SOME FOURTH ORDER NONLINEAR ELLIPTIC PROBLEMS RELATED TO EPITAXIAL GROWTH

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Abstract. This paper deals with some mathematical models arising in the theory of epitaxial growth of crystal. We focalize the study on a stationary problem which presents some analytical difficulties. We study the existence of solutions. The central model in this work is given by the following fourth order elliptic equation,

\[ \Delta^2 u = \det (D^2 u) + \lambda f, \quad x \in \Omega \subset \mathbb{R}^2 \]

The framework to study the problem deeply depends on the boundary conditions.

To the memory of James Serrin.

1. Introduction

In this work we are concerned with the stationary version of (5) below, which reads

\[ \left\{ \begin{array}{l}
\Delta^2 u = \det (D^2 u) + \lambda f, \\
\text{boundary conditions,}
\end{array} \right. \quad x \in \Omega \subset \mathbb{R}^2, \]

where \( \Omega \) has smooth boundary, \( n \) is the unit outward normal to \( \partial \Omega \), \( f \) is a function with a suitable hypothesis of summability and \( \lambda > 0 \). We will concentrate on Dirichlet boundary conditions, that is,

\[ u = 0, \quad \frac{\partial u}{\partial n} = 0 \text{ on } \partial \Omega \]

and Navier conditions

\[ u = 0, \quad \Delta u = 0 \text{ on } \partial \Omega, \]

This type of problems appears in a model of epitaxial growth.

Epitaxial growth is characterized by the deposition of new material on existing layers of the same material under high vacuum conditions. This technique is used in the semiconductor industry for the growth of thin films \[9\]. The mathematical description of epitaxial growth uses the function

\[ u : \Omega \subset \mathbb{R}^2 \times \mathbb{R}^+ \rightarrow \mathbb{R}, \]

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which describes the height of the growing interface at the spatial point \( x \in \Omega \subset \mathbb{R}^2 \) at time \( t \in \mathbb{R}^+ \). A basic modelling assumption is of course that \( u \) is an univalued function, a fact that holds in a reasonably large number of cases \([9]\). The macroscopic description of the growing interface is given by a partial differential equation for \( u \) which is usually postulated using phenomenological and symmetry arguments \([9, 22]\).

We will focus on one such an equation that was derived in the context of non-equilibrium surface growth \([15]\). It reads

\[
(5) \quad u_t = 2K_1 \det (D^2 u) - K_2 \Delta u + \xi(x, t).
\]

The field \( u \) evolves in time as dictated by three different terms: linear, nonlinear and non-autonomous ones. Both linear and nonlinear terms describe the dynamics on the interface. The nonlinear term is the small gradient expansion (which assumes \( |\nabla u| \ll 1 \)) of the Gaussian curvature of the graph of \( u \). The linear term is the bilaplacian, that can be considered as the linearized Euler-Lagrange equation of the Willmore functional \([20]\). This is in fact, see below, the proper interpretation in this context. Finally, the non-autonomous term takes into account the material being deposited on the growing surface. We can consider this equation as a sort of Gaussian curvature flow \([11, 6]\) which is stabilized by means of a higher order viscosity term. The derivation of this equation was actually geometric: it arose as a gradient flow pursuing the minimization of the functional

\[
(6) \quad \mathcal{V}(u) = \int_\Omega \left( K_1 H + \frac{K_2}{2} H^2 \right) \sqrt{1 + |\nabla u|^2} \, dx,
\]

where \( H \) is the mean curvature of the graph of \( u \), in the context of non-equilibrium statistical mechanics of surface growth \([22, 15]\). The actual terms on the right hand side of (5) are found after formally expanding the Euler-Lagrange equation corresponding to this functional for small values of the different derivatives of \( u \) and retaining only linear and quadratic terms. This type of derivation does not rely on a detailed modelling of the processes taking place in a particular physical system. Instead, because its goal is describing just the large scale properties of the growing interface, it is based on rather general considerations. This is the usual way of reasoning in this context, in which equations are derived as representatives of universality classes, this is, sets of physical phenomena sharing the same large scales properties \([9]\). In this sense, equation (6) has been claimed to be an exact representative of a well known universality class within the realm of non-equilibrium growth \([16]\). In particular, frequently used models of epitaxial growth belong to this universality class \([15]\).

We will devote this work to obtain some results on existence and multiplicity of solutions to the fourth order elliptic problem (1). One of the main consequences of this work is to exhibit the dependence on the boundary conditions, even in the formulation of the problems.

More precisely the organization of the paper is as follows.

In Section 2 we formulate some known results on properties of the Hessian of a function in the Sobolev space \( W^{2,2}(\mathbb{R}^N) \), that will be used in the article.

Section 3 is devoted to the variational formulation of the problem with Dirichlet boundary conditions and proving, via critical points arguments, existence and
multiplicity of solutions. We will use some arguments on minimization and a version by Ekeland, in [14] (see also [7]) of the Ambrosetti-Rabinowitz Mountain Pass Theorem (see [23]).

The Navier boundary conditions, as far as we know, have not a variational formulation. In Section 4 we prove existence of solution for the problem with Navier conditions by using some fixed point arguments. Finally, in Section 5 we present some related remarks and open problems.

2. Preliminaries. Functional setting

If \( v \) is a smooth function we have the following chain of equalities,

\[
\det (D^2 v) = v_{x_1 x_1} v_{x_2 x_2} - v_{x_1 x_2}^2 = (v_{x_1} v_{x_2 x_2}) x_1 - (v_{x_1} v_{x_2 x_1}) x_2 = \]

\[
(v_{x_1} v_{x_2}) x_1 x_2 - \frac{1}{2} (v_{x_2}^2) x_1 x_1 - \frac{1}{2} (v_{x_1}^2) x_2 x_2.
\]

From now on we will assume \( \Omega \subset \mathbb{R}^2 \) is open, bounded and has a smooth boundary. Notice that by density we can consider the above identities in \( D'(\Omega) \), the space of distributions. This subject is deeply related with a conjecture by J. Ball [8]:

If \( u = (u^1, u^2) \in W^{1,p}(\Omega, \mathbb{R}^2) \) consider \( \det(Du) = u_{x_1}^1 u_{x_2}^2 - u_{x_2}^1 u_{x_1}^2 \) and define

\[
\text{Det}(Du) = (u_{x_1}^1 u_{x_2}^2) x_1 - (u_{x_2}^1 u_{x_1}^2) x_2.
\]

When is it true that \( \det(Du) = \text{Det}(Du) \)?

A positive answer, among others results, was given by S. Müller in [23].

We will use Theorem VII.2, page 278 in [12], that we formulate as follows.

**Lemma 2.1.** Let \( v \in W^{2,2}(\mathbb{R}^2) \). Then,

\[
\det (D^2 v),
\]

\[
(v_{x_1} v_{x_2 x_2}) x_1 - (v_{x_1} v_{x_2 x_1}) x_2
\]

and

\[
(v_{x_1} v_{x_2}) x_1 x_2 - \frac{1}{2} (v_{x_2}^2) x_1 x_1 - \frac{1}{2} (v_{x_1}^2) x_2 x_2
\]

belong to the space \( H^1(\mathbb{R}^2) \) and are equal in it, where \( H^1(\mathbb{R}^2) \) is the Hardy space.

All the expressions involving third derivatives are understood in the distributional sense; then the result in the lemma is highly non-trivial and for the proof we refer to [12]. It is interesting to point out that this result deeply depends on Luc Tartar and François Murat arguments on compactness by compensation [27, 28] and [24] respectively.

For the reader convenience we recall the definition of the Hardy space in \( \mathbb{R}^N \) (see E. M. Stein G. Weiss [26]).

**Definition 2.2.** The Hardy space in \( \mathbb{R}^N \) is defined in an equivalent way as follows

\[
H^1(\mathbb{R}^N) = \{ f \in L^1(\mathbb{R}^N) | R_j(f) \in L^1(\mathbb{R}^N), j = 1, 2, \ldots, N \} = \{ f \in L^1(\mathbb{R}^N) | \sup |f * h_t(x)| | \in L^1(\mathbb{R}^N) \},
\]

where \( R_j \) is the classical Riesz transform, that is,

\[
R_j = \frac{\partial}{\partial x_j}(-\Delta)^{\frac{j}{2}}, \ j = 1, 2, \ldots, N,
\]

and

\[
h_t(x) = \frac{1}{t^N} h\left(\frac{x}{t}\right), \text{ where } h \in C_0^\infty(\mathbb{R}^N), \ h(x) \geq 0 \text{ and } \int_{\mathbb{R}^N} h \, dx = 1.
\]
Notice that, as a direct consequence of the definition, if \( f \in H^1(\mathbb{R}^N) \) and \( E_{1,N}(x) \) is the fundamental solution to the Laplacian in \( \mathbb{R}^N \), then
\[
u(x) = \int_{\mathbb{R}^N} E_{1,N}(x-y)f(y)dy
\]
verifies that \( \nu \in W^{2,1}(\mathbb{R}^N) \). See for instance [25].

In a similar way if we consider \( E_{2,N}(x) \), the fundamental solution to \( \Delta^2 \) in \( \mathbb{R}^N \), a direct calculation shows that \( D^\alpha E_{2,N}(x) \), \( |\alpha| = 4 \), are of the form
\[
D^\alpha E_{2,N}(x) = \frac{H_\alpha(\bar{x})}{|x|^N}, \quad \bar{x} = \frac{x}{|x|}
\]
where \( H_\alpha \) is a positively homogeneous function of zero degree and
\[
\int_{S^{N-1}} H_\alpha(\bar{x})d\bar{x} = 0,
\]
that is, \( D^\alpha E_{2,N}(x) \) is a classical Calderon-Zygmund kernel. For \( f \in H^1(\mathbb{R}^N) \) consider
\[
u(x) = \int_{\mathbb{R}^N} E_{2,N}(x-y)f(y)dy,
\]
then \( \nu \in W^{4,1}(\mathbb{R}^N) \).

D. C. Chang, G. Dafni, and E. M. Stein in [13] give the definition of Hardy space in a bounded domain \( \Omega \) in order to have the regularity theory for the Laplacian similar to the one in \( \mathbb{R}^N \).

The extension of this kind of regularity result to the bi-harmonic equation on bounded domains is of interest for the current problem. The application of such a result would allow obtaining extra regularity in the nonlinear setting studied in this work. We leave these questions as a subject of future research.

We use the following consequence which is a by product of the results in [12] and in [13].

**Lemma 2.3.** Let \( u \in W^{2,2}_0(\Omega) \). Then
\[
\text{Det}(D^2 u) = (u_{x_1} u_{x_2})_{x_1x_2} - \frac{1}{2} (u_{x_2}^2)_{x_1x_1} - \frac{1}{2} (u_{x_1}^2)_{x_2x_2}
\]
in \( L^1(\Omega) \cap h^1_1(\Omega) \). Here \( h^1_1(\Omega) \) is the class of function restrictions of \( H^1(\mathbb{R}^N) \) to \( \Omega \).

3. Variational settings: existence and multiplicity results for the Dirichlet conditions

We will study the following problem,

\[
\begin{aligned}
\Delta^2 u &= \det(D^2 u) + \lambda f, \quad x \in \Omega \subset \mathbb{R}^2, \\
u = 0, \quad \frac{\partial u}{\partial n} &= 0 \text{ on } \partial \Omega,
\end{aligned}
\]
(7)

this is, Dirichlet boundary conditions, where \( \Omega \) is a bounded domain with smooth boundary and \( f \in L^1(\Omega) \). The natural framework is the space \( W^{2,2}_0(\Omega) \), that is the completion of \( C_0^\infty(\Omega) \) with the norm of \( W^{2,2}(\Omega) \). This fact is the key to have in this case a variational formulation of the problem. We recall that the norm of the Hilbert space \( W^{2,2}_0(\Omega) \) is equivalent to the norm \( \|\Delta u\|_2 \). For all the functional framework we refer the reader to the precise, detailed and very nice monograph by F. Gazzola, H. Grunau and G. Sweers, [19]. See [3] and [4] as classical references for elliptic equations of higher order.
Remark 3.1. We could consider the inhomogeneous Dirichlet problem,

\[
\begin{cases}
\Delta^2 u = \det(D^2 u) + \lambda f, \\
u = \mu_1, \\ \frac{\partial u}{\partial n} = \mu_2 \text{ on } \partial \Omega,
\end{cases}
\]

where \(\mu_i, i = 1, 2\) are in suitable trace spaces. Considering the bi-harmonic function \(w\) with data \(\mu_1\) and \(\mu_2\) and defining \(v = u - w\) we reduce the problem to the homogeneous case with a linear perturbation and a new source term, that is,

\[
\begin{cases}
\Delta^2 v = \det(D^2 v) + w_{yy}v_{xx} + w_{xx}v_{yy} - 2w_{xy}v_{xy} + \det(D^2 u) + \lambda f, \\
v = 0, \\ \frac{\partial v}{\partial n} = 0 \text{ on } \partial \Omega.
\end{cases}
\]

This problem can be solved by using fixed point arguments for small data. See the next Section 4. The smallness of the data is only known so far in the radial framework. See [17].

We try to find a Lagrangian \(L(\nabla u, D^2 u)\) such that the critical points of the functional

\[
J_\lambda : W^{2,2}_0(\Omega) \rightarrow \mathbb{R}
\]

\[
u \rightarrow J_\lambda(u) = \frac{1}{2} \int_\Omega |\Delta u|^2 \, dx - \int_\Omega L(\nabla u, D^2 u) \, dx
\]

are solutions to (7).

3.1. Lagrangian for the Dirichlet conditions. We will try to obtain a Lagrangian for which the Euler first variation is the determinant of the Hessian matrix. Our ingredients will be the distributional identity

\[
\det(D^2 u) = (v_x v_{x2})_{x1x2} - \frac{1}{2}(v_{x2}^2)_{x1x1} - \frac{1}{2}(v_{x1}^2)_{x2x2}
\]

and the fact that \(C_0^\infty(\Omega)\) is dense in \(W^{2,2}_0(\Omega)\).

Consider \(\phi \in C_0^\infty(\Omega)\) and

\[
\int_\Omega \det(D^2 u) \phi \, dx = \int_\Omega \left[ -\frac{1}{2}(u_{x2}^2)_{x1x1} - \frac{1}{2}(u_{x1}^2)_{x2x2} + (u_{x1} u_{x2})_{x1x2} \right] \phi \, dx
\]

\[
= \int_\Omega \left[ \frac{1}{2} \phi_{x1}(u_{x2}^2)_{x1} + \frac{1}{2} \phi_{x2}(u_{x1}^2)_{x2} + u_{x1} u_{x2} \phi_{x1x2} \right] dx
\]

\[
= \left. \frac{d}{dt} G(u + t\phi) \right|_{t=0},
\]

where

\[
G(u) := \int_\Omega u_{x1} u_{x2} u_{x1x2} \, dx.
\]

Notice that by density we can take \(\phi \in W^{2,2}_0(\Omega)\) and by direct application of Lemma 2.1 above we find that the first variation of \(G(u)\) on \(W^{2,2}_0(\Omega)\) is

\[
\frac{\delta G(u)}{\delta u} = \det(D^2 u).
\]

Then we will consider as energy functional for problem (7) the following one

\[
J_\lambda(u) = \frac{1}{2} \int_\Omega |\Delta u|^2 \, dx - \int_\Omega u_{x1} u_{x2} u_{x1x2} \, dx - \lambda \int_\Omega f u \, dx,
\]

\[
J_\lambda : W^{2,2}_0(\Omega) \rightarrow \mathbb{R}
\]

\[
u \rightarrow J_\lambda(u) = \frac{1}{2} \int_\Omega |\Delta u|^2 \, dx - \int_\Omega L(\nabla u, D^2 u) \, dx
\]
defined in $W^{2,2}_0(\Omega)$.

As we will see $J_\lambda$ is unbounded from below and then we cannot use standard
minimization results but the general theory of critical points of functionals.

Remark 3.2. Notice that this Lagrangian is not useful for other boundary conditions. Indeed, consider
$\phi \in X = \{ \phi \in C^\infty(\Omega) \mid \phi(x) = 0 \text{ on } \partial\Omega \}$ and $u$ a smooth
function, then

$$
\frac{d}{dt} G(u + t\phi)_{|t=0} = \int_\Omega (u_{x_1} u_{x_2} \phi_{x_1} + u_{x_1} \phi_{x_2} u_{x_1} + \phi_{x_1} u_{x_2} u_{x_1} x_2) \, dx
$$

$$
= \int_\Omega \det (D^2 u) \phi \, dx - \frac{1}{2} \int_{\partial\Omega} u_{x_1} u_{x_2} (\phi_{x_1} + \phi_{x_2} \nu_{x_1} \nu_{x_2}) \, ds,
$$

and therefore for $\phi \in X$ the boundary term does not cancel.

This observation justifies the dependence of the problem on the boundary conditions.

3.2. The geometry of $J_\lambda$. Notice that by Hölder and Sobolev inequalities we find
the following estimate

$$
J_\lambda(u) \geq \frac{1}{2} \int_\Omega |\Delta u|^2 \, dx - c_1 \left( \int_\Omega |\Delta u| \, dx \right)^{\frac{2}{3}} - \lambda \|f\|_1 \|u\|_\infty \geq
$$

$$
\frac{1}{2} \int_\Omega |\Delta u|^2 \, dx - c_1 \left( \int_\Omega |\Delta u| \, dx \right)^{\frac{2}{3}} - \lambda c_2 \|f\|_1 \left( \int_\Omega |\Delta u| \, dx \right)^{\frac{2}{3}}
$$

$$
\equiv g (\|\Delta u\|_2),
$$

where

$$
g(s) = \frac{1}{2} s^2 - c_1 s^3 - \lambda c_2 \|f\|_1 \cdot s.
$$

Therefore we easily prove that for $0 < \lambda < \lambda_0$ small enough, the radial lower
estimate (in the Sobolev space), given by $g$ has a negative local minimum and a
positive local maximum. Moreover, it is easy to check that:

1. There exists a function $\phi \in W^{2,2}_0(\Omega)$ such that

$$
\int_\Omega f \phi \, dx > 0.
$$

2. There exists a function $\psi \in W^{2,2}_0(\Omega)$ such that

$$
\int_\Omega \psi_{x_1} \psi_{x_2} \psi_{x_1, x_2} \, dx > 0.
$$

For the function $\phi$ we just need it to be a local mollification of $f$. The case of $\psi$ is
a bit more involved but one still has many possibilities such as

$$
\psi = [(1 - |x|^2)^+]^4,
$$
where $|x| = \sqrt{x_1^2 + x_2^2}$ and $(\cdot)^+ = \max\{\cdot, 0\}$, that fulfils the positivity criterion even pointwise in a domain containing the unit ball. Then, in general, if $B_{2r}(x_0) \subset \Omega$ we consider $\psi_\Omega(x) = \psi\left(\frac{x - x_0}{r}\right)$.

Other suitable functions can be found by means of deforming this one adequately. Notice that the $\psi$ function we have chosen is in $C^2(\mathbb{R}^2)$.

According to the previous remark we find that $J_\lambda(t\phi) < 0$ for $t$ small enough and $J_\lambda(s\psi) < 0$ for $s$ large enough.

This behavior and the radial minorant (see Figure 3.1), suggest a kind of mountain pass geometry. See the classical paper by A. Ambrosetti and P. H. Rabinowitz, [5].

3.3. Palais-Smale condition for $J_\lambda$. As usual, we call $\{u_k\}_{k \in \mathbb{N}} \subset W^{2,2}_0(\Omega)$ a Palais-Smale sequence for $J_\lambda$ to the level $c$ if

i) $J_\lambda(u_k) \to c$ as $k \to \infty$

ii) $J'_\lambda(u_k) \to 0$ in $W^{-2,2}(\Omega)$.

We say that $J_\lambda$ satisfies the local Palais-Smale condition to the level $c$ if each Palais-Smale sequence to the level $c$, $\{u_k\}_{k \in \mathbb{N}}$, admits a strongly convergent subsequence in $W^{2,2}_0(\Omega)$.

We are able to prove the following compactness result.

Lemma 3.3. Assume a bounded Palais-Smale condition for $J_\lambda$, that is $\{u_k\}_{k \in \mathbb{N}} \subset W^{2,2}_0(\Omega)$ verifying

1) $J_\lambda(u_k) \to c$ as $k \to \infty$,

2) $J'_\lambda(u_k) \to 0$ in $W^{-2,2}$.

Then there exists a subsequence $\{u_k\}_{k \in \mathbb{N}}$ that converges in $W^{2,2}_0(\Omega)$.

Proof. Since $\{u_k\}_{k \in \mathbb{N}} \subset W^{2,2}_0(\Omega)$ is bounded, up to passing to a subsequence, we have:

i) $u_k \rightharpoonup u$ weakly in $W^{2,2}_0(\Omega)$,

ii) $\nabla u_k \rightharpoonup \nabla u$ strongly in $[L^p(\Omega)]^2$ for all $p < \infty$,

iii) $u_k \to u$ uniformly in $\Omega$.

We could write the condition $J'_\lambda(u_k) \to 0$ in $W^{-2,2}$ as

$$\Delta^2 u_k = \det(D^2 u_k) + \lambda f + y_k, \quad u_k \in W^{2,2}_0(\Omega) \text{ and } y_k \to 0 \text{ in } W^{-2,2}(\Omega).$$
Notice that multiplying (13) by \( (u_k - u) \), we have for all fixed \( k \)

\[
\int_{\Omega} \Delta(u_k) \Delta(u_k - u) \, dx =
\]

\[
\int_{\Omega} (u_k - u) \det(D^2 u_k) \, dx + \lambda \int_{\Omega} f(u_k - u) \, dx + y_k(u_k - u).
\]

The three terms on the right hand side go to zero as \( k \to \infty \) by the convergence properties \( i \) and \( iii \). Moreover adding in both terms of (14)

\[- \int_{\Omega} \Delta u \Delta(u_k - u) \, dx = o(1) \quad k \to \infty,
\]

we obtain,

\[
\int_{\Omega} |\Delta(u_k - u)|^2 \, dx =
\]

\[
\int_{\Omega} (u_k - u) \det(D^2 u_k) \, dx + \lambda \int_{\Omega} f(u_k - u) \, dx +
\]

\[
y_k(u_k - u) - \int_{\Omega} \Delta u \Delta(u_k - u) \, dx.
\]

As a consequence

\[
\int_{\Omega} |\Delta(u_k - u)|^2 \, dx \to 0 \text{ as } k \to \infty,
\]

that is, \( J_\lambda \) satisfies the Palais-Smale condition to the level \( c \).

3.4. The main multiplicity result. We now can prove the existence and multiplicity result.

**Theorem 3.4.** Let \( \Omega \subset \mathbb{R}^2 \) be a bounded domain with smooth boundary. Consider \( f \in L^1(\Omega) \) and \( \lambda > 0 \). Then there exists a \( \lambda_0 \) such that for \( 0 < \lambda < \lambda_0 \) problem (7) has at least two solutions.

**Proof.** By the Sobolev embedding theorem the functional \( J_\lambda \) is well defined in \( W_0^{2,2}(\Omega) \), is continuous and Gateaux differentiable, and its derivative is weak*-continuous (precisely the regularity required in the weak version by Ekeland of the mountain pass theorem in [7]).

We will try to prove the existence of a solution which corresponds to a negative local minimum of \( J_\lambda \) and a solution which corresponds to a positive mountain pass level of \( J_\lambda \).

**Step 1.-** \( J_\lambda \) has a local minimum \( u_0 \), such that \( J_\lambda(u_0) < 0 \).

We use the ideas in [18] to solve problems with concave-convex semilinear nonlinearities.

Consider \( \lambda_0 > 0 \) such that, if \( 0 < \lambda < \lambda_0 \), \( g \) attains its positive maximum at \( r_{\text{max}} > 0 \). Take \( r_0 \) the lower positive zero of \( g \) and \( r_0 < r_1 < r_{\text{max}} < r_2 \) such that \( g(r_1) > 0, g(r_2) > 0 \), where \( g \) is defined by (12). Now consider a cutoff function

\[
\tau : \mathbb{R}_+ \to [0, 1],
\]
such that \( \tau \) is nonincreasing, \( \tau \in C^\infty \) and it verifies
\[
\begin{cases}
\tau(s) = 1 & \text{if } s \leq r_0, \\
\tau(s) = 0 & \text{if } s \geq r_1.
\end{cases}
\]

Let \( \Theta(u) = \tau(\|\Delta u\|_2) \). We consider the truncated functional
\[
F_\lambda(u) = \frac{1}{2} \int_\Omega |\Delta u|^2 \, dx - \int_\Omega u_{x_1} u_{x_2} u_{x_1 x_2} \Theta(u) \, dx - \lambda \int_\Omega f u \, dx.
\]

As above, by Hölder and Sobolev inequalities we see \( F(u) \geq h(\|\Delta u\|_2) \), with
\[
h(s) = \frac{1}{2} s^2 - c_1 \tau(s) s^3 - \lambda ||f||_1 c_2 s.
\]

The principal properties of \( F \) defined by (16) are listed below.

**Lemma 3.5.**
1. \( F_\lambda \) has the same regularity as \( J_\lambda \).
2. If \( F_\lambda(u) < 0 \), then \( \|\Delta u\|_2 < r_0 \), and \( F_\lambda(u) = J_\lambda(u) \) if \( \|\Delta u\|_2 < r_0 \).
3. Let \( m \) be defined by \( m = \inf_{v \in W_0^{2,2}(\Omega)} F_\lambda(v) \).

Then \( F_\lambda \) verifies a local Palais-Smale condition to the level \( m \).

**Proof.** 1) and 2) are immediate. To prove 3), observe that all Palais-Smale sequences of minimizers of \( F_\lambda \), since \( m < 0 \), must be bounded. Then by Lemma 3.3 we conclude. □

Observe that, by 2), if we find some negative critical value for \( F_\lambda \), then we have that \( m \) is a negative critical value of \( J_\lambda \) and there exist \( u_0 \) local minimum for \( J_\lambda \).

**Step 2.** If \( \lambda \) is small enough, \( J_\lambda \) has a mountain pass critical point, \( u_* \), such that \( J_\lambda(u_*) > 0 \).

By the estimates in subsection 3.2, \( J_\lambda \) verifies the geometrical requirements of the Mountain Pass Theorem (see [3] and [7]). Consider \( u_0 \) the local minimum such that \( J_\lambda(u_0) < 0 \) and consider \( v \in W_0^{2,2}(\Omega) \) with \( \|\Delta v\|_2 > r_{max} \) and such that \( J_\lambda(v) < J_\lambda(u_0) \). We define
\[
\Gamma = \{ \gamma \in C([0,1], W_0^{2,2}(\Omega)) | \gamma(0) = u_0, \gamma(1) = v \},
\]

and the minimax value
\[
c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J_\lambda[\gamma(t)].
\]

Applying the Ekeland variational principle (see [13]), there exists a Palais-Smale sequence to the level \( c \), i.e. there exists \( \{u_k\}_{k \in \mathbb{N}} \subset W_0^{2,2}(\Omega) \) such that
\[
(1) \quad J_\lambda(u_k) \to c \text{ as } k \to \infty,
(2) \quad J'_\lambda(u_k) \to 0 \text{ in } W^{-2,2}.
\]

**Claim.** If \( \{u_k\}_{k \in \mathbb{N}} \subset W_0^{2,2}(\Omega) \) is a Palais-Smale sequence for \( J_\lambda \) at the level \( c \), then there exists \( C > 0 \) such that \( \|\Delta u_k\|_2 < C \).
Since the results in section 2 hold then if \( u \in W_0^{2,2}(\Omega) \), integrating by parts we find that

\[
\int_{\Omega} u \det(D^2 u) \, dx = \int_{\Omega} u \left[ (u_{x_1} u_{x_2 x_2})_{x_2} - (u_{x_1} u_{x_2 x_1})_{x_2} \right] \, dx \\
- \int_{\Omega} (u_{x_1})^2 u_{x_2 x_2} \, dx + \int_{\Omega} u_{x_1} u_{x_2 x_1} u_{x_2} \, dx = \\
2 \int_{\Omega} u_{x_1} u_{x_1 x_2} u_{x_2} \, dx + \int_{\Omega} u_{x_1} u_{x_2 x_1} u_{x_2} \, dx = \\
3 \int_{\Omega} u_{x_1} u_{x_2 x_1} u_{x_2} \, dx.
\]

Then if \( \{u_k\}_{k \in \mathbb{N}} \subset W_0^{2,2}(\Omega) \) is a Palais-Smale sequence for \( J_\lambda \) at the level \( c \) and calling \( \langle y_k, u_k \rangle = \langle J'_\lambda(u_k), u_k \rangle \)

\[
c + o(1) = J_\lambda(u_k) - \frac{1}{3} \langle J'_\lambda(u_k), u_k \rangle + \frac{1}{3} \langle y_k, u_k \rangle \geq \\
\geq \left( \frac{1}{2} - \frac{1}{3} \right) \int_{\Omega} |\Delta u_k|^2 \, dx - \frac{1}{3} |y_k|_{H^{-2}} \left( \int_{\Omega} |\Delta u_k|^2 \, dx \right)^{\frac{1}{2}} \\
- \frac{2}{3} \lambda C_S \|f\|_{L^1} \left( \int_{\Omega} |\Delta u_k|^2 \, dx \right)^{\frac{1}{2}},
\]

where \( C_S \) is a suitable Sobolev constant. This inequality implies that the sequence is bounded.

By using Lemma 3.3, \( J_\lambda \) satisfies the Palais-Smale condition to the level \( c \). Therefore

1. \( J_\lambda(u_*) = \lim_{k \to \infty} J_\lambda(u_k) = c \) (and then \( u_* \) is different from the local minimum, as in this case the value of the functional at this point is positive while in the other one was negative).
2. \( J'_\lambda(u_*) = 0 \), thus

\[
\Delta^2 u_* = \det(D^2 u_*) + \lambda f, \ u_* \in W_0^{2,2}(\Omega).
\]

In other words \( u_* \) is a \textit{mountain pass type} solution to the problem (17).

**Remark 3.6.** Notice that we cannot directly conclude that a bounded Palais-Smale sequence gives a solution in the distributional sense; indeed, we would need the convergence property

\[
\det(D^2 u_k) \rightharpoonup \det(D^2 u_*) \text{ at least in } L^1(\Omega).
\]

To have this property up to passing to a subsequence we need almost everywhere convergence (see the result by Jones and Journé in [21]). Notice that a.e. convergence for the second derivatives is only known after the proof of Lemma 3.3.

4. Some existence results including Navier boundary conditions

We will find a solution to our problem with Navier boundary conditions

\[
\Delta^2 u = \det(D^2 u) + \lambda f, \quad x \in \Omega \subset \mathbb{R}^2, \\
u = 0, \quad \Delta u = 0 \text{ on } \partial \Omega,
\]

(18)
SOME FOURTH ORDER NONLINEAR ELLIPTIC PROBLEMS

and also with Dirichlet boundary conditions

\[
\begin{cases}
\Delta^2 u &= \det(D^2u) + \lambda f, & x \in \Omega \subset \mathbb{R}^2, \\
u &= 0, & \Delta u = 0 \text{ on } \partial \Omega.
\end{cases}
\]

(19)

In this section we will prove the existence of at least one solution to problems (18) and (19) by means of fixed point methods.

First of all we need the following technical result

**Lemma 4.1.** For any functions \(v_1, v_2 \in W^{1,2}(\Omega)\) and \(v_3 \in W^{1,2}_0(\Omega) \cap W^{2,2}(\Omega)\) the following equality is fulfilled

\[
\int \det(\nabla v_1, \nabla v_2) v_3 \, dx = \int v_1 \nabla v_2 \cdot \nabla^\perp v_3 \, dx,
\]

where \(\nabla^\perp v_3 = (\partial_{x_2} v_3, -\partial_{x_1} v_3)\).

**Proof.** By Sobolev embedding we know \(v_3\) is bounded in \(L^\infty(\Omega)\) and consequently the left hand side of (20) is well defined. Now we take this expression and operate

\[
\int \det(\nabla v_1, \nabla v_2) v_3 \, dx = \int \nabla \cdot (v_1 \partial_{x_2} v_2, -v_1 \partial_{x_1} v_2) v_3 \, dx =
\]

(21)

\[
- \int (v_1 \partial_{x_2} v_2, -v_1 \partial_{x_1} v_2) \cdot \nabla v_3 \, dx = \int v_1 \det(\nabla v_2, \nabla v_3) \, dx
\]

The first equality is obviously correct for smooth functions, and its validity can be extended by approximation to functions \(v_1, v_2 \in W^{1,2}(\Omega)\) and \(v_3 \in C^1(\Omega)\) by considering the divergence on the right hand side in the distributional sense [23]. The second equality is a consequence of the definition of weak derivative and the fact that \(v_3\) is a traceless function. In this moment we can conclude both equalities are valid for \(v_3 \in W^{1,2}(\Omega) \cap W^{2,2}(\Omega)\) because \(C^1(\Omega)\) is dense in \(W^{1,2}(\Omega)\). The third and fourth equalities come from simple manipulations of the integrands. Sobolev embedding guarantees that the last three terms in this chain of equalities are well defined. □

Now we move to prove the main result of this section

**Theorem 4.2.** If \(\lambda > 0\) is small enough then:

a) There exists \(u \in W^{1,2}_0(\Omega) \cap W^{2,2}(\Omega)\) solution to problem (18).

b) There exists \(u \in W^{2,2}_0(\Omega)\) solution to problem (19).

**Proof.** As the proof is similar in both cases we skip the details of the case b).

We start considering the linear problems

(22)

\[
\begin{cases}
\Delta^2 u_1 &= \det(D^2u_1) + \lambda f, & x \in \Omega \subset \mathbb{R}^2, \\
u_1 &= 0, & \Delta u_1 = 0 \text{ on } \partial \Omega,
\end{cases}
\]

and

(23)

\[
\begin{cases}
\Delta^2 u_2 &= \det(D^2u_2) + \lambda f, & x \in \Omega \subset \mathbb{R}^2, \\
u_2 &= 0, & \Delta u_2 = 0 \text{ on } \partial \Omega,
\end{cases}
\]
where \(\varphi_1, \varphi_2 \in W^{1,2}_0(\Omega) \cap W^{2,2}(\Omega)\). Classical results guarantee the existence of a weak solution to both problems in \(W^{1,2}_0(\Omega) \cap W^{2,2}(\Omega)\) for given \(\varphi_1\) and \(\varphi_2\). Subtracting both equations we find

\[
\begin{align*}
\Delta^2 (u_1 - u_2) &= \det(D^2 \varphi_1) - \det(D^2 \varphi_2), \quad x \in \Omega \subset \mathbb{R}^2, \\
\Delta (u_1 - u_2) &= 0 \text{ on } \partial\Omega.
\end{align*}
\]

(24)

Now we note

\[
\det(D^2 \varphi_1) - \det(D^2 \varphi_2) = \det\{\nabla(\varphi_1)_{x_1}, \nabla[(\varphi_1)_{x_2} - (\varphi_2)_{x_2}]\} + \det\{\nabla[(\varphi_1)_{x_1} - (\varphi_2)_{x_1}], \nabla(\varphi_2)_{x_2}\}.
\]

(25)

Next we see that for any \(\varphi \in W^{1,2}_0(\Omega) \cap W^{2,2}(\Omega)\) we have

\[
\int \left[ \partial_{x_1} \varphi_1 \nabla(\partial_{x_2} \varphi_1 - \partial_{x_2} \varphi_2) \cdot \nabla^2 \varphi \right] dx - \int \left[ \partial_{x_2} \varphi_2 \nabla(\partial_{x_1} \varphi_1 - \partial_{x_1} \varphi_2) \cdot \nabla^2 \varphi \right] dx = 0,
\]

where we have used equality (25) together with Lemma 4.1. From the equalities (26), we multiply it by \((u_1 - u_2)\) and integrate by parts the left hand side twice to find

\[
\int |\Delta(u_1 - u_2)|^2 dx =
\]

\[
\int \left| \left\langle \det(D^2 \varphi_1) - \det(D^2 \varphi_2), u_1 - u_2 \right\rangle \right| dx \leq
\]

\[
\left( \|\nabla \varphi_1\|_4 + \|\nabla \varphi_2\|_4 \right) \left( \|D^2(\varphi_1 - \varphi_2)\|_2 \right) \|\nabla(u_1 - u_2)\|_4 \leq
\]

\[
C \left( \|\nabla \varphi_1\|_4 + \|\nabla \varphi_2\|_4 \right) \left( \|\Delta(\varphi_1 - \varphi_2)\|_2 \right) \|\nabla(u_1 - u_2)\|_4 \leq
\]

(28)

where we have used Sobolev and Poincaré inequalities and the embedding of the \(L^p\) spaces on bounded domains on the last step, and result (27) on the previous one. Simplifying the last chain of inequalities we arrive at

\[
|\Delta(u_1 - u_2)|_2 \leq C \left( \|\nabla \varphi_1\|_4 + \|\nabla \varphi_2\|_4 \right) \|\Delta(\varphi_1 - \varphi_2)\|_2
\]

for some suitable constant \(C\).

Now, by using the Sobolev embedding \(\|\nabla \varphi_i\|_4 \leq C\|\Delta \varphi_i\|_2\) for \(i = 1, 2\), we find

\[
|\Delta(u_1 - u_2)|_2 \leq C \left( \|\Delta \varphi_1\|_2 + \|\Delta \varphi_2\|_2 \right) \|\Delta(\varphi_1 - \varphi_2)\|_2.
\]

(30)

Consider \(v\) the solution to the problem

\[
\begin{align*}
\Delta^2 v &= \lambda f, \quad x \in \Omega \subset \mathbb{R}^2, \\
v &= 0, \quad \Delta v = 0 \text{ on } \partial\Omega.
\end{align*}
\]

(31)

Notice that

\[
|\Delta v|_2 \leq \lambda |f|_1,
\]
and then the first norm is small when \( \lambda \) is small.

If \( \varphi_i \in B_\rho(v) = \{ \varphi \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega) \mid ||\Delta(v - \varphi)||_2 \leq \rho \} \), then (30) becomes

\[
\|\Delta(u_1 - u_2)\|_2 \leq 2C(\rho + ||\Delta v||_2)\|\Delta(\varphi_1 - \varphi_2)||_2 \leq 2(\rho + \lambda ||f||_1)\|\Delta(\varphi_1 - \varphi_2)||_2 \leq \frac{1}{2} \|\Delta(\varphi_1 - \varphi_2)||_2
\]

for suitable \( \lambda \) and \( \rho \). Moreover, if \( \varphi_i \in B_\rho(v) \) and \( u_i \) is the corresponding solution to either problem (22) or (23), we find that

\[
||\Delta(u_i - v)||_2^2 \leq \int_\Omega |\det(D^2\varphi_i)| (u_i - v) dx \leq S \int_\Omega |\det(D^2\varphi_i)| dx ||\Delta(u_i - v)||_2,
\]

that is

\[
||\Delta(u_i - v)||_2 \leq S \int_\Omega |\det(D^2\varphi_i)| dx \leq C||\Delta\varphi_i||_2^2.
\]

But we can compute that

\[
||\Delta\varphi_i||_2^2 \leq 2(||\Delta(\varphi_i - v)||_2^2 + ||\Delta v||_2^2) \leq 2(\rho^2 + \lambda^2 ||f||_1^2) \leq \frac{\rho}{C}
\]

for \( \rho \) and \( \lambda \) small enough. Thus

\[
||\Delta(u_1 - u_2)||_2 \leq 2C(\rho + \lambda ||f||_1)\|\Delta(\varphi_1 - \varphi_2)||_2.
\]

For \( \varphi \in B_\rho(v) \), define \( u \) as the solution to

\[
\begin{cases}
\Delta^2 u &= \det(D^2\varphi) + \lambda f, & x \in \Omega \subset \mathbb{R}^2, \\
u &= 0, & \Delta u = 0 \text{ on } \partial\Omega,
\end{cases}
\]

and then the nonlinear operator

\[
T : B_\rho(v) \rightarrow B_\rho(v), \quad \varphi \rightarrow u.
\]

By using the estimates above and the classical Banach fixed point theorem, we find a unique fixed point \( u = T(u) \), which is a solution to problem (18).

\[\square\]

**Corollary 4.3.** The solution to problem (18) (respectively to problem (14)) is unique in the ball \( B_\rho(v) \subset W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega) \).

**Remark 4.4.** Notice that also the inhomogeneous Dirichlet problem can be solved by fixed point arguments. However in the homogenous Dirichlet problem with this kind of analysis we only find the trivial solution and not the mountain-pass type solution.

5. **Further remarks and open problems**

In this section we collect some further results that appear in the modelization by considering a more general functional than (6) and performing different truncation arguments. We also quote some results in [17] about the radial setting and propose some open problems.

5.1. **Some insights from the radial case.** In [17], among other results, there appears a numerical estimate of the value of \( \lambda \), or equivalently the size of the datum, for which we have solvability of the radial Dirichlet and Navier problems. The methods used are those of the dynamical systems theory and a shooting method of Runge-Kutta type.

It seems to be an open problem proving the existence of the corresponding threshold for the existence in general smooth domains.
5.2. A subcritical quasilinear problem. Let

\[ \mathcal{V}(u) = \int_{\Omega} \left( K_0 + K_1 H + \frac{K_2}{2} H^2 + \frac{K_3}{6} H^3 \right) \sqrt{1 + |\nabla u|^2} \, dx, \]

we can yet select another problem that appears by considering in the modelization the functional \( \mathcal{V}(u) \) with \( K_0 = K_2 = 0 \). The resulting equation is

\[
\begin{cases}
\Delta(|\Delta u|^2) = \det(D^2 u) + \lambda f & \text{in } \Omega \subset \mathbb{R}^2, \\
u|_{\partial \Omega} = 0, \quad \frac{\partial u}{\partial n}|_{\partial \Omega} = 0.
\end{cases}
\]

The main difficulty associated with this problem is that it is not elliptic in general. This is the same problem that appears for functional \( \mathcal{V}(u) \) when we assume that \( K_2 = K_3 = 0 \). Then the Euler-Lagrange equation is

\[ \Delta u + \det(D^2 u) = f \]

and performing the change of variable \( v = u + \frac{|x|^2}{2} \) (personal advise of N. Trudinger) we obtain the Monge-Ampère equation

\[ \det(D^2 v) = f + 1, \]

which is elliptic only if \( f + 1 \geq 0 \).

It seems to be an open problem finding the condition on the data \( f \) for the solvability of problem (35).

A problem formally close to (35) is the following,

\[
\begin{cases}
\Delta(|\Delta u| \Delta u) = \det(D^2 u) + \lambda f & \text{in } \Omega \subset \mathbb{R}^2, \\
u|_{\partial \Omega} = 0, \quad \frac{\partial u}{\partial n}|_{\partial \Omega} = 0,
\end{cases}
\]

which is elliptic. Despite it is a quasilinear problem, it is subcritical and then the variational formulation is easier. Indeed the energy functional is given by

\[ J(u) = \frac{1}{3} \int_{\Omega} |\Delta u|^3 \, dx - \int_{\Omega} u x_1 u x_2 u x_3 x_2 \, dx - \lambda \int f u \, dx, \]

and it is defined in \( W^{2,3}_0(\Omega) \). In this case, finding critical points could be done by means of the same arguments as before and \( f \in L^1(\Omega) \). For zeroth order nonlinearities this program was carried out in [10].

As in former cases, the problem

\[
\begin{cases}
\Delta(|\Delta u| \Delta u) = \det(D^2 u) + \lambda f & \text{in } \Omega \subset \mathbb{R}^2, \\
u|_{\partial \Omega} = 0, \quad \Delta u|_{\partial \Omega} = 0.
\end{cases}
\]

can be solved using a convenient fixed point argument.

The variational setting does not work for the same reasons as before, namely that the critical points of the functional do not correspond to solutions of our problem.
5.3. An extension of the Kardar-Parisi-Zhang equation with Navier boundary conditions. A nonvariational high order problem which is the counterpart of the Kardar-Parisi-Zhang equation in order 2, is the following problem

\[\begin{align*}
\Delta^2 u &= |\nabla(\Delta u)|^2 + \lambda f \quad \text{in } \Omega \subset \mathbb{R}^N, \\
u|_{\partial \Omega} &= 0, \quad \Delta u|_{\partial \Omega} = 0.
\end{align*}\]  

(39)

Notice that herein we will quote results that are valid for any arbitrary spatial dimension \(N\). We assume that \(f \in L^m(\Omega)\) with \(m \geq \frac{N}{2}\), \(f(x) \geq 0\).

We call \(-\Delta u = v\). Then we find the equivalent system given by

\[\begin{align*}
-\Delta u &= v \quad \text{in } \Omega, \quad u|_{\partial \Omega} = 0, \\
-\Delta v &= |\nabla v|^2 + \lambda f \quad \text{in } \Omega, \quad v|_{\partial \Omega} = 0.
\end{align*}\]  

(40)

The second equation has been studied extensively (see for instance [1] and [2] for the parabolic case) where necessary and sufficient conditions for the existence of solutions are established; moreover, a complete characterization of the solutions is presented. In particular if \(v \in H^1_0(\Omega)\) is a solution to the second equation of (40), then it verifies that

\[e^{\delta v} - 1 \in H^1_0(\Omega), \text{ for all } \delta \in \left(0, \frac{1}{2}\right),\]

and for \(\delta = \frac{1}{2}\) we reach a bounded solution. Then we obtain the following result for free.

**Theorem 5.1.** Problem (39) has infinitely many solutions in \(W^{3,2}(\Omega)\) such that if \(-\Delta u = v\) then

\[e^{\delta v} - 1 \in H^1_0(\Omega), \text{ for all } \delta \in \left(0, \frac{1}{2}\right).\]

If \(\delta = \frac{1}{2}\) then \(u \in W^{3,p}(\Omega)\) for all \(p < \infty\).

Notice that just the regularity of the solution gives sense to the second member of the equation defining boundary value problem (39).

On the contrary the equation in (39) with Dirichlet data, provides an interesting set of open problems.

5.4. Other boundary conditions. It would be interesting to analyze the Neumann problem, that is,

\[\begin{align*}
\Delta^2 u &= \det(D^2 u) + \lambda f \quad \text{in } \Omega \subset \mathbb{R}^2, \\
\frac{\partial u}{\partial n}|_{\partial \Omega} &= 0, \quad \frac{\partial \Delta u}{\partial n}|_{\partial \Omega} = 0.
\end{align*}\]  

(41)

Also it seems to be interesting to analyze inhomogeneous boundary conditions and, perhaps the more interesting from the physical view point, periodic boundary conditions.

We leave this analysis for the future.

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