Bypassing Markov Chains for Bayesian Generalized Linear Mixed Effects Models

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Abstract

Markov chain Monte Carlo (MCMC) is an all-purpose tool that allows one to generate dependent replicates from a posterior distribution for effectively any Bayesian hierarchical model. As such, MCMC has become a standard in Bayesian statistics. However, convergence issues, tuning, and the effective sample size of the MCMC are nontrivial considerations that are often overlooked or can be difficult to assess. Moreover, these practical issues can produce a significant computational burden. This motivates us to consider finding closed-form expressions of the posterior distribution that are computationally straightforward to sample from directly. We focus on a broad class of Bayesian generalized linear mixed-effects models (GLMM) that allows one to jointly model data of different types (e.g., Gaussian, Poisson, and binomial distributed observations). Exact sampling from the posterior distribution for Bayesian GLMMs is such a difficult problem that it is now arguably overlooked as a possible problem to solve. To solve this problem, we derive a new class of distributions that gives one the flexibility to specify the prior on fixed and random effects to be any conjugate multivariate distribution. We refer to this new distribution as the generalized conjugate multivariate (GCM) distribution, and several technical results are provided. The expression of the exact posterior distribution is given along with the steps to obtain direct independent simulations from the posterior distribution. These direct simulations have an efficient projection/regression form, and hence, we refer to our method as Exact Posterior Regression (EPR). Several theoretical results are developed that create the foundation for EPR. Illustrative examples are provided including a simulation study and an analysis of estimates from the U.S. Census Bureau’s American Community Survey (ACS).

Keywords: Bayesian hierarchical model; Big data; Gibbs sampler; Log-Linear Models; Markov chain Monte Carlo; Non-Gaussian; Nonlinear.

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1 Introduction

MCMC has become an invaluable tool in statistics and is covered in standard text books (Robert and Casella [2004]). MCMC is an all purpose strategy that allows one to obtain dependent samples from a generic posterior distribution. There are several theoretical considerations that one needs to consider when implementing MCMC to obtain samples from the posterior distribution including, ergodicity, irreducibility, and positive recurrence of the MCMC. In addition to theoretical considerations, practical implementation issues arise, including, a potential for high computational costs, assessing convergence (Gelman and Rubin [1992; Cowles and Carlin [1996]), tuning the MCMC (Roberts and Rosenthal [2009]), and computing the effective sample size of the Markov chain (Vats et al. [2019]), among other considerations. One of the current state-of-the-art techniques in MCMC is Hamiltonian Monte Carlo (HMC, Neal [2011]). HMC is a Metropolis–Hastings algorithm, where Hamiltonian dynamic evolution is used to propose a new value. In general, HMC leads to “fast mixing” (i.e., converges relatively quickly to the posterior distribution) because it provides a sample from the joint posterior distribution of all processes and parameters, and moreover, has been optimized efficiently using the software Stan (Carpenter et al. [2017]).

Of course, if one can directly sample independently from the joint posterior distribution the need for MCMC is not needed. In this article, we revisit the problem of directly sampling from the joint posterior distribution for a broad class of generalized linear mixed effects models (GLMM, McCulloch et al. [2008]). Much of the current literature does not consider solving this problem, since direct sampling from the exact posterior distribution for Bayesian GLMMs is a difficult problem, and MCMC can easily be adapted to many settings. We consider Bayesian GLMMs for Gaussian distributed data, binomial distributed data, and Poisson distributed data. Our approach uses the conjugate multivariate (CM) distribution (Bradley et al. [2020]) as the class of priors for fixed and random effects (e.g., see Chen and Ibrahim [2003] for the first use of conjugate priors in generalized linear regression). The samples from our proposed model are independently drawn,
and hence avoid issues with convergence, tuning, and positive autocorrelations in a MCMC. Moreover, replicates have an interpretable projection formulation. This regression-type projection can be computed efficiently using known block matrix inversion formulas (Lu and Shiou, 2002). Thus, we refer to our method as Exact Posterior Regression (EPR), which is the primary contribution of this article. Furthermore, we address unknown hyperparameters (i.e., variance parameters) when normal priors are used for fixed and random effects.

The conjugate prior distribution (or class of CM distribution) is often restricted to the data type. For example, for binomial, negative binomial, Bernoulli, and multinomial distributed data, the fixed and random effects are conjugate with the multivariate logit-beta distribution (Gao and Bradley, 2019; Bradley et al., 2019), which is the special case of the CM distribution. Similarly, Poisson and Weibull distributed data are conjugate with the multivariate log-gamma distribution (Bradley et al., 2018; Hu and Bradley, 2018; H.-C. Yang et al., 2019; Parker et al., 2020, 2021), another special case of the CM distribution. Finally, mixed effects models for Gaussian distributed data regularly make use of Gaussian priors for fixed and random effects (Gelman et al., 2013), which is also a type of CM distribution. Thus, our second major contribution is that we develop an extension of the CM distribution, which allows one to model fixed and random effects in a GLMM using any class of CM distributions. We call this new distribution the generalized CM (GCM) distribution, which allows for standard latent Gaussian process model specifications of GLMMs (e.g., see Gelfand and Schlief, 2016 for a recent discussion). Furthermore, we develop truncated versions and conditional distributions for GCM distributed random vectors.

We emphasize the high potential impact of the contributions of EPR and GCM, since much of the literature places a high consistent emphasis on using MCMC strategies to obtain exact (correlated) samples from the posterior distribution. For example, at the time of writing this manuscript the following papers use MCMC in a GLMM setting: Collins et al. (2021), Covernton et al. (2021), Gañan-Cardenas et al. (2021), Liu et al. (2021), Jobst and Rösch (2021), and Sapri et al. (2021), among others. All of these analyses can easily be adapted to be implemented using EPR, which
EPR allows one to efficiently jointly analyze several types of correlated data. In particular, we consider jointly modeling three “types of data,” namely, conditionally Gaussian, Poisson, and binomial distributed data. Joint analysis of multiple response-type datasets is often accomplished using Bayesian GLMMs with Gaussian priors (e.g., see Christensen and Amemiya 2002; Schliep and Hoeting 2013; Wu et al. 2015; Clarke et al. 2017; Todd et al. 2018 among several others) and make use of computationally expensive MCMC techniques. Recently, Bradley (2022) introduced an approach that transforms the data and makes use of the CM distribution, but models transformed data and does not model the data directly. Consequently, our approach, which allows for joint modeling of multiple data types directly, offers a contribution to this growing literature.

To summarize, the contributions of this article can be classified into two groups:

1. The first group of contributions of this article develops the GCM distribution. This includes expressions for the GCM distribution, the conditional GCM distribution, and truncated GCM distribution. The key literature on conjugate modeling began with Diaconis and Ylvisaker (1979)’s seminal paper which developed univariate conjugate models for the exponential family. Then Chen and Ibrahim (2003) developed Diaconis and Ylvisaker (1979)’s work in the context of fixed effects models and Bradley et al. (2020) developed Diaconis and Ylvisaker (1979)’s work in the context of mixed effects models. However, all of these papers require one to match the form of the prior distribution with that of the likelihood. The use of the GCM allows one to consider any class of CM distributions for models for fixed and random effects. Moreover, this literature often does not emphasize hyperparameters; however, our development explicitly addresses hyperparameters in the case of Gaussian priors.

2. The second group of contributions of this article allows one to use the GCM in a Bayesian GLMM context to produce what we call exact posterior regression (EPR). Much of the
Bayesian literature is shifting its’ focus on avoiding MCMC through the use of approximate Bayesian methods (e.g., see Wainwright and Jordan, [2008]; Rue et al., [2009] and additional discussion in Section [6]) or through exact sampling of the posterior distributions in special cases (Zhang et al., [2021]). EPR adds to this growing literature by allowing one to directly sample from the posterior from a broad class of GLMMs. By “broad” we mean that many existing GLMMs can be written in terms of our formulation, and we allow for multiple data types (e.g., Gaussian, binomial, Poisson, etc.). We show that the posterior distribution for fixed and random effects in a GLMM are either conditional GCM, GCM, or truncated GCM depending on the prior distribution on certain location parameters, which we refer to as “bypass location parameters.” Furthermore, we derive several properties of the GLMM including results on the fine-scale-random effect (see Theorem [4.2]) often used in the spatio-temporal statistics literature (Cressie and Wikle, [2011]), goodness-of-fit interpretations (see Theorem [4.6]), and matrix algebra techniques to aid in the computation of EPR (see Theorem [4.7]).

The remainder of the article proceeds as follows. In Section [2] we review the CM distribution and in Section [3] give an expression of the GCM, conditional GCM, and truncated GCM distributions. Then, in Section [4] we develop EPR, which includes EPR for Gaussian-only data and EPR for multiple data types, and we describe how to sample independent replicates from the joint posterior of the fixed effects, and random effects after marginalizing hyperparameters. Illustrations are provided in Section [5] which includes a simulation study and a multiple response-type ACS dataset. Proofs are given in the Appendix, and a discussion is given in Section [6].

2 Review: The Conjugate Multivariate Distribution

In this section, we give the reader a review of the CM distribution from Bradley et al. (2020). Suppose the observed data is distributed according to the natural exponential family (Diaconis
and Ylvisaker, 1979; Lehmann and Casella, 1998). That is, suppose that the probability density function/probability mass function (pdf/pmf) of the observed datum \( Z_k \) is given by,

\[
f(Z_k|Y_k) = \exp \{ Z_k Y_k - b_k \psi_k(Y_k) + c_k(Z_k) \}; \quad Z_k \in \mathcal{Z}_k, Y_k \in \mathcal{Y}_k,
\]

where \( f \) denotes a generic pdf/pmf, \( \mathcal{Z}_k \) is the support of \( Z_k \), \( \mathcal{Y}_k \) is the support of the unknown parameter \( Y_k \), \( b_k \) is a possibly known real-value, both \( \psi_k(\cdot) \) and \( c_k(\cdot) \) are known real-valued functions, and \( k = 1, \ldots, K \) is used to index the specific member of the exponential family (e.g., Gaussian, Poisson, binomial, etc.). The function \( b_k \psi_k(Y_k) \) is often called the log partition function (Lehmann and Casella, 1998). We focus on Gaussian responses, which sets \( \psi_1(Z) = Z^2 \), \( b_1 = 1/2\sigma^2 \), \( \mathcal{Z}_1 = \mathbb{R} \), and \( \mathcal{Y}_1 = \mathbb{R} \) with \( \sigma^2 > 0 \); Poisson responses, which sets \( \psi_2(Z) = \exp(Z) \), \( b_2 = 1 \), \( \mathcal{Z}_2 = \{0, 1, 2, \ldots\} \), and \( \mathcal{Y}_2 = \mathbb{R} \); and binomial responses, which sets \( \psi_3(Z) = \log\{1 + \exp(Z)\} \), \( b_3 = m \), \( \mathcal{Z}_3 = \{0, 1, \ldots, m\} \), and \( \mathcal{Y}_3 = \mathbb{R} \) with \( m \) a strictly positive integer.

It follows from Diaconis and Ylvisaker (1979) that the conjugate prior distribution for \( Y_k \) is given by,

\[
f(Y_k|\alpha_k, \kappa_k) = \mathcal{N}_k(\alpha_k, \kappa_k) \exp \{ \alpha_k Y_k - \kappa_k \psi_k(Y_k) \}; \quad Y_k \in \mathcal{Y}_k, \frac{\alpha_k}{\kappa_k} \in \mathcal{Z}_k, \kappa_k > 0, k = 1, \ldots, K, \]

where \( \mathcal{N}_k(\alpha_k, \kappa_k) \) is a normalizing constant. Let \( \text{DY}(\alpha_k, \kappa_k; \psi_k) \) denote a shorthand for the pdf in (2). Here “DY” stands for “Diaconis-Ylvisaker.” Of course, there are several special cases of the DY distribution other than the Gaussian (\( k = 1 \)), log-gamma (\( k = 2 \)), and logit-beta (\( k = 3 \)) distributions, several of which do not correspond to a member of the exponential family. For example, \( \alpha_4 = 0 \), \( \psi_4(Z) = \log\left\{ 1 + \frac{Z^2}{\nu} \right\} \), and \( \kappa_4 = (\nu + 1)/2 \) with \( \nu > 0 \), results in a Student-\( \nu \) distribution with degrees of freedom \( \nu \), which is not a member of the exponential family of distributions.

Diaconis and Ylvisaker (1979) proved that the pdf in (2) results in a valid probability measure, and is conjugate with (1). By conjugate we mean that the posterior distribution is from the same
family of distributions as the prior distribution. In the case of (1) and (2), we obtain conjugacy as

$$Y_k|Z_k, \alpha_k, \kappa_k \sim \text{DY}(\alpha_k + Z_k, \kappa_k + b_k; \psi_k).$$

(3)

Bradley et al. (2020) derived a multivariate version of \(\text{DY}(\alpha_k, \kappa_k; \psi_k)\). Specifically, define the \(n_k\)-dimensional random vector \(y_k\) using the following transformation:

$$y_k = \mu_k + V_kw_k; k = 1, \ldots, K,$$

(4)

where \(\mu_k\) is an \(n_k\)-dimensional real-valued vector called the “location parameter,” \(V_k\) is an \(n_k \times n_k\) real-valued invertible “covariance parameter matrix,” the elements of the \(n_k\)-dimensional random vector \(w_k\) are mutually independent, and the \(i\)-th element of \(w_k\) is \(\text{DY}(\alpha_k, \kappa_k; \psi_k)\) with \(\alpha_{k,i}/\kappa_{k,i} \in \mathcal{Z}_k\) and shape/scale (depending on \(k\)) \(\kappa_{k,i} > 0\), respectively. Straightforward change-of-variables of the transformation in (4) yields the following expression for the pdf of \(y_k\):

$$f(y_k|\mu_k, V_k, \alpha_k, \kappa_k) = \det(V_k^{-1}) \prod_{i=1}^{n} \mathcal{N}(\kappa_{k,i}, \alpha_{k,i}) \exp \left[ \alpha_k' V_k^{-1}(y_k - \mu_k) - \kappa_k' \psi_k \left\{ V_k^{-1}(y_k - \mu_k) \right\} \right],$$

(5)

where the \(j\)-th element of \(\psi_k \left\{ V_k^{-1}(y_k - \mu_k) \right\} \) contains \(\psi_k\) evaluated at the \(j\)-th element of the \(n_k\)-dimensional vector \(V_k^{-1}(y_k - \mu_k)\), “det” denotes the determinant function, \(\alpha_k \equiv (\alpha_{k,1}, \ldots, \alpha_{k,n_k})'\), and \(\kappa_k \equiv (\kappa_{k,1}, \ldots, \kappa_{k,n_k})'\). The density in (5) is referred to as the CM distribution, and we use the shorthand CM(\(\alpha_k, \kappa_k, \mu_k, V_k; \psi_k\)).
3 The Generalized Conjugate Multivariate Distribution

We now propose the generalized conjugate multivariate (GCM) distribution, which is defined by the transformation,
\[ y = \mu_M + V_M w_M, \] (6)
where the \( N \equiv \sum_{k=1}^{K} n_k \)-dimensional random vector \( y = (y'_1, \ldots, y'_K)' \), \( N \)-dimensional random vector \( w_M = (w'_1, \ldots, w'_K)' \) with \((k, i)\)-th element \( w_{k,i} \sim DY(\alpha_{k,i}, \kappa_{k,i}; \psi_k)\), the subscript “M” stands for “Multi-type,” the \( N \times N \) real-valued matrix \( V_M \) is an invertable covariance parameter matrix, and \( \mu_M \) is an unknown \( N \)-dimensional real-valued location parameter vector. The pdf for \( y \) and conditional distributions are stated in Theorems 3.1 and 3.2, respectively.

**Theorem 3.1.** Let \( y \) be defined as in (6). Then the pdf for \( y \) is given by,
\[
f(y|\mu_M, V_M, \alpha_M, \kappa_M) = \det(V_M^{-1}) \prod_{k=1}^{K} \left[ \prod_{i=1}^{n_k} N_k(\kappa_{k,i}, \alpha_{k,i}) \right] \exp\left[ \alpha'_M V_M^{-1}(y - \mu_M) - \kappa'_M \psi M \{ V_M^{-1}(y - \mu_M) \} \right], \] (7)
where \( y \in \mathcal{S} \), \( \mathcal{S} = \{ y : y = \mu_M + V_M c, c = (c_{k,i}), c_{k,i} \in \mathcal{Y}_k, i = 1, \ldots, n_k, k = 1, \ldots, K \} \), \( \alpha_{k,i}/\kappa_{k,i} \in \mathcal{D}_k, \kappa_{k,i} > 0 \), \( \psi_M \{ V_M(y - \mu_M) \} = (\psi_1 (J_1 V_M(y - \mu_M))', \ldots, \psi_K (J_K V_M(y - \mu_M))')' \), the \( n_k \times N \) matrix \( J_k = \left[ 0_{n_k, \sum_{j=1}^{k-1} n_j}, I_{n_k}, 0_{n_k, \sum_{j=k+1}^{K} n_j} \right] \), \( 0_{n,m} \) is an \( n \times m \) matrix of zeros, \( I_{n_k} \) is an \( n_k \times n_k \) identity matrix, the \( N \)-dimensional vector \( \alpha_M = (\alpha_1', \ldots, \alpha'_K)' \), and the \( N \)-dimensional vector \( \kappa_M = (\kappa'_1, \ldots, \kappa'_K)' \). Furthermore, the mean and covariance matrix of \( y \) is given by
\[
E(y|\mu_M, V_M, \alpha_M, \kappa_M) = \mu_M + V_M k(\alpha_M, \kappa_M)
\]
\[
\text{cov}(y|\mu_M, V_M, \alpha_M, \kappa_M) = V_M K(\alpha_M, \kappa_M) V_M',
\]
where the \( N \)-dimensional vector \( k(\alpha_M, \kappa_M) = \left( N_k(\kappa_{k,i}, \alpha_{k,i}), N_k^{(1)}(\kappa_{k,i}, \alpha_{k,i}) : i = 1, \ldots, n_k, k = 1, \ldots, K \)'.
the $N \times N$ diagonal matrix $K(\alpha, \kappa)$ has $(i,k)$-th diagonal element $\mathcal{N}_k(\kappa_{k,i}, \alpha_{k,i})$.

**Proof:** See the Appendix.

Equation (7) can be seen as a “stacked form” (stacked over data types) of the CM distribution in (5). We use the shorthand $GCM(\alpha, \kappa, \mu, \psi)$ for the density in (7). Our class of GLMMs will imply a GCM for the posterior predictive distribution for fixed effects, random effects, and location parameters. This form arises since the data model in a Bayesian hierarchical model has a similar form to that of a possibly non-Gaussian CM distribution, and when stacked over certain CM priors (e.g., Gaussian), we obtain the form of a GCM. The GCM allows one the flexibility to specify any CM prior (e.g., Gaussian, multivariate log-gamma, multivariate logit-beta, etc.) for fixed and random effects. It also allows you to specify the data model to be DY, which is not necessarily part of the exponential family. For example, one might consider $Z_{k,i}|Y_{k,i}$ to be distributed as Student-t, which is not part of the exponential family, but is part of the DY family.

**Theorem 3.2.** Let $y = (y^{(1)'}, y^{(2)'})' \sim GCM(\alpha, \kappa, \mu, \psi)$, where $y^{(1)}$ is $r$-dimensional and $y^{(2)}$ is $(N-r)$-dimensional. Also, let $V_M^{-1} = (H, Q)$, where $H$ is a $N \times r$ and $Q$ is $N \times (N-r)$. Then, it follows

$$f(y^{(1)} | y^{(2)}, \mu, V, \alpha, \kappa) \propto \exp \left[ \alpha M' Hy^{(1)} - \kappa M' \psi M \left\{ Hy^{(1)} - \mu M' \right\} \right],$$

where $\mu M' = V_M^{-1} \mu - Q y^{(2)}$.

**Proof:** See the Appendix.
Remark 1: It is not known how to simulate from a conditional GCM nor is it known how to directly simulate for a conditional CM distribution, except in the special case where \( k = K = 1 \) so that \( y \) is Gaussian distributed. This fact is one key reason why MCMC is needed for GLMMs. We provide more discussion in Section 4.3.

One can easily define truncated versions of the GCM similar to truncated versions of the multivariate normal distribution. In particular, let \( R \subset S \) such that

\[
\mathcal{P}(y \in R|\mu_M, V_M, \alpha_M, \kappa_M) = \int_R f(y|\mu_M, V_M, \alpha_M, \kappa_M) dy > 0.
\]

Then it is immediate from Theorem 3.1 that the truncated GCM is given by

\[
f(y|\mu_M, V_M, \alpha_M, \kappa_M) = \frac{I(y \in R)}{\mathcal{P}(y \in R|\mu_M, V_M, \alpha_M, \kappa_M)} \det(V^{-1}_M) \left\{ \prod_{k=1}^K \prod_{i=1}^{n_k} N_k(\kappa_{k,i}, \alpha_{k,i}) \right\} \exp \left[ \alpha'_M V^{-1}_M (y - \mu_M) - \kappa'_M \psi_M \left\{ V^{-1}_M (y - \mu_M) \right\} \right],
\]

where \( I(\cdot) \) is an indicator function and \( \mathcal{P}(y \in R|\mu_M, V_M, \alpha_M, \kappa_M) \) is the probability of a GCM random vector \( y \) belonging to the set \( R \). If \( \mathcal{P}(y \in R|\mu_M, V_M, \alpha_M, \kappa_M) \neq 0 \) then a rejection sampler is straightforward strategy to sample from the truncated GCM. Specifically, sample \( y \) from a GCM\((\mu_M, V_M, \alpha_M, \kappa_M; \psi_M)\), and if \( y \in R \) keep and otherwise reject. We consider the use of the truncated GCM in the context of Bayesian GLMMs in Section 4.3.

4 Exact Posterior Regression

In this section, we outline how to sample from the posterior predictive distribution of fixed and random effects from a general class of GLMMs. Various forms of the posterior predictive distribution are given, and are either a GCM, conditional GCM, or truncated GCM depending on certain prior
4.1 Gaussian Mixed Effects Models

Linear model assumptions are placed on the $n_1$-dimensional vectors $z_1 = (Z_1, \ldots, Z_{n_1})'$ and $y_1$ (McCullagh and Nelder, 1989) as follows

\begin{align*}
  z_1 &= y_1 + \varepsilon \\
  y_1 &= X_1\beta + G_1\eta + \xi_1,
\end{align*}

where the $n_1$-dimensional random vector $\varepsilon$ is Gaussian with mean zero and known $n_1 \times n_1$ positive definite covariance matrix $\text{cov}(\varepsilon|V_\varepsilon) = V_\varepsilon V'_\varepsilon$, $X_1$ is a $n_1 \times p$ matrix of known covariates, $\beta$ is an unknown $p$-dimensional vector of regression coefficients, $\beta$ is given a Gaussian prior with mean zero and unknown $p \times p$ covariance matrix $\text{cov}(\beta|V_\beta) = V_\beta V'_\beta$, $G_1$ is a $n_1 \times r$ matrix of coefficients for the $r$-dimensional random effects $\eta$. $\eta$ is assumed Gaussian with mean zero and unknown $r \times r$ positive definite covariance matrix $\text{cov}(\eta|V_\eta) = V_\eta V'_\eta$, and $\xi_1$ is an $n_1$-dimensional vector of random effects with mean zero and unknown $n_1 \times n_1$ covariance matrix $\text{cov}(\xi_1|V_{\xi,1}) = V_{\xi,1} V'_{\xi,1}$, where $V_{\xi,1}$ is a $n_1 \times n_1$ real-valued matrix. All fixed and random effects are assumed mutually independent. The real matrix-valued parameters $V_\varepsilon$, $V_\beta$, $V_\eta$, and $V_{\xi,1}$ are $n_1 \times n_1$, $p \times p$, $r \times r$, and $n_1 \times n_1$, respectively. In practice, we consider several structured specifications of $V_\varepsilon$, $V_\beta$, $V_\eta$, and $V_{\xi,1}$; for example, we consider $V_\varepsilon = \text{diag}(\sigma_i: i = 1, \ldots, n_1)$ and $V_{\xi,1} = \sigma_\xi I_{n_1}$, where $\sigma_i > 0$ and $\sigma_\xi > 0$.

In this article, $G_1$ will be set equal to a known pre-specified matrix of basis functions (e.g., splines (Wahba, 1990), wavelets (Novikov et al., 2005), Moran’s I basis functions (Hughes and
In general, \( G_1 \) could be unknown and set equal, for example, to the Cholesky decomposition of a known structured covariance matrix, so that \( \text{cov}(G_1\eta | V_\eta) = \sigma_\eta^2 G_1 G'_1 \), where \( V_\eta = \sigma_\eta I_{n_1} \) and \( \sigma_\eta > 0 \). It is fairly straightforward to sample directly from the posterior predictive distribution \( f(\xi_1, \beta, \eta | z_1, V_\varepsilon, V_\beta, V_\eta, V_{\xi_1}) \) using Theorem 4.1.

**Theorem 4.1.** Suppose \( z_1 \) and \( y_1 \) are distributed according to (8). Then the distribution of \( \xi_1 = (\xi'_1, \beta', \eta')' \) given \( z_1, V_\varepsilon, V_\beta, V_\eta, \) and \( V_{\xi_1} \) is Gaussian with mean \( (H'_1 H_1)^{-1} H'_1 (z'_1 V_{\varepsilon^{-1}'}, 0_{n_1+p+r})' \) and covariance matrix \( (H'_1 H_1)^{-1} \), where the \((2n_1+p+r) \times (n_1+p+r)\) matrix

\[
H_1 = \begin{pmatrix}
V_{\varepsilon^{-1}} & V_{\varepsilon^{-1}X_1} & V_{\varepsilon^{-1}G_1} \\
0_{p,n_1} & V_{\beta}^{-1} & 0_{p,r} \\
0_{r,n_1} & 0_{r,p} & V_{\eta}^{-1} \\
V_{\xi_1}^{-1} & 0_{n_1,p} & 0_{n_1,r}
\end{pmatrix}.
\]

**Proof:** See the Appendix.

To simulate from \( f(\xi_1 | z_1, V_\varepsilon, V_\beta, V_\eta, V_{\xi_1}) \) compute \( (H'_1 H_1)^{-1} H'_1 w_1 \), where \( w_1 \) is Gaussian with mean \( (z'_1 V_{\varepsilon^{-1}'}, 0_{n_1+p+r})' \) and identity covariance matrix. This type of ordinary least squares (OLS) representation has been seen in other less general settings. For example, if \( \beta \) is given an improper flat prior (i.e., \( f(\beta) = 1 \)), \( \eta \) and \( \xi_1 \) are removed from the linear model, and \( V_\varepsilon = \sigma I_{n_1} \) (with \( \sigma > 0 \)), then the posterior predictive distribution of \( \beta \) is Gaussian with mean \( (X'_1 X_1)^{-1} X'_1 z_1 \) with covariance \( \sigma^2 (X'_1 X_1)^{-1} \) (e.g., see Gelman et al. 2013, pg. 373).

**Remark 2:** The presence of the so-called “fine-scale” term \( \xi_1 \) is important to ensure that the implied covariance matrix of \( y_1 \) is full rank and nonsingular. Suppose \( n_1 > (p+r) \) and \( V_{\xi_1} = \sigma_\xi I_{n_1} \), where \( \sigma_\xi > 0 \). Then \( \text{cov}(y_1 | V_\beta, V_\eta, \sigma_\xi) = XV_\beta V_\beta' X' + \)
GV_\eta V_\eta' G' + \sigma_\zeta^2 I_{n_1}. Thus, if \sigma_\zeta = 0 then cov(y_1|V_\beta, V_\eta, \sigma_\zeta) is low rank and nonsingular. This is true of the posterior of y_1 as well; that is, if (p + r) < n_1 then cov(y_1|z_1, V_\beta, V_\eta, \sigma_\zeta = 0) is also not full-rank and hence not strictly positive definite. Thus, non-zero \sigma_\zeta is used to avoid a singular covariance specification for y_1 in both the prior and posterior distributions. However, one negative consequence is that incorporating \xi_1 leads to a model that can be interpreted as overparameterized. Marginalization is a common tool to account for overparameterization. That is, marginalize out \xi_1 so that z_1|\beta \xi, V_\epsilon, V_{\xi,1} is normal with mean X_1\beta + G_1 \eta and covariance matrix V_\epsilon V_\epsilon' + V_{\xi,1} V_{\xi,1}'.

Marginalizing \xi_1 is not the only strategy for addressing the overparameterization caused by \xi_1. Another strategy is to use a more informative prior; for example, set V_{\xi,1} = \sigma_\xi I_\xi with \sigma_\xi = 0, which effectively removes the term from the linear model. In general, \sigma_\xi should be restricted to be small as Theorem 4.2 suggests.

**Theorem 4.2.** Suppose z_1 and y_1 are distributed according to (8), but suppose we replace the Gaussian prior for \xi_1 with improper prior f(\xi_1) = 1. Then the distribution for \xi_1 given z_1, V_\epsilon, V_\beta, V_\eta, and V_{\xi,1} is Gaussian with mean H_1^{-1}(z' V_\epsilon^{-1}, 0_{1,p+r})' and covariance matrix H_1^{-1}(H_1'^{-1})^{-1}, where the \( (n_1 + p + r) \times (n_1 + p + r) \) matrix

\[
H_1^* = \begin{pmatrix}
V_\epsilon^{-1} & V_\epsilon^{-1} X_1 & V_\epsilon^{-1} G_1 \\
0_{p,n_1} & V_\beta^{-1} & 0_{p,r} \\
0_{r,n_1} & 0_{r,p} & V_\eta^{-1}
\end{pmatrix}
\]

and

\[
H_1'^{-1} = \begin{pmatrix}
V_\epsilon & -X_1 V_\beta & -G_1 V_\eta \\
0_{p,n_1} & V_\beta & 0_{p,r} \\
0_{r,n_1} & 0_{r,p} & V_\eta
\end{pmatrix}.
\]

**Proof:** See the Appendix.
Remark 3: To simulate from \( f(\xi_1|z_1,V_e,V_\beta,V_\eta,V_\xi) \) using the degenerate model in Theorem 4.2 compute \( H_{1}^{-1}w_1 \), where \( w_1 = (w'_e,w'_b,w'_h)' \) is Gaussian with mean \( (z'V_e^{-1},0_{p+r})' \) and identity covariance matrix such that \( w_e \) is \( n_1 \) dimensional, \( w_b \) is \( p \) dimensional, and \( w_h \) is \( r \) dimensional. Considering the structure of \( H_{1}^{-1} \), we see that the posterior replicates of \( \beta \) are set equal to \( V_\beta w_b \) and \( \eta \) equal to \( V_\beta w_h \), which are equal in distribution to replicates from their respective prior distributions. Additionally, in this setting posterior replicates of \( \xi_1 \) are equivalent to adding mean zero Gaussian distributed noise with covariance \( V_eV_e' \) to the residual \( z_1 - X_1V_\beta w_b - G_1V_\eta w_h \). Hence, the posterior mean of \( y_1 \) is simply \( z_1 \). This suggests that if one sets \( V_\xi_1 = \sigma_\xi I_{n_1} \) and places a prior on \( \sigma_\xi \), and the corresponding posterior replicates of \( \sigma_\xi \) are large (so that \( f(\xi_1|\sigma_\xi) \) is “close” to constant) then we tend to overfit the data when predicting \( y_1 \), while simultaneously having no Bayesian learning of \( \beta \) and \( \eta \). Consequently, in our illustrations we set \( V_\xi_1 = \sigma_\xi I_{n_1} \) with \( \sigma_\xi \) equal to a fixed value considered to be “small.” Additionally, to avoid overfitting the predictions, one might estimate \( y_1 \) using \( \tilde{y}_1 = X_1\beta + G_1\eta \), which implicitly estimates \( \xi_1 \) with a zero vector.

4.2 Generalized Linear Mixed Effects Models

Suppose we observe \( K \) different types of data from the exponential family, and we observe multiple observations of each data type. Let the total number observations of type \( k \) be denoted with \( n_k \). Denote the \( n_k \)-dimensional data vectors with \( z_k = (Z_{k,1},\ldots,Z_{k,n_k})' \), each of which depend on a latent unobserved vector \( y_k = (Y_{k,1},\ldots,Y_{k,n_k})' \). Then \( Z_{k,i}|Y_{k,i} \) is distributed independently according to \( (1) \) for \( k = 1,2,3,4 \). When \( k = 1 \) we again assumed \( Z_{1,i}|Y_{1,i},\sigma_i^2 \) is Gaussian distributed with mean \( Y_{1,i} \), but in what remains we will consider a different mixed effects model assumption on \( Y_{1,i} \) and different priors on fixed and random effects than the one presented in \( (1) \). Furthermore, we assume \( Z_{1,i}|Y_{1,i},\sigma_i^2 \) is conditionally independent across \( i \) with variance \( \sigma_i^2 \) so that \( V_e = \text{diag}(\sigma_i:i= \)
1, \ldots, n_1). Additionally, we assume $Z_{2,i}|Y_{2,i}$ is Poisson distributed with mean $\exp(Y_{2,i})$, $Z_{3,i}|Y_{3,i},m_i$ is binomial distributed with sample size $m_i$ and probability of success $\exp(Y_{3,i})/\{1 + \exp(Y_{3,i})\}$, the $n_3$-dimensional vector $m = (m_1, \ldots, m_{n_3})'$, and consider a shifted Student-$t$ specification $Z_{4,i}|Y_{4,i} \sim \text{CM}(0, \frac{\nu+1}{2}, Y_{4,i}, 1)$ with $\nu > 0$ known. Binomial distributed data allows for Bernoulli distributed data as a special case (i.e., $m_i = 1$), and multinomial distributed data when using a stick-breaking representation of the multinomial (e.g., see [Bradley et al. 2019] for stick-breaking in the context of CM prior distributions).

Consider the following overparameterized linear model assumption for the $n_k$-dimensional random vector $y_k$ (McCullagh and Nelder [1989]):

$$y_k = X_k \beta + G_k \eta + (\xi_k - \mu_k); k = 1, \ldots, K,$$

(9)

where we consider $K = 4$, $X_k$ is a $n_k \times p$ matrix of known covariates, and $\beta$ is an unknown $p$-dimensional vector of regression coefficients. Let $\beta$ have a CM prior with $p$-dimensional mean vector $\mu_\beta$, shape vector $\alpha_\beta$, scale vector $\kappa_\beta$, and $p \times p$ covariance parameter matrix $V_\beta$. Let $G_k$ be a $n_k \times r$ matrix of coefficients for the $r$-dimensional random effects $\eta$. We assume $\eta$ is CM with $r$-dimensional mean vector $\mu_\eta$, shape vector $\alpha_\eta$, scale vector $\kappa_\eta$, and $r \times r$ covariance parameter matrix $V_\eta$.

Stack the $n_k$-dimensional random vectors $\xi_k$ into a $N$-dimensional vector $\xi = (\xi_1', \ldots, \xi_k')'$. Let the prior on $\xi$ be proportional to a conditional GCM prior with $2N$-dimensional location parameter $(\mu_D' - \beta'X' - \eta'G', \mu_\xi)'$ and $2N \times N$ matrix-valued precision parameter matrix $(I_N, V_\phi')'$, where the $N$-dimensional vector $\mu_D = (\mu_1', \ldots, \mu_K')'$, the $N \times p$ matrix $X = (X_1', \ldots, X_K')'$, the $N \times r$ matrix $G = (X'_1, \ldots, X'_K)'$, the $N$-dimensional vector $\mu_\xi$ is real-valued, and the $N \times N$ matrix $V_\phi$ is real-valued. The shape parameter for the prior on $\xi$ is set to $\alpha_\xi = (0_{1,n_1}, \alpha_\xi 1_{1,n_2}, \alpha_\xi 1_{1,n_3}, 0_{1,N})'$ and the scale/shape parameter is set to $\kappa_\xi = (0_{1,n_1}, 0_{1,n_2}, 2\alpha_\xi 1_{1,n_3}, \frac{1}{2} 1_{1,N})'$, where $1_{r,N}$ is a $r \times N$ matrix of ones. The unit-log partition function $\psi_M$ associated with the conditional GCM prior for
\[ \psi_M(h) = (\psi_1(h_1), \ldots, \psi_1(h_{n_1}), \ldots, \psi_K(h_{\sum_{k=1}^{K-1} n_k+1}), \ldots, \psi_K(h_{N}), \psi_1(h_{N}^{*}), \ldots, \psi_1(h_{N}^{*}))', \]

for any \( h = (h_1, \ldots, h_N, h_1^*, \ldots, h_N^*)' \in \mathbb{R}^{2N} \). In this section, we allow the priors on \( \beta \) and \( \eta \) be generic CM, however in this article, we often choose these CM distributions to be Gaussian. Also, in practice we set \( V_\xi = I_{n_1} \) and \( \alpha_\xi = 0.5 \), which is chosen to ensure the posterior distribution has well-defined hyperparameters (i.e., do not fall on the boundaries of the parameter space). All fixed and random effects are assumed mutually independent.

Stack the location parameters into a \((2N + p + r)\)-dimensional vector \( \mu = (\mu_1', \ldots, \mu_K', \mu_\beta', \mu_\eta', \mu_\xi')' \). To ensure the location parameters are not confounded with the fixed and random effects, we assume that the location parameter vector is projected onto the orthogonal complement of the the \((2N + p + r) \times (N + p + r)\) posterior regressors,

\[
H = \begin{pmatrix}
    I_N & X & G \\
    0_{p,N} & V_\beta^{-1} & 0_{p,r} \\
    0_{r,N} & 0_{r,p} & V_\eta^{-1} \\
    V_\xi^{-1} & 0_{N,p} & 0_{N,r}
\end{pmatrix},
\]

where the \( N \times p \) matrix \( X = (X_1', \ldots, X_K')' \), and the \( N \times r \) matrix \( G^* = (G_1', \ldots, G_K')' \). Specifically, we assume \( \mu = -Qq \), where the \((2N + p + r) \times N\) matrix \( Q \) are the eigenvectors of the \((2N + p + r) \times (2N + p + r)\) matrix \( I_{2N+p+r} - H(H'H)^{-1}H' \) so that \( QQ' = I_{2N+p+r} - H(H'H)^{-1}H' \) and \( H'Q = 0_{N+p+r,N} \), where recall that idempotent matrices have eigenvalues equal to zero or one. We refer to \( q \) as the “bypass location parameters.” Several GLMMs in the literature set \( q = 0_{N+p+r,1} \) as done in the Gaussian mixed effects model in Section 4.1. However, the presence of non-zero \( q \) leads to a similar projection representation of posterior replicates of the \((N + p + r)\)-dimensional
vector $\zeta = (\xi', \beta', \eta')'$ used in the Gaussian mixed effects model setting as seen in Theorem 4.3 and Theorem 4.4 below.

**Theorem 4.3.** Suppose $Z_{k,i} | Y_{k,i}, \theta_D$ is independently distributed according to either (1) or (2) with $i = 1, \ldots, n_k$, and $k = 1, \ldots, 4$. Let $\theta_D$ be all data model parameters (e.g., when $k = 1 \theta_D$ contains $\{\sigma_1\}$ and when $k = 4 \theta_D$ contains $\nu$). Let $j = 1, \ldots, 4$ correspond to the choice of CM prior for $\beta | \mu_\beta, \alpha_\beta, \kappa_\beta, V_\beta$, and let $b = 1, \ldots, 4$ correspond to the choice of CM prior for $\eta | \mu_\eta, \alpha_\eta, \kappa_\eta, V_\eta$. Let $y_k$ be defined as in (9), $\mu = -Qq$, and assume the improper prior $f(q) = 1$. Then $(\xi', q') | z, \theta_D, V_\beta, V_\eta, V_\xi \sim GCM(\alpha_M, \kappa_M, \mu_M, (H, Q)^{-1}; \psi_M)$, where the $(2N + p + r)$-dimensional location and shape/scale parameter vectors,

$$
\mu_M = \left(\begin{array}{c}
(H'H)^{-1}H' \\
0_{n_1+n_2+n_3,1} \\
Q' \\
0_{N+p+r,1}
\end{array}\right) \left(\begin{array}{c}
z_4 \\
0_{N+p+r,1}
\end{array}\right)
$$

$$
\alpha_M = (z'_1 D'_\sigma, z'_2 + \alpha_\xi 1_{1,n_2}, z'_3 + \alpha_\xi 1_{1,n_3}, 0_{1,n_4}, \alpha'_\beta, \alpha'_\eta, 0_{1,N})'
$$

$$
\kappa_M = (\frac{1}{2}1_{1,n_1} D'_\sigma, 1_{1,n_2, m'} + 2 \alpha_\xi 1_{1,n_3, \nu} + \frac{1}{2}1_{1,n_4}, \kappa'_\beta, \kappa'_\eta, \frac{1}{2}1_{1,N})',
$$

where $D_\sigma = \text{diag}\left(\frac{1}{\sigma_i^2} : i = 1, \ldots, n_1\right)$, and $(2N + p + r)$-dimensional unit-log partition function

$$
\psi_M(h^*) = (\psi_1(h_1), \ldots, \psi_1(h_{n_1}), \psi_2(h_{n_1+1}), \ldots, \psi_2(h_{n_1+n_2}), \ldots, \psi_K(h_{n_1}), \psi_j(h_{j,1}^*), \ldots, \psi_j(h_{j,p}^*), \psi_b(h_{b,1}^*), \ldots, \psi_3(h_{b,r}^*), \psi_1(h_1^*), \ldots, \psi_1(h_N^*))',
$$

for $(2N + p + r)$-dimensional real-valued vector $h^* = (h_1, \ldots, h_N, h_{j,1}^*, \ldots, h_{j,p}^*, h_{b,1}^*, \ldots, h_{b,r}^*, h_1^*, \ldots, h_N^*)'$.

**Proof:** See the Appendix.
It is straightforward to adjust Theorem 4.3 when not all four data types are observed. For example, suppose we only observe Poisson distributed data and binomial distributed data. Then the posterior predictive distribution is GCM with \( \mu_\text{M} = 0_{2n_2+2n_3+p+r}, \alpha_\text{M} = (z_2^\prime + \alpha_\xi^\prime 1_{1,n_2}, z_3^\prime + \alpha_\xi^\prime 1_{1,n_3}, \alpha^\prime_\beta, \alpha^\prime_\eta, 0_{1,n_2+n_3})^\prime, \ k_\text{M} = (1_{1,n_2}, m^\prime + 2\alpha_\xi^\prime 1_{1,n_3}, \kappa^\prime_\beta, \kappa^\prime_\eta, \frac{1}{2} 1_{1,n_2+n_3})^\prime, \)

\[ \psi_M(h^*) = (\psi_2(h_1), \ldots, \psi_2(h_{n_2}), \psi_3(h_{n_2+1}), \ldots, \psi_3(h_{n_2+n_3}), \psi_j(h_{j,1}), \ldots, \psi_j(h_{j,p}), \]

\[ \psi_0(h_{b,1}^*), \ldots, \psi_0(h_{b,r}^*), \psi_1(h_1^*), \ldots, \psi_1(h_{n_2+n_3}^*) \]'

with \( h^* = (h_1, \ldots, h_{n_2+n_3}, h_{j,1}^*, \ldots, h_{j,p}^*, h_{b,1}^*, h_{b,r}^*, h_1^*, \ldots, h_{j,n_2+n_3})^\prime \in \mathbb{R}^{2n_2+2n_3+p+r} \), the first block row of \( H \) is replaced with \((1_{n_2+n_3}, (X_2^\prime, X_3^\prime)^\prime, (G_2^\prime, G_3^\prime)^\prime)\), and \( Q \) are the eigenvectors of the orthogonal complement of this adjusted expression of \( H \).

In Theorem 4.3 the presence of \( \xi \) makes \( \alpha_\text{M} \) and \( k_\text{M} \) strictly positive when elements of \( z_2 \) and \( z_3 \) are zero. Hence, similar to our discussion in Remark 3, the presence of a fine-scale term allows one to avoid the boundaries of the parameter space. Furthermore, it allows one to obtain replicates from the posterior predictive distribution \( f(\xi, q|z, \theta_D, V_\beta, V_\eta, V_\xi) \) as seen below in Theorem 4.4.

**Theorem 4.4.** Denote a replicate of \( \xi, q, \) and \( y \) using \( f(\xi, q|z, \theta_D, V_\beta, V_\eta, V_\xi) \) from Theorem (4.3) with \( \xi_{\text{rep}}, q_{\text{rep}}, \) and \( y_{\text{rep}} \). Then

\[ \xi_{\text{rep}} = (H^\prime H)^{-1} H^\prime w \]  
(10)

\[ q_{\text{rep}} = Q^\prime w, \]  
(11)

\[ y_{\text{rep}} = (I_N, 0_{N,N+p+r}) H \xi_{\text{rep}} + (I_N, 0_{N,N+p+r}) Q q_{\text{rep}} = (I_N, 0_{N,N+p+r}) w \]  
(12)

where the \((2N+p+r)\)-dimensional random vector \( w \) is GCM with location \((0_{1,n_1+n_2+n_3}, z_4^\prime, 0_{1,N+p+r})^\prime\), identity covariance parameter matrix, and the same shape parameter \( \alpha_\text{M} \), shape/scale parameter \( \kappa_\text{M} \), and unit-log partition function \( \psi_M \) defined in Theorem 4.3.
Proof: See the Appendix

**Remark 4:** Theorem 4.4 provides the motivation for including the location parameter $q$. Namely, this location parameter leads to Gaussian-like (i.e., the projection form similar to Theorem 4.1 and Remark 1) simulations from the posterior predictive distribution. However, the incorporation of $q$ leads to a model that is clearly overparameterized. Thus, a simple solution is to perform inference on $\zeta$ using replicates from (10), which generates values from the marginal distribution $f(\zeta | z, \theta, D, V_\beta, V_\eta, V_\xi)$. Then use the ad-hoc plug-in estimator of $q = 0_{N+p+r,1}$. This is the general strategy used in the CM literature (Bradley et al., 2020) implemented using a block Gibbs sampler. Let $\tilde{y}$ represent the profile of $y$ using the plug-in estimator $q = 0_{N+p+r,1}$, so that $\tilde{y}_{rep} = (I_N, 0_{N,N+p+r})H\zeta_{rep} = X\beta_{rep} + G\eta_{rep} + \xi_{rep}$, where $\zeta_{rep} = (\xi_{rep}', \beta_{rep}', \eta_{rep}')'$.

**Remark 5:** The random vector $w$ has a very important interpretation. If one assumes $Z_{ik} | Y_{ik}$ is distributed according to the exponential family in (1), and $Y_{ik}$ is independently distributed according to the DY distribution in (2) then we have that the implied posterior distribution for $Y_{ik}$ in (3) is equal in distribution to $w_{ik}$ in Theorem 4.4. Thus, $w$ represents a replicate from the posterior distribution from a saturated model. Recall in the goodness-of-fit literature that saturated models define a separate parameter for each datum and is meant to overfit the data, and then, measures of deviance from the saturated model are used to select more parsimonious models (Agresti, 2019; Bradley, 2022). This provides additional motivation for using the marginal distribution $f(\zeta | z, \theta, D, V_\beta, V_\eta, V_\xi)$ and $\tilde{y}_{rep}$ to perform inference on $y$, which implicitly implies the use of the ad-hoc plug-in estimator of $q = 0_{N+p+r,1}$. Moreover, when considering Remark 3, one might similarly use $\tilde{y} = X\beta + G\eta$ for inference on $y$, which would implicitly estimate both $q$ and $\xi$ to be zero.
4.3 An Informative Prior on the Bypass Location Parameters

Theorem 4.4 shows that posterior predictive inference on \( \zeta \) can be implemented analogously to Gaussian mixed effects models when using \( f(\zeta|z, \theta_D, V_\beta, V_\eta, V_\xi) \) for inference, which marginalizes across \( q \). In the GLMM literature it is more standard to use the informative point mass prior of \( f(q) = I(q = 0_{N+p+r,1}) \) leading one to use \( f(\zeta|z, \theta_D, V_\beta, V_\eta, V_\xi, q = 0_{N+p+r,1}) \) for inference (McCulloch et al., 2008; Banerjee et al., 2015).

Theorem 4.5. Suppose \( Z_{k,i}|Y_{k,i} \) is independently distributed according to either (1) or (2) with \( i = 1, \ldots, n_k, \) and \( k = 1, \ldots, K \). Let \( y_k \) be defined as in (9) with the added assumption that \( q = 0_{N+p+r,1} \). Then \( \zeta|z, \theta_D, V_\beta, V_\eta, V_\xi \) is conditional GCM with \( H, \mu_M, \alpha_M, \kappa_M, \) and \( \psi_M \) defined in Theorem 4.3.

Proof: See the Appendix

Theorem 4.5 shows that \( f(\zeta|z, \theta_D, V_\beta, V_\eta, V_\xi, q = 0_{N+p+r,1}) \) is a conditional GCM, however, as stated in Remark 1 it is currently unknown on how to directly simulated from the conditional GCM in many settings, and as such, simulating from this specification of the GLMM can only be immediately be solved with MCMC. Upon comparison of Theorem 4.3 and Theorem 4.5 we see that we “bypass” the use of MCMC by allowing for non-zero \( q \); hence the name, “bypass location parameters.” A compromise between the informative point mass at zero prior used in Theorem 4.5 and the constant improper non-informative prior in Theorem 4.3 would be to place a more informative prior on \( q \) forcing this parameter to be “close” to a zero vector.

Theorem 4.6. Suppose \( Z_{k,i}|Y_{k,i} \) is independently distributed according to either (1) or (2) with \( i = 1, \ldots, n_k, \) and \( k = 1, \ldots, K \). Let \( y_k \) be defined as in (9) with the added assumption that
\( f(q|\theta_D, V_\beta, V_\eta, V_\xi) = I\{g(q, Q) \leq \omega\} \), where \( \omega > 0 \), recall \( Q \) is a function of \( \{\theta_D, V_\beta, V_\eta, V_\xi\} \), and let

\[
g(q, Q) = \text{trace} \left[ (I_N, 0_{N+N+p+r})Qqq'Q'(I_N, 0_{N+N+p+r})' \right].
\]

Then \( (\zeta', q')|z, \theta_D, V_\beta, V_\eta, V_\xi \) is a truncated GCM with \( (H, Q)^{-1}, \mu_M, \alpha_M, \kappa_M, \) and \( \psi_M \) defined in Theorem 4.3 and \( R = \{q : g(q, Q) \leq \omega\} \). Additionally, \( g(q_{rep}, Q) = \{y_{rep} - \hat{y}_{rep}\}'\{y_{rep} - \hat{y}_{rep}\} \).

Proof: See the Appendix.

The set \( R \) defined in Theorem 4.6 can be interpreted as constraining a type of residual squared error to be small. Considering Remark 5, we see that \( g(q_{rep}, Q) = \{y_{rep} - \hat{y}_{rep}\}'\{y_{rep} - \hat{y}_{rep}\} \) suggests that goodness-of-fit is naturally built into the posterior predictive distribution in Theorem 4.6. In the case of the point mass prior (i.e., \( \omega = 0 \)) we have the most aggressive bound on the goodness-of-fit error \( g(q_{rep}, Q) \), and similarly, the non-informative prior (i.e., \( \omega = \infty \)) places no restrictions on the goodness-of-fit error \( g(q_{rep}, Q) \).

Depending on the choice of \( \omega \), the posterior predictive distribution in Theorem 4.6 is straightforward to simulate from using rejection sampling. That is first simulate \( y_{rep} = w, \zeta_{rep}, \) and \( \hat{y}_{rep} \) using Theorem 4.4. Then if \( g(q_{rep}, Q) \leq \omega \) keep \( y_{rep}, \zeta_{rep}, \) and \( \hat{y}_{rep} \), and otherwise reject this sample. In practice, we suggest choosing \( \omega \) as small as computationally feasible.

The three prior specifications for \( q \) cover three general strategies in the literature to handle overparameterization in a model, namely, sparsity (e.g., spike and slab [Ishwaran and Rao, 2005]), marginalization (e.g., see Remark 2), and near sparsity (e.g., the horseshoe prior [Carvalho et al., 2009]). The point mass prior \( f(q) = I(q = 0_{N+p+r,1}) \) makes use of sparsity. The non-informative prior \( f(q) = 1 \) requires one to marginalize out \( q \) for inference on \( \zeta \) as described in Remark 4, and use the sparse plug-in estimator \( q = 0_{N+p+r,1} \) for inference on \( y \). The informative prior \( f(q|\theta_D, V_\beta, V_\eta, V_\xi) = I\{g(q, Q) \leq \omega\} \) makes use of near-sparsity, where “near” is defined by the set \( R \) in Theorem 4.6.
4.4 Marginalizing Across Hyperparameters

Sampling from the posterior predictive distribution \( f(\zeta, q | z, \theta_D, V_\beta, V_\eta, V_\xi) \) is not the ultimate goal. We are primarily interested in sampling directly from the marginal posterior distribution \( f(\zeta | z) \) to perform inference on \( \zeta, y, \hat{y}, \) and \( \tilde{y}. \)

Theorem 4.2 and Remark 3 gave the motivation for assuming \( V_\xi = \sigma_\xi I_N \) with a point mass prior for \( \sigma_\xi \) on a “small positive value” in the Gaussian-only setting. A similar result exists for the GLMM setting, which we refer to as Result 1. We give the statement and proof of Result 1 in the Appendix for ease of exposition. As such, we assume \( V_\xi = \sigma_\xi I_N \) with a point mass prior for \( \sigma_\xi \) on a “small positive value.” Consequently, in what follows we drop \( V_\xi \) as an unknown term in our notations. We also consider the case where \( K = 3 \) so that we observe Gaussian data \((k = 1)\), Poisson data \((k = 2)\), and binomial data \((k = 3)\) with \( \theta_D = \{\sigma_i\} \), which corresponds to the standard deviations associated with the Gaussian data model. Additionally, we assume the CM priors on \( \beta \) and \( \eta \) are Gaussian so that the shape/scale parameters (denoted with \( \alpha \) and \( \kappa \)) are known \((0 \text{ and } 1/2)\). Thus, the hyperparameters in this setting consist of \( \{\sigma_i\}, V_\beta, \) and \( V_\eta \). When allowing for unknown variances, we set \( \mu = -Q_S q \), where “S” stands for standardized and \( Q_S \) is the \((2N + p + r) \times N\) eigenvectors of the orthogonal complement of,

\[
\begin{pmatrix}
I_N & X & G \\
0_{p,N} & I_p & 0_{p,r} \\
0_{r,N} & 0_{r,p} & I_r \\
I_N & 0_{N,p} & 0_{N,r}
\end{pmatrix}
\]
In general, we restrict ourselves to any prior on \(\{\{\sigma_i\}, V_\beta, V_\eta\}\) such that the marginal priors,

\[
f(\beta) = \int f(\beta | V_\beta) f(V_\beta) dV_\beta \\
f(\eta) = \int f(\eta | V_\beta) f(V_\eta) dV_\eta,
\]

are again CM distributions and

\[
f(z_1 | y_1) = \int \cdots \int \prod_{i=1}^{n_1} f(z_1 | y_1, \{\sigma_i^2\})^2 f(\{\sigma_i^2\}) d\sigma_1^2 \cdots d\sigma_{n_1}^2, \tag{14}
\]

is a distributed according to an exponential family or falls into the DY family of distributions. If the integrals in (13) and (14) are not respectively CM and CM/DY then one might use the CM distribution that minimizes the Kullback-Leibler (KL) divergence \cite{Kurz2016} to the integrals in (13) and (14), and hence, would satisfy our restrictions.

The specifications in (13) and (14) will allow us to use EPR to sample from the marginal posterior distribution,

\[
f(\zeta, q | z) \propto f(z | \beta, \eta, \xi, q) f(\beta) f(\eta) f(\xi | \eta, \beta) f(q).
\]

This strategy will allow us to perform inference on fixed and random effects while respecting hyperparameters; however it is naturally restrictive. In particular, one may be interested in performing inference on hyperparameters. If one is interested in inference on hyperparameters then of course one can derive the full-conditional distribution of the hyperparameters, which together with the marginal posterior, defines the joint posterior. However, in this article we consider these parameters as nuisance parameters and integrate them out as described above. If one uses the Kullback Leibler divergence to the marginal prior and data models in (13) and (14) to allow for non-conjugate hyperprior distributions, inference is naturally restricted to \(\zeta\) and \(y\). In Sections 4.5 and 4.6 we provide examples of two different ways to address hyperparameters.
4.5 Marginalizing Across Hyperparameters: Example 1

We now give an example where the integrals in (13) and (14) are known and respectively CM and DY. Consider $\beta$ and $\eta$ to have heterogeneous variances, where $V_\beta = \text{diag}(\sigma_{\beta,i} : i, \ldots, p)$, $V_\eta = \text{diag}(\sigma_{\eta,i} : i, \ldots, r)$, $\sigma_{\beta,i}^2 \sim \text{IG}(v_i/2, v_i/2)$, $\sigma_{\eta,i}^2 \sim \text{IG}(v_i/2, v_i/2)$, where $\text{IG}(v_i/2, v_i/2)$ is a shorthand for the inverse gamma prior with hyperparameter $v_i > 0$. Also, let $\sigma_i^2 \sim \text{IG}(v/2, v/2)$ with hyperparameter $v > 0$.

Then, for $\beta = (\beta_i : i = 1, \ldots, p)'$ we have,

$$f(\beta_i) = \int f(\beta_i | \sigma_{\beta,i}^2) f(\sigma_{\beta,i}^2) d\sigma_{\beta,i}^2$$

$$\propto \int \left( \frac{1}{\sigma_{\beta,i}^2} \right)^{1+(v_i+1)/2} \exp \left( -\frac{(\beta_i^2 + v_i)/2}{\sigma_{\beta,i}^2} \right) d\sigma_{\beta,i}^2$$

$$= \Gamma \left\{ \left( \frac{v_i+1}{2} \right) \left( \frac{v_i}{2} \right)^{(v_i+1)/2} \left( \frac{\beta_i^2}{v_i} + 1 \right)^{-(v_i+1)/2} \right\}$$

$$\propto \text{DY}(0, v_i; \psi_4),$$  \hspace{1cm} (15)

where $\Gamma$ is the gamma function and recall $\text{DY}(0, v_i; \psi_4)$ is a Student-$t$ distribution, which is easy to simulate from directly. Thus, for this prior specification the marginal prior is given by $\beta \sim \text{CM}(0_{p,1}, \nu_p, 0_{p,1}, 1_p; \psi_4)$, where $\nu_p = (c_1, \ldots, c_p)'$. Typically, we set $v_i \equiv c$ so that $\nu = c 1_p$. In a similar manner, $\eta \sim \text{CM}(0_{r,1}, \nu_r, 0_{r,1}, 1_r; \psi_4)$, where $\nu_r = (v_1, \ldots, v_r)'$. We set $v_i \equiv c$ so that $\nu_p = \nu 1_p$ and $\nu_r = \nu 1_r$. Now, we have

$$f(Z_{1,i}|Y_{1,i}) = \int f(Z_{1,i}|Y_{1,i}, \sigma_i^2) f(\sigma_i^2) d\sigma_i^2$$

$$\propto \int \left( \frac{1}{\sigma_i^2} \right)^{1+(v_i+1)/2} \exp \left( -\frac{(Z_{1,i} - Y_{1,i})^2 + v_i/2}{\sigma_i^2} \right) d\sigma_i^2$$

$$= \Gamma \left\{ \left( \frac{v_i+1}{2} \right) \left( \frac{v_i}{2} \right)^{(v_i+1)/2} \left\{ \left( \frac{Z_{1,i} - Y_{1,i}}{v_i} \right)^2 + 1 \right\}^{-(v_i+1)/2} \right\}$$

$$\propto \text{CM}(0, v_i, Y_{1,i}, 1; \psi_4),$$  \hspace{1cm} (16)
so that \( z_i | y_1 \sim \text{CM}(0_{n_1,1}, c1_{n_1}, y_1, I_{n_1}; \psi_4) \), which is a random vector consisting of independent shifted (by \( y_1 \)) Student-\( t \) random variables.

The specification in (16), \( \beta \sim \text{CM}(0_{p,1}, \nu_p, 0_{p,1}, I_p; \psi_4) \), and \( \eta \sim \text{CM}(0_{r,1}, \nu_r, 0_{r,1}, I_r; \psi_4) \) can be used to implement EPR and obtain samples from \( f(\zeta, q | z) \). Specifically, after integrating out variance parameters we are in the case where we observe Poisson distributed data, binomial distributed data, and shifted Student-\( t \) distributed data, and the CM priors for \( \beta \) and \( \eta \) are chosen to consist of independent Student-\( t \) random variables. This leads to the following adjustments to Theorem 4.3 to implement EPR: the first row of \( H_5 \) becomes \( (I_{n_1+n_2+n_3}, (X'_1, X'_2, X'_3)', (G'_1, G'_2, G'_3)') \), \( Q_5 \) are the eigenvectors of the orthogonal complement of this expression of \( H_5 \), and

\[
\begin{align*}
\alpha_M &= (0_{1,n_1}, z_2 + \alpha_5 1_{1,n_2}, z_3 + \alpha_5 1_{1,n_3}, 0_{1,n_1+n_2+n_3+p+r})' \\
\mu_M &= (z_1', 0_{1,n_1+2n_2+2n_3+p+r})' \\
\kappa_M &= (\frac{\nu+1}{2} 1_{1,n_1}, 1_{1,n_2}, \alpha_5 1_{1,n_3}, \frac{1}{2} (\nu_p + 1_{1,p,1})', \frac{1}{2} (\nu_r + 1_{1,r,1})', \frac{1}{2} 1_{1,n_1+n_2+n_3})', \\
\psi_M(h^*) &= (\psi_4(h_1), \ldots, \psi_4(h_{n_1}), \psi_2(h_{n_1+1}), \ldots, \psi_2(h_{n_1+n_2}), \psi_3(h_{n_1+n_2+1}), \ldots, \psi_3(h_{n_1+n_2+n_3}), \\
&\quad \psi_4(h_{j,1}^*), \ldots, \psi_4(h_{j,p}^*), \psi_4(h_{b,1}^*), \ldots, \psi_4(h_{b,r}^*), \psi_1(h_1^*), \ldots, \psi_1(h_{n_1+n_2+n_3}^*)')' \\
&\text{(17)}
\end{align*}
\]

with \( h^* = (h_1, \ldots, h_{n_1+n_2+n_3}, h_{j,1}^*, \ldots, h_{j,p}^*, h_{b,1}^*, h_{b,r}^*, h_1^*, \ldots, h_{j,n_1+n_2+n_3}^*)' \in \mathbb{R}^{2n_1+2n_2+2n_3+p+r} \).

Of course we are not the first to notice this relationship between the inverse gamma prior and the normal distribution (e.g., see [Gelman 2006] among others). This section only presents one example where the integrals in (13) and (14) are known, but several other prior specifications result in conjugate marginal prior distributions for fixed and random effects. For example, if the precision parameters follow a Pólya-gamma distribution, then the resulting marginal priors for \( \beta \) and \( \eta \) are logit-beta distributed ([Bradley et al. 2019]).
4.6 Marginalizing Across Hyperparameters: Example 2

One can write several examples of when the integrals in (13) are not respectively CM (i.e., non-conjugate). In this case, we suggest specifying your prior distributions for \( \beta \) and \( \eta \) to be the CM distributions that are “close” (e.g., in KL) to the desired non-conjugate specification. This strategy is not an approximate Bayesian approach. We emphasize this point since typical variational Bayesian approaches minimize the KL divergence between the posterior distribution and a target distribution. Instead, we use the KL divergence to calibrate our CM priors for \( \beta \) and \( \eta \) to non-conjugate priors before the data is used.

Consider the KL divergence (up to an additive constant) between \( f(\eta) \) in (13) and \( p(\eta | \ell, \Sigma) = N(\ell, \Sigma) \):

\[
- \int f(\eta) \log \{ p(\eta | \mu, \Sigma) \} \, d\eta \\
= \frac{1}{2} \log \{ \det(\Sigma) \} + \frac{r}{2} \log(2\pi) - E_f(\eta)'\Sigma^{-1}\ell + \frac{1}{2} \text{trace} \{ \Sigma^{-1}(\text{cov}_f(\eta) + E_f(\eta)E_f(\eta)') \} + \frac{1}{2} \ell'\Sigma^{-1}\ell,
\]

where \( N(\ell, \Sigma) \) is a shorthand for the normal distribution with mean \( \ell \) and covariance matrix \( \Sigma \). Taking derivatives, setting equal to zero, and solving produces the optimal values \( \ell = E_f(\eta) \) and \( \Sigma = \text{cov}_f(\eta) \). The subscript “\( f \)” indicates that the expected value and covariance is computed with respect to \( f(\eta) \). Provided that the non-conjugate prior \( f(\eta) \) can be sampled from directly and the first two moments exist, one can estimate \( E_f(\eta) \) and \( \text{cov}_f(\eta) \) using the sample mean and covariance from replicates from the prior distribution. In a similar manner, can derive a \( p\{ \beta | E_f(\beta), \text{cov}_f(\beta) \} \).

EPR can now be used to sample from the marginal posterior distribution,

\[
f(\zeta, q | z) \propto f(z | \beta, \eta, \xi, q) p\{ \beta | E_f(\beta), \text{cov}_f(\beta) \} p\{ \eta | E_f(\eta), \text{cov}_f(\eta) \} f(\xi | \eta, \beta) f(q).
\]

Note that the \( N \)-dimensional vector \( G\eta \) with \( \eta \sim N(E_f(\eta), \text{cov}_f(\eta)) \) is equal in distribution to
the $N$-dimensional vector $G \text{cov}_f(\eta)^{1/2} \eta$ with $\eta \sim N(0_r, I_r)$. Thus, we set $\mu = -Q_{KL} q$, where $Q_{KL}$ is the $(2N + p + r) \times N$ eigenvectors of the orthogonal complement of,

$$H_{KL} = \begin{pmatrix} I_N & X \text{cov}_f(\beta)^{1/2} & G \text{cov}_f(\eta)^{1/2} \\ \vdots & \vdots & \vdots \\ 0_{p,N} & I_p & 0_{p,r} \\ 0_{r,N} & 0_{r,p} & I_r \\ I_N & 0_{N,p} & 0_{N,r} \end{pmatrix},$$

so that $H_S$ and $H_{KL}$ have a similar structure except for a modification to the first block row. This leads to the following adjustments to Theorem 4.3 to implement EPR:

$$\mu_M = (0_{1,n_1+n_2+n_3}, z_4', 0_{1,N+p+r})',$$

$$\alpha_M = (z_1', D'_{1,2}, z_2 + \alpha_1 \xi_1, z_3' + \alpha_2 \xi_{1,3}, 0_{1,n_4+N+p+r})',$$

$$\kappa_M = (\frac{1}{2}1_{1,n_1}D_{1,2}', 1_{1,n_2}, m' + 2\alpha_3 \xi_{1,n_3}, \frac{v+1}{2}1_{1,n_4}, \frac{1}{2}1_{1,N+p+r})',$$

$$\psi_M(h^*) = \left( \psi_1(h_1), \ldots, \psi_1(h_{n_1}), \psi_2(h_{n_1+n_2}), \ldots, \psi_K(h_N), \psi_1(h^*_{\beta,1}), \ldots, \psi_1(h^*_{\beta,p}), \psi_1(h^*_{\eta,1}), \ldots, \psi_1(h^*_{\eta,r}), \psi_1(h^*_1), \ldots, \psi_1(h^*_N) \right)'$$

for $(2N + p + r)$-dimensional real-valued vector $h^* = (h_1, \ldots, h_N, h^*_{\beta,1}, \ldots, h^*_{\beta,p}, h^*_{\eta,1}, \ldots, h^*_{\eta,r}, h^*_1, \ldots, h^*_N)'$.

As an example, consider a type of multivariate conditional autoregressive model (CAR; Besag 1974, 1986, Besag et al. 1991) specification for $\eta$, which is assumed to be an $r(=N)$-dimensional random vector. Partition $\eta = (\eta_1', \eta_2')'$ into a $(N/2)$-dimensional vector $\eta_1'$ and $(N/2)$-dimensional vector $\eta_2'$, where $N$ is assumed even. Let $\eta_2 = \gamma \eta_1 + \epsilon_{\eta}$, where $\gamma$ is Student-$t$ with 3 degrees of freedom (to ensure the first two moments exist) and $\epsilon_{\eta}$ is Gaussian with mean zero and covariance matrix $\sigma^2_{\eta,2} I_{N/2}$ with $\sigma^2_{\eta,j} > 0$ for $j = 1, 2$. Then assume $\eta_2$ is Gaussian with
mean zero and covariance matrix \( \sigma^2_{\eta,1} (D_a - \rho A)^{-1} \), where \( (D_a - \rho A) \) is the precision matrix from a CAR model, \( A \) is a binary adjacency matrix, and the diagonal matrix \( D_a \) has \( i \)-th diagonal element set equal to the \( i \)-th row total of \( A \). The model between \( \eta_1 \) and \( \eta_2 \) is a type of a random walk model (e.g., see Cressie and Wikle 2011). Let \( \sigma^2_{\eta,j} \) be distributed according to an inverse gamma distribution with shape 3 and rate equal to 2 (to ensure first two moments exist), and \( \rho \) be distributed from a uniform distribution between \( 1/\lambda_N \) and \( 1/\lambda_1 \), where \( \lambda_1 \) and \( \lambda_N \) are the smallest and largest eigenvalues so that \( D_a - \rho A \) is positive definite (Banerjee et al., 2015). Then to simulate from \( f(\eta) \) first simulate \( \rho \) from its uniform prior, then simulate \( \sigma^2_{\eta,j} \) from their inverse gamma priors, and then simulate \( \eta \) from random walk/CAR model above with variances \( \sigma^2_{\eta,j} \) and range parameter \( \rho \). Once \( B \) replicates of \( \eta \) are drawn from \( f(\eta) \) use the sample mean and covariance to define \( \ell = E_f(\eta) \) and \( \Sigma = \text{cov}_f(\eta) \).

### 4.7 Computational Considerations

For large \( N \) the EPR formulation may not look practically feasible. However, standard block matrix inversion techniques can be used to reduce the order of operations to inverses of \( r \times r \) matrices, \( p \times p \) matrices, and \( N \times N \) diagonal matrices (Lu and Shiou 2002).

**Theorem 4.7.** Let \( N^* > p, N^* > r, X^* \) be a \( N^* \times p \) real-valued matrix, \( G^* \) be a \( N^* \times r \) real-valued matrix, \( V_\beta \) be a \( p \times p \) real-valued invertible matrix, \( V_\eta \) be a \( r \times r \) real-valued invertible matrix, \( V_\xi \) be a \( N^* \times N^* \) real-valued invertible matrix, and \( W \) be a \( N^* \times N^* \) real-valued invertible matrix. Let

\[
H^* = \begin{pmatrix}
W & X^* & G^* \\
0_{p,N^*} & V_\beta^{-1} & 0_{p,r} \\
0_{r,N^*} & 0_{r,p} & V_\eta^{-1} \\
V_\xi^{-1} & 0_{N^*,p} & 0_{N^*,r}
\end{pmatrix}.
\]
Then,

\[(H^*H)^{-1} = \begin{pmatrix}
A^{-1} + A^{-1}B(D - B'A^{-1}B)^{-1}B'A^{-1} & A^{-1}B(D - B'A^{-1}B)^{-1} \\
-(D - B'A^{-1}B)^{-1}B'A^{-1} & (D - B'A^{-1}B)^{-1}
\end{pmatrix}, \quad (19)
\]

where \(A = W^TW + V^{-1}_\xi V^{-1}_\xi\), the \(N^* \times (p + r)\) matrix \(B = (W^X^*, W^G^*)\), the \((p + r) \times (p + r)\) matrix

\[D = \begin{pmatrix}
X^*X^* + V^{-1}_\beta V^{-1}_\beta & X^*G^* \\
G^*X^* & G^*G^* + V^{-1}_\eta V^{-1}_\eta
\end{pmatrix}, \quad (20)
\]

the \((p + r) \times (p + r)\) matrix

\[(D - B'A^{-1}B)^{-1} = \begin{pmatrix}
A^{*-1} + A^{*-1}B^*(D^* - C^*A^{*-1}B^*)^{-1}C^*A^{*-1} & -A^{*-1}B^*(D^* - C^*A^{*-1}B^*)^{-1} \\
-(D^* - C^*A^{*-1}B^*)^{-1}C^*A^{*-1} & (D^* - C^*A^{*-1}B^*)^{-1}
\end{pmatrix},
\]

the \(p \times p\) matrix \(A^* = X^*X^* - X^*W^1AW^{-1}X^* + V^{-1}_\beta V^{-1}_\beta\), the \(p \times r\) matrix \(B^* = X^*G^* - X^*WA^{-1}WG^*\), the \(r \times p\) matrix \(C^* = G^*X^* - G^*WA^{-1}WX^*\), and the \(r \times r\) matrix \(D^* = G^*G^* - G^*WA^{-1}WG^* + V^{-1}_\eta V^{-1}_\eta\).

\textbf{Proof:} See the Appendix.

Theorem 4.7 allows us to reduce the inverse of the \((N^* + p + r) \times (N^* + p + r)\) matrix \(H^TH\) to the inverse of the \(N^* \times N^*\) matrix \(A\), the \(p \times p\) matrix \(A^*\), and the \(r \times r\) matrix \((D^* - C^*A^{*-1}B^*)^{-1}\).

In practice, these inverses exist when non-generate choices of \(V_\beta\) and \(V_\eta\) are used. Moreover, we consider the setting when \(A\) is diagonal and \(p\) and \(r\) are both “small,” and hence, each of these inverse are computationally efficient. For example, the Gaussian mixed effects model in Theorem 4.1 can be implemented efficiently when using Theorem 4.7 with \(H^* = H_1\). Similarly, Theorem 4.4 can be implemented efficiently with Theorem 4.7 when \(H^* = H, H_S, H_{KL}\).
Simulation from the posterior using EPR does not necessarily require first computing a matrix of the form \((H'^*H^*)^{-1}\), storing this matrix, and then computing a \((N + p + r)\)-dimensional vector of the form \((H'^*H^*)^{-1}H'^*w^*\), where \(w^* = (w^*_e, w^*_\beta, w^*_\eta, w^*_q)'\), \(w^*_e \in \mathbb{R}^{N^*} \), \(w^*_\beta \in \mathbb{R}^p \), \(w^*_\eta \in \mathbb{R}^r \), and \(w^*_q \in \mathbb{R}^r \). In fact this order of operations may require impractical storage, since the \((N^* + p + r) \times (N^* + p + r)\) matrix \((H'^*H^*)^{-1}\) may be high-dimensional. In fact, one can compute the \((N^* + p + r)\)-dimensional vector of the form \((H'^*H^*)^{-1}H'^*w^*\) that both avoids inversions of high-dimensional matrices and storage of high-dimensional matrices.

**Theorem 4.8.** Let \(N^* > p, N^* > r, X^*\) be a \(N^* \times p\) real-valued matrix, \(G^*\) be a \(N^* \times r\) real-valued matrix, \(V^*_\beta\) be a \(p \times p\) real-valued invertable matrix, \(V^*_\eta\) be a \(r \times r\) real-valued invertable matrix, \(V^*_\xi\) be a \(N^* \times N^*\) real-valued invertable matrix, and \(W\) be a \(N^* \times N^*\) real-valued invertable matrix. Also let \(w^* = (w^*_e, w^*_\beta, w^*_\eta, w^*_q)'\), \(w^*_e \in \mathbb{R}^{N^*} \), \(w^*_\beta \in \mathbb{R}^p \), \(w^*_\eta \in \mathbb{R}^r \), and \(w^*_q \in \mathbb{R}^r \). Let

\[
H^* = \begin{pmatrix}
W & X^* & G^* \\
0_{p,N^*} & V^{-1}_\beta & 0_{p,r} \\
0_{r,N^*} & 0_{r,p} & V^{-1}_\eta \\
V^{-1}_\xi & 0_{N^*,p} & 0_{N^*,r}
\end{pmatrix},
\]

Then,

\[
(H'^*H^*)^{-1}H'^*w^* = \begin{pmatrix}
(F - KL^{-1}K')^{-1}(R - KL^{-1}P) \\
-L^{-1}K'(F - KL^{-1}K')^{-1}(R - KL^{-1}P) + L^{-1}P
\end{pmatrix},
\]

where the \((N^* + p)\)-dimensional vector \(R = (w^*_eW + w^*_\beta'V^{-1}_\beta, w^*_\eta'X^* + w^*_\eta'V^{-1}_\eta)'\), the \(r\)-dimensional vector \(P = G'^*w^*_e + V^{-1}_\eta w^*_\eta\), the \((r \times (N^* + p))\) matrix \(K' = (G'^*W, G'^*X^*)\), the \((r \times r)\) matrix \(L = G'^*G^* + V^{-1}_\eta V^{-1}_\eta\), the \((N^* + p) \times (N^* + p)\) matrix \(F = \begin{pmatrix}
W'W + V^{-1}_\xi V^{-1}_\xi & W'X^* \\
X'^*W & X'^*X^* + V^{-1}_\beta V^{-1}_\beta
\end{pmatrix}
\),
and the \((N^* + p) \times (N^* + p)\) matrix

\[
F - KL^{-1}K' = \begin{pmatrix} F_1 & B_{12} \\ B'_{12} & F_2 \end{pmatrix}.
\]

The \((N^* + p) \times (N^* + p)\) matrix,

\[
(F - KL^{-1}K')^{-1} = \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix},
\]

where the \(N^* \times N^*\) matrix \(F_1 = W'W + V_{\xi}^{-1}V_{\xi}^{-1} - W'G^*L^{-1}G^*W\), the \(N^* \times p\) matrix \(B_{12} = W'X^* - W'G^*L^{-1}G^*X^*\), the \(p \times p\) matrix \(F_2 = X^*X^* + V_{\beta}^{-1}V_{\beta}^{-1} - X^*G^*L^{-1}G^*X^*\), the \(N^* \times N^*\) matrix \(F_{11} = F_1^{-1} + F_1^{-1}B_{12}(F_2 - B'_{12}F_1^{-1}B_{12})^{-1}B'_{12}F_1^{-1}\), the \(p \times N^*\) matrix \(F_{21} = (F_2 - B'_{12}F_1^{-1}B_{12})^{-1}B'_{12}F_1^{-1}\), the \(p \times p\) matrix \(F_{22} = (F_2 - B'_{12}F_1^{-1}B_{12})^{-1}\), and the \(N^* \times N^*\) matrix \(F_1^{-1} = (W'W + V_{\xi}^{-1}V_{\xi}^{-1})^{-1} + (W'W + V_{\xi}^{-1}V_{\xi}^{-1})^{-1}W'G^*(L - G^*W(W'W + V_{\xi}^{-1}V_{\xi}^{-1})^{-1}W'G^*)^{-1}G^*W(W'W + V_{\xi}^{-1}V_{\xi}^{-1})^{-1}.\)

**Proof:** See the Appendix.

Careful examination of the order of operations show that Theorem (4.8) allows one to compute the vector \((H^*H^*)^{-1}H^*w^*\) by storing/computing the \(N^* \times p\) matrix \(X^*\), the \(N^* \times r\) matrix \(G^*\), the \(r \times r\) matrix \(L^{-1}\), the \(r \times r\) matrix \((L - G^*W(W'W + V_{\xi}^{-1}V_{\xi}^{-1})^{-1}W'G^*)^{-1}\), the \(p \times p\) matrix \(F_2\), and the \(p \times p\) matrix \(F_{22}\). These computations are straightforward when \(r\) and \(p\) are “small.” One also needs to invert/store the \(N^* \times N^*\) matrix \(W'W + V_{\xi}^{-1}V_{\xi}^{-1}\), which we will specify to be a scalar times the identity, and hence, is not computationally expensive to work with.
4.8 Implementation

If \( \omega = \infty \) then simulation is not needed. From Theorem 3.1, \( E(\zeta | z) = (H'*H*)^{-1}H'*k(\alpha_M, \kappa_M) \) and \( \text{cov}(\zeta | z) = (H'*H*)^{-1}H'*K(\alpha_M, \kappa_M)H* (H'*H*)^{-1}, \) and Theorem 4.7 can be used to compute these quantities efficiently. Once these quantities are computed it follows that \( E(\hat{y} | z) = H*E(\zeta | z) \) and \( \text{cov}(\hat{y} | z) = H*\text{cov}(\zeta | z)H*. \)

If using the prior on the bypass location parameters implied by Theorem 4.6, or if \( (H'*H*)^{-1} \) is impractical to store, then simulation is required. The following gives step-by-step instructions on efficient exact sampling from the posterior distribution of fixed effects and random effects using Theorem 4.6. We consider Gaussian data with unknown non-constant variance, Poisson data, binomial data, and Gaussian priors on \( \beta \) and \( \eta \) with unknown non-constant variances. Minor adjustments to these steps are needed for other settings (e.g., Gaussian data with known variance, etc.).

1. Compute \( E_f(\eta), E_f(\beta), \text{cov}_f(\eta), \) and \( \text{cov}_f(\beta) \) when using the strategy in Section 4.6. Set \( H^* \) equal to \( H_S \) or \( H_{KL} \), and store/compute the \( N \times p \) matrix \( X \), the \( N \times r \) matrix \( G \), the \( r \times r \) matrix \( L^{-1} \), the \( r \times r \) matrix \( (L - \frac{1}{2}G'G)^{-1} \), the \( p \times p \) matrix \( F_2 \), and the \( p \times p \) matrix \( F_{22} \).

2. Simulate \( w \), which consists of independent DY random variables. The class of DY distribution each element follows is indicated by the corresponding element of the unit log partition function \( \psi_M \) (e.g., see (17) and (18)).

3. Use the matrices computed in Step 1 and Theorem 4.8 to compute \( \zeta_{rep}, y_{rep}, \) and \( \hat{y}_{rep} \) according to Theorem 4.4.

4. If \( g(q_{rep}, Q) \leq \omega \) reject and return to Step 2, otherwise, continue to Step 4.

5. Repeat Steps 2–4 \( B \) times.

Repeated matrix operations that one might see in a Gibbs sampler are avoided, since matrix inversions are only required \textit{a single time} in Step 1. Additionally, from Theorem 4.6 \( g(q_{rep}, Q) \) can be
written as the sums of squared difference between $y_{rep}$ and $\tilde{y}_{rep}$, and hence one does not need to compute $q_{rep}$ and $Q$ in Step 4. Additionally, $B$ does not have to be as large as what one requires for an MCMC, since one does not require a burn-in period, thinning, or have concerns of mixing and positive autocorrelations in the MCMC.

The choice of $\omega$ is a subjective portion of this algorithm. Ideally $\omega$ is “small,” but it cannot be chosen too small otherwise one would never obtain a sample. In practice, one can choose $\omega$ to give a desired acceptance rate. That is, one can repeat Steps 2, 3, and 4 a total of $\frac{1}{p}B$ times, and accept the $p$ percentage of replicates with the smallest $g(q_{rep}, Q)$, where the percentage $p$ is pre-specified (e.g., sample $2B$ times and accept the $B$ with the smallest $g(q_{rep}, Q)$). This is equivalent to using the $p$-th percentile of $B$ independent replicates of $g(q_{rep}, Q)$ as an estimate for $\omega$. Considering the well-known consistency properties of percentiles from i.i.d. replicates (Zielinski 1998), the added sampling variability of this estimated $\omega$ may be negligible for large $B$.

5 Illustrations

Sections 5.1 and 5.2 give demonstrations of the special cases of EPR described in Section 4.5 and Section 4.6, respectively. Section 5.1 provides a simulation study, where under various acceptance rates and $K = 3$, we investigate both the estimation performance of $\beta$, and the predictive performance of $y$ using posterior summaries of $y$, $\tilde{y}$, and $\hat{y}$. In Section 5.2 we demonstrate spatial prediction of $y$ for a dataset obtained from the US Census Bureau when $K = 2$. Code is provided on the Github page https://github.com/JonathanBradley28/CM.

5.1 Simulations

We now provide an illustration using simulations. We let $K = 3$, $n_1 = n_2 = n_3 = 5000$. We specify a $5000 \times 3$ matrix $M$ to have a column of ones, the second column consist of an equally spaced sequence from zero to one, and the third column consisting of the element-wise square of the sec-
ond column. Set $X = \text{blkdiag}(M, M, M)$, where “blkdiag” is the block diagonal operator. This sets up a simple quadratic expression for the mean, where there are different intercepts, linear effects, and quadratic effects per response type. Let the $5000 \times 100$ matrix $M_G$ consists of exponential radial basis functions, where the $(i, j)$-th element is given by $\exp\{-\text{abs}(i - j)\}$, where “abs” is the absolute value operator. Here, $j$ is equally spaced from 0 to 1. Let the $15000 \times 100$ matrix $G = \{I_{15000} - X(X'X)^{-1}X'\} (M'_G, M'_G, M'_G)'$, where the orthogonal complement of $X$ is introduced to avoid confounding. We arbitrarily fix $\beta_{\text{true}} = (9, -3, 3, -0.2, -1, 2, 2.6, -0.5, 2)'$ and $\eta_{\text{true}}$ is generated from a mean zero normal with variance equal to that implied by $X\beta_{\text{true}}$. This implies that the true value of $y$ in this simulation is $y_{\text{true}} = (y'_{\text{true},1}, y'_{\text{true},2}, y'_{\text{true},3})' = X\beta_{\text{true}} + G_{\text{true}}\eta_{\text{true}}$. Normal data are generated with constant variance $\sigma^2_{\text{true}} = 1$. For binomial data we set the $i$-th element of $m$ to be a sample from a Poisson distribution with mean 20. Figure 1 contains a plot of simulated replicate according to this design.

To evaluate the performance of EPR we consider the coverage of 95% joint credible intervals over independently replicated datasets, the root mean squared prediction error (RMSPE), and the root mean squared error (RMSE), where

$$\text{RMSPE} \{E(\tilde{y}_k|z)\} = \left[\frac{1}{5000} \left\{ y_{\text{true},k} - E(\tilde{y}_k|z) \right\}' \left\{ y_{\text{true},k} - E(\tilde{y}_k|z) \right\} \right]^{1/2} ; k = 1, 2, 3,$$

$$\text{RMSE}_{\beta} = \left[\frac{1}{9} \left\{ \beta_{\text{true}} - E(\beta|z) \right\}' \left\{ \beta_{\text{true}} - E(\beta|z) \right\} \right]^{1/2},$$

$\tilde{y}_k = X_k\beta + G_k\eta$, and we consider RMSPE $\{E(\tilde{y}_k|z)\}$, RMSPE $\{E(\tilde{y}_k|z)\}$, and RMSPE $\{E(y_k|z)\}$. Small values of RMSPE$_k$ and RMSE$_{\beta}$ are preferable. Coverage at least as big as 95% is preferable, where recall that 95% credible intervals are interpreted differently from confidence intervals so that there is no guarantee that this holds. The credible intervals are jointly 95%, and are of the form $E(\beta_i|z) \pm C \text{var}(\beta_i|z)^{1/2}$, where $C$ is chosen so that at least 95% of the replicates of $\beta$ are all contained within these intervals. Since we have independent replicates the effective sample size for our method is $B$ as there are no autocorrelations induced by a Markov chain. Moreover, there
Figure 1: In the first row we plot a single replicated dataset. The red line in the first row represents the true mean. The second row contains a plot of $y_{true,k}$ (red) and the posterior mean of $\tilde{y}_k$ (blue). In the second row, the gray lines represent the posterior mean of $\hat{y}_k$, which is left out when $k = 1$ for visualization purposes.

is no need for any additional computational overhead/subjectivity of examining trace plots and computing MCMC diagnostics.
Figure 2: We plot the posterior mean of $\beta - \beta_{true}$. The red lines represent the 95% joint credible region.

We implement EPR according to the steps outlined in Section 4.8 with the prior specifications in Section 4.5 and $\nu_i \equiv \nu = 2$. We use the prior specifications in Section 4.5. In Figure 1 we present a plot of $y_{true,k}$ and $E(\tilde{y}_k|z)$ for one simulated replicate, along with $E(y_k|z)$. In general, the predictions are fairly close to the truth in this example, and there is considerably more noise in the estimates based on $\tilde{y}_k$ instead of $y_k$. In Figure 2 we plot $E(\beta - \beta_{true}|z)$, which are close to zero. Moreover, the credible intervals for $\beta - \beta_{true}$ all contain zero, indicating reasonable estimates.

In Table 1 we present average (over 100 independent replicates of $z$) RMSE$_k$. In general, $E(\tilde{y}|z)$ produced smaller RMSPE than $E(y|z)$ followed by $E(y|z)$ for this simulation across all acceptance rates. The RMSPE is rather robust to the choice of acceptance rate. However, the RMSE$_\beta$ appeared sensitive to the choice of acceptance rate. The average across the 100 replicated datasets were respectively, 0.08, 0.07, and 0.06 with paired $t$-test $p$-values of 0.003 (comparing acceptance rate 1 to 1/2) and 0.29 (comparing acceptance rate 1 to 1/4). Thus, it appears that the acceptance rate 1/2 is reasonable when performing inference on $\beta$. The coverage rate of $\beta_{true}$, and
Table 1: Average RMSPE $\{E(\tilde{y}_k|z)\}$, RMSPE $\{E(\hat{y}_k|z)\}$, and RMSPE $\{E(y_k|z)\}$ over 100 independent replicates of $z$ by acceptance rate (i.e., percentile estimate of $\omega$) and $k$. EPR according to Section 4.8 is implemented with $B = 100$.

| Acceptance Rate | $k$ | RMSPE $\{E(\tilde{y}_k|z)\}$ | RMSPE $\{E(\hat{y}_k|z)\}$ | RMSPE $\{E(y_k|z)\}$ |
|-----------------|----|-------------------------------|-------------------------------|-------------------|
| 1               | 1  | 0.14                          | 0.54                          | 1.07              |
| 1/2             | 1  | 0.14                          | 0.53                          | 1.03              |
| 1/4             | 1  | 0.14                          | 0.52                          | 1.03              |
| 1               | 2  | 0.06                          | 0.13                          | 0.24              |
| 1/2             | 2  | 0.05                          | 0.13                          | 0.24              |
| 1/4             | 2  | 0.05                          | 0.13                          | 0.24              |
| 1               | 3  | 0.06                          | 0.26                          | 0.51              |
| 1/2             | 3  | 0.06                          | 0.26                          | 0.51              |
| 1/4             | 3  | 0.06                          | 0.26                          | 0.51              |

$y_{true}$ (using credible intervals for either $\tilde{y}$, $\hat{y}$, or $y$) over the 100 simulations were all one across all acceptance rates.

5.2 Analysis of American Community Survey Poverty Status and Mean Income Estimates

As an illustration, consider data downloaded from the U.S. Census Bureau’s American Community Survey (ACS; https://www.census.gov/). The ACS provides demographic statistics for the U.S. over 1-year and 5-year periods, and over several geographies. In Figure 3 we provide 2019 ACS 5-year period estimates of the population under the poverty threshold in Florida counties and mean income. We assume the population in poverty is Poisson distributed, and mean income is Gaussian distributed with known variances set equal to the direct survey variances. Thus, this dataset is a low-dimensional areal bivariate spatial dataset consisting of Gaussian and Poisson responses.
We make use of the Student-\(t\) distribution for \(\beta\) as described in (15). We use the full-rank random walk/ CAR prior for \(\eta\) in Section 4.6 with \(G = I_N\), where \(N = 134\), \(n_1 = n_2 = 67\). A log transformation is used to make the income estimates more symmetric, and the direct survey variances are adjusted using the delta method. Both responses appear to have a quadratic relationship with their respective direct survey variances from ACS, and consequently, we use the direct survey variance as covariates. We also define different intercepts for the different responses. We fit EPR according to Section 4.8 with acceptance rate equal to 1/8, and simulate 100 independent replicates from the posterior distribution.

The predicted mean and standard deviation of \(\tilde{y}_k\) versus the log-data are provided in Figure 4. In general, we see predictions that reflect the pattern of the data with spatial smoothing. The leave-one-out cross validation error (Wahba, 1990) is used to assess the predictive performance. Specifically, an observation is left out, and the model is used to predict this value. We compute the relative cross-validation error as 
\[
\text{CV}_1 \equiv \text{median}_{i \in \{1, \ldots, n_1\}} \left\{ \frac{\text{abs}(Z_{1,i} - E_{-i} [\tilde{Y}_{1,i}])}{\text{abs}(Z_{1,i})} \right\},
\]
and the Pearson’s correlation between the observations and the predictions based on leaving one observation out. Table 2 gives a table of both metrics by acceptance rate. We see that both metrics are robust to the choice of acceptance rate with acceptance rate equal to 1 consistently (albeit marginally) the least preferable. However, all relative cross-validation errors suggest that leave-one-out predictions are roughly within 4% of the hold-out observation indicating reasonable predictions, and in general the Pearson’s correlation coefficient suggests a strong relationship between the observations and the leave-one out predictions (roughly 90 to 95%).

6 Discussion

This paper describes how to efficiently sample directly from the posterior distribution for multiple-response type data modeled using a broad class of generalized linear mixed effects models. This development required the introduction of the GCM distribution, the conditional GCM distribution,
Figure 3: We plot the ACS estimates along with a plot of the log data versus their direct survey variance estimates, which display a quadratic pattern and truncated GCM distribution. The use of the GCM allows one to consider any class of CM’s for their prior distributions on fixed and random effects. Our development explicitly addresses hyper-parameters in the case of a specific class of Gaussian priors for fixed and random effects. We make
Figure 4: We plot the log ACS estimates (as a reference) along with a plot of the posterior mean and standard deviations computed from 100 independent replicates of EPR with acceptance rate of 1/8. The first row contains values for $k = 1$, and the second row contains values when $k = 2$. The third row provides the respective posterior standard deviations.
Table 2: Relative cross-validation error (CV) is $CV_k \equiv \text{median}_{i \in \{1, \ldots, n_1\}} \{\text{abs}(Z_{1,i} - E_{-i}[\tilde{Y}_{1,i}]) / \text{abs}(Z_{1,i})\}$, where $E_{-i}$ is the posterior expected value that leaves out the $i$-th observation. Let $CV_2 \equiv \text{median}_{i \in \{1, \ldots, n_2\}} \{\text{abs}(\log(Z_{2,i}) - E_{-i}[\tilde{Y}_{2,i}]) / \text{abs}(\log(Z_{2,i}))\}$, where “median” is the median function and “abs” is the absolute value function. Let “Corr” represent Pearson’s correlation between \{Z_{1,i}\} and \{E_{-i}[\tilde{Y}_{1,i}]\}, and \{\log(Z_{2,i})\} and \{E_{-i}[\tilde{Y}_{2,i}]\}, respectively.

use of the GCM in a GLMM context to produce what we call exact posterior regression, which represents an exact sample from the posterior distribution. This is a solution to a long-standing problem in Bayesian analysis. We show that the posterior distribution for fixed and random effects in a GLMM are either conditional GCM, GCM, or truncated GCM depending on the prior distribution on what we call bypass location parameters. Furthermore, we derive several properties of the GLMM including results on the fine-scale-random effect, goodness-of-fit interpretations, and matrix algebra techniques to aid in the computation of EPR.

The results in this paper solve an important problem for Bayesian analysis that is regularly overlooked (i.e., direct sampling from the posterior distribution in Bayesian GLMMs). Our solution also allows one to directly sample from the posterior predictive distribution, which avoids MCMC in empirical Bayesian settings as well. Specifically, Theorems 4.3 – 4.7 can be used with
estimates of $V_\epsilon$, $V_\beta$, $V_\eta$, and $V_\xi$ and the posterior predictive distribution can be used with plug-in estimators. However, plug-in estimators have unchecked sampling variability (provided that the plug-in estimator is a non-constant function of the data), and the development in Section 4.4 provides a straightforward solution that accounts for all sources of variability.

Although in this manuscript we are interested in the problem of obtaining an exact sample from the posterior distribution there are several approximate Bayesian methods that involve using an approximated posterior distribution for inference. A widely used approach is integrated nested Laplace approximations and its variations that use automatic differentiation (e.g., see Kristensen et al., 2015; Baydin et al., 2018 among others). Variational Bayes (VB) is another approximate Bayesian method (e.g., see Wainwright and Jordan, 2008), which is also available on Stan. VB is a general approach approximates the joint posterior distribution with pre-defined densities often chosen to be the product of univariate densities. There are several versions/extensions including those based on automatic differentiation, namely, Automatic Differentiation Variational Inference (ADVI Kucukelbir et al., 2017). Thus, to allow for inference on non-conjugate hyperparameters one might combine EPR with an approximate Bayesian technique, which we view as an exciting topic of future research.

To address hyperparameters we also needed to specify the distributions for $\beta$ and $\eta$ to be Gaussian or Student-$t$ distributed. However, one could also specify other CM distributions to allow for different tail behavior in the priors as done in several other papers (Gao and Bradley, 2019; Bradley et al., 2019, 2018; Hu and Bradley, 2018; H.-C. Yang et al., 2019; Parker et al., 2020, 2021). Incorporating hyperparameters in this case currently requires the use of MCMC.

While we feel that the results in this manuscript represent a significant advancement in Bayesian modeling of GLMMs, it is important to state that MCMC and approximate Bayesian tools will always be standard tools. This is because the GLMM specification we consider does not represent the wide variety of GLMMs used in the literature (e.g., allowing for mixture components, direct inference on hyperparameters, data models outside the class of distributions we
consider, etc.). However, we hope the theory developed in this article leads to further theoretical developments that allows one to directly sample from the posterior distribution in other settings.

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Appendix: Proofs

Proof of Theorem 3.1: Upon multiplying independent DY random variables we see that the distribution of $w_M$ is,

$$f(w_M|\alpha_M, \kappa_M) = \left\{ \prod_{k=1}^{K} \prod_{i=1}^{n_k} \mathcal{N}_k(k_{k,i}, \alpha_{k,i}) \right\} \exp\left\{ \alpha'_M w_M - \kappa'_M \psi_M(w_M) \right\} \cdot$$

The inverse transform is $w_M = V_M^{-1}(y - \mu_M)$, and the corresponding Jacobian is $\det(V_M^{-1})$. By standard change-of-variables (e.g., see Casella and Berger 2002), we have that,

$$f(y|\mu_M, V_M, \alpha_M, \kappa_M) = \det(V_M^{-1}) \left\{ \prod_{k=1}^{K} \prod_{i=1}^{n_k} \mathcal{N}_k(k_{k,i}, \alpha_{k,i}) \right\} \exp\left[ \alpha'_M V_M^{-1}(y - \mu_M) - \kappa'_M \psi_M \left\{ V_M^{-1}(y - \mu_M) \right\} \right] \cdot$$

Using the moment generating function of DY random variables (Bradley et al., 2020), we have that the mean and covariance matrix of $w_M$ is $k(\alpha_M, \kappa_M)$ and $K(\alpha_M, \kappa_M)$. This implies that
\( E(y|\mu_M, V_M, \alpha_M, \kappa_M) = \mu_M + V_M k(\alpha_M, \kappa_M) \) and \( \text{cov}(y|\mu_M, V_M, \alpha_M, \kappa_M) = V_M K(\alpha_M, \kappa_M) V_M' \).

**Proof of Theorem 3.2**: From Theorem 3.1, the conditional distribution is given by

\[
\begin{align*}
 f(y^{(1)}|y^{(2)}, \mu_M, V_M, \alpha_M, \kappa_M) &\propto f(y|\mu_M, V_M, \alpha_M, \kappa_M) \\
 &\propto \exp \left[ \alpha_M' (H Q) \begin{pmatrix} y^{(1)} \\ y^{(2)} \end{pmatrix} - \kappa_M' \psi_M \begin{pmatrix} y^{(1)} \\ y^{(2)} \end{pmatrix} - V_M^{-1} \mu_M \right], \\
&\propto \exp \left\{ \alpha_M' H y^{(1)} - \kappa_M' \psi_M \left( H y^{(1)} + Q y^{(2)} - V_M^{-1} \mu_M \right) \right\}, \\
&= \exp \left\{ \alpha_M' H y^{(1)} - \kappa_M' \psi_M \left( H y^{(1)} - \mu_M \right) \right\}; \quad y^{(1)} \in \mathbb{R}^n,
\end{align*}
\]

which proves the result.

**Proof of Theorem 4.1**: Upon vectorizing (1) and writing \( y_1 \) according to the mixed effects model representation in (8) leads to the following expression of the data model:

\[
\begin{align*}
 f(z_1|x_1, \beta, \eta, V_e) &\propto \exp \left[ z' V_e^{-1/2} \begin{pmatrix} V_e^{-1} X_1, V_e^{-1} G_1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \beta \\ \eta \end{pmatrix} - \frac{1}{2} \psi_1 \begin{pmatrix} V_e^{-1}, V_e^{-1} X_1, V_e^{-1} G_1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \beta \\ \eta \end{pmatrix} \right], \\
&\propto \exp \left[ z' V_e^{-1/2} \begin{pmatrix} V_e^{-1} X_1, V_e^{-1} G_1 \end{pmatrix} \xi_1 - \frac{1}{2} \psi_1 \begin{pmatrix} V_e^{-1} X_1, V_e^{-1} G_1 \end{pmatrix} \xi_1 \right],
\end{align*}
\]
where recall $\zeta_1 = (\xi_1', \beta', \eta')'$. The posterior predictive distribution of $\zeta_1$ is proportional to the product

$$f(z_1|\zeta_1, V_\epsilon)f(\xi_1|V_{\xi,1})f(\beta|V_\beta)f(\eta|V_\eta),$$

(22)

and when stacking vector and matrices in (22) leads to

$$f(\zeta_1|z_1, V_\beta, V_\eta, V_\xi) \propto \exp\left[\left(\begin{array}{c} z'V_\epsilon^{-1}\xi_1, V_\beta^{-1}\eta_1, V_\eta^{-1}\zeta_1 \end{array}\right) \left(\begin{array}{c} V_\epsilon^{-1}X_1, V_\epsilon^{-1}G_1 \end{array}\right) \left(\begin{array}{c} 0 \beta, 0 \eta \end{array}\right) \right].$$

Multiplying $(H_1'H_1)^{-1}(H_1'H_1)$ within the first term so that

$$(z'V_\epsilon^{-1}, 0_{1,n_1+p+r})H_1\zeta_1 = (z'V_\epsilon^{-1}, 0_{1,n_1+p+r})H_1(H_1'H_1)^{-1}(H_1'H_1)\zeta_1,$$
and completing the square gives

\[
\begin{align*}
  f(\zeta_1|z_1, V_\xi, V_\eta, V_\varepsilon) & \\
  \propto \exp \left[ -\frac{1}{2} \left( \zeta_1 - (H'_1H_1)^{-1}H'_1 \begin{pmatrix} V_\varepsilon^{-1}z_1 \\ 0_{n_1+p+r,1} \end{pmatrix} \right) \right] \left( \zeta_1 - (H'_1H_1)^{-1}H'_1 \begin{pmatrix} V_\varepsilon^{-1}z_1 \\ 0_{n_1+p+r,1} \end{pmatrix} \right),
\end{align*}
\]

which completes the result.

The posterior predictive distribution of \( \zeta_1 \) is proportional to the product

\[
  f(z_1|\zeta_1, V_\varepsilon) \propto \exp \left[ z'V_\varepsilon^{-1'} \begin{pmatrix} V_\varepsilon^{-1}, V_\varepsilon^{-1}X_1, V_\varepsilon^{-1}G_1 \end{pmatrix} \zeta_1 - \frac{1}{2} 1_{1,n_1} \psi_1 \left\{ (V_\varepsilon^{-1}, V_\varepsilon^{-1}X_1, V_\varepsilon^{-1}G_1) \zeta_1 \right\}.\]

The proof of Theorem 4.2: Upon vectorizing (1) and writing \( y_1 \) according to the mixed effects model representation in (8) leads to the following expression of the data model:

\[
\begin{align*}
  f(\zeta_1 & |z_1, \beta, \eta, V_\varepsilon) \\
  \propto \exp \left[ z'V_\varepsilon^{-1'} \begin{pmatrix} V_\varepsilon^{-1}, V_\varepsilon^{-1}X_1, V_\varepsilon^{-1}G_1 \end{pmatrix} \zeta_1 - \frac{1}{2} 1_{1,n_1} \psi_1 \left\{ (V_\varepsilon^{-1}, V_\varepsilon^{-1}X_1, V_\varepsilon^{-1}G_1) \zeta_1 \right\}.\]
\end{align*}
\]
and when stacking stacking vector and matrices in (23) leads to

\[
f(\zeta_1 | z_1, V_\beta, V_\eta) \propto \exp \left[ (z' V_{\epsilon}^{-1}, 0_{1,p+r}) \left( \begin{array}{ccc} V_{\epsilon}^{-1} & V_{\epsilon}^{-1} X_1 & V_{\epsilon}^{-1} G_1 \\ 0_{p,n_1} & V_\beta^{-1} & 0_{p,r} \\ 0_{r,n_1} & 0_{r,p} & V_\eta^{-1} \end{array} \right) \xi_1 \right] \left( \begin{array}{ccc} V_{\epsilon}^{-1} & V_{\epsilon}^{-1} X_1 & V_{\epsilon}^{-1} G_1 \\ 0_{p,n_1} & V_\beta^{-1} & 0_{p,r} \\ 0_{r,n_1} & 0_{r,p} & V_\eta^{-1} \end{array} \right) \xi_1^\prime \right] 
\]

\[
= \exp \left[ (z' V_{\epsilon}^{-1}, 0_{1,p+r}) H_1^* \zeta_1 - \frac{1}{2} 1_{1,n_1+p+r} \psi_1 (H_1^* \zeta_1) \right].
\]

Multiplying \( H_1^{\prime} (H_1^{\prime\prime})^{-1} H_1^* H_1^* \) within the first term so that

\[
(z' V_{\epsilon}^{-1}, 0_{1,p+r}) H_1^* \zeta_1 = (z' V_{\epsilon}^{-1}, 0_{1,p+r}) H_1^* H_1^{\prime} (H_1^{\prime\prime})^{-1} H_1^* H_1^* \zeta_1,
\]

and completing the square gives

\[
f(\zeta_1 | z_1, V_\beta, V_\eta) \propto \exp \left[ -\frac{1}{2} \left( \xi_1 - H_1^{\prime} \left( \begin{array}{c} V_{\epsilon}^{-1} z_1 \\ 0_{p+r,1} \end{array} \right) \right)' (H_1^{\prime\prime} H_1^*) \left( \begin{array}{c} \xi_1 - H_1^{\prime} \left( \begin{array}{c} V_{\epsilon}^{-1} z_1 \\ 0_{p+r,1} \end{array} \right) \right) \right] ,
\]

which completes the result. The expression of the inverse of \( H_1^* \) can be verified by both pre and post multiplying the two expressions for \( H_1^* \) and \( H_1^{\prime\prime} \) given in the statement of Theorem 4.2.
Proof of Theorem 4.3: The four data models can be written as:

\[
\begin{align*}
    f(z_1 | \xi_1, \beta, \eta, \{\sigma_i\}, \mu_1) &\propto \\
    \exp \left[ z_1'D' \sigma \begin{pmatrix} \xi_1 \\ \beta \\ \eta \end{pmatrix} - \mu_1 \right] - \frac{1}{2} 1_{1,n_1} D' \sigma \psi_1 \begin{pmatrix} \xi_1 \\ \beta \\ \eta \end{pmatrix} - \mu_1, \\
    f(z_2 | \xi_2, \beta, \eta, \mu_2) &\propto \\
    \exp \left[ z_2' \begin{pmatrix} \xi_2 \\ \beta \\ \eta \end{pmatrix} - \mu_2 \right] - 1_{1,n_2} \psi_2 \begin{pmatrix} \xi_2 \\ \beta \\ \eta \end{pmatrix} - \mu_2, \\
    f(z_3 | \xi_3, \beta, \eta, \mu_3) &\propto \\
    \exp \left[ z_3' \begin{pmatrix} \xi_3 \\ \beta \\ \eta \end{pmatrix} - \mu_3 \right] - m' \psi_3 \begin{pmatrix} \xi_3 \\ \beta \\ \eta \end{pmatrix} - \mu_3, \\
    f(z_4 | \xi_4, \beta, \eta, \nu, \mu_4) &\propto \\
    \exp \left[ - \frac{\nu + 1}{2} 1_{1,n_4} \psi_4 \begin{pmatrix} \xi_4 \\ \beta \\ \eta \end{pmatrix} - \mu_4 - z_4 \right].
\end{align*}
\]
Upon stacking the four data models we obtain:

\[
f(z|\xi, \beta, \eta, \{\sigma_i\}, \mu_1, \ldots, \mu_K) \propto \exp \left[ a' \left( I_N, X, G \right) \begin{bmatrix} \xi \\ \beta \\ \eta \end{bmatrix} - \mu_D^* \right] - b' \psi_D \left( I_N, X, G \right) \begin{bmatrix} \xi \\ \beta \\ \eta \end{bmatrix} - \mu_D^*,
\]

where \( a = (z_1' D_{\sigma}, z_2', z_3', 0_{1,1})' \), \( b = (\frac{1}{2} 1_{1,n_1} D_{\sigma}', 1_{1,n_2} m', \nu + 1 \xi z_{1,1})' \), \( \mu_D^* = \mu' = (\mu'_1, \ldots, \mu'_4 + z_4')' \), and \( \psi_D(h_D) = (\psi_1(h_1), \ldots, \psi_1(h_{n_1}), \psi_2(h_{n_1+1}), \ldots, \psi_2(h_{n_1+n_2}), \ldots, \psi_K(h_N))' \) for \( N \)-dimensional real-valued vector \( h_D = (h_1, \ldots, h_N)'. \) Equivalently

\[
f(z|\xi, \beta, \eta, \{\sigma_i\}, \mu_D) \propto \exp \left[ a' \left( I_N, X, G \right) \{ \xi - \mu_D^* \} - b' \psi_D \left( I_N, X, G \right) \{ \xi - \mu_D^* \} \right],
\]

where recall \( \xi = (\xi', \beta', \eta')' \). We reserve the right to re-order or drop blocks that define \( a, b, \mu_D^* \) to allow for fewer data types (e.g., Poisson, binomial, and Student-\( t \) distributed data only). The posterior predictive distribution of \( \xi \) and \( q \) is proportional to the product

\[
f(z|\xi, \beta, \eta, \{\sigma_i\}, \mu_D) f(\xi|\beta, \eta, \mu_2, \mu_3, \mu_4, V_\xi) f(\beta|V_\beta, \mu_\beta) f(\eta|V_\eta, \mu_\eta).
\]

Now,

\[
f(\xi|\beta, \eta, \mu_2, \mu_3, \mu_4, V_\xi) \propto \exp \left[ a_\xi' \left( \begin{bmatrix} I_N & X & G \\ V^{-1}_\xi & 0_{N,p} & 0_{N,r} \end{bmatrix} \right) \begin{bmatrix} \xi \\ \mu_\xi \end{bmatrix} - b_\xi' \psi_{D,\xi} \left( \begin{bmatrix} I_N & X & G \\ V^{-1}_\xi & 0_{N,p} & 0_{N,r} \end{bmatrix} \right) \begin{bmatrix} \xi \\ \mu_\xi \end{bmatrix} \right],
\]

with \( a_\xi = (0_{1,n_1}, \alpha_\xi 1_{1,n_2}, \alpha_\xi 1_{1,n_3}, 0_{1,n_4}, 0_{1,N})' \), \( b_\xi = (0_{1,n_1}, 0_{1,n_2}, 2\alpha_\xi 1_{1,n_3}, 0_{1,n_4}, 1_2 1_{1,N})' \), and \( \psi_{D,\xi}(h_{D,\xi}) = (\psi_1(h_1), \ldots, \psi_1(h_{n_1}), \psi_2(h_{n_1+1}), \ldots, \psi_2(h_{n_1+n_2}), \ldots, \psi_K(h_N), \psi_1(h_1'), \ldots, \psi_1(h_N'))' \) for \( 2N \)-dimensional
real-valued vector $\mathbf{h}_{D,\xi} = (h_1, \ldots, h_N, h_1', \ldots, h_N')'$. Notice that $\mu_1$ and $\mu_4 - z_4$ are contained within the expression of $\mu_D^*$, however these terms are multiplied by zero through $a_\xi$ and $b_\xi$. We write $f(\xi|\beta, \eta, \mu_2, \mu_3, \mu_\xi, V_\xi)$ in this way so that the expression of the following product is immediately given by,

$$f(z|\xi, \beta, \eta, \{\sigma_i\}, \mu_D)f(\xi|\beta, \eta, \mu_2, \mu_3, \mu_\xi, V_\xi) \propto \exp \left[ a'_Z \left\{ \begin{pmatrix} I_N & X & G \\ V^{-1}_\xi & 0_{N,p} & 0_{N,r} \end{pmatrix} \zeta - \begin{pmatrix} \mu_D^* \\ \mu_\xi \end{pmatrix} \right\} - b'_Z \psi_{D,\xi} \left\{ \begin{pmatrix} I_N & X & G \\ V^{-1}_\xi & 0_{N,p} & 0_{N,r} \end{pmatrix} \zeta - \begin{pmatrix} \mu_D^* \\ \mu_\xi \end{pmatrix} \right\} \right],$$

where

$$a_Z = (z'_1 D'_\sigma, z'_2 + \alpha_\xi 1_{1,n_2}, z'_3 + \alpha_\xi 1_{1,n_3}, 0_{1,n_4}, 0_{1,N})'$$

$$b_Z = (\frac{1}{2} 1_{1,n_1} D'_\sigma, 1_{1,n_2}, m' + 2 \alpha_\xi 1_{1,n_3}, \frac{v+1}{2} 1_{1,n_4}, \frac{1}{2} 1_{1,N})'.$$

Notice that the implied shape/scale parameters $a_Z$ and $b_Z$ are not on the boundary of the parameter space (when zero counts are present), where recall from Remark 3, is a motivation for including $\xi$ in the GLMM and in Gaussian mixed effects models. Multiplying by $f(\beta|V_\beta, \mu_\beta)f(\eta|V_\eta, \mu_\eta)$ and stacking stacking vector and matrices leads to

$$f(\xi, q|z, V_\beta, V_\eta, V_\xi) \propto \exp \left[ \alpha'_M \left\{ \begin{pmatrix} I_N & X & G \\ 0_{p,N} & V_\beta^{-1} & 0_{p,r} \\ 0_{r,N} & 0_{r,p} & V_\eta^{-1} \\ V^{-1}_\xi & 0_{N,p} & 0_{N,r} \end{pmatrix} \zeta - \begin{pmatrix} \mu_\beta \\ \mu_\eta \\ \mu_\xi \end{pmatrix} \right\} - \kappa'_M \psi_M \left\{ \begin{pmatrix} I_N & X & G \\ 0_{p,N} & V_\beta^{-1} & 0_{p,r} \\ 0_{r,N} & 0_{r,p} & V_\eta^{-1} \\ V^{-1}_\xi & 0_{N,p} & 0_{N,r} \end{pmatrix} \zeta - \begin{pmatrix} \mu_\beta \\ \mu_\eta \\ \mu_\xi \end{pmatrix} \right\} \right].$$
Substituting $\mu_D = -Qq$ leads to

$$f(\zeta, q|z, V_\beta, V_\eta, V_\zeta) \propto \exp \left[ \alpha_M' \left( \begin{pmatrix} H & Q \end{pmatrix} \begin{pmatrix} \zeta \\ q \end{pmatrix} - (H, Q)\mu_M \right) - \kappa_M' \psi_M \left( \begin{pmatrix} \zeta \\ q \end{pmatrix} - (H, Q)\mu_M \right) \right]$$

$$\propto \text{GCM}(\alpha_M, \kappa_M, \mu_M, (H, Q)^{-1}; \psi_M),$$

which completes the result.

**Proof of Theorem 4.4:** Equations (10) and (11) follows from (6), Theorem 4.3, and that 

$$(H, Q)^{-1} = \{ H(H'H)^{-1}, Q \}' .$$

Let $w_M$ be distributed according to a $\text{GCM}(\alpha_M, \kappa_M, 0_{2N+p+r}, (H, Q)^{-1}; \psi_M)$ so that,

$$w = w_M + \begin{pmatrix} 0_{n_1+n_2+n_3,1} \\ z_4 \\ 0_{N+p+r,1} \end{pmatrix} .$$

Then

$$\begin{pmatrix} \zeta_{rep} \\ q_{rep} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} H' \end{pmatrix}^{-1}H' \\ Q' \end{pmatrix} w = \begin{pmatrix} \begin{pmatrix} H' \end{pmatrix}^{-1}H' \\ Q' \end{pmatrix} \begin{pmatrix} 0_{n_1+n_2+n_3,1} \\ z_4 \\ 0_{N+p+r,1} \end{pmatrix} + \begin{pmatrix} \begin{pmatrix} H' \end{pmatrix}^{-1}H' \\ Q' \end{pmatrix} w_M .$$

50
From Theorem 3.1 it follows that \((\zeta_{\text{rep}}', q_{\text{rep}}')'\) is GCM\((\alpha_M, \kappa_M, \mu_M, (H, Q)^{-1}; \psi_M)\). Equation (12) follows from the fact that \(y = (I_N, 0_{N,N+p+r})H\zeta + (I_N, 0_{N,N+p+r})Qq\) so that

\[
y_{\text{rep}} = (I_N, 0_{N,N+p+r})H\zeta_{\text{rep}} + (I_N, 0_{N,N+p+r})Qq_{\text{rep}}
\]

\[
= (I_N, 0_{N,N+p+r})H(H'H)^{-1}H'w + (I_N, 0_{N,N+p+r})QQ'w
\]

\[
= (I_N, 0_{N,N+p+r})H(H'H)^{-1}H'w + (I_N, 0_{N,N+p+r})(I - H(H'H)^{-1}H')w
\]

\[
= (I_N, 0_{N,N+p+r})w.
\]

**Proof of Theorem 4.5:** Since \(f(\zeta | z, \theta_D, V_{\beta}, V_{\eta}, V_{\zeta}, q = 0_{N+p+r,1}) \propto f(\zeta, 0_{N+p+r,1}| z, \theta_D, V_{\beta}, V_{\eta}, V_{\zeta})\) it follows from Theorem 4.3 that \(f(\zeta | z, \theta_D, V_{\beta}, V_{\eta}, V_{\zeta}, q = 0_{N+p+r,1})\) is proportional to the conditional GCM stated in Theorem 4.5.

**Proof of Theorem 4.6:** The truncated GCM expression follows immediately from 4.3. From Theorem 4.4 and 4.6 we have that

\[
g(q_{\text{rep}}, Q) = \text{trace} \left[ (I_N, 0_{N,N+p+r})QQ'wq'QQ'(I_N, 0_{N,N+p+r})' \right]
\]

\[
= \text{trace} \left[ (I_N, 0_{N,N+p+r}) \{ w - H(H'H)^{-1}H'w \} \{ w - H(H'H)^{-1}H'w \}' \right] (I_N, 0_{N,N+p+r})'
\]

\[
= \text{trace} \left[ \{ y_{\text{rep}} - (I_N, 0_{N,N+p+r})H\zeta_{\text{rep}} \} \{ y_{\text{rep}} - (I_N, 0_{N,N+p+r})H\zeta_{\text{rep}} \}' \right]
\]

\[
= \{ y_{\text{rep}} - \hat{y}_{\text{rep}} \} \{ y_{\text{rep}} - \hat{y}_{\text{rep}} \}'.
\]

**Proof of Theorem 4.7:** From (Lu and Shiou 2002) the inverse of a 2\(\times\)2 block matrix is,

\[
\begin{pmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{pmatrix}^{-1} = 
\begin{pmatrix}
A_{11}^{-1} + A_{11}^{-1}A_{12}(A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1}A_{21}A_{11}^{-1} & -A_{11}^{-1}A_{12}(A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1} \\
-(A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1}A_{21}A_{11}^{-1} & (A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1}
\end{pmatrix},
\]

51
for generic real-valued $M \times M$ matrix $A_{11}$, $M \times (p+r)$ matrix $A_{12}$, $(p+r) \times M$ matrix $A_{21}$, and 
$(p+r) \times (p+r)$ matrix $A_{22}$. Equation (19) follows from applying this known inverse of $2 \times 2$ block matrices (Lu and Shiou 2002) to,

$$
(H^*H^*)^{-1} = \begin{pmatrix}
W'W + V^{-1}_\xi V^{-1}_\xi & W'X* & W'G* \\
X*W & X*X* + V^{-1}_\beta V^{-1}_\beta & X*'G* \\
G*W & G*X* & G*G* + V^{-1}_\eta V^{-1}_\eta
\end{pmatrix}.
$$

Similarly, Equation (20) follows from applying the same inverse identity to

$$(D - B'A^{-1}B) = \begin{pmatrix} A^* & B^* \\ C^* & D^* \end{pmatrix}.$$  

**Proof of Theorem 4.8:** (Lu and Shiou 2002) gave two identities for the inverse of a $2 \times 2$ block matrix,

$$
(A_{11} A_{12})^{-1} = \begin{pmatrix} A_{11}^{-1} + A_{11}^{-1} A_{12} (A_{22} - A_{21} A_{11}^{-1} A_{12})^{-1} A_{21} A_{11}^{-1} & -A_{11}^{-1} A_{12} (A_{22} - A_{21} A_{11}^{-1} A_{12})^{-1} \\
-(A_{22} - A_{21} A_{11}^{-1} A_{12})^{-1} A_{21} A_{11}^{-1} & (A_{22} - A_{21} A_{11}^{-1} A_{12})^{-1}
\end{pmatrix},
$$

and

$$
(A_{11} A_{12})^{-1} = \begin{pmatrix} (A_{11} - A_{12} A_{22}^{-1} A_{21})^{-1} & -(A_{11} - A_{12} A_{22}^{-1} A_{21})^{-1} A_{12} A_{22}^{-1} \\
-A_{22}^{-1} A_{21} (A_{11} - A_{12} A_{22}^{-1} A_{21})^{-1} A_{22}^{-1} + A_{22}^{-1} A_{21} (A_{11} - A_{12} A_{22}^{-1} A_{21})^{-1} A_{12} A_{22}^{-1}
\end{pmatrix},
$$
for generic real-valued $M \times M$ matrix $A_{11}$, $M \times (p + r)$ matrix $A_{12}$, $(p + r) \times M$ matrix $A_{21}$, and $(p + r) \times (p + r)$ matrix $A_{22}$. Apply this second identity to,

$$\begin{pmatrix} W'W + V_\xi^{-1}V_\xi^{-1} & W'X^* & W'G^* \\ X^*W & X^*X^* + V_\beta^{-1}V_\beta^{-1} & X^*G^* \\ G^*W & G^*X^* & G^*G^* + V_\eta^{-1}V_\eta^{-1} \end{pmatrix}$$

to produce

$$\begin{pmatrix} (F - KL^{-1}K')^{-1} & -(F - KL^{-1}K')^{-1}KL^{-1} \\ -L^{-1}K'(F - KL^{-1}K')^{-1} & L^{-1} + L^{-1}K'(F - KL^{-1}K')^{-1}KL^{-1} \end{pmatrix}$$

Since, we have that $H^*w^* = (R', P')'$, Equation (21) follows immediately. Apply the first block inverse identity to $F - KL^{-1}K'$ to obtain $F_{11}$, $F_{12}$, $F_{21}$, and $F_{22}$. Apply the Sherman-Morrison Woodbury identity (Cressie and Johannesson, 2008) to obtain $F_1^{-1}$.

**Statement and Proof of an Additional Result:** This result was discussed in Section 4.4, but not formally stated. We now state and prove the result here.

**Statement:** Suppose $Z_{k,i} | Y_{k,i}, \theta_D$ is independently distributed according to either (1) or (2) with $i = 1, \ldots, n_k$, and $k = 1, \ldots, K$. Set $\theta_D$ to be all data model parameters (e.g., when $k = 1$ $\theta_D$ contains $\{\sigma_i\}$ and when $k = 4$ $\theta_D$ contains $\nu$). Let $j = 1, \ldots, K$ correspond to the choice of CM prior for
$\beta \mid V_\beta, \alpha_\beta, \kappa_\beta$, and let $b = 1, \ldots, K$ correspond to the choice of CM prior for $\eta \mid V_\eta, \alpha_\eta, \kappa_\eta$. Define,

$$H^{**} = \begin{pmatrix} I_N & X & G \\ 0_{p,N} & V_\beta^{-1} & 0_{p,r} \\ 0_{r,N} & 0_{r,p} & V_\eta^{-1} \end{pmatrix},$$

and $Q^*$ be the eigenvectors of the orthogonal complement of $H^{**}$. Let $y_k$ be defined as in (9), $\mu = -Q^* q$, and assume the improper prior $f(q) = 1$. Let

$$f(\xi \mid \beta, \eta, \mu_2, \mu_3, V_\xi) \propto \exp \left[ a^* \xi \left( \begin{pmatrix} I_N & X & G \end{pmatrix} \xi - \left( \mu_D \right) \right) \right] - b^* \xi \left( \begin{pmatrix} I_N & X & G \end{pmatrix} \xi - \left( \mu_D^* \right) \right),$$

with $a^* \xi = (0_{1,n_1}, \alpha_\xi 1_{1,n_2}, \alpha_\xi 1_{1,n_3}, 0_{1,n_4})'$ and $b^* \xi = (0_{1,n_1}, 0_{1,n_2}, 2\alpha_\xi 1_{1,n_3}, 0_{1,n_4})'$. Then

$$(\xi', q') \mid \theta_D, V_\beta, V_\eta, V_\xi \sim \text{GCM}(\alpha_M^*, \kappa_M^*, \mu_M^*, (H^{**}, Q^*)^{-1}; \psi_M^*),$$

where the $(2N + p + r)$-dimensional location and shape/scale parameter vectors,

$$\mu_M^* = \begin{pmatrix} \left( H^{**} H^{**} \right)^{-1} H^{**} \xi \\ Q^* \xi \end{pmatrix} \begin{pmatrix} 0_{n_1+n_2+n_3,1} \\ z_4 \\ 0_{p+r,1} \end{pmatrix},$$

$$\alpha_M^* = (z'_1 D'_\sigma, z_2' + \alpha_\xi 1_{1,n_2}, z_3' + \alpha_\xi 1_{1,n_3}, 0_{1,n_4}, \alpha_\beta, \alpha_\eta)',$$

$$\kappa_M^* = \left( \frac{1}{2} 1_{1,n_1} D'_\sigma, 1_{1,n_2}, m' + 2\alpha_\xi 1_{1,n_3}, \frac{\nu + 1}{2} 1_{1,n_4}, \kappa_\beta', \kappa_\eta' \right),$$

54
where $D_\sigma = \text{diag} \left( \frac{1}{\sigma_i^2} : i = 1, \ldots, n_1 \right)$, and $(N + p + r)$-dimensional unit-log partition function

$$
\psi_M^*(h^*) = (\psi_1(h_1), \ldots, \psi_1(h_{n_1}), \psi_2(h_{n_1+1}), \ldots, \psi_2(h_{n_1+n_2}), \ldots, \psi_K(h_N), \psi_j(h_{j,1}^*), \ldots, \psi_j(h_{j,p}^*), \\
\psi_b(h_{b,1}^*), \ldots, \psi_3(h_{b,r}^*))',
$$

for $(N + p + r)$-dimensional real-valued vector $h^* = (h_1, \ldots, h_N, h_{j,1}^*, \ldots, h_{j,p}^*, h_{b,1}^*, \ldots, h_{b,r}^*)'$.

**Proof:** The proof proceeds the same as in Theorem 4.3 but replaces the prior for $\xi$ using the one stated in the Statement.

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