On the classification of non-integrable complex distributions

Toshikazu Ito & Bruno Scárdua

1 Introduction

Given a point \( p \in \mathbb{C}^n \) and a real number \( r > 0 \) we denote by \( B^{2n}(p, r) \) the open ball of radius \( r \) centered at \( p \) in \( \mathbb{C}^n \). The corresponding closed ball is denoted by \( B^{2n}[p, r] \) and its boundary sphere by \( S^{2n-1}(p, r) = \partial B^{2n}[p, r] \). We also write \( B^{2n}(1) = B^{2n}(0, 1) \), \( B^{2n}[1] = B^{2n}[0, 1] \) and \( S^{2n-1}(1) = \partial B^{2n}[1] \). Let \( \Omega \) be a germ of holomorphic one-form with an isolated singularity at the origin \( 0 \in \mathbb{C}^n \), \( n \geq 3 \). We address the problem of analytical classification of \( \Omega \) in the non-integrable case. Motivated by the geometrical-analytical classification of singularities in dimension 2 we consider the case where the kernel of \( \Omega \) generates a germ of distribution \( \text{Ker}(\Omega) \) transverse to small spheres \( S^{2n-1}(0, \varepsilon) \). This is one, though not the only, central motivation for this work. The problem of existence of integral manifolds for germs of singularities of integrable one-forms is an ancient problem already considered in the work of Briot-Bouquet. The existence results in [3] (for dimension \( n = 2 \)) and in [4] (for the non-dicritical case in dimension \( n = 3 \)) motivate the very basic question below:

**Question 1.** Is there a non-integrable germ of holomorphic one-form \( \Omega \) with an isolated singularity at the origin \( 0 \in \mathbb{C}^n \) such that \( \text{Ker}(\Omega) \) is transverse to the spheres \( S^{2n-1}(0, \varepsilon) \), for \( \varepsilon > 0 \) small enough and \( \text{Ker}(\Omega) \) admits no integral manifold through the origin?

Theorem 2 gives a positive answer to this question. Other motivations are related to our previous work in [9] and [10] where we study the obstructions to the integrability of \( \Omega \). Our first main result reads as:

**Theorem 1.** Let \( n \geq 3 \) and \( \Omega \) be a holomorphic one-form defined in a neighborhood of \( B^{2n}[1] \) and such that \( \text{sing}(\Omega) \cap S^{2n-1}(1) = \emptyset \). If there exists a holomorphic vector field \( \xi \) in a neighborhood of \( B^{2n}[1] \), transverse to \( S^{2n-1}(1) \), and such that \( \Omega \cdot \xi = 0 \), then \( \Omega \) is not integrable.

Let us give examples of distributions as in Theorem 1. Denote by \( A(2m) \) the set of all \( 2m \times 2m \) skew-symmetric complex matrices and by \( \Lambda(2m) \) the subset of nonsingular elements in \( A(2m) \). In [10] it is observed that if \( A = (a_{ij})_{i,j=1}^{2m} \) belongs to \( \Lambda(2m) \) then, for \( m \geq 2 \), the one-form \( \Omega_A = \sum_{i,j=1}^{2m} a_{ij} z_i \, dz_j \) defines a non-integrable holomorphic (linear) distribution transverse to the spheres \( S^{4m-1}(0, r) \subset \mathbb{C}^{2m} \), \( r > 0 \). Such a one-form will be called linear. A particular case is the one-form \( \Omega_{\mathcal{J}(2m)} = \sum_{j=1}^{m} (z_{2j-1} \, dz_{2j} - z_{2j} \, dz_{2j-1}) \). One may ask for non-linear examples. Given \( \ell = (\ell_1, \ldots, \ell_m) \in \mathbb{N}^m \), we introduce the corresponding non-integrable Poincaré-Dulac normal form
as $\Omega(\ell) = \sum_{j=1}^{m} \left[ z_{2j-1} \, dz_{2j} - (\ell_j z_{2j} + z_{2j-1}) \, dz_{2j-1} \right]$ in coordinates $(z_1, z_2, \ldots, z_{2m}) \in \mathbb{C}^{2m}$. We prove that also $\Omega(\ell)$ is not integrable for $m \geq 2$, singular only at the origin, and $\text{Ker}(\Omega(\ell))$ is transverse to $S^{4m-1}(0, r)$, $\forall r > 0$ small enough (see Example 11). The one-forms $\Omega_A, \Omega_{\mathcal{J}(2m)}$ and $\Omega(\ell)$ are our basic models in the classification we pursue (see §4).

Let $\text{Ker}(\Omega)$ be a codimension one holomorphic distribution on a complex manifold $V^n$. Let $p \in V^n$ be a singularity of $\text{Ker}(\Omega)$, that is, of $\Omega$. A germ of codimension one analytic subset $\Lambda_p$ at $p$ is an integral manifold of $\text{Ker}(\Omega)$ through $p$ if any vector $v_q \in T_q(V)$ which is tangent to $\Lambda_p$ at a point $q$ belongs to $\text{Ker}(\Omega)(q)$. This means that if $\Lambda$ is any representative of $\Lambda_p$ in a neighborhood $U$ of $p$ in $V^n$ and $\Lambda^*$ denotes the smooth part of $\Lambda$ then the tangent bundle $T \Lambda^*$ is a sub-bundle of $\text{Ker}(\Omega)|_{\Lambda^*}$ (see Definition 11). We shall always assume $\Lambda$ and $\Lambda_p$ to be irreducible, nevertheless we do not require that $\Lambda \cap \Lambda^* = \text{sing}(\Lambda)$ is contained in $\text{sing}(\text{Ker}(\Omega))$. Regarding the existence of integral manifolds for non-integrable distributions we have:

**Theorem 2.** Let $m \geq 2$. Given $A \in \mathbb{A}_{2m}$ and $\ell \in \mathbb{N}^m$ the distributions $\text{Ker}(\Omega_{\mathcal{J}(2m)}), \text{Ker}(\Omega_A)$ and $\text{Ker}(\Omega(\ell))$ admit no integral manifold through the origin.

In the course of the proof of Theorem 2 we obtain the following Darboux’s theorem type for (not necessarily integrable) polynomial distributions. This is actually a non-integrable version of the more precise Theorem 3.3 on page 102 of [11]:

**Proposition 1.** Let $\Omega$ be a (not necessarily integrable) polynomial one-form on $\mathbb{C}^n$, $n \geq 2$ and assume that $\text{cod} \text{sing}(\Omega) \geq 2$. If $\text{Ker}(\Omega)$ has infinitely many algebraic invariant hypersurfaces then $\Omega$ is integrable. Indeed $\Omega = P \, dQ - Q \, dP$ for some polynomials $P, Q$ with no common factors and, in particular, the leaves of the foliation $\mathcal{F}_\Omega$ defined by $\Omega$ are contained in the algebraic subvarieties $\{ \lambda P - \mu Q = 0 \}$ where $(\lambda, \mu) \in \mathbb{C}^2 - \{(0,0)\}$.

As already mentioned, the examples $\Omega_A$, $\Omega_{\mathcal{J}(2m)}$ and $\Omega(\ell)$ above constructed motivate the problem of analytical classification of germs of non-integrable one-forms defining distributions transverse to small spheres (see Questions 2 and 3 in [11]). In this direction we prove:

**Theorem 3.** Let $\Omega$ be a holomorphic one-form in a neighborhood $U$ of the closed ball $B^{4m}[1] \subset \mathbb{C}^{2m}$ and such that (1) $\Omega \cdot \vec{R} = 0$, where $\vec{R}$ is the radial vector field in $\mathbb{C}^{2m}$ and (2) $\text{sing}(\Omega) \cap S^{4m-1}(1) = \emptyset$. Then $\text{Ker}(\Omega)$ is homotopic to the linear distribution $\text{Ker}(\Omega_{\mathcal{J}(2m)})$ by distributions $\text{Ker}(\Omega_s)$, $0 \leq s \leq 1$, such that $\Omega_0 = \Omega$ and $\Omega_1 = \Omega_{\mathcal{J}(2m)}$, where $\Omega_s$ is holomorphic and satisfies (1) and (2) above.

**Acknowledgement.** We want to thank the anonymous referee for many valuable remarks, suggestions and for pointing out gaps and unclear points in the first version. Specially for suggesting clearer arguments and in particular for a drastic simplification in the proof of Theorem 2. Also mainly due to the referee is the discussion in Remark 11.

2 Invariant manifolds

In this section we discuss Question 11 and prove Theorem 2 and Proposition 11. First we prove the examples mentioned in the introduction.
Example 1. Let $A = (a_{ij}) \in \mathbb{A}(2m)$ and $\Omega_A := \sum_{i,j=1}^{2m} a_{ij} z_i z_j$ the corresponding linear one-form in $\mathbb{C}^{2m}$. Then $\text{sing}(\Omega_A) = \{0\} \subset \mathbb{C}^{2m}$ and $\Omega_A \cdot \vec{R} = 0$ for the radial vector field $\vec{R} = \sum_{j=1}^{2m} z_j \frac{\partial}{\partial z_j}$. This implies that $\text{Ker}(\Omega_A)$ is transverse to every sphere $S^{4m-1}(0, r), r > 0$. The non-integrability of $\Omega_A$ and $\Omega_\ell$ is a straightforward computation (cf. [9]). For the transversality of the non-integrable Poincaré-Dulac normal form $\Omega_\ell$ with small spheres $S^{4m-1}(0, \varepsilon)$ we observe that if $\xi(\ell) = \sum_{j=1}^{m} [(\ell_j z_j + z_{2j-1}) \frac{\partial}{\partial z_j} + z_{2j-1} \frac{\partial}{\partial z_{2j-1}}]$ then $\Omega_\ell \cdot \xi(\ell) = 0$ and, as it is well-known, the vector field $\xi(\ell)$ is transverse to the spheres $S^{4m-1}(0, \varepsilon)$ if $\varepsilon > 0$ is small enough.

Now we shall prove Theorem 2. Let us first precise some notions involved.

Definition 1. Let $\Omega$ be a holomorphic one-form on a complex manifold $V$. A smooth complex immersed submanifold $\Lambda^* \subset V$ is an integral manifold of $\Omega$ if $T\Lambda^* \subset \text{Ker}(\Omega)|_{\Lambda^*}$. In other words, any tangent vector to $\Lambda^*$ belongs to $\text{Ker}(\Omega)$. If $\Lambda \subset V$ is a possibly singular complex analytic submanifold we say that $\Lambda$ is an integral manifold of $\Omega$ if its regular part $\Lambda^* = \Lambda \setminus \text{sing}(\Lambda)$ is an integral manifold of $\Omega$.

The following lemma is found in the algebraic setting in [11], Section 3.1, page 99.

Lemma 1. Let $\text{Ker}(\Omega)$ be given by a holomorphic one-form $\Omega$ in $V$ with $\text{codim sing}(\Omega) \geq 2$ and let $\Lambda \subset V$ be a codimension one analytic subset given by a reduced equation $\Lambda : \{f = 0\}$ for some holomorphic $f : V \to \mathbb{C}$. The following conditions are equivalent:

1. $\Lambda$ is $\text{Ker}(\Omega)$-invariant.
2. $\Omega \wedge \frac{df}{f}$ is a holomorphic 2-form on $V$.

Proof. Since all objects involved are analytic we may consider the local case also at a generic (and therefore non-singular) point $p \in \Lambda^*$. In suitable local coordinates $(z_1, \ldots, z_n) = (z_1, \ldots, z_{n-1}, f)$ we have $p = 0$ and $\Lambda$ given by $\{f = z_n = 0\}$. Also we may write $\Omega = \sum_{j=1}^{n-1} a_j \, dz_j$. Suppose $\Lambda$ is $\text{Ker}(\Omega)$ invariant. Then, since $\Lambda$ is given by $\{z_n = 0\}$ we have $\Omega \cdot \left. \frac{\partial}{\partial z_j} \right|_{\{z_n = 0\}} = 0$ for all $j \in \{1, \ldots, n-1\}$. In other words $z_n$ divides $a_j$ in $\mathbb{C}\{z_1, \ldots, z_n\}$ for every $j \in \{1, \ldots, n-1\}$ and therefore $\Omega = \sum_{j=1}^{n-1} z_n \bar{a}_j \, dz_j + a_n \, dz_n$ for some holomorphic $\bar{a}_1, \ldots, \bar{a}_{n-1} \in \mathbb{C}\{z_1, \ldots, z_n\}$. Thus $\Omega \wedge \frac{dz_n}{z_n} = \sum_{j=1}^{n-1} \bar{a}_j \, dz_j \wedge dz_n$ is holomorphic. Conversely, if $\Omega \wedge \frac{dz_n}{z_n}$ is holomorphic then $\Omega = \sum_{j=1}^{n-1} z_n \bar{a}_j \, dz_j + a_n \, dz_n$ as above therefore $\Omega \cdot \left. \frac{\partial}{\partial z_j} \right|_{\{z_n = 0\}}$ vanishes on $\{z_n = 0\}$ for every $j = 1, \ldots, n-1$. Since $\{\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_{n-1}}\}$ generate $T\Lambda$ in a neighborhood of 0 we obtain that $T\Lambda \subset \text{Ker}(\Omega)$ in a neighborhood of $p$. Therefore $\Lambda$ is $\text{Ker}(\Omega)$-invariant. 

Proof of Proposition 1. Once we have Lemma 1 the proof is essentially the same given in [11], thus we will omit it and refer to [11].
2.1 Proof of Theorem 2

Let $\Omega_2^{(2m)} = \sum_{j=1}^{m} (z_{2j-1} dz_{2j} - z_{2j} dz_{2j-1})$ be given in $\mathbb{C}^{2m}$. Then $d\Omega_2^{(2m)} = 2 \sum_{j=1}^{m} dz_{2j-1} \wedge dz_{2j}$ is a non-degenerate 2-form on $\mathbb{C}^{2m}$ and $(d\Omega_2^{(2m)})^m$ is a non-zero 2m-form. Let $L$ be a complex manifold and $\zeta: L \to \mathbb{C}^{2m}$ a smooth embedding.

**Lemma 2.** The following statements are equivalent:

1. $T(\zeta(L)) \subset \text{Ker}(\Omega_2^{(2m)})|_{\zeta(L)}$.
2. $\zeta^*(\Omega_2^{(2m)}) = 0$.

**Proof.** To prove that (1) implies (2) we take any vector $v \in T(L)$, then $\zeta_*(v) \in T(\zeta(L)) \subset \text{Ker}(\Omega_2^{(2m)})|_{\zeta(L)}$ means $\Omega_2^{(2m)}(\zeta_*(v)) = 0$. Therefore we get $\zeta^*(\Omega_2^{(2m)})(v) = \Omega_2^{(2m)}(\zeta_*(v)) = 0$. To prove that (2) implies (1) take any vector $\tilde{v} = \zeta_*v \in T(\zeta(L)), v \in T(L)$. Then $\zeta^*(\Omega_2^{(2m)})(\tilde{v}) = \Omega_2^{(2m)}(\zeta_*(v)) = \zeta^*(\Omega_2^{(2m)})(v) = 0$. Thus $\tilde{v}$ belongs to $\text{Ker}(\Omega_2^{(2m)})|_{\zeta(L)}$. □

**Proposition 2.** Under the above notations, assume that $T(\zeta(L)) \subset \text{Ker}(\Omega_2^{(2m)})|_{\zeta(L)}$, then the complex dimension of $L$ is less than $m$ or equal to $m$, i.e., $\dim_{\mathbb{C}} L \leq m$.

**Proof.** First we observe that $\zeta^*(d\Omega_2^{(2m)}) = d(\zeta^*\Omega_2^{(2m)}) = 0$ on $L$. Assume that $\dim_{\mathbb{C}} L \geq m + 1$. Take $m + 1$ linearly independent vectors $v_1, ..., v_{m+1}$ in $T(L)$. Moreover, we take $m - 1$ vectors $u_1, ..., u_{m-1}$ in $T\mathbb{C}^{2m}$ such that $\zeta_*v_1, ..., \zeta_*v_{m+1}, u_1, ..., u_{m-1}$ are linearly independent in $T\mathbb{C}^{2m}$. Then we get $d(\Omega_2^{(2m)})(\zeta_*(v_1), ..., \zeta_*(v_{m+1}), u_1, ..., u_{m-1}) = 0$ because $\zeta^*(d\Omega_2^{(2m)}) = 0$. It is contradictory with the fact that $(d\Omega_2^{(2m)})^m$ is a non-zero 2m-form. □

**Corollary 1.** Let $m \geq 2$. The one-form $\Omega_2^{(2m)}$ has no integral manifold through the origin.

**Proof.** Assume that $\Lambda$ is an integral manifold of $\Omega_2^{(2m)}$. Then $\Lambda^* = \Lambda - \{0\}$ is a complex hypersurface, i.e., $\dim_{\mathbb{C}} \Lambda^* = 2m - 1$. By Proposition 2 we get $\dim_{\mathbb{C}} \Lambda^* \leq m$. We have a contradiction with the hypothesis $m \geq 2$.

This proves Theorem 2 for $\Omega_2^{(2m)}$. By the same argument, $\Omega_A$ and $\Omega_{(\ell)}$ have no integral manifold through the origin of $\mathbb{C}^{2m}$. □

Regarding the smoothness of invariant hypersurfaces of holomorphic distributions with an isolated singularity we have:

**Proposition 3.** Let $\Omega$ be a holomorphic one-form in a complex manifold $M$ and $p \in M$ and $\Lambda \subset M$ an integral manifold of $\Omega$ point. Given a point $p \in M$ where $\Omega$ is nonzero then $\Lambda$ is smooth at the point $p$.

**Proof.** In suitable local coordinates $(z_1, ..., z_n)$ centered at $p$ the Pfaffian equation $\Omega = 0$ is equivalent to an equation $dz_n = \sum_{j=1}^{n-1} g_j dz_j$ for some holomorphic functions $g_j$. Any integral manifold then writes as $z_n = f(z_1, ..., z_{n-1})$ for a holomorphic function $f$ satisfying $\partial f(z_1, ..., z_{n-1})/\partial z_j = g_j(z_1, ..., z_{n-1}, f(z_1, ..., z_{n-1}))$. This implies that $f$ cannot develop any singularities. □
3 Proof of Theorem 1

By hypothesis, $\xi$ defines a one-dimensional holomorphic foliation in a neighborhood of the closed ball $B^{2n}[1]$, transverse to the sphere $S^{2n-1}(1)$. According to [8] $\xi$ has a unique singularity $p$ in the open ball $B^{2n}(1)$ and this singularity is in the Poincaré domain: the vector field $\xi$ has a non-singular linear part at $p$ having eigenvalues $\lambda_1, \ldots, \lambda_n$ such that the origin does not belong to the convex hull of the set $\{\lambda_1, \ldots, \lambda_n\}$ in $\mathbb{R}^2$. On the other hand we have the following result from [10]:

**Theorem 4.** Let $\omega$ be a holomorphic one-form in an open subset $U \subset \mathbb{C}^n$, $n \geq 2$. Suppose that the distribution $\text{Ker}(\omega)$ is transverse to a sphere $S^{2n-1}(p, r) \subset U$, with $B^{2n}[p, r] \subset U$, $p \in U$, $r > 0$, then $n$ is even, $\omega$ has a single singular point in the ball $B^{2n}(p, r)$ and this is a simple singularity in the following sense: if we write $\omega = \sum_{j=1}^{n} f_j \, dz_j$ in local coordinates centered at the singularity then the matrix $(\partial f_j/\partial z_k)_{j,k=1}^{n}$ is non-singular at the singularity.

By the above result $\Omega$ has a unique singularity $q$ in $B^{2n}(1)$ and this is a simple singularity.

**Claim 1.** We have $p = q$.

**Proof.** Let $\Omega = \sum_{j=1}^{n} f_j \, dz_j$ in a neighborhood $U$ of $B^{2n}[1]$ on $\mathbb{C}^n$. Write $\xi = \sum_{i=1}^{n} A_i \frac{\partial}{\partial z_i}$, then $0 = \Omega \cdot \xi = \sum_{j=1}^{n} f_j A_j$ and therefore $0 = \frac{\partial}{\partial z_k}(\sum_{j=1}^{n} f_j A_j) = \sum_{j=1}^{n} \frac{\partial f_j}{\partial z_k} A_j + f_j \frac{\partial A_j}{\partial z_k}$. Thus $0 = \sum_{j=1}^{n} \frac{\partial f_j}{\partial z_k}(p) \cdot A_j(p)$, $\forall k$ and since the matrix $(\frac{\partial f_j}{\partial z_k}(p))_{j,k=1}^{n}$ is non-singular we have $A_j(p) = 0$, $j = 1, \ldots, n$. By the uniqueness of the singularity of $\xi$ we get $p = q$. This proves the claim.

Let us finish the proof. Suppose by contradiction that $\Omega$ is integrable. By a theorem of Malgrange [12], since codim sing($\Omega$) = $n \geq 3$ at $p$, the one-form $\Omega$ admits a holomorphic (Morse-type) first integral in a neighborhood of $p$, say $f : (W, p) \to (\mathbb{C}, 0)$. Then $\Omega \cdot \xi = 0$ implies that $\xi(f) = 0$. Because the germ of $\xi$ at 0 is in the Poincaré domain, this implies that $f$ is constant in a neighborhood of 0, contradiction with the fact that $f$ is of Morse type at 0. This ends the proof of Theorem 1. \qed

4 On the analytical classification

The problem of analytic classification of singularities of holomorphic one-forms in dimension two is a very well-developed topic. Recently (cf. [5]) the analytic classification was obtained for germs of reduced integrable one-forms at the origin $0 \in \mathbb{C}^3$. As far as we know, nothing is found regarding the non-integrable case. In the follow-up, "to classify" means to give a description in terms of objects which are completely understood. Obviously the class of germs of singular (non-integrable) holomorphic one-forms is too wide in order to be classified at a first moment and we also lack of geometric ingredients. This remark is one of the motivations for our approach in this section. Other motivation is given by the well-known results for holomorphic foliations with singularities in dimension two collected in the following omnibus theorem:

**Theorem 5 ([1], [2], [6], [8]).** Given a germ of singular holomorphic one-form $\Omega$ at $0 \in \mathbb{C}^2$ the following conditions are equivalent:
(1) $\Omega$ is in the Poincaré domain: $\Omega = A dy - B dx$ where the vector field $X = A \frac{\partial}{\partial x} + B \frac{\partial}{\partial y}$ has a singularity at $(0,0)$ whose linear part $DX(0,0)$ has non-zero eigenvalues $\lambda, \mu$ with quotient $\lambda/\mu \in \mathbb{C} \setminus \mathbb{R}^-$. 

(2) $\text{Ker}(\Omega)$ is transverse to some (and therefore to every) sphere $S^3(0,\varepsilon)$, for $\varepsilon > 0$ small enough.

(3) There are local analytic coordinates $(x, y) \in (\mathbb{C}^2, 0)$ such that $\Omega(x, y)$ is either linear $\Omega = \lambda x dy - \mu y dx$ with $\lambda/\mu \in \mathbb{C} \setminus \mathbb{R}^-$, or it is of the form $\Omega = x dy - (ny + x^n) dx$ where $n \in \mathbb{N}$ (called Poincaré-Dulac normal form).

Thus we have the following problem:

**Problem 1.** To obtain the local analytical classification of germs of non-integrable holomorphic one-forms $\Omega$ at $0 \in \mathbb{C}^n$ under the hypothesis of transversality with small spheres $S^{2n-1}(0, \varepsilon)$.

One may work with the following notions and models:

**Definition 2.** We shall say that a germ of singular non-integrable one-form $\Omega$ at the origin $0 \in \mathbb{C}^{2m}$ is in the Poincaré domain if $\text{Ker}(\Omega)$ is transverse to small spheres $S^{4m-1}(0, \varepsilon)$, $\varepsilon > 0$. A germ $\Omega$ will be called analytically linearizable if $f^* \Omega = \Omega_A$ for some germ of biholomorphism $f \in \text{Bih}(\mathbb{C}^{2m}, 0)$ fixing the origin and some $A \in \mathfrak{a}(2m)$. Finally we shall say that $\Omega$ is (analytically conjugate to) a non-integrable Poincaré-Dulac normal form if $f^* \Omega = \Omega_\ell$ for some $f \in \text{Bih}(\mathbb{C}^{2m}, 0)$ and some $\ell \in \mathbb{N}^m$.

**Question 2.** Are conditions (i), (ii) and (iii) below equivalent?

(i) $\Omega$ is the Poincaré-domain.

(ii) $\Omega$ is analytically linearizable or conjugate to a non-integrable Poincaré-Dulac normal form.

(iii) There is a holomorphic vector field $\xi$ transverse to the spheres $S^{4m-1}(0, r)$ for $r > 0$ small enough and such that $\Omega \cdot \xi = 0$?

**Remark 1.** The non-integrable examples we have given, do admit holomorphic sections $\xi$ transverse to small spheres $S^{4m-1}(0, r)$ (see Example 1). Let us observe that there are examples of non-integrable one-forms $\Omega$ with isolated singularity at the origin in $\mathbb{C}^n$, $n \geq 3$, which admit integral manifold. For example, take $f = \frac{1}{2} \sum_{j=1}^{2m} z_j^2$ and $\Omega = df + f \nu$ for some holomorphic one-form $\nu = \sum_{j=1}^{2m} \nu_j dz_j$ such that $d\nu(0)$ is non-degenerate. Then $\{f = 0\}$ is an integral manifold of $\Omega$ and also $\Omega(z) = 0$ if and only if $z_j + \sum_{k=1}^{2m} z_k^2 \nu_j(z) = 0$ for every $j$. This shows, because the Jacobian of the left hand side at $z = 0$ is the identity, that $\Omega$ has an isolated singularity at the origin $z = 0$. Finally, if $X_j \in \text{Ker}(\Omega)$, then $(d\Omega)(X_1, X_2) = f(d\nu)(X_1, X_2)$, we obtain that near the origin $\Omega$ is non-integrable at every point where $f \neq 0$. This example suggests that a more interesting question might be:

**Question 3.** Classify the non-integrable germs of holomorphic one-forms $\Omega$ with an isolated singularity at the origin and $\text{Ker}(\Omega)$ transverse to all small spheres centered at the origin, which admit an integral manifold through the origin.
The above construction suggests that if the integral manifold is taken in the form \( \{ f = 0 \} \) for a holomorphic function \( f \), then maybe we can write \( \Omega = g(df + f\nu) \) for some function \( g \) and a one-form \( \nu \), and we have to study the kernels of \( df + f\nu \).

### 4.1 Proof of Theorem 3

Now we shall prove Theorem 3. We need:

**Lemma 3.** The open subset \( \mathbb{A}(2m) \subset \mathcal{A}(2m) \) is arcwise-connected.

**Proof.** First we recall that \( \mathcal{A}(2m) \) is a complex vector space. Denote by \( f: \mathcal{A}(2m) \to \mathbb{C} \) the determinant function, i.e., \( f(A) = \text{Det}(A), \forall A \in \mathcal{A}(2m) \). Then \( f \) is holomorphic and \( f \neq 0 \) because if \( \mathcal{J}(2m) = \begin{bmatrix} [0 \cdot -1] \ldots \cdot [0 \cdot -1] \end{bmatrix} \) then \( \mathcal{J}(2m) \in \mathcal{A}(2m) \) and \( f(\mathcal{J}(2m)) = \text{Det}(\mathcal{J}(2m)) = (-1)^m \neq 0 \). Now \( \mathbb{A}(2m) = \{ A \in \mathcal{A}(2m) : f(A) \neq 0 \} = \mathcal{A}(2m) \setminus f^{-1}(0) \). Since \( \mathcal{A}(2m) \) is an affine space and \( f^{-1}(0) \) is a thin subset of \( \mathcal{A}(2m) \) it follows from [7] Corollary 4 page 20 that \( \mathbb{A}(2m) \) is (arcwise) connected. \( \square \)

We shall adopt the following natural definition:

**Definition 3.** Two codimension one holomorphic distributions \( \Delta_1, \Delta_2 \) on a complex manifold \( M^n \) are *homotopic* if there is a \( C^\infty \) family \( \{ P_s \}_{s \in [0,1]} \) of holomorphic distributions in \( M \) such that \( P_0 = \Delta_1 \) and \( P_1 = \Delta_2 \).

**Lemma 4.** Let \( \Omega_A \) be a linear one-form in \( \mathbb{C}^{2m} \) with \( A \in \mathbb{A}(2m) \). Then the distribution \( \text{Ker}(\Omega_A) \) is homotopic to the canonical distribution \( \Omega_{\mathcal{J}(2m)} = \sum_{j=1}^{2m} (z_{2j}dz_{2j} - z_{2j-1}dz_{2j}) \).

**Proof.** This is an immediate consequence of Lemma 3. Given a smooth path \( \alpha: [0,1] \to \mathbb{A}(2m) \) connecting \( \alpha(0) = A \) to \( \alpha(1) = \mathcal{J}(2m) \) we define a homotopy by setting \( P_s = \text{Ker}(\Omega_{\alpha(s)}) \) (recall that \( \alpha(s) \in \mathbb{A}(2m) \)). \( \square \)

Now we consider a holomorphic one-form \( \Omega \) defined in a neighborhood \( U \) of the closed unit ball \( B^{4m}[0,1] \subset \mathbb{C}^{2m} \) and such that \( \Omega \cdot \vec{R} = 0 \). Write \( \Omega = \sum_{\nu=1}^{+\infty} \omega_\nu \) where \( \omega_\nu \) is a homogeneous one-form of degree \( \nu \geq 1 \) in \( U \) (assume \( U \) is a ball). Then \( 0 = \Omega \cdot \vec{R} = \sum_{\nu=1}^{+\infty} \omega_\nu \cdot \vec{R} \) and since \( \omega_\nu \cdot \vec{R} \) is a homogeneous polynomial of degree \( \nu + 1 \) we conclude that \( \omega_\nu \cdot \vec{R} = 0, \forall \nu \geq 1 \). In particular, \( \omega_1 \cdot \vec{R} = 0 \) and therefore \( \omega_1 = \Omega_A \) for some \( A \in \mathcal{A}(2m) \). Assume now that \( \text{Ker}(\Omega) \) is transverse to \( S^{4m-1}(1) \) then it follows from Theorem 4 that \( \Omega \) has only one singularity which is simple. Because the group of holomorphic transformations of the unit ball acts transitively, we can assume by a holomorphic change of coordinates that the origin is the only singularity of \( \Omega \) in \( B^{4m}[0,1] \) and, since it is a simple singularity, \( A \) is non-singular, i.e., \( A \in \mathbb{A}(2m) \).

**Lemma 5.** Under the above hypothesis there is a real analytic deformation \( \{ \text{Ker}(\Omega^t) \}_{t \in [0,1]} \) of \( \text{Ker}(\Omega) \) into \( \text{Ker}(\Omega_A) \) by holomorphic distributions \( \text{Ker}(\Omega^t) \) transverse to \( S^{4m-1}(1) \) outside the intersection \( S^{4m-1}(1) \cap \text{sing}(\Omega^t) \).
Proof. We define $\Omega^t$ by $\Omega^t := t^{-1}\Omega(tz)$. Then $\Omega^t$ converges to a holomorphic one-form for each $t \in [0, 1]$ by canonical convergence criteria, indeed, $\Omega^t$ defines a holomorphic one-form in $U$ for each $t$ in the closed unit disc $\overline{D} \subset \mathbb{C}$. We have $\Omega^1 = \Omega$ and an easy computation shows that $\Omega^t$ converges for $t \to 0$ to $\Omega_A$ where $A$ is the Jacobi-matrix of $\Omega$ at the origin. Furthermore, because $\vec{R}(tz) = t\vec{R}(z)$, one has $\Omega^t(z) \cdot \vec{R}(z) = t^{-1}\Omega(tz) \cdot \vec{R}(z) = t^{-2}\Omega(tz) \cdot \vec{R}(z) = 0$ for every $t \neq 0$. Thus $\text{Ker}(\Omega^t)$ is transverse to $S^{4m-1}(1)$ outside $\text{sing}(\Omega^t) \cap S^{4m-1}(1)$.

Now we study the singular set $\text{sing}(\Omega^t)$.

Lemma 6. We have $\text{sing}(\Omega^t) = \{0\}$, $\forall t \in \overline{D}$.

Proof. Take any $z \in S^{4m-1}(1)$. Consider the complex line $\ell(z) = \{tz : t \in \mathbb{C}\}$ through $z$. If $0 < |t| < 1$, then $tz \neq 0$ belongs to $B^{4m}[1]$. Therefore $\Omega^t(z) = \frac{1}{t}\Omega(tz)$ is non-zero. If $t = 0$, then $\Omega^0(z) = \Omega_A(z)$ is non-zero on $S^{4m-1}(1)$. This proves the lemma.

Proof of Theorem 3 For the proof we just have to apply Lemmas 4, 5 and 6 above.

References

[1] A. D. Brjuno; Analytical form of differential equations, Trans. Moscow Math. Soc., vol. 25 (1971), 131–288.

[2] C. Camacho, N. Kuiper, J. Palis; The topology of holomorphic flows with singularity. Inst. Hautes Études Sci. Publ. Math., no. 48 (1978), 5–38.

[3] C. Camacho, P. Sad; Invariant varieties through singularities of holomorphic vector fields; Ann. of Math. 115 (1982), 579-595.

[4] F. Cano, D. Cerveau; Desingularization of holomorphic foliations and existence of separatrices; Acta Mathematica, Vol. 169, 1992, 429-446.

[5] D. Cerveau, J. Mozo-Fernandez; Classification analytique des feuilletages singuliers réduits de codimension 1 en dimension $n \geq 3$. Ergodic Theory Dynam. Systems 22 (2002), no. 4, 1041–1060.

[6] H. Dulac; Solutions d’un système de équations différentielle dans le voisinage des valeurs singulières, Bull. Soc. Math. France 40 (1912) 324–383.

[7] R.C. Gunning, H. Rossi; Analytic functions of several complex variables. Prentice Hall, N.J. 1965.

[8] T. Ito; A Poincaré-Bendixson type theorem for holomorphic vector fields; RIMS publication, Kyoto, no. 878, 1994.

[9] T. Ito, B. Scárdua; On codimension one foliations transverse to spheres, to appear in Moscow Mathematical Journal, 2005.

[10] T. Ito, B. Scárdua; A Poincaré-Hopf type theorem for holomorphic one-forms. Topology. Vol. 44, Fasc.1, pp.73-84, 2004.
[11] J.P. Jouanolou; Équations de Pfaff algébriques, Lecture Notes in Math. 708, Springer-Verlag, Berlin, 1979.

[12] B. Malgrange; Frobenius avec singularités I: codimension un. Pub. Math. IHES 46 (1976) 162-173.

Toshikazu Ito
Department of Natural Science
Ryukoku University
Fushimi-ku, Kyoto 612
JAPAN

Bruno Scárdua
Instituto de Matemática
Universidade Federal do Rio de Janeiro
Caixa Postal 68530
21.945-970 Rio de Janeiro-RJ
BRAZIL