Analysis of a slow-fast system near a cusp singularity

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Abstract

This paper studies a slow-fast system whose principal characteristic is that the slow manifold is given by the critical set of the cusp catastrophe. Our analysis consists of two main parts: first, we recall a formal normal form suitable for systems as the one studied here; afterwards, taking advantage of this normal form, we investigate the transition near the cusp singularity by means of the blow up technique. Our contribution relies heavily in the usage of normal form theory, allowing us to refine previous results.

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1. Introduction

A slow-fast system (SFS) is a singularly perturbed ordinary differential equation of the form

\[
\begin{align*}
\dot{x} & = f(x, z, \varepsilon) \\
\varepsilon \dot{z} & = g(x, z, \varepsilon),
\end{align*}
\]

where \( x \in \mathbb{R}^m, z \in \mathbb{R}^n \) are local coordinates and where \( \varepsilon > 0 \) is a small parameter. The over-dot denotes the derivative with respect to the time parameter \( t \). Throughout this text, we assume that the functions \( f \) and \( g \) are of class \( C^\infty \). In applications (e.g. [25]), \( z(t) \) represents states or measurable quantities of a process while \( x(t) \) stands for control parameters. The parameter \( \varepsilon \) models the difference of the rates of change between the variables \( z \) and \( x \). That is why systems like (1) are often used to model phenomena with two time scales. Observe that the smaller \( \varepsilon \) is, the faster \( z \) evolves with respect to \( x \). Therefore we refer to \( x \) (resp. \( z \)) as the slow (resp. fast) variable. The time parameter \( t \) is known as the slow time. For \( \varepsilon \neq 0 \), we can define a new time parameter \( \tau \) by the relation \( t = \varepsilon \tau \). With this time reparametrization (1) can be written as

\[
\begin{align*}
x' & = \varepsilon f(x, z, \varepsilon) \\
z' & = g(x, z, \varepsilon),
\end{align*}
\]

where now the prime denotes the derivative with respect to the rescaled time parameter \( \tau \), which we call the fast time. Since we consider only autonomous systems, we often omit to indicate the time dependence of the variables. In the rest of this document, we prefer to work with slow-fast systems presented as (2).

Observe that as long as \( \varepsilon \neq 0 \) and \( f \) is not identically zero, systems (1) and (2) are equivalent. A first approach to understand the qualitative behavior of slow-fast systems is to study the limit \( \varepsilon \to 0 \). The slow equation (1) restricted to \( \varepsilon = 0 \) reads as

\[
\begin{align*}
\dot{x} & = f(x, z, 0) \\
0 & = g(x, z, 0).
\end{align*}
\]

A system of the form (3) is called constrained differential equation (CDE) [14, 24]. On the other hand, in the limit \( \varepsilon \to 0 \), a system given by (2) becomes

\[
\begin{align*}
x' & = 0 \\
z' & = g(x, z, 0),
\end{align*}
\]

which is called the layer equation. Associated to both systems, (3) and (4), the slow manifold \( S \) is defined by

\[
S = \{(x, z) \in \mathbb{R}^m \times \mathbb{R}^n \mid g(x, z, 0) = 0 \},
\]

which serves as the phase space of the CDE (3) and as the set of equilibrium points of the layer equation (4). In the latter context, it is useful to recall the concept of Normally Hyperbolic Invariant Manifold (NHIM).

**Definition 1.1 (Normally Hyperbolic Invariant Manifold).** Consider a slow-fast system given by a vector field of the form

\[
X_\varepsilon = \varepsilon f(x, z, \varepsilon) \frac{\partial}{\partial x} + g(x, z, \varepsilon) \frac{\partial}{\partial z}.
\]

The associated slow (invariant) manifold \( S = \{g(x, z, 0) = 0\} \) is said to be normally hyperbolic if each point of \( S \) is a hyperbolic equilibrium point of \( X_0 \).
Figure 1: A schematic representation of the persistence of a NHIM under the perturbation of the corresponding vector field. $S$ denotes the slow manifold. Left-above: $S$ is a set of hyperbolic equilibrium points of the layer equation. Left-below: $S$ is the phase space of the constrained equation. Right: since $S$ is a NHIM, it persists as an invariant manifold $S_\varepsilon$ under small perturbations of the vector field.

NHIMs are relevant in the context of the geometric study of slow-fast systems, see for example [10]. It is known that compact NHIMs persist under $C^1$ small perturbation of the vector field [15, 16]. In the particular context presented above, a normally hyperbolic compact subset of the slow manifold $S$ persists as an invariant manifold of the slow-fast system $X_\varepsilon$. We show in fig. 1 a schematic of the previous description.

After this introduction, we turn to the subject of this paper. Our goal is to understand the dynamics of a particular slow-fast system which has one fast and two slow variables given as

$$X_\varepsilon = \varepsilon (1 + \varepsilon f_1) \frac{\partial}{\partial x_1} + \varepsilon^2 f_2 \frac{\partial}{\partial x_2} - (z^3 + x_2z + x_1 + \varepsilon f_3) \frac{\partial}{\partial z},$$

where the functions $f_i = f_i(x_1, x_2, z)$, for $i = 1, 2, 3$, are smooth and vanish at the origin. The corresponding slow manifold is defined by

$$S = \{(x_1, x_2, z) \in \mathbb{R}^3 | z^3 + x_2z + x_1 = 0\}.$$

**Remark 1.1.** The slow manifold $S$ can be regarded as the critical set of the cusp (or $A_3$) catastrophe, which is given as [1, 5]

$$V(x_1, x_2, z) = \frac{1}{4} z^4 + \frac{1}{2} x_2z^2 + x_1z.$$

We denote by $\Delta$ the set of points in $S$ at which $S$ is tangent to the fast direction, that is

$$\Delta = \{(x_2, z) \in S | 3z^2 + x_2 = 0\}.$$

In other words, $\Delta$ is the set of degenerate critical points of (9). See figure fig. 2 for a description of the slow manifold and the set $\Delta$.

Our interest in studying (7) is due to the fact that the origin $(x_1, x_2, z) = (0, 0, 0)$ is a non-hyperbolic equilibrium point of $X_0$. This implies that a compact subset, around the origin, of the slow manifold $S$ is not a NHIM of $X_0$, and therefore, the Geometric Singular Perturbation Theory [10, 15, 16] is not enough.
1.1. Motivation
There have been several studies, e.g. [18, 19], dealing with a SFS of the form

$$X_\varepsilon = \varepsilon(1 + f_1) \frac{\partial}{\partial x_1} - (z^2 + x_1 + \varepsilon h) \frac{\partial}{\partial z},$$

whose slow manifold is the critical set of the fold catastrophe. The next natural step is to consider the following case in the Thom list [22], i.e., a slow-fast system induced by the cusp catastrophe. That is

$$X_\varepsilon = \varepsilon(1 + f_1) \frac{\partial}{\partial x_1} + \varepsilon f_2 \frac{\partial}{\partial x_2} - (z^3 + x_2 z + x_1 + \varepsilon f_3) \frac{\partial}{\partial z},$$

In [4], the system (12) is studied in a qualitative way. Here, however, we aim to refine the results by heavily using techniques from normal form theory. Moreover, we remark that the methods presented here are applicable to a larger class of slow-fast system given by

$$X_\varepsilon = \varepsilon(1 + f_1) \frac{\partial}{\partial x_1} + \varepsilon f_2 \frac{\partial}{\partial x_2} - (z^3 + x_2 z + x_1 + \varepsilon f_3) \frac{\partial}{\partial z} + 0 \frac{\partial}{\partial \varepsilon},$$

which is called (regular) $A_k$-SFS, see [13].

1.2. Statement
We shall study the SFS

$$X_\varepsilon = \varepsilon(1 + f_1) \frac{\partial}{\partial x_1} + \sum_{i=2}^{k-1} \varepsilon f_i \frac{\partial}{\partial x_i} - \left( z^k + \sum_{j=1}^{k-1} x_j z^{j-1} - \varepsilon f_k \right) \frac{\partial}{\partial z},$$

where the functions $f_i = f_i(x_1, x_2, z, \varepsilon)$ are smooth. To avoid working with an $\varepsilon$-parameter family of vector fields as (14), it is customary to extend (14) by adding the trivial equation $\varepsilon' = 0$, and thus consider a smooth vector field in $\mathbb{R}^4$ which reads as

$$X = \varepsilon(1 + f_1) \frac{\partial}{\partial x_1} + \varepsilon f_2 \frac{\partial}{\partial x_2} - (z^3 + x_2 z + x_1 + \varepsilon f_3) \frac{\partial}{\partial z} + 0 \frac{\partial}{\partial \varepsilon}.$$

We regard (15) as a perturbation of “the principal part” $F$ which is given as

$$F = \varepsilon \frac{\partial}{\partial x_1} + 0 \frac{\partial}{\partial x_2} - (z^3 + x_2 z + x_1) \frac{\partial}{\partial z} + 0 \frac{\partial}{\partial \varepsilon}.$$

Note that in a qualitative sense, $F$ contains the essential elements of $X$. To state our main result, we first define the sections

$$\Sigma^- = \{ (x_1, x_2, z, \varepsilon) \in \mathbb{R}^4 \mid x_1 = -x_1^f \}$$

$$\Sigma^+ = \{ (x_1, x_2, z, \varepsilon) \in \mathbb{R}^4 \mid x_1 = x_1^f \},$$

where $x_1^f > 0$ and $x_1^f > 0$ are arbitrarily large constants. For $\varepsilon > 0$ but sufficiently small, the sections $\Sigma^-$ and $\Sigma^+$ are transversal to the flow of $X_\varepsilon$. Next, let $\Pi : \Sigma^- \to \Sigma^+$ be the Poincaré map induced by the flow of $X_\varepsilon$. We shall prove the following.
Figure 3: Description of our main result. We may choose appropriate coordinates at the sections $\Sigma^-$ and $\Sigma^+$ under which the invariant manifold $S_\varepsilon$ is given by $Z = 0$. Moreover from (16) we have that all other trajectories starting at $\Sigma^-$ are exponentially attracted to the invariant manifold $S_\varepsilon$. In this paper we provide quantitative information regarding this exponential contraction.

**Transition along the cusp** (see theorem 3.1). Consider a slow-fast system given by (15). Let $\Sigma^-$, $\Sigma^+$ and $\Pi : \Sigma^- \to \Sigma^+$ be defined as above. Then, we can choose coordinates in $\Sigma^-$ and in $\Sigma^+$ such that the map $\Pi$ reads as

$$\Pi(X_2, Z, \varepsilon) = (\tilde{X}_2, \tilde{Z}, \tilde{\varepsilon}),$$

(18)

where $\tilde{X}_2 = X_2 + H(X_2, \varepsilon)$ (with $H$ flat at $(X_2, \varepsilon) = (0, 0)$), $\tilde{\varepsilon} = \varepsilon$ and where

$$\tilde{Z} = \Phi(X_2, \varepsilon) + Z \exp \left( -\frac{1}{\varepsilon} (A(X_2, \varepsilon) + \varepsilon \Psi(X_2, Z, \varepsilon)) \right),$$

(19)

where $A(X_2, 0) > 0$. Details of the functions $\Phi$, $A$, and $\Psi$ are given in theorem 3.1. In an heuristic way, this result is described in fig. 3.

**1.3. Idea of the proof**

Our proof consists of two main steps.

1. From [12], it is known that there exists a formal transformation bringing (15) into

$$F = \varepsilon \frac{\partial}{\partial x_1} + 0 \frac{\partial}{\partial x_2} - \left( z^3 + x_2z + x_1 \right) \frac{\partial}{\partial z} + 0 \frac{\partial}{\partial \varepsilon}.$$  

(20)

Then, by Borel’s lemma [5], the vector field $F$ can be realized as a smooth normal form $X^N = F + R$ of (15) and where $R$ is flat at $(x_1, x_2, z, \varepsilon) = (0, 0, 0, 0)$. See more details in section 2.2.

2. Based the previous normalization, next we use the geometric desingularization or blow up method (as introduced in [9]) to study the flow of the normal form $X^N = F + R$. This is detailed in section 3.

**Remark 1.2.** With this document we aim at two goals:

1. To refine the results of [4]. This is, we do not only provide a qualitative description of the transition $\Pi$, but details on the differentiability of such a map is also presented.
2. To prepare a framework for the geometric desingularization of $A_k$ slow-fast systems. These are a generalization of (15) given as

$$X = \varepsilon (1 + f_1) \frac{\partial}{\partial x_1} + \sum_{i=1}^{k-1} \varepsilon f_i \frac{\partial}{\partial x_i} - \left(z^k + \sum_{j=1}^{k-1} x_j z^{j-1} + \varepsilon f_k\right) \frac{\partial}{\partial z} + 0 \frac{\partial}{\partial \varepsilon}. \quad (21)$$

The rest of this document is arranged as follows: in section 2 we provide a brief recollection of preliminary results that will simplify our later studies. Next, in section 3 we pose our result and prove it by means of the geometric desingularization method and the results of section 2. For readability purposes, many technicalities have been put in the appendix.

2. Preliminaries of slow-fast systems

In this section, we provide a number preliminary results that will be used later in section 3. First of all, we consider slow-fast systems along normally hyperbolic regions of the slow manifold. Afterwards, we recall a result from [12] dealing with the normal form of (15). We remark that we only consider SFS with one fast variable. Let us be more precise with the type of SFS that we shall study first.

**Definition 2.1.** A slow-fast system is said to be (locally) regular around a point $p_0$, if its corresponding slow manifold is normally hyperbolic in a some neighborhood of $p_0$.

2.1. The slow vector field

Let us consider a slow-fast system given by

$$X_\varepsilon = \sum_{i=1}^{m} \varepsilon f_i(x, z, \varepsilon) \frac{\partial}{\partial x_i} + H(x, z, \varepsilon) \frac{\partial}{\partial z}, \quad (22)$$

where $x \in \mathbb{R}^m$, $z \in \mathbb{R}$, and as usual $0 < \varepsilon \ll 1$. Furthermore, assume that $f(0, 0, 0) \neq 0$, $H(0, 0, 0) = 0$ and $\frac{\partial H}{\partial z}(0, 0, 0) < 0$. Thus $X_\varepsilon$ is regular around $0 \in \mathbb{R}^{m+2}$. The slow manifold associated to (22) is defined by

$$S = \{(x, z) \in \mathbb{R}^{m+1} | H(x, z, 0) = 0\}. \quad (23)$$

From the defining assumptions of (22), we have that $S$ is a NHIM in a neighborhood of the origin. By looking at the Jacobian of $X_\varepsilon$ at 0, it follows that there exists an $m+1$ dimensional a center manifold. Since $X$ is smooth, we can choose a $C^\ell$ center manifold $\mathcal{W}^{\ell}$ for any $\ell < \infty$. The manifold $\mathcal{W}^{\ell}$ is given as a graph $z = \phi(x, \varepsilon)$ where $\phi$ is a $C^\ell$ function.

**Remark 2.1.** Along the rest of the document we frequently make use of a finite class of differentiability. As it is customary in the present context, when we say that a manifold (or a map) is $C^\ell$, we mean that such a manifold (or map) is $\ell$-differentiable for $\ell$ as large as necessary.

The slow manifold $S$ is naturally given by the restriction $\mathcal{W}^{\ell}|_{\varepsilon=0} = S$. Next, let us consider the vector field $\frac{1}{\varepsilon} X_\varepsilon(x, \phi, \varepsilon)$. Since $\mathcal{W}^{\ell}$ is locally invariant, it follows that $\frac{1}{\varepsilon} X_\varepsilon$ is tangent to $\mathcal{W}^{\ell}$. Therefore the vector field

$$X^{\text{slow}} = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} X_\varepsilon(x, \phi, \varepsilon), \quad (24)$$

is tangent to $S$ at each point of $S$, and we call it the slow vector field. We remark that the slow vector field $X^{\text{slow}}$ is only well defined whenever $\phi$ is invertible.
2.1.1. The slow divergence integral

Associated to a regular slow-fast system and the corresponding slow vector field, the *slow divergence integral* is defined here. For this, let \( \Sigma^- \) and \( \Sigma^+ \) be two sections which are transversal to the flow of \( X_\varepsilon \) given by (22). For \( \varepsilon \neq 0 \) but sufficiently small, these sections are also transversal to the slow manifold \( S \). Let \( \gamma_\varepsilon \) be a solution curve of \( X_\varepsilon \) chosen along a center manifold \( W^C \), thus \( \gamma_\varepsilon \) is transversal to the sections \( \Sigma^- \) and \( \Sigma^+ \). In the limit \( \varepsilon = 0 \), the curve \( \gamma_0 \) is a curve along the slow manifold \( S \). The idea now is to borrow the well-known divergence theorem \([21]\) to get some sense on how the trajectories of \( X_\varepsilon \) are attracted to \( S \) (recall that we made the assumption \( \frac{\partial H}{\partial z} < 0 \)). The divergence of \( X_\varepsilon \) (given by (22)) reads as

\[
\text{div} \ X_\varepsilon = \frac{\partial H(x, z, \varepsilon)}{\partial z} + O(\varepsilon).
\]  

(25)

We can now take the integral of \( \text{div} \ X_\varepsilon \) along the orbit \( \gamma_\varepsilon \) of \( X_\varepsilon \) parametrized by the fast time \( \tau \), we have

\[
\int_{\gamma_\varepsilon} \text{div} \ X_\varepsilon \, d\tau = \int_{\gamma_\varepsilon} \left( \frac{\partial H(x, z, \varepsilon)}{\partial z} + O(\varepsilon) \right) d\tau.
\]  

(26)

The *slow divergence integral* is defined by

\[
I(t) = \int_{\gamma_0} \text{div} \ X_0 \, dt,
\]  

(27)

where \( t \) is the slow time defined by the slow vector field \( X^{\text{slow}} \). Our goal then is to relate the divergence integral (26) with \( I \).

**Proposition 2.1.** Under the assumptions made in this section, we have that

\[
\int_{\gamma_\varepsilon} \text{div} \ X_\varepsilon \, d\tau = \frac{1}{\varepsilon} \left( I(t) + o(1) \right),
\]  

(28)

where \( I(t) \) is the slow divergence integral.

**Proof.** Recall that the slow vector field reads as \( X^{\text{slow}} = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} X_\varepsilon \), where \( \phi = \phi(x, \varepsilon) \) is a \( C^\ell \) function. By our assumptions, the curve \( \gamma_\varepsilon \) is transversal to the sections \( \Sigma^- \) and \( \Sigma^+ \) for \( \varepsilon \) small enough.

Without loss of generality we can assume that \( \gamma_\varepsilon \) is parametrized by \( x_1 \). Then let \( x^-_1 \) and \( x^+_1 \) be defined by \( \gamma_\varepsilon(x^-_1) = \gamma_\varepsilon \cap \Sigma^- \) and \( \gamma_\varepsilon(x^+_1) = \gamma_\varepsilon \cap \Sigma^+ \). Next, the integral of the divergence of \( X_\varepsilon \) along \( \gamma_\varepsilon \) from \( \Sigma^- \) to \( \Sigma^+ \) reads as

\[
\int_{\gamma_\varepsilon} \text{div} \ X_\varepsilon \, d\tau = \frac{1}{\varepsilon} \int_{x^-_1}^{x^+_1} \left( \frac{\partial H(x, z, 0)}{\partial z} + O(\varepsilon) \right) \frac{dx_1}{f_1(x, z, 0) + o(1)}
\]

\[
= \frac{1}{\varepsilon} \left( \int_{x^-_1}^{x^+_1} \frac{\partial H(x, z, 0)}{\partial z} \frac{dx_1}{f_1(x, z, 0)} + o(1) \right)
\]

\[
= \frac{1}{\varepsilon} \left( \int_{\gamma_0} \text{div} \ X_0 \, dt + o(1) \right)
\]  

(29)

where \( t \) is the slow time induced by \( X^{\text{slow}} \), which in coordinates means that \( \frac{dx_1}{dt} = f_1 \).

Observe that the slow divergence integral is a first order approximation of the divergence along orbits of \( X_\varepsilon \). This will be useful when presenting our main result in section 3.
2.1.2. Normal form and transition of a regular slow-fast system

Now we consider the problem of finding a suitable normal form of a regular SFS. The following is a well-known result but we recall it here for completeness.

**Proposition 2.2.** Consider a regular slow-fast system on $\mathbb{R}^{m+3}$ given by

$$X_\varepsilon = \varepsilon (1 + f_1) \frac{\partial}{\partial u} + \sum_{j=1}^{m} \varepsilon g_j \frac{\partial}{\partial v_j} + H \frac{\partial}{\partial z},$$

(30)

where $(u,v_1,\ldots,v_m,z,\varepsilon) \in \mathbb{R}^{m+3}$; where the functions $f_1 = f_1(u,v,z,\varepsilon)$ and $g_j = g_j(u,v,z,\varepsilon)$, for $2 \geq j \geq k - 1$, are smooth and where the function $H = H(u,v,z,\varepsilon)$ is smooth with $H(0,0,0,0) = 0$ and $\frac{\partial H}{\partial z}(0,0,0,0) < 0$. Then, the vector field $X$ is $C^\ell$-equivalent to a normal form given by

$$X_N^\varepsilon = \varepsilon \frac{\partial}{\partial U} + \sum_{j=1}^{m} \varepsilon \tilde{G}_j \frac{\partial}{\partial V_j} - Z \frac{\partial}{\partial Z},$$

(31)

where $\{Z = 0\}$ corresponds to a choice of the center manifold $W^C$ of $X_\varepsilon$.

**Proof of proposition 2.2.** The first step is to divide the vector field $X$ by $1 + f_1$. In a sufficiently small neighborhood of the origin this is a smooth equivalence relation. That is $Y = \frac{1}{1 + f_1} X$ reads as

$$Y = \varepsilon \frac{\partial}{\partial u} + \sum_{j=1}^{m} \varepsilon \tilde{g}_j \frac{\partial}{\partial v_j} + \tilde{H} \frac{\partial}{\partial z},$$

(32)

where $\tilde{g}_j$, for $2 \geq j \geq k - 1$, and $\tilde{H}$ are smooth with $\tilde{H}(0) = 0$ and $\frac{\partial \tilde{H}}{\partial z}(0,0) < 0$. Now we note that the origin of $\mathbb{R}^{m+3}$ is a semi-hyperbolic equilibrium point with $(u,v,\varepsilon)$ being center coordinates and $z$ being the hyperbolic coordinate. We can now use Takens-Bonckaert results on normal forms of partially hyperbolic vector fields [2, 3, 23]. Thus, there exists a $C^\ell$ change of coordinates (maybe respecting some constraints if required) under which $Y$ is conjugated to

$$\tilde{Y} = \varepsilon \frac{\partial}{\partial U} + \sum_{j=1}^{m} \varepsilon \tilde{G}_j \frac{\partial}{\partial V_j} + \tilde{H} \frac{\partial}{\partial z},$$

(33)

where $\tilde{G}_j = \tilde{G}_j(U,V,\varepsilon)$, for $2 \geq j \geq k - 1$, and $\tilde{H} = \tilde{H}(U,V,\varepsilon)$ are $C^\ell$ functions, and where $\{Z = 0\}$ corresponds to a choice center manifold which we denote by $W^C$. We remark that in the vector field $\tilde{Y}$, the functions $\tilde{G}_j$ and $\tilde{H}$ are independent of $Z$. Furthermore we have

$$\tilde{H}(0,0,0) = \frac{\partial \tilde{H}}{\partial z}(0,0,0) < 0.$$  

(34)

This means that in a small neighborhood of the origin $\tilde{Y}$ can be divided by $|\tilde{H}|$. In other words, $\tilde{Y}$ is $C^\ell$-equivalent to

$$\bar{Y} = \varepsilon G \frac{\partial}{\partial U} + \sum_{j=1}^{m} \varepsilon \bar{K}_j \frac{\partial}{\partial V_j} - Z \frac{\partial}{\partial Z},$$

(35)

where $G(0,0,0) \neq 0$ and $\bar{K}_j = \bar{K}_j(U,V,\varepsilon)$, for $2 \geq j \geq k - 1$, are $C^\ell$. Next, since $W^C = \{Z = 0\}$ is invariant under the flow of $\bar{Y}$, we can study the restriction $\bar{Y}|_{Z = 0}$. This is

$$\bar{Y}|_{Z = 0} = \varepsilon G \frac{\partial}{\partial U} + \sum_{j=1}^{m} \varepsilon \bar{K}_j \frac{\partial}{\partial V_j}.$$  

(36)
For $\varepsilon \neq 0$, the vector field $Y|_{Z=0}$ is regular because $G(0,0,0) \neq 0$. Thus, by the flow-box theorem, there exists a change of coordinates, depending in a $C^\ell$ way on $\varepsilon$, under which $Y|_{Z=0}$ can be written as

$$
\varepsilon \frac{\partial}{\partial U} + \sum_{j=1}^{m} 0 \frac{\partial}{\partial V_j}.
$$

(37)

This implies that $Y$ is $C^\ell$-equivalent to

$$
X_{reg}^N = \varepsilon \frac{\partial}{\partial U} + \sum_{j=1}^{m} 0 \frac{\partial}{\partial V_j} - Z \frac{\partial}{\partial Z},
$$

(38)

as stated in the proposition.

Motivated by proposition 2.2 let us now discuss the dynamics of the vector field

$$
X_{reg}^N = \varepsilon \frac{\partial}{\partial U} + \sum_{j=1}^{m} 0 \frac{\partial}{\partial V_j} - Z \frac{\partial}{\partial Z}.
$$

(39)

The slow manifold $S$, corresponding to the normal form (39), is given by

$$
S = \{ \varepsilon = 0, Z = 0 \}.
$$

(40)

Furthermore, we can parametrize the solution of (39) by $U$. Let us define the sections

$$
\Sigma^- = \{(U,V,Z,\varepsilon) \in \mathbb{R} \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{R} \mid U = U^-\},
$$

$$
\Sigma^+ = \{(U,V,Z,\varepsilon) \in \mathbb{R} \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{R} \mid U = U^+\},
$$

(41)

where $U^- < U^+$. The sections $\Sigma^-$ and $\Sigma^+$ are transversal to the manifold $S$ and therefore, for $\varepsilon \neq 0$, are also transversal to the flow of (39). Associated to these sections, we define the transition

$$
\Pi : \Sigma^- \rightarrow \Sigma^+
$$

$$(V, Z, \varepsilon) \mapsto (\tilde{V}, \tilde{Z}, \tilde{\varepsilon}).
$$

(42)

To compute the component $\tilde{Z}$ we only need to integrate $\frac{d\tilde{Z}}{dU} = -\frac{1}{\varepsilon} Z$. Then it follows that $\tilde{Z} = Z(T)$, where $T$ is the time to go from $\Sigma^-$ to $\Sigma^+$, which is $T = U_f - U_i$. Then it follows that

$$
\tilde{V} = V
$$

$$
\tilde{Z} = Z \exp \left(-\frac{1}{\varepsilon}(U_f - U_i)\right)
$$

$$
\tilde{\varepsilon} = \varepsilon.
$$

(43)

Observe the particular format of the transition $\Pi$. The $Z$ component is an exponential contraction towards the center manifold $\{ Z = 0 \}$. Maps with this characteristic appear frequently in our text and also in several other cases where slow-fast systems are studied. Therefore, in appendix A we discuss in a rather general way, the properties of such maps.

2.2. Formal normal form of $A_k$ slow-fast systems

In this section we recall a normal form of the so-called $A_k$ slow-fast systems. A proof can be found in [12]. This normalization is important since it eliminates many unwanted terms from the system being studied here.
Definition 2.2. Let \( k \in \mathbb{N} \) with \( k \geq 2 \). An \( A_k \) slow-fast system (\( A_k\)-SFS) is an ODE of the form
\[
\begin{align*}
    x'_1 &= \varepsilon(1 + f_1) \\
    x'_j &= \varepsilon f_j \\
    z' &= -\left(z^k + \sum_{i=1}^{k-1} x_i z^{i-1}\right) + \varepsilon f_k \\
    \varepsilon' &= 0,
\end{align*}
\]
(44)
where \( j = 2, \ldots, k-1 \), and where the functions \( f_i = f_i(x_1, \ldots, x_{k-1}, z, \varepsilon) \), for \( 1 \leq i \leq k \), are smooth.

Remark 2.2.

- The system investigated in this work is an \( A_3\)-SFS.
- The slow manifold associated to an \( A_k\)-SFS is defined by
\[
S = \left\{(x, z) \in \mathbb{R}^k \mid z^k + \sum_{i=1}^{k-1} x_i z^{i-1} = 0\right\}.
\]
(45)
The manifold \( S \) can equivalently be defined as the critical set of an \( A_k \) catastrophe \([1]\). Hence the name \( A_k\)-SFS.

Locally, we can regard (44) as \( X = F + P \) where \( F \) and \( P \) are smooth vector fields of the form
\[
F = \varepsilon \frac{\partial}{\partial x_1} + \sum_{j=2}^{k-1} 0 \frac{\partial}{\partial x_j} + g \frac{\partial}{\partial z} + 0 \frac{\partial}{\partial \varepsilon},
\]
(46)
and
\[
P = \sum_{i=1}^{k-1} \varepsilon f_i \frac{\partial}{\partial x_i} + \varepsilon f_k \frac{\partial}{\partial z} + 0 \frac{\partial}{\partial \varepsilon},
\]
(47)
respectively and where \( g = -\left(z^k + \sum_{i=1}^{k-1} x_i z^{i-1}\right) \). We refer to \( F \) as the “principal part” and to \( P \) as the “perturbation”. Briefly speaking we want to eliminate, via a change of coordinates, the perturbation. The procedure of normalizing the vector field \( X \) is motivated by \([20]\), where normal forms of analytic perturbations of quasihomogeneous vector fields are investigated. The relevant result is the following

Theorem 2.1 (Formal normal form \([12]\)). Let \( k \geq 2 \) and let \( X = F + P \) be a smooth vector field where
\[
F = \varepsilon \frac{\partial}{\partial x_1} + \sum_{i=2}^{k-1} 0 \frac{\partial}{\partial x_i} - \left(z^k + \sum_{j=1}^{k-1} x_j z^{j-1}\right) \frac{\partial}{\partial z} + 0 \frac{\partial}{\partial \varepsilon},
\]
(48)
and where
\[
P = \sum_{i=1}^{k-1} P_i \frac{\partial}{\partial x_i} + P_k \frac{\partial}{\partial z} + 0 \frac{\partial}{\partial \varepsilon},
\]
(49)
where each \( P_i = P_i(x_1, \ldots, x_{k-1}, z, \varepsilon) \) is a smooth function. Assume that the following conditions are satisfied
1. \( P_i(x_1, \ldots, x_{k-1}, z, 0) = 0 \),
2. \( \rho(\hat{P}_i) \geq 2k - i + 1 \),

where \( \hat{P}_i \) denotes the Taylor expansion of \( P_i \) and \( \rho(\hat{P}_i) \) is the quasihomogeneous order of the polynomial \( \hat{P}_i \). Then, there exists a formal diffeomorphism \( \Phi \) such that \( \Phi_*\hat{X} = F \).

In words, theorem 2.1 shows that \( \hat{X} \) and \( F \) are conjugated via \( \Phi \). It follows that, by Borel’s lemma [5], the formal vector field \( \hat{X}^N = F + \hat{P} \) where \( \hat{P} \) is flat at \( (x, z, \varepsilon) = (0, 0, 0) \). This has important consequences in the geometric desingularization of an \( A_3 \)-SFS, presented in the following section.

3. Geometric desingularization of a slow-fast system near a cusp singularity

In this section we study an \( A_3 \) slow-fast system based on: a) the techniques introduced in section 2 and in appendix A, and b) the blow up method. To simplify the notation, let us now write the \( A_3 \)-SFS as

\[
X = \varepsilon(1 + f_1) \frac{\partial}{\partial a} + \varepsilon f_2 \frac{\partial}{\partial b} - (z^3 + bz + a + \varepsilon f_3) \frac{\partial}{\partial z} + 0 \frac{\partial}{\partial \varepsilon},
\]

(50)

where thanks to theorem 2.1, the smooth functions \( f_i = f_i(a, b, z, \varepsilon) \) are flat at the origin of \( \mathbb{R}^4 \). We investigate the transition associated to (50) between the sections

\[
\Sigma^- = \{(a, b, z, \varepsilon) \in \mathbb{R}^4 | a = -a^-, z > 0\}
\]

\[
\Sigma^+ = \{(a, b, z, \varepsilon) \in \mathbb{R}^4 | a = a^+, z < 0\},
\]

(51)

where \( a^- > 0 \) and \( a^+ > 0 \) are arbitrarily large constants. However, since the trajectories of \( X \) spend a long time along regular parts of \( S \), it will be useful to define the “entry” and “exit” sections

\[
\Sigma^{{en}} = \{(a, b, z, \varepsilon) \in \mathbb{R}^4 | a = -a_0, z > 0\}
\]

\[
\Sigma^{{ex}} = \{(a, b, z, \varepsilon) \in \mathbb{R}^4 | a = a_0, z < 0\},
\]

(52)

where \( a_0 \) is a positive but sufficiently small constant, for reference see fig. 4.

Figure 4: Qualitative representation of the investigation performed in this section. The sections \( \Sigma^{{en}} \) and \( \Sigma^{{ex}} \) are arbitrarily close to the cusp point. On the other hand the sections \( \Sigma^- \) and \( \Sigma^+ \) (not shown) are parallel to \( \Sigma^{{en}} \) and \( \Sigma^{{ex}} \) but far away from the cusp point. In a qualitative sense, we will construct an invariant manifold \( \mathcal{M}_\varepsilon \) and then extend it all the the way up to the sections \( \Sigma^- \) and \( \Sigma^+ \). Our analysis aims for simplicity and thus depends extensively on the usage of normal forms. This, of course, makes our results coordinate-dependant.

It will be clear from our analysis in the blow up space section 3.2 that the section \( \Sigma^- \) needs to be partitioned as follows.
Definition 3.1 (The inner layer and the lateral regions). Let $0 < L < M < \infty$ be constants. The inner layer $\Sigma^{\text{inner}} \subset \Sigma^-$ is defined as

$$
\Sigma^- \supset \Sigma^{\text{inner}} = \{(b, z, \varepsilon) \in \Sigma^- \mid |b| < M\varepsilon^{2/5}\}.
$$

(53)

On the other hand, the lateral regions are defined as

$$
\Sigma^- \supset \Sigma^{+b} = \{(b, z, \varepsilon) \in \Sigma^- \mid b > L\varepsilon^{2/5}\}
$$

and

$$
\Sigma^- \supset \Sigma^{-b} = \{(b, z, \varepsilon) \in \Sigma^- \mid -b > L\varepsilon^{2/5}\}.
$$

(54)

Note that the set $\{\Sigma^{\text{inner}}, \Sigma^{+b}, \Sigma^{-b}\}$ is an open cover of $\Sigma^-$, see fig. 5.

![Figure 5: The section $\Sigma^-$ needs to be partitioned into three subsections: the inner layer $\Sigma^{\text{inner}}$ and the lateral regions $\Sigma^{+b}$, $\Sigma^{-b}$. From a qualitative point of view, these three layers correspond to three different types of trajectories: 1. Trajectories starting at $\Sigma^{\text{inner}}$ pass close to the cusp point. Observe that $\lim_{\varepsilon \to 0} (\Sigma^{\text{inner}}) = \{b = 0\}$ and then corresponds to a solution of the associated CDE passing exactly through the cusp point. 2. Trajectories starting at $\Sigma^{+b}$ pass sufficiently away from the cusp point along the regular side of the manifold $S$. 3. Trajectories starting at $\Sigma^{-b}$ pass sufficiently away from the cusp point along the folded side of the manifold $S$.]

We are now in position to present our main result. In the following theorem, we characterize the transition $\Pi : \Sigma^- \to \Sigma^+$ under a suitable choice of coordinates at the section $\Sigma^-$ and $\Sigma^+$. Furthermore, we give details on the differentiability of this map according to the cover of $\Sigma^-$, see definition 3.1.

**Theorem 3.1** (Transition map of an $A_3$-SFS). Let $X$ be an $A_3$ slow-fast system. This is, $X$ is a vector field defined by

$$
X = \varepsilon(1 + f_1) \frac{\partial}{\partial a} + \varepsilon f_2 \frac{\partial}{\partial b} - (z^3 + bz + a + \varepsilon f_3) \frac{\partial}{\partial z} + 0 \frac{\partial}{\partial \varepsilon},
$$

(55)

where each $f_i = f_i(a, b, z, \varepsilon), i = 1, 2, 3$, is smooth. Let the sections $\Sigma^-, \Sigma^+$ be defined as above. Then we can choose suitable $C^\ell$-coordinates $(B, Z, \varepsilon)$ in $\Sigma^-$ and $C^\ell$-coordinates $(\tilde{B}, \tilde{Z}, \varepsilon)$ in $\Sigma^+$ such that the transition $\Pi : (B, Z, \varepsilon) \mapsto (\tilde{B}, \tilde{Z}, \varepsilon)$ is an exponential type map of the form

$$
\Pi(B, Z, \varepsilon) = \left( B + h, \phi(B, \varepsilon) + Z \exp \left(-\frac{A(B, \varepsilon) + \Psi(B, Z, \varepsilon)}{\varepsilon}\right), \varepsilon \right),
$$

(56)

where $h$ is flat at the origin, $A > 0$ is $C^\ell$, $\phi$ is $C^\ell$-admissible with $\phi(B, 0) = 0$, and $\Psi$ is $C^\ell$-admissible with $\Psi(B, Z, 0) = 0$, see appendix A for the definition of $C^\ell$-admissible. Moreover, we have the following properties of the function $A, \phi$ and $\Psi$. 12
1. $-A(B, 0) = I(B)$ where $I$ is the slow divergence integral associated to (55).

2. Restricted to $(B, Z, \varepsilon) \in \Sigma_{\text{inner}}$, there are functions $\tilde{\phi}$ and $\tilde{\Psi}$ such that

$$
\phi(B, \varepsilon) = \tilde{\phi} \left( \mu, \varepsilon^{1/5} \right)
$$

$$
\Psi(B, Z, \varepsilon) = \tilde{\Psi} \left( |B|^{1/2}, \varepsilon^{1/5}, \varepsilon \ln |B|, \mu, Z \right),
$$

where $\tilde{\phi}$ and $\tilde{\Psi}$ are $C^\ell$-functions with respect to monomials (see definition A.2) with $\mu = B \varepsilon^{-2/5}$. Note that in this domain, $\mu$ is well defined in the sense that $\mu$ is bounded by a constant as $\varepsilon \to 0$.

3. Restricted to $(B, Z, \varepsilon) \in \Sigma_{\text{b}}^+$, there is a function $\tilde{\Psi}$ such that

$$
\phi(B, \varepsilon) = 0
$$

$$
\Psi(B, Z, \varepsilon) = \tilde{\Psi} \left( |B|^{1/2}, \varepsilon^{1/5}, \varepsilon \ln(|B|), \sigma, Z \right),
$$

where $\tilde{\Psi}$ is a $C^\ell$-function with respect to monomials (see definition A.2) with $\sigma = \varepsilon |B|^{-5/2}$. Note that in this domain, $\sigma$ is well defined since $|B| > 0$.

4. Restricted to $(B, Z, \varepsilon) \in \Sigma_{\text{b}}^-$, there are functions $\tilde{\phi}$ and $\tilde{\Psi}$ such that

$$
\phi(B, \varepsilon) = \tilde{\phi} \left( |B|^{1/2}, \sigma \right)
$$

$$
\Psi(B, Z, \varepsilon) = \tilde{\Psi} \left( |B|^{1/2}, \varepsilon^{1/5}, \varepsilon \ln(|B|), \sigma \right),
$$

where $\tilde{\phi}$ and $\tilde{\Psi}$ are $C^\ell$-functions with respect to monomials (see definition A.2) with $\sigma = \varepsilon |B|^{-5/2}$. Note that in this domain, $\sigma$ is well defined since $|B| > 0$.

**Sketch of the proof.** The first step is to recall theorem 2.1, which shows that $X$ is formally conjugate to

$$
F = \varepsilon \frac{\partial}{\partial a} + 0 \frac{\partial}{\partial b} - \left( z^3 + bz + a \right) \frac{\partial}{\partial z} + 0 \frac{\partial}{\partial \varepsilon}.
$$

Next, by means of the Borel’s lemma [5], the vector field $F$ can be realized as a smooth vector field $X^N = F + \varepsilon H$ where $H$ is flat at $(a, b, z, \varepsilon) = (0, 0, 0, 0)$. Thus, from now on, we only treat an $A_3$-SFS given as

$$
X = \varepsilon (1 + \varepsilon \tilde{f}_1) \frac{\partial}{\partial a} + \varepsilon^2 \tilde{f}_2 \frac{\partial}{\partial b} - \left( z^3 + bz + a + \varepsilon \tilde{f}_3 \right) \frac{\partial}{\partial z} + 0 \frac{\partial}{\partial \varepsilon},
$$

where each $\tilde{f}_i = \tilde{f}_i(a, b, z, \varepsilon)$ is flat at $(a, b, z, \varepsilon) = (0, 0, 0, 0)$.

Another important ingredient of the proof is the blow up technique, which is described in section 3.1. This method provides several local vector fields whose corresponding transitions are of exponential type, refer to appendix A. Later all these local transitions are composed to produce an exponential type transition between the sections $\Sigma^-$ and $\Sigma^+$. Along the analysis of the local vector fields (in the blow up space) we will take advantage of the flatness of the higher order terms of $X$. The complete proof follows sections 3.1 to 3.5 and is given in section 3.6.

Now, assuming that the transition $\Pi$ is of the form (56), we can show that $A(B, 0)$ is given by the slow divergence integral of $X$. For this, let us recall the Poincaré-Leontovich-Sotomayor formula [7], which in general is given as follows.
Proposition 3.1. Let $X$ be a vector field on a manifold $M^n$ with a volume form $\Omega$. Let $\Sigma^-$ and $\Sigma^+$ be two open sections of $M$ and transverse to the flow of $X$. Let $\gamma_\varepsilon$ be an orbit of $X$ along a center manifold $W^C_\varepsilon$ of $X$, starting at $p = \gamma_\varepsilon \cap \Sigma^-$ and reaching $q = \gamma_\varepsilon \cap \Sigma^+$ in finite time. Let $\Pi : \Sigma^- \to \Sigma^+$ be the transition map defined in a neighborhood of $p$. If $\psi^- : U \to \Sigma^-$ and $\psi^+ : V \to \Sigma^+$, with $U \subset \mathbb{R}^{n-1}$ and $V \subset \mathbb{R}^{n-1}$, are coordinates in $\Sigma^-$ and in $\Sigma^+$ respectively, then

$$\det \left( D \left( (\psi^+)^{-1} \circ \Pi \circ \psi^- \right) \right) (s^-) = \frac{\langle \Omega(p), D\psi^-(s^-) \times X(p) \rangle}{\langle \Omega(q), D\psi^+(s^+) \times X(p) \rangle} \exp \left( \int_{\gamma_\varepsilon} \text{div} X \, d\tau \right),$$

where $s^- = (\psi^-)^{-1}(p)$ and $s^+ = (\psi^+)^{-1}(q)$. The integral is taken along the orbit $\gamma_\varepsilon$ from $p$ to $q$ parametrized by the fast time $\tau$.

So we have the following.

Proposition 3.2. Consider an $A_3$-SFS and assume that the transition $\Pi : \Sigma^- \to \Sigma^+$ is given by (56). Then $-A(B,0) = I(B)$, where $I(B)$ is the slow divergence integral associated to the $A_3$-SFS.

Proof. The only relevant component is $Z$, so denote by $\Pi_Z$ the $Z$-component of $\Pi$. The factor multiplying the exponential in (62) can be taken as a constant $C > 0$. Then we have that (62) for the vector field of theorem 3.1 reads as

$$\frac{\partial \Pi_Z}{\partial Z} = C \exp \left( \int_{\gamma_\varepsilon} \text{div}_\Sigma X \, d\tau \right).$$

Using the properties of the slow divergence integral described in section 2.1.1, and since $C \neq 0$, we have

$$\frac{\partial \Pi_Z}{\partial Z} = C \exp \left( \int_{\gamma_\varepsilon} \text{div}_\Sigma X \, d\tau \right)
= \exp \left( \frac{1}{\varepsilon} \left( \int_{\gamma_0} \text{div} X_0 \, dt + \varepsilon \ln C + o(1) \right) \right)$$

where $I$ is the slow divergence integral of $X$ along a curve in the slow manifold $S$ from $\Sigma^-$ to $\Sigma^+$. In principle, the limit $\varepsilon \to 0$ of (64) is not well defined. However, according to our theorem 3.1, we have by differentiating (56) w.r.t. $Z$

$$\frac{\partial \Pi_Z}{\partial Z} = \exp \left( -\frac{A(B,\varepsilon) + \varepsilon \Psi(B, Z, \varepsilon)}{\varepsilon} \right).$$

Identifying (64) with (65) and taking the limit $\varepsilon \to 0$ we have indeed that

$$\lim_{\varepsilon \to 0} (I + O(\varepsilon)) = \lim_{\varepsilon \to 0} (-A(B,\varepsilon) + \varepsilon \Psi(B, Z, \varepsilon)),$$

which shows the claim. Note that the slow divergence integral in the coordinates $(a, b, z)$ reads as

$$I(b) = \hat{I}(b, \zeta^+) - \hat{I}(b, \zeta^-),$$

where straightforward computations show that

$$\hat{I}(b, \zeta) = \frac{9}{5} \zeta^5 + 2 \zeta^3 b + b^2 \zeta,$$

and where $\zeta^\pm$ is a constant defined by $(a^\pm, b, \zeta^\pm) \in \Sigma^\pm \cap S$. 

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On the other hand, in normal coordinates and along regular parts of the slow manifold, the $A_3$-SFS can be written as (see section 2.1.2)

$$X(A, B, Z, \varepsilon) = \varepsilon \frac{\partial}{\partial A} + 0 \frac{\partial}{\partial B} - Z \frac{\partial}{\partial Z} + 0 \frac{\partial}{\partial \varepsilon}. \quad (69)$$

In these coordinates the slow divergence integral reads as

$$I = A^+ - A^-, \quad (70)$$

where $A^+$ and $A^-$ are the corresponding parametrizations of $\Sigma^+$ and $\Sigma^-$ (respectively) in the coordinates $(A, B, Z, \varepsilon)$.

### 3.1. Blow-up and charts

Let us briefly recall the blow up technique, for more details see e.g. [8, 9, 17]. The vector field $X$ (50) is quasihomogeneous [1, 12]. Therefore, it is convenient to use the quasihomogeneous blow up. This technique consists on performing a coordinate transformation defined by

$$a = r^3 \bar{a}, \ b = r^2 \bar{b}, \ z = r \bar{z}, \ \varepsilon = r^5 \bar{\varepsilon}, \quad (71)$$

which is called the blow up map, and where $\bar{a}^2 + \bar{b}^2 + \bar{z}^2 + \bar{\varepsilon}^2 = 1$ and $r \in [0, +\infty)$. That is $(\bar{a}, \bar{b}, \bar{z}, \bar{\varepsilon}, r) \in S^3 \times R^+$. Since $\bar{\varepsilon} \geq 0$, we can restrict the coordinates to $\bar{\varepsilon} \geq 0$. Note that $S^3 \times \{0\}$ is mapped, via the blow up map (71), to the origin of $R^4$. The powers or weights of the blow up map (71) are obtained from the type of quasihomogeneity of $X$.

Let us denote by $\Phi(\bar{a}, \bar{b}, \bar{z}, \bar{\varepsilon})$ the blow up map (71). This map induces a smooth vector field $\tilde{X}$ on $S^3 \times R^+$ defined by $\Phi \cdot \tilde{X} = X$. It is often the case in which the vector field $\tilde{X}$ is degenerate along $S^3 \times \{0\}$. Then one defines another vector field $\hat{X}$ by $\hat{X} = \frac{1}{r^m} \tilde{X}$ for a well chosen positive integer $m$ so that $\hat{X}$ is non-degenerate along $S^3 \times \{0\}$. Since $r \in R^+$, the phase portraits of $\tilde{X}$ and $\hat{X}$ are equivalent outside $S^3 \times \{0\}$, and therefore it is equally useful to study $\tilde{X}$ instead of $\hat{X}$. One obtains a complete description of the local flow of $\tilde{X}$ near the the cusp point by studying the flow of $\hat{X}$ for $(\bar{a}, \bar{b}, \bar{z}, \bar{\varepsilon}, r) \in S^3 \times [0, r_0)$ with $r_0 > 0$ sufficiently small. For problems of dimension greater than 2, performing computations in spherical coordinates becomes tedious. Therefore, it is more convenient to consider charts which parametrize hemispheres of the ball $S^3 \times [0, r_0)$.

In the present context, the useful charts are

$$K_{en} = \{ \bar{a} = -1 \}, \ K_{ex} = \{ \bar{a} = 1 \}, \ K_{\varepsilon} = \{ \bar{\varepsilon} = 1 \}, \ K_{\pm} = \{ \bar{b} = \pm 1 \} \quad (72)$$

and we always keep $r \in [0, r_0)$. The previous setting is also known as directional blow up. A qualitative picture of the charts is given in fig. 6. Briefly speaking, our analysis goes as follows: first, we perform a local analysis on each chart given in (72). Next, we compose (“glue”) the local results to provide a full description of the flow of $X$ (50) in a small neighborhood of the cusp point. In this way, we construct an invariant manifold from $\Sigma^{en}$ to $\Sigma^{ex}$. Later we “push away” this invariant manifold all the way up to the sections $\Sigma^-$ and $\Sigma^+$ along regular parts of the slow manifold $S$.

To avoid confusion of the coordinates we adopt the following notation. Any object $O$ defined in the chart $K_{en}$ is denoted by $O_1$. Similarly any object defined in the chart $K_{ex}$ is denoted by $O_3$. Finally, an object $O$ defined in either of the charts $K_{\varepsilon}$ or $K_{\pm}$ is denoted by $O_2$.

### 3.2. Analysis in the chart $K_{en}$

Taking into account our notation convention, the blow-up map in this chart is given by

$$a = -r_1^3, \ b = r_1^2 b_1, \ z = r_1^3 z_1, \ \varepsilon = r_1^5 \varepsilon_1. \quad (73)$$
The corresponding vector field in this chart (after multiplication by 3) has the form

\[
X_{en} : \begin{cases}
    r'_1 &= -\varepsilon_1 r_1 \left(1 + \bar{f}_1\right) \\
    b'_1 &= 2\varepsilon_1 b_1 \left(1 + \bar{f}_1\right) + r_1^6 \varepsilon_1^2 \bar{f}_2 \\
    z'_1 &= -3 \left(z_1^3 + b_1 z_1 - 1 - \frac{1}{3} \varepsilon_1 z_1\right) + r_1^6 \varepsilon_1 \bar{f}_3 \\
    \varepsilon'_1 &= 5\varepsilon_1^2 \left(1 + \bar{f}_1\right)
\end{cases}
\]

where the functions \(\bar{f}_i = f_i(r_1, b_1, z_1, \varepsilon_1)\) are flat along \(r_1 = 0\), recall that \(S^3 \times \{r = 0\} \mapsto 0 \in \mathbb{R}^4\) via the blow up map. We study a transition \(\Pi_1 : \Delta_{en}^1 \rightarrow \Delta_{ex}^1\) where

\[
\Delta_{en}^1 = \{(r_1, b_1, z_1, \varepsilon_1) \in \mathbb{R}^4 \mid r_1 = r_0, \varepsilon_1 < \delta, z_1 > 0\} \\
\Delta_{ex}^1 = \{(r_1, b_1, z_1, \varepsilon_1) \in \mathbb{R}^4 \mid \varepsilon_1 = \delta, r_1 < r_0\},
\]

where \(r_0\) and \(\delta\) are sufficiently small positive constants.

**Remark 3.1.** The section \(\Delta_{en}^1\) corresponds to \(\Sigma_{en}\) in the blow-up space, that is \(\Sigma_{en} = \Phi(\Delta_{en}^1)\), where \(\Phi\) is the blow-up map (73). This implies that trajectories of \(X\) crossing \(\Sigma_{en}\) correspond to trajectories of \(X_{en}\) crossing \(\Delta_{en}^1\).

Before going any further, let us provide a qualitative description of \(X_{en}\) as in [4]. This process can be repeated, following similar arguments, in all the local charts; however, for brevity we only detail it for the current one.

**Qualitative description of the flow of \(X_{en}\).** The subspaces \(\{r_1 = 0\}, \{\varepsilon_1 = 0\}\) and \(\{r_1 = 0\} \cap \{\varepsilon_1 = 0\}\) are invariant. Therefore, it is useful to study the flow of \(X_{en}\) restricted to the aforementioned subspaces.

**Restriction to \(\{r_1 = 0\} \cap \{\varepsilon_1 = 0\}\).** In this space \(X_{en}\) is reduced to

\[
\begin{align*}
    b'_1 &= 0 \\
    z'_1 &= -3 \left(z_1^3 + b_1 z_1 - 1\right).
\end{align*}
\]

The set

\[
\gamma_1 = \{(b_1, z_1) \mid z_1^3 + b_1 z_1 - 1 = 0\}
\]

is a curve of equilibrium points. The phase portrait of (76) is shown in figure 7.
Remark 3.2. All the trajectories of (76) restricted to an initial condition $z_0 > 0$ are attracted to the curve $\gamma_1|_{z_0 > 0}$. Furthermore, due to our definition of $\Delta_{en}^m$, we are interested only in trajectories satisfying this initial condition. Thus, from now on, we restrict our analysis to the subspace $\{z_1 > 0\}$.

**Restriction to $\{\varepsilon_1 = 0\}$.** In this space $X_{en}$ is reduced to
\begin{align*}
    r'_1 &= 0 \\
    b'_1 &= 0 \\
    z'_1 &= -3(z_1^3 + b_1 z_1 - 1). 
\end{align*}

(78)

The set $\Gamma_1 = \{(r_1, b_1, z_1) \mid z_0^3 + b_1 z_1 - 1 = 0\}$ is a surface of equilibrium points given by $\Gamma_1 = (r_1, \gamma_1)$. Since $r'_1 = 0$, the phase space of (78) is foliated by two dimensional leaves in which the flow looks like fig. 7.

**Restriction to $\{r_1 = 0\}$.** In this space $X_{en}$ is reduced to
\begin{align*}
    b'_1 &= 2\varepsilon_1 b_1 \\
    z'_1 &= -3(z_1^3 + b_1 z_1 - 1 - \frac{1}{3}\varepsilon_1 z_1) \\
    \varepsilon'_1 &= 5\varepsilon_1^2. 
\end{align*}

(79)

Once again, the set $\gamma_1 = \{(b_1, z_1, \varepsilon_1) \mid \varepsilon_1 = 0, z > 0, z_0^3 + b_1 z_1 - 1 = 0\}$ is a curve of equilibrium points. The Jacobian of (79) evaluated along $\gamma_1$ shows that, for small enough $\varepsilon_1$, there exists an invariant center manifold that passes through $\gamma_1$. Furthermore, the non-zero eigenvalue corresponding to the $z$-direction is negative along $\gamma_1$. The phase portrait of (79) is shown in figure 8.

Observe that the $b_1$ and the $\varepsilon_1$ directions are expanding. It is important to know the relation between such two expanding variables. We have
\begin{equation}
\frac{db_1}{d\varepsilon_1} = \frac{2}{5} \frac{b_1}{\varepsilon_1}, \tag{80}
\end{equation}

which has the solution
\begin{equation}
    b_1 = b^*_1 \left(\frac{\varepsilon_1}{\varepsilon^*_1}\right)^{2/5}, \tag{81}
\end{equation}

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Figure 7: The phase portrait of $X_{en}$ restricted to the invariant space $\{r_1 = 0\} \cap \{\varepsilon_1 = 0\}$. The shown curve is $\gamma_1$ and it comprises a set of equilibrium points. Note that locally, all trajectories with initial condition $z_1(0) > 0$ are attracted to $\gamma_1|_{z_1 > 0}$. 
where $b_1^* \leq b_1$ and $\varepsilon_1^* \leq \varepsilon_1$ are the initial conditions, that is $(b_1^*, \varepsilon_1^*) = (b_1, \varepsilon_1)|_{\Delta_1^{en}}$. It is important to look at the ratio of initial conditions $\frac{b_1^*}{(\varepsilon_1^*)^2/5}$. This ratio tells us that $b_1$ is bounded as $\varepsilon_1 \to 0$ (and therefore as $\varepsilon_1^* \to 0$) if and only if $b_1^* \in O\left((\varepsilon_1^*)^{2/5}\right)$. In other words, if the initial condition $b_1^*$ is not of order $O\left((\varepsilon_1^*)^{2/5}\right)$ then the value of $b_1$ at $\Delta_1^{ex}$ blows up as $\varepsilon_1^* \to 0$. This leads us to partition the section $\Delta_1^{en}$ into three open regions as follows.

$$\begin{align*}
\Delta_1^{en,inner} &= \Delta_1^{en}|_{b_1 < M\varepsilon_1^{2/5}} \\
\Delta_1^{en,b_1} &= \Delta_1^{en}|_{b_1 > K\varepsilon_1^{2/5}} \\
\Delta_1^{en,-b_1} &= \Delta_1^{en}|_{-b_1 > K\varepsilon_1^{2/5}},
\end{align*}$$

(82)

where $0 < K < M < \infty$. Observe that the open sets $\Delta_1^{en,inner}$, $\Delta_1^{en,b_1}$ and $\Delta_1^{en,-b_1}$ form an open cover of $\Delta_1^{en}$. Accordingly, these sets induce an open cover of the entry section $\Sigma^{en}$ via the blow up map (73). See fig. 9 for a representation of the aforementioned partition.

Based on the partition of the entry section $\Delta_1^{en}$, we define three transitions as follows

$$\begin{align*}
\Pi_1^{inner} &: \Delta_1^{en,inner} \to \Delta_1^{ex} \\
\Pi_1^{b_1} &: \Delta_1^{en,b_1} \to \Delta_1^{ex+b_1} \\
\Pi_1^{-b_1} &: \Delta_1^{en,-b_1} \to \Delta_1^{ex,-b_1},
\end{align*}$$

(83)

where
To finish with the qualitative description, note that there exists a (non-unique) 3-dimensional center manifold \( W_{\Gamma_1} \), which is shown to exist by evaluating the Jacobian of \( X_{en} \) all along the surface

\[
\Gamma_1 = \{(r_1, b_1, z_1, \varepsilon_1) \mid \varepsilon_1 = 0, z_1 > 0, z_1^3 + b_1 z_1 - 1 = 0\}. \tag{85}
\]

Moreover, by the analysis provided above, the center manifold \( W_{\Gamma_1} \mid z_1 > 0 \) is attracting for \( \varepsilon_1 \) small enough. Note that \( W_{\Gamma_1} \mid \varepsilon_1 = 0 = \Gamma_1 \). This means that \( W_{\Gamma_1} \) can be interpreted as a perturbation of the slow manifold \( S \), written in the coordinates of the current chart. See fig. 10 for a representation of the previous exposition.

Figure 10: Phase portrait of the trajectories of \( X_{en} \) depending on their initial condition. If the trajectories satisfy the estimate \( y \in O(\varepsilon_2^2/5) \), then they arrive to \( \Delta_{\Gamma_1}^{ex,\varepsilon_1} \) in finite time. If the estimate \( y \in O(\varepsilon_2^2/5) \) is not satisfied, then we must choose one of the outgoing sections \( \Delta_{\Gamma_1}^{ex,\pm b_1} \) in order to have a well defined transition map.

Let us recall that the vector field \( X_{en} \) is of the form

\[
X_{en} : \begin{cases}
r_1' = -\varepsilon_1 r_1 \left(1 + \tilde{f}_3\right) \\
b_1' = 2\varepsilon_1 b_1 \left(1 + \tilde{f}_1\right) + r_1^2 \varepsilon_1^2 \tilde{f}_2 \\
z_1' = -3 \left(z_1^3 + b_1 z_1 - 1 - \frac{1}{3} \varepsilon_1 z_1\right) + r_1^2 \varepsilon_1 \tilde{f}_3 \\
\varepsilon_1' = 5\varepsilon_1^2 \left(1 + \tilde{f}_1\right)
\end{cases} \tag{86}
\]

We now proceed to describe the transitions \( \Pi_1 \) given by (83). For this, first we write (86) in a suitable normal form. Next, based on this normal form, we compute the corresponding transition.

First of all, let us move the origin to the point \((r_1, b, z_1, \varepsilon_1) = (0, 0, 1, 0)\). This is done by defining a new variable \( \zeta_1 \) by \( \zeta_1 = z_1 - 1 \). With this variable we have a new local vector field \( Y_{en} \) which is defined by
\[ Y_{\text{en}} : \begin{cases} r'_1 = -\varepsilon_1 r_1 (1 + \tilde{f}_1) \\ b'_1 = 2\varepsilon_1 b_1 (1 + \tilde{f}_1) + r_1^6 \varepsilon_1^2 \tilde{f}_2 \\ \varepsilon'_1 = 5\varepsilon_1^2 (1 + \tilde{f}_1) \\ \zeta'_1 = -3G(b_1, \varepsilon_1, \zeta_1) + \varepsilon_1 \tilde{h}, \end{cases} \]  

(87)

where \( G(0, 0, 0) = 0 \) and \( \frac{\partial G}{\partial \zeta_1}(0, 0, 0) = 3 \). Now, we want to write \( Y_{\text{en}} \) in a suitable normal form. From proposition C.1, we know that \( Y_{\text{en}} \) is \( C^\ell \)-equivalent to \( X_{\text{en}} \):

\[ X_{\text{en}} : \begin{cases} r'_1 = -\varepsilon_1 r_1 \\ B'_1 = 2\varepsilon_1 B_1 \\ \varepsilon'_1 = 5\varepsilon_1^2 \\ Z'_1 = -9(1 + H_1(r_1, B_1, \varepsilon_1)) Z_1, \end{cases} \]  

(88)

where \( H_1 \) is a \( C^\ell \)-function vanishing at the origin. This normal form \( X_{\text{en}} \) is convenient since the chosen center manifold \( W_{\text{C}}^1 \) is now simply given by \( W_{\text{C}}^1 = \{ Z_1 = 0 \} \). Furthermore, from the format of \( X_{\text{en}} \), it is evident the “hyperbolic nature” of the flow restricted to the center manifold: the restriction of \( X_{\text{en}} \) to the center manifold \( W_{\text{C}}^1 \) has a simple structure, namely

\[ X_{\text{en}}|_{W_{\text{C}}^1} : \begin{cases} r'_1 = -\varepsilon_1 r_1 \\ B'_1 = 2\varepsilon_1 B_1 \\ \varepsilon'_1 = 5\varepsilon_1^2. \end{cases} \]  

(89)

Note that for \( \varepsilon_1 \neq 0 \), the vector field \( \frac{1}{\varepsilon_1} X_{\text{en}}|_{W_{\text{C}}^1} \) is hyperbolic.

The vector field \( X_{\text{en}} \) is of the form studied in proposition C.4, therefore we have that the transition

\[ \Pi_{1}^{\text{inner}} : (B_1, \varepsilon_1, z_1) \mapsto (\tilde{r}_1, \tilde{B}_1, \tilde{Z}_1) \]  

(90)

is of the form

\[ \tilde{r}_1 = r_0 \left( \frac{\varepsilon_1}{\delta} \right)^{1/5} \\
\tilde{B}_1 = B_1 \left( \frac{\delta}{\varepsilon_1} \right)^{2/5} \\
\tilde{Z}_1 = Z_1 \exp \left( -\frac{9}{5\varepsilon_1^2} \left( 1 + \alpha_1 \varepsilon_1 \ln \varepsilon_1 + \varepsilon_1^2 G_1 \right) \right), \]  

(91)

where \( \alpha_1 = \alpha_1(r_0|B_1|^{1/2}, r_0 \varepsilon_1^{1/5}) \) and \( G_1 = G_1(r_0|B_1|^{1/2}, r_0 \varepsilon_1^{1/5}, \mu) \) where \( \mu = B_1 \varepsilon_1^{-2/5} \). Recall that for this transition we have the condition \( B_1 \in O(\varepsilon_1^{2/5}) \) so \( \mu \) is well defined.

On the other hand, the transition

\[ \Pi_{1}^{\pm B_1} : (B_1, \varepsilon_1, Z_1) \mapsto (\tilde{r}_1, \tilde{\varepsilon}_1, \tilde{Z}_1) \]  

(92)
is (see proposition C.4) of the form
\[ \ddot{r}_1 = r_0 \left( \frac{B_1}{\eta} \right)^{1/2}, \]
\[ \ddot{\varepsilon}_1 = \varepsilon_1 \left( \frac{\eta}{B_1} \right)^{5/2}, \]
\[ \ddot{Z}_1 = Z_1 \exp \left( -\frac{9}{5\varepsilon_1}(1 + \beta_1 \varepsilon_1 \ln(|B_1|) + \varepsilon_1 H_1) \right), \]
where \( \beta_1 = \beta_1(r_0|B_1|^{1/2}, r_0 \varepsilon_1^{1/5}) \) and \( H_1 = H_2(r_0|B_1|^{1/2}, r_0 \varepsilon_1^{1/5}, \sigma) \), where \( \sigma = \varepsilon_1|B_1|^{-5/2} \). Note that since \( B_1 \notin O(\varepsilon_1^{2/5}) \), \( \sigma \) is well defined. We observe that the transitions \( \Pi_1^{\pm} \) and \( \Pi_1^{\pm B_1} \) are exponential type maps.

3.3. Analysis in the chart \( K_\varepsilon \)

Taking into account our notation convention, the blow-up map in this chart is given by

\[ a = r_2^3 a_2, \ b = r_2^2 a_2, \ z = r_2^2 z_2, \ \varepsilon = r_2^5. \]  
(94)

Then, the blown up vector field reads as

\[ X_\varepsilon : \begin{cases} 
   r_2' &= 0 \\
   a_2' &= 1 + \tilde{g}_1 \\
   b_2' &= r_0^5 \tilde{g}_2 \\
   z_2' &= -(z_2^2 + b_2 z_2 + a_2) + \tilde{g}_3,
\end{cases} \]
(95)

where the function \( \tilde{g}_1 = \tilde{g}_1(r_2, a_2, b_2, z_2) \) are flat along \( r_2 = 0 \). Note that in this chart \( r_2 \) acts as a parameter and that the flow is regular. Furthermore, note that \( X_\varepsilon \) is not a slow-fast system, but a regular vector field.

From the equation \( a_2' = 1 + \tilde{g}_1 \), we define the following “entry” and “exit” sections.

\[ \Delta_2^{en,\varepsilon} = \{(r_2, a_2, b_2, z_2) | a_2 = -A_0, \ z_2 \geq 0 \}, \]
\[ \Delta_2^{ex,\varepsilon} = \{(r_2, a_2, b_2, z_2) | a_2 = A_0, \ z_2 \leq 0 \}. \]
(96)

Therefore, we define a transition \( \Pi_2^{\varepsilon} \) as

\[ \Pi_2^{\varepsilon} : \Delta_2^{en,\varepsilon} \to \Delta_2^{ex,\varepsilon} \]
\[ (r_2, b_2, z_2) \mapsto (\tilde{r}_2, \tilde{b}_2, \tilde{z}_2). \]
(97)

Since (95) is regular, by the flow box theorem all trajectories starting at \( \Delta_2^{en,\varepsilon} \) arrive at \( \Delta_2^{ex,\varepsilon} \) in finite time. Moreover, the transition \( \Pi_2^{\varepsilon} \) is a diffeomorphism and then, from (95) we have that \( \Pi_2^{\varepsilon} \) reads as

\[ \Pi_2^{\varepsilon}(r_2, b_2, z_2) = (\tilde{r}_2, \tilde{b}_2, \tilde{z}_2) \]
\[ = (r_2, b_2 + h_{b_2} \phi_1(r_2, b_2) + \phi_2(r_2, b_2)(1 + \phi_0(r_2, b_2, z_2))) z_2, \]
(98)

where the \( \phi_i \)'s are smooth functions. Observe that in this chart, the transition is not an exponential type map.
3.4. Analysis in the chart \( K_{ex} \)

Taking into account our notation convention, the blow-up map in this chart is given by

\[
a = r_3^3, \quad b = r_3^2b_3, \quad z = r_3^2z_3, \quad \varepsilon = r_3^5\varepsilon_3.
\]  

(99)

Then, the blown up vector field reads as

\[
X_{ex} : \begin{cases}
r_{34} = \varepsilon_3 r_3 (1 + \tilde{f}_1) \\
b_{34} = -2\varepsilon_3 b_3 (1 + \tilde{f}_1) + r_3^2 \varepsilon_3 f_2 \\
\varepsilon_{34} = -3 (z_3^3 + b_3 z_3 + 1 + \frac{1}{3} \varepsilon_3 z_3) + r_3^2 \varepsilon_3 \tilde{f}_3 \\
e_{34} = -5\varepsilon_3^2 (1 + \tilde{f}_1)
\end{cases}
\]  

(100)

where the function \( \tilde{f}_1 = \tilde{f}_1(r_3, b_3, \varepsilon_3, z_3) \) are flat along \( r_3 = 0 \). Observe that the vector field \( X_{ex} \) resembles the vector field \( X_{en} \). Therefore, we have a similar behavior of the trajectories, the main difference is that in the case of \( X_{ex} \), there is one expanding \( (r_3) \) and three contracting \( (b_3, \varepsilon_3 \) and \( z_3) \) directions. The flow of \( X_{ex} \) is obtained following similar arguments as for the flow of \( X_{en} \).

From the fact that \( X_{ex} \) has three contracting and one expanding direction, we define the entry sections

\[
\Delta_{3}^{en,e} = \{(r_3, b_3, \varepsilon_3, z_3) : \varepsilon_3 = \delta, \ z_3 < 0, \ r_3 < r_0 \}
\]
\[
\Delta_{3}^{en+b3} = \{(r_3, b_3, \varepsilon_3, z_3) : b_3 = \eta, \ z_3 < 0, \ r_3 < r_0 \}
\]
\[
\Delta_{3}^{en-b3} = \{(r_3, b_3, \varepsilon_3, z_3) : b_3 = -\eta, \ z_3 < 0, \ r_3 < r_0 \},
\]  

(101)

where all the constants are positive and sufficiently small, and the exit section

\[
\Delta_{3}^{ex} = \{(r_3, b_3, \varepsilon_3, z_3) : r_3 = r_0, \ z_3 < 0, \ \varepsilon_3 < \delta, \ |b_3| < \eta \}.
\]  

(102)

Then, accordingly, we define three transition maps as follows

\[
\Pi_{3}^{\varepsilon_3} : \Delta_{3}^{en,e} \rightarrow \Delta_{3}^{ex} \\
: (r_3, b_3, z_3) \mapsto (\tilde{b}_3, \tilde{\varepsilon}_3, \tilde{z}_3)
\]
\[
\Pi_{3}^{+b3} : \Delta_{3}^{en+b3} \rightarrow \Delta_{3}^{ex} \\
: (r_3, \varepsilon_3, z_3) \mapsto (\tilde{b}_3, \tilde{\varepsilon}_3, \tilde{z}_3)
\]
\[
\Pi_{3}^{-b3} : \Delta_{3}^{en-b3} \rightarrow \Delta_{3}^{ex} \\
: (r_3, \varepsilon_3, z_3) \mapsto (\tilde{b}_3, \tilde{\varepsilon}_3, \tilde{z}_3).
\]  

(103)

Now we proceed to write \( X_{ex} \) in a normal form just as we did with \( X_{en} \) in section 3.2. Following Proposition C.1 we have that \( X_{ex} \) is \( C^\ell \) equivalent to

\[
X_{ex}^N : \begin{cases}
r_{34} = \varepsilon_3 r_3 \\
B_{34} = -2\varepsilon_3 B_3 \\
\varepsilon_{34} = -5\varepsilon_3^2 \\
Z_{34} = -9(1 + H_3) Z_3
\end{cases}
\]  

(104)
where $H_3 = H_3(r_3, B_3, \varepsilon_3)$ is a $C^4$ function vanishing at the origin. Just as in the chart $K_{en}$, there exists a three dimensional center manifold $\mathcal{W}^c_3$ associated to $X^X_{23}$ and which has been chosen such that $\mathcal{W}^c_3 = \{Z_3 = 0\}$. Since $r_3$ is the only expanding direction, we take as transition time $T_3 = \ln \left( \frac{r_0}{r_3} \right)$. This transition time is computed from the dynamics restricted to $\mathcal{W}^c_3$, that is, from the equation $r_3^3 = r_3$. In contrast to what happened in the chart $K_{en}$, the time $T_3$ is well defined for all the three transitions $\Pi^c_3$, $\Pi^b_3$ and $\Pi^{-b}_3$. Following proposition C.4 we have

$$\begin{align*}
\tilde{B}_3 &= B_3 \left( \frac{r_3}{r_0} \right)^2 \\
\tilde{\varepsilon}_3 &= \varepsilon_3 \left( \frac{r_3}{r_0} \right)^5 \\
\tilde{Z}_3 &= Z_3 \exp \left( -\frac{9}{5\varepsilon_3} \left( \left( \frac{r_0}{r_3} \right)^5 - 1 + \alpha_3 \varepsilon_3 \ln r_3 + \varepsilon_3 H_3 \right) \right),
\end{align*}$$

(105)

where $\alpha_3 = \alpha_3(r_3|B_3|^{1/2}, r_3 \varepsilon_3^{1/5})$ and $H_3 = H_3(r_3|B_3|^{1/2}, r_3 \varepsilon_3^{1/5}, r_3)$. Therefore, by taking the definitions of the entry sections we have

$$\begin{align*}
\Pi^c_3(r_3, B_3, Z_3) &= \left( B_3 \left( \frac{r_3}{r_0} \right)^2, \delta \left( \frac{r_3}{r_0} \right)^5, Z_3 \exp \left( -\frac{9}{5\varepsilon_3} \left( \left( \frac{r_0}{r_3} \right)^5 - 1 + \alpha_3 \delta \ln r_3 + \delta H_3 \right) \right) \right), \\
\Pi^{b\pm}_3(r_3, \varepsilon_3, Z_3) &= \left( \pm \eta \left( \frac{r_3}{r_0} \right)^2, \varepsilon_3 \left( \frac{r_3}{r_0} \right)^5, Z_3 \exp \left( -\frac{9}{5\varepsilon_3} \left( \left( \frac{r_0}{r_3} \right)^5 - 1 + \alpha_3 \varepsilon_3 \ln r_3 + \varepsilon_3 H_3 \right) \right) \right).
\end{align*}$$

(106)

Observe that these transitions are of exponential type.

### 3.5. Analysis in the charts $K_{+b}$

In this section we study the local flow at the charts $K_{+b}$ and $K_{-b}$. In a qualitative sense, these charts come into play when the initial condition $b_0 = b|_{\mathcal{S}^{en}}$ does not satisfy the estimate $b_0 \in O(\varepsilon^{3/5})$. This implies that the corresponding trajectory passes away from the cusp point. The chart $K_{+b}$ “sees” trajectories with initial condition $b|_{\mathcal{S}^{en}} > 0$ while $K_{-b}$ “sees” trajectories with initial condition $b|_{\mathcal{S}^{en}} < 0$.

**Analysis in the chart $K_{+b}$**

In this chart the blow-up maps reads

$$a = r_3^3 a_2, b = r_3^2, z = r_2 z_2, \varepsilon = r_2^5 \varepsilon_2.$$

(107)

Then we have that the blow-up vector field is given by

$$X_{+b} : \begin{cases}
  r'_2 &= \varepsilon_2 f, \\
  a'_2 &= \varepsilon_2 (1 + f_2 z_2) + \varepsilon_2 g a_2 \\
  \varepsilon'_2 &= -\varepsilon_2 f_2 z_2 \\
  z'_2 &= -(z_2^2 + z_2 + a_2) + \varepsilon_2 f_2 z_2
\end{cases}$$

(108)

where all the functions $f_i$ are flat along $\{r_2 = 0\}$. Observe that the set

$$\Gamma_2 = \{(r_2, a_2, \varepsilon_2, z_2) | \varepsilon_2 = 0, z_2^2 + z_2 + a_2 = 0\}$$

(109)
is a NHIM of $X_{+\delta}$. However, $X_{+\delta}$ is not exactly a slow-fast system since $\varepsilon_2 \neq 0$, but the restriction of $X_{+\delta}$ to $\{r_2 = 0\}$ is indeed a slow-fast system. This restriction reads as

$$X_{+\delta}|_{\{r_2 = 0\}}: \begin{cases} a'_2 = \varepsilon_2 \varepsilon'_2 = 0 \\ z'_2 = -z_2^3 + z_2 + a_2. \end{cases} \quad (110)$$

**Remark 3.3.** The subspace $\{r_2 = 0\}$ is invariant. Moreover, since $X_{+\delta}$ is a flat perturbation of $X_{+\delta}|_{\{r_2 = 0\}}$, it is equally useful to study the restriction $X_{+\delta}|_{\{r_2 = 0\}}$. After all, by regular perturbation theory, their flows are equivalent.

The slow manifold of $X_{+\delta}|_{\{r_2 = 0\}}$ is defined by $\Gamma_2|_{r_2 = 0}$ and is normally hyperbolic. Let us define the sections

$$\Delta_{en}^{\text{en+,b}_2} = \{(r_2, a_2, z_2) \in \mathbb{R}^4 | a_2 = -A_0\}$$
$$\Delta_{ex}^{\text{ex+,b}_2} = \{(r_2, a_2, z_2) \in \mathbb{R}^4 | a_2 = A_0\}. \quad (111)$$

Accordingly, we study the transition

$$\Pi_{2}^{+\text{b}_2}: \Delta_{en}^{\text{en+,b}_2} \rightarrow \Delta_{ex}^{\text{ex+,b}_2} \quad (r_2, \varepsilon_2, z_2) \mapsto (\tilde{r}_2, \tilde{\varepsilon}_2, \tilde{z}_2). \quad (112)$$

For a qualitative description of $X_{+\delta}|_{\{r_2 = 0\}}$ and the objects defined above see fig. 11.

![Figure 11](image-url)

Figure 11: Left: phase portrait of the corresponding layer equation of $X_{+\delta}|_{\{r_2 = 0\}}$. Center: phase portrait of the corresponding CDE of $X_{+\delta}|_{\{r_2 = 0\}}$. Right: Since the critical manifold is regular, by Fenichel theory we know that the manifold $\Gamma_2$ is perturbed to an invariant manifold $\Gamma_2, \varepsilon_2$ which is at distance of order $O(\varepsilon_2)$ from $\Gamma_2$.

We know from section 2.1.2 that for sufficiently small $\varepsilon_2$, there exists a $C^\ell$ change of coordinates that transforms $X_{+\delta}|_{\{r_2 = 0\}}$ into the vector field

$$Y^N: \begin{cases} a'_2 = \varepsilon_2 \\ \varepsilon'_2 = 0 \\ Z'_2 = -Z_2. \end{cases} \quad (113)$$

From the definition of the entry and exit sections (111), the time of integration is $T = 2A_0$. To obtain the component $Z_2$ of the transition $\Pi_{2}^{+\text{b}_2}|_{\{r_2 = 0\}}$ we need to integrate

$$Z'_2 = -\frac{1}{\varepsilon_2}Z_2, \quad (114)$$
and then $\tilde{Z}_2 = Z_2(T)$. Therefore we have that after choosing a center manifold $W_2^C$, the transition $\Pi_{2}^{+\text{b}_2}$ reads as
Note that $\Pi_2^{+b_2}$ is an exponential type map.

**Analysis in the chart $K_{-b}$**

In this chart the blow-up maps reads

$$a = r_2^3 a_2, \quad b = -r_2^2, \quad z = r_2 z_2, \quad \varepsilon = r^5 \varepsilon_2.$$  \hfill (116)

Then we have that the blow-up vector field is given by

$$X_{-b} : \begin{cases} r_2' = -\varepsilon_2 f_r \\ a_2' = \varepsilon_2 (1 + f_{a_2}) + \varepsilon_2 g_{a_2} \\ \varepsilon_2' = \varepsilon_2 f_{\varepsilon_2} \\ z_2' = -(z_2^3 - z_2 + a_2) + \varepsilon_2 f_{z_2} \end{cases} \quad \text{(117)}$$

where all the functions $f_r$ and $g_{a_2}$ are flat along $\{r_2 = 0\}$. Observe that, as in the previous section, the subspace $\{r_2 = 0\}$ is invariant. The restriction of $X_{-b}$ to this subspace reads as

$$X_{-b}|_{\{r_2 = 0\}} : \begin{cases} a_2' = \varepsilon_2 \\ \varepsilon_2' = 0 \\ z_2' = -(z_2^3 - z_2 + a_2) \end{cases} \quad \text{(118)}$$

The flow of $X_{-b}$ is a flat perturbation of the flow of $X_{-b}|_{\{r_2 = 0\}}$. Therefore, let us continue our analysis restricted to the invariant space $\{r_2 = 0\}$.

The manifold $\Gamma_2$, which is defined by

$$\Gamma_2 = \{(r_2, a_2, \varepsilon_2, z_2) \mid r_2 = 0, \varepsilon_2 = 0, z_2^3 - z_2 + a_2 = 0\} \quad \text{(119)}$$

is normally hyperbolic except at the two points $p_{\pm} = \pm \left(\frac{2}{3\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$. Let us define the sections

$$\Delta_2^{en,-b_2} = \{(r_2, a_2, \varepsilon_2, z_2) \in \mathbb{R}^4 \mid a_2 = -A_0\} \quad \text{(120)}$$

$$\Delta_2^{en,+b_2} = \{(r_2, a_2, \varepsilon_2, z_2) \in \mathbb{R}^4 \mid a_2 = A_0\},$$

where $A_0 > 0$ is a sufficiently large constant. We are interested in the transition

$$\Pi_2^{+b_2} : \Delta_2^{en,-b_2} \to \Delta_2^{en,+b_2}$$

$$(r_2, \varepsilon_2, z_2) \mapsto (\tilde{r}_2, \tilde{\varepsilon}_2, \tilde{z}_2) \quad \text{(121)}$$

For a qualitative description of $X_{-b}|_{\{r_2 = 0\}}$ and the objects defined above see fig. 12. Away from the fold points $p_{\pm}$, the manifold $\Gamma_2$ is regular and thus, Fenichel’s theory applies. However, we need to take care of the transition near the fold point $p_+$. The local transition of a slow-fast system near a fold point is investigated in e.g. [18]. However, in our current problem this transition is not essential. By this we mean that the passage through the fold point is seen as a flat perturbation of the trajectory along the stable branch of $\Gamma_2$. In a qualitative sense, this is due to the fact that the transition $\Pi_2^{+b_2}$ goes along a large NHIM, which fails to be normally hyperbolic only at one point.
Proposition 3.3. We can choose appropriate coordinates \((Z_2, \varepsilon_2)\) in \(\Delta_{en}^{en,-b_2}\) such that the transition \(\Pi_{2}^{b_2} : \Delta_{en}^{en,-b_2} \to \Delta_{ex}^{ex,-b_2}\), restricted to \(r_2 = 0\), is an exponential type map of the form

\[
\Pi_{2}^{b_2}(0, \varepsilon_2, Z_2) = \left(0, \varepsilon_2, \phi_2(\varepsilon_2) + Z_2 \exp \left(-\frac{1}{\varepsilon_2} (A_0 + \varepsilon_2 \psi_2(Z_2, \varepsilon_2)) \right) \right),
\]

where \(\phi_2\) are flat at \(\varepsilon_2 = 0\), \(\psi_2\) is \(C^\ell\)-admissible, and where \(A_0\) is given by the slow divergence integral of \(X_{-\delta}\{r_2=0\}\).

Proof. To prove that \(A_0\) is given by the slow divergence integral we proceed along the same reasoning as in proposition 3.2, so we do not repeat it here. In figure fig. 13 we see the three transitions that we must consider.

The three transitions are defined as

\[
\begin{align*}
\Pi_{2}^{reg_1} & : \Delta_{en}^{en,-b_2} \to \Omega^{en} \\
\Pi_{2}^{fold} & : \Omega^{en} \to \Omega^{ex} \\
\Pi_{2}^{reg_2} & : \Omega^{ex} \to \Delta_{ex}^{ex,-b_2},
\end{align*}
\]

where we define \(\Omega^{en}\) and \(\Omega^{ex}\) as
\[ \Omega^{en} = \{ (a_2, \varepsilon, Z_2) \in \mathbb{R}^3 \mid a_2 = -a_{2, en} \} \]
\[ \Omega^{ex} = \{ (a_2, \varepsilon, Z_2) \in \mathbb{R}^3 \mid Z_2 = -Z_{2, ex} \}, \]

where \(a_{2, en}\) and \(Z_{2, ex}\) are sufficiently small positive constants. The total transition \(\Pi_2^{+b_2}\) is given by \(\Pi_2^{+b_2} = \Pi_2^{reg} \circ \Pi_2^{old} \circ \Pi_2^{reg_1}\). Recall from appendix A that if we want to write the transition \(\Pi_2^{+b_2}\) as an exponential type map, we require that \(\Pi_2^{reg_1}\) is expressed as an exponential type map with no shift. The transition \(\Pi_2^{old}\) is studied in e.g. [13, 18]. In [13] is proved that there are local coordinates \((\tilde{Z}_2, \varepsilon)\) in \(\Omega^{en}\), and \((\tilde{a}_2, \tilde{\varepsilon})\) in \(\Omega^{ex}\), such that the transition \(\Pi_2^{old}\) is given by

\[ \Pi_2^{old}(\tilde{Z}_2, \varepsilon) = (\tilde{a}_2, \tilde{\varepsilon}_2). \]

Assume now that we have characterized an invariant manifold \(\mathcal{M}_2^{old}\) from \(\Omega^{en}\) to \(\Omega^{ex}\) via the map \(\Pi_2^{old}\). Now we want to “extend” \(\mathcal{M}_2^{old}\) all the way up to the sections \(\Delta_2^{en,-b_2}\) and \(\Delta_2^{ex,-b_2}\) via transitions along normally hyperbolic regions of \(\Gamma_2\). For this, it is more convenient to regard \(\mathcal{M}_2^{old}\) as a graph \(\zeta = \phi_{\varepsilon_2}(A_2)\) where \((\zeta_2, A_2)\) are local coordinates around the fold point \(p_+\) and where \(\phi_{\varepsilon_2}\) is a diffeomorphism for \(\varepsilon_2 > 0\). In this way we can equivalently express the map \(\Pi_2^{old}\) as

\[ \Pi_2^{old}(\zeta_2, \varepsilon_2) = (\tilde{\zeta}_2, \tilde{\varepsilon}_2). \]

where \(\psi_{\varepsilon_2}\) is a diffeomorphism for \(\varepsilon_2 > 0\) and only a homeomorphism for \(\varepsilon_2 = 0\). Next, following section 2.1.2 we can find coordinates \((Z_2, \varepsilon_2)\) in \(\Delta_2^{en,-b_2}\), and coordinates \((\tilde{Z}_2, \varepsilon_2)\) in \(\Delta_2^{ex,-b_2}\) in such a way that the transitions \(\Pi_2^{reg_1}\) and \(\Pi_2^{reg_2}\) are given as

\[ \Pi_2^{reg_1}(Z_2, \varepsilon_2) = \left( Z_2 \exp \left( -\frac{1}{\varepsilon_2} (A_0 - a_{2, en}) \right) \right) = (\tilde{Z}_2, \varepsilon_2) \]
\[ \Pi_2^{reg_2}(-Z_{2, ex}, \varepsilon_2) = \left( -\frac{1}{\varepsilon_2} (A_0 - \tilde{a}_2) \right) = (\tilde{Z}_2, \varepsilon_2). \]

**Remark 3.4.** Recall that along normally hyperbolic slow manifolds, it is possible to make a normal form transformation in such a way that this transformation respects certain constraint or structure of the vector field, [2, 3]. In this particular case, we respect the choice of the invariant manifold \(\mathcal{M}_2^{old}\).

Next, we can compute the composition \(\Pi_2^{-b_2} = \Pi_2^{reg_2} \circ \Pi_2^{old} \circ \Pi_2^{reg_1}\) by following appendix A and it thus follows that

\[ \Pi_2^{-b_2}(0, \tilde{Z}_2, \varepsilon_2) = \left( 0, \tilde{\psi}_{\varepsilon_2} + Z_2 \exp \left( -\frac{1}{\varepsilon_2} (A_1 + A_3 + \varepsilon_2 \psi_2) \right), \varepsilon_2 \right), \]

where \(\tilde{\psi}_{\varepsilon_2} = \psi_{\varepsilon_2}(0) \exp \left( -\frac{A_{-1}}{\varepsilon_2} \right)\) and where \(\psi_2 = \psi_2(Z, \varepsilon_2)\) is a \(C^l\)-admissible function. Note that \(\tilde{\psi}_{\varepsilon_2}\) is flat at \(\varepsilon_2 = 0\).
3.6. Proof of theorem 3.1

Let us first recall that, within the blow up space, we have three types of transitions according to the initial condition \( b_1|_{\Delta_{\text{en}}} \), namely

- If \( b_1|_{\Delta_{\text{en}}} \in O\left(\varepsilon_1^{2/5}\right) \) then we construct a transition passing through the charts \( K_{\text{en}} \to K_{\varepsilon} \to K_{\text{ex}} \).
- If \( b_1|_{\Delta_{\text{en}}} \notin O\left(\varepsilon_1^{2/5}\right) \) and \( b_1|_{\Delta_{\text{en}}} > 0 \) then we construct a transition passing through the charts \( K_{\text{en}} \to K_{\varepsilon}^{+} \to K_{\text{ex}} \).
- If \( b_1|_{\Delta_{\text{en}}} \notin O\left(\varepsilon_1^{2/5}\right) \) and \( b_1|_{\Delta_{\text{en}}} < 0 \) then we construct a transition passing through the charts \( K_{\text{en}} \to K_{\varepsilon}^{-} \to K_{\text{ex}} \).

In fig. 14 we give a qualitative diagram of the local transitions obtained and their relationship.

Let us only detail the transition through the inner layer \( \Delta_{\text{inner}} \) corresponding to \( b_1|_{\Delta_{\text{en}}} \in O\left(\varepsilon_1^{2/5}\right) \), the other cases follow the same lines.

The transition \( \Pi_{\text{inner}} : \Delta_{\text{inner}} \to \Delta_{\text{ex}}^{\text{2}} \) is given as

\[
\Pi_{\text{inner}} = \Pi_{\varepsilon}^{3} \circ M_{\text{ex}}^{\ast} \circ \Pi_{\varepsilon}^{2} \circ M_{\text{en}}^{\ast} \circ \Pi_{\text{inner}}^{1},
\]

where the matching maps are obtained from the blow-up map. For example, to obtain the matching map from the chart \( K_{\text{en}} \) to the chart \( K_{\varepsilon}^{\ast} \) we relate the two directional blow-up maps

\[
a = -r_{1}^{2}b_{1}, \quad b = r_{1}^{2}b_{1}, \quad z = r_{1}z_{1}, \quad \varepsilon = r_{1}^{5}\varepsilon_{1}
\]

and

\[
a = r_{1}^{2}a_{2}, \quad b = r_{2}^{2}b_{2}, \quad z = r_{2}z_{2}, \quad \varepsilon = r_{2}^{5}.
\]

Let us work out only with the \( z \)-component of the transitions as it is the only relevant one. Recall from section 3.2 that \( \Pi_{\text{inner}}^{1} \) is an exponential type map with no shift. Next, the composition \( \Pi_{\text{central}}^{1} = M_{\text{en}}^{\ast} \circ \Pi_{\varepsilon}^{2} \circ M_{\text{en}}^{\ast} \circ \Pi_{\text{inner}}^{1} \) yields a diffeomorphism as \( \Pi_{\varepsilon}^{2} \) is a diffeomorphism, and the matching maps are also diffeomorphisms on their domain of definition. Next, the last transition \( \Pi_{\varepsilon}^{3} \) is an exponential type map with no shift, see section 3.4. Therefore, following appendix A we have that \( \Pi_{\varepsilon}^{3} \circ \Pi_{\text{central}}^{1} \circ \Pi_{\text{inner}}^{1} \) is an exponential type map of the form

\[
\Pi_{\text{inner}}^{1} = \phi(B_{1}, \varepsilon_{1}) + Z_{1} \exp \left( -\frac{1}{\varepsilon_{1}} (A(B_{1}, \varepsilon_{1}) + \varepsilon_{1}\Psi(B_{1}, \varepsilon_{1}, Z_{1})) \right),
\]

where \( A > 0 \) and \( \phi \) and \( \Psi \) are \( C^{1} \) admissible functions. The differentiability of \( \phi \) and \( \Psi \) with respect to monomials is evident from the results of section 3.2. By blowing down we obtain that the transition \( \Pi_{\text{inner}}^{1} : \Sigma_{\text{en}} \to \Sigma_{\text{ex}} \) (in a small neighborhood of the cusp point and within the inner layer as domain) reads as

\[
\Pi_{\text{inner}}^{1} = \phi(B, \varepsilon) + Z \exp \left( -\frac{1}{\varepsilon} (A(B, \varepsilon) + \varepsilon\Psi(B, \varepsilon, Z)) \right).
\]

To obtain the transition \( \Pi : \Sigma^{-} \to \Sigma^{+} \) we now need to compose \( \Pi_{\text{inner}}^{1} \) with exponential type maps on the left and on the right corresponding to
Figure 14: All the transitions obtained in the charts. We have to compose all such transitions through the matching maps $M_i^j$. A matching map $M_i^j$ relates the coordinates between the charts $K_i$ and $K_j$. 
\[ \Pi^- : \Sigma^- \rightarrow \Sigma^{\text{en}} \\
\Pi^+ : \Sigma^{\text{ex}} \rightarrow \Sigma^+. \]  

However, we must proceed with care. In order to express the transition \( \Pi \) as an exponential type map, we need to choose appropriate coordinates on \( \Sigma^- \) and on \( \Sigma^+ \) that respect the already chosen coordinates in \( \Sigma^{\text{en}} \) and in \( \Sigma^{\text{ex}} \). Fortunately, this is possible with the extensions of Bonckaert [2, 3] to the normalization results of Takens [23].

For sake of clarity, let \((B_{\text{en}}, Z_{\text{en}})\) be coordinates in \( \Sigma^{\text{en}} \) and \((B_{\text{ex}}, Z_{\text{ex}})\) be coordinates in \( \Sigma^{\text{ex}} \). We have shown that these coordinates can be chosen in such a way that the “vertical” component of the transition map \( \Pi_{\text{inner}} : \Sigma^{\text{en}} \rightarrow \Sigma^{\text{ex}} \) reads as

\[
\Pi_{Z_{\text{en}}}(B_{\text{en}}, Z_{\text{en}}, \varepsilon) = Z_{\text{ex}} \\
= \phi(B_{\text{en}}, \varepsilon) + Z_{\text{en}} \exp\left(-\frac{1}{\varepsilon} \left(A(B_{\text{en}}, \varepsilon) + \varepsilon \Psi(B_{\text{en}}, \varepsilon, Z_{\text{en}})\right)\right). \tag{135}
\]

In this case the invariant manifold, say \( \mathcal{M}_{\varepsilon} \), is given by \( Z_{\text{en}} = 0 \). Using [2, 3] we can find suitable coordinates \((B_-, Z_-)\) in \( \Sigma^- \) in such a way that

\[
\Pi_{Z_-}^{-1}(B_-, Z_-, \varepsilon) = Z_- \exp\left(-\frac{1}{\varepsilon} (A_0)\right) = Z_{\text{en}}. \tag{136}
\]

In other words, there is a change of coordinates respecting the invariant manifold \( \mathcal{M}_{\varepsilon} \) under which the transition \( \Pi^- \) is an exponential type map with no shift and linear. Similar arguments hold for the choice of coordinates in \( \Sigma^+ \). Finally, following appendix A, the composition \( \Pi_{Z_+}^+ \circ \Pi_{Z_{\text{en}}} \circ \Pi_{Z_-}^+ \) leads to the result.

### A. Exponential type functions

In this section, we discuss a particular type of function which will be found and used frequently throughout the main text. First, however, let us give two preliminary definitions.

**Definition A.1** (\( C^\ell \)-admissible function). Let \( U \in \mathbb{R}^n \). A function \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) is said to be a \( C^\ell \)-admissible function if \( f \) is \( C^\ell \)-smooth away from the origin (for any \( \ell > 0 \), \( C^0 \) at the origin and if for all \( n_i \in \mathbb{N} \) and \( n_i < \ell \), there exists an \( N(n_i) \in \mathbb{N} \) such that

\[
\frac{\partial^{n_i} f}{\partial U_i^{n_i}} \in O\left(U_i^{-N(n_i)}\right), \quad \text{as } U_i \rightarrow 0. \tag{A.1}
\]

Now, we define a particular type of differentiability. For this we need to extend the common concept of monomial. In our context, a monomial, e.g. in two variables, \( \omega(u, v) \) is any expression of the form \( u^\alpha v^\beta \) or of the form \( u^\alpha (\ln v)^\beta \), with \( \alpha, \beta \in \mathbb{R} \). In general, if we let \( u \in \mathbb{R}^m \) and \( v \in \mathbb{R}^n \), we allow a monomial \( \omega \) to be any expression of the type \( u^p (\ln v)^q \), where \( u^p = u_{i_1}^{p_1} \cdots u_{i_m}^{p_m} \) and \( (\ln v)^q = (\ln v_1)^{q_1} \cdots (\ln v_n)^{q_n} \). We note that these monomials are admissible functions.

**Definition A.2** (\( C^\ell \)-function with respect to monomials). Let \((U, V) \in \mathbb{R}^m \times \mathbb{R}^n \). We say that a function \( f(U, V) \) is a \( C^\ell \)-function with respect to a monomial \( \omega \), if \( f \) is \( C^\ell \) w.r.t. \( V \) in a neighborhood of \( 0 \in \mathbb{R}^n \), and if there is a quadrant \( U = [0, u_1] \times \cdots \times [0, u_n] \subset \mathbb{R}^m \) where the monomial \( \omega \) is defined and such that the function \( f(\omega, U, V) = f(U, V) \) is \( C^\ell \) with respect to \( \omega \) in \( U \). Similarly, the function \( f \) is said to be a \( C^\ell \)-function with respect to the monomials \( \omega_1, \ldots, \omega_s \) if there is a quadrant \( U \) where the monomials are defined and such that the function \( f(\omega_1, \ldots, \omega_s, U, V) = f(U, V) \) is \( C^\ell \) with respect to \( \omega_1, \ldots, \omega_s \) in \( U \).
Observe that a function $f$ which is differentiable w.r.t monomials is an admissible function. As an example, consider $f(U) = U_1 \ln U_1 \phi(U)$ where $\phi(U)$ is smooth. This function is smooth away from $U = 0$ and $C^0$ at the origin. However, it is not differentiable w.r.t. $U_1$ at $U_1 = 0$ but it is differentiable with respect to $\omega = U_1 \ln U_1$ at $\omega = 0$.

Let $V \in \mathbb{R}^m$, $Z \in \mathbb{R}$, and as usual $\varepsilon$ denotes a small parameter.

**Definition A.3** (Exponential type function). A function $D(V, Z, \varepsilon)$ is called of exponential type if it has the following form

$$D(V, Z, \varepsilon) = B(V, \varepsilon) + Z \exp \left( - \frac{A(V, \varepsilon) + \Phi(V, Z, \varepsilon)}{\varepsilon} \right),$$

where $A$ and $B$ are $C^\ell$ admissible functions with $A > 0$, and $B(0, 0) = 0$; and where $\Phi$ is $C^\ell$ in $Z$ and $C^\ell$ w.r.t. monomials of $(V, \varepsilon)$ with $\Phi(V, 0, Z) = 0$. We distinguish two particular cases

1. The exponential type function $D$ is without shift if $B \equiv 0$.
2. The exponential type function $D$ is linear if $\Phi(V, Z, \varepsilon) \equiv \Phi(V, \varepsilon)$.

**Remark A.1.** Given a function $D$ and if it is of exponential type, the representation of $D$ is unique in the sense that all the functions in r.h.s of (A.2) are computable from $D$. In fact

$$\begin{align*}
\mathcal{B} &= D(V, 0, \varepsilon) \\
\mathcal{A} &= \lim_{Z \to 0} \left( -\varepsilon \ln \left( \frac{D(V, Z, \varepsilon) - D(V, 0, \varepsilon)}{Z} \right) \right) \\
\Phi &= -\varepsilon \ln \left( \frac{D(V, Z, \varepsilon) - D(V, 0, \varepsilon)}{Z} \right) - \mathcal{A}.
\end{align*}$$

We want to study the scenario where we have to compose $D$ with some other functions and want to keep the exponential type structure. To be more precise, we consider $D$ as an $(V, \varepsilon)$ parameter family of functions (in $Z$) and compose it with a $(V, \varepsilon)$ parameter family of diffeomorphisms $\Psi(V, \varepsilon)$ on $\mathbb{R}$.

**Proposition A.1** (Composition on the left). Let $\Psi_{(V, \varepsilon)} : \mathbb{R} \to \mathbb{R}$ be a family of diffeomorphisms, and let $D$ be an exponential type function. Then, the composition $\Psi_{(V, \varepsilon)} \circ D$ is also of exponential function of the form

$$\tilde{D} = \tilde{\mathcal{B}}(V, \varepsilon) + Z \exp \left( - \frac{\tilde{\mathcal{A}}(V, \varepsilon) + \tilde{\Phi}(V, Z, \varepsilon)}{\varepsilon} \right),$$

where $\tilde{\mathcal{B}}$ and $\tilde{\Phi}$ are admissible functions.

**Proof.** Let us simplify the notation by writing $\Psi = \Psi_{(V, \varepsilon)}$. Since $\Psi$ is a diffeomorphism we can write $\Psi(a + b) = \Psi(a) + C(1 + \psi(a, b))b$, near $b = 0$, with $\psi$ a $C^\ell$ function such that $\psi(a, 0) = 0$ and with $C > 0$. Then we have

$$\begin{align*}
\Psi \circ D(z) &= \Psi \left( \mathcal{B} + Z \exp \left( - \frac{\mathcal{A} + \Phi}{\varepsilon} \right) \right) \\
&= \Psi(\mathcal{B}) + C(1 + \psi(V, Z, \varepsilon))Z \exp \left( - \frac{\mathcal{A} + \Phi}{\varepsilon} \right) \\
&= \Psi(\tilde{\mathcal{B}}) + C(1 + \psi(V, Z, \varepsilon))Z \exp \left( - \frac{\tilde{\mathcal{A}} + \tilde{\Phi}}{\varepsilon} \right),
\end{align*}$$

where $\tilde{\mathcal{B}}$ and $\tilde{\Phi}$ are admissible functions.
Since $C > 0$ we can take the logarithm of $C(1 + \psi(V, Z, \varepsilon))$ and then we have

$$
\Psi \circ D(z) = \Psi(B) + \exp(\ln(C(1 + \psi))Z \exp\left(-\frac{A + \Phi}{\varepsilon}\right))
$$

$$
= \Psi(B) + Z \exp\left(-\frac{A + \Phi + \varepsilon \ln(C(1 + \psi))}{\varepsilon}\right).
$$

(A.6)

The result is obtained by setting $\tilde{B} = \Psi(B)$ and $\tilde{\Phi} = \Phi + \varepsilon \ln(C(1 + \psi))$.

**Proposition A.2** (Composition on the right). Let $\Psi(V, \varepsilon) : \mathbb{R} \to \mathbb{R}$ be a family of diffeomorphisms with no shift, that is $\Psi_V(0) = 0$ for all $(V, \varepsilon)$, and let $D$ be an exponential type function. Then, the composition $D \circ \Psi_V$ is also of exponential function of the form

$$
\tilde{D} = \tilde{B}(V, \varepsilon) + Z \exp\left(-\frac{A(V, \varepsilon) + \tilde{\Phi}(V, Z, \varepsilon)}{\varepsilon}\right),
$$

where $\tilde{B}$ and $\tilde{\Phi}$ are admissible functions.

**Proof.** Let us simplify the notation by writing $\Psi = \Psi_V$. Since $\Psi(0) = 0$ we can write $\Psi(z) = C(1 + O(z))z$ with $C > 0$. Then we have

$$
D \circ \Psi(z) = D(C(1 + O(z))z) = B(V, \varepsilon) + C(1 + O(z))z \exp\left(-\frac{A(V, \varepsilon) + \Phi(V, \varepsilon, \Psi)}{\varepsilon}\right)
$$

$$
= B(V, \varepsilon) + z \exp\left(-\frac{A(V, \varepsilon) + \Phi(V, \varepsilon, \Psi) + \varepsilon \ln(C(1 + O(z))))}{\varepsilon}\right).
$$

(A.8)

The result then is obtained by setting $\tilde{\Phi} = \Phi(V, \varepsilon, \Psi) + \varepsilon \ln(C(1 + O(z))).$

**Remark A.2.** If we want the composition $\Pi \circ \Psi_V$ to be of exponential type, the family $\Psi_V$ cannot be arbitrary. In order to preserve the “exponential structure”, $\Psi_V$ should satisfy the hypothesis of proposition A.2. In corollary A.2 we show a particular case in which the diffeomorphism $\Psi$ can have a shift and yet preserve the structure of the exponential type function.

Let us proceed by presenting a couple of useful corollaries.

**Corollary A.1.** Let $D_1$ and $D_2$ be two exponential type functions of the form

$$
D_1(V, Z, \varepsilon) = Z \exp\left(-\frac{A_1(V, \varepsilon) + \Phi_1(V, Z, \varepsilon)}{\varepsilon}\right)
$$

$$
D_2(V, Z, \varepsilon) = B_2(V, \varepsilon) + Z \exp\left(-\frac{A_2(V, \varepsilon) + \Phi_2(V, Z, \varepsilon)}{\varepsilon}\right),
$$

(A.9)

that is, $D_1$ is an exponential type function with no shift. Then $D_2 \circ D_1$ is an exponential type function.

**Corollary A.2.** Let $D_1$ and $D_2$ be two exponential type functions with $D_2$ linear, this is

$$
D_1(V, Z, \varepsilon) = B_1(V, \varepsilon) + Z \exp\left(-\frac{A_1(V, \varepsilon) + \Phi_1(V, Z, \varepsilon)}{\varepsilon}\right)
$$

$$
D_2(V, Z, \varepsilon) = B_2(V, \varepsilon) + Z \exp\left(-\frac{A_2(V, \varepsilon)}{\varepsilon}\right).
$$

(A.10)

Then the composition $D_2 \circ D_1$ is of exponential type.
It is useful to consider the following: let \( X(V,Z,\varepsilon) \) be a given vector field on \( \mathbb{R}^{m+2} \), and let \( \Sigma_0 \) and \( \Sigma_1 \) be codimension 1 subsets of \( \mathbb{R}^{m+2} \) which are transversal to the flow of \( X \). For the moment it is sufficient to think of a section \( \Sigma_i \) given by \( \{ V_j = v_0 \} \) or by \( \{ \varepsilon = \varepsilon_0 \} \) with \( v_0 \) and \( \varepsilon_0 \) fixed constants. Induced from definition A.3 we then have the following.

**Definition A.4 (Exponential type transition).** A transition \( \Pi : \Sigma_0 \to \Sigma_1 \) is called of exponential type if and only if its \( Z \)-component is an exponential type function. This is, an exponential type transition is of the form

\[
\Pi(V,Z,\varepsilon) = (G, D, H) = \left( G(V,\varepsilon), B(V,\varepsilon) + Z \exp \left( -\frac{A(V,\varepsilon) + \Phi(V,Z,\varepsilon)}{\varepsilon} \right), H(V,\varepsilon) \right),
\]

where \( G : \mathbb{R}^{m+1} \to \mathbb{R}^m \) and \( H : \mathbb{R}^{m+1} \to \mathbb{R} \) are \( C^\ell \)-admissible functions. The names exponential type transition with no shift and linear are inherited as well from the type of \( D \).

Suppose now that \( X \) is a given vector field on \( \mathbb{R}^{m+2} \), as above, and let \( \Sigma_i \) with \( i = 0, 1, 2, 3, 4, 5 \), be disjoint sections which are all transversal to the flow of \( X \). Assume that \( X \) induces exponential type transitions \( \Pi_i : \Sigma_{i-1} \to \Sigma_i \) with \( i = 1, 2, 3, 4, 5 \) of the following form

1. \( \Pi_1 \) is with no shift and linear
2. \( \Pi_2 \) is with no shift
3. \( \Pi_3 \) is a general diffeomorphism
4. \( \Pi_4 \) is with no shift
5. \( \Pi_5 \) is with no shift and linear.

We need to show that the composition of all these five maps is an exponential type transition.

**Proposition A.3.** Let \( \Pi_i : \Sigma_{i-1} \to \Sigma_i \) as described above. Then the composition \( \Pi = \Pi_5 \circ \Pi_4 \circ \Pi_3 \circ \Pi_2 \circ \Pi_1 \) is an exponential type map of the form

\[
\Pi = \left( \tilde{G}(V,\varepsilon), \tilde{B}(V,\varepsilon) + Z \exp \left( -\frac{\tilde{A}(V,\varepsilon) + \tilde{\Phi}(V,Z,\varepsilon)}{\varepsilon} \right), \tilde{H}(V,\varepsilon) \right),
\]

where \( \tilde{A} = A_1 + A_2 + A_4 + A_5 \).

**Proof.** Let us write each of the transitions as follows.

1. \( \Pi_1(V,Z,\varepsilon) = (G_1, D_1, H_1) = \left( G_1, Z \exp \left( -\frac{A_1(V,\varepsilon)}{\varepsilon} \right), H_1 \right) \)
2. \( \Pi_2(V,Z,\varepsilon) = (G_2, D_2, H_2) = \left( G_2, Z \exp \left( -\frac{A_2(V,\varepsilon) + \Phi_2(V,Z,\varepsilon)}{\varepsilon} \right), H_2 \right) \)
3. \( \Pi_3(V,Z,\varepsilon) = (G_3, D_3, H_3) \)
4. \( \Pi_4(V,Z,\varepsilon) = (G_4, D_4, H_4) = \left( G_4, Z \exp \left( -\frac{A_4(V,\varepsilon) + \Phi_4(V,Z,\varepsilon)}{\varepsilon} \right), H_4 \right) \)
5. \( \Pi_5(V,Z,\varepsilon) = (G_5, D_5, H_5) = \left( G_5, Z \exp \left( -\frac{A_5(V,\varepsilon)}{\varepsilon} \right), H_5 \right) \)
For brevity let $\Pi_2 \circ \Pi_1 = (\tilde{G}_2, \tilde{D}_2, \tilde{H}_2)$. Then we have

$$(\tilde{G}_2, \tilde{D}_2, \tilde{H}_2) = \left( G_2(G_1, H_1), D_1 \exp \left( -A_2(G_1, H_1) + \Phi_2(G_1, D_1, H_1) \right), H_2(G_1, H_1) \right). \quad (A.13)$$

Now, we take care only of the $Z$-component of the composition $\Pi_2 \circ \Pi_1$. From the hypothesis on $G_1$ and $H_1$ we can write $G_1 = V + O(\varepsilon)$ and $H_1 = \alpha \varepsilon (1 + O(\varepsilon))$ with $\alpha > 0$, then

$$\tilde{D}_2 = Z \exp \left( -A_1(V, \varepsilon) + A_2(V, \varepsilon) + \Phi_2(V, Z, \varepsilon) \over \varepsilon \right), \quad (A.14)$$

where we have gathered in $\Phi_2$ the function $\Phi_1$ and the terms resulting from taking $G_1 = V + O(\varepsilon)$ and $H_1 = \alpha \varepsilon (1 + O(\varepsilon))$. In a similar way, letting $\Pi_5 \circ \Pi_4 = (\tilde{G}_5, \tilde{D}_5, \tilde{H}_5)$ we get

$$\tilde{D}_5 = Z \exp \left( -A_4(\varepsilon) + A_5(\varepsilon) + \Phi_5(V, Z, \varepsilon) \over \varepsilon \right). \quad (A.15)$$

Next, and following similar arguments as above, we know from proposition A.1 that the composition $\Pi_{321} = \Pi_3 \circ \Pi_2 \circ \Pi_1$ is of exponential type with shift. Finally since the transition $\Pi_{54} = \Pi_5 \circ \Pi_4$ is of exponential type with no shift, and using proposition A.1, we have that $\Pi_{54} \circ \Pi_{321}$ is an exponential type transition as claimed in the proposition.

**Remark A.3.** In the case where $\Pi_3$ is an exponential type map, we get a similar result with $\tilde{A} = A_1 + A_2 + A_3 + A_4 + A_5$.

\[\square\]

**B. First order differential equations (by R. Roussarie)**

The contents of this section shall appear in greater detail in [6]. We reproduce some results here for completeness purposes and to use them in appendix C.1.

Let $X(x)$ be a smooth vector field defined on $W \subset \mathbb{R}^n$, for arbitrary $n \in \mathbb{N}$ (here we include the possible parameters). Let $G(x, y) : W \times \mathbb{R} \to \mathbb{R}$ be a smooth function. We shall study the solutions of the first order differential equation

$$X \cdot K(x) = G(x, K(x)), \quad (B.1)$$

where $K(x)$ is the unknown function. We assume the following

1. There exists an open section $\Sigma \subset W$ which is transverse to $X$.
2. Let $\phi(t, x)$ denote the flow of $X$. We can choose an open domain $W_\Sigma$ with the property that for any $x \in W_\Sigma$, there exists a unique smooth time $t(x)$ (possibly unbounded) such that $\phi(t(x), x) \in \Sigma$.
3. The vector field $Z(x, y) = X(x) + G(x, y) \partial_y$ has a complete flow.

The flow of $Z$ takes the form $(\phi(t, x), \psi(t, x, y))$, where $\phi$ is the flow of $X$. It follows that $K(x)$ is a solution of (B.1) if and only if the graph $\{y = K(x)\}$ is a surface tangent to the vector field $Z$. Then we have the implicit formula

$$\psi(t(x), x, K(x)) = 0. \quad (B.2)$$
In our applications, the function $G$ is affine in $y$, that is $G(x, y) = L(x)y + \Pi(x)$ where $L$ and $\Pi$ are smooth. If we write $\dot{L}(t, x) = L(\phi(t, x))$ and $\Pi(t, x) = \Pi(\phi(t, x))$ (where $\phi$ is the flow of $X$), we have for $\psi$ the following linear differential equation

$$\frac{d\psi}{dt}(t, x, y) = L(t, x)\psi(t, x, y) + \Pi(t, x). \tag{B.3}$$

Then we can integrate (B.3) with the initial condition $\psi(0, x, y) = y$ to obtain

$$\psi(t, x, y) = \exp\left(\int_0^t \dot{L}(\tau, x) d\tau\right) \left\{ y + \int_0^t \Pi(\tau, x) \left[ \exp\left(-\int_0^\tau \dot{L}(\sigma, x) d\sigma\right)\right] d\tau\right\}. \tag{B.4}$$

Since $\exp\left(\int_0^t \dot{L}(\tau, x) d\tau\right) > 0$ we can solve the implicit equation (B.2) obtaining

$$K(x) = -\int_0^t \Pi(\phi(\tau, x)) \left[ \exp\left(-\int_0^\tau L(\phi(\sigma, x)) d\sigma\right)\right] d\tau, \tag{B.5}$$

where we recall that $\phi$ is the flow of $X$ and $t(x)$ is the time to go from $x$ to the section $\Sigma$ along this flow.

Let us now assume that the vector field $X$ is partially hyperbolically attracting in the following sense: we assume coordinates $x = (a, b) \in \mathbb{R}^p \times \mathbb{R}^q$ and that the vector field $X$ has a decomposition $X(x) = U(x) + V(x)$ where $U$ is the component along $\mathbb{R}^p$ and $V$ is the component along $\mathbb{R}^q$. Moreover, we assume that $V = 0$ on $\mathbb{R}^p \times \{0\}$ (that is $X$ is tangent to $\mathbb{R}^p \times \{0\}$). We also assume that at each point $x = (a, b)$ it is satisfied that $D_VV(a, 0)$ has all its eigenvalues with strictly negative real part. We further suppose that $X$ is given on $W = D \times \Delta$ where $D$ is a domain diffeomorphic to a ball in $\mathbb{R}^p$ and $\Delta$ is a ball in $\mathbb{R}^q$. We choose $\Delta = \Delta_\rho_0$ for some $\rho_0 > 0$ where $\Delta_\rho = \{b \in \mathbb{R}^q \mid ||b|| < \rho\}$. It then follows that under a linear change of coordinates $(a, b) \mapsto (a, A(a)b)$, the vector field $X$ enters along $D \times \partial \Delta_\rho$ for $0 < \rho \leq \rho_0$ if we choose $\rho_0$ small enough. We now have the following

**Proposition B.1.** Assume that $D_VV(a, 0)$ has all its eigenvalues with a strictly negative real part and that $\rho_0$ is small enough as explained above. Let $B$ be any domain diffeomorphic to a closed ball inside the interior of $D$ and assume that the function $\Pi(x)$ is flat along $D \times \{0\}$. Then the equation

$$X \cdot K(x) = L(x)K(x) + \Pi(x) \tag{B.6}$$

has a smooth solution $K(x)$ in $B \times \Delta$ which is flat along $B \times \{0\}$.

**Proof.** Let $f(a) : \mathbb{R}^p \to [0, 1]$ be a smooth function which is equal to 1 on $B$ and equal to 0 on a neighborhood of $\partial D$. Define the vector field

$$T = V + fU. \tag{B.7}$$

This vector field $T$ coincides with $X$ on $B \times \Delta$. Moreover, $T$ is tangent along $\partial D \times \Delta$ and enters the domain $D \times \Delta$ along $D \times \partial \Delta$. Let $\phi(t, x) = (\phi_a(t, x), \phi_b(t, x)) \in \mathbb{R}^p \times \mathbb{R}^q$ denote the flow of $T$. It follows that $\phi(t, x) \in D \times \Delta$ for all $x \in D \times \Delta$ and all $t \geq 0$. From the assumption on $V$ we have that there exists a positive constant $E > 0$ such that

$$||\phi_b(t, x)|| \leq ||b||\exp(-Et), \tag{B.8}$$

for any $x = (a, b) \in D \times \Delta$ and $t \in [0, +\infty)$. We now want to use this flow $\phi$ in (B.5) noting that if the integral converges, then $K(x)$ is a solution to the equation $T \cdot K = LK + \Pi$ on $D \times \Delta$ and then to the equation $X \cdot K = LK + \Pi$ on $B \times \Delta$. In this setting (B.5) is written as

$$K(x) = -\int_0^\infty \Pi(\phi(\tau, x)) \left[ \exp\left(-\int_0^\tau L(\phi(\sigma, x)) d\sigma\right)\right] d\tau. \tag{B.9}$$
Now, we need to prove that \((B.9)\) defines a smooth function on \(D \times \Delta\) which is flat along \(D \times \{0\}\). In other words, we shall prove that \(K\) and all its partial derivatives are equal to 0 on \(D \times \{0\}\). As \(L\) is bounded, there exists a constant \(M_0 > 0\) such that
\[
\exp \left( - \int_0^\tau L(\phi(\sigma, x))d\sigma \right) \leq \exp(M_0 \tau). \tag{B.10}
\]

Next, let \(N \in \mathbb{N}\). Since \(\Pi\) is flat in \(v\), there exists a constant \(P_N > 0\) such that
\[
|\Pi(a, b)| \leq P_N ||b||^N, \tag{B.11}
\]
and then from \((B.8)\) it follows that
\[
|\Pi(\phi(\tau, x))| \leq P_N ||b||^N \exp(-NE\tau). \tag{B.12}
\]

Using these estimates we have that
\[
|K(x)| \leq P_N ||b||^N \int_0^{+\infty} \exp((M_0 - NE)\tau)d\tau. \tag{B.13}
\]
The integral in \((B.13)\) converges if \(N\) is large enough, strictly speaking if \(N > \frac{M_0}{E}\). This proves that by choosing \(N\) sufficiently large, the right hand side of \((B.9)\) defines a function which is continuous and equal to 0 on \(D \times \{0\}\).

Let us now consider any partial derivation \(\partial_\alpha K\) of \(K\). Let us write
\[
H(\tau, x) = \Pi(\phi(\tau, x)) \exp \left[ - \int_0^\tau L(\phi(\sigma, x))d\sigma \right], \tag{B.14}
\]
the integrand in \((B.9)\). Using chain rule on the derivative of \((B.9)\), we have to prove that the integral
\[
\int_0^{+\infty} \partial_\alpha H(\tau, x)d\tau \tag{B.15}
\]
is convergent and that there is an estimate similar to \((B.13)\) for \(N\) large enough. We do not want to give all the details here and refer the reader to [6]. The idea is that \(\partial_\alpha H(\tau, x)\) is a finite sum of terms such that each of these terms is a product of factors which are partial derivatives in \(x\) and are of one of the following forms

1. \(\partial_{\alpha_1}(\phi(\tau, x))\). Since \(\Pi\) is smooth and flat along \(D \times \{0\}\), this is also the case for \(\partial_{\alpha_1}(\phi(\tau, x))\). Therefore, for \(N\) sufficiently large, we can write an estimate of the form
\[
|\partial_{\alpha_1}(\phi(\tau, x))| \leq P_{N_{\alpha_1}} ||b||^N \exp(-NE\tau), \tag{B.16}
\]
for constants \(P_{N_{\alpha_1}} > 0\).

2. \(\partial_{\alpha_2}(\phi(\tau, x))\) (resp. \(\partial_{\alpha_2}(\phi(\sigma, x))\), note that \(0 \leq \sigma \leq \tau\)). By the usual variational method along trajectories, there exists constants \(E_{\alpha_2} > 0\) such that \(|\partial_{\alpha_2}(\phi(\tau, x))| \leq \exp(E_{\alpha_2} \tau)\) (resp. \(|\partial_{\alpha_2}(\phi(\sigma, x))| \leq \exp(E_{\alpha_2} \sigma)\) ).

3. \(\partial_{\alpha_3}(\phi(\tau, x))\). As \(L\) is smooth in \(D \times \Delta\), all these factors are bounded by a constant \(M_{\alpha_3}\).

4. \(\exp \left( - \int_0^\tau L(\phi(\sigma, x))d\sigma \right)\). This factor is bounded by \(\exp(M_0 \tau)\).

Next, by remarking that a factor of the first type appears in each term of the expansion of \(\partial_\alpha H\), and taking \(N\) large enough, it is possible to conclude that the integral \((B.15)\) converges and is equal to 0 for \(x \in D \times \{0\}\). Therefore, the partial derivative \(\partial_\alpha K(x)\) exists, is continuous and is equal to 0 on \(D \times \{0\}\).
C. Normal form and transition of a semi-hyperbolic vector field

In this section, we present a rather general framework for the computation of a \( C^\ell \) normal form and the corresponding transition of a vector fields with a semi-hyperbolic singularity. The contents of this section are not only relevant for the object studied in this document, but for more general systems as well, c.f. [13].

To make our computations simpler, we prove a lemma that allows us to “partition” a smooth function. As a simple example of this partition, let \( f(u,v) \) be a smooth function on \( \mathbb{R}^2 \). We show that \( f \) can be written as \( f(u,v) = f_1(uv,u) + f_2(uv,v) \), where \( f_1 \) and \( f_2 \) are smooth. This type of result becomes useful when computing the transition map that we present in appendix C.3.

C.1. Normal form

Here we provide a \( C^\ell \) normal form of a semi-hyperbolic vector field which frequently appears in the analysis of slow-fast systems. The goal of obtaining such a normal form is that the computation of the corresponding transition becomes simpler.

Proposition C.1. Let \( \alpha, \beta = (\beta_1, \ldots, \beta_m) \) and \( \gamma \) be non-zero constants, and consider the vector field \( X \) given by

\[
\begin{align*}
    u' &= \alpha w u (1 + f) + w g \\
    v_j' &= \beta_j w v_j (1 + f) \\
    w' &= \gamma w^2 (1 + f) \\
    z' &= -\Lambda + h,
\end{align*}
\]

where \( j = 1, 2, \ldots, m; \) where the functions \( f = f(u,v,w,z), g = g(u,v,w,z) \) and \( h = h(u,v,w,z) \) are smooth functions which are flat at the origin of \( \mathbb{R}^{m+3} \), and where \( \Lambda = \Lambda(u,v,w,z) \) is a smooth function such that \( \Lambda(0) = 0 \) and \( \frac{\partial \Lambda}{\partial z}(0) > 0 \). Then there exist a \( C^\ell \) coordinates \((U,V_1,\ldots,V_m,W,Z)\) under which \( X \) can be written as

\[
\begin{align*}
    U' &= \alpha W U \\
    V_j' &= \beta_j W V_j \\
    W' &= \gamma W^2 \\
    Z' &= -G Z,
\end{align*}
\]

where \( G = G(U,V,W) \) is a \( C^\ell \) function such that \( G(0) > 0 \).

Proof of proposition C.1. From the definition of the vector field \( X \) we note that the origin is a semi-hyperbolic singular point. The hyperbolic eigenspace is 1-dimensional while the center eigenspace is \((m+2)\)-dimensional. We now proceed in 4 steps as follows.

1. Define a new vector field \( Y \) by \( Y = \frac{1}{1+j} X \), which reads as

\[
\begin{align*}
    u' &= \alpha w u + w \bar{g} \\
    v_j' &= \beta_j w v_j \\
    w' &= \gamma w^2, \\
    z' &= -\Lambda + \bar{h},
\end{align*}
\]

where the functions \( \bar{g} \) and \( \bar{h} \) are flat at the origin of \( \mathbb{R}^{m+3} \). Note that in a small neighborhood of \((u,v,w,z) = (0,0,0,0)\) the vector fields \( X \) and \( Y \) are smoothly equivalent.
2. By looking at $DY(0)$, there exists an $(m + 2)$-dimensional center manifold $\mathcal{W}_2^C$ [11]. Let $M_0$ be the set of critical points of $Y$, that is

$$M_0 = \{(u, v, w, z) \mid \Lambda(u, v, 0, z) = 0\}. \quad \text{(C.4)}$$

By definition, the manifold $M_0$ is invariant and normally hyperbolic. Now, assume $|w| \ll 1$. This condition appears naturally in our applications. By Fenichel’s theory [10] the manifold $M_0$ persists as an invariant normally hyperbolic manifold $M_w$ for sufficiently small $w \neq 0$. We identify $M_w$ with $\mathcal{W}_2^C$. In other words, there exists a $C^\ell$ function $m = m(u, v, w)$ such that the center manifold $\mathcal{W}_2^C$ is given as a graph

$$\mathcal{W}_2^C = \text{Graph}(u, v, w, m). \quad \text{(C.5)}$$

Define $\zeta = z - m$, then $\zeta' = \zeta' - m'$. But we know, due to invariance of $\mathcal{W}_2^C$ under the flow of $Y$, that $\zeta'|_{\zeta=0} = 0$. This is, there exists a $C^\ell$ function $H = H(u, v, w, \zeta)$ such that $\zeta' = -H\zeta$. With $H(0) = 0$ and $\frac{\partial H}{\partial \zeta}(0) > 0$.

In conclusion of this step, there exists a $C^\ell$ transformation $\psi : (u, v, w, z) \mapsto (u, v, w, \zeta)$ that transforms the vector field $Y$ into

$$\tilde{Y} : \begin{cases} u' = \alpha_wu + \tilde{w}\tilde{g} \\ v'_j = \beta_jwv_j \\ w' = \gamma w^2 \\ \zeta' = -H\zeta, \end{cases} \quad \text{(C.6)}$$

where $H = H(u, v, w, \zeta)$ is a $C^\ell$ function such that $H(0) = 0$ and where $\frac{\partial H}{\partial \zeta}(0) = \frac{\partial \Lambda}{\partial z}(0) > 0$.

3. Observe that thanks to the previous step, the center manifold $\mathcal{W}_2^C$ has the simple expression $\mathcal{W}_2^C = \{\zeta = 0\}$. We now want to separate the variables on the center manifold (these are $(u, v, w)$) from those on the hyperbolic subspace $(z)$. Additionally, we want to keep the simple format that $\tilde{Y}$ has in the center direction. This amounts to find a change of coordinates along $\zeta$ only. For this we use an extension of Takens’s theorem on semi-hyperbolic vector fields [23] due to Bonckaert [2, 3]. With this, it is possible to show there exists a $C^\ell$ transformation, fixing the center coordinates, that conjugates $\tilde{Y}$ to the vector field

$$\tilde{Y} : \begin{cases} u' = \alpha_wu + \tilde{w}\tilde{g} \\ v'_j = \beta_jwv_j \\ w' = \gamma w^2, \\ Z' = -\tilde{H}Z, \end{cases} \quad \text{(C.7)}$$

where now the flat perturbation $\tilde{g}$ is independent of $Z$ and $\tilde{H} = \tilde{H}(u, v, w)$ is a $C^\ell$ function with $\tilde{H}(0, 0, 0) > 0$.

4. In this last step we eliminate the flat perturbation from $\tilde{Y}$, which appears only along $u$. Due to the previous step, the dynamics on the center manifold are independent of $Z$. The restriction of $\tilde{Y}$ to $\mathcal{W}_2^C$ reads as
\[ \dot{Y}|_{W_2^c} : \begin{cases} u' = awu + w\bar{g} \\ v'_j = \beta_jv_j \\ w' = \gamma w^2. \end{cases} \quad (C.8) \]

Note that for \( w \neq 0 \), the vector field \( \frac{1}{w}\dot{Y}|_{W_2^c} \) is hyperbolic. Let \( Y = \frac{1}{w}\dot{Y}|_{W_2^c} \), that is

\[ Y : \begin{cases} u' = \alpha u \\ v'_j = \beta_jv_j \\ w' = \gamma w. \end{cases} \quad (C.9) \]

Now we have the result that shows that there exists a change of coordinates, respecting the variables \((v, w)\) that kills the term \( \bar{g} \). Keeping the coordinate \( w \) fixed is important because we want to prove an equivalence relation with \( wY \) and not with \( Y \). The following proposition shall appear in a general context in [6].

**Proposition C.2** ([6]). There exists a diffeomorphism \( (u, v, w) \mapsto (u + H(u, v, w), v, w) \) with \( H \) flat at \((u, v, w) = 0\) which brings \( Y \) to

\[ \ddot{Y} : \begin{cases} u' = \alpha u \\ v'_j = \beta_jv_j \\ w' = \gamma w. \end{cases} \quad (C.10) \]

**Proof.** We shall use the path method to show that \( \ddot{Y} \) is conjugate to \( Y \). Let \( s \) be a parameter and let us define the \( s \)-parameter family of vector fields

\[ Y^s = Y + s\bar{g} \frac{\partial}{\partial u}. \quad (C.11) \]

We call \( Y^s \) the path between \( Y \) and \( Y + \bar{g} \frac{\partial}{\partial u} \). We now look for an \( s \)-parameter family of diffeomorphisms \( H^s \) with \( H^0 = \text{Id} \) such that for each \( s \) we have the conjugacy

\[ H^s Y = Y^s. \quad (C.12) \]

In such a case, the vector fields \( Y \) and \( Y + \bar{g} \frac{\partial}{\partial u} \) are conjugated by \( H^1 \). By derivation of the family \( H^s \) along \( s \), we obtain an \( s \)-parameter family of vector field \( \zeta^s \) satisfying

\[ \zeta^s(H^1) = \frac{\partial H^s}{\partial s}. \quad (C.13) \]

This implies that by derivation of (C.11) with respect to \( s \) we obtain

\[ [Y^s, \zeta^s] = \frac{\partial Y^s}{\partial s} = \bar{g} \frac{\partial}{\partial u}. \quad (C.14) \]

Therefore, if are able to find a solution \( \zeta^s \) of (C.14), the conjugacy \( H^s \) is obtained by integration of (C.13). In our particular case, we are looking for a solution along the \( u \)-direction, that is of the form \( \zeta^s = P_s \frac{\partial}{\partial u} \). It follows that
\[ [\mathcal{Y}, \xi] = \left[ (\alpha u + s\tilde{g}) + \beta v \frac{\partial}{\partial v} + \gamma w \frac{\partial}{\partial w}, P_s \frac{\partial}{\partial u} \right] \]

\[ = \left( \mathcal{Y}(P_s) - \left( \alpha + s \frac{\partial \tilde{g}}{\partial u} \right) P_s \right) \frac{\partial}{\partial u}. \]  

(C.15)

Therefore we have reduced our conjugacy problem to solving the differential equation

\[ \mathcal{Y}(P_s) - \left( \alpha + s \frac{\partial \tilde{g}}{\partial u} \right) P_s = \tilde{g}, \]  

(C.16)

where we recall that \( \tilde{g} = \tilde{g}(u, v, w) \) is flat at \( (u, v, w) = (0, 0, 0) \). We now want to use proposition B.1 to show that (C.16) has a solution \( P_s = P_s(u, v, w) \) which is flat at \( (u, v, w) = (0, 0, 0) \). For this, let \( G_s = \alpha + s \frac{\partial \tilde{g}}{\partial u} \). Now, we only need a small adaptation: in the setting and notation of proposition B.1 we may assume (under the suitable arrangement of coordinates) that \( \mathcal{Y} \) (or \( X \) in proposition B.1) is tangent to \( \mathbb{R}^d \times \{0\} \) and \( \{0\} \times \mathbb{R}^{n-d} \). Let \( \mathcal{M}_a^{\infty} \) and \( \mathcal{M}_b^{\infty} \) denote the space of germs of \( s \)-families of smooth functions that are flat at \( \{a = 0\} \) and \( \{b = 0\} \) respectively. Using a blowing-up at \( 0 \in \mathbb{R}^n \) it can be shown that \( \mathcal{M}_a^{\infty}(a, b) = \mathcal{M}_a^{\infty}(a) + \mathcal{M}_b^{\infty}(b) \) (see the arguments in lemma C.1). From this formula, it follows that it is sufficient to solve (C.16) in the spaces \( \mathcal{M}_a^{\infty}(a) \) and \( \mathcal{M}_b^{\infty}(b) \) respectively. Naturally, these two cases are equivalent up to the change of \( \mathcal{Y} \) by \( -\mathcal{Y} \) and \( G_s \) by \( -G_s \) in (B.6). In either case, the vector field \( \mathcal{Y} \) (or \( -\mathcal{Y} \)) of (C.16) satisfies the hypothesis of proposition B.1. Then for \( \tilde{g} \in \mathcal{M}_a^{\infty}(a) \) (resp. in \( \mathcal{M}_b^{\infty}(b) \)) and applying proposition B.1, we can solve (C.16) with \( P_s \) in \( \mathcal{M}_a^{\infty}(a) \) (resp. in \( \mathcal{M}_b^{\infty}(b) \)).

Thus, from proposition C.2, we have that \( \mathcal{Y} \sim \bar{Y} \) respecting \( w \), which implies \( w\mathcal{Y} \sim w\bar{Y} \). Therefore, we conclude that (C.7) can be written as stated in the proposition.

\[ \square \]

C.2. Partition of a smooth function

In this section we investigate the problem of partitioning a smooth function. The result presented below is important since it is used to simplify the computation of transition maps. To be more specific, let us give a brief example. Consider the three dimensional differential equation

\[ \begin{align*}
x' &= x \\
y' &= -y \\
z' &= g(x, y)z, \end{align*} \]  

(C.17)

where \( g \) is a smooth function. We want to take advantage from the fact that \( xy \) is a first integral. We show below that the function \( g \) can be partitioned as \( g(x, y) = g_1(xy, x) + g_2(xy, y) \). This makes the integration of \( z' \) simpler.

**Lemma C.1.** Let \( u \in \mathbb{R} \) and \( v \in \mathbb{R}^n \). Let \( f = f(u, v) \) be a smooth function such that \( f(0, 0) = 0 \). Then there exist smooth functions \( f_0 = f_0(uv, u) \) and \( f_1(uv, v) \) such that the function \( f \) can be written as

\[ f = f_0 + f_1, \]  

(C.18)

where \( f_0(0, 0) = 0 \) and \( f_1(0, 0) = 0 \).

**Proof of lemma C.1.** We proceed in two steps. The first consists in proving the formal version of the statement. The second step is to extend the formal result to the smooth case.
**Formal step**

Let \( \hat{f} \) denote the formal expansion of the smooth function \( f \). Let \( p \in \mathbb{N} \) and \( q \in \mathbb{N}^m \). We use the following notation:

- By \( q \geq 0 \) we mean \( q_i \geq 0 \) for all \( i \in [1, m] \).
- For a vector \( v \in \mathbb{R}^m \) we write \( v^q = v_1^{q_1} \cdots v_m^{q_m} \).
- The \( L_1 \) norm of \( q \) is denote by \( |q| \), and thus for \( q > 0 \) we have \( |q| = \sum_{j=1}^{m} q_j \).
- We denote by \( \tilde{q}_i \) the vector
  \[
  \tilde{q}_i = (q_1, \ldots, q_{i-1}, q_{i+1}, \ldots, q_m)
  \]
  (C.19)
  and therefore we have that \( v^{\tilde{q}_i} \) reads as
  \[
  v^{\tilde{q}_i} = v_1^{q_1} \cdots v_i^{q_i-1} v_{i+1}^{q_{i+1}} \cdots v_m^{q_m}.
  \]
  (C.20)

Besides, we have that the \( L_1 \) norm of \( \tilde{q}_i \) is given by
  \[
  |\tilde{q}_i| = |q| - q_i = \sum_{j=1, j \neq i}^{m} q_j.
  \]

The formal series expansion of \( f \) reads as

\[
\hat{f} = \sum_{p \geq 0, q \geq 0} a_{pq} u^p v^q,
\]
where \( a_{00} = 0 \). With the notation introduced above, we can partition \( \hat{f} \) as follows

\[
\hat{f} = \sum_{p \geq |q|} a'_{pq} (uv)^q u^{p-|q|} + \sum_{i=1}^{m} \sum_{q_i \geq p+|\tilde{q}_i|} a'_{pq} (uv_i)^q v_i^{p-|\tilde{q}_i|},
\]
(C.22)

where

\[
(uv)^q = (uv_1)^{q_1} \cdots (uv_m)^{q_m}
\]
and

\[
(v_i v_i^{\tilde{q}_i}) = v_i^{q_i}.
\]
(C.23)

and where \( a'_{pq} \in \mathbb{R} \) are suitable chosen coefficients. Let \( r \in \mathbb{N}^m, s \in \mathbb{N} \). Define the following formal polynomials

\[
\hat{h}(uv, u) = \sum_{r,s \geq 0} \alpha_{rs} (uv)^r u^s = \sum_{p \geq |q|} a'_{pq} (uv)^q u^{p-|q|},
\]
(C.24)

where \( \alpha_{rs} \in \mathbb{R} \), and

\[
\hat{g}_i(uv_i, v) = \sum_{r,s,t \geq 0} \beta_{rst} (uv_i)^s v^r
\]

\[
= \sum_{q_i \geq p+|\tilde{q}_i|} a'_{pq_i} (uv_i)^{q_i} v_i^{p-|\tilde{q}_i|},
\]
(C.25)
where $\beta_{irs} \in \mathbb{R}$. The coefficients $\alpha_{rs}$ and $\beta_{irs}$ are conveniently chosen to make the definitions hold. Let $uv = (uv_1, \ldots, uv_m)$. Define $\tilde{g} = \hat{g}(uv, v)$ by $\tilde{g}(uv, v) = \sum_{i=1}^{m} \hat{g}_i(uv_i, v)$, then we can write $\tilde{f}$ as

$$\tilde{f}(u, v) = \hat{h}(uv, u) + \hat{g}(uv, v). \quad (C.26)$$

This shows that the proposition holds for formal series.

**Smooth step**

By Borel’s lemma [5], there exist smooth functions $h = h(uv, u)$ and $g = g(uv, v)$ (whose formal series expansions are $\hat{h}$ and $\hat{g}$ respectively) such that

$$f = h + g + R, \quad (C.27)$$

where $R$ (reminder) is a flat function. We now show the following.

**Proposition C.3.** Let $u \in \mathbb{R}$, $v \in \mathbb{R}^m$, and $R(u, v)$ be a smooth flat function at $(0, 0) \in \mathbb{R} \times \mathbb{R}^m$. There exist flat functions $r_0 = r_0(uv, u)$ and $r_1 = r_1(uv, v)$ such that

$$R = r_0 + r_1. \quad (C.28)$$

**Remark C.1.** Proposition C.3 together with the formal step $\tilde{f} = \hat{h} + \hat{g}$ imply our result.

**Proof of proposition C.3.** For this proof we shall use the blow-up technique. Let $\Phi : S^m \times \{0\} \to \mathbb{R}^m$ be a blow-up map. The map $\Phi$ maps $S^m \times \{0\}$ to the origin in $\mathbb{R}^m$. Let $\tilde{R}$ be a function defined by $\tilde{R} = R \circ \Phi$. Since $R$ is flat at the origin, the function $\tilde{R}$ is flat along the sphere $S^m$. We assume that the function $R = R(u, v)$ is defined on a small neighborhood $R$ of the origin in $\mathbb{R} \times \mathbb{R}^m$; this neighborhood is defined as

$$R = \{|u| \leq A, |v_i| \leq B_i\}, \quad (C.29)$$

for some $A, B_i$ positive scalars. Let $0 < \delta < 1$. The sphere $S^m$ can be partitioned into $m + 1$ regions as follows:

$$\mathcal{U} = S^m \setminus \{|\bar{u}| \leq \delta\}$$
$$\mathcal{V}_i = S^m \setminus \{|\bar{v}_i| \leq \delta\}, \quad (C.30)$$

where $(\bar{u}, \bar{v}) = (\bar{u}_1, \bar{v}_1, \ldots, \bar{v}_m) \in S^m$. We can then take a partition of unity to split $\tilde{R}$ as

$$\tilde{R}(\bar{u}, \bar{v}) = \tilde{R}_0(\bar{u}, \bar{v}) + \sum_{i=1}^{m} \tilde{R}_i(\bar{u}, \bar{v}), \quad (C.31)$$

where $\text{Supp}(\tilde{R}_0) \subset \mathcal{U}$ and $\text{Supp}(\tilde{R}_i) \subset \mathcal{V}_i$ for $i \in [1, m]$. We define as $R_0$ and $R_i$ the corresponding functions on $\mathbb{R}^{m+1}$ flat at the origin given by the blow-up map $\Phi$, that is $R_j = R \circ \Phi_j$ for $j = 0, 1, \ldots, m$. Note that $R \to \tilde{R}$ is an isomorphism between the space of functions on $(u, v) \in \mathbb{R}^{m+1}$ flat at the origin, and the space of functions on $((\bar{u}, \bar{v}), \rho) \in S^m \times \mathbb{R}^+$ flat at $S^m \times \{0\}$. Therefore, the splitting (C.31) induces the splitting

$$R(u, v) = R_0(u, v) + \sum_{i=1}^{m} R_i(u, v) \quad (C.32)$$
of functions on $\mathbb{R}^{m+1}$. We will now prove that there exist flat functions $r_0$ and $r_i$ such that

$$R_0(u, v) = r_0(uv, u)$$
$$R_i(u, v) = r_i(uv_i, v).$$  \(\text{C.33}\)

Let us detail only the case of $R_0$. The other functions are obtained in a similar way.

The function $\tilde{R}_0$ has support in $\mathcal{U}$. We can parametrize $\mathcal{U}$ by the directional blow-up map $\Phi_u$ which reads

$$(\bar{u}, \bar{v}_1, \ldots, \bar{v}_m) \mapsto (\bar{u}, u\bar{v}_1, \ldots, u\bar{v}_m) = (u, v_1, \ldots, v_m).$$  \(\text{C.34}\)

Now, suppose that there exists a flat function $\tilde{P}_0$ defined by

$$\tilde{R}_0(u, \bar{v}) = \tilde{P}_0(u, u^2\bar{v}).$$  \(\text{C.35}\)

This implies that there is a function $\tilde{r}_0 = \tilde{P}_0 \circ \Phi_u^{-1}$ such that

$$R_0(u, v) = \tilde{r}_0(u, uv),$$  \(\text{C.36}\)

which is precisely what we want to prove. So, now we only need to show that indeed a function $\tilde{P}_0$ as above exists. For this let us define coordinates $(U, V_1, \ldots, V_m)$ given by

$$U = u, V_1 = u^2\bar{v}_1, \ldots, V_m = u^2\bar{v}_m,$$  \(\text{C.37}\)

and let $\tilde{P}_0(u, V)$ be a function defined as

$$\tilde{P}_0(u, V) = \tilde{R}_0 \left( \frac{V}{u}, u \right).$$  \(\text{C.38}\)

Note that $\tilde{P}_0$ is flat at $(u, V) = 0$. This is seen as follows. Since $\tilde{R}_0$ is flat along $\{u = 0\}$, it follows that $\tilde{P}_0(0, 0) = \tilde{R}_0|_{u=0} = 0$ and

$$\frac{\partial \tilde{P}_0}{\partial u}(0) = \frac{\partial \tilde{R}_0}{\partial u}|_{u=0} = 0$$
$$\frac{\partial \tilde{P}_0}{\partial V_i}(0) = \frac{1}{u^2} \frac{\partial \tilde{R}_0}{\partial \bar{v}_i}|_{u=0} = 0,$$  \(\text{C.39}\)

and so on for the higher order derivatives.

Finally, for convenience of notation we define $r_0(uv, u) = \tilde{r}_0(u, uv)$, thus we can write $R_0(u, v) = r_0(uv, u)$

Following similar arguments as above we find the functions $r_i = r_i(uv_i, v)$ such that $R_i(u, v) = r_i(uv_i, v)$ for $i \in [1, m]$. Then we define $r_1(uv, v) = \sum_{i=1}^m r_i(uv_i, v)$. It follows that

$$R(u, v) = r_0(uv, u) + r_1(uv, v).$$  \(\text{C.40}\)

With this last proposition we can now write the function $f$ as

$$f = h(uv, u) + g(uv, v) + R(u, v)$$
$$= h(uv, u) + g(uv, v) + r_0(uv, u) + r_1(uv, v).$$  \(\text{C.41}\)

Finally, to show the lemma we define the smooth functions $f_1, f_2$ of the statement by

$$f_1 = h + r_0$$
$$f_2 = g + r_1.$$  \(\text{C.42}\)
C.3. Transition

In this section we investigate the transitions for the vector field $X_{sh}^N$ computed in appendix C.1. Relabeling
the coordinates we recall that $X_{sh}^N$ reads as

\[
X_{sh}^N : \begin{cases}
    u' = \alpha w \\
    v_j' = \beta_j w v_j \\
    w' = \gamma w^2 \\
    Z' = -g Z,
\end{cases}
\]

(C.43)

where $j = 1, 2, \ldots, m$, and where $g = g(u, v, w)$ is a $C^\ell$ function such that $g(0) = \Lambda > 0$. We assume that
$w \in \mathbb{R}^+$. For our applications, we are interested in only two particular situations.

1. The saddle 1 case where $\alpha = -1$, $\beta_j > 0$ for all $j \in [1, m]$, and $\gamma > 0$.
2. The saddle 2 case where $\alpha = 1$, $\beta_j < 0$ for all $j \in [1, m]$, and $\gamma < 0$.

**Saddle 1**

In this case we investigate the transitions of a vector field of the form

\[
Y : \begin{cases}
    u' = -w u \\
    v_j' = \beta_j w v_j \\
    w' = \gamma w^2 \\
    Z' = -g Z,
\end{cases}
\]

(C.44)

where the coefficients $\beta_j, \gamma$ are positive. Observe that the flow in the direction of $u$ and $Z$ is a contraction
while it expands in all the other directions. Roughly speaking, this implies that a transition can go out at
any expanding direction $v_j$ of $w$.

We investigate two types of transitions that are used in our applications. For this, let us define the following sections

\[
\begin{align*}
    \Sigma_{en} &= \{(u, v, w, Z) \mid u = u_i\} \\
    \Sigma_{ex}^w &= \{(u, v, w, Z) \mid w = w_{out}\} \\
    \Sigma_{ex}^{\pm v_j} &= \{(u, v, w, Z) \mid v_j = v_{j, out}\}.
\end{align*}
\]

(C.45)

In this section we compute the transitions

\[
\Pi^w : \Sigma_{en} \rightarrow \Sigma_{ex}^w \\
(v, w, Z) \mapsto (\tilde{u}, \tilde{v}, \tilde{Z}),
\]

(C.46)

for all $i \in [1, m]$, and

\[
\Pi^{\pm v_j} : \Sigma_{i} \rightarrow \Sigma_{ex}^{\pm v_j} \\
(v, w, Z) \mapsto (\tilde{u}, \tilde{v}, \tilde{w}, \tilde{Z}),
\]

(C.47)

for all $i \in [1, m]$ with $i \neq j$.

**Proposition C.4.** Consider the vector field $Y$ given by (C.44) and let $\Sigma_{en}$, $\Sigma_{ex}^w$, $\Sigma_{ex}^{\pm v_j}$ and $\Pi^w$, $\Pi^{\pm v_j}$ be as above. Then
• The transition $\Pi^w$ is given by
\[
\tilde{u} = u \left( \frac{w}{w_{out}} \right)^{1/\gamma}, \quad \tilde{v}_i = v_i \left( \frac{w_{out}}{w} \right)^{\beta_i/\gamma}, \quad \tilde{Z} = Z \exp \left[ -\frac{\Lambda}{\gamma w} \left( 1 + \tilde{\alpha} w \ln(w) + w \tilde{G} \right) \right],
\]
where $\tilde{\alpha} = \tilde{\alpha}(uv_1^{1/\beta}, uw_1^{1/\gamma})$ and $\tilde{G} = \tilde{G}(uv_1^{1/\beta}, uw_1^{1/\gamma}, \mu_i)$ are $C^\ell$ functions with $\mu_i = v_i^{1/\beta} w^{-1/\gamma}$.

• The transition $\Pi^{\pm v_j}$ is given by
\[
\tilde{u} = \left( \frac{v_j}{\eta_j} \right)^{1/\beta}, \quad \tilde{v}_i = v_i \left( \frac{\eta_j}{v_j} \right)^{\beta_i/\beta_j}, \quad \tilde{w} = w \left( \frac{\eta_j}{v_j} \right)^{\gamma/\beta_j}, \quad \tilde{Z} = Z \exp \left[ -\frac{\Lambda}{\gamma w} \left( 1 + \tilde{\alpha}' w \ln(v_j) + w \tilde{G}' \right) \right],
\]
with $i \neq j$ and where
\[
\tilde{\alpha}' = \tilde{\alpha}'(uv_1^{1/\beta}, uw_1^{1/\gamma}) \quad \tilde{G}' = \tilde{G}'(uv_1^{1/\beta}, uw_1^{1/\gamma}, \mu_w, \mu_i)
\]
are $C^\ell$ functions with $\mu_w = w^{1/\gamma} v_j^{1/\beta_j}$ and $\mu_i = v_i^{1/\beta_j} v_j^{1/\beta_j}$.

Proof of proposition C.4. We detail first the computations for the transition $\Pi^w$. The transition $\Pi^{\pm v_j}$ is computed in a similar way so we only highlight the key parts of the computation.

The transition $\Pi^w$
In this case, the time of integration is $T = \ln \left( \frac{w_{out}}{w} \right)^{1/\gamma}$, where $w_{out} = w(t) |_{\Sigma_{w_{out}}}$ and $w = w(t) |_{\Sigma_{w}}$. This time of integration is obtained from the equation $w' = \gamma w$. We also make the assumption that $v_i \in O(w^{1/\beta_i})$. This assumption appears in our applications, but roughly speaking it ensures that $\tilde{v}_i$ is well defined when $w \to 0$. From the form of $Y$ we evidently have
\[
\begin{align*}
u(T) &= u(T) = \tilde{u} = u \left( \frac{w}{w_{out}} \right)^{1/\gamma}, \\
v_i(T) &= \tilde{v}_i = v_i \left( \frac{w_{out}}{w} \right)^{\beta_i/\gamma}.
\end{align*}
\]
It only remains to compute the transition for the $Z$ coordinate. Let us rewrite $Y$ as follows
\[
\begin{align*}
\frac{w'}{w} &= -u \\
v_i &= \beta_i v_i \\
w &= \gamma w \\
Z' &= -\frac{\Lambda + G(u, v, w)}{w} Z.
\end{align*}
\]
where $G$ is a $C^\ell$ function vanishing at the origin. Observe that we have the first integrals $u^b v_i$ and $u^\gamma w$. We shall take advantage of such a fact. We define new coordinates $(U, V, W)$ given by
\[
U = u, \quad V_i^{\beta_i} = v_i, \quad W^\gamma = w.
\]
In these new coordinates we have the system
\[
\begin{align*}
\frac{U'}{U} &= -u \\
V_i &= \beta_i v_i \\
W &= \gamma w \\
Z' &= -\frac{\Lambda + G(U, V, W)}{W} Z.
\end{align*}
\]
\[ U' = -U \]
\[ V_i' = V_i \]
\[ W' = W \]
\[ Z' = -\frac{\Lambda + G(U,V^\beta,W^\gamma)}{W^\gamma} Z. \]  
\text{(C.54)}

In the new coordinates, the time of integration is given as \( T = \ln \left( \frac{W}{W_o} \right) \). To have an idea of the expression of \( \tilde{Z} \), let us first study a simplified scenario.

The case \( G = 0 \)

Let us suppose \( G = 0 \). Therefore we have
\[ Z' = -\frac{\Lambda}{W^\gamma} z, \]
which has the solution
\[ Z(t) = Z(0) \exp \left( -\Lambda \int_0^t W(s)^{-\gamma} ds \right), \]
\text{(C.55)}

where \( W(s) = W(0) \exp(s) \). Substituting the time of integration \( T \) we have
\[ Z(T) = \tilde{Z} = Z \exp \left( -\Lambda \int_0^T \frac{\ln \left( \frac{W}{W_o} \right)}{W} e^{-\gamma s} ds \right) \]
\[ = Z \exp \left( -\frac{\Lambda}{\gamma W^\gamma} \left( 1 - \left( \frac{W}{W_o} \right)^{\gamma} \right) \right). \]
\text{(C.56)}

Observe that \( \tilde{Z} \to 0 \) as \( W \to 0 \). Let us now study the general case. We expect that the general case \( G \neq 0 \) is a perturbation of (C.56).

The case \( G \neq 0 \)

We now consider that \( G \neq 0 \), we have
\[ Z(T) = \tilde{Z} = Z \exp \left( I_0 + I_1 \right), \]
\text{(C.57)}

where
\[ I_0 = -\Lambda \int_0^T \frac{1}{W(s)} ds \]
\[ I_1 = \int_0^T \frac{G(U(s),V(s)^\beta,W(s)^\gamma)}{W(s)^\gamma} ds. \]
\text{(C.58)}
The integral \( I_0 \) has already been computed above. Let us write \( F(U,V,W) = \frac{G(U(s),V(s)^\beta,W(s)^\gamma)}{W(s)^\gamma} \). We can do this because \( G(U,0,0) = 0 \) and \( V^\beta \in O(W^\gamma) \). Now we estimate the integral \( I_1 \). Using lemma C.1, we can write
\[ I_1 = \int_0^T [F_1(s) + F_2(s)] ds, \]
\text{(C.59)}

where
\[ F_1 = F_1(UV_1, \ldots, UV_m, UW, U) \]
\[ F_2 = F_2(UV_1, \ldots, UV_m, UW, V_1, \ldots, V_m, W). \]
\text{(C.60)}
Observe that $UW$ and all the $UV_j$’s are first integrals. Let $J_1 = \int F_1$ and $J_2 = \int F_2$. Then we have

$$J_1 = \int_0^T F_1(UV, UW, U(s)) ds$$

$$= \int_0^T F_1(UV, UW, U e^{-s}) ds.$$  (C.61)

Let us make the change of variables $y = e^{-s}$, we obtain

$$J_1 = -\int_1^W F_1(UV, UW, Uy) \frac{dy}{y}.$$  (C.62)

We expand the function $F_1$ in power of $y$ that is

$$F_1(UV, UW, Uy) = F_1(UV, UW, 0) + O(y).$$  (C.63)

Then we have

$$J_1 = \alpha_1 \ln \left( \frac{W_0}{W} \right) + \tilde{F}_1,$$  (C.64)

where $\alpha_1 = \alpha_1(UV, UW)$ and $\tilde{F}_1 = \tilde{F}_1(UV, UW, Uy(T))$ is some (unknown) $C^\ell$ function. Finally we get

$$J_1 = \alpha_1 \ln \left( \frac{W_0}{W} \right) + \tilde{F}_1.$$  (C.65)

The function $\tilde{F}_1$ is $C^\ell$ but unknown, and $W_0$ is a fixed positive constant, then we can simplify the notation of $\tilde{F}_1$ as $\tilde{F}_1 = \tilde{F}_1(UV, UW)$.

Next we have

$$J_2 = \int_0^T F_2(UV, UW, V(s), W(s)) ds$$

$$= \int_0^T \frac{W}{W_0} F_2(UV, UW, V e^{\beta_1 s}, \ldots, V_m e^{\beta_m s}, W e^{\gamma s}) ds.$$  (C.66)

Let us make the change of variables $y = e^s$. Then we obtain

$$J_2 = \int_1^W F_2(UV, UW, V_1 y^{\beta_1}, \ldots, V_m y^{\beta_m}, W y^\gamma) \frac{dy}{y}.$$  (C.67)

As above, we expand in powers of $y$, that is

$$F_2 = \alpha_2 + O(y),$$  (C.68)

and then we have

$$J_2 = \alpha_2 \ln \left( \frac{W_0}{W} \right) + \tilde{F}_2,$$  (C.69)

where $\alpha_2 = \alpha_2(UV, UW)$, $\tilde{F}_2 = F_2(UV, UW, \mu_i)$ is a $C^\ell$ function with $\mu_i = V_i W^{-1}$ for all $i \in [1, m]$. Recall that since $v_i \in O(w^{\beta_i/\gamma})$ we also have that $V \in O(W)$, that is $\mu_i$ is well defined.
Now we can write the integral $I_1$ as
\[
I_1 = J_1 + J_2
= \alpha_1 \ln \left( \frac{W_0}{W} \right) + \tilde{F}_1 + \alpha_2 \ln \left( \frac{W_0}{W} \right) + \tilde{F}_2
= \alpha \ln \left( \frac{W_0}{W} \right) + \tilde{F},
\]
where $\alpha = \alpha(UV, UW)$ and $\tilde{F} = \tilde{F}(UV, UW, \mu_i)$ are $C^\ell$ functions. Finally we write $\tilde{Z}$ in the original coordinates as follows
\[
\tilde{Z} = Z \exp (I_0 + I_1)
= Z \exp \left[ -\frac{A}{\gamma w} \left( 1 - \frac{w}{w_{\text{out}}} \right) + \frac{1}{\gamma} \alpha \ln \left( \frac{w_{\text{out}}}{w} \right) + \tilde{F} \right]
= Z \exp \left[ -\frac{A}{\gamma w} \left( 1 + \tilde{\alpha} w \ln(w) + w \tilde{G} \right) \right],
\]
where $\tilde{\alpha} = \tilde{\alpha}(uw_1^{1/\beta_i}, uw^{1/\gamma})$ and $\tilde{G} = \tilde{G}(uw_1^{1/\beta_i}, uw^{1/\gamma}, \mu_i)$ are $C^\ell$ functions with $\mu_i = v_i w^{-\beta_i/\gamma}$.

**The transition $\Pi^{\pm v_j}$**

In this case the time of integration is given by $T = \ln \left( \frac{w}{v_j} \right)^{1/\beta_j}$. Such a time of integration is obtained from the equation $v_j' = \beta_j v_j$. We have
\[
\tilde{u} = u \left( \frac{v_j}{v_j} \right)^{1/\beta_j}
\tilde{v}_i = v_i \left( \frac{v_j}{v_j} \right)^{\beta_i/\beta_j}
\tilde{w} = w \left( \frac{v_j}{v_j} \right)^{\gamma/\beta_j}.
\]

It then only rests to compute $\tilde{Z}$. Following similar arguments as for the transition $\Pi^w$ we get in this case
\[
\tilde{Z} = Z \exp \left[ -\frac{A}{\gamma w} \left( 1 + \tilde{\alpha}' w \ln(v_j) + w \tilde{G}' \right) \right],
\]
where now
\[
\tilde{\alpha}' = \tilde{\alpha}'(uw_1^{1/\beta_i}, uw^{1/\gamma})
\tilde{G}' = \tilde{G}'(uw_1^{1/\beta_i}, uw^{1/\gamma}, \mu_w, \mu_i)
\]
are $C^\ell$ functions with $\mu_w = w v_j^{-\gamma/\beta_j}$ and $\mu_i = v_i v_j^{-\beta_i/\beta_j}$.

**Saddle 2**

In this case we investigate the transitions of a vector field of the form
\[
Y : \left\{ \begin{array}{l}
u' = uv \\v_j' = -\beta_j uv_j \\
w' = -\gamma w^2 \\
Z' = -gZ,
\end{array} \right. \tag{C.75}
\]
where the coefficients $\beta_j$, $\gamma$ are positive. We assume that $u \in \mathbb{R}^+$. Observe that now, in contrast with case 1, we only have one expanding direction, which is $u$. This makes the study of the transition easier. Due to the same reason, it is more convenient to study a transition

$$\Pi^u : \Sigma_{en} \rightarrow \Sigma_{ex},$$

where to be general, we let $\Sigma_{en}$ be any codimension 1 subset of $\mathbb{R}^{m+3}$ obtained by setting one of the coordinates $(v, w)$ to a constant and with $u < u_{out}$; and where

$$\Sigma_{ex} = \left\{ (u, \bar{v}, \bar{w}, \bar{Z}) \, | \, \bar{u} = u_{out} \right\}.$$  \hspace{1cm} (C.77)

**Proposition C.5.** Consider the vector field $Y$ given by (C.75) and let $\Sigma_{en}$, $\Sigma_{ex}$ and $\Pi^u$ be as above. Then

$$\bar{v}_i = v_i \left( \frac{u}{u_{out}} \right)^{\beta_i},$$

$$\bar{w} = w \left( \frac{u}{u_{out}} \right)^{\gamma}$$

$$\bar{Z} = Z \exp \left[ -\frac{\Lambda}{\gamma w} \left( \frac{u_{out}}{u} \right)^{\gamma} - 1 + \alpha w \ln(u) + w \bar{F} \right],$$

where $\alpha = \alpha(u^\beta v_i, u^\gamma w)$ and $\bar{F} = \bar{F}(u^\beta v_i, u^\gamma w, u)$ are $C^l$ functions.

**Proof of proposition C.5.** We have that the time of integration is $T = \ln \left( \frac{u_{out}}{u} \right)$. It follows that

$$\bar{v}_i = v_i \left( \frac{u}{u_{out}} \right)^{\beta_i},$$

$$\bar{w} = w \left( \frac{u}{u_{out}} \right)^{\gamma}.$$  \hspace{1cm} (C.79)

It only remains to compute $\bar{Z}$. Following similar arguments as in case 1 we have

$$\bar{Z} = Z \exp \left[ -\frac{\Lambda}{\gamma w} \left( \frac{u_{out}}{u} \right)^{\gamma} - 1 + \alpha w \ln(u) + w \bar{F} \right],$$

where $\alpha = \alpha(u^\beta v_i, u^\gamma w)$ and $\bar{F} = \bar{F}(u^\beta v_i, u^\gamma w, u)$ are $C^l$ functions. \hfill $\Box$

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**References**

[1] V.I. Arnold, S.M. Gusein-Zade, and A.N. Varchenko. *Singularities of Differentiable Maps, Volume I*, volume 17. Birkhäuser, 1985.

[2] P. Bonckaert. Partially hyperbolic fixed points with contraints. *Transactions of the American mathematical society*, 33, 1996.

[3] P. Bonckaert. Conjugacy of vector fields respecting additional properties. *Journal of dynamics and control systems*, 3:419–432, 1997.

[4] H. W. Broer, T. J. Kaper, and M. Krupa. Geometric Desingularization of a Cusp Singularity in Slow–Fast Systems with Applications to Zeeman’s Examples. *J. Dyn. Diff. Equat.*, 2013.

[5] Th. Bröcker. *Differentiable Germs and Catastrophes*, volume 17 of Lecture Note Series. Cambridge University Press, 1975.

[6] P. de Maëlsschack, F. Dumortier, and R. Roussarie. *Canard cycles from the birth to the transition*. Book in preparation.
[7] P. de Maesschalck and F. Dumortier. Time analysis and entry–exit relation near planar turning points. *Journal of Differential Equations*, 22(3–4):165 – 206, 2005.
[8] F. Dumortier and R. Roussarie. Geometric singular perturbation theory beyond normal hyperbolicity. In C.K.R.T. Jones and A. Khibnik, editors, *Multiple-Time-Scale Dynamical Systems*, volume 122, pages 29–65. Springer, 2001.
[9] Freddy Dumortier and Robert Roussarie. *Canard Cycles and Center Manifolds*, volume 121. American Mathematical Society, 1996.
[10] N. Fenichel. Geometric singular perturbation theory. *JDE*, pages 53–98, 1979.
[11] John Guckenheimer and Philip Holmes. *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*. Springer-Verlag, 1983.
[12] H. Jardón-Kojakhmetov. Formal normal form of $A_k$ slow fast systems. Submitted to *Les Comptes Rendus Mathematique de l’Académie des sciences*.
[13] H. Jardón-Kojakhmetov. Classification of constrained differential equations embedded in the theory of slow fast systems. PhD Thesis, University of Groningen, 2015.
[14] H. Jardón-Kojakhmetov and Henk W. Broer. Polynomial normal forms of constrained differential equations with three parameters. *Journal of Differential Equations*, 257(4):1012–1055, 2014.
[15] C. K. R. T. Jones. Geometric singular perturbation theory. In *Dynamical Systems*, LNM 1609, pages 44–120. Springer-Verlag, 1995.
[16] Tasso J. Kaper. An introduction to geometric methods and dynamical systems theory for singular perturbation problems. In *Symposia in Applied Mathematics*, volume 56, pages 85–131. AMS, 1999.
[17] M. Krupa and P. Szmolyan. Extending geometric singular perturbation theory to non hyperbolic points: fold and canard points in two dimensions. *SIAM J. Math. Anal.*, 33:286–314, 2001.
[18] M. Krupa and P. Szmolyan. Geometric analysis of the singularly perturbed planar fold. In *Multiple-Time-Scale Dynamical Systems*, LNM 1609, pages 89–116. Springer-Verlag, 2001.
[19] Martin Krupa and Martin Wechselberger. Local analysis near a folded saddle-node singularity. *Journal of Differential Equations*, 248(12):2841 – 2888, 2010.
[20] Eric Lombardi and Laurent Stolovitch. Normal forms of analytic perturbations of quasihomogeneous vector fields: rigidity, invariant analytic sets and exponentially small approximation. *Ann. Sci. Éc. Norm. Supér.*, 43(4), 2010.
[21] M. Spivak. *Calculus on manifolds*. Westview Press, 1965.
[22] Ian Stewart. Elementary catastrophe theory. *IEEE Transactions on Circuits and Systems*, CAS-30(8):578–586, 1983.
[23] F. Takens. Partially hyperbolic fixed points. *Topology*, 10:133–147, 1970.
[24] F. Takens. Constrained equations: a study of implicit differential equations and their discontinuous solutions. In *Structural Stability, the Theory of Catastrophes, and Applications in the Sciences*, LNM 525, pages 134–234. Springer-Verlag, 1976.
[25] E.C. Zeeman. Differential equations for the heart beat and nerve impulse. In *Towards a theoretical biology*, volume 4, pages 8–67. Edinburgh University Press.