Power structures over the Grothendieck ring of maps

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Abstract

A power structure over a ring is a method to give sense to expressions of the form \((1 + a_1 t + a_2 t^2 + \ldots)^m\), where \(a_i, i = 1, 2, \ldots\), and \(m\) are elements of the ring. The (natural) power structure over the Grothendieck ring of complex quasi-projective varieties appeared to be useful for a number of applications. We discuss new examples of \(\lambda\)- and power structures over some Grothendieck rings of varieties. Mostly we consider them over the Grothendieck ring of maps of complex quasi-projective varieties. We describe three natural \(\lambda\)-structures on it and determine which of them define an effective power structure over this ring.

1 Introduction

A \(\lambda\)-structure (called sometimes a pre-\(\lambda\)-structure) on a ring \(R\) is an additive-to-multiplicative homomorphism \(R \rightarrow 1 + tR[[t]]\) ([13]). A power structure over a ring \(R\) ([9]) is a method to give sense to expressions of the form \((1 + a_1 t + a_2 t^2 + \ldots)^m\), where \(a_i, i = 1, 2, \ldots\), and \(m\) are elements of the ring \(R\) (also as a series from \(1 + tR[[t]]\)). The notions of \(\lambda\)-structures and
power structures are closely related to each other, but are not equivalent. In particular, each \( \lambda \)-structure on a ring defines a power structure over it, but there are, in general, many \( \lambda \)-structures corresponding to one and the same power structure. A “natural” power structure over the Grothendieck ring \( K_0(\text{Var}_C) \) of complex quasi-projective varieties (see, e.g., [9]) was described in [9]. Its versions for the relative case (i.e., over the Grothendieck ring of complex quasi-projective varieties over a fixed variety) were defined in [11].

A power structure over the Grothendieck ring of complex quasi-projective varieties over an abelian monoid was defined in [14]. (A particular case of the Grothendieck ring of varieties over an abelian monoid when the monoid is the abelian group \( \mathbb{C} \) was considered in [3, 3, 16] under the name Grothendieck rings of varieties with exponentials.) Power structures over Grothendieck ring of varieties appear to be useful, in particular, for formulation and proof of formulae for the generating series of classes of some configuration spaces or of their invariants, see, e.g., [10, 1, 2, 14].

An important property of the power structure over the Grothendieck ring \( K_0(\text{Var}_C) \) which makes it useful for the mentioned applications is its effectiveness. This means that if all the coefficients \( a_i \) of the series \( A(t) = 1 + a_1 t + a_2 t^2 + \ldots \) and the exponent \( m \) are classes of complex quasi-projective varieties (not of virtual ones: differences of such classes), then all the coefficients of the series \( (1 + a_1 t + a_2 t^2 + \ldots)^m \) are also represented by classes of complex quasi-projective varieties. This is a somewhat special property of this power structure. Another natural power structure over the Grothendieck ring \( K_0(\text{Var}_C) \) (in fact up to now only two power structures over this ring are known) and also its natural “extension” to the Grothendieck ring of stacks ([12]) are not effective.

Here we discuss new examples of \( \lambda \)- and power structures over some Grothendieck rings of varieties. Mostly we consider them over the Grothendieck ring of maps of complex quasi-projective varieties. We describe three natural \( \lambda \)-structures on it and determine which of them define an effective power structure over this ring.

\section{Power structures and \( \lambda \)-structures}

A \emph{power structure} over a ring \( R \) is a way to give sense to expressions of the form \( (A(t))^m \), where \( A(t) = 1 + a_1 t + a_2 t^2 + \ldots \) is a formal series with the coefficients \( a_i \) from \( R \), the exponent \( m \) is also an element of \( R \).
Definition: ([9]) A power structure over a ring $R$ with unity is a map

$$(1 + tR[[t]]) \times R \to 1 + tR[[t]]$$

$$(A(t), m) \mapsto (A(t))^m$$

which possesses the properties of the exponential function, namely:

1. $(A(t))^0 = 1$,
2. $(A(t))^1 = A(t)$,
3. $(A(t) \cdot B(t))^m = (A(t))^m \cdot (B(t))^m$,
4. $(A(t))^{m+n} = (A(t))^m \cdot (A(t))^n$,
5. $(A(t))^m = ((A(t))^n)^m$,
6. $(1 + t)^m = 1 + mt + \text{terms of higher degree}$,
7. $(A(t^k))^m = (A(t))^m|_{t \to t^k}$.

Remark. In [9] the properties 6 and 7 were not demanded, though the constructed power structures possessed them.

Definition: A power structure over a ring $R$ is finitely determined if the fact that two series $A_1(t)$ and $A_2(t)$ from $1 + tR[[t]]$ differ by terms of degree $\geq k$ (i.e., $A_1(t) - A_2(t) \in \mathfrak{m}^k$, where $\mathfrak{m} = (t)$ is the maximal ideal in $R[[t]]$) implies that $(A_1(t))^m - (A_2(t))^m \in \mathfrak{m}^k$.

A natural power structure over the ring $\mathbb{Z}$ of integers (and the only one) is defined by the usual equation for an exponent of a series (see, e.g., [15], page 40)

$$1 + a_1t + a_2t^2 + \ldots)^m =$$

$$1 + \sum_{k=1}^{\infty} \left( \sum_{\{k_i\} : \sum_i k_i = k} \frac{m(m-1) \ldots (m - \sum_i k_i + 1) \times \prod_i a_i^{k_i}}{\prod_i k_i!} \right) \cdot t^k.$$

(1)

(Of course this power structure is finitely determined.)
The power structure over the Grothendieck ring $K_0(\text{Var}_C)$ of complex quasi-projective varieties defined in [9] is given by the equation

$$(1 + [A_1]t + [A_2]t^2 + \ldots)^{[M]} =$$

$$= 1 + \sum_{k=1}^{\infty} \left( \sum_{\{k_i\}, \sum_i k_i = k} \left( \left( \sum_i M^{\Sigma_i k_i} \backslash \Delta \right) \times \prod_i A_i^{k_i} \right) / \prod_i S_{k_i} \right) \cdot t^k,$$  \hspace{1cm} (2)

where $[A_i]$, $i = 1, 2, \ldots$, and $[M]$ are classes in $K_0(\text{Var}_C)$ of complex quasi-projective varieties, $\Delta$ is the “large diagonal” in $M^{\Sigma_i k_i}$, i.e., the set of (ordered) collections of $\sum_i k_i$ points from $M$ with at least two coinciding ones, the group $S_{k_i}$ of permutations on $k_i$ elements acts by simultaneous permutations on the components of the corresponding factor $M^{k_i}$ in $M^{\Sigma_i k_i} = \prod_i M^{k_i}$ and on the components of the factor $A_i^{k_i}$.

Except the Grothendieck ring of complex quasi-projective varieties one can consider the Grothendieck semiring $S_0(\text{Var}_C)$. It is defined in the same way as $K_0(\text{Var}_C)$ with the word group substituted by the word semigroup. Elements of the semiring $S_0(\text{Var}_C)$ are represented by “genuine” complex quasi-projective varieties, not by virtual ones (i.e., formal differences of varieties). Two complex quasi-projective varieties $X$ and $Y$ represent the same element of the semiring $S_0(\text{Var}_C)$ if and only if they are piece-wise isomorphic, i.e., if there exist decompositions $X = \coprod_i X_i$ and $Y = \coprod_i Y_i$ into Zariski locally closed subsets such that $X_i$ and $Y_i$ are isomorphic for $i = 1, \ldots, s$. One has a natural map (a semiring homomorphism) from $S_0(\text{Var}_C)$ to $K_0(\text{Var}_C)$.

A power structure over the Grothendieck ring $K_0(\text{Var}_C)$ is called effective if the fact that all the coefficients $a_i$ of the series $A(t)$ and the exponent $m$ are represented by classes of complex quasi-projective varieties (i.e., belong to the image of the map $S_0(\text{Var}_C) \to K_0(\text{Var}_C)$) implies that all the coefficients of the series $(A(t))^m$ are also represented by such classes. Roughly speaking this means that the power structure can be defined over the Grothendieck semiring $S_0(\text{Var}_C)$. The same concept is used for Grothendieck rings of complex quasi-projective varieties with additional structures. The effectiveness of the described power structure over the Grothendieck ring $K_0(\text{Var}_C)$ is clear from Equation (2).

An equation similar to (2) was given in [14] for a power structure over the Grothendieck ring of complex quasi-projective varieties over an abelian monoid used there.

Power structures over a ring are related to $\lambda$-structures on it. Let $R$ be a ring with a $\lambda$-structure, i.e., for each $a \in R$ there is defined a series
\( \lambda_a(t) = 1 + at + \ldots \in 1 + tR[[t]] \) so that \( \lambda_{a+b}(t) = \lambda_a(t)\lambda_b(t) \) (in other words one has an additive-to-multiplicative homomorphism \( R \to 1 + tR[[t]] \)); see, e.g., [13]. The \( \lambda \)-structure \( \lambda_a(t) \) defines the (finitely determined) power structure over \( R \) in the following way. Any power series \( A(t) \in 1 + tR[[t]] \) can be in a unique way represented as the product \( A(t) = \prod_{i=1}^{\infty} \lambda_{b_i}(t^i) \) with \( b_i \in R \).

Then one defines the series \((A(t))^m\) by

\[
(A(t))^m := \prod_{i=1}^{\infty} \lambda_{mb_i}(t^i). \tag{3}
\]

**Remark.** A \( \lambda \)-structure on a ring \( R \) also defines maps \( \text{Exp} : tR[[t]] \to 1 + tR[[t]] \) and \( \text{Log} : 1 + tR[[t]] \to tR[[t]] \) (inverse to each other) in the following way:

\[
\text{Exp}(b_1t + b_2t^2 + \ldots) := \prod_{k \geq 1} \lambda_{b_k}(t^k);
\]

if \( 1 + a_1t + a_2t^2 + \ldots = \prod_{k=1}^{\infty} \lambda_{b_k}(t^k) \), then

\[
\text{Log}(1 + a_1t + a_2t^2 + \ldots) := \sum_{k=1}^{\infty} b_k t^k.
\]

The map \( \text{Exp} \) is an additive-to-multiplicative homomorphism. The map \( \text{Log} \) is a multiplicative-to-additive homomorphism. Each of these maps determines the \( \lambda \)-structure on the ring, see e.g. [9].

One can show that the power structure (2) over the Grothendieck ring \( K_0(\text{Var}_C) \) corresponds to the \( \lambda \)-structure on it defined by the **Kapranov zeta function**

\[
\zeta_{[X]}(t) = 1 + [X]t + [S^2X]t^2 + [S^3X]t^3 + \ldots,
\]

where \( S^kX = X^k/S_k \) is the \( k \)th symmetric power of the variety \( X \). In terms of the power structure one has \( \zeta_{[X]}(t) = (1 + t + t^2 + \ldots)^{[X]} = (1 - t)^{-[X]} \).

There are many \( \lambda \)-structures corresponding to one and the same power structure over a ring \( R \). For any series \( \lambda_1(t) = 1 + a_2t^2 + \ldots \) the equation

\[
\lambda_a(t) := (\lambda_1(t))^a
\]

is satisfied.
gives a \( \lambda \)-structure on the ring \( R \). For example, except the Kapranov zeta function the power structure \([2]\) over \( K_0(\text{Var}_C) \) can be defined by the \( \lambda \)-structure

\[
\lambda_{[X]}(t) := (1 + t)^{|X|} = 1 + [X]t + [B_2X]t^2 + [B_3X]t^3 + \ldots,
\]

where \( B_kX := (X^k \setminus \Delta)/\Sigma_k \) is the configuration space of \( k \) distinct unordered points of \( X \).

Another “natural” \( \lambda \)-structure on the Grothendieck ring \( K_0(\text{Var}_C) \) (opposite to the one defined by the Kapranov zeta function \( \zeta_{[X]}(t) \)) is defined by the series \( \zeta_{[X]}(-t) \). One can show that the corresponding power structure over the ring \( K_0(\text{Var}_C) \) is not effective (see \([12]\)). (It seems that one knows no power structures over the ring \( K_0(\text{Var}_C) \) except the described two.)

Let \( R_1 \) and \( R_2 \) be rings with power structures over them. A ring homomorphism \( \varphi : R_1 \to R_2 \) induces the natural homomorphism \( R_1[[t]] \to R_2[[t]] \) (also denoted by \( \varphi \)) by \( \varphi(\sum_i a_it^i) = \sum_i \varphi(a_i)t^i \). One has the following statement.

**Proposition 1** \([10]\) If a ring homomorphism \( \varphi : R_1 \to R_2 \) is such that

\[
(1 - t) - \varphi(a) = \varphi((1 - t)^{-a}), \quad \text{then } \varphi((A(t))^m) = (\varphi(A(t)))^{\varphi(m)}.
\]

Equations written in terms of the power structure \([2]\) give equations for the Euler characteristics with compact support \( \chi(\bullet) \) and for the Hodge-Deligne polynomial \( e_\bullet(u,v) \) via the natural homomorphisms \( \chi : K_0(\text{Var}_C) \to \mathbb{Z} \) and \( e : K_0(\text{Var}_C) \to \mathbb{Z}[u,v] \). These homomorphisms are compatible with the power structures over the rings \( \mathbb{Z} \) (see Equation \([11]\)) and \( \mathbb{Z}[u,v] \), where the power structure over the latter one is defined as follows.

Let \( \mathbb{Z}[u_1, \ldots, u_r] \) be the ring of polynomials in \( r \) variables. Let \( P(u_1, \ldots, u_r) = \sum_{k \in \mathbb{Z}_{\geq 0}} p_k u_k \in \mathbb{Z}[u_1, \ldots, u_r] \), where \( k = (k_1, \ldots, k_r) \), \( u = (u_1, \ldots, u_r) \), \( u_k = u_1^{k_1} \cdots u_r^{k_r} \), \( p_k \in \mathbb{Z} \). Let

\[
\lambda_P(t) := \prod_{k \in \mathbb{Z}_{\geq 0}} (1 - u_k^t)^{-p_k},
\]

where the power (with an integer exponent \(-p_k\)) means the usual one. The series \( \lambda_P(t) \) defines a \( \lambda \)-structure on the ring \( \mathbb{Z}[u_1, \ldots, u_r] \) and therefore a power structure over it (with \( \lambda_P(t) = (1 - t)^P \)).
Let \( r = 2, u_1 = u, u_2 = v \). Let \( e : K_0(\text{Var}_\mathbb{C}) \to \mathbb{Z}[u,v] \) be the ring homomorphism which sends the class \([X]\) of a quasi-projective variety \( X \) to its Hodge–Deligne polynomial \( e_X(u,v) = \sum_{i,j} h_{ij}^X(-u)^i(-v)^j \). One can see that the homomorphism \( e \) respects the \( \lambda \)- and therefore the power structures over the source and over the target. This is shown in [4,10]: in terms of the power structures Proposition 1.2 therein can be rewritten as

\[
e \left((1 - t)^{-[X]}\right) = (1 - t)^{-e_X(u,v)}.
\]

3 Grothendieck ring of maps

Let us consider (regular) maps \( f : X \to Y \) between complex quasi-projective varieties.

Definition: Maps \( f : X \to Y \) and \( f' : X' \to Y' \) between complex quasi-projective varieties are equivalent if there exist isomorphisms \( h_1 : X \to X' \) and \( h_2 : Y \to Y' \) such that \( h_2 \circ f = f' \circ h_1 \).

The definition of the Grothendieck ring of maps almost literally repeats the one for the Grothendieck ring of varieties \( K_0(\text{Var}_\mathbb{C}) \). The only difference is the following one. Since \( \emptyset \times X = \emptyset \) for any \( X \), if one adapts the standard definition of the Grothendieck ring of varieties to maps (relations 1) and 2) below), one gets a sort of a degenerate ring with a lot of “obvious” divisors of zero. To avoid this inconvenience, it is reasonable to identify the classes of the maps \( \emptyset \to X \) with zero. (It is possible to say that one considers only surjective maps of varieties.)

Definition: The Grothendieck ring \( K_0(\text{Map}_\mathbb{C}) \) of maps between complex quasi-projective varieties is the free abelian group generated by the classes \([X \xrightarrow{f} Y]\) modulo the relations:

1) if two maps \( f : X \to Y \) and \( f' : X' \to Y' \) are equivalent, then \([X \xrightarrow{f} Y] = [X' \xrightarrow{f'} Y]\);

2) if \( f : X \to Y \) and \( Z \) is a Zariski closed subset of \( Y \), then

\[
[X \xrightarrow{f} Y] = [f^{-1}(Z) \xrightarrow{f|_{f^{-1}(Z)}} Z] + [f^{-1}(Y \setminus Z) \xrightarrow{f|_{f^{-1}(Y \setminus Z)}} Y \setminus Z];
\]

3) for every variety \( X \), \([\emptyset \to X] = 0\).
The second relation means that the summation in $K_0(\text{Map}_C)$ is defined by the disjoint union, that is

$$[X_1 \xrightarrow{f_1} Y_2] + [X_2 \xrightarrow{f_2} Y_2] := [X_1 \sqcup X_2 \xrightarrow{f_1 \sqcup f_2} Y_1 \sqcup Y_2].$$

The multiplication in $K_0(\text{Map}_C)$ is defined by the cartesian product:

$$[X_1 \xrightarrow{f_1} Y_2] \cdot [X_2 \xrightarrow{f_2} Y_2] := [X_1 \times X_2 \xrightarrow{f_1 \times f_2} Y_1 \times Y_2].$$

The unit $1$ in $K_0(\text{Map}_C)$ is represented by the identity map from a point to itself.

**Remark.** Since $X_{\text{red}}$ is a Zariski closed subset of $X$, then using 2) and 3) one has $[X \xrightarrow{id} X] = [X_{\text{red}} \xrightarrow{id} X_{\text{red}}]$.

The natural map from the Grothendieck ring $K_0(\text{Var}_C)$ of complex quasi-projective varieties to $K_0(\text{Map}_C)$: $[X] \mapsto [X \xrightarrow{id} X]$ is an injective ring homomorphism. Pay attention that the correspondence $X \mapsto (X \rightarrow \text{pt})$, where pt is a one point set, does not define a map from $K_0(\text{Var}_C)$ to $K_0(\text{Map}_C)$.

In the same way (substituting the word group by the word semigroup) one can define the Grothendieck semiring $S_0(\text{Map}_C)$ of maps between complex quasi-projective varieties. The elements in $S_0(\text{Map}_C)$ are represented by classes of genuine maps, not by virtual ones (that is differences of maps). The classes in $S_0(\text{Map}_C)$ of regular maps $f_1 : X_1 \rightarrow Y_1$ and $f_2 : X_2 \rightarrow Y_2$ are equal if and only if they are piece-wise isomorphic, i.e. if there exist partitions $Y_1 = \bigsqcup_{i=1}^n Y_{1,i}$ and $Y_2 = \bigsqcup_{i=1}^n Y_{2,i}$ such that the maps $f_1 : f_1^{-1}(Y_{1,i}) \rightarrow Y_{1,i}$ and $f_2 : f_2^{-1}(Y_{2,i}) \rightarrow Y_{2,i}$ are equivalent.

One can define the following invariants of elements of the Grothendieck ring $K_0(\text{Map}_C)$. For a map $f : X \rightarrow Y$ the Euler characteristic of the preimage $f^{-1}(y)$, $y \in Y$, is a constructible function on $Y$ with values in $\mathbb{Z}$. In this way one can define the Euler characteristic of an element $[f : X \rightarrow Y] \in K_0(\text{Map}_C)$ as the element $[Y, \chi(f^{-1}(\cdot))]$ of the Grothendieck ring of complex quasi-projective varieties over the group $\mathbb{Z}$. In the same way one defines the Hodge-Deligne polynomial of an element $[f : X \rightarrow Y] \in K_0(\text{Map}_C)$ as the element $[Y, e_{f^{-1}(\cdot)}(u, v)]$ of the Grothendieck ring of complex quasi-projective varieties over the group $\mathbb{Z}[u, v]$.

An interesting part of the Grothendieck ring of $K_0(\text{Map}_C)$ (useful, in particular, to show non-effectiveness of some power structures over this ring) consists of the classes of maps of zero-dimensional varieties: finite sets. Let us denote it by $K_0(f\text{-Map})$. (The part of Grothendieck ring $K_0(\text{Var}_C)$ of
complex quasi-projective varieties consisting of the classes of finite sets is isomorphic to the ring \( \mathbb{Z} \) of integers.) An element of \( K_0(\text{f-Map}) \) is defined by the numbers of points in the target with the preimages consisting of \( s \) points for \( s = 1, 2, \ldots \) (These numbers are integers, but may be negative.) It can be encoded by a (finite) expression (a function) of the form \( k_11^r + k_22^r + \ldots \). Moreover, in these notations the summation and the multiplication in the ring \( K_0(\text{f-Map}) \) are defined in the natural form: as the summation and the multiplication of the corresponding functions. One can see that in this way the ring \( K_0(\text{f-Map}) \) can be identified with the semigroup ring \( \mathbb{Z}[\tau][\mathbb{Z}_+ \cdot \cdot] \) of the semigroup of positive integers with respect to the multiplication. Instead of writing \( k_11^r + k_22^r + \ldots \), for short we shall sometimes write \( (k_1, k_2, \ldots) \).

4 \( \lambda \)-structures over the Grothendieck ring of maps

Let us describe “natural” \( \lambda \)-structures over the ring \( K_0(\text{Map}_C) \).

For a map \( f : X \to Y \), one has the natural map \( S^k f : S^k X \to S^k Y \) between the \( k \)th symmetric powers of \( X \) and \( Y \). (Pay attention that the map \( f \) does not define a map between the configuration spaces \( B_k X \) and \( B_k Y \) of \( k \) distinct points on \( X \) and \( Y \) respectively.)

Definition: The Kapranov zeta function of a map \( f : X \to Y \) is defined by

\[
\zeta_{[X \to Y]}(t) = 1 + \sum_{k \geq 1} [S^k X \xrightarrow{S^k f} S^k Y] \cdot t^k \in 1 + tK_0(\text{Map}_C)[[t]]. \tag{4}
\]

Proposition 2 The Kapranov zeta function defines a \( \lambda \)-structure on the ring \( K_0(\text{Map}_C) \).

Proof. It is necessary to show that, for two maps \( f_1 : X_1 \to Y_1 \) and \( f_2 : X_2 \to Y_2 \), one has

\[
\zeta_{[X_1 \sqcup X_2 \overset{f_1 \cup f_2}{\to} Y_1 \sqcup Y_2]}(t) = \zeta_{[X_1 \overset{f_1}{\to} Y_1]}(t) \cdot \zeta_{[X_2 \overset{f_2}{\to} Y_2]}(t).
\]

This follows from the following obvious relation

\[
S^k(X_1 \sqcup X_2) \xrightarrow{S^k(f_1 \cup f_2)} S^k(Y_1 \sqcup Y_2) = \bigsqcup_{i=0}^{k} \left( S^i X_1 \xrightarrow{S^i(f_1)} S^i Y_1 \right) \times \left( S^{k-i} X_2 \xrightarrow{S^{k-i}(f_2)} S^{k-i} Y_2 \right).
\]
For a map \( f : X \to Y \), let \( f^k : X^k \to Y^k \) be the corresponding map between the Cartesian powers of \( X \) and \( Y \) and let
\[
F(k)X := (f^k)^{-1}(Y^k \setminus \Delta) / S_k,
\]
where \( \Delta \) is the big diagonal in \( Y^k \), \( S_k \) acts in the natural way on \( X^k \supseteq (f^k)^{-1}(Y^k \setminus \Delta) \). (The variety \( F(k)X \) depends not only on \( X \), but on the map \( f \) as well.) One has the natural map \( F(k)f : F(k)X \to B_k Y \) to the configuration space \( B_k Y = (Y^k \setminus \Delta) / S_k \).

For a map \( f : X \to Y \) let
\[
\lambda'[X \to Y](t) := 1 + \sum_{k \geq 1} [F(k)X \to B_k Y] \cdot t^k \in 1 + tK_0(\text{Map}_\mathbb{C})[[t]].
\]

**Proposition 3** The series \( \lambda'[X \to Y](t) \) defines a \( \lambda \)-structure on the ring \( K_0(\text{Map}_\mathbb{C}) \).

**Proof.** It is necessary to show that, for two maps \( f_1 : X_1 \to Y_1 \) and \( f_2 : X_2 \to Y_2 \), one has
\[
\lambda'[X_1 \sqcup X_2 \to Y_1 \sqcup Y_2](t) = \lambda'[X_1 \to Y_1](t) \cdot \lambda'[X_2 \to Y_2](t).
\]
A \( k \)-tuple of distinct points of \( Y_1 \sqcup Y_2 \) consists of an \( i \)-tuple of distinct points of \( Y_1 \) and a \((k-i)\)-tuple of distinct points of \( Y_2 \), \( 0 \leq i \leq k \). Therefore
\[
F(k)(X_1 \sqcup X_2) \xrightarrow{F(k)(f_1 \sqcup f_2)} B_k(Y_1 \sqcup Y_2) = \bigcup_{i=0}^{k} \left( F(i)X_1 \xrightarrow{F(i)f_1} B_i Y_1 \right) \times \left( F(k-i)X_2 \xrightarrow{F(k-i)f_2} B_{k-i} Y_2 \right).
\]

For a map \( f : X \to Y \), one has the corresponding map \( B_k f : B_k X \to S^k Y \) from the configuration space of \( k \) distinct points on \( X \) to the \( k \)th symmetric power of the variety \( Y \). Let
\[
\lambda[X \to Y](t) := 1 + \sum_{k \geq 1} [B_k X \xrightarrow{B_k f} S^k Y] \cdot t^k \in 1 + tK_0(\text{Map}_\mathbb{C})[[t]].
\]

**Proposition 4** The series \( \lambda[X \to Y](t) \) defines a \( \lambda \)-structure on the ring \( K_0(\text{Map}_\mathbb{C}) \).
Proof. Just as in Propositions 2 and 3 for two maps \( f_1 : X_1 \to Y_1 \) and \( f_2 : X_2 \to Y_2 \), one has

\[
B_k (X_1 \sqcup X_2) \xrightarrow{B_k (f_1 \sqcup f_2)} S^k (Y_1 \sqcup Y_2) = \bigcup_{i=0}^k (B_i X_1 \xrightarrow{B_i f_1} S^i Y_1) \times (B_{k-i} X_2 \xrightarrow{B_{k-i} f_2} S^{k-i} Y_2).
\]

\( \square \)

It is useful to describe these \( \lambda \)-structures on the subring \( K_0(\text{f-Map}) \) of maps of finite sets. One has

\[
\zeta_{s^\tau}(t) = 1 + \sum_{k=1}^\infty \frac{(s+k-1)}{k-1} t^k;
\]

\[
\lambda'_{s^\tau}(t) = 1 + s^\tau t;
\]

\[
\lambda_{s^\tau}(t) = 1 + \sum_{k=1}^s \binom{s}{k} t^k.
\]

5 Non-effective power structures over the ring \( K_0(\text{Map}_C) \)

We have described three \( \lambda \)-structures on the Grothendieck ring \( K_0(\text{Map}_C) \) of maps. We will show that only one of them defines an effective power structure over the ring \( K_0(\text{Map}_C) \). (Except these three \( \lambda \)-structures one has the corresponding opposite ones. However, they cannot be effective since the corresponding \( \lambda \)-structure on the subring \( K_0(\text{Var}_C) \subset K_0(\text{Map}_C) \) ([\( X \mapsto X \)] \( [X] \mapsto [X \xrightarrow{id} X] \)) is not effective.)

Theorem 1 The power structure over the ring \( K_0(\text{Map}_C) \) defined by the Kapranov zeta function \( \zeta_{[X \to Y]}(t) \) is not effective.

Proof. Using the definition 3, one can show that the coefficient at \( t^2 \) in the series

\[
(1 + [X_1 \xrightarrow{f_1} Y_1]t + [X_2 \xrightarrow{f_2} Y_2]t^2 + \ldots)^{[M \xrightarrow{f} N]}
\]

is equal to

\[
[S^2(M \times X_1) \xrightarrow{S^2(f \times f_1)} S^2(N \times Y_1)] + [M \times X_2 \xrightarrow{f \times f_2} N \times Y_2] - [M \times S^2 X_1 \xrightarrow{f \times S^2 f_1} N \times S^2 Y_1].
\]
Let us consider this power structure on the Grothendieck ring $K_0(f\text{-Map})$ of maps of finite sets: a subring of $K_0(\text{Map}_C)$. Let $[X_1 \overset{f_1}{\to} Y_1] = 1 (= 1^\tau)$, $[X_k \overset{f_k}{\to} Y_k] = 0$ for all $k \geq 2$, $[M \overset{f}{\to} N] = 2^\tau$. (This means that $X_1$ and $Y_1$ are one-point sets (with the only map between them) $X_k = Y_k = \emptyset$ for $k \geq 2$, $M$ is a two-points set and $N$ is a one-point set.) One can see that the coefficient at $t^2$ in the series $(1 + t)^{2\tau}$ is equal to

$$3^\tau - 2^\tau.$$

Obviously this element does not belong to the semiring $S_0(\text{Map}_C)$. □

Theorem 2 The power structure over the ring $K_0(\text{Map}_C)$ defined by the series $\lambda'_{[X \overset{f}{\to} Y]}(t)$ is not effective.

Proof. One can show that the power structure defined by the series $\lambda'_{[X \overset{f}{\to} Y]}(t)$ “is effective up to degree 3” in the following sense: for maps $f_k : X_k \to Y_k$ and $f : M \to N$ of complex quasi-projective varieties, the coefficients at $t^2$ and at $t^3$ in the series

$$(1 + [X_1 \overset{f_1}{\to} Y_1]t + [X_2 \overset{f_2}{\to} Y_2]t^2 + \ldots)[M \overset{f}{\to} N]$$

belongs to (the image of) the semiring $S_0(\text{Map}_C)$. It is somewhat borrowing to compute the coefficient at $t^4$ in this series in the general case. It is not difficult to compute the series (8) for the case when all the coefficients and the exponent are from the Grothendieck ring $K_0(f\text{-Map})$ of maps of finite sets using a computer program. Making a number of computations for different coefficients and the exponent in the series (8), one can find an example with the coefficient at $t^4$ not in the image of the semiring $S_0(\text{Map}_C)$. After that it is easy to check the computations manually. In this way one can show that

$$(1 + (0, 0, 1, 1, 0, 1, \ldots)t + (1, 1, 0, 1, 0, 0, \ldots)t^2 + (1, 0, 1, 0, 0, 1, \ldots)t^3 + (0, 0, 0, 0, 0, -1, \ldots)t^4 + \ldots)$$

(see the explanation of the notations at the end of Section (8)). The coefficient at $t^4$ does not belong to the semiring $S_0(\text{Map}_C)$. This proves the statement. □
6 An effective power structures over the ring $K_0(\text{Map}_\mathbb{C})$

**Theorem 3** The power structure over the ring $K_0(\text{Map}_\mathbb{C})$ defined by the series $\lambda_{[X \xrightarrow{f} Y]}(t)$ is effective.

**Proof.** To prove the statement we shall give a direct (geometric) description of the corresponding power structure. For a series

$$A(t) := 1 + [X_1 \xrightarrow{f_1} Y_1]t + [X_2 \xrightarrow{f_2} Y_2]t^2 + \ldots \in 1 + tK_0(\text{Map}_\mathbb{C})[[t]],$$

and for an element $m = [M \xrightarrow{f} N] \in K_0(\text{Map}_\mathbb{C})$, let us define $(A(t))^m$ by

$$1 + \sum_{k=1}^{\infty} \left( \sum_{k_1, \ldots, k_i = k} \left[ \left( M^{\sum_i k_i} \setminus \Delta \xrightarrow{f_1^{\sum_i k_i}} N^{\sum_i k_i} \right) \times \prod_i (X_i \xrightarrow{f_i} Y_i)^{k_i} / \prod_i S_k \right] \right)^{t^k},$$

where $k = \{ k_i : i \in \mathbb{Z}_{>0}, k_i \in \mathbb{Z}_{\geq 0} \}$, $\Delta$ is the ”large diagonal” in $M^{\sum_i k_i}$ which consists of $(\sum_i k_i)$-tuples of points of $M$ with at least two coinciding ones, the permutation group $S_k$ acts simultaneously on the components of the factors $M^{k_i}$ and $N^{k_i}$ in $M^{\sum_i k_i} \setminus \Delta$ and in $N^{\sum_i k_i}$ and on the components of $(X_i \xrightarrow{f_i} Y_i)^{k_i}$. (Pay attention that the action of $\prod_i S_k$ on the corresponding space is free.)

We have to show that

1) the equation (9) defines a power structure over the ring $K_0(\text{Map}_\mathbb{C})$;

2) the equation (9) gives

$$\lambda_{[M \xrightarrow{f} N]}(t) = (1 + t)^{[M \xrightarrow{f} N]}.$$

Equation (10) is obvious since the only non-empty summand in the coefficient at $t^k$ in $(1 + t)^{[M \xrightarrow{f} N]}$ corresponds to the partition $k_1 = k, k_i = 0$ for $i > 1$, and is represented by the map

$$(M^{k_1} \setminus \Delta) / S_k = B_k M \rightarrow N^{k} / S_k = S^k N.$$
The fact that Equation (9) defines a power structure over the Grothendieck ring $K_0(\text{Map}_C)$ follows from the interpretation of the coefficient at $t^k$ in it similar to the one in [9]. Let $\Gamma$ be the disjoint union $\bigsqcup_{i=1}^{\infty}X_i$ and let $I : \Gamma \to \mathbb{Z}$ be the “tautological function” on $\Gamma$ which send the component $X_i$ to $i$. A representative of the coefficient at $t^k$ in (9) can be identified with the configuration space of pairs $(K, \psi)$, where $K$ is a finite subset of $M$, $\psi$ is a map from $K$ to $\Gamma$ such that $\sum_{x \in K} I(\psi(x)) = k$. (It is possible to say that one considers collections of (non-coinciding) particles on $M$ with different charges and with the space of internal states of a particle with charge $n$ parametrized by the variety $X_i$. The coefficient at $t^k$ in (9) is represented by the configuration space of collections of particles with the total multiplicity $k$; cf. [7].) These data correspond to the source of the map. The target is represented by a similar configuration space of particles on $N$ whose locations are permitted to coincide and whose spaces of internal states are parametrized by the varieties $Y_i$. The map from the source to the target is defined in the obvious way (and is determined by the maps $f$ and $f_i$). One can see that in this setting appropriate modifications of the arguments in the proof of Theorem 1 in [9] give the statement.

Let $L_v \in K_0(\text{Map}_C)$ be the class of the map $\mathbb{A}_C^1 \to \text{pt}$ from the complex affine line to a one point set. (Do not mix $L_v$ with the the image $L$ of the element $L \in K_0(\text{Var}_C)$ under the natural embedding. The element $L \in K_0(\text{Map}_C)$ is the class of the identity map $\mathbb{A}_C^1 \xrightarrow{id} \mathbb{A}_C^1$ from the complex affine line to itself.)

For a non-singular quasi-projective surface $X$, let $\text{Hilb}^k_X$ be the Hilbert scheme of zero-dimensional subschemes of length $k$ in $X$. One has the natural map $\pi_k : \text{Hilb}^k_X \to S^kX$. Proposition 5

**Proposition 5** In the Grothendieck ring $K_0(\text{Map}_C)$ one has

$$1 + \sum_{k=1}^{\infty} [\text{Hilb}^k_X \to S^kX] \cdot t^k = \left( \prod_{i=1}^{\infty} (1 - L_{i-1}^v i^i) \right)^{[X]}$$

**Proof.** For a partition $k$ of $k$ ($k = (k_1, k_2, \ldots)$, $\sum_i i k_i = k$), let $S^kX \subset S^kX$ be the space of $k$-tuples of points of $X$ with $k_i$ points of multiplicity $i$. One has $S^kX = \bigsqcup_{k=\sum_i i k_i = k} S^kX$. The arguments of [8], essentially give the equation

$$[\pi_k^{-1}(S^kX) \to S^kX] = L_{i-1}^v \sum_{i=1}^{k_i} [S^i X \to S^i X]$$
in \( K_0(\text{Map}_C) \). Now the proof goes in the same way as the one in \[9\] Statement 4. □

7 Versions of the described power structure

One can see that analogues of the power structure on the Grothendieck ring of maps \( K_0(\text{Map}_C) \) defined by Equations (9) and (7) hold in the following settings.

1. The relative setting. The Grothendieck group \( K_0(\text{Map}_{C/S}) \) of maps over a variety \( S \) is defined as the Grothendieck group generated by the classes of commutative diagrams of the form

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow & & \downarrow \\
S & \xrightarrow{\text{id}} & S
\end{array}
\]

with the natural analogues of the relations 1)-3). The multiplication in \( K_0(\text{Map}_{C/S}) \) is defined as the fibre product over \( S \). All the maps in Equations (9) are considered over the variety \( S \) (cf. [11]).

2. The equivariant setting. For a finite group \( G \), the Grothendieck ring \( K_0^G(\text{Map}_{C/S}) \) of \( G \)-equivariant maps is defined as the Grothendieck ring generated by the classes \([X \xrightarrow{f} Y]\), where \( X \) and \( Y \) are complex quasi-projective \( G \)-varieties and \( f \) is a \( G \)-equivariant map.

3. The relative setting over an abelian monoid. Let \( \mathcal{M} \) be an abelian monoid with zero. Just as in the relative setting above, the Grothendieck group \( K_0(\text{Map}_{C/\mathcal{M}}) \) of maps over the monoid \( \mathcal{M} \) is defined as the Grothendieck group generated by the classes of commutative diagrams of the form

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow_{P_X} & & \downarrow_{P_Y} \\
\mathcal{M} & \xrightarrow{\text{id}} & \mathcal{M}
\end{array}
\]
The difference is in the definitions of the multiplication in $K_0(\text{Map}_C/\mathcal{M})$ and of the maps to $\mathcal{M}$ of the summands in Equations (9). The multiplication is defined via the map $\mathcal{M} \times \mathcal{M} \to \mathcal{M}$ applied to the usual cartesian product (with the target $\mathcal{M} \times \mathcal{M}$). To define the map to $\mathcal{M}$ on the summands of (9), it is useful to use consider them as configuration spaces of particles on $M$ with some charges and some weights. The weights of a particle $s \in M$ of charge $n$ (and thus of the internal state $\phi$ from the variety $X_n$) is defined as $np_M(s) + p_{X_n}(\phi)$, where $p_M$ and $p_{X_n}$ are the maps from the corresponding varieties to $\mathcal{M}$. The weight of a collection of particles is defined as the sum of the weights of the individual particles. Cf. [14].

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