B-METRIC SPACES, FIXED POINTS AND LIPSCHITZ FUNCTIONS

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Abstract. The paper is concerned with b-metric and generalized b-metric spaces. One proves the existence of the completion of a generalized b-metric space and some fixed point results. The behavior of Lipschitz functions on b-metric spaces of homogeneous type, as well as of Lipschitz functions defined on, or with values in quasi-Banach spaces, is studied.

MSC2010: 54E25 54E35 47H09 47H10 46A16 26A16

Keywords: metric space, generalized metric space, b-metric space, completion, metrizability, fixed point, quasi-Banach space, Lipschitz mapping

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Introduction

There are a lot of extensions of the notions of metric and metric space – see for instance the books [32], [53], [71], or the survey papers [22], [51]. In this paper we concentrate on b-metric and generalized b-metric spaces, with emphasis on their topological properties, some fixed point results and Lipschitz functions on such spaces.

A part of the results from this paper are included in [24].

1. B-METRIC SPACES

In this section we present some results on b-metric spaces.

Date: March 15, 2022.
1.1. **Topological properties and metrizability.** A *b-metric* on a nonempty set $X$ is a function $d : X \times X \to [0, \infty)$ satisfying the conditions

\begin{align}
(i) \quad & d(x, y) = 0 \iff x = y; \\
(ii) \quad & d(x, y) = d(y, x); \\
(iii) \quad & d(x, y) \leq s[d(x, z) + d(z, y)],
\end{align}

for all $x, y, z \in X$, and for some fixed number $s \geq 1$. The pair $(X, d)$ is called a *b-metric space*. Obviously, for $s = 1$ one obtains a metric on $X$.

**Example 1.1.** If $(X, d)$ is a metric space and $\beta > 1$, then $d^\beta(x, y)$ is a b-metric.

Indeed,

$$d^\beta(x, y) \leq [d(x, z) + d(z, y)]^\beta \leq 2^\beta (\max\{d(x, z), d(z, y)\})^\beta \leq 2^\beta [d^\beta(x, y) + d^\beta(x, y)].$$

The $s$-relaxed triangle inequality implies

\begin{equation}
(\text{ii}) \quad d(x, x_n) \leq sd(x, x_1) + s^2d(x_1, x_2) + \cdots + s^{n-1}d(x_{n-2}, x_{n-1}) + s^{n-1}d(x_{n-1}, x_n),
\end{equation}

for all $n \in \mathbb{N}$ and all $x, x_1, \ldots, x_n \in X$.

Indeed, we obtain successively

$$d(x, x_n) \leq sd(x, x_1) + sd(x_1, x_1) \leq sd(x_1, x_2) + s^2d(x_2, x_2) \leq \cdots \leq sd(x_1, x_1) + s^2d(x_1, x_2) + \cdots + s^{n-1}d(x_{n-2}, x_{n-1}) + s^{n-1}d(x_{n-1}, x_n)$$

Along with the inequality (iii), called the *$s$-relaxed triangle inequality*, one considers also the *$s$-relaxed polygonal inequality*

\begin{equation}
(iv) \quad d(x, x_n) \leq s[d(x, x_1) + d(x_1, x_2) + \cdots + d(x_{n-1}, x_n)],
\end{equation}

for all $x, x_1, \ldots, x_n \in X$ and all $n \in \mathbb{N}$.

For $n = 2$ one obtains the inequality (iii). The following example shows that the converse is not true – there exist b-metrics that do not satisfy the relaxed polygonal inequality.

**Example 1.2** ([53], Theorem 12.10). Let $X = [0, 1]$ and $d(x, y) = (x - y)^2$, $x, y \in [0, 1]$. Then $d$ is a 2-relaxed metric on $X$ which is not polygonally $s$-relaxed for any $s \geq 1$.

Indeed, it is easy to check that $d$ satisfies the 2-relaxed triangle inequality. Suppose that, for some $s \geq 1$, $d$ satisfies the $s$-relaxed polygonal inequality. Taking $x_i = \frac{1}{n}$, $1 \leq i \leq n - 1$, we obtain

$$\frac{1}{s} = \frac{1}{s} \cdot d(0, 1) \leq d(0, x_1) + d(x_1, x_2) + \cdots + d(x_{n-1}, 1) = n \cdot \left(\frac{1}{n}\right)^2 = \frac{1}{n},$$

for all $n \in \mathbb{N}$, which is impossible.

One can consider also an ultrametric version of (iii):

\begin{equation}
(iii') \quad d(x, y) \leq \lambda \max\{d(x, z), d(y, z)\},
\end{equation}

for all $x, y, z \in X$. It is obvious that

\begin{align}
(iii') \implies (iii) & \iff s = \lambda; \\
(iii) \implies (iii') & \iff \lambda = 2s.
\end{align}

The condition

\begin{equation}
(iii'') \quad \max\{d(x, z), d(y, z)\} \leq \varepsilon \implies d(x, y) \leq 2\varepsilon,
\end{equation}

for all $\varepsilon > 0$ and $x, y, z \in X$, is equivalent to $\text{iii}'$ with $\lambda = 2$.

Let now $(X, d)$ be again a b-metric space. One introduces a topology on a b-metric space $(X, d)$ in the usual way. The “open” ball $B(x, r)$ of center $x \in X$ and radius $r > 0$ is given by

$$B(x, r) = \{y \in X : d(x, y) < r\}.$$
A subset $Y$ of $X$ is called open if for every $x \in Y$ there exists a number $r_x > 0$ such that $B(x, r_x) \subseteq Y$. Denoting by $\tau_d$ (or $\tau(d)$) the family of all open subsets of $X$ it follows that $\tau_d$ satisfies the axioms of a topology. This topology is derived from a uniformity $U_d$ on $X$ having as basis the sets

$$U_\varepsilon = \{(x,y) \in X \times X : d(x,y) < \varepsilon\}, \quad \varepsilon > 0.$$ 

The uniformity $U_d$ has a countable basis $\{U_1/n : n \in \mathbb{N}\}$ so that, by Frink’s metrization theorem (39), the uniformity $U_d$ is derived from a metric $\rho$, hence the topology $\tau_d$ as well. This was remarked in the paper [57]. In [37] it is shown that the topology $\tau_d$ satisfies the hypotheses of the Nagata-Smirnov metrizability theorem.

Concerning the metrizability of uniform and topological spaces, see the treatise [36].

There exist also direct proofs of the metrizability of the topology of a b-metric space. Let $(X, d)$ be a b-metric space. Put

$$\rho(x,y) = \inf \left\{ \sum_{k=0}^{n} d(x_{i-1}, x_i) \right\},$$

where the infimum is taken over all $n \in \mathbb{N}$ and all chains $x = x_0, x_1, \ldots, x_n = y$ of elements in $X$ connecting $x$ and $y$.

As remarked Frink [39], if a b-metric $d$ satisfies \(i\) for $\lambda = 2$, then formula (1.3) defines a metric equivalent to $d$. We present the result in the form given by Schroeder [72].

**Theorem 1.3** (A. H. Frink [39] and V. Schroeder [72]). If $d : X \times X \to [0, \infty)$ satisfies the conditions (i), (ii) from (1.1) and (ii) for some $1 \leq \lambda \leq 2$, then the function $\rho$ defined by (1.3) is a metric on $X$ satisfying the inequalities $\frac{1}{\lambda}d \leq \rho \leq d$.

V. Schroeder [72] also showed that for every $\varepsilon > 0$ there exists a b-metric $d$ satisfying (1.1).(iii) with $s = 1 + \varepsilon$ such that the mapping $\rho$ defined by (1.3) is not a metric. Other example showing the limits of Frink’s metrization method was given in An and Dung [10].

General results of metrizability were obtained in [2] and [64] by a slight modification of Frink’s technique.

Let $(X, d)$ be a b-metric space. For $0 < p \leq 1$ define

$$\rho_p(x,y) = \inf \left\{ \sum_{k=0}^{n} d(x_{i-1}, x_i)^p \right\},$$

where the infimum is taken over all $n \in \mathbb{N}$ and all chains $x = x_0, x_1, \ldots, x_n = y$ of elements in $X$.

The function $\rho_p$ defined by (1.4) satisfies the conditions

1. $\rho_p(x,y) = \rho_p(y,x)$,
2. $\rho_p(x,y) \leq \rho_p(x,z) + \rho_p(z,y)$,
3. $d^p(x,y) \leq \rho_p(x,y)$,

for all $x, y, z \in X$, i.e., $\rho$ is a pseudometric on $X$ and $d^p$ is dominated by $\rho$.

**Theorem 1.4** ([64]). Let $d$ be a b-metric on a nonempty set $X$ satisfying the $s$-related triangle inequality (1.1).(iii), for some $s \geq 1$. If the number $p \in (0,1]$ is given by the equation $(2s)^p = 2$, then the mapping $\rho_p : X \times X \to [0, \infty)$ defined by (1.4) is a metric on $X$ satisfying the inequalities

$$\rho_p(x,y) \leq d^p(x,y) \leq 2\rho_p(x,y),$$

for all $x, y \in X$.

The same conclusions hold if $d$ satisfies the conditions (i), (ii) from (1.1) and (iii) for some $\lambda \geq 2$. In this case $0 < p \leq 1$ is given by $\lambda p = 2$ and the metric $\rho_p$ satisfies the inequalities

$$\rho_p(x,y) \leq d^p(x,y) \leq 4\rho_p(x,y),$$

for all $x, y \in X$.

The inequalities (1.5) have the following consequences.
Corollary 1.5. Under the hypotheses of Theorem 1.4, \( \tau_d = \tau_p \), that is, the topology of any b-metric space is metrizable, and the convergence of sequences with respect to \( \tau_d \) is characterized in the following way:

\[ x_n \xrightarrow{\tau_d} x \iff d(x, x_n) \to 0, \]

for any sequence \( (x_n) \) in \( X \) and \( x \in X \).

Proof. The equality of topologies follows from the inclusions

\[ B_d(x, r^{1/p}) \subseteq B_p(x, r) \quad \text{and} \quad B_p(x, A^{-1}r^p) \subseteq B_d(x, r), \]

valid for all \( x \in X \) and \( r > 0 \).

The statement concerning sequences is a consequence of the equality \( \tau_d = \tau_p \) and of the inequalities (1.3). \( \square \)

Remark 1.6. In [2] the proof is given for a \( p \) satisfying the inequalities

\[ 1 \geq p \geq (\log_2(3s))^{-2}, \]

while in Theorem 1.4 the result holds for

\[ p = (\log_2(2s))^{-1}. \]

Putting

\[ \tilde{\rho}(p) = \sup\{p \in (0, 1] : \rho_p \text{ is a metric, Lipschitz equivalent to } d^p\}, \]

the estimation (1.7) yields \( \tilde{\rho}(p) \geq (\log_2(3s))^{-2} \), while from (1.8) one obtains the better evaluation \( \tilde{\rho}(p) \geq (\log_2(2s))^{-1} \), which cannot be improved, as it is shown by the example of the spaces \( \ell^p \) with \( 0 < p < 1 \).

A proof of Theorem 1.4 is also given in the book by Heinonen [42, Prop. 14.5], with the evaluation \( p \geq (\log_2 \lambda)^{-2} \), where \( \lambda \) is the constant from (1.1).

Remark 1.7. It follows that \( \tilde{\rho}(x, y) = \rho_p(x, y)^{1/p} \), \( x, y \in X \), is a b-metric on \( X \), Lipschitz equivalent to \( d \) and satisfying the inequality

\[ \tilde{\rho}(x, y)^p \leq \tilde{\rho}(x, z)^p + \tilde{\rho}(z, y)^p, \]

for all \( x, y, z \in X \). This is a well known fact in the theory of quasi-normed spaces, where a quasi-norm \( \| \cdot \| \) satisfying the inequality

\[ \|x + y\|^p \leq \|x\|^p + \|y\|^p, \]

for some \( 0 < p \leq 1 \) is called a \( p \)-norm (see Subsection 1.1).

Let \((X, d)\) be a b-metric space. The b-metric \( d \) is called

- \textit{continuous} if

\[ d(x_n, x) \to 0 \quad \text{and} \quad d(y_n, y) \to 0 \implies d(x_n, y_n) \to d(x, y), \]

- \textit{separately continuous} if the function \( d(x, \cdot) \) is continuous on \( X \) for every \( x \in X \), i.e.,

\[ d(y_n, y) \to 0 \implies d(x, y_n) \to d(x, y), \]

for all sequences \((x_n), (y_n)\) in \( X \) and all \( x, y \in X \).

The topology \( \tau_d \) generated by a b-metric \( d \) has some peculiarities – a ball \( B(x, r) \) need not be \( \tau_d \)-open and the b-metric \( d \) could not be continuous on \( X \times X \).

Remark 1.8. Let \((X, d)\) be a b-metric space and \( x \in X \). Then

\[ B(x, r) \text{ is } \tau_d\text{-open for every } r > 0 \iff d(x, \cdot) \text{ is upper semicontinuous on } X. \]

Consequently, if the b-metric is separately continuous on \( X \), then the balls \( B(x, r) \) are \( \tau_d \)-open.

The equivalence follows from the equality

\[ B(x, r) = d(x, \cdot)^{-1}((-\infty, r)). \]

We present now an example of a b-metric space where the balls are not necessarily open.
Remark 1.10. If, for some $0 < r > B > 0$ which can be proved as in the metric case (using (1.11)).

Example 1.9. Consider a fixed number $\varepsilon > 0$. For $X = \mathbb{N}_0 = \{0, 1, \ldots\}$ let $d : X \times X \rightarrow [0, \infty)$ be defined by

$$d(0, 1) = 1, \quad d(0, m) = 1 + \varepsilon \quad \text{for } m \geq 2$$

$$d(1, m) = \frac{1}{m}, \quad d(n, m) = \frac{1}{n} + \frac{1}{m} \quad \text{for } n \geq 2$$

and extended to $X \times X$ by $d(n, n) = 0$ and symmetry.

Then

$$d(m, n) \leq (1 + \varepsilon)[d(n, k) + d(k, m)],$$

for all $m, n, k \in X$, $B \left(0, 1 + \frac{\varepsilon}{2}\right) = \{0, 1\}$ and the ball $B(1, r)$ contains an infinity of terms for every $r > 0$, that is, for any $1 \in B \left(0, 1 + \frac{\varepsilon}{2}\right)$, $B(1, r) \not\subseteq B \left(0, 1 + \frac{\varepsilon}{2}\right)$ for every $r > 0$, showing that the ball $B \left(0, 1 + \frac{\varepsilon}{2}\right)$ is not $\tau_d$-open.

Other examples are given in [11].

Remark 1.11. It is obvious that, in general, there exists a metric $\rho$ on $X$ such that $\tau_d$ is not $\tau_\rho$-open. Moreover, the b-metric $d$ is continuous.

By Remark 1.7, the b-metric $\tilde{\rho}$ corresponding to the metric $\rho$ constructed in Theorem 1.4 satisfies the inequality (1.11).

Indeed, let $B(x, r)$ be a ball in $(X, d)$ and $y \in B(x, r)$. We have to show that there exists $r' > 0$ such that $B(y, r') \subseteq B(x, r)$. Taking $r' := (r^p - d(x, y)^p)^{1/p} > 0$, then $d(y, z) < r'$ implies

$$d(x, z)^p \leq d(x, y)^p + d(y, z)^p$$

$$< d(x, y)^p + r^p = r^p,$$

that is, $d(x, z) < r$. The continuity of the b-metric $d$ follows from the inequality

$$|d(x_n, y_n)^p - d(x, y)^p| \leq d(x_n, x)^p + d(y_n, y)^p,$$

which can be proved as in the metric case (using (1.11)).

Equivalence notions for b-metrics.

In connection to the metrizability of b-metric spaces, we mention the following notions of equivalence for b-metrics.

Let $d_1, d_2$ be two b-metrics on the same set $X$. Then $d_1, d_2$ are called

- **topologically equivalent** if $\tau_{d_1} = \tau_{d_2}$;
- **uniformly equivalent** if the identity mapping $I_X$ on $X$ is uniformly continuous both from $(X, d_1)$ to $(X, d_2)$ as well as from $(X, d_2)$ to $(X, d_1)$, i.e.

  $$\forall \varepsilon > 0, \exists \delta(\varepsilon) > 0 \text{ such that } \forall x, y \in X \left( d_1(x, y) \leq \delta(\varepsilon) \implies d_2(x, y) \leq \varepsilon, \right)$$

  $$\forall \varepsilon > 0, \exists \delta(\varepsilon) > 0 \text{ such that } \forall x, y \in X \left( d_2(x, y) \leq \delta(\varepsilon) \implies d_1(x, y) \leq \varepsilon \right).$$

- **Lipschitz equivalent** if there exist $c_1, c_2 > 0$ such that

  $$c_1 d_2(x, y) \leq d_1(x, y) \leq c_2 d_2(x, y),$$

  for all $x, y \in X$.

Of course, the above definitions apply to metrics as well, as particular cases of b-metrics.

Remark 1.11. It is obvious that, in general,

Lipschitz equivalence $\implies$ uniform equivalence $\implies$ topological equivalence.

For quasi-norms, topological equivalence is equivalent to Lipschitz equivalence.

So the expression “the topology $\tau_d$ generated by a b-metric $d$ on a set $X$ is metrizable” means that there exists a metric $\rho$ on $X$ topologically equivalent to $d$.

The problem of the existence of a metric that is Lipschitz equivalent to a b-metric was solved in [37], where this property was called **metric boundedness**.
Theorem 1.12 (37, see also 33, Theorem 12.9). Let \((X, d)\) be a \(b\)-metric space. Then \(d\) is Lipschitz equivalent to a metric if and only if \(d\) satisfies the \(s\)-relaxed polygonal inequality \((IV)\) for some \(s \geq 1\).

1.2. An axiomatic definition of balls in \(b\)-metric spaces. H. Aimar [2] found a set of properties characterizing balls in \(b\)-metric spaces.

For a nonempty set \(X\) consider a mapping \(U : X \times (0, \infty) \to \mathcal{P}(X)\) satisfying the following properties:

\[
\begin{align*}
\text{(i)} & \quad \bigcap_{r > 0} U(x, r) = \{x\}; \\
\text{(ii)} & \quad \bigcup_{r > 0} U(x, r) = X; \\
\text{(iii)} & \quad 0 < r_1 \leq r_2 \implies U(x, r_1) \subseteq U(x, r_2); \\
\text{(iv)} & \quad \text{there exists } c \geq 1 \text{ such that } y \in U(x, r) \implies U(x, r) \subseteq U(y, cr) \text{ and } U(y, r) \subseteq U(x, cr),
\end{align*}
\]

for all \(x \in X\) and \(r > 0\).

We call the sets \(U(x, r)\) formal balls.

Remark 1.13. By (i), \(x \in U(x, r)\) for all \(r > 0\) and \(x \in X\).

It is easy to check that if \((X, d)\) is a \(b\)-metric space, where the \(b\)-metric \(d\) satisfies the relaxed triangle inequality for some \(s \geq 1\), then the sets \(U(x, r) = B_d(x, r)\), \(x \in X, r > 0\) satisfy the properties from (1.12) with \(c = 2s\).

Conditions (i)–(iii) are easy to check. For (iv), if \(y \in B_d(x, r)\) and \(z \in B_d(x, r)\), then

\[d(y, z) \leq s(d(y, x) + d(x, z)) < 2sr,\]

i.e. \(z \in B_d(y, 2sr)\).

Similarly, \(y \in B_d(x, r)\) and \(z \in B_d(y, r)\) imply

\[d(x, z) \leq s(d(x, y) + d(y, z)) < 2sr,\]

i.e. \(z \in B_d(x, 2sr)\).

It can be shown that, conversely, a family of subsets of \(X\) satisfying the properties from (1.12) generates a \(b\)-metric on \(X\), the balls corresponding to \(d\) being tightly connected with the sets \(U(x, r)\).

Theorem 1.14 (2). For a nonempty set \(X\) and a mapping \(U : X \times (0, \infty) \to \mathcal{P}(X)\) satisfying the properties from (1.12), define \(d : X \times X \to [0, \infty)\) by

\[d(x, y) = \inf\{r > 0 : y \in U(x, r) \text{ and } x \in U(y, r)\}, \quad x, y \in X.\]

Then:

1. \(d\) is a \(b\)-metric on \(X\) satisfying the relaxed triangle inequality for \(s = c\);
2. the open balls \(B_d(x, r)\) corresponding to \(d\) and the sets \(U(x, r)\) are related by the inclusions

\[U(x, (\gamma c)^{-1} r) \subseteq B_d(x, r) \subseteq U(x, r),\]

for all \(x \in X\) and \(r > 0\), where \(\gamma > 1\).

Proof. We shall verify only the relaxed triangle inequality. Let \(x, y, z \in X\) and \(\varepsilon > 0\). By the definition (1.13) of \(d\) there exists \(r_1, r_2 > 0\) such that

\[0 < r_1 < d(x, z) + \varepsilon \text{ and } z \in U(x, r_1), \quad x \in U(z, r_1);\]

\[0 < r_2 < d(z, y) + \varepsilon \text{ and } z \in U(y, r_2), \quad y \in U(z, r_2).\]

Taking into account (iii) it follows

\[x, y \in U(z, r_1 + r_2) \text{ and } z \in U(x, r_1 + r_2) \cap U(y, r_1 + r_2).\]
Applying (iv) to \(x \in U(z, r_1 + r_2)\) one obtains

\[ y \in U(z, r_1 + r_2) \subseteq U(x, c(r_1 + r_2)). \]

Similarly, \(y \in U(z, r_1 + r_2)\) implies

\[ x \in U(z, r_1 + r_2) \subseteq U(y, c(r_1 + r_2)), \]

so that, by the definition of \(d\), \(d(x, y) \leq c(r_1 + r_2)\). But then, by adding the inequalities (1.15), one obtains

\[ d(x, y) \leq c(r_1 + r_2) < c(d(x, z) + d(z, y)) + 2c\varepsilon. \]

Since these hold for all \(\varepsilon > 0\), it follows

\[ d(x, y) \leq c(d(x, z) + d(z, y)). \]

Let us prove now the inclusions (1.14). If \(d(x, y) < r\), then there exists \(0 < r' < r\) such that \(y \in U(x, r')\) and \(x \in U(y, r')\). By (iii), \(U(x, r') \subseteq U(x, r)\), so that \(B_d(x, r) \subseteq U(x, r)\).

Let now \(y \in U(x, (\gamma c)^{-1}r)\). By Remark 1.13 and (iv),

\[ x \in U(x, (\gamma c)^{-1}r) \subseteq U(y, r/\gamma), \]

and

\[ y \in U(y, (\gamma c)^{-1}r) \subseteq U(x, r/\gamma). \]

By the definition of \(d\), \(d(x, y) \leq r/\gamma < r\), i.e. \(y \in B_d(x, r)\).

**Remark 1.15.** In [2] it is shown that \(d\) satisfies the relaxed triangle inequality with \(s = 2c\). Also, the first inclusion in (1.14) is proved for \(\gamma = 2\), i.e. it is shown that \(U(y, (2c)^{-1}r) \subseteq B_d(x, r)\).

In [1] a similar characterization is given in terms of some subsets of \(X \times X\). Denoting by \(\Delta\) the diagonal of \(X \times X\),

\[ \Delta = \{(x, x) : x \in X\}, \]

one considers a mapping \(V : (0, \infty) \rightarrow \mathcal{P}(X \times X)\) satisfying the properties:

\[
\begin{align*}
\text{(i)} & \quad \bigcap_{r > 0} V(r) = \Delta; \\
\text{(ii)} & \quad \bigcup_{r > 0} V(r) = X \times X; \\
\text{(iii)} & \quad 0 < r_1 \leq r_2 \implies V(r_1) \subseteq V(r_2); \\
\text{(iv)} & \quad \text{there exists } c \geq 1 \text{ such that } V(r) \circ V(r) \subseteq V(cr). \\
\end{align*}
\]

for all \(r > 0\).

By analogy with the case of uniform spaces we call the sets \(V(r)\) *antourages*. If \((X, d)\) is a \(b\)-metric space with \(d\) satisfying the relaxed triangle inequality for some \(s \geq 1\), then the sets

\[ W_d(r) = \{(x, y) \in X \times X : d(x, y) < r\}, \quad r > 0, \]

satisfy the conditions from (1.16) with \(c = 2s\). Indeed, the conditions (i)–(iii) are easily verified. To check (iv), suppose that \((x, y) \in W_d(r) \circ V(r)\). Then there exists \(z \in X\) such that \((x, z) \in W_d(r)\) and \((z, y) \in W_d(r)\), implying

\[ d(x, y) \leq s(d(x, z) + d(z, y)) < 2sr, \]

showing that (iv) holds with \(c = 2s\).

A converse result holds in this case too.

**Theorem 1.16 (1).** For a nonempty set \(X\) and a mapping \(V : (0, \infty) \rightarrow \mathcal{P}(X \times X)\) satisfying the properties from (1.16), define \(d : X \times X \rightarrow [0, \infty)\) by

\[ d(x, y) = \inf\{r > 0 : (x, y) \in V(r)\}, \quad x, y \in X. \]

Then:

1. \(d\) is a \(b\)-metric on \(X\) satisfying the relaxed triangle inequality for \(s = c;\)

The function \(d\) is Lipschitz continuous.
Again, we prove only the validity of the relaxed triangle inequality. Let \( x, y, z \in X \). By the definition of \( d \) there exist \( r_1, r_2 > 0 \) such that
\[
0 < r_1 < d(x, z) + \varepsilon \quad \text{and} \quad (x, z) \in V(r_1); \tag{1.19}
\]
\[
0 < r_2 < d(z, y) + \varepsilon \quad \text{and} \quad (z, y) \in V(r_2).
\]

By (iii) \( V(r_1) \cup V(r_2) \subseteq V(r_1 + r_2) \) so that, by (iv), \( (x, y) \in V(c(r_1 + r_2)) \), implying \( d(x, y) \leq c(r_1 + r_2) \).

Consequently, the inequalities (1.18) yield by addition
\[
d(x, y) \leq c(r_1 + r_2) < c(d(x, z) + d(z, y)) + 2\varepsilon.
\]

Since these hold for all \( \varepsilon > 0 \), it follows
\[
d(x, y) \leq c(d(x, z) + d(z, y)).
\]

The proof of the inclusions (1.18) is simpler than in the case considered in Theorem 1.14.

Indeed, \( (x, y) \in W(r) \) is equivalent to \( d(x, y) < r \). By the definition of \( d \), there exists \( 0 < r' < r \) such that \( (x, y) \in V(r') \). By (iii), \( V(r') \subseteq V(r) \), showing that \( (x, y) \in V(r) \), i.e. \( W(r) \subseteq V(r) \).

If \( (x, y) \in V(\gamma r) \), then
\[
d(x, y) \leq \gamma r < r,
\]
showing that \( (x, y) \in W_d(r) \), i.e. \( V(\gamma r) \subseteq W_d(r) \). \( \square \)

1.3. Strong b-metric spaces and completion. Let \((X, d)\) be a b-metric space. As we have seen, the topology \( \tau_d \) generated by the b-metric \( d \) has some drawbacks in what concerns the continuity property of \( d \) and the topological openness of the “open” balls. To remedy these shortcomings Kirk and Shahzad [53, §12.4] introduced a special class of b-metrics. A mapping \( d : X \times X \to [0, \infty) \) is called a strong b-metric if it satisfies the conditions (i) and (ii) from (1.1) and
\[
d(x, y) \leq d(x, z) + sd(y, z),
\]
for some \( s \geq 1 \) and all \( x, y, z \in X \). It is obvious that (v) is equivalent to
\[
d(x, y) \leq \min\{sd(x, z) + d(y, z), d(x, z) + sd(y, z)\},
\]
for all \( x, y, z \in X \), and that (v) implies the \( s \)-relaxed triangle inequality (iii) from (1.1).

The topology generated by a strong b-metric has good properties as, for instance, the openness of the balls \( B(x, r) \). Indeed, if \( y \in B(x, r) \) then
\[
d(y, z) \leq d(x, y) + sd(y, z) < \varepsilon,
\]
provided \( sd(y, z) < \varepsilon - d(x, y) \), that is \( B(y, r') \subseteq B(x, r) \), where \( r' = (\varepsilon - d(x, y))/s \).

Also the following inequality
\[
|d(x, y) - d(x', y')| \leq s[d(x, x') + d(y, y')], \tag{1.20}
\]
holds for all \( x, y, x', y' \in X \), implying the continuity of the b-metric: if \( d(x_n, x) \to 0 \) and \( d(y_n, y) \to 0 \), then the relations
\[
|d(x_n, y_n) - d(x, y)| \leq s[d(x_n, x) + d(y_n, y)] \to 0 \quad \text{as} \quad n \to \infty,
\]
show that \( d(x_n, y_n) \to d(x, y) \) as \( n \to \infty \).

A strong b-metric satisfies the \( s \)-polygonal inequality. Indeed,
\[
d(x_0, x_n) \leq sd(x_0, x_1) + d(x_1, x_n) \leq sd(x_0, x_1) + sd(x_1, x_2) + d(x_2, x_n) \leq \cdots \leq s\{sd(x_0, x_1) + sd(x_1, x_2) + \cdots + d(x_{n-1}, x_n)\}.
\]
Completeness and completion.

A Cauchy sequence in a b-metric space $(X, d)$ is a sequence $(x_n)$ in $X$ such that $\lim_{m,n \to \infty} d(x_n, x_m) = 0$. The inequality $d(x_n, x_m) \leq s [d(x_n, x) + d(x, x_m)]$ shows that every convergent sequence is Cauchy. The b-metric space $(X, d)$ is called complete if every Cauchy sequence converges to some $x \in X$. By a completion of a b-metric space $(X, d)$ one understands a complete b-metric space $(Y, \rho)$ such that there exists an isometric embedding $j : X \to Y$ with $j(X)$ dense in $Y$.

By an isometric embedding of a b-metric space $(X_1, d_1)$ into a b-metric space $(X_2, d_2)$ one understands a mapping $f : X_1 \to X_2$ such that

$$d_2(f(x), f(y)) = d_1(x, y),$$

for all $x, y \in X_1$. Two b-metric spaces $(X_1, d_1)$, $(X_2, d_2)$ are called isometric if there exists a surjective isometric embedding $f : X_1 \to X_2$.

The completeness is preserved by the uniform equivalence of b-metrics, but not by the topological equivalence.

A question raised in [53, p. 128] is:

Does every strong b-metric space admit a completion?

This question was answered in the affirmative in [12].

**Theorem 1.17.** Let $(X, d)$ be a strong b-metric space.

1. There exists a complete strong b-metric space $(\tilde{X}, \tilde{d})$ which is a completion of $(X, d)$.
2. The completion is unique up to an isometry, in the sense that if $(X_1, d_1)$, $(X_2, d_2)$ are two strong b-metric spaces which are completions of $(X, d)$, then $(X_1, d_1)$ and $(X_2, d_2)$ are isometric.

**Proof.** The proof follows the ideas from the metric case. On the family $\mathcal{C}(X)$ of all Cauchy sequences in $X$ one considers the equivalence relation

$$(x_n) \sim (y_n) \iff \lim_n d(x_n, y_n) = 0.$$ 

On the quotient space $\tilde{X} = \mathcal{C}(X)/\sim$ one defines $\tilde{d}$ by $\tilde{d}(\xi, \eta) = \lim_n d(x_n, y_n)$, where $(x_n) \in \xi$ and $(y_n) \in \eta$, and one shows that $(\tilde{X}, \tilde{d})$ is a complete strong b-metric space containing $X$ isometrically as a dense subset. □

**Remark 1.18.** As it is mentioned in [12], the existence of a completion of an arbitrary b-metric space is still an important open problem.

1.4. Spaces of homogeneous type. Completing some earlier results of Coifman and de Guzman [25], Macías and Segovia [57, 58] considered b-metrics (under the name quasi-distances) in connection with some problems in harmonic analysis.

The framework in [57] is the following. Let $(X, d)$ be a b-metric space. One considers a positive measure $\mu$ defined on a $\sigma$-algebra of subsets of $X$ containing the open sets and the balls $B(x, r)$ such that

$$(1.21) \quad 0 < \mu(B(x, ar)) \leq \beta \mu(B(x, r)),$$

for all $x \in X$ and $r > 0$, where $a > 1$ and $\beta > 0$ are fixed numbers. A b-metric space equipped with a measure $\mu$ satisfying (1.21) is called a space of homogeneous type and is denoted by $(X, d, \mu)$. If further, there exist $c_1, c_2 > 0$ such that

$$(1.22) \quad 0 < \mu(\{x\}) < r < \mu(X) \implies c_1 r \leq \mu(B(x, r)) \leq c_2 r,$$

for all $x \in X$, then the space $(X, d, \mu)$ of homogeneous type is called normal.

**Remark 1.19.** One can show that if $(X, d)$ is a b-metric space with a positive measure $\mu$ satisfying (1.22), then the space $(X, d, \mu)$ is of homogeneous type.

Concerning the openness of balls in b-metric spaces we mention the following result.
Theorem 1.20 \( [57] \). Let \((X,d)\) be a b-metric space. Then there exist a b-metric \(d'\) on \(X\), Lipschitz equivalent to \(d\), and the constants \(C > 0\) and \(0 < \alpha < 1\) such that
\begin{equation}
|d'(x, z) - d'(y, z)| \leq C r^{1-\alpha} d'(x, y)^\alpha,
\end{equation}
whenever \(\max\{d'(x, z), d'(y, z)\} < r\).

Remark 1.21. The inequality \( [1.23] \) can be written in the equivalent form
\begin{equation}
|d'(x, z) - d'(y, z)| \leq C r^{1-\alpha} (\max\{d'(x, z), d'(y, z)\})^{1-\alpha},
\end{equation}
and it is easy to check that the balls corresponding to a b-metric \(d'\) satisfying \( [1.24] \) are \(\tau_d\)-open.

Indeed, let \(B_d(x, r)\) be a ball. We have to show that for every \(y \in B_d(x, r)\) there exists \(r' > 0\) such that \(B_{d'}(y, r') \subseteq B_d(x, r)\), that is,
\[d'(y, z) < r' \implies d'(x, z) < r,\]
for any \(z \in X\). Supposing that \(d'\) satisfies the \(s'\)-relaxed triangle inequality for some \(s' \geq 1\), choose first \(0 < r' < r\). Then
\[d'(x, z) \leq s' d'(x, y) + s' d'(y, z) < 2s'r.\]

By \( [1.24] \),
\[
d'(x, z) \leq d'(x, y) + |d'(x, z) - d'(x, y)| \\
< d'(x, y) + C d'(y, z)^\alpha (\max\{d'(x, z), d'(y, z)\})^{1-\alpha} \\
< d'(x, y) + C(2s'r)^{1-\alpha}(r')^{\alpha}.
\]
Choosing \(0 < r' < r\) such that \(C(2s'r)^{1-\alpha}(r')^{\alpha} < r - d'(x, y)\), it follows \(d'(x, z) < r\).

Concerning the set of points \(x \in X\) with \(\mu(\{x\}) > 0\) we mention.

Proposition 1.22 \( [57] \). Let \((X,d,\mu)\) be a space of homogeneous type and
\[M = \{x \in X : \mu(\{x\}) > 0\}.\]

Then the set \(M\) is at most countable and for every \(x \in M\) there exists \(r > 0\) such that \(M \cap B(x, r) = \{x\}\).

We mention also the following result.

Theorem 1.23 \( [57] \). Let \((X,d,\mu)\) be a space of homogeneous type such that the balls are \(\tau_d\)-open. Let \(\delta : X \times X \to [0, \infty)\) be given by
\[\delta(x,y) = \inf\{\mu(B) : B\ is\ a\ ball\ containing\ x, y\},\]
if \(x \neq y\) and \(\delta(x,x) = 0\).

Then \(\delta\) is a b-metric on \(X\), \((X,\delta,\mu)\) is a normal space and \(\tau_\delta = \tau_d\).

1.5. Topological properties of \(f\)-quasimetric spaces. We present now, following \([14]\) a very general class of metric type spaces. On a nonempty set \(X\) consider a mapping \(d : X \times X \to \mathbb{R}_+\) satisfying only the condition
\begin{equation}
d(x,y) = 0 \iff x = y,\end{equation}
for all \(x,y \in X\), and call such a function \textit{distance}. One can define open balls with respect to \(d\) as usual
\(B(x, r) = \{y \in X : d(x,y) < r\}\),
for \(x \in X\) and \(r > 0\), and a topology \(\tau_d\) by
\[
G \in \tau_d \iff \forall x \in G, \exists r > 0, B(x, r) \subseteq G,
\]
where \(G \subseteq X\).

The topology \(\tau_d\) is \(T_1\) because \(X \setminus \{x\}\) is open, and so \(\{x\}\) closed. Indeed, if \(y \neq x\), then \(r := d(y, x) > 0\) and \(x \notin B(y, r)\).
Along with the distance $d$ one can consider the conjugate distance $\bar{d}(x, y) = d(y, x)$, $x, y \in X$. The $\bar{d}$-balls are given by

$$B_{\bar{d}}(x, r) = \{y \in X : \bar{d}(x, y) < r\} = \{y \in X : d(y, x) < r\} ,$$

and the corresponding topology is denoted by $\tau_{\bar{d}}$.

Consider now a function $f : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ such that

\begin{equation}
(1.26) \quad (t_1, t_2) \to (0, 0) \implies f(t_1, t_2) \to (0, 0) ,
\end{equation}

where $(t_1, t_2) \in \mathbb{R}_+ \times \mathbb{R}_+$. we say that a distance $d$ on a set $X$ is an $f$-quasimetric if it satisfies the inequality

\begin{equation}
(1.27) \quad d(x, y) \leq f(d(x, z), d(y, z)) ,
\end{equation}

for all $x, y, z \in X$.

**Example 1.24.** We present first some important particular cases of function $f$.

- $f(t_1, t_2) = t_1 + t_2$. In this case $d$ is a quasimetric (see [23]) and a metric if the distance $d$ is symmetric.
- $f(t_1, t_2) = s_1t_1 + s_2t_2$, for some $s_1, s_2 \geq 1$. In this case $d$ is called an $(s_1, s_2)$-quasimetric (see [13]), a b-quasimetric if $s_1 = s_2 = s$, respectively an $(s_1, s_2)$-metric, and b-metric if $d$ is symmetric.

From (1.27) one obtains the following result, called the asymptotic triangle inequality:

\begin{equation}
(1.28) \quad d(x_n, y_n) \to 0 \text{ and } d(y_n, z_n) \to 0 \implies d(x_n, z_n) \to 0 .
\end{equation}

Conversely, if a distance functions satisfies (1.28), then there exists a function $f$, satisfying (1.26), such that $d$ is an $f$-quasimetric.

Indeed, define $h : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$h(t) = \sup\{d(u, v) : u, v \in X, \exists w \in X, d(u, w) + d(w, v) \leq t\} .$$

The function $h$ is obviously nondecreasing and

$$\lim_{t \to 0} h(t) = 0 .$$

Indeed, if $t_n \to 0$, where $t_n \in \mathbb{R}_+, n \in \mathbb{N}$, then there exist $u_n, v_n, w_n \in X$ such that

$$d(u_n, w_n) + d(w_n, v_n) \leq t_n \quad \text{and} \quad d(u_n, v_n) > f(t_n) - \frac{1}{n} ,$$

for all $n \in \mathbb{N}$. The first inequality implies $d(u_n, w_n), d(w_n, v_n) \to 0$, so that, by (1.28), $d(u_n, v_n) \to 0$, which, by the second inequality from above, yields $f(t_n) \to 0$.

By the definition of the function $h$,

$$h(d(x, z) + d(z, y)) \geq d(x, y) ,$$

for all $x, y, z \in X$, so we can take $f(t_1, t_2) = h(t_1 + t_2)$.

Define the convergence of a sequence $(x_n)$ in a distance space $(X, d)$ to $x \in X$ by

$$x_n \xrightarrow{d} x \iff d(x, x_n) \to 0 ,$$

and, for $Z \subseteq X$ and $x \in X$ put

$$d(x, Z) = \inf\{d(x, z) : z \in Z\} .$$

We have the following useful characterizations of the interior and closure.

**Proposition 1.25.** Let $(X, d)$ be an $f$-quasimetric space and $Z \subseteq X$. Then

\begin{equation}
(1.29) \quad \text{int}(Z) = \{x \in X : d(x, X \setminus Z) > 0\} \quad \text{and} \quad \text{cl}(Z) = \{x \in X : d(x, A) = 0\} .
\end{equation}
Let
\[ \tilde{Z} := \{ x \in X : d(x, X \setminus Z) > 0 \} . \]

We show that
\begin{enumerate}[(i)]
  \item $\tilde{Z}$ is open;
  \item $\tilde{Z} \subseteq Z$;
  \item $\text{int}(Z) \subseteq \tilde{Z}$,
\end{enumerate}
which will imply that $\text{int}(Z) = \tilde{Z}$.

Suppose that $\tilde{Z}$ is not open. Then there exists $x \in \tilde{Z}$ such that
\[ B(x, n^{-1}) \not\subseteq \tilde{Z} \]
for all $n \in \mathbb{N}$. Hence, for each $n \in \mathbb{N}$, there exists $y_n \in X$ such that
\[ d(x, y_n) < 1/n \text{ and } d(y_n, X \setminus Z) = 0. \]
Then, for every $n \in \mathbb{N}$, there exists $w_n \in X \setminus Z$ such that
\[ d(y_n, w_n) < 1/n. \]
By (1.28), $d(x, w_n) \to 0$, implying $d(x, X \setminus Z) = 0$, in contradiction to the hypothesis that $x \in \tilde{Z}$.

The proof of (ii) is simple. If $x \not\in Z$, then $x \in X \setminus Z$, so that $d(x, X \setminus Z) = 0$, that is, $x \not\in \tilde{Z}$.

To prove (iii), suppose that $x \in \text{int}(Z)$. Then there exists $r > 0$ such that $B(x, r) \subseteq \text{int}(Z) \subseteq Z$.

But then, for any $y \in X \setminus Z$, $d(x, y) \geq r > 0$, that is, $x \in \tilde{Z}$.

The proof of the formula for closure is based on the equality
\[ \text{cl}(X \setminus Y) = X \setminus \text{int}(Y), \]
valid for any subset $Y$ of $X$. Then
\[ \text{cl}(Z) = X \setminus \text{int}(X \setminus Z) = X \setminus \{ x \in X : d(x, Z) > 0 \} \]
\[ = \{ x \in X : d(x, Z) = 0 \}. \]

Proposition 1.25 has some important consequences.

**Corollary 1.26.** Let $(X, d)$ be an $f$-quasimetric space.

1. For every $x \in X$ and $r > 0$, $x \in \text{int}(B(x, r))$, or, equivalently, $B(x, r)$ is a neighborhood of $x$.

2. The topology $\tau_d$ satisfies the first axiom of countability, i.e. every point has a countable base of neighborhood. Consequently, usual sequences suffice to characterize the topological properties of $X$.

3. The convergence of a sequence $(x_n)$ in $X$ to $x \in X$ with respect to $\tau_d$ is characterized in the following way:
\[ x_n \xrightarrow{\tau_d} x \iff d(x, x_n) \to 0. \]

**Proof.**

1. This follows from the following relations
\[ y \in X \setminus B(x, r) \iff d(x, y) \geq r \]
\[ \iff d(x, X \setminus B(x, r)) \geq r > 0 \]
\[ \iff x \in \text{int}(B(x, r)). \]

2. A countable base of neighborhoods of a point $x \in X$ is $B(x, n^{-1})$, $n \in \mathbb{N}$. If $V$ is a neighborhood of $x$, then there exists $G \in \tau_d$ such that
\[ x \in G \subseteq V. \]

By the definition of the topology $\tau_d$, there exists $r > 0$ such that $B(x, r) \subseteq G$, implying
\[ B(x, n^{-1}) \subseteq B(x, r) \subseteq G \subseteq V, \]
for some sufficiently large $n \in \mathbb{N}$. 


3. Suppose that $x_n \xrightarrow{\tau_d} x$ and let $r > 0$. Then, by 1, $B(x, r) \in \mathcal{V}(x)$ so there exists $n_r \in \mathbb{N}$ such that $x_n \in B(x, r)$, or, equivalently, $d(x, x_n) < r$ for all $n \geq n_r$. Consequently, $d(x, x_n) \to 0$.

Suppose now that $d(x, x_n) \to 0$ and let $V \in \mathcal{V}(x)$. Then $x \in \text{int}(V)$, so there exists $r > 0$ such that $B(x, r) \subseteq \text{int}(V) \subseteq V$. By hypothesis, there exists $n_0 \in \mathbb{N}$, such that $d(x, x_n) < r$ for all $n \geq n_0$, implying $x_n \in B(x, r) \subseteq V$ for all $n \geq n_0$. □

Remark 1.27. The convergence with respect to the topology $\tau_d$ generated by the conjugate $f$-quasimetric $\bar{d}(x, y) = d(y, x)$, $x, y \in X$, is characterized by

$$x_n \xrightarrow{\tau_d} x \iff d(x_n, x) \to 0.$$ \hspace{1cm} (1.31)

Since $x \in \text{int}(B(x, r))$, there exists $r' > 0$ such that $B(x, r') \subseteq \text{int}(B(x, r)) \subseteq B(x, r)$. A natural question is to find the biggest $r'$ such that $B(x, r') \text{ int}(B(x, r))$.

For $r > 0$ let

$$\Lambda(r) = \{ t_1 \geq 0 : \sup_{t_2 > 0} f(t_1, t_2) \geq r \} ,$$

and

$$\theta(r) = \begin{cases} \sup \Lambda(r) & \text{if } \Lambda(r) \neq \emptyset \\ r & \text{if } \Lambda(r) = \emptyset . \end{cases}$$

Here, by definition,

$$\limsup_{t_2 > 0} f(t_1, t_2) = \inf_{\delta > 0} \{ \sup\{ f(t_1, t_2) : 0 \leq t_2 < \delta \} \} .$$ \hspace{1cm} (1.32)

Remark 1.28. Observe that $\Lambda(r) = \emptyset$ implies $B(x, r) = X$.

Also $\theta(r) > 0$ in both cases.

Indeed, if $\Lambda(r) = \emptyset$, then putting $t_1 = d(x, y)$ for some arbitrary $y \in X$, it follows $\limsup_{t_2 > 0} f(t_1, t_2) < r$, so there exists $\delta > 0$ such that $\sup\{ f(t_1, t_2) : 0 \leq t_2 < \delta \} < r$, implying

$$d(x, y) \leq f(d(x, y), d(y, y)) = f(t_1, 0) < r ,$$

that is, $y \in B(x, r)$. If $\theta(r) = 0$, then $\Lambda(r) \neq \emptyset$, so there exists a sequence $(t^k)$ in $\Lambda(r)$ such that $t^k \to 0$ as $k \to \infty$. By (1.32) and the definition of $\Lambda(r)$, for every $k \in \mathbb{N}$ there exists $0 \leq t^k_2 < 1/k$ such that $f(t^k_1, t^k_2) > r/2$. Taking into account (1.26), one obtains the contradiction.

$$0 = \lim_{k \to \infty} f(t^k_1, t^k_2) \geq r/2 .$$

Proposition 1.29. Let $(X, d)$ be an $f$-quasimetric space. Then

$$B(x, \theta(r)) \subseteq \text{int}(B(x, r)) ,$$

for every $x \in X$ and $r > 0$.

Proof. By Remark 1.28 we can suppose $\Lambda(r) \neq \emptyset$.

Let $y \in B(x, \theta(r))$. Then $t_1 := d(x, y) < \theta(r)$, implying $t_1 \notin \Lambda(r)$ so that $\limsup_{t_2 > 0} f(t_1, t_2) < r$. By (1.32) there exists $\delta > 0$ such that

$$\sup\{ f(t_1, t_2) : 0 \leq t_2 < \delta \} < r .$$ \hspace{1cm} (1.33)

We show that

$$B(y, \delta) \subseteq B(x, r) .$$ \hspace{1cm} (1.34)

If $z \in B(y, \delta)$, then, by (1.32)

$$d(x, z) \leq f(d(x, y), d(y, z)) = f(t_1, d(y, z)) < r ,$$

because $d(y, z) < \delta$.

Since $B(y, \delta)$ is a neighborhood of $y$, the inclusion (1.34) shows that $B(x, r)$ is also a neighborhood of $y$, that is, $y \in \text{int}(B(x, r))$. □
Remark 1.30. If \((X,d)\) is an \((s_1,s_2)\)-quasimetric space, i.e. an \(f\)-quasimetric space for \(f(t_1,t_2) = s_1t_1 + s_2t_2\), then
\[
\theta(r) = r/s_1.
\]

Indeed,
\[
\theta(r) = \inf\{t_1 \geq 0 : \lim_{t_2 \to 0} (s_1t_1 + s_2t_2) \geq r\} = \inf\{t_1 \geq 0 : s_1t_1 \geq r\} = r/s_1.
\]

The authors define in [14] the notion of Cauchy sequence and completeness. A sequence in a distance space \((X,d)\) is called Cauchy if for every \(\varepsilon > 0\) there exists \(n_0 = n_0(\varepsilon)\) such that
\[
d(x_n,x_{n+k}) < \varepsilon,
\]
for all \(n \geq n_0\) and all \(k \in \mathbb{N}\). The distance space \((X,d)\) is called complete if every Cauchy sequence is convergent to some \(x \in X\).

The authors prove in [14] the validity of Baire category theorem in complete \(f\)-quasimetric spaces which satisfy the separation axiom \(T_3\) (i.e. are regular). As in the metric case, the proof is based on the nonemptiness of descending sequences of closed balls with radii tending to 0. They extend the metrization Theorems 1.3 and 1.4 to this setting proving the quasimetrizability of \(f\)-quasimetric spaces. In this case, the equivalent quasimetrics are also given by the formulae (1.3) and (1.4) and are denoted by \(\text{Inf} d\). They introduce the notion of weak symmetry of the \(f\)-quasimetric \(d\) by the condition
\[
d(x,x_n) \to 0 =\Rightarrow d(x_n,x) \to 0,
\]
for all sequences \((x_n)\) in \(X\) and \(x \in X\). The topology generated by a weakly symmetric \(f\)-quasimetric is normal and metrizable.

Remark 1.31. By (1.30) and (1.31), the condition (1.36) means that the identity mapping \(I : (X,\tau_d) \to (X,\tau_d)\) is continuous, or equivalently, \(\tau_d \subseteq \tau_d\), i.e. the topology \(\tau_d\) is finer than \(\tau_d\).

Remark 1.32. In the theory of quasimetric spaces (see Example 1.24) a sequence satisfying (1.35) is called "left \(K\)-Cauchy" and the corresponding notion of completeness, "left \(K\)-completeness". If the sequence \((x_n)\) satisfies the condition
\[
d(x,x_n) \to 0 =\Rightarrow d(x_n,x) \to 0,
\]
for all \(n \geq n_0\) and all \(k \in \mathbb{N}\), then it is called "right \(K\)-Cauchy", and the corresponding notion of completeness, "right \(K\)-completeness".

Some authors call a sequence \((x_n)\) satisfying (1.35) "forward Cauchy" and "backward Cauchy" if it satisfies (1.37). Also the convergence given by \(d(x,x_n) \to 0\) is called "backward convergence", while that given by \(d(x_n,x) \to 0\) is called "forward convergence" (see, e.g. [61]). Combining these notions of Cauchy sequence and convergence one obtains various notions of completeness: "forward-forward complete" meaning that every forward Cauchy sequence is forward convergent, with similar definitions for forward-backward, backward-forward, etc – completeness.

Due to the asymmetry of the quasimetric, there are several notions of Cauchy sequence (actually 7, see [68]), each of them agreeing with the usual notion of Cauchy (fundamental) sequence in the metric case. Considering \(d\)-convergence and \(\bar{d}\)-convergence, from these 7 notions of Cauchy sequence one obtains 14 notions of completeness (see the book [23]).

1.6. Historical remarks and further results. The relaxed triangle inequality and the corresponding spaces were rediscovered several times under various names – quasi-metric, near metric (in [72]), metric type, etc.

- (1970) Coifman and de Guzman [25] in connection with some problems in harmonic analysis (a b-metric is called "distance" function);
- (1979) the results of Coifman and de Guzman were completed by Macías and Segovia [67, 58].
• (1989) Bakhtin [17] called them “quasi-metric spaces” and proved a contraction principle for such spaces;
• (1993) Czerwik introduced them under the name “b-metric space”, first for \( s = 2 \) in [26], and then for an arbitrary \( s \geq 1 \) in [27], with applications to fixed points;
• (1998,2003) Fagin et al. [37] [38] considered distances satisfying the \( s \)-relaxed triangle and polygonal inequalities with applications to some problems in theoretical computer science;
• (2010) Khamis [50] introduced them under the name “metric type spaces” and remarked that if \( D \) is a cone metric on a set \( X \) with values in a Banach space ordered by a normal cone \( C \) with normality constant \( K \), then \( d(x,y) = \|D(x,y)\| \), \( x,y \in X \), is a b-metric on \( X \) satisfying the \( K \)-relaxed polygonal inequality.

Some topological properties of b-metric spaces (e.g. compactness) were studied in [52]. Xia studied the properties of the space \( C(T,X) \) of continuous functions from a compact metric space \( T \) to a b-metric space \( X \), and geodesics and intrinsic metrics in b-metric spaces. The results were applied to show that the optimal transport paths between atomic probability measures are geodesics in the intrinsic metric. An, Tuyen and Dung [11] extended to b-metric spaces Stone’s paracompactness theorem.

2. Generalized b-metric spaces

The notions of generalized metric, meaning a mapping \( d : X \times X \to [0,\infty] \) satisfying the axioms of a metric, and generalized metric space \((X,d)\) were introduced by W. A. J. Luxemburg in [53]–[56] in connection with the method of successive approximation and fixed points. These results were completed by A. F. Monna [62] and M. Edelstein [35]. Further results were obtained by J. B. Diaz and B. Margolis [34] [39] and C. F. K. Jung [43]. G. Dezső [33] considered generalized vector metrics, i.e. metrics with values in \( \mathbb{R}^m \cup \{+\infty\}^m \), and extended to this setting Perov’s fixed point theorem (see [65]–[66]) as well as other fixed point results (Luxemburg, Jung, Diaz-Margolis, Kannan).

For some recent results on generalized metric spaces see [18] and [28]. Recently, G. Beer and J. Vanderwerf [19] [20] considered vector spaces equipped with norms that can take infinite values, called “extended norms” (see also [30]).

Following these ideas, we consider here the notion of generalized b-metric on a nonempty set \( X \) as a mapping \( d : X \times X \to [0,\infty] \) satisfying the conditions (i)–(iii) from (1.1). If \( d \) satisfies further the condition (v), then \( d \) is called a generalized strong b-metric and the pair \((X,d)\) a generalized strong b-metric space.

Let \((X,d)\) be a generalized b-metric space. As in Jung [43], it follows that

\[
\tag{2.1}
x \sim y \iff d(x,y) < +\infty, \quad x,y \in X,
\]

is an equivalence relation on \( X \). Denoting by \( X_i, i \in I \), the equivalence classes corresponding to \( \sim \) and putting \( d_i = d|_{X \times X}, i \in I \), then \((X_i,d_i)\) is a b-metric space (a strong b-metric space if \((X,d)\) is a generalized strong b-metric space) for every \( i \in I \). Therefore, \( X \) can be uniquely decomposed into equivalence classes \( X_i, i \in I \), called the canonical decomposition of \( X \).

By analogy to [43] we have.

**Theorem 2.1.** Let \((X,d)\) be a generalized b-metric space and \( X_i, i \in I \), its canonical decomposition. Then the following hold.

1. The space \((X,d)\) is complete if and only if \((X_i,d_i)\) is complete for every \( i \in I \).
2. If \((Y_i,d_i), i \in I \), are b-metric spaces (with the same \( s \)) and \( Y_i \cap Y_j = \emptyset \) for all \( i \neq j \) in \( I \), then

\[
\tag{2.2}
d(x,y) := \begin{cases} d_i(x,y) & \text{if } x,y \in Y_i, \text{ for some } i \in I, \\ +\infty & \text{if } x \in Y_i \text{ and } y \in Y_j \text{ for some } i,j \in I \text{ with } i \neq j, \end{cases}
\]

is a generalized b-metric on \( Y = \bigcup_{i \in I} Y_i \), with \( \{Y_i : i \in I\} \) the family of equivalence classes corresponding to the equivalence relation (2.1).

The same results are true for generalized strong b-metric spaces.

2.1. The completion of generalized b-metric spaces. In this subsection we shall prove the existence of the completion of strong b-metric spaces. The existence of the completion of a generalized metric space was proved in [29].

We start with the following lemma.

Lemma 2.2. Let \((X, d)\) be a generalized b-metric space, \((Z, D)\) a complete generalized b-metric space, with continuous generalized b-metrics \(d, D\) and \(Y\) a dense subset of \(X\). Then for every isometric embedding \(f : Y \to Z\) there exists a unique isometric embedding \(F : X \to Z\) such that \(F|_Y = f\). If, in addition, \(X\) is complete and \(f(Y)\) is dense in \(Z\), then \(F\) is bijective (i.e. \(F\) is an isometry of \(X\) onto \(Z\)).

Proof. For the sake of completeness we include the simple proof of this result. For \(x \in X\) let \(\langle y_n \rangle\) be a sequence in \(Y\) such that \(d(y_n, x) \to 0\). Then \(\langle y_n \rangle\) is a Cauchy sequence in \((X, d)\) and the equalities \(D(f(y_n), f(y_m)) = d(y_n, y_m)\), \(m, n \in \mathbb{N}\), show that \(\langle f(y_n) \rangle\) is a Cauchy sequence in \((Z, D)\). Since \((Z, D)\) is complete, there exists \(z \in Z\) such that \(D(f(y_n), z) \to 0\). If \(\langle y'_n \rangle\) is another sequence in \(Y\) converging to \(x\), then \(\langle f(y'_n) \rangle\) will converge to an element \(z' \in Z\). By the continuity of the generalized b-metrics \(d\) and \(D\),

\[
D(z, z') = D(\lim_n f(y_n), \lim_n f(y'_n)) = \lim_n D(f(y_n), f(y'_n)) = \lim_n d(y_n, y'_n) = 0,
\]

showing that \(z = z'\). So we can unambiguously define a mapping \(F : X \to Z\) by \(F(x) = \lim_n f(y_n)\), where \(\langle y_n \rangle\) is a sequence in \(Y\) converging to \(x \in X\). For \(y \in Y\) taking \(y_n = y\), \(n \in \mathbb{N}\), it follows \(F(y) = y\).

For \(x, x' \in X\), let \(\langle y_n \rangle, \langle y'_n \rangle\) be sequences in \(Y\) converging to \(x\) and \(x'\), respectively. Then

\[
D(F(x), F(x')) = \lim_n D(f(y_n), f(y'_n)) = \lim_n d(y_n, y'_n) = d(x, x'),
\]

i.e. \(F\) is an isometric embedding.

If \(f(Y)\) is dense in \(Z\), then, for any \(z \in Z\), there exists a sequence \(\langle y_n \rangle\) in \(Y\) such that \(D(f(y_n), z) \to 0\). It follows that \(\langle f(y_n) \rangle\) is a Cauchy sequence in \(Z\) and so, as \(f\) is an isometry, \(\langle y_n \rangle\) will be a Cauchy sequence in \(X\). As the space \(X\) is complete, \(\langle y_n \rangle\) is convergent to some \(x \in X\). But then

\[
D(F(x), z) = \lim_n D(F(x), f(y_n)) = \lim_n d(x, y_n) = 0,
\]

showing that \(F(x) = z\).

\[\square\]

Remark 2.3. The proof can be adapted to show that, under the hypotheses of Lemma 2.2, every uniformly continuous mapping \(f : Y \to Z\) has a unique uniformly continuous extension to \(X\). The notion of uniform continuity for mappings between generalized b-metric spaces is defined as in the metric case.

Let \((X, d)\) be a generalized strong b-metric space with \(X_i, i \in I\), the family of equivalence classes corresponding to \([2.1]\). For every \(i \in I\), let \((Y_i, D_i)\) be a completion of the strong b-metric space \((X_i, d_i)\). Denote by \(T_i : (X_i, d_i) \to (Y_i, D_i)\) the isometric embedding with \(T_i(X_i)\) \(D_i\)-dense in \(Y_i\) corresponding to this completion.

Replacing, if necessary, \(Y_i\) with \(\overline{Y_i} = Y_i \times \{i\}\), \(D_i\) with \(\overline{D_i}(x, i, (y, i)) = D_i(x, y)\), for \(x, y \in Y_i\), and putting \(T_i(x, i) = (T_i(x), i)\), \(x \in Y_i\), we may suppose, without restricting the generality, that \(Y_i \cap Y_j = \emptyset\) for all \(i, j \in I\) with \(i \neq j\).

Put \(Y := \bigcup_{i \in I} Y_i\), and define

\[
D : Y \times Y \to [0, \infty]
\]

according to \([2.2]\) and \(T : X \to Y\) by

\[
T(x) := T_i(x),
\]

where \(i\) is the unique element of \(I\) such that \(x \in X_i\).

We have the following result.
\textbf{Theorem 2.4.} Let \((X, d)\) be a generalized strong b-metric space and \((Y, D)\) the generalized strong b-metric space defined above. Then

(i) \((Y, D)\) is a complete generalized strong b-metric space;
(ii) \(T : (X, d) \to (Y, D)\) is an isometric embedding with \(T(X)\) \(D\)-dense in \(Y\);
(iii) any other complete generalized strong b-metric space \((Z, \rho)\) that contains a \(\rho\)-dense isometric copy of \((X, d)\), is isometric to \((Y, D)\).

\textit{Proof.} Since each strong b-metric space \((Y_i, D_i)\) is complete, Theorem 2.1 implies that the generalized strong b-metric space \((Y, D)\) is complete.

Let \(x, y \in X\). If \(x, y \in X_i\), for some \(i \in I\), then

\[ D(T(x), T(y)) = D_i(T_i(x), T_i(y)) = d_i(x, y) = d(x, y). \]

If \(x \in X_i\), \(y \in X_j\) with \(i \neq j\), then

\[ T(x) = T_i(x) \in Y_i \text{ and } T(y) = T_j(x) \in Y_j, \]

so that

\[ D(T(x), T(y)) = D(T_i(x), T_j(y)) = +\infty = d(x, y). \]

Now for \(\xi \in Y\) there exists a unique \(i \in I\) such that \(\xi \in Y_i\). Since \(T_i(X_i)\) is dense in \((Y_i, D_i)\), there exists a sequence \((x_n)\) in \(X_i\) such that

\[ 0 = \lim_{n \to \infty} D_i(T_i(x_n), \xi) = \lim_{n \to \infty} D(T(x_n), \xi), \]

which means that \(T(X)\) is \(D\)-dense in \((Y, D)\).

Finally, to verify (iii), let \(S : (X, d) \to (Z, \rho)\) be an isometric embedding with \(S(X)\) dense in \(Z\). Define \(R : T(X) \to X\) by \(R(T(x)) = x, x \in X\). Then \(R\) is an isometry of \(T(X)\) onto \(X\) and \(S \circ R\) is an isometric embedding of \(T(X)\) into \(Z\). Since \(T(X)\) is dense in \(Y\) and \(S(R(T(X))) = S(X)\) is dense in \(Z\), Lemma 2.2 yields the existence of an isometry \(U\) of \(Y\) onto \(Z\), which ends the proof. \(\square\)

3. Fixed points in b-metric spaces

We shall prove some fixed point results in b-metric and in generalized b-metric spaces.

3.1. Fixed points in b-metric spaces. We start with the case of b-metric spaces. The second result is an extension to b-metric spaces of Theorem 4.1 from [11].

Let \((X, d)\) be a b-metric space with \(d\) satisfying the \(s\)-relaxed triangle inequality. We consider functions \(\varphi : \mathbb{R}^+ \to \mathbb{R}^+\) satisfying the conditions

\begin{align}
\text{(a)} \quad & \varphi \text{ is nondecreasing} \\
\text{(b)} \quad & \lim_{n \to \infty} \varphi^n(t) = 0 \text{ for all } t > 0.
\end{align}

\textbf{Remark 3.1.} If \(\varphi : \mathbb{R}^+ \to \mathbb{R}^+\) satisfies the conditions (a) and (b) from above, then

\begin{align}
\text{(c)} \quad & \varphi(t) < t \text{ for all } t > 0; \\
\text{(d)} \quad & \lim_{n \to \infty} \varphi^n(0) = 0 \text{ and } \varphi(0) = 0 = \lim_{t \to 0^+} \varphi(t).
\end{align}

Indeed, if \(\varphi(t) \geq t\) for some \(t > 0\), then, by (a), \(\varphi^2(t) \geq \varphi(t) \geq t\) and, in general \(\varphi^n(t) \geq t > 0\) for all \(n\), in contradiction to (b).

Also, \(0 \leq \varphi(0) \leq \varphi(1)\) implies \(0 \leq \varphi^2(0) \leq \varphi^2(1)\) and in general \(0 \leq \varphi^n(0) \leq \varphi^n(1)\). Since \(\lim_{n \to \infty} \varphi^n(1) = 0\), this yields (d).

Similarly, \(0 \leq \varphi(0) \leq \varphi(t) < t\) for any \(t > 0\), implies \(\varphi(0) = 0 = \lim_{t \to 0^+} \varphi(t)\).
Theorem 3.2. Let \((X,d)\) be a complete \(b\)-metric space, where \(d\) satisfies the \(s\)-relaxed triangle inequality and let \(\varphi : \mathbb{R}_+ \to \mathbb{R}_+\) be a function satisfying the conditions (a), (b) from (3.1). Then every mapping \(f : X \to X\) satisfying the inequality
\[
(3.2) \quad d(f(x), f(y)) \leq \varphi(d(x, y)),
\]
for all \(x, y \in X\), has a unique fixed point \(z\) and, for every \(x \in X\), the sequence \((f^n(x))_{n \in \mathbb{N}_0}\) converges to \(z\) as \(n \to \infty\).

Proof. We present the proof given in [44]. Let \(x \in X\). Put \(x_n = T^n x, n \in \mathbb{N}_0\), and let us show that \((x_n)_{n \in \mathbb{N}_0}\) is a Cauchy sequence.

Observe first that (3.3) implies
\[
(3.3) \quad d(T^n u, T^n v) \leq \varphi^n(d(u, v)),
\]
for all \(u, v \in X\) and \(n \in \mathbb{N}\). This implies
\[
(3.4) \quad d(T^n x_{kn}, x_{kn}) \leq \varphi^{kn}(d(x_n, x_0)),
\]
for all \(k, n \in \mathbb{N}\). Indeed, by (3.3),
\[
d(T^n x_{kn}, x_{kn}) = d(T^k x_n, T^k x_0) \leq \varphi^{kn}(d(x_n, x_0)).
\]

From (3.1) and (3.4), (b) one obtains
\[
(3.5) \quad \lim_{k \to \infty} d(T^n x_{kn}, x_{kn}) = 0,
\]
for every \(n \in \mathbb{N}\).

Let \(\varepsilon > 0\) be given.

Observe first that there exist \(\bar{k}, \bar{n} \in \mathbb{N}\) s.t.
\[
(i) \quad T^{\bar{n}}(B(x_{\bar{k}n}, \varepsilon)) \subseteq B(x_{\bar{k}n}, \varepsilon);
(ii) \quad x_{kn} \in B(x_{\bar{k}n}, \varepsilon) \quad \text{for all} \quad k \geq \bar{k};
(iii) \quad d(x_{k, \bar{n}}, x_{k, \bar{n}}) < 2s\varepsilon \quad \text{for all} \quad k_1, k_2 \geq \bar{k}.
\]

Indeed, by (3.1), (b), there exists \(\bar{n} \in \mathbb{N}\) s.t.
\[
\varphi^n(\varepsilon) < \varepsilon/(2s) \quad \text{for all} \quad n \geq \bar{n},
\]
and, by (3.5), there exists \(\bar{k} \in \mathbb{N}\) s.t.
\[
d(T^n x_{\bar{k}n}, x_{\bar{k}n}) < \varepsilon/(2s) \quad \text{for all} \quad k \geq \bar{k}.
\]

But then, \(d(u, x_{\bar{k}n}) < \varepsilon\) implies
\[
d(T^n u, T^n x_{\bar{k}n}) \leq \varphi^n(d(u, x_{\bar{k}n})) \leq \varphi^n(\varepsilon) < \varepsilon/(2s),
\]
so that
\[
d(T^n u, x_{\bar{k}n}) \leq \varphi^n(d(u, x_{\bar{k}n})) \leq \varphi^n(\varepsilon) < \varepsilon/(2s),
\]
showing that (3.6) (i) holds.

Since \(x_{\bar{k}n} \in B(x_{\bar{k}n}, \varepsilon)\) it follows that \(x_{(k+1)\bar{n}} = T^\bar{n} x_{\bar{k}n} \in B(x_{\bar{k}n}, \varepsilon)\) and, in general, by induction, \(x_{(k+j)\bar{n}} \in B(x_{\bar{k}n}, \varepsilon)\) for any \(j \in \mathbb{N}_0\).

Now, by (ii), \(x_{k, \bar{n}}, x_{k, \bar{n}} \in B(x_{\bar{k}n}, \varepsilon)\) for all \(k_1, k_2 \geq \bar{k}\), so that
\[
d(x_{k, \bar{n}}, x_{k, \bar{n}}) \leq s(d(x_{k, \bar{n}}, x_{\bar{k}n}) + d(x_{\bar{k}n}, x_{k, \bar{n}}) < 2s\varepsilon,
\]
showing that (iii) holds too.

By (3.5) for \(n = 1\) one obtains
\[
\lim_{k \to \infty} d(x_{k+1}, x_k) = 0.
\]
It is easy to check that this implies
\[ \lim_{k \to \infty} d(x_{kn+p}, x_{kn}) = 0 \quad \text{for} \quad p = 0, 1, \ldots, n - 1, \]
so there exists \( k_0 \in \mathbb{N} \) s.t.
\[ d(x_{kn+p}, x_{kn}) < \varepsilon \quad \text{for all} \quad k \geq k_0 \quad \text{and} \quad p = 0, 1, \ldots, n - 1. \]

Let now \( \tilde{k} := \max\{\tilde{k}, k_0\} \) and let \( m_1 = k_1 \tilde{n} + p_1, m_2 = k_2 \tilde{n} + p_2 \) with \( p_1, p_2 \in \{0, 1, \ldots, n - 1\} \) and \( k_1, k_2 \geq \tilde{k} \).

Combining (3.9) and (3.6)(iii) one obtains
\[
d(x_{m_1}, x_{m_2}) \leq sd(x_{k_1 \tilde{n} + p_1, x_{k_1 \tilde{n}}}) + s^2d(x_{k_1 \tilde{n}}, x_{k_2 \tilde{n}}) + s^3d(x_{k_2 \tilde{n}}, x_{k_2 \tilde{n} + p_2})
\leq (s + 2s^2 + s^3)\varepsilon \leq 4s^3\varepsilon,
\]
which shows that \( (x_n) \) is a Cauchy sequence.

The completeness of \( X \) implies the existence of a point \( z \in X \) s.t. \( \lim_{n \to \infty} d(x_n, z) = 0. \)

We have
\[ d(x_{n+1}, Tz) = d(Tx_n, Tz) \leq \varphi(d(x_n, z)) \leq d(x_n, z) \]
for all \( n \in \mathbb{N} \), so that
\[ d(z, Tz) \leq s [d(z, x_{n+1}) + d(x_{n+1}, Tz)] \leq s [d(z, x_{n+1}) + d(x_n, z)]. \]

Letting \( n \to \infty \) one obtains \( d(z, Tz) = 0 \), that is, \( z = Tz \).

The uniqueness follows in the following way. Suppose \( z_i = Tz_i, i = 1, 2 \). Then
\[ d(z_1, z_2) = d(Tz_1, Tz_2) \leq \varphi(d(z_1, z_2)). \]

By Remark 3.1(c) this can hold only for \( d(z_1, z_2) = 0 \), that is, for \( z_1 = z_2. \)

Let \( (X, d) \) be a \( b \)-metric space with \( d \) satisfying the \( s \)-relaxed triangle inequality for some \( s \geq 1 \).

An important particular case of a function \( \varphi \) satisfying the conditions (a),(b) from (3.1) is
\[ \varphi(t) = \alpha t, \quad t \geq 0. \]

If \( 0 < \alpha < 1 \), then
\[ \varphi^n(t) = \alpha^n t \to 0 \quad \text{as} \quad n \to \infty. \]

Since \( \varphi \) is also strictly increasing, it satisfies the conditions (a),(b) from (3.1).

The inequality (3.2) becomes in this case
\[ d(f(x), f(y)) \leq \alpha d(x, y), \]
for all \( x, y \in X. \)

So, Theorem 3.2 has as consequence the analog of Banach contraction principle in \( b \)-metric spaces. The following proposition also illustrates how various types of relaxed inequalities for the \( b \)-metric influence the form this principle takes.

**Proposition 3.3.** Let \( (X, d) \) be a complete \( b \)-metric space, where \( d \) satisfies the \( s \)-relaxed triangle inequality and \( f : X \to X \) a mapping such that, for some \( 0 < \alpha < 1, \)
\[ d(f(x), f(y)) \leq \alpha d(x, y), \]
for all \( x, y \in X. \) Then \( f \) has a unique fixed point \( z \) and, for every \( x \in X, \) the sequence \( \{f^n(x)\}_{n \in \mathbb{N}} \) converges to \( z \) as \( n \to \infty. \)

1. (17) If further \( 0 < \alpha < 1/s, \) then the following evaluation of the order of convergence holds
\[ d(x_n, z) \leq \frac{s^2d(x_0, x_1)}{1 - \alpha s} \cdot \alpha^n, \]
for all \( n \in \mathbb{N}. \)
2. \([\text{(3.13)}] \) If \(d\) satisfies the \(s\)-relaxed polygonal inequality, then the following evaluation of the order of convergence

\[
d(x_n, z) \leq \frac{s^2 d(x_0, x_1)}{1 - \alpha} \cdot \alpha^n, \quad n \in \mathbb{N},
\]

holds for any \(0 < \alpha < 1\).

**Proof.** Although, as we have remarked, the first statement of the proposition follows from Theorem \[3.2\] we show a proof based on Theorem \[1.4\]. Our presentation follows \[10\].

Suppose that \(d\) satisfies the \(s\)-relaxed triangle inequality, for some \(s \geq 1\). If \(0 < p \leq 1\) is given by the equation \((2s)^p = 1\), then, by Theorem \[1.4\] the functional \(\rho_p\) given by \[1.4\] is a metric on \(X\) satisfying the inequalities

\[
\rho_p(f(x), f(y)) \leq \sum_{i=0}^{n-1} d(y_i, y_{i+1})^p \leq \alpha^p \sum_{i=0}^{n-1} d(x_i, x_{i+1})^p.
\]

Since the inequality between the extreme terms in \[3.13\] holds for all chains \(x = x_0, x_1, \ldots, x_n = y, n \in \mathbb{N}\), connecting \(x\) and \(y\), it follows

\[
\rho_p(f(x), f(y)) \leq \alpha^p \rho_p(x, y),
\]

for all \(x, y \in X\), where \(0 < \alpha^p < 1\). Consequently, \(f\) is a contraction with respect to \(\rho_p\). The inequalities \[3.13\] and the completeness of \((X, d)\) imply the completeness of \((X, \rho_p)\) and so, by Banach’s contraction principle, \(f\) has a unique fixed point \(z \in X\) and the sequence of iterates \((f^n(x))_{n \in \mathbb{N}}\) is \(\rho_p\)-convergent to \(z\), for every \(x \in X\). Appealing again to the inequalities \[3.13\], it follows that \((f^n(x))_{n \in \mathbb{N}}\) is also \(d\)-convergent to \(z\) for every \(x \in X\).

1. The proof is similar to that of Banach’s contraction principle in the metric case. Observe first that, \[5.10\] implies

\[
d(f^n(x), f^n(y)) \leq \alpha^n d(x, y),
\]

for all \(n \in \mathbb{N}\) and \(x, y \in X\).

For \(x_0 \in X\) consider the sequence of iterates

\[
x_n = f(x_{n-1}) = f^n(x_0), \quad n \in \mathbb{N}.
\]

Let us prove that \((x_n)\) is a Cauchy sequence.

By \[1.2\] and \[3.15\],

\[
d(x_n, x_{n+k+1}) \leq s d(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + \ldots + s^k d(x_{n+k-1}, x_{n+k}) + s^{k+1} d(x_{n+k}, x_{n+k+1})
\]

\[
\leq \left( \alpha^n s + \alpha^{n+1} s^2 + \cdots + \alpha^{n+k-1} s^k \right) d(x_0, x_1) + \alpha^{n+k} s^{k+1} d(x_0, x_1)
\]

\[
= \alpha^n s \left( \frac{1 - (\alpha s)^k}{1 - \alpha s} + \alpha s^{k-1} \right) d(x_0, x_1)
\]

\[
= \alpha^n s \left( 1 - (\alpha s)^k \frac{1}{1 - \alpha s} \right) d(x_0, x_1)
\]

\[
= \alpha^n s \frac{1 - (\alpha s)^k - \alpha s + 1 - \alpha s}{1 - \alpha s} d(x_0, x_1)
\]

\[
< \alpha^n \frac{sd(x_0, x_1)}{1 - \alpha s}.
\]
for all \( n, k \in \mathbb{N} \). Since \( \lim_{n \to \infty} \alpha^n = 0 \), this shows that \((x_n)\) is a Cauchy sequence. By the completeness of \((X, d)\) there exists \( z \in X \) such that \( \lim_{n \to \infty} d(x_n, z) = 0 \). We have

\[
d(z, f(z)) \leq sd(z, x_{n+1}) + sd(x_{n+1}, f(z)) \leq sd(z, x_{n+1}) + sad(x_n, z) \to 0 \quad \text{as} \quad n \to \infty.
\]

Hence \( d(z, f(z)) = 0 \) and so \( z = f(z) \).

Taking into account (3.16),

\[
d(x_n, z) \leq sd(x_n, x_{n+k+1}) + sd(x_{n+k+1}, z) < \alpha^n \frac{s^2d(x_0, x_1)}{1 - \alpha s} + sd(x_{n+k+1}, z).
\]

Letting \( k \to \infty \), one obtains (3.11).

Suppose now that there exists two points \( z, z' \in X \) such that \( f(z) = z \) and \( f(z') = z' \). Then the relations

\[
d(z, z') = d(f(z), f(z')) \leq \alpha d(z, z')
\]

show that \( d(z, z') = 0 \), i.e. \( z = z' \).

2. Let \( x_0 \in X \) and \( x_n = f(x_{n-1}) \), \( n \in \mathbb{N} \). Taking into account the relaxed polygonal inequality and (3.15), we obtain

\[
d(x_n, x_{n+k}) \leq \sum_{i=0}^{k-1} d(x_{n+i}, x_{n+i+1}) \leq s(\alpha^n + \alpha^{n+1} + \cdots + \alpha^{n+k})d(x_0, x_1) = s\alpha^n \frac{1 - \alpha^{k+1}}{1 - \alpha}d(x_0, x_1) < \frac{sd(x_0, x_1)}{1 - \alpha}\alpha^n.
\]

Based on these relations the proof goes as in case 1. \( \square \)

**Remark 3.4.** The proof given here to statement 2 from Proposition 3.3 is simpler than that of Theorem 12.4 in [53].

**Remark 3.5.** The proofs given in [26] and [53] to Theorem 3.2 go in the following way.

Let \( x \) be a fixed element of \( X \) and \( \varepsilon > 0 \). By (3.11) (b) there exists \( m = m_\varepsilon \in \mathbb{N} \) such that

\[
\varphi^m(\varepsilon) < \frac{\varepsilon}{2s}.
\]

One considers the sequence \( x_k = f^{km}(x) \), \( k \in \mathbb{N} \), and one shows that there exists \( k_0 \in \mathbb{N} \) such that

\[
\varphi^{m'}(\varepsilon') < \frac{\varepsilon'}{2s},
\]

for all \( k, k' \geq k_0 \). One affirms that the inequality (3.18) shows that \((x_k)\) is a Cauchy sequence, which is not surely true, because the inequality is true only for this specific \( \varepsilon \).

Taking another \( \varepsilon \), say \( 0 < \varepsilon' < \varepsilon \), we find another number \( m' = m_{\varepsilon'} \) (possibly different from \( m \)), such that

\[
\varphi^{m'}(\varepsilon') < \frac{\varepsilon'}{2s}.
\]

The above procedure yields a sequence \( x'_k = f^{km'}(x) \), \( k \in \mathbb{N} \), satisfying, for some \( k_1 \in \mathbb{N} \),

\[
d(x_k, x_{k'}) < 2s\varepsilon',
\]

for all \( k, k' \geq k_1 \).

But the sequences \((x_k)\) and \((x'_k)\) can be totally different, so we cannot infer that the sequence \((x_k)\) is Cauchy.

As we have shown this flaw was fixed in the paper [44].
Remark 3.6. Berinde \cite{21} considers comparison functions satisfying a condition stronger than $0 < \alpha < 1/s$, namely $\sum_{k=1}^{\infty} \phi^k(t) < \infty$, allowing estimations of the order of convergence similar to (3.1). He also shows that the sequence $x_n = f^n(x_0)$, $n \in \mathbb{N}_0$, is convergent to a fixed point of $f$ if and only if it is bounded. For various kinds of comparison functions, the relations between them and applications to fixed points, see \cite{71} \[3.0.3].

3.2. Fixed points in generalized b-metric spaces. Theorem 3.2 admits the following extension to generalized b-metric spaces.

Theorem 3.7. Let $(X,d)$ be a complete generalized b-metric space and suppose that the mapping $f : X \rightarrow X$ is such that

$$(3.21) \quad d(f(x), f(y)) \leq \varphi(d(x, y)), \quad \text{for all } x, y \in X \text{ with } d(x, y) < \infty,$$

where the function $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ satisfies the conditions (a),(b) from (3.1).

Consider, for some $x \in X$, the sequence of successive approximations $(f^n(x))_{n \in \mathbb{N}_0}$. Then either

(A) $d(f^k(x), f^{k+1}(x)) = +\infty$ for all $k \in \mathbb{N}_0$, or

(B) the sequence $(f^n(x))_{n \in \mathbb{N}}$ is convergent to a fixed point of $f$.

Proof. Let $X = \bigcup_{i \in I} X_i$ be the canonical decomposition of $X$ corresponding to the equivalence relation \cite{21}. Assume that (A) does not hold. Then

$$d(f^m(x), f^{m+1}(x)) < +\infty,$$

for some $m \in \mathbb{N}_0$. If $i \in I$ is such that $f^m(x), f^{m+1}(x) \in X_i$, then

$$d(f^{m+1}(x), f^{m+2}(x)) \leq \varphi(d(f^m(x), f^{m+1}(x))) < \infty,$$

implies $f^{m+2}(x) \in X_i$, and so, by mathematical induction, $f^{m+k}(x) \in X_i$ for all $k \in \mathbb{N}_0$.

Since $z \in X_i \iff d(z, f^m(x)) < \infty$, the inequality

$$d(f(z), f^{m+1}(x)) \leq \varphi(d(z, f^m(x)) < \infty,$$

shows that the restriction $f_i = f|_{X_i}$ of $f$ to $X_i$ is a mapping from $X_i$ to $X_i$ satisfying

$$d(f_i(y), f_i(z)) \leq \varphi(d(y, z)),$$

for all $y, z \in X_i$.

By Theorem 2.1, $X_i$ is complete, so that, by Theorem 3.2, $(f^{m+k}(x))_{k \in \mathbb{N}_0}$ is convergent to a fixed point $z_i \in X_i$ of $f_i$, which is a fixed point for $f$. \qed

Remark 3.8. For $s = 1$ and $\varphi(t) = \alpha t$, $t \geq 0$, where $0 \leq \alpha < 1$, we get the Diaz and Margolis \cite{34} fixed point theorem of the alternative. At the same time this extends Theorem 2 from \cite{31} and give simpler proofs to Theorems 2.1 and 3.1 from \cite{16}.

Proposition 3.3 also admits extensions to this setting as results of the alternative. We formulate only one of these results.

Corollary 3.9. Let $(X,d)$ be a complete b-metric space, where $d$ satisfies the $s$-relaxed triangle inequality and let $f : X \rightarrow X$ be a mapping satisfying, for some $0 < \alpha < 1$, the inequality

$$d(f(x), f(y)) \leq \alpha d(x, y),$$

for all $x, y \in X$ with $d(x, y) < \infty$.

Then, for every $x \in X$, either

(A') $d(f^k(x), f^{k+1}(x)) = +\infty$ for all $k \in \mathbb{N}_0$, or

(B') the sequence $(f^n(x))_{n \in \mathbb{N}}$ is convergent to a fixed point of $f$. 

4. Lipschitz functions

In this section we shall discuss the behavior of Lipschitz functions defined on or taking values in quasi-normed spaces and of Lipschitz functions on spaces of homogeneous type.

4.1. Quasi-normed spaces. We start by a brief presentation of an important class of b-metric spaces – quasi-normed spaces. Good references are [47], [49], [54, pp. 156-166], [70].

A quasi-norm on a vector space \( X \) (over \( \mathbb{K} \) equal to \( \mathbb{R} \) or \( \mathbb{C} \)) is a functional \( \| \cdot \| : X \to \mathbb{R}_+ \) for which there exists a real number \( k \geq 1 \) so that:

\[
\begin{align*}
\text{(QN1)} & \quad \| x \| = 0 \iff x = 0; \\
\text{(QN2)} & \quad \| \alpha x \| = |\alpha| \| x \|; \\
\text{(QN3)} & \quad \| x + y \| \leq k(\| x \| + \| y \|),
\end{align*}
\]

for all \( x, y \in X \) and \( \alpha \in \mathbb{K} \). The pair \( (X, \| \cdot \|) \) is called a quasi-normed space. A complete quasi-normed space is called a quasi-Banach space.

If \( k = 1 \), then \( \| \cdot \| \) is a norm. The smallest constant \( k \) for which the inequality (QN3) is satisfied for all \( x, y \in X \) is called the modulus of concavity of the quasi-normed space \( X \).

For a linear operator \( T \) from a quasi-normed space \( (X, \| \cdot \|_X) \) to a normed space \( (Y, \| \cdot \|_Y) \) put

\[
\| T \| = \sup \{ \| Tx \|_Y : x \in X, \| x \|_X \leq 1 \}.
\]

In particular,

\[
\| x^* \| = \sup \{ |x^*(x)| : x \in X, \| x \|_X \leq 1 \}, \quad x^* \in X^*,
\]

is a norm on the dual space \( X^* = (X, \| \cdot \|_X)^* \).

It follows that \( T \) is continuous if and only if \( \| T \| < \infty \) and, in this case,

\[
\| Tx \|_Y \leq \| T \| \| x \|_X, \quad x \in X,
\]

\( \| T \| \) being the smallest number \( L \geq 0 \) for which the inequality \( \| Tx \|_Y \leq L \| x \|_X \) holds for all \( x \in X \). If \( Y \) is also a quasi-normed space, then \( \| \cdot \| \) is only a quasi-norm on the space \( \mathcal{L}(X, Y) \) of all continuous linear operators from \( X \) to \( Y \).

Two quasi-norms \( \| \cdot \|_1, \| \cdot \|_2 \) on a vector space \( X \) are called equivalent if they generate the same topology, or equivalently, if

\[
\| x_n - x \|_1 \to 0 \iff \| x_n - x \|_2 \to 0,
\]

for all sequences \( (x_n) \) in \( X \) and \( x \in X \). As in the case of norms, the equivalence of two quasi-norms \( \| \cdot \|_1, \| \cdot \|_2 \) on a vector space \( X \) is equivalent to the existence of two numbers \( \alpha, \beta > 0 \) such that

\[
\alpha \| x \|_1 \leq \| x \|_2 \leq \beta \| x \|_1,
\]

for all \( x \in X \).

A subset \( A \) of a topological vector space (TVS) \( (X, \tau) \) is called bounded if it is absorbed by any 0-neighborhood, i.e for every \( V \in \mathcal{V}_r(0) \) there exists \( t > 0 \) such that \( A \subseteq tV \). A TVS is called locally bounded if it has a bounded 0-neighborhood. A quasi-normed space \( (X, \| \cdot \|) \) is locally bounded, as the closed unit ball \( B_X = \{ x \in X : \| x \| \leq 1 \} \) is a bounded neighborhood of 0. One shows that, conversely, the topology of every locally bounded TVS is generated by a quasi-norm.

A quasi-normed space \( (X, \| \cdot \|) \) is normable (i.e. there exists a norm \( \| \cdot \|_1 \) on \( X \) equivalent to the quasi-norm \( \| \cdot \| \)) if and only if 0 has a bounded convex neighborhood (implying that \( X \) is locally convex).

**Definition 4.1.** An \( F \)-norm on a vector space \( X \) is a mapping \( \| \cdot \| : X \to \mathbb{R}_+ \) satisfying the conditions

\[
\begin{align*}
\text{(F1)} & \quad \| x \| = 0 \iff x = 0; \\
\text{(F2)} & \quad \| \lambda x \| \leq |\lambda| \| x \| \quad \text{for all } \lambda \in \mathbb{K} \text{ with } |\lambda| \leq 1; \\
\text{(F3)} & \quad \| x + y \| \leq \| x \| + \| y \|; \\
\text{(F4)} & \quad \| x_n \| \to 0 \iff \| \lambda x_n \| \to 0; \\
\text{(F5)} & \quad \lambda_n \to 0 \Rightarrow \| \lambda_n x \| \to 0,
\end{align*}
\]
for all \( x, y, x_n \in X \) \( \lambda, \lambda_n \in \mathbb{K} \). An \( F \)-space is a vector space equipped with a complete \( F \)-norm.

It follows that \( d(x, y) = \| y - x \|, x, y \in X \), is a translation-invariant metric on \( X \) defining a vector topology. It is known that the metrizability of a TVS \((X, \tau)\) is equivalent to the existence of a countable basis of 0-neighborhoods, and in this case there exists a translation-invariant metric \( d \) on \( X \) generating the topology \( \tau \).

One shows, see [54, p. 163], that the topology of a metrizable TVS can be always given by an \( F \)-norm. If \((X, \tau)\) is a TVS, then the topology \( \tau \) generates a uniformity \( \mathcal{W}_\tau \) on \( X \), a basis of it being given by the sets

\[
W_U = \{(x, y) \in X^2 : y - x \in U\},
\]

where \( U \) runs over a 0-neighborhood basis in \( X \). Any translation-invariant metric generating the topology \( \tau \) generates the same uniformity \( \mathcal{W}_\tau \), so that if \( X \) is complete with respect to \( \mathcal{W}_\tau \), then it is complete with respect to any translation-invariant metric generating the topology \( \tau \).

Typical examples of quasi-normed spaces are the spaces \( L^p[0, 1] \) and \( \ell^p \) with \( 0 < p < 1 \) equipped with the quasi-norms

\[
\| f \|_p = \left( \int_0^1 |f(t)|^p \, dt \right)^{1/p} \quad \text{and} \quad \| x \|_p = \left( \sum_{k=1}^{\infty} |x_k|^p \right)^{1/p},
\]

for \( f \in L^p[0, 1] \) and \( x = (x_k)_{k \in \mathbb{N}} \in \ell^p \), respectively.

The quasi-norms \( \| \cdot \|_p \) satisfy the inequalities

\[
\begin{align*}
\| f + g \|_p &\le 2^{(1-p)/p}(\| f \|_p + \| g \|_p) \quad \text{and} \\
\| x + y \|_p &\le 2^{(1-p)/p}(\| x \|_p + \| y \|_p),
\end{align*}
\]

for all \( f, g \in L^p[0, 1] \) and \( x, y \in \ell^p \).

The constant \( 2^{(1-p)/p} \) is sharp, i.e. the moduli of concavity of the spaces \( L^p[0, 1] \) and \( \ell^p \) are both equal to \( 2^{(1-p)/p} \).

To show this, we start with the elementary inequalities

\[
(a + b)^p \le a^p + b^p \le 2^{1-p}(a + b)^p,
\]

valid for all \( a, b > 0 \).

Let \( f, g \in L^p[0, 1] \). The first inequality from above implies

\[
|f(t) + g(t)|^p \le (|f(t)| + |g(t)|)^p \le |f(t)|^p + |g(t)|^p,
\]

for almost all \( t \in [0, 1] \), so that

\[
\| f + g \|_p \le \int_0^1 |f(t) + g(t)|^p \, dt \le \int_0^1 |f(t)|^p \, dt + \int_0^1 |g(t)|^p \, dt = \| f \|_p + \| g \|_p.
\]

This inequality and the second inequality from (4.4) yield

\[
\| f + g \|_p = \left( \| f + g \|_p^p \right)^{1/p} \le \left( \| f \|_p^p + \| g \|_p^p \right)^{1/p} \le 2^{(1-p)/p}(\| f \|_p + \| g \|_p).
\]

Similar calculations can be done to show that

\[
\| x + y \|_p \le 2^{(1-p)/p}(\| x \|_p + \| y \|_p),
\]

for all \( x, y \in \ell^p \).

To show that the constant \( 2^{(1-p)/p} \) is sharp take \( x = (1, 0, 0, \ldots) \) and \( y = (0, 1, 0, \ldots) \) in the case of the space \( \ell^p \). Then

\[
\| x + y \|_p = 2^{1/p} \quad \text{and} \quad 2^{(1-p)/p}(\| x \|_p + \| y \|_p) = 2^{(1-p)/p} \cdot 2 = 2^{1/p},
\]
that is, we have equality in the second inequality from (4.3). In the case of the space $L^p[0, 1]$ take $f = \chi_{[0, \frac{1}{2}]}$ and $g = \chi_{[\frac{1}{2}, 1]}$ to obtain equality in the first inequality from (4.3).

**Remark 4.2.** Apparently similar, the quasi-normed spaces $\ell^p$ and $L^p[0, 1]$ drastically differ. For instance, the space $L^p[0, 1]$ has trivial dual, $(L^p[0, 1])^* = \{0\}$, while $(\ell^p)^* = \ell^{\infty}$, see [54] pp. 156-158. D. Pallaschke [63] and Ph. Turpin [75] have shown that every compact endomorphism of $L^p$, $0 < p < 1$, is null. N. Kalton and J. H. Shapiro [48] showed that there exists a quasi-Banach space with trivial dual admitting non-trivial compact endomorphisms. The example is a quotient space of $H^p$, $0 < p < 1$. Here, $H^p$, $0 < p < 1$, denotes the classical Hardy quasi-Banach spaces of analytic functions in the unit disk of $\mathbb{C}$.

A p-norm, where $0 < p \leq 1$, is a mapping $\| \cdot \| : X \to \mathbb{R}_+$ satisfying (QN1) and (QN2) and (QN3) $\| x + y \| \leq \| x \| + \| y \|$, for all $x, y \in X$.

The quasi-norms of the spaces $L^p[0, 1]$ and $\ell^p$, $0 < p < 1$, a p-norms, i.e.

$$
\| f + g \|_p \leq \| f \|_p + \| g \|_p \\
\| x + y \|_p \leq \| x \|_p + \| y \|_p,
$$

for all $f, g \in L^p[0, 1]$ and $x, y \in \ell^p$.

A famous result of T. Aoki [13] and S. Rolewicz [69] says that on any quasi-normed space $(X, \| \cdot \|)$ there exists a $p$-norm equivalent to $\| \cdot \|$, where $p$ is determined from the equality $2^{1/p} = k$, $k$ being the constant from (QN3).

Let $0 < p \leq 1$. A subset $A$ of a vector space $X$ is called $p$-convex if $\alpha x + \beta y \in A$ for all $x, y \in A$ and all $\alpha, \beta \geq 0$ with $\alpha^p + \beta^p = 1$, and $p$-absolutely convex if $\alpha x + \beta y \in A$ for all $x, y \in A$ and all $\alpha, \beta \in \mathbb{K}$ with $|\alpha|^p + |\beta|^p \leq 1$. For $p = 1$ one obtains the usual convex and absolutely convex sets, respectively.

A TVS $X$ is $p$-normable if and only if it has a bounded $p$-convex 0-neighborhood, see [54] p. 161. One shows first that under this hypothesis there exists a bounded $p$-absolutely convex neighborhood $N$ of 0 and one defines the $p$-norm as the Minkowski functional corresponding to $N$, i.e. $\| x \| = \inf \{ t : t > 0, x \in tN \}$.

**Remark 4.3.** In Köthe [54] by a $p$-norm on a vector space $X$ one understands a mapping $\| \cdot \|' : X \to \mathbb{R}_+$ such that

$$
\| x \|' = 0 \iff x = 0, \quad \| \alpha x \|' = |\alpha| \| x \|' \quad \text{and} \quad \| x + y \|' \leq \| x \|' + \| y \|',
$$

for all $x, y \in X$ and $\alpha \in \mathbb{K}$. In this case the “$p$-norm” corresponding to a bounded absolutely $p$-convex neighborhood is given by $\| x \|' = \inf \{ t^p : t > 0, x \in tN \}$.

It follows that $\| \cdot \|$ is a $p$-norm in the sense given here if and only if $\| \cdot \|'$ is a $p$-norm in the sense given in [54].

**The Banach envelope.**

Let $(X, \| \cdot \|)$ be a quasi-Banach space and $B_X = \{ x \in X : \| x \| \leq 1 \}$ its closed unit ball. Denote by $\| \cdot \|_C$ the Minkowski functional of the set $C = \co(B_X)$. It is obvious that $\| \cdot \|_C$ is a seminorm on $X$ and a norm on the quotient space $X/N$, where $N = \{ x \in X : \| x \|_C = 0 \}$.

Since, for $x \neq 0$, $x' := x/\| x \| \in B_X \subseteq C$, it follows $\| x' \|_C \leq 1$, that is $\| x \|_C \leq \| x \|$. Denote by $\tilde{X}$ the completion of $X/N$ with respect to the quotient-norm $\| \cdot \|_C$ corresponding to $\| \cdot \|_C$, whose (unique) extension to $\tilde{X}$ is denoted also by $\| \cdot \|_C$. It follows $\| x \|_C \leq \| x \|$ for all $x \in X$, hence the embedding $j : X \to \tilde{X}$ is continuous and one shows that $j(X)$ is dense in $\tilde{X}$. The space $\tilde{X}$ is called the Banach envelope of the quasi-Banach space $X$.

We distinguish two situations.

1. $X$ has trivial dual: $X^* = \{0\}$.
   
   In this case $C = \co(B_X) = X$ (see [49] Proposition 2.1, p. 16]) and so $\| \cdot \|_C \equiv 0$, $N = X$ and $X/X = \{0\}$. It follows $\tilde{X} = \{0\}$ and $\tilde{X}^* = \{0\} = X^*$. In particular $\tilde{L}^p = \{0\}$, where $L^p = L^p[0, 1]$.

2. $X$ has a separating dual.
This means that for every \( x \neq 0 \) there exists \( x^* \in X^* \) with \( x^*(x) \neq 0 \) (e.g. \( X = \ell^p \) with \( 0 < p < 1 \)). In this case \( \| \cdot \|_C \) is a norm on \( X \) which can be calculated by the formula

\[
\|x\|_C = \sup\{|x^*(x)| : x^* \in X^*, \|x^*\| \leq 1\}.
\]

where the norm of \( x^* \in X^* \) is given by (4.1).

Consequently \( N = \{0\} \), \( X/N = X \) and we can consider \( X \) as a dense subspace of \( \hat{X} \) (in fact, continuously and densely embedded in \( \hat{X} \)). It follows that:

(i) every continuous linear functional on \((X, \| \cdot \|)\) has a unique norm preserving extension to \((\hat{X}, \| \cdot \|_{\hat{X}})\);

(ii) every continuous linear operator \( T \) from \((X, \| \cdot \|)\) to a Banach space \( Y \) has a unique norm preserving extension \( \hat{T} : (\hat{X}, \| \cdot \|_{\hat{X}}) \to Y \).

Consequently \((X, \| \cdot \|)^*\) can be identified with \((\hat{X}, \| \cdot \|_{\hat{X}})^*\) and the norm \( \| \cdot \|_{\hat{X}} \) can also be calculated by the formula (4.6) for all \( x \in \hat{X} \).

One shows that the Banach envelope of \( \ell^0 \) is \( \ell^1 \), for every \( 0 < p < 1 \).

Another way to define the Banach envelope in the case of a quasi-Banach space with separating dual is via the embedding \( j_X \) of \( X \) into its bidual \( X^{**} \) (see [39]). Since \( X^* \) separates the points of \( X \), it follows that \( j_X \) is injective of norm \( \| j_X \| \leq 1 \) (in this case one can not prove that \( \| j_X \| = 1 \) because the Hahn-Banach extension theorem may fail for non-locally convex spaces).

By (4.6)

\[
\| x \|_C = \sup\{|x^*(x)| : \|x^*\| \leq 1\} = \| j_X(x) \|_{X^{***}},
\]

so we can identify \( \hat{X} \) with the closure of \( j_X(X) \) in \((X^{**}, \| \cdot \|_{X^{**}})\).

4.2. Lipschitz functions and quasi-normed spaces. It turns out that some results concerning Banach space-valued Lipschitz functions fail in the quasi-Banach case and, in some cases, the validity of some of them forces the quasi-Banach space to be locally convex, i.e. a Banach space.

In this subsection we consider only spaces over \( \mathbb{R} \).

Let \((Z, d)\) be a b-metric space and \((Y, \| \cdot \|)\) a quasi-normed space. A function \( f : Z \to Y \) is called Lipschitz if there exists \( L \geq 0 \) (called a Lipschitz constant for \( f \)) such that

\[
\| f(z) - f(z') \| \leq Ld(z, z'),
\]

for all \( z, z' \in Z \). One denotes by \( \text{Lip}(Z, Y) \) the space of all Lipschitz functions from \( Z \) to \( Y \).

The \textit{Lipschitz norm} \( \| f \|_L \) of \( f \) is defined by

\[
\| f \|_L = \sup \left\{ \frac{\| f(z) - f(z') \|}{d(z, z')} : z, z' \in Z, z \neq z' \right\}.
\]

It follows that \( \| f \|_L \) is the smallest Lipschitz constant for \( f \).

Since \( \| f \| = 0 \) if and only if \( f = \text{const}, \| \cdot \| \) is actually only a seminorm on \( \text{Lip}(Z, Y) \). To obtain a norm, one considers a fixed element \( z_0 \in Z \) and the space

\[
\text{Lip}_0(Z, Y) = \{ f \in \text{Lip}(Z, Y) : f(z_0) = 0 \}.
\]

If \( Z = X \), where \( X \) is a quasi-normed space, then one take 0 for the fixed point \( z_0 \), and, in this case, \( \text{Lip}_0(X, Y) \) is a quasi-normed space, that is,

\[
\| f + g \|_L \leq k(\| f \|_L + \| g \|_L),
\]

for \( f, g \in \text{Lip}_0(X, Y) \), where \( k \geq 1 \) is the constant from (QN3). It is complete, provided \( Y \) is a quasi-Banach space.

If \( Y = \mathbb{R} \), then one uses the notation \( \text{Lip}(X) \), \( \text{Lip}_0(X) \) and \( \text{Lip}_0(X) \) is called the Lipschitz dual of the quasi-normed space \( X \).

We noted that the space \( L^p = L^p[0, 1] \) has trivial dual. F. Albiac [3] proved that it has also a trivial Lipschitz dual, i.e. \( \text{Lip}_0(L^p) = \{0\} \). Later he showed that this is a more general phenomenon.
Proposition 4.4 (F. Albiac [4]). Let \((X, \| \cdot \|)\) be a quasi-Banach space and
\[
\|\|x\|\| := \sup\{f(x) : f \in \operatorname{Lip}_0(X), \|f\|_L \leq 1\}, x \in X.
\]

Then

(i) \(\|\| \cdot \|\|\) is a seminorm on \(X\);
(ii) if \(\operatorname{Lip}_0(X)\) is nontrivial, then \(X\) has a nontrivial dual, i.e. \(X^* \neq \{0\}\);
(iii) if \(X\) has a separating Lipschitz dual, then \(X\) has a separating (linear) dual and \(\|\| \cdot \|\|\) is a norm on \(X\).

One says that a family \(\mathcal{F}\) of real valued functions on a quasi-normed space \(X\) is separating if for every \(x \neq y\) in \(X\) there exists \(f \in \mathcal{F}\) with \(f(x) \neq f(y)\).

It is known that every Lipschitz function \(f\) from a subset of a metric space \((X, d)\) to \(\mathbb{R}\) admits an extension to \(X\) which is Lipschitz with the same Lipschitz constant (McShane’s extension theorem). The following result shows that the validity of this result for every subset of a quasi-Banach space \(X\) forces this space to be Banach.

Proposition 4.5 ([4]). Let \((X, \| \cdot \|)\) be a quasi-Banach space. If for every subset \(Z\) of \(X\), every \(L\)-Lipschitz function \(f : Z \to \mathbb{R}\) admits an \(L'\)-Lipschitz extension, for some \(L' \geq L\), then the space \(X\) is locally convex, i.e. it is a Banach space.

It is known that every continuous linear operator from a quasi-Banach space \(X\) to a Banach space \(Y\) admits a norm preserving linear extension to the Banach envelope \(\hat{X}\) of \(X\) to \(Y\). F. Albiac [4] has shown that this is true for Lipschitz mappings too: every Lipschitz mapping \(f : X \to Y\) admits a unique Lipschitz extension with the same Lipschitz constant \(\hat{f} : \hat{X} \to Y\).

Moreover, if \(X, Y\) are normed spaces and \(f : X \to Y\) is Gâteaux differentiable on the interval \([x, y] := \{x + t(y - x) : t \in [0, 1]\}\), then
\[
(4.7) \quad \|f(x) - f(y)\| \leq \|x - y\| \sup\{\|f'(\xi)\| : \xi \in [x, y]\}.
\]

Proposition 4.6 ([4]). Let \((X, \| \cdot \|)\) be a quasi-Banach space. If every nonconstant Gâteaux differentiable Lipschitz function \(f : [0, 1] \to X\) satisfies the mean value inequality \((4.7)\) for all \(x, y \in [0, 1]\), then the space \(X\) is locally convex, i.e. it is a Banach space.

Let \(\alpha > 0\). A function \(f : (X_1, d_1) \to (X_2, d_2)\) between two b-metric spaces \(X_1, X_2\) is called Hölder of order \(\alpha\) if there exists \(L \geq 0\) such that
\[
(4.8) \quad d_2(f(x), f(y)) \leq Ld_1(x, y)^\alpha,
\]
for all \(x, y \in X_1\). As a consequence of the mean value theorem, every function \(f\) from \([0, 1]\) to a Banach space \(X\) which is Hölder of order \(\alpha > 1\) is constant, a fact that is no longer true if \(X\) is a quasi-Banach space.

Example 4.7. Let \(L^p = L^p[0, 1]\) for \(0 < p < 1\). The function \(f : [0, 1] \to L^p\) given by \(f(t) = \chi_{[0,t]}\) satisfies the equality
\[
\|f(s) - f(t)\|_p = |s - t|^{1/p},
\]
for all \(s, t \in [0, 1]\), where \(\| \cdot \|_p\) is the \(L^p\)-norm (see (4.2)).

Indeed, for \(0 \leq t < s \leq 1\),
\[
\|f(s) - f(t)\|_p = \left(\int_t^s \chi_{[t,s]}^p(u)du\right)^{1/p} = |s - t|^{1/p}.
\]

The Riemann integral of a function \(f : [a, b] \to X\), where \([a, b]\) is an interval in \(\mathbb{R}\) and \(X\) is a Banach space, can be defined as in the real case, by simply replacing the absolute value \(|\cdot|\) with the norm sign \(\|\cdot\|\), and has properties similar to those from the real case. For instance, the following result is true.

Proposition 4.8 ([4]). Let \(X\) be a Banach space. If \(f : [a, b] \to X\) is continuous, then
(i) \( f \) is Riemann integrable, and
(ii) the function
\[
F(t) = \int_a^t f(s)ds, \quad t \in [a, b],
\]
is differentiable with \( F'(t) = f(t) \) for all \( t \in [a, b] \).

**Remark 4.9.** However, there is a point where this analogy is broken, namely the Lebesgue criterion of Riemann integrability: a function \( f : [a, b] \to \mathbb{R} \) is Riemann integrable if and only if it is continuous almost everywhere on \([a, b]\) (i.e. excepting a set of Lebesgue measure zero). In the infinite dimensional case this criterion does not hold in general, leading to the study of those Banach spaces for which it, or some weaker forms, are true, see, for instance, [40], [73], [74] and the references quoted therein.

In the case of quasi-Banach spaces the situation is different. By a result attributed to S. Mazur and W. Orlicz [60] (see also [70, p. 122]) an \( F \)-space \( X \) is locally convex if and only if every continuous function \( f : [0, 1] \to X \) is Riemann integrable.

M. M. Popov [67] investigated the Riemann integrability of functions defined on intervals in \( \mathbb{R} \) with values in an \( F \)-space. Among other results, he proved that a Riemann integrable function \( f : [a, b] \to X \) is bounded and that the function \( F \) defined by (4.9) is uniformly continuous, but there exists a continuous function \( f : [0, 1] \to L^p \), where \( 0 < p < 1 \), such that the function \( F \) does not have a right derivative at \( t = 0 \). He asked whether any continuous function \( f \) from \([0, 1]\) to \( L^p[0, 1] \) with \( 0 < p < 1 \) (or more general, to a quasi-Banach space \( X \) with \( X^* = \{0\} \)) admits a primitive. This problem was solved by N. Kalton [40] who proved that if \( X \) is a quasi-Banach space with \( X^* = \{0\} \), then every continuous function \( f : [0, 1] \to X \) has a primitive. Kalton considered the space \( C^1_{Kal}(I, X) \), where \( I = [0, 1] \) and \( X \) is a quasi-Banach space, of all continuously differentiable functions \( f : I \to X \) such that the function \( \tilde{f} : I^2 \to X \) given for \( s, t \in I \) by \( \tilde{f}(t, s) = f'(t) \) and \( \tilde{f}(s, t) = (f(s) - f(t))/(s - t) \) if \( s \neq t \), is continuous. It follows that \( C^1_{Kal}(I, X) \) is a quasi-Banach space with respect to the quasi-norm
\[
\|f\| = \|f(0)\| + \|f\|_{L^p}.
\]

The notation \( C^1_{Kal}(I, X) \) was introduced in [5]; Kalton used the notation \( C^1(I, X) \).

Denote by \( C(I, X) \) the Banach space (with respect to the sup-norm) of all continuous functions from \( I \) to \( X \). The core of a quasi-Banach space \( X \) is the maximal subspace \( Z \) of \( X \) (denoted by \( \text{core}(X) \)) with \( Z^* = \{0\} \). One shows that such a subspace always exists, is unique and closed. Notice that \( \text{core}(X) = \{0\} \) implies only that \( X \) has a nontrivial dual, but not necessarily a separating one.

In [6] it is shown that if \( X \) is a quasi-Banach space with \( \text{core}(X) = \{0\} \), then there exists a continuous function \( f : [0, 1] \to X \) failing to have a primitive. Kalton, op. cit., called a quasi-Banach \( X \) a \( D \)-space if the mapping
\[
D : C^1_{Kal}(I, X) \to C(I, X)
\]
given by \( Df = f' \), is surjective and proved the following result.

**Theorem 4.10 ([60]).** Let \( X \) be a quasi-Banach with \( \text{core}(X) = \{0\} \). Then \( X \) is a \( D \)-space if and only if \( X \) is locally convex (or, equivalently, a Banach space).

It is known that every continuously differentiable function from an interval \([a, b] \subseteq \mathbb{R} \) to a Banach space \( X \) is Lipschitz with \( \|f\|_L = \sup \{\|f'(t)\| : t \in [a, b]\} \) (a consequence of the Mean Value Theorem, see (4.7)). As it was shown in [5] this in no longer true in quasi-Banach spaces.

**Theorem 4.11.** Let \( X \) be a non-locally convex quasi-Banach space \( X \). Then there exists a function \( F : I \to X \) such that:

(i) \( F \) is continuously differentiable on \( I \);
(ii) \( F' \) is Riemann integrable on \( I \) and \( F(t) = \int_a^t F'(s)ds, \quad t \in I; \)
(iii) $F$ is not Lipschitz on $I$.

In [7] it is proved that the usual rule of the calculation of the integral (called Barrow’s rule by the authors, known also as Leibniz rule) holds in the quasi-Banach case in the following form.

**Proposition 4.12.** Let $X$ be a quasi-Banach with separating dual. If $F : [a, b] \to X$ is differentiable with Riemann integrable derivative, then

$$\int_a^b F'(t) \, dt = F(b) - F(a).$$

Another pathological result concerning differentiability of quasi-Banach valued Lipschitz functions was obtained by N. Kalton [45, Theorem 3.3].

**Theorem 4.13.** Let $X$ be an $F$-space with trivial dual. Then for every pair of distinct points $x_0, x_1 \in X$ there exists a function $f : [0, 1] \to X$ such that $f(0) = x_0$, $f(1) = x_1$ and

$$\lim_{|s-t| \to 0} \frac{f(s) - f(t)}{s - t} = 0 \quad \text{uniformly for } s, t \in [0, 1].$$

In particular $f'(t) = 0$ for all $t \in [0, 1]$.

**Remark 4.14.** N. Kalton [45, Corollary 3.4] also remarked that if $X$ is an $F$-space and $x \in X \setminus \{0\}$, then in order to exist a function $f : [0, 1] \to X$ such that $f(0) = 0$, $f(1) = x$ and $f'(t) = 0$ for all $t \in [0, 1]$, it is necessary and sufficient that $x \in \text{core}(X)$.

If $X$ is a Banach space and $f : [0, 1] \to X$ is continuous then it is Riemann integrable and the average function $\text{Ave}[f] : [a, b] \times [a, b] \to X$, given by

$$\text{Ave}[f](s, t) = \begin{cases} \frac{1}{s-t} \int_s^t f(u) \, du & \text{if } a \leq s < t \leq b, \\ f(c) & \text{if } s = t = c \in [a, b], \\ \frac{1}{s-t} \int_s^t f(u) \, du & \text{if } a \leq t < s \leq b, \end{cases}$$

is jointly continuous on $[a, b] \times [a, b]$, and so, separately continuous and bounded. Some pathological properties of the average function in the quasi-Banach case are examined in [5], [8] and [67].

The analog of the Radon-Nikodým Property for quasi-Banach spaces and its connections with the differentiability of Lipschitz mappings and martingales are discussed in [9].

### 4.3. Lipschitz functions on spaces of homogeneous type

Let $(X, d, \mu)$ be a space of homogeneous type (see Subsection 1.4). By $B$ we shall denote balls of the form $B(x, r)$. If $\varphi$ is a function integrable on bounded sets, then the mean value of $\varphi$ on the ball $B$ is defined by

$$m_B(\varphi) = \mu(B)^{-1} \int_B \varphi(x) \, d\mu(x).$$

For $1 \leq q < \infty$ and $0 < \beta < \infty$ one denotes by $\text{Lip}(q, \beta)$ the set of all functions $\varphi$, integrable on bounded sets, for which there exists a constant $C \geq 0$ such that

$$\left( \frac{1}{\mu(B)} \int_B |\varphi(x) - m_B(\varphi)|^q \, d\mu(x) \right)^{1/q} \leq C \mu(B)^{\beta},$$

for all balls $B$. The least constant $C$ for which (4.11) holds will be denoted by $\|\varphi\|_{\beta, q}$.

We shall denote by $\text{Lip}(\beta)$ the set of all functions $\varphi$ on $X$ such that there exists a constant $C \geq 0$ satisfying

$$|\varphi(x) - \varphi(y)| \leq C d(x, y)^{\beta},$$

for all $x, y \in X$, i.e. it is Hölder of order $\beta$ (see (1.3)). The least $C \geq 0$ for which (4.12) holds is denoted by $\|\varphi\|_\beta$.

The following results concerning these classes of Lipschitz functions were proved in [57].
Theorem 4.15. Let $(X, d, \mu)$ be a space of homogeneous type. Then there exists a constant $C \geq 0$ (depending on $\beta$ and $q$ only) such that for every $\varphi \in \text{Lip}(\beta, q)$ there exists a function $\psi$ satisfying

(i) $\varphi(x) = \psi(x)$ a.e. on $X$, and

(ii) $|\psi(x) - \psi(y)| \leq C \|\varphi\|_{\beta, q} \mu(B)^{\beta}$,

for any ball $B$ containing the $x, y$.

Theorem 4.16. Let $(X, d, \mu)$ be a space of homogeneous type. Then, given $0 < \beta < \infty$, there exists a $b$-metric $\delta$ on $X$ such that $(X, \delta, \mu)$ is a normal space of homogeneous type and for every $1 \leq q < \infty$ we have

$\varphi \in \text{Lip}(\beta, q)$ of $(X, d, \mu) \iff \exists \psi \in \text{Lip}(\beta)$ of $(X, \delta, \mu)$ with $\varphi \approx \psi$.

Moreover, the norms $\|\varphi\|_{\beta, q}$ and $\|\psi\|_{\beta}$ are equivalent.

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