Humps for Dyck and for Motzkin paths

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Abstract

We calculate the total number of humps in Dyck and in Motzkin paths, and we give Standard-Young-Tableaux-interpretations of the numbers involved. One then observes the intriguing phenomena that the humps-calculations change the partitions in a strip to partitions in a hook.

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1 Introduction

Let $\lambda$ be a partition and denote by $f^{\lambda}$ the number of standard Young tableaux (SYT) of shape $\lambda$. The number $f^{\lambda}$ can be computed for example by the hook formula [9, Corollary 7.21.6].

1.1 The $(k, \ell)$-hook sums

We consider the SYT in the $(k, \ell)$ hook. More precisely, given integers $k, \ell, n \geq 0$ we denote

$$H(k, \ell; n) = \{ \lambda = (\lambda_1, \lambda_2, \ldots) | \lambda \vdash n \text{ and } \lambda_{k+1} \leq \ell \}$$

and

$$S(k, \ell; n) = \sum_{\lambda \in H(k, \ell; n)} f^{\lambda}.$$ 

We remark that classically, the partitions $\lambda \in \cup_{n \geq 0} H(k, 0; n)$ parametrize the irreducible representations of the Lie algebra $gl(k, \mathbb{C})$. Also, the partitions $\lambda \in \cup_{n \geq 0} H(k, \ell; n)$ parametrize those of the Lie super-algebra $pl(k, \ell)$ [1].

For the "strip" sums $S(k, 0; n)$ it is known [6] [9] that

$$S(2, 0; n) = \binom{n}{\lceil \frac{n}{2} \rceil} \quad \text{and} \quad S(3, 0; n) = \sum_{j \geq 0} \frac{1}{j+1} \binom{n}{2j} \binom{2j}{j}.$$ 

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Furthermore, Gouyon-Beauchamps \cite{Gouyon3} \cite{Gouyon9} proved that
\[
S(4, 0; n) = C_{\left\lfloor \frac{n}{2} \right\rfloor} \cdot C_{\left\lceil \frac{n}{2} \right\rceil} \quad \text{and} \quad S(5, 0; n) = 6 \sum_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n}{2j} \cdot C_j \cdot \frac{(2j + 2)!}{(j + 2)!(j + 3)!},
\]
where
\[
C_j = \frac{1}{j + 1} \binom{2j}{j}
\]
are the Catalan numbers. (1)

So far only the ”hook” sums \(S(1, 1; n)\) and \(S(2, 1; n) = S(1, 2; n)\) have been calculated: it is easy to see that \(S(1, 1; n) = 2^{n-1}\); for the sum \(S(2, 1; n)\) see Equation (11) below.

1.2 Humps-calculations of paths

1.2.1 Dyck paths

The Catalan numbers \(C_n\) count many combinatorial objects \cite{Gouyon9}. For example, they count the number of SYT of the \(2 \times n\) rectangular shape \((n, n)\), namely \(C_n = f_{(n,n)}\), which can be thought of as a SYT-interpretation of the Catalan numbers. It is also well known that \(C_n\) counts the number of Dyck paths of length \(2n\). A Dyck path of length \(2n\) is a lattice path, in \(\mathbb{Z} \times \mathbb{Z}\), from \((0, 0)\) to \((2n, 0)\), using up-steps \((1, 1)\) and down-steps \((1, -1)\) and never going below the \(x\)-axis.

A hump in a Dyck path is an up-step followed by a down-step. For example, there are 2 Dyck paths of length 4, with total number of humps being 3; and there are 5 Dyck paths of length 6, with total number of humps being 10. We denote by \(\mathcal{HC}_n\) the total number of humps in the Dyck paths of length \(2n\). Thus \(\mathcal{HC}_2 = 3\) and \(\mathcal{HC}_3 = 10\). Remark 2.2.1 gives a SYT-interpretation of the numbers \(\mathcal{HC}_n\).

1.2.2 Motzkin paths

A Motzkin path of length \(n\) is a lattice path from \((0, 0)\) to \((n, 0)\), using flat-steps \((1, 0)\), up-steps \((1, 1)\) and down-steps \((1, -1)\), and never going below the \(x\)-axis. The Motzkin number \(M_n\) counts the number of Motzkin paths of length \(n\). Recall also \cite{Gouyon6} \cite{Gouyon9} that
\[
M_n = S(3, 0; n) = \sum_{\lambda \in \mathcal{H}(3,0;n)} f^\lambda,
\]
which gives a SYT-interpretation of the Motzkin numbers \(M_n\). Bijective proofs of (2), as well as of related identities due to Zeilberger \cite{Zeilberger10}, have recently been given in \cite{Gouyon2}.

A hump in a Motzkin path is an up-step followed by zero or more flat-steps followed by a down-step. We denote by \(\mathcal{HM}_n\) the total number of humps in the Motzkin paths of length \(n\). Thus \(\mathcal{HM}_2 = 1\) and \(\mathcal{HM}_3 = 3\). Theorem 3.3 gives a SYT-interpretation of the numbers \(\mathcal{HM}_n\).
1.2.3 The main results

The main results here are explicit formulas for the total number of humps for the Dyck paths and for the Motzkin paths of a given length, together with SYT-interpretation of these numbers. One then observes the following intriguing phenomena. Theorem 2.1 below shows that the total number of humps $\mathcal{H}C_n$ for the Dyck paths of length $2n$ satisfies $\mathcal{H}C_n = f^{(n,1^n)}$. Together with $C_n = f^{(n,n)}$ this shows, roughly, that the humps-calculations correspond the $2 \times n$ rectangular shape $(n, n)$ to the $1-1$ hook shape $(n, 1^n)$. A somewhat similar phenomena occurs when studying humps in Motzkin paths: while $M_n = S(3,0;n)$, Theorem 3.3 asserts that $\mathcal{H}M_n = S(2,1;n) - 1$, which gives a SYT-interpretation of the numbers $\mathcal{H}M_n$. This shows, roughly, that the humps-calculations correspond the $(3,0)$ strip shape to the $(2,1)$ hook shape.

We also consider "super" such paths, which are allowed to also go below the $x$-axis, and show that their number is essentially twice the number of the corresponding humps, see Remarks 2.2 and 3.2.

In Section 4 we give a double Dyck path interpretation for the sums $S(4,0;n)$, together with the corresponding hump-numbers $\mathcal{H}S(4,0;n)$. Proposition 4.2 gives the intriguing identity $\mathcal{H}S(4,0;n) = \frac{n+3}{2} \cdot S(4,0;n)$.

Remark 1.1. It would be interesting to find bijective proofs to the identities in this paper.

2 Humps for Dyck paths

Recall that $C_n = f^{(n,n)} = \frac{1}{n+1} \binom{2n}{n}$, which is a SYT-interpretation of the Catalan numbers $C_n$.

Theorem 2.1. Let $\mathcal{H}C_n$ denote the total number of humps for all the Dyck paths from $(0,0)$ to $(2n,0)$, then

$$\mathcal{H}C_n = \binom{2n-1}{n}.$$  

(3)

Remark 2.2. 1. Let $\lambda$ be the $1-1$ hook shaped diagram $\lambda = (n, 1^n)$, then

$$f^{\lambda} = \binom{2n-1}{n} = \mathcal{H}C_n,$$

which gives a SYT-interpretation of the numbers $\mathcal{H}C_n$.

2. A super Dyck path is a Dyck path which is allowed to also go below the $x$-axis. Let $SD_n$ denote the number of super Dyck paths of length $2n$. By standard arguments it follows that

$$SD_n = \binom{2n}{n} = 2 \cdot \binom{2n-1}{n} = 2 \cdot \mathcal{H}C_n.$$
The proof of Theorem 2.1 clearly follows from the following two lemmas.

**Lemma 2.3.** We have $\mathcal{H}C_0 = \mathcal{H}C_1 = C_0 = C_1 = 1$, and the Catalan numbers $C_n$ and the numbers $\mathcal{H}C_n$ satisfy the following equation

$$\mathcal{H}C_n = \mathcal{H}C_{n-1} + \sum_{j=1}^{n-1} (\mathcal{H}C_{j-1} \cdot C_{n-j} + C_{j-1} \cdot \mathcal{H}C_{n-j}).$$

(4)

*Proof.* Given a Dyck path $D = D_n$ of length $2n$, we read it from left to right. Then $D_n$ meets the $x$-axis for the first time after $2j$ steps, $1 \leq j \leq n$.

*Case 1: $j = n$.* This implies that except for the endpoints $(0,0)$ and $(2n,0)$, $D$ does not meet the $x$-axis. Thus $D_n = s_1 \cdots s_n$ where $s_1$ is an up step, $s_n$ is a down step, and $s_2 \cdots s_{n-1}$ corresponds to a Dyck path $D_{n-1}$ in an obvious way. Also, the number of humps of $s_1 \cdots s_n$ and of $s_2 \cdots s_{n-1}$ is the same.

It follows that the Dyck paths of Case 1 contribute $\mathcal{H}C_{n-1}$ to the total number of humps $\mathcal{H}C_{n-1}$.

*Case 2: $j \leq n-1$.* Thus $D_n$ is the concatenation of two Dyck paths $D_n = D_{2j}D_{2(n-j)}$, where $D_{2(n-j)}$ is an arbitrary Dyck path of length $2(n-j)$ but $D_{2j}$, of length $2j$, is of the type studied in Case 1. Thus there are $C_{j-1}$ paths $D_{2j}$, with total humps-contribution being $\mathcal{H}C_{j-1}$. And there are $C_{2(n-j)}$ paths $D_{2(n-j)}$, with total humps-contribution being $\mathcal{H}C_{2(n-j)}$.

Equation (4) now follows, since the number of humps $\mathcal{H}(D_n)$ of $D_n = D_{2j}D_{2(n-j)}$ is the sum $\mathcal{H}(D_n) = \mathcal{H}(D_{2j}) + \mathcal{H}(D_{2(n-j)})$.

□

**Lemma 2.4.** Together with $\mathcal{H}C_0 = \mathcal{H}C_1 = 1$, Equation (4) implies Equation (3).

*Proof.* [D. Zeilberger] By induction on $n$, replace each $\mathcal{H}C_k$ on the right hand side of Equation (4) by $\binom{2k-1}{k}$. We obtain the following binomial identity:

$$\binom{2n-1}{n} = \binom{2n-3}{n-1} + \sum_{j=1}^{n-1} \left( \binom{2j-3}{j-1} \cdot C_{n-j} + C_j \cdot \binom{2n-2j-1}{n-j} \right)$$

(5)

This identity is easily provable by Gosper’s algorithm for indefinite summation (implemented in Maple’s sum command).

For a direct proof note first that $\binom{2k-1}{k} = \frac{k+1}{2} \cdot C_k$ if $k \geq 1$. Thus the right side of (5) is

$$2 \binom{2(n-1)-1}{n-1} + C_{n-1} + \sum_{j=1}^{n-1} \left( \frac{j}{2} \cdot C_{j-1} \cdot C_{n-j} + C_{j-1} \cdot \frac{n-j+1}{2} \cdot C_{n-j} \right) =$$

$$= \frac{n+1}{2} \cdot \sum_{j=1}^{n} C_{j-1} \cdot C_{n-j} = \frac{n+1}{2} \cdot C(n)$$

(by the defining relation for $C_n$: $C_n = \sum_{j=1}^{n} C_{j-1} \cdot C_{n-j}$), and this equals the left side of (4).

□
This completes the proof of Theorem 2.1.

3 Humps for Motzkin paths

Recall that a Motzkin path of length $n$ is a lattice path from $(0, 0)$ to $(n, 0)$, using flat-steps $(1, 0)$, up-steps $(1, 1)$ and down-steps $(1, -1)$ and never going below the $x$-axis. The Motzkin number $M_n$ counts the number of Motzkin paths of length $n$. Recall also [6] [9] that

$$M_n = \sum_{\lambda \in H(3,0;n)} f^\lambda,$$

which gives a SYT-interpretation of the Motzkin numbers $M_n$. A bijective proof of [2] has recently been given in [2].

Recall that a hump in a Motzkin path is an up-step followed by 0 or more flat-steps followed by a down-step.

**Theorem 3.1.** The number $H_n$ of humps in all Motzkin paths of length $n$ is given by

$$H_n = 1 \sum_{j \geq 1} \binom{n}{j} \binom{n-j}{j}.$$

**Remark 3.2.** A super Motzkin path is a Motzkin path which is allowed to also go below the $x$-axis. Let $SM_n$ denote the number of super Motzkin paths of length $n$, then one can prove the recurrence

$$SM_n = SM_{n-1} + 2 \sum_{k=2}^{n} M_{n-k} \cdot SM_{n-k},$$

which then implies that

$$SM_n = \sum_{j \geq 0} \binom{n}{j} \binom{n-j}{j}.$$

Together with Theorem 3.1 it implies that

$$SM_n = 2 \cdot H_n + 1.$$

**Proof of Theorem 3.1**

*Proof.* This result is stated in [8, sequence A097861], and it can be proved as follows. First argue as in Lemma 2.3. Given a Motzkin path $M = M_n$ of length $n$, we read it from left to right. Then $M_n$ meets the $x$-axis for the first time after $j$ steps, $1 \leq j \leq n$. The case $j = 1$ contributes $H_{n-1}$ to $H_n$. If $2 \leq j$, then $M_n = M_j M_{n-j}$ (concatenation) where $M_j$ starts with an up-step and ends with a down-step, while $M_{n-j}$ is an arbitrary Motzkin path of length $n - j$. Thus $M_j$ corresponds to a Motzkin path $M_{j-2}$ of length $j - 2$, with
the same number of humps as $\mathcal{M}_j$, except for the case that $\mathcal{M}'_{j-2}$ is a sequence of flat steps, in which case $\mathcal{M}_j$ contributes one hump while $\mathcal{M}'_{j-2}$ contributes zero humps.

Now $\mathcal{H}M_0 = \mathcal{H}M_1 = 0$, and for $n \geq 2$, similar to the proof of Lemma 2.3 the above argument implies the recurrence

$$\mathcal{H}M_n = \mathcal{H}M_{n-1} + \sum_{k=2}^{n} ((1 + \mathcal{H}M_{k-2}) \cdot M_{n-k} + M_{k-2} \cdot \mathcal{H}M_{n-k}).$$

(8)

Denote

$$B_n = \frac{1}{2} \sum_{j \geq 1} \binom{n}{j} \binom{n-j}{j},$$

(9)

then the proof of Theorem 3.1 follows – by induction on $n$ – from Equation (8) and from the following binomial identity

$$B_n = B_{n-1} + \sum_{k=2}^{n} ((1 + B_{k-2}) \cdot M_{n-k} + M_{k-2} \cdot B_{n-k}).$$

(10)

Equation (10) can be proved by the WZ method [5], [11].

Recall that $S(2, 1; n) = \sum_{\lambda \in \mathcal{H}(2, 1; n)} f^{\lambda}$. Here we prove the following intriguing identity.

**Theorem 3.3.** Let $\mathcal{H}M_n$ denote total number of humps for all the Motzkin paths of length $n$, then

$$\mathcal{H}M_n = S(2, 1; n) - 1.$$  

This gives a SYT-interpretation of the numbers $\mathcal{H}M_n$.

**Proof.** We prove Theorem 3.3 numerically. The following equation was proved in [7, Theorem 8.1]:

$$S(2, 1; n) = \frac{1}{4} \left( \sum_{r=0}^{n-1} \binom{n-r}{\lfloor \frac{n-r}{2} \rfloor} \binom{n}{r} + \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor-1} \frac{n!}{k! \cdot (k+1)! \cdot (n-2k-2)! \cdot (n-k-1) \cdot (n-k)} \right) + 1$$

(11)

Combining equations (7) and (11), the proof of Theorem 3.3 will follow once the following binomial identity – of interest on its own – is proved.

**Lemma 3.4.** For $n \geq 2$

$$2 \sum_{j \geq 1} \binom{n}{j} \binom{n-j}{j} = \frac{1}{4} \left( \sum_{r=0}^{n-1} \binom{n-r}{\lfloor \frac{n-r}{2} \rfloor} \binom{n}{r} + \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor-1} \frac{n!}{k! \cdot (k+1)! \cdot (n-2k-2)! \cdot (n-k-1) \cdot (n-k)} \right).$$

(12)

Equation (12) can be proved by the WZ method [5], [11].

This completes the proof of Theorem 3.3. 

□
4 Humps for $S(4, 0; n)$

4.1 Double Dyck paths interpretation for $S(4, 0; n)$

By [3],

$$S(4, 0; n) = C_{\lfloor n+1/2 \rfloor} \cdot C_{\lceil n+1/2 \rceil}.$$  \hspace{1cm} (13)

Therefore the Dyck paths interpretation for $C_m$ implies the following interpretation for $S(4, 0; n)$:

$S(4, 0; n)$ is the number of Dyck paths from $(0, 0)$ to $(2(n+1), 0)$ which go through $(2 \lfloor n+1 \rfloor, 0)$. Such a path is the concatenation of two Dyck paths, one of length $2 \lfloor n+1/2 \rfloor$ and one of length $2 \lceil n+1/2 \rceil$ (note that for any integer $m$, $\lfloor m/2 \rfloor + \lceil m/2 \rceil = m$). We call it a double-Dyck path. Clearly, the number of humps of such a concatenated path is the sum of the humps of its two parts. By arguments similar to those in the proof of Lemma 2.3 and of Theorem 3.1 this implies

Proposition 4.1. Let $\mathcal{HS}(4, 0; n)$ denote the total number of humps of the double-Dyck paths corresponding to $S(4, 0; n)$, then

$$\mathcal{HS}(4, 0; n) = HC_{\lfloor n+1/2 \rfloor} \cdot C_{\lfloor n+1/2 \rfloor} + C_{\lfloor n+1/2 \rfloor} \cdot HC_{\lceil n+1/2 \rceil}.$$  \hspace{1cm} (14)

For $n = 1, 2, \ldots$ this gives the following values for $\mathcal{HS}(4, 0; n)$:

2, 5, 12, 35, 100, 315, 980, 3234, 10584, 36036, 121968, 424710, etc.

We have the following intriguing identity:

Proposition 4.2. $\mathcal{HS}(4, 0; n) = \frac{n+3}{2} \cdot S(4, 0; n)$.

Proof. The proof is a straightforward calculations, applying Theorem 2.1, Equation (13) and Proposition 4.1. \hfill \square

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