Berezinian Construction of Super-Solitons in Supersymmetric Constrained KP Hierarchies

H. Aratyn¹, E. Nissimov²,³ and S. Pacheva²,³

¹ Department of Physics, University of Illinois at Chicago
845 W. Taylor St., Chicago, IL 60607-7059, U.S.A.

² Institute of Nuclear Research and Nuclear Energy
Boul. Tsarigradsko Chausee 72, BG-1784 Sofia, Bulgaria

³ Department of Physics, Ben-Gurion University of the Negev
Box 653, IL-84105 Beer-Sheva, Israel

November 11, 2018

Abstract

We consider a broad class $SKP_{2,2}$ of consistently reduced Manin-Radul supersymmetric KP hierarchies (MR-SKP) which are supersymmetric analogs of the ordinary bosonic constrained KP models. Compatibility of these reductions to $SKP_{2,2}$ with the MR fermionic isospectral flows is achieved via appropriate modification of the latter preserving their (anti-)commutation algebra. Unlike the general unconstrained MR-SKP case, Darboux-Bäcklund transformations do preserve the fermionic isospectral flows of $SKP_{2,2}$. This allows for a systematic derivation of explicit Berezinian solutions for the $SKP_{2,2}$ super-tau-functions (super-solitons).

Introduction

Manin-Radul supersymmetric KP (MR-SKP) integrable hierarchy of nonlinear evolution ("super-soliton") equations [1] and other related supersymmetric integrable hierarchies [2, 3, 4, 5, 6, 7, 8] attracted a lot of interest, both from purely mathematical point of view as supersymmetric generalizations of the inverse scattering method, bi-Hamiltonian structures, tau-functions and Sato Grassmannian approach, as well as in the context of theoretical physics due to their relevance in non-perturbative superstring theory [9].

In the present paper we will be specifically concerned with MR-SKP hierarchy [1], i.e., possessing $N = 1$ supersymmetry and being defined in terms of fermionic (Grassmann-odd) pseudo-differential Lax operator. In ref. [10] we have already started a systematic study of MR-SKP hierarchy with particular attention being paid to the proper treatment of the fermionic MR isospectral flows, which was lacking in the previous studies on the subject. In [11], we introduced an infinite algebra of commuting additional ("ghost") symmetries of MR-SKP hierarchy which were used to construct systematic reductions to a broad class of constrained supersymmetric KP hierarchies denoted as $SKP_{2,2}$ (see
Eq. (12) below; we will keep in the sequel the name MR-SKP to explicitly denote the full unconstrained hierarchy). The constrained SKP hierarchy possess correct evolution under the bosonic (Grassmann-even) isospectral flows. However, it turns out that the reductions from MR-SKP to SKP hierarchy are incompatible with the original MR-SKP fermionic (Grassmann-odd) isospectral flows. In [10] we provided a solution to this problem for the simplest case of constrained SKP hierarchy by appropriately modifying MR-SKP fermionic flows while preserving their original (anti-)commutation algebra, i.e., preserving the integrability of the constrained SKP system. One of the results of the present paper is the extension of this construction to all SKP hierarchies.

Our next result concerns the construction of Darboux-Bäcklund (DB) transformations preserving both types (even and odd) of the isospectral flows. As already pointed out in [10], DB transformations are always incompatible with the fermionic flows in the original unconstrained MR-SKP hierarchy. However, for constrained SKP hierarchies the compatibility of DB transformations is here achieved thanks to the above mentioned modification of the original MR-SKP fermionic flows.

Furthermore, we provide explicit expressions for the super-tau function and the super-eigenfunctions on DB-orbits of iterations of the DB transformations for arbitrary constrained SKP hierarchies, which are given in terms of Wronskian-like Berezinians. These Berezinian solutions constitute supersymmetric generalizations of the (multi-)soliton solutions in ordinary bosonic KP hierarchies.

**Background on Manin-Radul Supersymmetric KP Hierarchy**

MR-SKP hierarchy is defined through the fermionic $N = 1$ super-pseudo-differential Lax operator $\mathcal{L}$:

$$\mathcal{L} = \mathcal{D} + f_0 + \sum_{j=1}^{\infty} b_j \partial^{-j} \mathcal{D} + \sum_{j=1}^{\infty} f_j \partial^{-j}$$

where the coefficients $b_j, f_j$ are bosonic and fermionic superfield functions, respectively. We shall use throughout this paper the super-pseudo-differential calculus with the following notations: $\partial$ and $\mathcal{D} = \partial_x + \theta \partial$ denote operators, whereas the symbols $\partial_x$ and $\partial_\theta$ will indicate application of the corresponding operators on superfield functions. As usual, $(x, \theta)$ denote $N = 1$ superspace coordinates. For any super-pseudo-differential operator $A = \sum_j a_j/2 \mathcal{D}^j$ the subscripts $(\pm)$ denote its purely differential part ($A_+ = \sum_{j \geq 0} a_{j/2} \mathcal{D}^j$) or its purely pseudo-differential part ($A_- = \sum_{j \geq 1} a_{-j/2} \mathcal{D}^{-j}$), respectively. For any $\mathcal{A}$ the super-residuum is defined as $\text{Res}_+ \mathcal{A} = a_{-1}$.

The Lax evolution Eqs. for MR-SKP read:

$$\frac{\partial}{\partial t_1} \mathcal{L} = -[\mathcal{L}^{2i}, \mathcal{L}] = [\mathcal{L}^{2i}, \mathcal{L}]$$

$$D_n \mathcal{L} = -\{\mathcal{L}_{2n-1}^{2n}, \mathcal{L}\} = \{\mathcal{L}_{2n-1}^{2n}, \mathcal{L}\} - 2\mathcal{L}_{2n}^{2n}$$
with the short-hand notations:

\[ D_n = \frac{\partial}{\partial \theta_n} - \sum_{k=1}^{\infty} \theta_k \frac{\partial}{\partial t_{n+k-1}} \quad , \quad \{ D_k , D_l \} = -2 \frac{\partial}{\partial t_{k+l-1}} \quad (t, \theta) \equiv (t_1, t_2, \ldots; \theta_1, \theta_2, \ldots) \quad (4) \]

An important rôle in the present approach is played by the notion of (adjoint-) super-eigenfunctions (sEF's) \( \Phi = \Phi(t, \theta) \) and \( \Psi = \Psi(t, \theta) \) of MR-SKP hierarchy obeying:

\[ \frac{\partial}{\partial t_l} \Phi = \mathcal{L}^2_l(\Phi) , \quad D_n \Phi = \mathcal{L}^{2n-1}_n(\Phi) \quad , \quad \frac{\partial}{\partial t_l} \Psi = -\left(\mathcal{L}^2_l\right)^*(\Psi) , \quad D_n \Psi = -\left(\mathcal{L}^{2n-1}\right)^*(\Psi) \quad (6) \]

The (adjoint-)super-Baker-Akhiezer functions \( \psi^{(s)}_{BA} \) of MR-SKP are particular cases of (adjoint-)sEF's which satisfy the spectral equations \( (\mathcal{L}^2)^{(s)}_{BA} = \pm \lambda \psi^{(s)}_{BA} \) in addition to (6).

Finally, the super-tau-function \( \tau(t, \theta) \) is expressed in terms of the super-residues of powers of the super-Lax operator (1) as follows:

\[ \text{Res} \mathcal{L}^{2k} = \frac{\partial}{\partial \theta} \text{D}_0 \ln \tau , \quad \text{Res} \mathcal{L}^{2k-1} = D_k \text{D}_0 \ln \tau \quad (7) \]

### Constrained Supersymmetric KP Hierarchies

Let us consider an infinite set \( \{ \Phi_{j/2}, \Psi_{j/2} \}_{j=0}^{\infty} \) of pairs of (adjoint-)sEF's of \( \mathcal{L} \) where \( j \) indicates their Grassmann parity (integer indices correspond to bosonic, whereas half-integer indices correspond to fermionic parity). It was shown in [10] that the following infinite set of super-pseudo-differential operators:

\[ \mathcal{M}_{s/2} = \sum_{k=0}^{s-1} \Phi \frac{\partial}{\partial t_{s-k}} \mathcal{D}^{-1}_{s-k} \Psi , \quad s = 1, 2, \ldots \quad (8) \]

generate an infinite set of flows \( \bar{\partial}_{s/2} (\bar{\partial}_{n-\frac{1}{2}} \equiv \bar{D}_n , \quad \bar{\partial}_k \equiv \frac{\partial}{\partial \theta_k}) : \)

\[ \bar{D}_n \mathcal{L} = \left\{ \mathcal{M}_{n-\frac{1}{2}} , \mathcal{L} \right\} , \quad \frac{\partial}{\partial \theta_k} \mathcal{L} = \left[ \mathcal{M}_k , \mathcal{L} \right] \quad (9) \]

which (anti-)commute with the original isospectral flows \( \frac{\partial}{\partial \theta_l} , D_n \quad (3) \), i.e., \( \bar{\partial}_{s/2} \) define an infinite-dimensional algebra of additional “ghost” symmetries of MR-SKP hierarchy, obeying the (anti-)commutation relations:

\[ \left[ \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \theta_k} \right] = \left[ \frac{\partial}{\partial \theta}, \bar{D}_n \right] = 0 , \quad \left\{ \bar{D}_i , \bar{D}_j \right\} = -2 \frac{\partial}{\partial t_{i+j-1}} \quad (10) \]

The super-“ghost”-symmetry flows and the corresponding generating operators \( \mathcal{M}_{s/2} \quad (8)-(10) \) are used to construct a series of reductions of the MR-SKP hierarchy [10].
super-“ghost” flows obey the same algebra \([13]\), as the original MR-SKP flows \([1]\), one can identify an infinite subset of the latter with a corresponding infinite subset of the former:

\[
\partial_{t^\ell} = -\partial_{\ell^\ell} \quad , \quad \ell = 1, 2, \ldots ; \quad \partial_k \equiv \frac{\partial}{\partial t^k} \quad , \quad \partial_{k-\frac{1}{2}} \equiv D_k \quad ; \quad \partial_{k\ell} \equiv \frac{\partial}{\partial t^k t^\ell} \quad , \quad \partial_{k-\frac{1}{2} - \frac{1}{2} \ell} \equiv \bar{D}_k
\]

(11)

where \((r, m)\) are some fixed positive integers of equal parity, and retain only these flows as Lax evolution flows (this is a supersymmetric extension of the usual reduction procedure in the purely bosonic case \([11]\)). Eqs.\((11)\) imply the identification \(\text{SKP}^{[10]}\) we obtain:

for the (adjoint-)sEF’s, whereas the r.h.s. of (13) is manifestly non-zero. which leads to apparent contradiction, since the l.h.s. of (13) vanishes by virtue of Eqs.\((6)\)
of any \(\ell\). Therefore, the pertinent reduced (constrained) MR-SKP hierarchy, denoted as \(\text{SKP}_{\overline{r}, \overline{m}}\), is described by the following constrained super-Lax operator:

\[
\mathcal{L}(\overline{r}, \overline{m}) = \mathcal{D}^\prime + \sum_{i=0}^{r-1} \phi_i^{(r)} \mathcal{D}^i + \sum_{j=0}^{m-1} \Phi_{m-j+1} \mathcal{D}^{-1} \psi_{\overline{m}^j}
\]

(12)

which is the supersymmetric counterpart of the ordinary pseudo-differential Lax operator describing the bosonic constrained KP hierarchies \(c\text{KP}_{r,m}\) (for a detailed discussion and further references, see \([12]\)).

Henceforth we will restrict our attention to fermionic constrained \(\text{SKP}_{\overline{r}, \overline{m}}\) hierarchies, i.e., \([13]\) with \((r, m)\) being odd integers.

As already pointed out in \([10]\), the original MR fermionic flows \([3]\) are incompatible with the reduction of MR-SKP \([4]\) to fermionic constrained \(\text{SKP}_{\overline{r}, \overline{m}}\) hierarchies \([12]\). Namely, taking the \((-)\) part of Eqs.\((3)\) for fermionic constrained \(\mathcal{L}(\overline{r}, \overline{m})\) (Eq.\((13)\) with \(r, m = \text{odd}\)) and using a series of simple identities for super-pseudo-differential operators \([10]\) we obtain:

\[
\sum_{j=0}^{m-1} \left[ \left( D_n \Phi_{m-j+1} - \mathcal{L}^{2n-1} \Phi_{m-j+1} \right) \mathcal{D}^{-1} \psi_{\overline{m}^j} \Phi_{m-j+1} \mathcal{D}^{-1} \left( D_n \psi_{\overline{m}^j} + \left( \mathcal{L}^{2n-1} \right)^* \psi_{\overline{m}^j} \right) \right]
\]

\[
= -2 \sum_{j=0}^{m-1} \sum_{k=0}^{2n-1-k} \mathcal{L}^{2n-1-k} \Phi_{m-j+1} \mathcal{D}^{-1} \mathcal{L}^k \psi_{\overline{m}^j}
\]

(13)

which leads to apparent contradiction, since the l.h.s. of \((13)\) vanishes by virtue of Eqs.\((3)\) for the (adjoint-)sEF’s, whereas the r.h.s. of \((13)\) is manifestly non-zero.

Generalizing the argument given in \([10]\) for the simplest \(\text{SKP}_{\overline{1}, \overline{1}}\) case, we arrive at the following:

**Proposition 1** There exists the following consistent modification of MR-SKP flows \(D_n\) \([4]\) for constrained \(\text{SKP}_{\overline{r}, \overline{m}}\) hierarchy \((r, m = \text{odd})\):

\[
\mathcal{D}_k \mathcal{L} = - \left\{ \mathcal{L}^{2k-1} - X^{(2k-1)}, \mathcal{L} \right\} = \left\{ \mathcal{L}_+^{2k-1}, \mathcal{L} \right\} + \left\{ X^{(2k-1)}, \mathcal{L} \right\} - 2\mathcal{L}^{2k}
\]

(14)

\[
X^{(2k-1)} = 2 \sum_{j=0}^{m-1} \sum_{l=0}^{k-2} \mathcal{L}^{2(k-l)-3} \Phi_{m-j+1} \mathcal{D}^{-1} \left( \mathcal{L}^{2j+1} \right)^* \psi_{\overline{m}^j}
\]

(15)
\[ \mathcal{D}_k \Phi = \mathcal{L}_+^{2k-1}(\Phi) - 2\mathcal{L}_+^{2k-1}(\Phi) + X^{(2k-1)}(\Phi) \]  
(16)

\[ \mathcal{D}_k \Psi = -(\mathcal{L}_+^{2k-1})^*(\Psi) + 2(\mathcal{L}_+^{2k-1})^*(\Psi) - \left(X^{(2k-1)}\right)^*(\Psi) \]  
(17)

**The modified** \( \mathcal{D}_k \) **flows for** \( \text{SKP}_{\frac{1}{2}, \frac{1}{2}} \) **obey the same anti-commutation algebra** \( \{ \mathcal{D}_k, \mathcal{D}_l \} = -2 \frac{\partial}{\partial r(t_{k+j-1})} \) **as in the original unconstrained case** \( \{ \mathcal{D}_k, \mathcal{D}_l \} \) **(modulo** \( r \)).

**Remark.** For bosonic \( \text{SKP}_{\frac{1}{2}, \frac{1}{2}} \) **models** (Eq. [12] with \( r, m = \text{even} \)) there is no need to modify MR fermionic flows, since in this case the term in r.h.s. of [13] is absent.

**Berezinian Solutions for the Super-Tau Function**

It was demonstrated in \([1]\) that for the general MR-SKP hierarchy \([1]\) the Darboux-Bäcklund (DB) transformations \( \mathcal{L} = \mathcal{T}\mathcal{L}T^{-1} \), where \( \mathcal{T} = \chi \mathcal{T}^{-1} \) with \( \chi \) being a bosonic sEF \([1]\) of \( \mathcal{L} \), do not preserve the fermionic-flow Lax Eqs. [3]. Indeed, for the DB-transformed \( \mathcal{L} \) to obey the same MR flow Eqs. [2]–[3] as \( \mathcal{L} \), the DB-generating “gauge” transformation \( \mathcal{T} \) must satisfy:

\[ \frac{\partial}{\partial t_j} \mathcal{T}^{-1} + (\mathcal{T}^{2n-1}_+ \mathcal{T}^{-1}) = 0 \]  
(18)

The first Eq. [18] is exactly analogous to the purely bosonic case and implies that \( \chi \) must be a sEF \([1]\) of \( \mathcal{L} \) w.r.t. the even MR-SKP flows. However, the second Eq. [18] does not have solutions for \( \chi \) for the general MR-SKP hierarchy. In particular, if \( \chi \) would be a sEF also w.r.t. fermionic flows (cf. second Eq. [6]), then the l.h.s. of second Eq. [18] would become zero thereby leading to the contradictory relation: \( \left(\mathcal{L}^{2n-1}_+\right)_- = 0 \). This makes the standard DB method inapplicable to find solutions of the unconstrained MR-SKP.

On the other hand, for constrained fermionic \( \text{SKP}_{\frac{1}{2}, \frac{1}{2}} \) hierarchies it can easily be shown (extending the proof given in \([10]\) for the simplest \( \text{SKP}_{\frac{1}{2}, \frac{1}{2}} \) case), that auto-DB transformations (i.e., those preserving the constrained form \([12]\) of the initial \( \text{SKP}_{\frac{1}{2}, \frac{1}{2}} \) hierarchy) are compatible with the modified fermionic flows \([14]–[17]\). This latter result guarantees that any iteration of DB transformations of the initial \( \text{SKP}_{\frac{1}{2}, \frac{1}{2}} \) hierarchy (in particular, the “free” one with \( \mathcal{L}_{(\psi, \phi)} = \mathcal{D} \)) will yield new nontrivial solutions for \( \text{SKP}_{\frac{1}{2}, \frac{1}{2}} \), which obey the same isospectral flow Eqs. [2], [4], [6]–[7], i.e., both bosonic and fermionic, as the initial hierarchy.

Now, consider auto-DB transformations for arbitrary \( \text{SKP}_{\frac{1}{2}, \frac{1}{2}} \) (here \( L \equiv \mathcal{L}_{(\psi, \phi)} \equiv \mathcal{L}_{(\psi, \phi)}^{(0)}(\frac{1}{2}, \frac{1}{2}) \) ) :

\[ \mathcal{L} = \mathcal{T}_a \mathcal{L} \mathcal{T}_a^{-1} = \mathcal{L}_+ + \sum_{j=0}^{m-1} \mathcal{D}^{-1}_j \mathcal{T}_a \mathcal{D}^{j/2} \]  
(19)

\[ \mathcal{D}_a = \mathcal{T}_a \mathcal{L} \mathcal{T}_a^{-1} \]  
(20)
where $\mathcal{T}_a = \Phi_a D\Phi_a^{-1}$ with $a$ being a fixed integer (bosonic) index. Under DB transformations the super-tau function transforms as (cf. Eq.(3.4) in [10]):

$$\bar{\tau} = \Phi_a \tau^{-1}$$  \hspace{1cm} (21)

Before proceeding to the iteration of DB-transformations for SKP hierarchies (19)–(21), we will introduce some convenient short-hand notations for Wronskian-type Berezinians:

$$\operatorname{Ber}_{(k,l)}[\varphi_0, \ldots, \varphi_{k-1}; \varphi_{\frac{j}{2}}, \ldots, \varphi_{\frac{1}{2}}] \equiv \operatorname{Ber} \left( \begin{array}{c|c} W_{k,k}[\varphi_0, \ldots, \varphi_{k-1}] & W_{k,l}[\varphi_0, \ldots, \varphi_{k-1}] \\ \hline \hline & \hline W_{l,k}[D_\theta \varphi_0, \ldots, D_\theta \varphi_{k-1}] & W_{l,l}[D_\theta \varphi_0, \ldots, D_\theta \varphi_{k-1}] \end{array} \right)$$  \hspace{1cm} (22)

where $(\varphi_0, \ldots, \varphi_{k-1})$ and $(\varphi_{\frac{j}{2}}, \ldots, \varphi_{\frac{1}{2}})$ are sets of bosonic (fermionic) superfield functions, and where $W_{k,l}[f_1, \ldots, f_l]$ denotes a rectangular $k \times l$ Wronskian matrix:

$$W_{k,l}[f_1, \ldots, f_l] = \left\| \partial_x^{\alpha-1} f_{\beta} \right\|, \quad \alpha = 1, \ldots, k, \beta = 1, \ldots, l$$  \hspace{1cm} (23)

The derivation of the explicit form of the DB-orbit for the super-tau function and the (adjoint-)EF’s of SKP is based on the following Proposition:

**Proposition 2** The iteration of Darboux-Bäcklund-like transformations on arbitrary initial superfield functions ($\Phi$ – bosonic, $F$ – fermionic) can be expressed in a Berezinian form as follows:

$$\Phi^{(2n)} = \mathcal{T}_{\frac{\varphi_{n-1}}{\frac{j}{2}}}^{(2n-1)} \mathcal{T}_{\frac{\varphi_{n-2}}{\frac{j}{2}}}^{(2n-2)} \cdots \mathcal{T}_{\varphi_{\frac{3}{2}}}^{(1)} \mathcal{T}_{\varphi_{0}}^{(0)} \left( \Phi \right) = \left( \operatorname{Ber}_{(n,n)}[\varphi_0, \ldots, \varphi_{n-1}, \Phi; \varphi_{\frac{j}{2}}, \ldots, \varphi_{\frac{1}{2}}] \right)^{-1} \operatorname{Ber}_{(n+1, n)}[\varphi_0, \ldots, \varphi_{n-1}, \Phi; \varphi_{\frac{j}{2}}, \ldots, \varphi_{\frac{1}{2}}]$$  \hspace{1cm} (24)

$$F^{(2n+1)} = \mathcal{T}_{\varphi_{\frac{3}{2}}}^{(2n)} \mathcal{T}_{\varphi_{\frac{2}{2}}}^{(2n-1)} \cdots \mathcal{T}_{\varphi_{\frac{1}{2}}}^{(1)} \mathcal{T}_{\varphi_{0}}^{(0)} \left( F \right) = \left( \operatorname{Ber}_{(n+1, n+1)}[\varphi_0, \ldots, \varphi_{n}, \varphi_{\frac{j}{2}}, \ldots, \varphi_{\frac{1}{2}}, F] \right)^{-1} \left( \operatorname{Ber}_{(n+1, n+1)}[\varphi_0, \ldots, \varphi_{n-1}, \varphi_{\frac{j}{2}}, \ldots, \varphi_{\frac{1}{2}}] \right)$$  \hspace{1cm} (25)

where by definition:

$$\mathcal{T}_{\varphi_{\frac{j}{2}}}^{(j)} = \varphi_{\frac{j}{2}} D_{\varphi_{\frac{j}{2}}}^{-1}, \quad \varphi_{\frac{j}{2}}^{(j)} = \mathcal{T}_{\varphi_{\frac{j-1}{2}}}^{(j-2)} \mathcal{T}_{\varphi_{\frac{j-1}{2}}}^{(j-2)} \cdots \mathcal{T}_{\varphi_{\frac{2}{2}}}^{(0)} \mathcal{T}_{\varphi_{0}}^{(0)} \left( \varphi_{\frac{j}{2}} \right)$$  \hspace{1cm} (26)

Here and in what follows the superscripts in brackets indicate the step of iteration of DB(-like) transformations. Note that $F^{(2n+1)}$ (25) and $\varphi_{k+\frac{j}{2}}^{(2k+1)}$ (26) are bosonic although the initial $F, \varphi_{k+\frac{j}{2}}$ are fermionic.

The proof of Prop.2 relies on the observation, that both sides of (24) and (25) define monic super-differential operators acting on $\Phi$ and $F$, respectively, which share the same set of kernel elements, namely, the superfield functions $\varphi_0, \ldots, \varphi_{n-1}; \varphi_{\frac{j}{2}}, \ldots, \varphi_{\frac{1}{2}}$. 


Let us consider in more detail the DB-orbit of constrained SKP hierarchy with $r = 1$, i.e., $L \equiv \mathcal{L}_{(\frac{1}{2}, \frac{1}{2})} = \mathcal{D} + f_0 + \sum_{j=0}^{m-1} \Phi_{(m-1)j} \mathcal{D}^{-1} \Psi_{(m-1)j}$ (in the formulas below $m$ indicates the order of the pseudo-differential part of $L \equiv \mathcal{L}_{(\frac{1}{2}, \frac{1}{2})}$, the integer $k$ is $0 \leq k \leq m-1$, and $l$ is arbitrary non-negative integer):

$$
\Phi^{(m+l+k)} = \left( T^{(lm-1+k)} \Phi_{(lm-1)} \right) \cdots \left( T^{(1m)} \Phi_{(1)} \right) \left( T^{(1)} \Phi_{(1)} \right) \left( L^{l+1} \Phi_{(1)} \right)
$$

for $0 \leq j \leq k - 1$ \hspace{1cm} (27)

$$
\Phi^{(m+l+k)} = \left( T^{(lm-1+k)} \Phi_{(lm-1)} \right) \cdots \left( T^{(1m)} \Phi_{(1)} \right) \left( T^{(1)} \Phi_{(1)} \right) \left( L^{l} \Phi_{(1)} \right)
$$

for $k \leq j \leq m - 1$ \hspace{1cm} (28)

Eqs. (27)-(28) indicate that the DB-orbit is defined by successive iterations of DB-transformations w.r.t. all super-EF’s $\Phi_{j}$ ($j = 0, \ldots, m-1$) present in $L \equiv \mathcal{L}_{(\frac{1}{2}, \frac{1}{2})}$. Comparing (27)-(28) with the general formulas (24)-(25) we easily identify the functions $\varphi_k$ and $\varphi_{\frac{1}{2}}$ appearing in the latter with the super-EF’s $\Phi_{j}$ of $L \equiv \mathcal{L}_{(\frac{1}{2}, \frac{1}{2})}$ as follows:

$$
\varphi_{m+k} = L^l(\Phi_{j})
$$

(29)

Therefore, the explicit expressions for the super-tau functions on the DB-orbit (27)-(28), upon using (21) and (24)-(25), are given by:

$$
\tau^{(2n+1)} = \text{Ber}_{(n+1,n)}(\varphi_0, \varphi_1, \varphi_2, \ldots, \varphi_n, \varphi_{\frac{1}{2}}, \ldots, \varphi_{n-\frac{1}{2}}) \frac{1}{\tau^{(0)}}
$$

(30)

$$
\tau^{(2n)} = \left( \text{Ber}_{(n,n)}(\varphi_0, \varphi_1, \varphi_2, \ldots, \varphi_{n-1}, \varphi_{\frac{1}{2}}, \ldots, \varphi_{n-\frac{1}{2}}) \right)^{-1} \tau^{(0)}
$$

(31)

with the substitution (29) for $\varphi_k, \varphi_{\frac{1}{2}}$ in the r.h.s. of (30)-(31).

**Super-Soliton Solutions**

Now, let us provide some explicit examples of Berezin solutions for the SKP tau-function (21)-(11). We shall consider the simplest case of constrained SKP hierarchy and take the initial $\tau^{(0)} = \text{const}$, i.e., the initial super-Lax operator being $L \equiv \mathcal{L}_{(\frac{1}{2}, \frac{1}{2})} = \mathcal{D}$.

The initial super-EF $\Phi_0 \equiv \Phi^{(0)}$ satisfies according to (4) :

$$
\frac{\partial}{\partial t_k} \Phi_0 = \partial_k \Phi_0 \ , \ D_n \Phi_0 = -D_{\theta}^{2n-1} \Phi_0
$$

(32)

$$
\Phi_0(t, \theta) = \int d\lambda \left[ \varphi_B(\lambda) + \left( \theta - \sum_{n \geq 1} \lambda^{n-1} \partial_\theta \right) \varphi_F(\lambda) \right] e^{\sum_{n \geq 1} \lambda^i (t_i + \theta \theta_i)}
$$

(33)

where $\varphi_B(\lambda), \varphi_F(\lambda)$ are arbitrary bosonic (fermionic) “spectral” densities.
It is easy to show that for SKP case the Berezinian expressions (30)–(31), together with the substitution (29), which now \( m = 1, j = 0 \) becomes simply \( \varphi_x = D^j \Phi_0 \), reduce to the following ratios of ordinary Wronskians:

\[
\tau^{(2n)} = \frac{W_n[\partial_x \Phi_0, \ldots, \partial_x^n \Phi_0]}{W_n[\Phi_0, \ldots, \partial_x^{(n-1)} \Phi_0]}, \quad \tau^{(2n+1)} = \frac{W_{n+1}[\Phi_0, \ldots, \partial_x^{(n)} \Phi_0]}{W_n[\partial_x \Phi_0, \ldots, \partial_x^n \Phi_0]},
\]

(34)

where \( \Phi_0 \) is given by (33). In particular, choosing for the bosonic (fermionic) “spectral” densities in Eq.(33)

\[
\varphi_B(\lambda) = \sum_{i=1}^{N} c_i \delta(\lambda - \lambda_i), \quad \varphi_F(\lambda) = \sum_{i=1}^{N} \epsilon_i \delta(\lambda - \lambda_i),
\]

where \( c_i, \lambda_i \) and \( \epsilon_i \) are Grassmann-even and Grassmann-odd constants, respectively, we have for \( \Phi_0 \):

\[
\Phi_0 = \sum_{i=1}^{N} \left[ c_i + \left( \theta - \sum_{n \geq 1}^{\lambda_i n-1} \theta_n \right) \right] e^{\sum_{l \geq 1}^{\lambda_i l} (t_l + \theta \theta_l)}
\]

(35)

Substituting (35) into (34) we obtain the following “super-soliton” solutions for the super-tau function:

\[
\tau^{(2n+1)} = \frac{\sum_{1 \leq i_1 < \ldots < i_{n+1} \leq N} (N+1) \bar{c}_{i_1} \ldots \bar{c}_{i_{n+1}} E_{i_1} \ldots E_{i_{n+1}} \Delta_{n+1}^2(\lambda_{i_1}, \ldots, \lambda_{i_{n+1}})}{\sum_{1 \leq j_1 < \ldots < j_n \leq N} (N) \bar{c}_{j_1} \ldots \bar{c}_{j_n} E_{j_1} \ldots E_{j_n} \lambda_{j_1} \ldots \lambda_{j_n} \Delta_{n}^2(\lambda_{j_1}, \ldots, \lambda_{j_n})}
\]

\[
\bar{c}_i \equiv c_i + \left( \theta - \sum_{n \geq 1}^{\lambda_i n-1} \theta_n \right), \quad E_i \equiv e^{\sum_{l \geq 1}^{\lambda_i l} (t_l + \theta \theta_l)}
\]

\[
\Delta_n(\lambda_{i_1}, \ldots, \lambda_{i_n}) \equiv \det \left| \lambda_{ia}^{b-1} \right|_{a,b=1,\ldots,n}
\]

(36)

(37)

Outlook. There is a number of interesting issues, related to the present topic, which deserve further study such as: binary DB-transformations and new types of solutions for the super-tau-function; consistent formulation of supersymmetric two-dimensional Toda lattice and search for proper supersymmetric counterparts of random (multi-)matrix models, based on analogous approach [13] in the purely bosonic case.

Acknowledgements. The authors gratefully acknowledge support by NSF grant INT-9724747.

References

[1] Yu.Manin and A. Radul, Commun. Math. Phys. 98 (1985) 65
[2] M. Chaichain and P. Kulish, Phys. Lett. 78B (1978) 413; P. Di Vecchia and S. Ferrara, Nucl. Phys. B130 (1977) 93
[3] P. Mathieu, J. Math. Phys. 29 (1988) 2499; S. Bellucci, E. Ivanov, S. Krivonos and A. Pichugin, Phys. Lett. 312B (1993) 463, hep-th/9305078; F. Delduc, E. Ivanov and S. Krivonos, J. Math. Phys. 37 (1996) 1356, hep-th/9510033; F. Toppan, Int. J. Mod. Phys. A11 (1996) 3257, hep-th/9506133; Q.P. Liu and M. Mañas, in “Supersymmetry and Integrable Models”, H. Aratyn et.al. (eds.), Springer-Verlag, 1998 (Lecture Notes in Physics 502), solv-int/9711002
[4] V. Kac and J. van de Leur, *Ann. Inst. Fourier* **37** (1987) 99; V. Kac and E. Medina, *Letters in Math. Phys.* **37** (1996) 435; A. LeClair, *Nucl. Phys.* **B314** (1989) 425; M. Mulase, *J. Diff. Geom.* **34** (1991) 651; J. Rabin, *Commun. Math. Phys.* **137** (1991) 533

[5] L. Martinez Alonso and E. Medina Reus, *J. Math. Phys.* **36** (1995) 4898; A. Ibort, L. Martinez Alonso and E. Medina Reus, *J. Math. Phys.* **37** (1996) 6157; M. Takama, hep-th/9506167

[6] J.M. Figueroa-O’Farrill, J. Mas and E. Ramos, *Rev. Mod. Phys.* **3** (1991) 479; W. Oevel and Z. Popowicz, *Commun. Math. Phys.* **139** (1991) 441

[7] Z. Popowicz, *J. Physics* **A29** (1996) 1281, hep-th/9510183; J. C. Brunelli and A. Das, *Phys. Lett.* **337B** (1994) 303, hep-th/9406214; L. Bonora, S. Krivonos and A. Sorin, *Nucl. Phys.* **B477** (1996) 835, hep-th/9604165; F. Delduc and L. Gallot, *Commun. Math. Phys.* **190** (1997) 395, solv-int/9609008; J.-C. Shaw and M.-H. Tu, solv-int/9712009

[8] H. Aratyn and C. Criscione, *Phys. Lett.* **391B** (1997) 99, hep-th/9608107; H. Aratyn, A. Das and C. Criscione, *Mod. Phys. Lett.* **A12** (1997) 2623, solv-int/9704119; H. Aratyn, A. Das, C. Criscione, A.H. Zimerman, in “Supersymmetry and Integrable Models”, H. Aratyn et.al. (eds.), Springer-Verlag, 1998 (Lecture Notes in Physics 502); H. Aratyn and A. Das, *Mod. Phys. Lett.* **A13** (1998) 1185, solv-int/9710020

[9] L. Alvarez-Gaumé, H. Itoyama, J. Mañes and A. Zadra, *Int. J. Mod. Phys.* **A7** (1992) 5337; L. Alvarez-Gaumé, K. Becker, M. Becker, R. Emperan and J. Mañes, *Int. J. Mod. Phys.* **A8** (1993) 2297; S. Stanciu, *Commun. Math. Phys.* **165** (1994) 261, hep-th/9407189; J.M. Figueroa-O’Farrill and S. Stanciu, *Phys. Lett.* **316B** (1993) 282

[10] H. Aratyn, E. Nissimov and S. Pacheva, solv-int/9801021

[11] W. Oevel, *Physica* **A195** (1993) 533; Y. Cheng, W. Strampp and B. Zhang, *Commun. Math. Phys.* **168** (1995) 117

[12] H. Aratyn, E. Nissimov and S. Pacheva, *Int. J. Mod. Phys.* **A12** (1997) 1265-1340, hep-th/9607234

[13] H. Aratyn, E. Nissimov and S. Pacheva, *Phys. Lett.* **244A** (1998) 245, solv-int/9712012