Local Lie derivations on von Neumann algebras and algebras of locally measurable operators

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Abstract

Let \(A\) be a unital associative algebra and \(M\) be an \(A\)-bimodule. A linear mapping \(\phi\) from \(A\) into an \(A\)-bimodule \(M\) is called a Lie derivation if \(\phi[A, B] = [\phi(A), B] + [A, \phi(B)]\) for each \(A, B\) in \(A\), and \(\phi\) is called a local Lie derivation if for every \(A\) in \(A\), there exists a Lie derivation \(\phi_A\) (depending on \(A\)) from \(A\) into \(M\) such that \(\phi(A) = \phi_A(A)\). In this paper, we prove that every local Lie derivation on von Neumann algebras is a Lie derivation; and we show that if \(M\) is a type I von Neumann algebra with atomic lattice of projections, then every local Lie derivation on \(LS(M)\) is a Lie derivation.

Keywords: Lie derivation, local Lie derivation, von Neumann algebra, locally measurable operator.

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1 Introduction

Let \(A\) be a unital associative algebra over the complex field \(C\) and \(M\) be an \(A\)-bimodule. An linear mapping \(\delta\) from \(A\) into \(M\) is called a derivation if \(\delta(AB) = \delta(A)B + A\delta(B)\) for each \(A\) and \(B\) in \(A\). In particular, a derivation \(\delta_M\) defined by

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\[ \delta_M(A) = MA - AM \] for every \( A \) in \( \mathcal{A} \) is called an inner derivation, where \( M \) is a fixed element in \( \mathcal{M} \).

In [31], S. Sakai proves that every derivation on von Neumann algebras is an inner derivation. In [13], E. Christensen shows that every derivation on nest algebras on a Hilbert space \( \mathcal{H} \) is an inner derivation. For more information on derivations and inner derivations, we refer to [14, 15, 19].

In [23, 25], R. Kadison and D. Larson introduce the concept of local derivations. A linear mapping \( \delta \) from \( \mathcal{A} \) into \( \mathcal{M} \) is called a local derivation if for every \( A \) in \( \mathcal{A} \), there exists a derivation \( \delta_A \) (depending on \( A \)) from \( \mathcal{A} \) into \( \mathcal{M} \) such that \( \delta(A) = \delta_A(A) \).

In [23], R. Kadison proves that every continuous local derivation from a von Neumann algebra into its dual Banach module is a derivation. In [25], D. Larson and A. Sourour prove that if \( X \) is a Banach space, then every local derivation on \( B(X) \) is a derivation. In [21], B. Johnson shows that every local derivation from a \( C^* \)-algebra into its Banach bimodule is a derivation. In [17, 18], D. Hadwin and J. Li characterize local derivations on non self-adjoint operator algebras such as nest algebras and CDCSL algebras.

A linear mapping \( \varphi \) from \( \mathcal{A} \) into an \( \mathcal{A} \)-bimodule \( \mathcal{M} \) is called a Lie derivation if \( \varphi[A, B] = [\varphi(A), B] + [A, \varphi(B)] \) for each \( A \) and \( B \) in \( \mathcal{A} \), where \( [A, B] = AB - BA \) is the usual Lie product. A Lie derivation \( \varphi \) is said to be standard if it can be decomposed as \( \varphi = \delta + \tau \), where \( \delta \) is a derivation from \( \mathcal{A} \) into \( \mathcal{M} \) and \( \tau \) is a linear mapping from \( \mathcal{A} \) into \( Z(\mathcal{A}, \mathcal{M}) \) such that \( \tau[A, B] = 0 \) for each \( A \) and \( B \) in \( \mathcal{A} \), where \( Z(\mathcal{A}, \mathcal{M}) = \{ M \in \mathcal{M} : AM = MA \text{ for every } A \text{ in } \mathcal{A} \} \).

In [22], B. Johnson proves that every continuous Lie derivation from a \( C^* \)-algebra into its Banach bimodule is standard. In [28], M. Mathieu and A. Villena prove that every Lie derivation on a \( C^* \)-algebra is standard. In [12], W. Cheung characterizes Lie derivations on triangular algebras. In [27], F. Lu proves that every Lie derivation on a completely distributed commutative subspace lattice algebra is standard. In [14], D. Benkovič proves that every Lie derivation on matrix algebra \( M_n(\mathcal{A}) \) is standard, where \( n \geq 2 \) and \( \mathcal{A} \) is a 2-torsion free unital algebra.

Similar to local derivations, In [10], L. Chen, F. Lu and T. Wang introduce the concept of local Lie derivations. A linear mapping \( \varphi \) from \( \mathcal{A} \) into \( \mathcal{M} \) is called a local Lie derivation if for every \( A \) in \( \mathcal{A} \), there exists a Lie derivation \( \varphi_A \) (depending on \( A \)) from \( \mathcal{A} \) into \( \mathcal{M} \) such that \( \varphi(A) = \varphi_A(A) \).

In [10], L. Chen, F. Lu and T. Wang prove that every local Lie derivation on \( B(X) \) is a Lie derivation, where \( X \) is a Banach space of dimension exceeding 2. In [11], L. Chen and F. Lu prove that every local Lie derivation on nest algebras is a Lie derivation. In [26], D. Liu and J. Zhang prove that under certain conditions, every local Lie derivation on triangular algebras is a Lie derivation. In [20], J. He, J. Li, G. An and W. Huang prove that every local Lie derivation on some algebras such as finite von Neumann
algebras, nest algebras, Jiang-Su algebra and UHF algebras is a Lie derivation.

Compare with the characterizations of derivations on Banach algebras, investigation of derivations on unbounded operator algebras begin much later.

In [32], I. Segal studies the theory of noncommutative integration, and introduces various classes of non-trivial $\ast$-algebras of unbounded operators. In this paper, we mainly consider the $\ast$-algebra $S(\mathcal{M})$ of all measurable operators and the $\ast$-algebra $LS(\mathcal{M})$ of all locally measurable operators affiliated with a von Neumann algebra $\mathcal{M}$. In [32], I. Segal shows that the algebraic and topological properties of the measurable operators algebra $S(\mathcal{M})$ are similar to the von Neumann algebra $\mathcal{M}$. If $\mathcal{M}$ is a commutative von Neumann algebra, then $\mathcal{M}$ is $\ast$-isomorphic to the algebra $L^\infty(\Omega, \Sigma, \mu)$ of all essentially bounded measurable complex functions on a measure space $(\Omega, \Sigma, \mu)$; and $S(\mathcal{M})$ is $\ast$-isomorphic to the algebra $L^0(\Omega, \Sigma, \mu)$ of all measurable almost everywhere finite complex-valued functions on $(\Omega, \Sigma, \mu)$. In [5], A. Ber, V. Chilin and F. Sukochev show that there exists a derivation on $L^0(0, 1)$ is not an inner derivation, and the derivation is discontinuous in the measure topology. This result means that the properties of derivations on $S(\mathcal{M})$ are different from the derivations on $\mathcal{M}$.

In [1, 2], Albeverio, Ayupov and Kudaybergenov study the properties of derivations on various classes of measurable algebras. If $\mathcal{M}$ is a type I von Neumann algebra, in [1], the authors prove that every derivation on $LS(\mathcal{M})$ is an inner derivation if and only if it is $Z(\mathcal{M})$ linear; in [2], the authors give the decomposition form of derivations on $S(\mathcal{M})$ and $LS(\mathcal{M})$; they also prove that if $\mathcal{M}$ is a type $I_\infty$ von Neumann algebra, then every derivation on $S(\mathcal{M})$ or $LS(\mathcal{M})$ is an inner derivation. If $\mathcal{M}$ is a properly infinite von Neumann algebra, in [6], A. Ber, V. Chilin and F. Sukochev prove that every derivation on $LS(\mathcal{M})$ is continuous with respect to the local measure topology $t(\mathcal{M})$; and in [7], the authors show that every derivation on $LS(\mathcal{M})$ is an inner derivation. In [8], S. Albeverio and S. Ayupov give a characterization of local derivations on $S(\mathcal{M})$, where $\mathcal{M}$ is an abelian von Neumann algebra. In [16], D. Hadwin and J. Li prove that if $\mathcal{M}$ is a von Neumann algebra without abelian direct summands, then every local derivation on $LS(\mathcal{M})$ or $S(\mathcal{M})$ is a derivation. In [9], V. Chilin and I. Juraev show that every Lie derivation on $LS(\mathcal{M})$ or $S(\mathcal{M})$ is standard.

This paper is organized as follows. In Section 2, we recall the definitions of algebras of measurable operators and local measurable operators.

In Section 3, we generalize the Corollary 3.2 in [20] and prove that every local Lie derivation on von Neumann algebras is a Lie derivation.

In Section 4, we prove that if $\mathcal{M}$ is a type I von Neumann algebra with an atomic lattice of projections, then every local Lie derivation on $LS(\mathcal{M})$ is a Lie derivation.
2 Preliminaries

Let $\mathcal{H}$ be a complex Hilbert space and $B(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. Suppose that $\mathcal{M}$ is a von Neumann algebra on $\mathcal{H}$ and $\mathcal{Z}(\mathcal{M}) = \mathcal{M} \cap \mathcal{M}'$ is the center of $\mathcal{M}$, where

$$\mathcal{M}' = \{a \in B(\mathcal{H}) : ab = ba \text{ for every } b \in \mathcal{M}\}.$$ 

Denote by $\mathcal{P}(\mathcal{M}) = \{p \in \mathcal{M} : p = p^* = p^2\}$ the lattice of all projections in $\mathcal{M}$ and by $\mathcal{P}_{fin}(\mathcal{M})$ the set of all finite projections in $\mathcal{M}$. For each $p$ and $q$ in $\mathcal{P}(\mathcal{M})$, if we define the inclusion relation $p \subset q$ by $p \leq q$, then $\mathcal{P}(\mathcal{M})$ is a complete lattice. Suppose that $\{p_l\}_{l \in \lambda}$ is a family of projections in $\mathcal{M}$, we denote

$$\sup_{l \in \lambda} p_l = \bigcup_{l \in \lambda} p_l \mathcal{H} \quad \text{and} \quad \inf_{l \in \lambda} p_l = \bigcap_{l \in \lambda} p_l \mathcal{H}.$$ 

If $\{p_l\}_{l \in \lambda}$ is an orthogonal family of projections in $\mathcal{M}$, then we have that

$$\sup_{l \in \lambda} p_l = \sum_{l \in \lambda} p_l.$$ 

Let $x$ be a closed densely defined linear operator on $\mathcal{H}$ with the domain $\mathcal{D}(x)$, where $\mathcal{D}(x)$ is a linear subspace of $\mathcal{H}$. $x$ is said to be affiliated with $\mathcal{M}$, denote by $x\eta_{\mathcal{M}}$, if $u^*x u = x$ for every unitary element $u$ in $\mathcal{M}'$.

A linear operator affiliated with $\mathcal{M}$ is said to be measurable with respect to $\mathcal{M}$, if there exists a sequence $\{p_n\}_{n=1}^\infty \subset \mathcal{P}(\mathcal{M})$ such that $p_n \uparrow 1$, $p_n(\mathcal{H}) \subset \mathcal{D}(x)$ and $p_n^* = 1 - p_n \in \mathcal{P}_{fin}(\mathcal{M})$ for every $n \in \mathbb{N}$, where $\mathbb{N}$ is the set of all natural numbers. Denote by $S(\mathcal{M})$ the set of all measurable operators affiliated with the von Neumann algebra $\mathcal{M}$.

A linear operator affiliated with $\mathcal{M}$ is said to be locally measurable with respect to $\mathcal{M}$, if there exists a sequence $\{z_n\}_{n=1}^\infty \subset \mathcal{P}(\mathcal{Z}(\mathcal{M}))$ such that $z_n \uparrow 1$ and $z_n x \in S(\mathcal{M})$ for every $n \in \mathbb{N}$. Denote by $LS(\mathcal{M})$ the set of all locally measurable operators affiliated with the von Neumann algebra $\mathcal{M}$.

In [29], Muratov and Chilin prove that $S(\mathcal{M})$ and $LS(\mathcal{M})$ are both unital $*$-algebras and $\mathcal{M} \subset S(\mathcal{M}) \subset LS(\mathcal{M})$; the authors also show that if $\mathcal{M}$ is a finite von Neumann algebra or $\dim(\mathcal{Z}(\mathcal{M})) < \infty$, then $S(\mathcal{M}) = LS(\mathcal{M})$; if $\mathcal{M}$ is a type III von Neumann algebra and $\dim(\mathcal{Z}(\mathcal{M})) = \infty$, then $S(\mathcal{M}) = \mathcal{M}$ and $LS(\mathcal{M}) \neq \mathcal{M}$.

3 Local Lie derivations on von Neumann algebras
In this section, we consider local Lie derivations on von Neumann algebras. To prove our main theorem, we need the following lemma.

**Lemma 3.1.** Let \( A_1 \) and \( A_2 \) be two unital algebras and \( A = A_1 \bigoplus A_2 \). If the following five conditions hold:

1. each Lie derivation on \( A \) is standard;
2. each derivation on \( A \) is inner;
3. each local derivation on \( A \) is a derivation;
4. \( Z(A_1) \cap [A_1, A_1] = \{0\} \);
5. \( A_2 = [A_2, A_2] \),

then every local Lie derivation on \( A \) is a Lie derivation.

**Proof.** Denote the units of \( A, A_1 \) and \( A_2 \) by \( I, P \) and \( Q \), respectively. For each \( A \) in \( A \), we have that \( A = PA + QA = A_1 + A_2 \), where \( A_i \in \mathcal{A}_i \), \( i = 1, 2 \).

In the following we suppose that \( \varphi \) is a local Lie derivation on \( A \).

By the definition of local Lie derivation, we know that for every \( A_1 \) in \( A_1 \), there exists a Lie derivation \( \varphi_{A_1} \) on \( A \) such that \( \varphi(A_1) = \varphi_{A_1}(A_1) \). Since \( \varphi_{A_1} \) is standard and each derivation on \( A \) is inner, we can obtain that

\[
\varphi(A_1) = \varphi_{A_1}(A_1) = \delta_{A_1}(A_1) + \tau_{A_1}(A_1) = [A_1, T_{A_1}] + P\tau_{A_1}(A_1) + Q\tau_{A_1}(A_1),
\]

where \( \delta_{A_1} \) is a derivation on \( A_1 \), \( T_{A_1} \) is an element in \( A_1 \), and \( \tau_{A_1} \) is a linear mapping from \( A_1 \) into \( Z(A_1) \) such that \( \tau_{A_1}([A_1, A_1]) = 0 \).

It means that \( \varphi \) has a decomposition at \( A_1 \). Next we show that the decomposition at \( A_1 \) is unique. Assume there is another decomposition at \( A_1 \), that is

\[
\varphi(A_1) = \varphi'_{A_1}(A_1) = \delta'_{A_1}(A_1) + \tau'_{A_1}(A_1) = [A_1, T'_{A_1}] + P\tau'_{A_1}(A_1) + Q\tau'_{A_1}(A_1),
\]

where \( \delta'_{A_1} \) is a derivation on \( A_1 \), \( T'_{A_1} \) is an element in \( A_1 \) and \( \tau'_{A_1} \) is a linear mapping from \( A_1 \) into \( Z(A_1) \) such that \( \tau'_{A_1}([A_1, A_1]) = 0 \).

Then we have that

\[
[A_1, T_{A_1}] + P\tau_{A_1}(A_1) + Q\tau_{A_1}(A_1) = [A_1, T'_{A_1}] + P\tau'_{A_1}(A_1) + Q\tau'_{A_1}(A_1).
\]

Thus

\[
[A_1, T_{A_1}] - [A_1, T'_{A_1}] = P\tau_{A_1}(A_1) - P\tau'_{A_1}(A_1) + Q\tau_{A_1}(A_1) - Q\tau'_{A_1}(A_1).
\]

Since \( [A_1, T_{A_1}] - [A_1, T'_{A_1}] \) and \( P\tau_{A_1}(A_1) - P\tau'_{A_1}(A_1) \) belong to \( A_1 \), and \( Q\tau_{A_1}(A_1) - Q\tau'_{A_1}(A_1) \) belongs to \( A_2 \), we have that \( Q\tau_{A_1}(A_1) - Q\tau'_{A_1}(A_1) = 0 \). Moreover, we can obtain that

\[
[A_1, T_{A_1}] - [A_1, T'_{A_1}] = [A_1, PT_{A_1}] - [A_1, PT'_{A_1}] \in [A_1, A_1],
\]
and

\[ P\tau'_{A_1}(A_1) - P\tau_{A_1}(A_1) \in Z(A_1). \]

By condition (4), it follows that \([A_1, T_{A_1}] - [A_1, T'_{A_1}] = P\tau'_{A_1}(A_1) - P\tau_{A_1}(A_1) = 0.\) It implies that \(\delta_{A_1}(A_1) = \delta'_{A_1}(A_1)\) and \(\tau_{A_1}(A_1) = \tau'_{A_1}(A_1)\). Hence the decomposition is unique.

Now we have \(\varphi|_{A_1} = \delta_1 + \tau_1\), where \(\delta_1\) is a mapping from \(A_1\) into \(A_1\) such that \(\delta_1(A_1) = [A_1, S_{A_1}]\) for some element \(S_{A_1}\) in \(A_1\), and \(\tau_1\) is a mapping from \(A_1\) into \(Z(A)\) such that \(\tau_1([A_1, A_1]) = 0\).

Next we prove that \(\delta_1\) and \(\tau_1\) are linear mappings. For each \(A_1\) and \(B_1\) in \(A_1\), we have that

\[ \varphi(A_1) = \delta_1(A_1) + \tau_1(A_1) = [A_1, S_{A_1}] + \tau_1(A_1), \]

\[ \varphi(B_1) = \delta_1(B_1) + \tau_1(B_1) = [B_1, S_{B_1}] + \tau_1(B_1), \]

and

\[ \varphi(A_1 + B_1) = \delta_1(A_1 + B_1) + \tau_1(A_1 + B_1) = [A_1 + B_1, S_{A_1 + B_1}] + \tau_1(A_1 + B_1). \]

Since \(\varphi\) is additive, through a discussion similar to that before, it implies that

\[ [A_1 + B_1, S_{A_1 + B_1}] = [A_1, S_{A_1}] + [B_1, S_{B_1}] \]

and

\[ \tau_1(A_1 + B_1) = \tau_1(A_1) + \tau_1(B_1). \]

It means that \(\delta_1\) and \(\tau_1\) are additive mappings. Using the same technique, we can prove that \(\delta_1\) and \(\tau_1\) are homogeneous. Hence \(\delta_1\) and \(\tau_1\) are linear mappings.

For every \(A_2\) in \(A_2\), we have that

\[ \varphi(A_2) = \varphi_{A_2}(A_2) = \delta_{A_2}(A_2) + \tau_{A_2}(A_2) = [A_2, T_{A_2}] + \tau_{A_2}(A_2), \]

where \(\delta_{A_2}\) is a derivation on \(A\), \(T_{A_2}\) is an element in \(A\) and \(\tau_{A_2}\) is a linear mapping from \(A\) into \(Z(A)\) such that \(\tau_{A_2}([A, A]) = 0\). By condition (5), we have that \(\tau_{A_2}(A_2) = 0\). Thus \(\varphi(A_2) = [A_2, T_{A_2}] = [A_2, QT_{A_2}].\)

Let \(\varphi|_{A_2} = \delta_2\). Then we have \(\delta_2(A_2) = [A_2, S_{A_2}]\) for some element \(S_{A_2}\) in \(A_2\). And obviously, \(\delta_2\) is linear.

Define two linear mappings as follows:

\[ \delta(A) = \delta_1(A_1) + \delta_2(A_2), \quad \tau(A) = \tau_1(A_1), \]

for all \(A = A_1 + A_2 \in A\). By the previous discussion, \(\tau\) is a linear mapping from \(A\) into \(Z(A)\) such that \(\tau([A, A]) = 0\). In addition,

\[ \delta(A) = \delta_1(A_1) + \delta_2(A_2) = [A_1, S_{A_1}] + [A_2, S_{A_2}] = [A_1 + A_2, S_{A_1 + S_{A_2}}] = [A, S_{A_1 + S_{A_2}}]. \]
It means that $\delta$ is a local derivation. By condition (3), $\delta$ is a derivation. Notice that
\[
\varphi(A) = \varphi(A_1) + \varphi(A_2) = \delta_1(A_1) + \tau_1(A_1) + \delta_2(A_2) = \delta(A) + \tau(A).
\]
Hence $\varphi$ is a standard Lie derivation.

By Lemma 3.1, we have the following result.

**Theorem 3.2.** Every local Lie derivation on a von Neumann algebra is a Lie derivation.

**Proof.** Let $\mathcal{A}$ be a von Neumann algebra. It is well known that $\mathcal{A} = A_1 \oplus A_2$, where $A_1$ is a finite von Neumann algebra, and $A_2$ is a proper infinite von Neumann algebra.

By [28, Theorem 1.1], we know that every Lie derivation on $\mathcal{A}$ is standard, by [31, Theorem 1], we know that every derivation on $\mathcal{A}$ is inner, and by [21, Theorem 5.3], we know that every local derivation on $\mathcal{A}$ is a derivation. Since $A_2$ is a proper infinite von Neumann algebra, we known that $A_2 = [A_2, A_2]$ (see in [33]).

Hence it is sufficient to prove that $\mathcal{Z}(A_1) \cap [A_1, A_1] = \{0\}$. Since $A_1$ is finite and by [24, Theorem 8.2.8], it follows that there is a center-valued trace $\tau$ on $A_1$ such that $\tau(Z) = Z$ for every $Z$ in $\mathcal{Z}(A_1)$ and $\tau([A, B]) = 0$ for each $A$ and $B$ in $A_1$. Suppose that $A \in \mathcal{Z}(A_1) \cap [A_1, A_1]$, then we have that $\tau(A) = A$ and $\tau(A) = 0$. it implies that $A = 0$.

By Lemma 3.1 we know that every local Lie derivation on a von Neumann algebra is a Lie derivation.

4 Local Lie derivations on algebras of locally measurable operators

In this section, we mainly consider local Lie derivations on algebras of all locally measurable operators affiliated with a type I von Neumann algebra. To prove the main result, we need the following lemmas.

**Lemma 4.1.** Suppose that $\mathcal{A}$ is a commutative unital algebra and $\mathcal{J} = M_n(\mathcal{A})$. Then $\mathcal{Z}(\mathcal{J}) \cap [\mathcal{J}, \mathcal{J}] = \{0\}$

**Proof.** Let $\{e_{i,j}\}_{i,j=1}^n$ be the system of matrix units in $M_n(\mathcal{A})$. Then for every element $A$ in $\mathcal{J}$, we have that $A = \sum_{i,j=1}^n a_{ij} e_{ij}$, where $a_{ij} \in \mathcal{A}$.

Define a linear mapping $\tau$ from $\mathcal{J}$ into $\mathcal{A}$ by $\tau(A) = \sum_{i=1}^n a_{ii}$ for every $A = \sum_{i,j=1}^n a_{ij} e_{ij} \in \mathcal{J}$. Since $\mathcal{A}$ is commutative, it is not difficult to verify that $\tau([A, B]) = 0$ for each $A$ and $B$ in $\mathcal{J}$.
It should be noticed that \( \mathcal{Z}(\mathcal{J}) = \{ A : A = \sum_{i=1}^{n} a_{ii}, a \in \mathcal{A} \} \). Suppose that \( A = \sum_{i=1}^{n} a_{ii} \) is an element in \( \mathcal{Z}(\mathcal{J}) \cap [\mathcal{J}, \mathcal{J}] \), then by the definition of \( \tau \), we have that \( \tau(A) = na \) and \( \tau(A) = 0 \). It implies that \( A = 0 \).

**Lemma 4.2.** Suppose that \( \mathcal{A} = \prod_{i\in\Lambda} \mathcal{A}_i \). If \( \mathcal{Z}(\mathcal{A}_i) \cap [\mathcal{A}_i, \mathcal{A}_i] = \{0\} \) for every \( i \in \Lambda \), then we have that \( \mathcal{Z}(\mathcal{A}) \cap [\mathcal{A}, \mathcal{A}] = \{0\} \).

**Proof.** Let \( A = \{a_i\}_{i \in \Lambda} \) be an element in \( \mathcal{Z}(\mathcal{A}) \cap [\mathcal{A}, \mathcal{A}] \). Then for every \( i \in \Lambda \), we have that \( a_i \in \mathcal{Z}(\mathcal{A}_i) \cap [\mathcal{A}_i, \mathcal{A}_i] \). By assumption, it follows that \( a_i = 0 \). Hence \( A = 0 \). □

**Lemma 4.3.** Suppose that \( \mathcal{M} \) is a type I\(_\infty\) von Neumann algebra. Then \( \text{LS}(\mathcal{M}) = [\text{LS}(\mathcal{M}), \text{LS}(\mathcal{M})] \).

**Proof.** By [30], we know that for every \( x \in \text{LS}(\mathcal{M}) \), there exists a sequence \( \{z_n\} \) of mutually orthogonal central projections in \( \mathcal{M} \) with \( \sum_{n=1}^{\infty} z_n = I \), such that \( x = \sum_{n=1}^{\infty} z_nx \), and \( z_nx \in \mathcal{M} \) for every \( n \in \mathbb{N} \). Since \( \mathcal{M} \) is a proper infinite von Neumann algebra, it is well known that \( \mathcal{M} = [\mathcal{M}, \mathcal{M}] \). Thus we have that \( z_nx = \sum_{i=1}^{k}[a_i^n, b_i^n] \), where \( a_i^n, b_i^n \in \mathcal{M} \) for each \( n \) and \( i \).

Set \( s_i = \sum_{n=1}^{\infty} z_n a_i^n \) and \( t_i = \sum_{n=1}^{\infty} z_n b_i^n \). By the definition of locally measurable operators, it is easy to show that \( s_i \) and \( t_i \) are two elements in \( \text{LS}(\mathcal{M}) \).

Since that \( \{z_n\} \) are mutually orthogonal central projections, we can obtain that

\[
[s_i, t_i] = \left[ \sum_{n=1}^{\infty} z_n a_i^n, \sum_{n=1}^{\infty} z_n b_i^n \right] = \sum_{n=1}^{\infty} z_n [a_i^n, b_i^n],
\]

moreover, we have that

\[
\sum_{i=1}^{k} [s_i, t_i] = \sum_{i=1}^{k} \sum_{n=1}^{\infty} z_n [a_i^n, b_i^n] = \sum_{n=1}^{\infty} z_n \left( \sum_{i=1}^{k} [a_i^n, b_i^n] \right) = \sum_{n=1}^{\infty} z_n x = x.
\]

It follows that \( x \in [\text{LS}(\mathcal{M}), \text{LS}(\mathcal{M})] \). □

In the following we show the main result of this section.

**Theorem 4.4.** Suppose that \( \mathcal{M} \) is a type I von Neumann algebra with an atomic lattice of projections. Then every local Lie derivation from \( \text{LS}(\mathcal{M}) \) into itself is a Lie derivation.

**Proof.** By [24] Theorem 6.5.2, we know that \( \mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2 \), where \( \mathcal{M}_1 \) is a type \(^{1}\) finite von Neumann algebra and \( \mathcal{M}_2 \) is a type \(^{1}\) infinite von Neumann algebra. Hence by [2] Proposition 1.1], we have that \( \text{LS}(\mathcal{M}) \cong \text{LS}(\mathcal{M}_1) \oplus \text{LS}(\mathcal{M}_2) \).

In the following we will verify the conditions (1) to (5) in Lemma 3.1 one by one. By [9] Theorem 1], we know that every Lie derivation on \( \text{LS}(\mathcal{M}) \) is standard; by [2] Corollary 5,12, we know that every derivation on \( \text{LS}(\mathcal{M}) \) is inner for a von Neumann algebra with atomic lattice of projections.
It is proved in [17] that every local derivation on $LS(M)$ is a derivation for a von Neumann algebra without abelian direct summands. While for an abelian von Neumann algebra with atomic lattice of projections, by [3, Theorem 3.8] we know that every local derivation on $LS(M)$ is a derivation. Associated the two results, we can obtain each local derivation on $LS(M)$ is a derivation for a von Neumann algebra with atomic lattice of projections.

Since $M_1$ is a type $I_{finite}$ von Neumann algebra, we know that $M_1 = \bigoplus_{n=1}^{\infty} A_n$, where each $A_n$ is a homogenous type $I_n$ von Neumann algebra. Hence $LS(M_1) \cong \prod_{n=1}^{\infty} LS(A_n)$. Since $A_n$ is a homogenous type $I_n$ von Neumann algebra, by [2] we know that $LS(A_n) \cong M_n(Z(LS(A_n)))$. By Lemmas 4.1 and 4.2, we know that the condition (4) in Lemma 3.1 holds. And by Lemma 4.3 the condition (5) in Lemma 3.1 holds.

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