Cluster-tilted algebras as trivial extensions

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Abstract

Given a finite dimensional algebra $C$ (over an algebraically closed field) of
global dimension at most two, we define its relation-extension algebra to
be the trivial extension $C \ltimes \text{Ext}^2_C(DC, C)$ of $C$ by the $C$-$C$-bimodule
$\text{Ext}^2_C(DC, C)$. We give a construction for the quiver of the relation-
extension algebra in case the quiver of $C$ has no oriented cycles. Our
main result says that an algebra $\tilde{C}$ is cluster-tilted if and only if there ex-
ists a tilted algebra $C$ such that $\tilde{C}$ is isomorphic to the relation-extension
of $C$.

1 Introduction

Cluster categories were introduced in [6], and, for type $A_n$ also in [12], as a means
for a better understanding of the cluster algebras of Fomin and Zelevinsky [14, [15].
They are defined as follows: let $A$ be a hereditary algebra, and $\mathcal{D}^b(\text{mod } A)$
be the derived category of bounded complexes of finitely generated $A$-modules,
then the cluster category $\mathcal{C}_A$ is the orbit category of $\mathcal{D}^b(\text{mod } A)$ under the action
of the functor $F = \tau^{-1} [1]$, where $\tau$ is the Auslander-Reiten translation in
$\mathcal{D}^b(\text{mod } A)$ and $[1]$ is the shift.

In [7], Buan, Marsh and Reiten defined the cluster-tilted algebras as follows.
Let $A$ be a hereditary algebra, and $\tilde{T}$ be a tilting object in $\mathcal{C}_A$, that is, an
object such that $\text{Ext}^1_{\mathcal{C}_A}(\tilde{T}, \tilde{T}) = 0$ and the number of isomorphism classes of
indecomposable summands of $\tilde{T}$ equals the number of isomorphism classes of
simple $A$-modules. Then the endomorphism algebra $\text{End}_{\mathcal{C}_A}(\tilde{T})$ is called cluster-
tilted. Since then, these algebras have been the subject of many investigations,
see, for instance, [7, 8, 9, 10, 11, 12, 13, 24]. In several particular cases, it was
shown that the quiver of a cluster-tilted algebra was obtained from that of a
tilted algebra by replacing relations by arrows, see, for instance [10, 11]. Our
objective in this paper is to prove this statement in a more general context (not
depending on the representation type). This is achieved by looking at cluster-
tilted algebras as trivial extensions of tilted algebras by a bimodule which we
explicitly describe (compare [3]).

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For this purpose, we let $C$ be a finite dimensional algebra of global dimension two (over an algebraically closed field), and consider the $C$-$C$-bimodule $\text{Ext}^2_{C}(DC, C)$ with the natural action. The trivial extension $C \ltimes \text{Ext}^2_{C}(DC, C)$ is called the relation-extension algebra of $C$. Our first main result (Theorem 2.6) describes the quiver of the relation-extension of $C$ in the case where the quiver of $C$ has no oriented cycles: we prove that indeed this quiver is given by replacing each element in a (minimal) system of relations by an arrow (going in the opposite direction to the relation). We then prove the main result of this paper.

**Theorem 1.1** An algebra $\tilde{C}$ is cluster-tilted if and only if there exists a tilted algebra $C$ such that $\tilde{C}$ is the relation-extension of $C$.

We note that several tilted algebras may correspond to the same cluster-tilted algebra, so this mapping is not bijective. On the other hand, there clearly exist relation-extension algebras which are not cluster-tilted.

Combining the above theorem with Theorem 2.6 we deduce the construction of the quiver of a cluster-tilted algebra. This allows, for instance, as done in [11], to relate the list of tame concealed algebras of Happel and Vossieck [18] with Seven’s list of minimal infinite cluster quivers [23].

This paper consists of two sections. The first one describes relation-extension algebras and their quivers, and the second is devoted to the cluster-tilted algebras. Moreover, we give several examples.

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## 2 Relation-extension algebras

### 2.1 The definition

Throughout this paper, algebras are basic and connected finite dimensional algebras over a fixed algebraically closed field $k$. For an algebra $C$, we denote by $\text{mod } C$ the category of finitely generated right $C$-modules and by $\mathcal{D}^b(\text{mod } C)$ the derived category of bounded complexes over $\text{mod } C$. The functor $D = \text{Hom}_k(-, k)$ is the standard duality between $\text{mod } C$ and $\text{mod } C^{\text{op}}$. For facts about $\text{mod } C$ or $\mathcal{D}^b(\text{mod } C)$, we refer to [2, 22, 17].

Let $C$ be an algebra. We recall that the **trivial extension** of $C$ by a $C$-$C$-bimodule $M$ is the algebra $C \ltimes M$ with underlying $k$-vector space $C \oplus M = \{(c, m) \mid c \in C, m \in M\}$ and the multiplication defined by $(c, m)(c', m') = (cc', cm' + mc')$ for $c, c' \in C$ and $m, m' \in M$. For trivial extension algebras, we refer to [16, 11].

In this section, we introduce a particular class of trivial extension algebras which are useful for studying the cluster-tilted algebras.
Definition 2.1 Let $C$ be a finite dimensional algebra of global dimension at most two, and consider the $C$-$C$-bimodule $\text{Ext}^2_C(DC, C)$ (with the natural action). The trivial extension $C \ltimes \text{Ext}^2_C(DC, C)$ is called the relation-extension of $C$.

Clearly, any hereditary algebra is (trivially) the relation-extension of itself. On the other hand, if $C$ is of global dimension equal to two (thus not hereditary) there exist two simple $C$-modules $S$ and $S'$ such that $\text{Ext}^2_C(S, S') \neq 0$. Denoting by $I$ the injective envelope of $S$ and by $P'$ the projective cover of $S'$, the short exact sequences

$$0 \rightarrow S \rightarrow I \rightarrow I/S \rightarrow 0$$

and

$$0 \rightarrow \text{rad} P' \rightarrow P' \rightarrow S' \rightarrow 0$$

induce an epimorphism $\text{Ext}^2_C(I, P') \rightarrow \text{Ext}^2_C(S, S')$. Thus $\text{Ext}^2_C(I, P') \neq 0$ and consequently $\text{Ext}^2_C(DC, C) \neq 0$.

2.2 A system of relations

We wish to describe the bound quiver of a relation-extension algebra. Let $C$ be an algebra. It is well-known that there exists a (uniquely determined) quiver $Q_C$ and an admissible ideal $I$ of the path algebra $kQ_C$ of $Q_C$ such that $C \cong kQ_C/I$, see, for instance, [5]. We denote by $(Q_C)_0$ the set of points of $Q_C$ and by $(Q_C)_1$ its set of arrows. For each point $x \in (Q_C)_0$, we let $e_x$ denote the corresponding primitive idempotent of $C$, and by $S_x, P_x, I_x$ respectively, the corresponding simple, indecomposable projective and indecomposable injective $C$-module.

Following [4], we define a system of relations for $C \cong kQ_C/I$ to be a subset $R$ of $\bigcup_{x, y \in (Q_C)_0} e_x I e_y$ such that $R$, but no proper subset of $R$, generates $I$ as a two-sided ideal of $kQ_C$. Thus, for any $x, y \in (Q_C)_0$, the elements of $R \cap (e_x I e_y)$ are linear combinations of paths (of length at least two) from $x$ to $y$. We need the following result.

Lemma 2.2 (([1, 1.2])) Let $C \cong kQ_C/I$ be such that $Q_C$ has no oriented cycles and $R$ be a system of relations for $C$. Then, for each $x, y \in (Q_C)_0$, the cardinality of the set $R \cap (e_x I e_y)$ is independent of the chosen system of relations for $C$, and equals $\dim_k \text{Ext}^2_C(S_x, S_y)$.

2.3 The quiver of a trivial extension

We start with the following easy lemma.

Lemma 2.3 Let $C$ be an algebra, and $M$ be a $C$-$C$-bimodule. The quiver $Q_{C \ltimes M}$ of the trivial extension of $C$ by $M$ is constructed as follows:

1. $(Q_{C \ltimes M})_0 = (Q_C)_0$
2. For \( x, y \in (Q_C)_0 \), the set of arrows in \( Q_{C \times M} \) from \( x \) to \( y \) equals the set of arrows in \( Q_C \) from \( x \) to \( y \) plus

\[
\text{dim}_k \frac{e_x M e_y}{e_x M (\text{rad } C) e_y + e_x (\text{rad } C) M e_y}
\]

additional arrows from \( x \) to \( y \).

Proof. Since \( M \subset \text{rad}(C \ltimes M) \), the quivers of \( C \ltimes M \) and of \( C \) have the same points. The arrows in the quiver of \( C \ltimes M \) correspond to a \( k \)-basis of the vector space

\[
\text{rad}(C \ltimes M) / \text{rad}^2 (C \ltimes M).
\]

Now, as a vector space

\[
\text{rad}(C \ltimes M) = \text{rad } C \oplus M
\]

and since \( M^2 = 0 \) in \( C \ltimes M \),

\[
\text{rad}^2 (C \ltimes M) = \text{rad}^2 C \oplus [M (\text{rad } C) + (\text{rad } C) M].
\]

Since \( \text{rad}^2 C \subset \text{rad} C \) and \( M (\text{rad } C) + (\text{rad } C) M \subset M \) and since the arrows of \( Q_C \) correspond to a basis of \( \text{rad } C / \text{rad}^2 C \), the additional arrows of \( Q_{C \times M} \) correspond to a \( k \)-basis of \( M /[M (\text{rad } C) + (\text{rad } C) M] \). The arrows from \( x \) to \( y \) are obtained upon multiplying by \( e_x \) on the left and by \( e_y \) on the right.

2.4 The top of \( \text{Ext}^2_C(DC, C) \)

In the situation of section 2.3, the \( C \)-\( C \)-bimodule \( M (\text{rad } C) + (\text{rad } C) M \) is the radical of \( M \), and the quotient \( M /[M (\text{rad } C) + (\text{rad } C) M] \) is its top. In the case of relation-extension algebras, we are interested in the top of \( \text{Ext}^2_C(DC, C) \).

Lemma 2.4 Let \( C \) be an algebra of global dimension two. The top of the \( C \)-\( C \)-bimodule \( \text{Ext}^2_C(DC, C) \) is isomorphic to \( \text{Ext}^2_C(\text{soc } DC, \text{top } C) \).

Proof. The short exact sequences

\[
\begin{array}{cccccccc}
0 & \rightarrow & \text{rad } C & \rightarrow & C & \rightarrow & \text{top } C & \rightarrow & 0 \\
0 & \rightarrow & \text{soc } DC & \rightarrow & DC & \rightarrow & DC / \text{soc } DC & \rightarrow & 0
\end{array}
\]

where \( i, j \) are the inclusions, induce a commutative diagram with exact rows and columns (the zeros are obtained from the condition that the global dimension
of $C$ is two).

By the commutativity of the lower-right square, there exists an epimorphism $p : \Ext^2_C(DC, C) \to \Ext^2_C(DC, \text{top} C)$. We thus only need to show that the kernel of $p$ is isomorphic to the radical $\Ext^2_C(DC, C) (\text{rad} C) + (\text{rad} C) \Ext^2_C(DC, C)$. Now an easy diagram chasing yields

$$\text{Ker } p = \text{Im } j^* + \text{Im } i_* .$$

Thus, it suffices to prove that

$$\text{Im } i_* = (\text{rad } C) \Ext^2_C(DC, C) \quad \text{and} \quad \text{Im } j^* = \Ext^2_C(DC, C) (\text{rad } C).$$

We only show the first equality, the second is shown similarly. Let

$$0 \rightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} DC \rightarrow 0$$

be a projective resolution of $DC$. By definition

$$\Ext^2_C(DC, C) = \text{Hom}_C(P_2, C)/\text{Im } \text{Hom}_C(d_2, C).$$

We first claim that the image of the map $i_0 = \text{Hom}_C(P_2, i) : \text{Hom}_C(P_2, \text{rad } C) \to \text{Hom}_C(P_2, C)$ is equal to $(\text{rad } C) \text{Hom}_C(P_2, C)$. Indeed, the product $rf$ with $r \in \text{rad } C$ and $f \in \text{Hom}_C(P_2, C)$ is easily seen to factor through rad $C$. Therefore, we have $(\text{rad } C) \text{Hom}_C(P_2, C) \subset \text{Im } i_0$. On the other hand, there is an isomorphism of $k$-vector spaces

$$(\text{rad } C) \text{Hom}_C(P_2, C) \cong \text{Hom}_C(P_2, \text{rad } C).$$

Since $i_0$ is injective, this establishes our claim.

Now, the image of $i_*$ is generated by the residual classes (modulo the image of $\text{Hom}_C(d_2, C)$) of the products $ig$, with $g \in \text{Hom}_C(P_2, \text{rad } C)$. These are the residual classes of the elements in $\text{Im } i_0$ thus, by our claim above, the residual classes of the elements of the form $rf$ with $r \in \text{rad } C$ and $f \in \text{Hom}_C(P_2, C)$. We deduce that $\text{Im } i_* = (\text{rad } C) \Ext^2_C(DC, C)$, as required.
Remark 2.5 The proof of this lemma can easily be generalised to show that, for an algebra $C$ of global dimension at most $m$, the top of the bimodule $\text{Ext}_C^m(\text{DC}, C)$ is equal to $\text{Ext}_C^m(\text{soc DC}, \text{top C})$.

2.5 The quiver of a relation-extension

The following theorem states that the quiver of the relation-extension algebra is obtained from the quiver of the original algebra by adding, for each pair of points $x, y$, one arrow from $x$ to $y$ for each relation from $y$ to $x$. This justifies the name “relation-extension”.

Theorem 2.6 Let $C \cong kQ/I$ be an algebra of global dimension at most two, such that $Q_C$ has no oriented cycles, and let $R$ be a system of relations for $C$. The quiver of the relation-extension algebra $C \ltimes \text{Ext}_C^2(\text{DC}, C)$ is constructed as follows:

(a) $(Q_C \times \text{Ext}_C^2(\text{DC}, C))_0 = (Q_C)_0$

(b) For $x, y \in (Q_C)_0$, the set of arrows in $Q_C \times \text{Ext}_C^2(\text{DC}, C)$ from $x$ to $y$ equals the set of arrows in $Q_C$ from $x$ to $y$ plus $\text{Card}(R \cap (e_y I e_x))$ additional arrows.

Proof. Let $S_1, S_2, \ldots, S_n$ denote a complete set of representatives of the isomorphism classes of simple $C$-modules, and set $S = \oplus_{i=1}^n S_i$. Since $C$ is basic, the module $S$ is isomorphic to the top of $C$ and to the socle of $\text{DC}$. By Lemma 2.2 the relations of $R$ correspond to a $k$-basis of $\text{Ext}_C^2(S, S)$. By Lemma 2.3 the $C$-$C$-bimodule $\text{Ext}_C^2(S, S)$ is isomorphic to the top of $\text{Ext}_C^2(\text{DC}, C)$. Lemma 2.4 then implies that the number of additional arrows from $x$ to $y$ equals the $k$-dimension of the vector space $e_x \text{Ext}_C^2(S, S)e_y = \text{Ext}_C^2(S_y, S_x)$, and the result follows.

In particular, the quiver of a non-hereditary relation-extension algebra always contains oriented cycles.

2.6 The indecomposable projectives

It would be useful to know a system of relations for the relation-extension algebra $C \ltimes \text{Ext}_C^2(\text{DC}, C)$ starting from one for $C$. In actual examples, such a system is easily obtained once we know the indecomposable projective modules. In order to state the next lemma, we need a notation: for each $x \in (Q_C)_0$, we denote by $\tilde{P}_x$ the corresponding indecomposable projective $C \ltimes \text{Ext}_C^2(\text{DC}, C)$-module. Also, we note that $C$-modules can always be considered as $C \ltimes \text{Ext}_C^2(\text{DC}, C)$-modules under the standard embedding.

Lemma 2.7 Let $C$ be an algebra of global dimension at most two. Then, for each $x \in (Q_C)_0$, we have a short exact sequence in $\text{mod} \left(C \ltimes \text{Ext}_C^2(\text{DC}, C)\right)$

$$0 \to \text{Ext}_C^2(\text{DC}, P_x) \to \tilde{P}_x \xrightarrow{p_x} P_x \to 0$$

where $p_x$ is a projective cover.
Proof. Since both \( P_x \) and \( \tilde{P}_x \) admit \( S_x \) as a simple top, there indeed exists a projective cover morphism \( p_x : \tilde{P}_x \to P_x \). On the other hand, \( \operatorname{Ext}^2_C(DC, P_x) \cong e_x \operatorname{Ext}^2_C(DC, C) \) is clearly a submodule of the \( C \ltimes \operatorname{Ext}^2_C(DC, C) \)-module \( P_x \). The result then follows from the isomorphism of \( k \)-vector spaces

\[
\tilde{P}_x = e_x (C \ltimes \operatorname{Ext}^2_C(DC, C)) \cong P_x \oplus \operatorname{Ext}^2_C(DC, P_x).
\]

2.7 An example

Example 2.8 Let \( C \) be given by the quiver

\[
\begin{tikzcd}
1 & 2 & 3 \\
\alpha & & \\
\beta & 1 & \\
\end{tikzcd}
\]

bound by the relation \( \alpha \beta = 0 \). Thus

\[
C_C = 1 \oplus 2 \oplus 3 \quad \text{and} \quad (DC)_C = 2 \quad 3 \oplus 3 \oplus 3
\]

where the indecomposable projectives and injectives are represented by their Loewy series. It is easily seen that the global dimension of \( C \) is two. By Theorem 2.6, the quiver of \( C \ltimes \operatorname{Ext}^2_C(DC, C) \) is obtained by adding to \( Q_C \) a single arrow \( \delta : 1 \to 3 \).

\[
\begin{tikzcd}
1 & 2 & 3 \\
\alpha & & \\
\beta & 1 & \\
\delta & & \\
\end{tikzcd}
\]

We now compute the new indecomposable projective modules. A simple calculation yields

\[
\begin{align*}
\operatorname{Ext}^2_C(I_3, P_1) & \cong k , \\
\operatorname{Ext}^2_C(I_3, P_3) & \cong k \\
\operatorname{Ext}^2_C(I_1, P_1) & \cong k , \\
\operatorname{Ext}^2_C(I_1, P_3) & \cong k.
\end{align*}
\]

Since the projective dimension of \( I_2 \) is one and the injective dimension of \( P_2 \) is also one, this yields \( \dim_k \operatorname{Ext}^2_C(DC, C) = 4 \). Using Lemma 2.7, we get the new indecomposable projectives

\[
\begin{array}{ccc}
1 & 3 \\
3 & 2 & 1
\end{array}
\]

Thus, a system of relations for the relation-extension algebra is \( \alpha \beta = 0, \delta \alpha = 0, \beta \delta = 0 \) and \( \delta \gamma \delta = 0 \).
3 Cluster-tilted algebras

3.1 Preliminaries

Let $A$ be a hereditary algebra. The cluster category $C_A$ of $A$ is defined as follows. Let $F$ denote the automorphism of $D^b(\text{mod } A)$ defined as the composition $\tau^{-1}_{D^b(\text{mod } A)}$, where $\tau^{-1}_{D^b(\text{mod } A)}$ is the Auslander-Reiten translation in $D^b(\text{mod } A)$, and $[1]$ is the shift functor. Then $C_A$ is the quotient category $D^b(\text{mod } A)/F$. Its objects are the $F$-orbits $\tilde{X} = (F^iX)_{i \in \mathbb{Z}}$, where $X$ is an object in $D^b(\text{mod } A)$. The set of morphisms from $\tilde{X} = (F^iX)_{i \in \mathbb{Z}}$ to $\tilde{Y} = (F^iY)_{i \in \mathbb{Z}}$ in $C_A$ is given by

$$\text{Hom}_{C_A}(\tilde{X}, \tilde{Y}) = \bigoplus_{i \in \mathbb{Z}} \text{Hom}_{D^b(\text{mod } A)}(X, F^iY).$$

It is shown in [20], that $C_A$ is a triangulated category. Furthermore, the canonical functor $D^b(\text{mod } A) \to C_A$ is a functor of triangulated categories. We refer to [6] for facts about the cluster category.

An object $\tilde{T}$ in $C_A$ is called a tilting object provided $\text{Ext}^1_{C_A}(\tilde{T}, \tilde{T}) = 0$ and the number of isomorphism classes of indecomposable summands of $\tilde{T}$ equals the number of isomorphism classes of simple $A$-modules (that is, the number of points in the quiver of $A$). The algebra of endomorphisms $\tilde{C} = \text{End}_{C_A}(\tilde{T})$ is then called a cluster-tilted algebra [7].

Cluster-tilted algebras may also be expressed in terms of modules. We recall that an $A$-module $T$ is called a tilting module provided $\text{Ext}^1_A(T, T) = 0$ and the number of isomorphism classes of indecomposable summands of $T$ equals the number of isomorphism classes of simple $A$-modules. Denoting by $\tilde{T}$ the $F$-orbit of $T$, we have the following theorem.

**Theorem 3.1** (([6, 3.3])) Let $\tilde{C}$ be a cluster-tilted algebra, then there exist a hereditary algebra $A$ and a tilting $A$-module $T$ such that $\tilde{C} \cong \text{End}_{C_A}(\tilde{T})$.

We further recall that the endomorphism algebra of a tilting module over a hereditary algebra is called a tilted algebra, see, for instance, [22]. We need the following result.

**Theorem 3.2** (([17])) Let $A$ be a hereditary algebra, $T$ be a tilting $A$-module and $C = \text{End}_A(T)$ be the corresponding tilted algebra. Then

(a) The derived functor $\text{RHom}_A(T, -) : D^b(\text{mod } A) \to D^b(\text{mod } C)$ is an equivalence of categories which maps the $A$-module $T$ to the $C$-module $C$.

(b) $\text{RHom}_A(T, -)$ commutes with the Auslander-Reiten translations and the shifts in the respective categories.

3.2 Cluster-tilted algebras are trivial extensions

For any object $X$ in $D^b(\text{mod } A)$, the $k$-vector space $\text{Hom}_{D^b(\text{mod } A)}(X, FX)$ has a natural structure of $\text{End}_{D^b(\text{mod } A)}(X)$-$\text{End}_{D^b(\text{mod } A)}(X)$-bimodule under the
The following lemma is proved in [3, 3.1]. We include a simple proof for the convenience of the reader.

**Lemma 3.3** Let \( \tilde{C} \) be a cluster tilted algebra. Then, for each hereditary algebra \( A \) and tilting \( A \)-module \( T \) such that \( \tilde{C} = \text{End}_{C_A}(\tilde{T}) \), we have

\[
\tilde{C} \cong \text{End}_A(T) \times \text{Hom}_{D^b(\text{mod} A)}(T, FT).
\]

**Proof.** By definition of \( C_A \), we have

\[
\tilde{C} = \text{End}_{C_A}(\tilde{T}) = \bigoplus_{i \in \mathbb{Z}} \text{Hom}_{D^b(\text{mod} A)}(T, F^i T)
\]

as \( k \)-vector spaces, and the multiplication is given by

\[
(g_i)_{i \in \mathbb{Z}} (f_j)_{j \in \mathbb{Z}} = \left( \sum_{i+j=l} F^j g_i \circ f_j \right)_{l \in \mathbb{Z}}.
\]

Since \( A \) is hereditary, then, for any two \( A \)-modules \( M \) and \( N \), we have that \( \text{Hom}_{D^b(\text{mod} A)}(M, N[i]) = 0 \) for all \( i \geq 2 \). Therefore, as a \( k \)-vector space

\[
\tilde{C} = \text{End}_{C_A}(\tilde{T}) = \text{Hom}_{D^b(\text{mod} A)}(T, T) \oplus \text{Hom}_{D^b(\text{mod} A)}(T, FT).
\]

The multiplication of two elements \( f, g \in \text{End}_{C_A}(\tilde{T}) \) is given as follows. Assume \( f = (f_0, f_1) \) and \( g = (g_0, g_1) \), with \( f_0, g_0 \in \text{Hom}_{D^b(\text{mod} A)}(T, T) \) and \( f_1, g_1 \in \text{Hom}_{D^b(\text{mod} A)}(T, FT) \), then, since \( FG_1 \circ f_1 = 0 \),

\[
gf = (g_0 \circ f_0, Fg_0 \circ f_1 + f_0 \circ g_1).
\]

In view of the bimodule structure of \( \text{Hom}_{D^b(\text{mod} A)}(T, FT) \) defined above, this shows indeed that \( \tilde{C} = \text{End}_{C_A}(\tilde{T}) \) is the trivial extension of \( \text{End}_{D^b(\text{mod} A)}(T) = \text{End}_A(T) \) by the bimodule \( \text{Hom}_{D^b(\text{mod} A)}(T, FT) \).

Since the algebra \( \text{End}_A(T) \) of the lemma is tilted, any cluster-tilted algebra is a trivial extension of a tilted algebra. However, the hereditary algebra \( A \) and the \( A \)-module \( T \) above are not unique. Therefore, one cannot apply directly the lemma to construct a map from cluster tilted algebras to tilted algebras.

### 3.3 The main result

We are now able to prove the main theorem of this section.

**Theorem 3.4** An algebra \( \tilde{C} \) is cluster-tilted if and only if there exists a tilted algebra \( C \) such that \( \tilde{C} \) is the relation-extension of \( C \).
Proof. Let $C$ be a tilted algebra. Then there exist a hereditary algebra $A$ and a tilting $A$-module $T$ such that $C = \text{End}_A(T)$. Let $\tilde{T}$ denote as usual the $F$-orbit of $T$ in $\mathcal{D}^b(\text{mod} A)$. Then $\tilde{C} = \text{End}_{\mathcal{D}^b} (\tilde{T})$ is a cluster-tilted algebra. By Lemma 3.3, we have

$$\tilde{C} = \text{End}_{\mathcal{D}^b(\text{mod} A)} (T) \times \text{Hom}_{\mathcal{D}^b(\text{mod} A)} (T, FT).$$

(1)

By Theorem 3.2, the derived functor $R\text{Hom}_A(T, -)$ induces $C$-$C$-bimodule isomorphisms

$$\text{End}_{\mathcal{D}^b(\text{mod} A)} (T) \cong \text{End}_{\mathcal{D}^b(\text{mod} C)} (C) \cong \text{End}_C (C) \cong C$$

and

$$\text{Hom}_{\mathcal{D}^b(\text{mod} A)} (T, FT) \cong \text{Hom}_{\mathcal{D}^b(\text{mod} C)} (C, F'C)$$

where $F' = \tau_{\mathcal{D}^b(\text{mod} C)}^{-1} [1]$ is the functor corresponding to $F$ in the derived category $\mathcal{D}^b(\text{mod} C)$. Thus we get

$$\tilde{C} \cong C \times \text{Hom}_{\mathcal{D}^b(\text{mod} C)} (C, F'C).$$

Moreover, we have the following sequence of $C$-$C$-bimodule isomorphisms

$$\text{Hom}_{\mathcal{D}^b(\text{mod} C)} (C, F'C) \cong \text{Hom}_{\mathcal{D}^b(\text{mod} C)} (\tau_{\mathcal{D}^b(\text{mod} C)} C [1], C [2]) \cong \text{Hom}_{\mathcal{D}^b(\text{mod} C)} (DC, C [2]) \cong \text{Ext}^2_C (DC, C),$$

where the first is obtained by applying to both arguments the automorphism $\tau_{\mathcal{D}^b(\text{mod} C)} [1]$, the second uses the fact that $\tau_{\mathcal{D}^b(\text{mod} C)} C \cong DC[-1]$ and the third is a property of the derived category. This shows that the relation-extension $C \times \text{Ext}^2_C (DC, C)$ is a cluster-tilted algebra. Finally, by Lemma 3.3 every cluster-tilted algebra is obtained in this way.

3.4 Remarks and examples

(a) Since the quiver of a tilted algebra has no oriented cycles, it follows directly from Theorem 3.4 and Theorem 2.6 that we have a construction for the quiver of a cluster-tilted algebra $\tilde{C}$ starting from the quiver of a tilted algebra $C$. This construction is easily seen to generalise the one in [11, 4.1] and, thus, can be used to relate the Happel-Vossieck list of tame concealed algebras [18] with Seven’s list of minimal infinite cluster quivers [23].

(b) A different description, inspired from [11], of the relation-division algebra is sometimes useful. Consider the following doubly infinite matrix algebra

$$\hat{C} = \begin{bmatrix}
\ddots \\
& C_{i-1} & 0 \\
& M_i & C_i \\
& 0 & M_{i+1} & C_{i+1} \\
\end{bmatrix}$$
where matrices are assumed to have only finitely many non-zero coefficients, \( C_i = C \) and \( M_i = \text{Ext}^2_C(\text{DC}, C) \) for all \( i \in \mathbb{Z} \), all the remaining coefficients are zero. The addition is the usual addition of matrices while the multiplication is induced from the bimodule structure of \( \text{Ext}^2_C(\text{DC}, C) \) and the zero map \( \text{Ext}^2_C(\text{DC}, C) \otimes_C \text{Ext}^2_C(\text{DC}, C) \to 0 \). Clearly, \( \hat{C} \) is a Galois covering of \( C \ltimes \text{Ext}^2_C(\text{DC}, C) \) with group \( \mathbb{Z} \); the identity maps \( C_i \to C_{i+1}, M_i \to M_{i+1} \) induce an automorphism \( \eta \) of \( \hat{C} \) and \( \hat{C}/\eta \cong C \ltimes \text{Ext}^2_C(\text{DC}, C) \).

(c) As observed before, different tilted algebras \( C \) may correspond to the same cluster-tilted algebra \( \tilde{C} \) (thus, the surjective map \( C \to \tilde{C} \) is not injective). We give an example of such an occurrence.

**Example 3.5** Let \( C_1 \) be given by the quiver

```
  2  α
 / \ /
β 1 - δ  4  γ
 \ /  \\
 3  δ 4  γ
```

bound by \( \alpha \beta = \gamma \delta \). This is a tilted algebra of Dynkin type \( D_4 \), and the corresponding cluster-tilted (relation-extension) algebra \( \tilde{C}_1 \) is given by the quiver

```
  2  α
 / \ /
β 1 - δ  4  γ
 \ /  \\
 3  δ 4  γ
```

bound by \( \alpha \beta = \gamma \delta, \beta \epsilon = 0, \delta \epsilon = 0, \epsilon \alpha = 0, \epsilon \gamma = 0 \). Let now \( C_2 \) be the tilted algebra given by the quiver

```
    2  α
  / \ /
β 4 - ε 1
 \ /  \\
 3  β
```

bound by \( \epsilon \alpha = 0, \epsilon \beta = 0 \). Then it is easily seen that \( \tilde{C}_1 = \tilde{C}_2 \).

(d) Not surprisingly, it is possible that \( C \) is representation-finite whereas \( \tilde{C} \) is representation-infinite: it suffices to have two points \( x, y \in (Q_C)_0 \) such that \( \text{dim}_k \text{Ext}^2_C(I_y, P_x) > 1 \). We give an example of such a situation.
Example 3.6 Let $C$ be given by the quiver

\[
\begin{array}{cccc}
\beta & 2 & \alpha \\
\downarrow & \downarrow & \downarrow \\
1 & 3 & 4 \\
\end{array}
\]

bound by $\alpha \beta = 0, \gamma \delta = 0$. This is a representation-finite tilted algebra of euclidean type $\tilde{A}_3$. The injective resolution

\[
0 \to P_1 \to I_1 \to I_2 \oplus I_3 \to I_4 \oplus I_4 \to 0
\]

shows that $\dim_k \Ext^2_C(I_4, P_1) = 2$. The corresponding cluster-tilted algebra $\tilde{C}$ is given by the quiver

\[
\begin{array}{cccc}
\beta & 2 & \alpha \\
\downarrow & \downarrow & \downarrow \\
1 & \lambda & 4 \\
\end{array}
\]

bound by $\alpha \beta = 0, \gamma \delta = 0, \delta \lambda = 0, \lambda \gamma = 0, \beta \mu = 0, \mu \alpha = 0$. The indecomposable projective $\tilde{C}$-modules are given by

\[
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
4 & 4 & 1 & 1 \\
2 & 3 & 4 & 2 \\
4 & 4 & 1 & 3 \\
2 & 3 & 4 & 3 \\
\end{array}
\]

Clearly, $\tilde{C}$ is representation-infinite.

(e) The relation-extension algebra in Example 2.8 is not a cluster-tilted algebra. This follows from the fact that cluster-tilted algebras contain no oriented cycles of length two.

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