Topological ground-state excitations and symmetry
in the many-electron one-dimensional problem

J. M. P. Carmelo and N. M. R. Peres

Department of Physics, University of Évora, Apartado 94, P - 7001 Évora Codex, Portugal
and Institute for Scientific Interchange Foundation, Villa Gualino, I - 10133 Torino, Italy

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We consider the Hubbard chain in a magnetic field and chemical potential. We introduce a pseudohole basis where all states are generated from a single reference vacuum. This allows the evaluation for all sectors of Hamiltonian symmetry of the model of the expression of the $\sigma$ electron and hole operators at Fermi momentum $\pm k_{F\sigma}$ and vanishing excitation energy in terms of pseudohole operators. In all sectors and to leading order in the excitation energy the electron and hole are constituted by one $c$ pseudohole, one $s$ pseudohole, and one topological momenton. These three quantum objects are confined in the electron or hole and cannot be separated. We find that the set of different pseudohole types which in pairs constitute the two electrons and two holes associated with the transitions from the $(N_\uparrow, N_\downarrow)$ ground state to the $(N_\uparrow+1, N_\downarrow)$, $(N_\uparrow, N_\downarrow+1)$ and $(N_\uparrow-1, N_\downarrow)$, $(N_\uparrow, N_\downarrow-1)$ ground states, respectively, transform in the representation of the symmetry group of the Hamiltonian in the initial-ground-state sector of parameter space. We also find the pseudohole generators for the half-filling holon and zero-magnetic-field spinon. The pseudohole basis introduced in this paper is the only suitable for the extension of the present type of operator description to the whole Hilbert space.
I. INTRODUCTION

In contrast to three-dimensional Fermi liquids [1,2], the low-energy excitations of one-dimensional many-electron quantum problems are at zero magnetic field characterized by a charge-spin separation [3,4,5]. This can be interpreted in terms of holon and spinon modes [6]. On the other hand, at finite magnetic fields the charge and spin separation is replaced by a more exotic $c$ and $s$ separation [7]. Here $c$ and $s$ refer to orthogonal small-momentum and low-energy modes which couple both to the charge and spin channels.

In the case of many-particle problems solvable by Bethe ansatz (BA) [8,9,10] the low-energy spectrum can be studied explicitly. The BA solution of the Hubbard chain [11] at zero magnetic field and chemical potential has allowed the identification and study of the holon and spinon excitations and corresponding symmetry transformations [12]. (The spinon excitations of the Hubbard chain are similar to the corresponding spinon excitations of the spin $1/2$-isotropic Heisenberg chain [13].) However, the exact expression of the electron operator in terms of holon and spinon operators remains an open problem of some complexity. Its solution requires an operator representation for the generators of the holon and spinon excitations. Even if such operator representation is obtained, a second problem is expressing these generators in terms of electronic operators. Moreover, the relation between Hamiltonian symmetry and the transformation of the elementary excitations has not been studied for finite magnetic fields or (and) densities away of half filling. In this case the holon and spinon picture breaks down, as we show in this article.

For the general case of the Hubbard chain at finite magnetic field and chemical potential both the non-lowest-weight states (non-LWS’s) and non-highest weight states (non-HWS’s) of the eta-spin and spin algebras [14,15,16,17,18,19] have energy gaps relative to the corresponding canonical-ensemble ground state [20]. The LWS’s and HWS’s of these algebras can be classified into two types, the states I (or LWS’s I and HWS’s I) and the states II (or LWS’s II and HWS’s II). While the Hamiltonian eigenstates I are described only by real BA rapidities, all or some of the rapidities associated with the eigenstates II are complex and
non real. Since for finite magnetic field and chemical potential the states II have energy gaps relative to the corresponding canonical-ensemble ground state, in that case the low-energy physics is exclusively determined by the states I [20].

The pseudoparticle perturbation theory introduced in Refs. [21,22,23,24,25] and developed in a suitable operator basis in Refs. [7,26,27,28] refers to the Hilbert subspace spanned by the Hamiltonian eigenstates I. Rather than holons and spinons, at finite values of the magnetic field and chemical potential and at constant electronic numbers the low-energy excitations I are described by pseudoparticle-pseudohole processes relative to the canonical-ensemble ground state. The latter state, as well as all excited states I with the same electron numbers, are simple Slater determinants of pseudoparticle levels [7,20,26,27,28].

One of the goals of this paper is introducing an alternative pseudohole representation which generates both the LWS’s I and the HWS’s I from the same reference vacuum. Importantly, it will be shown elsewhere that all Hamiltonian eigenstates can be generated from the new pseudohole vacuum. In the case of the states I the present pseudohole picture is alternative to the pseudoparticle description. However, for the extension of our description to the whole Hilbert space the pseudohole picture is far more suitable. The description of the one-dimensional quantum problem in terms of forward-scattering-interacting pseudoparticles was introduced in Ref. [21] for contact-interaction soluble problems. Shortly after, the same kind of ideas were applied to $1/r^2$-interaction integrable models [29,30]. In the case of electronic models as the Hubbard chain such description is very similar to Fermi-liquid theory, except for two main differences: (i) the $\uparrow$ and $\downarrow$ quasiparticles are replaced by the $c$ and $s$ pseudoparticles and (ii) the discrete pseudoparticle momentum (pseudomomentum) is of the usual form $q_j = \frac{2\pi}{Na} I_j^a$ but the numbers $I_j^a$ are not always integers. They are integers or half-integers depending on whether the number of $\uparrow$ or $\downarrow$ electrons in the system is even or odd. This plays a central role in the ground-state – ground-state transitions [31] we study in Sec. III. The $c$ and $s$ pseudoparticles are non-interacting at the small-momentum and low-energy fixed point and the energy spectrum is described in terms of their bands. At higher energies and (or ) large momenta the pseudoparticles start to interact via zero-
momentum transfer forward-scattering processes. As in a Fermi liquid, these are associated with $f$ functions.

On the other hand, the transitions between states differing by odd electron numbers have a topological character. This follows from the changes from integers (or half integers) to half integers (or integers) of the pseudoparticle quantum numbers $I_\alpha^j$. Such topological excitations involve a global shift of the corresponding pseudo-Fermi sea combined with a process of removing (adding) or adding (removing) pseudoparticles (pseudoholes).

In general, all excitations involving states I can be decoupled into two types of transitions: (a) a topological ground-state – ground-state transition which changes the pseudoparticle (and electron) numbers and involves pseudo-Fermi sea global shifts, which we call topological momentons; and (b) a pseudoparticle - pseudohole excitation relative to the final ground state. In Ref. [31] it was shown that the generators of the transitions (a) are $\sigma$ quasiparticles or quasiholes. Except for a vanishing renormalization factor, these entities are low-energy electrons or holes, respectively. The presence of such factor implies that in the limit of vanishing excitation energy there is a singular relation between these quasiparticles (quasiholes) and the electrons (holes). By expressing these quasiparticles or quasiholes in terms of pseudoparticle operators one can then find the pseudoparticle contents of the electron or hole.

In this paper we introduce a pseudohole description which allows the generalization of the results of Ref. [31] concerning the electron – quasiparticle transformation to all sectors of symmetry of the Hubbard chain in a magnetic field and chemical potential [32]. Only this description is suitable for the study of the interplay between Hamiltonian symmetry and the transformation laws of the elementary excitations which constitute the electrons and holes of vanishing excitation energy. In contrast to the low-energy excitations at constant electron numbers, which at zero magnetic field and (or ) chemical potential can be states II or non-LWS’s and non-HWS’s, all ground states are states I [29]. This is also valid for canonical ensembles corresponding to zero magnetic field and (or ) chemical potential where the symmetry of the quantum problem is higher. Therefore, all ground-state – ground-state
transitions can be described in terms of $c$ and $s$ pseudoparticles or pseudoholes.

In the case of integrable models of simple Abelian $U(1)$ symmetry the elementary excitations are generated by a single type of pseudoparticles (pseudoholes)\(^3\). On the other hand, in the present case of the Hubbard chain we have shown\(^{20}\) that in each of the four sectors of Hamiltonian symmetry $U(1) \otimes U(1)$ there is one branch of $c$ pseudoparticles and one branch of $s$ pseudoparticles which describe the low-energy physics. In terms of pseudoholes the description of the states I involves four branches of $\alpha, \beta$ pseudoholes, where $\alpha = c, s$ and $\beta = \pm \frac{1}{2}$, as we discuss in future sections. In the present case of LWS’s I and (or) HWS’s I we have that $\beta = sgn(\eta_z)\frac{1}{2}$ for $\alpha = c$ and $\beta = sgn(S_z)\frac{1}{2}$ for $\alpha = s$. Therefore, in each $(l, l') = (sgn(\eta_z)1, sgn(S_z)1)$ sector of Hamiltonian symmetry $U(1) \otimes U(1)$\(^{20}\) only the $c, \frac{1}{2}$ and $s, \frac{1}{2}$ pseudohole branches are involved in the description of the corresponding states I.

We express the low-energy $\sigma$ electron and hole of momentum $\pm k_{F\sigma}$ in terms of pseudoholes for the nine sectors of Hamiltonian symmetry of the quantum problem. In all sectors both the electron and the hole are constituted by one topological momenton, one $c$ pseudohole, and one $s$ pseudohole which cannot be disassociated and are confined within the electron or hole. In the very particular limit of half filling and zero magnetic field we recover the holon and spinon picture and the associate symmetry transformation laws already found in Ref.\(^{12}\). In addition, we were able to find the operator generators of the holon and spinon excitations. The electron and hole contains one anti holon and holon, respectively, and one spinon. For instance, we find that the half-filling ($\eta = 1/2; S = 0; \eta_z = -1/2; S_z = 0$) holon [and the ($\eta = 1/2; S = 0; \eta_z = 1/2; S_z = 0$) anti holon]\(^{12}\) of lowest energy is constituted by one $c$ topological momenton and one $c, -\frac{1}{2}$ pseudohole [and one $c, \frac{1}{2}$ pseudohole]. The zero-magnetization ($\eta = 0; S = 1/2; \eta_z = 0; S_z = -1/2$) spinon of lowest energy [and the ($\eta = 0; S = 1/2; \eta_z = 0; S_z = 1/2$) spinon]\(^{12}\) is identified with one $s, \frac{1}{2}$ pseudohole [with one $s, -\frac{1}{2}$ pseudohole].

We also generalize the relation between the transformation laws of the elementary excitations and the symmetry of the Hamiltonian to all sectors of parameter space. We find
that the set of different pseudohole types which constitute the $\sigma$ electron and hole, ie which generate from the $(N_\uparrow, N_\downarrow)$ ground state the $(N_\uparrow + 1, N_\downarrow)$, $(N_\uparrow, N_\downarrow + 1)$, and $(N_\uparrow - 1, N_\downarrow)$, and $(N_\uparrow, N_\downarrow - 1)$ ground states, respectively, transform in the representation of the symmetry group of the Hamiltonian in the initial-ground-state sector of parameter space. Our results also reveal that the occurrence in the Hubbard chain of $\eta$ pairing \cite{14,16} at momentum $\pm \pi$ is a necessary condition for ground states differing in the number of $\sigma$ electrons by one to have as relative momentum the Fermi momentum $k_{F\sigma}$ or $-k_{F\sigma}$.

In Section II we introduce the pseudohole description for the four sectors of parameter space of Hamiltonian symmetry $U(1) \otimes U(1)$ \cite{7,20,27,28}. For the case of ground states this description refers to all nine Hamiltonian symmetry sectors. We also introduce the pseudohole vacuum which we find to be the $SO(4)$ ground state. In the pseudohole representation all ground states and remaining states I are simple Slater determinant of pseudohole levels. We also construct the momentum operator and evaluate the momentum expression for all ground states of the problem.

In Sec. III we express for all sectors of parameter space the momentum $\pm k_{F\sigma}$ electron and hole operators in terms of pseudohole operators and topological momentum operators. We also show that the usual holons and spinons are particular limits of our pseudohole excitations and study the interplay between Hamiltonian symmetry and the set of pseudoholes which describes the $\sigma$ electrons and holes in each canonical ensemble.

Finally, in Sec. IV we present the discussion and concluding remarks.

II. THE PSEUDOHOLE BASIS IN THE FOUR $U(1) \otimes U(1)$ SECTORS

We consider the Hubbard model \cite{11,23,27,28} in one dimension with a finite chemical potential $\mu$ and in the presence of a magnetic field $H$,

$$\hat{H} = \hat{H}_{SO(4)} + 2\mu \hat{n}_z + 2\mu_0 H \hat{S}_z,$$  \hspace{1cm} (1)

where
\[ \hat{H}_{SO(4)} = -t \sum_{j,\sigma} [c_{j+1\sigma}^\dagger c_{j\sigma} + c_{j\sigma}^\dagger c_{j+1\sigma}] + U \sum_{j} [c_{j\uparrow}^\dagger c_{j\uparrow} - 1/2] [c_{j\downarrow}^\dagger c_{j\downarrow} - 1/2], \] (2)

and

\[ \hat{\eta}_z = -\frac{1}{2} [N - \sum_\sigma \hat{N}_\sigma], \quad \hat{S}_z = -\frac{1}{2} \sum_\sigma \sigma \hat{N}_\sigma, \] (3)

are the diagonal generators of the \(SU(2)\) eta-spin and spin algebras, respectively \[16,18\].

In equations (1) – (3) the operator \(c_{j\sigma}^\dagger\) and \(c_{j\sigma}\) creates and annihilates, respectively, one electron of spin projection \(\sigma\) (\(\sigma\) refers to the spin projections \(\sigma = \uparrow, \downarrow\) when used as an operator or function index and is given by \(\sigma = \pm 1\) otherwise) at the site \(j\), \(\hat{N}_\sigma = \sum_j c_{j\sigma}^\dagger c_{j\sigma}\) is the number operator for \(\sigma\) spin-projection electrons, and \(t\), \(U\), \(\mu\), \(H\), and \(\mu_0\) are the first-neighbor transfer integral, the onsite Coulomb interaction, the chemical potential, the magnetic field, and the Bohr magneton, respectively.

There are \(N_{\uparrow}\) up-spin electrons and \(N_{\downarrow}\) down-spin electrons in the chain of \(N_a\) sites and with lattice constant \(a\) associated with the model (1). We use periodic boundary conditions and consider \(N_a\) to be even and when \(N = N_a\) (half filling) both \(N_{\uparrow}\) and \(N_{\downarrow}\) to be odd and employ units such that \(a = t = \mu_0 = \hbar = 1\). When \(N_\sigma\) is odd the Fermi momenta are given by

\[ k_{F\sigma}^\pm = \pm \frac{\pi}{N_a} [N_\sigma - 1], \] (4)

and when \(N_\sigma\) is even the Fermi momenta are given by

\[ k_{F\sigma}^\pm = \frac{\pi}{N_a} N_\sigma, \quad k_{F\sigma}^- = -\frac{\pi}{N_a} [N_\sigma - 2], \] (5)

or by

\[ k_{F\sigma}^\pm = \frac{\pi}{N_a} [N_\sigma - 2], \quad k_{F\sigma}^- = -\frac{\pi}{N_a} N_\sigma. \] (6)

Often we can ignore the \(\frac{1}{N_a}\) corrections in the right-hand side (rhs) of Eqs. (4) – (6) and consider \(k_{F\sigma}^\pm \simeq \pm k_{F\sigma} = \pm \pi n_\sigma\) and \(k_F = [k_{F\uparrow} + k_{F\downarrow}] / 2 = \pi n / 2\), where \(n_\sigma = N_\sigma / N_a\) and \(n = N / N_a\). The dimensionless onsite interaction is \(u = U / 4t\).
The two $SU(2)$ algebras – eta spin and spin – have diagonal generators given by Eq. (3) and off-diagonal generators \[ \hat{\eta}^- = \sum_j (-1)^j c_j^{\uparrow} c_j^{\downarrow}, \quad \hat{\eta}^+ = \sum_j (-1)^j c_j^{\downarrow} c_j^{\uparrow}, \] (7)

and

\[ \hat{S}^- = \sum_j c_j^{\uparrow} c_j^{\downarrow}, \quad \hat{S}^+ = \sum_j c_j^{\downarrow} c_j^{\uparrow}, \] (8)

respectively. In the absence of the chemical-potential and magnetic-field terms the Hamiltonian (1) reduces to (2) and has $SO(4) = SU(2) \otimes SU(2)/\mathbb{Z}_2$ symmetry \[ [12,18] \]. Since $N_a$ is even, the operator $\hat{\eta}_z + \hat{S}_z$ [see Eq. (3)] has only integer eigenvalues and all half-odd integer representations of $SU(2) \otimes SU(2)$ are projected out \[ [12,18] \].

For finite values of both the magnetic field and chemical potential the symmetry of the quantum problem is reduced to $U(1) \otimes U(1)$, with $\hat{\eta}_z$ and $\hat{S}_z$ commuting with $\hat{H}$. The eigenvalues $\eta_z$ and $S_z$ determine the different symmetries of the Hamiltonian (1). When $\eta_z \neq 0$ and $S_z \neq 0$ the symmetry is $U(1) \otimes U(1)$, for $\eta_z = 0$ and $S_z \neq 0$ (and $\mu = 0$) it is $SU(2) \otimes U(1)$, when $\eta_z \neq 0$ and $S_z = 0$ it is $U(1) \otimes SU(2)$, and at $\eta_z = 0$ and $S_z = 0$ (and $\mu = 0$) the Hamiltonian symmetry is $SO(4)$.

There are four $U(1) \otimes U(1)$ sectors of parameter space which correspond to $\eta_z < 0$ and $S_z < 0; \eta_z < 0$ and $S_z > 0; \eta_z > 0$ and $S_z < 0; \eta_z > 0$ and $S_z > 0$. We follow Ref. \[ [20] \] and refer these sectors by $(l, l')$ where

\[ l = sgn(\eta_z)1; \quad l' = sgn(S_z)1. \] (9)

The sectors $(-1, -1); (-1, 1); (1, -1); (1, 1)$ refer to electronic densities and spin densities $0 < n < 1$ and $0 < m < n; 0 < n < 1$ and $-n < m < 0; 1 < n < 2$ and $0 < m < (2 - n)$; and $1 < n < 2$ and $-(2 - n) < m < 0$, respectively.

There are two $(l')$ sectors of $SU(2) \otimes U(1)$ Hamiltonian symmetry [and two $(l)$ sectors of $U(1) \otimes SU(2)$ Hamiltonian symmetry] which correspond to $S_z < 0$ and $S_z > 0$ for $l' = -1$ and $l' = 1$, respectively, (and to $\eta_z < 0$ and $\eta_z > 0$ for $l = -1$ and $l = 1$, respectively).
There is one $SO(4)$ sector of parameter space [which is constituted only by the $\eta_z = 0$ (and $\mu = 0$) and $S_z = 0$ canonical ensemble].

In Ref. [20] we have considered the BA solution for the model (1) associated with each of the four sectors $(l, l')$ of Hamiltonian symmetry $U(1) \otimes U(1)$. As we have mentioned above, the pseudohole algebra introduced in the present paper is more suitable for the description of the non-LWS’s and non-HWS’s eta-spin and spin multiplets which are not contained in the BA solution and whose study will be presented elsewhere [32]. Although for the case of the Hilbert subspace spanned by the states I both the pseudoparticle representation of Ref. [20] and the present pseudohole description are valid, we use here the latter which also simplifies the description of these states which in the pseudohole basis are generated from a single reference vacuum. (Instead of four pseudoparticle vacua [20].)

In the Appendix A we discuss the connection of the present pseudohole description of the states I to the one of Ref. [20]. In that Appendix we also relate the BA equations to the pseudohole basis introduced in this section and present the Hamiltonian in that basis which includes the pseudohole dispersion relations and $f$ functions.

The pseudohole description we introduce below and in Appendix A includes four pseudohole branches which we denote in general by $\alpha, \beta$ pseudoholes. Here $\alpha = c, s$ [7,26,27,28] and $\beta = \pm \frac{1}{2}$. The colors $c$ and $s$ and quantum numbers $\beta = \pm \frac{1}{2}$ which label the four pseudohole branches also label the Hamiltonian eigenstates I which correspond to different $\alpha, \beta$ pseudohole distribution occupancies. (About the relation between the $\alpha, \beta$ pseudoholes and the $(l, l')$ pseudoparticles of Ref. [20] see Appendix A.)

Both in the case of the present states I and of the associate non-LWS’s and non-HWS’s [32] the $\alpha$ orbitals have $N_\alpha^*$ available pseudomomentum values. In the latter states these pseudomomentum values can either be empty (this means occupation by one $\alpha$ pseudoparticle), single occupied by one $\alpha, \frac{1}{2}$ pseudohole, or single occupied by one $\alpha, -\frac{1}{2}$ pseudohole. This reveals that in the general case we should consider $\alpha$ pseudoparticles and $\alpha, \beta$ pseudoholes but no $\alpha, \beta$ pseudoparticles. Therefore, in order to simplify the future generalization of the present results to the whole Hilbert space, we use in this paper the suitable $\alpha$ pseudopar-
article and $\alpha, \beta$ pseudohole description. In contrast to the above non-LWS’s and non-HWS’s, in the case of the states $I$ and $(l, l')$ sectors each of the $N^*_\alpha$ available pseudomomentum values can either be empty (this means occupation by one $\alpha$ pseudoparticle) or single occupied by one $\alpha, \beta$ pseudohole, where $\beta$ is fixed and given by $\beta = \frac{l}{2}$ for $\alpha = c$ or $\beta = \frac{l'}{2}$ for $\alpha = s$. The general expressions for $N^*_\alpha$ and number of $\alpha$ pseudoparticles, $N_\alpha$, are

$$N^*_c = N_a, \quad N^*_s = \frac{1}{2} [N_a - 2(\eta - S)],$$

and

$$N_c = N_a - 2\eta, \quad N_s = \frac{1}{2} [N_a - 2(\eta + S)],$$

respectively.

The results of Sec. III reveal that the existence of four types of $\alpha, \beta$ pseudoholes with $\alpha = c, s$ and $\beta = \pm \frac{1}{2}$ is consistent with the Hamiltonian symmetry. Note that because the description of a state $I$ of the $(l, l')$ sector involves only two out of these four branches, namely the $c, \beta = \frac{l}{2}$ and $s, \beta = \frac{l'}{2}$ pseudoholes, in the associate Hilbert subspace the pseudohole quantum number $\beta$ is directly related to and determined by the quantities of Eq. (9). This is not however a general property. For instance, in the case of the non-LWS’s and non-HWS’s we will study in Ref. [32] $\beta$ has no direct relation to the numbers of Eq. (9). Also the topological transitions of Sec. III connect states $I$ belonging to different sectors and involve three or four different branches of $\alpha, \beta$ pseudoholes.

Let us denote the pseudohole creation and annihilation operators by $a_{q,\alpha,\beta}^\dagger$ and $a_{q,\alpha,\beta}$, respectively. They obey the anticommutative algebra

$$\{a_{q,\alpha,\beta}^\dagger, a_{q',\alpha',\beta'}\} = \delta_{q,q'} \delta_{\alpha,\alpha'} \delta_{\beta,\beta'},$$

and

$$\{a_{q,\alpha,\beta}^\dagger, a_{q',\alpha',\beta'}^\dagger\} = \{a_{q,\alpha,\beta}, a_{q',\alpha',\beta'}\} = 0.$$

As we have mentioned in Sec. I, the discrete pseudomomentum values are
\[ q_j = \frac{2\pi}{N_\alpha} I_j^\alpha, \]

where \( I_j^\alpha \) are consecutive integers or half integers. There are \( N^*_\alpha \) possible \( I_j^\alpha \) values, the number of \( \alpha \) pseudomomentum orbitals \( N^*_\alpha \) being given in Eq. (10). From the point of view of the pseudoholes, a state \( I \) is specified by the distribution of \( N_\alpha \) unoccupied values over the \( N^*_\alpha \) available values. These unoccupied values correspond to the \( N_\alpha \) \( \alpha \) pseudoparticles, the number \( N_\alpha \) depending on \( \eta \) and \( S \) and being given in Eq. (11). The numbers \( I_j^c \) are integers (or half integers) for \( N_s \) even (or odd), and \( I_j^s \) are integers (or half integers) for \( N_s^* \) odd (or even).

There are \( N^h_\alpha = N^*_\alpha - N_\alpha \) occupied values, which following Eqs. (10) and (11) are given by

\[ N^h_c = 2\eta, \quad N^h_s = 2S. \]

In the general case including both the states \( I \) and the corresponding non-LWS’s and non-HWS’s we have that

\[ N^h_\alpha = \sum_{\beta=\pm\frac{1}{2}} N^h_{\alpha,\beta}, \]

where \( N^h_{\alpha,\beta} \) is the number of \( \alpha, \beta \) pseudoholes. In the present case of the states \( I \) and \( (l, l') \) sectors the summation of Eq. (16) simplifies to \( N^h_\alpha = N^h_{\alpha,\beta} \), with \( \beta = \frac{l}{2} \) for \( \alpha = c \) and \( \beta = \frac{l'}{2} \) for \( \alpha = s \). This selection rule also simplifies operator expressions including \( \beta \) summations, as we discuss in Appendix A.

In the four \( (l, l') \) sectors the Hamiltonian eigenstates \( I \) are simple Slater determinants of \( c, \beta \) and \( s, \beta \) pseudohole levels. As it becomes obvious from Eq. (15), the pseudohole vacuum is the \( SO(4) \) ground state, which is the only existing state \( I \) of the corresponding canonical ensemble. On the other hand, ground states of canonical ensembles belonging the two \( (l') \) sectors of Hamiltonian symmetry \( SU(2) \otimes U(1) \) and the two \( (l) \) sectors of Hamiltonian symmetry \( U(1) \otimes SU(2) \) are Slater determinants of only \( s, \frac{l'}{2} \) and \( c, \frac{l}{2} \) pseudohole levels, respectively, and have no \( c, \frac{l}{2} \) and \( s, \frac{l'}{2} \) pseudoholes, respectively.
One of the advantages of the pseudohole representation for the states I (and its multiplets \[32\]) is that while the pseudoparticle Slater determinants of Refs. \[20,27\] refer to a different pseudoparticle vacuum in each of the four \((l, l')\) sectors the corresponding pseudohole expressions involve a single and unique vacuum which is common to all sectors. This is the \(SO(4)\) ground state.

Since the electron and hole studies of Sec. III refer to ground-state – ground-state transitions, in this section we focus our attention on ground states. In terms of pseudoholes the general ground-state expression studied in Ref. \[20\] and associated with the \((l, l')\) sector of Hamiltonian symmetry \(U(1) \otimes U(1)\) reads

\[
|0; \eta_z, S_z\rangle = \prod_{q=q_c}^{\delta_{q_c}} a_{q,c}^\dagger \prod_{q=q_s}^{\delta_{q_s}} a_{q,s}^\dagger |0; 0, 0\rangle ,
\]

where \(l\) and \(l'\) are given in Eq. (9) and \(|0; 0, 0\rangle\) is the \(SO(4)\) \(\mu = 0\) and \(H = 0\) ground state. The pseudo-Fermi points \(q_{F\alpha}^{(\pm)}\) and corresponding limits of the pseudo-Brillouin zones \(q_{\alpha}^{(\pm)}\) are defined below. It is useful to introduce the pseudohole-Fermi points \(\bar{q}_{F\alpha}^{(\pm)}\) such that

\[
q_{F\alpha}^{(\pm)} = q_{F\alpha}^{(\pm)} \pm \frac{2\pi}{N_\alpha}.
\]

When \(N_\alpha\) is odd (or even) and \(I_j^\alpha\) are integers (or half integers) the pseudohole-Fermi points are symmetric and given by

\[
q_{F\alpha}^{(\pm)} = \frac{\pi}{N_\alpha} [N_\alpha + 1].
\]

On the other hand, when \(N_\alpha\) is odd (or even) and \(I_j^\alpha\) are half integers (or integers) we have that either

\[
q_{F\alpha}^{(\pm)} = \frac{\pi}{N_\alpha} [N_\alpha + 2], \quad \bar{q}_{F\alpha}^{(\pm)} = -\frac{\pi}{N_\alpha} N_\alpha ,
\]

or

\[
q_{F\alpha}^{(\pm)} = \frac{\pi}{N_\alpha} N_\alpha , \quad \bar{q}_{F\alpha}^{(\pm)} = -\frac{\pi}{N_\alpha} [N_\alpha + 2].
\]

The pseudo-Fermi points are defined by combining Eq. (18) with Eqs. (19) – (21).
The limits of the pseudo-Brillouin zones \( q_\alpha^{(\pm)} \) involve the number of \( \alpha \)-pseudomomentum orbitals \( N_\alpha^* \). When \( N_\alpha^* \) is odd (or even) and \( I_j^\alpha \) are integers (or half integers) the limits of the pseudo-Brillouin zones are symmetric and given by

\[
q_\alpha^{(+)} = -q_\alpha^{(-)} = \frac{\pi}{N_a} [N_\alpha^* - 1].
\] (22)

On the other hand, when \( N_\alpha^* \) is odd (or even) and \( I_j^\alpha \) are half integers (or integers) we have either that

\[
q_\alpha^{(+)} = \frac{\pi}{N_a} N_\alpha^*, \quad q_\alpha^{(-)} = -\frac{\pi}{N_a} [N_\alpha^* - 2],
\] (23)

or

\[
q_\alpha^{(+)} = \frac{\pi}{N_a} [N_\alpha^* - 2], \quad q_\alpha^{(-)} = -\frac{\pi}{N_a} N_\alpha^*.
\] (24)

For the topological excitations studied in Sec. III the terms of order \( \frac{1}{N_a} \) of the rhs of Eqs. (19) – (24) play an important role. However, for many quantities these corrections are in the thermodynamic limit unimportant and we can consider instead

\[
q_{F\alpha} = \frac{\pi N_\alpha}{N_a} \simeq \pm q_{F\alpha}^{(\pm)} \simeq \pm q_{F\alpha}^{(\pm)},
\] (25)

and

\[
q_\alpha = \frac{\pi N_\alpha^*}{N_a} \simeq \pm q_\alpha^{(\pm)}.
\] (26)

In all sectors of Hamiltonian symmetry there are states I. In the particular case of the \( SO(4) \) zero-chemical potential and zero-magnetic field canonical ensemble there is only one state I. The study of the spectrum for the states II and non-LWS’s and non-HWS’s multiplets reveals that this state I is nothing but the \( SO(4) \) ground state [35]. The same applies to the sectors of Hamiltonian symmetry \( SU(2) \otimes U(1) \) and \( U(1) \otimes SU(2) \), the ground state being always a state I. (In addition, in these sectors there is a large number of excited states I.)

While the description of the states II requires adding new “heavy” pseudoparticles onto the universal \( SO(4) \) pseudohole vacuum [35], all states I can be generated from that vacuum by
distribution occupancies of $\alpha, \beta$ pseudoholes only. In the case of the two ($l'$) $SU(2) \otimes U(1)$ sectors the ground state is both a LWS and HWS of the eta-spin algebra. Therefore, it is empty of $c$ pseudoholes and reads

$$|0; 0, S_z⟩ = \prod_{q=1}^{q_{p,c}^{(-)}} \prod_{q=1}^{q_{p,s}^{(+)}} a^+_q |0; 0, 0⟩ . \quad (27)$$

In the case of the ($l$) $U(1) \otimes SU(2)$ sector the ground state is both a LWS and a HWS of the spin algebra and is empty of $s$ pseudoholes. It reads

$$|0; \eta_z, 0⟩ = \prod_{q=1}^{q_{c,e}^{(-)}} \prod_{q=1}^{q_{c,s}^{(+)}} a^+_q |0; 0, 0⟩ . \quad (28)$$

Finally, the $\eta = \eta_z = 0$ (and $\mu = 0$) and $S = S_z = 0$ $SO(4)$ ground state is, at the same time, a LWS and HWS of both the eta-spin and spin algebras, i.e., following the notation of Ref. [20] it is a [LWS,LWS], a [LWS,HWS], a [HWS,LWS], and a [HWS,HWS]. Therefore, it is empty of both $c$ and $s$ pseudoholes and is the vacuum of the pseudohole theory. All the remaining states I can be described by Slater determinants of $\alpha, \beta$ pseudoholes, as shown in Refs. [20,27] in terms of pseudoparticles.

We close this section by constructing the momentum operator for the states I of all sectors of parameter space. (This question was not addressed in Ref. [20].) The $\pm \pi$ $\eta$-pairing [14,15,16] determines the value for the relative momentum of corresponding eta-spin LWSs and HWSs pairs of the same multiplet family (and thus having the same value of $\eta$). In the two sectors $(-1, \pm 1)$ we have that $0 < n < 1$ and the momentum operator has the usual form [27]

$$\hat{P} = \sum_q q \left[ 1 - a^+_q a_{q,c,-\frac{1}{2}} \right] + \sum_q q \left[ 1 - a^+_q a_{q,s,\pm\frac{1}{2}} \right] . \quad (29)$$

For each LWS I of the eta-spin algebra associated with the $(-1, l')$ sector there is one and only one HWS I of the eta-spin algebra in the corresponding $(1, l')$ sector which belongs to the same tower. That HWS I is generated by acting onto the corresponding LWS I a suitable number of times the operator $\hat{\eta}_+$ of Eq. (7). This operator has momentum $\pm \pi$, ie
when it acts once onto a state it generates a new state with momentum $\pm \pi$ relatively to the initial state.

The fact that the $\eta$-pairing operators (7) have in the present model momentum $\pm \pi$ implies that the relative momentum of the above LWS and HWS is either (a) a reciprocal-lattice momentum, $G = 2\pi j$ with $j = 0, \pm 1, \pm 2, \ldots$, or (b) a reciprocal lattice momentum plus $\pm \pi$. The occurrence of the cases (a) or (b) depends simply on the parity of the number of times needed to act the operator $\hat{\eta}_+\eta$ onto the LWS to obtain the corresponding HWS: the cases (a) and (b) correspond to an even and odd number of times, respectively. It is straightforward to find that when the total electron number $N$ associated with the canonical ensemble of the initial LWS (which has the same parity as the electron number of the final HWS) is either even or odd we have the case (a) or (b), respectively. Since we choose the momentum of the final HWS to belong the first Brillouin zone, this leads to the following momentum-operator expressions for the $(1, \pm 1)$ sectors

$$\hat{P} = \sum_q q \left[ 1 - a_{q,c,\frac{1}{2}}^{\dagger} a_{q,c,\frac{1}{2}} \right] + \sum_q q \left[ 1 - a_{q,s,\pm \frac{1}{2}}^{\dagger} a_{q,s,\pm \frac{1}{2}} \right], \quad N \ even, \quad (30)$$

and

$$\hat{P} = \pm \pi + \sum_q q \left[ 1 - a_{q,c,\frac{1}{2}}^{\dagger} a_{q,c,\frac{1}{2}} \right] + \sum_q q \left[ 1 - a_{q,s,\pm \frac{1}{2}}^{\dagger} a_{q,s,\pm \frac{1}{2}} \right], \quad N \ odd, \quad (31)$$

where in equation (31) we choose $\pi$ or $-\pi$ depending on which of these values provides the momentum $P$ of a given state in the first Brillouin zone. The momentum expressions (29) – (31) are valid for the Hilbert subspace spanned by the states I. Since all ground states of canonical ensembles belonging the nine sectors of Hamiltonian symmetry of the present model are states I, the operator expressions (29) – (31) provide the momenta of all ground states. These are always of the form (29), (30), or (31).

According to the integer or half-integer character of the $I_\alpha^{\eta}$ numbers we have four “topological” types of Hilbert subspaces I and corresponding ground states. Since that character depends on the parities of the $\eta$- and $S$-dependent numbers $N_{s}^\eta$ and $N_s$ of Eqs. (10) and (11), we refer these subspaces by the parities of $N_{s}^\eta$ and $N_s$, respectively, as: (A) even, even;
(B) even, odd; (C) odd, even; and (D) odd, odd. The ground-state momentum expression is different for each type of Hilbert sub space (A)-(D). While the ground states (A)-(C) are bi-degenerate, the ground states corresponding to the Hilbert sub space (D) have zero momentum and are non-degenerate. In Table I we present the ground-state momentum values for the Hilbert sub spaces (A)-(D) in the four \((l,l')\) sectors.

The \(SU(2) \otimes U(1)\) and \(U(1) \otimes SU(2)\) ground states (27) and (28), respectively, can belong to the Hilbert subspaces (A) and (C) only. These have momenta given in Table I [see (A) and (C)]. In the \(SU(2) \otimes U(1)\) case we should use \(2k_F = 2\pi - 2k_F = \pi\) in the momentum expressions of that Table. Finally, the \(SO(4)\) ground state refers to the Hilbert subspace (C). Its has zero momentum and is non degenerate.

As equations (10) – (11) reveal, the eta spin \(\eta\), spin \(S\) and the operators \(\hat{\eta}_z\) and \(\hat{S}_z\) have simple expressions in the pseudohole basis. The following expressions are both valid for the states I and for the non-LWS’s and non-HWS’s multiplets of these states [32]:

\[
\eta = \frac{1}{2} \sum_{\beta} N_{c,\beta}^h, \quad S = \frac{1}{2} \sum_{\beta} N_{s,\beta}^h.
\]

(32)

and

\[
\hat{\eta}_z = \sum_{\beta} \beta \hat{N}_{c,\beta}^h, \quad \hat{S}_z = \sum_{\beta} \beta \hat{N}_{s,\beta}^h.
\]

(33)

In the Hilbert sub space spanned by the states I and at energies smaller than the gaps for the non-LWS’s and non-HWS’s and for the states II the Hubbard model (1) can be written in the \(\alpha, \beta\) pseudohole basis. This expression is presented in Appendix A. Its normal-ordered expression with respect to the suitable ground state (17), Eqs. (A14)-(A15), has an infinite number of terms which correspond to increasing scattering orders. The perturbative character of the pseudohole basis [7,27,28] implies that at low energies only the two terms of lower pseudohole scattering order are relevant. For a detailed study of the present quantum problem in the Hilbert subspace spanned by the states I see Refs. [22,23,24,27,28] which consider the usual \((-1,-1)\) sector of Hamiltonian symmetry \(U(1) \otimes U(1)\).
III. GROUND-STATE TRANSITIONS AND PSEUDOHOLE SYMMETRIES

The study of the interplay between the Hamiltonian symmetry and the transformation laws of the elementary excitations reveals that symmetry is closely related to the low-energy electron and hole *pseudohole* content. The main aim of this section is thus to relate the Hamiltonian symmetry to the transformations of the set of pseudoholes which form the electrons and holes in each parameter-space sector. This requires the generalization to all sectors of parameter space of the present quantum problem of recent results [31] for the expression of the low-energy electron at the Fermi momentum \( \pm k_{F\sigma} \) if we use the discrete definition of Eqs. (4), (6) in terms of pseudoholes.

The studies of Ref. [31] referred to the \((-1, -1)\) sector and expressed the electron and quasiparticle in terms of pseudoparticles. However, only the introduction of our pseudohole basis allows the study of the relation of the results of Ref. [31] to Hamiltonian symmetry. In this section we express the \( \sigma \) electrons and holes in terms of pseudoholes for all nine sectors of parameter space. In the very particular limit of half filling and zero magnetization we recover the holon and spinon symmetry results of Ref. [12]. Moreover, in that \( SO(4) \) canonical ensemble our general operator study solves the problem of expressing the electron (the hole) in terms of anti holons and spinons (of holons and spinons). We find that the holons, anti holons, and spinons are closely related to limiting cases of our general pseudoholes.

To leading order in the excitation energy \( \omega \) the \( \sigma \) electron operator of momentum \( \pm k_{F\sigma} \) is the product of a \( \sigma \) quasiparticle operator of momentum \( \pm k_{F\sigma} \) and a vanishing renormalization factor [31]. In spite of the singular character of this electron – quasiparticle transformation, which justifies the perturbative nature of the quantum problem in the pseudohole basis, the renormalization factor is absorbed by the transformation. Therefore, expressing the quasiparticle in terms of pseudohole operators provides relevant physical information. This reveals that in terms of pseudoholes one electron is a topological excitation constituted by one \( c \) pseudohole, one \( s \) pseudohole, and one large-momentum many-pseudohole topological excitation, the topological momenton. These three quantum objects are confined in
the electron and cannot be separated.

Given a ground state with electron numbers \((N_\sigma, N_{-\sigma})\), we find in this section that the set of all pseudoholes of different type which constitute, in pairs, the \(\uparrow\) and \(\downarrow\) electrons and \(\uparrow\) and \(\downarrow\) holes associated with the ground-state transitions \((N_\uparrow, N_\downarrow) \rightarrow (N_\uparrow \pm 1, N_\downarrow)\) and \((N_\uparrow, N_\downarrow) \rightarrow (N_\uparrow, N_\downarrow \pm 1)\) transform as the symmetry group of the Hamiltonian (1) in the corresponding sector of parameter space.

The study of the ground-state momentum expressions of the previous section reveals that the relative momentum of ground states differing in the number of \(\sigma\) electrons by one equals the \(U = 0\) Fermi points, ie \(\Delta P = \pm k_{F\sigma}\). We define the quasiparticle operator, \(\tilde{c}_{k_{F\sigma},\sigma}^\dagger\), which creates one quasiparticle with spin projection \(\sigma\) and momentum \(k_{F\sigma}\) as \[\tilde{c}_{k_{F\sigma},\sigma}^\dagger|0; N_\sigma, N_{-\sigma}\rangle = |0; N_\sigma + 1, N_{-\sigma}\rangle.\] (34)

The quasiparticle operator defines a one-to-one correspondence between the addition of one electron to the system and the creation of one quasiparticle. Exactly as is expected from the Landau theory in three dimensions, in Appendix B we follow the \((-1, -1)\)-sector study of Ref. [31] and explain how the electronic excitation \(c_{k_{F\sigma},\sigma}^\dagger|0; N_\sigma, N_{-\sigma}\rangle\), defined at the Fermi momentum and small excitation energy \(\omega\), contains a single quasiparticle. (We measure the energy \(\omega\) from the initial-ground-state chemical potential.) In addition to the electron – quasiparticle transformation (B1), we also consider in that Appendix the hole – quasihole transformation [see Eq. (B4)]. In spite of the singular character of these transformations, the discussion of Appendix B reveals that one quasiparticle (quasihole) is basically one electron (hole). Therefore, we call often below the quasiparticle (quasihole) as electron (hole).

Let us then study the expression of the \(\sigma\) quasiparticle and quasihole operators in the pseudohole basis for all sectors of Hamiltonian symmetry. Since we are discussing the problem of addition or removal of one particle the boundary conditions play a crucial role [4][11]. When we add or remove one electron from the many-body system we have to consider the transitions between states with integer and half-integer quantum numbers \(I_j^\alpha\). The transition between two ground states differing in the number of electrons by one is then associated
with two different processes: a backflow in the Hilbert space of the \( \alpha, \beta \) pseudoholes with a shift of all the pseudomomenta by \( \pm \frac{\pi}{N_a} \) and the creation and (or) annihilation of one pair of \( c \) and \( s \) pseudoholes at the pseudo-Fermi points (or at the limit of the pseudo-Brillouin zone for the \( s \) pseudohole).

The backflow associated with a shift of all the pseudomomenta momenta by \( \pm \frac{\pi}{N_a} \) is described by a topological unitary operator such that

\[
V^\pm_\alpha a_{q,\alpha,\beta}^\dagger V^\mp_\alpha = a_{q\mp\frac{\pi}{N_a}\alpha,\beta}^\dagger, \tag{35}
\]

Obviously, the pseudohole vacuum is invariant under this operator, ie

\[
V^\pm_\alpha |0; 0, 0\rangle = |0; 0, 0\rangle. \tag{36}
\]

Using the same method as Ref. [31], we find

\[
V^\pm_\alpha = V_\alpha \left( \mp \frac{\pi}{N_a} \right), \tag{37}
\]

where

\[
V_\alpha (\delta q) = \exp \left\{ i\delta q G^h_\alpha \right\}, \tag{38}
\]

and

\[
G^h_\alpha = -i \sum_{q,\beta} \left[ \frac{\partial}{\partial q} a_{q,\alpha,\beta}^\dagger \right] a_{q,\alpha,\beta}, \tag{39}
\]

is the Hermitian generator of the \( \pm \frac{\pi}{N_a} \) topological pseudomomentum translation. Adding all pseudohole contributions gives a large momentum. This large-momentum excitation induced by the operator (37) is the \( \alpha \) topological momenton. That operator has the following discrete representation

\[
V^\pm_\alpha = \exp \left\{ - \sum_{q,\beta} a_{q\pm\frac{\pi}{N_a},\alpha,\beta}^\dagger a_{q,\alpha,\beta} \right\}. \tag{40}
\]

Note that in the present case of the Hilbert subspace spanned by the states I of the sector \( (l, l') \) only the value \( \beta = \frac{1}{2} \) for \( \alpha = c \) and the value \( \beta = \frac{l'}{2} \) for \( \alpha = s \) contributes to the \( \beta \) summation of Eqs. (39) and (40), as discussed in Sec. II and Appendix A.
In addition to the topological momenton, the quasiparticle or quasihole excitation includes creation and (or) annihilation of pseudoholes. The changes in the pseudohole and pseudoparticle numbers and the corresponding changes in the values of \( \eta \), \( \eta_z \), \( S \), and \( S_z \) are given in Tables II and III for the ground-state – ground-state transitions \((N_\uparrow, N_\downarrow) \rightarrow (N_\uparrow \pm 1, N_\downarrow)\) and \((N_\uparrow, N_\downarrow) \rightarrow (N_\uparrow, N_\downarrow \pm 1)\), respectively.

We consider below the expressions for the quasiparticles \( \tilde{c}_{kF,\uparrow}^\dagger \) and \( \tilde{c}_{kF,\downarrow}^\dagger \) associated with the transitions \((N_\uparrow, N_\downarrow) \rightarrow (N_\uparrow + 1, N_\downarrow)\) and \((N_\uparrow, N_\downarrow) \rightarrow (N_\uparrow - 1, N_\downarrow)\), respectively, and the quasiholes \( \tilde{c}_{kF,\uparrow} \) and \( \tilde{c}_{kF,\downarrow} \) associated with the transitions \((N_\uparrow, N_\downarrow) \rightarrow (N_\uparrow - 1, N_\downarrow)\) and \((N_\uparrow, N_\downarrow) \rightarrow (N_\uparrow, N_\downarrow + 1)\), respectively.

We emphasize that because the initial ground state for the above two quasiparticles and two quasiholes is the same, the \( \sigma \) quasiparticle and \( \sigma \) quasihole momenta differ by \( \pm \frac{2\pi}{N_a} \) [see Eqs. (4) – (6)]. Therefore, the corresponding quasiparticle and quasihole expressions are not related by an adjoint transformation. On the other hand, the operators \( \tilde{c}_{\pm kF,\sigma,\uparrow}^\dagger \) and \( \tilde{c}_{\pm kF,\sigma,\downarrow} \) associated with the transitions \((N_\sigma, N_{-\sigma}) \rightarrow (N_\sigma + 1, N_{-\sigma})\) and \((N_\sigma + 1, N_{-\sigma}) \rightarrow (N_\sigma, N_{-\sigma})\) are obviously related by such transformation. In this case the initial (final) ground state of the electrons (holes) is the final (initial) ground state of the holes (electrons). Moreover, let us consider the set of four operators \( \tilde{c}_{\pm kF,\uparrow,\uparrow}^\dagger \), \( \tilde{c}_{\pm kF,\uparrow,\downarrow}^\dagger \), \( \tilde{c}_{\pm kF,\downarrow,\uparrow} \), and \( \tilde{c}_{\pm kF,\downarrow,\downarrow} \) such that the creation operators act on the same initial ground state \((N_\uparrow, N_\downarrow)\) transforming it in the ground states \((N_\uparrow + 1, N_\downarrow)\) and \((N_\uparrow, N_\downarrow + 1)\), respectively, and the hole operators act on the corresponding latter states giving rise to the original ground state. Let us consider the reduced Hilbert subspace spanned by these three ground states, the Fermi-point discrete definitons (4) – (6), and the pseudo-Fermi points and pseudo-Brillouin-zone limits expressions (18) – (24). If we combine that with the electron and hole expressions introduced below, it is easy to show that the corresponding quasiparticle and quasihole operators \( \tilde{c}_{\pm kF,\uparrow,\uparrow}^\dagger \), \( \tilde{c}_{\pm kF,\uparrow,\downarrow}^\dagger \), \( \tilde{c}_{\pm kF,\downarrow,\uparrow} \), and \( \tilde{c}_{\pm kF,\downarrow,\downarrow} \) obey the usual anticommutation relations.

Generalization of the results of Ref. [31] leads to quasiparticle and quasihole operator expressions for all sectors. The two electrons and two holes refer to the same initial ground state. The pseudo-Fermi points and pseudo-hole-Fermi points of the expressions below refer
to that initial ground state. On the other hand, s pseudohole creation and annihilation operators at the limits of the pseudo-Brillouin zones refer to the final and initial ground states, respectively. In the case of the \((l, l')\) sectors of Hamiltonian symmetry \(U(1) \otimes U(1)\) we consider that the initial and final ground states belong the same sector of parameter space. In the case of the \((l')\) sectors of Hamiltonian symmetry \(SU(2) \otimes U(1)\) [or \((l)\) sectors of Hamiltonian symmetry \(U(1) \otimes SU(2)\)] we consider that the initial and final ground states belong to sectors of parameter space characterized by the same value of \((l')\) [or \((l)\)]. We present below the electron and hole expressions found for different initial ground states in the nine sectors of parameter space.

For initial ground states in the \((-1, -1)\) sector of Hamiltonian symmetry \(U(1) \otimes U(1)\) we find

\[
\tilde{c}_{\pm kF \uparrow}^\dagger = a_{q_{F \uparrow}, c, c, -\frac{1}{2}}^\dagger V_s \pm \frac{1}{2} a_{q_{s \uparrow}, s, s, -\frac{1}{2}}^\dagger, \quad \tilde{c}_{\pm kF \downarrow}^\dagger = V_c \pm \frac{1}{2} a_{q_{s \downarrow}, s, s, -\frac{1}{2}}^\dagger, \quad (41)
\]

for the electrons and

\[
\tilde{c}_{\pm kF \uparrow} = a_{q_{F \uparrow}, c, c, -\frac{1}{2}}^\dagger V_s \pm \frac{1}{2} a_{q_{s \uparrow}, s, s, -\frac{1}{2}}, \quad \tilde{c}_{\pm kF \downarrow} = V_c \pm \frac{1}{2} a_{q_{s \downarrow}, s, s, -\frac{1}{2}}, \quad (42)
\]

for the holes.

For the \((-1, 1)\) sector we find

\[
\tilde{c}_{\pm kF \uparrow}^\dagger = V_c \pm \frac{1}{2} a_{q_{F \uparrow}, c, c, -\frac{1}{2}}^\dagger V_s \pm a_{q_{s \uparrow}, s, s, -\frac{1}{2}}, \quad \tilde{c}_{\pm kF \downarrow}^\dagger = V_c \pm \frac{1}{2} a_{q_{s \downarrow}, s, s, -\frac{1}{2}}^\dagger a_{q_{s \downarrow}, s, s, -\frac{1}{2}}, \quad (43)
\]

for the electrons and

\[
\tilde{c}_{\pm kF \uparrow} = V_c \pm \frac{1}{2} a_{q_{F \uparrow}, c, c, -\frac{1}{2}}^\dagger V_s \pm a_{q_{s \uparrow}, s, s, -\frac{1}{2}}, \quad \tilde{c}_{\pm kF \downarrow} = V_c \pm \frac{1}{2} a_{q_{s \downarrow}, s, s, -\frac{1}{2}}^\dagger a_{q_{s \downarrow}, s, s, -\frac{1}{2}}, \quad (44)
\]

for the holes.

In the \((1, -1)\) sector the result is

\[
\tilde{c}_{\pm kF \uparrow}^\dagger = V_c \pm \frac{1}{2} a_{q_{F \uparrow}, c, c, -\frac{1}{2}}^\dagger V_s \pm a_{q_{s \uparrow}, s, s, -\frac{1}{2}}, \quad \tilde{c}_{\pm kF \downarrow}^\dagger = V_c \pm \frac{1}{2} a_{q_{s \downarrow}, s, s, -\frac{1}{2}}^\dagger a_{q_{s \downarrow}, s, s, -\frac{1}{2}}, \quad (45)
\]

for the electrons and
\[ \tilde{c}_{\pm k_{F}, \uparrow} = V_{c, \varepsilon}^{2, \sigma} a_{q_{F,c}, \varepsilon}^{(\pm)} a_{q_{F,s}, \sigma, -\frac{1}{2}}, \quad \tilde{c}_{\pm k_{F}, \downarrow} = a_{q_{F,c}, \varepsilon}^{(\pm)} V_{s, \varepsilon}^{2, \sigma} a_{q_{F,s}, \sigma, \frac{1}{2}} \]

for the holes.

The expressions for the (1, 1) sector are

\[ \tilde{c}_{\pm k_{F}, \uparrow} = a_{q_{F,c}, \varepsilon}^{(\pm)} V_{s, \varepsilon}^{2, \sigma} a_{q_{F,s}, \sigma, -\frac{1}{2}} \quad \tilde{c}_{\pm k_{F}, \downarrow} = V_{c, \varepsilon}^{2, \sigma} a_{q_{F,c}, \varepsilon}^{(\pm)} a_{q_{F,s}, \sigma, \frac{1}{2}} \]

for the electrons and

\[ \tilde{c}_{\pm k_{F}, \uparrow} = a_{q_{F,c}, \varepsilon}^{(\pm)} V_{s, \varepsilon}^{2, \sigma} a_{q_{F,s}, \sigma, -\frac{1}{2}} \quad \tilde{c}_{\pm k_{F}, \downarrow} = V_{c, \varepsilon}^{2, \sigma} a_{q_{F,c}, \varepsilon}^{(\pm)} a_{q_{F,s}, \sigma, \frac{1}{2}} \]

for the holes.

According to Eqs. (41) – (48) the \( \sigma \) quasiparticles and quasiholes are many-pseudohole objects which recombine the colors \( c \) and \( s \) (charge and spin in the limit \( m = n_{\uparrow} - n_{\downarrow} \to 0 \)) giving rise to spin projection \( \uparrow \) and \( \downarrow \) and have Fermi surfaces at \( \pm k_{F,\sigma} \).

Similar expressions can be derived for the sectors of parameter space where the Hamiltonian (1) has higher symmetry. We start by considering the sectors of Hamiltonian symmetry \( SU(2) \otimes U(1) \) where

\[ q_{F,c}^{(+)} = -q_{F,c}^{(-)} = q_{c}^{(-)} = -q_{c}^{(-)} = \pi[1 - \frac{1}{N_{a}}] \]

For ground states of the \( l' = -1 \) sector of Hamiltonian symmetry \( SU(2) \otimes U(1) \) the electrons read

\[ \tilde{c}_{\pm k_{F}, \uparrow} = V_{c, \varepsilon}^{2, \sigma} a_{q_{F,c}, \varepsilon}^{(\pm)} a_{q_{F,s}, \sigma, -\frac{1}{2}}, \quad \tilde{c}_{\pm k_{F}, \downarrow} = a_{q_{F,c}, \varepsilon}^{(\pm)} V_{s, \varepsilon}^{2, \sigma} a_{q_{F,s}, \sigma, \frac{1}{2}} \]

and the holes read

\[ \tilde{c}_{\pm k_{F}, \uparrow} = a_{q_{F,c}, \varepsilon}^{(\pm)} V_{s, \varepsilon}^{2, \sigma} a_{q_{F,s}, \sigma, -\frac{1}{2}}, \quad \tilde{c}_{\pm k_{F}, \downarrow} = V_{c, \varepsilon}^{2, \sigma} a_{q_{F,c}, \varepsilon}^{(\pm)} a_{q_{F,s}, \sigma, \frac{1}{2}} \]

For the \( l' = 1 \) sector of Hamiltonian symmetry \( SU(2) \otimes U(1) \) the electrons read

\[ \tilde{c}_{\pm k_{F}, \uparrow} = a_{q_{F,c}, \varepsilon}^{(\pm)} V_{s, \varepsilon}^{2, \sigma} a_{q_{F,s}, \sigma, \frac{1}{2}}, \quad \tilde{c}_{\pm k_{F}, \downarrow} = V_{c, \varepsilon}^{2, \sigma} a_{q_{F,c}, \varepsilon}^{(\pm)} a_{q_{F,s}, \sigma, -\frac{1}{2}} \]

and the holes read
\[ \tilde{c}_{\pm kF \uparrow, \uparrow} = V_c \tilde{a}_{q_{Fc}^{(\pm)}, c, \pm \frac{1}{2}, q_{Fs}^{(\pm)}, s, \pm \frac{1}{2}}, \quad \tilde{c}_{\pm kF \downarrow, \uparrow} = V_{s} \tilde{a}_{q_{Fc}^{(\pm)}, c, \mp \frac{1}{2}, q_{Fs}^{(\pm)}, s, \pm \frac{1}{2}}. \] (53)

In the sectors of Hamiltonian symmetry \( U(1) \otimes SU(2) \) we have that
\[ q_{Fs}^{(+)} = -q_{Fs}^{(-)} = q_{s}^{(+)} = -q_{s}^{(-)} = \pi \left[ \frac{n}{2} - \frac{1}{N_0} \right]. \] (54)

In the case of the \( l = -1 \) sector of Hamiltonian symmetry \( U(1) \otimes SU(2) \) the up-spin electron and hole read
\[ \tilde{c}_{\pm kF \uparrow, \uparrow}^{\dagger} = V_c \tilde{a}_{q_{Fc}^{(\pm)}, c, \pm \frac{1}{2}, q_{Fs}^{(\pm)}, s, \pm \frac{1}{2}}, \quad \tilde{c}_{\pm kF \downarrow, \uparrow}^{\dagger} = V_{c} \tilde{a}_{q_{Fc}^{(\pm)}, c, -\frac{1}{2}, q_{Fs}^{(\pm)}, s, \pm \frac{1}{2}}, \] (55)
and the down-spin electron and hole read
\[ \tilde{c}_{\pm kF \uparrow, \downarrow}^{\dagger} = V_c \tilde{a}_{q_{Fc}^{(\pm)}, c, \pm \frac{1}{2}, q_{Fs}^{(\pm)}, s, \mp \frac{1}{2}}, \quad \tilde{c}_{\pm kF \downarrow, \downarrow}^{\dagger} = V_{c} \tilde{a}_{q_{Fc}^{(\pm)}, c, -\frac{1}{2}, q_{Fs}^{(\pm)}, s, -\frac{1}{2}}. \] (56)

For the \( l = 1 \) the sector of Hamiltonian symmetry \( U(1) \otimes SU(2) \) the up-spin electron and hole read
\[ \tilde{c}_{\pm kF \uparrow, \uparrow}^{\dagger} = V_c \tilde{a}_{q_{Fc}^{(\pm)}, c, \pm \frac{1}{2}, q_{Fs}^{(\pm)}, s, -\frac{1}{2}}, \quad \tilde{c}_{\pm kF \downarrow, \uparrow}^{\dagger} = V_{c} \tilde{a}_{q_{Fc}^{(\pm)}, c, \mp \frac{1}{2}, q_{Fs}^{(\pm)}, s, -\frac{1}{2}}, \] (57)
and the down-spin electron and hole read
\[ \tilde{c}_{\pm kF \uparrow, \downarrow}^{\dagger} = V_c \tilde{a}_{q_{Fc}^{(\pm)}, c, \pm \frac{1}{2}, q_{Fs}^{(\pm)}, s, \frac{1}{2}}, \quad \tilde{c}_{\pm kF \downarrow, \downarrow}^{\dagger} = V_{c} \tilde{a}_{q_{Fc}^{(\pm)}, c, -\frac{1}{2}, q_{Fs}^{(\pm)}, s, \frac{1}{2}}. \] (58)

Finally, for the \( SO(4) \) initial ground state both Eq. (49) and the following equation
\[ q_{Fs}^{(+)} = -q_{Fs}^{(-)} = q_{s}^{(+)} = -q_{s}^{(-)} = \pi \left[ \frac{1}{2} - \frac{1}{N_0} \right], \] (59)
hold true and we find for the electrons
\[ \tilde{c}_{\pm kF \uparrow, \uparrow}^{\dagger} = V_c \tilde{a}_{q_{Fc}^{(\pm)}, c, \pm \frac{1}{2}, q_{Fs}^{(\pm)}, s, -\frac{1}{2}}, \quad \tilde{c}_{\pm kF \downarrow, \uparrow}^{\dagger} = V_{c} \tilde{a}_{q_{Fc}^{(\pm)}, c, -\frac{1}{2}, q_{Fs}^{(\pm)}, s, \pm \frac{1}{2}}, \] (60)
and for the holes
\[ \tilde{c}_{\pm kF \uparrow, \downarrow}^{\dagger} = V_c \tilde{a}_{q_{Fc}^{(\pm)}, c, \mp \frac{1}{2}, q_{Fs}^{(\pm)}, s, 1 \frac{1}{2}}, \quad \tilde{c}_{\pm kF \downarrow, \downarrow}^{\dagger} = V_{c} \tilde{a}_{q_{Fc}^{(\pm)}, c, -\frac{1}{2}, q_{Fs}^{(\pm)}, s, -\frac{1}{2}}. \] (61)

Equations (60) – (61) reveal that removing or adding electrons from the \( SO(4) \) ground state always involves creation of pseudoholes. Furthermore, while in the case of the \( (l, l') \)
sectors the initial and final ground states belong in general to the same sector, in the case of the $SO(4)$ ground state each of the four possible transitions associated with adding one up-spin or one down-spin quasiparticle or quasihole leads to four ground states belonging to a different $(l, l')$ sector. If the initial ground state belongs to the $(l') SU(2) \otimes U(1)$ sector [or to the $(l) U(1) \otimes SU(2)$ sector] then two of the final ground states belong to the $(1, l')$ [or to the $(l, 1)$] sector and the remaining two ground states to the $(-1, l')$ [or to the $(l, -1)$] sector.

Equations (32) and (33) tell us that the values of $\eta$ and $\eta_z$ are fully determined by the number of $c, \beta$ pseudoholes whereas the number of $s, \beta$ pseudoholes determines the values of $S$ and $S_z$. In addition, note that the quasiparticle and quasihole operators (41) – (48), (51) – (53), (55) – (58), and (60) – (61) involve always a change in the number of $c$ pseudoholes of one and a change in the number of $s$ pseudoholes also of one. Moreover, when acting on the suitable ground state these operators change the values of $\eta$ and $\eta_z$ by $\pm 1/2$ and $\pm sgn(\eta_z)1/2$, respectively, and the values of $S$ and $S_z$ by $\pm 1/2$ and $\pm sgn(S_z)1/2$, respectively. (The corresponding changes in the pseudohole numbers and in the values of $\eta$, $\eta_z$, $S$, and $S_z$ are shown in Tables II and III.) The analysis of the changes in the pseudohole numbers could lead to the conclusion that the $c, \pm \frac{1}{2}$ pseudoholes have quantum numbers ($\eta = 1/2; S = 0; \eta_z = \pm 1/2; S_z = 0$) and that the $s, \mp \frac{1}{2}$ pseudoholes have quantum numbers ($\eta = 0; S = 1/2; \eta_z = 0; S_z = \pm 1/2$). If this was true the $c$ and corresponding $\beta$ pseudohole quantum numbers could be identified with $\eta$ and $\eta_z$, respectively, and the $s$ and corresponding $\beta$ pseudohole quantum numbers could be identified with $S$ and $S_z$, respectively. However, this is not in general true. This holds true in the particular case of zero-momentum number operators. On the other hand, the above identities are also true for finite-momentum operators for $c, \pm \frac{1}{2}$ in the limit of zero chemical potential and for $s, \pm \frac{1}{2}$ in the limit of zero magnetic field, as we find below.

In order to confirm that the above equivalences are not in general true for finite-momentum fluctuations we consider the $\alpha$-pseudohole fluctuation operator.
\[ \rho_{\alpha}(k) = -\sum_{q,\beta} \beta a_{q,\alpha,\beta}^\dagger a_{q+k,\alpha,\beta}, \]  
and the \( \eta_z \)-(charge) and \( S_z \)-(spin) fluctuation operators

\[ \rho_{\eta_z}(k) = \sum_{k',\sigma} \left[ \frac{1}{2} \delta_{k,o} - c_{k'+k,\sigma}^\dagger c_{k',\sigma} \right], \]

and

\[ \rho_{S_z}(k) = \sum_{k',\sigma} \sigma c_{k'+k,\sigma}^\dagger c_{k',\sigma}, \]

respectively. From equations (32) and (33) we find

\[ \rho_c(0) = \rho_{\eta_z}(0) = N_a - N_\uparrow - N_\downarrow, \]

and

\[ \rho_s(0) = \rho_{S_z}(0) = N_\uparrow - N_\downarrow. \]

We then conclude that at zero momentum the above equivalences hold true. The electron numbers \( N_\uparrow \) and \( N_\downarrow \) are good quantum numbers of the many-electron system. Since the exact Hamiltonian eigenstates are simple Slater determinants of \( \alpha, \beta \)-pseudohole levels, the numbers of \( \alpha, \beta \) pseudoholes are thus required to be also good quantum numbers. They are such that Eqs. (65) and (66) are obeyed.

On the other hand, the conservation of electron and pseudohole numbers does not require the finite-momentum \( c \) and \( s \) fluctuations being bare finite-momentum charge and spin fluctuations, respectively. By simplicity, we consider the smallest momentum values, \( k \pm \frac{2\pi}{N_a} \). We emphasize that for \( k = \pm \frac{2\pi}{N_a} \), acting the operator \( \rho_{\alpha}(k) \) onto a ground state of general form (17) generates a singlet pair \( \alpha \)-pseudoparticle-pseudohole excitation where the \( \alpha, \beta \) pseudohole at \( q = \bar{q}_F^{(\pm)} \) moves to \( q = q_F^{(\pm)} \) [see Eq. (18)]. If in \( c, \beta \) the color \( c \) was eta spin and \( \beta = \eta_z \) and in \( s, \beta \) the color \( s \) was spin and \( \beta = S_z \), we should have that \( \rho_c(\pm \frac{2\pi}{N_a}) = \rho_{\eta_z}(\pm \frac{2\pi}{N_a}) \) and \( \rho_s(\pm \frac{2\pi}{N_a}) = \rho_{S_z}(\pm \frac{2\pi}{N_a}) \), respectively. However, the results of Ref. [7] show that this is not true for the sectors of Hamiltonian symmetry \( U(1) \otimes U(1) \). Although the pseudohole summations of Eqs. (32) and (33) give \( \eta, S, \eta_z, \) and \( S_z \) this does not require each
c pseudohole having eta spin 1/2 and spin 0 and each s pseudohole having eta spin 0 and spin 1/2. Also, the fact that the quasiparticle or quasihole of Eqs. (41) – (48), (50) – (53), (55) – (58), and (60) – (61) has \( \eta = 1/2; S = 1/2; \eta_z = sgn(\eta_z)1/2; S_z = sgn(S_z)1/2 \) does not tell how these values are distributed by the corresponding c pseudohole, s pseudohole, and topological momenton.

The studies of Ref. [7] reveal that for finite values of the chemical potential and magnetic field there is a c and s separation of the low-energy and small-momentum excitations but that the orthogonal modes c and s are not in general charge and spin, respectively [7]. On the other hand, in that reference it was found that in the limit of zero chemical potential the finite-momentum c fluctuations become real charge excitations and that in the limit of zero magnetic field the finite-momentum s fluctuations become real spin excitations. In the latter limit the c excitations are also real charge excitations and the c and s low-energy separation becomes the usual charge and spin separation [3, 4, 5, 6]. In these limits \( c, \beta \) becomes \( \eta, \eta_z \) and \( s, \beta \) becomes \( S, -S_z \).

It follows that in the case of the \( SO(4) \) canonical ensemble the set of pseudoholes involved in the description of the two electron and two hole operators (60) – (61), which are the \( c, +1/2; c, -1/2; s, +1/2; \) and \( s, -1/2 \) pseudoholes at the pseudo-Fermi points, transform in the \( \eta = 1/2 \) and \( S = 1/2 \) representation of the \( SO(4) \) group. Moreover, it can be shown from the changes in the BA quantum numbers and from the study of the pseudohole energies that the \( \eta = 1/2 \) and \( S = 1/2 \) elementary excitations studied in Ref. [12] are simple combinations of one of the ground-state – ground-state transitions generated by the operators (60) – (61) with a single pseudoparticle-pseudohole process relative to the final ground state. In addition, the usual half-filling holons and zero-magnetization spinons can be shown to be limiting cases of our pseudohole excitations. For instance, the \( (\eta = 1/2; S = 0; \eta_z = 1/2; S_z = 0) \) anti holon and \( (\eta = 1/2; S = 0; \eta_z = -1/2; S_z = 0) \) holon excitations of Ref. [12] are at lowest energy generated from the \( SO(4) \) ground state by the operators \( V^-_{c}a_{q_{FS}, \frac{\pi}{4}}^{\dagger} \) and \( V^+_{c}a_{q_{FS}, \frac{\pi}{4}}^{\dagger} \), respectively. Also at lowest energy, the two \( (\eta = 0; S = 1/2; \eta_z = 0; S_z = 1/2) \) and \( (\eta = 0; S = 1/2; \eta_z = 0; S_z = -1/2) \) spinons [12] are generated from that ground state.
by the operators $a^\dagger_{q F_s^{(\pm)},s,-\frac{1}{2}}$ and $a^\dagger_{q F_s^{(\pm)},s,\frac{1}{2}}$, respectively. The full spectrum of these excitations is obtained by adding to these generators a suitable single pseudoparticle-pseudohole-pair operator. The corresponding energy spectrum involves the pseudohole bands (A17) and (A18) and by use of the momentum expressions (29) – (31) and Hamiltonian expression (A15) recovers in the limit of zero chemical potential and magnetic field the expressions of Ref. [12]. Therefore, our expressions (B1) and (60) – (61) define the electron in terms of holons and spinons. Since only Hamiltonian eigenstates with integer values of $\eta_z + S_z$ are allowed, the holon – spinon pairs of Eqs. (60) – (61) cannot be separated. This also holds true in the general case, the electron being constituted by one $c$ pseudohole, one $s$ pseudohole, and one many-pseudohole topological momenton of large momentum, as confirmed by Eqs. (41) – (48), (50) – (53), and (55) – (58). Also in this case the fact that only Hamiltonian eigenstates with integer values of $\eta_z + S_z$ are allowed prevents these three excitations of being separated.

As for the $SO(4)$ ground state, we can relate the symmetry of the Hamiltonian (1) in a given canonical ensemble by looking at the pseudohole contents of the corresponding two electrons and two holes of Eqs. (41) – (48), (50) – (53), and (55) – (58). For instance, Eqs. (41) – (48) show that in the $(l, l')$ sectors of Hamiltonian symmetry $U(1) \otimes U(1)$ the two electrons and two holes involve one pair of the same type of pseudoholes, namely the corresponding $c, \frac{l}{2}$ and $s, \frac{l'}{2}$ pseudoholes. Each of these transforms in the representation of the group $U(1)$ and, therefore, the set of two pseudoholes transforms in the representation of the group $U(1) \otimes U(1)$.

In the case of the $(l')$ sectors of Hamiltonian symmetry $SU(2) \otimes U(1)$ $c$ is eta spin and the corresponding quantum number $\beta$ is $\eta_z$ and Eqs. (50) – (53) confirm that the two electrons and holes involve either one $c, \frac{1}{2}$ pseudohole or one $c, -\frac{1}{2}$ pseudohole combined with one $s, \frac{l'}{2}$ pseudohole. The $c, \frac{1}{2}$ and $c, -\frac{1}{2}$ pseudoholes transform in the $\eta = 1/2$ representation of the eta-spin $SU(2)$ group, whereas the $s, \frac{l'}{2}$ pseudohole transforms in the representation of the $U(1)$ group. Therefore, the set of $c, \frac{1}{2}; c, -\frac{1}{2};$ and $s, \frac{l'}{2}$ pseudoholes transforms in the $\eta = 1/2$
representation of the $SU(2) \otimes U(1)$ group.

In the case of the $(l)$ sectors of Hamiltonian symmetry $U(1) \otimes SU(2)$ $s$ is spin and the corresponding quantum number $\beta$ is $\beta = S_z$ and Eqs. (55)–(58) show that the two electrons and two holes are constituted by either one $s, \frac{1}{2}$ or one $s, -\frac{1}{2}$ pseudohole combined with one $c, \frac{1}{2}$ pseudohole. The $s, \frac{1}{2}$ and $s, -\frac{1}{2}$ pseudoholes transform in the $S = 1/2$ representation of the spin $SU(2)$ group and the $c, \frac{1}{2}$ pseudohole transforms in the representation of the $U(1)$ group. It follows that the set of the $c, \frac{1}{2}$; $s, \frac{1}{2}$; and $s, -\frac{1}{2}$ pseudoholes transforms in the $S = 1/2$ representation of the $U(1) \otimes SU(2)$ group.

IV. CONCLUDING REMARKS

In this paper we have introduced a pseudohole representation for the states I of the Hubbard chain in a magnetic field and chemical potential which is valid for all sectors of Hamiltonian symmetry. In the pseudohole picture all Hamiltonian eigenstates can be generated from a single pseudohole reference vacuum, the half-filling and zero-magnetic field ground state. This differs from the pseudoparticle description of Ref. [20] which requires four different reference vacua.

The introduction of the above pseudohole description has allowed the study of the interplay between Hamiltonian symmetry in each of the nine sectors of parameter space and the transformation laws of the set of pseudoholes which form the electrons and holes of vanishing excitation energy. This study has required the generalization of the $(-1, -1)$-sector results of Ref. [31] to all sectors of parameter space. For all the nine sectors we could express the Fermi-momentum $\pm k_{F, \sigma}$ electrons and holes in terms of one pair of pseudoholes and one topological momenton. These three quantum objects are confined in the electron or hole and cannot be separated. We have considered the particular set of the up-spin electron, down-spin electron, up-spin hole, and down-spin hole whose individual addition to an initial ground state leads to the four final ground states differing from it by one electron number. We found that the two, three, or four different types of $\alpha, \beta$ pseudoholes which are
contained (two in each electron or hole) in that set always transform in the representation of the symmetry group of the Hamiltonian in the sector of parameter space of the initial ground state.

We have also shown that the usual half-filling holons and zero-magnetization spinons are limiting cases of our pseudohole and topological momenton excitations. Our operator study has allowed the identification of the holon and spinon generators as well as the exact holon and spinon contents of the $SO(4)$ electrons and holes of vanishing excitation energy.

Finally, we will consider elsewhere an extension of the present pseudohole basis which refers to the whole Hilbert space of the Hamiltonian (1). In addition to the $\alpha, \beta$ pseudoholes, this requires the introduction of new branches of “heavy” pseudoparticles [31,35]. These heavy pseudoparticles are absent in the states I, the construction of the states II including their creation onto the pseudohole vacuum. Thus the universal pseudohole vacuum introduced in this paper and the associate pseudohole basis provide the correct and suitable starting point for the extension of our operator description to the whole Hilbert space.

Although the $\alpha, \beta$ pseudoholes (or, equivalently, the $\alpha$ pseudoparticles) associated with the states I are the transport carriers at low energy [23,24] and couple to external potentials [25], they refer to purely non-dissipative excitations, i.e. the Hamiltonian (1) commutes with the charge current operator in the Hilbert subspace spanned by the states I [24]. Therefore, the pseudohole currents give rise only to the coherent part of the conductivity spectrum, i.e. to the Drude peak [24]. The finite-frequency part is associated with the above “heavy” pseudoparticles. For instance, we will show elsewhere [31,35] that both the c, $\beta$ pseudoholes and some of the heavy pseudoparticles couple to external vector potentials in such a way that in the full Hilbert space the Hamiltonian does not commute with the charge current operator.
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APPENDIX A: THE PSEUDOHOLE BASIS AND THE BA SOLUTION

In this Appendix we discuss the pseudohole and pseudoparticle descriptions and relate the pseudohole number operator \( \hat{N}_{\alpha,\beta}^h(q) = a_{q,\alpha,\beta}^\dagger a_{q,\alpha,\beta} \) to the BA equations. We also present the Hamiltonian in the pseudohole basis and the associated dispersion relations.

In order to relate the present pseudoholes to the pseudoparticle description of Ref. [20], we emphasize that the \( \alpha(l, l') \) pseudoholes associated with the pseudoparticles introduced in that reference are such that \( c(l, 1) = c(l, -1) \) and \( s(1, l') = s(-1, l') \). This is related to the fact that for states \( I \) with the same values of \( \eta \) and \( S \) the \( \alpha(l, l') \) pseudomomentum-orbital numbers \( N_{\alpha(l,l')}^* \) [and \( \alpha(l, l') \) pseudoparticle numbers \( N_{\alpha(l,l')} \)] shown in Table I of Ref. [20] are for different \( (l, l') \) numbers equal. This refers to pairs of states \( I \) where one is a LWS and the other is the corresponding HWS of the same family of multiplets of eta-spin or spin algebras. In this way these numbers are \( l \)- and \( l' \)-independent [see Eq. (9)], ie do not depend on the signs of \( \eta_z \) and \( S_z \) but only on the corresponding values of \( \eta \) and \( S \). Therefore, we can refer them simply by \( N_{\alpha}^* \) and \( N_{\alpha} \). Their general \( \eta \)- and \( S \)-dependent expressions are given in Eqs. (10) and (11). This does not affect the conclusions and results of Ref. [20] which remain fully correct. The only consequence is the simplifying reduction of the problem to four pseudohole branches which we denote in general by \( \alpha, \beta \) pseudoholes. The colors \( c \) and \( s \) and quantum numbers \( \beta = \pm \frac{1}{2} \) which label the four pseudohole branches also label the Hamiltonian eigenstates \( I \), as discussed in Sec. II. Following that section, the description of the non-LWS’s and (or) non-HWS’s multiplets generated from the states \( I \) [32] reveals that the pseudoparticles associated with the \( \alpha, \beta \) pseudoholes should simply be denoted as \( \alpha \) pseudoparticles and not as \( \alpha, \beta \) pseudoparticles. Therefore, the \( \alpha \) pseudoparticle operators obey the following anticommuting algebra which does not include the pseudohole quantum number \( \beta \):

\[
\{ b_{q,\alpha}^\dagger, b_{q',\alpha'} \} = \delta_{q,q'}\delta_{\alpha,\alpha'} ,
\]

(A1)

and
\[ \{ b^\dagger_{q,\alpha}, b^\dagger_{q',\alpha'} \} = \{ b_{q,\alpha}, b_{q',\alpha'} \} = 0. \]  

Moreover, note that in the present case of the states I and \((l, l')\) sectors of parameter space the following selection rule is valid: out of the four \(\alpha, \beta\) pseudohole branches only the two branches \(c, \frac{l}{2}\) and \(s, \frac{l'}{2}\) contribute to the generators of Eq. (17). (The states I are constructed by acting these generators onto the pseudohole vacuum.) Since in this case there is in the \(\alpha\) orbital single occupancy by one of the two \(\beta\) pseudohole branches only, one could denote the associate \(\alpha\) pseudoparticles by \(\alpha, \beta\) pseudoparticles, with the fixed \(\beta\) value \(\frac{l}{2}\) or \(\frac{l'}{2}\) for \(c\) or \(s\), respectively. (These \(\alpha, \beta\) pseudoparticles are the holes of the corresponding \(\alpha, \beta\) pseudoholes – however, when there is in the same \(\alpha\) orbital occupation of both \(\alpha, \frac{1}{2}\) and \(\alpha, -\frac{1}{2}\) pseudoholes that notation is not allowed and their holes are to be denoted by \(\alpha\) pseudoparticles.) This together with the fact that our pseudoholes are related to the \(\alpha(l, l')\) pseudoholes of Ref. [20] as \(c, \beta = c(2\beta, 1) = c(2\beta, -1)\) and \(s, \beta = s(1, 2\beta) = s(-1, 2\beta)\) justifies the pseudoparticle notation of Ref. [20] which refers to states I only.

Let us denote by \(\mathcal{H}_I\) the Hilbert subspace spanned by the Hamiltonian eigenstates I. The above selection rule for \(\mathcal{H}_I\) and \((l, l')\) sectors simplifies the \(\beta\) summations which have only contributions from the \(\frac{l}{2}\) (for \(\alpha = c\)) and \(\frac{l'}{2}\) (for \(\alpha = s\)) \(\beta\) values. For instance, the Hamiltonian expression involves in \(\mathcal{H}_I\) the pseudomomentum distribution operator

\[ \hat{N}^h_\alpha(q) = \sum_\beta \hat{N}^h_{\alpha,\beta}(q) = \sum_\beta a^\dagger_{q,\alpha,\beta} a_{q,\alpha,\beta}. \]  

(A3)

It has the same information as the corresponding pseudoparticle operator

\[ \hat{N}_\alpha(q) = 1 - \hat{N}^h_\alpha(q) = b^\dagger_{q,\alpha} b_{q,\alpha}. \]  

(A4)

The \(\mathcal{H}_I\) and \((l, l')\)-sector selection rule allows Eq. (A3) to be simplified to

\[ \hat{N}^h_c(q) = \hat{N}^h_{c,\frac{l}{2}}(q) = a^\dagger_{q,c,\frac{1}{2}} a_{q,c,\frac{1}{2}}; \quad \hat{N}^h_s(q) = \hat{N}^h_{s,\frac{l'}{2}}(q) = a^\dagger_{q,s,\frac{1}{2}} a_{q,s,\frac{1}{2}}. \]  

(A5)

Equations (10), (11), and (16) can be shown to refer to a larger Hilbert space than \(\mathcal{H}_I\), which is spanned both by the states I and all their associate non-LWS’s and non-HWS’s. In the present case of \(\mathcal{H}_I\) these equations can be replaced by
\begin{align}
N_c^* &= N_a, \quad N_s^* = \frac{1}{2} \left[ N_a - 2(|\eta_z| - |S_z|) \right], \quad (A6) \\
N_c &= N_a - 2|\eta_z|, \quad N_s = \frac{1}{2} \left[ N_a - 2(|\eta_z| + |S_z|) \right], \quad (A7)
\end{align}

and
\begin{align}
N_c^h &= 2|\eta_z|, \quad N_s^h = 2|S_z|, \quad (A8)
\end{align}

respectively. The operator \( \hat{N}^h_\alpha \) can be written in terms of the pseudohole number operator \( \hat{N}^h_\alpha(q) \) as follows
\begin{align}
\hat{N}^h_\alpha = \hat{N}^*_\alpha - \hat{N}_\alpha &= \sum_q \hat{N}^h_\alpha(q). \quad (A9)
\end{align}

The pseudohole basis of the \((l, l')\) sectors is constructed from the BA solution precisely as in Ref. [27] for the particular case of the \((-1, -1)\) sector. Two differences are that (i) the \((-1, -1)\) numbers \( N^*_\alpha \) and \( N_\alpha \) of Eqs. (10) of Ref. [27] are here to be replaced by the general expressions (A6) and (A7); and (ii) we use here pseudoholes instead of pseudoparticles.

The operators (A3) commute with each other, i.e. \([\hat{N}^h_\alpha(q), \hat{N}^h_{\alpha'}(q')] = 0\). As in the \((-1, -1)\) case [27], in the pseudohole basis the \((l, l')\)-sector Hamiltonian expression involves the operator (A3) [or (A4)]. Furthermore, the Hamiltonian \textit{commutes} in \( \mathcal{H}_I \) with that operator. This plays a central role in this Hilbert subspace because all the Hamiltonian eigenstates which are LWS’s I or HWS’s I are also eigenstates of \( \hat{N}^h_\alpha(q) \). Let us denote such states I by \(|\eta_z, S_z\rangle\), where \( \eta_z \) and \( S_z \) are the eigenvalues which characterize the canonical ensemble. As for the LWS’s of the \((-1, -1)\) sector, these LWS’s I or (and) HWS’s I obey eigenvalue equations of the form
\begin{align}
\hat{N}^h_\alpha(q)|\eta_z, S_z\rangle = N^h_\alpha(q)|\eta_z, S_z\rangle, \quad (A10)
\end{align}

where \( N^h_\alpha(q) \) represents the eigenvalue of the operator (A3), which is given by 1 and 0 for pseudohole occupied and empty values of \( q \), respectively. The Hamiltonian reads
\begin{align}
\hat{H} = -\sum_q [1 - \hat{N}^h_c(q)] 2t \cos[\hat{K}(q)] + [2\mu - U] \hat{\eta}_z + 2\mu_0 H \hat{S}_z, \quad (A11)
\end{align}
where the expressions of the diagonal generators are given in Eq. (33). This is the exact expression of the Hamiltonian (1) in $\mathcal{H}_x$. At energy scales smaller than the gaps for the non-LWS’s and non-HWS’s multiplets, LWS’s II and HWS’s II, Eq. (A11) gives the exact expression of that Hamiltonian in the full Hilbert space.

Despite its simple appearance, the Hamiltonian (A11) describes a many-pseudohole problem. The reason is that the expression of the rapidity operator $\hat{K}(q)$ in terms of the operator $\hat{N}_h^{\alpha}(q)$ contains many-pseudohole interacting terms. As for the $(-1, -1)$ sector, in all the $(l, l')$ sectors the operator $\hat{K}(q)$ and associate rapidity operator $\hat{S}(q)$ obey the following two equations which are valid for any Hamiltonian eigenstate $I$

$$\hat{K}(q) - \frac{2}{N_a} \sum_{q'} [1 - \hat{N}_s^{h}(q')] \tan^{-1}\left(\hat{S}(q') - (4t/U) \sin[\hat{K}(q')]\right) |\eta_z, S_z\rangle = q|\eta_z, S_z\rangle$$  \hspace{1cm} (A12)

and

$$\frac{2}{N_a} \sum_{q'} [1 - \hat{N}_c^{h}(q')] \tan^{-1}\left(\hat{S}(q) - (4t/U) \sin[\hat{K}(q')]\right) - \sum_{q'} [1 - \hat{N}_s^{h}(q')] \tan^{-1}\left(\frac{1}{2} \left(\hat{S}(q') - \hat{S}(q')\right)\right) |\eta_z, S_z\rangle = q|\eta_z, S_z\rangle.$$  \hspace{1cm} (A13)

These equations fully define the rapidity operators in terms of the pseudomomentum distribution operator (A3). We note that the limits of the pseudo-Brillouin zones of Eqs. (A12) and (A13) pseudomomentum summations are given in Eqs. (22)−(24). These limits involve the numbers $N^{*}_a$ whose general expressions for the four $(l, l')$ sectors are given by Eqs. (10) and (A6). Otherwise the general Eqs. (A12) and (A13) have the same form as Eqs. (31) and (32) of Ref. [27] for the $(-1, -1)$ sector.

The normal-ordered Hamiltonian relatively to the suitable ground state of form (17) reads

$$\hat{H} := \sum_{i=1}^{\infty} \hat{H}^{(i)}$$  \hspace{1cm} (A14)

where to second pseudohole scattering order

$$\hat{H}^{(1)} = - \sum_{q, \alpha} \epsilon^0_{\alpha}(q) : \hat{N}_a^{h}(q) : + [2\mu - U] \sum_{q, \beta} \beta : \hat{N}_c^{h}(q) : + 2\mu_0 H \sum_{\beta} \beta : \hat{N}_s^{h}(q) :$$

$$\hat{H}^{(2)} = \frac{1}{N_a} \sum_{q, \alpha} \sum_{q', \alpha'} \frac{1}{2} f_{\alpha \alpha'}(q, q') : \hat{N}_a^{h}(q) : \hat{N}_a^{h}(q') :$$  \hspace{1cm} (A15)
Note that: $\hat{N}^h_\alpha (q) := \sum \beta : \hat{N}^h_{\alpha, \beta}(q) :$. In the present case of states $I$ and $(l, l')$ sectors of parameter space the $\beta$ selection rule has allowed us to write Eq. (A5) and also

$$
\sum \beta : \hat{N}^h_{c, \beta}(q) := \frac{l}{2} : \hat{N}^h_{c, l} :; \quad \sum \beta : \hat{N}^h_{s, \beta}(q) := \frac{l'}{2} : \hat{N}^h_{s, l'} : . \quad (A16)
$$

Equation (A15) includes the Hamiltonian terms which are relevant at low energy \cite{20, 21, 28}. Furthermore, it is shown in Ref. \cite{28} [for the $(-1, -1)$ sector] that at low energy and small momentum the only relevant term is the non-interacting term $\hat{H}^{(1)}$. This property justifies to the Landau-liquid character of the Hamiltonian (1) and plays a key role in the symmetries of the critical point.

The expressions of the bands are

$$
\epsilon^0_c(q) = -2t \cos K^{(0)}(q) + 2t \int_{Q^{(-)}}^{Q^{(+)}} dk \tilde{\Phi}_{cc} \left( k, K^{(0)}(q) \right) \sin k , \quad (A17)
$$

and

$$
\epsilon^0_s(q) = 2t \int_{Q^{(-)}}^{Q^{(+)}} dk \tilde{\Phi}_{cs} \left( k, S^{(0)}(q) \right) \sin k , \quad (A18)
$$

respectively. Here

$$
Q^{(\pm)} = K^{(0)}(q_{Fc}^{(\pm)}) , \quad (A19)
$$

$K^{(0)}(q)$ and $S^{(0)}(q)$ are the solutions of Eqs. (A12) and (A13) for the particular case of the suitable ground state (17), and the phase shifts $\tilde{\Phi}_{\alpha \alpha'}$ are given by

$$
\tilde{\Phi}_{cc}(k, k') = \tilde{\Phi}_{cc} \left( \frac{\sin k}{u}, \frac{\sin k'}{u} \right) ; \quad \tilde{\Phi}_{cs}(k, v') = \tilde{\Phi}_{cs} \left( \frac{\sin k}{u}, v' \right) , \quad (A20)
$$

$$
\tilde{\Phi}_{sc}(v, k') = \tilde{\Phi}_{sc} \left( v, \frac{\sin k'}{u} \right) ; \quad \tilde{\Phi}_{ss}(v, v') = \tilde{\Phi}_{ss} \left( v, v' \right) , \quad (A21)
$$

where the phase shifts $\Phi_{\alpha \alpha'}$ are defined by the following integral equations

$$
\tilde{\Phi}_{cc}(x, x') = \frac{1}{\pi} \int_{B_u^{(+)}} dy'' \frac{\Phi_{sc}(y'', x')}{1 + (x - y'')^2} , \quad (A22)
$$

$$
\tilde{\Phi}_{cs}(x, y') = -\frac{1}{\pi} \tan^{-1}(x - y') + \frac{1}{\pi} \int_{B_u^{(+)}} dy'' \frac{\Phi_{ss}(y'', y)}{1 + (x - y'')^2} , \quad (A23)
$$

35
\[ \Phi_{sc}(y, x') = -\frac{1}{\pi} \tan^{-1}(y - x') + \int_{\bar{\mu}(+)} \frac{1}{u} \, dy'' G(y, y'') \Phi_{sc}(y'', x') , \quad (A24) \]

\[ \Phi_{ss}(y, y') = \frac{1}{\pi} \tan^{-1}\left(\frac{y - y'}{2}\right) - \frac{1}{\pi^2} \int_{\sin Q(+)} \sin Q(-) \, dx'' \tan^{-1}\left(\frac{x'' - y'}{1 + (y - x'')^2}\right) \]

\[ + \int_{\bar{\mu}(-)} \frac{1}{u} \, dy'' G(y, y'') \Phi_{ss}(y'', y') . \quad (A25) \]

Here

\[ B^{(\pm)} / u = S^{(0)}(q_{F_s}^{(\pm)}) , \quad (A26) \]

and the kernel \( G(y, y') \) reads \[24\]

\[ G(y, y') = -\frac{1}{2\pi} \left[ \frac{1}{1 + ((y - y')/2)^2} \right] \left[ 1 - \frac{1}{2} \left( t(y) + t(y') + \frac{l(y) - l(y')}{y - y'} \right) \right] , \quad (A27) \]

where

\[ t(y) = \frac{1}{\pi} \left[ \tan^{-1}(y + \frac{\sin Q(+)}{u}) - \tan^{-1}(y + \frac{\sin Q(-)}{u}) \right] , \quad (A28) \]

and

\[ l(y) = \frac{1}{\pi} \left[ \ln(1 + (y + \frac{\sin Q(+)}{u})^2) - \ln(1 + (y + \frac{\sin Q(-)}{u})^2) \right] . \quad (A29) \]

The “Landau” \( f \) function, \( f_{\alpha\alpha'}(q, q') \), has universal form in terms of the two-pseudohole phase shifts \( \Phi_{\alpha\alpha'}(q, q') \) defined below and reads

\[ f_{\alpha\alpha'}(q, q') = 2\pi v_{\alpha}(q) \Phi_{\alpha\alpha'}(q, q') + 2\pi v_{\alpha'}(q') \Phi_{\alpha'\alpha}(q', q) \]

\[ + \sum_{j=\pm} \sum_{\alpha'\gamma = c,s} 2\pi v_{\alpha\alpha'} \Phi_{\alpha'\gamma}(q) \Phi_{\alpha'\gamma}(q') . \quad (A30) \]

The two-pseudohole phase shifts can be defined in terms of the phase shifts \( \bar{\Phi}_{\alpha\alpha'} \) as follows

\[ \Phi_{cc}(q, q') = \bar{\Phi}_{cc} \left( \frac{\sin K^{(0)}(q)}{u}, \frac{\sin K^{(0)}(q')}{u} \right) , \quad (A31) \]

\[ \Phi_{cs}(q, q') = \bar{\Phi}_{cs} \left( \frac{\sin K^{(0)}(q)}{u}, S^{(0)}(q') \right) , \quad (A32) \]
\[ \Phi_{sc}(q, q') = \Phi_{sc} \left( S^{(0)}(q), \frac{\sin K^{(0)}(q')}{u} \right) , \]  
\[ \Phi_{ss}(q, q') = \Phi_{ss} \left( S^{(0)}(q), S^{(0)}(q') \right) . \]  

Finally, the pseudohole group velocity appearing in the \( f \) function expression (A30) is given by

\[ v_\alpha(q) = \frac{d\epsilon_\alpha^0(q)}{dq} . \]  

In particular, the velocity

\[ v_\alpha \equiv v_\alpha(q_{F\alpha}) , \]  

plays a determining role at the critical point, representing the “light” velocities which appear in the conformal-invariant expressions [7,34].

We emphasize that the phase shifts expressions are the same as for the \((-1, -1)\) sector (see Ref. [24]), except that the present limits of pseudo-Brillouin zones and pseudo-Fermi points involve in the present general case the numbers (10)-(A6) and (11)-(A7), respectively [see Eqs. (18) – (24)].

Note that although the expressions for the bands (34) and (35), rapidity (36), \( f \) function (38), velocity (39), phase shifts (A18)-(A27) and (A39)-(A42) of Ref. [20] are absolutely correct they are \( l \)- and \( l' \)-independent: these expressions can be shown to depend only on \( \eta \) and \( S \) and not on the signs of \( \eta_z \) and \( S_z \) which for the states I determine the values of the numbers \( l \) and \( l' \) [see Eq. (9)]. Therefore, the corresponding pseudohole quantities presented in this Appendix have a simpler form than those of Ref. [20].
In this Appendix we follow the \((-1, -1)\)-sector study of Ref. [31] and present a short discussion of the electron - quasiparticle transformation which relates the electron operator \(c_{k_F,\sigma}^\dagger\) to the quasiparticle operator \(\tilde{c}_{k_F,\sigma}^\dagger\) in the limit of vanishing excitation energy. The latter operator is defined by Eq. (34). To leading order in that energy there is a singular transformation between these two operators which as for the \((-1, -1)\) sector [31] reads

\[
\tilde{c}_{\pm k_F,\sigma}^\dagger = \frac{1}{\sqrt{Z_\sigma}} c_{\pm k_F,\sigma}^\dagger,
\]

where the one-electron renormalization factor is given by 
\(Z_\sigma = \lim_{\omega \to 0} Z_\sigma(\omega)\) and \(Z_\sigma(\omega)\) is the small-\(\omega\) leading-order term of \(|\varsigma_\sigma|1 - \frac{\partial \text{Re} \Sigma(\pm k_F,\omega)}{\partial \omega}|^{-1}\). Here \(\Sigma(k,\omega)\) is the \(\sigma\) self energy. We emphasize that Eq. (B1) does not apply to the case when the starting state is the \(SO(4)\) or a \(SU(2) \otimes U(1)\) half-filling ground state. In this case a similar expression holds true where \(\omega\) is replaced by \(\omega - \Delta_c\). Here \(\Delta_c\) is the half-filling Mott-Hubbard gap [11,22]. In Ref. [31] it was found that in the Hilbert subspace spanned by the states I that self energy is given by

\[
\text{Re} \Sigma(\pm k_F,\omega) = \omega \left[1 - \frac{\omega^{-1-\varsigma_\sigma}}{c_0^\sigma + \sum_{j=1,2,3,...} c_j^\sigma \omega^j}\right],
\]

where \(c_j^\sigma\) with \(j = 0,1,2,...\) are constants and \(\varsigma_\sigma\) is a non-classical interaction dependent exponent such that \(-1 < \varsigma_\sigma < -1/2\). It follows that the function \(Z_\sigma(\omega)\) is of the form

\[
Z_\sigma(\omega) = c_0^\sigma \omega^{1+\varsigma_\sigma},
\]

and vanishes in the limit of \(\omega \to 0\). Thus, \(Z_\sigma = 0\). Although expression (B1) is very similar to the corresponding expression for a Fermi liquid, in the present one-dimensional many-electron problem there is no overlap between the quasiparticle and the electron, in contrast to a Fermi liquid.

The singular electron – quasiparticle transformation (B1) maps a non-perturbative electronic quantum problem in a perturbative quasiparticle problem, the factor \(\frac{1}{\sqrt{Z_\sigma}}\) being absorbed by that transformation. It maps a vanishing-spectral-weight electronic problem onto
a finite-spectral-weight quasiparticle problem. Following Eq. (34), the quasiparticle operator \( \tilde{c}^{\dagger}_{\pm k_F,\sigma} \) is the generator which transforms the ground state \( |0; N_\sigma, N_{-\sigma}\rangle \) onto \( |0; N_\sigma + 1, N_{-\sigma}\rangle \).

Apart from the factor \( \frac{1}{\sqrt{Z_\sigma}} \) absorbed by the transformation it is a \( \sigma \) electron of momentum \( \pm k_{F\sigma} \). Therefore, in Sec. III we refer often the quasiparticle by electron.

Similar results hold for the hole and corresponding quasihole which are related by the singular transformation

\[
\tilde{c}_{\pm k_F,\sigma} = \frac{1}{\sqrt{Z_\sigma}} c_{\pm k_F,\sigma},
\]

where \( Z_\sigma \) is the same as in Eq. (B1). As for the quasiparticle and the electron, in Sec. III we refer often the quasihole by hole. In that section we evaluate the quasiparticle and quasihole expressions in terms of pseudoholes and topological momentons for all nine sectors of parameter space.
REFERENCES

1 D. Pines and P. Nozières, in *The Theory of Quantum Liquids*, (Addison-Wesley, Redwood City, 1966 and 1989), Vol. I.

2 Gordon Baym and Christopher J. Pethick, in *Landau Fermi-Liquid Theory Concepts and Applications*, (John Wiley & Sons, New York, 1991).

3 J. Sólyom, Adv. Phys. 28, 201 (1979).

4 V. Meden and K. Schönhammer, Phys. Rev. B 46, 15753 (1992); J. Voit, Phys. Rev. B 47, 6740 (1993).

5 Walter Metzner and Carlo Di Castro, Phys. Rev. B 47, 16107 (1993).

6 P. W. Anderson and Y. Ren, in *High Temperature Superconductivity*, edited by K. S. Bedell, D. E. Meltzer, D. Pines, and J. R. Schrieffer (Addison-Wesley, Reading, MA, 1990).

7 J. M. P. Carmelo, A. H. Castro Neto, and D. K. Campbell, Phys. Rev. Lett. 73, 926 (1994).

8 H. A. Bethe, Z. Phys. 71, 205 (1931).

9 For one of the first generalizations of the Bethe ansatz to multicomponent systems see C. N. Yang, Phys. Rev. Lett. 19, 1312 (1967).

10 V. E. Korepin, N. M. Bogoliubov, and A. G. Izergin, *Quantum Inverse Scattering Method and Correlation Functions* (Cambridge University Press, 1993).

11 Elliott H. Lieb and F. Y. Wu, Phys. Rev. Lett. 20, 1445 (1968).

12 Fabian H. L. Essler and Vladimir E. Korepin, Phys. Rev. Lett. 72, 908 (1994); *ibid.* Nucl. Phys. B 426, (1994).

13 L. D. Faddeev and L. A. Takhtajan, Phys. Lett. 85A, 375 (1981).
14 O. J. Heilmann and E. H. Lieb, Ann. N. Y. Acad. Sci. **172**, 583 (1971).

15 E. H. Lieb, Phys. Rev. Lett. **62**, 1201 (1989).

16 C. N. Yang, Phys. Rev. Lett. **63**, 2144 (1989).

17 C. N. Yang and S. C. Zhang, Mod. Phys. Lett. B **4**, 759 (1990).

18 Fabian H. L. Essler, Vladimir E. Korepin, and Kareljan Schoutens, Phys. Rev. Lett. **67**, 3848 (1991); Nucl. Phys. B **372**, 559 (1992).

19 Stellan Östlund, Phys. Rev. Lett. **69**, 1695 (1992).

20 J. M. P. Carmelo and N. M. R. Peres, Phys. Rev. B **51**, 7481 (1995).

21 J. Carmelo and A. A. Ovchinnikov, Cargèse Lecture 1990 (unpublished); J. Phys.: Condens. Matter **3**, 757 (1991).

22 J. Carmelo, P. Horsch, P.-A. Bares, and A. A. Ovchinnikov, Phys. Rev. B **44**, 9967 (1991).

23 J. M. P. Carmelo, P. Horsch, and A. A. Ovchinnikov, Phys. Rev. B **45**, 7899 (1992).

24 J. M. P. Carmelo and P. Horsch, Phys. Rev. Lett. **68**, 871 (1992); J. M. P. Carmelo, P. Horsch, and A. A. Ovchinnikov, Phys. Rev. B **46**, 14 728 (1992).

25 J. M. P. Carmelo, P. Horsch, D. K. Campbell, and A. H. Castro Neto, Phys. Rev. B **48**, 4200 (1993).

26 J. M. P. Carmelo and A. H. Castro Neto, Phys. Rev. Lett. **70**, 1904 (1993).

27 J. M. P. Carmelo, A. H. Castro Neto, and D. K. Campbell, Phys. Rev. B **50**, 3667 (1994).

28 J. M. P. Carmelo, A. H. Castro Neto, and D. K. Campbell, Phys. Rev. B **50**, 3683 (1994).

29 F. D. M. Haldane, Phys. Rev. Lett. **66**, 1529 (1991).

30 E. R. Mucciolo, B. Shastry, B. D. Simons, and B. L. Altshuler, Phys. Rev. B **49**, 15 197 (1994).
31 J. M. P. Carmelo, A. H. Castro Neto, and N. M. R. Peres, preprint (1995); *ibid.* unpublished.

32 J. M. P. Carmelo, N. M. R. Peres, and D. K. Campbell, preprint (1995).

33 A. H. Castro Neto, H. Q. Lin, Y. -H. Chen, and J. M. P. Carmelo, Phys. Rev. B 50 14032 (1994).

34 Holger Frahm and V. E. Korepin, Phys. Rev. B 42, 10 553 (1990); *ibid.* 43, 5653 (1991).

35 J. M. P. Carmelo and N. M. R. Peres, to appear in 1996.
TABLE I – Values of the ground-state momentum in the four \((l, l')\) sectors of Hamiltonian symmetry \(U(1) \otimes U(1)\). The different momentum values correspond to the following parities of the numbers \(N_s^*\) and \(N_s\) of Eqs. (10) and (11), respectively: (A) even, even; (B) even, odd; (C) odd, even; and (D) odd, odd. In the case of the \((1, \pm 1)\) sectors, if \(k_{F\sigma} > \pi\) then \(\pm k_{F\sigma}\) should be replaced by the first-Brillouin-zone momenta \(\pm [2\pi - k_{F\sigma}]\).

| Sector | \((-1, -1)\) | \((-1, 1)\) | \((1, -1)\) | \((1, 1)\) |
|--------|---------------|---------------|---------------|---------------|
| (A) \(P\) | \(\pm 2k_F\) | \(\pm 2k_F\) | \(\pm [2\pi - 2k_F]\) | \(\pm [2\pi - 2k_F]\) |
| (B) \(P\) | \(\pm k_{F\uparrow}\) | \(\pm k_{F\uparrow}\) | \(\pm k_{F\uparrow}\) | \(\pm k_{F\uparrow}\) |
| (C) \(P\) | \(\pm k_{F\downarrow}\) | \(\pm k_{F\downarrow}\) | \(\pm k_{F\downarrow}\) | \(\pm k_{F\downarrow}\) |
| (D) \(P\) | 0 | 0 | 0 | 0 |

TABLE II – Changes in the numbers of pseudoholes, pseudoparticles, pseudoparticle orbitals and of the values of \(\eta, \eta_z, S,\) and \(S_z\) in the ground-state – ground-state transition \((N_{\uparrow}, N_{\downarrow}) \rightarrow (N_{\uparrow} \pm 1, N_{\downarrow})\).

| Increment | \((-1, -1)\) | \((-1, 1)\) | \((1, -1)\) | \((1, 1)\) |
|-----------|---------------|---------------|---------------|---------------|
| \(\Delta N^h_c\) | \(\mp 1\) | \(\mp 1\) | \(\pm 1\) | \(\pm 1\) |
| \(\Delta N_c\) | \(\pm 1\) | \(\pm 1\) | \(\mp 1\) | \(\mp 1\) |
| \(\Delta N^*_c\) | 0 | 0 | 0 | 0 |
| \(\Delta N^h_s\) | \(\pm 1\) | \(\mp 1\) | \(\pm 1\) | \(\mp 1\) |
| \(\Delta N_s\) | 0 | \(\pm 1\) | \(\mp 1\) | 0 |
| \(\Delta N^*_s\) | \(\pm 1\) | 0 | 0 | \(\mp 1\) |
| \(\Delta \eta\) | \(\mp 1/2\) | \(\mp 1/2\) | \(\pm 1/2\) | \(\pm 1/2\) |
| \(\Delta \eta_z\) | \(\pm 1/2\) | \(\pm 1/2\) | \(\pm 1/2\) | \(\pm 1/2\) |
| \(\Delta S\) | \(\pm 1/2\) | \(\mp 1/2\) | \(\pm 1/2\) | \(\mp 1/2\) |
| \(\Delta S_z\) | \(\mp 1/2\) | \(\mp 1/2\) | \(\mp 1/2\) | \(\mp 1/2\) |
|       | $(-1, -1)$ | $(-1, 1)$ | $(1, -1)$ | $(1, 1)$ |
|-------|------------|------------|------------|------------|
| $\Delta N^h_c$ | $\mp 1$ | $\mp 1$ | $\pm 1$ | $\pm 1$ |
| $\Delta N_c$ | $\pm 1$ | $\pm 1$ | $\mp 1$ | $\mp 1$ |
| $\Delta N^*_{c}$ | $0$ | $0$ | $0$ | $0$ |
| $\Delta N^h_s$ | $\mp 1$ | $\pm 1$ | $\mp 1$ | $\pm 1$ |
| $\Delta N_s$ | $\pm 1$ | $0$ | $0$ | $\mp 1$ |
| $\Delta N^*_{s}$ | $0$ | $\pm 1$ | $\mp 1$ | $0$ |
| $\Delta \eta$ | $\mp \pm 1/2$ | $\mp \pm 1/2$ | $\pm \pm 1/2$ | $\pm \pm 1/2$ |
| $\Delta \eta_z$ | $\pm \pm 1/2$ | $\pm \pm 1/2$ | $\pm \pm 1/2$ | $\pm \pm 1/2$ |
| $\Delta S$ | $\mp \pm 1/2$ | $\pm \pm 1/2$ | $\mp \pm 1/2$ | $\pm \pm 1/2$ |
| $\Delta S_z$ | $\pm \pm 1/2$ | $\pm \pm 1/2$ | $\pm \pm 1/2$ | $\pm \pm 1/2$ |

**TABLE III** – Changes in the numbers of pseudoholes, pseudoparticles, pseudoparticle orbitals and in the values of $\eta$, $\eta_z$, $S$, and $S_z$ in the ground-state – ground-state transition $(N^\uparrow, N^\downarrow) \rightarrow (N^\uparrow, N^\downarrow \pm 1)$.  

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