MULTIPERMUTATION SOLUTIONS OF THE YANG–BAXTER EQUATION

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Abstract. Set-theoretic solutions of the Yang–Baxter equation form a meeting-ground of mathematical physics, algebra and combinatorics. Such a solution consists of a set $X$ and a function $r : X \times X \to X \times X$ which satisfies the braid relation. We examine solutions here mainly from the point of view of finite permutation groups: a solution gives rise to a map from $X$ to the symmetric group $\text{Sym}(X)$ on $X$ satisfying certain conditions.

Our results include many new constructions based on strong twisted union and wreath product, with an investigation of retracts and the multipermutation level and the solvable length of the groups defined by the solutions; and new results about decompositions and factorisations of the groups defined by invariant subsets of the solution.

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1. Introduction

Let $V$ be a vector space over a field $k$. It is well-known that the “Yang–Baxter equations” on a linear map $R : V \otimes V \to V \otimes V$, the equation

$$R_{12}R_{23}R_{12} = R_{23}R_{12}R_{23}$$

(where $R_{i,j}$ denotes $R$ acting in the $i,j$ place in $V \otimes V \otimes V$), give rise to a linear representation of the braid group on tensor powers of $V$. When $R^2 = \text{id}$ one says that the solution is involutive, and in this case one has a representation of the symmetric group on tensor powers.

A particularly nice class of solutions is provided by set-theoretic solutions where $X$ is a set and $r : X \times X \to X \times X$ obeys similar relations on $X \times X \times X$. Of course, each such solution extends linearly to $V = kX$ with matrices in this natural basis having only entries from 0,1 and many other nice properties.

Associated to each set-theoretic solution are several algebraic constructions: the semigroup $S(X, r)$, the group $G(X, r)$, the semigroup algebra $kS(X, r) = kR[V]$ generated by $X$ with relations $xy = r(x, y)$ (where $\cdot$ denotes product in the free semigroup, resp free group) and the permutation group $\mathcal{G}(X, r) \subseteq \text{Sym}(X)$ defined by the corresponding left translations $y \mapsto \tau y$ for $x \in X$, where $r(x, y) = (\tau y, y\tau)$ (under assumptions which will be given later).

In this paper we study the special case when $(X, r)$ is a square-free solution, a square-free symmetric set of arbitrary cardinality. Our special interest is in the retractability of such solutions. We study multipermutation solutions and find close relation between the multipermutation level of such a solution and the properties of the associated algebraic objects $\mathcal{G}(X, r)$, $G(X, r)$, and $S(X, r)$. A feature of our approach is to give prominence to the group $G(X, r)$.

We introduce and study various constructions of solutions such as strong twisted unions of solutions, doubling of solutions, wreath product of solutions, and various other constructions, and study the multipermutation level of the new solutions and properties such as solvable length of their associated algebraic structures.

We now describe the contents of the paper in greater detail.

In Section 2 we recall basic definitions and basic results, and give a general description of the permutation groups $H \subseteq \text{Sym}(X)$ which can serve as a YB permutation group of some square-free solution $(X, r)$, see Proposition 2.17.

In Section 3 we recall the definitions and basic properties of homomorphisms and automorphisms of solutions, we give a general construction, the strong twisted union of a finite number of solutions, and describe strong twisted unions in terms of split maps, see Proposition 3.12.

In Section 4 we study various decompositions of square-free solutions $(X, r)$ into disjoint unions of a finite number of $r$-invariant subsets and the corresponding factorisation of $S(X, r)$, $G(X, r)$, and $\mathcal{G}(X, r)$, see Theorems 4.16, 4.19 and 4.21. We make essential use of the matched pairs approach to solutions (in the most general setting), developed in [GIM08].

Section 5 continues the study of multipermutation solutions of low levels which was initiated in [GIM07], and deepened in [GIM08] with detailed description of
solutions of multipermutation level 2. Using matched pair approach and results from [GIM08] we show in Proposition 5.6 that when \((X, r)\) has multipermutation level 2, the associated braided monoid \((S, r_S)\) and the associated braided group \((G, r_G)\) are symmetric sets which inherit some nice combinatorial conditions such as the cyclic condition and condition \(|ri|\) (see Definition 2.9) but they are not square-free. Furthermore, \(S\) (respectively \(G\)) acts on itself as automorphisms. Proposition 5.8 characterises the permutation groups \(H \subset \text{Sym}(X)\) which define (via the left action) square-free solutions \((X, r)\) with multipermutation level 2. As a corollary we obtain that every finite abelian group \(H\) is isomorphic to the permutation group \(\mathcal{G}(X, r)\) of some square-free solution with multipermutation level 2.

We characterise solutions with multipermutation level 3 in Proposition 5.12 and Corollary 5.13. We show that, for \((X, r)\) of arbitrary cardinality, each solution with multipermutation level 3 decomposes as a strong twisted union of a finite number of solutions with multipermutation level \(\leq 2\), and the permutation group \(\mathcal{G}\) decomposes as a product of abelian subgroups, see Propositions 5.15 and 5.12.

One can say a surprising amount about solutions \((X, r)\) for which \(\mathcal{G}(X, r)\) is abelian: this is the theme of Section 6. In Subsection 6.1 we develop a technique for computation with actions in the case of abelian \(\mathcal{G}\).

This technique and Theorem 7.10 are used to prove the main results of the section: Theorems 6.1 and 6.3. We assume \((X, r)\) is a square-free solution of arbitrary cardinality, \(\mathcal{G} = \mathcal{G}(X, r)\) is abelian, and \(X\) has finite number \(t\) of \(G\)-orbits. Theorems 6.1 shows that every such a solution is a multipermutation solution of level \(\text{mpl}\) \(X \leq t\), and \(X\) decomposes as a strong twisted union of its orbits. Furthermore, each orbit \(X_i\) is itself a trivial solution (so \(\text{mpl}\) \(X_i = 1\)). This confirms two conjectures of the first author (only for the case of abelian \(\mathcal{G}\), of course), see [GI] Conjecture I, Conjecture II, formulated for finite square-free solutions (see also Conjecture 2.26).

Theorem 6.3 proves that, under the assumption that \(\mathcal{G}(X, r)\) is abelian, a strong twisted union \(X = X_1 \upharpoonright X_2\) of two multipermutation solutions is itself a multipermutation solution with \(\text{mpl}\) \(X \leq \text{mpl}\) \(X_1 + \text{mpl}\) \(X_2\). In the general case, the question whether a strong twisted union \(X = X_1 \upharpoonright X_2\) of multipermutation solutions \(X_1, X_2\) is also a multipermutation solution remains open.

A resent result of Cedó, Jespers, and Okniński, [CJR09], shows that each finite square-free solution \((X, r)\) with \(\mathcal{G}\) abelian is retractable. The proof is combinatorial (and different from ours) and relies strongly on the finiteness of \(X\); there is no estimate of \(\text{mpl}\) \(X\). Our proof uses a general technique which is applicable to solutions with arbitrary cardinality of \(X\) (but a finite number \(t\) of \(G\)-orbits). In fact we show that \(t\) is an upper bound for the multipermutation level \(\text{mpl}\) \(X\).

Section 7 studies the general case of multipermutation square-free solutions. In Subsection 7.1 we recall basic notions and facts from [GI], and use them to develop a basic technique for dealing with retracts and retract classes. Theorem 7.10 gives an explicit identity in terms of action necessary and sufficient for \(\text{mpl}\) \(X = m\). This identity plays an essential role in the paper.

Subsection 7.2 contains some of the important results of this paper. We study the groups \(G = \mathcal{G}(X, r)\) and \(\mathcal{G} = \mathcal{G}(X, r)\) of multipermutation solutions. We show that
if \((X, r)\) is a square-free multipermutation solution (of arbitrary cardinality) the groups \(G\) and \(G^0\) are solvable. This was known for finite symmetric sets, see [ESS], (see also [G] for finite square-free solutions), but no information about the solvable length of \(G\) was known.

Lemma 7.15 and induction on \(m = \text{mpl}(X)\) allow us not only to prove solvability (without assuming \(X\) or \(G\) finite), but also to find an upper bound for the solvable lengths: \(\text{sl}(G) \leq m\) and \(\text{sl}(G) \leq m - 1\). The results of Section 8 show that these upper bounds are attained. Theorem 7.25 verifies that, whenever \((X, r)\) is a square-free solution of finite order, then \(\text{sl}(G) = \text{sl}(G^0) + 1\).

In Section 8 we define the notion of \textit{wreath product of solutions}, by analogy with the wreath products of permutation groups. Theorem 8.7 shows that wreath product of solutions of finite order, then \(\text{sl}(G)\) and \(\text{mpl}(X)\) are multipermutation solutions, one has \(\text{mpl}(Z, r) = \text{mpl}(X, r_{X_0}) + \text{mpl}(Y, r_Y) - 1\).

In Section 9 we construct an interesting sequence of explicitly defined solutions \((X_m, r_m)\), \(m = 0, 1, 2, \ldots\), such that \(\text{mpl}(X_m) = m\), \(\text{Ret}(X_{m+1}, r_{m+1}) \simeq (X_m, r_m)\), \(m = \text{sl}(G(X, r_m)) = \text{sl}(G(X, r_m)) + 1\): see Definition 9.9 and Theorem 9.11.

2. Preliminaries on set-theoretic solutions

There are many works on set-theoretic solutions and related structures, of which a relevant selection for the interested reader is [WX, GB, ESS, GI, GI04, GIM07, GIM08, GIM0806, LYZ, Ru, Ta, V]. In this section we recall basic notions and results which will be used in the paper. We shall use the terminology, notation and some results from [GI, GIM07, GIM08, GIM0806].

**Definition 2.1.** Let \(X\) be a nonempty set (not necessarily finite) and let \(r : X \times X \rightarrow X \times X\) be a bijective map. We refer to it as a \textit{quadratic set}, and denote it by \((X, r)\). The image of \((x, y)\) under \(r\) is presented as

\[
(r(x, y)) = (x^y, x^y).
\]

The formula (2.1) defines a “left action” \(\mathcal{L} : X \times X \rightarrow X\), and a “right action” \(\mathcal{R} : X \times X \rightarrow X\), on \(X\) as:

\[
\mathcal{L}_x(y) = x^y, \quad \mathcal{R}_y(x) = x^y,
\]

for all \(x, y \in X\). The map \(r\) is \textit{nondegenerate}, if the maps \(\mathcal{L}_x\) and \(\mathcal{R}_x\) are bijective for each \(x \in X\). In this paper we shall consider only the case where \(r\) is nondegenerate. As a notational tool, we shall often identify the sets \(X \times X\) and \(X^2\), the set of all monomials of length two in the free semigroup \(X\).

**Definition 2.2.**

1. \(r\) is \textit{square-free} if \(r(x, x) = (x, x)\) for all \(x \in X\).
2. \(r\) is a \textit{set-theoretic solution of the Yang–Baxter equation} or, shortly a \textit{solution} (YBE) if the braid relation

\[
r_{12}r_{23}r_{12} = r_{23}r_{12}r_{23}
\]

holds in \(X \times X \times X\), where the two bijective maps \(r_{ii+1} : X^3 \rightarrow X^3\), \(1 \leq i \leq 2\) are defined as \(r_{12} = r \times \text{id}_X\), and \(r_{23} = \text{id}_X \times r\). In this case we refer to \((X, r)\) also as a \textit{braided set}.

3. A \textit{braided set} \((X, r)\) with \(r\) involutive is called a \textit{symmetric set}.
Convention 2.3. By square-free solution we mean a nondegenerate square-free symmetric set \((X, r)\), where \(X\) is a set of arbitrary cardinality. Alternative definitions are given in terms of the left action, see Lemma 2.12 and Corollary 2.14.

To each quadratic set \((X, r)\) we associate canonical algebraic objects generated by \(X\) and with quadratic defining relations \(\mathcal{R} = \mathcal{R}(r)\) defined by
\[
xy = zt \in \mathcal{R}(r), \quad \text{whenever} \quad r(x, y) = (z, t).
\]

Definition 2.4. Let \((X, r)\) be a quadratic set.

(i) The unital semigroup \(S = S(X, r) = \langle X; \mathcal{R}(r) \rangle\), with a set of generators \(X\) and a set of defining relations \(\mathcal{R}(r)\), is called the semigroup associated with \((X, r)\).

(ii) The group \(G = G(X, r)\) associated with \((X, r)\) is defined as \(G = G(X, r) = \text{gr}(X; \mathcal{R}(r))\).

(iii) For arbitrary fixed field \(k\), the \(k\)-algebra associated with \((X, r)\) is defined as \(\mathcal{A}(k, X, r) = k\langle X; \mathcal{R}(r) \rangle\). (\(\mathcal{A}(k, X, r)\) is isomorphic to the monoidal algebra \(kS(X, r)\)).

(iv) To each nondegenerate braided set \((X, r)\) we also associate a permutation group, called the group of left action and denoted \(\mathcal{G} = \mathcal{G}(X, r)\), see Definition 2.7.

If \((X, r)\) is a solution, then \(S(X, r)\), resp. \(G(X, r)\), resp. \(\mathcal{G}(X, r)\), resp. \(\mathcal{A}(k, X, r)\) is called the Yang–Baxter semigroup, resp. the Yang–Baxter group, resp. the Yang–Baxter algebra, resp. the (Yang–Baxter) permutation group or shortly the YB permutation group, associated to \((X, r)\).

The YB permutation group \(\mathcal{G}(X, r)\) will be of particular importance in this paper.

Example 2.5. For arbitrary nonempty set \(X\), denote by \(\tau_X = \tau\) the flip map \(\tau(x, y) = (y, x)\) for all \(x, y \in X\). Then \((X, \tau)\) is a solution called the trivial solution. It is clear that \(\tau(X, r)\) is the trivial solution if and only if \(xy = y\) and \(xy = x\), for all \(x, y \in X\), or equivalently \(L_x = \text{id}_X = R_x\) for all \(x \in X\). In this case \(S(X, r)\) is the free abelian monoid, \(G(X, r)\) is the free abelian group, \(\mathcal{A}(k, X, r)\) the algebra of commutative polynomials in \(X\), and \(\mathcal{G}(X, r) = \{\text{id}_X\}\) is the trivial group.

Remark 2.6. Suppose \((X, r)\) is a nondegenerate quadratic set. It is well known, see for example [GIM08], that \((X, r)\) is a braided set (i.e. \(r\) obeys the YBE) if and only if the following conditions hold
\[
\begin{align*}
\text{li1} : & \quad \tau(yz) = \tau(y)x\tau(z), \\
\text{r1} : & \quad (xy)^z = (x^y)z^y, \\
\text{lr3} : & \quad (x^y)^z = (x^{y^z})z^{y^x},
\end{align*}
\]
for all \(x, y, z \in X\).

Clearly, conditions li1 imply that, for each nondegenerate braided set \((X, r)\), the assignment \(x \mapsto L_x\) for \(x \in X\) extends canonically to a group homomorphism
\[
\mathcal{L} : G(X, r) \longrightarrow \text{Sym}(X),
\]
which defines the canonical left action of \(G(X, r)\) on the set \(X\). Analogously r1 gives the canonical right action of \(G(X, r)\) on \(X\).
Definition 2.7. [GIM08] Let \((X, r)\) be a nondegenerate braided set, \(\mathcal{L} : G(X, r) \rightarrow \Sym(X)\) be the canonical group homomorphism defined via the left action. The image \(\mathcal{L}(G(X, r))\) is denoted by \(G(X, r)\). We call it the \emph{(permutation) group of left action.}

Remark 2.8. If \(X\) is a finite set, then \(G = \mathcal{L}(S(X, r))\). Indeed, for a finite group, generating sets as group and as semigroup coincide; for any element \(g\) of the group has finite order, say \(m\), and so its inverse \(g^{-1}\) can be expressed as a positive power \(g^{m-1}\).

The following conditions were introduced and studied in [GI, GIM07, GIM08]:

Definition 2.9. [GIM08] Let \((X, r)\) be a quadratic set.

1. \((X, r)\) is called \emph{cyclic} if the following conditions are satisfied
   
   \begin{align*}
   \text{cl1} : \quad & y^r x = yx \quad \text{for all } x, y \in X; \\
   \text{cr1} : \quad & x^ry = x^y, \quad \text{for all } x, y \in X; \\
   \text{cl2} : \quad & y^s x = yx, \quad \text{for all } x, y \in X; \\
   \text{cr2} : \quad & x^s y = x^y \quad \text{for all } x, y \in X.
   \end{align*}

   We refer to these conditions as \emph{cyclic conditions.}

2. Condition \(\text{lri}\) is defined as
   
   \[
   \text{lri} : \quad (y^r)^s = y = s(y^r) \quad \text{for all } x, y \in X.
   \]

   In other words \(\text{lri}\) holds if and only if \((X, r)\) is nondegenerate and \(R_x = L_x^{-1}\) and \(L_x = R_x^{-1}\).

2.1. Square-free solutions. In this paper the class of \emph{square-free solutions} will be of special interest. We now introduce these.

In the case when \((X, r)\) is a square-free solution \emph{of finite order} \(|X| = n > 2\), the algebras \(A(X, r)\) are \emph{binomial skew polynomial rings}, see [GI, GI04], which provided new classes of Noetherian rings [GI94, GI96-1], Gorenstein (Artin–Schelter regular) rings [GI96-2, GI00, GI04] and so forth. Artin–Schelter regular rings were introduced in [AS] and are of particular interest. The algebras \(A(X, r)\) are similar in spirit to the quadratic algebras associated to linear solutions, particularly studied in [Ma], but have their own remarkable properties. The semigroups \(S(X, r)\) were studied particularly in [GIM08] with a systematic theory of ‘exponentiation’ from the set to the semigroup by means of the ‘actions’ \(L_x, R_x\) (which in the process become a matched pair of semigroup actions) somewhat in analogy with the Lie theoretic exponentiation in [M90].

We shall recall some basic facts and recent results needed in this paper.

The following result is extracted from [GIM08], Theorem 2.34, where more equivalent conditions are pointed out. Note that in our considerations below (unless we indicate the contrary) the set \(X\) is not necessarily of finite order.

Facts 2.10. [GIM07]. Suppose \((X, r)\) is nondegenerate, involutive and square-free quadratic set (not necessarily finite). Then the following conditions are equivalent:

1. \((X, r)\) is a set-theoretic solution of the Yang–Baxter equation;
2. \((X, r)\) satisfies \(\text{II}\);
3. \((X, r)\) satisfies \(\text{r1}\);
4. \((X, r)\) satisfies \(\text{lri}\).
In this case \((X, r)\) is cyclic and satisfies \(\text{i} \text{ri}\).

**Corollary 2.11.** Every square-free solution \((X, r)\) satisfies \(\text{i} \text{ri}\), so it is uniquely determined by the left action \(\mathcal{L} : X \times X \to X\), more precisely, 
\[ r(x, y) = (\mathcal{L}_x(y), \mathcal{L}_y^{-1}(x)). \]
Furthermore it is cyclic.

The following is straightforward and gives an alternative definition of square-free solutions.

**Lemma 2.12.** Let \(X\) be a nonempty set and \(\mathcal{L}\) be a map 
\[ \mathcal{L} : X \to \text{Sym} X; \quad x \mapsto L_x \in \text{Sym} X. \]
Denote \(L_x(y) = x \cdot y\), \(L_x^{-1}(y) = y^x\) and define \(r : X \times X \to X \times X\) as \(r(x, y) = (x \cdot y, y^x \cdot x)\). Then \((X, r)\) is a square-free solution if and only if the following three conditions are satisfied for all \(x, y, z \in X\):

1. \(x^x = x\)
2. \((y^x)^x = y^x\)
3. \(x^{(y^x)}z = y^{(x^y)}z\)

**Remark 2.13.** Note that in the hypothesis of Lemma 2.12, condition (ii) implies \((X, r)\) involutive.

**Corollary 2.14.** [GIM08] In the hypothesis and notation of Lemma 2.12, \((X, r)\) is a square-free solution if and only if the following conditions are satisfied for all \(x, y, z \in X\):

1. \(x^x = x\)
2. \((y^x)^x = y^x\)
3. \(x^{(y^x)}z = y^{(x^y)}z\)

Recall that a quadratic set \((X, r)\) which satisfies condition (ii) of Corollary 2.14 is called a cyclic set, see [Rui] [GIM08].

**Definition 2.15.** The permutation group \(H \subseteq \text{Sym} X\) is called a YB permutation group, if it is isomorphic to \(G(X, r)\) for some square-free solution \((X, r)\).

**Open questions 2.16.** Let \(X\) be a nonempty set.

1. For which permutation groups \(H \subseteq \text{Sym} X\) is there a square-free solution \((X, r)\) with \(G(X, r) = H\) ?
2. Let \(m\) be a positive integer. For which permutation groups \(H \subseteq \text{Sym} X\) is there a square-free solution \((X, r)\) with \(G(X, r) = H\) and \(\text{mpl}(X, r) = m\) (see Definition 2.25) ?

The next result gives a translation of the first question which is not very easy to check.

**Proposition 2.17.** Let \(H \subseteq \text{Sym} X\) be a permutation group. Then \(H\) is a YB permutation group for some square-free solution \((X, r)\) if and only if there exists a map 
\[ f : X \to H; \quad x \mapsto f_x \]
such that the following conditions hold:
(1) $f(X)$ is a generating set for $H$;
(2) $f_x(x) = x$;
(3) $f_{f_y(x)} = f_y f_x$.

In this case the quadratic set $(X, r)$ with $r(x, y) = (f_x(y), (f_y)^{-1}(x))$ is a square-free solution, and $H \cong \mathcal{G}(X, r)$.

Proof. Corollary 2.14 implies that the quadratic set $(X, r)$ is a square-free solution. Clearly, in this case $L_x = f_x \in \text{Sym} X$. Condition 3 (together with 1ri) implies $f_x \cdot f_y = f_y f_x$, so $(f_x \cdot f_y)(a) = f_x(f_y(a))$ and the map $L_x \mapsto f_x$ extends to a group homomorphism $\varphi: \mathcal{G}(X, r) \to H$. We have $f_{x_1} \cdot \cdots \cdot f_{x_k}(y) = x_1 \cdot \cdots \cdot (x_k \cdot y) \cdots$ so the kernel of $\varphi$ is trivial.

In Proposition 5.8 we describe the permutation groups $H \subseteq \text{Sym} X$ which define square-free solutions $(X, r)$ with $\text{mpl}(X, r) = 2$, where $X$ is an arbitrary finite nonempty set.

**Definition 2.18.** Let $(X, r)$ be a braided set, $G = G(X, r), \mathcal{G} = \mathcal{G}(X, r)$. A subset $Y \subseteq X$ is said to be $r$-invariant if $r(Y \times Y) \subseteq Y \times Y$. Suppose $Y$ is an $r$-invariant subset of $(X, r)$. Then $r$ induces a solution $(Y, r_{Y \times Y})$. Denote $r_{Y \times Y} = r|_{Y \times Y}$. We call $(Y, r_{Y})$ the restricted solution (on $Y$). We say that $Y \subseteq X$ is a (left) $G$-invariant subset of $X$ or equivalently a $\mathcal{G}$-invariant subset, if $Y$ is invariant under the left action of $G$. Clearly, $Y$ is (left) $G$-invariant if and only if

$$L_a(Y) \subseteq Y, \quad \forall a \in X.$$ 

Right $G$-invariant subsets are defined analogously. In the case when $(X, r)$ is symmetric, and condition 1ri holds the subset $Y$ is left $G$-invariant if and only if it is right $G$-invariant. In this case we shall refer to it simply as a $\mathcal{G}$-invariant subset.

Clearly each $\mathcal{G}$-invariant subset $Y$ of $X$ is also an $r$-invariant subset, but, in general, an $r$-invariant subset may, or may not be $G$-invariant. The following holds:

**Lemma 2.19.** Let $(X, r)$ be a symmetric set with 1ri, $G = \mathcal{G}(X, r), \quad Y \subseteq X$. Denote by $Z$ the complement of $Y$ in $X$. The following conditions are equivalent.

(1) $Y$ is $G$-invariant;
(2) $Z$ is $G$-invariant;
(3) $Y$ and $Z$ are $r$-invariant complementary subsets of $X$.

Moreover, in this case $(X, r)$ decomposes as a disjoint union of $r$-invariant subsets $X = Y \cup Z$.

**Proof.** Suppose $Y$ is $\mathcal{G}$-invariant. Clearly, $x \in Y$ if and only if the $G$-orbit of $x$ is contained in $Y$.

It is easy to show now that $Z$ is also $G$-invariant. Indeed assume $z \in Z, a \in X, t = a z$. If we assume that $t$ is not in $Z$, this would imply $t \in Y$. By 1ri one has

$$t = a z \implies t^a = a^z a = z$$
But \( t \in Y \), and the \( G \)-orbit of \( t \) is contained in \( Y \), so \( z = t^{a} \in Y \), a contradiction. Thus \( Z \) is \( G \)-invariant. This proves \( (1) \implies (2) \).

The implication \( (2) \implies (1) \) is analogous. Clearly, then each of the conditions \( (1) \) and \( (2) \) implies that \( Y \) and \( Z \) are \( r \)-invariant.

Suppose now \( (X, r) \) is decomposed into two \( r \)-invariant disjoint subsets \( X = Y \cup Z \). We claim that \( (1) \) and \( (2) \) hold. It will be enough to verify \( (1) \). \( Y \) is \( r \)-invariant, therefore

\[
a^{y} \in Y \quad \forall a, y \in Y.
\]

It remains then to show that

\[
\bar{y} \in Y \quad \forall y \in Y, z \in Z.
\]

Assume the contrary, for some \( y \in Y, z \in Z \), one has \( \bar{y} \in Z \). This yield

\[
t = \bar{y} \in Z \implies y = (\bar{y})\bar{z} = t^{\bar{z}} \in Z,
\]

since \( Z \) is \( r \)-invariant. This contradicts \( y \in Y \).

It is easy to show that \( r_{Y} : Y \times Y \to Y \times Y \) is surjective, and therefore bijective. Clearly, \( (Y, r_{Y}) \) is involutive, hence \( (Y, r_{Y}) \) is a symmetric set. \( \square \)

Note that \( (X, r) \) is a square-free solution since in this case condition \( lri \) holds, see Corollary \( \ref{corollary:2.11} \).

**Remark 2.20.** Suppose \( (X, r) \) is a square-free solution. Then it satisfies \( lri \) (see Corollary \( \ref{corollary:2.11} \)) and therefore Lemma \( \ref{lemma:2.19} \) is in force. Clearly, each \( G \)-orbit \( X_{0} \) under the left action of \( G \) on \( X \) is \( G \)-invariant and therefore it is an \( r \)-invariant subset. In the case when \( G \) acts non-transitively on \( X \) (in particular, this holds when \( X \) is finite), \( (X, r) \) decomposes into a disjoint union \( X = X_{0} \cup Z \), of its \( r \)-invariant subsets \( X_{0} \) and \( Z \), where \( Z \) is the complement of \( X_{0} \) in \( X \).

Each finite involutive solution \( (X, r) \) with \( lri \) can be represented geometrically by its graph of the left action \( \Gamma(X, r) \). It is an oriented labeled multi-graph (although we refer to it as a graph). It was introduced in \( \cite{GIM00} \) for square-free solutions, see also \( \cite{GIM07} \), Section 5, and \( \cite{GIM08} \). Here we recall the definition.

**Definition 2.21.** \( \cite{GIM07} \) Let \( (X, r) \) be a finite symmetric set with \( lri \). We define the graph \( \Gamma = \Gamma(X, r) \) as follows. It is an oriented graph, which reflects the left action of \( G(X, r) \) on \( X \). The set of vertices of \( \Gamma \) is \( X \). There is a labeled arrow \( x \overset{a}{\to} y \) if \( x, y, a \in X \) and \( a^{x} = y \). An edge \( x \overset{a}{\to} y \) with \( x \neq y \) is called a nontrivial edge. We will often consider the simplified graph in which to avoid clutter we typically omit self-loops unless needed for clarity or contrast. Also for the same reason, we use the line type to indicate when the same type of element acts, rather than labeling every arrow. Clearly, \( x \overset{a}{\to} y \) indicates that \( a^{x} = y \) and \( a^{y} = x \). (One can make such graphs for arbitrary solutions but then it should be indicated which action is considered).

Note that two solutions are isomorphic if and only if their oriented graphs are isomorphic. Various properties of a solution \( (X, r) \) are reflected in the properties of its graph \( \Gamma(X, r) \), see for example the remark below.

**Remark 2.22.** Let \( (Z, r) \) be a symmetric set with \( lri \), \( \Gamma = \Gamma(Z, r) \).
\((Z,r)\) is a square-free solution if and only if \(\Gamma\) does not contain a nontrivial edge \(x \rightarrow y, x \neq y\); that is, the edge of type \(x\) leaving \(x\) is a loop.

(2) In this case, \((Z,r)\) is a trivial solution (or equivalently, \(\text{mpl}(Z,r) = 1\)) if and only if all the edges are loops.

(3) The \(G\)-orbits of \(X\) are in 1-1 correspondence with the connected components of \(\Gamma\).

(4) \([GIM07]\), Theorem 5.24 gives necessary and sufficient conditions for \(\text{mpl}(X,r) = 2\) in terms of the properties of \(\Gamma(X,r)\). It can be read off from \([GIM07]\), Theorem 5.22, Theorem 5.24 that, in this case, each nontrivial connected component \(\Gamma_i\) of \(\Gamma\) is a Cayley graph (see below).

**Definition 2.23.** If \(G\) is a group and \(S\) a subset of \(G\), the Cayley graph \(\text{Cay}(G,S)\) is the graph with vertex set \(G\) and directed edges \((g, gs)\) for all \(g \in G\) and \(s \in S\).

Note that the group \(G\) acts by left multiplication as automorphisms of the Cayley graph \(\text{Cay}(G,S)\). The graph is loopless if and only if \(\text{id} \not\in S\) and connected if and only if \(\langle S \rangle = G\).

Now any graph admitting a group \(G\) of automorphisms acting regularly on the vertices is a Cayley graph for \(G\). For the vertex set can be identified with \(G\) (with action by left multiplication); and, if \(S\) is the set of vertices \(s\) for which \((\text{id}, s)\) is an edge, then \((g, gs)\) is an edge for all \(g \in G\) since \(G\) acts by automorphisms.

In particular, every transitive abelian group action is regular, so a graph with a transitive abelian group of automorphisms is a Cayley graph.

The notions of retraction of symmetric sets and multipermutation solutions were introduced in the general case in \([ESS]\), where \((X,r)\) is not necessarily finite, or square-free.

In \([GI],[GIM07],[GIM08],[GIM0806]\) the multipermutation square-free solutions are studied; we recall some notions and results. Let \((X,r)\) be a nondegenerate symmetric set. An equivalence relation \(\sim\) is defined on \(X\) as

\[x \sim y \quad \text{if and only if} \quad L_x = L_y.\]

In this case we also have \(R_x = R_y\),

We denote by \([x]\) the equivalence class of \(x \in X\), \([X] = X/\sim\) is the set of equivalence classes.

**Lemma 2.24.** \([GIM08]\) Let \((X,r)\) be a nondegenerate symmetric set.

(1) The left and the right actions of \(X\) onto itself naturally induce left and right actions on the retraction \([X]\), via

\[(\alpha)[x] := [\alpha^x] \quad [\alpha]^x := [\alpha^x], \quad \text{for all } \alpha, x \in X.\]

(2) The new actions define a canonical map \(r_{[X]} : [X] \times [X] \rightarrow [X] \times [X]\)

where \(r_{[X]}([x],[y]) = ([x]^y,[x]^y])\).

(3) \((X,r_{[X]})\) is a nondegenerate symmetric set. Furthermore,

(4) \((X,r)\) cyclic \(\iff\) \((X,r_{[X]})\) cyclic.

(5) \((X,r)\) is lri \(\iff\) \((X,r_{[X]})\) is lri.

(6) \((X,r)\) square-free \(\iff\) \((X,r_{[X]})\) square-free.
Definition 2.25. [ESS] The solution $\text{Ret}(X, r) = ([X], [r])$ is called the retraction of $(X, r)$. For all integers $m \geq 1$, $\text{Ret}^m(X, r)$ is defined recursively as $\text{Ret}^m(X, r) = \text{Ret}(\text{Ret}^{m-1}(X, r))$.

$(X, r)$ is a multipermutation solution of level $m$, if $m$ is the minimal number (if any), such that $\text{Ret}^m(X, r)$ is the trivial solution on a set of one element. In this case we write $\text{mpl}(X, r) = m$.

By definition $(X, r)$ is a multipermutation solution of level 0 if and only if $X$ is a one element set.

The following conjecture was made by the first author in 2004.

Conjecture 2.26. [GI]

1. Every finite square-free solution $(X, r)$ is retractable.
2. Every finite square-free solution $(X, r)$ of finite order $n$ is multipermutation solution, with $\text{mpl}(X, r) < n$.

A more recent conjecture states

Conjecture 2.27. [GIM08] Suppose $(X, r)$ is a nondegenerate square-free multipermutation solution of finite order $n$. Then $\text{mpl}(X) < \log_2 n$.

Evidence for this conjecture will be given later in the paper.

3. Homomorphisms, automorphisms, strong twisted unions

In this section, we recall the definitions and basic properties of homomorphisms and automorphisms of solutions, see and give a general construction, the strong twisted union of solutions, see [GIM07] [GIM08].

Definition 3.1. [GIM07] Let $(X, r_X)$ and $(Y, r_Y)$ be arbitrary solutions (braided sets). A map $\varphi : X \rightarrow Y$ is a homomorphism of solutions, if it satisfies the equality

$$(\varphi \times \varphi) \circ r_X = r_Y \circ (\varphi \times \varphi).$$

A bijective homomorphism of solutions is called (as usual) an isomorphism. An isomorphism of the solution $(X, r)$ onto itself is an $r$-automorphism.

We denote by $\text{Hom}((X, r_X), (Y, r_Y))$ the set of all homomorphisms of solutions $\varphi : X \rightarrow Y$. The group of $r$-automorphisms of $(X, r)$ will be denoted by $\text{Aut}(X, r)$.

Clearly, $\text{Aut}(X, r)$ is a subgroup of $\text{Sym}(X)$.

Remark 3.2. [GIM07] Let $(X, r_X), (Y, r_Y)$ be finite square-free solutions. Every homomorphism of solutions $\varphi : (X, r_X) \rightarrow (Y, r_Y)$ induces canonically a homomorphism of their graphs:

$$\varphi_G : \Gamma(X, r_X) \rightarrow \Gamma(Y, r_Y).$$

Furthermore there is a one-to-one correspondence between $\text{Aut}(X, r)$ and $\text{Aut}(\Gamma(X, r))$, the group of automorphisms of the multigraph $\Gamma(X, r)$.

Lemma 3.3. [GIM07] Let $(X, r_X)$ and $(Y, r_Y)$ be braided sets.
A map $\varphi : X \rightarrow Y$ is a homomorphism of solutions if and only if
$$\varphi \circ L_x = L_{\varphi(x)} \circ \varphi \quad \text{and} \quad \varphi \circ R_x = R_{\varphi(x)} \circ \varphi \quad \text{for all} \ x \in X.$$ (1)

If both $(X, r_X)$ and $(Y, r_Y)$ satisfy $\text{lri}$, then $\varphi$ is a homomorphism of solutions if and only if
$$\varphi \circ L_x = L_{\varphi(x)} \circ \varphi, \quad \text{for all} \ x \in X.$$ (2)

If $(X, r)$ obeys $\text{lri}$, (in particular, if $(X, r)$ is a square-free solution) then $\sigma \in \text{Sym}(X)$ is an automorphism of $(X, r_X)$ if and only if
$$\sigma \circ L_x \circ \sigma^{-1} = L_{\sigma(x)}, \quad \text{for all} \ x \in X.$$ (3)

For example, if $(X, r)$ is the trivial solution then, clearly, $\text{Aut}(X, r) = \text{Sym}(X)$. More generally, (3.1) implies:

**Corollary 3.4.** \cite{GIM07} The group $\text{Aut}(X, r)$ is a subgroup of $\text{Nor}_{\text{Sym}(X)} \mathcal{G}(X, r)$, the normalizer of $\mathcal{G}(X, r)$ in $\text{Sym}(X)$.

**Corollary 3.5.** Suppose $(X, r)$ is a braided set with $\text{lri}$. Let $Y$ be an $r$-invariant subset, $(Y, r|_Y)$ be the restricted solution. Let $x \in X$. Then $L_x \in \text{Aut}(Y, r|_Y)$ if and only if
$$L_{\alpha} = (L_\alpha)_Y \quad \forall \alpha \in Y$$ (3.2)

**Corollary 3.6.** Suppose $(X, r)$ is a square-free solution. Then the following conditions hold.

1. For each $r$-invariant subset $Y$, and each $x \in X$, one has $L_x \in \text{Aut}(Y, r|_Y)$ if and only if (3.2) holds.
2. The intersection $\mathcal{G}_0 = \text{Aut}(X, r) \cap \mathcal{G}$ is an abelian subgroup of $\mathcal{G}$.

**Proof.** We know that each square-free solution satisfies $\text{lri}$, hence, Corollary 3.5 gives (1). By Corollary 2.14(ii) the following equality holds:
$$L_{\alpha} \circ L_x = L_{\alpha \circ x} \quad \forall x, \alpha \in X$$ (3.3)

Assume now that $L_x, L_y \in \text{Aut}(X, r)$. Then (1) implies
$$L_{\alpha \circ x} = L_x \quad \text{by} \quad L_x \in \text{Aut}(X, r)$$
$$L_{\alpha \circ y} = L_y \quad \text{by} \quad L_y \in \text{Aut}(X, r)$$ (3.4)

This together with (3.3) yield
$$L_x \circ L_y = L_y \circ L_x.$$

We shall now discuss a special class of extensions of solutions called **strong twisted unions of solutions**

**Definition 3.7.** \cite{GIM07}, \cite{GIM08} Let $(X, r)$ be an involutive quadratic set, suppose $X$ is a disjoint union $X = X_1 \cup X_2$ of $r$-invariant subsets. Suppose the restricted sets $(X_1, r_1), (X_2, r_2)$ are symmetric sets $(r_i = r|_{X_i}, i = 1, 2)$. The quadratic set $(X, r)$ is a **strong twisted union** of $(X_1, r_1)$ and $(X_2, r_2)$ if
Example 3.9. Let \((X_1, r_1)\) (and the associated monoid \(S(X_1, r_1)\)) on \(X_1\), and the assignment \(x \mapsto L_x \in \mathcal{X}_{L_r}X_2\), \(x \in X_1\) extends to a right action of the associated group of \(G(X_1, r_1)\) (and the associated monoid \(S(X_1, r_1)\)) on \(X_2\);

(2) The actions satisfy

\[
\text{stu} : \quad \alpha^x = \alpha^y = \alpha^z, \quad \text{for all } x, y, z \in X, \alpha, \beta \in Y.
\]

Note first that condition \(\text{stu}\) is equivalent to the following condition:

\[
(3.5) \quad \text{stu1} : \quad \alpha \cdot \mathcal{x} = \alpha \cdot \mathcal{y} = \alpha \cdot \mathcal{z}, \quad \text{for all } x, y, z \in X, \alpha, \beta \in X.
\]

We shall refer to it also as \(\text{stu}\) condition.

Secondly, note that by Definition 3.8 a strong twisted union of symmetric sets is not necessarily a solution (a symmetric set). We shall use notation

\[
(X, r) = X_1 \sharp X_2
\]

to denote that \((X, r)\) is a symmetric set which is a strong twisted union of its \(r\)-invariant subsets \(X_1\) and \(X_2\).

The strong twisted union \((X, r)\) of \((X_1, r_1)\) and \((X_2, r_2)\) is nontrivial if at least one of the actions in \((\text{I})\) is nontrivial. In the case when both actions \((\text{II})\) are trivial we write \((X, r) = X_1 \sharp_{\text{tr}} X_2\). In this case one has \(r(x, \alpha) = (\alpha, x)\), \(r(\alpha, x) = (x, \alpha)\) for all \(x \in X, \alpha \in X_2\).

In [GIM07], [GIM08] and [GIM0806] appear strong twisted unions of \(m\) disjoint symmetric sets, where \(m\) is arbitrary integer, \(m \geq 2\). Although a formal definition was not given, the notion of strong twisted union there is clear from the context. Here we give a formal definition.

**Definition 3.8.** Let \((X, r)\) be a symmetric set, which is a disjoint union of \(G\)-invariant subsets \(X_1, \ldots, X_m\), where \(m \geq 2\). For each pair \(i \neq j, 1 \leq i, j \leq m\), denote by \(X_{ij}\) the \(r\)-invariant subset \(X_{ij} = X_i \cup X_j\). We say that \((X, r)\) is a a strong twisted union of \((X_i, r_i), 1 \leq i \leq m\) and write

\[
X = X_1 \sharp X_2 \sharp \cdots \sharp X_m
\]

if

\[
X_{ij} = X_i \sharp X_j, \quad \forall i \neq j, 1 \leq i, j \leq m.
\]

Here as usual, \((X_i, r_i), 1 \leq i, j \leq m\) denotes the symmetric set with \(r_i\)-the restriction of \(r\) on \(X_i \times X_i\).

**Example 3.9.** Let \((X_1, r_1), \ldots, (X_s, r_s)\) be pairwise disjoint square-free solutions. Let \(X = \bigcup_{1 \leq i \leq s} X_i\). Let \(r : X \times X \mapsto X \times X\) be the extension of \(r_i, 1 \leq i \leq s\) satisfying \(r(x, \alpha) = (\alpha, x)\) whenever \(x \in X_i, \alpha \in X_j\), where \(1 \leq i, j \leq s, i \neq j\). Then clearly \((X, r)\) is a square-free solution and \((X, r) = X_1 \sharp_0 X_2 \sharp_0 \cdots \sharp_0 X_s\).

Note that when \((X, r)\) is a strong twisted union \(X = X_1 \sharp X_2 \sharp \cdots \sharp X_m\) of \(m\) \(G\)-invariant subsets, \(m > 2\), each set \(X_i\), and its complement \(Z_i\) are \(G\)-invariant. (Clearly, \(Z_i = \bigcup_{1 \leq j \leq m, j \neq i} X_j\)). However, the union \(X_i \cup Z_i\) may not be a strong twisted union of solutions.
Corollary 3.10. Let $(X, r)$ be a square-free solution. Suppose $X_1, X_2, \cdots, X_m$ are disjoint $G$-invariant subsets of $X$.

Then $(X, r)$ is a strong twisted union $X = X_1 \circledast X_2 \circledast \cdots \circledast X_m$ if and only if for each pair $i \neq j, 1 \leq i, j \leq m$ and each $x \in X_i$ one has

$$(L_x)_{X_j} \in \text{Aut}(X_j, r_j).$$

Definition 3.11. Suppose $(Z, r)$ is an extension (nondegenerate, involutive) of the square-free disjoint solutions $(X, r_X)$ and $(Y, r_Y)$. Consider the maps

$$f = f(X, Y) : Z \times Z \longrightarrow Z \times Z \quad g = g(Y, X) : Z \times Z \longrightarrow Z \times Z$$

defined for all $x, y \in X, \alpha, \beta \in Y$ as

$$f(\alpha, x) = (\alpha x, \alpha) \quad f(x, \alpha) = (x, x^\alpha) \quad f_{|X \times X} = r_X, \quad f_{|Y \times Y} = \tau_Y$$

$$g(\alpha, x) = (x, \alpha^x) \quad g(x, \alpha) = (x^\alpha, x) \quad g_{|X \times X} = \tau_X, \quad g_{|Y \times Y} = \tau_Y$$

(Here $\tau_X, \tau_Y$ are the corresponding flips, and the left and the right actions $^\alpha \cdot, \cdots, \cdot^x$ are the canonical actions defined via $r$).

We call $f$ and $g$, respectively, the associated $X$- and $Y$-split maps of $r$.

Proposition 3.12. Suppose the quadratic set $(Z, r)$ is an extension (nondegenerate, involutive) of the square-free disjoint solutions $(X, r_X)$ and $(Y, r_Y)$, let $f, g$ be respectively the associated $X$-, respectively, $Y$-split maps of $r$. Then the following condition holds.

(i) $f$ and $g$ are involutive maps.

(ii) There is an equality of maps $r = f \circ \tau \circ g$.

Suppose furthermore that $(Z, r)$ is a square-free solution.

(iii) $(Z, f)$ is a square-free solution $\iff$ $^{\alpha^y} x = ^\alpha x \quad \forall x, y \in X, \alpha \in Y$.

In this case $G(Y, r_Y)$ acts as automorphisms on $(X, r_X)$.

(iv) $(Z, g)$ is a square-free solution $\iff$ $^{x^\beta} \alpha = ^x \alpha \quad \forall x \in X, \alpha, \beta \in Y$

In this case $G(X, r_X)$ acts as automorphisms on $(Y, r_Y)$.

(v) $(Z, f)$ and $(Z, g)$ are square-free solutions $\iff (Z, r) = X \circledast Y$.

Proof. Let $x, y \in X$ and $\alpha, \beta \in Y$. We look at the diagrams. The left hand-side diagram contains all elements of the orbit of monomial $\alpha xy \in X^3$, under the action of the group $D(f) = g_\tau(f^{12}, f^{23})$. Analogously the right hand-side diagram contains the elements of the orbit of $x\alpha\beta \in X^3$ under the action of $D(g) =$
Suppose $f$ is a solution, then from the left hand-side diagram we obtain an equality of words in the free monoid $\langle X \rangle$:

$$\alpha x (\alpha y) (\alpha x)^\alpha = \alpha (\alpha x) (\alpha x)^\alpha,$$

and therefore

$$(3.7) \quad \alpha y = \alpha x y, \quad \forall x, y \in X, \alpha \in Y.$$

Assume now that $(Z, r)$ is a solution. Then direct computation to shows that (3.9) implies that $(Z, f)$ satisfies condition $l_1$, and therefore is a square-free solution.

Analogous argument shows that under the assumption that $(Z, r)$ is a solution, $(Z, g)$ is a square-free solution if and only if

$$(3.10) \quad x^\alpha = x^\beta \quad \forall x \in X, \alpha, \beta \in Y.$$

Clearly, both $(Z, f)$ and $(Z, g)$ are square free solutions if and only if both (3.9) and (3.10) hold, which by Corollary is equivalent to

$$(Z, r) = X \ast Y.$$

$\square$
4. Decomposition of solutions.

In this section we study various decompositions of square-free solutions \((X, r)\) into disjoint unions of a finite number of \(r\)-invariant subsets and the corresponding factorisation of \(S(X, r), G(X, r),\) and \(G(X, r)\). We use essentially the matched pairs approach to solutions (in the most general setting) developed in [GIM08]. We first recall the necessary notions and results from [GIM08].

4.1. The matched pairs approach to set-theoretic YBE.

The notion of a matched pair of groups in relation to group factorisation has a classical origin. By now there have been various works on matched pairs in different contexts and we refer to the text in [GIM08] and references therein. In particular, this notion was used by Lu, Yan and Zhu to study the set-theoretic solution of YBE and the associated ‘braided group’, see [LYZ] and the excellent review [Ta]. The notion of a matched pair of monoids, is developed in [GIM08] with additional refinements that disappear in the group case.

We now recall some notions and results from [GIM08].

**Definition 4.1.** \((S, T)\) is a matched pair of monoids if \(T\) acts from the left on \(S\) by \(\cdot\) and \(S\) acts on \(T\) from the right by \(\cdot\) and these two actions obey

- **ML0:** \(a^1 = 1, \; \; ^1u = u;\)
- **MR0:** \(1^v = 1, \; \; a^1 = a\)
- **ML1:** \((ab)u = a(ub),\)
- **MR1:** \(a^{(uv)} = (a^u)^v\)
- **ML2:** \(^u(v) = (^u)(^u)v,\)
- **MR2:** \((ab)^u = (a^u)(b^u),\)

for all \(a, b \in T, u, v \in S.\)

**Proposition 4.2.** [GIM08] A matched pair \((S, T)\) of monoids implies the existence of a monoid \(S \bowtie T\) (called the double cross product) built on \(S \times T\) with product and unit

\[(u, a)(v, b) = (u^a v, a^b), \; \; 1 = (1, 1), \; \; \forall u, v \in S, \; a, b \in T \]

and containing \(S, T\) as submonoids. Conversely, suppose that there exists a monoid \(R\) factorising into monoids \(S, T\) in the sense that (i) \(S, T \subseteq R\) are submonoids and (ii) the restriction of the product of \(R\) to a map \(\mu : S \times T \rightarrow R\) is bijective. Then \((S, T)\) are a matched pair and \(R \cong S \bowtie T\) by this identification \(\mu.\)

**Definition 4.3.** A strong monoid factorisation is a factorisation in submonoids \(S, T\) as above such that \(R\) also factorises into \(T, S\). We say that a matched pair is strong if it corresponds to a strong factorisation.

**Definition 4.4.** A braided monoid is a monoid \(S\) forming part of a matched pair \((S, S)\) such that

1. The equality \(uv = (^u v)(u^v)\) holds in \(S, \forall u, v \in S.\)
2. The associated map \(r_S\)
   \[r_S : S \times S \rightarrow S \times S\]
   defined by \(r_S(u, v) = (^u v, u^v)\)
   is bijective and obeys the YBE. A braided monoid is denoted by \((S, r_S)\).
3. The braided monoid \((S, r_S)\) is called a strong braided monoid if \((S, S)\) is a strong matched pair.
**Remark 4.5.** Matched pairs of groups and braided groups are defined analogously. Note that if the group $G$ forms a matched pair $(G,G)$ such that $uv = (u)v(u)v$ holds for all $u,v \in G$ then the associated map $r_G : G \times G \to G \times G$ with $r_G(u,v) := (u)v(u)v$ is a solution of YBE so $(G,r_G)$ is a braided group.

The following facts can be extracted from [GIM08] Theorems 3.6 and 3.14.

**Facts 4.6.** Let $(X,r)$ be a braided set and $S = S(X,r)$ the associated monoid. Then

1. The left and the right actions
   $$\cdot : X \times X \to X, \quad \cdot^{(1)} : X \times X \to X$$
   defined via $r$ can be extended in a unique way to a left and a right action
   $$\cdot : S \times S \to S, \quad \cdot^{(1)} : S \times S \to S.$$
   which make $(S,r_S)$ a strong (graded) braided monoid. (In particular $(S,r_S)$ is a set-theoretic solution of YBE).

2. $(S,r_S)$ is nondegenerate if and only if $(X,r)$ is nondegenerate.

3. $(S,r_S)$ is involutive if and only if $(X,r)$ is involutive.

The following is an interpretation of [GIM08], Remark 4.3 in our settings.

**Remark 4.7.** Suppose the square-free solution $(Z,r)$ decomposes into a disjoint union of its $r$-invariant (nonempty) subsets $X$ and $Y$. By definition $(Z,r)$ is non-degenerate, thus the equalities $r(x,y) = (y,x)^y$, $r(y,x) = (x,y)^x$, and the non-degeneracy of $r$, $r_X$, $r_Y$ imply that
   $$y^x,x^y \in X, \quad *^y,y^x \in Y,$$
   for all $x \in X, y \in Y$.
   Therefore, $r$ induces bijective maps
   $$(4.2) \quad \rho : Y \times X \to X \times Y, \quad \sigma : X \times Y \to Y \times X,$$
   and left and right “actions”
   $$(4.3) \quad \cdot^{(1)} : Y \times X \to X, \quad \cdot^{(1)} : Y \times X \to Y, \quad \text{projected from } \rho$$
   $$(4.4) \quad \triangleright : X \times Y \to Y, \quad \triangleleft : X \times Y \to X, \quad \text{projected from } \sigma.$$

In the general setting of [GIM08] extensions satisfying (4.2) are called regular extension.

### 4.2. Decompositions of square-free solutions, and factorisation of $S$, $G$ and $G$. From now on we keep the conventions and the usual notation of this paper $(X,r)$ will denotes a square-free solution, not necessarily finite (unless we indicate the contrary). $S = S(X,r)$, $G = G(X,r)$, $G = G(X,r)$ denote respectively the YB-monoid, YB-group and the YB permutation group associated with $(X,r)$. $L : G(X,r) \to \text{Sym}(X)$ is the canonical group homomorphism defined via the left action, see Remark 2.7 and by definition $G = L(G(X,r))$. One has $G = L(S(X,r))$ whenever $X$ is a finite set.

It is known that when $(X,r)$ is a finite square-free solution the group $G$, or equivalently $G$, acts intransitively on $X$. This follows from the decomposition theorem of Rump, [Ru]. Most of the statements in this and the later sections do not need necessarily the assumption that $X$ is of finite order.
Remark 4.11. The group $\mathcal{G}$ acts intransitively on $X$ whenever $(X, r)$ is a square-free solution with finite multipermutation level $m \geq 1$ (where $X$ is a set of arbitrary cardinality). In this case the number of orbits $t$ equals at least the cardinality of the $(m - 1)$th retract, $|\text{Ret}^{m-1}(X, r)|$, see Corollary 7.9.

When the set $X$ is infinite we shall often impose the restriction that the number $t$ of $\mathcal{G}$-orbits is finite, this will be clearly indicated.

Notation 4.9. $O_\mathcal{G}(x)$ will denote the $\mathcal{G}$-orbit of $x, x \in X$.

In all cases when $X$ has finite number of $\mathcal{G}$-orbits we shall denote these by $X_1, \ldots, X_t$.

Clearly, the orbits are $r$-invariant subsets of $X$, and each $(X_i, r_i), 1 \leq i \leq t$, where $r_i$ is the restriction $r_i = r_i|_{X_i}$, is also a square-free solution. For each $\alpha \in X_j$ there is an equality of sets

$$X_j = \{^a\alpha \mid u \in \mathcal{G}\}.$$

Notation 4.10. Let $(X, r)$ be a square-free solution, $Y \subset X$ an arbitrary subset. We shall use the following notation.

$$S(Y) := \text{the submonoid of } S(X, r) \text{ generated by } Y;$$

$$G(Y) := \text{the subgroup of } G(X, r) \text{ generated by } Y;$$

$$\mathcal{G}(Y) := \mathcal{L}(G(Y)).$$

This is a subgroup of $\mathcal{G}(X, r)$ generated by the permutations

$$\mathcal{L}_y \in \mathcal{G}(X, r), \text{ where } y \in Y :$$

$$\mathcal{G}(Y) = \langle \mathcal{L}_y \in \mathcal{G}(X, r) \mid y \in Y \rangle$$

Remark 4.11. Let $Y \subset X$, an arbitrary subset. Then $\mathcal{G}(Y) = 1$ if and only if $Y \subset \ker \mathcal{L}$.

Remark 4.12. Suppose $Y$ is $\mathcal{G}$-invariant. Then $(Y, r_Y)$ is a square-free solution, (as usual $r_Y$ denotes the restriction $r_{Y \times Y}$ of $r$). Then $S(Y) \simeq S(Y, r_Y), G(Y) \simeq G(Y, r_Y)$, (see Theorem 4.19). Note that in general, $\mathcal{G}(Y)$ is different from the permutation group $\mathcal{G}(Y, r|_Y) \leq \text{Sym}(Y)$. Furthermore, if $(X, r)$ is a finite solution and $Y$ is an $r$-invariant subset of $X$, then the group $\mathcal{G}(Y)$ is the image of $S(Y)$ under the map $\mathcal{L} : G(X, r) \rightarrow \text{Sym}(X)$.

Proposition 4.13. Let $(X, r)$ be a square-free solution, $S = S(X, r), G = G(X, r)$. Suppose $Y$ is a $G$-invariant subset of $X$. Then

(i) $^a u \in S(Y), \quad u^a \in S(Y) \quad \forall u \in S(Y), a \in S$;

(ii) $^a u \in G(Y), \quad u^a \in G(Y) \quad \forall u \in G(Y), a \in G$.

Under the hypothesis of the proposition we prove first the following key lemma.

Lemma 4.14. With the assumptions and notation of Proposition 4.13, the following are equalities in $G$

$$^a(y^{-1}) = (y^a)^{-1}; \quad (y^{-1})^a = (y^a)^{-1} \quad \forall a \in G, \quad y \in X.$$

Proof. Note that each element $a \in G$ can be presented as a monomial

$$a = \zeta_1 \zeta_2 \cdots \zeta_n, \quad \zeta_i \in X \cup X^{-1}.$$
We shall consider a reduced form of $a$, that is a presentation with minimal length $n$. We shall use induction on the length $n$ of the reduced form of $a$.

**Step 1.** $a \in X \cup X^{-1}$. Two cases are possible: (i) $a \in X$. By the cyclic condition we have $^ya = ay$. This implies

$$(^ya)(^ya)^{-1} = 1$$

Recall that $(G, G)$ is a matched pair of groups, thus $^a1 = 1$ for all $a \in G$. Consider the equalities

$$1 = ^ya1 = ^ya(yy^{-1}) : \text{by ML0}$$
$$= [^ya][(^ya)^y(y^{-1})] : \text{by ML2}$$
$$= ^ya.y.(a(y^{-1})) : \text{since } (^ya)^y = a$$

Hence

$$1 = (^ya)(^ya)^{-1} = ^ya.y.(a(y^{-1}))$$

which is an equality in the group $G$, therefore the left hand side of (4.5) holds. For the right hand side one uses analogous argument. By the (right) cyclic condition one has $y^a = y^a$, hence

$$(y^a)^{-1}(y^a) = 1$$

This time we act on the right-hand side:

$$1 = 1^a = (y^{-1}y)^a$$
$$= [(y^{-1})^a][y(y^{-1})] : \text{by MR0}$$
$$= [(y^{-1})^a][y(y^{-1})] : \text{by case i}$$

So $(y^{-1})^a(y^a) = (y^a)^{-1}(y^a)$ holds in $G$ and therefore $(y^{-1})^a = (y^a)^{-1}$. This verifies RHS of (4.5). (ii) $a \in X^{-1}$, or equivalently $a = \zeta^{-1}$, where $\zeta \in X$. Recall that there is an equality

$$\zeta^{-1} y = y^\zeta \quad \forall \zeta, y \in X.$$ 

Consider now the equalities:

$$1 = \zeta^{-1}[yy^{-1}] : \text{by ML0}$$
$$= [(\zeta^{-1})^{-1}][yy^{-1}] : \text{by case i}$$
$$= [(\zeta^{-1})^{-1}][yy^{-1}] : \text{by ML2}$$
$$= [y][\zeta^{-1}(y^{-1})] : \text{by case i}$$
$$= [\zeta^{-1} y][\zeta^{-1}(y^{-1})].$$

So the equality

$$[\zeta^{-1} y][\zeta^{-1}(y^{-1})] = 1$$

implies

$$[\zeta^{-1} y]^{-1} = (\zeta^{-1})y^{-1}$$

and therefore

$$(^a y)^{-1} = a(y^{-1}).$$

This proves the LHS of (4.5). Analogous argument verifies its RHS.
Step 2. Assume \((1.5)\) hold for each \(y \in X\) and each \(a \in G\) with reduced form of minimal length \(n\). Suppose \(a \in G\) has minimal length \(n + 1\). Then \(a = \zeta b\), where \(\zeta \in X \cup X^{-1}\), and \(b \in G\) has length \(n\). Then

\[
\begin{align*}
a^{-1}(y^{-1}) &= (\zeta b)^{-1}(y^{-1}) \\
&= \zeta^b(y^{-1})^{-1} : \text{by ML1} \\
&= \zeta(\zeta^b y)^{-1} : \text{by the inductive assumption} \\
&= ((\zeta^b y))^{-1} : \text{by the inductive assumption} \\
&= (a^{-1} y)^{-1} : \text{by ML1} \\
&= (a y)^{-1} : \text{by } a = \zeta b.
\end{align*}
\]

This proves the LHS of \((4.5)\), the remaining part is proven analogously. The lemma has been proved. \(\square\)

Proof of the proposition. We shall prove the implication

\begin{equation}
(4.7) \quad u \in G(Y), a \in G \implies a^u \in G(Y)
\end{equation}

This time we use induction on the length of \(u\). Lemma \ref{lem:corollary-4.14} gives the base for the induction. Assume now \((4.7)\) holds for all \(u \in G(Y)\) with length \(n\). Suppose \(u \in G(Y)\) has length \(n + 1\), so \(u = \zeta v\), where \(\zeta \in Y \cup Y^{-1}\), and \(v \in G(Y)\) has length \(n\). One has

\[
\begin{align*}
a^u &= a(\zeta v) \\
&= [^a\zeta]a^\zeta v] : \text{by ML2} \\
&\in G(Y) : \text{since } a^\zeta, a^\zeta v \in G(Y) \text{ by the inductive assumption.}
\end{align*}
\]

Analogous argument verifies \(u^a \in G(Y)\). This proves part (ii) of the proposition. The proof of (i) is analogous.

Corollary 4.15. In notation as above let \(Y\) be a \(G\)-invariant subset of \((X, r)\). Then

i) \(S(Y)\) is an \(r_S\)-invariant subset of the braided monoid \((S, r_S)\)

ii) \(G(Y)\) is an \(r_G\)-invariant subset of the braided group \((G, r_G)\)

Proof. We shall prove (ii) ((i) is analogous). We know that \((G, r_G)\) is a braided group and \(r_G\) is defined via the left and right actions on \(G = G(X, r)\). So we have

\[
r_G(u, v) = (^u v, v^u) \quad \forall u, v \in G.
\]

By Proposition \ref{prop:corollary-4.13} each pair \(u, v \in G(Y)\) satisfies \(^u v, v^u \in G(Y)\). This shows that \(G(Y)\) is \(r_G\)-invariant. \(\square\)

Theorem 4.16. Let \((X, r)\) be a square-free solution, which decomposes into a disjoint union \(X = Y \cup Z\) of \(r\)-invariant subsets. Let \((Y, r_Y), (Z, r_Z)\) be the restricted solutions, \(G = G(X, r)\), \(G_Y = G(Y, r_Y)\), \(G_Z = G(Z, r_Z)\). Then

1) \(G(Y) \cong G_Y, G(Z) \cong G_Z\).

2) \(G_Y, G_Z\) is a matched pair of groups with actions induced from the braided group \((G, r_G)\). \(G = G(X, r)\) is isomorphic to the double crossed products

\[
G \cong G_Y \bowtie G_Z \cong G_Z \bowtie G_Y.
\]

In particular, \(G\) factorises as:

\begin{equation}
(4.8) \quad G = G(X, r) = G_Y G_Z = G_Z G_Y.
\end{equation}
(3) $G$ decomposes as product of subgroups (which in general is not a factorisation):

\[ G = G(Y)G(Z) = G(Z)G(Y). \]

**Proof.** It follows from [GIM08] Proposition 4.25. that $G_Y, G_Z$ is a matched pair, so there is a factorisation $G = G_YG_Z$, and each $w \in G$ has unique presentation as

\[ w = ua \quad \text{with} \quad u \in G_Y, a \in G_Z. \]

On the other hand $(X, r)$ is a solution, thus $(G, r_G)$ is a braided group and the equality

\[ ua = (\overset{u}{a})(u^a) \quad \text{holds} \quad \forall u, a \in G. \]

$Y$ and $Z$ are $G$-invariant subsets of $X$, so by Proposition [LI13] $a \in G_Z, u \in G \implies \overset{u}{a} \in G_Z$.

Therefore each element $w \in G$ presents as

\[ w = ua = a_1u_1 \quad \text{where} \quad u, u_1 \in G_Y, a, a_1 \in G_Z, a_1 = \overset{u}{a}, u_1 = u^a. \]

The uniqueness of $a_1$ and $u_1$ in (4.11) follows from $G_Y \cap G_Z = 1$. This implies the factorisation $G = G_ZG_Y$, hence $G_Y, G_Z$ is a strong matched pair.

We apply the group homomorphism $L$ to (4.11). Now the equalities $L(G) = G$, $L(G_Y) = L(G(Y)) = G(Y)$ and $L(G_Z) = L(G(Z)) = G(Z)$ give the decomposition (4.9). Note that each $w \in G$ decomposes as a product $w = ua = a_1u_1$, where $u, u_1 \in G(Y), a, a_1 \in G(Z)$, but this presentation is possibly not unique. \qed

**Proposition 4.17.** With the assumptions and notation of Theorem 4.16 there are isomorphisms of monoids $S(Y) \simeq S(Y, r_Y)$, $S(Z) \simeq S(Z, r_Z)$. $S(X) \cap S(Y) = 1$.

Furthermore, $(S(Y), S(Z))$ is a strong matched pair of monoids, $S$ is isomorphic to the double crossed product

\[ S = S(X, r) \simeq S(Y) \bowtie S(Z) \simeq S(Z) \bowtie S(Y). \]

There is a factorisation of monoids

\[ S = S(Y)S(Z) = S(Z)S(Y), \]

where each $w \in S$ decomposes uniquely as

\[ w = ua = a_1u_1, \quad u, u_1 \in S(Y), a, a_1 \in S(Z). \]

The following lemma is straightforward. It verifies the associativity of bicross products for $G$-invariant subset of $(X, r)$.

**Lemma 4.18.** Notation as above. The double cross product on $G$-invariant disjoint subsets of $(X, r)$ is commutative and associative. More precisely, suppose $Y_1, Y_2, Y_3$ are pairwise disjoint $G$-invariant subsets of $X$. Let $Y = Y_1 \cup Y_2 \cup Y_3$. Then

\[ S(Y_1 \cup Y_j) \simeq S(Y_i) \bowtie S(Y_j), \quad 1 \leq i < j \leq s. \]

Analogous statement is true for the groups $G(Y_1 \cup Y_j), 1 \leq i < j \leq s$. Furthermore,

\[ S(Y) \simeq S(Y_1) \bowtie [S(Y_2) \bowtie S(Y_3)] \simeq [S(Y_1) \bowtie S(Y_2)] \bowtie S(Y_3) \]

\[ G(Y) \simeq G(Y_1) \bowtie [G(Y_2) \bowtie G(Y_3)] \simeq [G(Y_1) \bowtie G(Y_2)] \bowtie G(Y_3). \]
Theorem 4.19. Let \((X, r)\) be a nontrivial square-free solution. Suppose \(X\) decomposes into a disjoint union
\[
X = \bigcup_{1 \leq i \leq p} Y_i,
\]
of \(G\)-invariant subsets \(Y_1, Y_2, \ldots, Y_s\). Then
\[
S = S(X, r) \simeq S(Y_1) \bowtie S(Y_2) \bowtie \cdots \bowtie S(Y_s)
\]
and
\[
G = G(X, r) \simeq G(Y_1) \bowtie G(Y_2) \bowtie \cdots \bowtie G(Y_s).
\]
In particular,
\begin{enumerate}
\item \(S = S(X, r)\) factorises as a product of submonoids:
\[
S = S(X, r) = S(Y_1) S(Y_2) \cdots S(Y_s),
\]
where each \(u \in S\) has unique presentation \(u = u_1 u_2 \cdots u_s\), \(u_i \in S(Y_i), 1 \leq i \leq s\).
\item \(G = G(X, r)\) factorises as a product of subgroups:
\[
G = G(X, r) = G(Y_1) G(Y_2) \cdots G(Y_s),
\]
where each \(u \in G\) has unique presentation \(u = u_1 u_2 \cdots u_s\), \(u_i \in G(Y_i), 1 \leq i \leq s\).
\item \(G = G(X, r)\) factorises as a product of subgroups:
\[
G = G(X, r) = G(Y_1) G(Y_2) \cdots G(Y_s),
\]
in the sense that each \(a \in G\) is presented as a product \(a = a_1 a_2 \cdots a_s\), where \(a_i \in G(Y_i), 1 \leq i \leq s\), but this presentation is possibly not unique. At least one of the groups \(G(Y_i)\) is nontrivial. (We have \(G(Y_i) = 1\) if and only if \(Y_i \subset \ker \mathcal{L}\).)
\end{enumerate}

Corollary 4.20. Let \((X, r)\) be a nontrivial square-free solution, which is either finite, or infinite but with a finite set of \(G\)-orbits. Let \(X_1 \cdots X_t\) be the set of all orbits in \(X\) denoted so that the first \(t_0\) orbits are exactly the nontrivial ones. Then
\[
S = S(X, r) = S(X_1) S(X_2) \cdots S(X_i)
\]
\[
G = G(X, r) = G(X_1) G(X_2) \cdots G(X_i)
\]
\[
G = G(X, r) = G(X_1) G(X_2) \cdots G(X_n).
\]

For multipermutation square-free solutions \((X, r)\) with \(mpl X = m\) there is a natural and important decomposition: \(X\) decomposes as a disjoint union of its \((m - 1)\)th retract classes. The retract classes \([x^{(k)}], 1 \leq k, x \in X\) are introduced in section 7. They are disjoint \(r\)-invariant subsets of \((X, r)\) and behave nicely. Note that when \(mpl X = m\), and \(k < m - 1\) at least one of the \(k\)-th retract classes is not \(G\)-invariant. Moreover, each \((m - 1)\)-th retract class \([x^{(m-1)}]\) is \(G\)-invariant and contains the orbit \(O_G(x)\). More precisely, each retract class \([x^{(m-1)}]\) splits into a disjoint union of the orbits \(X_{ij}\), which intersect it nontrivially.

Theorem 4.21. Let \((X, r)\) be a nontrivial square-free solution, which is either finite, or is infinite but with a finite number of \(G\)-orbits. Suppose it has a finite multipermutation level \(mpl(X, r) = m\). Then
\begin{enumerate}
\item \(\text{Ret}^{(m-1)}(X, r)\) is a finite set of order \(\leq t\), where \(t\) is the number of \(G\)-orbits of \(X\). Let
\[
Y_1 = [x_1^{(m-1)}], \ldots, Y_s = [x_s^{(m-1)}]
\]
be the set of all distinct \((m - 1)\)-retract classes in \(X\). One has \(s \geq 2\), and 
\(|Y_i| \geq 2\) for some \(1 \leq i \leq s\).

(2) \(X\) is a disjoint union \(X = \bigcup_{1 \leq i \leq s} Y_i\). Each \(Y_i, 1 \leq i \leq s\), is \(G\)-invariant, 
and \(mpl(Y_i, r_i) \leq m - 1\), where \((Y_i, r_i)\) is the restricted solution.

(3) The monoid \(S = S(X, r)\) and the group \(G = G(X, r)\) have factorisations
as in (4.12) and (4.13), respectively. \(G = G(X, r)\) also decomposes as a
product of subgroups (4.14), but some pairs of these subgroups may have
nontrivial intersection.

Proof. Clearly \(X = \bigcup_{1 \leq i \leq s} Y_i\) is a disjoint union. It follows from Proposition 7.8
that \(s \geq 2\), \(Y_i\) is \(G\)-invariant for \(1 \leq i \leq s\), and contains each \(G\)-orbit which intersects
it nontrivially. If we assume \(|Y_i| = 1, 1 \leq i \leq s\), this would imply that all \(G\)-orbits
in \(X\) are trivial, and therefore \((X, r)\) is a trivial solution, a contradiction. It follows
than that \(|Y_i| \geq 2\) for some \(i, 1 \leq i \leq s\). The inequality \(mpl(Y_i, r_i) \leq m - 1\) follows
from F acts 7.4 2. We have proved (1) and (2). (3) follows straightforwardly from
Theorem 4.19. □

Remark 4.22. Proposition 5.15 shows that in the particular case when \(2 \leq m \leq 3\)
\((X, r)\) is a strong twisted union
\[X = Y_1 \# Y_2 \# \cdots \# Y_s,\]
and \(G\) decomposes as a product of abelian subgroups
\[G = G(Y_1)G(Y_2) \cdots G(Y_s),\]

5. Multipermutation solutions of low levels.

A natural question arises:

**Question 5.1.** Suppose \((X, r)\) is a multipermutation square-free solution. What is
the relation between the multipermutation level \(mpl(X, r)\) and the algebraic proper-
ties of \(S(X, r), G(X, r), kG(X, r), A(k, X, r)\)?

The following is straightforward.

**Lemma 5.2.** Suppose \((X, r)\) is a square-free solution of order \(\geq 2\) (but not neces-
sarily finite). Then the following conditions are equivalent.

1. \(mpl(X, r) = 1\).
2. \((X, r)\) is the trivial solution, i.e. \(r(x, y) = (y, x)\), for all \(x, y \in X\).
3. \(S(X, r)\) is the free abelian monoid generated by \(X\).
4. \(G(X, r)\) is the free abelian group generated by \(X\).
5. \(G(X, r) = \{\text{id}\}\), the trivial group.

5.1. Square-free solutions of multipermutation level 2. Detailed study of
square-free solutions of multipermutation level 2 is performed in [GIM0806] and
[GIM08]. We recall first some results from these works needed in the sequel.

**Proposition 5.3.** [GIM0806]. Let \((X, r)\) be a square-free solution of finite order, let
\(X_i, 1 \leq i \leq t\) be the set of all \(G(X, r)\)-orbits in \(X\) enumerated so that \(X_1, \cdots, X_{t_0}\)
is the set of all nontrivial orbits (if any). Then the following are equivalent.
(1) \((X, r)\) is a multipermutation solution of level 2.

(2) \(t_0 \geq 1\) and for each \(j, 1 \leq j \leq t_0\), \(x, y \in X_j\) implies \(L_x = L_y\).

(3) \(t_0 \geq 1\) and for each \(x \in X\) the permutation \(L_x\) is an \(r\)-automorphism, i.e., \(G(X, r) \subseteq \text{Aut}(X, r)\).

**Theorem 5.4.** [GIM08] Let \((X, r)\) be square-free solution of multipermutation level 2 and finite order, and \(X_i, 1 \leq i \leq t\) be the set of all \(G(X, r)\)-orbits as in Proposition 5.3. Let \((X_i, r_i), 1 \leq i \leq t\) be the restricted solution. Then:

1. \(G(X, r)\) is a nontrivial abelian group.
2. Each \((X_i, r_i), 1 \leq i \leq t_0\) is a trivial solution. Clearly in the case \(t_0 < t\), each \((X_j, r_j)\), with \(t_0 \leq j \leq t\) is a one element solution.
3. For any ordered pair \(i, j, 1 \leq i \leq t_0, 1 \leq j \leq t\), such that \(X_j\) acts nontrivially on \(X_i\), every \(x \in X_j\) acts via the same permutation \(\sigma_j^i \in \text{Sym}(X_i)\) which is a product of disjoint cycles of equal length \(d = d_j^i\):

   \[
   \sigma_j^i = (x_1 \cdots x_d)(y_1 \cdots y_d) \cdots (z_1 \cdots z_d),
   \]

   where each element of \(X_i\) occurs exactly once. Here \(d_j^i\) is an invariant of the pair \((X_i, X_j)\).
4. \(X\) is a strong twisted union \(X = X_1 \triangleleft X_2 \righttriangleleft \cdots \triangleleft X_t\).

**Fact 5.5.** [GIM08]. Proposition 2.5. Let \((X, r)\) be a quadratic set. Then any of the following two conditions imply the third. (i) \((X, r)\) is involutive; (ii) \((X, r)\) is nondegenerate and cyclic; (iii) \((X, r)\) satisfies \(\text{i}ri\).

**Proposition 5.6.** Suppose \((X, r)\) is a multipermutation solution of level 2. Then

1. The associated braided monoid \((S, r_S)\) is a symmetric set which satisfies the cyclic conditions and \(\text{i}ri\) (see Definition [ML2]). \((S, r_S)\) is not square-free.

   Furthermore, \(S\) acts on itself as automorphisms:

   \[
   a(uv) = (a^u)(a^v) \quad (uv)^a = (a^u)(v^a) \quad \forall \ a, u, v \in S.
   \]

2. An analogous statement is true for the associated braided group \((G, r_G)\).

**Proof.** By Facts 5.3 \((S, r_S)\) is an nondegenerate involutive set-theoretic solution of YBE, therefore it is a symmetric set. By Proposition 5.3 \(G(X, r) \subseteq \text{Aut}(X, r)\), thus, by Corollary 3.5 and by the definition of automorphism of solutions one has

\[
L_{ax} = L_x = L_{xa} \quad \forall \ x, a \in X
\]

\[
(a y) = (a x)(y^a) \quad \forall \ a \in S, x, y \in X.
\]

Using \(\text{ML2}\) and induction on the length \(|v|\) of \(v \in S\) one shows easily that

\[
a^b v = a^v \quad v^{a^b} = v^a, \quad \forall \ a, b, v \in S;
\]

\[
a^b v = a^v \quad v^{a^b} = v^a, \quad \forall \ a, b, v \in S.
\]

Replacing \(b\) with \(v\) in each of the above equalities one yields the cyclic conditions for the solution \((S, r_S)\). We have shown that \((S, r_S)\) satisfies (i) and (ii) of Fact 5.5 hence it satisfies (iii). This verifies \(\text{i}ri\) \((a^u)^a = u = a^v\) for all \(a, u \in S\). We will verify that the left action of \(S\) (see Facts 5.3) is as automorphisms (the proof for the right action is analogous). This follows from the equalities:

\[
a^a uv = (a^u)(a^v) \quad : \text{by ML2}
\]

\[
= (a^u)(v^a) \quad : \text{by (5.3)}.
\]
We claim that \((S, r_S)\) is not square-free, or equivalently there exist an \(a \in S\), such that \(^a a \neq a\). Assume the contrary. By hypothesis \((X, r)\) is not the trivial solution, so there exist \(x, y \in X\), with \(y x \neq x\). Let \(a = xy\)

\[
^a a = xy(x y) = (xy)(xy) = (x y)(y x) = (y x)(y x) = (x y)(x y).
\]

So \(xy = a = ^a a = (y x)(x y)\) are equalities in \(S\). The only quadratic relation in \(S\) involving \(xy\) is

\[
xy = x y x y.
\]

Therefore one of the following is an equality of words in the free monoid \(\langle X \rangle\):

\[
(5.4) \quad (^y x)(^x y) = xy \quad \text{in} \quad \langle X \rangle
\]

\[
(5.5) \quad (^y x)(^x y) = (y x)(x y) \quad \text{in} \quad \langle X \rangle.
\]

**Case a.** (5.4) holds. Then \(^y x = x\) which contradicts the choice of \(x\) and \(y\). **Case b.** (5.5) is in force. Hence \(^y x = (x y)\). We apply right action by \(y\) on both sides of this equality and obtain

\[
(5.6) \quad (^y x)y = (x y)y.
\]

Thus

\[
x = (^y x)y \quad \text{by lri}
\]

\[
= (x y)y \quad \text{by (5.6)}
\]

\[
= x (y y) \quad \text{by \(G\) abelian}
\]

\[
= x y \quad \text{by \((X, r)\) square-free}.
\]

Therefore \(x y = x = x y\). It follows then by the nondegeneracy of \((X, r)\) that \(y = x\), and \(^y x = y = x\). A contradiction with the choice of \(x, y\). We have verified the first part of the proposition. An analogous argument proves the statement for the braided group \((G, r_G)\).

It is natural to ask

**Questions 5.7.** 1) When can an abelian group of permutations \(H \leq \text{Sym}
\]

\(X\) be considered as a permutation YB group of a solution \((X, r)\)?

In particular,

2) When is \(H \leq \text{Sym}
\]

\(X\) a permutation YB group of a solution \((X, r)\) of multipermutation level 2?

An answer to the second question is given in the following Proposition 5.8. Furthermore, as shows Corollary 5.9 every finite abelian group is isomorphic to the permutation group of a solution \((X, r)\) with \(\text{mpl}
\]

\(X\) = 2.

**Proposition 5.8.** Let \(H\) be an abelian permutation group on a set \(X\). Then the following are equivalent:

(a) there is a solution \((X, r)\), with \(\text{mpl}(X, r) = 2\), such that \(G(X, r) = H\);

(b) there is a function \(f\) from the set of \(H\)-orbits on \(X\) to \(H\) with the properties

- \(f(X_i)\) fixes every point in \(X_i\);

- the image of \(f\) generates \(H\).
Proposition 5.11. Let

(iii) There exists a group $G$ that mpl($X, r$) $> 5.2$.

Proof. We use the following facts: If $H$ is an abelian permutation group on $X$, and if $h \in H$ fixes $x \in X$, then $H$ fixes every point of the $H$-orbit containing $X$. For, if $k \in H$, then

$$h(k(x)) = k(h(x)) = k(x).$$

Also, if a solution $(X, r)$ has mpl($X, r$) $= 2$, and $L_\alpha(x) = y$, then $L_x = L_y$. For in the first retract, $L_{[\alpha]}$ is the identity, so that $[x] = [y]$, which means precisely that $L_x = L_y$. It follows that if $x$ and $y$ lie in the same orbit of $G(X, r)$, then $L_x = L_y$.

Now suppose that $H = G(X, r)$ for some solution $(X, r)$ with mpl($X, r$) $= 2$. For any orbit $X_i$, choose $a \in X_i$, and let $f(X_i) = L_a$. This element fixes $a$, and hence fixes every point of its orbit. Moreover, the previous paragraph shows that $f(X_i)$ is independent of the choice of $a \in X_i$. Also, the image of $f$ consists of all permutations $L_a$, for $a \in X$; so it generates $G(X, r) = H$.

Conversely, suppose that we are given a function $f$ with the properties in the proposition. For all $a \in X_i$, we define $L_a = f(X_i)$, and then construct a map $r : X \times X \to X \times X$ in the usual way:

$$r(a, b) = (L_a(b), L_b^{-1}(a)).$$

By assumption, $L_a(a) = a$. Also, for any $a, b \in X$, $L_a(b) = \alpha b$ lies in the same orbit as $b$, and hence $L_{\alpha b} = L_b$; similarly $L_{\alpha b} = L_a$; and the fact that the group is abelian now implies that

$$L_{\alpha b} L_{\beta b} = L_b L_{\alpha a} = L_a L_{\beta b}.$$

It follows that we do have a solution. Moreover, the group $H$ is generated by all the maps $L_a$, for $a \in X$; so $H = G(X, r)$.

From this we can deduce the following.

Corollary 5.9. Let $H$ be a finite abelian group. Then there is a solution $(X, r)$ with mpl($X, r$) $= 2$ such that $G(X, r) \cong H$.

Proof. Let $h_1, \ldots, h_r$ generate $H$. Now let $X = H \cup \{a_1, \ldots, a_r\}$, where $H$ has its regular action on itself and fixes the points $a_1, \ldots, a_r$. Define a function $f$ by

$$f(H) = \text{id},$$

$$f(\{a_i\}) = h_i.$$

The conditions of the proposition are obviously satisfied by $f$.

5.2. Square-free solutions of multipermutation level 3. It is straightforward that mpl $X > 2$ if and only if Ret($X, r$) is a nontrivial solution, or equivalently

$$\exists \alpha, x \in X, \text{ such that } \langle \alpha x \rangle \neq [x].$$

Remark 5.10. The following are equivalent

i) mpl($X, r$) $> 2$

ii) There exists a first-retract class $[x] = [x^{(1)}]$ which is not $\mathcal{G}$-invariant.

(iii) There exists a $\mathcal{G}$-orbit $X_0$ in $X$ and a pair $\alpha, \beta \in X_0$ such that $L_\alpha \neq L_\beta$.

Proposition 5.11. Let $(X, r)$ be a square-free solution (of arbitrary cardinality).
Suppose now \( 1 \leq j \leq 3 \) and only if the following condition holds
\[
L(\alpha x) = L(x) \quad \forall \alpha, \beta, x \in X \quad \text{with} \quad O_G(\alpha) = O_G(\beta).
\]

(2) \( \text{mpl } X = 3 \), if and only if \((5.7)\) holds, and there exists a pair \( x, \alpha \in X \) such that
\[
L(\alpha x) \neq L_x
\]

**Proof.** The following implications hold:
\[
\text{mpl}(X, r) \leq 3 \iff \text{Ret}^2(X, r) = \{\alpha x = \beta x \mid \forall \alpha, \beta, x \in X, \forall \alpha, \beta \in G \}
\]
\[
\iff [\alpha] \sim [\beta] \quad \forall \alpha \in X, \forall \beta \in G
\]
\[
\iff [^\alpha x] = [^\beta x] \quad \forall \alpha, \beta, x \in X, \quad \text{with} \quad O_G(\alpha) = O_G(\beta)
\]
\[
\iff L_{\alpha x} = L_{\beta x} \quad \forall \alpha, \beta, x \in X, \quad \text{with} \quad O_G(\alpha) = O_G(\beta).
\]

This implies the first part of the proposition. Now (1) together Remark 5.10 imply (2).

**Proposition 5.12.** Let \( (X, r) \) be a nontrivial square-free solution with condition \((5.7)\). Suppose \( X_i, 1 \leq i \leq t \), is the set of all \( G(X, r) \)-orbits in \( X \) enumerated so that \( X_1, \ldots, X_{t_0}, (t_0 \geq 1) \) are exactly the nontrivial ones. Then the following are equivalent. Then the following conditions hold:

(1) There are equalities
\[
\beta x = \alpha \quad \forall x, y \in X_i, \forall \alpha, \beta \in X_j, 1 \leq i < j \leq t.
\]

(2) Each group \( G(X_j) \), \( 1 \leq j \leq t_0 \), is abelian. In the case when \( t_0 < t \), \( G(X_j) = \{1\} \), for all \( t_0 < j \leq t \).

(3) \( X \) is a strong twisted union \( X = X_1 \sharp X_2 \sharp \cdots \sharp X_t \).

(4) In particular, \text{mpl}(X, r) = 3 implies \((5.7)\) and conditions (1), (2), (3).

**Proof.** We apply the two sides of \((5.7)\) to the element \( \alpha \), and use the cyclic condition to yield: \( L_{\beta x} = L_{\alpha x} = L_x(\alpha) \). This interpreted in our typical notation gives:
\[
\beta x = \alpha \quad \forall x \in X, \forall \alpha, \beta \in X_j, 1 \leq j \leq t.
\]

We have verified the left hand-side equality of \((5.8)\). The right hand-side is analogous. This proves part (1).

To prove (2) it will be enough to show that for two arbitrary orbits \( X_i, X_j, 1 \leq i < j \leq t \) the set \( X_{ij} = X_i \cup X_j \) is \( r \)-invariant, and the restricted solution \( (X_{ij}, r|_{X_{ij}}) \) is a strong twisted union \( X_{ij} = X_i \sharp X_j \).

Consider two orbits \( X_i, X_j \), where \( 1 \leq i < j \leq t \). As a union of two \( G \)-invariant subsets of \( X \), the set \( X_{ij} = X_i \cup X_j \) is \( r \)-invariant, so \((5.8)\) implies that the restricted solution \( (X_{ij}, r|_{X_{ij}}) \) is a strong twisted union \( X_{ij} = X_i \sharp X_j \), which proves (3).

Suppose now \( 1 \leq j \leq t \). Without loss of generality we can assume that the group \( G(X_j) \) is nontrivial (by hypothesis \( (X, r) \) is a nontrivial solution). Now in the equality \((5.7)\) we set \( x = \alpha \) and since \( (X, r) \) is square-free \((^\alpha \alpha = \alpha)\) we obtain the left hand-side of the following
\[
L_{\beta x} = L_x, \quad L_{\beta \alpha} = L_{\alpha} \quad \forall \alpha, \beta \in X_j, 1 \leq j \leq t.
\]
The equality in the right hand-side is analogous. Recall that since \((X,r)\) is a solution one has
\[
\mathcal{L}_\beta \circ \mathcal{L}_\alpha = \mathcal{L}_\beta \circ \mathcal{L}_\alpha \quad \forall \alpha, \beta \in X,
\]
which together with (5.9) implies
(5.10)
\[
\mathcal{L}_\beta \circ \mathcal{L}_\alpha = \mathcal{L}_\alpha \circ \mathcal{L}_\beta, \quad \forall \alpha, \beta \in X_j, \quad 1 \leq j \leq t.
\]
By definition, \(\mathcal{G}(X_j)\) is the subgroup of \(\mathcal{G}\), generated by the set of all \(\mathcal{L}_\alpha, \alpha \in X_j\).
It follows then from (5.11) that each nontrivial \(\mathcal{G}(X_j), 1 \leq j \leq t\), is abelian. \(\square\)

**Corollary 5.13.** Let \((X,r)\) be a finite square-free solution with \(mpl X = 3\). Let \(X_1 \cdots X_t\) be the set of \(\mathcal{G}\)-orbits in \(X\). Then

1. Each group \(\mathcal{G}(X_j), 1 \leq j \leq t\), is abelian (\(\mathcal{G}(X_j) = \{e\}\) is also possible). There exists a \(j, 1 \leq j \leq t\) for which \(\mathcal{G}(X_j)\) is nontrivial. Without loss of generality we shall assume that the nontrivial groups are \(\mathcal{G}(X_j)\) with
   \(1 \leq j \leq t_0\).
2. Suppose \(1 \leq j \leq t_0\), so \(\mathcal{G}(X_j)\) acts on \(X\) nontrivially. Consider the set of \(\mathcal{G}(X_j)\)-orbits in \(X\). Then the elements of each \(\mathcal{G}(X_j)\)-orbit in \(X\) act equally on \(X_j\), that is:
   \[
   (\mathcal{L}_\alpha)|_{X_j} = (\mathcal{L}_\beta)|_{X_j} \quad \text{whenever} \quad \beta \in \mathcal{O}_{\mathcal{G}(X_j)}(\alpha).
   \]
   Moreover, \(X = X_1 \sqcup X_2 \sqcup \cdots \sqcup X_t\).
3. The group \(\mathcal{G}\) is a product of abelian subgroups
   \[
   \mathcal{G} = \mathcal{G}(X_1)\mathcal{G}(X_2) \cdots \mathcal{G}(X_{t_0}).
   \]

This corollary gives interesting information about the graph \(\Gamma\) of finite solutions \((X,r)\) with \(mpl X = 3\). Take an arbitrary \(j\) for which the group \(\mathcal{G}(X_j)\) is nontrivial. The action of \(\mathcal{G}(X_j)\) on \(X\) is represented graphically by taking the subgraph \(\Gamma^{(j)}\) of \(\Gamma\) with the same set of vertices, but only edges labeled by \(\alpha\), where \(\alpha \in X_j\), are considered. Then clearly, there is a \(1-1\) correspondence between the set of connected components in \(\Gamma^{(j)}\) and the set of \(\mathcal{G}(X_j)\)-orbits in \(X\). For each component \(\Gamma_x^{(j)}\) (\(x\) is an arbitrary vertex in the component) the corresponding orbit is \(\mathcal{O}_{\mathcal{G}(X_j)}(x)\). As a set it coincides with the set of all vertices in \(\Gamma_x^{(j)}\). It follows then from Corollary 5.13 that all vertices in a connected component of \(\Gamma^{(j)}, 1 \leq j \leq t_0\) have the same action on the set \(X_j\).

**Question 5.14.** In notation as above, suppose that \((X,r)\) is a finite square-free solution which satisfies condition (1) and (2) of Corollary 5.13. What additional “minimal” condition should be imposed on the actions (if any) to guarantee \(mpl X = 3\)? We are searching a condition which formally is weaker that the obviously sufficient condition (5.4).

**Proposition 5.15.** Let \((X,r)\) be a nontrivial finite square-free solution of multipermutation level \(m, 2 \leq m \leq 3\). Let
\[
Y_1 = [\xi_1^{(m-1)}], \cdots, Y_s = [\xi_s^{(m-1)}]
\]
be the set of all distinct \((m-1)\)-retract classes in \(X\). Let \(S(Y_j), \mathcal{G}(Y_j), \mathcal{G}(Y_j), 1 \leq j \leq s\), be as in Notation 4.16. Then

1. \(s \geq 2, |Y_i| \geq 2\) for some \(1 \leq i \leq s\). (We enumerate them so that the first \(s_0\) are exactly the nontrivial ones).
(2) Each retract class \( Y_i, 1 \leq i \leq s \), is \( G \)-invariant and the restricted solution \((Y_i, r_i)\) where \( r_i = r_{1Y_i} \) has multipermutation level \( \leq 2 \). (In the case when \( s_0 < i \leq s, \text{mpl} Y_j = 0 \)).

(3) \((X, r)\) is a strong twisted union

\[
X = Y_1 \sharp Y_2 \sharp \cdots \sharp Y_s.
\]

(4) All groups \( G(Y_j), 1 \leq j \leq s \), are abelian and \( G(Y_j) = 1 \) is possible for at most one retract class \( Y_j \) (with \( Y_j \subset \ker L \)). Furthermore,

\[
G = G(Y_1)G(Y_2)\cdots G(Y_s).
\]

**Proof.** We give a sketch of the proof for the case mpl \( X = 3 \), the case mpl \( X = 2 \) is analogous. \( 1 \) is clear. \( 2 \) follows from Facts \( 7.3 \). We shall prove \( 3 \). Clearly, \( X \) splits into a disjoint union \( X = \bigcup_{1 \leq j \leq s} Y_j \) of the \( G \)-invariant subsets \( Y_j, 1 \leq j \leq s \). By Definition \( 5.3 \), it will be enough to show that for each pair \( i \neq j, 1 \leq i, j \leq s \) the \( G \)-invariant subset \( Y_{ij} = Y_i \bigcup Y_j \) is a strong twisted union \( Y_{ij} = Y_i \sharp Y_j \). Suppose \( \alpha, \beta \in Y_i \), then \( \alpha(2) = \beta(2) \) and by Facts \( 7.2 \) \( 7.3 \) one has

\[
\alpha x \beta = \tau x \beta \quad \forall x \in X,
\]

which implies \( Y_{ij} = Y_i \sharp Y_j \). This proves \( 3 \).

By Corollary \( 5.7 \) each group \( G(Y_j), 1 \leq j \) is abelian. By Remarks \( 5.11 \), \( G(Y_j) = 1 \text{ iff } Y_j \subset \ker L \), and \( 7.6 \) implies that if such a retract class exists it contains the set \( X_0 = X \bigcap \ker L \), and therefore it is unique. Now \( 4 \) follows from the decomposition Theorem \( 4.21 \) (3). \( \square \)

We show in section 6 that for square-free solutions of arbitrary cardinality and finite multipermutation level \( m \) one has

\[
\text{sl} G(X, r) \leq m - 1, \quad \text{sl} G(X, r) \leq m
\]

Thus

\[
\text{mpl} X = 3 \implies 1 \leq \text{sl} G(X, r) \leq 2.
\]

We conclude the section with an example of a square-free solutions \((X, r)\) with \( \text{mpl}(X, r) = 3 \) and abelian YB permutation group \( G(X, r) \).

**Example 5.16.** Let \((X, r)\) be the square-free solution defined as

\[
X = X_1 \cup X_2 \cup X_3
\]

\[
X_1 = \{ x_i | 1 \leq i \leq 8 \} \quad X_2 = \{ a, c \} \quad X_3 = \{ b, d \}
\]

\[
L_a = (b d)(x_1 x_2)(x_3 x_4)(x_5 x_6)(x_7 x_8) \quad L_c = (b d)(x_1 x_5)(x_2 x_6)(x_3 x_7)(x_4 x_8)
\]

\[
L_b = (a c)(x_1 x_3)(x_2 x_4)(x_5 x_7)(x_6 x_8) \quad L_c = (a c)(x_1 x_8)(x_2 x_7)(x_3 x_6)(x_5 x_4)
\]

\[
L_{x_i} = \text{id}_X \quad 1 \leq i \leq 8.
\]

Then mpl \((X, r) = 3, G(X, r) \) is abelian, and the group \( G(x, r) \) is solvable of solvable length 2.
As usual (In this section we keep the notation and conventions from the previous sections.

We can say a surprising amount about solutions (X, r) each (X, r) where X is the number of -orbits in X. We still do not have examples of solutions with high multipermutation level and G abelian. Hence G = G(X, r) = ⟨L_a, L_b, L_c, L_d⟩ is abelian. Next, the equality L_a = L_d ≠ L_b, implies that mpl(X, r) ≥ 3. It is easy to see that 

G = ⟨L_a⟩ × ⟨L_b⟩ × ⟨L_c⟩ ≃ C_2 × C_2 × C_2.

We will show that mpl(X, r) = 3. For the retracts one has

Ret(X, r) = ([X], r|X|), where [X] = { [a], [b], [c], [d], [x_1] },

L_{[a]} = L_{[c]} = ([b] [d]), L_{[b]} = L_{[d]} = ([a] [c])

Ret^2(X, r) = ( [[[X]], r|[[X]]| ]), where [[[X]]] = { [a], [b], [x_1] }, L_{[[a]]} = L_{[[b]]} = L_{[[x_1]]} = e

Ret^3(X, r) is the one element solution on { [a] }.

So we have mpl(X, r) = 3. We have seen that G is abelian, or equivalently, sl G = 1. It is easy to see that sl G(X, r) = 2. Indeed, mpl(X, r) = 3 implies 2 ≤ sl G(X, r), by Theorem 7.23 sl G(X, r) ≤ sl G + 1 = 2.

□

**Question 5.17.** Are there multipermutation square-free solutions (X, r) of arbitrarily high multipermutation level, and with abelian permutation group G? If not, what is the largest integer M for which there exist solutions (X, r) with mpl(X, r) = M and G abelian.

We show, see Theorem 6.1 that assuming G(X, r) abelian, one has mpl(X, r) ≤ t, where t is the number of G-orbits in X. We still do not have examples of solutions with high multipermutation level and G abelian.

6. Solutions with abelian permutation group

We can say a surprising amount about solutions (X, r) for which G(X, r) is abelian. In this section we keep the notation and conventions from the previous sections.

As usual (X, r) is a square-free solution of arbitrary cardinality, G = G(X, r) denotes its YB permutation group. In the cases when we assume X finite this will be written explicitly. Without restriction to the case of necessarily finite solutions we shall assume that X has a finite number of G- orbits, as in the previous sections, X = { X_1, · · · , X_t } will denote the set of G-orbits in X. Clearly, in the case when X is finite this condition is always in force. As discussed in the previous sections, each (X, r), 1 ≤ i ≤ t, is also a square-free solution, where r_i is the restriction r_i = r|X_i. The main results of the section are the following theorems.
Theorem 6.1. Let \((X, r)\) be a square-free solution of arbitrary cardinality. Suppose its permutation group \(G\) is abelian, and \(X\) has a finite number of \(G\)-orbits, \(X_1, \ldots, X_t\), where \(t \geq 2\). Then the following conditions hold.

1. Each \((X_i, r_i), 1 \leq i \leq t\), is a trivial solution.
2. \((X, r)\) is a strong twisted union of its \(G\)-orbits (see Definition 3.8):

\[
X = X_1 \sharp X_2 \sharp \cdots \sharp X_t.
\]

3. \((X, r)\) is a multipermutation solution, with

\[
\text{mpl}(X, r) \leq t.
\]

Corollary 6.2. Every finite square-free solution \((X, r)\) with abelian permutation group \(G\) has multipermutation level \(\text{mpl}(X, r) \leq t\), where \(t\) is the number of its \(G\)-orbits.

Theorem 6.3. Let \((X, r)\) be a square-free solution with abelian permutation group \(G\). Suppose \(X\) is a strong twisted union, \((X, r) = X_1 \sharp X_2\) of the solutions \((X_1, r_{X_1})\), and \((X_2, r_{X_2})\). Then the three solutions are multipermutation solutions and

\[
\text{mpl} X \leq \text{mpl} X_1 + \text{mpl} X_2.
\]

Remark 6.4. Note that using a different argument Cedó, Jespers, and Okniński, [CJR09], have proven that each finite square-free solution \((X, r)\) with \(G\) abelian is retractable (and therefore multipermutation solution), but do not give estimation of the multipermutation level.

Remark 6.5. Theorem 6.1 confirms Conjectures I and II [GI] in the case when the finite square-free solutions \((X, r)\) has abelian permutation group \(G(X, r)\), or equivalently its YB group \(G(X, r)\) has solvable length 2.

Remark 6.6. Note that in the general case there is no relation between \(\text{mpl} X\) and the number of orbits \(t = t(X)\). As show Lemma 6.8 and Theorem 9.11, for every integer \(m \geq 3\) there exist finite square-free solutions \((X, r)\) with exactly 2 orbits and with \(\text{mpl} X = m\).

The proofs of the main results in this section heavily rely on a necessary and sufficient condition for \(\text{mpl}(X, r) = m\) given by Theorem 7.10 in the next section. This condition is given in terms of long actions, or as we call them informally towers of actions. We need to develop first some basic technique for computation with towers of actions. We end the section with a simple construction doubling of solutions which is also used to illustrate that a solutions \((X, r)\) may have exactly two orbits and arbitrarily large finite multipermutation level \(m = \text{mpl} X\).

Definition 6.7. Let \((X, r_X)\) and \((X', r_{X'})\) be a disjoint identical finite square-free solution, where \(X = \{x_1, \ldots, x_n\}\), \(X' = \{x'_1, \ldots, x'_n\}\). Let \(Y = \{\alpha\}\) be a one element set disjoint with \(X \cup X'\), and let \((Y, r_0)\) be the one element trivial solution. (By definition one has \(\text{mpl} Y = 0\).) Consider the extension of solutions \((Z, r)\)

\[
Z = (X \uplus X') \sharp \{\alpha\},
\]

where \(r\) is an extension of the YB maps, \(r_X, r_{X'}, r_Y\) defined (as usual) via the canonical isomorphism of solutions \(\mathcal{L}_\alpha = (x_1 x'_1) \cdots (x_n x'_n)\)

\[
\begin{align*}
& r_{|X} = r_X, \quad r_{|X'} = r_{X'} \\
& r(x_i, x'_j) = (x'_j, x_i) \quad 1 \leq i, j \leq n \\
& r(\alpha, x_j) = (x_j, \alpha), \quad r(\alpha, x'_j) = (x_j, \alpha).
\end{align*}
\]
We call \((Z, r)\) a canonical doubling of \((X, r_X)\), and denote it \(Z = X^{[2, \alpha]}\).

In the following lemma, \(wr\) denotes the wreath product of a group with a permutation group. We discuss wreath products more extensively below.

**Lemma 6.8.** Let \((X, r_X)\) be a square-free solution with \(mpl X = m\). Let \((Z, r) = X^{[2, y]}\) be a canonical doubling of \((X, r_X)\). Denote 
\[G_X = G(X, r_X), G_Z = G(Z, r)\].

Then

1. \(mpl(Z, r) = m + 1\) and clearly, \((Z, r)\) has exactly two \(G(Z, r)\) orbits, namely
   \[Z_1 = X \cup X', \quad Z_2 = \{y\}\].

2. There is an isomorphism of groups
   \[G_Z \cong (G_X \times G_X) \rtimes C_\infty \cong G_X \text{ wr } C_\infty\],
   where the generator of the infinite cyclic group interchanges the two factors.

3. There is an isomorphism of groups
   \[G(Z) \cong (G_X \times G_X) \rtimes C_2 \cong G_X \text{ wr } C_2\].

The proof is straightforward.

Clearly, \(\mathcal{L}_\alpha \in \text{Aut}(X \circ_0 X')\) but does not belong to the permutation group \(G(X \circ_0 X')\), so Lemma 6.8 is a particular case of Lemma 8.3.

**Remark 6.9.** It is straightforward to see that
\[\text{sl}(G_Z) = \text{sl}(G_X) + 1, \quad \text{sl}(G_Z) = \text{sl}(G_X) + 1,\]
where \(\text{sl}(G)\) denotes the solvable length of the group \(G\). We will see more general results later.

### 6.1. Computations with actions in \((X, r)\)

In cases when we have to write a sequence of successive actions we shall use one also well known notation
\[\alpha \triangleright x = ^\alpha x\]

**Definition 6.10.** Let \(\zeta_1, \zeta_2, \ldots, \zeta_m \in X\). The expression
\[\omega = (\cdots ((\zeta_m \triangleright \zeta_{m-1}) \triangleright \zeta_{m-2}) \triangleright \cdots \triangleright \zeta_2) \triangleright \zeta_1\]
will be called a tower of actions or shortly a tower.

Clearly, the result of this action has the shape \(^u \zeta_1\), where
\[u = (\cdots ((\zeta_m \triangleright \zeta_{m-1}) \triangleright \zeta_{m-2}) \triangleright \cdots \triangleright \zeta_3) \triangleright \zeta_2,\]
so it belongs to the \(G\)-orbit of \(\zeta_1\).

The following two remarks and lemma are straightforward and hold for the general case of square-free solutions, where \(G(X, r)\) is not necessarily abelian, and \(X\) is of arbitrary cardinality.

**Remark 6.11.** Let \(\Sigma_1, \Sigma_2\) be two disjoint alphabets, \(m_1, m_2\) be positive integers, \(m = m_1 + m_2 + 1\). Let \(\omega = \zeta_m \zeta_{m-1} \cdots \zeta_2 \zeta_1\) be a string (word) of length \(m\) in the alphabet \(\Sigma_1 \cup \Sigma_2\). Then one of the following conditions is satisfied

1) \(\omega\) contains a segment \(v\) of the shape \(v = \beta y_q \cdots y_2 y_1 \alpha\), where \(q \geq 1, \ y_k \in \Sigma_j, 1 \leq k \leq q, \ \alpha, \beta \in \Sigma_i, \text{ and } 1 \leq i \neq j \leq 2\).
W e shall refer (informally) to the procedure described in Lemma 6.13 as simply missing in the formulae above.

This remark has a transparent but very useful interpretation for towers of actions.

**Remark 6.12.** Let $X_1, X_2$ be disjoint subsets of the solution $(X, r)$, $m_1, m_2$ be positive integers, $m = m_1 + m_2 + 1$. Let $\zeta_1, \zeta_2, \ldots, \zeta_m \in X_1 \cup X_2$, and $\omega$ be the tower:

$$\omega = (\cdots (\zeta_m \triangleright \zeta_{m-1}) \triangleright \cdots \triangleright \zeta_2) \triangleright \zeta_1$$

Then either

i) $\omega$ contains a segment $(((\cdots \triangleright \alpha) \triangleright \cdots) \triangleright \alpha) \triangleright \alpha$, where $q \geq 1, y_k \in X_i, 1 \leq k \leq q, \alpha, \beta \in X_j$, and $1 \leq i \neq j \leq 2$; or

ii) $\omega$ has the shape

$$\omega = (((\cdots ((\alpha \triangleright y_k) \triangleright \cdots \triangleright y_2) \triangleright y_1) \triangleright \cdots \triangleright \alpha) \cdots \triangleright \alpha_1),$$

where $y_s \in X_j, 1 \leq s \leq q, \alpha_k \in X_i, 1 \leq k \leq p$, with $1 \leq i \neq j \leq 2$. Furthermore, either $p \geq m_i + 1$, or $q \geq m_j + 1$, $(p = 0, q = m_1 + m_2 + 1, q = m_1 + m_2 + 1, q = 0$ is also possible).

**Lemma 6.13.** Let $(X, r)$ be square-free solution, $y_1, \cdots, y_k \in X$, with $k \geq 1$, and let $Z$ be an $r$-invariant subset of $X$. Suppose there is an equality

$$(\cdots ((\alpha \triangleright y_k) \triangleright \cdots \triangleright y_2) \triangleright y_1) = (\cdots (\alpha \triangleright y_k) \triangleright \cdots \triangleright y_2) \triangleright y_1 \quad \forall \alpha \in Z$$

Then any longer tower

$$(\cdots (((\alpha \triangleright y_k) \triangleright \cdots) \triangleright \alpha) \triangleright (\cdots (\alpha \triangleright y_k) \triangleright \cdots) \triangleright \cdots \triangleright b_2) \triangleright b_1$$

with $a_1, \cdots, a_s, b_1, \cdots, b_p \in X$ and $\alpha \in Z$, can be simplified by “cutting” the leftmost sub-tower of length $s + 1$, that is there is an equality:

$$(\cdots (((\alpha \triangleright y_k) \triangleright \cdots) \triangleright \alpha) \triangleright \cdots \triangleright b_2) \triangleright b_1.$$

In the particular cases $s = 0$ (respectively $p = 0$) the $a$’s, (respectively the $b$’s) are simply missing in the formulae above.

The lemma follows straightforwardly from the clear implication

$$\alpha \in Z \implies ((\cdots (\alpha \triangleright a_{\alpha-1}) \triangleright \cdots) \triangleright a_1) \triangleright \alpha \in Z$$

We shall refer (informally) to the procedure described in Lemma 6.13 as truncation.

From now on we assume that $G(X, r)$ is abelian.

**Lemma 6.14.** Let $(X, r)$ be a square-free solution with abelian permutation group $G$. Suppose $Y, Z$ are $r$-invariant subsets of $X$ and the following $stu$-type condition is satisfied:

$$(\alpha \triangleright y) = (\alpha \triangleright y) \quad \forall \alpha, z \in Z \quad \text{and} \quad \forall y \in Y.$$

Then for every pair $\alpha, z \in Z$, and every finite sequence $y_1, \cdots, y_k \in Y$, $k \geq 1$ one has

$$(\cdots (\alpha \triangleright y_k) \triangleright \cdots \triangleright y_2) \triangleright y_1 = (\cdots (\alpha \triangleright y_k) \triangleright \cdots \triangleright y_2) \triangleright y_1 \triangleright z.$$
Proof. We shall prove (6.6) using induction on $k$. Clearly, (6.5) gives the base for the induction. Assume the statement of the lemma is true for $k - 1$ where $k > 1$. Suppose $\alpha, z \in Z, y_1, \cdots y_k \in Y$. For convenience we introduce the elements $y'_{k-1}, y''_{k-1}$ as follows:

\begin{equation}
(6.7) \quad y'_{k-1} = y_k \triangleright y_{k-1} = y_k y_{k-1} \quad y''_{k-1} = (\alpha \triangleright y_k) \triangleright y_{k-1} = (\alpha y_k) y_{k-1}
\end{equation}

Then the following equalities hold.

\begin{align*}
\alpha \triangleright y'_{k-1} &= \alpha (y_k y_{k-1}) \quad \text{(6.6)} \\
&= \alpha y_k (\alpha y_k y_{k-1}) \\
&= \alpha y_k (\alpha y_k y_{k-1}) \\
&= \alpha y_k \triangleright y''_{k-1} \quad \text{(6.7)}
\end{align*}

Thus

\begin{equation}
(6.8) \quad \alpha \triangleright y'_{k-1} = (\alpha y_k) \triangleright y''_{k-1}.
\end{equation}

Now consider the equalities

\begin{align*}
&(((\cdots ((\alpha \triangleright y_k) \triangleright y_{k-1}) \triangleright \cdots \triangleright y_2) \triangleright y_1) \triangleright z \\
&= (\cdots (y'_{k-1} \triangleright y_k) \triangleright \cdots \triangleright y_2) \triangleright y_1 \triangleright z \quad \text{(6.7)} \\
&= (\cdots ((\alpha y_k) \triangleright y'_{k-1} \triangleright \cdots \triangleright y_2) \triangleright y_1 \triangleright z \quad \text{by IH} \\
&= (\cdots ((\alpha \triangleright y_k) \triangleright y_{k-1}) \triangleright \cdots \triangleright y_2) \triangleright y_1 \triangleright z \quad \text{(6.8)} \\
&= (\cdots (y''_{k-1} \triangleright y_k) \triangleright \cdots \triangleright y_2) \triangleright y_1 \triangleright z \quad \text{(6.7)}
\end{align*}

where IH is the inductive assumption. This proves the Lemma. \qed

Remark 6.15. Note that in the hypothesis of Lemma 6.14 we do not assume that the sets $Y, Z$ are disjoint. Furthermore the stu-type condition is not imposed symmetrically on both sets, i.e. even if $Y$ and $Z$ are disjoint we do not assume that necessarily $Y \triangleleft Z$.

6.2. Proofs of the theorems.

Lemma 6.16. Let $(X, r)$ be a square-free solution, with abelian permutation group $G$. Then the following two conditions hold.

\begin{enumerate}
\item Let $Y$ be a $G$-orbit of $X$. Then for any $x \in Y$ one has $(L_x)|_Y = \text{id}_Y$
\item Suppose $Y, Z$ are two distinct $G$-orbits of $X$, $(Y, r_Y), (Z, r_Z)$ are the canonically induced solutions on $Y$ and $Z$. Then the actions satisfy the stu condition:

\begin{equation}
(L_{\alpha x})|_Y = (L_\alpha)|_Y: \quad (L_{\alpha x})|_Z = (L_x)|_Z \quad \text{for all} \quad x \in Y, \alpha \in Z
\end{equation}

Moreover, the induced solution $(T, r_T)$ on the union $T = Y \cup Z$ is a strong twisted union $T = Y \triangleleft Z$.
\end{enumerate}

Proof. Let $x \in Y$. To prove (1) it will be enough to show

\begin{equation}
(6.9) \quad x(t^t x) = t^t x, \quad \forall t \in X.
\end{equation}

Now the equalities

\begin{align*}
x(t^t x) &= t^{(t^t x)} \quad G \text{ abelian} \\
&= t^t x \quad (X, r) \text{ square-free}.
\end{align*}
By Lemma 6.13 there are equalities

\[ x^\alpha x = x, \quad \beta x^\alpha = x^\beta \alpha. \]

Consider the equalities

\[
\begin{align*}
\alpha x &= x^{(\alpha x)} \quad y^\alpha x \in Y \text{ and } (1) \\
\alpha x &= (\alpha x)^{(\alpha x)} \quad \text{by } \textbf{II} \\
(\alpha x)^x &= y^{\alpha x} \quad x, y^\alpha \in Y \text{ and } (1)
\end{align*}
\]

We have shown the left hand side equality in (6.10). Analogous argument gives the remaining equality.

**Corollary 6.17.** Let \((X, r)\) be a square-free solution of arbitrary cardinality and with abelian permutation group \(G\). Suppose \(G\) acts non transitively on \(X\) and splits it into a finite number of \(G\)-orbits \(X_1, \ldots, X_t\), \(t \geq 2\). Then each \((X_i, r_i)\) is a trivial solution and \(X\) is a strong twisted union

\[ X = X_1 \sharp X_2 \sharp \cdots \sharp X_t. \]

**Proposition 6.18.** Under the hypothesis of Theorem 6.14, \((X, r)\) is multipermutation solution with \(\text{mpl} X \leq t\), where \(t\) is the number of \(G\)-orbits of \(X\).

**Proof.** By Theorem 7.10 it will be enough to show that for each choice of \(y_1, \ldots, y_{m+1} \in X\) there is an equality

\[ \omega := (\cdots ((y_{m+1} \triangleright y_m) \triangleright \cdots y_2) \triangleright y_1) = (\cdots ((y_{m+1} \triangleright y_m) \triangleright \cdots y_2) y_1 =: \omega'. \]

Clearly, since the orbits are exactly \(m\), there will be some \(1 \leq \lambda < \lambda + \mu \leq m + 1\), such that \(y_\lambda, y_{\lambda+\mu}\) are in the same orbit, say \(X_i\).

Two cases are possible.

**Case 1.** \(\mu = 1\). In this case, \(\lambda + \mu = \lambda + 1 (y_{m+1} \triangleright y_m) \triangleright \cdots y_2) \triangleright y_1 = (\cdots ((y_{m+1} \triangleright y_m) \triangleright \cdots y_2) \triangleright y_1 = y_\lambda = y_\lambda, \)

thus

\[ \omega = (\cdots ((y_{m+1} \triangleright y_m) \triangleright y_{m-1}) \triangleright y_{\lambda+1}) \triangleright y_1 = (\cdots ((y_{m+1} \triangleright y_m) \triangleright y_{m-1}) \triangleright y_{\lambda+1}) \triangleright y_1 \]

Similarly,

\[ \omega' = (\cdots ((y_{m+1} \triangleright y_m) \triangleright y_{m-1}) \triangleright y_1 = (\cdots ((y_{m+1} \triangleright y_m) \triangleright y_{m-1}) \triangleright y_1 \]

So \(\omega = \omega'\), which proves (6.11).

**Case 2.** \(\mu > 1\). In this case we set \(y_{\lambda+\mu} = \alpha, y_\lambda = z\). The tower \(\omega\) contains the segment \((\cdots (\alpha \triangleleft y_{\lambda+\mu-1}) \triangleright \cdots y_{\lambda+1}) \triangleright z,\) with \(\alpha, z \in X_i\). By Lemma 6.13 we can cut \(\alpha\) from the tower to yield

\[ (\cdots (\alpha \triangleleft y_{\lambda+\mu-1}) \triangleright \cdots y_{\lambda+1}) \triangleright z = (\cdots (y_{\lambda+\mu-1} \triangleright y_{\lambda+\mu-2}) \triangleright \cdots y_{\lambda+1}) \triangleright z. \]

We shall assume \(\lambda + \mu < m + 1\) (The proof in the case \(\lambda + \mu = m + 1\) is analogous).

By Lemma 6.13 there are equalities

\[
\begin{align*}
\omega &= (\cdots ((\cdots ((y_{m+1} \triangleright y_m) \triangleright \cdots y_\lambda) \triangleright y_{\lambda+\mu-1}) \triangleright \cdots y_{\lambda+1}) \triangleright z) \triangleright \cdots \triangleright y_1 \\
&= (\cdots ((\cdots (y_{\lambda+\mu-1} \triangleright y_{\lambda+\mu-2}) \cdots y_{\lambda+1}) \triangleright z) \triangleright \cdots \triangleright y_1 \\
&= (\cdots ((\cdots (y_{m+1} \triangleright y_m) \triangleright \cdots y_\lambda) \triangleright y_{\lambda+\mu-1}) \triangleright \cdots y_{\lambda+1}) \triangleright z) \triangleright \cdots \triangleright y_1.
\end{align*}
\]

The proposition has been proved. \(\square\)
Theorem 6.1 follows straightforwardly from Corollary 6.17 and Proposition 6.18.

We shall now prove Theorem 6.3. Suppose the square free solution \((X, r)\) is a strong twisted union \((X, r) = X_1 \bowtie X_2\) and \(G(X, r)\) is abelian. Then by Theorem 6.1 \((X, r), (X_1, r_{X_1}), (X_2, r_{X_2})\) are multipermutation solutions. Let \(\text{mpl}(X_1) = m_1\), \(\text{mpl}(X_2) = m_2\). We claim that \(\text{mpl}(X) \leq m_1 + m_2\). Denote \(m = m_1 + m_2 + 1\). By Theorem 7.10 it will be enough to show that for any choice of \(\zeta_1, \ldots, \zeta_m \in X\) there is an equality

\[
\omega = \cdot \cdot \cdot (\zeta_m \triangleright \zeta_{m-1}) \triangleright \cdots \triangleright \zeta_2 \triangleright \zeta_1 = \omega' = \cdot \cdot \cdot (\zeta_{m-1} \triangleright \zeta_{m-2}) \triangleright \cdots \triangleright \zeta_2 \triangleright \zeta_1
\]

By Remark 6.12 two cases are possible.

**Case 1.** \(\omega\) contains a segment \((\cdot \cdot \cdot \triangleright \beta \triangleright y_q \triangleright \cdots \triangleright y_2 \triangleright y_1) \triangleright \alpha\), where \(q \geq 1\), \(y_k \in X_i, 1 \leq k \leq q\), \(\alpha, \beta \in X_1\), and \(1 \leq i \neq j \leq 2\). Since \(X\) is strong twisted union of \(X_1\) and \(X_2\) the hypothesis of Lemma 6.14 is in force, and therefore

\[
(\cdot \cdot \cdot \triangleright \beta \triangleright y_q \triangleright \cdots \triangleright y_2 \triangleright y_1) \triangleright \alpha = (\cdot \cdot \cdot (\triangleright y_q \triangleright y_{q-1}) \triangleright \cdots \triangleright y_2 \triangleright y_1) \triangleright \alpha
\]

Now apply Lemma 6.13 to deduce (6.12).

**Case 2.** \(\omega\) has the shape

\[
\omega = \cdot \cdot \cdot ((\cdot \cdot \cdot \triangleright y_p \triangleright y_{q-1}) \triangleright \cdots \triangleright y_2 \triangleright y_1 \triangleright \alpha_p) \cdots \triangleright \alpha_2 \triangleright \alpha_1
\]

where \(y_s \in X_j, 1 \leq s \leq q\), \(\alpha_k \in X_i, 1 \leq k \leq p\), with \(1 \leq i \neq j \leq 2\). Furthermore, either \(p \geq m_i + 1\), or \(q \geq m_j + 1\). Without loss of generality we may assume \(q \geq m_j + 1\). But \(\text{mpl}(X_j) = m_j\), so Theorem 7.10 implies the equality

\[
(\cdot \cdot \cdot (\triangleright y_q \triangleright y_{q-1}) \triangleright \cdots \triangleright y_2 \triangleright y_1) = (\cdot \cdot \cdot (\triangleright y_{q-1} \triangleright y_{q-2}) \triangleright \cdots \triangleright y_2) \triangleright y_1
\]

We apply again Lemma 6.13 to obtain (6.12). The case when \(p \geq m_i + 1\) is analogous and we leave it to the reader.

Theorem 6.3 has been proved. Q.E.D.

7. Multipermutation solutions of finite multipermutation level

7.1. General results. We shall use the notation from section 2. We also recall some notions and basic facts from [GI].

**Notation 7.1.** Let \((X, r)\) be a square-free solution. For each integer \(k \geq 0\) as usual, we shall use following notation.

1. \(\text{Ret}^k(X, r)\) denotes the \(k\)-th retract of \((X, r)\), but when \(k = 1\) it is convenient to use both notations \(\text{Ret}(X, r) = \text{Ret}^1(X, r)\) and \([X], r_{[X]}\) for the retract. For completeness we set \(\text{Ret}^0(X, r) = (X, r)\).

2. \(x^{(k)}\) denotes the image of \(x\) in \(\text{Ret}^k(X, r)\). The set

\[
[x^{(k)}] := \{\xi \in X \mid x^{(k)} = \xi^{(k)}\}
\]

is called the \textit{k-th retract class of x}. (In [GI] it is referred to as the \textit{k-th retract orbit of x}).

3. In the case when \(\text{mpl}(X, r) = m < \infty\), and \(X\) has a finite number of \(G\)-orbits, we let these orbits be

\[
X_1, \ldots, X_t.
\]
(4) We fix a notation for the elements of the \((m - 1)\)th retract:

\[
\text{Ret}^m(X, r) = \{\zeta_1^{(m-1)}, \ldots, \zeta_s^{(m-1)}\}.
\]

(By Corollary 7.9 one has \(s \leq t\)).

(5) The \(m - 1\)th retract classes will be denoted by

\[
Y_i := [\zeta_i^{(m-1)}], \ 1 \leq i \leq s.
\]

For each \(i, 1 \leq i \leq s\) we denote the set of all \(G\)-orbits of \(X\) which intersect \(Y_i\) nontrivially by

\[
X_{i1}, X_{i2}, \ldots, X_{it}.
\]

Remark 7.2. In the above notation, suppose that \(\text{mpl}(X, r) = m < \infty\), and that \(X\) has a finite number of \(G\)-orbits, say \(X_1, \ldots, X_t\). Then by Corollary 7.9 \(\text{Ret}^m(X, r)\) is a finite set of order \(s, 2 \leq s \leq t\).

Furthermore, it follows from Proposition 7.8 that for each pair \(i, j, 1 \leq i \leq s, 1 \leq j \leq t\) one has

\[
Y_i \cap X_j \neq \emptyset \iff X_j \subseteq Y_i.
\]

so each \((m - 1)\)th retract class \(Y_i, 1 \leq i \leq s\) is a disjoint union of the set of all \(G\)-orbits which intersect it nontrivially:

\[
Y_i = \bigcup_{1 \leq k \leq t_i} X_{ik}.
\]

The following corollary is straightforward from Lemma 2.24.

Corollary 7.3. For each integer \(k \geq 1\) the canonical map \((X, r) \to \text{Ret}^k(X, r), x \mapsto x^{(k)}\), is a homomorphism of solutions.

The following results are extracted from [GI], where they are stated for finite square-free solutions \((X, r)\). However, the argument does not rely on the finiteness of \(X\).

Facts 7.4. [GI]

(1) [GI, Lemma 8.10. For every \(\alpha, \beta, x \in X\), and \(k\) a positive integer,

\[
\alpha^{(k)} = \beta^{(k)} \implies (\alpha x)^{(k-1)} = (\beta x)^{(k-1)}.
\]

In particular,

\[
\alpha^{(2)} = \beta^{(2)} \implies \alpha x \sim \beta x \quad \forall x \in X, \quad \alpha \beta \sim \beta.
\]

(2) [GI, Lemma 8.9. It follows (7.1) that for any positive integer \(k\), and any \(x \in X\), the restriction \(r_{x,k}\) of \(r\) on \([x^{(k)}]\) is a bijective map

\[
r_{x,k} : [x^{(k)}] \times [x^{(k)}] \to [x^{(k)}] \times [x^{(k)}],
\]

so the \(k\)th retract class \(([x^{(k)}], r_{x,k})\) is itself a solution. Furthermore, \(([x^{(k)}], r_{x,k})\) is a multipermutation solution of level \(\leq k\). In particular, whenever \([x]\) has cardinality \(\geq 2\), \(([x], r_{x,1})\) is the trivial solution.

(3) [GI, Lemma 8.12.

\[
\alpha^{(2)} = \beta^{(2)} \implies x^{(2)} x = \beta \quad \forall x \in X, \quad (L_{\alpha x})^{|[\alpha^{(2)}]} = (L_x)^{|[\alpha^{(2)}]}.
\]

Remark 7.5. Note that Lemma 8.9 in [GI] states inaccurately that \(\text{mpl}([x^{(k)}], r_{x,k}) = k\). The correct statement is \(\text{mpl}([x^{(k)}], r_{x,k}) \leq k\).
Remark 7.6. Suppose that \(X_0 = X \cap \ker \mathcal{L} \neq \emptyset\) then \(G(X_0) = 1\). Let \(\zeta_0 \in X_0\). Then \(X_0 = [\zeta_0] \subseteq \mathcal{G}(\zeta_0)\), for all \(k \geq 1\). Let \(Y = [x^{(k)}]\) be a \(k\)-th retract class distinct from \([\zeta_0]^{(k)}\). Then this class generates a nontrivial permutation group \(G(Y)\). It follows then that \(G(Y) = 1\) is possible for at most one \(k\)-th retract class, namely \(Y = [\zeta_0]^{(k)}\), and this happens in the particular case \(Y = [\zeta_0]^{(k)} = X_0\).

Corollary 7.7. Suppose \(Y = [\zeta_0]^{(2)}\) is a second retract class in \(X\). Then the permutation group \(G(Y)\) is an abelian subgroup of \(G = G(X, r)\). \(G(Y) = 1\) iff \(Y \subset \ker \mathcal{L}\).

Proof. Let \(\alpha, \beta \in Y\). Then \(\alpha^{(2)} = \beta^{(2)}\), and by \((7.2)\) one has
\[
\alpha \beta \sim \beta \beta = \beta,
\]
or equivalently
\[
\text{\(L_{\alpha \beta} = L_\beta\)}
\]
\[
\text{\(L_{\alpha \beta} = L_\alpha\)}
\]
\[
\text{\(L_\alpha \circ L_\beta = L_\beta \circ L_\alpha = \iota \) in \(\{\lambda, \rho\}\),}
\]
hence \(G(Y)\) is abelian. \(\square\)

Proposition 7.8. The following conditions are equivalent:

1. \(\text{mpl}(X, r) = m\).
2. For every \(x \in X\) one has
\[
X \supset [x^{(m-1)}] \supset \mathcal{O}_G(x),
\]
where the left hand side inclusion is strict, and \(\mathcal{O}_G(x)\) is the \(G\)-orbit of \(x\).
3. For every \(x \in X\) the \(m-1\) retract class \([x^{(m-1)}]\) is a \(G\)-invariant proper subset of \(X\).

Proof. Note that \(\text{mpl}(X, r) = m\) iff \(\text{Ret}^{m-1}\) is a trivial solution with at least 2 elements. Clearly, \(\text{Ret}^{m-1}\) is a trivial solution iff
\[
(a^{(m-1)})(x^{(m-1)}) = x^{(m-1)} \quad \forall a, x \in X.
\]
The following are equalities in \(\text{Ret}^{m-1}\)
\[
\text{mpl}(X, r) = m \iff \text{Ret}^{m-1} \text{ is a trivial solution of order } \geq 2
\]
\[
\iff (a x)^{(m-1)} = x^{(m-1)} \quad \forall a, x \in X, [x^{(m-1)}] \subset X
\]
\[
\iff \mathcal{O}_G(x) \subseteq [x^{(m-1)}] \subset X \quad \forall a, x \in X
\]
\[
\iff [x^{(m-1)}] \text{ is a } G\text{-invariant proper subset of } X.
\]
\(\square\)

Corollary 7.9. Let \((X, r)\) be a square-free multipermutation solution of arbitrary cardinality. Suppose it is a multipermutation solution with \(2 \leq \text{mpl} X = m < \infty\). Then the number of \(G\)-orbits in \(X\) is at least the cardinality of \(\text{Ret}^{m-1}\). In particular, \(G\) acts intransitively on \(X\).
Proof. The \((m-1)\)st retract \(\text{Ret}^{m-1}\) is a trivial solution with at least 2 elements. By Proposition 7.8 each \((m-1)\)-retract class \([x^{(m-1)}]\) contains the \(G\)-orbit of \(x\). This proves the corollary. \(\square\)

**Theorem 7.10.** Let \((X, r)\) be an arbitrary square-free solution, not necessarily of finite cardinality. Then

i) \(\text{mpl}(X, r) \leq m\) if and only if the following equality holds

\[
(y_m \supseteq y_{m-1}) \supseteq \cdots \supseteq y_2 \supseteq y_1) \supseteq x,
\]

\(\forall x, y_1, \cdots, y_m \in X\).

ii) \(\text{mpl}(X, r) = m\) if and only if \(m\) is the minimal integer for which (7.6) holds.

Proof. We use induction on \(m\) to show the implication

\[
\text{mpl} X \leq m \iff (7.6).
\]

Clearly,

\[
\text{mpl} X \leq 2 \iff \{y\} = \{y\}, \forall y, z \in X \iff ^*y \cdot x = y \cdot x, \forall x, y, z \in X.
\]

This gives the base for the induction.

Assume the implications are true whenever \(\text{mpl} X \leq m\). Consider now the retract \([(X), r_{(X)}\]). Clearly \(\text{mpl} X = \text{mpl} [X] + 1\). Furthermore, by the inductive assumption the following equality holds if and only if \(\text{mpl} [(X), r_{(X)}] \leq m\).

\[
((\cdots ((y_{m+1} \supseteq y_m) \supseteq \cdots \supseteq y_2) \supseteq y_1) \supseteq (\cdots ((y_{m+1} \supseteq y_m) \supseteq \cdots \supseteq y_2) \supseteq y_1) \supseteq x,
\]

\(\forall y_1, \cdots, y_m, y_{m+1} \in X\).

(Here we enumerate differently: we write \(y_1\) instead of \(x\), etc.) By the obvious equalities

\[
(7.8) \quad [a \supseteq b] = [a] [b] = [a \supseteq b],
\]

\(7.7\) is equivalent to

\[
(7.9) \quad [\cdots ((y_{m+1} \supseteq y_m) \supseteq y_{m-1}) \supseteq \cdots \supseteq y_2) \supseteq y_1] = [\cdots ((y_{m+1} \supseteq y_m) \supseteq \cdots \supseteq y_2) \supseteq y_1]
\]

for all \(y_1, \cdots, y_m, y_{m+1} \in X\).

But \(7.8\) is equivalent to

\[
(7.10) \quad (\cdots ((y_{m+1} \supseteq y_m) \supseteq y_{m-1}) \supseteq \cdots \supseteq y_2) \supseteq y_1, j \supseteq (\cdots ((y_{m+1} \supseteq y_m) \supseteq \cdots \supseteq y_2) \supseteq y_1) \supseteq x,
\]

\(\forall x, y_1, \cdots, y_m, y_{m+1} \in X\).

This proves the equivalence \(\text{mpl} X \leq m + 1 \iff (7.10)\), which proves i).

(ii) follows straightforwardly from (i).

The theorem has been proved. \(\square\)
7.2. The groups $G(X, r)$ and $G(X, r)$. We recall more results.

**Facts 7.11.** Suppose $(X, r)$ is a square-free solution. Then

1. $G = G(X, r)$ is torsion free. Let $p$ be the least common multiple of the orders of all permutations $L_x$ for $x \in X$. Then the following conditions hold.
   - (i) $yx^p = ((xy)^p)y, \ x^py = y(x^y)^p \ \forall x, y \in X$. Thus the group $G$ acts via conjugation on the set $X^{(p)} = \{x^p \mid x \in X\}$. 
   - (ii) $x^py^p = y^px^p, \ \forall x, y \in X$. The subgroup $A$ of $G$ generated by the set $X^{(p)} = \{x^p \mid x \in X\}$ is isomorphic to the free abelian group in $n$ generators, (see [GIM08]).

2. It is known that each set-theoretic solution $(X, r)$ of YBE (a braided set) can be extended canonically to a solution $(S, r_S)$ on $S = S(X, r)$, see [GIM08] (respectively to a solution $(G, r_G)$, on $G = G(X, r))$, [LYZ] which makes $(S, r_S)$, a braided monoid (respectively $(G, r_G)$, a braided group. In other words the equality

$$uv = "v.u"$$

holds in for all $u, v$ in $S$ (respectively in $G$).

**Remark 7.12.** It follows from the results of [GIM08] that the extended solution $(G, r_G)$ satisfy

1. $(G, r_G)$ is involutive (i.e. $(G, r_G)$ is symmetric set) if and only if $(X, r)$ is involutive.
2. $(G, r_G)$ is nondegenerate if and only if $(X, r)$ is nondegenerate
3. In particular, if $(X, r)$ is a square-free solution then $(G, r_G)$ is a nondegenerate symmetric set (but in general it is not square-free). The notion of equivalence $u \sim v$ given by

$$u \sim v \iff (\forall g \in G)(u^g = v^g)$$

is well defined, and, as usual, $[u]$ denotes the equivalence class of $u$ in $G$.

**Lemma 7.13.** Let $G = G(X, r)$. The kernel $K_0 = \ker{\mathcal{L}}$ of the group homomorphism $\mathcal{L} : G \rightarrow \operatorname{Sym}(X)$ is a normal abelian subgroup of $G$ of finite index. In particular, $K_0$ contains the free abelian subgroup $A = \langle x_1^p, \ldots, x_n^p \rangle$, where $p$ is the least common multiple of all orders of permutations $L_x$, for $x \in X$.

**Proof.** Clearly $u \in K_0$ if and only if $L_u = \operatorname{id}_X$, and by lri the right action $R_u = (\mathcal{L}_u)^{-1} = \operatorname{id}_X$. This straightforwardly implies

$$u \in K_0 \iff u^a = a \ \forall a \in G \iff a^n = a, \ \forall a \in G$$

Assume now $u, v \in K_0$ Then $uv = ((a)^p)^v = ((a)^v)^p, \ uv = so K_0$ is abelian. Clearly, $\mathcal{L}_x^p = (\mathcal{L}_x)^p = \operatorname{id}_X$, so $x^p \in K_0$, for all $x \in X$, and therefore the free abelian group $A$ is contained in $K_0$.

In assumption and conventions as above we introduce more notation.

**Notation 7.14.**

1. $G_i = G(\text{Ret}^i(X, r)), \ G_0 = G(X, r) = G(\text{Ret}^0(X, r)) := (X, r)$.
2. $\mathcal{G}_i = G(\text{Ret}^i(X, r))$. 
(3) \( L^0 = L : G(X, r) \to G(X, r) \) is the usual epimorphism extending the assignment \( x \mapsto L_x, x \in X \)
\[ L^i = L : G_i \to G_i \] is the canonical epimorphism extending the assignment \( x^{(i)} \mapsto L_{x^{(i)}} \in \text{Sym}(\text{Ret}^i(X, r)), x \in X \).

(4) \( K_i \) is the pull-back of \( \ker L^i \) in \( G \), in particular \( K_0 = \ker L \).

(5) \( \mu_i : G_i \to G_{i+1} \) are the canonical epimorphisms extending \( x^{(i)} \mapsto x^{(i+1)} \), where \( 0 \leq i < \text{mpl} X \), see Lemma 7.15 and Proposition 7.17.

(6) \( N_i \) is the pull-back of \( \ker \mu_i \) in \( G \).

(7) \( \varphi_i : G_i \to G_{i+1} \) is the canonical epimorphism extending the assignments \( L_{x^{(i)}} \mapsto L_{x^{(i+1)}} \), \( x \in X \), see Lemma 7.15 and Proposition 7.17.

(8) \( H_i \) is the pull-back of \( \ker \varphi_i \) in \( G \).

Note that by definition, for \( 1 \leq i \leq m-1 \) one has
\[ K_1 = \{ u \in S \mid L_{[u]} = \text{id}_{[X]} \}, \quad K_i = \{ u \in S \mid L_{(u^{(i)})} = \text{id}_{\text{Ret}^i(X, r)} \}. \]

**Lemma 7.15.** In assumption and notation as above the following conditions hold.

1. The canonical epimorphism of solutions
   \[ \mu : (X, r) \to ([X], [r]); \quad x \mapsto [x], \]
   extends to a group epimorphism \( \mu_0 : G_0 \to G_1 \). Analogously there exists a group epimorphism \( \mu_1 : G_1 \to G_2 \).
2. There is a canonical epimorphism \( \varphi_0 : G_0 \to G_1 \) \( L_x \mapsto L_{[x]}, \forall x \in X \).
3. The kernels \( N_0 = \ker \mu_0, K_0 = \ker L^0, H_0 - the pull back of \( \ker \varphi_0 \) into \( G \), and \( K_1 \), the pull back of \( \ker L^1 \) into \( G \) satisfy
   \[ N_0 \subset K_0 \subset K_1 = H_0 \]
   \[ \ker \mu_1 \simeq N_1/N_0; \quad \ker L_1 \simeq K_1/N_0; \quad \ker \varphi_0 \simeq K_1/K_0. \]
4. In particular, \( N_0 \) is an abelian normal subgroup of \( G_0 \), and there is a canonical epimorphism of groups
   \[ f_0 : G_1 \to G_0 \mid [x] \mapsto L_{[x]}, \quad x \in X \quad \text{with} \quad \ker f_0 \simeq K_0/N_0. \]
5. There are short exact sequences:
   \[ 1 \to N_0 \to G \xrightarrow{\mu_0} G_1 \to 1 \quad 1 \to K_0 \to G \xrightarrow{L^0} G \to 1 \]
   \[ 1 \to K_1/N_0 \to G_1 \xrightarrow{\mu_1} G_2 \to 1 \quad 1 \to K_0/N_0 \to G_1 \xrightarrow{f_0} G \to 1 \]
   \[ 1 \to K_1/N_0 \to G_1 \xrightarrow{L^1} G_1 \to 1 \quad 1 \to K_1/K_0 \to G \xrightarrow{\varphi_0} G_1 \to 1, \]

Moreover, the following diagram is commutative:
Proof. Parts (1), (2) are clear. We shall verify (3). Clearly, the kernel \( N_0 = \text{Ker} \mu_G \) consists of all \( a \in G \), such that \( [a] = 1[G] \), hence it will be enough to show the implication

\[
[a] = 1[G] \implies L_a = \text{id}_X.
\]

Indeed, suppose \( [a] = 1[G] \). Then for an arbitrary \( x \in X \) one has \( [ax] = [a][x] = [x] \), so

\[
x = x \\
x = x \\
\text{(7.16)}
\]

In particular, (7.16) is true for \( y = x \), thus for an arbitrary \( x \in X \) one has

\[
x = x \\
x = x \\
\text{(7.17)}
\]

where the equality \( x = x \) follows from our assumption \((X, r)\) square-free. We have shown \( a^x = x \), for every \( x \in X \), thus \( L_a = \text{id}_X \). This verifies \( N_0 \subseteq K_0 \). By Lemma 7.15 the group \( K_0 \) is abelian, so is \( N_0 \).

The equality \( H_0 = K_1 \) follows from the implications:

\[
\begin{align*}
u \in H_0 & \iff L_{\mu} = \text{id}_X \\
& \iff \mu[x] = [\mu x] = [x] \quad \forall x \in X \\
& \iff xz = xz \quad \forall x, z \in X \\
& \iff u \in K_1.
\end{align*}
\]

The inclusions (7.13) for the three kernels are clear. This implies the second line in (7.12). The existence of the short exact sequences (7.14) is straightforward from (7.13). One easily sees that the diagram (7.15) is commutative.

We discuss some basic differences between the two kernels \( N_0 \) and \( K_0 \) below.

Remark 7.16. Suppose \((X, r)\) is a nontrivial square-free solution of finite order (so \( \text{mpl} X \geq 2 \)). Then \( K_0 \) is a normal subgroup of \( G \) of finite index \([G : K_0]\), and in contrast, the index \([G : N_0]\) of \( N_0 \) is not finite. Furthermore, \( A \subseteq K_0 \), but \( A \cap N_0 = e \). Indeed, by hypothesis \((X, r)\) is a nontrivial solution then, by Lemma 5.2 the set \([X]\) has order \( > 1 \). Furthermore the retract \(([X], r)\) is a braided set. Hence \([X]\) generates the group \( G_1 = G([X], r_{[X]}), [x] \neq 1_{G_1[X]}, \forall x \in X \). The group \( G_1 \) is torsion free as a YB group of square-free solution of order \( > 1 \), see Facts 7.11.
in particular, \([xp] = [x]^p \neq 1_{G_X}\), so \(\forall x \in X, xp\) is not in \(N_0\). On the other hand we have shown in Lemma 7.13 that \(xp \in K_0, \forall x \in X\).

The following proposition is an iteration of Lemma 7.15

**Proposition 7.17.** Let \((X, r)\) be a nontrivial square-free solution. Suppose \(mpl(X, r) = m\). Then the following conditions hold.

1. For all \(0 \leq j \leq m - 1\), there are canonical group epimorphisms
   \[
   \mu_j : G_j \rightarrow G_{j+1}, \quad \mu_j(x^{(j)}) \mapsto x^{(j+1)},
   \]
   \[
   \mathcal{L}_j : G_j \rightarrow G_j, \quad \mathcal{L}_j(x^{(j)}) \mapsto \mathcal{L}_{x^{(j)}},
   \]
   \[
   f_j : G_{j+1} \rightarrow G_j, \quad f_j(x^{(j+1)}) \mapsto \mathcal{L}_{x^{(j)}},
   \]
   \[
   \varphi_j : G_j \rightarrow G_{j+1}, \quad \varphi_j(x^{(j)}) \mapsto \mathcal{L}_{x^{(j+1)}}.
   \]

2. For \(0 \leq j \leq m - 1\) let \(N_j\), (respectively, \(K_j, H_j\)) be the pull-back in \(G\) of the kernel \(\ker \mu_j\), (respectively, the pull-back of \(\ker \mathcal{L}_j, \ker \varphi_j\)). Then there are inclusions
   \[
   N_0 \subset N_1 \subset N_2 \subset \cdots \subset N_j \subset N_{j+1} \subset \cdots
   \]
   \[
   K_0 \subset K_1 \subset K_2 \subset \cdots \subset K_j \subset K_{j+1} \subset \cdots
   \]
   \[
   H_0 \subset H_1 \subset \cdots \subset H_{j-1} \subset H_j \subset \cdots
   \]
   and
   \[
   \ker \mu_j \simeq N_j/N_{j-1}, \quad \ker \mathcal{L}_j \simeq K_j/K_{j+1}
   \]
   \[
   \ker f_j \simeq K_j/K_{j+1}, \quad \ker \varphi_j \simeq K_{j+1}/K_j.
   \]

3. **Remark 7.18.** Note that \(mpl(X, r) = m\) if and only if \(H_{m-1} = G\).

**Remark 7.19.** By assumption \((X, r)\) is a square-free solution, thus \(lri\) holds and the graph \(\Gamma(X, r)\) is well defined. \(\mu\) induces a homomorphism of graphs
   \[
   \mu : \Gamma(X, r_X) \rightarrow \Gamma([X], r_{[X]})
   \]
   The graph \(\Gamma([X], r_{[X]})\) is a homomorphic image of \(\Gamma(X, r_X)\), though not in general a retraction.

Recall that each solvable group \(G\) has a canonical solvable series, namely the derived series
   \[
   G \supset G' \supset G'' \supset \cdots \supset G^{(s)} = 1,
   \]
where the derived subgroups \(G^{(k)}\) are defined recursively. \(G'\) is the commutator of \(G\) (it is generated by the commutators \([x, y] = xyx^{-1}y^{-1}, x, y \in G\) and for all \(k \geq 1\), \(G^{(k+1)} = (G^{(k)})'\). Clearly, each \(G^k\) is a normal subgroup of \(G\). The length \(s\) of the derived series is called the solvable length of \(G\), it is the minimal length of solvable series for \(G\). We shall denote the solvable length of \(G\) by \(sl(G)\). The following fact is well known, and can be extracted, with a slight modification of the proof, from [MI, Proposition 6.6].
**Fact 7.20.** Let $N$ be a normal subgroup of $G$, and let $G/N$ be solvable of solvable lengths $m$ and $s$, respectively. Then the solvable length $\text{sl}(G)$ satisfies $\max(m, s) \leq \text{sl}(G) \leq m + s$.

**Remark 7.21.** It is known that the YB group $G(X, r)$ of every finite nondegenerate symmetric set is solvable, see [ESS], Theorem 2.15. In [GI], Theorem 7.10. is given a different proof for the case of finite square-free solutions.

We shall prove that for each multipermutation (square-free) solution $(X, r)$ of arbitrary cardinality the groups $G(X, r)$ and $G(X, r)$ are solvable, and the solvable length of $G(X, r)$ is at most $\text{mpl}(X, r)$.

**Proposition 7.22.** Let $(X, r)$ be a square-free solution of arbitrary cardinality, $G = G(X, r), G = G(X, r)$. Suppose $([X], r_{[X]}) = \text{Ret}(X, r), G_1 = G([X], r_{[X]})$. Then the following are equivalent

1. $G$ is solvable.
2. $G$ is solvable.
3. $G_1$ is solvable.

In this case the following inequalities hold:

\[
\text{sl}(G) \leq \text{sl}(G_1) \leq \text{sl}(G) \leq \text{sl}(G) + 1.
\]

In particular, if some of the retracts $\text{Ret}^i(X, r) (i \geq 0)$ is a finite set, then $G(X, r)$ is solvable.

**Proof.** We know that there is a short exact sequence

\[
1 \rightarrow K_0 \rightarrow G \xrightarrow{\mathcal{L}} G \rightarrow 1,
\]

where the kernel $K_0 = \ker \mathcal{L}$ is an abelian normal subgroup of $G$, see Lemma 7.13. Fact 7.20 implies then that

\[
\text{sl}(G) \leq \text{sl}(G) + 1.
\]

By Lemma 7.15 there is a short exact sequence

\[
1 \rightarrow N_0 \rightarrow G \xrightarrow{\mu_0} G_1 \rightarrow 1,
\]

where the kernel $N_0$ of $\mu_0$ is an abelian normal subgroup of $G$, so

\[
\text{sl}(G_1) \leq \text{sl}(G).
\]

By Lemma 7.15 $N_0 \subset K_0$ and there is a short exact sequence

\[
1 \rightarrow K_0/N_0 \rightarrow G_1 \rightarrow G \rightarrow 1,
\]

thus

\[
\text{sl}(G) \leq \text{sl}(G_1).
\]

We have verified the inequalities (7.20). Clearly this implies the equivalence of (1), (2), (3).

Assume now that for some $i$ the retract $\text{Ret}^i(X, r)$ is of finite order. Then by Remark 7.21 $G_i = G(\text{Ret}^i(X, r))$ is solvable, and therefore $G_{i-1} = G(\text{Ret}^{i-1}(X, r))$ is solvable. By decreasing induction on $i$ we deduce that $G_0 = G(X, r)$ is solvable.

$\square$
Theorem 7.23. Let \((X, r)\) be a square-free solution of arbitrary cardinality, \(G = G(X, r), \mathcal{G} = \mathcal{G}(X, r)\). Suppose \((X, r)\) is a multipermutation solution with \(\text{mpl}(X, r) = m\). Then \(G\) and \(\mathcal{G}\) are solvable with
\[
\text{sl}(G) \leq m \quad \text{and} \quad \text{sl}(\mathcal{G}) \leq m - 1.
\]
Furthermore,
\[
\text{mpl}(X, r) = 2 \implies \text{sl}(G) = 2 \quad \text{and} \quad \text{sl}(\mathcal{G}) = 1.
\]

Proof. We shall use induction on \(m\) to show that \(\text{sl}(G) \leq m\). Note that the retraction \(([X], [r])\) is a multipermutation square-free solution of level \(\text{mpl}([X], [r]) = m - 1\). Base for the induction, \(m = 1\). Then \((X, r)\) is the trivial solution, \(G = e\), and by Lemma 5.2 \(G\) is abelian, so \(\text{sl}G = 1 = \text{mpl}(X, r)\).

Suppose the statement is true for \(m \leq m_0\). Let \(\text{mpl}(X, r) = m_0 + 1\).

Using analogous argument one shows that \(\text{sl}\mathcal{G} \leq m - 1\), this time we use the short exact sequence
\[
1 \longrightarrow K_1/K_0 \longrightarrow G \longrightarrow G_1 \longrightarrow 1,
\]
where the kernel \(K_1/K_0\) is an abelian normal subgroup of \(G\), see Lemma 7.15 again.

Assume now that \(\text{mpl}(X, r) = 2\). This implies that \(G(X, r)\) is abelian, or equivalently \(\text{sl}G(X, r) = 1\), see Theorem 5.4. We have already shown that \(\text{sl}G(X, r) \leq \text{mpl}(X, r)(= 2)\). An assumption that there is a strict inequality \(\text{sl}G(X, r) < 2\) would imply \(G(X, r)\) is abelian, and therefore by Lemma 5.2 \(\text{mpl}(X, r) = 1\), a contradiction.

The theorem has been proved. \(\square\)

In the case when \((X, r)\) is of finite order we show that the solvable lengths of \(G\) and \(\mathcal{G}\) differ exactly with 1, see Theorem 7.25.

We need a preliminary lemma.

Lemma 7.24. Let \(A\) be a non-zero free abelian group of finite rank, and let \(H\) be a non-trivial finite group acting faithfully on \(A\). Let
\[
[H, A] = \{^h a - a : a \in A, h \in H\}.
\]
Then \([H, A]\) is non-zero, and \(H\) acts faithfully on \([H, A]\).

Proof. We begin by observing that \(H\) does indeed act on \([H, A]\). If \(k \in H\), then
\[
k(^h a - a) = ^{kk^{-1}} b - b, \quad \text{where} \quad b = ^k a \in A,
\]
so \(^k(^h a - a) \in [H, A]\).

Let \(\hat{A} = A \otimes \mathbb{Q}\). Then \(\hat{A}\) is a vector space over \(\mathbb{Q}\), with dimension equal to the rank of \(A\), and \(H\) acts faithfully on \(\hat{A}\). It suffices to prove the lemma with \(\hat{A}\) in place of \(A\), since elements of \([H, A]\) are multiples of elements of \([H, \hat{A}]\). The advantage
is that Maschke’s Theorem holds: if \( B \) is an \( H \)-submodule of \( \hat{A} \), then there is a complement \( C \), a \( H \)-submodule such that \( \hat{A} = B \oplus C \) (in other words, \( \hat{A}/B \cong C \)).

Now \([H, \hat{A}]\) is the smallest \( H \)-submodule \( B \) of \( \hat{A} \) such that \( H \) acts trivially on \( A/B \). So the complement guaranteed by Maschke’s Theorem is \( C_H(\hat{A}) = \{ a \in \hat{A} : a^h = a \}\). Since \( H \neq \{1\} \) and the action is faithful, \( C_H(\hat{A}) \neq \hat{A} \), so \([H, \hat{A}] \neq \{0\}\).

Finally, suppose that \( h \in H \) acts trivially on \([H, \hat{A}]\). Since also \( h \) acts trivially on \( C_H(\hat{A}) \) by definition, it acts trivially on the whole of \( \hat{A} \); since we assume that \( H \) acts faithfully on \( A \), we deduce that \( h = 1 \). The lemma is proved. □

**Theorem 7.25.** Let \((X, r)\) be a square-free solution of finite order. Then
\[
\text{sl}(G) = \text{sl}(\hat{G}) + 1.
\]

**Proof.** We know that there is a natural number \( p \) such that the subgroup of \( G \) generated by the \( p \)th powers of the generators is a free abelian group \( A \). Clearly \( A \) is isomorphic to the integral permutation module for \( G \) (in its action on \( X \)), so the action is faithful. (This uses the equation \( ba^p b^{-1} = (b a)^p \), which follows from \( ba^p = (b a)p b \), see Facts 7.11 1. i.)

Let \( A^{(n)} \) be defined inductively by \( A^{(0)} = A \) and
\[
A^{(n+1)} = [G^{(n)}, A^{(n)}]
\]
for \( n \geq 0 \). By lemma 7.24 and induction, if \( G^{(n)} \neq \{1\} \), then \( A^{(n+1)} \neq \{0\} \) and \( G^{(n)} \) acts faithfully on \( A^{(n+1)} \). So, if \( l = \text{sl} G \), then \( A^{(l)} \neq \{0\} \). But \( A^{(l)} \leq G^{(l)} \) (the \( l \)th derived group of \( G \)); so \( \text{sl} G > l \). By our previous observation, we know that \( \text{sl} G \leq l + 1 \); so in fact \( \text{sl} G = l + 1 \) holds, and the theorem is proved. □

We know that \( \text{mpl}(X, r) = 2 \) implies \( \text{sl}(G(X, r)) = 2 \). Example 5.16 shows that a gap between \( \text{mpl}(X, r) \) and \( \text{sl}(G(X, r)) \) can occur even for \( \text{mpl}(X, r) = 3 \).

**Question 7.26.** Suppose \((X, r)\) is a multipermutation square-free solution of finite order \(|X| > 1 \). The following questions are closely related.

1. Suppose \( \text{mpl}(X, r) = m \). Can we express a lower bound for \( \text{sl}(G(X, r)) \) in terms of \( m \)?
2. When there is an equality?
\[
\text{sl}(G(X, r)) = \text{mpl}(X, r) \quad \text{or equivalently} \quad \text{sl}(G(X, r)) = \text{mpl}(X, r) - 1 ?.
\]

In Section 8 we construct an infinite sequence of explicitly defined solutions \((X_m, r_m)\), \( m = 0, 1, 2, \ldots \), such that \( \text{mpl}(X_m) = m \), and \( m = \text{sl}(G(X, r_m)) = \text{sl}(G(X, r_m)) + 1 \), see Definition 9.9 and Theorem 9.11

**8. Wreath products of solutions**

In this section we define the notion of wreath product of solutions, by analogy with the wreath product of permutation groups.

The following result is true for arbitrary braided sets (without any further restrictions like being symmetric, finite, or square-free).
Theorem 8.1. \cite{GIM08} Let $(X, r_X)$, $(Y, r_Y)$ be disjoint solutions of the YBE, with YB-groups $G_X = G(X, r_X)$, and $G_Y = G(Y, r_Y)$. Let $(Z, r)$ be a regular YB-extension of $(X, r_X)$, $(Y, r_Y)$, with a YB-group $G_Z = G(Z, r)$. Then

- $G_X, G_Y$ is a matched pair of groups with actions induced from the braided group $(G_Z, r)$.
- $G_Z$ is isomorphic to the double crossed product $G_X \rtimes G_Y$.

Fact 8.2. \cite{ESS} Let $(X, r_X)$ and $(Y, r_Y)$ be symmetric sets. If $Z \in \text{Ext}^+(X, Y)$, then $G_Z \simeq G_Y \rtimes G_X$, where the semidirect product is formed using the action of $G_Y$ on $X$ via $\alpha \mapsto L_\alpha$.

$\text{Ext}^+(X, Y)$ is the set of all symmetric sets $(Z, r)$ which are solutions of $(X, r_1), (Y, r_2)$ with $r(x, \alpha) = (\alpha, x^\alpha)$.

Lemma 8.3. Let $(X, r)$ be square-free multipermutation solution, let $\tau \in \text{Aut}(X, r)$ be an automorphism of $(X, r)$. Let $(Y, r_0)$ be the trivial solution on the one element set $Y = \{\alpha\}$, where $\alpha$ is not in $X$. Let $(Z, r_Z) = X \bowtie Y$ be the strong twisted union defined via $L_\alpha = \tau$, $L_x\alpha = \text{id}_Y$, for all $x \in X$. (i.e. $\alpha x = \tau(x)x^\alpha = \alpha$, for all $x \in X$.) Then

1. $(Z, r_Z)$ is a square-free solution, so $Z = X \bowtie Y$.
2. $G(Z, r_Z)$ is the semidirect product $G(Z, r_Z) \simeq G(X, r) \rtimes C_\infty$.
3. Furthermore, if $(X, r)$ is a multipermutation solution of finite multipermutation level, and $\tau$ does not belong to $G(X, r)$ then

$$\text{mpl}(Z, r_Z) = \text{mpl}(X, r) + 1.$$ 

The following proposition can be deduced from \cite{GIM08}.

Proposition 8.4. Let $(X, r_x), (Y, r_y)$ be disjoint nondegenerate symmetric sets (not necessarily finite or square-free). Assume there is an injective map $Y \to \text{Sym}(X), \alpha \mapsto \sigma_\alpha \in \text{Sym}(X)$. Let $(Z, r)$ be the extension of $(X, r_X)$, and $(Y, r_Y)$ where $Z = X \bigcup Y$, $r$ extends $r_X$ and $r_Y$, and

$$r(\alpha, x) = (\sigma_\alpha(x), \alpha) \quad r(x, \alpha) = (\alpha, \sigma_\alpha^{-1}(x)) \quad \forall x \in X, \alpha \in Y.$$ 

Then $(Z, r)$ is a symmetric set if and only if the assignment $\alpha \mapsto \sigma_\alpha$ extends to a homomorphism $L_\alpha : G(Y, r_Y) \to \text{Aut}(X, r_X)$.

Lemma 8.5. Suppose $(X, r_x), (Y, r_y)$ are disjoint symmetric sets (most general setting). Let $\sigma \in \text{Sym} X, \rho \in \text{Sym} Y$. Let $(Z, r)$, be an extension of $(X, r_X), (Y, r_Y)$, such that

$$r(x, y) = (\rho(y), \sigma^{-1}(x)) \quad r(y, x) = (\sigma(x), \rho^{-1}(y))$$ 

Then $(Z, r)$ is involutive, nondegenerate quadratic set. Furthermore, $(Z, r)$ is a solution if and only if $\sigma \in \text{Aut} X$ and $\rho \in \text{Aut} Y$.

We now define the wreath product of solutions:

Definition 8.6. Suppose $(X_0, r_0)$, and $(Y, r_y)$ are disjoint square-free solutions. Let $\{(X_\alpha, r_\alpha) \mid \alpha \in Y\}$ be the set of $|Y|$ disjoint solutions $(X_\alpha, r_\alpha)$ indexed by $Y$, where each $(X_\alpha, r_\alpha)$ is an isomorphic copy of $(X_0, r_0)$ defined on a set $X_\alpha = \{t_{\alpha, x} \mid x \in X_0\}$, and $t_{\alpha, x}$ denotes the copy of $x$ in $X_\alpha$ ($t_{\alpha, x} \neq t_{\alpha, z}$ if and only if $x \neq z$),
and the map $r_\alpha$ and the associated actions are translations of $r_0$ and its associated actions to $X_\alpha$. Thus

$$r_\alpha(t_{\alpha,x}, t_{\alpha,z}) = (t_{(\alpha,x), t_{(\alpha,z)}}) \quad \forall x, z \in X.$$ 

Let $(X, r_X) = \bigotimes_{\alpha \in Y} X_\alpha$, be the trivial extension of all $X_\alpha$, for $\alpha \in Y$; that is, for all $\alpha, \beta \in Y$, we have

$$r|_{X_\alpha \times X_\alpha} = r_\alpha$$

$$r(x, z) = (z, x), \quad \forall x \in X_\alpha, z \in X_\beta, \text{ with } \alpha \neq \beta$$

Define the map $Y \to \text{Sym} X, \quad \beta \mapsto \sigma_\beta$, where the permutations $\sigma_\beta \in \text{Sym} X$ are defined as follows

$$(8.1) \quad \sigma_\beta : X_\alpha \to X_\beta, \quad \alpha \in Y$$

$$(8.2) \quad \sigma_\beta(t_{\alpha,x}) = t_{(\beta, \alpha), x}$$

Let $(Z, r)$ be the extension of $(X, r_X)$ and $(Y, r_Y)$ defined as follows

$$Z = X \cup Y,$$

$$r(\beta, (t_{\alpha,x})) = (\sigma_\beta(t_{\alpha,x}), \beta),$$

$$r((t_{\alpha,x}), \beta) = (\beta, (\sigma_\beta)^{-1}(t_{\alpha,x})), \forall \alpha, \beta \in Y, \quad t_{\alpha,x} \in X_\alpha.$$ 

We call $(Z, r)$ a wreath product of $(X_0, r_{X_0})$ and $(Y, r_Y)$, and denote it by $(Z, r) = (X_0, r_{X_0}) \text{ wr } (Y, r_Y)$.

**Theorem 8.7.**

1. The wreath product $(Z, r) = (X_0, r_{X_0}) \text{ wr } (Y, r_Y)$ is a square-free solution.

2. $G(Z, r) = G(X_0, r_{X_0}) \text{ wr } G(Y, r_Y)$.

3. Suppose $(X_0, r_{X_0})$ and $(Y, r_Y) = \text{ are multipermutation solutions of finite multipermutation level. Then}$

$$\text{mpl}(Z, r) = \text{mpl}(X_0, r_{X_0}) + \text{mpl}(Y, r_Y) - 1.$$ 

**Proof.** Note that $(Z, r)$ satisfies the hypothesis of Proposition 8.4 hence $(Z, r)$ is a solution if and only if the map $Y \to \text{Sym} X, \quad \beta \mapsto \sigma_\beta$, extends to a homomorphism $L_Y : G(Y, r_Y) \to \text{Aut}(X, r_X)$.

We show first that

$$(8.2) \quad \sigma_\beta \in \text{Aut}(X, r_X) \quad \forall \beta \in Y.$$ 

By Lemma 3.3 this is equivalent to

$$(8.3) \quad \sigma_\beta \circ L_{t_{\alpha,x}} = (L_{\sigma_\beta(t_{\alpha,x})}) \circ \sigma_\beta \quad \forall \alpha.$$ 

Note that by the definition of $r$ the associated left action on $Z$ satisfies:

$$(8.4) \quad L_{t_{\alpha,x}}(t_{\gamma,y}) = \begin{cases} t_{(\alpha, \gamma,y)} & \text{if } \gamma = \alpha \\ t_{\gamma,y} & \text{else} \end{cases}$$
Let \( t_{\gamma,y} \in X \). We apply both sides of (8.3) on \( t_{\gamma,y} \), and compute using (8.4) and (8.1).

**Case 1.** \( \gamma = \alpha \).

\[
\sigma_\beta \circ L_{t_{\alpha,x}}(t_{\alpha,y}) = \sigma_\beta(t_{\alpha,y}) \\
= t_{\alpha,y} \\
L_{\sigma_\beta}(t_{\alpha,x}) \circ \sigma_\beta(t_{\alpha,y}) = L_{t_{\alpha,y}}(t_{\alpha,y}) \\
= t_{\alpha,y},
\]

as desired.

**Case 2.** \( \gamma \neq \alpha \). In this case, due to the nondegeneracy of \( r_Y \) one has

\[
\beta \neq \gamma, \quad \forall \beta \in Y.
\]

Now our computation (applying (8.4), (8.1), and (8.6)) give:

\[
\sigma_\beta \circ L_{t_{\gamma,y}} = \sigma_\beta(t_{\gamma,y}) \\
= t_{\gamma,y} \\
L_{\sigma_\beta}(t_{\gamma,x}) \circ \sigma_\beta(t_{\gamma,y}) = L_{t_{\gamma,y}}(t_{\gamma,y}) \\
= t_{\gamma,y},
\]

We have verified (8.3), therefore (8.2) holds.

Next we show that the map \( L_0 : Y \rightarrow \text{Aut}(X, r) \quad \beta \mapsto \sigma_\beta \) extends to a homomorphism

\[
G(Y, r_Y) \rightarrow \text{Aut}(X, r).
\]

\( Y \) generates \( G(Y, r_Y) \), so it will be enough to show that \( L_0 \) respects the relations of \( G(Y, r_Y) \), or equivalently

\[
\sigma_\beta \circ \sigma_\gamma = \sigma_{\beta \gamma} \circ \sigma_\gamma, \quad \forall \beta, \gamma \in Y.
\]

Recall that since \( (Y, r_Y) \) is a solution, condition 11 holds, that is

\[
\beta(\gamma \alpha) = \beta(\gamma) \alpha \quad \forall \alpha, \beta, \gamma \in Y.
\]

Let \( t \in X \). Then \( t = t_{\alpha,x} \) for some \( \alpha \in Y, x \in X_0 \). We apply both sides of (8.8) on \( t \) and obtain:

\[
\sigma_\beta \circ \sigma_\gamma(t_{\alpha,x}) = \sigma_\beta(t_{\gamma \alpha,x}) \\
= t_{(\beta \gamma)(\alpha),x} \\
\sigma_\beta \circ \sigma_\gamma(t_{\alpha,x}) = t_{(\beta \gamma)(\alpha),x},
\]

which verifies (8.7). We have shown that the sufficient conditions for \((Z, r)\) being a solution are satisfied which proves part (1) of the theorem.

(2) The wreath product of a group \( G \) by a permutation group \( H \) (acting on the set \( Y \)) is defined to be the semidirect product of \( N \) by \( H \), where \( N \) (the base group) is the direct product of \(|Y|\) copies of \( G \), and \( H \) acts on \( N \) by permuting the direct factors in the same way as it permutes the indexing elements of \( Y \). If \( G \) is itself a permutation group on a set \( X \), then \( G \wr H \) is in a natural way a permutation group on \( X \times Y \) (regarded as the disjoint union of \(|Y|\) copies of \( X \)).

It is clear from our construction that \( G(Z, r) \) is isomorphic as abstract group to \( G(X_0, r_{X_0}) \wr G(Y, r_Y) \), and in fact acts on the union of copies of \( X_0 \) by the natural permutation action of the wreath product; it acts on \( Y \) according to the action of \( G(Y, r_Y) \) (that is, the base group is in the kernel of this action).
(3) Suppose now that \((X_0, r_0)\), and \((Y, r_Y)\) are multipermutation solutions with \(\text{mpl}(X_0, r_0) = m\) and \(\text{mpl}(Y, r_Y) = n\). Every retract \(\text{Ret}^k(X, r)\) is a trivial extension of the retracts \(\text{Ret}^k(X_\alpha, r_\alpha), \alpha \in Y\). It follows straightforwardly that \(\text{mpl}(X, r_X) = \text{mpl}(X_0, r_0) = m\).

We study the retracts \(\text{Ret}^N(Z, r), N = 1, 2, \cdots\). Note first that by the definition of the map \(r\) on \(Z \times Z\) it follows that
\[
\mathcal{L}_{x|X} = \mathcal{L}_{y|X} \iff \mathcal{L}_{x|Z} = \mathcal{L}_{y|Z} \quad x, y \in X
\]
(8.10)
\[
\mathcal{L}_{\alpha|Y} = \mathcal{L}_{\beta|Y} \iff \mathcal{L}_{\alpha|Z} = \mathcal{L}_{\beta|Z} \quad \alpha, \beta \in Y.
\]
Hence the retract \(\text{Ret}(Z, r)\) can be viewed as the extension of \(\text{Ret}(X, r_X)\) and \(\text{Ret}(Y, r_Y)\), induced from the original actions of \(X, Y\) onto \(X, Y\), but reduced after collapsing all elements \(x \in X\), with \(\mathcal{L}_x = \text{id}_X\) and \(\alpha \in Y\), with \(\mathcal{L}_\alpha = \text{id}_Y\) into a single point, say \([z_0]\). This does not have effect on \(\text{Ret}(X, r_X)\), \(\text{Ret}(Y, r_Y)\), but now these solutions might intersect in a single joint point (with trivial action). For each \(k, 1 \leq k \leq m, \alpha \in Y\), as usual, (see Notation 7.1) \(\alpha^{(k)}\) denotes the equivalence class of \(\alpha\) in \(\text{Ret}^k(Z, r)\). Denote \(Y_k = \{\alpha^{(k)} \mid \alpha \in Y\}\). Note that \((Y_k, r_k)\) (with \(r_k\) induced from \(r_Y\)) is isomorphic to \(\text{Ret}(Y, r_Y) = ([Y], r_{[Y]})\) for all \(k, 1 \leq k \leq m\). Note also that each retract \(\text{Ret}^k(Z, r)\), with \(1 \leq k \leq m - 1\) is an union of \(\text{Ret}^k(X, r_X)\) and \((Y_k, r_{Y_k})\) with possibly one joint point \(z_0^{(k)}\), the class of all \(\xi \in Z\) which act trivially on \(\text{Ret}^{k-1}(Z, r)\).

Indeed every such a retraction has the effect of obtaining the retraction \(\text{Ret}^k(X_\alpha, r_\alpha)\), where \(\alpha \in Y\), but these are disjoint sets and therefore for each pair \(\alpha, \beta \in Y\) the inequality of equivalence classes in \(Y, [\alpha] \neq [\beta]\) implies \((\mathcal{L}_{[\alpha]})_{|[X]} \neq (\mathcal{L}_{[\beta]})_{|[X]}\), and therefore \((\mathcal{L}_{[\alpha]})_{|[Z]} \neq (\mathcal{L}_{[\beta]})_{|[Z]}\).

This way exactly on the \(m\)-th retraction \(\text{Ret}^m(Z, r)\) all the elements of \(X\) collapse in a single element \(z_0^{(m)}\) (which could be also the unique joint element with \(Y_m\)). From now on the \(m + j\)-th retraction \(\text{Ret}^{m+j}(Z, r)\) has effect of \(j + 1\)-th retraction on \((Y, r_Y)\) in the usual way, since as we know \((Y_m, r_m)\) is isomorphic to \(\text{Ret}(Y, r_Y)\), all element of \(Y_m\) have trivial action on \(z_0^{(m)}\). It follows that we need exactly \(n - 2\) more retractions to obtain that \(\text{Ret}^{m+n-2}(Z, r)\) is a trivial solution of order \(\geq 2\), hence \(\text{Ret}^{m+n-1}(Z, r) = \{z_0^{(m+n-1)}\}\), is a one element solution. This verifies (3).

**Open questions 8.8.** Suppose the square-free solution \((Z, r) = X \triangleright Y\) is a strong twisted union of \((X, r_X)\) and \((Y, r_Y)\). Denote \(G_Z = G(Z, r), G_X = G(X, r_X), G_Y = G(Y, r_Y)\).

1) How are the groups \(G_Z, G_X, G_Y\) related?

Proposition 4.6. [GIM08], shows that for arbitrary braided set \((Z, r)\) which is an extension of two disjoint sets \((X, r_X)\) and \((Y, r_Y)\), \(G_X, G_Y\) is a matched pair of groups and \(G_Z\) is isomorphic to the double crossed product \(G_X \bowtie G_Y\).

Note that in the case when \(Z\) is a strong twisted union of \(X, Y\), the group \(G_X\) acts on \(G_Y\) via automorphisms and \(G_Y\) acts on \(G_X\) via automorphisms, so we expect the structure of \(G_Z \cong G_X \bowtie G_Y\) to be more special.

2) How are the groups \(G(Z, r), G(X, r_X), G(Y, r_Y)\) related?
3) Can we determine a upper bound for the solvable length of \( G(Z, r) \) in terms of the solvable lengths of \( G_X, G_Y \). Analogous question for the solvable lengths of \( G(Z, r), G_X, G_Y \)

Moreover, suppose both \((X, r_X)\) and \((Y, r_Y)\), are multipermutation solutions.

4) Is it true that \((Z, r)\) is always a multipermutation solution?

5) How is mpl(Z) related to mpl(X), mpl(Y)?

Clearly \( \max(\text{mpl}(X), \text{mpl}(Y)) \leq \text{mpl}(Z) \). Can we express an upper bound for \( \text{mpl}(Z) \) in terms of \( \text{mpl}(X), \text{mpl}(Y) \)?

6) Suppose \((Z, r)\) is a finite multipermutation square-free solution, with \( \text{mpl}(Z) = m \).

Is it true that \((Z, r)\) can be always presented as a strong twisted union of \( r \)-invariant subsets of multipermutation level \(< m \)?

**Lemma 8.9.** There exist two disjoint square-free symmetric sets \((X, r_X), (Y, r_Y)\), with \( \text{mpl}(X) = \text{mpl}(Y) = 2 \), and a strong twisted union \((Z, r) = X \uplus Y\), which is a multipermutation solution with \( \text{mpl}(Z, r) = \text{mpl}(X) + \text{mpl}(Y) = 4 \). Moreover, \( \text{sl}(G(Z, r)) = 4 = \text{sl}(G_X) + \text{sl}(G_Y) \), and \( \text{sl}(G(Z, r)) = 3 \).

**Proof.** We shall define \((X, r_X), (Y, r_Y)\) and \((Z, r)\) explicitly, via the left actions. Let \((X, r_X), (Y, r_Y)\) be the solutions defined as

\[
X = \bigcup_{1 \leq i \leq 4} X^i \quad \text{and} \quad Y = \left\{ a_1, a_2, a_3, a_4, a'_1, a'_2, a'_3, a'_4, b, c \right\},
\]

\[
\mathcal{L}_{x^i} = (x^i_1 x^i_2 x^i_3 x^i_4), \quad \mathcal{L}_{a_i} = (a_i a'_i), \quad \mathcal{L}_{a_i} = \text{id}_Y, \quad 1 \leq i \leq 4,
\]

\[
\mathcal{L}_b = (a_1 a_2)(a_3 a_4)(a'_1 a'_2)(a'_3 a'_4), \quad \mathcal{L}_c = (a_1 a_3)(a_2 a_4)(a'_1 a'_3)(a'_2 a'_4).
\]

Note that \( X \) is a trivial extension of the four isomorphic solutions \((X^i, r_i), 1 \leq i \leq 4\), with \( r_i \) defined, as usual, via the left action. It is easy to see that \( \text{mpl}(X) = 2 \)

Clearly, one has \( \text{mpl}(X) = 2 \). Direct computation shows that \( \text{mpl}(Y) = 2 \). We set \( Z = X \uplus Y \). In (8.11) we define a left action \( Z \times Z \rightarrow Z \) extending the given actions \( X \times X \rightarrow X \) and \( Y \times Y \rightarrow Y \). (All permutations bellow are considered as elements of Sym Z).

\[
(8.11)
\]

\[
\mathcal{L}_{x^i} = \mathcal{L}_{x^i} = (x^i_1 x^i_2 x^i_3 x^i_4)(a_i a'_i), \quad \mathcal{L}_{x^i} = \mathcal{L}_{x^i} = (x^i_1 x^i_2 x^i_3 x^i_4),
\]

\[
\mathcal{L}_{a_i} = \mathcal{L}_{a_i} = (a_i a'_i)(a'_i a'_i) \prod_{1 \leq i \leq 4} [(x^4_1 x^4_2)(x^4_3 x^4_4)],
\]

\[
\mathcal{L}_b = (a_1 a_2)(a_3 a_4)(a'_1 a'_2)(a'_3 a'_4) \prod_{1 \leq i \leq 4} [(x^4_1 x^4_2)(x^4_3 x^4_4)].
\]

Consider the map \( r : Z \times Z \rightarrow Z \times Z \), defined as \( r(z, t) = (\mathcal{L}_z(t), \mathcal{L}_z^{-1}(z)) \) where \( t, z \in Z, \mathcal{L}_z, \mathcal{L}_z \) as in (8.11). One verifies straightforwardly condition (11), hence \((Z, r)\) is a symmetric set. Furthermore, condition \( \text{st} \) holds, so \((Z, r) = (X, r_X) \uplus (Y, r_Y)\).

Direct computation shows that \( \text{mpl}(Z) = 4 \). Moreover, a short calculation with GAP shows that the group \( G(Z, r) \) has order 214 and solvable length 3.

(This can easily be seen directly, since it is the wreath product of the dihedral group of order 8 with the Klein group of order 4.) By Theorem 7.24, \( \text{sl} G = 4 \).
9. Infinite solutions

We consider solutions \((X, r)\) with \(X\) infinite but having finite multipermutation level.

**Question 9.1.** In the hypothesis and notation of Lemma 8.3, when can we express \(G(Z, r)\) as a wreath product \(G(X, r) \wr \langle \tau \rangle\) (where \(\tau'\) is some appropriate permutation deduced from \(\tau\))? (This is not true in general.)

Note that, in general, \(\text{Aut}(X, r) \subseteq G(X, r)\), is possible (so the existence of a \(\tau \notin G(X, r)\) is not automatic), as shown the following example.

**Example 9.2.** Let \((X, r)\) be the three element nontrivial solution, with \(X = \{x_1, x_2, x_3\}\) and \(r\) defined via the left actions \(L_{x_3} = (x_1x_2), L_{x_1} = L_{x_2} = \text{id}_X\). Then \(\text{mpl}(X, r) = 2\) and \(\text{Aut}(X, r) = G(X, r) = \{\text{id}_X, (x_1x_2)\}\).

**Example 9.3.** We shall construct an infinite sequence of explicitly defined square-free symmetric sets

\[
(X_0, r_0), (X_1, r_1), \ldots, (X_m, r_m), \ldots,
\]

such that

(i) for each \(m, m = 0, 1, 2, \ldots\), \(X_m \subset X_{m+1}\) is an \(r_{m+1}\)-invariant subset of \(X_{m+1}\).

Furthermore,

(ii) \((X_m, r_m)\) is a finite multipermutation solution of order \(|X| = 2^{m+1} - 1\), and

(iii) \(\text{mpl}(X_m, r_m) = m\), the solvable length of each \(G(X_m, r_m)\) is exactly \(m\).

(iv) \((X_{m+1}, r_{m+1})\) is a strong twisted union of a solution \((Y_m, r_m)\), with \(\text{mpl}(Y_m) = m, |Y_m| = 2(2^m - 1)\), and a trivial one-element solution.

As a starting point we chose an infinite countable set \(X = \{x_n \mid 1 \leq n\}\). We define the solutions \((X_m, r_m), m = 0, 1, 2, \ldots\) recursively.

- \((X_0, r_0)\) is the one element trivial solution with \(X_0 = \{x_1\}\).
- \((X_1, r_1)\) is the trivial solution on the set \(X_1 = \{x_1, x_2\}\).
- We set \((X_2, r_2) = X_1 \ast \{x_3\}\), where \(L_{x_3} = (x_1x_2)\). Clearly, \(\text{mpl}(X_2) = 2\).
- Construction of \((X_3, r_3)\). Let \((X_2', r_2')\) be an isomorphic copy of \((X_2, r_2)\), where \(X_2' = \{x_4, x_5, x_6\}\), and the map \(\tau : (X_2, r_2) \rightarrow (X_2', r_2')\) with \(\tau(x_i) = x_{i+3}, 1 \leq i \leq 3\), is an isomorphism of solutions. Let \((Y_2, r_{Y_2}) = X_2 \ast_0 X_2'\) be the trivial extension. We set \((X_3, r_3) = Y_2 \ast \{x_7\}\), where the map \(r_3\) is defined via the left action \(L_{x_7} = (x_1x_4)(x_2x_5)(x_3x_6)\). One has \(L_{x_7} \in \text{Aut}(Y_2, r_{Y_2}) \setminus G(Y_2, r_{Y_2})\), so \(\text{mpl}(X_3) = 3\).

Assume we have constructed the sequence \((X_0, r_0), (X_1, r_1), \ldots, (X_m, r_m)\), satisfying conditions (i) and (ii). We shall construct effectively \((X_{m+1}, r_{m+1})\) so that (i), (ii), and (iii) are satisfied. For \(N = 2^m - 1 = |X_m|\), let \(X'_m = \{x_{N+1}, \ldots, x_{2N}\}\) and let \((X_m, r_{X_m})\) be the solution isomorphic to \((X_m, r_m)\) via the isomorphism \(\tau : X_m \rightarrow X'_m\) with \(\tau(x_i) = x_{i+N}, 1 \leq i \leq N\). We denote by \((Y_m, r_{Y_m})\) the trivial extension \(X_m \ast_0 X'_m\), and set \((X_{m+1}, r_{X_{m+1}}) = (Y_m, r_{Y_m}) \ast \{x_{2N+1}\}\), where \(r_{m+1}\) is defined via the action \(L_{x_{2N+1}} = (x_1x_{N+1})(x_2x_{N+2}) \cdots (x_Nx_{2N})\). One can show that \(\text{mpl}(X_{m+1}, r_{m+1}) = m + 1\), and \(G(X_{m+1}, r_{m+1}) = G(X_m, r_m) \wr C_2\).
The following question was posed by Paul Martin.

**Question 9.4.** For each positive integer \( m \) denote by \( n_m \), the minimal integer so that there exists a square-free multipermutation solution \((X_m, r_m)\) of order \(|X_m| = n_m\), and with \( \text{mpl}(X_m, r_m) = m \). How does \( n_m \) depend on \( m \)?

In the proof of Theorem 9.11 we construct an infinite sequence of recursively defined explicit solutions \((X_m, r_m)\), \( m = 0, 1, 2, \ldots \), s.t. \( \text{mpl}(X_m) = m \), and \( |X_m| = 2^{m-1} + 1 \). Therefore \( n_m \leq 2^{m-1} + 1 \). By definition, the unique symmetric set with multipermutation level 0 is the one element solution, so \( n_0 = 1 \). Direct computation show that for \( 1 \leq m \leq 3 \) the square-free solutions \((X_m, r_m)\) in the Construction 9.11 are of minimal possible order, so \( n_m = 2^{m-1} + 1, m = 1, 2, 3 \).

**Question 9.5.** Is it true that \( n_m = 2^{m-1} + 1 \), for all integers \( m \geq 1 \)?

Let \( X = \{ x_i \mid i = 1, 2, 3, \ldots \} \) be an infinite countable set.

**Definition 9.6.** Let \( \rho = (x_{i_1}, x_{i_2}, \ldots, x_{i_k}) \) and \( \sigma = (y_{j_1}, y_{j_2}, \ldots, y_{j_k}) \) be disjoint cycles of length \( k \) in \( \text{Sym}(X) \), and let \( N \) be a natural number. Define
\[
\rho[N] = (x_{i_1+N}, x_{i_2+N}, \ldots, x_{i_k+N})
\]
\[
\rho \circ \sigma = (x_{i_1}, y_{j_1}, x_{i_2}, y_{j_2}, \ldots, x_{i_k}, y_{j_k})
\]

So \( \rho[N] \) is a cycle of the same length \( k \) as \( \rho \) and is obtained by shifting the indices \( N \)-steps to the right. \( \rho \circ \sigma \) is a cycle of length \( 2k \) and
\[
(\rho \circ \sigma)^2 = \rho \circ \sigma.
\]

**Definition 9.7.** For a pair \((x_i, x_j) \in X \times X\) we define \((x_i, x_j)[N] = (x_{i+N}, x_{j+N})\). Let \((Y, r)\) be a symmetric set, where \( Y = \{ x_1, x_2, \ldots, x_k \} \subset X \). For each integer \( N, N > k \) we define the quadratic set \((Y, r)[N]) = (Y[N], r[N])\), where \( Y[N] = \{ x_{1+N}, x_{2+N}, \ldots, x_{k+N} \} \), and
\[
r[N](x_{i+N}, x_{j+N}) = (r(x_i, x_j))[N]
\]

**Remark 9.8.** Clearly the left action induced by \( r[N] \) satisfies
\[
(r[N])(y[N]) = (r \circ y)[N]
\]
So \( 1 \) is satisfied and therefore \((Y[N], r[N])\) is a square-free solution. There is an isomorphism of solutions
\[
\varphi_N : (Y, r) \longrightarrow (Y[N], r[N]) \quad x_i \mapsto x_{i+N}.
\]

Furthermore,
\[
\mathcal{G}(Y[N], r[N]) = \mathcal{G}(Y, r)[N] = \mathcal{G}(\mathcal{L}_x[N] \mid x \in Y).
\]

**Definition 9.9.** For \( m = 0, 1, 2, \ldots \) define a sequence of cycles \( \sigma_m \in \text{Sym} X \), each of length \( 2^m \) as follows
\[
\sigma_1 = (x_1, x_2)
\]
\[
\sigma_2 = \sigma_1 \circ (\sigma_1[2]) = (x_1, x_3, x_2, x_4)
\]
\[
\sigma_3 = \sigma_2 \circ (\sigma_2[4])
\]
\[
\sigma_{m+1} = \sigma_m \circ (\sigma_m[2^m])
\]
For \( m = 0, 1, 2, \ldots \) we define the solutions \((X_m, r_m)\) and \((Y_m, r_m)\) recursively, as follows.

\[
\begin{align*}
(X_0, r_0) &= \{x_1\} & \text{the trivial solution on one element set } X_0 = \{x_1\} \\
(X_1, r_1) &= \{x_1, x_2\} & \text{the trivial solution on the set } X_1 = \{x_1, x_2\} \\
(Y_1, r_1) &= (X_1, r_1) & (\text{for completeness only}) \\
(X_2, r_2) &= Y_1 \circlearrowleft \{x_3\}, \quad \text{where } \mathcal{L}_{x_3} = \sigma_1 \\
Y_2 &= Y_1 \circlearrowleft Y_1[2], \quad \text{where } \mathcal{L}_{x_3} = \mathcal{L}_{x_4} = \sigma_1, \quad \mathcal{L}_{x_3} = \mathcal{L}_{x_2} = \sigma_1[2] \\
(X_3, r_3) &= Y_2 \circlearrowleft \{x_5\}, \quad \text{where } \mathcal{L}_{x_5} = \sigma_2 = \sigma_1 \vee (\sigma_1[2]) \\
Y_3 &= Y_2 \circlearrowleft Y_2[2], \quad \text{where } \mathcal{L}_{y_2} = \sigma_2, \quad \forall y \in Y_2[2], \quad \mathcal{L}_{x_3|y_2} = \sigma_2[4], \quad \forall x \in Y_2 \\
(X_4, r_4) &= Y_3 \circlearrowleft \{x_9\}, \quad \text{where } \mathcal{L}_{y_3} = \sigma_3 = \sigma_2 \vee (\sigma_2[4]) \\
Y_4 &= Y_3 \circlearrowleft Y_3[8], \quad \text{where } \mathcal{L}_{y_3|y_3} = \sigma_3, \quad \forall y \in Y_3[8], \quad \mathcal{L}_{x_3|y_3} = \sigma_3[8], \quad \forall x \in Y_3 \\
\end{align*}
\]

\[
Y_m = Y_{m-1} \circlearrowleft Y_{m-1}[2^{m-1}], \text{ where } \mathcal{L}_y = \sigma_{m-1}, \quad \forall y \in Y_{m-1}[2^{m-1}],
\]

and

\[
\mathcal{L}_{x|m-1} = \sigma_{m-1}[2^{m-1}], \quad \forall x \in Y_{m-1}
\]

\[
X_{m+1} = Y_m \circlearrowleft \{\xi\}, \quad \text{where } \xi = x_{2^m+1}, \quad \mathcal{L}_\xi = \sigma_m = \sigma_{m-1} \vee (\sigma_{m-1}[2^{m-1}]).
\]

Clearly \( X = \bigcup_{0 \leq m} X_m \).

**Definition 9.10.** Define the map

\[
r_X : X \times X \to X \times X \quad r(x, y) = r_m(x, y) \text{ where } x, y \in X_m.
\]

**Theorem 9.11.** In assumption and notation as above Let

\[
(X_0, r_0), (X_1, r_1), \ldots, (X_m, r_m), \ldots
\]

be the infinite sequence of quadratic sets defined in Definition 9.9. Then the following conditions hold for each \( m = 0, 1, 2, \ldots \):

\[
\begin{align*}
(1) & \quad (X_m, r_m) \text{ is a square-free solution of order } |X_m| = 2^{m-1} + 1 \\
(2) & \quad (X_{m+1}, r_{m+1}) \text{ is an extension of } (X_m, r_m). \\
(3) & \quad \text{Ret}(X_{m+1}, r_{m+1}) \approx (X_m, r_m). \quad (\text{i.e. } (X_{m+1}, r_{m+1}) \text{ is a blow up of } (X_m, r_m)). \\
(4) & \quad \text{mpl}(X_m, r_m) = m, \quad m = 0, 1, 2, \ldots. \\
(5) & \quad \text{Each group } G_{m+1} = G(X_{m+1}, r_{m+1}) \text{ is isomorphic to the wreath product } \\
& \quad G(X_m, r_m) \rtimes C_2, \text{ so } G_{m+1} = ((C_2 \rtimes \underbrace{C_2 \rtimes \cdots \rtimes C_2}) \rtimes \cdots) \rtimes C_2. \\
(6) & \quad \text{There are equalities } \text{sl}(G_m) = m, \quad \text{sl}(G_m) = m - 1. \\
(7) & \quad (X, r_X) \text{ is the inverse limit of the solutions } (X_m, r_m). \text{ For the retracts one has } \\
& \quad \text{Ret}^m(X, r_X) \neq \text{Ret}^{m+1}(X, r_X), \text{ and } \text{mpl}(X, r) = \infty. \text{ Furthermore, the group } G(X, r_X) \text{ acts nontransitively on } X.
\end{align*}
\]

**Proof.** Under the hypothesis of the theorem we prove first several preliminary statements.

**Remark 9.12.** Let \( m \) be an integer, \( m \geq 2, N_m = 2^m \)

\[
(1) \quad |Y_m| = 2^m \text{ and } |X_{m+1}| = 2^m + 1.
\]
(2) \( \sigma_m \) is well defined via \( \sigma_k, k \leq m - 1 \) and the shift \( y \mapsto y[N_{m-1}] \). We have the following explicit formulae:

\[
(\sigma_{m-1}[N_{m-1}])[y[N_{m-1}]] = (\sigma_{m-1}(y))[N_{m-1}] \quad \forall y \in Y_{m-1}
\]

\[
\sigma_m(y) = y[N_{m-1}] \in Y_{m-1}[N_{m-1}], \quad \forall y \in Y_{m-1}
\]

(9.5)

\[
\sigma_m(y[N_{m-1}]) = \sigma_{m-1}y \in Y_{m-1}, \quad \forall y \in Y_{m-1}
\]

(9.6)

\[
x[N_{m-1}]z[N_{m-1}] = (xz)[N_{m-1}]
\]

**Lemma 9.13.** For each integer \( m \geq 1 \), \( (Y_m, r_{Y_m}) \) is a square-free solution of order \( N_m = 2^m \) and \( \sigma_m \in \text{Aut}(Y_m, r_{Y_m}) \),

**Proof.** We shall prove the lemma using induction on \( m \). Base for the induction, \( m = 1 \). By definition \( (Y_1, r_{Y_1}) \) is the trivial solution \( \{x_1, x_2\} \) and clearly \( \sigma_1 = (x_1, x_2) \in \text{Aut}(Y_1, r_{Y_1}) \). Assume now that the lemma is true for all \( k \leq m \).

Clearly, by Definition of as a set \( Y_{m+1} \) is a disjoint union \( Y_{m+1} = Y_m \cup Y_m[N_m] \), so \( |Y_{m+1}| = 2|Y_m| = 2^{m+1} \). Furthermore (by definition), the quadratic set \( (Y_{m+1}, r_{m+1}) \) with \( r = r_{Y_{m+1}} \) satisfies

\[
r(x, y) = (\sigma_m[N_m](y), \sigma_m^{-1}(x)) \quad r(y, x) = (\sigma(x), \sigma_m[N_m]^{-1}(y) \quad \forall x \in Y_m, y \in Y_m[N_m].
\]

By assumption \( \sigma_m \in \text{Aut}(Y_m, r_{Y_m}) \), hence \( \sigma_m[N_m] \in \text{Aut}(Y_m[N_m], r_{Y_m[N_m]}) \). Hence by Lemma 5.5 \( (Y_{m+1}, r_{m+1}) \) is a solution.

We shall now prove that \( \sigma_{m+1} \in \text{Aut}(Y_{m+1}) \). By Lemma 5.3 it will be enough to show that for each \( x \in Y_m \)

\[
\sigma_{m+1} \circ \mathcal{L}_x = \mathcal{L}_{\sigma_{m+1}(x)} \circ \sigma_{m+1}
\]

is an equality of maps in \( Y_{m+1} \), or equivalently

(9.8)

\[
\sigma_{m+1}(xz) = \sigma_{m+1}(x)\sigma_{m+1}(z) \forall z \in Y_{m+1}.
\]

By the inductive assumption we have

(9.9)

\[
\sigma_m(xz) = \sigma_m(x)\sigma_m(z) \forall x, z \in Y_m.
\]

Let \( x, z \in Y_{m+1} \). By definition \( Y_{m+1} = Y_m \cup Y_m[N_m] \) (this is a disjoint union).

**Case 1.** \( x \in Y_m \). 1.a. \( z \in Y_m \)

\[
xz \in Y_m \quad \sigma_m(xz) = xz[N_m] \quad \text{by (9.5)}
\]

\[
\sigma_m(x) = x[N_m] \quad \sigma_m(z) = z[N_m] \quad \text{by (9.5)}
\]

\[
\sigma_m(x)\sigma_m(z) = x[N_m]z[N_m] = xz[N_m] \quad \text{by (9.6)}
\]
So in this case \((9.8)\) holds. 1.b.  \(z = y[N_m]\), where \(y \in Y_m\).

\[
xz = (\sigma_m[N_m])(z) = \sigma_m(y)[N_m] \in Y_m[N_m]
\]

by \((9.3)\)

\[
\sigma_m(xz) = \sigma_m(\sigma_m(y)[N_m]) = (\sigma_m)^2(y)
\]

by \((9.5)\)

\[
\sigma_m(x) = x[N_m] \quad \sigma_m(z) = \sigma_m(y[N_m]) = \sigma_m(y)
\]

by \((9.5)\)

\[
\sigma_m(x)\sigma_m(z) = x[N_m]\sigma_m(y) = \sigma_m(\sigma_m(y))
\]

by \((9.3)\).

Hence \((9.8)\) holds.

**Case 2.**  \(x = \xi[N_m] \in Y_m[N_m]\). 2.a. \(z \in Y_m\) In this case \((9.8)\) follows from the equalities

\[
xz = \sigma_m(z) = \xi[N_m]
\]

by \((9.3)\)

\[
\sigma_m(xz) = \sigma_m(\sigma_m(z)) = \sigma_m(z)[N_m]
\]

by \((9.5)\)

\[
\sigma_m(x) = \sigma_m(\xi[N_m]) = \sigma_m(\xi) \in Y_m \quad \sigma_m(z) = z[N_m] \in Y_m[N_m]
\]

by \((9.5)\)

\[
\sigma_m(x)\sigma_m(z) = \sigma_m(\xi)[z[N_m] = \sigma_m[N_m](z[N_m])
\]

by \((9.3)\)

\[
= (\sigma_m(z))[N_m]
\]

by \((9.4)\).

2.b. \(z = y[N_m], y \in Y_m\). Note that

\[
xz = \xi[N_m][y[N_m] = (\xi y)[N_m] \in Y_m[N_m]
\]

by \((9.2)\)

\[
\sigma_m(xz) = \sigma_m((\xi y)[N_m]) = \sigma_m(\xi y)
\]

by \((9.5)\)

\[
\sigma_m(x) = \sigma_m(\xi) \in Y_m \quad \sigma_m(z) = \sigma_m(y[N_m]) = \sigma_m(y) \in Y_m
\]

by \((9.5)\)

\[
\sigma_m(x)\sigma_m(z) = \sigma_m(\xi)\sigma_m(y) = \sigma_m(\xi y)
\]

by IH and \((9.8)\).

where IH denotes the inductive hypothesis.

We have shown that \(\sigma_{m+1} \in \text{Aut } Y_{m+1}\), which verifies the lemma. \(\square\)

The following corollary is straightforward from the recursive definition of the quadratic sets \((X_m, r_m)\) and Lemma 8.5.

**Corollary 9.14.** For each \(m = 0, 1, 2, \ldots\), \((X_m, r_m)\) is a square-free solution, of order \(|X_m| = 2^{m-1} + 1\). Furthermore, \((X_{m+1}, r_{m+1})\) is an extension of \((X_m, r_m)\).

The following lemma gives explicit recursive presentation of the left actions in \(X_m\), respectively \(Y_m\).

**Lemma 9.15.** Let \(m \geq 2\).
(1) Let \( x \in X_{m+1} \) then following equalities hold.

\[
\forall x \in Y_{m-1} \quad L_x|_{Y_m} = L_x|_{Y_{m-1}} \circ \sigma_{m-1}[N_{m-1}]
\]
\[
L_x|_{X_{m+1}} = L_x|_{Y_m}
\]
\[
\forall y \in Y_{m-1}[N_{m-1}], y = x[N_{m-1}], x \in Y_{m-1} \quad L_y|_{Y_m} = (L_x|_{Y_{m-1}})[N_{m-1}] \circ \sigma_{m-1}
\]
\[
L_y|_{X_{m+1}} = L_y|_{Y_m}
\]

(2) For all \( i, i = 2k - 1, 1 \leq k \leq 2^m - 1 \) one has

\[
L_{x_i}|_{X_{m+1}} = L_{x_{i+1}}|_{X_{m+1}}.
\]

(3) There is an isomorphism of solutions

\[
\text{Ret}(X_{m+1}, r_{m+1}) \simeq (X_m, r_m).
\]

Proof. The equalities \([9.10]\) follow from the recursive definition of \( Y_m \) and \( X_m, m = 0, 1, 2, \cdots \) see Definition \([9.3]\). The recursive definition and \([9.10]\) imply \([9.11]\). Then

\[
[X_{m+1}] = \{[x_1], [x_3], \cdots [x_{2m-2+1}], \cdots [x_{2m-1-1}]\}
\]

It is easy to see that \((X_{m+1}, r_{X_{m+1}}) \simeq (X_m, r_m)\). □

Now the statement of the theorem follows easily. Indeed, Corollary \([9.14]\) verifies parts \(1\) and \(2\). Part \(3\) follows from Lemma \([9.15]\). Clearly, induction on \( m \) and \(3\) straightforwardly imply \(1\). \(5\) is clear from the construction. \(5\) implies \(\text{sl}(G_m) = m - 1\). Hence, by Theorem \([7.25]\) we have \(\text{sl}(G_m) = \text{sl}(G_m) + 1\), which proves \(6\). Finally, \(7\) is clear. □

Construction 9.16. Let \( R \) be a finite ring and \( A \) a finite faithful \( R \)-module. Let \( \omega \) be a fixed unit in \( R \). For \( a \in A \), let \( L_a \) be the permutation \( x \mapsto \omega x + (1 - \omega)a \).

Note: We do not need to assume that \( R \) is commutative, since we work only in the subring generated by 1 and \( \omega \). Also, note that \( L_a \) has the effect of multiplying \( x - a \) by \( \omega \); so it is clearly a permutation fixing \( a \). (Its inverse is obtained by replacing \( \omega \) by its inverse.) If \( \omega = -1 \), it is the inversion in \( a \).

Proposition 9.17. The following are equivalent:

(a) the maps \( L_a \), for \( a \in A \), give a solution;
(b) the maps \( L_a \), for \( a \in A \), all commute;
(c) \((1 - \omega)^2 = 0\).

Proof. Clearly \( L_a(a) = a \) holds for all \( a \in A \). Thus, the maps form a solution if and only if

\[
L_{a_b}L_{a_b} = L_{a_b}L_{a_b},
\]
where $a^b = L_a^1(b)$ and $a^b = L_b^{-1}(a)$; and a short calculation (see below) shows that this condition and the condition $LaL_b = L_bL_a$ hold for all $a$ and $b$ if and only if $(1 - \omega)^2 = 0$.

If $\omega = 1$, then $L_a$ is the identity map for all $a$, and the multipermutation level is 1. Otherwise, $L_a = L_b$ if and only if $(1 - \omega)(a - b) = 0$; so the elements of the reduct are the cosets of the submodule $\{a \in A : (1 - \omega)a = 0\}$ (the annihilator of $1 - \omega$). Since $(1 - \omega)^2 = 0$, the element $(1 - \omega)$ acts as zero on the quotient module $A/I$, so the reduct has mpl 1, and the original solution has mpl 2.

Calculations. $L_a$ maps $x$ to $\omega x + (1 - \omega)a$, and so

$$L_a L_b x \mapsto \omega x + (1 - \omega)b
\mapsto \omega(\omega x + (1 - \omega)b) + (1 - \omega)a
= \omega^2 x + (1 - \omega)(a + \omega b).$$

Similarly $L_b L_a$ maps $x$ to $\omega^2 x + (1 - \omega)(b + \omega a)$. These are equal if and only if $(1 - \omega)^2(a - b) = 0$. For this to hold for all $a$ and $b$, it is necessary and sufficient that $(1 - \omega)^2 = 0$.

Now $a^b = \omega^{-1} a + (1 - \omega^{-1})b$ and $a^b = \omega b + (1 - \omega)a$, so we calculate that $L_a b L_a$ maps $x$ to

$$\omega^2 x + (1 - \omega)(2 - \omega)a + (1 - \omega)(2 \omega - 1)b.$$

This is equal to $L_a L_b(x)$ if and only if $(1 - \omega)^2(a - b) = 0$, and the conclusion follows as before. \quad \square

10. More about YB permutation groups

Let $(Z, r)$ be an YB extension of the disjoint solutions $(X, r_X)$ and $(Y, r_Y)$. Then for every $z \in Z$ the action $L_z$ splits

$$L_z = (L_z)|X \circ (L_z)|_{XY}.$$

Recall that if $(X, r_X)$ and $(Y, r_Y)$ are two disjoint solutions then the trivial extension $(Z, r)$ is defined as $Z = X \cup Y$, with $r(x, \alpha) = (\alpha, x)$, for all $x \in X, \alpha \in Y$.

**Proposition 10.1.** Let $(X, r_X)$ and $(Y, r_Y)$ be disjoint square-free solutions with $G_1 = G(X, r_X), G_2 = (Y, r_Y), G_1 = G(X, r_X), G_2 = G(Y, r_Y)$. Then the following conditions hold.

1. Suppose $(Z, r)$ is the trivial extension of $(X, r_X)$ and $(Y, r_Y)$. Then it is a square-free solution, and

$$G(Z, r) = G_1 \times G_2 \quad G(Z, r) = G_1 \times G_2$$

Conversely, if $G_1$ and $G_2$ are permutation YB groups, then $G = G_1 \times G_2$ is a permutation YB group.
(2) Suppose \((Z, r)\) is an involutive extension which satisfies
\[
(10.1) \quad r(\alpha, x) = (\alpha x, \alpha) \quad \forall x \in X, \alpha \in Y.
\]
Then \((Z, r)\) is a solution if and only if the assignment \(\alpha \mapsto (L_\alpha)_X\) extends to a homomorphism
\[
G(Y, r_Y) \longrightarrow \text{Aut}(X, r_X)
\]
In other words, \(G(Y, r_Y)\) acts as automorphisms on \((X, r)\). In this case \(G = G(Z, r)\) and the YB permutation group \(\mathcal{G} = \mathcal{G}(Z, r)\) are semidirect products:
\[
G = G_1 \rtimes G_2, \quad \mathcal{G} = \mathcal{G}_1 \rtimes \mathcal{G}_2.
\]

(3) Conversely, suppose there is an action of \(G(Y, r_Y)\) on \(G(X, r_X)\), such that \(X\) is invariant under this action. Then the formula \((10.7)\) induces canonically a solution \((Z, r)\) on \(Z\). In particular, a semidirect product \(G_1 \rtimes G_2\) of two permutation YB groups defined via an action of \(G(Y, r_Y)\) on \(G(X, r_X)\), which keeps \(X\) invariant is itself a permutation YB group.

(4) Suppose \((Z, r)\) is the wreath product of solutions \((Z, r) = (X, r_X) \wr (Y, r_Y)\), see Definition \([77]\). Then there are equalities
\[
G(Z, r) = G_1 \wr G_2, \quad \mathcal{G}(Z, r) = \mathcal{G}_1 \wr \mathcal{G}_2
\]

Question 10.2. Under what conditions is the wreath product \(G_1 \wr G_2\) of two permutation YB groups \(G_1\) and \(G_2\) isomorphic to a permutation YB group?

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References

[AS] Artin, M. and Schelter, W., Graded algebras of global dimension 3, Adv. Math. 66 (1987), 171–216.
[C] Peter J. Cameron, Permutation Groups, Cambridge University Press 45 (1999) 220pp.
[C08] Cameron, Peter J., Introduction to Algebra, Cambridge University Press (2008) 337pp.
[CJR08] Cedó, F., Jespers, E., and del Rio, A., Involutive Yang–Baxter groups, [arXiv:0803.4054v2 [math.QA]] 28 Mar 2008, 19 pp.
[CJR09] Cedó, F., Jespers, E., and Okišsik J., Retractability of set theoretic solutions of the Yang–Baxter equation, [arXiv:0903.3478v2 [math.GR]] 23 Mar 2009, 13 pp.
[Dri] Drinfeld, V., On some unsolved problems in quantum group theory, Lecture Notes in Mathematics 1510 (1992), 1–8.
[ESS] Etingof, P., Schedler, T. and Soloviev, A., Set-theoretical solutions to the quantum Yang–Baxter equation, Duke Math. J. 100 (1999), 169–209.
[GAP] The GAP Group, GAP – Groups, Algorithms, and Programming, Version 4.4.12; 2008, [http://www.gap-system.org]
[GI94] Gateva-Ivanova, T., Noetherian properties of skew polynomial rings with binomial relations, Trans. Amer. Math. Soc. 343 (1994), 203–219.
[GI96-1] Gateva-Ivanova, T., Skew polynomial rings with binomial relations, J. Algebra 185 (1996), 710–753.
[GI96-2] Gateva-Ivanova, T., Regularity of the skew polynomial rings with binomial relations, Preprint (1996).
[GI00] Gateva-Ivanova, T., *Set theoretic solutions of the Yang–Baxter equation*, Mathematics and education in Mathematics, Proc. of the Twenty Ninth Spring Conference of the Union of Bulgarian Mathematicians, Lovetch (2000), 107-117.

[GI04] T. Gateva-Ivanova, *Quantum binomial algebras, Artin-Schelter regular rings, and solutions of the Yang–Baxter equations*, Serdica Math. J. 30 (2004), 431-470.

[GI] Gateva-Ivanova, T., *A combinatorial approach to the set-theoretic solutions of the Yang–Baxter equation*, J.Math.Phys., 45 (2004), 3828–3858.

[GI08] Gateva-Ivanova, T., *Set-theoretic solutions of YBE, a combinatorial approach – Talk to the seminar Combinatorics and Statistical Mechanics, Isaac Newton Institute, March 2008.*

[GM08] T. Gateva-Ivanova, T. and Majid, S., *Matched pairs approach to set theoretic solutions of the Yang–Baxter equation*, J. Algebra 319 (2008) 1462-1529.

[GM07] T. Gateva-Ivanova, T. and Majid, S., *Set Theoretic Solutions of the Yang–Baxter Equations, Graphs and Computations*, J. Symb. Comp. 42 (2007) 1079-1112.

[GI0806] T. Gateva-Ivanova, T. and Majid, S., *Set-theoretic solutions of YBE, a combinatorial approach* - Talk to the seminar Combinatorics and Statistical Mechanics, Isaac Neuton Institute, March 2008.

[GIM08] T. Gateva-Ivanova, T. and Majid, S., *Matched pairs approach to set theoretic solutions of the Yang–Baxter equation*, J. Algebra 319 (2008) 1462-1529.

[GIM07] T. Gateva-Ivanova, T. and Majid, S., *Set Theoretic Solutions of the Yang–Baxter Equations, Graphs and Computations*, J. Symb. Comp. 42 (2007) 1079-1112.

[GI0806] T. Gateva-Ivanova, T. and Majid, S., *Set-theoretic solutions of YBE, a combinatorial approach* - Talk to the seminar Combinatorics and Statistical Mechanics, Isaac Neuton Institute, March 2008.

[L] G. Laffaille, *Quantum binomial algebras*, Colloquium on Homology and Representation Theory (Spanish) (Vaquéria, 1998). Bol. Acad. Nac. Cienc. (Córdoba) 65 (2000), 177–182.

[LYZ] J. Lu, M. Yan, Y. Zhu *On the set-theoretical Yang–Baxter equation*, Duke Math. J. 104 (2000) 1–18.

[M90] Majid, S., *Matched pairs of Lie groups associated to solutions of the Yang–Baxter equations. Pac. J. Math.*, 141 (1990) 311–332.

[M95] Majid, S., *Foundations of quantum group theory*, Cambridge Univ. Press (1995).

[MA] Manin, Yu., *Quantum Groups and Non Commutative Geometry Montreal University Report No. CRM- 1561, 1988*

[MJ] Milne, J.S., *Group Theory*, v2.11. (2003), [http://www.jmilne.org/math/](http://www.jmilne.org/math/)

[RU] Rump, W., *A decomposition theorem for square-free unitary solutions of the quantum Yang–Baxter equation*, ADVANCES IN MATHEMATICS, 193 (2005), 40–55.

[RTF] N.Yu. Reshetikhin, L.A. Takhtadzhyan, L.D. Faddeev *Quantization of Lie groups and Lie algebras (in Russian) , Algebra i Analiz 1 (1989), pp. 178–206; English translation in Leningrad Math.J. 1 (1990), pp. 193–225.*

[TS81] M. Takeuchi *Matched pairs of groups and bismash products of Hopf algebras*, Commun. Alg., 9 (1981) 841.

[TA] M. Takeuchi *Survey on matched pairs of groups. An elementary approach to the ESS-LYZ theory*, BANACH CENTER PUBL. 61 (2003) 305–331.

[V] A.P. Veselov *Yang-Baxter maps: dynamical point of view*, [arXiv:math/0612814v1 [math.QA]](http://arxiv.org/abs/math/0612814v1)

[WX] A. Weinstein and P. Xu *Classical solutions of the quantum Yang–Baxter equation*, Comm. Math. Phys. 148 (1992), pp. 309–343.

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