DIFFERENTIAL RENORMALIZATION OF THE WESS-ZUMINO MODEL

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Abstract

We apply the recently developed method of differential renormalization to the Wess-Zumino model. From the explicit calculation of a finite, renormalized effective action, the $\beta$-function is computed to three loops and is found to agree with previous existing results. As a further, nontrivial check of the method, the Callan-Symanzik equations are also verified to that loop order. Finally, we argue that differential renormalization presents advantages over other superspace renormalization methods, in that it avoids both the ambiguities inherent to supersymmetric regularization by dimensional reduction (SRDR), and the complications of virtually all other supersymmetric regulators.

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1. Introduction – Since it has been developed, differential renormalization has had some important successes. At considerable calculational (and conceptual) simplification over other regularization methods, it has been used to fully renormalize massless $\lambda\phi^4$ theory to three loops [1], to calculate the triangle anomaly [1,2], and the 1-loop $\beta$-function in Yang-Mills theory [1]. Further, mass effects in $\lambda\phi^4$ theory have also been included straightforwardly [1,2], and the translation of the method to a setting in which counterterms appear explicitly has also been achieved in the course of proving the unitarity of the method for massless $\lambda\phi^4$ theory [3]. Like for the $\lambda\phi^4$ model, the perturbation theory of the Wess-Zumino model is fairly simple and well-known, but at the same time sufficiently nontrivial for it to represent a good testing ground for differential renormalization, and we find that the method works as well in this case as it does in $\lambda\phi^4$ theory.

The basic idea of the method is as follows: rather than regularizing by altering the field content or the propagators of a theory (e.g., Pauli-Villars, higher derivatives, point splitting) or the space on which it is defined (e.g., dimensional regularization), it regularizes and renormalizes only the $x$-space amplitudes that are too divergent to have a Fourier transform into momentum space. In practice, this is done by writing amplitudes in $x$-space, expressing the divergent pieces as derivatives of less singular terms, and then formally performing partial integrations freely (which in fact will be precisely the implicit regularization and renormalization that the method performs). Counterterms do not appear at any point (although it is possible to keep track of them explicitly [3]), and one goes directly from bare amplitudes to renormalized ones, without a separate intermediate step of subtraction in the regularized theory. The simplest example occurs at one loop both for the 4-point function in $\lambda\phi^4$ theory and the 2-point function in the Wess-Zumino model: these amplitudes will contain the term $((x-x')^2)^{-2}$, and its Fourier transform into momentum space (of course, in the end one must go to momentum space to calculate scattering amplitudes) is:

$$\int d^4x \frac{e^{ip\cdot x}}{x^4} = \frac{4\pi^2}{p} \int_0^\infty dr \frac{1}{r^2} J_1(pr),$$

where $r = (x^2)^{\frac{1}{2}}$, and $J_1$ is the first-order Bessel function. For $r \to \infty$ the integrand converges fast enough, but for $r \to 0$, $J_1(pr) \sim \frac{1}{2}pr$, and we have a logarithmic UV divergence. To regularize it, differential renormalization then prescribes the use of the following identity:

$$\frac{1}{x^4} = -\frac{1}{4} \Box \ln x^2 M^2, \quad x \neq 0 \quad (1)$$
valid everywhere except at the origin. Now, the term on which the D’Alembertian is
acting is, as opposed to $\frac{1}{x^4}$, a well-defined distribution with a proper Fourier trans-
form. By formally integrating by parts we are then able to define a regularized Fourier
transform for $\frac{1}{x^4}$:

$$
\int d^4x e^{ip \cdot x} \frac{1}{x^4} = -\frac{1}{4} \int d^4x \frac{e^{ip \cdot x}}{x^4} \ln \frac{x^2 M^2}{x^2} \equiv \frac{p^2}{4} \int d^4x e^{ip \cdot x} \ln \frac{x^2 M^2}{x^2}
$$

$$
= -\pi^2 \ln \frac{p^2}{\bar{M}^2},
$$

where $\bar{M} = \frac{2M}{\gamma_E}$, and $\gamma_E = 1.78107...$ is the Euler-Mascheroni constant. A general fea-
ture of differential renormalization which can already be seen from the example above
is that in the process of expressing singular functions as derivatives of less singular func-
tions and then integrating by parts, thereby discarding (infinite) surface terms, we are,
on the one hand, removing the divergence of an amplitude and, on the other, acquiring
a new mass parameter $M$ which appears in the guise of an integration constant. Since
the method does not keep track explicitly of the infinities being subtracted, this mass
parameter will be crucial for the study of the RG invariance of renormalized amplitudes
and the consistency of the method.

Because Eq.(1) is an identity except at the origin, bare and renormalized amplitudes
will also be identical except at isolated points. The method does not, so to speak,
uniformly and blindly regularize an entire theory ab initio (like the regularization pro-
cedures mentioned above), but rather removes divergences singly where they appear in
amplitudes. In this sense, differential renormalization is a “minimal” regularization: it
alters the theory in the least amount possible, and in particular it does not disturb its
original symmetries.

To apply it to the Wess-Zumino model, we will use the manifestly supersymmetric
Feynman rules of Grisaru, Roček, and Siegel[4,5] (except for the fact that we do not go
into momentum space), to calculate loop corrections to the chiral propagator. For su-
perspace theories in general, using these Feynman rules, one ends up with the following
superspace effective action:

$$
\Gamma = \int d^4 \theta \sum_N \int d^4x_1...d^4x_N F_1(x_1, \theta) F_2(x_2, \theta)...F_N(x_N, \theta) G^{(N)}(x_1, ..., x_N),
$$

where the $F_i(x_i, \theta)$ are functions of superfields and their (supersymmetry) covariant
derivatives, and $G^{(i)}$ are translationally invariant functions of $i$ spacetime coordinates.
Because of the non-renormalization theorem\cite{4,5}, the Wess-Zumino model, in particular, does not have a genuine coupling constant renormalization (but only the one inherited from wavefunction renormalization)\cite{6,7}, and is entirely renormalized by removing the divergences from $G^{(2)}(x - x')$. That we do, up to three loops, with differential renormalization; with more ease than previous methods, we find a three-loop $\beta$-function which agrees with the existing results\cite{7,8,9}, and as further consistency checks of the method, we verify the Callan-Symanzik equation for $G_{\text{ren}}^{(2)}$, at each loop order, also to three loops.

2. The Model and Loop Calculations - The Wess-Zumino model is described by the following superspace action:

$$S = \int d^4x \left( \int d^2\theta d^2\bar{\theta} \phi \bar{\phi} - \frac{g}{3!} \left( \int d^2\theta \phi^3 + \int d^2\bar{\theta} \bar{\phi}^3 \right) \right),$$

(3)

where $(x^a, \theta^\alpha, \bar{\theta}^{\dot{\alpha}})$, $a = 1, 2, 3, 4$, $\alpha = +, -, \dot{\alpha} = \dot{+}, \dot{-}$ are coordinates of $d = 4, N = 1$ superspace, and $\phi$ ($\bar{\phi}$) is a chiral (antichiral) superfield:

$$\bar{D}_{\dot{\alpha}} \phi = \left( \frac{\partial}{\partial \theta^\alpha} + i \theta^\alpha \sigma^{\alpha\beta}_{\dot{\alpha}} \partial_{\dot{\beta}} \right) \phi(x, \theta, \bar{\theta}) = 0$$

$$D_{\alpha} \bar{\phi} = \left( \frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}} + i \sigma^{\alpha\beta}_{\dot{\alpha}} \bar{\theta}^{\dot{\beta}} \partial_{\dot{\alpha}} \right) \bar{\phi}(x, \theta, \bar{\theta}) = 0.$$

In component form,

$$\phi(x, \theta, \bar{\theta}) = \varphi(x) + \theta \psi(x) + \theta \theta F(x) + i \theta \sigma^a \bar{\theta} \partial_a \varphi(x) - \frac{i}{2} \theta \theta \partial_a \psi(x) \sigma^a \bar{\theta} + \frac{1}{4} \theta \theta \bar{\theta} \bar{\theta} \chi \varphi(x)$$

and

$$S = \int d^4x \left[ -\partial_a \varphi^a \varphi - \frac{i}{4} (\psi \sigma^a \partial_a \psi - \partial_a \psi \sigma^a \psi) + FF^* - \frac{g}{3!} (\varphi F + \frac{1}{2} \psi \psi A + c.c.) \right].$$

For our calculations in superspace, we use the Feynman rules of Grisaru, Roček and Siegel\cite{4,5}; in coordinate space, they are: i) the propagator is

$$\langle \bar{\phi}(x, \theta) \phi(x', \theta') \rangle = \frac{1}{4\pi^2(x - x')^2} \delta^{(4)}(\theta - \theta')$$

$$= \frac{1}{4\pi^2(x - x')^2} (\theta - \theta')^2 (\dot{\theta} - \dot{\theta}')^2;$$

ii) for each vertex, include a factor of $\frac{g}{3!}$; iii) for a chiral $(\bar{\phi}^3)$ vertex, include two factors of $-\frac{1}{4} \bar{D}^2$ acting on any two of the propagators for the lines arriving at the vertex; for
vertices containing an external line, include only one $-\frac{1}{4}D^2$ factor acting on any of the two other (internal) lines; *iv*) same for antichiral vertices ($\bar{\phi}^2$), with $-\frac{1}{4}D^2$ factors; *v*) integrate over internal superspace points ($x_{int}, \theta_{int}$) and external $\theta$'s, $\theta_{ext}$; *vi*) include appropriate symmetry factors for each diagram. Finally, the $D$-algebra and transfer rules are:

$$D^2D^2D^2 = 16\Box D^2 \quad , \quad \bar{D}^2D^2\bar{D}^2 = 16\Box \bar{D}^2,$$

$$D^3 = \bar{D}^3 = 0 \quad , \quad [D_\alpha, \bar{D}^2] = 4i\sigma^a_{\alpha\beta}\partial_\alpha \bar{D}^\beta,$$

$$D_\alpha(\theta', \partial_{x'})[\delta^{(4)}(\theta - \theta')f(x - x')] = -D_\alpha(\theta, \partial_x)[\delta^{(4)}(\theta - \theta')f(x - x')].$$

With these rules for calculating Feynman diagrams, the effective action takes the form (after all $\theta$-integrations but the last have been carried out):

$$\Gamma[\phi, \bar{\phi}] = \int d^4\theta \left[ \int d^4xd^4x' \phi(x, \theta, \bar{\theta})\bar{\phi}(x', \theta, \bar{\theta})G^{(2)}(x - x') \right]$$

$$+ \left[ \int d^2\theta \int d^4x_1d^4x_2d^4x_3\phi(x_1, \theta, \bar{\theta})\phi(x_2, \theta, \bar{\theta})\phi(x_3, \theta, \bar{\theta})G_{ch}^{(3)}(x_1, x_2, x_3) + c.c. \right]$$

$$+ \int d^4\theta \left[ \cdots \right], \quad (4)$$

where

$$G^{(2)}(x - x') = \delta^{(4)}(x - x') + \kappa_1(x - x') + \kappa_2(x - x') + \kappa_3(x - x') + \text{etc.} \quad (5)$$

$$G_{ch}^{(3)}(x_1, x_2, x_3) = -\frac{g}{3!^2} \left[ \delta^{(4)}(x_1 - x_2)\delta^{(4)}(x_1 - x_3) + 5 \text{perms.} \right] \quad (6)$$

and the dots in Eq.(4) refer to higher-point functions which are either finite or are renormalized by the renormalization of $G^{(2)}$. The diagrams contributing to $G^{(2)}$ are depicted in Figs.(1) and (2). $\kappa_1$ is the one-loop contribution from Fig.(1.a), $\kappa_2$ the two-loop contribution, Fig.(1.b), and $\kappa_3$ contains contributions from the four three-loop diagrams in Fig.(2).

The non-renormalization theorem determines that the renormalization of $g$ comes solely from wavefunction renormalization, and dictates the above form for $G_{ch}^{(3)}$, correct to all loops. By studying the Callan-Symanzik equation for $G_{ch}^{(3)}$, we find a simple relation between the $\beta$-function and the anomalous dimension $\gamma(g)$ of $\phi$:

$$\left( M \frac{\partial}{\partial M} + \beta(g) \frac{\partial}{\partial g} - 3\gamma(g) \right) G_{ch}^{(3)}(x_1, x_2, x_3; g) =$$

$$= \left( \beta(g) \frac{\partial}{\partial g} - 3\gamma(g) \right) G_{ch}^{(3)}(x_1, x_2, x_3; g) = 0.$$
\[ \Rightarrow \beta(g) = 3g\gamma(g). \]  

The Callan-Symanzik equation for the renormalized 2-point function then becomes:

\[
\left( M \frac{\partial}{\partial M} + \beta(g) \frac{\partial}{\partial g} - 2\gamma(g) \right) G_{\text{ren}}^{(2)}(x-x';g,M) = 0. 
\]

In general, \( M \) is a renormalization mass scale that governs the RG flow of the theory; in differential renormalization this scale appears through the process described in the introduction.

We now proceed to renormalize the theory. For the diagrams of Figs. (1) and (2) the \( D \)-algebra and \( \theta \)-integrations are straightforward, if tedious, and we present only the final \( x \)-space expressions to be renormalized:

\[
\kappa_1(x-x') = \frac{1}{2} \left( \frac{g}{4\pi^2} \right)^2 \frac{1}{(x-x')^4} 
\]

\[
\kappa_2(x-x') = \frac{1}{2} \left( \frac{g}{4\pi^2} \right)^4 \frac{1}{(x-x')^2} \int d^4x_1 \frac{1}{(x-x_1)^2} \frac{1}{(x_1-x')^2} 
\]

\[
\kappa_{3a}(x-x') = \frac{1}{8} \left( \frac{g}{4\pi^2} \right)^6 \left( \int d^4x_1 \frac{1}{(x-x_1)^4} \frac{1}{(x_1-x')^2} \right)^2 
\]

\[
\kappa_{3b}(x-x') = \frac{1}{2} \left( \frac{g}{4\pi^2} \right)^6 \frac{1}{(x-x')^2} \times 
\int d^4x_1 \int d^4x_2 \frac{1}{(x-x_1)^4} \frac{1}{(x_1-x_2)^2} \frac{1}{(x_2-x')^2} 
\]

\[
\kappa_{3c}(x-x') = \frac{1}{4} \left( \frac{g}{4\pi^2} \right)^6 \frac{1}{(x-x')^2} \times 
\int d^4x_1 \int d^4x_2 \frac{1}{(x-x_1)^2} \frac{1}{(x_1-x_2)^4} \frac{1}{(x_2-x')^4} 
\]

\[
\kappa_{3d}(x-x') = \frac{1}{2} \left( \frac{g}{4\pi^2} \right)^6 \frac{1}{(x-x')^2} \times 
\int d^4x_1 \int d^4x_2 \frac{1}{(x-x_1)^2} \frac{1}{(x_1-x_2)^2} \frac{1}{(x_2-x')^2} \frac{1}{(x-x_2)^2} 
\]

\[
= \frac{1}{2} \left( \frac{g}{4\pi^2} \right)^6 6\pi^4 \zeta(3) \frac{1}{(x-x')^4} 
\]

This last (finite) integral was done with the aid of the Gegenbauer polynomial technique [1,7]. Now, in the course of performing the above integrals, we use the
following differential renormalization identities[1] whenever singular terms appear (i.e., terms like the l.h.s. below):

\[
\frac{1}{x^4} = -\frac{1}{4} \Box \frac{\ln x^2 M^2}{x^2}, \quad x \neq 0 \tag{15}
\]

\[
\frac{\ln x^2 M^2}{x^4} = -\frac{1}{8} \frac{\ln^2 x^2 M^2 + 2 \ln x^2 M^2}{x^2}, \quad x \neq 0 \tag{16}
\]

\[
\frac{\ln^2 x^2 M^2}{x^4} = -\frac{1}{12} \frac{\ln^3 x^2 M^2 + 3 \ln^2 x^2 M^2 + 6 \ln x^2 M^2}{x^2}, \quad x \neq 0. \tag{17}
\]

With this, we are able to calculate the renormalized values of \(\kappa_i\):

\[
\kappa_1^{\text{ren}}(x) = -2g^2 \left( \frac{1}{4\pi} \right)^4 \Box \frac{\ln x^2 M^2}{x^2} \tag{18}
\]

\[
\kappa_2^{\text{ren}}(x) = g^4 \left( \frac{1}{4\pi} \right)^6 \Box \frac{\ln^2 x^2 M^2 + 2 \ln x^2 M^2}{x^2} \tag{19}
\]

\[
\kappa_3^{\text{ren}}_{\text{a}}(x) = -\frac{1}{6} g^6 \left( \frac{1}{4\pi} \right)^8 \Box \frac{\ln^3 x^2 M^2 + 3 \ln^2 x^2 M^2 + 6 \ln x^2 M^2}{x^2} \tag{20}
\]

\[
\kappa_3^{\text{ren}}_{\text{b}}(x) = -g^6 \left( \frac{1}{4\pi} \right)^8 \Box \frac{\ln^3 x^2 M^2 + 2 \ln^2 x^2 M^2 + 4 \ln x^2 M^2}{x^2} \tag{21}
\]

\[
\kappa_3^{\text{ren}}_{\text{c}}(x) = -\frac{1}{3} g^6 \left( \frac{1}{4\pi} \right)^8 \Box \frac{\ln^3 x^2 M^2 + 3 \ln^2 x^2 M^2 + 6 \ln x^2 M^2}{x^2} \tag{22}
\]

\[
\kappa_3^{\text{ren}}_{\text{d}}(x) = -12 \zeta(3) g^6 \left( \frac{1}{4\pi} \right)^8 \Box \frac{\ln x^2 M^2}{x^2}, \tag{23}
\]

where we have set \(x' = 0\) for simplicity. To calculate the \(\beta\)-function and verify the consistency of the method we apply the Callan-Symanzik equation, Eq.(4), to \(G^{(2)}_{\text{ren}} = \delta^{(4)} + \kappa_1^{\text{ren}} + \kappa_2^{\text{ren}} + \kappa_3^{\text{ren}}\). We write

\[
\beta(g) = \beta_1 g^3 + \beta_2 g^5 + \beta_3 g^7
\]

and separate the pieces in the Callan-Symanzik equation according to order in \(g^2\) and the coefficients of the \(\delta\)-function and the different powers of log. All of these have to vanish separately. At \(O(g^2)\), we get the one-loop \(\beta\)-function:

\[
\beta_1 g^3 = \frac{3}{2} \left( \frac{1}{4\pi} \right)^2 g^3. \tag{24}
\]
At $O(g^4)$, the coefficient of $\delta(x)$ vanishes for the following value of the two-loop $\beta$-function:

$$\beta_2 g^5 = -\frac{3}{2} \left( \frac{1}{4\pi} \right)^4 g^5,$$

(25)

and the coefficient of $\Box \ln \frac{x^2 M^2}{2}$ vanishes for the above value of $\beta_1$; this is a consistency check of the method. At $O(g^6)$, the vanishing of the coefficient of $\delta(x)$ gives the following three-loop $\beta$-function:

$$\beta_3 g^7 = \left( \frac{21}{4} + 9 \zeta(3) \right) \left( \frac{1}{4\pi} \right)^6 g^7.$$

(26)

Finally, the coefficients of $\Box \ln \frac{x^2 M^2}{2}$ and $\Box \ln^2 \frac{x^2 M^2}{2}$ will vanish for the above values of $\beta_1$ and $\beta_2$, and these are then two further consistency checks. Our final result for the $\beta$-function is then:

$$\beta(g) = g \left[ \frac{3}{2} \left( \frac{g}{4\pi} \right)^2 - \frac{3}{2} \left( \frac{g}{4\pi} \right)^4 + \left( \frac{21}{4} + 9 \zeta(3) \right) \left( \frac{g}{4\pi} \right)^6 \right].$$

(27)

We note that although the three-loop part is scheme-dependent, as opposed to the one- and two-loop results, the coefficient of $\zeta(3)$ should be universal (barring, of course, a coupling constant redefinition involving $\zeta(3)$) because the only diagram at three loops leading to the transcendental function $\zeta(3)$ is primitively divergent. While our coefficient of $\zeta(3)$ coincides with that obtained in [9], it does not agree with [7]. The correct result is that of [9], however[10]; in fact, because that is a four loop calculation, it is possible to verify the three-loop result by RG pole equations (which the authors of [9] in fact do).

Conclusions - We have seen that the method of differential renormalization is readily applicable to the superspace Wess-Zumino model and, in particular, the $\beta$-function calculations were considerably simpler than in SRDR [7,9]. Firstly, the integrations performed here were either trivial (integrations with $\delta$-functions) or very simple convolutions; secondly, we did not have to keep track of subtraction of subdivergences: at each loop order the renormalization is done in a single step, by the direct use of “differential renormalization identities” like Eqs.(12)-(14) whenever singular expressions are encountered (in SRDR, on the other hand (cf. Eq.(2.11) of [7]), a careful account needs to be kept of different renormalization constants corresponding to contributions
of subdivergences at higher loops).

We have attempted to describe the differential renormalization of the Wess-Zumino model in such a way that the extension to other superspace theories becomes obvious: one should again start with the general form for the effective action, Eq.(2), with the $G^{(N)}$ written in terms of $x$-space integrals, and apply differential renormalization identities to these integrals, thus regularizing and renormalizing the theory. Differential renormalization will take translation-invariant quantities $G^{(N)}$ into translation-invariant quantities $G^{(N)}_{\text{ren}}$, and will manifestly maintain supersymmetry. Naturally, we expect this procedure to be far simpler than several other existing regularization methods (e.g., the supersymmetric versions of Pauli-Villars, higher derivative and point-splitting methods). Furthermore, if the method proves to be as simple to implement in general as SRDR, it would have the advantage that the ambiguities associated with SRDR would be avoided: like for dimensional regularization, SRDR presents ambiguities due to the presence of intrinsically four-dimensional quantities, like $\epsilon^{\mu\nu\rho\sigma}$, in a dimensionally continued setting [5,11]. Differential renormalization, of course, never leaves four dimensions and would thus avoid these problems altogether. One initial indication of the further applicability of the method is the fact that gauge (vector) superfield propagators are identical to the massless chiral propagators we have considered here. Thus, calculations with vector superfields should not in principle present any new difficulty; for instance, the one-loop correction to the vector propagator in supersymmetric Yang-Mills theory coupled to a massless chiral superfield is again as simple to compute as in SRDR. Further work along these lines is in progress.

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References

[1] Freedman, D.Z., K. Johnson and J.I. Latorre, to be published in Nucl. Phys. B.
[2] Freedman, D.Z., Proceedings of the Stony Brook Conference on Strings and Symmetries, Spring 1991.
[3] Freedman, D.Z., R. Muñoz-Tapia, X. Vilasís-Cardona, manuscript in preparation.
[4] Grisaru, M.T., M. Roček and W. Siegel, Nucl. Phys. B159(1979)429.
[5] Gates, J., M.T. Grisaru, M. Roček and W. Siegel, Superspace, Benjamin-Cummings, 1983.
[6] Iliopoulos, J. and B. Zumino, Nucl. Phys. B76(1974)310.
     Ferrara, S., J. Iliopoulos and B. Zumino, Nucl. Phys. B77(1974)413.
Figures

Fig.(1): One- and two-loop contributions to \( G^{(2)}(x - x') \).

Fig.(2): Three-loop contributions to \( G^{(2)}(x - x') \).