Uniform in time lower bound for solutions to a quantum Boltzmann equation of bosons at low temperatures

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Abstract

We consider a quantum Boltzmann equation, which describes the growth of the condensate, or in other words, models the interaction between excited atoms and a condensate. In this work, the full form of the Bogoliubov dispersion law is considered, which leads to a detailed study of surface integrals inside the collision operator on energy manifolds. We prove that nonnegative radially symmetric solutions of the quantum Boltzmann equation are bounded from below by a Gaussian, uniformly in time.

Keyword: Low and high temperature quantum kinetics; Bose-Einstein condensate; quantum Boltzmann equation; Peierls equation; quantum theory of solids; quantum phonon equation.

MSC: 82C10, 82C22, 82C40.

1 Introduction

The discovery of Bose-Einstein condensation (BEC) in trapped ultracold atomic gases in 1995 [4, 5, 10] has led to an explosion of research on its properties. A kinetic equation for BECs was first derived by Kirkpatrick and Dorfmann [29, 31], by a mean field theory and the Green’s function method. Following the path of Kirkpatrick and Dorfmann, several authors have tried to derive kinetic equations to describe the dynamics of BECs ([2, 11, 14, 26, 7, 32, 4, 31, 30, 40, 23, 40, 25, 24]). In the series of papers [20, 28, 21, 27, 22, 13], C.W. Gardiner, P. Zoller and coauthors have formulated the Quantum Kinetic Theory, which is both a genuine kinetic theory and a genuine quantum theory, in terms of the Quantum Kinetic Master Equation (QKME) for bosonic atoms. In the Quantum Kinetic Theory, the significant quantum aspects are restricted to a few modes, the remaining modes being able to be described in the classical way as in the Boltzmann equation. Indeed, the kinetic aspect of the theory arises from the decorrelation between different momentum bands. The

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Quantum Kinetic Theory provides a fully quantum mechanical description of the kinetics of a Bose gas, including the regime of a Bose condensation. In particular, the QKME is capable of describing the formation of the Bose condensate. The QKME contains as limiting cases both the Uehling-Uhlenbeck equation \([15, 44, 36]\), the Gross-Pitaevskii equation, and the condensate growth term. The condensate growth term is indeed the principal term which gives rise to growth of the condensate, where the condensate grows by taking atoms out of the bath of warmer atoms.

Under the assumption \((18)\) that the condensate is very stable, the quantum Boltzmann equation describing the excited atoms and condensate interaction - the condensate growth term - reads

\[
\frac{\partial f}{\partial t} = Q[f]
\] (1.1)

for excited atom density distribution function \(f(t, p) \geq 0, \ t \geq 0 \text{ and } p \in \mathbb{R}^3\). Here, \(Q[f]\) denotes the collision integral operator describing the bosons-condensate interaction, defined by \([14, 26, 1, 2, 31, 30, 7, 9]\)

\[
Q[f](p) = \int_{\mathbb{R}^3 \times \mathbb{R}^3} \left( R(p, p_1, p_2) - R(p_1, p, p_2) - R(p_2, p_1, p) \right) dp_1 dp_2
\] (1.2)

in which, together with the convention of dropping the \(t\)-dependence in \(f(t, p)\),

\[
R(p, p_1, p_2) = \mathcal{K}(p, p_1, p_2) \left( f(p_1) f(p_2)(1 + f(p)) - (1 + f(p_1))(1 + f(p_2)) f(p) \right)
\]

with the collision kernel \(\mathcal{K}(p, p_1, p_2)\) defined by

\[
\mathcal{K}(p, p_1, p_2) = |\mathcal{M}(p, p_1, p_2)|^2 \delta \left( \mathcal{E}(p) - \mathcal{E}(p_1) - \mathcal{E}(p_2) \right) \delta \left( p - p_1 - p_2 \right).
\] (1.3)

Here, \(\delta\) denotes the Dirac delta function and \(\mathcal{E}(p)\) is the Bogoliubov dispersion law for particle energy, under the assumption that the external potential is zero

\[
\mathcal{E}(p) = \sqrt{\kappa_1 |p|^2 + \kappa_2 |p|^4}, \quad \kappa_1 = \frac{g n_c}{m} > 0, \quad \kappa_2 = \frac{1}{4m^2} > 0,
\] (1.4)

for \(m\) being the mass of the particles, \(g\) the interaction coupling constant, and \(n_c\) the density of particles in the Bose-Einstein condensate, which is supposed to be a constant. In this paper, we shall prove that positive radially symmetric solutions to the quantum Boltzmann equations \((1.1)-(1.4)\) are uniformly bounded below by a Gaussian.

The Dirac delta function in \((1.3)\) ensures the conservation of momentum and energy after collision:

\[
p = p_1 + p_2, \quad \mathcal{E}(p) = \mathcal{E}(p_1) + \mathcal{E}(p_2).
\] (1.5)
By supposing that the temperature $T$, the density $n_c$, and the interaction coupling constant $g$ are taken such that $k_B T$ is much smaller than $g n_c$, the transition probability $\mathcal{M}(p, p_1, p_2)$ can then be approximated by

$$|\mathcal{M}(p, p_1, p_2)|^2 = \kappa_0 |p||p_1||p_2|, \tag{1.6}$$

the approximation that we assume in this paper. Here, $\kappa_0$ is some positive physical constant could be found in [18, 26, 14]. This approximation is valid at low temperatures, in which only low momentum excitations are relevant [18, 26, 14]. In this low temperature regime, the interaction of excited bosons with a condensate is dominant, in comparison with the interaction between excited bosons themselves. Note that, unlike some of the previous works [7, 6, 8, 9], we do not have to assume that $|\mathcal{M}|^2$ is bounded from above by a positive constant and is truncated away from the origin (that is, multiplying $|\mathcal{M}|^2$ with the characteristic function of the set $\{|p|, |p_1|, |p_2| \geq \lambda\}$).

Next, let us emphasize that in this work the full form of energy functions (1.4) is considered, which significantly complicates the analysis in treating the collision integral operator $Q[f]$. The integrals are now reduced to the surface integral on the energy surfaces, dictated by the conservation laws (1.5), consisting of all points $p_1$ so that

$$E(p) = E(p_1) + E(p - p_1)$$

for each $p$; see Figures 1 and 2 for an illustration of these surfaces. In addition to the complication of dealing with the surface integrals, it is certainly not clear whether the second moment of $f$ on these surfaces is bounded, even if the second moment of $f$ in $\mathbb{R}^3$ is bounded. As a matter of fact, due to this very reason, the simplified energy functions

$$E = \frac{|p|^2}{2m} + g n_c \tag{1.7}$$

or $E(p) = \kappa_0 |p|$, have been used in the literature; see, for instance, [1, 2, 15, 8, 17] and the references therein. The latter energy law leads to line integrals, whereas the energy law (1.7) reduces to integrals on a sphere, as it is the case for the classical Boltzmann equations (e.g., [45, 35, 18]). Up to our knowledge, the current paper is the first time where the full energy of the form (1.4) and hence the surface integrals on the energy surfaces are studied.

In the quantum theory of solids, the quantum phonon Boltzmann equation ([38]) - the Peierls equation - is of the same formulation with the equation (1.1). To the best of our knowledge, in the context of the study of phonon interactions in anharmonic crystals [37], the model (1.1) is the first kinetic model of weak turbulence [47, 16, 43, 33]. In anharmonic crystals, electronic bands of dielectric crystals are completely filled and separated by an energy gap from the conduction band. As a consequence, electronic energy transport is suppressed and the vibrations of the atoms around their mechanical equilibrium position is the dominant contribution to heat transport. R. Peierls suggested the theoretical option of
considering the anharmonicities as a small perturbation to the perfectly harmonic crystal, which leads to a kinetic model of an interacting phonons in terms of a nonlinear Boltzmann equation. The phonon Boltzmann equation is then employed to carry on the actual computation of the thermal conductivity of dielectric crystals. The problem was also studied in [42], constrained to the torus in the space momentum.

As an attempt to build a mathematical theory for quantum kinetic equations, in [19] the authors have proved the existence, uniqueness and the convergence to equilibrium of solutions for a linearized model of (1.1). The discrete theory of the equation, based on a dynamical system approach, has been done in [12]. Most recently, existence and uniqueness of classical and radially symmetric solutions to the Cauchy problem (1.1) have been obtained in the work [3, 41]. Precisely, for radially symmetric initial data
\[ f_0(p) = f_0(|p|), \]
there exists a unique (radially symmetric) global-in-time solution in the function space \( C([0, \infty), L^1_m) \cap C^1([0, \infty), L^1_m), \) for sufficiently large \( m, \) in which \( L^1_m \) is defined by
\[
L^1_m(\mathbb{R}^3) := \left\{ f \mid \| f \|_{L^1_m} := \int_{\mathbb{R}^3} |f(p)| (1 + \mathcal{E}(p)^m) dp < \infty \right\}. \tag{1.8}
\]

In this paper, we are interested precisely in the properties of such a classical solution. Precisely, our main result is as follows.

**Theorem 1.1** Let \( f_0(p) = f_0(|p|) \) be initial data in \( L^1_{m_0}(\mathbb{R}^3), \) for some \( m_0 \geq 1. \) Suppose that (1.1) has a unique solution in \( C^1(\mathbb{R}_+, L^1_{m_0}(\mathbb{R}^3)) \cap C(\mathbb{R}_+ \times \mathbb{R}^3) \) under some suitable assumptions on the initial data. Assume that \( f_0(p) \geq \theta_0 \) on \( B_{R_0} = \{|p| \leq R_0\} \) for some positive constants \( \theta_0, R_0. \) Then, for any time \( T > 0, \) there exist positive constants \( \theta_1, \theta_2 \) such that radially symmetric solutions \( f(t,p) = f(t,|p|) \) to the quantum Boltzmann equations (1.1)-(1.6), with \( f(0,p) = f_0(p), \) satisfy
\[
f(t,p) \geq \theta_1 \exp(-\theta_2|p|^2), \quad \forall \ t \geq T. \tag{1.9}
\]

Physically speaking, Theorem 1.1 asserts that the collision operator \( Q \) prevents the excited atoms to all fall into the condensate. In other words, given a condensate and its thermal cloud, we can prove that there will be some portion of excited atoms which remain outside of the condensate and the density of such atoms will be greater than a Gaussian, uniformly in time \( t \geq \tau \) for any time \( \tau > 0. \) Similar results in the classical Boltzmann equation context have been obtained in [39, 34]. Note that the condition that initial data \( f_0(p) \) has positive mass near \( \{p = 0\} \) is necessary for such a lower bound by a Gaussian to hold, since otherwise \( f(t,0) \) would remain zero for all time, as a consequence of
\[
\partial_t f(t,0) = 0, \quad \forall \ t \geq 0.
\]

In Theorem 1.1, we are not concerned with the existence of positive and radially symmetric solutions to the quantum Boltzmann equations.
In order to prove Theorem 1.1 it is crucial that we derive bounds on the loss term in the collision operator, which then require bounds on the mass, second and third order energy moments. We then show that the second and third order energy moments of the radially symmetric solution of (1.1) are also created and propagated in time. Moreover, let us emphasize that unlike the classical Boltzmann equation, Equation (1.1) does not conserve the mass. However, we shall prove that the total of mass remains bounded uniformly in time.

The structure of the paper is as follows:

• In Section 2 we establish the conservation of momentum, energy and the H-theorem of (1.1). Notice that there is no conservation of mass.

• As mentioned above, we consider the full form of particle energy function - the Bogoliubov dispersion law (1.4) on the energy surfaces $S_p$, $S_p'$ and $S_p''$. In Section 3 we provide the technical estimates concerning these surfaces. The estimates in this section are the basic tools of the whole paper.

• In order to obtain a uniform in time lower bound for the solution $f(t,p)$ of (1.1), we need to estimate the loss part of the collision operator $Q$, which is based on controlling the mass of the excited atoms i.e. the integral $\int_{\mathbb{R}^3} f(t,p)dp$. Different from the classical Boltzmann equation, the mass of the solution of (1.1) is not conserved. The whole Section 4 is devoted to derive such a bound on the mass. The bound is proved in three smaller steps:

  – Subsection 4.1 is devoted to the creation and propagation of the second order energy moment of the radially symmetric solution of (1.1). We prove that the second order energy moment is bounded uniformly in time $t \geq \tau$ for all $\tau > 0$.

  – In Subsection 4.2, we prove that the third order energy moment of the radially symmetric solution of (1.1) is bounded uniformly in time $t \geq \tau$ for all $\tau > 0$.

  – In Subsection 4.2, we show that the mass $\int_{\mathbb{R}^3} f(t,p)dp$ of the radially symmetric solution of (1.1) is bounded uniformly in time.

• Finally, Section 5 is devoted to the proof of the main theorem, Theorem 1.1.

2 Conservation laws and the H-theorem

In this section, we obtain the basic properties of smooth solutions of (1.1).

**Lemma 2.1** For any smooth function $f(p)$, there holds

\[ \int_{\mathbb{R}^3} Q[f](p)\varphi(p)dp = \iiint_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3} R(p,p_1,p_2)(\varphi(p) - \varphi(p_1) - \varphi(p_2))dpdp_1dp_2. \]

for any smooth test function $\varphi$. 

5
Proof By the definition (1.2) of $Q[f](p)$, we have

$$\int_{\mathbb{R}^3} Q[f](p) \varphi(p) dp = \iiint_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3} \left( R(p, p_1, p_2) - R(p_1, p, p_2) - R(p_2, p_1, p) \right) \varphi(p) dp dp_1 dp_2.$$ 

By switching the variables $p \leftrightarrow p_1$ and $p \leftrightarrow p_2$ in the second and third integral, respectively, the lemma follows at once.

As a consequence, we obtain the following two corollaries.

**Corollary 2.1 (Conservation of momentum and energy)** Smooth solutions $f(t, p)$ of (1.1), with initial data $f(0, p) = f_0(p)$, satisfy

$$\int_{\mathbb{R}^3} f(t, p) dp = \int_{\mathbb{R}^3} f_0(p) dp \quad (2.1)$$

$$\int_{\mathbb{R}^3} f(t, p) \mathcal{E}(p) dp = \int_{\mathbb{R}^3} f_0(p) \mathcal{E}(p) dp \quad (2.2)$$

for all $t \geq 0$.

**Proof** This follows from Lemma 2.1 by taking $\varphi(p) = p$ or $\mathcal{E}(p)$. □

**Corollary 2.2 (H-Theorem)** Smooth solutions $f(t, p)$ of (1.1) satisfy

$$\frac{d}{dt} \int_{\mathbb{R}^3} \left[ f(t, p) \log f(t, p) - (1 + f(t, p)) \log(1 + f(t, p)) \right] dp \leq 0.$$ 

In addition, radially symmetric equilibria of (1.1) must have the following form

$$f(p) = \frac{1}{e^{c \mathcal{E}(p)} - 1}, \quad (2.3)$$

for some positive constant $c$.

**Proof** First notice that

$$\frac{d}{dt} \int_{\mathbb{R}^3} \left[ f(p) \log f(p) - (1 + f(p)) \log(1 + f(p)) \right] dp = \int_{\mathbb{R}^3} \partial_t f(p) \log \left( \frac{f(p)}{f(p) + 1} \right) dp.$$ 

We then rewrite

$$\int_{\mathbb{R}^3} Q[f](p) \varphi(p) dp$$

$$= \iiint_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3} K(p, p_1, p_2)(1 + f(p))(1 + f(p_1))(1 + f(p_2))$$

$$\times \left( \frac{f(p_1)}{f(p_1) + 1} \frac{f(p_2)}{f(p_2) + 1} - \frac{f(p)}{f(p) + 1} \right) \left[ \varphi(p) - \varphi(p_1) - \varphi(p_2) \right] dp dp_1 dp_2.$$
Using Lemma 2.1 with \( \varphi(p) = \log \left( \frac{f(p)}{f(p) + 1} \right) \) and the fact that \((a - b) \log \left( \frac{a}{b} \right) \geq 0\), with equality if and only if \(a = b\), we obtain

\[
\int_{\mathbb{R}^3} Q[f](p) \log \left( \frac{f(p)}{f(p) + 1} \right) dp \leq 0.
\]

This yields the claimed inequalities in the H-theorem. In the case of equality, we have

\[
\frac{f(p_1)}{f(p_1) + 1} \frac{f(p_2)}{f(p_2) + 1} - \frac{f(p)}{f(p) + 1} = 0,
\]

or equivalently, setting \( h(p) = \log \left( \frac{f(p)}{f(p) + 1} \right) \), where \( h \) is radially symmetric

\[
h(p_1) + h(p_2) = h(p) \quad (2.4)
\]

for all \((p, p_1, p_2)\) so that \( K(p, p_1, p_2) \neq 0\). In particular, by view of the conservation laws (1.5), the function \( h(p) \) satisfies \( h(p_1 + p_2) = h(p_1) + h(p_2) \), for all pairs \((p_1, p_2) \in \mathbb{R}^6\) so that

\[
\mathcal{E}(p_1 + p_2) = \mathcal{E}(p_1) + \mathcal{E}(p_2).
\]

This proves that \( h \circ \mathcal{E}^{-1}(a + b) = h \circ \mathcal{E}^{-1}(a) + h \circ \mathcal{E}^{-1}(b) \), for all \( p_1 = \mathcal{E}^{-1}(a) \) and \( p_2 = \mathcal{E}^{-1}(b) \), with the notice that \( h \) is radially symmetric. Since \( a, b \) may take arbitrary values in \( \mathbb{R} \), this yields \( h \circ \mathcal{E}^{-1}(a) = -ca \) for some positive constant \( c \) and for all \( a \geq 0 \), or equivalently \( h(p) = -c \mathcal{E}(p) \), for all \( p \in \mathbb{R}^3 \). This yields (2.3), and thus the H-theorem is proved for (1.1).

### 3 Energy surfaces

In this section, we study the surface integrals that arise in the collision operator, due to the conservation laws (1.5). For convenience, we introduce

\[
Q[f] = Q_1[f] + Q_2[f] \quad (3.1)
\]

with

\[
Q_1[f] := \int_{\mathbb{R}^3 \times \mathbb{R}^3} K(p, p_1, p_2) \left[ f(p_1)f(p_2) - f(p)(f(p_1) + f(p_2) + 1) \right] dp_1 dp_2
\]

\[
Q_2[f] := -2 \int_{\mathbb{R}^3 \times \mathbb{R}^3} K(p_1, p, p_2) \left[ f(p)f(p_2) - f(p_1)(f(p) + f(p_2) + 1) \right] dp_1 dp_2.
\]

We recall that the collision kernel is defined by

\[
K(p, p_1, p_2) = \kappa_0 |p||p_1||p_2| \left( \delta(\mathcal{E}(p) - \mathcal{E}(p_1) - \mathcal{E}(p_2)) \delta(p - p_1 - p_2) \right)
\]
for $\kappa_0$ being a positive constant; see (1.6). We also introduce the energy surfaces

$$
S_p := \left\{ p_\ast \in \mathbb{R}^3 : \mathcal{E}(p - p_\ast) + \mathcal{E}(p_\ast) = \mathcal{E}(p) \right\}
$$

$$
S'_p := \left\{ p_\ast \in \mathbb{R}^3 : \mathcal{E}(p + p_\ast) = \mathcal{E}(p) + \mathcal{E}(p_\ast) \right\}
$$

(3.2)

$$
S''_p := \left\{ p_\ast \in \mathbb{R}^3 : \mathcal{E}(p_\ast) = \mathcal{E}(p) + \mathcal{E}(p_\ast - p) \right\}
$$

for each $p \in \mathbb{R}^3 \setminus \{0\}$. Define

$$
K(p, p_1, p_2) = \kappa_0 |p| |p_1||p_2| \left( \delta(\mathcal{E}(p) - \mathcal{E}(p_1) - \mathcal{E}(p_2)) \right),
$$

by the nature of the Dirac delta function, the collision operators reduce to the following surface integrals

$$
Q_1[f] := \int_{S_p} K(p, p - p_2, p_2) \left[ f(p - p_2)f(p_2) - f(p)(f(p - p_2) + f(p_2) + 1) \right] d\sigma(p_2)
$$

$$
Q_2[f] := 2 \int_{S'_p} K(p + p_2, p, p_2) \left[ f(p + p_2)(f(p) + f(p_2) + 1) - f(p)f(p_2) \right] d\sigma(p_2).
$$

When convenient, we also write $Q[f]$ in term of gain and loss terms; namely,

$$
Q[f] = Q_{\text{gain}}[f] - Q_{\text{loss}}[f]
$$

(3.3)

with

$$
Q_{\text{gain}}[f] := \int_{S_p} K(p, p - p_2, p_2) f(p - p_2)f(p_2) d\sigma(p_2)
$$

$$
+ 2 \int_{S_p} K(p + p_2, p, p_2) f(p + p_2) \left( f(p) + f(p_2) + 1 \right) d\sigma(p_2)
$$

$$
Q_{\text{loss}}[f] := f(p) \int_{S_p} K(p, p - p_2, p_2) \left( f(p - p_2) + f(p_2) + 1 \right) d\sigma(p_2)
$$

$$
+ 2f(p) \int_{S'_p} K(p + p_2, p, p_2) f(p_2) d\sigma(p_2).
$$

The following two lemmas give estimates on the energy surface integrals.

**Lemma 3.1** Let $S_p$ be defined as in (3.2). There are positive constants $c_0, C_0$ so that

$$
c_0 |p|^2 \min\{1, |p|\} \leq \int_{S_p} d\sigma(w) \leq C_0 |p|^2 \min\{1, |p|\}
$$

uniformly in $p \in \mathbb{R}^3$.

Moreover, let $B$ be either the ball $B(p/2, |p|/2)$ or $B(0, |p|/2)$ the following estimate also holds

$$
\int_{S_p} ||p| - |w||^\gamma |w|^\gamma \chi_B d\sigma(w) \geq c_1 |p|^{2\gamma + 2} \min\{1, |p|\},
$$

(3.4)
where $\gamma$ is a non-negative constant and $\chi_B$ is the characteristic function of the ball $B$. In addition, for any radially symmetric function $G(\cdot)$, we have

$$
\int_{S_p} G(|w|) d\sigma(w) \leq c_2(1 + |p|) \int_0^{[p]} uG(u) du.
$$

(3.5)

**Proof** Recall that $S_p$ is the surface consisting of $w$ so that $E(p - w) + E(w) = E(p)$. It is clear that $\{0, p\} \subset S_p$. If we consider the energy function $E(p)$ as a function of $|p|$, this function is increasing. By the monotonicity, we have $|w| \leq |p|$ and $|p - w| \leq |p|$, for all $w \in S_p$, since $E(p - w) \leq E(p)$ and $E(w) \leq E(p)$. Hence, the surface $S_p$ is contained in $B(0, |p|) \cap B(p, |p|)$.

Next, let us set $G(w) := E(p - w) + E(w) - E(p)$. A direct computation yields

$$
\nabla_w G = \frac{w - p}{|p - w|} E'(p - w) + \frac{w}{|w|} E'(w).
$$

(3.6)

In particular, the directional derivative of $G$ in the direction of $q \neq 0$, with $p \cdot q = 0$, at the point $w = \alpha p + q, \alpha \in \mathbb{R}$, satisfies

$$
q \cdot \nabla_w G = \frac{1}{|q|^2} \left[ \frac{E'(p - w)}{|p - w|} + \frac{E'(w)}{|w|} \right] > 0,
$$

(3.7)

where

$$
E'(|\rho|) = \frac{2\kappa_1 + 4\kappa_2|\rho|^2}{\sqrt{\kappa_1 + \kappa_2|\rho|^2}^2}.
$$

That is, $G(w)$ is strictly increasing in any direction that is orthogonal to $p$. This, together with the fact that $G(\alpha p) < 0$ for $\alpha \in (0, 1)$ and $S_p \subset \overline{B(0, |p|) \cap B(p, |p|)}$, proves that the surface $S_p$ and the plane

$$
\mathcal{P}_\alpha = \left\{ \alpha p + q, p \cdot q = 0 \right\}
$$

intersect for each $\alpha \in [0, 1]$. In addition, $G(\alpha p + q)$ is a radial function in $|q|$, with $q \cdot p = 0$. This asserts that the intersection of $S_p$ and $\mathcal{P}_\alpha$ is precisely the circle centered at $\alpha p$ and of a finite radius $|q_\alpha|$, for each $\alpha \in [0, 1]$.

Let us give estimates on $q_\alpha$. It is clear that $|q_\alpha|$ is smooth with respect to $\alpha \in [0, 1]$ and there holds

$$
|q_\alpha| \leq |p|.
$$

(3.8)

In addition, set $w_\alpha = \alpha p + q_\alpha$. By definition, $G(w_\alpha) = 0$ and so

$$
0 = \partial_\alpha w_\alpha \cdot \nabla_w G = \frac{1}{2} \partial_\alpha |w_\alpha|^2 \left[ \frac{E'(p - w_\alpha)}{|p - w_\alpha|} + \frac{E'(w_\alpha)}{|w_\alpha|} \right] - |p|^2 \frac{E'(p - w_\alpha)}{|p - w_\alpha|}
$$

$$
= \frac{1}{2} \partial_\alpha |q_\alpha|^2 \left[ \frac{E'(p - w_\alpha)}{|p - w_\alpha|} + \frac{E'(w_\alpha)}{|w_\alpha|} \right] + \alpha |p|^2 \frac{E'(w_\alpha)}{|w_\alpha|} - (1 - \alpha) |p|^2 \frac{E'(p - w_\alpha)}{|p - w_\alpha|}
$$

(3.9)
upon using $|w_\alpha|^2 = \alpha^2|p|^2 + |q_\alpha|^2$. The above proves that

$$-\alpha|p|^2 \leq \frac{1}{2} \partial_\alpha|q_\alpha|^2 \leq (1 - \alpha)|p|^2$$  \hspace{1cm} (3.10)

uniformly for all $p$ and all $\alpha \in (0, 1)$. In addition, at $\alpha = \frac{1}{2}$, we have $|w_{1/2}| = |w_{1/2} - p|$ and hence $2\mathcal{E}(w_{1/2}) = \mathcal{E}(p)$, or equivalently with noting that $|w_{1/2}|^2 = \frac{1}{4}|p|^2 + |q_{1/2}|^2$,

$$\kappa_2 \left( \frac{1}{4}|p|^2 + |q_{1/2}|^2 \right)^2 + \kappa_1|q_{1/2}|^2 = \frac{\kappa_2}{4}|p|^4.$$  

This proves at once that

$$c_0|p|^2 \min \left\{ 1, |p|^2 \right\} \leq |q_{1/2}|^2 \leq C_0|p|^2 \min \left\{ 1, |p|^2 \right\}$$  \hspace{1cm} (3.11)

for some universal constants $c_0, C_0$, independent of $|p|$. Moreover,

$$|q_{1/2}| < \frac{1}{2}|p|, \text{ and } w_{1/2} \in B(p/2, |p|/2).$$  \hspace{1cm} (3.12)

Also, we can write

$$|q_\alpha|^2 = |q_{1/2}|^2 - \int_\alpha^\frac{1}{2} \partial_\alpha'|q_\alpha|^2 \, d\alpha'$$

Consider the case when $|p| \geq 1$. The above shows that there is a $c_0$ independent of $|p|$ so that

$$|q_\alpha|^2 \geq c_0|p|^2 - |\alpha - \frac{1}{2}|p|^2 \geq \frac{1}{2}c_0|p|^2$$  \hspace{1cm} (3.13)

for all $\alpha$ so that $|\alpha - \frac{1}{2}| \leq \frac{c_0}{2}$.

Let us consider the case when $|p|$ is sufficiently small. We recall that

$$\left( \mathcal{E}(w) + \mathcal{E}(p - w) \right)^2 - \mathcal{E}(p)^2 = \kappa_1(|p - w|^2 + |w|^2 - |p|^2) + \kappa_2(|p - w|^4 + |w|^4 - |p|^4) + 2\mathcal{E}(w)\mathcal{E}(p - w)$$

$$= 2\kappa_1w \cdot (w - p) + 2\kappa_2w \cdot (w - p) \left( |w|^2 + |w - p|^2 + |p|^2 \right) - 2\kappa_2|w|^2|p - w|^2 + 2\mathcal{E}(w)\mathcal{E}(p - w).$$

Thus, on $S_p$, the above gives

$$-w \cdot (w - p) \left( \kappa_1 + \kappa_2|w|^2 + \kappa_2|w - p|^2 + \kappa_2|p|^2 \right) = \mathcal{E}(w)\mathcal{E}(p - w) - \kappa_2|w|^2|p - w|^2$$  \hspace{1cm} (3.14)

in which the right hand side is

$$\mathcal{E}(w)\mathcal{E}(p - w) - \kappa_2|w|^2|p - w|^2$$

$$= |w||p - w| \sqrt{\left( \kappa_1 + \kappa_2|w|^2 \right)\left( \kappa_1 + \kappa_2|w - p|^2 \right) - \kappa_2|w|^2|p - w|^2}$$

$$= |w||p - w| \sqrt{\frac{\kappa_1 \left( \kappa_1 + \kappa_2|w|^2 + \kappa_2|w - p|^2 \right)}{(\kappa_1 + \kappa_2|w|^2)\left( \kappa_1 + \kappa_2|w - p|^2 \right) + \kappa_2|w||p - w|}}.$$  \hspace{1cm} (3.15)
Here and in what follows, $f = O(g)$ is to denote that $c_0 g \leq f \leq c_1 g$ for some universal positive constants $c_0, c_1$. Hence, for small $p$, there holds

$$
\sqrt{\left(1 + \frac{\kappa_2}{\kappa_1} |w|^2\right) \left(1 + \frac{\kappa_2}{\kappa_1} |w-p|^2\right)} = 1 + \frac{\kappa_2}{2 \kappa_1} (|w|^2 + |w-p|^2) + O(|p|^4),
$$

upon recalling that $|w| \leq |p|$ and $|w-p| \leq |p|$ for $w \in S_p$. We thus obtain

$$
E(w)E(p-w) - \kappa_2 |w|^2 |p-w|^2
\begin{align*}
&= |w||p-w|\left(\kappa_1 + \kappa_2 |w|^2 + \kappa_2 |w-p|^2\right) \\
&\quad \times \left(1 - \frac{1}{2 \kappa_1} \left(\kappa_1 + \kappa_2 |w|^2 + \kappa_2 |w-p|^2 - \kappa_2 w|w-p| + O(|p|^4)\right)\right) \\
&= |w||p-w|\left(\kappa_1 + \kappa_2 |w|^2 + \kappa_2 |w-p|^2 - \kappa_2 |w||w-p| + O(|p|^4)\right) \\
&= |w||p-w|\left(\kappa_1 + \kappa_2 |w|^2 + \kappa_2 |w-p|^2 + \kappa_2 |p|^2\right) \\
&\quad - \frac{\kappa_2}{2} |w||w-p|(|w|^2 + |w-p|^2 + 2 |w||w-p| + |p|^2)(1 + O(|p|^2))
\end{align*}
$$

Letting $\theta_\alpha$ be the angle between $w_\alpha$ and $w_\alpha - p$, the above and the fact that $w_\alpha \cdot (w_\alpha - p) = |w_\alpha||w_\alpha - p| \cos \theta_\alpha$ prove that

$$
1 + \cos \theta_\alpha = \frac{\kappa_2}{2} \frac{|w_\alpha|^2 + |w_\alpha - p|^2 + 2 |w_\alpha||w_\alpha - p| + |p|^2)(1 + O(|p|^2))}{\kappa_1 + \kappa_2 |w_\alpha|^2 + \kappa_2 |w_\alpha - p|^2 + \kappa_2 |p|^2} = O(|p|^2)
$$

or equivalently, $\pi - \theta_\alpha = O(|p|)$. Computing the area of the parallelogram formed by $w_\alpha$ and $w_\alpha - p$ yields

$$
2|p||q_\alpha| = |w_\alpha \times (w_\alpha - p)| = |w_\alpha||w_\alpha - p| \sin \theta_\alpha
$$

with $\sin \theta_\alpha = O(|p|)$. This proves that there is some universal constants $c_2, c_3$ so that

$$
c_2 \alpha (1 - \alpha)|p|^2 \leq |q_\alpha| \leq c_2 |p|^2
$$

for all $\alpha \in (0, 1)$.

To summarize, we can parametrize $S_p$ as follows; see Figure 1. Let $p^\perp$ be in $P_0 = \{p: q = 0\}$ and let $e_\theta$ be the unit vector in $P_0$ so that the angle between $p^\perp$ and $e_\theta$ is $\theta$. We parametrize $S_p$ by

$$
S_p = \left\{w(\alpha, \theta) = \alpha p + |q_\alpha|e_\theta : \theta \in [0, 2\pi], \alpha \in [0, 1]\right\}.
$$

Since $\partial_\theta e_\theta$ is orthogonal to both $p$ and $e_\theta$, we compute the surface area

$$
d\sigma(w) = |\partial_\alpha w \times \partial_\theta w| d\alpha d\theta = \left|(p + \partial_\alpha |q_\alpha|e_\theta) \times |q_\alpha|\partial_\theta e_\theta\right| d\alpha d\theta
\begin{align*}
&= \left|(q_\alpha|p| + \frac{1}{2} \partial_\alpha |q_\alpha|^2 e_\theta) \times \partial_\theta e_\theta\right| d\alpha d\theta \\
&= \sqrt{|p|^2 |q_\alpha|^2 + \frac{1}{4} |\partial_\alpha (|q_\alpha|^2)|^2} d\alpha d\theta.
\end{align*}
$$

(3.16)
Figure 1: Illustrated is the oval surface \( S_p \), centered at \( \frac{p}{2} \) and having 0 and \( p \) as its south and north poles, respectively.

Hence, for \( |p| \geq 1 \), the upper bound of the surface area of \( S_p \) follows at once from (3.8), (3.10) and (3.15), whereas the lower bound is due to the estimate (3.13). As for small \( p \), the upper and lower bounds follow from (3.10) and (3.15).

In order to obtain (3.4), we will study the intersections of \( B(0, |p|/2) \) and \( B(p/2, |p|/2) \) and \( S_p \). We show that there exists \( \gamma_1 \) independent of \( p \), such that for all \( \alpha \in [0, \gamma_1] \), \( w_\alpha \in B(0, |p|/2) \). By (3.9), we can see that

\[
0 \leq \partial_\alpha |w_\alpha|^2 \leq 2 |p|^2,
\]

which implies

\[
|w_\alpha|^2 = \int_0^\alpha \partial_\alpha |w_\alpha|^2 d\alpha' \leq 2\alpha |p|^2.
\]

As a consequence, we can choose \( \gamma_1 < \frac{1}{2} \).

Next, for \( w \in S_p \), the identity (3.14) and the fact that \( E(p) > \sqrt{\kappa_2} |p| \) yield \( w \cdot (w - p) < 0 \). Hence, on \( S_p \), we can compute

\[
|w - \frac{1}{2} p|^2 = \frac{1}{4} |w|^2 + \frac{1}{4} |w - p|^2 + \frac{1}{2} w \cdot (w - p) < \frac{1}{4} (|w|^2 + |w - p|^2) \leq \frac{1}{4} |p|^2
\]

in which the last inequality holds, since \( w \in S_p \). This proves that \( S_p \) is contained inside the ball \( B(p/2, |p|/2) \).

The estimate (3.4) then follows from the above arguments, (3.12), (3.13) and (3.15). As for the surface integral of a radial function \( G(|w|) \), we introduce the radial variable \( u = |w_\alpha| = \sqrt{\alpha^2 |p|^2 + |q_\alpha|^2} \). We compute \( 2udu = \partial_\alpha |w_\alpha|^2 d\alpha \) and hence

\[
d\sigma(w) = \frac{\sqrt{|p|^2 |q_\alpha|^2 + \frac{1}{4} |\partial_\alpha (|q_\alpha|^2)|^2}}{2\partial_\alpha |w_\alpha|^2} udu\theta.
\]
Using \(|w_\alpha|^2 = \alpha^2|p|^2 + |q_\alpha|^2\), we compute

\[
|p|^2|q_\alpha|^2 + \frac{1}{4} |\partial_\alpha(|q_\alpha|^2)|^2 \leq |p|^2|q_\alpha|^2 + 2\alpha^2|p|^4 + \frac{1}{2} |\partial_\alpha|w_\alpha|^2|^2 \\
\leq 2|w_\alpha|^2|p|^2 + \frac{1}{2} |\partial_\alpha|w_\alpha|^2|^2
\]

This proves that

\[
d\sigma(w) \leq \left(\frac{1}{4} + \frac{|w_\alpha||p|}{\partial_\alpha|w_\alpha|^2}\right) udud\theta. \tag{3.17}
\]

Recalling (3.9), together with the definition of \(E(w_\alpha)\), we have

\[
\frac{1}{\partial_\alpha|w_\alpha|^2} = \frac{(\kappa_1 + 2\kappa_2|p - w_\alpha|^2)E(w_\alpha) + (\kappa_1 + 2\kappa_2|w_\alpha|^2)E(w_\alpha - p)}{2(\kappa_1 + 2\kappa_2|w_\alpha - p|^2)E(w_\alpha)|p|^2}.
\]

We consider two cases. First, for \(|w_\alpha| \leq |w_\alpha - p|\), together with the fact that \(E(w_\alpha) + E(w_\alpha - p) = E(p)\), the above gives

\[
\frac{1}{\partial_\alpha|w_\alpha|^2} \leq \frac{1}{2|p|^2} + \frac{E(p)}{2E(w_\alpha)|p|^2} \leq \frac{1}{2|p|^2} + \frac{E(p)}{2\sqrt{\kappa_1|w_\alpha||p|^2}}.
\]

Whereas for \(|w_\alpha - p| \leq |w_\alpha|\), using the fact that \(\mathcal{E}'(w_\alpha)/|w_\alpha|\) is decreasing in \(|w_\alpha|\), we have

\[
\frac{1}{\partial_\alpha|w_\alpha|^2} \leq \frac{1}{2|p|^2} + \frac{(\kappa_1 + 2\kappa_2|w_\alpha|^2)E(w_\alpha - p)}{2(\kappa_1 + 2\kappa_2|w_\alpha - p|^2)E(w_\alpha)|p|^2} \leq \frac{1}{|p|^2}.
\]

Combining these into (3.17), we obtain

\[
d\sigma(w) \leq c_0(1 + |p|) udud\theta
\]

for some positive constants \(c_1, c_2\). This yields the upper bound on the surface integral. \(\blacksquare\)

**Lemma 3.2** Let \(S_\delta'\) be defined as in (3.2) and \(F\) be an arbitrary function satisfying

\[
\int_{\mathbb{R}^+} |u|F(u)du < \infty.
\]

There is a positive constant \(C_0\) so that

\[
\int_{S_\delta'} F(|w|) d\sigma(w) \leq C_0\|uF(\cdot)\|_{L^1(\mathbb{R}^+)} \tag{3.18}
\]

uniformly in \(p \in \mathbb{R}^3\). In addition, for any positive \(\delta\), there exists a \(c_\delta\) so that

\[
\int_{S_\delta'} F(|w|) d\sigma(w) \geq c_\delta \int_0^\infty \min\{\delta, |p| + u\} F(u) u \, du, \tag{3.19}
\]

for all \(p \in \mathbb{R}^3\). In particular, when \(|p| \geq \delta\), the above surface integral is bounded below by \(c_\delta\|uF(\cdot)\|_{L^1(\mathbb{R}^+)}\).
Remark 3.1 In case when $|p| \to 0$, the surface $S'_p$ tends to the two dimensional plane orthogonal to $e_p$, the direction to which $p$ tends to zero. Hence it is reasonable for the lower bound (3.19) to remain nonzero in the limit of $|p| \to 0$.

Proof [Proof of Lemma 3.2] Recall that $S'_p$ is the surface that consists of $w$ satisfying $\mathcal{E}(p + w) = \mathcal{E}(w) + \mathcal{E}(p)$. First, we compute

$$0 = \mathcal{E}(p + w)^2 - \left(\mathcal{E}(p) + \mathcal{E}(w)\right)^2$$

$$= \kappa_1 |p + w|^2 + \kappa_2 |p + w|^4 - \kappa_1 (|p|^2 + |w|^2)$$

$$- \kappa_2 (|p|^4 + |w|^4) - 2\mathcal{E}(p)\mathcal{E}(w)$$

$$= 2\kappa_1 w \cdot p + 2\kappa_2 w \cdot (|p|^2 + |w|^2 + |p + w|^2)$$

$$+ 2\kappa_2 |p|^2 |w|^2 - 2\mathcal{E}(p)\mathcal{E}(w).$$

It is clear that $\kappa_2 |p|^2 |w|^2 < \mathcal{E}(p)\mathcal{E}(w)$, since $\kappa_1 \neq 0$. This proves that if $w \in S'_p \setminus \{0\}$, then $w \cdot p > 0$. To describe the surface $S'_p$, we set $\overline{G}(w) := \mathcal{E}(p + w) - \mathcal{E}(w) - \mathcal{E}(p)$, and compute

$$\nabla_w \overline{G} = \frac{p + w}{|p + w|} \mathcal{E}'(|p + w|) - \frac{w}{|w|} \mathcal{E}'(|w|),$$

where

$$\mathcal{E}'(|\varrho|) = \frac{2\kappa_1 + 4\kappa_2 |\varrho|^2}{\sqrt{\kappa_1 + \kappa_2 |\varrho|^2}}.$$ 

Let us first evaluate the derivative at $w = \alpha p$ with $\alpha \in \mathbb{R}_+$. This yields

$$\partial_\alpha \overline{G} = \partial_\alpha w \cdot \nabla_w \overline{G}|_{w = \alpha p} = |p|\mathcal{E}'((1 + \alpha)p) - |p|\mathcal{E}'(\alpha p).$$

By using the convexity of the energy $\mathcal{E}(p)$ with respect to the length $|p|$, it follows that $\partial_\alpha \overline{G} > 0$ for all $\alpha > 0$. Since $\overline{G}(0) = 0$, $\overline{G}(\alpha p) > 0$ for all positive $\alpha$.

Now, let us consider the plane that consists of all the points $w_\alpha = \alpha p + q$, with $q \cdot p = 0$, for each fixed $\alpha > 0$. The directional derivative of $\overline{G}$ at $w_\alpha = \alpha p + q$ in the direction of $q \neq 0$ satisfies

$$q \cdot \nabla_w \overline{G} = |q|^2 \left[ \frac{\mathcal{E}'(p + w_\alpha)}{|p + w_\alpha|} - \frac{\mathcal{E}'(w_\alpha)}{|w_\alpha|} \right] < 0$$

in which we used the fact that $\mathcal{E}'(p)/|p|$ is strictly decreasing in $|p|$. By a view of (3.20), the sign of $\overline{G}(w)$, with $w_\alpha = \alpha p + q$, is the same as that of

$$\alpha |p|^2 \left( \kappa_1 + \kappa_2 (|p|^2 + |w_\alpha|^2 + |p + w_\alpha|^2) \right) + \kappa_2 |p|^2 |w_\alpha|^2 - \mathcal{E}(p)\mathcal{E}(w)$$

$$= \alpha |p|^2 \left( \kappa_1 + 2\kappa_2 (|p|^2 + \alpha |p|^2 + |w_\alpha|^2) \right)
$$

$$- \frac{(\kappa_1 |p|^2 + \kappa_2 |p|^4)(\kappa_1 |w_\alpha|^2 + \kappa_2 |w_\alpha|^4) - \kappa_2 |p|^4 |w_\alpha|^4}{\sqrt{\kappa_1 |p|^2 + \kappa_2 |p|^4} \sqrt{\kappa_1 |w_\alpha|^2 + \kappa_2 |w_\alpha|^4} + \kappa_2 |p|^2 |w_\alpha|^2}$$

$$= \alpha |p|^2 \left( \kappa_1 + 2\kappa_2 (|p|^2 + \alpha |p|^2 + |w_\alpha|^2) \right)
$$

$$- \frac{\kappa_2 |w_\alpha|^2 |p|^2 + \kappa_1 \kappa_2 |w_\alpha|^2 |p|^4 + \kappa_1 \kappa_2 |p|^2 |w_\alpha|^4}{\sqrt{\kappa_1 |p|^2 + \kappa_2 |p|^4} \sqrt{\kappa_1 |w_\alpha|^2 + \kappa_2 |w_\alpha|^4} + \kappa_2 |p|^2 |w_\alpha|^2}.$$
This yields that $G(\alpha p + q) < 0$ as long as

$$\alpha < \frac{\kappa_1(\kappa_1 + \kappa_2|p|^2) + \kappa_1\kappa_2|w_\alpha|^2}{\sqrt{\kappa_1|p|^2 + \kappa_2|p|^2} \sqrt{\kappa_1|w_\alpha|^2 + \kappa_2 + \kappa_2|p|^2} \left(\kappa_1 + 2\kappa_2(|p|^2 + \alpha|p|^2 + |w_\alpha|^2)\right)}.$$

Taking $|q| \to \infty$ (and so $|w_\alpha| \to \infty$), we obtain that $\lim_{q \to \infty} G(\alpha p + q) < 0$ if and only if

$$\alpha < \alpha_p := \frac{1}{2\kappa_2|p|^2 + \sqrt{\kappa_2|p|^2 + \kappa_2|p|^2}}.$$

In particular, we note that

$$\alpha_p |p|(1 + |p|) \leq C_0, \quad \forall p \in \mathbb{R}^3$$

for some positive constant $C_0$. Hence, for positive values of $\alpha$ satisfying (3.21), by monotonicity, $G(\alpha p) > 0$, and the fact that $G(\alpha p + q)$ is radial in $|q|$, there is a unique $|q_\alpha|$ so that $G(\alpha p + q) = 0$, for all $|q| = |q_\alpha|$. In addition, from the continuity of $G(w_\alpha)$, $|q_\alpha|$ is continuously differentiable with respect to $\alpha$. For $\alpha \geq \alpha_p$, $G(\alpha p + q) > 0$, for all $q$ so that $q \cdot p = 0$.

To summarize, the surface $S'_p$ is described as follows (see Figure 2):

$$S'_p = \left\{ w(\alpha, \theta) = \alpha p + |q_\alpha|e_\theta : \alpha \in [0, \alpha_p), \theta \in [0, 2\pi] \right\},$$

in which $\alpha_p$ and $|q_\alpha|$ are defined as above and $e_\theta$ denotes the unit vector rotating around $p$ and on the orthogonal plane to $p$. As in (3.16), we have

$$d\sigma(w) = \sqrt{|p|^2|q_\alpha|^2 + \frac{1}{4}|\partial_\alpha(|q_\alpha|^2)|^2} d\alpha d\theta$$

and hence, the surface integral is estimated by

$$\int_{S'_p} F(|w|) d\sigma(w) = \int_{[0,2\pi] \times [0,\alpha_p]} F(|\alpha p + q_\alpha|) \sqrt{|p|^2|q_\alpha|^2 + \frac{1}{4}|\partial_\alpha(|q_\alpha|^2)|^2} d\alpha d\theta.$$
Let us introduce the variable $u = |w_\alpha| = \sqrt{\alpha^2 |p|^2 + |q_\alpha|^2}$. We compute

$$2udu = \partial_\alpha |w_\alpha|^2 d\alpha$$

and hence

$$\int_{S^p} F(|w|) d\sigma(w) \leq 2\pi \int_0^\infty F(u) \frac{\sqrt{|p|^2 |q_\alpha|^2 + \frac{1}{4} |\partial_\alpha (|q_\alpha|^2)|^2}}{2 \partial_\alpha |w_\alpha|^2} u du.$$  \hspace{1cm} (3.25)

Since $\partial_\alpha |w_\alpha|^2 = 2\alpha |p|^2 + \partial_\alpha |q_\alpha|^2$, it is clear that

$$\frac{\sqrt{|p|^2 |q_\alpha|^2 + \frac{1}{4} |\partial_\alpha (|q_\alpha|^2)|^2}}{\partial_\alpha |w_\alpha|^2} \leq \frac{|q_\alpha| |p|}{\partial_\alpha |w_\alpha|^2} + \frac{1}{2} \frac{|\partial_\alpha (|q_\alpha|^2)|}{\partial_\alpha |w_\alpha|^2} \leq \frac{1}{2} + \frac{|q_\alpha| |p| + \alpha |p|^2}{\partial_\alpha |w_\alpha|^2}. \hspace{1cm} (3.26)$$

In the above inequality we have used the inequality $\partial_\alpha |w_\alpha|^2 > 0$. To prove that $\partial_\alpha |w_\alpha|^2 > 0$ and to bound the last term, we recall that $\mathcal{G}(w_\alpha) = 0$ and hence

$$0 = \partial_\alpha w_\alpha \cdot \nabla_w \mathcal{G} = \frac{1}{2} \partial_\alpha |w_\alpha|^2 \left[ \frac{\mathcal{E}'(p + w_\alpha)}{|p + w_\alpha|} - \frac{\mathcal{E}'(w_\alpha)}{|w_\alpha|} \right] + |p|^2 \frac{\mathcal{E}'(p + w_\alpha)}{|p + w_\alpha|}$$

which is the equality

$$|p|^2 \frac{\mathcal{E}'(p + w_\alpha)}{|p + w_\alpha|} = \frac{1}{2} \partial_\alpha |w_\alpha|^2 \left[ \frac{\mathcal{E}'(w_\alpha)}{|w_\alpha|} - \frac{\mathcal{E}'(p + w_\alpha)}{|p + w_\alpha|} \right]. \hspace{1cm} (3.27)$$

We shall give a lower bound and upper bound on the above left-hand side and right-hand side, respectively. Defining

$$G_1(|\varrho|) = \frac{\mathcal{E}'(|\varrho|)}{\varrho} = \frac{\kappa_1 + 2\kappa_2 |\varrho|^2}{\mathcal{E}(\varrho)} = \frac{\kappa_1 + 2\kappa_2 |\varrho|^2}{\sqrt{\kappa_1 |\varrho|^2 + \kappa_2 |\varrho|^4}},$$

it is straightforward that

$$G_1'(|\varrho|) = -G_2(|\varrho|) = -\frac{\kappa_2^2 |\varrho|^2}{\mathcal{E}^3(|\varrho|)} \leq 0.$$

In particular, $G_1(\cdot)$ is a decreasing function, and hence $\partial_\alpha |w_\alpha|^2 > 0$ by a view of (3.27) and the fact that $|w_\alpha| \leq |w_\alpha + p|$ (thanks to the monotonicity of $\mathcal{E}(p)$). In addition, it is easy to check that $G_2$ is also a decreasing function with respect to $|\varrho|$. A direct computation yields

$$0 \leq \frac{\mathcal{E}'(w_\alpha)}{|w_\alpha|} - \frac{\mathcal{E}'(p + w_\alpha)}{|p + w_\alpha|} = \frac{(\kappa_1 + 2\kappa_2 |w_\alpha|^2)\mathcal{E}(w_\alpha + p) - (\kappa_1 + 2\kappa_2 |w_\alpha + p|^2)\mathcal{E}(w_\alpha)}{\mathcal{E}(w_\alpha)\mathcal{E}(w_\alpha + p)} \leq \frac{\kappa_1(\mathcal{E}(w_\alpha + p) - \mathcal{E}(w_\alpha)) + 2\kappa_2 (|w_\alpha|^2 \mathcal{E}(w_\alpha + p) - |w_\alpha + p|^2 \mathcal{E}(w_\alpha))}{\mathcal{E}(w_\alpha)\mathcal{E}(w_\alpha + p)}$$
in which we note that the second term is nonpositive, since \(|w_\alpha| \leq |w_\alpha + p|\). Whereas, we estimate the first term by

\[
\frac{\kappa_1 (E(w_\alpha + p) - E(w_\alpha))}{E(w_\alpha)E(w_\alpha + p)} = \frac{\kappa_1 (|w_\alpha + p|^2 - |w_\alpha|^2)(\kappa_1 + \kappa_2 |w_\alpha|^2 + \kappa_2 |w_\alpha + p|^2)}{E(w_\alpha)E(w_\alpha + p)(E(w_\alpha + p) + E(w_\alpha))}
\leq \kappa_1 |p| (|w_\alpha + p| + |w_\alpha|)(\kappa_1 + \kappa_2 |w_\alpha|^2 + \kappa_2 |w_\alpha + p|^2)
\leq \sqrt{\kappa_1} |p| (\kappa_1 + 2\kappa_2 |w_\alpha + p|^2)
\leq \sqrt{\kappa_1} |p| \frac{E'(p + w_\alpha)}{E(w_\alpha)} \frac{E'(p + w_\alpha)}{|p + w_\alpha|}
\]

where we have used the triangle inequality, and again the inequality \(|w_\alpha| \leq |w_\alpha + p|\). Putting this estimate into (3.27), we obtain

\[
|p|^2 \frac{E'(p + w_\alpha)}{|p + w_\alpha|} \leq \frac{1}{2} \partial_\alpha |w_\alpha|^2 \frac{\sqrt{\kappa_1} |p| E'(p + w_\alpha)}{E(w_\alpha)} \frac{E'(p + w_\alpha)}{|p + w_\alpha|}
\]

or equivalently,

\[
\partial_\alpha |w_\alpha|^2 \geq \frac{2|p|E(|w_\alpha|)}{\sqrt{\kappa_1}} \geq 2|p||w_\alpha|
\]

upon using the fact that \(E(w_\alpha) \geq \sqrt{\kappa_1} |w_\alpha|\). Hence, by a view of (3.25) and (3.26), we have obtained

\[
\int_{S_p} F(|w_\alpha|) \, d\sigma(w_\alpha) \leq \pi \int_0^\infty F(u) \left( \frac{1}{2} + \frac{|q_0||p| + \alpha|p|^2}{\partial_\alpha |w_\alpha|^2} \right) u \, du
\leq \pi \int_0^\infty F(u) \left( \frac{1}{2} + \frac{|q_0||p| + \alpha|p|^2}{2|p||w_\alpha|} \right) u \, du
\leq \frac{3\pi}{2} \int_0^\infty F(u) u \, du
\]

upon using the fact that \(|w_\alpha|^2 = |q_0|^2 + \alpha^2 |p|^2\). This proves the claimed upper bound in (3.18) on the surface integral.

Finally, we prove the the lower bound on the surface integral: for any positive \(\delta\), there exists a \(c_\delta\) so that

\[
\int_{S_p} F(|w|) \, d\sigma(w) \geq c_\delta \int_0^\infty \min \{\delta, |p| + u\} F(u) u \, du, \quad \forall \ p \in \mathbb{R}^3. \tag{3.28}
\]
Directly from (3.24) and (3.25), we have
\[
\int_{S_p} F(|w|) \, d\sigma(w) = \int_{[0,2\pi] \times [0,\alpha_p]} F(|\alpha p + q_\alpha|) \sqrt{|p|^2 |q_\alpha|^2 + \frac{1}{4} |\partial_\alpha(|q_\alpha|^2)|^2} \, d\theta d\phi
\]
\[
= 2\pi \int_0^\infty F(u) \frac{\sqrt{|p|^2 |q_\alpha|^2 + \frac{1}{4} |\partial_\alpha(|q_\alpha|^2)|^2}}{2\partial_\alpha|w_\alpha|^2} \, u \, du,
\]
in which the change of variable \( u = |w_\alpha| = \sqrt{\alpha^2|p|^2 + |q_\alpha|^2} \) was made. Set
\[
I_\alpha := \frac{\sqrt{|p|^2 |q_\alpha|^2 + \frac{1}{4} |\partial_\alpha(|q_\alpha|^2)|^2}}{\partial_\alpha|w_\alpha|^2}. \tag{3.29}
\]
It suffices to give a lower bound on \( I_\alpha \). We note that
\[
0 \leq \partial_\alpha|w_\alpha|^2 = 2\alpha|p|^2 + \partial_\alpha|q_\alpha|^2 \leq 2\alpha|p|^2 + |\partial_\alpha|q_\alpha|^2|.
\]
On the other hand, we have
\[
\sqrt{|p|^2 |q_\alpha|^2 + \frac{1}{4} |\partial_\alpha(|q_\alpha|^2)|^2} \geq c_0 \left( |p||q_\alpha| + |\partial_\alpha(|q_\alpha|^2)| \right).
\]
This yields
\[
I_\alpha \geq c_0 \frac{|p||q_\alpha| + |\partial_\alpha(|q_\alpha|^2)|}{2\alpha|p|^2 + |\partial_\alpha|q_\alpha|^2|}. \tag{3.30}
\]
It is then clear that \( I_\alpha \geq 1 \) in the case when \( |q_\alpha| \gtrsim \alpha|p| \). It remains to check the validity of this latter inequality. Note that \(|p + w_\alpha|^2 \geq |w_\alpha|^2 + |p|^2\), since \( p \cdot \alpha > 0 \). In addition, from (3.20), we have
\[
0 < w_\alpha \cdot p = \frac{\mathcal{E}(p)\mathcal{E}(w_\alpha) - \kappa_2|p|^2|w_\alpha|^2}{\kappa_1 + \kappa_2(|p|^2 + |w_\alpha|^2 + |p + w_\alpha|^2)}
\]
\[
\leq \frac{\mathcal{E}(p)\mathcal{E}(w_\alpha)}{\kappa_1 + 2\kappa_2|p|^2 + 2\kappa_2|w_\alpha|^2}
\]
\[
\leq \frac{|p||w_\alpha|}{\sqrt{\kappa_1 + \kappa_2|w_\alpha|^2}} \frac{\sqrt{\kappa_1 + \kappa_2|w_\alpha|^2}}{\sqrt{\kappa_1 + 2\kappa_2|p|^2 + 2\kappa_2|w_\alpha|^2}}.
\]
This proves
\[
\cos \theta \leq \frac{\sqrt{\kappa_1 + \kappa_2|w_\alpha|^2}}{\sqrt{\kappa_1 + 2\kappa_2|p|^2 + 2\kappa_2|w_\alpha|^2}}
\]
for \( \theta \) being the angle between \( w_\alpha \) and \( p \). In addition,
\[
\sin^2 \theta = 1 - \cos^2 \theta \geq \frac{2\kappa_2|p|^2 + \kappa_2|w_\alpha|^2}{\kappa_1 + 2\kappa_2|p|^2 + 2\kappa_2|w_\alpha|^2}.
\]
Combining, we obtain
\[
\frac{\alpha|p|}{|q_\alpha|} = \frac{\cos \theta}{\sin \theta} \leq \frac{\sqrt{\kappa_1 + \kappa_2|w_\alpha|^2}}{\sqrt{2\kappa_1|p|^2 + \kappa_2|w_\alpha|^2}},
\]
(3.31)
for all \( p \) and all \( w_\alpha \in S'_p \).

**Case 1:** \(|p| \gtrsim 1\). In this case, the estimate (3.31) yields \(|q_\alpha| \gtrsim \alpha|p|\). This and (3.30) implies that \( I_\alpha \gtrsim 1 \), and hence, the inequality (3.28) is valid:
\[
\int_{S'_p} F(|w|)d\sigma(w) \geq c_0 \int_0^\infty F(u)u \, du,
\]
(3.32)
for all \(|p| \gtrsim 1\).

**Case 2:** \(|p| \ll 1\). In this case, if in addition \(|w_\alpha| \gtrsim 1\), it follows from (3.31) that \(|q_\alpha| \gtrsim \alpha|p|\), and hence \( I_\alpha \gtrsim 1 \). This proves that for any fixed positive \( \delta \), there is a \( c_\delta \) so that
\[
\int_{S'_p} F(|w|)d\sigma(w) \geq c_\delta \int_\delta^\infty F(u)u \, du,
\]
(3.33)
for all \( p \) (recalling that the bound is already valid for large \( p \)). It remains to study the case when \(|w_\alpha| \ll 1\). In this case, (3.31) yields
\[
|q_\alpha| \gtrsim \alpha|p|(|p| + |w_\alpha|).
\]
(3.34)
This proves \( I_\alpha \gtrsim |p| + |w_\alpha| \), for small \( p \) and \(|w_\alpha|\). Precisely, for any positive \( \delta \), there exists a \( c_\delta \) so that
\[
\int_{S'_p} F(|w|)d\sigma(w) \geq c_\delta \int_0^\delta \min\{\delta, |p| + u\} F(u)u \, du,
\]
(3.35)
for all \( p \). The lemma follows.

**Lemma 3.3** Let \( S''_p \) be defined as in (3.2) and \( F \) be an arbitrary function satisfying
\[
\int_{\mathbb{R}^+} |u|F(u)du < \infty.
\]
There are positive constants \( C_0 \) so that
\[
\int_{S''_p} F(|w|) \, d\sigma(w) \leq C_0 \|uF(\cdot)\|_{L^1(\mathbb{R}^+)}
\]
uniformly in \( p \in \mathbb{R}^3 \).
Proof We observe that

\[
S''_p = \{ p_* \mid E(p_*) = E(p) + E(p_* - p) \} \\
= \{ p_* + p \mid E(p_* + p) = E(p) + E(p_*) \} \\
= p + S'_p.
\]

The above identity shows that the same argument of Lemma 3.2 can be applied, yielding the lemma.

4 Moment estimates

In this section, we shall derive estimates on the energy moment and on the mass of nonnegative solutions of (1.1). In order to obtain the boundedness of the total mass (Proposition 4.3), which is crucial in the proof of the main theorem, we are obliged to bound the third order energy moment (Proposition 4.2), which in turn requires estimates on the second-order energy moment (Proposition 4.1).

4.1 Propagation and creation of second order energy moment

Proposition 4.1 Assume that initial data \( f_0(p) = f_0(|p|) \) have finite energy:

\[
\int_{\mathbb{R}^3} f_0(p) E(p) dp < +\infty.
\]

Then, nonnegative radially symmetric solutions \( f(t, p) \) of (1.1) with \( f(0, p) = f_0(p) \) satisfy

\[
\sup_{t \in [\tau, \infty)} \int_{\mathbb{R}^3} f(t, p) E^2(p) dp < +\infty
\]

for any \( \tau > 0 \). In particular, (4.2) holds for \( \tau = 0 \), if additionally

\[
\int_{\mathbb{R}^3} f_0(p) E^2(p) dp < +\infty.
\]

Proof In order to prove that the second-order energy moment is bounded, let us start with taking \( \varphi = E^2(p) \) to be the test function in Lemma 2.1, yielding

\[
\frac{d}{dt} \int_{\mathbb{R}^3} f E^2 dp = \iint_{\mathbb{R}^9} K(p, p_1, p_2)[f(p_1) f(p_2) - f(p) - f(p_1) f(p_2) - f(p) f(p_2)] \times \\
\times [E^2(p) - E^2(p_1) - E^2(p_2)] dpdp_1 dp_2 = \iint_{\mathbb{R}^9} K(p, p_1, p_2)[f(p_1) f(p_2) - f(p) - 2f(p) f(p_1)] \times \\
\times [E^2(p) - E^2(p_1) - E^2(p_2)] dpdp_1 dp_2
\]
Thus, with recalling

By Lemma 3.2, and the fact that \( f \)

For convenience, we set

and hence \( f(t, p) \) satisfies

We start with estimates on \( J_1 \). Recalling \( S_{p_1}^\prime \) to be the surface in \( \mathbb{R}^3 \) that consists of \( p_2 \) so that \( \mathcal{E}(p_1 + p_2) = \mathcal{E}(p_1) + \mathcal{E}(p_2) \), for each \( p_1 \), we can write the integral \( J_1 \) as

By Lemma 3.2 and the fact that \( f \) is radial, the surface integral is estimated by

Thus, with recalling \( \mathcal{E}(p) = \sqrt{\kappa_1|p|^2 + \kappa_2|p|^4} \geq |p| \), we obtain

\[
J_1 \leq C \int_{\mathbb{R}^6} |p_1|(|p_1| + |p_2|) \mathcal{E}(p_1) f(p_1) \mathcal{E}(p_2) dp_1 dp_2 \\
\leq C \left( \int_{\mathbb{R}^3} \mathcal{E}(p_1) + \mathcal{E}(p_1) \mathcal{E}(p_2) \right) f(p_1) \mathcal{E}(p_1) f(p_2) \mathcal{E}(p_2) dp_1 dp_2 \\
\leq C \left( \int_{\mathbb{R}^3} \mathcal{E}(p_1)^2 f(p_1) dp_1 \right) \left( \int_{\mathbb{R}^3} \mathcal{E}(p_2) f(p_2) dp_2 \right) + \left( \int_{\mathbb{R}^3} \mathcal{E}(p_1)^2 f(p_1) dp_1 \right)^2 .
\]
By using the conservation of energy: Corollary 2.1 and the assumption (4.1), the above yields
\[ J_1 \leq C \left( \int_{\mathbb{R}^3} f \mathcal{E}^2 dp \right)^2 + C \int_{\mathbb{R}^3} f \mathcal{E}^2 dp. \] (4.4)

Next, we estimate the integral \( J_2 \) in (4.3). Similarly, recalling \( S_p = \{ \mathcal{E}(p) = \mathcal{E}(p_1) + \mathcal{E}(p - p_1) \} \), we can write
\[ J_2 = -2 \int \int \int_{\mathbb{R}^3} \mathcal{K}(p, p_1, p_2) [f(p) + 2f(p)f(p_1)] \mathcal{E}(p_1) \mathcal{E}(p_2) dp dp_1 dp_2 \]
\[ = -2 \int_{\mathbb{R}^6} \mathcal{K}(p, p_1, p - p_1) [f(p) + 2f(p)f(p_1)] \mathcal{E}(p_1) \mathcal{E}(p - p_1) dp dp_1 \]
\[ = -2 \int_{\mathbb{R}^3} \int_{S_p} \kappa_0 ||p|| |p - p_1| [f(p) + 2f(p)f(p_1)] \mathcal{E}(p_1) \mathcal{E}(p - p_1) d\sigma(p_1) dp \]
\[ \leq -2\kappa_0 \int_{\mathbb{R}^3} \left( \int_{S_p} |p||p - p_1| \mathcal{E}(p_1) \mathcal{E}(p - p_1) d\sigma(p_1) \right) |p| f(p) dp \]
in which we recall that \( f \geq 0 \). We first estimate the surface integral. We observe that the product \( \mathcal{E}(p_1) \mathcal{E}(p - p_1) \) can be estimated as
\[ \mathcal{E}(p_1) \mathcal{E}(p - p_1) = \sqrt{\kappa_1 |p_1|^2 + \kappa_2 |p_1|^4} \sqrt{\kappa_1 |p - p_1|^2 + \kappa_2 |p - p_1|^4} \]
\[ \geq \theta_0 (|p_1||p - p_1| + |p_1|^2|p - p_1|^2) \]
\[ \geq \theta_0 (|p_1||p| - |p_1| + |p_1|^2||p| - |p_1||^2). \]
Hence, Lemma 3.1 yields
\[ \int_{S_p} |p||p - p_1| \mathcal{E}(p_1) \mathcal{E}(p - p_1) d\sigma(p_1) \]
\[ \geq \theta_0 \int_{S_p} (|p_1|^2||p| - |p_1||^2 + |p_1|^3||p| - |p_1||^3) d\sigma(p_1) \]
\[ \geq \theta_0 (|p|^6 + |p|^8) \min\{1, |p|\} \]
\[ \geq \theta_0 |p|^7, \]
for all \( p \in \mathbb{R}^3 \), and for some universal positive constant \( \theta_0 \). This proves
\[ J_2 \leq -\theta_0 \int_{\mathbb{R}^3} |p|^8 f(p) dp. \] (4.6)

We wish to replace the right-hand side by the second-order energy moment. To this end, using the Hölder inequality, we estimate
\[ \int_{\mathbb{R}^3} \mathcal{E}^2(p) f(p) dp \leq C \int_{\mathbb{R}^3} |p|^2 f(p) dp + C \int_{\mathbb{R}^3} |p|^4 f(p) dp \]
\[ \leq C \int_{\mathbb{R}^3} |p|^2 f(p) dp + C \left( \int_{\mathbb{R}^3} |p|^8 f(p) dp \right)^{\frac{1}{4}} \left( \int_{\mathbb{R}^3} |p|^2 f(p) dp \right)^{\frac{3}{4}}. \]
By definition, \( E(p) \gtrsim |p|^2 \). This and the conservation of energy prove that the second moment \( \int_{\mathbb{R}^3} |p|^2 f(p) dp \) remains bounded, yielding
\[
\int_{\mathbb{R}^3} \mathcal{E}^2(p) f(p) \, dp \leq C \left( \int_{\mathbb{R}^3} |p|^2 f(p) \, dp \right)^{\frac{3}{4}} + C \left( \int_{\mathbb{R}^3} |p|^8 f(p) \, dp \right)^{\frac{1}{4}}.
\]
Here again we used \( E(p) \gtrsim |p| \). Putting this back into (4.6), we obtain at once
\[
J_2 \leq -\theta_0 \left( \int_{\mathbb{R}^3} \mathcal{E}^2(p) f(p) \, dp \right)^{\frac{3}{4}} + C \int_{\mathbb{R}^3} \mathcal{E}^2(p) f(p) \, dp.
\]
(4.7)

Combining (4.4) and (4.7) into (4.3), we have obtained
\[
\frac{d}{dt} \int_{\mathbb{R}^3} f \mathcal{E}^2 dp \leq C \left( \int_{\mathbb{R}^3} f \mathcal{E}^2 dp \right)^2 + C \int_{\mathbb{R}^3} f \mathcal{E}^2 dp - \theta_0 \left( \int_{\mathbb{R}^3} f \mathcal{E}^2 dp \right)^{\frac{3}{2}}.
\]
Since \( a^2 \leq \epsilon a^3 + \frac{1}{\epsilon} a \) for arbitrary positive numbers \( a, \epsilon \), the above is further simplified into
\[
\frac{d}{dt} \int_{\mathbb{R}^3} f \mathcal{E}^2 dp \leq C_1 \int_{\mathbb{R}^3} f \mathcal{E}^2 dp \left[ 1 - \theta_1 \left( \int_{\mathbb{R}^3} f \mathcal{E}^2 dp \right)^2 \right]
\]
(4.8)
for some positive constants \( C_1, \theta_1 \). Since \( f \geq 0 \), the standard ODE argument applying to the differential inequality (4.8) yields at once the boundedness of \( \int_{\mathbb{R}^3} f \mathcal{E}^2 dp \); for instance, there holds
\[
\int_{\mathbb{R}^3} f(t,p) \mathcal{E}^2 dp \leq \max \left\{ \frac{1}{\sqrt{\theta_1}}, \int_{\mathbb{R}^3} f(\tau,p) \mathcal{E}^2 dp \right\}
\]
for all \( t \geq \tau \). The proposition follows.

4.2 Propagation and creation of third order energy moment

**Proposition 4.2** Assume that initial data \( f_0(p) = f_0(|p|) \) have finite energy
\[
\int_{\mathbb{R}^3} f_0(p) \mathcal{E}(p) dp < +\infty.
\]
Then, nonnegative radially symmetric solutions \( f(t,p) \) of (1.1) with \( f(0,p) = f_0(p) \) satisfy
\[
\sup_{t \in [\tau, \infty)} \int_{\mathbb{R}^3} f(t,p) \mathcal{E}^3(p) dp < +\infty
\]
(4.9)
for any \( \tau > 0 \). In particular, (4.9) holds for \( \tau = 0 \), if additionally
\[
\int_{\mathbb{R}^3} f_0 \mathcal{E}^3 dp < +\infty.
\]
Proof Without loss of generality, we assume that the second order energy moment
\[ \int_{\mathbb{R}^3} f_0(p) \mathcal{E}^2(p) dp \]
is finite. Indeed, the second moment is instantly created in time, thanks to Proposition 4.1. Thus, we could shift the time by \( \tau \) (from Proposition 4.1) to zero, and the above assumption is valid. In the case when the third moment is initially finite, so is the second moment, since
\[ \mathcal{E}^2 \leq \mathcal{E} + \mathcal{E}^3. \]

Similarly to the proof of Proposition 4.1, we choose \( \varphi = \mathcal{E}^3(p) \) to be the test function in Lemma 2.1 yielding
\[
\frac{d}{dt} \int_{\mathbb{R}^3} f \mathcal{E}^3 dp = \int \int \int_{\mathbb{R}^9} \mathcal{K}(p, p_1, p_2)[f(p_1)f(p_2) - f(p)f(p_1)] \\
\times [\mathcal{E}^3(p) - \mathcal{E}^3(p_1) - \mathcal{E}^3(p_2)] dp dp_1 dp_2 \\
= \int \int \int_{\mathbb{R}^9} \mathcal{K}(p, p_1, p_2)[f(p_1)f(p_2) - f(p)f(p_1)] \\
\times 3\mathcal{E}(p_1)\mathcal{E}(p_2)\mathcal{E}(p) dp dp_1 dp_2,
\]
in which we have used \( \mathcal{E}(p) = \mathcal{E}(p_1) + \mathcal{E}(p_2) \). As done in the previous section for \( J_1, J_2 \) (see (4.3)), we set
\[
H_1 : = 3 \int \int \int_{\mathbb{R}^9} \mathcal{K}(p, p_1, p_2)f(p_1)f(p_2)\mathcal{E}(p_1)\mathcal{E}(p_2)\mathcal{E}(p) dp dp_1 dp_2 \\
= 3 \int \int \mathcal{K}_0|p_1 + p_2||p_1||p_2|f(p_1)f(p_2)\mathcal{E}(p_1)\mathcal{E}(p_2)\mathcal{E}(p_1 + p_2) d\sigma(p_2) dp_1 \\
H_2 : = -3 \int \int \int_{\mathbb{R}^9} \mathcal{K}(p, p_1, p_2)[f(p) + 2f(p_1)f(p_1)]\mathcal{E}(p_1)\mathcal{E}(p_2)\mathcal{E}(p) dp dp_1 dp_2 \\
\leq -3 \int \int \mathcal{K}_0|p||p - p_1|f(p)\mathcal{E}(p_1)\mathcal{E}(p - p_1)\mathcal{E}(p) d\sigma(p_1) dp
\]
and hence
\[
\frac{d}{dt} \int_{\mathbb{R}^3} f \mathcal{E}^3 dp = H_1 + H_2 \quad (4.10)
\]

It remains to give estimates on \( H_1 \) and \( H_2 \). We start with the surface integral in \( H_1 \). Thanks to Lemma 3.2, for any radial function \( G(p_2) \), we estimate
\[
\int_{S_{p_1}} |p_1 + p_2||p_1||p_2|G(p_2) d\sigma(p_2) \leq C \int_{\mathbb{R}^+} (|p_1| + |p_2|)|p_1||p_2|^2 G(p_2) d(|p_2|) \\
\leq C \int_{\mathbb{R}^3} (|p_1| + |p_2|)|p_1|G(p_2) dp_2 \\
\leq C \int_{\mathbb{R}^3} \mathcal{E}(p_1)(1 + \mathcal{E}(p_2))G(p_2) dp_2
\]
in which the last inequality was due to the inequality $\mathcal{E}(p) \gtrsim |p| + |p|^2$. Using this with the radial function $G(p_2) = 3k_0 f(p_1) f(p_2) \mathcal{E}(p_1) \mathcal{E}(p_2)(\mathcal{E}(p_1) + \mathcal{E}(p_2))$, we obtain
\[
H_1 \leq C \int_{\mathbb{R}^6} f(p_1) f(p_2) \mathcal{E}(p_1)^2 \mathcal{E}(p_2)(\mathcal{E}(p_1) + \mathcal{E}(p_2))(1 + \mathcal{E}(p_2)) dp_1 dp_2
\]
\[
\leq C \int_{\mathbb{R}^6} f(p_1) f(p_2) \left[ \mathcal{E}^3(p_1) \mathcal{E}(p_2) + \mathcal{E}^2(p_1) \mathcal{E}^2(p_2) + \mathcal{E}^3(p_1) \mathcal{E}^2(p_2) + \mathcal{E}^2(p_1) \mathcal{E}^3(p_2) \right] dp_1 dp_2.
\]
This yields
\[
H_1 \leq C \left( \int_{\mathbb{R}^3} f(\mathcal{E} + \mathcal{E}^2) dp \right) \left( \int_{\mathbb{R}^3} f \mathcal{E}^3 dp \right) + C \left( \int_{\mathbb{R}^3} f \mathcal{E}^2 dp \right)^2.
\]
Applying the Hölder inequality: \( (\int_{\mathbb{R}^3} f \mathcal{E}^2 dp)^2 \leq (\int_{\mathbb{R}^3} f \mathcal{E} dp) \left( \int_{\mathbb{R}^3} f \mathcal{E}^3 dp \right) \), using the conservation of energy, and recalling the boundedness of the second-order energy moment obtained in Proposition 4.1, we obtain at once
\[
H_1 \leq C \int_{\mathbb{R}^3} f \mathcal{E}^3 dp.
\]

Next, we give estimates on $H_2$ in (4.10). Using the estimate (4.5) on the surface integral over $S_p$, we obtain
\[
H_2 \leq -3k_0 \int_{\mathbb{R}^3} \left( \int_{S_p} |p_1||p - p_1| \mathcal{E}(p_1) \mathcal{E}(p - p_1) d\sigma(p_1) \right) |p| f(p) \mathcal{E}(p) dp
\]
\[
\leq -\theta_0 \int_{\mathbb{R}^3} |p|^8 f(p) \mathcal{E}(p) dp
\]
\[
\leq -\theta_0 \int_{\mathbb{R}^3} |p|^{10} f(p) dp
\]
in which the inequality $\mathcal{E}(p) \gtrsim |p|^2$ was used. As before, we wish to insert the third-order energy moment into the right-hand side of the above inequality. To this end, we estimate
\[
\int_{\mathbb{R}^3} \mathcal{E}^3(p) f(p) dp \leq C \int_{\mathbb{R}^3} |p|^3 f(p) dp + C \int_{\mathbb{R}^3} |p|^6 f(p) dp
\]
in which the Hölder inequality and the fact that $\mathcal{E}(p) \gtrsim |p| + |p|^2$ yield
\[
\int_{\mathbb{R}^3} |p|^3 f(p) dp \leq \left( \int_{\mathbb{R}^3} \mathcal{E}^3(p) f(p) dp \right)^{\frac{1}{3}} \left( \int_{\mathbb{R}^3} \mathcal{E}^2(p) f(p) dp \right)^{\frac{1}{2}}
\]
\[
\int_{\mathbb{R}^3} |p|^6 f(p) dp \leq \left( \int_{\mathbb{R}^3} |p|^{10} f(p) dp \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^3} |p|^2 f(p) dp \right)^{\frac{1}{2}}.
\]
Thanks to the boundedness of the second-order energy moment obtained in Proposition 4.1, we thus obtain
\[
\int_{\mathbb{R}^3} \mathcal{E}^3(p) f(p) dp \leq C \left( \int_{\mathbb{R}^3} \mathcal{E}^3(p) f(p) dp \right)^{\frac{1}{2}} + C \left( \int_{\mathbb{R}^3} |p|^{10} f(p) dp \right)^{\frac{1}{2}}
\]
and hence
\[ H_2 \leq C \int_{\mathbb{R}^3} \mathcal{E}^3(p)f(p) \, dp - \theta_0 \left( \int_{\mathbb{R}^3} \mathcal{E}^3(p)f(p) \, dp \right)^2. \] (4.12)

Putting (4.11) and (4.12) into (4.10), we obtain
\[ \frac{d}{dt} \int_{\mathbb{R}^3} f \mathcal{E}^3 \, dp \leq C \int_{\mathbb{R}^3} f \mathcal{E}^3 \, dp \left[ 1 - \theta_0 \int_{\mathbb{R}^3} f \mathcal{E}^3 \, dp \right] \]
for some positive universal constants $C, \theta_0$, which yields the boundedness of $\int_{\mathbb{R}^3} f \mathcal{E}^3 \, dp$.

### 4.3 Bound of the mass

**Proposition 4.3** Suppose that initial data $f_0(p) = f_0(|p|)$ satisfy
\[ \int_{\mathbb{R}^3} f_0(p) \, dp < \infty, \quad \int_{\mathbb{R}^3} f_0(p) \mathcal{E}(p) \, dp < +\infty. \]
For any small time $\tau > 0$, there exists a constant $C_\tau$ depending on $\tau$, such that the mass of nonnegative solutions of (1.1) satisfies
\[ \int_{\mathbb{R}^3} f(t, p) \, dp \leq C_\tau \] (4.13)
for all $t \geq \tau$. Moreover, (4.13) is true for all $t \geq 0$, if in addition the third-order energy moment of the initial data is also bounded
\[ \int_{\mathbb{R}^3} f_0(p) \mathcal{E}^3(p) \, dp < \infty. \]

**Proof** We take $\varphi = 1$ to be the test function in Lemma 2.1, yielding
\[ \frac{d}{dt} \int_{\mathbb{R}^3} f \, dp = \iiint_{\mathbb{R}^3} \mathcal{K}(p, p_1, p_2) \left[ -f(p_1)f(p_2) + f(p) + 2f(p)f(p_1) \right] \, dp \, dp_1 \, dp_2. \]
For convenience, we set
\[ Y_0 := \iiint_{\mathbb{R}^3} \mathcal{K}(p, p_1, p_2) f(p_1)f(p_2) \, dp \, dp_1 \, dp_2 \]
\[ Y_1 := 2 \iiint_{\mathbb{R}^3} \mathcal{K}(p, p_1, p_2) f(p)f(p_1) \, dp \, dp_1 \, dp_2 \]
\[ Y_2 := \iiint_{\mathbb{R}^3} \mathcal{K}(p, p_1, p_2) f(p) \, dp \, dp_1 \, dp_2. \]
We shall give estimates on these $Y_j$. First, recalling that $S'_{p_2}$ is the surface that consists of $p_2$ satisfying $E(p_1 + p_2) = E(p_1) + E(p_2)$, we have

$$Y_0 = \int_{\mathbb{R}^6} K(p_1 + p_2, p_1, p_2) f(p_1) f(p_2) dp_1 dp_2$$

$$= \int_{\mathbb{R}^3} \left( \int_{S'_{p_1}} \kappa_0 |p_1 + p_2| f(p_2) d\sigma(p_2) \right) |p_1| f(p_1) dp_1$$

$$\geq \kappa_0 \int_{\mathbb{R}^3} \left( \int_{S'_{p_1}} |p_2| f(p_2) d\sigma(p_2) \right) |p_1|^2 f(p_1) dp_1,$$

noting that $|p_1 + p_2| \geq |p_1|$, since $p_2 \in S'_{p_1}$.

Let $\delta > 0$ to be determined. By (3.19), there exists a $c_\delta$ so that

$$\int_{S'_{p_1}} |p_2| f(p_2) d\sigma(p_2) \geq c_\delta \int_{\mathbb{R}^3} \min\{\delta, |p_1| + |u|\} |u|^2 f(u) du$$

$$\geq c_\delta \int_{\mathbb{R}^3} \min\{\delta, |p_1| + |p|\} f(p) dp$$

for all $p_1$. In particular, for $|p_1| \geq \delta$, the above reads

$$\int_{S'_{p_1}} |p_2| f(p_2) d\sigma(p_2) \geq c_\delta \int_{\mathbb{R}^3} f(p) dp. \quad (4.14)$$

Hence,

$$Y_0 \geq c_\delta \left( \int_{\mathbb{R}^3} f(p) dp \right) \left( \int_{\{|p| \geq \delta\}} |p|^2 f(p) dp \right). \quad (4.15)$$

Now, thanks to the conservation of momentum and energy, there are positive constants $E_1, E_2$ (independent of time) so that

$$E_1 \leq \int_{\mathbb{R}^3} \sqrt{\kappa_1} |p| f(p) dp, \quad E_2 = \int_{\mathbb{R}^3} \mathcal{E}(p) f(p) dp.$$

Recall our convention that $f(p) = f(t, p)$ and the above holds for all $t \geq 0$. Since $\sqrt{\kappa_1} |p| < \mathcal{E}(p)$ for all nonzero $p$ and $f(t, p)$ is not the Dirac delta function centered at the origin, the number

$$E_3 := E_2 - E_1$$

is strictly positive. A direct computation then yields

$$0 < E_3 = \int_{\mathbb{R}^3} (\mathcal{E}(p) - \sqrt{\kappa_1} |p|) f(p) dp = \int_{\mathbb{R}^3} \frac{\kappa_2 |p|^3}{\sqrt{\kappa_1 + \sqrt{\kappa_1 + \kappa_2}|p|^2}} f(p) dp$$

$$\leq \sqrt{\kappa_2} \int_{\mathbb{R}^3} |p|^2 f(p) dp.$$
Recalling that $\delta > 0$ was arbitrary and $\mathcal{E}(p) \geq \sqrt{\kappa_1}|p|$, we compute

$$E_3 \leq \sqrt{\kappa_2} \int_{\{|p|\leq \delta\}} |p|^2 f(p) dp + \sqrt{\kappa_2} \int_{\{|p|\geq \delta\}} |p|^2 f(p) dp$$

$$\leq \delta \kappa_1^{-1/2} \sqrt{\kappa_2} \int_{\mathbb{R}^3} \mathcal{E}(p) f(p) dp + \sqrt{\kappa_2} \int_{\{|p|\geq \delta\}} |p|^2 f(p) dp$$

$$\leq \delta \kappa_1^{-1/2} \sqrt{\kappa_2} E_2 + \sqrt{\kappa_2} \int_{\{|p|\geq \delta\}} |p|^2 f(p) dp.$$ 

We now take $\delta$ so that $\delta \kappa_1^{-1/2} \sqrt{\kappa_2} E_2 = \frac{1}{2} E_3$. Clearly, $\delta$ depends only on the initial momentum and energy constants $E_1, E_2$ and the parameters $\kappa_1, \kappa_2$. This yields

$$\int_{\{|p|\geq \delta\}} |p|^2 f(p) dp \geq \frac{1}{2} E_3 \kappa_2^{-1/2}$$

and together with (4.15),

$$Y_0 \geq c_0 \left( \int_{\mathbb{R}^3} f(p) dp \right)$$

for some positive universal constant $c_0$ (recalling that $\delta$ has been fixed).

We now turn to $Y_1$. Recalling the surface $S''_{p_1}$ defined by the constraint $\mathcal{E}(p) = \mathcal{E}(p_1) + \mathcal{E}(p - p_1)$, we may write

$$Y_1 = 2 \int_{\mathbb{R}^6} K(p, p_1, p - p_1) f(p) f(p_1) dp dp_1$$

$$= 2 \kappa_0 \int_{\mathbb{R}^3} \int_{S''_{p_1}} |p| |p - p_1| |p_1| f(p) f(p_1) d\sigma(p) dp_1$$

$$\leq 2 \kappa_0 \int_{\mathbb{R}^3} \left( \int_{S''_{p_1}} |p|^2 f(p) d\sigma(p) \right) |p_1| f(p_1) dp_1$$

in which we used $|p - p_1| \leq |p|$, due to the monotonicity of $\mathcal{E}(p)$. Regarding the surface integral, Lemma 3.3 yields

$$\int_{S''_{p_1}} |p|^2 f(p) d\sigma(p) \leq C \int_{\mathbb{R}} |p|^3 f(|p|) |dp| \leq C \int_{\mathbb{R}^3} |p| f(p) dp.$$ 

Hence,

$$Y_1 \leq C \left( \int_{\mathbb{R}^3} |p| f(p) dp \right)^2,$$

which is bounded, due to the conservation of energy, upon recalling that $\mathcal{E}(p) \geq |p|$.

Finally, we treat $Y_2$. By definition, we have

$$Y_2 = \kappa_0 \int_{\mathbb{R}^3} \left( \int_{S_p} |p - p_1| |p_1| d\sigma(p_1) \right) |p_1| f(p) dp$$

$$\leq \kappa_0 \int_{\mathbb{R}^3} \left( \int_{S_p} d\sigma(p_1) \right) |p|^3 f(p) dp.$$
in which we again used the monotonicity of $E(p)$ to deduce that $|p_1| \leq |p|$ and $|p - p_1| \leq |p|$. The surface integral over $S_p$ is then bounded by $C|p|^2$ by Lemma 3.1. This proves

$$Y_2 \leq C \int_{\mathbb{R}^3} |p|^5 f(p) dp$$

which is bounded by the third-order energy moment, since $E(p) \gtrsim |p| + |p|^2$.

Combining, we have proved the following differential inequality

$$\frac{d}{dt} \int_{\mathbb{R}^3} f dp \leq -c_0 \int_{\mathbb{R}^3} f dp + C + C \int_{\mathbb{R}^3} f \mathcal{E}^3 dp.$$

The proposition now follows easily. Indeed, for any $\tau > 0$, Proposition 4.2 shows that the third-order moment is bounded by a constant $C_\tau$. The Gronwall inequality then yields

$$\int_{\mathbb{R}^3} f(t,p) dp \leq c_0^{-1} C_\tau + e^{-c_0(t-\tau)} \int_{\{|p| \leq 1\}} f(\tau,p) dp + e^{-c_0(t-\tau)} \int_{\{|p| \geq 1\}} f(\tau,p) dp$$

for all $t \geq \tau$. The last integral is clearly bounded, thanks to the conservation of the first-order energy moment and the fact that $E(p) \gtrsim 1$ on $\{|p| \geq 1\}$. The boundedness of the mass of $f(\tau,p)$ on $\{|p| \leq 1\}$ follows by continuity in time. The bound (4.13) follows, and the proof of the proposition is complete. 

\section{Uniform lower bound}

In this section, we shall prove our main theorem, Theorem 1.1. First, we need the following lemmas on the gain and loss operators, defined in (3.3).

\textbf{Lemma 5.1} Let $F$ be any nonnegative smooth function so that $F(p) \leq G(|p|)$, for some radially symmetric function $G$ satisfying

$$\int_{\mathbb{R}_+} G(u)(|u|^2 + |u|^3) du \leq M.$$ 

Then, the loss operator $Q_{\text{loss}}[F]$, defined as in (3.3), satisfies

$$Q_{\text{loss}}[F] \leq L_0(|p|) F(p), \quad (5.1)$$

in which $L_0(|p|) := C_0(1 + M)(|p| + |p|^5)$, for some positive universal constant $C_0$.

\textbf{Proof} We recall from (3.3) that $Q_{\text{loss}}[F] = L[F](p) F(p)$, with

$$L[F](p) := \int_{S_p} K(p,p-p_2,p_2) \left( F(p-p_2) + F(p_2) + 1 \right) d\sigma(p_2)$$

$$+ 2 \int_{S_p} K(p + p_2,p,p_2) F(p_2) d\sigma(p_2)$$

$$=: I_1 + I_2.$$
Let us first estimate the surface integral over \( S_p \). We note that for \( p_2 \in S_p \), there hold \( |p_2| \leq |p| \) and \( |p - p_2| \leq |p| \) (by monotonicity of the energy function \( \mathcal{E}(p) \)). Hence, together with the assumption \( F(p) \leq G(|p|) \), the definition \([1.3], [1.6]\) of the collision kernel \( \mathcal{K}(p, p_1, p_2) \), and Lemma 3.1, we estimate

\[
I_1 \leq \int_{S_p} \kappa_0 |p| |p - p_2| |p_2| \left( G(|p - p_2|) + G(|p_2|) + 1 \right) d\sigma(p_2)
\]

\[
\leq \kappa_0 |p|^2 \int_{S_p} \left( 2G(|p_2|) + 1 \right) |p_2| d\sigma(p_2)
\]

\[
\leq C(|p|^2 + |p|^5) \left( 1 + \int_{\mathbb{R}^+} G(u) |u|^2 du \right)
\]

which is bounded by \( C(|p|^2 + |p|^5)(1 + \mathcal{M}) \). Let us treat the second integral \( I_2 \). Similarly, Lemma 3.2 yields

\[
I_2 \leq 2 \int_{S_p} \mathcal{K}(p + p_2, p, p_2) G(|p_2|) d\sigma(p_2)
\]

\[
\leq 2\kappa_0 \int_{S_p} |p| \left( |p| + |p_2| \right) |p_2| G(|p_2|) d\sigma(p_2)
\]

\[
\leq C \int_{\mathbb{R}} |p| \left( |p| + |u| \right) |u|^2 G(u) du
\]

which is bounded by \( C\mathcal{M}(|p| + |p|^2) \). The lemma follows at once.

**Lemma 5.2** Let \( \delta, \theta > 0 \), and \( F \) be any nonnegative smooth function so that \( F(p) \geq \theta \) on \( B_\delta := \{|p| \leq \delta\} \). Then, there exists a universal constant \( c_0 > 0 \) such that

\[
Q_{\text{gain}}[F](p) \geq c_0 |p| \min\{1, |p|\} \delta^4 \theta^2 \tag{5.2}
\]

for all \( p \in B_{\sqrt{2}\delta} \).

**Proof** By definition \([3.3]\) and the assumption on the lower bound on \( F \), we have

\[
Q_{\text{gain}}[F](p) = \int_{S_p} \mathcal{K}(p, p - p_2, p_2) F(p - p_2) F(p_2) d\sigma(p_2)
\]

\[
+ 2 \int_{S_p} \mathcal{K}(p + p_2, p, p_2) F(p + p_2) \left( F(p) + F(p_2) + 1 \right) d\sigma(p_2)
\]

\[
\geq \int_{S_p} \mathcal{K}(p, p - p_2, p_2) F(p - p_2) F(p_2) d\sigma(p_2)
\]

\[
\geq \kappa_0 |p| \theta^2 \int_{S_p \cap B(0, \delta) \cap B(p, \delta)} |p - p_2||p_2| d\sigma(p_2),
\]

in which we note again that \( p_2, p - p_2 \) are both in \( B_\delta \), thanks to the monotonicity of the energy function \( \mathcal{E}(p) \).
To proceed, we consider three cases. First, take $p \in B(0, \delta) \setminus B(0, \frac{\delta}{2})$. In this case, $B(\frac{p}{2}, \frac{|p|}{2}) \subset B(0, \delta) \cap B(p, \delta)$, and so we can estimate

$$Q_{\text{gain}}[F](p) \geq \kappa_0 |p| \theta^2 \int_{S_p \cap B(\frac{p}{2}, \frac{|p|}{2})} |p - p_2||p_2| d\sigma(p_2)$$

$$\geq c_0 |p|^5 \min\{1, |p|\} \theta^2 \geq c_1 |p| \min\{1, |p|\} \delta^4 \theta^2,$$

for some positive constants $c_0, c_1$, thanks to the lower bound (3.4) in Lemma 3.1, with $\gamma = 1$.

Next, for $p \in B(0, \frac{\delta}{2})$, we note that $B(0, \frac{\delta}{2}) \subset B(0, \delta) \cap B(p, \delta)$. Hence, in this case, we have, by the lower bound (3.4),

$$Q_{\text{gain}}[F](p) \geq \kappa_0 |p| \theta^2 \int_{S_p \cap B(0, \frac{\delta}{2})} |p - p_2||p_2| d\sigma(p_2)$$

$$\geq c_0 |p| \min\{1, |p|\} \delta^4 \theta^2.$$

The lemma is proved for $|p| \leq \frac{\delta}{2}$.

Finally, we consider the case when $p \in B(0, \sqrt{2} \delta) \setminus B(0, \delta)$. In this case, we check that $S_p \cap B(0, \delta) \cap B(p, \delta)$ has positive surface area. Indeed, let $D_p$ be the disk that is centered at $\frac{p}{2}$, of radius $\sqrt{\delta^2 - \frac{|p|^2}{4}}$, and is on the plane orthogonal to $p$. Let $x$ be a point on the boundary of $D_p$, then $|x - p/2| = \sqrt{\delta^2 - \frac{|p|^2}{4}}$ and $x - p/2$ is orthogonal to $p$. As a consequence, $|x|^2 = |x - p/2|^2 + |p/2|^2 = \delta^2$ and $|x - p|^2 = |x - p/2|^2 + |p/2|^2$. It is clear that $D_p$ belongs to the intersection $B(0, \delta) \cap B(p, \delta)$ and, since $\sqrt{\delta^2 - \frac{|p|^2}{4}} \geq \frac{|p|}{2}$, the surface $S_p$ crosses the interior of $D_p$. This proves that $S_p \cap B(0, \delta) \cap B(p, \delta)$ is non empty. Since $B(0, \delta) \cap B(p, \delta)$ has positive Lebesgue measure, the surface area of $S_p \cap B(0, \delta) \cap B(p, \delta)$ is bounded below from zero by a constant times $|p|$, since any geodesic on the surface starting from 0 to $p$ has a greater length than $|p|$. We can then compute

$$Q_{\text{gain}}[F](p) \geq \kappa_0 |p| \theta^2 \int_{S_p \cap B(0, \delta) \cap B(p, \delta)} |p - p_2||p_2| d\sigma(p_2)$$

$$\geq c_0 |p| \min\{1, |p|\} \delta^4 \theta^2,$$

due to the lower bound (3.4). This completes the proof of the lemma. ■

**Lemma 5.3** Let $\delta, \theta > 0$. Suppose that initial data $f_0(p) \geq \theta$ on $B_\delta = \{|p| \leq \delta\}$. Let $f(t, p)$ be a solution to (1.1) so that $f(t, p) \leq G(t, |p|)$ for all $t \geq 0$ and for some radially symmetric function $G$ so that

$$\int_{\mathbb{R}_+} G(t, u)(|u|^2 + |u|^3) du \leq \mathcal{M},$$

where $\mathcal{M}$ is a constant.
for some constant $\mathcal{M}$. Then, there holds the following uniform lower bound

$$f(t, p) \geq C_0 t e^{-tL_\star(\delta)} |p| \min\{1, |p|\} \delta^4 \theta^2, \quad \forall \ t \geq 0,$$

(5.4)

for all $p \in B_{\sqrt{2}\delta}$, with $L_\star(\delta) := c_0(\delta + \delta^5)$. Here, $c_0, C_0$ are some universal positive constants that might depend on $\mathcal{M}$, but are independent of $\delta, \theta$ and $p$.

**Proof** Using Lemma 5.1 with $F = f(t, p)$, we obtain

$$\partial_t f(t, p) + L_0(|p|) f(t, p) \geq Q_{\text{gain}}[f](t, p)$$

(5.5)

where $L_0(|p|)$ satisfies

$$L_0(|p|) = C_0(1 + \mathcal{M})(|p| + |p|^5).$$

The Duhammel’s representation applying to (5.5) yields

$$f(t, p) \geq f_0(p) e^{-tL_0(|p|)} + \int_0^t e^{-(t-\tau)L_0(|p|)} Q_{\text{gain}}[f](\tau, p) d\tau$$

(5.6)

for all $t \geq 0$. Since $Q_{\text{gain}}[f](p) \geq 0$ and $L_0(\cdot)$ is an increasing function, it follows that for $p \in B_\delta$, (5.6) yields

$$f(t, p) \geq f_0(p) e^{-tL_0(|p|)} \geq \theta e^{-tL_0(\delta)}, \quad t \geq 0.$$  

(5.7)

Next, for each fixed time $t \geq 0$, we now apply Lemma 5.2 for $F = f(t, p)$, with the new lower bound (5.7) on $B_\delta$, yielding

$$Q_{\text{gain}}[f](t, p) \geq C_0 |p| \min\{1, |p|\} \delta^4 \theta^2 e^{-2tL_0(\delta)},$$

for all $p \in B_{\sqrt{2}\delta}$. Putting this into (5.6), we obtain

$$f(t, p) \geq \int_0^t e^{-(t-\tau)L_0(|p|)} Q_{\text{gain}}[f](\tau, p) d\tau$$

$$\geq C_0 |p| \min\{1, |p|\} \delta^4 \theta^2 \int_0^t e^{-(t-\tau)L_0(\delta)} e^{-2\tau L_0(\delta)} d\tau$$

$$\geq C_0 |p| \min\{1, |p|\} \delta^4 \theta^2 e^{-2tL_0(\delta)} \theta$$

$$\geq C_0 |p| \min\{1, |p|\} \delta^4 \theta^2 e^{-tL_\star(\delta)},$$

in which we have set $L_\star(\delta) = 2L_0(\delta)$. This completes the proof of the lemma.

\[\square\]
5.1 Proof of Theorem 1.1

We are now ready to give the proof of Theorem 1.1. Let $\theta_0, R_0 > 0$ as in the assumption of Theorem 1.1 so that $f_0(p) \geq 2\theta_0$ on $B_{2R_0} = \{|p| \leq 2R_0\}$. Let $\tau$ be sufficiently small so that $f(\tau, p) \geq \theta_0$ on $B_{R_0}$, thanks to the continuity in time of the (classical) solution $f(t, p)$.

In the proof, we shall apply Lemma 5.3 repeatedly to the solution $f(t, p)$ of (1.1), with $G(t, |p|) = f(t, |p|)$. First, we note that since $f(t, p)$ is radially symmetric and $\mathcal{E}(p) \geq |p|$, we have

$$f(t, |p|)(1 + |p|)|\delta|d|p| \leq C_0 \int_{\mathbb{R}^3} f(t, p)(1 + \mathcal{E}(p)) \, dp \leq C_\tau,$$

for all $t \geq \tau$, thanks to Proposition 4.3 and the conservation of energy. This verifies the assumption (5.3) on $G(t, |p|) = f(t, |p|)$, made in Lemma 5.3.

Fix a positive and sufficiently small $\delta < R_0$, and a positive time $t_0$ so that

$$\sqrt{2}t_0 < 1. \quad (5.8)$$

Since $f_0(\tau, p) \geq \theta_0$ on $B_\delta$, applying Lemma 5.3 to the solution $f(t, p)$ of (1.1) with the initial data $f(\tau, p)$ yields

$$f(\tau + t_0, p) \geq t_0 e^{-t_0 L_*(\delta)} C_p \delta^4 \theta_0^2,$$  

for all $p \in B_{\sqrt{2}\delta}$, in which $L_*(\delta) = c_0(\delta + \delta^5)$ and

$$C_p := C_0|p| \min\{1, |p|\}. \quad (5.10)$$

We stress that $C_p$ does not depend on $\delta$ and $t_0$, and hence the estimate (5.9) can be iterated. Indeed, applying again Lemma 5.3 to the solution $f(t, p)$ of (1.1) with the initial data $f(\tau + t_0, p)$ satisfying (5.9), yielding

$$f(\tau + t_0 + t_1, p) \geq t_1 e^{-t_1 L_*(\sqrt{2}\delta)} C_p (\sqrt{2}\delta)^4 \left[t_0 e^{-t_0 L_*(\delta)} C_p \delta^4 \theta_0^2 \right]^2 \geq t_1 t_0^2 e^{-t_1 L_*(\sqrt{2}\delta)} e^{-2t_1 L_*(\delta)} 2^2 \left(C_p \delta^4 \right)^{1+2n^2} \theta_0^{2n^2}$$

for arbitrary positive time $t_1$ and for all $p \in B_{\sqrt{2}\delta}$. For each fixed integer $n \geq 2$, we iteratively apply Lemma 5.3, yielding

$$f(\tau + t_0 + \cdots + t_n, p) \geq t_n^2 \cdots t_{n-k}^2 t_{n-k}^2 \cdots t_0^2 e^{-t_n L_*(\sqrt{2}\delta)} \cdots e^{-2t_0 L_*(\delta)} \times 2^{(1+2+\cdots+2n-1)} \left(C_p \delta^4 \right)^{1+2+\cdots+2n} \theta_0^{2n^2},$$

for all $p \in B_{\sqrt{2}^{n+1}\delta}$. By using $1 + 2 + \cdots + 2^n = 2^{n+1} - 1$, the above is reduced to

$$f(\tau + t_0 + \cdots + t_n, p) \geq \frac{1}{2} t_n^2 \cdots t_{n-k}^2 \cdots t_0^2 \theta_0 \left(2C_p \delta^4 \theta_0 \right)^{2^{n+1}-1} E_n, \quad (5.11)$$

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for all $p \in B_{\sqrt{2}^{n+1} \delta}$, in which for convenience we have set
\[
E_n := e^{-t_n L_*(\sqrt{2}^n \delta)} \cdots e^{-2k t_{n-k} L_*(\sqrt{2}^{n-k} \delta)} \cdots e^{-2n t_0 L_*(\delta)}.
\] (5.12)

**Case 1:** $|p| > \sqrt{2} \delta$. Recall that $\delta, t_0$ are fixed. For each $p$ so that $|p| > \sqrt{2} \delta$, we take an integer $n$ satisfying
\[
\sqrt{2} ^n \delta < |p| \leq \sqrt{2} ^{n+1} \delta.
\] (5.13)

In particular, $p \in B_{\sqrt{2}^{n+1} \delta}$ and (5.11) holds for arbitrary positive time steps $t_k$. We now fix a time $t \in (\tau, t_*)$, with $t_* = 1/\sqrt{2}$. We take $t_k = t_0^k$ and choose $t_0$ so that $\sqrt{2} t_0 < 1$ and
\[
\sum_{k=0}^n t_k = t.
\]

Such a choice of $t_0$ is possible by the definition of $t_*$. The lower bound (5.11) then reads
\[
f(\tau + t, p) \geq \frac{1}{2} \theta_0 t_n t_{n-1}^2 \cdots t_{n-k}^2 \cdots t_0^n \left(2C_p \delta^4 \theta_0\right)^{2n+1-1} E_n,
\] (5.14)
for all $t \in (\tau, t_*)$ and all $|p| > \sqrt{2} \delta$, with $n$ being defined by (5.13).

Note in particular that $t_0 \geq T_\tau$ for some positive time $T_\tau$, since $t \geq \tau$. Using this, we can estimate
\[
t_n t_{n-1}^2 \cdots t_{n-k}^2 \cdots t_0^n \geq T_\tau^{n+2(n-1)+\cdots+2k(n-k)+\cdots+2n}
\geq T_\tau^{2n+\sum_{k=0}^n 2k(n-k)}
\geq T_\tau^{2n(1+\sum_{k=0}^n k^2-k)}
= C_0^{2n}
\]
in which $C_0 = T_\tau^{1+\sum_{k=0}^\infty k^2-k}$, which is finite and nonzero.

Next, by the definition (5.10) of $C_p$, we have $C_p \geq C_\delta$ for some positive constant $C_\delta$, since $|p| > \sqrt{2} \delta$, and hence
\[
\theta_0 \left(2C_p \delta^4 \theta_0\right)^{2n+1-1} \geq \theta_0 \left(2C_\delta \delta^4 \theta_0\right)^{2n+1-1} \geq C_1 C_2^{2n}
\]
for some positive constants $C_1$ and $C_2$, independent of $n, p$ and $t$.

Finally, we estimate the exponential term $E_n$ defined as in (5.12). Recalling that $\sqrt{2} t_0 < 1$, $t_k = t_0^k$ and $L_*(\delta) = c_0(\delta + \delta^5)$, we have
\[
e^{-2k t_{n-k} L_*(\sqrt{2}^{n-k} \delta)} \geq e^{-2^{k+1} t_0^{n-k} c_0(\sqrt{2}^{n-k} \delta + \sqrt{2}^{(n-k)} \delta^n)}
\geq e^{-2^\delta c_0 \sqrt{2}^{n+k} - 2^\delta c_0 (2\sqrt{2} t_0)^{n-k} 2^k}
= e^{-2^\delta c_0 \sqrt{2}^{n+k} - 2^\delta c_0 2^n (\sqrt{2} t_0)^{-k}}.
\]
Hence, we obtain

\[ E_n = \exp \left( -\sum_{k=0}^{n} 2^k t_{n-k} L_s(\sqrt{2}^{n-k} \delta) \right) \]
\[ \geq \exp \left( -2\delta c_0 \sqrt{2}^{n} \sum_{k=0}^{n} \sqrt{2}^k - 2\delta^5 c_0 2^n \sum_{k=0}^{n} (\sqrt{2} t_0)^{-k} \right) \]
\[ \geq \exp \left( -4\delta c_0 2^n - 2\delta^5 c_0 c_1 2^n \right) \]
\[ = C_3^{2n}, \]

for some positive constant \( \mathcal{E}_3 \), which is independent of \( n, p, \) and \( t \). In the above, \( c_1 = \sum_{k=0}^{n} (\sqrt{2} t_0)^{-k} \) which is bounded, since \( \sqrt{2} t_0 < 1 \).

Putting the above bounds into (5.14), we have obtained

\[ f(\tau + t, p) \geq \frac{1}{2} C_1 (C_0 C_2 C_3)^{2n} = \theta_1 e^{-\theta_2 2^n} \geq \theta_1 e^{-\theta_3 |p|^2} \]  
(5.15)

for all \( t \in [\tau, t_*] \) and all \( |p| > \sqrt{2}\delta \), with \( \theta_1 = \frac{1}{2} C_1, \theta_2 = \log \frac{1}{c_0 C_2 C_3}, \) and \( \theta_3 = \theta_2/(2\delta^2) \). Here, we stress that the constants \( \theta_j \) are independent of \( p \) and \( t \).

In the above proof, if we take \( t_0 < 1 \), we can also bound \( E_n \) as follows:

\[ E_n = \exp \left( -\sum_{k=0}^{n} 2^k t_{n-k} L_s(\sqrt{2}^{n-k} \delta) \right) \]
\[ \geq \exp \left( -(\delta + \delta^5) c_0' c_1' 2^{3n} \right) \]
\[ = C_3'^{23n}, \]

for some positive constant \( \mathcal{E}_3' \), which is independent of \( n, p, \) and \( t \). Putting the above bounds into (5.14), we have obtained

\[ f(\tau + t, p) \geq \frac{1}{2} C_1 (C_0 C_2 C_3')^{23n} = \theta_1' e^{-\theta_2' 2^{3n}} \geq \theta_1' e^{-\theta_3' |p|^6} \]  
(5.16)

for all \( t \in [\tau, t_*] \), where \( t_* = \sum_{k=0}^{t} t_0^k = \frac{1}{1-t_0} \). Since we can choose \( t_0 \) arbitrarily close to 1, \( t_* \) can be chosen arbitrarily large. As a consequence (5.16) holds true for all \( t \in [0, \infty) \).

Case 2: \( |p| \leq \sqrt{2}\delta \). In this case, we shall use the differential inequality (5.5):

\[ \partial_t f \geq Q_{\text{gain}}[f](p) - c_0(|p| + |p|^5) f(p). \]

Therefore

\[ f(t, p) \geq e^{-C(\delta)t(|p|+|p|^5)} f_0(t, p). \]
For each $p$, by repeating the same argument as in Case 1 (5.16), in which $\delta$ is replace by $|p|/8$, we can conclude that there exists $T_p$ and $b_{|p|}$ such that for all $t > T_p$, we have $f(t, p') > b_{|p|} > 0$ for all $|p'| > |p|$. We now show that there exist constants $c_5$ and $T_5$ such that for all $t > T_5$, the function $f(t, p)$ is bounded from below by $c_5$. Suppose the contrary that there exists an increasing sequence $\{t_n\}$, and the corresponding family of vectors $\{p_n\}$ such that $\lim_{n \to \infty} f(t_n, p_n) = 0$. We claim that there exists a sub-sequence $\{t_{n_k}, p_{n_k}\}$ of $\{t_n, p_n\}$ such that $\lim_{k \to \infty} p_{n_k} = 0$. In order to see this, let us suppose the contrary that there exists a constant $\gamma$ such that $|p_n| > \gamma > 0$ for all $n$. This implies $f(t_n, p_n)$ is bounded from below by $c_\gamma$ and this contradicts the fact that the limit of the sequence is 0. Since $\lim_{k \to \infty} p_{n_k} = 0$, we consider two cases. In the first case, if the increasing sequence $\{t_{n_k}\}$ has a limit $T_0$, then $f(T_0, 0) = 0$ by the continuity of $f$. However, we deduce from (1.1) that $\partial_t f(t, 0) = 0$ or $f(t, 0) = f_0(0) > 0$. This is a contradiction. In the second case, we have $\lim_{k \to \infty} t_{n_k} = \infty$, $\lim_{k \to \infty} p_{n_k} = 0$ and $\lim_{k \to \infty} f(t_{n_k}, p_{n_k}) = 0$. This still contracts the fact that $f(t, 0) = f_0(0) > 0$ and $f \in C(\mathbb{R}_+ \times \mathbb{R}^3)$. Finally, there exist constants $c_5$ and $T_5$ such that for all $t > T_5$, the function $f(t, p)$ is bounded from below by $c_5$ for all $|p| < \delta$.

**Iteration.** To conclude, we have obtained the Gaussian bound

$$f(t, p) \geq \theta_3 e^{-\theta_4 |p|^2}, \quad p \in \mathbb{R}^3, \quad t \in [2\tau, \tau + t_*],$$

(5.17)

for some universal constants $\theta_3, \theta_4$ that are independent of $p$ and $t$. Here, $t_* = 1/\sqrt{2}$. By induction, for each integer $k \geq 1$, we then repeat the above proof, starting with initial data at $t = kt_*$. This yields the same Gaussian bound on the each time interval $[\tau + kt_*, \tau + (k+1)t_*]$, upon noting that such a bound depends only on the initial mass near zero at $t = kt_*$, which is independent of $k^{th}$ iteration, since $f(t, 0) = f_0(0)$ for all $t \geq 0$; see Case 2. This proves the Gaussian lower bound for all time $t \geq 2\tau$, and hence the main theorem.

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