LOW-REGULARITY GLOBAL WELL-POSEDNESS FOR THE NONLINEAR WAVE EQUATION

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Abstract. We consider the problem of low-regularity global well-posedness for the nonlinear wave equation. In particular, we construct special analytic solutions with the optimal regularity predicted by Sogge and Lindblad which exist up to arbitrarily long times.

1. Introduction

The nonlinear wave equation (NLW) is the equation

\[ u_{tt} - \Delta u = \mu |u|^{p-1} u. \]

Here, \( u : \mathbb{R}^{d+1} \to \mathbb{C}, \mu \in \{-1, 1\}, \) and the exponent \( p \) satisfies \( 1 < p < \infty \). The case \( \mu = -1 \) is known as the defocusing case, while \( \mu = 1 \) is called the focusing case. This equation has a long history and many results are known; see [22] for a detailed exposition and references therein for details. With regards to the question of local well-posedness in the homogeneous Sobolev spaces \( H^s \), H. Lindblad and C. D. Sogge [18] showed that the optimal regularity in dimension \( d = 2 \) is given by

\[ s(p) := \left\{ \begin{array}{ll}
\frac{3}{4} - \frac{1}{p - 1}, & 3 \leq p \leq 5, \\
1 - \frac{2}{p - 1}, & p \geq 5.
\end{array} \right. \]

See also the refinements of M. Nakamura and T. Ozawa [19]. We remark that these results are stated for the homogeneous Sobolev spaces \( H^s \), but can be extended to the inhomogeneous Sobolev spaces \( \dot{H}^s \) by a simple integration in time argument.

The question of whether these local solutions exist globally in time has not been completely resolved; only partial results are known. Currently, the best known result is that of W. Han [11], who showed global well-posedness for arbitrary data in \( \dot{H}^s \) spaces for \( s > 5/11 \) and \( p = 3 \) by applying the I-method. This situation is quite common; many of the techniques used to establish a global result, such as conserved quantities, require significantly more regularity than the techniques used to establish local results. Thus, we often see gaps between the local theory and the global theory. To fill in these...
gaps (at least partially), one can consider special solutions with some additional properties which can be exploited to extend solutions to arbitrarily long times.

In the present work, we will consider the low regularity theory for special solutions to $\mathcal{L}$. In particular, we will prove the following theorem, which is our main result.

**Theorem 1.1.** Let $p \geq 3$ be an odd positive integer,

\[
s \geq \begin{cases} 
\frac{1}{2} - \frac{1}{p} & \text{if } d = 1, \\
\frac{3}{4} - \frac{1}{p-1} & \text{if } d = 2 \text{ and } 3 \leq p \leq 5, \\
1 - \frac{2}{p-1} & \text{if } d = 2 \text{ and } p \geq 5,
\end{cases}
\]

and $u$ be a local solution to the Cauchy problem

\[
\begin{cases} 
\ddot{u} - \Delta u + |u|^{p-1}u = 0, \\
\dot{u}(\cdot, 0) = u_0 \in H^s(\mathbb{R}^d), \\
\ddot{u}(\cdot, 0) = u_1 \in H^{s-1}(\mathbb{R}^d).
\end{cases}
\]

Suppose that for some $\sigma_0 > 0$, the initial data $u_0$ and $u_1$ admit analytic continuations on the subset $S_{\sigma_0}$ of $\mathbb{C}^d$ defined by

\[
S_{\sigma_0} := \{x + iy : x, y \in \mathbb{R}^d, |y| < \sigma_0\},
\]

such that, for $k = 1, 2$,

\[
\sup_{|y| < \sigma_0} \|u_k(x + iy)\|_{L_2^2(\mathbb{R}^d)} < \infty,
\]

where $s_0 := s$ and $s_1 := s - 1$. Then for any $T > 0$, the solution $u$ will satisfy

\[
\sup_{t \in [0, T]} \|u(\cdot, t)\|_{H^s(\mathbb{R}^d)} + \|\dot{u}(\cdot, t)\|_{H^{s-1}(\mathbb{R}^d)} < \infty.
\]

Thus, the solution exists globally.

In addition, we will also prove that the analyticity of solutions is preserved, at least for a short time.

**Theorem 1.2.** Let $s$ and $\sigma_0$ be as in Theorem 1.1. If $u_0 \in H^s(\mathbb{R}^d)$ and $u_1 \in H^{s-1}(\mathbb{R}^d)$ have holomorphic extensions on $S_{\sigma_0}$ satisfying (4), then there exists $\delta > 0$ such that the solution $u(\cdot, t)$ and its derivative $\dot{u}(\cdot, t)$ will also have holomorphic extensions on $S_{\sigma_0}$ for any $t \in [0, \delta)$. Thus, the analyticity of solutions persists, at least for short times.

The proof of Theorem 1.1 involves showing that our solutions belong to a subset of the Sobolev spaces known as the analytic Gevrey spaces. In Section 2, we introduce these spaces and their properties which are needed later. In Section 3, we use the results of the Gevrey spaces to show that the Sobolev norm of solutions remains uniformly bounded on arbitrarily large time intervals (Proposition 2.6). In Section 4, we present the proof of Theorem 1.2 which was reduced in Section 2 to Proposition 2.7.

We will denote constants which can be determined by known parameters in a given situation by $C$, but whose values are not crucial to the problem at hand and may differ from line to line. Such parameters would be, for example, $d$, $p$, $s$, $\sigma$. We also write $a \lesssim b$ as shorthand for $a \leq Cb$ and $a \sim b$ when $a \lesssim b$ and $b \lesssim a$. 
2. Gevrey-Sobolev spaces

We first recall some notation that will be used throughout. The Fourier transform of a function $f$ is denoted by

$$\hat{f}(\xi) := \mathcal{F}(f)(\xi) := \int_{\mathbb{R}^d} e^{-ix \cdot \xi} f(x) \, dx$$

and its inverse transform by

$$\tilde{f}(x) := \mathcal{F}^{-1}(f)(x) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix \cdot \xi} f(\xi) \, d\xi.$$ 

For any $s, \sigma \in \mathbb{R}$, we define the pseudodifferential operators $e^{\sigma|D|} \langle D \rangle^s$ and $|\nabla|^s$ by the Fourier multipliers

$$e^{\sigma|D|} \langle D \rangle^s f := \mathcal{F}^{-1}(e^{\sigma|\xi|} \langle \xi \rangle^s \hat{f})$$

and

$$|\nabla|^s f := \mathcal{F}^{-1}(|\xi|^s \hat{f}).$$

We begin our discussion with an introduction to an essential tool in our analysis, the Gevrey spaces $G^{\sigma}$. These spaces first appeared in the work of C. Foias and R. Temam [6] on the Navier-Stokes equation and are defined by the norm

$$\|f\|_{G^{\sigma}(\mathbb{R}^d)} := \|e^{\sigma|\xi|} \hat{f}(\xi)\|_{L^2_\xi(\mathbb{R}^d)}.$$ 

In the study of fluid mechanics, Gevrey spaces play an important role; see [1] and the references therein for a discussion of the physical interpretation of the parameter $\sigma$, and its importance in applications.

Related to the Gevrey spaces are the hybrid Gevrey-Sobolev spaces $G^{\sigma, s}$, which are defined by the modified Gevrey norm

$$\|f\|_{G^{\sigma, s}(\mathbb{R}^d)} := \|e^{\sigma|\xi|} \langle \xi \rangle^s \hat{f}(\xi)\|_{L^2_\xi(\mathbb{R}^d)},$$

where $\langle \xi \rangle := (1 + |\xi|^2)^{1/2}$. In recent years, many authors have considered the Cauchy problem for a variety of equations with initial data in $G^{\sigma, s}$ spaces; see, for example [2, 3, 7, 8, 9, 12, 13, 14, 15, 17, 20, 21, 23] for some of the more recent works on this subject. It should be noted that all of the works mentioned here are concerned with equations which are first-order in time. The only result known to the authors involving second-order equations is the result of Y. Guo and E. S. Titi [10], which deals with general nonlinear wave equations in the periodic setting.

2.1. Preliminary results. To use Gevrey-Sobolev spaces to establish global existence in Sobolev spaces, we need to establish several results connecting them. The most important of these is the following Paley-Wiener theorem, for which a discussion can be found in [16, pp. 174] for the case $d = 1$ and $s = 0$.

**Proposition 2.1.** Let $\sigma > 0$ and $s \in \mathbb{R}$. The following are equivalent:

1. $f \in G^{\sigma}(\mathbb{R}^d)$;
(2) \( f \) is the restriction to \( \mathbb{R}^d \subset \mathbb{C}^d \) of a function \( F \) which is holomorphic in the strip \( S_\sigma \) (see definition (3) above) and satisfies
\[
\sup_{|y|<\sigma} \|F(x + iy)\|_{C^{\sigma,s}(\mathbb{R}^d)} < \infty.
\]

We remark that the assumptions in Theorem 1.1 imply that our initial data satisfy
\[
u_0 \in G^{\sigma,s}(\mathbb{R}^d) \cap G^{\sigma_0}(\mathbb{R}^d) \quad \text{and} \quad u_1 \in G^{\sigma,s-1}(\mathbb{R}^d) \cap G^{\sigma_0}(\mathbb{R}^d).
\]

Next, we state an embedding result for Gevrey-Sobolev spaces which will be used repeatedly.

**Lemma 2.2.** Let \( s, s' \in \mathbb{R} \), and \( 0 \leq \sigma' < \sigma \). Then,
\[
\|f\|_{G^{\sigma,s}(\mathbb{R}^d)} \lesssim \|f\|_{G^{\sigma',s}(\mathbb{R}^d)},
\]
and hence \( G^{\sigma,s}(\mathbb{R}^d) \hookrightarrow G^{\sigma',s}(\mathbb{R}^d) \).

**Proof.** It is well-known that
\[
\langle \xi \rangle^{s'-s} \lesssim e^{(\sigma'-\sigma)|\xi|}
\]
for \( \sigma > \sigma' \). Multiplying by \( \langle \xi \rangle^s e^{\sigma|\xi|} \) we obtain
\[
\langle \xi \rangle^{s'} e^{\sigma'|\xi|} \lesssim \langle \xi \rangle^s e^{\sigma|\xi|},
\]
and the desired result immediately follows. \( \square \)

Another very useful inequality is the following generalized Sobolev product estimate from [5].

**Lemma 2.3.** Let \( d \geq 1 \), \( f \in H^{s_1}(\mathbb{R}^d) \) and \( g \in H^{s_2}(\mathbb{R}^d) \). Then, \( fg \in H^{-s_0}(\mathbb{R}^d) \) and
\[
\|fg\|_{H^{-s_0}(\mathbb{R}^d)} \lesssim \|f\|_{H^{s_1}(\mathbb{R}^d)} \|g\|_{H^{s_2}(\mathbb{R}^d)},
\]
provided that
\[
s_0 + s_1 + s_2 \geq \max\{s_0, s_1, s_2\} \quad \text{and} \quad s_0 + s_1 + s_2 \geq \frac{d}{2},
\]
but the equality cannot hold in both relations at the same time.

As a consequence of Lemma 2.3 we obtain

**Lemma 2.4.** Let \( \sigma \geq 0 \), \( p \geq 3 \) be an odd positive integer. Then, for every \( f \in G^{\sigma,s}(\mathbb{R}^d) \),
\[
\|f^p\|_{G^{\sigma,s-1}(\mathbb{R}^d)} \lesssim \|f\|_{G^{\sigma,s}(\mathbb{R}^d)}^p,
\]
whenever
\[
s \geq \begin{cases} \frac{1}{2} \left( d - \frac{1}{p} \right) & \text{if } d = 1; \\ \frac{d}{2} - \frac{1}{p} & \text{if } d = 2. \end{cases}
\]

**Proof.** By definition of the Gevrey spaces we have that
\[
\|f^p\|_{G^{\sigma,s-1}(\mathbb{R}^d)} = \|\langle \xi \rangle^{s-1} e^{\sigma|\xi|} \hat{f}^p(\xi)\|_{L^2(\mathbb{R}^d)},
\]
where \( \hat{f}^p(\xi) \) is the \( p \)-th Fourier power of \( f \).
On the other hand,
\[
|e^{\sigma|\xi|} \hat{f}^2(\xi)| = |e^{\sigma|\xi|} \hat{f} \ast \hat{f}(\xi)| \leq \int_{\mathbb{R}^d} e^{\sigma|\eta|} |\hat{\delta}(\eta)| e^{\sigma|\xi - \eta|} |\hat{f}(\xi - \eta)| \, d\eta
\]
\[
= \left( e^{\sigma|\xi|} |\hat{f}| \ast e^{\sigma|\xi|} |\hat{f}| \right)(\xi) = \left( \mathcal{F}(e^{\sigma|D|} \mathcal{F}^{-1}(|\hat{f}|)) \ast \mathcal{F}(e^{\sigma|D|} \mathcal{F}^{-1}(|\hat{f}|)) \right)(\xi)
\]
\[
= \mathcal{F} \left( \left( e^{\sigma|D|} \mathcal{F}^{-1}(|\hat{f}|) \right)^2 \right)(\xi).
\]
Iterating this relation, we can control (7) by
\[
\|f^p\|_{G^s,q-1,\mathbb{R}^d} \lesssim \|F^p\|_{H^{s-1},\mathbb{R}^d},
\]
where
\[
F(x) := e^{\sigma|D|} \mathcal{F}^{-1}(|\hat{f}|)(x).
\]
Then, in order to justify the estimate (6) it is enough to see that
\[
\|F^p\|_{H^{s-1},\mathbb{R}^d} \lesssim \|F\|_{H^s,\mathbb{R}^d}.
\]
As for (8), it suffices to check that
\[
\|F^{p-k}\|_{H^{s-1+\frac{k}{p}},\mathbb{R}^d} \lesssim \|F^{p-k-1}\|_{H^{s-1+\frac{k+1}{p}},\mathbb{R}^d} \|F\|_{H^s,\mathbb{R}^d}, \quad k = 0, \ldots, p - 1,
\]
but this follows from the product estimate in Lemma 2.3 for $s$ as in the statement of the lemma.

We conclude this section with a lemma which will be very helpful in Section 3 below.

**Lemma 2.5.** Let $\sigma \geq 0$, $p \geq 3$ be an odd number and let $\mathcal{L}$ be the operator given by
\[
\mathcal{L}f := |e^{\sigma|D|}|^{p-1} e^{\sigma|D|} f - e^{\sigma|D|}(|f|^{p-1} f).
\]
Then for $d = 1$, we have the estimate
\[
\|\mathcal{L}f\|_{L^2(\mathbb{R})} \lesssim \sigma^\theta \|f\|_{G^{\sigma,0}(\mathbb{R})} \|\nabla f\|_{G^{\sigma,0}(\mathbb{R})}, \quad 0 \leq \theta \leq 1,
\]
while for $d = 2$,
\[
\|\mathcal{L}f\|_{L^2(\mathbb{R}^2)} \lesssim \sigma^\theta \|f\|_{G^{\sigma,0}(\mathbb{R}^2)} \|\nabla f\|_{G^{\sigma,0}(\mathbb{R}^2)}, \quad 0 < \theta < 1.
\]

**Proof.** Let us introduce the notation
\[
F(x) := e^{\sigma|D|} f(x)
\]
and
\[
G(x) := \mathcal{F}^{-1}(|\hat{f}|)(x).
\]
Following the proof of Lemma 7 in [23], we have that
\[
\hat{\mathcal{L}} f(\xi) = \int_{\mathcal{H}} \left( 1 - e^{-\sigma (\sum_{k=1}^{p} |\eta_k| - |\xi|)} \right) \hat{F}(\eta_1) \hat{F}(\eta_2) \times \hat{F}(\eta_3) \hat{F}(\eta_4) \cdots \hat{F}(\eta_{p-2}) \hat{F}(\eta_{p-1}) \hat{F}(\eta_p) \, d\eta_1 \cdots d\eta_{p-1},
\]
where $\mathcal{H}$ is the hyperplane $\xi = \eta_1 + \cdots + \eta_p$. Next, we observe that
\[
1 - e^{-x} \leq 1 \quad \text{and} \quad 1 - e^{-x} \leq x.
\]
Interpolating, we see that
\[ 1 - e^{-x} \leq x^\theta, \quad 0 \leq \theta \leq 1. \]
Hence,
\[ 1 - e^{-\sigma(\sum_{k=1}^p |\eta_k| - |\xi|)} \leq \sigma^\theta \left( \sum_{k=1}^p |\eta_k| - |\xi| \right)^\theta \leq \sigma^\theta \sum_{k=1}^p |\eta_k|^\theta. \]
Plancherel’s identity then yields
\[ (10) \quad \| \mathcal{L} f \|_{L^2(\mathbb{R}^d)} \lesssim \sigma^\theta \| (|\nabla|^\theta G) G^{p-1} \|_{L^2(\mathbb{R}^d)}. \]

To proceed, we consider each dimension separately. For \( d = 1 \), we estimate the term on the right hand side of equation (10) by
\[ (11) \quad \| (|\nabla|^\theta G) G^{p-1} \|_{L^2(\mathbb{R})} \lesssim \| |\nabla|^\theta G \|_{L^2(\mathbb{R})} \| G^{p-1} \|_{L^\infty(\mathbb{R})}. \]
For the first norm above, we apply again Plancherel to see that
\[ \| |\nabla|^\theta G \|_{L^2(\mathbb{R})} = \| |\xi|^\theta \hat{G} \|_{L^2(\mathbb{R})} = \| |\xi|^\theta |\hat{F}| \|_{L^2(\mathbb{R})} = \| F \|_{\dot{H}^\theta(\mathbb{R})}. \]
Interpolating, we have that
\[ \| F \|_{\dot{H}^\theta(\mathbb{R})} \leq \| F \|_{L^2(\mathbb{R})} \| \nabla F \|_{L^2(\mathbb{R})}, \]
for \( 0 \leq \theta \leq 1 \). For the second term on the right hand side of equation (11), we apply the Gagliardo-Nirenberg inequality to obtain
\[ \| G \|_{L^\infty(\mathbb{R})} \lesssim \| G \|_{L^2(\mathbb{R})} \| G \|_{H^1(\mathbb{R})} = \| F \|_{L^2(\mathbb{R})} \| \nabla F \|_{L^2(\mathbb{R})}. \]
Combining previous estimates, we see that
\[ \| \mathcal{L} f \|_{L^2(\mathbb{R})} \lesssim \sigma^\theta \| F \|_{L^2(\mathbb{R})} \| \nabla F \|_{L^2(\mathbb{R})}, \]
which leads to the desired result.

For the case of \( d = 2 \), we cannot use the Gagliardo-Nirenberg inequality as before. Instead, we begin by applying Hölder’s inequality in (10) to obtain
\[ (12) \quad \| (|\nabla|^\theta G) G^{p-1} \|_{L^2(\mathbb{R}^2)} \leq \| |\nabla|^\theta G \|_{L^{2/\theta}(\mathbb{R}^2)} \| G^{p-1} \|_{L^{2(p-1)/\theta}(\mathbb{R}^2)}. \]
By Sobolev embedding, we have that
\[ \| |\nabla|^\theta G \|_{L^{2/\theta}(\mathbb{R}^2)} \lesssim \| |\nabla|^\theta G \|_{H^{1-\theta}(\mathbb{R}^2)} \lesssim \| \nabla f \|_{G^{\sigma,0}(\mathbb{R}^2)} \]
and
\[ \| G \|_{L^{2(p-1)/\theta}(\mathbb{R}^2)} \lesssim \| G \|_{H^\alpha(\mathbb{R}^2)} \]
with \( \alpha \) given by
\[ \alpha := \frac{p-2+\theta}{p-1}. \]
Observe that \( 0 < \alpha < 1 \). Thus, we can interpolate again to obtain
\[ \| G \|_{L^{2}(\mathbb{R}^2)} \lesssim \| G \|_{L^2(\mathbb{R}^2)} \| \nabla G \|_{L^2(\mathbb{R}^2)} = \| f \|_{G^{\sigma,0}(\mathbb{R}^2)} \| \nabla f \|_{G^{\sigma,0}(\mathbb{R}^2)}. \]
Substituting the above estimates into equation (12), we see that
\[ \|L f\|_{L^2(\mathbb{R}^2)} \lesssim \sigma^\theta \|\nabla f\|_{G^{\sigma,0}(\mathbb{R}^2)} \|f\|_{G^{\sigma,0}(\mathbb{R}^2)}, \]
as claimed.

\[ \Box \]

2.2. Reformulation of the global and local regularity results. Notice that in the special case \( \sigma' = 0 \), Lemma 2.2 give us
\[ (13) \quad G^{\sigma,s}(\mathbb{R}^d) \hookrightarrow H^{s'}(\mathbb{R}^d), \quad s, s' \in \mathbb{R}, \quad \sigma > 0. \]
To see how this will be used to obtain global results, we apply the following standard argument. Suppose that the maximal time of existence of the solution is \( T^* \). Then it must be the case that
\[ (14) \quad \sup_{t \in [0,T^*]} \|u(\cdot,t)\|_{H^{s'}(\mathbb{R}^d)} + \sup_{t \in [0,T^*]} \|u_t(\cdot,t)\|_{H^{s'-1}(\mathbb{R}^d)} = +\infty, \]
otherwise we would be able to apply the local existence theory again to continue the solution past time \( T^* \). To demonstrate that this will not happen, we may apply the inclusion (13) to control the right hand side of (14) by
\[ \sup_{t \in [0,T^*]} \|u(\cdot,t)\|_{G^{\sigma,s}(\mathbb{R}^d)} + \sup_{t \in [0,T^*]} \|u_t(\cdot,t)\|_{G^{\sigma,s-1}(\mathbb{R}^d)}, \]
for certain \( \sigma > 0 \) and some \( s \in \mathbb{R} \). If we can then justify that this norm is bounded on any interval of the form \([0,T]\), global existence in \( H^{s'} \) would follow. Thus, Theorem 1.1 is an immediate consequence of the following proposition.

**Proposition 2.6.** Suppose that the initial data \( u_0 \) and \( u_1 \) satisfy the assumptions of Theorem 1.1. Let \( u \) be the local solution to the Cauchy problem (2) which belongs to the space \( C([0,\delta); H^s(\mathbb{R}^d)) \cap C^1([0,\delta); H^{s-1}(\mathbb{R}^d)) \) for \( \delta > 0 \) sufficiently small. Then, for any \( T > 0 \),
\[ (15) \quad \sup_{t \in [0,T]} \|u(\cdot,t)\|_{G^{\sigma,1}(\mathbb{R}^d)} + \sup_{t \in [0,T]} \|u_t(\cdot,t)\|_{G^{\sigma,0}(\mathbb{R}^d)} < \infty, \]
provided that,
\[ 0 < \sigma \leq \min\left\{ \sigma_0, \frac{C}{(1 + T)^{(p+1)/2}} \right\} \]
when \( d = 1 \), and
\[ 0 < \sigma \leq \min\left\{ \sigma_0, \frac{C}{(1 + T)^{(p+1-\varepsilon)/(1-\varepsilon)}} \right\} \]
when \( d = 2 \), for any \( \varepsilon > 0 \) and some constant \( C > 0 \) which is independent of \( T \).

This will be shown in Section 3.

In addition to proving existence for all times of solutions associated to low-regularity analytic initial data, we will also show that the analyticity properties will persist, at least for short times. Based on Proposition 2.1 Theorem 1.2 will follow from the following proposition.
Proposition 2.7. The Cauchy problem \((2)\) is unconditionally locally well-posed in \(G^{\sigma,s}(\mathbb{R}^d) \times G^{\sigma,s-1}(\mathbb{R}^d)\), provided that \(s > d/2 - 1/p\) and \(\sigma > 0\). That is, for each \(u_0 \in G^{\sigma,s}(\mathbb{R}^d)\) and \(u_1 \in G^{\sigma,s-1}(\mathbb{R}^d)\), there exists \(\delta > 0\) such that equation \((2)\) has a unique solution

\[ u \in C([0,\delta]; G^{\sigma,s}(\mathbb{R}^d)) \cap C^1([0,\delta]; G^{\sigma,s-1}(\mathbb{R}^d)). \]

Moreover, the solution depends continuously on the initial data.

This result will be shown in Section 4.

Remark 2.8. Proposition 2.7 is by no means optimal. We expect that it is possible to reduce \(s\) below \(d/2 - 1/p\) in dimensions \(d \geq 2\) by applying Strichartz-type estimates, though this would then become a conditional well-posedness result. This will not be pursued here.

3. Proof of Proposition 2.6

In this section, we prove Proposition 2.6 from which Theorem 1.1 is a consequence. To begin, we first observe that

\[ \|u(\cdot,t)\|_{G^{\sigma,1}(\mathbb{R}^d)} \sim \|u(\cdot,t)\|_{G^{\sigma,0}(\mathbb{R}^d)} + \|\nabla u(\cdot,t)\|_{G^{\sigma,0}(\mathbb{R}^d)}. \]

This follows from the analogous result for Sobolev spaces (see, for example, Appendix A in [22]). To estimate the first term, we apply a simple integration in time argument to show that

\[ \|u(\cdot,t)\|_{G^{\sigma,0}(\mathbb{R}^d)} \leq \|u(\cdot,0)\|_{G^{\sigma,0}(\mathbb{R}^d)} + \int_0^t \|u_t(\cdot,\tau)\|_{G^{\sigma,0}(\mathbb{R}^d)} \, d\tau. \]

Next, we define the quantity

\[ E(\sigma)(t) := \frac{1}{2} \|\nabla u(\cdot,t)\|_{G^{\sigma,0}(\mathbb{R}^d)}^2 + \frac{1}{2} \|u_t(\cdot,t)\|_{G^{\sigma,0}(\mathbb{R}^d)}^2 + \frac{1}{p+1} \|c^{\sigma}\|_{L^{p+1}(\mathbb{R}^d)}. \]

We remark that \(E(\sigma)(0)\) is the conserved energy for the equation \((1)\). It is then easy to see that

\[ \|u(\cdot,t)\|_{G^{\sigma,0}(\mathbb{R}^d)} \leq \|u(\cdot,0)\|_{G^{\sigma,0}(\mathbb{R}^d)} + \int_0^t E(\sigma)^{1/2}(\tau) \, d\tau. \]

Moreover, since

\[ \|u_t(\cdot,t)\|_{G^{\sigma,0}(\mathbb{R}^d)} \leq E(\sigma)^{1/2}(t) \]

and

\[ \|\nabla u(\cdot,t)\|_{G^{\sigma,0}(\mathbb{R}^d)} \leq E(\sigma)^{1/2}(t), \]

in order to conclude \((15)\), it suffices to show that \(E(\sigma)(t)\) remains bounded in the interval \([0,T]\).

To show that this is the case, we will use a bootstrap argument, where the parameter \(\sigma\) will play a crucial role in “closing the bootstrap”. For any \(t \in [0,T]\), let \(H(t)\) and \(C(t)\) be the statements

- \(H(t)\): \(E(\sigma)(\tau) \leq 4E(\sigma)(0)\) for \(0 \leq \tau \leq t,\)
- \(C(t)\): \(E(\sigma)(\tau) \leq 2E(\sigma)(0)\) for \(0 \leq \tau \leq t.\)

To close the bootstrap, we must prove the following four statements:
a) \( H(t) \Rightarrow C(t) \);

b) \( C(t) \Rightarrow H(t') \) for all \( t' \) in a neighborhood of \( t \);

c) If \( \{t_n\}_{n \in \mathbb{N}} \) is a sequence in \([0, T]\) such that \( t_n \to t \in [0, T] \), with \( C(t_n) \) true for all \( t_n \), then \( C(t) \) is also true;

d) \( H(t) \) is true for at least one \( t \in [0, T] \).

**Proof of a).** Fix \( t \in [0, T] \) and assume that \( H(t) \) holds. Define

\[
U(x, t) := e^{\sigma|D|} u(x, t),
\]

where \( u \) is the local solution of the Cauchy problem \([2]\). It is clear that

\[
(17) \quad U_{tt} - \Delta U = -e^{\sigma|D|}(|u|^{p-1}u).
\]

Moreover, the modified energy \( E_\sigma(t) \) can be written as

\[
E_\sigma(t) = \frac{1}{2} \| \nabla U(\cdot, t) \|^2_{L^2(\mathbb{R}^d)} + \frac{1}{2} \| U(\cdot, t) \|^2_{L^2(\mathbb{R}^d)} + \frac{1}{p+1} \| U(\cdot, t) \|_{L^{p+1}(\mathbb{R}^d)}^{p+1}.
\]

Then, for \( \tau \leq t \), we use the Fundamental Theorem of Calculus, an integration by parts, equation \([17]\) and the Cauchy-Schwarz inequality to deduce

\[
E_\sigma(\tau) = E_\sigma(0) + \int_0^\tau \frac{dE_\sigma(\tau')}{d\tau'} d\tau'
\]

\[
= E_\sigma(0) + \int_0^\tau \int_{\mathbb{R}^d} \text{Re}\{[\Delta U + U_{tt} + |U|^{p-1}U]U_t\}(x, \tau') \, dx \, d\tau'
\]

\[
\leq E_\sigma(0) + \int_0^\tau \| [U|^{p-1}U - e^{\sigma|D|}(|u|^{p-1}u)](\cdot, \tau') \|_{L^2(\mathbb{R}^d)} \| U_t(\cdot, \tau') \|_{L^2(\mathbb{R}^d)} \, d\tau'.
\]

Furthermore, applying Lemma \([2,5]\) to the last line above, we obtain the estimates

\[
(19) \quad E_\sigma(\tau) \leq E_\sigma(0) + C \sigma^\theta \int_0^\tau \| U(\cdot, \tau') \|_{L^2(\mathbb{R}^d)}^{p+1+2\theta} E_\sigma^{p+1+2\theta}(\tau') \, d\tau', \quad 0 \leq \theta \leq 1,
\]

for \( d = 1 \), and

\[
(20) \quad E_\sigma(\tau) \leq E_\sigma(0) + C \sigma^\theta \int_0^\tau \| U(\cdot, \tau') \|_{L^2(\mathbb{R}^d)}^{p+1+\theta} E_\sigma^{p+1+\theta}(\tau') \, d\tau', \quad 0 < \theta < 1,
\]

for \( d = 2 \), where \( C > 0 \) is a generic constant. Here, we have also used the fact that

\[
\| \nabla U(\cdot, t) \|_{L^2(\mathbb{R}^d)} \leq E_\sigma^{1/2}(\tau) \quad \text{and} \quad \| U_t(\cdot, t) \|_{L^2(\mathbb{R}^d)} \leq E_\sigma^{1/2}(\tau).
\]

We first treat the case \( d = 1 \). Equation \([16]\) and the hypothesis \( H(t) \) imply that for \( 0 \leq \tau' \leq \tau \leq t \leq T \),

\[
\| U(\cdot, \tau') \|_{L^2(\mathbb{R})} \leq \| U(\cdot, 0) \|_{L^2(\mathbb{R})} \int_0^{\tau'} E_\sigma^{1/2}(z) \, dz
\]

\[
\leq \| U(\cdot, 0) \|_{L^2(\mathbb{R})} + (4E_\sigma(0))^{1/2}T
\]

\[
\leq \left( \| U(\cdot, 0) \|_{L^2(\mathbb{R})} + 2E_\sigma^{1/2}(0) \right)(1 + T).
\]
Inserting this into equation (19) and using again $H(t)$ we get
\[ E_\sigma(\tau) \leq E_\sigma(0) + C' \sigma^\theta (1 + T)^{\frac{p+1-2\theta}{2}} T. \]
(22)
where
\[ C' := C \left( \|U(\cdot,0)\|_{L^2(\mathbb{R})} + 2E_\sigma^{1/2}(0) \right)^{\frac{p+1-2\theta}{2}} (4E_\sigma(0))^{\frac{p+1+2\theta}{4}}. \]
It follows that
\[ E_\sigma(\tau) \leq 2E_\sigma(0), \]
provided that
\[ C' \sigma^\theta (1 + T)^{\frac{p+1-2\theta}{2}} \leq E_\sigma(0), \]
or, more simply,
\[ \sigma \leq C(1 + T)^{-\frac{p+1-2\theta}{2}}, \]
for some constant $C > 0$. It is easy to see that the exponent in the expression on the right-hand side is maximized when $\theta = 1$, which yields the desired result.

As for $d = 2$, we start from (20) and proceed similarly to (21) and (22) to get
\[ E_\sigma(\tau) \leq E_\sigma(0) + C'' \sigma^\theta (1 + T)^{p+\theta}, \]
where
\[ C'' := \left( \|U(\cdot,0)\|_{L^2(\mathbb{R}^2)} + 2E_\sigma^{1/2}(0) \right)^{p-1+\theta} (4E_\sigma(0))^{2-\theta}. \]
As in the previous case, the conclusion $C(t)$ follows if
\[ C'' \sigma^\theta (1 + T)^{p+\theta} \leq E_\sigma(0), \]
which holds if
\[ \sigma \leq C(1 + T)^{-\frac{p+\theta}{\theta}} \]
for some constant $C$. As before, the exponent in this expression is maximum when $\theta = 1$. Since we have the restriction $0 < \theta < 1$, we choose $\theta = 1 - \varepsilon$ for any $\varepsilon > 0$. The desired result follows.

Proof of b). Fix $t \in [0, T]$, and suppose that $E_\sigma(\tau) \leq 2E_\sigma(0)$ for $0 \leq \tau \leq t$. If $t = T$, then $H(t')$ holds for all $t'$ in any neighborhood of $t$, and there is nothing to prove. So assume that $0 \leq t < T$. Then for any $\delta > 0$ we have
\[ \sup_{\tau \in [t-\delta,t]} E_\sigma(\tau) \leq 2E_\sigma(0). \]
Thus, $H(t')$ holds for $t' \in (t-\delta,t]$. It remains to check $t' \in [t, t+\delta)$. From the definition of $E_\sigma$ and from equation (16) we have that
\[ \|u(\cdot, t)\|_{G^{p+1}(\mathbb{R}^d)} + \|u_t(\cdot, t)\|_{G^{p+1}(\mathbb{R}^d)} < \infty. \]
We may then apply the local existence theory from Section 4 to construct solutions which exist on an interval $[\tau, \tau + \delta) \subset [0, T]$ for some small $\delta > 0$. In particular, we can do this so that
\[ \sup_{\tau' \in [\tau, \tau+\delta]} E_\sigma(\tau') \leq 4E_\sigma(0). \]
Thus, \( H(t') \) is true for all \( t' \in (t - \delta, t + \delta) \).

**Proof of c.** Let \( \{t_n\}_{n \in \mathbb{N}} \) be a sequence in \( [0, T] \) such that \( t_n \to t \in [0, T] \). Suppose that \( C(t_n) \) holds for all \( n \in \mathbb{N} \). Then \( E_\sigma(t_n) \leq 2E_\sigma(0) \), for every \( n \in \mathbb{N} \). By construction, the \( H^1 \)-norm of \( u \) and the \( L^2 \)-norm of \( u_t \) are continuous functions in time. By the Sobolev embedding \( H^1 \to L^{p+1} \), we have that the \( L^{p+1} \)-norm of \( u \) is also continuous in time. It follows that \( E_\sigma(t) \) is continuous, so that

\[
E_\sigma(t) = \lim_{n \to \infty} E_\sigma(t_n) \leq 2E_\sigma(0).
\]

Consider now \( \tau \in [0, t) \). Since \( t_n \to t \), there exists \( n_0 \in \mathbb{N} \) so that \( 0 \leq \tau \leq t_{n_0} \). It follows that \( E_\sigma(\tau) \leq 2E_\sigma(0) \). Therefore, \( C(t) \) holds.

**Proof of d.** \( H(0) \) is obviously true.

Based on the above results, we may close the bootstrap, and it follows that \( C(t) \) holds for all \( t \in [0, T] \). Thus, we have proven Proposition 2.6.

### 4. Proof of Proposition 2.7

In this section we prove Proposition 2.7. Roughly speaking, we will show that if the initial data are holomorphic on the set \( S_\sigma \), then the solution will also be holomorphic on \( S_\sigma \), at least for short times. We proceed via a standard fixed-point argument, where the iteration takes place in the space

\[
C([0, \delta]; G^{\sigma,s}(\mathbb{R}^d)) \cap C^1([0, \delta]; G^{\sigma,s-1}(\mathbb{R}^d)).
\]

We start recalling the following result for inhomogeneous linear wave equations in Sobolev spaces ([22, p. 79]).

**Lemma 4.1.** Let \( s \in \mathbb{R}, \ d \geq 1 \), and assume that

\[
F \in L^1([0, T]; H^{s-1}(\mathbb{R}^d)).
\]

Suppose \( u \) is a solution to the Cauchy problem

\[
\begin{aligned}
&u_{tt} - \Delta u = F(x, t), \\
&u(\cdot, 0) = u_0 \in H^s(\mathbb{R}^d), \\
&u_t(\cdot, 0) = u_1 \in H^{s-1}(\mathbb{R}^d).
\end{aligned}
\]

(25)

Then \( u \) satisfies the energy estimate

\[
\sup_{t \in [0, T]} \|u(\cdot, t)\|_{H^s(\mathbb{R}^d)} + \sup_{t \in [0, T]} \|u_t(\cdot, t)\|_{H^{s-1}(\mathbb{R}^d)}
\]

\[
\lesssim \langle T \rangle \left( \|u_0\|_{H^s(\mathbb{R}^d)} + \|u_1\|_{H^{s-1}(\mathbb{R}^d)} + \int_0^T \|F(\cdot, \tau)\|_{H^{s-1}(\mathbb{R}^d)} \, d\tau \right).
\]

By applying the pseudodifferential operator \( e^{\sigma|D|} \) to all expressions in equation (25), we obtain the following corollary:

**Corollary 4.2.** Let \( s \in \mathbb{R}, \ \sigma \geq 0, \ \ d \geq 1 \), and assume that

\[
F \in L^1([0, T]; G^{\sigma,s-1}(\mathbb{R}^d)).
\]
Suppose \( u \) is a solution to the Cauchy problem (25) with \( u_0 \in G^{\sigma,s}(\mathbb{R}^d) \) and \( u_1 \in G^{\sigma,s-1}(\mathbb{R}^d) \). Then \( u \) satisfies the modified energy estimate

\[
\sup_{t \in [0,T]} \| u(\cdot,t) \|_{G^{\sigma,s}(\mathbb{R}^d)} + \sup_{t \in [0,T]} \| u_t(\cdot,t) \|_{G^{\sigma,s-1}(\mathbb{R}^d)}
\]

(26)

\[
\lesssim \langle T \rangle \left( \| u_0 \|_{G^{\sigma,s}(\mathbb{R}^d)} + \| u_1 \|_{G^{\sigma,s-1}(\mathbb{R}^d)} + \int_0^T \| F(\cdot,t) \|_{G^{\sigma,s-1}(\mathbb{R}^d)} \, dt \right).
\]

Next, we recall that solutions to equation (25) can be written in the Duhamel form

\[
u(x,t) = W'(t) * u_0(x) + W(t) * u_1(x) + \int_0^t W(t-s) * F(s,x) \, ds,
\]

where \( W(t) \) is the operator with symbol

\[
\widehat{W}(t) = \frac{\sin(t|\xi|)}{|\xi|}.
\]

For fixed \((u_0, u_1) \in G^{\sigma,s}(\mathbb{R}^d) \times G^{\sigma,s-1}(\mathbb{R}^d)\), define the mapping \( \Phi \) on the space

\[
C([0,\delta]; G^{\sigma,s}(\mathbb{R}^d)) \cap C^1([0,\delta]; G^{\sigma,s-1}(\mathbb{R}^d))
\]

by

\[
\Phi(u)(x,t) := W'(t) * u_0(x) + W(t) * u_1(x) + \int_0^t W(t-s) * \left( - |u(x,s)|^{p-1} u(x,s) \right) \, ds.
\]

To prove Proposition 2.7, it suffices to show that \( \Phi \) has a fixed point. For this, observe that \( \Phi(u) \) is a solution to the problem

\[
\begin{cases}
(\partial_t^2 - \Delta) \Phi(u) + |u|^{p-1} u = 0, \\
\Phi(u)(\cdot,0) = u_0, \\
\Phi(u)_t(\cdot,0) = u_1.
\end{cases}
\]

Moreover, for any \( u, v \) in a ball of radius \( R \) centered at 0 in the solution space, the difference \( \Phi(u) - \Phi(v) \) satisfies

\[
\begin{cases}
(\partial_t^2 - \Delta)(\Phi(u) - \Phi(v)) = |v|^{p-1} v - |u|^{p-1} u, \\
(\Phi(u) - \Phi(v))(x,0) = 0, \\
(\Phi(u) - \Phi(v))_t(x,0) = 0.
\end{cases}
\]

Applying the Gevrey energy estimate in equation (26), we see that

\[
\sup_{t \in [0,\delta]} \| \Phi(u)(\cdot,t) - \Phi(v)(\cdot,t) \|_{G^{\sigma,s}(\mathbb{R}^d)} + \sup_{t \in [0,\delta]} \| \Phi(u)_t(\cdot,t) - \Phi(v)_t(\cdot,t) \|_{G^{\sigma,s-1}(\mathbb{R}^d)}
\]

\[
\lesssim \langle \delta \rangle \int_0^\delta \| |v|^{p-1} v(\cdot,\tau) - |u|^{p-1} u(\cdot,\tau) \|_{G^{\sigma,s-1}(\mathbb{R}^d)} \, d\tau.
\]

A simple computation and a modification of the proof of Lemma 2.4 show that

\[
\| |v|^{p-1} v(\cdot,\tau) - |u|^{p-1} u(\cdot,\tau) \|_{G^{\sigma,s-1}(\mathbb{R}^d)}
\]

\[
\lesssim \left( \| u(\cdot,\tau) \|_{G^{\sigma,s}(\mathbb{R}^d)}^{p-1} + \| v(\cdot,\tau) \|_{G^{\sigma,s}(\mathbb{R}^d)}^{p-1} \right) \| u(\cdot,\tau) - v(\cdot,\tau) \|_{G^{\sigma,s}(\mathbb{R}^d)}.
\]
for $s \geq d/2 - 1/p$. Hence, it follows that
\[
\sup_{t \in [0, \delta]} \| \Phi(u)(\cdot, t) - \Phi(v)(\cdot, t) \|_{G^{s, s}(\mathbb{R}^d)} + \sup_{t \in [0, \delta]} \| \Phi(u)(\cdot, t) - \Phi(v)(\cdot, t) \|_{G^{s, s-1}(\mathbb{R}^d)} \\
\lesssim \delta(\delta)\left( \sup_{t \in [0, \delta]} \| u(\cdot, t) \|_{G^{s, s}(\mathbb{R}^d)} + \sup_{t \in [0, \delta]} \| v(\cdot, t) \|_{G^{s, s}(\mathbb{R}^d)} \right) \sup_{t \in [0, \delta]} \| u(\cdot, t) - v(\cdot, t) \|_{G^{s, s}(\mathbb{R}^d)} \\
\lesssim \delta(\delta)R^{n-1} \sup_{t \in [0, \delta]} \| u(\cdot, t) - v(\cdot, t) \|_{G^{s, s}(\mathbb{R}^d)}.
\]

If $\delta > 0$ is sufficiently small, we deduce that
\[
\sup_{t \in [0, \delta]} \| \Phi(u)(\cdot, t) - \Phi(v)(\cdot, t) \|_{G^{s, s}(\mathbb{R}^d)} + \sup_{t \in [0, \delta]} \| \Phi(u)(\cdot, t) - \Phi(v)(\cdot, t) \|_{G^{s, s-1}(\mathbb{R}^d)} \\
< \sup_{t \in [0, \delta]} \| u(\cdot, t) - v(\cdot, t) \|_{G^{s, s-1}(\mathbb{R}^d)} + \sup_{t \in [0, \delta]} \| u_t(\cdot, t) - v_t(\cdot, t) \|_{G^{s, s-1}(\mathbb{R}^d)},
\]
so that $\Phi$ is a contraction. The existence of a unique fixed point follows from the Banach fixed point theorem.

By a similar argument, we can shows that solutions depend continuously on the initial data. Thus, the Cauchy problem for the equation (2) is locally well-posed in $G^{s, s}(\mathbb{R}^d) \times G^{s, s-1}(\mathbb{R}^d)$.

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