Electromagnetic Contribution to the Proton-Neutron Mass Splitting

A. W. Thomas,¹ X. G. Wang,¹ and R. D. Young¹

¹ARC Centre of Excellence for Particle Physics at the Terascale and CSSM,
School of Chemistry and Physics, University of Adelaide 5005, Australia.

We study the electromagnetic contribution to the proton-neutron mass splitting by combining lattice simulations and the modified Cottingham sum rule of Walker-Loud, Carlson and Miller. This analysis yields an estimate of the isovector nucleon magnetic polarizability as a function of pion mass. The physical value, obtained by chiral extrapolation to the physical pion mass, is \(\beta_{p-n} = (-1.12 \pm 0.40) \times 10^{-4} \text{ fm}^3\), which is in agreement with the empirical result, albeit with a somewhat smaller error. As a result, we find \(\delta M_{p-n} = 1.04 \pm 0.11 \text{ MeV}\), which represents a significant improvement in precision.

The physical proton-neutron mass splitting has been measured extremely precisely [1, 2].

\[ M_n - M_p = 1.293322(4) \text{ MeV}. \]  (1)

Its separation into contributions from electromagnetic effects and the \(u - d\) quark mass difference is of enormous interest [3, 4]. Not only are the \(u - d\) masses critical parameters in the study of explicit chiral symmetry breaking in QCD but their precise values are vital to the discussion of mass generation (within the framework of grand unification), as well as the mechanism of \(CP\) violation. Clearly, if one of the two components of \(M_n - M_p\) can be determined accurately, the other may be inferred from the total.

Recently, Walker-Loud et al. (WLCM) [7] showed how to use the formal operator product expansion (OPE) analysis of Collins [8] to overcome an ambiguity in the original approach of Cottingham [9]. The WLCM analysis led to a significantly larger numerical value for the electromagnetic contribution to the p-n mass difference but with a rather large uncertainty,

\[ \delta M_{\gamma}^\gamma |_{p-n} = 1.30(03)(47) \text{ MeV}. \]  (2)

This may be compared with the value of Gasser and Leutwyler, based on the standard Cottingham sum-rule, of 0.76(30) MeV [4].

An alternative approach to this problem involves the direct calculation of the electromagnetic contribution to the mass shift using lattice QCD [10, 11]. Most recently, the BMW Collaboration [13] reported a value of 1.59(30)(35) MeV.

In the WLCM formalism, the total electromagnetic contribution to the p-n mass shift, denoted by \(\delta M^\gamma\), is written as the sum of five terms,

\[ \delta M^\gamma = \delta M^{el} + \delta M^{inel} + \delta M_{sub}^{el} + \delta M_{sub}^{inel} + \delta M^{\text{ct}}. \]  (3)

The terms \(\delta M^{el}\), \(\delta M^{inel}\) and \(\delta M_{sub}^{el}\) are uncontroversial and can be evaluated very accurately. Using the Kelly parametrization of the nucleon electromagnetic form factors [14] and modern knowledge of the structure functions [15, 16], one finds [7]

\[(\delta M^{el} + \delta M^{inel} + \delta M_{sub}^{el} + \delta M_{sub}^{inel} + \delta M^{\text{ct}})|_{p-n} = 0.83 \pm 0.04 \text{ MeV}, \]  (4)

where we have combined the uncertainties in quadrature. (Following WLCM, the counter term, \(\delta M^{\text{ct}}\), which is related to the \(\pi\)-N sigma commutator is set to zero with an uncertainty of \(\pm 0.02\) MeV.) On the other hand, the inelastic subtraction term, which as a matter of principle should be considered together with the very small counter term introduced by Collins, has been reported as

\[ \delta M_{\text{sub}}^{inel} |_{p-n} = 0.47 \pm 0.47 \text{ MeV}. \]  (5)

This large uncertainty, which in turn dominates the overall uncertainty on \(\delta M^\gamma\), results from the uncertainty associated with the isovector nucleon magnetic polarizability, which was taken to be

\[ \beta_{p-n} = (-1 \pm 1) \times 10^{-4} \text{ fm}^3. \]  (6)

In this Letter, we use data from the RBC Collaboration [11] for the electromagnetic mass shift as a function of quark mass to provide an improved constraint on the inelastic subtraction term of WLCM. In this way, the contribution from the term involving the isovector nucleon magnetic polarizability can be extracted as a function of pion mass. The result is a considerable improvement in the precision of the overall electromagnetic contribution to the mass splitting.

To begin, we consider the finite volume lattice QCD calculation of the RBC Collaboration [11]. This group has reported the electromagnetic mass difference as a function of quark mass for two lattice volumes, \(16^3 \times 8 \text{ fm}\) and \(24^3 \times 10 \text{ fm}\) with lattice cutoff \(a^{-1} \approx 1.78 \text{ GeV}\). In Fig. 1 we compare their results for the electromagnetic p-n mass splitting with the finite volume versions [12]

\[ \delta M^{el}(L) = \frac{2\pi \alpha}{L^3} \sum_{\bar{q}\neq q} \frac{1}{Q^2} \left[ \frac{3\sqrt{\tau_{el}} G_M^2}{2(1 + \tau_{el})} + \frac{G_E^2 - 2\tau_{el} G_M^2}{1 + \tau_{el}} \right] \times \left[ 1 + \tau_{el}^{3/2} - \tau_{el}^{3/2} - \frac{3}{2} \sqrt{\tau_{el}} \right], \]

\[ \delta M_{sub}^{el}(L) = -\frac{3\alpha \pi}{4M L^3} \sum_{\bar{q}\neq q} \frac{1}{Q^2} [2G_M^2 - 2F_1^2] \]  (7)

of the infinite volume expressions for the elastic contribu-
The dipole masses of the isovector magnetic and electric form factors are parametrized as

\[
\begin{align*}
(A_M^v)^2 &= \frac{12(1 + A_1 m_\pi^2)}{A_0 + \frac{\alpha_1}{m_\pi^2} \arctan(\mu/m_\pi) + \frac{\alpha_2}{2} \ln\left(\frac{m_\pi^2}{m_\pi^2 + \mu^2}\right)}, \\
(A_E^v)^2 &= \frac{12(1 + B_1 m_\pi^2)}{B_0 + \frac{\beta_1}{2} \ln\left(\frac{m_\pi^2}{m_\pi^2 + \mu^2}\right)},
\end{align*}
\]

where, once again, the leading non-analytic behavior of the charge and magnetic radii are chosen to agree with chiral perturbation theory:

\[
\begin{align*}
\chi_1 &= \frac{g_A^2 m_N}{8 \pi f_\pi^2}, & \chi_2 &= -\frac{5g_A^2 + 1}{8\pi^2 f_\pi^2},
\end{align*}
\]

with \(g_A = 1.27\) the axial coupling constant and \(f_\pi = 93\) MeV the pion decay constant. \(m_N = 940\) MeV is the nucleon mass and \(\kappa_v = 4.2\) is the isovector anomalous magnetic moment of the nucleon (in the chiral limit).

The isoscalar dipole masses are observed to be roughly linear in \(m_\pi^2\),

\[
(A_{E,M}^s)^2 = a_{E,M} + b_{E,M} m_\pi^2.
\]

The parameters are determined by fitting lattice data from the QCDSF Collaboration [21].

\[
\begin{align*}
A_0 &= 8.65, & A_1 &= 0.28, \\
B_0 &= 11.71, & B_1 &= 0.72,
\end{align*}
\]

in units of GeV\(^{-2}\) and

\[
\begin{align*}
a_E &= 1.09\, \text{GeV}^2, & b_E &= 0.85, \\
a_M &= 1.09\, \text{GeV}^2, & b_M &= 0.68,
\end{align*}
\]

with \(\mu = 0.14\) GeV. The electromagnetic form factors of the proton and neutron can be reconstructed through

\[
G^p = \frac{1}{2}(G^s + G^v), \quad G^n = \frac{1}{2}(G^s - G^v).
\]

Figure 1 shows the total elastic contribution to the proton-neutron mass difference, computed at two lattice volumes. While the general behavior of the calculation is in agreement with the lattice simulations, there is a clear discrepancy. This discrepancy is identified with the finite volume corrections to it will be well within the uncertainty quoted by WLCM and therefore we simply include the physical value (\(\delta M^{inel} = 0.057\) MeV) in our calculation.

Turning to the inelastic subtraction term, we note that the dipole form factor multiplying \(\beta_{p-n} Q^2\), which was used by WLCM, leads to a very large \(\log(Q^2_0)\) term. The magnitude is inconsistent with the \(\log(Q^2_0)\) behavior of \(\delta M^{el}\), which is the only term that will contribute to the asymptotic scaling. In order to avoid this problem, we choose to use a
Table I. The magnetic polarizability $\beta_{p-n}$ as a function of $m_\pi$, in units of $10^{-4}$ fm$^3$. $\delta M_{sub}^{inel}$ is given by Eq. (17).

| $m_\pi$ [GeV] | 0.279 | 0.394 | 0.558 | 0.683 |
|----------------|-------|-------|-------|-------|
| $n = 3$        |       |       |       |       |
| $16^3$         | $-0.246 \pm 0.103$ | $-0.258 \pm 0.040$ | $-0.294 \pm 0.030$ |       |
| $24^3$         | $-0.316 \pm 0.171$ | $-0.134 \pm 0.060$ | $-0.292 \pm 0.030$ | $-0.298 \pm 0.020$ |
| $n = 4$        |       |       |       |       |
| $16^4$         |       |       |       |       |
| $24^4$         | $-0.731 \pm 0.307$ | $-0.756 \pm 0.118$ | $-0.855 \pm 0.087$ |       |
|                | $-0.917 \pm 0.498$ | $-0.385 \pm 0.172$ | $-0.578 \pm 0.087$ | $-0.847 \pm 0.057$ |

The other two parameters, determined by fitting the results given in Tab. I are summarised in Tab. II and the results of those fits are illustrated in Fig. 2.

The physical values for $\beta_{p-n}$, obtained by extrapolating to the physical pion mass, are also shown in the last column of Tab. II. It is remarkable that even though the values of $\beta_{p-n}$ found at each value of the pion mass tend to be systematically smaller for the cubic form factor than for the quartic form factor, the values deduced at the physical pion mass are in fairly good agreement within their respective uncertainties. We make a conservative estimate by taking the average value of these two results,

$$\beta_{p-n} = (-1.12 \pm 0.40) \times 10^{-4} \text{ fm}^3 .$$

This value is of the right sign and order of magnitude compared with the experimental result $^{17}$, albeit with a significant smaller error.

In the infinite volume limit, the inelastic subtraction term contributes to the electromagnetic p-n mass splitting as

$$\delta M_{sub}^{inel}|_{p-n} = -\frac{3\beta_{p-n}}{8\pi} \int_0^\infty dQ^2 Q^2 \left( \frac{(\Lambda_M^p)^2}{(M_M^p)^2 + Q^2} \right)^3$$

$$= 0.30 \pm 0.04 \text{ MeV} , \quad (21a)$$

$$\delta M_{sub}^{inel}|_{p-n} = -\frac{3\beta_{p-n}}{8\pi} \int_0^\infty dQ^2 Q^2 \left( \frac{(\Lambda_M^n)^2}{(M_M^n)^2 + Q^2} \right)^4$$

$$= 0.12 \pm 0.04 \text{ MeV} . \quad (21b)$$

Again, we take the conservative approach of averaging these two results:

$$\delta M_{sub}^{inel}|_{p-n} = 0.21 \pm 0.11 \text{ MeV} , \quad (22)$$

where the dominant source of uncertainty comes from the model dependence arising from the choice of a cubic or quartic form factor in Eq. (17). Combining this with Eq. (4), we finally obtain the total electromagnetic contribution to the proton-neutron mass splitting,

$$\delta M_{p-n}^\gamma = 1.04 \pm 0.11 \text{ MeV} . \quad (23)$$

In summary, we have carried out an analysis of the RBC lattice simulations of the electromagnetic proton-neutron mass splitting using the modified Cottingham sum rule of WLMC. This provides an improved constraint on the inelastic subtraction term, which was the major source of uncertainty in their work. The isovector nucleon magnetic polarizability was extracted as a function of pion mass. The form factor with either cubic or quartic behavior. Thus the finite volume corrections missing in Fig. 1 are taken as:

$$\delta M_{sub}^{inel}(L) = -\frac{3\beta_{p-n}}{2} \sum_{n \neq 0} Q \left( \frac{(\Lambda_M^p)^2}{(M_M^p)^2 + Q^2} \right)^n$$

(17)

with $n = 3, 4$. Fitting the discrepancy between the curves and the RBC Collaboration data in Fig. 1 by adjusting $\beta_{p-n}$ leads to the extracted values for the isovector magnetic polarizability $\beta_{p-n}$ at each pion mass, shown in Tab. I.

The nucleon electromagnetic polarizabilities have been investigated in heavy baryon chiral perturbation theory $^{22, 23}$. The quantity $\beta_{p-n}$ does not depend on the unknown low energy constants $c_2$ and $c^+$, which appear in the separate expressions for $\beta_p$ and $\beta_n$. The $1/m_\pi$ terms in $\beta_p$ and $\beta_n$ cancel each other. Thus we finally obtain:

$$\beta_{p-n}(m_\pi) = c_1 \ln \frac{m_\pi}{M_N} + c_0 + c_1 \frac{m_\pi}{M_N} , \quad (18)$$

with the model independent coefficient, $c_1$, fixed by chiral perturbation theory $^{23}$.

$$c_1 = \frac{\alpha g_A^2}{4\pi^2 m_N f_\pi^2} (1 + \kappa) = 2.51 \times 10^{-4} \text{ fm}^3 , \quad (19)$$

Figure 2. Fit results for $\beta_{p-n}$ to the extracted values given by Tab. I. Red-solid and blue-dashed curves correspond to the sets of $n = 3$ and $n = 4$, respectively.
Table II. Fitted parameters and extrapolated $\beta_{p-n}$ at physical pion mass, in units of $10^{-4}$ fm$^3$.

| $n$ | $c_0$       | $c_1$       | $\chi^2_{d.o.f}$ | $\beta_{p-n}^{\text{phy}}$ |
|-----|-------------|-------------|------------------|-----------------------------|
| 3   | 4.83 ± 0.12 | -6.88 ± 0.27| 8.19/(7-2) = 1.64| -0.98 ± 0.12                |
| 4   | 4.68 ± 0.34 | -7.69 ± 0.78| 8.68/(7-2) = 1.74| -1.25 ± 0.36                |

physical value, obtained by chiral extrapolation to the physical pion mass, showed a significant improvement in precision in comparison with the current experimental value. Consequently, we were able to obtain the more accurate result for the overall electromagnetic contribution to the proton-neutron mass difference given in Eq. (23) [24]. This in turn allows us to deduce a more accurate value for the size of the contribution to the proton-neutron mass difference arising from the difference of the masses of the up and down quarks, namely $\delta m_{d-u} = 2.33 \pm 0.11$ MeV. It will be fascinating to explore the consequences of this new constraint.

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