The Heston Riemannian Distance Function

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Abstract

The Heston model is a popular stock price model with stochastic volatility that has found numerous applications in practice. In the present paper, we study the Riemannian distance function associated with the Heston model and obtain explicit formulas for this function using geometrical and analytical methods. Geometrical approach is based on the study of the Heston geodesics, while the analytical approach exploits the links between the Heston distance function and the sub-Riemannian distance function in the Grushin plane. For the Grushin plane, we establish an explicit formula for the Legendre-Fenchel transform of the limiting cumulant generating function and prove a partial large deviation principle that is true only inside a special set.

1. INTRODUCTION

There are two main protagonists in this paper: the Riemannian manifold associated with the Heston model of the stock price, and the Grushin plane, which is one of the best-known examples of a sub-Riemannian manifold. The present paper focuses on the Heston Riemannian distance and the Grushin sub-Riemannian distance and provides explicit formulas for these. The Heston distance and the Grushin distance are intimately related, and various facts concerning these distances can be easily transplanted from one setting into the other.

We will next briefly describe the main results obtained in this paper. Theorems 1 and 3 below contain explicit formulas for the Heston distance. The formulas in Theorem 1 are established using geometrical methods, while the proof of the distance formula in Theorem 3
uses certain links between the Heston and the Grushin distances and is more analytical. In the proof of Theorem 3, we compute and study the limiting cumulant generating function $\Lambda$ for the Grushin plane and the Legendre-Fenchel transform $\Lambda^*$ of the function $\Lambda$. One of the main results in the present paper is a partial large deviation principle for the Grushin plane (see Theorem 22). The word “partial” is used in the previous sentence because in the case of the Grushin plane the large deviation principle with $\Lambda^*$ as a rate function holds only inside a special subset of $\mathbb{R}^2 \times \mathbb{R}^2$. We would also like to bring the reader’s attention to the results concerning certain qualitative properties of the transcendental equations whose solution is involved in determining the Heston Riemannian distance function. These convexity and monotonicity properties established in Lemmas 8, 9, and 10 ensure that the equations can be efficiently and rapidly solved by Newton’s method or a bisection method. We also show in the present paper that it is crucial to distinguish two different regimes (the near and the far point regime) in the geometrical and analytical approaches to the Heston distance, each regime requiring its own analysis (see Theorems 1 and 3 and their proofs).

Let us expand on the financial motivations for considering the Heston Riemannian distance function. The Heston model is one of the most popular stock price models with stochastic volatility. This model was introduced in [19]. More information on the Heston model and stochastic volatility models can be found in [15, 16, 20, 31]. The stock price process $S$ and the variance process $V$ in the Heston model satisfy the following system of stochastic differential equations:

\[
\begin{align*}
    dS_t &= \mu S_t dt + \sqrt{V_t} S_t dW_t, \\
    dV_t &= (a - bV_t) dt + c \sqrt{V_t} dZ_t,
\end{align*}
\] (1)

where $a \geq 0$, $b \geq 0$, $c > 0$. In (1), $W$ and $Z$ are correlated standard Brownian motions such that $d\langle W, Z \rangle_t = \rho dt$ with $\rho \in (-1, 1)$.

Recently closely related models, the local-volatility Heston models, given by $dS_t = \mu S_t dt + \sqrt{V_t} \sigma(S_t) S_t dW_t$ for an appropriate function $\sigma$, have become objects of widespread interest among practitioners. Practitioners seek to come up with accurate approximations to the Black-Scholes implied volatility in such models and there is a considerable literature in this direction. In one of the approaches to this problem, initiated for another class of stochastic volatility models in [17], and in the Heston case by [11], [13], [14] a key element in determining the implied volatility is the Riemannian distance to a line $S = K$ in the $SV$-plane. We plan to address applications of
the results obtained in the present paper to the local volatility Hes-
ton models in future publications. This will include consideration of
heat kernel expansions, implied volatility expansions, and pricing of
exotic options in such models.

Let us consider the following uncorrelated Heston model:
\[ \begin{align*}
    dS_t &= S_t \sqrt{V_t} dW_t \\
    dV_t &= (a - bV_t) dt + \sqrt{V_t} dZ_t,
\end{align*} \tag{2} \]
where \(a \geq 0, b \geq 0,\) and \(W\) and \(Z\) are independent standard Brown-
ian motions. Denote by \(X\) the log-price process defined by \(X = \log S\).
Then the model in (2) transforms as follows:
\[ \begin{align*}
    dX_t &= -\frac{1}{2} V_t dt + \sqrt{V_t} dW_t \\
    dV_t &= (a - bV_t) dt + \sqrt{V_t} dZ_t.
\end{align*} \tag{3} \]
The state space for the process \((X, V)\) is the closed half-plane
\[ \mathcal{H} = \left\{ (x, v) \in \mathbb{R}^2 : v \geq 0 \right\}. \]

We will denote the initial condition for the process \((X, V)\) by \((x_0, v_0)\).

The Riemannian metric form associated with the Heston model is
defined on the interior \(\mathcal{H}^\circ\) of the closed half-plane \(\mathcal{H}\) as follows:
\[ ds^2 = v^{-1} \left( dx^2 + dv^2 \right). \tag{4} \]
The open half-plane \(\mathcal{H}^\circ\) equipped with the metric defined in (4) is
called the Heston manifold. The form in (4) generates the Riemann-
ian distance \(d_H\) on \(\mathcal{H}\). More examples of Riemannian distances aris-
ing in finance can be found in [20] (see also [22]).

In this paper, we discuss various explicit formulas for the Heston
distance \(d_H\). It is worth mentioning that the following two-sided
estimate for \(d_H\) is known (see [21], Proposition 4.3.2):
\[ D(x_0, v_0, x_1, v_1) \leq d_H((x_0, v_0), (x_1, v_1)) \leq 12D(x_0, v_0, x_1, v_1) \tag{5} \]
for all \((x_0, v_0) \in \mathcal{H}\) and \((x_1, v_1) \in \mathcal{H},\) where
\[ D(x_0, v_0, x_1, v_1) = \frac{\sqrt{(x_0 - x_1)^2 + (v_0 - v_1)^2}}{\sqrt{v_0} + \sqrt{v_1} + [(x_0 - x_1)^2 + (v_0 - v_1)^2]^{1/4}}. \tag{6} \]

We will next briefly explain how to obtain explicit formulas for
the distance function \(d_H\) in the general correlated Heston model de-
scribed in (1) from similar formulas for the distance function \(d_H\) in
the uncorrelated Heston model with the vol of vol coefficient equal
to one considered in the present paper. It is known that the principal
part of the generator of the diffusion \((X, V)\) with \(X = \log S\) in model (1) is given by

\[
v \left( \frac{\partial^2}{\partial x^2} + 2\rho c \frac{\partial^2}{\partial x \partial v} + c^2 \frac{\partial^2}{\partial v^2} \right).
\]

Let us show how to reduce this to the case of an uncorrelated Heston model, in which the metric is in the standard form presented in (4). First make the change of time \(\hat{t} = c^2 t\) which reduces the principal part of the generator of the diffusion to

\[
v \left( c^{-2} \frac{\partial^2}{\partial x^2} + 2\rho c^{-1} \frac{\partial^2}{\partial x \partial v} + \frac{\partial^2}{\partial v^2} \right).
\]

It is not hard to see that under such a change of time, the distance function \(\tilde{d}_H\) is multiplied by the constant \(c\). Next, we use the diffeomorphism

\[
\hat{x} = \frac{c}{\sqrt{1 - \rho^2}} x - \frac{\rho}{\sqrt{1 - \rho^2}} v,
\]

\[
\hat{v} = v.
\]

After this change of variables the principal part of the new diffusion operator is in the standard form

\[
\hat{v} \left( \frac{\partial^2}{\partial \hat{x}^2} + \frac{\partial^2}{\partial \hat{v}^2} \right).
\]

Taking into account the reasoning above, one can prove that

\[
\tilde{d}_H((x_0, v_0), (x_1, v_1)) = \frac{1}{c} d_H \left( \left( \frac{cx_0 - \rho v_0}{\sqrt{1 - \rho^2}}, v_0 \right), \left( \frac{cx_1 - \rho v_1}{\sqrt{1 - \rho^2}}, v_1 \right) \right).
\]  \quad (7)

Formula (7) shows how to adapt the distance formulas obtained in this paper to the case where the Heston model is correlated and given by (1).

Let us note that the drift terms in (3) do not affect the Heston distance. On the other hand, the Heston transition density \((x, v) \mapsto p^H_t((x_0, v_0), (x, v))\) associated with the process \((X, V)\) is influenced by the drift terms. Using the definition of the Riemannian distance and (4), we see that the distance \(d_H\) satisfies the following conditions:

\[
d_H((x_0, v_0), (x_1, v_1)) = d_H((0, v_0), (x_1 - x_0, v_1))
\]  \quad (8)

and

\[
d_H((ax_0, av_0), (ax_1, av_1)) = \sqrt{a} d_H((x_0, v_0), (x_1, v_1))
\]  \quad (9)
for all $\alpha > 0$.

It will be assumed throughout the paper that $x_0 \neq x_1$. In the case where $x_0 = x_1$, the geodesics joining the points $(x_0, v_0)$ and $(x_1, v_1)$ is a vertical line with length

$$\int_{\min(v_0, v_1)}^{\max(v_0, v_1)} \frac{1}{\sqrt{v}} dv = 2(\sqrt{\max(v_0, v_1)} - \sqrt{\min(v_0, v_1)}),$$

and hence

$$d_H((x_0, v_0), (x_0, v_1)) = 2(\sqrt{\max(v_0, v_1)} - \sqrt{\min(v_0, v_1)}).$$

It follows from the previous formula that the $x$-axis is at a finite distance from any point in $\mathcal{H}^o$, and hence the $x$-axis, being part of the boundary of the Heston manifold, is “at infinity”. Therefore, the Heston manifold is not complete, and we can not apply the Hopf-Rinow theorem to establish the existence of a length-minimizing geodesic joining two points in $\mathcal{H}$. Note that one difficulty in establishing such a result is proving that the length minimizing curve joining any two points is a true geodesic and not the union of broken geodesics (on the other hand for the metric $ds^2 = v(dx^2 + dy^2)$, Robert Bryant has communicated to us that only a subset of points in the upper half space can be joined by a non-broken geodesic).

It is interesting that for the metric defined by (4) the existence and uniqueness result for the length-minimizing geodesics has been essentially known for at least one century. We will next provide more information.

Let us consider the following metric form:

$$ds^2 = (2v)^{-1} (dx^2 + dv^2)$$

that is intimately related to the Heston metric. It is known that the length-minimizing geodesics for this metric are dilations and shifts of the standard cycloid given by

$$\begin{cases}
  x = s - \sin s \\
  v = 1 - \cos s.
\end{cases}$$

(10)

This was established by O. Bolza in 1904 (see [2], see also [18], Proposition I.2.1). Bolza proved that for any two points $C = (x_0, v_0)$ and $D = (x_1, v_1)$ in $\mathcal{H}$ with $x_0 \neq x_1$ there exists exactly one dilated and shifted standard cycloid joining $C$ and $D$ and such that no cusp of the cycloid lies between $C$ and $D$. Bolza states that the previous result was included without proof in unpublished lectures of K. T. W.
Weierstrass (1882). A special case was handled by Johann and Jacob Bernoulli at the very end of the 17th century.

It follows from the above-mentioned result of Bolza that the Heston geodesics can be obtained from the cycloid curve

\[
\begin{align*}
  x &= \frac{1}{\sqrt{2}} s - \sin \frac{s}{\sqrt{2}} \\
  v &= 1 - \cos \frac{s}{\sqrt{2}}
\end{align*}
\]  

(11)

by translation \( x \mapsto x + b \) and dilation \( x \mapsto cx, v \mapsto cv \), or are vertical lines. It is also clear that Bolza’s existence and uniqueness theorem holds for the Heston manifold.

2. The Heston distance

For every fixed \( C \in \mathbb{R} \) such that \( C \neq 0 \), define a function on the interval \([0, C^{-2}]\) by

\[
F(v, C) = \frac{\arcsin(C\sqrt{v})}{C^2} - \frac{\sqrt{v} \sqrt{1 - C^2 v}}{C}.
\]  

(12)

Now fix \( v_0 \geq 0 \) and \( v_1 \geq 0 \) and consider a function of the variable \( C \) given by

\[
F(v_0, v_1, C) = F(v_1, C) - F(v_0, C).
\]  

(13)

It is assumed in (13) that the variable \( C \) satisfies the condition

\[
0 < |C| \leq \min\left(v_0^{\frac{1}{2}}, v_1^{\frac{1}{2}}\right).
\]

Since \( \lim_{C \to 0} F(v_0, v_1, C) = 0 \) (use l’Hôpital’s rule twice), we can extend the function \( F \) continuously to an odd function on the interval

\[
I = [-\min(v_0^{\frac{1}{2}}, v_1^{\frac{1}{2}}), \min(v_0^{\frac{1}{2}}, v_1^{\frac{1}{2}})]
\]  

(14)

by putting \( F(v_0, v_1, 0) = 0 \).

We will next prove a theorem that provides explicit formulas for the Heston distance. Note that there are two expressions for the Heston distance in Theorem 1, depending on the location of the points \((x_0, v_0)\) and \((x_1, v_1)\) in the Heston half-plane.

**Theorem 1.** (i) Suppose the points \((x_0, v_0)\) and \((x_1, v_1)\) in the Heston half-plane satisfy the following condition:

\[
|x_1 - x_0| \leq F(\min(v_0, v_1), \max(v_0, v_1), \frac{1}{\sqrt{\max(v_0, v_1)}}),
\]  

(15)
where $F$ is given by (13). Then
\[
\begin{align*}
d_H((x_0, v_0), (x_1, v_1)) &= 2 \frac{\arcsin(C^* \sqrt{\max(v_0, v_1)}) - \arcsin(C^* \sqrt{\min(v_0, v_1)})}{C^*},
\end{align*}
\]
where $C^* = C^*((x_0, v_0), (x_1, v_1))$ is the unique solution to the transcendental equation
\[
\begin{align*}
x_1 - x_0 &= \frac{\sqrt{v_0} \sqrt{1 - C^2 v_0}}{C} - \frac{\arcsin(C \sqrt{v_0})}{C^2} \\
&\quad - \left[ \frac{\sqrt{v_1} \sqrt{1 - C^2 v_1}}{C} - \frac{\arcsin(C \sqrt{v_1})}{C^2} \right] 
\end{align*}
\]
on the interval $I$ defined by (14).

(ii) Suppose the points $(x_0, v_0)$ and $(x_1, v_1)$ in the Heston half-plane satisfy the following condition:
\[
|x_1 - x_0| \geq F(\min(v_0, v_1), \max(v_0, v_1), \frac{1}{\sqrt{\max(v_0, v_1)}}). 
\]
Then
\[
\begin{align*}
d_H((x_0, v_0), (x_1, v_1)) &= 2 \frac{\arccos(C^* \sqrt{v_0}) + \arccos(C^* \sqrt{v_1})}{C^*},
\end{align*}
\]
where $C^* = C^*((x_0, v_0), (x_1, v_1))$ is the unique solution to the transcendental equation
\[
\begin{align*}
|x_1 - x_0| &= \frac{\sqrt{v_0} \sqrt{1 - C^2 v_0}}{C} + \frac{\arccos(C \sqrt{v_0})}{C^2} \\
&\quad + \frac{\sqrt{v_1} \sqrt{1 - C^2 v_1}}{C} + \frac{\arccos(C \sqrt{v_1})}{C^2} 
\end{align*}
\]
on the interval $[0, \min(v_0^{-\frac{1}{2}}, v_1^{-\frac{1}{2}})]$.

Proof. We will first prove part (i). It is not hard to see that with no loss of generality we may assume that $x_0 < x_1$ and $v_0 < v_1$. Let $c$ be the dilation coefficient corresponding to the points $A = (x_0, v_0)$ and $B = (x_1, v_1)$ in Bolza’s description of the geodesics. Then we have $c > 0$. Define $C > 0$ from the equality $c = 2C^2$, and consider the geodesic, connecting $A$ with $B$, under the scaled parametrization $s \mapsto Cs$.

Let us first assume that the point $A$ and $B$ are both to the left of the apex of the arc of the geodesic passing through them. Then it is not
hard to see that the components \( s \mapsto x(s) \) and \( s \mapsto v(s) \) of the arc of the geodesic through \( A \) and \( B \) satisfy

\[
\begin{aligned}
\dot{x} &= Cv \\
\dot{v} &= \sqrt{v} \sqrt{1 - C^2 v},
\end{aligned}
\]

(21)

where the derivative in the system above is taken with respect to the arclength parameter and where \( v \in [0, C^{-2}] \). It is also clear that \( x \) is a function of \( v \) for \( v \in [0, C^{-2}] \) and we have

\[
\frac{dx}{dv} = \frac{C \sqrt{v}}{\sqrt{1 - C^2 v}}.
\]

(22)

Note that \( v = C^{-2} \) is the second component of the apex. Therefore \([v_0, v_1] \subset [0, C^{-2}]\).

It is easy to see that

\[
\frac{dF}{dv} = \frac{C \sqrt{v}}{\sqrt{1 - C^2 v}}.
\]

(23)

It follows from (22) and (23) that \( F(v, C) = x(v) + \alpha \), where \( \alpha \) is some constant. Plugging \( v = 0 \) into the previous equality, we get \( \alpha = b \), where \( b \) is the shift parameter in the description of the geodesic passing through \( A \) and \( B \). Hence,

\[
F(v, C) = x(v) + b.
\]

(24)

It follows from (24) that \( C \) satisfies the following condition:

\[
x_1 - x_0 = F(v_1, C) - F(v_0, C) = F(v_0, v_1, C).
\]

Moreover, the assumptions formulated above can be formulated as follows:

\[
x_1 - x_0 \leq F \left( v_0, v_1, \frac{1}{\sqrt{v_1}} \right).
\]

It is not hard to see using (22) that

\[
d_H(A, B) = \int_{v_0}^{v_1} \sqrt{\left( \frac{dx}{dv} \right)^2 + \frac{1}{\sqrt{v}}} dv = \int_{v_0}^{v_1} \frac{1}{\sqrt{1 - C^2 v}} dv \\
= 2 \frac{\arcsin(C \sqrt{v_1}) - \arccos(C \sqrt{v_0})}{C}.
\]

This establishes formula (16), and completes the proof of part (i) of Theorem 1 in the case where the points \( A \) and \( B \) are both to the left of the apex of the arc of the geodesic passing through them. The proof of part (i) in the case where \( A \) and \( B \) are to the right of the apex is similar.
Next suppose that one of the points is to the left of the apex, while the other one is to its right. This case is a combination of the previous two. With no loss of generality, we may assume that \((x_0, v_0)\) is to the left of the apex and \((x_1, v_1)\) is to the right of the apex. This happens if and only if condition (18) holds. It is not hard to see that under the restriction imposed above, we need to sum two contributions, one going from \((x_0, v_0)\) to the apex and the other going from the apex to \((x_1, v_1)\). Since the \(v\)-component of the apex equals \(C^{-2}\), we obtain

\[
x_{\text{apex}} - x_0 = \frac{\sqrt{v_0} \sqrt{1 - C^2 v_0}}{C} - \frac{\arcsin(C \sqrt{v_0})}{C^2} + \frac{\pi}{2C^2}
\]

To this we must add

\[
x_1 - x_{\text{apex}} = \frac{\sqrt{v_1} \sqrt{1 - C^2 v_1}}{C} - \frac{\arcsin(C \sqrt{v_1})}{C^2} + \frac{\pi}{2C^2}
\]

so, in total we obtain condition (20) for \(C\). In the same way we must add the corresponding distance formulas to get formula (19).

This completes the proof of Theorem 1.

We will say that the points \((x_0, y_0)\) and \((x_1, y_1)\) are \(C\)-close provided that the inequality in (15) holds. Similarly, if the inequality in (18) holds, then we will say that the points are \(C\)-far. The equations in (17) and (20) will be called the \(C\)-equations, while the formulas in (16) and (19) will be called the \(C\)-formulas.

**Remark 2.** Two different formulas appear in Theorem 1 because certain subtleties which underlie the geometry of the cycloid have to be dealt with. Note that formula (16) was suggested as the Heston distance formula in the book [20] by P. Henry-Labordère (see formula (6.66) in [20]). However, formula (16) holds only in the close-point regime (see part (i) of Theorem 1) and has to be replaced by formula (17) in the far-point regime. The presence of two different regimes was not taken into account in [20].

It follows from part (ii) of Theorem 1 that

\[
d_H((x_0, 0), (x_1, 0)) = 2 \sqrt{\pi |x_1 - x_0|}. \tag{25}
\]

The previous formula describes the Heston distance between any two points on the boundary of the Heston half-plane.
3. THE TOY HESTON MODEL AND THE GRUSHIN MODEL

Consider the following stochastic model:

\[
\begin{align*}
    dG_t &= H_t dW_t \\
    dH_t &= \frac{1}{2} dZ_t.
\end{align*}
\]  

(26)

We will call the model described by (26) the Grushin model because the Laplace operator associated with it is a special Grushin operator given by

\[
L = \frac{1}{4} \frac{\partial^2}{\partial y^2} + y^2 \frac{\partial^2}{\partial x^2}.
\]  

(27)

The heat kernel for the operator \( L \) will be denoted by \( p^G_t((x_0, y_0), (x_1, y_1)) \) where \( t > 0, (x_0, y_0) \in \mathbb{R}^2 \), and \( (x_1, y_1) \in \mathbb{R}^2 \), and the sub-Riemannian distance on \( \mathbb{R}^2 \) (a Carnot-Carathéodory distance), corresponding to the Grushin model will be denoted by \( d_G \). The plane \( \mathbb{R}^2 \) equipped with the distance \( d_G \) is called the Grushin plane. The heat kernel \( p^G \) for the Grushin plane satisfies the following partial differential equation:

\[
\frac{\partial p^G_t}{\partial t} = \frac{1}{8} \frac{\partial^2 p^G_t}{\partial y^2} + \frac{1}{2} y^2 \frac{\partial^2 p^G_t}{\partial x^2},
\]  

(28)

with the initial condition given by

\[
p^G_0((x_0, y_0), (x_1, y_1)) = \delta_{x_0}(x_1) \delta_{y_0}(y_1).
\]

More information on the geometry of the Grushin plane can be found in [6, 7, 8, 9, 27, 28]. Stochastic methods which are used in the study of Grushin type structures are discussed in [5].

Suppose the process \((G, H)\) is the solution to the system in (26) with the initial conditions \( g_0 \) and \( h_0 \), respectively. Then the process \((X, Y)\), where \( X = G \) and \( Y = H^2 \), solves the following system of stochastic differential equations:

\[
\begin{align*}
    dX_t &= \sqrt{Y_t} d\tilde{W}_t \\
    dY_t &= \frac{1}{2} dt + \sqrt{Y_t} d\tilde{Z}_t
\end{align*}
\]  

(29)

with initial conditions \( g_0 \) and \( h_0^2 \). In (29), the processes \( \tilde{W} \) and \( \tilde{Z} \) are new standard Brownian motions defined by \( \tilde{W}_t = \text{sign}(H_t) dW_t \) and \( \tilde{Z}_t = \text{sign}(H_t) dZ_t \). We will call the stochastic model described by (29) the toy Heston model. In this section, the Heston distance will be analyzed by using the following formula relating the Heston and the Grushin distances:

\[
d_H((x_0, v_0), (x_1, v_1)) = d_G((x_0, \sqrt{v_0}), (x_1, \sqrt{v_1})),
\]  

(30)
for all points $A = (x_0, v_0) \in \mathcal{H}$ and $B = (x_1, v_1) \in \mathcal{H}$. It is not hard to prove equality (30) when $A$ and $B$ belong to $\mathcal{H}^0$, and then extend the equality to $\mathcal{H}$ by continuity. The proof for $\mathcal{H}^0$ is based on the fact that the length minimizing Heston and Grushin geodesics for $A \in \mathcal{H}^0$ and $B \in \mathcal{H}^0$ are entirely contained in $\mathcal{H}^0$. We leave filling in the details as an exercise for the reader. It follows from (30) and (5) that

$$D(x_0, v_0, x_1, v_1) \leq d_G((x_0, \sqrt{v_0}), (x_1, \sqrt{v_1})) \leq 12D(x_0, v_0, x_1, v_1)$$

(31)

for all $(x_0, v_0) \in \mathcal{H}$ and $(x_1, v_1) \in \mathcal{H}$, where the function $D$ is given by (6).

We will next formulate a statement which provides an alternative formula for the Heston distance.

**Theorem 3.** For any two points $(x_0, v_0) \in \mathcal{H}$ and $(x_1, v_1) \in \mathcal{H}$ such that at least one of them is not on the boundary, the following formula holds:

$$d_H((x_0, v_0), (x_1, v_1)) = \frac{\delta}{\sin \left( \frac{\delta}{2} \right)} \sqrt{v_1 + v_0 - 2\sqrt{v_1v_0} \cos \left( \frac{\delta}{2} \right)},$$

(32)

where $\delta = \delta((x_0, v_0), (x_1, v_1))$ is the unique solution to the equation

$$\frac{(v_1 + v_0)(\delta - \sin(\delta)) - 2\sqrt{v_1v_0} \left( \delta \cos \left( \frac{\delta}{2} \right) - 2 \sin \left( \frac{\delta}{2} \right) \right)}{2 \sin^2 \left( \frac{\delta}{2} \right)} = x_1 - x_0,$$

(33)

satisfying the condition $-2\pi < \delta < 2\pi$.

**Remark 4.** In [27, 28], M. Paulat established a formula for the sub-Riemannian distance in a slightly different Grushin model given by:

$$\begin{cases} 
  dG_t = H_tdW_t \\
  dH_t = dZ_t.
\end{cases}$$

Paulat’s formula is equivalent to (32) (one formula can be obtained from the other using (30)). The ideas used in the proof of formula (32) are completely different from those employed in [27, 28]. Paulat analyzes sub-Riemannian geodesics in his proof, while the techniques used in the present paper are more analytical. In addition, the proof of Theorem 3 contains several new results, e.g., a partial large deviation principle for the Grushin model. We would like to thank M. Paulat for sending us his dissertation [28].
Note that there are two distance formulas in Theorem 1 (in the close point regime and in the far point regime), while Theorem 3 contains only one distance formula. An interesting fact is that in the \( \delta \)-environment, there is a special two-set partition of \( \mathcal{H} \times \mathcal{H} \) hidden in the background.

**Definition 5.** We will say that the points \((x_0, v_0)\) and \((x_1, v_1)\) are \( \delta \)-close provided that
\[
|x_1 - x_0| \leq \frac{\pi}{2} (v_1 + v_0) + 2\sqrt{v_1 v_0},
\]
and \( \delta \)-far if
\[
|x_1 - x_0| > \frac{\pi}{2} (v_1 + v_0) + 2\sqrt{v_1 v_0}.
\]

**Remark 6.** In terms of the parameter \( \hat{\delta} \), the description of the close-point \( \delta \)-regime and the far-point \( \delta \)-regime is \( 0 \leq |\hat{\delta}| \leq \pi \) and \( \pi < |\hat{\delta}| < 2\pi \), respectively.

**Remark 7.** It follows from Theorem 1 or Theorem 3 that
\[
d_H((x_0, v_0), (x_1, v_1)) = d_H((x_1, v_0), (x_0, v_1))
\]
and
\[
d_H((x_0, v_0), (x_1, v_1)) = d_H((x_0, v_1), (x_1, v_0)).
\]

The reasons, why the two regimes in Definition 5 are introduced, are rather subtle. It will be shown in the next sections that the close-point \( \delta \)-regime describes those pairs of points, for which formula (32) can be obtained by analytical methods. The far point \( \delta \)-regime is a proper part of the far point \( C \)-regime, and formula (32) in the far point regime can be established using formula (19) (see Lemma 14 below).

We will next derive formula (25) from Theorem 3. With no loss of generality we can assume \( x_0 < x_1 \). Let \( \varepsilon > 0 \), and take \( v_0 = 0, v_1 = \varepsilon \). Then Theorem 3 implies that
\[
d_H((x_0, 0), (x_1, \varepsilon)) = \hat{\delta} \left[ \sin \left( \frac{\hat{\delta}}{2} \right) \right]^{-1} \sqrt{\varepsilon}
\]
where
\[
\frac{\varepsilon (\hat{\delta} - \sin \hat{\delta})}{2 \sin^2 \frac{\hat{\delta}}{2}} = x_1 - x_0.
\]
Therefore,
\[
d_H((x_0, 0), (x_1, \varepsilon)) = \frac{\sqrt{2(x_1 - x_0)\hat{\delta}}}{\sqrt{\hat{\delta} - \sin \hat{\delta}}}. \tag{36}
\]
It is not hard to see that \( \hat{\delta} \to 2\pi \) as \( \varepsilon \downarrow 0 \). Taking the limit as \( \varepsilon \downarrow 0 \) in formula (36), we obtain formula (25).

4. SOLVABILITY AND CONVEXITY

In this section we discuss the unique solvability of the C-equations and of the \( \delta \)-equation. Let us start with the C-close point regime. It is clear from the definition of the function \( F \) in (13) that to study the unique solvability of equation (17), it suffices to assume \( x_1 > x_0 \) and \( v_1 > v_0 \). Then the equation becomes

\[
F(v_0, v_1, C) = x_1 - x_0,
\]

and we have to solve it on the interval \( 0 < C < v_1^{-\frac{1}{2}} \). Note that in the C-close point regime we have \( x_1 - x_0 \leq F(v_0, v_1, v_1^{-\frac{1}{2}}) \), and the function \( F \) maps the interval \([0, v_1^{-\frac{1}{2}}]\) onto the interval \([0, F(v_0, v_1, v_1^{-\frac{1}{2}})]\) (the monotonicity of the function \( F \) follows from the next lemma).

**Lemma 8.** For fixed \( v_1 > v_0 \), the function \( C \mapsto F(v_0, v_1, C) \) is strictly increasing on the interval \([0, v_1^{-\frac{1}{2}}]\) and convex on the interval \((0, v_1^{-\frac{1}{2}})\).

**Proof.** Consider the function

\[
F_1(v, C) = \frac{\arcsin(C\sqrt{v})}{C^2} - \frac{\sqrt{v}\sqrt{1 - C^2v}}{C}.
\]

By definition, the function \( F \) satisfies

\[
F(v_0, v_1, C) = F_1(v_1, C) - F_1(v_0, C).
\]

Therefore

\[
\frac{\partial^2 F(v_0, v_1, C)}{\partial C^2} = \frac{\partial^2 F_1}{\partial C^2}(v_1, C) - \frac{\partial^2 F_1}{\partial C^2}(v_0, C)
\]

\[
= \int_{v_0}^{v_1} \frac{\partial^3 F_1}{\partial v \partial C^2}(v, C)dy.
\]

By differentiating the function \( F_1 \), we get

\[
\frac{\partial F_1}{\partial C} = -\frac{2\arcsin(C\sqrt{v})}{C^3} + \frac{\sqrt{v}}{C^2\sqrt{1 - C^2v}}
\]

\[
+ \frac{\sqrt{v}\sqrt{1 - C^2v}}{C^2} + \frac{v\sqrt{1 - C^2v}}{\sqrt{1 - C^2v}}
\]

and

\[
\frac{\partial^2 F_1}{\partial C^2}(v, C) = \frac{2\sqrt{v}(3 + 4C^2v)}{C^3(1 - C^2v)^{\frac{3}{2}}} + \frac{6\arcsin(C\sqrt{v})}{C^4}.
\]
We also have
\[ \frac{\partial^3 F_1}{\partial C^2 \partial v}(v, C) = \frac{3Cv^3}{(1 - C^2v)^{5/2}}. \] (42)

This is quite remarkable because if we stop at the second derivative we have the more complicated expression given in (41). Now, it is not hard to see, using (39) and (42), that
\[ \frac{\partial^2 F(v_0, v_1, C)}{\partial C^2} > 0 \] (43)

for all \( C \in \left( 0, v_1^{-\frac{1}{2}} \right) \), and hence the convexity statement in Lemma 8 holds.

We will next prove that the function \( F \) is increasing. Using (38) and (40), and making tedious but straightforward computations, we obtain
\[ \frac{\partial F}{\partial C} = \lim_{C \to 0} \frac{\partial F}{\partial C}(v_0, v_1, C) = \frac{2}{3}(v_1^3 - v_0^3) > 0. \] (44)

Now the fact that the function \( F \) is increasing follows from (43), (44), and the equality
\[ \frac{\partial F}{\partial C}(v_0, v_1, C) = \int_0^C \frac{\partial^2 F}{\partial C^2}(v_0, v_1, u) \, du + \frac{\partial F}{\partial C}(v_0, v_1, 0). \]

This completes the proof of Lemma 8.

It follows from Lemma 8 that equation (37) is uniquely solvable for all pairs of points in the Heston half-plane which are C-close.

Our next goal is to prove a similar result for any pair of points \((x_0, v_0)\) and \((x_1, v_1)\) in the Heston half-plane which are C-far. With no loss of generality we assume \( x_0 < x_1 \) and \( v_0 < v_1 \). Recall that the C-far point regime is described by the following inequality:
\[ x_1 - x_0 \geq F \left( v_0, v_1, \frac{1}{\sqrt{v_1}} \right). \]

The C-equation in the far point regime is as follows:
\[ \tilde{F}(v_0, v_1, C) = x_1 - x_0, \] (45)

where
\[ \tilde{F}(v_0, v_1, C) = \sqrt{v_0} \sqrt{1 - C^2v_0} \frac{1}{C} + \arccos \left( C \sqrt{v_0} \right) \frac{C}{C^2} + \sqrt{v_1} \sqrt{1 - C^2v_1} \frac{1}{C} + \arccos \left( C \sqrt{v_1} \right) \frac{C}{C^2}, \] (46)
and we are looking for the solution $C^*$ satisfying $0 < C^* < v_1^{-\frac{1}{2}}$.

The function $\tilde{F}$ is decreasing on the interval $\left(0, v_1^{-\frac{1}{2}}\right)$ (see Lemma 9 below), and maps it onto the infinite interval $\left(F \left(v_0, v_1, \frac{1}{\sqrt{v_1}}\right), \infty\right)$. Indeed, it follows from the definition of $\tilde{F}$ that $\lim_{C \downarrow 0} \tilde{F}(v_0, v_1, C) = \infty$. We also have

$$\lim_{C \uparrow v_1^{-\frac{1}{2}}} \tilde{F}(v_0, v_1, C)$$

$$= \sqrt{\frac{v_0}{v_1}} \sqrt{1 - \frac{v_0}{v_1}} + v_1 \arccos \sqrt{\frac{v_0}{v_1}}$$

$$= \sqrt{\frac{v_0}{v_1}} \sqrt{1 - \frac{v_0}{v_1}} + v_1 \arcsin \sqrt{\frac{v_0}{v_1}} = F \left(v_0, v_1, \frac{1}{\sqrt{v_1}}\right).$$

It is clear from the previous discussion that equation (45) is uniquely solvable in the $C$-far point regime and the solution $C^*$ satisfies $0 < C^* < v_1^{-\frac{1}{2}}$.

**Lemma 9.** Let $v_0 < v_1$. Then the function $\tilde{F}$ defined by (46) is decreasing on the interval $\left(0, v_1^{-\frac{1}{2}}\right)$. It is locally convex near the point $C = 0$ and locally concave near the point $C = v_1^{-\frac{1}{2}}$.

**Proof.** Put

$$\tilde{F}_1(v, C) = \frac{\sqrt{v} \sqrt{1 - C^2 v}}{C} + \frac{\arccos(C \sqrt{v})}{C^2}.$$ (47)

Then we have

$$\tilde{F}(v_0, v_1, C) = \tilde{F}_1(v_0, C) + \tilde{F}(v_1, C).$$ (48)

By differentiating the function $\tilde{F}$ twice and simplifying, we obtain

$$\frac{\partial \tilde{F}_1}{\partial C}(v, C) = - \left[ \frac{2 \sqrt{v}}{C^2 \sqrt{1 - C^2 v}} + \frac{2 \arccos(C \sqrt{v})}{C^3} \right]$$ (49)

and

$$\frac{\partial^2 \tilde{F}_1}{\partial C^2}(v, C) = \frac{6 \sqrt{v} - 8 v^2 C^2}{C^3 (1 - C^2 v)^{\frac{3}{2}}} + \frac{6 \arccos(C \sqrt{v})}{C^4}.$$ (50)

It is clear from (49) that the function $C \mapsto \tilde{F}_1(v, C)$ decreases. Moreover, formula (50) shows that

$$\lim_{u \downarrow 0} \frac{\partial^2 \tilde{F}_1}{\partial C^2}(v, u) = \infty.$$
and
\[ \lim_{u \uparrow v^{-\frac{1}{2}}} \frac{\partial^2 \tilde{F}_1}{\partial v^2}(v, u) = -\infty. \]

Analyzing the previous equalities and taking into account (48), we see that Lemma 9 holds.

Let us next consider the equation in (33). We fix \( x_0 \in \mathbb{R}, x_1 \in \mathbb{R}, v_0 \geq 0, v_1 \geq 0 \) and assume that at least one of the numbers \( v_0 \) and \( v_1 \) is different from zero. Denote the function on the left-hand side of (33) by \( f(\delta) \). Then the equation in (33) can be rewritten as follows:
\[
 f(\delta) = x_1 - x_0. \tag{51}
\]

We have
\[
 f(\delta) = f(\delta, v_0, v_1) = \frac{1}{2} A(\delta) B(\delta) \tag{52}
\]
where
\[
 A(\delta) = A(\delta, v_0, v_1) = \sin^{-2}\left(\frac{\delta}{2}\right) \tag{53}
\]
and
\[
 B(\delta) = B(\delta, v_0, v_1) = (v_1 + v_0) (\delta - \sin(\delta)) - 2\sqrt{v_1 v_0} \left(\delta \cos\left(\frac{\delta}{2}\right) - 2 \sin\left(\frac{\delta}{2}\right)\right). \tag{54}
\]

The value of the function \( f \) at \( \delta = 0 \) is given by
\[
 f(0) = \frac{1}{2} \lim_{\delta \to 0} A(\delta) B(\delta) = 0. \tag{55}
\]

The function \( f \) is continuous and odd on the interval \((-2\pi, 2\pi)\). In order to prove that the equation in (33) is uniquely solvable on \((-2\pi, 2\pi)\), it suffices to assume that \( x_1 > x_0 \) and look for the unique solution belonging to the interval \((0, 2\pi)\). We will next show that the function \( f \) is positive on the interval \((0, 2\pi)\). Indeed, the functions \( \delta \mapsto \delta - \sin(\delta) \) and \( \delta \mapsto \delta \cos\left(\frac{\delta}{2}\right) - 2 \sin\left(\frac{\delta}{2}\right) \) are equal to zero at \( \delta = 0 \). Moreover, the former function is positive on \((0, 2\pi)\), while the latter one is decreasing (differentiate!) and hence negative on \((0, 2\pi)\). It follows from (54) that \( B(\delta) > 0 \) for all \( \delta \in (0, 2\pi) \). This shows that the function \( f \) is positive on the interval \((0, 2\pi)\).

We also have
\[
 f(\pi) = \frac{1}{2} \left[ \pi (v_1 + v_0) + 4\sqrt{v_1 v_0} \right].
\]

**Lemma 10.** The function \( f \) is strictly increasing and convex on the interval \((0, 2\pi)\). Moreover, it maps \((0, 2\pi)\) onto \((0, \infty)\).
Remark 11. Lemma 10 implies that there exists the unique solution to the equation in (33) belonging to the interval \((-2\pi, 2\pi)\).

Proof. For all \(\delta \in (0, 2\pi)\), we have

\[
A'(\delta) = -\sin^{-3}\left(\frac{\delta}{2}\right) \cos\left(\frac{\delta}{2}\right)
\]
and

\[
A''(\delta) = \frac{2 + \cos(\delta)}{2 \sin^4\left(\frac{\delta}{2}\right)}.
\]

\[
B'(\delta) = (v_1 + v_0) \left(1 - \cos(\delta)\right) + \sqrt{v_1 v_0} \delta \sin\left(\frac{\delta}{2}\right),
\]
and

\[
B''(\delta) = (v_1 + v_0) \sin(\delta) + \sqrt{v_1 v_0} \left(\sin\left(\frac{\delta}{2}\right) + \frac{1}{2} \delta \cos\left(\frac{\delta}{2}\right)\right).
\]

Next, using the product rule, l’Hôpital’s rule, and the formulas above, we see that

\[
\lim_{\delta \to 0} f'\left(\delta\right) = \frac{1}{3} (v_1 + v_0) + \frac{1}{3} \sqrt{v_1 v_0}.
\]

Now, (55), (60), and the fact that the function \(f\) is odd on \((-2\pi, 2\pi)\) imply that \(f\) is differentiable at \(\delta = 0\).

Our goal is to show that

\[
f''(\delta) > 0 \quad \text{for} \quad \delta \in (0, 2\pi).
\]

It is not hard to see that if (61) holds, then the function \(f\) is increasing on \((0, 2\pi)\). Indeed (61) implies that for all \(0 < \epsilon < \delta < 2\pi\),

\[
f'\left(\delta\right) = f'\left(\epsilon\right) + \int_{\epsilon}^{\delta} f''(u)du > f'\left(\epsilon\right).
\]

Therefore, (60) shows that the derivative of the function \(f\) is positive on \((0, 2\pi)\). Now the continuity of \(f\) on \([0, 2\pi]\) implies that the function \(f\) is increasing on \([0, 2\pi]\).

Our next goal is to prove (61). We have

\[
(AB)'' = A''B + 2A'B' + AB''.
\]

Using (53), (54), (56), (57), (58), and (59), we obtain

\[
A''B = \frac{2 + \cos(\delta)}{2 \sin^4\left(\frac{\delta}{2}\right)} \left[ (v_1 + v_0) (\delta - \sin(\delta)) - 2\sqrt{v_1 v_0} \left(\delta \cos\left(\frac{\delta}{2}\right) - 2 \sin\left(\frac{\delta}{2}\right)\right) \right],
\]
$$2A'B' = -\frac{2 \cos \left(\frac{\delta}{2}\right) \left[(v_1 + v_0) (1 - \cos(\delta)) + \sqrt{v_1 v_0} \delta \sin \left(\frac{\delta}{2}\right)\right]}{\sin^3 \left(\frac{\delta}{2}\right)},$$

and

$$AB'' = \frac{(v_1 + v_0) \sin(\delta) + \sqrt{v_1 v_0} \left(\sin \left(\frac{\delta}{2}\right) + \frac{1}{2} \delta \cos \left(\frac{\delta}{2}\right)\right)}{\sin^2 \left(\frac{\delta}{2}\right)}.$$
where $0 < \delta < \pi$. The following estimates, which can be easily derived using Taylor expansions of the sine and cosine functions, will be needed in the proof below:

\[
\sin \left( \frac{\delta}{2} \right) > \frac{\delta}{2} - \frac{\delta^3}{48}, \tag{64}
\]

\[
\cos \left( \frac{\delta}{2} \right) < 1 - \frac{\delta^2}{8} + \frac{\delta^4}{384}, \tag{65}
\]

and

\[
\sin \delta < \delta - \frac{\delta^3}{6} + \frac{\delta^5}{120} \tag{66}
\]

for all $\delta \in (0, \pi)$.

Using (64) and (65), we obtain

\[
C_2(\delta) = 6 \sin \left( \frac{\delta}{2} \right) \left( 1 + \cos^2 \left( \frac{\delta}{2} \right) \right) - \delta \cos \left( \frac{\delta}{2} \right) \left( 5 + \cos^2 \left( \frac{\delta}{2} \right) \right)
\]

\[
> \delta \left[ \left( 3 - \frac{\delta^2}{8} \right) \left( 1 + \cos^2 \left( \frac{\delta}{2} \right) \right) - \left( 1 - \frac{\delta^2}{8} + \frac{\delta^4}{384} \right) \left( 5 + \cos^2 \left( \frac{\delta}{2} \right) \right) \right]
\]

\[
= \delta \left[ \frac{\delta^2}{2} + 2 \cos^2 \left( \frac{\delta}{2} \right) - 2 - \frac{\delta^4}{384} \left( 5 + \cos^2 \left( \frac{\delta}{2} \right) \right) \right]
\]

\[
\geq \delta \left[ \frac{\delta^2}{2} - \frac{\delta^4}{64} - 2 \sin^2 \left( \frac{\delta}{2} \right) \right]. \tag{67}
\]

It will be shown next that the function

\[
h(\delta) = \frac{\delta^2}{2} - \frac{\delta^4}{64} - 2 \sin^2 \left( \frac{\delta}{2} \right)
\]

is positive on the interval $(0, \pi)$. Indeed, $h(0) = 0$ and

\[
h'(\delta) = \delta - \frac{\delta^3}{16} - \sin \delta.
\]

Now using (66) we get

\[
h'(\delta) > \delta^3 \left( \frac{5}{48} - \frac{\delta^2}{120} \right) > 0
\]

since $\delta \in (0, \pi)$. It follows that $C_2(\delta) > 0$ for $0 < \delta < \pi$. Finally, it is easy to check, using the definition of the function $f$, that $f$ maps $(0, 2\pi)$ onto $(0, \infty)$.

This completes the proof of Lemma 10.

**Lemma 12.** Suppose the points $(x_0, v_0) \in \mathcal{H}$ and $(x_1, v_1) \in \mathcal{H}$ are $\delta$-far. Then they are $C$-far.
Proof. It suffices to prove the lemma in the case where \( v_1 > 0, \)
\( v_0 \geq 0, \) and \( v_1 \geq v_0. \) The assumption in the formulation of Lemma 12 means that inequality (35) holds. To prove that the points are C-far, we have to establish estimate (18). For \( v_1 \geq v_0, \) this estimate is as follows:

\[
|x_1 - x_0| > \frac{\pi}{2} v_1 + \sqrt{v_1 v_0 - v_0^2} - v_1 \arcsin \sqrt{v_0 \over v_1}.
\] (68)

It is easy to see that condition (35) implies condition (68).

This concludes the proof of Lemma 12.

Let us define the following function:

\[
f(u, v_0, v_1) = \frac{\sin {u \over 2}}{\sqrt{v_1 + v_0 - 2 \sqrt{v_1 v_0} \cos {u \over 2}}}.\] (69)

It is assumed in (69) that \( u \in (\pi, 2\pi), \) \( v_0 \geq 0, \) and \( v_1 \geq 0. \) We also exclude the case where \( v_0 = v_1 = 0. \) It is not hard to see that for fixed \( v_0 \) and \( v_1, \) the function \( u \mapsto f(u, v_0, v_1) \) is strictly decreasing, continuous, and

\[
f : (\pi, 2\pi) \mapsto \left(0, {1 \over \sqrt{v_1 + v_0}}\right)\]
(70)

(the mapping is onto).

Suppose the points \((x_0, v_0) \in H \) and \((x_1, v_1) \in H \) are \( \delta \)-far. By Lemma 12, these points are also C-far. It is natural to ask whether there exists a relation between the numbers \( \delta^* \) and \( C^*, \) corresponding to the given points. The next statement answers the previous question.

**Lemma 13.** If \((x_0, v_0) \in H \) and \((x_1, v_1) \in H \) are \( \delta \)-far, then

\[
C^*((x_0, v_0), (x_1, v_1)) = f \left(|\hat{\delta}((x_0, v_0), (x_1, v_1))|, v_0, v_1\right),
\] (71)

where the function \( f \) is defined by (69).

Proof. Since the points are \( \delta \)-far, we have \( \pi < |\hat{\delta}| < 2\pi. \) With no loss of generality, we can assume that \( \pi < \hat{\delta} < 2\pi, \) \( x_0 \leq x_1, \) and \( v_0 \leq v_1. \) Note that condition (70) implies that all the expressions appearing in the proof of Lemma 13 are real numbers.
Let us denote $\alpha = \sqrt{\frac{v_0}{v_1}}$, and recall that the number $\hat{\delta}$ is the unique solution to equation (33). Then

$$x_1 - x_0 = v_1 \frac{(1 + \alpha^2)(\hat{\delta} - \sin(\hat{\delta})) - 2\alpha \left(\hat{\delta} \cos \left(\frac{1}{2} \hat{\delta}\right) - 2 \sin \left(\frac{1}{2} \hat{\delta}\right)\right)}{2 \sin^2 \left(\frac{1}{2} \hat{\delta}\right)}.$$  \hspace{1cm} (72)

Put

$$\tilde{C} = \sqrt{v_1} f \left(\hat{\delta}(x_0, v_0), (x_1, v_1), v_0, v_1\right) = \frac{\sin \left(\frac{1}{2} \hat{\delta}\right)}{\sqrt{1 + \alpha^2 - 2\alpha \cos \left(\frac{1}{2} \hat{\delta}\right)}}.$$  \hspace{1cm} (73)

It is not hard to see that equality (71) holds if and only if

$$x_1 - x_0 = v_1 \left[\alpha \sqrt{1 - \alpha^2 \tilde{C}^2} + \sqrt{1 - \tilde{C}^2} \frac{\arccos \tilde{C} + \arccos(\alpha \tilde{C})}{\tilde{C}^2}\right],$$  \hspace{1cm} (74)

where $\tilde{C}$ is defined by (73). Using the addition formula for the inverse cosines, we see that equality (74) is equivalent to the following equality:

$$\frac{x_1 - x_0}{v_1} = \alpha \sqrt{\frac{1}{\tilde{C}^2} - \alpha^2} + \sqrt{\frac{1}{\tilde{C}^2} - 1}$$

$$+ \frac{\arccos(\alpha \tilde{C}^2 - \sqrt{1 - \tilde{C}^2} \sqrt{1 - \alpha^2 \tilde{C}^2})}{\tilde{C}^2}.$$  \hspace{1cm} (75)

Since $\hat{\delta}$ solves equation (33), we have

$$x_1 - x_0 = v_1 \frac{(1 + \alpha^2)(\hat{\delta} - \sin(\hat{\delta})) + 2\alpha \left(2 \sin \left(\frac{1}{2} \hat{\delta}\right) - \hat{\delta} \cos \left(\frac{1}{2} \hat{\delta}\right)\right)}{2 \sin^2 \left(\frac{1}{2} \hat{\delta}\right)}.$$  \hspace{1cm} (76)

We will derive equality (75) from equality (77). It follows from (73) that

$$\tilde{C}^2 = \frac{1 - \cos^2 \left(\frac{1}{2} \hat{\delta}\right)}{1 + \alpha^2 - 2\alpha \cos \left(\frac{1}{2} \hat{\delta}\right)}.$$  \hspace{1cm} (78)
Solving the corresponding quadratic equation for \( \cos \left( \frac{1}{2} \delta \right) \) and taking into account that \( \cos \left( \frac{1}{2} \delta \right) < 0 \), we get

\[
\cos \left( \frac{1}{2} \delta \right) = \alpha \widetilde{C}^2 - \sqrt{\alpha^2 \widetilde{C}^4 - \widetilde{C}^2 - \alpha^2 \widetilde{C}^2} + 1.
\]

Therefore,

\[
\cos \left( \frac{1}{2} \delta \right) = \alpha \widetilde{C}^2 - \sqrt{1 - \widetilde{C}^2 \sqrt{1 - \alpha^2 \widetilde{C}^2}}.
\]  \hspace{1cm} (79)

Next, using (78) and (79), we see that equality (75) is equivalent to the following:

\[
\frac{x_1 - x_0}{v_1} = \alpha \sqrt{\frac{1 + \alpha^2 - 2 \alpha \cos \left( \frac{1}{2} \delta \right)}{\sin^2 \left( \frac{1}{2} \delta \right)}} - \alpha^2
\]

\[
+ \sqrt{\frac{1 + \alpha^2 - 2 \alpha \cos \left( \frac{1}{2} \delta \right)}{\sin^2 \left( \frac{1}{2} \delta \right)}} - 1 + \frac{\delta (1 + \alpha^2 - 2 \alpha \cos \left( \frac{1}{2} \delta \right))}{2 \sin^2 \left( \frac{1}{2} \delta \right)}.
\]  \hspace{1cm} (80)

Making simplifications in (80), we can show that equality (80) reduces to equality (77).

This completes the proof of Lemma 13.

**Lemma 14.** The \( \delta \)-formula for the Heston distance in the case, where the points \( (x_0, v_0) \in \mathcal{H} \) and \( (x_1, v_1) \in \mathcal{H} \) are \( \delta \)-far, follows from part (ii) of Theorem 3.

**Proof.** In the proof of Lemma 14, we will use the notation introduced in the proof of Lemma 13. Suppose the points \( (x_0, v_0) \in \mathcal{H} \) and \( (x_1, v_1) \in \mathcal{H} \) are \( \delta \)-far, and assume that part (ii) of Theorem 3 is valid. By Lemma 12, \( (x_0, v_0) \) and \( (x_1, v_1) \) are \( C \)-far, and it follows from part (ii) of Theorem 3 that

\[
d_H((x_0, v_0), (x_1, v_1)) = 2 \frac{\arccos(C^* \sqrt{v_0}) + \arccos(C^* \sqrt{v_1})}{C^*}.
\]

Taking into account (71), (73), and (79), we see that

\[
d_H((x_0, v_0), (x_1, v_1)) = 2 \sqrt{v_1} \frac{\arccos(a \widetilde{C}) + \arccos(\widetilde{C})}{C^*}
\]

\[
= 2 \sqrt{v_1} \frac{\arccos(a \widetilde{C}^2 - \sqrt{1 - \widetilde{C}^2 \sqrt{1 - \alpha^2 \widetilde{C}^2}})}{C^*} = \sqrt{v_1} \delta / \widetilde{C}.
\]
Next, using (73), we obtain

\[ d_H((x_0, v_0), (x_1, v_1)) = \frac{\delta \sqrt{v_1} \sqrt{1 + \alpha^2 - 2\alpha \cos(\frac{1}{2}\delta)}}{\sin(\frac{1}{2}\delta)}. \]

Finally, recalling that \( \alpha = \sqrt{\frac{v_0}{v_1}} \), we see that formula (32) holds. This completes the proof of Lemma 14.

5. The Limiting Cumulant Generating Function for the Grushin Model

It is not hard to see using the equation in (28) that the Laplace transform \( \tilde{p}_G^t((x_0, y_0), (w, y_1)) \) in the variable \( x_1 \) of the Grushin transition density \( p_G^T \) satisfies the following heat equation with quadratic potential:

\[
\frac{\partial \tilde{p}_G^t}{\partial t} = \frac{1}{8} \frac{\partial^2 \tilde{p}_G^t}{\partial y^2} + \frac{1}{2} w^2 y^2 \tilde{p}_G^t
\]

with the initial condition given by \( \tilde{p}_G^0((x_0, y_0), (w, y_1)) = \delta_{x_0}(x_1) e^{wy_0} \). The fundamental solution for such a heat equation is well-known (see, e.g., [4], Theorem 10.3). Using this fundamental solution, we get

\[
\tilde{p}_G^t(x_0, y_0, w, y_1) = \sqrt{\frac{w}{\pi \sin(\frac{wt}{2})}} \exp \left\{ -w \left[ x_0 \sin\left(\frac{wt}{2}\right) + (y_1^2 + y_0^2) \cos\left(\frac{wt}{2}\right) - 2y_1y_0 \right] \right\}. \quad (81)
\]

The function \( \tilde{p}_G^t \) given by (81) has a removable singularity at \( w = 0 \). The analyticity strip for \( \tilde{p}_G^t \) is given by \( -\frac{2\pi}{t} < \Re(w) < \frac{2\pi}{t} \). It is not clear yet what happens outside this strip.
We will next compute the Laplace transform of the function $\tilde{p}^G_t$ in the variable $y_1$, using formula (81). We have

$$
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{w x_1 + \beta y_1} p^G_t ((x_0, y_0), (x_1, y_1)) dy_1 dx_1
$$

$$
= \sqrt{\frac{w}{\pi}} \exp \left\{ \frac{-w}{\sin \left( \frac{wt}{2} \right)} \right\}
$$

$$
\int_{-\infty}^{\infty} e^{\beta y_1} \exp \left\{ \frac{-w}{\sin \left( \frac{wt}{2} \right)} \right\} dy_1
$$

Let us next replace $w$ by $\frac{\delta}{t}$ and $\beta$ by $\frac{\gamma}{t}$ in the previous equality. This gives the following:

$$
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp \left\{ \frac{1}{t} (\delta x_1 + \gamma y_1) \right\} p^G_t ((x_0, y_0), (x_1, y_1)) dy_1 dx_1
$$

$$
= \sqrt{\frac{\delta}{\pi t \sin \left( \frac{\delta}{t} \right)}} \exp \left\{ \frac{-\delta}{t \sin \left( \frac{\delta}{t} \right)} \right\}
$$

$$
\int_{-\infty}^{\infty} \exp \left\{ \frac{\gamma}{t} y_1 \right\} \exp \left\{ \frac{-\delta \left[ y_1^2 \cos \left( \frac{\delta}{t} \right) - 2y_1y_0 \right]}{t \sin \left( \frac{\delta}{t} \right)} \right\} dy_1.
$$

(82)

The new restrictions on the parameters are $-2\pi < \delta < 2\pi$ and $\gamma \in \mathbb{R}$.

Denote the expression on the left-hand side of (82) by $J_t(x_0, \delta, y_0, \gamma)$. Assume $-\pi < \delta < \pi$ and $\gamma \in \mathbb{R}$. After lengthy but straightforward computations, we obtain

$$
J_t(x_0, \delta, y_0, \gamma) = \frac{1}{\sqrt{\cos \left( \frac{\delta}{t} \right)}}
$$

$$
\times \exp \left\{ \frac{1}{t} \left( \frac{4\delta^2 y_0^2 + \gamma^2}{4\delta \cos \left( \frac{\delta}{t} \right)} \right) + \frac{4x_0\delta^2 y_0 + 4\gamma y_0}{4\delta \cos \left( \frac{\delta}{t} \right)} \right\}.
$$

(83)

For every pair $(\delta, \gamma) \in \mathbb{R}^2$ and $t > 0$, put

$$
\Lambda_t(\delta, \gamma) = t \log J_t(x_0, \delta, y_0, \gamma)
$$

(84)

and

$$
\Lambda(\delta, \gamma) = \lim_{t \to 0} \Lambda_t(\delta, \gamma)
$$

(85)
if the limit in (85) exists. Note that the functions $\Lambda_i$ and $\Lambda$ depend on $x_0$ and $y_0$. The function $\Lambda$ is called the limiting cumulant generating function associated with the Grushin model.

Let us assume that $-\pi < \delta < \pi$ and $\gamma \in \mathbb{R}$. Then, taking the logarithm of the integral on the left-hand side of (83), multiplying by $t$, and sending $t$ to infinity, we obtain

$$\Lambda(\delta, \gamma) = \frac{(4\delta^2 y_0^2 + \gamma^2) \sin \left(\frac{\delta}{2}\right) + 4x_0\delta^2 \cos \left(\frac{\delta}{2}\right) + 4y_0\gamma\delta}{4\delta \cos \left(\frac{\delta}{2}\right)}.$$

(86)

Let us next assume that $\delta \in (-2\pi, -\pi] \cup [\pi, 2\pi)$ and $\gamma \in \mathbb{R}$. Then, using (82), we see that $\Lambda_t(\delta, \gamma) = \infty$ for every $t > 0$. Here we take into account that under the restrictions imposed above,

$$\delta \cos \left(\frac{\delta}{2}\right) \sin^{-1} \left(\frac{\delta}{2}\right) \leq 0.$$ 

It follows from (85) that $\Lambda(\delta, \gamma) = \infty$ for all $\delta \in (-2\pi, -\pi] \cup [\pi, 2\pi)$ and $\gamma \in \mathbb{R}$. Now, using Hölder’s inequality, we see that $\Lambda(\delta, \gamma) = \infty$ for all $\delta \in (-\infty, -\pi] \cup [\pi, \infty)$. The limiting cumulant generating function $\Lambda$ is defined everywhere and convex on $\mathbb{R}^2$. This function is finite on the set $D = (-\pi, \pi) \times \mathbb{R}$ and identically infinite outside this set. Moreover, the function $\Lambda$ is continuous on the set $|\mathbb{R} \setminus \{-\pi\} \cup \{\pi\}| \times \mathbb{R}$. It is also continuous on the lines $\delta = \pi$ and $\delta = -\pi$ with the exception of the points $P_1 = (\pi, -2\pi y_0)$ and $P_2 = (-\pi, -2\pi y_0)$. More precisely, we have $\Lambda(P_1) = \Lambda(P_2) = \infty$. In addition,

$$\liminf_{(\delta, \gamma) \to P_1, (\delta, \gamma) \in D} \Lambda(\delta, \gamma) = \pi x_0$$

(87)

and

$$\liminf_{(\delta, \gamma) \to P_2, (\delta, \gamma) \in D} \Lambda(\delta, \gamma) = -\pi x_0.$$ 

(88)

Hence, the limiting cumulant generating function $\Lambda$ is lower semi-continuous everywhere in $\mathbb{R}^2$, except at the points $P_1$ and $P_2$.

Using the definition of the function $\Lambda$, we see that for all $(\delta, \gamma) \in D$,

$$\frac{\partial \Lambda}{\partial \delta} = \frac{4y_0^2 \delta^2 (\delta + \sin \delta) + \gamma^2 (\delta - \sin \delta) + 4\delta^2 \gamma y_0 \sin \left(\frac{\delta}{2}\right)}{8\delta^2 \cos^2 \left(\frac{\delta}{2}\right)} + x_0.$$

(89)
and
\[ \frac{\partial \Lambda}{\partial \gamma} = \frac{\gamma \sin \left( \frac{\delta}{2} \right) + 2\delta y_0}{2\delta \cos \left( \frac{\delta}{2} \right)}. \]  

(90)

Hence, the function \( \Lambda \) is continuously differentiable on the set \( D \). However, this function is not steep (the definition of the steepness of a function is given in [10], Definition 2.3.5). We will next prove the previous statement.

It follows from (90) that \( ||\nabla \Lambda (\delta, \gamma)|| \to \infty \) provided that \( (\delta, \gamma) \in D \) and \( (\delta, \gamma) \to (\delta_0, \gamma_0) \) with either \( \delta_0 = \pi \) and \( \gamma_0 \neq -2\pi y_0 \), or \( \delta_0 = -\pi \) and \( \gamma_0 \neq -2\pi y_0 \). The behavior of the gradient at the exceptional points \( P_1 = (\pi, -2\pi y_0) \) and \( P_2 = (-\pi, -2\pi y_0) \) can be described using (89), (90), and l’Hôpital’s rule. We have \( \frac{\partial \Lambda}{\partial \delta} \to \frac{y_0^2}{2\pi} \left( \pi^2 - 4 \right) + x_0 \) and \( \frac{\partial \Lambda}{\partial \gamma} \to -\frac{2y_0}{\pi} \) as \( (\delta, \gamma) \in D \) and \( (\delta, \gamma) \to P_1 \). Therefore,

\[ ||\nabla \Lambda (\delta, \gamma)|| \to \sqrt{\frac{4y_0^2}{\pi^2} + \left( \frac{y_0^2}{\pi^2} \left( \pi^2 - 4 \right) + x_0 \right)^2} \]

as \( (\delta, \gamma) \in D \) and \( (\delta, \gamma) \to P_1 \). Similarly,

\[ \frac{\partial \Lambda}{\partial \delta} \to \frac{y_0^2}{2\pi} \left( 4 - \pi^2 \right) + x_0, \quad \frac{\partial \Lambda}{\partial \gamma} \to -\frac{2y_0}{\pi}, \]

and

\[ ||\nabla \Lambda (\delta, \gamma)|| \to \sqrt{\frac{4y_0^2}{\pi^2} + \left( \frac{y_0^2}{\pi^2} \left( 4 - \pi^2 \right) + x_0 \right)^2} \]

as \( (\delta, \gamma) \in D \) and \( (\delta, \gamma) \to P_2 \). Therefore the steepness condition for the function \( \Lambda \) is satisfied everywhere on the boundary of the set \( D(\Lambda) \), with the exception of the points \( P_1 \) and \( P_2 \).

**Remark 15.** The absence of the lower semi-continuity and of the steepness property for the function \( \Lambda \) does not allow us to use the Gärtner-Ellis theorem (see Theorem 2.3.6 in [10]) to establish the large deviation principle for the Grushin model. It will be shown below that this principle is valid only in a special regime (see Theorem 22).
6. CRITICAL POINTS AND THE LEGENDRE-FENCHEL TRANSFORM

Let us consider the Legendre-Fenchel transform $\Lambda^*$ of the limiting cumulant generating function $\Lambda$. It is given by

$$\Lambda^*(x_0, y_0, x_1, y_1) = \sup_{\delta, \gamma \in \mathbb{R}} \left[ \langle x_1, \delta \rangle + \langle y_1, \gamma \rangle - \Lambda(\delta, \gamma) \right].$$

It is clear that

$$\Lambda^*(x_0, y_0, x_1, y_1) = \sup_{-\pi < \delta < \pi, \gamma \in \mathbb{R}} \left[ \langle x_1, \delta \rangle + \langle y_1, \gamma \rangle - \Lambda(\delta, \gamma) \right]. \quad (91)$$

A critical point $(\delta, \gamma) = (\delta^*, \gamma^*)$, $-\pi < \delta^* < \pi$, for the function $\langle x_1, \delta \rangle + \langle y_1, \gamma \rangle - \Lambda(\delta, \gamma)$ satisfies the following system of equations:

$$\frac{\partial \Lambda}{\partial \delta} = x_1, \quad \frac{\partial \Lambda}{\partial \gamma} = y_1. \quad (92)$$

It follows from (92) and (90) that

$$\gamma = \frac{2\delta \left( y_1 \cos \left( \frac{\delta}{2} \right) - y_0 \right)}{\sin \left( \frac{\delta}{2} \right)}. \quad (93)$$

Moreover, (92) and (89) imply that $\delta^*$ is a solution to the following equation:

$$\frac{4y_0^2 \delta^2 (\delta + \sin(\delta)) + \gamma^2 (\delta - \sin(\delta)) + 4\delta^2 \gamma y_0 \sin \left( \frac{\delta}{2} \right)}{8\delta^2 \cos^2 \left( \frac{\delta}{2} \right)} = x_1 - x_0. \quad (94)$$

The second component $\gamma^*$ of the critical point $(\delta^*, \gamma^*)$ can be found by plugging the solution $\delta^*$ to the equation (94) into (93). This gives

$$\gamma^* = \frac{2\delta^* \left( y_1 \cos \left( \frac{\delta^*}{2} \right) - y_0 \right)}{\sin \left( \frac{\delta^*}{2} \right)}. \quad (95)$$

Our next goal is to simplify the equation in (94) by taking into account (94) and (95). Tedium, but rather straightforward calculations show that (94) can be rewritten in the following form:

$$\tilde{f}(\delta) = \tilde{f}(\delta, y_0, y_1) = x_1 - x_0, \quad (96)$$

where

$$\tilde{f}(\delta) = \frac{1}{2} \tilde{A}(\delta) \tilde{B}(\delta) \quad (97)$$
with
\[
\tilde{A}(\delta) = A(\delta, y_0, y_1) = \sin^{-2} \left( \frac{\delta}{2} \right) \tag{98}
\]
and
\[
\tilde{B}(\delta) = B(\delta, y_0, y_1) = \left( y_1^2 + y_0^2 \right) (\delta - \sin(\delta))
- 2y_1y_0 \left( \delta \cos \left( \frac{\delta}{2} \right) - 2 \sin \left( \frac{\delta}{2} \right) \right) . \tag{99}
\]

It is assumed in (96) that \(-\pi \leq \delta \leq \pi\). The value of the function \(\tilde{f}\) at \(\delta = 0\) is computed as follows: \(\tilde{f}(0) = \frac{1}{2} \lim_{\delta \to 0} \tilde{A}(\delta) \tilde{B}(\delta) = 0\).

**Remark 16.** It follows from (97)-(99) that the function \(f\) is defined on the interval \((-2\pi, 2\pi)\). This fact will be used below.

**Remark 17.** The properties of the functions \(\tilde{f}, \tilde{A}, \text{ and } \tilde{B}\) are similar to those of the functions \(f, A, \text{ and } B\) defined by (52), (53), and (54), respectively. The previous statement follows from the equalities \(\tilde{f}(\delta, y_0, y_1) = f(\delta, y_0^2, y_1^2), \tilde{A}(\delta, y_0, y_1) = A(\delta, y_0^2, y_1^2), \tilde{B}(\delta, y_0, y_1) = B(\delta, y_0^2, y_1^2), \text{ and } \delta^*(x_0, y_0, x_1, y_1) = \hat{\delta}((x_0, y_0), (x_1, y_1)).\)

Let us fix \(x_0\) and \(y_0\). It follows from Lemma 2.3.9 in [10] that the Legendre-Fenchel transform \(\Lambda^*\) of \(\Lambda\) is a good rate function. Explicit formulas for the function \(\Lambda\) were found in Section 5. In the present section, we compute the function \(\Lambda^*\). A simple analysis of formula (91) defining the function \(\Lambda^*\) shows that to compute the supremum in (91) one has to take into account the input of the critical point \((\delta^*, \gamma^*)\), the boundary of the strip where the moment generating function is finite, and the boundary at infinity. Since \(\Lambda(\delta, \gamma) \to \infty \text{ as } \gamma \to \infty \text{ or } \gamma \to -\infty\), the input of the boundary at infinity can be ignored. Using formulas (87) and (88), we see that the the input of the exceptional points \(P_1\) and \(P_2\), more precisely, of sequences converging to those points, is given by the following expression:

\[
R_1 = \pi(|x_0 - x_1| - 2y_1y_0). \tag{100}
\]

Note that the number in (100) is positive if and only if \(|x_0 - x_1| > 2y_1y_0\).

Next, suppose \(\delta^* \not\in (-\pi, \pi)\). This means that

\[
\frac{1}{2} \left[ \pi \left( y_1^2 + y_0^2 \right) + 4y_1y_0 \right] \leq |x_1 - x_0|. \tag{101}
\]
In this case, there is no critical point inside the fundamental strip, and hence $\Lambda^*$ is given by

$$\Lambda^*(x_0, y_0, x_1, y_1) = \pi(|x_1 - x_0| - 2y_1y_0) = R_1. \quad (102)$$

On the other hand, if $\delta^* \in (-\pi, \pi)$ (this happens if the opposite inequality to the inequality in (101) holds), then the input that the critical point $(\delta^*, \gamma^*)$ brings to the computation of the supremum in the formula for $\Lambda^*$ is given by the following expression:

$$R_2 = (x_1 - x_0)\delta^* + y_1\gamma^*$$

$$- \frac{4(\delta^*)^2y_0^2 + (\gamma^*)^2}{4\delta^* \cos \left(\frac{\delta^*}{2}\right)} \sin \left(\frac{\delta^*}{2}\right) + 4\gamma^*\delta^*y_0. \quad (103)$$

Replacing $x_1 - x_0$ in formula (103) by the expression on the left-hand side of formula (96) and making simplifications, we obtain

$$R_2 = \frac{(\delta^*)^2}{2 \sin^2 \left(\frac{\delta^*}{2}\right)} \left[ y_1^2 + y_0^2 - 2y_1y_0 \cos \left(\frac{\delta^*}{2}\right) \right]. \quad (104)$$

Therefore the condition $|x_1 - x_0| \leq 2y_1y_0$ implies the equality

$$\Lambda^*(x_0, y_0, x_1, y_1) = R_2, \quad (105)$$

and the condition $2y_1y_0 < |x_1 - x_0| < \frac{1}{2} \left[ \pi \left( y_1^2 + y_0^2 \right) + 4y_1y_0 \right]$ gives

$$\Lambda^*(x_0, y_0, x_1, y_1) = \max \{R_1, R_2\}. \quad (106)$$

Our next goal is to compare all the inputs discussed above. We will next show that $R_2$ always dominates $R_1$.

**Lemma 18.** For all $(x_0, y_0) \in \mathbb{R}^2$ and $(x_1, y_1) \in \mathbb{R}^2$, the following inequality holds: $R_1 \leq R_2$. In addition, if

$$\frac{1}{2} \left[ \pi \left( y_1^2 + y_0^2 \right) + 4y_1y_0 \right] < |x_1 - x_0|, \quad (107)$$

then $R_1 < R_2$.

**Proof.** Taking into account the definition of $R_1$ and $R_2$ (see (100) and (103), respectively), replacing the expression $x_1 - x_0$ in formula (103) by the expression on the left-hand side of formula (96), and simplifying, we see that the inequality $R_1 \leq R_2$ can be derived from
the inequality
\[
\left( y_1^2 + y_0^2 \right) \left[ (\delta^*)^2 + \pi \sin (\delta^*) \right] + y_1 y_0 \left[ 2\pi \delta^* \cos \left( \frac{\delta^*}{2} \right) + 4\pi \sin^2 \left( \frac{\delta^*}{2} \right) \right] \geq \left( y_1^2 + y_0^2 \right) \pi \delta^* + y_1 y_0 \left[ 2(\delta^*)^2 \cos \left( \frac{\delta^*}{2} \right) + 4\pi \sin \left( \frac{\delta^*}{2} \right) \right]. \tag{108}
\]

It is easy to see that with no loss of generality we may assume that \( \delta^* > 0 \). We will prove that
\[
(\delta^*)^2 + \pi \sin (\delta^*) \geq \pi \delta^* \tag{109}
\]
and
\[
\pi \delta^* \cos \left( \frac{\delta^*}{2} \right) + 2\pi \sin^2 \left( \frac{\delta^*}{2} \right) \geq (\delta^*)^2 \cos \left( \frac{\delta^*}{2} \right) + 2\pi \sin \left( \frac{\delta^*}{2} \right) \tag{110}
\]
for all \( 0 < \delta^* \leq \pi \). Moreover, it will be shown that the condition \( \pi < \delta^* < 2\pi \) implies strict inequalities in (109) and (110). Note that condition (107) is equivalent to the condition \( \pi < \delta^* < 2\pi \). It is clear that Lemma 18 follows from the inequalities formulated above.

Suppose first that \( 0 < \delta^* \leq \pi \). We will next establish (109). The fact that the inequality in (110) is equivalent to the inequality
\[
\delta^* (\pi - \delta^*) \cos \left( \frac{\delta^*}{2} \right) \geq 2\pi \sin \left( \frac{\delta^*}{2} \right) \left( 1 - \sin \left( \frac{\delta^*}{2} \right) \right) \tag{111}
\]
will be used in the proof.

Let us assume that (109) holds for all \( 0 < \delta^* < \frac{\pi}{2} \), and let \( \frac{\pi}{2} < \delta^* < \pi \). Then \( \hat{\delta} = \pi - \delta^* \) satisfies (109), and it is easy to see that \( \delta^* \) also satisfies (109). It follows that it suffices to assume \( 0 < \delta^* < \frac{\pi}{2} \).

Using the Taylor series, we see that for \( 0 < \delta^* < \frac{\pi}{2} \),
\[
\frac{\sin(\delta^*)}{\delta^*} \geq 1 - \frac{(\delta^*)^2}{6}.
\]
Hence (109) can be derived from the inequality
\[
\pi \left( 1 - \frac{(\delta^*)^2}{6} \right) \geq \pi - \delta^*.
\]
The previous inequality is equivalent to \( \pi \delta^* \leq 6 \), which is of course correct. This establishes (109).
Our next goal is to prove (111). Let us first assume $0 < \delta^* < \frac{\pi}{2}$. Using the Taylor series, we obtain
\[
\cos \left( \frac{\delta^*}{2} \right) \geq 1 - \frac{(\delta^*)^2}{8} \quad \text{and} \quad \sin \left( \frac{\delta^*}{2} \right) \geq \frac{\delta^*}{2} - \frac{(\delta^*)^3}{48}.
\] (112)
Therefore, (111) can be derived from the following inequality:
\[
\delta^* (\pi - \delta^*) \left( 1 - \frac{(\delta^*)^2}{8} \right) \geq \pi \delta^* \left( 1 - \frac{\delta^*}{2} + \frac{(\delta^*)^3}{48} \right).
\]
The previous inequality is equivalent to
\[
6(\delta^*)^2 + 24\pi \geq 48 + 6\pi \delta^* + \pi (\delta^*)^2.
\] (113)
It is not hard to see that (113) can be rewritten as follows:
\[
(6 - \pi) \left[ \left( \delta^* - \frac{3\pi}{6 - \pi} \right)^2 + \frac{24\pi - 48}{6 - \pi} - \frac{9\pi^2}{(6 - \pi)^2} \right] \geq 0.
\] (114)
Since $\delta^* < \frac{\pi}{2}$, (114) follows from
\[
(6 - \pi) \left[ \left( \frac{3\pi}{6 - \pi} - \frac{\pi}{2} \right)^2 + \frac{24\pi - 48}{6 - \pi} - \frac{9\pi^2}{(6 - \pi)^2} \right] \geq 0.
\] (115)
Now, it is clear that (111) holds for all $0 < \delta^* < \frac{\pi}{2}$, since (115) can be easily checked.

We will next show that (111) also holds under the condition $\frac{\pi}{2} \leq \delta^* < \pi$. The following estimate will be needed in the sequel. For all $\tilde{\delta}$ with $0 < \tilde{\delta} < \frac{\pi}{2}$,
\[
\tilde{\delta}(\pi - \tilde{\delta}) \sin \left( \frac{\tilde{\delta}}{2} \right) \geq 2\pi \cos \left( \frac{\tilde{\delta}}{2} \right) \left( 1 - \cos \left( \frac{\tilde{\delta}}{2} \right) \right).
\] (116)
Dividing the both sides of (116) by $\cos \left( \frac{\tilde{\delta}}{2} \right)$ and using the inequality $\tan x \geq x$ for $0 < x < \frac{\pi}{2}$, we see that (116) can be derived from the inequality
\[
\tilde{\delta}^2 (\pi - \tilde{\delta}) \geq 4\pi \left( 1 - \cos \left( \frac{\tilde{\delta}}{2} \right) \right).
\] (117)
Next, using the first inequality in (112), we see that (117) follows from the inequality
\[
(\pi - \tilde{\delta}) \geq \frac{\pi}{2},
\] (118)
which clearly holds because $0 < \tilde{\delta} < \frac{\pi}{2}$.

Suppose that $\frac{\pi}{2} < \delta^* < \pi$ and denote $\tilde{\delta} = \pi - \delta^*$. Then $0 < \tilde{\delta} < \frac{\pi}{2}$, and hence inequality (118) holds for $\tilde{\delta}$. It is not hard to see that (118)
for $\tilde{\delta}$ is equivalent to (111) for $\delta^\ast$. This completes the proof of estimate (110).

It follows from (109) and (110) that estimate (108) holds. It has already been mentioned that (108) implies the inequality $R_1 \leq R_2$. Therefore, part of Lemma 18 in the case where $-\pi \leq \delta^\ast \leq \pi$ is valid.

Now let $\pi < \delta^\ast < 2\pi$. We will first establish that the strict inequality in (109) holds. It is clear that the function $\rho_1(u)$ on the left-hand side of (109) and the function $\rho_2(u)$ on the right-hand side equal $\pi^2$ at $u = \pi$. Moreover, $\rho_1'(u) = 2u + \pi \cos u$ and $\rho_2'(u) = \pi$. It is easy to see that $\rho_1'(u) > \rho_2'(u)$ for all $\pi < u < 2\pi$. It follows that the strict inequality in (109) holds when $\pi < \delta^\ast < 2\pi$. The proof of the strict inequality in (110) under the same restriction is similar. Here the functions $\rho_1$ and $\rho_2$ equal $2\pi$ at $u = \pi$. Moreover

$$\rho_1'(u) = \pi \cos \frac{u}{2} - \frac{\pi}{2}u \sin \frac{u}{2} + 2\pi \sin \frac{u}{2} \cos \frac{u}{2}$$

and

$$\rho_2'(u) = \pi \cos \frac{u}{2} - \frac{1}{2}u^2 \sin \frac{u}{2} + 2u \cos \frac{u}{2}.$$  

It is not hard to see that $\rho_1'(u) > \rho_2'(u)$ for all $\pi < u < 2\pi$. This implies the strict inequality in (110) in the case where $\pi < \delta^\ast < 2\pi$.

The proof of Lemma 18 is thus completed.

**Theorem 19.** Under the condition

$$|x_1 - x_0| \leq \frac{1}{2} \left[ \pi \left( y_1^2 + y_0^2 \right) + 4y_1y_0 \right], \quad (119)$$

the Legendre-Fenchel transform $\Lambda^\ast$ of the limiting cumulant generating function $\Lambda$ in the Grushin model is given by the following formula:

$$\Lambda^\ast(x_0, y_0, x_1, y_1) = \frac{(\delta^\ast)^2}{2 \sin^2 \left( \frac{\delta^\ast}{2} \right)} \left( y_1^2 - y_0 + 2y_1y_0 \cos \left( \frac{\delta^\ast}{2} \right) \right). \quad (120)$$

On the other hand, if

$$\frac{1}{2} \left[ \pi \left( y_1^2 + y_0^2 \right) + 4y_1y_0 \right] < |x_1 - x_0|, \quad (121)$$

then we have

$$\Lambda^\ast(x_0, y_0, x_1, y_1) = \pi \left( |x_1 - x_0| - 2y_1y_0 \right). \quad (122)$$
Remark 20. Recall that $\delta^*$ in formula (120) is the unique solution to the equation

$$
\frac{1}{2} \csc^2 \left( \frac{\delta}{2} \right) \left[ \left( y_1^2 + y_0^2 \right) \left( \delta - \sin(\delta) \right) - 2y_1y_0 \left( \delta \cos \left( \frac{\delta}{2} \right) - 2 \sin \left( \frac{\delta}{2} \right) \right) \right] = x_1 - x_0.
$$

The solution to this equation satisfies $-\pi < \delta^* < \pi$ if condition (119) holds.

Proof of Theorem 19. Formula (122) in Theorem 19 has already been established (see (102)). Formula (120) can be derived from (104), (105), (106), and Lemma 18.

Remark 21. Let us consider a special case where $x_1 = x_0$. Then $\delta^* = 0$ and therefore (120) gives

$$
\Lambda^*(x_0, y_0, x_1, y_1) = 2(y_1 - y_0)^2. 
$$

7. A PARTIAL LARGE DEVIATION PRINCIPLE FOR THE GRUSHIN MODEL

Let us recall that in the present paper we denoted by $\Lambda$ and $\Lambda^*$ the limiting cumulant generating function in the Grushin model and the Legendre-Fenchel transform of $\Lambda$, respectively. We will prove below that for any initial point $A = (x_0, y_0) \in \mathbb{R}^2$ the large deviation principle holds for the Grushin model in a certain open subset $M$ of the plane $\mathbb{R}^2$. The set $M$ consists of all the points which are $\delta$-close to $A$. In the formulation of the next theorem, the symbols $B^\circ$ and $\overline{B}$ stand for the interior and the closure of the set $B$, respectively.

Theorem 22. Let $x_0 \in \mathbb{R}$ and $y_0 \in \mathbb{R}$ be fixed, and consider the open set in $\mathbb{R}^2$ defined by

$$
M = \left\{ (x_1, y_1) \in \mathbb{R}^2 : |x_1 - x_0| < \frac{1}{2} \left[ \pi \left( y_1^2 + y_0^2 \right) + 4y_1y_0 \right] \right\}. 
$$

Then the large deviation principle with the rate function $\Lambda^*$ holds on the set $M$. More precisely, for any Borel subset $B$ of $M$,

$$
- \inf_{(x_1, y_1) \in B^\circ} \Lambda^*(x_0, y_0, x_1, y_1) \leq \liminf_{t \to 0} \left[ t \log P_t^G(x_0, y_0, B) \right] \leq \limsup_{t \to 0} \left[ t \log P_t^G(x_0, y_0, B) \right] \leq - \inf_{(x_1, y_1) \in \overline{B}} \Lambda^*(x_0, y_0, x_1, y_1). 
$$
In addition, if the set $B$ is such that $\overline{B^c} = B$, then
\[
\lim_{t \to 0} \left[ t \log p^G_t(x_0, y_0, B) \right] = -\inf_{(x_1, y_1) \in B} \Lambda^*(x_0, y_0, x_1, y_1).
\] (126)

Proof. We have shown in Section 6 that for all $(x_1, y_1) \in M$, the following equality holds: $\nabla \Lambda(\delta^*, \gamma^*) = (x_1, y_1)$. Therefore, Lemma 2.3.9 in [10] implies that any point $(x_1, y_1) \in M$ is an exposed point of $\Lambda^*$ with $(\delta^*, \gamma^*)$ being the exposing hyperplane for $(x_1, y_1)$ (see [10] for the definition of exposed points and exposing hyperplanes).

In other words, the set $M$ consists entirely of exposed points. Now it is not hard to see that all the conditions in the Gartner-Ellis theorem (see Theorem 2.3.6 in [10]) hold. Applying this theorem and taking into account the continuity of the function $\Lambda^*$, we establish Theorem 22.

8. THE DISTANCE FORMULA

Our goal in this section is to complete the proof of Theorem 3. Since we have already proved this theorem in the far $\delta$-regime, it remains to establish it for the pairs of points $(x_0, v_0) \in \mathcal{H}^c$ and $(x_1, v_1) \in \mathcal{H}^c$ in the close $\delta$-regime.

We will prove the following assertion:

**Theorem 23.** If condition (119) holds, then
\[
\frac{d^2_G((x_0, y_0), (x_1, y_1))}{2} = \Lambda^*(x_0, y_0, x_1, y_1),
\] (127)

while if condition (121) is valid, then
\[
\frac{d^2_G((x_0, y_0), (x_1, y_1))}{2} > \Lambda^*(x_0, y_0, x_1, y_1).
\] (128)

**Remark 24.** It is clear that Theorem 3 in the close $\delta$-regime follows from (30), (120) and (127). Inequality (128) is interesting because it is often expected that for the distances associated with stochastic models, the function $\frac{1}{2}d^2$ coincides with the Legendre-Fenchel transform $\Lambda^*$ of the limiting cumulant generating function $\Lambda$. Inequality (128) shows that this is not the case for the Grushin model when the points are in the far $\delta$-regime.

**Proof of Theorem 23.** Since the Grushin operator (27) is hypoelliptic, it follows from the results obtained by Léandre in [23], [24] (see also [25], [26]) that Varadhan’s equality, that is, the equality
\[
\lim_{t \to 0} \left[ t \log p^G_t((x_0, y_0), (x_1, y_1)) \right] = -\frac{d^2_G((x_0, y_0), (x_1, y_1))}{2}
\] (129)
holds uniformly on compact subsets of \( \mathbb{R}^2 \) (Varadhan’s results can be found in [29, 30]).

Fix a point \((x_0, y_0) \in \mathbb{R}^2\) and consider all the points \((x_1, y_1) \in \mathbb{R}^2\) such that condition (119) holds. Let \(B = B_\varepsilon(x_1, y_1)\), where \(B_\varepsilon(x_1, y_1)\) is the disk of radius \(\varepsilon\) in \(\mathbb{R}^2\) centered at \((x_1, y_1)\) and such that

\[
B_\varepsilon(x_1, y_1) \subseteq M
\]

(recall that \(M\) is defined in (124)). It follows from (126) that

\[
\lim_{t \to 0} \left[ t \log P_t(x_0, y_0, B_\varepsilon(x_1, y_1)) \right] = -\inf_{(x,y) \in B_\varepsilon(x_1, y_1)} \Lambda^*(x_0, y_0, x, y) \quad (130)
\]

for all small enough \(\varepsilon > 0\). Using the mean value theorem for integrals, we can prove that

\[
\left| t \log P_t^G(x_0, y_0, B_\varepsilon(x_1, y_1)) + \frac{d^2_G((x_0, y_0), (x_1, y_1))}{2} \right| \leq \left| t \log(\pi \varepsilon^2) \right|
\]

\[
+ \sup_{(x,y) \in B_\varepsilon(x_1, y_1)} \left| t \log P_t(x_0, x_1, x, y) + \frac{d^2_G((x_0, y_0), (x, y))}{2} \right|
\]

\[
+ \sup_{(x,y) \in B_\varepsilon(x_1, y_1)} \left| \frac{d^2_G((x_0, y_0), (x_1, y_1))}{2} - \frac{d^2_G((x_0, y_0), (x, y))}{2} \right|. \quad (131)
\]

It follows from (131) and (129) that

\[
\limsup_{t \to 0} \left| t \log P_t^G(x_0, y_0, B_\varepsilon(x_1, y_1)) + \frac{d^2_G((x_0, y_0), (x_1, y_1))}{2} \right|
\]

\[
\leq \sup_{(x,y) \in B_\varepsilon(x_1, y_1)} \left| \frac{d^2_G((x_0, y_0), (x_1, y_1))}{2} - \frac{d^2_G((x_0, y_0), (x, y))}{2} \right|. \quad (132)
\]

Taking the limit as \(\varepsilon \to 0\) in (132) and using the continuity of the function \(d^2\) (the continuity follows from (31)), we get

\[
\lim_{\varepsilon \to 0} \limsup_{t \to 0} \left| t \log P_t^G(x_0, y_0, B_\varepsilon(x_1, y_1)) + \frac{d^2_G((x_0, y_0), (x_1, y_1))}{2} \right| = 0. \quad (133)
\]
It is not hard to see that

\[
\frac{d^2_G((x_0, y_0), (x_1, y_1))}{2} - \Lambda^*(x_0, y_0, x_1, y_1) \leq t \log P_t^G(x_0, y_0, B_{\varepsilon}(x_1, y_1)) + \frac{d^2_G((x_0, y_0), (x_1, y_1))}{2}
\]

\[
+ t \log P_t^G(x_0, y_0, B_{\varepsilon}(x_1, y_1)) + \inf_{(x,y) \in B_{\varepsilon}(x_1, y_1)} \Lambda^*(x_0, y_0, x, y)
\]

\[
+ \inf_{(x,y) \in B_{\varepsilon}(x_1, y_1)} \Lambda^*(x_0, y_0, x, y) - \Lambda^*(x_0, y_0, x_1, y_1) + \inf_{(x,y) \in B_{\varepsilon}(x_1, y_1)} \Lambda^*(x_0, y_0, x, y) - \Lambda^*(x_0, y_0, x_1, y_1).
\] (134)

Taking the limit as $\varepsilon \to 0$ in (134) and using (130), (133), and the continuity of the function $\Lambda^*$, we obtain formula (127) for all pairs of points $(x_0, y_0)$, $(x_1, y_1)$ satisfying condition (119). In addition, it is not hard to see that formula (128) for all pairs of points $(x_0, y_0)$, $(x_1, y_1)$ satisfying condition (121) follows from (122), (100), (104), the second part of Lemma 18, and from Theorem 3 in the far $\delta$-regime. Note that we have already established Theorem 3 for points which are $\delta$-far.

This completes the proof of Theorem 23.

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