Existence and uniqueness for a class of nonlinear population diffusion system

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ABSTRACT
In this paper, we introduce a class of nonlinear population diffusion system. Existence and uniqueness of strong solution are proved for stochastic age-dependent population diffusion system in Hilbert space by use of Gronwall’s lemma and Burkholder–Davis–Gundy’s inequality.

1. Introduction
Stochastic differential equations has been more and more important in many fields, such as economics, biology, finance, ecology and other sciences (Caraballo & Liu, 1999; Chen, 2010, 2012; DaPrato & Zabczyk, 2014; Liu, 2005). There has been many interests in application of age-dependent population systems in recent years. For example, Chou and Greenman (2016) and Assas, Dennis, Elaydi, Kwessi, and Livadiotis (2015) investigated hierarchical age-dependent populations with intra-specific competition and predation. Zhang (2008) considered the exponential stability of numerical solutions of the age-dependent population with diffusion in. Hernández-Cerón, Feng, and van den Driessche (2013) and Brauer (1999) provided thorough theoretical foundations for discrete and continuous-time age-dependent models. The optimal control and harvesting of the age-structured population system was investigated by Fister and Lenhart (2004). Pollard (1966), Block and Allen (2000) studied the effects of adding stochastic terms to discrete-time age-dependent models that employed Leslie matrices. In Rathinasamy (2012), a class of split-step θ-method was proposed, and it is proved that the split-step θ-method converge to the analytical solutions of the equations under given conditions.

In this paper, we will focus on a class stochastic continuous time age-dependent model with diffusion. Actually, in Gurtnme (1973), the determined nonlinear age-dependent population dynamic with diffusion can be written in the following form.

$$\frac{\partial P}{\partial t} + \frac{\partial P}{\partial r} - k(r,t) \Delta P + \mu_0(r,t,x)P + \mu_e(r,t,x;P)P = f(r,t,x;P) \text{ in } Q,$$

$$P(0,t,x) = \int_0^\Lambda \beta(r,t,x;P) dr \text{ in } \Omega_\tau = [0,T] \times \Omega,$$

$$P(r,0,x) = P_0(r,x) \text{ in } \Omega_A = [0,A] \times \Omega,$$

$$P(r,t,x) = 0 \text{ in } \sum = [0,A] \times [0,T] \times \partial \Omega,$$

$$Y(t,x) = \int_0^\Lambda P(r,t,x) dr \text{ in } \Omega_r.$$

where $Q = (0,A) \times (0,T), r \in [0,A]$ denotes the age, $t \in [0,T]$ denotes the time, $0 < t < +\infty, x \in \Omega$ denotes the position variables in space, $\Omega \subset \mathbb{R}^N$ denotes a bounded region with smooth bounder $\partial \Omega; P = P(r,t,x) \geq 0$ denotes the age-density function of population at time $t$ age $r$ and position $x, Y(t,x)$ denotes the density function of population during $[0,A]$ at time $t$ and position $x, k(r,t) \geq 0$ is the coefficient of population diffusion; $\mu_0(r,t,x) \geq 0$ is the natural death rate function of population, $\mu_e(r,t,x;P) \geq 0$ is the extra death rate function of population, for example, misfortune death etc. $\Delta$ denotes the Laplace operator with respect to the space variable. $f(r,t,x;P)$ is the exterior disturbing function, as migration; $\beta(r,t,x;P)$ is the birth rate of population; $P_0(r,x) \geq 0$ is the initial distribution of age-density of population at time $t; A > 0$ is the top age that an individual can survive in population.
Suppose that \( f(r,t,x;P) \) is stochastically perturbed, with
\[
f(r,t,x;P) \to f(r,t,x;P) + g(r,t,x;P)\omega_t,
\]
where \( \omega_t = d\omega_t/dt \) is the white noise. Then this environmentally perturbed system can be described by the stochastic partial differential equation as follows:
\[
\frac{\partial P}{\partial t} dt - \frac{\partial P}{\partial r} dt + k(r,t)\Delta P dt
- \mu_0(r,t,x)P dt - \mu_e(r,t,x;P)\omega_t dt
+ f(r,t,x;P)\omega_t dt + g(r,t,x;P)\omega_t dt
\]
in \( Q \),
\[
P(0,r,x) = \int_0^A \beta(r,t,x;P)P(t,r,x) dt
\]
in \( \Omega_r = [0,T] \times \Omega_r \),
\[
P(r,0,x) = P_0(r,x)
\]
in \( \Omega_A = [0,A] \times \Omega_r \)
\[
\omega_t = \sum_{i=1}^{\infty} \beta_i(t)e_i,
\]
where \( \{e_i\}_{i \geq 0} \) is an orthonormal set of eigenvectors of \( \omega \), \( \beta_i(t) \) are mutually independent real Wiener processes with incremental covariance \( \lambda_i > 0 \), \( W_e = \lambda_i e_i \) and \( \text{tr}W = \sum_{i=1}^{\infty} \lambda_i \) (tr denotes the trace of an operator (Pardoux, 1975)). For an operator \( B \in C(K,H) \) to be the space of all bounded linear operators from \( K \) into \( H \), we denote by \( ||B||_2 \) the Hilbert–Schmidt norm, i.e.,
\[
||B||_2^2 = \text{tr}(BW^*B^*).
\]

In this paper, \( \omega_t \) is a real standard Wiener process. Let \( C = C([0,T],H) \) be the space of all continuous function from \([0,T] \) into \( H \) with sup-norm \( \|\cdot\|_C = \sup_{0 \leq t \leq T} |\psi(t)|_V \), \( L^p \|_V \) is a real standard Wiener process. Let
\[
P_t = \int_0^t \beta(r,s,x)P_s dt
\]
in \( Q \),
\[
P(0,r,x) = \int_0^A \beta(r,t,x;P)P(t,r,x) dt
\]
in \( \Omega_r = [0,T] \times \Omega_r \),
\[
P(0,0,x) = P_0(r,x)
\]
in \( \Omega_A = [0,A] \times \Omega_r \).

The rest of this paper is organized as follows. In Section 2, many definitions and probationary knowledge are given to be ready for our main results. In Section 3, the existence and uniqueness of strong solution are proven by using of Gronwall’s lemma and Burkholder–Davis–Gundy’s inequality. Conclusion will be proposed in Section 4.

### 2. Preliminaries

Let \( O = [0,A] \times \Omega_r \), and
\[
V = \left\{ \varphi \mid \varphi \in L^2(O), \frac{\partial \varphi}{\partial x_i} \in L^2(O), \right\}
\]
where \( \frac{\partial \varphi}{\partial x_i} \) is generalized partial derivatives.

\( V \) is a Sobolev space. \( H = L^2(O) \) such that
\[
V \hookrightarrow H \hookrightarrow H' \hookrightarrow V'.
\]
\( V' \) is the dual space of \( V \). We denote by \( \|\cdot\|_V \) and \( \|\cdot\|_H \)\) the norms in \( V, H \) and \( V' \) respectively; by \( \langle \cdot, \cdot \rangle \) the inner product; \( \langle \cdot, \cdot \rangle \) and \( \langle \cdot, \cdot \rangle \) the duality product between \( V, V' \), and by \( \langle \cdot, \cdot \rangle \) the scalar product in \( H \), and there exists a constant \( m \) such that
\[
|\varphi| \leq m\|\varphi\|_V, \quad \forall \varphi \in V.
\]

Let \( \omega_t \) be a Wiener process defined on complete probability space \( (\Xi, \mathcal{F}, P) \) and taking its values in the separable Hilbert space \( K \), with increment covariance operator \( W \). Let \( (\mathcal{F}_t)_{t \geq 0} \) be the \( \sigma \)-algebras generated by \( \{\omega_s \mid 0 \leq s \leq t\} \), then \( \omega_t \) is a martingale relative to \( (\mathcal{F}_t)_{t \geq 0} \) and we have the following representation of \( \omega_t \):
\[
\omega_t = \sum_{i=1}^{\infty} \beta_i(t)e_i,
\]
where \( \{e_i\}_{i \geq 0} \) is an orthonormal set of eigenvectors of \( \omega \), \( \beta_i(t) \) are mutually independent real Wiener processes with incremental covariance \( \lambda_i > 0 \), \( W_e = \lambda_i e_i \) and \( \text{tr}W = \sum_{i=1}^{\infty} \lambda_i \) (tr denotes the trace of an operator (Pardoux, 1975)). For an operator \( B \in C(K,H) \) to be the space of all bounded linear operators from \( K \) into \( H \), we denote by \( ||B||_2 \) the Hilbert–Schmidt norm, i.e.,
\[
||B||_2^2 = \text{tr}(BW^*B^*).
\]
(3) Equation (3) is satisfied for every $t \in [0, T]$ with probability one.

Let $T$ be replaced by $\infty$, $P_t$ is called a global strong solution of (3). The objective in this paper is that, we hopefully find a unique process $P_t \in P^0(0, T; V) \cap L^2(\Omega; C(0, T; H))$ such that (3) hold. For this objective, we assume the following conditions are satisfied:

(a.1) $f(r, t, 0) = 0, g(r, t, x, 0) = 0$;
(a.2) (Lipschitz condition) There exists a positive constant $K$ such that

$$
|f(r, t, x; p_1) - f(r, t, x; p_2)| + |g(r, t, x; p_1) - g(r, t, x; p_2)| \leq K|p_1 - p_2|,
$$

for all $p_1, p_2 \in C(0, T; H)$ a.e.

(a.3) $\mu(r, t, x)$ and $\beta(r, t, x)$ are continuous in $Q$, and there exists a positive constants $\bar{\mu}, \bar{\beta}, \bar{k}$ such that

$$
0 \leq \mu_x \leq \bar{\mu}, \quad 0 \leq \beta \leq \bar{\beta}, \quad k_0 \leq k(r, t) \leq \bar{k}
$$

We consider the equations

$$
P_t^1 = P_0 + \int_0^t \left[ -\frac{\partial P_s^1}{\partial r} - \frac{A\beta^2_s}{2} P_s^1 \right] ds, \quad t \in [0, T],
$$

$$
P_t^1(0, t, x) = \int_0^t \beta(r, t, r) P_{t_s}^1 dr, \quad t \in [0, T] \tag{4}
$$

$$
P_{t_{n+1}} = P_0 + \int_0^t \left[ -\frac{\partial P_{t_{n+1}}}{\partial r} - \frac{A\beta^2_{t_s}}{2} P_{t_s}^n \right] ds + \int_0^t k d\Delta_{t_s} ds
$$

$$
- \int_0^t \mu_0(r, s, x) P_s^0 d - \int_0^t \mu_0(r, s, x) P_s^0 ds d - \int_0^t g(r, s, x; P_s^n) d + \int_0^t g(r, s, P_s^n) d_{\omega_s},
$$

$$
t \in [0, T], \quad \forall n \geq 1,
$$

$$
P_{t_{n+1}}(0, t, x) = \int_0^t \beta(r, t, x) P_{t_s}^{n+1} dr, \quad t \in [0, T], \quad \forall n \geq 1. \tag{6}
$$

Lemma 3.1: ($P_t^n$) is a Cauchy sequence in $L^2(\Omega; C(0, T; H))$.

Proof: For $n > 1$ and the process $P_{t_{n+1}} - P_t^n$, it follows from Itô’s formula

$$
|P_{t_{n+1}} - P_t^n|^2 = 2 \int_0^t \left( -\frac{\partial P_{t_{n+1}}}{\partial r} + \frac{\partial P_t}{\partial r} - \frac{\partial P_{t_s}^n}{\partial r} - \frac{\partial P_{t_s}^n}{\partial r} \right) ds
$$

$$
+ 2 \int_0^t \int_0^t k(r, s) \Delta_{t_s} (P_{t_s}^{n+1} - P_{t_s}^n) (P_{t_s}^{n+1} - P_{t_s}^n) dx dr ds
$$

$$
- 2 \int_0^t \langle \mu_0(r, s, x) (P_{t_s}^{n+1} - P_{t_s}^n), P_{t_s}^{n+1} - P_{t_s}^n \rangle ds
$$

$$
- A\beta^2 \int_0^t |P_{t_{n+1}} - P_{t_s}^n|^2 ds
$$

$$
+ A\beta^2 \int_0^t \langle P_{t_{n+1}} - P_{t_s}^n, P_{t_s}^{n+1} - P_{t_s}^n \rangle ds
$$

$$
+ 2 \int_0^t \langle f(P_{t_s}^n) - f(P_{t_s}^n), P_{t_s}^{n+1} - P_{t_s}^n \rangle ds
$$

$$
+ 2 \int_0^t \langle \mu_0(r, s, x) (P_{t_s}^{n+1} - P_{t_s}^n), g(P_{t_s}^n) - g(P_{t_s}^n) \rangle ds
$$

$$
+ \int_0^t \|g(P_{t_s}^n) - g(P_{t_s}^{n+1})\|_2^2 ds
$$

$$
+ 2 \int_0^t \|\mu_0(r, s, x) (P_{t_s}^n - P_{t_s}^{n+1})\|_2^2 ds,
$$

where $P_{t_s}^n := P^n(0, t, x), f(P_{t_s}^n) := f(r, t, x; P_{t_s}^n), g(P_{t_s}^n) := g(r, t, x; P_{t_s}^n)$.}

3. Existence and uniqueness of solutions

3.1. Existence of strong solutions

In order to prove the existence of solutions for Equation (3), we shall first prove the following lemmas.
Since
\[
\left( -\frac{\partial \rho_{n+1}^s}{\partial r} + \frac{\partial \rho_n^s}{\partial r} \right)^2 dr^s \leq \frac{1}{2} A \beta^2 |\rho_{n+1}^s - \rho_0^s|^2.
\]

It is easy to deduce
\[
|\rho_{n+1}^s - \rho_0^s|^2
\]
\[
\leq |A \beta^2 - 2 \mu_0 - 2k_0 + 2 \mu | E \int_0^t \left| \rho_{n+1}^s - \rho_0^s \right| |\rho_{n}^s - \rho_{n-1}^s| ds
\]
\[
+ 2 \int_0^t \left| (\rho_{n+1}^s - \rho_0^s, g(\rho_{n}^s) - g(\rho_{n-1}^s)) \right| ds
\]
\[
+ 2 \int_0^t \left| (f(\rho_{n}^s) - f(\rho_{n-1}^s)) \right| |\rho_{n+1}^s - \rho_0^s| ds
\]
\[
+ E \int_0^t \left( |\rho_{n+1}^s - \rho_0^s| |\rho_{n}^s - \rho_{n-1}^s| \right)^\frac{1}{2} ds.
\]

Consequently, inequality (8) yields
\[
E \left[ \sup_{0 \leq r \leq t} |\rho_{n+1}^s - \rho_0^s|^2 \right]
\]
\[
\leq |A \beta^2 - 2 \mu_0 - 2k_0 + 2 \mu | E \int_0^t \left| \rho_{n+1}^s - \rho_0^s \right| |\rho_{n}^s - \rho_{n-1}^s| ds
\]
\[
+ 2 \int_0^t \left| (\rho_{n+1}^s - \rho_0^s, g(\rho_{n}^s) - g(\rho_{n-1}^s)) \right| ds
\]
\[
+ 2 \int_0^t \left| (f(\rho_{n}^s) - f(\rho_{n-1}^s)) \right| |\rho_{n+1}^s - \rho_0^s| ds
\]
\[
+ E \int_0^t \left( |\rho_{n+1}^s - \rho_0^s| |\rho_{n}^s - \rho_{n-1}^s| \right)^\frac{1}{2} ds.
\]

Now, we estimate the terms on the right-hand side of inequality (8) by using the inequality
\[
2ab \leq \frac{a^2}{p^2} + l^2 b^2, \quad l > 0.
\]

\[
|A \beta^2 - 2 \mu_0 - 2k_0 + 2 \mu | E \int_0^t \left| \rho_{n+1}^s - \rho_0^s \right| |\rho_{n}^s - \rho_{n-1}^s| ds
\]
\[
\leq \frac{1}{4} E \left[ \sup_{0 \leq r \leq t} |\rho_{n+1}^s - \rho_0^s|^2 \right]
\]
\[
+ (A \beta^2 - 2 \mu_0 - 2k_0 + 2 \mu)^2 T
\]
\[
\times E \left[ \sup_{0 \leq r \leq s} |\rho_{n}^s - \rho_{n-1}^s|^2 \right] ds.
\]

On the other hand, we can get from Lipschitz condition
\[
E \int_0^t \left( |g(\rho_{n}^s) - g(\rho_{n-1}^s)| \right)^\frac{1}{2} ds
\]
\[
\leq k^2 E \int_0^t \sup_{0 \leq r \leq s} |\rho_{r}^s - \rho_{r-1}^s|^2 ds.
\]
**Lemma 3.2:** The sequence \( \{P^n_t\} \) is bounded in \( P^0(0, T; V) \).

**Proof:** Indeed, applying Itô's formula to \( |P^n_t|^2 \) with \( n \geq 2 \) immediately yields

\[
E[|P^n_t|^2] = E[|P_0|^2] + 2E \int_0^T \left( - \frac{\partial P^n_t}{\partial r} P^n_t \right) \, ds
- A\beta^2 E \int_0^T |P^n_s|^2 \, ds
+ A\beta^2 E \int_0^T (P^{n-1}_s, P^n_s) \, ds
+ 2E \int_0^T \int_0^T k \Delta (P^{n-1}_s, P^n_s) \, dx \, dr \, ds
- 2E \int_0^T \mu_0 (r, s, x) (P^{n-1}_s, P^n_s) \, ds
- 2E \int_0^T \langle \mu_0 (r, s, x; P^{n-1}_s), P^n_s \rangle \, ds
+ 2E \int_0^T \langle f(s, r; P^{n-1}_s), P^n_s \rangle \, ds
+ E \int_0^T \| g(P^{n-1}_s) \|_2^2 \, ds. \tag{17}
\]

Therefore,

\[
- 2E \int_0^T \langle f(s, r; P^{n-1}_s), P^n_s \rangle \, ds
- E \int_0^T \| g(r, s, x; P^{n-1}_s) \|_2^2 \, ds
\leq E[|P_0|^2] + 2E \int_0^T \langle f(s, r; P^{n-1}_s) \rangle (|P^n_s| + |P^{n-1}_s|) \, ds
+ |A\beta^2 - 4\mu_0 - 2k_0| E \int_0^T |P^n_s| |P^{n-1}_s| \, ds. \tag{18}
\]

Since \( \{P^n\} \) is convergent in \( L^2(\Omega; C(0, T; H)) \), it will be bounded in this space. Now, it is not difficult to check that there exists a positive constant \( k' > 0 \) such that the right-hand side of inequality (18) is bounded by this constant. We will estimate one of those terms. Firstly, we observe that

\[
2E \int_0^T \langle f(s, r; P^{n-1}_s) \rangle (|P^n_s| + |P^{n-1}_s|) \, ds
\leq 2k_1 E \int_0^T \| P^{n-1}_s \|_C (|P^n_s| + |P^{n-1}_s|) \, ds
\leq k_1 E \int_0^T \| |P^{n-1}_s| \|^2 + (|P^n_s|^2 + |P^{n-1}_s|) \, ds
\leq Tk_1 E \left( \sup_{0 \leq \theta \leq T} |P^{n-1}_\theta|^2 \right)
+ k_1 \left[ E \left( \sup_{0 \leq \theta \leq T} |P^n_{\theta}|^2 \right) + E \left( \sup_{0 \leq \theta \leq T} |P^{n-1}_{\theta}|^2 \right) \right]
= Tk_1 \| P^{n-1}_t \|^2_{L^2(\Omega; C(0, T; H))} + k_1 T \| P^n_t \|^2_{L^2(\Omega; C(0, T; H))}
+ \| P^{n-1}_t \|^2_{L^2(\Omega; C(0, T; H))},
\]

which, in addition to inequality (18) and (H), leads to the following inequalities:

\[
\alpha \int_0^T E\| P^{n-1}_t \|^p \, ds
\leq -2E \int_0^T \langle f(P^{n-1}_t), P^n_t \rangle \, ds - E \int_0^T \| g(P^{n-1}_t) \|_2^2 \, ds
+ |\lambda| T \| P^{n-1}_t \|^2_{L^2(\Omega; C(0, T; H))} + \int_0^T \gamma(s) e^{\delta s} \, ds.
\]

Since \( \{P^n\} \) is convergent in \( L^2(\Omega; C(0, T; H)) \). Therefore, there exist a constant \( k' \) such that

\[
\int_0^T E\| P^{n-1}_t \|^p \, ds \leq k',
\]

and Lemma 3.2 is proved.

**Theorem 3.1:** Assume the preceding hypotheses hold, then there exists a process \( P_t \in P^0(0, T; V) \cap L^2(\Omega; C(0, T; H)) \) such that

\[
P_t = P_0 + \int_0^t \left( \frac{\partial P_s}{\partial r} + f_1(s) \right) \, ds + M(t),
\]

where \( f_1 \in \mathcal{L}(0, T, H) \), \( P_0 \in L^2(\Omega; \mathcal{F}_0; P; H) \) and \( M_t \) is an \( H \)-valued continuous, square integrable \( \mathcal{F}_t \)-martingale. In addition, the following energy equality also holds:

\[
|P_t|^2 = |P_0|^2 + 2 \int_0^t \langle \frac{\partial P_s}{\partial r}, P_s \rangle \, ds + 2 \int_0^t \langle f_1(s), P_s \rangle \, ds
+ 2 \int_0^t \langle P_s, dM_s \rangle + \text{tr}((\langle M \rangle)_t), \text{ P - a.s., } \forall t \in [0, T],
\]

where \( \langle \langle M \rangle \rangle_t \) denotes the quadratic variation of \( M_t \).

**Proof:** Use Lemmas 3.1 and 3.2, and use the same method in Métiver and Pellaumail (1980) we can get the conclusion.

### 3.2. Uniqueness of solutions

Now we will prove that there exists at most one solution of \( (3) \). This result will be deduced mainly from Itô’s formula.

**Theorem 3.2:** Assume the preceding hypotheses hold. Then, there exists at most one solution of \( (3) \) in \( P^0(0, T; V) \cap L^2(\Omega; C(0, T; H)) \).
Proof: Suppose that $P_{1t}, P_{2t} \in \mathcal{L}^2(\Omega; C(0, T; H))$ are two solutions of (3). Then, applying Itô's formula to $|P_{1t} - P_{2t}|^2$, we obtain

$$
|P_{1t} - P_{2t}|^2 \leq 2 \int_0^t \left( -\frac{\partial P_{1s}}{\partial t} + \frac{\partial P_{2s}}{\partial t} - \mu_0 (P_{1s} - P_{2s}) ight) ds + 2 \int_0^t (f(r, s, x; P_{1s}) - f(r, s, x; P_{2s}), P_{1s} - P_{2s}) ds + 2 \int_0^t (P_{1s} - P_{2s}, g(r, s, x; P_{1s}) - g(r, s, x; P_{2s})) ds + 2 \int_0^t \|g(r, s, x; P_{1s}) - g(r, s, x; P_{2s})\|^2 ds.
$$

Therefore, we get that

$$
|P_{1t} - P_{2t}|^2 \leq A\beta^2 \int_0^t \|P_{1s} - P_{2s}\|^2 ds + 2 \int_0^t \|P_{1s} - P_{2s}\|^2 ds - 4\mu_0 \int_0^t |P_{1s} - P_{2s}|^2 ds + \int_0^t \|g(r, s, x; P_{1s}) - g(r, s, x; P_{2s})\|^2 ds + 2 \int_0^t (P_{1s} - P_{2s}, g(r, s, x; P_{1s})) ds + 2 \int_0^t (P_{1s} - P_{2s}, g(r, s, x; P_{1s})) ds - g(r, s, x; P_{2s}) d\omega_s - 2k_0 \int_0^t \|P_{1s} - P_{2s}\|^2 ds.
$$

Now, it follows from condition (2) that for any $t \in [0, T]$,

$$
E \sup_{0 \leq s \leq t} |P_{1s} - P_{2s}|^2 \leq (|A\beta^2 - 4\mu_0| - 2k_0 + K + 1) \int_0^t E|P_{1s} - P_{2s}|^2 ds + 2E \sup_{0 \leq s \leq t} \int_0^s (P_{1\lambda} - P_{2\lambda}, g(r, \lambda, x; P_{1\lambda}) - g(r, \lambda, x; P_{2\lambda})) d\omega_s.
$$

However, by Burkholder–Davis–Gundy's inequality, we have

$$
E \left( \sup_{0 \leq s \leq t} \int_0^s (P_{1\lambda} - P_{2\lambda}, g(r, \lambda, x; P_{1\lambda}) - g(r, \lambda, x; P_{2\lambda})) d\omega_s \right) \leq \frac{1}{8} E \left( \sup_{0 \leq s \leq t} |P_{1s} - P_{2s}|^2 + K' \int_0^t E|P_{1s} - P_{2s}|^2 ds \right),
$$

where $K$ and $K'$ are positive constants. Thus, it follows from inequalities (19) and (20)

$$
E \sup_{0 \leq s \leq t} |P_{1s} - P_{2s}|^2 \leq \frac{4}{3} \left( |A\beta^2 - 4\mu_0| - 2k_0 + K + 1 \right) \int_0^t E|P_{1s} - P_{2s}|^2 ds + \int_0^t E \sup_{0 \leq \lambda \leq s} |P_{1\lambda} - P_{2\lambda}|^2 ds, \quad \forall t \in [0, T].
$$

Now, Gronwall's lemma obviously implies uniqueness, this completes the improvement.

4. Conclusion

This paper has considered the existence and uniqueness of strong solution of a class of age-structured diffusion...
population system. By using Burkholder–Davis–Gundy’s inequality and Itô’s formula in Hilbert space and iteration of Cauchy sequence, the strong solution’s existence and uniqueness of system (3) are obtained in $\mathcal{D}(0, T; V) \cap L^2(\Omega; C(0, T; H))$.

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