Free energy of an SU(2) monopole-antimonopole pair *

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60th October Anniversary Prospect 7a
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We induce an external $\mathbb{Z}_2$ monopole-antimonopole pair in an SU(2) lattice gauge system and measure its free energy as a way to probe the vacuum structure. We discuss the motivation and computational methodology of the investigation and illustrate our preliminary results.

It is well known that some Higgs theories with non-Abelian gauge group admit stable monopole solutions\cite{1,2}. With large gauge groups, as in grand unified theories, the residual unbroken gauge group can be non-Abelian. It is then interesting to determine the properties of the interaction induced among monopoles, or, as we will consider in this paper, between a monopole and an antimonopole, by the quantum fluctuations of the unbroken group. Beyond the relevance that such interaction may have for the original theory, it can shed light on the low energy properties of the residual gauge theory itself. From the point of view of the unbroken theory, to a very good approximation the monopoles act as static point sources. The way to incorporate such sources in an SU($N$) lattice gauge theory was spelled out in Refs.\cite{3,4}, which built on earlier results established in a seminal paper by 't Hooft \cite{5} and in Refs.\cite{6–8}. We will follow closely the treatment of Ref.\cite{4}. In a three dimensional theory, a monopole-antimonopole pair can be introduced within two cubes of the lattice by the replacement $\beta \text{Tr} U_P \rightarrow \beta \text{Tr} U_P \equiv z_n \beta \text{Tr} U_P$ for all the terms in the action corresponding to plaquettes traversed by a path joining the centers of the two cubes. $z_n$ stands here for a non-trivial element of the center of the group SU($N$) i.e. for one of the $N$th roots of unity: $z_n = \exp(2\pi in/N)$ with $n = 1\ldots N - 1$. In the case of SU(2) which we will study here the only non-trivial element of the center of the group is -1 and monopole and antimonopole coincide. By the redefinition $U \rightarrow z_n U$ of an appropriate set of the link variables, the path joining monopole and antimonopole can be deformed at will. The path is therefore unphysical and carries no free energy per se. But its end points cannot be moved without an accompanying change of free energy. A static monopole-antimonopole pair is introduced in the four dimensional theory by simply replicating the above construction in all time slices. Now the plaquettes with modified coupling are those transversed by a sheet joining the world lines of the monopoles. Again the sheet can be deformed at will. The insertion of a monopole-antimonopole pair can be reintrepeted in terms of the electric flux operator introduced in Ref.\cite{5} and several investigations have been devoted to the study of such operator in various contexts (see for example Refs.\cite{9,10}), but, to the best of our knowledge, no direct calculation of the free energy of a $\mathbb{Z}_2$ monopole-antimonopole pair has ever been attempted. If the vacuum is characterized by a monopole condensate, one would expect the behaviour of the free energy as function of separation to exhibit

\*Presented by C. Rebbi. This research was supported in part under DOE grant DE-FG02-91ER40676 and by the U.S. Civilian Research and Development Foundation for Independent States of FSU (CRDF) award RP1-187.
screening, whereas a $1/r$ behavior or a linearly rising behavior would characterize a Coulomb phase or a phase with condensation of electric charges, respectively. It is also noteworthy that the monopole and the antimonopole form two anchors for a center vortex. Recent investigations (cfr. Refs. [11][14]) have emphasized the role that such vortices play in confinement. The calculation which we present here can be reinterpreted as the calculation of the cost in free energy to create a center vortex spanning a certain distance within the lattice. If such excess free energy quickly saturates (screening), then the vacuum should indeed exhibit a condensate of center vortices.

The numerical calculation of a free energy is notoriously difficult. We have been able to obtain reasonably accurate results with acceptable amounts of CP time by combining a Monte Carlo simulation with the multihistogram method [15]. We consider a modified SU(2) lattice gauge theory with Wilson action, defined over a $N_x \times N_y \times N_z \times N_t$ hypercubical lattice with periodic boundary conditions. The modification consists in the fact that, for all the $x - y$ plaquettes $P'$ having a lower vertex with coordinates $x = 0, y = 0, 0 < z \leq d, 0 \leq t < N_t$, the coupling constant $\beta$ is replaced with $\beta'$. These are the plaquettes that cross the sheet joining the worldlines of the monopole and antimonopole at separation $r = da$ ($a$ being the lattice spacing). We denote the partition function of this system by $Z(\beta', \beta)$. We are interested in the free energy

$$F(r) = -\frac{1}{da} \log \left[ \frac{Z(-\beta, \beta)}{Z(\beta', \beta)} \right]$$  
(1)

Let us define

$$\rho(E) = \int dU [e^{-\sum_{P'} \text{Tr} U_{P'}}] e^{\sum_{P \neq P'} \beta \text{Tr} U_{P'}/2}$$  
(2)

If we perform a simulation with $\beta'$ set to a certain value $\beta_i$ and record in a histogram the frequency $n_i(E)$ of occurrences of a certain value of $E$, we will find

$$n_i(E) = \frac{\rho(E) e^{\beta_i E/2}}{Z(\beta_i, \beta)}$$  
(3)

If from $\rho(E)$ (as yet unknown) we get

$$Z(\beta_i, \beta) = \int dE \rho(E) e^{\beta_i E/2}$$  
(4)

The procedure we have followed consists in performing a number $N_h$ of simulations covering the interval $-\beta \leq \beta' \leq \beta$ with a sufficiently small step to guarantee a good overlap of the histograms. We take then

$$\rho(E) = \frac{1}{N_h} \sum_i n_i(E) e^{\beta_i E/2} Z_i^{(k)}$$  
(5)

We start from $Z_i^{(0)} = 1$ and iterate: the values of $Z_i^{(k)}$ at convergence are proportional to the corresponding $Z(\beta_i, \beta)$. As a technical improvement in the implementation of the histogram method, applicable to the case where the measured variable covers a continuous range, we have allocated the values of $E$ to the four neighboring end-points of the histogram intervals with the weights of a cubic interpolation. This procedure reduces substantially the total number of histogram subdivisions one must use to obtain accurate results.

We illustrate here the results we have obtained with $\beta = 2.6$, $N_x = N_y = 20$, $N_z = 40$ and the two time extents $N_t = 16$ and $N_t = 6$, which place the system in the confined and deconfined phases respectively. We have used a combined multi-hit Metropolis overrelaxation algorithm, with 5000 equilibrating iterations and 4000 to 20000 measurements separated by 50 iterations. The measurements themselves have been performed by averaging over 384 upgrading steps of the links in the plaquettes $P'$ as a variance reduction technique. In Figure 1 we show the histograms for a definite separation of the monopole-antimonopole pair. In Figure 2 we show our results for the free energy of the pair. The calculation is computer intensive, because one must perform separate calculations for all separations of the pair and for all the intermediate values of $\beta'$. However it is quite feasible with present day computer resources. We have written a Fortan 90 code which, paying some attention to the distribution of the data but without resorting to any special programming tricks like coding critical subroutines in assembler, runs at approx. 40% of peak speed on the SGI-Cray Origin 2000, with very satisfactory scaling. With this performance, the cost of the data presented in this paper is of the order of a few thousand processor hours, which is rather modest by today's
The results in Figure 2 show that the interaction of the pair is screened both in the confined and deconfined phases. The lines in the figure correspond to exponential fits $\exp(-d/l)$ with $l = 0.8300$ and $l = 0.7828$ for $N_t = 16$ and $N_t = 6$, respectively. We have also computed the free energy of a single monopole adopting free boundary conditions for the $z = 0, N_z$ boundaries of the lattice (we maintained periodic boundary conditions in all other directions) and the results are in agreement with the free energy of the pair for large separation. Our investigation is still in progress. We plan to repeat the calculation with a smaller value of $\beta$ to verify scaling and to study the behavior of the free energy of a single monopole as one goes across the deconfining transition. It would also be interesting to extend the calculation to other systems, especially to models which are expected to possess a varied structure of electric and magnetic confinement phases.

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