Asymptotic approach to the Schrödinger equation in the presence of a screened Coulomb potential and a uniform field

Marco Rosales-Vera

Department of Physics, Universidad Gabriela Mistral, Avenida Ricardo Lyon 1177, Santiago, Chile

E-mail: mrosa001@im2.codelco.cl

Received 25 September 2014, revised 14 January 2015
Accepted for publication 3 February 2015
Published 30 April 2015

Abstract

In this paper, the Schrödinger equation in the presence of a screened Coulomb potential and a uniform field is analysed using matched asymptotic expansions. When the cup well potential has a very short range, approximate analytical expressions for the energy levels and the lifetime of the system are found. The results are compared with those described in the literature. This paper may be helpful for undergraduate and graduate students in physics as an introductory problem in the application of asymptotic matching applied to quantum mechanics.

Keywords: Schrödinger equation, matched asymptotic expansions, quasi-stationary states

1. Introduction

Several authors have studied the stationary states of the one-dimensional cup well potential type:

\[ V(x) = -\frac{g}{2\varepsilon} e^{-\frac{|x|}{\varepsilon}}, \]  \hspace{1cm} (1)

where \( \varepsilon \) denotes the screening distance of the potential and \( g \) denotes the coupling constant that determines the depth of the cusp well potential. This potential becomes sharper as the...
shape parameter, $\varepsilon$, becomes smaller. In fact, the asymptotic limit, $\varepsilon \to 0$, $\delta \to -Vxg(x)$.

Potential (1) can also be regarded as a limit case of the Woods–Saxon potential well.

Znojil [1] suggests that this potential can be taken as a good description of heavy quarks. Rao and Kagali [2] obtain analytic solutions of a relativistic spinless particle in a one-dimensional screened Coulomb potential; they also illustrate the existence of several genuine bound states and make comparisons to the energy levels of bound states between the Schrödinger equation, Klein–Gordon equation, and Dirac equation. Dominguez-Adame et al. [3] solve the Dirac equation with the one-dimensional screened Coulomb potential. Villalba and Rojas [4] study the bound states of the Klein–Gordon equation with short-range potentials.

We use the one-dimensional superposition of a screened Coulomb potential and a uniform field, $F$, of the type

$$V(x) = -\frac{g}{2\varepsilon} e^{-\frac{|x|}{\varepsilon}} + Fx.$$  \hspace{1cm} (2)

With the presence of an external homogeneous field, the steady states become unstable. These previously bound states survive as (quasistationary) resonant states with a certain half-lifetime, which is to say they become states of decay with time. When the magnitude of the field is small, the average life resonate state is large and the energy is close to that of the atom undisturbed, but as the magnitude of field increases, the average life decreases and the shift in energy level becomes significant. Figure 1 shows the general shape of potential (2) in cases where $\varepsilon \sim O(1)$ (left) and $\varepsilon << 1$ (right).
All energy of a resonant state \([5]\) can be represented by \(E = \text{Re}(E) + i \text{Im}(E)\). The width, \(\Gamma\), of the resonance is related to the imaginary part of the energy through \(\Gamma = -2\text{Im}(E)\). The relationship between the half-life, \(\tau\), and the resonance width is \(\tau = \frac{\Gamma}{2}\).

In this paper we analyse the Schrödinger equation by matched asymptotic expansions to potential (2) in the case when the cup well potential has a very short range. This issue is closely linked to the Stark effect in hydrogen atoms \([6–8]\), where both systems have certain similar characteristics \([9]\). Beginning in the early 1990s, several authors introduced the complex scaling method to solve the problem of resonant states in quantum mechanics \([10, 11]\). Recently, Lin and Ho \([12]\) applied the complex scaling method to solve the Stark shifts and widths of the screened Coulomb potential.

2. Asymptotic analysis

The Schrödinger equation independent of the time for this potential is given by

\[
\frac{d^2 \psi(x)}{dx^2} + \frac{2m}{\hbar^2} \left( E + \frac{g}{2e} e^{-\frac{|x|}{\lambda}} - Fx \right) \psi(x) = 0.
\]  

(3)

If \(\varepsilon\) is a positive small parameter, the cup potential has a very short range, and the equation can be solved by matching between two asymptotic solutions using a technique similar to that performed in \([13]\). We divide the domain, \(x\), into two regions: the near field where \(|x| \sim \varepsilon\), and the far field where \(|x| > \varepsilon\). These two regions define two types of solutions: the inner solution and the outer solution.

2.1. Inner solution

In the region where \(|x| \sim O(\varepsilon)\), we defined the inner coordinate, \(\eta = \frac{x}{\varepsilon}\), so the following equation is obtained:

\[
\frac{d^2 \psi_\eta(\eta)}{d\eta^2} + \frac{2m}{\hbar^2} \left( \varepsilon^2 E + \varepsilon \frac{g}{2e} e^{-\eta} - \varepsilon^3 F \right) \psi_\eta(\eta) = 0 \quad \eta > 0.
\]  

(4)

The solution of (3.1) can be approximated to the following inner asymptotic expansion:

\[
\psi_\eta(\eta) = \phi_0^{(+)}(\eta) + \varepsilon \phi_1^{(+)}(\eta) + \varepsilon^2 \phi_2^{(+)}(\eta) + \ldots
\]  

(5)

\[
E = E_0 + \varepsilon E_1 + \varepsilon^2 E_2 + \ldots
\]  

(6)

Introducing the asymptotic expansions (5) and (6) into (4), we obtain the following equation for the leading-order and first-order corrections:

\[
\frac{d^2 \phi_0^{(+)}(\eta)}{d\eta^2} = 0,
\]  

(7)

\[
\frac{d^2 \phi_1^{(+)}(\eta)}{d\eta^2} + \frac{mg}{\hbar^2} e^{-\eta} \phi_0^{(+)}(\eta) = 0,
\]  

(8)

\[
\frac{d^2 \phi_2^{(+)}(\eta)}{d\eta^2} - \frac{2mE_0}{\hbar^2} \phi_0^{(+)}(\eta) + \frac{mg}{\hbar^2} e^{-\eta} \phi_1^{(+)}(\eta) = 0.
\]  

(9)
The solution of the system (7–9) is given by
\[ \phi_0^{(+)}(\eta) = a_0^{(+)} + b_0^{(+)} \eta \] (10)
\[ \phi_1^{(+)}(\eta) = -\frac{mg}{\hbar^2}e^{-\eta}a_0^{(+)} - \frac{mg}{\hbar^2}e^{-\eta}b_0^{(+)}(\eta + 2) + a_1^{(+)} + b_1^{(+)} \eta. \] (11)

\[ \varphi_2^{(+)}(\eta) = \frac{2mE_0}{\hbar^2} \left( \frac{1}{2}a_0^{(+)}\eta^2 + \frac{1}{6}b_0^{(+)}\eta^3 \right) + \frac{1}{4} \left( \frac{mg}{\hbar^2} \right)^2 a_0^{(+)}e^{-2\eta} + \frac{3}{4} \left( \frac{mg}{\hbar^2} \right)^2 b_0^{(+)}e^{-2\eta} + \frac{1}{4} \left( \frac{mg}{\hbar^2} \right)^2 b_0^{(-)}e^{-2\eta} \]
\[ + \frac{1}{4} \left( \frac{mg}{\hbar^2} \right)^2 b_0^{(+)}e^{-2\eta} - \frac{mg}{\hbar^2}b_1^{(+)}e^{-\eta}(\eta + 2) - \frac{mg}{\hbar^2}a_1^{(+)}e^{-\eta} + b_2^{(+)} + a_2^{(+)} \eta. \] (12)

Performing a similar analysis to the region \( \eta < 0 \), we have
\[ \varphi_0^{(-)}(\eta) = a_0^{(-)} + b_0^{(-)} \eta, \] (13)
\[ \phi_1^{(-)}(\eta) = -\frac{mg}{\hbar^2}e^{\eta}a_0^{(-)} - \frac{mg}{\hbar^2}e^{\eta}b_0^{(-)}(\eta - 2) + a_1^{(-)} + b_1^{(-)} \eta, \] (14)
\[ \phi_2^{(-)}(\eta) = \frac{2mE_0}{\hbar^2} \left( \frac{1}{2}a_0^{(-)}\eta^2 + \frac{1}{6}b_0^{(-)}\eta^3 \right) + \frac{1}{4} \left( \frac{mg}{\hbar^2} \right)^2 a_0^{(-)}e^{2\eta} - \frac{3}{4} \left( \frac{mg}{\hbar^2} \right)^2 b_0^{(-)}e^{2\eta} \]
\[ + \frac{1}{4} \left( \frac{mg}{\hbar^2} \right)^2 b_0^{(-)}e^{2\eta} - \frac{mg}{\hbar^2}b_1^{(-)}e^{\eta}(\eta - 2) - \frac{mg}{\hbar^2}a_1^{(-)}e^{\eta} + b_2^{(-)} + a_2^{(-)} \eta. \] (15)

The boundary conditions of continuity of \( \phi_0(\eta) \) and \( \phi_0(\eta) \) at \( \eta = 0 \) lead to
\[ a_0^{(+)} = a_0^{(-)} = a_0, \quad b_0^{(+)} = b_0^{(-)} = 0, \quad a_1^{(+)} = a_1^{(-)} = a_1 \]
\[ b_1^{(+)} - b_1^{(-)} = -2\frac{mg}{\hbar^2}a_0, \quad a_2^{(+)} - a_2^{(-)} = 2\frac{mg}{\hbar^2}(b_1^{(+)} + b_1^{(-)}) \]
\[ b_2^{(+)} - b_2^{(-)} = \frac{3}{2} \left( \frac{mg}{\hbar^2} \right)^2 a_0 - 2\frac{mg}{\hbar^2}a_1. \] (16)

2.2. Outer solution

In the region where \( \epsilon \ll |x| \), the cup potential is negligible, and we have
\[ \frac{d^2\phi_0(x)}{dx^2} + \frac{2m}{\hbar^2}(E - Fx)\phi_0(x) = 0. \] (18)

The general solution of equation (18) is given by
\[ \phi(x) = A\text{Ai}(y) + B\text{Bi}(y), \] (19)
where the functions \( \text{Ai}, \text{Bi} \) are the regular and irregular Airy functions, and the variable, \( y \), is defined as \( y = \left( \frac{2m}{\hbar^2} \right)^{1/3} \left( x - \frac{E}{F} \right) \). The wave, \( \phi(x) \), must become an outgoing wave as \( x \to -\infty \). From the asymptotic behaviour of the Airy functions [10], we have
\[ \text{Ai}(y) \sim \pi^{-1/2} (-y)^{-1/4} \sin \left( \xi + \frac{\pi}{4} \right) \quad \xi = \frac{2}{3}(-y)^{3/2} \quad y \to -\infty, \quad (20) \]

\[ \text{Bi}(y) \sim \pi^{-1/2} (-y)^{-1/4} \cos \left( \xi + \frac{\pi}{4} \right) \quad \xi = \frac{2}{3}(-y)^{3/2} \quad y \to -\infty. \quad (21) \]

This result shows that the outer solution in the region \( x < 0 \) is given by

\[ \phi(x) = b(\text{Ai}(y) - i \text{Bi}(y)) \quad -x \gg \varepsilon. \quad (22) \]

On the other hand, in the region \( x \to \infty \) we have \([14]\)

\[ \text{Ai}(y) \sim \frac{1}{2} \pi^{-1/2} y^{-1/4} e^{-\xi} \left( 1 - \frac{c_1}{y^{3/2}} + \frac{c_2}{y^3} + \cdots \right) \quad \xi = \frac{2}{3} y^{3/2} \quad y \to \infty, \quad (23) \]

\[ \text{Bi}(y) \sim \pi^{-1/2} y^{-1/4} e^{\xi} \left( 1 + \frac{c_1}{y^{3/2}} + \frac{c_2}{y^3} + \cdots \right) \quad \xi = \frac{2}{3} y^{3/2} \quad y \to \infty, \quad (24) \]

where \( c_1 = \frac{5}{48} \) and \( c_2 = \frac{385}{4608} \).

In this region, the physically acceptable solution is given by:

\[ \phi(x) = a \text{Ai}(y) \quad x \gg \varepsilon \quad (25) \]

since this function decays rapidly in this region.

### 2.3. Matching procedure

In the region \( \varepsilon \ll x \ll 1 \), the inner and outer solutions are valid, so \((5)\) and \((25)\) must make a match in it.

In the region \( \varepsilon \ll x \ll 1 \), the inner solution \((5)\) has the following asymptotic representation when \( \eta \to \infty \):

\[ \phi^{(+)}(\eta) \sim a_0 + \varepsilon \left( a_1^{(+)} + b_1^{(+)} \eta \right) + \varepsilon^2 \left( \frac{mE_0}{\hbar^2} a_0 \eta^2 + b_2^{(+)} \eta + a_2^{(+)} \right). \quad (26) \]

In the region \( x \ll 1 \), the outer solution \((25)\) has the following asymptotic representation:

\[ \phi(x) \sim a \text{Ai}(y_0) + a \text{Ai}'(y_0) \left( \frac{2m}{\hbar^2} F \right)^{1/3} x, \quad (27) \]

where \( y_0 = -\left( \frac{2m}{\hbar^2} \right)^{1/3} \frac{E}{F^{3/4}} \).

The matching between \((26)\) and \((27)\) delivers the following equations:

\[ a \text{Ai}(y_0) = a_0 + \varepsilon a_1^{(+)} + \varepsilon^2 a_2^{(+)} + \cdots \quad (28) \]

\[ a \text{Ai}'(y_0) \left( \frac{2m}{\hbar^2} F \right)^{1/3} = b_1^{(+)} + \varepsilon b_2^{(+)} + \cdots \quad (29) \]

On the other hand, in the region \( x < 0, \varepsilon \ll -x \ll 1 \), the inner and outer solutions are valid, so this solution also must make a match in it.

In the region \( \varepsilon \ll -x \), the inner solution has the following asymptotic representation:

\[ \phi_0^{(-)}(\eta) \sim a_0 + \varepsilon \left( a_1^{(-)} + b_1^{(-)} \eta \right) + \varepsilon^2 \left( \frac{mE_0}{\hbar^2} a_0 \eta^2 + b_2^{(-)} \eta + a_2^{(-)} \right). \quad (30) \]
In the region $-x \ll 1$, the outer solution has the following asymptotic representation:

$$\phi(x) = b \left( Ai(y_0) - i Bi(y_0) \right) + b \left( Ai'(y_0) - i Bi'(y_0) \right) \left( \frac{2m}{\hbar^2} F \right)^{1/3} x. \quad (31)$$

The matching between (30) and (31) delivers the following equations:

$$b \left( Ai(y_0) - i Bi(y_0) \right) = a_0 + \varepsilon a_1^{(-)} + \varepsilon^2 a_2^{(-)} + \ldots \quad (32)$$

$$b \left( Ai'(y_0) - i Bi'(y_0) \right) \left( \frac{2m}{\hbar^2} F \right)^{1/3} = b_1^{(-)} + \varepsilon b_2^{(-)} + \ldots \quad (33)$$

From equations (17), (28), and (32) we obtain

$$a Ai(y_0) - b \left( Ai(y_0) - i Bi(y_0) \right) = \varepsilon^2 \left( a_2^{(-)} - a_2^{(+)} \right) + O(\varepsilon^3). \quad (34)$$

From equations (17), (29), and (33) we obtain

$$\left[ a Ai'(y_0) - b \left( Ai'(y_0) - i Bi'(y_0) \right) \right] \left( \frac{2m}{\hbar^2} F \right)^{1/3} = -2 \frac{mg}{\hbar^2} (a_0 + \varepsilon a_1) + \left( \frac{mg}{\hbar^2} \right)^2 a_0 \varepsilon + O(\varepsilon^3). \quad (35)$$

With the aid of equation (28), equations (34) and (35) can be written to the first order in $\varepsilon$ as

$$a Ai(y_0) - b \left( Ai(y_0) - i Bi(y_0) \right) = 0 \quad (36)$$

$$\left[ a Ai'(y_0) - b \left( Ai'(y_0) - i Bi'(y_0) \right) \right] \left( \frac{2m}{\hbar^2} F \right)^{1/3} = -2 \frac{mg}{\hbar^2} \left( 1 - \frac{3}{2} \left( \frac{mg}{\hbar^2} \right)^2 \right) a Ai(y_0). \quad (37)$$

Removing the constants $a$ and $b$ of equations (36) and (37) and with the aid of the relationship:

$$Ai'(y_0) Bi(y_0) - Ai(y_0) Bi'(y_0) = \frac{1}{\pi}$$

the following equation is obtained:

$$Ai \left( y_0 \right) Bi'(y_0) - Ai'(y_0) Bi(y_0) = \frac{i}{\pi} \left( \frac{2m}{\hbar^2} \right)^{1/3} \frac{F^{1/3}}{k}, \quad (38)$$

where $k = \frac{2mg}{\hbar^2} \left( 1 - \frac{3}{2} \left( \frac{mg}{\hbar^2} \right)^2 \right) \varepsilon$.

3. Results and discussion

The solution of equation (38) leads the first-order correction to the values of energy to the quasistationary states in the case when the cup potential well has a very short range, $\varepsilon \ll 1$. Note that equation (38) has the same shape as that obtained by both Fernández and Castro [8] and Aquino [11] in their studies of the quasistationary states of a system with a delta potential and a uniform electric field.

To carry out the solution of equation (38), we consider the case where $-\left( \frac{2m}{\hbar^2} \right)^{1/3} \frac{E}{F^{1/3}} \gg 1$. Then, the Airy function in equation (38) can be expressed by its asymptotic representation
and (24), namely,
\[
\frac{1}{4} e^{-2\xi_0} \left(1 - \frac{2c_1}{\gamma_0^{3/2}}\right) - i \frac{1}{2} \left(1 + \left(2c_2 - c_1^2\right) \frac{1}{\gamma_0^{3/2}}\right) = -i \left(\frac{2m}{\hbar^2}\right)^{1/2} (-E)^{1/2} / \kappa. \tag{39}
\]

Assuming that the energy spectrum is given by \( E = \text{Re}(E) + i \text{Im}(E) \), we obtain the following solution to equation (39):
\[
\text{Re}(E) = -\frac{1}{2} \frac{m^2}{\hbar^2} \left(1 - 3 \left(\frac{m}{\hbar^2}\right)^2 \epsilon \right) \left(1 + \frac{5}{4} \left(\frac{\hbar^4}{m^2 g^3}\right)^2 F^2 + \ldots\right)
\]
\[
\text{Im}(E) = -\frac{1}{2} \frac{m^2}{\hbar^2} \left(1 - 3 \left(\frac{m}{\hbar^2}\right)^2 \epsilon \right) \left(1 - \frac{5}{12} \left(\frac{\hbar^4}{m^2 g^3}\right) F + \ldots\right) e^{-2\xi_0}, \tag{40}
\]

where \( \xi_0 = \frac{\sqrt{\pi} \sqrt{\epsilon}}{3 \hbar^2} \left(1 - \frac{9 m^2}{\hbar^2 \epsilon}\right) \).

In the case where \( F = 0 \) and \( \epsilon = 0 \), we have that \( \text{Im}(E) \to 0 \) exponentially; equation (40) represents the well-known value of the energy of the bound state with a delta potential.

In the case where \( F = 0 \) and \( \epsilon \ll 1 \), equation (40) represents the lowest value of the energy of the bound state with a screened Coulomb potential that has a very short range, but is finite. This result holds if \( \left(\frac{m}{\hbar^2}\right)^2 \epsilon \ll 1 \). In this case, equation (40) shows the existence of only a single bound quasistationary state with a slightly higher energy than that of a delta potential, \( \Delta E = \frac{3 m^2 \epsilon^2}{2 \hbar^2} \). This effect is in agreement with the numerical results obtained by Rao et al [2], showing that an increase in the parameter, \( \epsilon \), causes an increase in the ground state energy of the system.

Notably, in the case where \( F = 0 \), the problem has an exact solution, which is given by:
\[
\phi(\eta) = c_1 J_\nu \left(\beta e^{\frac{\pi}{2}}\right) + c_2 J_{-\nu} \left(\beta e^{-\frac{\pi}{2}}\right),
\]
where \( \nu^2 = \frac{\text{Im} \left| E \right| \epsilon^2}{\hbar^2} \), \( \beta^2 = \frac{4 \text{ange} \epsilon}{\hbar^2} \).

For odd parity states, the wave function it must satisfy, \( \phi(\eta) = 0 \), leads to
\[
\phi(\eta) = c_1 J_\nu (\beta) J_{-\nu} \left(\beta e^{-\frac{\pi}{2}}\right) - J_{-\nu} (\beta) J_\nu \left(\beta e^{-\frac{\pi}{2}}\right).
\]

Further, the eigenvalues of those solutions must satisfy the boundary condition, \( \eta \to \infty \), \( \lim_{\eta \to \infty} \phi(\eta) = 0 \). Then it follows from the behaviour of the Bessel function, \( J_\nu (x) \sim \left(\frac{x}{\nu}\right)^{\nu} \frac{1}{\Gamma(1 + \nu)} \) when \( x \ll 1 \), so the eigenvalues satisfy the condition \( J_\nu (\beta) = 0 \) in the limit \( \epsilon \to 0 \). We have \( J_\nu (\beta) \sim \left(\frac{\beta}{2}\right)^{\nu} \frac{1}{\Gamma(1 + \nu)} = 1 \), so the condition \( J_\nu (\beta) = 0 \) cannot be satisfied, which means that odd solutions do not exist in the case where \( \epsilon \to 0 \). For even-parity states, the wave function it must satisfy is \( \phi'(\eta) = 0 \). Further, with the boundary condition \( \lim_{\eta \to \infty} \phi(\eta) = 0 \), the following equation must be satisfied: \( J_{\nu-1} (\beta) = J_{\nu+1} (\beta) \). This equation in the limit \( \epsilon \to 0 \) leads to \( \nu = \frac{1}{2} \beta^2 \), having obtained a single-energy level given by
\[
E = -\frac{1}{2} \frac{m^2}{\hbar^2}, \text{ coinciding with (40) when } \epsilon = 0.
\]

The validity of equation (40) is verified in table 1, where the numerical results found by [2] for the ground state energy for the Schrödinger equation are compared with equation (40).
in the case where $F = 0$. Table 1 shows the binding energy in units of $mc^2$; $\varepsilon$ is expressed in units of $\hbar mc^2/(\hbar c)$ and $g$ in units of $\hbar c$.

Figure 2 shows the theoretical results of the ground state energy for different values of the screening distance of the potential, $\varepsilon$. The dashed line represents equation (40), and the points represent numerical values.

Figure 3. Typical behaviour of the wave function in the case where $\varepsilon = 0$ and $F \neq 0$.

Table 1. Binding energies of the Schrödinger equation in the presence of a screened Coulomb potential ($\varepsilon = \hbar/(mc)$).

| $\frac{\varepsilon}{\hbar c}$ | $E (mc^2)^{-1}$ [2] | $E (mc^2)^{-1}$ equation (40) |
|-------------------------------|---------------------|-------------------------------|
| 0.000 003                     | 0.000 00            | $-4.5 \times 10^{-12}$        |
| 0.001                         | $-4.9851 \times 10^{-7}$ | $-4.9850 \times 10^{-7}$     |
| 0.01                          | $-4.8557 \times 10^{-5}$ | $-4.850 \times 10^{-5}$     |
| 0.05                          | $-1.0929 \times 10^{-3}$ | $-1.10 \times 10^{-3}$     |
| 0.1                           | $-0.003 913$        | $-0.0035$                     |

in the case where $F = 0$. Table 1 shows the binding energy in units of $mc^2$; $\varepsilon$ is expressed in units of $\hbar/(mc)$ and $g$ in units of $\hbar c$.

Figure 2 shows the theoretical results of the ground state energy for different values of the screening distance of the potential, $\varepsilon$ (equation (40)), compared to numerical values found by direct integration of the Schrödinger equation. This calculation was performed by
choosing \( g = \hbar = m = 1 \). The results show that the theoretical result agrees quite well with the numerical calculation.

In the case where \( \varepsilon = 0 \) and \( F \neq 0 \), equations (40) and (41) are reduced to the solutions found by both Fernández and Castro [8] and Aquino [11] in their studies of the quasistationary states of a system with a delta potential and a uniform electric field. Figure 3 shows the typical behaviour of the wave function in this case, with an outgoing wave as \( x \to -\infty \). The wave function decays rapidly as \( x \to \infty \).

In the case where \( \varepsilon \ll 1 \) and \( F \neq 0 \), equations (40) and (41) provide a first-order correction if the width of the potential well is of short range but finite. As in the case of [9] and [15], equation (40) shows the existence of only a single bound quasistationary state with a slightly higher energy than that of a delta potential:

\[
\Delta E = \frac{3 m^2 g^3}{2 \hbar^2} \left( 1 + \frac{5}{4} \left( \frac{\hbar^4}{m^2 g^3} \right)^2 F^2 \right) \varepsilon.
\]

Figure 4 shows the energy level of the ground state as a function of the field strength for three values of the screening parameter, \( \varepsilon \) (equation (40)). This calculation was performed by choosing \( g = \hbar = m = 1 \).

Equations (40) and (41) have some similar characteristics with respect to other, more complex problems, such as the Stark effect in hydrogen atoms [5–8]. For example, the effect of the electric field at the ground state of energy is quadratic in \( F \); this same effect was found numerically by Lin et al [12] for the Stark effect in the Yukawa potential. Another similarity with the Stark effect in hydrogen atoms is the exponential factor, \( \exp \left( -\frac{2 m^2 g^3}{3 \hbar^2} \right) \), of the width, \( \Gamma \), of the resonance of equation (41). This behaviour also appears in the numerical results found by Lin et al [12].

Defined as \( \tau_0 \) to the half-life of this quasistationary state when \( \varepsilon = 0 \) and \( \tau \) to the half-life of this quasistationary state when \( \varepsilon \neq 0 \), we have

\[
\rho = \frac{\hbar}{m^2 g^3} \varepsilon = \frac{\hbar}{m^2 g^3} \left( 1 + \frac{5}{4} \left( \frac{\hbar^4}{m^2 g^3} \right)^2 F^2 \right) \varepsilon.
\]
$$\frac{\tau_e}{\tau_0} \equiv \exp \left( -\frac{3m^2g^4}{F \hbar^6} \varepsilon \right) \left( 1 - \frac{mg^3}{\hbar^2} \varepsilon \right).$$

(42)

Expanding to the first order in $\varepsilon$, we have

$$\frac{\tau_e}{\tau_0} \approx 1 + 3 \frac{mg^3}{\hbar^2} \frac{1 - m^2g^3}{F \hbar^4} \varepsilon.$$  

(43)

This result shows that for values $\frac{mg^3}{\hbar^2} < 1$, the half-life of the system increases for finite values of the width of the potential well, while for values $\frac{mg^3}{\hbar^2} > 1$, the half-life of the system decreases for finite values of the width of the potential well.

Students can see that the advantage of the asymptotic method described in this paper is that this method allows one to find the eigenvalues of energy directly in a very simple way, and also to find and build the total wave function with minimal computational cost compared to the usual numerical methods. A good description of the asymptotic methods can be found in [16, 17].

Students can extend the method of matched asymptotic expansions used in this work to other types of short-range potential, as well as also to other equations such as the Klein–Gordon and Dirac equations. Another possible extension of this work is the multidimensional formulation of the problem. To conduct this analysis, it is necessary to have an appropriate coordinate system for the separation of variables (parabolic coordinates), transforming the problem into two uncoupled differential equations. Thereafter, the problem is quite similar to those already solved.

References

[1] Znojil M 1983 Analytic green function and bound states for the screened Coulomb potential $V(r) = F/r^2 + G/r + H/(r+z^2)$ Phys. Lett. A 94 120–4
[2] Rao N A and Kagali B A 2002 Spinless particles in screened Coulomb potential Phys. Lett. A 296 192–6
[3] Dominguez-Adame F and Rodriguez A 1995 A one-dimensional relativistic screened Coulomb potential Phys. Lett. A 198 275–8
[4] Villalba V M and Clara R 2006 Bound states of the Klein–Gordon equation in the presence of short range potentials Int. J. Mod. Phys. A 21 313–25
[5] Landau L D and Lifshitz E (ed) 1978 Quantum Mechanics (Barcelona: Reverté)
[6] Damburg R J and Kolosov V V 1978 An asymptotic approach to the Stark effect for the hydrogen atom J. Phys. B: At. Mol. Phys. 11 1921
[7] Yamabe T, Tachibana A and Harris J S 1977 Theory of the ionization of the hydrogen atom by an external electrostatic field Phys. Rev. A 16 877
[8] Nicolaides C A and Spyros I T 1992 Theory of the resonances of the LoSurdo–Stark effect Phys. Rev. A 45 349
[9] Fernández F M and Eduardo A C 1985 Stark effect in a one-dimensional model atom Am. J. Phys. 53 757–60
[10] Moiseyev N 1998 Quantum theory of resonances: calculating energies, widths and cross-sections by complex scaling Phys. Rep. 302 212–93
[11] Karlsson H O and Osvaldo G 1994 A direct recursive residue generation method: application to photoionization of hydrogen in static electric fields J. Phys. B: At. Mol. Opt. Phys. 27 1061
[12] Lin C Y and Ho Y K 2011 Complex scaling in Lagrange-mesh calculations for Stark shifts and widths of the screened Coulomb potential J. Phys. B: At. Mol. Opt. Phys. 44 175001
[13] Rosales-Vera M 2012 Matched asymptotic expansions to the circular Sitnikov problem with long period J. Appl. Math. 2012 479093
[14] Abramowitz M and Stegun I A 1954 Handbook of Mathematical Functions (Washington, DC: US Govt Printing Office)
[15] Aquino N A 1990 Modelo unidimensional de un átomo en un campo eléctrico fuerte Rev. Mex. Fís. 36 471–7
[16] Bender C M and Steven A O 1999 Advanced Mathematical Methods for Scientists and Engineers: I. Asymptotic Methods and Perturbation Theory vol 1 (Berlin: Springer)
[17] Nayfeh A H 2011 Introduction to Perturbation Techniques (New York: Wiley)