Rotation, mass loss and pulsations of the star:
an analytical model

Zakir F. Seidov
Department of Physics, Ben-Gurion University, Beer-Sheva 84105, Israel
email: seidov@bgumail.bgu.ac.il

The characteristics of the star model with the "prescribed" density distribution $\rho = \rho_c [1 - (r/R)^\alpha]$ are analytically studied. The model has been first considered briefly in our 30-year old note of a restricted circulation [15].

Introduction

We choose the distribution of density $\rho(r)$ in a spherical-symmetric star in the form

$$\rho(r) = \rho_c (1 - x^\alpha), \quad x = \frac{r}{R};$$

(1)

here $\rho_c = \rho(0)$ is the central density, $r$ is a running radius, $R$ is the radius of the model, $\alpha$ is a free parameter, ($\alpha \geq 0$, as we require $d \rho/dr \leq 0$).

Central-to-mean density ratio

From Eq. (1) we have for a cumulative mass $m$, total mass $M$ and for the central-to-mean density ratio $\rho_c/\overline{\rho}$:

$$m(x) = 4\pi \rho_c R^3 \left( \frac{x^3}{3} - \frac{x^{3+\alpha}}{3+\alpha} \right); \quad M = m(1) = \frac{4\pi}{5} \rho_c R^5 \frac{\alpha}{5+\alpha}; \quad \frac{\rho_c}{\overline{\rho}} = \frac{3+\alpha}{\alpha}.$$  

(2)

Central moment of inertia

The moment of inertia of the star about its center is:

$$I = \int_0^M r^2 \, dm = \frac{4\pi}{5} \rho_c R^5 \frac{\alpha}{5+\alpha};$$

(3)

with help of Eq. (2) we may write down $I$ in terms of $M$ and $R$:

$$I = \frac{3}{5} \frac{3+\alpha}{5+\alpha} M R^2.$$  

(4)

Gravitational potential energy

The total potential energy of the star is:

$$W = -G \int_0^M \frac{m \, r}{r} \, dm = -\frac{16\pi^2}{15} \frac{\alpha^2 (11+2\alpha)}{(3+\alpha)(5+\alpha)(5+2\alpha)} GR^5 \rho_c^2 = \frac{-3(3+\alpha)(11+2\alpha)}{5(5+\alpha)(5+2\alpha)} \frac{GM^2}{R}.$$  

(5)

WUM-ratio

The central gravitational potential $U(0)$ and the potential at the surface $U(R)$ of the spherically-symmetric star are

$$U(0) = -4\pi G \int_0^R \rho(r) \, r \, dr; \quad U(R) = -\frac{GM}{R},$$

for the model (5):
\[ U(0) = \frac{-2 \alpha G \pi R^2 \rho_c}{2 + \alpha} = \frac{3}{2} \frac{\alpha}{2 + \alpha} U(R). \]  

(7)

Combining Eqs (2), (5), (7) we obtain the WUM-ratio \[ 13 \] for our model:

\[ W_{UM} = \frac{W}{U(0)M} = \frac{2 (2 + \alpha) (11 + 2 \alpha)}{5 (5 + \alpha) (5 + 2 \alpha)}. \]  

(8)

**Pressure-density relation**

Now we find the relation between pressure \( P \) and density \( \rho \) in the star, which has the density distribution \[ 1 \] at hydrostatic equilibrium. In other words, we find the equation of state of matter which leads to distribution of density \[ 1 \] in the star. We remind that a star with the equation of state \( P(\rho) \) which does not include a temperature is called pseudopolypotrope \[ 1 \].

The equation of hydrostatic equilibrium of spherically-symmetric star reads:

\[ \frac{1}{\rho} \frac{d}{d r} (\rho dP/dr) = -G \frac{m}{r^2}. \]  

(9)

Using Eqs (1, 2) and integrating Eq. (9) with initial condition \( P(0) = P_c \) (central pressure) we obtain the distribution of pressure and density within the star:

\[ P(x) = P_c - G \pi R^2 \rho_c^2 \left[ \frac{2 x^2}{3} - \frac{4}{3} \frac{(6 + \alpha) x^{2+\alpha}}{(2+\alpha)(3+\alpha)} + \frac{2 x^{2+2\alpha}}{(3+\alpha)(1+\alpha)} \right]; \]  

\[ \rho(x) = \rho_c (1 - x^\alpha); \quad x = r/R. \]  

(10)

From Eq. (10) using condition \( P(1) = 0 \), we obtain the following relations for the central pressure:

\[ P_c = \frac{2 \pi \alpha^2 (4 + \alpha)}{3 (1 + \alpha) (2 + \alpha) (3 + \alpha)} G R^2 \rho_c^2 = \frac{3 (3 + \alpha) (4 + \alpha)}{8 \pi (1 + \alpha) (2 + \alpha)} GM^2 R^4. \]  

(11)

From Eqs. (10, 11) we obtain the following relation between \( P \) and \( \rho \) ("equation of state", EOS):

\[ \frac{P}{P_c} = 1 + \frac{(1 - \frac{\rho}{\rho_c})^{2/\alpha}}{1 + \frac{2}{\alpha} \frac{\rho}{\rho_c} + \frac{3 (2 + \alpha)}{\alpha (1 + \alpha)} \left( \frac{\rho}{\rho_c} \right)^2}. \]  

(12)

Note that for any given pair of parameters \( \rho_c, P_c \) (and fixed \( \alpha \)) we have a particular EOS given by Eq. (12). Therefore these particular pseudopolypotropes have the two-parametric EOS, while e.g. the classic polytropes with given adiabatic index \( \gamma \) have one-parametric EOS: \( P = K \rho^\gamma \), the parameter \( K \) being related with polytropic temperature, see for more details \[ 13 \]. If we calculate a model with other values of central density \( \rho_c^* \) and pressure \( P_c^* \) with the same EOS (12), the resulted distribution of density will not coincide with the law \[ 1 \] see the next Section). Particular cases \( \alpha = 1 \) and \( \alpha = 2 \) give the more simple equations of state:

\[ \frac{P}{P_c} = \frac{1}{5} \left( \frac{\rho}{\rho_c} \right)^2 \left( 6 + 8 \frac{\rho}{\rho_c} - 9 \left( \frac{\rho}{\rho_c} \right)^2 \right); \quad \alpha = 1; \]  

(13)

\[ \frac{P}{P_c} = \frac{1}{2} \left( \frac{\rho}{\rho_c} \right)^2 \left( 1 + \frac{\rho}{\rho_c} \right); \quad \alpha = 2. \]  

(14)

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**FIG. 1.** Equations of state for different values of \( \alpha \): .... - \( \alpha = 1 \), --- - \( \alpha = 2 \), - - - - \( \alpha = 1/2 \), and solid line - polytropic EOS with \( \gamma = 4/3 \). Abscissae are values of density \( \frac{\rho}{\rho_c} \), ordinates are values of pressure \( \frac{P}{P_c} \).
Although these equations of state, and the more general EOS (12) are derived and are formally valid only for \( \rho / \rho_c \leq 1 \), we will loosely use them also at larger densities, see the next Section.

**Density distribution in the \( \alpha = 2 \) model**

Here we calculate density distribution of the particular model with \( \alpha = 2 \), which corresponds to EOS given by Eq. (14). With this EOS, the equation of hydrostatic equilibrium (9) is reduced to the next equation for the density distribution \( \rho(x) \):

\[
\frac{d}{dx} \left[ x^2 \left( \frac{3}{2} \rho + 1 \right) \frac{d\rho}{dx} \right] = -15 x^2 \rho. \tag{15}
\]

In deriving this equation, we used the first relation in Eq. (11). Important notice is that Eq. (15), at the case \( \rho(0) = 1 \), has the solution which corresponds to the law (1) (with \( \alpha = 2 \)), and all parameters corresponding to this case are taken to be equal to unit (that is, at \( \rho(0) = 1 \), we have \( R = 1, \ M = 1 \) and so on. Also, the dimensionless running radius in Eq. (15) \( x \) is expressed in units of the total radius of star \( R \), which is not equal to 1 at \( \rho(0) \neq 1 \).

![FIG. 2. Density distribution for EOS \( P = \rho^2 \left( \frac{\rho + 1}{2} \right) \) at three values of central density \( \rho(0) = 2, 1 \) and 1/2. Abscissae are values of dimensionless running radius \( x = r/R \), ordinates are values of \( \rho(x)/\rho(0) \).](image)

Eq. (15) was solved numerically for two values of \( \rho(0) \), 2 and 1/2. In these two cases the radius of the star is 1.169156 and .9070496, respectively. The resulting density distribution is shown in Fig. 2, where abscissae are \( x = r/R \) and ordinates are \( \rho(x)/\rho(0) \); \( \rho(0) = 2 \) case corresponds to the upper solid curve, while \( \rho(0) = 1/2 \) corresponds to the lower solid line. Also shown is the standard distribution - Eq. (1) with \( \alpha = 2 \), dash line in Fig. 2. For values of the mean-to-central density ratio \( \rho / \rho(0) \) we have: .363325, .4 and .442107 at \( \rho(0) \) equal to 1/2, 1 and 2, respectively. For larger central density, EOS (14) has larger adiabatic index \( \gamma \), EOS is more stiff, and this leads to more homogeneous configuration with larger mean-to-central density ratio.

**Pade approximants**

We briefly describe the Pade approximants method which we use further in this paper. From the segment of the series:

\[
s_4 = 1 + \sum_{k=1}^{k=4} a_k x^k,
\]

coefficients of Pade(2,2) approximants \[3\]

\[
PA = \frac{1 + A x + B x^2}{1 + C x + D x^2}
\]

are defined as follows \[3\]:

\[
D = \frac{a_3^2 - a_2 a_4}{a_2^2 - a_1 a_3}; \quad C = \frac{a_1 a_4 - a_2 a_3}{a_2^2 - a_1 a_3}; \quad B = a_2 + a_1 C + D; \quad A = a_1 + C.
\]

3
Pade approximants to density distribution

From Eq. (15) we have series expansion at the center:

\[ \rho(x) = \rho(0) [1 + \sum_{k=1}^{4} a_k x^k]; \quad a_1 = -\frac{5}{2+3 \rho(0)}; \quad a_2 = -\frac{15(-1+\rho(0))}{(2+3 \rho(0))^2}; \]

\[ a_3 = \frac{150(1-9 \rho(0))(-1+\rho(0))}{7(2+3 \rho(0))^3}; \quad a_4 = \frac{25(-1+\rho(0))(-5+22 \rho(0)-837 \rho(0)^2)}{7(2+3 \rho(0))^4}. \]  

(19)

Evidently, at \( \rho(0) = 1 \) the series is reduced to \( \rho(x) = 1 - x^2 \). Expressions for coefficients of Pade(2,2) approximants are too cumbersome at the general case of arbitrary \( \rho(0) \) so we present only one particular case:

\[ \rho(0) = 2 : \quad \rho_{\text{Pade}}(x) = 2 \frac{12816384}{13840960} - \frac{13840960}{x^2} + \frac{3314065}{12816384} x^4. \]  

(20)

Using this "analytic solution" we find the radius of the model equal to 1.17720 which is 1.0068 times the calculated value 1.169156. Also, we calculated mean-to-central density ratio according to (20) and found value 0.435602 which is 0.98528681 of the numerical value 0.442107.

Mean adiabatic index

The adiabatic index is defined as:

\[ \gamma = \frac{d \ln P}{d \ln \rho}. \]  

(21)

Distribution of adiabatic index within the model is:

\[ \gamma(x) = \frac{2 (1 + \alpha) (2 + \alpha) x^{2-\alpha} (3 + \alpha - 3 x^\alpha) (1 - x^\alpha)^2}{\alpha [4 \alpha^2 + \alpha^3 - (6 + 11 \alpha + 6 \alpha^2 + 3 \alpha^3) x^2 + (12 + 14 \alpha + 2 \alpha^2) x^{2+\alpha} - (6 + 3 \alpha) x^{2+2 \alpha}]} \].  

(22)

At the surface of the star (at \( x = 1 \)) \( \gamma = 2 \) for any \( \alpha \) while the value of \( \gamma \) at the center of the star depends on \( \alpha \): \( \gamma(0) = 0 \) at \( \alpha < 2 \), \( \gamma(0) = \infty \) at \( \alpha > 2 \), and \( \gamma(0) = 5/2 \) at \( \alpha = 2 \).

Moreover the function \( \gamma(x) \) is non-monotonic at \( \alpha < 2 \), while at \( \alpha \geq 2 \) the function \( \gamma(x) \) is monotonically decreasing with increasing \( x \), relative running radius, see Fig. 3.

![FIG. 3. Adiabatic index as function of \( x = r/R \), according to Eq. (22), curves from the lowest one to the upper one correspond to values of \( \alpha = 1/2 \), 1, 2, 2.1, respectively.](image)

\[ \gamma(0) = 0 \text{ at } \alpha < 2, \quad \gamma(0) = \infty \text{ at } \alpha > 2, \quad \gamma(0) = 5/2 \text{ at } \alpha = 2. \]

Therefore at \( \alpha < 2 \), in the inner parts of the star, the value of \( \gamma \) is less than 4/3, the critical value of adiabatic index, while in the outer regions \( \gamma \) is always larger than 4/3.

The stability of the star against radial perturbations is defined by the mean value of adiabatic index. That is why it is of interest to calculate the mean adiabatic index, \( \overline{\gamma} \), which is defined as follows:

\[ \overline{\gamma} = \frac{1}{\int_V P \, dv} \int_V \gamma P \, dv; \]  

(23)

here an integration is over the volume of star. We obtain at \( \alpha < 5 \):

\[ \overline{\gamma} = \frac{12 (7 + \alpha)}{(5 - \alpha)(11 + 2 \alpha)}. \]  

(24)

For any \( \alpha \geq 5 \), \( \overline{\gamma} \to \infty \) due to divergence of \( \gamma(r) P(r) r^2 \) at the center of the star. The function \( \overline{\gamma}(\alpha) \) is monotonically increasing and even at \( \alpha = 0 \), \( \overline{\gamma} = 84/55 > 4/3 \), that is the model is always stable against to the radial perturbations.
Ellipticity of slow rotating star

For the slow rotation, the ellipticity distribution inside the spherically-symmetric star is governed by the Clairaut’s equation
\begin{equation}
\frac{e''(x)}{x^2} - \frac{6e(x)}{x^4} \int_0^x \rho(t) t^2 \, dt + 2\rho(x) \left[ \frac{e(x)}{x} + e'(x) \right] = 0.
\end{equation}

Here \(x = r/R\) is dimensionless running radius, \(0 \leq x \leq 1\), and \(e(x)\) is an ellipticity of the equidensity surfaces within the star, \(e(x) = 1 - r_p(x)/r_{eq}\), with \(r_p\) and \(r_{eq}\) being the polar and equatorial radii of the equidensity surfaces which are assumed to be the biaxial ellipsoids of revolution.

The Equation (25) can be solved in terms of the series expansion at the center, which we write down here for two cases:
\begin{align}
\alpha = 1 : & \quad e(x) = \sum_{i=0}^\infty c_i x^i; \quad c_i = \frac{3^2 + 5i - 1}{4} \frac{1}{i(i+5)} c_{i-1}; \quad i \geq 1; \\
& \quad e(x) = 1 + \frac{x}{2} + \frac{15x^2}{112} + \frac{75x^3}{896} + \frac{25x^4}{448} + \frac{69x^5}{1792}; \\
\alpha = 2 : & \quad e(x) = \sum_{i=0}^\infty c_i x^{2i}; \quad c_{i+1} = \frac{3}{8i(i+1)(2i+7)} c_{i-1}; \quad i \geq 0; \\
& \quad e(x) = 1 + \frac{6x^2}{35} + \frac{13x^4}{179} + \frac{52x^6}{1375} + \frac{141x^8}{6875} + \frac{1974x^{10}}{171875}.
\end{align}

We solved Eq. (25) numerically for the model (1) with two values of \(\alpha\), see Figs. 4 and 5.

The values of the surface-to-central ellipticity ratio of ellipticity \(e(1)/e(0)\) are 1.6608 and 2.0488 at \(\alpha\) equal to 1 and 2, respectively. In the theory of (slowly) rotating stars the ratio of centrifugal acceleration \(w^2\) to the gravitational acceleration \(GM/R\) at the equator, \(m = \frac{w^2 R^3}{GM}\) is introduced. Also, the ratio \(m/e(R)\) ratio is introduced, which in the terms of functions \(e(x)\) and \(e'(x)\) is \(\frac{m}{e} = \frac{2}{3} (2 + \frac{e'(1)}{e(1)})\). For the model in question, we have \(\alpha = 1 : e(1) = 1.6608, e'(1) = 2.0488, m/e = 1.2936\), and \(\alpha = 2 : e(1) = 1.3312, e'(1) = 1.3785, m/e = 1.2142\).

Pade Approximants for the ellipticity distribution

For the ellipticity distribution inside the star, using the series expansion (26), we have Pade(2,2) approximant
\begin{equation}
\alpha = 2 : \quad e_{Pade}(x) = \frac{5 (25025 - 14036 x + 1138 x^2)}{7 (17875 - 13090 x + 1729 x^2)}.
\end{equation}
Even at the surface this expression is very accurate: $e_{Pade}(1)/e_{calc}(1) = 0.99891$. In fact [27] gives the analytical solution for the Clairaut’s equation for $\alpha = 2$ model.

For the $\alpha = 1$ model Pade(2,2) is also rather accurate:

$$e_{Pade}(x) = \frac{672 - 448 \, x + 41 \, x^2}{7 \, (96 - 88 \, x + 15 \, x^2)}; \quad \alpha = 1,$$

(28)

with $e_{Pade}(1)/e_{calc}(1) = 0.99108$.

**Rotation, contraction and mass loss**

The abovementioned analytical expressions for parameters of star may be applied to study of evolution of rotating star, or to evaluation of pulsational periods of star or to any other problem where the simplicity and explicity of evaluation justify the approximation certain inaccuracy.

We consider in this section the evolution of contracting and rotating star with mass loss. Let the rotating star be contracting in such a way that the distribution of density follows the law (1). At the same moment the condition for mass loss may be reached at the star’s equator. We assume that the star is spherically-symmetric and rotation is steady-state (that is angular rotation is constant over the volume of star, $\omega(r) = \text{constant}$) (that may occur at slow contraction and rapid rotation), then we may write the mass loss condition as:

$$\omega^2 = \frac{G \, M}{R^3}.$$

(29)

The rate of angular momentum loss is $dL = \omega \, R^2 \, dM$, and taking into account that $L = k \, \omega \, M \, R^2$, ($k$ is a structure-dependent parameter) we have:

$$\frac{M}{M_0} = \left( \frac{R}{R_0} \right)^{\beta}; \quad \beta = \frac{k}{2 - 3 \, k}.$$

(30)

As a result, a value of the final mass of the star depends only on parameter $k$, (see also [3]). The values of $k$ for polytropes, white dwarfs and “stepenars” are given in [5 - 8]. For the star with the density distribution (1):

$$k = \frac{3 + \alpha}{5 (5 + \alpha)};$$

(31)

note that in Eq. (4) the moment is central, and axial moment is $J_{axis} = 1/3 \, J_{center}$.

At $\alpha = .06$ that corresponds, by value of ratio $\frac{\rho_c}{\rho}$, to $n = 3$ polytrope, we have $k \simeq .012$ and $\beta \simeq .078$. Rotation-induced distortion of figure from sphere leads to $\beta \simeq .02$ [3]. For the star in the pre-white dwarf stage, a value of $\beta$ should be even less as the star in pre-white dwarf or pre-neutron star stage is a (very) hot star with elusive extent envelope and with small dense core (and with very small value of $k$). The envelope contains the large rotational momentum and the small mass therefore the mass loss per unit momentum loss is very small for a such star.

At the other hand for the homogeneous star, $k = .2$, $\beta = 1/7$ that is 7 times the value of $n = 3$ polytrope. Therefore if the pre-white dwarf star would be more homogeneous then at contraction the rotating star could lose more mass. However even in this case to reach a sizable value of mass loss at the pre-white dwarf stage, the star should have the rotational momentum by order of magnitude larger that the main sequence star, and for neutron star the difficulty is even larger. The presence of factors leading to the more homogeneous structure could lead to reducing the difficulties in explaining the origin of white dwarfs and neutron stars by the mass loss from ordinary stars.

**Pulsational periods of pre-white dwarf stars**

Ledoux and Pekeris [3] using the energy method obtained the following formula for the frequency of the fundamental mode of the adiabatic radial pulsations of the spherically-symmetric star:

$$\sigma^2 = (3 \, \gamma - 4) \frac{-W}{I}.$$

(32)

Using the formulas (4, 5, 12) we obtain

$$\sigma^2 = \frac{32 \, \pi \, (1 + \alpha)(4 + \alpha)}{3 \, (5 - \alpha)(5 + 2 \, \alpha) \, G \, \overline{\rho}}.$$

(33)
At \( \bar{\sigma} \approx 10^4 \) (a star with solar mass and radius \( \approx 1/20 \) the solar radius) and \( \alpha = 1 \), we have \( \sigma \approx 8.9 \times 10^2 \), or for the pulsational period \( P = 2 \pi / \sigma \approx 70 \) s.

Such and even larger periods of light variations occur in HL Tau stars, G 44-32 and R 548 [10], [11] which are apparently at the late stages of their evolution, and probably - at pre-white dwarf stage. Recent discussion of pre-white dwarf stars see in [14].

Radial pulsations

The differential equation for the adiabatic small radial pulsations of the spherically-symmetric star is:

\[
\frac{d}{dr} \left( r^4 \gamma P \frac{dy}{dr} \right) + y \left\{ \sigma^2 \rho r^4 + r^3 \frac{d}{dr} \left[ (3 \gamma - 4) P \right] \right\} = 0;
\]  

(34)

here \( y = \delta r / r \) is the ratio of the radial displacement to the radius, and \( \sigma \) is the frequency of the pulsations. At the center (at \( r = 0 \)), we have condition \( \delta r = r = 0 \), at the surface, \( r = R \), we require that amplitude is finite. We consider here the case of \( \alpha = 2 \) when Eq. (34) is reduced to the form:

\[
2 y(x) x (19 - \Sigma^2 - 15 x^2) + 2 y''(x) (-10 + 29 x^2 - 15 x^4) + y''(x) (25 - 3 x) = 0;
\]  

(35)

here \( x = r / R \), and \( \Sigma^2 \) is dimensionless value, expressed in units of \( \frac{P}{\rho c \sigma} = 2 \pi G \bar{\rho} / \gamma \), see Eqs. (2,11).

Equation (35) can be solved in terms of series expansion, and the recurrence relation can be written down for coefficients of the power series which may contain only even powers of \( x \). Note that Eq. (35) is the linear differential equation and the function \( y \) is defined up to an arbitrary factor, so we may take \( y(0) = 1 \). Then the first terms of the series expansion at the center are:

\[
y = 1 + \frac{19 - \Sigma^2}{25} x^2 + \frac{490 - 104 \Sigma^2 + 7 \Sigma^4}{1000} x^4 + \frac{2670 - 12610 \Sigma^2 + 239 \Sigma^4 - \Sigma^6}{60000} x^6.
\]  

(36)

At the surface \( (x = 1) \) we have the series expansion

\[
y(x) = y(1) \left[ 1 + \frac{\Sigma^2}{4} (1 - 1) (1 - x) + \frac{112 - 2 \Sigma^2 + \Sigma^4}{48} (1 - x)^2 \right].
\]

For arbitrary \( x \) the Eq. (35) may be solved numerically. We calculated the four lower modes of radial pulsations and found for the corresponding eigenvalues \( \Sigma^2 \) the values 10.325, 39.083, 81.04, and 136.1, see Fig. 6.

![Eigenvectors for 4 lower modes of radial adiabatic pulsations as functions of \( x = r / R \) for the star model with \( \alpha = 2 \), see Eq. (35). Curves from the lowest (fundamental) mode (no nodes) to upper mode (3 nodes) correspond to values of \( \Sigma^2 \) equal to 10.325, 39.083, 81.04 and 136.1, respectively.](image)

The lowest eigenvalue corresponds to \( \sigma^2 = 10.325 \times 2 / 3 \pi G \bar{\rho} = 6.88 \pi G \bar{\rho} = \), that only slightly differs from value obtained by energy method. \( 64 \pi / 9 G \bar{\rho} = 7.1 \pi G \bar{\rho} \), see Eq. (33) with \( \alpha = 2 \). Note that the evaluation of \( \sigma \) given by Eqs. (32, 33) for the frequency of the fundamental mode is to be either equal to or greater than the true value [1]. The evaluation of pulsational frequency of the lowest mode can also be obtained directly from Eq. (35), if we put \( y = const \) (which is not too rough assumption for the fundamental mode eigenfunction, see Fig. 6), then we have

\[
\Sigma^2 = 19 - 15 x^2.
\]  

(37)

We can take averaged value of right side of this equation with weight \( x^2 \):

\[
\Sigma^2 = \frac{1}{\int_0^1 x^2 \, dx} \int_0^1 (19 - 15 x^2) \, x^2 \, dx = 10,
\]  

(38)

which is very close to the "exact value" 10.325.
Conclusion

In conclusion, we present here the analytical model density distribution which allows to evaluate qualitatively and quantitatively many important characteristics of the star including the ellipticity distribution within the rotating star, the pulsational periods and the mass loss at the contraction stages of evolution. The certain inaccuracy of the approximation is the modest price paid for the large simplicity and explicitness of the model. Surely, as the first approximation or at least as the pedagogical tool the model is of some interest.

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