ABSTRACT

In this paper we present new ideas to accelerate the computation of the eigenvector of the transition matrix associated to the PageRank algorithm. New ideas are based on the decomposition of the matrix-vector product that can be seen as a fluid diffusion model, associated to new algebraic equations. We show through experimentations on synthetic data and on real data-sets how much this approach can improve the computation efficiency.

1. INTRODUCTION

In this paper, we are interested by the computation issue of the solution of the PageRank equation. PageRank is a link analysis algorithm that been initially introduced by [24] and used by the Google Internet search engine, that assigns a numerical value to each element of a hyper-linked set of nodes, such as the World Wide Web. The algorithm may be applied to any collection of entities (nodes) that are linked through directional relationships. The numerical value that it assigns to each node is called the PageRank of the node and as we will see below the rank value of the node is associated to a eigenvector problem.

The complexity of the problem for computing the eigenvector of a matrix increases rapidly with the dimension of the vector space. Efficient, accurate methods to compute eigenvalues and eigenvectors of arbitrary matrices are in general a difficult problem (cf. power iteration [26], QR algorithm [11, 18]).

In the particular case of PageRank equation, several specific solutions were proposed and analyzed [20, 21, 23], including power method [21] with adaptation [15] or extrapolation [12], iterative aggregation/disaggregation method [21, 23], adaptive on-line method [11], etc.

The approach proposed here is an improvement idea partially inspired from the algorithm proposed in [1]. We also proposed an algebraic proof of Lemma 2.3 in [1]. This approach can be also compared to the Gauss-Seidel iteration (cf. [25]): the Gauss-Seidel iteration is known to be faster than the Jacobi iteration (cf. [2]). However the approach can not be distributed because of the constraint in the order of the iteration (17). In our approach, the computation of each iteration uses the elements of the previous integrating the last update per component as with Gauss-Seidel method, as opposed to the power iteration (or to Jacobi method) where the computation of the whole vector is based on the previous vector, without taking into account the update (or partial update) at vector entry level.

As we will show below, our approach has the advantage of being iteration order independent and can be very naturally deployed using an asynchronous distributed computation.

Finally, if one can associate the Gauss-Seidel method to an operation of collection (one entry of vector is updated based on the previous vector based on the incoming links), our approach consists in an operation of diffusion (the fluid diffusion from one entry of the vector consists in updating all children nodes following the outgoing links) (cf. Figure 1).

The model is introduced in Section 2. Some theoretical results are presented in Section 3. Finally we compared the computation cost of our approach to existing methods through several data-sets in Section 4.

2. MODEL

2.1 Notations
We consider a non-negative matrix $P$ of size $N \times N$ such that each column sums up to one. In particular, we can associate such a matrix to a Markov chain on a state of size $N$, where $P$ would be the transition matrix with $P_{ij} = p(i,j)$ the transition probability from state $j$ to state $i$. In the following, we will also call $i$ as a node (from web graph and PageRank context).

The fact that each column of $P$ sums up to one means that in the Markov chain, each state $j$ has a positive probability to jump to at least one state $i$ (if such a condition is not true, we complete the matrix $P$ by replacing the zero columns with $1/N$ (by the personalization vector $V$ in a general case) to get $\mathcal{P}$. This corresponds to the dangling nodes completion [20], we will see below that such an adaptation is not required in our approach).

In this paper, we consider the iteration of equation of the form:

$$X_{n+1} = A.X_n$$

(1)

where $A$ is a matrix of size $N \times N$ which can be explicitly decomposed as:

$$A = dP + (1 - d)V.1^t$$

(2)

where $V$ is a normalized vector of size $N$ and $1^t$ is the column vector with all components equal to one. The equation (2) is an explicit Doeblin’s decomposition [9] [10]. So we have:

$$X_{n+1} = dPX_n + (1 - d)V.$$  

We assume that the stationary probability of $A$ is defined by the vector $X$ (its existence and uniqueness is straightforward e.g. from the contraction property of $A$). We have:

$$X = dPX + (1 - d)V.$$  

In the context of PageRank, the vector $V$ is a personalized initial condition cf. [20].

### 2.2 Main equations

We first define the vector $S_n$ by:

$$S_n = (1 - d) \sum_{k=0}^{n} d^k P^k V$$

So that:

$$S_\infty = (1 - d) \sum_{k=0}^{\infty} d^k P^k V = (1 - d)(I - dP)^{-1}V = X$$

The equation $S_n$ has been used in [7] to define formulae for PageRank derivatives of any order.

We define $J_k$ a matrix with all entries equal to zero except for the $k$-th diagonal term: $(J_k)_{kk} = 1$.

In the following, we assume given a deterministic or random sequence $I = \{i_1, i_2, \ldots, i_n, \ldots\}$ with $i_n \in \{1, \ldots, N\}$. We only require that the number of occurrence of each value $k \in \{1, \ldots, N\}$ in $I$ to be infinity.

Then, we define the fluid vector $F$ associated to $I$ by:

$$F_0 = (1 - d) V$$

$$F_n = dP J_{i_n} F_{n-1} + \sum_{k \neq i_n} J_k F_{n-1}$$

(4)

$$= (I_d - J_{i_n} + dP J_{i_n}) F_{n-1}$$

(5)

where $I_d$ is the identity matrix.

We define the history vector $H$ by:

$$H_n = \sum_{k=1}^{n} J_{i_k} F_{k-1}.$$  

(6)

By definition, $H_n$ is an increasing function (all components are positive).

The above fluid and history vector is associated to the following algorithm (ALGO-REF):

**Initialization:**

$H[i] := 0$;

$F[i] := (1-d)v_i$;

$R := 1-d$;

$k := 1$;

While ( $R/(1-d) > \text{Target\_Error}$ )

Choose $i_k$;

sent := $F[i_k]$;

$F[i_k] := 0$;

$H[i_k] += \text{sent}$;

For all child node $j$ of $i_k$:

$F[j] += \text{sent} \times p(j,i_k) \times d$;

$R -= \text{sent} \times (1-d)$;

$k++$;

**Remark 1.** If $d = 1$, the above algorithm is equal to the one defined in [3]. The role of $d$ is here important: in [1], the stopping condition is not defined, because in their algorithm, what they called cash (our fluid $F$) is not decreasing but constant. Also, because in our algorithm, the total fluid tends to zero, we can predict and control precisely the convergence to the limit, as we will see below.

### 3. THEORETICAL RESULTS

#### 3.1 Main equations

**Theorem 1.** We have the equality:

$$H_n + F_n = F_0 + dPH_n.$$  

(7)

**Proof.** The proof is straightforward by induction: assuming the equation (7) true for $n$:

$$H_{n+1} + F_{n+1} = H_n + J_{i_{n+1}} F_n + \sum_{k \neq i_{n+1}} J_k F_n + dP J_{i_{n+1}} F_n$$

$$= H_n + F_n + dP J_{i_{n+1}} F_n$$

$$= F_0 + dPH_n + dP J_{i_{n+1}} F_n$$

$$= F_0 + dPH_n$$

Note that the equation (7) is true for $d = 1$ with $F_0$ replaced by any initial vector.

**Theorem 2.** We have the equality:

$$S_\infty - H_n = \sum_{k=0}^{\infty} d^k P^k F_n = (I - dP)^{-1} F_n$$

(8)

and as a direct consequence, we have:

$$|S_\infty - H_n| = \frac{|F_n|}{1 - d}$$

(9)

where $\cdot$ is the $L_1$ norm.
This formulation may be directly exploited for an efficient computation of the sequence \( H_n \) (so of \( H_n \)). The equation (8) is obvious remarking that \( |P^k F_0| = |F_0| \).

Now, we assume that the sequence \( I \) is chosen such that at iteration \( n \): \( i_n = \arg \max_i (F_{n-1})_i \). Then we have the following result:

**Lemma 1.** If \( i_n = \arg \max_i (F_{n-1})_i \), we have:

\[
|F_n| \leq |F_{n-1}| \left(1 - \frac{1-d}{N}\right).
\]

**Proof.** The proof is straightforward, noticing that we suppressed \( J_{i_n} F_{n-1} \) and added \( J_{i_n} F_{n-1} \times d \). Then using \( J_{i_n} F_{n-1} \geq |F_{n-1}|/N \).

Thanks to Lemma 1 we have an exponential convergence to zero of \( F_n \). This lemma still holds if for \( i_n \) an entry of \( F_{n-1} \) which is larger or equal to \( |F_{n-1}|/N \) is chosen.

As a consequence of this lemma, we have:

\[
H_\infty = F_0 + dPH_\infty.
\]

Therefore, \( H_\infty = X \) and \( H_n \) is an increasing (component by component) function to \( X \).

**Remark 2.** From equations (9) and (10), we can obtain the following iteration equation on \( H \):

\[
H_n = (I_d - J_{i_n} (I_d - dP)) H_{n-1} + J_{i_n} (1-d)V.
\]

This formulation may be directly exploited for an efficient distributed computation of \( H_\infty \). This issue will be addressed in a future paper.

If \( d = 1 \), we still have:

\[
H_n = (I_d - J_{i_n} (I_d - P)) H_{n-1} + J_{i_n} X_0
\]

\[
X_n = PX_{n-1},
\]

and \( H_n \) converges to \( \sum_{k=0}^{\infty} X_k = \sum_{k=0}^{\infty} P^k X_0 \) after appropriate normalization (case of (7)), at least when all entries of \( P \) are non-negative (when the spectral radius of \( P \) is one and if there are negative entries in \( P \), it may converge or not converge); but this convergence is based on a Cesaro averaging which is known to be very slow (cf. results in Section 3).

In all cases, \( H_n \) can be simply interpreted as a specific way to compute or to estimate the power series \( \sum_{k=0}^{\infty} P^k X_0 \) (when there is convergence).

### 3.2 Updating equation

The fact that \( H_n \) converges to \( S_\infty \) for any arbitrary choice of the sequence \( I \) can be also exploited to compute more efficiently the new eigenvector in case of the graph modification (so of \( P \)).

**Theorem 3.** Assume the initial graph associated to \((P, H_\infty)\) is modified to the updated graph represented by \((P', H'_\infty)\), then we have:

\[
H'_\infty - H_\infty = (1 - dP')^{-1} d(P' - P) H_\infty.
\]

**Proof.** We have:

\[
(1 - dP')(H'_\infty - H_\infty) = F_0 - (1 - dP') H_\infty = F_0 - (1 - dP) H_\infty + d(P' - P) H_\infty = d(P' - P) H_\infty.
\]

In the above updating equation, \((P' - P) H_\infty\) may mix positive and negative terms. We can apply on them the operator \( F_\alpha \) separately or jointly.

Now, assume the equation \( H_\infty \) has been computed up to the iteration \( n_0 \) and that at that time, we are interested to compute the limit associated to \( P' \) (for instance, because the web graph has been modified/updated).

Then very naturally, we can apply our diffusion method with \( P' \), but with modified initial condition \( F'_0 = F_{n_0} + d(P' - P) H_\alpha \) for which we have:

\[
H'_\alpha + F'_0 = F_0 + dP'H_\alpha.
\]

We have the following very intuitive results:

**Theorem 4.** \( H_{n_0} + F'_\alpha \) is the limit of the equation (10) is the solution of the equation:

\[
X = dP' X + F_0.
\]

**Proof.** The limit of (10) satisfies:

\[
H'_\infty = F_0 + dP'H_\infty = F_{n_0} + d(P' - P) H_\alpha + dP'H_\alpha.
\]

Combining this with \( H_{n_0} + F_{n_0} = F_0 + dPH_\alpha \), we have obviously what we want.

The above result implies that one can continuously update the iterations when \( P \) is regularly updated by just injecting in the system a fluid quantity equal to \( d(P' - P) H_\alpha \) and then applying the new matrix \( P' \).

If a distributed computation approach was to be used, we just need to synchronize the time from which \( P' \) is applied.

### 3.3 About dangling nodes

In the algorithm ALGO-REF, we don’t need to complete the null columns of the matrix \( P \). We can simply add the following condition:

While ( \( R/(1-d) > \) Target_Error )

Choose \( i_k \);

sent := F\[i_k\];

\( H[i_k] += \) sent;

\( F[i_k] := 0; \)

If ( \( i_k \) has no child )

\( R -= \) sent;

else

For all child node \( j \) of \( i_k \):

\( F[j] += \) sent*P\( (j,i_k) \)*d;

\( R -= \) sent*(1-d);

++k;

The quantity \( R/(1-d) \) measures exactly (thanks to Theorem 2) the distance to the stationary probability. However when \( P \) includes dangling nodes, it is easy to see that \( R/(1-d) \) defines only an upper bound and that \( H_\infty \) need to be renormalized (dividing by the norm \( L_1 \) of \( H_\infty \)) to find the probability eigenvector satisfying the PageRank equation with the completed matrix \( \overline{P} \).
3.4 Asynchronous distributed computation

The proposed algorithm is very well suited for the asynchronous distributed computation (e.g. cf. [23]). Indeed, at any moment of the iteration, the fluid \( F_n \) can be split in any number of elements and be distributed per element, the \( L_1 \) norm of each element controlling exactly the error that can be reduced. The most natural way is to divide per component \( (F_n)_i \) of the vector, and an obvious particular example is to split the initial condition vector \( X_0 \) in \( N/m \) elements of size \( m \).

4. COMPARISON RESULTS

In the following, we will call:

- ALGO-MAX: if \( I \) is defined by \( i_n = \arg\max_i (F_{n-1}) \);
- ALGO-RAND: if \( i_n \) is randomly chosen from \( \{1, ..., N\} \);
- ALGO-PER: if \( i_n = n \mod N \) (periodic cycle);
- MAT-ITER: if the matrix product iteration [1] is used;
- OPIC: the one defined in [1] (Random version).

The stopping condition for ALGO-REF variants are based on \( R/(1-d) > \text{Target\_Error} \). For MAT-ITER, we use the well known condition: \( |X_n - X_{n-1}| \times d/(1-d) < \text{Target\_Error} \). For OPIC, we measured the convergence by comparing the distance to the precomputed limit (from MAT-ITER), since it does not define any stopping condition.

Below we set a simple simulation scenario to get a first evaluation of our proposed solution and comparison to existing solutions in the context of the original PageRank on the web graph. We don’t pretend to generate any realistic model, for more details on the web graph the readers may refer to [8, 19, 3, 4, 22].

4.1 Simulation scenario

We set \( N \) the total number of nodes (URLs) to be simulated. Then we create \( L \) random links (directional) to connect a node \( i \) to \( j \) as follow:

- the choice of the source node is done following a power-law: \( 1/k^\alpha \);
- the choice of the destination node is done following a power-law: \( 1/k^\alpha \).

For simplicity, we assumed no correlation between the number of incoming and outgoing links: for that purpose, to defined the distribution of the destination nodes by associating to node \( k \) a probability proportional to \( 1/k^\alpha \) followed by a large number (by default \( N \)) of permutations of randomly chosen pair of nodes \( (i, j) \). Then, to define the distribution of the destination nodes, we did the same operation. In this way, the order of the nodes does not introduce any bias for ALGO-PER.

The tables below summarize the characteristics of the 6 synthetic data we considered varying the number of links per node and the parameter \( \alpha \).

4.2 Simulation results analysis

We define the elementary step as an operation requiring the use of one non-zero entry of \( A \): for instance, if \( A \) has \( L \) entries that are not zero, the product \( AX \) would require \( L \) elementary steps. We already mentioned that ALGO-REF variants does not require the matrix completion \( P \). For MAT-ITER, we can also avoid such a completion, but the cost to pay is the vector renormalization. Therefore, we assumed here that the cost of \( AX \) in \( L \) is the number of not zero entries of the initial matrix \( P \).

For ALGO-RAND and ALGO-PER, when we have no fluid on the node \( i \) \( (F_{n_0} = 0) \), we go to the next step without incrementing the counter of the elementary steps.

For ALGO-MAX, there is of course a computation cost to find the arg max of \( F_n \). In order to show its full potential, its cost is not taken into account in the number of elementary steps. In the comparison tests, we found that taking the argmax is not necessary and some improvements were obtained replacing the argmax by iteration process where in one iteration all \( i \) such that \( (F_n)_i \) is above a threshold are all chosen, then scaling down the threshold progressively.

Figure 2, 3 and 4 shows the results for S1, S2 and S3. In these scenarios, because of \( \alpha \), there are the proportion of the dangling nodes are important. We can observe in all cases a substantial gain with our approach.

4.3 Comparison on web graph

| Scenario | L (nb links) | D (Nb dangling nodes) |
|----------|-------------|-----------------------|
| S1       | 2172        | 9552                  |
| S2       | 8081        | 8646                  |
| S3       | 28507       | 6252                  |

Table 1: Parameters: \( \alpha = 2.0, N = 10000 \)

| Scenario | L (nb links) | Nb dangling nodes |
|----------|-------------|-------------------|
| S1b      | 12624       | 7696              |
| S2b      | 61189       | 3197              |
| S3b      | 265245      | 33                |

Table 2: Parameters: \( \alpha = 1.5, N = 10000 \)

Figure 2: Scenario S1. \( L/N = 0.22 \), \( D/N = 0.96 \).

Figure 3 and 4 shows the results for S1b, S2b and S3b. In these scenarios, the proportion of the dangling nodes are less important. We can still observe in all cases a substantial gain with our approach.
Figure 3: Scenario S2. $L/N = 0.81$, $D/N = 0.86$.

Figure 4: Scenario S3. $L/N = 2.9$, $D/N = 0.62$.

Figure 5: Scenario S1b. $L/N = 1.3$, $D/N = 0.77$.

Figure 6: Scenario S2b. $L/N = 6.1$, $D/N = 0.32$.

Figure 7: Scenario S3b. $L/N = 26.5$, $D/N = 0.003$. 
In this section, we realized the same comparison tests than above on a web graph imported from the dataset gr0.California (available on http://www.cs.cornell.edu/Courses/cs685/2002fa/) and from the dataset uk-2007-05@1000000 (available on http://law.dsi.unimi.it/datasets.php).

The results for the dataset gr0.California is shown in Figure 8. In this figure, we added ALGO-OP a variant of ALGO-MAX where the argmax is replaced by

\[ i_n = \arg \max_i \left( \frac{F_{n-1} - 1}{\text{#out}_i + 1} \right) \]

with \#in (resp. \#out) equal to the number of incoming (resp. outgoing) links to (resp. from) the node \( i \). The intuition of this renormalization is:

- \#out: the cost of the diffusion of the fluid \( F \) is proportional to \#out;
- \#in: when there are many incoming links, it is worth to wait and aggregate the fluid before the fluid diffusion.

We see that we still have a significant gain and that ALGO-OP outperforms ALGO-MAX. The main reason for which the proposed approach outperforms greatly the original OPIC algorithm is the fact that we don’t have the fluctuation due to the Cesaro averaging as in OPIC (which converges as 1/sqrt(n)).

5. CONCLUSION

In this paper, we proposed a new algorithm to optimize the computation cost of the eigenvector of PageRank equation and showed the theoretical potential gain. We added here ALGO-OP2 based on:

\[ i_n = \arg \max_i \left( \frac{F_{n-1} - 1}{\text{#out}_i + 1} \right) \]

Figure 8: Dataset gr0.California: 16150 links on 9664 nodes (4637 dangling nodes).

Figure 9 shows the results for the dataset uk-2007-05@1000000. Here ALGO-MAX outperforms ALGO-OP. It shows that the optimization of the sequence choice \( I \) is still an open problem. We added here ALGO-OP2 based on:

\[ i_n = \arg \max_i \left( \frac{F_{n-1} - 1}{\text{#out}_i + 1} \right) \]

large data, in particular, using the asynchronous distributed computation approach.

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