On asymptotic properties of high moments of compound Poisson distribution

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Abstract
We study moments $M_k(\lambda)$ of the sum of random variables $X_1 + \cdots + X_{N_\lambda}$, where $N_\lambda$ follows the Poisson probability distribution with mean value $\lambda$ and $(X_j)$ is a family of i.i.d. random variables also independent from $N_\lambda$. We obtain an explicit expression for the leading term of the asymptotic expansion of $M_k(x)$ as $k \to \infty$. We show that if $\lambda_k$ is much smaller or proportional to $k$, then asymptotic behavior of $M_k(\lambda_k)$ is determined by the exponential generating function of $X_j$ while in the asymptotic regime when $\lambda_k$ is much greater than $k$, the leading term of $k^{-1} \ln M_k(\lambda_k) - \ln \lambda_k$, $k \to \infty$ depends on the first non-zero moment of $X_j$ only.

As a consequence, we establish a concentration property of maximal vertex degree of large weighted random graphs. Another application is related with a variable that arises in the studies of high moments of large random matrices. Finally, regarding three particular cases of probability distribution of $X_j$, we comment on the asymptotic behavior of certain combinatorial polynomials, including the Bell polynomials of even partitions.

1 Introduction, main result and discussion

Compound Poisson distribution is widely used in a number of applications in various areas, with the majority of recent applications in financial and risk modeling (see monograph [25] and references therein). This distribution can be associated with a random variable

$$Y_\lambda = \sum_{j=1}^{N_\lambda} X_j,$$ (1.1)

where $(X_j)_{j \in \mathbb{N}}$ is a family of i.i.d. random variables and random variable $N_\lambda$ independent from $(X_j)$ follows the Poisson law with mean value $\lambda$ [20]. We denote by $E$ the mathematical expectation with respect to a measure generated by the family of random variables $(N_\lambda, (X_j)_{j \in \mathbb{N}})$. We assume that all moments of $X_j$ exist and denote $V_i = E X_j^i$, $i \in \mathbb{N}$.

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It is known that the moments of $Y_\lambda$ are given by the following expression,

$$M_k(\lambda) = EY_\lambda^k = k! \sum_{(l_1, l_2, \ldots, l_k)} \prod_{i=1}^{k} \frac{(\lambda V_i)^{l_i}}{(i!)^{l_i}l_i!}, \quad k \in \mathbb{N},$$

(1.2)

where the sum runs over all $l_i \geq 0$ such that $l_1 + 2l_2 + \cdots + kl_k = k$. In the case when the random variables $X_j$ are all equal to 1, relation (1.2) determines the one-variable Bell polynomials $B_k(x)$ [7],

$$M_k(V_i=1)(x) = B_k(x) = k! \sum_{(l_1, l_2, \ldots, l_k)} \prod_{i=1}^{k} \frac{x^{l_i}}{(i!)^{l_i}l_i!}, \quad x \in \mathbb{R}.$$ (1.3)

If $x = 1$, then the right-hand side of (1.3) coincides with the $k$-th Bell number $B_k(1) = B^{(k)}$ [2, 3, 5, 13, 28],

$$B^{(k)} = \sum_{(l_1, l_2, \ldots, l_k)} B^{(k)}_{l_1,l_2,\ldots,l_k}, \quad B^{(k)}_{l_1,l_2,\ldots,l_k} = \frac{k!}{(1!)^{l_1}l_1!(2!)^{l_2}l_2!\cdots (k!)^{l_k}l_k!}.$$ (1.4)

The Bell number $B^{(k)}$ counts the number of all possible partitions $\pi(k)$ of the set of $k$ elements into non-empty subsets (blocks). Then $l_i$ represents the number of blocks of size $i$ and the sum $l_1 + \cdots + l_k = |\pi(k)|$ is the total number of blocks in the partition $\pi(k)$. According to this, we can say that polynomials $M_k(x)$ (1.2) represent a generalization of $B_k(x)$. This generalization can be referred as to the Bell polynomials of weighted partitions or simply as to the weighted Bell polynomials.

Limiting behavior of the Bell numbers $B^{(k)}$ has been studied since 50-s [13, 22, 29] (see also monograph [13]) while questions related with asymptotic properties of Bell polynomials represent more recent area of researches [10, 11, 31]. In paper [8], moments (1.2) of compound Poisson distribution have been studied and recurrent relations have been established for $M_k(\lambda)$ with given $k$.

It should be noted that probabilities of large deviations for sums of random number of i.i.d. random variables of the form (1.1) have been studied in the limit $\lambda \to \infty$ [21]. However, these results were obtained with no reference to the moments $M_k(\lambda)$ of the corresponding compound Poisson process. The use the high moments $M_k(\lambda)$ with increasing $k$ can be helpful also in studies of the concentration properties of sums $Y_\lambda^{(i)}$, $\lambda \to \infty$ (1.1) and their maxima that is in close relation with the maximal vertex degree of weighted random graphs. Also, the moments of $Y_\lambda$ mimic certain elements arising in studies of high moments of large random matrices of the Wigner ensemble. Here, the limiting transition when $k$ and $\lambda$ tend simultaneously to infinity is naturally motivated. Recently, high moments of the compound Poisson probability distribution have been used as upper bounds of cumulants of number of weighted walks on large random graphs [19]. We briefly discuss some of these questions at the end of the paper.

Up to our knowledge, asymptotic properties of weighted versions of Bell polynomials $M_k(x_k)$ with infinitely increasing $k$ have not been yet considered and the main statement
proved of this paper is new. formulate our results under assumption that there exists finite or infinite \( u_0 > 0 \) such that the following series converges,

\[
\forall u \in [0, u_0) : H(u) = \sum_{k=0}^{\infty} \frac{V_k}{k!} u^k < +\infty.
\] (1.5)

Function \( H(u) \) is known as the moment generating function of the probability distribution \( P_X \), \( H(u) = \mathbb{E} e^{uX} \). Clearly, all the derivatives of \( H(u) \) exist for all \( u \in [0, u_0) \) and the first derivative \( H'(u) \) is strictly increasing on \((0, u_0)\) [1]. In one of the three asymptotic regimes considered in this paper, we impose the following additional conditions on \( H(u) \):

i) there exists \( 0 < \alpha < 1/2 \) such that

either \( \frac{H(u + 1)}{(H''(u))^{1+\alpha}} \to 0, \ u \to \infty \) or \( \frac{H((u_0 + u)/2)}{(u_0 - u)^3(H''(u))^{1+\alpha}} \to 0, \ u \to u_0 \) (1.6)

and

\[
u^4 (H''(u))^{3+2\alpha} \ll H(u)^4, \ 	ext{when either } u \to \infty \text{ or } u \to u_0; \quad (1.7)
\]

ii) there exists positive \( w_0 \) and \( \beta \) such that

\[
\mathbb{E} e^{uW} 1_{\{W \geq w_0\}} \geq \beta H(u).
\] (1.8)

In our studies, we distinguish three major asymptotic regimes in dependence whether \( x_k \) is much less, proportional or much greater than \( k \) when \( k \) tends to infinity.

**Theorem 1.1.** Consider a sequence \((x_k)_{k \in \mathbb{N}}\) such that \( x_k = \chi_k, \ \chi > 0 \). Let \( u > 0 \) be determined by equation

\[
u H'(u) = \frac{1}{\chi}.
\] (1.9)

Then the following asymptotic equality is true,

\[
M_k(x_k) = \left( \frac{k}{u} \exp \left\{ \frac{H(u) - 1}{uH'(u)} - 1 \right\} \right)^k \frac{1 + o(1)}{\sqrt{\chi u(H'(u) + uH''(u))}}, \ k \to \infty.
\] (1.10)

If \((x_k)_{k \in \mathbb{N}}\) is such that \( x_k = \chi_k k \) with \( \chi_k \to \infty \) as \( k \to \infty \), then relation (1.10) is true with \( u \) replaced by \( u_k \) determined by equation

\[
u_k H'(u_k) = \frac{1}{\chi_k}, \ k \in \mathbb{N}.
\] (1.11)

If conditions (1.6), (1.7) and (1.8) are satisfied, then relation (1.10) remains valid in the limiting transition \( k \to \infty \) such that \( \chi_k \to 0 \) with \( x_k \) finite or tending to infinity and \((u_k)_{k \in \mathbb{N}}\) determined by (1.11).

**Remark.** Relation (1.10) remains valid also in the limit with vanishing \( x_k \) such that the following version of (1.6) holds,

either \( \frac{H(u_k + 1)}{\sqrt{x_k(H''(u_k))^{1+\alpha}}} \to 0 \) or \( \frac{H((u_0 + u_k)/2)}{\sqrt{x_k(u_0 - u_k)^3(H''(u_k))^{1+\alpha}}} \to 0, \ k \to \infty\).
with $u_k$ still determined by (1.11).

Theorem 1.1 generalizes results obtained in [18] for the one-variable Bell polynomials (1.3) with constant weight $X_j = 1$. To better see the novelties due to $X_j$, we present the following three corollaries of Theorem 1.1.

**Corollary A.** Let us consider first the asymptotic regime when $1 \ll k \ll x_k$. In this case $\chi_k \to \infty$ and we denote this limiting transition by

$$(k, x_k) \chi \to \infty. \quad (1.12)$$

It is not hard to deduce from (1.9) that in this case $u_k \to 0$ and Theorem 1.1 implies that

$$M_k(x_k) = \begin{cases} (x_k V_1 (1 + o(1)))^k, & \text{if } V_1 \neq 0, \\ \left(\frac{x_k V_2}{e} (1 + o(1))\right)^{k/2}, & \text{if } V_1 = 0. \end{cases} \quad (1.13)$$

Let us note that if the probability distribution of $X_j$ is symmetric, then Theorem 1.1 holds for the even moments $M_{2k}(x)$ only.

In the case of constant $X_j = 1$, the first relation of (1.13) has been obtained in [18] for Bell polynomials (1.3). The second is obtained for the case of restricted Bell polynomials [18] that concern the partitions (1.4) that have no blocks of size one.

**Corollary B.** Let us consider the second asymptotic regime when $x_k/k \to \chi$ as $k \to \infty$. We denote this limiting transition by

$$(k, x_k)_\chi \to \infty. \quad (1.14)$$

It follows from Theorem 1.1 that in this asymptotic regime

$$M_k(x_k) = \left(x_k e^{\Psi(\chi)/(1 + o(1))}\right)^k, \quad (k, x_k)_\chi \to \infty, \quad (1.15)$$

where

$$\Psi(\chi) = \frac{H(u) - 1}{u H'(u)} - 1 + \ln H'(u), \quad (1.16)$$

and $u$ is given by a solution of equation (1.9).

This result can be also reformulated as follows,

$$M_k(x_k) = \left(\frac{k}{e} A_\chi (1 + o(1))\right)^k, \quad (x, u)_\chi \to \infty, \quad (1.17)$$

where

$$A_\chi = \frac{1}{u} \exp \left\{\frac{H(u) - 1}{u H'(u)}\right\} \quad (1.18)$$

and $u$ is determined by (1.9). Relations (1.13) can be deduced from (1.15) and (1.16) in the limit $u \to 0$. 


Let us note that in the case of $X_j = 1$ we have obviously $H(u) = e^u$ and relation (1.9) takes the form of the Lambert equation [9]

$$ue^u = t, \quad t = \chi^{-1}. \quad (1.19)$$

Then (1.15) is transformed into asymptotic equality

$$M_k^{(X_j=1)}(x_k) = B_k(x_k) = \left( x_k e^{\psi(x)} \left( 1 + o(1) \right) \right)^k, \quad (1.20)$$

where

$$\psi(x) = \frac{e^x - 1}{ue^x} + u - 1. \quad (1.21)$$

This result has been previously obtained with the help of the ray method applied to the differential-difference equation satisfied by $B_k(x)$ [10, 11].

**Corollary C.** Let us consider the third asymptotic regime when $x = o(k), k \to \infty$. denote this limiting transition as

$$(k, x_k) \to \infty. \quad (1.22)$$

In this regime, dependence of the asymptotic behavior of $M_k(x_k)$ on the probability distribution of $X_j$ manifests itself in the most pronounced way. In Section 3 we consider several key particular cases of the probability distribution of $X$. Here we present a part of the results obtained.

In the case when $X_j$ have exponential gamma distribution $\Gamma(m, \theta)$ with $m > 8$, the asymptotic behavior of the moments $M_k(x_k)$ is similar to that given by (1.18),

$$M_k(x_k) = \left( \frac{k\theta}{e} \left( 1 + o(1) \right) \right)^k, \quad (k, x_k) \to \infty. \quad (1.23)$$

In the case when $X_j$ follow the Gaussian (normal) law $N(a, \sigma^2)$, we get asymptotic relation

$$M_k(x_k) = \left( \frac{k\sigma}{e \sqrt{2 \ln k - \ln x_k}} \left( 1 + o(1) \right) \right)^k, \quad (k, x_k) \to \infty. \quad (1.24)$$

In the case of the standard centered normal distribution, $X_j \sim N(0, 1)$, even moments show the following asymptotic behavior,

$$M_{2k}(x_k) = \left( \frac{\sqrt{2k}}{e \sqrt{\ln(2k) - \ln x_k}} \left( 1 + o(1) \right) \right)^{2k}, \quad (k, x_k) \to \infty. \quad (1.25)$$

The weighted Bell polynomials with weights given by centered Bernoulli random variables show the following asymptotic behavior,

$$M_{2k}(x_k) = \left( \frac{2k}{e \left( \ln(2k) - \ln x_k \right)} \left( 1 + o(1) \right) \right)^{2k}, \quad (k, x_k) \to \infty. \quad (1.26)$$
This asymptotic expression coincides with that obtained for the pure (non-weighted) Bell polynomials,
\[ B_k(x) = \left( \frac{k}{e(\ln k - \ln x_k)}(1 + o(1)) \right)^k, \quad (k, x_k)_0 \to \infty \] (1.27)
with \( k \) being replaced by \( 2k \) [10, 11, 18].

Regarding corollaries A, B and C of Theorem 1.1, we see that in the last asymptotic regime (1.22) when \( x_k \ll k \), the properties of the weights \( X_j \) crucially modify the asymptotic behavior of the moments \( M_k(x_k) \). In the second asymptotic regime (1.14) when \( x_k \) is proportional to \( l \), the moments \( M_k(x_k) \) exhibit almost the same asymptotic behavior (1.18), where the form of the probability distribution of \( X_j \) enters into the right-hand side of (1.18) via a constant \( A_\chi \). The same is true for the expression (1.15). Finally, we see that in the asymptotic regime (1.12) when \( x_k \gg k \), the moments \( M_k(x_k) \) are not sensible to the details of the probability distribution of the weights \( X_j \) and the leading term of their asymptotic expansion is almost universal. This universality can be explained by a kind of the law of large numbers that follow random variables \( Y_\lambda / \lambda \) in the limit \( \lambda \to \infty \). We discuss this topic in Section 4.

The paper is organized as follows. In Section 2, we formulate and prove our main technical result given by the Local Limit Theorem for a family of auxiliary random variables. Then we prove Theorem 1.1 as well as the corollaries A and B related with the asymptotic regimes (1.12) and (1.14), respectively. In Section 3, we consider the asymptotic behavior of the moments \( M_k(x_k) \) in the third limiting transition (1.22). In Section 4, we formulate and prove auxiliary statements and discuss a number of supplementary facts related to our main results.

2 Proof of Theorem 1.1

We prove Theorem 1.1 with the help of the method proposed in [29] to study the asymptotic behavior of the Bell numbers \( B^{(k)} \) and then modified in [18] in applications to the case of Bell polynomials \( B_k(x) \) (1.3). This method is based on the observation that the local limit theorem is valid for an auxiliary random variable \( Z \) such that the probability \( P(Z = k) \) is proportional to \( B_k \). The use of the local limit theorem in order to get the asymptotic properties of combinatorial items dates back to the works by E. A. Bender (see [4] and also [13, 14]). Further use of this technique developed in [4] and other papers would require proofs of more statements such as the log-concavity of the sequence \( M_k(x) \). We stay within the frameworks of the stochastic version of the method outlined in [29].

2.1 Random variables and Central Limit Theorem

Let us introduce a random variable \( Z^{(x,u)} \) that takes values in \( \mathbb{N} \) such that
\[ P(Z^{(x,u)} = k) = M_k(x) \frac{u^k}{k! G(x,u)}, \quad j \in \mathbb{N}, \ u > 0, \] (2.1)
where $G(x, u)$ is the normalization factor. It is well known that this exponential generating function $G(x, u)$ is determined by relations

$$G(x, u) = \sum_{j=0}^{\infty} M_j(x) \frac{u^j}{j!} = \exp \left\{ x (H(u) - 1) \right\}. \tag{2.2}$$

The last equality of (2.2) can be proved with the help of standard combinatorial arguments. This equality relates the moments of $X$ and $M_k(x)$. The main task of this paper is to see in what way and in what extension $H(u)$ determines the asymptotic behavior of $M_k(x)$ in one or another asymptotic regime. It is not surprising that relation (2.2) represents a keystone of the method in general and of the proof of our main results.

Elementary calculations based on (2.1), (2.2) and (2.3) show that

$$E Z(x, u) = \sum_{j=0}^{\infty} j P_j^{(x, u)} = x u H'(u) \tag{2.4}$$

and

$$Var(Z(x, u)) = \sigma^2_Z = x(uH'(u) + u^2 H''(u)). \tag{2.5}$$

The main technical statement given by the Local Central Theorem presented below says that if $k$ is large and not far from the mean value of $Z(x, u)$ (2.4), then the probability $P(Z(x, u) = k)$ is close to $(2\pi \sigma^2_Z)^{-1/2}$,

$$P(Z(x, u) = k) = \frac{1}{\sqrt{2\pi \sigma^2_Z}} (1 + o(1)), \quad E Z^{(x, u)} = k(1 + o(1)), \quad k \to \infty. \tag{2.6}$$

Rewriting (2.1) in the form

$$M_k(x) = \frac{k!}{u^k} \exp \left\{ x (H(u) - 1) \right\} P(Z^{(x, u)} = k) \tag{2.7}$$

and using (2.6), one can obtain the main result of Theorem 1.1 given by (1.10).

It is interesting to note that while the first part of relation (2.6) represents the principal result of this paper, the last factor of the right-hand side of (2.7), in the majority of cases, does not play a crucial role with respect to the asymptotic behavior of $M_k(x)$. In contrast with this, the second part of (2.6) that determines the value of $u$ in dependence of $k$ and $x$ contributes essentially to the form of asymptotic expansions of $M_k(x)$. We discuss these questions in more details in Section 3.

Let

$$Y^{(x, u)} = \frac{Z^{(x, u)} - E Z^{(x, u)}}{\sigma_Z}. \tag{2.8}$$
We consider an infinite sequence of \((x_k, \chi_k)\) and determine \(u_k\) by relation (1.9). We introduce a function
\[
\Phi_{Y_k}(t) = \mathbb{E}\exp\{-itY_k\},
\]
where \(Y_k = Y(x_k, u_k)\) and formulate our first statement given by the Central Limit Theorem for the sequence of random variables \(Y_k, k \to \infty\).

**Lemma 2.1.** For any given \(t \in \mathbb{R}\), the following asymptotic equality
\[
\Phi_{Y_k}(t) = \exp\{-t^2/2\}(1 + o(1))
\]
holds in the limiting transitions \((k, x_k)_\infty \to (1.12)\) and \((k, x_k)_\chi \to (1.14)\); if \(H(u)\) is such that conditions (1.6) and (1.8) are verified, then relation (2.9) holds also in the limiting transition \((x, u)_0 \to (1.22)\).

**Proof of Lemma 2.1.** Relations (2.1) and (2.2) obviously imply that
\[
\Phi_{Y}(t) = \exp\{-it\mathbb{E}Z/\sigma_Z\}F(e^{it/\sigma_Z}),
\]
where
\[
F(e^{it/\sigma_Z}) = \frac{G(x, ue^{it/\sigma_Z})}{G(x, u)} = \exp\left\{x \left(H(ue^{it/\sigma_Z}) - H(u)\right)\right\}.
\]
Here and below we omit the subscript \(k\) everywhere when no confusion can arise.

Taking into account (1.5), we conclude that the series \(H(z) = \sum_{k \geq 0} V_k z^{k}/k!\) converges uniformly in the open disk \(B(0, u_0) = \{z \in \mathbb{C} : |z| < u_0\}\) and therefore the following analog of the Taylor expansion holds,
\[
H(z) = \sum_{j=0}^{2} \frac{H^{(j)}(u)}{j!} (z-u)^j + R_2(z, u), \quad u \in [0, u_0)
\]  
(2.12)

with
\[
|R_2(z, u)| \leq h(u, r) \frac{(|z-u|/r)^3}{1 - |z-u|/r}, \quad h(u, r) = \max_{s \in \mathbb{C} : |s-u|=r} |H(s)|,
\]  
(2.13)

where one can take \(r = (u_0 - u)/2\) if \(u_0 < \infty\) and \(r = 1\) if \(u_0 = +\infty\). Relations (2.12) and (2.13) are proved in Section 4.

Regarding the right-hand side of (2.12) with \(z = ue^{it/\sigma_Z}\) we conclude that
\[
H(ue^{it/\sigma_Z}) - H(u) = uH'(u) \left(e^{it/\sigma_Z} - 1\right) + \frac{u^2 H''(u)}{2} \left(e^{it/\sigma_Z} - 1\right)^2 + R_2(ue^{it/\sigma_Z}, u),
\]
where, according to (2.13),
\[
|R_2(ue^{it/\sigma_Z}, u)| \leq h_0(u, r) \frac{u^3 |e^{it/\sigma_Z} - 1|^3}{r^3}, \quad \frac{1}{1 - u|e^{it/\sigma_Z} - 1|/r}
\]  
(2.14)

It follows from (2.4) and (2.5) that if \(k \to \infty\), then in all of the three asymptotic regimes (1.12), (1.14) and (1.22) we have \(\sigma_Z \to \infty\). Therefore for any given \(t\),
\[
|e^{it/\sigma_Z} - 1| = O(t/\sigma_Z), \quad (k, x_k) \to \infty.
\]  
(2.15)
Then we can write that
\[ H(ue^{it/\sigma^2}) - H(u) = uH'(u) \frac{st}{\sigma^2} - \frac{t^2}{2\sigma^2} \left( uH'(u) + u^2H''(u) \right) + R_2(ue^{it/\sigma^2}, u) + S_2(u, t) \] (2.16)

and
\[ \Phi_Y(t) = \exp \left\{ -\frac{t^2}{2} + xR_2(ue^{it/\sigma^2}, u) + xS_2(u, t) \right\}, \] (2.17)

where
\[ |xS_2(u, t)| = \left| \left( uH'(u) + u^2H''(u) \right) O \left( \frac{t^3}{\sigma^2} \right) \right| \leq \left( uH(u) + 1 \right). \] (2.18)

In (2.17) and (2.18) we have used twice definition of \( \sigma_Z \) (2.5).

Taking into account elementary upper estimate
\[ h_0(u, r) = \max_{\phi \in [0,2\pi]} |H(u + re^{i\phi})| \leq \max_{\phi \in [0,2\pi]} \sum_{k=0}^{\infty} V_k \frac{|u + re^{i\phi}|^k}{k!} \leq H(u + r), \]

and the upper bound (2.13), we can write that in the case of infinite \( u_0 = \infty, r = 1 \), starting form certain \( k_0 \),

\[ |xR_2(ue^{it/\sigma^2}, u)| \leq xQ^{(1)}(u) \frac{|t|^3}{\sigma^2}, \quad Q^{(1)}(u) = 2u^2H(u + 1). \] (2.19)

In the case of finite \( u_0 \), with the choice of \( r = (u_0 - u)/2 \), we can write that
\[ |xR_2(ue^{it/\sigma^2}, u)| \leq xQ^{(2)}(u) \frac{|t|^3}{\sigma^2}, \quad Q^{(2)}(u) = 16u_0^3H((u_0 + u)/2)). \] (2.20)

Taking into account (2.18) together with either (2.19) or (2.20), we conclude that two last terms of (2.17) vanish in the limiting transitions \( (k, x_k)_\infty \to \infty \) and \( (k, x_k)_\chi \to \infty \) and therefore relation (2.9) is true in this two asymptotic regimes.

Regarding the limiting transition \( (k, x_k)_0 \to \infty \) when \( u_k \to \infty \), we see that if (1.6) is true, then the right-hand side of either (2.19) or (2.19) vanishes. This completes the proof of Lemma 2.1. Let us note that Lemma 2.1 is also true in the limit \( (k, x_k)_0 \to \infty \), where the sequence \( x_k \to 0 \) in a way that (1.11) is verified. \( \square \)

### 2.2 Local Limit Theorem

Let us show that the random variables \( Z^{(x,u)} \) verify the Local Limit Theorem.

**Lemma 2.2.** Given a sequence of \( (x_k, \chi_k)_{k \in \mathbb{N}} \), we consider \( u_k \) such that (1.9) is verified for all \( k \in \mathbb{N} \). Then relation
\[ P(Z^{(x_k,u_k)} = k) = \frac{1}{\sqrt{2\pi \sigma_Z}} (1 + o(1)), \quad (k, x_k) \to \infty \] (2.21)
holds in the asymptotic regimes \((k, x_k) \to \infty (1.12)\) and \((k, x_k) \chi \to \infty (1.14)\). If \(H(u)\) is such that conditions (1.6), (1.7) and (1.8) are verified, then (2.21) is also true in the limiting transition \((k, x_k) \to \infty (1.22)\).

**Proof of Lemma 2.2.** We combine relation (2.18) with arguments developed by T. Tao in [26] for the proof of the Local Limit Theorem for sums of independent random variables. Taking mathematical expectation of both parts of equality

\[
I_{Z=k}(\omega) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{iyZ} e^{-iyk} \, dy,
\]

we get by the Fubini's theorem that

\[
P(Z = k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} E \left( e^{iy(Z - E_Z)} \right) e^{-iy(k - E_Z)} \, dy
= \frac{1}{2\pi\sigma_Z} \int_{-\pi\sigma_Z}^{\pi\sigma_Z} \Phi_Y(t) e^{it(k - E_Z)/\sigma_Z} \, dt,
\]

where \(Y = Y(x,u)\) is determined by (2.8). The key point of the proof is to show that the following difference vanishes,

\[
D_k = \int_{-\pi\sigma_Z}^{\pi\sigma_Z} \Phi_Y(t) e^{it(k - E_Z)/\sigma_Z} \, dt - \int_{-\pi\sigma_Z}^{\pi\sigma_Z} e^{it(k - E_Z)/\sigma_Z - t^2/2} \, dt = o(1)
\]

as \(k\) tends to infinity. Then (2.21) will follow from this convergence, classical identity

\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{it(k - E_Z)/\sigma_Z - t^2/2} \, dt = \frac{1}{\sqrt{2\pi}} e^{-(k - E_Z)^2/(2\sigma_Z^2)},
\]

and obvious estimate

\[
\left| \int_{|t| > \pi\sigma_Z} e^{-t^2/2 + iat} \, dt \right| = o(1), \quad \sigma_Z \to \infty.
\]

We split \(D_k\) into two parts and consider first the difference

\[
D_k^{(1)} = \int_{|t| \leq y_k} \left( \Phi_Y(t) - e^{-t^2/2} \right) e^{-it(k - E_Z)/\sigma_Z} \, dt.
\]

It follows from (2.17), (2.19) and (2.20) that

\[
|D_k^{(1)}| \leq \int_{|t| \leq y_k} e^{-t^2/2} \exp \{x_k R_2 + x_k S_2\} - 1 \, dt
= O \left( x_k Q^{(1)}(u_k) \frac{y_k}{\sigma_Z^4} \right) + O \left( \frac{x_k y_k^4}{\sigma_Z^4} \right) = o(1), \quad (k, x_k) \to \infty.
\]

It is clear that with the choice of \(y_k = \sigma_Z^{1/2}\) we have

\[
|D_k^{(1)}| = o(1)
\]
in the limiting transitions \((k, x_k)_\infty \to \infty\) and \((k, x_k)_t \to \infty\).

Regarding the second part of the integral \((2.23)\), we can write that

\[
|D^{(2)}_k| \leq \int_{y_k < |t| \leq \pi \sigma Z} e^{-t^2/2} dt + \int_{y_k < |t| \leq \pi \sigma Z} |\Phi_Y(t)| dt. \tag{2.25}
\]

Let us consider the last integral in the limiting transitions \((k, x)_k \to \infty\) \((1.12)\) and \((k, x)_t \to \infty\) \((1.14)\) when the parameter \(u_k\) remains finite as \(k \to \infty\). Using definitions \((1.5)\) and \((2.1)\) together with relations \((2.2)\) and \((2.11)\), we can write that

\[
|\Phi_Y(t)| = \left| \exp \left\{ x_k \sum V_j u_k^j \left( \cos \left( \frac{j t}{\sigma Z} \right) - 1 \right) + i \sin \left( \frac{j t}{\sigma Z} \right) \right\} \right|
= \exp \left\{ x_k \sum_{j=1}^2 u_k^j V_j \left( \cos \left( \frac{j t}{\sigma Z} \right) - 1 \right) / j! + x_k T_3(x_k, u_k, t) \right\}, \tag{2.26}
\]

where the remaining term

\[
T_3(x_k, u_k, t) = \sum_{j=3}^\infty V_j u_k^j \left( \cos \left( \frac{j t}{\sigma Z} \right) - 1 \right) / j!
\]

is negative, \(T_3(x_k, u_k, t) \leq 0\) for all \(t \in [-\pi \sigma Z, \pi \sigma Z]\).

Taking into account elementary inequalities

\[
\cos \left( \frac{j t}{\sigma Z} \right) \leq 1 - \frac{j^2 t^2}{24 \sigma^2 Z} \leq 1 - \frac{j^2 y_k^2}{24 \sigma^2 Z}, \ \forall t : y_k \leq |t| \leq \pi \sigma Z, \ j = 1, 2, \tag{2.27}
\]

we deduce from \((2.26)\) that

\[
\Delta_k = \int_{y_k < |t| \leq \pi \sigma Z} |\Phi_Y(t)| dt \leq 2 \pi \sigma Z \exp \left\{ -\frac{x_k u_k (V_1 + 4 u_k V_2) y_k^2}{24 \sigma^2 Z} \right\}.
\]

Remembering definition \((2.5)\) of \(\sigma^2_Z\) and the value \(y_k = \sigma^{1/4}_Z\), we conclude that

\[
\Delta_k \leq 2 \pi \sigma Z \exp \left\{ -\frac{V_1 + 4 u_k V_2}{24 (H'(u_k) + u_k H''(u_k)) \sigma^{1/4}_Z} \right\} = 2 \pi \sigma Z \exp \left\{ -A_k \sigma^{1/4}_Z \right\}.
\]

It is clear that in the limiting transition \((k, x_k)_t \to \infty\) when \(u_k\) converges as \(k \to \infty\) to a finite solution \(u > 0\) of equation \((1.17)\), we get \(\Delta_n = o(1)\).

Let us consider the limiting transition \((k, x_k)_\infty \to \infty\) \((1.12)\). In this case \(\chi_k \to \infty\) and therefore \(u_k \to 0\). If \(V_1 \neq 0\), then \(\lim_{k \to \infty} A_k = 1/24\) and if \(V_1 = 0\), then \(\lim_{k \to \infty} A_k = 1/6\). Then \(\Delta_k = o(1)\), \((k, x_k)_t \to \infty\). Returning to \((2.25)\), we conclude that in these two limiting transitions,

\[
|D^{(2)}_k| = o(1), \ (k, x_k) \to \infty.
\]

Combining this relation with \((2.24)\), we get \((2.23)\). Then \((2.22)\) implies \((2.21)\). The first part of Lemma 2.2 is proved.
Now let us consider the limiting transition \((k, x_k)_k \to \infty\) (1.22) when \(u_k \to \infty\). It follows from (2.17), (2.19) and (2.20) that

\[ |D_N^{(1)}| \leq \int_{|t| \leq y_N} |e^{xR_2 + xS_2} - 1|dt = \int_{|t| \leq y_N} |(xR_2 + xS_2)(1 + o(1))|dt \leq y_N^4 T_N(x, u), \]

where

\[ Q_H(x, u) = \begin{cases} xu^3H(u + 1)/\sigma_Z^3 + 1/\sigma_Z, & \text{if } u_0 = \infty, \\ xu^3H((u_0 + u)/2)/(u_0 - u)^3\sigma_Z^3 + 1/\sigma_Z, & \text{if } u_0 < \infty. \end{cases} \quad (2.28) \]

To estimate \(D_N^{(2)}\), we start with the first integral of the right-hand side of (2.25). Remembering that

\[ H(ue^{i\alpha}) = \mathbb{E}\exp\{u(\cos \alpha + i \sin \alpha)W\} = \mathbb{E}e^{uW\cos \alpha}(\cos(uW \sin \alpha) + i \sin(uW \sin \alpha)), \]

we write that

\[ |\exp \{xH(ue^{i\alpha}) - xH(u)\}| \leq \exp \left\{ xe^{uW}\left( (1 + (12\sigma_Z^2)) - 1 \right) \right\}, \]

where \(\alpha = t/\sigma_Z\).

Using the upper bound \(\cos s \leq 1 - s^2/12\), \(|s| \leq \pi\), we can write inequality

\[ |D_N^{(2)}| \leq 2\pi\sigma_Z \exp \left\{ xe^{uW}\left( (1 + (12\sigma_Z^2)) - 1 \right) \right\}. \]

Then

\[ |D_N^{(2)}| \leq 2\pi\sigma_Z \exp \left\{ xe^{uW}\left( (1 + (12\sigma_Z^2)) - 1 \right) I_{\{W \geq \sigma_0\}} \right\} \]

\[ \leq 2\pi\sigma_Z \exp \left\{ x \left( (1 + (12\sigma_Z^2)) - 1 \right) \beta H(u) \right\} \leq 2\pi \exp \left\{ -\frac{xu_0\beta H(u)}{24\sigma_Z^2}y^2 + \ln \sigma_Z \right\}. \]

In the last estimate, we have used elementary inequality

\[ e^{-s} - 1 \leq -\frac{s}{2} \]

that is true for sufficiently small \(s > 0\).

Gathering (2.28) and (2.29), we see that if the value of \(y_k\) infinitely increases as \(k \to \infty\) and and verifies the following chain of asymptotic estimates,

\[ \frac{\sigma_Z^4}{(x_ku_kH(u_k))^2} \left( \ln \sigma_Z \right)^2 \ll y_k^4 \ll \frac{1}{Q_H(x_k, u_k)}, \quad (k, x_k)_k \to \infty, \]

then (2.23) holds. It is not hard to see that conditions (1.7), (1.8) are sufficient to have the left-hand side of (2.30) much less than the right-hand side of (2.30). Lemma 2.2 is proved. □
2.3 Proof of Theorem 1.1 and Corollaries A and B

Taking into account relation (2.21) of Lemma 2.2 and using definitions (2.1), (2.2) and (2.7), we obtain the following asymptotic relation,

\[ M_k(x) = \frac{k!}{u_k} \cdot \frac{\exp \{ x_k (H(u_k) - 1) \}}{\frac{\sqrt{2\pi}\sigma_Z^{(k)}}{u_k}} (1 + o(1)), \quad (k, x_k) \to \infty, \quad (2.31) \]

where \( \sigma_Z^{(k)} = \sqrt{x_k u_k (H'(u_k) + u_k H''(u_k))} \) and \( u_k \) is determined by (1.9). Rewriting equality (1.9) in the form

\[ x_k = \frac{k}{u_k H'(u_k)}, \]

we deduce from (2.31) that

\[ M_k(x_k) = \frac{k!}{\sqrt{2\pi}\sigma_Z^{(k)}} \cdot \frac{\exp \left\{ \frac{k H(u_k) - 1}{u_k H'(u_k)} \right\}}{(1 + o(1))}, \quad (k, x_k) \to \infty. \quad (2.32) \]

Using the Stirling formula

\[ k! = \sqrt{2\pi} \left( \frac{k}{e} \right)^k (1 + o(1)), \quad k \to \infty \quad (2.33) \]

we transform relation (2.32) into the following asymptotic equality

\[ M_k(x_k) = \left( \frac{k}{u_k} \right)^k \exp \left\{ \frac{k (H(u_k) - 1)}{u_k H'(u_k)} - 1 \right\} \sqrt{\frac{k}{x_k u_k (H'(u_k) + u_k H''(u_k))}} (1 + o(1)), \quad (2.34) \]

that is valid in all of the three asymptotic regimes (1.12), (1.14) and (1.22). Then (1.10) obviously follows from (2.34). Theorem 1.1 is proved. \( \square \)

Let us prove two corollaries of Theorem 1.1 given by (1.13) and (1.15) in two asymptotic regimes (1.12) and (1.14), respectively.

A. Consider the case when \( 1 \ll k \ll x_k \). Then \( x_k \to \infty \) as \( k \to \infty \) and it follows from (1.9) that \( u_k \) converges to zero as \( k \to \infty \). Then we can write that

\[ H(u_k) - 1 - u_k H'(u_k) = -\frac{u_k^2 V_2}{2} + o(u_k^2), \quad (k, x_k) \to \infty. \]

If \( V_1 \neq 0 \), then \( H'(u_k) = V_1 (1 + o(1)) \) and

\[ \exp \left( \frac{H(u_k) - 1}{u_k H'(u_k)} \right) = 1 + o(1), \quad (k, x_k) \to \infty. \]

Remembering (1.9), we see that \( u_k = k (x_k V_1)^{-1} (1 + o(1)) \). Then we deduce from (2.34) relation

\[ M_k(x_k) = (x_k V_1 (1 + o(1)))^k, \quad (k, x_k) \to \infty. \quad (2.35) \]
If \( V_1 = 0 \), then \( H'(u_k) = V_2 u_k (1 + o(1)) \) and relation (1.9) implies equality

\[
u_k = \sqrt{\frac{k}{x_k V_2}} (1 + o(1)), \quad (k, x_k) \to \infty.
\]

Taking into account that

\[
\exp \left( \frac{H(u_k) - 1}{u_k H'(u_k)} - 1 \right) = e^{-1/2} (1 + o(1)), \quad (k, x_k) \to \infty,
\]

we deduce from (2.34) asymptotic equality

\[
M_k(x) = \left( \sqrt{\frac{k V_2}{e}} (1 + o(1)) \right)^k, \quad (k, x_k) \to \infty. \quad (2.36)
\]

Relations (2.35) and (2.36) give (1.13).

B. If \( x_k/k \to \chi > 0 \), then \( u_k \to u \) with \( u \) determined by the Lambert-type equation (1.9). It follows from (2.34) that in this case

\[
M_k(x_k) = \left( x_k H'(u) \exp \left( \frac{H(u) - 1}{u H'(u)} - 1 \right) (1 + o(1)) \right)^k, \quad (k, x_k) \to \infty. \quad (2.37)
\]

Then (1.15) and (1.16) follow from (2.37).

Let us note that if we determine \( x_k \) by relation \( x_k = k \chi \), then (2.34) implies the following asymptotic equality

\[
M_k(x_k) = \frac{1}{\sqrt{1 + x \chi u^2 H''(u)}} \left( \frac{k}{u} \exp \left( \frac{H(u) - 1}{u H'(u)} - 1 \right) (1 + o(1)) \right)^k, \quad (x, k) \to \infty \quad (2.38)
\]

that is more informative than (2.37).

3 The third asymptotic regime

In this section we consider asymptotic behavior of the moments \( M_k(x_k) \) in the limit \( (k, x_k) \to \infty \) (1.22) when \( x_k \) is much smaller than \( k \). We concentrate on several important particular cases of the probability distribution of the weights \( X_j \).

3.1 Gamma distribution

Assuming that \( X_j \) follows the Gamma distribution with density

\[
f^{(m, \theta)}(x) = \frac{x^{m-1} e^{-x/\theta}}{\theta^m \Gamma(m)}, \quad x > 0, \ m > 0, \ \theta > 0,
\]

where \( \Gamma(m) = \int_0^\infty x^{m-1} e^{-x} dx \) [20], it is not hard to see that (1.7) holds and that

\[
H(u) = \frac{1}{(1 - \theta u)^m}, \quad 0 \leq u < 1/\theta. \quad (3.9)
\]
We denote by \( M_k^{(\Gamma)}(x) \) the moments (1.5) with the weights \( V_j = k! G(x, u) \). Let us consider random variables \( Z(x,u) \) such that

\[
P(Z(x,u) = k) = M_k^{(\Gamma)}(x) \frac{u^k}{k! G(x, u)} = S_k(x) \frac{u^k}{G(x, u)}
\]  

(3.10)

with

\[
G(x, u) = \exp\{x(H(u) - 1)\} = \exp\{x ((1 - \theta u)^{-m} - 1)\}
\]

and

\[
EZ = \frac{m\theta xu}{(1 - \theta u)^{m+1}} \quad \text{and} \quad \sigma_Z^2 = \frac{m\theta xu}{(1 - \theta u)^{m+1}} + xu^2 \frac{m(m + 1)\theta^2}{(1 - \theta u)^{m+2}}.
\]

The Lambert-type equation (1.9) takes the form

\[
\frac{m\theta u}{(1 - \theta u)^{m+1}} = \frac{k}{x}.
\]  

(3.3)

Assume that conditions (1.6), (1.7) and (1.8) are verified. Then Lemma 2.2 is true in the asymptotic regime \((k, x_k)_0 \rightarrow \infty (1.22)\). In this case the solution of (3.3) admits the following expansion,

\[
u_k = \frac{1}{\theta} + \frac{1}{\theta} \left( \frac{mx_k}{k} \right)^{1/(m+1)} (1 + o(1)),
\]

Denoting by \( M_k^{(\Gamma)}(x) \) the moments (1.2) with the weights \( V_j = \int_0^\infty x^j f^{(m,\theta)}(x)dx \), we can write that

\[
M_k^{(\Gamma)}(x_k) = \left( \frac{mx_k}{k} \right)^{1/(m+1)} \left( \frac{k\theta}{e (1 + (mx_k/k)^{1/(m+1)} (1 + o(1)))} \right)^k (1 + o(1))
\]

\[
= \left( \frac{k\theta}{e (1 + o(1))} \right)^k, \quad (k, x_k)_0 \rightarrow \infty.
\]  

(3.4)

This proves proposition (1.23).

Let us show that conditions (1.6), (1.7) and (1.8) are verified when the Gamma distribution is such that \( m > 8 \). We start with (1.8). We can write that

\[
\frac{1}{\theta^{m\Gamma(m)}} \int_{x \geq w_0} e^{ux} x^{m-1} e^{-x/\theta} dx
\]

\[
\geq \frac{1}{(1/\theta - u)^m} \left( 1 - \frac{1}{\theta^{m\Gamma(m)}} \int_{s < w_0} s^{m-1} e^{-s} ds \right)
\]

\[
\geq \frac{1}{(1/\theta - u)^m} \left( 1 - \frac{1}{\theta^{m\Gamma(m)}} \int_{s < w_0} s^{m-1} ds \right).
\]

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We conclude that if \( w_0 \) is such that
\[
\frac{w_0^n}{m \theta^m \Gamma(m)} < \frac{1}{2},
\]
then relation (1.8) holds with \( \beta = 1/2 \).

Instead of (1.6), let us check whether relations (2.30) are satisfied. This happens when
\[
\sigma^2 (\ln \sigma^2)^2 \left( 16 x H \left( \frac{u + u_0}{2} \right) + (u_0 - u)^3 \sigma^2 \right) \ll (u_0 - u)^3 \left( xu H(u) \right)^2
\]
holds in the limit \( u \to u_0 \). Using equality \( H((u + u_0)/2)) = 2^m (1 - \theta u)^{-m} \) and observing that
\[
\sigma^2 = \frac{x m (m + 1) \theta^2}{(1 - \theta u)^m + 2} (1 + o(1)), \quad u \to 1/\theta,
\]
we conclude that (3.5) is verified when the following relation holds,
\[
\frac{2 m + 4 x^{3/2} \sqrt{m (m + 1) (m + 2)^2 \theta}}{(1 - \theta u)^{m + 2}} \left( \ln(1 - \theta u) \right)^2 \ll \frac{x^2}{(1 - \theta u)^{2 m - 3}}, \quad u \to 1/\theta.
\]
This is true under condition \( m > 8 \). It is interesting to note that condition (2.30) imposes restriction \( m > 8 \) on \( m \) only while the leading term of the asymptotic expression (3.4) depends on \( \theta \) only.

### 3.2 Normal (Gaussian) distribution

Let us consider first the centered random variables. If \( W_j \sim \mathcal{N}(0, V_2) \), then
\[
H(u) = e^{V_2 u^2/2}, \quad u \in \mathbb{R}.
\] (3.6)

In this case equation (1.9) takes the form of Lambert equation (1.19)
\[
se^s = t = \frac{1}{\chi},
\] (3.7)
where we denoted \( s = u^2 V_2/2 \). In the third asymptotic regime (1.22) when \( x_k = o(k) \), we have \( \chi = \chi_k \to 0, k \to \infty \). It is known that the asymptotic expansion of \( s = s(t) \) is as follows [9],
\[
s(t) = \ln(t) - \ln \ln(t) (1 + o(1)), \quad t \to \infty.
\] (3.8)

It is not hard to show that conditions (1.6), (1.7) and (1.8) are verified by \( H(u) \) (3.6). Then
\[
\chi_k = \sqrt{\frac{2 \ln(k/2) - \ln x_k}{V_2}} (1 + o(1)).
\] (3.9)

Taking into account that
\[
\frac{H(u_k) - 1}{u_k H'(u_k)} \to 0, \quad u_k \to \infty,
\]
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we deduce from (1.10) with the help of (3.9) that

\[ M_{2k}^{(N(0,V_2))}(x_k) = \left( \frac{2k}{ea} \right) (1 + o(1)) = \left( \frac{\sqrt{2V_2k}}{e \sqrt{\ln(2k) - \ln x_k}} (1 + o(1)) \right)^2, \quad (k, x_k)_0 \to \infty. \]  \hspace{1cm} (3.10)

Relation (1.25) is proved.

Now we consider the general case \( X_j \sim N(a, \sigma^2) \). It is easy to see that conditions (.16), (1.7) and (1.8) are verified. Then

\[ H(u) = e^{au + u^2 \sigma^2 / 2}, \quad u \in \mathbb{R} \]

and equation (1.9) takes the following form slightly different from (3.7),

\[ (au + \sigma^2 u^2) e^{au + u^2 \sigma^2 / 2} = t, \quad t = \frac{1}{\lambda}. \]  \hspace{1cm} (3.11)

To study asymptotic expansion of \( u \), we denote the left-hand side of (3.11) by \( \tilde{F}(u) \) and use the denotation \( \tilde{u} = \tilde{u}(t) \) for the solution of the corresponding equation \( \tilde{F}(u) = t \).

We introduce two auxiliary functions, \( \hat{F}(u) \) and \( \check{F}(u) \) by the following formulas,

\[ \hat{F}(u) = (au + \sigma^2 u^2) e^{au + u^2 \sigma^2 / 2} < \tilde{F}(u) < \check{F}(u) = (2au + \sigma^2 u^2) e^{au + u^2 \sigma^2 / 2}. \]

Then clearly

\[ \check{u}(t) \leq \tilde{u}(t) \leq \hat{u}(t), \]

where \( \check{u} \) and \( \hat{u} \) denote the solutions of equations \( \hat{F}(u) = t \) and \( \check{F}(u) = t \), respectively.

Remembering asymptotic expansion (3.8), we conclude that \( \check{u}(t) \) and \( \hat{u}(t) \) are such that

\[ a\check{u} + \sigma^2 \check{u}^2 / 2 = \ln t(1 + o(1)) \quad \text{and} \quad a\hat{u} + \sigma^2 \hat{u}^2 / 2 = \ln(t/2)(1 + o(1)). \]

Then

\[ \frac{\sqrt{2 \ln(t/2)}}{\sigma}(1 + o(1)) \leq \tilde{u}(t) \leq \frac{\sqrt{2 \ln t}}{\sigma}(1 + (1)) \]

and we can write that

\[ \tilde{u}(t) = \frac{\sqrt{2 \ln t}}{\sigma} (1 + o(1)) = \frac{\sqrt{2 \ln(k/x)}}{\sigma} (1 + o(1)). \]

Substituting this expression into (1.10) (see also (2.5)), we get relations

\[ M_k^{(N(a, \sigma^2))}(x_k) = \left( \frac{k \sigma}{e \sqrt{2 \ln(k) - \ln x_k}} (1 + o(1)) \right)^k, \quad (k, x_k)_0 \to \infty. \]  \hspace{1cm} (3.12)

Relation (1.24) is proved. Let us note that \( a \) does not alter the leading term of \( M_k(x_k) \). Moreover, we get the same asymptotic behavior as in the case of centered Gaussian distribution (3.10).
3.3 Centered Bernoulli and triangular distributions

Let us consider the case when independent random variables $X_j$ (1.1) follow the centered Bernoulli probability distribution,

\[ X_j = \begin{cases} 1, & \text{with probability } 1/2, \\ -1, & \text{with probability } 1/2, \end{cases}, \quad j \in \mathbb{N}. \quad (3.13) \]

In this case, the moments (1.2) take the form

\[ M_{2k}(x) = \sum_{(l_2,l_4,\ldots,l_{2k})^*} x^{l_2+l_4+\cdots+l_{2k}} \hat{B}_{2k}(l_2,l_4,\ldots,l_{2k}), \]

where

\[ \hat{B}_{2k}(l_2,l_4,\ldots,l_{2k}) = \frac{(2k)!}{(2!)^2 l_2! (4!)^4 l_4! \cdots ((2k)!)^{2k} l_{2k}!} \]

and the sum runs over $l_2 \geq 0$ such that $2l_2 + 4l_4 + \cdots + 2kl_{2k} = 2k$. Then

\[ M_{2k}(B_0)(1) = \hat{B}^{(2k)} = \sum_{(l_2,l_4,\ldots,l_{2k})^*} \hat{B}_{2k}(l_2,l_4,\ldots,l_{2k}), \quad (3.14) \]

represents the number of all possible partitions of a set of $2k$ elements into subsets of even cardinality [23]. One can refer to $\hat{B}^{(2k)}$ as to the Bell numbers of even partitions.

Definition (3.13) implies that

\[ H(u) = \mathbb{E}e^{uX} = \text{ch}(u) \]

and the Lambert-type equation (1.9) takes the form

\[ u \text{sh}(u) = t, \quad t = \frac{2}{\chi}. \quad (3.15) \]

Taking into account elementary bounds

\[ \frac{e^u}{4} \leq \text{sh}(u) \leq \frac{e^u}{2}, \quad u > 1, \]

it is easy to show with the help of the arguments of the previous subsection that (3.15) implies asymptotic equality

\[ u(t) = \ln t (1 + o(1)), \quad t \to \infty \]

It not hard to see that conditions (1.6), (1.7) and (1.8) are verified in the case of centered Bernoulli distribution (3.13). Then it follows from Theorem 1.1 that

\[ M_{2k}^{(B_0)}(x_k) = \left( \frac{2k}{e(\ln(2k) - \ln x_k)} (1 + o(1)) \right)^{2k}, \quad (k,x_k)_0 \to \infty. \quad (3.16) \]
Relation (3.16) implies (1.26). Relation (1.26), when taken with $x = 1$, coincides with the first terms of the asymptotic expression for Bell numbers $B(2^k)$ (1.27) in the limit $k \to \infty$, known since 50-s [22].

Let us consider the case when $X_j$ have triangular distribution $X_j \sim T(\nu)$ with the density of the form

$$f(\nu)(x) = \frac{1}{\nu} \begin{cases} 1 + x/\nu, & \text{if } x \in (-\nu, 0], \\ 1 - x/\nu, & \text{if } x \in (0, \nu], \\ 0, & \text{otherwise} \end{cases}.$$ 

It is easy to see that

$$V_{2i}^{(\nu)} = \int_{-\nu}^{\nu} x^{2i} f(\nu)(x) dx = \frac{\nu^{2i}}{(i + 1)(2i + 1)} = \nu^{2i} V_{2i}^{(1)}.$$ 

It follows from definition (1.2) that

$$M_{2k}^{(T(\nu))}(x) = \nu^{2k} M_{2k}^{(T(1))}.$$ 

Regarding the case of $\nu = 1$, we can write that

$$H(u) = \frac{2}{u^2} (\text{ch}(u) - 1)$$

and equation (1.9) takes the form

$$\tilde{F}(u) = t, \quad t = \frac{2k}{x} = \frac{2}{\chi},$$

where we denoted

$$\tilde{F}(u) = \frac{2}{u} \text{sh}(u) - \frac{4}{u^2} (\text{ch}(u) - 1).$$

In Section 4, we show that solution $\tilde{u}$ of (3.17) admits the following asymptotic expansion in the limit $\chi \to 0$,

$$\tilde{u} = \ln(k/x)(1 + o(1)).$$

Then

$$M_{2k}^{(T(1))}(x_k) = \left( \frac{2k}{e(\ln(2k) - \ln x_k)} (1 + o(1)) \right)^{2k}, \quad (k, x_k) \to \infty.$$ 

that coincides with (1.26) and (3.16). This coincidence is similar to that observed in the normal distribution, the right-hand side of (3.10) is the same of (3.12) with $k$ replace by $2k$.

Finally, we get for the the moments of compound Poisson distribution with $X_j \sim T(\nu)$ the following asymptotic expression,

$$M_{2k}^{(T(\nu))}(x_k) = \left( \frac{2k\nu}{e(\ln(2k) - \ln x_k)} (1 + o(1)) \right)^{2k}, \quad (k, x_k) \to \infty.$$ (3.20)
4 Auxiliary statements and supplementary results

In this section we collect auxiliary facts used in the proof and discuss a number of additional results related with Theorem 1.1.

4.1 Taylor expansion

In this subsection we prove relations (2.12) and (2.13). It follows from (1.5) that the series

\[ H(z) = \sum_{k=0}^{\infty} V_k \frac{z^k}{k!} \]

converges for any \( z \) from the open ball \( B(0, u_0) \) and by definition is analytic in \( B(0, u_0) \). It is (infinitely) holomorphic and the Cauchy integral formula is true,

\[ H^{(k)}(u) = \frac{k!}{2\pi i} \int_{C} \frac{H(s)}{(s-u)^{k+1}} ds, \quad s = u + r e^{i\phi}, \quad r \leq \frac{u_0 - u}{2}. \] (4.1)

Therefore we can write that for any \( z \) and \( u \) from \( B(0, u_0) \),

\[ H(z) = \sum_{j=0}^{\infty} H^{(j)}(z-u) \frac{1}{j!} = \sum_{j=0}^{2} H^{(j)}(z-u) \frac{1}{j!} + R_2(z,u), \]

where

\[ R_2(z,u) = \sum_{j=3}^{\infty} H^{(j)}(z-u) \frac{1}{j!}. \]

It follows from (4.1) that

\[ |H^{(j)}(u)| \leq \frac{j!h_0(u,r)}{r^k}, \quad h_0(u,r) = \max_{s:|s-u|=r} |H(s)|. \]

Finally, we obtain the following estimate,

\[ |R_2(z,u)| \leq \sum_{j=3}^{\infty} h_0(u,r) \frac{|z-u|^j}{r^j} = h_0(u,r) \frac{|z-u|^3}{r^3 (1-|z-u|/r)}. \]

This proves relations (2.12) and (2.13).

4.2 Lambert-type equation for triangular distribution

It follows from the definition (3.18) that

\[ \tilde{F}(u) = \sum_{l=1}^{\infty} \frac{4l}{(2l+2)!} u^{2l}. \]
Then we can write inequality
\[ \tilde{F}(u) \geq 4 \sum_{l=1}^{\infty} \frac{u^{2l}}{(2l+2)!} = \frac{4}{u^2}(\cosh(u) - 1) - 2. \] (4.2)

Combining (3.18) with (4.2), we conclude that
\[ \frac{4}{u^2}(\cosh(u) - 1) \leq \frac{\sinh(u)}{u} + 1 \]
and that
\[ \tilde{F}(u) \geq \frac{\sinh(u)}{u} - 1 = \hat{F}(u). \]

Denoting by \( \hat{u} = \hat{u}(t) \) solution of equation
\[ \hat{F}(u) = t, \] (4.3)
we conclude that \( \hat{u}(t) \leq \tilde{u}(t) \), where \( \tilde{u}(t) \) is the solution of equation (3.17).

Taking into account that for \( u \geq \ln 2 \) we have \( \sinh(u) \geq e^u/4 \), we can write that \( \hat{u}(t) \leq \bar{u}(t) \), where \( \bar{u}(t) \) is a solution of equation
\[ \frac{e^u}{u} = 4t + 4. \] (4.4)

It follows from (4.4) that
\[ u = \ln(4t + 4) + \ln u \leq \ln(4t + 4) + u/2. \]
The last inequality is true for sufficiently large \( t \) because \( u \geq \ln(4t + 4) \). Then \( u \leq 2\ln(4t + 4) \) and
\[ \ln u \leq \ln \ln(4t + 4) + \ln 2. \]

Then we get the upper bound
\[ \hat{u}(t) \leq \bar{u}(t) \leq \ln(4t + 4) + \ln \ln(4t + 4) + \ln 2. \] (4.5)

Let us find the lower bound for \( \hat{u}(t) \). Remembering definition (3.18), we can write that
\[ \tilde{F}(t) \leq \hat{F}(t) = \frac{2}{u} \sinh(u) \]
and therefore \( \hat{u}(t) \geq \bar{u}(t) \), where \( \bar{u}(t) \) is a solution of equation
\[ \tilde{F}(u) = t. \]

Taking into account inequality \( 2\sinh(u) \leq e^u \), we can write that \( \bar{u}(t) \geq \hat{u}(t) \), where \( \hat{u}(t) \) is a solution of equation
\[ \frac{e^u}{u} = t. \]

It is clear that \( \hat{u} = \ln t + \ln \hat{u} \) and therefore \( \hat{u}(t) \geq \ln t \). Then \( \bar{u}(t) \geq \ln t \). This relation together with (4.5) shows that
\[ \bar{u}(t) = \ln(t)(1 + o(1)), \quad t \to \infty. \] (4.6)

This relation implies (3.19).
4.3 Exponential distributions and combinatorial polynomials

If \( X_j \) follow the exponential distribution, \( X_j \sim \mathcal{E}(1) \), then \( V_k = k! \), \( k \in \mathbb{N} \) and

\[
H(u) = \sum_{k=0}^{\infty} V_k \frac{u^k}{k!} = \frac{1}{1-u}, \quad u \in [0,1).
\]  

(4.7)

It follows from (1.2) that

\[
M_k(x) = k! S_k(x),
\]

(4.8)

where

\[
S_k(x) = \sum_{(l_1, \ldots, l_k)^*} \frac{x^{l_1+\cdots+l_k}}{l_1! \cdots l_k!}.
\]

(4.9)

We deduce from Theorem 1.1 that if \( k \to \infty \) and \( x = \chi k \), then

\[
S_k(x) = \frac{1}{\sqrt{2\pi k}} \left( \frac{1}{1+u} \left( \frac{e^{ue^u}}{u^{ue^u}} \right)^k (1+o(1)) \right), \quad k \to \infty,
\]

(4.10)

where \( u \) is a solution of equation

\[
\frac{u}{(1-u)^2} = \frac{1}{\chi},
\]

and therefore

\[
u = \frac{2 + \chi - \sqrt{\chi(4+\chi)}}{2}.
\]

If \( k = o(x) \) and \( \chi \to \infty \), then \( u = k/x + 3/2 + o(1) \) and

\[
S_k(x) = \frac{1}{\sqrt{2\pi k}} \left( \frac{x e^{ue^u}}{k} (1+o(1)) \right)^k, \quad 1 \ll k \ll x.
\]

(4.11)

Using combinatorial identity ([24], p.183),

\[
\sum_{(l_1, \ldots, l_p)^=p} \frac{p!}{l_1! \cdots l_p!} = \binom{k-1}{p-1},
\]

(4.12)

it is easy to get the following expression for the polynomials \( S_k(x) \),

\[
S_k(x) = \sum_{p=1}^{k} \frac{x^p}{p!} \binom{k-1}{p-1}.
\]

(4.13)

Thus, relations (4.10) and (4.12) determine asymptotic behavior of combinatorial polynomials (4.13).

Let us consider \( M_k(x) \) (1.2) with \( V_k = (k-1)! \) for all \( k \in \mathbb{N} \). We can write that in this case \( M_k(x) = k! T_k(x) \), where

\[
T_k(x) = \sum_{(l_1, l_2, \ldots, l_k)^*} \frac{x^{l_1}}{l_1! l_2! \cdots l_k!}.
\]

(4.14)
We have
\[ H(u) = \sum_{k=0}^{\infty} V_k \frac{u^k}{k!} = 1 - \ln(1 - u). \] (4.14)

Introducing auxiliary random variables \( Z^{(x,u)} \) such that \( P(Z^{(x,u)} = k) = T_k(x) \frac{u^k}{G(x,u)} \) with
\[ G(x,u) = \exp \{ x(H(u) - 1) \} = \frac{1}{(1 - u)^x}, \quad u \in [0,1), \]
and
\[ E\mathbb{Z}^{(x,u)} = \frac{xu}{1-u} \quad \text{and} \quad \sigma^2_Z = \frac{xu}{1-u} + \frac{xu^2}{(1-u)^2}, \]
we deduce from Theorem 1.1 that if \( x_k = \chi k \), then
\[ T_k(x_k) = \frac{1}{\sqrt{2\pi\sigma_Z}} \frac{G(x_k,u)}{u^k} (1 + o(1)), \quad (k,x_k)_{\chi} \to \infty, \]
where \( u \) is determined by (1.9) with \( H'(u) = (1 - u)^{-1} \) (4.14) and therefore
\[ u_k = \frac{1}{1 + \chi} = \frac{k}{k + x_k}. \] (4.15)

Then
\[ T_k(x_k) = \frac{\sqrt{x_k}}{\sqrt{2\pi k(k+x_k)}} \left( 1 + \frac{x_k}{k} \right)^k \left( 1 + \frac{k}{x_k} \right)^{x_k} (1 + o(1)), \quad (k,x_k)_{\chi} \to \infty. \] (4.16)

From the other hand, using identity [24]
\[ T_k(x) = \sum_{(l_1,\ldots,l_k)^*} \frac{x^{l_1+\cdots+l_k}}{l_1! \cdots l_k!} = \frac{x(x+1)\cdots(x+k-1)}{k!}, \] (4.17)
we see that (4.16) can be obtained from (4.17) by simple use of the Stirling formula (2.33) in the case when \( x_k = \chi k, \ k \to \infty \). It follows from relation (4.17) that
\[ M_k(x) = \frac{x}{x+k} \frac{(x+k)!}{x!}. \]

Regarding the asymptotic regime \( 1 \ll x_k \ll k \), we can use the Stirling formula (2.33) and write that
\[ M_k(x) = \left( \frac{k + x}{e} \right)^k \left( \frac{x + k}{x} \right)^{x-1/2} (1 + o(1)). \]
Remembering (4.15), we get the following relation,
\[ P(Z^{(x,u)} = k) = \frac{M_k(x)}{k!} \frac{u^k}{(1-u)^x} \]
\[ = \frac{k^k}{e^{k!}} \left( \frac{x + k}{x} \right)^{x-1/2} \left( \frac{x}{k + x} \right)^{x} (1 + o(1)) = \frac{\sqrt{x}}{\sqrt{2\pi k(k+x)}} (1 + o(1)), \quad k \to \infty. \] (4.18)
From another hand, the definition of $\sigma_Z^2$ means that
\[
\sigma_Z = \frac{\sqrt{x-u}}{1-u} (1 + o(1)) = \frac{k}{\sqrt{x}} (1 + o(1)), \quad 1 \ll x \ll k.
\]
Comparing this expression with the right-hand side of (4.18), we conclude that (cf. (2.21))
\[
P(Z(x,u) = k) = \frac{1}{\sqrt{2\pi\sigma_Z}} \left(1 + o(1)\right), \quad k, x_k \to \infty
\]
and thus that the Local Limit Theorem holds for random variables $Z^{(x,u)}$. This means that the restriction $m > 8$ imposed on the Gamma distribution in subsection 3.1, or more generally, conditions (16), (1.7) and (1.8) could be of rather technical character.

### 4.4 Concentration property of normalized sums

Relations (1.13) is closely related with a law of large numbers for the random variable $Y_\lambda/\lambda$ as $\lambda \to \infty$ that is a known elementary fact. However, Theorem 1.1 gives more information about the limiting behavior of this variable. Indeed, given $y > 0$, we deduce from the first asymptotic equality of (1.13) that
\[
P \left( \frac{1}{\lambda_k} Y_{\lambda_k} > y \right) \leq \left( \frac{V_1}{y} (1 + o(1)) \right)^k, \quad (x, k)_0 \to \infty.
\]
(4.19)
The series of these probabilities converges for any $y > V_1$ and therefore by the Borel-Cantelli lemma,
\[
P \left( \limsup_{k \to \infty} \frac{Y_{\lambda_k}}{\lambda_k} \leq V_1 \right) = 1, \quad \lambda_k \gg k.
\]
(4.20)
Regarding the moments of centered random variables $\bar{Y}_\lambda = Y_\lambda - \lambda V_1$,
\[
\bar{M}_k(\lambda) = E \bar{Y}_\lambda^k
\]
we can prove analog of Theorem 1.1. Indeed, one can introduce auxiliary random variables $\bar{Z}^{(x,u)}$ by relation of the form (1.2), where $G(x, u)$ is replaced by
\[
\bar{G}(x, u) = \sum_{j=0}^{\infty} \bar{M}_j(x) \frac{w^j}{j!} = \exp \left\{ x (\bar{H}(u) - 1) \right\},
\]
where $\bar{H}(u) = H(u) - V_1 u$. Then all computations of the proof of Theorem 1.1 can be literally repeated. As a consequence, we can write in complete analogy with the second relation of (1.13) that for any $\varepsilon > 0$,
\[
P \left( \frac{1}{\lambda_k} |Y_{\lambda_k}| > \varepsilon \right) \leq \frac{1}{\varepsilon^{2k}} \bar{M}_{2k}(\lambda_k) = \left( \frac{2V_2k}{\varepsilon \lambda_k \varepsilon^2} (1 + o(1)) \right)^k, \quad 1 \ll k \ll \lambda_k.
\]
(4.21)
This upper estimate implies convergence of $Y_{\lambda_k}/\lambda_k$ to $V_1$ with probability 1 as $k \to \infty$ provided $k = o(\lambda_k)$ in this limit.
In the asymptotic regime when \( \lambda_k = \chi k \), \( k \to \infty \), we deduce from (1.15) the following version of (4.21),

\[
P\left( \frac{1}{\lambda_k} |\bar{Y}_{\lambda_k}| > y \right) \leq \left( \frac{e^{\bar{\Psi}(\chi)}}{y} \left(1 + o(1)\right) \right)^{2k},
\]

where

\[
\bar{\Psi}(\chi) = \frac{\bar{H}(u) - 1}{u\bar{H}'(u)} - 1 - \ln \bar{H}'(u)
\]

and \( u \) is determined by equation

\[u\bar{H}'(u) = \frac{1}{\chi}, \quad u > 0.\]

Upper bounds (4.20), (4.21) and (4.22) can be applied to a maximum of \( n \) independent random variables

\[
\frac{1}{\lambda} Y^{(i)}_{\lambda}, \quad i = 1, \ldots, n
\]

and their centered versions. Then one can obtain a number of statements in the spirit of the Erdős-Rényi limit theorem [12]. In particular, relation (4.22) will lead to the following upper bound

\[
P\left( \max_{i=1,\ldots,n} \frac{1}{\lambda_k} |Y^{(i)}_{\lambda_k}| > y \right) \leq n \left( \frac{e^{\bar{\Psi}(\chi)}}{y} \left(1 + o(1)\right) \right)^{2k}, \quad k \to \infty.
\]

It says that for any \( C > 0 \), \( k = C \log n \) and \( \lambda_k = \chi k \), there exists \( y = y(C, \chi) \) such that the superior limit of random variables

\[
T^{(n,\lambda_n)} = \max_{i=1,\ldots,n} \frac{1}{\lambda_n} Y^{(i)}_{\lambda_n}
\]

remains bounded with probability 1 when \( \lambda_n = \chi C \log n \) as \( n \to \infty \).

The Erdős-Rényi limit theorem says that the maximum of \( n \) random variables

\[
U^{(n,p)} = \max_{i=1,\ldots,n} \frac{1}{p} \sum_{j=1}^{p} X^{(i)}_{j},
\]

where \( \{X^{(i)}_{j}\} \) is a family of i.i.d. random variables converges, as \( n \to \infty \) and \( p = \tau \log n \) to a non-random limit \( \alpha \) determined by \( \tau \) and the exponential generating function of the moments of \( X^{(i)}_{j} \). A generalization of this statement to the case of random variables \( T^{(n,\lambda_n)} \) has been proved in [16, 17].

### 4.5 Maximal vertex degree of weighted random graphs

Let us consider a family of i.i.d. random variables

\[
\mathcal{A}^{(n,\rho)} = \{a^{(n,\rho)}_{ij}, \quad 1 \leq i < j \leq n\},
\]
where \( a_{ij}^{(n, \rho)} \) take values 1 and 0 with probability \( \rho/n \) and \( 1 - \rho/n \). Real symmetric random matrix whose elements above the diagonal are given by (4.26) and by zero otherwise can be regarded as an adjacency matrix of a random graph with \( n \) vertices known as the Erdős-Rényi ensemble of random graphs [6]. The weighted version is given by random symmetric matrices with the elements above the diagonal

\[
\left( A^{(n, \rho)} \right)_{ij} = a_{ij}^{(n, \rho)} X_j^{(i)} , \quad 1 \leq i < j \leq n
\]

and zero on the diagonal. Then random variables

\[
D_\rho^{(i)}(n) = \sum_{j=1,\ldots,n, j \neq i} a_{ij}^{(n, \rho)} X_j^{(i)}
\]

will play the role of the vertex degree of weighted random graphs. Regarding the random weights given by i.i.d. \( X_j^{(i)} \), \( D_\rho^{(i)}(n) \) can be regarded as a pre-limiting realization of random variables \( Y^{(i)} \) of (4.23). The only difference is that random variables \( D_\rho^{(i)}(n) \) and \( D_\rho^{(i')}(n) \) are not independent. However, this does not avoid the upper estimate of the deviation probabilities of the maximal vertex degree

\[
D_\rho^{(\text{max})}(n) = \max_{i=1,\ldots,n} D_\rho^{(i)}(n).
\]

In analogy with (4.24), we can prove that given a sequence \( \rho_n = \kappa \log n \), the following relation

\[
\lim_{n \to \infty} P \left( \left| \frac{D_\rho^{(\text{max})}(n)}{\rho_n} - V_1 \right| > s \right) = 0 \quad (4.27)
\]

for any \( s \) such that

\[
s > \hat{H}'(u) \exp \left\{ \frac{\hat{H}(u) - 1}{u \hat{H}'(u)} - \frac{1}{2} \right\},
\]

where \( u \) is determined by equation

\[
u \hat{H}'(u) = \frac{1}{\kappa}
\]

and

\[
\hat{H}(u) = H(u) - u \mathbb{E}(X_j^{(i)}), \quad H(u) = \sum_{k=0}^{\infty} \frac{u^k}{k!} \mathbb{E}(X_j^{(i)})^k.
\]

If \( \rho_n = \kappa_n \log n \) is such that \( \kappa_n \to \infty \) as \( n \to \infty \), then

\[
\lim_{n \to \infty} \frac{D_\rho^{(\text{max})}(n) / \rho_n}{V_1} = 1 \quad \text{with probability 1}. \quad (4.28)
\]

Moreover, one can show that if \( a \) is a decreasing sequence \( (s_n)_{n \in \mathbb{N}} \) such that

\[
s_n > \sqrt{e V_2 / \kappa_n} + V_1 / \kappa_n,
\]

then

\[
P \left( \limsup_{n \to \infty} \left| \frac{D_\rho^{(\text{max})}(n) / \rho_n - V_1}{s_n} \right| > 1 \right) = 0. \quad (4.29)
\]

Relations (4.27), (4.28) and (4.29) can be deduced from Theorem 1.1 with the help of standard arguments of probability theory. We do not present the proofs here.
4.6 Random matrices and even walks with multiple edges

In this subsection we describe one more situation when the moments of compound Poisson distribution can be useful in the random matrix theory. Random real symmetric matrix of the Wigner ensemble [30] is given by relation

\[
\left(W^{(n)}\right)_{ij} = \frac{1}{\sqrt{n}} w_{ij}, 1 \leq i < j, \quad \left(W^{(n)}\right)_{ii} = 0,
\]

where \(W = \{w_{ij}, 1 \leq i \leq j\}\) is a family of joint independent identically distributed random variables. The eigenvalue distribution of the matrices \(W^{(n)}\) can be studied with the help of the moments

\[
\mu_{i}^{(n)} = \mathbb{E} \left( \frac{1}{n} \text{Tr} \left(W^{(n)}\right)^{i}\right) = \frac{1}{n} \sum_{i=1}^{n} \sum_{i_2, i_3, \ldots, i_{k-1}} \mathbb{E} \left(W_{i_1 i_2}^{(n)} W_{i_2 i_3}^{(n)} \cdots W_{i_{k-1} i_k}^{(n)}\right).
\]

The last sum of (4.30) can be regarded as a sum over all possible sequences \(I^{(i_1)}_{l} = (i_2, i_3, \ldots, i_l)\) with corresponding weights. Adding the starting point \(i_1\), this sequence can be represented as a multi-graph with the set of vertices \(V_n = \{1, 2, \ldots, n\}\) and \(k\) oriented edges.

If one assumes that the probability distribution of \(W_{ij}\) is symmetric, then non-zero contributions to (4.30) are given by sequences \(I^{(i_1)}_{l-2}\), whose graphs have vertices connected by an even number of edges. Thus one has to take \(l = 2k\). A particular case of such an even sequence is given by

\[
\mathcal{I}^{(i_1)}_{2k} = (i_2, i_1, i_4, i_1, i_6, i_1, \ldots, i_1, i_k).
\]

In the corresponding graph, we start with the vertex \(i_1\), go to the vertex \(i_2\), then return to \(i_1\), then go to \(i_4\), etcetera. In this construction, all variables \(i_j\) with \(j \neq 1\) differ from \(i_1\).

It is not hard to see that the sum over all possible sequences \(\mathcal{I}^{(i_1)}_{2k-2}\) with corresponding weights

\[
\mathcal{P} \mathcal{I}^{(i_1)}_{2k} = \frac{1}{n^k} \mathbb{E} (w_{i_1 i_2} w_{i_2 i_3} w_{i_3 i_4} w_{i_4 i_1} w_{i_1 i_2} w_{i_2 i_3} \cdots w_{i_1 i_2} w_{i_2 i_{2k-1}})
\]

is given by expression of the form (1.2), also resembling (4.2A),

\[
\sum_{(i_2, i_4, i_6, \ldots, i_{2k}), i_{2j} \neq i_1} \mathcal{P} \mathcal{I}^{(i_1)}_{2k} = \frac{1}{n^k} \sum_{(i_2, i_4, i_6, \ldots, i_{2k})} \prod_{i=1}^{k} \frac{1}{i!} \frac{(n-1)W_{ij}}{i!} = \frac{1}{n^k} M_k^{(W)} (n - 1),
\]

where we denoted \(W_{2l} = \mathbb{E}(W_{ij})^{2l}, l \in \mathbb{N}\).

Assuming that the function \(H_{W}(u) = \sum_{j=0}^{\infty} u^{2j} W_{2j}/(2j)!\) exists, we can use the results of Theorem 1.1 and say that

\[
\frac{1}{n^k} M_k^{(W)} (n - 1) = (W_{2}(1 + o(1)))^k, \quad 1 \ll k \ll n
\]

and that

\[
\frac{1}{n^k} M_k^{(W)} (n - 1) = \left( e^{\psi_{W}(\chi)} (1 + o(1)) \right)^k, \quad k = \chi n, n \to \infty,
\]

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where \( \Psi_W(\chi) \) and \( u \) are determined by relations (1.16) and (1.17) with \( H(u) \) replaced by \( H_W(u) \). One can say that relation (4.32) considered with \( k = Cn^{2/3} \) represents a kind of a proof of a known result (see [27], relation (4.29) and also [15]), while the estimate from below \( \mu_{2k}^{(n)} \geq \left( e^{\Psi_W(\chi)}(1 + o(1)) \right)^k \) that follows from (4.33) can be regarded as a new result.

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