Chaotic flows and cosmic dynamos in anisotropic pseudo-Riemannian four-dimensional spacetime

by

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Abstract

It is shown that the existence of the cosmic kinematic fast dynamos in Bianchi type IX rotating cosmological models, faces severe difficulties, due to the fact that in these models rotation increases without bounds, which is strictly forbidden by CMB astronomical data which imposes strong bounds on its rotation with respect with its rapid expansion. The only way out of this difficulty is to assume that at least one of the expansion directions of this anisotropic universe decays as fast as the amplification rate of primordial magnetic fields. A solution is found where only one direction of the anisotropic universe expands while the other two remain constants. We compute an amplification of the seed magnetic field in the case where Bianchi IX degenerates into de Sitter metric, fields amplify from $10^{-6} G$ to $10^{-5} G$ in spiral galaxies for a cosmological constant of the order $|\Lambda| < 10^{33} s^{-2}$ and considering that the age of universe of the order of $10^{10} yrs$. Another example is given by the ABC chaotic flows in the pseudo-Riemannian spacetime representing the Kasner anisotropic nonsingular universe. PACS numbers: 02.40.Hw, Riemannian geometries
I Introduction

Earlier MacCallum [1] has given an outline review of the importance of anisotropic cosmological models of universe, and more recently J.D.Barrow [2] has investigate the anisotropic stresses in Einstein general relativistic homogeneous models where he showed that Kantowski-Sachs model is unstable against perturbations. In most astrophysical dynamos [3] rotation or most clearly differential rotation [4] is of utmost importance in the amplification process of the magnetic fields in galaxies [5] or accretion discs of Kerr black holes [6]. However recently, Brandenburg [7] have shown that Cowling antidynamo theorem also applies in accretion discs surrounding these very compact stellar objects, and dynamo effects are very limited. With these physical motivations in this paper we show that accretion discs around black holes are not the only astrophysical objects where dynamos present problems, but also on the cosmological setting cosmic dynamos do not fit well in some anisotropic universes as the Bianchi type IX rotating models considered here. Actually we show that the existence of dynamos in this universe implies the existence of an unbounded rotation, which is forbidden by cosmological data [8]. The only way out of this situation, saving the cosmic dynamo existence is to consider the isotropization of the model considering the irrotational Bianchi IX model or the de Sitter expanding universe model. Though great amplification of cosmic primordial magnetic field of the order of the several of orders of magnitude appear in galactic dynamos, slow dynamos can amplify a magnetic field of the order of $10^{-6}G$ to $10^{-5}G$ which happens in spiral galaxies. Actually is exactly this amplification that we show to be undertaken by the de Sitter cosmic dynamo here. Another way out is to consider turbulent dynamos [9] in anisotropic cosmological models. The paper is organised as follows: In section II we review the Bianchi IX model geometry in terms of tetrads and differential forms mathematical tools the problem, and show that fast decay of the anisotropic dynamo flow can solve. In section III we solve the self-induction equation and compute the amplification of galactic magnetic fields. In section III we show that Arnold-Beltrami-Childress chaotic flows or ABC flows for short are compatible with Kasner cosmological background. Conclusions are presented in section V.
II Bianchi type IX rotational cosmic dynamos

In this section we present a brief review of the mathematical background of cosmological solution of the equations of Einstein theory of gravitation. Let us start by by the Bianchi type IX line element [10]

\[ ds^2 = -dt^2 + g_{ij} \chi^i \chi^j \]  

(II.1)

where the indices \((i, j = 1, 2, 3)\) are summed by making use of Einstein convention of tensor calculus and \(\chi^i\) represent the synchronous frame system of Cartan differential forms. They are given explicitly by

\[ \chi^1 = -\sin x^3 dx^1 + \sin x^1 \cos x^3 dx^2 \]  

(II.2)

\[ \chi^2 = \cos x^3 dx^1 + \sin x^1 \sin x^3 dx^2 \]  

(II.3)

\[ \chi^3 = \cos x^1 dx^2 + dx^3 \]  

(II.4)

which after a simple algebra yields the relation

\[ \sum (\chi^i)^2 = \sum (dx^i)^2 \]  

(II.5)

This last relation is fundamental for the construction of the de Sitter metric bellow. In terms of the orthonormal tetrad \(\lambda^i = b^i_s \chi^s\) the Bianchi IX rotating metric becomes

\[ ds^2 = -dt^2 + \delta_{ij} \lambda^i \lambda^j \]  

(II.6)

where \(\delta_{ij}\) is the Krönecker delta function. These models expands or contracts according to the non-vanishing scalar \(\Theta := u^{\alpha \alpha}\), where \((\alpha = 0, 1, 2, 3)\) and \(u^\alpha\) is the four-velocity of the cosmic flow. In the synchronous system, with the help of geodesic equation one can express the scalar \(\Theta\) and the rotation tensor \(\omega_{ij}\) becomes

\[ \Theta = \frac{l_{ij} u^i u^j}{u_0} - u_0 l_{kk} \]  

(II.7)

\[ \omega_{ij} = -\frac{1}{2} u^k d^k_{ij} \]  

(II.8)

The metric tensor components \(g_{ij}\) are given in terms of triad coefficients \(b_{ij}\) by

\[ g_{ij} = b_i^l b_{lj} \]  

(II.9)
where \( b_{ij} \) satisfies the relation \( b_{[ij]} = 0 \). Coefficient \( d^k_{ij} \) is defined as

\[
d^k_{jk} := b_i^l \epsilon_{lmn} b^{-1}_{mj} b^{-1}_{nk} \quad (\text{II.10})
\]

The coefficients \( l_{jk} \) are expressed as

\[
l_{ij} := \dot{b}_{ij} b^{-1}_{lj} \quad (\text{II.11})
\]

the triad \( b_{ij} \) is defined as \( b_{ij} = b_{ij}(t) \) are solely a function of time. Here the Levi-Civita symbol \( \epsilon_{ijk}, \) where \( \epsilon_{123} = 1, \) is the totally skew symmetric object. With this mathematical tool we are able to express the vorticity vector as \( \omega^i = \epsilon^{ijk} \omega_{jk}. \) Let us now consider that the triad \( b_{ij} = b^0_{ij} e^{pt}. \) The main reason for this choice is that the magnetic field in tetrad components may be written as \( B^{(i)} = b^{0is} e^{pt} B_s, \) which is exactly the form generally used to investigate fast dynamos [9]. Substitution of our triad choice into the rotation tensor expression \( (\text{II.8}) \) one obtains

\[
\omega_{ij} = \frac{-1}{2} u^k d^0_{ij} e^{pt} \quad (\text{II.12})
\]

Note from the expression of the magnetic field that the existence of a kinematic fast cosmic dynamo would demand that \( p > 0 \) and so expression \( (\text{II.12}) \) shall tell us that the rotation of the cosmological fluid rotation increases without bounds as \( t \to \infty, \) which unfortunately is strictly forbidden from available COBE data for expanding universe, which tells us that rotation of the universe is several orders of magnitude lower than the expansion of the universe. The only way out of this difficulty would be to consider that the dynamo flow decays as \( u^k = (u^0)^k e^{-pt} \) which would neutralize the fast growing term \( e^{pt} \) in expression \( (\text{II.12}), \) and we end up with a constant vorticity for the Bianchi IX model. Let us now consider that only \( u^z \) component of the dynamo flow does not vanish and let us compute the self-induction equation

\[
\partial_t \vec{B} = \eta \nabla^2 \vec{B} + (\vec{u}.\nabla) \vec{B} - (\vec{B}.\nabla) \vec{u} \quad (\text{II.13})
\]

By considering a highly conductive cosmic fluid diffusion \( \eta \) vanishes and this equation reduces to

\[
\partial_t \vec{B} = (\vec{u}.\nabla) \vec{B} - (\vec{B}.\nabla) \vec{u} \quad (\text{II.14})
\]

to simplify matters we then imagine that the dynamo flow is homogeneous while the magnetic field also depends on space. This simplifies equation \( (\text{II.13}) \) even further to

\[
\partial_t \vec{B} = (\vec{u}.\nabla) \vec{B} \quad (\text{II.15})
\]
III De Sitter cosmic dynamos in spiral galaxies

In this section we show that the difficulty with the Bianchi IX model for the existence of cosmic dynamos in this cosmological setting does not appear in de Sitter inflationary cosmology [10]. By considering that the four-velocity of the cosmological fluid is orthogonal to the \( t = \text{constant} \), synchronous, space-like hypersurface, where \( u^0 = 1 \) and \( u^i = 0 \), so in this frame since we are dealing with diagonal metrics, the expression (II.8) tells us that the vorticity \( \omega^i = 0 \). The Friedmann metric belongs to this class of metrics, allows us to consider the de Sitter metric

\[
ds^2 = -dt^2 + e^{pt}[dx^2 + dy^2 + dz^2]
\]

according to the idea of the last section which allows us to write the triad \( \lambda^{(i)} \) as

\[
\lambda^{(i)} = (b^0)_s^i e^{pt} \chi^s
\]

Let us now show that if the homogeneous magnetic field obeys the magnetic self-induction equation, thus the triad obtained are exactly the triad generating de Sitter equation. This is easily accomplished by considering that homogeneous magnetic fields obey the self-induction equation

\[
\partial_t[(b)^i_s B_0] = 0
\]

This equation yields

\[
(\partial_t(b)^{is}) B_s + (b)^{is} \partial_t B_s = 0
\]

which is equivalent to

\[
(\partial_t(b)^{is}) + (b)^{is} p = 0
\]

where we have used the definition of the magnetic field given in the previous section. Solving equation (III.20) we obtain the triad \( b^i_s = \delta^i_j e^{pt} \) which yields exactly the de Sitter triad. Going back to de Sitter cosmology we see that \( p = \sqrt{\frac{\Lambda}{3}} \) where \( \Lambda \) is the cosmological constant. Taking the upper limit of \( |\Lambda| < 10^{-35} s^{-2} \) and the age of universe as \( 10^{10} yrs \) we are able to think that dynamo amplified field of de Sitter cosmic dynamo can be approximated as

\[
B_0 e^{pt} = B^0[1 - pt]
\]

where \( B_0 \) is the seed field to be amplified. From the above data, and assuming that we have a typical seed field of a spiral galaxy field of \( 10^{-6} G \) one obtains \( 10^{-5} G \).
IV ABC flows in Kasner universe

In 1981 Arnold, Zeldovich, Ruzmaikin and Sokoloff [11] a magnetic field in a stationary flow with stretching in a Riemannian three dimensional space, stationary flow which is exactly the so called Beltrami 1882 flow also lately investigated by Childress [9]. In this section, in a certain sense, we extend the ABC flows defined by the equations

\[ u_x = Asinz + Ccosy \]  
\[ u_y = Bsinx + Acosz \]  
\[ u_z = Csiny + Bcosx \]

(IV.22)  
(IV.23)  
(IV.24)

to a pseudo-Riemannian anisotropic spacetime called Kasner cosmological model. Note that the case \( A = B = 1 \) and \( C = 0 \), represents a fast dynamo. As we shall see bellow our case is \( A = C = 0 \) and \( B = 1 \). To this end we substitute the equations for the flow \( \vec{u} = (u_x, u_y, u_z) \) into the self-induction equation, assuming now that \( \vec{B} = \vec{B}_0 e^m \) where \( \vec{B}_0 := B_0^y \hat{j} + B_0^z \hat{k} \). This structure is encoded into the Kasner nonsingular cosmological spacetime

\[ ds^2 = -dt^2 + dx^2 + dy^2 + t^2 dz^2 \]  
(IV.25)

The equation \( \nabla.\vec{B} = 0 \) and self-induction equation becomes

\[ \partial_y B_y + \partial_z B_z = 0 \]  
(IV.26)

\[ mB_0^z = -cosx \partial_z B_0^z \]  
(IV.27)

Solution of the last equation yields

\[ B_0^z = B_2 e^{\frac{mz}{\cos x}} \]  
(IV.28)

substitution of this expression into (IV.26) yields

\[ \partial_y B_y = -\partial_z B_z = \frac{m}{\cos x} B_0^z \]  
(IV.29)

which can be easily solved to yield

\[ B_0^y = -B_1 e^{\frac{my}{\cos x}} \]  
(IV.30)

where \( B_1 \) and \( B_2 \) are integration constants. This shows that the magnetic field is amplified if \( m > 0 \) and the flow and the magnetic field are spatially periodic.
V Conclusions

In conclusion, we have investigated a chaotic flows, such as ABC chaotic flow, and the cosmic dynamo in Einstein’s gravitational equations in four-dimensional anisotropic spacetime or in anisotropic pseudo-Riemannian space, which seems to certain extent be a generalization of the chaotic non-relativistic ABC flow in a three-dimensional Riemannian space investigated by Arnold and his group. Of course other types of chaotic flows in isotropic and anisotropic rotating models such as the Gödel model can be investigated elsewhere.

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