \circ E(\bar{x}) \neq \bar{x}) = \theta(\bar{x} \notin A_{\text{pass}}) = \theta(H(P_C(\bar{x})) > R) . \] (68)

Thus, for CK universal coding,

\[ p_{err} = \sum_{\bar{x}} Q(\bar{x}) \theta(H(P_C(\bar{x})) > R) . \] (69)

Applying Reduction Formula 1 to the RHS of the previous equation yields

\[ p_{err} \approx r(n, N_{n}) \int_{pd(S_{n})} D \frac{e^{-nD(P/Q)}}{\sqrt{\Pi\{P\}}} \theta(H(P) > R) . \] (70)

In the previous equation, the exponential inside the integral reaches its maximum value when \( D(P/Q) = 0 \). If we approximate \( P \) by \( Q \) in the theta function of the integrand, then we can pull the theta function out of the integral. Doing this yields

\[ p_{err} \approx \theta(H(Q) > R) . \] (71)

In other words, if the compression factor \( R \) is larger (ditto, smaller) than \( H(Q) \), then the probability of error is zero (ditto, one). The next few sections of this paper will be dedicated to improving this estimate of \( p_{err} \).

### 4.2 Old Approximation for \( p_{err} \)

In this section, we will review the standard calculation (see \[3\]) of the error exponent for CK universal coding. In the next section, we will calculate the error exponent (and much more) using CCRG.

The standard way of finding the error exponent for CK universal coding is equivalent to using Laplace’s Method to find the leading term of an asymptotic expansion of Eq.(70). To apply Laplace’s Method, we must minimize \( D(P/Q) \) over all \( P \in pd(S_{n}) \), subject to the inequality constraint \( H(P) > R \).

To obtain a minimum point \( \bar{x}^* \in \text{Reals}^n \) of a smooth, real-valued function \( f(\bar{x}) \), subject to equality constraints \( c_j(\bar{x}) = 0 \) for \( j \in C_{eq} \), one can use the well known method of Lagrange multipliers. But suppose that, in addition to these equality constraints, \( \bar{x}^* \) must also satisfy inequality constraints \( c_j(\bar{x}) \geq 0 \) for \( j \in C_{geq} \). To
obtain a minimum $\bar{x}^*$ in this more complicated case, one can generalize the method of Lagrange multipliers. Kuhn and Tucker, among others, have done this. Let $J = C_{eq} \cup C_{geq}$, and define the Lagrangian function $\mathcal{L} = f(\bar{x}) - \sum_{j \in J} \lambda_j c_j(\bar{x})$. According to Kuhn-Tucker, the minimum point $\bar{x}$ and the Lagrange multipliers $(\lambda_j)_{j \in J}$ must satisfy the Kuhn-Tucker conditions\textsuperscript{[5]} given by (1) $\nabla_{\bar{x}} \mathcal{L} = 0$, (2) $\forall j \in C_{eq}$, one has $c_j(\bar{x}) = 0$ (3)$\forall j \in C_{geq}$, one has $c_j(\bar{x}) \geq 0$, $\lambda_j \geq 0$ and $\lambda_j c_j(\bar{x}) = 0$.

Let

$$\mathcal{L} = D(P//Q) - \lambda(H(P) - R) + \mu\left(\sum_{x} P(x) - 1\right).$$

(72)

For the problem we are considering here, the Kuhn-Tucker conditions are (1) $\forall x$, $\frac{\partial \mathcal{L}}{\partial P(x)} = 0$, (2)$\sum_{x} P(x) = 1$, (3) $H(P) - R \geq 0$, $\lambda \geq 0$, $\lambda(H(P) - R) = 0$. We will assume that the inequality constraint is “active”\textsuperscript{[5]}, in which case condition (3) reduces to

Figure 4: $A_{pass}$ when $H(Q)$ is greater or smaller than $R$. Strictly speaking, $A_{pass}$ is a set of $\bar{x}$, and what we are showing is $P(A_{pass})$ instead of $A_{pass}$.
\[ H(P) = R. \] Condition (1) implies:

\begin{align*}
0 &= \ln \frac{P(x)}{Q(x)} + 1 - \lambda(-\ln P(x) - 1) + \mu \\
&= \ln(P^{1+\lambda}(x)) - \ln Q(x) + 1 + \lambda + \mu. \quad (73a)
\end{align*}

The previous equation is satisfied by

\[ P^{(\lambda)}(x) = \frac{Q(x)^{1+\lambda}}{Z}, \quad (74)\]

where

\[ Z = \sum_x Q(x)^{1+\lambda}. \quad (75)\]

This value for \( P^{(\lambda)} \) satisfies \( \sum_x P^{(\lambda)}(x) = 1 \), but does not yet satisfy \( H(P^{(\lambda)}) = R \). The equation \( H(P^{(\lambda)}) = R \) defines a unique value of \( \lambda \).

Define

\[ \gamma(\lambda) = \min_{P,\mu} \mathcal{L} = D(P^{(\lambda)} \| Q). \quad (76)\]

Substituting the value for \( P^{(\lambda)} \) given by Eq.(74) into \( D(P^{(\lambda)} \| Q) \) yields:

\[ \gamma(\lambda) = \lambda R - (1 + \lambda) \ln Z. \quad (77)\]

\( P^{(\lambda)} \) and \( \gamma(\lambda) \) still depend on a parameter \( \lambda \) which is specified implicitly by the equation \( H(P^{(\lambda)}) = R \). In fact, one can show that \( H(P^{(\lambda)}) = R \) iff \( \frac{d\gamma(\lambda)}{d\lambda} = 0 \).

Define the error exponent \( \gamma \) by

\[ \gamma = \max_{\lambda \geq 0} \gamma(\lambda). \quad (78)\]

It is now clear that \( p_{err} \) given by Eq.(70) can be approximated by:

\[ p_{err} \approx e^{-n\gamma}, \text{ where } \gamma = \max_{\lambda \geq 0}[\lambda R - (1 + \lambda)Z(\lambda)]. \quad (79)\]

Eq.(79) is the traditional asymptotic approximation for the probability of error for CK universal coding.

### 4.3 New (CCRG) Approximation for \( p_{err} \)

In this section and the next one, we will use CCRG to calculate the probability of error for compression using the CK universal code. In this section, we will calculate \( p_{err} \) as given by Eq.(70), assuming that we have rescaled the variables of the RHS of
Eq. (70) so that the integrand is a Gaussian. In the next section, we will derive the RG equations that characterize this rescaling.

Let $P = P - Q$, $\Delta H(P) = H(P) - H(Q)$, and $\Delta R = R - H(Q)$. Hence, $	heta(H(P) > R) = \theta(\Delta H(P) > \Delta R)$.

Let
\[ \mathcal{L} = D(P||Q) - \lambda(\Delta H(Q) - \Delta R) + \mu(\sum_x P(x) - 1) . \] (80)

Minimizing this Lagrangian with respect to $P, \lambda, \mu$ gives the saddle (or boundary) point $P^*$ that dominates the integral given by Eq. (70). Unfortunately, finding an explicit expression for $P^*$ is not possible.

Define test fractions $\Phi_0$ and $\Phi_1$ by
\[ \Phi_0 = \left| \frac{D(P^*||Q)}{\sum_x \frac{\Delta P^*(x)^2}{2Q(x)}} - 1 \right| , \] (81)

\[ \Phi_1 = \left| \frac{H(P^*) - H(Q)}{-\sum_x \Delta P^*(x) \ln Q(x)} - 1 \right| . \] (82)

$\Phi_0$ (ditto, $\Phi_1$) measures how much $D(P^*||Q)$ (ditto, $\Delta H(P^*)$) differs from the leading term of its Taylor expansion about $Q$. (See Appendix B for a compendium of Taylor expansions related to Information Theory).

Suppose we have rescaled the variables in the RHS of Eq. (70) so that after rescaling, we are in the “Gaussian region”: $\Phi_0 << 1$ and $\Phi_1 << 1$. Then Eq. (70) can be approximated by

\[ p_{err} \approx \frac{r(n, N_x)}{\sqrt{\Pi(P^*)}} \int D\Delta P \delta(\sum_x \Delta P(x)) \exp\left[-n \sum_x \frac{[\Delta P(x)]^2}{2Q(x)} \right] \theta\left(-\sum_x \Delta P(x) \ln Q(x) > \Delta R \right) . \] (83)

(For large $n$, if $Q$ is not too close to the boundary of the probability simplex, then the constraint $\theta(P \geq 0)$ can be ignored.)

In the Gaussian region, we can also approximate Eq. (80) by

\[ \mathcal{L} = \sum_x \frac{[\Delta P(x)]^2}{2Q(x)} - \lambda(-\sum_x \Delta P(x) \ln Q(x) - \Delta R) + \mu(\sum_x P(x) - 1) . \] (84)

Minimizing this Lagrangian with respect to $P, \lambda, \mu$ gives the point $P^*$ that dominates the integral given by Eq. (83). Finding an explicit expression for $P^*$ in the Gaussian region is possible.

\[ \frac{\partial \mathcal{L}}{\partial P(x)} = 0 \text{ gives:} \]
\[ \frac{\Delta P(x)}{Q(x)} + \lambda \ln Q(x) + \mu = 0 . \] (85)
Enforcing the constraints \(- \sum_x \Delta P(x) \ln Q(x) = \Delta R\) and \(\sum_x P(x) = 1\) then yields

\[
\Delta P^*(x) = B(x) \Delta R ,
\]

where

\[
B(x) = \frac{\beta(x) Q(x)}{\langle \beta^2 \rangle} ,
\]

\[
\beta(x) = -[\ln Q(x) + H(Q)] ,
\]

\[
\langle \beta \rangle = \sum_x Q(x) \beta(x) = 0 ,
\]

\[
\langle \beta^2 \rangle = \sum_x Q(x) \beta^2(x) .
\]

On the RHS of Eq.(83), we can apply the Gaussian Integration Formulae of Appendix C. We can also substitute there the value for \(P^*\) given by Eq.(86). Doing so finally gives

\[
\rho_{\text{err}} \approx \frac{1}{2u} \text{erfc} \left( \frac{\Delta R \sqrt{\frac{n}{2\langle \beta^2 \rangle}}}{\sqrt{\prod \left\{ 1 + \frac{\beta(x) \Delta R}{\langle \beta^2 \rangle} \right\}} \right) ,
\]

where

\[
u = \sqrt{\prod \left\{ 1 + \frac{\beta(x) \Delta R}{\langle \beta^2 \rangle} \right\}} \forall x .
\]

Appendix A reviews some basic properties of the Error Function \(\text{erf()}\) and its complement \(\text{erfc()}\).

### 4.4 RG Equations

In this section, we will calculate the RG equations for compression using CK universal coding.

Important: In this section, \(\Delta P^{(s)}\) describes the motion, upon successive rescalings, of the point that dominates the integral of Eq.(70).

Consider the argument of the exponential in the integrand of Eq.(70). It should be invariant under a change of scale:

\[
n^\wedge D^\wedge (P//Q) = n D(P//Q) .
\]

If for some \(\delta s\) such that \(0 \leq \delta s << 1\),

\[
n^\wedge = e^{\delta s} n ,
\]
then

\[ D^\wedge (P//Q) = D(P^\wedge //Q) = e^{-\delta s} D(P//Q). \quad (95) \]

Define

\[ P^\wedge (x) = P^{(\delta s)}(x) = (1 - \gamma_0 \delta s)P(x) + (\gamma_0 \delta s)Q(x). \quad (96) \]

Then, for \( s > 0 \),

\[ \frac{\partial \Delta P^{(s)}(x)}{\partial s} = -\gamma_0 (P^{(s)}, Q) \Delta P^{(s)}(x), \quad (97) \]

where

\[ \gamma_0 (P, Q) = \lim_{s \to 0} \frac{(-1) \frac{\partial \Delta P^{(s)}}{\partial s}}{\Delta P^{(s)}}. \quad (98) \]

By virtue of Eq. (95),

\[ \frac{\partial D(P^{(s)}//Q)}{\partial s} = -D(P^{(s)}//Q), \quad (99) \]

where

\[ 1 = \lim_{s \to 0} \frac{(-1) \frac{\partial D(P^{(s)}//Q)}{\partial s}}{D(P^{(s)}//Q)}. \quad (100) \]

Note that

\[ \lim_{s \to 0} \frac{\partial D(P^{(s)}//Q)}{\partial s} = \lim_{s \to 0} \sum_x \frac{\partial P^{(s)}}{\partial s} (\ln \frac{P^{(s)}(x)}{Q(x)} + 1) \quad (101a) \]

\[ = -\gamma_0 \sum_x \Delta P(x)(\ln \frac{P(x)}{Q(x)} + 1) \quad (101b) \]

\[ = -\gamma_0 [D(P//Q) + D(Q//P)]. \quad (101c) \]

Thus

\[ \gamma_0 (P, Q) = \frac{D(P//Q)}{D(P//Q) + D(Q//P)}. \quad (102) \]

Now consider the theta function in the integrand of Eq. (70). It too should be invariant under a change of scale:

\[ \theta(\Delta H^\wedge (P) > \Delta R^\wedge) = \theta(\Delta H(P) > \Delta R). \quad (103) \]

If for some \( \delta s \) such that \( 0 \leq \delta s << 1 \),

\[ \Delta R^\wedge = e^{-\gamma_1 \delta s} \Delta R, \quad (104) \]
then
$$
\Delta H^\wedge(P) = \Delta H(P^\wedge) = e^{-\gamma_1 \delta s} \Delta H(P).
$$

Eqs. (104) and (105) imply
$$
\frac{\partial V^{(s)}}{\partial s} = -\gamma_1 (P^{(s)}, Q)V^{(s)},
$$

where
$$
V^{(s)} = \begin{pmatrix}
\Delta R^{(s)} \\
\Delta H(P^{(s)})
\end{pmatrix},
$$

and
$$
\gamma_1(P, Q) = \lim_{s \to 0} \frac{(-1) \frac{\partial \Delta R^{(s)}}{\partial s}}{\Delta R^{(s)}} = \lim_{s \to 0} \frac{(-1) \frac{\partial \Delta H(P^{(s)})}{\partial s}}{\Delta H(P^{(s)})}.
$$

Note that
$$
\Delta H(P) = -\sum_x \Delta P(x) \ln P(x) + D(Q//P).
$$

Interchanging $P$ and $Q$ in the previous equation also yields:
$$
-\Delta H(P) = +\sum_x \Delta P(x) \ln Q(x) + D(P//Q).
$$

Note that
$$
\lim_{s \to 0} \frac{\partial \Delta H(P^{(s)})}{\partial s} = (-1) \lim_{s \to 0} \sum_x \frac{\partial P^{(s)}}{\partial s} (\ln P^{(s)}(x) + 1) = \gamma_0 [-\Delta H(P) + D(Q//P)].
$$

Thus,
$$
\gamma_1(P, Q) = \left(1 - \frac{D(Q//P)}{\Delta H(P)}\right) \gamma_0(P, Q).
$$

We will call $\gamma_0$ and $\gamma_1$ the critical exponents for $\Delta P^{(s)}$ and $\Delta R^{(s)}$, respectively. Note that $\gamma_0(P, Q)$ and $\gamma_1(P, Q)$ both tend to $\frac{1}{2}$ as $P \to Q$. Note also that $\gamma_0$ and $\gamma_1$ are related to the test fraction $\Phi_1$ as follows. Define $\phi \geq 0$ by
$$
\phi = \frac{\gamma_1}{\gamma_0} - 1 = \left|\frac{D(Q//P)}{\Delta H(P)}\right|.
$$
\[ \Phi_1 = \frac{\Delta H(P) + \sum_x \Delta P(x) \ln Q(x)}{\sum_x \Delta P(x) \ln Q(x)} \]  
(114a)

\[ = \frac{D(P//Q)}{\Delta H(P) + D(P//Q)} \]  
(114b)

\[ \approx \frac{\phi}{1 + \phi} . \]  
(114c)

In conclusion, we must solve the following pair of coupled RG equations,

\[ \frac{\partial \Delta P(s)(x)}{\partial s} = -\gamma_0(Q + \Delta P(s), Q)\Delta P(s)(x) \]  
(115)

for all \( x \in S_x \), and

\[ \frac{\partial \Delta R(s)}{\partial s} = -\gamma_1(Q + \Delta P(s), Q)\Delta R(s) . \]  
(116)

We must solve this pair of RG equations subject to the following pair of boundary conditions: At \( s = 0 \):

\[ \Delta R^{(0)} = \Delta R , \]  
(117)

and at \( s = s_{\text{fin}} \):

\[ \Delta P^{(s_{\text{fin}})}(x) = B(x)\Delta R^{(s_{\text{fin}})} . \]  
(118)

\( s_{\text{fin}} \) is defined as any \( s \) large enough for the following to be true: \( \Phi_0(P^{(s_{\text{fin}})}, Q) \ll 1 \) and \( \Phi_1(P^{(s_{\text{fin}})}, Q) \ll 1 \).

Section 6 describes a computer program called WimpyRG-C1.0 that solves these RG equations.

## 5 Noisy Coding

In this section, we will discuss Noisy Coding (i.e., a coding used in channel transmission). In particular, we will calculate the probability of error, in the limit of large word size \( n \), for channel transmission using random encoding and maximum-likelihood decoding.

### 5.1 Error Model

In this section we will review the error model for channel transmission using random encoding and maximum-likelihood decoding. Subsequent sections will apply CCRG to it.
Suppose that, as shown in Fig. 5: (1) Each message \( m \in Z_{1,M} \) is mapped by an encoder function \( E \) into a word \( \vec{x} \in S^n_x \). (2) A channel \( Q(\vec{y}|\vec{x}) \) gives the probability that word \( \vec{x} \in S^n_x \) is mapped into word \( \vec{y} \in S^n_y \). (3) Each word \( \vec{y} \) is then mapped by a decoder function \( D \) into message \( m' \in Z_{0,M} \). We assume a discrete memoryless channel, by which we mean that

\[
Q(\vec{y}|\vec{x}) = \prod_{i} Q(y_i|x_i) \quad \forall \vec{x} \in Z^{n}_{x},
\]

An \((M,n)\) channel code is characterized by its encoding function \( E \), the conditional probability of its channel \( Q(\vec{y}|\vec{x}) \), and its decoding function \( D \).

Let \( p_{\text{err}} | m \) be the probability of error when message \( m \in Z_{1,M} \) exits the encoder. Then

\[
p_{\text{err}} | m = Pr\{D(\vec{y}) \neq m | \vec{x} = E(m)\}.
\]

The code rate \( R \) of the encoder is defined by

\[
R = \frac{\ln M}{n} = (\ln 2) \frac{\log_2 M}{n}.
\]

Note that if \( N_x = 2 \), then \( \frac{\log_2 M}{n} = \frac{n_{\text{out}}}{n_{\text{in}}} \) (careful: for noiseless coding \( \frac{\log_2 M}{n} = \frac{n_{\text{out}}}{n_{\text{in}}} \) instead), where \( n_{\text{out}} \) (ditto, \( n_{\text{in}} \)) is the encoder output (ditto, input) measured in bits.

The maximum achievable rate \( R_{\text{max.ach.}} \) is defined by:

\[
R_{\text{max.ach.}} = \lim_{\epsilon \to 0} \lim_{M \to \infty} \sup \left\{ \frac{\ln M}{n} : \exists (n, E, D) \forall m, p_{\text{err}} | m(n, E, D) < \epsilon \right\}.
\]
The information capacity $C$ is defined by:

$$C = \max_{Q_x \in pd(S_x)} H_{Q_x, y}(x : y) .$$  \hspace{1cm} (123)

The fact that $R_{\text{max,ach.}} = C$ is essentially Shannon’s Noisy Coding (or “Second”) Theorem.

Eq. (120) can be re-expressed as

$$p_{\text{err}|m} = \sum_{\vec{y}} \Pr\{D(\vec{y}) \neq m|\vec{x} = E(m), \vec{y} = \vec{y}\} \Pr\{\vec{y} = \vec{y}|\vec{x} = E(m)\} \hspace{1cm} (124a)$$

$$= \sum_{\vec{y}} \theta(D(\vec{y}) \neq m) Q(\vec{y}|\vec{x}(m)) \hspace{1cm} (124b)$$

$$= 1 - \sum_{\vec{y}} \theta(D(\vec{y}) = m) Q(\vec{y}|\vec{x}(m)) . \hspace{1cm} (124c)$$

A random encoder $E$ is defined by choosing each component of $\vec{x} = E(m)$ independently from the other components and according to the probability distribution $Q(x)$. With such an encoder,

$$p_{\text{err}} = \sum_{m \in Z_{1,M}, E} p_{\text{err}|m,E} P(E)P(m) \hspace{1cm} (125a)$$

$$= 1 - \sum_{m \in Z_{1,M}} P(m) \prod \left\{ \sum_{\vec{x}(m_1) \in S_n^y} Q(\vec{x}(m_1)) \right\}_{m_1 \in Z_{1,M}} \sum_{\vec{y}} \theta(D(\vec{y}) = m) Q(\vec{y}|\vec{x}(m)) . \hspace{1cm} (125b)$$

Suppose $\Gamma: S_{\vec{y}}^n \times Z_{1,M} \rightarrow \{\text{true, false}\}$ is a condition, and $\text{Good}(\Gamma)$ is the set of all $\vec{y}$ for which there is a unique $m \in Z_{1,M}$ that satisfies $\Gamma(\vec{y}, m) = \text{true}$. Also let $\text{Bad}(\Gamma) = S_{\vec{y}}^n - \text{Good}(\Gamma)$. One can define the decoding function $D$ implicitly in terms of the condition $\Gamma$ as follows:

$$D(\vec{y}) = \begin{cases} 
\text{unique } m \text{ such that } \Gamma(\vec{y}, m) = \text{true}, & \text{if } \vec{y} \in \text{Good}(\Gamma) \\
0 & \text{if } \vec{y} \in \text{Bad}(\Gamma) 
\end{cases} . \hspace{1cm} (126)$$

Hence, for $m \in Z_{1,M}$ and $\vec{y} \in \text{Good}(\Gamma),$

$$\theta(D(\vec{y}) = m) = \theta(\Gamma(\vec{y}, m)) . \hspace{1cm} (127)$$

The maximum likelihood (ML) decoder is defined by the condition

$$\Gamma(\vec{y}, m) = \left( \frac{Q(\vec{y}|\vec{x}(m))}{Q(\vec{y}|\vec{x}(m'))} > 1 \forall m' \in Z_{1,M}, m' \neq m \right) . \hspace{1cm} (128)$$
(As illustrated in Fig. 6 we assume that \textit{Bad}(\Gamma) is negligibly small, in the sense that, for all \( m \in Z_{1,M}, \sum_{\bar{y} \in \text{Bad}(\Gamma)} Q(\bar{y}|\bar{x}(m)) << 1 \) ) Actually, the ML decoder is not optimal. It can be shown\cite{3} that the optimal decoder is one for which

\[
\Gamma(\bar{y}, m) = \left( \frac{Q(\bar{x}(m)|\bar{y})}{Q(\bar{x}(m')|\bar{y})} > 1 \forall m' \in Z_{1,M}, m' \neq m \right).
\]

\[
(129)
\]

Figure 6: Intuitive picture of condition Eq.(128) for Maximum Likelihood decoder.

For each \((\bar{x}, \bar{y}) \in S^n_x \times S^n_y\), define functions \(v\) and \(f\) by

\[
v(\bar{x}, \bar{y}) = \sum_{\bar{x}' \in S^n_x} \theta \left( \frac{Q(\bar{y}|\bar{x})}{Q(\bar{y}|\bar{x}')} > 1 \right) Q(\bar{x}') ,
\]

\[
(130)
\]

and

\[
f = 1 - v .
\]

(\text{mnemonic: } v \text{ stands for victory and } f \text{ for failure}).

If we substitute into Eq.(125b) the value of \( \theta(D(\bar{y}) = m) \) for ML decoding, one finds for random encoding and ML decoding:

\[
p_{\text{err}} = 1 - \sum_{\bar{x}, \bar{y}} Q(\bar{y}|\bar{x}) Q(\bar{x})[v(\bar{x}, \bar{y})]^{M-1} .
\]

\[
(132)
\]

Later on, we will show that \( f \approx e^{-nC} \). Since \( M = e^{nR} \), it follows that for random encoding and ML decoding

\[
p_{\text{err}} \approx 1 - (1 - e^{-nC})^M \approx 1 - \exp(-Me^{-nC}) \approx 1 - \exp[-e^{n(R-C)}] \approx 1 - \theta(R - C \leq 0) = \theta(R > C) .
\]

\[
(133a)
(133b)
(133c)
(133d)
\]
In Eq. (133), we went from line (c) to (d) by using the following easy to prove identity: For all \( x \neq 0, \)
\[
\theta(x > 0) = \lim_{n \to 0} \exp[-\exp(-nx)] . \tag{134}
\]
According to Eq. (133d), if the code rate \( R \) is larger (ditto, smaller) than the channel capacity \( C \), then the probability of error is one (ditto, zero). The next few sections of this paper will be dedicated to improving this estimate of \( p_{\text{err}} \).

### 5.2 New (CCRG) Approximation for \( p_{\text{err}} \)

In this section and the next one, we will use CCRG to calculate the probability of error for channel transmission using random encoding and ML decoding. This section will calculate \( p_{\text{err}} \) as given by Eq. (132), assuming that we have rescaled the variables on the RHS of Eq. (132) so that the integrand is Gaussian. The next section will calculate the RG equations that characterize this rescaling.

In what follows, we will use \( Q(x, y) \) to mean \( Q(x, y) = Q(y|x)Q(x) \), where \( Q(y|x) \) (ditto, \( Q(x) \)) is the probability distribution that specifies the transmission channel (ditto, the random encoding). We will also use the following abbreviations:

\[
C_1 = \sum_{x,y} Q(x,y) \ln \left( \frac{Q(x,y)}{Q(x)Q(y)} \right) = H_{Q(x,y)} , \tag{135}
\]
\[
\Delta R = R - C_1 , \tag{136}
\]
\[
\Delta P(x,y) = P(x,y) - Q(x,y) , \quad \Delta P(x|y) = P(x|y) - Q(x|y) , \tag{137}
\]
\[
L_{xy} = \ln \left( \frac{Q(x,y)}{Q(x)Q(y)} \right) . \tag{138}
\]

Note that \( C_1 \) is not equal to the channel capacity \( C \), but \( C = \max_{Q_{xy} \in pd(S_{xy})} C_1 \).

Applying Reduction Formula 1 to Eq. (132) yields

\[
p_{\text{err}} = 1 - \sum_{\vec{x}, \vec{y}} Q(\vec{x}, \vec{y}) v^M \approx 1 - r(n, N_{\geq 2}) \int_{pd(S_{\geq 2})} D_{P_{\geq 2}} \exp\left[ -nD(P_{\geq 2}/Q_{\geq 2}) \right] \sqrt{\prod P_{\geq 2}} v^M . \tag{139b}
\]

For \( n >> 1 \), and fixed \( R, M = e^{nR} >> 1 \). Later on we will show that \( 0 \leq f << 1 \). The inequalities \( M >> 1 \), and \( 0 \leq f << 1 \), and Eq. (134) imply

\[
v^M = (1 - f)^M \approx e^{-Mf} = e^{-\exp(nR+\ln f)} \approx \theta(R + \frac{\ln f}{n} < 0) . \tag{140}
\]
Our next goal is to calculate \( \ln(f) \). One has

\[
f(\vec{x}, \vec{y}) = \sum_{\vec{x}', \vec{y}'} \delta(\vec{y}, \vec{y}') \theta \left( \frac{Q(\vec{y}|\vec{x})}{Q(\vec{y}'|\vec{x}')} < 1 \right) Q(\vec{x}') .
\] (141)

Henceforth, we will abbreviate the probability distributions for the CC’s \( C \left( \frac{\vec{x}}{\vec{y}} \right) \) and \( C \left( \frac{\vec{x}'}{\vec{y}'} \right) \) as follows:

\[
P \left( \frac{\vec{x}}{\vec{y}} \right) \rightarrow P_{\vec{x}y}, P \left( \frac{\vec{x}'}{\vec{y}'} \right) \rightarrow \tilde{P}_{\vec{x}y} .
\] (142)

Using these abbreviations, one has

\[
\theta \left( \frac{Q(\vec{y}|\vec{x})}{Q(\vec{y}'|\vec{x}')} < 1 \right) = \theta \left( \exp[n \sum_{x,y} P(x,y) \ln Q(y|x)] \frac{\exp[n \sum_{x,y} \tilde{P}(x,y) \ln Q(y|x)] - 1}{\exp[n \sum_{x,y} P(x,y) \ln Q(y|x)]} \right).
\] (143a)

\[
= \theta \left( \sum_{x,y} [P - \tilde{P}](x,y) \ln Q(y|x) < 0 \right) .
\] (143b)

Substituting Eq. (143b) into Eq. (141) and applying Reduction Formula 2 yields

\[
f(\vec{x}, \vec{y}) = \frac{r(n, N_{\vec{x}y} - N_{\vec{y}})}{[\prod \{P_y\}]^{\frac{1}{2}(N_{\vec{x}y} - 1)} \int D\tilde{P}_{\vec{x}y} \theta(\tilde{P}_{\vec{x}y} \geq 0) \prod_\{\delta(P(y) - P(y)\}_{\vec{y}} \exp[n \sum_{x,y} \tilde{P}(x,y) \ln Q(x)] \sqrt{\prod \{\tilde{P}(x|y)\}_{\vec{x}, \vec{y}}} \theta \left( \sum_{x,y} [P - \tilde{P}](x,y) L_{xy} < 0 \right) .
\] (144)

Note that

\[
D(\tilde{P}_{\vec{x}y}/Q_{\vec{x}y}) = \sum_{x,y} \tilde{P}(x,y) \left[ \ln \frac{\tilde{P}(x,y)}{Q(x,y)} + \ln \frac{Q(x,y)}{Q(x)Q(y)} + \ln \frac{Q(y)}{P(y)} \right] .
\] (145a)

\[
= D(\tilde{P}_{\vec{x}y}/Q_{\vec{x}y}) + C_1 + \sum_{x,y} \Delta \tilde{P}(x,y) L_{xy} - D(\tilde{P}_{\vec{y}}/Q_{\vec{y}}) .
\] (145b)

Hence,

\[
f(\vec{x}, \vec{y}) = \frac{r(n, N_{\vec{x}y} - N_{\vec{y}}) \exp[-nC_1 + nD(P_y/Q_y)]}{[\prod \{P_y\}]^{\frac{1}{2}(N_{\vec{x}y} - 1)}}
\]
\[
\int \mathcal{D}\tilde{P}_{x,y} \theta(\tilde{P}_{x,y} \geq 0) \prod_y \left\{ \delta(\tilde{P}(y) - P(y)) \right\}_y \\
\exp[-nD(\tilde{P}_{x,y}/Q_{x,y}) - n \sum_{x,y} \Delta \tilde{P}(x,y)L_{xy}] \\
\sqrt{\prod_y \{\tilde{P}(x|y)\}}_{x,y} \\
\theta \left( \sum_y [P - \tilde{P}](x,y)L_{xy} < 0 \right) . 
\] (146)

We will assume that, in the integrand of the previous equation, the inequality constraint is active; i.e., that \( \sum_{x,y} \Delta P(x,y)L_{xy} = \sum_{x,y} \Delta \tilde{P}(x,y)L_{xy} \). Therefore, we can simplify Eq. (146) by pulling \( e^{-n \sum_{x,y} \Delta \tilde{P}(x,y)L_{xy}} \) outside the integral to get

\[
f(\vec{x}, \vec{y}) = \frac{r(n, N_{x,y} - N_y)}{[\prod \{P_y\}]^{\frac{1}{2}(N_x-1)}} \\
\int \mathcal{D}\tilde{P}_{x,y} \theta(\tilde{P}_{x,y} \geq 0) \prod_y \left\{ \delta(\tilde{P}(y) - P(y)) \right\}_y \\
\exp[-nD(\tilde{P}_{x,y}/Q_{x,y})] \\
\sqrt{\prod_y \{\tilde{P}(x|y)\}}_{x,y} \\
\theta \left( \sum_{x,y} [P - \tilde{P}](x,y)L_{xy} < 0 \right) . 
\] (147)

To find \( \ln(f) \) to leading order in \( n \), we need to find the point \( \tilde{P}^*(x,y) \) that dominates the integral on the RHS of Eq. (147). To find \( \tilde{P}^* \), we must minimize the following Lagrangian with respect to \( \tilde{P} \), \( \lambda \), and \( \mu_y \):

\[
\mathcal{L} = D(\tilde{P}_{x,y}/Q_{x,y}) - \lambda \left( \sum_{x,y} (P - \tilde{P})(x,y)L_{xy} \right) + \sum_y \mu_y (P - \tilde{P})(y) . 
\] (148)

The Gaussian approximation for the previous Lagrangian is:

\[
\mathcal{L} = \sum_{x,y} \frac{\Delta \tilde{P}(x,y)^2}{2Q(x,y)} - \lambda \left( \sum_{x,y} (P - \tilde{P})(x,y)L_{xy} \right) + \sum_y \mu_y (P - \tilde{P})(y) . 
\] (149)

Assume that the exact Lagrangian of Eq. (148) is well approximated by its Gaussian approximation. (This assumption is not necessary and will be removed later, in Appendix E.) Let

\[
\alpha_{xy} = L_{xy} - \sum_{x'} Q(x'|y)L_{x'y} , 
\] (150)
\[ \langle \alpha \rangle = \sum_{x,y} Q(x,y) \alpha_{xy} = 0 , \]  \hspace{1cm} (151) \\
\[ \langle \alpha^2 \rangle = \sum_{x,y} Q(x,y) \alpha_{xy}^2 = \sum_{x,y} Q(x,y) L_{xy} \alpha_{xy} . \]  \hspace{1cm} (152a) \\
Minimizing Eq.(149) with respect to \( \tilde{P} \), \( \lambda \), and \( \mu_y \) yields \\
\[ \lambda = - \sum_{x,y} P(y) \Delta P(x|y) L_{xy} \langle \alpha^2 \rangle , \]  \hspace{1cm} (153) \\
and \\
\[ \Delta \tilde{P}^*(x, y) = - \lambda Q(x,y) \alpha_{xy} - Q(x|y) \Delta P(y) . \]  \hspace{1cm} (154) \\
If \( \mathcal{L}^* \) is the value of \( \mathcal{L} \) at the extremum, then \\
\[ \mathcal{L}^* = \sum_y \frac{[\Delta P(y)]^2}{2Q(y)} + t , \]  \hspace{1cm} (155) \\
where, to lowest order in \( \Delta P \), \( t \) is given by \\
\[ t = \frac{\epsilon^2}{2 \langle \alpha^2 \rangle} , \]  \hspace{1cm} (156) \\
where \\
\[ \epsilon = \sum_{x,y} P(y) \Delta P(x|y) L_{xy} . \]  \hspace{1cm} (157) \\
Now that we know \( \mathcal{L}^* \), we can apply Laplace’s Method to the integral on the RHS of Eq.(147) to get \\
\[ \ln(f) \approx -nC_1 + nD(P_{x,y}/Q_{x,y}) - n \sum_{x,y} \Delta P(x,y) L_{xy} - n\mathcal{L}^* \]  \hspace{1cm} (158a) \\
\[ \approx -n \left( C_1 + \sum_{x,y} \Delta P(x,y) L_{xy} + t \right) . \]  \hspace{1cm} (158b) \\
This value for \( \ln(f) \) can be inserted into Eqs.(139) and (140) to get \\
\[ p_{err} \approx 1 - r(n, N_{x,y}) \int_{pd(S_{x,y})} \mathcal{D}P_{x,y} \\
\exp[-nD(P_{x,y}/Q_{x,y})] \sqrt{\prod\{P_{x,y}\}} \theta(\Delta R - \sum_{x,y} \Delta P(x,y) L_{xy} - t < 0) . \]  \hspace{1cm} (159)
Assume that the integral of the previous equation has been rescaled so that its integrand is in the Gaussian regime. Then

\[
p_{\text{err}} \approx 1 - \frac{r(n, N_{x,y})}{\sqrt{\prod \{P_{x,y}^*\}}} \int_{pd(s_{x,y})} D P_{x,y} 
\exp[-n \sum_{x,y} \left( \frac{\Delta P(x, y)^2}{2Q(x, y)} \right)] \theta(\sum_{x,y} \Delta P(x, y)L_{xy} \geq \Delta R) .
\]

Let

\[
\beta_{xy} = L_{xy} - \sum_{x,y} Q(x, y)L_{xy} = L_{xy} - C_1 ,
\]

\[
\langle \beta \rangle = \sum_{x,y} Q(x, y)\beta_{xy} = 0 ,
\]

\[
\langle \beta^2 \rangle = \sum_{x,y} Q(x, y)\beta_{xy}^2 .
\]

Applying the Gaussian Integration Formulae of Appendix C to the RHS of Eq. (160) yields

\[
p_{\text{err}} \approx 1 - \frac{1}{2u} \text{erfc}\left( \Delta R \sqrt{\frac{n}{2\langle \beta^2 \rangle}} \right) ,
\]

where

\[
u = \sqrt{\frac{\prod \{P_{x,y}^*\}}{\prod \{Q_{x,y}^*\}}} .
\]

To find the dominant point \( P_{x,y}^* \) alluded to in Eq. (165), one must minimize the following Lagrangian with respect to \( P, \lambda \) and \( \mu \):

\[
\mathcal{L} = \sum_{x,y} \left[ \frac{\Delta P(x, y)^2}{2Q(x, y)} \right] + \lambda(\Delta R - \sum_{x,y} \Delta P(x, y)L_{xy}) + \mu \sum_{x,y} \Delta P(x, y) .
\]

One finds that the extremum is at

\[
\Delta P^*(x, y) = B(x, y)\Delta R ,
\]

where

\[
B(x, y) = \frac{\beta_{xy}Q(x, y)}{\langle \beta^2 \rangle} .
\]
\[ u = \sqrt{\prod \left\{ 1 + \frac{\beta_{xy} \Delta R}{\langle \beta^2 \rangle} \right\}}_{\forall (x,y)} . \]  

(169)

Note that this paper has exposed a close analogy between noiseless and noisy coding, as far as \( p_{\text{err}} \) is concerned. For example, Eq. (70) for noiseless coding is analogous to Eq. (159) for noisy coding. Likewise, Eq. (91) is analogous to Eq. (164).

5.3 RG Equations

In this section, we will calculate the RG equations for channel transmission using random encoding and ML decoding.

For noiseless coding, the RG equations arose from rescaling Eq. (70). In the case we are now considering, that of noisy coding, the RG equations arise from rescaling Eq. (159). Note the close resemblance between these two equations.

In the noiseless coding case, we found a RG equation for \( P_x \) by assuming that the argument \( nD(P_x/Q_x) \) of the exponential in the integrand of Eq. (70) was invariant under a change of scale. In analogy, for noisy coding, we find a RG for \( P_{x,y} \) by assuming that the argument \( nD(P_{x,y}/Q_{x,y}) \) of the exponential in the integrand of Eq. (159) is invariant under a change of scale. We get

\[ \frac{\partial \Delta P^{(s)}(x)}{\partial s} = -\gamma_0 (P^{(s)}, Q) \Delta P^{(s)}(x) , \]  

(170)

where

\[ \gamma_0 (P, Q) = \frac{D(P//Q)}{D(P//Q) + D(Q//P)} . \]  

(171)

In the noiseless coding case, we found a RG equation for \( \Delta R \) by assuming that the theta function in the integrand of Eq. (70) was invariant under a change of scale. In analogy, for noisy coding, we find a RG for \( \Delta R \) by assuming that the theta function in the integrand of Eq. (159) is invariant under a change of scale. We get

\[ \frac{\partial \Delta R^{(s)}(s)}{\partial s} = -\gamma_1 (P^{(s)}, Q) \Delta R^{(s)}(s) , \]  

(172)

where

\[ \gamma_1 (P, Q) = \lim_{s \to 0} \left( \frac{-1}{T(P^{(s)}, Q)} \right) \frac{\partial T(P^{(s)}, Q)}{\partial s} , \]  

(173)

where

\[ T(P, Q) = T_0 + t , \]  

(174)

where
\[ T_0 = \sum_{x,y} \Delta P(x,y)L_{xy}. \]  

(175)

For any real valued function \( f(s) \) of \( s \geq 0 \), define

\[ Df = \lim_{s \to 0} \left( -\frac{1}{\gamma_0} \right) \frac{\partial f}{\partial s}. \]  

(176)

Note that \( DP(s) = \Delta P \) and \( \gamma_1 = \gamma_0 \frac{DT}{T} \). Substituting Eq.(174) into Eq.(173) gives

\[ \gamma_1(P,Q) = (1 + \frac{-t + Dt}{T}) \gamma_0(P,Q). \]  

(177)

Eq.(156) gives \( t \) to lowest order in \( \epsilon \). It is easy to show that for such a \( t \), \( Dt = 2t \), so \( \gamma_1 = (1 + \frac{1}{T}) \gamma_0 \). In Appendix E, we find \( t \) and \( \gamma_1 \) to all orders in \( \epsilon \).

5.4 Coda to Error Model

It is customary to end a discussion of noisy coding with random encoding with the following 3 observations.

Replace \( C_1 \) by Capacity. In \( C_1 \), \( Q(x) \) and \( Q(y|x) \) are independent. The capacity is defined by \( C = \max_{Q_x \in pd(S_x)} C_1 \). Let \( Q^*_{x} \in pd(S_x) \) be the probability distribution \( Q_x \) that maximizes \( C_1 \) at fixed \( Q(y|x) \). The \( p_{err} \) that we derived for random encoding depends on \( C_1 \). It is advantageous to set \( Q_x = Q^*_x \) in \( p_{err} \) since \( p_{err}(C) \leq p_{err}(C_1) \).

Keep Best Codebook. The \( p_{err} \) that we derived for random encoding was averaged over all possible codebooks \( \kappa \) (there are \( N^n_M \) of them). There must exist a “best” codebook \( \kappa_{\text{best}} \) among these such that \( p_{err}(\kappa_{\text{best}}) \leq p_{err}(\kappa) \) for all \( \kappa \), and therefore \( p_{err}(\kappa_{\text{best}}) \leq \text{mean of } (p_{err}(\kappa))_\kappa \).

Keep Ruly Half of Codebook. Suppose \( x_1 \leq x_2 \leq \ldots \leq x_N \) is a monotonically non-decreasing sequence of real numbers. Define partial sums \( S_{a,b} = x_a + x_{a+1} + \ldots + x_b \) for \( a \leq b \). The mean of the sequence is \( \mu = S_{1,N}/N \) and its median is \( x_{\frac{N}{2}} \). It is easy to prove by contradiction that \( x_{\frac{N}{2}} \leq 2\mu \).

Define the “unruly half” \( S_{m}^{\text{unruly}} \) of a codebook to be the set of all \( m \in S_{m}^{\text{all}} \) for which \( p_{err|m} \) is larger than the median of \( (p_{err|m})_{m \in S_{m}^{\text{all}}} \). Thus, \( S_{m}^{\text{ruly}} \cup S_{m}^{\text{unruly}} = S_{m}^{\text{all}} \). If we remove the “unruly half” of a codebook, then we end up with a new codebook with half as big an \( M \); symbolically, \( M_{\text{ruly}} = \frac{M_{\text{all}}}{2} \). In the limit of large codeword size \( n \), this does not affect the rate \( R \) too much. Indeed, \( R_{\text{ruly}} = \frac{1}{n} \ln\left(\frac{M_{\text{ruly}}}{2}\right) = R_{\text{all}} - \frac{1}{n} \ln(2) \to R_{\text{all}} \). The advantage of keeping only the ruly half of a codebook is that \( p_{err|m} \) for all \( m \in S_{m}^{\text{ruly}} \) is bounded above by \( 2p_{err}(\text{all}) \).
6 Computer Results

In this section, we will describe the algorithms used by the computer program WimpyRG-C1.0 to solve the equations of this paper, and we will give examples of typical inputs and outputs of said program. For more information about WimpyRG, see its source code and accompanying documentation.

6.1 Old-Noiseless Approximation of $p_{err}$

First, let us describe how WimpyRG calculates the old fashioned approximation for $p_{err}$, in the case of noiseless coding.

We shall indicate derivatives by primes. Previously, we defined

$$Z(\lambda) = \sum_{x} Q(x)^{\frac{1}{1+\lambda}}$$

(178)

$$\gamma(\lambda) = \lambda R - (1 + \lambda) \ln Z(\lambda)$$

(179)

$$\gamma = \max_{\lambda \geq 0} \gamma(\lambda)$$

(180)

and we showed that the probability of error is approximated by

$$p_{err} = e^{-n\gamma}.$$ 

(181)

To maximize the function $\gamma(\lambda)$, WimpyRG uses the simple Newton Raphson (NR) method as follows. Note that only the range $R \in (0, \ln N_x)$ is of interest. It is easy to show that for all $\lambda \geq 0$, if $R \in (H(Q), \ln N_x)$, then $\gamma(\lambda)$ has a negative second derivative and $\gamma'(0) = \Delta R > 0$. Hence, for $R \in (H(Q), \ln N_x)$, $\gamma(\lambda)$ has a unique maximum at some point $\lambda = \lambda_0 > 0$. The NR method is way of finding the zeros of a function $f : \text{Reals} \rightarrow \text{Reals}$. Suppose that $f(x) = 0$ at $x = a$. We can Taylor expand $f(x)$ to first order about this zero: $f(x) \approx f(a) + f'(a)(x-a)$. Thus, $f(x) = 0$ implies $x = a - f(a)/f'(a)$. This suggest the recursion relation: $x_{n+1} = x_n - f(x_n)/f'(x_n)$ for $n = 0, 1, 2, \ldots$. Replacing $x$ by $\lambda$, and $f(x)$ by $\gamma'(\lambda)$, one gets

$$\lambda_{n+1} = \lambda_n - \frac{\gamma'(\lambda_n)}{\gamma''(\lambda_n)}.$$ 

(182)

WimpyRG uses the previous recursion relation to find the maximum of $\gamma(\lambda)$. This algorithm requires that we know the functions $\gamma'(\lambda)$ and $\gamma''(\lambda)$. These two derivatives can be computed explicitly as follows. Define

$$Z_n(\lambda) = \sum_{x} Q(x)^{\frac{1}{1+\lambda}}[\ln Q(x)]^n.$$ 

(183)

Note that $Z = Z_0$. It is easy to show that
\[
\gamma'(\lambda) = R - \ln Z_0 + \frac{Z_1}{(1 + \lambda)Z_0},
\]  
(184)

and

\[
\gamma''(\lambda) = \frac{-(Z_0Z_2 - Z_1^2)}{(1 + \lambda)^3Z_0^2}.
\]  
(185)

### 6.2 New-Noiseless and New-Noisy Approximations of \( p_{err} \)

Next, let us describe how WimpyRG calculates the new (CCRG) approximation for \( p_{err} \), in the case of either noiseless or noisy coding.

For both noiseless and noisy coding, we must solve the following pair of coupled RG equations. For \( s \geq 0 \),

\[
\frac{\partial \Delta R^{(s)}}{\partial s} = -\gamma_1(P^{(s)}, Q)\Delta R^{(s)},
\]  
(186)

and

\[
\frac{\partial \Delta P^{(s)}(X)}{\partial s} = -\gamma_0(P^{(s)}, Q)\Delta P^{(s)}(X)
\]  
(187)

for all \( X \in S_X \), where \( S_X = S_\lambda \) for noiseless coding and \( S_X = S_{\lambda, y} \) for noisy coding. We must solve this pair of RG equations subject to the following pair of boundary conditions: At \( s = 0 \):

\[
\Delta R^{(0)} = \Delta R,
\]  
(188)

and at \( s = s_{\text{fin}} \):

\[
P^{(s_{\text{fin}})}(X) = Q(X) + B(X)\Delta R^{(s_{\text{fin}})},
\]  
(189)

for all \( X \). \( \gamma_0 \) and \( \gamma_1 \) are known functions of \( P \) and \( Q \). \( \gamma_0 \) is the same for both noiseless and noisy coding, but \( \gamma_1 \) is different. \( \Delta R \) is assumed to be known. \( \Delta R \) equals \( R - H(Q) \) for noiseless coding and \( R - C_1 \) for noisy coding. The test fractions \( \Phi_0(P, Q) \) and \( \Phi_1(P, Q) \) are also known functions of \( P \) and \( Q \). \( s_{\text{fin}} \) is defined as any \( s \) large enough for the following to be true: \( \Phi_0(P^{(s_{\text{fin}})}, Q) << 1 \) and \( \Phi_1(P^{(s_{\text{fin}})}, Q) << 1 \). \( B(x) \) is also a known function. It depends on \( Q \) but not \( P \), and it differs for noiseless and noisy coding.

Eqs. (186) and (187) can be solved recursively by performing the following steps:

1. **Move Backwards** (from \( s = s_{\text{fin}} \) to \( s = 0 \)) This step will be performed either at the beginning of the algorithm, or after performing step (2) below. If this step is being performed after step (2), then step (2) has just yielded a fresh value of \( \Delta R^{(s_{\text{fin}})} \). On the other hand, if this step is being performed at the beginning of the algorithm, take \( \Delta R^{(s_{\text{fin}})} = 10^{-12} \).
Substituting $\Delta R^{(s_{\text{fin}})}$ into Eq. (189) gives $\Delta P^{(s_{\text{fin}})}$. Hence we can solve Eq. (187) numerically (using the Fourth Order Runge Kutta algorithm) to get $\Delta P^{(s)}(x)$ for all $x \in S_x$ and all $s \in [0, s_{\text{fin}}]$. These $\Delta P^{(s)}(x)$ values can in turn be used to calculate $\gamma_1(P^{(s)}, Q)$ for each $s \in [0, s_{\text{fin}}]$.

(2) Move Forwards (from $s = 0$ to $s = s_{\text{fin}}$) After step (1), we have a fresh value of $\gamma_1(P^{(s)}, Q)$ for each $s \in [0, s_{\text{fin}}]$. By virtue of Eq. (188), $\Delta R^{(0)}$ is also known. Hence we can solve Eq. (186) numerically (again, using the Fourth Order Runge Kutta algorithm) to get $\Delta R^{(s_{\text{fin}})}$.

One performs steps (1), (2), (1), (2), ..., until the difference between two successive values of $\Delta R^{(s_{\text{fin}})}$ is very small.

Let

$$
\mathcal{E} = \frac{1}{2} \text{erfc} \left( \frac{\Delta R^{(s_{\text{fin}})}}{\sqrt{n^{(s_{\text{fin}})} / 2 \langle \beta^2 \rangle}} \right),
$$

where $n^{(s_{\text{fin}})} = e^{s_{\text{fin}}} n$. The probability of error $p_{\text{err}}$ is approximately equal to $\mathcal{E}$ for noiseless coding and to $1 - \mathcal{E}$ for noisy coding. However, the quantities $\Delta R^{(s_{\text{fin}})}$ and $\langle \beta^2 \rangle$ that appear in $\mathcal{E}$ have different definitions for noiseless and noisy coding.

6.3 Examples of WimpyRG Input and Output

![Figure 7: A plot of WimpyRG output for noiseless coding.](image-url)
Fig. 7 is a plot of WimpyRG output for noiseless coding. It gives $p_{err}$ as a function of $R - H(Q)$, for $n = 20$ and $\vec{Q}_2 = (\cdot20, 30, 13, 37)$. $H(Q) = 1.316$. The maximum possible $R$ is $\ln(N_\Sigma) = 1.386$. Curve \text{Old}, the old approximation of $p_{err}$, is a plot of Eq. (181). Let $\mathcal{E}$ be given by Eq. (190). Curve \text{Unren}, the unrenormalized approximation of $p_{err}$, is a plot of $\mathcal{E}$ with $s_{\text{fin}} = 0$ (hence $n^{(s_{\text{fin}})} = e^{s_{\text{fin}}n} = 20$). Curve \text{Ren}, the renormalized approximation of $p_{err}$, is a plot of $\mathcal{E}$ with $s_{\text{fin}} = 7.5$ (hence $n^{(s_{\text{fin}})} = e^{s_{\text{fin}}n} = 36160.8$).

It appears from Fig. 7 that curve \text{Unren} is always higher or equal to curve \text{Ren}. As expected, both the \text{Old} and \text{Ren} curves plummet towards $p_{err} = 0$ at $R = \ln N_\Sigma$.

Curve \text{Old} is not expected to be a good approximation for $p_{err}$ when $R$ is close to $H(Q)$. Indeed, for $R = H(Q)$, $\gamma = 0$, so $e^{-n\gamma}$ is indeterminate because $n\gamma = \infty \cdot 0$ . On the other hand, curve \text{Ren} is expected to behave best when $R$ is near $H(Q)$, in the sense that the closer $R$ is to $H(Q)$, the lower the value of $s_{\text{fin}}$ that is required to reach the quadratic regime.

While generating the points $(\Delta R, p_{err})$ plotted in Fig. 7, WimpyRG also generated certain figures of merit for each point. For example, when generating the point $(\Delta R, p_{err}) = (-0.15825, 0.925769)$, WimpyRG also generated:

\begin{verbatim}
number of cycles (max is 100) = 6
test fraction 0 (initial, final) = 0.15137, 0.00234084
test fraction 1 (initial, final) = 0.397863, 0.0105788
n (initial, final) = 20, 36160.8
Delta R (initial, final) = -0.15825, -0.00271302
R, unrenormalized error_prob, error_prob = 1.15793, 0.976272, 0.925769
\end{verbatim}

In this output, “initial” always refers to $s = 0$ and “final” to $s = s_{\text{fin}} = 7.5$. A “cycle” is defined as a single application of the Backward/Forward steps defined previously. A cycle takes the computer program from $s = s_{\text{fin}}$ to $s = 0$ and back again. The “number of cycles” is how many cycles were required before reaching a reasonably constant (i.e. varying no more than 0.1% between successive cycles) value for $\Delta R^{(s_{\text{fin}})}$. Notice that test fractions $\Phi_0$ and $\Phi_1$ decreased substantially whereas $n$ increased substantially in going from $s = 0$ to $s = s_{\text{fin}}$. Hurray!

Fig. 8 is a plot of WimpyRG output for noisy coding. It gives $p_{err}$ as a function of $R - C$, for $n = 20$. The channel probability $Q(y|x)$ for these plots is $Q(0|0) = Q(1|1) = 0.99, Q(1|0) = Q(0|1) = 0.01$ (a symmetric binary channel). The source distribution $Q(x)$ is $Q(0) = Q(1) = 0.5$, as required to make $C_1 = C$ for a binary symmetric channel. For this $Q(y|x)$ and $Q(x)$, $C = 0.637 \text{nats}$ (or $C = .919 \text{bits}$ if one uses base 2 logs). Let $\mathcal{E}$ be given by Eq. (190). Curve \text{Unren}, the unrenormalized approximation of $p_{err}$, is a plot of $1 - \mathcal{E}$ with $s_{\text{fin}} = 0$ (hence $n^{(s_{\text{fin}})} = e^{s_{\text{fin}}n} = 20$). Curves \text{Ren2}, \text{Ren3} and \text{Ren4}, renormalized approximations of $p_{err}$, are plots...
Figure 8: A plot of WimpyRG output for noisy coding.

of $1 - \mathcal{E}$ with $s_{\text{fin}} = 7.5$ (hence $n^{(s_{\text{fin}})} = e^{s_{\text{fin}}} n = 36160.8$.) To obtain curve $\text{Ren}_j$ for $j \in \{2, 3, 4\}$, we used an approximation for $t$ that included terms up to and including order $\epsilon^j$. See Appendix E.

Fig. 9 is a magnified view of a part of Fig. 8, the part with the smallest values of $\Delta R$. Each renormalized curve $\text{Ren}_j$ for $j \in \{2, 3, 4\}$ has endpoints $a_j$ and $b_j$ such that the curve is shown only for $\Delta R \in [a_j, b_j]$. We found that our algorithm for obtaining $\text{Ren}_j$ broke down for $\Delta R < a_j$ and $\Delta R > b_j$. There is no guarantee that the Runge Kutta algorithm that we use for solving the RG equations will not produce unphysical values such as $P^{(s)}(X) \notin [0, 1]$ or a $\gamma_1 < 0$ at some intermediate step. Such unphysical values for $P^{(s)}(X)$ or $\gamma_1$ were obtained by WimpyRG for $\Delta R < a_j$ or $\Delta R > b_j$ but not for $a_j < \Delta R < b_j$. We conjecture that a curve $\text{Ren}_\infty$ that used $t$ to all orders in $\epsilon$ would reach $p_{\text{err}} = 0$ and $p_{\text{err}} = 1$ at finite values of $\Delta R$. 
Appendix: Error Function

This appendix reviews well known properties of the Error Function\textsuperscript{7}.

The Error Function is defined for real $x$ by

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x d\xi \ e^{-\xi^2}.$$ \hfill (191)

$\text{erf}(x)$ can be analytically continued to complex $x$, but we have no need to consider such an extension in this paper. The complement of the Error Function is defined by

$$\text{erfc}(x) = 1 - \text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} d\xi \ e^{-\xi^2}.$$ \hfill (192)

See Fig\textsuperscript{10} for a plot of $\text{erf}(x)$ and $\text{erfc}(x)$. Under reflection $x \rightarrow -x$, $\text{erf}(x)$ obeys

$$\text{erf}(-x) = -\text{erf}(x),$$ \hfill (193)

and $\text{erfc}(\cdot)$ obeys

$$\text{erfc}(-x) = 1 - \text{erf}(-x) = 2 - \text{erfc}(x).$$ \hfill (194)
For real \( x \) such that \(|x| << 1\),

\[
\text{erf}(x) = \frac{2}{\sqrt{\pi}} \left( x - \frac{x^3}{3 \cdot 1!} + \frac{x^5}{5 \cdot 2!} - \frac{x^7}{7 \cdot 3!} + \ldots \right). \tag{195}
\]

For real \( x \) such that \(|x| >> 1\),

\[
\text{erfc}(x) = 2\theta(x < 0) + \frac{e^{-x^2}}{x\sqrt{\pi}} \left( 1 - \frac{1}{2x^2} + \frac{1 \cdot 3}{(2x^2)^2} - \frac{1 \cdot 3 \cdot 5}{(2x^2)^3} + \ldots \right). \tag{196}
\]

**Claim A.1** For \( a, b, \Lambda \in \text{Reals} \) with \( \Lambda \), \( a > 0 \),

\[
\text{erfc}\left( \frac{b}{2\sqrt{a}} \right) = \frac{1}{\pi i} \int_{\Lambda - i\infty}^{\Lambda + i\infty} \frac{d\lambda}{\lambda} \exp(a\lambda^2 - b\lambda). \tag{197}
\]

**proof:**

\[
erfc(x) = \frac{2}{\sqrt{\pi}} \int_{-\infty}^{+\infty} d\xi \, e^{-\xi^2} \theta(\xi > x) \tag{198a}
\]

\[
= \frac{1}{\pi i} \int_{\Lambda - i\infty}^{\Lambda + i\infty} \frac{d\lambda}{\lambda} \int_{-\infty}^{+\infty} d\xi \exp(-\xi^2 + \lambda\xi - \lambda x) \tag{198b}
\]

\[
= \frac{1}{\pi i} \int_{\Lambda - i\infty}^{\Lambda + i\infty} \frac{d\lambda}{\lambda} \exp\left(\frac{\lambda^2}{4} - \lambda x\right). \tag{198c}
\]

In Eq. (198), we went from line (a) to (b) by using the integral representation of the theta function, as given by Eq. (20). Now make the replacements \( \lambda \rightarrow 2\sqrt{a}\lambda, x \rightarrow \frac{b}{2\sqrt{a}} \) in Eq. (198c). QED
Appendix: Taylor Expansions Related to Information Theory

This handy appendix collects in one place several Taylor expansions that arise frequently in Information Theory.

For real $x$ such that $|x| < 1$,

$$
\ln(1 + x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^n}{n}
$$

(199a)

$$
= x - \frac{x^2}{2} + \frac{x^3}{3} + \ldots
$$

(199b)

Thus, for $|\frac{h}{x}| < 1$,

$$
\ln(x + h) = \ln[x(1 + \frac{h}{x})] = \ln x + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(\frac{h}{x})^n}{n}
$$

(200a)

$$
= \ln x + \frac{h}{x} - \frac{h^2}{2x^2} + \ldots
$$

(200b)

$$(x + h)\ln(x + h) = x\ln x + h(\ln x + 1) + h\sum_{n=2}^{\infty} \frac{(-1)^n}{n(n-1)} \left(\frac{h}{x}\right)^{n-1}
$$

(201a)

$$
= x\ln x + h(\ln x + 1) + \frac{h^2}{2x} + \ldots
$$

(201b)

Let $\Delta P(x) = P(x) - Q(x)$. Then

$$
H(P) = -\sum_x P(x) \ln P(x)
$$

(202a)

$$
= H(Q) - \sum_x \Delta P(x) \ln Q(x) - \sum_x \frac{[\Delta P(x)]^2}{2Q(x)} + O((\Delta P)^3),
$$

(202b)

and

$$
D(P//Q) = \sum_x P(x) \ln \frac{P(x)}{Q(x)}
$$

(203a)

$$
= \sum_x \frac{[\Delta P(x)]^2}{2Q(x)} + O((\Delta P)^3).
$$

(203b)
Appendix: Gaussian Integration Formulae

In this appendix, we present certain integration formulae that contain a Gaussian times a delta or a theta function in the integrand.

The following lemma will be used to prove Claim C.1, which is the main result of this appendix.

**Lemma C.1** Suppose \( A \in \text{Reals}^{n \times n} \) is invertible, \( v \in \text{Reals}^{n \times 1} \), \( v^T A^{-1} v \neq 0 \), \( 0 < \epsilon << 1 \), and

\[
B = A + \frac{vv^T}{\epsilon}.
\]

Then the inverse and determinant of \( B \) are given by

\[
B^{-1} = A^{-1} - A^{-1} \tilde{A} A^{-1} \text{ where } \tilde{A} = \frac{vv^T}{v^T A^{-1} v},
\]

and

\[
det B = det(A) \frac{v^T A^{-1} v}{\epsilon}.
\]

**proof:**

It is easy to show that if \( u \) and \( v \) are \( n \) dimensional column vectors and

\[
B = A + uv^T,
\]

then

\[
B^{-1} = A^{-1} - \frac{A^{-1}uv^T A^{-1}}{1 + v^T A^{-1} u}
\]

satisfies \( BB^{-1} = B^{-1} B = 1 \). Setting \( u = v/\epsilon \) and taking the limit \( \epsilon \to 0 \) yields Eq.(205).

To show Eq.(206), recall that

\[
\ln(det A) = \text{tr}(\ln A).
\]

(This well known identity is obvious when \( A \) is diagonal. The proof is also very simple when \( A \) is non-diagonal but diagonalizable.) If the entries of \( A \) are taken to be independent variables, then Eq.(209) implies

\[
\delta \ln(det A) = \text{tr}(A^{-1} \delta A) = \sum_{i,j} (A^{-1})_{ij} \delta A_{ji}.
\]

Therefore,

\[
(A^{-1})_{ij} = \frac{\partial}{\partial A_{ji}} \ln \det A = \frac{1}{\det A} \frac{\partial (\det A)}{\partial A_{ji}}.
\]
This is just the usual expansion of $A^{-1}$ in terms of cofactors. For definiteness, suppose $A$ is a $3 \times 3$ matrix with columns $\vec{a}_1, \vec{a}_2, \vec{a}_3$. Suppose $u$ and $v$ are also $3 \times 1$ column vectors. Then

$$\det(A + uv^T) = \det[\vec{a}_1 + v^1 \vec{u}, \vec{a}_2 + v^2 \vec{u}, \vec{a}_3 + v^3 \vec{u}]$$  \hspace{1cm} (212a)

$$= \det A + \det[v^1 \vec{u}, \vec{a}_2, \vec{a}_3] + \det[\vec{a}_1, v^2 \vec{u}, \vec{a}_3] + \det[\vec{a}_1, \vec{a}_2, v^3 \vec{u}]$$  \hspace{1cm} (212b)

$$= \det(A) + \sum_{i,j} u^i \frac{\partial(\det A)}{\partial A_{ji}} v^j$$  \hspace{1cm} (212c)

$$= \det(A)(1 + v^T A^{-1} u) .$$  \hspace{1cm} (212d)

In Eq.\((212)\), we went from line (a) to (b) by using the fact that determinants are linear functions of each column. We also used the fact that determinants with a pair of proportional columns are zero, so that, for example,

$$\det[v^1 \vec{u}, v^2 \vec{u}, \vec{a}_3] = 0 .$$  \hspace{1cm} (213)

Now setting $u = v/\epsilon$ in Eq.\((212)\) yields

$$\det(B) = \det(A) \left(1 + \frac{v^T A^{-1} u}{\epsilon}\right)$$  \hspace{1cm} (214a)

$$\approx \det(A) \left(\frac{v^T A^{-1} u}{\epsilon}\right) .$$  \hspace{1cm} (214b)

QED

Claim C.1 For $x, b \in \text{Reals}^{N \times 1}$ and $A \in \text{Reals}^{N \times N}$, define a measure $dG(x)$ so that for any real valued function $f(x)$,

$$\int dG(x) \ f(x) = \prod_{j \in \mathbb{Z}_{1,N}} \left\{ \int_{-\infty}^{+\infty} dx_j \right\} \exp \left( -\frac{x^T A x}{2} + b^T x \right) f(x) .$$  \hspace{1cm} (215)

Suppose $A$ is a real, positive definite, symmetric matrix. Suppose $u, v \in \text{Reals}^{N \times 1}$ and $\alpha \in \text{Reals}$. Define

$$\tilde{A} = \frac{vv^T}{v^T A^{-1} v} , \quad B^{-1} = A^{-1} - A^{-1} \tilde{A} A^{-1} .$$  \hspace{1cm} (216)

Then

$$\int dG(x) 1 = \left(\frac{2\pi}{\sqrt{\det A}}\right)^N \exp \left( \frac{b^T A^{-1} b}{2} \right) ,$$  \hspace{1cm} (217a)
\[ \int dG(x) \delta(v^T x) = \left[ \int dG(x) \right] \frac{1}{\sqrt{2\pi v^T A^{-1} v}} \exp \left( -\frac{b^T A^{-1} \tilde{A} A^{-1} b}{2} \right), \quad (217b) \]

\[ \int dG(x) \theta(u^T x - \alpha \geq 0) = \left[ \int dG(x) \right] \frac{1}{2} \text{erfc} \left( \frac{\alpha - u^T A^{-1} b}{\sqrt{2u^T A^{-1} u}} \right), \quad (217c) \]

\[ \int dG(x) \delta(v^T x) \theta(u^T x - \alpha \geq 0) = \left[ \int dG(x) \delta(v^T x) \right] \frac{1}{2} \text{erfc} \left( \frac{\alpha - u^T B^{-1} b}{\sqrt{2u^T B^{-1} u}} \right). \quad (217d) \]

**proof of Eq. (217a):**

Since \( A \) is symmetric, it can be diagonalized. By diagonalizing \( A \), one can convert \( \int dG(x) 1 \) into a product of one dimensional Gaussian integrals.

**proof of Eq. (217b):**

For \( 0 < \epsilon << 1 \),

\[ \delta(v^T x) \approx \frac{1}{\sqrt{2\pi \epsilon}} \exp \left( -\frac{(v^T x)^2}{2\epsilon} \right). \quad (218) \]

Define \( B \) by

\[ B = A + \frac{v^T v}{\epsilon}. \quad (219) \]

Then

\[ \int dG(x) \delta(v^T x) = \frac{1}{\sqrt{2\pi \epsilon}} \prod_{\lambda \in \mathbb{Z}_{1,N}} \left\{ \int_{-\infty}^{+\infty} dx_j \right\} \exp \left( \frac{-x^T B x}{2} + b^T x \right) \quad (220a) \]

\[ = \frac{1}{\sqrt{2\pi \epsilon} \sqrt{\det B}} \exp \left( \frac{b^T B^{-1} b}{2} \right). \quad (220b) \]

Now use the values for \( B^{-1} \) and \( \det B \) calculated in Lemma C.1.

**proof of Eq. (217c):**

\[ \int dG(x) \theta(u^T x - \alpha \geq 0) = \int dG(x) \frac{1}{2\pi i} \int_{\Lambda - i\infty}^{\Lambda + i\infty} d\lambda \frac{e^{\lambda(u^T x - \alpha)}}{\lambda} \quad (221a) \]

\[ = \frac{1}{2\pi i} \int_{\Lambda - i\infty}^{\Lambda + i\infty} d\lambda \sqrt{\det A} \exp \left( \frac{(b + \lambda u)^T A^{-1} (b + \lambda u)}{2} - \lambda \alpha \right) \quad (221c) \]
\[
\frac{1}{2\pi i \sqrt{\det A}} \exp \left( \frac{b^T A^{-1} b}{2} \right) \int_{\Lambda - i\infty}^{\Lambda + i\infty} \frac{d\lambda}{\lambda} \exp \left( \lambda^2 \left( \frac{u^T A^{-1} u}{2} \right) + \lambda (u^T A^{-1} b - \alpha) \right)
\]

\[
= \left[ \int dG(x) \right] \frac{1}{2} \text{erfc} \left[ \frac{\alpha - u^T A^{-1} b}{\sqrt{2} u^T A^{-1} u} \right].
\]

In Eq. (221), line (a), we used the integral representation of the theta function given by Eq. (20). In Eq. (221), we went from line (b) to (c) by applying Eq. (217a). We went from line (d) to (e) by applying Eq. (197).

Proof of Eq. (217d):
This proof is similar to that of Eqs. (220) (a), (b) and (c) so it is left to the reader. QED

**D Appendix: An Integral Over All Joint Probability Distributions with a Fixed Marginal**

In this appendix, we will show how to convert (1) to (2) where (1) is an integral over all joint probability distributions \( P_{x,y} \) with the same marginal \( P_y \), and (2) is an integral over all conditional probability distributions \( P_{x|y} \).

Claim D.1

\[
\int \mathcal{D}P_{x,y} \prod \delta(P(y) - Q(y)) \prod_{y} \theta(P_{x,y} \geq 0) f(P_{x,y}) = \prod \left\{ [Q(y)]^{N_x-1} \right\} \prod_{x} \int \prod \{dP(x|y)\} \prod_{y} \prod \left\{ \delta(\sum_{x} P(x|y) - 1) \right\} \theta(P_{x,y} \geq 0) f(P_{x,y}).
\]

Proof:
Let RHS (ditto, LHS) stand for the right (ditto, left) hand side of Eq. (222). Suppose \( 0 \in S_x \). Then

\[
LHS = \int \prod \{dP(x,y)\} \prod \left\{ \theta(0 \leq \sum_{x: x \neq 0} P(x,y) \leq Q(y)) \right\} \prod_{y} \theta(P_{x,y} \geq 0) f(P) \quad (223a)
\]

\[
= \prod \left\{ [Q(y)]^{N_x-1} \right\} \prod_{y} \int \prod \{dP(x|y)\} \prod_{x: x \neq 0} \theta(0 \leq \sum_{x: x \neq 0} P(x,y) \leq Q(y)) \prod_{y} \theta(P_{x,y} \geq 0) f(P).
\]
\[
\prod \left\{ \theta(0 \leq \sum_{x: x \neq 0} P(x|y) \leq 1) \right\} \; \forall y \; \theta(P_{\neq 0} \geq 0) f(P) = \text{RHS}.
\]

QED

E Appendix: Perturbation Expansion of \( t \)

In Eq. (156), we gave \( t \) to lowest order in \( \Delta P \). In this appendix, we show how to calculate \( t \) exactly, as a Taylor series in powers of \( \Delta P \).

The point \( \tilde{P}^* \) that dominates the integral Eq. (147) is an extremum of the Lagrangian Eq. (148). In Section 5.2, we approximated the Lagrangian Eq. (148) by its quadratic approximation Eq. (149). This gave us the dominant point \( P^* \) only to lowest order in \( \Delta P \). This time we will use the exact Lagrangian and get the exact dominant point. Let us re-state the exact Lagrangian:

\[
\mathcal{L} = D(\tilde{P}_{x,y}/Q_{x,y}) - \lambda \left( \sum_{x,y} (P - \tilde{P})(x,y) L_{xy} \right) + \sum_{y} \mu_y (P - \tilde{P})(y).
\]

Minimizing this Lagrangian with respect to \( \tilde{P} \), \( \lambda \) and \( \mu_y \) gives

\[
\tilde{P}^*(x,y) = \frac{Q(x|y) \exp(-\lambda L_{xy})}{Z_y(\lambda)} P(y),
\]

where

\[
Z_y(\lambda) = \sum_x Q(x|y) \exp(-\lambda L_{xy}).
\]

The parameter \( \lambda \) in Eq. (225) is specified implicitly by the equation:

\[
\sum_{x,y} P(x,y) L_{xy} = \sum_{x,y} \frac{Q(x|y) \exp(-\lambda L_{xy})}{Z_y(\lambda)} P(y) L_{xy} = -\sum_y P(y) \frac{d \ln Z_y(\lambda)}{d\lambda}.
\]

The previous equation can be rewritten as

\[
0 = \epsilon + F(\lambda),
\]

where \( \epsilon \) and \( F(\lambda) \) are defined by
\[ \epsilon = \sum_{x,y} P(y) \Delta P(x|y) L_{xy}, \]  

and

\[ F(\lambda) = \sum_y P(y) \left[ \frac{d \ln Z_y(\lambda)}{d\lambda} - \left( \frac{d \ln Z_y(\lambda)}{d\lambda} \right)_{\lambda=0} \right]. \]  

Next we will solve Eq. (228) for \( \lambda \) by expressing \( \lambda \) as a Taylor series in powers of \( \epsilon \). We begin by expressing the RHS of Eq. (226) as a Taylor series in powers of \( \lambda \):

\[ Z_y(\lambda) = \sum_{k=0}^{\infty} \frac{A_k(y)(-\lambda)^k}{k!}, \]  

where

\[ A_k(y) = \sum_x Q(x|y)(L_{xy})^k \]  

for \( k = 0, 1, 2, \ldots \). It follows that

\[ \ln Z_y(\lambda) = a_1 \lambda + a_2 \frac{\lambda^2}{2} + a_3 \frac{\lambda^3}{3} + \ldots, \]  

where

\[ a_1 = -A_1, \]  

\[ a_2 = -A_1^2 + A_2, \]  

\[ a_3 = -A_1^3 + \frac{3}{2} A_1 A_2 - \frac{1}{2} A_3, \]  

\[ a_4 = -A_1^4 + 2 A_2 A_1^2 - \frac{2}{3} A_1 A_3 - \frac{1}{2} A_2^2 + \frac{1}{6} A_4. \]

Define

\[ \alpha_k = \sum_y P(y) a_k(y) \]  

for \( k = 1, 2, 3, \ldots \). If we express \( F(\lambda) \) as a Taylor series in powers of \( \lambda \)

\[ F(\lambda) = F_1 \lambda + F_2 \lambda^2 + F_3 \lambda^3 + \ldots, \]

then, by virtue of Eqs. (230), (233) and (235),

\[ F_k = \alpha_{k+1} \]  

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for \( k = 1, 2, 3, \ldots \) Eq. (228) can be expressed as a Taylor series in powers of \( \lambda \):

\[
0 = \epsilon + F_1 \lambda + F_2 \lambda^2 + F_3 \lambda^3 + \ldots .
\]

(238)

\( \lambda \) itself can be expressed as a Taylor series in powers of \( \epsilon \):

\[
\lambda = \lambda_1 \epsilon + \lambda_2 \epsilon^2 + \lambda_3 \epsilon^3 + \ldots .
\]

(239)

Substituting Eq. (239) into Eq. (238) yields an equation for each power of \( \epsilon \). These equations for each power of \( \epsilon \) imply:

\[
\lambda_1 = -\frac{1}{F_1}, 
\]

(240a)

\[
\lambda_2 = -\frac{F_2}{F_1} , 
\]

(240b)

\[
\lambda_3 = \frac{F_3 F_1 - 2 F_2^2}{F_1^3} ,
\]

(240c)

\[
\lambda_4 = -\frac{5 F_2^3 + 5 F_3 F_2 F_1 - F_4 F_2^2}{F_1^7} .
\]

(240d)

Now that we know \( \tilde{P}_\ast_{x,y} \) explicitly (in terms of Eq. (225), where \( \lambda \) is expressed as a Taylor series in powers of \( \epsilon \)), we can find explicitly \( \mathcal{L} \) given by Eq. (224) evaluated at \( \tilde{P}_\ast_{x,y} \).

\[
\mathcal{L}_\ast = D(P_y/Q_y) + \sum_{x,y} \tilde{P}_\ast(x,y) \ln \left( \frac{\tilde{P}_\ast(x|y)}{Q(x|y)} \right)
\]

(241a)

\[
= D(P_y/Q_y) + \sum_{x,y} \tilde{P}_\ast(x,y) \ln \left( \exp(-\lambda L_{xy}) \right) 
\]

(241b)

\[
= D(P_y/Q_y) - \lambda \sum_{x,y} P(x,y) L_{xy} - \sum_{y} P(y) \ln Z_y(\lambda) .
\]

(241c)

Expanding the ln \( Z_y(\lambda) \) in the previous equations in powers of \( \lambda \) yields

\[
\mathcal{L}_\ast = D(P_y/Q_y) - \lambda \sum_{x,y} P(x,y) L_{xy}
\]

\[
- (\lambda \alpha_1 + \lambda^2 \frac{\alpha_2}{2} + \lambda^3 \frac{\alpha_3}{3} + \cdots)
\]

(242a)

\[
= D(P_y/Q_y) - (\lambda \epsilon + \lambda^2 \frac{\alpha_2}{2} + \lambda^3 \frac{\alpha_3}{3} + \cdots) .
\]

(242b)

Expanding \( \lambda \) in the previous equation in powers of \( \epsilon \) yields
\[ \mathcal{L}^* = D(P_y / Q_y) + t, \tag{243} \]

where

\[ t = t_1 \epsilon + t_2 \epsilon^2 + t_3 \epsilon^3 + \cdots, \tag{244} \]

and

\[ t_1 = 0, \tag{245a} \]
\[ t_2 = \frac{1}{2 \alpha_2}, \tag{245b} \]
\[ t_3 = \frac{\alpha_3}{3 \alpha_2^2}, \tag{245c} \]
\[ t_4 = \frac{2 \alpha_3^2 - \alpha_2 \alpha_3}{4 \alpha_2^4}. \tag{245d} \]

Now that we know \( t \) to all orders in \( \epsilon \), we can also find \( \gamma_1 \) to all orders in \( \epsilon \). Recall from Section 5.3 that for any real valued function \( f(s) \) of \( s \geq 0 \),

\[ Df = \lim_{s \to 0} \left( -\frac{1}{\gamma_0} \right) \frac{\partial f}{\partial s}, \tag{246} \]

so that \( DP^{(s)} = \Delta P \). When \( f \) is the \( k \)th power of \( \epsilon \),

\[ D\epsilon^k = k \epsilon^{k-1} \sum_{x,y} [\Delta P(x,y) - Q(x|y)\Delta P(y)]L_{xy}, \tag{247a} \]

\[ = k \epsilon^k. \tag{247b} \]

From Eq. (244) one gets

\[ Dt = \begin{cases} 
  t_1 \epsilon + t_2 2 \epsilon^2 + t_3 3 \epsilon^3 + \cdots \\
  + \epsilon Dt_1 + \epsilon^2 Dt_2 + \epsilon^3 Dt_3 + \cdots 
\end{cases}. \tag{248} \]

One can use Eqs. (245) to calculate \( Dt_k \) in terms of \( \{ \alpha_k \}_{\forall k} \) and \( \{ D\alpha_k \}_{\forall k} \). For example, \( Dt_2 = \frac{1}{2 \alpha_2^2} D\alpha_2 \). By Eq. (235),

\[ D\alpha_k = \sum_y \Delta P(y)a_k(y), \tag{249} \]

for \( k = 1, 2, 3, \ldots \). Once we know \( t \) and \( Dt \) to all orders in \( \epsilon \), we can use Eq. (177) to find \( \gamma_1 \) to all orders in \( \epsilon \).
References

[1] Nigel Goldenfeld, *Lectures on Phase Transitions and the Renormalization Group* (1992, Perseus Books).

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[5] R. Fletcher, *Practical Methods of Optimization* (2000, John Wiley).

[6] An alternative method of getting a good trial value for \( \Delta R^{(s_{\text{fin}})} \) is as follows. Note that \( \gamma_0(P, Q) \) and \( \gamma_1(P, Q) \) both tend to \( \frac{1}{2} \) as \( P \to Q \). Thus, a good trial value for \( \Delta R^{(s_{\text{fin}})} \) is \( e^{-s_{\text{fin}}} \Delta R \). Plug this value of \( \Delta R^{(s_{\text{fin}})} \) into \( \Delta P(X) = B(X) \Delta R^{(s_{\text{fin}})} \) and check that it gives \( P^{(s_{\text{fin}})}(X) \in [0, 1] \) for all \( X \). If not, then continue halving the trial value of \( \Delta R^{(s_{\text{fin}})} \) until \( P^{(s_{\text{fin}})}(X) \in [0, 1] \) for all \( X \). This occurs eventually, assuming \( Q(X) \neq 0 \) for all \( X \).

[7] M. Abramowitz, I.A. Stegun, *Handbook of Mathematical Functions* (1974, Dover).