Solvable models of Bose-Einstein condensates: a new algebraic Bethe ansatz scheme

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Abstract

A new algebraic Bethe ansatz scheme is proposed to diagonalise classes of integrable models relevant to the description of Bose-Einstein condensation in dilute alkali gases. This is achieved by introducing the notion of \( Z \)-graded representations of the Yang-Baxter algebra.

1 Introduction

Theoretical studies into the behaviour of Bose-Einstein condensates (BECs) continue at a prolific rate, motivated by the experimental successes of producing condensates of atomic alkali gases [1, 2] and superpositions of atomic-molecular alkali gases [3, 4]. Many of the theoretical results to date have been obtained through use of the Gross-Pitaevskii mean-field theory and generalisations (e.g., see [5, 6, 7, 8, 9]). However, such mean-field theory approaches have limited applicability in regions of the parameter space where quantum fluctuations dominate. In these cases, only an exact treatment of the model will give a reliable description of the physics.

In our recent work we have shown that one model describing Josephson tunneling between two coupled BECs [10, 11], and another that models coherent superpositions of atomic and molecular BECs [12], can both in fact be solved exactly in the framework of the algebraic Bethe ansatz. Our intention here is to develop a new mathematical approach which allows us to extend this method to establish that very general classes of
Hamiltonians for BECs admit exact solutions. These classes of Hamiltonians cannot be solved in the usual form of the algebraic Bethe ansatz.

In this Letter, three classes of solvable models relevant to Bose-Einstein condensates of dilute alkali gases are determined. This is achieved by formulating a new scheme for the algebraic Bethe ansatz by introducing the notion of \( \mathbb{Z} \)-graded representations of the Yang-Baxter algebra. The first model we will present is a six-parameter generalisation of the canonical Josephson Hamiltonian [7] (which can also be considered as a two site Bose-Hubbard model) describing a tunnel-coupled pair of trapped Bose-Einstein condensates. The Hamiltonian is also applicable to model solid state Josephson junctions and coupled Cooper pair boxes [13]. The second model we present has six free parameters and the third one has ten parameters. They both describe coherent coupling between atomic and diatomic molecular BEC’s with additional interactions such as \( S \)-wave scattering between the atoms, between the molecules, and between atoms and molecules. Such effects were not included in [9, 12] but are important for a quantitative description of experiments [14]. Finally, we formulate the Slavnov formula for the scalar products between a Bethe eigenstate and an arbitrary Bethe vector, which facilitates the exact computation of form factors and correlations functions analogous to the results of [10].

2 Bethe ansatz for \( \mathbb{Z} \)-graded representations of the Yang-Baxter algebra

The main ingredient in the study of exactly solvable quantum systems through the algebraic Bethe ansatz [15, 16] is the Yang-Baxter equation

\[
R_{12}(u - v)R_{13}(u)R_{23}(v) = R_{23}(v)R_{13}(u)R_{12}(u - v).
\]

Here \( R_{jk}(u) \) denotes the matrix in \( \text{End}(V \otimes V \otimes V) \) acting non-trivially on the \( j \)-th and \( k \)-th spaces and as the identity on the remaining space. The \( R \)-matrix solution may be viewed as the structural constants for the Yang-Baxter algebra, denoted \( \mathcal{A} \), generated by the monodromy matrix \( T(u) \)

\[
R_{12}(u - v)T_1(u)T_2(v) = T_2(v)T_1(u)R_{12}(u - v).
\]

The simplest case is that for the \( sl(2) \) invariant \( R \)-matrix, which will be the subject of our study, given by

\[
R(u) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & b(u) & c(u) & 0 \\
0 & c(u) & b(u) & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
\]

with the rational functions \( b(u) = u/(u + \eta) \) and \( c(u) = \eta/(u + \eta) \).

Setting

\[
T(u) = \begin{pmatrix}
A(u) & B(u) \\
C(u) & D(u)
\end{pmatrix},
\]
it follows from the defining relations \(^{(2)}\) that

\[
[A(u), A(v)] = [D(u), D(v)] = 0, \\
[B(u), B(v)] = [C(u), C(v)] = 0, \\
A(u)C(v) = \frac{u - v + \eta}{u - v} C(v)A(u) - \frac{\eta}{u - v} C(u)A(v), \\
D(u)C(v) = \frac{u - v - \eta}{u - v} C(v)D(u) + \frac{\eta}{u - v} C(u)D(v).
\]

(5)

Note that there are many more relations satisfied by the generators of the Yang-Baxter algebra. However, those given above are the only ones needed for the algebraic Bethe ansatz procedure which we investigate below. For convenience, we extend \(A\) by a unit element \(I\) which we will represent by the identity matrix in any representation.

We also introduce an auxiliary operator \(Z\), called the grading operator, which satisfies the relations

\[
[Z, X(u)] = p\{X(u)\}.X(u),
\]

(6)

where \(X = A, B, C\) or \(D\) and \(p\{A(u)\} = p\{D(u)\} = 0, p\{B(u)\} = 1\) and \(p\{C(u)\} = -1\).

We call \(p\{X(u)\} \in Z\) the gradation of \(X(u)\) and extend the gradation operation to the entire algebra by the requirement

\[
p\{\theta.\phi\} = p\{\theta\} + p\{\phi\} \quad \forall \theta, \phi \in A.
\]

This definition for the grading operator is consistent with the defining relations of \(A\) governed by \(^{(2)}\).

Let us now define a new class of representations of the Yang-Baxter algebra which we call \(Z\)-graded representations. We say that a vector space \(V\), equipped with an endomorphism \(z\), is a \(Z\)-graded vector space, denoted \((V, z)\), if it admits a decomposition into subspaces

\[
V = \bigoplus_{k=-\infty}^{\infty} V_k
\]

such that

\[
zV_k = k.V_k, \quad k \in \mathbb{Z}.
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\]

Note that some of the \(V_k\) may be trivial subspaces. Formally, the grading operator can be used to define the following projection operators

\[
P_k = \prod_{\substack{j=-\infty \atop j \neq k}}^{\infty} \frac{(z - jI)}{(k - j)}
\]

(7)

such that

\[
P_kP_l = \delta_{kl}P_l, \quad P_kV_j = \delta_{kj}V_k.
\]
We say that a \( \mathbb{Z} \)-graded vector space
\[
V' = \bigoplus_{k=-\infty}^{\infty} V'_k
\]
is equivalent to \( V \) if for some \( j \in \mathbb{Z} \) there exist a vector space isomorphism between \( V'_k \) and \( V_{j+k} \) for all \( k \). This terminology is motivated by the fact that for a given \( (V, z) \) one can always generate another \( \mathbb{Z} \)-graded space \( (V', z') \) through the mappings \( V'_k \to V_{j+k}, z' \to z - jI \) for any \( j \in \mathbb{Z} \).

For a given \( \mathbb{Z} \)-graded \( V \) we say that \( \pi : \mathcal{A} \to \text{End} V \) provides a \( \mathbb{Z} \)-graded representation of \( \mathcal{A} \) if \( \pi(Z) = z \) and the relations (2, 6) are preserved. In such a case we can write
\[
\pi(X(u)) = \sum_{j=-\infty}^{\infty} X(u, j)
\]
where the matrices \( X(u, j) \) satisfy
\[
X(u, j)V_k = 0 \quad \text{for } j \neq k.
\]
More specifically, this means that for \( |\psi_k\rangle \in V_k \) we have
\[
\pi(X(u)Y(v))|\psi_k\rangle = X(u, k + p\{Y(u)\})Y(v, k)|\psi_k\rangle.
\]
In view of the equivalence of \( \mathbb{Z} \)-graded vector spaces defined above, there can also exist equivalent representations. We can define a representation \( \pi' \) equivalent to \( \pi \) by specifying some \( k \in \mathbb{Z} \) such that
\[
\pi'(Z) = \pi(Z - kI)
\]
and for
\[
\pi'(X(u)) = \sum_{j=-\infty}^{\infty} X'(u, j)
\]
the matrices \( X'(u, j) \) are defined by
\[
X'(u, j) = X(u, j + k) \quad \forall j \in \mathbb{Z}.
\]
For any \( \mathbb{Z} \)-graded representation it follows from (3) that the following hold:
\[
\begin{align*}
[A(u, j), A(v, j)] & = [D(u, j), D(v, j)] = 0, \\
B(u, j)B(v, j - 1) & = B(v, j)B(u, j - 1), \\
C(u, j)C(v, j + 1) & = C(v, j)C(u, j + 1), \\
A(u, j)C(v, j + 1) & = \frac{u - v + \eta}{u - v} C(v, j + 1)A(u, j + 1) \\
& - \frac{\eta}{u - v} C(u, j + 1)A(v, j + 1), \\
D(u, j)C(v, j + 1) & = \frac{u - v - \eta}{u - v} C(v, j + 1)D(u, j + 1) \\
& + \frac{\eta}{u - v} C(u, j + 1)D(v, j + 1),
\end{align*}
\]
From the defining relations (2) the transfer matrix defined by \( \tau(u) = A(u) + D(u) \) commutes for different values of the spectral parameter \( u \); viz.

\[ [\tau(u), \tau(v)] = 0. \]

Moreover, we may express the representation \( \pi(\tau(u)) \) of the transfer matrix as

\[ \pi(\tau(u)) = \sum_{j=-\infty}^{\infty} \tau(u,j) \]

such that

\[ \tau(u,j)V_k = 0 \quad \text{for } j \neq k \]

and

\[ [\tau(u,j), \tau(v,k)] = 0 \quad \forall \ j, k. \]

Since \( p\{\tau(u)\} = 0 \), the diagonalisation of \( \pi(\tau(u)) \) is thus reduced to the diagonalisation of each of the matrices \( \tau(u,j) \) on the \( \mathbb{Z} \)-graded component \( V_j \), where we have

\[ [\tau(u,j), \tau(v,j)] = 0. \]

We may restrict our attention to the case of \( \tau(u,0) \), as each \( \tau(u,j) \) is equivalent to some \( \tau'(u,0) \) through the use of equivalent representations as introduced earlier.

In order to formulate the algebraic Bethe ansatz solution for this class of representations, we assume the existence of a pseudovacuum vector \( |\chi\rangle \in V_k \) such that

\[
A(u,k)|\chi\rangle = \alpha(u,k)|\chi\rangle \\
B(u,k)|\chi\rangle = 0 \\
C(u,k)|\chi\rangle \neq 0 \\
D(u,k)|\chi\rangle = \delta(u,k)|\chi\rangle.
\]

The above implies that \( |\chi\rangle \) is a maximal weight vector with respect to \( \mathbb{Z} \). Without loss of generality we can choose \( k = M \), again due to the equivalence of representations discussed earlier, and look for Bethe states defined by

\[
\Psi(v_1, \ldots, v_M) \equiv \Psi(\{v_i\}) = C(v_1, 1)C(v_2, 2) \cdots C(v_M, M)|\chi\rangle.
\]

(9)

It is easy to check that this Bethe state is symmetric with respect to the variables \( v_i \), a feature which plays a crucial role below. Acting \( A(u,0) \) and \( D(u,0) \) on the Bethe state we have

\[
A(u,0)\Psi(\{v_i\}) = \alpha(u,M) \prod_{i=1}^{M} \frac{u - v_i + \eta}{u - v_i} \Psi(\{v_i\})
\]

5
\[
+ \sum_{i=1}^{M} \mathcal{M}_i(u, \{v_j\}) \Psi(v_1, \ldots, v_{i-1}, u, v_{i+1}, \ldots, v_M),
\]
\[
D(u, 0) \Psi(v_1, \ldots, v_M) = \delta(u, M) \prod_{i=1}^{M} \frac{u - v_i - \eta}{u - v_i} \Psi(\{v_i\})
\]
\[
+ \sum_{i=1}^{M} \mathcal{N}_i(u, \{v_j\}) \Psi(v_1, \ldots, v_{i-1}, u, v_{i+1}, \ldots, v_M),
\]
with
\[
\mathcal{M}_i(u, \{v_j\}) = -\frac{\eta}{u - v_i} \alpha(v_i, M) \prod_{j \neq i}^{M} \frac{v_i - v_j + \eta}{v_i - v_j},
\]
\[
\mathcal{N}_i(u, \{v_j\}) = \frac{\eta}{u - v_i} \delta(v_i, M) \prod_{j \neq i}^{M} \frac{v_i - v_j - \eta}{v_i - v_j}.
\]
Requiring
\[
\mathcal{M}_i(u, \{v_j\}) + \mathcal{N}_i(u, \{v_j\}) = 0
\]
forces \(\Psi(v_1, \ldots v_M)\) to be an eigenstate of \(\tau(u, 0)\) and leads to the Bethe ansatz equations
\[
\frac{\alpha(v_i, M)}{\delta(v_i, M)} = \prod_{j \neq i}^{M} \frac{v_i - v_j - \eta}{v_i - v_j + \eta}, \quad i = 1, \ldots, M.
\]
The corresponding eigenvalue of the matrix \(\tau(u, 0)\) is
\[
\Lambda(u, \{v_i\}) = \alpha(u, M) \prod_{i=1}^{M} \frac{u - v_i + \eta}{u - v_i} + \delta(u, M) \prod_{i=1}^{M} \frac{u - v_i - \eta}{u - v_i}.
\]

3 Explicit \(\mathbb{Z}\)-graded realisations

Next we give two nontrivial \(\mathbb{Z}\)-graded realisations of the algebra \(A\). One is expressible in terms of two Heisenberg algebras with generators \(a_i, a_i^\dagger, i = 1, 2\) and reads \(X(u, j) = \tilde{X}(u, j)P_j\) with
\[
\tilde{A}(u, j) = u^2 + \eta u N + \eta^2 N_1 N_2 - \eta (N_1 - N_2) \omega(N + j I)
- \omega^2 (N + j I) + a_1^\dagger a_1,
\]
\[
\tilde{B}(u, j) = (u + \omega(N + j I) + \eta N_1) a_2 + \eta^{-1} a_1,
\]
\[
\tilde{C}(u, j) = a_1^\dagger (u - \omega(N + j I) + \eta N_2) + \eta^{-1} a_2^\dagger,
\]
\[
\tilde{D}(u, j) = a_1^\dagger a_2 + \eta^{-2}.
\]
Above, $P_j$ are the projections defined by (8), $N_i = a_i^\dagger a_i$, $N = N_1 + N_2$ and $\omega(x)$ is an arbitrary polynomial function of $x$. Note that in the case when $\omega(x)$ is constant, the above realisation reduces to that discussed in [10, 11] and is factorizable into two local representations of the Yang-Baxter algebra expressible in terms of the two Heisenberg algebras. It is important to note that for generic $\omega(x)$ no such factorisation exists.

The representation acts on the infinite dimensional Fock space spanned by the vectors

$$|m, n\rangle = (a_1^\dagger)^m (a_2^\dagger)^n |0\rangle, \quad m, n = 0, 1, 2, \ldots, \infty.$$  \hspace{1cm} (14)

For this representation, we choose the pseudovacuum $|\chi\rangle$ as the Fock vacuum $|0\rangle$. The representation of the grading operator $Z$ is chosen to be

$$\pi(Z) = M.I - N.$$  

We then have

$$\alpha(u, M) = u^2 - \omega^2(M), \quad \delta(u, M) = \eta^{-2}$$  \hspace{1cm} (15)

and the Bethe ansatz equations become

$$\eta^2 (v_i^2 - \omega^2(M)) = \prod_{j \neq i}^{M} \frac{v_i - v_j - \eta}{v_i - v_j + \eta}$$  \hspace{1cm} (16)

for the diagonalisation of the matrix $\tau(u, 0)$. The eigenstates (9) in this instance are also eigenstates of the total particle number $N$ with eigenvalue $M$.

Another $\mathbb{Z}$-graded realisation of the Yang-Baxter algebra is $X(u, j) = \tilde{X}(u, j)P_j$ with

$$\tilde{A}(u, j) = -\eta u^2 + u (1 - \eta^2(K_z + N_c) - \eta \omega(K_z + N_c + jI)) + \eta K_z - \eta^2 K_z \omega(K_z + N_c + jI) - \eta^3 N_c K_z + \eta^2 c K_+,$$
$$\tilde{B}(u, j) = \eta(1 - \eta u - \eta \omega(K_z + N_c + jI) - \eta^2 N_c)K_- - \eta c(u - \eta K_z),$$
$$\tilde{C}(u, j) = \eta c^\dagger(u + \eta K_z) - \eta K_+,$$
$$\tilde{D}(u, j) = u - \eta K_z + \eta^2 c^\dagger K_-.$$  \hspace{1cm} (17)

Above, the operators $c, c^\dagger$ form a Heisenberg algebra, with $N_c = c^\dagger c$, and the operators $K_z, K_+, K_-$ satisfy the relations of the $su(1, 1)$ algebra $[K_z, K_+] = \pm K_\pm, \quad [K_+, K_-] = -2K_z$. As in the previous example, $\omega(x)$ is an arbitrary polynomial function of $x$ and the above realisation is factorisable only in the case when $\omega(x)$ is constant.

For this representation, we choose the pseudovacuum $|\chi\rangle$ as the tensor product of the Fock vacuum $|0\rangle$ with a lowest weight state for the algebra $su(1, 1)$ of weight $\kappa$. The representation of the grading operator may be chosen as

$$\pi(Z) = M.I - K_z - N_c.$$  

Then,

$$\alpha(u, M) = (1 - \eta u - \eta \omega(M))(u + \eta \kappa), \quad \delta(u, M) = u - \eta \kappa$$  \hspace{1cm} (18)

and the Bethe ansatz equations are

$$(1 - \eta v_i - \eta \omega(M)) \left(\frac{v_i + \eta \kappa}{v_i - \eta \kappa}\right) = \prod_{j \neq i}^{M} \frac{v_i - v_j - \eta}{v_i - v_j + \eta}.$$  \hspace{1cm} (19)
4 Three models of Bose-Einstein condensates

4.1 Model 1: Two coupled BECs

Consider the following general Hamiltonian describing Josephson tunneling between two coupled Bose-Einstein condensates

\[ H = U_{11}N_1^2 + U_{12}N_1N_2 + U_{22}N_2^2 + \mu_1 N_1 + \mu_2 N_2 \]

\[-\frac{\mathcal{E}_J}{2}(a_1^+a_2 + a_2^+a_1). \quad (20)\]

The above Hamiltonian generalises the canonical Josephson Hamiltonian studied in [10, 11] in that the couplings \( U_{11}, U_{22} \) for the \( S \)-wave scattering terms can be chosen arbitrarily. It also describes a pair of Cooper pair boxes with capacitive coupling [13]. In the limit \( U_{22} \to 0 \), then \( \langle N_2 \rangle > > \langle N_1 \rangle \), which can be considered as a single Cooper pair box coupled to a reservoir.

It is an algebraic exercise to show that the Hamiltonian is related with the matrix \( \tilde{\tau}(u, 0) = \tilde{A}(u, 0) + \tilde{D}(u, 0) \) through

\[ H = -\frac{\mathcal{E}_J}{2} [\tilde{\tau}(0,0) - \eta^{-2} + (\alpha N + \beta)^2 - \eta \sigma N - \eta \delta N^2]. \]

Here we have chosen \( \omega(N) = \alpha N + \beta \) and the coupling constants are identified as

\[ \eta^2 = \frac{2(U_{11} + U_{22} - U_{12})}{\mathcal{E}_J}, \]

\[ \alpha = \frac{U_{11} - U_{22}}{\eta \mathcal{E}_J}, \]

\[ \beta = \frac{\mu_1 - \mu_2}{\eta \mathcal{E}_J}, \]

\[ \sigma = \frac{\mu_1 + \mu_2}{\eta \mathcal{E}_J}, \]

\[ \delta = \frac{U_{11} + U_{22}}{\eta \mathcal{E}_J}. \]

Noting that

\[ N = \eta^{-1} \frac{d\tilde{\tau}}{du}(0,0), \]

the above demonstrates that the Hamiltonian (20) is expressible solely in terms of the matrix \( \tilde{\tau}(u, 0) \) and its derivative.

Since \([H, N] = 0\), the Hamiltonian is block diagonal on the Fock basis (14). Thus on a subspace of the Fock space with fixed particle number \( N \), the diagonalisation of \( \tilde{\tau}(u, 0) \) is equivalent to the diagonalisation of \( \tau(u, 0) \) presented earlier in the Bethe ansatz framework. We then deduce that the solution of (20) for the energy spectrum is

\[ E = -\frac{\mathcal{E}_J}{2} \left[ \eta^{-2} \prod_{i=1}^{N} \frac{v_i + \eta}{v_i} - (\alpha N + \beta)^2 \prod_{i=1}^{N} \frac{v_i - \eta}{v_i} \right]. \]
\[-\eta^{-2} + (\alpha N + \beta)^2 - \eta \sigma N - \eta \delta N^2 \]  

where the parameters \( \{v_i\} \) are subject to the Bethe ansatz equations

\[ \eta^2 (v_i^2 - (\alpha N + \beta)^2) = \prod_{j \neq i}^{N} \frac{v_i - v_j - \eta}{v_i - v_j + \eta}. \]

4.2 Model 2: Homo-atomic-molecular BECs

Next we turn our attention to a two-mode model for an atomic-molecular Bose-Einstein condensate with identical atoms. The Hamiltonian takes the form

\[
H = U_{aa} N_a^2 + U_{ac} N_a N_c + U_{cc} N_c^2 + \mu_a N_a + \mu_c N_c \\
+ \Omega (a^\dagger a^\dagger c + c^\dagger a a)
\]  

(22)

which acts on a basis of Fock states analogous to (14). For the case of \(^{87}\)Rb, all of these parameters have been estimated from experiment (see [8]). In the experiment described in [4], the parameter \( U_{aa} \) was varied significantly with a magnetic field.

The Hamiltonian commutes with the total atom number \( N = N_a + 2N_c \). In terms of a realisation of the algebra \( su(1,1) \) through

\[
K_+ = \left(\frac{a^\dagger)^2}{2}, \ K_- = \frac{a^2}{2}, \ K_z = \frac{2N_a + 1}{4}, \]

(23)

one may establish the relation between the Hamiltonian and the corresponding transfer matrix \( \tilde{\tau}(u, 0) = A(u, 0) + D(u, 0) \) arising from the the realisation (17) of the Yang-Baxter algebra is

\[
H = \sigma + \delta (N/2 + 1/4) + \gamma (N/2 + 1/4)^2 + 2\eta^{-2}\Omega \tilde{\tau}(0, 0),
\]

with

\[
\frac{d\tilde{\tau}}{du}(0, 0) = 2 - \eta(\eta + \alpha)(N/2 + 1/4) - \eta \beta.
\]

Above we have chosen

\[
\omega(K_z + N_c) = \alpha(K_z + N_c) + \beta = \alpha(N/2 + 1/4) + \beta
\]

and the following identification has been made for the coupling constants

\[
\begin{align*}
\eta &= \frac{4U_{aa} + U_{cc} - 2U_{ac}}{2\Omega}, \\
\alpha &= \frac{U_{cc} - 4U_{aa}}{2\Omega}, \\
\beta &= \frac{2\mu_c - 4\mu_a + 4U_{aa} - U_{ac}}{4\Omega},
\end{align*}
\]
\[ \sigma = \frac{U_{aa} - 2\mu_a}{4}, \]
\[ \delta = \frac{2\mu_c - U_{ac}}{2}, \]
\[ \gamma = U_{cc}. \]

By the same argument as before, we conclude that the exact solution for the energy spectrum of (22) is determined by

\[ E = \sigma + \delta(M + \kappa) + \gamma(M + \kappa)^2 \]
\[ + 2\eta^{-1}\kappa\Omega \left[ (1 - \eta(\alpha(M + \kappa) + \beta)) \prod_{i=1}^{M} \frac{v_i - \eta}{v_i} - \prod_{i=1}^{M} \frac{v_i + \eta}{v_i} \right], \quad (24) \]

where the parameters \( v_i \) satisfy the Bethe ansatz equations

\[ [1 - \eta v_i - \eta(\alpha(M + \kappa) + \beta)] \left( \frac{v_i + \eta\kappa}{v_i - \eta\kappa} \right) = \prod_{j \neq i}^{M} \frac{v_i - v_j - \eta}{v_i - v_j + \eta}. \quad (25) \]

For the representation (23) of the \( su(1,1) \) algebra there are two lowest weight vectors; viz. the Fock vacuum \(|0\rangle\) and the one particle state \( a^\dagger |0\rangle \). It follows that the allowed values for \( \kappa \) in (23,24) are \( \kappa = 1/4, 3/4 \). This demonstrates that the solution of the model depends on whether the total particle number \( N = 2M + 2\kappa - 1/2 \) is even or odd, the effects of which on the energy spectrum can be seen through numerical analysis (cf. [12]).

### 4.3 Model 3: Hetero-atomic-molecular BECs

The previous construction can be extended to model an atomic-molecular Bose-Einstein condensate with two distinct species of atoms, denoted \( a \) and \( b \). For this case the Hamiltonian takes the form

\[ H = U_{aa}N_a^2 + U_{bb}N_b^2 + U_{cc}N_c^2 + U_{ab}N_aN_b + U_{ac}N_aN_c + U_{bc}N_bN_c + \mu_aN_a + \mu_bN_b + \mu_cN_c + \Omega(a^{\dagger b^{\dagger}}c + c^{\dagger}ba) \quad (26) \]

which commutes with the total atom number \( N = N_a + N_b + 2N_c \) and \( \mathcal{I} = N_a - N_b \). Here the model acts on the Fock space spanned by the vectors

\[ |l, m, n\rangle = (a^{\dagger})^l(b^{\dagger})^m(c^{\dagger})^n |0\rangle. \]

In order to show the solvability of this model, we adopt the realisation of the \( su(1,1) \) algebra given by

\[ K_+ = a^{\dagger}b^{\dagger}, \quad K_- = ab, \quad K_z = \frac{N_a + N_b + 1}{2}, \quad (27) \]

and observe that the operator \( \mathcal{I} \) commutes with the \( su(1,1) \) algebra in this representation, hence taking a constant value in any irreducible representation. Due to the symmetry
upon interchanging the labels $a$ and $b$, we can assume without loss of generality that the eigenvalues of $\mathcal{I}$ are non-negative. In particular, note then that the lowest weight states for this realisation are of the form

$$|m\rangle = (a^\dagger)^m |0\rangle, \quad m = 0, 1, 2, \ldots$$

and $K_z |m\rangle = (m/2 + 1/2) |m\rangle$. We conclude that the lowest weight labels $\kappa$ can be taken from the set \{1/2, 1, 3/2, \ldots\} and the eigenvalue of $\mathcal{I}$ on the irreducible representation labelled by $\kappa$ is $2\kappa - 1$.

For this case the relation between the Hamiltonian and the corresponding matrix $\tilde{\tau}(u, 0)$ is

$$H = \sigma + \delta (N/2 + 1/2) + \lambda (N/2 + 1/2)^2 + \mu \mathcal{I} + \nu \mathcal{I}^2 + \xi \mathcal{I}(N/2 + 1/2) + \eta^{-2} \Omega \tilde{\tau}(0, 0)$$

with

$$\frac{d\tilde{\tau}}{du}(0, 0) = 2 - \eta(\eta + \alpha)(N/2 + 1/2) - \eta\beta \mathcal{I} - \eta\gamma.$$ 

Above we have chosen

$$\omega(K_z + N_b) = \alpha(K_z + N_c) + \beta(2\kappa - 1) + \gamma = \alpha(N/2 + 1/2) + \beta \mathcal{I} + \gamma$$

and the coupling constants are related through the relations

$$\eta = \frac{U_{aa} + U_{bb} + U_{cc} + U_{ab} - U_{ac} - U_{bc}}{\Omega},$$

$$\alpha = \frac{U_{cc} - U_{aa} - U_{bb} - U_{ab}}{\Omega},$$

$$\beta = \frac{2U_{bb} - 2U_{aa} + U_{ac} - U_{bc}}{2\Omega},$$

$$\gamma = \frac{2U_{aa} + 2U_{bb} + 2U_{ab} - U_{ac} - U_{bc} + 2\mu_c - 2\mu_a - 2\mu_b}{2\Omega},$$

$$\sigma = \frac{2\mu_c - U_{ac} - U_{bc}}{4},$$

$$\delta = \frac{4}{2},$$

$$\lambda = U_{cc},$$

$$\rho = \frac{U_{bb} - U_{aa} + \mu_a - \mu_b}{2},$$

$$\nu = \frac{U_{aa} + U_{bb} - U_{ab}}{4},$$

$$\xi = \frac{U_{ac} - U_{bc}}{2}.$$
The exact solution in this instance reads
\[
E = \sigma + \delta(M + \kappa) + \lambda(M + \kappa)^2 \\
+ \rho(2\kappa - 1) + \nu(2\kappa - 1)^2 + \xi(2\kappa - 1)(M + \kappa) \\
+ \eta^{-1}\kappa\Omega \left[ (1 - \eta(\alpha(M + \kappa) + \beta(2\kappa - 1) + \gamma)) \prod_{i=1}^{M} \frac{v_i - \eta}{v_i} \\
- \prod_{i=1}^{M} \frac{v_i + \eta}{v_i} \right],
\]
where the parameters \(v_i\) satisfy the Bethe ansatz equations
\[
[1 - \eta v_i - \eta (\alpha(M + \kappa) + \beta(2\kappa - 1) + \gamma)] \left( \frac{v_i + \eta \kappa}{v_i - \eta \kappa} \right) = \prod_{j \neq i}^{M} \frac{v_i - v_j - \eta}{v_i - v_j + \eta}
\]
and the total atom number is given by \(N = 2M + 2\kappa - 1\).

5 Wave function scalar products

Recall that in the usual algebraic Bethe ansatz for the algebra \(\mathcal{A}\) there is a formula originally due to Slavnov [17] (see also [13, 18]) for the wave function scalar products. The Slavnov formula still applies in the \(\mathbb{Z}\)-graded case and takes the usual form
\[
S_M(\{u_j\}, \{v_k\}) = \Phi(\{u_j\})\Psi(\{v_k\}) = \frac{\det T(\{u_j\},\{v_k\})}{\det V(\{u_j\},\{v_k\})}
\]
with the entries of the \(M \times M\) matrices \(T\) and \(V\) given by
\[
T_{ab} = \frac{\partial}{\partial v_a} \Lambda(u_b, \{v_k\}), V_{ab} = \frac{1}{u_b - v_a}, a, b = 1, \ldots, M
\]
\(\Phi(\{u_i\})\) is the left vector
\[
\Phi(u_1, \ldots, u_M) = \langle \chi| B(u_M, M) \ldots B(u_1, 1).
\]
Above, we have adopted the usual convention to scale the Yang-Baxter algebra such that \(\delta(u, M) = 1\). Also, \(\{v_k\}\) provide a solution of the Bethe ansatz equation (12) and the parameters \(\{u_j\}\) can be chosen arbitrarily.

The Yang-Baxter algebra \(\mathcal{A}\) admits a conjugation operation \(\dagger : \mathcal{A} \rightarrow \mathcal{A}\) defined by
\[
A(u)^\dagger = A(u), \quad B(u)^\dagger = C(u), \quad C(u)^\dagger = B(u), \quad D(u)^\dagger = D(u)
\]
and extended to all of $\mathcal{A}$ through

$$(\theta, \phi)^\dagger = \phi^\dagger \theta^\dagger, \quad \forall \theta, \phi \in \mathcal{A}$$

such that the defining relations (2) are preserved. Consequently the right vector $\Phi(v_1, \cdots, v_M)^\dagger$ is also an eigenvector of the transfer matrix whenever the Bethe ansatz equations for the parameters $\{v_i\}$ are satisfied. However, it is apparent that the $\mathbb{Z}$-graded representations (13, 17) we have introduced are not unitary, and generally

$$\Phi(\{v_i\})^\dagger \neq \Psi(\{v_i\}).$$

On the other hand, numerical analysis we have undertaken for the above models indicates that for fixed particle numbers, and generic values of the coupling parameters, the energy spectrum is free of degeneracies. This is presumably due to the fact that the only Lie algebra symmetries for these models are $u(1)$ invariances corresponding to conservation of particle numbers, and the non-degenerate spectra are examples of Hund’s non-crossing rule [19, 20]. Whenever this is the case, we can conclude that

$$\Phi(\{v_i\})^\dagger = K \Psi(\{v_i\})$$

for some constant $K$ and the Slavnov formula can still be invoked for the computation of form factors and correlation functions (cf. the example of [10] where it was found $K = \pm 1$).

6 Conclusion

In conclusion we have introduced a new scheme for the algebraic Bethe ansatz to diagonalise three classes of integrable models relevant to Bose-Einstein condensates of dilute alkali gases. The extension of this construction to other types of models, such as the Jaynes-Cummings model [21], and generalised Tavis-Cummings model discussed in [22], is straightforward.

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