Finite temperature scaling theory for the collapse of Bose-Einstein condensate

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(Dated: March 23, 2022)

We show how to apply the scaling theory in an inhomogeneous system like harmonically trapped Bose condensate at finite temperature. We calculate the temperature dependence of the critical number of particles by a scaling theory within the Hartree-Fock approximation and find that there is a dramatic increase in the critical number of particles as the condensation point is approached.

PACS numbers: 03.75.Hh, 03.75.-b, 05.30.Jp

I. INTRODUCTION

Experimental[1, 2, 3, 4, 5, 6, 7] and theoretical[8, 9, 10, 11, 12] studies of Bose-Einstein condensation(BEC) reveal the fact that Bose-Einstein condensates of trapped ultracold atomic gases are influenced by atomic interaction[8, 9, 10, 12, 13, 14] which in general is characterized by s-wave scattering length($a_s$) and can be tuned[15, 16, 17, 18] by an external magnetic field. Stability and collapse of the Bose gas with negative scattering length has been observed in the clouds of ultracold $^7$Li[17, 18] and $^8$Rb[19, 20]. If the interaction is attractive, the gas tends to increase the density of the central region of the trap. This tendency is opposed by the zero-point energy and thermal energy of the atoms. If the number of atoms is greater than a critical number($N_c$), the central density increases strongly and the zero-point and thermal energies are no longer able to avoid the collapse of the gas.

Let us first consider a many particle system of 3-D isotropic harmonically trapped ideal Bose gas of $N$ alkali atoms. The system is in equilibrium with its surroundings at temperature($T$). Let the mass of each particle be $m$. The ground state of a single particle is macroscopically populated below a certain temperature which is called the Bose-Einstein condensation temperature and is given by $\frac{\hbar}{k_B}T_c = \frac{\hbar^2a_s}{4m\omega^2} \left(\frac{\mu}{\hbar^2}\right)^{1/3}$, where $\omega$ is the angular frequency of the harmonic oscillators and $k$ is the Boltzmann constant. Now we consider the gas to be interacting weakly and attractively by the two body interacting potential $V_{int}(r) = g\delta^3(r)$, where $g = -\frac{4\pi \hbar^2a}{m}$ is the coupling constant and $a = a_s$ is the absolute value of the s-wave scattering length. For $0 \leq T < T_c$ the length scale of the system becomes $l = \sqrt{\frac{\hbar^2a}{2m\mu}}$, which is the length scale of the ground state of the harmonic oscillators. The typical two body interaction energy for $N$ number of particles is $\frac{\hbar^2}{2m} N^2a/2V$, where $V$ is the volume of the system. For the attractive interaction, the gas tends to increase the density of the central region of the trap. For $T = 0$, this tendency is resisted by the zero-point energy of the atoms. If the number of atoms is greater than a critical number, the central density increases so much that the zero-point energy is no longer able to avoid the collapse of the gas. In the critical situation, the typical oscillator energy must be comparable to the typical interaction energy and as a consequence we have $\frac{\hbar^2a}{2m} \sim 1$.

However, for $0 < T < T_c$, although length scale of the system is $l$ yet the thermal fluctuation dominates over the quantum fluctuation. In this situation, the collapse of the attractive Bose gas is opposed by the thermal energy($\frac{\hbar^2}{2m} l^4 \zeta(4) \sim \hbar^2 N^{4/3}$) of the atoms and consequently, we get the estimation of the critical number as $\frac{N^2}{l^3} \sim \frac{\hbar^2a}{2m} \sim \frac{\hbar^2a}{2m} \frac{1}{\zeta(4)} = \frac{1}{\zeta(4)} > 1$.

However, close to the condensation temperature, the length scale of the system becomes $\frac{\hbar^2a}{2m} l^3 \sim \frac{\hbar^2a}{2m} l^3 \frac{1}{\zeta(4)} \gg 1$. For this reason the estimation of the critical number would be $\frac{N^2}{l^3} \sim \frac{\hbar^2a}{2m} \frac{1}{\zeta(4)} > 1$.

In this paper we will explicitly calculate the critical numbers by a scaling theory within the Hartree-Fock(H-F) approximation and will explicitly show its temperature dependence. Before doing that, let us see how the critical number for $T = 0$, was calculated by the scaling theory of Baym and Pethick [13].

II. ZERO TEMPERATURE SCALING THEORY FOR THE COLLAPSE OF BEC

For $T \to 0$, the system of $N$ weakly interacting Bose particles is well described by the Gross-Pitaevskii(G-P) equation

$$i\hbar \frac{\partial \Psi_0(r, \tau)}{\partial \tau} = \left(-\frac{\hbar^2\nabla^2}{2m} + V(r) + g \left| \Psi_0(r, \tau) \right|^2 \right) \Psi_0(r, \tau),$$

where $V(r) = \frac{1}{2} m \omega^2 r^2$ is the harmonic potential and $\Psi_0(r, \tau)$ is the order parameter for Bose condensate and is a function of space($r$) and time($\tau$). A stationary solution of the G-P equation have the form $\Psi_0(r, \tau) = \psi_0(r) e^{-i \mu \tau}$, where $\mu$ is the chemical potential. The stationary solution of this form can be obtained by mini-
mizing the energy functional

\[ E = \int \left( \frac{\hbar^2}{2m} | \nabla \Psi_0(r) |^2 + V(r) | \Psi_0(r) |^2 + \frac{g}{2} | \Psi_0(r) |^4 \right) d^3r \]  

(2)

for a fixed number of particles \( N = \int | \Psi_0 |^2 \, dr \). The stationary solution without current gives \( \Psi_0(r) = \sqrt{n(r)} \), where \( n(r) \) is the number density of particles at the position \( r \). For stationary and for no current condition, we have the energy functional from the Eqn.(2) as

\[ E(n) = \int dr \left( \frac{\hbar^2}{2m} | \nabla \sqrt{n} |^2 + nV + \frac{g}{2} n^2 \right). \]  

(3)

For the noninteracting gas \( g = 0 \), the solution of G-P equation must be the ground state of harmonic oscillators. Hence, for \( g = 0 \), the solution of the G-P equation is \( n(r) \sim e^{-r^2/\lambda^2} \). For \( g < 0 \), the solution of G-P equation was obtained by a scaling theory(variational approach) based on the Gaussian function \([10, 14]\). From this scaling theory we can write \( \Psi_0(r) = \sqrt{n} = \sqrt{\nu} \frac{N}{\sqrt[3]{\pi \nu}} e^{-r^2/2\nu^2} \), where \( \nu \) is a scaling parameter which fixes the width of the condensate. Since the attractive interaction causes the reduction of the width of the trap, \( \nu \) must be less than one. With this solution, the energy functional of the Eqn.(3) can be written in units of \( Nh \omega \) as \([14]\)

\[ X_0(\nu) = \frac{E(\nu)}{Nh \omega} = \left[ \frac{3}{4} \frac{1}{\nu^2} + \nu^2 - \frac{Na}{\sqrt{2\pi l^3 \nu^3}} \right]. \]  

(4)

According to FIG. 1, this energy functional has a minimum and a maximum below a critical number of particles \( N_c \). In equilibrium, the system must be at the minimum of the energy functional. About this minimum, \( \nu \) has a stable value. Above the critical number of particles, \( E(\nu) \) has no stable minimum. In this situation, the energy of the system arbitrarily reduces to take \( \nu = 0 \) and the gas collapses to vanish its width. \( N_c \) is calculated by requiring that the first and second derivative vanish at the critical point. From this scaling theory we can have \([8, 10, 14]\) \( N_c \frac{a}{\lambda} = 0.671 \). According to the numerical result \([21]\) \( N_c \frac{a}{\lambda} = 0.575 \) and according to the experimental result \([20]\) \( N_c \frac{a}{\lambda} = 0.459 \). However, the same problem was also attacked by many authors \([22, 23, 21, 25]\). Kagan, Surkov and Shlyapnikov discussed the dynamic properties of a trapped Bose-condensed gas under variations of the confining field and found analytical scaling solutions for the evolving condensate \([22]\). Pitaevskii addressed the condition of collapse from the dynamical point of view \([24]\). Yukalov and Yukalova calculated the optimal trap shape allowing for the condensation of the maximal number of atoms with negative scattering lengths \([24]\). Savage, Robins and Hope solved the Gross-Pitaevskii equation numerically for the collapse, induced by a switch from positive to negative scattering lengths \([25]\). Although the G-P equation is extensively used for the Bose gas with attractive interaction, yet it is to be noted that, it was rigorously derived for repulsive interaction \([26]\).

### III. SCALING ASSUMPTION FOR THE COLLAPSE OF BEC AT FINITE TEMPERATURE

Although the problem of an attractive Bose gas at finite temperature was discussed by many authors \([27, 28, 29]\), yet the temperature dependence of critical number of particles with a scaling theory has not been calculated. The authors of the Refs. \([27, 28]\) considered the collapse of the condensate only. The authors of the Ref. \([29]\) explored the collapse for \( T = 0 \) and for \( T \geq T_c \). But we are considering the collapse for \( 0 \leq T \leq T_c \) and are considering the fact that both the condensed cloud and the thermal cloud of the Bose gas would collapse due to the attractive type of interaction. In the fifth section we will calculate \( N_c \) by the following scaling theory within the Hartree-Fock(H-F) approximation which was discussed extensively for Bose gas in the Ref. \([8]\).

For the interacting Bose gas, average occupation numbers \( n_i \) as well as single particle wave functions \( \psi_i \) are determined \([8]\) by minimizing the grand potential \( \Omega = (E - TS - \mu N) \), where \( S = k \sum_i (1 + n_i) \ln (1 + n_i) - (n_i) \ln (n_i) \) is the entropy and \( i \) runs from 0 to \( \infty \). The entropy \( S \) and the total number of particles \( N = \sum_i n_i \) both are the functions of \( \{ n_i \} \). Minimizing the grand potential with respect to the occupation numbers \( n_i \) we get the average occupation number as \( \bar{n}_i = \frac{1}{e^{(\epsilon_i - \mu)/\hbar \omega} - 1} \).
Since the entropy and the total number of particles depend on the occupation numbers \( \{ n_i \} \), the single particle wave functions \( \{ \phi_i \} \) are simply obtained by minimizing the energy functional with proper normalization constraint \( \int d^3 r \ | \phi_i \|^2 = 1 \). Within the Hartree-Fock approximation, we have the expression of energy functional as\[8\]

\[
E = \int d^3 r \left( \frac{\hbar^2}{2m} \nabla \phi_i \cdot \nabla \phi_i + V(r) n_0(r) + V(r) n_T(r) + g \frac{\nu^2}{2} \phi_i^2(r) \right)
\]

Minimizing the above energy functional of the Eqn.(5) we have the following equations[8].

\[
\begin{align*}
\left[ -\frac{\hbar^2}{2m} \nabla^2 + V(r) + g(n_0(r) + 2n_T(r)) \right] \phi_i &= \mu \phi_i \\
\left[ -\frac{\hbar^2}{2m} \nabla^2 + V(r) + 2gn_0(r) \right] \phi_i &= \epsilon_i \phi_i
\end{align*}
\]

From the Eqns.(6) and (7) we can determine the wave functions \( \{ \phi_i \} \) and the energy eigenvalues \( \{ \epsilon_i \} \) for a given \( n_0(r) \) and \( n_T(r) \). In our scaling theory we introduce a scaling parameters \( \nu \), which would fix the width of the condensed and thermal clouds as well as would take proper choices of \( n_0(r) \) and \( n_T(r) \). Eventually the choices of the wave functions \( \{ \phi_i \} \) are denoted by the scaling parameter \( \nu \). Since the minimum of the grand potential corresponds to the equilibrium of the system, the minimum of the energy functional for a certain choice of \( \{ \phi_i \} \) would correspond to the equilibrium of the system. So, for equilibrium condition of the system, the energy functional would be minimized with respect to the scaling parameter(\( \nu \)). In the FIG. 2, we will see that above a critical number of particles(\( N_c \)), the energy functional has no stable minimum. If the energy functional as well as the grand potential has no stable minimum, the grand potential as well as the energy functional would arbitrarily decrease until the width of the system becomes zero. Under this condition the system is said to be collapsed.

In absence of the interaction, the condensed cloud is described by the wave function \( \phi_0(r) = \frac{1}{\pi^{3/2}} e^{-r^2/2} \) and by the density \( n_0(r) = \frac{1}{\pi^{3/2}} e^{-r^2/2} \). In absence of interaction, the number density of the excited particles is\[8\]

\[
n_T(r) = \int_0^\infty \frac{1}{\nu^{3/2}} e^{-(x^2 + \frac{\nu^2 r^2}{2})} \frac{4\pi^2 dp}{(2\pi h)^2} = \frac{\lambda_T}{\lambda_T} g_2(\nu) \]

where \( \lambda_T = \sqrt{\frac{2\pi^2 m}{\nu^2}} \), and \( g_2(x) = x + x^2/2 + x^3/6 + \ldots \) is Bose-Einstein function of a real variable \( x \). The length scale of the excited particles in thermal equilibrium is \( L_T \sim \sqrt{\frac{\hbar^2}{\nu m}} \). Since the length scale(\( L_T \)) of the thermal cloud is proportional to the length scale(\( l = \sqrt{\hbar/m\nu} \)) of the condensed cloud, the two length scale would reduce by the same factor due to the attractive interaction.

In presence of the attractive interaction the wave function of the condensed would become

\[
\phi_0(r) = \frac{1}{\nu^{3/2}} e^{-r^2/2\nu^2/l^2}
\]

and the density of the thermal cloud would become

\[
n_T(r) = \frac{1}{\nu^{3/2}} g_2(\nu) e^{-\frac{\nu^2 r^2}{2l^2}}.
\]

The same scaled form of \( n_T(r) \) is also obtained from the variation of B-E statistics such that \( \bar{n}(p,r) = \frac{e^{(\nu^2 p^2 + \frac{\nu^2 r^2}{2})/kT} - 1}{\nu^3 \lambda_T^3} \). With this variational form of statistics we have the total number of excited particles as

\[
N_r = \int d^3 r \frac{1}{\nu^{3/2}} g_2(\nu) e^{-\frac{\nu^2 r^2}{2l^2}} = \frac{1}{\nu^3 \lambda_T^3} \psi(\nu^3 \lambda_T^3)
\]

The second term of the energy functional is the kinetic energy of the thermal cloud and is approximately obtained as

\[
\int_0^\infty \int_0^\infty \frac{p^2}{2m} e^{-(\frac{\mu^2 p^2}{kT} + \frac{\nu^2 r^2}{2})} \frac{4\pi^2 dp}{(2\pi h)^2} \frac{4\pi^2 dr}{(2\pi h)^2}
= \frac{3}{2} \frac{1}{\nu^3 \lambda_T^3} \left( \frac{T}{T_c} \right)^4 \zeta(4) \hbar \omega.
\]
The third term of the energy functional is the potential energy of the condensed cloud and is obtained as

$$\int_0^\infty \frac{1}{2} m \omega^2 r^2 \left[ \sqrt{\frac{n_0}{\nu^3 \pi^{3/2}}} e^{-r^2/2\nu^2 r^2} \right]^2 4\pi r^2 dr = \frac{3}{4} \nu^2 \omega N \left[ 1 - \left( \frac{T}{T_c} \right)^3 \right].$$

(12)

The fourth term of the energy functional is the potential energy of the thermal cloud and is obtained as

$$\frac{1}{\nu^3 \lambda_T^3} \int_0^\infty [g \left( e^{-\frac{r^2}{2\nu^2 r^2}} \right)]^2 \frac{1}{2} m \omega^2 r^2 4\pi r^2 dr = \frac{3}{2} \nu^2 \left( \frac{N}{\zeta(3)} \right)^{1/3} \left( \frac{T}{T_c} \right)^4 \zeta(4) \hbar \omega.$$  

(13)

The fifth term of the energy functional is the interaction energy of the particles solely in the condensed cloud and is obtained as

$$\frac{2 \pi \hbar^2 a}{m} \frac{n_0^2}{\nu^3 \lambda_T^3} \int_0^\infty e^{-\frac{r^2}{2\nu^2 r^2}} 4\pi r^2 dr = \frac{N^2}{\sqrt{2\pi \nu^3}} \hbar \omega \left[ 1 - \left( \frac{T}{T_c} \right)^3 \right]^2.$$  

(14)

The sixth term of the energy functional is the interaction energy of the particles in the condensed and thermal clouds and is obtained as

$$\frac{32 \pi^2 \hbar^2 a}{m} \frac{n_0}{\nu^3 \lambda_T^3} \int_0^\infty e^{-\frac{r^2}{2\nu^2 r^2}} g \left( e^{-\frac{r^2}{2\nu^2 r^2}} \right)^2 r^2 dr = \frac{8 \pi \hbar^2 a n_0}{m \nu^3 \lambda_T^3} \sum_{i=1}^\infty \left( \frac{2kT \nu^2}{i^2 m \omega^2 r^2 + i^2 kT} \right)^{3/2} \approx \frac{2^{3/2} \sqrt{\zeta(3)/2}}{\sqrt{\pi \nu^3}} a \frac{N^{3/2}}{T} \left[ \frac{T}{T_c} \right]^{3/2} \hbar \omega.$$  

(15)

The seventh term of the energy functional is the interaction energy of the particles solely in the thermal cloud and is obtained as

$$\frac{4 \pi \hbar^2 a}{m} \frac{4 \pi}{\nu^3 \lambda_T^3} \sum_{i,j=1}^\infty \int_0^\infty e^{-\frac{r^2}{2\nu^2 r^2}} \frac{(i+j)^2 r^2}{j^{3/2} i^{3/2}} dr = \sqrt{\frac{2}{\pi \nu^3 \zeta(3)^{3/2}}} \frac{a}{T} \hbar \omega \left( \frac{T}{T_c} \right)^{9/2}.$$  

(16)

where $S' = \sum_{i,j=1}^\infty \left( \frac{(i+j)^2}{i^{3/2} j^{3/2}} \right) \approx 0.6534$.

The energy functional $E$ of the Eqn.(5) is now a function of the variational parameter $\nu$. Above the critical number of particles, $E(\nu)$ has the lowest value ($-\infty$) at $\nu = 0$. So, above the critical number of particles, the width of the system would be zero to signalize the collapse. Let us define the energy per particle in units of $\hbar \omega$ as $X_t(\nu) = \frac{E(\nu)}{\nu \hbar \omega}$. Let us also write $\frac{T}{T_c} = t$. Now collecting all the terms of the energy functional from the Eqn.(10) to (16), we have

$$X_t(\nu) = \frac{3}{4} \left[ \frac{1}{\nu^2} + \nu^2 \right] [1 - t^3] + \frac{3 N^{1/3} \zeta(4)}{2} \frac{1}{\zeta(3)^{3/4}} t^4 \left( \frac{1}{\nu^2} + \nu^2 \right)$$

$$- \frac{1}{\sqrt{2\pi T}} \frac{N a}{T} [1 - t^3] + \sqrt{\frac{8 \zeta(3/2)}{\pi}} \frac{N^{1/2} a}{T} t^{3/2} [1 - t^3]$$

$$+ S' \sqrt{\frac{2}{\pi \zeta(3)^{3/2}}} N^{1/2} a t^{3/2} \frac{1}{\nu^3} \left( \frac{1}{\nu^2} + \nu^2 \right)$$

$$= c_1(t) \left[ \frac{1}{\nu^2} + \nu^2 \right] - c_2(t) \frac{1}{\nu^3}$$  

(17)

where

$$c_1(t) = \frac{3}{4} \left[ 1 - t^3 \right] + \frac{3 N^{1/3} \zeta(4)}{2} \frac{1}{\zeta(3)^{3/4}} t^4$$  

(18)

and

$$c_2(t) = \frac{1}{\sqrt{2\pi T}} \frac{N a}{T} [1 - t^3] + \sqrt{\frac{8 \zeta(3/2)}{\pi}} \frac{N^{1/2} a}{T} t^{3/2} [1 - t^3]$$

$$+ S' \sqrt{\frac{2}{\pi \zeta(3)^{3/2}}} N^{1/2} a t^{3/2} \frac{1}{\nu^3}.$$  

(19)

From the Eqns.(18) and (29), and putting $\frac{T}{T_c} = 0.0066$ and $t = 0.8$, in the Eqn.(17) we can write

$$X_{0.8}(\nu) = (0.366 + 0.5203 N^{1/3} \left[ \frac{1}{\nu^2} + \nu^2 \right] - \frac{0.0069 N^{1/2} + 0.000627 N}{\nu^3}$$  

(20)

The Eqn.(20) represents the FIG. 2. In the FIG. 2 we see that below a critical number of particles, $X_t(\nu)$ has a stable minimum and an unstable maximum. At the critical number of particles, the minimum and maximum coincide. Above the critical number of particles, there is no stable minimum and the energy of the system arbitrarily decreases to $-\infty$ and its width($\nu l$) becomes zero to signalize the collapse of the gas.

V. CRITICAL CONDITIONS FOR THE COLLAPSE

For a minimum of $X_t(\nu)$ as well as for the equilibrium condition of the system, we must have $\frac{\partial X_t}{\partial \nu} = 0$. At the critical point($\nu_c$), the minimum and maximum coincide. So, at the critical point, we must have $\frac{\partial X_t}{\partial \nu} |_{\nu_c} = 0$. For the critical point($\nu_c$), the Eqn.(17) gives following equations

$$\frac{\partial X_t}{\partial \nu} |_{\nu_c} = -c_1(t) \frac{2}{\nu^3} + c_1(t) \nu + c_2(t) \frac{3}{\nu^4} = 0$$  

(21)
From the Eqns.(21) and (22) we have \( \nu_c = (1/5)^{1/4} = 0.66874 \) and \( c_2(t) = \frac{8}{15} \nu_c C_1(t) \), which represents the equation for the critical number \( N_c \) and can be recast as

\[
\frac{N_c a}{\sqrt{2\pi l}} \left[ 1 - t^3 \right]^{1/2} + \sqrt{\frac{8\zeta(3/2)}{\pi}} \frac{N_c^{1/2} a t^{3/2}}{l} \left[ 1 - t^3 \right] \\
+ 3 \int \frac{N_c^{1/3} \zeta(4) t^4}{l} + \frac{3}{2} \frac{N_c^{1/3} \zeta(4) t^4}{\left[ \zeta(3) \right]^{1/3}}.
\]

(23)

From the Eqn.(23) we get the expression of \( \frac{N_c a}{l} \) for \( t = 0 \) as

\[
\frac{N_c a}{l} = \sqrt{\frac{2\pi}{2}} \left[ \frac{4\nu_c}{5} \right] = 0.671
\]

(24)

Now we see our scaling theory for \( T > 0 \) to be consistent with that given by Baym and Pethick for \( T = 0 \).

For \( 0 < t < 1 \), we can consider the second and third terms of the left hand side and the first term of the right hand side of the Eqn.(23) as the perturbing terms. We expand its left hand side with respect to its first term and similarly, we expand its right hand side with respect to its second term. Now we get the critical number from the Eqn.(23) up to three leading orders in \( l/a \) as

\[
\frac{N_c a}{l} = \left[ \frac{\sqrt{2\pi \zeta(4)}^{3/2}}{\left[ \zeta(3) \right]^{1/2}} \right] \frac{4\nu_c}{5} \left[ \frac{a}{l} \right]^{1/2} \left[ 1 - t^3 \right]^{3/2}
\]

\[
+ 3 \int \frac{2\pi^{1/2}}{\sqrt{\zeta(3)}} \left[ \frac{4\nu_c}{5} \right] \left[ \frac{a}{l} \right]^{1/4} \left[ 1 - t^3 \right]^{7/2}
\]

\[
= 1.210 \frac{a}{l}^{1/2} t^6 + \frac{1.006}{1 - t^3} - \frac{10.66 t^3}{1 - t^3}.
\]

(25)

The Eqn.(25) represents the behavior of the critical number with temperature. For \( \frac{a}{l} = 0.0066 \), the change of critical number \( N_c \) with \( t \) is shown in FIG. 3. From this figure and according to the Eqn.(25), we get \( \frac{N_c a}{l} = 27.89 \) for \( t = 0.8 \). The Eqn.(25) represent an approximate relation between \( N_c \) and \( t \). However, from the FIG. 2, we see that \( \frac{N_c a}{l} = 27.78 \) for \( t = 0.8 \).

At the condensation temperature i.e. as \( t \to 1 \), only the third term of the left hand side of the Eqn.(23) con-
tributes to give
\[
\frac{N_c a}{l} = 2.253\left(\frac{l}{a}\right)^5 + \frac{\pi^3 3 \zeta(3) \zeta(4)^6 4\nu_c}{5 S^6}
\]

From the above equation we see that near the condensation temperature \( N_c^{3/6} = \frac{T}{T_c} = 1.145 \) is of the order of unity and is also expected from the qualitative point of view.

VI. CONCLUSIONS

Initially we explained the physics of the collapse of the attractive atomic Bose gas. Then we qualitatively estimated \( N_c a/l \) for the various range of temperatures \( 0 \leq T \leq T_c \). Finally we calculate \( N_c a/l \) by a scaling theory within the Hartree-Fock approximation. Our calculation supports the qualitative estimation of \( N_c a/l \).

In the FIG. 3, we see that the temperature dependence of \( N_c \) is not significant for \( 0 \leq T \leq 0.5 T_c \). If experiment of collapse is performed within this range of temperature, the zero temperature theory would well satisfy the experimental result.

Here we consider an attractive type of delta potential which is not an actual potential but a model potential \[8, 10, 14\]. The problem with this model potential is that, no bound state is formed for a finite number of particles in a three dimensional delta potential. So, for a true delta potential, the collapse of a finite number of particles is impossible. However, once we do the scaling assumption, the collapse is not impossible for a delta potential and we can calculate the critical numbers.

It is a coincidence of our scaling theory that the grand potential and the energy are minimized simultaneously for a certain choice of the scaling parameter. However, in general, the grand potential and the energy are not minimized simultaneously. If we want to calculate the critical number by some other theory, the minimization of the energy may not be sufficient for the calculation of the critical number. In that case, some modification would come from the entropy and chemical potential.

Although the Hartree-Fock approximations are extensively used \[8, 13\] even for the critical region \( (T_c) \) of Bose systems, yet the Hartree-Fock-Bogoliubov (H-F-B) approximation is a more reliable one \[28\]. Any mean-field approximation (H-F, H-F-B, etc.) is also known to be less suitable for the critical region.

We considered the interaction as \( V_{int}(r) = g\delta^3(r) \). For repulsive interaction, the density of the gas in the central region decreases. This would cause a slight decrease in \( T_c \) as shown theoretically in the Ref. \[13\] and experimentally in the Refs. \[2, 5\]. On the other hand, for attractive interaction, the density in the central region of the trap would increase and we expect slight increase in \( T_c \). However, our scaling theory does not predict any \( T_c \) shift. This is a limitation of our scaling theory. We should look for a better theory which would simultaneously predict the critical numbers and the \( T_c \) shift.

VII. ACKNOWLEDGMENT

Several useful discussions with J. K. Bhattacharjee of SNBNCBS, Deepak Dhar of TIFR and Koushik Ray of IACS are gratefully acknowledged. We also acknowledge the hospitality of the Centre for Nonlinear Studies in Hong Kong Baptist University.

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