Winning combinations of history-dependent games

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The Parrondo effect describes the seemingly paradoxical situation in which two losing games can, when combined, become winning [Phys. Rev. Lett. 85, 24 (2000)]. Here we generalize this analysis to the case where both games are history-dependent, i.e. there is an intrinsic memory in the dynamics of each game. New results are presented for the cases of both random and periodic switching between the two games.

I. INTRODUCTION

The Parrondo effect is the counter-intuitive situation whereby individually losing games somehow ‘cooperate’ to produce a winning game. In particular, these losing games can be combined randomly and yet the effect still emerges. The intriguing aspect is that randomness in this system is acting in a constructive way. Possible applications of this effect have been suggested in several fields including biogenesis, molecular transport, random walks and biological systems. Even in the social sciences, ‘winning’ models for investment have been reported.

Consider a gambling game in which the player has a time-dependent capital $X(t)$ where $t = 0, 1, 2, \ldots$, and whose evolution is determined by tossing biased coins. The rules as to which coins to toss, and hence the probability of winning, are determined by the history, i.e. the game is history-dependent. The game can be divided into three regimes: winning, losing and fair (for which $\langle X(t) \rangle$ is respectively an increasing, decreasing or constant function of $t$). Parrondo et al. considered combinations of such a history-dependent game B, as described above, and a simple biased coin toss (i.e. game A which is history-independent and hence has no memory). In Parrondo et al’s study, game A is defined by the probability $p$ of $X(t)$ increasing, where $p = \frac{1}{2} - \epsilon$. Hence game $A$ is a losing game for $\epsilon > 0$. Game B is defined by the probabilities of four biased coins: $\{p_1, p_2, p_3, p_4\}$. The particular coin played at a given time step depends upon the history of the game as shown in Table I. Parrondo et al showed that two losing games A and B can be combined to yield a winning game, if the games are alternated either periodically or at random.

The reason that Parrondo’s paradox arises for combined A-B games is that losing cycles in game B are effectively broken up by the memoryless behavior, or ‘noise’, of game A. The question therefore arises: what happens if both games are of type B, and hence have losing cycles? Can the losing cycle in one game break up the losing cycle in the other in order to produce ‘winning dynamics’? Since the answer is not obvious, and since the Parrondo effect promises to have a variety of applications, it is important to establish whether two history-dependent games will indeed produce a Parrondo effect. This provides the motivation for the present study.

In this paper, we generalize the analysis of Ref. to the case where both games are history-dependent, i.e. there is an intrinsic memory in the dynamics of each game. We find specific regimes which do indeed exhibit a Parrondo effect. New results are presented for the cases of both random and periodic switching between the two games. The paper is organized as follows. In Sec. II we investigate random combinations of two games of type B. In Sec. III we consider periodic combinations of such games. In Sec. IV we investigate the effect of varying the switching probability. Section V provides a summary.

II. RANDOM COMBINATIONS OF HISTORY-DEPENDENT GAMES

We now extend the analysis of Parrondo et al to the case of two history-dependent games of type B. We define $\{p_i - \epsilon\}$ and $\{q_i - \epsilon\}$ as the probability sets defining the B games and $\{r_i - \epsilon\}$ as the probability set defining the combined game. We follow Parrondo et al in only considering losing games which result by subtracting a small quantity $\epsilon$ from each of the probabilities that define a fair game. As in Ref., we can define a vector Markovian process $Y(t)$ based on the capital $X(t)$ as follows:

$$Y(t) = \begin{pmatrix} X(t) - X(t-1) \\ X(t-1) - X(t-2) \end{pmatrix}$$ (1)
If the two B games are combined randomly, the probability set for the combined game is given by:

\[ r_i = \alpha p_i + (1 - \alpha) q_i \]  

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The third condition in Eq. (3) now becomes:

\[ (2 - p_4 - q_4)(2 - p_3 - q_3) < (p_1 + q_1)(p_2 + q_2) \]  

Given that we require the initial games to be fair for \( \epsilon = 0 \), we can use the first two conditions in Eq. (3) to substitute for \( p_1 \) and \( q_1 \). Hence:

\[ (2 - p_4 - q_4)(2 - p_3 - q_3) < \left( \frac{(1 - p_2)(1 - p_3)}{p_2} + \frac{(1 - q_2)(1 - q_3)}{q_2} \right) (p_2 + q_2). \]  

A. Special case \( p_2 = p_3, q_2 = q_3 \)

In order to reduce the number of free variables so that the different regions of the parameter space can be displayed in a three dimensional figure, Parrondo et al. made the restriction \( p_2 = p_3 \). Here we are going to reduce the number of free variables by appealing to the first two conditions in Eq. (3). These conditions give \( p_1 \) and \( q_1 \) in terms of \( \{p_i\}, \{q_i\} \) \((j = 2, 3, 4)\) such that both games are fair when \( \epsilon = 0 \).

We choose a particular game \( \{p_i\} \) and then plot the regions in the parameter space \((q_2, q_3, q_4)\) which enclose all games \( \{q_i\} \) for which the Parrondo effect is observed. Initially we treat the special case introduced by Parrondo et al. using the parameter space \((q_2 = q_3, q_4)\), taking for the first B game:

\[ \{p_i\} = \left\{ \frac{9}{10}, \frac{1}{4}, \frac{1}{4}, \frac{7}{10} \right\} \]  

Rearranging Eq. (3) gives:

\[ q_1 = \frac{(1 - q_4)(1 - q_3)}{q_2} \]  

\( q_1 \) is a probability and is thus subject to the restriction \( 0 < q_1 < 1 \). Therefore in order to be physically realized, the game \( \{q_i\} \) must be restricted as follows:

\[ q_4 > 1 + \frac{q_2}{q_3 - 1} \]  

and hence, in the special case where \( q_3 = q_2 \):

\[ q_4 > 1 + \frac{q_2}{q_2 - 1}. \]  

From Eq. (3) the condition that defines the regions of the parameter space in which two fair games combine to yield a winning game is given by:

\[ q_4 \left\{ \begin{array}{ll} > 1 + \frac{(p_4 - 1)}{p_2} q_2 & \text{if } q_3 > q_2 \\ < 1 + \frac{(p_4 - 1)}{p_2} q_2 & \text{if } q_2 > q_3. \end{array} \right. \]  

Figure 1 depicts the regions of parameter space defined by Eqs. (10) and (11) for \( \{p_i\} \) given by Eq. (3). In particular, Fig. 1 shows the region in which two fair games combine to yield a winning game for some value of \( \epsilon > 0 \). In the appendix we derive an expression for the maximum value of \( \epsilon \) for which this remains true, \( \epsilon_{\text{max}} \).

Figure 2 shows \( \epsilon_{\text{max}} \), given by Eq. (A4) using the same game set \( \{p_i\} \) as in Fig. 1. The value of \( \epsilon_{\text{max}} \) is shown for all possible games \( \{q_i\} \) given \( \{p_i\} \) defined by Eq. (3). This plot demonstrates the robustness of the Parrondo effect in the present case of two history-dependent games.

B. General case \( p_2 \neq p_3, q_2 \neq q_3 \)

Now we drop Parrondo et al’s restriction to \( p_2 = p_3 \) and treat the general case. From Eq. (3):

\[ q_4 \left\{ \begin{array}{ll} < 1 + \frac{(p_4 - 1)}{p_2} q_2 & \text{if } q_3 < 1 + \frac{p_4 - 1}{p_2} q_2 \\ > 1 + \frac{(p_4 - 1)}{p_2} q_2 & \text{if } q_3 > 1 + \frac{p_4 - 1}{p_2} q_2. \end{array} \right. \]  

Figure 3 depicts the regions of parameter space defined by Eqs. (11) and (12) for \( \{p_i\} \) given by Eq. (3). Equation (11) defines two regions (labelled “I” and “II” in Fig. 3). Equation (12) excludes almost all of region I.

| \( Y(t) \) | State |
|------------|-------|
| \((-1, -1)\) | 1     |
| \((+1, -1)\) | 2     |
| \((-1, +1)\) | 3     |
| \((+1, +1)\) | 4     |

Table II: Labels for the four possible states of the Markovian process \( Y(t) \), where \( Y(t) \) is defined in terms of the capital \( X(t) \) as prescribed by Eq. (1).
in this case. An expression for $\epsilon_{\text{max}}$ in the general case is derived in the appendix, Eq. (A7), given that $\{q_i\}$ is a fair game for $\epsilon = 0$. In principle we could plot this over the 3D axes of Fig. 3. This would be the generalization of Fig. 2.

The original combination of game A and game B due to Parrondo et al (see Section I) represents a special case of our more general treatment. The game considered in Ref. 6 corresponds to combining $\{p_i\}$ as defined by Eq. (3) with $\{q_i\}$ as defined by Eq. (6). In Figures 1 and 2 the black dot represents Parrondo et al’s original game as described in Ref. 6 for which $\{q_1 = \frac{1}{2}, q_2 = \frac{1}{2}, q_3 = \frac{1}{2}, q_4 = \frac{1}{2}\}$. In Figures 1 and 2 the black dot represents Parrondo et al’s original game as described in Ref. 6 for which $\{q_1 = \frac{1}{2}, q_2 = \frac{1}{2}, q_3 = \frac{1}{2}, q_4 = \frac{1}{2}\}$.

III. PERIODIC COMBINATIONS OF HISTORY-DEPENDENT GAMES

Next we investigate periodic combinations of games. Rather than randomly selecting the game to be played at each time step, game $\{p_i\}$ is played $a$ times and then game $\{q_i\}$ is played $b$ times. This cycle is repeated periodically. Figure 4 shows the capital after 500 times steps, resulting from a combination of two games for a range of values of $a$ and $b$. The capital is greater if the games are switched more frequently, as found by Parrondo et al for the combination of a simple game A and a history-dependent game B. The analysis for the periodic case is more complex than for the random case because we can no longer appeal to a single game formed from a weighted average of two games.

Let the elements of the vector $u_i$, labelled $u_{ij}$, be the probability of the game being in state $j$ at time $t = i$. The evolution of the game from $u_i$ to $u_{i+a+b}$ can be described...
we can calculate the overall probability of a win at time steps $t = (a + b)i + n$ by taking the dot product of the stationary state of the transition matrix $T_n$, with a vector formed from the probabilities of each of the coins from the game played at that time step. These vectors are $p = (p_1 p_2 p_3 p_4)$ and $q = (q_1 q_2 q_3 q_4)$, where $p$ corresponds to $A$ in Eq. (13) and $q$ corresponds to $B$ in Eq. (14). The matrix to the right of the product in $T_n$ corresponds to the game that will be played at time step $t = (a + b)i + n$. Therefore, if the matrix to the right is $A$ we must take the dot product with $p$. If it is $B$, we must take the dot product with $q$.

An expression for the average probability $P_{\text{win}}$ of a win for the combined game, can thus be found by averaging over all possible cyclic permutations of $T_0$. The gradient, grad$[\langle X_c(t) \rangle]$, is then given by Eq. (13), as before. The resulting expressions are lengthy. Each set of values of $a$ and $b$ yields an expression for grad$[\langle X_c(t) \rangle]$ in terms of $\{p_i\}, \{q_i\}$, where $i = 1, 2, 3, 4$. These expressions are too complicated to set out here explicitly. However, we can numerically plot the analytic equivalent of Figure 4: this is what we have essentially done in Figure 5. The lines show the analytic prediction for the average capital after 500 time steps, $\langle X_c(500) \rangle$, found by multiplying Eq. (13) by 500. Each line corresponds to a slice through the surface in Fig. 4 at constant $b$. The error bars indicate one standard deviation on the mean over ten ensemble averages of the numerical game. Each ensemble average comprises an average over 50,000 individual runs.

We can see that the numerical and analytic results agree to within one standard deviation. This confirms that the equations generated by the analysis presented in this section are indeed correct. Thus, we have derived expressions for the robustness of the Parrondo effect when two history-dependent games are combined periodically.

### IV. VARYING THE SWITCHING PROBABILITY IN THE RANDOM CASE

We now examine the dependence of the capital on the switching probability in the case that the games are randomly combined. Figure 5 shows the capital after 500 iterations plotted against the probability per iteration $\alpha$ that the game $\{p_i\}$ will be chosen. The curve is symmetric and demonstrates that the capital is greatest if the games are switched with equal probability. When imple-
menting the games it is necessary to assign values to the results of the coin tosses at times $t = -2, -1$ in order to seed the game. This arbitrary choice introduces transients which can slightly bias the final results. However, by allowing the game to first run for 100 iterations, this effect can be eliminated.

The curve in Fig. 7 represents the capital predicted by Eq. (A3) plotted for all $\alpha$ with the same $\{p_i\}$ and $\{q_i\}$. The error bars show one standard deviation either side of the mean capital, averaged over an ensemble of 10 runs. The agreement between the theoretical curve and numerical data is therefore better than one standard deviation.

V. SUMMARY

We have demonstrated that the apparently paradoxical effect of two losing games combining to produce a winning game also applies to combinations of two history-dependent games. We derived expressions for the regions of the parameter space in which the effect is observed for both random and periodic combinations of these games.

We derived expressions for the gradient of the average capital and hence the robustness of the Parrondo effect for games combined randomly or periodically.

Our work has therefore expanded the understanding of the Parrondo effect by demonstrating its existence for new combinations of history-dependent games. We are now faced with the more general question as to what property of the constituent games guarantees that the Parrondo effect will be observed. In addition if we were to combine many games of a more general nature than those used to date, how could we predict whether the effect would emerge? We hope that the present work will stimulate further research on such questions, in addition to pursuing applications of the Parrondo effect itself.

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APPENDIX A: DERIVATION OF $\epsilon_{\text{max}}$ FOR RANDOM COMBINATIONS OF GAMES

In this appendix we derive expressions for the maximum value of $\epsilon$, $\epsilon_{\text{max}}$, for which two games which are fair for $\epsilon = 0$ combine to yield a winning game.

Let $P_{\text{win}}$ and $P_{\text{lose}}$ be the probabilities, per iteration for the combined game, that $X_c(t)$ will increase or decrease respectively. Then the gradient of the capital line for the combined game is given by:

$$\text{grad}(X_c(t)) = 2P_{\text{win}} - 1.$$ (A1)
One can derive the following expression for $P_{\text{win}}$ for a

\[ P_{\text{win}} = \frac{r_1(r_2 + 1 - r_4)}{(1 - r_4)(2r_1 + 1 - r_3) + r_1r_2}. \quad \text{(A2)} \]

Substituting for \( \{r_i\} \) from Eq. (B) yields:

\[ \text{grad}[(X_e(t))] = \frac{2 \times \ldots}{[(1 - \alpha)q_1 + \alpha p_1] \ldots \ldots [1 + (1 - \alpha)q_2 - (1 - \alpha)q_4 + \alpha (p_2 - p_4)] - 1} \]

Now we reintroduce $\epsilon$ via the transformations $p_i \rightarrow p_i - \epsilon$, $q_i \rightarrow q_i - \epsilon$ to obtain:

\[ \text{grad}[(X_e(t))] = \frac{2 \times \ldots}{[\alpha p_1 + (1 - \alpha)q_1 - \epsilon] \ldots \ldots [\alpha p_4 + (1 - \alpha)q_4 - \epsilon]} - 1 \quad \text{(A4)} \]

Any games \( \{p_i\}, \{q_i\} \) defined as above will be losing

\[ \text{grad}[(X_e(t))] = \frac{2 \times \ldots}{[\alpha p_1 + (1 - \alpha)q_1 - \epsilon] \ldots \ldots [\alpha p_4 + (1 - \alpha)q_4 - \epsilon]} - 1 \quad \text{(A4)} \]

which \( \text{grad}[(X_e(t))] = 0 \). We shall consider games combined with equal probability, therefore $\alpha = \frac{1}{2}$. Setting \( \text{grad}[(X_e(t))] \) equal to zero in Eq. (A4) gives:

\[ \epsilon_{\text{max}} = \frac{-4 + p_1 p_2 + 2 p_3 - p_3 p_4 + p_2 q_1 + \ldots}{2(4 + p_1 + p_2 - p_3 - p_4 + \ldots)} \quad \text{(A5)} \]

Appealing to the condition (Eq. (F)) that \( \{q_i\} \) is a fair game for $\epsilon = 0$ and in the special case where $q_2 = q_3$ and $p_2 = p_3$, this becomes:

\[ \epsilon_{\text{max}} = \frac{p_2 [1 + q_2 (1 + p_1 - p_4) - q_4] + \ldots}{2(1 - q_4) + \ldots} \quad \text{(A6)} \]

Similarly in the general case, $\epsilon_{\text{max}}$ is given by:

\[ \epsilon_{\text{max}} = \frac{p_2 [1 - q_4 + q_2 p_1 + q_3 (q_4 - 1)] + \ldots}{2 - 2 q_4 + \ldots} \quad \text{(A7)} \]

\[ \epsilon_{\text{max}} = \frac{2 q_2 (q_2 - q_3 - q_4) + \ldots}{2 q_2 (q_2 - q_3 -q_4) + \ldots} \quad \text{(A7)} \]

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