New Look at QED$_4$: the Photon as a Goldstone Boson and the Topological Interpretation of Electric Charge.

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Abstract

We develop the dual picture for Quantum Electrodynamics in 3+1 dimensions. It is shown that the photon is massless in the Coulomb phase due to spontaneous breaking of the magnetic symmetry group. The generators of this group are the magnetic fluxes through any infinite surface $\Phi_S$. The order parameter for this symmetry breaking is the operator $V(C)$ which creates an infinitely long magnetic vortex. We show that although the order parameter is a stringlike rather than a local operator, the Goldstone theorem is applicable if $\langle V(C) \rangle \neq 0$. If the system is properly regularized in the infrared, we find $\langle V(C) \rangle \neq 0$ in the Coulomb phase and $V(C) = 0$ in the Higgs phase. The Higgs - Coulomb phase transition is therefore understood as condensation of magnetic vortices. The electric charge in terms of $V(C)$ is topological and is equal to the winding number of the mapping from a circle at spatial infinity into the manifold of possible vacuum expectation values of a magnetic vortex in a given direction. Since the vortex operator takes values in $S^1$ and $\Pi_1(S^1) = \mathbb{Z}$, the electric charge is quantized topologically.

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1 Introduction

Gauge theories play a dominant role in modern elementary particle physics. It is clear beyond reasonable doubt that all the interactions of elementary particles known to date are described by a gauge theory. As a consequence in some physicists minds gauge symmetry attained a status of a philosophical principle. It must be noted however, that the reason for this is purely empirical. The "gauge principle" does not have the same deep philosophical roots as, say the equivalence principle of general relativity or the uncertainty principle of quantum mechanics. Mainly this is because it pertains to the form of description, the "language" in which one describes a physical law rather than to the physical law itself. This language proved to be indispensable when formulating renormalizable theories of interacting vector particles. In many instances it is also very convenient for actual calculations since the degrees of freedom used in this description are almost free and the perturbation theory can be easily applied. Such is the case in QED and the ultraviolet region of QCD.

In some cases however, although a neat mathematical construction, the gauge symmetry in fact obscures rather than highlights the underlying non-perturbative physics. The main conceptual difficulty with the gauge description is that it makes use of redundant nonphysical quantities which often makes the interpretation of a calculation almost as difficult as performing the calculation itself.

One example of such a situation is the understanding of the (constituent) quark degrees of freedom in QCD. On the large nonphysical Hilbert space those appear as multiplets of the "global color" $SU(3)$ group. However this group acts trivially on the physical gauge invariant states of the theory. Hence the problem of understanding in physical terms what is precisely meant by the color and its confinement.

It is of course possible in principle to fix the gauge completely. However in a completely gauge fixed formulation the fields that appear in the Lagrangian are as a rule nonlocal. For example in QED, fixing axial gauge turns the matter fields into variables localized on a line rather than at a point

$$
\phi_{axial}(x) = \phi(x) \exp\{ie \int_x^\infty dy^3 A^3(y)\} \tag{1}
$$

where the initial fields $\phi(x)$ and $A_\mu(x)$ are "local" but on the nonphysical
Hilbert space. In the Coulomb gauge the matter field

\[ \phi_{\text{coulomb}}(x) = \phi(x) \exp\left\{ i e \int d^3y A^i(y) \frac{x^i - y^i}{|x - y|^3} \right\} \]  \hspace{1cm} (2)

creates the electric field of a point charge and has therefore a nonvanishing support everywhere in space. When written in terms of these fields, the Lagrangian is nonlocal and the theory looks very different from a local field theory which would usually please one’s eye.

Because of this unfortunate feature there are several interesting physical questions already in the simplest, abelian gauge theories which either do not arise naturally or tend to be ignored in the framework of the standard gauge description. Here are several of these, which motivated us in the present research.

1. Exact masslessness of a photon. In the standard formulation the masslessness of a photon is almost a consequence of kinematics. However, we know that the photon is not always massless in the Higgs model, and that this is in fact a profound dynamical effect. When one discovers a massless particle the natural question is: what keeps it massless. The simplest possible explanation is that it is the Goldstone theorem. So there is a question whether the photon in QED a Goldstone boson and if yes of what symmetry.

2. The nature of the Higgs - Coulomb phase transition. The Higgs - Coulomb phase transition is usually described as due to spontaneous breaking of the electric charge in the Higgs phase. This description is however not without a flaw. Electric charge, being equal to a surface integral of the electric field at spatial infinity does not have a local order parameter. (This is the reason why the Goldstone theorem is not applicable in the Higgs phase.) Consequently the Coulomb and the Higgs phase differ only in expectation values of nonlocal fields. In this situation however, there is no physical argument that tells us that the two phases must be separated by a phase transition. In fact in the similar situation in $Z_N$ gauge theories it is known that the phases are analytically connected [1]. In QED, however the two phases are separated by a genuine phase transition which is second or first order depending on the values of parameters. The question is whether one can give a different, complementary characterization of the Higgs and Coulomb phases in QED which will make clear that those are really distinct phases. Usually such an explanation involves spontaneous breaking of some global symmetry.
3. Topological nature of the electric charge. The electric current in QED is trivially conserved: \( \partial_\mu J^\mu = \partial_\mu (\partial_\nu F^{\mu\nu}) = 0 \). In the quantum theory the charge is also quantized. Both these features would automatically follow if the electric charge could be represented as a topological charge associated with nontrivial homotopy of a vacuum manifold. The possibility that the electric charge could be topological is not so unnatural. One can measure the charge by making local measurements of electric field far from it making use of Gauss law. For a nontopological charge this would be impossible. Maybe it is possible to find in QED a set of variables in terms of which the degeneracy of the vacuum and the topological nature of the electric charge are explicit.

It would be very interesting to find an alternative formulation of a gauge theory, or at least (since the exact reformulation turns out to be very difficult) an alternative basis in which these questions become natural and the answers to them are relatively straightforward.

In fact in 2+1 dimensions there exists a “dual” representation that allows to answer all of these questions in the affirmative. There one is able to define such a variable: the complex vortex operator \( V(x) \). Although it is defined in terms of an exponential of a line integral of the electric field, it can be shown that it is a local scalar field. It is an eigenoperator of the conserved charge - the magnetic flux through the plane. In the Coulomb phase the field \( V(x) \) has a nonvanishing expectation value and the flux symmetry is spontaneously broken. This results in the appearance in the spectrum of a massless Goldstone boson - the photon. The electric charge when expressed in terms of the vortex field has the form of a topological charge associated with the homotopy group \( \Pi_1(S^1) \): 

\[
Q \propto \int d^2x \epsilon_{ij} \partial_i (i V^* \partial_j V + c.c.)
\]

In the Higgs phase \( < V > = 0 \). Consequently the charges are completely screened and there is no massless particle in the spectrum.

In this picture it is clear that the Higgs and the Coulomb phases must be separated by a genuine phase transition line. The relevant symmetry, the magnetic flux symmetry, is restored in the Higgs phase. The Nielsen-Olesen (NO) vortices exist in this phase as particles that carry the corresponding charge. The Coulomb - Higgs phase transition can be thought of as condensation of the NO vortices in the Coulomb phase. Moreover on the basis of universality one concludes, that whenever it is second order, the Higgs - Coulomb phase transition must be in the universality class of the XY model.

In 1+1 dimensional QED the dual representation also exists. Since in
1+1 dimensions there is no massless photon and no Coulomb - Higgs phase transition, only the third question is can be asked. In this case the electric charge can be represented as topological in terms of the dual field $\sigma$:
\[
Q = \int dx \partial \sigma = \sigma(+\infty) - \sigma(-\infty).
\]
For the massless and massive Schwinger model the standard bosonization procedure leads to exact reformulation of the theory in terms of the field $\sigma$ only[2] and thereby to the exact dual Lagrangian. In the scalar Higgs model, the dual transformation can be only performed approximately [3], but the topological interpretation of the electric charge is nevertheless achieved.

The aim of this paper is to develop a similar picture for the 3+1 dimensional Higgs model. In Section 2 we discuss the analog of the flux symmetry in 3+1 dimensions. The conserved currents of this magnetic symmetry are the components of the dual field strength tensor $\tilde{F}_{\mu\nu}$. Because of the constraint $\partial_i B_i = 0$ no local order parameter can be found. The 3+1 dimensional analog of $V(x)$ are stringlike operators $V(C)$ which create infinitely long magnetic vortex lines along a curve $C$.

In section 3 we show that although these operators are not local they are still good order parameters, in the sense that the Goldstone theorem is applicable in the phase where they have a nonzero expectation value. We also show that in this phase the electric charge is topological in terms of $V(C)$ and is thereby quantized. An electrically charged state carries a unit wind of the phase of $V(C_{\vec{u}})$ where $V(C_{\vec{u}})$ is the set of all magnetic vortices associated with straight lines in the direction $\vec{u}$. In the classical approximation, indeed $V(C) \neq 0$ in the Coulomb phase.

Due to the infinite length of the vortex line $V(C)$ can not have a finite expectation value beyond classical approximation. Infrared divergences due to phase fluctuations of $V(C)$ lead to vanishing of the VEV even in the Coulomb phase, not unlike vanishing of an order parameter in 1+1 dimensional theories with a “classically broken” continuous symmetry. In the present case however one can define a regularized model in which one of the spatial dimensions is compact. Vortex lines parallel to this direction will then have finite expectation value in the Coulomb phase and the Goldstone theorem and the topological interpretation of the electric charge will be retained. This is discussed in Section 4.

Section 5 is devoted to a brief discussion of the dual picture and possible extension of this approach to nonabelian theories.
2 The magnetic symmetry, the Coulomb - Higgs phase transition and the vortex operator in 3+1 dimensions.

Approaches to all the three questions mentioned in the introduction which we are addressing in this paper have one common element. They all require a construction of a sufficiently local (gauge invariant) order parameter. Let us briefly recall how this was constructed in 2+1 dimensions.

The symmetry which is broken in the Coulomb phase of the 2+1 dimensional Higgs model is the magnetic flux symmetry generated by

\[ \Phi = \int d^2 x B(x) \]  

with the conserved current \( \tilde{F}_\mu \). The defining relation for the vortex operator \( V(x) \) therefore was the commutation relation with the magnetic field

\[ [B(x), V(y)] = -g\delta^2(x - y)V(y) \]  

One also insisted on the locality of \( V(x) \): it had to commute with all gauge invariant local fields at space - like separations.

\[ [V(x), O(y)] = 0, \quad x \neq y \]  

These conditions determined \( V(x) \) up to a multiplicative local gauge invariant factor as

\[ V(x) = C \exp\left\{ \frac{i}{e} \int d^2 y [\epsilon_{ij} \frac{(x - y)_j}{(x - y)^2} E_i(y) + \Theta(x - y)J_0(y)] \right\} \]  

where \( \Theta(x) \) is an angle between the vector \( x_1 \) and the \( x_1 \) axis, \( 0 < \Theta < 2\pi \). The requirement of locality lead in particular to a quantization condition on possible eigenvalues of the magnetic flux \( g \): \( g = \frac{2\pi n}{e} \).

The operator \( V(x) \) has a simple physical meaning: it creates a pointlike magnetic vortex of the strength \( g \). In the Higgs phase the magnetic flux symmetry is not broken and the NO vortices behave like particles. In the Coulomb phase they condence. This breaks the flux symmetry spontaneously and leads to the appearance of the massless photon.
2.1 The vortex operator.

Let us now try to implement the same program in the 3+1 dimensional Higgs model. The analog of the conserved flux current in 3+1 dimensions is the dual magnetic field strength tensor $\tilde{F}_{\mu\nu}$. Its conservation equation is again just the homogeneous Maxwell equation of electrodynamics. It was shown in [10] that the matrix element of $\tilde{F}_{\mu\nu}$ between the vacuum and the one photon state in the Coulomb phase has the characteristic form of a matrix element of a spontaneously broken current between the vacuum and a state with one Goldstone boson. In the circular polarization basis

$$<0|\tilde{F}_{0i}(0)|e^\lambda_{\pm}, \vec{p}> = \mp \frac{i}{(2\pi)^{3/2}} \sqrt{\frac{p_0}{2}} e^{\pm}_i \lim_{p^2 \to 0} \frac{1}{1 - \Pi(p^2)}$$  \quad (7)$$

$$<0|\tilde{F}_{ij}(0)|e^\lambda_{\pm}, \vec{p}> = -\frac{i}{(2\pi)^{3/2}} \sqrt{\frac{p_0}{2}} \epsilon_{ijk} e^{\pm}_k \lim_{p^2 \to 0} \frac{1}{1 - \Pi(p^2)}$$  \quad (8)

where $\Pi(p^2)$ is the vacuum polarization.

One can define many conserved charges associated with the currents $\tilde{F}_{\mu\nu}$. In particular the magnetic flux through any infinite surface $S$

$$\Phi_S = \int_S dS^i B_i$$  \quad (9)

is time independent. Since the magnetic flux through any closed surface vanishes, the set of independent $\Phi_S$ is given by the set of boundaries (at spatial infinity) rather than the set of surfaces themselves. It will suffice for our purposes however to consider only the planes perpendicular to the three coordinate axes. We define the magnetic charge $\Phi_i$ as the average magnetic flux through a plane perpendicular to the $i$-th axis

$$\Phi_i \equiv \lim_{L \to \infty} \frac{1}{2L} \int_{-L}^L dx^i \int_{S(x^i)} dS \tilde{B}(x^i)$$  \quad (10)

where $S(x^i)$ is the plane perpendicular to the $i$-th axis with the $i$-th coordinate $x^i$.

We now construct an order parameter for $\tilde{F}_{0i} = B_i$. This is an operator which creates magnetic vortices. Clearly in 3+1 dimensions no local operator of this kind can be constructed. Since the magnetic field is divergenceless, the
magnetic flux must either form closed loops or infinitely long lines. Closed loops however do not carry any global charge. The best one can do therefore is to construct an operator creating an infinite vortex line. This operator should be "line - local".

Any gauge invariant local field should commute with $V(C)$ at all points but on the line $C$. In particular

$$[B_i(x), V(C)] = g l_i(x, C)V(C), \quad l_i(x, C) = \int d\tau \delta(x - \bar{x}(\tau)) \frac{d\bar{x}_i(\tau)}{d\tau}$$  \hspace{1cm} (11)

The commutator of $V(C)$ and $J_i(x)$ should also vanish for $x \notin C$. Analogously to 2+1 dimensions, these two conditions determine $V(C)$ as

$$V_n(C) = \exp \left\{ i\frac{n}{e} \int d^3y [a_i(y - x)E_i(y) + b(x - y)J_0(y)] \right\} \hspace{1cm} (12)$$

where $a_i(x)$ is a vector potential of an infinitesimally thin magnetic vortex along $C$ : $\epsilon_{ijk}\partial_ja_k(x) = l_i(x, C)$ and the function $b(x)$ satisfies $\partial_i[b(x)]_{\text{mod}2\pi} = a_i(x)$. Since $a_i(x)$ has a nonvanishing curl, the function $b(x)$ must have a surface of singularities ending at the curve $C$. For example, for a straight line $C$ running along the $x_3$ axis one has $a_i(x) = \epsilon_{ij} \frac{x_i}{x_1^2 + x_2^2}, \quad i = 1, 2; \quad a_3(x) = 0$ and $b(x) = \Theta(x)$ with $\Theta$ the polar angle in the $x_1 - x_2$ plane (Fig.1.). As in 2+1 dimensions, the operator $V(C)$ is an operator of a singular gauge transformation with the gauge function $n b(x)$. This ensures the commutativity of $V(C)$ with any local gauge invariant operator outside the line of singularities $C$.

The single valuedness of the gauge transformation imposes the quantization condition on possible fluxes in a vortex $g = \frac{2\pi n}{e}$. This of course corresponds to the well known fact that the Abrikosov vortices in the Higgs phase carry quantized flux. From now on we will concentrate on the elementary vortex operator $n = 1$.

Choosing $C_i$ as a straight line parallel to the i-th axis we have

$$[V(C_i), \Phi_j] = \delta_{ij} \frac{2\pi}{e} V(C_i)$$  \hspace{1cm} (13)

Using the Gauss’ law and integrating by parts one can recast the vortex operator into the following form

$$V(C) = \exp \left\{ i\frac{2\pi}{e} \int_{S: \partial S = C} dS^i E^i \right\} \hspace{1cm} (14)$$
Figure 2: The vortex operator $V(C_3)$ which creates the magnetic flux tube parallel to the third axis.

where the integration is over the half plane bounded by $C$ (Fig.2).

### 2.2 The vortex operator and the dual vector potential.

Let us now digress a little bit and show how the vortex operator can be represented in terms of the dual vector potential. This is particularly simple in the case of a free photon. The Gauss’s law in this case reduces to $\partial_i E_i = 0$ and can be solved by introducing the dual vector potential $\chi_i$ via

$$E_i = \epsilon_{ijk} \partial_j \chi_k$$  \hspace{1cm} (15)

To reproduce the correct equal time commutation relations we must also have

$$B_i(x) = \pi_i(x)$$  \hspace{1cm} (16)

where $\pi_i$ is canonically conjugate to $\chi_i$. Of course the field $\chi_i$ is determined by eq. (15) only up to a gradient of a scalar function. The transformation

$$\chi_i \rightarrow \chi_i + \partial_i \lambda$$  \hspace{1cm} (17)

is generated by $\partial_i B_i$ and is in fact a magnetic gauge symmetry due to the constraint

$$\partial_i B_i = 0$$  \hspace{1cm} (18)

As in the case of the electric gauge symmetry, one should however be careful. The transformations eq.(17) with gauge functions $\lambda$ that satisfy $\lim_{x \to \infty} \lambda(x) \to 0$ are indeed generated by $\exp\{i \int \lambda(x) \partial_i B_i(x)\}$ and are therefore gauge symmetries. However if the function $\lambda$ does not vanish somewhere at the spatial infinity, the transformation eq.(17) is generated by $\exp\{i \int \partial_i (\lambda B_i)\}$. The operator of the transformation is not a unit operator on the constraint eq.(18) and the transformation therefore is a true physical symmetry. So, for example if $\lambda(x) = a \delta^2(x - x(V_S))$ where $x(V_S)$ are points inside a half space bounded by the surface $S$, one has a global transformation generated by $\Phi_S$ of eq.(19). The magnetically gauge invariant operators are the t’Hooft’s loops $V(C)$ (the dual analogs of Wilson loops) or infinite t’Hooft’s lines

$$V(C) = \exp\{ig \int_C dl_i \chi_i\}$$  \hspace{1cm} (19)
In a theory of a free photon the constant \( g \) is not quantized.

In the interacting theory the Gauss’s law is

\[
\partial_i E_i - J_0 = 0
\]  

(20)

We can now define the dual potential in the following way. Since the electric current is conserved it can be potentiated

\[
J_\mu = \epsilon_{\mu\nu\lambda\rho} \partial_\nu K_{\lambda\rho}
\]  

(21)

The tensor potential \( K_{\mu\nu} \) is defined up to a Kalb-Ramond gauge transformation

\[
K_{\mu\nu} \rightarrow K_{\mu\nu} + \partial_\mu M_\nu
\]  

(22)

Let us fix this Kalb-Ramond gauge symmetry by requiring that for any surface \( S \)

\[
\int_S dS^i \epsilon_{ijk} K_{jk} = en(S)
\]  

(23)

where \( e \) is the electric coupling constant and \( n(S) \) is an integer which depends on the surface. In QED this is an admissible gauge fixing, since the divergenceless part of \( \epsilon_{ijk} K_{ij} \) can be changed arbitrarily by a choice of \( M_i \) and the charge inside any closed surface is an integer of \( e \). The dual vector potential is then defined by

\[
E_i - \epsilon_{ijk} K_{jk} = \epsilon_{ijk} \partial_j \chi_k
\]  

(24)

With this definition one has

\[
\exp\{i \frac{2\pi n}{e} \int_C dl_i \chi_i \} = \exp\{i \frac{2\pi n}{e} \int_{S;\partial S=C} dS^i E^i \}
\]  

(25)

Comparing this with eq.(14) we find

\[
V(C) = \exp\{i \frac{2\pi}{e} \int_C dl_i \chi_i \}
\]  

(26)
3 The Goldstone theorem and the topological interpretation of the electric charge.
The classical approximation.

3.1 The Goldstone theorem.

We have now constructed eigenoperators of the magnetic symmetry in 3+1 dimensions. The first natural question to ask is whether their vacuum expectation value vanish. First let us consider the classical approximation. We will calculate the quantum corrections in the next section. Although in the infinite system they change the results in a very important way, we will see in the next section that in the IR regularized system where some of the dimensions are compact the classical result is indeed qualitatively correct.

In the classical approximation the electric field as well as the electric charge density in the vacuum vanish. Therefore the dual vector potential has a "pure gauge" form

$$\chi_i = \partial_i \lambda$$

As discussed earlier the dual potentials $\chi_i$ that are given by the functions $\lambda$ with different values on spatial infinity are not gauge equivalent.

Therefore in this approximation the QED vacuum is infinitely degenerate with degeneracy that corresponds to global transformations generated by $\Phi_s$. The expectation value of a vortex operator in any of these vacua also does not vanish and is given by

$$< V(C) > = \exp \left\{ \frac{2\pi}{e} \int d_i \chi_i \right\}$$

This still does not answer the question whether the Goldstone theorem apply if $V(C)$ has a nonvanishing expectation value. The problem is that $V(C)$ are nonlocal operators and no local order parameters of $\Phi_i$ exist. The situation with the electric charge in the Higgs phase seems similar. There the Goldsone theorem was not applicable and no massless particle existed for the following reason. Suppose one has a spontaneously broken charge $Q$. For the Goldstone theorem to hold \cite{14} there must exist an operator $O$ which satisfies the following two conditions

$$\lim_{V \to \infty} < [Q_V, O] > \neq 0$$

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Here $Q_V \equiv \int_V d^D x J_0(x)$ is the spontaneously broken charge in the volume $V$. To satisfy eq. (29) it is sufficient to find any order parameter of $Q$ with non-vanishing expectation value. It is less trivial to satisfy the second equation. If the operator $O$ is local, then eq. (30) is satisfied automatically:

$$\lim_{V \to \infty} \langle [\dot{Q}_V(t), O] \rangle = 0$$

(30)

In the case of magnetic symmetry it turns out however, that the Goldstone theorem is indeed applicable even though the order parameter is nonlocal. To see this let us consider the charge $\Phi_3$ - the magnetic flux in the $z$ direction. The corresponding order parameter is $V(C_3)$ of eq. (13). The regularized flux operator $\Phi_3(L)$ is defined as in eq. (10) but without taking the limit $L \to \infty$. For any finite $L$, $V(C_3)$ is still an order parameter. Therefore if $\langle V(C_3) \rangle \neq 0$, eq. (29) is satisfied. Furthermore, only the boundary of the volume $V$ in which $\Phi_3(L)$ is defined which is perpendicular to the axis $x_3$ is crossed by the fluxon created by $V(C_3)$. Therefore the only nonvanishing contribution to eq. (30) can arise from the commutator of $V(C_3)$ with the third component of the magnetic current. However, the third component of the current is $\tilde{F}_{33}$ and vanishes identically at all times due to antisymmetry of $\tilde{F}_{\mu\nu}$. Therefore eq. (30) is satisfied also and the Goldstone theorem is applicable. The corresponding Goldstone boson is the linearly polarized photon with magnetic field in the $x_3$ direction.

Clearly the same argument applies to all the charges $\Phi_i$ if one chooses $V(C_i)$ as corresponding order parameters. In this way photons with any direction of the wave vector and polarization should have a gapless dispersion relation e.g. to be massless.

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1 One can also restrict the integration in the perpendicular directions $x_1$ and $x_2$ to a finite domain, but this is irrelevant to our argument
3.2 Topological interpretation of electric charge.

Let us now show that the electric charge has an explicit topological interpretation in terms of the vortex operators. First let us explain what do we mean by this. As we mentioned in the Introduction, the electric current is trivially conserved. In the usual representation of the electrodynamics via gauge fields the charge however is not explicitly given as some “winding number” but rather as a surface integral of the electric field.

Consider a state with a pointlike charged particle at the origin. We know that if we place an infinite magnetic vortex somewhere in space and move it adiabatically around the charge, no matter how large the distance between the vortex and the charge is, the Aharonov - Bohm (or rather the Aharonov - Casher) \[15\] phase will be accumulated. In order for that to happen, the vortex must complete the rotation around the charge. This means then, that although locally the charged state at spatial infinity is indistinguishable from the vacuum (for example \( F_{\mu\nu}, J_\mu \) and \( T_{\mu\nu} \) all vanish) there exists some global characteristics which does distinguish between them.

Remembering that the QED vacuum is in fact degenerate, the natural possibility is that locally at every point at infinity the charged state is similar to one of the vacua, but moving from point to point we actually move from one vacuum to another. If this is the case, then when we complete the rotation we should, of course come back to the same vacuum. If this closed path in the manifold of vacuum states is incontractible there should be a topological winding number associated with it. Given the fact that the electric charge in QED has features characteristic of a topological charge (trivially conserved and quantized) it is natural to expect that it is identical to this winding number.

Note that although the notion of topology of the manifold of the vacua originates in the classical field theory, it has a precise quantum meaning. Suppose one has a vacuum degeneracy in the quantum theory due to spontaneous breaking of some symmetry group \( G \). The different vacuum states will differ not only in expectation value of the order parameter \( O \) but also all its correlators and, in fact all operators which are nontrivial representations of \( G \). However, since the vacuum degeneracy is only due to the spontaneous breaking of \( G \), the VEV of any operator \( O_i \) on which the action of \( G \) is free, unambiguously determines the values of all the other correlators. Therefore the possible values of this order parameter \( O \) can be taken to parametrize the
manifold of the quantum vacua and the topology is identical to the classical one.

Analogously one defines a notion of a topological soliton which "interpolates between different vacuum states" at spatial infinity. Usually (when the broken symmetry has a local order parameter) this is the state with the following properties. Consider a chunk of space $T$ of linear dimension $a$ at a distance $R$ from the soliton core, so that $\frac{a}{R} \to 0$. Then the expectation value of any local operator $O(x)$, $x \in T$ and any correlator of local operators $O_1(x_1)...O_n(x_n)$, $x_1,...,x_n \in T$ will be equal to their vacuum expectation values in one of the vacua. In another chunk $T_1$ which is also very far from the soliton core but also far from $T$ so that $|x_T - x_{T_1}| = o(R)$ the values of these correlators are given by their expectation values in another vacuum state. So that in this soliton state at each "point" at infinity one has a vacuum state in the sence that all local and quasilocal operators (operators with finite support) have expectation values equal to their VEVs in a vacuum. These vacua are however different at different points at infinity and the mapping from the spatial boundary into the vacuum manifold is not homotopic to a trivial map. The soliton charge in this case is equal to the winding number corresponding to the homotopy group $\Pi_{D-1}(M)$, where $D - 1$ is the dimension of a spatial boundary and $M$ is the manifold of the vacua.

This was precisely the picture in QED in 2+1 dimensions. The vacuum manifold was $S_1$ corresponding to the phase of the VEV of the vortex operator $V(x)$. In a charged state with charge $n$ the configuration of the vortex field looked asymptotically like a hedgehog : $V(x) \to_{x \to \infty} e^{in\Theta}$, and the electric charge was equal to the winding number corresponding to the homotopy group $\Pi_1(S^1) = \mathbb{Z}$.

The situation in $QED_{3+1}$ is slightly different. The vacuum (at least in the classical approximation) is still degenerate. However the broken symmetry group is represented trivially on all local operators. The only operators that carry the broken charges and whose VEVs therefore distinguish between different vacua are the infinitely long magnetic vortex lines $V(C)$. It is clear therefore, that in any soliton like state (if it exists) all quasilocal operators will have the same VEV at all points at spatial infinity. The soliton is not characterized by $\Pi_2(M)$, or rather $\Pi_2(M) = 0$. To see the difference between different regions of space far from the soliton core one has to calculate the VEV of $V(C)$. One therefore has to divide the spatial infinity not into quasipointlike regions but rather into quasitringlike regions and compare
VEVs of $V(C)$ and their correlators (which fit into one such region).

The set of all the vortex operators is overcomplete. Like in the case when a local order parameter exists, it is enough to pick the minimal set of operators so that every group element of the spontaneously broken group be represented nontrivially. In our case the set of broken charges is $\{\Phi_S\}$. The most convenient choice for $\{V(C)\}$ is the set of all straight lines.

The operators whose VEVs one compares should be transformable into each other by translation. The operation of translation does not change either the orientation or the form of a string. Moreover, all the points on a string should be far from the soliton core. Therefore for a given straight vortex line the set of operators to which it should be compared can be chosen as the set of all straight vortices having the same direction and the same distance from the soliton core, in the limit where this distance becomes infinite. We see therefore that the relevant homotopy is the first rather than the second homotopy group $\Pi_1(M)$. As we have discussed earlier, the vacuum manifold is infinitely dimensional, corresponding to infinite number of the broken charges $\Phi_S$ and therefore this homotopy group is huge. However if we only consider rotationally symmetric solitons, the things simplify considerably. Since the straight lines in different directions can be all transformed into each other by a rotation, for the rotationally symmetric soliton the winding numbers for all sets of straight lines is the same. Since the vortex operator creating a straight line in a given direction takes values in $S^1$, we see that for rotationally symmetric configurations the soliton charges must take values in $\Pi_1(S^1) = Z$.

Let us now calculate expectation values of the magnetic vortex lines in the third direction in a state with electric charge at the origin. We again do this in the classical approximation. The vortex operator which creates a fluxon parallel to the third axis with coordinates $(X_1, X_2)$ is

$$V(X_1, X_2) = \exp\left\{i\frac{2\pi}{e} \int_{-\infty}^{\infty} dx_3 \int_{X_1}^{\infty} dx_1 E_2(x_1, X_2, x_3) \right\}$$

In the classical approximation the phase factor is proportional to the electric flux through the half plane ($x_2 = X_2, x_1 > X_1$). For the spherically symmetric configuration of a pointlike electric charge this is proportional to the planar angle $\Theta$ between the vector $(X_1, X_2)$ and the axis $x_1$. Since the total flux is equal to $e$, we find

$$< V(R, \Theta) > = \exp\{i\Theta\}$$

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For pointlike electric charge this expression is valid for any $R$. If the charged state has some charge distribution, the expression eq. (33) will be still valid asymptotically for $R \to \infty$. In the state with electric charge $eN$, one clearly has

$$< V(R, \Theta) >= \exp\{iN\Theta\}$$

(34)

So we see that electrically charged states realize the nontrivial windings of the vortex operators. The electric charge is equal to the winding number.

It is quite easy to construct states with winding numbers corresponding to more general elements of $\Pi_1(M)$ and not only $\Pi_1(S^1)$. For example consider a charged state which is not spherically symmetric but has all the electric flux lines asymptotically parallel to the $(x_1x_3)$ plane. In this case all the operators considered earlier will have a unit expectation value, since no flux crosses the $(x_1x_3)$ plane. However the fluxons in the direction $x_2$, for example will still have a winding number 1. So this state has a nonzero winding with respect to transformations generated by $\Phi_2$, but is trivial with respect to transformations generated by $\Phi_3$.

However the mere fact that one can construct a state with a given topological charge does not mean that it is necessarily realized in the theory. It must also pass the test of having a finite energy. Electrically charged states which are not asymptotically rotationally invariant have infinite energy and are of no interest in QED.

As a final comment in this section we note that in the classical approximation the Higgs phase can not be studied. Since the vortex operator is defined as a unitary operator, classically its VEV can not be zero and therefore we are always in the Coulomb phase. This is similar to the nonlinear $\sigma$ - model, where in the classical approximation one does not see the unbroken phase. Quantum corrections of course induce the phase transition there. In the present case as we will see in the following section, the same phenomenon occurs.

4 Quantum corrections and the infrared regularizrarion.
4.1 Quantum corrections to $< V(C) >$.

We will now calculate the $< V(C_3) >$ taking into account the lowest order quantum fluctuations.

Let us start with the Coulomb phase. The lowest order in $e$ correction to the classical result is

$$< V(C_3) >= \exp\left\{ -\frac{1}{2} \left(\frac{2\pi}{e}\right)^2 \int d^4 k a_i(k) a_j(-k) G_{ij}(k) \right\}$$

(35)

where

$$a_i(k) = \delta_{i2} \frac{1}{k_1} \delta(k_3)$$

(36)

and $G_{ij}(k)$ is the propagator of the electric field

$$G_{ij}(k) = i \left\{ \frac{k_0^2}{k^2} (\delta_{ij} - \frac{k_i k_j}{k_0^2}) - \delta_{ij} \right\}$$

(37)

The integral in eq.(35) is both ultraviolet and infrared divergent. Introducing the ultraviolet cutoff $\Lambda$ and the infrared cutoff (in real space) $L$, we find

$$< V(C_3) >= \exp\left\{ -\left(\frac{2\pi}{e}\right)^2 \Lambda L \right\}$$

(38)

The reason for both divergencies is intuitively clear. The ultraviolet divergence appears since the vortex line created by $V(C)$ has zero thickness. It can be dealt with by either regularizing the vortex itself (making it finite in cross section) or by multiplicative renormalization \[11\]. The infrared divergence comes about because of the infinite length of the vortex line.

So we find that in one very important respect the quantum corrections change the classical result. Now in the limit $L \to \infty$ we have $< V(C) > \to 0$. The situation is similar in some sense to 1+1 dimensional field theories having a continuous global symmetry. There too, in the classical approximation one can have nonvanishing order parameter which, however is found to vanish when quantum fluctuations are taken into account. As a result a continuous symmetry is never broken in 1+1 dimensions.

There is however a very important physical difference between the two cases. In the 1+1 dimensional models the order parameter is local. The

\[2\] Since the calculation is analogous to the corresponding one in 2+1 dimensions we skip the details that can be found in \[\]
vanishing of its expectation value therefore persists also for a finite infrared
cutoff as long as the cutoff theory preserves the symmetry. Technically,
the system is disordered by the zero mode. If the infrared regularization is
done in such a way that the ”spontaneously broken” symmetry is not broken
explicitly, the zero mode is still present and it still leads to the vanishing of
the VEV of the order parameter. If the regularization is such that the zero
mode is given a finite mass, the VEV can be nonzero, but the symmetry is
then broken explicitly. In the case at hand though, the order parameter is
nonlocal and this nonlocality, rather than the zero mode contribution is the
factor which leads to the vanishing of the VEV. This can be seen explicitly
by taking a massive rather than a massless propagator in eq. (37). The result
is still linearly infrared divergent. Therefore it is clear that one can find
an infrared regularization which without explicit breaking of the magnetic
symmetry will yield a finite expectation value for the vortex operator. In this
regularized theory the arguments concerning the Goldstone theorem and the
topological interpretation of the electric charge presented in the previous
section in the classical approximation will be valid also in the full quantum
theory.

But before doing that let us calculate $< V >$ in the Higgs vacuum. The
most convenient way to do this is using the euclidean path integral formalism.
The expectation value $< V(C_3) >$ can be written in the following form [13]:

$$< V(C_3) > = \int \mathcal{D}\phi \exp\left\{ -\frac{1}{4e^2} (\tilde{F}_{\mu\nu} - \tilde{f}_{\mu\nu})^2 + |D_{\mu}\phi|^2 + U(\phi^*\phi) \right\} \quad (39)$$

where

$$\tilde{f}_{\mu\nu}(x) = \delta_{[\mu} \delta_{\nu]} \delta(x_0) \delta(x_2) \theta(x_1)$$

with $\theta(z)$ a step function. For any given $x_3$ the field $\tilde{f}$ satisfies

$$\partial_\nu \tilde{f}_{\mu\nu} = \delta_{3\mu} \delta^3(x) \quad (40)$$

If we now view $x_3$ as the euclidean time, $\tilde{f}$ is the magnetic field of the
Dirac string of a static magnetic monopole propagating in time. At the tree
level therefore the VEV is given by a Euclidean action of a static magnetic
monopole in the Higgs phase. In the Higgs phase magnetic monopoles are
linearly confined and the energy of a single monopole diverges linearly with
the dimension of a system. The action therefore diverges quadratically and
we obtain

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\[< V(C_3) > = e^{-\alpha L_1 L_3} \]  

(41)

where \(\alpha\) is a dimensional constant, and \(L_1\) and \(L_3\) are infrared cutoffs on the first and the third directions respectively.

We see that the VEV vanishes much faster in the infrared than in the Coulomb phase. In fact even if we make the system finite in the direction of the vortex line, the VEV still vanishes in the limit \(L_1 \to \infty\). So even though in the infinite system the VEV of the vortex operator vanishes in both phases, there is a qualitative difference in the dependence on the infrared cutoff. This difference will be reflected in the VEV of a closed vortex loop (t’Hooft loop). Evidently the large loops in the Coulomb phase will have a perimeter law behaviour, while in the Higgs phase - the area law. This result of course coincides with t’Hooft’s discussion of expected behaviour of vortex loops [12].

4.2 The infrared regularization.

Let us now describe the simplest infrared regularized theory which has a finite VEV of the vortex operator in the Coulomb phase. Consider QED defined on a spatial manifold which is compact in the direction of the \(x_3\) axis. This means that all the gauge invariant fields \((B_i, E_i, J_0\) etc.) must at all times obey the periodic boundary condition

\[O(x_1, x_2, x_3) = O(x_1, x_2, x_3 + L)\]  

(42)

In this theory the magnetic flux \(\Phi_3\) is still a conserved charge. We will concentrate on it and on its order parameter the vortex line \(V(C_3)\). As previously, \(V(C_3)\) is a well defined operator, except that now the vortex line it creates has a finite length \(L\). The calculation of \(< V(C_3) >\) is the same as previously. The only alteration is that the photon’s propagator must be modified according to the new boundary condition, so that \(k_3\) in eq.(37) takes discrete values \(k_3 = \frac{2\pi n}{L}\). The exact form of the propagator, however does not matter as we have seen earlier and we obtain

\[< V(C_3) > = e^{-\alpha L} \]  

(43)

The proof of the Goldstone theorem goes through in precisely the same way as in the unbounded case since the dual field strength tensor remains antisymmetric. The Goldstone bosons that appear due to spontaneous breaking
of $\Phi_3$ are the linearly polarized photons with magnetic field pointing in the direction $x_3$.

Note that in the Higgs phase $< V(C_3) > = 0$ because of the infinite extent of our system in the direction $x_1$. The Higgs - Coulomb phase transition therefore is attributed to the spontaneous breaking of $\Phi_3$.

One also immediatelly realizes that an electrically charged state has a nonzero winding of $V(C_3)$. The configuration of the electric field of a point charge near the location of a charge is the same as in the unbounded case. However since the electric flux can not escape through the boundary due to periodic boundary conditions, near the boundary the electric flux lines get squeezed and become parallel to the $x_1x_2$ plane, so that all the flux escapes to infinity (Fig.3.).

Repeating now the calculation of a previous section we find that the surface associated with $V(C_3)$ collects the electric flux proportional to the planar angle, and therefore the eq.(33) is still valid.

If we would have taken free instead of periodic boundary conditions, part of the electric flux would have escaped through the boundary and we wouldn’t have had a complete wind of $V(C_3)$. However in this case the electric charge would also not be conserved, since charged particles would be able to leave the system freely through the boundary. And of course, if a charge is not conserved it can not be topological.

Other infrared regularizations are possible. For example, one could take the system to be finite in two directions $x_3$ and $x_2$. Then both $< V(C_3) >$ and $< V(C_2) >$ would be nonvanishing in the Coulomb phase. The masslessness of photons with two linear polarizations would then follow by Goldstone’s theorem. If the boundary conditions preserve the $\pi/2$ rotations around the first axis, the finite energy electrically charged states will carry a unit winding of both $V(C_2)$ and $V(C_3)$. Note however that the regularization in which all three directions are made compact is illegal since in this case $V(C)$ can not be defined. The reason is that the surface of singularities associated with $V(C)$ must be infinite and there are no such surfaces in a completely finite system. The same is true in $2 + 1$ dimensions where one can not define the
local vortex field $V(x)$ in a finite system with periodical boundary conditions.

We see therefore, that any sensible infrared regularization leads to a non-vanishing VEV of the vortex operators. The Goldstone’s theorem for the photon and the identity of the electric charge with the winding number hold for any finite value of the infrared cutoff. In this sense both results are also true in the unbounded theory although the actual VEV of the order parameter vanishes.

Note that $QED_4$ differs from a typical field theory in the following respect. Usually the tendency of a smaller system is towards a restoration of any broken symmetries, since the potential barrier between the degenerate vacua becomes smaller. In a certain sense this is why the high temperature phase is usually the one where all the symmetries are restored. In $QED_4$ as we have seen the opposite happens. Due to the nonlocality of the order parameter, its expectation values is actually larger for a smaller system. This might be connected to the fact that in $QED_4$ the high temperature phase is the Coulomb phase, in which the flux symmetry is broken whereas the low temperature, Higgs (superconducting) phase has the symmetry restored.

5 Discussion.

In this paper we looked at QED from an unconventional point of view. Instead of concentrating our attention on the standard degrees of freedom like photons and charged particles, we have analysed the behaviour of the dual variables - the magnetic vortex lines. The picture that transpires from this point of view is somewhat similar to 2+1 dimensional electrodynamics.

In the Coulomb phase the operators creating infinitely long vortex lines $V(C)$ have “finite expectation value per unit length”. What this means is that the expectation value of such an operator in a system with a finite infrared cutoff in the direction of the vortex line behaves as $e^{-\alpha L}$ in the Coulomb phase. The operators $V(C)$ are eigenoperators of the magnetic symmetry generators $\Phi_S$. Therefore in the Coulomb phase the magnetic symmetry group is spontaneously broken. Although the only order parameters for $\Phi_S$ are the nonlocal vortex operators, the Goldstone theorem is still applicable and the spontaneous breaking of this symmetry leads to exact masslessness of the photon.

The electric charge in this picture is topological and corresponds to the
homotopy group $\Pi_1(S^1)$ of possible string configurations.

In the Higgs phase the VEV $\langle V(C) \rangle$ vanishes. The magnetic symmetry is restored and no massless excitations are present. The Higgs - Coulomb phase transition is driven by condensation of the magnetic vortices.

According to the standard lore a theory near a phase transition (and also away from the phase transition but at low energies) should be describable in terms of the Landau - Ginzburg type Lagrangian for the order parameter. In the case at hand this would not be a standard field theory but rather a string theory of the vortex lines $V(C)$. An approximate derivation of this string theory is given in [16]. In fact the dual field strength tensor $\tilde{F}_{\mu\nu}$ can be expressed via $V(C)$ in the same way as $F_{\mu\nu}$ is expressed via the Wilson line [17]

$$\tilde{F}_{\mu\nu}(x) = V^\dagger \frac{\delta}{\delta S_{\mu\nu}(x)} V(C)$$

(44)

where $\frac{\delta}{\delta S_{\mu\nu}(x)}$ is the area derivative at the point $x$. The kinetic term therefore can be rewritten in terms of the vortex creation operator as

$$\frac{\delta}{\delta S_{\mu\nu}} V^\dagger(C) \frac{\delta}{\delta S_{\mu\nu}} V(C)$$

(45)

There is an interesting possibility that the exact dual reformulation of QED$_4$ exists. Then QED should be exactly equivalent to an interacting string theory. Clearly the weakly interacting QED will be described by a strongly interacting dual string theory. This must be so, since the spectrum of a free string theory contains an infinite number of particles, whereas the spectrum of QED contains just the familiar excitations: the photon and the charged particles. It should also be noted that a massless photon will arise in this string theory in a way very different from massless gauge particles in a free string theory, since it is massless in the phase in which the strings are condensed.

There are many further questions which have been asked in the context of 2+1 dimensional gauge theories which we did not address in this paper. For example, what elements of the dual picture should be modified if the matter fields are fermionic. But the most interesting one is, perhaps can this picture be generalized to nonabelian theories. It would be very rewarding to have a simple qualitative picture of confinement based on topological interpretation of electric charge similar to the one available in 2+1 dimensions [18]. There constituent quarks can be understood as topological defects
like electric charges in QED, but the flux symmetry is broken explicitly (by the nonperturbative monopole instanton effects). As a result of this explicit breaking the vacuum is nondegenerate (or has a finite degeneracy) and the topological defects are linearly confined. In nonabelian theories in 3+1 dimensions there are also nonperturbative effects due to magnetic monopoles (which now are particles rather than instantons). The appearance of the monopoles again breaks explicitly the magnetic symmetry since the dual field strength is not conserved anymore. It is interesting to see whether this explicit breaking leads to linear confinement of the topological defects as in 2+1 dimensions, although the defects now are of quite a different nature. It is also worth noting that in $SU(N)$ theories with adjoint matter fields only, the monopoles carry $N$ units of the elementary Dirac quantum. Therefore the discreet subgroup of the magnetic group will still survive (just like in 2+1 dimensions). The phase transitions between the ”completely broken” Higgs phase and a confinement phase can then be attributed to the spontaneous breaking of this discreet symmetry.

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