Abstract

An equational axiomatisation of probability functions for one-dimensional event spaces in the language of signed meadows is expanded with conditional values. Conditional values constitute a so-called signed vector meadow. In the presence of a probability function, equational axioms are provided for expected value, variance, covariance, and correlation squared, each defined for conditional values.

Finite support summation is introduced as a binding operator on meadows which simplifies formulating requirements on probability mass functions with finite support. Conditional values are related to probability mass functions and to random variables. The definitions are reconsidered in a finite dimensional setting.

Keywords and phrases: Boolean algebra, signed meadow, vector meadow, probability function, probability mass function, conditional value.

1 Introduction

In [4] a proposal is made for a loose algebraic specification probability functions in the context of signed meadows. The objective of this paper is to proceed on the basis of the results of [4] and to provide an account of some basic elements of probability calculus including probability mass functions, probability functions, expected value operators, variance, covariance, correlation, independence, sample space, and random variable. Ample use is made of the special properties of meadows, most notably $1/0 = 0$, and the presentation optimises the match with the equational and axiomatic setting of meadows.

A conventional ordering of the introduction of concepts in probability theory is as follows: (i) given a sample space $S$, an event space $E$ is introduced as a subset of the power set of $S$. Then (ii) probability functions are defined over event spaces and (iii) discrete random variables are introduced as real functions on $S$ with a countable support. Given these ingredients,
(iv) expected value and variance are defined for discrete random variables and covariance, and correlation are defined for pairs of random variables. (v) Subsequently probability mass functions are derived from random variables, (vi) multivariate discrete random variables are introduced as vectors of random variables on the same event space, and (vii) joint probability mass functions are derived from multivariate random variables, (viii) independence is defined for joint probability mass functions, and (ix) marginalisation is defined as a transformation on joint probability mass functions. (x) The development of concepts and definitions is redone for the continuous case with probability distributions replacing probability mass functions and general random variables replacing discrete random variables.

Below the same topics are discussed, though under restrictive conditions, and in a different order. A central role is played by probability mass functions with finite support. These are meadow valued functions taking nonzero values for finitely many arguments, and such that the sum of the nonzero values adds up to one. As a conceptual cornerstone the space (sort) of conditional values is introduced.

As a consequence of these choices the definition for the expected value operator, and the definitions of (co)variance and correlation which directly depend on expected values, will be repeated in three different settings: (i) for probability mass functions with finite support, (ii) for an event space equipped with a sort of conditional values and a probability function, and (iii) for a multidimensional event space equipped with a sort of conditional values and a family of multivariate probability functions.

With this order of presentation an adequate match is obtained with meadow based equational axiomatisations. The account of probability mass functions is independent of probability theory. By defining expected values and derived quantities on conditional values over an event structure the incentive for introducing a sample space is avoided, thus avoiding the introduction of a proper subsort of samples for the sort of events, and thereby maintaining the simplicity of the use of a loose equational specification for probability functions.

1.1 Survey of the paper

In Section 2 meadows are discussed and so-called signed vector meadows are introduced. A novel binding operator, called finite support summation (FSS) is introduced and examples of its use are provided.

In Section 3 the notion of a probability mass function (PMF) with finite support is introduced and its formal specification in the setting of meadows is provided with the help of finite support summation. By default a PMF is assumed to be univariate.

Marginalisation is defined as a family of transformations from a PMF with more than one argument (i.e. a multivariate PMF) to a PMF with a smaller number of arguments. Expected value and variance are defined as functionals on univariate PMFs and covariance and correlation are defined as functionals on bivariate PMFs.

Having developed an account of PMFs independently of axioms for probability functions, Section 4 proceeds with a recall from 4 of the combination of an event space (a Boolean algebra) and value space (a meadow), and the equational specification of a probability function. Two versions of Bayes’ rule are considered and the relative position of these statements w.r.t. the various axioms is examined.
A conditional operator is applied to events and the results of the operator are collected in an additional sort $V_C$ of so-called conditional values (CVs), which constitutes a so-called finite dimensional vector space meadow.

Thinking in terms of outcomes of a probabilistic process one may assume that the process produces as an outcome an entity of some sort. Events from an event space $E$ represent assessments about the outcome. It is plausible that besides Boolean assessments also values, for instance rationals or reals, are considered attributes of an outcome. A CV directly relates values to events. In the presence of a probability function two equations specify the expected value of a CV.

From [4] the specification of probability function families relative to an arity family is imported, and in Section 5 an corresponding axiomatisation for expected value operators is provided for the finite dimensional case.

According to [4] the equations of $\text{BA} + \text{Md} + \text{Sign} + \text{ABS} + \text{PFBC}_P + \text{PFA}_P$ constitute a finite equational basis for the class of Boolean algebra based, real valued probability functions, and the proof theoretic results, viz. soundness and completeness, concerning signed meadows of $\mathbb{R}$ extend to the case with Boolean algebra based probability functions. The axiom system $\text{BA} + \text{Md} + \text{Sign} + \text{ABS} + \text{PFBC}_P + \text{PFA}_P$ is merely a particular formalisation of Kolmogorov’s axioms for probability theory phrased in the context of meadows and the completeness result asserts the completeness of this particular formalisation w.r.t. its standard model. The main result of the paper is to provide an extension of this axiomatisation with conditional values and expected value operators $E_P$.

1.2 On the use of equational logic

By working in first order equational logic, I am able to make use of an axiomatic style when developing basic elements of the theory of probability. The exposition therewith is somewhat biased towards formalisation.

We will develop an incremental collection of equational specifications. These specifications can be understood in three different ways: (a) as formalisation of a preferred underlying mathematical reality, (b) as axioms determining what is claimed with one or more underlying mathematical realities primarily as a principled method for justification, (c) as a combination of (a) and (b) in which both views (a) and (b) have been assimilated, thus following the terminology of [14], in such a manner that when appropriate, or when required for clarification explicit dis-assimilation of both views (again following [14]) is an option.

The objective of formalisation and axiomatisation in this paper is not inherited from an overarching intention to avoid mistakes, as is the most prominent rationale for formalisation in computer science. Instead the objective is to use the axiomatic approach to obtain a uniform degree of clarity about assumptions, working hypotheses, patterns of reasoning, and patterns of calculation.

I will not distinguish between names for constants and functions of meadows and their mathematical counterparts. Rather than writing say $\mathbb{R}_0 \models t = r$, in cases where ordinary mathematics suggests writing $t = r$ and provided there is no risk of confusion, that is a preferred arithmetical datatype is supposed to be known to the reader, “$t = r$” is preferred. On the other hand sort names, e.g. $E$ for events, will be distinguished from the corresponding...
(x + y) + z = x + (y + z) \quad (1)

x + y = y + x \quad (2)

x + 0 = x \quad (3)

x + (-x) = 0 \quad (4)

(x \cdot y) \cdot z = x \cdot (y \cdot z) \quad (5)

x \cdot y = y \cdot x \quad (6)

1 \cdot x = x \quad (7)

x \cdot (y + z) = x \cdot y + x \cdot z \quad (8)

(x^{-1})^{-1} = x \quad (9)

x \cdot (x \cdot x^{-1}) = x \quad (10)

Table 1: Md: axioms for a meadow

carriers, (e.g. \(|E|\) in the case of \(E\)) and a specific probability function with intended to serve as an interpretation of \(P\) will be referred to as \(\hat{P}\).

In summary: below equational logic is applied with the following objectives in mind: (i) to demonstrate that an axiomatic approach in terms of equational logic to elementary probability calculus is both feasible and attractive, (ii) to illustrate the compatibility of an axiomatic approach to probability calculus with conventional mathematical style and notation, and (iii) to provide optimal clarity about the assumptions which underly the various definitions, while (iv) using meadows as a tool throughout the presentation.

1.3 Assimilation and dis-assimilation: a plurality of options

In [] the idea is put forward that persons working with mathematical text may or may not at a certain stage assimilate, that is perceive as one and the same, different notations. For instance it is common not to distinguish 1 and +1, but one may well imagine a stage in which these notations are distinguished. I mentioned already that not making a distinction between constants 0 and 1 and the corresponding values in a chosen mathematical domain may be understood as a result of assimilation which may be reversed temporarily by way of implicit or explicit dis-assimilation.

In Paragraph 4.3 below so-called conditional values are introduced as elements of a sort \(V_C\). These are understood as a version of the numbers, though with conditions, the connection between conditional values and (ordinary, i.e. unconditional) values is made via an embedding \(v: V \rightarrow CV\). Here it is an option to assimilate conditional values and unconditional values and to drop occurrences of \(v(\cdot)\). I have chosen not to do so, and to leave unconditional values and conditional values dis-assimilated in order to allow a better focus on the fact that the conditional values constitute a vector space meadow rather than a meadow.
\[ x^2 = x \cdot x \]  
\[ \frac{x}{y} = x \cdot y^{-1} \]  
\[ 1(x) = x/x \]  
\[ 0(x) = 1 - x/x \]  
\[ x < y \succ z = 1(y) \cdot x + 0(y) \cdot z \]  

**Table 2: DO: axioms for derived operators**

\[ s(1(x)) = 1(x) \]  
\[ s(0(x)) = 0(x) \]  
\[ s(-1) = -1 \]  
\[ s(x^{-1}) = s(x) \]  
\[ s(x \cdot y) = s(x) \cdot s(y) \]  
\[ 0(s(x) - s(y)) \cdot s(x + y) = 0(s(x) - s(y)) \cdot s(x) \]

**Table 3: Sign: axioms for the sign operator**

2 Meadows and vector space meadows

Numbers will be viewed as elements of a meadow rather than as elements of a field. For the introduction of meadows and elementary theory about meadows I refer to \([7, 2, 3]\) and the papers cited there. I will copy the tables of equational axioms for meadows and for the sign function which plays a central role below.

Below \( \mathbb{R} \) will denote some specific choice of a structure for real numbers. The domain \(|\mathbb{R}|\) of \( \mathbb{R} \) is a particular set, the choice of which depends on how one prefers to define real numbers. There is no preference for a specific choice. \( \mathbb{R}_0 \) is the unique expansion of \( \mathbb{R} \) with a zero-totalised inverse function \(-1\). \( \mathbb{R}_0 \) is referred to as the meadow of reals.

With \((\mathbb{R}_0, s)\) the expansion of the meadow \( \mathbb{R}_0 \) with the sign function is denoted. The following completeness result was obtained in \([3]\).

**Theorem 1.** A conditional equation in the signature of signed meadows is valid in \((\mathbb{R}_0, s)\) if and only if it is provable from the axiom system \(\text{Md} + \text{DO} + \text{Sign}\).

The axioms in Table 1 specify the variety of meadows, while Table 2 introduces some function symbols by means of defining equations serving as explicit definitions for derived operations. Table 3 specifies the sign function, and Table 4 introduces the absolute value function. Following \([2]\), a meadow that satisfies the (nonequational) implication IL from Table 5 is called a cancellation meadow.
The meadow $M$ is the expansion of $s$-potents. Now $R$ is the object of the mentioned identities involving the $e$-dimensional vector space meadows of different dimension is possible.

### Table 4: ABS: defining axiom the absolute value operator

\[ |x| = s(x) \cdot x \]  

### Table 5: IL: inverse law

\[ x \neq 0 \rightarrow x \cdot x^{-1} = 1 \]

#### 2.1 Signed vector space meadows

Let $e_1, \ldots, e_n$ be a series of pairwise distinct objects outside the meadow $M$, and outside $\Sigma_{Md}$. The meadow $M(e_1, \ldots, e_n)$ is defined as a direct sum of copies of $M$:

\[ M(e_1, \ldots, e_n) = e_1 \cdot M \oplus \ldots \oplus e_n \cdot M \]

Here the $e_i$ serve as new constants for orthogonal ($e_i \cdot e_j = 0$ for $i \neq j$) idempotents ($e_i \cdot e_i = e_i$) such that the set \{ $e_1, \ldots, e_n$ \} is complete ($e_1 + \ldots + e_n = 1$). Moreover it is assumed that $s(e_i) = e_i$. Elements of this structure are given by sequences $(l_1, \ldots, l_n) \in M^n$ representing the object $e_1 \cdot l_1 + \ldots + e_n \cdot l_n$. The meadow operations and sign are performed coordinate-wise, e.g. $s(e_1 \cdot l_1 + \ldots + e_n \cdot l_n) = e_1 \cdot s(l_1) + \ldots + e_n \cdot s(l_n)$, thus obtaining an $n$-dimensional vector space over $M$. For $n = 1$ the construction brings noting new: $M(e_1) \cong M$. For $n > 1$, and assuming that $M$ is non-trivial ($M \models 0 \neq 1$) the resulting structures are not cancellation meadows, i.e. $M(e_1, \ldots, e_n) \not\models IL$.

$\Sigma_{Md,e_1,\ldots,e_n}$ is the signature $\Sigma_{Md}$ expanded with constants $e_1, \ldots, e_n$. $M_{e_1,\ldots,e_n}(e_1, \ldots, e_n)$ is the expansion of $M(e_1, \ldots, e_n)$ with (the $e_i$ serving as names for the new orthogonal idempotents. Now $\mathbb{R}_0(s)e_1,\ldots,e_n(e_1, \ldots, e_n) \models Md + Sign + E_{e_1,\ldots,e_n}$, where $E_{e_1,\ldots,e_n}$ captures the mentioned identities involving the $e_i$: idempotence for the $e_i$, orthogonality for $e_i$ and $e_j$ with $i \neq j$, completeness, and the equations for $s(-)$.

**Problem 1.** *Is the axiom system Md + Sign + E_{e_1,\ldots,e_n} complete for the equational theory of the structure $\mathbb{R}_0(s)e_1,\ldots,e_n(e_1, \ldots, e_n)$?*

With disjunctive assertions (among which IL $\equiv 1(x) = 0 \lor 1(x) = 1$) discrimination between vector space meadows of different dimension is possible.

Let $\phi \equiv \lambda x f \cdot x \cdot x = x \land y \cdot y = y \land x + y = 1 \land x \cdot y = 0 \rightarrow (x = 0 \lor y = 0)$. Then $\mathbb{R}_0(s)\langle e_1, e_2 \rangle \not\models \phi$ while : $\mathbb{R}_0(s)\langle \rangle \models \phi$.

#### 2.2 Representing functions by expressions

The expression language may be extended with lambda abstraction thereby introducing $\lambda x.t$ as an expression denoting the function which maps $v \in V$ to $[v/x]t$, i.e. the result of substituting $v$ for $x$ in $t$. A disadvantage of this approach is that it imports typed $\lambda$-calculus, definitely a non-trivial subject.
Another option is to use $L x. t$ to represent the same function. Now if $y$ does not occur freely in $t$, then $L y. [y/x] t$ constitutes a different representation for the same function, i.e. unlike in the $\lambda$-calculus alpha conversion does not apply to $L x. t$.

In statistical theory Jeffrey’s notation $t[\bullet]$, with $t[-]$ a context with zero or more “holes”, stands for $\lambda x. t[x]$, with $x$ a fresh variable. Finally function abstraction may be left implicit when a specific binding mechanism is employed.

When summation over a bound variable $x$ is applied to a term $t$ or to a context $t[-]$, these four options lead to different notations: $\sum^r (\lambda x. t)$, $\sum^r (L x. t)$, $\sum^r t[\bullet]$ and $\sum^r t$, respectively. There is no need to choose a single convention from these four options and below it is supposed to be clear from the context which one of these conventions is used in each particular case.

## 2.3 Finite support summation

Given a meadow $M$ and a term $t$ in which variable $x$ may or may not occur it may be useful to determine the summation of all substitutions (or rather interpretations) $[v/x] t$ with $v \in |M|$. This sum is unambiguously defined, however, if the support in $M$ of $L x. t$ is finite, that is if there are only finitely many values $v \in |M|$ such that $[v/x] t$ is nonzero.

The expression $\sum^r t$ denotes in $M$ the sum of all $[v/x] t$ if at most finitely many of these substitutions $[v/x] t$ yield a non-zero value and 0, otherwise. The $\sum^r$ operator will be referred to as finite support summation (FSS). At this stage we have little information about the logical properties of this binding mechanism on terms but it is semantically unproblematic, being well-defined in each meadow, and it will be used below for presenting several definitions. We first notice some technical facts concerning FSS, assuming the interpretation of equations is performed in an arbitrary cancellation meadow $M$.

1. $L x. t$ has finite support iff $L x. t/t$ has finite support.
2. $\sum^r 0 = 0$, $\sum^r 0(x) = 1$,
3. $\sum^r 1 = 0$. To see this first notice that in an infinite meadow 1 is nonzero for infinitely many $x$ and thus $\sum^r 1 = 0$. A finite meadow has nonzero characteristic (say $p$) and $\sum^r 1$ counts up to the cardinality of the structure, which is a multiple of $p$ and therefore vanishes modulo $p$.
4. $\sum^r 1(x) = 0$ if and only if $M$ is infinite.
5. $\sum^r 1(x) = -1$ if and only if $M$ is finite.
6. $\sum^r (t + 0(x)) = (\sum^r t) + 1$ if and only if $L x. t$ has finite support.
7. $\sum^r (t + 0(x)) = \sum^r t$ if and only if $L x. t$ has infinite support.
8. If $x \not\in \text{FV}(t)$ then $\sum^r (r \cdot t) = (\sum^r r) \cdot t$.
9. If $x \not\in \text{FV}(t)$ then $\sum^r (x \cdot 0(t - x)) = t$ and $\sum^r (x \cdot 1(t - x)) = (\sum^r t) - [0/x] t$.
10. If both $L x. t$ and $L x. t$ have finite support then $(\sum^r t) + (\sum^r r) = \sum^r (t + r)$. 


If, moreover, \( \mathbb{M} \) is signed:

1. \( \sum_x^* 1(x) = 0 \), because a signed meadow is infinite.

2. Consider context \( C[-] \) with \( C[X] = 1(\sum_x^* 1(X)) < (\sum_x^* (X + 0(x)) - \sum_x^* X) \triangleright 1 \), then \( C[t] = 1 \) if and only if the support of \( L x.t \) is nonempty, and otherwise \( C[t] = 0 \).

3. \( C[t] \cdot (\sum_x^*(t + 0(x)) - (\sum_x^* t)) \cdot 0(1 - \sum_x^* s(t)) = 0 \) if and only if the support of \( L x.t \) is a singleton.

**Proposition 1.** \( L x.t \) has finite support in \( \mathbb{Q}_0 \) if and only if it has finite support in \( \mathbb{R}_0 \).

**Proof.** Because \( \mathbb{Q}_0 \) is a substructure of \( \mathbb{Q}_0 \) the number of non-zero values of \( \lambda x.t \) in \( \mathbb{Q}_0 \) cannot exceed the number of nonzero values in \( \mathbb{R}_0 \) so the if part is immediate. Now for “only if” suppose that \( \lambda x.t \) has infinitely many non-zero values in \( \mathbb{R}_0 \). In [2] it is shown that non-zero \( t(x) \) is provably equal to a sum of simple fractions, i.e. fractions for which numerator and denominator are each nonzero-polynomials. This implies that \( \lambda t(x) \) is discontinuous on at most finitely many arguments so that it must be nonzero at some real argument \( r \) where it is continuous at the same time. This implies that \( \lambda t(x) \) is nonzero in some neighbourhood \( (r - \epsilon, r + \epsilon) \) of \( r \) so that it is nonzero on the infinitely many rational arguments in this same neighbourhood. It follows that \( \lambda x.t \) has infinite support in \( \mathbb{Q}_0 \).

**Problem 2.** Is there a context \( C[-] \) (not involving \( s \)) so that for all meadow expressions without sign and for all cancellation meadows (in particular those with non-zero characteristic) \( C[t] = 0 \) equals 0 if \( t \) has empty support and \( C[t] = 1 \) otherwise?

**Problem 3.** Consider the meadows \( \mathbb{R}_0 \) enriched with FSS. Is equality between closed terms for this structure computably enumerable, and if so is it decidable?

**Problem 4.** Consider the meadows \( \mathbb{Q}_0 \) enriched with FSS. Is equality between closed terms for this structure decidable?

### 2.4 Multivariate finite support summation

The multivariate case of FSS operations requires separate definitions for each number of variables because a stepwise reduction to the definition for the univariate case is unfeasible. To demonstrate this difficulty we consider the bivariate case only, the case with three or more variables following the same pattern. In a meadow \( \mathbb{M} \), \( \sum_{x,y}^* t \) produces 0 if for infinitely many pairs of values \( a, b \in |\mathbb{M}| \) the value of \([a/x][b/y] t\) is nonzero, otherwise it produces the sum of the finitely many nonzero values thus obtained.

The need for expressions of the form \( \sum_{x,y}^* t \) transpires from an elementary example, which demonstrates that a 2-dimensional FSS cannot be simply reduced to a composition of 2 occurrences of a 1-dimensional FSS. Let \( t(x, y) = 0(x) \cdot 0(y) + 0(1 - x) \). Because \( t(1, y) = 1 \) for all \( y \), \( t(x, y) \) is nonzero on infinitely many pairs of values, so that

\[
\sum_{x,y}^* t(x, y) = 0.
\]
Now notice that $\sum_y y(0, y) = 1$, $\sum_y y(1, y) = 0$, and if $x \neq 0 \land x \neq 1$, $\sum_y y(x, y) = 0$. It follows that

$$\sum_x \sum_y y(x, y) = 1.$$ 

### 3 Probability mass functions with finite support

The main application of FSS in this paper is to enable the following definition of what it means for a term to represent a finitely supported probability mass function. Probability mass function will be abbreviated as PMF. Finitely supported PMFs constitute a special case of “arbitrary” PMFs, a more general notion which cannot easily be defined on an arbitrary signed meadow, and which will not be used in the sequel.

**Definition 1.** Given a signed meadow $M$, a pair $(t; x)$ consisting of a term and variable $x$, represents a PMF with finite support $M$ if $M \models |t| = |t|$ and $M \models \sum^* x t = 1$.

A PMF with finite support is also called a finitary PMF or a finitely supported PMF. The use of terminology from probability theory requires some justification. Indeed PMFs occur in probability theory where these comprise precisely all nonnegative functions from reals to reals with a countable support so that the sum of all non-zero values equals 1. With this fact in mind, and working in the signed meadow $\mathbb{R}_0(s)$, the two requirements of Definition 1 indeed guarantee that the function represented by $L x.t$ is a PMF with finite support according to standard terminology.

The property of being a representative of a finitely supported PMF is sensitive to the meadow at hand. For instance consider the expression $t$ given by

$$t = 0(x^2 - 2) \cdot ((1 + s(x)) \cdot x + (1 - s(x)) \cdot (2 - x))/4.$$ 

In $\mathbb{R}_0$ the function description $L x.t$ represents a finitary PMF. To see this notice that $L x.t$ takes non-zero values only in $-\sqrt{2}$ and $\sqrt{2}$ where it has values $1 - 1/2 \cdot \sqrt{2}$ and $1/2 \cdot \sqrt{2}$ respectively, so that $L x.t$ represents a finitary PMF, while in $\mathbb{Q}_0$ it is not the case that $L x.t$ represents a finitary PMF because $t(q)$ vanishes for all $q \in \mathbb{Q}_0$ with the implication that $\sum^*_x t = 0$. On the other hand when considering $t'(x) = t(x) + 0(x)$ it turns out that $L x.t$ represents a finitely supported PMF in $\mathbb{Q}_0$ while it fails to do so in $\mathbb{R}_0$.

### 3.1 Multivariate PMFs with finite support

Given a signed cancellation meadow $\mathbb{M}$, a joint PMF with finite support of arity $n$ is a function $L x_1, \ldots, x_n, F(x_1, \ldots, x_n)$ from $\mathbb{M}^n$ to $\mathbb{M}$ which satisfies these two conditions:

1. $\sum_{x_1, \ldots, x_n} F(x_1, \ldots, x_n) = 1$, and
2. for all $x_1, \ldots, x_n \in \mathbb{R}_0^n$, $F(x_1, \ldots, x_n) = |F(x_1, \ldots, x_n)|$.

For example assuming that information about the graph of a joint PMF with finite support, with exception of argument vectors for which the result vanishes, is encoded in a set of triples:
\[ E_{pmf}(F) = \sum_{x} (x \cdot F(x)) \]  
(Expected value of \( F \))

\[ VAR_{pmf}(F) = \sum_{x} (x^2 \cdot F(x)) - E_{pmf}(F)^2 \]  
(variance of \( F \))

\[ COV_{pmf}(G) = \sum_{x,y} (x \cdot y \cdot G(x,y)) - E_{pmf}(G(1)) \cdot E_{pmf}(G(2)) \]  
(covariance of \( G \))

\[ CORR_{pmf}^2(G) = \frac{COV_{pmf}(G)^2}{VAR_{pmf}(G(1)) \cdot VAR_{pmf}(G(2))} \]  
(correlation of \( G \) squared)

Table 6: expected value, (co)variance, and correlation

\( \{(y_{1,1}, y_{2,1}, z_1), \ldots, (y_{1,n}, y_{2,n}, z_n)\} \), a corresponding function expression \( F \) for the same joint PMF with key variables \( x_1 \) and \( x_2 \) is as follows:

\[ F(x_1, x_2) = \sum_{i=1}^{n} (0(x_1 - y_{1,i}) \cdot 0(x_2 - y_{2,i}) \cdot z_i) \]

### 3.2 Marginalisation and independence

Given a finitely supported joint PMF \( G \) with \( n \) variables \( x_1, \ldots, x_n \), marginalisation can be defined to each subset \( x_{i_1}, \ldots, x_{i_k} \) with \( 1 \leq i_1 < \ldots < i_k \leq n \). Let \( x_{j_1}, \ldots, x_{j_{n-k}} \) be an enumeration without repetition of the variables in \( x_{i_1}, \ldots, x_{i_k} \) that are not listed in \( x_{i_1}, \ldots, x_{i_k} \), then \( G_{(i_1,\ldots,i_k)} \) represents a joint PMF with \( k \) variables \( x_{i_1}, \ldots, x_{i_k} \) as follows:

\[ G_{(i_1,\ldots,i_k)}(x_{i_1}, \ldots, x_{i_k}) = \sum_{x_{j_1} \ldots x_{j_{n-k}}} G(x_1, \ldots, x_n) \]

For a bivariate PMF \( G(x, y) \) independence is defined as independence of its two marginalisations.

\[ IND(G) \equiv_{def} \forall x, y \in V.G(x, y) = G(1)(x) \cdot G(2)(y). \]

### 3.3 Expectation, (co)variance, and correlation for PMFs

Now \( F(x) \) is assumed to be a term representing a finite support PMF with \( x \) as the key variable, while \( G(x, y) \) represents a joint PMF with finite support with \( x \) as the first and \( y \) as the second key variable. Two PMFs \( G(1) \) and \( G(2) \) are derived from \( G \) by marginalization: \( G(1)(x) = \sum_{y} G(x, y) \) and \( G(2)(y) = \sum_{x} G(x, y) \). The expected value \( E_{pmf}(F) \) of \( F \) and related operations are given in Table 6. The square of correlation is included in order not to burden the present exposition with the equational specification of a square root operator. In the context of meadows the square root can be made total, and the equationally specified, by
Table 7: BA: a self-dual equational basis for Boolean algebras

\[(x \lor y) \land y = y\]  \hspace{1cm} (23)
\[(x \land y) \lor y = y\]  \hspace{1cm} (24)
\[x \land (y \lor z) = (y \land x) \lor (z \land x)\]  \hspace{1cm} (25)
\[x \lor (y \land z) = (y \lor x) \land (z \lor x)\]  \hspace{1cm} (26)
\[x \land \neg x = \bot\]  \hspace{1cm} (27)
\[x \lor \neg x = \top\]  \hspace{1cm} (28)

These definitions admit a justification on the basis of the conventional use of the defined terminology, the details which are worth mentioning. Given a PMF \(F\) with finite support, its support, say \(S\), may be viewed as a sample space so that in conventional terminology \(id_S\), the identity function of type \(S \rightarrow \mathbb{R}_0\), qualifies as a random variable, say \(X\). The power set of \(S\) serves as an event space, say \(E_S\). Let the probability function \(P\) be generated by \(P(\{s\}) = F(s)\) for \(s \in S\). Now \(P(X = x) = F(x)\) and \(E_{pmf}(F) = E_P(X) = \sum_{s \in S} (X(s) \cdot P(X = s)) = \sum_s (x \cdot F(x))\).

### 4 Event spaces and probability functions

From [4] I will recall equations for Boolean algebras, (signed) meadows, and probability functions. A Boolean algebra \((B, +, -, \cdot, 1, 0)\) may be defined as a system with at least two elements such that \(\forall x, y, z \in B\) the well-known postulates of Boolean algebra are valid. In order to avoid overlap with the operations of a meadow, Boolean algebras are equipped with notation from propositional logic, thus consider \((B, \lor, \land, \neg, \top, \bot)\) and adopt the axioms as presented in Table 7. In [15] it was shown that the axioms in Table 7 constitute an equational basis for the equational theory of Boolean algebras. In the setting of probability functions the elements of the underlying Boolean algebra are referred to as events. Events are closed under \(- \lor -\) which represents alternative occurrence and \(- \land -\) which represents simultaneous occurrence. The term “value” will refer to an element of a cancellation meadow, mainly the meadow of reals and the meadow of rationals. A probability function from events to the values in a signed meadow. An expression of sort \(E\) is an event expression or an event term, an expression of type \(V\) is a value expression or equivalently a value term. In this paper considerations are limited to structures involving a single name for a probability function only, the function symbol \(P\), at least in the 1-dimensional case. Table 8 provides axioms that determine generally agreed boundary conditions for a probability function. Table 9 contains the axiom for additivity that is included in the axiomatisation of [4]. Together with the axioms for signed meadows and for Boolean algebras we find the following set of axioms: \(BA + Md + DO + Sign + ABS + PFBC_P + PFA_P\).

Table 10 provides explicit definitions of some useful conditional probability operators made
Table 8: PFBC$_P$: boundary conditions for a probability function

$$P(\top) = 1$$  \hspace{1cm} (29)
$$P(\bot) = 0$$  \hspace{1cm} (30)
$$P(x) = |P(x)|$$  \hspace{1cm} (31)

Table 9: PFA$_P$: additivity axiom for a named probability function

$$P(x \lor y) = P(x) + P(y) - P(x \land y)$$  \hspace{1cm} (32)

4.1 Soundness and completeness of axioms for probability functions

The reader is assumed to be familiar with the concept of a probability function, say $\hat{P}$ with name $P$, on an event space $E$, where $\hat{P}$ is supposed to comply with the informal Kolmogorov axioms of probability theory. Being based on the availability of real numbers, sets, and measures on sets, the Kolmogorov axioms are more easily understood as providing a mathematical definition, that is a set of requirements, governing which functions are considered probability functions than as constituting a formal system of axioms. The axiom system $\text{BA} + \text{Md} + \text{DO} + \text{Sign} + \text{ABS} + \text{PFBC}_P + \text{PFA}_P$ may be considered a formalisation of the Kolmogorov axioms for probability functions.

A probability function structure over an event space $E$ is a two sorted structure having $E$ (events) and $V$ (values) as sorts with $E$ interpreted by a Boolean algebra, now denoted $E$, and $V$ interpreted as the real numbers $\mathbb{R}$ as chosen in Section 2, enriched with a probability function $\hat{P}$ from $E$ to $V$. The Kolmogorov axioms specify precisely which functions are total by choosing a value in case the condition has probability 0.

Table 10: conditional probability operators

$$P^0(x \mid y) = \frac{P(x \land y)}{P(y)}$$  \hspace{1cm} (33)
$$P^1(x \mid y) = P^0(x \mid y) \leq P(y) \triangleright 1$$  \hspace{1cm} (34)
$$P^*(x \mid y) = P^0(x \mid y) \land P(y) \triangleright P(x)$$  \hspace{1cm} (35)
probability functions. I will assume that \( V \) is the domain of the meadow of reals, i.e. that the meadow version of real numbers is used. With \( \text{EPV}(\mathcal{E}, \mathbb{R}_0(s), P) \) the class of probability function structures over a fixed event structure \( \mathcal{E} \) is denoted, with values taken in \( [\mathbb{R}_0(s)] \). For a specific PMF \( \hat{P} \) the pertinent structure is denoted by \( \text{EPV}(\mathcal{E}, \mathbb{R}_0(s), \hat{P}) \). \( \text{EPV}(\mathcal{A}, \mathbb{R}_0(s), P) \) denotes the union of all collections \( \text{EPV}(\mathcal{E}, \mathbb{R}_0(s), P) \) for all \( \mathcal{E} \) with \( \mathcal{E} \models \mathcal{A} \). It is apparent from the construction that \( \text{EPV}(\mathcal{E}, \mathbb{R}_0(s), \hat{P}) \models \mathcal{A} + \mathcal{B} + \text{ABS} + \text{PFB}_P + \text{PFA}_P \).

A completeness result for \( \mathcal{A} + \mathcal{B} + \mathcal{D} + \text{ABS} + \text{PFB}_P + \text{PFA}_P \) is taken from [4].

**Theorem 2.** \( \mathcal{A} + \mathcal{B} + \mathcal{D} + \text{ABS} + \text{PFB}_P + \text{PFA}_P \) is sound and complete for the equational theory of \( \text{EPV}(\mathcal{A}, \mathbb{R}_0(s), P) \).

It is a corollary of the completeness proof in [4] that the same axioms are complete for the class \( \text{EPV}(\mathcal{A}, \mathbb{R}_0(s), P) \) containing those probability function structures which are expansions of a finite event structure. In [10] first order axioms are provided for probability calculus, and corresponding completeness is shown making use of the completeness result for the first order theory of real numbers, a fact which also underlies the result in [4].

4.2 BR and BRs, two forms of Bayes’ rule

As a comment to the specification of probability functions an excursion to Bayes’ rule is worthwhile. First consider the following equation:

\[
P(x \land y) \cdot P(y) \cdot P(y)^{-1} = P(x \land y)
\]  

(EQ1)

Equation EQ1 follows from \( \mathcal{A} + \mathcal{B} + \mathcal{D} + \text{ABS} + \text{PFB}_P + \text{PFA}_P \). This fact is a consequence of Theorem 4 above. A direct proof reads as follows.

\( \phi(u, v) \equiv 0(|u| + |v|) \cdot u \). Now \( (\mathbb{R}_0, s) \models \phi(u, v) = 0 \), and using the completeness theorem of [4] one obtains that \( \mathcal{A} + \mathcal{B} + \text{Sign} \models \phi(u, v) = 0 \). Substituting \( P(y \land x) \) for \( u \) and \( P(y \land \neg x) \) for \( v \) one derives: \( \vdash 0 = \phi(P(y \land x), P(y \land \neg x)) = 0(|P(y \land x)| + |P(y \land \neg x)|) \cdot P(y \land x) = 0(P(y \land x) + P(y \land \neg x)) \cdot P(y \land x) = 0(P(y)) \cdot P(y \land x) \), from which the required result follows by expanding \( 0(P(y)) \).

Bayes’ rule, also known as Bayes’ theorem, occurs in different forms. The conditional operator \( P^0 \) of Table [10] is used for its presentation below. The simplest form of Bayes’ rule, is an equation here referred to as BR:

\[
P^0(x \mid y) = \frac{P^0(y \mid x) \cdot P(x)}{P(y)}
\]  

(BR)

In [4] it is shown that BR follows from the specification \( \mathcal{A} + \mathcal{B} + \mathcal{D} + \text{ABS} + \text{PFB}_P + \text{EQ1} \). As it turns out BR implies equation EQ1. This fact is shown as follows: by substituting \( x \land y \) for \( y \) in BR one obtains: \( P^0(x \mid x \land y) = (P^0(x \land y \mid x) \cdot P(x)) / P(x \land y) \).

Multiplying both sides with \( P(x \land y) \) gives \( L = R \) with \( L = P^0(x \land y \mid x) \cdot P(x \land y) \) and \( R = ((P^0(x \land y \mid x) \cdot P(x)) / P(x \land y)) \cdot P(x \land y) \). Now \( L = (P(x \land y \mid x) / P(x \land y)) \cdot P(x \land y) = (P(x \land y) \cdot P(x \land y) / P(x \land y) = P(x \land y), \) and \( R = (((P(x \land y) \mid x) / P(x)) \cdot P(x)) / P(x \land y)) \cdot P(x \land y) = (P(x \land y) / P(x \land y)) \cdot P(x \land y) \cdot P(x) / P(x) = P(x \land y) \cdot P(x) \cdot P(x)^{-1} \).

**Proposition 2.** The axiom system \( \mathcal{A} + \mathcal{B} + \mathcal{D} + \text{ABS} + \text{PFB}_P + \text{EQ1} \) is strictly weaker than \( \mathcal{A} + \mathcal{B} + \mathcal{D} + \text{ABS} + \text{Sign} + \text{PFB}_P + \text{PFA}_P \).
Proof. Consider a four element event space generated by an atomic event $e$ and choose $\hat{P}$ as follows: $\hat{P}(\top) = \hat{P}(e) = \hat{P}(\neg e) = 0$ and $\hat{P}(\bot) = 1$. The equations of PFBC$_p$ and EQ1 are satisfied while PFAP is not satisfied.

This weakness persists if EQ1 is replaced by BR. A second and equally well-known form of Bayes’ rule is BRs from Table 11. BR follows from BA + PFBC$_p$ + BRs by taking $z = \top$.

**Proposition 3.** BA + PFBC$_p$ + BRs implies PFAP.

Proof. It suffices to derive the following equation [EQ2]

$$P(y) = P(y \land z) + P(y \land \neg z)$$

This suffices because, according to [4], it is the case that EQ2 in combination with BA + Md + DO + Sign + ABS + PFBC$_p$ entails PFAP$_p$. To this end, set $x = y$ in BR$_2$, thereby obtaining $P^0(y|y) = (P^0(y|y) \cdot P(y))/(P^0(y|z) \cdot P(z) + P^0(y|\neg z) \cdot P(\neg z))$.

To derive equation EQ2, notice $P^0(y|y) = P(y \land y)/P(y) = P(y)/P(y)$, take the inverse at both sides thus obtaining $L = R$ with $L = P(y)/P(y)$ and $R = (P^0(y|z) \cdot P(z) + P^0(y|\neg z) \cdot P(\neg z))P(y)$. Then multiplying $L$ and $R$ with $P(y)$ yields $L \cdot P(y) = R \cdot P(y)$. Now $L \cdot P(y) = (P(y)/P(y)) \cdot P(y) = P(y)$ and $R \cdot P(y) = ((P^0(y|z) \cdot P(z) + P^0(y|\neg z) \cdot P(\neg z))/P(y)) \cdot P(y) = ((P(y \land z)/P(z)) \cdot P(z) + (P(y \land \neg z)/P(\neg z)) \cdot P(\neg z)) \cdot (P(y)/P(y)) = (P(y \land z) + P(y \land \neg z)) \cdot (P(y)/P(y)) = P(y \land z) \cdot (P(y)/P(y)) + P(y \land \neg z) \cdot (P(y)/P(y)) = P(y \land z) + P(y \land \neg z).

It may be concluded that Table 11 provides an adequate substitute of PFAP. This observation suggests an alternative axiomatisation BA + Md + DO + Sign + ABS + PFBC$_p$ + PFAP based on BRs as given in Table 11.

For BR, however, there seems to be no role as an axiom in the axiomatic framework of this paper. For instance one may wonder if BR provides an implicit definition of conditional probability.

**Proposition 4.** It is not the case that in the presence of BA + Md + DO + Sign + ABS + PFBC$_p$ + PFAP, though in the absence of the definitions of Table 11, BR serves as an implicit definition of $P^0$.

Proof. Let $Q(x, y) = 1(P(y)) \cdot P(x)$. Then $Q(x, y)$ differs from $P^0(x|y)$ in all but exceptional cases. However, $Q(\neg, \neg)$ satisfies BR considered as a requirement on $P^0(\neg|\neg)$:

$$\frac{Q(y, x) \cdot P(x)}{P(y)} = \frac{1(P(x)) \cdot P(y) \cdot P(x)}{P(y)} = 1(P(y)) \cdot 1(P(x)) \cdot P(x) = 1(P(y)) \cdot P(x) = Q(x, y).$$

**4.3 A signed vector space meadow of conditional values**

A third sort named $V_C$ containing so-called conditional values will be introduced. $V_C$ is generated by an embedding $v: V \rightarrow V_C$ and a conditional operator $\rightarrow: E \times V_C \rightarrow V_C$. $V_C$ is equipped with all meadow operations while $v(0)$ serves as 0 and $v(1)$ serves as
A specification is given by combining (i) the axioms UCV of Table 12 with (ii) \( \text{Md}_{cv} \) = \( \text{Md}_{v(0)/0,v(1)/1} \), i.e. the equations of Table 11 however with variables \( X, Y, Z \) now ranging over \( V_C \), and with \( v(0) \) substituted for 0 and \( v(1) \) substituted for 1, (iii) \( \text{Sign}_{cv} \), the equations of Table 3, but now with its variables ranging over \( V_C \) and writing \( v(0) \) for 0 and \( v(1) \) for 1, and (iv) the specification \( \text{Cond} \) of the conditional operator \( - \rightarrow - : E \times V_C \rightarrow V_C \) as specified in Table 13.

Given a Boolean algebra \( E \) and a signed meadow \( M(s) \) there is a three sorted algebra \( \text{UCV}(E, M(s), \text{CV}(E, M(s))) \) with the domain \( \text{CV}(E, M(s)) \) for sort \( V_C \) freely generated from \( E \) and \( M(s) \), of which includes a sort \( V_C \), the conditional operator on \( E \times V_C \), and the embedding \( v \) from \( V \) into \( V_C \).

For a Boolean algebra \( E \) the subset \( E_{at} \) consists of the atomic elements of \( |E| \), where \( a \in |E| \) is atomic if \( a \neq \perp \) and whenever for \( b \) and \( c \) in \( |E| \), \( E \models (\neg b \lor a) \land (\neg c \lor a) = \top \) then \( E \models \neg b \lor a = \top \) or \( E \models \land \neg c \lor a = \top \). \( E_{at} \) contains the maximally consistent elements of the Boolean algebra.

To each closed term \( X \) of type \( V_C \) of the extended signature a mapping \( [X] : E_{at} \rightarrow V \) is assigned, with the rules of Table 14. The equivalence relation \( \equiv_{at} \) on closed \( V_C \) terms is given by \( X \equiv_{at} Y \iff \forall a \in E_{at}(\llbracket X \rrbracket(a) = \llbracket Y \rrbracket(a)) \). \( \equiv_{at} \) is a congruence relation which meets all requirements imposed by \( \text{UCV} + \text{Sign}_{cv} + \text{Md}_{cv} + \text{Cond} \) and \( \text{CV}(E, M) \) can be defined as the free term algebra for sort \( V_C \) in the extended signature modulo \( \equiv_{at} \). This construction guarantees the consistency of the given construction of the structure for \( V_C \) as for arbitrary \( a \in E_{at} : \llbracket v(0) \rrbracket(a) = 0 \neq 1 = \llbracket v(1) \rrbracket(a) \).

**Proposition 5.** If \( M \) is nontrivial (that is \( M \not\models 0 = 1 \)) and \( |E| \) has more than two elements then \( \text{CV}(E, M) \) is not a cancellation meadow (that is \( \text{CV}(E, M) \not\models X \neq 0 \rightarrow X \cdot X^{-1} = 1 \)).

**Proof.** The proof works by finding an \( X \) which differs from \( v(0) \) modulo \( \equiv_{at} \) and so that \( X \cdot X^{-1} \) differs from \( v(1) \) modulo \( \equiv_{at} \). Indeed If \( |E| > 2 \) then \( E_{at} \) is non-empty, and let \( a \) be an atom. Now \( a \rightarrow v(1) \), violates IL. First notice that \( [\perp \rightarrow v(1)](a) = 0 \neq 1 = [a \rightarrow v(1)](a) \) so that

\[
P^0(x \mid y) = \frac{P^0(y \mid x) \cdot P(x)}{P^0(y \mid z) \cdot P(z) + P^0(y \mid \neg z) \cdot P(\neg z)} \quad \text{(BRs)}
\]

**Table 11: PFA\(_P^0\): alternative axiom for additivity**

\[
v(-x) = -v(x) \tag{36}
v(x^{-1}) = v(x)^{-1} \tag{37}
v(x + y) = v(x) + v(y) \tag{38}
v(x \cdot y) = v(x) \cdot v(y) \tag{39}
v(s(x)) = s(v(x)) \tag{40}
\]

**Table 12: UCV: axioms for unconditional values; \( x, y \) range over \( V \).**

1. A specification is given by combining (i) the axioms UCV of Table 12 with (ii) \( \text{Md}_{cv} = \text{Md}_{v(0)/0,v(1)/1} \), i.e. the equations of Table 11 however with variables \( X, Y, Z \) now ranging over \( V_C \), and with \( v(0) \) substituted for 0 and \( v(1) \) substituted for 1, (iii) \( \text{Sign}_{cv} \), the equations of Table 3, but now with its variables ranging over \( V_C \) and writing \( v(0) \) for 0 and \( v(1) \) for 1, and (iv) the specification \( \text{Cond} \) of the conditional operator \( - \rightarrow - : E \times V_C \rightarrow V_C \) as specified in Table 13.
Moreover, if

Definition 2.

An expression \( X = e \) if and only if \( \bot \):

Definition 3.

expression.

Proposition 9. If we fix

Proposition 8.

X

Proposition 6.

collection of conditions, used in the same order.

Two non-overlapping flat expressions generated from \( V_C \) are similar if both involve the same collection of conditions, used in the same order.

Proposition 6. For each closed \( V_C \) expression \( X \) there is a non-overlapping flat \( V_C \) expression \( Y \) such that \( X = e \) and \( Y = e \).

Proposition 7. For closed \( V_C \) expressions \( X \) and \( Y \) similar non-overlapping flat expressions \( X' \) and \( Y' \) can be found so that \( X = X' \) and \( Y = Y' \).

Proposition 8. If we fix \( E \) as some finite minimal event space with \( E \models \top \neq \bot \), then the \( V_C \) expressions generated from \( E \) and \( R_0 \) constitute a signed vector meadow with dimension \( \#(E_{at}) \). If \( \#(E_{at}) \geq 2 \) then the meadow of conditional values is not a cancellation meadow. Instead it is a vector space meadow (see Paragraph \ref{vector-space}). Elements of the form elements of \( V_C \). CVs \( e \) and \( f \) are orthogonal if and only if \( e \land f = \bot \) in \( E \). If \( a_1, \ldots, a_n \) enumerates \( E_{at} \) without repetition then \( V_C \cong \mathbb{R}_0(a_1, \ldots, a_n) \).

Proposition 9. Given closed \( V_C \) expressions in flat form \( X = \sum_{i=1}^n e_i : \rightarrow v(t_i) \) and \( Y = \sum_{j=1}^m f_j : \rightarrow v(r_j) \), a flat form representation for \( X \cdot Y \) is: \( \sum_{i=1}^n \sum_{j=1}^m (e_i \land f_j) : \rightarrow v(t_i \cdot r_j) \). Moreover, if \( X \) and \( Y \) are non-overlapping then so is the given expression for \( X \cdot Y \).

\[ \top \rightarrow X = X \quad \bot \rightarrow X = v(0) \]
\[ e : \rightarrow (X + Y) = (e : \rightarrow X) + (e : \rightarrow Y) \]
\[ e : \rightarrow (X \cdot Y) = (e : \rightarrow X) \cdot Y \]
\[ e : \rightarrow (-X) = -(e : \rightarrow X) \]
\[ e : \rightarrow (X^{-1}) = (e : \rightarrow X)^{-1} \]
\[ (e \lor f : \rightarrow X) = (e : \rightarrow X) + (f : \rightarrow X) - (e \land f : \rightarrow X) \]
\[ e \land f : \rightarrow X = e : \rightarrow (f : \rightarrow X) \]
\[ s(e : \rightarrow X) = e : \rightarrow s(X) \]

Table 13: \( \text{Cond} \): axioms for the conditional operator

\[ \bot : \rightarrow v(1) \neq_{at} a : \rightarrow v(1) \text{, and similarly by application to } -a \text{ that } a : \rightarrow v(1) \neq_{at} \top : \rightarrow v(1). \]

Now \( (a : \rightarrow v(1))^\bot = a : \rightarrow v(1)^\bot = a : \rightarrow v(1) \text{ whence } (a : \rightarrow v(1)) \cdot (a : \rightarrow v(1)^\bot) = (a : \rightarrow v(1)) \cdot (a : \rightarrow v(1)) = a : \rightarrow v(1) \neq_{at} \top : \rightarrow v(1) \neq_{at} a : \rightarrow v(1). \]

Definition 2. An expression \( X = e_1 : \rightarrow v(t_1) + \ldots + e_n : \rightarrow v(t_n) \) of type \( V_C \) is a flat \( V_C \) expression.

Definition 3. A flat \( V_C \) expression \( X = e_1 : \rightarrow v(t_1) + \ldots + e_n : \rightarrow v(t_n) \) is non-overlapping if for all \( 1 \leq i, j \leq n \) with \( i \neq j \), it is the case that provably \( e_i \land e_j = \bot \).

Definition 4. Two non-overlapping flat \( V_C \) expressions are similar if both involve the same collection of conditions, used in the same order.

F
4.4 Combining CVs with a probability function: expected values

A $V_C$ expression, say $X$, denotes a value which is conditional on an event, that is it depends on the actual event $e$ chosen from $E$. Therefore CVs are well-suited for defining an expected value, denoted with $E_P(X)$. The concept of an expectation lies at the basis of further definitions of probabilistic quantities such as variance, covariance, and correlation. Defining the expected value for a conditional value can be done if a besides a probability function, say $P$, $V_C$ expression in flat form is available, say $\sum_{i=1}^{n} e_i \mapsto v(t_i)$.

$$ E_P(\sum_{i=1}^{n} e_i \mapsto v(t_i)) = \sum_{i=1}^{n} (P(e_i) \cdot t_i). $$

These identities provide an axiom scheme for the function $E_P : CV \rightarrow V$.

Given a probability function structure $EPV(E, R_0(s), \hat{P})$ and a CV structure involving the same event space, say $ECV(E, R_0(s), CV(E, R_0(s)))$ a joint expansion exists. Denoting the joint expansion with $EPCV(E, R_0(s), CV(E, R_0(s)), \hat{P})$ it can be further expanded with an expected value operator named $E_P$, interpreted in compliance with the mentioned scheme, to a structure $EPCV(E, R_0(s), CV(E, R_0(s)), \hat{P}, \hat{E}_P)$. Taken together for all event spaces $E$ and for all probability functions $\hat{P}$ the latter structures constitute a class of probability structures $K(\hat{BA})$.

Instead of using an axiom scheme, a finite axiomatisation of $E_P(\cdot)$ is given in Table 14 from which each instance of the scheme can be derived. The equations (named $EV_P$) of Table 14 determine $E_P(\cdot)$ on all $V_C$ expressions not involving variables of sort $V_C$.

Grouping together the axioms collected thus far one finds an equational theory: $MBPC_P =$

| $[v(m)](a) = m$ |
| $[-t](a) = -([t](a))$ |
| $[t^{-1}](a) = ([t](a))^{-1}$ |
| $[t + r](a) = [t](a) + [r](a)$ |
| $[t \cdot r](a) = [t](a) \cdot [r](a)$ |
| $[e : \rightarrow t](a) = [t](a), \text{if } E \models \neg a \lor e = \top$ |
| $[e : \rightarrow t](a) = 0, \text{if } E \models a \land e = \bot$. |

Table 14: Definition of $[[t]](a)$ for $a \in E_{at}$

$$ E_P(X + Y) = E_P(X) + E_P(Y) \quad (50) $$

$$ E_P(x \mapsto v(y)) = P(x) \cdot y \quad (51) $$

Table 15: $Ev_P$, axioms for the expected value operator, $x$ ranges over $E$, $y$ over $V$
\[
VAR_P(X) = E_P(X^2) - (E_P(X))^2 \\
COV_P(X, Y) = E_P(X \cdot Y) - E_P(X) \cdot E_P(Y) \\
CORR_P(X, Y) = \frac{COV_P(X, Y)^2}{VAR_P(X) \cdot VAR_P(Y)}
\]

Table 16: EV\(_P\), axioms for variance, covariance, and correlation for conditional values

\(BA + Md + DO + Sign + ABS + PFBC_P + PFA_P + UCV + Sign_{cv} + Md_{cv} + Cond + EV_P\) (meadow based probability calculus). A plausible class of models for MBPC\(_P\) is \(K(\text{BA})\). With a proof similar to that of Theorem 2, it follows that MBPC\(_P\) is complete for such equations w.r.t. validity in \(K(\text{BA})\).

\(E_P\) can be eliminated from expressions of sort \(V\) without free variables of sort \(V\). Therefore an expression of sort \(V\) without free variables of sort \(V\) is provably equal within MBPC\(_P\) to an expression not involving subterms of sort \(V\).

### 4.5 Variance, covariance, and correlation for conditional values

On the basis of a definition of expectation, variance, covariance, and correlation on conditional values can be introduced as derived operators as in Table 16.

Let \(X\) and \(Y\) be \(V\) expressions with flat forms \(X = \sum_{i=1}^{n} e_i : \to v(t_i)\) and \(Y = \sum_{i=1}^{m} f_i : \to v(r_i)\). The equations in Table 15 provide explicit definitions of variance, covariance, and correlation for \(X\), resp. \(Y\).

There is no novelty to these definitions except for the effort made to make each definition fit a framework that has been setup on the basis of an algebraic specification. By proceeding in this manner an axiomatic framework is obtained for equational reasoning about each of these technical notions.

Forgetting the subscript for \(E_P\), that is using \(E(X)\) instead of \(E_P(X)\), and similarly for the other operators, is common practice in probability theory. Doing so, however requires that it is apparent from the context which probability function is used. Moreover it must be assumed that for \(X\) and for \(Y\) the same probability function applies.

### 4.6 Extracting a probability mass function from a conditional value

Given a conditional value \(X = \sum_{i=1}^{n} e_i : \to v(t_i)\) in non-overlapping flat form, and a probability function \(P\) the probability mass function, \(\lambda x. P(X = x)\) for \(X\) is supposed to yield for each value \(x\) the probability that \(X\) takes value \(x\). An explicit definition for the PMF of \(X\) is as follows:

\[
Pmf_P(X) = \lambda x \in V. \sum_{i=1}^{n} (0(t_i - x) \cdot P(e_i)).
\]
This specification of $Pmf_P$ is schematic and for that reason does not achieve the simplicity found for the expected value operation.

**Problem 5.** Can $Pmf_P$ be specified by means of a fixed and finite number of equations rather than with an axiom scheme involving an equation for each non-overlapping closed $VC$ expression?

**Proposition 10.** Equivalence of definitions for expectation and variance for $CV$ expressions in non-overlapping flat form via (joint) PMFs extraction.

1. $E_P(X) = E_{pmf}(Pmf_P(X))$,
2. $VAR_P(X) = VAR_{pmf}(Pmf_P(X))$.

**Proof.** Let $X = \sum_{i=1}^n e_i \xrightarrow{v} t_i$ be a non-overlapping flat $VC$ expression. Making use of the facts listed in Paragraph 2.3 one obtains:

\[
E_{pmf}(L_x.P(X = x)) = \sum_x \sum_{i=1}^n (0(t_i-x) \cdot P(e_i)) = \sum_{i=1}^n \sum_x (0(t_i-x) \cdot P(e_i)) = \sum_{i=1}^n (t_i \cdot P(e_i)) = E_P(X). \quad \Box
\]

### 4.7 Joint PMF extraction for event sharing conditional values

Two conditional values are event sharing if both have conditions over the same domain. Extraction of a joint PMF from event sharing conditional values works as follows. Given two $VC$ expressions $X$ and $Y$ with similar nonoverlapping flat forms $\sum_{i=1}^n (e_i \xrightarrow{t} t_i)$ and $\sum_{i=1}^n (e_i \xrightarrow{r} r_i)$ the joint PMF for these conditional values, denoted by $P(X = x, Y = y)$, is defined by

\[
P(X = x, Y = y) = \sum_{i=1}^n (0(t_i-x) \cdot 0(r_i-y) \cdot P(e_i)).
\]

Extending Proposition 4.6 the following connections between definitions involving a conditional value and definitions involving a PMF or a joint PMF can be found.

**Proposition 11.** Equivalence of definitions for covariance and correlation (squared) via $CV$s and via (joint) PMFs.

1. $COV_P(X, Y) = COV_{pmf}(L_x, y.P(X = x, Y = y))$,
2. $CORR_{sq}^P(X, Y) = CORR_{sq}^{pmf}(L_x, y.P(X = x, Y = y))$.

### 5 The multidimensional case

In the multidimensional case the event space is considered a product of event spaces. In the multi-dimensional case $CV$s occurring in a vector of $CV$s are supposed by default not to be event space sharing and the notion of a joint probability function working over a tuple of event spaces enters the picture.
The multi-dimensional case becomes relevant once tuples (vectors) of CVs are considered in combination with a plurality of joint probability functions for product spaces of higher dimensional event space corresponding to various vectors of CVs such that there may not exist a joint probability function for the full product space.

5.1 Multidimensional probability functions

Let $D = \{a_1, \ldots, a_n\}$ be a finite set. The elements of $D$ will be called dimensions. $D$ is called a dimension set, and it is assumed that $n = \#(D)$.

**Definition 5.** (Arities over $D$) $ar_D$, the collection of arities over dimension set $D$, denotes the set of finite non-empty sequences of elements of $D$ without repetition.

Elements of $ar_D$ will serve as arities of probability functions on multi-dimensional event spaces. $l(w)$ denotes the length of $w \in ar_D$.

**Definition 6.** (Arity family) Given an event space $E$, and a name $P$ for a probability function, an arity family (for $E$ and $P$) is a finite subset $W$ of $ar_D$ which is (i) closed under permutation, and (ii) closed under taking non-empty subsequences, and (iii) which contains for each $d \in D$ the arity $(d)$, that is the one-dimensional arity consisting of dimension $d$ only.

For each dimension $d \in D$ the presence of a sort $E_d$ of events for dimension $d$ is assumed. For simplicity of notation it is assumed that these sorts are identical, so that only a sort $E$ is required.

**Definition 7.** A probability family (denoted $PFF_W$) for an arity family $W \subseteq ar_D$ consists of a probability function $P_w : E^{l(w)} \to V$ for each $w \in W$, such that for all $w \in W$ each the axioms in Table 17 (taken from [4]) are satisfied.

The axioms of Table 17 case correspond to the axioms for a probability function of Table 9 in the one dimensional case.

Because in an arity repetition of dimensions is disallowed these axioms reduce to what we had already in the case of a single dimension.

5.2 Multivariate conditional values

Just as in the one-dimensional case, multivariate conditional values are the elements of sort $V_C$. $V_C$ has, besides the embedding $v$ from $V$ into $V_C$ (which must meet the requirements of Table 12), for each $d \in D$ a constructor $- : \to_d$ of type $E \times V_C \to V_C$. $- : \to_d$ must satisfy the requirements $\text{Cond}_d$ which result from $\text{Cond}$ in Table 14 by replacing operator $- : \to -$ by $- : \to_d -$ in all equations. In addition to these requirements the equations $\text{Cond}_{mv}$ of Table 18 must be satisfied for all different pairs $a, b \in D$.

5.3 Expected value operators

For a specification of the expected value operator it is assumed that $d_1, \ldots, d_n$ is an enumeration without repetitions of $D$. For each $w \in W$ a separate expected value operator $E^w_P$ arises.
\[ P_{d,u,e,u'} (y_1, x_1, \ldots, x_l, y_2, z_1, \ldots, z_{l'}) = P_{e,u,d,u'} (y_2, x_1, \ldots, x_l, y_1, z_1, \ldots, z_{l'}) \] (55)

\[ P_d (\top) = 1 \] (56)

\[ P_d (\bot) = 0 \] (57)

\[ P_{d,w} (\top, x_1, \ldots, x_n) = P_w (x_1, \ldots, x_n) \] (58)

\[ P_{d,w} (\bot, x_1, \ldots, x_n) = 0 \] (59)

\[ P_w (x_1, \ldots, x_n) = |P_w (x_1, \ldots, x_n)| \] (60)

\[ P_{d,u} (x \lor y, x_1, \ldots, x_l) = P_{d,u} (x, x_1, \ldots, x_l) + P_{d,u} (y, x_1, \ldots, x_l) \]
\[ - P_{d,u} (x \land y, x_1, \ldots, x_l) \] (61)

Table 17: \textit{PFF}_{W,P}: axioms for a probability function family with name \( P \) (with \( d, e \in D \), \( w, (d, u), (e, u, d, u') \in W, n = l(w) \), and \( u, u' \in \text{ar}_D \cup \{\epsilon\}, l = l(u), l' = l(u') \).

\[ e :\rightarrow_a (f :\rightarrow_b X) = f :\rightarrow_b (e :\rightarrow_a X) \] (62)

Table 18: \textit{Cond}_{mv}: commuting multivariate condition constructors

Each operator is specified by means of two equations as displayed in Table 19.

Given the multi dimensional expected value operator, corresponding operators for variance, covariance, and correlation can be derived in the usual manner.

5.4 Summing up

Collecting the equations mentioned thus far for the multidimensional setting the axiom system \( \text{MBPC}^W_P = \text{BA} + \text{Md} + \text{DO} + \text{Sign} + \text{ABS} + \text{UCV} + \text{Sign}_{cv} + \text{Md}_{cv} + \text{Cond}_{d (d \in D)} + \text{PFF}_{W,P} + \text{EV}_{P,w (w \in W)} \) is obtained.

Completeness of these axiomatisations can be shown with the same methods as for the 1D case. The design of these structures can be somewhat simplified if for each subset of \( D \) at most a single probability function is admitted, having the arguments for the different dimensions in a fixed order. When adopting this alternative, Table 17 needs to be redesigned as follows: permutation axioms are dropped and axioms involving the first argument must be replicated.

\[ E^w_P (X + Y) = E^w_P (X) + E^w_P (Y) \] (63)

\[ E^w_P (x_1 :\rightarrow_{d_1} \ldots (x_n :\rightarrow_{d_n} v(y) \ldots)) = P_w (x_1, \ldots, x_n) \cdot y \] (64)

Table 19: \textit{EV}_{P,w}, axioms for the expected value operator for arity \( w \)
for each argument position.

## 6 Concluding remarks

This paper is a sequel to [4] where a meadow based approach to the equational specification of probability functions was proposed. In [3] probabilistic choice is formalised with the meadow of reals as a number system. The equations in that paper demonstrate, just as well as the equations in Table 9, an attractive compatibility between the requirements of probability calculus and the treatment of division in a meadow.

In [16] an extensive survey is presented of the history leading up to Kolmogorov’s choice of axioms, and to Kolmogorov’s claim that these axioms are what probability is about. The equations in PFBCP + PFAP do not take the 6th axiom into account, however, which asserts that if \((e_i)_{i \in \mathbb{N}}\) is an infinite descending chain of events such that only \(\bot\) is below each element of the chain, then \(\lim_{i \to \infty} P(e_i) = 0\). A closer resemblance with Kolmogorov’s original axioms is found if the equation in Table 9 is replaced by the conditional equation \(e \land f = \bot \rightarrow P(e \lor f) = P(e) + P(f)\). This replacement produces a logically equivalent axiom system. The equation of Table 9 is preferred because it is logically simpler than a conditional equation.

Conditional values play the role of a discrete random variables with finite range. By working with conditional values the use of a sample space underlying the event space is avoided which helps to maintain the style and simplicity of the axiomatisation of probability functions of [4]. Instead of including an additional sort \(V_C\), the conditional values might be viewed as an extension of the sort \(V\). A reason for not doing so, however, is to prevent \(P\) from taking values of the form say \(P(e) = f : \rightarrow v(1/2)\).

Regarding the choice of terminology, the presence of alternative options must be mentioned, for instance in [8] a probability function is referred to as a probability law.

As a technical tool finite support summation is introduced, a novel binding operator on meadows. Finite support summation is of independent interest for the theory of meadows and it gives rise to intriguing new questions. Further it is worth mentioning that working with \(1/0 = 0\) in matrix theory is pursued in e.g. [13].

For the derived operations \(1(-)\) and \(0(-)\) of Table 2 the original notation from [2][3] is \(1(x) = 1 \times x\), resp. \(0(x) = 0 \times x\), which notations may still be used as alternatives. The chosen notation is preferable if a sizeable expression is substituted for \(x\). Table 1 makes use of inversive notation. The phrase “inversive notation” was coined in [5] where it stands in contrast with “divisive notation” which involves a two place division operator symbol. In [5] the equivalence of both notations is discussed. Two place division is provided as a derived operation in Table 2. Division commonly appears in a plurality of syntactical forms: \(x:y, x/y, x/y\), and \(\frac{z}{y}\). These diverse forms are not in need of a separate defining equation, just as much as in the specification of a meadow no mention is made of the existing notational variation for multiplication (viz. \(x \times y, x \cdot y, x \cdot y\) and \(xy\)).

**Acknowledgement**  
Yoram Hirschfeld, Kees Middelburg and Alban Ponse gave useful comments on a previous version of the paper.
References

[1] D. Barber. *Bayesian Reasoning and Machine Learning*. Cambridge University Press, 2012. (ISBN 0521518148, 9780521518147). On-line version available at [http://web4.cs.ucl.ac.uk/staff/D.Barber/pmwiki/pmwiki.php?n=Brml.Online](http://web4.cs.ucl.ac.uk/staff/D.Barber/pmwiki/pmwiki.php?n=Brml.Online) (consulted version: 18 June 2013).

[2] J.A. Bergstra, I. Bethke, and A. Ponse. Cancellation meadows: a generic basis theorem and some applications. *The Computer Journal*, 56(1):3–14, 2013.

[3] J.A. Bergstra, I. Bethke, and A. Ponse. Equations for formally real meadows. *Journal of Applied Logic*, 13(2):1–23, 2015.

[4] Jan A. Bergstra and Alban Ponse. Probability functions in the context of signed involutive meadows. *in: Recent Trends in Algebraic Development Techniques*, Eds. Philip James & Markus Roggenbach, Proc. 23th IFIP WG1.2 International Workshop WADT, Springer LNCS 10644, 73–87, (also [https://arxiv.org/pdf/1307.5173.pdf](https://arxiv.org/pdf/1307.5173.pdf)), 2017.

[5] J.A. Bergstra and C.A. Middelburg. Inversive meadows and divisive meadows. *Journal of Applied Logic*, 9(3): 203–220, 2011.

[6] J.A. Bergstra and C.A. Middelburg. Probabilistic thread algebra. *SACS*, 25(2): 211–243, 2015.

[7] J. A. Bergstra and J. V. Tucker. The rational numbers as an abstract data type. *J. ACM*, 54, 2, Article 7 (April 2007) 25 pages, 2007.

[8] D.P. Bertsekas and J.N. Tsitsiklis. *Introduction to Probability*, Athena Scientific, Nashua USA, ISBN 978-1-886529-23-6, 2008.

[9] D. Davidson and P. Suppes. A Finitistic Axiomatization of Subjective Probability and Utility. *Econometrica* 24 (3) 264-275, 1956.

[10] J.Y. Halpern. An analyisis of first-order logics of probability. *Artificial Intelligence* 46, 311-350, 1990.

[11] Khan Academy. Random variables and probability distributions. [https://www.khanacademy.org/math/probability/random-variables-topic/random-variables-prob-dist/v/discrete-and-continuous-random-variables](https://www.khanacademy.org/math/probability/random-variables-topic/random-variables-prob-dist/v/discrete-and-continuous-random-variables), (consulted July 9 2016).

[12] C.P.J. Koymans, and J. L. M. Vrancken. Extending process algebra with the empty process. Electronic, report LGPS 1. Dept. of Philosophy, State University of Utrecht, The Netherlands (1985).

[13] T. Matsuura and S. Saitoh. Matrices and Division by Zero. *Advances in Linear Algebra & Matrix Theory* 6: 51-58 [http://dx.doi.org/10.4236/alamt.2016.62007](http://dx.doi.org/10.4236/alamt.2016.62007), (2016).

[14] J-. F. Nicaud, D. Bouhineau, and J-. M. Gelis. Syntax and semantics in algebra. *Proc. 12th ICMI Study Conference, The University of Melbourne, 2001*. HAL archives-ouvertes [https://hal.archives-ouvertes.fr/hal-00962023/document](https://hal.archives-ouvertes.fr/hal-00962023/document) (2001).
A Random variables

The notion of a random variable plays a central role in many presentations of probability theory. In the presentation of the current paper the role of random variables is played by a conditional values (CVs) instead. In this Appendix it will be outlined how to view a CV as a random variable provided that the event space is finite.

A.1 From implicit sample space to explicit sample space

Given event space \( \mathcal{E} \), the subset of its domain \( \mathcal{E}_{at} \) consisting of atoms as defined in Paragraph 4.3 can be taken for the corresponding sample space and then a random variable is supposed to be a function from sample space to values. Viewing \( \mathcal{E}_{at} \) as a sample space, for each close conditional value expression \( X \), the function \( \llbracket X \rrbracket \), as specified in Table 14, qualifies as a random variable.

I prefer not to have \( \mathcal{E}_{at} \) as a sort because the resulting setting with \( \mathcal{E}_{at} \) as a subsort of \( \mathcal{E} \) is not easily reconciled with equational logic. Logical difficulties with the equational logic of subsorts persist in spite of the presence of many works that have been devoted to that particular complication.

Now summation over the sample space \( \mathcal{E}_{at} \) is specified as follows. For an event space \( \mathcal{E} \) and a term \( t \) of sort \( V \), then \( \sum_{\alpha \in \mathcal{E}_{at}} t = 0 \) if there are either none or infinitely many atomic events in \(|\mathcal{E}|\) and otherwise

\[
\sum_{\alpha \in \mathcal{E}_{at}} \ast t = [a_1/\alpha]t + \ldots + [a_k/\alpha]t
\]

with \( a_1, \ldots, a_k \) an enumeration without repetitions of the atomic events of \( \mathcal{E} \). Provided \( \mathcal{E} \) is finite, the expectation of \( \llbracket X \rrbracket \) can be defined by summation over the sample space, using an identity which lies outside first order equational logic:

\[
E_P([X]) = \sum_{\alpha \in \mathcal{E}_{at}} \ast ([X](\alpha) \cdot P(\alpha))
\]

A.2 Random variables in colloquial language

Random variables play a key role in many accounts of probability theory. However, the concept of a random variable seems to be rather informal and its use is often cast in colloquial
language. A common wording states that “a random variable is the outcome of a stochastic process”. Complicating an understanding of a random variable, however, is the fact that the mathematical definition of it, which reads “a function from sample space to reals” makes no reference to any variable or variable name, or to a probability function, or to a stochastic mechanism. In [17] it is asserted about a random variable that it is:

... a variable whose value is subject to variations due to chance (i.e. randomness, in a mathematical sense).... A random variable can take on a set of possible different values (similarly to other mathematical variables), each with an associated probability, in contrast to other mathematical variables.

In [11] a random variable is explained as a mapping from “outcomes” to values which provides quantification, while the main argument put forward for the introduction of a random variable is about the use of its name, and at the same time the suggestion is made that a random variable is linked to a probability function. In [8] it is stated that

A discrete random variable has an associated probability mass function..

In the introductory probability refresher of [1] the domain of a variable is said to be the set of states it can take, while the relation between (random) variables and events is explained as follows:

For our purposes, events are expressions about random variables, such as Two heads in 6 coin tosses.