Darwinism in quantum systems?

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Abstract

We investigate the role of quantum mechanical effects in the central stability concept of evolutionary game theory i.e. an Evolutionarily Stable Strategy (ESS). Using two and three-player symmetric quantum games we show how the presence of quantum phenomenon of entanglement can be crucial to decide the course of evolutionary dynamics in a population of interacting individuals.

1 Introduction

Many interesting results in the recently developed quantum game theory [1, 2, 3, 4] are about the most fundamental idea in noncooperative game theory i.e. the Nash equilibrium (NE). A strategy profile is a NE [5] if no player can gain by unilaterally deviating from it. The implicit assumption behind NE is that players make their choices simultaneously and independently. This idea also assumes that each player participating in a game behaves rational and searches to maximize his own payoff. In situations where evolution of complex behavior occurs further refinements of NE concept are required [6], especially, when multiple NE appear in the analysis of a game. A refinement is then needed to prefer some NE over the others. Due attention has already been given to NE in the recent works [4, 5] on quantum games and this development has motivated us to study certain refinement notions of NE in quantum games. Refinements of NE in classical game theory are popular as well as numerous [6]. Speaking historically, the set of refinements became so large that eventually almost any NE could be justified in terms of someone or other’s refinement [7].

An interesting, and fruitful as well, refinement of NE was introduced by Maynard Smith in 1970’s that became the central notion of evolutionary game theory. In his book Evolution and the Theory of Games [8] he diverted attention away from elaborate definitions of rationality and presented an evolutionary approach in classical game theory. The evolutionary approach can be seen as a
large population model of adjustment to a NE i.e. an adjustment of population segments by evolution as opposed to learning. Contrary to classical game theory, in evolutionary game theory the individuals of a population, subject to evolution, are not assumed to act consciously and rationally. Many successful applications of evolutionary game theory appeared in mathematical biology to predict the behavior of bacteria and insects that can hardly be said to think at all. The most important feature of evolutionary game theory is that the assumption of rational players, originating from classical game theory, does not remain crucial. This important aspect appears when players’ payoffs are equated to their reproductive success.

The concept of evolutionary stability stimulated the development of evolutionary game theory that establishes a link between game theory and the theory of evolution. Presently the ESS theory is the central model of evolutionary dynamics of a populations of interacting individuals. It asks, and finds answer to it, a basic question: which states, during the course of selection process, of a given population are stable against perturbations induced by mutations. The ESS theory is based on Darwin’s idea of natural selection; which is shown to be describable as an algorithm called replicator dynamic. Iterations of selections from randomly mutating replicators is an important feature of the dynamic. Speaking the language of game theory, the replicator dynamic says that in a population the proportion of players which play better strategies increase with time. When replicator dynamic is underlying process of a game the ESSs are shown to be stable against perturbations. In other words ESSs are, then, rest points of the replicator dynamic.

Recent developments in quantum games provide an incentive to look at the mathematical theory of evolution, with the central idea of an ESS, in a broader picture given by Hilbert structure of strategy space in the new theory. This incentive is driven by a question: how game-theoretical models, of evolutionary dynamics in a population, shape themselves in the new settings provided to the game theory recently by quantum mechanics? Our motivation is that quantum mechanical effects, especially entanglement, may have decisive role in the evolutionary dynamics that have already been successfully, and also quite rigorously, modelled using the classical game theory. To study evolution in quantum settings we have chosen the ESS idea mostly for simplicity and beauty of the concept. We ask questions like, how ESSs are affected when a classical game played in a population changes itself to one of its quantum forms? How pure and mixed ESSs are distinguished from one another when such a change in form of a game takes place? And most importantly, how evolutionary dynamics becomes linked to quantum entanglement present in games that are played in quantum settings?

In earlier papers we showed that the presence of entanglement, in asymmetric as well as symmetric bimatrix games, can disturb the evolutionary stability expressed by the idea of ESS. So that, evolutionary stability of a symmetric NE can be made to appear or disappear by controlling entanglement in symmetric and asymmetric bimatrix games. We found a consideration of symmetric games more appropriate because the notion of an ESS was origi-
nally investigated for pair-wise symmetric contests. In present letter we find examples of two and three-player games where entanglement changes evolutionary stability of a symmetric NE. We assume initial quantum states in the same, originally suggested, simpler form presented in the scheme that tells how to play a quantum game. With initial states in this form, we show that two and three-player games are distinguished in an interesting aspect i.e. entanglement can change evolutionary stability of pure strategies in two-player games. However, for a mixed strategy it can be done when the number of players are increased from two to three. Before coming to quantum settings we first describe, mathematically, the concept of an ESS in classical evolutionary game theory.

2 Evolutionary stability

Maynard Smith introduced the idea of an Evolutionarily Stable Strategy (ESS) in a seminal paper ‘The logic of animal conflict’. In rough terms, an ESS is a strategy which, if played by almost all members of a population, cannot be displaced by a small invading group playing any alternative strategy. Suppose pairs of individuals are repeatedly drawn at random from a large population to play a symmetric two-person game. Let the game between the individuals be a symmetric bimatrix game represented by the expression $G = (M, M^T)$ where $M$ represents the payoff matrix and $T$ its transpose. In usual notation $P(x, y)$ gives the payoff to a $x$-player against a $y$-player for a symmetric pair-wise contest. A strategy $x$ is said to be an ESS if for each mutant strategy $y$ there exists a positive invasion barrier such that if the population share of individuals playing the mutant strategy $y$ falls below this barrier, then $x$ earns a higher expected payoff than $y$. Mathematically speaking, $x$ is an ESS when for each strategy $y \neq x$ the inequality $P[x, (1 - \epsilon)x + \epsilon y] > P[y, (1 - \epsilon)x + \epsilon y]$ should hold for all sufficiently small $\epsilon > 0$; where, for example, the expression on the left-hand side is payoff to strategy $x$ when played against the mixed strategy $(1 - \epsilon)x + \epsilon y$. This condition for an ESS can be shown easily to be equivalent to the following two requirements

1. $P(x, x) > P(y, x)$
2. If $P(x, x) = P(y, x)$ then $P(x, y) > P(y, y)$

(1)

It becomes apparent that an ESS is a symmetric NE but also possesses a stability property against mutations. Condition 1 in the above definition shows that $(x, x)$ is a NE for the bimatrix game $G = (M, M^T)$ if $x$ is an ESS. However, the reverse is not true. If $(x, x)$ is a NE, then $x$ is an ESS only if $x$ satisfies condition 2 in the definition. The ESS condition gives a refinement on the set of symmetric Nash equilibria. Its essential feature is that, apart from being a symmetric NE, it is robust against a small number of mutants appearing in a population playing an ESS.

Now we consider quantum settings, involving two players, to investigate the concept of evolutionary stability.
3 Two-player case

We use the quantization scheme suggested by Marinatto and Weber for the two-player quantum game of Battle of Sexes. In this scheme the players’ tactics consist of deciding the classical probabilities of applying two unitary and Hermitian operators (the identity $I$ and the inversion operator $C$) on an initial quantum strategy in $2 \otimes 2$ dimensional Hilbert space; that can be obtained from a system of two qubits. The tactics phase is similar to probabilistic choice between pure strategies in the classical game theory. Interestingly, the classical form of the game is reproduced by making the initial quantum state unentangled. There are some different opinions concerning the use of the term strategy in Marinatto and Weber’s scheme. Earlier Eisert, Wilkins, and Lewenstein proposed a scheme where choosing a move corresponds to a strategy. Expanding on this work, Marinatto and Weber presented a different approach and preferred to call tactics the process of choosing a move when an initial strategy, in the form of a quantum state, is forwarded to the players. In this paper we will refer to Marinatto and Weber’s tactics as strategies or moves and their initial strategy as initial quantum state. We then find how entanglement affects evolutionary stability in the circumstances that a quantum version of a game can be reduced to its classical form by removing entanglement. Because the classical game corresponds to an unentangled initial quantum state, a comparison between ESSs in classical and quantized versions of the game can be made by maneuvering the initial quantum state, in some particular form. The scheme for two-player quantum game is shown in fig. 1.

Consider a two-player symmetric game given by the matrix

$$
\begin{array}{cc}
S_1 & S_2 \\
\hline
S_1 & (\alpha, \alpha) & (\beta, \gamma) \\
S_2 & (\gamma, \beta) & (\delta, \delta)
\end{array}
$$

(2)

and played via the initial state $|\psi_{in}\rangle = a |S_1 S_1\rangle + b |S_2 S_2\rangle$ where $|a|^2 + |b|^2 = 1$. A unitary and Hermitian operator $C$ used in the scheme is defined as $C |S_1\rangle = |S_2\rangle$, $C |S_2\rangle = |S_1\rangle$ and $C^\dagger = C = C^{-1}$. Let one of the players chooses his strategy by implementing the identity operator $I$ with probability $p$ and the operator $C$ with probability $(1 - p)$, on the initial state $\rho_{in}$ that corresponds to $|\psi_{in}\rangle$. Similarly, suppose the second player applies the operators $I$ and $C$ with probabilities $q$ and $(1 - q)$ respectively. The final density matrix is written as

$$
\rho_{fin} = \sum_{U=I,C} \Pr(U_A) \Pr(U_B) [U_A \otimes U_B \rho_{in} U_A^\dagger \otimes U_B^\dagger]
$$

(3)

where the unitary and Hermitian operator $U$ is either $I$ or $C$. $\Pr(U_A)$, $\Pr(U_B)$ are the probabilities with which players $A$ and $B$ apply the operator $U$ on
Figure 1: The scheme to play a two-player quantum game.

the initial state, respectively. It is seen that $\rho_{fin}$ corresponds to a convex combination of all possible quantum operations. Payoff operators for Alice and Bob are $\rho_{oper} = \alpha, \alpha \langle S_1 S_1 \rangle + \beta, \gamma \langle S_1 S_2 \rangle + \beta, \beta \langle S_2 S_1 \rangle + \delta, \delta \langle S_2 S_2 \rangle$.

The payoffs are then obtained as mean values of these operators i.e. $P_{A,B} = Tr [(P_{A,B})_{oper} \rho_{fin}]$. Because the quantum game is symmetric using the initial state $|\psi_{in}\rangle$ and the payoff matrix (2), there is no need for subscripts. We, therefore, write the payoff to a $p$ player against a $q$ player as $P(p,q)$. When $p$ is a NE we find the following payoff difference $\frac{1}{2}$.

$$P(\hat{p}, \hat{p}) - P(p, \hat{p}) = (\hat{p} - p) [a^2 (\beta - \delta) + |b|^2 (\gamma - \alpha) - \hat{p} ((\beta - \delta) + (\gamma - \alpha))]$$ (4)

Now the ESS conditions for the pure strategy $p = 0$ are given as

1. $|b|^2 ((\beta - \delta) - (\gamma - \alpha)) > (\beta - \delta)$
2. If $|b|^2 ((\beta - \delta) - (\gamma - \alpha)) = (\beta - \delta)$
   then $q^2 ((\beta - \delta) + (\gamma - \alpha)) > 0$ (5)
where 1 is the NE condition. Similarly the ESS conditions for the pure strategy \( p = 1 \) are

1. \( |b|^2 \{ (\gamma - \alpha) - (\beta - \delta) \} > (\gamma - \alpha) \)
2. If \( |b|^2 \{ (\gamma - \alpha) - (\beta - \delta) \} = (\gamma - \alpha) \)
   then \( (1 - q)^2 \{ (\beta - \delta) + (\gamma - \alpha) \} > 0 \) (6)

Because these conditions for both the pure strategies \( p = 1 \) and \( p = 0 \) depend on \( |b|^2 \), therefore, there can be examples of two-player symmetric games for which the evolutionary stability of pure strategies can be changed while playing the game using initial state in the form \( |\psi_{in}\rangle = a |S_1 S_1\rangle + b |S_2 S_2\rangle \). However, for the mixed NE, given as \( \hat{p} = \frac{|a|^2 (\beta - \delta) + |b|^2 (\gamma - \alpha)}{(\beta - \delta) + (\gamma - \alpha)} \), the corresponding payoff difference becomes identically zero. From the second condition of an ESS we find, for the mixed NE \( \hat{p} \), the difference

\[
P(\hat{p}, \hat{q}) - P(q, q) = \frac{1}{(\beta - \delta) + (\gamma - \alpha)} \times \\
\{ (\beta - \delta) - q \{ (\beta - \delta) + (\gamma - \alpha) \} \} - |b|^2 \{ (\beta - \delta) - (\gamma - \alpha) \} \}^2 \]  

(7)

Therefore, the mixed strategy \( \hat{p} \) is an ESS when \( \{ (\beta - \delta) + (\gamma - \alpha) \} > 0 \). This condition, making the mixed NE \( \hat{p} \) an ESS, is independent of \( |b|^2 \). So that, in this symmetric two-player quantum game, evolutionary stability of the mixed NE \( \hat{p} \) can not be changed when the game is played using initial quantum states of the form \( |\psi_{in}\rangle = a |S_1 S_1\rangle + b |S_2 S_2\rangle \).

Therefore, evolutionary stability of only the pure strategies can be affected, with the chosen form of the initial states, for the two-player symmetric games. Examples of the games with this property are easy to find. The class of games for which \( \gamma = \alpha \) and \( (\beta - \delta) < 0 \) the strategies \( p = 0 \) and \( p = 1 \) remain NE for all \( |b|^2 \in [0, 1] \); but the strategy \( p = 1 \) is not an ESS when \( |b|^2 = 0 \) and the strategy \( p = 0 \) is not an ESS when \( |b|^2 = 1 \). In an earlier letter \[12\] we found an example of a class of games for which a pure strategy, that is an ESS classically, does not remain ESS for a particular value of \( |b|^2 \), even though it remains a NE for all possible range of \( |b|^2 \).

To find examples of symmetric quantum games, where evolutionary stability of the mixed strategies may also be affected by controlling the entanglement, we now increase the number of players from two to three.

### 4 Three-player case

In extending the two-player scheme to a three-player case, we assume that three players \( A, B, \) and \( C \) play their strategies by applying the identity operator \( I \) with the probabilities \( p, q \) and \( r \) respectively on the initial state \( |\psi_{in}\rangle \). Therefore,
they apply the operator $C$ with the probabilities $(1 - p), (1 - q)$ and $(1 - r)$ respectively. The final state then corresponds to the density matrix

$$\rho_{\text{fin}} = \sum_{U=I,C} \Pr(U_A) \Pr(U_B) \Pr(U_C) \left[ U_A \otimes U_B \otimes U_C \rho_{\text{in}} U_A^\dagger \otimes U_B^\dagger \otimes U_C^\dagger \right]$$ \hspace{1cm} (8)$$

where the 8 basis vectors are $|S_i S_j S_k \rangle$, for $i, j, k = 1, 2$. Again we use initial quantum state in the form $|\psi_{\text{ini}}\rangle = a |S_1 S_1 S_1 \rangle + b |S_2 S_2 S_2 \rangle$, where $|a|^2 + |b|^2 = 1$. It is a quantum state in $2 \otimes 2 \otimes 2$ dimensional Hilbert space that can be prepared from a system of three two-state quantum systems or qubits. Similar to the two-player case, we define the payoff operators for the players $A$, $B$, and $C$ as

$$(P_{A,B,C})_{\text{oper}} =$$

$$\alpha_1, \beta_1, \eta_1 \ |S_1 S_1 S_1 \rangle \langle S_1 S_1 S_1 | + \alpha_2, \beta_2, \eta_2 \ |S_2 S_1 S_1 \rangle \langle S_2 S_1 S_1 | + \alpha_3, \beta_3, \eta_3 \ |S_1 S_2 S_1 \rangle \langle S_1 S_2 S_1 | + \alpha_4, \beta_4, \eta_4 \ |S_1 S_1 S_2 \rangle \langle S_1 S_1 S_2 | + \alpha_5, \beta_5, \eta_5 \ |S_1 S_2 S_2 \rangle \langle S_1 S_2 S_2 | + \alpha_6, \beta_6, \eta_6 \ |S_2 S_1 S_2 \rangle \langle S_2 S_1 S_2 | + \alpha_7, \beta_7, \eta_7 \ |S_2 S_2 S_1 \rangle \langle S_2 S_2 S_1 | + \alpha_8, \beta_8, \eta_8 \ |S_2 S_2 S_2 \rangle \langle S_2 S_2 S_2 |$$ \hspace{1cm} (9)$$

where $\alpha_l, \beta_l, \eta_l$ for $1 \leq l \leq 8$ are 24 constants of the matrix of this three-player game. Payoffs to the players $A$, $B$, and $C$ are then obtained as mean values of these operators

$$P_{A,B,C}(p, q, r) = \text{Trace} \left[ (P_{A,B,C})_{\text{oper}} \rho_{\text{fin}} \right]$$ \hspace{1cm} (10)$$

Here, similar to two player case, the classical payoffs can be obtained by making the initial quantum state unentangled and fixing $|b|^2 = 0$. To get a symmetric game we define $P_A(x, y, z)$ as the payoff to player $A$ when players $A$, $B$, and $C$ play the strategies $x,y$ and $z$ respectively. Following relations make payoffs to the players a quantity that is identity independent but depends only on their strategies

$$P_A(x, y, z) = P_A(x, z, y) = P_B(y, x, z) = P_B(z, x, y) = P_C(y, z, x) = P_C(z, y, x)$$ \hspace{1cm} (11)$$

For these relations to hold we need following replacements for $\beta_i$ and $\eta_i$

$$\begin{align*}
\beta_1 &\rightarrow \alpha_1 & \beta_2 &\rightarrow \alpha_3 & \beta_3 &\rightarrow \alpha_2 & \beta_4 &\rightarrow \alpha_3 \\
\beta_5 &\rightarrow \alpha_6 & \beta_6 &\rightarrow \alpha_5 & \beta_7 &\rightarrow \alpha_6 & \beta_8 &\rightarrow \alpha_8 \\
\eta_1 &\rightarrow \alpha_1 & \eta_2 &\rightarrow \alpha_3 & \eta_3 &\rightarrow \alpha_3 & \eta_4 &\rightarrow \alpha_2 \\
\eta_5 &\rightarrow \alpha_6 & \eta_6 &\rightarrow \alpha_6 & \eta_7 &\rightarrow \alpha_5 & \eta_8 &\rightarrow \alpha_8 \end{align*}$$ \hspace{1cm} (12)$$
Also, it is now necessary that we should have $\alpha_6 = \alpha_7$ and $\alpha_3 = \alpha_4$. A symmetric game between three players, therefore, can be defined by only six constants. We take these to be $\alpha_1, \alpha_2, \alpha_3, \alpha_5, \alpha_6$ and $\alpha_8$. The payoff to a player now becomes only a strategy dependent quantity and becomes identity independent. No subscripts are therefore, needed. Payoff to a player, when other two players play $q$ and $r$, can now be written as $P(p, q, r)$. A symmetric NE $\hat{p}$ can be found from the Nash condition $P(\hat{p}, \hat{p}, \hat{p}) - P(p, \hat{p}, \hat{p}) \geq 0$ i.e.

$$P(\hat{p}, \hat{p}, \hat{p}) - P(p, \hat{p}, \hat{p}) = (\hat{p} - p)\left[\hat{p}^2(1 - 2|b|^2)(\sigma + \omega - 2\eta) + 2\hat{p}\left(|b|^2(\sigma + \omega - 2\eta) - \omega + \eta\right) + \left(\omega - |b|^2(\sigma + \omega)\right)\right] \geq 0$$

(13)

where $(\alpha_1 - \alpha_2) = \sigma$, $(\alpha_3 - \alpha_6) = \eta$, and $(\alpha_5 - \alpha_8) = \omega$. Three possible NE are found as

$$\hat{p}_1 = \left\{\frac{(\omega - \eta) - |b|^2(\sigma + \omega - 2\eta)}{(1 - 2|b|^2)(\sigma + \omega - 2\eta)}\right\} \pm \sqrt{\left((\sigma + \omega)^2 - (2\eta)^2\right)\left[|b|^2(\sigma + \omega - 2\eta)^2 + (\eta^2 - \sigma^2)\right]}$$

$$\hat{p}_2 = 0$$

$$\hat{p}_3 = 1$$

(14)

Clearly the mixed NE $\hat{p}_1$ makes the difference $P(\hat{p}, \hat{p}, \hat{p}) - P(p, \hat{p}, \hat{p})$ identically zero and two values for $\hat{p}_1$ can be found for a given $|b|^2$. $\hat{p}_2$, $\hat{p}_3$ are pure strategy NE. We notice that $\hat{p}_1$ comes out as a NE without imposing further restrictions on the matrix of the symmetric three-player game. However, the pure strategies $\hat{p}_2$ and $\hat{p}_3$ can be NE when further restriction are imposed on the matrix of the game. For example, $\hat{p}_2$ can be a NE provided $\sigma \geq (\omega + \sigma)|b|^2$ for all $|b|^2 \in [0, 1]$. Similarly $\hat{p}_3$ can be NE when $\omega \leq (\omega + \sigma)|b|^2$.

Now the question we ask: how the evolutionary stability of these three NE can be affected while playing the game via the initial quantum states given in the form $|\psi_{in}\rangle = a|S_1 S_1 S_1\rangle + b|S_2 S_2 S_2\rangle$. For the two-player asymmetric game of Battle of Sexes we showed that out of the three NE only two can be evolutionarily stable [8]. In classical evolutionary game theory the concept of an ESS is well known to be extended to multi-player case. When mutants are allowed to play only one strategy the definition of an ESS for three-player case is written as [8]

1. $P(p, p, p) > P(q, p, p)$
2. If $P(p, p, p) = P(q, p, p)$ then $P(p, q, p) > P(q, q, p)$

(15)

Here $p$ is a NE if it satisfies the condition 1 against all $q \neq p$. For our case the ESS conditions for the pure strategies $\hat{p}_2$ and $\hat{p}_3$ can be written as follows. For example, $\hat{p}_2 = 0$ is an ESS when
An interesting class of three-player games is one with $\eta$ out of four possible NE in this three-player game only three can be ESs. Similarly, when $|\eta| < \eta_2$, we have only one mixed NE, that is not an ESS. Therefore, out of the two possible roots $p_2$ becomes zero we have only one mixed NE that is not an ESS. However for all $|\eta| > \eta_2$, we generally obtain two NE out of which one can be an ESS.

Therefore, out of the two possible roots $(p_1)_1$ and $(p_1)_2$, that make the difference $P(p_1,q,p_1) - P(q,q,p_1)$ greater than and less than zero respectively, of the quadratic equation

$$p_1^2(1 - 2|b|^2)(\sigma + \omega - 2\eta) + 2p_1\left\{(|b|^2(\sigma + \omega - 2\eta) - \omega + \eta) + \omega - |b|^2(\sigma + \omega)\right\} = 0 \quad (19)$$

only $(p_1)_1$ can exist as an ESS. When the square root term in the equation (18) becomes zero we have only one mixed NE, that is not an ESS. Therefore, out of four possible NE in this three-player game only three can be ESs. An interesting class of three-player games is one with $\eta^2 = \sigma \omega$. For these games the mixed NE are $p_1 = \frac{(1 - 2|b|^2)(\sigma + \omega - 2\eta)}{\pm|a||b|(\sigma - \omega)}$, and, when played classically, we can get only one mixed NE that is not an ESS. However for all $|b|^2$, different from zero, we generally obtain two NE out of which one can be an ESS.

Similar to the two-player case, the NE in a three-player symmetric game important from the point of view of evolutionary stability are those that survive
a change between two initial states; one being unentangled corresponding to the classical game. Suppose $\hat{p}_1$ remains a NE for $|b|^2 = 0$ and some other non-zero $|b|^2$. It is possible when $(\sigma - \omega)(2\hat{p}_1 - 1) = 0$. One possibility is the strategy $\hat{p} = \frac{1}{2}$ remaining a NE for all $|b|^2 \in [0, 1]$. It reduces the defining quadratic equation for $\hat{p}_1$ to $\sigma \omega + 2 \eta = 0$ and makes the difference $P(\hat{p}_1, q, \hat{p}_1) - P(q, q, \hat{p}_1)$ independent of $|b|^2$. Therefore the strategy $\hat{p} = \frac{1}{2}$, even when remaining a NE for all $|b|^2 \in [0, 1]$, can not be an ESS in only one version of the symmetric three-player game. For the second possibility $\sigma = \omega$ the defining equation for $\hat{p}_1$ is reduced to

\[
(1 - 2|b|^2) \left\{ \hat{p}_1 - \frac{(\eta - \sigma) - \sqrt{\eta^2 - \sigma^2}}{2(\eta - \sigma)} \right\} \left\{ \hat{p}_1 - \frac{(\eta - \sigma) + \sqrt{\eta^2 - \sigma^2}}{2(\eta - \sigma)} \right\} = 0
\]

(20)

for which

\[
P(\hat{p}_1, q, \hat{p}_1) - P(q, q, \hat{p}_1) = \pm 2(\hat{p}_1 - q)^2 \left| |b|^2 - \frac{1}{2} \right| \sqrt{\eta^2 - \sigma^2}
\]

(21)

Here the difference $P(\hat{p}_1, q, \hat{p}_1) - P(q, q, \hat{p}_1)$ still depends on $|b|^2$ and becomes zero for $|b|^2 = \frac{1}{2}$. Therefore for the class of games for which $\sigma = \omega$ and $\eta > \sigma$, one of the mixed strategies $(\hat{p}_1)_1, (\hat{p}_1)_2$ remains a NE for all $|b|^2 \in [0, 1]$ but not an ESS when $|b|^2 = \frac{1}{2}$. In this class of three-player quantum games the evolutionary stability of a mixed strategy can, therefore, be changed while the game is played using initial quantum states in the form $|\psi_{in}\rangle = a|S_1S_1S_1\rangle + b|S_2S_2S_2\rangle$.

5 Discussion

The fact that classical games are played in natural macroscopic world is well known for a long time. Evolutionary game theory is a subject, growing out of such studies and, dealing mostly with games played in the animal world. Recent work in biology [20] suggests nature playing classical games at micro-level. Bacterial infections by viruses have been presented as classical game-like situations where nature prefers the dominant strategies. The concept of evolutionary stability, without assuming rational and conscious individuals, gives game-theoretical models of stable states for a population of interacting individuals. Darwinian idea of natural selection provides physical ground to these models of rationality studied in evolutionary game theory. Why there is need to study evolutionary stability in quantum games? We find it interesting that some entirely quantum aspect like entanglement can have a deciding role about which stable states of the population should survive and others should not. It means that the presence of quantum interactions, in a population undergoing evolution, can alter its stable states resulting from evolutionary dynamics.
When entanglement decides the evolutionary outcomes, the role for quantum mechanics clearly increases from just keeping atoms together that constitute molecules. This new role may now also include to define and maintain complexity emerging from quantum interactions among a collection of molecules. It becomes even more interesting with reference to the example of equilibrium in a mixture of chemicals presented above. When quantum nature of molecular interactions decides the equilibria that the mixture of Schuster et al. \[21\] should be able to attain, there is a clear possibility for the quantum mechanical role in the models of self-organization in matter and the evolution of macromolecules before the advent of life.

We have two suggestions where this finding, about quantum effects deciding evolutionary outcomes in a population of interacting entities, can have a relevance

5.1 Genetic code evolution

The genetic code is the relationship between the sequence of the bases in the DNA and the sequence of amino acids in proteins. Recent work \[22\] about evolvability of the genetic code suggests that the code, like all other features of organisms, was shaped by natural selection. The question about the process and evolutionary mechanism by which the genetic code was optimized is still unanswered. Two major suggested possibilities are \(a\). A large number of codes existed out of which the adaptive one was selected. \(b\). Adaptive and error-minimizing constraints gave rise to an adaptive code via code expansion and simplification. The second possibility of code expansion from earlier simpler forms is now thought to be supported by much empirical and genetic evidence \[23\] and results suggest that the present genetic code was strongly influenced by natural selection for error minimization. Recently Patel \[24\] suggested quantum dynamics played a role in the DNA replication and the optimization criteria involved in genetic information processing. He considers the criteria involved as a task similar to an unsorted assembly operation where the Grover’s database search algorithm \[25\] fruitfully applies; given the different optimal solutions for classical and quantum dynamics. The assumption underlying this approach, as we understood it, is that an adaptive code was selected out of a large numbers that existed earlier. Recent suggestions about natural selection being the process for error minimization in the mechanism of adaptive code evolution suggests, instead, an evolutionary approach for this optimization problem. We suggest that, in the evolution and expansion of the code from its earlier simpler forms, quantum dynamics too has played important role. The mechanism, however, leading now to this optimization will be completely different. Our result that stable outcomes, of an evolutionary process based on natural selection, depend on presence or absence of quantum entanglement clearly implies the possibility that other quantum interactions may also have deciding role in obtaining some optimal outcome of evolution for a system of molecules constituting a population. The mathematically rigorous representation of stability in an evolutionary dynamics, based on Darwinian selection, is the concept of an
ESS. We believe that the code optimization is a problem having close similarities with the problem of evolutionary stability. By showing that entanglement can bring or take away evolutionary stability from a symmetric NE we have indicated that the code optimization was probably achieved by forces and effects that were quantum mechanical in nature.

5.2 Quantum evolutionary algorithms

A polynomial time algorithm that can solve an NP problem is not known yet. A viable alternative approach, shown to find acceptable solutions within a reasonable time period, is the evolutionary search [26]. Iteration of selection based on competition, random variation usually called mutation, and exploration of the fitness landscape of possible solutions are the basic ingredients of many distinct paradigms of evolutionary computing [27]. On the other hand superposition of all possible solution states, unitary operator exploiting interference to enhance the amplitude of the desired states, and final measurement extracting the solution are the components of quantum computing. These two approaches in computing are believed to represent different philosophies [28]. Finding ESSs can be easily formulated as an evolutionary algorithm with mutations occurring within only a small proportion of the total population. In fact ESSs also constitute an important technique in evolutionary computation. Our proposal that entanglement has a role in the theory of ESSs suggests that the two philosophies, considered different, may have some common grounds that possibly unites them. It also hints the possibility of other evolutionary algorithms that utilize or even exploit quantum effects. In an evolutionary algorithm, exploiting quantum effects, we may have, for example, fitness functions depending on the amount of entanglement present. The interesting question then is: how the population will evolve towards a solution or an equilibrium in relation to the entanglement?

6 Conclusion

In this paper we have shown that the idea of evolutionary stability, the central stability concept of evolutionary game theory, can be related to quantum entanglement. We investigated ESSs in three-player quantum games and compared them to two-player games played by a proposed scheme where two, Hermitian and unitary, operators are applied on an initial quantum state with classical probabilities. We used the initial quantum state in the same form proposed by Marinatto and Weber [3]. In two-player symmetric games we found that evolutionary stability of a mixed strategy cannot be changed by a unitary maneuver of the initial quantum state. However, for a class of three-player symmetric games it becomes possible to do so. It shows that the presence of quantum mechanical effects may have a deciding role on the outcomes of evolutionary dynamics in a population of interacting entities. We suggested that a relevance of these ideas may be found, for example, in the studies of the evolution of genetic code at the dawn of life. Another suggestion is designing evolutionary
algorithms where interactions between individual of a population are governed by quantum effects. The nature of these quantum effects, influencing the course of evolution, will also determine the evolutionary outcome. Quantum mechanics playing a role in the theory of ESSs implies that Darwin’s idea of natural selection has a relevance even for quantum systems.

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An alternative possibility is that we adjust $|b|^2 = (\beta - \sigma) - q(\gamma - \alpha)$ and it makes $P(p, q) - P(q, q)$ zero and the mixed strategy $\hat{p}$ does not remain an ESS. However such ‘mutant dependent’ adjustment of $|b|^2$ is not reasonable because the mutant strategy $q$ can be anything in the range $[0, 1]$. 

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These conditions required to make the strategy $\hat{p}_3 = 1$ an ESS can also be written as: (1): $\sigma > (\omega + \sigma)|b|^2$ (2): If $\sigma = |b|^2(\omega + \sigma)$ then $\frac{(\omega - \sigma)}{(\omega + \sigma)} > 0$. For the strategy $\hat{p}_2 = 0$ the ESS conditions reduce to: (1). $\omega < (\omega + \sigma)|b|^2$ (2). If $\omega = |b|^2(\omega + \sigma)$ then $\frac{(\omega - \sigma)}{(\omega + \sigma)} > 0$.

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Preparation of a two-qubit quantum state

Initial quantum state \( \Psi_{in} = |S1 S1\rangle + |S2 S2\rangle \)

Probabilistic combination of I and C

Unitary & Hermitian operators

Probabilistic combination of I and C

Alice

Two qubits

Measurement

Bob's payoff

Final state

Alice's payoff

Two qubits