The Petrov type D equation on genus $> 0$ sections of isolated horizons

Denis Dobkowski-Rylko, Wojciech Kamiński, Jerzy Lewandowski and Adam Szereszewski
Faculty of Physics, University of Warsaw, ul. Pasteura 5, 02-093 Warsaw, Poland
(Dated: June 8, 2018)

The Petrov type D equation imposed on the 2-metric tensor and the rotation scalar of a cross-section of an isolated horizon can be used to uniquely distinguish the Kerr -(anti) de Sitter spacetime in the case the topology of the cross-section is that of a sphere. In the current paper we study that equation on closed 2-dimensional surfaces that have genus $> 0$. We derive all the solutions assuming the embeddability in 4-dimensional spacetime that satisfies the vacuum Einstein equations with (possibly 0) cosmological constant. We prove all of them have constant Gauss curvature and zero rotation. Consequently, we provide a quasi-local argument for a black hole in 4-dimensional spacetime to have a topologically spherical cross-section.

I. INTRODUCTION

A quasilocal local theory generalizes black holes to surfaces that have some properties of black hole horizons [1, 2]. They are apparent horizons [3], non-expanding horizons, isolated horizons [4–9], Killing horizons [2, 10]. Typically, the generalized black holes have infinite set of local degrees of freedom as opposed to the finite dimensional families of the Kerr, Kerr-de Sitter, and Kerr-anti de Sitter spacetimes [11, 12]. For example, the internal geometry of every vacuum non-extremal isolated horizon $H$ is determined by a metric tensor $g$ and a rotation 1-form potential $\omega$ induced on a spacelike section $S \subset H$ [6]. They are unconstraint, a priori free (in the extremal case that changes drastically [13]). On that data we imposed "the Petrov type D equation" implied by assuming that the spacetime Weyl tensor is of the Petrov type D at the horizon, and that it is Lie dragged by the null symmetry of the horizon geometry [14, 15]. The equation was solved explicitly in the case of $S$ topologically equivalent to two-dimensional sphere $S_2$ and axially symmetric data $(g, \omega)$ [14, 16]. The corresponding isolated horizons are embeddable isometrically in the Kerr, Kerr-de Sitter, or Kerr-anti de Sitter spacetimes [12] depending on the cosmological constant, or in the near extremal Killing horizon limit spacetimes known under the name Near Horizon Geometry [17–22]. If the isolated horizon is bifurcated and both the components are of the
Petrov type D, then the geometry of the horizon is necessarily axially symmetric [23, 24]. That is proven locally, without the rigidity theorem. In that way our results become part of the wider context of local characterizations that can be used to distinguish those globally defined spacetimes [25–29].

The Petrov type D equation also turns out to be a necessary integrability condition for the Near Horizon geometry equation [13].

In the current paper, we consider the Petrov type D equation for the case when $S$ is a two-dimensional closed surface of the genus $> 0$. We derive a general solution for arbitrary value of the cosmological constant.

II. THE PETROV TYPE D EQUATION

The Petrov type D equation is imposed on a Riemannian metric tensor $g_{AB}$ and a 1-form $\omega_A$ defined on a two-dimensional manifold $S$. The equation involves two scalar invariants of that data: the Gaussian curvature $K$ of the metric $g_{AB}$ and the pseudo-scalar $\Omega$ of $\omega$,

$$R_{AB} := K g_{AB}, \quad d\omega := \Omega \eta,$$

where $R_{AB}$ and $\eta$ are the Ricci tensor and the area 2-form, respectively, of the metric tensor $g_{AB}$. The equation uses the complex structure defined by $g_{AB}$, hence it is convenient to express it by a complex null co-frame $m_A$, such that

$$g_{AB} = m_A \bar{m}_B + m_B \bar{m}_A, \quad \eta_{AB} = i (\bar{m}_A m_B - \bar{m}_B m_A).$$

With that notation, and with the covariant derivative $D_A$ defined by $g_{AB}$ such that,

$$D_A g_{BC} = 0, \quad (D_A D_B - D_B D_A) f = 0$$

for every function $f$, the Petrov type D equation reads

$$\bar{m}^A m^B D_A D_B \left( K - \frac{\Lambda}{3} + i \Omega \right)^{-\frac{1}{2}} = 0,$$

where $\Lambda$ is a constant. We are also assuming the non-degeneracy condition

$$K - \frac{\Lambda}{3} + i \Omega \neq 0$$

at every point of $S$.

The equation is invariant with respect to the gauge transformations

$$\omega \mapsto \omega + dh, \quad h \in C^m(S)$$
and with respect to every diffeomorphism $\Phi : S \to S$, 
\[(g, \omega) \mapsto (\Phi^* g, \Phi^* \omega).\] (6)

A two-dimensional surface $S$ equipped with a metric tensor $g_{AB}$ and a 1-form $\omega^A$ determines the spacetime geometry at a non-extremal stationary to the second order null surface $H$ diffeomorphic to $S \times \mathbb{R}$ and contained in a 4-dimensional spacetime $M$ endowed with a metric tensor $g_{\mu\nu}$ that satisfies the vacuum Einstein equations with a cosmological constant $\Lambda$:
\[R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 0.\] (7)

By stationary to the second order we mean that there exists a vector field $\ell$ in a neighborhood of $H$ tangent to and null at $H$, that Lie drags along $H$ the metric tensor $g_{\mu\nu}$, the spacetime covariant derivative $\nabla_\mu$ and the Riemann tensor $R_{\mu\nu\alpha\beta}$:
\[\mathcal{L}_\ell g_{\mu\nu}|_H = [\mathcal{L}_\ell, \nabla_\nu]|_H = \mathcal{L}_\ell R_{\mu\nu\alpha\beta}|_H = 0.\] (8)

A consequence of those assumptions is that $\ell$ is self parallel at $H$,
\[\ell^\mu \nabla_\mu \ell^\nu|_H = \kappa \ell^\nu,\] (9)
and by the zeroth law of non-expanding null surfaces thermodynamics [6], the function $\kappa$ (surface gravity) is constant
\[\kappa = \text{const.}\] (10)

By non-extremal we mean that
\[\kappa \neq 0.\] (11)

In this construction the 2-surface $S$ is a section of $H$ transversal to the vector field $\ell$. The metric $g_{AB}$ is induced in $S$ by the spacetime metric $g_{\mu\nu}$. The 1-form $\omega^A$ is defined by the spacetime covariant derivative of $\ell$ in the directions tangent to $S$, namely, for every vector $X$ tangent to $S$,
\[X^A \nabla_A \ell^\mu = X^A \omega_A \ell^\mu.\] (12)

By the stationarity assumption [8] and the Einstein equations (7), $g_{AB}$ and $\omega_A$ induced on $S$ determine at $H$: $g_{\mu\nu}, \nabla_\mu$, and $R_{\mu\nu\alpha\beta}$. If $S'$ is another section of $H$, then
\[g'_{AB} = g_{AB}, \quad \omega'_A = \omega_A + \kappa D_A f\] (13)
where we identify $S$ and $S'$ by using the null geodesics in $H$, and $f$ is a function on $S'$. If we hold $H, \ell$ and $S \subset H$ fixed, and vary the spacetime metrics $g_{\mu\nu}$ such that (7 8 11) then the data $(g_{AB}, \omega_A)$ ranges all possible metric tensors and 1-forms.

In the sense explained above, the data $(g_{AB}, \omega_A)$ is free on a section $S$ of $H$. In particular it determines the spacetime Weyl tensor $C_{\mu\nu\alpha\beta}$ at $H$. Now, the Weyl tensor $C_{\mu\nu\alpha\beta}$ at $H$ is of the Petrov type D if and only if $g_{AB}$ and $\omega_A$ satisfy the Petrov type D equation [3].
III. THE PETROV TYPE D EQUATION ON A TWO-DIMENSIONAL TORUS

In this section the 2-manifold $S$ a is a two-dimensional torus $T_2$,

$$S = T_2 = S_1 \times S_1,$$  \hspace{1cm} (14)

where $S_1$ is a circle. We fix on the first copy of $S_1$ a coordinate $\phi \in [0,2\pi)$ and on the second copy a coordinate $\psi \in [0,2\pi)$. They set coordinates

$$(x^A) = (\phi, \psi)$$
on $S$. The coordinates are defined globally on $S$, except for that they are not continuous at $\phi, \psi = 0$. However, the cotangent frame $d\phi, d\psi$ and the dual tangent frame $\partial_\phi, \partial_\psi$ defined globally on $S$, are continuous and smooth everywhere. That property will be important below. Modulo the diffeomorphisms $[6]$, every $C^n$ metric tensor $g_{AB}$ on $S$ can be written in the following form,

$$g_{AB} dx^A dx^B = \frac{1}{P^2} \left( a^2 d\phi^2 + 2ab d\phi d\psi + (1 + b^2)d\psi^2 \right),$$  \hspace{1cm} (15)

where $a, b$ are constants and $P \in C^n(S)$ nowhere vanishes. For the compatibility with the Petrov type D equation (3) we assume $n \geq 4$.

To introduce the complex null basis we define complex coordinates $(z, \bar{z})$,

$$z = \frac{a\phi + b\psi + i\psi}{\sqrt{2}}.$$  \hspace{1cm} (16)

In the new coordinates, the metric tensor $g_{AB}$ reads

$$g_{AB} dx^A dx^B = \frac{2}{P^2} dz d\bar{z},$$  \hspace{1cm} (17)

whereas the area 2-form is

$$\eta = i \frac{1}{P^2} dz \wedge d\bar{z}.$$  \hspace{1cm} (18)

The null tangent and co-tangent frame respectively is

$$m^A \partial_A = P \partial_z, \quad m_A dx^A = \frac{1}{P} d\bar{z}.$$  \hspace{1cm} (19)

Notice, that despite of the discontinuity of the coordinates $(z, \bar{z})$, the tangent frame $(m^A, \bar{m}^A)$, the cotangent frame $(\bar{m}_A, m_A)$, the complex valued vector fields $\partial_z, \partial_{\bar{z}}$ and the complex valued 1-forms $dz, d\bar{z}$ are all globally defined on $S$. 
In those coordinates the differential operator featuring in (3) is a composition (denoted by "\circ") of three operators

\[ \overline{m}^A \overline{m}^B D_A D_B = \partial_{\overline{z}} \circ P^2 \circ \partial_{\overline{z}}. \]  

(19)

Hence, the type D equation (3) may be written now in the following explicit form:

\[ \partial_{\overline{z}} (P^2 \partial_{\overline{z}} f) = 0, \]  

(20)

where

\[ f = \left( K - \frac{\Lambda}{3} + i\Omega \right)^{-\frac{1}{2}}. \]  

(21)

The first consequence of the eq. (20) and of the global properties of the function \( P \) and the vector field \( \partial_{\overline{z}} \) is that \( P^2 \partial_{\overline{z}} f \) is an entire holomorphic function on all of \( S \). Hence, due the compactness of \( S \),

\[ P^2 \partial_{\overline{z}} f = F_0 = \text{const}, \]  

(22)

equivalently

\[ \partial_{\overline{z}} f = \frac{F_0}{P^2}. \]  

(23)

Next, we perform the following calculation

\[ F_0 \int_S \eta = i \int_S \partial_{\overline{z}} f dz \wedge d\overline{z} = -i \int_S d(f dz) = 0, \]  

(24)

where again, the last equation follows from the compactness of \( S \). That implies

\[ F_0 = 0. \]  

(25)

Furthermore, going back to the eq. (23) we conclude that \( f \) is an entire holomorphic function on \( S \), hence it has to be constant,

\[ f = \text{const}. \]

That is a general solution of the eq. (20). Recalling, that in our case \( f \) is given by the eq. (21), we can see that also

\[ K = K_0 = \text{const}, \quad \Omega = \Omega_0 = \text{const}. \]  

(26)

But since \( S \) is a torus, the Gauss-Bonet theorem implies about the Gauss curvature

\[ K_0 \int_S \eta = \int_S K \eta = 0, \]
meaning

\[ K_0 = 0. \]

Similar integral law applies to the rotation scalar \( \Omega \), namely

\[ \Omega_0 \int_S \eta = \int_S \Omega \eta = \int_S d\omega = 0. \]

In conclusion, if \( \Lambda \neq 0 \), then a general solution \((g_{AB}, \omega_A)\) to the Petrov type D equation on

\[ S = T_2 \]

is a flat metric tensor \( g_{AB} \) and a closed 1-form \( \omega \), whereas in the case \( \Lambda = 0 \) there are no solutions because of (4). From the point of view of the reconstruction of a stationary to the second order null surface from \((g, \omega)\), the 1-form \( \omega \) is meaningful only modulo the gauge transformations (15). A 2-torus, however, admits non-trivial de Rham cohomology group. Modulo the diffeomorphisms (6) and the gauge transformations (5), a general solution has the form

\[ g_{AB} dx^A dx^B = \frac{1}{P_0^2} \left(a^2 d\phi^2 + 2 ab d\phi d\psi + (1 + b^2) d\psi^2\right), \quad \omega = A d\phi + B d\psi \quad (27) \]

where \( P_0 > 0, a > 0, b, A, B \) are arbitrary constants.

IV. THE PETROV TYPE D EQUATION ON HIGHER GENUS SURFACES

If \( S \) is an arbitrary two-dimensional surface endowed with a 2 metric tensor \( g \), we can cover it with charts, such that in each of them complex coordinates \((z, \bar{z})\) are defined such that

\[ g_{AB} dx^A dx^B = \frac{2}{P^2} dz d\bar{z}, \quad (28) \]

and the equation

\[ \bar{m}^A \bar{m}^B D_A D_B f = 0, \quad (29) \]

for arbitrary unknown function \( f \) takes the form

\[ \partial_z \left(P^2 \partial_z f\right) = 0. \quad (30) \]

The entry of the parentheses can be considered as a component of the complex vector field

\[ (P^2 \partial_z f) \partial_z = (g^{zA} \partial_A f) \partial_z. \quad (31) \]
The anti-holomorphic derivative $\bar{\partial}$ acts in the vector space of all the vector fields $X^z\partial_z$ with arbitrary coefficients $X^z$ as follows

$$\bar{\partial}X = \partial_z X^z \partial_z \otimes d\bar{z}. \quad (32)$$

The space of vector fields

$$X = X^a \partial_z$$

as well as that operator $\bar{\partial}$ is invariant with respect to all the holomorphic coordinate transformations

$$z' = h(z), \quad h, \bar{h} = 0,$$

meaning that it is independent on choice of the coordinates $(z, \bar{z})$ such that (28). Solutions $X^z\partial_z$ to the equation

$$\bar{\partial}X = 0$$

are called holomorphic vector fields. For every compact orientable 2-surface $S$ the dimension of the space of the holomorphic vector fields is known [33]. In particular, if the genus $> 1$ the dimension is 0,

$$X = 0.$$

Hence, the eq. (30) implies

$$\partial_z f = 0. \quad (33)$$

Then due to the compactness of $S$,

$$f = \text{const.} \quad (34)$$

Hence, as in the case of torus, via (21) it implies

$$K = \text{const}, \quad \Omega = \text{const}. \quad (35)$$

Since on every compact $S$ the rotation scalar $\Omega$ satisfies the constraint

$$\int_S \Omega = 0$$

as before we conclude

$$\Omega = 0. \quad (36)$$
On the other hand $K$ is proportional to the inverse of the area $A$ with a coefficient given by the Gauss-Bonet theorem

$$K = \frac{4\pi (1 - \text{genus})}{A}$$  \hspace{1cm} (37)$$

except for

$$K = \frac{\Lambda}{3}$$  \hspace{1cm} (38)$$

which violates the condition [4].

V. SUMMARY AND DISCUSSION

We derived all the metric tensors $g$ and 1-forms $\omega$ defined on a compact, orientable 2-surface of genus $\geq 1$ that are solutions to the Petrov type D equation [3].

**Theorem 1** A pair $(g, \omega)$ is a solution to the Petrov type D equation with a cosmological constant $\Lambda$ if and only if $g$ has constant Gauss curvature (Ricci scalar)

$$K = \text{const} \neq \frac{\Lambda}{3}$$

and $\omega$ is closed

$$d\omega = 0.$$  

Assumption about the type D could be relaxed to possible type O in some degenerate subsets of $S$. Then more solutions can possibly exist.

The solutions were easy to guess therefore the main result is that there are no other solutions. Still, the family of the solutions is more than zero dimensional. For example, for $S = T_2$ the family of solutions is 5 dimensional. The corresponding isolated horizons (stationary to the second order) are non-rotating - their angular momentum $J = 0$. Therefore, one may conclude, that

**Theorem 2** Every rotating Petrov type D isolated horizon stationary to the second order and contained in a 4-dimensional spacetime that satisfies the vacuum Einstein equations with possibly non-zero cosmological constant, has spacelike section of the topology of a 2-sphere.

Due to the vanishing of the rotation scalar $\Omega$, each of the solutions satisfies also the conjugate Petrov type D equation

$$m^A m^B D_A D_B \left( K - \frac{\Lambda}{3} + i\Omega \right)^{-\frac{3}{2}} = 0.$$
Therefore, by the black hole holograph technique [30–32] one can construct from each $S$ and $(g,\omega)$ a spacetime

$$M = S \times \mathbb{R} \times \mathbb{R}$$

that contains a bifurcated horizon of the bifurcation surface $S$ and such that the Petrov type of the spacetime Weyl tensor is D at the horizon [24].

The Petrov type D equation is a necessary integrability condition for the near horizon geometry equation. Therefore all the solutions to the NHG equation on the 2-surfaces of genus $> 0$ belong to the solutions described by Theorem 1.

**Acknowledgements:** This work was partially supported by the Polish National Science Centre grant No. 2015/17/B/ST2/02871.
10

[17] J. M. Bardeen, G.T. Horowitz, The Extreme Kerr throat geometry: A Vacuum analog of $AdS(2) \times S^2$, Phys. Rev. D 60 (1999), 104030, [arXiv:hep-th/9905099].

[18] T. Pawłowski, J. Lewandowski, J. Jezierski, Spacetimes foliated by Killing horizons, Class. Quantum Grav. 21 (2004), 1237–1252, [arXiv:gr-qc/0306107].

[19] H. K. Kunduri, J. Lucietti, Classification of Near-Horizon Geometries of Extremal Black Holes, Living Rev. Rel. 16 (2013), 8, [http://www.livingreviews.org/lrr-2013-8], arXiv:abs/1306.2517.

[20] J. Podolský, M. Žofka, General Kundt spacetimes in higher dimensions, Class. Quantum Grav. 26 (2009), 105008.

[21] J. Podolský, R. Švarc, Physical interpretation of Kundt spacetimes using geodesic deviation, Class. Quantum Grav. 30 (2013), 205016.

[22] J. Lewandowski, A. Szereszewski, P. Waluk, Spacetimes foliated by non-expanding and Killing horizons: higher dimension, Phys. Rev. D 94 (2016), 064018.

[23] I. Rácz, private communication.

[24] J. Lewandowski, A. Szereszewski, The axial symmetry of Kerr without the rigidity theorem, [arXiv:1803.09241]

[25] M. Mars, A spacetime characterization of the Kerr metric, Class. Quantum Grav. 16 (1999), 2507–2523.

[26] M. Mars, Uniqueness properties of the Kerr metric, Class. Quantum Grav. 17 (2000), 3353–3373.

[27] A. Coley, D. McNutt, Identification of black hole horizons using scalar curvature invariants, Class. Quantum Grav. 35 (2018), 025013.

[28] D. Brooks, P.C. Chavy-Waddy, A.A. Coley, A. Forget, D. Gregoris, M.A.H. MacCallum, D.D. McNutt, Cartan invariants and event horizon detection, Gen. Rel. Gravit. 50:37 (2018), [arXiv:1709.03362 [gr-qc]].

[29] A.A. Coley, D.D. McNutt, A.A. Shoom, Geometric horizons, Phys. Lett. B 771 (2017), 131–135.

[30] I. Rácz, Stationary black holes as holographs, Class. Quantum Grav. 24 (2007), 5541–5571.

[31] I. Rácz, Stationary black holes as holographs II., Class. Quantum Grav. 31 (2014), 035006.

[32] J. Lewandowski, I. Rácz, A. Szereszewski, Near Horizon Geometries and Black Hole Holograph, Phys. Rev. D 96 (2017), 044001.

[33] P. Griffiths, J. Harris Principles of algebraic geometry, John Wiley & Sons (1978).