Proof of the weak cosmic censorship conjecture for several extremal black holes

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Abstract
We show explicitly, for different types of extremal black holes, that test fields satisfying the null energy condition at the event horizon cannot violate the weak cosmic censorship conjecture. This is done by checking, in each case, that the hypotheses for a general theorem proved in a previous paper are satisfied.

Keywords Weak cosmic censorship · Extremal black holes · Black hole thermodynamics · BTZ black holes · Quintessence · Gauss–Bonnet-AdS black holes · Nonlinear electrodynamics · Born–Infeld-AdS black holes · Stringy black holes · MTZ black holes

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1 Introduction

In 1965, Penrose published the proof of the first singularity theorem \[1\], soon to be followed by increasingly more sophisticated results \[2,3\]. The singularities predicted by these theorems as a result of gravitational collapse do not need, in principle, to be hidden inside a black hole event horizon, that is, they could conceivably be naked singularities. This would signal a major breakdown of general relativity, as the physics at the singularity (and consequently in its causal future) cannot be predicted by this theory.

In response to this problem, Penrose formulated, in 1969, the weak cosmic censorship conjecture \[4,5\], hypothesizing that singularities resulting from gravitational collapse must always be hidden inside a black hole event horizon. This would prevent access to the singularity by observers in the exterior of the black hole, meaning that the singularity could not possibly influence the physics of the outside universe.

The weak cosmic censorship conjecture remains unproven. In an attempt to violate it, Wald proposed a thought experiment to turn extremal Kerr–Newman black holes into naked singularities by sending in spinning and/or charged particles \[6\]. He discovered that particles with large enough angular momentum or charge to overspin or overcharge the black hole never went in, preserving the conjecture. The same was found to be true for scalar and electromagnetic fields \[7–10\], and, more generally, for any kind of test matter or fields \[11\]. A similar result also holds for near-extremal Kerr–Newman black holes, if backreaction is taken into consideration \[12\].

In this paper we explore the weak cosmic censorship conjecture further by showing explicitly, using a general result in \[11\], that the conjecture holds for several other types of extremal black holes which have been recently studied in the literature.

2 Weak cosmic censorship conjecture

The proof in \[11\] that extremal Kerr–Newman or Kerr–Newman-AdS black holes cannot be destroyed by interacting with test fields can be adapted to other extremal black holes; this will be done in the present paper in several examples. As already noted in \[11\], the conditions that must be satisfied for the proof to go through for a generic black hole are the following:

1. The Killing generator for the event horizon is of the form

\[ Z = K + \sum_i \Omega_i Y_i, \]  

where \( K \) is an asymptotically timelike Killing vector field which determines the physical mass of the black hole, \( Y_i \) are angular Killing vectors giving the angular momenta of the black hole, and \( \Omega_i \) are the thermodynamic angular velocities.

2. The physical mass \( M \), entropy \( S \), angular momenta \( J_i \) and electric charge \( Q \) can be related through a Smarr formula

\[ M = M(S, J_i, Q), \]
yielding a first law of black hole thermodynamics

\[ dM = TdS + \sum_i \Omega_i dJ_i + \Phi dQ, \] (3)

where \( T \) is the black hole temperature and \( \Phi \) is the event horizon’s electric potential.

3. Extremal black holes (that is, black holes with \( T = 0 \)) are given by \( M = M_{ext}(J_i, Q) \), and subextremal black holes by \( M > M_{ext}(J_i, Q) \).

4. The test fields satisfy the null energy condition at the event horizon, and appropriate boundary conditions at infinity (guaranteeing zero flux of conserved charges at infinity for suitably defined spacelike hypersurfaces).

5. The interactions with the test fields preserve the black hole type, that is, after the interaction the spacetime settles to a stationary solution characterized by the same parameters \((M, J_i, Q)\).

Condition 5 was not explicitly stated in [11], but will be important here, since we will be working with black holes for which (in general) there are no uniqueness theorems. As a simple example, when working with Reissner–Nordström black holes only one must consider interactions that preserve spherical symmetry (e.g. interactions with spherically symmetric charged fields), as otherwise one may end up with a Kerr–Newman spacetime. Even when working with Kerr–Newman black holes, one should be aware that in general the black hole resulting from the interaction with the test field is a Kerr–Newman spacetime with a different symmetry axis, that is, that the black hole’s angular momentum may rotate; this is usually not discussed because only the component of the absorbed angular momentum along the initial symmetry axis is relevant. Indeed, if \((0, 0, J)\) is the black hole’s initial ADM angular momentum and \((j_1, j_2, j_3)\) is the absorbed ADM angular momentum then, to first order,

\[ [(0, 0, J) + (j_1, j_2, j_3)]^2 = J^2 + 2Jj_3 = (J + j_3)^2, \] (4)

that is, the final spacetime has angular momentum \( J + j_3 \) along its new symmetry axis.

If conditions 1–5 are satisfied then the test fields cannot destroy the black hole, in the sense that if the black hole absorbs energy \( \Delta M \), angular momenta \( \Delta J_i \) and electric charge \( \Delta Q \) by interacting with the test fields, then the metric corresponding to the physical quantities \((M + \Delta M, J_i + \Delta J_i, Q + \Delta Q)\) represents a subextremal (or at worse an extremal) black hole, rather than a naked singularity. We now proceed to check that these conditions hold for several extremal black holes in the literature which have been recently studied in connection with the weak cosmic censorship conjecture.

3 BTZ black hole

The BTZ metric is given by [13–16]

\[ ds^2 = -f dt^2 + \frac{1}{f} dr^2 + r^2 \left( d\phi - \frac{J}{2r^2} dt \right)^2, \] (5)
where

\[ f(r) = -M + \frac{r^2}{l^2} + \frac{J^2}{4r^2} \quad (6) \]

and

\[ \Lambda = -\frac{1}{l^2}. \quad (7) \]

Here \( \Lambda \) is the cosmological constant, \( M \) and \( J \) are the mass and angular momentum, respectively, and \( r \) is the radial coordinate. There are two horizons, which coincide when the black hole becomes extremal (the inner horizon radius \( r_- \) becomes equal to the outer event horizon radius \( r_+ \)). Since the metric is axisymmetric and stationary, we have two Killing vectors, \( X = \frac{\partial}{\partial t} \) and \( Y = \frac{\partial}{\partial \phi} \), and thus the most general Killing vector will be a linear combination of \( X \) and \( Y \).

The entropy \( S \), angular velocity \( \Omega \), Hawking temperature \( T \) and mass \( M \) are given by [13,14]

\[ S = 4\pi r_+, \quad \Omega = \frac{J}{2r_+^2}, \quad \Omega_\infty = 0 \quad (8) \]
\[ T = \frac{r_+^2}{2\pi l^2} - \frac{J^2}{8\pi r_+^3}, \quad M = \frac{r_+^2}{l^2} + \frac{J^2}{4r_+^2}, \quad (9) \]

and satisfy the first law of black hole thermodynamics

\[ dM = TdS + \Omega dJ. \quad (10) \]

The angular velocity of the black hole horizon is given by [17,18]

\[ \Omega_H = \left. \frac{g_{t\phi}}{g_{\phi\phi}} \right|_{r=r_+} = \frac{J}{2r_+^2} \quad (11) \]

that is, the angular velocity of the event horizon is equal to the thermodynamic angular velocity. Moreover, the angular velocity of observers at infinity, with respect to the Killing vector \( X \), is given by

\[ \Omega_\infty = \left. \frac{g_{t\phi}}{g_{\phi\phi}} \right|_{r \to \infty} = 0, \quad (12) \]

meaning that the asymptotically timelike Killing vector field which determines the physical mass of the black hole is \( K = X \). Therefore the event horizon \( \mathcal{H}^+ \) Killing generator is given by

\[ Z = X + \Omega_H Y = K + \Omega Y, \quad (13) \]
and so the first condition of the theorem is satisfied.

From (8) and (9) it is clear that the mass $M$ of the BTZ black hole can be entirely determined by the entropy $S$ and angular momentum $J$ of the event horizon through a Smarr formula of the form

$$M = M(S, J),$$

meaning that the second condition of the theorem is also satisfied.

For the black hole to be extremal we must have $T = 0$, which is equivalent to

$$\frac{\partial M}{\partial S}(S, J) = 0.$$  \hspace{1cm} (15)

Solving this equation for the entropy leads to the entropy, hence to the mass, of an extremal black hole as a function of its angular momentum,

$$S = S_{\text{ext}}(J) \Rightarrow M_{\text{ext}} = M(S_{\text{ext}}(J), J),$$

which satisfies the third condition of the theorem.

If we assume that the test fields satisfy the null energy condition and preserve the black hole type, then the five conditions of the theorem are satisfied. Therefore, the BTZ black hole cannot be overspun to create a naked singularity, that is, the weak cosmic censorship conjecture holds for BTZ black holes.

4 Quintessence RN-AdS black hole

The metric for the spherically symmetric Reissner–Nordström-AdS black hole surrounded by quintessence dark energy is given by [20]

$$ds^2 = -f dt^2 + \frac{1}{f} dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2),$$

where

$$f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} + \frac{r^2}{l^2} - \frac{a}{r^{3\omega+1}}$$ \hspace{1cm} (18)

and

$$\Lambda = -\frac{3}{l^2}.$$  \hspace{1cm} (19)

Here $M$ and $Q$ are the mass and electric charge of the black hole, respectively, $r$ is the radial coordinate, $a$ is a normalization factor related to the density of quintessence dark energy, $l$ is the AdS radius, $\Lambda$ is the cosmological constant, and $-1 < \omega < -\frac{1}{3}$ so that we have quintessence dark energy.
Since the metric is static and spherically symmetric, the black hole is nonrotating and there is no angular momentum to consider; consequently, the thermodynamic angular velocity can be set to zero. The timelike Killing vector \( X = \frac{\partial}{\partial t} \) determines the black hole mass and is also the Killing generator of the event horizon \( \mathscr{H}^+ \), whose angular velocity is then also zero. In other words, \( X = K = Z \), and so the first condition of the theorem is satisfied.

We also have \[20,21\]

\[
\Phi = \frac{Q}{r_+}, \quad S = \pi r_+^2, \tag{20}
\]

\[
T = \frac{f'(r_+)}{4\pi} = \frac{1}{4\pi} \left[ \frac{2M}{r_+^2} - \frac{2Q^2}{r_+^3} + \frac{2r_+}{I_2} + \frac{(3\omega + 1)a}{r_+^{3\omega+2}} \right], \tag{21}
\]

where \( r_+ \), \( \Phi \), \( S \) and \( T \) are the radius, electric potential, entropy and Hawking temperature of the event horizon. For fixed \( \Lambda \) and \( a \), we have the first law of thermodynamics \[20–22\]

\[
dM = T dS + \Phi dQ. \tag{22}
\]

Moreover, from (18) it is clear that

\[
f(r_+) = 0 \Leftrightarrow M = M(r_+, Q), \tag{23}
\]

and using (20) we obtain a Smarr relation in the form

\[
M = M(S, Q). \tag{24}
\]

Hence the second condition of the theorem is satisfied.

For the black hole to be extremal we must have \( T = 0 \), which is equivalent to

\[
\frac{\partial M}{\partial S} (S, Q) = 0. \tag{25}\]

Solving this equation for the entropy leads to the entropy, hence to the mass, of an extremal black hole as a function of its charge,

\[
S = S_{ext}(Q) \Rightarrow M_{ext} = M(S_{ext}(Q), Q), \tag{26}
\]

which satisfies the third condition of the theorem.

Finally, assuming that the test fields satisfy the null energy condition and preserve the black hole type, we have just shown that it is not possible to violate the weak cosmic censorship with such test fields, as all conditions in the theorem are met.
5 Gauss–Bonnet-AdS black hole

We now apply the theorem to the Gauss–Bonnet-AdS black hole in $d$ dimensions. The metric for this black hole can be written in the form $[23,24]$

$$ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2h_{ij}dx^idx^j.$$  \hspace{1cm} (27)

with

$$f(r) = \kappa + \frac{r^2}{2\alpha} - \frac{r^2}{2\alpha}\sqrt{1 + \frac{64\pi\alpha M}{(d-2)\Sigma_\kappa r^{d-1}} - \frac{2\alpha Q^2}{(d-2)(d-3)r^{2d-4}} - \frac{4\alpha}{l^2}}$$ \hspace{1cm} (28)

and

$$\Lambda = -\frac{(d-1)(d-2)}{2l^2}.$$ \hspace{1cm} (29)

Here $\alpha = (d-3)(d-4)\alpha_{GB}$ is the redefined Gauss–Bonnet coefficient (and $\alpha_{GB}$ the Gauss–Bonnet coefficient), $M$ and $Q$ are the mass and electric charge of the black hole, respectively, $\Lambda$ is the cosmological constant (which we assume to be fixed), and $r^2h_{ij}dx^idx^j$ is the line element of the $(d-2)$-dimensional maximally symmetric Einstein space with constant curvature $(d-2)(d-3)\kappa$ and volume $\Sigma_\kappa$. The parameter $\kappa$ defines the topology of the black hole event horizon; for our purposes we will focus on the case of spherical topology, corresponding to $\kappa = 1$ $[24,25]$. We note that because the Gauss–Bonnet term is a topological invariant for $d < 5$, we must assume $d \geq 5$.

Since the metric is static and spherically symmetric, the first condition in the theorem is satisfied, as shown in Sect. 4. The electric potential at the event horizon $\Phi$, Hawking temperature $T$ and entropy $S$ are given by $[23]$

$$\Phi = \frac{Qr^+}{16\pi(d-3)}, \hspace{1cm} T = \frac{f'(r_+)}{4\pi}, \hspace{1cm} S = r_+^{d-4}(2\alpha(d-2) + (d-4)r_+^2)\Sigma_\kappa.$$ \hspace{1cm} (30)

For fixed $\Lambda$ and $\alpha$, the quantities above satisfy the first law of thermodynamics $[23,24]$

$$dM = TdS + \Phi dQ.$$ \hspace{1cm} (31)

By setting $f(r) = 0$ we obtain two solutions, $r_-$ and $r_+$, and thus the black hole has an inner and outer horizon, which coincide as we take it to be extremal. From (28) we have

$$f(r_+) = 0 \Leftrightarrow M = M(r_+, Q),$$ \hspace{1cm} (32)

and using (30) we obtain a Smarr relation in the form

$$M = M(S, Q),$$ \hspace{1cm} (33)
which fulfills the second condition of the theorem. The third condition in the theorem then follows by the same argument as in Sect. 4.

Finally, assuming that the test fields satisfy the null energy condition and preserve the black hole type, we have thus shown it is not possible to violate the weak cosmic censorship with such test fields, as all conditions in the theorem are met.

6 RN-AdS black hole in nonlinear electrodynamics

For a $d$-dimensional static, spherically symmetric, electrically charged AdS black hole solution in general nonlinear electrodynamics (NLED) theories we have the metric and NLED field given by [26]

$$\begin{align*}
ds^2 &= -f(r)dt^2 + f(r)^{-1}dr^2 + r^2d\Omega^2_{d-2}, \\
A &= A_t(r)dt,
\end{align*}$$

where $d\Omega^2_{d-2}$ is the metric on the unit $(d-2)$-sphere, $A$ is the nonlinear electromagnetic potential one-form and $f(r)$ is given by

$$f(r) = 1 - \frac{m}{r^{d-2}} + \frac{r^2}{l^2} - \frac{4}{d-2} \frac{1}{r^{d-3}} \int_r^\infty r^{d-2} \left[ \mathcal{L}(s; a_i) - A'(r) \frac{q}{r^{d-2}} \right] dr. \quad (36)$$

Here $m$ is a constant related to the black hole mass, $l$ is the AdS radius, related to the cosmological constant by $\Lambda = -\frac{(d-1)(d-2)}{2l^2}$, $q$ is a constant related to the black hole charge, $\mathcal{L}(s; a_i)$ is the generic NLED Lagrangian, with $a_i$ characterizing the nonlinearity of the electrodynamics, and $s$ an independent nontrivial scalar. Note that the metric approaches the usual RN-AdS metric when $r \to \infty$.

Since the metric is static and spherically symmetric, the first condition in the theorem is satisfied, as shown in Sect. 4. The electric potential at the event horizon $\Phi$, Hawking temperature $T$, black hole mass $M$, entropy $S$ and black hole charge $Q$ are given by [26]

$$\begin{align*}
\Phi &= -A_t(r_+), \\
T &= \frac{f'(r_+)}{4\pi}, \\
M &= \frac{d-2}{16\pi} \omega_{d-2} m, \\
S &= \frac{r_+^{d-2} \omega_{d-2}}{4}, \\
Q &= \frac{q}{4\pi} \omega_{d-2},
\end{align*}$$

where $\omega_{d-2}$ is the volume of the unit $(d-2)$-sphere. Assuming the cosmological constant and the parameters $s$, $a_i$ from the NLED theory to be fixed, we have the first law of black hole thermodynamics [26]

$$dM = \Phi dQ + T dS. \quad (39)$$

From (36) together with (37) and (38) we see that

$$f(r_+) = 0 \Leftrightarrow M = M(r_+, Q), \quad (40)$$
and using (38) we obtain a Smarr relation in the form

\[ M = M(S, Q), \quad (41) \]

thus satisfying the second condition of the theorem. The third condition in the theorem then follows by the same argument as in Sect. 4.

Finally, assuming the test fields satisfy the null energy condition and preserve the black hole type, we have shown it is not possible to violate the weak cosmic censorship with such test fields, as all conditions in the theorem are met.

**7 BTZ black hole in nonlinear electrodynamics**

The metric for the \((2 + 1)\)-dimensional static BTZ black hole coupled with nonlinear electrodynamics (NLED) is given by [27]

\[ ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2d\phi^2, \quad (42) \]

where

\[ f(r) = \frac{r^2}{l^2} - m - q^2 \log \left( \frac{q^2}{a^2r^2} + \frac{r^2}{l^2} \right). \quad (43) \]

Here \(l\) is the AdS radius, related to the cosmological constant \(\Lambda\) by

\[ \Lambda = -\frac{1}{l^2}, \quad (44) \]

\(m\) and \(q\) are parameters that relate to the black hole mass \(M\) and charge \(Q\) by

\[ M = \frac{m}{8}, \quad Q = 8q, \quad (45) \]

and \(a\) is a free parameter. The electromagnetic potential one-form is given by [27]

\[ A = -\frac{a^2qr^2 - q^3 \log(q^2 + a^2r^2)}{32\pi} dt. \quad (46) \]

Since the metric is static and spherically symmetric, the first condition in the theorem is satisfied, as shown in Sect. 4. The electric potential at the event horizon \(\Phi\), Hawking temperature \(T\), black hole mass \(M\) and entropy \(S\) can be expressed as [27]

\[ \Phi = -\frac{Q(l^2Q^2 + (l^2Q^2 + 64a^2r_+^2) \log(\frac{Q^2}{64a^2} + \frac{r_+^2}{l^2}) \log(\frac{Q^2}{64a^2} + \frac{r_+^2}{l^2})}{256(l^2Q^2 + 64a^2r_+^2)} \], \quad (47) \]

\[ T = \frac{r_+}{128l^2\pi} \left( 64 - \frac{64a^2Q^2r_+^2}{l^2Q^2 + 64a^2r_+^2} \right), \quad (48) \]
\[
M = \frac{r_+^2}{8l^2} - \frac{Q^2}{512} \log \left[ \frac{1}{l^2} \left( \frac{Q^2}{64a^2} + r_+^2 \right) \right],
\]
\[
S = \frac{1}{2} \pi r_+.
\]

where \( r_+ \) is the radius of the event horizon. For fixed \( a \) and \( \Lambda \), we have the first law of thermodynamics [27]

\[
dM = TdS + \Phi dQ.
\]

From (43) we have

\[
f(r_+) = 0 \iff M = M(r_+, Q),
\]

and using (43) we obtain a Smarr relation in the form

\[
M = M(S, Q),
\]

which satisfies the second condition in the theorem. The third condition in the theorem then follows by the same argument as in Sect. 4.

Finally, assuming that the test fields satisfy the null energy condition and preserve the black hole type, we just have shown it is not possible to violate the weak cosmic censorship with such test fields, as all conditions in the theorem are met.

8 Born–Infeld-AdS black holes

The \( d \)-dimensional Born–Infeld-AdS black hole solution spacetime may be written as [28,29]

\[
ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2d\Omega_{d-2}^2,
\]

with \( f(r) \) given by

\[
f(r) = 1 - \frac{m}{r^{d-3}} + \frac{r^2}{l^2} + \frac{4b^2r^2}{(d-1)(d-2)} \left( 1 - \sqrt{1 + \frac{(d-2)(d-3)q^2}{2b^2r^{2d-4}}} \right) + \frac{2(d-2)q^2}{(d-1)r^{2d-6}} \left( \frac{d-3}{2d-4}, \frac{1}{2}, \frac{3d-7}{2d-4} \right) - \frac{(d-2)(d-3)q^2}{2b^2r^{2d-4}},
\]

where \( \text{\( \text{2F1} \) is the hypergeometric function and \( b \) is the Born–Infeld parameter characterizing the nonlinearity of the electromagnetic field. For this spacetime we have a negative constant cosmological constant given by}

\[
\Lambda = -\frac{(d-1)(d-2)}{2l^2},
\]
and the parameters \( m \) and \( q \) relate to the black hole mass \( M \) and charge \( Q \) by

\[
M = \frac{(d-2)\omega_{d-2}}{16\pi} m \quad \text{and} \quad Q = \frac{q\omega_{d-2}}{4\pi} \sqrt{\frac{(d-2)(d-3)}{2}},
\]

with \( \omega_{d-2} \) the volume of the unit \((d-2)\)-dimensional sphere. The black hole mass is given as a function of the event horizon radius \( r_+ \) by

\[
M = \frac{(d-2)\omega_{d-2}}{16\pi} r_+^{d-3} + \frac{(d-2)\omega_{d-2}}{16\pi} r_+^{d-1}
\]

\[
+ \frac{b^2\omega_{d-2}}{4\pi(d-1)} r_+^{d-1} \left( 1 - \sqrt{1 + \frac{16\pi^2 Q^2}{b^2\omega_{d-2}^2 r_+^{2d-4}}} \right)
\]

\[
+ \frac{4\pi(d-2)Q^2}{(d-1)(d-3)\omega_{d-2} r_+^{d-3}} 2 F_1 \left[ \frac{d-3}{2d-4}, \frac{1}{2}, \frac{3d-7}{2d-4}, -\frac{16\pi Q^2}{b^2\omega_{d-2}^2 r_+^{2d-4}} \right],
\]

the Hawking temperature as

\[
T = \frac{1}{4\pi} \left[ \frac{d-3}{r_+} + (d-1)r_+ + \frac{4b^2 r_+}{d-2} \left( 1 - \sqrt{1 + \frac{16\pi^2 Q^2}{b^2\omega_{d-2}^2 r_+^{2d-4}}} \right) \right],
\]

and the entropy and electric potential at the event horizon are then

\[
S = \frac{\omega_{d-2} r_+^{d-2}}{4},
\]

\[
\Phi = \frac{q}{\sqrt{\frac{2(d-3)}{d-2}}} r_+^{d-2} 2 F_1 \left[ \frac{d-3}{2d-4}, \frac{1}{2}, \frac{3d-7}{2d-4}, -\frac{(d-2)(d-3)q^2}{2b^2 r_+^{2d-4}} \right].
\]

Since the metric is static and spherically symmetric, the first condition in the theorem is satisfied, as shown in Sect. 4. Assuming the Born–Infeld parameter \( b \) and the cosmological constant \( \Lambda \) to be fixed, we have the first law of black hole of thermodynamics

\[
dM = T dS + \Phi dQ.
\]

From (55) we have

\[
f(r_+) = 0 \Leftrightarrow M = M(r_+, Q),
\]

and using (60) we obtain a Smarr relation in the form

\[
M = M(S, Q),
\]
thus satisfying the second condition of the theorem. The third condition in the theorem then follows by the same argument as in Sect. 4.

Finally, assuming that the test fields satisfy the null energy condition and preserve the black hole type, we just have shown that test fields cannot destroy extremal Born–Infeld-AdS black holes, as all conditions in the theorem are met.

9 Charged toroidal black holes

The metric of a charged toroidal black hole is given by [30,31]

$$ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2(d\theta^2 + d\psi^2)$$

(65)

with

$$f(r) = -\frac{\Lambda r^2}{3} - \frac{2M}{\pi r} + \frac{4Q^2}{\pi r^2},$$

(66)

where the cosmological constant is negative, $\Lambda < 0$. The electromagnetic potential one-form is given by

$$A = -\frac{4Q}{r}dt.$$  

(67)

These black holes have two horizons, which coincide when the black hole becomes extremal (the inner horizon radius $r_-$ becomes equal to the outer event horizon radius $r_+$). The Hawking temperature can be written as [30]

$$T = \frac{f'(r_+)}{4\pi} = \frac{-12Q^2 + 3Mr_+ - \pi r_+^4\Lambda}{6\pi^2 r_+^3},$$

(68)

the black hole entropy as

$$S = \frac{\pi^2 r_+^2}{6\pi^2 r_+^3},$$

(69)

and the electric potential at the event horizon is given by

$$\Phi = A_t(\infty) - A_t(r_+) = \frac{4Q}{r_+}.$$  

(70)

Since the metric is static, the black hole is nonrotating and there is no angular momentum to consider; consequently, the thermodynamic angular velocity can be set to zero. The timelike Killing vector $X = \frac{\partial}{\partial t}$ determines the black hole mass and is also the Killing generator of the event horizon $\mathcal{H}^+$, whose angular velocity is then also zero. In other words, $X = K = Z$, and so the first condition of the theorem is satisfied.
For a fixed cosmological constant, we have the first law of black hole thermodynamics given as \([30]\)

\[
\frac{dM}{dt} = T dS + \Phi dQ.
\] (71)

The black hole mass can be obtained from (66) in the form \(M = M(r_+, Q)\):

\[
f(r_+) = 0 \Leftrightarrow M = \frac{12Q^2 - \pi r_+^4 A}{6r_+}.
\] (72)

Using (69) we obtain a Smarr relation in the form

\[
M = M(S, Q),
\] (73)

which satisfies the second condition of the theorem. The third condition in the theorem then follows by the same argument as in Sect. 4.

Finally, assuming that the test fields satisfy the null energy condition and preserve the black hole type, we just have shown that it is not possible to violate the weak cosmic censorship with such test fields, as all conditions in the theorem are met.

10 Charged black holes in string theory

The metric for the RN black hole analogue solution in the low energy limit of heterotic string theory can be written as \([32–35]\),

\[
ds_E^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r\left(r - \frac{Q^2}{M}\right)d\Omega
\] (74)

where

\[
f(r) = 1 - \frac{2M}{r},
\] (75)

\(M\) is the black hole mass and \(Q\) is the magnetic charge (related to the black hole charge \(q\) by \(q = \frac{1}{\sqrt{4\pi Q}}\)) \([34]\). For this solution we have the electromagnetic potential one-form \([32]\)

\[
A = -\frac{Q}{r}dt
\] (76)

and the dilaton field \(\phi\) given by

\[
e^{2\phi} = 1 - \frac{Q^2}{Mr}.
\] (77)
Here we assume spacetime to be asymptotically flat, thus with a zero cosmological constant and a dilaton field $\phi$ vanishing at infinity \cite{32}. In order to make sure that the action reduces to the usual Einstein action with scalar field when the Maxwell field vanishes, we make the scaling on the metric $g_{\mu\nu}^E = e^{-2\phi}g_{\mu\nu}$ \cite{32–35}.

Since the metric is static and spherically symmetric, the first condition in the theorem is satisfied, as shown in Sect. 4. The Hawking temperature $T$ and electric potential $\Phi$ at the event horizon are

$$T = \frac{f'(r_+)}{4\pi} = \frac{M}{2\pi r_+^2} = \frac{1}{8\pi M},$$  \hspace{1cm} (78)

$$\Phi = A_t(\infty) - A_t(r_+) = \frac{Q}{r_+},$$  \hspace{1cm} (79)

where in the last step to obtain the temperature we used the fact that the event horizon radius is $r_+ = 2M$. The black hole mass $M$ and entropy $S$ are then given by \cite{32,35}

$$M = \frac{Q^2}{r_+}, \quad S = \frac{A}{4} = \pi r_+^2 - 2\pi Q^2 = 4\pi M^2 - 2\pi Q^2,$$  \hspace{1cm} (80)

where $A$ is the event horizon’s area. These parameters are then related through the first law of black hole thermodynamics \cite{35}

$$dM = TdS + \Phi dQ,$$  \hspace{1cm} (81)

(assuming a fixed cosmological constant). From (80) is clear that we have a Smarr relation in the form

$$M = M(r_+, Q) = M(S, Q),$$  \hspace{1cm} (82)

thus satisfying the second condition of the theorem. The third condition in the theorem then follows by the same argument as in Sect. 4.

Finally, assuming that the test fields satisfy the null energy condition and preserve the black hole type, we just have shown that it is not possible to violate the weak cosmic censorship with such test fields, as all conditions in the theorem are met.

\section{11 5D charged rotating minimally gauged supergravity black hole}

The metric for the 5-dimensional rotating minimally gauged supergravity black hole solution is given by \cite{36}
\[ ds^2 = - \left( dt - a \sin^2 \theta d\phi - b \cos^2 \theta d\psi \right) \left[ f \left( dt - a \sin^2 \theta d\phi - b \cos^2 \theta d\psi \right) + \frac{2q}{\Sigma} \left( b \sin^2 \theta d\phi + a \cos^2 \theta d\psi \right) \right] + \Sigma \left( \frac{r^2 dr^2}{\Delta} + d\theta^2 \right) + \frac{\sin^2 \theta}{\Sigma} \left[ a dt - \left( r^2 + a^2 \right) d\phi \right]^2 + \frac{\cos^2 \theta}{\Sigma} \left[ b dt - \left( r^2 + b^2 \right) d\psi \right]^2 + \frac{1}{r^2 \Sigma} \left[ a b dt - b \left( r^2 + a^2 \right) \sin^2 \theta d\phi - a \left( r^2 + b^2 \right) \cos^2 \theta d\psi \right]^2, \]

with

\[ f(r, \theta) = \frac{(r^2 + a^2)(r^2 + b^2)}{r^2 \Sigma} - \frac{\mu \Sigma - q^2}{\Sigma^2}, \]

\[ \Sigma(r, \theta) = r^2 + a^2 \cos^2 \theta + b^2 \sin^2 \theta, \]

\[ \Delta(r) = \left( r^2 + a^2 \right) \left( r^2 + b^2 \right) + 2abq + q^2 - \mu r^2. \]

Here \( \mu, q, a \) and \( b \) are the mass, charge and angular momentum per unit mass parameters, respectively, and are related to the physical mass \( M \), charge \( Q \) and angular momenta \( J_\phi \) and \( J_\psi \) by [36]

\[ \mu = \frac{8M}{3\pi}, \quad q = \frac{4Q}{\sqrt{3} \pi}, \quad a + b = \frac{4}{\pi} \frac{J_\phi + J_\psi}{\mu + q}. \]

The electromagnetic potential one-form can be written as

\[ \mathbf{A} = -\frac{\sqrt{3}q}{2\Sigma} \left( dt - a \sin^2 \theta d\phi - b \cos^2 \theta d\psi \right). \]

The Killing generator of the event horizon is given by [36]

\[ Z = \frac{\partial}{\partial t} + \Omega^{(\phi)} \frac{\partial}{\partial \phi} + \Omega^{(\psi)} \frac{\partial}{\partial \psi}, \]

with

\[ \Omega^{(\phi)} = \frac{a \left( r^2_+ + b^2 \right) + bq}{(r^2_+ + a^2) \left( r^2_+ + b^2 \right) + abq}, \]

\[ \Omega^{(\psi)} = \frac{b \left( r^2_+ + a^2 \right) + aq}{(r^2_+ + a^2) \left( r^2_+ + b^2 \right) + abq}. \]
We have the first law of black hole thermodynamics \[36\]

\[
dM = \frac{\kappa}{8\pi} dA + \Omega^{(\phi)} dJ_\phi + \Omega^{(\psi)} dJ_\psi + \Phi dQ, \tag{91}
\]

where \(A\) is the area of the event horizon,

\[
A = \frac{2\pi^2}{r_+} \left( \mu r_+^2 - abq - q^2 \right), \tag{92}
\]

\(\kappa\) is the surface gravity at the event horizon,

\[
\kappa = \frac{(2r_+^2 + a^2 + b^2 - \mu) r_+}{\mu r_+^2 - abq - q^2}, \tag{93}
\]

and \(\Phi\) is the electric potential at the event horizon,

\[
\Phi = \frac{\sqrt{3}qr_+^2}{\mu r_+^2 - abq - q^2}. \tag{94}
\]

This shows that the first condition of the theorem is satisfied, since \(\Omega^{(\phi)}\) and \(\Omega^{(\psi)}\) are the thermodynamic angular velocities. From \(83\) we see that

\[
f(r_+) = 0 \iff M = M(r_+, J_\phi, J_\psi, Q), \tag{95}
\]

and using \(92\) and the Bekenstein–Hawking formula \(19\), we obtain a Smarr relation in the form

\[
M = M(S, J_\phi, J_\psi, Q), \tag{96}
\]

thus satisfying the second condition of the theorem.

When the black hole becomes extremal by definition we have \(k = 0\), or equivalently \(T = 0\), leading to

\[
\frac{\partial M}{\partial S}(S, J_\phi, J_\psi, Q) = 0. \tag{97}
\]

Solving for the entropy leads to the entropy, hence to the mass, of an extremal black hole as a function of its angular momentum and charge,

\[
S = S_{ext}(J_\phi, J_\psi, Q) \Rightarrow M_{ext} = M(S_{ext}(J_\phi, J_\psi, Q), J_\phi, J_\psi, Q), \tag{98}
\]

satisfying the third condition of the theorem.

Finally, assuming the test fields interacting with the black hole satisfy the null energy condition and preserve the black hole type the black hole type, all the conditions of the theorem are met and thus the weak cosmic censorship is preserved for such test fields.
12 (2 + 1)-dimensional MTZ black holes

The metric for the (2 + 1)-dimensional MTZ black hole is given by [37]

\[ ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2d\phi^2, \tag{99} \]

with

\[ f(r) = r^2 - M - \left(\frac{Q}{2}\right)^2 \ln(r^2), \tag{100} \]

and the electromagnetic potential one-form given by

\[ A = -Q \ln r. \tag{101} \]

Since the metric is static and spherically symmetric, the first condition in the theorem is satisfied, as shown in Sect. 4. The Hawking temperature and electric potential at the event horizon can we written as

\[ T = \frac{f'(r_+)}{4\pi} = \frac{r_+}{2\pi} - \frac{Q^2}{8\pi r_+}, \tag{102} \]

\[ \Phi = \left(\frac{\partial M}{\partial Q}\right)_{r=r_+} = -Q \ln r_+. \tag{103} \]

We have

\[ f(r_+) = 0 \iff M = r_+^2 - \left(\frac{Q}{2}\right)^2 \ln r_+^2 = M(r_+, Q). \tag{104} \]

Noting that in (2 + 1)-dimensions the area \( A \) of the black hole horizon is the perimeter, \( A = 2\pi r_+ \), and using the Bekenstein–Hawking formula [19], we obtain the Smarr relation in the form

\[ M = M(S, Q), \tag{105} \]

yielding the first law of black hole thermodynamics

\[ dM = TdS + \Phi dQ, \tag{106} \]

hence satisfying the second condition of the theorem. The third condition in the theorem then follows by the same argument as in Sect. 4.

Finally, assuming the test fields satisfy the null energy condition and preserve the black hole type, all the conditions of the theorem are met and we conclude that such test fields cannot destroy extremal (2 + 1)-dimensional MTZ black holes.
13 Discussion

We have shown, using the general result in [11], that the weak cosmic censorship holds for various extremal black holes in the literature; more precisely, it is impossible to create naked singularities by overcharging or overspinning these extremal black holes, as long as the test particles or fields interacting with them satisfy the null energy condition. Although we chose a small subset of the many extremal black holes in the literature, the theorem in [11] can be used to check the validity of the conjecture for other extremal black holes as well (recent interesting examples include [38,39]). Our results seem to indicate that as long as the interacting test particles or fields satisfy the null energy condition and only exchange energy, angular momentum and electric charge with the black hole then the conclusion that weak cosmic censorship holds is almost automatic (assuming that there is a first law of thermodynamics relating the changes in these quantities). In fact, the best strategy to look for possible violations of weak cosmic censorship appears to be either to use test fields which do not satisfy the null energy condition (such as fermion fields [40,41]) or to use test fields that exchange other types of conserved charges with the black hole.

We should emphasize that we assumed that the interaction with the test fields preserves the black hole type, meaning that after the interaction the spacetime settles to a stationary black hole solution characterized by the same parameters. In this sense our proof is not exhaustive, since there is always the possibility of violating the weak cosmic censorship conjecture via interactions that change the black hole type.

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