STAR PRODUCT FORMULA OF THETA FUNCTIONS

HIROSHIGE KAJIURA

Abstract. As a noncommutative generalization of the addition formula of theta functions, we construct a class of theta functions which are closed with respect to the Moyal star product of a fixed noncommutative parameter. These theta functions can be regarded as bases of the space of holomorphic homomorphisms between holomorphic line bundles over noncommutative complex tori.

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1. Introduction

Theta functions are associated with various algebraic relations. One of them is the addition formula, which also appears in the context of the homological mirror symmetry [14] for elliptic curves [14, 20], Abelian varieties [14] and also noncommutative real two tori with complex structures [9, 19, 12, 10].

It is known that the bases of the space of sections for a line bundle on an abelian variety is described by theta functions. However, in the context of homological mirror symmetry, theta functions are regarded rather as the bases of the space of holomorphic homomorphisms between two line bundles. The composition of two holomorphic homomorphisms is just the product of two theta functions, which by the addition formula turns out to be a linear combination of theta functions. Homological mirror symmetry then asserts that such formulas can be reproduced in a geometric way by the mirror dual symplectic torus (see subsection 3.3).

A noncommutative extension of these stories is given in the case of elliptic curves [9, 19, 12, 10] based on A. Schwarz’s framework of noncommutative complex tori [21, 3]. However, the conclusion is that the structure constants of the product are independent of the noncommutative parameter $\theta$. 

Date: October, 2005.
which implies that the derived category of holomorphic vector bundles on a noncommutative real
two-torus is independent of $\theta$ \textsuperscript{[19]}.

Thus, in order to obtain noncommutative deformations of the structure constants, one should
discuss higher dimensional complex tori. In this case, again, an extension of the framework of
A. Schwarz’s noncommutative complex tori gives various explicit noncommutative deformations \textsuperscript{[11]},
which includes the deformations described in more familiar terminologies by the Moyal star product
of theta functions. In this paper, we present the noncommutative deformation of the addition formula
of theta functions for higher dimensional tori (Theorem 4.1). For more categorical set-up describing
this phenomena, see \textsuperscript{[11]}. To explore geometric interpretations of this theorem from the mirror dual
side should be especially interesting. We hope to discuss on it elsewhere.

In section 2, we begin with the commutative case; we present explicitly the addition formula
of theta functions corresponding to the holomorphic line bundles on the $n$-dimensional complex
torus $T^{2n} := \mathbb{C}^n/(\mathbb{Z}^n \oplus \sqrt{-1} \mathbb{Z}^n)$. In section 3, we explain various aspects of the addition formula.
Though the readers can move ahead to section 4 directly, this section provides us with interesting
and pedagogical backgrounds on the product of these theta functions, together with an introduction
to the approach by noncommutative complex tori. In subsection 3.1, we explain the relation of
these theta functions with the theta vectors introduced by A. Schwarz \textsuperscript{[21, 3]} (see also \textsuperscript{[2]}) in the
framework of (non)commutative complex tori. In subsection 3.2, these theta functions or the theta
vectors are interpreted in terms of holomorphic line bundles on complex tori. In subsection 3.3,
we give an explicit geometric realization of the addition formula in the commutative case by the
mirror dual symplectic torus based on the homological mirror symmetry \textsuperscript{[14]}. In section 4, we give
a noncommutative generalization of this addition formula (Theorem 4.1). Of course, we can replace
the product of the addition formula in the commutative case by the Moyal star product. However,
the result is no longer described by any linear combination of the theta functions. The important
point is that we should and in fact can find a class of theta functions which are closed with respect
to the Moyal star product. Finally, an example of these noncommutative theta functions in the case
of complex two-tori is presented in section 5.

Throughout this paper, any (graded) vector space stands for the one over the field $k = \mathbb{C}$.

Acknowledgments: I would like to thank A, Kato, T. Kawai and K. Saito for valuable discussions
and useful comments. The author is supported by JSPS Research Fellowships for Young Scientists.

2. Commutative theta functions

The theta function $\vartheta : (\mathbb{R}^n/\mathbb{Z}^n \times \mathbb{R}^n/\mathbb{Z}^n) \times \mathfrak{H} \times \mathbb{C}^n \rightarrow \mathbb{C}$ is defined by

$$\vartheta[c_1, c_2](\Omega, z) := \sum_{m \in \mathbb{Z}^n} \exp(\pi \sqrt{-1}(m + c_1)^t \Omega (m + c_1) + 2 \pi \sqrt{-1}(m + c_1)^t \cdot (z + c_2)),$$ \hspace{1cm} (2.1)

where $c_1, c_2 \in \mathbb{R}^n/\mathbb{Z}^n$ and $\mathfrak{H}$ is the Siegel upper half plane, that is, the space of $\mathbb{C}$ valued $n$ by $n$
symmetric matrices whose imaginary parts are positive definite. Here, for two symmetric matrices
For any \( a, b \in \text{Mat}_n(\mathbb{Z}) \) such that \( A_{ab} := A_b - A_a \) is positive definite, we define

\[
\epsilon_{ab}^\mu(z) = \frac{1}{\sqrt{\det(A_{ab})}} \vartheta[0, -A_{ab}^{-1} \mu](\sqrt{-1} A_{ab}^{-1} x, z), \quad \mu \in \mathbb{Z}_n / A_{ab} \mathbb{Z}^n ,
\]

(2.2)

where \( \vartheta(\mathbb{Z}_n / A_{ab} \mathbb{Z}^n) = \det(A_{ab}) \). One obtains the following addition formula:

**Theorem 2.1.** Given three symmetric matrices \( A_a, A_b, A_c \in \text{Mat}_n(\mathbb{Z}) \) such that \( A_{ab}, A_{bc} \) are positive definite, the following product formula holds:

\[
(\epsilon_{ab}^\mu \cdot \epsilon_{bc}^\nu)(z) = \sum_{\rho \in \mathbb{Z}_n / A_{ac} \mathbb{Z}^n} C^{\mu\nu}_{abc, \rho} \epsilon_{ac}^\rho(z) ,
\]

where the structure constant \( C^{\mu\nu}_{abc, \rho} \in \mathbb{C} \) is given by

\[
C^{\mu\nu}_{abc, \rho} = \sum_{u \in \mathbb{Z}^n} \delta_{[A_{ab}]_{-u+\rho}} \delta_{[A_{bc}]_u} \exp\left(-\pi(u - A_{bc} A_{ac}^{-1} \rho)'(A_{ab}^{-1} + A_{bc}^{-1})(u - A_{bc} A_{ac}^{-1} \rho)\right) .
\]

(2.3)

As explained in the next section, in particular, in subsection 3.2 the collection of these theta functions \( \{\epsilon_{ab}^\mu\}_{\mu \in \mathbb{Z}_n / A_{ab} \mathbb{Z}^n} \) can be interpreted as the basis of holomorphic homomorphisms between a line bundle specified by \( A_a \) and the one specified by \( A_b \) on the \( n \)-dimensional complex torus \( T^{2n} = \mathbb{C}^n / (\mathbb{Z}^n + \sqrt{-1} \mathbb{Z}^n) \). The addition formula above is then interpreted as the composition of the holomorphic homomorphisms.

Let \( \text{Ob} := \{a, b, \cdots \} \) be a finite collection of labels, where any \( a \in \text{Ob} \) is associated with a nondegenerate symmetric matrix \( A_a \in \text{Mat}_n(\mathbb{Z}) \) such that, for any \( a, b \in \text{Ob} \), \( A_{ab} \) is nondegenerate if \( a \neq b \). For any \( a, b \in \text{Ob} \), define a vector space \( H^0(a, b) \) over \( \mathbb{C} \) as follows:

- If \( A_{ab} \) is positive definite, \( H^0(a, b) \) is the \( \det(A_{ab}) \)-dimensional vector space spanned by the theta functions \( \{\epsilon_{ab}^\mu\} \).
- If \( a = b \), then \( H^0(a, b) := \mathbb{C} \).
- If otherwise, then we set \( H^0(a, b) := 0 \).

For any \( a, b \in \text{Ob} \), \( \text{Hom}(a, a) \) and \( \text{Hom}(b, b) \) act on \( \text{Hom}(a, b) \) from the left and the right, respectively, as the trivial algebraic structure by complex numbers. Then, the product formula in Theorem 2.1 defines an algebraic structure on \( \oplus_{a,b \in \text{Ob}} H^0(a, b) \). This can in fact be described by the zero-th cohomology of an appropriate differential graded category (see [11]).

The main result of this paper is a noncommutative extension of Theorem 2.1 by the Moyal star product (Theorem 4.1).

For the proof of Theorem 2.1, it is convenient to prepare the following notion.

**Definition 2.2.** Given two symmetric matrices \( A_a, A_b \in \text{Mat}_n(\mathbb{Z}) \) such that \( A_{ab} \) is nondegenerate, let \( \mu \) be an element in \( \mathbb{Z}_n / A_{ab} \mathbb{Z}^n \) and \( T^{\mu}_{A_{ab}} : S(\mathbb{R}^n) \to C^\infty(T^n) \) a linear map defined by

\[
(T^{\mu}_{A_{ab}} \xi)(x) = \sum_{w \in \mathbb{Z}^n} \xi(x + w - A_{ab}^{-1} \mu) , \quad x \in \mathbb{R}^n .
\]

Here, \( S(\mathbb{R}^n) \) is the Schwartz space, that is, the space of functions on \( \mathbb{R}^n \) which tend to zero faster than any power of \( |x|, x \in \mathbb{R}^n \).
Lemma 2.3. Let $A_a, A_b, A_c \in \text{Mat}_n(\mathbb{Z})$ be symmetric matrices such that $A_{ab}$, $A_{bc}$ and $A_{ac}$ are nondegenerate. For $\xi_{ab}, \xi_{bc} \in S(\mathbb{R}^n)$, the following formula holds:

$$(T_{A_{ab}}^\mu \xi_{ab}) \cdot (T_{A_{bc}}^\nu \xi_{bc}) = \sum_{\rho \in \mathbb{Z}^n/A_{ac}\mathbb{Z}^n} (T_{A_{ac}}^\rho \xi_{ac}^\rho),$$

where $\xi_{ac}^\rho \in S(\mathbb{R}^n)$ is defined by

$$\xi_{ac}^\rho(x) := \sum_{u \in \mathbb{Z}^n} \delta_{[A_{ab}]_{-u+\rho}} \delta_{[A_{bc}]_u} \xi_{ab}(x + A_{ab}^{-1}(u - A_{bc}A_{ac}^{-1}\rho)) \cdot \xi_{bc}(x - A_{bc}^{-1}(u - A_{bc}A_{ac}^{-1}\rho)).$$

Here, $\delta_{[A_{ab}]_\rho}$ is the Kronecker’s delta mod $\mathbb{Z}^n/A_{ab}\mathbb{Z}^n$, that is,

$$\delta_{[A_{ab}]_\rho} = \begin{cases} 1 & \rho - \mu \in A_{ab}\mathbb{Z}^n, \\ 0 & \text{otherwise}. \end{cases}$$

Proof. By direct calculation, the left hand side is

$$(T_{A_{ab}}^\mu \xi_{ab}) \cdot (T_{A_{bc}}^\nu \xi_{bc})(x) = \sum_{v \in \mathbb{Z}^n} \delta_{[A_{ab}]_v} \xi_{ab}(x + A_{ab}^{-1}v) \sum_{v' \in \mathbb{Z}^n} \delta_{[A_{bc}]_{-v'}} \xi_{bc}(x + A_{bc}^{-1}v').$$

By the transformation

$$\begin{pmatrix} v \\ v' \end{pmatrix} = \begin{pmatrix} 1_n & A_{ab} \\ -1_n & A_{bc} \end{pmatrix} \begin{pmatrix} u \\ w \end{pmatrix} - \begin{pmatrix} \rho \\ 0_n \end{pmatrix},$$

the equation above is rewritten as

$$(T_{A_{ab}}^\mu \xi_{ab}) \cdot (T_{A_{bc}}^\nu \xi_{bc})(x) = \sum_{\rho \in \mathbb{Z}^n/A_{ac}\mathbb{Z}^n} \sum_{u \in \mathbb{Z}^n/u \cdot w \in \mathbb{Z}^n} \delta_{[A_{ab}]_{-u+\rho}} \delta_{[A_{bc}]_u} \xi_{ab}(x + w + A_{ab}^{-1}(u - \rho)) \xi_{bc}(x + w - A_{bc}^{-1}u).$$

On the other hand, the right hand side can be computed directly as

$$(T_{A_{ac}}^\rho \xi_{ac}^\rho)(x) = \sum_{\rho \in \mathbb{Z}^n/A_{ac}\mathbb{Z}^n} \sum_{u \in \mathbb{Z}^n} \delta_{[A_{ab}]_{-u+\rho}} \delta_{[A_{bc}]_u} \xi_{ab}(x + w - A_{ac}^{-1}\rho + A_{ab}^{-1}(u - A_{bc}A_{ac}^{-1}\rho)) \cdot \xi_{bc}(x + w - A_{ac}^{-1}\rho - A_{bc}^{-1}(u - A_{bc}A_{ac}^{-1}\rho))
= \sum_{u \in \mathbb{Z}^n} \delta_{[A_{ab}]_{-u+\rho}} \delta_{[A_{bc}]_u} \xi_{ab}(x + w + A_{ab}^{-1}(u - \rho)) \cdot \xi_{bc}(x + w - A_{bc}^{-1}u).$$

Thus, the left hand side coincides with the right hand side.

For $A_a, A_b$ such that $A_{ab} \in \text{Mat}_n(\mathbb{Z})$ is positive definite, define a function $e_{ab} \in S(\mathbb{R}^n)$ by

$$e_{ab}(x) = \exp \left( -\pi x^t A_{ab} x \right).$$

Then, by the Poisson resummation formula (see [17], p195-197), one can rewrite the theta functions $\xi_{ab}^\mu(z)$ as

$$\xi_{ab}^\mu(z) := T_{A_{ab}}^\mu (e_{ab})(z), \quad \mu \in \mathbb{Z}^n/A_{ab}\mathbb{Z}^n,$$

where, for $T_{A_{ab}}^\mu (e_{ab}) \in S(\mathbb{R}^n)$, $T_{A_{ab}}^\mu (e_{ab})(z)$ stands for the holomorphic extension.

Thus, for symmetric matrices $A_a, A_b, A_c \in \text{Mat}_n(\mathbb{Z})$ such that $A_{ab}$ and $A_{bc}$ are positive definite, apply Lemma 2.3 with $\xi_{ab} = e_{ab}$, $\xi_{bc} = e_{bc}$, and the holomorphic extension leads to Theorem 2.1.
3. Various interpretations of the product formula

In this section, we give various interpretations of Theorem 2.1

3.1. The tensor product of Heisenberg modules. Theorem 2.1 can be understood directly in A. Schwarz’s framework of noncommutative complex tori [21, 3]. A noncommutative torus $\mathcal{A}_\theta^d$ is an algebra defined by unitary generators $U_1, \ldots, U_d$ with relations

$$U_i U_j = e^{-2\pi i \theta_{ij}} U_j U_i, \quad \theta_{ij} = -\theta_{ji} \in \mathbb{R}$$

(3.1)

for $i, j = 1, \ldots, d$. Now, we shall consider $2n$-dimensional commutative torus $\mathcal{A}_\theta^{2n} := \mathcal{A}_{\theta = 0}$. Namely, $\mathcal{A}_\theta^{2n}$ is thought of as the space of functions on a $2n$-dimensional commutative torus $T^{2n}$. Thus, the generators $U_1, \ldots, U_{2n}$ now commute with each other.

A pair $E_a := (E_{A_a}, \nabla_a)$ of a finitely generated projective module $E_{A_a}$, called a Heisenberg module (see [13]), with a constant curvature connection $\nabla_a$ is constructed as follows. The Heisenberg module is defined by

$$E_{A_a} := S(\mathbb{R}^n \times (\mathbb{Z}^n/\mathbb{A}_a\mathbb{Z}^n))$$

for a fixed nondegenerate symmetric matrix $A_a \in \text{Mat}_n(\mathbb{Z})$. The right action of $\mathcal{A}_\theta^{2n}$ on $E_{A_a}$ is defined by specifying the right action of each generator; for $\xi_a \in E_{A_a}$, it is given by

$$(U_i \xi_a)(x; \mu) = e^{2\pi i \sqrt{-1}(x_i + (A_n^{-1})_{i\mu} \mu)} \xi_a(x; \mu),$$

$$U_{n+i} \xi_a(x; \mu) = \xi_a(x + A_n^{-1} t_i; \mu - t_i), \quad i = 1, \ldots, n,$$

(3.2)

where $x := (x_1 \cdots x_n)^t \in \mathbb{R}^n$ ($^t$ indicates the transpose), $\mu \in \mathbb{Z}^n/\mathbb{A}_a\mathbb{Z}^n$ and $t_i \in \mathbb{R}^n$ is defined by $(t_1 \cdots t_n) = 1_n$. A constant curvature connection $\nabla_{a,i} : E_{A_a} \to E_{A_a}$, $i = 1, \ldots, 2n$, is given by

$$(\nabla_{a,1} \cdots \nabla_{a,2n}) (x; \mu) = \left( \frac{1_n}{-A_n} \right) \left( \frac{\partial_x}{2\pi \sqrt{-1} x} \right),$$

(3.3)

where $\partial_x := (\frac{\partial_x}{2\pi \sqrt{-1}}, \ldots, \frac{\partial_x}{2\pi \sqrt{-1}})^t$, whose curvature $F_a := \{ \nabla_{a,i}, \nabla_{a,j} \}_{i,j = 1, \ldots, 2n}$ is

$$F_a := \left( 0_n, A_n, -A_n, 0_n \right).$$

The generators of the endomorphism algebra is the same as $U_i$, $i = 1, \ldots, 2n$:

$$(\xi_a Z_i)(x; \mu) = \xi_a(x; \mu) e^{2\pi i \sqrt{-1}(x_i + (A_n^{-1})_{i\mu} \mu)} ,$$

$$(\xi_a Z_n+i)(x; \mu) = \xi_a(x + A_n^{-1} t_i; \mu - t_i), \quad i = 1, \ldots, n.$$  

Namely, the endomorphism algebra also forms a commutative torus $\mathcal{A}_\theta^{2n}$.

Given $E_a$ and $E_b$ such that $A_{ab}$ is nondegenerate, the space $\text{Hom}(E_a, E_b)$ is defined again as the Schwartz space $\text{Hom}(E_a, E_b) := S(\mathbb{R}^n \times (\mathbb{Z}^n/\mathbb{A}_{ab}\mathbb{Z}^n))$. For $\xi_{ab} \in \text{Hom}(E_a, E_b)$, the right action of $\mathcal{A}_\theta^{2n}$, generated by $U_i$, $i = 1, \cdots, 2n$, and the left action of $\mathcal{A}_\theta^{2n}$, generated by $Z_i$, $i = 1, \cdots, 2n$, are defined by

$$(U_i \xi_{ab})(x; \mu) = e^{2\pi i \sqrt{-1}(x_i + (A_{ab}^{-1})_{i\mu} \mu)} \xi_{ab}(x; \mu),$$

$$(U_{n+i} \xi_{ab})(x; \mu) = \xi_{ab}(x + A_{ab}^{-1} t_i; \mu - t_i),$$

$$(\xi_{ab} Z_i)(x; \mu) = \xi_{ab}(x; \mu) e^{2\pi i \sqrt{-1}(x_i + (A_{ab}^{-1})_{i\mu} \mu)} ,$$

$$(\xi_{ab} Z_n+i)(x; \mu) = \xi_{ab}(x + A_{ab}^{-1} t_i; \mu - t_i),$$

$$F_{ab} := \left( 0_n, A_n, -A_n, 0_n \right).$$
when \( \mu \in \mathbb{Z}^n/A_{ab}\mathbb{Z}^n \). In fact, all these generators \( U_i \) and \( Z_i \), \( i = 1, \cdots, 2n \), commute with each other. The constant curvature connection \( \nabla_i : \text{Hom}(E_a, E_b) \to \text{Hom}(E_a, E_b), i = 1, \cdots, 2n \), is given by

\[
(\nabla_1 \cdots \nabla_{2n})^t := \left( \begin{array}{cc} 1_n & \partial_i x \\ -A_{ab} & 2\pi \sqrt{-1} x \end{array} \right).
\]

For \( \xi_{ab} \in \text{Hom}(E_a, E_b) \) and \( \xi_{bc} \in \text{Hom}(E_b, E_c) \), the tensor product \( m : \text{Hom}(E_a, E_b) \otimes \text{Hom}(E_b, E_c) \to \text{Hom}(E_a, E_c) \) is defined by

\[
m(\xi_{ab}, \xi_{bc})(x, \rho) = \sum_{u \in \mathbb{Z}^n} \xi_{ab}(x + A_{ab}^{-1}(u - A_{bc}A_{ac}^{-1} \rho), -u + \rho) \cdot \xi_{bc}(x - A_{bc}^{-1}(u - A_{bc}A_{ac}^{-1} \rho), u).
\]

One can see that this tensor product formula is just the definition of \( \xi_{ac}^\rho \) in eq. (2.21). This tensor product is in fact associative and the connection \( \nabla_i : \text{Hom}(E_a, E_b) \to \text{Hom}(E_a, E_b) \) satisfies the Leibniz rule with respect to this product (see [11]).

Now suppose we consider a \( n \)-dimensional complex torus \( T^{2n} := \mathbb{C}^n/(\mathbb{Z}^n \oplus \sqrt{-1} \mathbb{Z}^n) \). For \( E_a = (E_{A\alpha}, \nabla_a) \) a Heisenberg module with the constant curvature connection, the holomorphic structure \( \nabla_{a,i} : E_{A\alpha} \to E_{A\alpha}, i = 1, \cdots, n \), is defined by

\[
\nabla_{a,i} = \nabla_{a,i} + \sqrt{-1} \nabla_{a,n+i}.
\]

Also, for given \( E_a, E_b \), the holomorphic structure \( \nabla_i : \text{Hom}(E_a, E_b) \to \text{Hom}(E_a, E_b), i = 1, \cdots, n \), is defined in the same way:

\[
\nabla := \nabla_i + \sqrt{-1} \nabla_{n+i}, \quad i = 1, \cdots, n.
\]

When \( A_{ab} \) is positive definite, the space \( H^0(E_a, E_b) := \bigcap_{i=1}^n \text{Ker}(\nabla_i : \text{Hom}(E_a, E_b) \to \text{Hom}(E_a, E_b)) \) forms a \( \text{det}(A_{ab}) \)-dimensional vector space. The bases \( e_{ab}^{\mu}, \mu \in \mathbb{Z}^n/A_{ab}\mathbb{Z}^n \), are called A. Schwarz’s theta vectors [21] (see also [2]), which are just the function \( e_{ab} \in \mathcal{S}(\mathbb{R}^n) \) defined in eq. (2.3):

\[
e_{ab}^{\mu}(x, \rho) = \delta_{[A_{ab}]\rho} \exp(-\pi x^t A_{ab} x).
\]

The Leibniz rule of \( \nabla \) then guarantees that the tensor product \( m(e_{ab}^{\mu}, e_{bc}^{\nu}) \) turns out to be the linear combination of \( e_{ac}^{\mu}, \rho \in \mathbb{Z}^n/A_{ac}\mathbb{Z}^n \).

This approach by Heisenberg modules allows us various noncommutative deformations of these structures (see [11]), but some of such deformations can be lifted to theta functions as the Moyal star product; the consequence is the one presented in section 4.

### 3.2. Holomorphic line bundles on tori.

In this subsection, the theta functions \( \{e_{ab}^{\mu}\} \) in eq. (2.4), or equivalently, the theta vectors \( \{e_{ab}^{\mu}\} \) in eq. (3.5), are interpreted in terms of holomorphic line bundles on complex tori.

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1In [11], left modules in this paper is flipped to be right modules. The relation of the conventions between this paper and [11] is as follows. First, consider a bimodule \( \text{Hom}(E_a, E_b) \) in this paper. Replace \( A_a \) by \( -A_b \) and \( A_b \) by \( -A_a \). Then, one gets a bimodule in [11]. In both cases, a left/right module \( E_{A_b} \) is obtained by setting \( A_a = 0 \).
Given a \( d \)-dimensional-torus \( T^d = \mathbb{R}^d / \Lambda \), \( \Lambda := \mathbb{Z}^d \), let \( \pi : \mathbb{R}^d \to \mathbb{R}^d / \mathbb{Z}^d \) be the projection. A vector bundle \( p : E \to T^d \) is constructed as the pullback \( \pi^* E \) together with the action of \( \Lambda \), where

\[
\pi^* E = \{(x, \xi) \in \mathbb{R}^{2n} \times E \mid \pi(x) = p(\xi)\}.
\]

When \( p : E \to T^d \) is a rank \( q \) vector bundle, \( \pi^* E \) is a rank \( q \) trivial vector bundle over \( \mathbb{R}^d \). The \( \Lambda \) \( \lambda \) action on the sections of \( \pi^* E \simeq \mathbb{R}^d \times \mathbb{C}^q \) is defined by

\[
\xi(x + \lambda) := c_\lambda(x)\xi(x), \quad \xi \in \Gamma(\pi^* E) \simeq (C^\infty(\mathbb{R}^d))^\oplus q, \quad c_\lambda \in U(q; C^\infty(\mathbb{R}^d)).
\]

We require that this \( c \) satisfies the following condition:

\[
c_{\lambda'}(x + \lambda)c_\lambda(x) = c_{\lambda + \lambda'}(x).
\]

Thus, \( c_\lambda \) is regarded as a transition function of the vector bundle \( E \). A connection \( \nabla_i : \Gamma(\pi^* E) \to \Gamma(\pi^* E) \), \( i = 1, \ldots, d \), is defined so that the following compatibility conditions hold:

\[
(\nabla_i)(x + \lambda) = c_\lambda(x)(\nabla_i)(x)c_{\lambda}^{-1}(x),
\]

where the curvature is defined by

\[
F = \{F_{ij}\}_{i,j=1,\ldots,d}, \quad F_{ij} := \frac{\sqrt{-1}}{2\pi}[\nabla_i, \nabla_j].
\]

Now, let us consider a complex torus \( T^{2n} := \mathbb{C}^n / (\mathbb{Z}^n \oplus \sqrt{-1}\mathbb{Z}^n) \), where we denote the coordinates of the covering space \( \mathbb{C}^n \) by \( z := (z_1 \cdots z_n)^t \), \( z_i := x_i + \sqrt{-1}y_i \), \( i = 1, \ldots, n \). For a nondegenerate symmetric matrix \( A_a \in \text{Mat}_n(\mathbb{Z}) \), the space of sections \( \tilde{E}_{A_a} \) of a line bundle \( (q = 1 \text{ case}) \) on \( T^{2n} \) is constructed by setting

\[
c_{(\lambda_x,0)}(x,y) = 1, \quad c_{(0,\lambda_y)}(x,y) = e^{-2\pi \sqrt{-1}x^t A_a \lambda_y}.1,
\]

where \( x := (x_1 \cdots x_n)^t \), \( y := (y_1 \cdots y_n)^t \) and \( \lambda_x, \lambda_y \in \mathbb{Z}^n \) such that \( \lambda = (\lambda_x, \lambda_y) \in \Lambda \). In order to show that this transition function \( c_\lambda \) satisfies the condition (3.7), it is enough to check

\[
c_{(\lambda,0)}^{-1}(x + \lambda_x, y)c_{(0,\lambda_y)}^{-1}(x, y + \lambda_y)c_{(\lambda_x,0)}(x, y + \lambda_y)c_{(0,\lambda_y)}(x, y) = 1.
\]

The general form of sections in \( \tilde{E}_{A_a} \) is described as a function on the covering space \( \mathbb{R}^{2n} \) with coordinates \( (x,y) \) satisfying eq. (3.8); it is given as a natural extension of the two dimensional case (5, 7, 15) and see 13, the vector bundles constructed there are called twisted bundles:

\[
\tilde{\xi}_a(x,y) = \sum_{w \in \mathbb{Z}^{2n}} \sum_{\mu \in \mathbb{Z}^n/\mathbb{A}_a \mathbb{Z}^n} \exp\left(2\pi \sqrt{-1}y^t (-A_a(x+w)+\mu)\right) \xi_a^\mu(x+w-A_a^{-1}\mu), \quad \xi_a^\mu \in \mathcal{S}(\mathbb{R}^n).
\]

For \( \xi_a^\mu(x) := \xi_a(x,\mu), \xi_a \in \mathcal{S}(\mathbb{R}^n \otimes (\mathbb{Z}^n/A_a \mathbb{Z}^n)) = E_{A_a}, \) we regard \( \sim \) in the formula above as the isomorphism from \( E_{A_a} \) to \( \tilde{E}_{A_a} \) which sends \( \xi_a \) to \( \tilde{\xi}_a \). This line bundle can be equipped with the following constant curvature connection \( \{\nabla_{a,i} : \tilde{E}_{A_a} \to \tilde{E}_{A_a}\}_{i=1,\ldots,2n} \) with its curvature \( F_a \):

\[
(\nabla_{a,1}, \ldots, \nabla_{a,n})^t = \partial_x + 2\pi \sqrt{-1} A_y, \quad (\nabla_{a,n+1}, \ldots, \nabla_{a,2n})^t = \partial_y, \quad F_a = \begin{pmatrix} 0_n & A_a \\ -A_a & 0_n \end{pmatrix},
\]

\( 0_n \)
where \( \partial_{a,x} := (\frac{\partial}{\partial x_1} \cdots \frac{\partial}{\partial x_n})^t, \ \partial_{a,y} := (\frac{\partial}{\partial y_1} \cdots \frac{\partial}{\partial y_n})^t \). Let us define the generators of the space \( \text{C}^\infty(T^{2n}) \) of functions by

\[
\bar{U}_i = e^{\pi \sqrt{-1} t_i}, \quad \bar{U}_{n+i} = e^{\pi \sqrt{-1} y_i}, \quad i = 1, \ldots, n.
\]

Then, the relationship of \( \bar{E}_a := (\bar{E}_{A_a}, \nabla_a) \) with \( E_a = (E_{A_a}, \nabla_a) \) in the previous subsection can be summarized as follows: for \( \xi_a \in E_{A_a}, \)

\[
\bar{U}_i \xi_a = \bar{U}_i \xi_a, \quad \bar{U}_{n+i} \xi_a = \bar{U}_{n+i} \xi_a, \quad \nabla_a \xi_a = \nabla_a \xi_a, \quad \nabla_{a,n+i} \xi_a = \nabla_{a,n+i} \xi_a, \quad i = 1, \ldots, n.
\]

In a similar way, for given \( \bar{E}_a \) and \( \bar{E}_b \) such that \( A_{ab} \) is nondegenerate, the space \( \text{Hom}(\bar{E}_a, \bar{E}_b) \) of homomorphisms from \( \bar{E}_a \) to \( \bar{E}_b \) is the space whose elements are described of the form:

\[
\xi_{ab}(x, y) = \sum_{w \in \mathbb{Z}^n} \sum_{\mu \in \mathbb{Z}^n/A_{ab} \mathbb{Z}^n} \exp(2\pi \sqrt{-1} y^t (-A_{ab}(x + w) + \mu)) \xi_{ab}^\mu(x + w - A_{ab}^{-1} \mu),
\]

(3.9)

for \( \xi_{ab}^\mu \in S(\mathbb{R}^n) \), where the compatible constant curvature connection \( \nabla_i : \text{Hom}(\bar{E}_a, \bar{E}_b) \rightarrow \text{Hom}(\bar{E}_a, \bar{E}_b), i = 1, \ldots, n \), is given by

\[
(\nabla_1, \cdots, \nabla_n)^t := \partial_x + 2\pi \sqrt{-1} A_{ab} y, \quad (\nabla_{n+1}, \cdots, \nabla_{2n})^t := \partial_y, \quad F_{ab} = \begin{pmatrix} 0_n & A_{ab} \\ -A_{ab} & 0_n \end{pmatrix}.
\]

Again, for \( \xi_{ab}(x) =: \xi_{ab}(x, \mu), \xi_{ab} \in S(\mathbb{R}^n \otimes (\mathbb{Z}^n/A_{ab} \mathbb{Z}^n)) = \text{Hom}(E_a, E_b), \) in eq. (3.9) is regarded as the isomorphism from \( \text{Hom}(E_a, E_b) \) to \( \text{Hom}(\bar{E}_a, \bar{E}_b) \) which sends \( \xi_{ab} \) to \( \tilde{\xi}_{ab} \).

Actually, for \( E_a, E_b, E_c, \xi_{ab} \in \text{Hom}(E_a, E_b), \xi_{bc} \in \text{Hom}(E_b, E_c) \) and the corresponding elements \( \tilde{\xi}_{ab} \in \text{Hom}(\bar{E}_a, \bar{E}_b), \tilde{\xi}_{bc} \in \text{Hom}(\bar{E}_b, \bar{E}_c) \), the pointwise product \( \tilde{\xi}_{ab} \cdot \tilde{\xi}_{bc} \) turns out to be

\[
\tilde{\xi}_{ab} \cdot \tilde{\xi}_{bc} = m(\tilde{\xi}_{ab}, \tilde{\xi}_{bc}),
\]

where \( m \) is the tensor product of the Heisenberg modules defined in eq. (3.14). The proof is essentially the same as that of Lemma 2.2.

Now, for \( T^{2n} \) as a complex torus, the holomorphic structure \( \{ \bar{\nabla}_{a,i} : \bar{E}_a \rightarrow \bar{E}_a \}_{i=1, \ldots, n} \) is defined by \( \bar{\nabla}_{a,i} := \nabla_{a,i} + \sqrt{-1}\nabla_{a,n+i} \). Similarly, given \( \bar{E}_a \) and \( \bar{E}_b \), the holomorphic structure \( \{ \bar{\nabla}_i : \text{Hom}(\bar{E}_a, \bar{E}_b) \rightarrow \text{Hom}(\bar{E}_a, \bar{E}_b) \}_{i=1, \ldots, n} \) is defined by \( \bar{\nabla}_i := \nabla_i + \sqrt{-1}\nabla_{n+i} \). The space of holomorphic sections in \( \text{Hom}(\bar{E}_a, \bar{E}_b) \) is then defined by \( H^0(\bar{E}_a, \bar{E}_b) := \bigcap_{i=1}^n \text{Ker}(\bar{\nabla}_i : \text{Hom}(\bar{E}_a, \bar{E}_b) \rightarrow \text{Hom}(\bar{E}_a, \bar{E}_b)) \). This space \( H^0(\bar{E}_a, \bar{E}_b) \) forms a \( \text{det}(A_{ab}) \)-dimensional vector space spanned by \( \{ \xi_{ab}^\mu \} \), the extension of the theta vectors \( \{ \xi_{ab}^\mu \}_{\mu \in \mathbb{Z}^n/A_{ab} \mathbb{Z}^n} \) in (3.5) by eq. (3.9). Also, the explicit relation of these \( \xi_{ab}^\mu \) with the theta functions \( \epsilon_{ab}^\mu \) is given by

\[
\epsilon_{ab}^\mu(z) = \exp(\pi y^t A_{ab} y) \cdot \xi_{ab}^\mu(x, y).
\]

3.3. Lagrangian submanifolds and triangles. The homological mirror symmetry [14] asserts that the product \( m(\epsilon_{ab}^\mu, \epsilon_{bc}^\nu) \) can also be derived from geometry of the mirror dual torus \( \tilde{T}^{2n} \), a symplectic 2n-dimensional torus with the symplectic structure

\[
\omega = \begin{pmatrix} 0_n & -1_n \\ 1_n & 0_n \end{pmatrix}.
\]

(3.10)
For the covering space $\mathbb{R}^{2n}$ of $\hat{T}^{2n}$, let $\pi: \mathbb{R}^{2n} \to \hat{T}^{2n}$ be the natural projection. The coordinates for $\mathbb{R}^{2n}$ is denoted $(x_1, \ldots, x_n, y_1, \ldots, y_n)$.

The affine lagrangian submanifold mirror dual to $E_a = (E_{A_a}, \nabla_a)$ over $\mathcal{A}^{2n}$, the space of functions on $T^{2n}$, is defined by the image of the affine subspace in $\mathbb{R}^{2n}$

$$L_a : \hat{y} = A_a x$$

by the projection $\pi: \mathbb{R}^{2n} \to \hat{T}^{2n}$. Thus, we have

$$\pi^{-1} \pi(L_a) = \{ \hat{y} = A_a x + c_a, \ c_a \in \mathbb{Z} \} .$$

Let us define the space of morphisms $\text{Hom}(L_a, L_b)$ which is isomorphic to the $\mathbb{Z}^n \to A_{ab} \mathbb{Z}^n$-dimensional vector space $H^0(E_a, E_b)$ in subsection 3.1. Denote the basis of $\text{Hom}(L_a, L_b)$ by $v_{\mu}^{ab}$, $\mu \in \mathbb{Z}^n \to A_{ab} \mathbb{Z}^n$, to which is associated the image of the intersection point of $\hat{y} = A_a x + \mu$ with $\hat{y} = A_a x$ in $\mathbb{C}^n$ by $\pi: \mathbb{C}^n \to \hat{T}^{2n}$. One can see that actually the intersection point of $\pi(L_a)$ and $\pi(L_b)$ in $\hat{T}^{2n}$ is $\mathbb{Z}^n \to A_{ab} \mathbb{Z}^n = \text{det}(A_{ab})$. For a bases $v_{\mu}^{ab}$ of $\text{Hom}(L_a, L_b)$, we denote the corresponding point in $\hat{T}^{2n}$ also by $v_{\mu}^{ab}$, which defines the set $\hat{V}_a := \pi^{-1}(v_{\mu}^{ab})$ of points in the covering space $\mathbb{R}^{2n}$.

The structure constant $C_{abc, \rho}^{\mu
u} \in \mathbb{C}$ can be identified with the sum of the exponentials of the symplectic areas of the triangles $\hat{v}_a \hat{v}_{bc} \hat{v}_{ac}$ for any $\hat{v}_a \in \hat{V}_a$, $\hat{v}_{bc} \in \hat{V}_{bc}$ and $\hat{v}_{ac} \in \hat{V}_{ac}$ with respect to the symplectic structure $\omega$ in eq. (3.10), where the triangles related by parallel transformations on the covering space $\mathbb{R}^{2n}$ are identified with each other.

It is calculated as follows. Consider three affine subspaces $L'_a, L'_b, L'_c$ in $\mathbb{R}^2$ as follows:

$$L'_a : \hat{y} = A_a x + c_a , \quad L'_b : \hat{y} = A_b x + c_b , \quad L'_c : \hat{y} = A_c x + c_c .$$

If $A_{ab}$ is nondegenerate, the intersection of $L'_a$ and $L'_b$ is a point $v_{ab}$; the coordinates $(\frac{x}{y})$ are:

$$v_{ab} = \left( - (A_{ab})^{-1} (c_b - c_a) \right) \left( - A_{ab}^{-1} c_b + A_{bc} A_{ac}^{-1} c_a \right) .$$

Now, assume that $A_{ab}$ and $A_{bc}$ are positive definite. Then, $A_{ac}$ is also positive definite. The three intersection points $v_{ab}, v_{bc}, v_{ac}$ form a triangle, where the edges $(v_{ab} v_{bc}), (v_{bc} v_{ac}), (v_{ac} v_{ab})$ belong to $L'_b, L'_c$ and $L'_a$, respectively. The symplectic area of the triangle is defined by

$$(v_{ab} - v_{ac})^t \omega(v_{bc} - v_{ac}) = \left( (c_c - c_a)^t \ (c_b - c_a)^t \right) \left( A_{bc}^{-1} \ A_{ac}^{-1} \ A_{ab}^{-1} \right) \left( c_b - c_c \right) .$$

Let us put $c_a = 0$, $c_b = u'$ and $c_c = -\rho$ so that $\pi(v_{ac}) = v_{bc}^{u'}$. Then, consider

$$\sum_{u'} \delta_{[A_{ab}]^\mu - u'} \delta_{[A_{bc}]^\nu - u' + \rho} \exp \left( (- u') u' \right) \left( A_{bc}^{-1} \ A_{ac}^{-1} \ A_{ab}^{-1} \right) \left( u' + \rho \right) ,$$

where $\delta_{[A_{ab}]^\mu - u'}$ and $\delta_{[A_{bc}]^\nu - u' + \rho}$ correspond to the condition of $\pi(v_{ab}) = v_{ab}^\mu$ and $\pi(v_{bc}) = v_{bc}^{u'}$, respectively. One can see that, by the replacement $u' + \rho =: u$, this coincides with the structure constant $C_{abc, \rho}^{\mu\nu}$ of the product of the theta functions in eq. (3.4).
4. Noncommutative theta functions

The Moyal star product ([16]) is an associative noncommutative product on functions on a flat space. It gives the first example of deformation quantization [1] and is also be used as a building block of deformation quantization on arbitrary symplectic manifolds (see [18, 6]). A Moyal star product on functions on \( \mathbb{C}^n \) is defined by

\[
(f \ast g)(z) = f(z)e^{-\frac{\sqrt{-1}}{\pi} \sum_i \frac{\partial}{\partial z_i} \cdot \frac{\partial g}{\partial z_i}}
\]

where \( \frac{\partial}{\partial z_i} \cdot \frac{\partial}{\partial z_j} := \sum_{i,j=1}^n \frac{\partial}{\partial z_i} \theta_{ij} \frac{\partial}{\partial z_j} \). Note that this skewsymmetric matrix \( \theta \in \text{Mat}_n(\mathbb{R}) \) can be thought of as the restriction of the \( \theta = \{ \theta_{ij} \}_{i,j=1, \ldots, n} \) in eq. (3.1) to \( \theta = \{ \theta_{ij} \}_{i,j=1, \ldots, n} \). \(^2\)

Now, for two symmetric matrices \( A_a, A_b \in \text{Mat}_n(\mathbb{C}) \) such that \( A_{ab} \) is nondegenerate, the following matrix \( M_{ab} \in \text{Mat}_n(\mathbb{C}) \),

\[
M_{ab} := \left( 1_n + \frac{\sqrt{-1}}{2} A_{ab}^\dagger \theta A_{ab} \right)^{-1} A_{ab}, \quad A_{ab}^\dagger := A_a + A_b,
\]
is symmetric if and only if the the following condition holds:

\[
A_a \theta A_a = A_b \theta A_b.
\]

For \( A_a, A_b \in \text{Mat}_n(\mathbb{C}) \) satisfying the condition (4.1), the real part of \( M_{ab} \) is positive definite if and only if \( A_{ab} \) is positive definite (see [8], p.5). For two matrices \( A_a, A_b \in \text{Mat}_n(\mathbb{C}) \) such that \( A_{ab} \) is positive definite, define theta functions \( \epsilon_{ab}^\mu, \mu \in \mathbb{Z}^n/A_{ab} \mathbb{Z}^n \), by

\[
\epsilon_{ab}^\mu(z) = \frac{\det(1_n + \sqrt{-1} A_a \theta)^{\frac{1}{2}} \det(1_n + \sqrt{-1} A_b \theta)^{\frac{1}{2}}}{\det(A_{ab})^{\frac{1}{2}}} \vartheta[0, -A_{ab} \mu](\sqrt{-1} M_{ab}^{-1}, z).
\]

It is clear that these theta functions actually coincides with those in eq. (2.2) if \( \theta = 0 \).

Then, we get the \( \ast \) product formula of these noncommutative theta functions.

**Theorem 4.1.** For a fixed \( \theta \), consider a set of symmetric matrices \( A_a, A_b, A_c \in \text{Mat}_n(\mathbb{Z}) \) such that \( A_a \theta A_a = A_b \theta A_b = A_c \theta A_c \) and \( A_{ab}, A_{bc} \in \text{Mat}_n(\mathbb{C}) \) are positive definite. Then, the following product formula holds:

\[
(\epsilon_{ab}^\mu \ast \epsilon_{bc}^\nu)(z) = \sum_{\mu \in \mathbb{Z}^n/A_{ac} \mathbb{Z}^n} C_{abc, \rho}^{\mu \nu} \epsilon_{ac}^\rho(z),
\]

\[
C_{abc, \rho}^{\mu \nu} := \sum_{u \in \mathbb{Z}^n} \delta_{[A_{ab}]}^{\mu -u + \rho} \delta_{[A_{bc}]}^\nu \exp(-\pi(u - A_{bc} A_{ac}^{-1} \rho)^t ((A_{ab}^{-1} + A_{bc}^{-1})(1 + \sqrt{-1} A_b \theta)^{-1} (u - A_{bc} A_{ac}^{-1} \rho)).
\]

Note that the matrix \( (A_{ab}^{-1} + A_{bc}^{-1})(1 + \sqrt{-1} A_b \theta)^{-1} \in \text{Mat}_n(\mathbb{C}) \) is already symmetric.

**Proof.** Again, by the Poisson resummation formula, the theta functions \( \{ \epsilon_{ab}^\mu \} \) in eq. (4.2) can be rewritten as \( \epsilon_{ab}^\mu(z) = T_{A_{ab}}^{\mu}(e_{ab})(z) \), where

\[
e_{ab}(x) := C_{ab} \cdot e^{-\pi x^t M_{ab} x}, \quad C_{ab} := \frac{\det(1_n + \sqrt{-1} A_a \theta)^{\frac{1}{2}} \det(1_n + \sqrt{-1} A_b \theta)^{\frac{1}{2}}}{\det(1_n + \frac{\sqrt{-1}}{2} A_{ab}^\dagger \theta A_{ab})^{\frac{1}{2}}},
\]

\(^2\)This skewsymmetric matrix \( \theta \in \text{Mat}_n(\mathbb{R}) \) corresponds to \( \theta_1 \) in [11].
As in the commutative case in subsection 3.1, one can consider the corresponding Heisenberg modules with a constant curvature connection $\nabla$, where the tensor product is given just by replacing the product $\cdot$ in the right hand side of eq. (2.4) by the star product, the constant curvature connection $\nabla$ satisfies the Leibniz rule with respect to the tensor product, and the theta vectors are obtained just as the function $e_{ab}$ above [11]. The Leibniz rule of $\nabla$ then guarantees that the tensor product $m(e^\mu_{ab}, e^\nu_{bc})$ is a linear combination of $e^\rho_{ac}$. The appropriate coefficients $C_{ab} \in \mathbb{C}$ and the structure constant $C_{\mu\nu,\rho} \in \mathbb{C}$ are obtained by direct calculations. □

In the same way as in the commutative ($\theta = 0$) case, the product formula above leads to the following. Let $\text{Ob} := \{a, b, \cdots\}$ be a finite collection of labels, where any $a \in \text{Ob}$ is associated with a nondegenerate symmetric matrix $A_a \in \text{Mat}_n(\mathbb{Z})$ such that for any $a, b \in \text{Ob}$ the condition (4.1) holds and $A_{ab}$ is nondegenerate if $a \neq b$. For any $a, b \in \text{Ob}$, define a vector space $H^0(a, b)$ as follows:

- If $A_{ab}$ is positive definite, $H^0(a, b)$ is the $|\det(A_{ab})|$-dimensional vector space spanned by the theta functions $\{e_{ab}^\mu\}$.
- If $a = b$, then $H^0(a, b) := \mathbb{C}$.
- If otherwise, then we set $H^0(a, b) = 0$.

Then, the product formula in Theorem 4.1 defines an algebraic structure on $\oplus_{a,b \in \text{Ob}} H^0(a, b)$. The condition (4.1) has an interpretation in a categorical setting of these structures (see [11]).

5. An example

We end with showing an example for the case of noncommutative complex two-torus ($n = 2$). In this case, for any fixed $\theta$, the condition $A_a \theta A_a = A_b \theta A_b$ reduces to

$$\det(A_a) = \det(A_b).$$

In general there exist infinite symmetric matrices $A \in \text{Mat}_2(\mathbb{Z})$ for a fixed $\det(A)$. For instance, diagonal matrices $A \in \text{Mat}_2(\mathbb{Z})$ with $\det(A) = -4$ are

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & -4 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix},$$

and $A_1' := -A_1$, $A_2' := -A_2$, $A_3' := -A_3$. Since $H^0(i, j') = H^0(i', j) = 0$ for any $i, j = 1, 2, 3$, let us concentrate on the one side $\{1, 2, 3\}$. Then, one obtains $H^0(i, j) \neq 0$ if and only if $i \leq j$ and in particular

$$\dim(H^0(1, 2)) = 2, \quad \dim(H^0(2, 3)) = 2, \quad \dim(H^0(1, 3)) = 9.$$ 

Thus, one obtains the following quiver:

$$\begin{array}{c}
1 \\
\downarrow 2 \\
\downarrow 2 \\
\downarrow 9 \\
2 \\
\rightarrow 3 \\
\end{array}$$

However, there exist infinite symmetric matrices $A \in \text{Mat}_2(\mathbb{Z})$ with $\det(A) = -4$, since the matrix $g^t A g$ has $\det(A) = -4$ for any $SL(2, \mathbb{Z})$ element $g$. 
References

[1] F. Bayen, M. Flato, C. Frensdal, A. Lichnerowicz, D. Sternheimer, “Deformation theory and quantization I, II,” Ann. Phys. 111 (1978), 61-110, 111-151.

[2] L. Dabrowski, T. Krajewski and G. Landi, “Some properties of non-linear sigma models in noncommutative geometry,” Int. J. Mod. Phys. B 14 (2000) 2367 arXiv:hep-th/0003099.

[3] L. Dabrowski, T. Krajewski and G. Landi, “Non-linear sigma-models in noncommutative geometry: Fields with values in finite spaces,” Mod. Phys. Lett. A 18 (2003) 2371 arXiv:math.qa/0309143.

[4] M. Dieng and A. Schwarz, “Differential and complex geometry of two-dimensional noncommutative tori,” math.QA/0203160.

[5] K. Fukaya, “Mirror symmetry of abelian varieties and multi theta functions,” J. Algebraic Geom. 11 (2002), 393–512, preprint, Kyoto University, 1998.

[6] O. J. Ganor, S. Ramgoolam and W. I. Taylor, “Branes, fluxes and duality in M(atrix)-theory,” Nucl. Phys. B 492 (1997) 191, hep-th/9611202.

[7] S. Gutt, and J. Rawnsley, “Equivalence of star products on a symplectic manifold; an introduction to Deligne’s Cech cohomology classes,” J. Geom. Phys. 29 (1999) 347–392.

[8] P. Ho, “Twisted bundle on quantum torus and BPS states in matrix theory,” Phys. Lett. B 434 (1998) 41, hep-th/9803166.

[9] J. Igusa, Theta functions, Die Grundlehren der mathematischen Wissenschaften, Band 194. Springer-Verlag, New York-Heidelberg, 1972. x+232 pp.

[10] H. Kajiura, “Kronecker foliation, D1-branes and Morita equivalence of noncommutative two-tori,” JHEP 0208 (2002) 050 arXiv:hep-th/0207097.

[11] H. Kajiura, “Homological mirror symmetry on noncommutative two-tori,” hep-th/0406233.

[12] E. Kim and H. Kim, “Moduli Spaces of Standard Holomorphic Bundles on a Noncommutative Complex Torus,” math.AG/0312228.

[13] A. Polishchuk and A. Schwarz, “Categories of holomorphic line bundles on higher dimensional noncommutative complex tori,” math.QA/0510119.

[14] M. Kontsevich, “Homological algebra of mirror symmetry,” Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994), 120–139, Birkhäuser, Basel, 1995.

[15] B. Morariu and B. Zumino, “Super Yang-Mills on the noncommutative torus,” hep-th/9807198.

[16] J.E. Moyal, “Quantum mechanics as a statistical theory,” Proc. Cambridge Phil. Soc. 45 (1949), 99–124.

[17] D. Mumford, “ Tata lectures on theta. I,” With the assistance of C. Musili, M. Nori, E. Previato and M. Stillman, Progress in Mathematics, 28. Birkhäuser Boston, Inc., Boston, MA, 1983. xii+235 pp.

[18] H. Omori, Y. Maeda and A. Yoshioka, “Weyl manifolds and deformation quantization,” Adv. in Math. 85 (1991), 225–255.

[19] A. Polishchuk and A. Schwarz, “Categories of holomorphic vector bundles on noncommutative two-tori,” Commun. Math. Phys. 236 (2003) 135 arXiv:math.QA/0211262.

[20] A. Polishchuk and E. Zaslow, “Categorical mirror symmetry: the elliptic curve,” Adv. Theor. Math. Phys. 2 (1998), 443–470. math.AG/9801119.

[21] A. Schwarz, “Theta functions on noncommutative tori,” Lett. Math. Phys. 58 (2001), 81–90. math.QA/0107186.

YUKAWA INSTITUTE FOR THEORETICAL PHYSICS, KYOTO UNIVERSITY, KYOTO 606-8502, JAPAN
E-mail address: kajiura@yukawa.kyoto-u.ac.jp