General post-Minkowskian expansion of time transfer functions

Pierre Teyssandier¹ and Christophe Le Poncin-Lafitte¹,²

¹ Département Systèmes de Référence Temps et Espace, CNRS/UMR 8630, Observatoire de Paris, 61 avenue de l’Observatoire, F-75014 Paris, France
² Lohrmann Observatory, Dresden, Technical University, Mommsenstr. 13, D-01062 Dresden, Germany

Received 22 April 2008, in final form 29 May 2008
Published 30 June 2008
Online at stacks.iop.org/CQG/25/145020

Abstract
Modeling most of the tests of general relativity requires us to know the function relating light travel time to the coordinate time of reception and to the spatial coordinates of the emitter and the receiver. We call such a function the reception time transfer function. Of course, an emission time transfer function may as well be considered. We present here a recursive procedure enabling us to expand each time transfer function into a perturbative series of ascending powers of the Newtonian gravitational constant $G$ (general post-Minkowskian expansion). Our method is self-sufficient in the sense that neither the integration of null geodesic equations nor the determination of Synge’s world function is necessary. To illustrate the method, the time transfer function of a three-parameter family of static, spherically symmetric metrics is derived within the post-linear approximation.

PACS numbers: 04.20.Cv, 04.25.Nx, 04.80.–y

1. Introduction

In many tests of general relativity, it is crucial to explicitly know the function relating light travel time to the coordinate time of reception (or emission) and to the spatial coordinates of the emitter and the receiver. As the accuracy of measurements improves over the years, it will soon be indispensable to take into account the effects on the propagation of light predicted within the post-linear approximation of the relativistic theories of gravity. It will be the case, e.g., if experiments such as the laser astrometric test of relativity (LATOR) [1] or the astrodynamical space test of relativity using optical devices (ASTROD) [2] are carried out.

In most of the studies devoted to determining light travel time, the relativistic effects are calculated by integrating the null geodesic equations (see, e.g., [3, 4]). General results have been obtained within the linear regime [5–8]. However, the implementation of this method comes up against two problems. First, the integration of the null geodesic equations leads
to heavy calculations in the post-post-Minkowskian approximation, even in the simple case of a static, spherically symmetric space-time [9–12]. Second, once the general solution is obtained, it is not easy to deduce from it the null geodesic path which joins an emitter and a receiver that have specified spatial positions.

An alternative procedure enabling us to overcome these two difficulties has recently been developed in [13–15]: the method consists of determining Synge’s world function Ω(x_A, x_B) for any pair of points-events x_A and x_B [16], and then deducing light travel time from the equation Ω(x_A, x_B) = 0 which expresses that x_A and x_B are linked by a null geodesic path. This procedure works well, but presents an unpleasant drawback: once the general post-Minkowskian expansion of the world function is known, obtaining the corresponding expansion of light travel time still requires a lot of additional calculations (cf [15]). It is the purpose of this paper to present a stand-alone method which totally avoids performing the calculation of the world function.

The present work is organized as follows. In section 2 we summarize the notations and conventions used in the paper. In section 3 we recall the definition of the reception time transfer function and of the emission time transfer function, and we show that each of these functions obeys a Hamilton–Jacobi-like partial differential equation. In section 4 we define the reception time delay function and the emission time delay function, and we show that each of these functions satisfies an integro-differential equation. Using this result, we derive in section 5 the general post-Minkowskian expansions of the time transfer functions. In section 6 we focus on the case of a stationary spacetime. In section 7 we apply our method to the gravitational field of a static, spherically symmetric body treated within the second post-Minkowskian approximation. We present some concluding remarks in section 8.

2. Notations

In this work the signature of the Lorentzian metric g is (+, −, −, −). We suppose that spacetime is covered by some global coordinate system (x^μ) = (x^0, x). We put x^0 = ct, c being the speed of light in a vacuum. Greek indices run from 0 to 3 and Latin indices run from 1 to 3. Einstein’s convention on repeated indices is used here for expressions like a^i b^i as well as for expressions like A^μ B^μ or a^i c^i. Bold letters denote ordered triples. Given two triples a = (a^1, a^2, a^3) = (a^i) and b = (b^1, b^2, b^3) = (b^i), we use a · b to denote a^i b^i. The quantity |a| stands for the ordinary Euclidean norm of a: |a| = (δ^i_j a^i a^j)^1/2. When it seems necessary for the sake of legibility, a quantity f(x) is denoted by [f]_x or f_x. The indices in parentheses characterize the order of perturbation. These indices are set up or down, depending on the convenience.

G is the Newtonian gravitational constant.

3. Time transfer functions

We assume that spacetime is globally regular with the topology of \( \mathbb{R} \times \mathbb{R}^3 \), that is without horizon. We suppose in addition that the coordinate system is chosen in such a way that the curves of equation \( x = x_A \) are timelike for any given \( x_A \), which means that \( \partial / \partial x^0 \) is a timelike vector field, i.e. \( g_{00} > 0 \) everywhere. In agreement with these assumptions, we suppose that the past null cone at a given point \( x_B = (ct_B, x_B) \) intersects the world line \( x = x_A \) at one and only one point \( x_A = (ct_A, x_A) \). The difference \( t_B - t_A \) is the (coordinate) travel time of a light ray connecting the emission point \( x_A \) and the reception point \( x_B \). This quantity may be considered either as a function of the instant of reception \( t_B \) and of \( x_A, x_B \), or as a function...
of the instant of emission \( t_A \) and of \( x_A \) and \( x_B \). So it is possible to define two time transfer functions, \( T_r \) and \( T_e \) by putting
\[
t_B - t_A = T_r(x_A, t_B, x_B) = T_e(t_A, x_A, x_B).
\]

We call \( T_r \) the reception time transfer function and \( T_e \) the emission time transfer function. These functions are distinct except in a stationary spacetime in which the coordinate system is chosen so that the metric does not depend on \( x^0 \).

Let us denote by \( k^\mu \) the vector \( dx^\mu/d\zeta \) tangent to the null geodesic path connecting \( x_A \) and \( x_B \), \( \zeta \) being an arbitrary affine parameter of this geodesic. It was shown in [14] that the covariant components of \( k^\mu \) satisfy the relations
\[
\left( \frac{k_0}{k^0} \right)_{x_A} = c \frac{\partial T_r(x_A, t_B, x_B)}{\partial x^i_A} \tag{2}
\]
and
\[
\left( \frac{k_0}{k^0} \right)_{x_B} = -c \frac{\partial T_e(t_A, x_A, x_B)}{\partial x^i_B}. \tag{3}
\]

The following theorem can be inferred from equations (2) and (3).

**Theorem 1.** The time transfer functions \( T_r(x_A, t_B, x_B) \) and \( T_e(t_A, x_A, x_B) \) satisfy the Hamilton–Jacobi-like equations
\[
g^{00}(x^0_B - c T_r, x_A) + 2c g^{0i}(x^0_B - c T_r, x_A) \frac{\partial T_r}{\partial x^i_A} + c^2 g^{ij}(x^0_B - c T_r, x_A) \frac{\partial T_r}{\partial x^i_A} \frac{\partial T_r}{\partial x^j_A} = 0 \tag{4}
\]
and
\[
g^{00}(x^0_A + c T_e, x_B) - 2c g^{0i}(x^0_A + c T_e, x_B) \frac{\partial T_e}{\partial x^i_B} + c^2 g^{ij}(x^0_A + c T_e, x_B) \frac{\partial T_e}{\partial x^i_B} \frac{\partial T_e}{\partial x^j_B} = 0, \tag{5}
\]
respectively.

**Proof of theorem 1.** The covariant components of the vector tangent to \( \Gamma_{AB} \) at \( x_A \) satisfy the equation
\[
(g^{\mu\nu} k_\mu k_\nu)_{x_A} = 0. \tag{6}
\]

Dividing equation (6) side by side by \( (k_0)^2 \), and then taking equation (2) into account yield equation (4).\(^5\) The same reasoning using \( (g^{\mu\nu} k_\mu k_\nu)_{x_B} = 0 \) and equation (3) leads to equation (5).

In what follows, we give detailed proofs only for the reception time transfer function because similar procedures may be applied to the emission time transfer function.

---

\(^3\) Note that the order of the arguments of \( T_r \) in equation (1) is different from that given in [14, 15].

\(^4\) Equations (4) and (5) can also be derived from the general-relativistic version of Fermat’s principle established in [17].

\(^5\) The covariant component \( k_0 \) of a null vector \( k \) cannot vanish in the chosen coordinate system since the vector \( \partial / \partial x^0 \) is assumed to be timelike everywhere.
4. Integro-differential equations satisfied by the time delay functions

Henceforth, we suppose that the metric takes the form

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

throughout spacetime, with $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$. Then the contravariant components of the metric may be decomposed as

$$g^{\mu\nu} = \eta^{\mu\nu} + k^{\mu\nu},$$

the quantities $k^{\mu\nu}$ being determined by the relations

$$k^{\mu\nu} + \eta^{\mu\rho} \eta^{\nu\sigma} h_{\rho\sigma} + \eta^{\mu\rho} h_{\rho\sigma} k^{\nu\sigma} = 0.$$  

According to equation (7) the reception time transfer function may be written as

$$T_r(x_A, t_B, x_B) = \frac{1}{c} R_{AB} + \frac{1}{c} \Delta_r(x_A, t_B, x_B),$$

where

$$R_{AB} = |x_B - x_A|$$

and $\Delta_r(x_A, t_B, x_B)$ is of the order of the gravitational perturbation $h_{\mu\nu}$. The function $\Delta_r/c$ defined by equation (10) may be called the reception time delay function. It is well known that this quantity is $> 0$ in Schwarzschild spacetime, which explains its designation.

Replace now $x_A$ by a variable $x$ and consider $t_B, x_B$ as fixed parameters. Inserting $T_r(x, t_B, x_B) = |x_B - x|/c + \Delta_r(x, t_B, x_B)/c$ into equation (4) taken at $x$ instead of $x_A$, we get an equation which may be written in the form

$$2N^i \frac{\partial \Delta_r(x, t_B, x_B)}{\partial x^i} = -W(x, t_B, x_B),$$

where $N^i$ is defined as

$$N^i = \frac{x_B - x^i}{|x_B - x|}$$

and $W(x, t_B, x_B)$ is given by

$$W(x, t_B, x_B) = k^0(x_\nu) - 2N^i k^i(x_\nu) + N^i N^j k^{ij}(x_\nu) + 2[k^0(x_\nu) - N^j k^{ij}(x_\nu)] \frac{\partial \Delta_r(x, t_B, x_B)}{\partial x^i} + [\eta^{ij} + k^{ij}(x_\nu)] \frac{\partial \Delta_r}{\partial x^i} \frac{\partial \Delta_r}{\partial x^j},$$

$x_\nu$ being the point-event defined by

$$x_\nu = \left(x^0_B - |x_B - x| - \Delta_r(x, t_B, x_B), x \right).$$

Since $x$ is a free variable, consider the case where $x$ is varying along the straight segment joining $x_A$ and $x_B$. Then we have

$$N^i = N^i_{AB},$$

where $N^i_{AB}$ is by definition

$$N^i_{AB} = \frac{x_B^i - x_A^i}{R_{AB}}.$$

and we can put

$$x = z_-(\lambda),$$
where
\[ z_-(\lambda) = x_B - \lambda R_{AB} N_{AB}, \quad 0 \leq \lambda \leq 1. \] (18)

A straightforward calculation shows that the total derivative of \( \Delta_r(z_-(\lambda), t_B, x_B) \) with respect to \( \lambda \) is given by
\[
\frac{d}{d\lambda} \Delta_r(z_-(\lambda), t_B, x_B) = -R_{AB} N_{AB} \partial \Delta_r \partial x^i \left( z_-(\lambda), t_B, x_B, x_B \right),
\] (19)

where \( \partial \Delta_r \partial x^i \left( z_-(\lambda), t_B, x_B, x_B \right) \) denotes the partial derivative of \( \Delta_r(x, t_B, x_B) \) w.r.t. \( x^i \) taken at \( x = z_-(\lambda) \).

Taking equation (15) into account, and then comparing equation (19) with equation (12), it may be seen that \( \Delta_r(z_-(\lambda), t_B, x_B) \) is governed by the differential equation
\[
\frac{d}{d\lambda} \Delta_r(z_-(\lambda), t_B, x_B) = \frac{1}{2} R_{AB} W(z_-(\lambda), t_B, x_B)
\] (20)

with the boundary condition
\[
\Delta_r(z_-(0), t_B, x_B) = 0
\] (21)

which follows from the obvious requirement \( \Delta_r(x_B, t_B, x_B) = 0 \) and from \( x_B = z_-(0) \). As a consequence \( \Delta_r(z_-(\lambda), t_B, x_B) \) is such that
\[
\Delta_r(z_-(\lambda), t_B, x_B) = \frac{1}{2} R_{AB} \int_{\lambda=0}^{\lambda} W(z_-(\lambda'), t_B, x_B) d\lambda'.
\] (22)

Noting that \( z_-(1) = x_A \), we deduce a theorem as follows from equations (22), (13) and (14).

**Theorem 2.** The function \( \Delta_r \) defined by equation (10) satisfies the integro-differential equation
\[
\Delta_r(x_A, t_B, x_B) = \frac{1}{2} R_{AB} \int_0^1 \left[ (k^{00} - 2N_{AB}^0 k^{0i} + N_{AB}^0 N_{AB}^{ij} k^{ij}) \right]_{\tilde{z}_-(\lambda)}
+ 2(k^{00} - N_{AB}^0 k^{ij})_{\tilde{z}_-(\lambda)} \frac{\partial \Delta_r}{\partial x^i} \left( z_-(\lambda), t_B, x_B \right)
+ \left[ \eta^{ij} + k^{ij} \left( \tilde{z}_-(\lambda) \right) \right] \left( \frac{\partial \Delta_r}{\partial x^i} \frac{\partial \Delta_r}{\partial x^j} \right)_{\left( z_-(\lambda), t_B, x_B \right)} d\lambda,
\] (23)

where \( \tilde{z}_-(\lambda) \) is the point-event defined by
\[
\tilde{z}_-(\lambda) = \left( x_B^0 - \lambda R_{AB} - \Delta_r(z_-(\lambda), t_B, x_B), z_-(\lambda) \right),
\] (24)

\( z_-(\lambda) \) being given by equation (18).

Of course, if we define the emission time delay function \( \Delta_e(t_A, x_A, x_B)/c \) by the equation
\[
\tau_e(t_A, x_A, x_B) = \frac{1}{c} R_{AB} + \frac{1}{c} \Delta_r(t_A, x_A, x_B),
\] (25)
a similar theorem may be stated for \( \Delta_e \).

**Theorem 3.** The function \( \Delta_e \) defined by equation (25) satisfies the integro-differential equation
\[
\Delta_e(t_A, x_A, x_B) = \frac{1}{2} R_{AB} \int_0^1 \left[ (k^{00} - 2N_{AB}^0 k^{0i} + N_{AB}^0 N_{AB}^{ij} k^{ij}) \right]_{\tilde{z}_-(\lambda)}
- 2(k^{00} - N_{AB}^0 k^{ij})_{\tilde{z}_-(\lambda)} \frac{\partial \Delta_e}{\partial x^i} \left( t_A, x_A, z_+(\mu) \right)
+ \left[ \eta^{ij} + k^{ij} \left( z_+(\mu) \right) \right] \left( \frac{\partial \Delta_e}{\partial x^i} \frac{\partial \Delta_e}{\partial x^j} \right)_{\left( t_A, x_A, z_+(\mu) \right)} d\mu,
\] (26)
where \( \tilde{z}_+(\mu) \) is the point-event defined by
\[
\tilde{z}_+(\mu) = (x_0^A + \mu R_{AB} + \Delta_\varepsilon(t_A, x_A, z_+(\mu)), z_+(\mu))
\] (27)

with
\[
z_+(\mu) = x_A + \mu R_{AB} N_{AB},
\] (28)

and \( \partial \Delta_\varepsilon(t_A, x_A, z_+(\mu))/\partial x^i \) denotes the partial derivative of \( \Delta_\varepsilon(t_A, x_A, x) \) w.r.t. \( x^i \) taken at \( x = z_+(\mu) \).

Integro-differential equations (23) and (26) are very convenient to obtain the general post-Minkowskian expansion of each of the time transfer functions, as we shall see in the following section.

5. General post-Minkowskian expansion of time delay functions

Henceforth, we suppose that each perturbation term \( h_{\mu\nu} \) is represented at any point \( x \) by a series in ascending powers of the Newtonian gravitational constant \( G \)
\[
h_{\mu\nu}(x, G) = \sum_{n=1}^{\infty} G^n g_{(n)}^{(\mu\nu)}(x).
\] (29)

The concomitant expansion of \( k^{\mu\nu} \) is then given by
\[
k^{\mu\nu}(x, G) = \sum_{n=1}^{\infty} G^n g^{(\mu\nu)}_{(n)}(x),
\] (30)

where the set of quantities \( g_{(n)}^{(\mu\nu)} \) can be recursively determined by using the relations
\[
g_{(1)}^{(\mu\nu)} = -\eta^{\mu\rho} \eta^{\nu\sigma} (1)
\] (31)

and
\[
g_{(n)}^{(\mu\nu)} = -\eta^{\mu\rho} \eta^{\nu\sigma} g_{(n)}^{(\rho\sigma)} - \sum_{p=1}^{n-1} \eta^{\mu\rho} g_{(p)}^{(\rho\sigma)} g_{(n-p)}^{(\nu\sigma)}
\] (32)

for \( n \geq 2 \).

As a consequence, the reception time delay function admits the expansion
\[
\Delta_r(x, t_B, x_B, G) = \sum_{n=1}^{\infty} G^n \Delta^{(n)}_r(x, t_B, x_B).
\] (33)

It follows from equations (30) and (24) that each term \( k^{\mu\nu} \) involved on the right hand side of equation (23) may be written as
\[
k^{\mu\nu}(\tilde{z}_-(\lambda), G) = \sum_{n=1}^{\infty} G^n g_{(n)}^{(\mu\nu)}(x_0^B - \lambda R_{AB} - \Delta_\varepsilon(\tilde{z}_-(\lambda), t_B, x_B, G), z_-(\lambda)).
\] (34)

The general post-Minkowskian expansion of \( k^{\mu\nu}(\tilde{z}_-(\lambda), G) \) is obtained by substituting for \( \Delta_\varepsilon \) from equation (33) into equation (34), and then performing Taylor’s expansion about the point \( z_-(\lambda) \) defined by
\[
z_-(\lambda) = (x_0^B - \lambda R_{AB}, z_-(\lambda)).
\] (35)

Let us put
\[
\Phi^{(m,k)}(x, t_B, x_B) = \frac{(-1)^k}{k!} \sum_{l_1 + \cdots + l_k = m - k} \left[ \prod_{j=1}^{k} \Delta^{(l_j+1)}_r(x, t_B, x_B) \right],
\] (36)
where \( l_1, l_2, \ldots, l_k \) are either positive integers or zero \((m \geq 1 \text{ and } 1 \leq k \leq m)\). A straightforward calculation yields

\[
k^{\mu} (\zeta_-(\lambda), G) = \sum_{n=1}^{\infty} G^n g^{\mu\nu} (\zeta_-(\lambda), t_B, x_B),
\]

where the quantities \( \hat{g}^{\mu\nu} (\zeta_-(\lambda), t_B, x_B) \) are given by

\[
\hat{g}^{\mu(1)} (\zeta_-(\lambda)) = g^{(1)} (\zeta_-(\lambda))
\]

and

\[
\hat{g}^{\mu\nu} (\zeta_-(\lambda), t_B, x_B) = \frac{g^{\langle m\rangle}(\zeta_-(\lambda)) + \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} \Phi_{\mu\nu}^{(m,k)} (\zeta_-(\lambda), t_B, x_B) \left[ \frac{\partial^{k+2} g^{\mu\nu} (\zeta_-(\lambda), t_B, x_B)}{\partial x^k} \right]}{g^{(m-k)}}
\]

for \( n \geq 2 \). Substituting for \( k^{\mu} (\zeta_-(\lambda), G) \) from equation (37) into equation (23), we get the theorem which follows.

**Theorem 4.** When the perturbation term \( h_{\mu\nu} \) in the metric is represented by expansion (29), the function \( \Delta_r \) is given by equation (33), namely

\[
\Delta_r (x_A, t_B, x_B) = \sum_{n=1}^{\infty} G^n \Delta_r^{(n)} (x_A, t_B, x_B).
\]

where

\[
\Delta_r^{(1)} (x_A, t_B, x_B) = \frac{1}{2} R_{AB} \int_0^1 \left[ g_0^{(1)} - 2N_{AB}^i g_0^{(1)} + N_{AB}^i N_{AB}^j g_0^{(1)} \right] d\lambda,
\]

\[
\Delta_r^{(2)} (x_A, t_B, x_B) = \frac{1}{2} R_{AB} \int_0^1 \left[ \hat{g}_0^{(2)} - 2N_{AB}^i \hat{g}_0^{(2)} + N_{AB}^i N_{AB}^j \hat{g}_0^{(2)} \right] d\lambda,
\]

\[
\Delta_r^{(n)} (x_A, t_B, x_B) = \frac{1}{2} R_{AB} \int_0^1 \left[ \hat{g}_0^{(n)} - 2N_{AB}^i \hat{g}_0^{(n)} + N_{AB}^i N_{AB}^j \hat{g}_0^{(n)} \right] d\lambda.
\]

for \( n \geq 3 \), the quantities \( \hat{g}^{\mu\nu} \) being defined by equations (38) and (39).

It may be noted that the integral expressions occurring in equations (40)–(42) are line integrals taken along the zeroth-order null geodesic of parametric equation \( x = \zeta_-(\lambda) \), where \( \zeta_-(\lambda) \) is defined by equation (35).

7
A similar reasoning works for the function $\Delta_e$, which admits the expansion

$$\Delta_e(t_A, x_A, x_B, G) = \sum_{n=1}^{\infty} G^n \Delta_e^{(n)}(t_A, x_A, x_B).$$

(43)

We define the functions $\Phi^{(m,k)}_e(t_A, x_A, x)$ as

$$\Phi^{(m,k)}_e(t_A, x_A, x) = \frac{1}{k!} \sum_{l_1+\ldots+l_k=m-k} \left[ \prod_{j=1}^{k} \Delta^{(l_j + 1)}_{\Delta_1}(t_A, x_A, x) \right].$$

(44)

where $l_1, l_2, \ldots, l_k$ are either positive integers or zero. Then, defining $z_e(\mu)$ as the point-event

$$z_e(\mu) = (x_A^0 + \mu R_{AB}, z_e(\mu)),$$

(45)

where $z_e(\mu)$ is given by equation (28), we put

$$\tilde{g}^{\mu\nu}_{\Delta_1}(t_A, x_A, z_e(\mu)) = \delta_{1(1)}(z_e(\mu))$$

(46)

and

$$\tilde{g}^{\mu\nu}_{\Delta_1}(t_A, x_A, z_e(\mu)) = g^{\mu\nu}_{(1)}(z_e(\mu)) + \sum_{m=1}^{n-1} \sum_{k=1}^{m} \Phi^{(m,k)}_{\Delta_1}(t_A, x_A, z_e(\mu)) \left[ \frac{\partial^k g^{\mu\nu}_{(n-m)}}{\partial x^k} \right]_{z_e(\mu)}$$

(47)

for $n \geq 2$. Thus we can formulate a theorem as follows.

**Theorem 5.** On the assumption of theorem 4, the function $\Delta_e$ is given by expansion (43), where

$$\Delta_e^{(1)}(t_A, x_A, x_B) = \frac{1}{2} R_{AB} \int_{0}^{1} \left[ g^{(0)}_{(1)} - 2 N_{AB}^{(0)} \right] + N_{AB}^{(i)} N_{AB}^{(j)} \left[ \frac{\partial \Delta_e^{(1)}}{\partial x^i} \right]_{(t_A, x_A, z_e(\mu))} d\mu,$$

(48)

$$\Delta_e^{(2)}(t_A, x_A, x_B) = \frac{1}{2} R_{AB} \int_{0}^{1} \left[ g^{(0)}_{(2)} - 2 N_{AB}^{(0)} \tilde{g}^{\mu\nu}_{\Delta_1} + N_{AB}^{(i)} N_{AB}^{(j)} \tilde{g}^{\mu\nu}_{\Delta_1} \right]_{(t_A, x_A, z_e(\mu))} \left[ \frac{\partial \Delta_e^{(1)}}{\partial x^i} \right]_{(t_A, x_A, z_e(\mu))} + \frac{\partial \Delta_e^{(1)}}{\partial x^j} \right]_{(t_A, x_A, z_e(\mu))} d\mu$$

(49)

and

$$\Delta_e^{(n)}(t_A, x_A, x_B) = \frac{1}{2} R_{AB} \int_{0}^{1} \left[ g^{(0)}_{(n)} - N_{AB}^{(0)} N_{AB}^{(i)} N_{AB}^{(j)} \tilde{g}^{\mu\nu}_{\Delta_1} \right]_{(t_A, x_A, z_e(\mu))} \left[ \frac{\partial \Delta_e^{(1)}}{\partial x^i} \right]_{(t_A, x_A, z_e(\mu))} \left[ \frac{\partial \Delta_e^{(1)}}{\partial x^j} \right]_{(t_A, x_A, z_e(\mu))} + \sum_{p=1}^{n-1} \eta^{ij} \left[ \frac{\partial \Delta_e^{(p)}}{\partial x^i} \right]_{(t_A, x_A, z_e(\mu))} \left[ \frac{\partial \Delta_e^{(n-p)}}{\partial x^j} \right]_{(t_A, x_A, z_e(\mu))}$$

(50)

for $n \geq 3$, the quantities $\tilde{g}^{\mu\nu}_{\Delta_1}$ being defined by equations (46) and (47).
The integrals are now taken along the zeroth-order null geodesic defined by the parametric equation \( x = z_+(\mu) \).

As far as we know, the above results are new for \( n \geq 2 \). On the other hand, the expressions obtained here for \( \Delta^{(1)} \) and \( \Delta^{(2)} \) are equivalent to the well-known expressions derived by other methods. Taking equation (31) into account, it may be seen that the formulae (40) and (48) coincide with the expressions obtained, e.g., in [14] for \( cT_r^{(i)}(1) \) and \( cT_e^{(i)}(1) \), respectively.

6. Case of a stationary spacetime

In the case of a stationary spacetime, one can choose coordinates \((x^\alpha)\) such that the components of the metric do not depend on \( x^0 \). Then the two time transfer functions reduce to a single one, so that we can write

\[
T_r(x_A, t_B, x_B) \equiv T_e(t_A, x_A, x_B) \equiv T(x_A, x_B).
\]

(51)

As a consequence, the functions \( \Delta_r \) and \( \Delta_e \) are identical and depend only on \( x_A \) and \( x_B \), which implies that

\[
\Delta^{(n)}(x_A, t_B, x_B) \equiv \Delta^{(n)}(t_A, x_A, x_B) \equiv \Delta^{(n)}(x_A, x_B)
\]

(52)

for any \( n \geq 1 \).

Moreover, the stationary character of the metric implies that

\[
\hat{g}^{(\mu)}_{\lambda\lambda}(z_-(\lambda), t_B, x_B) \equiv \hat{g}^{(\mu)}_{\lambda\lambda}(z_+(\lambda))
\]

(53)

and

\[
\hat{g}^{(\mu)}_{\lambda\lambda}(t_A, x_A, z_+(\mu)) \equiv \hat{g}^{(\mu)}_{\lambda\lambda}(z_+(\mu)).
\]

(54)

So equations (40)–(42) and (48)–(50) simplify. For example, equations (48) and (49) may be written as

\[
\Delta^{(1)}(x_A, x_B) = \frac{1}{2} R_{AB} \int_0^1 \left[ g^{(0)}_{\lambda\lambda}(1) - 2N^i_{AB} g^{(0)}_{\lambda\lambda}(1) + N^i_{AB} N^j_{AB} g^{(i)}_{\lambda\lambda}(1) \right] z_+(\mu) \, d\mu
\]

(55)

and

\[
\Delta^{(2)}(x_A, x_B) = \frac{1}{2} R_{AB} \int_0^1 \left\{ [ g^{(0)}_{\lambda\lambda}(2) - 2N^i_{AB} g^{(0)}_{\lambda\lambda}(2) + N^i_{AB} N^j_{AB} g^{(i)}_{\lambda\lambda}(2) ] z_+(\mu) 

- 2 [ g^{(0)}_{\lambda\lambda} - N^j_{AB} g^{(j)}_{\lambda\lambda}(1) ] z_+(\mu) \frac{\partial \Delta^{(1)}}{\partial x^i}(x_A, z_+(\mu)) + \eta^{ij} \left[ \frac{\partial \Delta^{(1)}}{\partial x^i} \frac{\partial \Delta^{(1)}}{\partial x^j} \right]_{(x_A, z_+(\mu))} \right\} d\mu,
\]

(56)

respectively.

A straightforward calculation shows that making the change of variable

\[
\lambda = -\mu
\]

transforms the right-hand side of equations (55) and (56) into the expressions deduced from equations (40) and (41), respectively when equations (52) and (53) are taken into account. It is worthy of note that the integrals reduce in this case to line integrals taken along the segment joining \( x_A \) and \( x_B \).
7. Application to a static, spherically symmetric spacetime

To illustrate the previous results, let us consider the gravitational field outside a static, spherically symmetric body of mass \( M \). Choosing spatial quasi-Cartesian isotropic coordinates and putting \( r = |x| \), we suppose that the metric has the form

\[
\begin{align*}
  ds^2 &= \left( 1 - \frac{2GM}{c^2r} + 2\beta \frac{G^2M^2}{c^4r^2} + \cdots \right) (dx^0)^2 - \left( 1 + 2\gamma \frac{GM}{c^2r} + \frac{3}{2} \delta \frac{G^2M^2}{c^4r^2} + \cdots \right) \delta_{ij} dx^i dx^j,
\end{align*}
\]

where \( \beta \) and \( \gamma \) are the usual post-Newtonian parameters, \( \delta \) is a post-post-Newtonian parameter and \( \cdots \) means that terms of order \( G^3 \) are neglected (\( \beta = \gamma = \delta = 1 \) in general relativity).

Furthermore, we suppose that points \( x_A \) and \( x_B \) are such that the geodesic path connecting them is entirely outside the body. We use the notations \( r_A = |x_A|, r_B = |x_B| \).

The contravariant components of the metric are given by

\[
\begin{align*}
  g^{00}_{(1)} &= \frac{2M}{c^2r}, & g^{0i}_{(1)} &= 0, & g^{ij}_{(1)} &= 2\gamma \frac{M}{c^2r} \delta_{ij},
  g^{00}_{(2)} &= 2(2 - \beta) \frac{M^2}{c^4r^2}, & g^{0i}_{(2)} &= 0, & g^{ij}_{(2)} &= \left( \frac{3}{2} \delta - 4\gamma^2 \right) \frac{M^2}{c^4r^2} \delta_{ij}.
\end{align*}
\]

Taking equations (58) into account, we immediately deduce from equation (55) that \( \Delta_1^{(1)} \) is given by the well-known formula

\[
\Delta_1^{(1)}(x_A, x_B) = \frac{(\gamma + 1)M}{c^2} \int_0^1 \frac{d\mu}{|z_+(\mu)|} = \frac{(\gamma + 1)M}{c^2} \ln \left( \frac{r_A + r_B + R_{AB}}{r_A + r_B - R_{AB}} \right),
\]

which is equivalent to the expression of the time delay found by Shapiro [18].

Substituting now equations (58) and (59) into equation (56), we get

\[
\begin{align*}
  \Delta_1^{(2)}(x_A, x_B) &= \frac{1}{4} R_{AB} \int_0^1 \left\{ (8 - 4\beta - 8\gamma^2 + 3\delta) \frac{M^2}{c^4|z_+(\mu)|^2} \right. \\
  &\left. + 8\gamma \frac{M}{c^2|z_+(\mu)|} N_{AB}^{(1)} \frac{\partial \Delta_1^{(1)}}{\partial x^i}(x_A, z_+(\mu)) - 2 \sum_{i=1}^{3} \left( \frac{\partial \Delta_1^{(1)}}{\partial x^i}(x_A, z_+(\mu)) \right)^2 \right\} d\mu.
\end{align*}
\]

We deduce from equation (60) that

\[
N_{AB}^{(1)} \frac{\partial \Delta_1^{(1)}}{\partial x^i}(x_A, z_+(\mu)) = (\gamma + 1) \frac{M}{c^2|z_+(\mu)|}
\]

and

\[
\sum_{i=1}^{3} \left( \frac{\partial \Delta_1^{(1)}}{\partial x^i}(x_A, z_+(\mu)) \right)^2 = 2(\gamma + 1) \frac{M^2}{c^4} \int \frac{d\mu}{|z_+(\mu)|} \left[ \frac{\mu}{r_A|z_+(\mu)|} + \mu x_A \cdot (x_B - x_A) + r_A^2 \right].
\]

Substituting equations (62) and (63) into equation (61), and then noting that

\[
\int_0^1 \frac{d\mu}{|z_+(\mu)|^2} = \frac{\arccos(n_A \cdot n_B)}{r_A r_B \sqrt{1 - (n_A \cdot n_B)^2}},
\]
where

\[ n_A = \frac{x_A}{r_A}, \quad n_B = \frac{x_B}{r_B} \]  \hspace{1cm} (65)

we obtain by a straightforward calculation

\[ \Delta_1^{(2)}(x_A, x_B) = \frac{M^2 R_{AB}}{c^4 r_A r_B} \left[ \frac{8 - 4\beta + 8\gamma + 3\delta}{4\sqrt{1 - (n_A \cdot n_B)^2}} \arccos(n_A \cdot n_B) - \frac{(1 + \gamma)^2}{1 + (n_A \cdot n_B)} \right]. \]  \hspace{1cm} (66)

Finally, using the formulae (60) and (66), we get an expression as follows for the time transfer function

\[ T(x_A, x_B) = \frac{R_{AB}}{c} + \frac{(\gamma + 1)GM}{c^3} \ln \left( \frac{r_A + r_B + R_{AB}}{r_A + r_B - R_{AB}} \right) + \frac{G^2 M^2 R_{AB}}{c^5 r_A r_B} \times \left[ \frac{(8 - 4\beta + 8\gamma + 3\delta)}{4\sqrt{1 - (n_A \cdot n_B)^2}} \arccos(n_A \cdot n_B) - \frac{(1 + \gamma)^2}{1 + (n_A \cdot n_B)} \right] + O(G^3). \]  \hspace{1cm} (67)

We recover a result previously derived by different approaches [10, 14] (see also [11, 12] in the case where \( \beta = \gamma = \delta = 1 \)).

8. Concluding remarks

Equations (40)–(42) and (48)–(50) are the main results of this paper. These equations show that the time transfer functions can be obtained within the \( n \)th post-Minkowskian approximation by a recursive procedure which spares the trouble of solving the geodesic differential equations and avoids determining Synge’s world function. It is remarkable that any \( n \)th-order perturbation term is an integral taken along a zeroth-order null straight line. The derivation of the time transfer function performed here for a static, spherically symmetric spacetime is significantly more simple than the calculation carried out in [14].

As a final remark, it may be noted that since the time transfer functions are sufficient to determine the deflection and the frequency shift of a light signal, the systematic method developed here will be very convenient to tackle the relativistic problems raised by highly accurate astrometry and time/frequency metrology.

References

[1] Turyshhev S G, Shao M and Nordtvedt K L 2008 Science, technology, and mission design for the laser astrometric test of relativity Lasers, Clocks and Drag-Free Control: Exploration of Relativistic Gravity in Space (Springer Series in Astrophysics and Space Science Library vol 349) ed H Dittus, C Lämmerzahl and S G Turyshhev (Berlin: Springer) p 473 (Preprint gr-qc/0601035)
[2] Ni W T 2008 Int. J. Mod. Phys. at press (Preprint arXiv:0712.2492)
[3] Will C M 1993 Theory and Experiment in Gravitational Physics 2nd edn (Cambridge: Cambridge University Press)
[4] Blanchet L, Salomon C, Teyssandier P and Wolf P 2001 Astron. Astrophys. 370 320
[5] Kopeikin S M 1997 J. Math. Phys., NY 38 2587
[6] Kopeikin S M and Schäfer G 1999 Phys. Rev. D 60 124002
[7] Kopeikin S M and Mashhoon B 2002 Phys. Rev. D 65 064025
[8] Ciufolini I, Kopeikin S, Mashhoon B and Ricci F 2003 Phys. Lett. A 308 101
[9] Richter G W and Matzner R A 1982 Phys. Rev. D 26 2549
[10] Richter G W and Matzner R A 1983 Phys. Rev. D 28 3007
[11] John R W 1975 Exp. Tech. Phys. 23 127
[12] Brumberg V A 1987 Kinematics Phys. Celest. Bodies 3 6
[13] Linet B and Teyssandier P 2002 Phys. Rev. D 66 024045
[14] Le Poncin-Lafitte C, Linet B and Teyssandier P 2004 Class. Quantum Grav. 21 4463
[15] Teyssandier P, Le Poncin-Lafitte C and Linet B 2008 A universal tool for determining the time delay and the frequency shift of light: Synge’s world function Lasers, Clocks and Drag-Free Control: Exploration of Relativistic Gravity in Space (Springer Series in Astrophysics and Space Science Library vol 349) ed H Dittus, C Lämmerzahl and S G Turyshev (Berlin: Springer) p 153 (Preprint arXiv:0711.0034)
[16] Synge J L 1964 Relativity: The General Theory (Amsterdam: North-Holland)
[17] Bel Ll and Martín J 1994 Gen. Rel. Grav. 26 567
[18] Shapiro I I 1964 Phys. Rev. Lett. 13 789