Cyclicity in rank-1 perturbation problems

Evgeny Abakumov, Constanze Liaw and Alexei Poltoratski

Abstract
The property of cyclicity of a linear operator, or equivalently the property of simplicity of its spectrum, is an important spectral characteristic that appears in many problems of functional analysis and applications to mathematical physics. In this paper, we study cyclicity in the context of rank-1 perturbation problems for self-adjoint and unitary operators. We show that for a fixed non-zero vector the property of being a cyclic vector is not rare, in the sense that for any family of rank-1 perturbations of self-adjoint or unitary operators acting on the space, that vector will be cyclic for every operator from the family, with a possible exception of a small set with respect to the parameter. We discuss applications of our results to Anderson-type Hamiltonians.

1. Introduction
Consider a self-adjoint operator $T$ on a separable Hilbert space $\mathcal{H}$. A vector $\varphi \in \mathcal{H}$ is called cyclic for an operator $T$, if

$$\mathcal{H} = \text{clos span}\{ (T - \lambda I)^{-1} \varphi : \lambda \in \mathbb{C} \setminus \mathbb{R} \}.$$ 

An operator $T$ is called cyclic, if there exists a cyclic vector. For a bounded operator $T$, an equivalent definition is that

$$\mathcal{H} = \text{clos span}\{ T^n \varphi : n \in \mathbb{N} \cup \{0\} \},$$

that is, the span of the orbit of $\varphi$ under $T$ is dense in the Hilbert space. Cyclicity of an operator is equivalent to the property that the operator has simple spectrum. The property of simplicity of the spectrum often appears in problems originating from physics.

In this note, we study cyclicity in the context of rank-1 perturbation problems for self-adjoint and unitary operators. If $A$ is a self-adjoint operator and $\varphi$ is its cyclic vector, then one can consider the family of rank-1 perturbations

$$A_\alpha = A + \alpha (\cdot, \varphi) \varphi \quad \text{for } \alpha \in \mathbb{R}. \quad (1.1)$$

Similar families can be defined for unitary operators, see Subsections 2.2 and 2.3 for the definitions. We show that the property of cyclicity for a fixed non-zero vector in the Hilbert space is not a rare event, in the sense that for any family of cyclic rank-1 perturbations a fixed vector is cyclic for all operators in the family with an exception of some small sets of parameters.

While similar statements exist in the literature (see [7, 8]), we attempt to make these statements more precise.

In the rank-1 setting, for the singular part we prove that the exceptional set of parameters has Lebesgue measure zero; for the absolutely continuous part, we prove ‘all but countably many’; and in some more special cases even ‘all but possibly one’.
In particular, in Section 3 we prove that an arbitrary non-zero vector $\varphi$ in a Hilbert space is cyclic for all but countably many operators $(A_\alpha)_{ac}$ for any family of self-adjoint (or unitary) rank-1 perturbations $A_\alpha$. An arbitrary non-zero vector is also cyclic for almost all operators $(A_\alpha)_{s}$ for any such family. Here $(A_\alpha)_{ac}$ and $(A_\alpha)_{s}$ denote the absolutely continuous and singular parts of the operators $A_\alpha$, respectively, see Subsection 2.1 for the definitions.

In Subsection 3.4, we discuss applications of our results to Anderson-type Hamiltonians and deduce and improve some of the results from [7, 8, 20].

In Section 4, we show that if a vector belongs to a certain natural class of vectors, associated with the family $A_\alpha$, then we have cyclicity for all except possibly one operator $A_\alpha$.

The theory of rank-1 perturbations of self-adjoint and unitary operators, and its applications to Anderson-type models became an active area of research over the last 20 years. The interest to this part of perturbation theory is caused, to a large degree, by connections to the famous problem of Anderson localization.

In 1958, Anderson (see [1]) suggested that sufficiently large impurities in a semi-conductor could lead to spatial localization of electrons, called Anderson localization. Although most physicists consider the problem solved, many mathematical questions with striking physical relevance remain open. The field has grown into a rich mathematical theory (see, for example, [5, 6, 10] for different Anderson models and [12, 19] for refined notions of Anderson localization).

While the property of localization for a random Anderson-type operator has many different definitions, one of the ‘weaker’ definitions of localization is equivalent to the property that the spectrum of the operator is almost-surely singular. It is well known that, if an Anderson-type Hamiltonian is almost-surely singular, then it is almost-surely cyclic. Equivalently, if such an operator is not cyclic with positive probability, then it is delocalized.

The study of spectral behavior under rank-1 perturbations proves to be one of the main tools in spectral analysis of Anderson-type models, in particular in problems concerning cyclicity; see, for instance, [7, 8, 20]. This connection served as one of the motivating factors for the current paper.

1.1. Notation/beware

We consider three main classes of families of perturbations: rank-1 self-adjoint, rank-1 unitary perturbations and Anderson-type Hamiltonians. Throughout we use the notation $A_\alpha$, $U_\gamma$ and $H_\omega$ to denote the corresponding families of perturbations, respectively. While all three classes are somewhat closely related, the two types of rank-1 perturbations ($(2.1)$ and $(2.6)$) are almost interchangeable via the Cayley transform: The Aleksandrov–Clark theory and all its basic results discussed in Subsection 2.3 can be equivalently re-stated in the case of the real line (upper half-plane). For example, the Cauchy transform in $\mathbb{D}$, see the second equation of $(2.7)$, is replaced with its analog in $\mathbb{C}_+$, see equation $(2.2)$ for the definition. Similarly, results on rank-1 perturbations of self-adjoint operators can be re-formulated for the families of unitary rank-1 perturbations; see, for instance, [16]. It is a well-known feature of complex function theory that some of the proofs of the half-plane statements look more natural in the settings of the unit disk and vice versa. Similarly, in this paper we utilize both self-adjoint and unitary settings in our statements and proofs.

2. Preliminaries

2.1. Cyclicity for normal operators

Recall that an operator in a separable Hilbert space is called normal if $T^*T = TT^*$. By the spectral theorem, the operator $T$ is unitarily equivalent to $M_z$, multiplication by the
independent variable $z$, in a direct sum of Hilbert spaces

$$\mathcal{H} = \bigoplus \int \mathcal{H}(z) \, d\mu(z),$$

where $\mu$ is a scalar positive measure on $\mathbb{C}$. The measure $\mu$ is called a spectral measure of $T$.

If $T$ is a unitary or self-adjoint operator, then its spectral measure $\mu$ is supported on the unit circle or on the real line, respectively. Via Radon decomposition, $\mu$ can be decomposed into a singular and an absolutely continuous part $\mu = \mu_s + \mu_{ac}$. The singular component $\mu_s$ can be further split into singular continuous and pure point parts. For unitary or self-adjoint $T$, we denote by $T_{ac}$ the restriction of $T$ to its absolutely continuous part, that is, $T_{ac}$ is unitarily equivalent to

$$M_t |_{\bigoplus \int \mathcal{H}(t) \, d\mu_{ac}(t)}.$$  

Similarly, define the singular, singular continuous and the pure point parts of $T$, denoted by $T_s$, $T_{sc}$ and $T_{pp}$, respectively.

In terms of the spectral representation described above, the property of cyclicity, as defined in Section 1, is equivalent to the property that all Hilbert spaces $\mathcal{H}(t)$ are one-dimensional and the space

$$\bigoplus \int \mathcal{H}(t) \, d\mu(t)$$

can be identified with $L^2(\mu)$. Cyclic vectors for $T$ correspond to functions with full support in $L^2(\mu)$, that is, those functions that are non-zero almost everywhere (a.e.) with respect to $\mu$.

2.2. **Self-adjoint rank-1 perturbations**

Let $T$ be a normal operator on a Hilbert space $\mathcal{H}$ and let $\varphi \in \mathcal{H}$ be a non-zero vector. An alternative definition of the spectral measure of $T$ can be given as follows. Note that there exists a unique measure $\mu$ on $\mathbb{C}$ such that

$$\langle (T - \lambda I)^{-1} \varphi, \varphi \rangle_{\mathcal{H}} = \int \frac{d\mu(t)}{t - \lambda},$$

for all $\lambda$ outside of the spectrum of $T$. If $\mu$ is such a measure, then we say that $\mu$ is the spectral measure of $T$ with respect to the vector $\varphi$. Note that such a measure is unique, once $T$ and $\varphi$ are fixed. The operator $T$ is bounded if and only if $\mu$ is compactly supported.

Let $A$ be a self-adjoint operator and let $\varphi$ be its cyclic vector. Consider the family of rank-1 perturbations

$$A_\alpha = A + \alpha \langle \cdot, \varphi \rangle \varphi \quad \text{for} \quad \alpha \in \mathbb{R}. \quad (2.1)$$

It is not difficult to show that then $\varphi$ will be a cyclic vector for $A_\alpha$ for all $\alpha \in \mathbb{R}$. Denote by $\mu_\alpha$ the spectral measure of $A_\alpha$ with respect to $\varphi$. In this notation, $\mu = \mu_0$.

In virtue of the spectral theorem, one can always assume that $\mathcal{H} = L^2(\mu)$, $A = M_t$ and $\varphi = 1 \in L^2(\mu)$. Denote by $V_\alpha$ the operator of spectral representation for $A_\alpha$, that is, the unitary operator $V_\alpha : L^2(\mu) \to L^2(\mu_\alpha)$ such that $V_\alpha A_\alpha = M_t V_\alpha$ and $V_\alpha \varphi = 1$. An explicit formula for $V_\alpha$ was recently derived in [13].

For unbounded $A$ (that is, not compactly supported $\mu$), we always assume that the spectral measure $\mu$ corresponding to $\varphi$ satisfies

$$\int_{\mathbb{R}} \frac{d\mu(t)}{1 + |t|} < \infty.$$

Using the standard terminology, this means that we consider the class of singular form bounded perturbations and assume that $\varphi \in \mathcal{H}_{-1}(A) \supset \mathcal{H}$, that is, that $A \varphi \in \mathcal{H}$. Note that if
\( \varphi \notin \mathcal{H}_{-1}(A) \), then the formal expression (2.1) does not possess a unique self-adjoint extension; see, for instance, [11].

The Aronszajn–Donoghue theory analyzes the spectrum of the perturbed operator under rank-1 perturbations. We will use the following well-known statement.

**Theorem 2.1** [21]. For non-equal coupling constants \( \alpha \neq \beta \), the singular parts \((\mu_{\alpha})_s\) and \((\mu_{\beta})_s\) are mutually singular.

Of fundamental importance to many spectral problems is the Cauchy transform. If \( \tau \) is the spectral measure of a self-adjoint operator \( A \) corresponding to the vector \( \varphi \), then the Cauchy transform of \( \tau \),

\[
K_{\tau}(z) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{d\tau(t)}{t-z}, \quad z \in \mathbb{C} \setminus \mathbb{R},
\]

is equal to the corresponding resolvent function of \( A \):

\[
K_{\tau}(z) = ((A - zI)^{-1}\varphi, \varphi) = \int \frac{d\mu(t)}{t-z}.
\]

This connection allows one to apply complex analysis in spectral problems.

We use the notation

\[
(K_{\tau})_+(x) = \lim_{y \to 0} K_{\tau}(x + iy) \quad \text{and} \quad (K_{\tau})_-(x) = \lim_{y \to 0} K_{\tau}(x - iy),
\]

for \( x \in \mathbb{R} \). By a theorem of Privalov,

\[
(K_{\tau})_-(x) - (K_{\tau})_+(x) = 2\pi i \frac{d\tau}{dx}(x),
\]

for Lebesgue-a.e. \( x \in \mathbb{R} \).

2.3. Aleksandrov–Clark theory and unitary rank-1 perturbations

By \( H^p \) we will denote the standard Hardy spaces in the unit disk. Recall that a function \( \theta \in H^\infty \) is called inner, if \( |\theta(z)| = 1 \) for a.e. \( |z| = 1 \). The (scalar valued) model space \( K_\theta \) is defined as \( K_\theta = H^2 \ominus \theta H^2 \). Such spaces play an important role in complex function theory and functional analysis; see, for instance, [14].

The model operator on a space \( K_\theta \) is defined as \( S_\theta = P_\theta S \), where \( P_\theta \) denotes the orthogonal projection onto \( K_\theta \), while \( S \) is the shift operator given by \( Sf(z) = zf(z) \) for \( f \in H^2 \). The adjoint to the shift operator is the so-called backward shift operator defined as

\[
S^* f = \frac{f(z) - f(0)}{z}.
\]

Let \( \theta \) be an inner function. To simplify the formulas, we will assume that \( \theta(0) = 0 \). In [4], Clark showed that the family of rank-1 perturbations

\[
\tilde{U}_\gamma = S_\theta + \gamma(\cdot, S^* \theta) 1 \quad \text{for} \quad \gamma \in \mathbb{T},
\]

consists of unitary operators on \( K_\theta \), and, vice versa, that all unitary rank-1 perturbations of the model operator \( S_\theta = P_\theta S|_{K_\theta} \) are given by (2.4).

It is well known that the vector \( 1 \in K_\theta \) is cyclic for all operators \( \tilde{U}_\gamma \) in the above family. By \( \sigma_{\gamma} \), denote the spectral measures of \( \tilde{U}_\gamma \) with respect to the function \( 1 \), that is, such measures that the identity

\[
((\tilde{U}_\gamma + zI)(\tilde{U}_\gamma - zI)^{-1}1,1) = \int_{\mathbb{T}} \frac{\xi + z}{\xi - z} d\sigma_{\gamma}(\xi)
\]
holds true for all $\gamma \in \mathbb{T}$. In fact, every inner function $\theta$ with $\theta(0) = 0$ corresponds in a one-to-one fashion with a family of Clark measures $\{\sigma_\gamma\}_{\gamma \in \mathbb{T}}$. Further, for inner functions $\theta$, the measures $\sigma_\gamma$ are purely singular for all $\gamma \in \mathbb{T}$; and vice versa.

One of the main results of the Clark theory says that the spectral measures of $\tilde{U}_\gamma$ are defined by the identity

$$
\frac{\theta + \gamma}{\theta - \gamma} = \int_{\mathbb{T}} \frac{\xi + z}{\xi - z} d\sigma_\gamma(\xi).
$$

(2.5)

The Clark operator is the unitary operator $\Phi_\gamma : K_\theta \to L^2(\sigma_\gamma)$ such that $\Phi_\gamma \tilde{U}_\gamma = M_z \Phi_\gamma$, where $M_z$ is the operator that acts as multiplication by the independent variable in $L^2(\sigma_\gamma)$. In other words, the Clark operator is the spectral representation of $\tilde{U}_\gamma$.

Note that the spectral representation of $V_\alpha : L^2(\sigma) \to L^2(\sigma_\alpha)$ from the previous subsection on self-adjoint rank-1 perturbations corresponds to the composition operator $\Phi_\gamma \Phi_1^* : L^2(\sigma_1) \to L^2(\sigma_\gamma)$ in the case of unitary rank-1 perturbations.

The situation becomes more complicated without the assumption that the spectral measure is purely singular: In the case of non-trivial absolutely continuous spectrum, the model space consists of pairs of functions analytic inside and outside of the unit disk, see [15].

However, many of the formulas of the Aleksandrov–Clark theory remain valid in the case where the spectral measures are not purely singular. If $\mu$ is a positive finite measure on the unit circle, then denote by $U_1$ the operator of multiplication by $z$ in $L^2(\mu)$. Let $\theta$ be a bounded holomorphic function in the unit disk $D$ that satisfies (2.5) for $\gamma = 1$ and $\sigma_1 = \mu$. If $\mu$ is not singular, then $\theta$ is not inner, but still belongs to the unit ball of $H^\infty$, that is, $|\theta| \leq 1$ in $D$. Nonetheless, one can still consider a family of measures $\sigma_\gamma$ defined by (2.5). This family will consist of spectral measures of unitary rank-1 perturbations of $U_1$ corresponding to the vector $1 \in L^2(\mu)$, with $U_\gamma$ defined as

$$
U_\gamma = U_1 + (\gamma - 1)(\gamma, U_1^* 1) 1, \quad \gamma \in \mathbb{T},
$$

(2.6)

see [18].

We will also use the following two theorems from the Aleksandrov–Clark theory. Let $m$ denote the Lebesgue measure on $\mathbb{T}$.

**Theorem 2.2** Aleksandrov’s spectral averaging; see, for example, [18]. For $f \in L^1(T, dm)$, we have

$$
\int f \, dm = \int \left( \int f \, d\sigma_\gamma \right) \, dm(\gamma).
$$

It is well known that the adjoint $\Phi_\gamma^* : L^2(\sigma_\gamma) \to K_\theta$ of the Clark operator can be represented using the normalized Cauchy transform

$$
\Phi_\gamma^* h = \frac{\mathcal{K}_h \sigma_\gamma}{\mathcal{K}_{\sigma_\gamma}},
$$

where $\mathcal{K}$ stands for the Cauchy transform in $D$:

$$
\mathcal{K}_{\sigma_\gamma}(z) = \int_{\mathbb{T}} \frac{d\sigma_\gamma(\xi)}{1 - \xi z} \quad \text{and} \quad \mathcal{K}_{h \sigma_\gamma}(z) = \int_{\mathbb{T}} \frac{h(\xi) \, d\sigma_\gamma(\xi)}{1 - \xi z}.
$$

(2.7)

**Theorem 2.3** [17]. For any $f \in L^1(\sigma_\gamma)$,

$$
\lim_{r \to 1} \frac{\mathcal{K}_f \sigma_\gamma(r z)}{\mathcal{K}_{\sigma_\gamma}(r z)} = f(z) \quad \text{for} \ (\sigma_\gamma)_s \text{-a.e.} \ z \in \mathbb{T}.
$$
Theorems 2.2 and 2.3 hold true, even if the family of spectral measures possesses non-trivial absolutely continuous parts, although the normalized Cauchy transform of an arbitrary function in \( L^2(\sigma_\gamma) \) cannot be interpreted as a vector from \( K_\theta \).

Via the standard agreement, every function from a Hardy space \( H^p \) is identified with its boundary values on the circle \( \mathbb{T} \). It follows from Theorem 2.3 that the boundary values of any \( f \in K_\theta \) exist \( \sigma_\gamma \)-a.e for any \( \gamma \in \mathbb{T} \) and the Clark operator \( \Phi_\gamma : K_\theta \to L^2(\sigma_\gamma) \) simply sends \( f \) into its boundary values.

For a function \( f \in K_\theta \), denote by \( \tilde{f} \) the function \( \theta \bar{f} \) on \( \mathbb{T} \). Note that \( f \in K_\theta \) and \( f(0) = 0 \) imply that \( \theta \bar{f} \in K_\theta \). A function \( f \in K_\theta \) is called a Hermitian element, if \( \tilde{f} = f \). Note that Hermitian functions satisfy \( f(0) = 0 \).

The following simple statement plays an important role in Section 4.

**Theorem 2.4** [17]. Let \( f \in K_\theta \). Then \( f \) is a Hermitian element if and only if
\[
\arg(\Phi_\gamma f) = \frac{\arg \gamma}{2} \pmod{\pi}, \quad \sigma_\gamma \text{-a.e., (2.8)}
\]
and \( \int f \, d\sigma_\gamma = 0 \) for some \( \gamma \in \mathbb{T} \). If \( f \) is a Hermitian element, then \( f \) satisfies (2.8) and \( \int f \, d\sigma_\gamma = 0 \) for any \( \gamma \in \mathbb{T} \).

2.4. **Spaces of Paley–Wiener functions**

For \( a > 0 \), the class of Paley–Wiener functions on \( \mathbb{R} \) is given by
\[
\text{PW}_a = \{ \hat{\tilde{f}} : f \in L^2(-a, a) \},
\]
where \( \hat{f}(z) = \int e^{-itz} f(t) \, dt \) denotes the classical Fourier transform of \( f \). Alternatively, the Paley–Wiener space can be characterized as the space of entire functions of exponential type at most \( a \) whose boundary values on the real line are square summable with respect to Lebesgue measure.

The Paley–Wiener space \( \text{PW}_a \) is closely related to the model space \( K_\theta \) for the inner function
\[
\theta_a(z) = \theta(z) = e^{-2a(1+z)/(1-z)}
\]
in the unit disk. To establish the connection, consider the conformal map
\[
\psi(z) = \frac{z - i}{z + i},
\]
from \( \mathbb{C}_+ \) to \( \mathbb{D} \). Denote \( \vartheta(z) = \vartheta_a(z) = e^{2iaz} \). Note that
\[
\vartheta_a(z) = \theta_a(\psi(z)).
\]
By \( K_{\theta}^{\mathbb{R}} \) denote the space obtained from \( K_\theta \) by composing all functions from \( K_\theta \) with \( \psi \), that is,
\[
K_{\theta}^{\mathbb{R}} = \{ f(\psi) : f \in K_\theta \}.
\]
Then the space
\[
e^{-iaz} K_{\theta}^{\mathbb{R}} = \{ e^{-iaz} f(\psi) : f \in K_\theta \}
\]
is equal to the space of entire functions of exponential type at most \( a \) and with boundary values on \( \mathbb{R} \) that are square summable with respect to the measure \( (1 + x^2)^{-1} \, dx \).

Hence, we have
\[
\text{PW}_a \subset e^{-iaz} K_{\theta}^{\mathbb{R}} \quad \text{for } 0 < a.
\]
Further, one can prove that the codimension is equal to 1 and 

\[ e^{-iaz}K_{\theta_a} \oplus PW_a \]

consists of constant functions.

3. **Arbitrary non-zero vectors yield cyclic vectors for almost all parameters**

Let \( A \) be a self-adjoint (possibly unbounded) operator on a separable Hilbert space \( \mathcal{H} \) and let \( \varphi \) be a cyclic vector for \( A \). Define the family of self-adjoint rank-1 perturbations of \( A \), \( A_\alpha \), as in Subsection 2.2. Recall that \( (A_\alpha)_{ac} \) and \( (A_\alpha)_s \) denote the absolutely continuous part and the singular part of the operator \( A_\alpha \), respectively.

Further note that \( A_\alpha = (A_\alpha)_s \oplus (A_\alpha)_{ac} \), since \( (\mu_\alpha)_s \perp (\mu_\alpha)_{ac} \).

**Theorem 3.1.** Let \( A_\alpha \) be a family of self-adjoint rank-1 perturbations in a Hilbert space \( \mathcal{H} \) given by (2.1). Let \( 0 \neq f \in \mathcal{H} \). Then the following conditions are satisfied.

1. The function \( f \) is a cyclic vector for \( (A_\alpha)_{ac} \) for all but a countable number of \( \alpha \in \mathbb{R} \).
2. The function \( f \) is a cyclic vector for \( (A_\alpha)_s \) for Lebesgue-a.e. \( \alpha \in \mathbb{R} \).

In the following subsection, we prove an equivalent reformulation of this theorem in terms of its spectral representation.

3.1. **Proof of Theorem 3.1**

As we mentioned before, in view of the spectral theorem, instead of dealing with a general family of self-adjoint rank-1 perturbations given by equation (2.1), one can consider the self-adjoint rank-1 perturbations \( A_\alpha = M_t + \alpha(\cdot, \mathbf{1})_{L^2(\mu)} \mathbf{1} \) on \( L^2(\mu) \). Let the spectral operator \( V_\alpha : L^2(\mu) \to L^2(\mu_\alpha) \) be as defined in Subsection 2.2. In these settings, Theorem 3.1 can be stated as follows.

**Theorem 3.2.** Let \( 0 \neq f \in L^2(\mu) \). Then the following conditions are satisfied.

1. The function \( f_\alpha = V_\alpha f \in L^2(\mu_\alpha) \) is not equal to zero \((\mu_\alpha)_{ac}\)-a.e. for all but a countable number of \( \alpha \in \mathbb{R} \).
2. The function \( f_\alpha = V_\alpha f \in L^2(\mu_\alpha) \) is not equal to zero \((\mu_\alpha)_s\)-a.e. for Lebesgue-a.e. \( \alpha \in \mathbb{R} \).

**Remarks.** (a) Obviously, one cannot expect to obtain the above conclusion of cyclicity for all \( \alpha \in \mathbb{R} \). It is always possible to start with \( f \) that is zero on a set of positive \( \mu_0 \)-measure, that is, therefore, not cyclic for \( \alpha = 0 \). In fact, under the conditions of Theorem 3.2, we cannot replace ‘countable’ with ‘finite’.

(b) In the case of purely singular spectral measures for some natural classes of \( f \), the conclusion can be strengthened to ‘all but one’ \( \alpha \), see Section 4.

Our next example says that, in general, if \( f \) is not Hermitian, then \( f \) can be non-cyclic for uncountably many corresponding rank-1 perturbations. In particular, in the conclusion of Theorem 3.2 the distinction between the singular and the absolutely continuous is necessary. As was mentioned above, throughout the rest of the paper we will switch between self-adjoint and unitary settings as a matter of convenience. An analogous discussion can always be carried out in the other case.
EXAMPLE 1. Consider the setting of rank-1 unitary perturbation described in Subsection 2.3. We will construct a bounded holomorphic function $\theta$ such that for the family of spectral measures $(\sigma_\gamma)_{\gamma \in T}$ defined by (2.5) and the corresponding family of unitary operators $U_\gamma$ there exists a non-zero function $f \in L^2(\sigma_1)$ such that $f_\gamma = \Phi_\gamma \Phi^*_1 f$ is non-cyclic in $L^2(\sigma_\gamma)$ for an uncountable set of values of $\gamma$. Note that in this example $\theta$ is not inner and the corresponding operators have non-trivial absolutely continuous parts.

Let $C$ be a Cantor (closed uncountable) subset of the unit circle $T$. Let $w$ be the continuous function on $T$ defined by $w(\xi) = \text{dist}^2(\xi, C)$. Denote by $(\sigma_\gamma)_{\gamma \in T}$ the system of probability measures which is the family of Clark measures for some inner function $\theta$, and such that the measure $\sigma_1$ coincides, up to a multiplicative constant, with the measure $w\, dm$, where $m$ is the Lebesgue measure on $T$. Note that, by definition of $\sigma_1$, we have

$$\int \frac{1}{|x-y|^2} \, d\sigma_1(y) < \infty,$$

for any $x \in C$. It is well known (see, for example, [3]) that the last condition implies that each point of $C$ is a point mass for one of the measures $\sigma_\gamma$. Since $C$ is uncountable, we conclude that uncountably many measures $\sigma_\gamma$ have atoms on the set $C$.

Let now $F$ be an outer function with modulus equal to $w$-a.e. on $T$. Consider $f = F/w$. Then $f$ is a unimodular function on $T$, and we have $f\sigma_1 = fw = F$, hence $K_{f\sigma_1} = 0$ on the set $C$. So we have

$$\frac{K_{f\sigma_1}}{K_{\sigma_1}} = 0 \quad \text{on } C.$$

In virtue of the following Lemma 3.3, it follows that $\Phi_\gamma \Phi^*_1 f = 0$ on the (uncountable) set of those $\gamma \in T$ for which $\sigma_\gamma$ has a point mass on $C$. Hence, $f$ is not cyclic for uncountably many operators $U_\gamma$.

**Lemma 3.3 (Aronszajn–Krein-type formula).** Under the hypotheses of Theorem 3.2, we have

$$K_{f_\alpha \mu_\alpha} = \frac{K_{f_\beta \mu_\beta}}{1 + (\alpha - \beta)K_{\mu_\beta}} \quad \text{and} \quad \frac{K_{f_\alpha \mu_\alpha}}{K_{\mu_\alpha}} = \frac{K_{f_\beta \mu_\beta}}{K_{\mu_\beta}}. \quad (3.1)$$

**Proof.** First consider the case where $1 \in \mathcal{H} = L^2(\mu)$. Let $z \in \mathbb{C}\setminus\mathbb{R}$. Combining the second resolvent equation and the fact that $A_\alpha = A_\beta + (\alpha - \beta)(\cdot, 1)$, we obtain

$$(A_\beta - z1)^{-1} \cdot (A_\alpha - z1)^{-1} = (\alpha - \beta)((A_\alpha - z1)^{-1} \cdot 1)(A_\beta - z1)^{-1}1.$$

Application to a vector $f \in \mathcal{H}$ and pairing with $1$ yields

$$((A_\beta - z1)^{-1} f, 1) - ((A_\alpha - z1)^{-1} f, 1) = (\alpha - \beta)((A_\alpha - z1)^{-1} f, 1)((A_\beta - z1)^{-1}1, 1).$$

Recall that $V_\alpha A_\alpha = M_f V_\alpha, V_\alpha 1 = 1$ and $V_\alpha f = f_\alpha$. With this, we obtain

$$K_{f_\beta \mu_\beta} - K_{f_\alpha \mu_\alpha} = (\alpha - \beta)K_{f_\alpha \mu_\alpha} K_{\mu_\beta},$$

or equivalently,

$$K_{f_\alpha \mu_\alpha} = \frac{K_{f_\beta \mu_\beta}}{1 + (\alpha - \beta)K_{\mu_\beta}} \frac{K_{f_\alpha \mu_\alpha}}{K_{\mu_\beta}} \frac{K_{\mu_\beta}}{K_{\mu_\beta}} \frac{K_{f_\beta \mu_\beta}}{K_{\mu_\beta}} = \frac{K_{f_\beta \mu_\beta}}{K_{\mu_\beta}} \frac{K_{\mu_\beta}}{K_{\mu_\beta}}. \quad (3.2)$$

In the last equality, we used the well-known Aronszajn–Krein formula, which can be obtained from the first equality of (3.2) by using $f = \varphi$ (or equivalently $f_\alpha = 1$).
To obtain the second formula of (3.1), we divide both sides by $K_{\mu_\alpha}$.

If $1 \in H_{-1}(A) \setminus H$, then the resolvent formula is slightly more complicated

$$(A_\alpha - \lambda I)^{-1}f = (A_\beta - \lambda I)^{-1}f - \frac{(\alpha - \beta)}{1 + (\alpha - \beta)(K_{f_{\alpha}} + f_{\alpha} - f_{\beta})}I(A_\beta - \lambda I)^{-1}1,$$

for $f \in H_{-1}(A)$; see, for example, [11]. When paired with the vector $1$ this yields

$$K_{f_{\alpha}} = \left(1 - \frac{(\alpha - \beta)K_{\mu_\alpha}}{1 + (\alpha - \beta)K_{\mu_\alpha}}\right)K_{f_{\beta}} = \frac{K_{f_{\beta}1}}{1 + (\alpha - \beta)K_{\mu_\alpha}}.$$

The remainder of the proof for $1 \in H_{-1}(A) \setminus H$ now follows similarly to the case of regular perturbations $1 \in H$.

**Proof of part (1) of Theorem 3.2.** Define the set

$$\Sigma_\alpha = \{x \in \text{supp}(\mu_\alpha)_{ac} : f_{\alpha}(x) = 0\}.$$ 

The goal is to show that $(\mu_\alpha)_{ac}(\Sigma_\alpha) = 0$ for all but a countable number of parameters $\alpha$.

Assume that $f_{\alpha}$ is not cyclic for uncountably many $\alpha \in \mathbb{R}$, that is, assume that for some $S \subset \mathbb{R}$, $S$ uncountable, we have $(\mu_\alpha)_{ac}(\Sigma_\alpha) > 0$ for all $\alpha \in S$. Then $|\Sigma_\alpha| > 0$ for all $\alpha \in S$. Since $S$ is uncountable

$$|\Sigma_\alpha \cap \Sigma_\beta| > 0 \quad \text{for some } \alpha, \beta \in S \text{ with } \alpha \neq \beta. \quad (3.3)$$

Let us fix $\alpha$ and $\beta$ satisfying (3.3) and investigate the jump behavior in the first equation of (3.1). By Fatou’s jump theorem, see equation (2.3), we have

$$\left(K_{f_{\alpha}}\right)_-(x) - \left(K_{f_{\alpha}}\right)_+(x) = 2\pi i \frac{d(f_{\alpha})}{dx}(x) = 0 \quad \text{Lebesgue-a.e. } x \in (\Sigma_\alpha \cap \Sigma_\beta),$$

because $f_{\alpha} = 0$ on $\Sigma_\alpha$.

Similarly $K_{f_{\beta}}$ has no jump Lebesgue-a.e. on $\Sigma_\alpha \cap \Sigma_\beta$. On the other hand, $K_{\mu_\alpha}$ has a non-zero jump Lebesgue-a.e. on $(\Sigma_\alpha \cap \Sigma_\beta) \subset \text{supp}(\mu_\alpha)_{ac}$.

Hence, while the right-hand side in the first equation of (3.1) has a jump, the left-hand side does not jump Lebesgue-a.e. on $\Sigma_\alpha \cap \Sigma_\beta$ where we have (3.3), and we arrive at a contradiction. Therefore, the assumption that $f_{\alpha}$ is not cyclic for uncountably many $\alpha \in \mathbb{R}$ cannot be maintained.

**Proof of part (2) of Theorem 3.2.** Denote by $S_\alpha$ the essential support of the measure $(\mu_\alpha)_{ac}$ defined as the set of points where the Radon derivative of $(\mu_\alpha)_{ac}$ is infinite. It follows from the Aronszajn–Donoghue theorem, Theorem 2.1, that the sets $S_\alpha$ are disjoint. Define the set

$$\Omega_\alpha = \{x \in S_\alpha : f_{\alpha}(x) = 0\}.$$

The goal is to show that $(\mu_\alpha)_{ac}(\Omega_\alpha) = 0$ for Lebesgue-a.e. $\alpha \in \mathbb{T}$.

Assume that $f_{\alpha}$ is not cyclic for some set $\alpha$ with positive Lebesgue measure, that is, assume that for some $S \subset \mathbb{R}$, $|S| > 0$ we have

$$\mu_\alpha(\Omega_\alpha) > 0 \quad \text{for all } \alpha \in S.$$

In virtue of Theorem 2.3 (or, more precisely, its analog for the real line), we have for the boundary values

$$\frac{K_{f_{\alpha}}}{K_{\mu_\alpha}}(x + iy) \mapsto 0,$$

(3.4)
for \((\mu_\alpha)_{\cdot}\)-a.e. \(x \in \Omega_\alpha\) and all \(\alpha \in S\). Therefore, there exists a set \(M \subset \mathbb{R}, |M| > 0\) such that for all \(x \in M\) there exists an \(\alpha\) such that (3.4) is satisfied.

Using Lemma 3.3, the analytic function
\[
\frac{K_{f_0,\mu_\alpha}}{K_{\mu_\alpha}} = \frac{K_{f_0,\mu_0}}{K_{\mu_0}}
\]
has zero boundary values on \(M\), a set of Lebesgue measure greater than zero. 

Hence,
\[
\frac{K_{f_0,\mu_0}}{K_{\mu_0}} \equiv 0,
\]
and we must have \(f_0 \equiv 0\). But this contradicts the hypothesis that \(f\) is a non-zero vector.

Let us prove the lemma that was used in the above proofs. Recall the definition (2.2) of the Cauchy transform.

3.2. A Corollary of Lemma 3.3

Let us mention another consequence of Lemma 3.3, although we will not use this fact later in this paper. Consider a family of Aleksandrov–Clark measures \(\{\sigma_\gamma\}_{\gamma \in \Gamma}\).

**Corollary 3.4.** We have
\[
\frac{\mathcal{K}_{f_\gamma,\sigma_\gamma}}{\mathcal{K}_{f_\sigma_1}}(z) = \frac{1 - \theta(z)}{1 - \bar{\gamma}\theta(z)}, \quad m\text{-a.e. } z \in \mathbb{C}\setminus \mathbb{T}.
\]

In particular, the function
\[
\frac{\mathcal{K}_{f_\gamma,\sigma_\gamma}}{\mathcal{K}_{f_\sigma_1}}
\]
is independent of the choice of \(f = f_1 \in L^2(\sigma_1)\).

**Remark.** It was R. G. Douglas who observed the independence of the expression (3.5) from the choice of \(f \in L^2(\sigma_1)\).

**Proof.** In order to see the second statement note that by Lemma 3.3, the expression (3.5) is independent from the choice of \(f \in L^2(\sigma_1)\), that is,
\[
\frac{\mathcal{K}_{f_\gamma,\sigma_\gamma}}{\mathcal{K}_{f_\sigma_1}} = \frac{\mathcal{K}_{g_\gamma,\sigma_\gamma}}{\mathcal{K}_{g_\sigma_1}} \quad \text{for all } f, g \in L^2(\sigma_1).
\]

We obtain the first statement by expanding
\[
\frac{\mathcal{K}_{f_\gamma,\sigma_\gamma}}{\mathcal{K}_{f_\sigma_1}} = \left(\frac{\mathcal{K}_{f_\gamma,\sigma_\gamma}}{\mathcal{K}_{f_\sigma_1}}/\mathcal{K}_{\sigma_\gamma}\right)\mathcal{K}_{\sigma_\gamma},
\]
and use Lemma 3.3 for \(g = 1\) to cancel the fractions in the numerator and denominator. Further apply equation
\[
\mathcal{K}_{\sigma_\gamma}(z) = \frac{1}{1 - \bar{\gamma}\theta(z)}
\]
(confer of [3]) to the remaining \(\mathcal{K}_{\sigma_\gamma}\) in the numerator as well as \(\mathcal{K}_{\sigma_1}\) in the denominator.

3.3. Anderson-type Hamiltonians

The following operator is a generalization of most Anderson models discussed in literature.
For $n = 1, 2, \ldots$ consider the probability space $\Omega_n = (\mathbb{R}, B, \mu_n)$, where $B$ is the Borel sigma-algebra on $\mathbb{R}$ and $\mu_n$ is a Borel probability measure on $\mathbb{R}$. Let $\Omega = \prod_{n=1}^{\infty} \Omega_n$ be a product space with the probability measure $\mathbb{P}$ on $\Omega$ introduced as the product measure of the corresponding measures on $\Omega_n$ on the product sigma-algebra $\mathcal{A}$. The elements of $\Omega$ are points in $\mathbb{R}^\infty$, $\omega = (\omega_1, \omega_2, \ldots)$, $\omega_n \in \Omega_n$.

Let $\mathcal{H}$ be a separable Hilbert space. Consider a self-adjoint operator $H$ on $\mathcal{H}$ and let $\varphi_1, \varphi_2, \ldots$ be a countable collection of non-zero vectors in $\mathcal{H}$. For each $\omega \in \Omega$, define an Anderson-type Hamiltonian on $\mathcal{H}$ as a self-adjoint operator formally given by

$$H_\omega = H + V_\omega, \quad V_\omega = \sum_n \omega_n (\cdot, \varphi_n) \varphi_n. \quad (3.6)$$

We will suppose that the operator $H_\omega$ is densely defined $\mathbb{P}$-almost surely. Except for degenerate cases, the perturbation $V_\omega$ is almost-surely a non-compact operator. It is hence not possible to apply results from classical perturbation theory to study the spectra of $H_\omega$; see, for example, [2, 9].

In the case of an orthonormal sequence $\{\varphi_n\}$, this operator was studied in [7, 8].

Probably the most important special case of an Anderson-type Hamiltonian is the discrete random Schrödinger operator on $l^2(\mathbb{Z}^d)$

$$Hf(x) = -\Delta f(x) = - \sum_{|n|=1}^\infty (f(x + n) - f(x)),$$

$$\varphi_n(x) = \delta_n(x) = \begin{cases} 1 & \text{if } x = n \in \mathbb{Z}^d, \\ 0 & \text{else}. \end{cases}$$

### 3.4. An application of Theorem 3.1 to Anderson-type Hamiltonians

Let $H_\omega$ be the Anderson-type Hamiltonian introduced in equation (3.6). Fix $\omega_0 \in \Omega$. Assume that $\varphi \in \mathcal{H}_{-1}(H_{\omega_0})$ is a cyclic vector for the self-adjoint operator $H_{\omega_0}$. Consider operators $H_{\omega_0} + \alpha (\cdot, \varphi) \varphi, \alpha \in \mathbb{R}$.

Then (by Theorem 3.1) any non-zero $f \in \mathcal{H}$ is cyclic for $H_{\omega_0} + \alpha (\cdot, \varphi) \varphi$ for almost all $\alpha \in \mathbb{R}$.

In particular, for Lebesgue-a.e. $\alpha$, the operators $H_{\omega_0} + \alpha (\cdot, \varphi) \varphi$ are cyclic.

In the case where $\varphi \in \text{clos span} \{\varphi_n\}$, $\varphi = \sum a_n \varphi_n$, we say that $\varphi$ corresponds to the (possibly non-unique) sequence $a = (a_1, a_2, \ldots)$. Further, the operators $H_{\omega_0} + \alpha (\cdot, \varphi) \varphi$ correspond to $\omega$ belonging to the one-dimensional affine subspace

$$l(\omega_0, a) = \{\omega_0 + \alpha(a_1, a_2, a_3, \ldots) \mid \alpha \in \mathbb{R}\}.$$ 

Cyclicity of the operators for a.e. $\omega$ in any one-dimensional affine subspace is a stronger statement than $\mathbb{P}$-almost-sure cyclicity that can be found in the literature for some particular cases of our model. In terms of almost-sure cyclicity, we obtain the following result.

If $l(\omega_0, a)$ is a one-dimensional affine subspace of $\mathbb{R}^\infty$, then one can introduce Lebesgue measure on $l$ as

$$m(S) = |\{\alpha \mid \omega_0 + \alpha(a_1, a_2, a_3, \ldots) \in S\}|,$$

for any Borel subset $S$ of $l$.

**Corollary 3.5.** Suppose that $H_{\omega_0}$ is self-adjoint for some $\omega_0$ and that $\sum a_n \varphi_n, a = (a_1, a_2, \ldots)$, is cyclic for $H_{\omega_0}$. Consider a one-dimensional affine subspace of $\mathbb{R}^\infty$, $l = l(\omega_0, a)$. Then any non-zero vector $\varphi$ is cyclic for all $(H_\omega)_a$, $\omega \in l$, except possibly countably many $\omega$, and cyclic for a.e. $(H_\omega)_a, \omega \in l$, with respect to Lebesgue measure on $l$. 


In particular, suppose that the probability measure $\mathbb{P}$ is a product of absolutely continuous measures, $H_\omega$ is self-adjoint $\mathbb{P}$-almost surely and some $\varphi_n$ is cyclic for $H_\omega$, $\mathbb{P}$-almost surely. Then any non-zero $\varphi \in \mathcal{H}$ is cyclic for $H_\omega$, $\mathbb{P}$-almost surely.

It is well known that if an Anderson-type Hamiltonian is singular almost-surely, then it is cyclic almost-surely. The proof of almost-sure cyclicity of the singular part $(H_\omega)_s$ and almost-sure cyclicity of certain specific vectors can be found in [8] and for the discrete Schrödinger operator in [20]. The second part of Corollary 3.5 extends (from the singular part to the full Anderson-type Hamiltonian $H_\omega$) these results showing that if one of the vectors $\varphi_n$ is almost sure cyclic, then any non-zero vector possesses that property.

Proof of Corollary 3.5. The first statement follows immediately from Theorem 3.1.

Let $\omega$ be such that $\varphi_n$ is a cyclic vector for $H_\omega$. For $a = (0, \ldots , 0, a_n, 0, \ldots ) \in \mathbb{R}^\infty$ every $0 \neq \varphi \in \mathcal{H}$ is cyclic for a.e. point in $l(\omega, a)$. Since the union of such subspaces covers $\mathbb{P}$-almost all points of $\mathbb{R}^\infty$, we obtain the statement. $\square$

4. General Hermitian elements and rank-1 perturbations

Let $U$ be a unitary operator. For a vector $\varphi$, consider the space $X$ defined as the closure of the set of real finite linear combinations of elements of the form

$$(U + U^*)^n \varphi \quad \text{and} \quad \frac{1}{i}(U - U^*)^n \varphi \quad \text{for } n \in \mathbb{Z}.$$  

Then a vector $f \in \mathcal{H}$ is Hermitian with respect to $U$ and the vector $\varphi$, if $f \in X$ and $f \perp \varphi$.

An analogous definition can be given for self-adjoint operators. For a bounded self-adjoint operator $A$ on a separable Hilbert space $\mathcal{H}$ and a vector $\varphi$, let $(\Re A)\varphi$ denote the closure of the space of linear combinations of $A^n\varphi$, $n \in \mathbb{N}$ with real coefficients. We say that a vector $f \in \mathcal{H}$ is Hermitian with respect to the operator $A$ and the vector $\varphi$, if $f \in (\Re A)\varphi$ and $f \perp \varphi$. For general (unbounded) operators $(\Re A)\varphi$ can be defined as the closed span of

$$((A - zI)^{-1} + (A - \bar{z}I)^{-1})\varphi, \quad z \in \mathbb{C}_+.$$  

Note that in the settings of Subsection 2.3, the space $X$ defined above is the set of Hermitian functions from $K_\theta$ defined there, see the proof of Theorem 4.1.

Let $U$ be a unitary operator on a separable Hilbert space $\mathcal{H}$. Consider the family

$$U_\gamma = U + (\gamma - 1)(\cdot , U^{-1}b)_{\mathcal{H}}b$$  

of rank-1 perturbations, $\gamma \in \mathbb{T}$, $b \in \mathcal{H}$ with $\|b\|_{\mathcal{H}} = 1$. It is well known that $U_\gamma$ is unitary for all $\gamma \in \mathbb{T}$. Clearly, we have $U = U_1$. Without loss of generality, assume that $b$ is cyclic for $U$, that is,

$$\text{clos span}\{U^k b : k \in \mathbb{Z}\} = \mathcal{H}.$$  

Theorem 4.1. Consider the family $U_\gamma$ of rank-1 unitary perturbations given by (4.1). Assume that $U_\gamma$ has purely singular spectrum for some (all) $\gamma \in \mathbb{T}$. Let $0 \neq f \in \mathcal{H}$ be Hermitian with respect to $U = U_1$ and $b$. Fix a constant $c \in \mathbb{C}\{0\}$. Then the vector $f - cb$ is cyclic for $U_\gamma$ for all $\gamma \in \mathbb{T}\{e^{2\pi i \frac{c}{\theta}}\}$.

4.1. Proof of Theorem 4.1

Via the spectral theorem, without loss of generality one can assume that $U = U_1$ is an operator of multiplication by $z$ in $L^2(\sigma_1)$, $\sigma_1$ is the spectral measure of $U$ corresponding to $b$ and
b = 1 ∈ L^2(σ_1). Define the inner function \( \theta \) so that \( \sigma_1 \) is its Clark measure. Let \( K_\gamma, \gamma, \sigma_\gamma \) as well as \( \Phi_\gamma \) and \( \Phi_\gamma^* \) be as defined in Subsection 2.3. Consider operators \( \tilde{U}_\gamma \) defined in (2.4). Then \( U_\gamma = \tilde{U}_\gamma \).

Recall that \( f ∈ K_\theta \) is called Hermitian if \( f = \tilde{f} \), where \( \tilde{f} = \theta \bar{f} \).

Using Theorem 2.4, one can show that this definition is equivalent to that of a Hermitian element with respect to operator \( U = U_1 \) and vector \( b = S^*\theta \). The condition \( f(0) = 0 \) (for \( f ∈ K_\theta \)) is translated into \( f \perp b \).

Recall that by Theorem 2.3, the non-tangential limit of \( f ∈ K_\theta \) exist \( \sigma_\gamma \)-a.e. for all \( \gamma \in \mathbb{T} \). Let us denote this non-tangential limit by \( f_\gamma(z) = \lim_{\xi \to z} f(\xi), \sigma_\gamma \)-a.e.

In fact, we can identify these boundary values with one function, say \( f \) (slightly abusing notation), on the circle which is defined \( \sigma_\gamma \)-a.e. for all \( \gamma \). Indeed, we restricted ourselves to purely singular measures and by the Aronszajn–Donoghue theorem, Theorem 2.1, we have \( \sigma_\gamma \perp \sigma_\eta \) for \( \gamma \neq \eta \). Our statement now follows from Theorem 4.2.

**Theorem 4.2.** Let \( 0 \neq f ∈ K_\theta \) be a Hermitian function and fix any constant \( c ∈ \mathbb{C} \setminus \{0\} \). Then the level sets
\[
\{z \in \mathbb{T} : f(z) = c\}
\]
have zero \( \sigma_\gamma \)-measure for all \( \gamma ∈ \mathbb{T} \setminus \{e^{2i\arg c}\} \). In particular, the function \( f - c \) is cyclic for \( \tilde{U}_\gamma \) for all \( \gamma ∈ \mathbb{T} \setminus \{e^{2i\arg c}\} \).

**Proof of Theorem 4.2.** Pick \( f \) and \( c \) according to the hypotheses of the theorem. By Theorem 2.4, we have
\[
\arg f = \frac{\arg \gamma}{2} (\mod \pi),
\]
with respect to \( \sigma_\gamma \)-a.e.

If \( \gamma \) is such that \( f = c \) on a set \( S ⊂ \mathbb{T} \) with \( \sigma_\gamma(S) > 0 \), then we have
\[
\frac{\arg \gamma}{2} (\mod \pi) = \arg c,
\]
and therefore \( \gamma = e^{2i\arg c} \).

**Remark.** In the statement of Theorem 4.2, and therefore Theorem 4.1, the constant \( c \) cannot be equal to 0. Indeed, consider \( K_{z^n} \) that consists of all polynomials of degree less than \( n \). Let \( \beta_1, \ldots, \beta_{n-2} \) be points on \( \mathbb{T} \) such that \( \beta_k^{\beta_k} = \gamma_k \) are different points. Let
\[
p(z) = a_{n-1}z^{n-1} + \cdots + a_1z
\]
be a polynomial with roots at 0, \( \beta_1, \ldots, \beta_{n-2} \). Then
\[
\tilde{p}(z) = \bar{a}_1z^{n-1} + \cdots + \bar{a}_{n-1}z
\]
has roots at the same points. Note that \( p + \tilde{p} \) is a Hermitian element of \( K_{z^n} \) whose zero set \( Z = \{0, \beta_1, \ldots, \beta_{n-2}\} \) satisfies \( \sigma_{\gamma_k}(Z) = 1/n > 0 \) for \( k = 1, 2, \ldots, n - 2 \).

Let us mention the following examples that illustrate Theorem 4.2.

**Level sets of self-reciprocal polynomials.** A polynomial
\[
p(z) = \sum_{m=0}^{n-1} a_m z^m
\]
of degree less than \( n \) is a Hermitian element of \( K_{\theta^n} \), if and only if
\[
a_0 = 0 \quad \text{and} \quad a_m = \overline{a_{n-m}}, \quad m = 1, \ldots, n - 1. \tag{4.2}
\]
Polynomials that satisfy (4.2) are called self-reciprocal. Note that the Clark measure \( \sigma_\gamma \) of \( \theta = z^n \) is concentrated on the set of \( n \)th roots of \( \gamma \).

Hence, if \( c \neq 0 \) and \( p \) is a self-reciprocal polynomial, then by Theorem 4.2 all roots of the equation \( p = c \) on \( T \) must be contained in a set of \( n \)th roots of \( \gamma \) for \( \gamma \in T \) given in the statement of the theorem.

Naturally, this simple fact can also be proved directly. If \( z \) is such that \( p(z) = c \), then
\[
c = p(z) = z^n p(z) = z^n \overline{c}.
\]
Hence, \( z^n = c/\overline{c} \) where \( |c/\overline{c}| = 1 \) and \( \arg(c/\overline{c}) = 2 \arg c \). The proof of Theorem 4.2 can be viewed as a generalization of this argument.

The non-zero level sets of Paley–Wiener functions. Recall the definition of Paley–Wiener functions from Subsection 2.4.

The following statement is an analog to Euler’s Formula \( e^{i\theta} = \cos \theta + i \sin \theta \) for Paley–Wiener functions.

**Proposition 4.3.** Let \( f \in \text{PW}_a \). Then \( e^{iaz} f = g_1 + ig_2 \) where \( g_1, g_2 \) are entire functions such that \( g_1, g_2 \in e^{iaz} \text{PW}_a \) and each level set
\[
\{ x \in \mathbb{R} \mid g_i(x) = c \}, \quad i = 1, 2; \quad c \neq 0
\]
is contained in the arithmetic progression
\[
\left\{ 2 \arg c + \frac{2\pi n}{a} \right\}_{n \in \mathbb{Z}}.
\]

**Proof.** Recall that \( K^\mathbb{R}_{\theta_a} \) was defined as the function space obtained by ‘mapping’ the model space \( K_\theta \), where
\[
\theta_a(z) = \theta(z) = e^{-2a(1+z)/(1-z)}, \quad 0 < a \leq 1,
\]
from \( \mathbb{D} \) to \( \mathbb{R} \) using the standard conformal map \( \psi : \mathbb{C}_+ \to \mathbb{D} \), see Subsection 2.4. Then
\[
e^{iaz} \text{PW}_a \subset K^\mathbb{R}_{\theta_a}.
\]
Hence, we have \( e^{iaz} f \in K^\mathbb{R}_{\theta_a} \).

Without loss of generality \( a = 1 \). For the inner function \( \theta = e^{-(1+z)/(1-z)} \) in \( \mathbb{D} \), the Clark measure \( \sigma_\gamma \gamma \in T \) is concentrated on the sequence
\[
\psi(\{ \arg \gamma + 2\pi n \}) \subset T.
\]

Like in the remark following Theorem 4.2, we can decompose \( f \) into \( f = g_1 + ig_2 \) where the translations \( \tilde{g}_1 \) and \( \tilde{g}_2 \) of the functions \( g_1 \) and \( g_2 \) from the upper half-plane to the disk are Hermitian in \( K_\theta \). By Theorem 4.2,
\[
\{ z \in T : \tilde{g}_i = c \} \subset \psi(\{ \arg \gamma + 2\pi n \}), \quad \arg \gamma = 2 \arg c.
\]

Hence, the level sets of the functions \( g_i \) are contained in arithmetic progressions given in the statement.

**References**

1. P. W. Anderson, ‘Absence of diffusion in certain random lattices’, Phys. Rev. 109 (1958) 1492–1505.
2. M. S. Birman and M. Z. Solomjak, *Spectral theory of self-adjoint operators in Hilbert space* (Springer Kluwer Academic Publishers, 1987).
3. J. A. Cima, A. L. Matheson and W. T. Ross, *The Cauchy transform*, Mathematical Surveys and Monographs 125 (American Mathematical Society, Providence, RI, 2006).
4. D. N. Clark, ‘One dimensional perturbations of restricted shifts’, *J. Anal. Math.* 25 (1972) 169–191.
5. F. Germinet, A. Klein and J. H. Schenker, ‘Dynamical delocalization in random Landau Hamiltonians’, *Ann. of Math.* (2) 166 (2007) 215–244.
6. F. Ghribi, P. D. Hislop and F. Klopp, ‘Localization for Schrödinger operators with random vector potentials’, *Contemp. Math.* 447 (2007) 123–138.
7. V. Jakšić and Y. Last, ‘Spectral structure of Anderson type Hamiltonians’, *Invent. Math.* 141 (2000) 561–577.
8. V. Jakšić and Y. Last, ‘Simplicity of singular spectrum in Anderson-type Hamiltonians’, *Duke Math. J.* 133 (2006) 185–204.
9. T. Kato, *Perturbation theory for linear operators*, Classics in Mathematics (Springer, Berlin, 1995), Reprint of the 1980 edition.
10. W. Kirsh, ‘An invitation to Random Schrödinger operators’, *Random Schrödinger operators*, Panoramas et Synthèses 25 (Soc. Math. France, Paris, 2008) 1–119.
11. P. Kurasov, ‘Singular and supersingular perturbations: Hilbert space methods’, *Spectral theory of Schrödinger operators* (2004), Contemporary Mathematics 340 (Lecture Notes from a Workshop on Schrödinger Operator Theory, 2003) 185–216.
12. Y. Last, ‘Exotic spectra: a review of Barry Simon’s central contributions’, *Proc. Sympos. Pure Math.* 76.2. (2007) 697–712.
13. C. Liaw and S. Treil, ‘Rank one perturbations and singular integral operators’, *J. Funct. Anal.* 257 (2009) 1947–1975.
14. N. K. Nikolski, *Treatise on the shift operator*, Translated from the Russian by Jaak Peetre, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences] 273 (Springer, Berlin, 1986) xii+491 pp.
15. N. K. Nikolski, ‘Operators, functions, and systems: an easy reading’, *Model operators and systems*, vol. 2, Translated from the French by Andreas Hartmann and revised by the author, Mathematical Surveys and Monographs 93 (American Mathematical Society, Providence, RI, 2002) xiv+439 pp.
16. A. Poltoratski, ‘The Krein spectral shift and rank one perturbations of spectra’, *St. Petersburg Math. J.* 10 (1999) 833–859.
17. A. G. Poltoratski, ‘Boundary behavior of pseudocontinuable functions’, *Algebra i Analiz* 5 (1993) 189–210 (Russian), *St. Petersburg Math. J.* 5 (1994) 389–406 (English).
18. A. Poltoratski and D. Sarason, ‘Aleksandrov–Clark measures’, *Recent advances in operator-related function theory*, Contemporary Mathematics 393 (American Mathematical Society, Providence, RI, 2006) 1–14.
19. R. del Rio, S. Jitomirskaya, Y. Last and B. Simon, ‘Operators with singular continuous spectrum. IV. Hausdorff dimensions, rank-one perturbations, and localization’, *J. Anal. Math.* 69 (1996) 153–200.
20. B. Simon, ‘Cyclic vectors in the Anderson model’, *Rev. Math. Phys.* 6 (1994) 1183–1185, Special issue dedicated to Elliott H. Lieb.
21. B. Simon, ‘Spectral analysis of rank-one perturbations and applications’, *Mathematical quantum theory II: Schrödinger operators* (Vancouver, BC, 1993), CRM Proceeding Lecture Notes 8 (American Mathematical Society, Providence, RI, 1995) 109–149.

Evgeny Abakumov  
LAMA (UMR 8050)  
UPCEMLV, UPEC, CNRS  
Université Paris-Est  
F-77454 Marne-la-Vallée  
France  
evgueni.abakoumou@univ-mlv.fr

Alexei Poltoratski  
Department of Mathematics  
Texas A&M University  
Mailstop 3368  
College Station, TX 77843  
USA  
alexeip@math.tamu.edu

Constanze Liaw  
Department of Mathematics  
Baylor University  
One Bear Place #97328  
Waco, TX 76798  
USA  
Constanze.Liaw@baylor.edu