Minimum Reload Cost Cycle Cover in Complete Graphs

Yasemin Büyükçolak\textsuperscript{a}, Didem Gözüpek\textsuperscript{b,∗}, Sibel Özkan\textsuperscript{a}

\textsuperscript{a}Department of Mathematics, Gebze Technical University, Kocaeli, Turkey
\textsuperscript{b}Department of Computer Engineering, Gebze Technical University, Kocaeli, Turkey

Abstract

The reload cost refers to the cost that occurs along a path on an edge-colored graph when it traverses an internal vertex between two edges of different colors. Galbiati et al. \cite{1} introduced the \textit{Minimum Reload Cost Cycle Cover} problem, which is to find a set of vertex-disjoint cycles spanning all vertices with minimum reload cost. They proved that this problem is strongly \textbf{NP}-hard and not approximable within $1/\epsilon$ for any $\epsilon > 0$ even when the number of colors is 2, the reload costs are symmetric and satisfy the triangle inequality. In this paper, we study this problem in complete graphs having equitable or nearly equitable 2-edge-colorings, which are edge-colorings with two colors such that for each vertex $v \in V(G)$, $||c_1(v)| - |c_2(v)|| \leq 1$ or $||c_1(v)| - |c_2(v)|| \leq 2$, respectively, where $c_i(v)$ is the set of edges with color $i$ that is incident to $v$. We prove that except possibly on complete graphs with fewer than 13 vertices, the minimum reload cost is zero on complete graphs with nearly equitable 2-edge-colorings by proving the existence of a monochromatic cycle cover. Furthermore, we provide a polynomial-time algorithm that constructs a monochromatic cycle cover in complete graphs with an equitable 2-edge-coloring except possibly in a complete graph with four vertices. Our algorithm also finds a monochromatic cycle cover in complete graphs with a nearly equitable 2-edge-coloring except some special cases.

\textit{Keywords:} Reload cost, minimum reload cost cycle cover, complete graph, equitable edge-coloring, nearly equitable edge-coloring, monochromatic cycle cover.

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\textsuperscript{∗}Corresponding author.

\textit{Email addresses:} y.buyukcolak@gtu.edu.tr (Yasemin Büyükçolak), didem.gozupek@gtu.edu.tr (Didem Gözüpek), s.ozkan@gtu.edu.tr (Sibel Özkan)
1. Introduction

Edge-colored graphs can be used to model various network design problems. In this work, we consider an optimization problem with the *reload cost* model. The reload cost occurs along a path on an edge-colored graph while traversing through an internal vertex via two consecutive edges of different colors. That is, the reload cost depends only on the colors of the incident traversed edges. In addition, the reload cost of a path or a cycle on an edge-colored graph is the sum of the reload costs that arise from traversing its internal vertices between edges of different colors. Because of practical reasons, it is generally assumed that the reload costs are symmetric and satisfy the triangle inequality. The reload cost concept is used in many areas such as transportation networks, telecommunication networks, and energy distribution networks. For instance, in a cargo transportation network, each carrier can be represented by a color and the reload costs arise only at points where the carrier changes, i.e., during transition from one color to another. In telecommunication networks, the reload costs arise in several settings. For instance, switching among different technologies such as cables, fibers, and satellite links or switching between different providers such as different commercial satellite providers in satellite networks correspond to reload costs. In energy distribution networks, the reload cost corresponds to the loss of energy while transferring energy from one form to another, such as the conversion of natural gas from liquid to gas form.

Although the reload cost concept has significant applications in many areas, only few papers about this concept have appeared in the literature. Wirth and Steffan [2] and Galbiati [3] studied the minimum reload cost diameter problem, which is to find a spanning tree with minimum diameter with respect to reload cost. Amaldi et al. [4] presented several path, tour, and flow problems under the reload cost model. They also focused on the problem of finding a spanning tree that minimizes the total reload cost from a source vertex to all other vertices. The works in [5, 6, 7, 8] focused on the minimum changeover cost arborescence problem, which is to find an arborescence rooted at a given vertex such that the total reload cost is minimized. The work in [9], on the other hand, focused on problems related to finding a proper edge coloring of the graph so that the total reload cost is minimized.

Galbiati et al. [1] introduced *Minimum Reload Cost Cycle Cover* (MinRC3) problem, which is to find a set of vertex-disjoint cycles spanning all vertices with minimum reload cost. They proved that it is strongly NP-hard and not approximable within $1/\varepsilon$ for any $\varepsilon > 0$ even when the number of colors is 2, the reload costs are symmetric and satisfy the
triangle inequality. To the best of our knowledge, MinRC3 problem has previously not been investigated for special graph classes.

In this work we focus on a special case of MinRC3 problem, which is MinRC3 in complete graphs. Our motivation for working on complete graphs is to avoid the feasibility issue, since a complete graph of any order has a cycle cover. We show that MinRC3 is strongly NP-hard and is not approximable within $1/\varepsilon$ for any $\varepsilon > 0$ for complete graphs, even when the reload costs are symmetric. We are then interested in MinRC3 problem in complete graphs having equitable 2-edge-coloring, which is an edge-coloring with two colors such that for each vertex $v \in V(G)$, $||c_1(v)| - |c_2(v)|| \leq 1$, where $c_i(v)$ is the set of edges with color $i$ that is incident to $v$. To the best of our knowledge, this paper is the first one focusing on the MinRC3 problem in a special graph class. In particular, we present the first positive (polynomial-time solvability) result for this problem.

Feasibility of equitable edge-colorings received some attention in the literature. In 2008, Xie et al. [10] showed that the problem of finding whether an equitable $k$-edge-coloring exists is NP-complete in general. Indeed, if $k = \Delta$, where $\Delta$ is the maximum degree of the given graph, then this problem becomes equivalent to the well-known NP-complete problem of classifying Class-1 graphs. In 1994, Hilton and de Werra [11] proved the following sufficiency condition on equitable $k$-edge-colorings: if $k \geq 2$ and $G$ is a simple graph such that no vertex in $G$ has degree equal to a multiple of $k$, then $G$ has an equitable $k$-edge-coloring. In 1971, de Werra [12] found the following necessary and sufficient condition to have an equitable 2-edge-coloring in a connected graph: a connected graph $G$ has an equitable 2-edge-coloring if and only if it is not a connected graph with odd number of edges and all vertices having an even degree. Furthermore, a nearly equitable $k$-edge-coloring is an edge-coloring with $k$ colors such that for each vertex $v \in V(G)$ and for each pair of colors $i, j \in \{1, 2, ..., k\}$, $||c_i(v)| - |c_j(v)|| \leq 2$, where $c_i(v)$ is the set of edges with color $i$ that is incident to $v$. The notion of nearly equitable edge-coloring was introduced in 1982 by Hilton and de Werra [13], who also proved that for each $k \geq 2$ any graph has a nearly equitable $k$-edge-coloring.

In this paper, we focus on the MinRC3 problem in complete graphs having equitable or nearly equitable 2-edge-coloring. Recall that the reload cost is zero between two edges of the same color. The reload cost of a monochromatic cycle, i.e. a cycle having all edges with the same color, is clearly zero. We then investigate the existence of a monochromatic cycle cover in complete graphs having equitable or nearly equitable 2-edge-coloring. In the literature, there exist various results about covering a $k$-edge-colored graph with monochromatic subgraphs such as cycles, paths, and trees (see [14, 15]). In 1983, Gyárfás
proved that the vertex set of any 2-edge-colored complete graph can be covered by two monochromatic cycles that have different colors and intersect in at most one vertex. In 2010, Bessy and Thomassé proved that the vertex set of any 2-edge-colored complete graph can be partitioned into two monochromatic cycles having different colors, i.e. it has a vertex-disjoint monochromatic cycle cover with two different colors. However, unlike in the MINRC3 problem, a vertex \((K_1)\) and an edge \((K_2)\) are considered to be cycles in almost all works in the literature (including 16) about monochromatic cycle covers. Clearly, both a vertex and an edge have zero reload cost, yet in this paper, we do not allow cycles to have less than three vertices.

We prove in this paper that except possibly in a complete graph with fewer than 13 vertices, in a complete graph with a nearly equitable 2-edge-coloring, there exists a cycle cover that is either a monochromatic Hamiltonian cycle or consists of exactly two monochromatic cycles both having the same color and whose sizes differ by at most one; therefore, the value of the minimum reload cost cycle cover is zero in such a case. In addition, we show that except possibly in a complete graph with 4 vertices, there exists a monochromatic cycle cover in complete graphs with equitable edge coloring. Our constructive proof leads to a polynomial time algorithm to solve the MINRC3 problem on complete graphs with an equitable 2-edge-coloring. Our proof also leads to a polynomial time algorithm to solve this problem on complete graphs with nearly equitable 2-edge coloring except some special cases.

2. Preliminaries

An undirected graph \(G = (V(G), E(G))\) is given by a pair of a vertex set \(V(G)\) and an edge set \(E(G)\), which consists of 2-element subsets \(\{u, v\}\) of \(V(G)\). An edge \(\{u, v\}\) between two vertices \(u\) and \(v\) is denoted by \(uv\) in short. In this work, we consider only simple graphs, i.e., graphs without loops or multiple edges. The order of \(G\) is denoted by \(|V(G)|\) and the degree of a vertex \(v\) of \(G\) is denoted by \(d(v)\). In addition, \(\delta(G)\) and \(\Delta(G)\) denote the minimum and maximum degree of \(G\), respectively. When the graph \(G\) is clear from the context, we omit it from the notations and write \(V, E, \delta\) and \(\Delta\).

Given two graphs \(G = (V, E)\) and \(G' = (V', E')\), if \(G\) is isomorphic to \(G'\), we denote it by \(G \cong G'\). We define the union, \(G \cup G'\), of \(G\) and \(G'\) as the graph obtained by the union of their vertex and edge sets, i.e., \(G \cup G' = (V \cup V', E \cup E')\). When \(V\) and \(V'\) are disjoint, their union is referred to as the disjoint union and denoted by \(G + G'\). The join \(G \vee G'\) of \(G\) and \(G'\) is the disjoint union of graphs \(G\) and \(G'\) together with all the edges
joining \( V \) and \( V' \). Formally, \( G \lor G' = (V \cup V', E \cup E' \cup \{V \times V'\}) \). The complement of a graph \( G = (V, E) \) is the graph \( \bar{G} \) with the same vertex set \( V \) but whose edge set \( E' \) consists of 2-element subsets of \( V \) that are not in \( E \). That is, \( E \cap E' = \emptyset \) and \( E \cup E' \) contains all possible edges on the vertex set of \( G \).

A cycle on \( n \) vertices is denoted by \( C_n \). A cycle cover of a graph \( G \) is a collection of cycles such that every vertex in \( G \) is contained in at least one such cycle. If the cycles of the cover have no vertices in common, the cover is called vertex-disjoint. Unless otherwise stated, the cycle covers are always assumed to be vertex-disjoint in this work. A graph is \( r \)-regular if all of its vertices have degree \( r \). We say that a graph \( H \) is an \( r \)-factor of a graph \( G \) when \( V(H) = V(G) \) and \( H \) is \( r \)-regular. Notice here that a cycle cover of a graph \( G \) is equivalent to a 2-factor of \( G \).

A complete graph on \( n \) vertices is denoted by \( K_n \). The closure of a graph \( G \) with \( n \) vertices, denoted by \( cl(G) \), is the graph obtained from \( G \) by repeatedly adding edges between non-adjacent vertices whose degrees sum to at least \( n \), until no such vertices exist. The degree sequence of a graph \( G \) is the non-decreasing sequence of its vertex degrees.

An independent set in a graph \( G \) is a subset of pairwise nonadjacent vertices in vertex set \( V(G) \). A maximum independent set is an independent set of largest size for a given graph \( G \). The size of a maximum independent set is called the independence number of \( G \) and is denoted by \( \alpha(G) \). Besides, a complete subgraph of a given graph \( G \) is called a clique of \( G \).

A Hamiltonian cycle of a graph \( G \) is a cycle passing through every vertex of \( G \) exactly once, and a graph \( G \) containing a Hamiltonian cycle is called Hamiltonian. Some fundamental results on hamiltonicity used in this paper are as follows:

**Theorem 1. (Dirac [17])** If \( G \) is a graph of order \( n \geq 3 \) such that \( \delta(G) \geq n/2 \), then \( G \) is Hamiltonian.

**Theorem 2. (Büyükçolak et al. [18])** Let \( G \) be a connected graph of order \( n \geq 3 \) such that \( \delta(G) \geq \lfloor n/2 \rfloor \). Then \( G \) is Hamiltonian unless \( G \) is a graph \( K_{\lfloor n/2 \rfloor} \cup K_{\lceil n/2 \rceil} \) with one common vertex or a graph \( K_{\lceil n/2 \rceil} \lor G_{\lfloor n/2 \rfloor} \) for odd \( n \), where \( G_n \) is a not necessarily connected simple graph on \( n \) vertices.

**Theorem 3. (Bondy-Chvátal [19])** A graph \( G \) is Hamiltonian if and only if its closure \( cl(G) \) is Hamiltonian.

**Theorem 4. (Chvátal [20])** Let \( G \) be a simple graph with degree sequence \((d_1, d_2, \ldots, d_n)\),
where \( d_1 \leq d_2 \leq \ldots \leq d_n \). If there is no \( m < n/2 \) such that \( d_m \leq m \) and \( d_{n-m} < n - m \), then \( cl(G) \) is a complete graph and therefore \( G \) is Hamiltonian.

**Theorem 5.** (Nash-Williams [21]) Let \( G \) be 2-connected graph of order \( n \) with \( \delta \geq \max\{(n + 2)/3, \alpha(G)\} \). Then \( G \) is Hamiltonian.

**Theorem 6.** (Moon-Moser [22]) Let \( G \) be a bipartite graph with two disjoint vertex sets \( V_1 \) and \( V_2 \) such that \( |V_1| = |V_2| = m \). If \( \min\{d(u) + d(v)\} u \in V_1, v \in V_2, u \text{ and } v \text{ are nonadjacent} \geq m + 1 \), then \( G \) is Hamiltonian.

A \( k \)-edge-coloring of a graph \( G \) is an assignment of \( k \) colors to edges of \( G \), which is represented by a mapping \( \chi: E(G) \rightarrow C \), where \( C = \{c_1, c_2, \ldots, c_k\} \) is a set of \( k \) colors. Given a \( k \)-edge-coloring of \( G \) with \( k \) colors \( c_1, \ldots, c_k \), \( c_i(v) \) denotes the set of edges incident to \( v \) colored with \( c_i \) for \( v \in V(G) \), where \( 1 \leq i \leq k \). Given a \( k \)-edge-coloring of \( G \), for each color \( c_i \), the graph \( G(c_i) \) denotes the subgraph of \( G \) induced by all edges of \( G \) colored \( c_i \). A \( k \)-edge-coloring of \( G \) is said to be equitable if \( ||c_i(v)\| - ||c_j(v)\| \leq 1 \) for each vertex \( v \in V(G) \) and for each pair of colors \( i, j \in \mathbb{Z}_k \). \( G \) is said to be nearly equitable if \( ||c_i(v)\| - ||c_j(v)\| \leq 2 \) for each vertex \( v \in V(G) \) and for each pair of colors \( i, j \in \mathbb{Z}_k \).

A reload cost function is a function \( \rho: C \times C \rightarrow \mathbb{N}_0 \) such that for all pairs of colors \( c_1, c_2 \in C \),

1. \( c_1 = c_2 \Rightarrow \rho(c_1, c_2) = 0 \),
2. \( c_1 \neq c_2 \Rightarrow \rho(c_1, c_2) > 0 \).

The reload cost incurs at a vertex while traversing two consecutive edges of different colors and \( \rho(e_1, e_2) = \rho(\chi(e_1), \chi(e_2)) \) by definition where \( e_1 \) and \( e_2 \) are incident edges. The reload cost of a path is the sum of the reload costs that occur at its internal vertices, i.e. \( \rho(P) = \sum_{i=2}^{n} \rho(e_{i-1}, e_i) \), where \( P = (e_1 - e_2 - \ldots - e_n) \) is a path of length \( n - 1 \). The reload cost of a cycle is \( \rho(C) = \rho(e_1, e_n) + \sum_{i=2}^{n} \rho(e_{i-1}, e_i) \), where \( C \) is a cycle consisting of edges \( e_1, e_2, \ldots, e_n \) in this cyclic order. Note that a monochromatic path or cycle, i.e. a path or cycle having all edges with the same color, clearly has zero reload cost. Besides, the reload cost of a cycle cover is the sum of the reload costs of each cycle component of the cycle cover, i.e. \( \rho(C) = \sum_{i=1}^{n} \rho(C_i) \), where \( C = C_1 + C_2 + \ldots + C_n \).

The **Minimum Reload Cost Cycle Cover** (MINRC3) problem is an optimization problem which aims to span all vertices of an edge-colored graph by a set of vertex-disjoint cycles with minimum reload cost. Formally,
MinRC3 \((G, C, \chi, \rho)\)

**Input:** A graph \(G = (V, E)\) with an edge coloring function \(\chi : E \rightarrow C\) and a reload cost function \(\rho : C \times C \rightarrow \mathbb{N}_0\).

**Output:** A cycle cover \(C\) of \(G\).

**Objective:** Minimize \(\rho(C)\).

The previous results on MinRC3 are as follows:

**Theorem 7.** (Galbiati et al. [1]) MinRC3 is strongly NP-hard even if the number of colors is 2, the reload costs are symmetric, and satisfy the triangle inequality.

**Corollary 8.** (Galbiati et al. [1]) MinRC3 is not approximable within \(1/\epsilon\), for any \(\epsilon > 0\), even if the number of colors is 2, the reload costs are symmetric, and satisfy the triangle inequality.

A monochromatic cycle cover is composed of cycles such that the colors of the edges of a particular cycle are the same; however, the colors of edges in different cycles may differ in general. In this work, we investigate the MinRC3 problem in equitably or nearly equitably 2-edge-colored complete graphs and prove that the minimum reload cost is zero in such graphs by constructing a monochromatic cycle cover in a single color.

**3. MinRC3 in Complete Graphs**

By Theorem 7 and Corollary 8, we already know that MinRC3 problem is in general strongly NP-hard and not approximable within \(1/\epsilon\) for any \(\epsilon > 0\), even when the number of colors is 2. In the following theorem, we prove a hardness result for complete graphs:

**Theorem 9.** The MinRC3 problem is strongly NP-hard and not approximable within \(1/\epsilon\) for any \(\epsilon > 0\) for complete graphs even if the reload costs are symmetric.

**Proof.** The proof is by reduction from the problem itself. Given an instance \(I\) of MinRC3 \((G, C, \chi, \rho)\), where \(G = (V(G), E(G))\), we construct an instance \(I'\) of MinRC3 \((G', C', \chi', \rho')\) as follows: \(G' = (V(G), E(G'))\) is a complete graph such that \(E(G') = E(G) \cup \bigcup_{uv \in E(G)} uv, C' = C \cup \bigcup_{uv \in E(G)} \chi(\text{uv}),\) and for all \(c \in C'\) and \(uv \notin E(G)\), we set \(\rho(\chi(\text{uv}), c) = \rho(c, \chi(\text{uv})) = M\), where \(M\) is a very large integer. In other words, for every \(uv \notin E(G)\), \(\chi(\text{uv})\) is a new color in \(G'\) having a very large reload cost value with all other colors in \(C'\). This reduction shows that \(I\) is a satisfiable instance of MinRC3 if and only if \(I'\) is a satisfiable instance of MinRC3 in complete graphs. Furthermore, in the case
where \( I \) is a satisfiable instance, we clearly have \( \text{OPT}(G) = \text{OPT}(G') \), where \( \text{OPT}(G) \) and \( \text{OPT}(G') \) denote the reload cost of an optimum solution of \( G \) and \( G' \), respectively. Let \( A' \) be a \( 1/\epsilon \) approximation algorithm for \( \text{MinRC3} \) problem in complete graphs for some \( \epsilon > 0 \). Then since \( A'(I) \leq (\text{OPT}(I')/\epsilon) = \text{OPT}(I)/\epsilon \), \( I' \) is also a \( 1/\epsilon \) approximation algorithm for \( \text{MinRC3} \) problem in general, contradicting Corollary 8. Hence, the theorem holds.

Having proved that \( \text{MinRC3} \) is in general inapproximable within \( 1/\epsilon \) for any \( \epsilon > 0 \) in complete graphs, now we investigate \( \text{MinRC3} \) in complete graphs with equitable and nearly equitable 2-edge-colorings.

3.1. Complete Graphs with Equitable 2-Edge-Coloring

Since a monochromatic cycle cover has zero reload cost, it is sufficient to show that there exists a partition of vertices of a complete graph having an equitable 2-edge-coloring into monochromatic vertex-disjoint cycles.

The following lemma given in [12] implies the existence of an equitable 2-edge-coloring in complete graphs except \( K_{4k+3} \):

**Lemma 10.** [12] A connected graph \( G \) has an equitable 2-edge-coloring if and only if it is not a connected graph with odd number of edges and all vertices having an even degree.

**Corollary 11.** A complete graph has an equitable 2-edge-coloring if and only if it is not a complete graph \( K_{4k+3} \) with \( k \geq 0 \).

Now, we analyze the cases of \( K_n \) for even \( n \) and odd \( n \) separately. For odd \( n \), by Corollary 11 it suffices to examine complete graphs with order \( n = 4k+1 \). The following lemma shows that a complete graph \( K_{4k+1} \) having an equitable 2-edge-coloring has a monochromatic Hamiltonian cycle for both colors:

**Lemma 12.** For a complete graph \( K_{4k+1} \), where \( k \geq 1 \), with an equitable 2-edge-coloring, there exists a monochromatic cycle cover of the form \( C_{4k+1} \) for both colors; in particular, there exist monochromatic Hamiltonian cycles for both colors.

**Proof.** Let \( \chi \) be an equitable 2-edge-coloring in the complete graph \( K_{4k+1} \), \( k \geq 1 \), with colors, say red and blue. In an equitable 2-edge-coloring of \( K_{4k+1} \), each vertex is incident to \( 2k \) red edges and \( 2k \) blue edges.

Let \( K_{4k+1}^r \) and \( K_{4k+1}^b \) denote the red and blue subgraphs, respectively. Both \( K_{4k+1}^r \) and \( K_{4k+1}^b \) are regular graphs on \( 4k+1 \) vertices. Note that a \( 2k \)-regular graph on \( 4k+1 \) vertices cannot be disconnected because otherwise
each component has to have at least $2k + 1$ vertices, contradicting with the order $4k + 1$. Both $K_{4k+1}^r$ and $K_{4k+1}^b$ are connected $2k$-regular graphs on $4k + 1$ vertices. Hence, they have Hamiltonian cycles by Theorem 2 since they are neither $K_{2k+1} \cup K_{2k+1}$ with one common vertex nor $\overline{K}_{2k+1} \lor G_k$. Therefore, there exists a monochromatic cycle cover of the form $C_{4k+1}$ in both colors.

For even $n \geq 6$, the following lemma shows that a complete graph $K_n$ having an equitable 2-edge-coloring has a monochromatic cycle cover with at most two cycles having the same size and the same color:

**Lemma 13.** For a complete graph $K_{2k}$, $k \geq 3$, with an equitable 2-edge-coloring, there exists a monochromatic cycle cover in a single color of the form $C_k + C_k$ or $C_{2k}$. In particular, there exists a cycle cover $C_k + C_k$ in a single color if $K_{2k}$ has a disconnected or 1-connected color induced subgraph, and a Hamiltonian cycle $C_{2k}$ otherwise.

**Proof.** Let $\chi$ be an equitable 2-edge-coloring of the complete graph $K_{2k}$ with red and blue. In an equitable 2-edge-coloring, each vertex is incident to either $k$ red edges and $k - 1$ blue edges or $k$ blue edges and $k - 1$ red edges. We consider the subgraphs $K_{2k}^r$ and $K_{2k}^b$ in $K_{2k}$ induced by red and blue, respectively. For both of them, the minimum degree is at least $k - 1$ and the maximum degree is at most $k$; i.e. $\delta \geq k - 1$ and $\Delta \leq k$.

Note that the only disconnected graph on $2k$ vertices with $\delta \geq k - 1$ is the disjoint union of two $K_k$, i.e. $K_k + K_k$, which is a $(k-1)$-regular graph. Let $K_{2k}^r$ be a disconnected subgraph induced by red on $2k$ vertices with $\delta \geq k - 1$; i.e. $K_{2k}^r \cong K_k + K_k$. Since both components of $K_{2k}^r$ are complete graphs, $K_{2k}^r$ has a cycle cover of the form $C_k + C_k$. Hence, $K_{2k}^b$ is a complete bipartite graph $K_{k,k}$ since it is the complement of $K_k + K_k$.

Clearly, $K_{k,k}$ has a Hamiltonian cycle by Theorem 1. In this case, we therefore have a monochromatic cycle cover of the form $C_{2k}$ in both induced subgraphs $K_{2k}^r$ and $K_{2k}^b$.

We now suppose that both $K_{2k}^r$ and $K_{2k}^b$ are connected graphs with $\delta \geq k - 1$ and $\Delta \leq k$. Assume that both $K_{2k}^r$ and $K_{2k}^b$ are regular graphs, i.e. $k - 1 \leq \delta = \Delta \leq k$ for both graphs. Hence, one of the graphs is a $k$-regular graph on $2k$ vertices, whereas the other is a $(k-1)$-regular graph on $2k$ vertices. By Theorem 1 there is a Hamiltonian cycle on the $k$-regular graph on $2k$ vertices. Therefore, in the case where color induced subgraphs are regular, we have a monochromatic cycle cover of the form $C_{2k}$ in the $k$-regular subgraph induced by one of the colors.

We then consider the case where both $K_{2k}^r$ and $K_{2k}^b$ are connected and are not regular graphs. In other words, both of them have $\delta = k - 1$ and $\Delta = k$. Let the degree sequences of $K_{2k}^r$ and $K_{2k}^b$ be $(r_1, r_2, ..., r_{2k})$ and $(b_1, b_2, ..., b_{2k})$, respectively, where $r_1 \leq r_2 \leq ... \leq
vertices, i.e. vertices of degree $k$ of each other. Without loss of generality (w.l.o.g.), assume that at least half of the vertices have degree $k$. Let us consider the degrees of the remaining vertices, i.e. vertices of degree $k - 1$, in $K_{2k}^r$:

1. Assume that the number of vertices having degree $k - 1$ is less than $k - 1$, i.e. $r_{k-1} = k$. Then the closure of $K_{2k}^r$ is a complete graph; therefore, by Theorem 3, $K_{2k}^r$ has a Hamiltonian cycle.

2. Assume that the number of vertices having degree $k - 1$ is at least $k - 1$, i.e. $r_{k-1} = k - 1$. If there exists a pair of non-adjacent vertices $u$ and $v$ having both degree $k$, then the closure of $K_{2k}^r$ must contain the edge $uv$ by definition. The degrees of $u$ and $v$ become $k + 1$ and then they must be adjacent to all other vertices in the closure of $K_{2k}^r$. Iteratively adding edges between non-adjacent vertices whose degrees sum to at least $n$, we obtain the complete graph $K_{2k}$ as the closure of $K_{2k}^r$. Then, $K_{2k}^r$ has a Hamiltonian cycle by Theorem 3. Otherwise, i.e. there is no pair of non-adjacent vertices having both degree $k$, then all vertices having degree $k$ are adjacent to each other. It follows that the vertices of degree $k$ form a clique of size at least $k$ and at most $k + 1$ in $K_{2k}^r$. If the vertices of degree $k$ form a clique of size $k + 1$ in $K_{2k}^r$, then it contradicts with the fact that $K_{2k}^r$ is a connected graph. Hence, we deduce that there are $k$ vertices having degree $k$ in $K_{2k}^r$ and these vertices form a clique of size $k$ in $K_{2k}$. Thus, all vertices having degree $k - 1$ in $K_{2k}^r$ form an independent set of size $k$ in $K_{2k}^b$. Besides, $k$ must be even in order to satisfy the relation $2E(K_{2k}^r) = \sum_{i=1}^n d(v_i)$ for $v_i \in V(K_{2k}^r)$.

Claim 13.1. For a graph $G$ on $n$ vertices with $\delta \leq n/2$ and $\Delta \leq n - \delta$, the independence number $\alpha$ of $G$ satisfies the inequality $\alpha \leq n - \delta$ and the equality holds only for the complete bipartite graph $K_{n-\delta,\delta}$ with $\Delta = n - \delta$ and $\alpha = n - \delta$.

Proof of Claim 13.1. Assume to the contrary that $I$ is an independent set of $G$ with size greater than $n - \delta$ and let $J$ be the remaining vertices in $G$, i.e. $|I| > n - \delta$ and $|J| < \delta$. Each vertex in $I$ can be adjacent only to the vertices in $J$ since $I$ is an independent set in $G$. However, then each vertex in $I$ has degree less than $\delta$ in $G$, which is a contradiction since $\delta$ is the minimum degree. When $|I| = n - \delta$ and $|J| = \delta$, each vertex in $I$ must be adjacent to every vertex in $J$ to attain the
Corollary 14. Except possibly $K_4$, a complete graph $K_n$ with an equitable 2-edge-coloring has a monochromatic cycle cover in a single color with at most two cycles. In particular, such a graph contains two cycles of the same size and the same color, or a monochromatic Hamiltonian cycle.
Therefore, we obtain the first main theorem of this section as follows:

**Theorem 15.** Except possibly $K_4$, the solution of the MinRC3 problem equals zero for complete graphs having an equitable 2-edge-coloring.

**Remark 16.** In $K_4$ with an equitable 2-edge-coloring, the only case where the solution of the MinRC3 problem is nonzero is when both colors induce a path on three vertices.

### 3.2. Complete Graphs with Nearly Equitable 2-Edge-Coloring

We now analyze MinRC3 problem in complete graphs having a nearly equitable 2-edge-coloring, which is an edge-coloring with two colors such that for each vertex $v \in V(G)$ $||c_1(v)| - |c_2(v)|| \leq 2$, where $c_i(v)$ is the set of edges with color $i$ that is incident to $v$. Clearly, every equitable 2-edge-coloring is indeed a nearly equitable 2-edge-coloring. Therefore, it is sufficient to study MinRC3 problem in complete graphs having a nearly equitable 2-edge-coloring that is not an equitable 2-edge-coloring.

By Lemma 11, we see that the complete graph $K_{4k+3}$ cannot have an equitable 2-edge-coloring. On the other hand, in 1982 Hilton and de Werra proved the following:

**Lemma 17.**[13] Any graph $G$ has a nearly equitable edge-coloring with $r$ colors, where $r \geq 2$.

In this subsection, we show that there exists a partition of vertices of complete graphs having a nearly equitable 2-edge-coloring into monochromatic cycles with a single color.

In the following lemma, we prove that a complete graph $K_{2k}$, where $k \geq 2$, cannot have a nearly equitable 2-edge-coloring which is not an equitable 2-edge-coloring.

**Lemma 18.** In a complete graph $K_{2k}$, where $k \geq 2$, any nearly equitable 2-edge-coloring is indeed an equitable 2-edge-coloring.

**Proof.** Assume that in the complete graph $K_{2k}$ we have a nearly equitable 2-edge-coloring $\chi$ that is not equitable; that is, there exist a vertex $v$ such that $||\chi(v)| - |\beta(v)|| = 2$, implying that the degree of $v$ is even. However, this contradicts with the fact that the degree of $v$ is $2k - 1$ in $K_{2k}$.

In the following lemma, we also show that a complete graph $K_{2k+1}$, where $k \geq 2$, may have nearly equitable 2-edge-colorings that are not equitable. Note here that a nearly equitable edge coloring is not possible when $k = 1$.

**Lemma 19.** A complete graph $K_{2k+1}$, where $k \geq 2$, may have a nearly equitable 2-edge-coloring that is not equitable.
Proof. By Lemma [17] and Corollary [11] we deduce that complete graphs $K_{4k+3}$ have a nearly equitable 2-edge-coloring, but do not have an equitable 2-edge-coloring.

In a complete graph $K_{4k+1}$, we can obtain a nearly equitable 2-edge-coloring that is not equitable as follows: let $\chi_2$ be a nearly equitable edge-coloring with colors, say red and blue, in $K_{4k+1}$ such that some vertices are incident to $2k-1$ red edges and $2k+1$ blue edges, while other vertices are incident to $2k$ red edges and $2k$ blue edges. It is easy to see that $\chi_2$ is a nearly equitable 2-edge-coloring which is not equitable.

Furthermore, Lemma [20] shows that for odd $n \geq 13$ any complete graph $K_n$ having a nearly equitable 2-edge-coloring has a monochromatic cycle cover with at most two cycles of sizes $\lfloor n/2 \rfloor$ and $\lceil n/2 \rceil$ with a single color.

Lemma 20. For a complete graph $K_{2k+1}$, $k \geq 6$, having a nearly equitable 2-edge-coloring, there exists a monochromatic cycle cover in a single color of the form $C_{k+1} + C_k$ or $C_{2k+1}$. In particular, there exists a cycle cover $C_{k+1} + C_k$ in a single color if $K_{2k+1}$ has a subgraph of the form $K_{k+1} + K_k$ induced by a color, and a Hamiltonian cycle $C_{2k+1}$ otherwise.

Proof. If the complete graph $K_{2k+1}$ has a nearly equitable 2-edge-coloring that is also an equitable 2-edge-coloring, then by Lemma [12] there exists a monochromatic Hamiltonian cycle $C_{2k+1}$.

Let $\chi$ be a nearly equitable 2-edge-coloring that is not an equitable 2-edge-coloring in the complete graph $K_{2k+1}$. Since all vertices of $K_{2k+1}$ have even degree $2k$, $|r(v)| - |b(v)|$ is either 0 or 2. Notice that there must be at least one vertex $v$ with $|r(v)| - |b(v)| = 2$ since $\chi$ is a nearly equitable, but not equitable, 2-edge-coloring.

We consider the subgraphs $K_{r_{2k+1}}^r$ and $K_{r_{2k+1}}^b$ induced by red and blue, respectively, in $K_{2k+1}$. For both of them, the minimum degree is at least $k-1$ and the maximum degree is at most $k+1$; i.e. $\delta \geq k-1$ and $\Delta \leq k+1$. In the case where both $K_{2k+1}^r$ and $K_{2k+1}^b$ are regular graphs, i.e. without loss of generality $(k-1)$-regular and $(k+1)$-regular graphs respectively, by Theorem [11] we have a Hamiltonian cycle in the $(k+1)$-regular graph $K_{2k+1}^r$. Therefore, we have a monochromatic cycle cover of the form $C_{2k+1}$ in this case.

We then suppose that neither $K_{2k+1}^r$ nor $K_{2k+1}^b$ are regular graphs, i.e. we have $\delta \neq \Delta$ for both subgraphs.

Case 1: Assume that $\delta = k$ for one of $K_{2k+1}^r$ and $K_{2k+1}^b$, say $K_{2k+1}^r$. If $K_{2k+1}^r$ is disconnected, then each component has to have at least $k+1$ vertices, which contradicts
with the order being $2k + 1$. Hence, such a graph has to be connected. Moreover, $K_{2k+1}^r$ has $\Delta = k + 1$ because it is not regular. Thus, $K_{2k+1}^r$ is a connected graph with $\delta = k$ and $\Delta = k + 1$ on $2k + 1$ vertices. By Theorem 2, $K_{2k+1}^r$ has a Hamiltonian cycle if it is neither the union of two complete graphs $K_{k+1}$ with one common vertex nor the join of an independent set of size $k + 1$ with any graph $G_k$ with order $k$. Since $\Delta = k + 1$, $G_k$ has to be an independent set, thus making the second exceptional graph family the complete bipartite graph $K_{k+1,k}$ in this case. We analyze these two cases separately as follows:

- $K_{2k+1}^r$ cannot be the union of two complete graphs $K_{k+1}$ with one common vertex since $\Delta(K_{2k+1}^r) = k + 1$.

- Since $K_{2k+1}$ has odd order, any cycle cover of it has to have at least one odd cycle. Since bipartite graphs do not have an odd cycle, if $K_{2k+1}^r$ is the complete bipartite graph $K_{k+1,k}$, then it cannot have a cycle cover. Now consider $K_{2k+1}^b$, i.e. the complement of $K_{k+1,k}$, which is the disjoint union of two complete graphs $K_{k+1}$ and $K_k$. In this case, there exists a monochromatic cycle cover of the form $C_{k+1} + C_k$ in $K_{2k+1}^b$.

Therefore, we have a monochromatic cycle cover of the form either $C_{2k+1}$ or $C_{k+1} + C_k$ in this case. In particular, there exists a monochromatic cycle cover $C_{k+1} + C_k$ if $K_{2k+1}$ has a subgraph $K_{k+1} + K_k$ induced by a color.

**Case 2:** Assume that $\delta = k - 1$ for both $K_{2k+1}^r$ and $K_{2k+1}^b$. Then, both of them have $\Delta = k + 1$ since they are complements of each other in $K_{2k+1}$. Thus, both $K_{2k+1}^r$ and $K_{2k+1}^b$ are connected graphs with $\delta = k - 1$ and $\Delta = k + 1$. Assume that such a graph is disconnected. Since $\delta = k - 1$, each component has at least $k$ vertices. Since $\Delta = k + 1$, at least one component has $k + 2$ vertices. Then the order of the graph has to be at least $2k + 2$, contradiction. Therefore, a graph with $\delta = k - 1$ and $\Delta = k + 1$ on $2k + 1$ vertices has to be connected.

**Case 2a:** Assume that at least one of $K_{2k+1}^r$ and $K_{2k+1}^b$ has connectivity 1; that is, there exists a cut vertex $x$ in $K_{2k+1}^r$. Then, the subgraph $K_{2k+1}^r - x$ has exactly two components since each component has to have at least $k - 1$ vertices and $k \geq 6$. Then $K_{2k+1}^r - x$ has two components $A$ and $B$ such that $k - 1 \leq |A|, |B| \leq k + 1$ and $|A| + |B| = 2k$. Hence, we have $\delta \geq k - 2$ and $\Delta \leq k$ for both $A$ and $B$. We now consider two disjoint and complementary cases:

- Assume that $|A| = k - 1$ and $|B| = k + 1$. Then $A$ is the complete graph $K_{k-1}$ whose all vertices are adjacent to $x$ in $K_{2k+1}^r$, since $\delta(K_{2k+1}^r) = k - 1$. That is, we
have a complete subgraph $K_k$ in $K_{2k+1}^r$, and hence a cycle $C_k$, that consists of all vertices of $A$ and $x$. On the other hand, $B$ is then a subgraph with $\delta \geq k - 2$ on $k + 1$ vertices. Since $k \geq 6$, by Theorem 1, $B$ has a Hamiltonian cycle $C_{k+1}$.

- Assume that $|A| = |B| = k$. Since $\delta \geq k - 2$ for both $A$ and $B$ and $k \geq 6$, both $A$ and $B$ have a Hamiltonian cycle $C_k$. Besides, since $|A| = |B| = k$, all vertices in $A$ and $B$ have degree at most $k$. Then, the only vertex having maximum degree $k + 1$ in $K_{2k+1}^r$ is the cut vertex $x$. It follows that $x$ is adjacent to at least $\lceil (k + 1)/2 \rceil$ vertices of either $A$ or $B$, say $A$. Since $x$ is adjacent to more than half of the vertices of $A$, $x$ must be adjacent to two consecutive vertices $y_1$ and $y_2$ of the Hamiltonian cycle $C_k$ in $A$. By using the path $y_1 - x - y_2$ instead of the edge $y_1 - y_2$ in $C_k$, we can construct a cycle $C_{k+1}$ covering all vertices of $A$ and $x$ in $K_{2k+1}^r$.

Therefore, in this case we have a monochromatic cycle cover of the form $C_k + C_{k+1}$.

**Case 2b:** Assume that both subgraphs $K_{2k+1}^r$ and $K_{2k+1}^b$ have connectivity at least 2. Recall that $\delta = k - 1$ and $\Delta = k + 1$. Let $\alpha^r$ and $\alpha^b$ be the independence numbers of $K_{2k+1}^r$ and $K_{2k+1}^b$, respectively. By Claim 13.1, $\alpha^r$ and $\alpha^b$ must be strictly less than $n - \delta = (2k + 1) - (k - 1) = k + 2$ because none of $K_{2k+1}^r$ and $K_{2k+1}^b$ can be the complete bipartite graph $K_{k+2,k-1}$ since $\Delta = k + 1$. We now consider the following disjoint and complementary cases:

- Assume that at least one of $\alpha^r$ and $\alpha^b$ is at most the minimum degree $\delta = k - 1$, say $\alpha^r \leq k - 1$. Then, since $k \geq 6$, we have $\delta \geq \max\{(n + 2)/3, \alpha^r\}$. Therefore, by Theorem 5, $K_{2k+1}^r$ has a Hamiltonian cycle for $k \geq 6$.

- Assume that $\alpha^r = \alpha^b = k$. Then, there exists an independent set of size $k$ and a clique of size $k$ in $K_{2k+1}^r$. Let us consider a spanning bipartite subgraph $\tilde{K}_{2k+1}^r$ with the partite set $V_1$, which is an independent set of size $k$, and the partite set $V_2$, which consists of the remaining $k + 1$ vertices. Notice that $\tilde{K}_{2k+1}^r$ is obtained by removing all edges among the vertices of $V_2$ in $K_{2k+1}^r$. Besides, all vertices except possibly one vertex of the clique of size $k$ must lie in $V_2$ since $V_1$ is an independent set in $K_{2k+1}^r$. Then, each vertex of this clique lying in $V_2$ is adjacent to at most three vertices in $V_1$. Recall that $k \geq 6$. It follows that $\tilde{K}_{2k}^r$ may have at most $3(k - 1) + 2k = 5k - 3$ edges by counting edges leaving $V_2$, and at least $k(k - 1) = k^2 - k$ edges by counting edges leaving $V_1$. Since the inequality $k^2 - k > 5k - 3$ always holds when $k \geq 6$, i.e. the minimum number of edges leaving $V_1$ is greater than the maximum number of edges leaving $V_2$ when $k \geq 6$, we have a contradiction. Therefore, we cannot have $\alpha^r = \alpha^b = k$. 

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• Assume that one of $\alpha_r$ and $\alpha_b$ is $k + 1$ and the other is $k$, say $\alpha_r = k + 1$ and $\alpha_b = k$.

Then, there exists an independent set of size $k + 1$ and a clique of size $k$ in $K_{2k+1}^r$. Let us consider a spanning bipartite subgraph $\tilde{K}_{2k+1}^r$ with the partite set $V_1$, which is an independent set of size $k + 1$, and the partite set $V_2$, which consists of the remaining $k$ vertices. Since all vertices except possibly one vertex of the clique of size $k$ must lie in $V_2$, each vertex of this clique lying in $V_2$ is adjacent to at most three vertices in $V_1$. It follows that $\tilde{K}_{2k}^r$ may have at most $3(k - 1) + (k + 1) = 4k - 2$ edges by counting edges leaving $V_2$, and at least $(k + 1)(k - 1) = k^2 - 1$ edges by counting edges leaving $V_1$. Since the inequality $k^2 - 1 > 4k - 2$ always holds when $k \geq 6$, we have a contradiction.

• Assume that $\alpha_r = \alpha_b = k + 1$. In a similar fashion to the previous cases, there exists an independent set of size $k + 1$ and a clique of size $k + 1$ in both $K_{2k+1}^r$ and $K_{2k+1}^b$. Let us consider a spanning bipartite subgraph $\tilde{K}_{2k+1}^b$ with the partite set $V_1$, which is an independent set of size $k + 1$, and the partite set $V_2$, which consists of the remaining $k$ vertices. Since all vertices except exactly one vertex of the clique of size $k + 1$ must lie in $V_2$, each vertex of this clique lying in $V_2$ is adjacent to at most two vertices in $V_1$. It follows that $\tilde{K}_{2k}^b$ may have at most $2k$ edges by counting edges leaving $V_2$. Furthermore, since at least one vertex of $V_1$ is adjacent to all vertices in $V_2$ forming a clique of size $k + 1$, $\tilde{K}_{2k}^b$ may have at least $k(k - 1) + k = k^2$ edges by counting edges leaving $V_1$. Since the inequality $k^2 > 2k$ always holds when $k \geq 3$, we have a contradiction.

Therefore, at least one of $\alpha_r$ and $\alpha_b$, i.e. the independence numbers of $K_{2k+1}^r$ and $K_{2k+1}^b$ respectively, must be less than or equal to the minimum degree $\delta = k - 1$. Hence, in this case we have a monochromatic cycle cover of the form $C_{2k+1}$ where $k \geq 6$.

By combining Lemmas 18 and 20, we obtain the following result:

**Corollary 21.** A complete graph $K_n$ with $n \geq 13$ and a nearly equitable 2-edge-coloring has a monochromatic cycle cover in a single color with at most two cycles, which have sizes $\lfloor n/2 \rfloor$ and $\lceil n/2 \rceil$.

Hence, we obtain the second main theorem of this section as follows:

**Theorem 22.** The solution of the MinNC3 problem equals zero for complete graphs with at least 13 vertices and a nearly equitable 2-edge-coloring.
4. Algorithm for MinRC3

In this section we present an algorithm MonochromaticCycleCoverAlgorithm (MCCA) that, given a complete graph \( K_n \) with a 2-edge-coloring, returns either a monochromatic cycle cover \( C \) or “NONE”. Although MCCA may in general return “NONE” for a complete graph with a 2-edge-coloring \( \chi \), we will show that it always returns a monochromatic cycle cover if \( \chi \) is an equitable 2-edge-coloring. Furthermore, except some special cases, MCCA mostly (but not always) returns a monochromatic cycle cover if \( \chi \) is a nearly equitable 2-edge-coloring.

We first consider a complete graph \( K_n \) of even order \( n \), say \( n = 2k \). By Lemma 18 any nearly equitable 2-edge-coloring is indeed an equitable 2-edge-coloring in \( K_{2k} \). Hence, the algorithm MCCA works identically for both equitable and nearly equitable 2-edge-colorings in \( K_{2k} \). We then consider the case where \( K_{2k} \) has an equitable 2-edge-coloring. Given a subgraph \( G \) of \( K_{2k} \) induced by a color, the algorithm tests \( G \) for \( \delta \geq n/2 \), which is Dirac’s sufficiency condition for hamiltonicity given in Theorem 1. Once \( G \) passes the test, the algorithm constructs a Hamiltonian cycle via the function \textsc{DirachHamiltonian}, which builds a Hamiltonian cycle by following the proof of Theorem 1. According to the proof of Lemma 13 a monochromatic cycle cover, in particular a Hamiltonian cycle, is obtained when the subgraph \( G \) induced by a color is a disconnected or a regular graph in this case. If \( G \) fails to satisfy the condition \( \delta \geq n/2 \), then the algorithm builds the closure \( G^* \) of \( G \) and tests \( G^* \) for being a complete graph. It is Bondy-Chvátal’s hamiltonicity condition given in Theorem 3. Once \( G^* \) passes the test, the algorithm construct a Hamiltonian cycle via the function \textsc{ClosureHamiltonian}, which builds a Hamiltonian cycle by following the proof of Theorem 3. Indeed, in the rest of proof of Lemma 13 we use Theorems 3, 4 and 6, which give sufficiency conditions for closure and hamiltonicity of \( G \). Hence, the function \textsc{ClosureHamiltonian} will be sufficient to construct a monochromatic cycle cover, in particular a Hamilton cycle, in order to complete the rest of this case.

We now consider a complete graph \( K_n \) of odd order \( n \), say \( n = 2k + 1 \). In this case, we first assume that \( K_{2k+1} \) has an equitable 2-edge-coloring. Indeed, since the complete graph \( K_{4k+3} \) does not have an equitable 2-edge-coloring by Corollary 11 we only need to consider a complete graph \( K_{4k+3} \) with an equitable 2-edge-coloring. Given a subgraph \( G \) of \( K_{4k+3} \) induced by a color, the algorithm tests \( G \) for \( \delta \geq \lfloor n/2 \rfloor \), which is Büyukçoılak’s sufficiency condition for hamiltonicity given in Theorem 2. Once \( G \) passes the test, the algorithm constructs a Hamilton cycle via the function \textsc{ExtensionDirachHamiltonian},
which builds a Hamiltonian cycle by following the proof of Theorem 2 given in [18]. According to the proof of Lemma 12, a monochromatic cycle cover, particularly a Hamiltonian cycle, is obtained.

Let us consider the case where a complete graph $K_{2k+1}$ has a nearly equitable 2-edge-coloring. By Lemma 19 and 20, a complete graph $K_{4k+1}$ may have equitable and nearly equitable 2-edge-colorings in different forms, whereas a complete graph $K_{4k+3}$ can only have a nearly equitable 2-edge-coloring. Therefore, if $K_{4k+1}$ has a nearly equitable 2-edge-coloring $\chi$, which is also an equitable 2-edge-coloring, then the algorithm constructs a monochromatic cycle cover, in particular a Hamiltonian cycle, by considering $\chi$ as an equitable 2-edge-coloring. We then consider the case where a complete graph $K_{2k+1}$ has a nearly equitable 2-edge-coloring that is not an equitable 2-edge-coloring. Given a subgraph $G$ of $K_{2k+1}$ induced by a color, the algorithm works in the following way:

- The algorithm tests $G$ for $\delta \geq n/2$ and constructs a Hamilton cycle via the function \textsc{DiracHamiltonian} if $G$ passes the test. According to the proof of Lemma 20 in this case a monochromatic cycle cover, in particular a Hamiltonian cycle, is obtained when the subgraph $G$ induced by a color is a regular graph.

- Otherwise, the algorithm then tests $G$ for $\delta \geq \lfloor n/2 \rfloor$, which is B"uy"ukcolak’s sufficient condition for hamiltonicity given in Theorem 2. If $G$ passes the test, then the algorithm constructs either a cycle cover $C_{k+1} + C_k$ by following the proof of Theorem 2 given in [18] or a Hamilton cycle via the function \textsc{ExtensionDiracHamiltonian}. According to the proof of Lemma 20, a monochromatic cycle cover of the form $C_{k+1} + C_k$ is obtained in the complement of $G$ when the subgraph $G$ is a complete bipartite graph $K_{k+1,k}$ in this case. Indeed, if $G = K_{k+1,k}$ then the algorithm constructs two cycles $C_1$ and $C_2$ with the vertices of degree $\lceil n/2 \rceil$ and the vertices of degree $\lfloor n/2 \rfloor$, respectively. Notice that the order of vertices in $C_1$ and $C_2$ makes no difference for Hamiltonian cycle, since these sets of vertices form two distinct complete graphs in the complement of $G$.

- Otherwise, the algorithm tests $G$ for a cut vertex $x$. If $G$ passes the test, then the algorithm constructs a cycle cover of the form $C_{k+1} + C_k$ in two different ways by using the structure of $G - x$ given in the proof of Lemma 20.

Notice that the algorithm returns “NONE” for a complete graph with a nearly equitable 2-edge-coloring $\chi$ if both subgraphs induced by the colors are 2-connected with $\delta = k - 1$ and $\Delta = k + 1$ (Case 2b in the proof of Lemma 20).
Algorithm 1 MonochromaticCycleCoverAlgorithm (MCCA)

Require: A complete graph $K_n$ of order $n$ with a 2-edge-coloring $\chi$

Ensure: $C$ is a monochromatic cycle cover of $K_n$ in a single color

1: $G_1 \leftarrow$ the subgraph of $K_n$ induced by red.
2: $G_2 \leftarrow$ the subgraph of $K_n$ induced by blue.
3: $\delta_1 \leftarrow$ the minimum degree of $G_1$.
4: $\delta_2 \leftarrow$ the minimum degree of $G_2$.
5: $\Delta_1 \leftarrow$ the maximum degree of $G_1$.
6: $\Delta_2 \leftarrow$ the maximum degree of $G_2$.
7: for $i = 1$ to 2 do
8: if $\delta_i \geq n/2$ then
9: $C \leftarrow$ DirachHamiltonian($G_i, \delta_i$)
10: if $C \neq$ NONE then return $C$.
11: $G_i^* \leftarrow$ closure of $G_i$
12: if $G_i^*$ is a complete graph then
13: $C \leftarrow$ ClosureHamiltonian($G_i, G_i^*$)
14: if $C \neq$ NONE then return $C$.
15: if $\delta_i = \lfloor n/2 \rfloor$ then
16: if $G_i$ is a complete bipartite graph $K_{\lfloor n/2 \rfloor, \lfloor n/2 \rfloor}$ then
17: $C_1 \leftarrow$ the vertices of degree $\lfloor n/2 \rfloor$
18: $C_2 \leftarrow$ the vertices of degree $\lfloor n/2 \rfloor$
19: $C \leftarrow C_1 + C_2$
20: if $|V(C)| = |V(G_i)|$ then return $C$.
21: else
22: $C \leftarrow$ ExtensionDirachHamiltonian($G_i, \delta_i$)
23: if $C \neq$ NONE then return $C$. 
if $G_i$ has a cut vertex $x$ and if $k \geq 4$ then

Let $A$ and $B$ be two components of $G_i - x$ such that $|B| \geq |A|$.

if $|B| > |A|$ then

$C_1 \leftarrow$ the vertices of $A$ and $x$

$C_2 \leftarrow$ DiracHamiltonian($B, \delta_i - 1$)

$C \leftarrow C_1 + C_2$

if $|V(C)| = |V(G_i)|$ then return $C$

else if $|B| = |A|$ then

$C_1 \leftarrow$ DiracHamiltonian($A, \delta_i - 1$)

$C_2 \leftarrow$ DiracHamiltonian($B, \delta_i - 1$)

Let $x$ be adjacent to more vertices of $A$ than $B$.

Let $P = x_0 x_1 ... x_{k-1} \leftarrow C_1$.

for $j = 0$ to $k - 1$ do

if $xx_j \in E(G_i)$ and $xx_{j+1} \in E(G_i)$ then

return $C_1 = (x_0 ... x_j xx_{j+1} ... x_{k-1})$.

$C \leftarrow C_1 + C_2$

if $|V(C)| = |V(G_i)|$ then return $C$.

return NONE

Although Dirac’s original proof for Theorem 1 was obtained by a contradiction, in [23] a polynomial-time algorithm is also presented for finding Hamiltonian cycles in graphs that satisfy the condition of Theorem 1, i.e. having at least three vertices and minimum degree at least half the total number of vertices. For the sake of completeness, in Algorithm 2 we give a function which produces a Hamiltonian cycle under the condition of Theorem 1.

In Algorithm 2 the function DiracHamiltonian first builds a maximal path by starting with an edge and then extending it in both directions as long as this is possible. Afterwards, the function closes the path to a cycle and then tries to find a larger path by adding to the cycle a new vertex and opening it back to a path. By the minimum degree condition $\delta \geq n/2$, any maximal path can be closed to a cycle and it is possible to extend a closed cycle to a larger path. Finally, the function builds a Hamiltonian path and then a Hamiltonian cycle.
Algorithm 2 DiracHamiltonian

1: function DiracHamiltonian\((G, \delta)\)

Require: \(\delta \geq |V(G)|/2\)

Ensure: return a Hamiltonian cycle \(C\) or “NONE”

2: \(P \leftarrow\) a trivial path in \(G\).
3: repeat
4: while \(P\) is not maximal do
5: Append an edge to \(P\). \(\triangleright P\) is a maximal path in \(G\).
6: Let \(P = x_0x_1...x_k\).
7: for \(i = 0\) to \(k - 1\) do
8: if \(x_0x_{i+1} \in E(G)\) and \(x_ix_k \in E(G)\) then
9: return \(C = (x_0, x_{i+1}...x_{k-1}, x_k, x_i, x_{i-1}...x_1, x_0)\).
10: \(\triangleright\) The existence of such an index \(i\) is guaranteed by minimum degree condition.
11: if \(C \neq \text{NONE}\) and \(|V(C)| \neq |V(G)|\) then
12: Let \(e\) be an edge with exactly one endpoint in \(C\).
13: Let \(e'\) be an edge of \(C\) incident to \(e\) \(\triangleright\) There are two such edges.
14: \(P \leftarrow C + e - e'\)
15: until \(|V(C)| = |V(G)|\) or \(C = \text{NONE}\)
16: return \(C\).

As a result of the constructive nature of Bondy-Chvátal’s proof for Theorem 3, there exists a polynomial-time algorithm which produces a Hamiltonian cycle in graphs having complete closure. For the sake of completeness, we give such an algorithm in Algorithm 3.

In Algorithm 3, the function first arbitrarily arranges all vertices in a cycle since the closure is complete. Note that this cycle is Hamiltonian since it contains all vertices of the graph. If all edges of the cycle are already in the graph, then we are done. Otherwise, there exists an edge \(e\) which is in the closure but not in the graph. The function opens this Hamiltonian cycle to a Hamiltonian path \(P = x_0x_1...x_k\) by removing \(e\), and builds a Hamiltonian cycle from this Hamiltonian path using edges \(x_0x_{i+1}\) and \(x_kx_i\) in the graph for some \(2 \leq i \leq k - 1\). The existence of such an edge is guaranteed by definition of closure, i.e. \(d(x_0) + d(x_k) \geq n\). After repeating this process for each edge which is in the closure but not in the graph, the function constructs a Hamiltonian cycle in the graph.
Algorithm 3 ClosureHamiltonian

1: function ClosureHamiltonian(G, cl(G))

Require: cl(G) is a complete graph

Ensure: return a Hamiltonian cycle C or “NONE”

2: C ← a Hamiltonian cycle in cl(G).

3: \[\triangleright C\text{ can be obtained by arbitrarily arranging all vertices.}\]

4: for \(j = 1\) to \(k\) do

5: if \(e_i\) is not an edge in \(G\) then

6: \(P \leftarrow C - e_j\) \[\triangleright P\text{ is a Hamiltonian path in } \tilde{G}.\]

7: Let \(P = x_0x_1...x_k_x_0\).

8: for \(i = 2\) to \(k - 1\) do

9: if \(x_0x_{i+1} \in E(G)\) and \(x_ix_k \in E(G)\) then

10: \(C \leftarrow (x_0,x_{i+1}...x_{k-1},x_k,x_i,x_{i-1}...x_1,x_0).\)

11: repeat

12: Let \(E = \{e_1,e_2...e_k\}\) be edge set of \(C.\)

13: until \(E \subseteq E(G)\) or \(C = \text{NONE}\)

14: return \(C.\)

Since the function ExtensionDiracHamiltonian and its constructive structure is explicitly stated in \cite{18}, we do not give the algorithm ExtensionDiracHamiltonian here. We refer to \cite{18} for details.

5. Conclusion

In this work, we show that there exists a monochromatic cycle cover in complete graphs with at least 13 vertices and a nearly equitable 2-edge-coloring. Hence, we conclude that the minimum reload cost is zero in these graphs. In general, all proofs except one part in this paper are constructive. Then, we provide a polynomial-time algorithm that constructs a monochromatic cycle cover, in particular a Hamiltonian cycle or two cycles whose sizes differ by at most one, in complete graphs with a nearly equitable 2-edge-coloring. This algorithm builds a monochromatic cycle cover in all complete graphs with an equitable 2-edge-coloring, whereas it may remain inconclusive in some complete graphs with a nearly equitable 2-edge-coloring. In particular, the algorithm remains inconclusive for the case where both subgraphs induced by a color in a complete graph of odd order \(2k+1\) with a nearly equitable 2-edge-coloring is 2-connected with \(\delta = k - 1\) and \(\Delta = k + 1.\)
We believe that MinRC3 problem may have solution zero for other types of 2-edge colorings in complete graphs because of the insight provided by this work. As a future work, we first aim to make the non-constructive part of the proofs in this work constructive. We then plan to study the MinRC3 problem in complete graphs with 2-edge coloring in general and design an algorithm that not only determines whether a monochromatic cycle cover exists but also constructs a monochromatic cycle cover whenever it exists.

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