Experimental assessment of eigenfrequencies and stiffness of the elastically supported body

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Abstract. A presumption for the dynamic analysis and assessment of the driving quality of railway vehicles is an important experimental determination and verification of the basic characteristics of the suspension respectively. A method for determining these characteristics for two-axle vehicles has been developed and applied. This method requires knowledge of the position of the center of gravity and the main central moments of inertia. Vehicle bodies, chassis frames, and wheelsets are assumed to be rigid bodies under the investigation of these characteristics. The characteristics of the springs are linear or can be linearized. The vehicle body performs a general spatial movement. The result of the investigation is an analytical method to obtain the required characteristics needed for the dynamic analysis of the vehicle.

1. Introduction

The solution to the problem of vehicle dynamics, both road and rail, is complex and is based on several parameters and conditions, which are determined by calculation or experimentally. This includes, for example, knowledge of the position of the center of gravity, the main central axes of inertia and their corresponding moments of inertia and natural frequency, both empty and loaded vehicles. Knowledge of the distribution of the load during operation is also important. The main problem is the fact that the vehicles are not symmetrical in operation, they exhibit both structural and operational asymmetries. The influence of different types and methods of mechanical properties of suspension and dissipative elements, geometry of their support, etc., and the resulting legitimacy of assumptions of asymmetry of different kind in most vehicles, but also other constructions, is solved.

On the other hand, the usual assumption of symmetry in the design of a structure presents a number of problems in operation, including mechanical ones. Another important aspect is the asymmetry and unevenness of the road not only in road but also in rail vehicles.

The investigation of the effect of asymmetry on the vertical vibration of a vehicle model must therefore be performed by analyzing the various causes and consequences on three-dimensional vehicle models.

When analyzing the effect of asymmetry on the vertical vibration of vehicles, it is necessary to distinguish three basic cases of asymmetry with respect to the axes of geometric symmetry. The axes of geometric symmetry are determined by two mutually perpendicular axes of symmetry of the vehicle wheel spacing and wheelbase and intersecting in the geometric center of the vehicle.
1. asymmetry of the vehicle mass distribution with respect to the axes of geometric symmetry, position of the center of gravity, directions of the main central axes of inertia of both the vehicle structure itself and the loaded vehicle

2. asymmetry of the geometry of distribution of elastic and dissipative elements of connections of individual bodies of the vehicle system and their mechanical properties, spring stiffness, intensity of viscous damping, assuming linear connections of individual quantities and small displacements and rotation of system parts

3. asymmetry of the kinematic excitation i.e., the field of unevenness of the road surface, or tracks, defining the kinematic excitation of the system at the point of wheel-road contact, or wheel-rails.

These types of asymmetry can exist individually or together. The third case occurs in operation the most often.

The solution dealt with both the theoretical justification and the computational and experimental solution of vehicle vibration when crossing over inequalities. The basic premise of investigating the effect of asymmetry is the choice of a suitable simple spatial model. Quarter, half and exceptionally full models are usually used. Since relevant results (due to asymmetry - design and operational) are given only by the full model, this model was also chosen for our solution. The solution was based on the analytical derivation of equations of motion with different excitation methods, symmetry, or asymmetry. Experimental, numerical and simulation methods were also used. The result of the solution was to find the time course of quantities affecting the stability of the vehicle, such as vertical displacement of the center of gravity \( w \), rotation of the vehicle \( \phi_x \), \( \phi_y \) about axes passing through the center of gravity and especially vertical displacement at any point of the vehicle model - mechanical system. To derive the equations of motion, we used Lagrange’s equations for a elastically supported body, or a system of bodies supported and connected by elastic and dissipative elements [1]. The excitation was kinematic.

In the following text, we will show the procedure for solving a two-axle railway vehicle with knowledge of the basic characteristics (suspension, position of the center of gravity, main and central moments of inertia. The solution is based on the fact that the housing, chassis frame and wheelsets are rigid parts.

2. Solution

For the investigation of vertical displacements of a rail two-axle vehicle, under specific assumptions it is possible to choose as a calculation model a rigid rectangular plate. The mass of plate is asymmetrically distributed with respect to the geometric axes of symmetry of the median surface. The plate is spatially elastically supported on 4 coil springs. Investigation of vertical plate displacements for various cases of asymmetry in the kinematic excitation of a system with three degrees of freedom, which correspond to the vertical displacement of the center of gravity \( w \), rotation \( \phi_x \), \( \phi_y \) around the central axes of plate, has been reported in several papers, e.g. [2, 3, 4].

The vertical displacements of the two-axle vehicle model are represented by rigid plate with the influence of energy dissipation during oscillating movements. The elastic plate support is combined with a parallel viscous damper (Fig. 1).

In the case of spatial support of plate, the equations of motion can be written in matrix form

\[
\begin{bmatrix}
\mathbf{m} & 0 & 0 \\
0 & \mathbf{J}_x & -D_{xy} \\
0 & -D_{xy} & \mathbf{J}_y
\end{bmatrix}
\begin{bmatrix}
\ddot{\mathbf{w}} \\
\mathbf{\phi}_x \\
\mathbf{\phi}_y
\end{bmatrix}
+ \begin{bmatrix}
\mathbf{\mu}_{11} & \mathbf{\mu}_{12} & \mathbf{\mu}_{13} \\
\mathbf{\mu}_{21} & \mathbf{\mu}_{22} & \mathbf{\mu}_{23} \\
\mathbf{\mu}_{31} & \mathbf{\mu}_{32} & \mathbf{\mu}_{33}
\end{bmatrix}
\begin{bmatrix}
\dot{\mathbf{w}} \\
\dot{\mathbf{\phi}}_x \\
\dot{\mathbf{\phi}}_y
\end{bmatrix}
+ \begin{bmatrix}
\mathbf{\kappa}_{11} & \mathbf{\kappa}_{12} & \mathbf{\kappa}_{13} \\
\mathbf{\kappa}_{21} & \mathbf{\kappa}_{22} & \mathbf{\kappa}_{23} \\
\mathbf{\kappa}_{31} & \mathbf{\kappa}_{32} & \mathbf{\kappa}_{33}
\end{bmatrix}
\begin{bmatrix}
\mathbf{w} \\
\mathbf{\phi}_x \\
\mathbf{\phi}_y
\end{bmatrix}
= \begin{bmatrix}
\mathbf{F}_z(t) \\
\mathbf{F}_x(t) \\
\mathbf{F}_y(t)
\end{bmatrix}
\tag{1}
\]

where \( \mathbf{J}_x, \mathbf{J}_y \) are inertia moments and \( D_{xy} \) deviation moment to central axes of plate, \( F_z(t), F_x(t), F_y(t) \) are function of external excitation including kinematic loading.
Figure 1. Investigated model - scheme

Legend: x, y, z - axes, e – eccentricity in axis (index x, y), l - distance, A – position of spiring, B – position of dumper, C₁ – geometric center of vehicle, C₂ – geometric center of chassis 1, C₃ – geometric center of chassis 2, T – center of gravity, m - weight, xxyz – position of spring (dumper) in axis x, yxyz - position of spring (dumper) in axis y, φₓ, φᵧ – rotation angle

The system of equations (1) can be arranged to

\[
\begin{align*}
| 1 & 0 & 0 | \begin{bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \\ \ddot{y}_3 \end{bmatrix} & + & \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} & + & \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix} \begin{bmatrix} \dddot{y}_1 \\ \dddot{y}_2 \\ \dddot{y}_3 \end{bmatrix} & = & \begin{bmatrix} F_1(t) \\ F_2(t) \\ F_3(t) \end{bmatrix} \\
0 & 1 & s_{23} & \dddot{y}_2 & + & \begin{bmatrix} b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \begin{bmatrix} y_2 \\ y_3 \end{bmatrix} & + & \dddot{k}_{23} & \dddot{y}_2 & = & F_2(t) \\
0 & s_{32} & 1 & \dddot{y}_3 & + & \begin{bmatrix} b_{31} & b_{32} & b_{33} \end{bmatrix} \begin{bmatrix} y_3 \end{bmatrix} & + & \dddot{k}_{33} & \dddot{y}_3 & = & F_3(t)
\end{align*}
\]

(2)

where elements of mass matrix \( s_{23} = -\frac{D_{xy}}{J_x} \), \( s_{32} = -\frac{D_{xy}}{J_y} \) determine the effect of asymmetry of mass distribution and rotation of the main axes of inertia relative to the central axes of inertia.

By applying the Laplace transform, assuming zero initial conditions, a set of linear algebraic equations is obtained

\[
\begin{bmatrix} p^2 + b_{21} + p & b_{12} + p & b_{13} \\ b_{21} & p^2 + b_{22} + p & b_{23} \\ b_{31} & b_{32} & p^2 + b_{33} + p \end{bmatrix} \begin{bmatrix} \dddot{y}_1(p) \\ \dddot{y}_2(p) \\ \dddot{y}_3(p) \end{bmatrix} = \begin{bmatrix} F_{1}'(p) \\ F_{2}'(p) \\ F_{3}'(p) \end{bmatrix}
\]

(3)

where \( \dddot{y}_1(p), \dddot{y}_2(p), \dddot{y}_3(p), F_{1}'(p), F_{2}'(p), F_{3}'(p) \) are images of functions \( y_1(t) \rightarrow w(t), y_2(t) \rightarrow \varphi_x(t), y_3(t) \rightarrow \varphi_y(t), F_1(t) \rightarrow F_{1}(t), F_2(t) \rightarrow F_{2}(t), F_3(t) \rightarrow F_{3}(t) \).

Solution of a system of linear algebraic equations (3) \( j = 1, 2, 3 \) (Cramer’s rule)
\[ \tilde{y}_j(p) = \frac{D_j(p)}{D(p)} = \sum_{i=1}^{n-3} (-1)^{i+j} F_j(p) \frac{D_j(p)}{D(p)} \]  

(4)

where \( D(p) \) is determinant of matrix of equations system (3)

\[ D(p) = C_A \sum_{i=0}^{2n-1} A_{2n-i} p^{2n-i} \]  

for \( n = 3 \)  

(5)

where \( C_A = 1 - s_{23} s_{32} \), and for real coefficients \( A_{2n-i} \) following equations were derived

\[
A_6 = 1 \quad \quad A_9 = C_A \left[ \sum_{i=4}^{n-3} b_{ji} - s_{23} s_{32} b_{14} - s_{23} b_{12} - b_{32} b_{23} \right]
\]

\[
A_4 = C_A \left[ \sum_{i=1}^{n-3} k_{ji} + \sum_{i=j+1}^{n-2} b_{ji} b_{ij} \right] - s_{23} s_{32} k_{14} - s_{23} a_{12} - s_{32} a_{13} - s_{23} b_{12} - s_{32} b_{13} - s_{23} b_{23}
\]

\[
A_3 = C_A \left[ \sum_{i=1}^{n-2} \sum_{j=i+1}^{n-3} \left[ k_{ji} b_{ij} + b_{ji} b_{ij} \right] + b_{11} b_{12} b_{13} \right] - s_{23} \left[ \begin{array}{ccc} k_{11} & k_{12} & k_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{array} \right]
\]

\[
A_2 = C_A \left[ \sum_{i=1}^{n-3} \sum_{j=i+1}^{n-2} \left[ k_{ji} b_{ij} + b_{ji} b_{ij} \right] + b_{11} b_{12} b_{13} \right] - s_{23} \left[ \begin{array}{ccc} k_{11} & k_{12} & k_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{array} \right]
\]

\[
A_1 = C_A \left[ \begin{array}{ccc} k_{11} & k_{12} & k_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{array} \right] \quad A_0 = C_A \left[ \begin{array}{ccc} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{array} \right]
\]

Determinant \( D(p) \) in equation (4) is created by replacing the \( j \)-th column of the determinant \( D(p) \) with the vector \( \tilde{F}_i(p) \) for \( i = 1, 2, 3 \) from equation (3). Determinant \( D_i(p) \) is an algebraic complement of the series of the determinant \( D(p) \) according to the \( i \)-th element of the \( j \)-th column - vector \( \tilde{F}_i(p) \).

For inverse image transformation \( \tilde{y}_j(p) \) the ratio of determinants \( D_i(p) \) and \( D(p) \) needs to be adjusted to make the product by adjusting the obtained expression and image \( \tilde{F}_i(p) \) allows to application of the formula for image convolution. Assuming subcritical damping, the polynomial (5) has three complex conjugate roots \( p_i \) for \( i = 1, 2, 3 \), whose real part \( \chi_i = Re p_i < 0 \) and imaginary \( \omega_i > 0 \).

The polynomial (5) can be unambiguously decomposed into the product of three quadratic polynomials with real coefficients \( \chi_i \) and \( \omega_i \).
\[ \sum_{i=0}^{2n} A_{2i-n-1} p^{2(n-i)} = \prod_{i=1}^{n} \left[ (p + \chi_i)^2 + \omega_i^2 \right] \quad \text{for } n = 3 \] (7)

The calculation of unknown coefficients is simplified by using the
\[ \omega_i^2 = \omega_{0i}^2 - \chi_i^2 \] (8)
where \( \omega_{0i} \) – angular frequency of dumped movement, \( \omega_{oi} \) – angular frequency of undumped movement, \( \chi_i \) – coefficient of subcritical damping.

We adjust the quadratic polynomials in equation (7) using (8)
\[ \prod_{i=1}^{3} \left[ (p + \chi_i)^2 + \omega_i^2 \right] = \prod_{i=1}^{3} \left[ (p^2 + 2\chi_i p + \chi_i^2 + \omega_i^2) \right] = \prod_{i=1}^{3} \left( p^2 + 2\chi_i p + \omega_{0i}^2 \right) \] (8a)

By making the product of three arranged quadratic polynomials, a polynomial of degree 2n is obtained
\[ \prod_{i=1}^{3} \left( p^2 + 2\chi_i p + \omega_{0i}^2 \right) = \sum_{i=0}^{2n} B_{2(n-i)} p^{2(n-i)} \quad \text{for } n = 3 \] (9)
where coefficients \( B_{2(n,i)} \) are given by relations
\[ B_5 = 1, \quad B_5 = 2 \sum_{i=1}^{3} \chi_i, \quad B_4 = \sum_{i=1}^{3} \omega_{0i} + 4 \sum_{i=1}^{3} \sum_{j=i+1}^{3} \chi_i \chi_j \] (10)
\[ B_3 = 2, \quad B_3 = 2 \sum_{i=1}^{3} \chi_i + 2 \chi_3 \sum_{i=1}^{2} \omega_{0i}^2 + 2 \omega_{03}^2 \sum_{i=1}^{3} \chi_i \]
\[ B_2 = 2 \sum_{i=1}^{3} \omega_{0i}^2 \omega_{0j}^2 + 4 \left( \chi_1 \chi_2 \omega_{03}^2 + \chi_1 \chi_3 \omega_{02}^2 + \chi_2 \chi_3 \omega_{01}^2 \right) \]
\[ B_1 = 2 \left( \omega_{01}^2 \omega_{02}^2 \chi_3 + \omega_{02}^2 \omega_{01}^2 \chi_2 + \omega_{03}^2 \omega_{01}^2 \chi_1 \right) \quad B_0 = \prod_{i=1}^{3} \omega_{0i}^2 \]

Unknown coefficients \( \chi_i, \omega_{0i}, \omega_{0i}^2 \) are determined from the equality of the coefficients for the same powers in relation (7) from relation (10) \( A_{2(n,i)} \) and the coefficients \( B_{2(n,i)} \) from relation (10).

Since the unknown circular frequencies \( \omega_{0i} \) belong to the undamped motion, i.e. \( b_{ij} = 0 \) in equation (3), or in relations (6) and therefore also \( \chi_i = 0 \) in relations (10), it is possible to determine them under these assumptions from the equality of coefficients for the same exponents in relations (6) a (10), i.e. from equations
\[ B_4 = A_4; \quad B_2 = A_2; \quad B_0 = A_0 \]

From equations
\[ \sum_{i=1}^{3} \omega_{0i}^2 = C_A^{-1} \left[ \sum_{i=1}^{3} k_{ii} - s_{23} s_{32} k_{ii} + s_{23} k_{32} - s_{32} k_{23} \right] = A_4 \]
\[ \sum_{i=1}^{2} \sum_{j=i+1}^{3} \omega_{0i}^2 \omega_{0j}^2 = C_A^{-1} \left[ \sum_{i=1}^{2} \sum_{j=i+1}^{3} k_{ii} k_{jj} - s_{23} k_{11} k_{32} + s_{32} k_{31} k_{13} - s_{32} k_{11} k_{33} \right] = A_2 \] (11)
\[
\prod_{i=1}^{3} \omega_i^2 = C_A^{-1} \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix} = A_0
\]

after adjustment from the frequency equation of undamped oscillation

\[
\omega_{01}^2 - A_1 \omega_{02}^2 + A_2 \omega_{03}^2 - A_3 = 0
\]

To determine the circular frequencies \( \omega_{0i} \) it is possible from the equality of coefficients \( A_i = B_i \) of the polynomials (7) for damped oscillations, i.e. \( b_{ij} \neq 0 \), \( \chi_i \neq 0 \) obtain the equations for determining the damping coefficients \( \chi_i \):

\[
B_3 = A_4; \quad B_3 = A_5; \quad B_1 = A_1
\]

where \( \chi_i \) is the real root of the cubic equation

\[
-8 \chi_i^3 (Q_2 - 1) Q_2 + 4 \chi_i^2 \left[ Q_i (Q_2 - 1) + Q_2 (Q_i - A_i) \right] + 2 \chi_i \left[ C_0 (Q_2 - 1) - Q_i (Q_i - A_i) \right] + \left( \omega_{01}^2 + \omega_{02}^2 \right) A_5 = Q_i, C_0 - A_3 = 0
\]

where

\[
C_0 = \omega_{01}^2 + \omega_{02}^2 - \omega_{03}^2 \quad \quad Q_i = \frac{A_i - A_j \omega_{01}^2 \omega_{02}^2}{\omega_{01}^2 (\omega_{03}^2 - \omega_{01}^2)}
\]

after determining of \( \chi_1 \) rest roots are obtained from relations

\[
\chi_2 = \frac{Q_1}{2} - \chi_1 Q_2 \quad \quad \chi_3 = \frac{A_5}{2} - \chi_1 \chi_2
\]

By determining the coefficients of quadratic polynomials \( \chi_i \) and \( \omega_{0i}^2 \) in relation (9) it is possible to modify the determinant \( D(p) \) according to relations (7), (8), (9) into the form

\[
D(p) = C_A \prod_{i=1}^{3} \left( p^2 + 2 \chi_i p + \omega_{0i}^2 \right)
\]

In relation (4) it is necessary to evaluate the determinants \( D_j(p) \) – algebraic complements of the determinants \( D_i(p) \), in the investigated case the determinants of the second order. The determinants \( D_j(p) \) are determined by series of the determinants \( D_i(p) \) according to the i-th element of the j-th column. For example, \( j = 1 \) are \( D_0(p) \)

for \( i = 1 \)

\[
\begin{vmatrix}
    p^2 + b_{22} p + k_{22} & s_{22} p^2 + b_{22} p + k_{22} \\
    s_{32} p^2 + b_{32} p + k_{32} & p^2 + b_{33} p + k_{33}
\end{vmatrix}
\]

for \( i = 2 \)

\[
\begin{vmatrix}
    b_{12} p + k_{12} & b_{13} p + k_{13} \\
    s_{32} p^2 + b_{32} p + k_{32} & p^2 + b_{33} p + k_{33}
\end{vmatrix}
\]

for \( i = 3 \)

\[
\begin{vmatrix}
    b_{12} p + k_{12} & b_{13} p + k_{13} \\
    p^2 + b_{22} p + k_{22} & s_{32} p^2 + b_{32} p + k_{32}
\end{vmatrix}
\]

The determinants \( D_j(p) \) can be expressed in the form of a polynomial

\[
D_j(p) = \sum_{i=0}^{n} C_{n-i} p^{n-i}
\]
where $n$ is the highest power of the parameter $p$, the coefficients $C_{n, l}$ are determined from the difference of the products of the elements - the polynomials of the parameter $p$ of the main and secondary diagonals of the determinants $D_{ij}(p)$. For example

$$D_{ij}(p) = C_A \sum_{i=1}^{n} C_{n-1, i} p^{n-1}$$

where $C_4 = 1$

$$C_2 = C_A^{-1} \left[ k_{02} + k_{33} + \left( k_{22} b_{33} - k_{32} b_{22} \right) \right]$$

$$C_1 = C_A^{-1} \left[ k_{03} + k_{23} + \left( k_{22} b_{33} - k_{32} b_{22} \right) \right]$$

$$C_0 = C_A^{-1} \left[ k_{03} b_{22} - k_{32} b_{22} \right]$$

Ratio of determinants in equation (4) can be written for $j = 1, 2, 3$ and $i = 1, 2, 3$ in form

$$\frac{D_{ij}(p)}{D(p)} = \frac{\sum_{l=0}^{n} C_{n-1, i} p^{n-1}}{\prod_{i=1}^{n} \left[ p^2 + 2 \kappa_i + \omega_{0i}^2 \right]}$$

Since this ratio of determinants is a purely polynomial rational function, it can be expressed in the form of partial fractions

$$\frac{D_{ij}(p)}{D(p)} = \frac{K_{i1} p + L_{i1}}{p^2 + 2 \kappa_1 + \omega_{01}^2} + \frac{K_{i2} p + L_{i2}}{p^2 + 2 \kappa_2 + \omega_{02}^2} + \frac{K_{i3} p + L_{i3}}{p^2 + 2 \kappa_3 + \omega_{03}^2}$$

where the constants $K_{i\alpha}$ and $L_{i\alpha}$ are determined by the method of indeterminate coefficients. After adding the fractio\ns on the right side of equation (14), a polynomial is obtained in the numerator \n
$$\sum_{l=0}^{n} D_{n-1, l} p^{n-1},$$

where $n = 5$ is again the highest power of the parameter $p$ and equation (14) can be modified by the relation (13)

$$\sum_{l=0}^{n} C_{n-1, l} p^{n-1} = \sum_{l=0}^{5} D_{n-1, l} p^{n-1}$$

By comparing the coefficients $C_{n, l}$ and $D_{n, l}$, a set of nine linear algebraic equations is obtained for the calculation of nine unknown constants $K_{i\alpha}$ and $L_{i\alpha}$.

$$\begin{vmatrix}
    a_{11} & \ldots & a_{16} & K_{i1} & C_5 \\
    \vdots & \ddots & \vdots & K_{i2} & C_4 \\
    \vdots & \vdots & \ddots & K_{i3} & C_3 \\
    \vdots & \vdots & \vdots & K_{i4} & C_2 \\
    a_{61} & \ldots & a_{66} & K_{i6} & C_0 \\
\end{vmatrix} = K_{i\alpha}$$

Index $s = 1, 2, 3$ belongs to the partial fraction corresponding to the pole of the determinant $D_i(p)$ see (15). Index $i = 1, 2, 3$ belongs to the vector element of the excitation function $\vec{F}_i(p)$ and index $j = 1, 2, 3$. 

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belongs to the quantities $y_i$, the calculation of which is the goal of the solution. It is obvious that a total of 27 constants $K_{ii}$ and $L_{ii}$ must be determined for this, for which it is necessary to determine the coefficients $C_{n,i}$ corresponding to the determinants $D_{p}(p)$, see (13), (15). By determining the constants $K_{ii}$ and $L_{ii}$, the relation (4) for calculating the image $\bar{y}_j(p)$ with respect to the relations (8) and (8a) can be adjusted to the form

$$\bar{y}_j(p) = \sum_{i=1}^{3} (-1)^{j+1} F_i(p) \sum_{i=1}^{3} \frac{K_{ii} p + L_{ii}}{(p + \chi_i)^2 + \omega_i^2}$$

for the inverse transformation, it is appropriate to modify this relationship to a form

$$\bar{y}_j(p) = \sum_{i=1}^{3} (-1)^{j+1} F_i(p) \sum_{i=1}^{3} \frac{K_{ii} p}{(p + \chi_i)^2 + \omega_i^2} + \frac{L_{ii} - \chi_i K_{ii}}{\omega_i} \left[ F_i(p) e^{-\chi_i(t-t) \cdot \cos \omega_i(t-t) dt} + \frac{L_{ii} - \chi_i K_{ii}}{\omega_i} \frac{\omega_i}{(p + \chi_i)^2 + \omega_i^2} \right]$$

After the inverse transformation, the resulting relation for the calculation of the demanded quantities $y_j(t)$ is obtained

$$y_j(t) = \sum_{i=1}^{3} (-1)^{j+1} \sum_{i=1}^{3} K_{ii} \int_0^t F_i(\tau) e^{-\chi_i(t-t)} \cdot \cos \omega_i(t-t) dt + \frac{L_{ii} - \chi_i K_{ii}}{\omega_i} \int_0^t F_i(\tau) e^{-\chi_i(t-t)} \cdot \sin \omega_i(t-t) dt$$

which is the solution of equation (2), and by adjustment we get

$$y_1(t) \Rightarrow w(t) \quad y_2(t) \Rightarrow \varphi_x(t) \quad y_3(t) \Rightarrow \varphi_y(t)$$

$$F_1(t) \Rightarrow F_x(t) \quad F_2(t) \Rightarrow F_y(t) \quad F_3(t) \Rightarrow F_y(t)$$

relations for the required quantities of the system of equations (1) are obtained. For the case of undamped oscillations

$$b_{ij} = 0 \Rightarrow \chi_i = 0 \quad a \quad \omega_i = \omega_{ss} \Rightarrow K_{ii} = 0$$

$$\bar{y}_j(p) = \sum_{i=1}^{3} (-1)^{j} F_i(p) \sum_{i=1}^{3} \frac{L_{ii}}{p^2 + \omega_i^2}$$

thence after the inverse

$$y_j(t) = \sum_{i=1}^{3} (-1)^{j} \sum_{i=1}^{3} \frac{L_{ii}}{\omega_i} \int_0^t F_i(t) \cdot \sin \omega_{ss}(t-t) dt \cdot$$

3. Conclusion

With respect to the general formulation of the problem, the calculation of oscillations, i.e. the vertical displacement of the general point of a body with generally distributed mass, spatially generally supported on vertical springs and hydraulic dampers, caused by the excitation function of the general time course, and due to the use of known calculation methods, it is possible to implement this procedure advantageously in computer programs MAPLE, event. MATLAB (analytical solution) or use simulation programs (ADAMS, SIMPACK) etc.

SIMPACK and ADAMS were used in the solution itself. Experiments were also performed on a simple model and a real vehicle (railway platform car Smmps) - but this already exceeds the scope of the article.
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References
[1] Soukup, J., Volek, J. et al (2008) Vibration of mechanical systems - vehicles. Analysis of the effect of asymmetry. Acta Universitatis Purkynae 138, Studia Mechanica. University of J. E. Purkyne in Ústí nad Labem, p. 283, ISBN978-80-7414-020-4
[2] Soukup, J., Skočilas, J., Skočilasová, B. (2014) Central inertia moments and gravity center of large volume and weight bodies. International Journal of Dynamics and Control Dissertation. Springer-Verlag Berlin Heidelberg. ISSN 2195-268X, Int. J. Dynam. Control, DOI 10.1007/s40435-014-0101-x
[3] Volek, J., Soukup, J., Kout, J. (2003) Investigation of influence of asymmetry in case of the vibration a resiliently supported plate – application on the vibration of vehicle III. In.: National Conference with International Participation Engineering Mechanics 2003, extended abstract proc. pp. 386-387, CD ROM 6 pages. UTAM AV ČR, Praha, 2003, ISBN 80-86246-18-3- vehicles.
[4] Rektorys, K. at al (2000) Overview of applied mathematics I and II. Prometheus publishing house, Prague