Singular solutions of conformal Hessian equation

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Abstract. We show that for any \( \varepsilon \in ]0,1[ \) there exists an analytic outside zero solution to a uniformly elliptic conformal Hessian equation in a ball \( B \subset \mathbb{R}^5 \) which belongs to \( C^{1,\varepsilon}(B) \setminus C^{1,\varepsilon+}(B) \).

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1 Introduction

In this paper we study a class of fully nonlinear second-order elliptic equations of the form

\[
F(D^2 u, Du, u) = 0
\]

(1)

defined in a domain of \( \mathbb{R}^n \). Here \( D^2 u \) denotes the Hessian of the function \( u \), \( Du \) being its gradient. We assume that \( F \) is a Lipschitz function defined on a domain in the space \( \text{Sym}_2(\mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R} \), \( \text{Sym}_2(\mathbb{R}^n) \) being the space of \( n \times n \) symmetric matrices and that \( F \) satisfies the uniform ellipticity condition, i.e. there exists a constant \( C = C(F) \geq 1 \) (called an ellipticity constant) such that

\[
C^{-1}||N|| \leq F(M + N) - F(M) \leq C||N||
\]

for any non-negative definite symmetric matrix \( N \); if \( F \in C^1(\text{Sym}_2(\mathbb{R}^n)) \) then this condition is equivalent to

\[
\frac{1}{C'}|\xi|^2 \leq F_{u_{ij}}(\xi,\xi) \leq C'|\xi|^2, \forall \xi \in \mathbb{R}^n.
\]

Here, \( u_{ij} \) denotes the partial derivative \( \partial^2 u/\partial x_i \partial x_j \). A function \( u \) is called a classical solution of (1) if \( u \in C^2(\Omega) \) and \( u \) satisfies (1). Actually, any classical solution of (1) is a smooth \( (C^{\alpha+3}) \) solution, provided that \( F \) is a smooth \( (C^\alpha) \) function of its arguments.

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More precisely, we are interested in conformal Hessian equations (see, e.g. [9], pp. 5-6) i.e. those of the form

\[ F[u] := f(\lambda(A^u)) = \psi(u, x) \tag{2} \]

\( f \) being a Lipschitz function on \( \mathbb{R}^n \) invariant under permutations of the coordinates and

\( \lambda(A^u) = (\lambda_1, \ldots, \lambda_n) \)

being the eigenvalues of the conformal Hessian in \( \mathbb{R}^n \):

\[ A^u := uD^2u - \frac{1}{2}|Du|^2I_n \tag{3} \]

where \( n \geq 3, u > 0 \).

In this case \( F \) is invariant under conformal mappings \( T: \mathbb{R}^n \rightarrow \mathbb{R}^n \), i.e. transformations which preserve angles between curves. In contrast to the case \( n = 2 \), for \( n \geq 3 \) any conformal transformation of \( \mathbb{R}^n \) is decomposed into a finitely many Möbius transformations, that is mappings of the form

\[ Tx = y + \frac{kA(x - z)}{|x - z|^a}, \]

with \( x, z \in \mathbb{R}^n, k \in \mathbb{R}, a \in \{0, 2\} \) and an orthogonal matrix \( A \). In other words, each \( T \) is a composition of a translation, a homothety, a rotation and (may be) an inversion. If \( T \) is a conformal mapping and \( v(x) = J_T^{-1/n}u(Tx) \), where \( J_T \) denotes the Jacobian determinant of \( T \) then \( F[v] = F[u] \). Note that this class of equations is very important in geometry, see [1] and references therein.

We are interested in the Dirichlet problem

\[
\begin{cases}
F(D^2u, Du, u) = 0, u > 0 & \text{in } \Omega \\
u = \varphi & \text{on } \partial\Omega,
\end{cases}
\tag{4}
\]

where \( \Omega \subset \mathbb{R}^n \) is a bounded domain with a smooth boundary \( \partial\Omega \) and \( \varphi \) is a continuous function on \( \partial\Omega \).

Consider the problem of existence and regularity of solutions to the Dirichlet problem (4) which has always a unique viscosity (weak) solution for fully nonlinear elliptic equations. The viscosity solutions satisfy the equation (1) in a weak sense, and the best known interior regularity ([1], [2], [8]) for them is \( C^{1+\varepsilon} \) for some \( \varepsilon > 0 \). For more details see [2], [8]. Recall that in [5] the authors constructed a homogeneous singular viscosity solution in 5 dimensions for Hessian equations of order \( 1 + \delta \) for any \( \delta \in [0, 1] \), that is, of any order compatible with the mentioned interior regularity results. In fact we proved in [5] the following result.
Theorem 1.1.

The function

\[ w_{5, \delta}(x) = P_5(x)/|x|^{1+\delta}, \quad \delta \in [0, 1] \]

is a viscosity solution to a uniformly elliptic Hessian equation \( F(D^2w) = 0 \) with a smooth functional \( F \) in a unit ball \( B \subset \mathbb{R}^5 \) for the isoparametric Cartan cubic form

\[ P_5(x) = x_1^3 + \frac{3x_1^2}{2} (z_1^2 + z_2^2 - 2z_3^2 - 2x_2^2) + \frac{3\sqrt{3}}{2} (x_2z_1^2 - x_2z_2^2 + 2z_1z_2z_3) \]

with \( x = (x_1, x_2, z_1, z_2, z_3) \).

which proves the optimality of the interior \( C^{1+\varepsilon} \)-regularity of viscosity solutions to fully nonlinear equations in 5 and more dimensions.

In the present paper we show that the same singularity result remains true for conformal Hessian equations.

Theorem 1.2.

Let \( \delta \in [0, 1] \). The function

\[ u(x) := c + w_{5, \delta}(x) = c + \frac{P_5(x)}{|x|^{1+\delta}}, \]

is a viscosity solution to a uniformly elliptic conformal Hessian equation (1) in a unit ball \( B \subset \mathbb{R}^5 \) for a sufficiently large positive constant \( c \) (\( c = 240000 \) is sufficient for \( \delta = \frac{1}{2} \)).

Notice also that the result does not hold for \( \delta = 0 \) and we do not know how to construct a non-classical \( C^{1,1} \)-solution to a uniformly elliptic conformal Hessian equation.

The rest of the paper is organized as follows: in Section 2 we recall some necessary preliminary results and we prove our main results in Section 3; to simplify the notation we suppose that \( \delta = \frac{1}{2} \) in Section 3; for any \( \delta \) the proof is along the same line, but more cumbersome. The proof in Section 3 uses MAPLE to verify some algebraic identities but is completely rigorous (and is human-controlled for \( \delta = \frac{1}{2} \)).

2 Preliminary results

Notation: for a real symmetric matrix \( A \) we denote by \( |A| \) the maximum of the absolute value of its eigenvalues.

Let \( u \) be a strictly positive function on \( B_1 \). Define the map

\[ A : B_1 \rightarrow \lambda(S) \in \mathbb{R}^n. \]

\( \lambda(S) = \{ \lambda_1 \geq \ldots \geq \lambda_n \} \in \mathbb{R}^n \) being the (ordered) set of eigenvalues of the conformal Hessian

\[ A^u := uD^2u - \frac{1}{2}|Du|^2I_n. \]
The following ellipticity criterion can proved similarly to Lemma 2.1 of [6].

**Lemma 2.1.** Suppose that the family
\[
\{A^u(a) - O^{-1} \cdot A^u(b) \cdot O : a, b \in B_1, O \in SO(n)\} \setminus \{0\}
\]
is uniformly hyperbolic, i.e. if \(\mu_1(a, b, O) \geq \ldots \geq \mu_n(a, b, O)\) is the ordered spectrum of \(A^u(a) - O^{-1} \cdot A^u(b) \cdot O \neq 0\) then
\[
\forall a, b \in B_1, \forall O \in SO(n), \quad C^{-1} \leq -\frac{\mu_1(a, b, O)}{\mu_n(a, b, O)} \leq C
\]
for some constant \(C > 1\). Then \(u\) is a viscosity solution in \(B_1\) of a uniformly elliptic conformal Hessian equation (1).

We recall then some properties of the function \(w := w_{5, \delta}(x) = \frac{P_5(x)}{|x|^{1+\delta}},\) and its Hessian \(D^2w\) proved in [5].

**Lemma 2.2.**
There exists a 3-dimensional Lie subgroup \(G_P\) of \(SO(5)\) such that \(P\) is invariant under its natural action and the orbit \(G_P S_1^1\) of the circle \(S_1^1 = \{(\cos(\chi), 0, \sin(\chi), 0, 0) : \chi \in \mathbb{R}\} \subset S_1^4\) under this action is the whole \(S_1^4\).

**Lemma 2.3.**
(i) Let \(x \in S_1^4\), and let \(x \in G_P(p, 0, r, 0, 0)\) with \(p^2 + r^2 = 1\). Then
\[
\text{Spec}(D^2w_{5, \delta}(x)) = \{\mu_{1, \delta}, \mu_{2, \delta}, \mu_{3, \delta}, \mu_{4, \delta}, \mu_{5, \delta}\}
\]
for
\[
\mu_{1, \delta} = \frac{p(p^2\delta + 6 - 3\delta)}{2},
\]
\[
\mu_{2, \delta} = \frac{p(p^2\delta - 3 - 3\delta) + 3\sqrt{12 - 3p^2}}{2},
\]
\[
\mu_{3, \delta} = \frac{p(p^2\delta - 3 - 3\delta) - 3\sqrt{12 - 3p^2}}{2},
\]
\[
\mu_{4, \delta} = -\frac{p\delta(6 - \delta)(3 - p^2) + \sqrt{D(p, \delta)}}{4},
\]
\[
\mu_{5, \delta} = -\frac{p\delta(6 - \delta)(3 - p^2) - \sqrt{D(p, \delta)}}{4},
\]
and
\[
D(p, \delta) := (6 - \delta)(4 - \delta)(2 - \delta)\delta(p^2 - 3)^2p^2 + 144(\delta - 2)^2 > 0.
\]
(ii) Let \( \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_5 \) be the ordered eigenvalues of \( D^2 w_{5,\delta}(x) \). Then

\[
\lambda_1 = \mu_{2,\delta}, \quad \lambda_5 = \mu_{3,\delta}, \\
\lambda_2 = \begin{cases} \\
\mu_{4,\delta} & \text{for } p \in [-1, p_0(\delta)], \\
\mu_{1,\delta} & \text{for } p \in [p_0(\delta), 1], \\
\end{cases} \\
\lambda_3 = \begin{cases} \\
\mu_{5,\delta} & \text{for } p \in [-1, -p_0(\delta)], \\
\mu_{1,\delta} & \text{for } p \in [-p_0(\delta), p_0(\delta)], \\
\mu_{4,\delta} & \text{for } p \in [p_0(\delta), 1], \\
\end{cases} \\
\lambda_4 = \begin{cases} \\
\mu_{1,\delta} & \text{for } p \in [-1, -p_0(\delta)], \\
\mu_{5,\delta} & \text{for } p \in [-p_0(\delta), p_0(\delta)], \\
\mu_{4,\delta} & \text{for } p \in [p_0(\delta), 1], \\
\end{cases}
\]

where

\[
p_0(\delta) := \frac{3^{1/4} \sqrt{1 - \delta}}{(3 + 2\delta - \delta^2)^{1/4}} = \frac{3^{1/4} \sqrt{\varepsilon}}{(4 - \varepsilon^2)^{1/4}} \in ]0, 1].
\]

Note the oddness property of the spectrum:

\[
\lambda_1,\delta(-p) = -\lambda_5,\delta(p), \quad \lambda_2,\delta(-p) = -\lambda_4,\delta(p), \quad \lambda_3,\delta(-p) = -\lambda_3,\delta(p).
\]

**Proposition 2.1.**

Let \( N_\delta(x) = D^2 w_\delta(x) \), \( 0 \leq \delta < 1 \). Suppose that \( a \neq b \in B_1 \setminus \{0\} \) and let \( O \in O(5) \) be an orthogonal matrix s.t.

\[
N_\delta(a, b, O) := N_\delta(a) - ^t O \cdot N_\delta(b) \cdot O \neq 0.
\]

Denote \( \Lambda_1 \geq \Lambda_2 \geq \ldots \geq \Lambda_5 \) the eigenvalues of the matrix \( N_\delta(a, b, O) \). Then

\[
\frac{1}{C} \leq -\frac{\Lambda_1}{\Lambda_5} \leq C
\]

for \( C := C(\delta) := \frac{1000(\delta+1)(3-\delta)}{3(1-\delta)^2} \), for \( k \in [\frac{1}{2}, 1] \) one can choose \( C = 1000 \).

**Corollary 2.1.**

\[
\Lambda_1 \geq \frac{|N_\delta(a, b, O)|}{C(\delta)}, \quad |\Lambda_5| \geq \frac{|N_\delta(a, b, O)|}{C(\delta)}.
\]

We need also the following classical Weyl’s result:

**Lemma 2.4.**

Let \( A, B \) be two real symmetric matrices with the eigenvalues \( \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \) and \( \lambda'_1 \geq \lambda'_2 \geq \ldots \geq \lambda'_n \) respectively. Then for the eigenvalues \( \Lambda_1 \geq \Lambda_2 \geq \ldots \geq \Lambda_n \) of the matrix \( A - B \) we have

\[
\Lambda_1 \geq \max_{i=1,\ldots,n} (\lambda_i - \lambda'_i), \quad \Lambda_n \leq \min_{i=1,\ldots,n} (\lambda_i - \lambda'_i).
\]
3 Proofs

Let $n = 5$, $u(x) = c + w_{5,\delta}(x)$. We begin with $\delta = 0$ and show that the result is false in this case. Indeed let $a = (1,0,0,0,0)$, $b = (\frac{1}{2},0,0,0,0)$, $O = I_5$. Then

$$w(a) = 1, \quad w(b) = \frac{1}{2}, \quad |Du(a)| = |Dw(a)| = 9, \quad |Du(b)| = |Dw(b)| = \frac{9}{4},$$

and

$$D^2 u(a) = D^2 w(a) = D^2 u(b) = D^2 w(b),$$

which is negative since the spectrum of $D^2 w(a)$ is $(2, 2, 2, -7, -7)$. The reason is clearly that $D^2 w(a)$ for $\delta = 0$ is homogeneous order 0 and depends only on the direction vector $a/|a|$.

Suppose now that $\delta \in [0, 1]$. As we mentioned before, we set $\delta = \frac{1}{2}$; in this case $c = 240000$. First we spell out Lemma 2.3 for $\delta = \frac{1}{2}$.

Lemma 3.1.

(i) Let $x \in S^4_1$ and let $x \in G_P(p, 0, r, 0, 0)$ with $p^2 + r^2 = 1$. Then

$$\text{Spec}(D^2 u(x)) = \text{Spec}(D^2 w(x)) = \{\mu_1, \mu_2, \mu_3, \mu_4, \mu_5\}$$

for

$$\mu_1 = \frac{3p(p^2 + 1)}{4},$$

$$\mu_2 = \frac{3p(p^2 - 5) + 6\sqrt{12 - 3p^2}}{4},$$

$$\mu_3 = \frac{3p(p^2 - 5) - 6\sqrt{12 - 3p^2}}{4},$$

$$\mu_4 = \frac{27p(p^2 - 3) + 3\sqrt{105p^6 - 630p^4 + 945p^2 + 64}}{16},$$

$$\mu_5 = \frac{27p(p^2 - 3) - 3\sqrt{105p^6 - 630p^4 + 945p^2 + 64}}{16}.$$  

(ii) Let $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_5$ be the ordered eigenvalues of $\text{Spec}(D^2 u(x)) = \text{Spec}(D^2 w(x))$. Then

$$\lambda_1 = \mu_2, \quad \lambda_5 = \mu_3,$$

$$\lambda_2 = \begin{cases} \mu_4 & \text{for } p \in [-1, p_0], \\ \mu_1 & \text{for } p \in [p_0, 1], \end{cases}$$

$$\lambda_3 = \begin{cases} \mu_5 & \text{for } p \in [-1, -p_0], \\ \mu_1 & \text{for } p \in [-p_0, p_0], \\ \mu_4 & \text{for } p \in [p_0, 1], \end{cases}$$
\[
\lambda_4 = \begin{cases} 
\mu_1 & \text{for } p \in [-1, -p_0], \\
\mu_5 & \text{for } p \in [-p_0, 1], 
\end{cases}
\]

where

\[p_0 = 5^{1/4} \approx 0.6687403050.\]

We will need also the derivatives of the eigenvalues.

**Lemma 3.2.** Let \( d_i(p) := \frac{d(\mu_i)}{dp} \). Then

\[
d_1(p) = \frac{3(3p^2 + 1)}{4},
\]

\[
d_2(p) = -\frac{3(5 - 3p^2)}{4} + \frac{9p}{2\sqrt{12 - 3p^2}},
\]

\[
d_3(p) = \frac{3(5 - 3p^2)}{4} - \frac{9p}{2\sqrt{12 - 3p^2}},
\]

\[
d_4(p) = \frac{81(1 - p^2)}{16} \left( \frac{35p(3 - p^2)}{3\sqrt{105p^6 - 630p^4 + 945p^2 + 64}} - 1 \right),
\]

\[
d_5(p) = -\frac{81(1 - p^2)}{16} \left( \frac{35p(3 - p^2)}{3\sqrt{105p^6 - 630p^4 + 945p^2 + 64}} + 1 \right).
\]

Simple calculus gives

**Corollary 3.1.**

\[D := \max\{|d_i(p)| : p \in [-1, 1], i = 1, \ldots, 5\} < 10.\]

Below we denote \( D_i(p) := \frac{d(\lambda_i)}{dp} \); the relation of \( D_i(p) \) and \( d_i(p) \) is clear from Lemma 3.1 (ii); for example, \( D_1(p) = d_2(p) \), \( D_5(p) = d_3(p) \).

The proof of Theorem 1.2 is based on the following lemmas. Let

\[a, b \in B_1 \setminus \{0\}, |a| = s \leq 1, |b| = t \leq 1, O \in O(5),\]

\[a' := \frac{a}{s} \in G_P(p, 0, r, 0, 0), b' := \frac{b}{t} \in G_P(q, 0, r', 0, 0).\]

Below we denote

\[K := K(p, q, s, t) = |s - t| + |p - q|,\]

\[M_1 := M_1(a, b, O) := D^2 u(a) - O^{-1} D^2 u(b) \cdot O,\]

\[M_2 := M_2(a, b, O) := w(a) D^2 u(a) - O^{-1} w(b) D^2 u(b) \cdot O.\]

**Lemma 3.3.**

\[\left| |Du(a)|^2 - |Du(b)|^2 \right| \leq 16K.\]
Proof. First, $|Du(a)|^2 = |Dw(a)|^2, |Du(b)|^2 = |Dw(b)|^2$. Since $P = P_t(x)$ can be represented as the generic traceless norm in the Jordan algebra $\text{Sym}_4(\mathbb{R})$ it verifies the eiconal equation $|DP|^2 = |x|^4$, see e.g. [7]. Therefore, an easy calculation gives

$$|Du(a)|^2 = \frac{9s(16 - 3p^2(p^2 - 3)^2)}{32}, |Du(b)|^2 = \frac{9t(16 - 3q^2(q^2 - 3)^2)}{32},$$

$$|Du(a)|^2 - |Du(b)|^2 \leq \left| \frac{9s(16 - 3p^2(p^2 - 3)^2)}{32} - \frac{9t(16 - 3q^2(q^2 - 3)^2)}{32} \right| + \left| \frac{9t(16 - 3p^2(p^2 - 3)^2)}{32} - \frac{9t(16 - 3q^2(q^2 - 3)^2)}{32} \right| = \left| \frac{9(s - t)(16 - 3p^2(p^2 - 3)^2)}{32} \right| + \left| \frac{27t(p - q)(p + q)((q^2 - 3)^2 - (p^2 - 3)^2)}{32} \right| \leq \left| \frac{9(s - t)}{2} \right| + \left| \frac{243(p - q)}{16} \right| \leq 16K.$$ 

Lemma 3.4. Let $M := |M_1| = |D^2u(a) - O^{-1} \cdot D^2u(b) \cdot O|$. Then

$$M \geq \frac{K}{8}.$$ 

Proof. If one replaces $a$ by $a' = a/s$ and $b$ by $b'' = b/s$ the quantity $M$ gets bigger and $K$ gets smaller. Therefore we can suppose that $|a| = s = 1$. Then we have

$$D^2u(a) - O^{-1} \cdot D^2u(b) \cdot O = D^2u(a) - \frac{O^{-1} \cdot D^2u(b') \cdot O}{\sqrt{t}}.$$ 

By Lemma 2.4 we have

$$M \geq \max \left\{ \lambda_i(p) - \frac{\lambda_i(q)}{\sqrt{t}} : i = 1, \ldots, 5 \right\},$$

$$M \geq \min \left\{ \lambda_i(p) - \frac{\lambda_i(q)}{\sqrt{t}} : i = 1, \ldots, 5 \right\}.$$ 

Suppose first $p \geq q$. If $q \geq -\frac{24}{25} = -0.96$ then

$$\forall p' \in [q, p], D_1(p') < -1/4 = -0.25, \lambda_1(p) > \frac{3}{2}$$

(by a simple calculation using the explicit formulas for $D_1, \lambda_1$). Therefore

$$\lambda_1(p) - \frac{\lambda_1(q)}{\sqrt{t}} = \lambda_1(p) - \lambda_1(q) + \lambda_1(q) - \frac{\lambda_1(q)}{\sqrt{t}} \leq - \frac{p - q}{4} - \frac{3}{2} \left( \frac{1}{\sqrt{t}} - 1 \right) < -\frac{K}{4}.$$ 

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If \( q < -0.96 \) but \( p \geq -\frac{24}{25} = -0.92 \) then

\[
\lambda_1(p) - \frac{\lambda_1(q)}{\sqrt{t}} = \lambda_1(p) - \lambda_1(q) + \lambda_1(q) - \frac{\lambda_1(q)}{\sqrt{t}} \leq \lambda_1(p) - \lambda_1 \left( \frac{24}{25} \right) + \lambda_1(q) - \frac{\lambda_1(q)}{\sqrt{t}}
\]

\[
- \frac{p + 0.96}{4} - \frac{3}{2} \left( \frac{1}{\sqrt{t}} - 1 \right) < - \frac{p - q}{8} - \frac{3}{2} \left( \frac{1}{\sqrt{t}} - 1 \right) < - \frac{K}{8}.
\]

Suppose then that \( q < -0.96, p < -0.92 \). In this case we have

\[
\forall p' \in [q, p], \; d_2(p') > \frac{5}{2}, \; \lambda_2(p') < -\frac{3}{2}
\]

and thus

\[
\lambda_2(p) - \frac{\lambda_2(q)}{\sqrt{t}} = \lambda_2(p) - \lambda_2(q) + \lambda_2(q) - \frac{\lambda_2(q)}{\sqrt{t}} \geq \frac{5(p - q)}{2} + \frac{3}{2} \left( \frac{1}{\sqrt{t}} - 1 \right) \geq \frac{3K}{4}
\]

which finishes the proof for \( p \geq q \). The case \( q \geq p \) is treated similarly (replace \( \lambda_1 \) by \( \lambda_5 \) and \( \lambda_2 \) by \( \lambda_4 \)).

**Lemma 3.5.**

\[
|M_2| = |w(a)D^2u(a) - O^{-1}w(b)D^2u(b)\cdot O| \leq 10K
\]

**Proof.** Indeed, let \( a' := a/s, b' := b/s \) then by homogeneity

\[
|w(a)D^2w(a) - O^{-1}w(b)D^2w(b)\cdot O| = |sD^2w(a') - O^{-1}tD^2w(b')\cdot O| \leq
\]

\[
\leq s|D^2w(a') - O^{-1}\cdot D^2w(b')\cdot O| + |s - t| \cdot |O^{-1}D^2w(b')| \leq
\]

\[
\leq \max_{p,i}\{|D_i(p)|\} |p - q| + 7|s - t| = \max_{p,i}\{|d_i(p)|\} |p - q| + 7|s - t| \leq 10K.
\]

**Remark 3.1.** These results remain true for any \( \delta \in [0, 1] \) if one replaces the respective constants 16, 1/8 and 10 in Lemmas 3.3, 3.4 and 3.5 by appropriate positive constants depending on \( \delta \). On the contrary, Lemma 3.4 is false for \( \delta = 0 \).

We can now prove the uniform hyperbolicity of \( M(a, b, O) \) and thus the theorem. In fact we show that one can take \( C = 6007 \) in Lemma 3.1.

Indeed,

\[
|M(a, b, O)| = |A^u(a) - O^{-1}\cdot A^u(b)\cdot O| = |cM_1 + M_2 - (|Du(a)|^2 - |Du(b)|^2) I_5|.
\]

Therefore,

\[
|A_5| \geq c|A_5(M_1)| - 10K - 16K \geq \frac{c|M_1|}{1000} - 26K \geq 240|M_1| - 26K \geq 4K,
\]

\[
|A_1| \geq cA_1(M_1) - 10K - 16K \geq \frac{c|M_1|}{1000} - 26K \geq 240|M_1| - 26K \geq 4K,
\]

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\[ |M(a, b, O)| \leq c |M_1| + |M_2| + ||Du(a)||^2 - ||Du(b)||^2 | \leq c |M_1| + 26K. \]

Thus
\[
\frac{1}{C} < \frac{4}{240026} \leq \frac{240|M_1| - 26K}{c |M_1| + 26K} \leq \frac{|A_5|}{|A_1|} \leq \frac{c |M_1| + 26K}{240|M_1| - 26K} \leq \frac{240026}{4} < C
\]

which finishes the proof. Notice that we can take \( C = 1000 + \varepsilon \) for \( \delta \leq \frac{1}{2} \) if \( c \) is sufficiently large; in the case \( \frac{1}{2} < \delta < 1 \) for sufficiently large \( c \) one gets \( C = C(\delta) + \varepsilon = \frac{1000(\delta+1)(3-\delta)}{(3-\delta)^2} + \varepsilon \).

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