Analytical model of fracture propagation in carbonate reservoirs

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Abstract. The fracture propagation length and propagation angle can be calculated quickly using the fracture propagation analytical model. But for fractured-vuggy carbonate reservoirs, the addition of a discontinuous medium in the analytical model makes it difficult to obtain accurate results via calculation. In this paper, an analytical solution model for fracture propagation in carbonate reservoirs is established. On the premise of ensuring calculation accuracy, the discontinuous displacement algorithm is used to introduce the fracture and hole characteristics into the calculation. The model comprehensively considers the interference of in-situ stress, fracture tip stress, and stress field of fractures and holes on fracture propagation. The calculation accuracy of the model in a fractured-vuggy carbonate reservoir is verified by a true triaxial fracturing physical simulation experiment. The model can be used for subsequent research on the fracture propagation law of carbonate reservoirs containing discontinuous bodies, such as fractures and caves.

Key words: Carbonate reservoir; Fracture propagation; Numerical analysis

1. Introduction
Fracture-cavity carbonate reservoirs are low in porosity and permeability. In addition, karst cave is the main reservoir of oil and gas. Therefore, the key to increasing production is to connect fractures with as many karst caves as possible. The fracture propagation length and propagation angle can be calculated quickly using an analytical model. However, for fractured-vuggy carbonate reservoirs, the addition of a discontinuous medium in the analytical model makes it difficult to obtain accurate results via calculation. Kaiser et al. [1,2] simulated the intersection of hydraulic fractures with karst caves and revealed that fracture propagation is affected by the difference between the maximum and minimum in-situ stresses and the difference between the minimum horizontal and vertical ground stresses. Additionally, Cheng et al. [3] established a numerical model of the interaction between hydraulic fractures and vugs, solved the stress field and the pressure field of hydraulic fracturing with the extended finite element method, and simulated how vugs affect fracture propagation. Cai [4] studied the accurate calculation of the stress around the cave and the crack tip, and creatively proposed an effective calculation method for solving the elastic stress solution around the polygonal cave in the infinite plane under any load. Xu et al. [5] established the plane stress model of circular holes and crack coexistence by using the generalized parameters of circular singular zone and analyzed the influence of circular hole position and geometric parameters on the stress intensity factor at the tip of I–II mixed type crack.

However, there is no analytical method to calculate the fracture propagation. Therefore, in the
study of fracture propagation in a fractured-vuggy carbonate reservoir, fractures and holes are added for the first time, and the stress state of the fracture tip area of the composite fracture is considered. The stress field of the fracture tip of the composite fracture is deduced and calculated.

2. Establishment of analytical model

The analytical model established in this paper uses the fracture initiation model around the wellbore to determine whether the fracture is initiated. According to the principle of stress superposition, the distribution of ground stress field around the wellbore can be expressed as the following formula [6]:

\[ \sigma_{\theta\theta}' = (\sigma_x + \sigma_y + \sigma_{z\theta}) + 2(\sigma_x + \sigma_y - \sigma_{x\theta}) \cos 2\theta' - 2(\sigma_x - \sigma_y)(\cos 2\theta + 2 \cos 2\theta \cos 2\theta') - 4\sigma_{xy}(1 + 2 \cos 2\theta) \sin 2\theta - 4\sigma_{z\theta} \cos 2\theta' - 2p_w(1 + \cos 2\theta') + 2\delta\left(\frac{1 - 2\nu}{1 - \nu} - \frac{\phi}{1 - \nu}\right)(p_w - p_0)(1 + \cos 2\theta') \]

\[ \sigma_r = p_w - \delta\phi(p_w - p_0) \]

\[ \sigma_{z\theta} = \sigma_z - 2\nu(\sigma_s - \sigma_y) \cos 2\theta - 4\nu\sigma_y \sin 2\theta \]

\[ -c p_w - \delta\left(\frac{1 - 2\nu}{1 - \nu} - \frac{\phi}{1 - \nu}\right)(p_w - p_0) \]

\[ \tau_{yx} = 2(-\tau_{xx} \sin \theta + \tau_{yz} \cos \theta) \]

\[ \tau_{xz} = \tau_{z\theta} = 0 \]

\[ \tau_{zy} = 2\tau_z \]

\[ \tau_{x\theta} = \tau_{\theta z} = \tau_{r z} = 0 \]

\[ \sigma_r \] is the axial stress on the shaft wall; \( \sigma_{\theta\theta}' \) is the circumferential shear stress of sidewall when the angle is \( \theta \); \( \sigma_{z\theta} \) is axial stress of shaft lining at angle \( \theta \); \( \tau_{r\theta} \), \( \tau_{\theta z} \), \( \tau_{r z} \) is shear stress of shaft lining in cylindrical coordinate system; \( \sigma_x \), \( \sigma_y \), \( \sigma_z \) is normal stress of shaft lining under Cartesian coordinate system. \( \tau_{xy} \), \( \tau_{yz} \), \( \tau_{xx} \) is the tangential stress on the shaft wall under Cartesian coordinate system. \( p_w \) is bottomhole pressure; \( \nu \) is Poisson's ratio.

The maximum tensile stress at any angle \( \theta' \) on the hole wall is the second principal stress:

\[ \sigma(\theta') = \frac{1}{2} \left[ (\sigma_{\theta\theta}' + \sigma_{x\theta}) + \sqrt{(\sigma_{\theta\theta}' - \sigma_{x\theta})^2 + 4\tau_{z\theta}^2} \right] \]

According to the maximum circumferential stress criterion, the shaft wall breaks when \( \sigma(\theta') \) reaches the maximum value \( \sigma(\theta_{0})_{\text{max}} \). Considering the influence of pore pressure, when the maximum tensile stress of rock at the angle \( \theta_0 \) is greater than the tensile strength \( \sigma_t \), the rock will break.

After determining the crack initiation pressure, this value can be taken as the initial value of the stress at the crack tip in the analytical model of wellbore crack initiation and near-well expansion.

The fracture propagation is based on the maximum circumferential stress theory proposed by Erdogan, F and Sih, G. C [7]. This theory holds that when the maximum circumferential tensile stress \( \sigma_\theta \) at the crack tip reaches the critical value \( \sigma_{\theta c} \), the fracture expands, and its extension direction is parallel to the maximum circumferential tensile stress action surface.

Since there is generally a turning in the process of crack propagation, the hydraulic fracture should be considered as a composite fracture of type I and type II. For type I fracture, the formula is

\[ \sigma^I_\theta = \frac{K_I}{\sqrt{2\pi r}} \cos^3 \frac{\theta}{2} \]

For type II fractures, the formula is

\[ \sigma^{II}_\theta = -\frac{K_{II}}{\sqrt{2\pi r}} \cos^3 \frac{\theta}{2} \sin \theta \]

The circumferential stress \( \sigma_\theta \) of composite cracks can be obtained by superimposing the circumferential stress generated by type I and type II cracks.
When the circumferential stress at the crack tip takes the maximum value, \( \frac{\partial \sigma_\theta}{\partial \theta} = 0 \) and \( \frac{\partial^2 \sigma_\theta}{\partial \theta^2} < 0 \), the obtained \( \theta \) angle is the crack propagation angle.

\[
\cos \frac{\theta}{2} [K_I \sin \theta + K_{II}(3 \cos \theta - 1)] = 0
\]

where \( K_I \) and \( K_{II} \) are the stress intensity factors of I and II cracks.

When \( \cos \frac{\theta}{2} = 0 \), \( \theta = \pm \pi \) is the fracture surface, which is inconsistent with the actual situation.

Therefore, the crack propagation angle \( \theta_0 \) can be determined according to the following formula:

\[
K_I \sin \theta_0 + K_{II}(3 \cos \theta_0 - 1) = 0
\]

Equation (3.11) is only a necessary condition for calculating the crack tip propagation angle. To make \( \sigma_\theta \) take the maximum value, it is necessary to satisfy the equation \( \frac{\partial^2 \sigma_\theta}{\partial \theta^2} < 0 \). Based on the above equation, the crack propagation angle can be obtained by trigonometric function transformation:

\[
\theta_0 = (K_I, K_{II}) = \begin{cases} 0 & \text{if } K_{II} = 0 \\ 2 \arctan \left( \frac{1}{4} \left( \frac{K_I}{K_{II}} - sgn(K_{II}) \sqrt{\frac{K_I^2}{K_{II}^2} + 8} \right) \right) & K_{II} \neq 0 \end{cases}
\]

When the tensile stress at the crack tip along the direction of \( \theta_0 \) reaches the critical value \( \sigma_{\theta_c} \), the crack extends. According to the definition of stress intensity factor, \( \sigma_{\theta_c} \sqrt{2\pi r} = K_{IC} \) can be written as follows:

\[
\frac{1}{2} \cos \frac{\theta_0}{2} [K_I(1 + \cos \theta_0) - 3K_{II}\sin \theta_0] = K_{IC}
\]

The equivalent stress intensity factor \( K_e \) at the crack tip is

\[
K_e = \frac{1}{2} \cos \frac{\theta_0}{2} [K_I \sin \theta + K_{II}(3 \cos \theta - 1)]
\]

The criterion of crack propagation is as follows: for I–II type composite cracks, when \( K_e = K_{IC} \), the crack propagates along the direction of \( \theta_0 \) with the original crack surface.

The innovation and advantage of the above calculation method is that when calculating the composite expansion of hydraulic fractures, I and II, the stress state of the crack tip region of the composite fracture is considered in the analytical model creatively, and the stress field at the crack tip of the composite fracture is deduced and calculated.

In the study of fracture propagation in fractured-vuggy reservoirs, the stress field at the crack tip produces obvious stress concentration due to the influence of hydraulic action, and the stress state near the crack tip should be considered. For I–II composite fractures, the stress field at the crack tip can be expressed as:

\[
\begin{align*}
\sigma_x &= \frac{K_I}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \left( 1 - \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \right) + O(r^{1/2}) \\
\sigma_y &= \frac{K_I}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \left( 1 + \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \right) + O(r^{1/2}) \\
\tau_{xy} &= \frac{K_{II}}{\sqrt{2\pi r}} \sin \frac{\theta}{2} \cos \frac{\theta}{2} \cos \frac{3\theta}{2} + O(r^{1/2}) \\
\sigma_x' &= \frac{K_{II}}{\sqrt{2\pi r}} \sin \frac{\theta}{2} \left( 2 + \cos \frac{\theta}{2} \cos \frac{3\theta}{2} \right) + O(r^{1/2}) \\
\sigma_y' &= \frac{K_{II}}{\sqrt{2\pi r}} \sin \frac{\theta}{2} \cos \frac{\theta}{2} \cos \frac{3\theta}{2} + O(r^{1/2}) \\
\tau_{xy}' &= \frac{K_{II}}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \left( 1 - \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \right) + O(r^{1/2})
\end{align*}
\]

The stress field between I–II composite fractures in the above formula is introduced into the discontinuous Crouch calculation method for calculating the displacement of fracture propagation length and angle. The mechanical characteristics of natural cracks and holes are introduced in the analytical calculation.

For natural fractures in the formation, affected by in-situ stress and tectonic stress, there will be
stress field between natural fractures, which is called inter-fracture stress field. The stress expression of stress field between natural fractures is as follows:

\[ \sigma_n^i = \sum_{j=1}^{N} \left( A_{nj}^i D_s^j + A_{nj}^i D_n^j \right) \]  \hspace{1cm} (17)

\[ \sigma_l^i = \sum_{j=1}^{N} \left( A_{nj}^i D_s^j + A_{nj}^i D_n^j \right) \]  \hspace{1cm} (18)

When calculating the hydraulic fracture propagation in fractured-vuggy reservoirs, the hydraulic fracture is close to the natural fracture, and the stress concentration area at the crack tip and the stress field between natural fractures have an impact on the fracture propagation. At this time, the two stress fields need to be superimposed and calculated.

Therefore, the stress field during the crack propagation constitutes the far-field stress, the stress concentration area at the crack tip, and the stress area between natural cracks.

The calculation method is as follows: the hydraulic fracture width is fixed, and the flow field is calculated by power law fluid. According to the Newton–Raphson iteration method, the PKN model concentration area at the crack tip, and the stress area between natural cracks.

The calculation method is as follows: the hydraulic fracture width is fixed, and the flow field is calculated by power law fluid. According to the Newton–Raphson iteration method, the PKN model

\[ \frac{\partial \sigma_n}{\partial x} = -64 \frac{q u}{\pi w H} \frac{\partial q}{\partial x} = -\frac{\sigma_n}{4} \frac{\partial w}{\partial t} \]  \hspace{1cm} (19)

\[ \frac{\partial \sigma_l}{\partial x} - \frac{\partial w}{\partial t} = 0 \]  \hspace{1cm} (20)

\[ A(t) = \frac{Q w}{4 \pi c^2} \left( e^{-\frac{r}{R}} - 1 \right) \]  \hspace{1cm} (22)

\[ x = \frac{2 \sqrt{\pi t}}{w} \]  \hspace{1cm} (23)

The analytical model of crack propagation length considering fluid filtration, is presented.

Combined with stress intensity factor and rock fracture toughness, judge whether can continue to expand and expansion angle. At this time, the stress intensity factor of crack tip is solved by using the stress and displacement equations of the crack surface. The influence of fracture tip stress field, in-situ stress, and natural fracture stress field is introduced in the calculation of stress intensity factor, which makes the model accurate.

\[ F_s = \sum_{j=1}^{N} A_{sj}^i D_s^j + \sum_{j=1}^{N} A_{nj}^i D_n^j \]  \hspace{1cm} (24)

\[ F_n = \sum_{j=1}^{N} A_{nj}^i D_s^j + \sum_{j=1}^{N} A_{nj}^i D_n^j \]  \hspace{1cm} (25)

\[ K_i = \sqrt{\pi a F_n} = \frac{n D_n}{(1-\nu)\sqrt{a}} \]  \hspace{1cm} (26)

\[ K_{II} = \sqrt{\pi a F_n} = \frac{D_n}{D_s \sqrt{1-\nu} \sqrt{a}} \]  \hspace{1cm} (27)

\[ K_I = \sqrt{\frac{GD_n}{2(1-\nu)\sqrt{\pi a}}} \]  \hspace{1cm} (28)

\[ K_{II} = \sqrt{\frac{GD_n}{2(1-\nu)\sqrt{\pi a}}} \]  \hspace{1cm} (29)

The correction coefficient is used to eliminate singularity, and the I–II crack strength factor is solved. Thus, the angle of propagation direction is calculated, and the in-situ stress term is included in the strength factors of crack tip of type I and type II above.

After linear superposition of in-situ stress and stress field near hydraulic fracture tip:

\[ \sigma_{xx} = -\sigma_4 + \frac{K_1}{\sqrt{2\pi r_c}} \cos \frac{\theta}{2} \left( 1 - \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \right) \]  \hspace{1cm} (30)

\[ \sigma_{yy} = -\sigma_3 + \frac{K_1}{\sqrt{2\pi r_c}} \cos \frac{\theta}{2} \left( 1 + \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \right) \]  \hspace{1cm} (31)

The maximum tensile stress at the tip \( \theta = 0 \) of type I hydraulic fracture is

\[ \sigma_{yy} \big|_{\theta=0} = -\frac{K_1}{\sqrt{2\pi r_c}} \sigma_3 \]  \hspace{1cm} (32)
When the fracture stress satisfies equation (3.32), the hydraulic fracture will penetrate the natural fracture. Otherwise, natural fractures occur at tangential slip; hydraulic fractures will not penetrate natural fractures.

\[ \sigma_{yy}\big|_{\theta=0} = T_0 \quad (32) \]

\[ |\tau_\theta| < \tau_0 - \mu\sigma_\theta \]

In the formula above, the total normal stress and the total shear stress acting on the natural fracture surface when \( \sigma_0 \) and \( \tau_0 \) approach the angle of 0 respectively. \( T_0 \) and \( \tau_0 \) are tensile strength and cohesion of rock mass. \( \mu \) is the friction coefficient of natural fracture surface.

The stress distribution around the hydraulic fracture approaching the cavity is

\[ \sigma_z = \sigma_v - 2\nu(\sigma_H - \sigma_h) \cos 2\theta \]

\[ \sigma_r = \frac{1}{(V_0 + \pi R^2 h)}(V_0 P_0 + \pi R^2 h P_p) \]

\[ \sigma_\theta = \sigma_H + \sigma_h - 2(\sigma_H - \sigma_h) \cos 2\theta - \frac{1}{V_0 + \pi R^2 h}(V_0 P_0 + \pi R^2 h P_p) \quad (33) \]

When the tensile stress \( \sigma_\theta \) produced by hydraulic cracks on the hole wall reaches the tensile strength \( S_t \) of rock, the hydraulic cracks can propagate into the hole. Otherwise, it will stop expanding.

\[ \sigma_\theta < -S_t \]

\[ 3\sigma_h - \sigma_H + S_t - \frac{1}{V_0 + \pi R^2 h}(V_0 P_0 + \pi R^2 h P_p) < 0 \quad (34) \]

### 3. Model calculation and experimental verification

The indoor large-scale true triaxial experiment was performed to verify the accuracy of the analytical model in Chapter 2 through the experimental results. Firstly, the artificial rock samples containing fractures and holes are constructed, and the experiments are carried out by true triaxial fracturing instrument. The communication between artificial and natural fractures and holes is counted. The experimental parameters are as Table 3.1:

| Number | Fracture angle | Difference between maximum and minimum in-situ stress/MPa | Delivery capacity | Viscosity/mPa · s |
|--------|---------------|----------------------------------------------------------|------------------|-----------------|
| 01     | 25°           | 6                                                        | 20ml/min         | 1               |

The experimental instrument is illustrated to simulate the real reservoir stress (see Fig. 3.1)
Figure 3.1. Schematic diagram of hydraulic fracturing simulation experimental device.

Rock samples after fracturing are as follows (see Fig. 3.2):

Figure 3.2. Sample model diagram after fracturing.

According to the fracturing operation parameters, the test displacement, viscosity and fracturing fluid injection are determined, and the extension state of hydraulic fractures in the formation is calculated. The crack stress state under the influence of multiple cracks is calculated, and the propagation patterns of hydraulic cracks with in-situ stresses, different natural crack lengths, and dip angles are judged (see Fig. 3.3).

Figure 3.3. Analytical simulation results

Corresponding to the experimental and simulation results, the fracture in the simulation did not penetrate after communicating with the natural fracture and extended along the direction of the natural fracture to the next fracture. The experimental results also showed the same communication mode, which proved that the fracture network morphology produced by artificial fracture and natural fracture, and the interaction trend was consistent. The accuracy of the model was verified at the laboratory scale.
4. The influence of different factors on fracture propagation simulated by analytical model

Different parameters are set in the model, and the viscosity of the fracturing fluid is changed to determine the expansion of the fracture in the formation under the same other conditions. Fig. 4.1 (a) shows the simulation of the fracture expansion when the viscosity of the high fracturing fluid is 30 mpa·s, and Fig. 4.1 (b) shows the simulation of the fracture expansion when the viscosity of the low fracturing fluid is 10 mpa·s.

![Figure 4.1](image)

**Figure 4.1.** Crack propagation under different fracturing fluid viscosities

It can be seen from the figure that the total range of natural fractures communicated is the largest when the viscosity of low fracturing fluid is 10 mpa·s, which is easier to form a complex fracture network structure and achieve better fracturing effect. When the viscosity of fracturing fluid is 30 mpa·s the number of fractures and caves communicated is less, and the range of fracturing transformation is also small. Therefore, low viscosity fracturing fluid is more effective in transforming reservoirs.

5. Conclusion

1. In this paper, an analytical solution model for fracture propagation in carbonate reservoirs is established. On the premise of ensuring the calculation accuracy, the discontinuous displacement algorithm is used to introduce the fracture and hole characteristics into the calculation.

2. The model comprehensively considers the interference of in-situ stress, fracture tip stress, and stress field of fractures and holes on fracture propagation.

3. The calculation accuracy of the model in fractured-vuggy carbonate reservoirs is verified via a true triaxial fracturing physical simulation experiment.

4. Through the analytical model, the fracture propagation under different viscosities of fracturing fluids is obtained, which proves that it is easier for fracturing fluids with low viscosities to
communicate with more natural fractures and vugs.

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