New Regularization Using Domain Wall

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Abstract

We present a new regularization method, for d dim (Euclidean) quantum field theories in the continuum formalism, based on the domain wall configuration in (1+d) dim space-time. It is inspired by the recent progress in the chiral fermions on lattice. The wall "height" is given by 1/M, where M is a regularization mass parameter and appears as a (1+d) dim Dirac fermion mass. The present approach gives a thermodynamic view to the domain wall or the overlap formalism in the lattice field theory. We will show qualitative correspondence between the present continuum results and those of lattice. The extra dimension is regarded as the (inverse) temperature t. The domains are defined by the directions of the "system evolvement", not by the sign of M as in the original overlap formalism. Physically the parameter M controls both the chirality selection and the dimensional reduction to d dimension (domain wall formation). From the point of regularization, the limit Mt $\rightarrow$ 0 regularize the infra-red behaviour whereas the condition on the momentum $(k^\mu)$ integral, $|k^\mu| \leq M$, regularize the ultra-violet behaviour.

To check the new regularization works correctly, we take the 4 dim QED and 2 dim chiral gauge theory as examples. Especially the consistent and covariant anomalies are correctly obtained. The choice of solutions of the higher dim Dirac equation characterizes the two anomalies. The projective

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properties of the positive and negative energy free solutions are exploited in calculation. Some integral functions, the incomplete gamma functions and the generalized hypergeometric functions characteristically appear in this new regularization procedure.

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1 Introduction

Regularizing quantum theories respecting the chirality has been a longstanding problem both in the discrete and in the continuum field theories. The difficulty originates from the fact that the chiral symmetry is a symmetry strongly bound to the space-time dimension and is related to the discrete symmetry of parity and to the global features of the space-time topology. Non-continuous property is usually difficult to regularize. Ordinary regularizations, such as the dimensional regularization, often hinder controlling the chirality. The symmetry should be compared with others such as the gauge symmetry of the internal space and the Lorentz symmetry of the space-time. In the lattice field theory, the difficulty appears as the doubling problem of fermions [1] (see a text, say, [2]) and as the Nielsen-Ninomiya no-go theorem [3]. The recent very attractive progress in the lattice chiral fermion tells us the domain wall configuration in one dimension higher space(-time) serves as a good regularization, at least, as far as vector theories are concerned [4, 5]. It was formulated as the overlap formalism [6, 7] and was further examined by [8, 9, 10]. The corresponding lattice models were analyzed by [11, 12, 13, 14, 15]. The numerical data also look to support its validity [16]. Most recently the overlap Dirac operator by Neuberger [17], which satisfies Ginsparg-Wilson relation [18], and Lüschers’s chiral symmetry on lattice [19] makes the present direction more and more attractive.

The present motivation is to find a counter-part, in the continuum, of the above regularization on lattice. Through the analysis we expect to clarify the essence of the regularization mechanism more transparently than on lattice. We see some advanced points over the ordinary regularizations in the continuum field theories. The main goal is to develop a new feasible regularization, in the continuum formalism, which is compatible with the chiral symmetry.

The overlap formalism has been newly formulated using the heat-kernel [20]. The heat-kernel formalism is most efficiently expressed in the coordinate space [21], and which enables us to do comparison with the lattice formalism. We will often compare the present results with those obtained by the lattice domain wall approach. The present formalism is based on three key points:

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2 We do not touch on the chiral gauge theories except some argument in the second paragraph of Sect.8.
1. We utilize the characteristic relation of heat and temperature, that is, heat propagates from the high temperature to the low temperature (the second law of thermodynamics). In the system which obey the heat equation, there exists a fixed direction in the system evolvement. We regard the heat equation for the spinor system, after the Wick rotation, as the Dirac equation in one dimension higher space-time. And we consider the (1+4 dim classical) configuration which has a fixed direction in time. This setting is suitable for regularizing the dynamics (in 4 dim Euclidean space) with control of the chirality.

2. Anti-commutativity between the system operator $\hat{D}$ and the chiral matrix, that is, $\gamma_5 \hat{D} + \hat{D} \gamma_5 = 0$ plays the crucial role to separate the whole configuration into two parts (we will call them "(+)-domain" and "(-)-domain") which are related by the sign change of the "time"-axis. This is contrasted with the original formulation of the overlap where the difference of two vacua, one is constructed from the (+) sign regularization (1+4 dim fermion) mass and the other from the (-) sign, distinguishes the two domains.

3. Taking the small momentum region compared to the regularization mass scale $M$ regularize the ultra-violet divergences and, at the same time, controls the chirality.

In this paper we further examine the new approach and strengthen its basis for the establishment. In order to show that the present approach is a new regularization for general field theories, the regularization mechanism is systematically presented. Three kinds of wall configurations appear depending on the choice of propagators (regularization). In the regularization of momentum-integral, some characteristic functions (sine integral functions, incomplete gamma functions, etc) appear. Because the presented perturbation calculation is not so familiar, the description of calculation is rather explanatory so that readers can follow them. The main points in the present analysis are as follows.

1. The extra axis is interpreted as the (inverse) temperature. This formalism gives a thermodynamic view on the domain wall algorithm. As the extra axis, it should be a half line (not a line) like the temperature.
2. The characteristic condition of the present regularization: \(|k^\mu| \leq |M| \ll 1/t\), is naturally obtained and its role is closely examined.

3. The reason why the “overlap” of \(|+\rangle\) and \(|-\rangle\) should be taken in the anomaly calculation is manifest. The “overlap” in the partition function corresponds to a “difference” in the effective action.

4. This new regularization is applied to 4 dim Euclidean QED and 2 dim chiral gauge theory. Especially, in the latter model, both consistent and covariant anomalies appear depending on the choice of solutions in the 1+4 dim Dirac equation. It is a new characterization of the two anomalies.

In Sec.2 the heat-kernel method is reviewed taking 4 dim Euclidean QED as an example. In Sec.3 the new regularization formalism of domain wall approach is explained. We apply it, in Sec.4, to the model of Sec.2 and reproduce the result using the domain wall regularization. In Sec.5 the ultraviolet regularization of the momentum integral is closely explained. Some characteristic functions appear. The new regularization is applied to 2 dim chiral gauge theory in Sec.6. The consistent and covariant anomalies are newly characterized. In Sec.7 we consider the case of massive fermion. Finally we conclude in Sec.8. Three appendices are in order. App.A describes the present notation. Some useful integral formulae are listed in App.B. Projective properties of free solutions (positive and negative energy) of 1+4 dim Dirac equation are displayed in App.C.

2 QED in the Heat-Kernel Approach

We review 4 dim Euclidean QED using the heat-kernel approach and fix the present notation. The results will be compared with the domain wall approach in the following sections. In 1951 Schwinger [22] did the heat-kernel analysis of the (1+3 dim) QED and succeeded in calculating the radiative corrections in the covariant way. Physically some interesting phenomenon of the vacuum polarization in the strong magnetic field was pointed out.

We consider the massless Euclidean case and focus on its anomaly aspect. The lagrangian is given by

\[
\mathcal{L} = i \bar{\psi} \slashed{D} \psi \quad , \quad \slashed{D} = \gamma_\mu (\partial_\mu + ieA_\mu) \quad , \quad (i \slashed{D})^\dagger = i \slashed{D} \quad .
\]  

(1)
Our convention of the gamma matrices is given in App.A. The (1-loop) effective action is usually evaluated as

\[
\ln Z[A] = \ln \int \mathcal{D}\psi\mathcal{D}\bar{\psi}e^{-\int d^4x\mathcal{L}} = \text{Tr} \ln i\mathcal{D} = \frac{1}{2}\text{Tr} \ln(-\hat{\mathcal{D}}^2)
\]

\[
= -\frac{1}{2}\text{Tr} \int_0^\infty \frac{e^{-t\hat{D}}}{t} dt + \text{const} \quad , \quad \hat{D} = -\hat{\mathcal{D}}^2 \quad , \quad (2)
\]

where \(\hat{\mathcal{D}}\) is a quadratic differential operator and has positive (semi)definite eigenvalues, hence the \(t\)-integral converges well. The heat kernel is introduced as

\[
G(x, y; t) \equiv \langle x|e^{-t\hat{D}}|y \rangle \quad ,
\]

\[
(\partial_t + \hat{D})G(x, y; t) = 0 \quad , \quad \lim_{t \to 0^+} G(x, y; t) = \delta^4(x - y) \quad ,
\]

\[
\ln Z[A] = -\frac{1}{2}\int_0^\infty \frac{1}{t}\text{Tr} G(x, y; t) dt + \text{const} \quad , \quad (3)
\]

where \(\langle x\rangle\) and \(|y\rangle\) are the x-representation of \(\hat{D}\) (Dirac’s bra- and ket-vectors respectively) which can be well-defined by the complete set of eigenfunctions of \(\hat{D}\): \(\{f_n(x); n = 0, 1, \cdots | \hat{D}f_n(x) = \lambda_n f_n(x)\}\). The boundary condition equation in (3) shows the heat kernel regularization. The heat equation is usually solved in perturbation around the free solution.

\[
\hat{D} = -\partial^2 - \hat{V} \quad , \quad (\partial_t - \partial^2)G(x, y; t) = \hat{V} G(x, y; t) \quad ,
\]

\[
\hat{V} = 2ieA_\mu \partial_\mu - e^2 A^2 + ie\partial_\mu A_\mu + \frac{ie}{4}[\gamma_\mu, \gamma_\nu]F_{\mu\nu} \quad , \quad (4)
\]

where \(F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\). For the anomaly calculation in this model, the relevant order is \(O(A^2)\). In terms of the free solution \(G_0(x; t)\) and the propagator \(S(x; t)\) defined by

\[
(\partial_t - \partial^2)G_0(x; t) = 0 \quad , \quad \lim_{t \to 0^+} G_0(x; t) = \delta^4(x)I_4 \quad ,
\]

\[
(\partial_t - \partial^2)S(x; t) = \delta^4(x)\delta(t)I_4 \quad , \quad (5)
\]

the formal solution \(G(x, y; t)\) is given as

\[
G(x, y; t) = G_0(x - y; t) + \int d^4z \int_{-\infty}^\infty ds \ S(x - z; t - s)\hat{V}(z)G(z, y; s) \quad ,
\]
\[
G_0(x - y; t) = \frac{1}{(4\pi t)^2} e^{-\frac{(x-y)^2}{4t}} I_4 , \quad S(x - y; t) = \theta(t)G_0(x - y; t) ,
\]
where \( I_4 \) is the 4 by 4 unit matrix. From these we obtain the boundary condition on \( G(x, y; t) \).
\[
\lim_{t \to +0} G(x, y; t) = \delta^4(x - y) I_4 .
\]

We note here the following things.

1. The propagator has the form: the theta function \( \theta(t) \times \) free solution \( G_0(x; t) \).

2. The factor \( \theta(t) \) in \( S(x - y; t) \) guarantees that the system evolves in the forward direction in the proper time \( t \).

These properties are characteristic of the heat propagation and will be utilized in the following sections. The analogy to the heat is the central idea of this paper\[20\].

Under the infinitesimal chiral U(1) transformation:
\[
\delta_\alpha \psi = i\gamma_5 \alpha(x) \psi , \quad \delta_\alpha \bar{\psi} = \bar{\psi} i\gamma_5 \alpha(x)
\]
the lagrangian \( \mathcal{L} \) transforms as
\[
\delta_\alpha \mathcal{L} = - \partial_\mu \alpha \bar{\psi}\gamma_\mu \gamma_5 \psi .
\]
Under the infinitesimal Weyl transformation
\[
\delta_\omega \psi = \omega(x) \psi , \quad \delta_\omega \bar{\psi} = \bar{\psi} \omega(x)
\]
it changes as
\[
\delta_\omega \mathcal{L} = i \partial_\mu \omega \bar{\psi}\gamma_\mu \psi + 2 \omega \cdot \mathcal{L} .
\]

The ”naive” Ward-Takahashi(WT) identities for the chiral and Weyl transformations are derived from (8) and (10). The deviation from the ”naive” WT identities, which originates from the measure change\[23\], is identified as the anomaly. They are given by
\[
J_{ABJ} \equiv \left[ \frac{\partial (\psi + \delta_\alpha \psi, \bar{\psi} + \delta_\alpha \bar{\psi})}{\partial (\psi, \bar{\psi})} \right] , \quad \delta_\alpha \ln J_{ABJ} = \text{Tr} \ i\alpha(x)\gamma_5 \left( \delta^4(x - y) + \delta^4(x - y) \right) ,
\]
\[
\frac{1}{2} \delta \ln J_{ABJ}|_{\alpha = 0} = i \lim_{t \to +0} \int d^4x \text{tr} \gamma_5 G(x, x; t) = \int d^4x \frac{e^2}{32\pi^2} i F_{\mu\nu} \tilde{F}_{\mu\nu} ,
\]
where $\tilde{F}_{\mu\nu} = \epsilon_{\mu\nu\lambda\sigma} F_{\lambda\sigma}$, for the Adler-Bell-Jackiw (chiral U(1)) anomaly and

$$J_W \equiv \left| \frac{\partial(\psi + \delta_\omega \psi, \bar{\psi} + \delta_\omega \bar{\psi})}{\partial(\psi, \bar{\psi})} \right|,$$

$$\delta_\omega \ln J_W = \text{Tr} \omega(x) (\delta^4(x-y) + \delta^4(x-y)),$$

$$\frac{\delta}{2 \delta \omega(x)} \ln J_{\omega=0} = \lim_{t \to +0} \int d^4 x \text{tr} G(x, x; t) = \frac{1}{2} \int d^4 x \beta(e) F_{\mu\nu} F^{\mu\nu}$$

$$\beta(e) = \frac{e^2}{12\pi^2}, \quad (13)$$

for the Weyl anomaly. In (12) and (13), the delta functions are regularized using the relation (7). $\beta(e)$ is the 1-loop $\beta$-function of QED. A useful formula for the calculation of $G(x, x; t = +0)$ in (12) and (13) is given in [24]. (For the Yang-Mills theory, $\beta$-function was obtained in this way in [25].)

### 3 Domain Wall Approach

Let us do the analysis of the previous section (the 4 dim massless Euclidean QED) in the domain wall approach. First we express the effective action in terms of $i\gamma_\mu(\partial_\mu + ieA_\mu)$ itself, not its square as in Sec.2..

$$L = \bar{\psi} \hat{D} \psi, \quad \hat{D} = i\gamma_\mu(\partial_\mu + ieA_\mu), \quad \hat{D}^\dagger = \hat{D}. \quad (14)$$

Formally we have

$$\ln Z[A] = \ln \int D\psi D\bar{\psi} e^{-\int d^4xL} = \text{Tr} \ln \hat{D} = -\text{Tr} \int_0^\infty \frac{e^{-rt\hat{D}}}{t} dt + \text{const}$$

$$= -\int_0^\infty dt \text{Tr} \left[ \frac{1}{2}(1 + i\gamma_5)e^{it\gamma_5\hat{D}} + \frac{1}{2}(1 - i\gamma_5)e^{-it\gamma_5\hat{D}} \right] + \text{const}. \quad (15)$$

Because the eigenvalues of $\hat{D}$ are both negative and positive, the $t$-integral above is divergent. We clearly need regularization to make it meaningful. We should notice here that the final equality above relies only on the following properties of $\hat{D}$ and $\gamma_5$:

$$\gamma_5 \hat{D} + \hat{D} \gamma_5 = 0 \quad , \quad (\gamma_5)^2 = 1 \quad . \quad (16)$$

Note that, in the final expression of (15), the signs of the eigenvalues of $\hat{D}$ become less important (than the case of the previous section) for the $t$-integral.
convergence. This is because the exponential operator $e^{-t\hat{D}}$ is replaced by the oscillating operators $e^{\pm it\gamma_5\hat{D}}$ due to the relation (16). Here we introduce two regularization parameters $M$ and $M'$, which are most characteristic in this approach.

\[
\ln Z = -\lim_{M \to 0} \int_0^\infty \frac{dt}{t} \left(1 - i\frac{\partial}{t \partial M}\right) \text{Tr} G^5(x, y; t) \\
- \lim_{M' \to 0} \int_0^\infty \frac{dt}{t} \left(1 - i\frac{\partial}{t \partial M'}\right) \text{Tr} G^5_{-}(x, y; t),
\]

where

\[
G^5_{+}(x, y; t) \equiv \langle x | \exp\{+it\gamma_5(\hat{D} + iM)\} | y \rangle,
\]

\[
G^5_{-}(x, y; t) \equiv \langle x | \exp\{-it\gamma_5(\hat{D} + iM')\} | y \rangle.
\]  

(17)

$M$ and $M'$ can be regarded as the "sources" for $\gamma_5$. Through this procedure we can treat $\gamma_5$ within the new heat kernels $G^5_{+}$ and $G^5_{-}$. From its usage above, the limit $M \to 0, M' \to 0$ should be taken in the following way before $t$-integral:

\[
|M|t \ll 1, \quad |M'|t \ll 1.
\]  

(18)

Very interestingly, the above heat-kernels satisfy the same (except masses) 1+4 dim Minkowski Dirac equation after the following Wick rotations for $t$.

\[
(i\partial - M)G^5_{+} = i\varepsilon_\gamma G^5_{+}, \quad (X^a) = (-it, x^\mu),
\]

\[
(i\partial - M')G^5_{-} = i\varepsilon_\gamma G^5_{-}, \quad (X^a) = (+it, x^\mu),
\]  

(19)

where $\varepsilon_\gamma = \gamma_\mu A_\mu(x)$, $\partial \equiv \Gamma^a \frac{\partial}{\partial x^a}$. ($\mu$ is the 4 dim Euclidean space indices and runs from 1 to 4, while $a$ is the 1+4 dim Minkowski space-time indices and runs from 0 to 4. A slash '/' is used for 4 dim gamma matrix ($\gamma_\mu$) contraction, whereas a backslash '\\' for 1+4 dim one ($\Gamma_\alpha$). See App.A for the present notation.) Note here that the signs of the Wick-rotation is different for $G^5_{+}$ and $G^5_{-}$. $G^5_{+}$ and $G^5_{-}$ turn out to correspond to (+)-domain and (-)-domain in the original formulation [3, 4] and we also call them in this statement is a disguise at the present stage. It is correct after the Wick rotations for $t$ at (13). Concretely saying, "cos" and "sin" functions will appear in Subsect.4.2 and Sect. 6(ii).

\[3\] This statement is a disguise at the present stage. It is correct after the Wick rotations for $t$ at (13). Concretely saying, "cos" and "sin" functions will appear in Subsect.4.2 and Sect. 6(ii).

\[4\] In a sense we generalize the relation (16) through this procedure. In order to derive eq.(17) from eq.(15) for an infinitesimally small value of $M (= M')$, some $M$-dependent terms should appear in the right hand side of the first equation of (16). It looks to correspond to a kind of Ginsparg-Wilson relation [8].

\[5\] See the explanation in the first paragraph after eq.(13).
the same way.

Eq. (19) says $G^5_M$ and $G^5_{-M'}$ are given by the solutions of the 1+4 dim Dirac equation. The perturbative solutions are given by the standard textbooks [23]. For simplicity we consider the case $M = M' > 0$ in the following. Both $G^5_M$ and $G^5_{-M}$ are obtained in the same form $G^5_M$ specified by $(G_0, S)$.

\[(i\partial - M)G^5_M = i\epsilon A G^5_M \quad ,
\]
\[G^5_M(X,Y) = G_0(X,Y) + \int d^5Z S(X,Z) i\epsilon A(z) G^5_M(Z,Y) \quad ,
\]

(20)

where $(X^a) = (x^0, x^\mu = x^\mu)$ and $G_0(X,Y)$ is the free solution and $S(X,Z)$ is the propagator:

\[(i\partial - M)G_0(X,Y) = 0 \quad , \quad (i\partial - M)S(X,Y) = \delta^5(X - Y) \quad .
\]

(21)

There are four choices of the above propagator $S(X,Y)$. See Fig.1. From them we make three solutions and discuss them separately in Subsec.3.1 and 3.2 below. They are obtained by some combinations of the positive and negative energy free solutions:

\[G^p_0(X,Y) \equiv -i \int \frac{d^4k}{(2\pi)^4} \Omega_+(k) e^{-i\tilde{K}(X - Y)} \equiv \int \frac{d^4k}{(2\pi)^4} G^p_0(k) e^{-ik(x-y)} \quad ,
\]
\[\Omega_+(k) \equiv \frac{M + \tilde{K}}{2E(k)} \quad , \quad G^p_0(k) \equiv -i\Omega_+(k) e^{-iE(k)(x^0 - y^0)}
\]

(20)
\[ G_0^n(X,Y) \equiv -i \int \frac{d^4k}{(2\pi)^4} \Omega_-(k)e^{+iK(X-Y)} \equiv \int \frac{d^4k}{(2\pi)^4} G_0^n(k)e^{-ik(x-y)}, \]

\[ \Omega_-(k) \equiv \frac{M - \bar{K}}{2E(k)}, \quad G_0^n(k) \equiv -i\Omega_-(k)e^{iE(k)(X^0 - Y^0)} \quad (22) \]

where \( E(k) = \sqrt{k^2 + M^2}, (\bar{K}^a) = (\bar{K}^0 = E(k), \bar{K}^\mu = K^\mu = -k^\mu), (\bar{K}^a) = (\bar{K}^0 = E(k), \bar{K}^\mu = -K^\mu = k^\mu). \) \( k^\mu \) is the momentum in the 4 dim Euclidean space. \( \Omega_\pm(k) \) have projective property with their hermite conjugate. The relations are listed in App.C and will be efficiently used in the calculations in Sec.4 and 6.

The final important stage is regularization of the (1-loop) ultraviolet divergences. (We will explain it in Sec.5 in detail.) Corresponding to the 1-loop quantum evaluation, the determinant (23) finally involves one momentum \( (k^\mu) \)-integral (besides \( t \)-integral). We will take the analytic continuation method (Sect.5(ii)) in order to avoid introducing further regularization parameters and to avoid breaking the gauge invariance. As will be explained in (60) in Sect.5(i), it is essentially equivalent to restricting the integral region from \( 0 \leq |k^\mu| < \infty \) to

\[ \text{Chiral Condition : } 0 \leq |k^\mu| \leq M \quad (24) \]

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6 The relation between 4 dim quantities \((x^\mu \text{ and } k^\mu)\) and 1+4 dim ones \((X^a \text{ and } K^a)\):
\( (X^a) = (X^0, X^\mu = x^\mu), (K^a) = (K^0, K^\mu = -k^\mu), \quad K_a X^a = K_0 X^0 - K^\mu X^\mu = K_0 X^0 + k^\mu x^\mu. \) Only for 1+4 dim quantities (capital letters), the upper and lower indices have meaning.

7 Useful relations: \(-i\bar{K}X = -iE(k)X^0 - ikx, \quad i\bar{K}X = iE(k)X^0 - ikx, \quad M + \bar{K} = M + E(k)\gamma_5 + i\bar{k}, \quad M - \bar{K} = M - E(k)\gamma_5 + i\bar{k}. \)

8 The overlap Dirac operator \( D \) on lattice is
\[ S_D = a^4 \sum_{x \in \text{all sites}} \bar{\psi}(x)D\psi(x), \quad aD = 1 + \gamma_5 \frac{H}{\sqrt{m}}. \]
\[ \gamma_5H = \sum_{\mu} \left( \gamma^5 \nabla_\mu + \nabla^\mu \gamma_5 - \frac{i}{2} \gamma^5 \nabla_\mu \right) - M, \]
where \( a \) is a lattice spacing. If we ignore the \( \nabla^\mu \nabla_\mu \) term (Wilson term), the form is quite similar to \((33)\).
This looks similar to the usual Pauli-Villars procedure in the point of ultra-violet regularization. $M$ plays the role of the momentum cut-off. We should stress that this restriction condition (24) on the momentum integral, at the same time, controls the chirality as explained in the following. (This point is a distinguished property of the domain wall regularization.) We call (24) chiral condition. In fact, taking the extreme chiral limit:

$$\text{Extreme Chiral Limit : } \frac{M}{|k^\mu|} \rightarrow \infty,$$

(25) in the present case implies the chirality selection. In the original temperature coordinate $t \ (X^0 - Y^0 = \mp it)$, $G^p_0(k)$ and $G^n_0(k)$ behave as, in the extreme chiral limit (e.c.l.) $M/|k^\mu| \rightarrow +\infty$,

- for (+)-domain $(X^0 = -it)$
  $$iG^p_0(k) \rightarrow \frac{1 + \gamma^5}{2} e^{-Mt}, \quad iG^n_0(k) \rightarrow \frac{1 - \gamma^5}{2} e^{+Mt},$$
- for (-)-domain $(X^0 = +it)$
  $$iG^p_0(k) \rightarrow \frac{1 + \gamma^5}{2} e^{+Mt}, \quad iG^n_0(k) \rightarrow \frac{1 - \gamma^5}{2} e^{-Mt}.$$

(26)

This result will be used for characterizing different configurations with respect to the chirality. We use (24) instead of (25) in concrete calculations. (25) is too restrictive to keep the dynamics. Loosening the extreme chiral limit (25) to the chiral condition (24) can be regarded as a part of the present regularization. This situation looks similar to the introduction of the Wilson term, in the lattice formalism, in order to break the chiral symmetry. (See the last paragraph of Sec.8.)

Let us reexamine the condition (18). As read from the above result, the domain is characterized by the exponential damping behaviour which has the “width” $\sim 1/M$ around the origin of the extra $t$-axis. (18) restricts the region of $t$ as $t \ll 1/M$. This is for considering only the massless mode as purely as possible. In the lattice formalism, this corresponds to taking the zero mode (surface state) limit, in order to avoid the doubling problem, by

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9 Instead of the analytic continuation, we can take the higher derivative regularization. This corresponds to the Wilson term in lattice: $i \partial_t - M \rightarrow i \partial_t \pm (\partial^2 - M)$, $r$: ”Wilson term” coefficient. In this case the unitarity problem, rather than the gauge invariance, should be clarified.
introducing many "flavor" fermions (or adding an extra dimension) and many bosonic Pauli-Villars fields to kill the heavy fermions contribution. Besides the extreme chiral limit (25), we often consider, corresponding to (18), the following limit:

\[ M|X^0 - Y^0| \to +0 . \]  

(27)

\( G^p_0(X,Y) \) and \( G^n_0(X,Y) \) behave, in this limit, as

\[ iG^p_0(X,Y) \to \int \frac{d^4k}{(2\pi)^4} \Omega_+(k)e^{-ik(x-y)} , \]

\[ iG^n_0(X,Y) \to \int \frac{d^4k}{(2\pi)^4} \Omega_-(k)e^{-ik(x-y)} , \]

\[ i(G^p_0(X,Y) - G^n_0(X,Y)) \to \gamma_5 \delta^4(x - y) , \]

\[ i(G^p_0(X,Y) + G^n_0(X,Y)) \to \int \frac{d^4k}{(2\pi)^4} \left( \frac{M + i\frac{k}{|M + i\frac{k}|}}{M + i\frac{k}} \right) e^{-ik(x-y)} . \]  

(28)

The factor \( \frac{M + i\frac{k}{|M + i\frac{k}|}}{M + i\frac{k}} \) can be regarded as a "phase" operator depending on configuration. The above result will be used to find the boundary conditions of the full solutions (20).

In the lattice numerical simulation, the best fit value of the regularization mass \( M \) looks restricted both from the below and from the above depending on the simulation "environment" [28, 16]. The similar one occurs in the present regularization. The "double" limits (18) and (24) or (25) imply

\[ |k^\mu| \ll M \ll \frac{1}{t} \text{ or } |k^\mu| \leq M \ll \frac{1}{t} . \]  

(29)

In the standpoint of the extra dimension, the limit \( M \ll \frac{1}{t} (Mt \to +0) \) corresponds to, combined with the condition on \( |k^\mu|/M \), taking the dimensional reduction from 1+4 dim to 4 dim (Domain wall picture of 4 dim space). In the regularization view, this limit plays the role of the infrared regularization. The condition \( |k^\mu| \leq M \) should be basically regarded as a control of chirality as explained above. Its role, from the view of regularization, is the control of the high momentum region in the divergent momentum integral.

The relation (29) is the most characteristic one of the present regularization. It should be compared with the usual heat-kernel regularization

\[^{10} \text{In lattice the corresponding bound on } M, \text{ from the requirement of no doublers, has been known since the original works}[4, 6].\]
in (12) and (13) where only the limit $t \to +0$ is taken and the ultraviolet regularization is done by the simple subtraction of divergences. Eq.(29) shows the delicacy in taking the limit in the the present 1+4 dimensional regularization scheme. It implies, in the lattice simulation, $M$ should be appropriately chosen depending on the regularization scale (say, lattice size) and the momentum-region of 4 dim fermions.

### 3.1 Feynman Path

In this subsection and the next, we obtain some solutions of 1+4 dim Dirac equation which are specified by $(G_0, S)'s$ through the general form (20)-(21). First we consider the Feynman path (F) in Fig.1. Then the propagator is given by the Feynman propagator:

$$S_F(X,Y) = \theta(X^0 - Y^0)G^p_0(X,Y) + \theta(Y^0 - X^0)G^n_0(X,Y).$$

It has both the retarded and advanced parts. Now we remind ourselves of the fact that there exists a fixed direction in the system evolvement when the temperature parameter works well (See the statement around (6) and (7) in Sec.2). Let us regard the extra axis, after the Wick-rotations, as a temperature. Assuming the analogy holds here, we try to adopt the following solution, imitating the form of the Weyl anomaly solution (6).

**Retarded solution for $G_5^+M$:**

$$G_0(X,Y) = G^p_0(X,Y), \quad S(X,Y) = \theta(X^0 - Y^0)G^p_0(X,Y) \equiv S^+_F(X,Y) \quad (30)$$

**Advanced solution for $G_5^-M$:**

$$G_0(X,Y) = G^n_0(X,Y), \quad S(X,Y) = \theta(Y^0 - X^0)G^n_0(X,Y) \equiv S^-_F(X,Y) \quad (31)$$

This is chosen in such a way that the $t$-integral converges. (The "opposite" choice will be considered in the last part of this subsection.) Because we have "divided" a full solution into two chiral parts in order to introduce a fixed direction in the system evolvement, $S^\pm_F$ above does not satisfy the proper propagation equation (21). Instead it satisfies the following ones:

$$\left(i\partial - M\right)S^+_F = \frac{1 + \gamma_5 \delta^5(X - Y)}{2} \gamma_5 \delta(X^0 - Y^0) \int \frac{d^4k}{(2\pi)^4} \left(\frac{i k}{2M} + O(\frac{k^2}{M^2})\right)e^{-ik(x-y)},$$

$$\frac{1}{2} \gamma_5 \delta(X^0 - Y^0) \int \frac{d^4k}{(2\pi)^4} \left(\frac{i k}{2M} + O(\frac{k^2}{M^2})\right)e^{-ik(x-y)},$$
Fig. 2 Domain Wall structure read from the momentum spectrum of the Feynman path propagators \((30,31)\) at the extreme chiral limit (e.c.l.) \((25)\). \(S_F^\pm\) for \(|+\rangle\) and \(S_F^\mp\) for \(|-\rangle\).

\[
(i\partial - M)S_F^+ = \frac{1 - \gamma_5}{2}\delta^5(X - Y)
- \frac{\gamma_5}{2}\delta(X^0 - Y^0)\int \frac{d^4k}{(2\pi)^4}\left(i \frac{k}{2M} + O(\frac{k^2}{M})\right)e^{-ik(x-y)}, \tag{32}
\]

The equations above say \(S_F^\pm\) above satisfy the "chiral" propagator equation at the extreme chiral limit: \(M/|k^\mu| \to +\infty\). \(G^{5M}_\pm\) defined by \((30,31)\), through the second equation of \((20)\), do not satisfy \((i\partial - M)G^{5M}_\pm = ieAG^{5M}_\pm\) but satisfy

\[
M \to +\infty, \quad
(i\partial - M)G^{5M}_+ = ie\frac{1 + \gamma_5}{2}AG^{5M}_+ + O(\frac{1}{M}), \tag{33}
\]

\[
(i\partial - M)G^{5M}_- = ie\frac{1 - \gamma_5}{2}AG^{5M}_- + O(\frac{1}{M}).
\]

They correspond to the determinant of the chiral QED: \(\hat{D}_\pm \equiv i(\partial + ie\frac{1 + \gamma_5}{2}A)\) instead of \(\hat{D}\) of \((14)\). Taking the extreme chiral limit in the momentum spectrum of the propagators \(S_F^\pm\) \((30,31)\), we can read off the domain wall structure as in Fig.2. The full solutions \(G^{5M}_\pm\) \((20)\), made by the solutions
satisfy the same boundary condition as the free ones (28) from its construction:

\[ G_+^M(X,Y) \text{ (Retarded)} \rightarrow -i \int \frac{d^4k}{(2\pi)^4} \Omega_+(k) e^{-ik(x-y)} \text{ as } M(X^0 - Y^0) \rightarrow +0, \]

\[ G_-^M(X,Y) \text{ (Advanced)} \rightarrow -i \int \frac{d^4k}{(2\pi)^4} \Omega_-(k) e^{-ik(x-y)} \text{ as } M(X^0 - Y^0) \rightarrow -0. \]

From this we obtain

\[ i(G_+^M(X,Y) - G_-^M(X,Y)) \rightarrow \gamma_5 \delta^4(x - y) \text{ as } M|X^0 - Y^0| \rightarrow +0, \]

\[ i(G_+^M(X,Y) + G_-^M(X,Y)) \rightarrow \int \frac{d^4k}{(2\pi)^4} \frac{M + i\bar{k}}{|M + i\bar{k}|} e^{-i\bar{k}(x-y)} \text{ as } M|X^0 - Y^0| \rightarrow +0, \]

where \(|M + i\bar{k}| \equiv E(\bar{k})\). Taking into account the boundary conditions above, we should take, in the anomaly calculation (12,13), as

\[ \frac{1}{2} \delta_\alpha \ln J_{ABJ} = \lim_{M|X^0 - Y^0| \rightarrow +0} \text{Tr} i\alpha(x)i(G_+^M(X,Y) - G_-^M(X,Y)), \]

\[ \frac{1}{2} \delta_\omega \ln J_W = \lim_{M|X^0 - Y^0| \rightarrow +0} \text{Tr} \omega(x)i\gamma_5(G_+^M(X,Y) - G_-^M(X,Y)). \] (36)

The meaning of the choice of Feynman path solution (30,31) is subtle (but interesting) \(^\text{11}\), because the solution does not satisfy \((i\partial - M)G_+^5 = ie\mathcal{A}G_+^5\). The clear separation of right and left (Fig.2) and its calculational simplicity fascinate us to examine this solution. One purpose of this paper is to find whether this solution works correctly as regularization or not. (As will be seen in the following sections, it works well except a simple factor as far as anomalies are concerned.) The same thing can be said about the next paragraph.

As a final comment of this subsection, we refer to the opposite choice in (30) and (31). We call this choice anti-Feynman path solution. If we take the anti-Feynman (F') path in Fig.1, we obtain

\[ \text{Retarded solution for } G_+^5 \text{ :} \]

\[ G_0(X,Y) = G_0^0(X,Y), \quad S(X,Y) = \theta(X^0 - Y^0)G_0^n(X,Y) \] (37)

\[ \text{Advanced solution for } G_-^5 \text{ :} \]

\[ G_0(X,Y) = G_0^0(X,Y), \quad S(X,Y) = \theta(Y^0 - X^0)G_0^b(X,Y) \] (38)

\(^{11}\) As for the possibility of defining the chiral theory using this solution, see the argument in the second paragraph of Sect.8.
Fig. 3 Domain Wall structure read from the momentum spectrum of the anti-Feynman path propagators \( \langle \downarrow \uparrow \rangle \) at the extreme chiral limit (e.c.l.) \( (25) \).

Taking the extreme chiral limit \( \frac{|k^\mu M_\mu|}{M} \ll 1 \) in the chosen propagator above, we can read the domain wall structure as in Fig. 3. It shows the domain wall not at the origin \( (t = 0) \) but at the infinity \( (t = \infty) \) for each domain. The above solutions satisfy the boundary condition:

\[
G^5_+(X, Y) \text{ (Retarded)} \to -i \int \frac{d^4k}{(2\pi)^4} \Omega_-(k) e^{-ik(x-y)} \text{ as } M(X^0 - Y^0) \to +0 , \\
G^5_-(X, Y) \text{ (Advanced)} \to -i \int \frac{d^4k}{(2\pi)^4} \Omega_+(k) e^{-ik(x-y)} \text{ as } M(X^0 - Y^0) \to -0 . (39)
\]

From this we obtain

\[
i(G^5_+(X, Y) - G^5_-(X, Y)) \to -\gamma_5 \delta^4(x - y) \text{ as } M|X^0 - Y^0| \to +0 , \\
i(G^5_+(X, Y) + G^5_-(X, Y)) \to \int \frac{d^4k}{(2\pi)^4} \frac{M + i k}{M + i k} e^{-ik(x-y)} \text{ as } M|X^0 - Y^0| \to +0 . (40)
\]

The regularization using this solution turns out to give the same result as the Feynman path solution. The different point is that, due to the presence of the exponentially growing factor \( e^{+E(k)t} \), we must do calculation in the \( X^0 \)-coordinate. [20]
3.2 Symmetric Path

Let us consider the symmetric paths $S_a$ and $S_b$ in Fig.1. In this case we are led to take the following solution.

Symmetric retarded solution for $G^5_-$:

$$G_0(X,Y) = G_0^p(X,Y) - G_0^R(X,Y) ,$$

$$S(X,Y) = \theta(X^0 - Y^0) (G_0^p(X,Y) - G_0^R(X,Y)) \equiv S^+_\text{sym}(X,Y) ; \quad (41)$$

Symmetric advanced solution for $G^5_+$:

$$G_0(X,Y) = G_0^R(X,Y) - G_0^p(X,Y) ,$$

$$S(X,Y) = \theta(Y^0 - X^0) (G_0^R(X,Y) - G_0^p(X,Y)) \equiv S^-_\text{sym}(X,Y) . \quad (42)$$

$S^\pm_\text{sym}$ satisfy the propagator equation properly,

$$(i\partial - M)S^\pm_\text{sym} = \delta^5(X-Y) , \quad (43)$$

which should be compared with $S^\pm_\text{F}$ of (42). Taking the extreme chiral limit $|k^\mu| \ll 1$ in the spectrum of the chosen propagator above, we can read off the symmetric wall structure (one wall at the origin and the other at the infinity) as in Fig.4. The above solutions satisfy the following boundary condition:

$$G^5_- (\text{Retarded}) \to -i\gamma_5 \delta^4(x-y) \quad \text{as} \quad M(X^0 - Y^0) \to +0 ,$$

$$G^5_+ (\text{Advanced}) \to +i\gamma_5 \delta^4(x-y) \quad \text{as} \quad M(X^0 - Y^0) \to -0 ,$$

$$\frac{i}{2} (G^5_+ - G^5_-) \to \gamma_5 \delta^4(x-y) \quad \text{as} \quad M |X^0 - Y^0| \to +0 . \quad (44)$$

In this case, the measure change (12,13) is regularized as

$$\delta_\alpha \ln J_{ABJ} = \lim_{M(X^0 - Y^0) \to +0} \text{Tr} \ i\alpha(x)iG^5_+(X,Y)$$

$$+ \lim_{M(X^0 - Y^0) \to -0} \text{Tr} \ i\alpha(x)(-i)G^5_-(X,Y)$$

$$= \lim_{M |X^0 - Y^0| \to +0} \text{Tr} \ i^2\alpha(x) \{G^5_+(X,Y) - G^5_- (X,Y)\} ,$$

$^{12}$ The proper solution of (24) with the initial condition (14), which is the Cauchy problem of 1+4 dim Dirac equation, is given here. Note, in the present case, that the time axis is a half line $X^0 > 0$ or $X^0 < 0$ with $Y^0 = 0$. With this note, the results (11) and (12) coincide with 1+4 dim version of (3.29) of [27].
(ii) Symmetric, Walls at Origin and Infinity

Fig.4 Domain Wall structure read from the momentum spectrum of the Symmetric path propagators \( \{11,12\} \) at the extreme chiral limit (e.c.l.) \( (25) \). \( S^+_{\text{sym}} \) for \(+>\) and \( S^-_{\text{sym}} \) for \(->\)

\[
\delta \omega \ln J_W = \lim_{M(X^0-Y^0) \to +0} \text{Tr} \omega(x) i \gamma_5 G^5_{+M}(X,Y) \]
\[
+ \lim_{M(X^0-Y^0) \to -0} \text{Tr} \omega(x)(-i) \gamma_5 G^5_{-M}(X,Y) \]
\[
= \lim_{M|X^0-Y^0| \to +0} \text{Tr} i \omega(x) \gamma_5 \{ G^5_+(X,Y) - G^5_-(X,Y) \} \quad . \quad (45)
\]

The figures of Fig.2,3 and 4 schematically show the present regularization can control the chirality well because the separation between the left and the right is the key point. Both in (36) and in (45), the anomalies are expressed by the "difference" between \( G^5_{+M} \) and \( G^5_{-M} \) contributions. This exactly corresponds to the "overlap" equation in the original formalism. The "difference" in the effective action corresponds to the "product" in the partition function between (+) part and (-) part, that is the "overlap". This is the reason we can name \( G^5_{\pm} \) as \((\pm)-\)domains in \((17)\) and \((19)\).

\footnote{Especially for the Feynman path case \((36)\), taking the difference is indispensable to regularize \( \gamma_5 \delta^4(x-y) \).}
So far we have mainly explained the formalism of the domain wall regularization. In the following sections, we will explicitly evaluate the chiral anomalies in 4 dim QED and 2 dim chiral gauge theory using this new domain wall formalism. Surely the known results are reproduced. This shows the correctness of the present regularization. Some different regularizations appear depending on the choice of solutions of the 1+4 dim massive Dirac equation. Each choice has its characteristic aspect. The Feynman and anti-Feynman pathes are advantageous in that the calculation is simple. Especially for the Feynman path, the evaluation can be done in the original $t$-coordinate (no need for Wick rotation). The clear separation of left and right chirality is also advantageous. The chiral version of the original theory, which is non-chiral (hermitian), is automatically treated at the limit $M \to +\infty$. It is disadvantageous (at least, at present) that the regularization relies on the "approximate" solution of the Dirac equation as shown in (32). On the other hand the symmetric path is advantageous in that it relies on the proper solution of the Dirac equation as shown in (33). The configuration of two walls (one at the origin and the other at the infinity for each domain) is similar to the lattice situation. Its disadvantageous point is the calculational complexity. We must take into account both positive and negative energy states for every propagator. We will later see, in Sec.6, another important difference, between the above two kinds pathes, in relation to the consistent and covariant anomalies.

4 Anomalies of 4 Dim Euclidean QED Using Domain Wall Regularization

In this section we explicitly evaluate chiral anomalies. In the process some divergent integrals will appear. They correspond to the ultraviolet divergences in the local field theories. Its regularization is one of key points of the present approach and is separately examined in Sec.5. Only the 2-nd order (with respect to $A_\mu$) perturbation contributes.

\footnote{In lattice, the wall at the infinity is often called "anti-wall". This is the chiral partner of the other at the origin in the symmetric solution. Do not confuse it with the similar terminology : the "anti-Feynman" path in Subsec.3.1.}
Fig. 5 Abelian gauge theory, $G_+^{5M}$, $O(AA)$, (i) Feynman path and (ii) symmetric path.

### 4.1 Feynman Path

The first term of (36) is evaluated as, taking $X_0 > Y_0 \equiv 0$, 

$$
G_+^{5M}|_{AA} = \int_0^{X^0} dZ^0 \int_0^{Z^0} dW^0 \int d^4Z \int d^4W
$$

$$
\times (\tilde{K}^*)_A(z)G_0^o(Z)^* \tilde{A}^o(z)G_0^o(W)^* \tilde{A}^o(w)G_0^o(W,Y)
$$

$$
= \int_0^{X^0} dZ^0 \int_0^{Z^0} dW^0 \int d^4Z \int d^4W \int d^4k \int d^4l \int d^4q
$$

$$
\times \exp \{-i\tilde{K}(X-Z) - i\tilde{L}(Z-W) - i\tilde{Q}(W-Y)\}
$$

(46)

where $\Omega_+(q) \equiv \frac{M_+\tilde{K}}{2E(k)}$. See Fig. 5(i). As we see from the lower-ends of $Z^0$ and $W^0$-integrals, we have made here an important assumption about the extra axis: the axis is a half line (not a (straight) line) like the temperature ($t$) axis of Sec. 2. Instead of $z^\mu(=Z^\mu)$ and $w^\mu(=W^\mu)$, we take shifted variables $z'^\mu$ and $w'^m$ (As for $Z^0$ and $W^0$, we keep them.), and expand $A_\mu(z)$ and $A_\mu(w)$ in a series of $z'^\mu$ and $w'^m$.

15 If we take the extra axis as a (straight) line, we have to introduce an infrared cut-off in the $Z^0$ and $W^0$-integrals in (46), and the final anomaly result depends on it.
A_{\mu}(w)$ around the “center” $(x + y)/2$.

$$z = z' + \frac{x + y}{2},$$

$$A_{\mu}(z) = A_{\mu}(\frac{x + y}{2}) + \partial_\alpha A_{\mu}\frac{z'\alpha}{x + y} + \frac{1}{2} \partial_\alpha \partial_\beta A_{\mu}\frac{z'\alpha z'\beta}{x + y} + O(z'^3). \quad (47)$$

The same is for $A_{\nu}(w) = A_{\nu}(w' + \frac{x + y}{2})$. As a typical calculation example, we show the procedure briefly. For simplicity we consider the chiral anomaly. Among terms in (46), only $\gamma_5 \times (\partial A)^2$-terms contribute to it.

$$\text{Tr} \, \alpha(x) G_+^{5M}(X, Y) \mid_{(\partial A)^2} \sim \int_0^{X^0} \, dZ^0 \int_0^{Z^0} \, dW^0 \int \frac{d^4q}{(2\pi)^4} (-ie^2) \int d^4x \alpha(x) \partial_\alpha A_{\mu} \cdot \partial_\beta A_{\nu} \times e^{-iE(q)X^0}$$

$$\times \text{tr} \left[ \gamma_\mu \Omega_+(q) \gamma_\nu \left\{ \frac{\partial \omega(q)}{\partial q^\beta} + \Omega_+(q) \frac{\partial E(q)}{\partial q^\alpha} i(-W^0) \right\} \right]$$

$$\times \left\{ \frac{\partial \omega(q)}{\partial q^\alpha} + \Omega_+(q) \frac{\partial E(q)}{\partial q^\beta} (-i)(X^0 - Z^0) \right\}$$

$$\sim \frac{1}{2} (X^0)^2 \int \frac{d^4q}{(2\pi)^4} (-ie^2) \int d^4x \alpha(x) \partial_\alpha A_{\mu} \cdot \partial_\beta A_{\nu}$$

$$\times \left( \frac{1}{E(q)^4} \epsilon_{\mu \nu \alpha \tau} q^\tau q^\beta + \frac{1}{2E(q)^2} \epsilon_{\mu \nu \beta \alpha} E^2_5(Mt) \right) e^{-iE(q)X^0}, \quad (48)$$

where $\omega(k) \equiv \frac{M+i\lambda}{2E(k)}$. The notation ”$\sim$”, here and in the following, means ”equal up to irrelevant terms”. We do the momentum $(q^\mu)$ integral in $t$-coordinate (not in $X^0$-coordinate, $X^0 = -it$). The final result is obtained by taking the limit $Mt \to +0$.

$$\text{Tr} \, \alpha(x) G_+^{5M}(X, Y) \mid_{(\partial A)^2} \sim -\frac{t^2}{2} (-ie^2) \int d^4x \alpha(x) \partial_\alpha A_{\mu} \cdot \partial_\beta A_{\nu}$$

$$\times \frac{M^2}{8\pi^2} \epsilon_{\mu \nu \alpha \beta} \frac{\delta_{\tau \beta}}{4} E^2_4(Mt) + \frac{1}{2} \epsilon_{\mu \nu \beta \alpha} E^3_2(Mt)$$

$$\rightarrow \frac{ie^2}{64\pi^2} \times 4 \int d^4x \alpha(x) F_{\alpha \beta} \tilde{F}_{\alpha \beta}, \quad Mt \to +0, \quad (49)$$

where the function $E^r_n(a)$ is defined in App.B and the results $a^2E^5_n(a) \to 1, a^2E^3_2(a) \to 1$ are used (see (101)). In the above evaluation in $t$-coordinate,
the momentum \((q^\alpha)\) integral is convergent. This is because we take only positive (negative) energy states for (+)-domain ((-)-domain). This is one advantageous point of the Feynman path. We can obtain the same result in the \(X^0\)-coordinate \[^{20}\]. In this case, however, the momentum integral must be regularized and we take the analytic continuation. (This is always necessary for the anti-Feynman and symmetric path.) Since it is one of important points of the present regularization, we explain it separately in the next section.

\(G_{\Delta M}^{\Delta M} \mid_{AA}\) is similarly calculated (see Fig.4(i)).

\[
X^0 < Z^0 < W^0 < Y^0 = 0
\]

\[
G_{\Delta M}^{\Delta M} \mid_{AA} = \int_{X^0}^{0} dZ^0 \int_{Z^0}^{0} dW^0 \int d^4Z \int d^4W \times G_0^\alpha(X, Z)ieA(z)G_0^\alpha(Z, W)ieA(w)G_0^\alpha(W, Y)
\]

\[
= \int_{X^0}^{0} dZ^0 \int_{Z^0}^{0} dW^0 \int d^4Z \int d^4W \int d^4k \int d^4l \int d^4q \
\times (-i) \Omega_-(k)ieA(z)(-i) \Omega_-(l)ieA(w)(-i) \Omega_-(q)
\times \exp\{i\vec{K}(X - Z) + i\vec{L}(Z - W) + i\vec{Q}(W - Y)\} ,
\]

where \(\Omega_-(k) \equiv \frac{M - \vec{K}}{2E(k)}\). Using this expression, \(\text{Tr } \alpha(x)G_{\Delta M}^{\Delta M}(X, Y)\mid_{(\partial A)^2}\) turns out to be the same as \[^{19}\] except the sign.

Finally we obtain the ABJ anomaly as one fourth of \[^{12}\]. \[^{16}\]

### 4.2 Symmetric Path

We sketch the derivation of ABJ anomaly of 4 dim QED using the symmetric path. The first eq. of \[^{13}\] is evaluated using the following expression(see Fig.5(ii)).

\[
G_{\Delta M}^{\Delta M} \mid_{AA} = \int_{X^0}^{0} dZ^0 \int_{Z^0}^{0} dW^0 \int d^4Z \int d^4W (G_0^\alpha(X, Z) - G_0^\alpha(X, Z))ieA(z)
\]

\[^{16}\] The discrepant factor 1/4 comes from the fact that the Feynman path is not the proper solution as commented in Subsec.3.1. If we allowed to consider in the extreme chiral limit, hoping that the chiral anomaly itself is a topological object and does not depend on the continuous parameter \(M\), we can explain the factor as follows. For each vertex, instead of \(A = \frac{1 + 2\beta}{2}A + \frac{1 - 2\beta}{2}\bar{A}\), we consider "half" of it (the right-part for Feynman). Therefore \((\frac{1}{2})^2\) factor appears for \(O(A^2)\) contribution.
Fig. 6 Abelian Gauge Theory, \( G^5_M, O(AA), \) Feynman Path.

\[
(G^0_0(Z, W) - G^0_0(Z, W))i e A(w)(G^0_0(W, Y) - G^0_0(W, Y)) = \int_0^{X^0} dZ^0 \int_0^{Z^0} dW^0 \int d^4 z \int d^4 w \int \frac{d^4 k}{(2\pi)^4} \frac{d^4 l}{(2\pi)^4} \frac{d^4 q}{(2\pi)^4} \\
\times (-i)(\Omega_+(k)e^{-iE(k)(X^0-Z^0)} - \Omega_-(k)e^{iE(k)(X^0-Z^0)})i e A(z) \\
\times (-i)(\Omega_+(l)e^{-iE(l)(Z^0-W^0)} - \Omega_-(l)e^{iE(l)(Z^0-W^0)})i e A(w) \\
\times (-i)(\Omega_+(q)e^{-iE(q)(W^0-Y^0)} - \Omega_-(q)e^{iE(q)(W^0-Y^0)})e^{-ik(x-z) - il(z-w) - iq(w-y)}. \quad (51)
\]

We will realize the properties of \( \Omega_\pm \) presented in App.C efficiently works here. We evaluate the measure change (45): \( \delta \alpha \ln J_{ABJ} = \lim Tr \, i^2 \alpha(x)(G^5_M(X, Y) - G^5_M(X, Y)) \). For later use we call the following procedure ”Standard Calculation (SC) Procedure”.

1) expanding \( A_\mu(z) \) and \( A_\nu(w) \) around the ”center” \( (x + y)/2 \),
2) integrating out the shifted coordinates: \( w' = w - \frac{x+y}{2}, \ z' = z - \frac{x+y}{2} \),
3) integrating out the momenta \( q_\mu, l_\mu, \)
4) partial integrations,
5) taking the trace for 4 dim space coordinate \( x=y, \)

After applying the above procedure to \( Tr \, \alpha(x)G^5_M(X, Y) \) using the expression (51), its relevant part is given by

\[
Tr \alpha(x)G^5_M(X, Y)|_{(\partial A)^2} = \int_0^{X^0} dZ^0 \int_0^{Z^0} dW^0 \int d^4 x \alpha(x)(-ie^2)\partial_\alpha A_\mu \cdot \partial_\beta A_\nu
\]
\[ F_{\alpha\mu\beta\nu} \equiv \text{tr} \left\{ \partial_\beta (\Omega_+ e^{-iEw^0} - \Omega_- e^{iEw^0}) \cdot \partial_\alpha (\Omega_+ e^{-iE(X^0 - Z^0)} - \Omega_- e^{iE(X^0 - Z^0)}) \cdot \gamma_\mu \times (\Omega_+ e^{-iE(Z^0 - W^0)} - \Omega_- e^{iE(Z^0 - W^0)}) \cdot \gamma_\nu \right\} \]

where \( \partial_\alpha = \frac{\partial}{\partial x^\alpha} \), \( \Omega_\pm = \Omega_\pm(k) \), \( E = E(k) \), and \( Y^0 = 0 \). Here we focus only on \((\partial A)^2\)-part because it is sufficient for the ABJ anomaly. Now we use the following relations:

\[
\partial_\alpha (\Omega_+ e^{-iE(X^0 - Z^0)}) = \{ \partial_\alpha \omega - i \partial_\alpha E \cdot (X^0 - Z^0) \} \Omega_+ e^{-iE(X^0 - Z^0)},
\]

\[
\omega = \omega(k) = \frac{M + i k}{2E},
\]

\[
\Omega_+ \Omega_+ = \frac{M}{E} \Omega_+, \quad \Omega_- \Omega_- = \frac{M}{E} \Omega_-, \quad \Omega_+ \Omega_- = \frac{M}{E} \Omega_+ - \gamma_5 \Omega_-(-k), \quad \Omega_- \Omega_+ = \frac{M}{E} \Omega_- + \gamma_5 \Omega_+(-k).
\]

The full list of useful relations involving \( \Omega_\pm \) are given in App.C. Especially the projective property between \( \Omega_+ \) and \((\Omega_-)^\dagger\) and between \( \Omega_- \) and \((\Omega_+)^\dagger\) should be noted.

\( F_{\alpha\mu\beta\nu} \) is rewritten as

\[
F_{\alpha\mu\beta\nu} = \text{tr} \left[ \partial_\beta \omega \cdot \partial_\alpha \omega (\gamma_\mu \Omega_+ \gamma_\nu e^{-iEX^0} - \gamma_\mu \Omega_- \gamma_\nu e^{iEX^0}) \right.
\]

\[
- \{ \partial_\beta \omega \cdot \partial_\alpha \omega - i \partial_\beta \omega \cdot \partial_\alpha E \cdot (X^0 - Z^0) \} \Omega_+ - i \partial_\beta E \cdot W^0 \Omega_+ \partial_\alpha \omega \}
\times \gamma_\mu \Omega_- \gamma_\nu e^{-iE(X^0 - 2Z^0 + 2W^0)}
\]

\[
+ \{ \partial_\beta \omega \cdot \partial_\alpha \omega + i \partial_\beta \omega \cdot \partial_\alpha E \cdot (X^0 - Z^0) \} \Omega_- + i \partial_\beta E \cdot W^0 \Omega_- \partial_\alpha \omega \}
\times \gamma_\mu \Omega_+ \gamma_\nu e^{iE(X^0 - 2Z^0 + 2W^0)}
\]

\[
- \{ \partial_\beta \omega \cdot \partial_\alpha \omega + i \partial_\alpha E \cdot (X^0 - Z^0) \} \gamma_\mu \Omega_+ \gamma_\nu e^{iE(X^0 - 2Z^0)}
\]

\[
+ \{ \partial_\beta \omega \cdot \partial_\alpha \omega - i \partial_\beta E \cdot W^0 \Omega_+ \partial_\alpha \omega \} \gamma_\mu \Omega_- \gamma_\nu e^{-iE(X^0 - 2Z^0)}
\]

\[
- \{ \partial_\beta \omega \cdot \partial_\alpha \omega + i \partial_\beta E \cdot W^0 \Omega_- \partial_\alpha \omega \} \gamma_\mu \Omega_+ \gamma_\nu e^{-iE(X^0 - 2Z^0)} \right].
\]

where the following relations are used to eliminate some terms. As for the contribution to the chiral anomaly, we can confirm

\[
\text{tr} \Omega_\pm \gamma_\mu \Omega_\pm \gamma_\nu \sim 0 \quad \text{(arbitrary choice for} \pm) ,
\]

\[25\]
\[ \text{tr} \gamma_5 \Omega_{\pm} (-k) \gamma_\mu \Omega_{\pm} (k) \gamma_\nu \sim 0 \quad \text{(arbitrary choice for } \pm \text{)} , \]
\[ \text{tr} \partial_\alpha \omega \cdot \Omega_+ \gamma_\mu \Omega_+ \gamma_\nu \sim 0 , \quad \text{tr} \partial_\alpha \omega \cdot \Omega_- \gamma_\mu \Omega_- \gamma_\nu \sim 0 , \]
\[ \text{tr} \Omega_+ \partial_\alpha \omega \cdot \gamma_\mu \Omega_+ \gamma_\nu \sim 0 , \quad \text{tr} \Omega_- \partial_\alpha \omega \cdot \gamma_\mu \Omega_- \gamma_\nu \sim 0 , \quad (55) \]

where all other terms than the \( \epsilon \)-tensor term are ignored in the right-hand sides of above equations. Further evaluation goes with the help of the following useful relations valid for the chiral anomaly (\( \epsilon \)-tensor) part.

\[ \text{tr} \partial_\alpha \omega \cdot \Omega_+ \gamma_\mu \Omega_- \gamma_\nu \sim + \frac{k^\tau}{E^2} \epsilon_{\tau \alpha \mu \nu} , \quad \text{tr} \partial_\alpha \omega \cdot \Omega_- \gamma_\mu \Omega_+ \gamma_\nu \sim - \frac{k^\tau}{E^2} \epsilon_{\tau \alpha \mu \nu} , \]
\[ \text{tr} \Omega_+ \partial_\alpha \omega \cdot \gamma_\mu \Omega_+ \gamma_\nu \sim - \frac{k^\tau}{E^2} \epsilon_{\tau \alpha \mu \nu} , \quad \text{tr} \Omega_- \partial_\alpha \omega \cdot \gamma_\mu \Omega_- \gamma_\nu \sim + \frac{k^\tau}{E^2} \epsilon_{\tau \alpha \mu \nu} , \]
\[ \text{tr} \partial_\beta \omega \cdot \partial_\alpha \omega \cdot \gamma_\mu \Omega_\pm \gamma_\nu \sim \mp \frac{1}{2} \left( 1 - \frac{k^2}{2 E^4} \right) \epsilon_{\nu \beta \alpha \mu} \quad \text{(corresponding choice for } \pm \text{)} . \quad (56) \]

Finally \( \text{Tr} \alpha(x) G_+^{5M} \) reduces to

\[ \text{Tr} \alpha(x) G_+^{5M} (X, Y)|(\partial A)^2 \sim \]
\[ \int d^4 x \alpha(x)(-ie^2) \partial_\alpha A_\mu \cdot \partial_\beta A_\nu \int_0^{X^0} dZ^0 \int_0^{Z^0} dW^0 \]
\[ \times \int \frac{d^4 k}{(2\pi)^4} \left[ \frac{1}{2} \left( \frac{1}{E^2} - \frac{k^2}{2E^4} \right) \epsilon_{\alpha \mu \beta \nu} \times 2 \{ \cos E X^0 + \cos E (X^0 - 2Z^0 + 2W^0) \}
\]
\[ - \cos E (X^0 - 2Z^0) - \cos E (X^0 - 2W^0) \}
\[ + i \frac{k^\tau}{E^2} \left[ \left\{ (X^0 - Z^0) \partial_\alpha E \cdot \epsilon_{\tau \beta \mu \nu} - W^0 \partial_\beta E \cdot \epsilon_{\tau \alpha \mu \nu} \right\} (-2i) \sin E (X^0 - 2Z^0 + 2W^0) \]
\[ + (X^0 - Z^0) \partial_\alpha E \cdot \epsilon_{\tau \beta \mu \nu} 2i \sin E (X^0 - 2Z^0) + W^0 \partial_\beta E \cdot \epsilon_{\tau \alpha \mu \nu} 2i \sin E (X^0 - 2Z^0) \right] \]
\[ = \int d^4 x \alpha(x)(-ie^2) \partial_\alpha A_\mu \cdot \partial_\beta A_\nu \int \frac{d^4 k}{(2\pi)^4} \times \]
\[ \left[ \frac{1}{2} \left( \frac{1}{E^2} - \frac{k^2}{2E^4} \right) \epsilon_{\beta \nu \alpha \mu} \left( (X^0)^2 \cos E X^0 - \frac{X^0}{E} \sin E X^0 \right) \]
\[ + \frac{1}{2E^5} (-E (X^0)^2 \cos E X^0 + X^0 \sin E X^0) \times \frac{1}{4} k^2 \epsilon_{\alpha \beta \mu \nu} \times 2 \right] \]
\[ = \int d^4 x \alpha(x)(-ie^2) \partial_\alpha A_\mu \cdot \partial_\beta A_\nu \epsilon_{\beta \nu \alpha \mu} \times \frac{2\pi^2}{(2\pi)^4} \]

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\[
\times \left\{ \frac{1}{2} (MX^0)^2 (C_2^3(MX^0) - \frac{1}{2} C_4^5(MX^0)) - \frac{1}{2} MX^0 (S_3^3(MX^0) - \frac{1}{2} S_5^5(MX^0)) \right. \\
\left. + \frac{1}{4} (MX^0)^2 C_4^5(MX^0) - \frac{1}{4} MX^0 S_5^5(MX^0) \right\}
\rightarrow \int d^4 x \alpha(x) (-ie^2) \frac{2\pi^2}{(2\pi)^4} \left\{ \frac{1}{2} \left( (-1) - \frac{1}{2} (-1) \right) \\
\frac{1}{2} \left( 1 - \frac{1}{2} \times 1 \right) + \frac{1}{4} (-1) - \frac{1}{4} \times 1 \right\} \frac{1}{4} F_{\mu\nu} \tilde{F}_{\mu\nu}
\]

\[
= \int d^4 x \alpha(x) (ie^2) \frac{1}{16\pi^2} \frac{1}{2} F_{\mu\nu} \tilde{F}_{\mu\nu} \quad \text{as} \quad MX^0 \to +0 , \quad (57)
\]

where \(S_r(a)\) and \(C_r(a)\) are defined in App.B. Adding the (-)-domain contribution \(\mathbb{Tr} \alpha(x) G_{M}^{-}(X,Y)\), which is the same as above except the sign, we obtain the correct value of ABJ anomaly (12).

The Weyl anomaly can be evaluated using the second equation of (34) (Feynman) or (45) (Symmetric) in the similar way above. In this case we need to consider the parity-even terms instead of the odd ones (\(\gamma_5\) terms) in the trace. We also need to take into account \(A \partial \partial A\)-terms besides \((\partial A)^2\)-terms. It is one of the present advantage that both Weyl and chiral anomalies can be treated in a common framework. Details have been found in [33].

5 Regularization of Momentum Integral

As shown in the cut-off parameter \(M\) (24), the regularization in the momentum integral is one of most important points in the present approach. On the one hand the explanation about the chiral condition can be clearly stated in terms of the cutoff parameter \(M\) as we have done in Sect.3. From this viewpoint, the present regularization is explained in (i) part of the following. On the other hand we cannot explicitly introduce the cutoff parameter \(M\) in the momentum integral because it breaks the gauge invariance. The present regularization is re-examined in (ii) from this viewpoint. ( Another possible regularization is the higher-derivative one, which corresponds to the Wilson term as commented in a footnote of (24). ) We take two characteristic
momentum integrals which are divergent:
\[
F_s(a) \equiv \int_0^\infty \frac{dx}{\sqrt{x^2 + 1}} \frac{x(x^2 + 2)}{(\sqrt{x^2 + 1})^3} \sin(a\sqrt{x^2 + 1}) = \int_1^\infty \frac{dy}{y^2} \frac{(y^2 + 1)}{y^2} \sin(ay) ,
\]
\[
F_c(a) \equiv \int_0^\infty \frac{dx}{\sqrt{x^2 + 1}} \frac{x(x^2 + 2)}{(\sqrt{x^2 + 1})^3} \cos(a\sqrt{x^2 + 1}) = \int_1^\infty \frac{dy}{y^2} \frac{(y^2 + 1)}{y^2} \cos(ay) , \tag{58}
\]
where \( a > 0 \), \( y = \sqrt{x^2 + 1} \) and \( x \) appears, in the concrete calculation, as \( x = \sqrt{k^\mu k^\mu}/M \) (\( k^\mu \) : momentum in the 4 dim Euclidean space; \( M : 1+4 \) dim fermion mass). In terms of the notations in App.B, these two integrals are \( F_s(a) = S_3^3(a) + 2S_3^1(a) \), \( F_c(a) = C_3^3(a) + 2C_3^1(a) \).

(i) Use of exponentially damping factor

We can regularize above ones as
\[
F_c(a) + iF_s(a) = \lim_{\epsilon \to +0} \int_1^\infty \frac{dy}{y^2} \frac{(y^2 + 1)}{y^2} e^{i(a + i\epsilon)y} = -\frac{\sin a}{a} + \cos a - a \int_1^\infty \frac{\sin y}{y}dy \\
+ i\left\{ \frac{\cos a}{a} + \sin a + a \int_1^\infty \frac{\cos y}{y}dy \right\} , \tag{59}
\]
where \( \epsilon (\to +0) \) is a positive regularization parameter for the convergence.

As for the correspondence with the chiral condition (24), the above regularization is essentially equivalent to taking the following condition.
\[
x = \frac{\sqrt{k^2}}{M} \sim y \leq 1 . \tag{60}
\]

We use some formulæ
\[
s_i(a) \equiv - \int_a^\infty \frac{\sin y}{y}dy = \text{Si}(a) - \frac{\pi}{2} , \\
\text{Si}(a) \equiv \int_0^a \frac{\sin y}{y}dy = \sum_{n=0}^\infty \frac{(-1)^n a^{2n+1}}{(2n+1)!(2n+1)} , \\
c_i(a) \equiv \text{Ci}(a) \equiv - \int_a^\infty \frac{\cos y}{y}dy = \gamma + \ln a + \int_0^a \frac{\cos y - 1}{y}dy , \\
\cos a + a \text{Si}(a) = 1F_2(-\frac{1}{2}; \frac{1}{2}, \frac{1}{2}; -\frac{a^2}{4}) , \\
\sin a - a \int_0^a \frac{\cos y - 1}{y}dy = a + \frac{a^3}{12} \text{F}_3(1,1;2,2,\frac{5}{2}; -\frac{a^2}{4}) \tag{61} ,
\]
where $\text{si}(a)$, $\text{Si}(a)$, $\text{ci}(a)$ and $\text{Ci}(a)$ are integral functions and $\varphi F_q$ is the generalized hypergeometric function (see App.B). The appearance of those functions clearly distinguishes the present regularization from other ones (dimensional, usual Pauli-Villars, etc.). We finally obtain exact expressions.

$$F_s(a) = a - a\gamma + \frac{\cos a}{a} - a\ln a + \frac{a^3}{12} 2F_3(1, 1; 2, 2; \frac{5}{2}; -\frac{a^2}{4}) ,$$

$$F_c(a) = -\frac{\pi}{2} a - \frac{\sin a}{a} + i 2F_2(-\frac{1}{2}; \frac{1}{2}, \frac{1}{2}; -\frac{a^2}{4}) ,$$

(62)

where $2F_3$ and $1F_2$ are regular at $a \to +0$ and has the following forms:

$$1F_2(-\frac{1}{2}; \frac{1}{2}, \frac{1}{2}; -\frac{a^2}{4}) = 1 - \frac{1}{2} \sum_{n=1}^{\infty} \frac{(-1)^n a^{2n}}{(2n-1)!(2n-1)n} = 1 + \frac{1}{2} a^2 + O(a^4) ,$$

$$2F_3(1, 1; 2, 2; \frac{5}{2}; -\frac{a^2}{4}) = 6 \sum_{n=0}^{\infty} \frac{(-1)^n a^{2n}}{(2n+3)!(n+1)} = 1 - \frac{1}{40} a^2 + O(a^4) .$$

(63)

Therefore we have the following limit:

$$a F_s(a) \to 1 \quad , \quad F_c(a) \to 0 \quad , \quad \text{as} \quad a \to +0 \quad .$$

(64)

(ii) Analytical Continuation

Instead of (i) we can do the same thing by the analytic continuation, which donot need to introduce an additional regularization parameter. We start with a convergent integral :

$$F_e(a) \equiv \int_0^{\infty} dx \frac{x(x^2 + 2)}{(\sqrt{x^2 + 1})^3} e^{-a \sqrt{x^2 + 1}}$$

$$= \frac{e^{-a}}{a} + e^{-a} + a(\gamma + \ln a + \sum_{n=1}^{\infty} \frac{(-1)^n a^n}{n!n}) , \quad a > 0 \quad ,$$

(65)

where $F_e(a) = E_3^a(a) + 2E_3^1(a)$. Let us define $F_c(a)$ and $F_s(a)$ by the following analytic continuation.

$$a \to -ia \quad , \quad F_e(a) \to F_e(-ia) = F_c(a) + iF_s(a) \quad .$$

(66)

In this case we must specify a branch $N = -1$ in

$$\ln(-ia) = (\frac{3}{2} + 2N)\pi i + \ln a \quad \text{in order to obtain the results} \quad (62) .$$

Note that the final limit $(a \to +0)$ is not affected by this ambiguity $N$. 

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Fig. 7 Graphs of \( f(x, a = 10.0; \epsilon = 0.0) = \frac{x(x^2 + 2)}{(\sqrt{x^2 + 1})^3} \sin(10.0\sqrt{x^2 + 1}) \) (above)
and its regularized one
\[ f(x, a = 10.0; \epsilon = 1.0) = \frac{x(x^2 + 2)}{(\sqrt{x^2 + 1})^3} \sin(10.0\sqrt{x^2 + 1})e^{-\sqrt{x^2 + 1}} \] (below).
\[ 0 \leq x \leq 5. \] See the integrand of (58) and its regularized one (67).

In Fig. 7, we plot the integrand of \( F_s(a) \) of (58) and its regularized one:
\[ f(x, a; \epsilon) = \frac{x(x^2 + 2)}{(\sqrt{x^2 + 1})^3} \sin(a\sqrt{x^2 + 1})e^{-\epsilon\sqrt{x^2 + 1}}, \quad a = 10.0, \quad \epsilon = 1.0 \quad \text{(67)} \]

The present regularization, in the momentum integral, typically do the following things: 1) (Exponentially) divergent functions, due to negative eigenvalues, are first replaced by oscillating ones (Fig. 7, above) using the Wick rotation (\( X^0 = \mp it \)) and then 2) the large momentum (\(|k^\mu| \geq M\)) region is made to be exponentially damped (Fig. 7, below) by the \( \epsilon \)-factor (or the analytic continuation, or the chiral condition (24)). Finally we take 3) the limit \( Mt \to +0 \) or \( MX^0 \to \pm 0. \) Steps 1) and 2) are for ultra-violet regularization whereas 3) is for infrared one.

## 6 2 Dim Non-Abelian Anomaly

Let us consider 2 dim chiral non-Abelian gauge theory and analyze its anomaly in the present domain wall approach. Its consistent and covariant anomalies were examined in the ordinary heat-kernel by \cite{29, 30} and in the ordinary domain wall approach by \cite{16, 31, 32}.

\[ \mathcal{L} = \bar{\psi} \hat{D} \psi, \]
\begin{equation}
\hat{D} = i\gamma_\mu (\partial_\mu + iP_+ R_\mu) \quad \hat{D}^\dagger = i\gamma_\mu (\partial_\mu + iP_- R_\mu) \quad P_\pm = \frac{1}{2}(1 \pm \gamma_5) ,
\end{equation}

where \( R_\mu \) is the chiral (right-handed) gauge field. \( \hat{D} \) is not hermitian, which is a different point from the (4 dim) QED of Sec.3. The lagrangian has the chiral gauge symmetry.

\begin{equation}
\psi' = e^{iP_+ \lambda(x)} \psi \quad \bar{\psi}' = \bar{\psi} e^{-iP_- \lambda(x)} \quad \lambda(x) = T^a \lambda^a(x) \quad R_\mu' = U(\lambda) R_\mu U^{-1}(\lambda) + i\partial_\mu U(\lambda) \cdot U^{-1}(\lambda) \quad U(\lambda) = e^{i\lambda(x)} ,
\end{equation}

where \( \lambda(x) \) is the gauge parameter and \( T^a \) is the generators of the symmetry group. In the above notation the field strength and its transformation is given by

\begin{equation}
F_{\mu\nu} = \partial_\mu R_\nu - \partial_\nu R_\mu + i[R_\mu, R_\nu] \quad F_{\mu\nu}' = U(\lambda) F_{\mu\nu} U^{-1}(\lambda) .
\end{equation}

The variation of the Jacobian for the change of variables \((\psi, \bar{\psi} \rightarrow \psi', \bar{\psi}')\) is given by

\begin{equation}
\delta_\lambda \ln J_{N,A} = \delta_\lambda \ln \left| \frac{\partial(\psi', \bar{\psi}')}{\partial(\psi, \bar{\psi})} \right|
= \begin{cases} 
\text{Tr} i \lambda(x) (P_+ \delta^2(x - y) - P_- \bar{\delta}^2(x - y)) + O(\lambda^2) , \\
\text{or} \\
\text{Tr} i \lambda(x) \gamma_5 \delta^2(x - y) + O(\lambda^2) , \\
\text{or} \\
\frac{1}{2} \text{Tr} i \lambda(x) (\gamma_5 \delta^2(x - y) + \gamma_5 \bar{\delta}^2(x - y)) + O(\lambda^2) ,
\end{cases}
\end{equation}

We have different choices here. \( \delta^2(x - y) \) stresses that its regularization form is not necessarily the same as that of \( \delta^2(x - y) \) at the intermediate stage. Because both \( \hat{D} \) and \( \hat{D}^\dagger \) satisfy the relation (16), we have

\begin{align}
\ln Z[R] &= \ln \int D\psi D\bar{\psi} e^{-\int dt d\xi} = \text{Tr} \ln \hat{D} = -\text{Tr} \int_0^\infty \frac{e^{-t\hat{D}}}{t} dt + \text{const} \\
&= -\int_0^\infty \frac{dt}{t} \text{Tr} \left[ \frac{1}{2}(1 + i\gamma_5)e^{it\gamma_5 \hat{D}} + \frac{1}{2}(1 - i\gamma_5)e^{-it\gamma_5 \hat{D}} \right] + \text{const} , \\
&= -\int_0^\infty \frac{dt}{t} \text{Tr} \left[ \frac{1}{2}(1 + i\gamma_5)e^{it\gamma_5 \hat{D}^\dagger} + \frac{1}{2}(1 - i\gamma_5)e^{-it\gamma_5 \hat{D}^\dagger} \right] + \text{const} .
\end{align}
Some heat-kernels are naturally introduced.

\[
G_{\pm}^5M(x, y; t) \equiv \langle x | \exp\{\pm it\gamma_5(\hat{D} + iM)\} | y \rangle , \\
G_{c\pm}^5M(x, y; t) \equiv \langle x | \exp\{\pm it\gamma_5(\hat{D}^\dagger + iM)\} | y \rangle , \\
G_{h\pm}^5M(x, y; t) \equiv \langle x | \exp\{\pm it\gamma_5(\hat{D}_h + iM)\} | y \rangle , \\
\hat{D}_h = i\gamma_\mu(\partial_\mu + iR_\mu) , \quad \hat{D}_h = (\hat{D}_h)^\dagger .
\]

(72)

We notice the existence of different choices for the present regularization. See Fig.8. Among them we consider two representative ones which correspond to the consistent and covariant anomalies.

(i) Consistent Anomaly
Let us first take the Feynman path and \( G_{\pm}^5M \). From the middle eq. of (70) and the boundary condition of Feynman path solution (34), we obtain

\[
\delta_\lambda \ln J_{NA} = \lim_{Mt \to +0} \text{Tr} \left( i\gamma_\mu(\partial_\mu + iR_\mu) \right) (iG_{+}^5M(x, y; t) - iG_{-}^5M(x, y; t)) + O(\lambda^2) .
\]

(74)

\footnote{We should not take \( \frac{1}{2}(\hat{D} + \hat{D}^\dagger) = i\gamma_\mu(\partial_\mu + \frac{1}{2}R_\mu) \) as the hermitian operator \( \hat{D}_h \) because, in this case, \( R_\mu \) transforms as \( R_\mu / 2 = U(\lambda)(R_\mu / 2)U^{-1}(\lambda) + i\partial_\mu U(\lambda) \cdot U^{-1}(\lambda) \), not as in (69).}

Fig.8 Vertices corresponding to \( G_{\pm}^5M, G_{c\pm}^5M, G_{h\pm}^5M \) which are defined by (73).
We consider the 1-st and 2-nd order (with respect to \( R_\mu \)) perturbations. The first term of (74) is evaluated from

\[
G_+^{5M} \big|_R = \int_0^{X^0} dZ^0 \int d^2Z G_0^{\rho}(X, Z) i \bar{R}(z) P_+ G_0^{\rho}(Z, Y) \\
= \int_0^{X^0} dZ^0 \int d^2z \int \frac{d^2k}{(2\pi)^2} \frac{d^2l}{(2\pi)^2} \\
\times (-i)\Omega_+(k) i \bar{R}(z) P_+ (-i)\Omega_+(l) e^{-i\bar{K}(X-Z)} e^{-i\bar{L}(Z-Y)} . \tag{75}
\]

See Fig.9(i). Using the SC-procedure given in Sec.4, the relevant part \((\epsilon\text{-tensor } \times \partial R)\) is given as

\[
\text{Tr } i^2 \lambda(x)G_+^{5M}(x, y; t)|_{\partial R} \\
\sim i \int_0^{X^0} dZ^0 \int d^2x \text{tr } \lambda(x) \partial_\mu R_\nu \\
\times \int \left[ -\frac{1}{i} \left\{ \frac{\partial}{\partial k^\mu} \omega(k) - i \frac{\partial E(k)}{\partial k^\mu} (X^0 - Z^0) \Omega_+(k) \right\} \gamma_\nu P_+ \Omega_+(k) \right] e^{-iE(k)X^0} \\
\sim - \int_0^{X^0} dZ^0 \int d^2x (\text{tr } \lambda(x) \partial_\mu R_\nu) \times \int \frac{d^2k}{(2\pi)^2}
\]

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we finally obtain the non-Abelian anomaly up to the first order of $R$

\[
\left[ \frac{M}{2E^4} - \frac{1}{4E^3} \right] \epsilon_{\mu \nu} k^\mu k^\nu + \left( \frac{M}{4E^2} + \frac{1}{4E} \right) \epsilon_{\mu \nu} - i(X^0 - Z^0) \frac{M}{2E^3} \epsilon_{\mu \nu} k^\lambda k^\mu \right] e^{-iE(k)X^0}
\]

\[
\sim - \int d^2x (\text{tr} \lambda(x) \partial_\mu R_\nu) \times
\]

\[
\frac{1}{2\pi} \int_0^\infty dk \cdot k \left[ M t \frac{k^2}{4E^4} + t \frac{k^2}{8E^3} - M t \frac{1}{4E^2} - t \frac{1}{4E} + M t^2 \frac{k^2}{8E^3} \right] \epsilon_{\mu \nu} e^{-E(k)t}
\]

\[
= - \frac{i}{2\pi} \int d^2x (\text{tr} \lambda(x) \partial_\mu R_\nu) \epsilon_{\mu \nu}
\]

\[
\times \left[ \frac{1}{4} M t E_4^3(Mt) + \frac{1}{8} M t E_3^3(Mt) - \frac{1}{4} M t E_2^3(Mt) - \frac{1}{4} M t E_1^3(Mt) + \frac{1}{8} (M t^2) E_3^3(Mt) \right]
\]

\[
\rightarrow - \frac{i}{2\pi} \left( \frac{1}{4} \times 0 + \frac{1}{8} \times 1 - \frac{1}{4} \times 0 - \frac{1}{4} \times 1 + \frac{1}{8} \times 0 \right) \int d^2x \epsilon_{\mu \nu} \text{tr} \lambda(x) \partial_\mu R_\nu
\]

\[
= + \frac{i}{16\pi} \int d^2x \epsilon_{\mu \nu} \text{tr} \lambda(x) (\partial_\mu R_\nu - \partial_\nu R_\mu)
\]

where we use $Y^0 = 0$, $X^0 = -it$, $a E_4^3(a) \to 0$, $a E_3^3(a) \to 1$, $a E_2^3(a) \to 0$, $a E_1^3(a) \to 1$, $a^2 E_3(a) \to 0(a \to +0)$ (see App.B). The same result is obtained in the $X^0$-coordinate. Taking into account the $G_5^{5M}$ contribution, we finally obtain the non-Abelian anomaly up to the first order of $R$.

\[
\delta \ln J_{NA} = \lim \text{Tr} \, i \lambda(x) \left( i G_5^{5M}(x, y; t) - i G_5^{5M}(x, y; t) \right)
\]

\[
= \int d^2x \left[ + \frac{i}{16\pi} \epsilon_{\mu \nu} \text{tr} \lambda(x) (\partial_\mu R_\nu - \partial_\nu R_\mu) \right].
\]

This turns out to be true even after the second order is taken into account.

The next order is evaluated from

\[
G_5^{5M}|_{RR} = \int_0^{X^0} dZ^0 \int_0^{Z^0} dW_0 \int d^2Z \int d^2W
\]

\[
G_0^0(X, Z) i \tilde{R}(z) P_+ G_0^0(Z, W) i \tilde{R}(w) P_+ G_0^0(W, Y)
\]

\[
= \int_0^{X^0} dZ^0 \int_0^{Z^0} dW_0 \int d^2Z \int d^2W \int \frac{dk^2}{(2\pi)^2} \frac{d^2l}{(2\pi)^2} \frac{d^2q}{(2\pi)^2}
\]

\[
\times (-i) \Omega_+(k) i \tilde{R}(z) P_+ (-i) \Omega_+(l) i \tilde{R}(w) P_+ (-i) \Omega_+(q)
\]

\[
\times \exp \left\{-i \tilde{K}(X - Z) - i \tilde{L}(Z - W) - i \tilde{Q}(W - Y) \right\}
\]

See Fig.9(ii). The contribution to the chiral anomaly ($\epsilon$-tensor $\times RR$) is evaluated as

\[
\text{Tr} \, i^2 \lambda(x) G_5^{5M}(x, y; t)|_{\epsilon RR}
\]
\[ \sim i^2 \frac{1}{2} (X^0)^2 i^2 (\frac{i}{2})^3 \int d^2 x (\text{tr} \lambda(x) R_\mu R_\nu) \times \int \frac{d^2 k}{(2\pi)^2} \text{tr} \left[ P_+ \frac{i k}{E(k)} \gamma_\nu (2\Omega_+(k))^2 \gamma_\mu \right] e^{-iE(k)X^0} \]
\[ = \frac{i}{16} (X^0)^2 \int d^2 x (\text{tr} \lambda(x) R_\mu R_\nu) \int \frac{d^2 k}{(2\pi)^2} \times 4\text{tr} \left[ P_+ \frac{i k}{E(k)} \gamma_\nu \frac{M}{E(k)} \Omega_+(k) \gamma_\mu \right] e^{-iE(k)X^0} \]
\[ \sim -\frac{i}{4} i^2 \int d^2 x (\text{tr} \lambda(x) R_\mu R_\nu) \int \frac{d^2 k}{(2\pi)^2} \times 4\text{tr} \left[ \frac{i k}{E(k)} \gamma_\nu \frac{M}{E(k)} (-\frac{1}{2} \frac{i k}{2E(k)} \gamma_5) \gamma_\mu \right] e^{-E(k)t} \]
\[ = 0. \quad (79) \]

where \( \text{tr} \ k_{\mu \nu} k_{\gamma \delta} \gamma_{\mu} = 0 \) is used. Similarly we have
\[ \text{Tr} \ i^2 \lambda(x) G^{5M}_-(x,y,t) |_{RR} \sim 0. \]
Thus we have confirmed the absence of \( \epsilon_{\mu \nu} \text{tr} R_\mu R_\nu \), which characterize the 2 dim consistent anomaly (non-covariant). The result (77) is true even after the second order correction and is the half of the consistent anomaly. If we take the symmetric path, instead of the Feynman, in the above, we indeed have the consistent anomaly.

\[ \delta \lambda \ln J_{NA} = \frac{1}{2} \lim M \rightarrow +0 \text{Tr} \ i\lambda(x) \left( iG^{5M}_+(x,y,t) - iG^{5M}_-(x,y,t) \right) \]
\[ = \int d^2 x \left[ + \frac{i}{8\pi} \epsilon_{\mu \nu} \text{tr} \lambda(x) (\partial_\mu R_\nu - \partial_\nu R_\mu) \right]. \quad (80) \]

(As for the past literature, see, for example, (13.68) or (13.128) of [32], where we should take \( a = 1 \).)

(ii) Covariant Anomaly
We take the symmetric path and \( G^{5M}_{\pm} \). From (70) and the boundary condition of Symmetric path solution (74), we obtain
\[ \delta \lambda \ln J_{NA} = \frac{1}{2} \lim_{M(X^0-Y^0) \rightarrow +0} \text{Tr} \ i\lambda(x) iG^{5M}_{h+}(x,y,t) \]
\[ + \frac{1}{2} \lim_{M(X^0-Y^0) \rightarrow -0} \text{Tr} \ i\lambda(x) (-i)G^{5M}_{h-}(x,y,t) + O(\lambda^2) \quad . \quad (81) \]

The first term is evaluated as
\[ G^{5M}_{h+}|_R = \int_0^{X^0} dZ^0 \int d^2 Z (G^0_0(X,Z) - G^0_0(X,Z)) i \mathcal{R}(z) (G^0_0(Z,Y) - G^0_0(Z,Y)) \]
Fig. 10 Non-Abelian Chiral Gauge Theory, $G_{h^+}^{5M}$ with Symmetric Path, $O(R)$ and $O(RR)$.

\[
\begin{align*}
&= \int_{Z^0}^{X^0} dZ^0 \int d^2 z \int \frac{d^2 k}{(2\pi)^2} \frac{d^2 l}{(2\pi)^2} \\
&\times (-i)(\Omega_+(k)e^{-iE(k)(X^0-Z^0)} - \Omega_-(k)e^{iE(k)(X^0-Z^0)})i\tilde{R}(z) \\
&\times (-i)(\Omega_+(l)e^{-iE(l)(Z^0-Y^0)} - \Omega_-(l)e^{iE(l)(Z^0-Y^0)})e^{-ik(x-z)-il(z-y)} .
\end{align*}
\]

(82)

See Fig. 10(i).

After the SC-procedure of Sec. 4, we are led to the following one, as the relevant part for the anomaly,

\[
\frac{1}{2}\text{Tr } i^2 \lambda(x)G_{h^+}^{5M}(x, y; t)\frac{i}{\partial R} \sim \frac{i^2}{2} \int d^2 x (\text{tr } \lambda(x)\partial_\mu R_\nu)\epsilon_{\mu\nu} \\
\times -\frac{i}{4\pi} MX^0 \{S_3^3 (MX^0) + 2S_3^1 (MX^0) + S_3^3 (MX^0)\} \\
\rightarrow + \frac{i}{4\pi} \int d^2 x \epsilon_{\mu\nu} \text{tr } \lambda(x)\partial_\mu R_\nu \quad (MX^0 \rightarrow +0) ,
\]

(83)

Taking into account $G_{h^-}^{5M}$, we obtain

\[
\delta_\lambda \ln J_{NA} = + \frac{i}{4\pi} \int d^2 x \epsilon_{\mu\nu} \text{tr } \lambda(x)\left(\partial_\mu R_\nu - \partial_\nu R_\mu + O(R^2)\right) .
\]

(84)
The coefficient is two times of the previous case, which is the well-known relation between 2D consistent and covariant anomalies.

Next order contribution is evaluated from

\[ G_{h+}^{5M} |_{RR} = \int_0^{X^0} dZ^0 \int_0^{Z^0} dW^0 \int d^2 Z \int d^2 W (G_0^0(X,Z) - G_0^n(X,Z)) i \Omega (z) \]

\[ (G_0^0(Z,W) - G_0^n(Z,W)) i W (G_0^0(W,Y) - G_0^n(W,Y)) \]

\[ = \int_0^{X^0} dZ^0 \int_0^{Z^0} dW^0 \int d^2 z \int d^2 w \int \frac{d^2 k}{(2\pi)^2} \frac{d^2 l}{(2\pi)^2} \frac{d^2 q}{(2\pi)^2} \]

\[ \times (-i) (\Omega_+(k) e^{-i E(k)(X^0 - Z^0)} - \Omega_-(k) e^{i E(k)(X^0 - Z^0)}) i \Omega (z) \]

\[ \times (-i) (\Omega_+(l) e^{-i E(l)(Z^0 - W^0)} - \Omega_-(l) e^{i E(l)(Z^0 - W^0)}) i \Omega (w) \]

\[ \times (-i) (\Omega_+(q) e^{-i E(q)(W^0 - Y^0)} - \Omega_-(q) e^{i E(q)(W^0 - Y^0)}) e^{-i k(x-z) - i l(z-w) - i q(w-y)} \]

(85)

See Fig.10(ii). After the use of SC-procedure, we obtain

\[ \text{Tr} \lambda(x) G_{h+}^{5M} |_{eRR} \sim (-i) \int_0^{X^0} dZ^0 \int_0^{Z^0} dW^0 \int d^2 x \int \frac{d^2 k}{(2\pi)^2} \text{tr} \left[ \lambda(x) R_{\mu}(x) R_{\nu}(x) \right] \]

\[ \times \{ \omega(k)(-2i) \sin E(k)(X^0 - Z^0) + \gamma_5 \cos E(k)(X^0 - Z^0) \} \gamma_{\mu} \]

\[ \times \{ \omega(k)(-2i) \sin E(k)(Z^0 - W^0) + \gamma_5 \cos E(k)(Z^0 - W^0) \} \gamma_{\nu} \]

\[ \times \{ \omega(k)(-2i) \sin E(k)(W^0 - Y^0) + \gamma_5 \cos E(k)(W^0 - Y^0) \} \]

\[ \sim (-i) \int d^2 x \text{tr} \left[ \lambda(x) R_{\mu}(x) R_{\nu}(x) \right] \]

\[ \times \int \frac{d^2 k}{(2\pi)^2} \left[ (\text{tr} \gamma_5 \gamma_{\mu} \gamma_5 \gamma_{\nu} \gamma_5) \frac{1}{4} \left\{ \frac{1}{2} (X^0)^2 \cos EX^0 + \frac{3X^0}{2E} \sin EX^0 \right\} \right. \]

\[ \left. + (\text{tr} \gamma_5 \gamma_{\nu} \gamma_{\mu})(-2i)^2 \right\} \]

\[ \times \left\{ \frac{M^2}{4E^2} (- \cos E(X^0 - Z^0) \sin EW^0 - \sin E(X^0 - Z^0) \cos EW^0) \sin E(Z^0 - W^0) \right. \]

\[ + \frac{M^2 - k^2}{4E^2} \sin E(X^0 - Z^0) \cos E(Z^0 - W^0) \sin EW^0 \right\} \]

\[ \quad \quad \quad = (-i) \int d^2 x \text{tr} \left[ \lambda(x) R_{\mu}(x) R_{\nu}(x) \right] \]

\[ \times 2i \epsilon_{\mu\nu} \left\{ \frac{1}{4} \left\{ \frac{1}{2} \frac{(MX^0)^2}{2\pi} C_0^1(MX^0) + \frac{3}{2} \frac{MX^0}{2\pi} S_1(MX^0) \right\} \right. \]

\[ \left. + (-2i)^2 \frac{1}{4} \left\{ \frac{MX^0}{8\pi} S_3^1(MX^0) + \frac{(MX^0)^2}{16\pi} (C_2^1(MX^0) - C_2^3(MX^0)) \right\} \quad \right\} \]
\[
\int d^2 x \text{tr} \left[ \lambda(x) R_\mu(x) R_\nu(x) \right] \times (-i) \times 2 \epsilon_{\mu\nu} \times \left( + \frac{1}{4\pi} \right) .
\] (86)

Adding \( G_5^{M-} \) contribution, the above result gives indeed \( O(R^2) \) part of the covariant anomaly.

\[
\delta \lambda \ln J_{NA} = + \frac{i}{4\pi} \int d^2 x \epsilon_{\mu\nu} \text{tr} \left( \partial_\mu R_\nu - \partial_\nu R_\mu + i [ R_\mu, R_\nu ] \right) ,
\] (87)

which agrees with (13.72) of [12].

We have checked if we take the Feynman path, even with \( G_5^{M-} \), the result leads to the consistent anomaly.

### 7 Massive Fermion

So far we have examined the regularization only for the massless fermion. Let us consider here the case where the (4 dim) fermion has a small mass \( m \). QED is taken as the example.

\[
\mathcal{L} = \bar{\psi} \hat{D}_m \psi , \quad \hat{D}_m = \hat{D} - m , \quad \hat{D}_m^\dagger = \hat{D}_m .
\] (88)

where \( \hat{D} = i\gamma_\mu (\partial_\mu + ieA_\mu) \) has been introduced in (14). In the same way as eq.(15), the effective action is given by

\[
\ln Z_m[A] = - \text{Tr} \int_0^\infty \frac{e^{-t\hat{D}_m}}{t} dt + \text{const}
\]

\[
= - \int_0^\infty \frac{dt}{t} \text{Tr} \left[ \frac{1}{2} (1 + i\gamma_5) \text{Tr} e^{it\gamma_5(\hat{D} - i\gamma_5m)} + \frac{1}{2} (1 - i\gamma_5) \text{Tr} e^{-it\gamma_5(\hat{D} + i\gamma_5m)} \right] + \text{const} .
\] (89)

The corresponding heat-kernels satisfy the following 1+4 dim massive Dirac equation.

\[
\ln Z_m = - \lim_{M \to 0} \int_0^\infty \frac{dt}{t} \frac{1}{2} \left( 1 - i \frac{\partial}{t \partial M} \right) \text{Tr} G_{5}^{M}(x, y ; t)
\]

\[
- \lim_{M' \to 0} \int_0^\infty \frac{dt}{t} \frac{1}{2} \left( 1 - i \frac{\partial}{t \partial M'} \right) \text{Tr} G_{5}^{M'}(x, y ; t) ,
\]

where
\[ G_{m+}^{M}\equiv x| \exp\{+it\gamma_5(\hat{D} - i\gamma_5m + iM)\}|y > , \]
\[ (i\partial - M)G_{m+}^{M} = (ie\mathcal{A} - \gamma_5m)G_{m+}^{M} , \quad (X^a) = (-it, x^\mu) \]
\[ G_{m-}^{M'}\equiv x| \exp\{-it\gamma_5(\hat{D} + i\gamma_5m + iM')\}|y > , \]
\[ (i\partial - M')G_{m-}^{M'} = (ie\mathcal{A} + \gamma_5m)G_{m-}^{M'} , \quad (X^a) = (+it, x^\mu) . \]  

In the lattice approach, the superiority of the domain wall fermion (to the ordinary Wilson fermion) is no need to fine-tune the hopping parameter which suffers from the renormalization due to the gauge interaction. In addition to the wave-function and coupling renormalizations, the mass renormalization is one crucial test of the superiority. We can calculate such an effect using the above formulae. The full consideration can be done only after the fermion and the gauge fields are treated on the equal footing. It has recently been performed in [33].

8 Conclusion and Discussion

Inspired by the recent progress of the chiral fermion on lattice, we have presented a new regularization, for the continuum Euclidean field theories, which is based on the domain wall configuration in one dimension higher (Minkowski) space. We have verified the analogous aspect to the lattice case, such as the domain wall structure, the condition on the 1+4 dim Dirac mass (the regularization parameter \(M\)), the overlap Dirac operator, etc. Applying the proposed regularization to some models (4 dim QED, 2 dim chiral gauge theory), the known anomalies are correctly reproduced. It shows the present regularization correctly works.

Comparing Fig.2 (or 3) with Fig.4, we can imagine that the choice of (anti-)Feynman path solution perturbatively defines the chiral version of the original theory, for example, the chiral gauge theory. As far as anomaly calculation is concerned, it holds true. In order to show the statement definitely, we must clarify the following things. The ’’ordinary’’ chiral symmetry appears only in the limit : \[ \frac{|k|}{M} \to +0, \] as shown in [33]. But this

\[ 18 \text{ If we write as } ie\mathcal{A} \equiv \gamma_5m = -e\mathcal{A} , \quad (A^\mu) = (\pm \frac{e}{M}, A_\mu), \quad \text{the theory looks like the 1+4 dim QED in a } \text{temporal gauge } (A^0 = \pm \frac{e}{M}). \] In the lattice approach, however, such an extended standpoint does not seem to successfully work at present [11].
limit can not be taken because it "freezes" the dynamics and anomalies do not appear. It seems we must introduce some new "softened" version of the chiral symmetry which keeps the dynamics. One standpoint taken in this paper is to replace $|k^\mu|/M \ll 1$ by $|k^\mu|/M \leq 1$ in (33). It breaks the "ordinary" chiral symmetry. It could, however, be possible that this replacement can avoid the breaking by changing (generalizing) the "ordinary" chiral symmetry. In this case, the new chiral Lagrangian has infinitely many higher-derivative terms. We should explain this new "deformed" Lagrangian from some generalized chiral symmetry. This situation looks similar to what Lüscher [19] did for the chiral lattice.

The present regularization is characterized by the introduction of the fifth coordinate $t$, and the regularization mass $M$. The limit (29) is quite impressive as a new regularization. Because we are, in this article, mainly concerned with anomalies, the analysis need not to touch upon the $t$-integral. (We examine only the variation of the partition function under the chiral transformation, or the measure change. We do not examine the partition function itself.) For the evaluation of the partition function, we know $t$-integral is usually regularized as

$$\epsilon \leq t \leq T \quad \left( \frac{1}{\epsilon} \geq \frac{1}{t} \geq \frac{1}{T} \right), \quad \frac{T}{\epsilon} \gg 1,$$

where another two regularization parameters $\epsilon$ (ultraviolet) and $T$ (infrared) appear. The consistency with (29) requires

$$M \ll \frac{1}{\epsilon}.$$  

The full treatment of the partition function has been consistently done in a recent work [33], where the renormalization is formulated with the care for the quantum effect of both fermions and gauge bosons.

We stress characteristic points of the present domain wall regularization.

- The higher dimensional Dirac field mass $M$ is the only regularization parameter used in this paper. It should satisfy the condition (29) in order to do the following role: 1) controlling the chirality, 2) forming the domain wall configuration (dimensional reduction), 3) Ultraviolet and infrared regularization for the momentum integral.
• There appears different regularizations depending on the solutions of the higher dim Dirac equation. The Feynman path is (practically) appropriate for the analysis of chiral properties, although overall factors of anomalies deviate from the right ones. The calculation is most simple. Whereas the symmetric path gives the proper solution of the Dirac equation and the regularization with the path gives the correct anomaly including the coefficient. Especially the covariant anomaly is obtained only by the choice of the symmetric path and the hermitian vertex. Other choices essentially lead to the consistent anomaly.

• The characteristic functions appearing in the present regularization, such as the incomplete gamma function, indicate the advance over other ordinary regularizations, say, the dimensional one where only the gamma function appears.

The chiral problem itself does not depend on the interaction. It looks a kinematical problem in the quantization of fields. How do we treat the different propagations of free solutions depending on the boundary conditions (with respect to the Wick-rotated time) is crucial to the problem. In the standpoint of the operator formalism (the Fock-space formalism) it corresponds to how to treat the "delicate" structure (due to the ambiguity of the fermion mass sign) of the vacuum of the free fermion theory. The present paper insists the following prescription: First we go to 1+4 dim Minkowski space by the Wick-rotation of the inverse temperature $t$, and take the "directed" solution as in Sect.3. The anomaly phenomena concretely reveal the chiral problem. The proposed prescription passes the anomaly test.

The recent progress in the lattice formalism, using the Neuberger’s overlap Dirac operator\[17\] and Lüscher’s chiral symmetry on lattice\[19\], has revealed the importance of the condition on $M$, where the anomaly coefficient varies depending on the different conditions on $M$. $M$ is bounded by the Wilson parameter in some ways\[34, 35, 36, 37\]. It shows whether the zero mode (surface state) is correctly picked up in the regularization crucially depends on the choice of $M$. Note that there is not the Wilson term in the present formalism. Even such case we have the similar condition (29) which produces the correct coefficient. This strongly
implies the present regularization effectively does the same thing as the
Wilson term does in the lattice formalism.

Appendix A. Notations

In the analysis of the chiral property, much care should be paid to ±1 and
±i. The physical interpretation of the final result heavily relies on such
delicacy. In this circumstance, we decide to present the present notation in
this appendix. We adopt the convention of Ref. [7] for the gamma matrices
(γµ) and the metric in 2 and 4 dim Euclidean space and for those (Γa) in 5
dim Minkowski space.

(ia) 2 dim Euclidean: µ = 1, 2.

γ1 = \begin{pmatrix} 0, & 1 \\ 1, & 0 \end{pmatrix} = σ_1 , \gamma_2 = \begin{pmatrix} 0, & i \\ -i, & 0 \end{pmatrix} = -σ_2 , \gamma_5 = iγ_1γ_2 = \begin{pmatrix} 1, & 0 \\ 0, & -1 \end{pmatrix} = σ_3 ,

\gamma_5^† = γ_5 , \quad \text{Tr} \gamma_µγ_νγ_5 = -2iε_{µν} , \quad ε_{12} = 1 ,

\{γ_µ, γ_ν\} = 2δ_{µν} , \quad γ_µ^† = γ_µ , \quad (δ_{µν}) = \text{diag}(1, 1)

\bar{ψ} = ψ^†γ_2 . (93)

where σ_i(i = 1, 2, 3) are Pauli matrices.

(ib) 1+2 dim Minkowski: a = 0, 1, 2; \quad µ = 1, 2.

\{Γ_a, Γ_b\} = 2η_{ab} , \quad (η_{ab}) = \text{diag}(1, -1, -1) , \quad Γ_0 = Γ^0 = γ_5 , \quad Γ_µ = -Γ^µ = -iγ_µ ,

Γ_0^† = Γ_0 , \quad Γ_µ^† = -Γ_µ , \quad \bar{ψ} = ψ^†Γ_0 = ψ^†γ_5 . (94)

(iia) 4 dim Euclidean: µ = 1, 2, 3, 4.

γ_µ = \begin{pmatrix} 0, & σ_µ \\ σ_µ^†, & 0 \end{pmatrix} , \quad \{γ_µ, γ_ν\} = 2δ_{µν} , \quad γ_µ^† = γ_µ , \quad (δ_{µν}) = \text{diag}(1, 1, 1, 1)

\gamma_5 = γ_1γ_2γ_3γ_4 = \begin{pmatrix} 1_{2×2}, & 0 \\ 0, & -1_{2×2} \end{pmatrix} , \quad γ_5^† = γ_5 , \quad (γ_5)^2 = 1 ,

\text{Tr} γ_µγ_νγ_λγ_σγ_5 = 4ε_{µνλσ} , \quad ε_{1234} = 1 ,

\text{chiral projection operator:} \quad P_± \equiv \frac{1}{2}(1_{4×4} ± γ_5)

\bar{ψ} = ψ^†γ_4 . (95)
where $\sigma_i (i = 1, 2, 3)$ are Pauli matrices defined in (ia), and $\sigma_4 = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$.

(ii) 1+4 dim Minkowski: $a = 0, 1, 2, 3, 4; \mu = 1, 2, 3, 4$.

$$\{\Gamma_a, \Gamma_b\} = 2\eta_{ab}, \quad (\eta_{ab}) = \text{diag}(1, -1, -1, -1),$$
$$\Gamma_0 = \Gamma^0 = \gamma_5, \quad \Gamma_\mu = -i\gamma_\mu; \quad \bar{\psi} = \psi^\dagger \Gamma_0 = \psi^\dagger \gamma_5$$ (96)

Appendix B. Integral Formulae

In Sec.5 of the text, we explain the momentum integral. We list here some useful integral formulae. We define $E_n^r(a), S_n^r(a), C_n^r(a)$ as

$$E_n^r(a) \equiv \int_0^\infty dx \frac{x^r}{(\sqrt{x^2 + 1})^n} e^{-ax\sqrt{x^2 + 1}}$$,
$$S_n^r(a) \equiv \int_0^\infty dx \frac{x^r}{(\sqrt{x^2 + 1})^n} \sin(ax\sqrt{x^2 + 1})$$,
$$C_n^r(a) \equiv \int_0^\infty dx \frac{x^r}{(\sqrt{x^2 + 1})^n} \cos(ax\sqrt{x^2 + 1})$$,

$$E_n^r(-ia) = C_n^r(a) + i S_n^r(a).$$ (97)

Some exact expressions are

$$E_3^3(a) = \int_0^\infty dx \frac{x^3}{(\sqrt{x^2 + 1})^3} e^{-ax\sqrt{x^2 + 1}} = \frac{1}{a} \Gamma(1, a) - a \Gamma(-1, a)$$
$$= \frac{1}{a} (1 - 2a + O(a^2, a^2 \ln a))$$,

$$E_3^3(a) + 2E_3^1(a) = \int_0^\infty dx \frac{x(x^2 + 2)}{(\sqrt{x^2 + 1})^3} e^{-ax\sqrt{x^2 + 1}} = \frac{1}{a} \Gamma(1, a) + a \Gamma(-1, a)$$
$$= \frac{1}{a} (1 + 0 \times a + O(a^2)).$$
\[ E_2^1(a) = \int_0^\infty dx \frac{x}{x^2 + 1} e^{-a\sqrt{x^2 + 1}} = -\text{Ei}(-a) = \Gamma(0, a) = -\ln a - \gamma + O(a), \]

\[ S^5_5(a) + 2S^3_5(a) = \int_0^\infty dx \frac{x^3(x^2 + 2)}{(x^2 + 1)^5} \sin(a\sqrt{x^2 + 1}) \]

\[ = \frac{1}{720a}\{-360a^2 + 220a^4 - 120a^4\gamma + 720\cos a + 3a^6\, _3F_3(1, 1; 2, 3, 7/2; -a^2/4) - 120a^4\ln a\} \]

\[ = \frac{1}{a}(1 + O(a^2)), \]

\[ C^3_4(a) + 2C^1_4(a) = \int_0^\infty dx \frac{x(x^2 + 2)}{(x^2 + 1)^2} \cos(a\sqrt{x^2 + 1}) \]

\[ = \frac{1}{2} - \frac{3}{4}a^2 + \frac{a^2}{2}\gamma - \text{Ci}(a) - \frac{a^4}{48}\, _2F_3(1, 1; 2, 5/2, 3; -a^2/4) + \frac{a^2}{2}\ln a \]

\[ = \frac{1}{2} - \gamma - \ln a + O(a^2), \quad (98) \]

where \( \gamma = 0.57721\cdots \) is the Euler constant, \( \Gamma(*, *), \text{Ei}(\cdot), \text{Ci}(\cdot) \) and \( _pF_q(*; *; *) \) are the incomplete gamma function, the exponential integral function, the cosine integral function and the (Pochhammer’s) generalized hypergeometric function respectively. They are defined as

\[ \Gamma(z, p) = \int_p^\infty e^{-t}t^{z-1}dt, \quad \Gamma(z) = \int_0^\infty e^{-t}t^{z-1}dt = \Gamma(z, p = 0), \quad \Re z > 0; \]

\[ \Gamma(1 - z, p)\Gamma(z) = p^{-z}\int_0^\infty \frac{e^{-(p+t)}t^{z-1}}{p+t}dt, \quad \Re z > 0; \]

\[ \text{Ei}(-x) = -\int_x^\infty \frac{e^{-t}}{t}dt = -\Gamma(0, x), \quad x > 0; \]

\[ \text{Ci}(x) = -\int_x^\infty \frac{\cos t}{t}dt = \gamma + \ln x + \int_0^x \frac{\cos t - 1}{t}dt, \quad x > 0; \]

\[ _pF_q(\alpha_1, \alpha_2, \cdots, \alpha_p; \beta_1, \beta_2, \cdots, \beta_q; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_p)_n z^n}{(\beta_1)_n \cdots (\beta_q)_n n!}, \quad (\alpha)_n = \frac{\Gamma(\alpha + n)}{\Gamma(\alpha)}, (\alpha)_0 = 1, |z| < 1; (99) \]

where \( \Gamma(z) \) is the (Euler) gamma function. Some other formulae useful for
the weak expansion are

\[ \text{Re } z > 0, \quad \Gamma(z, p) = \Gamma(z) - \gamma(z, p), \quad \gamma(z, p) = \int_0^p e^{-t^2} dt = \sum_{n=0}^{\infty} \frac{(-1)^n p^{n+1}}{n!(z + n)}. \]

\[ \text{Ei}(-x) = \ln x + \gamma + O(x), \quad 1 \gg x > 0. \]

The following asymptotic expressions for \( a \to +0 \) are used in the text.

\[
\begin{align*}
 a E_1^1(a) & \to 1, \quad a E_1^2(a) \to 0, \quad a^2 E_2^3(a) \to 1, \quad a E_3^1(a) \to 1, \quad a E_3^1(a) \to 0, \\
 a^2 E_3^3(a) & \to 0, \quad a E_4^1(a) \to 0, \quad a E_3^3(a) \to 0, \quad a^2 E_3^3(a) \to 0, \quad a^2 E_4^4(a) \to 1, \\
 a(S_3^3(a) + 2S_3^1(a)) & \to 1, \quad a(C_3^3(a) + 2C_3^1(a)) \to 0, \\
 a(C_4^3(a) + 2C_4^1(a)) & \to 0, \quad a(S_3^3(a) + 2S_3^3(a)) \to 1, \\
 a(S_3^3(a) + 2S_3^3(a)) & \to 1, \quad a(E_4^1(a)) \to 0, \quad a(E_3^3(a)) \to 0, \quad a(E_4^3(a)) \to 1.
\end{align*}
\]

Appendix C. Projective Properties of \( \Omega_\pm(k) \)

Here we list some useful relations of \( \Omega_\pm(k) \), which correspond to free solutions of 1+4 dim Dirac equation.

\[ E(k) = \sqrt{k^2 + M^2}, \quad (K^a) = (E(k), K^\mu = -k^\mu), \quad (\bar{K}^a) = (E(k), -K^\mu = k^\mu). \]

\( k^\mu \) is the momentum in the 4 dim Euclidean space. \( \hat{K} \) and \( \bar{K} \) are on-shell momenta, of 1+4 dim Dirac equation, \( (\hat{K}^2 = \bar{K}^2 = M^2) \), which correspond to the positive and negative energy states respectively.

\[ \Omega_+(k) \equiv \frac{M + \hat{K}}{2E(k)} = \omega(k) + \frac{\gamma_5}{2}, \quad \omega(k) \equiv \frac{M + i \hat{k}}{2E(k)}, \]

\[ \Omega_-(k) \equiv \frac{M - \bar{K}}{2E(k)} = \omega(k) - \frac{\gamma_5}{2} = \Omega_+(k) - \gamma_5, \]

\[ \left( \frac{\gamma_5}{2} \right)^2 = \omega(k)\omega^\dagger(k) = \omega^\dagger(k)\omega(k) = \frac{1}{4} I_{4 \times 4}, \]

\[ \omega(k)^\dagger = \omega(-k) = \frac{M - i k}{2E(k)}, \]

\[ \Omega_+(k)^\dagger = \Omega_+(k) = \frac{M - i k + E\gamma_5}{2E(k)}, \quad \Omega_-(k)^\dagger = \Omega_-(k) = \frac{M - i k - E\gamma_5}{2E(k)}. \]
In the following we will omit the momentum \((k)\) dependence of \(\Omega_\pm(k), \omega(k)\) unless we would like to stress the momentum coordinate.

\[
\Omega_+ - (\Omega_+)^\dagger = \Omega_- - (\Omega_-)^\dagger = \omega - \omega^\dagger = \frac{i}{E} k , \\
\Omega_+ + (\Omega_+)^\dagger = \frac{M}{E} + \gamma_5 , \Omega_- + (\Omega_-)^\dagger = \frac{M}{E} - \gamma_5 , \omega + \omega^\dagger = \frac{M}{E} , \\
\Omega_+ - (\Omega_-)^\dagger = iE/k , \Omega_- + (\Omega_-)^\dagger = \gamma_5 , \omega^\dagger = \gamma_5 , \Omega_- + (\Omega_-)^\dagger = \gamma_5 , \omega^\dagger = \gamma_5 , \\
(1A) \quad \Omega_+ + (\Omega_-)^\dagger = (\Omega_+)^\dagger \Omega_- = 0 , \\
(2A) \quad \Omega_+ - (\Omega_-)^\dagger = \frac{i}{E} k + \gamma_5 , \quad (2A') \quad (\frac{i}{E} k + \gamma_5)^2 = (\frac{M}{E})^2 , \\
(2B) \quad \Omega_- - (\Omega_+)^\dagger = \frac{i}{E} k - \gamma_5 , \quad (2B') \quad (\frac{i}{E} k - \gamma_5)^2 = (\frac{M}{E})^2 , \\
(3) \quad \Omega_+ + \Omega_- = 2\omega , \quad \Omega_+ - \Omega_- = \gamma_5 , \\
(4A) \quad \Omega_+ \Omega_+ = \frac{M}{E} \Omega_+ , \quad (4B) \quad \Omega_- \Omega_- = \frac{M}{E} \Omega_- , \\
\Omega_+ \gamma_5 \Omega_+ = \Omega_+ , \quad \Omega_- \gamma_5 \Omega_- = -\Omega_- , \\
\Omega_+ \Omega_- = \frac{M}{E} \Omega_+ (k) - \gamma_5 \Omega_+ (-k) = -\frac{k^2}{2E^2} + \frac{i}{2E}(\frac{M}{E} + \gamma_5) k , \\
\Omega_- \Omega_+ = \frac{M}{E} \Omega_- (k) + \gamma_5 \Omega_- (-k) = -\frac{k^2}{2E^2} + \frac{i}{2E}(\frac{M}{E} - \gamma_5) k , \\
[\gamma_5 , \omega] = \frac{i}{E} \gamma_5 k , \quad [\gamma_\mu , \omega] = \frac{k^\nu}{E} \sigma_{\mu\nu} , \sigma_{\mu\nu} \equiv \frac{i}{2} [\gamma_\mu , \gamma_\nu] , \quad (\sigma_{\mu\nu})^\dagger = \sigma_{\nu\mu} , \\
\omega \gamma_5 = \gamma_5 \omega^\dagger , \Omega_+ \gamma_5 = \gamma_5 (\Omega_+)^\dagger , \Omega_- \gamma_5 = \gamma_5 (\Omega_-)^\dagger , \\
[\gamma_\mu , \Omega_+] = \frac{k^\nu}{E} \sigma_{\mu\nu} + \gamma_\mu \gamma_5 , \quad [\gamma_\mu , \Omega_-] = \frac{k^\nu}{E} \sigma_{\mu\nu} - \gamma_\mu \gamma_5 , \\
\{\gamma_\mu , \Omega_\pm\} = \{\gamma_\mu , \omega\} = \frac{M \gamma_\mu + ik^\mu}{E} (103) \]

The numbered equations above show projective property between \(\Omega_+\) and \(\Omega_+^\dagger\) and between \(\Omega_-\) and \(\Omega_-^\dagger\).

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