On interface conditions for flows in coupled free-porous media

The derived interface conditions are summarized along with a pictorial description of the problem, which pertains to the flow of an incompressible fluid in coupled free-porous media. $\Psi$ is the power expended density along the interface. $v_{\text{free}}$ and $v_{\text{por}}$ are the velocities in the free and porous regions, respectively. A superposed asterisk on a (vectorial) quantity denotes its tangential component along the interface. $v_n$ is the normal component of the velocity at the interface from the free region into the porous region. $T_{\text{extra}}^{\text{free}}$ and $T_{\text{extra}}^{\text{por}}$, respectively, denote the extra Cauchy stresses in the free and porous regions. $t_{\text{free}}$ and $t_{\text{por}}$, respectively, denote the tractions on the free and porous sides of the interface with outward normals $\hat{n}_{\text{free}}$ and $\hat{n}_{\text{por}}$. A unit tangential vector along the interface is denoted by $\hat{s}$.  

\[ v_{\text{free}}^{(n)}(x) + v_{\text{por}}^{(n)}(x) = 0 \]
\[ t_{\text{free}}(x) \cdot \hat{n}_{\text{free}}(x) + \frac{\partial \Psi}{\partial v_n} = t_{\text{por}}(x) \cdot \hat{n}_{\text{por}}(x) \]
\[ \hat{s}(x) \cdot T_{\text{extra}}^{\text{free}} \hat{n}_{\text{free}}(x) = -\hat{s}(x) \cdot \frac{\partial \Psi}{\partial N_{\text{free}}} \]
\[ \hat{s}(x) \cdot T_{\text{extra}}^{\text{por}} \hat{n}_{\text{por}}(x) = -\hat{s}(x) \cdot \frac{\partial \Psi}{\partial N_{\text{por}}} \]
On interface conditions for flows in coupled free-porous media

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ABSTRACT. Many processes in nature (e.g., physical and biogeochemical processes in hyporheic zones, and arterial mass transport) occur near the interface of free-porous media. A firm understanding of these processes needs an accurate prescription of flow dynamics near the interface which (in turn) hinges on an appropriate description of interface conditions along the interface of free-porous media. Although the conditions for the flow dynamics at the interface of free-porous media have received considerable attention, many of these studies were empirical and lacked a firm theoretical underpinning. In this paper, we derive a complete and self-consistent set of conditions for flow dynamics at the interface of free-porous media. We first propose a principle of virtual power by incorporating the virtual power expended at the interface of free-porous media. Then by appealing to the calculus of variations, we obtain a complete set of interface conditions for flows in coupled free-porous media. A noteworthy feature of our approach is that the derived interface conditions apply to a wide variety of porous media models. We also show that the two most popular interface conditions – the Beavers-Joseph condition and the Beavers-Joseph-Saffman condition – are special cases of the approach presented in this paper. The proposed principle of virtual power also provides a minimum power theorem for a class of flows in coupled free-porous media, which has a similar mathematical structure as the ones enjoyed by flows in uncoupled free and porous media.

PROBLEM STATEMENT

Let us consider a domain which consists of two non-overlapping regions: a porous region and a free flow region. The interface is the surface that demarcates these two regions. Fig. 1 provides a pictorial description. Now consider the situation in which an incompressible fluid flows in this domain with the porous solid to be rigid. The central question pertaining flows in coupled free-porous media is:

Given the domain, free flow and porous regions, boundary conditions on the external boundaries, properties of the incompressible fluid (e.g., the coefficient of dynamic viscosity, true density), and properties of the rigid porous medium (e.g., porosity, permeability), what is the set of conditions appropriate at the interface?

1. INTRODUCTION AND MOTIVATION

1.1. Motivation. Many important science and engineering problems involve flows in a domain which comprises free flow and porous regions. In these problems, a plethora of vital processes

Key words and phrases. coupled free-porous media; principle of virtual power; interface conditions; internal constraints; calculus of variations; minimum power principle.
Figure 1. A pictorial description of coupled free-porous media. The free flow region \( K_{\text{free}} \) and the porous region \( K_{\text{por}} \) share a common interface \( \Gamma_{\text{int}} \). The outward unit normal vector to \( K_{\text{free}} \) at the interface is denoted by \( \hat{n}_{\text{free}}(x) \). A similar notation holds for \( \hat{n}_{\text{por}}(x) \), which is equal to \(-\hat{n}_{\text{free}}(x)\). The side of \( \Gamma_{\text{int}} \) that shares with \( K_{\text{free}} \) is noted by \( \Gamma_{\text{free}} \), and a similar notation holds for \( \Gamma_{\text{por}} \). The external boundaries of the free and porous regions are, respectively, denoted by \( \partial K_{\text{ext}}^{\text{free}} \) and \( \partial K_{\text{ext}}^{\text{por}} \). The corresponding unit outward normals to these external boundaries are denoted by \( \hat{n}_{\text{ext}}^{\text{free}} \) and \( \hat{n}_{\text{ext}}^{\text{por}} \). A unit tangent vector on the interface is denoted by \( \hat{s} \).

\( K_{\text{free}} \) takes place near the interface of free flow and porous regions. One has to capture these processes accurately to discern the overall dynamics and all the interactions in the entire domain. We now discuss two such problems, which have motivated us to undertake the research presented in this paper.\(^1\)

The first problem pertains to surface-subsurface interactions of large water systems. Groundwater and surface water interactions between rivers and streams are vital to flora and fauna, water distribution, and environmental factors which all affect the whole food chain [Jones and Holmes, 1996; Sophocleous, 2002]. For example, mixing at the interface of groundwater and surface water is critical for nutrient transport and the carbon & nitrogen (C&N) cycles; both are vital to an ecosystem [Dwivedi et al., 2017]. The interactions between groundwater and surface water greatly depend on the flow dynamics in the hyporheic zone (see Fig. 2). Several physical and biogeochemical processes take place in the hyporheic zone, and these processes are in turn coupled with the processes that take place in the free and subsurface zones. Therefore, the success of a predictive

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\(^1\)Professor Lallit Anand has informed us that an appropriate set of conditions at the interface of porous and free regions is also important in the studies on Lithium-ion batteries.
modeling of surface-subsurface interactions will rest on the accurate modeling of the flow dynamics at the interface of free-porous regions.

The second problem pertains to the arterial mass transfer — the transport of atherogenic macromolecules, such as low-density lipoproteins (LDL), from bulk blood flow into artery walls and vice versa [Sun et al., 2006; Wada and Karino, 1999]. Accumulation of LDL at the interface of bulk blood flow and the endothelial layer – the part of lumen next to the blood flow – is a primary cause of various cardiovascular diseases; for example, atherosclerotic lesions within the intima of arteries [Caro et al., 1971; Hoff et al., 1975]. A firm understanding of this complex process will enable physicians to administer better therapeutic procedures. Mechanics can play an important role to gain a good understanding of this problem. However, any such an effort has to address accurately the complex flow dynamics at the interface of bulk blood flow and (porous) arterial walls.

Figure 2. The top figure (adapted from the US Geological Survey [Thomas et al., 1998]) shows a typical hyporheic zone. The size of a hyporheic zone can vary tens of meters vertically to hundreds of meters laterally. The bottom figure (adapted from the British Environment Agency [Buss, 2009]) depicts a myriad of important processes that take place in a hyporheic zone which affect the processes at the watershed scale (varying from tens to hundreds of kilometers) and hence affect the overall ecosystem.

However, obtaining a self-consistent, independent and complete set of conditions at the interface — which we will refer to as the interface conditions\(^2\) — for flows in coupled free-porous media is far

\(^2\)We believe that the usage of interface conditions is more appropriate than the two alternatives: jump conditions and boundary conditions. As discussed in Appendix A, the jump conditions (which are the balance laws across a singular surface) do not furnish a workable set of conditions for flows in coupled free-porous media; especially when the porous solid is rigid, which is the case in this paper. Moreover, the set of conditions derived in this paper (given
from settled. Before we elaborate on some prior works and present our approach, we now outline what should be the nature of the interface conditions. We portray the character of interface conditions as follows:

(i) Interface conditions may directly stem from the balance laws and the associated jump conditions. For example, the no-penetration boundary condition at a stationary impervious wall, commonly employed in fluid mechanics, stems from the jump condition associated with the balance of mass.

(ii) Alternatively, they may be constitutive specifications. If this is the case, they should be compatible with the balance laws and satisfy the essential invariance properties (e.g., the principle of material frame-indifference or the Galilean invariance).

(iii) It is needless to say that they should agree with the experiments.

(iv) They should apply to a wide variety of problems.

(v) They should give rise to mathematical models (i.e., boundary value problems and initial boundary value problems) that are mathematically well-posed.

This paper fills the gap in our understanding of interface conditions for flows in coupled free-porous media. The specific aims of this paper are twofold. First, to develop a framework for obtaining appropriate conditions for coupled flow dynamics at the interface of free-porous media. Second, to recover some popular conditions available in the literature for coupled flows as special cases of the proposed framework. Our approach will utilize the principle of virtual power and the theory of interacting continua, invoke a geometric argument to enforce the internal constraints, impose the principle of material frame-indifference on all the constitutive relations and use the standard results from the calculus of variations.

Over the last three decades, the principle of virtual power has been extended with respect to its domain of applicability, which was re-ignited by Germain [1973] and was further developed by Maugin [1980]. Currently, the principle of virtual power has been employed for a wide variety of problems in mechanics, ranging from viscoplasticity [Anand and Su, 2005], gradient theories [Gurtin and Anand, 2005] to coupled problems [Fried and Gurtin, 2007]. A significant extension of this principle is to pose on an arbitrary subset of the domain and obtain the Cauchy’s fundamental theorem for the stress (which relates the Cauchy stress with the traction on a surface) as a consequence [Podio-Guidugli, 2009; Fosdick, 2011]. Although such an extension (defining the principle on an arbitrary subset) is not essential to derive the interface conditions, we will still show how to extend the proposed framework to recover the Cauchy’s fundamental theorem but will relegate such a discussion to one of the appendices.

The theory of interacting continua, TIC, (also known as the mixture theory) is a mathematical framework to develop continuum models for a homogenized response of a mixture of (interacting) constituents [Bowen, 1976]. The overall idea of TIC is to model a mixture of constituents as a collection of superposed continua. Two inherent assumptions of TIC are the treatment of each constituent as a continuum and the coexistence of all constituents in the space occupied by the mixture [Truesdell, 2012]. The second assumption can be thought as follows: at every point in the by equation (4.16)) does not entirely stem from the jump conditions. Since the interface Γ_{int} is not an external boundary to the domain of interest (which consists of both free and porous regions, i.e., Ω), it is not appropriate to refer to these conditions as boundary conditions for coupled flows.

3Influenced by the lecture “The Character of Physical Law” given by Feynman [1967], we mimic the terminology and employ the phrase: the character of interface conditions, in our discussion on the general physical and mathematical nature of interface conditions.
space occupied by the mixture, there is a particle from each of its constituents. Each constituent has balance laws similar to that of a single continuum. However, the balance laws will contain terms which account the interactions due to the presence of other constituents. We will appeal to the TIC framework to model the porous media.

The last piece in our proposed framework is to systematically enforce internal constraints, which in our case arise due to the incompressibility of the fluid in both regions, under the principle of virtual power. There are several approaches proposed in the literature to enforce internal constraints. The most popular approach, which is commonly referred to as the Truesdell-Noll approach [Truesdell and Noll, 2013, §30], is built upon two a priori constitutive assumptions: (i) the stress is decomposed into active and reactive components, and (ii) the reactive component performs no work under a motion consistent with the constraint. Alternatively, we employ the approach put forth by Carlson et al. [2004] to enforce the internal constraint. An attractive feature of this approach is that the two assumptions made under the Truesdell-Noll approach can be obtained as mathematical consequences rather than a priori constitutive assumptions. This approach hinges on the direct sum provided by the projection theorem; however, the approach can be easily explained by a simple geometrical argument: If a vector $a$ is perpendicular to every vector $b$ that is (in turn) perpendicular to a vector $c$ then $a$ and $c$ are collinear. Carlson et al. [2004] have employed the geometric argument in the context of hyperelasticity (which is a non-dissipative model) by utilizing the underlying energy balance formalism. Herein, we show the principle of virtual power nicely blends with the geometric argument for flows in coupled free-porous media.

1.2. Scope and an outline of this paper. The plan for the rest of this paper is as follows. We will first outline some of the experimental observations and discuss some important prior works (§2). We propose a principle of virtual power for coupled flows by taking into account the virtual power expended at the interface of free-porous media (§3). Using this principle, we obtain a complete set of interface conditions which capture the prior experimental observations (§4). We then show the popular conditions – Beavers-Joseph and Beavers-Joseph-Saffman conditions – to be special cases of the proposed framework. This discussion will particularly reveal the assumptions and the validity of these popular conditions for flows in coupled free-porous media (§5). We also show that a class of flows in coupled free-porous media enjoys a minimum power theorem (§6). We then employ the minimum power theorem to establish the uniqueness of solutions under certain assumptions on the internal dissipation and the power expended density along the interface (§7). We end the paper with a discussion on the main findings (§8).

2. EXPERIMENTAL OBSERVATIONS AND PRIOR WORKS

The two most popular approaches are the Beavers-Joseph (BJ) condition [Beavers and Joseph, 1967] and the Beavers-Joseph-Saffman (BJS) condition [Saffman, 1971]. The experiments conducted by Beavers and Joseph [1967] provided the following two pieces of information regarding flows near the interface of coupled free-porous media:

(i) The no-slip condition, commonly used for free flows at a boundary, is no longer satisfied at the interface.

The coexistence of all constituents at a point in space may seem like a violation of the reality. However, it is no different from the fact that a spatial point in a continuum description is (in reality) made of several atoms, electrons, and elementary particles. It is thus essential to be aware of the scale at which the modeling is done and at the same time recognize that TIC is a form of homogenized theory.

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(ii) There is a jump in the tangential components of velocity on either side of the interface. Beavers and Joseph [1967] also proposed an empirical relation, which advocates that the shear stress on the free flow side of the interface is linearly proportional to the jump in the tangential velocities across the interface. Based on the velocity profile and the notation introduced in Fig. 3, the BJ condition takes the following form:

\[ u_B - Q = \left( \frac{k^{1/2}}{\alpha} \right) \frac{\partial u}{\partial y} \bigg|_{y=0^+} \]  \hspace{1cm} (2.1)

where \( y = 0^+ \) is the boundary limit point from the free flow region, \( k \) denotes the isotropic permeability of the porous medium, and \( \alpha \) is a constant that depends only on the properties of the fluid and the porous material.

Later, Saffman [1971] performed a statistical analysis and suggested a modification to the BJ condition, and this new condition is popularly referred to as the BJS condition. Specifically, using a one-dimensional geometrical setting and assuming uniform pressure gradient in the porous medium, Saffman [1971] argued that the velocity on the porous medium side is a higher-order term compared to the velocity on the free flow side of the interface, and hence one can neglect the higher-order term. The BJS condition takes the following form:

\[ u_B = \left( \frac{k^{1/2}}{\alpha} \right) \frac{\partial u}{\partial y} \bigg|_{y=0^+} + O(k) \]  \hspace{1cm} (2.2)

where \( O(\cdot) \) is the standard “big O notation,” which describes the limiting behavior of a function when the argument tends towards a particular value.
Although these two approaches have laid the foundation for much of the works in this field, they suffer from some drawbacks, which became clear because of new experimental and numerical studies. First, the slip coefficients under the BJ and BJS conditions are independent of the velocities in the free flow and porous regions. However, Liu and Prosperetti [2011] have shown the linear dependence of the slip coefficient on the Reynolds number, so the slip coefficient can depend on the velocities. Second, their primary interest is free flows in a region with a part of its boundary to be pervious due to a juxtaposed porous medium. Their approaches were aimed at replacing the slip condition with an alternate boundary condition which is appropriate for free flows due to a pervious boundary. Their intended aim is also clear from the titles of these works. Thus their treatments do not provide sufficient information to study flows in coupled free-porous media, as there was no discussion on appropriate boundary conditions for the flows in the porous region. Third, their treatment of the boundary conditions is rather ad hoc and are not amenable to generalization to other porous media models.

One can find in the literature great efforts towards extending these two empirical conditions; for example, see [Larson and Higdon, 1987; Sahraoui and Kaviany, 1992]. However, to the authors’ best knowledge, the literature does not address all the issues laid out earlier under the character of interface conditions. For example, do the BJ/BJS conditions stem from the jump conditions, are they constitutive specifications, or do they combine jump conditions and constitutive specifications? If they are constitutive specifications, what is the rationale behind them? Are they compatible with the balance laws? Are these conditions valid for other porous media models? In the subsequent sections, we will answer all these questions and present a framework for getting a complete set of interface conditions (not just boundary conditions for free flows due to the presence of a pervious boundary) suitable for modeling flows in coupled free-porous media.

3. THE PROPOSED FRAMEWORK

3.1. Notation and definitions. Consider a domain \( \Omega \subset \mathbb{R}^{nd} \) in which an incompressible fluid flows, where “\( nd \)” denotes the number of spatial dimensions and \( \mathbb{R} \) denotes the set of real numbers. A spatial point in the domain is denoted by \( \mathbf{x} \). The gradient and divergence operators with respect to \( \mathbf{x} \) are, respectively, denoted by \( \text{grad}[\cdot] \) and \( \text{div}[\cdot] \). The domain consists of two non-overlapping but adjoining regions: a porous region and a free flow region. See Fig. 1 for a pictorial description.

3.1.1. The interface. The interface – the surface that demarcates the two regions – is denoted by \( \Gamma_{\text{int}} \). The face of \( \Gamma_{\text{int}} \) that is adjacent to the free flow region is denoted by \( \Gamma_{\text{free}} \), and the face of \( \Gamma_{\text{int}} \) that is adjacent to the porous region is denoted by \( \Gamma_{\text{por}} \). Note that \( \Gamma_{\text{int}} \), for our purposes, has a zero thickness, and the faces \( \Gamma_{\text{free}} \) and \( \Gamma_{\text{por}} \) have been introduced for mathematical convenience. The unit outward normal on \( \Gamma_{\text{free}} \) emanating away from the free flow region is denoted by \( \hat{n}_{\text{free}} \).

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5The title of the paper by Beavers and Joseph is “Boundary conditions at a naturally permeable wall,” and the title of the paper by Saffman is “On the boundary condition at the surface of a porous medium.”

6To quote from [Beavers and Joseph, 1967, p. 199]: “…we relate the slip velocity to the exterior flow by the ad hoc boundary condition

\[
\frac{du}{dy}\bigg|_{y=0^+} = \beta(u_B - Q)
\]  

where \( 0^+ \) is a boundary limit point from the exterior fluid.”

7In some applications involving multiphase fluids and heterogeneous mixtures, it will be necessary to treat the thickness of an interface to be of finite-size (albeit small) across which material and thermodynamic properties change sharply. For example, see [Berg, 2010].
Similarly, the unit outward normal on \( \Gamma_{\text{por}} \) emanating away from the porous region is denoted by \( \hat{n}_{\text{por}} \). Clearly, these normals on the interface satisfy:

\[
\hat{n}_{\text{free}}(x) + \hat{n}_{\text{por}}(x) = 0 \quad \forall x \in \Gamma_{\text{int}}
\]  

(3.1)

A unit tangent vector on \( \Gamma_{\text{int}} \) is denoted by \( \hat{s} \).

3.1.2. Free flow region. We denote the region in which free flow occurs by \( \mathcal{K}_{\text{free}} \), and its whole boundary and external boundary are, respectively, denoted by \( \partial \mathcal{K}_{\text{free}} \) and \( \partial \mathcal{K}_{\text{free}}^{\text{ext}} \). We thus have:

\[
\partial \mathcal{K}_{\text{free}} = \partial \mathcal{K}_{\text{free}}^{\text{ext}} \cup \Gamma_{\text{free}} \quad \text{and} \quad \partial \mathcal{K}_{\text{free}}^{\text{ext}} \cap \Gamma_{\text{free}} = \emptyset
\]  

(3.2)

The unit outward normal to the external boundary \( \mathcal{K}_{\text{free}}^{\text{ext}} \) is denoted by \( \hat{n}_{\text{free}}^{\text{ext}} \). We denote the velocity vector field in the free flow region by \( \mathbf{v}_{\text{free}}(x) \), and the corresponding pressure field by \( p_{\text{free}}(x) \). Mathematically, \( \mathbf{v}_{\text{free}} : \mathcal{K}_{\text{free}} \cup \partial \mathcal{K}_{\text{free}} \rightarrow \mathbb{R}^{nd} \) and \( p_{\text{free}} : \mathcal{K}_{\text{free}} \cup \partial \mathcal{K}_{\text{free}} \rightarrow \mathbb{R} \). We denote the specific body force and the stress tensor in the free flow region by \( \mathbf{b}_{\text{free}}(x) \) and \( \mathbf{T}_{\text{free}} \), respectively. The external boundary \( \partial \mathcal{K}_{\text{free}}^{\text{ext}} \) is divided into two parts: \( \Gamma_{\text{free}}^{v} \) and \( \Gamma_{\text{free}}^{t} \), such that

\[
\Gamma_{\text{free}}^{v} \cup \Gamma_{\text{free}}^{t} = \partial \mathcal{K}_{\text{free}}^{\text{ext}} \quad \text{and} \quad \Gamma_{\text{free}}^{v} \cap \Gamma_{\text{free}}^{t} = \emptyset
\]  

(3.3)

\( \Gamma_{\text{free}}^{v} \) is the part of the external boundary of the free flow region on which velocity boundary condition is prescribed, and \( \Gamma_{\text{free}}^{t} \) is that part of the external boundary of the free flow region on which traction boundary condition is prescribed. We thus have:

\[
\partial \mathcal{K}_{\text{free}} = \partial \mathcal{K}_{\text{free}}^{\text{ext}} \cup \Gamma_{\text{free}}^{v} = \Gamma_{\text{free}}^{v} \cup \Gamma_{\text{free}}^{t} \cup \Gamma_{\text{free}}
\]  

(3.4)

We denote the prescribed velocity vector on \( \Gamma_{\text{free}}^{v} \) by \( \mathbf{v}_{\text{free}}^{p}(x) \), and the prescribed traction on \( \Gamma_{\text{free}}^{t} \) by \( t_{\text{free}}^{p}(x) \).

3.1.3. Porous region. We denote the porous region by \( \mathcal{K}_{\text{por}} \), and its whole boundary and external boundary are, respectively, denoted by \( \partial \mathcal{K}_{\text{por}} \) and \( \partial \mathcal{K}_{\text{por}}^{\text{ext}} \). Similar to the free flow region, we have

\[
\partial \mathcal{K}_{\text{por}} = \partial \mathcal{K}_{\text{por}}^{\text{ext}} \cup \Gamma_{\text{por}} \quad \text{and} \quad \partial \mathcal{K}_{\text{por}}^{\text{ext}} \cap \Gamma_{\text{por}} = \emptyset
\]  

(3.5)

The unit outward normal to the external boundary \( \mathcal{K}_{\text{por}}^{\text{ext}} \) is denoted by \( \hat{n}_{\text{por}}^{\text{ext}} \). The porous solid is assumed to be rigid, and its motion can be ignored.\(^8\) We denote the porosity by \( \phi_{\text{por}}(x) \). We denote the discharge velocity and the pressure of the fluid in the porous region by \( \mathbf{v}_{\text{por}}(x) \) and \( p_{\text{por}}(x) \), respectively. It is important to note that the discharge velocity is equal to the true (or seepage) velocity times the porosity. We denote the specific body force and the stress of the fluid in the porous region by \( \mathbf{b}_{\text{por}}(x) \) and \( \mathbf{T}_{\text{por}}(x) \), respectively.

\(^8\)Some terms pertaining to the porous region will tacitly involve the velocity of the porous solid. Three such cases will be the virtual velocities in the virtual power expended due to the interactions (3.20), the interaction term \( i_{\text{por}} \) itself and the power expended density along the interface \( \Psi \). A quantity that appears in these cases will be the velocity of the fluid in the porous region with respect to the velocity of the porous solid. If the motion of the porous solid is taken to be zero, which can be done by choosing a specific frame of reference, its explicit dependence will not be apparent. For example, the interaction force under the Darcy model (which assumed the porous solid to be rigid) is commonly written as

\[
i_{\text{por}} = \mu K^{-1} \mathbf{v}_{\text{por}}
\]

but in fact it needs to be interpreted as

\[
\mu K^{-1} (\mathbf{v}_{\text{por}} - \mathbf{v}_{\text{por}}^{(\text{solid})})
\]

(In the above equations, \( \mathbf{v}_{\text{por}}^{(\text{solid})} \) is the velocity of the porous solid, \( K \) is the permeability of the porous region, and \( \mu \) is the coefficient of viscosity.) Noting the dependence on the velocity of the porous solid will be particularly important when we invoke a change of observer to obtain constitutive restrictions, or when we require the internal virtual power expended to vanish under a superimposed rigid body motion on the virtual velocities. In such cases, the actual velocity of the porous solid and its virtual counterpart will not be zero.
The following properties are satisfied:

where

and the following decomposition:

\[ v(x) = w_{\text{free}}(x) + w_{\text{por}}(x) \]  

\[ w_{\text{free}}(x) = w_{\text{free}}^{(n)}(x) n_{\text{free}}(x) \]

\[ w_{\text{por}}(x) = w_{\text{por}}^{(n)}(x) n_{\text{por}}(x) \]

where \( w_{\text{free}}^{(n)}(x) \) and \( w_{\text{por}}^{(n)}(x) \) denote the corresponding tangential components of the vector fields.

We refer to a pair of vector fields \((w_{\text{free}}(x), w_{\text{por}}(x))\) ∈ \( \mathcal{W} \) to be \textit{kinematically admissible} if the following properties are satisfied:

(i) \( \text{div}[w_{\text{free}}] = 0 \) in \( \mathcal{K}_{\text{free}} \) and \( \text{div}[w_{\text{por}}] = 0 \) in \( \mathcal{K}_{\text{por}} \),

(ii) \( w_{\text{free}}^{(n)}(x) + w_{\text{por}}^{(n)}(x) = 0 \) on the interface \( \Gamma_{\text{int}} \), and

(iii) \( w_{\text{free}}(x) \) and \( w_{\text{por}}(x) \) satisfy the velocity boundary conditions on the external boundary (i.e.,

on \( \Gamma_{\text{free}}^{u} \) and \( \Gamma_{\text{por}}^{u} \), respectively).

\footnote{Under Darcy equations, only the normal component of the velocity vector field can be prescribed on the boundary. In such cases, the velocity boundary condition will be of the form: \( v_{\text{por}}(x) \cdot n_{\text{por}}(x) = v_{\text{por}}^{p}(x) \) on \( \Gamma_{\text{por}}^{t} \). On the other hand, under mathematical models like the Darcy-Brinkman model, the whole velocity vector field can be prescribed on the boundary. The mathematical reason is that the Darcy equations contain at most first-order spatial derivative of the velocity field. On the other hand, the Darcy-Brinkman model gives rise to governing equations which contain a second-order spatial derivative of the velocity field.}
We denote the set of all kinematically admissible pairs of vector fields by \( \mathcal{V} \). Certainly, the exact velocity fields are kinematically admissible; that is \( (v_{\text{free}}(x), v_{\text{por}}(x)) \in \mathcal{V} \).

We refer to a pair of vector fields \( (w_{\text{free}}(x), w_{\text{por}}(x)) \in \mathcal{W} \) to be a pair of virtual vector fields if the first two properties under kinematical admissibility are met, and \( w_{\text{free}}(x) \) and \( w_{\text{por}}(x) \) vanish on \( \Gamma'_{\text{free}} \) and \( \Gamma'_{\text{por}} \), respectively. We denote the set of all pairs of virtual vector fields by \( \mathcal{V}' \).

3.1.6. Fields under a rigid body motion. Consider a superimposed rigid body motion of the entire domain\(^{10}\):

\[
x'(t) \leftarrow Q(t)x + c(t) \quad \forall x \in \Omega
\]  

(3.11)

where \( t \) denotes the time, \( c(t) \) is a translational vector, and \( Q(t) \in \text{SO}(3) \) is a rotation at each instance of time\(^{11}\). The subspace \( \mathcal{W}_{\text{rigid}} \subseteq \mathcal{W} \) that is spanned by the vector fields generated by such a rigid body motion at a given instance of time \( t \) takes the following form:

\[
\mathcal{W}_{\text{rigid}} := \left\{ (w_{\text{free}}(x), w_{\text{por}}(x)) \in \mathcal{W} \mid w_{\text{free}}(x) = v(x, t) \big|_{x \in K_{\text{free}} \cup \partial K_{\text{free}}}, \right. \\
\left. w_{\text{por}}(x) = v(x, t) \big|_{x \in K_{\text{por}} \cup \partial K_{\text{por}}} \text{ where } v(x, t) = \dot{Q}(t)(x - x_0) + v_0 \right\} 
\]

(3.12)

where \( \dot{Q}(t) \) denotes the time derivative of \( Q(t) \). It is important to note that, at each instance of time, the vector fields \( (w_{\text{free}}(x), w_{\text{por}}(x)) \in \mathcal{W}_{\text{rigid}} \) satisfy:

\[
\text{grad}[w_{\text{free}}] = \dot{Q}(t)Q^T(t) \in \text{skew}[K_{\text{free}}] \quad \text{and} \quad \text{grad}[w_{\text{por}}] = \dot{Q}(t)Q^T(t) \in \text{skew}[K_{\text{por}}]
\]

(3.13)

where \( \text{skew}[:] \) denotes the space of skew-symmetric tensor fields on the indicated spatial region.

3.1.7. Other notation for convenience. We occasionally use the following notation:

\[
L_{\text{free}} = \text{grad}[v_{\text{free}}], L_{\text{por}} = \text{grad}[v_{\text{por}}], D_{\text{free}} = \frac{1}{2}(L_{\text{free}} + L_{\text{free}}^T) \quad \text{and} \quad D_{\text{por}} = \frac{1}{2}(L_{\text{por}} + L_{\text{por}}^T)
\]

(3.14)

3.2. Proposed principle of virtual power. The mathematical statement of the proposed principle of virtual power for flows in coupled free-porous media, which will be in the form of balance of virtual power, can be written as follows:

Find \( (v_{\text{free}}(x), v_{\text{por}}(x)) \in \mathcal{V} \) such that the following two properties are met:

\[
(P1) \quad \mathcal{P}^{(\text{internal})} = \mathcal{P}^{(\text{external})} \quad \forall (w_{\text{free}}(x), w_{\text{por}}(x)) \in \mathcal{V}'
\]

(3.15)

\[
(P2) \quad \mathcal{P}^{(\text{internal})} = 0 \quad \forall (w_{\text{free}}(x), w_{\text{por}}(x)) \in \mathcal{W}_{\text{rigid}}
\]

(3.16)

where the internal virtual power expended (i.e., virtual stress power) in the free flow region is given by

\[
\mathcal{P}_{\text{free}}^{(\text{internal})} := \int_{K_{\text{free}}} T_{\text{free}} \cdot \text{grad}[w_{\text{free}}] \, d\Omega
\]

(3.17)

The internal virtual power expended in the porous region is written as follows:

\[
\mathcal{P}_{\text{por}}^{(\text{internal})} := \mathcal{P}_{\text{por, stress}}^{(\text{internal})} + \mathcal{P}_{\text{por, interactions}}^{(\text{internal})}
\]

(3.18)

---

\(^{10}\)One should not confuse the expression (3.11) with that of a Euclidean transformation between two frames of reference (i.e., two observers). We will deal the latter aspect in a subsequent section when we discuss the principle of material frame-indifference for constitutive relations. For the current discussion, it is important to note that a single observer looks at two motions \( (x' \text{ and } x) \) which differ by a rigid body motion.

\(^{11}\)SO(3) is a group of all rotations about the origin of \( \mathbb{R}^3 \) – the three-dimensional Euclidean space – under the operation of composition; e.g., see [Marsden and Ratiu, 2013].
where virtual stress power in the porous region is defined as follows:

\[
\mathcal{P}_{\text{por, stress}}^{(\text{internal})} := \int_{K_{\text{por}}} T_{\text{por}} \cdot \text{grad}[w_{\text{por}}] \, d\Omega
\]  

(3.19)

and the internal virtual power expended due to interactions between the constituents in the porous region is written as follows:

\[
\mathcal{P}_{\text{por, interactions}}^{(\text{internal})} := \int_{K_{\text{por}}} i_{\text{por}} \cdot \left( w_{\text{por}} - w_{\text{por}}^{(\text{solid})}\right)^0 \, d\Omega = \int_{K_{\text{por}}} i_{\text{por}} \cdot w_{\text{por}} \, d\Omega
\]  

(3.20)

In the above equation, \( w_{\text{por}}^{(\text{solid})} \) denotes the vector field associated with the porous solid. Since we assumed the porous solid to be rigid and neglected its motion, this term becomes zero. However, when we invoke vanishing of the internal virtual power under a rigid body motion of the entire domain, it becomes important to acknowledge the presence of this term, as it will not be zero in that situation. The internal virtual power expended at the interface is written as follows:

\[
\mathcal{P}_{\text{int}}^{(\text{internal})} := \int_{\Gamma_{\text{int}}} \delta \Psi \, d\Gamma
\]  

(3.21)

where \( \delta \Psi \) denotes the virtual power expended density at the interface and depends on both the true velocity fields and their virtual counterparts. The total internal virtual power expended takes the following form:

\[
\mathcal{P}^{(\text{internal})} := \mathcal{P}^{(\text{internal})}_{\text{free}} + \mathcal{P}^{(\text{internal})}_{\text{por}} + \mathcal{P}^{(\text{internal})}_{\text{int}}
\]  

(3.22)

The total external virtual power expended takes the following form:

\[
\mathcal{P}^{(\text{external})} := \int_{K_{\text{free}}} \gamma b_{\text{free}} \cdot w_{\text{free}} \, d\Omega + \int_{\Gamma_{\text{free}}} t_{\text{free}}^p \cdot w_{\text{free}} \, d\Gamma
\]

external virtual power expended on the free flow region

\[
+ \int_{K_{\text{por}}} \gamma \phi_{\text{por}} b_{\text{por}} \cdot w_{\text{por}} \, d\Omega + \int_{\Gamma_{\text{por}}} t_{\text{por}}^p \cdot w_{\text{por}} \, d\Gamma
\]

external virtual power expended on the porous region

(3.23)

We will show that an appropriate set of interface conditions can be derived by prescribing a functional form for \( \delta \Psi \), and this prescription will be a constitutive specification. We place the following restrictions on \( \delta \Psi \), and these restrictions are based on either invariance requirements, physical properties or convenience.

(i) **Exact differential.** In equation (3.21), \( \delta \Psi \) need not be an exact differential. However, for convenience we assume \( \delta \Psi \) to be an exact differential. This implies that there exists a functional \( \Psi \), which will be referred to as the power expended at the interface, such that \( \delta \Psi \) is a Gâteaux variation of \( \Psi \). Mathematically, if \( \delta \Psi \) depends on a set of variables, which is collectively denoted by \( \chi \), and a set of the corresponding virtual variables, \( \delta \chi \), then

\[
\delta \Psi[\chi; \delta \chi] = \left[ \frac{d}{dc} \Psi[\chi + c \delta \chi] \right]_{c=0} = \frac{\partial \Psi}{\partial \chi} \cdot \delta \chi
\]  

(3.24)
In the case of an exact differential, the Vainberg’s theorem [Vainberg, 1964; Hjelmstad, 2005] implies that

$$\Psi[\chi] = \int_0^1 \delta \Psi[\tau \chi; \chi] d\tau$$ (3.25)

where \( \tau \) is a dummy variable introduced for integration.

(ii) Positive semi-definiteness. The total power expended at the interface should be physically non-negative. This can be ensured by assuming \( \Psi \) to be a positive semi-definite functional. Mathematically,

$$\Psi[\chi] \geq 0 \quad \forall \chi$$ (3.26)

(iii) Dependence of \( \Psi \) on velocities. We take the set of variables for the functional dependence of \( \Psi \) as follows:

$$\chi = \{v^*_\text{free}(x), v^*_\text{por}(x), v_n(x)\}$$ (3.27)

where

$$v_n(x) := v^{(n)}_{\text{free}}(x)$$ (3.28)

Recall that the tangential velocities have been defined in equation (3.10). Since the true fluid densities in the porous and free flow regions are assumed to be the same, the balance of mass across the interface implies that

$$v_n(x) = -v^{(n)}_{\text{por}}(x)$$ (3.29)

The chosen functional dependence will imply that

$$\delta \Psi = \frac{\partial \Psi}{\partial v^*_\text{free}} \cdot \delta v^*_\text{free} + \frac{\partial \Psi}{\partial v^*_\text{por}} \cdot \delta v^*_\text{por} + \frac{\partial \Psi}{\partial v_n} \cdot \delta v_n$$ (3.30)

Noting that \( \delta v^*_{\text{free}} \) and \( \delta v^*_{\text{por}} \) are relative velocities with respect to the rigid porous solid, they vanish under a rigid body motion of the entire domain. Hence, \( \delta \Psi \) vanishes under a rigid body motion of the virtual velocities. This point is important to satisfy the statement (P2) under the proposed principle of virtual power.

(iv) Invariance. We require the constitutive relations emanating from the functional \( \Psi \) to satisfy the principle of material frame-indifference. Following [Leigh, 1968; Svendsen and Bertram, 1999; Bertram and Svendsen, 2001], this amounts to enforcing form invariance on the functional and invariance under a Euclidean transformation between frames (i.e., observers). Before proceeding further, we will first recall that \( v^*_{\text{free}}(x) \) and \( v^*_{\text{por}}(x) \) are relative velocities with respect to the rigid porous solid, which is assumed to be at rest. It is important to realize that \( v^*_{\text{free}} \) and \( v^*_{\text{por}} \) are relative velocities between two constituents at the same point in the space and they are not relative velocities (of the same constituent) between two different points in the space. Such a distinction is germane to TIC and is paramount to our discussion, as the former quantities are invariant under a Euclidean transformation between frames of reference, while the later ones are not. In fact, a relative velocity between two points in the space is not an invariant even under a Galilean transformation between frames.

Now consider two frames of reference, \((x', t')\) and \((x, t)\), which differ by a Euclidean transformation:

$$x' \leftarrow Q(t)x + c(t) \quad \text{and} \quad t' \leftarrow t + t_0$$ (3.31)
where $c(t)$ is a translation vector, and $Q(t) \in SO(3)$ is a rotation for each $t$. Under this Euclidean transformation, the tangential and the normal components of these (relative) velocity fields satisfy:

$$v'_\text{free} = Q(t)v_{\text{free}}, \quad v'_\text{por} = Q(t)v_{\text{por}} \quad \text{and} \quad v'_n = v_n$$

where the quantities with a prime are under $x'$ frame of reference. The above expressions (3.32) and the form invariance of the function together imply that

$$\Psi[v'_\text{free}, v'_\text{por}, v'_n] = \Psi[Q(t)v_{\text{free}}, Q(t)v_{\text{por}}, v_n] = \Psi[v_{\text{free}}, v_{\text{por}}, v_n] \quad \forall Q(t) \in SO(3)$$

which implies that $\Psi$ is an isotropic functional of its arguments. From the representation theory, we further assert that $\Psi$ can depend only on the following individual and joint invariants [Smith, 1971]:

$$\Psi[v'_\text{free} \cdot v'_\text{free}, v'_\text{por} \cdot v'_\text{por}, v_{\text{free}} \cdot v_{\text{por}}, v_{\text{free}} \cdot v_{\text{por}} \text{ and } v_n]$$

4. DERIVATION OF INTERFACE CONDITIONS AND FIELD EQUATIONS

Under a rigid body motion, $\mathcal{S}_\text{(internal)} = 0$ and $\mathcal{S}_\text{por. interactions} = 0$, as they (linearly) depend on the relative virtual velocities. Recall that under a rigid body motion $\text{grad}[w_{\text{free}}]$ and $\text{grad}[w_{\text{por}}]$ are skew symmetric tensor fields. Thus, the main consequence of the statement (P2) is the symmetry of the Cauchy stresses in both the regions, which is equivalent to the balance of angular momentum. That is,

$$T_{\text{free}}(x) = T^T_{\text{free}}(x) \quad \forall x \in K_{\text{free}} \quad \text{and} \quad T_{\text{por}}(x) = T^T_{\text{por}}(x) \quad \forall x \in K_{\text{por}}$$

4.1. Handling internal constraints. We extend the approach put forth by Carlson et al. [2004] for handling internal constraints to the case of flows in coupled free-porous media. Consider a constraint manifold for the motion in $K_{\text{free}}$:

$$\mathcal{G}_{\text{free}} := \{L_{\text{free}} \mid \Upsilon_{\text{free}}(L_{\text{free}}) = 0 \text{ in } K_{\text{free}}\}$$

where the constraint function is:

$$\Upsilon_{\text{free}} : \text{Lin}(K_{\text{free}}) \to \mathbb{R}$$

Herein, we have employed the standard notation for $\text{Lin}(K)$ to denote the linear space of all (second-order) tensors defined over $K$. The normal space to $\mathcal{G}_{\text{free}}$ at $L_{\text{free}} \in \text{Lin}(K_{\text{free}})$ can be written as follows:

$$\text{Norm}(\mathcal{G}_{\text{free}}) := \text{Lsp}\{\text{Grad}[\Upsilon_{\text{free}}(L_{\text{free}})]\}$$

where $\text{Lsp}\{\cdot\}$ denotes the linear space spanned by its argument. It is easy to check that $\text{Norm}(\mathcal{G}_{\text{free}})$ is a subspace of $\text{Lin}(K_{\text{free}})$. Then the orthogonal complement of the normal space (which is commonly referred to as the tangent space) at $L_{\text{free}}$ can be defined as follows:

$$\text{Tan}(\mathcal{G}_{\text{free}}) = (\text{Norm}(\mathcal{G}_{\text{free}}))^\perp := \{A_{\text{free}} \in \text{Lin}(K_{\text{free}}) \mid A_{\text{free}} \cdot B_{\text{free}} = 0 \forall B_{\text{free}} \in \text{Norm}(\mathcal{G}_{\text{free}})\}$$

The projection theorem implies the following direct sum decomposition:

$$\text{Lin}(K_{\text{free}}) = \text{Norm}(\mathcal{G}_{\text{free}}) \oplus \text{Tan}(\mathcal{G}_{\text{free}})$$

---

12 Also see footnote 10.

13 $\text{Grad}[\Upsilon_{\text{free}}(L_{\text{free}})]$ means gradient of $\Upsilon_{\text{free}}$ with respect to its argument $L_{\text{free}}$. 
This implies that for each \( \mathbf{A}_{\text{free}} \in \text{Lin}(\mathbf{K}_{\text{free}}) \) we have
\[
\mathbf{A}_{\text{free}} = \mathbf{A}^\perp_{\text{free}} + \mathbf{A}^\parallel_{\text{free}}
\] (4.7)
where \( \mathbf{A}^\perp_{\text{free}} \in \text{Norm}(\mathbf{C}_{\text{free}}) \) and \( \mathbf{A}^\parallel_{\text{free}} \in \text{Tan}(\mathbf{C}_{\text{free}}) \) are, respectively, the active and reactive components of \( \mathbf{A}_{\text{free}} \). Similarly, one can define the constraint manifold \( \mathbf{C}_{\text{por}} \) in terms of \( \mathbf{L}_{\text{por}} \) for the region \( \mathbf{K}_{\text{por}} \) and the corresponding \( \text{Norm}(\mathbf{C}_{\text{por}}) \) and \( \text{Tan}(\mathbf{C}_{\text{por}}) \) subspaces of \( \text{Lin}(\mathbf{K}_{\text{por}}) \).

Specifically in our case, the constraint functions are:
\[
\Upsilon_{\text{free}}(\mathbf{L}_{\text{free}}) = \text{tr}[\mathbf{L}_{\text{free}}] = 0 \quad \text{and} \quad \Upsilon_{\text{por}}(\mathbf{L}_{\text{por}}) = \text{tr}[\mathbf{L}_{\text{por}}] = 0
\] (4.8)
where \( \text{tr}[:] \) denotes the standard trace of second-order tensors. The corresponding normal spaces take the following form:
\[
\text{Norm}(\mathbf{C}_{\text{free}}) = \text{Lsp}\{\mathbf{I}\} \quad \text{and} \quad \text{Norm}(\mathbf{C}_{\text{por}}) = \text{Lsp}\{\mathbf{I}\}
\] (4.9)

The direct sum decomposition form the projection theorem implies that the Cauchy stresses under the constrained motion due to internal constraints can be written as follows\(^{14}\):
\[
\mathbf{T}_{\text{free}}(\mathbf{x}) = -p_{\text{free}}(\mathbf{x})\mathbf{I} + \mathbf{T}^{\text{extra}}_{\text{free}}(\mathbf{x}) \\
\mathbf{T}_{\text{por}}(\mathbf{x}) = -p_{\text{por}}(\mathbf{x})\mathbf{I} + \mathbf{T}^{\text{extra}}_{\text{por}}(\mathbf{x})
\] (4.10a)
(4.10b)
where the extra stresses, \( \mathbf{T}^{\text{extra}}_{\text{free}} \) and \( \mathbf{T}^{\text{extra}}_{\text{por}} \), belong to the tangent spaces and should be prescribed through constitutive specifications\(^{15}\). Moreover, the no-work by the active components will be a trivial mathematical consequence. To wit,
\[
\mathbf{T}^\perp_{\text{free}} \cdot \mathbf{L}_{\text{free}} = -p_{\text{free}}\mathbf{I} \cdot \mathbf{L}_{\text{free}} = -p_{\text{free}}\text{tr}[\mathbf{L}_{\text{free}}] = 0
\] (4.11)
A similar reasoning holds for \( \mathbf{T}^\perp_{\text{por}} \). The following relations will also be mathematical consequences:
\[
\mathbf{T}_{\text{free}} \cdot \mathbf{L}_{\text{free}} = \mathbf{T}^\parallel_{\text{free}} \cdot \mathbf{L}_{\text{free}} = \mathbf{T}^{\text{extra}}_{\text{free}} \cdot \mathbf{L}_{\text{free}}
\] (4.12)

### 4.2. Consequences of (P1) statement.

Using Green’s identity and noting that the virtual velocity fields vanish on \( \Gamma^v_{\text{free}} \) and \( \Gamma^v_{\text{por}} \), the (P1) statement (3.15) can be rewritten as follows:
\[
\int_{\Gamma^t_{\text{free}}} \mathbf{w}_{\text{free}} \cdot \{\mathbf{T}_{\text{free}}\mathbf{n}^{\text{ext}}_{\text{free}} - \mathbf{t}^p_{\text{free}}\} \, d\Gamma - \int_{\mathbf{K}_{\text{free}}} \mathbf{w}_{\text{free}} \cdot \{\text{div}[\mathbf{T}_{\text{free}}] + \gamma\mathbf{b}_{\text{free}}\} \, d\Omega \\
+ \int_{\Gamma^t_{\text{por}}} \mathbf{w}_{\text{por}} \cdot \{\mathbf{T}_{\text{por}}\mathbf{n}^{\text{ext}}_{\text{por}} - \mathbf{t}^p_{\text{por}}\} \, d\Gamma - \int_{\mathbf{K}_{\text{por}}} \mathbf{w}_{\text{por}} \cdot \{\text{div}[\mathbf{T}_{\text{por}}] + \gamma\phi_{\text{por}}\mathbf{b}_{\text{por}} - \mathbf{i}_{\text{por}}\} \, d\Omega \\
+ \int_{\Gamma_{\text{int}}} \left\{ \mathbf{w}_{\text{free}} \cdot \mathbf{T}_{\text{free}}\mathbf{n}_{\text{free}} + \mathbf{w}_{\text{por}} \cdot \mathbf{T}_{\text{por}}\mathbf{n}_{\text{por}} + \mathbf{w}_{\text{free}} \cdot \frac{\partial \Psi}{\partial \mathbf{v}_{\text{free}}} + \mathbf{w}_{\text{por}} \cdot \frac{\partial \Psi}{\partial \mathbf{v}_{\text{por}}} + \mathbf{w}_{\text{n}} \cdot \frac{\partial \Psi}{\partial \mathbf{v}_{\text{n}}} \right\} \, d\Gamma = 0
\] (4.13)

\(^{14}\)The minus sign is introduced for convenience so that \( p_{\text{free}} \) and \( p_{\text{por}} \) will be the mechanical pressures.

\(^{15}\)See [O’Reilly and Srinivasa, 2001] for an insightful discussion on a related issue in the context of particle dynamics. They discussed active and reactive components due to a constraint, what flexibility a dynamicist will have as a part of constitutive specifications, and the relation between the prescription for the reactive component and the Gauss’s principle of least constraint.
We now invoke the arbitrariness of the fields $w_{\text{free}}(x)$ and $w_{\text{por}}(x)$ but respecting the requirements of kinematic admissibility. The first two terms give rise to the following governing equations for the free flow region except along the part of the boundary that shares with the interface:

\[
\begin{align*}
\text{div}[\mathbf{T}_{\text{free}}] + \gamma b_{\text{free}} &= 0 \quad \text{in } K_{\text{free}} \\
\text{div}[\mathbf{v}_{\text{free}}] &= 0 \quad \text{in } K_{\text{free}}
\end{align*}
\]  

(4.14a) (4.14b)

\[
\mathbf{T}_{\text{free}}\hat{n}_{\text{ext}}_{\text{free}}(x) = t^p_{\text{free}}(x) \quad \text{on } \Gamma^l_{\text{free}}
\]  

(4.14c)

\[
\mathbf{v}_{\text{free}}(x) = v^p_{\text{free}}(x) \quad \text{on } \Gamma^v_{\text{free}}
\]  

(4.14d)

The third and fourth terms give rise to the following governing equations for the porous region except along the part of the boundary that shares with the interface:

\[
\begin{align*}
\text{div}[\mathbf{T}_{\text{por}}] + \gamma \phi \mathbf{b}_{\text{por}} - i_{\text{por}} &= 0 \quad \text{in } K_{\text{por}} \\
\text{div}[\mathbf{v}_{\text{por}}] &= 0 \quad \text{in } K_{\text{por}}
\end{align*}
\]  

(4.15a) (4.15b)

\[
\mathbf{T}_{\text{por}}\hat{n}_{\text{ext}}_{\text{por}}(x) = t^p_{\text{por}}(x) \quad \text{on } \Gamma^l_{\text{por}}
\]  

(4.15c)

\[
\mathbf{v}_{\text{por}}(x) = v^p_{\text{por}}(x) \quad \text{on } \Gamma^v_{\text{por}}
\]  

(4.15d)

Noting the decomposition given in equation (3.10), the fifth term gives rise to the following interface conditions on $\Gamma_{\text{int}}$:

\[
\begin{align*}
v_{\text{free}}^{(n)}(x) + v_{\text{por}}^{(n)}(x) &= 0 \quad \text{Equation (4.16a)} \\
\hat{n}_{\text{free}}(x) \cdot \mathbf{T}_{\text{free}}(x)\hat{n}_{\text{free}}(x) + \frac{\partial \Psi}{\partial v_n} &= \hat{n}_{\text{por}}(x) \cdot \mathbf{T}_{\text{por}}(x)\hat{n}_{\text{por}}(x) \quad \text{(4.16b)} \\
\hat{s}(x) \cdot \mathbf{T}_{\text{free}}^{\text{extra}}\hat{n}_{\text{free}}(x) &= -\hat{s}(x) \cdot \frac{\partial \Psi}{\partial v_{\text{free}}} \quad \text{(4.16c)} \\
\hat{s}(x) \cdot \mathbf{T}_{\text{por}}^{\text{extra}}\hat{n}_{\text{por}}(x) &= -\hat{s}(x) \cdot \frac{\partial \Psi}{\partial v_{\text{por}}} \quad \text{(4.16d)}
\end{align*}
\]

Equation (4.16a) is in fact the jump condition corresponding to the balance of mass (cf. equation (A.8a) in §A). The other three conditions are in general not jump conditions and they stem from a constitutive specification in the form of a prescription for the functional $\Psi$. If $\Psi$ is independent of $v_n$ (which is assumed in §5 to obtain some popular conditions like the BJ and BJS conditions) then the second condition (4.16b) will reduce to the normal component of the jump condition for the balance of linear momentum\textsuperscript{16}. To summarize, the complete set of governing equations for flows in coupled free-porous media is:

- the equations in the free flow region along with the boundary conditions on the external boundary (not including $\Gamma_{\text{int}}$) of the region (4.14a)-(4.14d),
- the equations in the porous region along with the boundary conditions on the external boundary (not including $\Gamma_{\text{int}}$) of the region (4.15a)-(4.15d),
- the symmetry of Cauchy stresses (4.1),
- the decomposition of Cauchy stresses (4.10a)-(4.10b),
- the interface conditions (4.16a)-(4.16d) and
- the (prescribed) constitutive specifications for $T_{\text{extra}}^{\text{free}}$, $T_{\text{extra}}^{\text{por}}$, $i_{\text{por}}$ and $\Psi$.

\textsuperscript{16}The second interface condition can be interpreted in a more familiar form using tractions, see Appendix B.
The solution fields will be $v_{\text{free}}(x)$, $v_{\text{por}}(x)$, $p_{\text{free}}(x)$ and $p_{\text{por}}(x)$.

5. SPECIAL CASES

We now show the BJ and BJS conditions, and the no-slip condition (which is commonly employed in the fluid mechanics for free flows) are, respectively, special cases and a limiting case of the proposed framework. The following assumptions will be common to all the mentioned conditions:

(A1) The normal component of the velocity at the interface does not contribute towards the power expended density at the interface. That is, $\Psi$ is independent of $v_n$.

(A2) $\Psi$ is a quadratic functional of the tangential (relative) velocities, and the invariance requirements demand that this functional has to be in terms of individual and joint invariants of the tangential (relative) velocities. Thus, mathematically, we write the functional as follows:

$$\Psi[v_{\text{free}}, v_{\text{por}}, v_n] = \alpha_{11}v_{\text{free}} \cdot v_{\text{free}} + 2\alpha_{12}v_{\text{free}} \cdot v_{\text{por}} + \alpha_{22}v_{\text{por}} \cdot v_{\text{por}}$$

where $\alpha_{11}$, $\alpha_{12}$ and $\alpha_{22}$ are constants, and $v_{\text{free}}^*$ and $v_{\text{por}}^*$ are the tangential velocities.

(A3) The non-negativity of $\Psi$ is enforced by assuming that

$$\alpha_{11}\alpha_{22} \geq \alpha_{12}^2$$

(A4) The Stokes model is assumed to describe the flow in the free flow region. That is, the flow in the free flow region is assumed to be a creeping flow, which implies the following:

$$T_{\text{extra}}^{\text{free}} = 2\mu D_{\text{free}}$$

The above assumptions give rise to the following interface conditions for the tangential component of the tractions:

$$\hat{s} \cdot T_{\text{free}}^{\text{extra}}\hat{n}_{\text{free}} = -\frac{\partial \Psi}{\partial v_{\text{free}}^*} \cdot \hat{s} = -2(\alpha_{11}v_{\text{free}}^* + \alpha_{12}v_{\text{por}}^*) \cdot \hat{s} \quad \text{on } \Gamma_{\text{free}}$$  \hspace{1cm} (5.4a)

$$\hat{s} \cdot T_{\text{por}}^{\text{extra}}\hat{n}_{\text{por}} = -\frac{\partial \Psi}{\partial v_{\text{por}}^*} \cdot \hat{s} = -2(\alpha_{12}v_{\text{free}}^* + \alpha_{22}v_{\text{por}}^*) \cdot \hat{s} \quad \text{on } \Gamma_{\text{por}}$$  \hspace{1cm} (5.4b)

where $\hat{s}(x)$ denotes an arbitrary unit tangent vector field along the interface.

5.1. Beavers-Joseph condition. The BJ condition can be obtained by further making the following choices:

$$\alpha_{11} = \alpha_{22} = \frac{\alpha\mu\sqrt{3}}{2\sqrt{\text{tr}[K]}} \quad \text{and} \quad \alpha_{12} = \frac{-\alpha\mu\sqrt{3}}{2\sqrt{\text{tr}[K]}}$$

where $\text{tr}[\cdot]$ denotes the trace of a second-order tensor. Then equation (5.4a) will reduce to:

$$\hat{s} \cdot (-2\mu D_{\text{free}})\hat{n}_{\text{free}} = \frac{\alpha\mu\sqrt{3}}{\sqrt{\text{tr}[K]}} \hat{s} \cdot (v_{\text{free}} - v_{\text{por}})$$

which is the “boundary” condition proposed in [Beavers and Joseph, 1967] for the free flow region due to the presence of a pervious boundary. By aligning the coordinate axes similar to the one shown in Fig. 3 and by taking the x-component of $v_{\text{por}}$ to be $Q$, one will get an expression similar to the one provided in [Beavers and Joseph, 1967] (cf. equation (2.1)). It should be however noted that Beavers and Joseph do not provide a corresponding condition for the flow in the porous media, which lies on the other side of the interface.
On the other hand, using the proposed framework, one can obtain a corresponding condition for
the flow on the other side of the interface (i.e., the porous medium); which is needed if one wants
to simulate a coupled flow in both free and porous regions. Using equation (5.4b), the interface
condition on $\Gamma_{\text{por}}$ can be written as follows:

$$\hat{s} \cdot T^\text{extra}_\text{por} \hat{n}_\text{por} = \frac{\alpha \mu \sqrt{3}}{\sqrt{\text{tr}[K]}} \hat{s} \cdot (\mathbf{v}_\text{free} - \mathbf{v}_\text{por})$$

(5.7)

### 5.1.1. A discussion on the BJ condition.

The velocity field in the porous region is assumed to be known a priori. Moreover, the flow in the porous region is tacitly assumed to be uniform beyond a boundary layer (see Fig. 3). But the velocity field in the porous region is seldom known a priori and this is particularly true in the case of flows in coupled free-porous media. Even if the velocity field in the porous region is known, this field will not be uniform due to spatial heterogeneity of medium properties (e.g., permeability). (Heterogeneity is inherent to the two application problems that we discussed in the introduction.) This will create an ambiguity in assigning a value to $Q$ (cf. equation (2.1)). Specifically, at what depth one has to sample the (horizontal or tangential) velocity to specify $Q$ (cf. Fig. 3).

Last but not least, the BJ condition may not be compatible with all porous media model. For example, if the flow in the porous region is modeled using the Darcy model, for which, $T^\text{extra}_\text{por} = 0$. Equation (5.7) will then imply that

$$\hat{s} \cdot (\mathbf{v}_\text{free} - \mathbf{v}_\text{por}) = 0$$

which, based on the BJ condition (5.6), will further imply that

$$\hat{s} \cdot D_\text{free} \hat{n} = 0$$

But this condition will not be met in general, as, for example, the horizontal velocity can depend on the $y$-coordinate or the vertical velocity can depend on the $x$-coordinate.

### 5.2. Beavers-Joseph-Saffman condition.

In addition to the aforementioned four assumptions (A1)–(A4), we make the following choices to obtain the BJS condition:

$$\alpha_{11} = \alpha_{22} = \frac{\alpha \mu \sqrt{3}}{2 \sqrt{\text{tr}[K]}} \quad \text{and} \quad \alpha_{12} = 0$$

(5.8)

Then, using equation (5.4a), the boundary condition at $\Gamma_\text{free}$ for the flow in the free region due to a juxtaposed porous region takes the following form:

$$\hat{s} \cdot (-2 \mu D_\text{free}) \hat{n}_\text{free} = \frac{\alpha \mu \sqrt{3}}{\sqrt{\text{tr}[K]}} \hat{s} \cdot \mathbf{v}_\text{free}$$

(5.9)

Using equation (5.4b), the interface condition on $\Gamma_\text{por}$ takes the following form:

$$\hat{s} \cdot T^\text{extra}_\text{free} \hat{n}_\text{free} = \frac{\alpha \mu \sqrt{3}}{\sqrt{\text{tr}[K]}} \hat{s} \cdot \mathbf{v}_\text{por}$$

(5.10)

### 5.2.1. A discussion on the BJS condition.

Since the BJS condition (5.9) does not contain $Q$ (the mean velocity in the porous region beyond the boundary layer), it does not assume the velocity field in the porous region is neither known a priori nor uniform. However, the BJS condition need not be compatible with all porous media models. If one again considers the Darcy model to describe the flow in the porous region, equation (5.10) implies that $\mathbf{v}_\text{por} = 0$ – the no-slip boundary condition.
for the porous region along the interface – which is not what has been observed in the experiments [Beavers and Joseph, 1967].

On the other hand, if one uses the Darcy-Brinkman model, for which \( T^\text{extra}_{\text{por}} = 2\mu D_{\text{por}} \), the BJS condition will be compatible with the chosen model. Saffman did recognize that his condition is actually compatible with the Darcy-Brinkman model and not the Darcy model\(^{17}\). However, by using asymptotic analysis, he argued that solutions from the Darcy-Brinkman model and the Darcy model do not differ significantly outside the boundary layer, and the size of the boundary layer is in the order of the square-root of the (trace of) permeability.

### 5.3. No-slip condition.

The classical no-slip condition can be obtained by making the following choices for the constants:

\[
\alpha_{11} = \frac{\alpha}{2\sqrt{\text{tr}[K]}} , \quad \alpha_{22} = 0 \quad \text{and} \quad \alpha_{12} = 0 \tag{5.11}
\]

and then by letting \( \text{tr}[K] \to 0 \). To wit, based on the choices made in equation (5.11), the interface condition (5.4a) reduces to the following:

\[
\hat{s} \cdot \vec{v}_{\text{free}} = -\left( \frac{\sqrt{\text{tr}[K]}}{\alpha} \right) \hat{s} \cdot T^\text{extra}_{\text{free}} \hat{n}_{\text{free}} \tag{5.12}
\]

By letting \( \text{tr}[K] \to 0 \) and noting that \( \hat{s} \) is an arbitrary tangent vector along the interface, one can conclude that \( \vec{v}_{\text{free}} = 0 \) on \( \Gamma_{\text{free}} \), which is the no-slip condition. Note that \( \text{tr}[K] \to 0 \) basically implies that the boundary is impervious, and the no-slip boundary condition is typically enforced at an impervious boundary in an uncoupled free flow.

### 6. MINIMUM POWER THEOREM FOR A CLASS OF COUPLED FLOWS

It is well-known that an uncoupled creeping flow, which is governed by the incompressible Stokes equations, enjoys a minimum power theorem [Guazzelli and Morris, 2011]. It has also been established that an uncoupled flow through porous media based on either Darcy equations or Darcy-Brinkman equations enjoys a minimum power theorem [Shabouei and Nakshatrala, 2016]. It is thus natural to ask whether a flow in coupled free-porous media enjoys a minimum power theorem.

We now show that the answer to this question is affirmative for a class of coupled flows. This class of flows is characterized by these two requirements:

- (R1) There exists two potentials, \( \Phi_{\text{free}} \) and \( \Phi_{\text{por}} \), with the following properties:
  - (i) They satisfy the form-invariance and the invariance under a Euclidean transformation (i.e., they satisfy the principle of material frame indifference). Specifically these potentials can be expressed as \( \Phi_{\text{free}}[D_{\text{free}}] \) and \( \Phi_{\text{por}}[D_{\text{por}}, \vec{v}_{\text{por}}] \).\(^{18}\)
  - (ii) They provide the constitutive relations of the following form for the extra Cauchy stresses and the interaction term:

\[
T^\text{extra}_{\text{free}} = \frac{\partial \Phi_{\text{free}}}{\partial D_{\text{free}}} \tag{6.1a}
\]

\[
T^\text{extra}_{\text{por}} = \frac{\partial \Phi_{\text{por}}}{\partial D_{\text{por}}} \tag{6.1b}
\]

\(^{17}\)See [Saffman, 1971, equation (2.18)] and the text below that equation.

\(^{18}\)\( \vec{v}_{\text{por}} \) should be interpreted with respect to the velocity of the porous solid, and hence it is objective under a Euclidean transformation.
(i) Each of the potentials has a positive definite Hessian\textsuperscript{19}.

(R2) The functional $\Psi$ has a positive definite Hessian.

The requirement (R2) is in addition to the properties that outlined in §3 for $\Psi$ to satisfy. It is easy to construct $\Psi$ to have a positive definite Hessian; the functional (5.1) satisfying the condition (5.2) is one such example.

6.1. On construction of the potentials. For many popular uncoupled free flow models (e.g., Stokes equations) and porous media models (e.g., Darcy equations, Darcy-Brinkman equations), the rate of internal dissipation density satisfies the conditions (6.1a)–(6.1c). One can take the same approach to construct the potentials $\Phi_{\text{free}}$ and $\Phi_{\text{por}}$ even for the case of coupled flows. This approach can be best illustrated by the following examples.

Under the Stokes model, the Cauchy stress and the extra Cauchy stress are given by
\begin{equation}
T_{\text{free}} = -p_{\text{free}} I + 2\mu D_{\text{free}} = -p_{\text{free}} I + T_{\text{free}}^{\text{extra}}
\end{equation}
and the rate of internal dissipation density is given by
\begin{equation}
2\mu D_{\text{free}} \cdot D_{\text{free}}
\end{equation}
Clearly, by choosing the potential $\Phi_{\text{free}}$ to be
\begin{equation}
2\Phi_{\text{free}}[v_{\text{free}}] = 2\mu D_{\text{free}} \cdot D_{\text{free}}
\end{equation}
one can satisfy the requirement (6.1a). Similarly, under the Darcy model, the extra Cauchy stress and interaction term are, respectively, given by
\begin{equation}
T_{\text{por}}^{\text{extra}} = 0 \quad \text{and} \quad i_{\text{por}} = \mu K^{-1} v_{\text{por}}(x)
\end{equation}
By choosing the potential $\Phi_{\text{por}}$ to be
\begin{equation}
2\Phi_{\text{por}}[v_{\text{por}}] = \mu K^{-1} v_{\text{por}}(x) \cdot v_{\text{por}}(x)
\end{equation}
one can satisfy the requirements (6.1b) and (6.1c). Under the Darcy-Brinkman model, the extra Cauchy stress and interaction term are, respectively, given by
\begin{equation}
T_{\text{por}}^{\text{extra}} = 2\mu D_{\text{por}} \cdot D_{\text{por}} \quad \text{and} \quad i_{\text{por}} = \mu K^{-1} v_{\text{por}}(x)
\end{equation}
By choosing the potential $\Phi_{\text{por}}$ to be
\begin{equation}
2\Phi_{\text{por}}[v_{\text{por}}] = 2\mu D_{\text{por}} \cdot D_{\text{por}} + \mu K^{-1} v_{\text{por}}(x) \cdot v_{\text{por}}(x)
\end{equation}
one can satisfy the requirements (6.1b) and (6.1c).

If the coupled flow is modeled based on Stokes-Darcy equations (i.e., Stokes model is used for the free flow region, and Darcy model is used for the porous region), then the two potentials for the coupled flow can be chosen based on equations (6.3) and (6.5), which are for uncoupled flows. Similarly, if the coupled flow is based on Stokes-Darcy-Brinkman equations (i.e., Stokes model is

\textsuperscript{19}The Hessian of a functional is the Jacobian matrix containing the second derivatives of the functional with respect to its input arguments. A positive definite Hessian means that the Jacobian matrix is positive definite. In other words, the second variation of the functional is positive under all non-zero variations of its input arguments.
used for the free flow region and Darcy-Brinkman model is used for the porous region), then the
two potentials for the coupled flow can be chosen based on equations (6.3) and (6.7).

6.2. Minimum power theorem. We define the total mechanical power functional as follows:

\[
P_{\text{coupled}}[z_{\text{free}}(x), z_{\text{por}}(x)] := \int_{K_{\text{free}}} \Phi_{\text{free}}[z_{\text{free}}(x)] \, d\Omega + \int_{K_{\text{por}}} \Phi_{\text{por}}[z_{\text{por}}(x)] \, d\Omega \\
+ \int_{\Gamma_{\text{int}}} \Psi[x_{\text{free}}(x), x_{\text{por}}(x), n(x)] \, d\Gamma \\
- \int_{K_{\text{free}}} \gamma b_{\text{free}}(x) \cdot z_{\text{free}}(x) \, d\Omega - \int_{\Gamma_{\text{free}}} t_{\text{free}}^p(x) \cdot z_{\text{free}}(x) \, d\Gamma \\
- \int_{K_{\text{por}}} \gamma \phi_{\text{por}}(x) b_{\text{por}}(x) \cdot z_{\text{por}}(x) \, d\Omega - \int_{\Gamma_{\text{por}}} t_{\text{por}}^p(x) \cdot z_{\text{por}}(x) \, d\Gamma
\] (6.8)

where \(z_{\text{free}} : K_{\text{free}} \to \mathbb{R}^{nd}\) and \(z_{\text{por}} : K_{\text{por}} \to \mathbb{R}^{nd}\) are vector fields; \(z_{\text{free}}^*\) and \(z_{\text{por}}^*\) denote, respectively the tangential components of \(z_{\text{free}}\) and \(z_{\text{por}}\); and

\[z_n(x) := z_{\text{free}}(x) \cdot \hat{n}_{\text{free}}(x)\]

We then establish the following result with a proof provided under the supplementary material.

**Theorem 6.1** (Minimum power theorem for coupled flows). For the class of coupled flows satisfying the requirements (R1)–(R2), any pair of kinematically admissible vector fields \((\tilde{v}_{\text{free}}(x), \tilde{v}_{\text{por}}(x))\) satisfies

\[P_{\text{coupled}}[v_{\text{free}}(x), v_{\text{por}}(x)] \leq P_{\text{coupled}}[\tilde{v}_{\text{free}}(x), \tilde{v}_{\text{por}}(x)]\] (6.9)

in which \(v_{\text{free}}(x)\) is the velocity field in the free flow region and \(v_{\text{por}}(x)\) is the velocity field in the porous region.

7. UNIQUENESS OF SOLUTIONS

We will use the minimum power theorem to establish the uniqueness of solutions under the proposed interface conditions. For brevity, we will show for the case of coupled Stokes-Darcy-Brinkman equations; however, with straightforward alterations, one can show for the case of Darcy equations coupled with the Stokes equations. We establish the uniqueness under the following functional form for \(\Psi\), which is (slightly) more general than the one considered in §5:

\[\Psi[v_{\text{free}}^*, v_{\text{por}}^*, v_n(x)] = \alpha_{11} v_{\text{free}}^* \cdot v_{\text{free}}(x) + 2\alpha_{12} v_{\text{free}}^* \cdot v_{\text{por}}^* + \alpha_{22} v_{\text{por}}^* \cdot v_{\text{por}}(x) + \beta v_n(x) \cdot v_n(x)\] (7.1)

with

\[\alpha_{11} \alpha_{22} \geq \alpha_{12}^2 \quad \text{and} \quad \beta \geq 0\] (7.2)

To establish uniqueness under more general conditions (e.g., a more general functional form for \(\Psi\)), one needs to resort to techniques from functional analysis, which is beyond the scope of this paper. We establish the following theorem with a proof provided under the supplementary material.

**Theorem 7.1** (Uniqueness). Under the prescribed data given by \(b_{\text{free}}(x), b_{\text{por}}(x), v_{\text{free}}^p(x), v_{\text{por}}^p(x), t_{\text{free}}^p(x)\) and \(t_{\text{por}}^p(x)\); and under \(\Psi\) given by equation (7.1); the solution to the coupled Stokes-Darcy-Brinkman equations is unique up to an arbitrary constant for the pressures.
8. CONCLUDING REMARKS

We have considered the flows of incompressible fluids in coupled free-porous media. We have presented a theoretical framework to obtain a complete set of self-consistent conditions, which describes the flow dynamics at the interface of free flow and porous regions. The interface conditions are essential for the closure of the mathematical model. The framework is primarily built upon the principle of virtual power, theory of interacting continua, and a geometric argument for enforcing internal constraints, which in our case is the incompressibility of the fluid. The central idea in the proposed principle of virtual power is to account for the power expended at the interface and thereby making it possible to circumvent the need to estimate the partial stress in the porous solid.

Under the proposed framework, the set of interface conditions is a combination of jump conditions and a constitutive specification, which is provided by prescribing the physically meaningful power expended density at the interface. We have also shown that the jump conditions by themselves do not provide a workable set of conditions, which is because of the inability to quantify the traction taken by the rigid porous solid under the theory of interacting continua. The salient features of the proposed framework of obtaining interface conditions are: (i) The framework enjoys a strong theoretical underpinning. (ii) The resulting interface conditions make the resulting mathematical model well-posed. Specifically, we have shown that the resulting mathematical model has a unique solution. (iii) The framework is amenable to generalizations, and the resulting interface conditions are valid for a wide variety of porous media models. (iv) Several popular conditions in the literature are special cases of the proposed framework. (v) Similar to uncoupled free flows and uncoupled flows in porous media, the flows in coupled free-porous media under the proposed interface conditions also enjoy a minimum power theorem.

In closure, the proposed principle of virtual power for flows in coupled free-porous media encapsulates the balance of linear momentum, the balance of angular momentum, internal constraints, Cauchy’s fundamental theorem for the stress, and interface conditions!

Appendix A. ON JUMP CONDITIONS

It can be tempting to treat the interface as a singular surface, obtain the jump conditions across the singular surface and consider them as an appropriate set of interface conditions. We will now show why the jump conditions will not render a useful set of conditions at the interface for flows in coupled free-porous media, especially when the porous solid is assumed to be rigid.

The jump conditions (which are the balance laws across a singular surface) in the context of a single constituent can be found in many standard texts on continuum mechanics (e.g., [Chadwick, 2012; Liu, 2013]). But the problem central to this paper involves a porous medium, which is not a single constituent. A jump condition for a mixture (i.e., a continuum with multiple constituents) will be a bit more than the balance laws, as one need to make additional assumptions on defining quantities for the mixture on the whole in terms of the corresponding quantities of its constituents. We first present the jump conditions in the most familiar setting of a single constituent and then extend to the case of multiple constituents using TIC. Only the jump conditions pertaining to the balance of mass and the balance of linear momentum will be relevant here.

Consider a singular surface \( \Gamma \) which evolves with a velocity vector \( \mathbf{v}_\Gamma \). The regions on the either sides of \( \Gamma \) and the corresponding quantities are indicated by “+” and “−” (see Fig. 4). The velocity vector of the interface, in general, need not be along the normal to the interface. That is, the unit vector \( \hat{\mathbf{m}} \) need not be parallel to \( \hat{\mathbf{n}}^+ \) or \( \hat{\mathbf{n}}^- \). However, only the normal component of
the interface velocity manifests in the jump conditions. To this end, without loss of generality, we define the normal component of the interface velocity as follows:

\[ V_Γ := v_Γ \cdot \hat{n} - \] (A.1)

We define the jump operator acting on a quantity \( \eta \) as follows:

\[ J[\eta] = \eta^+ - \eta^- \] (A.2)

**A.1. Jump conditions for a single constituent.** The jump condition for the balance of mass across \( Γ \) reads:

\[ [γ(V_Γ - v \cdot \hat{n})] = 0 \] (A.3)

which when expanded reads as follows:

\[ (γ^+ - γ^-)V_Γ + γ^+ v^+ \cdot \hat{n}^+ + γ^- v^- \cdot \hat{n}^- = 0 \] (A.4)

The jump condition for the balance of linear momentum across \( Γ \) reads:

\[ [γ(V_Γ - v \cdot \hat{n})v] + t^- + t^+ = 0 \] (A.5)

where \( t^- \) and \( t^+ \) denote the tractions on the either side of the singular surface.

**A.2. Multiple constituents.** For the coupled free-porous media, we associate, without loss of generality, the “−” region with the free flow region and the “+” region with the porous region. The jump condition for the balance of the mass for the fluid takes the following form:

\[ (γ_{por} - γ_{free})V_Γ + γ_{free}v^{(n)}_{free}(x) + γ_{por}v^{(n)}_{por}(x) = 0 \] (A.6)

where \( v^{(n)}_{por} \) is the normal component of the discharge velocity, which is equal to the product of the (surface) porosity and the seepage velocity.

Figure 4. A singular surface \( Γ \) evolves with a velocity \( v_Γ \) along the direction given by the unit vector \( \hat{m} \). The regions on the either side of \( Γ \) are denoted by “+” and “−”. The corresponding quantities are denoted using these signs as superscripts. The tractions are denoted by \( t^+ \) and \( t^- \), and the unit outward normals are denoted by \( \hat{n}^+ \) and \( \hat{n}^- \). The jump conditions are balance laws across such a singular surface.
In order to write the jump condition for the balance of linear momentum, the multi-constituent nature of the porous medium needs to be considered and an additional assumption on the total traction of the mixture needs to be made. Even in the simplest case as considered in this paper, a porous medium consists of two constituents; one of them being the porous solid and the other one is the fluid in the pores. Although different definitions are employed under TIC to define a quantity of a mixture in terms of the corresponding quantities of its constituents [Hansen et al., 1991], it is however common to assume that the total traction of a mixture is the sum of the partial tractions of its constituents. Thus, the total traction in the porous medium (consisting of a fluid and a solid constituents) is taken as
\[
t_{\text{por}}^{(\text{fluid})} + t_{\text{por}}^{(\text{solid})}
\]
where \(t_{\text{por}}^{(\text{fluid})}\) and \(t_{\text{por}}^{(\text{solid})}\) are, respectively, the partial tractions in the fluid and solid constituents.

The jump condition for the balance of linear momentum for the entire mixture (i.e., all the constituents) across \(\Gamma\) can be written as follows:
\[
\gamma_{\text{free}} \left( V_{\Gamma} - v_{\text{free}} \cdot \hat{n}_{\text{free}} \right) v_{\text{free}} + \gamma_{\text{por}} \left( V_{\Gamma} - v_{\text{por}} \cdot \hat{n}_{\text{por}} \right) v_{\text{por}} + t_{\text{free}} + \left( t_{\text{por}}^{(\text{fluid})} + t_{\text{por}}^{(\text{solid})} \right) = 0 \quad (A.7)
\]

We now specialize to the case when the singular surface is stationary (which implied \(V_{\Gamma} = 0\)) and the true density of the fluid across the singular surface is the same (i.e., \(\gamma_{\text{free}} = \gamma_{\text{por}}\)). The jump conditions for the balance of mass and the balance of linear momentum can be compactly written as follows:
\[
v_{\text{free}}^{(n)}(x) + v_{\text{por}}^{(n)}(x) = 0 \quad (A.8a)
\]
\[
t_{\text{free}} \cdot \hat{n}_{\text{free}} = \left( t_{\text{por}}^{(\text{fluid})} + t_{\text{por}}^{(\text{solid})} \right) \cdot \hat{n}_{\text{por}} \quad (A.8b)
\]
\[
\gamma v_{\text{h}}(v_{\text{free}} - v_{\text{por}}) \cdot \hat{s} = t_{\text{free}} \cdot \hat{s} + \left( t_{\text{por}}^{(\text{fluid})} + t_{\text{por}}^{(\text{solid})} \right) \cdot \hat{s} \quad (A.8c)
\]

Equations (A.8b) and (A.8c) are, respectively, the normal and tangential components of equation (A.7). Equation (A.8a) has been invoked in obtaining equations (A.8b) and (A.8c).

A.3. Discussion. We now compare the above set of three jump conditions with the set of four interface conditions (4.16a)–(4.16d). The following are the similarities and the notable differences:
(a) The jump condition pertaining to the balance of mass (A.8a) is exactly the same as the first interface condition (4.16a), which is the reason why we mentioned earlier that the interface condition (4.16a) stems from the jump conditions.
(b) There is only one jump condition involving the tangential part of the tractions. On the other hand, two interface conditions are related to the tangential components of the tractions.
(c) The jump conditions (A.8b)–(A.8c) involve \(t_{\text{por}}^{(\text{solid})}\) but the interface conditions (4.16c)–(4.16d) involve the functional \(\Psi\) instead.

Let us now focus on equation (A.8b). The total traction in the porous medium is distributed among these two constituents: the porous solid and the fluid in the pores. If the porous solid is rigid, one cannot estimate what part of the total traction is taken up by the porous solid, and hence one will not be able to find the traction taken by the fluid in the pores. A similar case exists even with the condition (A.8c). Thus, the jump condition related to the balance of linear momentum does not provide a workable condition. This type of difficulty (i.e., finding the partial tractions of the individual constituents from the total traction) is inherent to porous media models which are
Appendix B. RECOVERING CAUCHY’S FUNDAMENTAL THEOREM

To recover the Cauchy’s fundamental theorem for the stress, one need to enforce the balance of virtual power on arbitrary subsets of the domain. To this end, we replace statement (P1) in the principle of virtual power (3.15) with the following:

\[(P1') \quad \mathcal{P}^{(\text{internal})}(\mathcal{B}) = \mathcal{P}^{(\text{external})}(\mathcal{B}) \quad \forall (\mathbf{w}_{\text{free}}, \mathbf{w}_{\text{por}}) \in \widetilde{\mathcal{W}} \text{ and } \forall \mathcal{B} \subseteq \Omega \quad (B.1)\]

where \(\mathcal{B}\) is an arbitrary subset of the domain \(\Omega\) and

\[
\mathcal{P}^{(\text{internal})}(\mathcal{B}) := \int_{\mathcal{K}_{\text{free}} \cap \mathcal{B}} \mathbf{T}_{\text{free}} \cdot \text{grad}[\mathbf{w}_{\text{free}}] \, d\Omega + \int_{\mathcal{K}_{\text{por}} \cap \mathcal{B}} \mathbf{T}_{\text{por}} \cdot \text{grad}[\mathbf{w}_{\text{por}}] \, d\Omega + \int_{\kappa_{\text{por}} \cap \mathcal{B}} \mathbf{i}_{\text{por}} \cdot \mathbf{w}_{\text{por}} \, d\Omega + \int_{\Gamma_{\text{int}} \cap \mathcal{B}} \delta \Phi \, d\Gamma \quad (B.2)
\]

\[
\mathcal{P}^{(\text{external})}(\mathcal{B}) := \int_{\partial \mathcal{K}_{\text{free}} \cap \mathcal{B}} \mathbf{t}_{\text{free}} \cdot \mathbf{w}_{\text{free}} \, d\Gamma + \int_{\mathcal{K}_{\text{free}} \cap \mathcal{B}} \gamma^{b}_{\text{free}} \cdot \mathbf{w}_{\text{free}} \, d\Omega + \int_{\partial \mathcal{K}_{\text{por}} \cap \mathcal{B}} \mathbf{t}_{\text{por}} \cdot \mathbf{w}_{\text{por}} \, d\Gamma + \int_{\mathcal{K}_{\text{por}} \cap \mathcal{B}} \gamma^{b}_{\text{por}} \cdot \mathbf{w}_{\text{por}} \, d\Omega \quad (B.3)
\]

In the above expression, \(\mathbf{t}_{\text{free}}\) and \(\mathbf{t}_{\text{por}}\) denote the tractions, respectively, on \(\partial \mathcal{K}_{\text{free}}\) and \(\partial \mathcal{K}_{\text{por}}\).

By taking the subset \(\mathcal{B}\) to be entirely within \(\mathcal{K}_{\text{free}}\) and by using a similar approach taken in the previous sections (e.g., Green’s identity, the fundamental lemma of calculus of variations), one can establish:

\[
\mathbf{t}_{\text{free}}(\mathbf{x}) = \mathbf{T}_{\text{free}}(\mathbf{x}) \hat{n}(\mathbf{x}) \quad (B.4)
\]

on any surface in the free flow region \((\mathcal{K}_{\text{free}} \cup \partial \mathcal{K}_{\text{free}})\) with the unit outward normal \(\hat{n}(\mathbf{x})\). Similarly, by taking the subset \(\mathcal{B}\) to be entirely within \(\mathcal{K}_{\text{por}}\), one can establish:

\[
\mathbf{t}_{\text{por}}(\mathbf{x}) = \mathbf{T}_{\text{por}}(\mathbf{x}) \hat{n}(\mathbf{x}) \quad (B.5)
\]

on any surface in the porous region \((\mathcal{K}_{\text{por}} \cup \partial \mathcal{K}_{\text{por}})\) with the unit outward normal \(\hat{n}(\mathbf{x})\). The relations (B.4) and (B.5), respectively, represent the Cauchy’s fundamental theorem for the stress for the free flow region and the porous region. Using the traction-stress relations, the second interface condition (4.16b) takes the following more familiar form:

\[
\mathbf{t}_{\text{free}}(\mathbf{x}) \cdot \hat{n}_{\text{free}}(\mathbf{x}) + \frac{\partial \Psi}{\partial v_n} = \mathbf{t}_{\text{por}}(\mathbf{x}) \cdot \hat{n}_{\text{por}}(\mathbf{x}) \quad \forall \mathbf{x} \in \Gamma_{\text{int}} \quad (B.6)
\]

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Appendix C. SUPPLEMENTARY MATERIAL

C.1. A proof of the minimum power theorem. Based on the first-order optimality condition it will suffice to show that

\[
\delta P_{\text{coupled}}[v_{\text{free}}, v_{\text{por}}; \delta v_{\text{free}}, \delta v_{\text{por}}] := \left[ \frac{d}{d\epsilon} P_{\text{coupled}}[v_{\text{free}} + \epsilon \delta v_{\text{free}}, v_{\text{por}} + \epsilon \delta v_{\text{por}}] \right]_{\epsilon=0} = 0 \\
\forall (\delta v_{\text{free}}, \delta v_{\text{por}}) \in \mathcal{W}  \quad (C.1)
\]

The positive definite Hessians will ensure that the extremum is in fact a minimum. The Gâteaux variation can be written as follows\(^{20}\):

\[
\delta P_{\text{coupled}}[v_{\text{free}}, v_{\text{por}}; \delta v_{\text{free}}, \delta v_{\text{por}}] = \int_{\Omega_{\text{free}}} \frac{\partial \Phi_{\text{free}}}{\partial D_{\text{free}}} \cdot \delta D_{\text{free}} \, d\Omega + \int_{\Gamma_{\text{por}}} \left( \frac{\partial \Phi_{\text{por}}}{\partial v_{\text{por}}} \cdot \delta v_{\text{por}} + \frac{\partial \Phi_{\text{por}}}{\partial D_{\text{por}}} \cdot \delta D_{\text{por}} \right) \, d\Gamma \\
+ \int_{\Gamma_{\text{int}}} \left( \frac{\partial \Psi}{\partial v_{\text{por}}} \cdot \delta v_{\text{free}} + \frac{\partial \Psi}{\partial v_{\text{por}}} \cdot \delta v_{\text{por}} \cdot \delta v_{\text{por}} + \frac{\partial \Psi}{\partial v_{\text{por}}} \cdot \delta v_{\text{por}} \right) \, d\Gamma \\
- \int_{\Omega_{\text{free}}} \gamma b_{\text{free}} \cdot \delta v_{\text{free}} \, d\Omega - \int_{\Gamma_{\text{free}}} t_{\text{free}}^{p}(x) \cdot \delta v_{\text{free}}(x) \, d\Gamma \\
- \int_{\Omega_{\text{por}}} \gamma \phi_{\text{por}} b_{\text{por}} \cdot \delta v_{\text{por}} \, d\Omega - \int_{\Gamma_{\text{por}}} t_{\text{por}}^{p}(x) \cdot \delta v_{\text{por}}(x) \, d\Gamma  \quad (C.2)
\]

Using the conditions (6.1a)–(6.1c) under the requirement (R1), we obtain the following:

\[
\delta P_{\text{coupled}}[v_{\text{free}}, v_{\text{por}}; \delta v_{\text{free}}, \delta v_{\text{por}}] = \int_{\Omega_{\text{free}}} T_{\text{extra}}^{\text{free}} \cdot \delta D_{\text{free}} \, d\Omega + \int_{\Omega_{\text{por}}} (i_{\text{por}} \cdot \delta v_{\text{por}} + T_{\text{extra}}^{\text{por}} \cdot \delta D_{\text{por}}) \, d\Omega \\
+ \int_{\Gamma_{\text{int}}} \left( \frac{\partial \Psi}{\partial v_{\text{por}}} \cdot \delta v_{\text{free}} + \frac{\partial \Psi}{\partial v_{\text{por}}} \cdot \delta v_{\text{por}} \cdot \delta v_{\text{por}} + \frac{\partial \Psi}{\partial v_{\text{por}}} \cdot \delta v_{\text{por}} \right) \, d\Gamma \\
- \int_{\Omega_{\text{free}}} \gamma b_{\text{free}} \cdot \delta v_{\text{free}} \, d\Omega - \int_{\Gamma_{\text{free}}} t_{\text{free}}^{p}(x) \cdot \delta v_{\text{free}}(x) \, d\Gamma \\
- \int_{\Omega_{\text{por}}} \gamma \phi_{\text{por}} b_{\text{por}} \cdot \delta v_{\text{por}} \, d\Omega - \int_{\Gamma_{\text{por}}} t_{\text{por}}^{p}(x) \cdot \delta v_{\text{por}}(x) \, d\Gamma  \quad (C.3)
\]

Noting the internal constraints (4.14b) and (4.15b), utilizing the decomposition of the Cauchy stresses (4.10), and invoking the Green’s identity, we obtain the following:

\[
\delta P_{\text{coupled}}[v_{\text{free}}, v_{\text{por}}; \delta v_{\text{free}}, \delta v_{\text{por}}] = - \int_{\Omega_{\text{free}}} (\text{div}[T_{\text{free}}] + \gamma b_{\text{free}}) \cdot \delta v_{\text{free}} \, d\Omega \\
= 0 \text{ due to (4.14a)} \\
- \int_{\Omega_{\text{por}}} (\text{div}[T_{\text{por}}] + \gamma \phi_{\text{por}} b_{\text{por}} - i_{\text{por}}) \cdot \delta v_{\text{por}} \, d\Omega \\
= 0 \text{ due to (4.15a)} \\
+ \int_{\partial \Omega_{\text{free}}} (T_{\text{free}} \hat{n}_{\text{free}}) \cdot \delta v_{\text{free}}(x) \, d\Gamma - \int_{\Gamma_{\text{free}}} t_{\text{free}}^{p}(x) \cdot \delta v_{\text{free}}(x) \, d\Gamma \\
+ \int_{\partial \Omega_{\text{por}}} (T_{\text{por}} \hat{n}_{\text{por}}) \cdot \delta v_{\text{por}}(x) \, d\Gamma - \int_{\Gamma_{\text{por}}} t_{\text{por}}^{p}(x) \cdot \delta v_{\text{por}}(x) \, d\Gamma \\
+ \int_{\Gamma_{\text{int}}} \left( \frac{\partial \Psi}{\partial v_{\text{por}}} \cdot \delta v_{\text{free}} + \frac{\partial \Psi}{\partial v_{\text{por}}} \cdot \delta v_{\text{por}} \cdot \delta v_{\text{por}} + \frac{\partial \Psi}{\partial v_{\text{por}}} \cdot \delta v_{\text{por}} \right) \, d\Gamma  \quad (C.4)
\]

\(^{20}\delta D_{\text{free}} := \frac{1}{2}(\text{grad}[\delta v_{\text{free}}] + \text{grad}[\delta v_{\text{free}}]^T)\) and \(\delta D_{\text{por}} := \frac{1}{2}(\text{grad}[\delta v_{\text{por}}] + \text{grad}[\delta v_{\text{por}}]^T)\)
Noting the decomposition of the boundaries $\partial K_{\text{free}}$ and $\partial K_{\text{por}}$, given by equations (3.4) and (3.5), we obtain the following:

$$\delta P_{\text{coupled}}[v_{\text{free}}, v_{\text{por}}; \delta v_{\text{free}}, \delta v_{\text{por}}] = \int_{\Gamma_{\text{free}}} (T_{\text{free}} \hat{n}_{\text{free}} - t^p_{\text{free}}(x)) \cdot \delta v_{\text{free}}(x) \, d\Gamma + \int_{\Gamma_{\text{free}}} (T_{\text{free}} \hat{n}_{\text{free}}) \cdot \delta v_{\text{free}}(x) \, d\Gamma$$

$$= 0 \text{ due to (4.14c)}$$

$$+ \int_{\Gamma_{\text{por}}} (T_{\text{por}} \hat{n}_{\text{por}} - v^p_{\text{por}}(x)) \cdot \delta v_{\text{por}}(x) \, d\Gamma + \int_{\Gamma_{\text{por}}} (T_{\text{por}} \hat{n}_{\text{por}}) \cdot \delta v_{\text{por}}(x) \, d\Gamma$$

$$= 0 \text{ due to (4.15c)}$$

$$+ \int_{\Gamma_{\text{int}}} (T_{\text{free}} \hat{n}_{\text{free}}) \cdot \delta v_{\text{free}}(x) \, d\Gamma + \int_{\Gamma_{\text{int}}} (T_{\text{por}} \hat{n}_{\text{por}}) \cdot \delta v_{\text{por}}(x) \, d\Gamma$$

$$+ \int_{\Gamma_{\text{int}}} \left( \frac{\partial \Psi}{\partial v_{\text{free}}} \cdot \delta v_{\text{free}} + \frac{\partial \Psi}{\partial v_{\text{por}}} \cdot \delta v_{\text{por}} \right) \, d\Gamma$$  \hspace{1cm} (C.5)

Invoking that $\delta v_{\text{free}}(x)$ and $\delta v_{\text{por}}(x)$, respectively, vanish on $\Gamma_{\text{free}}^\nu$ and $\Gamma_{\text{por}}^\nu$ (see §3.1.5), and using the first interface condition (4.16a) and the notation introduced in (3.28), we obtain the following:

$$\delta P_{\text{coupled}}[v_{\text{free}}, v_{\text{por}}; \delta v_{\text{free}}, \delta v_{\text{por}}] = \int_{\Gamma_{\text{int}}} (\hat{n}_{\text{free}} \cdot T_{\text{free}} \hat{n}_{\text{free}} - \hat{n}_{\text{por}} \cdot T_{\text{por}} \hat{n}_{\text{por}} + \frac{\partial \Psi}{\partial v_{\text{por}}}) \cdot \delta v_{\text{por}} \, d\Gamma$$

$$+ \int_{\Gamma_{\text{int}}} (T_{\text{free}} \hat{n}_{\text{free}} + \frac{\partial \Psi}{\partial v_{\text{free}}}) \cdot \delta v_{\text{free}} \, d\Gamma$$

$$+ \int_{\Gamma_{\text{int}}} (T_{\text{por}} \hat{n}_{\text{por}} + \frac{\partial \Psi}{\partial v_{\text{por}}}) \cdot \delta v_{\text{por}} \, d\Gamma$$  \hspace{1cm} (C.6)

Finally, by utilizing the interface conditions (4.16b)–(4.16d) we have established that the first variation of $P_{\text{coupled}}$ vanishes.

### C.2. A proof of the uniqueness theorem.

On the contrary, assume that

$$\{v_{\text{free}}^{(1)}(x), p_{\text{free}}^{(1)}(x), v_{\text{por}}^{(1)}(x), p_{\text{por}}^{(1)}(x)\} \text{ and } \{v_{\text{free}}^{(2)}(x), p_{\text{free}}^{(2)}(x), v_{\text{por}}^{(2)}(x), p_{\text{por}}^{(2)}(x)\}$$

are two solutions to the coupled Stokes-Darcy-Brinkman equations for the prescribed data. That is, $\{v_{\text{free}}^{(1)}(x), p_{\text{free}}^{(1)}(x)\}$ and $\{v_{\text{free}}^{(2)}(x), p_{\text{free}}^{(2)}(x)\}$ satisfy the Stokes equations in $K_{\text{free}}$, and $\{v_{\text{por}}^{(1)}(x), p_{\text{por}}^{(1)}(x)\}$ and $\{v_{\text{por}}^{(2)}(x), p_{\text{por}}^{(2)}(x)\}$ satisfy the Darcy-Brinkman equations in $K_{\text{por}}$. Moreover, $v_{\text{free}}^{(1)}$, $v_{\text{free}}^{(2)}$, $v_{\text{por}}^{(1)}$ and $v_{\text{por}}^{(2)}$ satisfy

$$\text{div} \left[ v_{\text{free}}^{(1)} \right] = 0 \text{ and } \text{div} \left[ v_{\text{free}}^{(2)} \right] = 0 \text{ in } K_{\text{free}}$$

$$\text{div} \left[ v_{\text{por}}^{(1)} \right] = 0 \text{ and } \text{div} \left[ v_{\text{por}}^{(2)} \right] = 0 \text{ in } K_{\text{por}}$$  \hspace{1cm} (C.7)

$$\text{div} \left[ v_{\text{free}}^{(1)} \right] = 0 \text{ and } \text{div} \left[ v_{\text{free}}^{(2)} \right] = 0 \text{ in } K_{\text{por}}$$  \hspace{1cm} (C.8)

Since the pairs $\{v_{\text{free}}^{(1)}(x), v_{\text{por}}^{(1)}(x)\}$ and $\{v_{\text{free}}^{(2)}(x), v_{\text{por}}^{(2)}(x)\}$ are both kinematically admissible, the minimum power theorem implies that:

$$P_{\text{coupled}} \left[ v_{\text{free}}^{(1)}(x), v_{\text{por}}^{(1)}(x) \right] = P_{\text{coupled}} \left[ v_{\text{free}}^{(2)}(x), v_{\text{por}}^{(2)}(x) \right]$$  \hspace{1cm} (C.9)

Using the definition of $P_{\text{coupled}}$ given by equation (6.8), the above equation can be expanded as follows:

$$\frac{1}{2} \left( \Phi_{\text{free}} \left[ v_{\text{free}}^{(1)} - v_{\text{free}}^{(2)} \right] + \frac{1}{2} \left( \Phi_{\text{por}} \left[ v_{\text{por}}^{(1)} - v_{\text{por}}^{(2)} \right] \right) \right) + \int_{\Gamma_{\text{int}}} \left( \Psi \left[ v_{\text{free}}^{(1)} v_{\text{por}} v_{\text{int}}^{(1)} + v_{\text{por}} v_{\text{free}} v_{\text{int}}^{(2)} \right] \right) \, d\Gamma$$

28
\[ = \int_{K_{\text{free}}} \gamma b_{\text{free}} \cdot (v_{\text{free}}^{(1)} - v_{\text{free}}^{(2)}) \, d\Omega + \int_{\Gamma_{\text{free}}^T} t_{\text{free}}^p \cdot (v_{\text{free}}^{(1)} - v_{\text{free}}^{(2)}) \, d\Omega \\
+ \int_{K_{\text{por}}} \gamma \phi_{\text{por}} b_{\text{por}} \cdot (v_{\text{por}}^{(1)} - v_{\text{por}}^{(2)}) \, d\Omega + \int_{\Gamma_{\text{por}}^T} t_{\text{por}}^p \cdot (v_{\text{por}}^{(1)} - v_{\text{por}}^{(2)}) \, d\Omega \quad (\text{C.10}) \]

Noting the rate of internal dissipation in the Stokes model, it is easy to establish the following:
\[ \frac{1}{2} \left( \Phi_{\text{free}} \left[ v_{\text{free}}^{(1)} \right] - \Phi_{\text{free}} \left[ v_{\text{free}}^{(2)} \right] \right) = \frac{1}{2} \Phi_{\text{free}} \left[ v_{\text{free}}^{(1)} - v_{\text{free}}^{(2)} \right] + \int_{K_{\text{free}}} 2\mu D_{\text{free}}^{(2)} : \left( D_{\text{free}}^{(1)} - D_{\text{free}}^{(2)} \right) \, d\Omega \quad (\text{C.11}) \]

Using equation (C.8), the above equation can be written as follows:
\[ \frac{1}{2} \left( \Phi_{\text{free}} \left[ v_{\text{free}}^{(1)} \right] - \Phi_{\text{free}} \left[ v_{\text{free}}^{(2)} \right] \right) = \frac{1}{2} \Phi_{\text{free}} \left[ v_{\text{free}}^{(1)} - v_{\text{free}}^{(2)} \right] + \int_{K_{\text{free}}} T_{\text{free}}^{(2)} : \left( D_{\text{free}}^{(1)} - D_{\text{free}}^{(2)} \right) \, d\Omega \quad (\text{C.12}) \]

where
\[ T_{\text{free}}^{(2)} = -p_{\text{free}}^{(2)} I + 2\mu D_{\text{free}}^{(2)} \quad (\text{C.13}) \]

On similar lines, one can establish the following relation:
\[ \frac{1}{2} \left( \Phi_{\text{por}} \left[ v_{\text{por}}^{(1)} \right] - \Phi_{\text{por}} \left[ v_{\text{por}}^{(2)} \right] \right) = \frac{1}{2} \Phi_{\text{por}} \left[ v_{\text{por}}^{(1)} - v_{\text{por}}^{(2)} \right] + \int_{K_{\text{por}}} T_{\text{por}}^{(2)} : \left( D_{\text{por}}^{(1)} - D_{\text{por}}^{(2)} \right) \, d\Omega \]
\[ + \int_{K_{\text{por}}} \mu K^{-1} v_{\text{por}}^{(2)} : \left( v_{\text{por}}^{(1)} - v_{\text{por}}^{(2)} \right) d\Omega \quad (\text{C.14}) \]

where
\[ T_{\text{por}}^{(2)} = -p_{\text{por}}^{(2)} I + 2\mu D_{\text{por}}^{(2)} \quad (\text{C.15}) \]

We note the fields under the second solution satisfy the balance of linear momentum; that is:
\[ \text{div} \left[ T_{\text{free}}^{(2)} \right] + \gamma b_{\text{free}} = 0 \quad \text{in } K_{\text{free}} \quad (\text{C.16}) \]
\[ \text{div} \left[ T_{\text{por}}^{(2)} \right] + \gamma \phi_{\text{por}} b_{\text{por}} = \mu K^{-1} v_{\text{por}}^{(2)} \quad \text{in } K_{\text{por}} \quad (\text{C.17}) \]

and the prescribed tractions on the external boundary; that is:
\[ t_{\text{free}}^{(2)} := T_{\text{free}}^{(2)} n_{\text{free}} = t_{\text{free}}^p \quad \text{on } \Gamma_{\text{free}}^T \quad (\text{C.18}) \]
\[ t_{\text{por}}^{(2)} := T_{\text{por}}^{(2)} n_{\text{por}} = t_{\text{por}}^p \quad \text{on } \Gamma_{\text{por}}^T \quad (\text{C.19}) \]

Using equations (C.12)–(C.17) and the interface conditions (4.16b)–(4.16d), equation (C.10) reduces to the following:
\[ \frac{1}{2} \Phi_{\text{free}} \left[ v_{\text{free}}^{(1)} - v_{\text{free}}^{(2)} \right] + \frac{1}{2} \Phi_{\text{por}} \left[ v_{\text{por}}^{(1)} - v_{\text{por}}^{(2)} \right] + \int_{\Gamma_{\text{int}}} \left( \Psi \left[ s^{(1)} v_{\text{free}}, s^{(1)} v_{\text{por}}, v_{\text{n}}^{(1)} \right] - \Psi \left[ s^{(2)} v_{\text{free}}, s^{(2)} v_{\text{por}}, v_{\text{n}}^{(2)} \right] \right) d\Gamma \]
\[ = \int_{\Gamma_{\text{int}}} \left( \frac{\partial \Psi}{\partial v_{\text{free}}} \left[ s^{(2)} v_{\text{free}} - s^{(1)} v_{\text{free}} \right] + \frac{\partial \Psi}{\partial v_{\text{por}}} \left[ s^{(1)} v_{\text{por}} - s^{(2)} v_{\text{por}} \right] + \frac{\partial \Psi}{\partial v_{\text{n}}} \left[ v_{\text{n}}^{(1)} - v_{\text{n}}^{(2)} \right] \right) d\Gamma \quad (\text{C.20}) \]

Noting the functional form of \( \Psi \), the above equation reduces to the following:
\[ \frac{1}{2} \Psi_{\text{free}} \left[ v_{\text{free}}^{(1)} - v_{\text{free}}^{(2)} \right] + \frac{1}{2} \Psi_{\text{por}} \left[ v_{\text{por}}^{(1)} - v_{\text{por}}^{(2)} \right] + \int_{\Gamma_{\text{int}}} \Psi \left[ s^{(1)} v_{\text{free}} - s^{(2)} v_{\text{free}}, s^{(1)} v_{\text{por}} - s^{(2)} v_{\text{por}}, v_{\text{n}}^{(1)} - v_{\text{n}}^{(2)} \right] d\Gamma = 0 \quad (\text{C.21}) \]

Using the fact that \( \Phi_{\text{free}}[\cdot] \), \( \Phi_{\text{por}}[\cdot] \) and \( \Psi[\cdot] \) are individually norms (and hence individually nonnegative), each term in the above equation is individually zero. This further implies that
\[ v_{\text{free}}^{(1)}(x) = v_{\text{free}}^{(2)}(x) \quad \forall x \in K_{\text{free}} \quad (\text{C.22a}) \]
\( v^{(1)}_{\text{por}}(x) = v^{(2)}_{\text{por}}(x) \quad \forall x \in \mathcal{K}_{\text{por}} \)  
(C.22b)

\( \ast^{(1)} v^{(1)}_{\text{free}} = \ast^{(2)} v^{(2)}_{\text{free}} \quad \forall x \in \Gamma_{\text{free}} \)  
(C.22c)

\( v^{(1)}_{\text{por}} = v^{(2)}_{\text{por}} \quad \forall x \in \Gamma_{\text{por}} \)  
(C.22d)

\( v^{(1)}_{n}(x) = v^{(2)}_{n}(x) \quad \forall x \in \Gamma_{\text{int}} \)  
(C.22e)

The balance of linear momentum in \( \mathcal{K}_{\text{free}} \) and \( \mathcal{K}_{\text{por}} \), respectively, implies that:

\[
\text{grad}\left[ p^{(1)}_{\text{free}}(x) - p^{(2)}_{\text{free}}(x) \right] = 0 \quad \forall x \in \mathcal{K}_{\text{free}}
\]  
(C.23a)

\[
\text{grad}\left[ p^{(1)}_{\text{por}}(x) - p^{(2)}_{\text{por}}(x) \right] = 0 \quad \forall x \in \mathcal{K}_{\text{por}}
\]  
(C.23b)

which further implies that:

\[
p^{(1)}_{\text{free}}(x) = p^{(2)}_{\text{free}}(x) + C_1 \quad \forall x \in \mathcal{K}_{\text{free}} \quad \text{and} \quad p^{(1)}_{\text{por}}(x) = p^{(2)}_{\text{por}}(x) + C_2 \quad \forall x \in \mathcal{K}_{\text{por}}
\]  
(C.24)

where \( C_1 \) and \( C_2 \) are arbitrary constants. Using the interface condition given by equation (4.16b) and noting that the velocity fields are continuous fields, we conclude that \( C_1 = C_2 = C \) and

\[
p^{(1)}_{\text{free}}(x) = p^{(2)}_{\text{free}}(x) + C \quad \text{and} \quad p^{(1)}_{\text{por}}(x) = p^{(2)}_{\text{por}}(x) + C \quad \forall x \in \Gamma_{\text{int}}
\]  
(C.25)

Physically, the constant \( C \) fixes the datum for the pressure field. This completes the proof.