HOMOGENEOUS SOLUTIONS TO FULLY NONLINEAR ELLIPTIC EQUATIONS

NIKOLAI NADIRASHVILI AND YU YUAN

Abstract. We classify homogeneous degree $d \neq 2$ solutions to fully nonlinear elliptic equations.

In this note, we show that any homogeneous degree other than 2 solution to fully nonlinear elliptic equations must be “harmonic”. Consider the fully nonlinear elliptic equation $F(D^2 u) = 0$ with $\mu I \leq (F_{ij}) = (F_{M_{ij}}(M)) \leq \mu^{-1}I$. Nirenberg [N] derived the a priori $C^{2,\alpha}$ estimates for the above equation in dimension 2 in 1950s. Krylov [K] and Evans [E] showed the same a priori estimates for the above equations in general dimensions under the assumption that $F$ is convex. As a modest investigation of a priori estimates for general fully nonlinear elliptic equations without convexity condition, we study the homogeneous solutions.

Theorem 0.1. Let $u$ be a continuous in $\mathbb{R}^n \setminus \{0\}$ homogeneous degree $d \neq 2$ solution to the elliptic equation $F(D^2 u) = 0$ in $\mathbb{R}^n$ with $F \in C^1$. Then $u$ is harmonic in a possible new coordinate system in $\mathbb{R}^n$, namely

$$\sum_{i,j=1}^n F_{ij}(0) D_{ij} u(x) = 0.$$ 

Consequently, $u \equiv 0$ if $-(n-2) < d < 0$ or $d$ is not an integer; otherwise $u$ is a homogeneous harmonic polynomial with integer degree $d$.

In contrast to the variational problem, Sverak and Yan [SY] constructed homogeneous degree less than 1 minimizers to some strongly convex functional. Also Safonov [S] constructed homogeneous order $\alpha \in (0,1)$ solutions to linear non-divergence elliptic equations with variable coefficients earlier on.

As one simple application to special Lagrangian equations [HL] $F(D^2 u) = \sum_{i=1}^n \arctan \lambda_i - c = 0$, where $\lambda_i$s are the eigenvalues of the Hessian $D^2 u$. It follows from our theorem that any homogeneous degree other than 2 solutions must be a harmonic polynomial (and it also forces $c = 0$).

When $d \in [0, 1 + \alpha(n, \mu))$, our theorem follows from Krylov-Safonov $C^{\alpha}$ estimates (cf. [CC, corollary 5.7]). The missing case $d = 2$ is delicate. One only knows that any homogeneous degree 2 solution to the above fully nonlinear elliptic equation in dimension 3 is quadratic [HNY, p. 426].

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Now we show our theorem.

Proof. We first consider the case that \( u \) is smooth in \( \mathbb{R}^n \setminus \{0\} \). Set \( \sum = \left\{ |x|^{d-2} D^2 u \left( \frac{x}{|x|} \right) | x \in \mathbb{R}^n \setminus \{0\} \right\} \) and \( \Gamma = \{ M | F(M) = 0 \} \). For the homogeneous order \( d \) function \( u(x) \), \( D^2 u(x) = |x|^{d-2} D^2 u \left( \frac{x}{|x|} \right) \). Let \( |x| \to 0 \) for \( d > 2 \) or \( |x| \to \infty \) for \( d < 2 \), we see that \( 0 \in \sum \). Also \( u \) is a solution to \( F(D^2 u) = 0 \), then the cone \( \sum \subseteq \Gamma \).

Now \( F \in C^1 \) and \( (F_{ij}(0)) > 0 \), we know that the unique tangent plane of \( \Gamma \) at 0 includes \( \sum \). It follows that \( \sum \perp (F_{ij}(0)) \), or

\[
\sum_{i,j=1}^n F_{ij}(0) D_{ij} u(x) = 0.
\]

Without loss of generality, we assume \( (F_{ij}(0)) = I \) throughout the proof, then

\[
0 = \Delta u(x) = |x|^{d-2} \left[ d(d+n-2) u \left( \frac{x}{|x|} \right) + \Delta_{S^{n-1}} u \left( \frac{x}{|x|} \right) \right].
\]

The remaining conclusion of the theorem follows.

Next we show the regularity of the viscosity solution \( u \) away from 0. Set \( \lambda = d(d+n-2) \) and \( \theta = x/|x| \). To start, we prove that \( u(\theta) \) is a viscosity solution to

\[
(0.1) \quad \Delta_{S^{n-1}} u + \lambda u = 0.
\]

Let any smooth \( \varphi(\theta) \) touch \( u \) from the above at \( \theta_0 \),

\[
\varphi \geq u \quad \text{in a neighborhood of } \theta_0
\]

\[
\varphi(\theta_0) = u(\theta_0).
\]

then

\[
|x|^d \varphi \left( \frac{x}{|x|} \right) \geq |x|^d u \left( \frac{x}{|x|} \right) \quad \text{in a neighborhood of } \theta_0
\]

\[
|x|^d \varphi(\theta_0) = |x|^d u(\theta_0).
\]

From our assumption that \( u \) is a viscosity (sub) solution, it follows that

\[
F \left( D^2 \left( |x|^d \varphi \left( \frac{x}{|x|} \right) \right) \right) \geq 0
\]

or

\[
F \left( |x|^{d-2} D^2 x \varphi(\theta) \right) \geq 0.
\]

Let \( |x| \to 0 \) for \( d > 2 \) or \( |x| \to \infty \) for \( d < 2 \), we see that \( F(0) \geq 0 \). If we use the fact \( u \) is also a viscosity (super) solution, we can derive that \( F(0) \leq 0 \). So \( F(0) = 0 \), and

\[
\frac{F(t D^2 \varphi(\theta)) - F(0)}{t} \geq 0.
\]
Let \( t \to 0 \), we see that
\[
\sum_{i,j=1}^{n} F_{ij}(0) D_{ij} \varphi \left( \frac{x}{|x|} \right) \geq 0
\]
or
\[
\Delta_{S^{n-1}} \varphi + \lambda \varphi \geq 0.
\]
Thus \( u \) is a viscosity sub solution to (0.1). Similarly, \( u \) is a viscosity super solution to the same equation.

Let \( N_\varepsilon \) be an \( \varepsilon \) neighborhood of any \( \theta_0 \) on \( S^{n-1} \), with \( \varepsilon \) small enough so that \( N_\varepsilon \) is in a narrow strip, then there exists positive smooth function \( h \) on \( N_\varepsilon \) such that
\[
\Delta_{S^{n-1}} h + \lambda h \leq 0.
\]
Let \( \psi \) be the smooth solution to (0.1) in \( N_\varepsilon \) with the boundary value \( u \) on \( \partial N_\varepsilon \), then \( q \equiv \psi - u \) is a viscosity solution to
\[
\Delta_{S^{n-1}} q + 2 \frac{\nabla h}{h} \cdot \nabla q + \frac{\Delta_{S^{n-1}} h + \lambda h}{h} q = 0,
\]
where \( \nabla h \cdot \nabla q \) simply denotes some linear combinations of first order derivatives of \( q \) in some local coordinates for \( N_\varepsilon \), which we avoid for the sake of simple notation. Now that the coefficient \( \frac{\Delta_{S^{n-1}} h + \lambda h}{h} \leq 0 \), it follows from [W,Corollary 3.20] that \( q = 0 \) in \( N_\varepsilon \). Therefore, \( u \) is smooth in \( N_\varepsilon \) and then on the whole \( S^{n-1} \).

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Department of Mathematics, University of Chicago, 5734 S. University Ave., Chicago, IL 60637

Current address: LATP, Centre de Mathématiques et Informatique, 39, rue F. Joliot-Curie, 13453 Marseille Cedex, France

E-mail address: nicholas@math.uchicago.edu

Department of Mathematics, Box 354350, University of Washington, Seattle, WA 98195

E-mail address: yuan@math.washington.edu