On the representation of $A_\kappa(2)$ algebra and $A_\kappa(d)$ algebra

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In this paper the representation of $A_\kappa(2)$ algebra given by Daoud and Kibler [M.Daoud and M.Kibler , J.Phys.A 43 115303 (2010) , 45 244036 (2012)] is investigated. It is shown that the new generators are necessary for consistency of the algebra. The multi-mode extension of $A_\kappa(2)$ algebra, which is called $A_\kappa(d)$ algebra, is also obtained. The positivity condition of the energy spectrum is also investigated for $A_\kappa(2)$ algebra and $A_\kappa(d)$ algebra.

I. INTRODUCTION

In two decades, many deformations of a boson algebra have been accomplished. Some of them are constructed by using the Jackson’s $q$-calculus [1], while others are not. The deformed boson algebra through $q$-calculus is called a $q$-boson algebra, which was firstly accomplished by Arik and Coon [2] and lately accomplished by Macfarlane [3] and Biedenharn [4] by using the $q$-calculus which was originally introduced by Jackson in the early 20th century [1]. In the study of the basic hypergeometric function Jackson invented the Jackson derivative and integral, which is now called $q$-derivative and $q$-integral. Jackson’s pioneering research enabled theoretical physicists and mathematician to study the new physics or mathematics related to the $q$-calculus. Much was accomplished in this direction and work is under way to find the meaning of the deformed theory.

Recently Daoud and Kibler [5,6,7] introduced the most interesting algebra which is not related to the $q$-calculus. Their algebra is called a $A_{\{\kappa\}}(1)$ algebra or a generalized Weyl-Heisenberg algebra. It is a polynomial algebra which depends on some parameters. The $A_{\{\kappa\}}(1)$ algebra is defined as

$$\{a, a^\dagger\} = N \prod_{i=1}^{r} [1 + \kappa_i (N - 1)], \quad [N, a] = -a, \quad [N, a^\dagger] = a^\dagger,$$

where $\{k\}$ implies the set of real parameters as follows:

$$\{k\} = \{\kappa_1, \kappa_2, \cdots, \kappa_r\} \quad (2)$$

The representation of this algebra is well known in the ref [5,6,7]. They obtained a finite dimensional representation and an infinite dimensional representation for this algebra. They also considered the extension of $A_{\kappa}(1)$ (with one degree of freedom ) to $A_{\kappa}(2)$ involving two degrees of freedom. They insisted that the $A_{\kappa}(2)$ algebra is spanned by $a, a^\dagger$ and $N_i$ satisfying

$$[a_i, a_j^\dagger] = 1 + \kappa (N_1 + N_2 + N_i), \quad [N_i, a_j] = -\delta_{ij} a_j, \quad [N_i, a_j^\dagger] = -\delta_{ij} a_j^\dagger,$$

$$[a_i, a_j] = 0, \quad [a_i^\dagger, a_j^\dagger] = 0, \quad [a_i, [a_i, a_j^\dagger]] = [a_i^\dagger, [a_i^\dagger, a_j]] = 0, \quad (i \neq j), \quad (3)$$

where $\kappa$ is real and $i, j \in \{1, 2\}$. However, Daoud and Kibler did not find the concrete form of the commutation relation between $a_i$ and $a_j^\dagger$ for $i \neq j$ case.

In this paper we will construct the concrete form of the commutation relation between $a_i$ and $a_j^\dagger$ for the $A_{\kappa}(2)$ algebra. In this process, it is shown that the new generators , which I call $X_{12}$ and $X_{21}$, are necessary. Moreover, we will extend our work to the $A_{\kappa}(d)$ algebra including $d$ degrees of freedom.
II. REPRESENTATION OF $A_{\kappa}(2)$ ALGEBRA

In this section, we will find the representation of $A_{\kappa}(2)$ algebra. To do so, let us introduce the eigenvector of two number operators as follows:

$$N_1|n_1, n_2> = (n_1 + \nu)|n_1, n_2>, \quad N_2|n_1, n_2> = (n_2 + \nu)|n_1, n_2>, \quad n_1, n_2 = 0, 1, 2, \ldots, \quad (4)$$

where $N_1, N_2$ are hermitian and the ground state $|0, 0>$ obeying

$$a_1|0, 0> = a_2|0, 0> = 0 \quad (5)$$

From the algebra (3), we have following representation

$$a_1|n_1, n_2> = f_1(n_1, n_2)|n_1 - 1, n_2>, \quad a_1^\dagger|n_1, n_2> = f_1(n_1 + 1, n_2)|n_1 + 1, n_2> \quad (6)$$

$$a_2|n_1, n_2> = f_2(n_1, n_2)|n_1, n_2 - 1>, \quad a_2^\dagger|n_1, n_2> = f_2(n_1, n_2 + 1)|n_1, n_2 + 1>$$

Inserting the eq.(6) into the first relation of the eq.(3), we have

$$f_1^2(n_1 + 1, n_2) - f_1^2(n_1, n_2) = 1 + \kappa(2(n_1 + \nu) + n_2 + \nu), \quad (7)$$

Solving the recurrence relation (7), we have

$$f_1^2(n_1, n_2) = \kappa(n_1 + \nu)^2 + \kappa(n_1 + \nu)(n_2 + \nu) + (1 - \kappa)(n_1 + \nu) + g_1(n_2), \quad (8)$$

$$f_2^2(n_1, n_2) = \kappa(n_2 + \nu)^2 + \kappa(n_1 + \nu)(n_2 + \nu) + (1 - \kappa)(n_2 + \nu) + g_2(n_1),$$

where $g_1(n_2), g_2(n_1)$ are arbitrary functions.

To factorize the right hand side of the eq.(8), we choose

$$g_1(n_2) = (1 - \kappa)n_2, \quad g_2(n_1) = (1 - \kappa)n_1 \quad (9)$$

Then we have the following representation:

$$a_1|n_1, n_2> = \sqrt{(n_1 + n_2 + 2\nu)(\kappa(n_1 + \nu) + 1 - \kappa)}|n_1 - 1, n_2>,$$

$$a_1^\dagger|n_1, n_2> = \sqrt{(n_1 + n_2 + 2\nu + 1)(\kappa(n_1 + \nu) + 1)}|n_1 + 1, n_2>$$

$$a_2|n_1, n_2> = \sqrt{(n_1 + n_2 + 2\nu)(\kappa(n_2 + \nu) + 1 - \kappa)}|n_1, n_2 - 1>,$$

$$a_2^\dagger|n_1, n_2> = \sqrt{(n_1 + n_2 + 2\nu + 1)(\kappa(n_2 + \nu) + 1)}|n_1, n_2 + 1> \quad (10)$$

and the operator relation is given by

$$a_i^\dagger a_i = (N_1 + N_2)(\kappa N_i + 1 - \kappa) \quad (11)$$

Because $|0, 0>$ is annihilated by $a_1, a_2$, we have

$$\nu = 1 - \frac{1}{\kappa}$$

For the representation (10), we can easily check that

$$[a_1, a_2] = 0, \quad [a_1^\dagger, a_2^\dagger] = 0 \quad (12)$$
From the fact that
\[ [a_2, a_1^\dagger] = (\kappa N_1 + 1 - \kappa)a_2, \quad [a_1, a_2^\dagger] = (\kappa N_2 + 1 - \kappa)a_1, \]
we should impose the following relation:
\[ [a_2, a_1^\dagger] = X_{21}, \quad [a_1, a_2^\dagger] = X_{12}, \]
where the new operator \( X_{21} \) and \( X_{12} \) are defined as
\[ X_{21}a_1 = (\kappa N_1 + 1 - \kappa)A_2, \quad X_{12}a_2 = (\kappa N_2 + 1 - \kappa)A_1. \]

We can see that the new generators \( X_{21} \) and \( X_{12} \) also satisfy the following commutation relation:
\[ [X_{12}, a_1] = [X_{21}, a_2^\dagger] = [X_{21}, a_2] = [X_{12}, a_2^\dagger] = 0 \]

Acting the eq.(15) on the eigenvector of the number operators, we obtain the matrix representation of \( X_{21} \) and \( X_{12} \) as follows:
\[ X_{21}|n_1, n_2\rangle = \sqrt{(\kappa(n_1 + \nu) + 1)(\kappa(n_2 + \nu) + 1 - \kappa)n_1 + 1, n_2 - 1}, \]
\[ X_{12}|n_1, n_2\rangle = \sqrt{(\kappa(n_1 + \nu) + 1 - \kappa)(\kappa(n_2 + \nu) + 1)n_1 - 1, n_2 + 1}, \]

It can be easily checked that the relation (14) obey the following:
\[ [a_i, [a_i, a_j^\dagger]] = [a_i^\dagger, [a_i^\dagger, a_j]] = 0, \quad (i \neq j) \]

Now let us consider the deformed harmonic potential problem. If we define the deformed harmonic Hamiltonian \( H \) as
\[ H = X_{12}^2 + X_{21}^2 + P_1^2 + P_2^2, \]
we have
\[ H = \kappa(N_1 + N_2)^2 + \left(2 - \frac{\kappa}{2}\right)(N_1 + N_2) + 1 \]

Acting the hamiltonian on the number eigenvectors, we have
\[ H|n_1, n_2\rangle = E_{n_1, n_2}|n_1, n_2\rangle \]
where
\[ E_{n_1, n_2} = \kappa(n_1 + n_2)^2 + (2 - \frac{\kappa}{2})(n_1 + n_2) + 1 \]

The energy eigenvalue \( E_{n_1, n_2} \) depends on the value of \( n_1 + n_2 \). The ground state \(|0, 0\rangle\) is non-degenerate, but all the excited states are degenerate and the degeneracy is \( 2H_n = n+1C_n \). If we set \( n_1 + n_2 = n \), the energy eigenvalue is defined as
\[ E_n = \kappa n^2 + (2 - \frac{\kappa}{2})n + 1, \]
where \( n \geq 0 \).

Now let us discuss the positivity condition of the energy eigenvalue. Because the energy of the harmonic Hamiltonian is positive in an ordinary quantum mechanics. Thus we should demand the positivity of the energy spectrum for the deformed harmonic Hamiltonian. This problem depends on the sign of \( \kappa \).

**Case I : \( \kappa < 0 \)**
In this case, the positivity of the energy is not guaranteed for all $n$.

**Case II : $\kappa = 0$**

In this case, the positivity of the energy is guaranteed for all $n$.

**Case III : $\kappa > 0$**

In this case, the positivity of the energy is guaranteed for all $n$ only when $\kappa \geq 12 - 8\sqrt{2}$.

### III. REPRESENTATION OF $A_\kappa(d)$ ALGEBRA

In this section we will extend the result of section II to the multi-mode case. We will set the number of modes to $d$. Then the algebra has $d$ degrees of freedom and we will denote this algebra by $A_\kappa(d)$ algebra. The $A_\kappa(d)$ algebra is defined as

\[
[a_i, a_j^\dagger] = 1 + \kappa \left( \sum_{k=1}^{d} N_k + N_i \right), \quad [N_i, a_j] = -\delta_{ij} a_j, \quad [N_i, a_j^\dagger] = -\delta_{ij} a_j^\dagger,
\]

\[
[a_i, a_j] = 0, \quad [a_i^\dagger, a_j^\dagger] = 0, \quad [a_i, [a_i, a_j]] = [a_j^\dagger, [a_i^\dagger, a_j]] = 0, \quad (i \neq j),
\]

where $\kappa$ is real and $i, j \in \{1, 2, \cdots, d\}$.

Let us introduce the eigenvector of number operators as follows:

\[
N_i |\vec{n} > = (n_i + \nu) |\vec{n} >, \quad n_i = 0, 1, 2, \cdots,
\]

where $N_i$ are hermitian and the ground state $|\vec{0} >$ obeying

\[
a_i |\vec{0} > = 0
\]

and $|\vec{n} >$ denotes $|n_1, n_2, \cdots, n_d >$ and $|\vec{0} >$ denotes $|0, 0, \cdots, 0 >$.

From the algebra (24), we have following representation

\[
a_i |\vec{n} >= \sqrt{\left( \sum_{k=1}^{d} n_k + d\nu \right) \left( \kappa \left( n_i + \nu \right) + 1 - \kappa \right)} |\vec{n} - n_i \vec{e}_i >, \\
a_i^\dagger |\vec{n} >= \sqrt{\left( \sum_{k=1}^{d} n_k + d\nu + 1 \right) \left( \kappa \left( n_i + \nu \right) + 1 \right)} |\vec{n} + n_i \vec{e}_i >,
\]

where $|\vec{n} - n_i \vec{e}_i >$ and $|\vec{n} + n_i \vec{e}_i >$ is defined as follows:

\[
|\vec{n} - n_i \vec{e}_i > = |n_1, n_2, \cdots, n_i - 1, n_{i+1}, \cdots, n_d > \\
|\vec{n} + n_i \vec{e}_i > = |n_1, n_2, \cdots, n_i + 1, n_{i+1}, \cdots, n_d >
\]

and the operator relation is given by

\[
a_i^\dagger a_i = \left( \sum_{k=1}^{d} N_k \right) (\kappa N_i + 1 - \kappa)
\]
Because $|\vec{0}>$ is annihilated by $a_i$'s, we have

$$\nu = 1 - \frac{1}{\kappa}$$

For the representation (27), we can easily check that

$$[a_i, a_j] = 0, \quad [a_i^+, a_j^+] = 0$$

(30)

From the fact that

$$[a_j, a_i^+ a_i] = (\kappa N_i + 1 - \kappa) a_j,$$

(31)

we should impose the following relation:

$$[a_j, a_i^+] = X_{ji},$$

(32)

where the new operator $X_{ji}$ is defined as

$$X_{ji}a_i = (\kappa N_i + 1 - \kappa) a_j$$

(33)

and $X_{ji}^* = X_{ij}$. The new generators $X_{ji}$ also satisfy the following commutation relation:

$$a_i^+ X_{ij} = a_j^+ (\kappa N_i + 1 - \kappa), \quad [X_{ji}, a_j] = [X_{ij}, a_i^+] = 0$$

(34)

Acting the eq.(34) on the eigenvector of the number operators, we obtain the matrix representation of $X_{ji}$ as follows:

$$X_{ji}|\vec{n}> = \sqrt{(\kappa(n_i + \nu) + 1)(\kappa(n_j + \nu) + 1 - \kappa)|\vec{n} + \vec{e}_i - \vec{e}_j>$$

(35)

It can be easily checked that the relation (32) obey the following:

$$[a_i, [a_i, a_j^+]] = [a_i^+, [a_i, a_j]] = 0, \quad (i \neq j)$$

(36)

If we define the deformed harmonic Hamiltonian $\hat{H}$ as

$$H = \sum_{i=1}^{d}(X_i^2 + P_i^2),$$

(37)

we have

$$H = \kappa \left(\sum_{k=1}^{d}N_k\right)^2 + \left(d - \frac{d-1}{2}k\right) \sum_{k=1}^{d}N_k + \frac{d}{2}$$

(38)

Acting the hamiltonian on the number eigenvectors, we have

$$H|\vec{n}> = E_{\vec{n}}|\vec{n}>,$$

(39)

where

$$E_{\vec{n}} = \left(\kappa \left(\sum_{k=1}^{d}n_k\right)^2 + (d - \frac{d-1}{2}k) \sum_{k=1}^{d}n_k + \frac{d}{2}\right)|\vec{n}>$$

(40)

The energy eigenvalue $E_{\vec{n}}$ depends on the value of $\sum_{k=1}^{d}n_k$. The ground state $|\vec{0}>$ is non-degenerate, but all the excited states are degenerate and the degeneracy is $d_H = d + n - 1 C_n$. If we set $\sum_{k=1}^{d}n_k = n$, the energy eigenvalue is defined as

$$E_n = \kappa n^2 + \left(d - \frac{d-1}{2}k\right) n + 1,$$

(41)

where $n \geq 0$. 


Now let us discuss the positivity condition of the energy eigenvalue. Because the energy of the harmonic Hamiltonian is positive in an ordinary quantum mechanics. Thus we should demand the positivity of the energy spectrum for the deformed harmonic Hamiltonian. This problem depends on the sign of $\kappa$.

Case I : $\kappa < 0$

In this case, the positivity of the energy is not guaranteed for all $n$.

Case II : $\kappa = 0$

In this case, the positivity of the energy is guaranteed for all $n$.

Case III : $\kappa > 0$

In this case, the positivity of the energy is guaranteed for all $n$ only when $\kappa \geq \frac{2d(d+1)-4d\sqrt{d}}{(d-1)^2}$.

Conclusion

In this paper we discussed the representation of $A_\kappa(2)$ algebra given by Daoud and Kibler [5, 6, 7]. We found that the new generators are necessary for consistency of the algebra. We extended $A_\kappa(2)$ algebra to the multi-mode case, which is called $A_\kappa(d)$ algebra. We also found the positivity condition of the energy spectrum for the deformed harmonic Hamiltonian for $A_\kappa(2)$ algebra and $A_\kappa(d)$ algebra.

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