Holographic Bounds on the UV Cutoff Scale in Inflationary Cosmology

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Abstract

We discuss how holographic bounds can be applied to the quantum fluctuations of the inflaton. In general the holographic principle will lead to a bound on the UV cutoff scale of the effective theory of inflation, but it will depend on the coarse-graining prescription involved in calculating the entropy. We propose that the entanglement entropy is a natural measure of the entropy of the quantum perturbations, and show which kind of bound on the cutoff it leads to. Such bounds are related to whether the effects of new physics will show up in the CMB.

1 Introduction and Summary

Due to the increasing accuracy of cosmological data, there is interest as to whether we can learn something about fundamental physics from inflation [1–6]. The holographic principle is thought to be a generic feature of fundamental level theories which include gravity, and might thus play an important model independent role in such studies. The holographic principle is basically a statement about the number of fundamental degrees of freedom. On the other hand, it has lately become customary to view inflation as an effective field theory, with a cutoff at energy scales where fundamental physics becomes important. The number of degrees of freedom will depend on the cutoff scale, hence holography might quite reasonably, in some way or another, be connected with an upper bound on the cutoff scale of the effective theory.

In this paper we explore this possibility in more detail. We do not find any strong constraints from holography itself, but it instead tells us something about the entropy of cosmic perturbations. Only indirectly, does holography lead to a bound on the cutoff scale. The bound appears to be weaker than the one in [7], and no stronger than one would expect from ordinary back-reaction considerations. One issue that we shall discuss

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is the classical versus quantum nature of the entropy. In previous studies of holography, the entropy of the cosmic perturbations has usually been treated at classical level, based on some coarse-graining prescription. However, the modes on sub-horizon scales are inherently quantum, so any bounds on the entropy inside the horizon of a cosmic observer should involve a semiclassical analysis\(^1\). The holographic bounds are typically based on a statement about the entropy of a matter system before it is lost across the horizon. It then seems clear that it is the entropy of the quantum modes, before they exit the horizon, that should have a holographic bound. This being the case, we propose that a natural concept of entropy of the modes on sub-horizon scales is the entanglement entropy\(^2\). It naturally scales like the horizon area, and is thus often also called the geometric entropy. It also depends explicitly on the cutoff scale.

We then move on to consider the shift of the horizon area, due to the backreaction of the density perturbations on the geometry, or due to the time evolution of the Hubble parameter during the slow roll. In both cases, the change of the area leads to a flux of entropy across the horizon. However, according to the holographic bound, the entropy flux must be smaller than the slow-roll change of the horizon area. This leads to an ”indirect” bound on the cutoff scale.

This paper is organized as follows. In sections 2 and 3 we first review some basic facts of holographic bounds and the theory of density perturbations. In the end of section 3 we discuss the coarse-grained and the geometric entropy as the entropy of the perturbations. In section 4 we analyze the backreaction of the perturbations and the ensuing shift in the horizon area. In section 5 we discuss the direct and the indirect entropy bound, and apply them to density fluctuations in section 6. The different bounds are then summarized in a figure in section 6. We use the definitions \(M_p = 1/\sqrt{8\pi G N}\), \(l_p = \sqrt{G N}\), and set \(\hbar = c = 1\).

## 2 Holography and Cosmological Density Perturbations

If the entropy of a matter system could be arbitrary large while its total energy would be fixed, the generalized second law (GSL) could be violated. As a gedanken experiment, drop the matter system into a black hole. The entropy of the matter system is lost, while the increase in the entropy of the black hole is proportional to the increase of its horizon area, which only depends on the total energy of the matter system. Thus, if the entropy of the matter system was high enough, the total entropy will decrease in the process. However, there are strong indications that the generalized second law is a true principle of Nature. This led Bekenstein to suggest an entropy bound on matter systems coupled

\(^1\)On an even more sophisticated level, there are coarse-graining prescriptions taking into account interaction with environment and the ensuing decoherence [8]. For holography, one should then analyze bounds on the total entropy of cosmic perturbations plus the environment. However, previous studies of holography and the present study restrict to a toy model level, focusing only on the cosmic perturbations.

\(^2\)Quantum entanglement across the horizon in de Sitter space has also been discussed in a different context in [9].
to gravity. It has later been developed into a more general entropy bound, called the covariant entropy bound [10].

One can apply the same reasoning in inflation where cosmological perturbations carry entropy and redshift to superhorizon scales. The gedanken experiment which led to the entropy bounds on matter, suggests that one should consider the dynamical process in which entropy is lost to the horizon and compensated by an increase in the horizon area. This might lead to interesting constraints in the theory of inflation.

The first natural application to consider is known as the D-bound. Gibbons and Hawking have shown that de Sitter horizon is associated with an entropy [11]

\[ S_0 = \frac{1}{4} A_0 = 4\pi \frac{1}{H^2} \]  

(1)

where \( A_0 \) is the area of the de Sitter horizon and \( H \) is the Hubble parameter. Now, if we place a freely falling matter system inside the horizon of an observer, she will see the matter system redshift away. The horizon area grows, as the accessible space-time region again converts into empty de Sitter space. The entropy of the final state is \( S_0 \), while the entropy of the initial state is the sum of the matter entropy \( S_m \) and the Gibbons-Hawking entropy, one quarter of the initial area \( A_c \) of the horizon:

\[ S = S_m + \frac{1}{4} A_c \]  

(2)

From the generalized second law \( (S \leq S_0) \), one obtains a bound on the entropy of the matter

\[ S_m \leq (A_0 - A_c) \]  

(3)

This is called the D-bound [12].

The horizon area also changes during slow-roll inflation. However, the change in the horizon area must be small, for the slow-roll approximation to be valid. The expansion rate of the horizon area is given by the slow-roll parameter \( \epsilon \). It is natural to expect that the D-bound (3) will lead to an \( \epsilon \)-dependent upper bound on the entropy carried by the cosmological perturbation modes, as they redshift to superhorizon scales. Whether this is a non-trivial bound, depends on how one calculates the entropy of the modes. We will next review some of the suggested methods, and suggest a new particularly natural one.

### 3 Entropy of Perturbation Modes

For convenience and to collect together some useful formulas, we begin with a short review of the theory of density perturbations.

#### 3.1 The Origin of Density Perturbations

One can show that the evolution of scalar (or tensor) perturbations is described by a minimally coupled massless scalar field \( \phi \). For brevity, we will only consider a spatially
flat FLRW universe, with the metric
\[ ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j. \] (4)

After the field redefinition \( \mu \equiv a\phi \) the equation of motion for the Fourier modes \( \mu_k \) in de Sitter space takes the form
\[ \mu_k'' + \left[ \omega^2(k) - \frac{a''}{a} \right] \mu_k = 0, \] (5)

where prime denotes derivative with respect to the conformal time \( \eta \) defined as \( dt \equiv a(t)d\eta \).

It is convenient to introduce a cutoff scale \( \Lambda \) in Fourier space to parameterize our ignorance of physics beyond this scale\(^3\). As a first approximation, one could consider it to be of the order of the Planck scale \( M_p \). However, the main issue is if one can derive tighter bounds for it from holographic considerations. The possible effect of such a cutoff on the observed CMB spectrum was investigated in [1, 2, 5, 13]. We will assume that the physics below the cutoff scale obeys the standard linear dispersion relation \( \omega^2(k) = k^2 \). Thus, below the cutoff the solution to the field equation is given by
\[ \mu_k(\eta) = \frac{\alpha_k}{\sqrt{2k}} e^{-i\eta} + \frac{\beta_k}{\sqrt{2k}} e^{i\eta}, \] (6)

with \( |\alpha_k|^2 - |\beta_k|^2 = 1 \) from the usual Wronskian normalization condition. The origin of density fluctuations is a quantum effect. At quantum level the field \( \mu \) is a pure state
\[ \tilde{\rho} = |0, in\rangle \langle 0, in|, \] (7)
corresponding to initial absence of density fluctuations. In the above case, where trans-Planckian effects have been excluded (the dispersion relation is linear), the initial vacuum \( |0, in\rangle \) is the one with respect to modes with \( \beta_k = 0 \), the Bunch-Davies vacuum. However, due to the time-dependence of the background, at later times the physical vacuum becomes a different one, with respect to \( \mu_k \) above. With respect to the new vacuum and particle states, the state \( \tilde{\rho} \) appears as a many-particle (squeezed) state
\[ \tilde{\rho} = \prod_{k,k',n_k,m_{k'}} \tilde{\rho}(n_k, m_{k'}) |m_{k'}\rangle \langle n_k|, \] (8)

with the average occupation number per wavenumber
\[ \bar{n}_k = \sum_{n_k=0}^{\infty} \tilde{\rho}(n_k, n_k) n_k = |\beta_k|^2. \] (9)

\(^3\)Note that the cutoff must be in physical momentum \( (p < \Lambda) \) in order not to be macroscopic at the end of inflation. In comoving momentum the cutoff is \( k < a\Lambda \)
These excitations correspond to density fluctuations with respect to the new vacuum. In de Sitter space, for a massless scalar field in the initial vacuum, the average occupation number is found to be

$$\bar{n}_k \sim (aH)^2/(2k)^2.$$  \hfill (10)

One can also introduce the squeezing parameter $r_k$ and denote

$$\bar{n}_k = \sinh^2(r_k).$$  \hfill (11)

If $\bar{n}_k \gg 1$ (large squeezing), the population of modes is so large that they can be approximated by a classical fluid. In the opposite limit, $\bar{n}_k \ll 1$, the density perturbations behave quantum mechanically.

On sub-horizon scales the curvature $a'/a$ can be neglected and the energy density is

$$\rho = \langle 0, in|T^0_0|0, in \rangle = \frac{1}{4\pi^2a^4} \int_{aH}^{aA} dkk^3 \left( \frac{1}{2} + |\beta_k|^2 \right).$$  \hfill (12)

After subtracting the zero-point energy,

$$\rho = \frac{1}{4\pi^2a^4} \int_{aH}^{aA} dkk^2 \omega(k) \bar{n}_k.$$  \hfill (13)

Substituting the result (10) in (13), one finds

$$\rho = \frac{1}{32\pi^2} \Lambda^2 H^2.$$  \hfill (14)

Inside the horizon, when $|k\eta| \gg 1$ and the space-time is effectively flat, the equation of state becomes $p = (1/3)\rho$, that of a relativistic fluid or a free massless scalar field in Minkowski space. Since the density fluctuations now behave like a fluid, it becomes natural to associate an entropy with the mean occupation number. However, strictly speaking the ensemble is still in a pure state with zero entropy, so the entropy of the fluid will involve some amount of coarse graining.

### 3.2 The Entropy of the Density Perturbations

The aim of this paper is to apply holographic entropy bounds to the inflationary quantum fluctuations. However, it is clear that the modes are not classical before they exit the horizon and one should be careful which entropy is assigned to those modes. The entropy bounds, like the D-bound, are bounds on the true degrees of freedom of the fundamental theory. When they are applied to the inflationary quantum fluctuations, unnecessarily strong constraints may result simply due to excessive coarse-graining. We now take a brief look at the standard coarse-graining prescriptions.
Coarse-grained entropy. A popular approximation in the literature is that on superhorizon scales, where the average occupation number is much greater than one, the entropy per comoving momentum is given in terms of the squeezing parameter \( r_k \) as

\[
S_k = 2r_k ,
\]

since it agrees with the classical concepts of entropy. In [14], it was proposed that the entropy of a quantized field with average occupation number \( \bar{n}_k \) is in general given by

\[
S = \sum_k g_k \left[ (\bar{n}_k + 1) \ln(\bar{n}_k + 1) - \bar{n}_k \ln \bar{n}_k \right] ,
\]

where \( g_k \) is the degeneracy. This expression for the entropy has the same form as that of a thermal distribution, and it reproduces equation (15) in the limit of large squeezing \( r_k \gg 1 \). It is tempting to take the limit \( \bar{n}_k \ll 1 \). With \( g_k = (4\pi/3a^3)Vk^2dk \), one finds

\[
S = -\frac{4\pi V}{a^3} \int d^3k \bar{n}_k \ln \bar{n}_k .
\]

or, alternatively, the entropy per mode \( S_k \sim -\bar{n}_k \ln \bar{n}_k \). However, the considerations in [14] do not necessarily apply in the \( r_k \ll 1 \) limit. References [15, 16] followed another coarse-graining approach. It was argued that the expression for entropy per mode in equation (15) is valid for all values of the squeezing parameter, \( r_k \).

Entanglement entropy. The reference [13] looked for a more detailed understanding by going back to the origin of the entropy. It was argued that from the point of a local observer there is a natural bound on the freedom to coarse-grain the system. Imagine that the full system is divided into two subsystems, with \( N_1 \) and \( N_2 \) degrees of freedom, such that the first subsystem and degrees of freedom are accessible to the local observer. If the total system is in a pure state, when we trace over the second subsystem (inaccessible to the local observer), we find that the entropy of the second subsystem is bounded by the number of the traced-over degrees of freedom, \( S_1 < \ln N_2 \). So it was concluded that from the local point of view the production of entropy is limited by the ability to coarse-grain.

Let us elaborate this viewpoint. What is essentially proposed in the above argument is that the maximum entropy of the observable subsystem (the density perturbations within the horizon) is defined as the entropy of entanglement, or the geometric entropy [17]. Trace over the degrees of freedom outside the horizon radius \( r_H \) and define a reduced density matrix

\[
\tilde{\rho}_{in} = Tr_{r>r_H}(\hat{\rho}) ,
\]

this only describes the system within the horizon. But now \( \tilde{\rho}_{in} \) is a mixed state, with an associated entropy

\[
S_{in} = -Tr_{r<r_H}(\tilde{\rho}_{in} \ln \tilde{\rho}_{in}) .
\]

Müller and Lousto have shown [18] that the entropy is

\[
S_{in} \approx 0.3 \left( r_H/\varepsilon \right)^2
\]
where $r_H = 1/Ha$ is the radius in the comoving frame and $\varepsilon$ is an UV cutoff scale defining the sharpness of the 'cut'. The cutoff scale is of course ambiguous, making the entropy ill-defined and formally divergent as the scale approaches zero. However, in the present context, in our treatment of the density fluctuations, we have already explicitly introduced an UV cutoff $\Lambda$ in the Fourier space, and the entropy or the fluctuations is expected to have an explicit dependence of that scale. Therefore it is logical to set $\varepsilon = 1/a\Lambda$. (The cutoff is imposed in the physical frame, so in the comoving frame a scale factor is included.)

One virtue of the entanglement entropy is that $S_{in}$ is now directly associated with the degrees of freedom inside the horizon. The "flow" of entropy across the horizon can now be viewed as follows. Consider a set of perturbations with comoving wavenumber $k > k_0$. At very late times, they have essentially all crossed well outside the horizon, see Figure 1, and can then be approximated by a classical fluid. On the other hand, at very early times ($a < k_0/H$) they are all within the horizon of (comoving) radius $r_H = 1/Ha$. Then the average occupation number per mode is of the order of one, and the system is far from classical. At this point it is natural to describe the system by $\tilde{\rho}_{in}$ as above, so the initial entropy assigned to these seeds of perturbations inside the horizon is (20). Consider then the same set of density perturbations at late times, when they are (practically) all grown outside the horizon (Figure 1). Now, in order to describe the degrees of freedom outside the horizon, it is natural to trace over the interior of the horizon to obtain a reduced density matrix $\tilde{\rho}_{out}$ for the outside degrees of freedom. The resulting entropy turns out to be the same:

$$S_{out} = S_{in} \equiv S_{ent}. \tag{21}$$

Naively the entropy appears to have decreased, since we have been using the comoving frame where $r_H = 1/Ha$ decreases in time. However, in the physical frame the entropy is a constant,

$$S_{ent} = 0.3 \left( \frac{\Lambda}{H} \right)^2. \tag{22}$$

As the perturbations have grown well outside, the average occupation number per mode $\bar{n}_k \gg 1$ and they can be approximated by a classical fluid. We could then assign the above entropy for the fluid. But, at earlier times when the modes were inside the horizon, they had the same entropy. Let us compare $S_{ent}$ with the coarse-grained entropy in the sub-horizon region following the prescription of Giovaninni and Gasperini [16] (See also [19]). They suggested that the entropy per comoving mode is given by $S_k = 2r_k$. For small squeezing, $r_k \approx |\tilde{\beta}_k| = aH/(2k)$, hence the entropy contained in the relativistic particles sub-horizon ($n_k \ll 1$) is given by

$$S_{cg} = \frac{V}{\pi^2a^3} \int_{aH}^{a\Lambda} dk \, k^2 \, \frac{aH}{2k} \approx \frac{1}{3\pi} \left( \frac{\Lambda}{H} \right)^2, \tag{23}$$

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4It would also be interesting to perform a refined derivation of the entanglement entropy formula in the context where trans-Planckian modifications have been incorporated into field theory, for a more sophisticated analysis of the cutoff dependence. Indeed, such a consideration has just appeared [30].
Figure 1: In the Penrose diagram above, modes corresponding to constant comoving radius which are inside the apparent horizon (AH) at time $t_0$ are shown. (The future event horizon (FEH) is also depicted in the figure.) The diagram shows the inflation, radiation and matter dominated eras. It is illustrated that calculating the entropy of those modes by tracing over the exterior of the apparent horizon on the time slice $t_0$, is equivalent to later calculating their entropy by tracing over the interior of the horizon at time $t_1$.

where $V = (4/3)\pi H^{-3}$. This is in agreement with the entanglement entropy, up to an irrelevant numerical factor of order one. In the coarse-grained approach it is not evident why the prescription should be valid in the sub-horizon region where the population of the modes is small (and the system is essentially quantum). Further, the coarse-graining process drops out information of the system which is not manifestly associated with a region of spacetime. However, the fact that the result agrees with the entanglement entropy gives it further credibility. The coarse-grained entropy saturates the bound arising from the entanglement of the subsystems\(^5\).

To summarize, we considered density perturbations at early and late times. At late times they are essentially classical, outside the horizon, and it is natural to describe their entropy as that of a classical fluid (defined by a suitable coarse graining). However, we argued that considering the quantum mechanical origin of the perturbations, the most natural notion of entropy is actually the entanglement entropy — and that one is the same as the entropy of the perturbations at very early times when they are all within the horizon. We just choose to trace over opposite regions of space. The various definitions of coarse-grained entropy must be consistent with the entanglement entropy, which in turn naturally scales like the area. Then, it is natural that the holographic bound is satisfied. In the prescription for the entropy, it is also explicit why there is a bound on the number

\(^5\)On the other hand, in a more realistic analysis where one takes into account other fields as an environment, the decoherence-induced coarse-grained entropy of the cosmic perturbations is less [8] than the bound. But then a proper count of the total entropy should include the contribution from the environment as well.
of degrees of freedom inside the horizon. This in turn is somewhat less satisfying in the coarse-graining descriptions: first one performs a coarse-graining procedure which makes no reference to a specific region in spacetime, then the "put-in-by-hand" entropy should correspond to at most a specific number of degrees of freedom inside the horizon dictated by the holographic bound.

4 Perfect Fluid with a Cutoff in Quasi de Sitter Space

In the previous section, we showed that the entropy of the inflationary quantum fluctuations scales like the horizon area times the square of the UV cutoff. Next, we look at the backreaction and the shift in horizon area due to the loss of fluctuations across the horizon. Before introducing the entropy bounds in section 4, it is useful to discuss this relationship in more detail.

To this end, let us review some of the discussion in [21] (see also [22]). It is argued that the change in the horizon area is given by the energy flux through the horizon $\delta E$ according to the first law of thermodynamics

$$TdS = \delta E ,$$

(24)

where $S$ is the geometrical entropy. In [21] the energy flux is written as

$$\delta E = \delta \int d\Sigma_\mu T^\mu_\nu \zeta^\nu ,$$

(25)

where $d\Sigma_\mu$ is the 3-volume of the horizon and in the metric

$$ds^2 = -dt^2 + e^{2Ht} (dr^2 + r^2 d\Omega^2) ,$$

(26)

$\zeta^\mu = (1, -Hr, 0, 0)$ is the approximate Killing vector which is null at the horizon. The coordinate position of the horizon is given by $r = 1/aH$. By equating $\delta E/T$ to the change in the geometrical entropy given by the variation in the horizon area, one obtains

$$\dot{H} = -4\pi G T_{\mu\nu} \zeta^\mu \zeta^\nu .$$

(27)

It was then shown that for the energy momentum tensor of the background inflaton field, the energy flux through the horizon is $T_{\mu\nu} \zeta^\mu \zeta^\nu = \dot{\phi}$, which leads to the usual Einstein equation

$$\dot{H} = -4\pi G \dot{\phi}^2 .$$

(28)

On other hand, the density fluctuations can be viewed as a perfect fluid with the energy momentum tensor

$$T^\nu_\mu = diag(-\rho, p, p, p) ,$$

(29)

with $p = \rho/3$. In this way one finds

$$T_{\mu\nu} \zeta^\mu \zeta^\nu = \rho + p .$$

(30)
As expected, a cosmological constant $\rho = -p$ leads to no change in the horizon area, while a constant energy density $p = (1/3)\rho$ leads to

$$\dot{H} = -\frac{16\pi}{3}G\rho.$$  \hspace{1cm} (31)

This agrees with what we would expect from the Einstein equations. Introducing a small number $\tilde{\epsilon}$, we can parameterize the energy density as

$$\rho = \frac{\tilde{\epsilon}}{2}3H^2M_p^2.$$  \hspace{1cm} (32)

Then, (31) can be rewritten in the form

$$\dot{H} \simeq -\tilde{\epsilon}H^2.$$  \hspace{1cm} (33)

The energy flux across the horizon from the constant radiation fluid leads to a small slow-roll-like change in the horizon area.

There might seem to be a contradiction between the continuity equation

$$\dot{\rho} + 3H(\rho + p) = 0,$$  \hspace{1cm} (34)

which implies that a constant energy density is always associated by a negative but equal pressure, and the equation of state of a massless scalar field $p = (1/3)\rho$. However, one should note that the energy conservation equation is violated by the fixed cutoff, since the cutoff acts as an energy reservoir from which energy is pumped into the system.

This can also be understood in terms of a source term. For a constant energy density, $T^{\mu\nu}_{\mu\nu} = 0$ would imply that $\rho = -p$ and $\dot{H} = 0$. But obviously $T^{\mu\nu}_{\mu\nu} = 0$ is not valid since modes redshift across the cutoff and energy in this way leaks into the system that is no longer closed. In fact, we should include a source term modifying the continuity equation to $T^{\mu\nu}_{\mu\nu} = -4H\rho$.

Heuristically, even if the energy density is constant, the energy density of relativistic particles which were present at any given time $t'$ will fall off like $1/a^4$ and thus behave like relativistic fluid. Those modes will eventually redshift away and if no new modes had been created to sustain the pressure, the horizon area would have changed by $\Delta A$ which then limits their entropy. Even if new modes are created continuously to make up a constant energy density, we have seen that it is natural to assume that the modes that do redshift (exit the horizon), will change the horizon area proportional to the entropy they carry according to the generalized second law\footnote{A related application of the GSL was considered also in [23].}. This change must be smaller than the observed slow-roll change. This is the indirect bound that we will discuss in the next section.

## 5 Entropy Bounds

Now we are ready to discuss two specific entropy bounds on inflationary quantum fluctuations; a direct and an indirect entropy bound. The first one is just an application of
the D-bound introduced in section 2, and as we will argue, it could be viewed as a consistency check which is trivially satisfied if one has performed the right amount of coarse graining. The second one is a bound on the inflationary quantum fluctuations from the known background evolution of the geometry.

5.1 The Direct Entropy Bound

Consider a massless scalar field with the energy density inside the horizon (13). The energy density carried in these modes must be much smaller than the critical energy density. Like in equation (32) above, it is useful to parameterize the energy density in terms of $\tilde{\epsilon} \ll 1$.

In [13] it was shown that this amount of relativistic matter embedded in a pure de Sitter space would change the horizon by $\Delta A$ where

$$\Delta A = A_f - A_0 = \frac{\tilde{\epsilon}}{2} A_f .$$

(35)

Here $A_0$ is the horizon area of the initial de Sitter space with $\rho$ included and $A_f = 4\pi H^{-2}$ is the horizon area of the final pure de Sitter space after the matter component as exited the horizon. The D-bound [24] then implies that the entropy contained in the matter is limited by

$$S \leq \frac{1}{4} \frac{\Delta A}{H^2} = 4\pi^2 \frac{M_p^2}{H^2}. $$

(36)

By using equation (32), we can eliminate $\tilde{\epsilon}$. The D-bound (direct bound) then takes the form

$$S \leq \frac{8\pi^2}{3} \frac{\rho}{H^4}.$$  

(37)

Using that the vacuum energy of the de Sitter space is $\rho_\Lambda = 3H^2M_p^2$, we can write the direct bound in equation (37) in a more transparent way

$$S \leq 24\pi^2 \frac{M_p^4}{\rho_\Lambda} \frac{\rho}{\rho_\Lambda} .$$

(38)

Now, the D-bound should not be viewed as something intrinsically related to de Sitter space, but only a way to use de Sitter space to obtain a bound on the entropy of matter, much like the Schwarzschild solution leads to the Bekenstein bound. Thus, a reasonable choice of coarse graining should render the bound in equation (37) trivial. If one obtains a nontrivial constraint from the bound in equation (37), it is a hint that the counting of entropy is unphysical. Indeed, the geometrical entropy of the inflationary quantum fluctuations trivially satisfies the bound. We will return to the examples in section 6.

5.2 The Indirect Entropy Bound

In a slow-roll set-up, inflation is driven by the inflaton potential $V$, and

$$H^{-1} = \frac{M_p}{\sqrt{3V}}.$$  

(39)
During slow-roll the potential is almost constant and in one e-fold, it changes maximally by
\[ \Delta V \simeq \frac{\epsilon}{2} V, \tag{40} \]
where \( \epsilon = 1/2 \) \((V'/V)^2\) is the slow-roll parameter and prime denotes derivative with respect to the inflaton field \( \phi \). This also implies that during one e-fold the horizon area changes by
\[ \Delta A \simeq \frac{\epsilon}{2} A = 4\pi \frac{\epsilon}{H^2}. \tag{41} \]
The entropy that exits the horizon during one e-fold\(^7\) is then limited by
\[ \Delta S_{\text{exit}} \lesssim \frac{1}{4} \frac{\Delta A}{l_p^2} = 8\pi^2 \epsilon \frac{M_p^2}{H^2}. \tag{42} \]

This is what we refer to as the indirect entropy bound. It is of course weaker than the D-bound, since it implicitly assumes \( \tilde{\epsilon} \ll \epsilon \). However, it bounds the flux of entropy across the horizon by the actual change in the geometry in a cosmological setting. This analogous to some of the considerations in also \cite{7} and \cite{23}.

### 6 Applications to Quantum Fluctuations in the Early Universe

In this section we apply the considerations above explicitly to the theory of cosmological perturbations in quasi de Sitter space.

As a first very trivial consistency check, consider the coarse-grained or entanglement entropy
\[ S \simeq C \cdot \left( \frac{\Lambda}{H} \right)^2, \tag{43} \]
where \( C \) is a numerical factor of order one, and the D-bound. It can be seen by comparing equations (14) and (37), that the D-bound is trivially satisfied if \( C \) is of the order one. The only difference between the bound applied to the geometric versus to the coarse-grained entropy is a conceptual one. In the latter case the bound is a more arbitrary consistency check. There seemed to be no direct understanding as to why the discarded information in the coarse-graining scheme should be related to a holographic bound on the degrees of freedom inside the horizon. In the case of the geometric entropy, the correspondence is more transparent.

The indirect bound is a bit more interesting. From equations (58) and (59) it is easy to see that in one e-folding, the amount of entropy that exit the horizon is given by
\[ \Delta S_{\text{exit}} = \frac{1}{\pi} \frac{\Lambda^2}{H^2}. \tag{44} \]

\(^7\)The entropy that exits the horizon is equal to the loss of entropy inside the horizon due to redshift. See the discussion after eq.(60) in the appendix.
From the indirect entropy bound in equation (42), we obtain

\[ \Lambda \lesssim \sqrt{8\pi^3} \epsilon \Lambda \]  

(45)

Using equation (14), one can see that the indirect bound in this case turns out to be similar to requiring \( \bar{\epsilon} \lesssim \epsilon \), with the definition of \( \bar{\epsilon} \) given in equation (32). Note that this result depends on the coarse-graining prescription. If one approximate the entropy per mode by an almost constant, one reproduces the stronger bound of Albrecht and Kaloper [7] (see also the footnote).

This is an interesting bound, but in many cases \( \epsilon \sim 1/N \) where \( N \) is the number of e-foldings till the end of inflation. If inflation lasts about 100 e-foldings, this bound is no stronger than \( \Lambda \ll \Lambda \). On the other hand, if inflation lasts much longer than 100 e-foldings then the bound is potentially interesting. The bound is illustrated in figure 2.

![Figure 2: The slow-roll parameter \( \epsilon = 10^6 \) vs. the cutoff scale \( \Lambda \). For \( H \sim 10^{-4} \Lambda \), the range in which trans-Planckian effects are observable are also indicated. It is argued in [4] that if \( H/\Lambda > 10^{-2} \), then the trans-Planckian effects can be observable. If the cutoff scale is too high, the perturbations become important for the background evolution. This is indicated by the dark-gray region. The region below the additional line is the region allowed by the bound obtained from the \( \alpha \)-vacua scenario.](image)

We also note that, if the vacuum is chosen to be the Minkowski vacuum for each mode as they exits the trans-Planckian regime as in [2], such that

\[ |\beta_k| \simeq \frac{1}{2} \frac{H}{\Lambda} \]  

(46)

on sub-horizon scales, then the total entropy inside the horizon, calculated by using \( S_k = 2r_k \), will still scale as \( \Lambda^2/H^2 \). Also the energy density will still be given by equation (14). Hence, the direct and the indirect bounds will not change significantly.

\[ ^{8} \text{Note that if the total entropy that has left the horizon during the first few e-foldings is given by } S \simeq 4\pi \Lambda^3/H^3 \text{ and if it dominates the total entropy in modes that have left the horizon, as argued in [7], then we reproduce the constraint in the cutoff in } [7] \text{ } \Lambda \lesssim \epsilon^{1/3} (HM_p^2)^{1/3}. \text{ This estimate was obtained by using, as a measure for the entropy in the lost modes, their entropy just as they stretch to superhorizon scales. Note, however, that in the coarse-graining approach the loss of entropy due to redshift is distributed on all modes inside the horizon, as also discussed in the appendix.} \]
As a final example we mention the $\alpha$-vacuum as suggested in [25] (for a discussion of the $\alpha$-vacuum, see [26]). It was suggested that the modes below the cutoff should be placed in the $\alpha$-vacuum with a trans-Planckian signature determined by

$$|\beta_k| \simeq \frac{HM_p}{\Lambda^2}.$$  \hspace{1cm} (47)

In the present set-up, the direct bound would again be trivial, while the indirect bound leads to the more interesting constraint

$$\Lambda \lesssim 6\pi^2\epsilon M_p.$$  \hspace{1cm} (48)

This constraint is also illustrated in figure 2.

In the present approach we have ignored the modes above the cutoff. The above-the-cutoff region instead acts as a reservoir. If one has an explicit model for incorporating the trans-Planckian physics, like in the approach of [3] or [6], one can in principle avoid the use of the explicit cutoff. Instead it will appear as the scale controlling the strength of the modifications of the standard quantum field theory.

For example in [3], the effect of fundamental physics on the evolution of perturbation modes above the cutoff was modeled by incorporating effectively a minimum length (mimicking string theory) into the field theory at the level of modified commutation relations. Using this approach, it was also discussed how the modes are created in the trans-Planckian regime as an effect of the novel feature of trans-Planckian damping.

Since the modes above the new physics scale will also contribute to the energy and entropy inside the horizon, we expect that the indirect bound on the new physics scale will be at least marginally more tight depending on the model. This is a possible direction for further investigation.

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**A Relation to covariant entropy bound**

Since the covariant entropy bound is supposed to be the generalization of the entropy bound discussed until now, it is interesting to briefly exemplify how it is related to the discussion above.

Following [27–29], the integral of the entropy flux $s$ over the lightsheet between two spatial surfaces $B$ and $B'$ can be written as

$$\int_{L(B-B')} s = \int_B d^2x \sqrt{h(x)} \int_0^1 d\lambda s(x, \lambda) A(x, \lambda).$$  \hspace{1cm} (49)
We have chosen a coordinate system \((x^1, x^2)\) on \(B\) and \(h(x)\) is the determinant of the induced metric on \(B\). The affine parameter \(\lambda\) is normalized such that it takes the value one at \(B'\). The function \(\mathcal{A}(x, \lambda)\) is the area decrease factor for the geodesic that begins at the point \(x\) on \(B\).

As explained also in [27–29], the physical meaning is simply as follows: As we parallel propagate a infinitesimal area \(d^2 x \sqrt{h(x)}\) from the point \((x, 0)\) on \(B\) to the point \((x, \lambda)\) on the lightsheet, the area contracts to \(d^2 x \sqrt{h(x)}(x, \lambda)\). Thus the proper infinitesimal volume on the lightsheet is \(d\lambda d^2 x \sqrt{h(x)}(x, \lambda)\). The infinitesimal volume times the entropy flux, all of it integrated, then gives the entropy over the lightsheet. As stated in [27–29], the generalized covariant bound now takes the form

\[
\int_0^1 d\lambda s(\lambda) \mathcal{A}(\lambda) \leq \frac{1}{4} (1 - \mathcal{A}(1))
\]

for each geodesic of the lightsheet.

Now let us make that into a differential bound and define

\[
A_B = \int_B d^2 x \sqrt{h(x)}.
\]

Now assume that \(\mathcal{A}(\lambda)\) and \(s(\lambda)\) does not depend on \(x\). Then we can write equation (49) as

\[
\int_{L(B-B')} s = A_B \int_0^1 d\lambda s(\lambda) \mathcal{A}(\lambda).
\]

and equation (50) can be written

\[
A_B \int_0^1 d\lambda s(\lambda) \mathcal{A}(\lambda) \leq \frac{1}{4} A_B (1 - \mathcal{A}(1))
\]

or in infinitesimal form

\[
A_B s(0) \mathcal{A}(0) d\lambda \leq \frac{1}{4} dA_B.
\]

Since the matter is at rest, we can construct the entropy flux vector \(s_a\) by multiplying \(\sigma\) with the four-vector \(u_a = (-1, 0, 0, 0)\),

\[
s_a = \sigma u_a.
\]

Analogous to how we calculated the energy flux across the horizon, we calculate the entropy flux across the horizon by multiplying \(s_a\) by the tangent vector to the null congruence generating the horizon \(k^a\). It is related to the approximate Killing vector \(\zeta^a = -H\lambda k^a\), where \(\zeta^a = (1, -Hr, 0, 0)\) in spherical coordinates. Thus, the flux across the horizon is \(s = s_a k^a = \sigma/(H\lambda)\). Using \(d\lambda \propto adt\), as can easily be verified, one can apply \(d\lambda/\lambda = Hdt\) in order to obtain

\[
A_B \sigma \leq \frac{1}{4} \frac{dA_B}{dt}.
\]
Now let us apply the equation above to our cosmological setup. Let the total entropy inside the horizon be denoted by $S$. Then we can construct the entropy density $\sigma$ by dividing by the Hubble volume

$$\sigma = \frac{3H^3S}{4\pi}.$$  

(57)

Inserted into equation (56), we find using $A_H = 4\pi H^{-2}$,

$$3HS \leq \frac{1}{4} \frac{dA_H}{dt}.$$  

(58)

The left-hand-side is basically the total entropy flux across the horizon, so equation (58) is analogous to the indirect bound in equation (42) if the dominant time-dependence of the horizon area is taken to be due to the slow-roll variation.

To compare, we consider how entropy is red-shifted away inside the horizon. If one considers the entropy present in the radiation fluid at some given moment, that entropy would dilute like $\sim a^{-3}$ if no new entropy was generated. Thus the entropy density would redshift as $a^{-3}$. This agrees with the familiar statement that the entropy density of an equilibrium gas in an expanding universe, dilutes as $a^{-3}$ as can be seen from conservation of entropy per comoving volume $S_c$ in an adiabatic evolution.

This means that if the total entropy inside the horizon is given by $S$, then there will be a loss in entropy inside the horizon due to the red-shifting

$$-\frac{dS}{dt} = 3HS.$$  

(59)

Now comparing equation (58) and equation (59) we obtain

$$-\frac{dS}{dt} \leq \frac{1}{4} \frac{dA_H}{dt}.$$  

(60)

which is just the generalized second law. Thus, for our particular case, the generalized covariant bound reduces to the GSL which underlies all of our discussion. In particular we see that the flow of entropy through the horizon is the same as the loss of entropy due to redshift inside the horizon (Compare left-hand-side of equation (58) with the right-hand-side of equation (59)).

Similarly we can, as a consistency check, show that the redshift of modes inside the horizon agrees with the energy flux through the horizon area. The energy-density of radiation redshift as $a^{-4}$, such that the loss in energy density due to the red-shifting is

$$-\frac{d\rho}{dt} = 4H\rho.$$  

(61)

Thus, the total loss in energy due to red-shift inside the horizon is

$$\frac{dE}{dt} = V \frac{d\rho}{dt} = \frac{-16\pi}{3} H^{-2}\rho.$$  

(62)

On the other hand, we calculated in section , that the energy flux across the horizon is

$$T_{\mu\nu} \xi^\mu \xi^\nu = \rho + p = \frac{4}{3}\rho,$$  

(63)
where we used the equation of state for radiation. This means that the total flux of energy across the horizon is given by

\[
\frac{dE}{dt} = A_H T_{\mu\nu} \zeta^\mu \zeta^\nu = (4\pi H^{-2}) \frac{4}{3} \rho = \frac{16\pi}{3} H^{-2} \rho
\]

(64)

Again the expression in equations (62) and (64) agree.

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