PRESERVATION OF DISCRETE STRUCTURES. A METRIC POINT OF VIEW

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† Dedicated to the memory of Ivo G. Rosenberg

Abstract. In the early 80’s, Alain Quilliot presented an approach of ordered sets and graphs in terms of metric spaces, where instead of positive real numbers, the values of the distance are elements of an ordered monoid equipped with an involution. This point of view was further developed in a series of papers by Jawhari, Misane, Pouzet, Rosenberg and Kabil [35, 67, 41]. Some results are currently published by Rosenberg, Kabil and Pouzet, Bandelt, Pouzet and Saidane, Khamsi and Pouzet [42, 4, 5, 46]. Special aspects were developed by the authors of the present paper [55, 20]. A survey on generalized metric spaces is in print [43]. In this paper, we review briefly the salient aspects of the theory of generalized metric spaces, then we illustrate the properties of the preservation, by operations, of sets of relations, notably binary relations, and particularly equivalence relations.

1. Presentation

We survey some aspects of the preservations by operations of discrete structures like graphs, ordered sets and transition systems. In those aspects, the influence of Ivo Rosenberg was prominent to say the less. Despite the sadness of his passing, we are pleased to give homage to his influence.

We consider graphs, ordered sets and transition systems as generalized metric spaces. Instead of positive real numbers, the values of the distance are elements of an ordered monoid equipped with an involution; the notion of non-expansive map particularly fit in this frame. Several notions and results about ordinary metric spaces and their non-expansive maps extend to these structures, e.g., the characterization of injective objets by Aronszajn and Panitchpakdi [2], as well as the fixed point result of Sine and Soardi [80, 82] (see Espinola-Khamsi [26] for a survey of classical results on metric spaces). As an illustration, we mention that in the category of oriented reflexive graphs with Quilliot’s zigzag distance, the absolute retracts are the retracts of products of zigzags (see [4, 5]) and, on those which are bounded, any set of graph homomorphism which commute pairwise has a fixed point ([46]). Also, as a byproduct of the description of the injective envelope of a two-element transition system, Kabil, Rosenberg and the third author of this paper [42] have shown that the monoid of the final sections of the free monoid of words on a finite alphabet is also free.

The approach of graphs and ordered sets as metric spaces has its origin in Alain Quilliot’s work done in the early 80’s [74, 75]. It was developed by Jawhari, Misane and the third author of this paper [35]. Later on, Rosenberg and the third author of this paper [67] developed a more general approach, seeing operations preserving relations as kind of non-expansive maps. They put some emphasis on the study of operations preserving systems of equivalence relations on a set, viewing these systems as ultra metric spaces. More recently, this lead to the study of semirigid structures (for which the only unary self maps are the identity and the constants), notably those made of systems of three equivalence relations [55, 20].

This paper is composed as follows. We introduce directed graphs equipped with the zigzag distance as a motivating example. Then, we present briefly a survey of properties of generalized metric spaces over a Heyting algebra, alias a dual integral involutive quantale (in short a $D^2I$-quantale). We illustrate the main results with ordinary metric spaces, ultrametric spaces, graphs (directed or oriented), ordered sets and
transition systems. We use the notion of injective envelope to prove the freeness of the algebra of nonempty final segments of the set of words on an alphabet, a recent result of Kabil, Rosenberg and the third author of this paper [42]. We defer the proofs to [35] and the forthcoming survey [41]. We introduce to the duality between relations and operations, via the notion of preservation. We focus on the case of binary relations and more specifically on equivalence relations. Sublattices of the lattice of equivalence relations on a set play an important role in algebra, basic examples being congruence lattices of algebras. Among those are arithmetical lattices. They lead to generalized ultrametric spaces close to the hyperconvex ones (they are finitely hyperconvex). A typical arithmetical lattice is the lattice of congruences of the additive group \( \mathbb{Z} \). We present the result of Cégieski, Grigorieff and Guessarian (CGG), 2014 [14, 15] describing the operations preserving this lattice; we give a short proof of the crucial argument due to the first author of this paper. The case of the additive group of \( \mathbb{Z} \times \mathbb{Z} \) is completely different. While on the lattice of congruences of \( \mathbb{Z} \) the number of operations preserving the congruences is the continuum, there are only countably many operations preserving the congruences of \( \mathbb{Z} \times \mathbb{Z} \). In fact preserving three congruences to preserve all is enough. Next, we go to the opposite direction: the study of semirigidity. A relational structure is semirigid if the only operations preserving it are the constant and projection maps. In the case of a set of equivalence relations this amount to the fact that the self maps preserving it are the constant maps and the identity map. We recall Zadori’s result proving that for \( n \neq 4, 2 \) there is a semirigid set of three equivalence relations on an \( n \)-element set. Using geometric properties of the plane, we show that for each cardinal \( \kappa \), \( \kappa \notin \{2, 4\} \) and \( \kappa \leq 2^{2^{60}} \), there exists a semirigid system of three equivalences on a set of cardinality \( \kappa \). We leave the question for \( \kappa > 2^{2^{60}} \) as a conjecture.

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2. Motivation: distances on graphs

We present the zigzag distance on directed graphs introduced by Alain Quilliot in 1983 [74, 75].

Let \( G \) be an undirected graph, that is, a pair \((V,E)\), where \( V \) is a set of elements called vertices and \( E \) is a set of pairs of (distinct) vertices called edges. If \( x \) and \( y \) are two vertices, the distance from \( x \) to \( y \), denoted by \( d_G(x,y) \), is 0 if \( x = y \), \( m \) if \( m \) is the length of the shortest path joining \( x \) to \( y \) and \( \infty \) if there is no path between \( x \) and \( y \) (that is \( x \) and \( y \) belong to two different connected components of \( G \)).

This notion of distance with integer values plays a basic role in the study of graphs. In [74, 75], Alain Quilliot proposed an extension of this notion to ordered sets and directed graphs.

A directed graph \( G \) is a pair \((V,E)\), where \( E \) is a binary relation on \( V \). We say that \( G \) is reflexive if \( E \) is reflexive and we say that \( G \) is oriented if \( E \) is antisymmetric (that is \( (x,y) \) and \( (y,x) \) cannot be in \( E \) simultaneously except if \( x = y \)). Typical examples of oriented graphs are ordered sets.

If \( E \) is symmetric, and reflexive, we may identify it with a subset of pairs of distinct elements of \( V \), so we get an undirected graph.

2.1. From paths to zigzags. Let us recall that a finite path is an undirected graph \( L := (V,E) \) such that one can enumerate the vertices into a non-repeating sequence \( v_0, \ldots, v_n \) such that edges are the pairs \( \{v_i, v_{i+1}\} \) for \( i < n \).
Recall that if $G := (V, \mathcal{E})$ and $G' := (V', \mathcal{E}')$ are two graphs, a \textit{homomorphism} from $G$ to $G'$ is a map $h : V \to V'$ such that $(h(x), h(y)) \in \mathcal{E}'$ whenever $(x, y) \in \mathcal{E}$ for every $(x, y) \in V \times V$. If $G'$ has loops, a homomorphism can collapse an edge, otherwise it cannot. We suppose that our graphs are reflexive, i.e., have a loop at every vertex, this simplifies considerably the study.

With these two definitions, the distance from a vertex $x$ to a vertex $y$ in an undirected graph $G$ is the least integer $n$ such that there an homomorphism from a path $L$ of length $n$ into $G$ and sending an extremity of $L$ on $x$ and the other on $y$, otherwise it is infinite.

This definition extends to directed graphs as follows.

2.2. Zigzags. A \textit{reflexive zigzag} is a reflexive graph $Z$ such that the symmetric hull is a path.

Figure 1. A reflexive oriented zigzag

Figure 2. A reflexive directed zigzag

If $x$ and $y$ are two vertices of a graph $G$, we may look at all the zigzags $Z$ which can be mapped into $G$ by a homomorphism sending the extremities of $Z$ on $x$ and $y$ respectively.

Figure 3. A morphism of an oriented zigzag into a directed graph

In order to simplify, we consider only reflexive oriented zigzag.
If \( L \) is a reflexive oriented zigzag, we may enumerate the vertices in a non-repeating sequence \( v_0 := x, \ldots, v_i, \ldots, v_n := y \) and to this enumeration we may associate the word \( ev(L) := \alpha_0 \cdots \alpha_i \cdots \alpha_{n-1} \) over the alphabet \( \Lambda := \{+, -\} \), where \( \alpha_i := + \) if \( (v_i, v_{i+1}) \) is an edge and \( \alpha_i := - \) if \( (v_{i+1}, v_i) \) is an edge. If the path has just one vertex, the word will be the empty set and will be denoted by \( \Box \). Conversely, to a word \( u := \alpha_0 \cdots \alpha_i \cdots \alpha_{n-1} \) we may associate the reflexive zigzag \( L_u := \{0, \ldots, n\}, \delta_u \) with extremities 0 and \( n \) (where \( n \) is the length \( \ell(u) \) of \( u \)) such that \( \delta_u(i, i+1) := u_i \) for \( i < n \).

2.3. The zigzag distance. Let \( G := (V, E) \) be a reflexive directed graph. For each pair \((x, y) \in V \times V\), the zigzag distance from \( x \) to \( y \) is the set \( d_G(x, y) \) of words \( u \) such that there is graph homomorphism \( h \) from \( L_u \) into \( G \) which sends 0 on \( x \) and \( \ell(u) \) on \( y \).

This notion is due to Quilliot (Quilliot considered reflexive directed graphs, not necessarily oriented, and in defining the distance, considered only oriented paths). A general study is presented in Jawhari-Misane-Pouzet 1986 [35]; some developments appear in Pouzet-Rosenberg 1994 [67] and in Kabil-Pouzet 1998 [41].

Let us see that this map has the properties of a distance.

2.4. The set of values. The set \( d_G(x, y) \) is a subset of \( \Lambda^* \), the set of words over the alphabet \( \Lambda := \{+, -\} \). Because of the reflexivity of \( G \), every word obtained from a word belonging to \( d_G(x, y) \) by inserting letters will be also into \( d_G(x, y) \). That is, \( d_G(x, y) \) is a subset of \( \Lambda^* \), the set of final segments of \( \Lambda^* \) ordered by the subword ordering. To see that the map \( d_G : V \times V \to \Lambda^* \) has properties similar to a distance, let us put in evidence some properties of \( \Lambda^* \).

Extend the involution on \( \Lambda \) exchanging \( + \) and \( - \) to \( \Lambda^* \) by setting \( \overline{\alpha} := \Box \) and \( \overline{u_0 \cdots u_{n-1}} := \overline{u_{n-1} \cdots u_0} \) for every word in \( \Lambda^* \). Next, set \( X := \{ \overline{u} : u \in X \} \) for any set \( X \) of words and note that \( X \) belongs to \( \Lambda^* \), whenever \( X \) belongs to \( \Lambda^* \). Order \( \Lambda^* \) by reverse of the inclusion, denote by 0 its least element (that is \( \Lambda^* \)), set \( X \oplus Y \) for \( X \cdot Y := \{uv : u \in X, v \in Y\} \).

2.5. The distance. The map \( d_G \) satisfies the following properties:

(i) \( d_G(x, y) = 0 \) iff \( x = y \);

(ii) \( d_G(x, y) \leq d_G(x, z) \oplus d_G(z, y) \);

(iii) \( d_G(x, y) = d_G(x, y) \).

Several categorical properties of (reflexive) graphs and their homomorphisms depend upon some properties of \( \Lambda^* \).

A crucial one is that this set is an ordered monoid equipped with an involution and satisfying the following distributivity condition:

\[
\bigcup_{\alpha \in A, \beta \in B} U_\alpha \oplus V_\beta = \bigcup_{\alpha \in A} U_\alpha \oplus \bigcup_{\beta \in B} V_\beta
\]

for all \( U_\alpha \in \Lambda^* \) (\( \alpha \in A \)) and \( V_\beta \in \Lambda^* \) (\( \beta \in B \)).

As this will be shown below, it turns out from this distributivity condition that the distance set \( \Lambda^* \) can be equipped with a graph structure, say \( G_{\Lambda^*} \); the zigzag distance \( d_{\Lambda^*} \) associated with this graph is such that every metric space over \( \Lambda^* \) can be embedded isometrically into a power of the space \( (\Lambda^*), d_{\Lambda^*} \), furthermore, every (directed) graph can be isometrically embedded into a power of \( G_{\Lambda^*} \).

These facts hold for generalized metric spaces over any ordered monoid satisfying this distributivity condition. In the next section, we present shortly a general frame and review the most salient facts. For the proofs see [35] and, for more details, see the forthcoming survey [43].

3. Generalized metric space

Generalizations of the notion of a metric space are as old as the notion of ordinary metric space and arise from geometry and logic, as well as probability (see [8], [9], [10], [53]). The generalization we consider
here, originating in Jawhari et al. [35], is one among several. The basic object, called a Heyting algebra, is an ordered monoid equipped with an involution satisfying a distributivity condition. It was pointed out recently to the third author that the study of these Heyting algebras goes back to the late 1930’s and the work of M. Ward and R. P. Dilworth (1939) [56] and also that a more appropriated term would have been a dual integrale involutive quantale (in short a $D^2I$-quantale) see [59]. The term quantale was introduced in 1984 by C. J. Mulvey [56] as a combination of ”quantum logic” and ”locale”, see Rosenthal [77] and the recent book of Eklund et al [25] about quantales.

3.1. The objects. Let $\mathcal{H}$ be an ordered monoid equipped with an involution. We denote by $\oplus$ the monoid operation, by $0$ its neutral element and by $\neg$ the involution, so that $p + q = \neg \oplus \neg$ for all $p, q \in \mathcal{H}$.

From now on, we suppose that the neutral element of the monoid $\mathcal{H}$ is the least element of $\mathcal{H}$ for the ordering.

Following Pouzet-Rosenberg, 1994 [67], we say that a set $E$ equipped with a map $d$ from $E \times E$ into $\mathcal{H}$ and which satisfies properties (i), (ii), (iii) stated below is a $\mathcal{H}$-distance, and the pair $E := (E, d)$ is a $\mathcal{H}$-metric space.

(i) $d(x, y) = 0$ iff $x = y$;
(ii) $d(x, y) \leq d(x, z) \oplus d(z, y)$;
(iii) $d(y, x) = d(x, y)$.

In the Encyclopedia of distances [23] (cf. p.82) the corresponding $\mathcal{H}$-metric spaces are called generalized distance spaces and the maps $d$ are called generalized metrics.

3.2. The morphisms. If $E := (E, d)$ is a $\mathcal{H}$-metric space and $A$ a subset of $E$, the restriction of $d$ to $A \times A$, denoted by $d_{|A}$, is a $\mathcal{H}$-distance and $A := (A, d_{|A})$ is a restriction of $E$. As in the case of ordinary metric spaces, if $E := (E, d)$ and $E' := (E', d')$ are two $\mathcal{H}$-metric spaces, a map $f : E \rightarrow E'$ is a non-expansive map (or a contraction) from $E$ to $E'$ provided that $d'(f(x), f(y)) \leq d(x, y)$ holds for all $x, y \in E$. The map $f$ is an isometry if $d'(f(x), f(y)) = d(x, y)$ for all $x, y \in E$.

3.3. Retracts and fix-point property. The space $E := (E, d)$ is a retract of $E' := (E', d')$, if there are two non-expansive maps $f : E \rightarrow E'$ and $g : E' \rightarrow E$ such that $g \circ f = \text{id}_E$ (where $\text{id}_E$ is the identity map on $E$). In this case, $f$ is a coretraction and $g$ a retraction. If $E$ is a subspace of $E'$, then clearly $E$ is a retract of $E'$ if there is a non-expansive map from $E'$ to $E$ such $g(x) = x$ for all $x \in E$. We can easily see that every coretraction is an isometry. A generalized metric space $E$ is an absolute retract w.r.t. isometries if it is a retract of every isometric extension. This notion is related to the fixed-point property.

Claim 1. If a metric space has the fixed-point property (f.p.p.), i.e., every non-expansive self map has a fixed point, then every retract has it.

Consequence: if there are plenty of metric spaces with the f.p.p. then absolute retracts are good candidates.

3.4. Injectivity, extension property, hyperconvexity. We introduce three other notions. Let $E$ be a metric space. This space is said to be injective if for all spaces $E'$ and $E''$, each non-expansive map $f : E' \rightarrow E$, and every isometry $g : E' \rightarrow E''$ there is a non-expansive map $h : E'' \rightarrow E$ such that $h \circ g = f$.

The metric space $E$ has the extension property if for every space $E'$, every non-expansive map $f$ from a subset $A$ of $E'$ into $E$ extends to a non-expansive map from $E'$ to $E$. With the help of Zorn’s lemma, this amounts to the fact that every non-expansive map defined on a subset $A$ of $E'$ extends to every $x \in E' \setminus A$ to a non-expansive map from the subspace of $E'$ induced on $A \cup \{x\}$ to $E$.

Let us say that a closed ball in $E$ is any subset of $E$ of the form $B(x, r) := \{y \in E : d(x, y) \leq r\}$. We say that the space $E$ is hyperconvex if the intersection of every family of closed balls $\{B(x_i, r_i)\}_{i \in I}$ is non-empty whenever $d(x_i, y_i) \leq r_i \oplus r_j$ for all $i, j \in I$.
3.5. How these notions relate?

Claim 2. For a generalized metric space, hyperconvexity implies extension property; extension property is equivalent to injectivity, and injectivity imply that the space is an absolute retract.

Hyperconvexity is equivalent to the conjunction of the following conditions:
1) Convexity: for all \( x, y \in E \) and \( p, q \in \mathcal{H} \) such that \( d(x, y) \leq p \oplus q \) there is \( z \in E \) such that \( d(x, z) \leq p \) and \( d(z, y) \leq q \).
2) 2-Helly property, also called the 2-ball intersection property: The intersection of every set (or, equivalently, every family) of closed balls is non-empty provided that their pairwise intersections are all non-empty.

3.6. Heyting algebra alias dual integrale involutive quantale.

Claim 3. The four notions above are equivalent provided that the set \( \mathcal{H} \) of values of the distances is a complete lattice and satisfies the following distributivity condition:

\[
\bigwedge_{\alpha \in A, \beta \in B} u_\alpha \oplus v_\beta = \bigwedge_{\alpha \in A} u_\alpha \bigoplus \bigwedge_{\beta \in B} v_\beta
\]

for all \( u_\alpha \in \mathcal{H} \ (\alpha \in A) \) and \( v_\beta \in \mathcal{H} \ (\beta \in B) \).

In this case, we say that \( \mathcal{H} \) is an involutive Heyting algebra. The proof of Claim 3 relies on Proposition 3.2 below.

3.7. Metrisation of the set of values. On an involutive Heyting algebra \( \mathcal{H} \), we may define a \( \mathcal{H} \)-distance. This is the most important fact about generalized metric spaces. It relies on the classical notion of residuation. Let \( v \in \mathcal{H} \). Given \( \beta \in \mathcal{H} \), the sets \( \{ r \in \mathcal{H} : v \leq r \oplus \beta \} \) and \( \{ r \in \mathcal{H} : v \leq \beta \oplus r \} \) have least elements, that we denote respectively by \( [v \oplus -\beta] \) and \( [-\beta \oplus v] \) and call the right and left quotient of \( v \) by \( \beta \) (note that \( [\beta \oplus v] = [v \oplus -\beta] \)). It follows that for all \( p, q \in \mathcal{H} \), the set:

\[
D(p, q) := \{ r \in \mathcal{H} : p \leq q \oplus \bar{r} \text{ and } q \leq p \oplus r \}
\]

has a least element, namely \( [\bar{p} \oplus -\bar{q}] \vee [-p \oplus q] \), that we denote by \( d_\mathcal{H}(p, q) \).

Lemma 3.1. If \( \mathcal{H} \) is a Heyting algebra then for every metric space \( (E, d) \) over \( \mathcal{H} \), and for all \( x, y \in E \), the following equality holds:

\[
d(x, y) = \bigvee_{z \in E} d_\mathcal{H}(d(z, x), d(z, y)) .
\]

Let \( ((E_i, d_i))_{i \in I} \) be a family of \( \mathcal{H} \)-metric spaces. The direct product \( \prod_{i \in I} (E_i, d_i) \), is the metric space \( (E, d) \) where \( E \) is the cartesian product \( \prod_{i \in I} E_i \) and \( d \) is the "sup" (or \( \ell^\infty \)) distance defined by \( d((x_i)_{i \in I}, (y_i)_{i \in I}) = \bigvee_{i \in I} d_i(x_i, y_i) \). We recall the following result.

Proposition 3.2. (Proposition II.2.7 of [35]) The map \( (p, q) \mapsto d_\mathcal{H}(p, q) \) over a Heyting algebra \( \mathcal{H} \) is a distance on \( \mathcal{H} \), in fact \( (\mathcal{H}, d_\mathcal{H}) \) is an hyperconvex metric space and every metric space over \( \mathcal{H} \) embeds isometrically into a power of \( (\mathcal{H}, d_\mathcal{H}) \).

Hyperconvex spaces enjoy the extension property, hence they are injective. Since hyperconvexity is preserved under the formation of products, Proposition 3.2 ensures that every metric space embeds isometrically into an injective object. This fact is shortly expressed by saying that the category of metric spaces over \( \mathcal{H} \) has enough injectives. From that follows an important structural property of the category of metric spaces over a Heyting algebra.

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Proposition 3.3. (Theorem 1, section II-2.9 of [35]) In the category of metric spaces over a Heyting algebra \( \mathcal{H} \), injective, absolute retracts, hyperconvex spaces, spaces with the extension property and retracts of powers of \((\mathcal{H}, d_\mathcal{H})\) coincide.

3.8. **Injective envelope.** In the category of metric spaces (over the non-negative reals), every metric space has an injective envelope (also called an injective hull), a major fact due to Isbell [34]. One can view an injective envelope of a metric space \( E \) as an hyperconvex isometric extension \( F \) of \( E \), which is minimal with respect to inclusion (that is, there is no proper hyperconvex subspace of \( F \) containing isometrically \( E \)). One can note that those minimal extensions are isometric via the identity on \( E \). These facts extend to generalized metric spaces.

Proposition 3.4. (Theorem 2, section II-3.1 of [35]). Every metric space over a Heyting algebra has an injective envelope.

3.9. **Fixed point property.** An element \( v \in \mathcal{H} \) is self-dual if \( \overline{v} = v \), it is accessible if there is some \( r \in \mathcal{H} \) with \( v \not\leq r \) and \( v \leq r \oplus \overline{r} \) and inaccessible otherwise. Clearly, 0 is inaccessible; every inaccessible element \( v \) is self-dual (otherwise, \( \overline{v} \) is incomparable to \( v \) and we may choose \( r := \overline{v} \)). We say that a space \( E \) is bounded if 0 is the only inaccessible element below the diameter \( \delta(E) \) of \( E \) (the diameter of \( E \) is \( \delta(E) := \sqrt{\{d(x, y) : x, y \in E\}} \)).

Theorem 3.5. If a generalized metric space over a Heyting algebra is bounded and hyperconvex then every commuting family of non expansive self maps has a common fixed point.

This result was obtained by J.B. Baillon [3] for ordinary metric spaces. His proof applies without much changes. It follows from a much more general result that we present in the next section.

3.10. **Compact normal structure.** A generalization to spaces with a compact and normal structure of the fixed point theorem of Sine and Soardi was obtained by Kirk [47], then M.A.Khamsi [45] extended to these spaces the result of Baillon. We extend first the notion of Penot [62] of compact normal structure to our spaces.

A generalized metric space \( E \) has a compact structure if the intersection of every family of closed balls is nonempty provided that the intersections of finite subfamilies are nonempty.

The diameter \( \delta_E(A) \) of a subset \( A \) of a metric space \( E \) is \( \sqrt{\{d(x, y) : x, y \in A\}} \). The radius \( r_E(A) \) of a subset \( A \) is \( \Lambda \{ r \in \mathcal{H} : A \subseteq B(x, r) \text{ for some } x \in A \} \).

A subset \( A \) of a metric space \( E \) is equally centered if \( r_E(A) = \delta_E(A) \). For an example, if \( A \) is the empty set and \( E \) is nonempty then \( A \) is not equally centered. If \( A \) a singleton, say \( a \), then \( A \) is equally centered.

The space \( E \) has a normal structure if no intersection of closed balls \( A \) distinct from a singleton is equally centered. Equivalently, if \( |A| \neq 1 \) then \( r_E(A) \neq \delta_E(A) \).

Example 3.6. If \( E \) is a hyperconvex metric space, it has a compact structure; if it is bounded it has a normal structure (in fact, the radius is half the diameter).

Lemma 3.7. Let \( A \) be an intersection of balls of \( E \). If \( \delta_E(A) \) inaccessible then \( A \) is equally centered; the converse holds if \( E \) is hyperconvex.

Khamsi and the third author of this paper obtained the following result [46]:

Theorem 3.8. If a generalized metric space has a compact and normal structure then every commuting family of non expansive self maps has a common fixed point.

The key of their proof is the notion of one-local retract, already used in [45]. We say that \( A := (A, d_\uparrow A) \) is a one-local retract of \( E \) if it is a retract of \( (A \cup \{x\}, d_\uparrow A \cup (x)) \) (via the identity map) for every \( x \in E \).

Adapting the intersection process discovered by Baillon [3], they proved:

Theorem 3.9. If a generalized metric space \( E \) has a compact normal structure, then, the intersection of every down-directed family \( F \) of one-local retracts of \( E \) is a nonempty one-local retract of \( E \).
We stop here the description of categorical properties of generalized metric spaces. The interested reader will find more, notably on Hole-preserving maps and one-local retracts in [35] and in [43].

In [67] there is a study of more general metric spaces for which the neutral element, 0 of the set \( H \) of values is not necessarily the least element and condition (i) for the distance is replaced \( d(x, y) = 0 \) iff \( x = y \). Despite of the scope and applicability we voluntarily omitted it.

4. FROM GENERALIZED METRIC SPACES TO GRAPHS, ORDERED SETS AND AUTOMATA

We illustrate the notion of generalized metric spaces with graphs, ordered sets and a special kind of transition systems. We conclude this section with an application of the notion of injective envelope to the freeness of a monoid of final segments.

4.1. The case of ordinary metric spaces. Let \( \mathbb{R}^+ \) be the set of non negative reals with the addition and natural order, the involution being the identity. Let \( H \) be \( \mathbb{R}^+ \cup \{+\infty\} \). Extend to \( H \) the addition and order in a natural way. Then, metric spaces over \( H \) are direct sums of ordinary metric spaces (the distance between elements in different components being \( +\infty \)). The set \( H \) is a Heyting algebra and the distance \( d_H \) once restricted to \( \mathbb{R}^+ \) is the absolute value. The inaccessible elements are 0 and \( +\infty \) hence, if one deals with ordinary metric spaces, unbounded spaces in the above sense are those which are unbounded in the ordinary sense. If one deals with ordinary metric spaces, infinite products can yields spaces for which \( +\infty \) is attained. Thus, one has to replace powers of \( \mathbb{R}^+ \) by \( \ell^\infty \)-spaces (if \( I \) is any set, \( \ell^\infty(I) \) is the set of bounded families \( (x_i)_{i \in I} \) of reals numbers, endowed with the sup-distance). With that, the notions of absolute retract, injective, hyperconvex and retract of some \( \ell^\infty(I) \) space coincide. This is the well known result of Aronszjan-Panitchpakdi [2]. The existence of an injective envelope was proved by Isbell [34]. The injective envelope of a 2 element ordinary metric space is a bounded closed interval of the real line; injective envelopes of ordinary metric spaces consisting of few many elements have been described [24]; for applications see [18].

The existence of a fixed point for a non-expansive map on a bounded hyperconvex space is the famous result of Sine and Soardi [80, 82]. Theorem 3.5 applied to a bounded hyperconvex metric space is Baillon’s fixed point theorem [3]. Applied to a metric space with a compact normal structure, this is the result obtained by Khamsi [45](1996).

The results presented about generalized metric spaces over a Heyting algebra apply to ultrametric spaces over \( \mathbb{R}^+ \cup \{+\infty\} \). Indeed, with a the join operation, the distributivity condition holds, hence \( \mathbb{R}^+ \cup \{+\infty\} \) is a Heyting algebra. A similar characterization to ours was obtained in [7]; a description of the injective envelope is also given. The paper [67] contains a study of ultrametric spaces over a complete lattice satisfying this distributivity condition, called an op-frame. Metric spaces over op-frame are studied in [1]. Ultrametric spaces over a lattice and their connexion with collections of equivalence relations have been recently studied in [13]. More general ultrametric spaces have been studied in [69, 70, 71]. Due to their interest, we devote most of the last section of this paper to their study.

4.2. Directed graphs, transition systems and ordered sets. Let us equip directed graphs with the zigzag distance. The set \( F(\Lambda^+) \) of values of the distance is an involutive Heyting algebra. We may apply the results of the theory.

**Lemma 4.1.** A map from a reflexive directed graph \( G \) into an other is a graph-homomorphism iff it is non-expansive w.r.t. the zigzag distance.

**Lemma 4.2.** The distance \( d \) of a metric space \( (E,d) \) over \( F(\Lambda^+) \) is the zigzag distance of some reflexive directed graph \( G := (E,E) \) iff it satisfies the following property for all \( x,y,z \in E \), \( u,v \in F(\Lambda^+) \): \( u,v \in d(x,y) \) implies \( u \in d(x,z) \) and \( v \in d(z,y) \) for some \( z \in E \). When this condition holds, \( (x,y) \in E \) iff \( u \in d(x,y) \).

Due to Lemma 4.2 above, the various metric spaces mentionned above (injective, absolute retracts, etc.) are graphs equipped with the zigzag distance; in particular, the distance \( d_{F(\Lambda^+)} \) defined on \( F(\Lambda^+) \) is the
zigzag distance of some graph. This later fact leads to a fairly precise description of absolute retracts in the category of reflexive directed graphs (see [44]).

4.3. **Transition systems.** Instead of a two-letters alphabet, we may consider a finite one, say $A$. The analog of directed graphs are transition systems. A transition system is a pair $T := (Q, T)$, where the elements $q \in Q$ are called states and the elements of $T$, the transitions, are triples $(p, a, q) \in Q \times A \times T$.

If $x$ and $y$ are two states, we may define the distance from $x$ to $y$ as the set $d_T(x, y)$ of words coding the paths from $x$ to $y$. In automata theory this is simply the language accepted by the automaton made of $T$, in-state $x$ and out-state $y$.

To mimic the case of directed graphs, we could equip the alphabet $A$ of an involution – and impose our transition systems to be involutive that is for every letter $a$, $(p, a, q) \in T$ iff $(p, \bar{a}, q) \in T$. This is a cosmetic change in the usual theory of languages. We could impose the system to be reflexive that $(p, p, p) \in T$ for every state $p$, letter $a$. This is a strong requirement, about the same than imposing our graphs to be reflexive. We refer to [67] for more.

4.4. **Ordered sets.** In this case, zigzags reduce to fences. There are two fences of length $n$: the up-fence; and the down-fence. The first one starts with $x_0 < x_1 > ...$, the second with $x_0 > x_1 < ...$. So one can express the distance as the pair $(n, m)$ of integers such that $n$ is the shortest length of an up-fence from $x$ to $y$ and $m$ the shortest length of a down-fence from $x$ to $y$. For more, see Nevermann-Rival, [57] 1985 and Jawhari-al [35] 1986.

4.5. **The case of oriented graphs.** Oriented graphs and directed graphs behave differently. Oriented graphs cannot be modeled over a Heyting algebra (Theorem I V-3.1 of [35] is erroneous), but the absolute retracts in this category can be (this was proved by Bandelt, Saïdane and the third author of this paper and included in Saïdane’s thesis [78]). The appropriate Heyting algebra is $\mathbb{N}(\Lambda^*)$, the MacNeille completion of $\Lambda^*$.

The MacNeille completion of $\Lambda^*$ is in some sense the least complete lattice extending $\Lambda^*$. The definition goes as follows. If $X$ is a subset of $\Lambda^*$ ordered by the subword ordering then

$$X^\Delta := \bigcap_{x \in X} \uparrow x$$

is the upper cone generated by $X$, and

$$X^\nabla := \bigcap_{x \in X} \downarrow x$$

is the lower cone generated by $X$.

The pair $(\Delta, \nabla)$ of mappings on the complete lattice of subsets of $\Lambda^*$ constitutes a Galois connection. Thus, a set $Y$ is an upper cone if and only if $Y = Y^\nabla \Delta$, while a set $W$ is a lower cone if and only if $W = W \Delta \nabla$. This Galois connection $(\Delta, \nabla)$ yields the MacNeille completion of $\Lambda^*$. This completion is realized as the complete lattice \( \{ W^\nabla : W \subseteq \Lambda^* \} \) ordered by inclusion or alternatively \( \{ Y^\Delta : Y \subseteq \Lambda^* \} \) ordered by reverse inclusion. We choose as completion the set \( \{ Y^\Delta : Y \subseteq \Lambda^* \} \) ordered by reverse inclusion that we denote by $\mathbb{N}(\Lambda^*)$. This complete lattice is studied in detail in Bandelt and Pouzet, 2018 [4].

We recall the following characterization of members of the MacNeille completion of $\Lambda^*$.

**Proposition 4.3.** [4] Corollary 4.5. A member $Z$ of $\mathbb{F}(\Lambda^*)$ belongs to $\mathbb{N}(\Lambda^*)$ if and only if it satisfies the following cancellation rule: if $u + v \in Z$ and $u - v \in Z$ then $uv \in Z$.

The concatenation, order and involution defined on $\mathbb{F}(\Lambda^*)$ induce an involutive Heyting algebra on $\mathbb{N}(\Lambda^*)$ (see Proposition 2.2 of [4]). Being an involutive Heyting algebra, $\mathbb{N}(\Lambda^*)$ supports a distance $d_{\mathbb{N}(\Lambda^*)}$ and this distance is the zigzag distance of a graph $G_{\mathbb{N}(\Lambda^*)}$. But it is not true that every oriented graph embeds isometrically into a power of that graph. For example, an oriented cycle cannot be embedded. The following result characterizes graphs which can be isometrically embedded, via the zigzag distance, into products of reflexive and oriented zigzags. It is stated in part in Subsection IV-4 of [35].
Theorem 4.4. For a directed graph \( G \) equipped with the zigzag distance, the following properties are equivalent:

(i) \( G \) is isometrically embeddable into a product of reflexive and oriented zigzags;
(ii) \( G \) is isometrically embeddable into a power of \( G_{\Lambda^*} \);
(iii) The values of the zigzag distance between vertices of \( V \) belong to \( \Lambda^* \).

Theorem 4.5. An oriented graph \( G \) is an absolute retract in the category of oriented graphs if and only if it is a retract of a product of directed zigzags.

We just give a sketch. For details, see Chapter V of [78] and the forthcoming paper of Bandelt, Pouzet, Saïdane[5]. The proof has three steps. Let \( G \) be an absolute retract. First, one proves that \( G \) has no 3-element cycle. Second, one proves that the zigzag distance between two vertices of \( G \) satisfies the cancellation rule. From Proposition 4.3 it belongs to \( \Lambda^* \); from Theorem 4.4 \( G \) isometrically embeds into a product of directed zigzags. Since \( G \) is an absolute retract, it is a retract of that product.

As illustrated by the results of Tarski and Sine and Soardi, absolute retracts are appropriate candidates for the fixed point property. Reflexive graphs with the fixed point property must be antisymmetric, i.e., oriented. Having described absolute retracts among oriented graphs, we derive from Theorem 3.5 that the bounded ones have the fixed point property.

We start with a characterization of accessible elements of \( \Lambda^* \). The proof is omitted.

Lemma 4.6. Every element \( v \) of \( \Lambda^* \setminus \{\Lambda^*,\varnothing\} \) is accessible.

Theorem 4.7. If a graph \( G \), finite or not, is a retract of a product of reflexive and directed zigzags of bounded length then every commuting set of endomorphisms has a common fixed point.

Proof. We may suppose that \( G \) has more than one vertex. The diameter of \( G \) equipped with the zigzag distance belongs to \( \Lambda^* \setminus \{\Lambda^*,\varnothing\} \). According to Lemma 4.6, it is accessible, hence as a metric space, \( G \) is bounded. Being a retract of a product of hyperconvex metric spaces it is hyperconvex. Theorem 3.5 applies.

If we consider zigzags of length one we get Tarski’s fixed point theorem [84].

4.6. An illustration: the freeness of \( F(A^*) \) and \( \Lambda(A^*) \). Instead of a two letter alphabet, we consider an arbitrary alphabet \( A \), not necessarily finite. We suppose that the alphabet \( A \) is ordered. We order \( A^* \) with the Higman ordering: if \( \alpha \) and \( \beta \) are two elements in \( A^* \) such \( \alpha := a_0 \ldots a_{n-1} \) and \( \beta := b_0 \ldots b_{m-1} \) then \( \alpha \leq \beta \) if there is an injective and increasing map \( h \) from \( \{0,\ldots,n-1\} \) to \( \{0,\ldots,m-1\} \) such that for each \( i, 0 \leq i \leq n-1 \), we have \( a_i \leq h(i) \). Then \( A^* \) is an ordered monoid with respect to the concatenation of words. A final segment of \( A^* \) is any subset \( F \subseteq A^* \) such that \( \alpha \leq \beta, \alpha \in F \) implies \( \beta \in F \). Initial segments are defined dually. Let \( X \) be a subset of \( A^* \); then

\[ \uparrow X := \{ \beta \in A^* : \alpha \leq \beta \text{ for some } \alpha \in X \} \]

is the upper set generated by \( X \) and

\[ \downarrow X := \{ \alpha \in A^* : \alpha \leq \beta \text{ for some } \beta \in X \} \]

is the lower set generated by \( X \).

Let \( F(A^*) \) be the set of final segments of \( A^* \). The concatenation of words extends to \( F(A^*) \); this operation defined by \( XY := \{ \alpha \beta : \alpha \in X, \beta \in Y \} \) induces an operation on \( F(A^*) \) for which the set \( A^* \) is neutral. Hence \( F(A^*) \) is a monoid. Since it contains the empty set \( \varnothing \) and \( \varnothing \) has several decompositions (e.g., \( \varnothing = A^* \varnothing \)), this monoid is not free. Let \( F^0(A^*) := F(A^*) \setminus \{\varnothing\} \) be the set of non-empty final segments of \( A^* \). This is submonoid of \( F(A^*) \).

Theorem 4.8. \( F^0(A^*) \) is a free monoid.
The following illustration of Theorem 4.8 was proposed to us by J. Sakarovitch. An antichain of $A^*$ is any subset $X$ of $A^*$ such that any two distinct elements $\alpha$ and $\beta$ of $X$ are incomparable w.r.t. the Higman ordering. The set $\text{Ant}(A^*)$ of antichains of $A^*$ and the set $\text{Ant}_{\omega}(A^*)$ of finite antichains of $A^*$ are submonoids of $\mathcal{P}(A^*)$; the sets $\text{Ant}^0(A^*) := \text{Ant}(A^*) \setminus \{\emptyset\}$ and $\text{Ant}^0_{\omega}(A^*) := \text{Ant}_{\omega}(A^*) \setminus \{\emptyset\}$ of non-empty antichains are also submonoids. From Theorem 4.8, we deduce:

**Theorem 4.9.** The monoids $\text{Ant}^0(A^*)$ and $\text{Ant}^0_{\omega}(A^*)$ are free.

4.6.1. **Well-quasi-ordered alphabets.** Note that if $A$ is well-quasi-ordered (w.q.o)(that is to say that every final segment of $A$ is finitely generated) then the monoids $\text{Ant}(A^*)$ and $\text{Ant}_{\omega}(A^*)$ are equal and isomorphic to the monoid $F(A^*)$, thus Theorem 4.9 reduces to Theorem 4.8. Indeed, if $A$ is w.q.o. then, according to a famous result of Higman [33] 1952, $A^*$ is w.q.o. too, that is every final segment $F$ of $A^*$ is generated by $\text{Min}(F)$ the set of minimal elements of $F$. Since $\text{Min}(F)$ is an antichain and in this case a finite one, our claim follows.

4.6.2. **The MacNeille completion of $A^*$.** Let $\mathcal{N}(A^*)$ be the MacNeille completion of the poset $A^*$, that we may view as the collection of intersections of principal final segments of $A^*$. The MacNeille completion of $\mathcal{N}(A^*)$ is a submonoid of $\mathcal{F}(A^*)$. From Theorem 4.8, we derive:

**Theorem 4.10.** Let $A$ be an ordered alphabet. The monoid $\mathcal{N}^0(A^*) := \mathcal{N}(A^*) \setminus \{\emptyset\}$ is free.

We recall that a member $F$ of $\mathcal{F}(A^*)$ is irreducible if it is distinct from $A^*$ and is not the concatenation of two members of $\mathcal{F}(A^*)$ distinct of $F$ (note that with this definition, the empty set is irreducible). For an example, if $F = \uparrow \{u, v\}$ with $u$ incomparable to $v$, then $F$ is irreducible iff $u$ and $v$ do not have a common prefix nor a common suffix. The fact that $\mathcal{F}^0(A^*)$ is free amounts to the fact that each member decomposes in a unique way as a concatenation of finitely many irreducible elements.

4.6.3. **An interpretation.** We interpret the freeness of these monoids by means of injective envelopes of $2$-element metric spaces.

We suppose that $A$ equipped with an involution (this is not a restriction: we may choose the identity on $A$ as our involution). We consider a notion of metric spaces with values in $\mathcal{F}(A^*)$. The category of these spaces over $\mathcal{F}(A^*)$, with the non-expansive maps as morphisms, has enough injectives (meaning that every metric space extends isometrically to an injective one). For every final segment $F$ of $A^*$, the $2$-element metric space $E := (\{x, y\}, d)$ such that $d(x, y) = F$, has an injective envelope $S_F$ (that is a minimal extension to an injective metric space).

Since $S_F$ is injective, corresponds to it a transition system $M_F$ on the alphabet $A$, with transitions $(p, a, q)$ if $a \in d(p, q)$. The automaton $A_F := (M_F, \{x\}, \{y\})$ with $x$ as initial state and $y$ as final state accepts $F$. A transition system yields a directed graph whose arcs are the ordered pairs $(x, y)$ linked by a transition. The transition system $M_F$ being reflexive and involutive, the corresponding graph $G_F$ is undirected and has a loop at every vertex. For an example, if $F = A^*$, $S_F$ is the one-element metric space and $G_F$ reduces to a loop. If $F = \emptyset$, $S_F$ is the two-elements metric space $E := (\{x, y\}, d)$ with $d(x, y) = \emptyset$ and $G_F$ has no edge.

The gluing of two injectives by a common vertex yields an injective; we will say that an injective which is not the gluing of two proper injectives is irreducible.

With the notion of cut vertex and block borrowed from graph theory, Kabil, Pouzet and Rosenberg [42] proved:

**Theorem 4.11.** Let $F$ be a final segment of $A^*$, distinct from $A^*$. Then $F$ is irreducible if and only if $S_F$ is irreducible if and only if $G_F$ has no cut vertex. If $F$ is not irreducible, the blocks of $G_F$ are the vertices of a finite path $C_0, \ldots, C_{n−1}$ with $n \geq 2$, whose end vertices $C_0$ and $C_{n−1}$ contain respectively the initial state $x$ and the final state $y$ of the automaton $A_F$ accepting $F$. Furthermore, $F = F_0 \cdots F_i \cdots F_{n−1}$, the automaton $A_F$, accepting $F_i$ being isomorphic to $(M_F \uparrow C_i, \{x_i\}, \{y_i\})$, where $x_i := x$ if $i = 0$, $y_i := y$ if $i = n−1$ and $\{x_i\} = C_{i−1} \cap C_i$, $\{y_i\} = C_i \cap C_{i+1}$, otherwise.
5. Preservation of binary relations by operations

We recall the duality between relations and operations. We consider then the special case of binary relational structures, particularly equivalence relations, leading to generalized metric and ultrametric spaces.

5.1. Duality between relations and operations. Let $E$ be a set. For $n \in \mathbb{N}^* := \mathbb{N} \setminus \{0\}$, a map $f : E^n \to E$ is an $n$-ary operation on $E$, whereas a subset $\rho \subseteq E^n$ is an $n$-ary relation on $E$. Denote by $O^{(n)}$ (resp. $R^{(n)}$) the set of $n$-ary operations (resp. relations) on $E$ and set $O := \bigcup \{O^{(n)} : n \in \mathbb{N}^*\}$ (resp $R := \bigcup \{R^{(n)} : n \in \mathbb{N}^*\}$). For $n, i \in \mathbb{N}^*$ with $i \leq n$, define the $i^{th}$ $n$-ary projection $e^n_i$ by setting $e^n_i(x_1, \ldots, x_n) := x_i$ for all $x_1, \ldots, x_n \in E$ and set $\mathcal{P} := \{e^n_i : i, n \in \mathbb{N}^*\}$. An operation $f \in O$ is constant if it takes a single value, it is idempotent provided $f(x, \ldots, x) = x$ for all $x \in E$. We denote by $\mathcal{C}$ (resp. $\mathcal{I}$) the set of constant, (resp. idempotent) operations on $E$.

Let $m, n \in \mathbb{N}^*$, $f \in O^{(m)}$ and $\rho \in R^{(n)}$. Then $f$ preserves $\rho$ if:

$$(3) \quad (x_1, 1, \ldots, x_{1, n}) \in \rho, \ldots, (x_{m, 1}, \ldots, x_{m, n}) \in \rho \implies (f(x_1, 1, \ldots, x_{1, n}), \ldots, f(x_{m, 1}, \ldots, x_{m, n})) \in \rho$$

for every $m \times n$ matrix $X := (x_{i,j})_{i=1, \ldots,n}$ of elements of $E$.

If $\rho$ is binary and $f$ is unary, then $f$ preserves $\rho$ means:

$$(4) \quad (x, y) \in \rho \implies (f(x), f(y)) \in \rho$$

for all $x, y \in E$.

If $\mathcal{F}$ is a set of operations on $E$, let $\text{Inv}(\mathcal{F})$, resp. $\text{Inv}_n(\mathcal{F})$ be the set of relations, resp. $n$-ary relations, preserved by all $f \in \mathcal{F}$. Dually, if $\mathcal{R}$ is a set of relations on $E$, let $\text{Pol}(\mathcal{R})$, resp. $\text{Pol}_n(\mathcal{R})$, be the set of operations, resp. $n$-ary operations, which preserve all $\rho \in \mathcal{R}$. The operators $\text{Inv}$ and $\text{Pol}$ define a Galois correspondence. The study of this correspondence is the theory of clones.

Two basic problems have been considered:
1) Describe the sets of the form $\text{Inv}(\mathcal{F})$.
2) Describe the sets of the form $\text{Pol}(\mathcal{R})$.

A solution is due to Bodnaručuk, Kalužnin, Kotov, and Romov. In concrete cases, this description does not help much. For example, given $\mathcal{R}$, decide if $\text{Inv}(\text{Pol}(\mathcal{R})) = \mathcal{R}$?

5.2. Towards generalized metric spaces. We restrict our attention to the case of unary operations and binary relations. We recall that if $\rho$ and $\tau$ are two binary relations on the same set $E$, then their composition $\rho \circ \tau$ is the binary relation made of pairs $(x, y)$ such that $(x, z) \in \tau$ and $(z, y) \in \rho$ for some $z \in E$. It is customary to denote it $\tau \cdot \rho$.

The set $\text{Inv}_2(\mathcal{F})$ of binary relations on $E$ preserved by all $f$ belonging to a set $\mathcal{F}$ of self maps has some very simple properties that we state below (the proofs are left to the reader). For the construction of many more properties by means of primitive positive formulas, see.

**Lemma 5.1.** Let $\mathcal{F}$ be a set of unary operations on a set $E$. Then the set $\mathcal{R} := \text{Inv}_2(\mathcal{F})$ of binary relations on $E$ preserved by all $f \in \mathcal{F}$ satisfies the following properties:

- (a) $\Delta_E \in \mathcal{R}$;
- (b) $\mathcal{R}$ is closed under arbitrary intersections; in particular $E \times E \in \mathcal{R}$;
- (c) $\mathcal{R}$ is closed under arbitrary unions;
- (d) If $\rho, \tau \in \mathcal{R}$ then $\rho \cdot \tau \in \mathcal{R}$;
- (e) If $\rho \in \mathcal{R}$ then $\rho^{-1} \in \mathcal{R}$.

Let $\mathcal{R}$ be a set of binary relations on a set $E$ satisfying items (a), (b), (d) and (e) of the above lemma (we do not require (c)). To make things more transparent, denote by $0$ the set $\Delta_E$, set $\rho \oplus \tau := \rho \cdot \tau$. Then $\mathcal{R}$ becomes a monoid. Set $\overline{\rho} := \rho^{-1}$, this defines an involution on $\mathcal{R}$ which reverses the monoid operation.
With this involution \( \mathcal{R} \) is an involutive monoid. With the inclusion order, that we denote \( \leq \), this involutive monoid is an involutive complete ordered monoid.

With these definitions, we have immediately:

**Lemma 5.2.** Let \( \mathcal{R} \) be an involutive complete ordered monoid of the set of binary relations on \( E \) and let \( d \) be the map from \( E \times E \) into \( \mathcal{R} \) defined by

\[
d(x, y) := \bigcap \{ \rho \in \mathcal{R} : (x, y) \in \rho \}.
\]

Then, the following properties hold:

(i) \( d(x, y) \leq 0 \) iff \( x = y \);
(ii) \( d(x, y) \leq d(x, z) \oplus d(z, y) \);
(iii) \( d(y, x) = d(x, y) \).

In [67], a set \( E \) equipped with a map \( d \) from \( E \times E \) into an involutive ordered monoid \( \mathcal{H} \) (for which 0 is not necessarily the least element of the monoid) and which satisfies properties (i), (ii), (iii) stated in Lemma 5.2 is called a \( \mathcal{H} \)-distance, and the pair \( (E, d) \) a \( \mathcal{H} \)-metric space. A study of metric spaces and non-expansive mappings along the lines of the one developed in Section 5 is in [67]. It is no more than the study of systems of binary relations and homomorphisms. Indeed to a metric space \( E := (E, d) \) over an ordered monoid \( \mathcal{H} \), we may associate the relational structure \( \mathcal{R}_d := (E, (d_v)_{v \in \mathcal{H}}) \) where \( d_v := \{(x, y) : d(x, y) \leq v\} \). If \( f \) is a map from \( E := (E, d) \) into \( E' := (E', d') \), then \( f \) is non-expansive iff \( f \) is a homorphism from \( \mathcal{R}_d \) into \( \mathcal{R}'_{d'} \); that is for every \( v \in \mathcal{H}, (x, y) \in d_v \) implies \( (f(x), f(y)) \in d'_v \). Lemma 5.2 justifies that we write \( d(x, y) \leq \rho \) for the fact that a pair \( (x, y) \) belongs to a binary relation \( \rho \) on the set \( E \). Hence, one can use notions borrowed from the theory of metric spaces in the study of binary relational structures. An illustration is given in [46].

In the next section, we consider equivalence relations and generalized ultrametric spaces.

### 6. Preservation of equivalence relations

The presentation of this section is borrowed from [68].

A binary relation \( \rho \) on a set \( A \) is an equivalence relation if it is reflexive, symmetric and transitive. It decomposes \( A \) into blocks. Two elements in the same block are equivalent; whereas two elements into two different blocks are inequivalent. The fact that a binary operation \( f \) preserves \( \rho \) means that if \( x \) and \( y \) belong to blocks \( X \) and \( Y \) respectively, then the block containing \( f(x, y) \) does not depends upon the particular choice of \( x \) and \( y \). This allows to define an operation on the set of blocks that mimicks \( f \).

A pair \( A_F \) made a set \( A \) and a collection \( F \) of operations on \( A \) is called an algebra. Equivalence relations preserved by all members of \( F \) are called congruences and their set denoted by \( \text{Cong}(A_F) \). The study of the relationship between the set of congruences of an algebra and the set of maps which preserve all congruences is one of the goals of universal algebra.

#### 6.1. Algebra and congruences

If \( F \) is a set of maps on \( A \), the set \( \text{Cong}(A_F) \) is a subset of the set \( \text{Equiv}(A) \) of equivalence relations on \( A \); this set is closed under intersection and union of chains. Ordered by inclusion this is an algebraic lattice. It was show by Grätzer and Schmidt [28] that every algebraic lattice is isomorphic to the congruence lattice of some algebra.

One of the oldest unsolved problem in universal algebra is "the finite lattice representation problem":

**Problem 6.1.** Is every finite lattice isomorphic to the congruence lattice of a finite algebra?

See Grätzer [31] 2007 and Pálfy [60] 2001 for an overview. Say that a lattice \( L \) is representable as a congruence lattice if it is isomorphic to the lattice of congruences of some algebra. Say that it is strongly representable if every sublattice \( L' \) of some \( \text{Equiv}(A) \) (with the same 0 and 1 elements) which is isomorphic to \( L \) is the lattice of congruences of some algebra on \( A \). As shown in [73], not every representable lattice is strongly representable. The first step in the positive direction for the representation problem is the fact...
that every finite lattice embeds as a sublattice of the lattice of equivalences on a finite set, a famous and non trivial result of Pudlák and Tuma \cite{72} solving an old conjecture of Birkhoff.

For an integer \( n \), let \( M_n \) be the lattice made of a bottom and a top element and an \( n \)-element antichain. Let \( M_3 \) be the lattice made of a 3-element antichain and a top and bottom. This lattice is representable (as the set of congruences of the group \( \mathbb{Z}/2 \cdot \mathbb{Z} \times \mathbb{Z}/2 \cdot \mathbb{Z} \)) but not strongly representable. We may find sublattices \( L \) of \( \text{Equiv}(A) \) isomorphic to \( M_3 \) that such the only unary maps preserving \( L \) are the identity and constants (see Section \ref{9}). Hence, the congruence lattice of the algebra on \( A \) made of these unary maps is \( \text{Equiv}(A) \). The sublattices \( L \) of \( \text{Equiv}(A) \) such that \( \text{Cong}(A_L) = \text{Equiv}(A) \) (where \( A_L := (A, \text{Pol}^1(L)) \)) are said to be dense. The fact that, as a lattice, \( M_3 \) has a dense representation in every \( \text{Equiv}(A) \) with \( A \) finite on at least five elements, amounting to a Zádori’s result \cite{87} given is Section \ref{9} appears in \cite{22} as Proposition 3.3.1 on page 20. It is not known if \( M_n \) is representable for each integer \( n \) (it is easy to see that \( M_n \) is representable if \( n = q + 1 \) where \( q \) is a power of a prime. The case \( n = 7 \) was solved by W.Feit, 1983).

### 6.2. Orthogonal systems of equivalence relations.

Two equivalence relations \( \rho \) and \( \tau \) on the same set \( E \) are orthogonal if their intersection \( \rho \cap \tau \) is the equality relation \( \Delta_E \) and their join in the lattice \( \text{Equiv}(E) \) of equivalence relations on \( E \) is the full relation \( E \times E \) (Note that in \cite{55} p.397 this terminology was used for the fact that \( \rho \cap \tau = \Delta_E \)).

**Problems 6.2.**

1. Given an integer \( n \), find the largest number of pairwise orthogonal equivalence relations on a set of size \( n \).
2. Find the largest number of pairwise orthogonal equivalence relations on a set of size \( 3m \) whose blocks have three elements?
3. More generally, given two integers \( k \) and \( m \), find the largest integer \( o(k, m) \) such that there are \( o(k, m) \) pairwise orthogonal relations on a set of size \( km \) whose blocks have \( k \) elements?

The first question amounts to find the largest \( m \) such that the lattice \( M_m \) embeds as a sublattice of the lattice \( \text{Equiv}(E) \) of equivalences on a set of cardinality \( n \). The second question was asked to us by Rosenberg in October 2013. It is worth noticing that for \( k = 2 \), it was conjectured by Kotzig \[49] in 1963 that \( o(2, m) \) is \( 2m - 1 \). This conjecture, still unsolved, is known under the name of Perfect one-factorisation (PIF) of the complete graph \( K_{2m} \). A one-factor of a graph is a set of pairwise disjoint edges whose union covers all vertices; a factorisation is a covering all the edges by pairwise disjoint one-factors; a one-factorization is perfect if the union of any two one-factors forms a Hamiltonian cycle. Kotzig’s conjecture is known to hold if \( m \) or \( 2m - 1 \) is prime and also if \( 2m \) is among some particular set of values; eg a PIF of \( K_{56} \) was obtained only this year \cite{63}.

### 6.3. Generalized ultrametric spaces.

Generalized ultrametric spaces provide natural sets of equivalence relations. We restrict ourselves to the case of ultrametric spaces over a join-semilattice \( V \), and consider first the case where \( V \) is complete and completely meet-distributive.

A join-semilattice is an ordered set in which two arbitrary elements \( x \) and \( y \) have a join, denoted by \( x \lor y \), defined as the least element of the set of common upper bounds of \( x \) and \( y \).

Let \( V \) be a join-semilattice with a least element, denoted by 0. A pre-ultrametric space over \( V \) is a pair \( E := (E, d) \) where \( d \) is a map from \( E \times E \) into \( V \) such that for all \( x, y, z \in E \):

\[
d(x, x) = 0, \quad d(x, y) = d(y, x) \quad \text{and} \quad d(x, y) \leq d(x, z) \lor d(z, y).
\]

The map \( d \) is an ultrametric distance over \( V \) and \( E \) is an ultrametric space over \( V \) if \( E \) is a pre-ultrametric space and \( d \) satisfies the separation axiom:

\[
d(x, y) = 0 \implies x = y.
\]

Any family \( R := (E, (\rho_i)_{i \in I}) \) of equivalence relations on a set \( E \) can be viewed as a pre-ultrametric space on \( E \). Indeed, given a set \( I \), let \( \mathcal{P}(I) \) be the power set of \( I \). Then \( \mathcal{P}(I) \), ordered by inclusion, is a join-semilattice (in fact a complete Boolean algebra) in which the join is the union, and 0 the empty set.
Proposition 6.3. Let \( R := (E, (\rho_i)_{i \in I}) \) be a family of equivalence relations. For \( x, y \in E \), set \( d_R(x, y) := \{ i \in I : (x, y) \notin \rho_i \} \). Then the pair \( E_R := (E, d_R) \) is a pre-ultrametric space over \( \emptyset(I) \).

Conversely, let \( E := (E, d) \) a pre-ultrametric space over \( \emptyset(I) \). For every \( i \in I \) set \( \rho_i := \{ (x, y) \in E \times E : i \notin d(x, y) \} \) and let \( R := (E, (\rho_i)_{i \in I}) \). Then each \( \rho_i \) is an equivalence relation on \( E \) and \( d_R = d \).

Furthermore, \( E_R \) is an ultrametric space if and only if \( \bigcap_{i \in I} \rho_i = \Delta_E := \{ (x, x) : x \in E \} \).

For a join-semilattice \( V \) with a 0 and for two pre-ultrametric spaces \( E := (E, d) \) and \( E' := (E', d') \) over \( V \), a non-expansive mapping (or contracting map) from \( E \) to \( E' \) is any map \( f : E \to E' \) such that for all \( x, y \in E \):

\[
d'(f(x), f(y)) \leq d(x, y).
\]

Pre-ultrametric spaces with their non-expansive mappings and systems of equivalence relations with their relational homomorphisms are two faces of the same coin. Indeed:

Proposition 6.4. Let \( R := (E, (\rho_i)_{i \in I}) \) and \( R' := (E', (\rho'_i)_{i \in I}) \) be two family of equivalence relations. A map \( f : E \to E' \) is a homomorphism from \( R \) into \( R' \) if and only if \( f \) is a non-expansive mapping from \( E_R \) into \( E'_R \).

The proof is immediate and left to the reader.

6.4. Hyperconvexity. Most of the results of this subsection are borrowed from [67].

Let \( V \) be a join-semilattice with a least element 0. Let \( d_\vee : V \times V \to V \) defined by \( d_\vee(x, y) = x \vee y \) if \( x \neq y \) and \( d_\vee(x, y) = 0 \) if \( x = y \).

Lemma 6.5. The map \( d_\vee \) is a ultrametric distance over \( V \) satisfying:

\[
d_\vee(0, x) = x
\]

for all \( x \in V \).

This is the largest ultrametric distance over \( V \) satisfying [8].

Proof. Let \( x, y, z \). If two of these elements are equal, the triangular inequality holds. Otherwise we have trivially \( d_\vee(x, y) = x \vee y \vee z = d_\vee(x, z) \vee d_\vee(z, y) \). This proves that \( d_\vee \) is an ultrametric distance. If \( d \) is any ultrametric distance satisfying [8] then \( d(x, y) \leq d(x, 0) \vee d(0, y) = x \vee y \) for every \( x, y \in V \). If \( x \neq y \) we get \( d(x, y) \leq d_\vee(x, y) \) and if \( x = y \) we get \( d(x, y) = 0 = d_\vee(x, y) \).

Let \( x, y \) be two elements of \( V \).

If \( d \) is any ultrametric distance over \( V \) we have:

\[
x \leq y \vee d(x, y)
\]

and

\[
y \leq x \vee d(x, y).
\]

Let \( D(x, y) := \{ z \in V : x \leq y \vee z \} \). If \( V \) is a distributive lattice then \( D(x, y) \) is a filter. Indeed, if \( x \leq y \vee z_1 \) and \( x \leq y \vee z_2 \), then \( x \leq (y \vee z_1) \land (y \vee z_2) = y \vee (z_1 \land z_2) \). Hence, if \( V \) is finite then \( D(x, y) \) has a least element, the residual of \( x \) and \( y \).

In full generality, the residual of two elements \( x, y \) of a join-semilattice \( V \) (or even a poset) is the least element \( x \wedge y \), if it exists, of the set \( D(x, y) \). If \( V \) is a Boolean algebra, this is the ordinary difference of \( x \) and \( y \). We say that \( V \) is residuated if the residual of any two elements exists.

Lemma 6.6. Let \( V \) be a join semilattice with a least element \( 0 \). If \( V \) is residuated, then the map \( d_\vee : V \times V \to V \) defined by \( d_\vee(x, y) := (x \wedge y) \vee (y \wedge x) \) is an ultrametric distance over \( V \), and in fact the least possible distance \( d \) satisfying condition [8].
Proof. Clearly, $d_V(x, y) = 0$ iff $x = y$ and $d_V(x, y) = d_V(y, x)$. Let $x \in V$. We have $0 \leq x = 0$ and $x \leq 0 = x$ hence $d_V(0, x) = x$. Let $x, y \in L$. Clearly, $x \leq y \lor (x \land y) \leq d_V(x, y)$. Furthermore, $d_V(x, y) \leq x \lor y$. Hence the triangular inequality holds for $\{0, x, y\}$. Now, let $z \in V$. We have:

$$x \land y \leq (x \lor z) \lor (z \land y).$$

Indeed, this inequality amounts to $x \leq y \lor ((x \lor z) \lor (z \land y))$. An inequality which follows from the inequalities $x \leq z \lor (x \land z)$ and $z \leq y \lor (z \land y)$.

The triangular inequality follows easily. \qed

We say that a complete lattice $V$ is $\kappa$-meet-distributive if for every subset $Z \subseteq V$ with $|Z| \leq \kappa$ and $y \in V$,

$$\wedge\{y \lor z : z \in Z\} = y \lor \bigwedge Z.$$

It is completely meet-distributive if it is $|V|$-meet-distributive (beware, this terminology has other meanings).

A meet-distributive lattice is also called a op-frame. This is a Heyting algebra w.r.t. to the join as the binary operation, and the involution equal to the identity. We have:

**Lemma 6.7.** Let $V$ be complete lattice. Then $V$ is residuated if and only if it completely meet-distributive.

**Proof.** Suppose that $V$ is residuated. Let $y \in V$ and $Z \subseteq V$. Let $x := \wedge\{y \lor z : z \in Z\}$ and let $x \land y$ be the residual of $x$ and $y$. Trivially $y \lor \bigwedge Z$ is a lower bound of $\{y \lor z : z \in Z\}$. Hence, $y \lor \bigwedge Z \leq \wedge\{y \lor z : z \in Z\} = x$. We claim that conversely $x \leq y \lor \bigwedge Z$. It will follows that $\bigwedge\{y \lor z : z \in Z\} = y \lor \bigwedge Z$ as required. Indeed, from the fact that $x$ is a lower bound of $\{y \lor z : z \in Z\}$ we get that $x \land y$ is a lower bound of $Z$ and thus $x \land y \leq \bigwedge Z$. It follows that $x \land y \leq \bigwedge Z$, proving our claim.

Suppose that $V$ is complete and completely meet-distributive. Let $x, y \in V$ and $Z := D(x, y)$. Since $V$ is complete, $\bigwedge Z$ exists. Due to complete meet-distributivity, we have $y \lor \bigwedge Z = \bigwedge\{y \lor z : z \in Z\} \geq x$, hence $\bigwedge Z$ is the least element of $Z$, proving that this is $x \land y$. \qed

A residuated lattice does not need to be complete. For an example, if $V$ is a Boolean algebra, then $V$ is residuated and the distance over $V$, $d_V(a, b)$ is equal to $a \Delta b$, the symmetric difference of $a$ and $b$. There is a huge literature about Boolean algebra viewed as metric spaces (e.g. [8, 9, 10]). However, from Lemma 6.7 we have:

**Corollary 6.8.** A finite lattice is residuated iff it is distributive.

From Lemma 6.6, 6.13, and Proposition 3.2, we have

**Theorem 6.9.** If a join-semilattice $V$ is completely meet-distributive then it can be endowed with an ultrametric distance $d_V$ for which it becomes hyperconvex. Furthermore, every ultrametric metric space over $V$ embeds isometrically into a power of $V$.

Rewriting Proposition 3.3, we obtain that in the category of ultrametric spaces over a completely meet-distributive lattice $V$, injective, absolute retracts, hyperconvex spaces, spaces with the extension property and retracts of powers of $(V, d_V)$ coincide. Also, with Proposition 3.4. Every ultrametric space has an injective envelope.

This result obtained for general Heyting algebras in [35] has been independently obtained by [7] in 1987 for ordinary ultrametric spaces, see [11] for generalizations.

6.5. **Preservation.** Let $E := (E, d)$ be a metric space over a join-semilattice $V$. For each $r \in V$ set $\equiv_r := \{(x, y) \in E : d(x, y) \leq r\}$. Let $\text{Equ}_{d}(E) := \{\equiv_r : r \in V\}$. Let $F := \text{Hom}(E, E)$ be the set of non-expansive maps from $E$ into itself, let $\mathcal{E}_F := (E, F)$ be the algebra made of unary operations $f \in F$ and let $\text{Cong}_{d}(E) := \text{Cong}(\mathcal{E}_F)$ be the set of congruences of this algebra, that is the set of all equivalence relations on $E$ preserved by all contractions from $E$ into itself.
**Proposition 6.10.** Let \( E := (E, d) \) be an ultrametric space over a join-semilattice \( V \) with a least element 0. Then:

1. \( \text{Equiv}_d(E) \subseteq \text{Cong}_d(E) \).
2. A map \( f : E \to E \) is a non-expansive map of \((E, d)\) into itself iff it preserves all members of \( \text{Equiv}_d(E) \).
3. If the meet of every non-empty subset of \( V \) exists, then \( \text{Equiv}_d(E) \) is an intersection closed subset of \( \text{Equiv}(E) \), the set of equivalence relations on \( E \).
4. The set \( \text{Cong}_d(E) \) is an algebraic lattice; furthermore, for every \((x, y) \in E \times E\), the least member \( \delta(x, y) \) of \( \text{Cong}_d(E) \) containing \((x, y)\) is a compact element of \( \text{Cong}_d(E) \) and \( \delta(x, y) \) is included into \( \equiv_r \), where \( r := d(x, y) \).
5. Any two members of \( \text{Equiv}_d(E) \) commute and \( \equiv_r \circ \equiv_s \circ \equiv_r = \equiv_{r \vee s} \) for every \( r, s \in V \) iff \( E \) is convex.
6. \( E \) is hyperconvex iff \( \text{Equiv}_d(E) \) is a completely meet-distributive sublattice of the lattice \( \text{Equiv}(E) \) of equivalence relations on \( E \).

**Proof.** The first two item are immediate. Trivially, each \( \equiv_r \) is an equivalence relation and it is preserved by all contracting maps. Item (3). Let \( \equiv_{r_i}, i \in I \) be a family of members of \( \text{Equiv}_d(E) \) then \( \bigcap_{i \in I} \equiv_{r_i} \) equals \( \equiv_r \) where \( r := \bigwedge \{ r_i : i \in I \} \). Item (4). Since \((x, y) \in \equiv_r \) and \( \equiv_r \) is preserved by all contractions, we have \( \delta(x, y) \subseteq \equiv_r \). Since \( \text{Cong}_d(E) \) is the congruence lattice of an algebra it is algebraic. The fact that \( \rho(x, y) \) is algebraic follows from the algebraicity of \( \text{Cong}_d(E) \). Item (5) is Proposition 3.6.7 of [67]. We recall the proof. Let \( r, s \in V \). Due to the triangular inequality, we have \( \equiv_s \circ \equiv_r \subseteq \equiv_{r \vee s} \). We claim that the equality holds whenever \( E \) is convex. Let \( t := r \vee s \) and \((x, y) \in \equiv_t \). Since \( d(x, y) \leq t = r \vee s \) and \( E \) is convex, the closed balls \( B(x, r) \) and \( B(y, s) \) intersect. If \( z \) belongs to this intersection, then \( d(x, z) \leq r \) and \( d(y, s) \leq t \) hence \((x, y) \in \equiv_s \circ \equiv_r \). This proves our claim. Conversely, let \( B(x, r) \) and \( B(y, s) \) with \( d(x, y) \leq r \vee s \), that is \((x, y) \in \equiv_{r \vee s} \). We have \( \equiv_r \circ \equiv_s \subseteq \equiv_{r \vee s} \) and since \( r \) and \( s \) commute, \( r \circ s = s \circ r \subseteq \equiv_r \subseteq \equiv_s \). Due to our assumption \( \equiv_r \circ \equiv_s = \equiv_{r \vee s} \), hence \( \equiv_r \circ \equiv_s \subseteq \equiv_{r \vee s} \). From item (4) of Proposition 6.10 it follows that \( \delta(x, y) = \equiv_r \). Also, \( \rho \) is the union of all \( \equiv_r \) it contains.

**Corollary 6.11.** If \( E := (E, d) \) is convex the map \( r \mapsto \equiv_r \) is a lattice homomorphism from \( V \) into \( \text{Equiv}_d(E) \).

**Theorem 6.12.** If an ultrametric space \( E := (E, d) \) is hyperconvex, then every member of \( \text{Cong}_d(E) \) is a join of equivalence relations of the form \( \equiv_r \), for \( r \in V \).

**Proof.** Let \( \rho \) be an equivalence relation on \( E \). Let \((x, y) \in \rho \) and \( r := d(x, y) \). We claim that if \( \rho \) is preserved by every contracting map then \( \equiv_r \subseteq \rho \). Indeed, let \((x', y') \in \equiv_r \). The (partial) map \( f \) sending \( x \) to \( x' \) and \( y \) to \( y' \) is contracting. Since \( E \) is hyperconvex, it extends to \( E \) to a non-expansive map \( \overline{f} \). Since \( \rho \) must be preserved by \( \overline{f} \), and \((x, y) \in \rho \), we have \((x', y') \in \rho \). This proves our claim. From item (4) of Proposition 6.10 it follows that \( \delta(x, y) = \equiv_r \). Also, \( \rho \) is the union of all \( \equiv_r \) it contains.

**Lemma 6.13.** If \( L \) is an algebraic lattice then the residual of two compact elements (provided it exists) is compact.

**Proof.** Suppose \( x \) and \( y \) compact. Suppose \( x \wedge y \leq \bigvee Z \) for some subset \( Z \) of \( L \). We have \( x \leq y \bigvee Z \). Since \( x \) is compact, \( x \leq y \bigvee Z' \) for some finite \( Z' \subseteq Z \). Since \( x \wedge y \) is the least \( z \) such that \( x \leq y \bigvee z \), we have \( x \wedge y \leq \bigvee Z' \) proving that \( x \wedge y \) is compact.

**Theorem 6.14.** Let \( L \) be an algebraic lattice and \( K(L) \) be the join-semilattice of compact elements of \( L \). If \( L \) is completely meet-distributive then \( K(L) \) has an ultrametric structure and \( L \) is isomorphic to the set of equivalence relations on \( K(L) \) preserved by all contracting maps on \( K(L) \).

**Proof.** Due to Lemma 6.6 and 6.13 we may define on \( V := K(L) \) the distance \( d_V \). Due to meet-distributivity, \( V \) is hyperconvex. According to Theorem 6.12 each equivalence relation preserved by all contracting operation is a join of equivalence relations of the form \( \equiv_r \) for some \( r := d_V(a, b) \).
Corollary 6.15. If $V$ is a finite distributive lattice, then $V$ is isomorphic to the lattice of equivalence relations preserved by all contracting maps from $V$ into itself, $V$ being equipped with the distance $d_V$.

Hence, $V$ is representable as the lattice of congruences of some algebra. In fact it is strongly representable [73]. Dilworth proved that it is representable as the lattice of congruences of some lattice [28]. Define an arithmetic lattice as a sublattice of the lattice $\text{Equiv}(E)$ of equivalence relations on a set $E$ which is distributive and such that these equivalence commutes (see Section 7 below). Then from (6) of Proposition 6.10 follows that every finite distributive lattice is representable as an arithmetic lattice.

7. Arithmetical lattices

Let $\text{Equiv}(E)$ be the lattice of equivalence relations on a set $E$. A sublattice $L$ of $\text{Equiv}(E)$ is arithmetical (see [64]) if it is distributive and pairs of members of $L$ commute with respect to composition, that is

$$\rho \circ \theta = \theta \circ \rho \text{ for every } \theta, \rho \in L.$$  

This second condition amounts to the fact that the join $\theta \lor \rho$ of $\theta$ and $\rho$ in the lattice $L$ is their composition.

A basic example of arithmetic lattice is the lattice of congruences of $(\mathbb{Z}, +)$. The fact that pairs of congruences commute is easy (and interesting). If $\theta$ and $\rho$ are two congruences, take $(x, y) \in \rho \circ \theta$. Then, there is $z \in \mathbb{Z}$ such that $(x, z) \in \theta$ and $(z, y) \in \rho$. Let $r, t \in \mathbb{N}$ such that $\theta = \equiv_r$ and $\rho = \equiv_t$, then there are $k, \ell \in \mathbb{Z}$ such that $z = x + k.r$ and $y = z + \ell.t$. Set $z' := x + \ell.t$ then $x \equiv_t z' \equiv_r y$ hence $(x, y) \equiv_r \circ \equiv_t = \theta \circ \rho$. Thus $\rho \circ \theta = \theta \circ \rho$ as claimed.

As it is well known, if $\theta$ and $\rho$ are two congruences, $\theta = \equiv_i$ and $\rho = \equiv_r$ with $r, t \in \mathbb{N}$, then $\theta \lor \rho = \equiv_{\text{lcm}(t, r)}$ whereas, $\theta \land \rho = \equiv_{\text{lcm}(t, r)}$. Distributivity follows.

As it is well known (see [64]), arithmetic lattices can be characterized in terms of the Chinese remainder condition.

We say that a sublattice $L$ of $\text{Equiv}(E)$ satisfies the Chinese remainder condition if:

- for each finite set of equivalence relations $\theta_1, \ldots, \theta_n$ belonging to $L$ and elements $a_1, \ldots, a_n \in A$, the system:

$$x \equiv a_i(\theta_i), i = 1, \ldots, n$$

is solvable iff for all $1 \leq i, j \leq n$

$$a_i \equiv a_j(\theta_i \lor \theta_j).$$

Recall the following classical result:

Theorem 7.1. A sublattice $L$ of $\text{Equiv}(E)$ is arithmetical iff it satisfies the Chinese remainder condition.

7.1. Chinese remainder condition and metric spaces. Chinese remainder condition can be viewed as a property of balls in a metric space. For an example, in the case of $\mathbb{Z}$, if we may view the congruence class of $a_i$ modulo $r_i$ as the (closed) ball $B(a_i, r_i) := \{x \in E : d(a_i, x) \leq r_i\}$ in a metric space $(E, d)$, we are looking for an element of the intersection of these balls. As we have seen in Section 3 conditions ensuring that such element exists were considered in metric spaces (generalized or not), Helly property and convexity being the keywords. In our case, we may observe that $\mathbb{Z}$ has a structure of ultrametric space, but the set of values of the distance is not totally ordered. Ordering $\mathbb{N}$ by the reverse of divisibility: $n \leq m$ if $n$ is a multiple of $m$, we get a (distributive) complete lattice, the least element being 0, the largest 1, the join $n \lor m$ of $n$ and $m$ being the largest common divisor. Replace the addition by the join and for two elements $a, b \in \mathbb{Z}$, set $d(a, b) := |a - b|$. Then $d(a, b) = 0$ iff $a = b$; $d(a, b) = d(b, a)$ and $d(a, b) \leq d(a, c) \lor d(c, b)$ for all $a, b, c \in \mathbb{Z}$. With this definition, closed balls are congruence classes. In an ordinary metric space, a necessary condition for the non-emptiness of the intersection of two balls $B(a_i, r_i)$ and $B(a_j, r_j)$ is that the distance
between centers is at most the sum of the radii, i.e. \( d(a_i, a_j) \leq r_i + r_j \). Here this yields \( d(a_i, a_j) \leq r_i \lor r_j \) that is \( a_i \) and \( a_j \) are congruent modulo \( \text{lcd}(r_i, r_j) \). When this condition suffices for the non-emptiness of the intersection of any family of balls they are said hyperconvex and finitely hyperconvex if it suffices for any finite family. Hence, Chinese remainder theorem of arithmetic is the finite hyperconvexity of \( \mathbb{Z} \) viewed as an ultrametric space.

This generalizes.

Let \( L \) be a sublattice of \( \text{Equiv}(E) \) that contains 0 and is stable under the intersection of arbitrary meets. Let \( E := (E, d) \) where \( d : E \times E \to L \) is such that \( d(x, y) \) is the least member of \( L \) containing \( x \) and \( y \). Then, trivially, \( E \) is a generalized ultrametric space over \( L \).

Naturally, we obtain:

**Theorem 7.2.** \( L \) is arithmetical if and only if \( E \) is finitely hyperconvex.

In Proposition [3.3] was stated that hyperconvexity and one-extension property were equivalent provided that the set of values is Heyting. If it is not, a weakening is still valid. Kaarli [37] obtained the following two results:

**Corollary 7.3.** If \( L \) is arithmetical (and stable by arbitrary meets) then every partial function \( f : B \to A \) where \( B \) is a finite subset of \( A \) which preserves all members of \( L \) extends to any element \( z \) of \( A \setminus B \) to a function with the same property.

We recall the proof.

**Proof.** Our aim is to find \( x \in A \) such that for each \( \theta \in L \) and \( b \in B \), if \( b \equiv z(\theta) \) then \( f(b) \equiv x(\theta) \). Let \( B' := f(B) \). For each \( b' \in B' \), let \( \theta_{b'} \) be the least element of \( L \) such that

\[
(15) \quad b \equiv z(\theta_{b'})
\]

for all \( b \) such that \( f(b) = b' \).

We claim that the system \( x \equiv b'(\theta_{b'}) \) is solvable and next that any solution yields the element we are looking for. \( \square \)

**Corollary 7.4.** If \( L \) is arithmetical on a finite or countable set \( A \), then every partial function \( f : B \to A \) where \( B \) is a finite subset of \( A \) which preserves all members of \( L \) extends to a total function \( \overline{f} \) with the same property.

**Proof.** Enumerate the elements of \( A \setminus B \) in a list \( z_0, \ldots, z_n \ldots \). Set \( B_n := B \cup \{ z_m : m < n \} \). Define \( f_n : B_n \to A \) in such a way that \( f_0 = f \) and \( f_{n+1} \) extends \( f_n \) to the element \( z_n \) and to no other. Set \( \overline{f} := \bigcup_n f_n \).

We note that \( \mathbb{N} \) ordered by reverse of divisibility is not meet-distributive. Still, it can be equipped with a distance (given by the absolute value). It is finitely hyperconvex, but it is not hyperconvex. Indeed, an infinite set of equations does not need to have a solution while every finite subset has one (for an example, let \( a_{2n} := 2, r_{2n} := 2^n, a_{2n+1} := 3, r_{2n+1} := 3^n \), then \( d(a_{2n}, a_{2m}) = 0 \leq r_{2n} \lor r_{2m}, d(a_{2n}, a_{2m+1}) = 1 \leq r_{2n} \lor r_{2m+1} = \text{lcd}(2^n, 3^m) = 1 \).

### 7.2. Operations preserving all the congruences of \( (\mathbb{Z}, +) \).

Equip the set \( \mathbb{Z} \) of relative integers with the operation \( + \). This algebra is a commutative group. As in every commutative group, a congruence is determined by the class of 0 (the others being translates). Such a class is a subgroup. And this subgroup is of the form \( r \mathbb{Z} \) for some non-negative \( r \). Hence, a congruence on \( \mathbb{Z} \) is determined by a non-negative integer \( r \) and is defined by \( x \equiv_r y \) if \( x - y \) is a multiple of \( r \).

Cęgielski, Grigorieff and Guessarian (CGG), 2014 [14][15] handled the description of (unary) maps preserving all congruences of \( (\mathbb{Z}, +) \). Their description is given in terms of Newton expansion. If falls in the scope of the study of non expansive maps of an ultrametric space. Indeed, we may see \( \mathbb{Z} \) as an ultrametric
space, values of the distance being the integers, ordered by multiplication, the least element being 0 and the largest element 1.

The proof of CGC result is by no means trivial. The first author found in 2016 a few lines proof of the main argument. It was presented with a proof of GCC result in [68]. We will give it in Lemma 7.7 below.

Let $C$ be the set of maps $f : \mathbb{Z} \to \mathbb{Z}$ which preserve all congruences on $\mathbb{Z}$. It is closed under product, hence it contains all polynomials with integer coefficients. But it contains others (e.g. the polynomial $g(x) := \frac{x^2(x-1)^2}{2}$ is a congruence preserving map on $\mathbb{Z}$). It is locally closed, meaning that $f \in C$ iff for every finite subset $A$ of $\mathbb{Z}$, (in fact, every 2-element subset of $\mathbb{Z}$), the map $f$ coincides on $A$ with some $g \in C$ (in topological terms, $C$ is a closed subset of the topological space $\mathbb{Z}^2$ of maps $f : \mathbb{Z} \to \mathbb{Z}$ equipped with the pointwise convergence topology, the topology on $\mathbb{Z}$ being discrete).

Let $n$ be a non-negative integer, let $\text{lcm}(n) := 1$ if $n = 0$, otherwise let $\text{lcm}(n)$ be the least common multiple of $1, \ldots, n$, i.e. $\text{lcm}(n) := \text{lcm}\{1, \ldots, n\}$. If $X$ is an indeterminate (as well as a number) we set $X^0 := 1$, $X^1 := X$, $X^n := X \cdot (X - 1) \cdot \cdots \cdot (X - n + 1)$. The binomial polynomial is $\binom{X}{n} := \frac{X^n}{n!}$.

CGG’s result can be expressed as follows:

**Theorem 7.5.** (1) Polynomial functions of the form $\text{lcm}(n) \cdot \binom{X}{n}$ preserve all congruences;
(2) Every polynomial function which preserves all congruences is a finite linear sum with integer coefficients of these polynomials;
(3) The set $C$ of maps $f : \mathbb{Z} \to \mathbb{Z}$ which preserve all congruences is the local closure of the set of polynomials preserving all congruences.

A more compact form is given in (b) of Lemma 7.9 below. Note that being closed in the set $\mathbb{Z}^2$ of all maps $f : \mathbb{Z} \to \mathbb{Z}$ endowed with the pointwise convergence topology, the set $C$ is a Baire subset of $\mathbb{Z}^2$. Hence, it is uncountable (apply Lemma 7.9, or observe that it has no isolated point and apply Baire theorem). In particular, it contains functions which are not polynomials. A striking example using Bessel functions is given in CGG’s paper.

We give the proof of Theorem 7.5 below. Note first this:

**Lemma 7.6.** Let $f(x) := \lambda_k \cdot \binom{x}{k}$. If $f$ preserves the congruences $\equiv_i$ for all $i := 0, \ldots, k$ then $\lambda_k$ is a multiple of $\text{lcm}(k)$.

*Proof.* For $i := 0, 1, \ldots, k - 1$, we have $f(i) = 0$. If $f$ preserves $\equiv_{k - i}$, $f(k) = f(k) - f(i)$ is a multiple of $k - i$, hence $f(k)$ is a multiple of $k, k - 1, \ldots, 1$. Since $f(k) = \lambda_k$, the result follows.

Next, we prove that (1) of Theorem 7.5 holds.

**Lemma 7.7.** Let $n$ be a non-negative integer and $f_n(x) := \text{lcm}(n) \cdot \binom{x}{n}$. Then $f$ preserves all congruences.

*Proof.* This means that $f_n(x + k) - f_n(x)$ is divisible by $k$ for every non-zero $k$.

This follows from the equalities:

\[(x + k)^n \mod n = \sum_{i=1}^{n} \binom{x}{n - i} \cdot k^i \cdot \binom{k - 1}{i - 1}.
\]

Indeed, $\binom{x}{n}$ is an integer for every $i = 1, \ldots, n$. To prove that the first equality holds, it suffices to check that its holds for infinitely many values of $x$. So suppose $x, k \in \mathbb{N}$. In this case, the left hand side counts the number of $n$-element subsets $Z$ of a $x + k$-element set union of two disjoints set $X$ and $K$ of size $x$ and $k$, each $Z$ meeting $K$. Dividing this collection of subsets according to the size of their intersection with $K$ yields the right hand side of this equality.

We go to the proof of (2) of Theorem 7.5.

We first recall the description of polynomial functions with integer values given by Polya in 1915 (cf Theorem 22 page 794 in Bhargava [6]).
**Lemma 7.8.** Polynomial functions from \( \mathbb{Z} \) to \( \mathbb{Z} \) are finite linear sums with integer coefficients of polynomial functions of the form \( \binom{x}{k} \).

**Proof.** Let \( P \) be a polynomial of degree \( n \) over the reals. Since the \( \binom{x}{k}, k \in \mathbb{N}, \) have different degrees, they form a basis, hence

\[
P := \lambda_0 + \cdots + \lambda_k \cdot \binom{X}{k} + \cdots + \lambda_n \cdot \binom{X}{n}
\]

for some reals \( \lambda_0, \ldots, \lambda_n \).

Since \( \binom{x}{k} \) is a binomial coefficient (for \( k \leq m \)), every linear combination with integer coefficients of these polynomials takes integer values. Thus, if the \( \lambda_k \)'s are integers, \( P \) takes integer values. Conversely, suppose that the values of \( P \) are integers for \( X := 0, \ldots, n \). A trivial recurrence on the degree will show that the coefficients are integers. Indeed, let

\[
Q := \lambda_0 + \cdots + \lambda_k \cdot \binom{X}{k} + \cdots + \lambda_{n-1} \cdot \binom{X}{n-1}
\]

Since \( Q(k) = P(k) \) for all \( k \leq n - 1 \), each \( Q(k) \) is an integer. Hence induction applies to \( Q \) and yields that all \( \lambda_0, \ldots, \lambda_{n-1} \) are integers. Now, \( P(n) = Q(n) + \lambda_n \cdot \binom{X}{n} \). Since \( \lambda_0, \ldots, \lambda_{n-1} \) are integers, \( Q(n) \) is an integer; since \( \binom{X}{n}(X = n) = 1 \), it follows that \( \lambda_n \) is an integer. This proves our affirmation about the integrality of the coefficients.

One can say a bit more:

**Lemma 7.9.** (a) Every map \( f \) from a non-empty finite subset \( A \) of \( \mathbb{Z} \) and values in \( \mathbb{Z} \) extends to a polynomial function with integer values and degree at most \( n \) where \( n + 1 \) is the cardinality of the smallest interval containing \( A \). (b) For every map \( f : \mathbb{Z} \to \mathbb{Z} \) there are integer coefficients \( a_n, n \in \mathbb{N} \), such that

\[
f(x) = \sum_{n=0}^{\infty} a_n \cdot P_n(x)
\]

for every \( x \in \mathbb{Z} \), where \( P_n \) the polynomial equal to \( \binom{x+k}{2k} \) if \( n = 2k \) and equal to \( \binom{x+k}{2k+1} \) if \( n = 2k + 1 \).

The proof is a bit tedious but not difficult, we leave it to the reader (see [63] for details). Beware, Lagrange approximation will not do (e.g., in order to extend a map defined on a 2-element subset, we may need a polynomial of large degree).

We adapt the proof of Lemma 7.8 in order to prove (2) of Theorem 7.5.

**Lemma 7.10.** Polynomial functions from \( \mathbb{Z} \) to \( \mathbb{Z} \) which preserve all congruences are finite linear sums with integer coefficients of polynomial functions of the form \( \text{lcm}(k) \cdot \binom{x}{k} \).

**Proof.** Let \( P \) be a polynomial from \( \mathbb{Z} \) to \( \mathbb{Z} \). According to Lemma 7.8,

\[
P := \lambda_0 + \cdots + \lambda_k \cdot \binom{X}{k} + \cdots + \lambda_n \cdot \binom{X}{n}
\]

where \( \lambda_0, \ldots, \lambda_n \) are integers. Suppose that \( P(k) - P(k') \) is a multiple of \( k - k' \) for all \( k, k' : 1, \ldots, n \). We prove by induction on the degree that \( \lambda_k \) is a multiple of \( \text{lcm}(k) \) for each \( k = 1, \ldots, n \). Let

\[
Q := \lambda_0 + \cdots + \lambda_k \cdot \binom{X}{k} + \cdots + \lambda_{n-1} \cdot \binom{X}{n-1}
\]

We have \( Q(k) = P(k) \) for all \( k \leq n - 1 \). Hence, \( Q \) satisfies the property, induction applies and yields that all \( \lambda_k \) are integer multiples of \( \text{lcm}(k) \) for \( k \leq n - 1 \). Now, \( P(n) = Q(n) + \lambda_n \cdot \binom{X}{n} \). Since \( \lambda_k \) is a multiple of \( \text{lcm}(k) \) for \( k \leq n - 1 \), it follows from Lemma 7.7 that \( Q \) preserves all congruences, in particular \( Q(n) - Q(k) \) is a multiple of \( n - k \); since \( P(n) - P(k) \) is a multiple of \( n - k \), \( P(n) - Q(n) = \lambda_n \cdot \binom{X}{n} = \lambda_n \) is a multiple of \( n - k \). Hence \( \lambda_n \) is a multiple of \( 1, \ldots, n \). Proving that \( \lambda_n \) is a multiple of \( \text{lcm}(n) \).
The proof yields:

**Corollary 7.11.** If a polynomial of degree \( n \) preserves all congruences of the form \( \equiv_k \) for \( k := 1, \ldots, n \), it preserves all congruences.

**Lemma 7.12.** Every map \( f \) from a finite subset \( A \) of \( \mathbb{Z} \) and values in \( \mathbb{Z} \) which preserves the congruences extends to every \( a \in \mathbb{Z} \setminus A \) to a map with the same property.

**Proof.** This follows from the Chinese remainder theorem (see Corollary 7.3 in the previous subsection). \( \Box \)

Lemma 7.9 becomes: **Lemma 7.13.** (a) Every map \( f \) from a finite subset \( A \) of \( \mathbb{Z} \) and values in \( \mathbb{Z} \) which preserves all congruences extends to a polynomial function preserving all congruences. (b) Every map \( f : \mathbb{Z} \rightarrow \mathbb{Z} \) which preserves all congruences is of the form

\[
\sum_{n=0}^{\infty} a_n \cdot P_n
\]

where each \( a_n \) is an integer multiple of \( \text{lcm}(n) \).

**Proof.** We extend \( A \) to a finite interval \( \overline{A} \). With Lemma 7.12 we extend \( f \) to \( \overline{A} \) to a map \( \overline{f} \) which preserves all congruences. A proof as in Lemma 7.9 will apply. We only need to check that \( a_n \) is a multiple of \( \text{lcm}(n) \) for each \( n \in \mathbb{N} \). We do that by induction. We suppose \( a_i \) is a multiple of \( \text{lcm}(i) \) for each \( i < n \). We need to prove that \( a_n \) is a multiple of \( \text{lcm}(n) \). The map \( \overline{f} \mid_{\overline{A}_{n+1}} \) preserves the congruences \( \equiv_1, \ldots, \equiv_n \), hence, by the proof of Lemma 7.6, \( a_n \) is a multiple of \( \text{lcm}(n) \). \( \Box \)

8. OPERATIONS PRESERVING THE CONGRUENCES OF OTHER GROUPS AND MONOIDS

There are many results on the preservation of congruences of groups and monoids, this in relation with studies about polynomial completeness, see [38]. In this section we discuss the case of the group \( \mathbb{Z}^n \).

Let \( n \) be a non-negative integer and \( (\mathbb{Z},+)^n \) the \( n \)-power of the additive group \( \mathbb{Z} \). Results about the preservation of congruences for \( n \geq 2 \) and for \( n = 1 \) are completely different. For \( n \geq 2 \) there are only countably many operations preserving the congruences: these operations are affine [58], while for \( n = 1 \) there are uncountably many as we have shown in the previous section.

Suppose \( n \geq 2 \). Let \( k, l < n \), set \( \mathbb{Z}_{(k)} := \{(x_i)_{i<n} : x_i = 0 \text{ for all } i \neq k\} \), set \( \mathbb{Z}_{(k,l)} := \{(x_i)_{i<n} : x_i = 0 \text{ for all } i \neq \{k,l\} \text{ and } x_k = -x_l\} \). 

**Theorem 8.1.** Let \( n \) be a non-negative integer and \( f : (\mathbb{Z},+)^n \rightarrow (\mathbb{Z},+)^n \). The following properties are equivalent:

(i) \( f \) preserves all congruences of \( (\mathbb{Z},+)^n \);
(ii) \( f \) preserves the congruences associated with the subgroups \( \mathbb{Z}_{(k)} \) and \( \mathbb{Z}_{(k,l)} \) for \( k, l < n \);
(iii) There are \((a_i)_{i<n} \in (\mathbb{Z},+)^n\) and \( m \in \mathbb{Z} \) such that \( f((x_i)_{i<n}) = (a_i)_{i<n} + (m \cdot x_0, \ldots, m \cdot x_{n-1}) \) for all \((x_0, \ldots, x_{n-1}) \in (\mathbb{Z},+)^n\).

The proof relies on the properties of the three equivalence relations \( \equiv_i \), for \( i \leq 2 \), on the square \( A \times A \) of an abelian group \( A \). These equivalences relations are: \((x, y) \equiv_1 (x', y')\) if \( x = x' \); \((x, y) \equiv_2 (x', y')\) if \( y = y' \); \((x, y) \equiv_0 (x', y')\) if \( x + y = x' + y' \).

We recall the following result:

**Theorem 8.2.** (Theorem 3.2 of [20]) A map \( f : A \times A \rightarrow A \times A \) preserves \( \equiv_i \) for \( i = 0, 1, 2 \) iff \( f(x, y) := (x_0, y_0) + (h(x), h(y)) \) for some \((x_0, y_0) \in A \times A \) and some additive map \( h \) on \( A \) (i.e., satisfying \( h(x+y) = h(x) + h(y) \) for all \( x, y \in A \)).

**Proof of Theorem 8.1** Implications (i) \( \Rightarrow \) (ii) and (iii) \( \Rightarrow \) (i) are obvious. We prove (ii) \( \Rightarrow \) (iii). Set \( g((x_i)_{i<n}) := f((x_i)_{i<n}) - f(0) \). Let \( k, l < n \). The map \( g \) preserves \( \mathbb{Z}_{(k)} \), \( \mathbb{Z}_{(l)} \) and \( \mathbb{Z}_{(k,l)} \); setting \( A := \mathbb{Z} \) we may identify the direct sum \( \mathbb{Z}_{(k)} \oplus \mathbb{Z}_{(l)} \) with \( A \times A \) and the restriction of \( g \) to the map \( \overline{g}_{k,l} \) from \( A \times A \)
into itself associating to each pair \((x, y)\) the pair \((u, v)\) such that \(u = g((x_i)_{i \in n})_k\) and \(v = g((x_i)_{i \in n})_l\) for \((x_i)_{i \in n} \in \mathbb{Z}^{(k)} \oplus \mathbb{Z}^{(l)}\) such that \(x_k = x, x_l = y\). This map preserves the three equivalences \(x_i\) above. Hence \(g_{k,l}(x, y) = (h_{k,l}(x), h_{k,l}(y))\) where \(h_{k,l}\) is an homomorphism of \(\mathbb{Z}\). Hence, there is some \(m_{k,l} \in \mathbb{Z}\) such that \(h_{k,l}(z) = m_{k,l} \cdot z\) for every \(z \in A\). If \(n = 2\) this fact yields the result. If \(n > 2\) then \(m_{k,l}\) is independent of \(k\) and then of \(l\). This also yields the result. □

Instead of preserving congruences of a group we may preserve the congruences of a monoid. For the free commutative monoid \(\mathbb{N}^n\) the results are similar (see [17]). We conclude by mentioning the following beautiful result of Cégielski, Grigorieff and Guissarian about the free monoid.

**Theorem 8.3.** [16] Let \(n \geq 3\). A map \(g\) preserves all congruences on the free monoid \(A^*\) with at least three generators iff there exists \(n \in \mathbb{N}\) and \(a_0, \ldots, a_{n-1} \in A^*\) such that \(g(x) = a_0xa_1 \ldots xa_{n-1}\) for every \(x \in A^*\).

No simple proof is known. The case of the free monoid with two generators is open.

9. **RIGIDITY AND SEMIRIGIDITY**

A relation, and more generally a set \(\mathcal{R}\) of relations, on a set \(E\) is **rigid** if the only self-map \(f\) of \(E\) preserving this relation, or this set of relations, is the identity. It is **strongly rigid** if the only operations preserving \(\mathcal{R}\) are the projections. If the relations are reflexive then they are also preserved by the constant maps. So we say that a relation or a set \(\mathcal{R}\) of relations is **semirigid** if the only self-maps \(f\) of \(E\) preserving this relation, or this set of relations, is the identity or a constant map. It is **strongly semirigid** if the only operations preserving \(\mathcal{R}\) are the projections or the constant maps. We refer to [50, 51, 76, 85] for sample of results about these notions and to [19] for some generalizations.

A set \(\mathcal{R}\) of equivalence relations on a set \(E\) is strongly semirigid if and only if it is semirigid. Zádorí in 1983 has shown in 1983 that there are semirigid systems of three equivalence relations on finite sets of size different from 2 and 4. We present a construction given in [20]. It leads to the examples given by Zádorí and to many others and also extends to some infinite cardinalities.

We recall the following lemma.

**Lemma 9.1.** (Pierce) If a set \(\mathcal{R}\) generates by means of joins and meets the lattice of equivalences relations on \(E, |E| \neq 2\), then \(\mathcal{R}\) is semirigid.

The proof is simple, still it is illustrative.

**Proof.** For \(x \neq y\), let \(<x, y>\) be the equivalence relation such that \(\{x, y\}\) is the only non singleton class. We prove that if \(f\) preserves all the \(<x, y>\)-equivalences then \(f\) is the identity or a constant. Indeed, suppose by way of contradiction that \(f\) is neither the identity, nor a constant map. Not being the identity, there is some \(x\) with \(f(x) \neq x\). The elements \(x\) and \(f(x)\) are equivalent w.r.t. \(<x, f(x)\>\) hence \(f(x)\) and \(f(f(x))\) must be equivalent w.r.t. \(<x, f(x)\>\). This gives two cases 1) \(f(f(x)) = f(x)\) and 2) \(f(f(x)) = x\). If 1) holds, then since \(f\) is not constant, there is some \(z\) such that \(f(z) \neq f(x)\); but since \(z\) and \(x\) are equivalent w.r.t. \(<z, x>\), \(f(z)\) and \(f(x)\) must be equivalent w.r.t. \(<z, x>\). This implies \(f(z) = f(x)\), a contradiction.

If 2) holds, pick any \(z \notin \{x, f(x)\}\). From the preservation of \(<x, z>\), get \(f(z) = f(x)\); from the preservation of \(<f(x), z>\) get \(f(z) = x\), which is impossible. □

According to Strietz [83] 1977, if \(E\) is finite with at least four elements, four equivalences are needed to generate the lattice of equivalence relations on \(E\) by means of joins and meets. In that respect, Zádorí result proving that one can find semirigid systems made of three equivalence relations is interesting. His examples seem a bit mysterious. Our construction shows that this is not the case.

9.1. **Zádorí’s examples.** Zádorí’s result reads as follows.
**Theorem 9.2.** Let \( A := \{0, \ldots, n - 1\} \) with \( n = 3 \) or \( n > 4 \). The following system of three equivalence relations \( \rho, \sigma, \tau \) on \( A \) is semirigid.

\[
\rho = \{\{0\}, \{1, 2, \ldots, k\}, \{k + 1, \ldots, 2k + 1\}\}, \\
\sigma = \{\{0, 1, k + 1\}, \{2, k + 2\} \ldots, \{k, 2k\}\}, \\
\tau = \{\{1, k + 2\}, \{2, k + 3\} \ldots, \{0, k, 2k + 1\}\}.
\]

One of Zádori’s examples: case \( n \) even.

![Figure 4. Zádori’s example: case \( n \) even](image)

For \( n \) odd, simply delete the node 0 which is located on top of the graph and relabel conveniently the vertices.

9.2. **A geometric construction.** We describe a general construction of semirigid systems of three equivalence relations that includes Zádori’s examples

Let \( \mathbb{R} \) denotes the real line, let \( \mathbb{R} \times \mathbb{R} \) denotes the real plane. Let \( p_1 \) and \( p_2 \) be the first and second projection from \( \mathbb{R} \times \mathbb{R} \) onto \( \mathbb{R} \) and let \( p_0 : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) the map \( p_0 := p_1 + p_2 \). For \( i = 0, 1, 2 \), let \( z_i \) be the kernel of \( p_i \), i.e., for all \( u, v \in \mathbb{R} \times \mathbb{R} \), \( u \sim_i v \) if \( p_i(u) = p_i(v) \). Set \( R := (\mathbb{R} \times \mathbb{R}, (z_0, z_1, z_2)) \).

The system \( R \) is not semirigid. But, there are many subsets \( C \) of the plane for which the system \( R \upharpoonright C \) is semirigid. As in [20], let us introduce the notion of **monogenic subset** of the plane.

We define **triangles** of the planes: there are the trivial ones: the singletons, the non-trivial ones are the 3-element subsets \( \{u_0, u_1, u_2\} \) of \( E \) such that \( u_0 \cong u_1, u_1 \cong u_2 \) and \( u_2 \cong u_0 \). The plane with the collection of triangles is an hypergraph and maps which preserve the three equivalences send triangles on triangles (possibly trivial). A subset \( X \) of a subset \( C \) of the plane **generates** \( C \) if for every (nontrivial) triangle \( T \) included in \( C \) there is a finite sequence of triangles \( T_0, \ldots, T_n = T \) such that \( |X \cap T_0| = |T_i \cap T_{i+1}| = 2 \) for \( i < n \).

We say that \( C \) is **monogenic** if some subset of \( C \) with at most two elements generates \( C \).

Using Theorem 3.2 of [20] recalled as Theorem 8.2, one can prove.

**Theorem 9.3.** (Theorem 1.3 [20]) If a finite subset \( C \) of \( \mathbb{R} \times \mathbb{R} \) is monogenic and has no center of symmetry then \( R \upharpoonright C \) is semirigid.

The proof is based on Theorem 3.2 of [20] recalled as Theorem 8.2.

A simple example, which is at the origin of this result, is the following.
For $n \in \mathbb{N}$ set $T_n := \{(i, j) \in \mathbb{N} \times \mathbb{N} : i + j \leq n\}$. Then $T_n$ satisfies the hypotheses of Theorem 9.3. Hence $R \upharpoonright T_n$ is semirigid. This yields an example of a semirigid system of three pairwise isomorphic equivalence relations on a set having $(n+1)(n+2)/2$ elements.

Set $T_{n,2} := \{(i, j) \in T_n : i + j \in \{n - 1, n\}\}$ and $T'_{n,2} := T_{n,2} \cup \{(0, 0)\}$.

Both sets satisfy the hypotheses of Theorem 9.3 hence the induced $R \upharpoonright T_{n,2}$ and $R \upharpoonright T'_{n,2}$ are semirigid. They are isomorphic to those of Zádori. The technique of the proof of Theorem 9.3 yields:

**Theorem 9.4.** For each cardinal $\kappa$, $\kappa \notin \{2, 4\}$ and $\kappa \leq 2^{\aleph_0}$, there exists a semirigid system of three equivalences on a set of cardinality $\kappa$.

For an example, let $A := \mathbb{Z}$, $E := A \times A$ and

$$B := \{(x, y) \in \mathbb{Z} \times \mathbb{Z} : x + y \in \{1, 2\}\} \cup \{(0, 0)\}.$$  

The system $R \upharpoonright B$ induced by the system $R$ of three equivalence relations on $\mathbb{R} \times \mathbb{R}$ is semirigid.

We recall that a subset $X$ of $\mathbb{R}$ is *dense* if for every $x < y$ in $\mathbb{R}$ there is some $z \in X$ such that $x < z < y$. We also recall that every additive subgroup $D$ of $\mathbb{R}$ is either discrete, in which case $D = \mathbb{Z} \cdot r$ for some $r \in \mathbb{R}$, or dense. Let $D$ be an additive subgroup of $\mathbb{R}$ containing $\mathbb{Z}$. Set $\Delta := \{(x, y) \in D \times D : 0 \leq x, 0 \leq y, x + y \leq 1\}$. If $D$ is a dense subgroup of $\mathbb{R}$ including $\mathbb{Q}$ then the system $R \upharpoonright (B \cup \Delta)$ induced by the system $R$ is semirigid.

Figure 9.2 shows the set $B \cup \Delta$. The set $\Delta$ (shadowed) is represented schematically and the elements of the infinite set $B$ of two-dimensional integer points by circles.

**Problem 9.5.** Does the conclusion of Theorem 9.4 extend to every cardinal $\kappa > 2^{\aleph_0}$?

Related problems:
- Describe the semirigid systems of the plane.
- No useful characterization of semirigid systems is known.
- No algorithm known to decide in reasonable time whether or nor a system of $k$ equivalence relations on a set of size $n$ is semirigid or not.
FIGURE 6. An infinite band with a triangle

FIGURE 7. A semirigid system $B \cup \Delta$. 
10. CONCLUSION

With the present metric approach we have surveyed some categorical aspects of systems of reflexive binary relations. We skipped the study of non necessarily reflexive relations and also the case of \( n \)-ary relations. This latter case was considered in [67]. It could be studied in the same spirit as generalized metric spaces. Recent work on \( n \)-distances by Marichal and his collaborators [48] point in this direction.

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