Increasing stability of the continuation for general elliptic equations of second order.

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To memory of a good colleague and dear friend Alfredo Lorenzi

1 Introduction

The Cauchy Problem (or equivalently the continuation of solutions) for partial differential equations has a long and rich history, starting with the Holmgren-John theorem on uniqueness for equations with analytic coefficients. It is of great importance in the theories of boundary control and of inverse problems. In 1938 T. Carleman introduced a special exponentially weighted energy (Carleman type) estimates to handle non analytic coefficients. These estimates imply in addition some conditional Hölder type stability estimates for solutions of this problem. In 1960 [11] F. John showed that for the continuation for the Helmholtz equation from inside of the unit disk onto any larger disk the stability estimate which is uniform with respect to the wave numbers is still of logarithmic type. Logarithmic stability is quite damaging for numerical solution of many inverse problems. In recent papers [1], [2], [4], [9], [6], [7] it was shown that in a certain sense stability is always improving for larger \( k \) under (pseudo) convexity conditions on the geometry of the domain and of the coefficients of the elliptic equation.

In this paper we attempt to eliminate any convexity type condition on the elliptic operator or the domain. Due to John’s counterexample, one can not expect increasing stability for all solutions. We show that (near Lipschitz) stability holds on a subspace of ("low frequency") solutions which is growing with the wave number \( k \) under some mild boundedness constraints on complementary "high frequency" part.

We will consider the Cauchy problem

\[
(A + ck + k^2)u = f \text{ in } \Omega,
\]

with the Cauchy data

\[
u = u_0, \partial_\nu u = u_1 \text{ on } \Gamma_0 \subset \partial \Omega,
\]

where

\[
Au = \sum_{j,m=1}^n a_{jm} \partial_j \partial_m u + \sum_{j=1}^n a_j \partial_j u + au
\]

is the general partial differential operator of second order satisfying the ellipticity condition

\[
\varepsilon_0 |\xi|^2 \leq \sum_{j,l=1}^n a_{jl}(x)\xi_j \xi_l
\]

for some positive number \( \varepsilon_0 \) and all \( x \in \Omega \) and \( \xi \in \mathbb{R}^n \). We assume that \( a_{jm}, \partial_\nu a_{jm}, a_j, a, c \in L^\infty(\Omega) \).

We consider bounded open \( \Omega \subset \mathbb{R}^{n-1} \times (0,1) \), \( \Gamma_0 = \partial \Omega \cap \{x_n = 0\} \), \( \Gamma_1 = \partial \Omega \cap \{x_n = 1\} \), and \( \Gamma = \partial \Omega \cap (\mathbb{R}^{n-1} \times [0,1]) \). Let \( V \) be a neighborhood of \( \Gamma \) and \( \omega = \Omega \cap V \).

We use the classical Sobolev spaces \( H^{(p)}(\Omega) \) with the standard norm \( \| \cdot \|_{(p)}(\Omega) \).

In what follows \( C \) denote generic constants depending only on \( \Omega, \Gamma_0, \omega, \Gamma_1, A, c, \) and \( \varepsilon \).

Under additional a priori constraints on \( u \) near \( \Gamma \) and on a "high frequency" part of \( u \) we can claim
Theorem 1.1. There are a monotone family of closed subspaces $H^{(1)}(\Omega; k)$ of $H^{(1)}(\Omega)$ with $\bigcup_k H^{(1)}(\Omega; k) = H^{(1)}(\Omega)$, linear continuous operators $P_k$ from $H^{(1)}(\Omega)$ onto $H^{(1)}(\Omega; k)$ with $P_k u = u$ for $u \in H^{(1)}(\Omega; k)$, a semi norm $||| \cdot |||(1;k)(\Omega)$ on $H^{(1)}(\Omega)$ which is zero on $H^{(1)}(\Omega; k)$ and decreasing with respect to $k$, and a constant $C$ such that

$$
||u||_0(\Omega \setminus \bar{V}) \leq C(||u||_0(\Omega) + k^{-1}||f||_0(\Omega) + ||u||_0(\Omega)) + ||u||_0(\Omega) + ||u||_0(\Omega)) + \theta > 0.
$$

where $u = P_k u$, and

$$
||u||_0(\Omega \setminus \bar{V}) \leq C(||u||_0(\Omega) + k^{-1}||f||_0(\Omega) + ||u||_0(\Omega) + \theta > 0.
$$

for all $u \in H^{(2)}(\Omega)$ solving (1.1), (1.2).

In the next result we will partially replace the Cauchy data on $\Gamma_0$ by a function $u$ in $\omega = \Omega \cap V$.

Theorem 1.2. Let $\theta > 0$.

There are a monotone family of closed subspaces $H^{(2)}(\Omega; k)$ of $H^{(1)}(\Omega)$ with $\bigcup_k H^{(2)}(\Omega; k) = H^{(2)}(\Omega)$, linear continuous operators $P_k$ from $H^{(2)}(\Omega)$ onto $H^{(2)}(\Omega; k)$ with $P_k u = u$ for $u \in H^{(2)}(\Omega; k)$, a semi norm $||| \cdot |||(2;k)(\Omega)$ on $H^{(2)}(\Omega)$ which is zero on $H^{(2)}(\Omega; k)$ and decreasing with respect to $k$, and a constants $C, C(\theta)$ depending on $\theta > 0$ such that

$$
||u||_0(\Gamma \setminus \bar{V}) + ||u||_0(\Omega) \leq C F(\Omega) + \theta > 0.
$$

where $u = P_k u$, and

$$
||u||_0(\Gamma \setminus \bar{V}) = \sqrt{\theta > 0.}
$$

for all $u \in H^{(2)}(\Omega)$ solving (1.1), (1.2) where $F = \sqrt{\theta > 0}$.

Let $\chi$ be $C^\infty$ function, $\chi = 1$ on $\Omega \setminus V$, $\chi = 0$ near $\Gamma$. We let $v = \chi u$ in $\Omega$ and $v = 0$ on $\mathbb{R}^{n-1} \times (0, 1)$. Obviously, $\sum_{j,m=0}^{n-1} a_{jm}(x) \xi_j \xi_m \leq E(\Omega) \xi^2$ for some number $E > 0$ and all $x \in \mathbb{R}^{n-1} \times (0, 1), \xi \in \mathbb{R}^{n-1}$. We introduce low and high frequency projectors

$$
v_0(x) = \mathcal{F}^{-1}(\chi_k \mathcal{F}v(x)), \quad v_h = v - v_0,
$$

where $\mathcal{F}$ is the (partial) Fourier transformation with respect to $x' = (x_1, ..., x_{n-1}, 0)$, $\chi_k(\xi') = 1$ when $|\xi'|^2 < (1 - \varepsilon)\frac{k^2}{k^2}$ and $\chi_k(\xi') = 0$ otherwise. We define $u = v_0$ on $\Omega \setminus \bar{V}$.

As can be seen from the proofs of Theorems 1.1, 1.2,

$$
||u||_{(m,k)}(\Omega) = \sum_{j=1}^{n-1} \sum_{m=0}^{n-1} |\partial_j v_h|_{(m-1)}(\Omega) \frac{k^2}{k^2}.
$$

Corollary 1.3. Let $\Omega$ be a $C^2$-diffeomorphic image of the unit ball, $V$ be a neighborhood of a boundary point of $\Omega$, $\Gamma_1 = \partial \Omega \setminus V$, and $\omega = \Omega \cap V$. Let $\theta > 0$.

There are a monotone family of closed subspaces $H^{(2)}(\Omega; k)$ of $H^{(2)}(\Omega)$ with $\bigcup_k H^{(2)}(\Omega; k) = H^{(2)}(\Omega)$, linear continuous operators $P_k$ from $H^{(2)}(\Omega)$ onto $H^{(2)}(\Omega; k)$ with $P_k u = u$ for $u \in H^{(2)}(\Omega; k)$, a semi norm $||| \cdot |||(2;k)(\Omega)$ on $H^{(2)}(\Omega)$ which is zero on $H^{(2)}(\Omega; k)$ and decreasing with respect to $k$, and a constants $C, C(\theta)$ such that

$$
||u||_0(\Gamma \setminus \bar{V}) + ||u||_0(\Omega) \leq C F(\Omega) + \theta > 0.
$$

where $u = P_k u$, and

$$
||u||_0(\Gamma \setminus \bar{V}) = \sqrt{\theta > 0.}
$$

for all $u \in H^{(2)}(\Omega)$ solving (1.1), where $F = \sqrt{\theta > 0}$. 

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This corollary shows that the geometrical condition of Theorem 1.2 can be substantially relaxed. We will show that it directly follows from Theorem 1.2.

Since the operator $A$ preserves ellipticity, we can use $C^2$ diffeomorphic substitution and hence we can assume that $\Omega = \{ x : |x - e(n)| = 1 \}$, where $e(n) = (0,...,0,1)$, and the origin is contained in $V$. To make use of Theorem 1.2 we will use the inversion $y = -2|x|^{-2}x$. In $y$ coordinates $\Gamma$ will be a (bounded) part of the (hyper)plane $\{ y_n = -1 \}$, $\Omega$ will be the lower half-space $\{ y_n < -1 \}$, and $\omega$ will contain the unbounded part of ball $\{ y : y_n < -1, |y| > R \}$. After a scaling, a translation, and possible shrinking $\omega$ we can assume that $\Omega = \{ y : 0 < y_n < 1, |y - e(n)| < 0.5 \}$ and $\omega = \{ y : 0 < y_n < 1, 0.4 < |y - e(n)| < 0.5 \}$. Now we can apply Theorem 1.2 (with void $\Gamma_0$), and complete the derivation of Corollary.

Now, also for a particular $\Omega$ in $\mathbb{R}^2$ we will eliminate the constraint on $u$ in $\omega$. Let $\Omega = \{ x : |x| < R \}$, $\Gamma_0 = \{ x : |x| = 1 \}$, and $\Gamma_1 = \{ x : |x| = R \}$. The principal part of the operator $A$ in the polar coordinates $(\varphi, r)$ is $a^{22}\partial^2_\varphi + 2a^{12}\partial_\varphi \partial_r + a^{11}\partial^2_r$. Let $E = \sup(a^{11})^{-\frac{1}{2}}$ over $\Omega$ and $\varepsilon_0 > 0$. We will write the (angular) Fourier series

$$u(\varphi) = \sum_m u(m)e^{i\varphi m},$$

where $e(\varphi m) = \frac{1}{\sqrt{2\pi}} e^{im\varphi}$ and introduce the low frequency part of $u$

$$u_l(\varphi) = \sum_{|m|^2 < E^2(1-\varepsilon)k^2} u(m)e(\varphi m). \quad (1.11)$$

Under a constraint on the high frequency component of $u$ we have

**Theorem 1.4.** Let $\theta > 0$.

There are $C, C(\theta)$ such that

$$k\|u\|_0(\Gamma_1) + \|\nabla u\|_0(\Gamma_1) + \|u\|_1(\Omega) \leq C(k\|u_0\|_0(\Gamma_0) + \|u_1\|_0(\Gamma_0) + \|f\|_0(\Omega)) + C(\theta)k^{-\frac{1}{2}+\theta}\|u - u_l\|_2(\Omega) \quad (1.12)$$

and

$$k\|u_l\|_0(\Gamma_1) + \|\partial_r u_l\|_0(\Gamma_1) + \|u_l\|_1(\Omega) \leq C(k\|u_0\|_0(\Gamma_0) + \|u_1\|_0(\Gamma_0) + \|f\|_0(\Omega)) + C(\theta)k^{-\frac{1}{2}+\theta}\|u - u_l\|_2(\Omega) \quad (1.13)$$

for all $u \in H^{(2)}(\Omega)$ solving (1.1), (1.2).

## 2 Proof under ”high frequency” and local energy a priori constraints

In this section we will prove Theorem 1.1.

Since $v = \chi u$ from (1.1) by using the Leibniz formula we yield

$$(\sum_{j,m=1}^n a_{jm} \partial_j \partial_m v + \sum_{j=1}^n a_j \partial_j v + a + kc + k^2)v = \chi f + A_1 u, \quad (2.14)$$

where $A_1 = 2 \sum_{j,m=1}^n a_{jm} \partial_j \chi \partial_m u + \sum_{j=1}^n a_j \partial_j \chi u$.

Observe that

$$a_{nn}\partial^2_n v \partial_n v e^{-\tau x_n} = \frac{1}{2} \partial_n (a_{nn}(\partial_n v)^2 e^{-\tau x_n}) + \tau a_{nn}(\partial_n v)^2 e^{-\tau x_n} - \frac{1}{2} (\partial_n a_{nn})(\partial_n v)^2 e^{-\tau x_n},$$

$$2a_{jn} \partial_j \partial_n v \partial_n v e^{-\tau x_n} = \partial_j (a_{jn}(\partial_n v)^2 e^{-\tau x_n}) - (\partial_j a_{jn})(\partial_n v)^2 e^{-\tau x_n}, j = 1, \ldots, n - 1. \quad (2.15)$$

Integrating by parts with respect to $x_j$,

$$\int_{\mathbb{R}^{n-1} \times (0,1)} \sum_{j=1}^{n-1} a_{jm} \partial_j \partial_m v \partial_n v e^{-\tau x_n} = \text{...}$$
We have
\[
\sum_{j,m=1}^{n-1} a_{jm} \partial_m v \partial_j v \sigma_n e^{-\tau x_n} = \frac{1}{2} \sum_{j,m=1}^{n-1} \partial_n (a_{jm} \partial_m v \partial_j v e^{-\tau x_n}) + \\
\frac{\tau}{2} \sum_{j,m=1}^{n-1} a_{jm} \partial_m v \partial_j v e^{-\tau x_n} - \frac{1}{2} \sum_{j,m=1}^{n-1} (\partial_n a_{jm}) \partial_m v \partial_j v e^{-\tau x_n},
\]
(2.17)
due to symmetry of $a_{jm}$.

To form an energy integral we multiply the both sides of (2.14) by $\partial_n v e^{-\tau x_n}$ and integrate by parts over $\mathbb{R}^{n-1} \times (0,\theta), 0 < \theta \leq 1$, with using (2.15), (2.16), and (2.17) to yield
\[
\frac{1}{2} \int_{\mathbb{R}^{n-1}} a_{nn}(\partial_n v)^2(\theta, e^{-\tau \theta}) - \frac{1}{2} \int_{\mathbb{R}^{n-1}} a_{nn}(\partial_n v)^2(0) + \frac{\tau}{2} \int_{\mathbb{R}^{n-1} \times (0,\theta)} a_{nn}(\partial_n v)^2 e^{-\tau x_n} - \\
\frac{1}{2} \int_{\mathbb{R}^{n-1}} \sum_{j,m=1}^{n-1} a_{jm} \partial_j v \partial_m v(\theta) e^{-\tau \theta} + \frac{1}{2} \int_{\mathbb{R}^{n-1}} \sum_{j,m=1}^{n-1} a_{jm} \partial_j v \partial_m v(0) - \\
\frac{\tau}{2} \int_{\mathbb{R}^{n-1} \times (0,\theta)} \sum_{j,m=1}^{n-1} a_{jm}(x, x_n) \partial_j v \partial_m v(x, x_n) e^{-\tau x_n} + \int_{\mathbb{R}^{n-1} \times (0,\theta)} c k v \partial_n v(x, x_n) e^{-\tau x_n} + \\
\frac{k^2}{2} \int_{\mathbb{R}^{n-1}} v^2(\theta) e^{-\tau \theta} - \frac{k^2}{2} \int_{\mathbb{R}^{n-1}} v^2(0) + \frac{\tau k^2}{2} \int_{\mathbb{R}^{n-1} \times (0,\theta)} v^2 e^{-\tau x_n} + \ldots = \\
\int_{\mathbb{R}^{n-1} \times (0,\theta)} \partial_n v \chi e^{-\tau x_n} + \int_{\mathbb{R}^{n-1} \times (0,\theta)} \partial_n v A_1 u e^{-\tau x_n},
\]
(2.18)
where ... denotes the sum of terms bounded by
\[
C \int_{\mathbb{R}^{n-1} \times (0,1)} \left( \sum_{j=1}^{n} (\partial_j v)^2 + k^2 v^2 \right) e^{-\tau x_n}.
\]

We have
\[
\sum_{j,m=1}^{n-1} a_{jm}(x, x_n) \partial_j v \partial_m v(x, x_n) = \sum_{j,m=1}^{n-1} a_{jm}(x, x_n) \partial_j (v_l + v_h) \partial_m (v_l + v_h)(x, x_n) = \\
\sum_{j,m=1}^{n-1} (a_{jm}(x, x_n) \partial_j v_l \partial_m v_l(x, x_n) + 2 a_{jm}(x, x_n) \partial_j v_l \partial_m v_h(x, x_n) + a_{jm}(x, x_n) \partial_j v_h \partial_m v_l(x, x_n)) \leq \\
\sum_{j,m=1}^{n-1} a_{jm}(x, x_n) \partial_j v \partial_m v(x, x_n) + C \delta \sum_{j=1}^{n-1} (\partial_j v_l)^2(x, x_n) + C \delta^{-1} \sum_{j=1}^{n-1} (\partial_j v_h)^2(x, x_n),
\]
(2.19)
where we used the elementary inequality $AB \leq \frac{\delta}{2} A^2 + \frac{1}{2\delta} B^2$ with $A = \partial_j v_l, B = \partial_m v_h$ and assumed that $0 < \delta < 1$.

According to the definition of $E$,
\[
- \int_{\mathbb{R}^{n-1}} \sum_{j,m=1}^{n-1} a_{jm}(x, x_n) \partial_j v \partial_m v_l(x, x_n) \geq - \int_{\mathbb{R}^{n-1}} E^2 \sum_{j=1}^{n-1} (\partial_j v_l)^2(x, x_n) = - \\
- \int_{\mathbb{R}^{n-1}} E^2 \sum_{j=1}^{n-1} \epsilon_j^2 |V^2 v_l(x, x_n) | \geq - \int_{\mathbb{R}^{n-1}} k^2 (1 - \varepsilon_1) |V^2 v_l(x, x_n) | = -(1 - \varepsilon_1) k^2 \int_{\mathbb{R}^{n-1}} v_l^2(x, x_n),
\]
(2.20)
where we used that \( \mathcal{F}v_l(\xi', x_n) = 0 \) when \(-E^2|\xi'|^2 < (1 - \varepsilon)k^2\), due to (1.7), and utilized the Parseval’s identity. Similarly,

\[
\int_{\mathbb{R}^{n-1}} \left( \sum_{j=1}^{n-1} \partial_j v_l \right)^2 \leq Ck^2 \int_{\mathbb{R}^{n-1}} v_l^2. \tag{2.21}
\]

Therefore, using (2.19) and (2.20) we obtain

\[- \int_{\mathbb{R}^{n-1}} \sum_{j,m=1}^{n-1} a_{jm}(x_n) \partial_j v_l \partial_m v_l(x_n) \geq - (1 - \varepsilon)k^2 \int_{\mathbb{R}^{n-1}} v_l^2(x_n) - C\delta k^2 \int_{\mathbb{R}^{n-1}} v_l^2(x_n) - \frac{C}{\delta} \int_{\mathbb{R}^{n-1}} \sum_{j=1}^{n-1} (\partial_j v_h)^2(x_n). \tag{2.22}
\]

Since

\[
\int_{\mathbb{R}^{n-1}} v_l v_h(x_n) = 0, \quad \int_{\mathbb{R}^{n-1}} \partial_j v_l \partial_j v_h(x_n) = 0, \quad j = 1, \ldots, n,
\]

we have

\[
\int_{\mathbb{R}^{n-1}} v_l^2(x_n) = \int_{\mathbb{R}^{n-1}} (v_l + v_h)^2(x_n) = \int_{\mathbb{R}^{n-1}} v_l^2(x_n) + \int_{\mathbb{R}^{n-1}} v_h^2(x_n),
\]

\[
\int_{\mathbb{R}^{n-1}} (\partial_j v_l)^2(x_n) = \int_{\mathbb{R}^{n-1}} (\partial_j v_l)^2(x_n) + \int_{\mathbb{R}^{n-1}} (\partial_j v_h)^2(x_n). \tag{2.23}
\]

Hence from (2.18) and (2.22) by using the inequalities \(2AB \leq A^2 + B^2\) and \(\frac{1}{C} < a_{nn}\) (due to the ellipticity of \(A\)) we conclude that

\[
\frac{1}{C} \int_{\mathbb{R}^{n-1}} (\partial_n v_l)^2(\theta)e^{-\tau \theta} + \frac{\tau}{C} \int_{\mathbb{R}^{n-1} \times (0, \theta)} (\partial_n v_l)^2e^{-\tau x_n} + \frac{1}{2} \int_{\mathbb{R}^{n-1}} v_l^2 \leq C \left( \int_{\mathbb{R}^{n-1}} ((\partial_n v_l)^2(0) + k^2 v_l^2(0)) + \int_{\Omega} f^2 e^{-\tau x_n} + \int_{\omega} (A_1 u)^2 e^{-\tau x_n} \right) + C \left( \int_{\mathbb{R}^{n-1}} \sum_{j=1}^{n-1} (\partial_j v_h)^2(\theta)e^{-\tau} + \int_{\mathbb{R}^{n-1} \times (0, \theta)} \frac{1}{\delta} \sum_{j=1}^{n-1} (\partial_j v_l)^2 + (\partial_n v_l)^2 + k^2 v_l^2 e^{-\tau x_n} \right).
\]

Let \(\delta = \frac{k^2}{\tau}\) and use (2.23) again, then we yield the inequality

\[
\int_{\mathbb{R}^{n-1}} (\partial_n v_l)^2(\theta)e^{-\tau \theta} + \tau \int_{\mathbb{R}^{n-1} \times (0, \theta)} (\partial_n v_l)^2e^{-\tau x_n} + \frac{1}{2} \int_{\mathbb{R}^{n-1}} v_l^2 \leq C \left( \int_{\mathbb{R}^{n-1}} ((\partial_n v_l)^2(0) + k^2 v_l^2(0)) + \int_{\Omega} f^2 e^{-\tau x_n} + \int_{\omega} (A_1 u)^2 e^{-\tau x_n} \right) + C \left( \sum_{j=1}^{n-1} (\partial_j v_h)^2(1)e^{-\tau} + \int_{\mathbb{R}^{n-1} \times (0, 1)} \sum_{j=1}^{n-1} (\partial_j v_h)^2 + \sum_{j=1}^{n-1} (\partial_j v_l)^2 + (\partial_n v_l)^2 + k^2 v_l^2 e^{-\tau x_n} \right). \tag{2.24}
\]

Choosing and fixing sufficiently large \(\tau\) (depending on the same parameters as \(C\)) to absorb the three last terms on the right side in (2.24) by the left side we obtain

\[
\int_{\mathbb{R}^{n-1}} (\partial_n v_l)^2(\theta) + \int_{\mathbb{R}^{n-1} \times (0, \theta)} (\partial_n v_l)^2 + k^2 \int_{\mathbb{R}^{n-1}} v_l^2(\theta) + k^2 \int_{\mathbb{R}^{n-1} \times (0, \theta)} v_l^2 \leq
\]
By increasing $C$, we have

$$C\left(\int_{\mathbb{R}^{n-1}} ((\partial_n v)^2, \mathbb{R}^{n-1}) + k^2 v^2, \mathbb{R}^{n-1}) + \int_{\Omega} f^2 + \int_{\omega}(A_1 u)^2 + \int_{\mathbb{R}^{n-1}} \sum_{j=1}^{n-1} (\partial_j v_h)^2, (\partial_\theta) + \int_{\mathbb{R}^{n-1} \times (0,1)} \sum_{j=1}^{n-1} (\partial_j v_h)^2. \right) \tag{2.25}$$

Integrating the inequality (2.25) with respect $\theta$ over $(0,1)$, dropping the first two terms on the left side, and recalling that $v = \chi u$ we yield

$$k^2 \|v\|_{(0)}^2(\Omega) \leq C(\|u_1\|_{(0)}^2(\Gamma_0) + k^2 \|u_0\|_{(0)}^2(\Gamma_0) + \|f\|_{(0)}^2(\Omega) + \|u\|_{(1)}^2(\omega) + \sum_{j=1}^{n-1} \|\partial_j v_h\|_{(0)}^2(\mathbb{R}^{n-1} \times (0,1)))$$

Recalling the definition of a high frequency norm (1.8), using that $u = v$ on $\Omega \setminus V$, and dividing by $k^2$ we obtain (1.4).

Due to (2.24), (1.4) follows from (1.3).

### 3 Proof under high frequency constraints

In this section we will prove Theorem 1.2.

From the proof of Theorem 1.1, (2.25), we have

$$\int_{\mathbb{R}^{n-1}} (\partial_n v)^2, (\partial_\theta) + \int_{\mathbb{R}^{n-1} \times (0,1)} (\partial_n v)^2 + k^2 \int_{\mathbb{R}^{n-1}} v^2, (\partial_\theta) + k^2 \int_{\mathbb{R}^{n-1} \times (0,1)} v^2 \leq C(\int_{\mathbb{R}^{n-1}} ((\partial_n v)^2, (0) + k^2 v^2, (0)) + \int_{\Omega} f^2 + \int_{\omega}(A_1 u)^2 + \int_{\mathbb{R}^{n-1}} \sum_{j=1}^{n-1} (\partial_j v_h)^2, (1) + \int_{\mathbb{R}^{n-1} \times (0,1)} \sum_{j=1}^{n-1} (\partial_j v_h)^2).$$

Using (2.21) and (2.23) we obtain

$$\int_{\mathbb{R}^{n-1}} (\partial_n v)^2, (1) + \int_{\mathbb{R}^{n-1}} \sum_{j=1}^{n-1} (\partial_j v_h)^2, (1) + \int_{\mathbb{R}^{n-1} \times (0,1)} (\partial_\theta) + \int_{\mathbb{R}^{n-1} \times (0,1)} (\partial_\theta) + \sum_{j=1}^{n-1} (\partial_j v_h)^2 \leq C(\int_{\mathbb{R}^{n-1}} ((\partial_n v)^2, (0) + k^2 v^2, (0)) + \int_{\Omega} f^2 + \int_{\omega}(A_1 u)^2 + \int_{\mathbb{R}^{n-1}} \sum_{j=1}^{n-1} (\partial_j v_h)^2, (1) + \int_{\mathbb{R}^{n-1} \times (0,1)} \sum_{j=1}^{n-1} (\partial_j v_h)^2).$$

By increasing $C$ and using (2.23) again it gives

$$\int_{\mathbb{R}^{n-1}} \sum_{j=1}^{n} (\partial_j v)^2, (1) + \int_{\mathbb{R}^{n-1} \times (0,1)} \sum_{j=1}^{n} (\partial_j v)^2 + k^2 \int_{\mathbb{R}^{n-1}} v^2, (1) + k^2 \int_{\mathbb{R}^{n-1} \times (0,1)} v^2 \leq C(\int_{\mathbb{R}^{n-1}} ((\partial_n v)^2, (0) + k^2 v^2, (0)) + \int_{\Omega} f^2 + \int_{\omega}(A_1 u)^2 + \int_{\mathbb{R}^{n-1}} \sum_{j=1}^{n-1} (\partial_j v_h)^2, (1) + \int_{\mathbb{R}^{n-1} \times (0,1)} \sum_{j=1}^{n-1} (\partial_j v_h)^2).$$
Therefore,
\[
\| \partial_n v(1) \|_1^2_{(0)}(\mathbb{R}^{n-1}) + \| v(1) \|_1^2_{(1)}(\mathbb{R}^{n-1}) + \| v \|_1^2_{(1)}(\mathbb{R}^{n-1} \times (0,1)) + \\
k^2 \| v(1) \|_{(0)}(\mathbb{R}^{n-1}) + k^2 \| v(1) \|_{(0)}(\mathbb{R}^{n-1} \times (0,1)) \leq \\
C(\| u_1 \|_1^2_{(0)}(\Gamma_0) + \| u_0 \|_1^2_{(1)}(\Gamma_0) + \| f \|_{(0)}^2(\Omega) + \| u \|_{(1)}^2(\omega) + \\
\| v_h(1) \|_{(1)}^2(\mathbb{R}^{n-1}) + \| v_h \|_{(1)}^2(\mathbb{R}^{n-1} \times (0,1)).
\]
(3.26)

Let \( v_h^* \) be the extension of \( v_h \) onto \( \mathbb{R}^n \) constructed in [12], p.14. As follows from [12], with natural choice of functional spaces,
\[
\| v_h^* \|_{(2)}(\mathbb{R}^n) \leq C\| v_h \|_{(2)}(\mathbb{R}^{n-1} \times (0,1))
\]
(3.27)
From the construction in [12] it follows that \((v^*)_h = (v^*)_h\), i.e. that the Fourier transform of \((v^*)_h\) with respect to \((x_1, ..., x_{n-1})\) is zero when \(|\xi'| \leq \sqrt{1 - \frac{k}{2}}\). By known trace theorems for Sobolev spaces, [12], p.42,
\[
\| v_h(1) \|_{1}^2(\mathbb{R}^{n-1}) \leq C(\theta)\| v_h^* \|_{(n-\theta)}(\mathbb{R}^n).
\]
(3.28)
Let \( V_h^*(\xi) \) be the Fourier transform of \( v_h^*(x) \). As known, [12], p.30, for Sobolev norms,
\[
\| v_h^* \|_{(n-\theta)}^2(\mathbb{R}^n) = \int (1 + |\xi'|^2 + \xi_n^2)^{1+\theta}|V_h^*(\xi)|^2d\xi \leq \\
k^{-1+2\theta} \int k^{-1-2\theta}(1 + |\xi'|^2 + \xi_n^2)^{-\frac{1}{2}+\theta}(1 + |\xi'|^2 + \xi_n^2)^{2}|V_h^*(\xi)|^2d\xi \leq \\
Ck^{-1+2\theta} \int (1 + |\xi'|^2 + \xi_n^2)^{2}|V_h^*(\xi)|^2d\xi = \\
Ck^{-1+2\theta} \| v_h^* \|_{(2)}^2(\mathbb{R}^{n-1}) \leq Ck^{-1+2\theta} \| v_h \|_{(2)}^2(\mathbb{R}^{n-1} \times (0,1)),
\]
due to (3.27). Here we used that \( k^{-1-2\theta}(1 + |\xi'|^2 + \xi_n^2)^{-\frac{1}{2}+\theta} \leq C \) on the actual integration domain (where \(|V_h^*(\xi)| > 0\) and hence \( \frac{k}{2} < |\xi'| \)). So using (3.28) we yield
\[
\| v_h(1) \|_{(2)}^2(\mathbb{R}^{n-1}) \leq Ck^{-1+2\theta} \| v_h \|_{(2)}^2(\mathbb{R}^{n-1} \times (0,1)).
\]
(3.29)
Similarly,
\[
\| v_h \|_{(1)}^2(\mathbb{R}^{n-1} \times (0,1)) \leq \| v_h^* \|_{(1)}^2(\mathbb{R}^n) = \int (1 + |\xi'|^2 + \xi_n^2)|V_h^*(\xi)|^2d\xi = \\
k^{-2} \int (k^2(1 + |\xi'|^2 + \xi_n^2)^{-1}(1 + |\xi'|^2 + \xi_n^2)^2)|V_h^*(\xi)|^2d\xi \leq \\
Ck^{-2} \int (1 + |\xi'|^2 + \xi_n^2)^2|V_h^*(\xi)|^2d\xi = \\
Ck^{-2} \| v_h^* \|_{(2)}^2(\mathbb{R}^n) \leq Ck^{-2} \| v_h \|_{(2)}^2(\mathbb{R}^{n-1} \times (0,1)),
\]
due to (3.27). So
\[
\| v_h \|_{(1)}^2(\mathbb{R}^{n-1} \times (0,1)) \leq Ck^{-2} \| v_h \|_{(2)}^2(\mathbb{R}^{n-1} \times (0,1)).
\]
(3.30)
Using that \( v = \chi u \), from (3.27), (3.29), and (3.30), we obtain (3.6).
As in the proof of Theorem 1.1, (1.5) follows from (1.6) because of (2.28).
The proof is complete.
4 Proof for annular domains

In this section we will prove Theorem 1.4. We will use polar coordinates and the operator $A$ in these coordinates.

From (1.1) we yield

$$a^{22} \partial^2_r u + 2a^{12} \partial_r \partial_u + a^{11} \partial^2_r u + a^1 \partial_r u + a^2 \partial_r u + au + kcu + k^2 u = f \text{ in } [0, 2\pi] \times (1, R). \quad (4.31)$$

Repeating the argument from the proof of Theorem 1.1 (multiplying the both parts of (4.31) by $\partial_r e^{-\tau \rho}$ and integrating by parts over $\Omega$ with using angular periodicity) we will have

$$\frac{1}{2} \int_{[0, 2\pi]} a^{22}(\partial_r u)^2(R) e^{-\tau \rho R} - \frac{1}{2} \int_{[0, 2\pi]} a^{22}(\partial_r u)^2(1) e^{-\tau} + \frac{\tau}{2} \int_{\Omega} a^{22}(\partial_r u)^2 e^{-\tau \rho} -$$

$$\frac{1}{2} \int_{[0, 2\pi]} a^{11}(\partial_r u)^2(R) e^{-\tau \rho} R - \frac{1}{2} \int_{[0, 2\pi]} a^{11}(\partial_r u)^2(1) e^{-\tau} - \frac{\tau}{2} \int_{\Omega} a^{11}(\partial_r u)^2 e^{-\tau \rho} +$$

$$\frac{k^2}{2} \int_{[0, 2\pi]} u^2(R) e^{-\tau \rho} R - \frac{k^2}{2} \int_{[0, 2\pi]} u^2(1) + \frac{\tau k^2}{2} \int_{\Omega} u^2 e^{-\tau \rho} + \ldots = \int_{\Omega} \partial_r ue^{-\tau \rho}. \quad (4.32)$$

where $\ldots$ denotes the sum of terms bounded by

$$C \int_{\Omega} ((\partial_r u)^2 + (\partial_r u)^2 + k^2 u^2)e^{-\tau \rho}.$$

To handle the negative terms on the left side of (4.32), we use that

$$- \int_{[0, 2\pi]} a^{11}(\partial_r u)^2(R) e^{-\tau \rho} R - \int_{[0, 2\pi]} a^{11}(\partial_r u + \partial_r u_k)^2(R) \geq$$

$$- \int_{[0, 2\pi]} a^{11}(\partial_r u)^2(R) - \delta \int_{[0, 2\pi]} (\partial_r u)^2(R) - C \delta \int_{[0, 2\pi]} (\partial_r u_k)^2(R).$$

As in the proof of Theorem 1.1, using (1.11), from the Parseval’s identity for the Fourier series, we have

$$- \int_{[0, 2\pi]} a^{11}(\partial_r u)^2(R) \geq - \int_{[0, 2\pi]} E^2(\partial_r u)^2(R) \geq -(1 - \varepsilon)k^2 \int_{[0, 2\pi]} u^2(R) \geq -(1 - \varepsilon)k^2 \int_{[0, 2\pi]} u^2(R) \quad (4.33)$$

and

$$- \int_{[0, 2\pi]} (\partial_r u)^2(R) \geq - Ck^2 \int_{[0, 2\pi]} (u)^2(R) \geq - Ck^2 \int_{[0, 2\pi]} u^2(R).$$

Hence from (4.32) we conclude that

$$\frac{1}{2} \int_{(0, 2\pi)} a^{22}(\partial_r u)^2(R) e^{-\tau R} + \frac{\tau}{2} \int_{(0, 2\pi) \times (1, R)} a^{22}(\partial_r u)^2 e^{-\tau \rho} +$$

$$(\varepsilon - C \delta) \frac{k^2}{2} \int_{(0, 2\pi)} u^2(R) e^{-\tau R} + \frac{\tau(\varepsilon - C \delta)k^2}{2} \int_{(0, 2\pi) \times (1, R)} u^2 e^{-\tau \rho} \leq$$

$$C \left( \frac{1}{2} \int_{(0, 2\pi)} ((\partial_r u)^2 (1) + k^2 u^2(1)) + \int_{(0, 2\pi) \times (1, R)} f^2 e^{-\tau \rho} + \right.$$

$$\left. \frac{C}{\delta} \int_{(0, 2\pi)} ((\partial_r u) + (\partial_r u_k)^2)(R) e^{-\tau R} + \frac{\tau}{(0, 2\pi) \times (1, R)} ((\partial_r u)^2 + (\partial_r u_k)^2) e^{-\tau \rho} + \right.$$

$$\int_{(0, 2\pi) \times (1, R)} ((\partial_r u)^2 + (\partial_r u)^2 + k^2 u^2) e^{-\tau \rho}).$$
Choosing $\delta = \frac{\epsilon}{2\tau}$ and using that $\frac{1}{\tau} < a^{2\tau}$ we yield

$$
\int_{(0,2\pi)} ((\partial_r u)^2 (R) + k^2 u^2 (R)) e^{-\tau R} + \tau \int_{(0,2\pi) \times (1, R)} ((\partial_r u)^2 + k^2 u^2) e^{-\tau r} \leq
\int_{(0,2\pi)} ((\partial_r u)^2 (1, 1) + k^2 u^2 (1)) + \int_{(0,2\pi) \times (1, R)} f^2 e^{-\tau r} +
$$

$$
\int_{(0,2\pi)} e^{-\tau R} + \tau \int_{(0,2\pi) \times (1, R)} ((\partial_r u)^2 + (\partial_{xx} u)^2 + k^2 u^2) e^{-\tau r}. \tag{4.34}
$$

From the definition of $u_1$ we have

$$
\int_{(0,2\pi) \times (1, R)} ((\partial_r u)^2 e^{-\tau r} = \int_{(0,2\pi) \times (1, R)} ((\partial_r u)^2 + (\partial_{xx} u)^2) e^{-\tau r} \leq
\int_{(0,2\pi) \times (1, R)} ((\partial_r u)^2 + k^2 u^2) e^{-\tau r} \leq
$$

$$
C k^2 \int_{(0,2\pi) \times (1, R)} u^2 + \int_{(0,2\pi) \times (1, R)} ((\partial_r u)^2 + (\partial_{xx} u)^2) e^{-\tau r},
$$

when we apply (4.33). So from (4.33) we obtain

$$
\int_{(0,2\pi)} ((\partial_r u)^2 (R) + k^2 u^2 (R)) e^{-\tau R} + \tau \int_{(0,2\pi) \times (1, R)} ((\partial_r u)^2 + k^2 u^2) e^{-\tau r} \leq
\int_{(0,2\pi)} ((\partial_r u)^2 (1, 1) + k^2 u^2 (1)) + \int_{(0,2\pi) \times (1, R)} f^2 e^{-\tau r} +
$$

$$
\int_{(0,2\pi)} ((\partial_r u)^2 + (\partial_{xx} u)^2) e^{-\tau R} + \tau \int_{(0,2\pi) \times (1, R)} ((\partial_r u)^2 + (\partial_{xx} u)^2) e^{-\tau r} +
\int_{(0,2\pi) \times (1, R)} ((\partial_r u)^2 + k^2 u^2) e^{-\tau r}).
$$

Now, choosing and fixing $\tau$ sufficiently large (but depending on the same quantities as $C$) to absorb the last term on the right side by the left side we yield

$$
\int_{(0,2\pi)} ((\partial_r u)^2 (R) + k^2 u^2 (R)) + \tau \int_{(0,2\pi) \times (1, R)} ((\partial_r u)^2 + k^2 u^2) \leq
\int_{(0,2\pi)} ((\partial_r u)^2 (1, 1) + k^2 u^2 (1)) + \int_{(0,2\pi) \times (1, R)} f^2 +
\int_{(0,2\pi)} ((\partial_r u)^2 + (\partial_{xx} u)^2) (, R) + \int_{(0,2\pi) \times (1, R)} ((\partial_r u)^2 + (\partial_{xx} u)^2)). \tag{4.35}
$$

By trace theorems for Sobolev spaces

$$
\|u_h (R)\|_{(1)(1)} + ||(\partial_r u_h (R)||_{(0)(1)}) \leq C(\theta)\|u_h \|_{(1, 1 + \theta)} (\Omega).
$$

For the high frequency part

$$
\|u_h \|_{(1, 1 + \theta)} (\Omega) \leq C k^{2\theta - 1} \|u_h \|_{(2)} (\Omega), \tag{4.36}
$$

so from (4.35) we obtain (1.12).

Since the Sobolev norms of $u_1$ are bounded by Sobolev norms of $u$, (1.13) follows from (1.12). The proof is complete.
\section{Conclusion}

We think that the results of this paper can extended onto higher order elliptic equations and systems. An important question is about minimal a priori constraints on the high frequency part of a solution. It is feasible that semi norms \( ||| \cdot |||_{(m,k)}(\Omega) \), \( m = 2 \) in Theorem 1.2 can be replaced by a similar semi norm with \( m = 1 \), imposing only natural energy constraints on the high frequency part of \( u \). Moreover, a complete justification of increasing stability can be obtained by proving that there are growing invariant subspaces where the solution of the Cauchy problem (1.1), (1.2) is Lipschitz stable. We will give one of related conjectures.

Let \( \Omega \) be a Lipschitz bounded domain introduced in Theorem 1.1. Let us assume that there is an unique solution \( u \in H^{(1)}(\Omega) \) of the following Neumann problem

\[
Au + cku + k^2 u = 0 \text{ in } \Omega, \quad \partial_{\nu}u = 0 \text{ on } \partial\Omega \setminus \Gamma_1, \quad \partial_{\nu}u = g \in H^{(-\frac{1}{2})}(\Gamma_1), \text{ on } \Gamma_1.
\]

The operator \( B \) mapping \( g \) into \( u_0 = u \) on \( \Gamma_0 \) is compact from \( L^2(\Gamma_1) \) into \( L^2(\Gamma_0) \). Hence it admits the singular value decomposition consisting of complete orthonormal system of functions \( g_m, m = 1, 2, \ldots \) in \( L^2(\Gamma_1) \) and corresponding singular values \( \sigma_m \geq \sigma_{m+1} > 0 \) (eigenfunction and square roots of eigenvalues of \( B^*B \)). The conjecture is that there are positive numbers \( \delta_1, \delta_2 \) depending only on \( A, c \) and \( \Omega \) (but not on \( k \)) such that \( \sigma_m > \delta_1 \) when \( m < \delta_2 k \). This conjecture for some interesting plane \( \Omega \) was numerically confirmed in [9].

Use of low frequency zone does not need any convexity type assumptions and for this reason is very promising for applications. In the recent paper [9] we studied this phenomenon in more detail and gave regularization schemes for numerical solution incorporating the increasing stability. We gave several numerical examples of increasing stability for the Helmholtz equation in some interesting plane domains, admitting or not admitting explicit analytical solution and complete analytic justification. It is important to collect numerical evidence of the increasing stability for more complicated geometries and for systems.

The increasing stability is expected in the inverse source problem, where one looks for \( f \) in the Helmholtz equation \( (\Delta + k^2)u = f \) (not depending on \( k \)) in \( \Omega = \{ x : 1 < |x| < R \} \) from the Cauchy data \( u, \partial_{\nu}u \) on \( \Gamma = \{ x : |x| = 1 \} \), \( k_* < k < k^* \). One needs to obtain stability estimates improving with growing \( k^* \) and to give a numerical evidence of better resolution for larger \( k^* \).

It was (numerically) observed, that use of only low frequency zone can produce a stable solution of the inverse problem, where one looks for a speed of the propagation from all possible boundary measurements. One can look at the linearized problem: find \( f \) (supported in \( \Omega \subset \mathbb{R}^3 \)) from

\[
\int_{\Omega} f(y) e^{ki(|x-y| - |z-y|)} |x-y| |z-y| dy
\]

given for \( x, z \in \Gamma \subset \partial\Omega \). The closest analytic results on improving stability are obtained in [8], citeIW for the Schrödinger potential.

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