1. INTRODUCTION

In this article we want to describe a long-term research project that the authors plan to carry out over a period in the immediate future. We would like to outline the basic ideas of the project and give a few preliminary calculations the bear on the validity of our ideas as well as some speculations on where the research will lead. We chose this volume to do this because it occurred to us that each of the components of our plan lies in a field that Charlie has touched on at some point in his varied career. These components are the Hamiltonian formulation of the gravitational field equations, path integral quantization, and quantum cosmology. Such a wide-ranging list shows how much we all owe to Charlie as a scientist and we hope that our efforts will demonstrate our debt to his lead in these fields.

The genesis of our paper lies in a long-term concern that both of us have had about the structure of quantum mechanics that has been the subject of years of blackboard discussions between us (the speculations of one of us [M.R.] go all the way back to fondly-remembered discussions with Charlie and other Charlie-students in Maryland). The focal point of our ideas has always been the cookbook nature of quantum mechanics as based on a Hamiltonian action functional for the classical equations of motion and canonical commutation relations derived from the Poisson bracket relations of the variables in the action. Perhaps the main among many objections to this procedure are that not all classical equations of motion are derivable from an action
principle and the well-known fact that not all canonical phase-space variables \((q_a, p_a)\) give a reasonable quantum theory when simple quantization procedures are applied to them. In the quantum mechanics of mechanical systems and some fields experiment has given enough guidance to be able to sidestep many of these difficulties, but in the quantization of the gravitational field there are no such experiments, and it is much more important to have a well-formulated quantum theory that is as free from ambiguities as possible.

One of us (S.H.) has spent some time studying the Lagrangian and Hamiltonian formulations of classical systems with an eye toward quantization, and especially the quantization of the gravitational field. The other has studied quantum cosmologies as models for quantum gravity. We have already combined these studies in a paper with Dario Núñez on the quantization of Bianchi-V cosmological models using non-standard Hamiltonian techniques\(^1\). The project we have in mind has several objectives and in this article we plan to give examples from quantum cosmology of several of the possible lines of investigation that we hope to pursue.

Basically, we hope to give a better foundation to the idea of quantization using non-standard Hamiltonian techniques by studying the possible meaning of using very general phase-space variables as a basis for quantization. In the current paper we will not use the Heisenberg picture (although this is not excluded as a future possibility), but will instead confine ourselves to Schrödinger and path-integral methods. One of the problems with Schrödinger quantization in non-standard phase space variables is that some operators are realized as derivatives, and it is vital to consider the function space on which these derivatives operate. As Bargmann\(^2\) showed in the context of phase-space variables related to the usual momentum and coordinate variables of the harmonic oscillator by a complex “canonical” transformation, this function space need not be a Hilbert space, but that one needs only demand that the space be capable of being mapped onto a Hilbert space in a meaningful way. While we are not in a position to develop a general mapping scheme here, we will mention possible forms that such a mapping might take.

With these goals in mind we will, in Sec. 2, sketch the problem of quantization using non-standard phase-space variables, with some emphasis on the difficulties of Schrödinger quantization. In Sec. 3 we will use a Bianchi type-I cosmological model as a model for quantum gravity and quantize it by means of BRST path integral methods using a simple set of non-standard phase space variables. Because of the speculative nature of the article we will not carry any of our discussions too far, but only attempt enough to give the broad outline of a picture we hope to fill in in the future.
2. PHASE SPACE VARIABLES IN QUANTUM MECHANICS.

The usual formulation of quantum mechanics begins with an action principle

\[ S = \int_{t_0}^{t_1} L(q_a, \dot{q}_a, t) dt = \int_{t_0}^{t_1} \left[ p_a \dot{q}_a - H(p_a, q_a) \right] dt, \tag{2.1} \]

which in principle gives the classical equations of motion when one varies \( S \) with respect to \((q_a, \dot{q}_a)\) or \((p_a, q_a)\). The standard procedure is to convert the phase space variables \((q_a, p_a)\) into operators \((\hat{q}_a, \hat{p}_a)\) and convert the Poisson bracket relations \(\{q_a, q_b\} = 0 = \{p_a, p_b\}, \{q_a, p_b\} = \delta_{ab}\) into operator commutator relations \([\hat{q}_a, \hat{q}_b] = 0 = [\hat{p}_a, \hat{p}_b], [\hat{q}_a, \hat{p}_b] = i\delta_{ab}\). One also takes any function \(O(q_a, p_b)\) and converts it to an operator \(\hat{O}(\hat{q}_a, \hat{p}_b)\) which has (modulo factor -ordering problems) the same functional form as \(O\). The most important function for the quantum dynamics of the system is the Hamiltonian operator \(\hat{H}(\hat{q}_a, \hat{p}_b)\). In this article we will concern ourselves mainly with the Schrödinger picture, where the momentum or coordinate variables are realized as derivative operators on a function space consisting of functions of eigenvalues of the other operators and the time \(t\). Note that we say “function space” rather than Hilbert space; this is deliberate.

This procedure leads to a Schrödinger equation

\[ \hat{H}\Psi(q_a, t) = i\frac{\partial \Psi}{\partial t}(q_a, t) \tag{2.2} \]

[or the equivalent for a momentum space wave function, \(\Phi(p_a, t)\)]. If the solutions to (2.2) form a Hilbert space, we can define a positive-definite probability density \(\rho(q_a, t) = \Psi^\ast \Psi(q_a, t)\) on the space of solutions.

This simple-minded precis of quantum theory ignores a large number of well-known problems, some of which have at least tentative solutions, and we will assume the reader is familiar with these. The quantization of constrained systems, the quantum theory of relativistic systems, and factor ordering difficulties are among them. The problems we plan to discuss in most detail are basically difficulties associated with the process of Schrödinger quantization we outlined above. It is well known (and usually cheerfully ignored) that the process is only valid for equations of motion that admit an action principle of the form (2.1), and the complete procedure requires a Hamiltonian form of the action, and that the phase space formulation where one realizes operators as derivatives is only supposed to be valid for a restricted class of phase space coordinates.

One of the best known examples of the failure of straightforward Schrödinger quantization is the motion of a particle in a central potential. In Cartesian coordinates
we have
\[ S = \int [p_x \dot{x} + p_y \dot{y} + p_z \dot{z} - \left\{ \frac{1}{2}(p_x^2 + p_y^2 + p_z^2) + V(x^2 + y^2 + z^2) \right\}] dt \] (2.3)

and if we realize \( \hat{p}_x, \hat{p}_y, \hat{p}_z \) as \( -i\partial_x, -i\partial_y, -i\partial_z \), then we arrive at a correct Schrödinger equation, \( i\partial\Psi / \partial t = H\Psi \). If we make the canonical transformation to spherical coordinates, \( x, y, z \rightarrow r, \theta, \phi \), \( p_x, p_y, p_z \rightarrow p_r, p_\theta, p_\phi \), the action becomes
\[ S = \int [p_r \dot{r} + p_\theta \dot{\theta} + p_\phi \dot{\phi} - \left\{ \frac{1}{2}(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2 \sin^2 \theta}) + V(r) \right\}] dt, \] (2.4)

and if we try to realize the momentum operators \( \hat{p}_r, \hat{p}_\theta, \hat{p}_\phi \) as \( -i\partial_r, -i\partial_\theta, -i\partial_\phi \), the resulting Schrödinger equation is “incorrect”, at least given a naive interpretation of the resulting wave function \( \Psi(r, \theta, \phi, t) \). Since it can be shown that any infinitesimal canonical transformation is equivalent to a unitary transformation acting on the corresponding operators, is is only transformations such as those to spherical coordinates that cannot be built up from infinitesimal transformations that will cause such problems.

In this article we would like to speculate on possible solutions to these problems. We have discussed problems that can be broken down into three main points: 1) How can one quantize a system described by any phase space variables? 2) How can one quantize systems whose equations of motion do not come from an action principle?; and 3) Can one quantize a system in the Schrödinger representation in non-standard phase-space variables? In order to make at least some steps toward answering these questions, we will begin with an outline of the symplectic approach to the canonical formulation of the equations of motion.

In principle one needs a first order formulation of the equations of motion of a system described by a set of variables \( x^a, a = 1, \cdots, 2n \). It is usual to suppose that half of these variables are coordinates and half are velocities or momenta. The equations of motion are
\[ \dot{x}^a = f^a(x^b). \] (2.5)

A canonical set of equations that reproduces these is
\[ \{x^a, x^b\} = J^{ab}; \] (2.6a)
\[ \dot{x}^a = \{x^a, H\} = J^{ab} \frac{\partial H}{\partial x^b} = f^a, \] (2.6b)

where for consistency of the equations of motion \( J^{ab} \) must obey the Jacobi identity,
\[ J^{ab},_d J^{dc} + J^{bc},_d J^{da} + J^{ca},_d J^{db} = 0. \] (2.7)
The symplectic structure matrix $J^{ab}$ is usually assumed to have the form

$$J^{ab} = \begin{bmatrix} 0 & I_N \\ -I_N & 0 \end{bmatrix},$$

(2.8)

where $I_N$ is the $n \times n$ unit matrix, so that half of the $x^a$ are momenta and the other half configuration variables. Of course, one must still be able to find a Hamiltonian $H$ that gives (2.6b).

In an unpublished work reported by Dyson, Feynman recognized that the form of $J^{ab}$ given by (2.8) is very restrictive and that relaxing this requirement leads to easier solution of certain problems, but dropped the idea because he felt it was unphysical. Recently Hojman and Shepley have attempted to use Feynman’s extension of symplectic theory to find new methods of quantization. They have shown that there exists a Hamiltonian $H$ for any system of the form (2.6), and a $J^{ab}$ can be constructed for such a system if one knows $2n$ constants of the motion $C_i$, $(2n-1$ of which do not depend explicitly on time), that is known them explicitly as functions of the coordinates (a fairly strong requirement, equivalent to knowing the full classical solution). This $J^{ab}$ may be constructed by summing elements of the basic form

$$J^{ab}_1 = \lambda(x^c) \varepsilon^{ab\mu_1\ldots\mu_{2n-3}} C_1,\mu_1 \cdots C_{2n-2},\mu_{2n-2},$$

(2.9)

where $C'_\mu \equiv \frac{\partial C_i}{\partial x^\mu}$, $\varepsilon^{ab\mu_1\ldots\mu_{2n-3}}$ is the $2n$-index Levi-Civita symbol, and $\lambda(x^c)$ is a function of the phase space coordinates that will be explained below. This $J^{ab}$ satisfies the Jacobi identity. The $C_1 \cdots C_{2n-1}$ are time independent constants of the motion. The Hamiltonian is defined as $H = C_{2n-1}$, along with $C_{2n} = t + d_{2n}$, where $d_{2n}$ is time independent. This can always be achieved by a change of coordinates. It is easy to realize that $\lambda(x^c)$ may always be chosen so that $J^{ab}_1 \frac{\partial H}{\partial x^b} = f^a$. There is considerable freedom in this formalism in selecting the Hamiltonian $H$. The main advantage of this formulation is that it allows one to find Hamiltonians for systems that do not admit an action principle (an example that will be mentioned below is the set of Class B Bianchi models when the isometries are imposed directly in the Hilbert action). A disadvantage (which we feel may not be a true disadvantage) is that the Hamiltonian is not unique.

Perhaps the best way of characterizing this concept is to imagine that the form of $J^{ab}$ given in (2.8) is the phase space equivalent of a “Minkowski metric”, and that phase space “general coordinate transformations” as opposed to ordinary canonical transformations, the equivalent of Lorentz transformations, will naturally change the metric form while preserving the equations of motion (2.6). Notice that one can make very general phase space transformations, including transformations to sets of variables that are all constants of the motion, the equivalent of the procedure that leads to the Hamilton-Jacobi formulation of classical mechanics.
In the future we are planning to use these ideas to attempt to build up a quantum theory and to apply it to a list of problems. Perhaps one of the most interesting possibilities will be its application to quantum gravity. We have already made a first attempt, applying it to the quantization of Bianchi Type V models in quantum cosmology.

There are essentially three paths we could take. One would be the use of the Heisenberg picture, since the equations of motion (2.6) are written in a form that is particularly amenable to this representation. While we plan to investigate this line later, our interest in quantum gravity has, given the direction that field has taken lately, led us to consider first the two other major methods of quantization, both of which are a little more difficult in the context of Eqns. (2.6). One is Schrödinger quantization and the other is path integral quantization. We will discuss the difficulties of Schrödinger quantization below, while path integral quantization will be the subject of Sec. 3. The symplectic structure language that we use is reminiscent of that used in geometric quantization, although the aims and methods are different, and only canonical transformations (“Lorentz”) transformations are allowed there. It is possible that there may be some natural points of contact between our ideas and that theory.

In principle, Schrödinger quantization is not difficult, at least mechanically. We must select half of the operators $\hat{x}^a$ as multiplication operators so the “wave function” will be a function of the eigenvalues of these operators and, in principle, time. The rest of the operators will be realized as derivative operators with respect to their “conjugates”. We can then construct a Schrödinger equation

$$i\frac{\partial \Psi(x^a, t)}{\partial t} = \hat{H}(x^a, -i\partial_{x^a})\Psi(x^a, t). \quad (2.10)$$

There are two main problems here, one obvious and the other a bit more subtle. One is that when $J^{ab}$ is no longer of the form (2.8) it becomes a tricky problem to decide which variables are conjugate to which. In the examples we have considered so far this problem can be handled, but we do not yet have a general procedure. The second problem is: To what function space do the solutions $\Psi(x^a, t)$ belong? There is no guarantee that they form a Hilbert space, and this is often considered a fatal defect. However, the original Bargmann formulation for the harmonic oscillator, that is the germ of the Ashtekar variables, addressed exactly this question. The complex “canonical” transformation $\Pi \equiv \frac{1}{\sqrt{2}}(p + iq)$, $X \equiv \frac{1}{\sqrt{2}}(q + ip)$ leads to an action (total derivatives dropped)

$$S = \int [\Pi \dot{X} + iX\Pi] dt, \quad (2.11)$$

which can be quantized in the Schrödinger representation by realizing $\Pi$ as $-i\partial/\partial X$. 
This leads to a very simple Schrödinger equation of the form
\[
\hat{H}\Psi(X, t) = -X\frac{\partial\Psi}{\partial X} = i\frac{\partial\Psi}{\partial t},
\] (2.12)
first given by Fock\(^8\). What Bargmann showed was that the function space upon which \(\partial/\partial X\) operates is not a Hilbert space, but that there exists a kernel \(K(q, X)\) which maps the space of functions \(F(X)\) of solutions to (2.12) onto the usual Hilbert space \(H(q)\), that is
\[
\psi(q, t) = \int K(q, X)\Psi(X, t)dX,
\] (2.13)
where \(dX\) means \(dpdq\). This, of course, means that wave functions \(\Psi(q_a, p_a, t)\) need not be wave functions in the usual sense. However, we have achieved simpler equations (note that [2.13] is much simpler than the usual harmonic-oscillator Schrödinger equation) at the cost of finding the kernel \(K\) that maps the new wave functions into useful wave functions that have the properties usually associated with them. In the case of the harmonic oscillator, finding \(K\) is relatively easy, but for more general phase space variables it may become quite difficult. We plan to investigate this in the future.

In a first attempt to apply some of our ideas to quantum gravity, the authors and Dario Núñez quantized Bianchi type V cosmological models as quantum cosmologies with one-sixth the logarithm of the determinant of the three-metric as an internal time\(^1\). The Einstein equations in this case do not result from the variation of the Hilbert action restricted by symmetry, but we were able to find a Hamiltonian \(H\) in the sense of Eqns. (2.6) and use it for Schrödinger quantization of the theory. We used a simple-minded mapping of the resulting wave function to one that could be used to construct a Hilbert space rather than attempt to construct a Bargmann-like kernel. We argued that given the confused state of the current interpretation of the wave function of the universe, that our solution has as much claim as any other to be a viable candidate for such a wave function.

We will not give the details here of the Bianchi V calculation, but will retreat to a simpler model, the Bianchi type I cosmologies. In the section that follows we will go on to the next part of the program outlined in the Introduction and apply BRST path-integral quantization to these models in non-standard phase space coordinates. We will again use a simple mapping of the wave function we obtain rather than attempt to develop here the more complicated integral mapping outlined above.

3. PATH INTEGRAL QUANTIZATION OF BIANCHI TYPE I MODELS.
Part of our plans for future research are aimed at path-integral quantization of gravity using non-standard phase space variables. In general this will be difficult to achieve
since Eqs. (2.6) do not necessarily admit an action which can be calculated for different histories. In the present article we will do a preliminary calculation in order to show how such a quantization might be expected to come out using a diagonal Bianchi type I cosmology where the results of standard path-integral quantization are known and the Hilbert action calculated for the model is a valid action for its equations of motion.

The form of the metric is

$$ds^2 = -N^2 d\alpha^2 + e^{2\alpha} e^{2\beta} dx^i dx^j,$$  \hspace{1cm} (3.1)

where $\beta_{ij} = \text{diag}(\beta_+ + \sqrt{3}\beta_-, \beta_+ - \sqrt{3}\beta_-, -2\beta_+)$, and $\frac{1}{6} \ln g = \frac{1}{6} \ln[\det(g_{ij})] = \alpha$ is taken as an internal time. The action can be reduced to the form

$$I = \int [p_+ \dot{\beta}_+ + p_- \dot{\beta}_- - p_\alpha - \tilde{N}(-p_\alpha^2 + p_+^2 + p_-^2)] d\alpha,$$  \hspace{1cm} (3.2)

where $\cdot \equiv d/d\alpha$ and $\tilde{N}$ is a normalized lapse function such as that used by Berger and Voegli\textsuperscript{10}. Variation of this action with respect to $p_\pm, \beta_\pm, p_\alpha$ and $\tilde{N}$ along with the equation $\dot{p}_\alpha = 0$ give the full set of equations for the model. Our non-standard variable set will be $p_\pm, \beta_\pm^{(0)}, K$, and $C = -p_\alpha^2 + p_+^2 + p_-^2$, where the $\beta_\pm^{(0)}$ are constants of motion that are the initial values of $\beta_\pm$, i.e. $\beta_\pm(\alpha = 0)$, $p_\pm$ are unchanged, $C$ is the non-linear combination given above, and $K$ is an extension of the phase space that we need to obtain the full equations of motion. In terms of these variables an action that gives the correct classical equations of motion is

$$I = \int [p_+ \dot{\beta}_+^{(0)} + p_- \dot{\beta}_-^{(0)} + C \dot{K} - \tilde{N} C] d\alpha.$$  \hspace{1cm} (3.3)

Varying $I$ with respect to $p_\pm, \beta_\pm^{(0)}, C, K$ and $\tilde{N}$ we find

$$\dot{p}_\pm = \dot{\beta}_\pm^{(0)} = 0,$$  \hspace{1cm} (3.4a)

$$\dot{C} = \dot{K} = 0, \quad C = 0.$$  \hspace{1cm} (3.4b)

The classical map between these variables and $\beta_\pm, p_\alpha$ (for $p_\pm$ being unchanged) gives the usual relations

$$p_\pm = p_\pm^{(0)} = \text{const.}, \quad p_\alpha = \sqrt{p_+^{(0)^2} + p_-^{(0)^2}}, \quad \beta_\pm = \beta_\pm^{(0)} + \frac{p_\pm^{(0)} \alpha}{\sqrt{p_+^{(0)^2} + p_-^{(0)^2}}}.$$  \hspace{1cm} (3.5)

The BRST path integral quantization of this system is relatively easy to carry out. What we would like to do is use a different coordinate system on the space of paths than that generated by skeletonization. That is, we would like to expand the possible paths in Fourier series, something that is rarely done except for the harmonic oscillator, but, of course, is possible for any continuous path.
For BRST quantization we have to extend the phase space to a set of both normal and anticommuting (ghost) variables. As is usual in phase space path-integral quantization, one must treat “momentum” and “coordinate” variables differently, and the decision about how to do this is somewhat of an art. In the Fourier-series formulation one way to handle this is to decide on the type of Fourier series to be used for each variable. We will not discuss this problem in detail here, but it will become obvious in the type of series we choose. The phase space is extended to include $\Pi$, a momentum conjugate to $\tilde{N}$ and $\rho, \bar{\rho}, c, \bar{c}$, four anticommuting functions of $\alpha$, which is all we need since there is only one constraint, $C = 0$ and one gauge fixing. We will take the proper time gauge, so the gauge function $\chi(p_\pm, \beta_{\pm}^{(0)}, K, C, \tilde{N}, \alpha) \equiv \tilde{N}$ is zero, so the gauge-fixing potential $\Phi$ is simply $\bar{\rho}\tilde{N}$, while the BRST charge $\Omega$ has the form $\Omega = cC + \rho\Pi_{11}$. With all of these choices the Batalin-Fradkin-Vilkovisky action for the system described by (3.3),

$$I_B = \int [p_+ \dot{\beta}_+^{(0)} + p_- \dot{\beta}_-^{(0)} + C \dot{K} - \tilde{N}C + \bar{\rho}\dot{c} + \bar{c}\dot{\rho} + \Pi \dot{\tilde{N}} - \{\Phi, \Omega\}]d\alpha,$$

becomes

$$I_B = \int [p_+ \dot{\beta}_+^{(0)} + p_- \dot{\beta}_-^{(0)} + C \dot{K} - \tilde{N}C + \bar{\rho}\dot{c} + \bar{c}\dot{\rho} + \Pi \dot{\tilde{N}} - \bar{\rho}\rho]d\alpha.$$  

(3.6)

The Fourier series we need for each of the variables which describe paths between some value of the variables at $\alpha = 0$ and other values at $\alpha = T$ are

$$\Pi = \sum_{n=1}^{\infty} \Pi_n \sin \left(\frac{n\pi\alpha}{T}\right); \quad \tilde{N} = N_0 + \sum_{n=1}^{\infty} N_n \cos \left(\frac{n\pi\alpha}{T}\right);$$

(3.8a)

$$c = \sum_{n=1}^{\infty} c_n \sin \left(\frac{n\pi\alpha}{T}\right); \quad \rho = \rho_0 + \sum_{n=1}^{\infty} \rho_n \cos \left(\frac{n\pi\alpha}{T}\right);$$

(3.8b)

$$\bar{c} = \sum_{n=1}^{\infty} \bar{c}_n \sin \left(\frac{n\pi\alpha}{T}\right); \quad \bar{\rho} = \bar{\rho}_0 + \sum_{n=1}^{\infty} \bar{\rho}_n \cos \left(\frac{n\pi\alpha}{T}\right);$$

(3.8c)

$$\beta_{\pm}^{(0)} = \beta_{\pm}^{(0)}(\alpha) + \sum_{n=1}^{\infty} \beta_{\pm}^{(n)} \sin \left(\frac{n\pi\alpha}{T}\right); \quad K = K_c(\alpha) + \sum_{n=1}^{\infty} K^{(n)} \sin \left(\frac{n\pi\alpha}{T}\right);$$

(3.8d)

$$p_{\pm} = p_{\pm}^{(0)} + \sum_{n=1}^{\infty} p_{\pm}^{(n)} \cos \left(\frac{n\pi\alpha}{T}\right); \quad C = C^{(0)} + \sum_{n=1}^{\infty} C^{(n)} \cos \left(\frac{n\pi\alpha}{T}\right),$$

(3.8e)

where the Fourier coefficients of the anticommuting variables are anticommuting numbers, and, in principle functions with subscript $c$ are the classical solutions for those variables. Since the variables that appear in (3.8d) have classical solutions that are constant, $\beta_{\pm}^{(0)}$ and $K_c$ would be constants. These classical solutions give unphysical results, so we will use a linear solution for each of them connecting $\beta_{\pm}^{(0)}$, $K_0$ at $\alpha = 0$ with $\beta_{\pm}^{(0)}$, $K_1$ at $\alpha = T$, or

$$\beta_{\pm}^{(0)} = \frac{1}{T}(\beta_{\pm}^{(0)} - \beta_{\pm}^{(0)}(\alpha) + \beta_{\pm}^{(0)}), \quad K_c = \frac{1}{T}(K_1 - K_0)\alpha + K_0.$$  

(3.9)
The ghost part of the action \( I_{gh} = \int_0^T [ \bar{\rho} \dot{c} + \dot{\bar{\rho}} \rho - \bar{\rho} \rho] d\alpha \) becomes

\[
I_{gh} = \sum_{n=1}^{\infty} \left[ \bar{\rho}_n \alpha \frac{n\pi}{2} - \bar{c}_n \rho_n \frac{n\pi}{2} - \bar{\rho}_n \rho_n T/2 \right] - \bar{\rho}_0 \rho_0 T. \tag{3.10}
\]

The ghost part of the propagator \( \int \Pi_n \Pi_n d\rho \Pi_n d\bar{\rho} \Pi_n d\bar{\rho} d\rho_0 d\bar{\rho}_0 e^{it_{gh}} \) can be shown, using Berezin integration (with the normalization \( \int \theta d\theta = 1 \)) to be

\[
T[-i \Pi_n (\frac{n^2 \pi^2}{4})].
\]

The rest of the action, \( I_A \) is

\[
I_A = \int_0^T \left\{ \left\{ \sum_{n=1}^{\infty} \cos \left( \frac{n\pi \alpha}{T} \right) \right\} \left\{ \frac{1}{T} \left( \beta_{+1}^{(0)} - \beta_{+0}^{(0)} \right) + \sum_{n=1}^{\infty} \frac{n\pi}{T} \beta_{+}^{(n)} \cos \left( \frac{n\pi \alpha}{T} \right) \right\} +
\]

\[
+ \left\{ \sum_{n=1}^{\infty} \cos \left( \frac{n\pi \alpha}{T} \right) \right\} \left\{ \frac{1}{T} \left( \beta_{-1}^{(0)} - \beta_{-0}^{(0)} \right) + \sum_{n=1}^{\infty} \frac{n\pi}{T} \beta_{-}^{(n)} \cos \left( \frac{n\pi \alpha}{T} \right) \right\} +
\]

\[
+ \left\{ \sum_{n=1}^{\infty} C^{(n)} \cos \left( \frac{n\pi \alpha}{T} \right) \right\} \left\{ \frac{1}{T} (K_1 - K_0) + \sum_{n=1}^{\infty} \frac{n\pi}{T} K^{(n)} \cos \left( \frac{n\pi \alpha}{T} \right) \right\} -
\]

\[
- \sum_{n=1}^{\infty} \Pi_n \sin \left( \frac{n\pi \alpha}{T} \right) \sum_{m=1}^{\infty} N_m \frac{m\pi}{T} \sin \left( \frac{m\pi \alpha}{T} \right) -
\]

\[
- \left\{ N_0 + \sum_{n=1}^{\infty} N_n \cos \left( \frac{n\pi \alpha}{T} \right) \right\} \{ C^{(0)} + \sum_{n=1}^{\infty} C^{(n)} \cos \left( \frac{n\pi \alpha}{T} \right) \} d\alpha
\]

\[
= \left\{ \begin{array}{c}
p_{+}(\beta_{+1}^{(0)} - \beta_{+0}^{(0)}) + \sum_{n=1}^{\infty} p_{+}^{(n)} \frac{n\pi}{2} + p_{-}^{(0)} (\beta_{-1}^{(0)} - \beta_{-0}^{(0)}) +
\]

\[
+ \sum_{n=1}^{\infty} p_{-}^{(n)} \frac{n\pi}{2} + C^{(0)} (K_1 - K_0) + \sum_{n=1}^{\infty} C^{(n)} K^{(n)} \frac{n\pi}{2} +
\]

\[
+ \sum_{n=1}^{\infty} \Pi_n N_n \frac{n\pi}{2} - N^{(0)} C^{(0)} T + \sum_{n=1}^{\infty} N_n C^{(n)} \frac{n\pi}{2}.
\] \tag{3.11}
\]

Integrating \( e^{it_A} \) over \( \Pi_n \) (we will always use the Liouville measure, e.g. for \( \Pi_n \), \( d\Pi_n/2\pi \)) from \(-\infty\) to \(+\infty\) gives an infinite product of delta functions of the form \( \delta(\frac{n^2 \pi^2}{4} N_n) \) which when integrated over the \( N_n \) removes the last term of (3.11) at the cost of a term \( \Pi_n (1/n\pi^2) \) as an overall factor. Similar integrations over \( C^{(n)} \), \( p_{+}^{(n)} \) and subsequently \( K^{(n)} \) and \( \beta_{+}^{(n)} \) remove all of the summations at the cost of three more \( \Pi_n (1/n\pi^2) \) factors. The integration over \( N^{(0)} \) gives \( \delta(C^{(0)} T) = (1/T) \delta(C^{(0)}) \) which removes the factor of \( T \) in the ghost integration and also gets rid of the \( C^{(0)} (K_1 - K_0) \) term. The final form of the propagator is

\[
< \beta_{+1}^{(0)}, K_1, T | \beta_{+0}^{(0)}, K_0, 0 > = -i \prod_n \left( \frac{n^2 \pi^2}{4} \right) \left\{ \left( \prod_{\ell} \frac{1}{\ell \pi^2} \right)^4 \right\} \times
\]

\[
 \times \int_{-\infty}^{\infty} \frac{dp_{+}^{(0)}}{2\pi} \frac{dp_{-}^{(0)}}{2\pi} e^{ip_{+}^{(0)} (\beta_{+1}^{(0)} - \beta_{+0}^{(0)})} e^{ip_{-}^{(0)} (\beta_{-1}^{(0)} - \beta_{-0}^{(0)})}
\]
\[
N \delta (\beta_+^{(0)} - \beta_0^{(0)}) \delta (\beta_-^{(0)} - \beta_0^{(0)}).
\] (3.12)

The constant normalization \(N\) can be treated as one over the Jacobian of the transformation from skeletonization coordinates to Fourier-series coordinates on the space of paths as Feynman does for the case of the harmonic oscillator.

Notice that the Fourier series coordinates are simpler to use than skeletonization for linear equations of motion because the coefficients are nicely grouped to give \(\delta\)-functions that kill terms on further integration. Unfortunately, this simplicity disappears for more complicated motions unless one can calculate explicitly the Fourier series for the motion and integrate easily over the resulting coefficients.

The propagator above can be compared to the Green function for the Schrödinger quantization of the system, and they agree, since the solution for \(\Psi(\beta_+^{(0)}, K, \alpha) = \Psi(\beta_-^{(0)})\), that is, an arbitrary function of \(\beta_-^{(0)}\) that is independent of time. This wave function is perhaps the ultimate in frozen dynamics, but, of course, this fact means little until one knows how to map such a wave function into a true Hilbert space function. In principle one must develop a kernel such as those mentioned in Sec. 2 to map \(\Psi(\beta_-^{(0)})\) to \(\psi(\beta_-, \alpha)\), but we will leave this to future work. Here we will appeal to an argument similar that used in our Bianchi-V work, where we argued that the \(\partial/\partial t\) that appears in the Schrödinger equation is a partial derivative that implies holding certain variables constant\(^1\). However, since the variables \(\beta_-^{(0)}\) depend on \(\alpha\) implicitly, the function \(\Psi\) can depend on \(\alpha\) through this dependence. That is, since \(\beta_-^{(0)} = \beta_- - (p_-^{(0)}/\sqrt{p_-^{(0)2} + p_+^{(0)2}}) \alpha\), \(\Psi\) can be written as

\[
\Psi = \Psi \left( \beta_- - \frac{p_-^{(0) \alpha}}{\sqrt{p_-^{(0)2} + p_+^{(0)2}}} \right).
\] (3.13)

In fact, the product of eigenstates of \(p_-\) with eigenvalue \(p_-^{(0)}\) becomes

\[
\Psi = e^{i(p_-^{(0)} \beta_- + p_+^{(0)} \beta_+) - \sqrt{p_-^{(0)2} + p_+^{(0)2}} \alpha},
\] (3.14)

which is the solution found by Charlie in his original study of quantum cosmology\(^9\).

As we mentioned in the Introduction, we have not attempted to carry this path integral formulation too far. We have only tried to give the flavor of the quantization of the gravitational field using non-standard phase space variables by presenting this simple model. Path integral quantization in non-standard phase space variables still faces many problems, especially for systems that do not come from a variational principle, where even the definition of the process will require new ideas, and it may even be impossible to achieve a consistent theory.
REFERENCES
1. S. Hojman, D. Núñez, and M. Ryan, Phys. Rev. D 45, 3523 (1992).
2. V. Bargmann, Comm. Pure and App. Math. 14, 2960 (1961).
3. F. Dyson, Am. J. Phys. 58, 209 (1990).
4. S. Hojman and L. Shepley, J. Math. Phys. 32, 142 (1991)
5. S. Hojman, in press
6. See, for example, D. Simms and N. Woodhouse, Lectures on Geometric Quantization (Springer, Berlin, 1977);
   N. Woodhouse, Geometric Quantization (Oxford U. P., Oxford, 1981).
7. A. Ashtekar et al., New Perspectives in Canonical Gravity (Bibliopolis, Naples, 1988).
8. V. Fock, Z. Phys. 49, 339 (1928).
9. C. Misner, Phys. Rev. 186, 1319 (1969).
10. B. Berger and C. Voegli, Phys. Rev. D 32, 2477 (1985).
11. See J. Guven and M. Ryan, Phys. Rev. D 45, 3559 (1992), and references therein.
12. L. Fradkin and G. Vilkovisky, CERN Report TH 2332 (1977); I. Batalin and G. Vilkovisky, Phys. Lett. 69B, 309 (1977).
13. F. Berezin, The Method of Second Quantization (Academic Press, New York, 1966).