A New Method for Constructing the Coefficients of Pressure Correction Equation for Colocated Unstructured Grids

R. Rafee and H. Rahimzadeh
Department of Mechanical Engineering,
Amirkabir University of Technology, Hafez Ave., P.O. Box 15875-4413, Tehran, Iran

Abstract: One of the important equations in numerical solution of Navier Stokes equations is the pressure correction equation. In this article, a new method for constructing the coefficients of this equation for colocated unstructured meshes is proposed. This method is based on momentum interpolation. The method is compared with the approach adopted in literature. The complete discretization of the Navier Stokes equations using finite volume method is presented. An algorithm similar to SIMPLE is used to evaluate the rate of convergence on two sample problem. The results show that by using rectangular grids, two methods have the same performance, but when triangular meshes are applied, the new method increases the rate of convergence.

Key words: Unstructured mesh, colocated arrangement, navier stokes equations, pressure correction equation, momentum interpolation

INTRODUCTION

In the last two decades, solution of Navier Stokes equations using colocated arrangement has received great interests. On the other hand, unstructured grids are popular for solution of flow field in complex geometries. Colocated arrangement has some obvious advantages over staggered grids especially in case of non-orthogonal meshes. In colocated arrangement, all variables share the same location, hence only one set of volumes is considered. Also, by applying the colocated grids, the convection contribution to the coefficients in discretized equations, are the same for all variables. Finally, Cartesian velocity components can be used in conjunction with non-orthogonal grids for complex geometries which result simpler discretized equations.

After the original work by Rhie and Chow[1], Peric[2] generalized the same idea for three dimensional flows and calculated several two and three dimensional flow situations. Majumdar[3] studied the role of under relaxation parameter in momentum interpolation for calculation of flow with colocated grids. Lien[4] used the colocated arrangement for unstructured grids successfully. He applied the momentum interpolation as a concept for derivation of pressure correction equation. However, different authors have different ideas to construct the pressure correction equation. For example the approach used by Thomadakis et al.[5] for constructing the coefficients is quiet different from that used by Lien[4]. Here the emphasis is on the fact that these coefficients can affect the numerical solution. Therefore, a new form of the coefficients is proposed and it's rate of convergence and performance is examined.

In this study, two examples are solved by structured and unstructured grids:

- Laminar flow in a lid-driven square cavity
- Two dimensional parallel flow

FLOW EQUATIONS

For incompressible Newtonian fluid flows, the conservation equation for mass and momentum are as:

\[ \nabla \cdot (\rho \vec{V}) = 0 \] (1)

\[ \frac{\partial}{\partial t}(\rho u) + \nabla \cdot (\rho \vec{V} u) = \nabla \cdot (\mu \vec{V} u) - \frac{\partial p}{\partial x} \] (2)

\[ \frac{\partial}{\partial t}(\rho v) + \nabla \cdot (\rho \vec{V} v) = \nabla \cdot (\mu \vec{V} v) - \frac{\partial p}{\partial y} \] (3)

In above equation, \( u \) and \( v \) are Cartesian components of velocity vector.

Transport equation for a general variable, such as \( \phi \), can be written as:

\[ \frac{\partial}{\partial t}(\rho \phi) + \nabla \cdot (\rho \vec{V} \phi) = \nabla \cdot (\mu \vec{V} \phi) + S \]
Fig. 1: Data Structure for triangular mesh

\[ \frac{\partial}{\partial t}(\rho \phi) + \frac{\partial}{\partial x_j} (\rho u_j \phi) = \frac{\partial}{\partial x_j} (\Gamma \frac{\partial \phi}{\partial x_j}) + q_\phi \]  \hspace{1cm} (4)

**DATA STRUCTURE**

As mentioned before, colocated arrangement for velocity components and pressure within an arbitrary finite volume is adopted here. The basic idea of this data structure is shown in Fig. 1. As shown, forming points are the vertices of control volume and the center of control volume is assumed for storage of all variables.

In Fig. 1, each cell has some adjacent cells with two or three shared nodes. For example, in triangular meshes, the adjacent cells have two shared nodes and in tetrahedral meshes, they have three shared nodes.

The forming points and adjacent of a cell are numbered in the counterclockwise direction suitable for divergence theorem.

**FINITE VOLUME DISCRETIZATION**

By integrating the Eq. 5, over a control volume, using divergence theorem, the general transport equation in integral form is obtained:

\[ \frac{\partial}{\partial t} \int \rho \phi d\Omega + \int \rho \phi \vec{V} \cdot d\vec{S} = \int \Gamma \nabla \phi \cdot d\vec{S} + \int q_\phi d\Omega \]  \hspace{1cm} (5)

where, \( \vec{S} \) is the surface vector with the Cartesian components, namely \( S_x, S_y \). This integral form consists of four parts; transient term, convection term, diffusion and source term.

On triangular meshes, the convective term can be approximated as:

\[ \int \rho \phi \vec{V} \cdot d\vec{S} = \sum_{i=1}^{3} C_{ai} \phi_i \]  \hspace{1cm} (6)

The parameter \( C_{ai} \) is the mass flux over the i’th surface of control volume. The mass fluxes are:

Variables \( u_i \) and \( v_i \) are the components of velocities at the i’th face of the control volume.

The value of \( \phi \) on the faces (i.e. \( \phi_{ai} \)) can be approximated by the following second-order upwind scheme\(^{[1]}\):

\[ \phi_{ai} = \begin{cases} \phi_0 + \frac{\vec{V} \cdot \vec{r}_{0-ai}}{h_{0-C_i}} & \text{if } C_i > 0 \\ \phi_0 + \frac{\vec{V} \cdot \vec{r}_{i-0}}{h_{i-C_0}} & \text{if } C_i < 0 \end{cases} \]  \hspace{1cm} (7)

The vectors of \( \vec{r}_{0-ai} \) and \( \vec{r}_{i-0} \) are shown in Fig. 2. In Fig. 2, \( C_0 \) is the Center of the control volume \( \Omega \) and \( C_i \) is the centre of its i’th adjacent cell.

Diffusion term can be approximated as follows:

\[ \int \Gamma \nabla \phi \cdot d\vec{S} = \int \frac{\partial \phi}{\partial n} dS = \sum_{i=1}^{3} \Gamma_i \frac{\partial \phi}{\partial n} S_i \]  \hspace{1cm} (8)

The normal gradient of any variable can be approximated as below:

\[ \frac{\partial \phi}{\partial n} = \frac{\phi_0 - \phi_i}{h_{C_0-C_i}} = \frac{\phi_0 + (\vec{V} \cdot \vec{r}_{C_0-C_i}) - (\vec{V} \cdot \vec{r}_{C_0-C_i})}{h_{C_0-C_i}} \]  \hspace{1cm} (9)

The normal gradient of any variable can be approximated as below:

\[ \frac{\partial \phi}{\partial n} = \frac{\phi_0 - \phi_i}{h_{C_0-C_i}} = \frac{\phi_0 + (\vec{V} \cdot \vec{r}_{C_0-C_i}) - (\vec{V} \cdot \vec{r}_{C_0-C_i})}{h_{C_0-C_i}} \]  \hspace{1cm} (10)

where, \( h_{C_0-C_i} \) is the summation of distances \( h_{C_0-S_i} \) and \( h_{C_i-S_i} \) in Fig. (2). Inserting Eq. 10 in Eq. 9, results:

\[ \int \Gamma \nabla \phi d\vec{S} = \sum_{i=1}^{3} \left( \frac{\Gamma_i S_i}{h_{C_0-C_i}} \phi_0 - \sum_{i=1}^{3} \frac{\Gamma_i S_i}{h_{C_i-C_0}} \phi_i \right) \]  \hspace{1cm} (11)

The \( S_i \) is the magnitude of face vector \( \vec{S}_i \).

The pressure source terms can be approximated by using Green’s theorem. For example:
The transient term is approximated as:
\[
\frac{\partial}{\partial t} \int_{s} \rho \, d\Omega = \int_{s} \rho \, dS = -\sum_{i=1}^{3} \rho_{i} S_{u} = -S_{pe} \tag{12}
\]

The pressure source term can be obtained from:
\[
S_{pe} = \sum_{i=1}^{3} \rho_{i} p_{i} \frac{\Gamma_{\Omega}}{h_{c_{i} - c}} \tag{17}
\]

Convection and diffusion source terms are:
\[
S_{sp} = \sum_{i=1}^{3} \rho_{i} \left( [ \nabla \cdot \mathbf{u} ] + \frac{\Gamma_{\Omega}}{h_{c_{i} - c}} \right) \cdot \mathbf{S} \tag{18}
\]

In above equations $\Gamma_{\Omega}$ is diffusion coefficient at $i$th face and $S_{i}$ is the magnitude of a face vector and $S_{qi}$ denotes that component of face vector which is parallel with $\mathbf{q}$. For example in calculation of $S_{pu}$, the term $S_{qi}$ is $S_{xi}$ and for calculation of $S_{pv}$, the term $S_{qi}$ is $S_{yi}$.

**PRESSURE CORRECTION EQUATION**

The pressure correction equation is derived from integral form of the continuity equation. This integral form of can be written as:
\[
\int_{s} \rho \nabla \cdot \mathbf{V} \, dS = \int_{s} \rho \mathbf{V} \cdot d\mathbf{S} = \sum_{i=1}^{3} \rho \left( u_{i} S_{u,i} + v_{i} S_{v,i} + w_{i} S_{w,i} \right) = 0 \tag{19}
\]

The pressure and velocity link is very important. According to the momentum interpolation method, the face velocities are approximated as:
\[
u_{i} = \frac{1}{2} \left[ \nu + \left( \frac{\Omega}{A_{i}} \frac{\partial \mathbf{p}}{\partial x_{i}} + \nu_{i} + \left( \frac{\Omega}{A_{i}} \frac{\partial \mathbf{p}}{\partial x_{i}} \right) \right) \right] \tag{20}
\]

By considering below notations (Eq. 15):
\[
\hat{u}_{i} = \frac{1}{A_{i}} \sum_{i=1}^{3} \left( \nu + \left( \frac{\Omega}{A_{i}} \frac{\partial \mathbf{p}}{\partial x_{i}} \right) \right) S_{pe} - S_{v} \tag{21}
\]

Using momentum interpolation concept, the face velocity can be written as:
\[
u_{i} = \frac{1}{2} \left[ \nu + \left( \frac{\Omega}{A_{i}} \frac{\partial \mathbf{p}}{\partial x_{i}} \right) \right] \tag{22}
\]

Different approximations can be employed for estimating the face value for $\left( \frac{\Omega}{A_{i}} \right)$ in the above equations.

Lien[4], proposed the face value of $\left( \frac{\Omega}{A_{i}} \right)$ is estimated as below:
\[
\left( \frac{\Omega}{A_{i}} \right) = \left[ \frac{2 \Omega}{\left( A_{i} + \frac{\partial \mathbf{p}}{\partial x_{i}} \right)} \right] \tag{23}
\]

Here, we propose another approximation, i.e.:
\[
\left( \frac{\Omega}{A_{i}} \right) = \beta \left( \frac{\Omega}{A_{i}} \right) + \left( 1 - \beta \right) \left( \frac{\Omega}{A_{i}} \right) \tag{24}
\]

The weight factor ($\beta$) is defined by:
\[
\beta = \frac{\left| S_{c_{i} - c_{i}} \right|}{\left| S_{c_{i} - c_{i}} \right| + \left| S_{c_{i} - c_{i}} \right|} \tag{25}
\]

By replacing the above approximations, the integral form of continuity equation can be written as:
\[
\int_{s} \rho \mathbf{V} \cdot d\mathbf{S} = \sum_{i=1}^{3} \rho \left( \hat{u}_{i} \right) - \left( \frac{\Omega}{A_{i}} \right) \frac{\partial \mathbf{p}}{\partial x_{i}} \right] S_{u,i} + \sum_{i=1}^{3} \rho \left( \hat{v}_{i} \right) - \left( \frac{\Omega}{A_{i}} \right) \frac{\partial \mathbf{p}}{\partial y_{i}} \right] S_{v,i} = 0 \tag{26}
\]
And by introducing perturb mass flux \((C'_{Si})\) as:

\[
C'_{Si} = \rho S_{h_{Si}}(\Omega_{h_{Si}})(\frac{\partial p'}{\partial n})_{Si}
\]  

(28)

The continuity equation becomes:

\[
\sum_{i=1}^{n} (C_{Si} + C'_{Si}) = 0
\]  

(29)

Therefore, the final form of pressure correction is:

\[
A_{np}p' = \sum_{i=1}^{n} A_{np}p'_{Si} - \sum_{i=1}^{n} C'_{Si}
\]  

(30)

Where:

\[
A_{np} = \frac{\rho S}{h_{c_{x-x'}}}(\Omega_{h_{Si}})_{Si}
\]  

for \(i = 1,2,3\)  

(31)

And:

\[
A_{np} = A_{np} + A_{np} + A_{np}
\]  

(32)

According to the Lien [4] Proposal, The \(A_{np}\) coefficients are:

\[
A_{np} = \frac{2\rho S^2}{(A_{h_{Si}} + (A_{Si}))}
\]  

for \(i = 1,2,3\)  

(33)

And according to Eq. 24, The New form for \(A_{np}\) coefficients can be written as:

\[
A_{np} = \frac{\rho S}{h_{c_{x-x'}}}(\beta(\Omega_{h_{Si}}) + (1-\beta)(\frac{\Omega_{h_{Si}}}{A_{Si}}))
\]  

for \(i = 1,2,3\)  

(34)

New pressure and velocity field can be calculated from:

\[
p = p' + \alpha p'
\]  

(35)

\[
u_0 = \frac{\Omega_{h_{Si}}}{(A_{h_{Si}})} \frac{\partial p'}{\partial x}
\]  

and

\[
v_0 = \frac{\Omega_{h_{Si}}}{(A_{h_{Si}})} \frac{\partial p'}{\partial y}
\]  

(37)

where, \(\alpha\) is under relaxation factor for \(p'\). In this study, the value of 0.3 is used as relaxation factor for \(p'\).

The algorithm of solution is:

- Guessing the pressure field \(p\)
- Calculation of mass flux (Eq. 7)
- Calculation of coefficients (Eq. 15, 16, 17)
- Calculation of velocities (Eq. 14)
- Calculation of pressure correction (Eq. 30, 36)
- Assuming the corrected pressure as a new guessed pressure and returning to step (2), then repeating the whole procedure until a converged solution is obtained.

This is quite similar to SIMPLE method used by Patankar[6] for staggered grids.

CONVERGENCE CRITERIA'S AND DEFINITION OF RESIDUALS

After discretization, the residual for a general variable \(\varphi\) at a cell can be written as:

\[
R_{\varphi} = \sum_{i=1}^{n} \left( \frac{\rho S}{A_{h_{Si}}} \frac{\partial \varphi}{\partial n} \right)_{Si} - \sum_{i=1}^{n} (S_{e_{i}e_{i-1}} - S_{e_{i}e_{i+1}} - S_{e_{i}e_{i}} - S_{e_{i}e_{i}}) - A_{np} \varphi_{Si}
\]  

(38)

A different definition is used for residual of continuity equation, which is defined by:

\[
b_i = \sum_{j=1}^{n} C_{ij}
\]  

(39)

In above equation \(b_i\) is mass imbalance over the \(i\)th cell. According to above equation, the total mass imbalance is defined as:

\[
b = \sum_{i=1}^{n} b_i
\]  

(40)
APPLICATIONS

Lid driven cavity: The first case for comparison of two different pressure correction equations is a 1×1m² lid driven cavity. The velocity at upper edge is 0.001 m sec⁻¹ and the fluid is water (with the viscosity of 0.001 Ns m⁻² and the density of 1000 kg m⁻³). The Reynolds number is 1000 based on the cavity length.

To solve the problem, two different meshes are generated (Fig. 3). The problem is solved using two different pressure correction equations mentioned in previous section.

The pressure contours and the streamlines for structured grid are shown in Fig. 4. These results are achieved by using the new method (based on Eq. 34) for pressure correction equation.

In Fig. 5a, dimensionless profiles of u component of velocity vector at the midline of the cavity (at x = 0.5) for both structured and unstructured meshes are compared with Ghia et al. results[7]. The results are achieved using Eq. 34. There is a satisfactory agreement between the results, but in triangular mesh results deviation is more. Some error resources such as non-orthogonality are important for triangular meshes. Using deferred correction term can reduce the effects of non-orthogonality[8].

In Fig. 5b, profiles of u component of velocity vector are plotted for different methods on structured grids. Figure 5b also shows that the results are equal for two methods.

Total mass imbalance (defined by equation 40), vs. the number of iterations are plotted for different cases in Fig. 6a and b. The important feature of these results is that, the values of total mass imbalance (b) can be interpreted as a good criterion for overall convergence of the problem. The only difficulty in this way is that

Fig. 3: Rectangular and triangular meshes used for calculation of cavity flow a: rectangular mesh 40x40(1600 cells) b: triangular mesh (1644 cells)

Fig. 4: Pressure contours and streamlines of cavity flow at Re = 1000 Using rectangular mesh (by using Eq. 34) a: pressure contours, b: streamlines
the value of total mass imbalance can vary over a wide range for different problems, especially in unstructured grids. As shown in this Fig. 6a, the value of total mass imbalance are equal for two methods in rectangular meshes, but new method needs less number of iterations to converge in triangular meshes (Fig. 6b).

It can be understood that the rate of convergence by using new method (Eq. 34) is more, in the case of triangular meshes. The study of the results for parallel flow between two plates will be presented in the subsequent section as another sample problem.

Two-dimensional parallel flow: This case has an analytical solution in fully developed region. The geometry and properties of the fluid are shown in Fig. 7.

For solving the problem, two types of rectangular and triangular meshes are generated. The specifications of the meshes are given in Table 1.

| Type of Mesh | No. of intervals in x direction | No. of intervals in y direction | Total No. of CV |
|--------------|---------------------------------|---------------------------------|-----------------|
| Rectangular  | 80                              | 50                              | 4000            |
| Triangular   | 400                             | 20                              | 18144           |

In Fig. 8, contours of u and v components of the velocity are shown. These results are achieved using rectangular mesh and new method pressure correction equation.
Fig. 8: Contours of u and v component of velocity using rectangular mesh A: contours of u component B: contours of v component

Fig. 9: Profiles of dimensionless x component of velocity vector in developing region compared with results of MacDonald et al.\textsuperscript{[9]}. a: at x = 0.5H; b: at x = 2H

Fig. 10: Comparison of total mass imbalance (b) (defined by equation 40) for flow between parallel plate a: for rectangular grid b: for triangular grid

Figure 9 shows the profile of u component of velocity vector, compared with MacDonald et al.\textsuperscript{[9]} in two sections of developing region. This results show the correctness of written code.
Total mass imbalance (defined by equation 40), vs. the number of iterations are plotted for different cases in Fig. 10a, b.

As shown in this Fig. 10a, the value of total mass imbalance are equal for two methods in rectangular meshes, but in triangular meshes new method needs less number of iterations to converge (Fig. 10b).

SUMMARY AND CONCLUSION

By studying two sample problems, it is evident that new method (defined by Eq. 34) can decrease the residuals faster than by Lien\cite{4} proposed method. Two methods are approximately equal when rectangular meshes are used (Fig. 6a, 10a), but by using triangular meshes, new method needs less number of iterations to converge (Fig. 6b, 10b). This phenomenon is due to non-orthogonality of triangular meshes in which, weak pressure correction can’t damp the oscillation generated during the solution procedure. Therefore the method of construction of coefficients in Pressure correction equation can affect the rate of convergence. It must be emphasized that, using new method, the calculation time is less, because the residual of continuity equation and total mass imbalance decrease faster for this approach (Fig. 6b, 10b).

REFERENCES

1. Rhie, C.M. and W.L. Chow, 1983. A numerical study of turbulent flow past an isolated airfoil with trailing edge separation. AIAA J., 21: 1525-1532.
2. Peric, M., 1985. A finite volume method for the prediction of three dimensional flows in complex ducts. Ph. D. Thesis, University of London.
3. Majumdar, S., 1988. Role of under relaxation in momentum interpolation for calculation of flow with non-staggered grids. Numerical Heat Transfer, 13: 125-132.
4. Lien, F.S., 2000. A pressure based unstructured grid method for all speed flows. Int. J. Numerical Methods Fluids, 33: 355-374.
5. Thomadakis, M. and M. Leschziner, 1996. A pressure-correction method for the solution of incompressible viscous flows on unstructured grids. Int. J. Numerical Methods Fluids, 22: 581-601.
6. Patankar, S.V., 1980. Numerical heat transfer and fluid flow. McGraw-Hill, New York, pp: 126-131.
7. Ghia, U., N. Ghia and C.T. Shin, 1982. High resolution for incompressible flow using the navier stokes equations and multigrid method. J. Comput. Phys., 48: 387-411.
8. Freziger, J.H. and M. Peric, 2002. Computational methods for fluid dynamics. Springer. New York, pp: 232-238.
9. McDonald, J.W., V.E. Denny and A.F. Mills, 1972. Numerical solutions for navier-stokes equations in inlet regions. J. Applied Mech., 39 Ser. E, No.4, pp: 873-878.