Gauge invariance and the detection of gravitational radiation

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The detection of gravitational radiation raises some subtle issues having to do with the coordinate invariance of general relativity. This paper explains these issues and their resolution by using an analogy with the Aharonov-Bohm effect of quantum mechanics.

I. INTRODUCTION

Gravitational radiation is one of the most important predictions of Einstein’s general theory of relativity. It has been detected indirectly through its effect on the orbital period of the Hulse-Taylor binary pulsar but it has not yet been directly detected. Several gravitational radiation detectors have been built in an attempt to perform such a direct detection. These detectors are essentially large laser interferometers, and the usual explanation of how they work goes as follows: an interferometer, by using the interference of light beams that travel along different paths is very sensitive to changes in the lengths of those paths. According to general relativity, gravity is a distortion in the geometry of space. When a gravitational wave passes through the detector, it changes the lengths of the arms of the interferometer and this change is detected through its effect on the the relative phase of the two light rays. At first glance, this explanation sounds simple and clear. But on reflection some issues arise: one issue comes from thinking about the usual explanation for cosmological redshift, which is that the expansion of the universe causes a corresponding expansion in the wavelength of light. Applying this concept to the interferometer, if the wavelength of the light expands as much as the interferometer arm does, then there should be no change in phase and therefore no detection. Other issues arise from the fact that general relativity, as a theory of gravity, doesn’t just give predictions for the geometry of space, but also for the propagation of light and the motion of material objects. In addition to changes in the lengths of the interferometer arms then, one might expect additional effects due to the effect of gravity on the propagation of the light as it moves along the interferometer arms. Furthermore, the mirrors at each end of the arms are also subject to gravity, so one might expect an additional effect due to motion of these mirrors under the effects of the gravitational wave. Why are these additional effects not discussed in the usual explanation of how gravitational wave detectors work? Are these additional effects small enough to be negligible? But if so, then why are they? Are these additional effects absent? But again, if so, why are they absent?

It turns out that these questions can be answered by a careful consideration of the role of coordinate invariance in general relativity. Though coordinate invariance in general relativity is not a subject easily at the command of most physicists, it turns out that it is analogous to gauge invariance in electrodynamics. In fact the line of reasoning used to resolve the issue of the properties of gravitational wave detectors is the same as that used to understand the role of gauge invariance in the Aharonov-Bohm effect. In this paper, we will first look at the issues that come up for the Aharonov-Bohm effect and the resolution of those issues. We will then show how the same line of reasoning applied to gravitational wave detectors serves to resolve the issues raised here.

II. AHARONOV-BOHM EFFECT

The simplest of the Maxwell equations of electrodynamics is the statement that the magnetic field is divergence free: $\nabla \cdot \vec{B} = 0$. Soon after learning this equation, we learn a calculational trick to solve it: since the divergence of a curl is zero, we simply introduce a vector potential $\vec{A}$ such that $\vec{B} = \nabla \times \vec{A}$ and it then follows that for any $\vec{A}$ the $\vec{B}$ given by this formula is automatically divergence free. However, we also quickly learn that the vector potential is not unique: since the curl of a gradient is zero, it follows that for any complex scalar $\chi$, if we replace $\vec{A}$ by $\vec{A} + \nabla \chi$ then the magnetic field is unchanged. In other words, $\vec{A} + \nabla \chi$ is just as good a vector potential as $\vec{A}$. This property, the invariance of $\vec{B}$ under the transformation $\vec{A} \rightarrow \vec{A} + \nabla \chi$, is called gauge invariance. All this is not particularly disturbing. Since $\vec{A}$ is a calculational trick rather than a physical field, there is no reason for it to be unique.

However, the situation changes when we consider the quantum mechanics of charged particles in a magnetic field. A standard example of quantum mechanical behavior is the interference pattern of the two slit experiment. It is therefore natural to ask how this interference pattern changes when the particles are charged and a magnetic field is applied. The Aharonov-Bohm effect consists of this case further specialized so that the magnetic field is localized in a region between the two slits but away from the paths that the particles would take in going from the slits to the...
screen. The wavefunction satisfies the Schrodinger equation for a particle of charge $q$ in a magnetic field $\vec{B}$ which is

$$\frac{1}{2m} \left( \frac{\hbar}{i} \nabla - \frac{q}{c} \vec{A} \right)^2 \psi = E\psi$$  \hspace{1cm} (1)$$

where $\vec{A}$ is the vector potential associated with $\vec{B}$.

At first sight, this equation seems crazy. There is no such thing as the vector potential associated with $\vec{B}$. Rather there is a class of vector potentials, all equally good. But if we choose a different $\vec{A}$ it appears that Schrodinger’s equation will change. Nonetheless, despite appearances Schrodinger’s equation really is gauge invariant. It is just that whenever we make the change $\vec{A} \to \vec{A} + \nabla \chi$ we also have to change the wave function by

$$\psi \to \psi \exp(i q \chi / \hbar c)$$  \hspace{1cm} (2)$$

That is, the wave function must change by a phase.

The best way to understand this is to reverse the order of the reasoning. We know that quantum mechanics has global phase invariance: multiply the wave function by $\exp(ia)$ where $a$ is a real constant and nothing physical changes. However, it turns out that the quantum mechanics of charged particles also has local phase invariance: multiply the wave function by $\exp(if(x))$ and nothing physical changes. In order for nothing to change under a local change of phase, the theory must contain a vector field $\vec{A}$ coupled to the wave function in a particular way and the vector field must change by

$$\vec{A} \to \vec{A} + \frac{\hbar c}{q} \nabla f$$  \hspace{1cm} (3)$$

whenever $\psi$ is multiplied by $\exp(if(x))$. The gauge invariance of electrodynamics is then not the byproduct of a particular mathematical trick to calculate the magnetic field. Instead gauge invariance is a rather deep property having to do with the local phase invariance of quantum mechanics.

Now we calculate the effect of the magnetic field on the interference pattern. The particles come from a source at point $a$, go through either slit 1 or slit 2, and arrive at a point $b$ on the screen. By superposition, the wavefunction at point $b$ takes the form $\psi = \psi_1 + \psi_2$ where $\psi_1$ is the value that the wave function would have if only slit 1 were open, and correspondingly for $\psi_2$. Now, let $\psi_10$ be the value that $\psi_1$ would have if the magnetic field were turned off, and correspondingly for $\psi_20$. For all points $\vec{x}$ on the far side of the slits, define the function $W_1$ by

$$W_1(\vec{x}) = \int_{C_1} \vec{A} \cdot d\vec{l}$$  \hspace{1cm} (4)$$

Here the curve $C_1$ is essentially two straight lines: one from $a$ to slit 1 and the other from slit 1 to $\vec{x}$, though we round off the curve at slit 1 to make it smooth. Here $\vec{x}$ is any point on the far side of the slits, though in particular we will apply equation 11 to points $b$ on the screen. Correspondingly define $W_2$. The point of this somewhat clumsy looking definition is that since $W_1$ is the line integral of $\vec{A}$ it follows that $\nabla W_1 = \vec{A}$. Then using $\chi = -W_1$ in the gauge transformation $\vec{A} \to \vec{A} + \nabla \chi$ transforms $\vec{A}$ to zero in a region that includes points $a$ and $b$ and slit 1. It then follows from gauge invariance that

$$\psi_1 = \psi_{10} \exp(i q W_1 / \hbar c)$$  \hspace{1cm} (5)$$

and correspondingly for $\psi_2$. Let $\Delta \phi$ be the change in phase difference between $\psi_2$ and $\psi_1$ due to the magnetic field. Then it follows that

$$\Delta \phi = \frac{q}{\hbar c} (W_2 - W_1) = \frac{q}{\hbar c} \int_C \vec{A} \cdot d\vec{l} = \frac{q}{\hbar c} \int_S \vec{B} \cdot d\vec{a}$$  \hspace{1cm} (6)$$

Here the closed curve $C$ is the curve $C_2$ from $a$ to $b$ followed by the curve $C_1$ backwards from $b$ to $a$. The surface $S$ is one whose boundary is the curve $C$. Note that $\int_S \vec{B} \cdot d\vec{a}$ is simply the magnetic flux $\Phi$ through that surface. Thus we have

$$\Delta \phi = \frac{q}{\hbar c} \Phi$$  \hspace{1cm} (7)$$

In this problem the vector potential $\vec{A}$ and its gauge invariance properties were sources of both conceptual confusion and some helpful calculation techniques. Note however that $\vec{A}$ is completely absent in the final result. Instead, the final result relates the physically measured quantity, the phase difference, to a gauge invariant quantity, the magnetic flux. For conceptual clarity, this is the sort of formula that we would like in a theory that has gauge. Though we may use a particular gauge to do the calculation; things are much more clear if the final result is explicitly gauge invariant. It is this sort of result that we will derive for gravitational wave detection in the next section.
III. GRAVITATIONAL WAVE DETECTORS

General relativity is the theory of the metric \( g_{\mu\nu} \) of curved spacetime. In a strong gravitational field, including those where the gravitational waves of interest are produced, the field equations of general relativity are highly nonlinear and somewhat complicated. However, once the gravitational waves get to the detector, they are quite weak. The metric can then be expressed as

\[
g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}
\]

Here \( h_{\mu\nu} \) is small and \( \eta_{\mu\nu} \) is the flat metric of special relativity. That is in the usual coordinates \((t, x, y, z)\) we have \( \eta_{tt} = -1 \) and \( \eta_{xx} = \eta_{yy} = \eta_{zz} = 1 \) and all other components vanish. Since \( h_{\mu\nu} \) is small, we can write all equations in first order perturbation theory. Thus, we can treat gravity as the theory of a tensor field \( h_{\mu\nu} \) in special relativity in much the same way that electrodynamics is the theory of a vector field \( A_\mu \) (the four-vector version of the vector potential). We use the relativity conventions that greek indicies stand for spacetime components \((t, x, y, z)\), latin indicies for spatial components \((x, y, z)\) that repeated upper and lower indices are summed over and that units are chosen so that the speed of light is equal to one. We now consider the motion of material particles and light rays.

We usually think of a trajectory as giving the spatial coordinates \( \vec{x} \) as a function of time. For our purposes, it will be helpful to instead give all four coordinates \( x^\alpha \) as functions of a parameter \( \lambda \) called the affine parameter. For material particles this affine parameter will be the proper time, that is the time elapsed on a clock carried along that particle’s trajectory. For a light ray, the affine parameter will be the phase. Let an overdot denote derivative with respect to \( \lambda \) and define \( u^\alpha = \dot{x}^\alpha \). Then the equation of motion is

\[
\dot{u}^\alpha + \Gamma^\alpha_{\beta\gamma} u^\beta u^\gamma = 0
\]

Here the Christoffel symbols \( \Gamma^\alpha_{\beta\gamma} \) are given by

\[
\Gamma^\alpha_{\beta\gamma} = \frac{1}{2} \eta^{\alpha\delta} (\partial_\beta h_{\gamma\delta} + \partial_\gamma h_{\beta\delta} - \partial_\delta h_{\beta\gamma})
\]

where \( \partial_\alpha \) is an abbreviation for the partial derivative with respect to \( x^\alpha \).

Since general relativity is invariant under general coordinate transformations, some form of this invariance must be inherited by the weak field form of the theory. A (small) coordinate change takes the form \( x^\alpha \rightarrow x^\alpha - \xi^\alpha \). Where the \( \xi^\alpha \) are functions of the \( x^\alpha \). Under this coordinate change the \( h_{\alpha\beta} \) transform as

\[
h_{\alpha\beta} \rightarrow h_{\alpha\beta} + \partial_\alpha \xi_\beta + \partial_\beta \xi_\alpha
\]

This is the gauge transformation of weak field gravity. Note that the Christoffel symbols are not gauge invariant and therefore the equations of motion are not gauge invariant. Thus, in particular, the question “what is the trajectory of a material particle or a light ray?” is not a question with a gauge invariant answer. We will thus have to be a bit careful about what sort of questions we ask.

What then is gauge invariant in weak field gravity? The answer is a quantity called the Riemann tensor. This can be expressed in terms of the Christoffel symbols as

\[
R_{\alpha\beta\gamma\delta} = \eta_{\epsilon\lambda} (\partial_\delta \Gamma^\epsilon_{\alpha\gamma} - \partial_\gamma \Gamma^\epsilon_{\alpha\delta})
\]

or in terms of \( h_{\alpha\beta} \) as

\[
R_{\alpha\beta\gamma\delta} = \frac{1}{2} (\partial_\delta \partial_\gamma h_{\alpha\beta} + \partial_\alpha \partial_\lambda h_{\beta\gamma} - \partial_\beta \partial_\gamma h_{\alpha\lambda} - \partial_\alpha \partial_\lambda h_{\beta\gamma})
\]

Using equation 11 in equation 12 shows that the Riemann tensor does not change under a gauge transformation. The Riemann tensor has a straightforward physical interpretation in terms of tidal force. Recall that in a non-uniform gravitational field, objects at different positions acquire different accelerations. It is this effect, from the gravitational field of the moon, that leads to ocean tides. For two (slowly moving) objects with separation \( \Delta \vec{s} \), their relative acceleration \( \Delta \vec{a} \) is given by

\[
\Delta a^i = -R_{ki\lambda\delta} \Delta s^k
\]

Recall that the relative phase of the light rays in the interferometer is a physically measured quantity. Our goal, in analogy with the previous section of the paper is to find a formula that expresses that quantity in terms of the Riemann tensor. That formula will then be manifestly gauge invariant. Though we may end up introducing a particular gauge to do the calculation, once we have the result we will no longer need to worry about issues of gauge.
The ability to make gauge transformations is the ability to choose a gauge that makes the calculations convenient. For our purposes, a convenient gauge is the so-called “radiation gauge” in which (among other properties of the gauge) $h_{\mu\nu}$ has only spatial components, in other words where $h_{tt}$ and $h_{ti}$ vanish. For gravitational radiation, $h_{\mu\nu}$ can always be put in radiation gauge. From equation (10) it follows that $T_{tt}^i$ vanishes in radiation gauge. Therefore, a solution to the equation of motion for material particles is that the spatial coordinates are constant (and therefore that $u' = 0$). Thus the mirrors of the interferometer do not change their spatial coordinates under the influence of the gravitational wave. Note that this is not the same as saying that “the mirrors do not move.” It is only a statement about the coordinates of the mirrors in a particular gauge. From equation (13) it follows that (in this gauge)

$$R_{tkti} = -\frac{1}{2}\partial_t\partial_i h_{k\bar{i}}$$ (15)

We now treat the propagation of light in the interferometer. Though the interferometer is large (arm lengths of 4 km for LIGO), it is small compared to the wavelength of the gravitational radiation that it is designed to detect (the peak sensitivity of LIGO is at about 200 Hz corresponding to a wavelength of about 1500 km)\(^{2}\). This means that during a single trip of a light ray in an arm of the interferometer, the components of $h_{\mu\nu}$ can be regarded as constant: having no dependence on space or time. (Note that this is not true of the proposed space based gravitational wave detector LISA, which is much larger and for which the analysis of this paper would be more complicated). Therefore for the purposes of calculating the properties of a single light ray trip, the $g_{\mu\nu}$ are constant with $g_{tt} = -1$ and $g_{t\bar{i}} = 0$. That is, the metric is just the usual metric of special relativity with $t$ as the usual time. However the $x^i$ are not the usual spatial coordinates because the $g_{ij}$, though constant, no longer have their flat space values of 1 for the diagonal components and 0 for the off-diagonal ones. Instead, the usual spatial coordinates are some linear combination of the $x^i$. Thus, the only tools that we will need to find the phase difference of the two light rays are the ordinary rules of light propagation of special relativity. These rules say that for light of angular frequency $\omega$ the phase of the wave is $\phi = \omega(t - l)$ where $t$ is the special relativity time (and because of the radiation gauge, also our coordinate time) and where $l$ is the spatial distance in the direction along which the light propagates. The phase difference between the two light rays is then equal to $\omega$ multiplied by the difference in their travel times and is therefore also equal to $\omega$ multiplied by the difference in their travel distances.

Choose the origin of coordinates to be the position of the beam splitter and the $x$ and $y$ axes to contain the two mirrors at the ends of the interferometer arms. Then the difference in phase between the two light rays when they recombine at the beam splitter is

$$\Delta\phi = \omega(\Delta x(2 + h_{xx}) - \omega\Delta y(2 + h_{yy})$$ (16)

Here $\Delta x$ is the $x$ coordinate position of the mirror on the $x$ axis, and correspondingly for $\Delta y$. The first term in each parenthesis is much larger and the second. In the absence of gravitational radiation, this term is just the usual expression for the phase in an interferometer having to do with the difference in the lengths of the arms. However, this term is time independent; so we can get rid of it by taking a time derivative. In fact we will take two time derivatives of equation (16) yielding

$$\frac{d^2\Delta\phi}{dt^2} = \omega(\Delta x\partial_t h_{xx} - \Delta y\partial_t h_{yy})$$ (17)

Since terms involving $h_{\mu\nu}$ are first order, we can replace $\Delta x$ and $\Delta y$ by their zeroth order expressions: $\Delta x = \Delta y = L$ where we are assuming (approximately) equal lengths $L$ for the interferometer arms. Since we are in radiation gauge, we can replace second derivatives of the metric with the Riemann tensor. We thus have

$$\frac{d^2\Delta\phi}{dt^2} = 2\omega L(R_{t\bar{t}g\bar{t}} - R_{t\bar{t}xx})$$ (18)

Equation (18) is the main result of this paper. It expresses a physically measurable quantity, the second time derivative of the phase difference, in terms of a gauge invariant quantity, the Riemann tensor. It is thus the analog for gravitational radiation of equation (7) for the Aharonov-Bohm effect.

We are now in a position to address the issues raised in the introduction. The cosmological redshift is due to the expansion of the universe between the time the light is emitted and the time it is absorbed. We have neglected this sort of effect in the interferometer by treating the $h_{\mu\nu}$ as constants between the time the beam is split and the time it recombines. A more exact treatment not making this approximation does yield an additional effect. However, this additional effect is smaller by a factor of $L/\lambda$ (where $L$ is the interferometer arm length and $\lambda$ is the wavelength of the gravitational radiation) than the effect we have calculated. As for the effect of gravitational radiation on the motion of the mirrors, though the mirrors are at constant spatial coordinates, their separation is not constant. The acceleration
of their separation is given by equation (14). Whether one regards the change in separation as a consequence of motion of the mirrors or the expansion (and contraction) of space is a matter of semantics. What are not matters of semantics are observable quantities like the phase difference between the two light beams.

The examples of this section and the previous one illustrate an important principle of physics that recurs throughout both quantum mechanics and general relativity: a physical theory is required to answer only those questions that correspond to a measurement. When the theory in question is a gauge theory, the answer about the result of a measurement is gauge invariant. However, for convenience the calculations are often done in a particular gauge. Often the results are simply presented in that gauge, and if one forgets that a particular gauge choice has been made it seems as if a physical quantity (the result of the measurement) has been equated to a quantity without an unambiguous meaning. Another way to put this issue is that it is important to remember that the results of a newer theory (general relativity) cannot be “shoehorned” into the categories of the old theory (Newtonian mechanics) even though it is tempting to try to do so. What is physically meaningful in one theory may be mere gauge in the other. This temptation is particularly dangerous when the results of a calculation in a particular gauge seem to have a nice physical interpretation using the old theory categories. One might then regard the calculation as the “physical” explanation of the phenomenon and become confused when a calculation in a different gauge seems to point to a different “physical” explanation. To allay this confusion, it is sometime helpful to present the results in a way that is manifestly gauge invariant by expressing the result of the measurement in terms of gauge invariant quantities. This is what has been done in this paper for gravitational radiation detectors.

IV. ACKNOWLEDGEMENTS

I would like to thank Alberto Rojo for posing the question that this paper answers. This work was partially supported by NSF grant PHY-0456655 to Oakland University.

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