Estimating Renyi entropies of a multiparticle system from event-by-event fluctuations

A.Bialas and K. Zalewski
M.Smoluchowski Institute of Physics
Jagellonian University, Cracow

and

Institute of Nuclear Physics, Polish Academy of Sciences

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Abstract

Recent improvements in the method of estimating Renyi entropies from measurements of coincidences between the events observed in high energy collisions are reviewed. A new, more precise, formulation of the method is presented and its accuracy analyzed.

1 Introduction

Entropy of a system produced in high-energy collisions is an interesting object, very useful for understanding the physics of the process in question. This is particularly important for the search for quark-gluon plasma in heavy ion collisions. It is not easy, however, to obtain information on entropy directly from data (without additional assumptions about the properties of the system). A window which may open such a possibility is to study the coincidences between observed events. As was suggested in [1], such measurements may allow to estimate of the Renyi entropies of the system [2] and thus, by extrapolation, give information on its Shannon entropy. This simple idea (based on an old suggestion by Ma [3]) is, however, difficult to implement

* Dedicated to Adriano Di Giacomo on the occasion of his 70th birthday.
† Address: Reymonta 4, 30 059 Krakow, Poland, e-mail: bialas@th.if.uj.edu.pl, zalewski@th.if.uj.edu.pl

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and its accuracy hard to determine. These two problems were studied recently in a series of papers by W. Czyz and the present authors [5, 6]. In this note we compile and summarize these results. Although no new results are presented (all can be found in [5] and [6]), we feel that such a compilation in a single place will be convenient for the reader and may be useful for future applications.

The object of our study is the M-particle semi-inclusive distribution. It is defined by considering a collection of events in which exactly \( M \) particles were observed in a given region of the momentum space. We shall call them \( M \)-particle events (independently of how many particles were actually produced)\(^1\). These events can be described by the normalized \( M \) particle Wigner function \( W_M(X, K) \) with \( X = X_1, ..., Z_M, K = K_{x_1}^{(1)}, ..., K_{z_M}^{(M)} \), which we shall interpret as the \( M \)-particle phase-space distribution [7].

It should be emphasized that the phase-space distribution \( W_M \), describing the semi-exclusive distribution, refers only to particles actually measured in a given experiment and in a given momentum region. It gives no direct information about the particles which are not registered by the detector. To discuss the phase-space density of all produced particles, additional assumptions (e.g. of thermodynamic equilibrium) are necessary.

At this point it is also important to realize that the phase-space distribution of particles produced in high-energy scattering is not a precisely defined quantity. Apart from the standard problems with the uncertainty principle, one has to take into account that particles may be produced at different times. In the present paper, following [8], we shall consider the time-averaged distribution.

The aim of this paper is to discuss (i) how the moments of \( W_M(X, K) \) can be estimated from the measured coincidences of the observed events and, (ii) how these moments are related to Renyi entropies and thus also to the Shannon entropy of the system.

To this end we first introduce the effective coincidence probabilities \( \hat{C}_M(l) \) of order \( l \), related to the moments of the phase-space distribution by\(^2\) [5]

\[
\hat{C}(l) = (2\pi)^{3M(l-1)} \int d^3M X \int d^3M K [W(X, K)]^l.
\] (1)

These quantities are interesting because, as was shown in [5] (and will be

\(^1\)This terminology is often used in experimental descriptions of multiparticle processes. The proper technical terms are: exclusive distribution if all particles are observed, and semi-inclusive distribution if besides a given number of observed particles there is an unspecified number of other particles. This should not be confused with inclusive \( M \)-particle distributions.

\(^2\)To simplify the formulae we shall from now on omit the index \( M \) in all quantities. Since we are discussing solely \( M \)-particle events, this should not lead to any confusion.
explained in the next section), for a rather large class of phase-space densities, \( \hat{C}(l) \) defined above can be approximated by the measured coincidence probability \( C^{\text{exp}} \) of the M-particle events [11] [9]

\[
C^{\text{exp}}(l) = \frac{N_l}{N(N-1)...(N-l+1)/l!}
\]  

(2)

where \( N_l \) is the number of the observed l-plets of identical events and \( N \) is the total number of events. \( N(N-1)...(N-l+1)/l! \) is the total number of l-plets of events\(^3\). One sees that the measurement of \( C^{\text{exp}} \) reduces to the count of the number of coincidences between the observed events.

As the next step we investigate the relation between \( \hat{C}(l) \), as defined by (1), and the coincidence probabilities of the states of the system, \( C(l) \), given by

\[
C(l) \equiv \sum_i [P_i]^l = \text{Tr}[\rho^l]
\]  

(3)

where the sum runs over all states of the system, \( P_i \) is the probability of a state \( i \) to occur and \( \rho \) is the density matrix of the system. The second part of this equality is obvious in the representation where the density matrix is diagonal. Since the trace of a matrix is independent of the representation, the result is generally valid.

\( C(l) \) defines the Renyi entropy of order \( l \), \( H(l) \), by the formula

\[
H(l) = \frac{1}{1-l} \log C(l)
\]  

(4)

and thus opens a window to the true entropy of the system. Indeed, as is well known and easy to show

\[
S = \lim_{l \to 1} H(l)
\]  

(5)

where \( S = -\sum_i P_i \log P_i = -\text{Tr}[\rho \log \rho] \) is the Shannon entropy.

Unfortunately, since measurement of coincidences can only provide information on \( H(l) \) for integer \( l \geq 2 \) and, in practice, only for \( l = 2, 3 \) and perhaps \( l = 4 \), the extrapolation procedure is rather uncertain [11]. However, since for \( l \geq 1 \) [12],

\[
S \geq H(l) \geq H(l + 1)
\]  

(6)

Renyi entropies provide an exact lower limit for \( S \), a quantity very important for understanding the properties of the quark-gluon plasma [13].

\(^3\)For \( l=2 \) formula [2] was first suggested, in a different context, by Ma [3]. See also [10].
We show that $C(l)$ and $\hat{C}(l)$ are equal to each other in the limit of infinite size of the system. Also the finite volume corrections are studied and shown to fall with inverse square of the smallest (linear) size. When combined with the previous result, one obtains a reliable method to measure, with a good control of error, the Renyi entropies of the system.

In the next section an Ansatz for the particle phase-space distribution is introduced. In Section 3 the corresponding formulae for the effective coincidence probabilities $\hat{C}(l)$ are written down and the optimal binning procedure is obtained by comparing them with $C^{\exp}(l)$. In Section 4 the true coincidence probabilities $C(l)$ are analyzed in the same framework and relation between $\hat{C}(l)$ and $C(l)$ is explained. Our conclusions and outlook are given in the last section.

2 The phase-space distribution

To proceed we consider a rather general form of the phase-space distribution

$$W(X, K) = \frac{1}{(L_x L_y L_z)^M} \exp[-v(K)]$$

with $X/L \equiv (X_1 - \bar{X}_1)/L_x, \ldots, (Z_M - \bar{Z}_M)/L_z$, $K = K_1, \ldots, K_M$. The function $G$ satisfies the normalization conditions

$$\int d^3 M u_i G(u) = 1 \rightarrow \int dX G(X/L) = (L_x L_y L_z)^M;$$
$$\int d^3 M uu_i G(u) = 0 \rightarrow <X_i, Y_i, Z_i> = \bar{X}_i, \bar{Y}_i, \bar{Z}_i;$$
$$\int d^3 M u(u_i)^2 G(u) = 1 \rightarrow <(X_i - \bar{X}_i)^2, \ldots > = L_x^2, \ldots$$

The first condition insures that $e^{-v(K)}$ is the properly normalized (multidimensional) momentum distribution\(^4\), the second defines the central values of the particle distribution in configuration space and the third defines $L_x, L_y, L_z$ as the root mean square sizes of the distribution in configuration space. Both sizes and central positions may depend on the particle momenta\(^5\). The form of the function $G$ describes the shape of the multiparticle distribution in configuration space.

The form (7) for the time-averaged phase space density is satisfied in a variety of models \([14]\). It is obviously valid for models which assume thermal

\(^4\)By definition of the phase-space distribution, the momentum distribution is given by $\int dX W(X, K)$.

\(^5\)They may be also different for different kinds of particles.
equilibrium. It can also incorporate expansion of the system, provided $\bar{X}'_i$s depend on $K'_i$s (the Hubble-like expansion is obtained for $\bar{X}_i \sim K_i$). It is general enough to incorporate any multiparticle momentum distribution.

In our further discussion we shall restrict somewhat this general form by taking the function $G(u)$ as a Gaussian:

$$\begin{equation}
G(u) = \frac{1}{(2\pi)^{3M/2}} e^{-\sum_{m=1}^{M} \sum_{\alpha} [u_{m\alpha}]^2 / 2}
\end{equation}$$

where $m$ labels the particles and $\alpha$ labels the space directions. This restriction can be avoided at the cost of some complications of the algebra. Since the exact shape of the particle emission region is not well determined and since, moreover, it is not in obvious disagreement with the data from quantum interference, we shall stick to it.

3 Moments of phase-space distribution and experimental coincidence probabilities

Using the Ansatz for the phase space density given by (7) and (9), we discuss in this section the relation of experimental coincidence probabilities (2) to the effective coincidence probabilities, determined by the moments of the phase-space semi-inclusive densities (1).

To discuss $C^{exp}(l)$ we have to face the problem of discretization. The point is that the measured events are characterized by particle momenta which are continuous variables. Therefore, the definition (2) is not directly applicable: a binning is necessary. Once discretized, the identical events can be defined as those which have the same population of the predefined bins and thus counting of coincidences becomes straightforward\(^6\). The counting of identical events obviously depends on the binning, however, so the procedure is ambiguous \cite{1, 4, 9, 10}. In order to obtain a viable estimate of $\hat{C}(l)$, we thus have to select the binning in such a way that the result of (2) is as close as possible to that given by (1).

Let us denote the $3M$-dimensional momentum bins by $j = 1, \ldots, J$ and their volumes by $\omega_j$. As the first step we express the measured $l$-fold coincidences (2) by the $3M$-dimensional distribution of momenta

$$\begin{equation}
e^{-v_3K} = \int dX W(K, X)
\end{equation}$$

\(^6\)A detailed description of this procedure was given in \cite{4} and applied in \cite{15}. 

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and the binning \( \omega_j \). This is clearly possible because the observed coincidences depend only on the momentum distribution and on binning. The relevant formulae are derived in [5].

Next, we consider \( \hat{C}(l) \), defined in (1). Using (7) and (9), a formula for \( \hat{C}(l) \) can be written down in form of an integral

\[
\hat{C}(l) = \int d^3K_1...d^3K_M \hat{\Omega}(K_1, ..., K_M; l)
\]

where

\[
\hat{\Omega}(K_1, ..., K_M; l) = \left( \frac{2\pi}{l} \right)^{3M(l-1)/2} \frac{e^{-lv(K_1, ..., K_M)}}{(L_x L_y L_z)^{(l-1)M}}.
\]

The derivation can be found in [5].

The integral (11) can be of course replaced by a sum over all bins \( \omega_j \).

The last step is to compare the formulae for \( C_{\text{exp}}(l) \) and \( \hat{C}(l) \). When this is done, one observes that if the bin sizes are

\[
\omega_j = \prod_{m=1}^{M} \left[ \frac{2\pi}{l^{3/2}(L_x L_y L_z)^{(m)}} \right] = \prod_{m=1}^{M} \prod_{\alpha=x, y, z} \left[ \frac{(2\pi)^{(l-1)/2}}{l^{1/2}(L_\alpha_j)^{(m)}} \right]
\]

where \( \alpha = x, y, z \), we obtain

\[
\hat{C}(l) = C_{\text{exp}}(l) \frac{\sum_j \frac{1}{\omega_j} \int_{\omega_j} dK e^{-v(K)}}{\sum_j \frac{1}{\omega_j} \int_{\omega_j} dK e^{-v(K)} [l]}.
\]

Note that the product \( L_x L_y L_z \) is related to the volume of the system in configuration space.

Eq. (13) is very general and can be applied to an entirely arbitrary discretization procedure. In the simple (but most useful in practice) case when \( L_x, L_y, L_z \) do not depend on \( \vec{K} \), the components \( K_x, K_y, K_z \) can be divided into bins of constant lengths \( \Delta_x, \Delta_y, \Delta_z \). Then the condition for the size of the bin is

\[
\Delta_x \Delta_y \Delta_z = \left( \frac{2\pi}{l} \right)^{(l-1)/2} \frac{1}{L_x L_y L_z}.
\]

and \( \omega_j = [\Delta_x \Delta_y \Delta_z]^M \).

Equations (14) and (15) define the method of estimating the effective coincidence probabilities \( \hat{C}(l) \) from the observed coincidence probabilities \( C_{\text{exp}} \).
The formula (14) can be rewritten in a somewhat more intuitive form

\[ \hat{C}_M(l) = C_M^{\text{exp}}(l) \frac{\sum_{\text{bins}} < e^{-l v(K)} >}{\sum_{\text{bins}} < e^{-v(K)} >^l}. \]  

which explicitly shows that in the limit when bins are so small that the momentum distribution inside each bin can be treated as a constant, \( \hat{C}(l) = C^{\text{exp}}(l) \). This implies that, as discussed in detail in [5], the accuracy of the method improves for large volume \((L_x L_y L_z)\) of the system. It also shows that the method of estimating the Renyi entropies proposed in [1] is only an approximation, valid at a very large volume of the system.

For smaller systems the correction factor

\[ \Phi \equiv \frac{\sum_{\text{bins}} < e^{-l v(K)} >}{\sum_{\text{bins}} < e^{-v(K)} >^l} \]  

may be estimated either from the measured single-particle distribution and correlation functions\(^7\) or, more precisely, by Monte Carlo simulations.

4 Moments of phase-space distribution and Renyi entropies

In this section we discuss the relation between the effective coincidence probabilities \( \hat{C}(l) \) (defined by the moments of the phase-space distribution \((1))\), and the coincidence probabilities of the states of the system (defined in terms of the density matrix \((3))\). To this end we have to evaluate the trace of the \(l\)-th power of the density matrix and compare it with the formula for \( \hat{C}(l) \).

The density matrix can be obtained from the phase-space distribution (Wigner function) by the Fourier transform \([7]\):

\[
\rho_M(p; p') \equiv \rho(p_1, ..., p_M; p'_1, ..., p'_M) = \int dX e^{iqX} W(X, K) \equiv \int d^3 X_1 ... d^3 X_M e^{i[q_1 X_1 + ... + q_M X_M]} W(X_1, ..., X_M, K_1, ..., K_M) = e^{-v(K_1, ..., K_M)} e^{-\frac{1}{2} \sum_{m=1}^M \sum_{a} L^2 q_{ma}^2 + i \sum_{m=1}^M \sum_{a} q_{ma} X_{ma}(K)}
\]

where \( q = p - p' \) and \( K = (p + p')/2 \), and where we have explicitly used the Gaussian form \((6)\) of \( G(u) \).

It is seen from this formula that \( \rho(p; p') \) may be diagonal in \( p, p' \) only if \( W[X, K] \) does not depend on \( X \), a condition which can be realized only if the volume of the system extends to infinity.

\(^7\)\( \Phi \) depends only on momentum distribution.
The non-diagonal nature of the density matrix is in fact the fundamental reason for all complications. When (18) is introduced into (3) we obtain $C(l)$ in form of a multidimensional integral. In [6] the first two terms in the asymptotic form of this integral at large $L_x, L_y, L_z$ were investigated. The results are summarized in the formula

$$C(l) = \int d^3K_1...d^3K_M \Omega(K_1, ..., K_M; l)$$

where

$$\Omega(K_1, ..., K_M; l) = \frac{\hat{\Omega}(K_1, ..., K_M; l)}{\Theta(K_1, ..., K_M; l)}.$$  

(20)

$\hat{\Omega}(K_1, ..., K_M; l)$ is defined in [12] and

$$\Theta(K_1, ..., K_M; l) = \text{Det} \left[ 1 + \sum_{s=1}^{2s} a_s T^s \right]$$

(21)

with

$$a_s = \frac{1}{2^{2s} (2s + 1)! (l - 2s - 1)!}$$

(22)

and $T$ is the $3M \times 3M$ matrix

$$T_{\alpha \beta} = \frac{1}{L_{\alpha}} V_{\alpha \beta} \frac{1}{L_{\beta}}.$$  

(23)

where

$$V_{\alpha \beta} = \partial_{\alpha \beta} v(K_1, ..., K_M); \quad V_{\alpha \beta} = \partial_{\alpha} \partial_{\beta} v(K_1, ..., K_M).$$  

(24)

The indices $m, n = 1, ..., M$ denote particles and $\alpha, \beta = x, y, z$ denote directions.

Note that the sum over $s$ is finite, because all $a_s$ vanish for $2s > l - 1$. In particular, for $l = 2$ all $a_i = 0$ and we simply have $\Omega(K; 2) = \hat{\Omega}(K; 2)$.

Comparing (19) with (11) one sees that for $l \geq 3 \hat{C}(l)$ and $C(l)$ differ only by the factor $\Theta(K_1, ..., K_M; l)$ under the integral. The first observation is that in the limit when all $L_x, L_y, L_z$ are very large, the matrix $T_{\alpha \beta}$ tends to 0 and thus the correction factor $\Theta$ approaches 1. Consequently, the difference between $\hat{C}(l)$ and $C(l)$ disappears. For finite size of the system, using (20), the difference can be explicitly calculated from the M-particle momentum distribution.
5 Discussion

When combined together, the results reported in Sections 3 and 4, provide a substantial improvement on the method of estimating the Renyi entropies of a multiparticle system suggested originally in [1]. First, the discretization procedure, necessary to give a precise meaning to the coincidence measurement, is properly formulated. Second, the role of the size of the system in configuration space for the accuracy of the measurement is explained. Finally, the corrections due to the finite size of the system are explicitly evaluated and can be used to improve the precision of the method. In effect we obtain a practical and reliable method of determining the Renyi entropies of the multiparticle systems and thus also the lower limit for its Shannon entropy.

Several comments are in order.

(i) One sees from (13) that the optimal size of the bin does not depend on the average position of the particles at freezeout \( \bar{X}(K) \). One also sees from (24) - (23) that the correction factor \( \Theta \) does not depend on it. This implies that the momentum-position correlations induced by the \( K \)-dependence of \( \bar{X} \) do not influence significantly the measurement of Renyi entropies.

(ii) It is also seen from (13) that only the volume of the bin \( \omega_{jm} = (\Delta x \Delta y \Delta z)_{jm} \), but not its shape, matters in the determination of the optimal discretization. One can use this freedom to improve the accuracy of the measurement by taking bins large in the directions with weak momentum dependence and small in the direction where the momentum dependence is significant.

(iii) Our analysis can be applied to any part of the momentum space. This is important for two reasons. First, for large systems, when the optimal size of the bins is small, a reliable measurement of coincidences in full momentum space may require a prohibitively large statistics. Thus restriction to a small part of phase space may be necessary. Second, it allows to measure the local entropy density in momentum space (integrated over all configuration space). In case of strong momentum-position correlations, the selection of a given momentum region can induce, however, a selection of a corresponding region in configuration space.

(iv) The accuracy of the measurement depends on the correct estimate of the size of the system. Information from HBT measurements allowing to determine the parameters \( L_x, L_y, L_z \) (and, hopefully, also the shape of the emission region [16]) is therefore necessary. One should keep in mind, however, that the interpretation of the results from quantum interference is far from unique [17]. Thus some modelling may be needed.

(v) It is interesting to note that the effect of the finite size of the system tend to cancel when the Eqs. (16) and (20) are combined. Indeed, increasing
$L$ implies smaller bins and thus decreasing $C^{\exp}(l)$. Thus also $\hat{C}(l)$ evaluated from (16) decreases. But one sees from (20) that then $C(l)$ increases. Thus the obtained value of Renyi entropy is less sensitive to a change in size of the system than Eqs. (16) and (20), taken separately, suggest.

(vi) Through this paper we have only discussed the M-particle systems (at fixed $M$). The coincidence probabilities including all multiplicities can obtained from the relation

$$C(l) = \sum_{M} [P(M)]^l C_M(l) \tag{25}$$

where $P(M)$ is the multiplicity distribution. One sees from this formula that at large $l$ only multiplicities close to the most probable one contribute effectively to $C_l$.

(vii) It should be emphasized that the method we propose takes into account all correlations between particles measured in the experiment. This is to be contrasted with the estimate of entropy from the single particle inclusive distribution [18] where, by definition, the correlations between particles are neglected (the assumption of equilibrium is used instead).

To summarize, we have reviewed recent developments [5, 6] of the method of estimating the Renyi entropies from measurement of coincidences between events observed in high energy collisions [1]. As discussed in detail, the accuracy of the method depends crucially on the size of the system in configuration space. It turns out that the original idea is strictly correct only for systems of very large size. The finite size corrections are derived. As shown in [6] they are negligible for systems encountered, e.g., in heavy ion collisions. For smaller systems they are more important but seem not to be prohibitive even for systems of the size as small as 1 fm. One thus obtains a new, reliable, tool for studies of the effective degrees of freedom in multiparticle phenomena.

The proposed method does not demand any assumptions about the thermodynamic properties of the system, in particular it does not assume thermodynamic equilibrium. It thus may be of particular interest for testing the standard assumptions of the models of quark-gluon plasma. Moreover, it can serve as a quantitative measure of the deviations from equilibrium.

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References

[1] A.Bialas and W.Czyz, Phys. Rev. D61 (2000) 074021.

[2] A.Renyi, Proc. 4-th Berkeley Symp. Math. Stat. Prob. 1960, Vol.1, Univ. of California Press, Berkeley-Los Ageles 1961, p.547.

[3] S.K.Ma, Statistical Mechanics, World Scientific, Singapore 1985; S.K.Ma, J. Stat. Phys. 26 (1981) 221.

[4] A.Bialas and W.Czyz, Acta Phys. Pol. B31 (2000) 687.

[5] A.Bialas, W.Czyz and K.Zalewski, [hep-ph/0506233], Acta Phys. Pol. B36 (2005) 3109 [hep-ph/0508289].

[6] A. Bialas and K. Zalewski, [hep-ph 0512248] to be published in Acta Phys.Pol. B; A.Bialas, W.Czyz and K.Zalewski, [hep-ph/0512293].

[7] For a discussion of the physical meaning of the Wigner function see, e.g., M.Hillery, R.F.O’Connell, M.O.Scully and E.P.Wigner, Phys.Rept. 106 (1984) 121.

[8] G.F.Bertsch, Phys. Rev. Letters 72 (1994) 2349; 77 (1996) 789 (E); D.A.Brown, S.Y.Panitkin and G.F.Bertsch, Phys. Rev. C62 (2000) 014904.

[9] A.Bialas and W.Czyz, Acta Phys. Pol. B31 (2000) 2803; B34 (2003) 3363.

[10] A.Bialas, W. Czyz and J.Wosiek, Acta Phys. Pol.B30 (1999) 107 .

[11] K.Zyczkowski, Open Sys. and Information Dyn. 10 (2003) 297.

[12] C.Beck and F.Schloegl, Thermodynamics of chaotic systems, Cambridge U. Press, Cambridge (1993).

[13] For a recent discussion see, e.g. B.Muller and K.Rajagopal, [hep-ph/0502174] and references therein.
[14] For a review of models see, e.g., U.A. Wiedemann and U. Heinz, Phys. Rep. 319(1999)145; U. Heinz and B. Jacak, Ann. Rev. Nucl.Part.Sci. 49(1999)529; R.M. Weiner, Phys. Rep. 327(2000)250; T.Csorgo, H.I.Phys. 15 (2002)1.

[15] K.Fialkowski and R.Wit, Phys.Rev. D62 (2000) 114016; NA22 coll, M. Atayan et al., Acta Phys. Pol. B36 (2005) 2969.

[16] See, D.A.Brown and P.Danielewicz, Phys. Lett. B398 (1997) 252; Phys. Rev D58 (1998) 094003; S.Y.Panitkin and D.A.Brown, Phys. Rev C61 (1999) 021901; G.Verde et al, Phys. Rev. C65 (2002) 054609; P.Danielewicz et al., Acta Phys. Hung. A19 (2004) nucl-th/0407022.

[17] See, e.g., A.Bialas and K.Zalewski, Phys. Rev. D72 (2005) 036009.

[18] S.Pal and S.Pratt, Phys. Lett. B578 (2004) 310.