Harmonic oscillator with finite mass variation under asymmetry condition

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Abstract: Classical as well as quantum mechanical analysis have been carried out on harmonic oscillator with asymmetric position dependent mass. Classical and quantum phase space analysis are reflected for a plausible understanding of the subject.

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I. Introduction

Position dependent mass (PDM) has drawn considerable attention due to their applications in semiconductor physics [1-4], quantum dots [5], quantum liquids [6], nonlinear oscillators [7-9] etc. However, one needs to take proper care in defining the kinetic energy operator in PDM due to noncommutivity of momentum and position and the same can be avoided if one chooses suitable combination of position and momentum at classical level [3]. The authors [3] suggested the PDM is of the form

\[ m(x) = m e^{\frac{a x + \frac{1}{2} b x^2}{}}. \]  

(1)

The PDM stated above is of asymmetric in nature. In fact, it has been seen that mass is the asymmetric function of position \((x)\) in semiconductor physics. If \(a\) and \(b\) are negative then the \(m(x)\) (Eq. (1)) will be 0 at large value of \(x\). In other words a massive particle undergoes a drastic change and becomes a photon, which can hardly interact with any matter. Secondly, if the mass behaves as

\[ m(x) = m(1 + x^2) \]. \]  

(2)

Then at large distance mass becomes infinite. If the mass becomes large while moving with distance then it becomes meaningless to discuss physics behind an infinite mass. In order to avoid this ambiguity, we choose the PDM is of the form

\[ m(x) = \frac{m}{1 + e^{-x^2\lambda^2}} \]  

(3)

where the parameter \(\lambda\) is of weak in nature. The purpose of considering this mass is that asymmetry in variation of mass and is confined within two finite limits such as (i) \(\frac{m}{2}\) and (ii) \(m\) (Fig. 1). Hence considering the above facts, we confined our focus on finite mass. Now we feel to introduce this simple asymmetry mass in Harmonic oscillator and study the system both classically and quantum mechanically.
II. Classical Analysis

The Hamiltonian of the oscillator considered here is

\[ H = \frac{p_x^2}{2m(x)} + \frac{m(x)}{2} x^2. \]  

(4)

The Lagrangian is related to the Hamiltonian as:

\[ H = \sum_i p_i \dot{q}_i - L. \]  

(5)

In our case we have just one generalized coordinate \( x \), and one generalized momenta \( p_x \). Therefore we have

\[ H = p_x \dot{x} - L; \]

\[ \frac{p_x^2}{2m(x)} + \frac{m(x)}{2} x^2 = p_x \dot{x} - L. \]  

(6)

On solving Eq. (6), one can find the expression of Lagrangian \( L \) as

\[ L = p_x \dot{x} - \frac{p_x^2}{2m(x)} - \frac{m(x)}{2} x^2. \]  

(7)
Using \( \frac{dx}{dt} = \frac{\partial H}{\partial p_x} \Rightarrow \dot{x} = \frac{p_x}{m(x)} \), and \( p_x = \dot{x}m(x) \). Thus Eq. (7) becomes:

\[
L = \frac{m(x)}{2} (\dot{x}^2 - x^2). \tag{8}
\]

Our next step is deriving the equation of motion (i.e., known as: Euler Lagrange Equation). Substituting Eq. (8) into the following relation

\[
\frac{\partial L}{\partial q_i} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) = 0 \text{ with } q_i = x, \]

we got:

\[
-xm(x) + \frac{(x^2 - x^2)}{2} \frac{dm(x)}{dx} + \dot{x}m(x) - x \frac{dm(x)}{dt} = 0. \tag{9}
\]

Substituting the expression of mass (Eq. (1)) into Eq. (9), we get the following equation of motion, which in a simplified form can be written as

\[
\ddot{x} + x - \left( \frac{x^2 + x^2}{2} \right) \left( \frac{1 + 2\lambda x}{1 + e^{-x-\lambda x^2}} \right) e^{-x-\lambda x^2} = 0. \tag{10}
\]

**II.1 Analytical Study:**

In order to solve the above equation analytically (Eq. (10)), we follow the He’s frequency formulation on ancient Chinese method [10-12] to get the frequency of oscillation using the following condition:

\[
\omega^2 = \frac{\omega_1^2 R_2(0) - \omega_2^2 R_1(0)}{R_2(0) - R_1(0)}. \tag{11}
\]

In the above formalism, the boundary condition for the solution \( x(t) \) will be \( x(t = 0) = 1 \) and \( \dot{x}(t = 0) = 0 \). We therefore considered

\[
x(t) = A \cos \omega t \tag{12}
\]

and following the He’s formalism [10-12] as stated above, we get the frequency of oscillation as

\[
\omega = \sqrt{1 - \frac{A}{2} \left( \frac{1 + 2\lambda A}{1 + e^{-A-\lambda A^2}} \right) e^{-A-\lambda A^2}}. \tag{13}
\]
The variation of $x(t)$ vs time ($t$), $p(t)$ vs $t$ and trajectory of phase space ($p$ vs $x$) are shown in Figure 2, 3 and 4 respectively. In classical analysis of phase space, we notice the clear asymmetry behaviour.

**Fig. 2:** Variation of $x(t)$ with time ($t$) for different amplitude.

**Fig. 3:** Variation of $p(t)$ with time ($t$) for different amplitude.
Fig. 4: Trajectory of classical phase space obtained analytically.

II.2 Numerical Study:

In order to get better information on the classical dynamics of the system (Eq. (10)), the numerical study on the said system is undertaken. We can rewrite the Eq. (10) as

\[ \ddot{x} - Q\dot{x}^2 - Qx^2 + x = 0 \]  

(14)

where

\[ Q = \frac{1}{2} \left( \frac{1 + 2\lambda x}{1 + e^{-x-\lambda t}} \right) e^{-x-\lambda t} = \frac{1 + 2\lambda x}{2(1 + e^{x+\lambda t})} \]  

(15)

Here, we also use the initial condition \( x(t=0) = 1 \) and \( \dot{x}(t=0) = 0 \) for the values of \( \lambda = 0.1 \). Using the algorithm (Appendix-A), we find the numerical solution of Eq. (14). The variation of \( x \) and \( p \) with respect to time \( t \) and phase space trajectory (\( p \) with respect to \( x \)) obtained numerically for \( \lambda = 0.1 \) are shown in Figure 5 and 6 respectively.
Figure 5: Variation of $x$ and $p$ obtained numerically for $\lambda = 0.1$ with respect to time $t$.

Figure 6: Variation of $p$ obtained numerically for $\lambda = 0.1$ with respect to $x$. 
III. Quantum Mechanical Analysis

Now we analyze the above problem quantum mechanically. In quantum analysis, we solve the eigenvalue relation [13-15] as

\[ H|\Psi_n\rangle = E|\Psi\rangle. \]  (16)

Where

\[ |\Psi\rangle = \sum_n A_n |\phi_n\rangle \]  (17)

and \( |\phi_n\rangle \) satisfies the eigenvalue relation for position independent harmonic oscillator as

\[ H_0|\phi_n\rangle = \left[ p^2 + x^2 \right]|\phi_n\rangle = (2n+1)|\phi_n\rangle. \]  (18)

Using the above formalism [13-15], we get the recurrence relation as

\[ \sum_{k=2,4,6,...} P_n^k A_{n-k} + S_n A_n + R_n^k A_{n+k} = 0. \]  (19)

Where

\[ P_n^k = \langle n | H | n-k \rangle. \]  (20)

\[ R_n^k = \langle n | H | n+k \rangle. \]  (21)

\[ S_n = \langle n | H | n \rangle - E. \]  (22)

The eigenvalues of the Harmonic oscillator Hamiltonian (Eq. (4)) with position dependent mass (Eq. (1)) are obtained following above mentioned procedure for \( \lambda = 0.1 \) and \( m = 1 \) (Table 1). In this case, we also plot quantum mechanical phase space (Fig. 7) considering \( E = H \) of respective states. Comparing the trajectory of quantum phase space (Fig. 7) with that of classical phase space obtained both analytically (Fig. 4) and numerically (Fig. 6), we noticed a great similarity between classical and quantum.
**Fig. 7:** Trajectory of quantum phase space.

**Table 1:** Eigenvalues of Asymmetry mass Harmonic oscillator.

| Quantum number (n) | Computed Eigenvalues |
|-------------------|----------------------|
| 0                 | 0.5                  |
| 1                 | 1.2789               |
| 2                 | 2.2610               |
| 3                 | 3.3735               |
| 4                 | 4.5412               |

**IV. Conclusion**

We discuss about the finite mass variation under asymmetric condition. Interested reader will see that mass of the system varied asymmetrically between two finite values. Further, classical phase space nature remains practically the same as
quantum phase space. We believe that apart from theoretical analysis, the present model could be suitable for semiconductor physics where asymmetry is an important factor.

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Appendix-A: The proposed method for to find the solution for Eq. (14) using MATLAB

1 function [MatrixEq]= SolveMainEq(lambda, x0, xprime0, m0, tmin, tmax)
2 % gamma : [gamma];
3 % Eq1(r, c): results of main equation ;
4 [xlambda, ylambda]=size(lambda);
5 column=2;
6 for j=1:ylambda
7 f1=@(t, x)[x(2); (1+2_lambda(j)_x(1))(x(2))^2/(2_(1+exp(x(1) + ...
8 lambda(j)_x(1)))) + ...
9 (1+2_lambda(j)_x(1))(x(1))^2/(2_(1+exp(x(1) + ...)
10 lambda(j)_x(1)^2)) - x(1)];
11 [tr, xr]=ode23(f1, [tmin tmax], [x0, xprime0]);
12 maxrow=max(size(tr));
13 for k=1:maxrow
14 MatrixEq(k, 1)=k;
15 MatrixEq(k, column)= tr(k);
16 MatrixEq(k, column+1)= round(xr(k, 1),6);
17 MatrixEq(k, column+2)= round(xr(k, 2),6);
18 MatrixEq(k, column+3)= round((1+2_lambda(j)_xr(k, 1))/(2_(1+ ...
19 exp(xr(k, 1)+ lambda(j)_xr(k, 1)^2))),6);
20 mx = round(m0/(1+ exp((-1)_xr(k, 1) - lambda(j)_xr(k, 1)^2)),6);
21 MatrixEq(k, column+4)=mx;
22 px = round(mx_xr(k, 2),6);
23 MatrixEq(k, column+5)=px;
24 MatrixEq(k, column+6)=round( px^2/(2_mx) + mx_xr(k, 1)/2,6);
25 end;
26 column=column+7;
27 end;
28 end
Table A1: Numerical results of Eq. 14 for $\lambda = 0.1$ and $t \in [0,10]$.

| Sl. No. | $t$      | $x$     | $\dot{x}$ | $Q(x)$   | $m(x)$   | $p_x$ |
|---------|----------|---------|-----------|----------|----------|-------|
| 1       | 0        | 1       | 0         | 0.14984  | 0.75026  | 0     |
| 2       | 9.41004E-5 | 1       | -8E-5     | 0.14984  | 0.75026  | -6E-5 |
| 3       | 5.64602E-4 | 1       | -4.8E-4   | 0.14984  | 0.75026  | -3.8E-4|
| 4       | 0.00292  | 1       | -0.00248  | 0.14984  | 0.75026  | -0.00186|
| 5       | 0.01468  | 0.99991 | -0.01248  | 0.14985  | 0.75024  | -0.00936|
| 6       | 0.07349  | 0.9977  | -0.06242  | 0.1501   | 0.74974  | -0.0468|
| 7       | 0.19277  | 0.98425 | -0.16281  | 0.15158  | 0.74671  | -0.12157|
| 8       | 0.35773  | 0.94619 | -0.29727  | 0.15578  | 0.73802  | -0.21939|
| 9       | 0.56268  | 0.86908 | -0.45197  | 0.16432  | 0.72003  | -0.32543|
| 10      | 0.8076   | 0.73825 | -0.61014  | 0.17879  | 0.68842  | -0.42003|
| 11      | 1.10091  | 0.53734 | -0.7489   | 0.20051  | 0.63789  | -0.47772|
| 12      | 1.38849  | 0.30989 | -0.82162  | 0.22344  | 0.5792   | -0.47588|
| 13      | 1.64128  | 0.10004 | -0.83008  | 0.24213  | 0.52524  | -0.43599|
| 14      | 1.78967  | -0.02192| -0.81097  | 0.25163  | 0.49453  | -0.40105|
| 15      | 1.93806  | -0.13977| -0.77465  | 0.25973  | 0.4656   | -0.36068|
| 16      | 2.10593  | -0.26498| -0.714    | 0.26712  | 0.43587  | -0.31121|
| 17      | 2.32048  | -0.40736| -0.60902  | 0.27394  | 0.40353  | -0.24576|
| 18      | 2.58348  | -0.54654| -0.44499  | 0.27895  | 0.37363  | -0.16626|
| 19      | 2.86371  | -0.64272| -0.23898  | 0.28147  | 0.35402  | -0.08461|
| 20      | 3.10574  | -0.67728| -0.04656  | 0.28219  | 0.34719  | -0.01616|
| 21      | 3.25751  | -0.67501| 0.07604   | 0.28215  | 0.34764  | 0.02644|
| 22      | 3.40929  | -0.65423| 0.197     | 0.28172  | 0.35173  | 0.06929|
| 23      | 3.59514  | -0.60423| 0.33906   | 0.28055  | 0.36176  | 0.12266|
| 24      | 3.82111  | -0.50932| 0.4966    | 0.27777  | 0.38145  | 0.18943|
| 25      | 4.08596  | -0.35637| 0.65049   | 0.27169  | 0.41492  | 0.2699|
| 26      | 4.35082  | -0.16817| 0.76142   | 0.26152  | 0.45876  | 0.34931|
| 27      | 4.52112  | -0.03435| 0.80632   | 0.25253  | 0.49144  | 0.39626|
| 28      | 4.69141  | 0.10522 | 0.82871   | 0.2417   | 0.52656  | 0.43636|
| 29      | 4.83679  | 0.22596 | 0.82932   | 0.23124  | 0.55751  | 0.46235|
| 30      | 5.0241   | 0.37944 | 0.80462   | 0.21668  | 0.59721  | 0.48052|
| 31      | 5.24693  | 0.55211 | 0.7385    | 0.19895  | 0.64166  | 0.47387|
| 32      | 5.50401  | 0.7272  | 0.61608   | 0.18001  | 0.6857   | 0.42244|
| 33      | 5.7995   | 0.88195 | 0.42394   | 0.16289  | 0.72307  | 0.30654|
| 34      | 6.07287  | 0.96924 | 0.21135   | 0.15323  | 0.7433   | 0.1571|
| 35      | 6.2513   | 0.99372 | 0.06257   | 0.15054  | 0.74885  | 0.04685|
| 36      | 6.3851   | 0.99451 | -0.05066  | 0.15045  | 0.74902  | -0.03794|
| 37      | 6.51889  | 0.98019 | -0.16292  | 0.15202  | 0.74579  | -0.1215|
| 38      | 6.68406  | 0.94209 | -0.297    | 0.15623  | 0.73708  | -0.21891|
| 39      | 6.88928  | 0.86499 | -0.45123  | 0.16477  | 0.71906  | -0.32446|
| 40      | 7.1345   | 0.73425 | -0.60881  | 0.17923  | 0.68744  | -0.41852|
| 41      | 7.42818  | 0.53357 | -0.74691  | 0.20091  | 0.63692  | -0.47572|
| 42      | 7.71554  | 0.30698 | -0.81893  | 0.22372  | 0.57845  | -0.47371|

12
|   |       |       |       |       |       |
|---|-------|-------|-------|-------|-------|
| 43| 7.96804 | 0.0981 | -0.82707 | 0.24229 | 0.52475 | -0.434 |
| 44| 8.11563 | -0.02277 | -0.80802 | 0.25169 | 0.49432 | -0.39942 |
| 45| 8.26323 | -0.13957 | -0.77202 | 0.25972 | 0.46565 | -0.35949 |
| 46| 8.43119 | -0.26442 | -0.71157 | 0.26709 | 0.436 | -0.31024 |
| 47| 8.64579 | -0.40635 | -0.60695 | 0.27389 | 0.40376 | -0.24506 |
| 48| 8.90881 | -0.54508 | -0.44347 | 0.27891 | 0.37394 | -0.16583 |
| 49| 9.189 | -0.64091 | -0.23819 | 0.28143 | 0.35438 | -0.08441 |
| 50| 9.43098 | -0.67535 | -0.04643 | 0.28215 | 0.34757 | -0.01614 |
| 51| 9.58271 | -0.6731 | 0.07572 | 0.28211 | 0.34801 | 0.02635 |
| 52| 9.73444 | -0.65241 | 0.19624 | 0.28168 | 0.35209 | 0.0691 |
| 53| 9.92024 | -0.60261 | 0.3378 | 0.28051 | 0.36209 | 0.12231 |
| 54| 10 | -0.57335 | 0.39547 | 0.27973 | 0.36808 | 0.14557 |