FUNDAMENTAL THEOREM OF ALGEBRA - A NEVANLINNA THEORETIC PROOF

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ABSTRACT. There are several proofs of the Fundamental Theorem of Algebra, mainly using algebra, analysis and topology. In this article, we have shown that the Fundamental Theorem of Algebra can be proved using Nevanlinna’s first fundamental theorem as well as second fundamental theorem also.

1. Introduction

One of the celebrated theorems in mathematics is Fundamental theorem of algebra, which tells that every non constant polynomial over \( \mathbb{C} \) has a root in \( \mathbb{C} \).

Some mathematicians, like Peter Roth (Arithmetica Philosophica, 1608), Albert Girard (L’invention nouvelle en l’Algèbre, 1629), and René Descartes told some version of Fundamental Theorem of Algebra in the early 17th century in their writings.

But first attempt at proving of this theorem was made in 1746 by Jean Le Rond d’Alembert, unfortunately his proof was incomplete. Also attempts were made by Euler(1749), de Foncenex(1759), Lagrange(1772) and Laplace(1795) but Carl Friedrich Gauss is credited with producing the first correct proof in his doctoral dissertation of 1799, although in that proof also had some small gaps.

A rigorous proof was first produced by Argand in 1806 and the first textbook containing the proof is Cours d’analyse de l’École Royale Polytechnique(1821) due to Cauchy.

Now a days, there are many proofs of the Fundamental Theorem of Algebra, mainly using algebra, analysis and topology [2].

There are several analytical proofs using complex analysis, for example, proof based on Liouville’s Theorem, Rouche’s Theorem, the Maximum Principle, Picard’s Theorem, and the Cauchy Integral Theorem, Open mapping theorem etc.

The aim of this article is to produce another analytical proof of Fundamental Theorem of Algebra, using Nevanlinna theory.

2. Basic notations of Nevanlinna Theory

The theory of meromorphic functions was greatly developed by Rolf Nevanlinna [13] during the 1920’s. In both its scope and its power, his approach greatly surpasses previous results, and in his honor the field is now also known as Nevanlinna theory.

Before going to describe our new proof, we briefly state some definitions, notations, estimations like the Second Main Theorem of Nevanlinna [11, 18] which we used vividly throughout our journey.

Let \( f(z) \) be a function that is meromorphic (i.e., regular except poles) and non-constant in the complex plane. We denote by \( n(r, a; f) \) the number of \( a \)-points, with due count of multiplicity, of \( f(z) \) in \( |z| < r \) for \( a \in \mathbb{C} \cup \{ \infty \} \), where an \( a \)-point is counted according to its multiplicity. We put

\[
(2.1) \quad N(r, a; f) = \int_0^r \frac{n(t, a; f) - n(0, a; f)}{t} dt + n(0, a; f) \log r,
\]

2010 Mathematics Subject Classification: Primary 30D35; Secondary 30D20.

Key words and phrases: Fundamental Theorem of Algebra, Nevanlinna Theory, Mathematical induction.
where \( n(0, a; f) \) denotes the multiplicity of \( a \)-points of \( f(z) \) at the origin. Similarly we define 
\( \overline{N}(r, a; f) \) where \( a \)-points of \( f \) are counted without multiplicity. Next we define

\[
m(r, \infty; f) = m(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})|d\theta,
\]

(2.2)

where \( \log^+ x := \max\{\log x, 0\} \) for \( x \geq 0 \).

Also the Nevanlinna’s characteristic function of \( f \) is defined as

\[ T(r, f) = m(r, f) + N(r, f). \]

The basic estimate is the First Fundamental Theorem of Nevanlinna, for every complex number \( a \), finite or infinite

\[
T(r, f) = m(r, af) + N(r, a; f) + O(1)
\]

as \( r \to \infty \). This result provides an upper bound to the number of roots of the equation \( f(z) = a \) for all \( a \).

But the difficult question of lower bounds of the number of roots of the equation \( f(z) = a \) is answered by Second Fundamental Theorem of Nevanlinna.

Suppose that \( f(z) \) is a non constant meromorphic function in whole complex plane. A quantity \( \Delta \) is said to be \( S(r, f) \) if

\[ \Delta T(r, f) \to 0 \]

as \( r \to \infty \) out side of a set \( E \) in \((0, \infty)\) with finite linear measure.

3. Lemmas

In this section, we state some results which we need in due course of time.

**Lemma 3.1.** ([11]) For a non-constant polynomial function \( P \)

\[ T(r, P) = \text{deg}(P) \log r + O(1), \]

as \( r \to \infty \), where \( \text{deg}(P) \) is the degree of the polynomial \( P \) and \( O(1) \) is a bounded quantity \( a \).

**Lemma 3.2.** ([IS]) (Nevanlinna’s First Fundamental Theorem) Let \( f(z) \) be a non-constant meromorphic function defined in \(|z| < R \) \((0 < R \leq \infty)\) and let \( a \in \mathbb{C} \cup \{\infty\} \) be any complex number. Then for \( 0 < r < R \)

\[ T\left(r, \frac{1}{f-a}\right) = T(r, f) + O(1), \]

where \( O(1) \) is a bounded quantity depending on \( f \) and \( a \) but not on \( r \).

**Lemma 3.3.** ([IS]) (page no. 23) (Second Fundamental Theorem) Suppose that \( f(z) \) is a non-constant meromorphic function in the complex plane and \( a_1, a_2, ..., a_q \) are \( q \geq 3 \) distinct values in \( \mathbb{C} \cup \{\infty\} \). Then

\[ (q-2)T(r, f) < \sum_{j=1}^{q} N(r, a_j; f) + S(r, f) \]

where \( S(r, f) \) is a quantity such that \( \frac{S(r, f)}{T(r, f)} \to 0 \) as \( r \to +\infty \) out side of a set \( E \) in \((0, \infty)\) with finite linear measure.

**Lemma 3.4.** A polynomial of the form

\[ Q(z) = b_0z^m + b_1z^{m-1} + b_2z^{m-2} + ... + b_{m-l}z^l + b_m \]

has at least one zero, where \( m \) and \( l \) are positive integers satisfying \( m > 2 \), \( m > l \geq 2 \) and \( b_i \in \mathbb{C} \) for \( i = 1, 2, 3, ..., m - 2, m \) with \( b_0 \neq 0 \).
Proof. On contrary, we assume that the polynomial $Q(z)$ has no zero, then by definition, $\overline{N}(r, 0; Q) = 0$.

Next we define $F(z) := z^l R(z)$ where $R(z) = b_0 z^{m-l} + b_1 z^{m-l-1} + b_2 z^{m-l-2} + ... + b_{m-2}$. Then

\begin{equation}
Q(z) = F(z) + b_m,
\end{equation}

Thus applying the Second Fundamental Theorem and Lemma 3.1 we have

\begin{align*}
T(r, F) &< \overline{N}(r, \infty; F) + \overline{N}(r, 0; F) + \overline{N}(r, -b_m; F) + S(r, F) \\
&< \overline{N}(r, 0; z^l) + \overline{N}(r, 0; R(z)) + \overline{N}(r, 0, Q(z)) + S(r, F) \\
&\leq \frac{1}{l} N(r, 0; z^l) + N(r, 0; R(z)) + S(r, F) \\
&= \log r + (m-l) \log r + S(r, F) \\
&= \left( \frac{m-l+1}{m} \right) T(r, F) + S(r, F),
\end{align*}

which is absurd, hence our assumption is wrong. Thus $Q(z)$ has at least one zero. Hence the proof. \hfill \Box

4. Proof of the Fundamental Theorem of Algebra

Assume $S(n)$ denotes the following statement :

Any $n$-degree polynomial over $\mathbb{C}$ has at least one zero for any $n \in \mathbb{N}$.

Thus it is sufficient to prove that the statement $S(n)$ is true for all $n \in \mathbb{N}$, and we have to prove that the statement $S(n)$ is true by using mathematical induction on $n$.

It is obvious that the statements $S(1)$ and $S(2)$ are true, and assume that $S(k)$ is true for any natural number $k \geq 3$. Now we have to show that $S(k+1)$ is true.

For this, we consider any polynomial of degree $k+1$ over $\mathbb{C}$ as

\[ P(z) = a_0 z^{k+1} + a_1 z^k + a_2 z^{k-1} + \ldots + a_k z + a_{k+1} \]

with $a_0 \neq 0$.

Since the statement $S(k)$ is true, so that we can find a suitable complex number $h$ such that the coefficient of $z$ in $P(z+h)$ is zero. So by Lemma 3.3 $P(z+h)$ has at least one zero. Consequently, $P(z)$ has at least one zero. Hence the statement $S(k+1)$ is true.

Thus by mathematical induction, any $n$-degree non-constant polynomial over $\mathbb{C}$ has at least one zero for any $n \in \mathbb{N}$. Hence the proof.

**Remark 4.1.** If $P(z)$ be a non-constant polynomial having no zero in $\mathbb{C}$, then by First Fundamental Theorem, one can deduce that

\begin{equation}
\log r = O(1),
\end{equation}

for sufficiently large $r$ where $z := re^{i\theta}$ and $O(1)$ is a bounded term dependent on $P(z)$. But equation (4.1) is absurd. Thus one can also deduce Fundamental Theorem Algebra from First Fundamental Theorem.

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