Propagation of Anti-plane Shear Waves in a Cracked Graded Strip with Viscous Damping

S.M. Mousavi, S.J. Fariborz

Department of Mechanical Engineering, Amirkabir University of Technology (Tehran Polytechnic), 424, Hafez Avenue, Tehran 158754413, Iran

Abstract

The dislocation-distributed technique is utilized to study the elastodynamic fracture behavior of a graded isotropic layer with viscous damping. By investigation of the stress components due to the dislocation, the familiar Cauchy singularity is detected at the location of dislocation. Then the dislocation is utilized for the formation of cracks in the strip. The stress components of dislocation and time-harmonic antiplane point force leads to the integral equations. These equations results in the stress intensity factors (SIF) for the crack configuration in the strip.

Keywords: functionally graded material; fracture; viscous damping; strip.

1. Introduction

The Functionally graded materials (FGMs) have special behavior which makes them useful in some mechanical applications. The development of the application of these materials, necessitates the analysis of them in different fields such as fracture mechanics. The static fracture analysis of these materials is the subject of many articles recently [1]. The elastodynamic fracture analysis of FGM has been performed in some simple configurations of cracks. Zhou et al. considered a finite crack in FGM and performed the analysis by use of the Schmidt method [2]. Ma et al. considered two collinear cracks and determined the stress intensity factor for them [3]. In another article, Two cracks in a functionally graded layer bonded to dissimilar half-planes subjected to anti-plane incident harmonic stress wave is analysed by Ma et al. [4]

1 Corresponding author: Tel.: +98 21 6454 3460; fax: +98 21 641 9736.
E-mail address: sjfariborz@yahoo.com

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An off-center vertical crack in a functionally graded material strip was the subject of the study by Li et al. [5]. Obviously these researches have been done on some limited configurations of cracks and the need for a comprehensive method in the analysis of multiple cracks is vivid. There are some solution methods which are useful for arbitrary configuration of crack. For example, Element Free Galerkin Method is used by Wen and Aliabadi in order to determine the SIF for arbitrary oriented cracks [6]. This method may be used in elastodynamic fracture mechanics. Also the distributed dislocation technique is proved to be a capable method in the static [7] and dynamic [8-10] fracture mechanics for various configurations of cracks and cavities. This method has been utilized for the static analysis of FGM strip containing cracks [1].

In the present article, distributed dislocation technique is used to perform elastodynamic analysis of cracked FGM strip with damping. Any configuration of cracks may be tackled and the SIF may be determined. Due to brevity, only one configuration is considered in this article and the results for SIF are depicted. More Examples will be published as a benchmark in a future publication by the authors.

2. Isotropic FGM strip with screw dislocation

An isotropic FGM layer with thickness $h$ is under consideration, Fig. (1). $y$ axis is taken as direction of variation of material properties. A Volterra-type screw dislocation with the line of dislocation coinciding with the $y$-axis is located at distance $h_1$ from the $x$-axis, Fig. 1.

![Fig. 1: Schematic view of the FGM layer with screw dislocation](image)

The equation for anti-plane motion of a body with internal viscous dissipation reads

\[
\frac{\partial \hat{\sigma}_{xz}}{\partial x} + \frac{\partial \hat{\sigma}_{zy}}{\partial y} = \rho(y) \frac{\partial^2 W}{\partial t^2} + \eta(y) \frac{\partial W}{\partial t}
\]

where $\rho(y)$ and $\eta(y)$ are the mass density and the viscous damping coefficient per unit volume of material, respectively and $W$ is the anti-plane displacement component. The stress components are

\[
\hat{\sigma}_{xz} = \mu(y) \frac{\partial W}{\partial x}, \quad \hat{\sigma}_{zy} = \mu(y) \frac{\partial W}{\partial y}
\]

Substitution of Eq. (2) into Eq. (1) yields

\[
\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2} + \frac{\mu'(y)}{\mu(y)} \frac{\partial W}{\partial y} = \frac{\rho(y)}{\mu(y)} \frac{\partial^2 W}{\partial t^2} + \frac{\eta(y)}{\mu(y)} \frac{\partial W}{\partial t}
\]

For the layer depicted in Fig. 1, the boundary, continuity and limiting conditions may be expressed as
\[
\begin{aligned}
\sigma_{yz}(x,0,t) &= 0, \quad \sigma_{yz}(x,h,t) = 0, \quad W(0^+,y,t) - W(0^-,y,t) = B_z(t)H(y - h_1), \\
\sigma_{xz}(0^+,y,t) &= \sigma_{xz}(0^-,y,t), \quad \lim_{|t| \to \infty} W = 0 
\end{aligned}
\]  
(4)

where \( B_z(t) \) is the dislocation Burgers vector and \( H(.) \) is the Heaviside step-function. Under the assumption of steady-state deformation and time-harmonic excitation with angular frequency \( \omega \), the displacement, stress fields and Burgers vector may be written as

\[
\begin{bmatrix}
W(x,y,t), \sigma_{xz}(x,y,t), \sigma_{yz}(x,y,t), B_z(t)
\end{bmatrix} = \begin{bmatrix}
w(x,y), \sigma_{xz}(x,y), \sigma_{yz}(x,y), b_z \end{bmatrix} e^{i \omega t}
\]  
(5)

Equation (3) in view of Eqs (5) becomes

\[
\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\mu'(y)}{\mu(y)} \frac{\partial w}{\partial y} + \frac{\omega}{\mu(y)} \left[ \omega \rho(y) - i \eta(y) \right] w = 0
\]  
(6)

By virtue of the anti-symmetry of problem with respect to the y-axis, Eq. (6) should be solved in the semi-infinite strip \( x > 0 \) subject to the following boundary and limiting conditions

\[
\begin{aligned}
\sigma_{yz}(x,0) &= 0, \quad \sigma_{yz}(x,h) = 0, \quad w(0^+,y) = \frac{b}{2} H(y - h_1), \quad \lim_{x \to \infty} w = 0
\end{aligned}
\]  
(7)

To solve Eq. (6) with conditions (7), it is convenient to divide the strip into two regions \( 0 < y < h_i \) and \( h_i < y < h \). The continuity conditions necessary for the two regions are

\[
w(x,h_i^-) = w(x,h_i^+) \quad \text{and} \quad \sigma_{yz}(x,h_i^-) = \sigma_{yz}(x,h_i^+)
\]  
(8)

The material properties of the graded layer is assumed as

\[
\begin{bmatrix} \mu(y), \rho(y), \eta(y) \end{bmatrix} = \begin{bmatrix} \mu_0, \rho_0, \eta_0 \end{bmatrix} e^{\gamma xy}
\]  
(9)

Equation (6) in view of Eqs (9) becomes

\[
f \mu \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + 2\kappa \frac{\partial w}{\partial y} + k_T^2 w = 0
\]  
(10)

where \( f = \mu_{0x} / \mu_{0y}, k_T = -|k_T| \exp(-i \theta / 2) \),

\[
|k_T| = \frac{\omega}{C_y} \sqrt{\frac{\omega^2}{C_y} + \frac{\eta_0^2}{\mu_0 \rho_0}}, \quad \theta = \tan^{-1} \left( \frac{\eta_0}{\rho_0 \omega} \right), \quad 0 \leq \theta < \frac{\pi}{2}
\]  
(11)

and \( C_y = \sqrt{\mu_0 / \rho_0} \) is the shear wave velocity in the y-direction.

The solution to Eq. (10) in the two regions \( 0 < y < h_i \) and \( h_i < y < h \) is achieved by means of the Fourier sine transform. Following a routine methodology, the displacement field in the whole region may be obtained and utilizing equation (2), the stress components are obtained as

\[
\sigma_{xz} = -\frac{\mu_{0x} f \mu b_z}{\pi} e^{x(h_1+y)} \int_0^\infty \frac{\lambda^2 \cos(\lambda x)}{\beta (k_T^2 - f \mu \lambda^2)} \left[ \beta \cos h(\beta y) + \kappa \sin h(\beta y) \right] \sin h(\beta (h - h_1)) \left[ \beta \cos h(\beta y) + \kappa \sin h(\beta y) \right] \sin h(\beta h) d\lambda
\]  
\[0 < y \leq h_i \]
\[\sigma_{xz} = \frac{\mu_0 f \mu_\beta}{\pi} e^{\kappa(h+y)} \int_0^\infty \frac{\lambda^2 \cos(\lambda x)}{\beta(k_y^2 - f^2)\lambda^2} \left[ \beta \cos h(\beta(y-h)) + \kappa \sin h(\beta(y-h)) \right] \sin h(\beta h) d\lambda \]

\[\sigma_{yy} = -\frac{\mu_0 b \mu_\beta}{\pi} \frac{e^{\kappa(h+y)}}{2f_\mu} \sin \left( k_x x / \sqrt{f_\mu} \right) \]

where \( \kappa(x) \) designates the sign function and \( \beta = \sqrt{\kappa^2 + \lambda^2 - k_y^2} \). The asymptotic analysis of Eqs (12) shows that stress fields exhibit the familiar Cauchy type singularity at dislocation location.

3. Strip under anti-plane point-force

Consider a strip without any defect and under a pair of self-equilibrating time-harmonic anti-plane concentrated loads of magnitude \( p \) represented by

\[\hat{\sigma}_{yz}(x,0,t) = p \delta(x) e^{i\omega t} \]

\[\hat{\sigma}_{yz}(x,h,t) = p \delta(x) e^{i\omega t} \]

where \( \delta(x) \) is the Dirac delta function. Equation (3) should be solved subjected to the above boundary conditions. The application of Fourier transform to Eq. (3) and carrying out a procedure similar to that for dislocation solution leads to the following stress components:

\[\sigma_{xx}(x,y) = \frac{f \mu}{\pi} e^{-3k_y} \int_0^\infty \frac{\beta \cosh[\beta(y-h)] + \kappa \sinh[\beta(y-h)]}{(\beta^2 - \kappa^2) \sinh(\beta h)} \lambda \sin(\lambda x) d\lambda \]

\[\sigma_{yy}(x,y) = \frac{3p \mu}{\pi} e^{-3k_y} \int_0^\infty \frac{\beta \cosh[\beta(y-h)] + \kappa \sinh[\beta(y-h)]}{(\beta^2 - \kappa^2) \sinh(\beta h)} \cos(\lambda x) d\lambda \]

where \( sgn(x) \) designates the sign function and \( \beta = \sqrt{\kappa^2 + \lambda^2 - k_y^2} \).

The asymptotic analysis of Eqs (12) shows that stress fields exhibit the familiar Cauchy type singularity at dislocation location.
4. Crack formulation-Results and discussion

The solutions of dislocation and point load are utilized in crack formation and stress intensity factors are determined for various crack configurations [7]. As an example, the interaction of kinked and straight central cracks in an isotropic FGM layer is analyzed. The influence of kink-angle $\theta$ on stress intensity factors for a graded strip is illustrated in Figs (2). The interaction of cracks for $\theta = \pi / 2$ and $\theta = 3\pi / 2$ is weak and enhances as the distance between $R_1$ and $L_2$ diminishes. At $\theta = \pi$, the local extremum of SIF at all cracks tips occur and as it was expected, $\left( k_{III} \right)_{L_1} = \left( k_{III} \right)_{R_2}$ and $\left( k_{III} \right)_{R_1} = \left( k_{III} \right)_{L_2}$.

Fig. 2: Interaction of a kinked and a straight crack in an isotropic FGM layer
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