Fourier Transform of the Orbital Angular Momentum of a Single Photon

Jaroslav Kysela,1,2 Xiaoqin Gao,3,1,2 and Borivoje Dakić1,2

1Institute for Quantum Optics and Quantum Information (IQOQI), Austrian Academy of Sciences, Boltzmanngasse 3, 1090 Vienna, Austria.
2Faculty of Physics, University of Vienna, Boltzmanngasse 5, 1090 Vienna, Austria.
3National Mobile Communications Research Laboratory, Quantum Information Research Center, Southeast University, Sipailou 2, 210096 Nanjing, China.

(Dated: April 28, 2020)

Optical networks implementing single-qudit quantum computation gates may exhibit superior properties to those for qubits as each of the optical elements in the network can work in parallel on many optical modes simultaneously. We present an important class of such networks, that implements in a deterministic and efficient way the quantum Fourier transform (QFT) in an arbitrarily large dimension. These networks redistribute the initial quantum state into the path and orbital angular momentum (OAM) degrees of freedom and exhibit two modes of operation. Either the OAM-only QFT can be implemented, which uses the path as an internal auxiliary degree of freedom, or the path-only QFT is implemented, which uses the OAM as the auxiliary degree of freedom. The resources for both schemes scale linearly $O(d)$ with the dimension $d$ of the system, beating the best known bounds for the path-encoded QFT. While the QFT of the orbital angular momentum states of single photons has been applied in a multitude of experiments, these schemes require specially designed elements with non-trivial phase profiles. In contrast, we present a different approach that utilizes only conventional optical elements.

INTRODUCTION

The field of quantum computation has gained ever-increasing attention thanks to the invention of the quantum factoring algorithm due to Shor [1–3], which utilizes as its key part a quantum Fourier transform. The quantum Fourier transform (QFT), or quantum Hadamard gate, has been since then used in many areas of quantum computation and communication [4] for systems of qubits as well as high-dimensional qudits. The application areas of the high-dimensional QFT acting on a single photon’s state include, but are not limited to, generation of mutually unbiased bases in the quantum state tomography [5–8] and quantum key distribution [9, 10]; generation of angular states [11–14]; sorting of spatial modes of a photon [15]; and representation of multiphoton devices employed in Bell test experiments [16]. Single photon’s high-dimensional QFTs can be also used as building blocks of programmable universal multi-port arrays [17–19].

The orbital angular momentum (OAM) of single photons is a quantized property with infinite-dimensional Hilbert space, which allows for construction of qudits in arbitrarily high dimension [20–22]. By manipulating a single photon’s OAM state the universal quantum computation is possible [23–24]. The Fourier transform of the OAM modes of single photons has been demonstrated in a number of experiments [10–14, 25–28] using free-space propagation and specially designed optical elements imparting non-trivial phase profiles. Alternative experimental schemes have been presented in special cases [29].

In this paper we demonstrate a completely different general approach, which works with in principle 100% efficiency for arbitrarily large dimension of the OAM state of single photons. Our scheme decomposes the Fourier transform into a series of elementary operations that can be directly implemented with basic commercially-available optical elements such as beam-splitters, mirrors and Dove prisms. Such an explicit decomposition reveals how individual components participate in the evolution of different OAM modes and allows for modifications, such as miniaturization of the setup to a micro-chip level. The scheme’s implementation is recursive and makes use of an interplay between the OAM and path degrees of freedom. The number of beam-splitters required scales linearly in the dimension $O(d)$, as opposed to $O(d \log d)$ scaling of the setup using path-encoded qudits [30–32]. This is made possible by the fact that a single passive optical element can act on many OAM modes at the same time leading to a heavily parallelized operation of the network of optical elements. Moreover, the setup for the OAM Fourier transform can be modified to act as the path-only Fourier transform. In such a scheme, the OAM is present only in the inner workings of the transform. This OAM-enhanced setup preserves the linear scaling of the number of beam-splitters, which shows a clear advantage of the present scheme over the setup that uses only the path degree of freedom.

One of the scheme’s main components is the OAM-Path swap operator, which interchanges the OAM and path degrees of freedom of a photon’s state. To the best of our knowledge, we demonstrate for the first time the implementation of such an operator in terms of conventional optical elements. The OAM-Path swap represents a multiport generalization of the OAM sorter and its implementation features efficient deployment of the OAM parity sorter. Each instance of the parity sorter functions simultaneously as a series of many conventional beam-splitters for different OAM modes.

The manuscript is organized as follows. In the first sec-
tion we present the theoretical background for the
construction of the Fourier transform. Then we present the
setup implementing the Fourier transform acting on the
OAM state of single photons. We summarize the prop-
eties of the OAM sorter, a key part of the setup, in the
third section. In the fourth section, we demonstrate how
to generalize the OAM sorter into the OAM-Path swap
operator. In the fifth section the scaling of our scheme
is presented. In the sixth section, we discuss the rel-
ation of our scheme to the recursive scheme for the path-only
Fourier transform and summarize our results in the last
section.

FOURIER TRANSFORM

In this section, we present a recursive scheme for the
construction of the Fourier transform acting on the OAM
of a single photon. Suppose the dimension \(d\) is a com-
posite number and the initial OAM state of a photon reads
\[
\sum_{j=0}^{d-1} \alpha_j |j\rangle.
\]
The Fourier image of such a state is then a superposition
\[
\sum_{k=0}^{d-1} \beta_k |k\rangle,
\]
where coefficients \(\beta_k\) satisfy
\[
\beta_k = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} e^{i \frac{\pi}{d} \frac{k}{j}} \alpha_j.
\]

Our implementation of the Fourier transform is inspired
by the classical fast Fourier transform algorithms, an idea
already applied in the field of quantum information for
the path degree of freedom \([31][35]\).

To implement the \(d\)-dimensional Fourier transform in
OAM, we first decompose the total \(d\)-dimensional OAM
space of a photon into a tensor product \(\mathcal{H} = \mathcal{H}_O \otimes \mathcal{H}_P\),
as shown in Fig. 1(a). There, a \(d_O\)-dimensional subspace
\(\mathcal{H}_O\) is spanned by a subset of original OAM modes and
\(\mathcal{H}_P\) is a \(d_P\)-dimensional subspace represented by a path
degree of freedom, such that \(d = d_O \times d_P\). For each
division of dimension \(d\) into a pair of smaller dimensions
\(d_O\) and \(d_P\) there is a one-to-one correspondence between
an index \(j \in \{0, \ldots, d-1\}\) and a pair of numbers \((m, l)\)
with \(0 \leq m < d_O\) and \(0 \leq l < d_P\) such that
\[
|j\rangle = |m \times d_P + l\rangle.
\]

The Fourier transform in dimension \(d\) is then obtained
in four steps as demonstrated in Fig. 1(b). In the first
step, a \(d_O\)-dimensional Fourier transform is applied only
on the OAM subspace \(\mathcal{H}_O\). Then a controlled phase gate
acts on both subspaces, followed by a \(d_P\)-dimensional
Fourier transform applied only on the path subspace. As
the last step, a swap gate is used, which effectively ex-
changes the OAM and path subspaces. This procedure
can be mathematically summarized by the formula
\[
F^{(d)} = \text{SWAP} \cdot F^{(d_P)} \cdot \text{CZ} \cdot F^{(d_O)},
\]
where the phase gate CZ acts as a high-dimensional
controlled-Z gate on the OAM degree of freedom
\[
\text{CZ}(|\beta\rangle_{\mathcal{O}} |\gamma\rangle_{\mathcal{P}}) = (Z^l |\beta\rangle_{\mathcal{O}} |\gamma\rangle_{\mathcal{P}}) = \omega^{|m|_{\mathcal{O}} \times |l|_{\mathcal{P}}}
\]
with \(\omega = \exp (2\pi i / d)\). The action of the swap on an
input mode reads SWAP(|\beta\rangle_{\mathcal{O}} |\gamma\rangle_{\mathcal{P}}) = |\gamma\rangle_{\mathcal{O}} |\beta\rangle_{\mathcal{P}}\). Note that
when \(d_O \neq d_P\) the swap operator effectively changes
dimensions of the two subspaces \(\mathcal{H}_O\) and \(\mathcal{H}_P\). This imposes
nevertheless no restrictions on our implementation. The
resulting mode \(|\beta\rangle_{\mathcal{O}} |\gamma\rangle_{\mathcal{P}}\) can be now identified with
a single index \(|k\rangle\) in a way analogous to Eq. 2 as
\[
|\beta\rangle_{\mathcal{O}} |\gamma\rangle_{\mathcal{P}} \Leftrightarrow |k\rangle = |q \times d_O + r\rangle.
\]

In the end, one obtains the Fourier image of the OAM
state of a single photon leaving the device along a single
output path. For details see Appendix A.

Even though generalizations of our scheme for an ar-
bitrary dimension \(d\) are possible, in the following discus-
sion we consider only dimensions of the form \(d = 2^M\) for
\(M \in \mathbb{N}\). The recursive scheme allows us to choose the
optimal factorization of \(d\) into \(d_O\) and \(d_P\) in each recursion
and reduce thus the number of experimental components.
The optimal factorization scenario is presented in section
Scaling properties.

IMPLEMENTATION

The general scheme of our implementation of the
Fourier transform is depicted in Fig. 2(a). The scheme
comprises six separate modules, each of which is de-
scribed below.

1. In the first module, the initial \(d\)-dimensional OAM
state is split into a superposition of OAM states,
each of which contains smaller number \(d_O\) of OAM
terms and propagates along one of \(d_P\) different
paths. This splitting, corresponding to relabeling

![Figure 1](image-url)
FIG. 2. General scheme of the Fourier transform in the orbital angular momentum (OAM) of single photons. a) The Fourier transform in dimension $d = d_O \times d_P$ is constructed recursively making use of Fourier transforms in smaller dimensions $d_O$ and $d_P$. The OAM state of a photon is at first split by an OAM sorter $S_{(d_P)}$ into a superposition of OAM states, each containing $d_O$ modes of the form $|0\rangle$, $|d_P\rangle$, ..., $|d_P(d_O - 1)\rangle$ and propagating along one of $d_P$ different paths denoted by $p_0$ through $p_{d_P-1}$. The Fourier transform itself is then performed in four steps. The first step consists in application of the $d_O$-dimensional Fourier transform $F_{OAM}$ on each path. Then a phase gate is applied on all paths, which is implemented by a series of Dove prisms. In the third step a single $d_P$-dimensional path-only Fourier transform $F_{Path}$ is applied on the path degree of freedom for all OAM modes. All beam-splitters in the path-only Fourier transform are supplemented by two mirrors as demonstrated in b). The fourth step is represented by the OAM-Path swap gate. Finally, all states are recombined into a single output path by the second OAM sorter $S^{-1}_{(d_O)}$, which is operated in reverse. b) The building block of the recursive scheme – Fourier transform for $d = 2$ – consists of two elementary OAM sorters, and a single beam-splitter that is complemented by two mirrors such that the OAM value of the incoming modes is not affected by reflection off the beam-splitter’s interface.

$|j\rangle \rightarrow |m\rangle_O |l\rangle_P$ in Eq. 2 is performed by the $d_P$-dimensional OAM sorter. The structure and operation of the OAM sorter are described in detail in the following section.

2. The second module consists of a collection of identical $d_O$-dimensional OAM Fourier transforms, each of which is applied onto a different path to implement $F_{OAM}$ in Eq. 3. The $d_O$-dimensional Fourier transform itself is constructed recursively following the same pattern as the one presented in this section when one replaces $d$ with the value of $d_O$. The elementary building block – the Fourier transform for $d = 2$ – is depicted in Fig. 2 b).

3. The third module, which represents the phase gate CZ in Eq. 4, is implemented by a series of properly rotated Dove prisms, which impart additional phases to OAM modes passing through.

4. The fourth module comprises a single path-only $d_P$-dimensional Fourier transform, which performs transformation $F_{Path}$ in Eq. 3. This transform affects only the path degree of freedom and acts identically onto each OAM mode. The implementation of the path-only Fourier transform in terms of beam-splitters is given in Refs. 31, 32. To compensate for the OAM inversion $|m\rangle_O \rightarrow |-m\rangle_O$ due to reflection off the beam-splitter’s interface, each beam-splitter in the path-only Fourier transform has to be complemented by two extra mirrors as demonstrated in Fig. 2 b).

5. The fifth module is the SWAP gate which reorders the coefficients of the joint OAM-path state such that the coefficient for mode $|m\rangle_O |l\rangle_P$ becomes the coefficient for mode $|l\rangle_O |m\rangle_P$. The swap gate, whose structure and working principle are one of the main results of this paper, is described in detail in a separate section. Without the swap gate, the OAM sorter in the sixth module would also perform undesirable additional permutation of the output OAM modes.

6. In the sixth module, all parts of the OAM state are recombined into a single output path by the $d_O$-dimensional OAM sorter, which is operated in reverse. This action corresponds to relabeling in Eq. 5.
The first stage of the Fourier transform implementation consists of an OAM sorter. The OAM sorter is a device that transforms the OAM state of a photon, represented by a superposition of OAM modes, into a superposition of propagation paths along which the photon leaves the device $[36, 39]$. For a fixed dimension $d_P$ the sorter sorts input modes $|0\rangle_O$ through $|d_P - 1\rangle_O$ which enter the first path $p_0$, into separate output paths $p_0$ through $p_{d_P - 1}$. All such input modes leave the sorter in OAM mode $|0\rangle_O$. The propagation of higher OAM modes is analogous except that the output mode is no longer zero. It turns out that the input OAM mode $|m\rangle_O$ gets transformed according to relations

$$S_{(d_P)}(|m\rangle_O |0\rangle_P) = |d_P \frac{m}{d_P} \rangle_O \bigg| m \text{ mod } d_P \bigg|_P , \quad (6)$$

where $[x]$ is the integral part of $x \in \mathbb{R}$ and $S_{(d_P)}$ denotes a $d_P$-dimensional OAM sorter. This modulo property of the OAM sorter is summarized in Fig. 3 a) and corresponds exactly to relabeling introduced in Eq. 2.

The sorter is constructed as a binary-tree-like network of OAM-manipulating elements in a way shown in Fig. 3 b). Each of these elements, henceforth referred to as OAM exchangers, is an interferometric device composed of an OAM-parity sorter $[36, 39]$ and two holograms, which shift the OAM value of the input mode $38$. Note that additional permutation of paths is necessary in Fig. 3 b) to comply with the order of paths depicted in Fig. 3 a).

The OAM exchanger $EX_k$ exhibits three modes of operation based on its order $k$ and on the OAM value and the input path of the incoming photon, see Fig. 3 c). A photon in OAM state $|mk\rangle$ that enters the upper port of the exchanger $EX_k$ of order $k$ either leaves its upper or lower output port depending on the parity of $m \in \mathbb{Z}$. All other input OAM modes, which are not multiples of $k$, are left in a superposition of both output ports, where OAM values in the two ports differ by $k$. A single exchanger therefore behaves either as an identity, or as a switch, or as a beam-splitter with varying splitting ratio for different OAM modes. This beam-splitter-like property was first utilized in Ref. [41] for the special case when the order of the exchanger is $k = 2$. In this paper we show that a single exchanger of order $k$ works effectively as $4k$ different beam-splitters simultaneously. For a more detailed description of the exchanger refer to Appendix B.

**OAM-PATH SWAP**

In the final part of the setup of the Fourier transform, the swap operator is utilized, whose operation on individual modes can be summarized as

$$\text{SWAP}(|m\rangle_O |l\rangle_P) = |l\rangle_O |m\rangle_P , \quad (7)$$
an OAM mode is injected to any of the input ports \( p_k \) with \( k \geq 3 \), at some point the mode assumes the form \( |m k + \Delta k\rangle_O \) with \( 0 < \Delta k < k \). From that point onward, all exchangers work as beam-splitters with varying splitting ratios leading to emergence of superpositions in the output state. Specifically, for mode \( |2m k + \Delta k\rangle_O \) entering the upper port \( p_a \) of the OAM exchanger \( \text{EX}_k \) of order \( k \) one obtains (omitting a global phase)

\[
\text{EX}_k(|2m k + \Delta k\rangle_O |a\rangle_P) = \left( \begin{array}{c} \cos \left( \frac{\pi \Delta k}{2k} \right) \, |2m k + \Delta k\rangle_O |a\rangle_P \\ + \sin \left( \frac{\pi \Delta k}{2k} \right) \, |(2m - 1)k + \Delta k\rangle_O |b\rangle_P \end{array} \right),
\]

and analogously for modes \( |(2m + 1) k + \Delta k\rangle_O \) and the lower input port \( p_b \).

This undesirable behavior is avoided when we augment the network with a collection of holo-beam-splitters. A holo-beam-splitter \( \text{HBS}_{(\alpha, k)} \) of order \( (\alpha, k) \) is a passive optical device consisting of a conventional beam-splitter with a splitting ratio \( \alpha \pi/(2k) \) and two holograms of opposite values \( k \) and \( -k \). Its operation on modes entering its upper port is summarized in Fig. 4(b) and its detailed structure is presented in Appendix B. A crucial observation is that one can force the OAM exchanger of order \( k \) to work as a mere identity or switch even for modes \( |m k + \Delta k\rangle_O \) that are not multiples of \( k \). One can do so by prepending a holo-beam-splitter of order \( (-\Delta k, k) \) to the exchanger, as demonstrated in Fig. 4(c) [13]. We obtain transformation rules (again omitting a global phase)

\[
\left( \text{EX}_k \cdot \text{HBS}_{(-\Delta k, k)} \right)(|2m k + \Delta k\rangle_O |a\rangle_P) = |2m k + \Delta k\rangle_O |a\rangle_P,
\]

where analogous relations hold also for modes \( |(2m + 1) k + \Delta k\rangle_O \) and the lower input port \( p_b \).

One can stack multiple setups in Fig. 4(c) to create a larger network. This way we arrive at the setup that implements an OAM-Path swap operator for general dimensions \( d_O \) and \( d_P \). The swap gate consists of a network of exchangers and a series of networks of holo-beam-splitters of increasing size as shown in Fig. 4(d) for the case of \( d_P = d_O = 8 \). For the detailed explanation of the construction and structure of the swap operator refer to Appendix C. The resulting network works as a proper sorter for all input ports, where the OAM value of output modes contains information about the path the original mode was injected to.

### SCALING PROPERTIES

Relation \( d = d_O \times d_P \) allows for different values of \( d_O \) and \( d_P \) to be chosen for a fixed dimension \( d \) of the input OAM state. One can search for such a combination of \( d_P \) and \( d_O \) that leads to the lowest number of beam-splitters.
required for implementation of the Fourier transform in $d$ dimensions. Our simulations show that the optimal number of beam-splitters scales approximately linearly as $6.1412 \times d$. This optimal scenario tends to prefer choices with $d_P \approx d_O$. Moreover, already for $d = 16$ is our scheme more resource-efficient than an analogous implementation of QFT in the path-encoding. In Appendix D the linear scaling is analytically confirmed for a subset of dimensions. The Fourier transform setup reported in Ref. [24] relates to the number of elementary gates, not actual optical elements. When implementing the proposal in Ref. [24] with beam-splitters, their number scales as $O(d \log_2(d))$.

The linear scaling of the whole scheme is made possible by an efficient implementation of the swap gate, which requires approximately $3d \log_2(d)/2$ beam-splitters, see Appendix D. There are alternative brute-force implementations of the swap gate, but these require asymptotically larger number of beam-splitters.

**OAM VS. PATH**

The presented scheme makes use of an interplay between the OAM and path degrees of freedom to efficiently perform the Fourier transform in the OAM degree of freedom. No other properties, such as the polarization, of incoming photons are affected. Nevertheless, the scheme can be after slight modification used also as a path-only Fourier transform, where the OAM degree of freedom plays the role of an intermediary that does not appear either in the input or the output state. Specifically, the first part of the recursive scheme, see Fig. 2 a), represented by a series of OAM sorters of decreasing dimensions, can be removed completely. We are then left with $d$ input ports. The last part of the scheme has to be adjusted by removing the very last reverted sorter and adding a series of additional sorters to obtain $d$ output ports. This OAM-enhanced scheme of the path-only Fourier transform shows better scaling properties in terms of the number of beam-splitters than the scheme presented in Refs. [31, 32]. The OAM-enhanced scheme requires $O(d)$ beam-splitters as opposed to $O(d \log_2(d))$ beam-splitters of the original scheme with the crossing point from which OAM-enhanced scheme prevails occurring at $d = 8192 = 2^{13}$. This improvement is made possible by optimal redistribution of coefficients of the quantum state between the OAM and path degrees of freedom.

**CONCLUSION**

We presented a scheme for an efficient implementation of the Fourier transform acting on the OAM state of single photons. Only commercially accessible optical elements are used in our scheme. An integral component of the scheme is the OAM-Path swap operator, which is a generalization of the OAM sorter for multiple input ports. In its implementation a heavy use is made of non-trivial parallel operation of OAM exchangers, which are elementary building blocks of the OAM sorter. A single exchanger, a passive element composed among others of two conventional beam-splitters, can work as many beam-splitters with varying splitting ratio at the same time. This property may be used in a more general framework, where each OAM mode undergoes a different complex, yet precisely tailored, evolution by propagating through the identical network of standard optical elements. Even though algorithms for decomposition of a general unitary into separate gates for path and OAM degrees of freedom exist [14–16], our scheme is to our knowledge the first explicit example of such a network.

The number of beam-splitters required in our scheme scales as $O(d)$, which is an improvement over the scaling $O(d \log_2(d))$ of the Fourier transform setup acting on path-encoded qudits [31, 32]. Our scheme can be after a slight modification used also to implement the path-only Fourier transform while preserving the scaling properties $O(d)$. The modulo property of the scheme allows one to use different $d$-dimensional OAM subspaces, such as $\{\lvert -d/2 + 1 \rangle, \ldots, \lvert d/2 \rangle\}$, which is naturally produced in the process of parametric down-conversion and which imposes less stringent requirements on the precision of the OAM-manipulating elements.

**ACKNOWLEDGEMENT**

The authors thank Anton Zeilinger, Mario Krenn, Manuel Erhard, Armin Hochrainer, Marcus Huber, Robert Fickler and Elizabeth Aguado for valuable discussions. XQG thanks Bin Sheng and Zaichen Zhang for support. This work was supported by the Austrian Academy of Sciences (OeAW), the European Research Council (SIQS Grant No. 600645 EU-FP7-ICT), and the Austrian Science Fund (FWF): F40 (SFB FoQuS) and W 1210-N25 (CoQuS). XQG acknowledges support from the National Natural Science Foundation of China (No. 61501109). B. D. acknowledges support from an ESQ Discovery Grant of the Austrian Academy of Sciences (OAW) and the Austrian Science Fund (FWF) through BeyondC (F71).

J. K. and XQG contributed equally to this work.

---

[1] P.W. Shor, “Algorithms for quantum computation: discrete logarithms and factoring,” in Proceedings 35th Annual Symposium on Foundations of Computer Science (IEEE Comput. Soc. Press, 1994) pp. 124–134.
[2] Artur Ekert and Richard Jozsa, “Quantum computation and Shor’s factoring algorithm,” Reviews of Modern Physics 68, 733–753 (1996).

[3] W. Shor, “Polynomial-Time Algorithms for Prime Factorization and Discrete Logarithms on a Quantum Computer,” SIAM Journal on Computing 26, 1484–1509 (1997).

[4] Andrew M. Childs and Wim van Dam, “Quantum algorithms for algebraic problems,” Reviews of Modern Physics 82, 1–52 (2010).

[5] William K. Wootters and Brian D. Fields, “Optimal state-determination by mutually unbiased measurements,” Annals of Physics 191, 363–381 (1989).

[6] Stephen Brierley, Stefan Weigert, and Ingemar Bengtsson, “All mutually unbiased bases in dimensions two to five,” Quantum Info. Comput. 10, 803920 (2010).

[7] Thomas Durt, Berthold-Georg Englert, Ingemar Bengtsson, and Karol Życzkowski, “On mutually unbiased bases,” International Journal of Quantum Information 08, 535–640 (2010).

[8] D. Giovannini, J. Romero, J. Leach, A. Dudley, A. Forbes, and M. J. Padgett, “Characterization of High-Dimensional Entangled Systems via Mutually Unbiased Measurements,” Physical Review Letters 110, 143601 (2013).

[9] Simon Gröblacher, Thomas Jennewein, Alipasha Vaziri, Gregor Weihs, and Anton Zeilinger, “Experimental quantum cryptography with qutrits,” New Journal of Physics 8, 75–75 (2006).

[10] Meiul Malik, Malcolm O’Sullivan, Brandon Rodenburg, Mohammad Mirhosseini, Jonathon Leach, Martin P. J. Lavery, Miles J. Padgett, and Robert W. Boyd, “Influence of atmospheric turbulence on optical communications using orbital angular momentum for encoding,” Optics Express 20, 13195 (2012).

[11] Stephen M. Barnett and D. T. Pegg, “Quantum theory of rotation angles,” Physical Review A 41, 3427–3435 (1990).

[12] Sonja Franke-Arnold, Stephen M Barnett, Eric Yao, Jonathan Leach, Johannes Courtial, and Miles Padgett, “Uncertainty principle for angular position and angular momentum,” New Journal of Physics 6, 103–103 (2004).

[13] Eric Y. Franke-Arnold, Johannes Courtial, Stephen Barnett, and Miles Padgett, “Fourier relationship between angular position and optical orbital angular momentum,” Optics Express 14, 9071 (2006).

[14] Yu Wang, Václav Potoček, Stephen M. Barnett, and Xue Feng, “Programmable holographic technique for implementing unitary and nonunitary transformations,” Physical Review A 95, 033827 (2017).

[15] Radu Ionițiu, “Sorting quantum systems efficiently,” Scientific Reports 6, 23536 (2016).

[16] Marek Żukowski, Anton Zeilinger, and Michael A. Horne, “Realizable higher-dimensional two-particle entanglements via multiphoton beam splitters,” Physical Review A 55, 2564–2579 (1997).

[17] Vctor J. Lepé-Pastor, Jeff S. Lundeen, and Florian Marquardt, “Arbitrary optical wave evolution with fourier transforms and phase masks,” (2019), arXiv:1912.04721.

[18] M. Yu. Saygin, I. V. Kondratyev, I. V. Dyakonov, S. A. Mironov, S. S. Straupe, and S. P. Kulik, “Robust Architecture for Programmable Universal Unitaries,” Physical Review Letters 124, 010501 (2020).

[19] Luciano Pereira, Alejando Rojas, Gustavo Caas, Gustavo Lima, Aldo Delgado, and Adn Cabello, “Universal multi-port interferometers with minimal optical depth,” (2020), arXiv:2002.01371.

[20] L. Allen, M. W. Beijersbergen, R. J. C. Spreeuw, and J. P. Woerdman, “Orbital angular momentum of light and the transformation of Laguerre-Gaussian laser modes,” Physical Review A 45, 8185–8189 (1992).

[21] Mario Krenn, Marcus Huber, Robert Fickler, Radek Lapkiewicz, Sven Ramelow, and Anton Zeilinger, “Generation and confirmation of a (100 x 100)-dimensional entangled quantum system,” Proceedings of the National Academy of Sciences 111, 6243–6247 (2014).

[22] Manuel Erhard, Robert Fickler, Mario Krenn, and Anton Zeilinger, “Twisted photons: new quantum perspectives in high dimensions,” Light: Science & Applications 7, 17146–17146 (2018).

[23] Juan Carlos García-Escartin and Pedro Chamorro-Posada, “Universal quantum computation with the orbital angular momentum of a single photon,” Journal of Optics 13, 064022 (2011).

[24] Xiaojin Gao and Zhengwei Liu, “Universal Quantum Computation by a Single Photon,” (2019), arXiv:1909.09535.

[25] Martin P. J. Lavery, David J. Robertson, Gregorius C. G. Berkhout, Gordon D. Love, Miles J. Padgett, and Johannes Courtial, “Refractive elements for the measurement of the orbital angular momentum of a single photon,” Optics Express 20, 2110 (2012).

[26] Malcolm N. O’Sullivan, Mohammad Mirhosseini, Mehul Malik, and Robert W. Boyd, “Near-perfect sorting of orbital angular momentum and angular position states of light,” Optics Express 20, 24444 (2012).

[27] Mohammad Mirhosseini, Mehul Malik, Zhimin Shi, and Robert W. Boyd, “Efficient separation of the orbital angular momentum eigenstates of light,” Nature Communications 4, 2781 (2013).

[28] Florian Brandt, Markus Hiekkamäki, Frédéric Bouchard, Marcus Huber, and Robert Fickler, “High-dimensional quantum gates using full-field spatial modes of photons,” Optica 7, 98 (2020).

[29] Xingbing Song, Yifan Sun, Pengyuan Li, Hongwei Qin, and Xianglong Zhang, “Bell’s measure and implementing quantum Fourier transform with orbital angular momentum of classical light,” Scientific Reports 5, 14113 (2015).

[30] Michael Reck, Anton Zeilinger, Herbert J. Bernstein, and Philip Bertani, “Experimental realization of any discrete unitary operator,” Physical Review Letters 73, 58–61 (1994).

[31] P. Törnä, I. Jex, and S. Stenholm, “Beam splitter realizations of totally symmetric mode couplers,” Journal of Modern Optics 43, 245–251 (1996).

[32] Ronen Barak and Yacob Ben-Aryeh, “Quantum fast Fourier transform and quantum computation by linear optics,” Journal of the Optical Society of America B 24, 231 (2007).

[33] James W. Cooley and John W. Tukey, “An algorithm for the machine calculation of complex Fourier series,” Mathematics of Computation 19, 297–297 (1965).

[34] S. Zhang, C. Lei, A. Vourdas, and J. A. Dunningham, “Applications and implementation of Fourier multiphoton devices,” Journal of Physics B: Atomic, Molecular and Optical Physics 39, 1625–1637 (2006).

[35] Gélo Noel M. Tabia, “Recursive multiphoton schemes for implementing quantum algorithms with photonic integrated circuits,” Physical Review A 93, 012323 (2016).
After the sequential application of the phase gate, the path-only Fourier transform, and the swap gate the state on the right-hand side of Eq. 10 transforms into

$$\frac{1}{\sqrt{d_O d_P}} \sum_{r=0}^{d_O-1} \sum_{q=0}^{d_P-1} e^{i \frac{2\pi}{d_O} m r + i \frac{2\pi}{d_P} r l + i \frac{2\pi}{d} l q} |q\rangle_O |r\rangle_P .$$  

It is easy to see that the exponent simplifies into

$$\frac{2\pi}{d} m r + \frac{2\pi}{d} r l + \frac{2\pi}{d} l q = \frac{2\pi}{d} (m d_P + l) (q d_O + r) \equiv \frac{2\pi}{d} j k \pmod{2\pi},$$

where we defined $|k\rangle = |q d_O + r\rangle = |q\rangle_O |r\rangle_P$ in accordance with Eq. 5 and we used relation $d = d_O \times d_P$. As a result we obtain the transformation rule

$$|j\rangle \rightarrow \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} e^{i \frac{2\pi}{d} j k} |k\rangle ,$$

which is equivalent to formula in Eq. 1. This completes the proof.

**APPENDIX B: OPTICAL ELEMENTS**

The OAM sorter as well as OAM-Path swap consist of two kinds of passive optical elements—OAM exchangers and holo-beam-splitters. Their structure as well as operation are depicted in Fig. 5. Whereas the holo-beam-splitter has a fixed splitting ratio for all OAM modes, the exchanger splits the incoming OAM modes into the two output ports according to a splitting ratio that depends on the OAM mode. As a result, a single exchanger of order $k$ works as 4k different conventional beam-splitters with splitting ratios $0, \pi/(2k), 2\pi/(2k), \ldots, (4k-1)\pi/(2k)$.

The trivial example of an OAM-Path swap is a single exchanger of order $k$. OAM modes $|m\rangle$ entering its input port, where $m$ is a multiple of $k$, are either not affected by the exchanger, or rerouted to the other output port. All remaining OAM modes $|mk + \Delta k\rangle$ with $0 < \Delta k < k$ leave the exchanger in a superposition of the two output ports. The same exchanger can be used to reroute such non-multiple OAM modes as well if input modes are first shifted by $-\Delta k$ with a hologram as shown on the left-hand side in Fig. 5.a. In such a case though, the superpositions are introduced to the multiple OAM modes instead. As an alternative to such an approach we note that the composition of an exchanger of order $k$ and a holo-beam-splitter of order $\alpha$ with $\alpha = \Delta k$ performs the identical rerouting operation, see Fig. 5.b. The advantage of the latter approach is that additional exchangers can be inserted between the two elements. These additional exchangers can reroute unwanted terms away from the setup and inject wanted terms in instead. By using this idea iteratively, the OAM-Path swap operator can be constructed.
FIG. 5. OAM exchangers and holo-beam-splitters. a) The OAM exchanger EX$_k$ of order $k$ is an interferometric device made out of two holograms with opposite values and the OAM parity sorter [36]. Its operation is captured by the transformation formulas depicted at the bottom. One exchanger of order $k$ effectively works as $4^k$ different beam-splitters with varying splitting ratio for different OAM modes. Note the reversed order of sine and cosine functions in the formula for $|m,p\rangle$. b) The holo-beam-splitter HBS$\left(\alpha,k\right)$ of order $\left(\alpha,k\right)$ comprises a beam-splitter with splitting ratio $\alpha \pi / (2k)$ accompanied by two holograms with opposite values $k$ and $-k$. In the diagram, the implementation of the variable splitting ratio beam-splitter is demonstrated with help of two 50:50 beam-splitters. The operation of the holo-beam-splitter is captured by the transformation formulas depicted at the bottom.

APPENDIX C: GENERAL STRUCTURE OF THE OAM-PATH SWAP

To illustrate the operating principle of the OAM-Path swap for general dimensions $d_P$ and $d_O$, let us focus first on the simplest case with $d_P = d_O = 4$, which is depicted in Fig. 6a)–d). When the network of OAM exchangers is used to reroute OAM modes entering different input ports, undesirable superpositions of modes are created by higher-order exchangers. All modes entering the first or the second path are rerouted correctly as in the case of the OAM sorter. Nevertheless, for all other paths, the incoming mode leaves the network in a superposition. Utilizing the observation in Fig. 6a) we can insert holo-beam-splitters as demonstrated in Fig. 7c) to fix the undesirable splitting of modes by the last column of exchangers (see also the previous section). Applying identity from Fig. 6b) we can swap the first column of exchangers with the holo-beam-splitters. This way we fixed behavior of exchangers for modes entering the third and forth input ports. Before the addition of holo-beam-splitters, modes injected into the first and second port were rerouted correctly. This property would be destroyed had we left the upper holo-beam-splitter in place. When we remove it and keep only the lower holo-beam-splitter, we arrive at the setup depicted in Fig. 7d), that sorts correctly all OAM modes injected to any of the input ports.

The construction of the OAM-Path swap for general dimensions is recursive and relies heavily on the property illustrated in Fig. 6a). Let us assume for the moment that $d_P = d_O \equiv \bar{d}$. The swap in dimension $\bar{d} = 2^M$ is constructed from two swaps in dimension $\bar{d}/2 = 2^{M-1}$
FIG. 7. The structure of OAM-Path swap operator. a), b) A specific case of a network of OAM exchangers for $d = 4$ and propagation of OAM modes $|2\rangle$ and $|3\rangle$ entering the third and forth paths. The OAM modes leave the network in a superposition. c) When holo-beam-splitters are inserted as suggested in the figure the undesirable splitting of modes is avoided. d) The first column of exchangers can be swapped with the holo-beam-splitters. Furthermore, the upper holo-beam-splitter can be removed such that it does not affect modes injected into the first and second port. As a result, we obtain a setup that sorts correctly all OAM modes injected to any of the input ports. e) The idea of inserting holo-beam-splitters can be generalized to higher dimensions. We demonstrate the general idea for $d = 2^3$. The setup then consists of two setups for $d = 4$ from d) that are connected by four exchangers of order 1. Notice also that orders of exchangers and holo-beam-splitters in the two setups have to be multiplied by two. To comply with the formula in Eq. 7 an additional permutation of paths and a series of properly rotated Dove prisms are necessary in the final part of the setup. As the last part does not alter the evolution of modes through the network we omit it in the following. f) The holo-beam-splitters from the $d/2$-dimensional swaps can be moved to the left thanks to identity in Fig. 6 b). At this point, all OAM modes that enter any of the first $d/2 = 4$ ports are sorted correctly. For the other four input ports the OAM modes undergo more complex evolution and leave the setup in a superposition. In analogy to c), appropriately chosen holo-beam-splitters are inserted into the setup to preclude creation of such superpositions. g) Using identities from Fig. 6 b) and c) all holo-beam-splitters can be aggregated in front of the network of exchangers. h) Holo-beam-splitters aggregated this way in the first four paths nevertheless negatively affect OAM modes propagating through these paths. We can remove these holo-beam-splitters to restore the sorting properties of the network for these input paths. In the case of the last four paths, we can use identity from Fig. 6 d) to merge all holo-beam-splitters originating in the setup for $d = 4$ with the aggregated holo-beam-splitters. For $d = 4$ there is only one holo-beam-splitter. We can merge it with its neighbour as made clear in the figure. i) This way we obtained the setup for the OAM-Path swap in $d = 8$. All OAM modes that enter any of the eight input ports are sorted correctly. We can identify two conceptually different parts of the swap operator. The network of exchangers, henceforth referred to as an $E$ block and a series of networks of increasing size made out of holo-beam-splitters, which we refer to as $H$ blocks.

that are connected by a layer of additional exchangers. For a specific example in $d = 8$ refer to Fig. 7 e), where the $d/2$-dimensional swap is presented in Fig. 7 d). All OAM modes entering first $d/2$ ports are sorted correctly due to properties of $d/2$-dimensional swaps. Nevertheless, modes that are injected into the other $d/2$ ports leave the network in a superposition of $d/2$ output ports. The reason is that the layer of additional exchangers adds one quantum of OAM to such modes and their value is thus no longer a multiple of the order of exchangers in the remaining layers. These exchangers then act not as switches, but rather as genuine beam-splitters.

As follows from Fig. 6 a), the switch-like behaviour can be restored if specifically chosen holo-beam-splitters are added to the remaining layers of exchangers, see also Fig. 7 f). Making use of identities in Fig. 6 b) and c) all
such holo-beam-splitters can be aggregated at the beginning of the network as shown in Fig. 7(g). At this point, one has to recall that the holo-beam-splitters were introduced to correctly reroute modes entering last $d/2$ ports. Modes entering first $d/2$ ports should not be affected by the additional holo-beam-splitters. All such holo-beam-splitters are therefore removed from the first $d/2$ paths. This step is illustrated in Fig. 7(h). To save resources, one can merge the additional holo-beam-splitters in last $d/2$ paths with those holo-beam-splitters that are there due to the $d/2$ dimensional swap. At the end, we obtain the OAM-Path swap in dimension $d$ that correctly sorts OAM modes injected to any of its $d$ input ports as shown in Fig. 7(i). The network of exchangers forms the routing part of the swap operator, which we refer to as an $E$ block, and the presence of superpositions in output modes is corrected for by a series of networks of increasing size made out of holo-beam-splitters. We refer to these smaller networks as $H$ blocks.

In order for the setup to implement the swap transformation in the sense of Eq. (4) an additional permutation of paths has to be appended to the setup in each recursion (cf. Fig. 7(c)). This permutation reroutes modes from paths $(p_0, p_1, p_2, \ldots, p_{d-1})$ to paths $(p_0, p_2, p_4, \ldots, p_{d-2}, p_1, p_3, \ldots, p_{d-1})$. As the last stage, a Dove prism rotated through an angle of $\pi/4$ has to be inserted in each path (in each recursion; compare again with Fig. 7(e), where two recursions have been applied). These Dove prisms correct for alternating phases of modes leaving the network. Let us note that $H$ blocks can be simplified even more as identity similar to that in Fig. 6(b) exists for beam-splitters and holograms. This way, all holograms present in holo-beam-splitters can be put to the sides of the block, whose middle part is then formed merely by conventional beam-splitters.

In cases when $d_O \neq d_P$ one constructs the network for dimension $d = \max(d_O, d_P)$ and then uses only first $d_P$ input ports and first $d_O$ output ports of the network. Obviously, some elements then do not enter the evolution of injected OAM modes and can be removed from the network with no effect on the swap functionality.

**APPENDIX D: SCALING OF THE NUMBER OF BEAM-SPLITTERS FOR HIGH DIMENSIONS**

The number of beam-splitters present in the setup reflects the complexity of the interferometric structure of the setup. In this section we estimate how this number scales with the dimension of the form $d = 2^{k^2}$ for a specific choice $d_P = d_O = \sqrt{d}$. It is not hard to calculate the number of beam-splitters required to implement the OAM sorter and path-only Fourier transform in dimension $d_P$ is $N_{BS}^{(\text{sort})}(d_P) = 2(d_P - 1)$ and $N_{BS}^{(\text{FFT})}(d_P) = \frac{d_P}{d} \log_2(d_P)$ [31, 32], respectively. The OAM-Path swap comprises $E$ blocks with $N_{BS}^{(E)}(d_P) = d_P \log_2(d_P)$ and a series of $H$ blocks, for which we can determine the total number of beam-splitters as

$$N_{BS}^{(H)}(d_P) = \sum_{k=1}^{\log_2(d_P)-1} \frac{2^k}{2} \log_2(2^k) = \frac{d_P}{2} \log_2(d_P) - d_P + 1.$$ (14)

The number of beam-splitters implementing the OAM-Path swap in dimension $d_P$ is therefore $N_{BS}^{(\text{swap})}(d_P) = N_{BS}^{(H)}(d_P) + N_{BS}^{(\text{sort})}(d_P) + N_{BS}^{(\text{FFT})}(d_P) = \frac{d_P}{2} \log_2(d_P) - d_P + 1$. The number of conventional beam-splitters required in our implementation of the Fourier transform for dimensions of the form $d_k := 2^{2^k}(\sqrt{d} - d_k - 1)$ is then

$$N_{BS}^{(\text{FT})}(d_k) = 2N_{BS}^{(\text{sort})}(d_k-1) + N_{BS}^{(\text{FFT})}(d_k-1) + N_{BS}^{(\text{swap})}(d_k-1) + d_k - 3 + d_k - 1 - N_{BS}^{(\text{sort})}(d_k-1).$$ (15)

The last recursive relation can be solved when we define $c_k := 2d_k \log_2(d_k) + 3d_k - 3$ such that

$$N_{BS}^{(\text{FT})}(d_k) = c_{k-1} + d_k - 1 - N_{BS}^{(\text{FT})}(d_k-1)$$

$$= \sum_{j=2}^{k-1} \prod_{l=j}^{m} d_l \frac{c_j - 1 + \prod_{l=1}^{k-1} d_l}{N_{BS}^{(\text{FT})}(d_1)}$$

$$= \sum_{j=2}^{k} 2^{2^j - 2^j} c_j - 1 + 2^{2^j - 2} N_{BS}^{(\text{FT})}(d_1)$$

$$= 2d_k \sum_{j=1}^{k-1} \frac{c_j - 1 + \prod_{l=1}^{j} d_l}{N_{BS}^{(\text{FT})}(d_1)}$$

$$= 2d_k \sum_{j=1}^{k-1} \frac{2^j}{2^{2j}} + 3 \left(\frac{d_k}{4} - 1\right) + \frac{d_k}{4} N_{BS}^{(\text{FT})}(d_1).$$

We can bound the sum in the previous formula as

$$\sum_{j=1}^{k-1} 2^j \leq \sum_{j=1}^{k-1} \frac{2^j}{2^{2j}} \leq \sum_{j=1}^{\infty} \frac{1}{2^j} = 1,$$ (16)

and also directly calculate $N_{BS}^{(\text{FT})}(d_1) = 16$ so that in the end we arrive at

$$N_{BS}^{(\text{FT})}(d_k) \leq 7d_k.$$ (17)

| $d$ | $d_P$ | $d_O$ | $F_{OAM}^{(d)}$ | $F_{Path}^{(d)}$ |
|-----|------|------|----------------|----------------|
| 1   | 2    | 1    | 5              | 5              |
| 2   | 4    | 1    | 16             | 16             |
| 3   | 8    | 2    | 40             | 40             |
| 4   | 16   | 4    | 88             | 92             |
| 5   | 32   | 8    | 184            | 204            |
| 6   | 64   | 16   | 380            | 444            |
| 7   | 128  | 32   | 780            | 956            |
For the overall scaling of the Fourier transform in dimension \( d = 2^k \) as given in this paper we therefore obtain

\[
N_{BS}^{(FT)}(d) \sim O(d). \tag{18}
\]

The exact total numbers of beam-splitters required by our scheme are shown in Tab. I for several lowest dimensions of the form \( d = 2^M \). Optimal choices of \( d_P \) and \( d_O \) for these dimensions were found by an optimization algorithm that searched for a setup with the lowest number of beam-splitters when a given dimension \( d \) was fixed.

The brute force approach for the implementation of the OAM Fourier transform is to use a \( d \)-dimensional OAM sorter to transform the OAM state of a photon into the path encoding; then apply the path-only Fourier transform; and in the end apply another \( d \)-dimensional OAM sorter operated in reverse. For comparison, the number of beam-splitters required in such a brute-force approach are also shown in Tab. I.