Localized Gravity on Branes in anti–de Sitter Spaces

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ABSTRACT

We discuss the conditions under which 4D gravity is localized on domain walls in 5D anti–de Sitter spaces. Our approach is based on considering the limits in which the localized gravity decouples. We find that gravity is localized if the wall is located a finite distance from the boundary of the anti–de Sitter space and has a finite tension. In addition, it has to be a δ–function source of gravity.

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1 Introduction

The only consistent theory of quantum gravity is superstring theory which lives in ten dimensions. Traditionally the extra six dimensions are compactified on a manifold of string or Planck size in order to render them unobservable. Four dimensional gravity is obtained from ten dimensional gravity by dimensional reduction on the small six manifold. Recently, two interesting modifications to this scenario have been investigated. On one hand, the existence of D–branes in string theory with only super Yang–Mills (SYM) interactions on their world–volume was discovered. The SYM theory arises from the low energy physics of the open strings which live on the branes. Gravity does not propagate on D–branes but only in the bulk of space which surrounds them. On the other hand, it was noticed that the internal compact six dimensional manifold does not have to be string or Planck scale if only gravity can propagate in the bulk. In fact, present day limits on Newton’s law can be accomodated with two or more rather large dimensions of micron to milimeter size\[1\]. In this case, the weakness of gravity relative to the gauge interactions is a result of the large compact dimensions.

More recently, it was argued that there is no need to compactify extra dimensions at all in order to get four dimensional gravity. In ref. \[2\] it was shown that gravity can be localized on domain walls in anti–de Sitter (AdS) space. In this manner, one can eliminate compactification with all its well–known problems like moduli, stability etc. At first sight this seems to contradict the well–known AdS–CFT duality which relates a bulk gravitational theory to a boundary field theory. In particular on the boundary of the \(AdS\) space there is no gravity. This work arose from trying to clarify this apparent contradiction.

In this letter, we find the necessary conditions for localization of gravity on domain walls in AdS space. In the following, we will use the terms brane or domain wall interchangably. The branes in this language do not necessarily refer to D–branes of string theory. We analyze the conditions under which the four dimensional gravity decouples and interpret the absence of these conditions as the necessary conditions for localization of gravity. We find that gravity is localized on branes in AdS space if the brane is a finite distance away from the boundary of the AdS space and has a finite tension. It is also crucial for the brane to contribute to the stress–energy tensor as a \(\delta\)–function source as for domain walls in supergravity. These are the conditions that seem necessary for localizing gravity on the brane world–volume. The reason is related to the nonnormalizable bulk graviton modes in AdS space. In pure AdS space there is no gravity on the boundary since the nonnormalizable graviton modes cannot be excited there. However, in the presence of a brane and under the above conditions these modes are eliminated. The new space with the brane at its boundary is not AdS but a truncated version of it which is symmetrical on both sides of the brane. As a result, gravity can propagate on the boundary of the new space given by the brane world–volume. The above also applies to more general cases such as intersections of branes in higher dimensional spaces\[3, 4\].
Recently, there have been a number of papers investigating different aspects of this and related scenarios[5]-[9].

The letter is organized as follows. In the next section we review the solution for the 3–brane in $AdS_5$ and localization of gravity in four dimensions. In section 3 we show that 4D gravity decouples in certain limits. The absence of these are interpreted as the necessary conditions for localized gravity. Section 4 includes are conclusions and a discussion of our results.

2 Localized Gravity on Branes

We begin by reviewing the solution for a 3D domain wall in $AdS_5$ located a finite distance away from the boundary of $AdS_5$. In this configuration the five dimensional graviton has a massless normalizable zero mode on the world–volume of the brane[2],[3]. Therefore gravity is localized or trapped in four dimensions. Consider a 3D domain wall in $AdS_5$ a finite distance from the boundary. We assume that the brane couples only gravitationally to the 5D bulk theory. The action specifying the dynamics of the brane-bulk system is

$$S = \int_M d^5x \sqrt{g} \left\{ \frac{R}{2\kappa_5^2} + \Lambda \right\} - \int_{\partial M} d^4x \sqrt{g} \sigma$$

where the bulk cosmological constant, $\Lambda$, is negative (since the bulk is $AdS_5$), the brane tension $\sigma$ is positive and $\kappa_5^2 = 8\pi/M_5^3$ where $M_5$ is the fundamental mass scale of the bulk theory.

The equations of motion which follow from (3) are

$$R^\mu_\nu - \frac{1}{2} g^\mu_\nu R = -\kappa_5^2 \sigma \delta(z-l) \text{diag}(1,1,1,1,0) + \kappa_5^2 \Lambda \delta^\mu_\nu$$

where the stress-energy tensor is a combination of the domain wall situated at $z = l$ and the $AdS_5$ bulk terms. Note that the wall contribution is given by a $\delta$–function in the $z$ direction which cuts all bulk gravitational modes on the wall. We are looking for solutions to the above equations which reflect the symmetries of the 3-brane, i.e. $3+1$ dimensional Lorentz symmetry. The solution is given by (in Poincare coordinates, in terms of the shifted coordinate $z' = z + l$)[2],[3]

$$ds_5^2 = \frac{L^2}{(|z'| + l)^2} (\eta_{\mu\nu} dx^\mu dx^\nu + dz'^2)$$

Here $L$ is related to the bulk cosmological constant and $M_5$ by $L^2 = 6/\kappa_5^2 \Lambda$ and $l$ is the distance of the brane to the boundary of $AdS_5$. This is similar to but not quite $AdS_5$ due to the absolute value sign and the nonzero distance $l$. In fact, in this metric the portion of $AdS_5$ from the brane to the boundary is deleted and the bulk geometry is symmetric.
on both sides of the brane due to the absolute value sign. The difference between this space and $AdS_5$ are essential for localization of gravity on its boundary, i.e. the brane world–volume. The metric in eq. (3) is a solution of Einstein’s equations only if the tension of the wall is fine tuned to be $\sigma = 6\kappa_5^2 l^2 / L^3$.

In order to show that gravity is localized on the brane we solve the equation of motion for the linearized graviton in the above background

$$\left[ \frac{1}{2} \Box_4 + \left( -\frac{1}{2} \nabla_z^2 + V(z) \right) \right] \hat{h}(x, z) = 0,$$

where

$$V(z) = \frac{3}{2L^2} \Omega(z)^2 - \frac{3}{2L} \Omega(z) \delta(z).$$

and $h = \Omega(z)^{-3/2} \hat{h}$ with $\Omega(z) = L/(|z| + l)$.

The massless four dimensional mode of the graviton is obtained by setting $\hat{h}(x, z) = e^{ipx} \psi(z)$ and solving the effective Schrödinger equation

$$\left( -\frac{1}{2} \nabla_z^2 + V(z) \right) \psi_\lambda = \frac{1}{2} m_\lambda^2 \psi_\lambda,$$

where $\lambda$ labels the eigenfunctions. This potential has a repulsive piece which goes to zero for $|z| \gg L$, and an attractive $\delta$ function. The 4D massless graviton corresponds to a bound state with the wavefunction

$$\psi(z) = \sqrt{\frac{l^2}{L^3}} \Omega(z)^{3/2}.$$

In this case, the 4D gravitational action is given by

$$S = M_5^3 \int dz \Omega^3 \times \int d^4x \sqrt{g^{(4)}} R^{(4)}$$

Thus, the four dimensional graviton couples with the strength

$$M_P^2 = M_5^2 \int dz \Omega^3 = \frac{M_5^3 L^3}{l^2}.$$

The result for $M_P$ is identical to the case in which 5d gravity is compactified on a circle of size $L^3/l^2$. It can also be shown that gravity behaves as if it is four dimensional down to distances $\sim L^3/l^2$ whereas for shorter distances it becomes the higher dimensional bulk gravity. We see that gravity behaves as if it is compactified on a circle whereas in reality the fifth dimension is infinite.

As long as $l \neq 0$ i.e. the brane is not on the boundary of $AdS_5$ all values of $l$ give equivalent but different solutions. This can be seen from the transformation $x^\mu \rightarrow \lambda x^\mu$, $z \rightarrow \lambda^{-1}z$ which is a broken symmetry (by the nonzero value of $l$) of the solution in eq. (3).
This transformation takes \( l \to l/\lambda \) which corresponds to a new solution. Different values of \( l \) are superselection sectors of the theory since the brane world–volume is infinite. If we want we can conveniently choose \( \lambda \) so that \( l/\lambda = L \) in which case the metric becomes that of ref. [2]. (In the following we assume that \( l \neq L \) in order to clarify the role of the distance to the boundary \( l \).) This symmetry is closely related to the well–known scaling–radial translation equivalence of the AdS–CFT duality. In that context, it is known that a scale transformation on the boundary is equivalent to a radial displacement in the bulk. This is precisely the transformation we considered above.

3 Decoupling Localized Gravity

In this section we find the necessary conditions for localizing bulk gravity on a lower dimensional brane world–volume. We only consider 3–branes in bulk \( AdS_5 \) however our results can be easily generalized to higher dimensions. The method we employ is to find out under what conditions or in which limits the localized 4D gravity decouples. We interpret the absence of these conditions as necessary conditions for localizing gravity. Without a brane in \( AdS_5 \) bulk there is no gravity on the boundary as is well–known from the AdS–CFT duality which relates a gravitational bulk theory to a field theory on the boundary. This is due to the fact that there are nonnormalizable bulk gravitational modes which cannot be excited on the boundary. Localization of gravity on the brane is related to the fact that the presence of the brane eliminates these nonnormalizable modes.

We first consider the limit \( l \to 0 \). In this limit the brane approaches the boundary and overlaps with it. From the solution in eq. (3) we see that in this limit the space–time goes to \( AdS_5 \) which has no gravity on its boundary. This means that localized 4D gravity decouples in this limit. We also see from eq. (9) that when \( l \to 0 \), \( M_P \) diverges. This is another indication that 4D gravity decouples in this limit. Thus, the first condition for localization of gravity is the presence of a brane at a finite distance from the boundary of \( AdS_5 \).

We can also see this from the bound state wavefunction in the transverse \( z \) direction given by \( \psi(z) \sim l \Omega(z)^{3/2} \). It is easy to see that when \( l \neq 0 \), the bound state wavefunction \( \psi(z) \) is normalizable and gravity is localized. On the other hand, when \( l = 0 \), \( \psi(z) \) is not normalizable. Therefore there is no bound state in the transverse direction and gravity cannot be localized on the brane. The full geometry of the brane in \( AdS_5 \) is a truncated version of \( AdS_5 \) which eliminates the nonnormalizable bulk graviton modes.

The second limit we consider is \( l \) finite and fixed but \( \sigma \to 0 \), i.e. the limit in which the tension of the brane vanishes. This is a nontrivial limit because in order for eq. (3) to be a solution to Einstein’s equations we need \( \sigma = 6/\kappa_5^2 L \) or using the definition of \( L \), \( \sigma = \sqrt{6\Lambda}/\kappa_5 \). Because of this relation we cannot take the \( \sigma \to 0 \) limit and keep the bulk curved. In order to be able to take \( \sigma \to 0 \) and keep \( \Lambda \) fixed we consider a system of two branes at a distance with positive tensions \( \sigma_1 \) and \( \sigma_2 \). The two branes can be at a finite
distance only if $6\Lambda > \kappa_5^2 \sigma_1 \sigma_2$. Note that due to this condition the brane world–volumes are $AdS_4$ slices of the 5D metric with negative overall energy density. The metric which corresponds to this case is given by\[ ds_5^2 = a^2(w)(dx^2 + e^{2Hx}(-dt^2 + dy^2 + dz^2)) + dw^2 \] with the warp factor\[ a = \cosh(\sqrt{\frac{\kappa_5^2 \Lambda}{6}}w) - \frac{\kappa_5 \sigma}{\sqrt{6\Lambda}} \sinh(\sqrt{\frac{\kappa_5^2 \Lambda}{6}}|w|) \] Here $H = iH$ where\[ H^2 = \frac{\kappa_5^2 \sigma^2 - 6\Lambda}{36} \] Due to the presence of the two branes at $w = 0$ and $w = w_c$ and the absolute value sign in the metric the identifications $w \sim -w$ and $w \sim w + w_c$ have to be made in (10). This is precisely what would be obtained by compactifying the transverse direction on a circle of size $w_c$ and orbifolding by $Z_2$. This metric is equivalent to the one in eq. (3) when $H = 0$ with the redefinition of the transverse coordinate $L \exp(|w|/L) = |z| + l$. Note that the condition $H = 0$ can be written as $\sigma = 6/\kappa_5^2 L$ using the definition of $L$. It can be shown that the branes are separated by a distance $w_c$ given by\[ \tanh(\sqrt{\frac{\kappa_5^2 \Lambda}{6}}w_c) = \frac{\kappa_5 \sqrt{6\Lambda}(\sigma_1 + \sigma_2)}{6\Lambda + \kappa_5^2 \sigma_1 \sigma_2} \] We would like to use this configuration with two branes to reach a configuration with only one brane which does not satisfy the relation $\sigma = 6/\kappa_5^2 L$. In order to get rid of the second brane we take the distance between the branes to infinity. We see that $w_c \to \infty$ requires\[ (\sqrt{6\Lambda} - \kappa_5 \sigma_1)(\sqrt{6\Lambda} - \kappa_5 \sigma_2) = 0 \] By choosing the tension of the second brane to satisfy $6\Lambda = \kappa_5^2 \sigma_2^2$ we send it to infinity. As a result, we are left with only one brane with positive tension $\sigma_1$ in $AdS_5$, but now there is no correlation between $\sigma_1$ and $\Lambda$. This is precisely the situation we wanted to examine for the limit $\sigma_1 \to 0$. In this picture, the second brane with a tension fixed by $\Lambda$ plays a role similar to a regulator. Now consider a very large but finite distance between the two branes. Then using\[ M_P^2 = M_5^3 \int_0^1 d\vartheta w_c a^2(w_c \vartheta) \] where $a(w)$ is the warp factor we find that\[ M_P^2 = \frac{6M_5^6}{\sigma_1} \]
We see that in the limit $\sigma_1 \to 0$, $M_P$ diverges and 4D gravity decouples.

The third condition for localizing gravity on branes is the requirement that the brane contribute as a $\delta$-function to the stress–energy tensor which is the source of gravity as in eq. (2). In supergravity in $AdS$ space with a domain wall this is indeed the case. Consider for example the effective low–energy bosonic action (in $D$ space–time dimensions)

$$L = \sqrt{g} \left( \frac{R}{2\kappa_D^2} - \frac{1}{2}(\partial \phi)^2 + \Lambda e^{-a\phi} \right)$$

(17)

which appears in dimensionally reduced supergravity where

$$a^2 = \Delta + \frac{2(D-1)}{D-2}$$

(18)

is a constant. Note that this is similar to eq.(1) without the brane source term but with a dilaton coupling. For $a = 0$ there is the well–known solution which is anti–de Sitter space. For $a \neq 0$ however, the solution is a $D-2$ dimensional domain wall [10]

$$ds^2 = \frac{H^4/\Delta}{(D-2)} \eta_{\mu\nu} dx^\mu dx^\nu + H^{4(D-1)/\Delta(D-2)} dy^2$$

(19)

situated at $y = 0$ with $H(y) = 1 + k|y|$. (here $H$ is $\Omega$ of the previous solution with $k = 1/L$.) The curvature is smooth everywhere except at $y = 0$ where it has a $\delta$–function singularity. This is due to the brane tension which must be included in eq. (17) for consistency. This $\delta$-function source cuts all graviton modes on both sides of the brane, in particular it cuts the nonnormalizable modes of the bulk graviton which allow gravity to be localized on the brane.

Finally, there can be domain wall configurations which look like eq. (3) but with a power $p$ of $\Omega$ different than two as in eq. (19)[11]. Such configurations appear in theories obtained by dimensional reduction of higher dimensional supergravity. Then from eq. (9)

$$M_P^2 = M_5^3 \int dz \Omega^{3p}$$

(20)

Thus $M_P$ is finite and there is 4D gravity only if $p < -1/3$. The same bound also guarantees that the bound state wavefunction in eq. (7) is normalizable.

4 Conclusions and Discussion

In this letter, we found the necessary conditions for localizing gravity on 3D branes (or domain walls) in $AdS_5$. Gravity is localized on the brane if 3D brane is located a finite distance away from the boundary of $AdS_5$. The four dimensional Planck scale is inversely proportional to this distance. As a result, the smaller the distance to the boundary the weaker is 4D gravity. When the brane is on the boundary, 4D gravity decouples as is
expected from the well–known AdS–CFT duality. Another condition for localizing gravity is a finite brane tension. Since for these solutions, the brane tension is proportional to the cosmological constant this cannot be examined in a trivial manner. Thus, we started from two positive tension branes in $AdS_5$ and sent one of them to infinity by fixing its tension. Then we found that 4D Planck scale is inversely proportional to the tension of the remaining brane. As a result, if the tension of the brane goes to zero 4D gravity decouples. The third condition for localization of gravity is that the contribution of the brane to the stress–energy tensor must be a $\delta$–function. This is indeed the case for domain walls in supergravity. The $\delta$–function cuts all momentum modes of the bulk graviton including the ones which are nonnormalizable in full $AdS_5$ without the brane. We stress that the geometry with the brane in the bulk of $AdS_5$ is a modified space where the part of $AdS_5$ from the brane to the boundary is deleted and the space is symmetric on both sides of the brane. Since the bulk graviton modes are now normalizable on the brane they can be excited and this gives rise to 4D gravity. This can also be seen by inspecting the normalizable graviton wavefunction in the fifth dimension transverse to the brane.

In refs. [2], [3], it was shown that gravity is localized on the brane by solving the linearized Einstein equations in the background of the brane in $AdS_5$. One finds that there is a graviton zero mode in 4D which is localized around the brane. This analysis is clearly perturbative since it considers only weak fields. It is not clear whether the same result can be obtained for strong gravity. In particular, it is not clear how to think about black holes either in the bulk or on the brane in this context. For example, if there is a 5D black hole inthe bulk, it is not clear how to reduce it to 4D or whether it will look like a 4D black hole. Perturbative 4D gravity looks like 5D bulk gravity compactified on a circle of size $L^3/l^2$; however it does not seem that the same is true for 5D black holes. Similarly, it is not clear how 4D black holes which must exist look like in the bulk.

Another issue is whether localization of gravity which was shown in (super)gravity can be realized in string theory. This has been the subject of refs. [2] to some degree. In string theory a possible setup would be IIB string theory on $AdS_5 \times S^5$ with a D3 brane in the bulk. The part of $AdS_5$ between the brane and the boundary can be deleted by an orbifold about the brane position. Naively, the appearance of gravity on the D3 brane world–volume is very surprising since the D3 brane world–volume theory is supersymmetric Yang–Mills theory at low energy. If gravity can be localized in string theory it would be important to understand whether there is a bulk/boundary duality as in the AdS–CFT case. In the AdS–CFT duality the dimension transverse to the boundary is holographic, i.e. it is related to the energy scale of the boundary theory. It is not clear that this will continue to be the case when gravity is localized on the brane.

On the more phenomenological side it is not clear how inflation can be realized. Previously considered mechanisms for inflation on branes such as brane inflation, asymmetric inflation and D–term inflation[12] cannot be easily used in this context. This is either due to the absence of the scalar (parametrizing the distance between branes) and radion fields or due to the absence of an anomalous $U(1)$ symmetry on the brane. In [3, 4] it was
shown that if there is an overall nonzero vacuum energy on the brane it will inflate. This requires a time dependent negative cosmological constant in the bulk, e.g. a bulk scalar with a negative vacuum energy and a potential suitable for inflation. Such scalars seem to exist in gauged supergravity and may indeed lead to inflation on the brane\cite{13}. It is important to see whether these scalars have potentials which can satisfy all the requirements of acceptable inflation.

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