$n$-H-CLOSED SPACES

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Abstract. In this paper we extend the theory of H-closed extensions of Hausdorff spaces to a class of non-Hausdorff spaces, defined in [2], called $n$-Hausdorff spaces. The notion of H-closed is generalized to an $n$-H-closed space. Known construction for Hausdorff spaces $X$, such as the Katetov H-closed extension $\kappa X$, are generalized to a maximal $n$-H-closed extension denoted by $n-\kappa X$.

Keywords: $n$-Hausdorff spaces; H-closed spaces; Katetov H-closed extension.

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1. Introduction

A Hausdorff space that is closed in every Hausdorff space in which it is embedded is called H-closed - short for Hausdorff-closed. H-closed spaces were introduced in 1924 by Alexandroff and Urysohn [1] and characterized as those Hausdorff spaces in which every open cover has a finite subfamily whose union is dense. It is straightforward to show that every open cover of a space $X$ has a finite subfamily whose union is dense iff for every open filter $F$ on $X$, $aF \neq \emptyset$ iff for every open ultrafilter $U$ on $X$, $aU \neq \emptyset$ (Hausdorff is not needed). Most of the basic properties of H-closed spaces appear in [3].

The theory of H-closed extensions of a non H-closed Hausdorff space $X$ is well developed (see [3]). The Katetov H-closed extension $\kappa X$ defined on the set $X \cup \{U : U$ is a free open ultrafilter on $X\}$ is the projective maximum among all H-closed extensions of $X$. The Fomin extension $\sigma X$ is $\sigma X = (\kappa X)^\#$, where $Y^\#$ is the strict extension defined in §3, is another important $H$-closed extension of $X$.

In 2013, Bonanzinga [2] expanded the property of Hausdorff to $n$-Hausdorff where $n \in \omega$, $n \geq 2$ by defining a space to be $n$-Hausdorff if for each distinct points $x_1, ..., x_n \in X$, there exist open subsets $U_i$ of
X containing \(x_i\) for every \(i = 1, ..., n\) such that \(x_i \in U_i\) and \(\bigcap_{i=1}^{n} U_i = \emptyset\). Clearly, we have Hausdorff when \(n = 2\).

In this study, we develop the theory of \(n\)-H-closed spaces, for every \(n \in \omega, n \geq 2\) that generalizes the theory of H-closed spaces. This theory is developed within the context of \(n\)-Hausdorff spaces, as H-closed spaces are studied within the class of Hausdorff spaces. A maximal \(n\)-H-closed extension \(n\)-\(\kappa X\) is constructed as well as the related Fomin extension \(n\)-\(\sigma X = (n\)-\(\kappa X)^\#\).

In §2, \(n\)-qH-closed spaces are defined and characterized in terms of filters and covering properties, likewise with \(n\)-H-closed spaces. In §3, dense embeddings are constructed for \(n\)-Hausdorff spaces, and in §4 we study maximal extensions, denoted by \(n\)-\(\kappa X\), and look at the partitions of \(n\)-\(\sigma X \setminus X\). We use standard notation as in [3].

2. \(n\)-qH-CLOSED SPACES AND \(n\)-H-CLOSED SPACES

We recall the following.

**Definition 1.** Let \(\mathcal{F}\) be a filter on a space \(X\). The set \(\{\text{cl}_X(F) : F \in \mathcal{F}\}\) is called the adherence of \(\mathcal{F}\) and it is denoted by \(a\mathcal{F}\). The set \(\{G \in \mathcal{P}(X) : G \supseteq F\text{ for some } F \in \mathcal{F}\}\) is denoted by \(\langle \mathcal{F} \rangle\). We say that \(\mathcal{F}\) converges to \(p \in X\) if \(N_p\) (the family of all the neighborhoods of \(p\)) is contained in \(\langle \mathcal{F} \rangle\). Let \(c(\mathcal{F})\) denote the set of all convergent points of \(\mathcal{F}\).

**Definition 2.** [2] Let \(n \in \omega, n \geq 2\). A space \(X\) is \(n\)-Hausdorff if for each distinct points \(x_1, ..., x_n \in X\), there exist open subsets \(U_i\) of \(X\) containing \(x_i\) for every \(i = 1, ..., n\) such that \(x_i \in U_i\) and \(\bigcap_{i=1}^{n} U_i = \emptyset\). Clearly, we have Hausdorff when \(n = 2\).

**Proposition 1.** [2] A space \(X\) is 3-Hausdorff iff \(\Delta_3\) is closed in \(Y \cup \Delta_3\), where \(\Delta_3\) is the diagonal in \(X^3\) and \(Y = \{(x, y, z) : x, y, z\text{ are distinct points of } X\}\).

The following theorem gives some characterizations of \(n\)-Hausdorffness and also represents a generalization of the previous proposition.

**Theorem 1.** Let \(n \in \omega, n \geq 2\), and \(X\) be a space. The following are equivalent:

(a) \(X\) is \(n\)-Hausdorff

(b) if \(\mathcal{U}\) is an open ultrafilter on \(X\), then \(|a\mathcal{U}| \leq n - 1\)
(c) if $\mathcal{F}$ is an open filter base on $X$, $c(\mathcal{F})$ contains at most $n - 1$ points

(d) $\Delta_n$ is closed in $Y \cup \Delta_n$, where $\Delta_n$ is the diagonal in $X^n$ and $Y = \{(x_1, x_2, ..., x_n) : x_1, x_2, ..., x_n$ are distinct points of $X\}$. 

Proof. (a) ⇒ (b): Assume that $|a\mathcal{U}| \geq n$, then by (a), there are open sets $\{U_a : a \in a\mathcal{U}\}$ such that $\cap_{a\in a\mathcal{U}} U_a = \emptyset$. By the ultrafilter property of $\mathcal{U}$, $U_a \in \mathcal{U}$, and $\mathcal{U}$ has the finite intersection property (fip). So, $\cap_{a\in a\mathcal{U}} U_a \neq \emptyset$, a contradiction. Hence, $|a\mathcal{U}| \leq n - 1$.

(b) ⇒ (c): Let $\mathcal{F}$ be an open filter base on $X$. Then $\mathcal{F}$ extends to an open ultrafilter $\mathcal{U}$. By (b), $|a\mathcal{U}| \leq n - 1$. Let $x \in c(\mathcal{F})$. Then $N_x \subseteq \mathcal{F} \subseteq \mathcal{U}$, so $x \in a\mathcal{U}$. Then $|c(\mathcal{F})| \leq |a\mathcal{U}| \leq n - 1$.

(c) ⇒ (d): Let $x_1, x_2, ..., x_n$ be distinct points of $X$. If $N_{x_1} \cup \cdots \cup N_{x_n}$ has the finite intersection property, we have that $N_{x_1} \cup \cdots \cup N_{x_n}$ is a filter converging to $x_1, x_2, ..., x_n$, contradicting (c). So there are $N_i \in N_{x_i}$, where $i \in \{1, ..., n\}$ such that $\cap_{i=1}^n N_i = \emptyset$. Then $N_1 \times ... \times N_n$ is a neighbourhood of $(x_1, ..., x_n)$ and $(N_1 \times ... \times N_n) \cap \Delta_n = \emptyset$.

(d) ⇒ (a): Similar to (c) ⇒ (d). □

Definition 3. Let $n \in \omega$, $n \geq 2$. An extension $Y$ of $X$ is said to be $n$-Hausdorff except for $X$ if for points $p \in Y \setminus X$ and $q_1, ..., q_{n-1} \in Y$, there are open sets $U, V_1, ..., V_{n-1}$ in $Y$ such that $p \in U$ and $q_i \in V_i$, $i = 1, ..., n - 1$ and $U \cap V_1 \cap ... \cap V_{n-1} = \emptyset$.

Theorem 2. Let $n \in \omega$, $n \geq 2$. The following are equivalent for any space $X$:

(a) For every open filter $\mathcal{F}$ on $X$, $|a\mathcal{F}| \geq n-1$;

(b) For every $A \in [X]^{<n-1}$ and for family of open subsets $\mathcal{U}$ of $X$ such that $X \setminus A \subseteq \bigcup \mathcal{U}$, there exists $\mathcal{V} \in \mathcal{U}^{<\omega}$ such that $X = \bigcup_{V \in \mathcal{V}} V$.

(c) $X$ is closed in every extension of $X$ that is $n$-Hausdorff except for $X$.

Proof. (a) ⇒ (b) Let $A \in [X]^{<n-1}$. Suppose there exists $\mathcal{C}$ a family of open subsets of $X$ containing $X \setminus A$ such that for every finite subfamily $\mathcal{V}$ of $\mathcal{C}$, $X \neq \bigcup_{V \in \mathcal{V}} V$. $F = \{U : U$ is open and $U \supseteq X \setminus \bigcup_{V \in \mathcal{V}} V$ for some finite subfamily $\mathcal{V}$ of $\mathcal{C}\}$ is an open filter. As $a\mathcal{F} = \bigcap_{U \in \mathcal{C}} U \subseteq \bigcap_{V \in \mathcal{C}^{<\omega}} X \setminus \bigcup_{V \in \mathcal{V}} V \subseteq \bigcap_{V \in \mathcal{V}^{<\omega}} X \setminus V \subseteq X \setminus \mathcal{C} = X \setminus (X \setminus A) = A$, we have $|a\mathcal{F}| < n-1$. A contradiction.
Proposition 2. Clearly we have: 

Definition 5. one (and hence all) of the conditions of Theorem 2.

is open in \( X \) is Hausdorff (hence

Proposition 3. \( (c) \) Assume every filter \( F \) has \( |\alpha F| \geq n - 1 \). By the way of contradiction, let \( Y \) be an extension of \( X \) such that \( X \neq \text{cl}_Y(X) \) and \( Y \) is \( n \)-Hausdorff except for \( X \). Let \( p \in \text{cl}_Y(X) \setminus X \). Then \( F = \{ U \cap X : p \in U \in \tau(Y) \} \) is an open filter on \( X \). Extend \( F \) to an open ultrafilter \( U \). Then \( |\text{cl} U| \geq n - 1 \). Let \( x_1, \ldots, x_{n-1} \in \text{cl} U \). As \( Y \) is \( n \)-Hausdorff except for \( X \), there exists open sets in \( Y U_i, U \) such that \( x_i \in U_i, i = 1, \ldots, n - 1, p \in U \) and \( \bigcap_{i=1}^{n} U_i \cap U = \emptyset \). We can observe that \( X \cap U_i \in U \) for every \( i = 1, \ldots, n - 1 \) and \( U \cap X \in U \), then \( \bigcap_{i=1}^{n-1} (X \cap U_i) \cap (U \cap X) \in U \). Therefore \( \bigcap_{i=1}^{n-1} U_i \cap U \neq \emptyset \); a contradiction.

(a) \( \Rightarrow \) (c) Assume every filter \( F \) has \( |\alpha F| \geq n - 1 \). By the way of contradiction, let \( Y \) be an extension of \( X \) such that \( X \neq \text{cl}_Y(X) \) and \( Y \) is \( n \)-Hausdorff except for \( X \). Let \( p \in \text{cl}_Y(X) \setminus X \). Then \( F = \{ U \cap X : p \in U \in \tau(Y) \} \) is an open filter on \( X \). Extend \( F \) to an open ultrafilter \( U \). Then \( |\text{cl} U| \geq n - 1 \). Let \( x_1, \ldots, x_{n-1} \in \text{cl} U \). As \( Y \) is \( n \)-Hausdorff except for \( X \), there exists open sets in \( Y U_i, U \) such that \( x_i \in U_i, i = 1, \ldots, n - 1, p \in U \) and \( \bigcap_{i=1}^{n} U_i \cap U = \emptyset \). We can observe that \( X \cap U_i \in U \) for every \( i = 1, \ldots, n - 1 \) and \( U \cap X \in U \), then \( \bigcap_{i=1}^{n-1} (X \cap U_i) \cap (U \cap X) \in U \). Therefore \( \bigcap_{i=1}^{n-1} U_i \cap U \neq \emptyset \); a contradiction.

(c) \( \Rightarrow \) (a) Let \( F \) be an open filter such that \( |\alpha F| < n - 1 \), consider the extension \( Y = X \cup \{ \emptyset \} \), and define a set \( U \subseteq Y \) to be open if \( U \cap X \) is open in \( X \) and \( F \in U \) implies \( U \cap X \in F \). It is easy to prove that \( Y \) is Hausdorff (hence \( n \)-Hausdorff) except for \( X \) but \( X \) is not closed in \( Y \) (see also Proposition 4.8 (b) in [3].)

Definition 4. Let \( n \in \omega, n \geq 2 \). A space is \( n \)-qH-closed if \( X \) satisfies one (and hence all) of the conditions of Theorem 2.

Definition 5. 2-qH-closed spaces are called qH-closed spaces.

Clearly we have:

Proposition 2. Let \( n \in \omega, n \geq 2 \). Every \( (n+1) \)-qH-closed space is \( n \)-qH-closed.

We can notice the following behaviour of \( n \)-qH-closedness with respect to regular closed subsets.

Proposition 3. Let \( n \in \omega, n \geq 2 \). If \( X \) is a \( n \)-qH-closed space and \( U \) is an open subset of \( X \) then \( \text{cl}_X(U) \) is \( n \)-qH-closed.

Proof. We put \( A = \text{cl}_X(U) \) and let \( F \) be an open filter of \( A \). As \( \mathcal{G} = \{ W \subseteq X : W \text{ is an open subset of } X, W \supseteq F \cap U, \forall F \in F \} \) is an open filter on \( X \), we have \( |\alpha X \mathcal{G}| \geq n - 1 \). We have \( A X \mathcal{G} \subseteq A A F \) then \( |\alpha A F| \geq n - 1 \). By Theorem 2 \( A \) is \( n \)-qH-closed.

The following represents a generalization of the concept of H-closedness.
Definition 6. Let $n \in \omega$, $n \geq 2$. An $n$-Hausdorff space $X$ is called $n$-H-closed if $X$ is closed in every $n$-Hausdorff space $Y$ in which $X$ is embedded.

Clearly a space $X$ is 2-H-closed iff $X$ is H-closed.

Obviously, for every $n \in \omega$, $n \geq 2$, every $n$-Hausdorff space is $n + 1$-Hausdorff. In Example 4 in \cite{2}, a $(n + 1)$-Hausdorff non $n$-Hausdorff space is given, for every $n \in \omega$. Also we have the following

Example 1. An H-closed non 3-H-closed space.

Let $Y = 3 \cup (\omega \times 3)$, where $3 = \{0, 1, 2\}$, be the space topologized as follows: all points from $\omega \times 3$ are isolated; a basic neighborhood of $i \in 3$ takes the form $U(i, N) = \{i\} \cup \{(m, j) : j \neq i, m \geq N\}$, where $N \in \omega$. $Y$ is 3-Hausdorff non Hausdorff. Consider the subspace $X = 2 \cup (\omega \times 2) \subset Y$. $X$ is compact and Hausdorff, then H-closed but $X$ is not closed in $Y$.

The following is easy to verify.

Proposition 4. An $n$-Hausdorff space $X$ is $n$-H-closed iff $X$ is closed in every $n$-Hausdorff space $Y$ in which $X$ is embedded where $|Y \setminus X| = 1$.

Now we consider some characterizations of $n$-H-closed spaces.

Theorem 3. Let $n \in \omega$, $n \geq 2$. For a $n$-Hausdorff space $X$ the following are equivalent

(a) $X$ is $n$-H-closed

(b) for each open ultrafilter $\mathcal{U}$ on $X$, $|a(\mathcal{U})| = n - 1$

(c) $X$ is $n$-qH-closed

(d) for every $\mathcal{F}_1, \ldots, \mathcal{F}_{n-1}$ open filters on $X$ such that $\bigcup_{i=1}^{n-1} \mathcal{F}_i$ has the finite intersection property, we have $|a(\mathcal{F}_1 \lor \ldots \lor \mathcal{F}_{n-1})| \geq n - 1$

Proof. (b) $\Rightarrow$ (a) Suppose for each open ultrafilter $\mathcal{U}$ on $X$, $|a(\mathcal{U})| = n - 1$, and $X$ is embedded in an $n$-Hausdorff space $Y$ such that $Y \setminus X = \{p\}$. Assume $p \in cl_Y X$. Let $\mathcal{B}_p = \{U \cap X : p \in U \in \tau(Y)\}$ and $\mathcal{U}_p$ be an open ultrafilter in $X$ containing $\mathcal{B}_p$. By hypothesis, $|a_X(\mathcal{U}_p)| = n - 1$. Let $\mathcal{W}_p$ be the unique open ultrafilter in $cl_Y(X)$ containing $\mathcal{U}_p$ and $T \in \mathcal{W}_p$. If $p \in V \in \tau(Y)$, then $V \cap X \in \mathcal{B}_p$ and $(V \cap X) \cap (T \cap X) \neq \emptyset$ for each $T \in \mathcal{W}_p$. Thus, $V \cap T \neq \emptyset$. This shows that $p \in a(\mathcal{W}_p)$ and $|a(\mathcal{W}_p)| = n$, a contradiction.
(a) ⇒ (b) Suppose \( X \) is \( n \)-H-closed and assume there is an open ultrafilter \( \mathcal{U} \) on \( X \) such that \( |a\mathcal{U}| < n-1 \). Let \( Y = X \cup \{p\} \) where \( p \notin X \).

We define the topology on \( Y \) by \( U \) is open in \( Y \) if \( U \cap X \) is open in \( X \) and if \( p \in U \), then \( U \cap X \) is also open in \( Y \). Thus, \( p \in cl_Y(X) \). To obtain a contradiction, we will show that \( Y \) is \( n \)-Hausdorff. Let \( A \subseteq Y \) such that \( |A| = n \).

If \( A \subseteq X \), there are open sets \( U_a \) in \( X \) for each \( a \in A \) such that \( \bigcap_{a \in A} U_a = \emptyset \) and as \( X \) is open in \( Y \), \( A \) is also open in \( Y \). Otherwise, \( p \in A \) and \( |A \setminus \{p\}| = n-1 \). As \( |a\mathcal{U}| < n-1 \), there is \( q \in V \) and \( V \cap U = \emptyset \). Both \( V \) and \( \{p\} \cup U \) are disjoint open sets in \( Y \) and \( \{p,q\} \subseteq A \). This completes the proof that \( Y \) is \( n \)-Hausdorff, a contradiction.

(a) ⇒ (c) Let \( F \) be an open filter on \( X \). As \( F \) is contained in some open ultrafilter \( \mathcal{U} \) on \( X \), we have \( aF \supseteq a\mathcal{U} \) and by (a) we have \( |aF| \geq n-1 \).

(c) ⇒ (a) Let \( \mathcal{U} \) be an open ultrafilter on \( X \), by (b) we have \( |\mathcal{U}| \geq n-1 \). By Theorem 1, \( |\mathcal{U}| \leq n-1 \) and \( |\mathcal{U}| = n-1 \).

(b) ⇒ (d) The open filter \( F_1 \vee \ldots \vee F_{n-1} \) is contained in an open ultrafilter \( \mathcal{U} \) and \( |a\mathcal{U}| = n-1 \) by (b). As \( |a(F_1 \vee \ldots \vee F_{n-1})| \geq |a\mathcal{U}| \), it follows that \( |a(F_1 \vee \ldots \vee F_{n-1})| \geq n-1 \).

(d) ⇒ (c) is obvious. \( \square \)

Note that the space \( X \) of Example 1 is an H-closed not 3-H-closed space. We can also observe that an H-closed space \( X \) is not \( n \)-H-closed for \( n > 2 \). As every open ultrafilter on an \( n \)-H-closed space \( X \) has \( n-1 \) adherence points but in an H-closed space, every open ultrafilter has an unique adherent point.

**Remark 1.** Let \( n \in \omega, n \geq 2 \). For a \( n \)-Hausdorff space \( X \), an open ultrafilter \( \mathcal{U} \) on \( X \) is said to be full if \( |a\mathcal{U}| = n-1 \). So, by Theorem 1, a \( n \)-Hausdorff space \( X \) is \( n \)-H-closed iff every open ultrafilter on \( X \) is full. Consider the simple space \( X = \omega \cup \{a,b\} \) where a set \( U \subseteq X \) is defined to be open if \( a \in U \) or \( b \in U \), then \( U \cap \omega \) is cofinite in \( \omega \), and each point \( n \in \omega \) is isolated. The compact space \( X \) is \( 3 \)-Hausdorff but not \( 3 \)-H-closed for if \( \mathcal{U} \) is an open ultrafilter containing the open set \( \{1\} \), then \( a\mathcal{U} = \{1\} \) and \( \mathcal{U} \) is not full.

3. **Embeddings**

In 1924, Alexandroff and Urysohn asked if every Hausdorff space can be embedded densely in an H-closed space. Katětov and Stone answered
Let $Y$ be an extension of a space $X$. For $p \in Y$, let $O^p = \{ U \cap X : p \in U \in \tau(Y) \}$ and for $U \in \tau(X)$, let $oU = \{ p \in Y : U \in O^p \}$. Note that for $U, V \in \tau(X)$, $o(U \cap V) = oU \cap oV$, $o(\emptyset) = \emptyset$, and $oX = Y$. So, $\{ oU : U \in \tau(X) \}$ forms a basis for a topology, denoted as $\tau^+(Y)$, on $Y$. Denote by $\tau^+(Y)$ the topology on $Y$ generated by the base $B = \{ U \cap \{ p \} : U \in O^p$ and $p \in Y \}$. We have $\tau^+(Y) \subseteq \tau(Y) \subseteq \tau^+(Y)$, and $Y^\#$ (resp. $Y^+$) is used to denote the set $Y$ with $\tau^+(Y)$ (resp. $\tau^+(Y)$). $Y^+$ is called simple extension of the space $X$ and $Y^\#$ is called strict extension of the space $X$.

**Proposition 5.** Let $Y$ be an extension of a space $X$. Then $Y^+$ and $Y^\#$ are extensions of $X$ with the following properties:

(a) $(Y^+)^+ = Y^+$ and $(Y^\#)^+ = Y^\#$;

(b) $Y$ is qH-closed iff $Y^+$ is qH-closed iff $Y^\#$ is qH-closed;

(c) $Y$ is Hausdorff iff $Y^+$ is Hausdorff if $Y^\#$ is Hausdorff;

(d) If $U \in \tau(X)$, $cl_Y U = cl_{Y^+} U = cl_{Y^\#} U$;

(e) If $p \in Y$, $O^p_Y = O^p_{Y^+} = O^p_{Y^\#}$;

(f) If $\sigma$ is a topology on the set $Y$, $(Y, \sigma)$ is an extension of $X$, and $O^p_Y = O^p_{Y^\#}$ for all $p \in Y$, then $\sigma = \tau^+(Y)$;

(g) The subspace $Y^+ \setminus X$ is discrete.

**Proposition 6.** Let $Y$ be an extension of a space $X$, $\mathcal{U}$ an open ultrafilter on $X$, and $\mathcal{U}^e = \{ V \in \tau(Y) : V \cap X \in \mathcal{U} \}$. Then:

(a) $\mathcal{U}^e$ is an open ultrafilter on $Y$,

(b) if $\mathcal{V}$ is an open ultrafilter on $Y$ such that $\{ V \cap X : V \in \mathcal{V} \} \subseteq \mathcal{U}$, then $\mathcal{U}^e = \mathcal{V}$ and $\mathcal{V} = \bigcap V cl_Y U \supseteq \mathcal{U}$.

**Proposition 7.** Let $n \in \omega$, $n \geq 2$, and $Y$ be a $n$-Hausdorff extension of a space $X$. Then both the simple extension $Y^+$ and strict extension $Y^\#$ of $X$ are $n$-Hausdorff as well as $X$. 

This (see page 307 in [3]). We ask a similar question: can every n-Hausdorff space be densely embedded in an n-H-closed space? We will use open ultrafilters in answering this question in the affirmative. We start the construction of an extension of $X$ by adding one open ultrafilter that is not full. This is a modification of the proof of (a) $\iff$ (b) in Theorem 3.
Proof. As $\tau(Y) \subseteq \tau(Y^+)$, it is immediate that $Y^+$ is also $n$-Hausdorff. Likewise, it is easy to verify that $X$ is also $n$-Hausdorff. To show that $Y^\#$ is $n$-Hausdorff, let $A \subseteq Y^\#$ such that $|A| = n$. As $Y$ is $n$-Hausdorff extension, there is a family of open sets $\{V_a : a \in A\}$ such that $a \in V_a$ and $\bigcap_{a \in A} V_a = \emptyset$. So, $\bigcap_{a \in A} (V_a \cap X) = \emptyset$. Thus, $\emptyset = o(\emptyset) = o(\bigcap_{a \in A} (V_a \cap X)) = \bigcap_{a \in A} o(V_a \cap X)$. This shows that $Y^\#$ is $n$-Hausdorff. \hfill \Box

Proposition 8. Let $n \in \omega$, $n \geq 2$. If $Y$ is an extension of $X$, then $Y$ is $n$-H-closed iff $Y^\#$ is $n$-H-closed.

Proof. This is an application of Proposition\[\ref{prop:extension-properties}\], Proposition\[\ref{prop:n-h-closed}\] and Proposition\[\ref{prop:n-h-closed-extension}\].\hfill \Box

Extension Construction Technique 1. Let $n \in \omega$, $n \geq 2$, and $X$ be an $n$-Hausdorff space and $\mathcal{U}$ an open ultrafilter on $X$ such that $|\mathcal{U}| < n - 1$. Let $k = n - 1 - |\mathcal{U}|$, and $Y = X \cup \{p_1, p_2, \ldots, p_k\}$ where $\{p_1, p_2, \ldots, p_k\} \cap X = \emptyset$. A set $V$ is defined to be open in $Y$ if $V \cap X$ is open in $X$ and if $p_i \in V$ for $1 \leq i \leq k$, $V \cap X \in \mathcal{U}$. Now, $Y \setminus X = \{p_1, p_2, \ldots, p_k\}$, and a basic open set containing $p_i$ is $\{p_i\} \cup T$ where $T \in \mathcal{U}$. It is straightforward to verify the following result using the technique developed in the proof of Theorem\[\ref{thm:n-h-closed-extension}\].

Lemma 1. Let $n \in \omega$, $n \geq 2$, $X$ be an $n$-Hausdorff space and $Y$ the space defined in the above construction. The space $Y$ is an $n$-Hausdorff space that contains $X$ as a dense subspace and if $\mathcal{V}$ is the unique open ultrafilter on $Y$ containing $\mathcal{U}$, $|\mathcal{V}| = n - 1$. $\mathcal{W}$ is an open ultrafilter on $Y$ iff $\mathcal{W} = \mathcal{V}$ or $\mathcal{W}$ is an open ultrafilter on $X$ other than $\mathcal{U}$.

Extension Construction Technique 2. Let $n \in \omega$, $n \geq 2$, $X$ be an $n$-Hausdorff space and $\mathcal{U} = \{\mathcal{U} : \mathcal{U}$ is an open ultrafilter such that $|\mathcal{U}| < n - 1\}$. That is, $\mathcal{U}$ is the set of open ultrafilters on $X$ that are not full. We indexed $\mathcal{U}$ by $\mathcal{U} = \{\mathcal{U}_\alpha : \alpha \in |\mathcal{U}|\}$. For each $\alpha \in |\mathcal{U}|$, let $k\alpha = n - 1 - |\mathcal{U}_\alpha|$ and $\{p_{\alpha i} : 1 \leq i \leq k\alpha\}$ a set of distinct points disjoint from $X$. Let $Y = X \cup \{p_{\alpha i} : 1 \leq i \leq k\alpha, \alpha \in |\mathcal{U}|\}$. A set $V$ is defined to be open in $Y$ if $V \cap X$ is open in $X$ and if $p_{\alpha i} \in V$ for $1 \leq i \leq k\alpha$, $V \cap X \in \mathcal{U}_\alpha$. The space $Y$ is an extension of $X$ such that $Y$ is an $n$-Hausdorff space in which every open ultrafilter on $Y$ is full. That is, $Y$ is $n$-H-closed. These properties are summarized in the following result.

Theorem 4. Let $n \in \omega$, $n \geq 2$. An $n$-Hausdorff space can be densely embedded in an $n$-H-closed space.
In the next result, we present the basic extension properties of $O^p$ and $oU$ and $cl_Y U$ for $U \in \tau(X)$ for this particular extension $Y$.

**Proposition 9.** Let $n \in \omega$, $n \geq 2$, $X$ be $n$-Hausdorff space and $Y$ the construction of an $n$-H-closed simple extension of $X$. Then:

(a) For $p = p_{a\alpha} \in Y \setminus X$, $O^{p_{a\alpha}} = \{ V \cap X : p \in V \in \tau(Y) \} = \mathcal{U}_\alpha$;

(b) For $p \in X$, $O^p = \{ V \cap X : p \in V \in \tau(Y) \} = \{ U \in \tau(X) : p \in U \}$;

(c) For $U \in \tau(X)$, $oU = \{ p \in Y : U \in O^p \} = U \cup \{ p_{a\alpha} : U \in \mathcal{U}_\alpha, 1 \leq i \leq k\alpha \}$;

(d) For $U \in \tau(X)$, $cl_Y U = cl_X U \cup oU$.

**Proof.** The proof of (a), (b), and (c) are straightforward and left to the reader. To show (d), let $p_{a\alpha} \in cl_Y U \setminus X$. Then $V \cap U \neq \emptyset$ for each $V \in \mathcal{U}_\alpha$ implying $U \in \mathcal{U}_\alpha$ and that $p_{a\alpha} \in oU$ by (a) and (c). Note that $cl_Y U \cap X = cl_X U$. Thus, $cl_Y U \subseteq cl_X U \cup oU$. Clearly, $cl_X U \subseteq cl_Y U$ and by (a) and (c), $oU \subseteq cl_Y U$. This completes the proof of (d). \qed

**Remark 2.** Let $n \in \omega$, $n \geq 2$. The above construction of an $n$-H-closed extension works whenever $X$ is $n$-Hausdorff. Consider the example of a 3-Hausdorff space $X = \omega \cup \{ a, b \}$ described in Remark 1 that is not 3-H-closed. Notice that every open ultrafilter on $X$ containing only infinite subsets is an open ultrafilter $\mathcal{U}$ on $\omega$ and $a\mathcal{U} = \{ a, b \}$. The only open ultrafilters remaining are those containing a single point set like $\{ n \}$. To construct a 3-H-closed space $Y$ that contains $X$ as a dense subspace let $Y = X \cup \{ c_n : n \in \omega \}$. A set $U \subseteq Y$ is defined to be open if $U \cap X$ is open in $X$ and if $c_n \in U$, then $n \in U$. The space $Y$ is 3-Hausdorff, and $X$ is dense in $Y$. Let $U$ be an open ultrafilter on $X$ and $V = \{ U : U$ is open in $Y$ and $U \cap X \in \mathcal{U} \}$ the unique open ultrafilter on $Y$ that contains $\mathcal{U}$. If $U$ is infinite for each $U \in \mathcal{U}$, then $\{ a, b \} = aV$. If some $U$ is finite for $U \in \mathcal{U}$, then there is an unique $n \in \omega$ such that $U = \{ n \}$, and it follows that $\{ c_n, n \} = aV$. Thus, $Y$ is a 3-H-closed space. The space $Y$ provides an example that shows the size of $a\mathcal{F}$ can be quite large. For $n \in \omega$, the open ultrafilter $\mathcal{U}_n = \{ U \in \tau(Y) : n \in U \}$ has the property that $|a\mathcal{U}_n| = 2$. However, the open filter $\mathcal{F} = \bigcap_{1 \leq i \leq n} \mathcal{U}_n$ on $Y$ has the property that $|a\mathcal{F}| = 2n$. This shows that the adherence of an open filter $\mathcal{F}$ in an $n$-H-closed space can be large.
4. Theory of $n$-H-closed Extensions.

**Theorem 5.** Let $n \in \omega$, $n \geq 2$, $X$ be $n$-Hausdorff space and $Y$ the simple, $n$-H-closed extension of $X$ constructed in Extension Construction Technique 2. If $Z$ is an $n$-H-closed extension of $X$, there is a continuous surjection $f : Y \to Z$ such that $f(x) = x$ for all $x \in X$.

**Proof.** Let $\hat{u} = \{ \mathcal{U} : \mathcal{U} \text{ is an open ultrafilter on } X \text{ such that } |a| < n - 1 \}$. For $\mathcal{U} \in \hat{u}$, the collection $\mathcal{V}_\mathcal{U} = \{ V \in \tau(Z) : V \cap X \in \mathcal{U} \}$ is an open ultrafilter on $Z$. Note that $a_Z \mathcal{V}_\mathcal{U} \setminus X = a_Z \mathcal{V}_\mathcal{U} \setminus a_X \mathcal{U} = \{ p_1, p_2, \ldots, p_k \}$ where $k = n - 1 - |a_X \mathcal{U}|$ and $Z = X \cup \bigcup_{u \in \hat{u}} a_Z \mathcal{V}_\mathcal{U} \setminus a_X \mathcal{U}$.

For $\mathcal{U} \in \hat{u}$, let $\mathcal{V}_\mathcal{U} = \{ V' \in \tau(Y) : V' \cap X \in \mathcal{U} \}$ be an open ultrafilter on $Y$. The set $a_Y \mathcal{V}_\mathcal{U} \setminus a_X \mathcal{U} = \{ q_1, q_2, \ldots, q_k \}$ where $k = n - 1 - |a_X \mathcal{U}|$. Note that $Y = X \cup \bigcup_{u \in \hat{u}} a_Y \mathcal{V}_\mathcal{U} \setminus a_X \mathcal{U}$.

Define $f_{\mathcal{U}}(q_i) = p_i$ for $1 \leq i \leq k$; $f_{\mathcal{U}}$ is a bijection. Define $f : Y \to Z$ as follows: for $x \in X$, $f(x) = x$ and for $q_i \in \mathcal{V}_\mathcal{U}$, $f(q_i) = f_{\mathcal{U}}(q_i) = p_i$. The function $f$ is onto, $f(x) = x$ for $x \in X$, but not necessarily one-to-one. As $X$ is open in $Y$, $f$ is continuous for $x \in X$. For $\mathcal{U} \in \hat{u}$ and $q_i \in a_Y \mathcal{V}_\mathcal{U} \setminus a_X \mathcal{U}$, $f(q_i) = f_{\mathcal{U}}(q_i) = p_i$; let $p_i \in V$ for some $V$ open in $Z$. Then $V \cap X \in \mathcal{U}$ and $f([q_i] \cup (V \cap X)) \subseteq V$. As $\{ q_i \} \cup (V \cap X)$ is open in $Y$, it follows that $f$ is continuous at $q_i$. □

**Remark 3.** Let $n \in \omega$, $n \geq 2$, $S$ and $T$ be $n$-H-closed extensions of an $n$-Hausdorff space $X$. We say $S$ is **projectively larger** than $T$ if there is a continuous surjection $f : S \to T$ such that $f(x) = x$ for $x \in X$. This projectively larger function may not be unique.

The proof of Theorem 5 shows that the $n$-H-closed extension $Y$ of $X$ is projectively larger than every $n$-H-closed extension of $X$. The space $Y$ from Theorem 5 has an interesting uniqueness property as noted in the next result.

**Theorem 6.** Let $n \in \omega$, $n \geq 2$, $X$ be an $n$-Hausdorff space and $Y$ the $n$-H-closed extension of $X$ described in Theorem 5. Let $f : Y \to Y$ be a continuous surjection such that $f(x) = x$ for all $x \in X$. Then $f$ is a homeomorphism.

**Proof.** From Theorem 5 we have that $Y = X \cup \bigcup_{u \in \hat{u}} a_Y \mathcal{V}_\mathcal{U} \setminus a_X \mathcal{U}$ where if $\mathcal{S}$, $\mathcal{T} \in \hat{u}$ are distinct open ultrafilters on $X$, then $(a_Y \mathcal{V}_\mathcal{U} \setminus a_X \mathcal{S}) \cap (a_Y \mathcal{V}_\mathcal{T} \setminus a_X \mathcal{T}) = \emptyset$. Let $\mathcal{U} \in \mathcal{P}$ and $\mathcal{V}$ be the unique open ultrafilter on $Y$ such that $\mathcal{U} \subseteq \mathcal{V}$. Then $f[a \mathcal{V}] = f[\bigcap_{V \in \mathcal{V}} \operatorname{cl}_Y V] \subseteq \bigcap_{V \in \mathcal{V}} f[\operatorname{cl}_Y V] = \bigcap_{V \in \mathcal{V}} f[\operatorname{cl}_Y (V \cap X)] \subseteq \bigcap_{V \in \mathcal{V}} \operatorname{cl}_Y f[V \cap X] = \bigcap_{V \in \mathcal{V}} \operatorname{cl}_Y V = a \mathcal{V}$. Now, $f[a \mathcal{V}] = f[a \mathcal{V}] \cap a \mathcal{U} = f[a \mathcal{V}] \cap a \mathcal{U} \subseteq a \mathcal{V} \cap a \mathcal{U} \subseteq a \mathcal{V} \cap a \mathcal{U} \cap a \mathcal{U}$. If $\mathcal{T} \in \hat{u}$,
$\mathcal{T} \neq \mathcal{U}$, and $\mathcal{V}$ is the unique open ultrafilter on $Y$ such that $\mathcal{T} \subseteq \mathcal{V}$, then $f[a\mathcal{V}] \subseteq a\mathcal{V}\setminus a\mathcal{T} \cup a\mathcal{T}$. But $a\mathcal{V}\setminus a\mathcal{T} \cap a\mathcal{V}\setminus a\mathcal{T} = \emptyset$. Thus, if $f$ is onto, $f[a\mathcal{V}\setminus a\mathcal{T}] = a\mathcal{V}\setminus a\mathcal{T}$. Since $a\mathcal{V}\setminus a\mathcal{T}$ is finite, then $f|_{a\mathcal{V}\setminus a\mathcal{T}} : a\mathcal{V}\setminus a\mathcal{T} \to a\mathcal{V}\setminus a\mathcal{T}$ is also one-to-one. This completes the proof that $f$ is a bijection. To show that $f$ is open first note that for an open set $V \subseteq X$, $f[V] = V$ is open. Let $p \in a\mathcal{V}\setminus a\mathcal{T}$ and $p \in V \in \mathcal{V}$. Then $\{p\} \cup V \cap X$ is a basic open set in $Y$ containing $p$. Now, $f[\{p\} \cup V \cap X] = \{f(p)\} \cup f[V \cap X] = \{f(p)\} \cup (V \cap X)$ is a basic open set in $Y$ containing $f(p)$. This completes the proof that $f$ is open.

Remark 4. In the setting of Hausdorff spaces, projective maximums and projectively larger functions (defined in [3]) are unique. Sometimes this is a problem in non-Hausdorff spaces. From Theorem 6 we see that a form of uniqueness for the class of $n$-H-closed extensions is possible. We extend the definition of projective maximum in [3] as follows: Let $\mathcal{E}(X)$ be a collection of extensions of a space $X$. We say $Y \in \mathcal{E}(X)$ is a projective maximum of $X$ if $Y$ is projective larger than each $Z \in \mathcal{E}(X)$ and if $f : Y \to Y$ is a continuous surjection such that $f(x) = x$ for each $x \in X$, then $f$ is a homeomorphism. By Theorem 6 the $n$-H-closed space $Y$ constructed in Theorem 5 for a $n$-Hausdorff space $X$ is a projective maximum. We denote this projective maximum by $n$-$\kappa X$ and call it the $n$-Katětov extension of $X$.

4.1. Fomin H-closed Extension. Let $n \in \omega$, $n \geq 2$, $X$ be a $n$-Hausdorff space and $Y$ denote $n$-$\kappa X$ from Theorem 5. In the setting of Hausdorff spaces, the combination of the Katětov and Fomin extensions provide the major support for the theory of $H$-closed extensions. In the class of $n$-Hausdorff spaces, the Fomin extension is defined as $Y^\#$ and denoted as $n$-$\sigma X$. By Proposition 8 $n$-$\sigma X$ is $H$-closed, and by Proposition 5(f), the identity function $id : n$-$\kappa X \to n$-$\sigma X$ is a continuous bijection such that $id(x) = x$ for all $x \in X$. Recall that for spaces $S, T$, a function $f : S \to T$ is $\theta$-continuous if for each $p \in S$ and open set $V \in \tau(T)$ such that $f(p) \in V$, there is an open set $U \in \tau(S)$ such that $f[cl_S U] \subseteq cl_T V$. By Proposition 5(d), it follows that $id : n$-$\sigma X \to n$-$\kappa X$ is $\theta$-continuous. If $Z$ is an $n$-$H$-closed extension of $X$, there is a continuous surjection $f : n$-$\kappa X \to Z$ such that $f(x) = x$ for $x \in X$. So, the composition $f \circ id : n$-$\sigma X \to Z$ is $\theta$-continuous, onto, and $(f \circ id)^{-1}[Z \setminus X] = n$-$\sigma X \setminus X$. The next result presents some interesting properties of $n$-$\sigma X \setminus X$.

Theorem 7. Let $n \in \omega$, $n \geq 2$, $X$ be an $n$-Hausdorff space and $Y^\#$ denote the Fomin extension $n$-$\sigma X$ of $X$. Then:

(a) For $U \in \tau(X)$, $cl_{Y^\#} o U = cl_{Y^\#} U = cl_X U \cup o U$ and $cl_{Y^\#} U \setminus X =$
Proof. For (a), let $U \in \tau(X)$. Then $\text{cl}_{Y^\#} U = \text{cl}_Y U = \text{cl}_X U \cup oU$ by Propositions \[d\] and \[d\]. (b) and (c) follow from (a). To show (d), let $\{o(U_a) : a \in A\}$ be an open cover of $K$. As $K$ is closed in $n$-$\sigma X$, there is a family of basic open sets $\{o(U_b) : b \in B\}$ such that $o(U_b) \cap K = \emptyset$ for each $b \in B$. As $n$-$\sigma X$ is $n$-H-closed by Proposition \[h\] then $n$-$\sigma X$ is $qH$-closed by Theorem \[i\]. There are finite families $A' \subseteq A$ and $B' \subseteq B$ such that $n$-$\sigma X = \bigcup_{a \in A'} \text{cl}_{Y^\#} o(U_a) \cup \bigcup_{b \in B'} \text{cl}_{Y^\#} o(U_b) = \bigcup_{a \in A'} \text{cl}_X U_a \cup o(U_a) \cup \bigcup_{b \in B'} \text{cl}_X U_b \cup o(U_b)$. As $K \subseteq n$-$\sigma X \setminus X$, we have that $K \subseteq \bigcup_{a \in A'} \text{cl}_X o(U_a)$. To show (e), first note that $X \setminus A$ is open and dense in $X$. So, $o(X \setminus A) = (X \setminus A) \cup n$-$\sigma X \setminus X$, and $n$-$\sigma X \setminus o(X \setminus A) = n$-$\sigma X \setminus (X \setminus A) \cap n$-$\sigma X \setminus (n$-$\sigma X \setminus X) = n$-$\sigma X \setminus (X \setminus A) \cap X = A$. \(\square\)

Remark 5. Let $n \in \omega$, $n \geq 2$, and $X$ be an $n$-Hausdorff space. By Theorem 9(c), the space $n$-$\sigma X \setminus X$ is completely regular (Tychonoff without being Hausdorff) in addition to being zero-dimensional. In fact, by Theorem 9(b), $n$-$\sigma X \setminus X$ is close to being extremally disconnected for if $V \in \tau(Y^\#)$, $\text{cl}_{Y^\#} X$ is clopen in $Y^\# \setminus X$.

4.2. Partitions of $n$-$\sigma X \setminus X$. The theory of H-closed extensions starts with these two results that distinguish it from the theory of Hausdorff compactifications.

Let $X$ be a Hausdorff space and $Z$ an H-closed extension of $X$. Let $f : \kappa X \rightarrow Z$ be the continuous surjection such that $f(x) = x$ for $x \in X$. From \[j\] we have the following facts:

(a) Then $\mathcal{P} = \{f^{-1}(p) : p \in Z \setminus X\}$ is a partition of $\sigma X \setminus X$ into nonempty compact spaces.

(b) If $\mathcal{P}$ is a partition of $\sigma X \setminus X$ into nonempty compact spaces, there is an H-closed extension $Z$ of $X$ such that if $f : \kappa X \rightarrow Z$ such that $f(x) = x$ for $x \in X$, then $\mathcal{P} = \{f^{-1}(p) : p \in Z \setminus X\}$.

The question is whether this can be done with $n$-H-closed extensions. We give a partial answer to this problem considering the $n$-H-closed extensions that are Hausdorff except for $X$. The following Lemma \[k\] Theorems \[k\] and \[l\] show that it works in this very restricted setting.
Lemma 2. Let $n \in \omega$, $n \geq 2$, and $Z$ be an extension of a space $X$.

(a) If $U$ and $V$ are open in $Z$, then $U \subseteq o(U \cap X)$ and if $U \cap V = \emptyset$, then $o(U \cap X) \cap o(V \cap X) = \emptyset$.

(b) If $Z$ is Hausdorff except for $X$, then $Z^\#$ is Hausdorff except for $X$.

(c) If $Z$ is Hausdorff except for $X$ and $\mathcal{U}$ is an open ultrafilter on $X$, then $|a_\mathcal{U} \setminus a_X \mathcal{U}| \leq 1$.

(d) If $Z$ is Hausdorff except for $X$ and $Z$ is $n$-$H$-closed, then $n$-$\kappa X$ is Hausdorff except for $X$.

Proof. The proof of (a) is straightforward. For (b), let $p, q \in Z \setminus X$. There are open sets $U, V$ such that $p \in U$, $q \in V$, and $U \cap V = \emptyset$. Then $U \subseteq o(U \cap X)$, $V \subseteq o(V \cap X)$, $p \in o(U \cap X)$, $q \in o(V \cap X)$, and $o(U \cap X) \cap o(V \cap X) = \emptyset$. Similar proof for $p \in Z \setminus X$.

(c) If $p, q \in a_\mathcal{U} \setminus a_X \mathcal{U}$, there are disjoint open sets $U, V$ in $Z$ such that $p \in U$ and $q \in V$. Then $U \cap X, V \cap X \in \mathcal{U}$, a contradiction as $U \cap V = \emptyset$.

(d) Let $p, q \in n$-$\kappa X \setminus X$. Suppose $p, q \in \mathcal{U}$, an open ultrafilter on $X$. Then $a_X \mathcal{U} \leq n - 3$. However, by (c), $a_X \mathcal{U} \geq n - 2$, a contradiction. So, $p \in a_X \mathcal{U}$ and $q \in a_X \mathcal{U}$ where $\mathcal{U}$ and $\mathcal{V}$ are distinct. There are $U \in \mathcal{U}$ and $V \in \mathcal{V}$ such that $U \cap V = \emptyset$. Thus, points of $n$-$\kappa X \setminus X$ can be separated by disjoint open sets. For a point $p \in n$-$\kappa X \setminus X$ and a point $x \in X$, $f(p)$ and $x$ can be separated by disjoint open sets. As $f$ is continuous, it follows that $p$ and $x$ can be separated by disjoint open sets.

Definition 7. A partition $\mathcal{P}$ of a subset of a space $X$ is said to be Hausdorff if $A, B \in \mathcal{P}$, $A \neq B$, there are open sets $U, V$ in $X$ such that $A \subseteq U$, $B \subseteq V$, and $U \cap V = \emptyset$.

Theorem 8. Let $n \in \omega$, $n \geq 2$, and $Z$ be an $n$-$H$-closed extension of a space $X$ that is Hausdorff except for $X$ and $f : n$-$\kappa X \to Z$ a continuous surjection such that $f(x) = x$. Then $\mathcal{P} = \{f^{-1}(p) : p \in Z \setminus X\}$ is a Hausdorff partition of compact subsets of $n$-$\sigma X \setminus X$ with the property that each $f^{-1}(p)$ in $\mathcal{P}$ is Hausdorff separated from each point in $X$.

Proof. Fix $p \in Z \setminus X$, and let $K = f^{-1}(p)$. First, we show that $K$ is closed in $n$-$\sigma X$. Let $r \in n$-$\sigma X \setminus K$. As $Z$ is Hausdorff except for $X$, there are open sets $U, V \in \tau(Z)$ such that $f(r) \in V$ and $p \in U$ and $U \cap V = \emptyset$. As $f$ is $\theta$-continuous, there is an open set $T \in \tau(n$-$\sigma X)$ such that $f[cl(T)] \subseteq cl(V)$ and $r \in T$. As $p \in U$, $f(r) \in f[cl(T)] \subseteq cl(V)$ and $U \cap cl(V) = \emptyset$, $cl(T) \cap K = \emptyset$. A similar proof works when $r \in X$.

This shows that $K$ is closed in $n$-$\sigma X$ and that $K$ is Hausdorff separated
from each point in $X$. By Theorem 7(e), $K$ is compact as well as closed in $n\sigma X$. \hfill \square

**Theorem 9.** Let $n \in \omega$, $n \geq 2$, and $X$ be an $n$-Hausdorff space. Let $\mathcal{P}$ be a Hausdorff partition of compact subsets of $n\sigma X \setminus X$ that are closed in $n\sigma X$ and Hausdorff separated from each point of $X$.

There is an $n$-H-closed extension $Z$ of $X$ that is Hausdorff except for $X$ and a continuous surjection $f : n\sigma X \to Z$ such that $f(x) = x$ and $\mathcal{P} = \{f^{-1}(p) : p \in Z \setminus X\}$.

**Proof.** Let $\mathcal{P} = \{P_a : a \in A\}$ and $Z = X \cup A$ where $X \cap A = \emptyset$. Define $f : n\sigma X \to Z$ by $f(x) = x$ for $x \in X$ and if $p \in P_a$, $f(p) = a$. The function $f$ is onto.

We define $U \subseteq Z$ to be open if $U \cap X$ is open in $X$ and if $f(p) = a \in U$, $\{p\} \cup (U \cap X)$ is open in $n\sigma X$ for each $p \in P_a$. Note that $Z$ is a simple extension of $X$, $f$ is continuous, and if $V$ is open in $X$ and $\{p\} \cup (V)$ is open in $n\sigma X$ for each $p \in P_a$, $\{a\} \cup V$ is open in $Z$. Conversely, if $\{a\} \cup V$ is open in $Z$, then $\{p\} \cup (V)$ is open in $n\sigma X$ for each $p \in f^{-1}(a)$.

Next, we show that $Z$ is Hausdorff except for $X$: Let $a, b \in Z \setminus X$. There are open sets $U, V$ in $n\sigma X$ such that $P_a \subseteq U$, $P_b \subseteq V$, and $U \cap V = \emptyset$. Then $\{a\} \cup U \cap X$ and $\{b\} \cup V \cap X$ are open and disjoint in $Z$. A similar proof shows that $a \in A$ and $x \in X$ can be separated by disjoint disjoint open sets in $Z$ using the hypothesis that $P_a$ and $x$ can be separated by disjoint open sets in $n\sigma X$.

Next, we will show that $Z$ is $n$-Hausdorff. Let $A \subseteq Z$ such that $|A| = n$. If $A \setminus X \neq \emptyset$, then there is a $a \in A \setminus X$. Now $a$ can be separated by disjoint open sets from any point in $A \setminus X \cup \{a\}$ and any point in $X$.

The only remaining case is when $A \subseteq X$. As $X$ is $n$-Hausdorff, there are open sets $\{U_a : a \in A\}$ in $X$ such that $a \in U_a$ and $\bigcap_{a \in A} U_a = \emptyset$.

Finally, we will show that $Z$ is $n$-H-closed. Let $\mathcal{U}$ be an open ultrafilter on $Z$, by Theorem 3 it suffices to show that $|aZ\mathcal{U}| = n - 1$. Let $\mathcal{V} = \{U \cap X : U \in \mathcal{U}\}$ is an open ultrafilter on $X$. Then $|aZ\mathcal{U}| = |aZ\mathcal{V}|$. The goal is to show that $|a_{n\sigma X}\mathcal{V}| = |a\mathcal{V}|$, more precisely, that $|a_{n\sigma X}\mathcal{V}\setminus a_X\mathcal{V}| = |aZ\mathcal{V}\setminus a_X\mathcal{V}|$. Let $p \in a_{n\sigma X}\mathcal{V}\setminus a_X\mathcal{V}$ and $f(p) = a \in Z \setminus X$. As $p \in a_{n\sigma X}\mathcal{V}\setminus a_X\mathcal{V}$, $O_P^n_{n\sigma X} = \mathcal{V}$. As $f(p) = a$ and $f$ is continuous, $O^a_Z \subseteq O_P^n_{n\sigma X} = \mathcal{V}$. Thus, $f(p) = a \in a\mathcal{V}\setminus a_X\mathcal{V}$. Conversely, suppose $a \in a\mathcal{V}\setminus a_X\mathcal{V}$. As $Z$ is $n$-Hausdorff, $|a\mathcal{V}| \leq n - 1$. Thus $|a_X\mathcal{V}| \leq n - 2$. But, $|a_{n\sigma X}\mathcal{V}| = n - 1$. So, there is some $p \in a_{n\sigma X}\mathcal{V}\setminus a_X\mathcal{V}$. \hfill \square
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REFERENCES

[1] P. Alexandroff and P. Urysohn, Zur theorie der topologischen Räume, Math Ann 92(1924), 258-262.
[2] M. Bonanzinga, On the Hausdorff Number of a Topological Space, Houston J. Math 39 (2013), 1013-1030.
[3] J. Porter and R. Woods, Extensions and Absolutes of Hausdorff Spaces, Springer, Berlin (1988).

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