On Optimal Mechanisms in the Two-Item Single-Buyer Unit-Demand Setting

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Abstract

We consider the problem of finding an optimal mechanism in the two-item, single-buyer, unit-demand setting so that the expected revenue to the seller is maximized. The buyer’s valuation for the two items \( (z_1, z_2) \) is assumed to be uniformly distributed in an arbitrary rectangle \([c, c + b_1] \times [c, c + b_2]\) in the positive quadrant, having its left-bottom corner on the line \( z_1 = z_2 \). The exact solution in the setting without the unit-demand constraint can be computed using the dual approach designed in [15]. However, in the unit-demand setting, computing the optimal solution via the dual approach turns out to be a much harder, nontrivial problem; the dual approach does not offer a clear way of finding the dual measure. In this paper, we first show that the structure of the dual measure shows significant variations for different values of \((c, b_1, b_2)\) which makes it hard to discover the correct dual measure, and hence to compute the solution. We then nontrivially extend the virtual valuation method of [22] to provide a complete, explicit solution for the problem considered. In particular, we prove that the optimal mechanism is structured into five simple menus. Finally, we conjecture, with promising preliminary results, that the optimal mechanism when the valuations are uniformly distributed in an arbitrary rectangle \([c_1, c_1 + b_1] \times [c_2, c_2 + b_2]\) is also structured according to similar menus.

Keywords: Game theory, Economics, Optimal Auctions, Stochastic Orders, Convex Optimization.

1. Introduction

*Optimal mechanism design* is the problem of designing a mechanism to
maximize the expected revenue to the seller. The solution to the problem is well known when the buyer’s bid is one-dimensional (Myerson [21]). The problem however becomes much harder when the buyer’s bid is multi-dimensional. Though many partial results are available in the literature, finding the general solution remains open in the two-item setting, be it with or without the unit-demand constraint.

In this paper, we consider the problem of optimal mechanism design in the two-item one-buyer unit-demand setting, when the valuations of the buyer are uniformly distributed in arbitrary rectangles in the positive quadrant, having its left-bottom corner on the line $z_1 = z_2$. Observe that this is a setting that occurs often in practice. For example, consider a setting where two homes in a locality are auctioned. The seller is aware of a minimum and a maximum price for each house. Further, the buyer has a unit-demand, i.e., he is in need of only one of the homes, but submits his bids for both the homes. We consider that the buyer’s valuations are uniform in the rectangle formed by those intervals. We compute the optimal mechanism for all cases when the minimum price for both the homes are the same.

1.1. Prior Work

Consider the setting where the buyer is not restricted by the unit-demand constraint. Daskalakis et al. [13, 14, 15] provided a solution when the buyer’s valuation vector $z$ arises from a rich class of distribution functions that give rise to a so-called “well-formed canonical partition” of the support set of the distribution. The authors of these papers formulate this problem as an optimization problem, identify its dual as a problem of optimal transport, and exploit its solution to obtain a primal solution. Giannakopoulos and Koutsoupias [18] computed the solution for the multi-item setting, but only when the valuations for each item is uniformly distributed in $[0, 1]$. Giannakopoulos and Koutsoupias [19] also provided closed form solutions in the two-item setting, when the distribution satisfies some sufficient conditions, by using a dual approach similar to [13, 14, 15]. In a companion paper [26] (see also [25]), we used the same approach of solving the optimal transport problem as in [15] to obtain the solution when $z \sim \text{Unif}[c, c + 1]^2$. The exact solution in the unrestricted setting has largely been computed using the dual approach designed in [15].

The exact solution in the unit-demand setting, on the other hand, has been computed using various other methods. Pavlov [22] obtained a solution both in the unrestricted setting and in the restricted setting of unit-demand constraint, when $z \sim \text{Unif}[c, c + 1]^2$. The above paper used a marginal profit function $V$, whose properties are analogous to the virtual valuation function in [21], to compute the exact solution. We thus call this method the virtual valuation method. The function however depends on the region of zero allocation, and is thus not as straightforward to compute as the
virtual valuation function in [21] for the single item case. Lev [20] provided a solution for the unit-demand setting when the distribution is uniform in certain polygons containing the co-ordinate axes; the approach involves analyzing the utility function of the optimal mechanism at the edges of the polygon. We are not aware of any work that computes the exact solution in the unit-demand setting using the duality approach.

Wang and Tang [27], [24] proved that when the distributions are uniform in any rectangle in the positive quadrant, the optimal mechanism is a menu with at most five constant allocation regions. However, the exact menus and associated allocations were left open. Chawla et al. [11] proved that if the distribution over the items are independent, then setting a price for each item gives at least a constant fraction of the optimal revenue. But if the distributions are correlated, then Briest et al. [2, 3] proved that the revenue from a deterministic auction can be an arbitrary fraction of the optimal revenue, in case the number of items is more than 4. In another work, Chawla et al. [12] proved that even when the distributions have a certain kind of positive correlation, the randomness only contributes to a small factor of revenue increase.

There has been some interest in finding approximately optimal solutions when the distribution of the buyer’s valuation satisfies certain conditions. For example, Bhattacharya et al. [1] provided constant factor approximations when the distributions over the items are independent and satisfy the monotone hazard rate condition. Daskalakis and Weinberg [16, 17] and Cai and Daskalakis [4, 5] provided polynomial-time approximate solutions. Cai et al. [6, 7, 8] compute the optimal or approximately optimal solutions when every valuation in the support set is listed along with the probability with which the valuation occurs. Cai et al. [9] and Cai and Zhao [10] provided a unified view of many kinds of approximation algorithms, and also proved that the deterministic mechanism or the VCG auction with per-bidder entry fees gives at least a constant fraction of the optimal revenue. In this paper however we shall focus on exact solutions.

1.2. Our Contributions

Our contribution in this work is three-fold:

(i) We identify the correct dual problem to the primal problem of optimal auction in the restricted unit-demand setting. This is an optimal transport problem as in [15], but the transportation cost differs from the unrestricted setting.

(ii) We argue that the computation of the dual measure in the unit-demand setting using the approach of optimal transport in [15] is intricate. Specifically, we consider three examples: $z \sim \text{Unif}[1.26, 2.26]^2$, $z \sim \text{Unif}[1.5, 2.5]^2$, and $z \sim \text{Unif}[0, 1] \times [0, 1.2]$, and show that the optimal dual varies significantly with variation in $c$, thus making it hard to discover the correct dual measure.
Figure 1: A phase diagram of the optimal mechanism when $b_2 \leq b_1 \leq 2b_2$. The optimal mechanisms for other values of $b_1$ are also indicated in the figure.

(iii) Motivated by the above, we explore the virtual valuation method in [22] and nontrivially extend this method to compute the exact solution when $z \sim \text{Unif}([c, c + b_1] \times [c, c + b_2])$, for arbitrary nonnegative values of $(c, b_1, b_2)$. We establish that the structure of the optimal mechanism comprises of five simple menus, each having at most five constant allocation regions. We also make some remarks on the general case $[c_1, c_1 + b_1] \times [c_2, c_2 + b_2]$.

The optimal menus for various values of $(c, b_1, b_2)$ are mentioned in Theorem [12]. The phase diagram in Figure 1 represents how the optimal menu changes when the values of $(c, b_1, b_2)$ change. We interpret the solution as follows.

- Beyond the no sale region, the allocation probabilities are the same for all $z$ falling in the same $45^\circ$ line (Theorem [9]). Observe that this is in sharp contrast with the unrestricted setting, where the allocation probabilities are the same either for all $z$ falling in the same vertical line or the same horizontal line (see [25, Fig. 1–3]). This is because, in the unit-demand case, the buyer demands at most one of the two items, and thus the seller decides the item to be sold based on the difference of valuations on the items.

- Consider the case when $c$ is low. The seller then knows that the buyer possibly could have very low valuations, and thus sets a high threshold $\delta_i$ to sell item $i$. When $z_i \geq \delta_i$, i.e., when the buyer’s valuation for item $i$ goes beyond the threshold, the seller finds it optimal to sell one of the items with probability 1 (see Menu I in Figure 2).

- When $c$ increases, the threshold to sell item $i$ decreases. But the seller now finds it optimal to set a second threshold, below which he sells item $i$ with some probability $a_i \in (0, 1)$, and above which he sells item
i with probability 1 and the other item with probability 0.  (see menus II and III in Figures 3 and 4).

- When \( c \) increases further, the seller sells item \( i \) only when \( z_i \) is very high compared to \( z_{-i} \). In case the difference is not so high, then the seller finds it optimal to allocate randomly one or the other item (see menus IV and IV’ in Figures 5 and 7).

- When \( c \) is very high, the seller finds no reason to withhold the items for any valuation profile, because even the lowest valuation profile of the buyer \((c, c)\) is high enough to grant him a good revenue. Thus he finds it optimal to sell item 1 in case the difference between the valuations, \((z_1 - z_2)\), is beyond a threshold, and item 2 in case it is below the threshold (see menus V and V’ in Figures 6 and 8).

\[
\begin{align*}
(c + b_2, c) & \quad (c + b_1, c) \\
(0, 1) & \quad (1, 0) \\
(0, 0) & \quad (c, c)
\end{align*}
\]

Figure 2: Menu I

\[
\begin{align*}
(c + b_2, c) & \quad (c + b_1, c) \\
(0, 1) & \quad (1 - a, a) \\
(0, 0) & \quad (c, c)
\end{align*}
\]

Figure 3: Menu II

\[
\begin{align*}
(c + b_2, c) & \quad (c + b_1, c) \\
(0, 1) & \quad (1 - a, a) \\
(0, 0) & \quad (c, c)
\end{align*}
\]

Figure 4: Menu III

\[
\begin{align*}
(c + b_2, c) & \quad (c + b_1, c) \\
(0, 1) & \quad (1, 0) \\
(0, 0) & \quad (c, c)
\end{align*}
\]

Figure 5: Menu IV

\[
\begin{align*}
(c + b_2, c) & \quad (c + b_1, c) \\
(0, 1) & \quad (1, 0) \\
(0, 0) & \quad (c, c)
\end{align*}
\]

Figure 6: Menu V

\[
\begin{align*}
(c + b_2, c) & \quad (c + b_1, c) \\
(0, 1) & \quad (1 - a, a) \\
(0, 0) & \quad (c, c)
\end{align*}
\]

Figure 7: Menu IV’

\[
\begin{align*}
(c + b_2, c) & \quad (c + b_1, c) \\
(0, 1) & \quad (1, 0) \\
(0, 0) & \quad (c, c)
\end{align*}
\]

Figure 8: Menu V’
1.3. Our Method

Our method is as follows. We initially formulate the problem at hand (in the unit-demand setting) into an optimization problem, and derive its dual. As expected, the dual problem turns out to be an optimal transport problem that transfers mass from the support set $D$ to itself. Mass transfer must occur subject to the constraint that the difference between the mass densities before and after the transfer convex-dominates a signed measure that depends only on the distribution of the valuations. The dual problem is similar to that in [15] for the unrestricted setting, but differs in the transportation cost.

The key challenge in solving the dual problem lies in constructing the “shuffling measure” that convex-dominates 0, and in finding the location in the support set $D$ where the shuffling measure sits. The shuffling measure was always added at fixed locations in the unrestricted setting, and had a fixed structure for the uniform distribution of valuations over any rectangle in the positive quadrant (see [23]). In the unit-demand setting, however, we show that both the locations and the structures of the shuffling measure vary significantly for different values of $c$. There is as yet no clear understanding on how to construct the shuffling measure, and hence on how to compute the optimal solution via the dual method.

Motivated by the above, we explore the virtual valuation method by Pavlov [22]. Pavlov [22] computed the optimal mechanism when the buyer’s valuations are given by $z \sim \text{Unif}[c, c+1]$; the optimal mechanism was obtained only for distributions that are symmetric across the two items. When compared with the case of symmetric distributions, the case of asymmetric distributions poses the following challenge. The optimal menu is symmetric along a diagonal in the case of symmetric distributions. For asymmetric distributions, the menu must be computed over the larger region of the entire support set. The asymmetry leads to more parameters, more conditions to check for optimality, and a more complex variety of solutions determined, as we will soon see, by a larger number of polynomials. All these make the computation more difficult.

In this paper, we demonstrate how to compute the optimal mechanism for asymmetric distributions, when $z \sim \text{Unif}[c, c+b_1] \times [c, c+b_2]$. Specifically, we do the following.

- Taking cue from the result in [27] that the optimal mechanism is a menu with at most five constant allocation regions, we first construct some possible menus, parametrized by at most four parameters.

- We find the relation between the parameters using the sufficient conditions on the marginal profit function $V$. We show that the parameters can be computed by simultaneously solving at most two polynomials, each of degree at most 4.
• We then use continuity of the polynomials to prove that there exists a solution having desired values for all parameters. We then prove that the optimal mechanism is in the form of five simple menus for arbitrary nonnegative values of \((c, b_1, b_2)\) (see Theorem 12).

• We conjecture that the optimal menus have a similar structure even when \(z \in [c_1, c_1 + b_1] \times [c_2, c_2 + b_2]\) for all \((c_1, c_2, b_1, b_2) \geq 0\). We provide preliminary results to justify the conjecture (see Theorem 17).

Proofs of some case use Mathematica to verify certain algebraic inequalities. This is because (i) the parameters turn out to be solutions that simultaneously satisfy two polynomials of degree at most 4; and (ii) the solutions are complicated functions of \((c, b_1, b_2)\) involving fifth roots and eighth roots of some expressions. Verifying that these expressions satisfy some bounds were automated via the Mathematica software. The results that use Mathematica have been marked with an asterisk in the statement of Theorem 12. We believe that all of these results can be proved in the strict mathematical sense; but we leave this for the future in the interest of timely dissemination of our conclusions and observations. The skeptical reader could proceed by interpreting the Mathematica-based conclusions as conjectures.

Our work thus provides insights into two well-known approaches to solve representative problems on optimal mechanisms in the multi-item setting, besides solving, in the process, one such problem for asymmetric distributions. Specifically, our work clarifies under what situations the duality approach is likely to work well, and the intrinsic difficulties in using that approach in some other settings. Furthermore, the special cases that we solve provide insights into various possible structures of the optimal menus which, we feel, would act as a guideline to solve the problem of computing good menus in practical settings. We believe that our work is an important step towards understanding the applicability of the two different approaches, and a useful step addition to the growing canvas of canonical problems in multi-dimensional optimal auctions.

The rest of the paper is organized as follows. In Section 2, we first formulate an optimization problem under the unit-demand setting. We next derive its dual and solve it for three representative examples of \((c, b_1, b_2)\). The main purpose behind these examples is to bring out the variety in structure, and therefore the difficulty in guessing and computing, the dual measure for more general settings. In Section 3, we nontrivially extend the virtual valuation method of [22] to provide a complete and explicit solution for the case of asymmetric distributions. In particular, we prove that the optimal mechanism is structured into five simple menus. In Section 4, we conjecture, with promising preliminary results, that the optimal mechanism when the valuations are uniformly distributed in an arbitrary rectangle \([c_1, c_1 + b_1] \times [c_2, c_2 + b_2]\) is also structured according to similar menus. In
Section 5, we conclude the paper and provide some directions for future work.

2. Exploring The Dual Approach

Consider a two-item, single-buyer, unit-demand setting. The buyer’s valuation is \( z = (z_1, z_2) \) for the two items, sampled according to the joint density \( f(z) = f_1(z_1)f_2(z_2) \), where \( f_1(z_1) \) and \( f_2(z_2) \) are marginal densities. The support set of \( f_1 \) is defined as \( D_1 := \{ z_1 : f_1(z_1) > 0 \} \), and the support set of \( f \) is defined as \( D := D_1 \times D_2 \). Throughout the paper, we consider \( D_i = [c, c + b_i] \), where \((c, b_1, b_2)\) are nonnegative. A quasilinear mechanism comprises an allocation function \( q : D \to \{(q_1, q_2) : 0 \leq q_1, q_2, q_1 + q_2 \leq 1\} \) and a payment function \( t : D \to \mathbb{R}_+ \) that represent, respectively, the probabilities of allocation of the items to the buyer and the amount of transfer from the buyer to the seller. If the buyer’s true valuation is \( z \), and he reports \( \hat{z} \), his realized utility is \( \hat{u}(z, \hat{z}) := z \cdot q(\hat{z}) - t(\hat{z}) \), which is the valuation minus the payment.

A quasilinear mechanism satisfies incentive compatibility (IC) when truth telling is a weakly dominant strategy for the buyer, i.e., \( \hat{u}(z, z) \geq \hat{u}(z, \hat{z}) \) for every \( z, \hat{z} \in D \). In this case the buyer’s realized utility is \( u(z) := \hat{u}(z, z) = z \cdot q(z) - t(z) \). An incentive compatible mechanism satisfies individually rationality (IR) if the buyer is not worse off by participating in the mechanism, i.e., \( u(z) \geq 0 \) for every \( z \in D \).

An optimal mechanism is one that maximizes the expected revenue to the seller subject to IC and IR constraints. The optimal mechanism can be computed by solving the following functional optimization problem

\[
\max_u \int_D (z \cdot \nabla u(z) - u(z))f(z) \, dz + u(c, c) \tag{1}
\]

subject to

(a) \( u \) convex,

(b) \( \nabla u(z) \in [0, 1]^2, \nabla u(z) \cdot 1 \in [0, 1] \), a.e. \( z \in D \).

Using integration by parts, the objective function of problem (1) can be written as

\[
\int_D u(z)\mu(z) \, dz + \int_{\partial D} u(z)\mu_s(z) \, d\sigma(z) + u(c, c)\mu_p(c, c),
\]

where

\[
\mu(z) := -z \cdot \nabla f(z) - 3f(z), \quad z \in D; \quad \mu_s(z) := (z \cdot n(z))f(z), \quad z \in \partial D;
\]

\[
\mu_p(z) = \delta_{\{(c, c)\}}(z). \tag{2}
\]

The vector \( n(z) \) is the normal to the surface \( \partial D \) at \( z \) if it is defined, and 0 otherwise (at corners). We regard \( \mu \) as the density of a signed measure on the support set \( D \) that is absolutely continuous with respect to (w.r.t.)

\[1\text{For more details, see [23] for an analog of } (1) \text{ in the unrestricted setting. The extra condition in the restricted setting is } \nabla u(z) \cdot 1 \in [0, 1].\]
the two-dimensional Lebesgue measure \(dz\). Further, we regard \(\mu_s\) as the density of a signed measure on \(\partial D\) that is absolutely continuous w.r.t. the surface measure \(d\sigma(z)\). We regard \(\mu_p\) as a point measure. Often, by an abuse of notation, we use \(\mu\) and \(\mu_s\) to represent the measures instead of just the densities. By taking \(u(z) = 1\ \forall z \in D\), we observe that

\[
\int_D \mu(z) \, dz + \int_{\partial D} \mu_s(z) \, d\sigma(z) + \mu_p(c,c) = \int_D u(z) \, \mu(z) \, dz + \int_{\partial D} u(z) \, \mu_s(z) \, d\sigma(z) + \mu_p(c,c) = 0. \tag{3}
\]

Defining \(\bar{\mu} := \mu + \mu_s + \mu_p\), we observe that \(\bar{\mu}(D) = 0\). (By \(\bar{\mu} = \mu + \mu_s + \mu_p\), we mean \(\bar{\mu}(A) = \mu(A) + \mu_s(A \cap \partial D) + \mu_p(A \cap \partial D)\).) The optimization problem (1) can now be written as

\[
\max_u \int_D u \, d\bar{\mu} \quad \text{subject to} \quad (a) \ u \ \text{convex}, \\
(b) \ u(z_1, z_2) - u(z'_1, z'_2) \leq (z_1 - z'_1)_+, \forall z_1, z'_1 \in D_1, \forall z_2 \in D_2, \\
(c) \ u(z_1, z_2) - u(z_1, z'_2) \leq (z_2 - z'_2)_+, \forall z_1 \in D_1, \forall z_2, z'_2 \in D_2, \\
(d) \ u(z_1, z_2) - u(z'_1, z_2 - z_1 + z'_1) \leq (z_1 - z'_1)_+, \\
\forall z_1, z'_1 \in D_1, \forall z_2 \in D_2.
\]

Observe that the constraints (b)–(d) are equivalent to the following constraint:

\[
u(z) - u(z') \leq \max((z_1 - z'_1)_+, (z_2 - z'_2)_+), \forall z, z' \in D.
\]

So the optimization problem can now be written as

\[
\max_u \int_D u \, d\bar{\mu} \quad \text{subject to} \quad (a) \ u \ \text{convex, increasing}, \\
(b) \ u(z) - u(z') \leq \|z - z'\|_\infty, \forall z, z' \in D.
\] \tag{4}

We now recall the definition of the convex ordering relation. A function \(f\) is increasing if \(z \geq z'\) component-wise implies \(f(z) \geq f(z')\).

**Definition 1.** (See for e.g., [14]) Let \(\alpha\) and \(\beta\) be measures defined on a set \(D\). We say \(\alpha\) convex-dominates \(\beta\) (\(\alpha \succeq_{\text{cvx}} \beta\) if \(\int_D f \, d\alpha \geq \int_D f \, d\beta\) for all convex and increasing \(f\).

We now proceed to find the dual of (4).
Theorem 2. The answer to problem (4) is the same as the answer to the following problem:

$$\min_{\gamma(\cdot) \geq 0} \int_{D \times D} \| z - z' \|_\infty \, d\gamma(z, z')$$

subject to

$$\gamma(\cdot, D) = \gamma_1, \ \gamma(D, \cdot) = \gamma_2, \ \gamma_1 - \gamma_2 \succeq \text{cvx} \bar{\mu}.$$ (5)

Proof. The proof of this theorem is along the lines of the proof of [14, Thm. 2] with appropriate changes to handle the $\| \cdot \|_\infty$ norm in our setting, see (4(b)), in place of the $\| \cdot \|_1$ norm. We relegate the details to Appendix A. □

The dual problem (5) can be interpreted as an optimal transport problem, where (i) the density $\gamma(z, z')$ refers to the differential mass to be transported from $z$ to $z'$, (ii) $\bar{\mu}$ refers to the supply-demand profile of the points in $D$, and (iii) $\| z - z' \|_\infty$ refers to the cost of transporting unit mass from $z$ to $z'$. The problem aims at minimizing the cost of transport. The constraint $\gamma_1 - \gamma_2 \succeq \text{cvx} \bar{\mu}$ indicates that the mass transfer need not meet the supply-demand constraint exactly, but the difference function, $\gamma_1 - \gamma_2 - \bar{\mu}$, must convex-dominate 0.

We now derive the complementary slackness conditions. Consider $u^*$ and $\gamma^*$ to be feasible for the primal (4) and dual (5) problems respectively. Then

$$\int_{D \times D} \| z - z' \|_\infty \, d\gamma^*(z, z') \geq \int_{D \times D} (u^*(z) - u^*(z')) \, d\gamma^*(z, z')$$

$$= \int_D u^*(z) \, d\gamma^*_1(z) - \int_D u^*(z') \, d\gamma^*_2(z') = \int_D u^*(z) \, d(\gamma^*_1 - \gamma^*_2)(z) \geq \int_D u^* \, d\bar{\mu}$$

where the first inequality follows from (4(b)), and the last inequality follows because (i) $u^*$ is convex, and (ii) $\gamma^*_1 - \gamma^*_2 \succeq \text{cvx} \bar{\mu}$. This chain of inequalities immediately implies a sufficient condition for strong duality.

Proposition 3. Let $u^*$ and $\gamma^*$ be feasible for the aforementioned primal (4) and dual (5) problems, respectively. Then the objective functions of (4) and (5) with $u = u^*$ and $\gamma = \gamma^*$ are equal iff (i) $\int_D u^* \, d(\gamma^*_1 - \gamma^*_2) = \int_D u^* \, d\mu$, and (ii) $u^*(z) - u^*(z') = \| z - z' \|_\infty$, hold $\gamma^* - a.e.$

We now present a few examples to indicate why it is hard to compute the optimal mechanism using this dual approach. We first compute the components of $\bar{\mu}$ (i.e., $\mu, \mu_s, \mu'_s$), with $f(z) = \frac{1}{b_1 b_2}$ for $z \in D = [c, c + b_1] \times \ldots$
Then, the optimal menu is given by Figure 9, with \( \delta_2 \).

**Example 1:** Consider the case when \( z \in D \).

Proof. Pavlov [22] proved this via virtual valuations. We shall use the dual method. To prove this theorem, we must find a feasible allocation \( q \) and show that they satisfy the constraints of Proposition 3. We define the this, since

\[
\alpha = 1 + 2 \ \text{area density} \quad \mu(z) = -3/(b_1 b_2), \quad z \in D,
\]

\[
\text{(line density)} \quad \mu_a(z) = \sum_{i=1}^{2} (-c_1(z_i = c) + (c + b_i)(z_i = c + b_i)) / (b_1 b_2),
\]

\[
\text{(point measure)} \quad \mu_p(z) = \delta{(c,c)}(z).
\]

(6)

**2.1. Example 1:** \( z \sim \text{Unif}[1.26, 2.26]^2 \)

**Theorem 4.** [22] Consider the case when \( c = 1.26, \) and \( b_1 = b_2 = 1 \). Then, the optimal menu is given by Figure 9 with \( \delta_1 = \delta_2 = 20/63 \) and \( a_1 = a_2 = a = 0.6615 \).

\[
\begin{align*}
\text{(c, c + b_2)} & \quad (c + 2/3, c + 1) \\
(0, 0) & \quad (1, 0) \\
(0, 1) & \quad (1 - a_1, a_1) \\
(1, 1 - a_1) & \quad (c + b_1, c) \\
\end{align*}
\]

Figure 9: Optimal menu for \( c = 1.26, \) \( b_1 = b_2 = 1 \).

Figure 10: Optimal menu for \( c = 1.5, \) \( b_1 = b_2 = 1 \).

**Proof.** Pavlov [22] proved this via virtual valuations. We shall use the dual method. To prove this theorem, we must find a feasible \( u \) and a feasible \( \gamma \), and show that they satisfy the constraints of Proposition 3. We define the allocation \( q \) as given in Figure 9. The primal variable \( u \) can be derived from this, since \( \nabla u = q \).

We now define functions \( \alpha^{(1)}, \beta^{(1)} : D \rightarrow \mathbb{R} \) as follows (see Figures 11 and 12).

\[
\alpha^{(1)}(c + t, c + t') = \begin{cases} 
3t - 1 & (t, t') \in ([0, 2/3], 1), \\
0 & \text{otherwise}.
\end{cases}
\]

(7)

\[
\beta^{(1)}(c + t, c + t') = \begin{cases} 
3t - 1 & (t, t') \in ([2/3, 1 - \delta_2], 1), \\
3t + 3a(1 - t - \delta_2) - c - 1 & (t, t') \in ([1 - \delta_2, 1], 1), \\
0 & \text{otherwise}.
\end{cases}
\]

(8)
The functions $\alpha^{(2)}$ and $\beta^{(2)}$ are defined similarly on the intervals $(c+1, [c, c+2/3])$ and $(c+1, [c+2/3, c+1])$ respectively.

We now construct the dual variable $\gamma$ as follows. First, let $\gamma_1 := \gamma_1^Z + \gamma_1^{D\setminus Z}$, where $Z$ is the region having allocation $(0,0)$, $\gamma_1^Z = \bar{\mu}^Z$, the $\bar{\mu}$ measure restricted to $Z$, and $\gamma_1^{D\setminus Z} = (\bar{\mu}^{D\setminus Z} + \sum_i (\alpha^{(i)} + \beta^{(i)}))$. So $\gamma_1$ is supported on $Z \cup ([1.26, 2.26], 2.26) \cup (2.26, [1.26, 2.26])$. Often, by abuse of notation, we use $\gamma_1$ to represent the density instead of the measure. Now we specify a transition probability kernel $\gamma(\cdot \mid x)$ for every $x$ in the support of $\gamma_1$.

\[ \alpha^{(1)}(1.26 + t, 2.26) \quad \beta^{(1)}(1.26 + t, 2.26) \]

(a) For $x \in Z$, we define $\gamma(y \mid x) = \delta_x(y)$. This is interpreted as no mass being transferred.

(b) For $x \in ([1.26, 2.26], 2.26) \cup (2.26, [1.26, 2.26])$, we define $\gamma(y \mid x) = (\mu(y) + \mu_\alpha(y)) - / \gamma_1(x)$ if $y \in \{y \in D \setminus Z : y_1 - y_2 = x_1 - x_2\}$, and zero otherwise. This is interpreted as a transfer of $\gamma_1(x)$ from the boundary point $x$ to (the 45° line segment) \{y \in D \setminus Z : y_1 - y_2 = x_1 - x_2\}, which has $x$ as one end-point.

We then define $\gamma(F) = \int_{(x,y) \in F} \gamma_1(dx) \gamma(dy \mid x)$ for any measurable $F \in D \times D$. It is now easy to check that $\gamma_2^Z = \bar{\mu}^Z$, and $\gamma_2^{D\setminus Z} = (\bar{\mu}^{D\setminus Z} + \sum_i (\alpha^{(i)} + \beta^{(i)})) -$. Thus we have $(\gamma_1 - \gamma_2)^Z = 0$, and $(\gamma_1 - \gamma_2)^{D\setminus Z} = \bar{\mu}^{D\setminus Z} + \sum_i (\alpha^{(i)} + \beta^{(i)})$.

We now verify that $\gamma$ is feasible. Observe that the components of $\bar{\mu}^Z$ are positive only at the left-bottom corner of $D$ (i.e., at $(c,c)$) and negative elsewhere, and that $\bar{\mu}_+(Z) = 1 = \bar{\mu}_-(Z)$ (the second equality requires some calculations). So we have $\int_Z f(z) d\mu \leq 0$ for any increasing function $f$, and thus $\bar{\mu}^Z \preceq \limsup 0 = (\gamma_1 - \gamma_2)^Z$. We next prove that $(\gamma_1 - \gamma_2)^{D\setminus Z} \preceq \limsup \bar{\mu}^{D\setminus Z}$. Since $(\gamma_1 - \gamma_2 - \bar{\mu})^{D\setminus Z} = \sum_i (\alpha^{(i)} + \beta^{(i)})$, it suffices to prove that $\sum_i (\alpha^{(i)} + \beta^{(i)}) \preceq \limsup 0$. We do this in the next lemma.

**Lemma 5.** (i) The measure $\alpha^{(1)}$ is such that $\alpha^{(1)}([1.26, 1.26 + 2/3], 2.26) = 0$ and $\int_{1.26}^{1.26 + 2/3} (t - 1.26) \alpha^{(1)}(dt, 2.26) \geq 0$. Hence for any $f$ constant on $[1.26, 1.26 + 2/3]$, we have $\int_{1.26}^{1.26 + 2/3} f(t) d\alpha^{(1)}(dt, 2.26) = 0$. Further, $\alpha^{(1)} \preceq \limsup 0$. A similar result holds for $\alpha^{(2)}$.  

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\((ii) \ \beta^{(1)}([1.26 + 2/3, 2.26], 2.26) = 0 \) and \( f_{1.26+2/3}^{2.26}(t - 1.26) \beta^{(1)}(dt, 2.26) = 0. \) Hence we have \( f_{1.26+2/3}^{2.26}(t) \beta^{(1)}(dt, 2.26) = 0 \) for any affine \( f \) on
\([1.26 + 2/3, 2.26]. \) Further, \( \beta^{(1)} \preceq_{\text{conv}} 0. \) A similar result holds for \( \beta^{(2)}. \)

**Proof.** See Appendix A. □

We have thus established that \( \gamma_1 - \gamma_2 \preceq_{\text{conv}} 0. \) We now verify if \( u \) and \( \gamma \)
satisfy the conditions in Proposition 3.

\[
\int_D u \, d(\gamma_1 - \gamma_2) = \int_Z u \, d(\gamma_1 - \gamma_2) + \int_{D \setminus Z} u \, d(\gamma_1 - \gamma_2)
\]

\[
= \int_{D \setminus Z} u \, (\bar{\mu} + \sum_i (\alpha_i + \beta_i)) = \int_{D \setminus Z} u \, d\bar{\mu} = \int_D u \, d\bar{\mu},
\]

where the second equality holds because \((\gamma_1 - \gamma_2)^2 = 0\); the third equality holds because \(u(z)\) is a constant when \(z \in ([1.26, 1.26 + 2/3], 2.26) \cup (2.26, [1.26, 1.26 + 2/3], 2.26) \cup (2.26, [1.26 + 2/3, 2.26]);\) and the last equality holds because \(u(z) = 0\) when \(z \in Z.\) To see why \(u(z) - u(z') = \|z - z'\|_\infty\) holds \(\gamma\)-a.e., it suffices to check this condition for those \((z, z')\) for which \(\gamma(\cdot \mid z)\) is nonzero, as in the cases (a) and (b) above. For \(z, z'\) in (a), \(z = z'\) and hence \(u(z) - u(z') = 0;\) in (b), \((z' z')\)
lie on a 45° line, and hence \(u(z) - u(z') = (z_1 - z'_1) = (z_2 - z'_2) = \|z - z'\|_\infty.\) Thus \(u(z) - u(z') = \|z - z'\|_\infty\) holds \(\gamma\)-a.e. □

The dual measure \(\gamma\) was defined so that the measure \(\gamma_1 - \gamma_2 - \bar{\mu},\) called the shuffling measure, convex-dominates 0. Our key challenge in computing the optimal menu lies in constructing the shuffling measure. The next example shows the need of a different shuffling measure.

### 2.2. Example 2: \(z \sim \text{Unif } [1.5, 2.5]^2\)

**Theorem 6.** Consider the case when \(c = 1.5, \) and \(b_1 = b_2 = 1. \) Then, the optimal menu is given by Figure 10 with \(\delta_1' = \delta_2' = \sqrt{5/3} - 1.\)

**Proof.** We define \(q\) as given in Figure 10 and construct \(u\) such that \(\nabla u = q. \) We now construct the shuffling measure \(\lambda + \sum (\alpha_i + \beta_i)\) as follows. We define \(\alpha_i\) and \(\beta_i\) same as in (17) and (18) respectively, but with \(\delta = \delta_2 = \frac{(3 + \sqrt{33})/8 - 1}{(27 - 3\sqrt{33})/32} > \delta_2'\) and \(a = (27 - 3\sqrt{33})/32. \) We define \(\lambda : D \rightarrow \mathbb{R},\) as follows (see Figure 13):

\[
\lambda(c + (t - 1 + \delta_2)/2, c + \delta_2 - (t - 1 + \delta_2)/2) = \lambda(c + \delta_2 - (t - 1 + \delta_2)/2, c + (t - 1 + \delta_2)/2)
\]

\[
= \begin{cases} 3a(t - 1 + \delta_2) + c, & t \in [1 - \delta_2, 1 - \delta_2'], \\ 3t(a - 1/2) + 3/2(1 - \delta_2') - 3a(1 - \delta_2), & t \in [1 - \delta_2', 1]. \end{cases}
\]
\( \lambda \) is defined to be 0 at every other point in \( D \). Observe that the function is defined on the line \( z_1 + z_2 = 2c + \delta_2 \), and thus is symmetric about the line \( z_1 = z_2 \).

We now construct \( \gamma \) as follows. Let \( \gamma_1 = \gamma_1^Z + \gamma_1^{D\backslash Z} \), with \( \gamma_1^Z = \bar{\mu}^Z \) and \( \gamma_1^{D\backslash Z} = (\bar{\mu}^{D\backslash Z} + \sum \alpha(i) + \beta(i) + \lambda_+) \). This is supported on \( Z \cup ([1.5,2.5], 2.5 \cup (2.5,[1.5,2.5]) \), and additionally on points where \( \lambda > 0 \).

Now we specify \( \gamma(x \mid x) \) for every \( x \) in the support of \( \gamma_1 \).

(a) For \( x \in Z \), we define \( \gamma(y \mid x) = \delta_x(y) \). This is interpreted as no mass being transferred.

(b) For \( x \in ([1.5,2.5], 2.5 \cup (2.5,[1.5,2.5]) \), we define \( \gamma(y \mid x) = (\mu(y) + \mu_+(y) + \lambda(y)) - \gamma_1(x) \) when \( y \in \{ y \in QRS\delta_2p_2\delta_1Q : y_1 - y_2 = x_1 - x_2 \} \), and zero otherwise (see Figure 14). (By an abuse of notation, we denote the values of \( \delta_1 \) and \( \delta_2 \) as points marked in the figure.) This is interpreted as transfer of \( \gamma_1(x) \) from the boundary to the above line segment.

(c) For \( x \) where \( \lambda(x) > 0 \), we define \( \gamma(y \mid x) = (\mu(y) + \mu_+(y)) - \lambda(x) \) when \( y \in \{ y \in (\delta_1p_1\delta_1\delta_1) \cup (\delta_2p_2\delta_2\delta_2) : y_1 - y_2 = x_1 - x_2 \} \), and zero otherwise (see Figure 14). This is interpreted as transfer of \( \lambda(x) \) from the point \( x \) on the line \( x_1 + x_2 = 2c + \delta_2 \) to the above line segment.

We then define \( \gamma(F) = \int_{(x,y) \in F} \gamma_1(dx)\gamma(dy \mid x) \) for any measurable \( F \in D \times D \). It is now easy to check that \( \gamma_2^Z = \bar{\mu}^Z \) and \( \gamma_2^{D\backslash Z} = (\bar{\mu}^{D\backslash Z} + \sum \alpha(i) + \beta(i) + \lambda_+) \). Thus we have \( (\gamma_1 - \gamma_2)^Z = 0 \), and \( (\gamma_1 - \gamma_2)^{D\backslash Z} = \bar{\mu}^{D\backslash Z} + \sum \alpha(i) + \beta(i) + \lambda_+ \).

The proof that \( \gamma \) satisfies all the required conditions of Proposition 3 traces the same steps as in the proof of Theorem 4. The extra step here is to show that \( \lambda \geq_{\text{cvx}} 0 \). We now compute

\[
\int_{1-\delta_2}^1 \lambda(c + (t - 1 + \delta_2)/2, c + \delta_2 - (t - 1 + \delta_2)/2) \, dt \\
= \int_0^{\delta_2 - \sqrt{5}/3 + 1} (3at + c) \, dt \\
+ \int_{1-(\sqrt{5}/3-1)}^1 (3t(a - 1/2) + 3/2(1 - (\sqrt{5}/3 - 1)) - 3a(1 - \delta_2)) \, dt \\
= 3/2(\delta_2 - \sqrt{5}/3 + 1)^2 + c(\delta_2 - \sqrt{5}/3 + 1) \\
+ 3/2(a - 1/2)(1 - (2 - \sqrt{5}/3)^2) \\
+ (\sqrt{5}/3 - 1)(3/2(2 - \sqrt{5}/3) - 3a(1 - \delta_2)) \] \\
= 0
\]
where the last equality follows by substituting $c$ and $\delta_2$. We also have

$$
\int_{1-\delta_2}^{1} (t-1)\lambda(c + (t - 1 + \delta_2)/2, c + \delta_2 - (t - 1 + \delta_2)/2) \, dt
$$

$$
+ \int_{1-\delta_1}^{1} (t-1)\lambda(c + \delta_1 + (t - 1 + \delta_1)/2, c - (t - 1 + \delta_1)/2) \, dt = 0
$$

which follows because (i) $\lambda$ is symmetric about the line $t = 1$, and (ii) $(t-1)$ is an odd function about the line $t = 1$. The proof of $\lambda \succeq_{\text{cvx}} 0$ now traces the same steps as in Lemma 5. \hfill \Box

![Figure 13: The measure $\lambda$. The y-axis expression for the left and the right portions of the graph is indicated using $L$ and $R$. The measure is symmetric because we have $\delta_1 = \delta_2$.](image)

![Figure 14: The menu in Figure 10 magnified near its left-bottom corner. The full support set $D$ is denoted by $PQRS$. $P_1$, $P_2$ are points where the dotted line intersects with the line denoted $\delta_1^\prime \delta_2^\prime$.](image)

The results of Theorems 4 and 6 are parts of a more general result shown in [22]. Pavlov’s proof uses a virtual valuation method, but we have proved using the dual approach. We now solve another example via the dual approach, going beyond those considered in [22].

2.3. Example 3: $z \sim \text{Unif}[0,1.2] \times [0,1]$

**Theorem 7.** Consider the case when $c = 0$, $b_1 = 1.2$, and $b_2 = 1$. Then, the optimal menu is as in Figure 17, with $(\delta_1, \delta_2)$ simultaneously solving

$$
-3\delta_1\delta_2 - c(\delta_1 + \delta_2) + b_1b_2 = 0.
$$

$$
-\frac{3}{2}\delta_2^2 + 2b_2\delta_2 - b_2^2 + (c - 2b_2 + 3\delta_2)\delta_1 = 0.
$$

The values of $(\delta_1, \delta_2)$ can be solved numerically to be

$$(\delta_1, \delta_2) \approx (0.678837, 0.589243).$$

**Proof.** We define $q$ as given in Figure 13 and construct $u$ such that $\nabla u = q$. Defining $\delta^* := \delta_1 - \delta_2$, we now construct the shuffling measure $\alpha +
\( \alpha^{(o)} + \alpha^{(h)} \) as follows. The superscripts \((o)\) and \((h)\) stand for 'oblique' and 'horizontal'.

\[
\alpha(c + t, c + t') := \begin{cases} 1/2 & (t, t') \in ([0, 1 - \delta_2], 1), \\ 3t - 1 & (t, t') \in ([1 - \delta_2, 1 + \delta^*], 1), \\ 3(1 - \delta_2) - c - 1 & (t, t') \in ([0, 1 - \delta_2], 1), \\ 0 & \text{otherwise.} \end{cases}
\]

\[
\alpha^{(o)}(c + t, c + t') := \begin{cases} 1/2 & (t, t') \in (1.2, [0, 1.2 - \delta_1]), \\ 3t - 1.2 & (t, t') \in ([1.2, 1.2 - \delta_1], 1), \\ 2(1.2) - 3\delta_1 & (t, t') \in ([1.2, 1.2 - \delta_1], 1), \\ 0 & \text{otherwise.} \end{cases}
\]

\[
\alpha^{(h)}(c + t, c + t') := \begin{cases} 1/2 & (t, t') \in (1.2, [\delta_1 - 0.2, \delta_2]), \\ 3(t - \delta_1 + 0.2) & (t, t') \in ([\delta_1 - 0.2, \delta_2], 1), \\ 3(0.2 - \delta^*) & (t, t') \in ([\delta_2, 2/3], 1), \\ 0 & \text{otherwise.} \end{cases}
\]

We now construct \( \gamma \) as follows. Let \( \gamma_1 = \gamma^Z_1 + \gamma^D_1 Z \), with \( \gamma^Z_1 = \bar{\mu}^Z \) and \( \gamma^D_1 Z = (\bar{\mu}^{D \setminus Z} + \alpha + \alpha^{(o)} + \alpha^{(h)})_+ \). This is supported on \( Z \cup ([0, 1.2], 1) \cup (1.2, [0, 1]) \). Now we specify \( \gamma(\cdot | x) \) for every \( x \) in the support of \( \gamma_1 \).

(a) For \( x \in Z \), we define \( \gamma(y \mid x) = \delta_x(y) \). This is interpreted as no mass being transferred.

(b) For \( x \in ([0, 1 + \delta^*], 1) \cup (1.2, [0, 1 - 0.2] \cup [2/3, 1]) \), we define \( \gamma(y \mid x) = (\mu(y) + \mu_s(y)) - \gamma_1(x) \) when \( y \in \{y \in D \setminus Z : y_1 - y_2 = x_1 - x_2\} \), and zero otherwise. This is interpreted as transfer of \( \gamma_1(x) \) from the boundary to the above line segment.

(c) For \( x \in ([1 + \delta^*, 1.2], 1) \), we define \( \gamma(y \mid x) = (\mu(y) + \mu_s(y)) - \gamma_1(x) \) when \( \{y_1 - y_2 = x_1 - x_2, y_2 \in [2/3, 1]\} \), and zero otherwise. Again, this is interpreted as transfer of \( \gamma_1(x) \) from the boundary to the above line segment.

(d) For \( x \in ([1.2, 1 + \delta^*, 2/3]) \), we define \( \gamma(y \mid x) = (\mu(y) + \mu_s(y)) - \gamma_1(x) \), when \( y \in \{y \in D \setminus Z : y_1 - y_2 = x_1 - x_2\} \cup \{y_2 = x_2, y_1 - y_2 \in [\delta^*, b_1 - b_2]\} \), and zero otherwise. This is interpreted as a transfer of \( \gamma_1(x) \) from the boundary to two line segments – one is a 45° line segment contained within \( D \setminus Z \), and the other is a horizontal line contained within \( \{y_1 - y_2 \in [\delta^*, b_1 - b_2]\} \). The transfers occur respectively due to the shuffling measures \( \alpha^{(o)} \), oblique transfer, and \( \alpha^{(h)} \), horizontal transfer.

We then define \( \gamma(F) = \int_{(x,y) \in F} \gamma_1(dx) \gamma(dy \mid x) \) for any measurable \( F \in D \times D \). It is now easy to check that \( \gamma_2^Z = \bar{\mu}^Z \), and \( \gamma_2^{D \setminus Z} = (\bar{\mu}^{D \setminus Z} + \alpha + \alpha^{(o)} + \alpha^{(h)})_+ \). Thus we have \( (\gamma_1 - \gamma_2)^Z = 0 \), and \( (\gamma_1 - \gamma_2)^{D \setminus Z} = (\bar{\mu}^{D \setminus Z} + \alpha + \alpha^{(o)} + \alpha^{(h)}) \).
The proof that $\gamma$ satisfies all the required conditions of Proposition 3 traces the same steps as in the proof of Theorem 4. The extra step here is to show that $\alpha \succeq \alpha^{(0)} + \alpha^{(h)} \succeq 0$. We now have

$$\alpha([0, 1 + \delta^*], 1) = \int_0^{1-\delta_2} (3t - 1) dt + \int_1^{1+\delta^*} (2 - 3\delta_2) dt = 3/2(1 - \delta_2)^2 - (1 - \delta_2) + \delta_1(2 - 3\delta_2) = 0.$$  

$$\int_0^{1+\delta^*} t \, d\alpha(t, 1)$$

$$= \int_0^{1-\delta_2} t(3t - 1) dt + \int_1^{1+\delta^*} t(2 - 3\delta_2) dt$$

$$= (1 - \delta_2)^3 - (1 - \delta_2)^2/2 + (2 - 3\delta_2)((1 + \delta^*)^2 - (1 - \delta_2)^2)/2$$

$$\approx 0.103227 \geq 0.$$  

$$\alpha^{(0)} + \alpha^{(h)}(1.2, [0, 1])$$

$$= \int_0^{1-\delta_1} (3t - 1.2) dt + \int_{1.2-\delta_1}^{1} (2.4 - 3\delta_1) dt$$

$$+ \int_{\delta_1-0.2}^{\delta_2} 3(t - \delta_1 + 0.2) dt + \int_{\delta_2}^{2/3} 3(0.2 - \delta^*) dt$$

$$= 3/2(1.2 - \delta_1)^2 - 1.2(1.2 - \delta_1) + (\delta_1 - 0.2)(2.4 - 3\delta_1)$$

$$+ 3/2(0.2 - \delta^*)^2 + (2 - 3\delta_2)(0.2 - \delta^*)$$

$$= 0.$$
\[
\begin{align*}
\int_0^1 t \, d(\alpha^{(o)} + \alpha^{(h)})(1.2, t) \\
= \int_0^{1.2-\delta_1} t(3t - 1.2) \, dt + \int_{1.2-\delta_1}^1 t(2.4 - 3\delta_1) \, dt \\
+ \int_{\delta_1}^{\delta_2} 3t(t - \delta_1 + 0.2) \, dt + \int_{\delta_2}^{2/3} 3t(0.2 - \delta^*) \, dt \\
= (1.2 - \delta_1)^3 - 0.6(1.2 - \delta_1)^2 + (1.2 - 3\delta_1/2)(1 - (1.2 - \delta_1)^2) \\
+ (\delta_2^3 - (\delta_1 - 0.2)^3) - 3/2(\delta_1 - 0.2)(\delta_2^2 - (\delta_1 - 0.2)^2) \\
+ 3/2(4/9 - \delta_2^3)(0.2 - \delta_1 + \delta_2) \approx 0.137171 \geq 0.
\end{align*}
\]

The proofs of \( \alpha \geq_{\text{cvx}} 0, \alpha^{(o)} + \alpha^{(h)} \geq_{\text{cvx}} 0 \) trace the same steps of the proof of Lemma 5. \( \square \)

We have computed the optimal menus in the two-item unit-demand setting for three representative examples, using the dual approach. The challenge in each of the examples was to construct the shuffling measure \( \gamma_1 - \gamma_2 - \bar{\mu} \) so that it convex-dominates 0. We now argue that arriving at a general algorithmic procedure to construct the shuffling measure may be difficult.

- The locations of the shuffling measure exhibit significant variations in our examples. For instance, the shuffling measure was non-zero only at the top boundary and the right boundary of \( D \) in Theorems 4 and 7 whereas, it was non-zero additionally on the line \( z_1 + z_2 = 2c + \delta_2 \) in Theorem 6.
- The structures of the shuffling measure also exhibit significant variations. The variations can be observed from the structures in Figures 13 and 16. This is in contrast to the unrestricted setting solved in [25], where the shuffling measures were added at a fixed location and had a fixed structure.
- In the case of \( c = 0, b_1 = 1.2, b_2 = 1 \), the shuffling measure had to be constructed partly for a mass transfer along the 45° line segment, and partly for a transfer along the horizontal line segment (see point (d) in the proof of Theorem 7). The example thus had two shuffling measures: \( \alpha^{(o)} \) and \( \alpha^{(h)} \).

The examples above suggest that it may be difficult to come up with a general algorithmic method to construct a shuffling measure, even for the restricted setting of uniform distributions. This motivates us to tackle the general problem using the virtual valuation method in [22].

3. Exploring The Virtual Valuation Method

Recall that we consider the problem of optimal mechanism design in a two-item, one-buyer, unit-demand setting. In this section, we compute the
optimal mechanism when the buyer’s valuation \( z \sim \text{Unif}[c, c+b_1] \times [c, c+b_2] \), using the virtual valuation method in [22]. We start with the following general result from [22].

**Theorem 8.** [22] Prop. 1] If the distribution \( f \) satisfies

\[
3f_1(z)f_2(z) + z_1f'_1(z)f_2(z) + z_2f_1(z)f'_2(z) \geq 0 \forall z \in D,
\]

then the allocation function \( q \) in the optimal mechanism is such that \( q_1 + q_2 \in \{0, 1\} \).

The theorem states that if \( f \) satisfies some sufficient conditions, then for every \( z \in D \setminus Z, q(z) \) satisfies \( q_1(z) + q_2(z) = 1 \) (recall that \( Z \) is the region where the allocation is \((0,0)\)). Observe that the sufficient condition in Theorem 8 is clearly satisfied for the uniform distributions. So when \( z \sim \text{Unif}[c, c+b_1] \times [c, c+b_2] \), the utility of the buyer in \( D \setminus Z \) can be written as \( u(z) = (z_1 - z_2)q_1(z) + z_2 - t(z) \). We have used \( q_2 = 1 - q_1 \). Defining \( \delta := z_1 - z_2 \), we have \( \delta \in [-b_2, b_1] \) for the case under consideration. The following theorem from [22] reduces the domains of \( q \) and \( t \) from two-dimensions to one-dimension.

**Theorem 9.** [22] Prop. 2] In the optimal mechanism, the allocations and the payments, \((q, t)\), can be rewritten so that they are a constant for every \( \{z \in D \setminus Z : z_1 - z_2 = \delta\} \).

The theorem indicates that if \( Z \) is fixed, then the domains of \((q, t)\) become one-dimensional, in the region \( D \setminus Z \). So we now have \( t : [-b_2, b_1] \rightarrow \mathbb{R}_+, \quad q_1 : [-b_2, b_1] \rightarrow [0,1], \) and \( q_2 = 1 - q_1 \). As done in [22], define \( u_1 : [-b_2, b_1] \rightarrow \mathbb{R}, \quad u_1(\delta) := \delta q_1(\delta) - t(\delta) \), and define \( g(u_1(\delta), \delta) := \int_{z_1-z_2=\delta} f(z) \, dz \). Observe that \( u(z) = u_1(z_1 - z_2) + z_2 \), and thus the expression \( u_1(\delta) + z_2 \geq 0 \) denotes \( \{z : u(z) \geq 0\} \).

Consider the problem of maximizing the expected revenue subject to IC and IR constraints. The IC constraint, from [21, Lem. 2], can equivalently be written as (i) \( q_1 \) increasing, and (ii) \( u_1(\delta) \) has the representation \( u_1(\delta) = u_1(-b_2) + \int_{b_2}^{\delta} q_1(\delta) \, d\delta \) for every \( \delta \in [-b_2, b_1] \). The optimal mechanism can thus be computed by solving the following optimization problem.

\[
\max_{q_1(\cdot), u_1(\cdot)} \int_{-b_2}^{b_1} (\delta q_1(\delta) - u_1(\delta)) g(u_1(\delta), \delta) \, d\delta \quad (9)
\]

subject to

\[
(a) \quad q_1(\delta) \in [0,1] \forall \delta \in [-b_2, b_1]; \quad q_1 \text{ increasing};
(b) \quad u_1(\delta) = u_1(-b_2) + \int_{b_2}^{\delta} q_1(\delta) \, d\delta \forall \delta \in [-b_2, b_1].
\]

The IR constraint is taken into account because \( g(u_1(\delta), \delta) \) is taken to be nonzero only in \( D \setminus Z \).
Observe that the problem (9) is similar to the optimization problem in [21, Lem. 3]. To solve the problem in a similar way, we now search for an equivalent of the virtual valuation function \( \phi \), in our setting.

Define the marginal profit function \( \bar{V} : [-b_2, b_1] \to \mathbb{R} \) as

\[
\bar{V}(\delta) = \delta g(u_1(\delta), \delta) - \int_{\delta}^{b_1} g(u_1(\delta), \bar{\delta}) \, d\bar{\delta} + \int_{\delta}^{b_1} (\bar{\delta} q_1(\bar{\delta}) - u_1(\bar{\delta})) \frac{\partial}{\partial q_1} g(u_1(\bar{\delta}), \bar{\delta}) \, d\bar{\delta}.
\]

Notice that in Myerson’s setting, we have \( g \) sufficiently small.

(iii) Consider \( \bar{V}(\delta) \) to be as in Figure 18. Then, condition 3(d) is equivalent to \( \int_0^{\delta'} \bar{V}(\delta) \, d\delta = k \geq 0 \) for all \( x \in [\delta', \delta''] \). Recall that in Myerson’s optimal mechanism, the transition of \( q \) from 0 to 1 occurs at some \( \delta \) that is a rising zero. Condition 2(d) thus ensures among the rising zeros, no \( \delta \) other than \( \delta'' \) can be the point where \( q_1 \) transits from 0.

(ii) Consider \( \bar{V}(\delta) \) to be as in Figure 18. Then, condition 3(d) is equivalent to \( \int_{a_0}^{\delta'} \bar{V}(\delta) \, d\delta \leq 0 \). This condition ensures that \( a_0 \) (and other rising zeros) cannot be the transition point.

(iii) Consider \( \bar{V}(\delta) \) to be as in Figure 19. Then, condition 4(d) is equivalent to \( \int_{\delta'}^{\delta''} \bar{V}(\delta) \, d\delta \geq 0 \) for \( x = \{a_0, a_1, a_2\} \). This condition ensures that among the rising zeros, no \( \delta \) other than \( \delta' \) can be the point where \( q_1 \) transits to 1.

---

2 A point \( x \) is called a rising zero, if (i) \( \bar{V}(x) = 0 \), and (ii) \( \bar{V}(x + h) > 0 \) for all \( h > 0 \) sufficiently small.
The similarities between the ironing conditions of $\phi$ and $\bar{V}$ can be observed from Figures 17, 18, and 19. The key difference between $\phi$ and $\bar{V}$ is that the former depends only on $f$, whereas the latter depends on $u_1(\delta)$ in addition, which is known only when the optimal menu is known. So the computation of $\bar{V}$ requires the knowledge of the menu itself. However, given a mechanism, we can use the theorem to determine if the mechanism is optimal or not.

We now simplify the computation of the marginal profit function. We define virtual valuation function $V : [-b_2, b_1] \rightarrow \mathbb{R}$ as $V(\delta) := \bar{\mu}(\{ z : z_1 - z_2 \geq \delta \} \setminus Z)$ where $\bar{\mu} = \mu + \mu_s + \mu_p$, with $\mu$, $\mu_s$, and $\mu_p$ defined as in (2). We then have $\bar{\mu}(D) = 0$ (see (3)). The following lemma shows that $V$ is equal to the marginal profit function $\bar{V}$.

**Lemma 11.** Let the allocation function $q$ be such that there exists a $u : D \rightarrow \mathbb{R}$ with $\nabla u = q$. Then, the functions $V$ and $\bar{V}$ are one and the same.

**Proof.** See Appendix B.

Observe that the virtual valuation function $V$ can be computed if the region of no allocation $Z$ is known. In the rest of the paper, we propose menus for all possible values of $(c, b_1, b_2) \geq 0$, and then prove that they are indeed the optimal mechanisms, using Theorem 10.

### 3.1. Optimal Mechanisms for the Uniform Distribution on a Rectangle

Without loss of generality, we assume $b_1 \geq b_2$. The following theorem asserts that the optimal mechanism falls within one of the menus depicted in Figures 2–8.

**Theorem 12.** Consider $z \sim \text{Unif } [c, c + b_1] \times [c, c + b_2]$. The optimal menu in the unit demand setting is described as follows.
Remark 1. The starred portions in the theorem statement indicate that we have used Mathematica to verify certain inequalities in proving those parts.

Remark 2. The values of \( \alpha_1 \) fall in the interval \([b_2, tb_2]\), where \( t = (37 + 3\sqrt{465})/176 \approx 1.733379 \). Similarly, the values of \( \alpha_2 \in [kb_2, tb_2] \) where \( k \geq 1 \) is the root of \( 32k^3 - 54k^2 + 19 = 0 \) \((k \approx 1.37214)\), and the values of \( \beta \in [tb_2, 2b_2] \). See Figure 1

The following is a pictorial representation of the results in Theorem 12. It depicts the regions in \((c, b_1, b_2)\) space at which each of the menus I, II, III, IV, IV', V and V' turns out to be optimal.
Figure 20: When \((c, b_1, b_2)\) falls in the shaded region in the left, the optimal menu is Menu I depicted in the right.

Figure 21: When \((c, b_1, b_2)\) falls in the shaded region in the left, the optimal menu is Menu II depicted in the right.
Figure 22: When \((c, b_1, b_2)\) falls in the shaded region in the left, the optimal menu is Menu III depicted in the right.

**Remark 3.** The menus IV and IV’ (given below) differ only in that the line separating the regions with allocations \((1-a, a)\) and \((1,0)\) falls to the right of the line \(z_1 - z_2 = b_1 - b_2\) in the former, and to the left of it in the latter. The menus meet at \(b_1 = 3b_2/2\) when the line of separation exactly falls at \(z_1 - z_2 = b_1 - b_2\).

Figure 23: When \((c, b_1, b_2)\) falls in the shaded region in the left, the optimal menu is Menu IV depicted in the right.
Remark 4. Observe that the menus II, III, IV, and IV’ meet at $b_1 = \frac{3}{2}b_2$, $c = tb_2$. The menus meet because at this $(c,b_1,b_2)$, the parameter $h$ (in Menus II and III) turn 0, and $\delta^* = \delta_1 = b_1/2 - b_2/4 = b_1/3 = b_1 - b_2$.

Remark 5. The menus V and V’ (given below) differ only in that the line separating the regions with allocations $(0,1)$ and $(1,0)$ falls to the right of the line $z_1 - z_2 = b_1 - b_2$ in the former, and to the left of it in the latter. The menus meet at $b_1 = \frac{3}{2}b_2$ when the line of separation exactly falls at $z_1 - z_2 = b_1 - b_2$.

Figure 24: When $(c,b_1,b_2)$ falls in the shaded region in the left, the optimal menu is Menu IV’ depicted in the right.

Figure 25: When $(c,b_1,b_2)$ falls in the shaded region in the left, the optimal menu is Menu V depicted in the right.
Remark 6. The menus IV, IV’, V, and V’ meet at \( b_1 = 3b_2/2, c = (243/38)b_2 \). The menus meet because at this \((c, b_1, b_2)\), the parameter \( a \) (in menus IV and IV’) turn 0, and \( b_1/2 - b_2/4 = b_1/3 = b_1 - b_2 \).

For a summarizing phase diagram see Figure 1. To see a portrayal of all possible menus, see Figures 2–8.

We now proceed to prove Theorem \( \text{12} \). We consider every menu separately, and go through the following steps in order to prove that the menu is optimal:

Step 1: We compute the virtual valuation function \( V(\delta) \) for every \( \delta \in [-b_2, b_1] \).

Step 2: We find the relation between the parameters – \((\delta_1, \delta_2, \delta^*, h, a_1, a_2)\) – using the equality conditions in Theorem \( \text{10} \).

Step 3: We evaluate bounds for the parameters and prove that the solution is meaningful.

Step 4: We verify if all the inequality conditions of Theorem \( \text{10} \) hold.

We now proceed to prove parts 1(a) and 2(a) of Theorem \( \text{12} \).

Theorem 13. Let \( c \in [0, b_2] \). Then the optimal menu is Menu I as depicted in Figure 20. The values of \( \delta_1 \) and \( \delta_2 \) are computed by solving the following equations simultaneously.

\[
-3\delta_1\delta_2 - c(\delta_1 + \delta_2) + b_1b_2 = 0. \tag{16}
\]

\[
-\frac{3}{2}\delta_2^2 + 2b_2\delta_2 - \frac{b_2^2}{2} + (c - 2b_2 + 3\delta_2)\delta_1 = 0. \tag{17}
\]

Proof. Step 1: We compute the virtual valuation function for Menu I depicted in Figure 20. Since \( \mu(D) = 0 \), we compute \( V \) using the formula

\[
V(\delta) = -\bar{\mu}(\{z : z_1 - z_2 < \delta\} \cup Z). \tag{18}
\]
When $\delta$ for every $V(2)$ when $c$ line between the points $(c - 2b_2 + 3\delta_2)(\delta + \delta_2) \delta \in [-\delta_2, \delta^*]$

\[
V(\delta) = \begin{cases} 
\mu(Z) + \frac{3}{2} \delta^2 + 2b_2\delta + \frac{b_2^2}{2} & \delta \in [-b_2, -\delta_2] \\
V(-\delta_2) - (c - 2b_2 + 3\delta_2)(\delta + \delta_2) & \delta \in [-\delta_2, \delta^*] \\
V(\delta^*) - (c - 2b_2)(\delta - \delta^*) + \frac{2}{3}(\delta_1 - \delta^*)^2 - \delta_2^2 & \delta \in [\delta^*, b'] \\
V'(b') - (c - 2b_1 + 3\delta_1)(\delta - b_1 + b_2) & \delta \in [b', \delta_1] \\
-\frac{3}{2} \delta^2 + 2b_1\delta - \frac{b_1^2}{2} & \delta \in [\delta_1, b_1] 
\end{cases}
\]  

(19)

where $b_1 - b_2$ is denoted as $b'$. For ease of notation, we drop the factor $\frac{1}{b_1b_2}$ in the rest of the paper.

**Step 2:** Menu I has three unknowns: $\delta^*$, $\delta_1$, and $\delta_2$. Observe that the line between the points $(c + b_2 + \delta^*, c + b_2)$ and $(c + \delta^*, c)$ passes through $(c + \delta_1, c + \delta_2)$. So we have $\delta^* = \delta_1 - \delta_2$.

We now proceed to compute $\delta_1$ and $\delta_2$. We do so by equating $\mu(Z) = 0$ and $V(\delta^*) = 0$. The latter follows from Theorem 10 because $q_1 = 0$ for $\delta \in [-b_2, \delta^*]$. We thus obtain equations (16) and (17).

**Step 3:** We show that there exists a $\delta_2 \in [\frac{b_2}{3}, \frac{2b_2-c}{3}]$ and $\delta_1 \in [\frac{b_1}{2} - \frac{b_2}{6}, \frac{2b_1-c}{3}]$, as a simultaneous solution to (16) and (17).

**Step 3a:** In this step, we show the bounds on $\delta_2$. Substituting $\delta_1 = \frac{b_1b_2-c\delta_2}{b_1b_2+c}$ from (16) in (17) and simplifying, we get

$$9\delta_1^2 + \delta_2^2(9c-12b_2) + \delta_2(2c^2 - 8b_2c + 3b_2^2 - 6b_1b_2) + b_2^2c - 2b_1b_2c + 4b_1b_2^2 = 0.$$  

(20)

When $\delta_2 = \frac{2b_2-c}{3}$, the left-hand side of (20) equals $-\frac{2}{3}b_2(b_2^2 - c^2) \leq 0$, and when $\delta_2 = \frac{b_2}{3}$, it equals $2b_2(b_1 - c/3)(b_2 - c) \geq 0$. From the continuity of the left-hand side of (20), we conclude that there exists a solution $\delta_2 \in [\frac{b_2}{3}, \frac{2b_2-c}{3}]$ for every $c \in [0, b_2]$.

**Step 3b:** In this step, we show the bounds on $\delta_1$. Substituting $\delta_2 = \frac{b_1b_2-c\delta_1}{b_1b_2+c}$ from (16) in (17) and simplifying, we have

$$-\frac{3}{2}b_1^2\delta_1^2 + 2b_1 b_2^2c - \frac{1}{2}b_2^2c^2 + (6b_1 b_2^2 + 6b_1 b_2 c - 3b_2^2c - 4b_2 c^2 + 3c^3)\delta_1$$

$$+ (9b_1 b_2^2 - \frac{9}{2}b_2^2 - 18b_2 c + \frac{3}{2}c^2)\delta_1^3 - 18b_2\delta_1^3 = 0.$$  

(21)

When $\delta_1 = \frac{2b_1-c}{3}$, the left-hand side of (21) equals $\frac{1}{6}(-8b_1^3b_2 + 3b_1^2b_2^2 + 4b_1^2c^2 + 2b_1b_2c^2 - c^4)$. We claim that this expression is negative for $b_1 \geq b_2$, $c \in [0, b_2]$. If $c \in [0, b_2]$, it is easy to see that the expression attains its maximum when $c = b_2$. At $c = b_2$, the expression equals $b_2(b_1 - b_2)(-8b_1^2 - b_1 - b_2^2)$ which clearly is nonpositive when $b_1 \geq b_2$. We have proved our claim.

Now when $\delta_1 = \frac{b_1}{2} - \frac{b_2}{1}$, the left-hand side of (21) equals $\frac{1}{6}(b_2 - c)(27b_1^2 b_2^2 - 18b_1 b_2^2 - b_2^2 + (42b_1 b_2 - 9b_1^2 - b_2^2)c + 4(b_2 - 3b_1)c^2)$. Observe that this expression is a quadratic in $c$, with the coefficient of $c^2$ being negative. So to prove that this expression is nonnegative for $c \in [0, b_2]$, it suffices
to prove that it is nonnegative at $c = 0$ and $c = b_2$. At $c = 0$, the expression equals $27b_2^2 - 18b_1b_2^2 - b_2^3 \geq 0$ for $b_1 \geq b_2$, and at $c = b_2$, it equals $18b_1b_2^2 + 12b_2^2 + 2b_2^3 \geq 0$. We have thus shown that there exists a solution $\delta_1 \in [\frac{b_2}{3}, -\frac{b_2}{3}, \frac{2b_2-c}{3}]$ for every $c \in [0, b_2]$.  

**Step 4:** We now proceed to prove parts (c) and (d) in Theorem 10 (2) and 11 (4). Observe that the proof is complete if we prove that $V(\delta) \leq 0$ when $\delta \in [-\delta_2, \delta^*]$, and $V(\delta) \geq 0$ when $\delta \in [\delta^*, b_1]$. We now compute $V'(\delta)$ for almost every $\delta \in [-\delta_2, b_1]$.

$$V'(\delta) = \begin{cases} 
3\delta + 2b_2 & \delta \in (-\delta_2, -\delta_1) \\
-(c - 2b_2 + 3\delta_2) & \delta \in (-\delta_2, \delta^*) \\
-(c - 2b_2) - 3(\delta_1 - \delta) & \delta \in [\delta^*, b_1 - b_2) \\
-(c - 2b_1 + 3\delta_1) & \delta \in (b_1 - b_2, \delta_1) \\
-3\delta + 2b_1 & \delta \in (\delta_1, b_1). 
\end{cases} \quad (22)$$

Observe that $V'(\delta)$ is negative when $\delta \in [-\delta_2, -\frac{2b_2}{3}]$, and positive when $\delta \in [-\frac{2b_2}{3}, \delta^*]$ (follows because $\delta_2 \leq \frac{2b_2-c}{3}$). We also have $V(-b_2) = V(\delta^*) = 0$. So $V(\delta) = V(-b_2) + \int_{-b_2}^{\delta} V'(\delta) \, d\delta \leq 0$ for all $\delta \in [-\delta_2, \delta^*]$, and hence $\int_{-b_2}^{\delta} V(\delta) \, d\delta \leq 0$, and $\int_{-b_2}^{\delta} V(\delta) \, d\delta \geq \int_{-b_2}^{\delta_1} V(\delta) \, d\delta$ for all $x \in [-\delta_2, \delta^*]$.

We now claim that $V'(\delta)$ is positive when $\delta \in [\delta^*, \frac{2b_2}{3}]$, and negative when $\delta \in [\frac{2b_2}{3}, b_1]$. Observe that $V'(\delta)$ is continuous at $\delta = \delta^*$, and that it increases in the interval $[\delta^*, b_1 - b_2]$. So $V'(\delta) \geq 0$ when $\delta \in [\delta^*, b_1 - b_2]$. Also, $V'(\delta) \geq 0$ when $\delta \in [b_1 - b_2, \delta_1]$ because $\delta_1 \leq \frac{2b_2-c}{3}$. That $V'(\delta)$ is positive when $\delta \in [\delta_1, \frac{2b_2}{3}]$, and negative when $\delta \in [\frac{2b_2}{3}, b_1]$ is obvious. We have proved our claim.

Since we also have $V(\delta^*) = V(b_1) = 0$, it follows that $V(\delta) = V(\delta^*) + \int_{\delta^*}^{\delta} V'(\delta) \, d\delta \geq 0$ for all $\delta \in [\delta^*, b_1]$. So we have $\int_{\delta^*}^{b_1} V(\delta) \, d\delta \geq 0$ and $\int_{-b_2}^{\delta} V(\delta) \, d\delta \leq \int_{\delta_1}^{b_1} V(\delta) \, d\delta$ for all $x \in [\delta^*, b_1]$. \qed

With the above theorem, we have completely solved the $c \leq b_2$ case. We now analyze the case at which the transition occurs. At $c = b_2$, when we solve (16) and (17) simultaneously, we obtain $\delta_2 = \frac{b_2}{3} = \frac{2b_2-c}{3}$ and $\delta_1 = \frac{b_2}{2} - \frac{b_2}{6}$. When $c > b_2$, the left-hand side of (20) still continues to change sign at $\delta_2 = \frac{b_2}{3}$ and $\delta_2 = \frac{2b_2-c}{3}$, but since $\frac{b_2}{3} > \frac{2b_2-c}{3}$, the solution $\delta_2$ now belongs to the interval $[\frac{2b_2-c}{3}, \frac{b_2}{3}]$. We thus have (i) $V(-\frac{b_2}{3}) = 0 = V(\delta^*)$, and (ii) $V'(\delta) \geq 0$ when $\delta \in [-\frac{2b_2}{3}, -\delta_2]$ and $V'(\delta) \leq 0$ when $\delta \in [-\delta_2, \delta^*]$. These both imply that $V(\delta) \geq 0$ when $\delta \in [-\frac{b_2}{3}, \delta^*]$. So the minimum of $\int_{-b_2}^{\delta} V(\delta) \, d\delta$ can never occur at $x = \delta^*$, causing the condition in part (d) of Theorem 11 (2) to fail.

At $c = b_2$, a transition occurs from Menu I to Menu II. We now proceed to prove the optimality of Menu II, i.e., parts 1(b) and 2(b) in Theorem 12.
\textbf{Theorem 14.} Let \( c \in [b_2, \beta] \) if \( b_1 \geq 3b_2/2 \) and let \( c \in [b_2, \alpha_1] \) if \( b_1 \in [b_2, 3b_2/2] \) with \( \alpha_1 \) and \( \beta \) as defined in Theorem 12. Then, the optimal menu is Menu II as depicted in Figure 21. The values of \( h \) and \( \delta^* \) are obtained by solving (10) and (11) simultaneously, and the values of \( (\delta_1, \delta_2) \) are given by

\[ (\delta_1, \delta_2) = \left( h + \delta^*, \frac{b_1b_2 - (3h/2 + c)(h + \delta^*)}{3/2(h + \delta^*) + c} \right). \]

The probability of allocation \( a_2 \) is given by \( a_2 = \frac{b + \delta^*}{\delta_2 + \delta^*}. \)

\textbf{Proof.} \textbf{Step 1:} We compute the virtual valuation function for Menu II depicted in Figure 21.

\[ V(\delta) = \begin{cases} V(-\delta_2) - (c - 2b_2 + 3\delta_2)(\delta + \delta_2) + \frac{2}{2}(\frac{h - b}{\delta_2 + \delta})^2 & \delta \in [-\delta_2, \delta^*] \\ V(\delta^*) - (c - 2b_2)(\delta - \delta^*) + \frac{2}{2}((\delta_1 - \delta)^2 - h^2) & \delta \in [\delta^*, b'] \end{cases} \]

where \( b_1 - b_2 \) is denoted by \( b' \). The expression for \( V(\delta) \) when \( \delta \in [-\delta_2, -\delta_2] \cup [b_1 - b_2, b_1] \) remains the same as in (15).

\textbf{Step 2:} Menu II has five parameters: \( h, \delta^*, \delta_1, \delta_2, \) and \( a_2 \). Observe that the \( 45^\circ \) line segment joining the points \((c + b_2 + \delta^*, c + b_2)\) and \((c + \delta^*, c)\) passes through \((c + \delta_1, c + h)\). So we have \( \delta_1 = h + \delta^* \). Since \( q = \nabla u \), a conservative field, we must have the slope of the line separating \((0, 0)\) and \((1 - a_2, a_2)\) allocation regions satisfying \(-1/a_2 = \frac{h - \delta^*}{h + \delta^*} \). This yields \( a_2 = \frac{h + \delta^*}{\delta_2 + \delta^*}. \)

We now proceed to compute \( h, \delta_2 \) and \( \delta^* \). We do so by equating \( \hat{\mu}(Z) = 0 \), \( V(\delta^*) = 0 \), and \( \int_{\delta_2}^{\delta^*} V(\delta) \, d\delta = 0 \). The latter two conditions follow from Theorem 10 3(b) and 3(c) because \( q_1(\delta) = 1 - a_2 \in (0, 1) \) for \( \delta \in [-\delta_2, \delta^*] \).

We then have the following implications.

\[ \hat{\mu}(Z) = 0 \Rightarrow \frac{3}{2}(h + \delta^*)(h + \delta_2) - c(\delta_2 + h + \delta^*) + b_1b_2 = 0. \]  \hfill (23)

From (18), we see that \( V(\delta^*) \) is the negative of \( \hat{\mu} \) measure of the nonconvex pentagon bound by \((c, c), (c + b_2), (c + b_2 + \delta^*, c + b_2), (c + \delta_1, c + h)\), and \((c + \delta_1, c)\). Thus

\[ V(\delta^*) = 0 \Rightarrow \frac{3}{2}h^2 - ch - \frac{3}{2}b_2(2b_2 + 2\delta^*) + b_2(2b_2 + \delta^*) + b_1b_2 = 0 \]  \hfill (24)

\[ \Rightarrow h = \frac{-c + \sqrt{c^2 + 3b_2(2b_1 - b_2 - 4\delta^*)}}{3}. \]  \hfill (25)

Next,

\[ \int_{\delta_2}^{\delta^*} V(\delta) \, d\delta = 0 \Rightarrow \int_{-\delta_2}^{\delta_2} V(\delta) \, d\delta + \int_{-\delta_2}^{\delta^*} V(\delta) \, d\delta = 0 \]

\[ \Rightarrow b_2(\delta_2^2 - b_2^2/9) + \frac{1}{2}(b_2^2/27 - \delta_2^2) + b_2^2/2(b_2/3 - \delta_2) \]

\[- (2b_2\delta_2 - 3\delta_2^2/2 - b_2^2/2)(\delta^* + \delta_2) - (c - 2b_2 + 2\delta_2 + h)(\delta^* + \delta_2)^2/2 = 0 \]

\[ \Rightarrow \frac{1}{54}(4b_2 + 3\delta^*)(b_2 + 3\delta^*)^2 - \frac{(c + h + \delta^*)}{2}(\delta^* + \delta_2)^2 = 0. \]  \hfill (26)
The values of \( h, \delta^* \), and \( \delta_2 \) can be obtained by solving (23), (24), and (26) simultaneously. We now proceed to prove that \((\delta, \delta^*)\) can be obtained by solving \((10)\) and \((11)\) simultaneously. From (24), we get

\[
3h^2/2 + ch + 2b_2\delta^* - b_1b_2 + b_2^2/2 = 0
\]

which is \((10)\). We next find an expression for \(\delta_2 + \delta^*\). Rearranging (23), we get

\[
\delta_2 = \frac{b_1b_2 - (3h/2 + c)(h + \delta^*)}{3/2(h + \delta^*) + c} = \frac{2b_2\delta^* + b_2^2/2 - \delta^*(3h/2 + c)}{3/2(h + \delta^*) + c}
\]

where we have used (27). Thus

\[
\delta_2 + \delta^* = \frac{(b_2 + 3\delta^*)(b_2 + \delta^*)/2}{3/2(h + \delta^*) + c}.
\]

Plugging this into (26), we eliminate \(\delta_2\), and obtain

\[
27(c + h + \delta^*)(b_2 + \delta^*)^2 - 4(4b_2 + 3\delta^*)(3(h + \delta^*)/2 + c)^2 = 0
\]

which is \((11)\). It is thus clear that \((h, \delta^*)\) can be obtained by simultaneously solving \((10)\) and \((11)\).

**Step 3:** We now bound the parameters \( h, \delta^* \), and \(\delta_2\). In Step 3a, we prove the bounds on \((h, \delta^*)\) when \(b_1 \geq 3b_2/2\). In Step 3b, we prove the bounds on \((h, \delta^*)\) when \(b_1 \in [b_2, 3b_2/2]\). In Step 3c, we prove the bounds on \(\delta_2\) for all \(b_1\).

**Step 3a:** Consider the case when \(b_1 \geq 3b_2/2\). We claim that there exists a \(\delta^* \in \left[\frac{c^2 + 6b_1b_2 - 7b_2^2}{12b_2}, \frac{b_1}{2} - \frac{b_2}{4}\right]\) and a \(h \in [0, \frac{2b_2 - c}{3}]\) that solves (24) and (29) simultaneously. Observe that \(h\) is a decreasing function of \(\delta^*\) (see (25)), and that the pairs \((\delta^*, h) = \left(\frac{c^2 + 6b_1b_2 - 7b_2^2}{12b_2}, \frac{2b_2 - c}{3}\right)\) and \((\delta^*, h) = \left(\frac{b_1}{2} - \frac{b_2}{4}, 0\right)\) satisfy (24). The choice \(h = \frac{2b_2 - c}{3}\) will be motivated later. It suffices now to indicate that it is to satisfy condition 3(d) of Theorem 10. We now prove that the left-hand side of (11) has opposite signs at these pairs of \((\delta^*, h)\).

Substituting \((\delta^*, h) = \left(\frac{c^2 + 6b_1b_2 - 7b_2^2}{12b_2}, \frac{2b_2 - c}{3}\right)\), we obtain

\[
\frac{(c - b_2)(6b_1b_2^2 + b_2^3 + 6b_1b_2c + 9b_2^2c + 2b_2c^2 + c^3)}{4b_2} \leq 0
\]

for every \(c \geq b_2\). Substituting \((\delta^*, h) = \left(\frac{b_1}{2} - \frac{b_2}{4}, 0\right)\), we obtain

\[
\frac{1}{16}(72b_2^2b_2 + 144b_1b_2^2 - 90b_3^2 + (-36b_2^2 + 84b_1b_2 + 399b_2^2)c - (96b_1 + 208b_2)c^2)
\]

which is nonnegative for every \(c \in [0, \beta]\). We have thus proved our claim.
Step 3b: Consider the case when \( b_1 \in [b_2, 3b_2/2] \). We claim that there exists a \( \delta^* \in \left[ \frac{c^2+6b_1b_2-7b_2^2}{12b_2}, b_1 - b_2 \right] \) and \( h \in \left[ -c+\sqrt{c^2+3b_2(3b_2-2b_1)}, \frac{2b_2-c}{3} \right] \) simultaneously solving (27) and (29). As before, substitution of \((\delta^*, h) = \left( \frac{c^2+6b_1b_2-7b_2^2}{12b_2}, 2b_2-c \right)\) yields (30). We now substitute the other pair of \((\delta^*, h)\) in the left-hand side of (29), and obtain

\[
9b_1^2 \left( 3b_1 - 3b_2 + 2c + \sqrt{9b_2^2 - 6b_1b_2 + c^2} \right) - (3b_1 + b_2) \left( 3b_1 - 3b_2 + c + \sqrt{9b_2^2 - 6b_1b_2 + c^2} \right)^2 \quad (31)
\]

We now show that this expression is nonnegative for every \( b_1 \in [b_2, 3b_2/2] \), \( c \in [b_2, \alpha_1] \). We do so by the following steps: (a) We first differentiate the expression with respect to \( c \) and show that the differential is nonpositive; (b) We then evaluate the expression at \( c = 2(t-1)(b_1 - b_2) + b_2 \) (recall from Remark 2 that \( t = 3(37 + 3\sqrt{465})/176 \)) and show that it is nonnegative; and (c) We finally show that \( \alpha_1 \leq 2(t-1)(b_1 - b_2) + b_2 \).

Fix \( v = \sqrt{9b_2^2 - 6b_1b_2 + c^2} \). When \( b_1 \in [b_2, 3b_2/2] \) and \( c \geq b_2 \), we have

(i) \( v = \sqrt{9b_2^2 - 6b_1b_2 + c^2} \geq \sqrt{9b_2^2 - 6(3b_2/2)b_2 + c^2} = c \),

(ii) \( v = \sqrt{9b_2^2 - 6b_1b_2 + c^2} \leq \sqrt{9b_2^2 - 6(b_2)b_2 + c^2} = \sqrt{3b_2^2 + c^2} \leq 2c \).

So we have \( c \leq v \leq 2c \). Differentiating (31) with respect to \( c \), we have

\[
18b_1^2 + \frac{9b_1^2c}{v} - 2(3b_1 + b_2)(-3b_2 + 3b_1 + c + v)(1 + c/v)
= \frac{18b_1^2 v + 9b_1^2 c - 2(3b_1 + b_2)(c + v)^2 - (18b_1^2 + 2(-6b_1b_2 - 3b_2^2)(c + v))}{v}
= \frac{-9b_1^2 c + 2(c + v)(3b_2(2b_1 + b_2) - (3b_1 + b_2)(c + v))}{v}
= \frac{-9b_1^2 c + 2(c + v)(2b_1 + b_2)(2b_2 - c - v) + b_2(2b_1 + b_2 - b_1)(c + v))}{v}
\leq \frac{-9b_1^2 c + 2(c + v)b_2^2}{v} \leq \frac{-6b_1^2 c + 6c b_2^2}{v} \leq 0
\]

where the first inequality follows from \( c + v \geq 2c \geq 2b_2 \), the second inequality from \( c + v \leq 3c \), and the third inequality from \( b_2 \leq b_1 \).

We now evaluate the expression at \( c = 2(t-1)(b_1 - b_2) + b_2 \). Substituting
\[ c = 2(t - 1)(b_1 - b_2) + b_2 \] in (31), we now verify if
\[
\frac{15(117\sqrt{465} - 4189)b_1^2 + 13(13417 - 225\sqrt{465})b_1^2b_2}{1936} \\
- \frac{(70269 - 981\sqrt{465})b_1b_2^2 + 9(5021 - 21\sqrt{465})b_2^3}{1936} \\
+ \left( \frac{-201 + 27\sqrt{465})b_1^2 + (134 + 18\sqrt{465})b_1b_2 + (111 + 9\sqrt{465})b_2^3}{44} \right)
\]
\[
\sqrt{9b_2^2 - 6b_1b_2 + \left( \frac{3(37 + 3\sqrt{465})}{176} - 1 \right)(b_1 - b_2) + b_2} \geq 0
\]

Writing the above expression as \( X + Y\sqrt{Z} \), we note that (i) \( X \leq 0 \) when \( b_1 \in [1, 1.03873] \), and \( X \geq 0 \) when \( b_1 \in [1.03873, 1.5] \); (ii) \( Y \geq 0 \) when \( b_1 \in [1, 1.04088] \), and \( Y \leq 0 \) when \( b_1 \in [1.04088, 1.5] \). So we now verify if \( X^2 - Y^2Z \leq 0 \) when \( b_1 \in [1, 1.03873] \), and if \( X^2 - Y^2Z \geq 0 \) when \( b_1 \in [1.04088, 1.5] \). That \( X + Y\sqrt{Z} \geq 0 \) when \( b_1 \in [1.03873, 1.04088] \) is clear since both \( X \) and \( Y \) are positive in that interval. Evaluating \( X^2 - Y^2Z \), we have
\[
\frac{9}{42592}(b_1 - b_2)(3b_2 - 2b_1)((20196\sqrt{465} - 447876)b_2^4 \\
+ (108900\sqrt{465} - 2234628)b_1b_2^3 + (32337\sqrt{465} - 952857)b_1^2b_2^2 \\
+ (4841141 - 276237\sqrt{465})b_1^3b_2 + (140940\sqrt{465} - 1820460)b_1^4)
\]
which is negative in the interval \( b_1 \in [1, 1.03977] \) and positive when \( b_1 \in [1.03977, 1.5] \). We have thus shown that the expression in (14) is nonnegative when \( b_1 \in [b_2, 3b_2/2], b_2 \leq c \leq 2(t - 1)(b_1 - b_2) + b_2 \). That \( \alpha_1 \leq 2(t - 1)(b_1 - b_2) + b_2 \) is shown via Mathematica (see Appendix C.1).4)

Step 3c: For both the cases, we now claim that \( \delta_2 \in [\frac{2b_2 - c}{3}, \frac{b_2}{3}] \). We first show that \( \delta_1 \), as a function of \( \delta^* \), decreases with increase in \( \delta^* \). Differentiating the expression for \( \delta_1 = (h + \delta^*) \) with \( h \) as in (25), we get
\[ 1 - \frac{2b_2}{c^2 + 3b_2(2b_1 - b_2 - 4b^*)} \]
which is nonpositive for \( \delta^* \geq \frac{c^2 + 6b_1b_2 - 7b_2^2}{12b_2} \). But this is exactly the lower bound that we computed for \( \delta^* \). The highest value of \( \delta_1 \) thus occurs at \( (h, \delta^*) = \left( \frac{2b_2 - c}{3}, \frac{c^2 + 6b_1b_2 - 7b_2^2}{12b_2} \right) \). Using these expressions, we get \( \delta_1 = (h + \delta^*) \leq \frac{c^2 + 6b_1b_2 + b_2^2 - 4b_2c}{12b_2} \). We now have from (28).
that
\[
\delta_2 = \frac{b_1b_2 - (3h/2 + c)(h + \delta^*)}{3(h + \delta^*)/2 + c}
\geq \frac{b_1b_2 - (b_2 + c/2)(c^2 + 6b_1b_2 + b_2^2 - 4b_2c)/(12b_2)}{(c^2 + 6b_1b_2 + b_2^2 + 4b_2c)/(8b_2)}
= \frac{2b_2 - c}{3} + \frac{4b_2(c^2 - b_2^2)}{3(c^2 + 6b_1b_2 + b_2^2 + 4b_2c)}
\geq \frac{2b_2 - c}{3}
\]

where the first inequality occurs from the upper bound \( h \leq (2b_2 - c)/3 \) and the above upper bound on \((h + \delta^*)\), and the second inequality from \( c \geq b_2 \). We have thus shown the lower bound. We have also shown that the probability of allocation \( a_2 = \frac{h + \delta^*}{\delta_2 + \delta^*} \leq 1 \), since \( \delta_2 \geq \frac{2b_2 - c}{3} \geq h \). The upper bound \( \delta_2 \leq \frac{b_2}{3} \) is shown via Mathematica (see Appendix C.3.2).

**Step 4:** We now proceed to prove parts (c) and (d) of Theorem 10 (2)–(4). The expression for \( V'(\delta) \) is the same as in the proof of Theorem 10 except in \([-\delta_2, \delta^*] \), where it is given by
\[
V'(\delta) = -(c - 2b_2 + 3\delta_2) + 3\frac{\delta_2 - h}{\delta_2 + \delta^*}(\delta + \delta_2), \forall \delta \in [-\delta_2, \delta^*].
\] (32)

From (22), observe that \( V'(\delta) \) is negative when \( \delta \in [-b_2, -\frac{b_2 - c}{3}] \) and positive when \( \delta \in [-\frac{2b_2}{3}, -\frac{b_2}{3}] \). We also have from (19) that \( V(-b_2) = V(-\frac{b_2}{3}) = 0 \). So \( V(\delta) = V(-b_2) + \int_{-b_2}^{\delta} V'(\delta) \, d\delta \leq 0 \) for all \( \delta \in [-b_2, -\frac{b_2}{3}] \). It follows that \( \int_{-b_2}^{x} V(\delta) \, d\delta \leq 0 \), and that \( \int_{-b_2}^{x} V(\delta) \, d\delta \geq \int_{-b_2}^{x} V(\delta) \, d\delta \) for all \( x \in [-b_2, -\frac{b_2}{3}] \). Thus condition (2) of Theorem 10 is verified.

We now prove that \( \int_{-b_2}^{x} V(\delta) \, d\delta \geq 0 \) for every \( x \in [\frac{b_2}{3}, \delta^*] \). Observe that \( V(\delta) \) is positive when \( \delta \in [-\frac{b_2}{3}, -\delta_2] \), negative when \( \delta \in [-\delta_2, b_2] \) for some \( b_2 \in [-\delta_2, \delta^*] \), and positive when \( \delta \in [b_2, \delta^*] \). These statements follow from (i) \( \delta_2 \geq \frac{2b_2 - c}{3}, \) (ii) \( V'(\delta) \) increasing in the interval \([-\delta_2, \delta^*] \), and (iii) \( h \leq \frac{2b_2 - c}{3} \), all of which can be obtained from (32). We also have \( V(-\frac{b_2}{3}) = V(\delta^*) = \int_{-\frac{b_2}{3}}^{\delta^*} V(\delta) \, d\delta = 0 \), which we used to derive the parameters \( h, \delta_2, \) and \( \delta^* \). It follows that \( \int_{-b_2}^{x} V(\delta) \, d\delta \geq 0 \) for all \( x \in [-\frac{b_2}{3}, \delta^*] \). Thus condition (3) of Theorem 10 is verified.

The proof that the conditions of Theorem 10 (4) are satisfied trace the same steps as in the proof of Theorem 13 provided \( \delta_1 \leq \frac{2b_1 - c}{3} \). If \( \delta_1 > \frac{2b_1 - c}{3} \), then \( V'(\delta) \) is no more positive in the interval \([b_1 - b_2, \delta_1] \). We consider two cases.

Let \( b_1 \geq 3b_2/2 \). Then we claim that \( V(\delta) \geq 0 \) holds for all \( \delta \in [\delta^*, b_1] \), even when \( V'(\delta) \leq 0 \) for \( \delta \in [b_1 - b_2, \delta_1] \). Observe that (i) \( V(\delta) = \frac{1}{3}(3\delta -
\(b_1(b_1 - \delta) \geq 0\) for all \(\delta \in [\max(b_1 - b_2, \frac{b_1}{3}), b_1]\), and (ii) \(\delta_1 \geq b_1 - b_2 \geq \frac{b_1}{3}\), when \(b_1 \geq 3b_2/2\). So, \(V(\delta) \geq 0\) for all \(\delta \in [b_1 - b_2, b_1]\). Now \(V(\delta) \geq 0\) also holds in the interval \(\delta \in [\delta^*, b_1 - b_2]\) since \(V'(\delta) \geq 0\) in that interval (see the discussion following (22)), and since \(V(\delta^*) = 0\). We have proved our claim.

We now consider the case when \(b_1 \in [b_2, 3b_2/2]\). \(V(\delta)\) could possibly be negative at some values of \(\delta\). We now evaluate \(\int_{\delta^*}^{b_1} V(\delta) \, d\delta\):

\[
\int_{\delta^*}^{b_1} V(\delta) \, d\delta = \int_{b_1 - b_2}^{\delta_1} V(\delta) \, d\delta + \int_{b_1 - b_2}^{\delta_1} V(\delta) \, d\delta
\]

\[
= -\frac{2}{27} b_1^3 - b_2(\delta^*)^2 + b_2\delta^* (b_1 - b_2/2) + b_1 b_2 h - \frac{b_2 h}{2} - 2b_2 h \delta^* - \frac{ch^2}{2} - h^3
\]

\[
= -\frac{2}{27} b_1^3 + \frac{(c + h)}{2} h^2 - b_2(\delta^*)^2 + b_2 \delta^* (b_1 - b_2/2) + h V(\delta^*)
\]

where \(V(\delta^*)\) is obtained from (24). The last expression is the same as (12), since \(V(\delta^*) = 0\). From Mathematica, (12) is nonnegative for all \(c \in [b_2, \alpha_1]\) (see Appendix C.1(3)). Since \(\int_{b_1}^{b_1} V(\delta) \, d\delta = \frac{2}{27} b_1^3 \geq 0\), we have \(\int_{\delta^*}^{b_1} V(\delta) \, d\delta \geq 0\). This verifies condition 4(c) of Theorem 10.

Observe that \(V'(\delta) \leq 0\) only when \(\delta \in [b_1 - b_2, \delta_1]\). Also, \(V(\delta^*) = 0 = V(\frac{b_1}{3})\). So \(V(\delta)\) can be negative only when \(\delta\) is in some subset of \([b_1 - b_2, \frac{b_1}{3}]\), say in the interval \([\delta^*, \frac{b_1}{3}]\). Observe that \(V(\delta)\) appears as in Figure 19, with rising zeros occurring only when \(\delta \in \{\delta^*, \frac{b_1}{3}\}\). The integral \(\int_{\delta^*}^{b_1} V(\delta) \, d\delta\) thus attains its maximum either at \(\delta^*\) or at \(\frac{b_1}{3}\). But we just evaluated \(\int_{\delta^*}^{b_1} V(\delta) \, d\delta \geq 0\), and so the maximum cannot be at \(x = \frac{b_1}{3}\). Thus we have \(\int_{\delta^*}^{b_1} V(\delta) \, d\delta \leq \int_{\delta^*}^{b_1} V(\delta) \, d\delta\) for all \(x \in [\delta^*, b_1]\). Hence the result. \(\square\)

Observe that at \(c = \alpha_1\), we have \(\int_{\delta^*}^{b_1} V(\delta) \, d\delta = 0\). When \(c > \alpha_1\), the quantity turns negative, causing the condition in Theorem 10 4(d) to fail. A transition occurs from Menu II to Menu III. We now proceed to prove the optimality of Menu III, i.e., part 1(c) of Theorem 12.

**Theorem 15.** Consider the case when \(b_1 \in [b_2, 3b_2/2]\), and \(c \in [\alpha_1, \alpha_2]\), where \(\alpha_1\) and \(\alpha_2\) are as defined as in Theorem 12. Then the optimal menu is Menu III as depicted in Figure 22. The values of \(h\) and \(\delta^*\) are found by solving (17) and (18) simultaneously, and the values of \(\delta_1\) and \(\delta_2\) are given by

\[
(\delta_1, \delta_2) = \left(\delta^* + \frac{b_1 b_2 - 2 b_2 \delta^* - b_2^2/2}{3h/2 + c}, \frac{b_1 b_2 - (3h/2 + c) \delta_1}{3/2(h + \delta^*) + c}\right).
\]

The values of \(a_1\) and \(a_2\) are given by \((a_1, a_2) = \left(\frac{h}{\delta_1 + \delta^*}, \frac{h + \delta^*}{\delta_2 + \delta^*}\right)\).

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Theorem 16. (i) Consider the case when \( c \in [\beta, \frac{216b_2^2}{108b_2 - 108b_1 b_2 - 5b_2^2}] \), and \( b_1 \geq 3b_2/2 \), where \( \beta \) is as defined in Theorem 12. Then the optimal menu is Menu IV’ as depicted in Figure 24. The values of \( \delta_1 \) and \( \delta_2 \) are computed by solving the following equations simultaneously.

\[
-\frac{3}{2}\delta_1\delta_2 - c(\delta_1 + \delta_2) + b_1 b_2 = 0.
\]

\[
-\frac{2}{27}b_2^3 + \frac{1}{2}\delta_1\delta_2(\delta_2 - \delta_1) + \frac{c}{2}(\delta_2^2 - \delta_1^2) + \frac{1}{16}b_2(2b_1 - b_2)^2 = 0.
\]

The value of \( a \) is given by \( a = \frac{\delta_1}{\delta_1 + \delta_2} \). If \( c \geq \frac{216b_2^2}{108b_2 - 108b_1 b_2 - 5b_2^2} \), then the optimal menu is Menu V’ as depicted in Figure 24.

(ii) Consider the case when \( b_1 \in [b_2, 3b_2/2] \), and \( c \in [\alpha_2, \frac{27b_2^2}{4(b_1 - b_2)}] \), where \( \alpha_2 \) is as defined in Theorem 12. Then the optimal menu is Menu IV as depicted in Figure 23. The values of \( \delta_1 \) and \( \delta_2 \) are computed by solving the following equations simultaneously.

\[
-\frac{3}{2}\delta_1\delta_2 - c(\delta_1 + \delta_2) + b_1 b_2 = 0.
\]

\[
\frac{2}{27}(b_1^2 - b_2^2) + \frac{1}{2}\delta_1\delta_2(\delta_2 - \delta_1) + \frac{c}{2}(\delta_2^2 - \delta_1^2) = 0.
\]

The value of \( a \) is given by \( a = \frac{\delta_1}{\delta_1 + \delta_2} \). If \( c \geq \frac{27b_2^2}{4(b_1 - b_2)} \), then the optimal menu is Menu V as depicted in Figure 23.

Proof. See Appendix B. This too relies on Mathematica for verification of certain inequalities.

Consider \( b_1 \in [b_2, 3b_2/2] \). The proof (in Appendix B) indicates that at \( c = \alpha_2 \), we have \( a_1 + a_2 = 1 \), and that when \( c > \alpha_2 \), we have \( a_1 + a_2 < 1 \). This causes the monotonicity of \( q_1 \) to fail (recall that \( q_1 \) increasing is one of the constraints of Problem (9)). Further, when \( a_1 + a_2 = 1 \), the slope of the line segment joining \((c,c + \delta_2)\), \((c + h + \delta_2, c + h)\), and the slope of the line segment joining \((c + h + \delta_2, c + h)\), \((c + \delta_1, c)\), are equal, i.e., \( -\frac{1-a_2}{a_2} = -\frac{a_1}{1-a_1} \). The two line segments thus turn into a single line segment that joins \((c,c + \delta_2)\), \((c + \delta_1, c)\). A transition thus occurs from Menu III to Menu IV, with \( a_2 = 1 - a_1 = a \).

Consider \( b_1 \geq 3b_2/2 \). At \( c = \beta \), we have \( h = 0 \). Thus a transition occurs from Menu II to Menu IV’.

We now proceed to prove the optimality of the menus IV, IV’, V, V’, i.e., parts 1(d)–(e) and 2(c)–2(d) of Theorem 12.

Proof. See Appendix B
4. On Extending to Uniform Distributions on General Rectangles

We have computed the optimal mechanism in the two-item unit-demand setting when \( z \sim \text{Unif}[c, c + b_1] \times [c, c + b_2] \) for every nonnegative \((c, b_1, b_2)\). Our computation used the method based on the virtual valuation function designed in [22]. We can now ask if there is a generalization of this method for more general distributions, specifically for uniform distributions on rectangles \([c_1, c_1 + b_1] \times [c_2, c_2 + b_2]\), when \( c_1 \neq c_2 \). We conjecture that the optimal menus would be similar to the five menus as in the case of \( c_1 = c_2 \). We now report some promising preliminary results that support this conjecture.

**Theorem 17.** Consider the case when \( b_1 \geq b_2 \). Let \( c_2 \geq 0, c_1 \geq c_2, \) and \( 2c_1 - c_2 \leq b_2 \). Then, the optimal menu is Menu I as depicted in Figure [20]. The values of \( \delta_1 \) and \( \delta_2 \) are computed by solving the following equations simultaneously.

\[
-3\delta_1\delta_2 - c_2\delta_1 - c_1\delta_2 + b_1b_2 = 0.
\]

\[
-3\delta_2^2 + 2b_2\delta_2 - \frac{b_2^2}{2} - d(b_2 - \delta_2) + (c_2 - 2b_2 + 3\delta_2)\delta_1 = 0.
\]

**Proof.** See Appendix B. The proof traces the same steps as in the proof of Theorem [13]. \( \square \)

5. Conclusion and Future Work

We solved the problem of computing the optimal mechanism for the two-item one-buyer unit-demand setting, when the buyer’s valuation \( z \sim \text{Unif}[c, c + b_1] \times [c, c + b_2] \) for arbitrary nonnegative values of \((c, b_1, b_2)\). Our results show that a wide range of menus arise out of different values of \( c \). When the buyer guarantees that his valuations for the items are at least \( c \), the seller offers different menus based on the guaranteed minimum \( c \) and the upper bounds \( c + b_i, i = 1, 2 \).

Taking a cue from the solution method in the unrestricted setting [25], we initially attempted to solve the problem using the duality approach in [13], but constructing a dual measure in the unit-demand turned out to be intricate. We then used the virtual valuation method used in [22] to compute the solution. We now characterize the pros and cons of these approaches.

The duality approach could not be pursued systematically because the construction of a shuffling measure that both convex-dominates 0 and spans over more than one line segment appears to be difficult. Observe that in each of Example-2 and Example-3, there exists some constant allocation region that is a part of both the top boundary and the right boundary of \( D \). So the shuffling measure had to be constructed so that it spans over two line segments connected at the top-right corner of \( D \). To get around this issue, we had to construct (i) a shuffling measure on the line \( z_1 + z_2 = 2c + \delta_2 \) in
Example-2, and (ii) a shuffling measure that transfers mass horizontally in Example-3. The problem of constructing a “generalized” shuffling measure that both convex-dominates 0 and also spans over two segments, thereby rendering the dual approach practical, is a possible direction for future work.

The virtual valuation method on the other hand, did not pose any issue when constant allocation regions span over the top-right corner. The approach provides a generalized procedure to verify if the menu at hand is optimal or not, under the (only) constraint that the distribution satisfies the negative power rate condition (stated in Theorem 8). So unlike the duality approach, we cannot use this approach to solve the problem for general distributions. But our results for \( z \sim \text{Unif}[c, c + b_1] \times [c, c + b_2] \) and the extension to general rectangles suggest that this approach can be used to solve the problem of computing the optimal mechanism for all distributions satisfying the negative power rate condition. The key challenge in solving these problems is to find the region of no allocation \( Z \) for arbitrary distributions, so that we can use Theorem 10 to verify if the menu is optimal or not. Coming up with a generalized procedure to compute \( Z \) is a possible direction for future work.

Our work considers an auction scenario in which the seller designs a revenue maximizing mechanism. One could also consider the procurement setting where the buyer designs a cost minimizing mechanism. Characterization of the possible menus and the transitions in such a procurement setting is a possible direction for future work.

Our proofs used Mathematica to verify certain algebraic inequalities that turn out to be complicated functions of \((c; b_1, b_2)\) involving fifth roots and eighth roots of some expressions. This leads us to the following questions. From a rather abstract perspective, Pavlov’s sufficient conditions lead to the identification of a family of polynomial equalities and inequalities in the variables \((h, \delta^*, \delta_1, \delta_2)\) in Figures 20–26, indexed by the parameters \((c; b_1, b_2)\). In a nutshell, our work is a careful analysis of the solution space \( L_{c,b_1,b_2} \) associated with the polynomial equalities and inequalities. We argued that \( L_{c,b_1,b_2} \) is nonempty for every parameter \((c; b_1, b_2)\). We also captured the transitions of \( L_{c,b_1,b_2} \) as the parameters vary. Can this view provide a more systematic procedure to solve the case of uniform distribution on any rectangle in the positive quadrant, or more generally, the case of any distribution of valuations on the positive quadrant? Alternatively, can the procedure of this paper (both existence of solutions and capture of transitions) be automated on Mathematica or other similar tool? These are some computation related problems that might be of interest to the computer scientists.

Appendix

Appendix A. Proofs from Section 2

Proof of Theorem 2: We use the Fenchel-Rockafellar duality, given
by the following theorem:

**Theorem 18.** [23, Thm. 1] Let $E$ be a normed vector space, and let $\Xi, \Theta : E \to \mathbb{R} \cup \{\infty\}$ be convex functions. If $\Theta$ is continuous at some $f \in E$ where both $\Theta$ and $\Xi$ are finite, then

$$\inf_{f \in E} (\Theta(f) + \Xi(f)) = \max_{\gamma \in E^*} (-\Theta^*(-\gamma) - \Xi^*(\gamma))$$

where $\Xi^*$ and $\Theta^*$ are the Fenchel duals of $\Xi$ and $\Theta$ respectively.

We first prove the following theorem:

**Theorem 19.** The answer to the problem

$$\sup_{\phi, \psi} \left( \int_D \phi \, d\bar{\mu}_+ - \int_D \psi \, d\bar{\mu}_- \right)$$

(A.1)

subject to

(a) $\phi, \psi$ continuous, convex, increasing,
(b) $\phi(z) - \psi(z') \leq \|z - z'\|_{\infty}$, $\forall z, z' \in D$,

is the same as the answer to the problem

$$\min_{\gamma(\cdot, \cdot) \geq 0} \int_{D \times D} \|z - z'\|_{\infty} \, d\gamma(z, z')$$

(A.2)

subject to

$\gamma(z, D) \geq_{cvx} \bar{\mu}_+$, $\gamma(D, \cdot) \leq_{cvx} \bar{\mu}_-.$

**Proof.** Consider $E$ to be the vector space containing all continuous bounded functions on $D \times D$ with $\|\cdot\|_{\infty}$-norm. Then $E^*$ is the set of all Radon measures. Define the functions $\Xi$ and $\Theta$ as follows:

$$\Xi(f) := \begin{cases} 
\int_D \psi \, d\bar{\mu}_- - \int_D \phi \, d\bar{\mu}_+ & \text{if } f(z, z') = \psi(z') - \phi(z) \text{ for some } \psi, \phi \\
\infty & \text{otherwise.}
\end{cases}$$

$$\Theta(f) := \begin{cases} 
0 & \text{if } f(z, z') \geq -\|z - z'\|_{\infty} \forall (z, z') \in D \\
\infty & \text{otherwise.}
\end{cases}$$

We now derive the conjugates of the functions $\Theta$ and $\Xi$.

$$\Theta^*(-\gamma) = \sup_{f} \left( -\int_{D \times D} f(z, z') \, d\gamma(z, z') - \Theta(f) \right)$$

$$= \sup_{f: \Theta(f) = 0} \left( -\int_{D \times D} f(z, z') \, d\gamma(z, z') \right)$$

$$= \begin{cases} 
\int_{D \times D} \|z - z'\|_{\infty} \, d\gamma(z, z') & \text{if } \gamma(z, z') \geq 0 \\
\infty & \text{otherwise}
\end{cases}$$
where the last equality arises because if $\gamma(z, z') < 0$ for some $z, z'$, then $f$ can be scaled up to $\infty$ so that the supremum equals $\infty$. We now derive the conjugate of $\Xi$.

$$\Xi^*(\gamma) = \sup_f \left( \int_{D \times D} f(z, z') d\gamma(z, z') - \Xi(f) \right)$$

$$= \sup_{f: \Xi(f) < \infty} \left( \int_{D \times D} f(z, z') d\gamma(z, z') + \int_D (\phi d\mu_+ - \psi d\mu_-) \right)$$

$$= \sup_{\phi, \psi \text{ satisfying } (A.1(a))} \left( \int_D (\phi d(\mu_+ - \gamma_1) + \psi d(\gamma_2 - \bar{\mu}_-)) \right)$$

$$= \begin{cases} 
0 & \text{if } \gamma_1 \geq_{\text{cvx}} \bar{\mu}_+, \gamma_2 \leq_{\text{cvx}} \bar{\mu}_- \\
\infty & \text{otherwise.} 
\end{cases}$$

The last equality occurs because in case $\int_D \phi d(\mu_+ - \gamma_1) > 0$ for some convex, increasing function, as in the case when $\gamma_1 \not\leq_{\text{cvx}} \bar{\mu}_+$, then $\phi$ may be scaled up so that the supremum equals $\infty$. A similar argument holds if $\gamma_2 \not\leq_{\text{cvx}} \bar{\mu}_-$. It is easy to verify that the functions $\Xi$ and $\Theta$ are convex. Also, at $f = 1$, we have $\Xi(f) = \bar{\mu} - (D)$, and $\Theta(f) = 0$. We now argue that $\Theta$ is continuous at $f = 1$. For any $f \geq 0$, we have $\Theta(f) = 0$. So for every $g$ such that $\|g - 1\|_\infty$, we have $\Theta(1) = \Theta(g) = 0$. So $\Theta$ is continuous at $f = 1$. We can now apply the Fenchel-Rockafellar theorem (Theorem 18) to get

$$\inf_{f \in E} (\Theta(f) + \Xi(f)) = \max_{\gamma \in E^*} (-\Theta^*(-\gamma) - \Xi^*(\gamma)).$$

It can now be easily verified that the left-hand side is the same as problem (A.1), and the right-hand side is same as problem (A.2). Hence the result. □

**Theorem 20.** The answer to problem (A.1) is same as the answer to the following problem:

$$\sup_{u \text{ satisfying } (A.1), \|u - 1\|_\infty} \int_D u d\mu.$$ **Proof.** In order to prove this result, we show that the supremum in problem (A.1) occurs when $\phi$ and $\psi$ are equal to some feasible function $u$. Let us consider a feasible $\phi$ and $\psi$, $\phi \neq \psi$. We now construct $u : D \to \mathbb{R}$ such that (a) $\phi \leq u \leq \psi$ point-wise, (b) $u$ is convex, increasing, and 1-Lipschitz with respect to $L_\infty$-norm. Observe that the objective function of (A.1) only increases when $\phi$ and $\psi$ are replaced by $u$, and thus the theorem is proved if we can construct such a $u$. Consider

$$u(z) := \inf_{z' \in D} (\psi(z') + \|z - z'\|_\infty).$$ (A.3)
That \( u \geq \phi \) follows because \( \phi \) is a feasible point of problem (A.1) and satisfies (A.1(b)). Similarly, \( u \leq \psi \) is clear because \( u(z) = \inf_{z' \in D}(\psi(z') + \|z - z'\|_\infty) \leq \psi(z)\).

We now proceed to prove that \( u \) is convex, increasing, and 1-Lipschitz with respect to \( L_\infty \)-norm. We first claim that in (A.3), it suffices to take \( z' \leq z \). Indeed, if \( z' \not\leq z \), then \( I := \{i : z'_i > z_i\} \in \{1, 2\} \) is non-empty.

Observe that \( \psi(z') = \psi(z'_I, z'_{-I}) \geq \psi(z_I, z'_{-I}) \) due to monotonicity of \( \psi \).

Also, \( \|z - z'\|_\infty \geq \|z - (z_I, z'_{-I})\|_\infty \), since the latter max is over a smaller set of indices. So, \( \psi(z') + \|z - z'\|_\infty \geq \psi(z_I, z'_{-I}) + \|z - (z_I, z'_{-I})\|_\infty \), and \( (z_I, z'_{-I}) \leq z \) yields a smaller value in (A.3).

We now prove that \( u \) is increasing. Consider \( y, y' \in D, y \leq y' \). Let \( u(y) = \psi(x) + \|y - x\|_\infty \) for some \( \{x \in D : x \leq y\} \). Fix \( x' = (x'_1, x'_2) = (y'_1 + x_1 - y_1, y'_2 + x_2 - y_2) \in D \), and observe that \( (x - y') = (x' - y') \). Further, since \( x \leq y \leq y' \), we have \( x \leq x' \) and \( x' \in D \). By monotonicity of \( \psi \), we get \( \psi(x') + \|y - x\|_\infty \leq \psi(x') + \|y' - y\|_\infty \). Hence \( u(y) \leq u(y') \).

Next, \( u \) is 1-Lipschitz with respect to \( L_\infty \)-norm:

\[
\begin{align*}
u(z) - u(z') &= \inf_x (\psi(x) + \|z - x\|_\infty) - \inf_y (\psi(y) + \|z' - y\|_\infty) \\
&= \inf_x (\psi(x) + \|z - x\|_\infty) - (\psi(y^*) + \|z' - y^*\|_\infty) \\
&\leq \|z - z'\|_\infty
\end{align*}
\]

where the second equality follows by considering \( y^* \) as the point where the infimum is attained, the first inequality follows by substituting \( x = y^* \), and the last inequality follows from triangle inequality.

We now prove that \( u \) is convex. It suffices to prove that \( u \) is mid-point convex, since continuity of \( u \) is clear from the fact that \( u \) is 1-Lipschitz. Consider \( y, y', y'' \in D \) such that \( y = \frac{y + y''}{2} \). Let \( u(y') = \psi(x') + \|x' - y\|_\infty \) for some \( \{x' \in D : x' \leq y\} \), and \( u(y'') = \psi(x'') + \|x'' - y''\|_\infty \) for some \( \{x'' \in D : x'' \leq y''\} \). Fix \( x = \frac{x' + x''}{2} \in D \). Observe that \( 2(y_i - x_i) = (y'_i - x'_i) + (y''_i - x''_i) \) for \( i = \{1, 2\} \), and so it is clear that \( 2\|y - x\|_\infty \leq \|y' - x''\|_\infty + \|y'' - x''\|_\infty \). Further by convexity, \( 2\psi(x) \leq \psi(x') + \psi(x'') \). Adding these two inequalities, we get

\[
2\psi(x) + 2\|y - x\|_\infty \leq \psi(x') + \|y' - x''\|_\infty + \psi(x'') + \|y'' - x''\|_\infty.
\]

It thus follows that \( 2u(y) \leq u(y') + u(y'') \), which is mid-point convexity. \( \square \)

We are now ready to prove Theorem 2. Theorems A.2 and A.3 together imply that the answers to the problems (4) and (A.2) are the same. We now prove that the answers to the problems (5) and (A.2) are the same. Observe that this will complete the proof of Theorem 2.

The difference between the problems (5) and (A.2) is that the constraint
is tighter in the latter problem. So it follows that

\[
\min_{\gamma(\cdot) \geq 0, \gamma_1 - \gamma_2 \leq \bar{\mu}} \int_{D \times D} \|z - z'\|_\infty \, d\gamma(z, z') \\
\leq \min_{\gamma(\cdot) \geq 0, \gamma_1 \leq \bar{\mu}^+, \gamma_2 \leq \bar{\mu}^-} \int_{D \times D} \|z - z'\|_\infty \, d\gamma(z, z').
\]

To see the other inequality,

\[
\min_{\gamma(\cdot) \geq 0} \int_{D \times D} \|z - z'\|_\infty \, d\gamma(z, z') \\
\geq \sup_{u \text{ satisfying } (4(a),(b))} \int_D u \, d\bar{\mu} \\
= \sup_{\phi, \psi \text{ satisfying } (A.1(a),(b))} \int_D \phi \, d\bar{\mu}_+ - \int_D \psi \, d\bar{\mu}_- \\
= \min_{\gamma(\cdot) \geq 0, \gamma_1 \leq \bar{\mu}^+, \gamma_2 \leq \bar{\mu}^-} \int_{D \times D} \|z - z'\|_\infty \, d\gamma(z, z')
\]

where the first inequality follows from weak duality, the first equality from Theorem 20, and the final equality from Theorem 19. It still remains to be proved that the supremum is achieved for some \( u^* \) in problem (4). The proof follows the same steps as in [14, Appendix A.4]. Hence the result. □

**Proof of Lemma 5** We first compute the quantities \( \alpha^{(1)}([1.26, 1.26 + 2/3], 2.26) \) and \( \int_0^{2/3} t \, d\alpha^{(1)}([1.26 + t, 2.26], 2.26) \).

\[
\alpha^{(1)}([1.26, 1.26 + 2/3], 2.26) = \int_0^{2/3} (3t - 1) \, dt = (3/2)(2/3)^2 - 2/3 = 0.
\]

\[
\int_0^{2/3} t \, d\alpha^{(1)}([1.26 + t, 2.26], 2.26) = \int_0^{2/3} t(3t - 1) \, dt = \frac{2^3}{3^2} - \frac{1}{2} \cdot \frac{2^2}{3^2} = 0.
\]

We compute the same quantities for \( \beta^{(1)} \).

\[
\beta^{(1)}([1.26 + 2/3, 2.26], 2.26) \\
= \int_{2/3}^{43/63} (3t - 1) \, dt + \int_{43/63}^1 (t(1.0155) + (1.9845)(43/63) - 2.26) \, dt \\
= (3/2)((43/63)^2 - (2/3)^2) - 1/63 + 1.0155(1 - (43/63)^2)/2 \\
+ (20/63)(1.9845(43/63) - 2.26) = 0.
\]
\[
\int_{2/3}^{1} t \, d\beta^{(1)}([1.26 + t, 2.26], 2.26)
\]
\[
= \int_{2/3}^{43/63} t(3t - 1) \, dt + \int_{43/63}^{1} t(t(1.0155) + (1.9845)(43/63) - 2.26) \, dt
\]
\[
= (43/63)^{3} - (2/3)^{3} - ((43/63)^{2} - (2/3)^{2})/2 + (1.0155)(1 - (43/63)^{3})/3
\]
\[
+ (1 - (43/63)^{2})(1.9845(43/63) - 2.26)/2 = 0.
\]

Now consider \( h \) to be the affine shift of any increasing convex function \( g \) (i.e., \( h = \theta_{1}g + \theta_{2}, \theta_{1} > 0, \theta_{2} \in \mathbb{R} \)) such that \( h(t) = t \) for \( t = 43/63 \) and \( t = 2.26 - 1.9845 \times 43/63 \approx 0.891679 \). Observe that \( \beta^{(1)}(1.26 + t, 2.26) \geq 0 \) when \( t \in [2/3, 43/63] \cup [0.891679, 1] \), and \( \beta^{(1)}(1.26 + t, 2.26) < 0 \) when \( t \in (43/63, 0.891679) \). So we have \( h(t) \leq t \) when \( \beta^{(1)} < 0 \), and \( h(t) \geq t \) when \( \beta^{(1)} > 0 \). Now,

\[
\int_{2/3}^{1} g(t) \, d\beta^{(1)}([1.26 + t, 2.26], 2.26)
\]
\[
= \frac{1}{\theta_{1}} \int_{2/3}^{1} h(t) \, d\beta^{(1)}([1.26 + t, 2.26], 2.26)
\]
\[
= \frac{1}{\theta_{1}} \left( \int_{2/3}^{1} (h(t) - t) \, d\beta^{(1)}([1.26 + t, 2.26], 2.26) \right)
\]
\[
= \frac{1}{\theta_{1}} \left( \int_{2/3}^{1} (h(t) - t) \, d\beta^{(1)}([1.26 + t, 2.26], 2.26) \right)
\]
\[
\geq 0
\]

where the first equality follows from \( \beta^{(1)}([1.26 + t, 2.26], 2.26) = 0 \), the third equality follows from \( \int_{2/3}^{1} t \, d\beta^{(1)}([1.26 + t, 2.26], 2.26) = 0 \), and the last inequality follows because \( \text{sgn}(h(t) - t) = \text{sgn} (\beta^{(1)}(t)) \) for every \( t \in [2/3, 1] \). The proof of \( a^{(1)} \geq_{\text{conv}} 0 \) is similar. Hence the result. \( \square \)

Appendix B. Proofs from Section 3 and Section 4

**Proof of Lemma 11.** Recall that the marginal profit function is defined as \( \nu(\delta) = \delta g(u_1(\delta), \delta) - \int_{\delta}^{b_1} g(u_1(\delta), \delta) \, d\delta + \int_{\delta}^{b_1} (\delta q_1(\delta) - u_1(\delta)) \frac{\partial}{\partial u_1} g(u_1(\delta), \delta) \, d\delta. \)

Consider the term \( \int_{\delta}^{b_1} g(u_1(\delta), \delta) \, d\delta. \)

\[
\int_{\delta}^{b_1} g(u_1(\delta), \delta) \, d\delta = \int_{\delta}^{b_1} \int_{z: z \in D \setminus Z, z_1 - z_2 = \delta} f(z) \, dz \, d\delta
\]
\[
= \int_{z: z \in D \setminus Z, z_1 - z_2 \geq \delta} f(z) \, dz.
\]
We use integration by parts on $\int_X (z \cdot \nabla h(z) - h(z)) f(z) \, dz$, and we obtain $\int_X h(z) \nu(z) \, dz + \int_{\partial X} h(z) \nu_s(z) \, d\sigma(z)$. Here,

$$\nu(z) := -z \cdot \nabla f(z) - 3f(z), \quad z \in X; \quad \nu_s(z) := (z \cdot n(z)) f(z), \quad z \in \partial X.$$  

We regard $\nu$ as the density of a two-dimensional measure, and $\nu_s$ as the density of a one-dimensional measure. Defining the measure $\bar{\nu} = \nu + \nu_s$, and substituting $h(z) = 1 \forall z \in X$, we get

$$\int_X (z \cdot \nabla h(z) - h(z)) f(z) \, dz = \int_X -f(z) \, dz = \bar{\nu}(X).$$  

We now compare the components of the measures $\bar{\mu}$ and $\bar{\nu}$ in some set $X \subseteq D$. The area measures $\mu$ and $\nu$ are clearly equal. The line measure $\nu_s$ has nonzero densities in every $z \in \partial X$, whereas $\mu_s$ has nonzero densities only when $z \in (X \cap \partial D)$. In other words, $\nu_s$ has extra nonzero densities for every $z \in (\partial X \setminus \partial D)$. We now show that the terms $\delta g(u_1(\delta), \delta)$ and $\int_0^{b_1} (\delta q_1(\delta) - u_1(\delta)) \frac{\partial}{\partial u_1} g(u_1(\delta), \delta) \, d\delta$ cancel those “extra” densities. In other words, we show that

$$\delta g(u_1(\delta), \delta) + \int_0^{b_1} (\delta q_1(\delta) - u_1(\delta)) \frac{\partial}{\partial u_1} g(u_1(\delta), \delta) \, d\delta = (\bar{\mu} - \bar{\nu})(z : z \in D \setminus Z, z_1 - z_2 \geq \delta), \quad (B.1)$$

and this completes our proof. We show this for the menu depicted below in Figure B.27 with the no allocation region $Z$ being a convex, decreasing set. Observe that all the menus depicted in Figures 2–8 have this property.

\begin{figure}[h]
  \centering
  \includegraphics[width=0.5\textwidth]{menu.png}
  \caption{The structure of a typical menu. The variables marked in the boundary denote the values of $\delta$.}
\end{figure}

Define $z_2^* : [-\delta_2, \delta_1] \rightarrow [c, c + \delta_2]$ as

$$z_2^*(\delta) = \{ z_2 \in [c, c + b_2] : u(z_2 + \delta, z_2) = 0, \quad u(z_2 + \delta + h, z_2 + h) > 0 \text{ for all } h > 0 \text{ sufficiently small} \}.  \quad 43$$
Observe that \( z_2^*(\delta) \) is the value of \( z_2 \) in the curve \((-\delta_2\delta_1)\) that separates \( Z \) and \( D\setminus Z \), when \( z_1 - z_2 = \delta \). So \( z_2^* \) is decreasing with \( \delta \), with \( z_2^*(-\delta_1) = c + \delta_2 \), and \( z_2^*(\delta_1) = c \). Also, the curve \((-\delta_2\delta_1)\) can be represented by the points \( \{ (\delta + z_2^*(\delta), z_2^*(\delta)), \delta \in [-\delta_2, \delta_1] \} \). We now compute \( u_1(\delta) \) for every \( \delta \in [-\delta_2, \delta_1] \). We use the fact that \( u(z) = 0 \) when \( z \in Z \), and \( u(z) = u_1(z_1 - z_2) + z_2 \).

\[
u \delta \begin{cases} 
-\nu z_2^*(-\delta_1) + \nu_1(\delta) & \text{if } \delta \in [-\delta_2, \delta_1] \\
-\nu z_2^*(\delta_1) - \nu_1(\delta) & \text{if } \delta \in [-\delta_2, -\delta_1] 
\end{cases}
\]

Using the values of \( u_1(\delta) \), we now compute \( g(u_1(\delta), \delta) \).

For ease of notation, we drop the factor \( 1/(b_1b_2) \) in the rest of the proof.

To show \( (B.1) \), we consider the following three cases: (i) \( \delta \in [\delta_1, b_1] \), (ii) \( \delta \in [-\delta_2, \delta_1] \), and (iii) \( \delta \in [-\delta_2, -\delta_1] \) (see Figure B.27). In case (i), consider \( \delta = \delta^*(1) \). The measures \( \mu \) and \( \nu \) differ only in that \( \nu \) has an extra nonzero line measure on the line segment \( \{ z \in D \setminus Z : z_1 - z_2 = \delta^*(1) \} \). We thus have

\[
(\mu - \nu)(z \in D \setminus Z : z_1 - z_2 \geq \delta^*(1)) = -\nu_\delta(z \in D \setminus Z : z_1 - z_2 = \delta^*(1)) = \int_{z_1 - z_2 = \delta^*(1)} f(z) \, dz.
\]

Now observe that \( \frac{\partial}{\partial \delta} g(u_1(\delta), \delta) = 0 \) when \( \delta \in [\delta_1, b_1] \). So \( (B.1) \) holds.

In case (ii), consider \( \delta = \delta^*(2) \). Then, \( \nu \) has an extra nonzero line measure on (i) the line segment \( \{ z \in D \setminus Z : z_1 - z_2 = \delta^*(2) \} \), and (ii) the curve \( \delta_1 x \).

Now we have

\[
(\mu - \nu)(z \in D \setminus Z : z_1 - z_2 \geq \delta^*(2)) = -\nu_\delta(z \in D \setminus Z : z_1 - z_2 = \delta^*(2)) - \nu_\delta(z \in \text{curve } \delta_1 x)
\]

\[
= \int_{z_1 - z_2 = \delta^*(2)} f(z) \, dz - \int_{z \in \text{curve } \delta_1 x} (z \cdot n(z)) f(z) \, dz
\]

\[
= \delta^*(2) g(u_1(\delta^*(2)), \delta^*(2)) + \int_{\delta^*(2)} \left( z_2^*(\delta) + \delta q_1(\delta) + z_2^*(\delta)(1 - q_1(\delta)) \right) d\delta
\]

\[
= \delta^*(2) g(u_1(\delta^*(2)), \delta^*(2)) + \int_{\delta^*(2)} \left( z_2^*(\delta) + \delta q_1(\delta) \right) d\delta.
\]
where the third equality follows because (i) $\nabla u = q$, and (ii) $q_1(\delta) + q_2(\delta) = 1$ for $z \in D \setminus Z$. Now observe that we have $\frac{\partial}{\partial u_1} g(u_1(\delta), \delta) = 1$ when $\delta \in [-\delta_2, \delta_1]$. Therefore,

$$
\delta^{(2)} g(u_1(\delta^{(2)}), \delta^{(2)}) + \int_{\delta^{(2)}}^{b_1} (\delta q_1(\delta) - u_1(\delta)) \frac{\partial}{\partial u_1} g(u_1(\delta), \delta) \, d\delta
$$

$$
= \delta^{(2)} g(u_1(\delta^{(2)}), \delta^{(2)}) + \int_{\delta^{(2)}}^{b_1} (z_2^*(\delta) + \delta q_1(\delta)) \, d\delta.
$$

The equation (B.1) thus holds for case 2.

In case (iii), consider $\delta = \delta^{(3)}$. Then, $\tilde{\nu}$ has an extra nonzero line measure on (i) the line segment $\{z \in D \setminus Z : z_1 - z_2 = \delta^{(3)}\}$, and (ii) the curve $(-\delta_2\delta_1)$. Now by an analysis similar to case 2, it follows that

$$
(\tilde{\mu} - \tilde{\nu})(z \in D \setminus Z : z_1 - z_2 \geq \delta^{(3)})
$$

$$
= \delta^{(3)} g(u_1(\delta^{(3)}), \delta^{(3)}) + \int_{-\delta_2}^{b_1} (z_2^*(\delta) + \delta q_1(\delta)) \, d\delta
$$

$$
= \delta^{(3)} g(u_1(\delta^{(3)}), \delta^{(3)}) + \int_{\delta^{(3)}}^{b_1} (\delta q_1(\delta) - u_1(\delta)) \frac{\partial}{\partial u_1} g(u_1(\delta), \delta) \, d\delta.
$$

\[\Box\]

**Proof of Theorem 15**

**Step 1:** We compute the virtual valuation function Menu III depicted in Figure 22.

$$
V(\delta) = \begin{cases} 
V(-\delta_2) - (c - 2b_2 + 3\delta_2)(\delta + \delta_2) + \frac{3}{2} \frac{\delta_2 - h}{\delta_2 + \delta^*} (\delta + \delta_2)^2 & \delta \in [-\delta_2, \delta^*] \\
V(\delta^*) - (c - 2b_2)(\delta - \delta^*) + \frac{3h}{2} (\delta_1 - \delta^*)^2 - \frac{h}{2} (\delta_1 - \delta^*) & \delta \in [\delta^*, b_1] 
\end{cases}
$$

where $b_1 - b_2$ is indicated as $b'$. The expression for $V(\delta)$ when $\delta \in [-b_2, -\delta_2] \cup [\delta_1, b_1]$ remains the same as in [19], and the expression when $\delta \in [b', b_1]$ is given by

$$
V(\delta) = V(b') - (c - 2b_1)(\delta - b') - \frac{3}{2}(\delta^2 - (b')^2) - \frac{h}{\delta_1 - \delta^*}((\delta_1 - \delta)^2 - (\delta_1 - b')^2)).
$$

**Step 2:** Menu III has six unknowns $- h, \delta^*, \delta_1, \delta_2, a_1, \text{ and } a_2$. Since $q = \nabla u$, a conservative field, we must have the slope of the line separating $(0, 0)$ and $(1 - a_2, a_2)$ allocation regions satisfying $-\frac{1 - a_2}{a_2} = \frac{h - \delta_2}{\delta_2 + \delta^*}$, which yields $a_2 = \frac{h + \delta^*}{\delta_2 + \delta^*}$. Similarly, the slope of the line separating $(0, 0)$ and $(a_1, 1 - a_1)$ allocation regions must satisfy $-\frac{a_1}{1 - a_1} = \frac{h}{\delta_1 - \delta^* - h}$, which yields $a_1 = \frac{h}{\delta_1 - \delta^* - h}$.

We compute the other four unknowns by equating $\bar{\mu}(Z) = 0$, $V(\delta^*) = 0$, $\int_{-\delta^*}^{b_1} V(\delta) \, d\delta = 0$, and $\int_{\delta^*}^{b_1} V(\delta) \, d\delta = 0$. The latter three conditions follow
from Theorem 11(3(b) and 3(c) because $q_1(\delta) = 1 - a_2$ for $\delta \in [-\frac{b_2}{3}, \delta^*]$, and $q_1(\delta) = a_1$ for $\delta \in [\delta^*, \frac{b_1}{3}]$. We then have the following implications.

$$\bar{\mu}(Z) = 0 \Rightarrow -(3h/2 + c)(\delta_1 + \delta_2) - 3\delta_2\delta^*/2 + b_1b_2 = 0. \quad (B.2)$$

From (22), we see that $V(\delta^*)$ is the negative of $\bar{\mu}$ measure of the nonconvex pentagon bound by $(c, c), (c, c + b_2), (c + b_2 + \delta^*, c + b_2), (c + h + \delta^*, c + h)$, and $(c + \delta_1, c)$. Thus

$$V(\delta^*) = 0 \Rightarrow -(3h/2 + c)(\delta_1 - \delta^*) - 2b_2\delta^* - b_2^2/2 + b_1b_2 = 0. \quad (B.3)$$

The expression for $\int_{-\frac{b_2}{2}}^{\delta^*} V(\delta) d\delta$ remains the same as in (27).

$$\int_{-\frac{b_2}{2}}^{\delta^*} V(\delta) d\delta = 0 \Rightarrow \frac{1}{54}(4b_2 + 3\delta^*)(b_2 + 3\delta^*)^2 - \frac{c + h + \delta^*}{2}(\delta^* + \delta_2)^2 = 0. \quad (B.4)$$

Next

$$\int_{\delta^*}^{b_1 - b_2} V(\delta) d\delta = 0$$

$$\Rightarrow \int_{\delta^*}^{b_1 - b_2} V(\delta) d\delta + \int_{b_1 - b_2}^{\delta_1} V(\delta) d\delta + \int_{\delta_1}^{b_2} V(\delta) d\delta = 0$$

$$\Rightarrow -\frac{2}{27}b_1^3 + b_1b_2\delta_1 + \frac{1}{2}(-b_2^2\delta_1 + 2b_2\delta^*(-2\delta_1 + \delta^*) - (\delta_1 - \delta^*)^2(c + 2h)) = 0$$

$$\Rightarrow \frac{c + h}{2}(\delta_1 - \delta^*)^2 - b_2(\delta^*)^2 + \frac{b_2\delta^*}{2}(2b_1 - b_2) - \frac{2}{27}b_1^3 + (\delta_1 - \delta) V(\delta^*) = 0. \quad (B.5)$$

The values of $h$, $\delta^*$, $\delta_1$ and $\delta_2$ can be obtained by solving these four equations simultaneously. We now proceed to prove that $(h, \delta^*)$ can be computed by solving (11) and (13) simultaneously.

We first find an expression for $\delta_2 + \delta^*$. Rearranging (B.2), we have

$$\delta_2 = \frac{b_1b_2 - (3h/2 + c)\delta_1}{3/2(h + \delta^*) + c} \quad (B.6)$$

Similarly, rearranging (B.3), we have $\delta_1 = \delta^* + \frac{b_1b_2 - 2b_2\delta^* - b_2^2/2}{3h/2 + c}$. Substituting $\delta_1$ in (B.6), we get

$$\delta_2 + \delta^* = \frac{b_1b_2 - (3h/2 + c)(\delta_1 - \delta^*) + \frac{3}{2}(\delta^*)^2}{3/2(h + \delta^*) + c} = \frac{(b_2 + 3\delta^*)(b_2 + \delta^*)/2}{3/2(h + \delta^*) + c}.$$

Plugging this into (B.4), we eliminate $\delta_2$. Similarly, plugging $(\delta_1 - \delta^*) = \frac{b_1b_2 - 2b_2\delta^* - b_2^2/2}{3h/2 + c}$ (obtained by rearranging (B.3)) in (B.5), we eliminate $\delta_1$. We
thus solve the following equations:

\[
27(c + h + \delta^*)(b_2 + \delta^*)^2 - 4(4b_2 + 3\delta^*)(3/2(h + \delta^*) + c)^2 = 0
\]

\[
(2b_1^2/27 + b_2(\delta^*)^2) - b_2\delta^*(b_1 - b_2/2))(3h/2 + c)^2
\]

\[
- (c + h)/2(2b_2\delta^* + b_2^2/2 - b_1b_2)^2 = 0
\]

which are (11) and (13), respectively.

**Step 3:** We now proceed to bound the parameters. In Step 3a, we first prove that the condition $q_1$ increasing in Problem (9) is satisfied only when the left-hand side of (14) is nonnegative. In Steps 3b–3e, we prove the bounds on $h$, $\delta^*$, $\delta_1$ and $\delta_2$, respectively.

**Step 3a:** We compute the values of $c$ where monotonicity of $q_1$ holds. Observe that monotonicity of $q_1$ holds when $1 - a_2 \leq a_1$, and that of $q_2$ holds when $1 - a_1 \leq a_2$. We thus verify if $a_1 + a_2 \geq 1$. On substituting the expressions for $a_1$ and $a_2$, we obtain

\[
(h + \delta^*)(3/2(h + \delta^*) + c) \geq \frac{h(3h/2 + c)}{(b_1b_2 - 2b_2\delta^* - b_2^2/2)} \geq 1
\]

\[
\Rightarrow (b_2^2 + 4b_2\delta^* - 3\delta^* h)(b_2^2 + 4b_2\delta^* - 2c\delta^* - 3\delta^* h) - 2b_1b_2(b_2^2 + 4b_2\delta^* - 2c(\delta^* + h) - 3h(2\delta^* + h)) \geq 0.
\]

The monotonicity condition thus amounts to verifying if the left-hand side of (14) is nonnegative. We verify via Mathematica that the expression is nonnegative for $c \in [b_2, \alpha_2]$, and that $\alpha_2 \leq 2(t - 1.4)(b_1 - b_2) + 1.4b_2$ (see Appendix C.2(6–7)). We thus compute the bounds of $h$, $\delta^*$, $\delta_1$, and $\delta_2$ when $c \in [b_2, 2(t - 1.4)(b_1 - b_2) + 1.4b_2]$.

**Step 3b:** We now prove that there exist a $h \in [0, \frac{2b_2 - c}{3}]$ and $\delta^* \in [0, b_1 - b_2]$ that simultaneously solves (11) and (13). Substituting $h = \frac{2b_2 - c}{3}$ in the left-hand side of (13), we obtain

\[
\delta^*_{\frac{2b_2 - c}{3}} = \frac{b_1}{2} - \frac{b_2}{4} \frac{(3b_2 - 2b_1)(2b_2 + c)\sqrt{b_2(8b_1 - 3b_2)(2b_2 - c)(2b_2 + 3c)}}{(12b_2(2b_2 - c)(2b_2 + 3c))}.
\]

We first prove that $\delta^*_{\frac{2b_2 - c}{3}} \geq -\frac{2b_2}{3}$.

\[
\delta^*_{\frac{2b_2 - c}{3}} \geq \frac{2b_2}{3}
\]

\[
\Rightarrow \frac{b_1}{2} - \frac{b_2}{4} - \frac{(3b_2 - 2b_1)(2b_2 + c)\sqrt{b_2(8b_1 - 3b_2)(2b_2 - c)(2b_2 + 3c)}}{(12b_2(2b_2 - c)(2b_2 + 3c))} \geq \frac{2b_2}{3}
\]

\[
\Rightarrow (6b_1 + 5b_2)^2b_2(2b_2 - c)(2b_2 + 3c) \geq (3b_1 - 2b_2)^2(2b_2 + c)^2(8b_1 - 3b_2)
\]

Consider $b_2 \leq c \leq tb_2$, and $b_1 \in [b_2, 3b_2/2]$ (recall that $t = 3(37 + 3\sqrt{465})/176$). The inequality then clearly holds, since we have (a) $6b_1 + 5b_2 \geq 8b_1 - 3b_2$, (b)
(6b_1 + 5b_2)(2b_2 - c) \geq (6b_1 + 5b_2)b_2/4 \geq b_2^2 \geq (3b_2 - 2b_1)^2, (c) b_2 \geq 3b_2 - 2b_1, and (d) 2b_2 + 3c \geq 2b_2 + c. We have thus shown that \( \delta^*_{2b_2-c} \geq -\frac{2b_2}{3}. \)

Substituting \((h, \delta^*) = \left( \frac{2b_2-c}{3}, \delta^*_{2b_2-c} \right)\) in the left-hand side of (11), we get

\[
(b_2 - c)(2b_2^2 + 4b_2c + 3(b_2 + c)\delta^*_{2b_2-c}) \leq 0
\]

for \( c \leq b_2 \), since \( \delta^*_{2b_2-c} \geq -\frac{2b_2}{3}. \) Substituting \( h = 0 \) in the left-hand side of (13), we obtain

\[
\delta_0^0 = \frac{b_1}{2} - \frac{b_2}{4} - \frac{(3b_2 - 2b_1)\sqrt{3b_2c(8b_1 - 3b_2)(2b_2 - c)}}{36b_2(2b_2 - c)}
\]

Substituting \((h, \delta^*) = (0, \delta_0^0)\) in the left-hand side of (11), we get

\[
27b_2c - 16b_2c^2 + (27b_2^2 + 6b_2c - 12b_2^2)\delta_0^0 + 9(2b_2 - c)(\delta_0^0)^2.
\]

From Mathematica, this is nonnegative when \( c \in [b_2, 2(t - 1.4)(b_1 - b_2) + 1.4b_2] \) (see Appendix C.2(2)). We have thus shown that \( h \in [0, 2b_2-c] \).

**Step 3c:** We now prove that \( \delta^* = 0 \) in (13), we obtain \( 2b_2(3h/2 + c)^2/27 - (c + h)b_2^2(b_1 - b_2/2)^2/2 = 0. \) Let \( h = h_0 \) solve this equation. Substituting \((h, \delta^*) = (h_0, 0)\) in (13), we get \( 27(c + h_0)b_2 - 16(3h_0/2 + c)^2. \) We now prove that this expression is nonpositive.

\[
27(c + h_0)b_2 - 16(3h_0/2 + c)^2 = 27(c + h_0)b_2 - \frac{108(c + h_0)b_2^2(b_1 - b_2/2)^2}{b_1^2}
\]

\[
= \frac{27(c + h_0)b_2}{b_1^2}(b_1^3 - 4b_2(b_1 - b_2/2)^2)
\]

\[
= \frac{27(c + h_0)b_2}{b_1^2}(b_1 - b_2)(b_1^2 - 3b_1b_2 + b_2^2) \leq 0
\]

when \( b_1 \in [b_2, 3b_2/2] \). The first equality occurs since \( h_0 \) solves \( 2b_2(3h/2 + c)^2/27 - (c + h)b_2^2(b_1 - b_2/2)^2/2 = 0. \) Now substituting \( \delta^* = b_1 - b_2 \) in (13), we obtain

\[
h_{b_1-b_2} = \frac{9b_2^2 - 4c(b_1 + 3b_2) + 3b_2\sqrt{9b_2^2 + 4c(b_1 + 3b_2)}}{6(b_1 + 3b_2)}.
\]

Substituting \((h, \delta^*) = (h_{b_1-b_2}, b_1 - b_2)\) in (11), we get

\[
27b_1^2(b_1 - b_2 + c + h_{b_1-b_2}) - (3b_1 + b_2)(3b_1 - 3b_2 + 2c + 3h_{b_1-b_2})^2.
\]

From Mathematica, this is nonnegative when \( c \in [b_2, 2(t - 1.4)(b_1 - b_2) + 1.4b_2] \) (Appendix C.2(2)). We have thus shown that \( \delta^* \in [0, b_1 - b_2] \).

**Step 3d:** We now prove that \( \delta_1 \in [h + \delta^*, \frac{b_2}{3}] \). To prove \( \delta_1 \geq h + \delta^* \), we first assume the contrapositive, and do the following. (a) Solving (10)
and (I) simultaneously, we obtain \((h_{II}, \delta_{II}^*)\) in Menu II. We prove that 
\((h_{II}, \delta_{II}^*) < (h_{III}, \delta_{III}^*)\); (b) We then show \(\int_{\delta_{II}}^b \nu_{II}(\delta) \, d\delta \geq \int_{\delta_{III}}^b \nu_{III}(\delta) \, d\delta = 0\). But \(\int_{\delta_{II}}^b \nu_{II}(\delta) \, d\delta\) is negative for \(c \in [a_1, tb]\) (from Appendix C.1(3)), which is a contradiction. We now proceed to prove our claim.

Observe that when \(\delta < h + \delta^*\), we have \(a_1 > 1\), and the menu appears as depicted (in solid lines) in Figures B.28 and B.29. We now solve the problem for the parameters \((h_{II}, \delta_{II}^*)\) in Menu II. We first prove that if \((h, \delta^*)\) satisfy (I), then \(h\) increases with increase in \(\delta^*\). Solving (I) for \(h\), we get

\[
h = \frac{9b_2^2 - 16b_2c - 6\delta^*(b_2 + 2c) - 9(\delta^*)^2}{6(4b_2 + 3\delta^*)} + \frac{3(b_2 + \delta^*) \sqrt{9b_2^2 + 16b_2c + 6\delta^*(3b_2 + 2c) + 9(\delta^*)^2}}{6(4b_2 + 3\delta^*)}.
\]

Denoting \(X := 9b_2^2 + 16b_2c + 6\delta^*(3b_2 + 2c) + 9(\delta^*)^2\), and differentiating with respect to \(\delta^*\), we get

\[
\frac{\partial h}{\partial \delta^*} = \frac{(4b_2 + 3\delta^*)(-6b_2 + 2c + 3\delta^* + 3\sqrt{X} + 3(4b_2 + 3\delta^*)(9b_2^2 + 6c + 9\delta^*))}{6(4b_2 + 3\delta^*)^2}
\]

\[
-3 \left(\frac{9b_2^2 - 16b_2c - 6\delta^*(b_2 + 2c) - 9(\delta^*)^2}{6(4b_2 + 3\delta^*)} + \frac{3(b_2 + \delta^*) \sqrt{X}}{6(4b_2 + 3\delta^*)}ight)
\]

\[
= \frac{-51b_2^2 - 72b_2 \delta^* - 27(\delta^*)^2 + 3b_2 \sqrt{X} + 9(4b_2 + 3\delta^*)(b_2 + \delta^*)(3b_2 + 2c + 3\delta^*)}{6(4b_2 + 3\delta^*)^2}
\]

\[
\geq 0
\]

if \((9(\delta^*)^2 + 24b_2 \delta^* + 17b_2^2) \sqrt{X} \leq b_2 \cdot X + 3(4b_2 + 3\delta^*)(b_2 + \delta^*)(3b_2 + 2c + 3\delta^*)\). Squaring this expression on both sides and simplifying, we have

\[
4(4b_2 + 3\delta^*)^2(b_2 - c)(b_2^2(9b_2 + 25c) + 6b_2 \delta^*(3b_2 + 5c) + 9(\delta^*)^2(b_2 + c)) \leq 0
\]

which clearly is true for \(c \geq b_2\). This proves that if \((h, \delta^*)\) satisfy (I), then \(h\) increases monotonically in \(\delta^*\).

We now claim that \((h_{II}, \delta_{II}^*) < (h_{III}, \delta_{III}^*)\). This is because if not, then (i) \((h_{II}, \delta_{II}^*) > (h_{III}, \delta_{III}^*)\) must hold, since \((h, \delta^*)\) in both the menus satisfy (I), and \(h\) monotonically increases in \(\delta^*\); (ii) If \((h_{II}, \delta_{II}^*) > (h_{III}, \delta_{III}^*)\), then \(\tilde{\mu}(Z) = 0\) cannot be true for both the menus simultaneously (see Figure B.28). We have proved our claim.

We now evaluate \(\int_{\delta_{II}}^b \nu_{II}(\delta) \, d\delta - \int_{\delta_{III}}^b \nu_{III}(\delta) \, d\delta\). From (I), we have \(V'(\delta) = -\tilde{\mu}(z : z \in D \setminus Z, z_1 - z_2 = \delta)\). Observe from Figure B.29 that \(V'_{II}(\delta) < V'_{III}(\delta)\) when \(\delta \in (l, \delta_{II}^*)\) for some \(l \in [\delta_{II}^*, \delta_{III}^*]\), \(V'_{II}(\delta) > V'_{III}(\delta)\) when \(\delta \in [\delta_{II}^*, l]\), and \(V'_{II}(\delta) = V'_{III}(\delta)\) when \(\delta \in [\delta_{II}^*, b]\cup\{l\}\). Since we
have \( V_{II}^{*} = V_{III}^{*} = V_{II}^{(\frac{b_1}{3})} = V_{II}^{(\frac{b_1}{3})} = 0 \), we conclude that \( V_{II}(\delta) \geq V_{III}(\delta) \) when \( \delta \in [\delta_{II}^{*}, \frac{b_1}{3}] \). Further, from \( V'(\delta) = -(c - 2b_2) - 3(\delta_{II}^{*} - \delta) = -(c - 2b_2 + 3\delta_{II}^{*}) + 3\delta \) when \( \delta \in [\delta_{II}^{*}, b_1 - b_2] \), we have \( V_{II}'(\delta) \geq 0 \) in that interval, and thus \( V_{II}(\delta) = V_{II}(\delta_{II}^{*}) + \int_{\delta_{II}^{*}}^{\delta} V_{II}'(\delta) \, d\delta \geq 0 \). Therefore,

\[
\int_{\delta_{II}^{*}}^{b_1} V_{II}(\delta) \, d\delta - \int_{\delta_{II}^{*}}^{b_1} V_{III}(\delta) \, d\delta = \int_{\delta_{II}^{*}}^{b_1} (V_{II}(\delta) - V_{III}(\delta)) \, d\delta + \int_{\delta_{II}^{*}}^{b_1} V_{II}(\delta) \, d\delta \, d\delta \geq 0.
\]

This proves that \( \delta_1 \geq h + \delta^{*} \), and also that \( a_1 \leq 1 \). We then verify the upper bound \( \delta_1 \leq \frac{b_1}{3} \) via Mathematica (see Appendix C.2.5).

**Step 3**: We now prove that \( \delta_2 \geq \frac{2b_2 - a_1}{3} \). Suppose that \( \delta_2 < \frac{2b_2 - a_1}{3} \). Then, from \( V'(\delta) = -(c - 2b_2 + 3\delta_2) + 3(\frac{\delta_2 - h}{\delta_{II}^{*} + \delta_2}) (\delta + \delta_2) \) when \( \delta \in [-\delta_2, \delta^{*}] \), we have \( V'(-\delta_2) = -(c - 2b_2 + 3\delta_2) > 0 \). Also, \( V'(\delta^{*}) \geq 0 \) holds since \( h \leq \frac{2b_2 - a_1}{3} \). So we have \( V'(\delta) \geq 0 \) when \( \delta \in [-\frac{b_1}{3}, \delta^{*}] \). This implies that \( V(\delta^{*}) > 0 \), a contradiction. So we have \( \delta_2 \geq \frac{2b_2 - a_1}{3} \geq h \), and thus \( a_2 \leq 1 \). The upper bound \( \delta_2 \leq \frac{b_1}{3} \) is verified via Mathematica (see Appendix C.2.4).

**Step 4**: We now proceed to prove that the conditions in Theorem 10 (2)–(4) are satisfied. Observe from the expressions of \( V(\delta) \) it is nonpositive when \( \delta \in [-b_2, \frac{-b_2}{3}] \) (i.e., in the interval where \( q_1 = 0 \), and nonnegative when \( \delta \in [\frac{b_1}{3}, b_1] \) (i.e., in the interval where \( q_1 = 1 \)). This proves the conditions of Theorem 10 (2) and 10 (4).

We now prove the conditions in Theorem 10 (3). The values of \( V'(\delta) \) can be computed as

\[
V'(\delta) = \begin{cases} 
-(c - 2b_2 + 3\delta_2) + 3(\frac{\delta_2 - h}{\delta_{II}^{*} + \delta_2}) (\delta + \delta_2) & \delta \in (-\delta_2, \delta^{*}] \\
-(c - 2b_2) - 3(\frac{h}{\delta_{II}^{*} - \delta}) (\delta_1 - \delta) & \delta \in [\delta^{*}, b_1 - b_2) \\
-(c - 2b_1 + 3\delta) - 3(\frac{h}{\delta_{II}^{*} - \delta}) (\delta_1 - \delta) & \delta \in (b_1 - b_2, \delta_1)
\end{cases}
\]
The values of \( V'(\delta) \) when \( \delta \in [-b_2, \delta_2) \cup (\delta_1, b_1] \) is the same as \( \text{(22)} \).

The proof of \( \int_{\delta}^{b_1} V(\delta) \, d\delta \geq 0 \) for every \( x \in [-b_2^*, \delta^*] \), is the same as that in the proof of Theorem 14. So we proceed to prove \( \int_{\delta}^{b_1} V(\delta) \, d\delta \geq 0 \) for every \( x \in [\delta^*, b_1^*] \). Observe that \( V'(\delta) \) is positive when \( \delta \in [\delta^*, b_1 - b_2) \cup (\delta_1, b_1/3] \). This is because (i) \( V(\delta^*) \geq 0 \) since \( h \leq \frac{2b_2 - c}{3} \), (ii) \( V'(\delta) \) increasing in the interval \([\delta^*, b_1 - b_2)\), and (iii) \( \delta_1 \leq b_1/3 \). We now claim that \( V'(\delta) \leq 0 \) in some continuous subset of \([b_1 - b_2, \delta_1]\). This is because (i) \( V'(\delta) \) decreases in the interval \((b_1 - b_2, \delta_1)\), and so when \( V'(\delta) = 0 \) at some \( l_1 \in [b_1 - b_2, \delta_1]\), then \( V'(\delta) \leq 0 \) for every \( \delta \in [l_1, \delta_1]\); (ii) if \( V'(\delta) > 0 \) for every \( \delta \in (b_1 - b_2, \delta_1)\), then \( V(\delta) \geq 0 \) throughout the interval, and thus \( \int_{\delta}^{b_1} V(\delta) \, d\delta = 0 \) cannot be true. We have proved the claim.

Combining the fact that \( V(\delta^*) = V(\delta^*) = \int_{\delta}^{b_1} V(\delta) \, d\delta = 0 \), with \( V'(\delta) \) being nonnegative everywhere other than some continuous subset of \( \delta \in (b_1 - b_2, \delta_1) \), it is now easy to see that \( \int_{\delta}^{b_1} V(\delta) \, d\delta \geq 0 \) for all \( x \in [\delta^*, b_1^*] \).

**Proof of Theorem 16 (i):**

**Step 1:** We compute the virtual valuation function for Menu IV’ depicted in Figure 24.

\[
V(\delta) = \begin{cases} 
\bar{\mu}(Z) + \frac{3}{2} \delta^2 + 2b_2 \delta + \frac{b_2^2}{2} & \delta \in [-b_2, -\delta] \\
V(-\delta_2) - (c - 2b_2 + 3d_2)(\delta + \delta_2) + \frac{3}{2} \delta_1 + \delta_2) (\delta + \delta_2)^2 & \delta \in [-\delta_2, \delta_1] \\
V(\delta_1) + 2b_2(\delta - \delta_1) & \delta \in [\delta_1, b'] \\
V(b') - b_1(\delta - b_1 + b_2) - \frac{3}{2} ((b_1 - \delta)^2 - b_2^2) & \delta \in [b', b_1]
\end{cases}
\]

where \( b_1 - b_2 \) is denoted as \( b' \).

**Step 2:** Menu IV’ has three unknowns – \( \delta_1, \delta_2 \) and \( a_1 \). Since \( q = \nabla u \), a conservative field, we must have the slope of the line separating \((0, 0)\) and \((1 - a, a)\) allocation regions satisfying \(-\frac{1 - a}{a} = -\frac{\delta_2}{\delta_1}\). This yields \( a = \frac{\delta_1}{\delta_1 + \delta_2} \).

We now compute the other two parameters by equating \( \bar{\mu}(Z) = 0 \) and \( \int_{\delta}^{b_1} V(\delta) \, d\delta = 0 \). The latter condition follows from Theorem 16 3(c) because \( q_1(\delta) = 1 - a \in (0, 1) \) for \( \delta \in [-b_2/3, b_2/3 - b_2/4] \).

\[
\bar{\mu}(Z) = 0 \Rightarrow -\frac{3}{2} \delta_1 \delta_2 - c(\delta_1 + \delta_2) + b_1 b_2 = 0.
\]
\[\int_{-\frac{b_3}{2}}^{\frac{b_3}{2}} V(\delta) \, d\delta = 0\]

\[
\Rightarrow \int_{-\delta_2}^{-\delta_2} V(\delta) \, d\delta + \int_{\delta_2}^{\delta_1} V(\delta) \, d\delta + \int_{\delta_1}^{\frac{b_3}{2} - \frac{b_3}{2}} V(\delta) \, d\delta = 0
\]

\[
\Rightarrow b_2(\delta_2^2 - b_2^2/9) + \frac{1}{2}(b_2^3/27 - \delta_2^3) + \frac{b_2^2}{2}(b_2/3 - \delta_2)
\]

\[
- (3\delta_1\delta_2/2 + c(\delta_1 + \delta_2) - b_2^2/2)(2b_1 - b_2 - 4\delta_1)/4 + b_2(2b_1 - b_2)/4 - b_2\delta_1^2
\]

\[
- (2b_2\delta_2 - 3\delta_2^2/2 - b_2^2/2)(\delta_1 + \delta_2) - (c - 2b_2 + 2\delta_2)(\delta_1 + \delta_2)^2/4 = 0
\]

\[
\Rightarrow \frac{2b_3^3}{27} + \frac{\delta_1 - \delta_2}{2}(\delta_1\delta_2 + c(\delta_1 + \delta_2)) - \frac{b_2}{16}(2b_1 - b_2)^2 - \frac{(2b_1 - b_2)\mu(Z)}{4} = 0.
\]

(B.8)

The values of \(\delta_1\) and \(\delta_2\) can be obtained by solving (B.7) and (B.8) simultaneously.

**Step 3:** We now prove the bounds on parameters. In Steps 3a–3b, we prove the bound on \(\delta_1\) and \(\delta_2\) respectively.

**Step 3a:** We prove that \(\delta_1 \in [0, \frac{b_3}{2} - \frac{b_3}{4}]\). We fix \(\delta_2 = \frac{b_3(b_3 - c\delta_1)}{c + 3b_2/2}\) from (B.7), and substitute \(\delta_1 = \frac{b_3}{2} - \frac{b_3}{4}\) in the left-hand side of (B.8), to obtain

\[
- \left(\frac{(6b_1 + b_2)^2}{864(6b_1 - 3b_2 + 8c)^2}\right)
\]

\[
(72b_1^2b_2 + 144b_1b_2^2 - 90b_2^3 + (-36b_1^2 + 84b_1b_2 + 399b_2^2)c - (96b_1 + 208b_2)c^2)
\]

which is nonnegative for all \(c \geq \beta\). We now substitute \(\delta_1 = 0\), and obtain

\[
\frac{1}{432c} \frac{b_2}{c}(108b_1b_2c + 5b_2^2c - 216b_1^2b_2 - 108b_1^2c) \leq 0
\]

for \(b_1 \geq 3b_2/2\). This proves \(\delta_1 \in [0, \frac{b_3}{2} - \frac{b_3}{4}]\).

**Step 3b:** We prove that \(\delta_2 \in [0, \frac{b_3}{3}]\). We fix \(\delta_1 = \frac{b_3(b_3 - c\delta_1)}{c + 3b_2/2}\) from (B.7), and substitute \(\delta_2 = 0\), to obtain

\[
\frac{1}{432c} \frac{b_2}{c}(216b_1^2b_2 - 108b_1^2c + 108b_1b_2c + 5b_2^3) \geq 0
\]

for all \(c \leq \frac{216b_1^2b_2}{108b_1^2 - 108b_1b_2 - 5b_2^2}\). We now substitute \(\delta_2 = \frac{b_3}{3}\), and obtain

\[
\frac{b_2(6b_1 + b_2)^2(5b_2^2 + 12b_2c - 12c^2)}{432(b_2 + 2c)^2} \leq 0
\]

for \(c \geq 3b_2/2\). Observe that the value of \(c\) in the theorem statement is at least \(3b_2/2\), since \(c = 3b_2/2\) makes the left-hand side of (15) positive. This proves \(\delta_2 \in [0, \frac{b_3}{3}]\).
Similarly, the proof that parts (2) and (4) now traces the same steps as in Theorem 13. Observe that $V(2)–(4)$ are satisfied. We now show that the optimal menu is Menu $V'$ as depicted in Figure 26, where $c$ is the same as in Theorem 14. This completes the proof of optimality of $V'$.

At $c = \frac{216b_1b_2}{108b_1b_2 - 5b_2^4}$, we obtain $\delta_2 = 0$, when we solve (B.7) and (B.8) simultaneously. A transition thus occurs from Menu IV' to Menu V'. We now show that the optimal menu is Menu $V'$ as depicted in Figure 26 when $c \geq \frac{216b_1b_2}{108b_1b_2 - 5b_2^4}$.

**Step 1:** We first consider the zero allocation region to be $Z = (\{c, c + \frac{b_1b_2}{c}\}, c)$, and compute the virtual valuation function as follows.

$$
V(\delta) = \begin{cases} 
\bar{\mu}(Z) + \frac{3}{2}\delta^2 + 2b_2\delta + \frac{b_2^2}{2} & \delta \in [-b_2, 0] \\
V(0) - (c - 2b_2)\delta & \delta \in [0, \frac{b_1b_2}{c}] \\
V(\frac{b_1b_2}{c}) + 2b_2(\delta - \frac{b_1b_2}{c}) & \delta \in [\frac{b_1b_2}{c}, b'] \\
V(b') - b_1(\delta - b_1 + b_2) - \frac{3}{2}(b_1 - \delta)^2 - b_2^2 & \delta \in [b', b_2]
\end{cases}
$$

where $b_1 \in b_2$ is denoted by $b'$. Observe that the zero allocation region only consists of a portion of the bottom boundary. So when we modify $q(z)$ to be $(0, 1)$ instead of $(0, 0)$ for every $z \in Z$, the utility $u(z)$ remains the same for every $z \in D$. The expected revenue, $E_{z\sim f}[z \cdot q(z) - u(z)]$, also remains unchanged, because $\int_Z f(z)dz = 0$. So we continue our analysis of Menu $V'$, assuming the zero allocation region $Z$ to be non-empty.

The menu does not have any unknowns to compute. So steps 2 and 3 are not necessary. We move straightaway to step 4.

**Step 4:** We now prove that the conditions in Theorem 10 (2) and (4) are satisfied. Since $V'(\delta)$ changes sign from positive to negative only at $\delta = \max(\frac{2b_2}{3}, b_1 - b_2)$ in the interval where $q_1 = 1$, the proof for Theorem 10 (4) traces the same steps as in Theorem 13. But $V'(\delta)$ changes sign at three values of $\delta$ in the interval where $q_1 = 0$. So proving the other condition needs more work.

We have $V(-b_2) = V(-\frac{b_2}{3}) = 0$, and $V(\delta) \leq 0$ when $\delta \in [-b_2, -\frac{b_2}{3}]$. We now evaluate $\int_{-\frac{b_2}{3}}^{-b_2} V(\delta) d\delta$.

$$
\int_{-\frac{b_2}{3}}^{-b_2} V(\delta) d\delta = \frac{2}{27}b_2^2 + \frac{b_1^2b_2^2}{2c} - \frac{1}{16}b_2(2b_1 - b_2)^2 \leq 0 \quad (B.9)
$$

when $c \geq \frac{216b_1b_2}{108b_1b_2 - 5b_2^4}$. So we have $\int_{-\frac{b_2}{3}}^{-b_2} V(\delta) d\delta \leq 0$. 

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Observe that $V(\delta)$ is negative when $\delta \in [-b_2, -\frac{b_2}{3}]$, positive when $\delta \in [-\frac{b_2}{2}, \frac{b_2}{2(3c-2b_2)}]$, and negative again when $\delta \in [\frac{b_2}{2(3c-2b_2)}, -\frac{b_2}{2} - \frac{b_2}{4}]$. So $V(\delta)$ appears as in Figure [17] with rising zeros occurring only when $\delta \in \{-\frac{b_2}{3}, \frac{b_2}{2} - \frac{b_2}{4} \}$. The integral $\int_{-\frac{b_2}{2}}^{x} V(\delta) \, d\delta$ thus attains its minimum either at $-\frac{b_2}{3}$ or at $\frac{b_2}{2} - \frac{b_2}{4}$. But from (B.9), we have $\int_{-\frac{b_2}{2}}^{x} V(\delta) \, d\delta \leq 0$, and so the minimum cannot occur at $-\frac{b_2}{3}$. Therefore, $\int_{-\frac{b_2}{2}}^{x} V(\delta) \, d\delta \geq \int_{-\frac{b_2}{2}}^{\frac{b_2}{2} - \frac{b_2}{4}} V(\delta) \, d\delta$ holds for all $x \in [-b_2, \frac{b_2}{2} - \frac{b_2}{4}]$. Hence the result.

**Proof of Theorem 16(ii):** Consider case (ii), where $b_1 \in [b_2, 3b_2/2]$. The values of $V(\delta)$ and the expression for $\bar{\mu}(Z) = 0$ are the same as in the proof of Theorem 16(i). We thus skip step 1.

**Step 2:** We now compute the expression for $-\int_{-\frac{b_2}{2}}^{\frac{b_2}{2} - \frac{b_2}{4}} V(\delta) \, d\delta = 0$.

$$
- \int_{-\frac{b_2}{2}}^{\frac{b_2}{2} - \frac{b_2}{4}} V(\delta) \, d\delta = 0
$$

$$
\Rightarrow - \int_{-\frac{b_2}{2}}^{-\delta_2} V(\delta) \, d\delta - \int_{-\delta_2}^{\delta_1} V(\delta) \, d\delta - \int_{\delta_1}^{b_2-b_2} V(\delta) \, d\delta - \int_{b_2-b_2}^{b_2} V(\delta) \, d\delta = 0
$$

$$
\Rightarrow \frac{2}{27}(b_1^3 - b_2^3) - \frac{\delta_1 - \delta_2}{2} \left( b_1 b_2 - \frac{\delta_1 \delta_2}{2} \right) - \left( b_1 - b_2 - \frac{\delta_1 - \delta_2}{2} \right) \bar{\mu}(Z) = 0.
$$

(B.10)

The values of $(\delta_1, \delta_2)$ can be obtained by solving (B.7) and (B.10) simultaneously.

**Step 3:** We prove the bounds on the parameters. In Steps 3a–3b, we prove the bounds on $\delta_1$ and $\delta_2$, respectively.

**Step 3a:** We prove that $\delta_1 \in [0, \frac{b_1}{3}]$. We fix $\delta_2 = \frac{b_1 b_2 - c_1\delta_1}{c + 3b_2/2}$, and substitute $\delta_1 = \frac{b_1}{3}$ in the left-hand side of (B.10), to obtain

$$
\frac{4b_1^5 - 16b_1 b_2^3 c - 16b_2^3 c^2 + b_1^3 (-6b_2 + 15c)}{54(b - 1 + 2c)^2} + \frac{12b_1^3 (3b_2^3 - 3b_2c + c^2) - 4b_1^2 b_2 (b_2^3 - 27b_2c + 18c^2)}{54(b_1 + 2c)^2}
$$

We now verify if this expression is nonpositive for every $c \geq \alpha_2, b_1 \in [b_2, 3b_2/2]$. We first observe from Mathematica that $\alpha_2 \geq 1.36 b_2 + 2(t - 1.36)(b_1 - b_2)$, with $t = 3(37 + 3\sqrt{465})/176$ (see Appendix C.2(7)). We now prove that the expression is nonpositive when $c = 1.36 b_2 + 2(t - 1.36)(b_1 - b_2), b_1 \in [b_2, 3b_2/2]$, and that it is decreasing in $c$. Substituting $c = 1.36 b_2 +$
We now verify if this expression is nonnegative for every $\delta = 0$ when we simultaneously solve (B.7) and (B.10) simultaneously. Differentiating the numerator with respect to $c$, we have

$$-16b_1b_2^2 - 32b_3c + 15b_1^4 - 36b_1^3b_2 + 108b_1^2b_2(b_2 - c) + 24b_1^4c(b_1 - 3b_2/2) \leq 0$$

for every $c \geq b_2, b_1 \in [b_2, 3b_2/2]$. We have thus proved that $\int_{\frac{b_1}{c \pm \frac{1}{2}}} V(\delta) \leq 0$ when $\delta = 0$, we have $\delta_2 = b_1b_2/c$, and thus $\int_{-\frac{b_1}{c \pm \frac{1}{2}}} V(\delta) = 2(b_1^3 - b_2^3)/27 + (b_1b_2)/2c \geq 0$. This proves that there exists a $\delta_1 \in [0, \frac{b_1}{c}]$ when we simultaneously solve (B.7) and (B.10) simultaneously.

Step 3b: We now prove that $\delta_2 \in [0, \frac{b_1}{c}]$. We fix $\delta_1 = \frac{b_1b_2 - c\delta_2}{c^3b_2^2}$, and substitute $\delta_2 = 0$ in (B.10) to obtain $2/27(b_1^3 - b_2^3) - b_1b_2^2/2c \leq 0$ when $c \leq \frac{2b_1b_2^2}{3(b_1 - b_2)}$. Substituting $\delta_2 = \frac{b_1}{c}$, we get

$$\frac{4b_1^3(b_2 + 2c)^2 - 36b_1^2b_2^2(b_2 + 3c) + 6b_1b_2(b_2^2 + 6b_2c + 12c^2)}{54(b - 2 + 2c)^2} - \frac{b_1^2(4b_2^2 + 15b_2c + 12c^2)}{54(b + 2c)^2} \leq 0$$

We now verify if this expression is nonnegative for every $c \geq \alpha_2, b_1 \in [b_2, 3b_2/2]$. We verify this for $c \geq 1.36b_2 + 2(t - 1.36)(b_1 - b_2)$, and prove that it is increasing in $c$. Substituting $c = 1.36b_2 + 2(t - 1.36)(b_1 - b_2)$, we have

$$2(t - 1.36)(b_1 - b_2), \text{ we have }$$

$$(21.8931)b_1^3 - (52.8447)b_1^4b_2 + (33.1421)b_1^3b_2^2 + (14.2829)b_1^2b_2^3 - (24.4661)b_1b_2^4 - (6.01705)b_2^5 \leq 0$$

for $b_1 \in [b_2, 3b_2/2]$. We consider two cases: (a) $c \geq 2b_2$, (b) $c \in [\alpha_2, 2b_2]$. Consider
that \( \delta \) case (i), and as in Figure B.31 for case (ii). Appendix C.2(6)). Thus the menus III and IV appear as in Figure B.30 for \( V \) and \( V \) than \( \delta \) cases: (i) \((\mu, c)\)

\[ V(x) \] in Theorem 10 holds in Menu III, the proof is complete.

Consider case (b). At \( c = \alpha_2 \), we have \( \int_{\delta_{III}}^{\delta_{IV}} V(\delta) \, d\delta = \int_{\delta_{III}}^{\delta_{IV}} V(\delta) \, d\delta = 0 \). So when \( \delta \in [-\frac{b_1}{2}, \frac{b_1}{2}] \), \( V(\delta) \) appears as in Figure B.31 with \( a_0 = \delta^* \). We thus do the following. We compare Menu IV with Menu III. We first prove that \((\delta_{IV}^1, \delta_{IV}^2)\) obtained by solving (B.7) and (B.10) in Menu IV, is at most the value of \((\delta_{III}^1, \delta_{III}^2)\) values obtained in Menu III. We then argue that \( \int_{\delta_{III}}^{\delta_{IV}} (V_{IV}(\delta) - V_{III}(\delta)) \geq 0 \) for every \( x \in [-\frac{b_1}{2}, \frac{b_1}{2}] \). Since we know that condition 3d in Theorem [1] holds in Menu III, the proof is complete.

We now prove that \((\delta_{IV}^1, \delta_{IV}^2) < (\delta_{III}^1, \delta_{III}^2)\). Suppose not. We have two cases: (i) \((\delta_{IV}^1, \delta_{IV}^2) > (\delta_{III}^1, \delta_{III}^2)\), (ii) One of \((\delta_{IV}^1, \delta_{IV}^2)\), say \( \delta_{IV}^1 \), is greater than \( \delta_{III}^1 \). From Mathematica, we have \( a_1 + a_2 < 1 \) when \( c \in [\alpha_2, 2b_2) \) (see Appendix C.2(6)). Thus the menus III and IV appear as in Figure B.30 for case (i), and as in Figure B.31 for case (ii).

Figure B.30: The menus III and IV superimposed on each other, when \((\delta_{III}^1, \delta_{III}^2) < (\delta_{IV}^1, \delta_{IV}^2)\). Menu III is denoted by dotted lines, and Menu IV by solid lines.

Figure B.31: The menus III and IV superimposed on each other, when \( \delta_{III}^1 < \delta_{IV}^1 \), and \( \delta_{III}^1 < \delta_{IV}^2 \). Menu III is denoted by dotted lines, and Menu IV by solid lines.

Figure B.32: The menus III and IV superimposed on each other, when \((\delta_{III}^1, \delta_{III}^2) > (\delta_{IV}^1, \delta_{IV}^2)\). Menu III is denoted by dotted lines, and Menu IV by solid lines.

Consider case (i). We have \( \bar{\mu}_{III}(Z) = \bar{\mu}_{IV}(Z) \) — a negative number > \( \bar{\mu}_{III}(Z) \). So \( \bar{\mu}(Z) = 0 \) cannot hold simultaneously for both the menus, a contradiction. Consider case (ii). We then have \( V'_{III}(\delta) > V'_{IV}(\delta) \) for \( \delta \in (-\delta_{IV}^1, l_1) \) for some \( l_1 \in [-\delta_{III}^1, \delta_{IV}^1] \), \( V'_{III}(\delta) < V'_{IV}(\delta) \) for \( \delta \in (l_1, \delta_{III}^1) \), and \( V'_{III}(\delta) = V'_{IV}(\delta) \) for \( \delta \in [-\frac{b_2}{3}, -\delta_{IV}^1] \cup \{l_1\} \cup [\delta_{III}^1, \frac{b_1}{3}] \). We also have \( V_{III}(-\frac{b_2}{3}) = V_{III}(\frac{b_1}{3}) = V_{IV}(-\frac{b_2}{3}) = V_{IV}(\frac{b_1}{3}) = 0 \). So \( V_{III}(\delta) - V_{IV}(\delta) = 56 \)
This completes the proof of optimality of Menu IV. Thus \( \int_{-\frac{b_3}{3}}^{\frac{b_1}{3}} (V_{III}(\delta) - V_{IV}(\delta)) \, d\delta > 0 \) for all \( \delta \in (-\frac{b_3}{3}, \frac{b_1}{3}) \).

We thus have \((\delta_1^{III}, \delta_2^{III}) > (\delta_1^{IV}, \delta_2^{IV})\), and the menus appear as in Figure 13.32. \( V_{IV}'(\delta) \) for \( \delta \in (-\delta_2^{III}, l_1) \cup (l_2, \delta_1^{III}) \) for some \( l_1 \in [-\delta_2^{IV}, \delta_1^{IV}] \) and \( l_2 \in [l_1, \delta_1^{IV}] \). We also have \( V_{III}(-\frac{b_3}{3}) = V_{IV}(-\frac{b_3}{3}) \) when \( \delta \in [-\frac{b_3}{3}, \frac{b_1}{3}] \). We also have \( V_{III}(\frac{b_1}{3}) = V_{IV}(\frac{b_1}{3}) = \int_{-\frac{b_3}{3}}^{\frac{b_1}{3}} V_{III}(\delta) \, d\delta = \int_{-\frac{b_3}{3}}^{\frac{b_1}{3}} V_{IV}(\delta) \, d\delta = 0 \). We now have a series of observations.

1. \( V_{IV}(\delta) - V_{III}(\delta) = V_{IV}(-\frac{b_3}{3}) - V_{III}(-\frac{b_3}{3}) + \int_{-\frac{b_3}{3}}^{\delta} (V_{IV}'(\delta) - V_{III}'(\delta)) \, d\delta \geq 0 \) when \( \delta \in [-\frac{b_3}{3}, l_1] \).

2. \( V_{IV}(\delta) - V_{III}(\delta) = V_{IV}(\frac{b_1}{3}) - V_{III}(\frac{b_1}{3}) - \int_{\frac{b_1}{3}}^{\delta} (V_{IV}'(\delta) - V_{III}'(\delta)) \, d\delta \leq 0 \) when \( \delta \in [l_2, \frac{b_1}{3}] \).

3. By a similar argument, it is easy to see that \( V_{IV}(\delta) - V_{III}(\delta) \) is non-negative when \( \delta \in [l_1, m] \) for some \( m \in [l_1, l_2] \), and is nonpositive when \( \delta \in [m, l_2] \). We thus have \( V_{IV}(\delta) \geq V_{III}(\delta) \) when \( \delta \in [-\frac{b_3}{3}, m] \), and \( V_{IV}(\delta) \leq V_{III}(\delta) \) when \( \delta \in [m, \frac{b_1}{3}] \).

4. \( \int_{-\frac{b_3}{3}}^{x} (V_{IV}(\delta) - V_{III}(\delta)) \, d\delta \geq 0 \) when \( x \in [-\frac{b_3}{3}, m] \).

5. \( \int_{\frac{b_1}{3}}^{x} (V_{IV}(\delta) - V_{III}(\delta)) \, d\delta = -\int_{-\frac{b_3}{3}}^{x} (V_{IV}(\delta) - V_{III}(\delta)) \, d\delta \geq 0 \) for any \( x \in [m, \frac{b_1}{3}] \).

6. Notice that \( \int_{-\frac{b_3}{3}}^{x} V_{III}(\delta) \, d\delta \geq 0 \) holds for any \( x \in [-\frac{b_3}{3}, \frac{b_1}{3}] \), and thus \( \int_{-\frac{b_3}{3}}^{x} V_{IV}(\delta) \, d\delta \geq 0 \) now follows.

This completes the proof of optimality of Menu IV.

For the proof of optimality of Menu V when \( c \geq \frac{27b_2b_3}{4(b_1^3 - b_2^3)} \), we note that the proof is exactly the same as in the proof of Theorem 16(i). except for the term \( \int_{-\frac{b_3}{3}}^{\frac{b_1}{3}} V(\delta) \, d\delta = \frac{2}{27}(b_3^3 - b_1^3) + \frac{b_2^2b_3^3}{2c} \). The expression clearly is negative when \( c \geq \frac{27b_2b_3}{4(b_1^3 - b_2^3)} \). \( \square \)

**Proof of Theorem 17:** We fix \( c_1 - c_2 = d \). Observe that the domain of the functions \( (q_1, t) \) is the interval \([d - b_3, d - b_1]\). But it can be verified that all the results hold even for a shifted version of the domain. So we redefine \( \delta = z_1 - z_2 - d \), and retain the domain to be \([-b_2, b_1]\).
\textbf{Step 1:} We compute the virtual valuation function for Menu I depicted in Figure [20]

\[ V(\delta) = \begin{cases} 
\hat{\mu}(Z) + \frac{3}{2}\delta^2 + 2b_2\delta + \frac{b_2^2}{2} + d(\delta + b_2) & \delta \in [-b_2, -b_2] \\
V(\delta_2) - (c_2 - 2b_2 + 3\delta_2)(\delta + \delta_2) & \delta \in [-\delta_2, \delta_*] \\
V(\delta_1) - (c_2 - 2b_2)(\delta - \delta_*) + \frac{3}{2}((\delta_1 - \delta)^2 - \delta_*^2) & \delta \in [\delta_*, b_1 - b_2] \\
V(b_1 - b_2) - (c_1 - 2b_1 + 3\delta_1)(\delta - b_1 + b_2) & \delta \in [b_1 - b_2, \delta_1] \\
-\frac{3}{2}\delta^2 + 2b_2\delta - \frac{b_2^2}{2} - d(\delta - b_2) & \delta \in [\delta_1, b_1] 
\end{cases} \]

\textbf{Step 2:} Menu I has three unknowns \(-\delta^*, \delta_1, \) and \(\delta_2\). Observe that the line between the points \((c_1 + b_2 + \delta^*, c_2 + b_2)\) and \((c_1 + \delta^*, c_2)\) passes through \((c_1 + \delta_1, c_2 + \delta_2)\). So we have \(\delta^* = \delta_1 - \delta_2\).

We now proceed to compute \(\delta_1\) and \(\delta_2\). We do so by equating \(\hat{\mu}(Z) = 0\) and \(V(\delta^*) = 0\). The latter follows from Theorem [10] because \(q_1 = 0\) for \(\delta \in [-b_2, \delta^*]\).

\[ \hat{\mu}(Z) = 0 \Rightarrow -3\delta_1\delta_2 - c_2\delta_1 - c_1\delta_2 + b_1b_2 = 0. \tag{B.11} \]

\[ V(\delta^*) = 0 \Rightarrow -\frac{3}{2}\delta^2 + 2b_2\delta - \frac{b_2^2}{2} - d(\delta - b_2) + (c_2 - 2b_2 + 3\delta_2)\delta_1 = 0. \tag{B.12} \]

The values of \(\delta_1\) and \(\delta_2\) can be computed by solving (B.11) and (B.12) simultaneously.

\textbf{Step 3:} We now prove the bounds on \(\delta_1\) and \(\delta_2\). In Steps 3a–3b, we show the bounds on \(\delta_1\) and \(\delta_2\) respectively. Specifically, we show that there exists a \(\delta_2 \in [\frac{b_2 + 2d}{3}, \frac{2b_2 - c_2}{3}]\) and \(\delta_1 \in [\frac{b_1}{2} - \frac{c_1}{3} + \frac{c_1 c_2}{6b_2}, \frac{2b_2 - c_1}{3}]\), as a simultaneous solution to (B.11) and (B.12).

\textbf{Step 3a:} Substituting \(\delta_1 = \frac{b_1b_2 - c_1\delta_2}{3b_2 + c_2}\) from (B.11) in (B.12) and simplifying, we get

\[ 9\delta_2^3 + \delta_2^2(9c_2 - 12b_2) + \delta_2(2c_2^2 - 10b_2c_2 + 2b_2c_1 + 3b_2^2 - 6b_1b_2) + \frac{b_2^2}{4}c_2 - 2b_1b_2c_2 + 4b_1b_2^2 + 2b_2c_2(c_1 - c_2) = 0. \tag{B.13} \]

When \(\delta_2 = \frac{b_2 - c_1}{3}\), the left-hand side of (B.13) equals \(-\frac{2}{3}b_2(b_2 + c_2)(b_2 - 2c_1 + c_2) \leq 0\) for \(2c_1 - c_2 \leq b_2\). When \(\delta_2 = \frac{b_2 + 2d}{3}\), the left-hand side of (B.13) equals \(\frac{3}{2}(3b_1b_2 - c_1(b_2 + 2d))(b_2 - 2c_1 + c_2) \geq 0\) for \(c_1 \leq b_2, d \leq b_2/2,\) and \(2c_1 - c_2 \leq b_2\). This shows that there exists a solution \(\delta_2 \in [\frac{b_2 + 2d}{3}, \frac{2b_2 - c_2}{3}]\) for every \(c_1, c_2\) in the theorem statement.

\textbf{Step 3b:} Substituting \(\delta_2 = \frac{b_1b_2 - c_1\delta_2}{3b_2^2 + c_2}\) from (B.11) in (B.12) and simplifying, we have

\[ -36b_2\delta_1^3 + (9b_2(2b_1 - b_2) + 6b_2(c_2 - 7c_1) + 3c_2^2)\delta_1^2 + (12b_1b_2(b_2 + c_1) - 2b_2c_1(3b_2 + 8c_1)) + 8b_2c_1c_2 + 2c_1c_2^2)\delta_1 - 3b_1b_2^2 + 4b_1b_2^2c_1 + 2b_1b_2^2c_1^2 - b_2^2c_1^2 - b_2^2c_2^2 - 2b_1b_2c_1c_2 + 2b_2c_1^2c_2 = 0. \tag{B.14} \]
When $\delta_1 = \frac{2b_1 - c_1}{3}$, the left-hand side of (B.14) equals $\frac{1}{3}(-8b_2^3b_2^2 + 2b_1b_2c_1c_2 - c_1^2c_2^2 + b_1^2(3b_2^2 - 8b_2c_1 + 8b_2c_2 + 4c_2^2))$. We now prove that this expression is negative for all $c_1, c_2$ under consideration.

\[-8b_1^3b_2 + 2b_1b_2c_1c_2 - c_1^2c_2^2 + b_1^2(3b_2^2 - 8b_2(c_1 - c_2) + 4c_2^2)\]
\[\leq -8b_1^3b_2 + 2b_1b_2c_1c_2 - c_1^2c_2^2 + b_1^2(3b_2^2 + 4c_2^2)\]
\[\leq -8b_1^3b_2 + 4b_1^3b_2^2 + 4b_1^2c_2^2 \leq -8b_1^3b_2(b_1 - b_2) \leq 0\]

where the first inequality follows from $c_1 \geq c_2$; the second inequality occurs because the expression is maximized when $c_1 = \frac{b_1}{c_2}$; the third inequality follows because when $c_2 \in [0, b_2]$, the expression is maximized at $c_2 = b_2$; and the final inequality occurs since $b_1 \geq b_2$.

Now when $\delta_1 = \frac{b_1}{2} - \frac{c_1}{3} + \frac{c_1^2}{6b_2}$, the left-hand side of (B.14) equals $\frac{1}{12b_2}(b_2 + c_2)(b_2 - 2c_1 + c_2)(3b_1b_2 + c_1c_2)^2 \geq 0$ for $2c_1 - c_2 \leq b_2$. This shows that there exists a solution $\delta_1 \in \left[\frac{b_1}{2} - \frac{c_1}{3} + \frac{c_1^2}{6b_2}, \frac{2b_1 - c_1}{3}\right]$.

**Step 4:** We now proceed to prove parts (c) and (d) in Theorem III (2) and III (4). We first compute $V'(\delta)$ for almost every $\delta \in [-b_2, b_1]$.

\[V'(\delta) = \begin{cases} 
3\delta + 2b_2 + d & \delta \in (-b_2, -\delta_2) \\
-(c_2 - 2b_2 + 3b_2) & \delta \in (-\delta_2, \delta^*) \\
-(c_2 - 2b_2) - 3(\delta_1 - \delta) & \delta \in [\delta^*, b_1 - b_2) \\
-(c_1 - 2b_1 + 3\delta_1) & \delta \in (b_1 - b_2, \delta_1) \\
-3\delta + 2b_1 - d & \delta \in (\delta_1, b_1) 
\end{cases}\]

Observe that $V'(\delta)$ is negative when $\delta \in [-b_2, -\frac{2b_1 + d}{3}]$, and positive when $\delta \in [-\frac{2b_1 + d}{3}, -\delta_2]$ (follows because $\delta_2 \leq \frac{2b_1 + c_2}{3}$). We also have $V(-b_2) = V(\delta^*) = 0$. So $V(\delta) = V(-b_2) + \int_{-b_2}^{\delta} V'(\delta) \, d\delta \leq 0$ for all $\delta \in [-b_2, \delta^*]$, and hence $\int_{-b_2}^{\delta^*} V(\delta) \, d\delta \leq 0$, and $\int_{-b_2}^{\delta^*} V(\delta) \, d\delta \geq \int_{-b_2}^{\delta^*} V(\delta) \, d\delta$ for all $x \in [-b_2, \delta^*]$.

We now claim that $V'(\delta)$ is positive when $\delta \in [\delta^*, \frac{2b_1 + d}{3}]$, and negative when $\delta \in [\frac{2b_1 + d}{3}, b_1]$. Observe that $V'(\delta)$ is continuous at $\delta = \delta^*$, and that it increases in the interval $[\delta^*, b_1 - b_2]$. So $V'(\delta) \geq 0$ when $\delta \in [\delta^*, b_1 - b_2]$. Also, $V'(\delta) \geq 0$ when $\delta \in [b_1 - b_2, \delta_1]$ because $\delta_1 \leq \frac{2b_1 - c_1}{3}$. That $V'(\delta)$ is positive when $\delta \in [\delta_1, \frac{2b_1 - d}{3}]$, and negative when $\delta \in [\frac{2b_1 - d}{3}, b_1]$ is obvious. We have proved our claim.

Since we also have $V(b_1) = 0 = V(\delta^*)$, it follows that $V(\delta) = V(\delta^*) + \int_{\delta^*}^{\delta} V'(\delta) \, d\delta \geq 0$ for all $\delta \in [\delta^*, b_1]$. So we have $\int_{\delta^*}^{b_1} V(\delta) \, d\delta \geq 0$ and $\int_{\delta^*}^{b_1} V(\delta) \, d\delta \leq \int_{\delta^*}^{b_1} V(\delta) \, d\delta$ for all $x \in [\delta^*, b_1]$. □

**Appendix C. Proofs using Mathematica**

**Appendix C.1. Expressions used in Theorem I**

1. We first find the expression for $h$ that solves (11) and (11) simultaneously.
Mathematica Input:
\[ \delta^* = (b_1 b_2 - b_2^2/2 - 3h^2/2 - ch)/(2b_2); \]
Solve\[27(c+h+\delta^*)(b_2+\delta^*)^2-4(4b_2+3\delta^*)(3/2(h+\delta^*)+c)^2 == 0, h] \]

Mathematica Output:
\[
\{h \to \text{Root}[\text{-}72b_1^2b_2^2 - 144b_1 b_3^2 + 90b_2^2 + 36b_1^2b_2^2c - 84b_1b_3^2c - 399b_2^4c + 96b_1b_2^3c^2 + 208b_3^2c^2 + (108b_1^2b_2^2 + 36b_1b_3^2 - 477b_1^2 + 432b_1b_3^2c + 768b_2^2c - 72b_1b_2^2c^2 + 84b_3^2c^2 - 96b_2c^3)\#1 + (432b_1b_2^2 + 68b_2^4 - 324b_1b_2c + 90b_2^2c - 504b_2c^2 + 36c^3)\#1^2 + (-324b_1b_2 - 54b_2^2 - 84b_2c + 216c^2)\#1^3 + (-48b_2 + 405c)\#1^4 + 243\#1^5 & \in [1]],\} \} \} (all five roots) \}

In this subsection, we verify (i) \( \delta_2 \leq b_2/3 \) when \( b_1 \geq b_2, c \in [b_2, 2b_2], \)
(ii) the left-hand side of \( (12) \) is nonnegative when \( b_1 \in [b_2, 3b_2/2], c \in [b_2, \alpha_1], \)
and (iii) \( 2(t-1)(b_1 - b_2) + b_2 \geq \alpha_1, \) where \( t = 3(37 + 3\sqrt{465})/176. \) We will use bullet (1) above.

2. We now proceed to verify if \( \delta_2 \leq b_2/3. \) From \( (28), \) we have \( \delta_2 = h\#2-(3h/2+c)(h+\delta^*)/h. \) Observe that this is in terms of \( (h, \delta^*) \) that are obtained by solving \( (10) \) and \( (11). \) We thus initialize the values of \( h \) and \( \delta^* \) using expressions from bullet (1) above, and then find the values of \( (c, b_1, b_2) \) for which \( \delta_2 \leq b_2/3 \) holds.

Mathematica Input:
\[ h = \text{Root}[\text{-}72b_1^2b_2^2 - 144b_1 b_3^2 + 90b_2^2 + 36b_1^2b_2^2c - 84b_1b_3^2c - 399b_2^4c + 96b_1b_2^3c^2 + 208b_3^2c^2 + (108b_1^2b_2^2 + 36b_1b_3^2 - 477b_1^2 + 432b_1b_3^2c + 768b_2^2c - 72b_1b_2^2c^2 + 84b_3^2c^2 - 96b_2c^3)\#1 + (432b_1b_2^2 + 68b_2^4 - 324b_1b_2c + 90b_2^2c - 504b_2c^2 + 36c^3)\#1^2 + (-324b_1b_2 - 54b_2^2 - 84b_2c + 216c^2)\#1^3 + (-48b_2 + 405c)\#1^4 + 243\#1^5 & \in [3]],\} \]
\[ \delta^* = (b_1 b_2 - 3/2h^2 - ch - b_2^2)/2b_2; \delta_2 = b_1 b_2 (3h/2 + c)(h + \delta^*)/(3(h + \delta^*)/2 + c); \]
Reduce\[\delta_2 \leq b_2/3 \& \& 0 \leq b_2 \leq b_1 \& \& b_2 \leq c \leq 2b_2, \{b_2, b_1, c\}] 

Mathematica Output:
\[ b_2 > 0 \& \& b_1 \geq b_2 \& \& b_2 \leq c \leq 2b_2 \]
The output indicates that \( \delta_2 \leq b_2/3 \) holds for every \( b_1 \geq b_2, c \in [b_2, 2b_2]. \)

3. We then find the values of \( c \) for which the left-hand side of \( (12) \) is nonnegative.

Mathematica Input:
\[ \text{Reduce}[-2b_1^3/27 - b_2(\delta^*)^2 + b_2\delta^*(b_1 - b_2/2) + (c + h)h^2/2 \geq 0 \& \& 0 \leq b_2 \leq b_1 \leq 3b_2/2 \& \& b_2 \leq c \leq tb_2, \{b_2, b_1, c\}] \]
Appendix C.2. Expressions used in Theorem 15:

1. We first prove that the left-hand side of (11) is nonnegative when

\[ \alpha = \text{Root}[f_{c-I1}(c) \& \& 2] \]

Here, \( f_{c-I1}(c) \) is a polynomial of degree 12. We have not written it here since it is too long. Let \( \alpha_1 = \text{Root}[f_{c-I1}(c) \& \& 2] \). Then this proves that the left-hand side of (12) is nonnegative for every \( c \in [b_2, \alpha_1] \).

4. To prove that \( \alpha_1 \leq 2(t-1)(b_1-b_2)+b_2 \), with \( t = 3(37 + 3\sqrt{465})/176 \), we again find the values of \( c \) for which the left-hand side of (12) is nonnegative, but with \( c \) restricted to \( c \in [b_2, 2(t-1)(b_1-b_2)+b_2] \).

**Mathematica Input:**

\[ t = \frac{3}{176}(37 + 3\sqrt{465}); \text{Reduce}[\frac{-2b_2^3}{27} - b_2(\delta^*)^2 + b_2\delta^*(b_1 - b_2/2) + (c + h)h^2/2 \geq 0 \& \& 0 \leq b_2 \leq 1 \leq 3b_2/2 \& \& b_2 \leq c \leq 2(t-1)(b_1-b_2) + b_2, \{b_2, b_1, c\}] \]

**Mathematica Output:**

\[ b_2 > 0 \& \& b_2 \leq 1 \leq 5b_2 \& \& b_2 \leq c \leq \text{Root}[f_{c-I1}(c) \& \& 2] \]

This proves that \( \alpha \leq 2(t-1)(b_1-b_2) + b_2 \). This completes all the proofs in Theorem 14.

**Appendix C.2. Expressions used in Theorem 15:**

1. We first prove that the left-hand side of (11) is nonnegative when

\( (h, \delta^*) = (0, \delta^*_0) \).

**Mathematica Input:**

\[ \delta^* = \frac{b_2}{2} - \frac{b_2}{4} - \frac{(3b_2-2b_1)}{6b_2}\sqrt{b_2c(8b_1-3b_2)(2b_2-c)} \]

\[ t = 3(37 + 3\sqrt{465})/176; \text{Reduce}[27b_2^2c-16b_2c^2+(27b_2^2+6b_2c-12c^2)\delta^*+(18b_2-9c)(\delta^*)^2 \geq 0 \& \& 0 \leq b_2 \leq b_1 \leq 1.5b_2 \& \& b_2 \leq c \leq 2(t-1.4b_2)(b_1-b_2) + 1.4b_2, \{b_2, b_1, c\}] \]

**Mathematica Output:**

\[ b_2 > 0 \& \& b_2 \leq b_1 \leq 1.5b_2 \& \& b_2 \leq c \leq 0.66676b_1 + 0.73324b_2 \]

2. We now prove that the left-hand side of (11) is nonnegative when

\( (h, \delta^*) = (h_{b_1-b_2}, b_1-b_2) \).

**Mathematica Input:**

\[ h = \frac{9b_2^2-4c(b_1+3b_2)+3b_2\sqrt{9b_2^2-4c(b_1+3b_2)}}{6(b_1+3b_2)} \]

\( \text{Reduce}[(27b_2^2(b_1-b_2+c+h)-(3b_1+b_2)(3b_1-3b_2+2c+3h)^2) \geq 0 \& \& 0 \leq b_2 \leq b_1 \leq 1.5b_2 \& \& b_2 \leq c \leq 2(t-1.4)(b_1-b_2) + \)

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1.4b₂, \{b₂, b₁, c\}

Mathematica Output:

\[ b₂ > 0 \&\& b₂ ≤ b₁ ≤ 1.5b₂ \&\& b₂ ≤ c ≤ 0.66676b₁ + 0.73324b₂ \]

3. We now find the expression for \(δ^*\) that solves (11) and (13) simultaneously.

Mathematica Input:

\[ h = \frac{96b₂^3 - 16b₂c - 6δ(b₂ + 2c) - 9(δ^*)^2}{6(4b₂ + 3δ^*)} \]
\[ + \frac{3(b₂ + δ^*)\sqrt{9b₂^2 + 16b₂c + 6δ^*(3b₂ + 2c) + 9(δ^*)^2}}{6(4b₂ + 3δ^*)}; \]

\[ Solve \left[ \frac{b₂}{3} - \frac{b₃}{4} - \frac{(3b₂ - 2b₁)(2c + 3h)}{12} \right] = \delta^*, \delta^* \]

Mathematica output:

\{ \{δ^* \to \{-4b₂/3\}\} (twice), \{δ^* \to \text{Root}\{-16(b₁ - b₂)^3b₂^1(b₂^3 - 3b₁b₂ + b₂^2)^2c^2 + (-4(b₁ - b₂)δ(b₂^3 - 3b₁b₂ + b₂^2)(108b₂^3b₁ - 108b₂ b₂^3 + 27b₂^3 + 24b₁b₂c + 96b₁^2b₂c^2 - 96b₁b₂^3c + 24b₂c + 16b₁^3c^2 - 24b₁b₂^2c^2 + 44b₁b₂^2c^2 - 16b₁^3c^2))\#1 + (144b₁^2b₂^4 + 176b₁^2b₂^3 - 13752b₁b₂^5 + 15660b₂b₂^6 - 7218b₂b₂^7 + 1152b₂b₂^8 - 256b₂b₂^9 + 512b₁b₂^10 + 4384b₁b₂^11 - 18064b₁b₂^12 + 21888b₁b₂^13 - 10368b₁b₂^14 + 1680b₂b₂^15 - 96b₂b₂^16c + 320b₁b₂^17c - 144b₂b₂^18c - 1392b₂b₂^19c + 1884b₂b₂^20c - 768b₂b₂^21c + 96b₂c^2\#1^2 + (192b₁b₂^2 + 624b₁b₂^3 + 4128b₁b₂^4 - 24828b₁b₂^5 + 42984b₁b₂^6 - 27342b₁b₂^7 + 5580b₂b₂^8 - 224b₂b₂^9c + 768b₂b₂^10c + 4960b₂b₂^11c + 58680b₂b₂^12c - 38808b₁b₂^13c + 8064b₂b₂^14c - 64b₂b₂^15c + 96b₂b₂^16c + 1216b₂b₂^17c - 4120b₂b₂^18c + 7632b₂b₂^19c - 5056b₂b₂^20c + 1016b₂c^2\#1^3 + (-64b₂b₂^2 + 288b₂b₂^3 + 2400b₁b₂^4 - 20496b₁b₂^5 + 5512b₁b₂^6 + 52656b₂b₂^7 + 14364b₂b₂^8 - 64b₂b₂^9c + 288b₂b₂^10c + 2688b₂b₂^11c - 23560b₂b₂^12c + 69120b₂b₂^13c - 70320b₂b₂^14c + 20040b₂b₂^15c - 16b₂^2c + 1344b₂b₂^3c - 3280b₂b₂^4c - 2592b₂b₂^5c - 4020b₂b₂^6c + 144b₂b₂^7c)\#1^4 + (576b₂b₂^2 - 6864b₁b₂^3b₂ + 33696b₂b₂^4b₂ + 56628b₂b₂^5b₂ + 24696b₂b₂^6b₂ + 576b₂b₂^7b₂ - 7152b₂b₂^8b₂ + 37584b₂b₂^9b₂ + 10380b₂b₂^10b₂ + 33360b₂b₂^11b₂ + 432b₂b₂^2c^2 - 1506b₂b₂^2c^2 - 5184b₁b₂^3b₂^2 + 10224b₁b₂^3b₂^2 + 2616b₂^2c^2)\#1^5 + (-576b₂b₂^2 + 7776b₂b₂^3b₂ - 32832b₁b₂^4b₂ + 27396b₂b₂^5b₂ - 576b₂b₂^6b₂ + 7776b₂b₂^7b₂ - 36720b₂b₂^8b₂ + 34272b₂b₂^9b₂ - 432b₂b₂^2c^2 - 2916b₂b₂^2c^2 + 15876b₂b₂^3c^2 + 9729b₂^2c^2)\#1^6 + (-108b₂b₂^2b₂^2c^2 + 36b₂b₂^2b₂^2c^2 + 18b₂b₂^2c^2 + 29b₂b₂^2c^2)\#1^7 + (972b₂b₂^2b₂^2c^2 + 2b₂b₂^2c^2 + 3c)\#1^8 \&\& 1\} \{\text{all eight roots}\}; \]

We verify (i) \(δ₂ ≤ b₂/3\) when \(b₁ \in \{b₂, 3b₂/2\}, c \in \{b₂, 2b₂\}\), (ii) \(δ₂ ≤ b₁/3\) when \(b₁ \in \{b₂, 3b₂/2\}, c \in \{b₁, 2b₂\}\), (iii) the left-hand side of (14) is nonnegative when \(b₁ \in \{b₂, 3b₂/2\}, c \in \{b₂, α₂\}\), and (iv) \(2(t - 1.36)/(b₁ - b₂) + 1.36b₂ ≤ α₂ ≤ 2(t - 1.4)(b₁ - b₂) + 1.4b₂\), where \(t = 3(37 + 3\sqrt{465})/176\). We will use bullet (3) above.

4. We now verify if \(δ₂ ≤ b₂/3\). From the statement of Theorem 13, we have \(δ₂ = \frac{b₂^2/2 + (2b₂ - c - 3b₂/2)δ^*}{3(h + δ^*)/2 + c}\). We now initialize \((h, δ^*)\) as in bullet (3), and
find the values of \((c, b_1, b_2)\) for which \(\delta_2 \leq b_2/3\).

Mathematica Input:

\[
\delta^* = \text{Root}[\neg 16(b_1 - b_2)^2 b_2^2 (b_1^2 - 3b_1 b_2 + b_2^2)^2 c^2 + \neg 4(b_1 - b_2) b_2^3 (b_1^2 - 3b_1 b_2 + b_2^2) (108b_1 b_2^3 - 108b_1 b_2^2 + 27b_2^2 + 24b_2 b_2 c + 96b_1 b_2 c - 96b_1 b_2^2 c + 24b_2^2 c + 16b_1^2 c^2 - 24b_2^2 b_2 c^2 + 44b_1^2 b_2^2 c^2 - 16b_1^2 c^2) \#1^1 + \neg 144b_1 b_2^2 + 4176b_1 b_2^2 - 13752b_1 b_2^2 + 15600b_1 b_2^2 - 7218b_1 b_2^2 + 1152b_1 b_2^2 + 512b_1^2 b_2^2 + 4384b_1^2 b_2^2 - 1806b_1 b_2^2 c + 21888b_1 b_2^2 c - 10368b_1 b_2^2 c + 1608b_1^2 c - 96b_1^2 b_2^2 c^2 + 320b_1^2 b_2^2 c^2 - 144b_1^2 b_2^2 c^2 - 1392b_1^2 b_2^2 c^2 + 1884b_1^2 b_2^2 c^2 - 768b_1 b_2^2 c^2 + 96b_1^2 c^2) \#1^1 + \neg 192b_0 b_2^3 + 624b_0 b_2^3 + 4128b_0 b_2^3 - 24828b_0 b_2^3 + 4298b_0 b_2^3 - 27432b_0 b_2^3 + 5580b_0^3 - 224b_0^3 b_2^2 c + 768b_0^3 b_2 c + 4960b_0^3 b_2^2 c - 31904b_0^3 b_2^3 c + 58680b_0^3 b_2^4 c - 38808b_0^3 b_2^5 c + 8064b_0^3 c - 64b_0^3 b_2^2 c^2 + 96b_0^3 b_2^2 c^2 + 1216b_0^3 b_2^2 c^2 - 4120b_0^3 b_2^2 c^2 + 7632b_0^3 b_2^2 c^2 - 5056b_0^3 b_2^2 c^2 + 1016b_0^3 b_2^2 c^2) \#1^3 + \neg 64b_0^3 b_2^2 + 288b_0^3 b_2^2 + 2400b_0^3 b_2^2 + 20496b_0^3 b_2^2 + 55512b_0^3 b_2^2 - 52650b_0^3 b_2^2 + 14364b_0^3 - 64b_0^3 b_2 c + 288b_0^3 b_2 c + 2688b_0^3 b_2 c - 23560b_0^3 b_2 c - 70320b_0^3 b_2 c + 20040b_0^3 b_2 c - 16b_0^3 c^2 + 1344b_0^3 b_2 c^2 - 3280b_0^3 b_2^2 c^2 + 2592b_0^3 b_2^2 c^2 - 4020b_0^3 b_2^2 c^2 + 1424b_0^3 b_2^2 c^2) \#1^4 + (576b_0^3 b_2^2 - 6864b_0^3 b_2^2 + 33696b_0^3 b_2^5 - 56628b_0^3 b_2^5 + 24696b_0^3 b_2^5 + 576b_0^3 b_2^5 c - 7152b_0^3 b_2^5 c + 37584b_0^3 b_2^5 c - 70380b_0^3 b_2^5 c + 33360b_0^3 b_2^5 c + 432b_0^3 b_2^2 c^2 - 156b_0^3 b_2^2 c^2 - 5184b_0^3 b_2^2 c^2 + 10224b_0^3 b_2^2 c^2 - 2616b_0^3 b_2^2 c^2) \#1^5 + (-576b_0^3 b_2^5 + 7776b_0^3 b_2^5 b_2^2 - 32832b_0^3 b_2^5 b_2^2 + 27396b_0^3 b_2^5 b_2^2 c - 7776b_0^3 b_2^5 b_2^2 c - 36720b_0^3 b_2^5 b_2^2 c + 34272b_0^3 b_2^5 b_2^2 c - 432b_0^3 b_2^2 c^2 - 2916b_0^3 b_2^2 c^2 + 15876b_0^3 b_2^2 c^2 - 9729b_0^3 b_2^2 c^2) \#1^6 + \neg 108b_0^3 b_2^2 (2b_2 + 3c)(36b_0 b_2 - 76b_2^2 - 18b_1 c + 29b_2 c)) \#1^7 + (972b_0^3 (2b_2 - c)(2b_2 + 3c)) \#1^8 \& 5];
\]

\[
h = \frac{9b_2 - 16b_2 c - 6\delta^*(b_2 + 2c) - 9\delta^*}{6(4b_2 + 3c)};
\]

\[
\delta_2 = \frac{b_2^2 + 2b_2 c - 3b_2 c + 2c}{(3(b_2 + c))/2 + c};
\]

Reduce\(\delta_2 - b_2/3 \leq 0 \& c \leq b_2 < b_1 < 1.5b_2 \& c \leq b_2 < 2b_2, \{b_2, b_1, c\}\)

Mathematica Output:

\(b_2 > 0 \& c \leq b_1 < 1.5b_2 \& c \leq b_2 < 2b_2\)

5. We now verify if \(\delta_1 \leq b_1/3\). We use \(\delta_1 = \delta^* + \frac{b_1 b_2 - 2b_2 \delta^* - b_2^2/2}{3b_1/2 + c}\) from the statement of Theorem 13.

Mathematica Input:

\[
\delta_1 = \delta^* + \frac{b_1 b_2 - 2b_2 \delta^* - b_2^2/2}{3b_1/2 + c};
\]

Reduce\(\delta_1 - b_1/3 \leq 0 \& c \leq b_2 < b_1 < 1.5b_2 \& c \leq b_2 < 2b_2, \{b_2, b_1, c\}\)

Mathematica Output:

\(b_2 > 0 \& c \leq b_1 < 1.5b_2 \& c \leq b_2 < 2b_2\)

6. We now verify the monotonicity of \(q\), i.e., verify if the left-hand side
of \((14)\) is nonnegative when \(c \in [b_2, \alpha_2]\).

**Mathematica Input:**
\[
\text{Reduce}\left[ (b_2^2 + 4b_2\delta^* - 3\delta^*h)(b_2^2 + 4b_2\delta^* - 2c\delta^* - 3\delta^*h) - 2b_1b_2(b_2^2 + 4b_2\delta^* - 2c\delta^* + h - 3h(2\delta^* + h)) \geq 0 \&\& 0 \leq b_2 \leq b_1 \leq 1.5b_2 \&\& b_2 \leq c < 2b_2, \{b_2, b_1, c\}\right]
\]

**Mathematica Output:**
\[b_2 > 0 \&\& b_2 \leq b_1 \leq 1.5b_2 \&\& b_2 \leq c \leq \text{Root}[f_{c-III}(c) &, 3]\]

Here, \(f_{c-III}(c)\) is a humongous polynomial running for several pages. Define \(\alpha_2 := \text{Root}[f_{c-III}(c) &, 3]\). Then this proves that the left-hand side of \((14)\) is nonnegative for every \(c \in [b_2, \alpha_2]\).

7. We finally verify the bounds on \(\alpha_2\). We again find the values of \(c\) for which the left-hand side of \((14)\) is nonnegative, but with \(c\) restricted to \(2(t - 1.36)(b_1 - b_2) + 1.36b_2, 2(t - 1.4)(b_1 - b_2) + 1.4b_2\).

**Mathematica Input:**
\[
t = \frac{3}{176} \left(37 + 3\sqrt{465}\right) ; \text{Reduce}\left[ (b_2^2 + 4b_2\delta^* - 3\delta^*h)(b_2^2 + 4b_2\delta^* - 2c\delta^* - 3\delta^*h) - 2b_1b_2(b_2^2 + 4b_2\delta^* - 2c(\delta^* + h) - 3h(2\delta^* + h)) \geq 0 \&\& 0 \leq b_2 \leq b_1 \leq 1.5b_2 \&\& 2(t - 1.36)(b_1 - b_2) + 1.36b_2 \leq c \leq 2(t - 1.4)(b_1 - b_2) + 1.4b_2, \{b_2, b_1, c\}\right]
\]

**Mathematica Output:**
\[b_2 > 0 \&\& b_2 \leq b_1 \leq 1.5b_2 \&\& 0.74675801 + 0.613242b_2 \leq c \leq \text{Root}[f_{c-III}(c) &, 3]\]

This completes all the proofs in Theorem 15.

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