POISSON SEMIGROUPS FOR THE MODIFIED BESSEL OPERATOR, WITH MORSE POTENTIAL AND ON THE HYPERBOLIC SPACE

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Abstract

In this paper we find explicit formulas for the Poisson semigroups associated to the modified Bessel operator, the Schrödinger operator with the Morse potential on $\mathbb{R}$ and on the hyperbolic spaces $\mathbb{H}^n$.

1 Introduction

In the last decades the Poisson semigroups for many second differential operators have been studied and computed explicitly and there is many interesting papers published in this area of reaserch. The differential operators of Bessel type, the Schrödinger operator with Morse Potential and the Laplace-Beltrami operator on the hyperbolic space are known as very important operators in analysis and its applications. For recent works on Bessel operator (see [2, 3, 7, 14]). The Schrödinger operator with Morse potential is considered more recently by [10, 11, 4, 5, 13]. This paper deals with the Poisson semigroups associated to these second order differential operators.
The main objective of this paper is to solve explicitly the following three Poisson problems

\[
\begin{cases}
L^a u(y, x) = -\frac{\partial^2}{\partial y^2} u(y, x), (y, x) \in \mathbb{R}^+ \times \mathbb{R} \\
u(0, x) = u_0(x), u_0 \in C_0^\infty(\mathbb{R}^+) 
\end{cases}
\tag{1.1}
\]

\[
\begin{cases}
M^a U(y, X) = -\frac{\partial^2}{\partial y^2} U(y, X), (y, X) \in \mathbb{R}^+ \times \mathbb{R} \\
u(0, X) = U_0(X), U_0 \in C_0^\infty(\mathbb{R}^+) 
\end{cases}
\tag{1.2}
\]

and

\[
\begin{cases}
\mathcal{L}_n v(y, w) = -\frac{\partial^2}{\partial y^2} v(y, w), (y, w) \in \mathbb{R}^+ \times \mathbb{H}^n \\
v(0, w) = v_0(w), v_0 \in C_0^\infty(\mathbb{H}^n) 
\end{cases}
\tag{1.3}
\]

where

\[
L^a = x^2 \frac{\partial^2}{\partial x^2} + x \frac{\partial}{\partial x} - a^2 x^2
\tag{1.4}
\]

\[
M^a = \frac{\partial^2}{\partial X^2} - a^2 \exp 2X
\tag{1.5}
\]

and

\[
\mathcal{L}_n = x_n^2 \Delta_{n-1} + x_n^2 \frac{\partial^2}{\partial x_n^2} + (2 - n)x_n \frac{\partial}{\partial x_n} + \frac{(n - 1)^2}{4}
\tag{1.6}
\]

are respectively the Bessel operator on \(\mathbb{R}^+\), the Schrödinger operator with the Morse potential on \(\mathbb{R}\) and the Laplace-Beltrami operator on the half space model of the hyperbolic space \(\mathbb{H}^n\).

\section{Poisson semigroup associated to Bessel operator}

In this section we give explicit formulas for the Poisson semigroup associated to the Bessel operator and for this we prove the following theorem.
Theorem 2.1. If \( a \in \mathbb{R}^n \) the problem (2.1) has the unique solution given by

\[
 u(y, x) = \int_0^\infty p_a(y, x, x')u_0(x') \frac{dx'}{x'}
\]  

with

\[
p_a(y, x, x') = \frac{|a| x' \sin y K_1 \left( |a| \sqrt{x^2 + x'^2 - 2xx' \cos y} \right)}{\pi \sqrt{x^2 + x'^2 - 2xx' \cos y}}
\]

and \( K_1 \) is the modified Bessel functions of second kind.

Proof. To see that the function \( u(y, x) \) satisfies the partial differential equation in (2.1) set \( \varphi(y, x, x') = \phi(z) \) with \( z = x^2 + x'^2 - 2xx' \cos y \), then we have

\[
\frac{\partial \varphi}{\partial x} = (2x - 2x' \cos y) \frac{\partial \phi}{\partial z}, \quad \frac{\partial^2 \varphi}{\partial x^2} = (2x - 2x' \cos y)^2 \frac{\partial^2 \phi}{\partial z^2} + 2 \frac{\partial \phi}{\partial z}
\]

\[
\frac{\partial \varphi}{\partial y} = 2xx' \sin y \frac{\partial \phi}{\partial z}, \quad \frac{\partial^2 \varphi}{\partial y^2} = (2xx' \sin y)^2 \frac{\partial^2 \phi}{\partial z^2} + 2xx' \cos y \frac{\partial \phi}{\partial z}
\]

\[
\left( L^a + \frac{\partial^2}{\partial y^2} \right) \varphi = 4x^2 \left( z \frac{\partial^2 \phi}{\partial z^2} + \frac{\partial \phi}{\partial z} - \frac{a^2}{4} \phi \right)
\]

and we see that the first equation in the problem (2.1) is equivalent to

\[
z^2 \phi_{zz} + z \phi_z - \frac{a^2}{4} z \phi = 0.
\]

This is a particular case of Lommel differential equation for modified Bessel functions

\[
\left[ z^2 \frac{\partial^2 \phi}{\partial z^2} + (1 - 2\alpha)z \frac{\partial \phi}{\partial z} - (\beta \gamma z) \phi \right] + (\alpha^2 - \nu^2 \gamma^2) \phi = 0
\]

with \( \alpha = 0, \nu = 0, \beta = 1 \) and \( \gamma = 1/2 \), and an appropriate solution is \( \phi(z) = cK_0(z^{1/2}) \) where \( K_0 \) is the modified Bessel function of second kind.
That is \( \phi(z) = cK_0\left(|a|\sqrt{x^2 + x'^2 - 2xx'\cos y}\right) \) is a solution of the equation (2.1). Using the formula \( K'_0(z) = -K_1(z) \) it is clear that
\[
p_a(y, x, x') = -\frac{1}{\pi} \frac{\partial}{\partial y} K_0\left(|a|\sqrt{x^2 + x'^2 - 2xx'\cos y}\right)
\]
also satisfies the Poisson equation in (2.1). To finish the proof of Theorem 2.1 it remain to show the limit conditions. For this set \( z = x^2 + x'^2 - 2xx'\cos y = 2xx'(\frac{x^2 + x'^2}{2xx'} - \cos y) \) and \( x = e^X \) and \( x' = e^{X'} \) to obtain \( z = 4e^{X+X'}\{\sinh^2\left(\frac{X-X'}{2}\right) + \sin^2(y/2)\} \) in (2.1) and we can write

\[
U(y, X) = \int_{0}^{\infty} P_a(y, X, X')U_0(X')dX'
\]

with
\[
P_a(y, X, X') = \frac{|a|e^{(X+X')/2}}{\pi} \sin yK_1\left(|a|\sqrt{\sinh^2\left(\frac{X-X'}{2}\right) + \sin^2(y/2)}\right) \left(\frac{\sqrt{1 + s^2}}{\sqrt{1 + s^2 \sin^2(y/2)}}\right) U_0(X + arg \sinh(s \sin y/2))2ds.
\]

Inserting this in the formula (2.8) and setting \( \sinh\left(\frac{X'-X}{2}\right) = s \sin y/2 \) or \( X' = X + 2\arg \sinh(s \sin y/2) \) we can write

\[
u(y, x) = U(y, X) = \frac{|a|}{\pi} \int_{-\infty}^{\infty} e^{X+\arg \sinh(s \sin y/2)} \sin y \frac{K_1\left(2|a|e^{X+\arg \sinh(s \sin y/2)} \sin(y/2)\sqrt{1 + s^2}\right)}{\sqrt{1 + s^2 \sin^2(y/2)}} U_0(X + \arg \sinh(s \sin y/2))2ds.
\]

Now we use the asymptotic formula for the modified Bessel function of second kind see Lebedev [8] p.136 \( K_v(z) \sim \frac{2^{v-1}\Gamma(v)}{z^v}, z \to 0 \) to obtain

\[
\lim_{y \to 0} u(y, x) = U(y, X) = U_0(X) \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{ds}{1 + s^2} = U_0(X) = u_0(x)
\]
and this finishes the proof of Theorem 2.1.
3 Poisson equation with Morse Potential

The main objective of this section is first to solve the Poisson problems (1.2) associated to the Schrödinger operator with Morse potential on the real line \( \mathbb{R} \).

**Theorem 3.1.** For a real number the problem (1.2) has the unique solution given by

\[
U(y, X) = \int_{0}^{\infty} P_a(y, X, X') U_0(X') dX' 
\]

with

\[
P_a(y, X, X') = \frac{|a| e^{X+X'} \sin y K_1\left(|a| \sqrt{\sinh^2 \frac{(X-X')}{2} + \sin^2 y/2}\right)}{\pi \sqrt{\sinh^2 \frac{(X-X')}{2} + \sin^2 y/2}} \tag{3.2}
\]

where \( K_1 \) is the modified Bessel functions of second kind.

**Proof.** Set \( X = \ln x \) the problem (1.2) is transformed into the problem (2.1) and it is not hard to see the result of theorem (2.1).

4 Poisson equation on the hyperbolic space

In this section we consider the Poisson equation on the hyperbolic space modelled by the upper half space: \( \mathbb{H}^n = \{ z = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n, x_n > 0 \} \) endowed with the usual hyperbolic metric

\[
\tilde{ds}^2 = x_n^{-2}[dx_1^2 + dx_1^2 + dx_2^2 + \ldots + dx_n^2] \tag{4.1}
\]

the metric \( \tilde{ds} \) is invariant with respect to the group \( G = SO(n, 1) \) with the hyperbolic surface form \( \tilde{d\mu}(z) \)

\[
\tilde{d\mu}(z) = \frac{1}{x_n^n} dx_1 dx_2 \ldots dx_n, \tag{4.2}
\]
and the hyperbolic distance $\rho(z, z')$ given respectively by
\[
\cosh^2(\rho(z, z')/2) = \frac{|z - z'|^2 + (x_n + x'_n)^2}{4x_n x'_n},
\]
(4.3)
with the Laplace-Beltrami operator
\[
\mathcal{L}_n = x_n^2 \Delta_n + (2 - n) \frac{\partial}{\partial x_n} + ((n - 1)/2)^2
\]
(4.4)
where
\[
\Delta_n = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}
\]
(4.5)

Before giving the main results of this section we prove the following lemma giving the Fourier transform of the Poisson semigroup for the Bessel operator with respect to the parameter $|\xi|

**Lemma 4.1.** Let $p_{|\xi|}(y, x, x_n')$ is kernel of the Poisson semigroup for Bessel operator given in (2.2)
\[
[Fp_{|\xi|}(y, x_n, x'_n)](x) = \frac{2^{(n-1)/2}\Gamma((n + 1)/2)}{\pi} \frac{x_n x'_n \sin y}{(z^2 + |x|^2)^{(n+1)/2}},
\]
(4.6)

**Proof.**
\[
[F^{-1}p_{|\xi|}(y, x_n, x'_n)](x) = \frac{x_n x'_n \sin y}{\pi z} |x|^{3-n}/2
\]
\[
\int_0^{+\infty} K_1(r|z) J_{n-3/2}(r|x|) r^{n-1} dr
\]
(4.7)
\[
\int_0^{+\infty} K_1(r|z) J_{n-3/2}(r|x|) r^{n+1} dr
\]
(4.8)
\[
\int_0^{+\infty} K_1(r|z) J_{n-3/2}(r|x|) r^{n+1} dr
\]
(4.9)

using formula\[11\] p.365
\[
\int_0^{+\infty} x^{\alpha-1} J_\mu(bx) K_\nu(cx) dx = A^{\alpha}_{\mu,\nu}
\]
(4.10)
\begin{align}
A_{\mu,\nu}^\alpha &= 2^{\alpha-2}b^\mu c^{-(\alpha+\mu)} \Gamma((\alpha + \mu + \nu)/2) \Gamma((\alpha + \mu - \nu)/2) \\
2F_1((\alpha + \mu + \nu)/2, (\alpha + \mu - \nu)/2, \mu + 1, -\frac{b^2}{c^2}) \tag{4.11}
\end{align}

\begin{align}
\alpha &= (n + 3)/2, \mu = (n - 3)/2, \nu = 1, b = |x|, c = z \\
\int_0^{+\infty} K_1(r|z)J_{\frac{n-3}{2}}(r|x|)r^{\frac{n+1}{2}} dr = A^{(n+3)/2}_{1,(n-3)/2} \tag{4.12}
\end{align}

\begin{align}
A^{(n+3)/2}_{1,(n-3)/2} &= \frac{|x|^{(3-n)/2}}{\pi z} 2^{(n-1)/2} \frac{|x|^{(n-3)/2}}{z^n} \Gamma((n + 1)/2) \tag{4.13} \\
F \left( ((n + 1)/2, (n - 1)/2, (n - 1)/2, -\frac{|x|^2}{z^2}) \right). \tag{4.14}
\end{align}

Now with the formula $F(a, b, b, z) = (1 - z)^{-a}$ we obtain the result. \qed

**Theorem 4.1.** The Poisson equation in the hyperbolic space has the unique solution

\begin{align}
v(y, w) &= \int_0^\infty p'_n(y, w, w') v_0(w') d\mu(z') \tag{4.15}
\end{align}

with

\begin{align}
p'_n(y, w, w') &= \frac{\Gamma((n + 1)/2)}{\pi^{(n+1)/2}} \frac{\sin y}{(2 \cosh d(w, w') - 2 \cos y)^{(n+1)/2}} \tag{4.16}
\end{align}

It is not hard to see that

\begin{align}
x_n^{-(n-1)/2} L_n x_n^{-(n-1)/2} = x_n^2 \frac{\partial^2}{\partial x_n^2} + x_n \frac{\partial}{\partial x_n} - x_n^2 \Delta_{n-1}. \tag{4.17}
\end{align}

Set $a = |\xi|$ where $\xi = (\xi_1, \xi_2, \ldots, \xi_{n-1}) \in \mathbb{R}^{n-1}$, and set $u(y, x) = x_n^{(n-1)/2} \phi(y, x)$ the problem (4.18) is transformed into

\begin{align}
\left\{ \begin{array}{l}
\mathcal{F} x_n^{-(n-1)/2} L_n x_n^{-(n-1)/2} \phi \left( \xi_1, \ldots, \xi_{n-1} \right) = -\frac{\partial^2}{\partial y^2} \mathcal{F} \phi \left( y, \xi \right), (y, w) \in \mathbb{R}^+ \times \mathbb{H}^n \\
\mathcal{F} \phi \left( 0, \xi \right) = x^{-1/2} \mathcal{F} u_0 (\xi), u_0 \in C_0^\infty (\mathbb{H}^n) \end{array} \right. \tag{4.18}
\end{align}
that is
\[
\begin{cases}
L^{[\xi]}_{x_n} [\mathcal{F} \phi] (\xi_1, \ldots, \xi_{n-1}) = -\frac{\partial^2}{\partial y^2} [\mathcal{F} \phi] (y, \xi), (y, w) \in \mathbb{R}^+ \times \mathbb{H}^n \\
[\mathcal{F} \phi] (0, \xi) = x^{(1-n)/2} [\mathcal{F} u_0] (\xi), u_0 \in C_0^\infty (\mathbb{H}^n)
\end{cases}
\] (4.19)

where \( L^{[\xi]}_{x_n} \) is the Bessel operator (1.4).

Using Theorem 2.1 we have using the above formulas the Poisson problem in hyperbolic space is transformed into the Bessel Poisson problem (2.1) with 

\[
u (y, x_n) = x^{(n-1)/2} [\mathcal{F} \phi] (0, \xi) = x^{(1-n)/2} \mathcal{F} \phi (y, x_n) \] (4.20)

with

\[
p^{[\xi]} (y, x_n, x'_n) = \frac{|\xi|}{\pi} x_{x_n} x'_{x_n} \sin y K_1 \left( |\xi| \sqrt{x^2_n + x'^2_n - 2x_n x'_n \cos y} \right) \] (4.21)

\[
\phi (y, x_n) (|x|) = \int_0^\infty [\mathcal{F}^{-1} p^{[\xi]} (y, x_n, x'_n)] (x) * u_0 (x, x'_n) x^{(1-n)/2} x'_n \frac{dx'_n}{x'_n} \] (4.22)

\[
\phi (y, x_n) (|x|) = (2\pi)^{-(n-1)/2} \times
\int_0^\infty [\mathcal{F}^{-1} p^{[\xi]} (y, x_n, x'_n)] (x) * u_0 (x, x'_n) x^{(1-n)/2} x'_n \frac{dx'_n}{x'_n} \] (4.23)

\[
u (y, w) = \int_{\mathbb{H}^n} P^{H} (y, w') u_0 (w') d\mu (w'), \] (4.24)

with

\[
P^{H} (y, w') = \frac{\Gamma \left( (n+1)/2 \right)}{\pi^{(n+1)/2}} \frac{\sin y}{(2 \cosh d (w, w') - 2 \cos y)^{(n+1)/2}}. \] (4.25)
5 Applications

In this section we give some applications of our results. The sphere $S^n$ the
Poisson semigroup on $S^n$

**Corollary 5.1.** The Poisson equation in the sphere $S^n$ has the unique solution given by

$$u(y, \omega) = \int_{S^n} P_n^S(y, \omega, \omega') u_0(\omega')d\mu(\omega') \quad (5.1)$$

with

$$P_n^S(y, \omega, \omega') = \frac{\Gamma((n+1)/2)}{\pi^{(n+1)/2}} \frac{\sin y}{(2 \cos d(\omega, \omega') - 2 \cos y)^{(n+1)/2}} \quad (5.2)$$

This corollary can be seen from the theorem 4.18 by an argument of analytic continuation. Note that the result of the corollary agrees with the formula (4.9) of [12] p. 114.

Note that the wave equation on hyperbolic space $IH^n$ is studied in [6].

**Corollary 5.2.** If $a \in i\mathbb{R}^*$ and $a = ib$ the problem (2.1) has the unique solution given by

$$v(y, x) = \int_0^\infty q_b(y, x, x')v_0(x')dx' \quad (5.3)$$

with

$$q_b(y, x, x') = -|b|x' \sin y H^{(1)}_1 \left( \frac{|b|\sqrt{x^2 + x'^2 - 2xx'\cos y}}{\sqrt{x^2 + x'^2 - 2xx'\cos y}} \right) \quad (5.4)$$

$$q_b(y, x, x') = \frac{\partial}{\partial y} H^{(1)}_0 \left( |b|\sqrt{x^2 + x'^2 - 2xx'\cos y} \right) \quad (5.5)$$

with $H^{(1)}_1, H^{(1)}_0$ are the Bessel function of the third kind.

Proof. \hfill \Box
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