On discrete time hedging in $d$-dimensional option pricing models

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ABSTRACT

We study the approximation of certain stochastic integrals with respect to a $d$-dimensional diffusion by corresponding stochastic integrals with piece-wise constant integrands i.e. an approximation of the form $\sum_{k=1}^{d} \int_{0}^{T} N_{k} dX_{k} \approx \sum_{i=1}^{n} \int_{t_{i-1}}^{t_{i}} (X_{k}^{t_{i}} - X_{k}^{t_{i-1}})$. In finance this corresponds to replacing a continuously adjusted portfolio by discretely adjusted one. The approximation error is measured with respect to $L^{2}$ and it is shown that under certain assumptions the approximation rate is $n^{-1/2}$ when one optimizes over deterministic but not necessarily equidistant time-nets $0 = t_{0} \leq t_{1} \leq \cdots \leq t_{n} = T$.

Key Words: approximation, discrete time hedging, rate of convergence, stochastic integral.

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1. INTRODUCTION

Assume a Borel-function $f : \mathbb{R}^{d} \to \mathbb{R}$, $T > 0$ and a stochastic process $(X_{t})_{t \in [0,T]}$ defined as a solution of

$$X_{t}^{i} = x_{0}^{i} + \int_{0}^{t} b_{i}(X_{u}) du + \sum_{j=1}^{d} \int_{0}^{t} \sigma_{ij}(X_{u}) dW_{u}^{j}, \quad i \in \{1, \ldots, d\},$$

where $(W_{t})_{t \in [0,T]}$ is a $d$-dimensional Brownian motion and the functions $b$ and $\sigma$ satisfy certain assumptions (cf. Chapter 2).

Consider the problem that a trader has to hedge, by a self-financing strategy, a European type option with maturity $T > 0$, where the pay-off of the option is described by a random variable $f(X_{T})$. The perfect hedging strategy is determined by the process $(N_{u})_{u \in [0,T]}$ in a stochastic integral representation of $f(X_{T})$,

$$f(X_{T}) = V_{0} + \sum_{k=1}^{d} \int_{0}^{T} N_{u}^{k} dX_{u}^{k},$$

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where $V_0$ is the initial capital. In practice the continuous strategy has to be replaced by a discretely adjusted one. This leads to an approximation
\[
\sum_{k=1}^d \int_0^T N_k^u dX_k^u \approx \sum_{k=1}^d \sum_{i=1}^n N_k^u (X_{t_i}^k - X_{t_{i-1}}^k),
\]
where $0 = t_0 \leq t_1 \leq t_2 \leq \cdots \leq t_n = T$ is a deterministic but not necessarily equidistant time-net.

We will measure and (to some extent) optimize the error of this approximation in $L^2$. Our interest lies in the rate of convergence of the approximation, when the approximation error is minimized over all time-nets with at most $n + 1$ time-knots. This means that we are interested in the quantity
\[
(1.2) \quad \inf_{\tau \in \mathcal{T}_n} \left\| \sum_{k=1}^d \int_0^T N_k^u dX_k^u - \sum_{k=1}^d \sum_{i=1}^m N_k^u (X_{t_i}^k - X_{t_{i-1}}^k) \right\|_{L^2}
\]
as $n$ tends to infinity, where
\[
\mathcal{T}_n := \{(t_i)_{i=0}^m : 0 = t_0 < t_1 < \cdots < t_m = T, m \leq n\}.
\]

Let us recall some results from the literature. Among others, the 1-dimensional case has been considered by Zhang [13], Gobet-Temam [8] and Geiss [5]. Geiss considered the approximation problem for general deterministic nets, which are not necessarily equidistant, and a closed form formula for the $L^2$-error was obtained. Based on this, in [7] several classes of examples were given, where the optimal rate of convergence $n^{-1/2}$ is attained by general deterministic nets (but, in general, not by equidistant ones). The result from [5] and [7] cannot be straightforwardly extended to the multi-dimensional case because part of the arguments from the 1-dimensional case do not seem to apply in the multi-dimensional situation.

The multi-dimensional case was, for example, studied by Zhang [13] and Temam [12] for equidistant nets. For $C^1$-functions with derivatives of polynomial growth, cf. Proposition 3.1.6 and Corollary 3.3.3], Zhang established the rate $n^{-1/2}$. On the other side, Temam [12] proved the rate $n^{-1/4}$ for the European digital option.

The aim of this paper is to improve the approximation rate of the European digital option in the multi-dimensional case from $n^{-1/4}$ to $n^{-1/2}$ by replacing the equidistant nets by general deterministic nets.

The paper is organized as follows: In section 2 we explain the setting we are working with. Section 3 introduces Theorem 3.1 which is our main result. In Theorem 3.1 we show for a certain class of functions $f$, including European digital option, that one gets the $L^2$-approximation rate of $n^{-1/2}$ by optimizing over all deterministic nets of cardinality $n + 1$. Our Theorem also allows a drift term in the underlying diffusion.
process (which is sometimes remarked, but not carried out, in the literature). Section 4 gives some examples illustrating Theorem 3.1.

2. Preliminaries

In this chapter we introduce the setting we are working with and recall some known facts that are needed in order to prove our results.

We shall use the standard assumptions from stochastic calculus, i.e. we assume a complete probability space \((\Omega, F, \mathbb{P})\) and, for \(T > 0\), a right-continuous filtration \((F_t)_{t \in [0,T]}\) generated by a standard \(d\)-dimensional Brownian motion \(W = (W_t)_{t \in [0,T]}\) such that \(F_T = F\) and \(F_0\) contains all null-sets of \(F\) (cf. [9]). By \(||x||\) we denote the Euclidean norm of a vector \(x \in \mathbb{R}^d\). A Borel-function \(\varphi : B \rightarrow \mathbb{R}\) on some set \(B \subset \mathbb{R}\) will be extended to \(B^d \subset \mathbb{R}^d\) by the notation \(\varphi(x) := (\varphi(x_1), \varphi(x_2), \ldots, \varphi(x_d)), x \in B^d\).

We consider a diffusion

\[
X^i_t = x^i_0 + \int_0^t b_i(X_u)du + \sum_{j=1}^d \int_0^t \sigma_{ij}(X_u)dW^j_u, \quad i = 1, \ldots, d, \text{ a.s.}
\]

where \(x_0 \in \mathbb{R}^d\). The process \(X\) is obtained through \(Y\) given as the unique path-wise continuous solution of (cf. [10, Corollary 2.2.1 on p. 101])

\[
Y^i_t = y^i_0 + \int_0^t \hat{b}_i(Y_u)du + \sum_{j=1}^d \int_0^t \hat{\sigma}_{ij}(Y_u)dW^j_u, \quad i = 1, \ldots, d, \text{ a.s.}
\]

where

\[
\hat{b}_i(x), \hat{\sigma}_{ij}(x) \in C^\infty_b(\mathbb{R}^d)
\]

and \(\hat{\sigma} \hat{\sigma}^T\), where \((\hat{\sigma} \hat{\sigma}^T)_{ij}(x) = \sum_{k=1}^d \hat{\sigma}_{ik}(x) \hat{\sigma}_{jk}(x)\), is uniformly elliptic i.e.

\[
\sum_{i,j=1}^d (\hat{\sigma} \hat{\sigma}^T)_{ij}(x)\xi_i \xi_j \geq \lambda ||\xi||^2, \quad \text{for all } x, \xi \in \mathbb{R}^d \text{ and some } \lambda > 0.
\]

Under these assumptions the process \(Y\) has a transition density \(\Gamma\) with appropriate tail estimates (see Theorem 5.1 in Appendix).

We consider two cases. The first case

\((C1)\) \(x_0 = y_0 \in \mathbb{R}^d, \hat{b}_i(x) := b_i(x), \hat{\sigma}_{ij}(x) := \sigma_{ij}(x), X_t = Y_t,\)

is related to the Brownian motion and the second case

\((C2)\) \(x_0 = e^{y_0} \in (0, \infty)^d, \hat{b}_i(y) := \frac{b_i(e^{y})}{e^{y_i}} - \frac{1}{2} \sum_{j=1}^d \hat{\sigma}_{ij}^2(y), \quad \hat{\sigma}_{ij}(y) := \frac{\sigma_{ij}(e^{y})}{e^{y_i}}\) and

\(X_t = e^{Y_t} \).
with the convention \( e^y = (e^{y_1}, \ldots, e^{y_d}) \) for \( y \in \mathbb{R}^d \), is close to the geometric Brownian motion. In both cases we have

\[
E \sup_{t \in [0,T]} ||X_t||^p < \infty
\]

for any \( p > 0 \) (cf. [10] Corollary 2.2.1 on p. 101).

To summarize the above, we start with the process \( X \) by choosing the matrix \( \sigma \) and the vector \( b \) such that the matrix \( \hat{\sigma} \) and the vector \( \hat{b} \) satisfy the required conditions above. In this way we obtain the process \( Y \) and deduce properties of the process \( X \) from the properties of \( Y \).

To handle both of these cases simultaneously, we define functions \( Q_i : \mathbb{R}^d \rightarrow \mathbb{R} \) for \( i = 1, \ldots, d \) by

\[
Q_i(x) := \begin{cases} 
1, & \text{in case (C1)} \\
x_i, & \text{in case (C2)}.
\end{cases}
\]

In what follows we assume, for some \( q \in [2, \infty) \) and \( C > 0 \), that

\[
|f(x)| \leq C (1 + ||x||^q), \quad x \in E,
\]

where the \( f : E \rightarrow \mathbb{R} \) is a Borel-function and the set \( E \) is defined by

\[
E := \begin{cases} 
\mathbb{R}^d, & \text{in case (C1)} \\
(0, \infty)^d, & \text{in case (C2)}.
\end{cases}
\]

Through the function \( f \) we define the function \( g \) on \( \mathbb{R}^d \) by

\[
g(y) := \begin{cases} 
f(y), & \text{in case (C1)} \\
f(e^y), & \text{in case (C2)}.\end{cases}
\]

Applying Theorem 5.1 to the stochastic differential equation

\[
\begin{align*}
Z_i^t &= Z_i^0 + \sum_{j=1}^d \int_0^t \hat{\sigma}_{ij}(Z_u)dW_u^j, \\
Z_i^t &= Z_i^0 - \int_0^t \left( \frac{1}{2} \sum_{j=1}^d \hat{\sigma}_{ij}^2(Z_u) \right) du + \sum_{j=1}^d \int_0^t \hat{\sigma}_{ij}(Z_u)dW_u^j,
\end{align*}
\]

in case (C1)

\[
\begin{align*}
Z_i^t &= Z_i^0 + \frac{1}{2} \sum_{k,l=1}^d \left( \hat{\sigma} \hat{\sigma}^T(y) \right)_{kl} \frac{\partial^2}{\partial y_k \partial y_l} G(t,y), \\
Z_i^t &= Z_i^0 - \sum_{i=1}^d \left( \frac{1}{2} \sum_{j=1}^d \hat{\sigma}_{ij}^2(y) \right) \frac{\partial}{\partial y_i} + \frac{1}{2} \sum_{k,l=1}^d \left( \hat{\sigma} \hat{\sigma}^T(y) \right)_{kl} \frac{\partial^2}{\partial y_k \partial y_l} G(t,y)
\end{align*}
\]

in case (C2)

\[
gives a transition density \( \Gamma_0 \) such that we can define the function \( G \in C^\infty([0,T) \times \mathbb{R}^d) \) by

\[
G(t,y) := \int_{\mathbb{R}^d} \Gamma_0(T-t, y, \xi) g(\xi) d\xi, \quad 0 \leq t < T
\]

so that

\[
\begin{align*}
\left( \frac{\partial}{\partial t} + \frac{1}{2} \sum_{k,l=1}^d \left( \hat{\sigma} \hat{\sigma}^T(y) \right)_{kl} \frac{\partial^2}{\partial y_k \partial y_l} \right) G(t,y) &= 0 \quad \text{(C1)} \\
\left( \frac{\partial}{\partial t} - \sum_{i=1}^d \left( \frac{1}{2} \sum_{j=1}^d \hat{\sigma}_{ij}^2(y) \right) \frac{\partial}{\partial y_i} + \frac{1}{2} \sum_{k,l=1}^d \left( \hat{\sigma} \hat{\sigma}^T(y) \right)_{kl} \frac{\partial^2}{\partial y_k \partial y_l} \right) G(t,y) &= 0 \quad \text{(C2)}.
\end{align*}
\]

Now we can define the function \( F \) on \([0,T) \times E\) by

\[
F(t,x) := \begin{cases} 
G(t,x), & \text{in case (C1)} \\
G(t,\log(x)), & \text{in case (C2)}.
\end{cases}
\]
Assumption (2.5) together with Theorem 5.1 implies that for $0 \leq t \leq T' < T$

$$|Q_i(x)| \left| \frac{\partial}{\partial x_i} F(t, x) \right| \leq C_{d,T'}(1 + ||x||^q), \ x \in E, \ i = 1, \ldots, d$$

and

$$|Q_i(x)||Q_j(x)| \left| \frac{\partial^2}{\partial x_i \partial x_j} F(t, x) \right| \leq C_{d,T'}(1 + ||x||^q), \ x \in E, \ i, j = 1, \ldots, d.$$ 

Let

$$A := \frac{\partial}{\partial t} + \frac{1}{2} \sum_{k,l=1}^d A_{kl}(x) \frac{\partial^2}{\partial x_k \partial x_l}$$

where

$$A_{kl}(x) := \sum_{j=1}^d \sigma_{kj}(x) \sigma_{lj}(x).$$

From the definition of $F$ and equation (2.6) it follows that

$$AF(t, x) = 0 \text{ on } [0, T) \times E.$$ 

Moreover, Itô’s formula gives that

$$F(t, X_t) = F(0, X_0) + \sum_{k=1}^d \int_0^t \frac{\partial}{\partial x_k} F(u, X_u) dX_u^{k}, \ a.s. \ t \in [0, T).$$

Finally, Theorem 5.1 gives that

$$F(t, X_t) \to f(X_T) \text{ in } L^2 \text{ as } t \uparrow T$$

and

$$f(X_T) = F(0, X_0) + \sum_{k=1}^d \int_0^T \frac{\partial}{\partial x_k} F(u, X_u) dX_u^{k} \text{ a.s.}$$

3. Results

In the rest of the paper we assume the setting from Chapter 2. We start this chapter by stating our main result Theorem 3.1. It implies that under certain conditions the convergence rate for the supremum of the approximation error is bounded by $n^{-1/2}$, when one optimizes over all deterministic time-nets of cardinality $n + 1$. Two examples where Theorem 3.1 is applied to are presented in Chapter 4.
Theorem 3.1. Assume that for all \( x \in E \)
\[
\left| \frac{\partial^s}{\partial x_\alpha \partial x_\beta} \sigma_{ij}(x) \right| \leq C_1 \frac{Q_1(x)}{Q_2^r(x)} Q_\alpha^r(x), \text{ where } q + r = s, \ q, r, s \in \{0, 1, 2\},
\]
\[
|b_1(x)| \leq C_1 Q_1(x) \text{ and } A_{ii}(x) \geq \frac{1}{C_1} Q_2^2(x) \text{ for } i \in \{1, \ldots, d\} \text{ and some fixed } C_1 > 0.
\]
Moreover, assume that
\[
(3.1)
\]
\[
\text{Assume that for all } x \text{ and }
\]
\[
(3.2)
\]
\[
\text{Assume that for all } x \text{ and }
\]
\[
(3.3)
\]
\[
\text{Assume that for all } x \text{ and }
\]
\[
(3.4)
\]

Then we have the following two cases:

(L1) In the case that \( \theta \in [0, 3/4) \), we have, for any sequence of time-nets \( 0 = t_0^n \leq t_1^n \leq \ldots \leq t_n^n = T \) with \( \sup_{i=1, \ldots, n} (t_i^n - t_{i-1}^n) \leq C_T/n, C_T > 0 \), that
\[
(3.5)
\]

(L2) If \( \theta \in [3/4, 1) \), then we have that
\[
(3.6)
\]
The constant $D_2 > 0$ depends at most on $C_1, C_2, C_H, d$ and $T$.

Remark 3.2. (1) In the case that the process $(X_t)_{t \in [0,T]}$ does not have a drift, it follows from Doob’s inequality that inequalities (3.4) and (3.5) can be replaced by

$$\liminf_{n \to \infty} \sqrt{n} \left( \mathbb{E} \left[ \sum_{i=1}^{n} \sum_{k=1}^{d} \int_{t_{i-1}^{n}}^{t_i^{n}} \left( \frac{\partial}{\partial x_k} F(u, X_u) - \frac{\partial}{\partial x_k} F(t_{i-1}^{n}, X_{t_{i-1}^{n}}) \right) dX_k^u \right]^2 \right)^{\frac{1}{2}} \geq \frac{1}{4D_2}.$$ 

Of course in case of (3.7) we set $t_i^n = t_i^{\eta,n}$

(2) In (L2) we have the lower bound only for time-nets $\tau_n$. Compared to (L1) this does not seem natural, since larger $\theta$ should correspond to a worse approximation. We need this restriction for technical reason (but believe that it can be removed).

(3) Under the setting of the Chapter 2 the assumptions in Theorem 3.1 which concern the estimates of the matrices $A$ and $\sigma$ and the vector $b$ by the functions $Q_i$ are always satisfied for some $C_1 > 0$.

(4) It follows by a simple calculation that

$$H^2(u) = \mathbb{E} \sum_{\alpha,\beta=1}^{d} \left( \sum_{\alpha,\beta=1}^{d} \sigma_{\alpha m}(X_u)\sigma_{\beta n}(X_u) \frac{\partial^2}{\partial x_\alpha x_\beta} F(u, X_u) \right)^2.$$ 

Now because of (2.8) we have that $H^2(u) \in [0, \infty)$, for $u \in [0, T)$.

(5) If the matrix $A$ defined in (2.10) is a diagonal matrix, then

$$H^2(u) = \mathbb{E} \sum_{\alpha,\beta=1}^{d} A_{\alpha\alpha}(X_u)A_{\beta\beta}(X_u) \left( \frac{\partial}{\partial x_\alpha x_\beta} F(u, X_u) \right)^2$$

and thus it is equivalent to the function

$$\sup_{\alpha,\beta} \mathbb{E} \left[ A_{\alpha\alpha}(X_t)A_{\beta\beta}(X_t) \left( \frac{\partial}{\partial x_\alpha x_\beta} F(t, X_t) \right)^2 \right]$$

considered for the upper bound in Theorem 3.1. In the 1-dimensional case our function $H$ is the same as the function $H$ controlling the approximation error in [5].

Now turn to the proof of Theorem 3.1. We deal with a multi-step approximation error i.e. the stochastic integral $\sum_{k=1}^{d} \int_{t_{i-1}}^{t_i} \frac{\partial}{\partial x_k} F(u, X_u) dX_k^u$ is approximated by the stochastic integral $\sum_{k=1}^{d} \sum_{i=1}^{n} \sum_{j_{i-1}}^{t_i} \frac{\partial}{\partial x_k} F(t_{i-1}, X_{t_{i-1}}) dX_k^u$. In order to estimate the multi-step error we need to have information about the one-step error occurring in a time interval $[t_{i-1}, t_i]$. Here Proposition 3.3 and Proposition 3.4 below are needed. Proposition 3.3 gives the upper bound for the one-step error. It is an extension of
Temam [12] for the upper estimate and replaces the limit arguments by the inequality (3.6), which can be applied to any fixed time-net to get an upper bound for the approximation error. From Proposition 3.4 we get the lower bound for the one-step error. In the proof of Proposition 3.4 we use the same principal decomposition as in [12], but apply it to non-equidistant nets. We conclude the proof of our main result by considering multi-step error starting on page 15.

Proposition 3.3. If for all $x \in E$

$$\left| \frac{\partial^s}{\partial x_\beta \partial x_\alpha} \sigma_{ij}(x) \right| \leq C_3 \frac{Q_i(x)}{Q_j^\beta(x) Q_i(x)}, \quad q + r = s, \quad q, r, s \in \{0, 1, 2\},$$

$|b_i(x)| \leq C_3 Q_i(x)$ and $A_i(x) \geq \frac{1}{C_3} Q_i^2(x)$ for $i \in \{1, \ldots, d\}$ and for some $C_3 > 0$, then for $0 \leq a \leq u < T$ it holds

$$\sum_{l=1}^d \sum_{k=1}^d \mathbb{E} \left( \frac{\partial}{\partial x_k} F(u, X_u) - \frac{\partial}{\partial x_k} F(a, X_a) \right)^2 \sigma_{kl}(X_u)^2 \leq D_3 \int_a^u \sup_{\alpha, \beta} \mathbb{E} \left[ A_{\alpha \alpha}(X_t) A_{\beta \beta}(X_t) \left| \frac{\partial^2}{\partial x_\alpha \partial x_\beta} F(t, X_t) \right|^2 \right] dt,$$

where $D_3 > 0$ depends at most on $C_3$, $d$ and $T$.

Proof. To keep the notation simple, we allow in the following that the constant $C > 0$ may change from line to line.

Set

$$v_a := \left( \frac{\partial}{\partial x_k} F(a, X_a) \right)_{k=1}^d \quad \text{and} \quad \phi_{kl}(u, x) := \left( \frac{\partial}{\partial x_k} F(u, x) - v_k^a \right) \sigma_{kl}(x).$$

Using this notation the assertion can be re-written as

$$\sum_{l=1}^d \sum_{k=1}^d \mathbb{E} \phi_{kl}^2(u, X_u) \leq D_3 \int_a^u \sup_{\alpha, \beta} \mathbb{E} \left[ A_{\alpha \alpha}(X_t) A_{\beta \beta}(X_t) \left| \frac{\partial^2}{\partial x_\alpha \partial x_\beta} F(t, X_t) \right|^2 \right] dt.$$

By the definition of $\phi_{kl}$ we have that

$$\sum_{l=1}^d \sum_{k=1}^d \phi_{kl}^2(u, x) = \sum_{l=1}^d \sum_{k=1}^d \left( \frac{\partial}{\partial x_k} F(u, x) - v_k^a \right)^2 \sigma_{kl}^2(x)$$

$$= \sum_{k=1}^d \left( \frac{\partial}{\partial x_k} F(u, x) - v_k^a \right)^2 \sum_{l=1}^d \sigma_{kl}^2(x) \right].$$

The assumptions on $\sigma$ give that

$$\frac{Q_k^2(x)}{C_3} \leq A_k(x) = \sum_{l=1}^d \sigma_{kl}^2(x) \leq d \quad C_3 Q_k^2(x).$$
This implies the equivalence
\begin{align}
\frac{1}{C_3} \sum_{k=1}^{d} \left[ \left( \frac{\partial}{\partial x_k} F(u, x) - v^k_a \right)^2 Q_k^2(x) \right] & \leq \sum_{l=1}^{d} \sum_{k=1}^{d} \phi^2_{kl}(u, x) \\
& \leq d C_3 \sum_{k=1}^{d} \left[ \left( \frac{\partial}{\partial x_k} F(u, x) - v^k_a \right)^2 Q_k^2(x) \right].
\end{align}

Lemma 5.5 allows us to use the stopping argument from Lemma 5.4, which implies that
\begin{equation}
\mathbb{E} \phi^2_{kl}(u, X_u) = \int_a^u \mathbb{E} \left( \mathcal{A} \phi^2_{kl}(v, X_v) \right) dv + \sum_{m=1}^{d} \int_a^u \mathbb{E} \left( \frac{\partial}{\partial x_m} \phi^2_{kl}(v, X_v) \right) b_m(X_v) dv.
\end{equation}

To prove our theorem we need to compute an upper bound for \( \mathcal{A} \phi^2_{kl}(u, x) \) and for \( \frac{\partial}{\partial x_m} \phi^2_{kl}(u, x) b_m(x) \). First we consider the term \( \mathcal{A} \phi^2_{kl} \):
\begin{align}
|\mathcal{A} \phi^2_{kl}(u, x)| &= \left| 2 \phi_{kl}(u, x) \left( \mathcal{A} \phi_{kl} \right)(u, x) + \sum_{\alpha, \beta=1}^{d} \sum_{j=1}^{d} \left( \sigma_{\alpha j}(x) \frac{\partial}{\partial x_\alpha} \phi_{kl}(u, x) \right) \left( \sigma_{\beta j}(x) \frac{\partial}{\partial x_\beta} \phi_{kl}(u, x) \right) \right| \\
& \leq \phi^2_{kl}(u, x) + (\mathcal{A} \phi_{kl})^2(u, x) + d \sum_{\alpha=1}^{d} \sum_{j=1}^{d} \left( \sigma_{\alpha j}(x) \frac{\partial}{\partial x_\alpha} \phi_{kl}(u, x) \right)^2.
\end{align}

Hence equation (3.8) implies that
\begin{align}
\mathbb{E} \phi^2_{kl}(u, X_u) & \leq \int_a^u \mathbb{E} \phi^2_{kl}(v, X_v) dv + \int_a^u \mathbb{E} \left( \mathcal{A} \phi^2_{kl}(v, X_v) \right) dv \\
& \quad + \sum_{\alpha=1}^{d} \sum_{j=1}^{d} \int_a^u \mathbb{E} \left( \sigma_{\alpha j}(X_v) \frac{\partial}{\partial x_\alpha} \phi_{kl}(v, X_v) \right)^2 dv \\
& \quad + \sum_{m=1}^{d} \int_a^u \mathbb{E} \left| \left( \frac{\partial}{\partial x_m} \phi^2_{kl}(v, X_v) \right) b_m(X_v) \right| dv,
\end{align}
where the right-hand side is finite because of Lemma 5.5. From Gronwall’s Lemma (see Theorem 5.3 in Appendix) it follows that
\begin{align}
\mathbb{E} \phi^2_{kl}(u, X_u) & \leq \int_a^u \mathbb{E} \left( \mathcal{A} \phi^2_{kl}(v, X_v) \right) dv + \sum_{\alpha=1}^{d} \sum_{j=1}^{d} \int_a^u \mathbb{E} \left( \sigma_{\alpha j}(X_v) \frac{\partial}{\partial x_\alpha} \phi_{kl}(v, X_v) \right)^2 dv.
\end{align}
To continue we need to find an upper bound for the above expression. We start with $A_{\phi_{kl}}$ and have, by definition, that

$$A_{\phi_{kl}}(u, x) = \left( \frac{\partial^2}{\partial x_k \partial x} F(u, x) \right) \sigma_{kl}(x) +$$

$$+ \frac{1}{2} \sum_{\alpha, \beta = 1}^{d} A_{\alpha \beta}(x) \left[ \left( \frac{\partial}{\partial x_\alpha x_\beta} \sigma_{kl}(x) \right) \left( \frac{\partial}{\partial x_k} F(u, x) - v_a^k \right) \right.$$

$$\left. + \left( \frac{\partial}{\partial x_\beta} \sigma_{kl}(x) \right) \left( \frac{\partial^2}{\partial x_\alpha x_k} F(u, x) \right) \right. + \left( \frac{\partial}{\partial x_\alpha} \sigma_{kl}(x) \right) \left( \frac{\partial^2}{\partial x_\beta x_k} F(u, x) \right)$$

$$\left. + \sigma_{kl}(x) \left( \frac{\partial^3}{\partial x_\alpha x_\beta x_k} F(u, x) \right) \right].$$

Taking the derivative with respect to $x_k$ in the partial differential equation (2.11) we get that

$$\frac{\partial}{\partial x_k} \frac{\partial}{\partial t} F(u, x) + \frac{1}{2} \sum_{\alpha, \beta = 1}^{d} A_{\alpha \beta}(x) \frac{\partial^3}{\partial x_k x_\alpha x_\beta} F(u, x)$$

$$= - \frac{1}{2} \sum_{\alpha, \beta = 1}^{d} \left( \frac{\partial}{\partial x_k} A_{\alpha \beta}(x) \right) \frac{\partial^2}{\partial x_\alpha x_\beta} F(u, x).$$

Now we can replace the derivative with respect to $t$ and the third order derivatives in the formula for $A_{\phi_{kl}}(u, x)$ by second order derivatives:

$$A_{\phi_{kl}}(u, x) = \frac{1}{2} \sum_{\alpha, \beta = 1}^{d} A_{\alpha \beta}(x) \left[ A_{\alpha \beta}(x) \left( \frac{\partial^2}{\partial x_\alpha x_\beta} \sigma_{kl}(x) \right) \left( \frac{\partial}{\partial x_k} F(u, x) - v_a^k \right) \right.$$

$$\left. + A_{\alpha \beta}(x) \left( \frac{\partial}{\partial x_\beta} \sigma_{kl}(x) \right) \left( \frac{\partial^2}{\partial x_\alpha x_k} F(u, x) \right) \right. + A_{\alpha \beta}(x) \left( \frac{\partial}{\partial x_\alpha} \sigma_{kl}(x) \right) \left( \frac{\partial^2}{\partial x_\beta x_k} F(u, x) \right)$$

$$\left. - \sigma_{kl}(x) \left( \frac{\partial}{\partial x_k} A_{\alpha \beta}(x) \right) \left( \frac{\partial^3}{\partial x_\alpha x_\beta x_k} F(u, x) \right) \right].$$

It follows from the definition of the matrix $A$ and the assumption on the matrix $\sigma$ that

$$|A_{\alpha \beta}(x)| \leq CQ_\alpha(x)Q_\beta(x)$$
and
\[
\left| \frac{\partial}{\partial x_k} A_{\alpha\beta}(x) \right| \leq C \frac{Q_\alpha(x) Q_\beta(x)}{Q_k(x)}.
\]

Now we can bound the function \((\mathcal{A} \phi_{kl})^2(u, x)\) from the above by
\[
(\mathcal{A} \phi_{kl})^2(u, x) \leq C \sum_{\alpha, \beta=1}^d \left[ \left( \frac{\partial}{\partial x_k} F(u, x) - v_a^k \right)^2 Q_k^2(x) + Q_{\alpha a}^2(x) Q_k^2(x) \left( \frac{\partial^2}{\partial x_\alpha x_k} F(u, x) \right)^2 + Q_{\beta a}^2(x) Q_k^2(x) \left( \frac{\partial^2}{\partial x_\beta x_k} F(u, x) \right)^2 + Q_{\alpha a}^2(x) Q_{\beta a}^2(x) \left( \frac{\partial^2}{\partial x_\alpha x_\beta} F(u, x) \right)^2 \right].
\]

For \(\left( \sigma_{\alpha j}(x) \frac{\partial}{\partial x_\alpha} \phi_{kl}(u, x) \right)^2\) we get that
\[
\left( \sigma_{\alpha j}(x) \frac{\partial}{\partial x_\alpha} \phi_{kl}(u, x) \right)^2 \leq C \left( Q_k^2(x) \left( \frac{\partial}{\partial x_k} F(u, x) - v_a^k \right)^2 + Q_{\alpha a}^2(x) Q_k^2(x) \left( \frac{\partial^2}{\partial x_\alpha x_k} F(u, x) \right)^2 \right).
\]

The term including \(b_m\) can be bounded as follows:
\[
\left| \left( \frac{\partial}{\partial x_m} \phi_{kl}^2(u, x) \right) b_m(x) \right| \leq 2 \left| \frac{\partial}{\partial x_k} F(u, x) - v_a^k \right| |\sigma_{kl}(x)| |b_m(x)| \times \left( \left| \frac{\partial}{\partial x_m} \sigma_{kl}(x) \left( \frac{\partial}{\partial x_k} F(u, x) - v_a^k \right) \right| + |\sigma_{kl}(x)| \left| \frac{\partial^2}{\partial x_m x_k} F(u, x) \right| \right) \leq C \left| \frac{\partial}{\partial x_k} F(u, x) - v_a^k \right| Q_k(x) Q_m(x) \times \left( Q_k(x) \left| \frac{\partial}{\partial x_k} F(u, x) - v_a^k \right| + Q_k(x) \left| \frac{\partial^2}{\partial x_m x_k} F(u, x) \right| \right) \leq C \left( \left| \frac{\partial}{\partial x_k} F(u, x) - v_a^k \right|^2 Q_k^2(x) + Q_k^2(x) Q_m^2(x) \left| \frac{\partial^2}{\partial x_m x_k} F(u, x) \right|^2 \right),
\]
(3.13)
where we used that
\[
\left| \frac{\partial}{\partial x_k} F(u, x) - v_a^k \right| Q_k^2(x) Q_m(x) \left| \frac{\partial^2}{\partial x_m x_k} F(u, x) \right| \\
\leq \left| \frac{\partial}{\partial x_k} F(u, x) - v_a^k \right|^2 Q_k^2(x) + Q_k^2(x) Q_m(x) \left| \frac{\partial^2}{\partial x_m x_k} F(u, x) \right|^2.
\]

Now the expectation of $\phi_{kl}^2(u, X_u)$ can be bounded by
\[
\mathbb{E}\phi_{kl}^2(u, X_u) \leq C \int_a^u \mathbb{E} \left( \frac{\partial}{\partial x_k} F(v, X_v) - v_a^k \right)^2 Q_k^2(X_v) dv \\
+ C \int_a^u \sup_{\alpha, \beta} \mathbb{E} Q_{\alpha}^2(X_v) Q_{\beta}^2(X_v) \left( \frac{\partial^2}{\partial x_\alpha x_\beta} F(v, X_v) \right)^2 dv,
\]
where we use (3.10), (3.11), (3.12) and (3.13). From the above and (3.7) we get
\[
\sum_{k=1}^d \mathbb{E} \left( \frac{\partial}{\partial x_k} F(u, X_u) - v_a^k \right)^2 Q_k^2(X_u) \leq C \sum_{l=1}^d \sum_{k=1}^d \mathbb{E}\phi_{kl}^2(u, X_u)
\]
\[
\leq C \int_a^u \sum_{k=1}^d \left( \frac{\partial}{\partial x_k} F(v, X_v) - v_a^k \right)^2 Q_k^2(X_v) dv + \\
+ C \int_a^u \sup_{\alpha, \beta} \mathbb{E} Q_{\alpha}^2(X_v) Q_{\beta}^2(X_v) \left( \frac{\partial^2}{\partial x_\alpha x_\beta} F(v, X_v) \right)^2 dv.
\]
Gronwall’s lemma (Theorem 5.3) gives
\[
(3.14) \quad \sum_{k=1}^d \mathbb{E} \left( \frac{\partial}{\partial x_k} F(u, X_u) - v_a^k \right)^2 Q_k^2(X_u) \leq e^{C(u-a)} C \int_a^u \sup_{\alpha, \beta} \mathbb{E} Q_{\alpha}^2(X_v) Q_{\beta}^2(X_v) \left( \frac{\partial^2}{\partial x_\alpha x_\beta} F(v, X_v) \right)^2 dv
\]
and the assertion follows from (3.7).

\[\square\]

**Proposition 3.4.** If for all $x \in E$
\[
\left| \frac{\partial^s}{\partial x_\beta \partial^r x_\alpha} \sigma_{ij}(x) \right| \leq C_4 \frac{Q_i(x)}{Q_{\beta}^q(x) Q_{\alpha}^r(x)}, \quad q + r = s, \quad q, r, s \in \{0, 1, 2\},
\]
and
\[
| b_i(x) | \leq C_4 Q_i(x)
\]
for $i \in \{1, \ldots, d\}$ and for some $C_4 > 0$, then for $0 \leq a < t < T$ it holds that
\[
\sum_{j=1}^d \mathbb{E} \left( \sum_{i=1}^d \left( \frac{\partial}{\partial x_i} F(t, X_t) - \frac{\partial}{\partial x_i} F(a, X_a) \right) \sigma_{ij}(X_u) \right)^2 \geq \int_a^t H^2(u) du
\]
\[
- D_4 \int_a^t \left( \int_a^u \sup_{\alpha, \beta} \mathbb{E} Q_{\alpha}^2(X_v) Q_{\beta}^2(X_v) \left( \frac{\partial^2}{\partial x_\alpha x_\beta} F(v, X_v) \right)^2 dv \right)^{\frac{1}{2}} \times
\]
\[
\times \left( \sup_{\alpha, \beta} \mathbb{E} Q_\alpha^2(X_u) Q_\beta^2(X_u) \left| \frac{\partial^2}{\partial x_\alpha x_\beta} F(u, X_u) \right|^2 \right)^{\frac{1}{2}} + \\
+ \int_a^u \sup_{\alpha, \beta} \mathbb{E} Q_\alpha^2(X_v) Q_\beta^2(X_v) \left| \frac{\partial^2}{\partial x_\alpha x_\beta} F(v, X_v) \right|^2 dv \right] du,
\]

where the function \( H \) is defined in Theorem 3.1 and \( D_4 > 0 \) depends at most on \( C_4, d \) and \( T \).

Proof.

To abbreviate the notation we assume again that \( C > 0 \) may change from line to line. We let

\[
\phi^{ij}_u := \left( \frac{\partial}{\partial x_i} F(u, X_u) - v^i_a \right) \sigma^{ij}(X_u),
\]

where \( u \in [a, t] \) and \( v^i_a := \frac{\partial}{\partial x_i} F(a, X_a) \). Itô’s formula gives that, a.s.,

\[
\phi^{ij}_t = \int_a^t \frac{\partial^2}{\partial t x_i} F(u, X_u) \sigma^{ij}(X_u) du \\
+ \sum_{\alpha=1}^d \int_a^t b_\alpha(X_u) \left[ \frac{\partial^2}{\partial x_\alpha x_i} F(u, X_u) \sigma^{ij}(X_u) \\
+ \left( \frac{\partial}{\partial x_i} F(u, X_u) - v^i_a \right) \frac{\partial}{\partial x_\alpha} \sigma^{ij}(X_u) \right] du \\
+ \sum_{\alpha=1}^d \sum_{n=1}^d \int_a^t \left[ \frac{\partial^2}{\partial x_\alpha x_i} F(u, X_u) \sigma^{ij}(X_u) \\
+ \left( \frac{\partial}{\partial x_i} F(u, X_u) - v^i_a \right) \frac{\partial}{\partial x_\alpha} \sigma^{ij}(X_u) \right] \sigma^{\alpha n}(X_u) dW^n_u \\
+ \frac{1}{2} \sum_{\alpha, \beta=1}^d \int_a^t \left[ \frac{\partial^3}{\partial x_\beta x_\alpha x_i} F(u, X_u) \sigma^{ij}(X_u) + \frac{\partial^2}{\partial x_\alpha x_i} F(u, X_u) \frac{\partial}{\partial x_\beta} \sigma^{ij}(X_u) + \\
+ \frac{\partial^2}{\partial x_\beta x_i} F(u, X_u) \frac{\partial}{\partial x_\alpha} \sigma^{ij}(X_u) + \\
+ \left( \frac{\partial}{\partial x_i} F(u, X_u) - v^i_a \right) \frac{\partial^2}{\partial x_\alpha x_i} \sigma^{ij}(X_u) \right] A_{\alpha \beta}(X_u) du.
\]

From the above we deduce

\[
d\langle \phi^{ij}, \phi^{kj} \rangle_u \\
= \sum_{\alpha=1}^d \sum_{\beta=1}^d \left[ \left( \frac{\partial^2}{\partial x_\alpha x_i} F(u, X_u) \sigma^{ij}(X_u) + \left( \frac{\partial}{\partial x_i} F(u, X_u) - v^i_a \right) \frac{\partial}{\partial x_\alpha} \sigma^{ij}(X_u) \right) \times \\
\times \left( \frac{\partial^2}{\partial x_\beta x_k} F(u, X_u) \sigma^{kj}(u, X_u) + \left( \frac{\partial}{\partial x_k} F(u, X_u) - v^k_a \right) \frac{\partial}{\partial x_\beta} \sigma^{kj}(X_u) \right) A_{\alpha \beta}(X_u) \right] du
\]
and, using the equality \( AF = 0 \) (cf. (2.9) and (2.11)), we get that

\[
d\phi^{ij}_u
\]

\[
= \sum_{\alpha=1}^{d} \sum_{n=1}^{d} \left( \frac{\partial^2}{\partial x_{\alpha} x_i} F(u, X_u) \sigma_{ij}(X_u) + \left( \frac{\partial}{\partial x_i} F(u, X_u) - v_i^a \right) \frac{\partial}{\partial x_{\alpha}} \sigma_{ij}(X_u) \right) \sigma_{\alpha n}(X_u) dW^n_u
\]

\[
+ \frac{1}{2} \sum_{\alpha, \beta=1}^{d} A_{\alpha \beta}(X_u) \left[ \frac{\partial^2}{\partial x_{\alpha} x_i} F(u, X_u) \frac{\partial}{\partial x_{\beta}} \sigma_{ij}(X_u) + \frac{\partial^2}{\partial x_{\beta} x_i} F(u, X_u) \frac{\partial}{\partial x_{\alpha}} \sigma_{ij}(X_u) \right.
\]

\[
+ \left( \frac{\partial}{\partial x_i} F(u, X_u) - v_i^a \right) \frac{\partial}{\partial x_{\alpha}} \sigma_{ij}(X_u) \right] du
\]

\[- \frac{1}{2} \sum_{\alpha, \beta=1}^{d} \left[ \frac{\partial}{\partial x_i} A_{\alpha \beta}(X_u) \frac{\partial^2}{\partial x_{\alpha} x_i} F(u, X_u) \sigma_{ij}(X_u) \right] du
\]

\[
+ \sum_{\alpha=1}^{d} b_{\alpha}(X_u) \left[ \frac{\partial^2}{\partial x_{\alpha} x_i} F(u, X_u) \sigma_{ij}(X_u) \right.
\]

\[
+ \left( \frac{\partial}{\partial x_i} F(u, X_u) - v_i^a \right) \frac{\partial}{\partial x_{\alpha}} \sigma_{ij}(X_u) \right] du.
\]

Let

\[
S_N := \inf \{ u \geq a : |\phi^{ij}_u| \geq N \text{ or } ||X_u|| \geq N \text{ for some } i, j \in \{1, \ldots, d\} \} \wedge t
\]

for \( N = 1, 2, \ldots \). Because of \( S_N \nearrow t \) a.s. as \( N \to \infty \) and Lemma 5.5 one has that

\[
\sum_{j=1}^{d} \mathbb{E} \left( \sum_{i=1}^{d} \phi^{ij}_t \right)^2 = \lim_{N \to \infty} \sum_{j=1}^{d} \mathbb{E} \left( \sum_{i=1}^{d} \phi^{ij}_S \right)^2.
\]

Using the integration by parts formula for semi-martingales, we get

\[
\sum_{j=1}^{d} \mathbb{E} \left( \sum_{i=1}^{d} \phi^{ij}_{S_N} \right)^2 = \sum_{j=1}^{d} \mathbb{E} \int_a^{S_N} \sum_{i,k=1}^{d} \phi^{kj}_u d\phi^{ij}_u + \sum_{i,k=1}^{d} \mathbb{E} \int_a^{S_N} d\langle \phi^{ij}, \phi^{kj} \rangle_u.
\]

Because of the choice of the stopping time \( S_N \), the expected value of the "\( dW^n_u\)-terms" vanishes. For the rest, we obtain as main term

\[
\mathbb{E} \int_a^{S_N} \sum_{\alpha, \beta, i, k=1}^{d} A_{\alpha \beta}(X_u) A_{ik}(X_u) \frac{\partial^2}{\partial x_{\alpha} x_i} F(u, X_u) \frac{\partial^2}{\partial x_{\beta} x_k} F(u, X_u) du
\]

and (after some computation) terms of the type

\[
\mathbb{E} \int_a^{S_N} A_{\alpha \beta}(X_u) \sigma_{kj}(X_u) \frac{\partial}{\partial x_{\beta}} \sigma_{ij}(X_u) \left( \frac{\partial}{\partial x_k} F(u, X_u) - v_k^a \right) \frac{\partial}{\partial x_{\alpha}} F(u, X_u) du
\]

and

\[
\mathbb{E} \int_a^{S_N} \left( \frac{\partial}{\partial x_k} F(u, X_u) - v_k^a \right) \sigma_{kj}(X_u) b_{\alpha}(X_u) \left( \frac{\partial}{\partial x_i} F(u, X_u) - v_i^a \right) \frac{\partial}{\partial x_{\alpha}} \sigma_{ij}(X_u) du.
\]
Using the assumptions on the matrix $\sigma$ and the vector $b$ we can bound these terms by
\[
\left| A_{\alpha\beta}(X_u)\sigma_{kj}(X_u)\frac{\partial}{\partial x_{j\beta}}\sigma_{ij}(X_u) \left( \frac{\partial}{\partial x_k}F(u, X_u) - \nu_a^k \right) \frac{\partial^2}{\partial x_{\alpha}x_i}F(u, X_u) \right|
\]
\[
\leq CQ_k(X_u) \left| \frac{\partial}{\partial x_k}F(u, X_u) - \nu_a^k \right| Q_\alpha(X_u)Q_i(X_u) \left| \frac{\partial^2}{\partial x_{\alpha}x_i}F(u, X_u) \right|
\]
and
\[
\left| \left( \frac{\partial}{\partial x_k}F(u, X_u) - \nu_a^k \right) \sigma_{kj}(X_u)b_\alpha(X_u) \left( \frac{\partial}{\partial x_i}F(u, X_u) - \nu_a^i \right) \frac{\partial}{\partial x_{\beta}}\sigma_{ij}(X_u) \right|
\]
\[
\leq CQ_k(X_u) \left| \frac{\partial}{\partial x_k}F(u, X_u) - \nu_a^k \right| Q_i(X_u) \left| \frac{\partial}{\partial x_i}F(u, X_u) - \nu_a^i \right| .
\]
Hölder’s inequality and (3.14) give
\[
\mathbb{E}Q_k(X_u) \left| \frac{\partial}{\partial x_k}F(u, X_u) - \nu_a^k \right| Q_\alpha(X_u)Q_i(X_u) \left| \frac{\partial^2}{\partial x_{\alpha}x_i}F(u, X_u) \right|
\]
\[
\leq \left( \mathbb{E}Q_k^2(X_u) \left| \frac{\partial}{\partial x_k}F(u, X_u) - \nu_a^k \right|^2 \right)^{\frac{1}{2}} \left( \sup_{\alpha',\beta'} \mathbb{E}Q_{\alpha'}^2(X_u)Q_{\beta'}^2(X_u) \left| \frac{\partial^2}{\partial x_{\alpha'}x_{\beta'}}F(u, X_u) \right|^2 \right)^{\frac{1}{2}}
\]
\[
\leq C \left( \int_a^u \sup_{\alpha',\beta'} \mathbb{E}Q_{\alpha'}^2(X_v)Q_{\beta'}^2(X_v) \left| \frac{\partial^2}{\partial x_{\alpha'}x_{\beta'}}F(v, X_v) \right|^2 \, dv \right)^{\frac{1}{2}} \times
\]
\[
\left( \sup_{\alpha',\beta'} \mathbb{E}Q_{\alpha'}^2(X_u)Q_{\beta'}^2(X_u) \left| \frac{\partial^2}{\partial x_{\alpha'}x_{\beta'}}F(u, X_u) \right|^2 \right)^{\frac{1}{2}} .
\]
Moreover
\[
\mathbb{E}Q_k(X_u) \left| \frac{\partial}{\partial x_k}F(u, X_u) - \nu_a^k \right| Q_i(X_u) \left| \frac{\partial}{\partial x_i}F(u, X_u) - \nu_a^i \right|
\]
\[
\leq \left( \mathbb{E}Q_k^2(X_u) \left| \frac{\partial}{\partial x_k}F(u, X_u) - \nu_a^k \right|^2 \right)^{\frac{1}{2}} \left( \mathbb{E}Q_i^2(X_u) \left| \frac{\partial}{\partial x_i}F(u, X_u) - \nu_a^i \right|^2 \right)^{\frac{1}{2}}
\]
\[
\leq C \int_a^u \sup_{\alpha,\beta} \mathbb{E}Q_{\alpha}^2(X_v)Q_{\beta}^2(X_v) \left| \frac{\partial^2}{\partial x_{\alpha}x_{\beta}}F(v, X_v) \right|^2 \, dv
\]
and the assertion follows by $N \to \infty$. \hfill \Box

**Proof of Theorem 3.1.**

Also in this proof, we use the same notation for different constants. First we consider the upper bound for the approximation error. Let $\varepsilon \in (0, T)$. Using Doob’s inequality together with Hölder’s inequality we see that
\[
\left( \mathbb{E} \sup_{t\in[0,T-\varepsilon]} \left| \sum_{i=1}^d \sum_{k=1}^{d} \int_{t_{i-1}}^{t_i} \left( \frac{\partial}{\partial x_k}F(u, X_u) - \frac{\partial}{\partial x_k}F(t_{i-1}, X_{t_{i-1}}) \right) dX_u^k \right|^2 \right)^{\frac{1}{2}}
\]
Inequality (3.14) gives that

\[
\begin{align*}
&\leq \left( \mathbb{E} \sup_{t \in [0, T - \varepsilon]} \left| \sum_{i=1}^{n} \sum_{k=1}^{d} \int_{t_{i-1} \wedge t}^{t_i \wedge t} \left( \frac{\partial}{\partial x_k} F(u, X_u) - \frac{\partial}{\partial x_k} F(t_{i-1}, X_{t_{i-1}}) \right) b_k(X_u) du \right| \right)^{\frac{1}{2}} + \\
&+ \left( \mathbb{E} \sup_{t \in [0, T - \varepsilon]} \left| \sum_{i=1}^{n} \sum_{l=1}^{d} \sum_{k=1}^{d} \int_{t_{i-1} \wedge t}^{t_i \wedge t} \left( \frac{\partial}{\partial x_k} F(u, X_u) - \frac{\partial}{\partial x_k} F(t_{i-1}, X_{t_{i-1}}) \right) \sigma_{kl}(X_u) dW^l_u \right| \right)^{\frac{1}{2}} \\
&\leq \left( \mathbb{E} \left| \sum_{i=1}^{n} \sum_{k=1}^{d} \int_{t_{i-1} \wedge t}^{t_i \wedge t} \left| \frac{\partial}{\partial x_k} F(u, X_u) - \frac{\partial}{\partial x_k} F(t_{i-1}, X_{t_{i-1}}) \right| \left| b_k(X_u) \right| du \right|^2 \right)^{\frac{1}{2}} + \\
&+ 2 \left( \mathbb{E} \left| \sum_{i=1}^{n} \sum_{l=1}^{d} \sum_{k=1}^{d} \int_{t_{i-1} \wedge t}^{t_i \wedge t} \left( \frac{\partial}{\partial x_k} F(u, X_u) - \frac{\partial}{\partial x_k} F(t_{i-1}, X_{t_{i-1}}) \right) \sigma_{kl}(X_u) dW^l_u \right|^2 \right)^{\frac{1}{2}} \\
&\leq \sum_{k=1}^{d} \sqrt{T} \left( \mathbb{E} \left| \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \left| \frac{\partial}{\partial x_k} F(u, X_u) - \frac{\partial}{\partial x_k} F(t_{i-1}, X_{t_{i-1}}) \right|^2 \left| b_k(X_u) \right|^2 du \right) \right)^{\frac{1}{2}} + \\
&+ 2 \left( \mathbb{E} \left| \sum_{i=1}^{n} \sum_{l=1}^{d} \sum_{k=1}^{d} \int_{t_{i-1} \wedge t}^{t_i \wedge t} \left( \frac{\partial}{\partial x_k} F(u, X_u) - \frac{\partial}{\partial x_k} F(t_{i-1}, X_{t_{i-1}}) \right) \sigma_{kl}(X_u) dW^l_u \right|^2 \right)^{\frac{1}{2}} \\
&:= B_1 + B_2.
\end{align*}
\]

Inequality (3.14) gives that

\[
B_1 \leq C_1 \sqrt{T} \sum_{k=1}^{d} \left( \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \mathbb{E} \left( \left. \left. \frac{\partial}{\partial x_k} F(u, X_u) - \frac{\partial}{\partial x_k} F(t_{i-1}, X_{t_{i-1}}) \right|_{\alpha=\beta} \right. \right. \right)^2 Q_k^2(X_u) du \right)^{\frac{1}{2}} \\
\leq \sqrt{T} C \sum_{k=1}^{d} \left( \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \sup_{\alpha, \beta} Q_{\alpha}(X_u) Q_{\beta}(X_v) \left( \frac{\partial^2}{\partial x_\alpha x_\beta} F(v, X_v) \right)^2 dv du \right)^{\frac{1}{2}}.
\]

For \( B_2 \) the Itô-isometry and the orthogonality of stochastic integrals give

\[
B_2^2 = 4 \mathbb{E} \left| \sum_{i=1}^{n} \sum_{l=1}^{d} \int_{t_{i-1} \wedge t}^{t_i \wedge t} \sum_{k=1}^{d} \left( \frac{\partial}{\partial x_k} F(u, X_u) - \frac{\partial}{\partial x_k} F(t_{i-1}, X_{t_{i-1}}) \right) \sigma_{kl}(X_u) dW^l_u \right|^2 \\
= 4 \sum_{i=1}^{n} \sum_{l=1}^{d} \mathbb{E} \left| \int_{t_{i-1} \wedge t}^{t_i \wedge t} \sum_{k=1}^{d} \left[ \frac{\partial}{\partial x_k} F(u, X_u) - \frac{\partial}{\partial x_k} F(t_{i-1}, X_{t_{i-1}}) \right] \sigma_{kl}(X_u) dW^l_u \right|^2 \\
= 4 \sum_{i=1}^{n} \sum_{l=1}^{d} \int_{t_{i-1} \wedge t}^{t_i \wedge t} \mathbb{E} \left| \sum_{k=1}^{d} \left[ \frac{\partial}{\partial x_k} F(u, X_u) - \frac{\partial}{\partial x_k} F(t_{i-1}, X_{t_{i-1}}) \right] \sigma_{kl}(X_u) \right|^2 du \\
\leq 4d \sum_{i=1}^{n} \sum_{l=1}^{d} \sum_{k=1}^{d} \int_{t_{i-1}}^{t_i} \left( \frac{\partial}{\partial x_k} F(u, X_u) - \frac{\partial}{\partial x_k} F(t_{i-1}, X_{t_{i-1}}) \right)^2 \sigma_{kl}(X_u) du.
\]
Letting $\varepsilon \downarrow 0$ we get by monotone convergence that
\[
\left( \mathbb{E} \sup_{t \in [0,T]} \left| \sum_{i=1}^{n} \sum_{k=1}^{d} \int_{t_{i-1}}^{t_{i}} \left( \frac{\partial}{\partial x_k} F(u, X_u) - \frac{\partial}{\partial x_k} F(t_{i-1}^{a}, X_{t_{i-1}^{a}}) \right) X_u^k \right|^2 \right) \frac{1}{2} 
\leq \sqrt{T \mathbb{E}} \sum_{k=1}^{d} \left( \sum_{i=1}^{n} \int_{t_{i-1}}^{t_{i}} \int_{t_{i-1}^{a}}^{u} \sup \mathbb{E} Q^2(X_v) Q^2(X_v) \left( \frac{\partial^2}{\partial x_\alpha \partial x_\beta} F(v, X_v) \right)^2 dv \right) \frac{1}{2} + 
\left( 4d \sum_{i=1}^{n} \sum_{l=1}^{d} \sum_{k=1}^{d} \int_{t_{i-1}}^{t_{i}} \mathbb{E} \left( \frac{\partial}{\partial x_k} F(u, X_u) - \frac{\partial}{\partial x_k} F(t_{i-1}^{a}, X_{t_{i-1}^{a}}) \right)^2 \sigma_{kl}(X_u)^2 du \right) \frac{1}{2}.
\]

The assertion for the upper bound follows from Proposition 3.3 and Lemma 5.6 in Appendix.

Now we continue with the lower bound of the approximation error. Let $[A, B]$ be a subinterval of $(r, s)$ such that
\[
0 < \frac{(B - A) C_B}{(T - B)^2 g} \leq \frac{C_H}{4},
\]
where $C_H$ is taken from (3.2) and the constant $C_B > 0$ satisfies (cf. (3.1) and (3.14))
\[
d \sum_{k=1}^{d} \mathbb{E} \left( \frac{\partial}{\partial x_k} F(u, X_u) - \frac{\partial}{\partial x_k} F(a, X_a) \right)^2 b_k^2(X_u) \leq C_B \int_{a}^{u} \frac{1}{(T - u)^2 g} dv
\]
for $A \leq a < u \leq B$. Let us now consider the approximation error inside the interval $[A, B]$. Denote $I_n := \{ i : A \leq t_i^{n} \leq t_i^{n-1} \leq B \}$ and denote in both cases for the lower estimate (cf. Theorem 3.1 cases (L1) and (L2)) the sequence of time-nets by $(t_i^{n})_{i=0}^{n}$. Note that for large $n$ the set $I_n$ is not an empty set because, in both cases, we have that $\sup_{i=1,\ldots,n}(t_i^{n} - t_i^{n-1}) \leq C_t/n$. Now on $[A, B]$ we get that
\[
\mathbb{E} \left| \sum_{i \in I_n} \sum_{k=1}^{d} \int_{t_{i-1}^{n}}^{t_i^{n}} \left( \frac{\partial}{\partial x_k} F(u, X_u) - \frac{\partial}{\partial x_k} F(t_{i-1}^{n}, X_{t_{i-1}^{n}}) \right) X_u^k \right|^2 \frac{1}{2}
\geq \mathbb{E} \left| \sum_{i \in I_n} \sum_{l=1}^{d} \sum_{k=1}^{d} \int_{t_{i-1}^{n}}^{t_i^{n}} \left( \frac{\partial}{\partial x_k} F(u, X_u) - \frac{\partial}{\partial x_k} F(t_{i-1}^{n}, X_{t_{i-1}^{n}}) \right) \sigma_{kl}(X_u)dW_u^l \right|^2 \frac{1}{2} - 
\mathbb{E} \left| \sum_{i \in I_n} \sum_{k=1}^{d} \int_{t_{i-1}^{n}}^{t_i^{n}} \left( \frac{\partial}{\partial x_k} F(u, X_u) - \frac{\partial}{\partial x_k} F(t_{i-1}^{n}, X_{t_{i-1}^{n}}) \right) b_k(X_u)du \right|^2 \frac{1}{2}.
\]
This implies that
\[
\mathbb{E} \left| \sum_{i \in I_n} \sum_{k=1}^{d} \int_{t_{i-1}^{n}}^{t_i^{n}} \left( \frac{\partial}{\partial x_k} F(u, X_u) - \frac{\partial}{\partial x_k} F(t_{i-1}^{n}, X_{t_{i-1}^{n}}) \right) X_u^k \right|^2.
\]
Let us now consider the lower bound for

$$\sum_{i \in I_n} \sum_{k=1}^d \int_{t_i}^{t_i^n} \sum_{k=1}^d \left( \frac{\partial}{\partial x_k} F(u, X_u) - \frac{\partial}{\partial x_k} F(t_i^n, X_{t_i^n}) \right) \sigma_k(X_u) dW_u^t \mid^2$$

Now (3.15) and (3.16) give that

$$B_3 := \sum_{i \in I_n} \sum_{k=1}^d \int_{t_i}^{t_i^n} \sum_{k=1}^d \left( \frac{\partial}{\partial x_k} F(u, X_u) - \frac{\partial}{\partial x_k} F(t_i^n, X_{t_i^n}) \right) \sigma_k(X_u) dW_u^t \mid^2$$

Using the Itô-isometry, for $0 \leq a < b < T$, we get that

$$B_3 \geq \int_a^b \int_a^t H^2(u) du dt - D_4 \int_a^b \int_a^t \left( \int_a^u \frac{1}{(T-v)^{2\theta}} dv \right)^\frac{1}{2} \left( \frac{1}{(T-u)^{2\theta}} \right)^\frac{1}{2} du dt$$

Assuming $b-a \leq 1$, Proposition 3.4 together with (3.1) imply that

$$B_3 \geq \int_a^b \int_a^t H^2(u) du dt - b \int_a^b \int_a^t \frac{1}{(T-v)^{2\theta}} dv du dt$$
Considering the multi-step error for the approximation we get that

\[
\mathbb{E} \left\| \sum_{i \in I_n} \sum_{k=1}^{d} \int_{t_{i-1}}^{t_{i}} \left( \frac{\partial}{\partial x_k} F(u,X_u) - \frac{\partial}{\partial x_k} F(t_{i-1}^n, X_{t_{i-1}^n}) \right) dX_u^k \right\|^2 \\
\geq \frac{1}{2} \mathbb{E} \left\| \sum_{i \in I_n} \sum_{k=1}^{d} \int_{t_{i-1}}^{t_{i}} \int_{t_{i-1}}^{t_{i}} \left( \frac{\partial}{\partial x_k} F(u,X_u) - \frac{\partial}{\partial x_k} F(t_{i-1}^n, X_{t_{i-1}^n}) \right) \sigma_k(X_u) dW_u^k \right\|^2 \\
- (B - A) \sum_{i \in I_n} d \sum_{k=1}^{d} \mathbb{E} \int_{t_{i-1}}^{t_{i}} \left( \frac{\partial}{\partial x_k} F(u,X_u) - \frac{\partial}{\partial x_k} F(t_{i-1}^n, X_{t_{i-1}^n}) \right)^2 b_k^2(X_u) du \\
\geq \frac{1}{2} \sum_{i \in I_n} \int_{t_{i-1}}^{t_{i}} \int_{t_{i-1}}^{t_{i}} H^2(u) du dt - C \int_{t_{i-1}}^{t_{i}} \int_{t_{i-1}}^{t_{i}} \frac{\sqrt{t_{i-1}^n - t_{i-1}}}{(T - u)^{2\theta}} du dt \\
- \sum_{i \in I_n} \int_{t_{i-1}}^{t_{i}} \int_{t_{i-1}}^{t_{i}} \frac{H^2(u)}{4} du dt \\
\geq \sum_{i \in I_n} \int_{t_{i-1}}^{t_{i}} (t_{i-1}^n - t) \frac{H^2(t)}{4} dt - C \sqrt{n} \sum_{i \in I_n} \int_{t_{i-1}}^{t_{i}} \frac{1}{(T - u)^{2\theta}} du dt.
\]

In the case (L1) for our lower estimates Remark 6.6 implies that

\[
\frac{1}{\sqrt{n}} \sum_{i \in I_n} \int_{t_{i-1}}^{t_{i}} \int_{t_{i-1}}^{t} \frac{1}{(T - u)^{2\theta}} du dt \leq \frac{C}{\sqrt{n} \ n^{1/2 + \varepsilon}}
\]

for some \( \varepsilon > 0 \). In the case (L2), Lemma 5.6 gives

\[
\frac{1}{\sqrt{n}} \sum_{i \in I_n} \int_{t_{i-1}}^{t_{i}} \int_{t_{i-1}}^{t} \frac{1}{(T - u)^{2\theta}} du dt \leq \frac{C}{\sqrt{n} \ n}
\]

The term containing \( H^2 \) can be bounded from below as

\[
\liminf_{n \to \infty} n \sum_{i \in I_n} \int_{t_{i-1}}^{t_{i}} (t_{i-1}^n - t) \frac{H^2(t)}{4} dt \geq \liminf_{n \to \infty} \frac{C_H}{4} n \sum_{i \in I_n} \frac{(t_{i-1}^n - t_{i-1}^n)^2}{2} \\
\geq \liminf_{n \to \infty} \frac{C_H}{8} \left( \sum_{i \in I_n} (t_{i-1}^n - t_{i-1}^n) \right)^2 \\
= \frac{C_H}{8} (B - A)^2.
\]

This proves the estimate. \( \square \)
4. Examples

In this chapter we give two examples as an application of our results. For simplicity, we consider a diffusion

\[ X^i_t - x^i_0 + \sum_{j=1}^d \int_0^t \sigma_{ij}(X_u) dW^j_u, \quad i = 1, \ldots, d \]

in the case (C2). Let \( 0 = t_0 \leq t_1 \leq \cdots \leq t_n = T \) be any deterministic time-net on \( [0, T] \). By \((t^n_i)_{i=0}^n\) we denote the time-net

\[(t^n_i)_{i=0}^n = \left( T \left( 1 - \left( 1 - \frac{i}{n} \right)^{\frac{1}{\eta}} \right) \right)_{i=0}^n \quad \text{and} \quad \begin{cases} \eta = 0, & \theta \in [0, \frac{1}{2}) \\ \eta \in (2\theta - 1, 1), & \theta \in [\frac{1}{2}, 1) \end{cases} \]

where \( \theta \) is from Theorem 3.1 equation (3.1).

**Example 4.1.** For a European digital option with strike price \( K > 0 \),

\[ f(x) := \mathbb{1}_{\sum_{i=1}^d \lambda_i x_i \geq K}(x), \quad \text{where} \quad \lambda_1, \ldots, \lambda_d > 0, \]

the approximation rate is \( n^{-1/4} \) if equidistant time-nets are used [12, Theorem 2.1]. Our main result Theorem 3.1 gives that this option can be approximated by the rate \( n^{-1/2} \), more precisely

\[
\left( \mathbb{E} \sup_{t \in [0, T]} \left| \sum_{i=1}^n \sum_{k=1}^d \int_{t_{i-1}}^{t_i} \left( \frac{\partial}{\partial x_k} F(u, X_u) - \frac{\partial}{\partial x_k} F(t^n_{i-1}, X_{t_{i-1}}) \right) dX_u^k \right|^2 \right)^{\frac{1}{2}} \leq D_1 \sqrt{n}
\]

and

\[
\frac{1}{D_2} \leq \liminf_{n \to \infty} \sqrt{n}\left( \mathbb{E} \left| \sum_{i=1}^n \sum_{k=1}^d \int_{t_{i-1}}^{t_i} \left( \frac{\partial}{\partial x_k} F(u, X_u) - \frac{\partial}{\partial x_k} F(t^n_{i-1}, X_{t_{i-1}}) \right) dX_u^k \right|^2 \right)^{\frac{1}{2}}
\]

for all \( \eta \in (1/2, 1) \). Assumption (3.1) follows from [12, Proposition 3.3] for \( \theta = 3/4 \) and (3.2) is due to [12, equation (4.2)].

**Example 4.2.** In this example we will show how our main result Theorem 3.1 applies to the case of \( \sigma \) being a \( d \times d \) diagonal matrix with assumption \( \sigma_{ii}(x) = \sigma_{ii}(x_i) \) and \( d \geq 2 \). As a special case of mixing different type of options we also apply Theorem 3.1 to the pay-off function

\[ f(x) := (x_1 - K_1)_+ (x_2 - K_2)_+ \mathbb{1}_{[K_3, \infty)}(x_3) \]

and show that for this pay-off one has the approximation rate \( n^{-1/2} \) when approximation is optimized over deterministic time-nets of cardinality \( n+1 \).
Let us first turn to the case of $d \times d$ diagonal matrix. If $d \geq 2$, $(\sigma_{ij})_{i,j=1}^{d}$ is a diagonal matrix, and if $\sigma_{ii}(x) = \sigma_{i}(x)$, then the transition density of the process $Y$ can be written as the product of the transition densities of the process $Y^i$, i.e.

$$
\Gamma_Y(t, y, \xi) = \prod_{i=1}^{d} \Gamma_{Y^i}(t, y_i, \xi_i).
$$

Assume that $f(x) = \prod_{i=1}^{d} f_i(x_i)$, where the functions $f_i$ are of at most polynomial growth. The definition of $F$ implies that

$$
F(t, x) = \prod_{i=1}^{d} F_i(t, x_i)
$$

with

$$
F_i(t, x_i) := \int_{\mathbb{R}} \Gamma_{Y^i}(T - t, \log x_i, \xi_i) f_i(e^{\xi_i}) d\xi_i,
$$

and the second order derivatives of the function $F$ can be written as

$$
\frac{\partial^2}{\partial x_i \partial x_j} F(t, x) = \left\{ \frac{\partial}{\partial x_i} F_i(t, x_i) \frac{\partial}{\partial x_j} F_j(t, x_j) \prod_{m=1, m \neq i, m \neq j}^{d} F_m(t, x_m) \right\} i \neq j,
$$

$$
\frac{\partial^2}{\partial x_i \partial x_i} F_i(t, x_i) \prod_{m=1, m \neq i}^{d} F_m(t, x_m), \qquad i = j.
$$

Assume that there exist $C > 0$ and $\theta_i \in [0, 1)$ for all $i = 1, \ldots, d$ such that

$$
\mathbb{E} \left[ Q_i^2(X_i) \frac{\partial^2}{\partial x_i \partial x_i} F_i(t, X_i^i) \right]^2 \leq \frac{C}{(T - t)^{2\theta_i}}.
$$

Theorem 3 [Theorem 2.3] gives that

$$
\sup_{t \in [0, T)} (T - t)^{\delta} \sqrt{t} \left( \mathbb{E} \left[ \sigma_{ii}(X_i) \frac{\partial}{\partial x_i} F_i(t, X_i^i) \right]^2 \right)^{\frac{1}{2}} < \infty
$$

if and only if

$$
\sup_{t \in [0, T)} (T - t)^{\delta} \left( \int_{0}^{t} \mathbb{E} \left[ \sigma_{ii}^2(X_u) \frac{\partial^2}{\partial x_i \partial x_i} F_i(u, X_u^i) \right]^2 du \right)^{\frac{1}{2}} < \infty,
$$

where $\delta \in [0, 1/2)$. Now we assume, without loss of generality, that $\theta_i \in (1/2, 1)$. For $\alpha = \beta$ we get that

$$
\mathbb{E} \left[ A_{\alpha\alpha}(X_t) A_{\beta\beta}(X_t) \left( \frac{\partial^2}{\partial x_{\alpha} \partial x_{\beta}} F(t, X_t^i) \right)^2 \right]
$$

$$
= \mathbb{E} \left[ \sigma_{\alpha\alpha}^2(X_t^\alpha) \left| \frac{\partial^2}{\partial x_{\alpha} \partial x_{\alpha}} F(t, X_t^\alpha) \right|^2 \right] \prod_{m=1, m \neq \alpha}^{d} \mathbb{E} |F_m(t, X_t^m)|^2
$$

$$
\leq \mathbb{E} \left[ \sigma_{\alpha\alpha}^2(X_t^\alpha) \left| \frac{\partial^2}{\partial x_{\alpha} \partial x_{\alpha}} F(t, X_t^\alpha) \right|^2 \right] \prod_{m=1, m \neq \alpha}^{d} \mathbb{E} |f_m(X_T^m)|^2
$$
which is at most of order \((T - t)^{2\theta_\alpha}\). For \(\alpha \neq \beta\) we get that
\[
\mathbb{E} \left[ A_{\alpha\alpha}(X_t)A_{\beta\beta}(X_t) \left( \frac{\partial^2}{\partial x_\alpha \partial x_\beta} F(t, X_t) \right)^2 \right] 
\leq \mathbb{E} \left[ \sigma_{\alpha\alpha}(X_t) \left| \frac{\partial}{\partial x_\alpha} F(t, X_t) \right|^2 \right] \mathbb{E} \left[ \sigma_{\beta\beta}(X_t) \left| \frac{\partial}{\partial x_\beta} F(t, X_t) \right|^2 \right] \times 
\prod_{m=1, m \neq \alpha, m \neq \beta}^d \mathbb{E} |f_m(X_T^m)|^2.
\]

The implication (4.3) \(\Rightarrow (4.2)\) implies, for \(\delta_i := (\theta_i - 1/2) \in (0, 1/2)\), that
\[
\sup_{t \in [0, T]} (T - t)^{\delta_i} \sqrt{t} \left( \mathbb{E} \left| \sigma_{ii}(X_t) \frac{\partial}{\partial x_i} F_i(t, X_t) \right|^2 \right)^{1/2} < \infty.
\]
Using [3] Lemma 5.2] one can remove the factor \(\sqrt{t}\), so that
\[
\sup_{t \in [0, T]} (T - t)^{2(\delta_i + \delta_\beta)} \mathbb{E} \left| A_{\alpha\alpha}(X_t)A_{\beta\beta}(X_t) \frac{\partial^2}{\partial x_\alpha \partial x_\beta} F(t, X_t) \right|^2 < \infty.
\]

Putting all estimates together, we find a \(\theta \in [0, 1)\) such that
\[
\mathbb{E} \left[ A_{\alpha\alpha}(X_t)A_{\beta\beta}(X_t) \left( \frac{\partial^2}{\partial x_\alpha \partial x_\beta} F(t, X_t) \right)^2 \right] \leq \frac{D}{(T - t)^{2\theta}}.
\]

Looking at the above computations, one can take \(\theta := \max\{\theta_1, \ldots, \theta_d, 1/2\}\) without the assumption \(\theta_i \in (1/2, 1)\).

Let us go back to our 3-dimensional example of mixing different type of options. Assume that \(\sigma\) is a 3 \(\times\) 3 matrix defined by
\[
\sigma_{ij}(x) = \begin{cases} 
0, & i \neq j, \\
x_i, & i = j,
\end{cases}
\]
and that \(x_0 = (1, 1, 1)\). Define the pay-off function \(f\) as above by
\[
f(x_1, x_2, x_3) = f_1(x_1)f_2(x_2)f_3(x_3) := (x_1 - K_1)_+ (x_2 - K_2)_+ 1_{[K_3, \infty)}(x_3),
\]
where \(K_i > 0, i = 1, 2, 3\) and \(\alpha \in (0, 1/2)\). For \(F_1\) one can compute
\[
\frac{\partial^2}{\partial x_1 x_1} F_1(t, x_1) = \frac{1}{x_1 \sqrt{T - t}} \frac{1}{\sqrt{2\pi}} \exp \left[ -\left( \frac{\log \left( \frac{x_1}{K_1} \right) + \frac{T - t}{\sqrt{2\pi}}}{\sqrt{T - t}} \right)^2 \right]
\]
and
\[
\mathbb{E} \left[ Q_1^2(X_t) \left( \frac{\partial^2}{\partial x_1 x_1} F_1(t, X_t^1) \right)^2 \right] = \frac{K_1}{2 \pi \sqrt{T - t}^2} \exp \left[ -\frac{(T/2 + \log(K_1))^2}{T + t} \right].
\]
This implies that one can choose \( \theta_1 = \frac{1}{4} \). For \( F_2 \), we can choose \( \theta_2 = \frac{3 - 2\alpha}{4} \) and for \( F_3, \theta_3 = \frac{3}{4} \) (cf. \[ \text{Lemma 1 and Lemma 2} \]). Now Theorem \[3.1 \] gives that

\[
\left( \mathbb{E} \sup_{t \in [0,T]} \left| \sum_{i=1}^{d} \sum_{k=1}^{d} \int_{t_{i-1}}^{t_i} \left( \frac{\partial}{\partial x_k} F(u, X_u) - \frac{\partial}{\partial x_k} F(t_{i-1}^{\eta}, X_{t_{i-1}}^{\eta}) \right) dX_u^k \right|^2 \right)^{\frac{1}{2}} \leq \frac{D_1}{\sqrt{n}}
\]

for all \( \eta \in (1/2, 1) \). Under the assumptions of this example we have that

\[
H^2(u) = \mathbb{E} \sum_{\alpha, \beta = 1}^{3} \left( \sigma_{\alpha \beta}(X_u) \frac{\partial^2}{\partial x_\alpha x_\beta} F(u, X_u) \right)^2
\]

\[
\geq \mathbb{E} \sum_{\alpha = 1}^{3} \left( \sigma_{\alpha \alpha}(X_u) \frac{\partial^2}{\partial x_\alpha x_\alpha} F(u, X_u) \right)^2
\]

\[
\geq \frac{1}{C_1} \sum_{\alpha = 1}^{3} \mathbb{E} \left| X_u^\alpha \right|^4 \left| \frac{\partial^2}{\partial x_\alpha x_\alpha} F(u, X_u^\alpha) \right|^2 \prod_{m \neq \alpha} \mathbb{E} \left| F_m(u, X_m^u) \right|^2.
\]

Since \( f_1, f_2 \) and \( f_3 \) are not almost surely linear and since

\[
u \mapsto \mathbb{E} \left| X_u^\alpha \right|^4 \left| \frac{\partial^2}{\partial x_\alpha x_\alpha} F_\alpha(u, X_u^\alpha) \right|^2
\]

is continuous and increasing, \[ \text{[5, proof of Proposition 2.1], [5, Theorem 4.6]} \] implies

\[
\sup_{u \in [0,T]} \mathbb{E} \left| F_\alpha(u, X_u^\alpha) \right|^2 > 0.
\]

Moreover,

\[
\lim_{u \to T} \mathbb{E} \left( F_\alpha(u, X_u^\alpha) \right)^2 = \mathbb{E} f_\alpha^2(X_T^u) > 0.
\]

Hence Theorem \[3.7 \] gives that

\[
\frac{1}{D_2} \leq \lim \inf_{n \to \infty} \sqrt{n} \left( \mathbb{E} \sup_{t \in [0,T]} \left| \sum_{i=1}^{d} \sum_{k=1}^{d} \int_{t_{i-1}}^{t_i} \left( \frac{\partial}{\partial x_k} F(u, X_u) - \frac{\partial}{\partial x_k} F(t_{i-1}^{\eta}, X_{t_{i-1}}^{\eta}) \right) dX_u^k \right|^2 \right)^{\frac{1}{2}}.
\]

If we take \( f(x_1, x_2) = f_1(x_1) f_2(x_2) \), then we can choose \( \theta = \theta_2 < 3/4 \) and get

\[
\frac{1}{D_2} \leq \lim \inf_{n \to \infty} \sqrt{n} \left( \mathbb{E} \sup_{t \in [0,T]} \left| \sum_{i=1}^{n} \sum_{k=1}^{d} \int_{t_{i-1}}^{t_i} \left( \frac{\partial}{\partial x_k} F(u, X_u) - \frac{\partial}{\partial x_k} F(t_{i-1}^{\eta}, X_{t_{i-1}}^{\eta}) \right) dX_u^k \right|^2 \right)^{\frac{1}{2}},
\]

for any sequence of time-nets with \( \sup_{i=1,...,n} (t_i^n - t_{i-1}^n) \leq C/n \).
Theorem 5.1 (Theorem 8. p. 263 in [1], Theorem 5.4. p. 149 in [2]). For \( \hat{b}, \hat{\sigma} \) with \( \hat{\sigma}\hat{\sigma}^T \) uniformly elliptic, there exists a transition density \( \Gamma : (0, T] \times \mathbb{R}^d \times \mathbb{R}^d \to [0, \infty) \) in \( C^\infty \) such that \( \mathbb{P}(Y_t \in B) = \int_B \Gamma(t, y, \xi) d\xi, \) for \( t \in (0, T] \) and \( B \in \mathcal{B}(\mathbb{R}^d) \), where \( Y = (Y_t)_{t \in [0, T]} \) is the strong solution of the SDE (2.2) starting from \( y \): Moreover, the following are satisfied:

(i) For \( (s, y, \xi) \in (0, T] \times \mathbb{R}^d \times \mathbb{R}^d \) one has

\[
\frac{\partial}{\partial s} \Gamma(s, y, \xi) = \frac{1}{2} \sum_{k, l=1}^{d} \sum_{j=1}^{d} \hat{\sigma}_{kj}(y) \hat{\sigma}_{lj}(y) \frac{\partial^2}{\partial y_k \partial y_l} \Gamma(s, y, \xi) + \sum_{i=1}^{d} \hat{b}_i(y) \frac{\partial}{\partial y_i} \Gamma(s, y, \xi).
\]

(ii) For \( a \in \{0, 1, 2, \ldots\} \) and multi-indices \( b \) and \( c \) there exist positive constants \( C \) and \( D \), depending only on \( a, b, c \) and \( d \), such that

\[
\left| \frac{\partial^{a+|b|+|c|}}{\partial t^a \partial y^b \partial \xi^c} \Gamma(t, y, \xi) \right| \leq \frac{C}{t^{(d+2a+|b|+|c|)/2}} e^{-D|y-\xi|^2/t},
\]

where \( ||\cdot|| \) is the Euclidean norm.

Remark 5.2. In references [1] and [2] it is assumed that functions \( \hat{b} \) and \( \hat{\sigma} \) are Lipschitz continuous. Here it follows from the assumption (2.3).

Theorem 5.3 (Gronwall’s Lemma, [11]). If, for \( t_0 \leq t \leq t_1 \), \( \phi(t) \geq 0 \) is a continuous function such that

\[
\phi(t) \leq K + L \int_{t_0}^{t} \phi(s) ds
\]

for \( t_0 \leq t \leq t_1 \) where \( K, L \geq 0 \), then

\[
\phi(t) \leq Ke^{L(t-t_0)}
\]
on \( t_0 \leq t \leq t_1 \).

Lemma 5.4. Let \( 0 \leq a < b < T \) and define

\[
\phi_{kl}(u, x) := \left( \frac{\partial}{\partial x_k} F(u, x) - \nu_a^k \right) \sigma_{kl}(x), \quad u \in [0, T), \quad x \in E,
\]

where \( \nu_a^k \) is an \( \mathcal{F}_a \)-measurable random variable and assume that

\[
\mathbb{E} \sup_{u \in [a, b]} \left[ \phi_{kl}^2(u, X_u) + \left| (A\phi_{kl}^2)(u, X_u) \right| + \sum_{m=1}^{d} \left| \frac{\partial}{\partial x_m} \phi_{kl}^2(u, X_u) b_m(X_u) \right| \right] < \infty.
\]

Then for \( s \in [a, b] \) one has

\[
\mathbb{E}\phi_{kl}^2(s, X_s) = \mathbb{E}\phi_{kl}^2(a, X_a) + \int_a^{s} \mathbb{E} \left( A\phi_{kl}^2 \right)(u, X_u) du + \int_a^{s} \mathbb{E} \sum_{m=1}^{d} \left( \frac{\partial}{\partial x_m} \phi_{kl}^2(u, X_u) \right) b_m(X_u) du.
\]
Proof. By Itô’s formula we obtain
\[
\phi_{kl}^2(s, X_s) = \phi_{kl}^2(a, X_a) + \int_a^s (A\phi_{kl}^2)(u, X_u)du + \sum_{m=1}^d \int_a^s \frac{\partial}{\partial x_m}\phi_{kl}^2(u, X_u)dX^m_u.
\]

Define
\[
S^m_n := \inf \left\{ r \in [a, s] \middle| \int_a^r \sum_{j=1}^d \left[ \frac{\partial}{\partial x_m}\phi_{kl}^2(u, X_u) \right]^2 \sigma_{mj}^2(X_u)du > n \right\} \wedge s
\]
and
\[
S_n := \min \{ S^m_n, m \in \{1, \ldots, d\} \}.
\]
This implies that
\[
\int_a^{S_n} \sum_{j=1}^d \left[ \frac{\partial}{\partial x_m}\phi_{kl}^2(u, X_u) \right]^2 \sigma_{mj}^2(X_u)du \leq n
\]
and
\[
\mathbb{E} \int_a^{S_n} \frac{\partial}{\partial x_m}\phi_{kl}^2(u, X_u)\sigma_{mj}(X_u)dW^j_u = 0,
\]
for \( n \in \mathbb{N}, m \in \{1, \ldots, d\} \) and \( j \in \{1, \ldots, d\} \). Dominated convergence gives
\[
\mathbb{E}\phi_{kl}^2(s, X_s) = \lim_{n \to \infty} \mathbb{E}\phi_{kl}^2(S_n, X_{S_n})
\]
\[
= \lim_{n \to \infty} \mathbb{E}\phi_{kl}^2(a, X_a) + \mathbb{E}\int_a^{S_n}(A\phi_{kl}^2)(u, X_u)du + \mathbb{E}\sum_{m=1}^d \int_a^{S_n} \frac{\partial}{\partial x_m}\phi_{kl}^2(u, X_u)dX^m_u
\]
\[
= \mathbb{E}\phi_{kl}^2(a, X_a) + \mathbb{E}\int_a^{s}(A\phi_{kl}^2)(u, X_u)du + \mathbb{E}\sum_{m=1}^d \int_a^{s} \frac{\partial}{\partial x_m}\phi_{kl}^2(u, X_u)b_m(X_u)du.
\]
\[\square\]

Lemma 5.5. If for all \( x \in E \)
\[
\left| \frac{\partial^r}{\partial x_i \partial x_j^{r}} \sigma_{ij}(x) \right| \leq C \frac{Q_i(x)}{Q^r_\beta(x)Q^r_\alpha(x)}, \quad q + r = s, \quad q, r, s \in \{0, 1, 2\},
\]
for some \( C > 0 \), then for all \( 0 \leq a \leq b < T \) and \( k, l \in \{1, \ldots, d\} \) we have that
\[
\mathbb{E} \sup_{u \in [a, b]} \phi_{kl}^2(u, X_u) < \infty,
\]
\[ \mathbb{E} \sup_{u \in [a, b]} \left| Q_m(X_u) \frac{\partial}{\partial x_m} \phi_{kl}(u, X_u) \right|^2 < \infty, \ m = 1, \ldots, d, \]

\[ \mathbb{E} \sup_{u \in [a, b]} \left| Q_m(X_u) \frac{\partial}{\partial x_m} \phi_{kl}^2(u, X_u) \right| < \infty, \ m = 1, \ldots, d, \]

and

\[ \mathbb{E} \sup_{u \in [a, b]} \left| (A\phi_{kl}^2)(u, X_u) \right| < \infty, \]

where

\[ \phi_{kl}(u, x) = \left( \frac{\partial}{\partial x_k} F(u, x) - \frac{\partial}{\partial x_k} F(a, X_a) \right) \sigma_{kl}(x), \ u \in [a, b]. \]

**Proof.** This proof uses the same notation for different constants. Equation (2.7) implies that the random variable \( \phi_{kl}^2(u, X_u) \) can be bounded by

\[
\phi_{kl}^2(u, X_u) = \left( \frac{\partial}{\partial x_k} F(u, X_u) - \frac{\partial}{\partial x_k} F(a, X_a) \right)^2 \sigma_{kl}(X_u)
\]

\[
\leq C \left( \frac{\partial}{\partial x_k} F(u, X_u) - \frac{\partial}{\partial x_k} F(a, X_a) \right)^2 Q_k^2(X_u)
\]

\[
\leq C \left( \left| Q_k(X_u) \frac{\partial}{\partial x_k} F(u, X_u) \right|^2 + \left| Q_k(X_a) \frac{\partial}{\partial x_k} F(a, X_a) \right|^2 \frac{Q_k^2(X_u)}{Q_k^2(X_a)} \right)
\]

\[
\leq C \left[ \sup_{u' \in [a, b]} (1 + ||X_{u'}||^q)^2 \right] \left( 1 + \frac{Q_k^2(X_u)}{Q_k^2(X_a)} \right).
\]

Applying Hölder’s inequality we get that

\[
\mathbb{E} \sup_{u \in [a, b]} \phi_{kl}^2(u, X_u) \leq C \mathbb{E} \left[ \sup_{u \in [a, b]} (1 + ||X_u||^q)^2 \sup_{u \in [a, b]} \left( 1 + \frac{Q_k^2(X_u)}{Q_k^2(X_a)} \right) \right]
\]

\[
\leq C \left( \mathbb{E} \sup_{u \in [a, b]} (1 + ||X_u||^q)^4 \right)^{\frac{1}{2}} \left( \mathbb{E} \sup_{u \in [a, b]} \left( 1 + \frac{Q_k^2(X_u)}{Q_k^2(X_a)} \right)^2 \right)^{\frac{1}{2}}.
\]

Equation (2.4) gives that \( \mathbb{E} \sup_{u \in [a, b]} (1 + ||X_u||^q)^4 \) is finite. In the case (C1) it is trivial that the latter term is finite. Let us now turn to the case (C2). Theorem 5.1 implies that

\[
(5.1) \quad \mathbb{E}(X_u^k)^{-p} = \mathbb{E} e^{-pX_u^k}
\]

\[
= \int_{\mathbb{R}^d} e^{-py_k} \Gamma_Y(u, y, y) dy
\]

\[
\leq C \exp \left[ -py_0 + \frac{1}{2} \frac{u}{2D} p^2 \right] < \infty
\]
for all \( p \in [0, \infty) \) and some \( C > 0 \) and \( D > 0 \). Hölder’s inequality now implies that

\[
\mathbb{E} \sup_{u \in [a,b]} \left( 1 + \frac{Q_k^2(X_u)}{Q_k^2(X_a)} \right)^2 \leq 2 \left( 1 + \mathbb{E} \sup_{u \in [a,b]} \frac{Q_k^4(X_u)}{Q_k^4(X_a)} \right)
\]

\[
\leq 2 \left( 1 + \left( \mathbb{E} \sup_{u \in [a,b]} (X_u^k)^{\frac{8}{k}} \right)^{\frac{k}{4}} \left( \mathbb{E} \left( \frac{1}{X_a^k} \right)^{\frac{k}{4}} \right)^{\frac{k}{4}} \right),
\]

which is finite by (2.4) and (5.1).

Straightforward calculation gives that

\[
\mathbb{E} \sup_{u \in [a,b]} \left| Q_m(X_u) \left( \frac{\partial}{\partial x_m} \phi_{kl} \right)(u, X_u) \right|^2
\]

\[
= \mathbb{E} \sup_{u \in [a,b]} \left[ Q_m^2(X_u) \left| \frac{\partial^2}{\partial x_m x_k} F(u, X_u) \sigma_{kl}(X_u) \right|^2 + \left( \frac{\partial}{\partial x_k} F(u, X_u) - \frac{\partial}{\partial x_k} F(a, X_a) \right) \left( \frac{\partial}{\partial x_m \sigma_{kl}} (X_u) \right)^2 \right]
\]

\[
\leq C \left( \mathbb{E} \sup_{u \in [a,b]} Q_m^2(X_u) Q_k^2(X_u) \left| \frac{\partial^2}{\partial x_m x_k} F(u, X_u) \right|^2 + \mathbb{E} \sup_{u \in [a,b]} Q_k^2(X_u) \left| \frac{\partial}{\partial x_k} F(u, X_u) - \frac{\partial}{\partial x_k} F(a, X_a) \right|^2 \right),
\]

and this is finite by equations (2.4) and (2.8) and the above argument.

Hölder’s inequality together with the above gives that

\[
\mathbb{E} \sup_{u \in [a,b]} \left| Q_m(X_u) \left( \frac{\partial}{\partial x_m} \phi_{kl}^2 \right)(u, X_u) \right|
\]

\[
= 2 \mathbb{E} \sup_{u \in [a,b]} \left| Q_m(X_u) \phi_{kl}(u, X_u) \left( \frac{\partial}{\partial x_m} \phi_{kl} \right)(u, X_u) \right|
\]

\[
\leq 2 \left( \mathbb{E} \sup_{u \in [a,b]} |\phi_{kl}(u, X_u)|^2 \right)^{\frac{k}{4}} \left( \mathbb{E} \sup_{u \in [a,b]} Q_m^2(X_u) \left| \left( \frac{\partial}{\partial x_m} \phi_{kl} \right)(u, X_u) \right|^{2} \right)^{\frac{k}{4}} < \infty.
\]

For the last part of the proof, equation (3.9) gives that

\[
\mathbb{E} \sup_{u \in [a,b]} \left| (A\phi_{kl})^2(u, X_u) \right| < \infty
\]

if

\[
\mathbb{E} \sup_{u \in [a,b]} (A\phi_{kl})^2(u, X_u) < \infty.
\]

This follows from equation (3.11) and the above arguments. \( \square \)
Lemma 5.6. Assume that a Borel-measurable function \( \varphi : [0, T) \to [0, \infty) \) satisfies
\[
\varphi(u) \leq \frac{C}{(T - u)^\theta}, \quad u \in [0, T),
\]
for some \( C > 0 \) and some \( \theta \in [0, 1) \). Then there exists a constant \( C' > 0 \) such that
\[
\sum_{t_i \in \mathcal{T}_n} \int_{t_{i-1}}^{t_i} \int_{t_{i-1}}^{u} \varphi^2(s) ds du \leq \frac{C'}{n},
\]
where
\[
\tau_n = (t_i^n)^{i=0}_n := \left( T \left( 1 - \left( 1 - \frac{i}{n} \right)^{1-n} \right) \right)^{n} \quad \text{and} \quad \begin{cases} 
\eta = 0, & \theta \in [0, \frac{1}{2}) \\
\eta \in (2\theta - 1, 1), & \theta \in \left[ \frac{1}{2}, 1 \right).
\end{cases}
\]

Proof. Lemma follows from [4, Lemma 4.14 and Proposition 4.16].

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