Functional relations for the order parameters of the chiral Potts model: low-temperature expansions

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Abstract

This is the third in a series of papers in which we set up and discuss the functional relations for the “split rapidity line” correlation function in the $N$–state chiral Potts model. The order parameters of the model can be obtained from this function. Here we consider the case $N = 3$ and write the equations explicitly in terms of the hyper-elliptic functions parametrization. We also present four-term low-temperature series expansions, which we hope will cast light on the analyticity properties needed to solve the relations. The problem remains unsolved, but we hope that this will prove to be a step in the right direction.

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1 Introduction

In a previous paper we used the method pioneered by Jimbo, Miwa and Nakayashiki to obtain functional relations for a generalized one-spin correlation function for the planar solvable $N$–state chiral Potts model. The correct solution of this would give the
spontaneous magnetization - a long-standing problem which is all the more tantalizing as there is an elegant conjecture for the result \[3\] – \[5\].

Unfortunately, the functional relations are really just symmetry and periodicity conditions and have more than one solution. One needs to supplement them with an appropriate analyticity requirement. For other models this is easy: one goes to the elliptic (or trigonometric) parametrization that uniformizes the equations and manifests the rapidity-difference-property. There is then an obvious vertical strip in the complex plane of the spectral parameter variable within which the function is analytic and Fourier transformable in the vertical direction. The equations can then be solved directly and uniquely by taking such a transform, in much the same way as the inversion relations for the free energy can be solved \[6\].

For the chiral Potts model with \(N > 2\), there is no such parametrization. In one sense one can uniformize the equations, but at the price of introducing hyperelliptic functions with more than one argument, the arguments being related to one another in a complicated manner. We have discussed these functions (particularly for the case \(N = 3\)) earlier \[7\] – \[10\], and in \[11\] have obtained formulae for the coefficients in the functional relations. Here we write the relations in terms of this parametrization, focussing on the case \(N = 3\). We discuss their properties and present some low-temperature expansions of the functions, obtained by direct calculation from their definitions. We still have not succeeded in obtaining the appropriate solution of the functional relations, but hope that the data presented herein will assist in the search. Certainly the solution must agree with the low-temperature expansions.

## Functional relations

The chiral Potts model is explained in \[1\]. Spins live on the sites of a square lattice, oriented diagonally. Each spin takes the values \(0, \ldots, N-1\). Adjacent spins \(a, b\) interact with weight function \(W_{pq}(a-b)\) on SW \(\rightarrow\) NE edges, \(W_{pq}(a-b)\) on SW \(\rightarrow\) NE edges. The symbol \(p\) denotes a point (a “rapidity”) \(a_p, b_p, c_p, d_p\) on the projective curve

\[
a_p^N + k' b_p^N = k d_p^N \quad , \quad k' a_p^N + b_p^N = k c_p^N ,
\]

(1)
where \( k, k' \) are real constants (moduli) satisfying
\[
k^2 + k'^2 = 1.
\] (2)

The symbol \( q \) denotes another such point and
\[
W_{pq}(n) = \prod_{j=1}^{n} \frac{d_p b_q - a_p c_q \omega^j}{b_p d_q - c_p a_q \omega^j}, \quad W_{pq}(n) = \prod_{j=1}^{n} \frac{\omega a_p d_q - d_p a_q \omega^j}{c_p b_q - b_p c_q \omega^j}.
\] (3)

Operators \( R \) and \( M \) act on the rapidities, being defined by
\[
(a_{Rp}, b_{Rp}, c_{Rp}, d_{Rp}) = (b_p, \omega a_p, d_p, c_p),
\]
\[
(a_{Mp}, b_{Mp}, c_{Mp}, d_{Mp}) = (\omega a_p, b_p, c_p, \omega d_p).
\] (4) (5)

They satisfy \( RM = M^{-1}R \) and ensure that
\[
\overline{W}_{pq}(n) = W_{q,Rp}(n), \quad W_{pq}(n) = \overline{W}_{q,Rp}(-n).
\] (6)

In [1] we consider a square lattice with one spin deep inside it fixed at some value \( a \). The horizontal rapidity line immediately beneath this spin is broken, the left and right segments having different rapidities \( p, q \), respectively. The boundary spins are fixed to zero. Writing the corresponding partition function as \( Z_{pq}(a) \), the probability that a free spin in this position has value \( a \) is simply
\[
F_{pq}(a) = Z_{pq}(a)/[Z_{pq}(0) + \cdots + Z_{pq}(N-1)].
\] (7)

In the limit of a large lattice, it is independent of the rapidities of all other lines (because of \( Z \)-invariance [12]). By definition, \( F_{pq}(0) + \cdots + F_{pq}(N-1) = 1 \).

In [1] we show that the Yang-Baxter relations imply that \( F_{pq}(a) \) satisfies the following functional relations:
\[
F_{Rp,Rq}(a) = F_{pq}(-a), \quad F_{Rq,p}(a) = F_{Rp,q}(a), \quad F_{p,Rq}(a) = \xi_{pq} \sum_{b=0}^{N-1} \overline{W}_{pq}(a-b) F_{q,Rp}(b),
\]
\[
F_{M,p,q}(a) = F_{p,M^{-1}q}(a) = \eta_{pq} \omega^a F_{pq}(a),
\] (8)

where
\[
\xi_{pq} = \sum_{n=0}^{N-1} \overline{W}_{pq}(n), \quad \eta_{pq} = \sum_{a=0}^{N-1} \omega^a F_{pq}(a).
\] (9)
Let $\omega = \exp(2\pi i/N)$. It is natural to introduce the discrete Fourier transforms (for any integer)

$$
\tilde{F}_{pq}(r) = \sum_{a=0}^{N-1} \omega^{ra} F_{pq}(a) ,
$$

$$
\nabla_{pq}(r) = \sum_{a=0}^{N-1} \omega^{ra} W_{pq}(a) .
$$

(10)

Then $\tilde{F}_{pq}(0) = \tilde{F}_{pq}(N) = 1$ and the relations (8) become

$$
\tilde{F}_{Rp,Rq}(r) = \tilde{F}_{pq}(N−r) ,
\tilde{F}_{Rq,p}(r) = \tilde{F}_{Rp,q}(r) ,
\tilde{F}_{p,Rq}(r) = \nabla_{pq}(r) \tilde{F}_{q,Rp}(r)/\nabla_{pq}(0) ,
\tilde{F}_{M,p,q}(r) = \tilde{F}_{p,M−1,q}(r) = \tilde{F}_{pq}(r+1)/\tilde{F}_{pq}(1) .
$$

(11)

Without loss of generality, in these equations we can take $r = 1, \ldots, N − 1$.

It can be convenient to work with the ratios:

$$
G_{pq}(r) = \tilde{F}_{pq}(r)/\tilde{F}_{pq}(r−1)
$$

(12)

and with the product

$$
L_{pq}(r) = G_{pq}(r)G_{Rq,Rp}(r) .
$$

(13)

In terms of $L_{pq}(r)$, the functional relations become, for $r = 1, \ldots, N$, with $L_{pq}(N + 1) = L_{pq}(1)$:

$$
L_{Rp,Rq}(r) = L_{pq}(r) = 1/L_{pq}(N−r+1) ,
L_{p,Rq}(r) = \nabla_{pq}(r) L_{q,Rp}(r)/\nabla_{pq}(r−1) ,
L_{pq}(1) \cdots L_{pq}(N) = 1 ,
L_{M,p,q}(r) = L_{p,M−1,q}(r) = L_{pq}(r+1) .
$$

(14)

It follows that

$$
L_{R2N,p,q}(r) = L_{p,R2N,q}(r) = (-1)^{N−1}L_{pq}(r) ,
$$

(15)

which is a periodicity (or anti-periodicity) property.

The function $F_{pp}(a)$ is the probability, for the regular square lattice, that a central spin has value $a$. This should be independent of the rapidity $p$, so $F_{pp}(a), \tilde{F}_{pp}(r),$
\(G_{pp}(r), L_{pp}(r)\) should all be independent of \(p\). The order parameters are the expectation values \(M_r\) of \(\omega^r\) for \(r = 1, \ldots, N - 1\). From the above definitions of \(\tilde{F}_{pq}(r), G_{pq}(r), L_{pq}(r)\), we see that

\[
M_r = \tilde{F}_{pp}(r) = G_{pp}(1) \cdots G_{pp}(r) = \sqrt{L_{pp}(1) \cdots L_{pp}(r)}.
\]

The conjectured result for \(M_r\) is

\[
M_r = \langle \omega^r \rangle = (1 - k'^2)^{r(N-r)/2N^2},
\]

(eqns 3.13 of [3], eqn. 1.20 of [4], eqn. 15 of [5], \(\beta\) and \(\lambda\) therein being the \(k'\) of this paper. This \(k'\) can be thought of as a temperature variable: small at low temperatures and increasing to one at criticality.

3 Hyperelliptic function parametrization

The hyperelliptic function parametrization is given in [7] and [11]. There are \(N - 1\) constants \(\rho_1, \ldots, \rho_{N-1}\) (positive pure imaginary) which are defined solely by \(N, k\) and \(k'\), and satisfy \(\rho_\alpha = \rho_{N-\alpha}\). For \(\alpha, \beta = 1, \ldots, N - 1\), define

\[
\tau_{\alpha\beta} = \rho_\alpha + \rho_\beta - \rho_{|\alpha-\beta|},
\]

taking \(\rho_0 = \rho_N = 0\). Let \(s = \{s_1, \ldots, s_{N-1}\}\) be a set of variables. Then the hyperelliptic \(\Theta\) function is defined by

\[
\Theta(s) = \sum_m \exp\{2\pi i \sum_\alpha m_\alpha s_\alpha + \pi i \sum_\alpha \sum_\beta m_\alpha \tau_{\alpha\beta} m_\beta\},
\]

the inner sums being over \(\alpha, \beta = 1, \ldots, N - 1\), and the outer sum over all values of the integers \(m = \{m_1, \ldots, m_{N-1}\}\).

We also define the constant sets \(g, \rho\) by \(g = \{g_1, \ldots, g_{N-1}\}, \rho = \{\rho_1, \ldots, \rho_{N-1}\}\), where \(g_\alpha = \alpha/N\). From (18) and (19) the \(\Theta\) function satisfies various quasi-periodicity and symmetry relations, in particular

\[
\Theta(s) = \Theta(-s) = \Theta(\tilde{s}) = \Theta(s + Ng) = \exp\{\pi i \sum_\alpha (2s_\alpha + N\rho_\alpha)\} \Theta(s + N\rho),
\]

where \(\tilde{s} = \{s_{N-1}, s_{N-2}, \ldots, s_1\}\) is the sets in reverse order.
For each rapidity \( p \) there is a set of such \( s \)-variables, which we write as \( s_p \) (Greek, numerical or “\( N \)” suffixes refer to the position of an entry in the set, while lower-case Roman suffixes – usually \( p \) or \( q \) – refer to the rapidity dependence). We write the entry \( \alpha \) of \( s_p \) as \( (s_p)_\alpha \). The entries \( (s_p)_1, \ldots, (s_p)_{N-1} \) are not independent: they are functions of the single rapidity variable \( p \) and satisfy the \( N-2 \) relations (36) of \([7]\), or equivalently (10) of \([11]\).

Set \( t = \{ t_1, \ldots, t_{N-1} \} \), where

\[
t_\alpha = s_1 + \cdots + s_\alpha - \alpha(s_1 + \cdots + s_{N-1})/N.
\]

(21)

As with \( s \), we write \( t_p \) for the set \( t \) corresponding to the rapidity \( p \), and \( (t_p)_\alpha \) for its entry \( \alpha \). Then

\[
(t_{Rp})_\alpha = (t_p)_{N-\alpha} + \frac{1}{2} \rho_\alpha,
\]

(22)

\[
t_{Rp} = t_p + \rho, \quad t_{Mp} = t_p - g.
\]

(23)

(choosing \( M \) so that \( M^{-1} \) is the operator \( M_1^{(1)} \cdots M_{N-1}^{(1)} \) of \([7]\) and \([11]\).)

The relation (22), together with the \( s, \tilde{s} \) symmetry in (20), implies that

\[
\Theta(t_{Rq} - t_{Rp} + mg + n\rho) = \Theta(t_q - t_p - mg + n\rho)
\]

(24)

for all integers \( m, n \).

From (3) and (10),

\[
\nabla_{pq}(r)/\nabla_{pq}(r-1) = (c_pb_q - \omega^r a_p d_q)/(b_pc_q - \omega^r d_p a_q)
\]

so

\[
\nabla_{p,Rq}(r)/\nabla_{p,Rq}(r-1) = (\omega c_p a_q - \omega^r a_p c_q)/(b_p d_q - \omega^r d_p b_q).
\]

(25)

In eqn (26) of \([11]\) we give a formula for the RHS of (25). This formula has not been proven for \( N > 3 \), but we have tested it numerically and believe it should be proveable by Liouville’s theorem, along the lines indicated in \([11]\). Using it, we obtain

\[
\nabla_{p,Rq}(r)/\nabla_{p,Rq}(r-1) = \psi_{p,Rq}(r)/\psi_{pq}(r),
\]

(26)

where

\[
\psi_{pq}(r) = \Theta(t_q - t_p + r\rho)/\Theta(t_q - t_p + (r-1)\rho).
\]

(27)

Replacing \( q \) by \( Rq \), the second line of equation (14) is (using the first equation)

\[
L_{p,Rq}(r) = \psi_{p,Rq}(r)L_{pq}(r)/\psi_{pq}(r).
\]

(28)
A spurious solution

Obviously a solution of (28) is

\[ L_{pq}(r) = \psi_{pq}(r) \]  

Further, using (23) and (24), we find that it satisfies all the other equations (14). For \( N = 2 \), it is indeed the correct solution, but for general \( N \) the formula (25) of [11] (also unproven for \( N > 3 \), but believed to be “proveable”) tells us that

\[ \psi_{pq}(r) = k^{(N-1)/N} \prod_{m=1}^{N-1} \left\{ \frac{b_p d_q - \omega^j m a_q - \omega^j m b_q}{c_p d_q - \omega^j m a_q} \right\}^{m/N} \]  

This is precisely the wrong solution discussed in [1]. It gives a result for \( \mathcal{M}_r \) that is in error by a power \( N/2 \), and disagrees with low-temperature series expansions. Yet it is such a simple and elegant (in terms of our hyperelliptic functions) solution of (14). This illustrates well the difficulty with functional relations such as (14): they do not by themselves define the function. One has to incorporate the correct analyticity properties.

4 The case \( N = 3 \)

If \( N = 3 \), then \( \rho_1 = \rho_2 = \rho \), so if we define

\[ x = e^{2\pi i \rho} \ , \ |x| < 1 \]  

then

\[ \Theta\{s_1, s_2\} = \Phi(e^{2\pi i s_1}, e^{2\pi i s_2}) \]  

where

\[ \Phi(\alpha, \beta) = \sum_{m,n} x^{m^2 + mn + n^2} \alpha^m \beta^n \]  

the sum being over all integers \( m, n \). (Here we regard \( x \) as a given constant and do not usually explicitly exhibit the dependence of functions on it.) The function \( \Phi(\alpha, \beta) \) satisfies the symmetry and quasi-periodicity properties (34)–(36) of [11], in particular

\[ \Phi(\alpha, \beta) = \Phi(1/\alpha, 1/\beta) = \Phi(\alpha, \alpha/\beta) \]  

7
The nome $x$ is related to the modulus $k$ by [3]

$$(k'/k)^2 = 27x \left[ Q(x^3)/Q(x) \right]^{12},$$

where

$$Q(x) = \prod_{n=1}^{\infty} (1 - x^n).$$

At low temperatures both $k$ and $x$ are small.

If we define

$$\alpha = \alpha_{pq} = \exp\{2\pi i[(s_q)_{1} - (s_p)_{1}]\},$$

$$\beta = \beta_{pq} = \exp\{2\pi i[(s_q)_{2} - (s_p)_{2}]\},$$

$$u = u_{pq} = \exp\{2\pi i[(s_q)_{1} - 2(s_q)_{2} - (s_p)_{1} + 2(s_p)_{2}] / 3\},$$

$$\rho(z) = \prod_{n=1}^{\infty} (1 - x^{3n-2}z)(1 - x^{3n-1}/z),$$

$$\phi(z) = \rho(z^{-1}) / \rho(z),$$

then [3]

$$\alpha = u^3 \beta^2,$$ \quad (39)

$$W_{pq}(1) = u^2 \beta \phi(\alpha) \phi(\alpha/\beta),$$

$$W_{pq}(2) = u \beta \phi(\alpha) \phi(\beta).$$ \quad (40)

For any function, we can in principle eliminate the two degrees of freedom $p$ and $q$ in favour of the variables $\alpha, \beta$ (regarding $k, k', \rho, x$ as given constants). The resulting expressions will not necessarily be simple: they may be multi-valued functions of $\alpha$ and $\beta$, but the simple form of (40) gives some encouragement that this may be a useful parametrization for our functions. In this spirit, let us define a function $\psi(\alpha, \beta)$ such that

$$F_{pq}(1)/F_{pq}(0) = xu^{-1} \beta^{-1} \psi(\alpha, \beta).$$ \quad (41)

We have expanded $\psi(\alpha, \beta)$ to third order in the low-temperature parameter $x$, working directly from the definition (7) and performing finite-size lattice calculations. The results are given below. We used a Fortran computer program, working with series in $x$ with double-precision numerical coefficients for various given values of $\alpha$ and $\beta$. The method is not rigorous, depending both on observing that a given coefficient stabilized once the lattice was sufficiently large (the biggest lattice we considered was about eight by eight), and on numerically fitting the results to postulated expressions...
with unknown integer coefficients. However, we believe the results to be correct. The key observation is that for $n > 0$ the coefficient of $x^n$ in the expansion is a Laurent polynomial in $\alpha$ and $\beta$, divided by $(\alpha - 1)^{2n-1}$. The coefficient of $x^0$ is one.

The variables $\alpha_{pq}, \beta_{pq}, u_{pq}$ satisfy the relations

$$
\alpha_{Rp,Rq} = \alpha_{pq} \quad , \quad \beta_{Rp,Rq} = \alpha_{pq}/\beta_{pq} \quad , \quad u_{Rp,Rq} = 1/u_{pq} \quad ,
$$

$$
\alpha_{qp} = 1/\alpha_{pq} \quad , \quad \beta_{qp} = 1/\beta_{pq} \quad , \quad u_{qp} = 1/u_{pq} \quad ,
$$

$$
\alpha_{p,R^2q} = x^2\alpha_{pq} \quad , \quad \beta_{p,R^2q} = x\beta_{pq} \quad , \quad u_{p,R^2q} = u_{pq} \\
(42)
$$

$$
\alpha_{Mp,q} = \alpha_{p,M^{-1}q} = \alpha_{pq} \quad , \quad \beta_{Mp,q} = \beta_{p,M^{-1}q} = \beta_{pq} \quad , \\
u_{Mp,q} = u_{p,M^{-1}q} = \omega^{-1}u_{pq} .
$$

Using these, the first of the equations (8) is equivalent to

$$
F_{pq}(2)/F_{pq}(0) = xu^{-2}\beta^{-1}\psi(\alpha,\alpha/\beta) .
(43)
$$

Using (42), (40) and (6), the second and third of the relations (8) become

$$
x\alpha\psi(x^{-2}/\alpha,x^{-1}\beta/\alpha) = \psi(\alpha,\beta) ,
(44)
$$

$$
\Delta(\alpha,\beta) \psi(x^2/\alpha,x/\beta) = \phi(\alpha)\phi(\beta) + (x/\alpha)\psi(\alpha,\alpha/\beta) + (x/\beta)\phi(\alpha)\phi(\alpha/\beta)\psi(\alpha,\beta) ,
(45)
$$

where

$$
\Delta(\alpha,\beta) = 1 + x\phi(\alpha)\phi(\alpha/\beta)\psi(\alpha,\alpha/\beta) + x\phi(\alpha)\phi(\beta)\psi(\alpha,\beta) .
(46)
$$

The last two of the relations (8) (those involving $Mp$ and $M^{-1}q$) are automatically satisfied.

**Series expansions for $\psi(\alpha,\beta)$**

Define

$$
v = \frac{\beta - 1}{\beta(\alpha - 1)} \quad , \quad \overline{v} = \frac{\alpha - \beta}{\beta(\alpha - 1)} \quad , \quad \mu = \beta v\overline{v} .
(47)\]
Then, using the configuration of the rapidity lines $p$ and $q$ mentioned above, taking $\alpha$ and $\beta$ to be of order one, to third order in an expansion in powers of $x$ we find that:

$$
\psi(\alpha, \beta) = 1 + x[v\alpha(\beta - 1) - 2 + \alpha] + x^2[v\mu(\alpha^2 - \beta) + v(1 + 4\alpha + 2\alpha^2 - 2\beta - 5\alpha\beta + 13 - 6\alpha) + x^3[v\mu^2\alpha(3\beta - \alpha\beta - \alpha - \alpha^2) + v\mu(2\beta/\alpha + 5\beta + 3\alpha\beta + 1 - 2\alpha - 8\alpha^2 - \alpha^3) + v(-6/\beta/\alpha + 12\beta + 28\alpha\beta + \alpha^2\beta - 3 - 21\alpha - 10\alpha^2 - \alpha^3) + 5/\alpha - 71 + 38\alpha + \alpha^2] + O(x^4). \quad (48)
$$

We also considered the configurations where one of the lines $p, q$ was rotated through $180^\circ$, as discussed in [1]. This gives results for four cases where the arguments of the function $\psi$ are themselves proportional to non-zero powers of $x$, namely:

$$
\psi(x\alpha, x\beta) = 1 + x[(1 - \alpha)/\beta - 2] + x^2[(1 - \alpha)/\beta^2 + (-4 + 6\alpha - \alpha^2)/\beta + 1/\alpha + 9 + 3\alpha] + x^3[(1 - \alpha)/\beta^3 + (-5 + 7\alpha - \alpha^2 - \alpha^3)/\beta^2 + (2/\alpha + 17 - 32\alpha + 11\alpha^2 - \alpha^3)/\beta - 10/\alpha - 49 - 23\alpha + 2\alpha^2 + (2/\alpha + 4 - \alpha)\beta] + O(x^4), \quad (49)
$$

$$
\psi(x\alpha, \beta) = 1 - 2x + x^2[(1 - \alpha)/\beta + 1/\alpha + 9 + 3\alpha + (2/\alpha + 4 - \alpha)\beta - (1 - \alpha)^2\beta^2/\alpha^2] + x^3[(-4 + 6\alpha - \alpha^2)/\beta - 10/\alpha - 49 - 23\alpha + 2\alpha^2 + (3/\alpha^2 - 19/\alpha - 29 + 14\alpha - \alpha^2)\beta + (-1/\alpha^3 + 13/\alpha^2 - 20/\alpha + 11 - 3\alpha)\beta^2 - 2(1 - \alpha)^2\beta^3/\alpha^3] + O(x^4), \quad (50)
$$

$$
\psi(\alpha/x, \beta) = 1 + \alpha + x[(1 - \alpha)^2/\beta - 4 - 4\alpha + \beta] + x^2[(1 - \alpha)(1 - \alpha^2)/\beta^2 + (1/\alpha - 8 + 13\alpha - 8\alpha^2 + \alpha^3)/\beta - 2/\alpha + 25 + 25\alpha - 2\alpha^2 + (1/\alpha - 5 + \alpha)\beta] + x^3[(1 - \alpha)(1 - \alpha^3)/\beta^3 + (3/\alpha - 14 + 11\alpha + 11\alpha^2 - 14\alpha^3 + 3\alpha^4)/\beta^2 + (1/\alpha^2 - 14/\alpha + 58 - 79\alpha + 58\alpha^2 - 14\alpha^3 + \alpha^4)/\beta - 2/\alpha^2 + 19/\alpha - 146 - 146\alpha + 19\alpha^2 - 2\alpha^3 + (1/\alpha^2 - 8/\alpha + 30 - 8\alpha + \alpha^2)\beta] + O(x^4), \quad (51)
$$

$$
\psi(\alpha/x, \beta/x) = 1 + \alpha + \beta + x[-4 - 4\alpha + (1/\alpha - 5 + \alpha)\beta] +
$$
\[
x^2 \left[ (1 - \alpha)^2 / \beta - 2 / \alpha + 25 + 25 \alpha - 2 \alpha^2 + (1 / \alpha^2 - 8 / \alpha + 30 - 8 \alpha + \alpha^3) \beta \right] + x^3 \left[ (1 / \alpha - 8 + 13 \alpha - 8 \alpha^2 + \alpha^3) / \beta - 2 / \alpha^2 + 19 / \alpha - 146 - 146 \alpha + 19 \alpha^2 - 2 \alpha^3 + (1 / \alpha^3 - 11 / \alpha^2 + 56 / \alpha - 175 + 56 \alpha - 11 \alpha^2 + \alpha^3) \beta + (1 / \alpha^3 - 2 / \alpha^2 + 2 / \alpha + 2 - 2 \alpha + \alpha^2) \beta^2 + (-1 / \alpha^3 + 3 / \alpha^2 - 4 / \alpha + 3 - \alpha) \beta^3 \right] + O(x^4) .
\] (52)

In all of the expansions (48) - (52), the variables \( \alpha, \beta \) are of order one.

We can regard these expressions as giving the expansion of the function \( \psi(\alpha, \beta) \) in various domains in the \((\alpha, \beta)\) plane. We have verified that the results for neighbouring domains are consistent with one another. For example (49) and (50) are consistent in that if one replaces \( \beta \) in (50) by \( x \beta \), then one obtains (49), except for terms in (49) that are of order higher than 3 when \( \beta \) is of order \( x^{-1} \). Similarly, (48) is consistent with both (50) and (51), and (51) is consistent with (52).

Taking \( \alpha = O(x^{-1}) \) and \( \beta = O(1) \) or \( O(x^{-1}) \) in (44), one can readily check that (51) and (52) each has the consequent symmetry property. Also, taking \( \alpha = O(x) \) and \( \beta = O(1) \) or \( O(x) \) in (45), we find that the relation is satisfied by (49) and (50).

If \( q \to p \), the \( p \) variables remaining finite, then \( \alpha, \beta \to 1 \) while \( v \) and \( \bar{v} \) remain finite. The coefficients of the terms proportional to \( v \) and \( \bar{v} \) in (48) then vanish, so we obtain the unique result:

\[
\psi(1, 1) = 1 - x + 7x^2 - 27x^3 + O(x^4) .
\] (53)

From (41) and (43) it follows that \( F_{pp}(1)/F_{pp}(0) = F_{pp}(2)/F_{pp}(0) = x \psi(1, 1) \), so from (16)

\[
\mathcal{M}_1 = \mathcal{M}_2 = \frac{1 - F_{pp}(1)/F_{pp}(0)}{1 + 2F_{pp}(1)/F_{pp}(0)} = 1 - 3x + 9x^2 - 45x^3 + 231x^4 + O(x^5) ,
\] (54)
in agreement with the conjecture (17).

**Fourier transformed relations**

To obtain the relations (11), (14) in terms of the variables \( \alpha, \beta \) – or rather in terms of \( u, \beta \) – define:

\[
A(u, \beta) = 1 + x \beta^{-1} u^{-1} \psi(\alpha, \beta) + x \beta^{-1} u^{-2} \psi(\alpha, \alpha / \beta) ,
\] (55)
\[ B(u, \beta) = A(u, \beta) A(u, u^{-3}\beta^{-1}) \quad , \] 
\[ \zeta(u, \beta) = \left[ 1 + u\beta \phi(\alpha) \phi(\beta) + u^2\beta \phi(\alpha) \phi(\alpha/\beta) \right] / \Delta(\alpha, \beta) \quad . \] 

Then \( F_{pq}(0) = 1/A(u, \beta) \),
\[ \tilde{F}_{pq}(r) = A(\omega^{-r}u, \beta)/A(u, \beta) \quad , \] 
\[ L_{pq}(r) = B(\omega^{-r}u, \beta)/B(\omega^{1-r}u, \beta) \quad , \] 

and, from (44) and (45),
\[ A(1/u, u^3\beta) = A(1/u, x^{-1}\beta^{-1}) = A(u, \beta) \quad , \] 
\[ A(1/u, x/\beta) = \zeta(u, \beta) A(u, \beta) \quad , \] 
\[ B(u^{-1}, u^3\beta) = B(u, u^{-3}\beta^{-1}) = B(u, \beta) \quad , \] 
\[ B(u^{-1}, x\beta^{-1}) = \zeta(u, \beta) B(u^{-1}, \beta^{-1}) \quad . \] 

Together with (42) and the fact that \( \Delta(\alpha, \beta) \) is a single valued function of \( u^3 \) and \( \beta \), the relations (60) imply (11), while (61) imply (14). One does not need the definition (46) of \( \Delta(\alpha, \beta) \).

Our function \( \rho(z) \) is the \( \tilde{\psi}(z) \) of [11]. Another function discussed therein is
\[ h(\alpha, \beta) = \Phi(u, u^2\beta) \Phi(\omega u, \omega^2 u^2\beta) \Phi(\omega^2 u, \omega u^2\beta) \quad . \] 

Using also eqns (32) and (58) of [11], it follows that
\[ \zeta(u, \beta) = \tau(\alpha, \beta) \Phi(u, u^2\beta/x)/\Phi(u, u^2\beta) \quad , \] 
where
\[ \tau(\alpha, \beta) = \frac{h(\alpha, \beta)}{Q(x)^2 Q(x^3)^4 \rho(\alpha) \rho(\beta) \rho(\alpha/\beta) \Delta(\alpha, \beta)} \quad . \] 

Obviously (58) and (59) are unaffected by multiplying the functions \( A(u, \beta) \), \( B(u, \beta) \) by factors which are single-valued functions of \( u^3, \beta \) (i.e. of \( \alpha, \beta \)), such as \( h(\alpha, \beta) \) and \( \tau(\alpha, \beta) \). To within such factors, using the symmetry property (44), we see that a simple solution of (61) is \( B(u, \beta) = \Phi(u, u^2\beta) \). This implies that \( L_{pq}(r) = \Phi(\omega^{-r}u, \omega^r u^2\beta)/\Phi(\omega^{1-r}u, \omega^{r-1}u^2\beta) \), which is the spurious solution (29) mentioned above.
Thus all factors to the simple form $u,\alpha,\beta$ while from (49) - (52), for $u,\beta$ and $\alpha$ consistent with one another in the sense discussed above, provided we remember that some coefficients have poles when $\alpha = 1$. The equations (61) then reduce precisely (including all factors) to the simple form

$$D(1/u, u^3 \beta) = D(u, u^{-3} \beta^{-1}) = D(u, \beta) = D(u, x \beta) \ .$$

From our series expansion (48), for $u,\alpha,\beta$ of order 1, we find to order $x^4$ that

$$D(u, 1) = 1 - (u + u^{-1})x + (1+2u+2u^{-1})x^2 + (u^3/3 + u^{-3}/3 - 11u - 11u^{-1} - 4) x^3 + (-4u^4/3 - 4u^{-4}/3 - 2u^3 - 2u^{-3} - 10u^2/3 - 10u^{-2}/3 + 56u + 56u^{-1} + 26) x^4$$

and

$$D(u, \beta)/D(u, 1) = 1 + [(\alpha+1-2\beta)u + (\alpha+1-2\alpha/\beta)/u] \mu x^3 + [(2\alpha\beta + 2\beta - 4\alpha)\mu]^2 + (2\alpha^2/\beta + 2\alpha/\beta - 4\alpha)\mu^2/\mu + (14\beta - 8\alpha - 8 + \beta^2/\alpha)\mu + (14\alpha/\beta - 8\alpha - 8 + \alpha^2/\beta^2 + \alpha/\beta^2)\mu/\mu + (1 - \beta)(\alpha - \beta)/(\beta^2 + 3u/\alpha + \alpha^{-1}/u + 3\beta^{-2}/u) x^4 \ ,$$

while from (61) - (62), for $u,\alpha,\beta$ of orders $x^{1/3}, x, 1$, respectively, we find to order $x^4$ that

$$D(u, 1) = 1 - (u + u^{-1})x + (1+2u+2u^{-1})x^2 + (u^3/3 + u^{-3}/3 - 11u - 11u^{-1} - 4) x^3 + (-4u^4/3 - 2u^3 - 10u^2/3 + 56u^{-1} + 26) x^4 + (2u^{-5} + 47u^{-4}/3 + 58u^{-3}/3)x^5 - (3u^{-7} + 7u^{-6}/9) x^6$$

and

$$D(u, \beta)/D(u, 1) = 1 + (1 - \beta)(1 - \beta x/\alpha) \left\{(u - 2\beta u + u^{-1} - 3\alpha u^{-1}/\beta + 3\alpha u^{-1})x^3 + (-\beta^2 u/\alpha - 2\beta u/\alpha - 10u^{-1} + 3\beta u^{-1} - 3u^{-1}/\beta) x^4 + 3(1 + \beta + \beta^2)u^{-1}x^5/\alpha \right\} \ .$$

Applying the symmetries (66) to these expansions, we can obtain $D(u, \beta)$ to order $x^4$ for $u$ of order 1, $x^{1/3}$ and $x^{-1/3}$, and for all $\beta$. We have verified that the expansions are consistent with these symmetries, and the results for neighbouring domains are consistent with one another in the sense discussed above, provided we remember that some coefficients have poles when $a = 1, x^2, x^{-2}, \ldots$. 

13
Order parameters in terms of $D(u, \beta)$

From (16) and the above equations we find that the order parameters $M_1, M_2$ are given in terms of the function $D(u, \beta)$ by the simple relation

$$M_1^2 = M_2^2 = \frac{k^{2/3} D(\omega^{\pm 1}, 1)}{D(1, 1)},$$

the factor $k^{2/3}$ being the contribution from the spurious result obtained by taking $D(u, \beta) = 1$, i.e. by using (29). The right-hand side is to be evaluated in the limit $q \to p$, i.e. when the arguments $u, \beta$ of $D(u, \beta)$ behave so that $u^3, \beta \to 1$, the ratio $(\beta - 1)/(u^3 \beta^2 - 1)$ remaining finite.

Using the expansion (71), we again obtain (54).

5 Discussion

If we can solve the functional relations for the “split rapidity line” correlation function $F_{pq}(a)$, then we can calculate the order parameters $M_j$. For the $N = 2$ chiral Potts model, i.e. the Ising model, this can be done straightforwardly, using a uniformizing elliptic function parametrization. For $N \geq 3$ it is still an unsolved problem. For $N = 3$ we have written down the equations explicitly using a hyperelliptic function parametrization: in (44) - (46) in terms of the function $\psi(\alpha, \beta)$, and in (66) in terms of $D(u, \beta)$. We have also presented the first few terms in a series expansion, obtained from finite lattice calculations.

The problem is how to solve the equations. One modest approach is to try to guess the general form of the coefficients in the series expansions, and then see if the relations determine the coefficients precisely, at least successively term-by-term. Such an approach works well for the Ising model in terms of its original variables (the Boltzmann weights) [14, eqns 2.15 to 2.30]. As a first step, we tried replacing every numerical coefficient in (48) – (52) by an arbitrary number, substituted the expansions into (44) - (46), and attempted to systematically solve the resulting equations for the unknown numbers. This works for $\psi(\alpha, \beta)$ to zero and first order in $x$, but at second order we were left with four undetermined coefficients, making it impossible to proceed further. (We did try looking ahead to see if the higher-order equations fixed these four
unknowns, but with no success.)

This approach appears to be even less successful when applied to the relations (66): for $D(u, \beta)$ to first order, which corresponds to zero order for $\psi(\alpha, \beta)$, we easily see that a solution is $D(u, \beta) = 1 - \mu(u + u^{-1})x$, where $\mu$ is an arbitrary constant. In fact it is painfully obvious that the equations are unaffected by multiplying $D(u, \beta)$ by any function $f(u)$ satisfying $f(u) = f(u^{-1})$, $f(u)f(\omega u)f(\omega^2 u) = 1$. This can certainly affect $M_1$ and $M_2$.

It could be very useful to merely know the functions to leading order for all values of $\alpha, \beta$, or $u, \beta$: this would provide valuable clues as to the locations of any zeros or poles. This would be very significant if the functions are single-valued functions of their arguments, but we have no reason to suppose this, and know of no way of obtaining even the leading-order behaviour except near the “physical regime” where the Boltzmann weights are real and positive (and those regimes which map to it by the symmetries used to derive the functional relations).

It may not be necessary to consider the full complex $u$ and $\beta$ planes. If we discard the first equality in (66), we have two equations in which we can regard $u$ as a fixed “constant”. They relate the values of $D$ with second argument $\beta$, $x\beta$ and $u^{-3}/\beta$, so are much like inversion relations [3], the inversion points being $\beta = u^{-3/2}$ and $\beta = x^{1/2}u^{-3/2}$. If $D$ were analytic in an annulus in the complex $\beta$ plane, centre the origin, including these points, then it would be easy to solve (66) using a Laurent expansion in this annulus. In fact the solution would be a constant, i.e. $D(u, \beta)$ would be independent of $\beta$.

To second order in $x$ this is the case, but from (68) we see that at third order there are terms proportional to $\mu$, which from (17) and (38) has a second-order pole at $\beta = u^{-3/2}$. (At fourth order there are terms proportional to $\mu^2$.) So $D$ is not analytic at the inversion points and does depend on $\beta$.

We are left with a tantalizing puzzle: undoubtedly the functional relations do contain information, but they need to be supplemented with a knowledge of the analyticity properties at the “inversion point”. Perhaps we should be using the individual hyper-
elliptic function variables \[3\]
\[
z_p = x^{1/2} \exp[2\pi i(s_p)_1] , \quad w_p = \exp[2\pi i(s_p)_2] ,
\]
(72)
in terms of which \(\alpha = z_q/z_p\), \(\beta = w_q/w_p\). They can be chosen to be of order unity when \(x\) is small, and to leading order \(w_p = z_p + 1 = (\alpha - 1)/(\alpha - \beta)\), \(w_q = z_q + 1 = \beta(\alpha - 1)/(\alpha - \beta)\). Higher-order terms can be obtained from equations (4.5) and (4.6) of \[8\].

A third variable that enters the hyperelliptic parametrization of the Boltzmann weights is \(\gamma = w_p w_q/z_p\) \[11\]. The variables \(\alpha, \beta, \gamma\) all have the property that they are unchanged by simultaneously replacing \(p, q\) by \(R^2 p, R^2 q\), so they automatically incorporate this symmetry of the generalized correlation function. One can write the expansion in (48) (at least to the order given) in a form where the coefficients are Laurent polynomials in \(\alpha, \beta, \gamma\). The trouble is that this expansion is not unique, since \(\alpha, \beta, \gamma\) are related to one another. Defining \(\mu\) as in (47), i.e. \(\mu = \beta^{-1}(\beta - 1)(\alpha - \beta)/(\alpha - 1)^2\), we find that
\[
\mu = \gamma^{-1} + [\gamma/\alpha - 3 - 3\alpha^{-1} + (2\alpha + 2\alpha^{-1} - 1)/\gamma + (1 + \alpha)/\gamma^2] x + O(x^2) .
\]

Of course it may be that this hyperelliptic parametrization is not helpful at all, but this seems unduly pessimistic since (a) the parametrization does provide convenient and simple expressions for the relevant Boltzmann weight functions and their Fourier transforms \[11\], and (b) the coefficients in the expansions are much simpler than they are if one uses the original variables \(a_p, \ldots, d_p, a_q, \ldots, d_q\). While it is true that the coefficients in our series expansions contain negative powers of \(\alpha - 1\), there do not appear to be any negative powers of \(\beta - 1\) or \(\alpha - \beta\) (which occur in, say, the expansion of \(\gamma\)). It also appears from (67) that such negative powers disappear completely when \(\beta = 1\), which is the case of interest for calculating the order parameters.

The above results are presented in the hope that they may be a step towards verifying the conjecture (17) for the chiral Potts model order parameters.

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