CRITICAL GAUSSIAN MULTIPLICATIVE CHAOS REVISITED

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Abstract. We present new, short and self-contained proofs of the convergence (with an adequate renormalization) of four different sequences to the critical Gaussian Multiplicative Chaos: (a) the derivative martingale (b) the critical martingale (c) the exponential of the mollified field (d) the subcritical Gaussian Multiplicative Chaos.

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1. Introduction

We consider $K: \mathbb{R}^d \times \mathbb{R}^d \to (-\infty, \infty]$ to be a positive definite kernel on $\mathbb{R}^d$ ($d \geq 1$ is fixed) which admits a decomposition in the following form

$$K(x, y) := \log \frac{1}{|x - y|} + L(x, y),$$

where $L$ is a continuous function ($\log 0 = \infty$ by convention). A kernel $K$ is positive definite if for any $\rho \in C_c(\mathbb{R}^d)$ ($\rho$ continuous with compact support) $\int_{\mathbb{R}^d} K(x, y) \rho(x) \rho(y) dx dy \geq 0$. Given a centered Gaussian field $X$ with covariance $K$, the critical Gaussian Multiplicative Chaos (or critical GMC) is heuristically defined as

$$e^{\alpha X(x)} dx \quad \text{with} \quad \alpha = \sqrt{2d}. \tag{1.2}$$

The main difficulty that comes up when trying to give an interpretation to the expression in (1.2) is that a field with a covariance of the type (1.1) can be defined only as a random distribution and thus $X$ is not defined pointwise. The problem of rigorously defining the GMC was first considered by Kahane in [14]. The standard procedure (for any $\alpha \geq 0$) is to use a sequence of approximation of $X$ and then pass to the limit. Mostly two kinds of approximation of $X$ have been considered in the literature:

(A) A mollification of the field, $X_\varepsilon$, via convolution with a smooth kernel on scale $\varepsilon$,

(B) A martingale approximation, $X_t$, via an integral decomposition of the kernel $K$.

Rigorous definitions of $X_\varepsilon$ and $X_t$ are given in the next subsections. We define $M^{\alpha}_\varepsilon$ by

$$M^{\alpha}_\varepsilon(f) := \int_{\mathbb{R}^d} f(x) e^{\alpha X_\varepsilon(x)} - \frac{\alpha^2}{2} \overline{\mathbb{E}[X_\varepsilon(x)]} dx \tag{1.3}$$

and $M^{\alpha}_t$ in the same manner using the martingale approximation $X_t$. Then we let either $\varepsilon \to 0$ or $t \to \infty$. When $\alpha \in [0, \sqrt{2d})$, it has been proved [5, 20] that both $M^\alpha_t$ and $M^\alpha_\varepsilon$ converge to a nontrivial limiting measure $M^\alpha$, which does not depend on the mollifier. We refer the reader to the introduction of [5] or to the review [18] for an account of the steps leading to this result. We also refer to [18, Section 5] for motivations to study GMC coming from various fields in including theoretical physics.

When $\alpha = \sqrt{2d}$, the limiting behavior is different. We have $\lim_{\varepsilon \to 0} M^{\sqrt{2d}}_\varepsilon(f) = 0$ for any bounded measurable $f$ and a renormalization procedure is required to obtain a nontrivial
limit. More precisely it has been proved that $\sqrt{\pi \log(1/\varepsilon)/2M_\varepsilon^{2d}}$ converges to a nontrivial limit that does not depend on the mollifier. This was achieved in three steps, starting with the martingale approximation $X_t$:

(i) In [6], the convergence of the random (signed) measure

$$D_t(dx) := (\sqrt{2d}\mathbb{E}[X_t(x)] - X_t)e^{\sqrt{2d}X_t(x) - d\mathbb{E}[X_t(x)]}dx,$$ \hspace{1cm} (1.4)

$$\varepsilon$$

to a nontrivial nonnegative measure $D_\infty$ was established.

(ii) In [7], it was proved that $\sqrt{\pi t/2M_\varepsilon^{2d}}$ converges to the same limit.

(iii) The convergence of $\sqrt{\pi \log(1/\varepsilon)/2M_\varepsilon^{2d}}$ towards $D_\infty$ was shown in [12].

Additionally two other convergence results have been proved:

(iv) In [4, 17], the convergence of $(\sqrt{2d} - \alpha)M_\alpha$ towards $2D_\infty$ when $\alpha \uparrow \sqrt{2d}$.

(v) In [16], the convergence of $D_\varepsilon$ (defined as the mollified analogue of $D_t$) to $D_\infty$.

We refer to [17] for a thorough review of these results. The objective of this paper is to present new simple and short proofs of statements (i) to (iv) above with a presentation as self-contained as possible. We believe that the technique we present can also be adapted to provide an alternative proof of (v). Along the way we present proof for an asymptotic upper bound for log-correlated fields (Proposition 2.4) which is much shorter that what has appeared so far in the literature.

While this manuscript is focused on the Gaussian case, let us mention that analogous random measures have been obtained considering the exponential of nongaussian log-correlated fields [9, 10, 11]. In particular [10] establishes results which are very similar to those mentioned above for the exponentiation of the square root of the local time of a planar Brownian Motion at the critical parameter.

1.1. Mollification of the field and limit.

Log-correlated fields defined as distributions. Since $K$ is infinite on the diagonal, it is not possible to define a Gaussian field indexed by $\mathbb{R}^d$ with covariance function $K$. We consider instead a process indexed by test functions. We define $\tilde{K}$, a bilinear form on $C_c(\mathbb{R}^d)$, by

$$\tilde{K}(\rho, \rho') = \int_{\mathbb{R}^d} K(x, y)\rho(x)\rho'(y)dxdy.$$ \hspace{1cm} (1.5)

Since $\tilde{K}$ is positive definite, we can consider $X = \langle X, \rho \rangle_{\rho \in C_c(\mathbb{R}^d)}$ a centered Gaussian field indexed by $C_c(\mathbb{R}^d)$ with covariance kernel given by $\tilde{K}$.

Convolution approximation. The random distribution $X$ can be approximated by a sequence of functional fields - fields indexed by $\mathbb{R}^d$ - by the mean of convolution with a smooth kernel. Consider $\theta$ a nonnegative function in $C_c^\infty(\mathbb{R}^d)$ whose compact support is included in $B(0,1)$ (for the remainder of the paper $B(x,r)$ denotes the closed Euclidean ball of center $x$ and radius $r$) and which satisfies $\int_{B(0,1)} \theta(x)dx = 1$. We define for $\varepsilon > 0$, $\theta_\varepsilon := \varepsilon^{-d}\theta(\varepsilon^{-1} \cdot)$, set and consider $(X_\varepsilon(x))_{x \in \mathbb{R}^d}$, the mollified version of $X$, that is

$$X_\varepsilon(x) := \langle X, \theta_\varepsilon(x - \cdot) \rangle$$ \hspace{1cm} (1.6)

From (1.5), the field $X_\varepsilon(\cdot)$ has covariance

$$K_\varepsilon(x,y) := \mathbb{E}[X_\varepsilon(x)X_\varepsilon(y)] = \int_{\mathbb{R}^{2d}} \theta_\varepsilon(x - z_1)\theta_\varepsilon(y - z_2)K(z_1, z_2)dz_1dz_2.$$ \hspace{1cm} (1.7)
We set $K_\varepsilon(x) := K_\varepsilon(x, x)$ and use a similar convention for other covariance functions. Since $K_\varepsilon$ is infinitely differentiable, by Kolmogorov’s Continuity Theorem (e.g. [13, Theorem 2.9]) there exists a continuous modification of $X_\varepsilon(\cdot)$. In the remainder of the paper, we always consider the continuous modification of a process when it exists, this ensures that integrals such as the one appearing in (1.9) are well defined. We let $B_b$ denote the bounded Borel subsets of $\mathbb{R}^d$ and $B_b$ denote the bounded Borel functions with bounded support

$$B_b = B_b(\mathbb{R}^d) = \left\{ f \text{ measurable} : \sup_{x \in \mathbb{R}^d} |f(x)| < \infty, \{ x : |f(x)| \neq 0 \} \in B_b \right\}. \quad (1.8)$$

We define the measure $M_\varepsilon^\alpha$ by setting for $f \in B_b$ (recall (1.3))

$$M_\varepsilon^\alpha(f) := \int_{\mathbb{R}^d} f(x) e^{\alpha X_\varepsilon(x) - \frac{\alpha^2}{2} K_\varepsilon(x)} \, dx. \quad (1.9)$$

We set $M_\varepsilon^\alpha(E) := M_\varepsilon^\alpha(1_E)$ for $E \in B_b$ and keep a similar convention for other measures. When considering the the value $\alpha = \sqrt{2d}$ (which is most of the time) we drop the dependence in $\alpha$ in the notation, we set $M_\varepsilon = M_\varepsilon^{\sqrt{2d}}$. Before stating the main results, we need to mention an important assumption that need to make on $K$.

1.2. Star-scale invariance and our assumption on $K$. We assume, in most of the paper, that the kernel $K$ has an almost star-scale invariant part. This assumption might seem at first quite restrictive, but it has been shown in [13] that it is locally satisfied as soon as $L$ is sufficiently regular. We explain in Appendix C how this allows to extend the results to log-correlated field on arbitrary domains, provided that $L$ satisfies the required regularity assumption. Following a terminology introduced in [13], we say that a the kernel $K$ defined on $\mathbb{R}^d$ is almost star-scale invariant if it can be written in the form

$$\forall x, y \in \mathbb{R}^d, K(x, y) = \int_0^\infty (1 - \eta_1 e^{-\eta_2 t}) \kappa(e^{t}(x - y)) \, dt, \quad (1.10)$$

where $\eta_1 \in [0, 1]$ and $\eta_2 > 0$ are constants and the function $\kappa \in C_{c,0}(\mathbb{R}^d)$ is radial, nonnegative and definite positive. More precisely we assume the following:

(i) $\kappa \in C_{c,0}(\mathbb{R}^d)$ and there exists $\hat{\kappa} : \mathbb{R}^+ \to [0, \infty]$ such that $\kappa(x) := \hat{\kappa}(|x|),$

(ii) $\hat{\kappa}(0) = 1$ and $\hat{\kappa}(r) = 0$ for $r \geq 1$,

(iii) The mapping $(x, y) \mapsto \kappa(x - y)$ defines a positive definite-kernel on $\mathbb{R}^d \times \mathbb{R}^d$.

We say furthermore that a kernel $K$ has an almost star-scale invariant part, if

$$\forall x, y \in \mathbb{R}^d, K(x, y) = K_0(x, y) + \overline{K}(x, y) \quad (1.11)$$

where $\overline{K}(x, y)$ is an almost star-scale invariant kernel and $K_0$ is Hölder continuous on $\mathbb{R}^{2d}$ and positive definite. Given $K$ with an almost star-scale invariant part, and using the decomposition (1.10) for $\overline{K}$, we set

$$Q_t(x, y) := \kappa(e^{t'}(x - y)) \quad (1.12)$$

where $t'$ is defined as the unique positive solution of $t' - \frac{\eta_2}{\eta_1} (1 - e^{-\eta_2 t'}) = t$. We set

$$K_t(x, y) := K_0(x, y) + \int_0^t Q_s(x, y) \, ds \quad (1.13)$$

$$= K_0(x, y) + \int_0^t (1 - \eta_1 e^{-\eta_2 s}) \kappa(e^{s}(x - y)) \, ds =: K_0(x, y) + \overline{K}_t(x, y).$$
Note that we have \( \lim_{t \to \infty} K_t(x, y) = K(x, y) \) and \( \overline{\mathcal{T}}_t(x) = t. \) If \( K \) satisfies (1.10) then
\[
L(x, y) := K(x, y) + \log |x - y|,
\]
(1.14)
can be extended to a continuous function on \( \mathbb{R}^{2d} \), so that a kernel \( K \) with an almost star-scale invariant part can always be written in the form (1.1).

**Remark 1.1.** There is an obvious conflict of notation between \( K_t \) introduced above and \( K_\varepsilon \) introduced in (1.7) and the same can be said about \( X_t \) and \( M_t \) introduced in the next section. However this abuse should not cause any confusion since we keep using the letter \( \varepsilon \) for quantities related to the mollified field \( X_\varepsilon \) and latin letters for quantities related to the martingale approximation \( X_t \).

### 1.3. Convergence of the convolution approximation.

The main result whose proof we wish to present in this paper is the convergence of the measure \( M_\varepsilon \) - properly rescaled - towards a limit \( M' \). The principal task is to show that \( M_\varepsilon \) converges for any fixed \( E \).

**Theorem 1.2.** If \( X \) is a centered Gaussian field whose covariance kernel \( K \) has an almost star-scale invariant part and \( E \subset B_b \) is of positive Lebesgue measure then there exists an a.s. positive random variable \( M' \) such that the following convergence holds in probability
\[
\lim_{\varepsilon \to 0} \sqrt{\frac{\pi \log(1/\varepsilon)}{2}} M_\varepsilon(E) = M'(E).
\]
(1.15)
The limit does not depend on the specific mollifier \( \theta \) used to define \( X_\varepsilon \).

In order to extend the statement to a convergence of measure, we need to specify a topology. We say that a sequence of locally finite measures \( (\mu_n) \) converges weakly to \( \mu \) if
\[
\forall f \in C_c(\mathbb{R}^d), \quad \lim_{n \to \infty} \int_{\mathbb{R}^d} f(x) \mu_n(dx) = \int_{\mathbb{R}^d} f(x) d\mu(dx).
\]
(1.16)
The topology of weak convergence for locally finite nonnegative measures is metrizable and separable, hence we can associate to it a notion of convergence in probability for a sequence of random measure. An example of metric generating the topology is given in Appendix B. For \( E \subset B_b \) we let \(|E| := \max_{x,y \in E} |x - y|\) denote the diameter of \( E \). We introduce the gauge function \( \phi \) defined on \([0, \infty)\) by
\[
\phi(u) = \begin{cases} 
\log \log(1/u)[\log(1/u)]^{-1/4} & \text{ if } u \leq e^{-e}, \\
e^{-1/4} & \text{ if } u \geq e^{-e}.
\end{cases}
\]
(1.17)

**Theorem 1.3.** There exists a locally finite random measure \( M' \) with dense support such that
\[
\lim_{\varepsilon \to 0} \sqrt{\frac{\pi \log(1/\varepsilon)}{2}} M_\varepsilon = M',
\]
weakly in probability. Furthermore \( M' \) has no atoms and we have any fixed \( R > 0 \)
\[
\sup_{E \subset B(0,R)} M'(E)/\phi(|E|) < \infty \quad \mathbb{P} - \text{a.s.}
\]
(1.18)

**Remark 1.4.** Most of our efforts are dedicated to the proof of Theorem 1.2 and of (1.18). The (standard) task of showing that the convergence of \( M_\varepsilon(E) \) for every \( E \) implies weak convergence of measures is performed in Appendix B for completeness. The fact that \( M' \) has dense support is a consequence of the a.s. positivity of \( M'(E) \) in Theorem 1.2.
Remark 1.5. Theorem 1.3 extends Theorem 1.3 to all log-correlated fields defined on an open domain $\mathcal{D}$ provided that the function $L$ in (1.1) has sufficient regularity. This condition on $L$ is satisfied for most if not all known applications of Gaussian Multiplicative Chaos [1, Section 5], and in particular by planar free fields considered in [1, 13, 8]. Without any modification to the proof, it is also possible to extend our result to chaos defined with a reference measure which is not Lebesgue measure, provided that it is “sufficiently spread out”, see the discussion in Section 1.7.

1.4. The martingale decomposition of $X$. Given $K$ a kernel on $\mathbb{R}^d$ with an almost star-scale invariant part, we define $(X_t(x))_{x \in \mathbb{R}^d, t \geq 0}$ to be a centered Gaussian field with covariance given by (using the notation $D$ to denote the natural filtration associated with $X(\cdot)$). The process $X$ indexed by $C_c(\mathbb{R}^d)$ and defined by $\langle X, \rho \rangle = \lim_{t \to \infty} \int_{\mathbb{R}^d} X_t(x) \rho(x) \, dx$, is a centered Gaussian field, so that $X_t$ is an approximation sequence for a log-correlated field with covariance $K$. We define the measure $M_t^\alpha$ by setting for $f \in B_b$ (recall (1.8))

$$M_t^\alpha(f) := \int_{\mathbb{R}^d} f(x) e^{\alpha X_t(x) - \frac{\alpha^2}{2} K_t(x)} \, dx,$$

and set $M_t = M_t^{\sqrt{2d}}$. By independence of the increments of $X$ (see Section 2.2), $M_t(f)$ is an $(\mathcal{F}_t)$-martingale. We also consider the derivative martingale $D_t$, defined by

$$D_t(f) = \int_{\mathbb{R}^d} f(x) \left( \sqrt{2d} K_t(x) - X_t(x) \right) e^{\sqrt{2d} X_t(x) - dK_t(x)} \, dx. \quad (1.21)$$

We have $D_t(f) := -\partial_\alpha M_t^\alpha(f)|_{\alpha = \sqrt{2d}}$ (hence the name). To prove Theorem 1.2 we first establish the convergence of $D_t$. We let $\lambda$ denote the Lebesgue measure and define

$$\overline{D}_x(f) := \limsup_{t \to \infty} D_t(f). \quad (1.22)$$

The following result, that is, the convergence of $D_t$ to a non-trivial limit, is also the first step to prove Theorem 1.2.

Theorem 1.6. For any fixed $E \in B_b$ we have almost surely

$$\overline{D}_x(E) = \lim_{t \to \infty} D_t(E) \in [0, \infty), \quad (1.23)$$

If $\lambda(E) > 0$, then $\mathbb{P}[\overline{D}_x(E) > 0] = 1$. Recalling (1.17), for every $R > 0$ we have

$$\sup_{E \in B(0, R)} \overline{D}_x(E)/\phi(|E|) < \infty. \quad (1.24)$$

Applying Proposition 1.1 to the positive part $D_t$ (it coincides with $D_t$ for large $t$ by Proposition 2.4) we can deduce from the above the weak convergence of the measure $D_t$.

Theorem 1.7. There exists a measure $D_\infty$ such that almost surely, $D_t$ converges weakly to $D_\infty$. Furthermore we have for every $f \in B_b$,

$$\mathbb{P}[\overline{D}_x(f) = D_\infty(f)] = 1. \quad \text{The measure } D_\infty \text{ has dense support and satisfies } (1.24) \text{ (in particular it has no atoms).}$$

We also prove that $M_t$, when appropriately rescaled, converges to the same limit.
Theorem 1.8. For any fixed $E \in \mathcal{B}_b$ we have the following convergence in probability,

$$\lim_{t \to \infty} \sqrt{\pi t} M_t(E) = D_{\infty}(E).$$

(1.25)

Furthermore $\sqrt{\pi t} M_t$ converges weakly in probability to $D_{\infty}$.

1.5. **Convergence of subcritical chaos when $\alpha$ tends to $\sqrt{2d}$.** The last result we wish to present is the convergence of subcritical chaos, when properly renormalized, towards the same limit $D_{\infty}$ when $\alpha \uparrow \sqrt{2d}$. Let us recall the definition of the subcritical chaos $M^\alpha$.

**Theorem A** ([5, Theorem 1]). Let $X$ be a log-correlated field (with covariance $K$ of the form (1.1)) and $M^\alpha_e$ be defined by (1.9). There exists a random measure $M^\alpha$ such that for every $E \in \mathcal{B}_b$ the following convergence holds in $L^1$ (for any choice of mollifier)

$$\lim_{\varepsilon \to 0} M^\alpha_{\varepsilon}(E) = M^\alpha(E).$$

(1.26)

If $K$ is an almost-star scale invariant kernel, and $X_t$ is the martingale approximation of $X$ then for every $E \in \mathcal{B}_b$, the following convergence holds in $L^1$ and almost surely

$$\lim_{t \to \infty} M^\alpha_t(E) = M^\alpha(E).$$

(1.27)

We can now formulate our result. In the statement below $M^\alpha$ and $M'$ are the limiting measures obtained in Theorems A and 1.3 respectively. The notation $\alpha \uparrow \sqrt{2d}$ means that the limit is taken with $\alpha < \sqrt{2d}$.

**Theorem 1.9.** If $K$ is an almost-star scale invariant kernel and $X$ is a field with covariance $K$, then for every $E \in \mathcal{B}_b$ we have the following convergence in probability

$$\lim_{\alpha \uparrow \sqrt{2d}} \frac{M^\alpha(E)}{\sqrt{2d - \alpha}} = 2M'(E).$$

(1.28)

1.6. **Review of the literature and organization of the paper.** The results presented above are not new. Theorems 1.6-1.7 correspond to [6, Theorem 4] Theorem 1.8 corresponds to [7, Theorem 5], Theorems 1.2-1.3 can be obtained by combining this last result with [12, Theorem 1.1 and Theorem 4.4] and Theorem 1.9 corresponds to [17, Theorem 3.1]. The aim of this paper is to expose shorter and simpler proofs, with a presentation as self-contained and elementary as possible.

In Section 2 we present a variety of technical results which are needed for our proofs. This includes standard probability textbook results, as well as technical estimates which are specific to our problem. The proof of the latter are displayed in Appendix A Sections 3-6. Sections 3 and 6 respectively display the proofs of Theorems 1.6-1.2 and 1.1-1.3. Appendix B explains how the convergence result for the measure of a fixed Borel set implies the weak convergence of the measure. In Appendix C we replicate an argument from [13] to regular log-correlated fields on arbitrary domains.

Our starting point to prove the convergence of $D_t$ and $M_t$ is the same as in [6, 7], we consider truncated versions $D^{(q)}_t$ and $M^{(q)}_t$ of $D_t$ and $M_t$ (introduced in Section 2.3) which ignore the contribution of $X$ with atypically high values. We extend this approach, defining also a truncated version $M^{(q)}_e$ for $M_e$. Proposition 2.4 justifies this procedure by showing that for large $q$, with high probability, $M^{(q)}_t$ coincides with $M_t$. We present in Appendix A.1 a proof of Proposition 2.4. It relies on standard Gaussian tools and classic ideas.
developed for the study of the maximum of log-correlated fields and branching random walk see for e.g. [1, 3], but is much shorter than what has appeared so far in the literature.

The next steps are based on a couple of novel ideas. The key point is Lemma 3.2 which relates $M_t^{(q)}$ to $D_t^{(q)}$ using conditional expectation. First, this allows to transfer second moment estimates from $M_t^{(q)}$ to $D_t^{(q)}$, to show that $D_t^{(q)}$ is bounded in $L^2$ (see the proof of Proposition 3.1 in Section 3) and hence that $D_t$ converges. More importantly, with very little additional efforts, we can use this show to that $\pi t^{\frac{1}{2}}M_t^{(q)}$ is close to $D_t^{(q)}$ in $L^2$ for large values of $t$ and hence that both sequences converge in probability to the same limit, see Section 4. In Section 5 and 6 using an analogue strategy (using Lemma 5.2/6.2 instead of Lemma 3.2), we prove that $M_\varepsilon$ and $M^\alpha$ (appropriately normalized) converge to $D_X$.

1.7. Critical chaos with a different reference measure. In the definition of $M_\varepsilon$, $M_t$ and $D_t$ (1.9), (1.20) and (1.21), it is possible replace the Lebesgue measure on $\mathbb{R}^d$ with an arbitrary locally finite measure $\mu$ by setting

$$M_\varepsilon := \int_{\mathbb{R}^d} f(x) e^{\sqrt{2d}X_\varepsilon(x) - dK_\varepsilon(x)} \mu(dx).$$

(and similarly for $M_t$ and $D_t$), and ask whether the sequences $\sqrt{\pi \log(1/\varepsilon)/2}M_\varepsilon$, $\sqrt{\pi t/2}M_t$ and $D_t$ converge weakly in that setup. It turns out that the method we exposed below can give an answer to the question slightly beyond the case of Lebesgue measure. More precisely, all our proofs (except that of (1.18) where one needs to modify the gage function $\varphi$) extend without modification to the case where $\mu$ is a nonnegative measure that satisfies

$$\forall x \in \mathbb{R}^d, \int_{B(x,\varepsilon^{-2})} \left( \frac{\log \log \frac{1}{|x-y|}}{|x-y|^d} \right) \mu(dx) \mu(dy) < \infty,$$

with $\gamma \in (0, 1/2)$. The above condition originates from Proposition 2.4, more precisely it is required to deduce (2.26) from (2.21)- (2.24) (the necessity for the log log term in the numerator comes from the second term in (A.27)).

Not that while (1.30) allows to consider measures that are singular w.r.t. to Lebesgue measure, it still requires that the measure to be in a certain sense almost as “spread out” as the Lebesgue measure. It is worth noting for instance that (1.30) is not satisfied if $d = 2$ and $\mu = \mu_T$ is taken to be the occupation measure of a two dimensional Brownian Motion $(B_t)_{t<0,T}$. Defining the critical GMC over this occupation measure is the first step of the construction of Liouville Brownian Motion at criticality [19].

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2. Technical preliminaries

2.1. Gaussian tools. Let us recall here a couple of basic results concerning Gaussian processes. Firstly, the Cameron-Martin formula, that indicates how the distribution of a Gaussian field is affected by an exponential tilt.
Proposition 2.1. Let \((Y(z))_{z \in \mathcal{Z}}\) be a centered Gaussian field indexed by a set \(\mathcal{Z}\). We let \(H\) denote its covariance and \(P\) denote its law. Given \(z_0 \in \mathcal{Z}\) let us define \(\widehat{P}_{z_0}\) the probability obtained from \(P\) after a tilt by \(Y(z_0)\) that is
\[
\frac{d\widehat{P}_{z_0}}{dP} := e^{Y(z_0) - \frac{1}{2} H(z_0, z_0)}
\]  
(2.1)

Under \(\widehat{P}_{z_0}\), \(Y\) is a Gaussian field with covariance \(H\), and mean \(E_{z_0}[Y(z)] = H(z, z_0)\).

Secondly, bounds on the probability for a Brownian Motion to remain below a threshold. The first two estimates are direct consequences of the reflection principle, the third one is obtained by applying the optional stopping Theorem to the martingale \(e^{2abB_t - 2at}\).

Lemma 2.2. Let \(B\) be a standard Brownian Motion and let \(P\) denote its distribution. Setting \(g_t(a) := \int_0^a e^{-\frac{z^2}{2t}} \, dz\), we have
\[
P\left[ \sup_{s \in [0, t]} B_s \leq a \right] = \sqrt{\frac{2\pi t}{2\pi}} g_t(a) \leq \sqrt{\frac{2\pi t}{2\pi}} a.
\]  
(2.2)

If \(\text{P}_a[\cdot | B_t = b]\) denote the distribution of the Brownian bridge of length \(t\) starting from \(a\) and ending at \(b\) with \(ab \geq 0\) then
\[
\text{P}_a[\forall s \in [0, t], B_s \geq 0 \mid B_t = b] = \left(1 - e^{-\frac{2ab}{t}}\right) \leq 1 - \frac{2ab}{t}.
\]  
(2.3)

If \(a, b > 0\) then,
\[
P[\forall t > 0, B_t \leq at + b] = 1 - e^{-2ab} \leq 2ab.
\]  
(2.4)

2.2. Simple observations concerning our Gaussian fields. We compile here a list of useful facts concerning the covariance and dependence structure of \(X_t\) and \(X_\varepsilon\). We introduce a third field which appears when considering conditional expectations of \(M_\varepsilon\). We let \(X_{t, \varepsilon}\) denote the mollification of \(X_t\)
\[
X_{t, \varepsilon}(y) = \int_{\mathbb{R}^d} X_t(y) \theta_\varepsilon(x - y) \, dy = \mathbb{E}[X_\varepsilon \mid \mathcal{F}_t].
\]  
(2.5)

By construction the increments of \(X_t\) are orthogonal in \(L^2\) and hence independent. Setting
\[
X^{(s)}_t(x) = X_{t+s}(x) - X_s(x) \quad \text{and} \quad \overline{X}_t(x) = X^{(0)}_t(x),
\]  
(2.6)

the field \((\overline{X}(x))_{x \in \mathbb{R}^d, t \geq 0}\) has covariance \(\overline{K}_{s, t}(x, y)\) (recall (1.13)). In particular, since \(\overline{K}_t(x, x) = t\), this implies that for every fixed \(x \in \mathbb{R}^d\) and \(s \geq 0\), \(t \mapsto X^{(s)}_t(x)\) is a standard Brownian Motion independent of \(\mathcal{F}_s\).

Furthermore, recalling (1.12), we necessarily have \(t' \in [t, t + \eta_1/\eta_2]\). Our assumption that \(\kappa\) is supported on \(B(0, 1)\) implies that \(Q_u(x, y) = 0\) if \(|x - y| \geq e^{-u}\). In particular if \(|x - y| \geq e^{-s}\), \(X^{(s)}(x)\) and \(X^{(s)}(y)\) are independent Brownian Motions. The increments of the convoluted fields also have a finite range dependence for the same reason and \((X_\varepsilon - X_{s, \varepsilon})(x)\) is independent of \((X_\varepsilon - X_{s, \varepsilon})(y)\) and of \(X^{(s)}(y)\) when \(|x - y| \geq e^{-s} + 2\varepsilon\).

We let \(K_{t, \varepsilon}\) and \(\overline{K}_{t, \varepsilon, 0}\) denote the covariance of \(X_{t, \varepsilon}\) and the cross-covariance with \(\overline{X}_t\),
\[
K_{t, \varepsilon}(x, y) := \mathbb{E}[X_{t, \varepsilon}(x)X_{t, \varepsilon}(y)] = \int_{\mathbb{R}^d} K_t(z_1, z_2) \theta_\varepsilon(x - z_1) \theta_\varepsilon(x - z_1) \, dz_1 dz_2,
\]
(2.7)

\[
\overline{K}_{t, \varepsilon, 0}(x, y) := \mathbb{E}[X_{t, \varepsilon}(x)\overline{X}_t(y)] = \mathbb{E}[X_\varepsilon(x)\overline{X}_t(y)] = \int_{\mathbb{R}^d} K_t(z, y) \theta_\varepsilon(x - z) \, dz.
\]
Setting \( \log_+ u = \max(\log u, 0) \), there exists a constant \( C_R \) which is such that

\[
\left| K_{t, \varepsilon}(x, y) - t \wedge \log_+ \frac{1}{|x - y| \vee \varepsilon} \right| \leq C_R
\]  
(2.8)

The same bound are valid for \( K_t, \bar{K}_t \) and \( \bar{K}_{t, \varepsilon, 0} \), (with \( \varepsilon = 0 \) in the two first cases). For the upper bound on \( \bar{K}_t \) no constant is needed and we have

\[
\bar{K}_t(x, y) \leq t \wedge \log_+ \frac{1}{|x - y|}.
\]  
(2.9)

The two bounds (2.8)-(2.9) are easily obtained from the definition (2.7) and (1.12). We include a proof in Appendix A.3 for completeness.

2.3. Truncation of \( M_t, M_\varepsilon \) and \( D_t \). We introduce here truncated versions of our processes that are the starting point of all our proofs. Given \( q > 0 \), and \( R > 0 \), we set

\[
A^{(q)}_t(x) := \{ \forall s \in [0, t], \bar{X}_s(x) < \sqrt{2}ds + q \},
\]

\[
A^{(q)}_R := \{ \sup_{t > 0} \sup_{|x| \leq R} \left( \bar{X}_t(x) - \sqrt{2}dt \right) < q \} = \bigcap_{|x| \leq R} \bigcap_{t > 0} A^{(q)}_t(x).
\]  
(2.10)

Setting \( t_\varepsilon := (\log 1/\varepsilon) \), we define \( M^{\alpha, (q)}_t, M^{\varepsilon, (q)}_t \) and \( D^{(q)}_t \) as follows

\[
M^{\alpha, (q)}_t(E) := \int_E e^{\alpha X_t(x) - \frac{d}{2}K_t(x)} 1_{A^{(q)}_t(x)} \, dx =: \int_E W^{\alpha, (q)}_t(x) dx,
\]

\[
M^{(q)}_t(E) := \int_E e^{\sqrt{2d}X_t(x) - dK_t(x)} 1_{A^{(q)}_t(x)} \, dx =: \int_E W^{(q)}_t(x) dx,
\]

\[
M^{\varepsilon, (q)}_t(E) := \int_E e^{\sqrt{2d}X_t(x) - dK_t(x)} 1_{A^{(q)}_t(x)} \, dx =: \int_E W^{\varepsilon, (q)}_t(x) dx,
\]

\[
D^{(q)}_t(E) := \int_E (\sqrt{2d}t + q - \bar{X}_t(x)) W^{(q)}_t(x) dx =: \int_E Z^{(q)}_t(x) dx.
\]  
(2.11)

We further set

\[
\overline{D}^{(q)}_\infty(E) := \limsup_{t \to \infty} D^{(q)}_t(E) \quad \text{and} \quad \overline{M}^{\alpha, (q)}_\infty(E) := \limsup_{t \to \infty} M^{\alpha, (q)}_t(E).
\]  
(2.12)

It is not difficult to check that \( D^{(q)}_t(E) \) is a nonnegative martingale and \( M^{\alpha, (q)}_t \) a nonnegative supermartingale. Hence almost surely the lim sups in (2.12) are limits. Let us register these observations as a lemma (proved at the end of this section for completeness).

**Lemma 2.3.** For any fixed \( E \in \mathcal{B}_0 \), \( D^{(q)}_t(E) \) is a nonnegative martingale for the filtration \( (\mathcal{F}_t) \). In particular we have almost-surely

\[
\overline{D}^{(q)}_\infty(E) = \lim_{t \to \infty} D^{(q)}_t(E) < \infty.
\]

For any fixed \( E \in \mathcal{B}_0, \alpha \geq 0, M^{\alpha, (q)}_t(E) \) is a nonnegative supermartingale for the filtration \( (\mathcal{F}_t) \). When \( \alpha < \sqrt{2d} \), we have \( \lim_{t \to \infty} M^{\alpha, (q)}_t(E) = \overline{M}^{\alpha, (q)}_\infty(E) \) in \( L^1 \).

A crucial result that validates the truncation approach is that for large values \( q \) the event \( A^{(q)}_R \) has large probability. We provide a proof in Appendix A.1.
Proposition 2.4. For any \( R > 0 \) we have almost surely
\[
\sup_{t \to 0} \sup_{|x| \leq R} \left( X_t - \sqrt{2d} t + \frac{\log t}{2\sqrt{2d}} - \frac{4\log \log t}{\sqrt{2d}} \right) < \infty. \tag{2.13}
\]
In particular we have \( \lim_{q \to \infty} P \left[ A_R^{(q)} \right] = 1 \). As a consequence given \( R \) there exists a random \( q_0 \in \mathbb{N} \) such that for every \( q \geq q_0 \), \( \alpha \in (0, \sqrt{2d}] \), \( t > 0 \), \( \varepsilon \in (0, 1] \) and \( E \subset B(0, R) \)
\[
M_t^{\alpha, (q)}(E) = M_t^\alpha(E) \quad \text{and} \quad M_t^{(q)}(E) = M_t^\varepsilon(E). \tag{2.14}
\]
Let us present a couple of easy consequences of Proposition 2.4.

Corollary 2.5. We for any \( E \in \mathcal{B}_0 \) we have almost surely
\[
\lim_{t \to \infty} M_t(E) = 0. \tag{2.15}
\]
Given \( R > 0 \), there exists a random \( q_0 \in \mathbb{N} \) such that a.s. for every \( q \geq q_0 \),
\[
\lim_{t \to \infty} \sup_{E \subset R} |D_t(E) - D_t^{(q)}(E)| = 0. \tag{2.16}
\]

Proof of Corollary 2.5. \( M_t(E) \) is a nonnegative martingale and hence it converges almost-surely. It suffices thus to show that (2.15) holds in probability. We prove that
\[
\lim_{t \to \infty} \mathbb{E} \left[ M_t(E) \mathbf{1}_{A_R^{(q)}} \right] = 0. \tag{2.17}
\]
Using Proposition 2.4, we see that (2.17) implies convergence in probability. We have
\[
\mathbb{E} \left[ M_t(E) \mathbf{1}_{A_R^{(q)}} \right] \leq \mathbb{E} \left[ M_t^{(q)}(E) \right] = \int_E \mathbb{E} \left[ e^{\sqrt{2d} X_t(x) - dK_t(x)} \mathbf{1}_{A_t^{(q)}(x)} \right] dx. \tag{2.18}
\]
Using Cameron-Martin formula (Proposition 2.1), we see that the exponential tilt shifts \( \overline{X}_s(x) \) by an amount \( \sqrt{2d} ds \) and thus using Lemma 2.2 we have
\[
\mathbb{E} \left[ e^{\sqrt{2d} X_t(x) - dK_t(x)} \mathbf{1}_{A_t^{(q)}(x)} \right] = \mathbb{P} \left[ \sup_{s \in [0, t]} \overline{X}_s(x) \leq q \right] \leq \sqrt{\frac{2}{\pi t}} q. \tag{2.19}
\]
Since \( E \) has finite Lebesgue measure, this allows to conclude the proof of (2.15). Next, using Proposition 2.4 there exists \( q_0 \in \mathbb{N} \) such that \( A_R^q \) holds for \( q \geq q_0 \), and hence
\[
|D_t^{(q)}(E) - D_t(E)| = \left| \int_E (q - X_0(x)) e^{\sqrt{2d} X_t(x) - dK_t(x)} dx \right|
\leq \left( q + \sup_{|x| \leq R} |X_0(x)| \right) M_t(B(0, R)). \tag{2.20}
\]
Since \( X_0 \) is continuous the prefactor is finite and hence (2.16) follows from (2.15) \( \square \)

Proof of Lemma 2.3. It is sufficient to prove that \( Z_t^{(q)}(x) \) is a martingale for any \( x \). Indeed since \( \mathbb{E}[Z_t^{(q)}(x)] = q \) and \( E \) is bounded, integrability conditions allow for the exchange of integral and conditional expectation. We obtain that for \( s < t \)
\[
\mathbb{E} \left[ \int_E Z_t^{(q)}(x) dx \mid \mathcal{F}_s \right] = \int_E \mathbb{E}[Z_t^{(q)}(x) \mid \mathcal{F}_s] dx = \int_E Z_s^{(q)}(x) dx.
\]
Since \( Z_0^{(q)}(x) = e^{\sqrt{2d}X_0(x) - dK_0(x)} \) is integrable and independent of \( Z_t^{(q)}(x) := Z_t^{(q)}(x)/Z_0^{(q)}(x) \), it is sufficient to show that \( Z_t^{(q)}(x) \) is a martingale. Since \((X_t(x))_{t \geq 0}\) is an \( \mathcal{F}_t \)-Brownian motion, we only have to prove that if \((B_t)_{t \geq 0}\) is a Brownian motion then

\[
N_t^{(q)} := (\sqrt{2d}t + q - B_t)e^{\sqrt{2d}B_t - dt}1_{\{y \in [0,t], B_s \leq \sqrt{2d}s + q\}}
\]

is a martingale for its natural filtration. The reader can check, using Itô formula, that \( N_t^{(q)} \) is a local martingale and using a direct computation that \( N_t^{(q)} \in L^2 \) for every \( t > 0 \).

To show that \( M_t^{\alpha,(q)} \) is a supermartingale, we proceed similarly and observe that

\[
Q_t^{\alpha,(q)} := e^{\sqrt{2d}B_t - dt}1_{\{y \in [0,t], B_s \leq \sqrt{2d}s + q\}}
\]

is a supermartingale. When \( \alpha \in [0,\sqrt{2d}] \), since \( M_t^{\alpha,(q)}(E) \leq M_t^{\alpha}(E) \) and \( M_t^{\alpha}(E) \) is uniformly integrable (by Theorem A), so is \( M_t^{\alpha,(q)} \). Hence it converges in \( L_1 \).

\( \square \)

2.4. Moment estimates. Lemma 2.3 ensures the convergence of \( D_t^{(q)}(E) \), but we still need to prove uniform integrability to show that the limit is nontrivial. Our strategy is to compute the second moments. This is the purpose of the following technical estimates. Their proof is not conceptually difficult, but requires a couple of lengthy computation. We postpone it to Appendix A.1 for this reason.

**Proposition 2.6.** Given \( x, y \in \mathbb{R}^d \), \( t \geq 0 \) and \( \varepsilon \in (0,1) \), we set

\[
w(x, y) := \log \frac{1}{|x - y|} \wedge 1, \quad u(x, y, t) := w \wedge t, \quad v(\varepsilon, x, y) := w \wedge \log(1/\varepsilon),
\]

and \( t_\varepsilon = \log(1/\varepsilon) \) Given \( q \) and \( R \) positive there exists a constant \( C_{q,R} \) such that the following holds for every \( x, y \in B(0,R) \), \( t \geq 0 \), \( \varepsilon \in (0,1) \) and \( \alpha \in (0,\sqrt{2d}) \),

\[
\mathbb{E} \left[ W_t^{(q)}(x)W_t^{(q)}(y) \right] \leq Ce^\frac{du}{u+1} - \frac{3}{2}(t - u + 1)^{-1}, \quad (2.21)
\]

\[
\mathbb{E} \left[ W_t^{(q)}(x)W_\varepsilon^{(q)}(y) \right] \leq Ce^{\frac{du}{u+1} - \frac{3}{2}(t_\varepsilon - v + 1)^{-1}}, \quad (2.22)
\]

\[
\mathbb{E} \left[ W_t^{\alpha,(q)}(x)W_t^{\alpha,(q)}(y) \right] \leq Ce^\frac{du}{u+1} - \frac{3}{2}, \quad (2.23)
\]

\[
\lim_{\delta \to 0} \mathbb{E} \left[ W_s^{\alpha,(q)}(x)W_s^{\alpha,(q)}(y) \right] \leq C(\sqrt{2d} - \alpha)^2e^\frac{du}{u+1} - \frac{3}{2}. \quad (2.24)
\]

As consequences we have (with a possibly different constant \( C_{q,R} \))

(i) Setting \( \varphi(u) = u^{d}[1 \vee (\log(1/u))^{1/2} \) for all \( E \subset B(0,R) \)

\[
\lim_{t \to \infty} \sup_{E} \mathbb{E} \left[ t(M_t^{(q)}(E))^2 \right] \leq C\varphi(|E|). \quad (2.25)
\]

(ii) For any \( E \in B_b \),

\[
\lim_{\delta \to 0} \sup_{E} \int_{E^2} \mathbb{E} \left[ tW_t^{(q)}(x)W_t^{(q)}(y) \right] 1_{|x-y| \leq \delta} \, dx \, dy = 0,
\]

\[
\lim_{\delta \to 0} \sup_{E} \int_{E^2} \mathbb{E} \left[ \log(1/\varepsilon)W_\varepsilon^{(q)}(x)W_\varepsilon^{(q)}(y) \right] 1_{|x-y| \leq \delta} \, dx \, dy = 0, \quad (2.26)
\]

\[
\lim_{\delta \to 0} \sup_{\alpha \in (0,\sqrt{2d})} \int_{E^2} \mathbb{E} \left[ (\sqrt{2d} - \alpha)^{-2}W_t^{\alpha,(q)}(x)W_t^{\alpha,(q)}(y) \right] 1_{|x-y| \leq \delta} \, dx \, dy = 0.
\]
3. The convergence of $D_t$

We first prove a quantitative uniform integrability statements for $D_t^{(q)}(E)$. Using the observations of the previous section, we deduce our first main result from it.

**Proposition 3.1.** For any fixed $E \in \mathcal{B}_b$, and $q \geq 0$, the martingale $D_t^{(q)}(E)$ is bounded in $L^2$. More precisely, there exists $C_{R,q}$ such that for any $E \subset B(0,R)$ and $t \geq 0$,

$$
\mathbb{E} \left[ D_t^{(q)}(E)^2 \right] \leq C_{R,q} \phi(|E|).
$$

**(3.1)**

*Proof of Theorem 1.6.* Combining Lemma 2.3 and (2.16) in Corollary 2.5, there exists a random $q_0 \in \mathbb{N}$ such that for every $q \geq q_0$ and $E \subset B(0,R)$

$$
\overline{D}_x(E) = \lim_{t \to \infty} D_t(E) = \overline{D}_x^{(q)}(E).
$$

**(3.2)**

Next we prove that $\overline{D}_x(E) > 0$ a.s. if $\lambda(E) > 0$. We first establish a zero-one law. Let $u > 0$ be fixed and recall the definition (2.6). Setting $s = t - u$, we have

$$
D_t(E) = \int_E \left( \sqrt{2}ds - X_s^{(u)}(x) \right) e^{\sqrt{2}dX_s^{(u)}(x)-ds} e^{\sqrt{2}dX_t(x)-du} dx
$$

$$
+ \int_E \left( \sqrt{2}du - X_u(x) \right) e^{\sqrt{2}dX_t(x)-du} dx.
$$

**(3.3)**

The field $\overline{X}_u(\cdot)$ is continuous and thus bounded on $E$, recalling (2.15) this implies that the second exponential factor in the first line of (3.3) is bounded from above and below, hence the event $\{\overline{D}_x(E) = 0\}$ coincides with $\limsup_{s \to \infty} \int_E \left( \sqrt{2}d - X_s^{(u)}(x) \right) e^{\sqrt{2}dX_s^{(u)}(x)-ds} dx = 0$.

In particular, recalling Section 2.2, $\overline{D}_x(E) = 0$ is independent of $\mathcal{F}_u$ for every $u > 0$. Since it is measurable w.r.t. $\mathcal{F}_x$, it is independent of itself and $\mathbb{P}[\overline{D}_x(E) = 0] \in \{0,1\}$.

To prove that this last probability is equal to zero, since $\overline{D}_x^{(q)}(E)$ is increasing to $D_x(E)$ when $q$ increases (cf. (3.2)), it is sufficient to show that $\mathbb{P}[\overline{D}_x^{(q)}(E) > 0] > 0$ for some value of $q$. Using Proposition 3.1 the uniform integrability of the martingale yields $\mathbb{E}[\overline{D}_x^{(q)}(E)] = \mathbb{E}[\overline{D}_0^{(q)}(E)] = q\lambda(E) > 0$.

Finally we prove (1.24). Using (3.2), it is sufficient to prove the statement with $\overline{D}_t$ replaced by $\overline{D}_x^{(q)}$. For simplicity we prove it for subsets of $[0,1]^d$ rather than of $B(0,R)$. For $n \geq 1$, we partition $[0,1]^d$ in $k_n := 2^{dn}$ cube of sidelength $\delta_n = 2^{-2n}$, call them $(I_i^{(n)})_{i=1}^{k_n}$. Given $E \in \mathcal{B}_b$, if $n$ is such that $|E| \in (\delta_{n+1}, \delta_n]$ then $E$ can be fitted in the union of at most four $I_i^{(n)}$s. Hence it suffices to show that for all $n$ sufficiently large

$$
\max_{i \in [1,k_n]} \overline{D}_x^{(q)}(I_i^{(n)}) \leq n 2^{-n/4}.
$$

**(3.4)**

Using the union bound we obtain that

$$
\mathbb{P} \left[ \max_{i \in [1,k_n]} \overline{D}_x^{(q)}(I_i^{(n)}) \geq n 2^{-n/4} \right] \leq k_n \mathbb{P} \left[ \overline{D}_x^{(q)}(I_1^{(n)}) \geq n 2^{-n/4} \right]
$$

$$
\leq k_n n^{2n/2} 2^{-n} \mathbb{P} \left[ \left( \overline{D}_x^{(q)}(I_1^{(n)}) \right)^2 \right] \leq C n^{-2}.
$$

**(3.5)**

For the last inequality, we combined (3.1) and Fatou. We conclude with Borel-Cantelli. ☐
To prove (3.7), we deduce from our uniform bound (2.25) on the second moment of $\sqrt{t}M_t^{(q)}$ a bound for the second moment of $D_t^{(q)}$, using conditional expectation.

**Lemma 3.2.** Given $E \in \mathcal{B}_b$, for any fixed $s \geq 0$ we have

$$D_s^{(q)}(E) = \lim_{t \to \infty} \mathbb{E} \left[ \sqrt{\frac{\pi(t-s)}{2}} M_t^{(q)}(E) \mid \mathcal{F}_s \right]. \quad (3.6)$$

Furthermore, the r.h.s. is is nondecreasing in $t$ and convergence holds in $L^2$.

**Proof.** It is sufficient to show that for any $x \in E$

$$Z_x^{(q)}(x) = \lim_{t \to \infty} \mathbb{E} \left[ \sqrt{\frac{\pi(t-s)}{2}} W_t^{(q)}(x) \mid \mathcal{F}_s \right] \quad (3.7)$$

monotonically. The dominated convergence theorem used twice (first for $\int_E$ and then for $\mathbb{E}[(D_s^{(q)} - \ldots)^2]$) implies the desired $L^2$ convergence. Omitting the variable $x$ for readability, factoring out $W_t^{(q)}$ (which is $\mathcal{F}_s$ measurable) and then using the Cameron-Martin formula for the Brownian Motion $(X^{(s)}(x))$ and Lemma 2.2 we obtain that

$$\mathbb{E} \left[ W_t^{(q)} \mid \mathcal{F}_s \right] = W_s^{(q)} \mathbb{E} \left[ e^{\sqrt{\pi(2ds - d(t-s))}} 1_{\{\forall u \in [0,t-s], \ X_u^{(s)} \leq \sqrt{2ds + q - X_s} \mid \mathcal{F}_s \}} \mid \mathcal{F}_s \right]$$

$$= W_s^{(q)} \sup_{u \in [0,t-s]} X_u^{(s)} \leq \sqrt{2ds + q - X_s} \mid \mathcal{F}_s$$

$$= W_s^{(q)} \sqrt{\frac{2}{\pi(t-s)}} g_{t-s}(\sqrt{2ds + q - X_s}). \quad (3.8)$$

The result is thus a consequence of the fact that $\lim_{r \to \infty} g_r(u) = u_+$ monotonically. \hfill \Box

**Proof of Proposition 3.7** Using Lemma 3.2 we have

$$\mathbb{E} \left[ (D_t^{(q)}(E))^2 \right] = \lim_{t \to \infty} \mathbb{E} \left[ \sqrt{\frac{\pi(t-s)}{2}} M_t^{(q)}(E) \mid \mathcal{F}_s \right]^2$$

$$\leq \limsup_{t \to \infty} \mathbb{E} \left[ \left( \sqrt{\frac{\pi(t-s)}{2}} M_t^{(q)}(E) \right)^2 \right] = \limsup_{t \to \infty} \mathbb{E} \left[ \frac{\pi t}{2} (M_t^{(q)}(E))^2 \right]. \quad (3.9)$$

We can then conclude using the estimate (2.25). \hfill \Box

**Remark 3.3.** The use of (3.6) in the above proof is convenient since the moments of $\sqrt{t}M_t^{(q)}$ is slightly easier to compute than that of $D_t^{(q)}$, but the most important application of Lemma 3.2 comes in the next section. It allows to deduce almost without efforts the convergence of $\sqrt{t}M_t^{(q)}$ (which has no martingale structure) from that of $D_t^{(q)}$.

4. **The convergence of $M_t$**

We prove Theorem 1.8 by showing that $\sqrt{\pi t/2}M_t^{(q)}$, converges to the same limit as $D_t^{(q)}$. 

Proposition 4.1. For any $E \in \mathcal{B}_b$ and $q \geq 0$, we have the following convergence in $L^2$

$$
\lim_{t \to \infty} \sqrt{\frac{\pi t}{2}} M_t^{(q)}(E) = D_{\infty}^{(q)}(E) \tag{4.1}
$$

Proof of Theorem 1.8. Using (3.2) and Proposition 2.4 there exists a random $q_0 \in \mathbb{N}$ such that for $q \geq q_0$,

$$
D_{\infty}^{(q)}(E) = D_{\infty}(E) \quad \text{and} \quad M_t^{(q)}(E) = M_t(E).
$$

Using Proposition 4.1 we conclude that $\sqrt{\pi t/2} M_t(E)$ converges to $D_{\infty}(E)$ in probability. To obtain the weak convergence of the measures, we apply Proposition B.1. □

Proof of Proposition 4.7. We drop $E$ from the notation for better readability. Since $D_s^{(q)}$ converge in $L^2$, it is sufficient to prove that

$$
\lim_{s \to \infty} \lim_{t \to \infty} \mathbb{E} \left[ \left| \sqrt{\frac{\pi t}{2}} M_t^{(q)} - D_s^{(q)} \right|^2 \right] = 0. \tag{4.2}
$$

Using the conditional expectation to make an orthogonal decomposition in $L^2$ we have

$$
\mathbb{E} \left[ \left| \sqrt{\frac{\pi t}{2}} M_t^{(q)} - D_s^{(q)} \right|^2 \right] = \frac{\pi t}{2} \mathbb{E} \left[ |M_t^{(q)} - \mathbb{E} [M_t^{(q)} \mid \mathcal{F}_s]|^2 \right] + \mathbb{E} \left[ \left( \mathbb{E} \left[ \sqrt{\frac{\pi t}{2}} M_t^{(q)} \mid \mathcal{F}_s \right] - D_s^{(q)} \right)^2 \right]. \tag{4.3}
$$

From Lemma 3.2 the second term in the r.h.s. of (4.3) tends to zero when $t \to \infty$. To control the first term, we set (recall (2.11))

$$
\xi_{s,t}(x) := W_t^{(q)}(x) - \mathbb{E} \left[ W_t^{(q)}(x) \mid \mathcal{F}_s \right].
$$

Expanding the square we obtain

$$
\mathbb{E} \left[ |M_t^{(q)} - \mathbb{E} [M_t^{(q)} \mid \mathcal{F}_s]|^2 \right] = \int_{E^2} \mathbb{E} \left[ \xi_{s,t}(x) \xi_{s,t}(y) \right] \, dx \, dy. \tag{4.4}
$$

Recalling Section 2.2 (in particular the definition (2.6)) we note that $\xi_{s,t}(x)$ is measurable with respect to $\mathcal{F}_s \vee \sigma((X_u^{(s)}(x))_{u \geq 0})$. If $|x - y| \geq e^{-s}$ then $X^{(s)}(x)$ and $X^{(s)}(y)$ are independent and both processes are independent of $\mathcal{F}_s$. This implies the conditional independence of $\xi_{s,t}(x)$ and $\xi_{s,t}(y)$ given $\mathcal{F}_s$ and hence

$$
\mathbb{E} [\xi_{s,t}(x) \xi_{s,t}(y) \mid \mathcal{F}_s] = \mathbb{E} [\xi_{s,t}(x) \mid \mathcal{F}_s] \mathbb{E} [\xi_{s,t}(y) \mid \mathcal{F}_s] = 0. \tag{4.5}
$$

On the other hand when $|x - y| \leq e^{-s}$, we have

$$
\mathbb{E} [\xi_{s,t}(x) \xi_{s,t}(y)] \leq \mathbb{E} \left[ W_t^{(q)}(x) W_t^{(q)}(y) \right]. \tag{4.6}
$$

Thus combining (4.5) and (4.6) we have

$$
\mathbb{E} \left[ |M_t^{(q)} - \mathbb{E} [M_t^{(q)} \mid \mathcal{F}_s]|^2 \right] \leq \int_{E^2} \mathbb{E} \left[ W_t^{(q)}(x) W_t^{(q)}(y) \right] 1_{\{ |x - y| \leq e^{-s} \}} \, dx \, dy, \tag{4.7}
$$

Using Equation (2.26) (first line) from Proposition 2.6 we obtain that

$$
\lim_{s \to \infty} \lim_{t \to \infty} \frac{\pi t}{2} \mathbb{E} \left[ |M_t^{(q)} - \mathbb{E} [M_t^{(q)} \mid \mathcal{F}_s]|^2 \right] = 0. \tag{4.8}
$$
Recalling (1.3) this concludes the proof. □

5. The convergence of \( M_\varepsilon \)

The strategy of the previous section can be adapted to prove the convergence of \( M_\varepsilon \). We show that \( \sqrt{\varepsilon} \log(1/\varepsilon)/2M^{(q)}_\varepsilon(E) \) converges to the same limit as \( D^{(q)}_t(E) \) and \( M^{(q)}_t(E) \).

**Proposition 5.1.** For any \( E \in \mathcal{B}_0 \) and \( q \geq 0 \), we have the following convergence in \( L_2 \)

\[
\lim_{t \to \infty} \sqrt{\varepsilon} \log(1/\varepsilon)/2M^{(q)}_\varepsilon(E) = D^{(q)}_\infty(E)
\] (5.1)

**Proof of Theorem 1.8.** A first important observation is that the veracity of the statement “\( M_\varepsilon(E) \) converges in probability to the same limit for all choices of mollifier” only depends on the distribution of the process \( X \) (recall that this is a process indexed by \( C_t(\mathbb{R}^d) \) with covariance given by (1.5)). Hence without loss of generality, it is sufficient to prove it in the case were the probability space contains the martingale approximation sequence \( (X_\varepsilon) \) described in Section 1.3. In that case, combining Propositions 2.4 and 5.1 in the same way as in the proof of Theorem 1.8 we obtain (1.15) with \( M'(E) = D_\infty(E) \). In particular the limit does not depend on the mollifying kernel. □

The proof of Proposition 5.1 is very similar to that of Proposition 4.1 but we need the following replacement for Lemma 3.2

**Lemma 5.2.** For any fixed \( s \geq 0 \) we have the following convergence in \( L^2 \)

\[
D^{(q)}_s(E) = \lim_{\varepsilon \to 0} \mathbb{E} \left[ \left( \frac{\pi \log(1/\varepsilon)}{2} M^{(q)}_\varepsilon(E) \right) | \mathcal{F}_s \right].
\] (5.2)

**Proof.** As there is no monotonicity in \( \varepsilon \), the proof of the \( L^2 \) convergence is more technical than that of Lemma 3.2. Using the convention \( t = t_\varepsilon = (\log 1/\varepsilon) \), we need to show that

\[
\lim_{\varepsilon \to 0} \sup_{x \in E} \mathbb{E} \left[ \left( Z^{(q)}_s(x) - \mathbb{E} \left[ \frac{\pi t}{2} W^{(q)}_\varepsilon(x) | \mathcal{F}_s \right] \right)^2 \right] = 0.
\] (5.3)

The uniformity in (5.3) implies that the convergence in \( L^2 \) is maintained after integrating w.r.t. \( x \) over \( E \). Taking \( \mathcal{F}_s \)-measurable terms out of the expectation we obtain

\[
\mathbb{E} \left[ \frac{\pi t}{2} W^{(q)}_\varepsilon(x) | \mathcal{F}_s \right] = W^{(q)}_s(x) \times V^{(q)}_s(x),
\] (5.4)

where

\[
W^{(q)}_s(x) := e^{\sqrt{2d}X_{s,x}(x) - dK_{s,x}(x)} A^{(q)}_{s,x}(x),
\]

\[
V^{(q)}_s(x) := \sqrt{\frac{\pi t}{2}} \mathbb{E} \left[ e^{\sqrt{2d}X_{s,x}(x) - dK_{s,x}(x)} 1_{\{y \in [s,t], \ X_u(x) \leq q + \sqrt{2du} \}} | \mathcal{F}_s \right].
\] (5.5)

In view of the inequality \( \mathbb{E}[(A_tB_t - AB)^2] \leq 4 (\mathbb{E}[(A_t - A)^2] \mathbb{E}[B^4_t] + \mathbb{E}[(B_t - B)^4] \mathbb{E}[A^4_t]) \) (valid for arbitrary random variables \( A, B, A_t, B_t \), to prove (5.3), it is sufficient to prove the two following convergences, and that \( W^{(q)}_s(x) \) and \( (q + \sqrt{2ds - X_s(x)})^2 \) are uniformly bounded in \( L^4 \) when \( x \) varies

\[
\lim_{\varepsilon \to 0} \sup_{x \in E} \mathbb{E} \left[ (W^{(q)}_s(x) - W^{(q)}_s(x))^2 \right] = 0,
\]

\[
\lim_{\varepsilon \to 0} \sup_{x \in E} \mathbb{E} \left[ (V^{(q)}_s(x) - (q + \sqrt{2ds - X_s(x))^2} \right] = 0.
\] (5.6)
The first line in (5.6) follows from the the uniform convergence of $K_{s,\varepsilon}(x, y)$ and $K_{s,\varepsilon,0}(x, y)$ towards $K_s(x, y)$, which implies that

$$\limsup_{\varepsilon \to 0} \mathbb{E} \left[ \left( e^{\sqrt{2d} X_{s,\varepsilon}(x) - dK_{s,\varepsilon}(x)} - e^{\sqrt{2d} X_s(x) - dK_s(x)} \right)^4 \right] = 0. \quad (5.7)$$

For the second line in (5.6), by translation invariance of $X$, the quantity in the supremum does not depend on $x$. Thus we do not need to worry about the sup and omit $x$ in the notation. We rewrite the event appearing in the definition of $V_{s,\varepsilon}^{(q)}$ as

$$\{ \forall u \in [0, t - s], \ X_u^{(s)} \leq q + \sqrt{2d}(s + u) - X_s \}.$$  

Using the Cameron-Martin formula (Proposition 2.1), the exponential tilt in $f_{s,\varepsilon}^{(q)}$ has the effect of shifting the mean of $X_u^{(s)}(x)$ by an amount (recall (2.7))

$$\mathbb{E}[(X_{s,\varepsilon} - X_{s,\varepsilon})(x)|X_u^{(s)}(x)] = K_{s+u,\varepsilon,0}(x) - K_{s,\varepsilon,0}(x) =: K_{u,\varepsilon,0}^{(s)}.$$  

Hence we have

$$V_{s,\varepsilon}^{(q)} = \sqrt{\frac{\pi t}{2}} \mathbb{P} \left[ \forall u \in [0, t - s], \ X_u^{(s)} \leq \sqrt{2d} \left( u - K_{u,\varepsilon,0}^{(s)} \right) + (q + \sqrt{2d}s - X_s) \mid F_s \right]. \quad (5.8)$$

Since $K_{u,\varepsilon,0}^{(s)} = \int_s^{s+u} \left( \int_{2d} Q_{v}(0, z) \theta \varepsilon(z) \, dz \right) \, dv$ and $Q_v(0, z) \leq 1$, we have $K_{u,\varepsilon,0}^{(s)} \leq u$. Injecting this bound in (5.8), we obtain from Lemma 2.2

$$V_{s,\varepsilon}^{(q)} \geq \sqrt{\frac{t}{t - s}} \left( q + \sqrt{2d}s - X_s \right). \quad (5.9)$$

To obtain a bound in the other direction, recalling (1.12) and the regularity of $\kappa$ we obtain that for some universal constant $C$ we have $(1 - Q_v(0, z)) \leq C(e^{\varepsilon}z)^2$. Setting $\mathcal{T}_\varepsilon = \log(1/\varepsilon) - \sqrt{\log 1/\varepsilon}$, we have for any $\varepsilon$ sufficiently large and $u \leq \mathcal{T} - s$

$$(u - K_{u,\varepsilon,0}^{(s)}) = \int_s^{s+u} \left( \int_{2d} (1 - Q_v(0, z)) \theta \varepsilon(z) \, dz \right) \, dv$$

$$\leq C \int_s^{u+s} e^{2\varepsilon v^2} \, dv \leq (C/2) e^{2\varepsilon^2} \leq \frac{\delta_\varepsilon}{\sqrt{2d}}, \quad (5.10)$$

where $\delta_\varepsilon = C \sqrt{d/2} e^{-\sqrt{\log(1/\varepsilon)}}$. Hence we have

$$V_{s,\varepsilon}^{(q)} \leq \sqrt{\frac{\pi t}{2}} \mathbb{P} \left[ \forall u \in [0, \mathcal{T} - s], \ X_u^{(s)} \leq \delta_\varepsilon + q + \sqrt{2d}s - X_s \mid F_s \right]$$

$$\leq \sqrt{\frac{t}{t - s}} \mathbb{P} \left[ \mathcal{T}_\varepsilon - s (q + \sqrt{2d}s - X_s + \delta_\varepsilon) \right]. \quad (5.11)$$

Both upper (5.11) and lower bound (5.9) converge to $(q + \sqrt{2d}s - X_s)_+$ in $L^4$ as $\varepsilon \to 0$ which conclude the proof of (5.6) and hence of the lemma.

Proof of Proposition 5.7 Proceeding like for Proposition 4.1 it is sufficient to prove that

$$\limsup_{s \to \infty} \limsup_{\varepsilon \to 0} \mathbb{E} \left[ \left( \frac{\pi \varepsilon}{2} \mathbb{E} \left[ \mathcal{M}^{(q)}_s - D^{(q)}_s \right] \right)^2 \right] = 0. \quad (5.12)$$
After a decomposition like (4.3) and using Lemma 5.2 we only need to prove the following
\[
\lim_{s \to \infty} \limsup_{\varepsilon \to 0} \log(1/\varepsilon) \mathbb{E} \left[ \left| M_{\varepsilon}^{(q)} - \mathbb{E} \left[ M_{\varepsilon}^{(q)} \mid \mathcal{F}_s \right] \right|^2 \right] = 0. \tag{5.13}
\]
Setting \( \xi_{s,\varepsilon}(x) := W_{\varepsilon}^{(q)}(x) - \mathbb{E}[W_{\varepsilon}^{(q)}(x) \mid \mathcal{F}_s] \) (recall (2.11)) we have like for (4.4)
\[
\mathbb{E} \left[ \left| M_{\varepsilon}^{(q)} - \mathbb{E} \left[ M_{\varepsilon}^{(q)} \mid \mathcal{F}_s \right] \right|^2 \right] = \int_{E^2} \mathbb{E} \left[ \xi_{s,\varepsilon}(x) \xi_{s,\varepsilon}(y) \right] \, dx \, dy. \tag{5.14}
\]
Recalling the observations of Section 2.2 \( (X^{(s)}(x), (X_\varepsilon - X_{\varepsilon,s})(x)) \) and \( (X^{(s)}(y), (X_\varepsilon - X_{\varepsilon,s})(y)) \) are independent and independent of \( \mathcal{F}_s \) whenever \( |x - y| \geq e^{-s} + 2\varepsilon \). Hence \( \xi_{s,\varepsilon}(x) \) and \( \xi_{s,\varepsilon}(y) \) are conditionally independent given \( \mathcal{F}_s \) and like for (4.5) we have
\[
|x - y| \geq e^{-s} + 2\varepsilon \implies \mathbb{E} \left[ \xi_{s,\varepsilon}(x) \xi_{s,\varepsilon}(y) \right] = 0.
\]
Proceeding as in (4.7), this implies
\[
\mathbb{E} \left[ \left| M_{\varepsilon}^{(q)} - \mathbb{E} \left[ M_{\varepsilon}^{(q)} \mid \mathcal{F}_s \right] \right|^2 \right] \leq \int_{E^2} \mathbb{E} \left[ W_{\varepsilon}^{(q)}(x)W_{\varepsilon}^{(q)}(y) \right] 1_{|x - y| \leq e^{-s} + 2\varepsilon} \, dx \, dy. \tag{5.15}
\]
Using Equation (2.26) (second line) we conclude the proof of (5.13). \( \square \)

6. The convergence of subcritical chaos

As in the previous section, we establish the convergence for the truncated version of \( M^\alpha \).

**Proposition 6.1.** For any \( E \in \mathcal{B}_b \) we have the following convergence in \( L^2 \)
\[
\lim_{\alpha \uparrow \sqrt{2d}} \frac{M_{\alpha}^{\alpha,q}(E)}{\sqrt{2d - \alpha}} = 2D_{\alpha,q}^{(q)}(E) \tag{6.1}
\]

**Proof of Theorem 1.9.** As in the proof Theorem 1.2, we first observe that the event “\((\sqrt{2d} - \alpha)M^\alpha(E) \) and \( \sqrt{2\pi/(\alpha \varepsilon)}M_\varepsilon^{\sqrt{2d}}(E) \)” converge towards the same limit” only depends on the distribution of the process \( X \). Hence, we can focus on the case where the probability space includes a martingale approximation sequence (the framework of Proposition 6.1). To conclude we observe that from Proposition 2.4 (taking the limit of (2.11)) and (3.2), there exists a random \( q_0 \) such that for \( q \geq q_0 \) we have for every \( \alpha \in (0, \sqrt{2d}) \)
\[
\overline{M}_{\alpha}^{\alpha,q}(E) = M^\alpha(E) \text{ and } \overline{D}_{\alpha}^{(q)}(E) = D_{\alpha,q}^{(q)}(E).
\]

To prove Proposition 6.1 we use the following replacement for Lemma 3.2

**Lemma 6.2.** Given \( E \in \mathcal{B}_b(E) \), and \( s \geq 0 \) the following convergences holds in \( L^2 \).
\[
\mathbb{E} \left[ \overline{M}_{\alpha}^{\alpha,q}(E) \mid \mathcal{F}_s \right] = \lim_{t \to \infty} \mathbb{E} \left[ M_t^{\alpha,q}(E) \mid \mathcal{F}_s \right], \tag{6.2}
\]
\[
\lim_{\alpha \uparrow \sqrt{2d}} \mathbb{E} \left[ \frac{M_{\alpha}^{\alpha,q}(E)}{\sqrt{2d - \alpha}} \mid \mathcal{F}_s \right] = 2D_{s}^{(q)}(E). \tag{6.3}
\]

**Proof.** Since \( M_t^{\alpha,q}(E) \) converges in \( L^1 \) (cf. Lemma 2.3), we already know that (6.2) holds in \( L^1 \). Furthermore since the sequence \( \mathbb{E} \left[ M_t^{\alpha,q}(E) \mid \mathcal{F}_s \right] \) is decreasing (from the supermartingale property) and since \( M_{s}^{\alpha,q}(E) \in L^2 \), the convergence also holds in \( L^2 \) by
dominated convergence. We now move to the proof of (6.3). Exchanging the integral over $E$ with conditional expectation in (6.2) we obtain

$$\mathbb{E}\left[\overline{M}^{\alpha,(q)}(E) \mid \mathcal{F}_s\right] = \lim_{t \to \infty} \int_E \mathbb{E}\left[W_t^{\alpha,(q)}(x) \mid \mathcal{F}_s\right] dx$$  \hspace{1cm} (6.4)$$

Dropping $x$ in the notation for readability we obtain from Cameron-Martin formula that

$$\mathbb{E}\left[W_t^{\alpha,(q)}(x) \mid \mathcal{F}_s\right] = W_s^{\alpha,(q)} \mathbb{P}\left[\forall u \in [0, t - s], \ X_u^{(s)} \leq q + \sqrt{2ds - X_s} + (\sqrt{2d - \alpha})u \mid \mathcal{F}_s\right].$$

The r.h.s. is nondecreasing in $t$, and from Lemma 2.2 we have

$$\lim_{t \to \infty} \mathbb{E}\left[W_t^{\alpha,(q)}(x) \mid \mathcal{F}_s\right] = W_s^{\alpha,(q)}(x) \left(1 - e^{-2(\sqrt{2d - \alpha})(q + \sqrt{2ds - X_s}(x))}\right).$$  \hspace{1cm} (6.5)$$

Using (6.4) and dominated convergence (our integrand is bounded by $W_s^{\alpha,(q)}(x)$) we obtain

$$\mathbb{E}\left[\overline{M}^{\alpha,(q)}(E) \mid \mathcal{F}_s\right] = \int_E W_s^{\alpha,(q)}(x) \left(1 - e^{-2(\sqrt{2d - \alpha})(q + \sqrt{2ds - X_s}(x))}\right) dx.$$  \hspace{1cm} (6.6)$$

The integrand on the r.h.s. converges to $2D_s^{(q)}(x)$ when $\alpha \uparrow \sqrt{2d}$. It is also smaller than $|\sqrt{2ds + q - X_s(x)}|e^{\sqrt{2d}X_s(x)}$ for every $\alpha \in (0, \sqrt{2d})$. We conclude the proof of (6.3) using dominated convergence. The domination also ensures boundedness in $L^p$ for $p > 2$ and thus the convergence holds in $L^2$.

**Proof of Proposition 6.7** Proceeding like for Propositions 4.1 and 5.1 using first approximation of $\overline{D}_\infty^{(q)}$ with $D_s^{(q)}$ then an orthogonal decomposition conditioning to $\mathcal{F}_s$ and finally Lemma 6.2 we are left with proving

$$\lim_{s \to \infty} \limsup_{\alpha \uparrow \sqrt{2d}} \mathbb{E}\left[\left|\overline{M}_\infty^{\alpha,(q)}(E) - \mathbb{E}\left[\overline{M}_\infty^{\alpha,(q)}(E) \mid \mathcal{F}_s\right]\right|^2\right] = 0.$$  \hspace{1cm} (6.7)$$

We omit the dependence in $E$ for readability. To apply our usual scheme of proof we want to replace $\infty$ by $t$ in (6.7). Fatou implies that $\mathbb{E}\left[\left(M_t^{\alpha,(q)}\right)^2\right] \leq \liminf_{t \to \infty} \mathbb{E}\left[\left(M_t^{\alpha,(q)}\right)^2\right]$, combined with (6.2) this yields

$$\mathbb{E}\left[\overline{M}_\infty^{\alpha,(q)} - \mathbb{E}\left[\overline{M}_\infty^{\alpha,(q)} \mid \mathcal{F}_s\right]\right]^2 \leq \liminf_{t \to \infty} \mathbb{E}\left[\left|M_t^{\alpha,(q)} - \mathbb{E}\left[M_t^{\alpha,(q)} \mid \mathcal{F}_s\right]\right|^2\right].$$  \hspace{1cm} (6.8)$$

Setting $\xi_{s,t}^{\alpha}(x) := W_t^{\alpha,(q)}(x) - \mathbb{E}\left[W_t^{\alpha,(q)}(y) \mid \mathcal{F}_s\right]$ and repeating the argument (4.4)-(4.7) we obtain

$$\mathbb{E}\left[\left|M_t^{\alpha,(q)} - \mathbb{E}\left[M_t^{\alpha,(q)} \mid \mathcal{F}_s\right]\right|^2\right] = \int_{E^2} \mathbb{E}[\xi_{s,t}^{\alpha}(x)\xi_{s,t}^{\alpha}(y)]dxdy$$

$$= \int_{E^2} 1_{\{|x-y| \leq e^{-s}\}}\mathbb{E}[\xi_{s,t}^{\alpha}(x)\xi_{s,t}^{\alpha}(y)]dxdy = \int_{E^2} 1_{\{|x-y| \leq e^{-s}\}}\mathbb{E}[W_t^{\alpha,(q)}(x)W_t^{\alpha,(q)}(y)]dxdy.$$  \hspace{1cm} (6.9)$$

Combining (6.7), (6.8) and (6.9), we can conclude using the last line of (2.2b).
APPENDIX A. PROOF OF TECHNICAL ESTIMATES

A.1. Proof of Proposition 2.4. For simplicity we replace \( B(0, R) \) by \([0, 1]^d\) but this entails no loss of generality. Like in [6], the first idea is to restrict ourselves to a discrete set instead of checking the inequality at every coordinates. We define

\[
I_n := \{ i \in \mathbb{Z}^d : ii = [0, 1]^d \}, \quad A_{n, i} := [n, n+1) \times (i e^{-n} + [0, e^{-n}])_d, 
\]

\[
Y_{n, i} := \mathbb{E}_n(i e^{-n}), \quad Z_{n, 1} := \max_{(t, x) \in A_{n, i}} (X_t(x) - Y_{n, i}). \tag{A.1}
\]

The statement we want to prove can be restated as follows

\[
\sup_{n \geq 0} \max_{i \in I_n} \left[ Y_{n, i} + Z_{n, 1} - \sqrt{2d} n + \frac{\log n}{2\sqrt{2d}} - \frac{4(\log \log n)}{\sqrt{2d}} \right] < \infty. \tag{A.2}
\]

The following estimate (whose proof we postpone to the end of the section) ensures that the tails of \( Z_{n, i} \) are uniformly subgaussian.

Lemma A.1. There exists a constant \( c > 0 \) such that for all \( n \geq 0, i \in \mathbb{Z}^d \) and \( \lambda \geq 0 \)

\[
\mathbb{P}[Z_{n, i} \geq \lambda] \leq 2 \exp \left( -c\lambda^2 \right). \tag{A.3}
\]

Combining (A.3) and the fact that \( Y_{n, i} \) is a centered Gaussian with variance \( n \), we obtain

\[
\mathbb{P}[Y_{n, i} + Z_{n, i} \geq A] \leq \frac{2}{\sqrt{2\pi n}} \int_{\mathbb{R}} \exp \left( -\frac{u^2}{2n} - c(A - u^2) \right) du. \tag{A.4}
\]

This implies that

\[
\sum_{n \geq 0} \sum_{i \in I_n} \mathbb{P} \left[ Y_{n, i} + Z_{n, i} - \sqrt{2d} n + \frac{\log n}{\sqrt{2d}} \geq 0 \right] \leq \sum_{n \geq 0} \sum_{i \in I_n} C n^{-3/2} e^{-dn} \leq C' \sum_{n \geq 0} n^{-3/2} < \infty.
\]

Hence using Borel-Cantelli we obtain the following rough bound

\[
\sup_{n \geq 0} \sup_{i \in I_n} \left[ Y_{n, i} + Z_{n, i} - \sqrt{2d} n + \frac{\log n}{\sqrt{2d}} \right] < \infty. \tag{A.5}
\]

We now build on (A.5) to obtain (A.2). We set for \( q \in \mathbb{N} \)

\[
B_q := \bigcap_{n \geq 0} \bigcap_{i \in I_n} \left\{ Y_{n, i} + Z_{n, i} \leq \sqrt{2d} n + \frac{\log n}{\sqrt{2d}} + q \right\}
\]

From (A.5) we have \( \lim_{q \to \infty} \mathbb{P}[B_q] = 1 \). Using the bound (2.3) together with the fact that

\[
B_q \subseteq \left\{ \forall t \in [0, n], X_t(i e^{-n}) \leq \sqrt{2d} + \frac{\log n}{\sqrt{2d}} + q \right\}
\]

we can compute an upper bound on the density distribution of \( Y_{n, i} \) restricted to \( B_q \). Using improper but unambiguous notation we have

\[
\mathbb{P}[Y_{n, i} \in du \cap B_q] \leq \mathbb{P} \left[ B_n \in du : \forall t \in [0, n], B_t \leq \sqrt{2d} t + \frac{\log n}{\sqrt{2d}} + q \right] \leq \frac{du}{\sqrt{2\pi n}} e^{-\frac{n^2}{2}} (q + \log n)(q + \log n + \sqrt{2d}n - u_+). \tag{A.6}
\]
Similarly to (A.4), setting $A_n := \sqrt{2dn - \log n} + \frac{4(\log \log n)}{\sqrt{2d}}$ we obtain that

$$
P[Y_{n, i} + Z_{n, i} - A_n \geq 0 \mid B_q] \leq C_q \frac{(\log n)}{n} \int (q + \log n + \sqrt{2dn} - u)^+ e^{-\frac{u^2}{2n} - c(A_n - u)^2} \, du \leq C_q \frac{e^{-dn}}{n(\log n)^2}. \tag{A.7}
$$

Using Borel-Cantelli after summing over $i \in I_n$ and $n \geq 0$, we obtain that for each $q \in \mathbb{N}$

$$
\left( \sup_{n \geq 0} \sup_{i \in I_n} Y_{n,i} + Z_{n,i} - \sqrt{2dn} + \frac{\log n}{2\sqrt{2d}} - \frac{4(\log \log n)}{\sqrt{2d}} \right) 1_{B_q} < \infty. \tag{A.8}
$$

Letting $q \to \infty$ we conclude the proof. □

Proof of Lemma A.7. The Borrel-TIS inequality (see [2] Theorem 2.1) implies that the fluctuation of the maximum of a Gaussian field around its mean are sub-Gaussian, that is

$$
P[Z_{n,i} \geq \mathbb{E}[Z_{n,i}] + u] \leq 2e^{-\frac{u^2}{2\sigma_n^2}} \tag{A.9}
$$

where $\sigma_n := \max_{(t,x) \in A_n} \mathbb{E}[(X_t(x) - Y_{n,i})^2] = 1 + \max_{x \in [0,e^{-n}]} \mathbb{E}[(X_n(x) - X_n(0))^2]$. The maximized quantity is simply equal $2(n - K_n(0, x))$, and thus using (A.11), we have $\sup_{n \geq 0} \sigma_n < \infty$. To deduce (A.3) from (A.9), we want a uniform bound on $\mathbb{E}[Z_{n,i}]$ that is

$$
\sup_{n \geq 0} \sup_{i \in I_n} \mathbb{E}[Z_{n,i}] = \sup_{n \geq 0} \mathbb{E} \left[ \max_{(t,x) \in [0,1]^{d+1}} X_{n+t}(e^{-n}x) \right] < \infty \tag{A.10}
$$

(the equality comes from translation invariance and the fact that $Y_{n,i}$ is centered). To achieve this, we need to control, uniformly in $n$, the modulus of continuity of the canonical metric associated with the field $(X_{n+t}(e^{-n}x))_{(t,x) \in [0,1]^{d+1}}$, which is defined by

$$
d_n((s, x), (t, y)) := \mathbb{E} \left[ (X_{n+s}(e^n x) - X_{n+t}(e^n y))^2 \right]^{1/2}.
$$

It is sufficient to prove that $d_n$ is uniformly Hölder continuous. Given $t \geq s$ we have

$$
\mathbb{E} \left[ (X_s(x) - X_t(y))^2 \right] = (t - s) + 2(s - K_s(x, y)) \leq (t - s) + 2 \int_0^s (1 - Q_u(x, y)) \, du
$$

$$
\leq (t - s) + C \int_0^s e^{2|u|} |x - y|^2 \, du \leq (t - s) + Ce^{2|s|} |x - y|^2. \tag{A.11}
$$

Hence for any $(s, x), (t, y) \in [0, 1]^{d+1}$ we have (for a different constant $C$),

$$
d_n((s, x), (t, y))^2 \leq |t - s| + C |x - y|^2. \tag{A.12}
$$

Using either Fernique’s majorizing measure techniques (applying [2] Theorem 4.1 with Lebesgue measure on $[0, 1]^{d+1}$) or Garantia-Rodemich-Rumsey inequality (see for instance [21] Corollary 4) we obtain that (A.12) implies (A.10) and conclude the proof. □

A.2. Proof of Propositions 2.6. Let us first prove (2.21). By Cameron-Martin formula, the $e^{\sqrt{2d}(X_t(x) + X_t(y))}$ term shifts the mean of $\overline{X}_s(x)$ and $\overline{X}_s(y)$ by $\sqrt{2d}(s + K_s(x, y))$ for $s \in [0, t]$. As a result we obtain that

$$
\mathbb{E} \left[ W^{(q)}(x) W^{(q)}(y) \right] = e^{2dK_t(x,y)} \mathbb{P} \left[ \forall s \in [0, t], \overline{X}_s(x) \vee \overline{X}_s(y) \leq q - \sqrt{2dK_s(x, y)} \right]. \tag{A.13}
$$

To estimate the probability on the right-hand side, we perform a couple of simplifications. Recalling the definition of $u(x, y, t)$, from (2.8), there exists $C > 0$ such that

$$
\overline{K}_s(x, y) \geq s \wedge u - C. \tag{A.14}
$$
For the remainder of the proof, we set $q' = q + C\sqrt{2d}$. Another observation is that $\overline{X}(x)$ and $\overline{X}(y)$ play symmetric roles so that adding the restriction $\overline{X}_u(x) \leq \overline{X}_u(y)$ only changes the probability of the event in the r.h.s. of (A.13) by a factor $1/2$. It is thus smaller than

$$2\mathbb{P}\left[ \forall s \in [0, t], \overline{X}_s(x) \vee \overline{X}_s(y) \leq q' - \sqrt{2d(u \wedge s)} ; \overline{X}_u(x) \leq \overline{X}_u(y) \right] \leq 2\mathbb{P}\left[ F_1 \cap F_2 \cap F_3 \right] = 2\mathbb{E}\left[ 1_{F_1} \mathbb{P}[F_2 \cap F_3 \mid \mathcal{F}_u] \right].$$

(A.15)

where the events $(F_i)_{i=1}^3$ are defined by

$$F_1 := \{ \forall s \in [0, u], \overline{X}_s(x) \leq q' - \sqrt{2ds} \},$$

$$F_2 := \{ \forall s \in [0, t - u], X_s^{(u)}(x) \leq q' - \sqrt{2du} - \overline{X}_u(x) \},$$

$$F_3 := \{ \forall s \in [0, t - u], X_s^{(u)}(y) \leq q' - \sqrt{2du} - \overline{X}_u(x) \}.$$

The $\overline{X}_u(x)$ appearing in the definition of $F_3$ is not a typo: the restriction $\overline{X}_u(x) \leq \overline{X}_u(y)$ implies that $F_3$ (like $F_1$ and $F_2$) is included in the event in the l.h.s. of (A.15). From the observations of Section 2.2, $X^{(u)}(x)$ and $X^{(u)}(y)$ are independent and independent of $\mathcal{F}_u$. Thus $F_2$ and $F_3$ are conditionally independent given $\mathcal{F}_u$, and using Lemma 2.2 we have

$$\mathbb{P}\left[ F_2 \cap F_3 \mid \mathcal{F}_u \right] \leq 1 \wedge \left( \frac{2(q' - \sqrt{2du} - \overline{X}_u(x))^2}{\pi(t-u)} \right).$$

(A.17)

To compute $\mathbb{E}\left[ 1_{F_1} \mathbb{P}[F_2 \cap F_3 \mid \mathcal{F}_u] \right]$, we use (2.3) and obtain that

$$\mathbb{E}[F_1 \mid \overline{X}_u(x)] \leq 1 \wedge \left( \frac{2q'(q' - \sqrt{2du} - \overline{X}_u(x))_+}{u} \right).$$

(A.18)

Putting together (A.17) and (A.18) and integrating w.r.t. $z = (q' - \sqrt{2du} - \overline{X}_u(x))$ we have

$$\mathbb{P}[F_1 \cap F_2 \cap F_3] \leq \int_0^\infty e^{-\frac{(z-q'+\sqrt{2du})^2}{2u}} \left( 1 \wedge \frac{2q'z}{u} \right) \left( 1 \wedge \frac{z^2}{\pi(t-u)} \right) dz \leq Ce^{-du} \frac{1}{(u+1)^{3/2}[(t-u)+1]}.$$ 

(A.19)

Since in (A.13) we have $e^{2dK(x,y)} \leq Ce^{2du}$ (cf. (2.8)) this concludes the proof of (2.21).

To prove (2.22) we notice that from Cameron Martin formula we have

$$\mathbb{E}\left[ W_t^{(q)}(x)W_t^{(q)}(y) \right] = e^{2dK_t(x,y)} \mathbb{P}\left[ \forall s \in [0, t], \overline{X}_s(x) \vee \overline{X}_s(y) \leq q' - \sqrt{2dK_{s,\varepsilon,0}(x,y)} \right].$$

Recalling the definition of $v(x,y,\varepsilon)$, the bound (2.8) implies that $K_t(x,y) \leq v + C$ and $K_{s,\varepsilon,0}(x,y) \geq s \wedge v - C$. We can thus repeat the proof of (2.21) with $u$ replaced by $v$.

To prove (2.23), we set $\beta = \sqrt{2d} - \alpha$. Using (A.14) and Cameron-Martin formula

$$\mathbb{E}\left[ W_t^{\alpha,q}(x)W_t^{\alpha,q}(y) \right] \leq e^{\alpha^2 K_t(x,y)} \mathbb{P}\left[ \forall s \in [0, t], \overline{X}_s(x) \vee \overline{X}_s(y) \leq q' + \beta s - \alpha(u \wedge s) \right].$$

We use (2.8) to bound the exponential prefactor. We obtain (omitting $\alpha$ and $q$ in the r.h.s. for readability)

$$\mathbb{E}\left[ W_t(x)W_t(y) \right] \leq Ce^{\alpha^2 u} \mathbb{P}\left[ \forall s \in [0, u], \overline{X}_s(x) \leq q' - (2\alpha - \sqrt{2d})s \right],$$

$$\lim_{t \to \infty} \mathbb{E}\left[ W_t(x)W_t(y) \right] \leq Ce^{\alpha^2 u} \mathbb{P}\left[ \forall s \geq 0, \overline{X}_s(x) \vee \overline{X}_s(y) \leq q' + \beta s - \alpha(w \wedge s) \right].$$

(A.20)
We introduce the following events (for $r \geq 0$)
\begin{align*}
G'_1 := & \{ \forall s \in [0, r], \, \bar{X}_s(x) \leq q' - (2\alpha - \sqrt{2d})s \}, \\
G_2 := & \{ \forall s \geq 0, \, X'_t(u)(x) \leq q' - (2\alpha - \sqrt{2d})u - \bar{X}_u(x) + \beta s \}, \\
G_3 := & \{ \forall s \geq 0, \, X'_t(y)(x) \leq q' - (2\alpha - \sqrt{2d})u - \bar{X}_u(x) + \beta s \}.
\end{align*}
(A.21)

To prove (2.23), we assume that $u \geq 1$ and $\alpha > \sqrt{2d}/3$ (if not then the bound $Ce^{\alpha^2 u^2}$ from (A.20) is good enough). Integrating over $\bar{X}_u(x)$ and using Lemma 2.2 we have
\begin{equation}
\mathbb{P}[G_u] \leq \frac{1}{\sqrt{2\pi u}} \int_{\mathbb{R}} e^{-\frac{1}{2} \frac{2q'(q' - (2\alpha - \sqrt{2d})u - z)^2}{u}} dz \leq Ce^{-\frac{(2\alpha - \sqrt{2d})^2 u}{2} u^{-3/2}}. \tag{A.22}
\end{equation}

We can conclude that (2.23) holds by observing that
\begin{equation}
\alpha^2 - \frac{(2\alpha - \sqrt{2d})^2}{2} = d - (\alpha - \sqrt{2d})^2 \leq d. \tag{A.23}
\end{equation}

For (2.24) starting with the second line in (A.20) we proceed as in (A.15) and obtain
\begin{equation}
\lim_{t \to \infty} \mathbb{E}[W_t(x)W_t(y)] \leq 2Ce^{\alpha^2 w} \mathbb{P}[G_1 \cap G_2 \cap G_3] \tag{A.24}
\end{equation}

Repeating the steps leading to (A.19) and replacing (A.18) by (cf. Lemma 2.2)
\begin{equation}
\mathbb{P}[G_2 \cap G_3 \mid F_u] \leq 4\beta^2 (q' - (2\alpha - \sqrt{2d})w - \bar{X}_w(x)) \tag{A.25}
\end{equation}

we obtain (the variable of integration is $z = q' - (2\alpha - \sqrt{2d})w - \bar{X}_w(x)$)
\begin{equation}
\mathbb{P}[F_1 \cap G_2 \cap G_3] \leq \int_{0}^{\infty} \frac{e^{-\frac{2q'z}{w} + \frac{(2\alpha - \sqrt{2d})z}{w}}}{\sqrt{2\pi w}} \left(1 + \frac{2q'z}{w}\right) 4\beta^2 z^2 dz \leq \frac{C\beta^2 e^{\frac{(2\alpha - \sqrt{2d})^2 w}{2} (w + 1)^{3/2}}}{(w + 1)^{3/2}}. \tag{A.26}
\end{equation}

Recalling (A.20) and (A.23), this is sufficient to conclude the proof of (2.24).

To prove (2.25), we set $\delta = |E|$. Without loss of generality we can assume that $|\delta| \leq 1$. Expanding the second moment, and the change of variable $r = -\log 1/|x - y|$ when integrating over $y$ we obtain
\begin{align*}
\mathbb{E}[M_t^{(q)}(E)^2] \leq & \int_{E} \left( \int_{B(x, \delta)} W_t^{(q)}(x)W_t^{(q)}(y) \right) dx \\
\leq & C\lambda(E) \int_{0}^{\infty} \frac{e^{-dr} e^{-\lambda r \wedge t} dr}{r \wedge t + 1^{3/2} (t - r \wedge t + 1)}. \tag{A.26}
\end{align*}

The factor $\lambda(E)$ can be bounded by $\delta^d$. Assuming that $t \geq 2\log(1/\delta)$ we have
\begin{align*}
\frac{e^{-dr} e^{-\lambda r \wedge t} dr}{r \wedge t + 1^{3/2} (t - r \wedge t + 1)} & \leq \frac{2}{t} \int_{t/2}^{t} \frac{dr}{(r + 1)^{3/2}} + \frac{1}{(t/2 + 1)^{3/2}} \int_{t/2}^{t} \frac{dr}{(t - r + 1)^{3/2}} + \frac{1}{(t + 1)^{3/2}} \int_{t}^{\infty} e^{-d(r-t)} dr \\
& \leq C \left( t^{-1/2} \log(1/\delta)^{1/2} + t^{-3/2} \log t \right). \tag{A.27}
\end{align*}

This proves (2.25) and the two first lines of (2.26) can be obtained similarly.

For the third line of (2.26), using (2.23) (which of course valid if also $u$ is replaced by $w$), and applying Fatou to the positive integrable function $(Ce^{dw}(1 + w)^{3/2} - \mathbb{E}[W_t(x)W_t(y)])$
we obtain that the lim sup of in the l.h.s. of (A.28) is smaller than the integral of the lim sup of the integrand. With (2.24) this yields (recall that \( \beta = \sqrt{2d - \alpha} \))

\[
\limsup_{t \to \infty} \int_{E^2} \mathbb{E} \left[ W_t^{\alpha, (q)}(x)W_t^{\alpha, (q)}(y) \right] 1_{|x - y| \leq \delta} \, dx \, dy \leq C \beta^2 \int_{E^2} e^{\mu} 1_{|x - y| \leq \delta} \, dx \, dy, \quad (A.28)
\]

and it is easy to conclude from there.

A.3. Estimates on the covariance. Let us start with (2.9) Since from (1.12) we have \( Q_u(x, y) \leq 1_{|x - y| \leq e^{-u}} \) we obtain that for \( |x - y| \leq 1, \)

\[
\mathcal{K}_t(x, y) \leq \int_0^t 1_{|x - y| \leq e^{-u}} \, du = t \wedge \log_+ \frac{1}{|x - y|}.
\]

(A.29)

Now for a lower bound, using the \( C^2 \) regularity of \( \kappa \) around 0, (1.12) implies that

\[
Q_u(x, y) \geq 1 - Ce^{2u}|x - y|^2
\]

This implies that if \( |x - y| \leq e^{-s} \) then

\[
\mathcal{K}_s(x, y) \geq s - C \int_0^t e^{2u}|x - y|^2 \geq s - C'
\]

Using the monotonicity in \( s \), applying this bound for \( s = t \wedge \log_+(1/|x - y|) \) we obtain

\[
\mathcal{K}_s(x, y) \geq t \wedge \log_+(1/|x - y|) - C.
\]

(A.30)

The upper and lower bound remain valid with an additional constant when adding \( K_0(x, y) \) which is continuous and hence locally bounded, and convoluting the upper or the lower bound with \( \theta_\varepsilon \) on one or both variables yields (2.8) for \( K_{t, \varepsilon} \) and \( \mathcal{K}_{t, \varepsilon, 0} \).

Appendix B. Weak convergence of measure

Let us first present a metric which corresponds to the topology of weak convergence on \( \mathbb{R}^d \) (for locally finite, nonnegative measures). We let \( \pi_R \) denote the Lévy-Prokhorov metric between two finite measures on \( B(0, R) \)

\[
\pi_R(\mu, \nu) := \inf \{ \varepsilon > 0 : \forall A \in \mathcal{B}_R, \ \mu(A^\varepsilon) \leq \nu(A) + \varepsilon \quad \text{and} \quad \nu(A^\varepsilon) \leq \mu(A) + \varepsilon \} \quad (B.1)
\]

where \( \mathcal{B}_R \) is collection of Borel subsets of \( B(0, R) \) and

\[
A^\varepsilon := \{ x \in B(0, R) : \exists y \in A, |y - x| \leq \varepsilon \}.
\]

For locally finite measures \( \mu \) and \( \nu \) on \( \mathbb{R}^d \), the metric \( d \) defined by

\[
d(\mu, \nu) := \sum_{R=1}^{\infty} 2^{-R} \max(\pi_R(\mu, \nu), 1),
\]

where in the summand \( \mu \) and \( \nu \) are identified with their restriction on \( B(0, R) \), generates the topology of weak convergence of (1.16). We say that a sequence of random measure \( M_n \) converges weakly in probability if

\[
\forall \varepsilon \in (0, 1], \ \lim_{n \to \infty} \mathbb{P} \left[ d(M_n, M) > \varepsilon \right] = 0. \quad (B.2)
\]

Since the topology of weak convergence is separable, convergence in probability implies that there exists a subsequence that converges weakly almost surely. Recall that \( \lambda \) denote the Lebesgue measure.
Proposition B.1. Let $M_n$ be a sequence of non-negative random measures and let us assume that for any fixed $E \in \mathcal{B}_b$, $M_n(E)$ converges in probability towards a finite limit $\overline{M}(E)$. Then the following holds

(i) For any $f \in \mathcal{B}_b$, $M_n(f)$ converges in probability towards a finite limit $\overline{M}(f)$.

(ii) There exists a random measure $M$ towards which $M_n$ converges in probability.

(iii) If additionally there exist $K > 0$ such that $\mathbb{E}[M_n(E)] \leq K\lambda(E)$ for all $E \in \mathcal{B}_b$ and $n$, then for all $f \in \mathcal{B}_b$, $\mathbb{P}[\overline{M}(f) = M(f)] = 1$.

If the convergence of $M_n(E)$ holds a.s. then the convergence (i) and (ii) also hold a.s.

Proof. For (i) we observe that the statement is obviously true for simple functions with bounded support. Given $f \in \mathcal{B}_b$, and $k \geq 1$, setting $E := \{x : f(x) \neq 0\}$ we can find an increasing sequence of simple functions $(f_k)_{k \geq 1}$ such that $0 \leq (f - f_k) \leq k^{-1}1_E$. This entails

$$0 \leq M_n(f) - M_n(f_k) \leq k^{-1}M_n(E)$$

and hence that $M_n(f)$ converges to $\lim_{k \to \infty} \overline{M}(f_k)$. For (ii) note that (i) already entails the convergence of $(M_n(f))_{f \in F}$ for any finite family $F \subset C_c(\mathbb{R}^d)$. Hence it is sufficient to prove that the sequence $(M_n)_{n \geq 0}$ is tight for the weak convergence. For almost-sure convergence we must check that $\sup_n M_n(B(0,R))$ is finite for every $R \in \mathbb{N}$ and for convergence in probability we need to check that for every $R \in \mathbb{N}$ the sequence $M_n(B(0,R))$ is tight. In both cases this is a consequence the fact that, by assumption $M_n(B(0,R))$ converges.

For (iii), given $g \in C_c(\mathbb{R}^d)$ nonnegative, since by (ii), $M_n(g) \to M(g)$ Fatou implies that

$$\mathbb{E}[M(g)] \leq K \int g(x)dx.$$  \hspace{1cm} (B.3)

This implies that the measure $E \mapsto \mathbb{E}[M(E)]$ is absolutely continuous with respect to Lebesgue with density bounded above by $K$. Now given $f \in \mathcal{B}_b$ and $\delta$ we let $g \in C_c(\mathbb{R}^d)$ be such that $|f - g|dx \leq \delta$ we have

$$\mathbb{E} \left[ |M(f) - \overline{M}(f)| \right] \leq \mathbb{E} \left[ |M(f) - M(g)| + \overline{M}(f) - M(g)| \right].$$  \hspace{1cm} (B.5)

Our observation concerning the density of $\mathbb{E}[M(\cdot)]$ implies that the first term is smaller than $K\delta$. The second term is smaller than $\mathbb{E} \overline{M}(\{f - g\})$ which by Fatou is smaller than

$$\liminf_{n \to \infty} \mathbb{E} \left[ M_n(\{|f - g|\}) \right] \leq K\delta.$$  \hspace{1cm} (B.6)

Since $\delta$ is arbitrary this is sufficient to conclude. \qed

Appendix C. Extending the main result beyond star-scale invariance

Using a decomposition result from [13], we show that our main theorems can be extended to sufficiently regular kernels defined on an arbitrary domain $\mathcal{D}$. Let us recall the definition for the Sobolev space with index $s \in \mathbb{R}$ on $\mathbb{R}^k$ which is the Hilbert space of complex valued function associated with the norm

$$\|\varphi\|_{H^s(\mathbb{R}^k)} := \left( \int_{\mathbb{R}^k} (1 + |\xi|^2)^s |\hat{\varphi}(\xi)|^2 d\xi \right)^{1/2},$$  \hspace{1cm} (C.1)

where $\hat{\varphi}(\xi)$ is the Fourier transform of $\varphi$ defined for $\varphi \in C_c^\infty(\mathbb{R}^k)$ by $\hat{\varphi}(\xi) = \int_{\mathbb{R}^k} e^{i\xi x} \varphi(x) dx$. For $U \subset \mathbb{R}^k$ open, the local Sobolev space $H^s_{\text{loc}}(U)$ denotes the function which belongs to $H^s(U)$ after multiplication by an arbitrary smooth function with compact support

$$H^s_{\text{loc}}(U) := \left\{ \varphi : U \to \mathbb{R} \mid \rho \varphi \in H^s(\mathbb{R}^d) \text{ for all } \rho \in C_c^\infty(U) \right\}.$$  \hspace{1cm} (C.2)
where above $\rho \varphi$ is identified with its extension by zero on $\mathbb{R}^k$. The following result is a consequence of [3] Theorem 4.5.

**Proposition C.1.** If $K$ is a positive definite kernel on $\mathcal{D}$ that can be written in the form (1.1) with $L \in H^s_{\text{loc}}(\mathcal{D}^2)$ with $s > d$, then for every $z \in \mathcal{D}$, there exist $\delta_z > 0$ such that the restriction of $K$ to $B(z, \delta_z)$ has an almost star-scale invariant part.

A consequence of the above is the following generalization of Theorem 1.3. A similar generalization can be made for Theorem 1.9. The set of continuous resp. measurable functions on $\mathcal{D}$ vanishing outside a compact is denoted by $C_c(\mathcal{D})$ resp. $B_b(\mathcal{D})$. If $K$ is a positive definite kernel on $\mathcal{D}$ that can be written in the form (1.1) then we define a centered Gaussian field $X$ indexed by $C_c(\mathcal{D})$ as in Section 1.1 and given $\varepsilon \in (0,1]$ set $X_\varepsilon(x) := \langle X, \theta_\varepsilon \rangle$ for $x \in \mathcal{D}_\varepsilon$ where $\mathcal{D}_\varepsilon := \{ x \in \mathcal{D} : \forall y \in \mathbb{R}^d \setminus \mathcal{D}, |x - y| > 2\varepsilon \}$ (the definition ensures that $\theta_\varepsilon$ can be identified with an element of $C_c(\mathcal{D})$). Then we define for $f \in B_b(\mathcal{D})$

$$M_\varepsilon(f) := \int_{\mathcal{D}_\varepsilon} e^{\frac{1}{2} \varepsilon^2 X_\varepsilon(x) - dE[X_\varepsilon(x)^2]} dx.$$ (C.3)

**Theorem C.2.** If $K$ is a positive definite kernel on $\mathcal{D}$ that can be written in the form (1.1) with $L \in H^s_{\text{loc}}(\mathcal{D}^2)$ with $s > d$. Then there exists a random measure $M'$ on $\mathcal{D}$ with dense support such that for any choice of mollifier $\theta$, $\sqrt{\pi \log(1/\varepsilon)}/2 M_\varepsilon$ converges weakly in probability towards a limit $M'$. For every $f \in B_b(\mathcal{D})$ we have $\lim_{\varepsilon \to 0} M_\varepsilon^{(\sqrt{2d})}(f) = M'(f)$ in probability and for any compact subset $D \subset \mathcal{D}$, we have (recall (1.17))

$$\sup_{E \subset D} M'(E)/\phi(|E|) < \infty.$$ (C.3)

**Sketch of proof.** By Proposition [3.1] it is sufficient to prove the convergence in probability of $M_\varepsilon^{(\sqrt{2d})}(E)$ for every $E \in B_b(\mathcal{D})$. Since $\overline{E}$, the topological closure of $E$ is compact, using Proposition [C.1] we can cover it by a finite number of balls $(B(z_i, \delta_i))_{i \in I}$. We can associate to this cover a partition of unity $(\rho_i)_{i \in I}$ (the $\rho_i$ are continuous, vanish outside $B(z_i, \delta_i)$ and $\sum_{i \in I} \rho_i(x) = 1$ for $x \in \overline{E}$). Applying Theorem 1.2 and Proposition [B.1] we obtain that $M_\varepsilon^{(\sqrt{2d})}(\rho_i 1_E)$ converges in probability for every $i \in I$ and hence $M_\varepsilon^{(\sqrt{2d})}(\rho_i 1_E)$ converges for every $i$, and thus $M_\varepsilon^{(\sqrt{2d})}(E)$. We deduce (C.3) from (1.18) in a similar fashion. \[\square\]

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