Monogamous property of generalized W states in three-qubit systems in terms of relative entropy of entanglement

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Abstract

Because of the difficulty in getting the analytic formula of relative entropy of entanglement, it becomes troublesome to study the monogamy relations of relative entropy of entanglement for three-qubit pure states. However, we find that all generalized W states have the monogamous property for relative entropy of entanglement by calculating the relative entropy of entanglement for the reduced states of the generalized W states in three-qubit systems.

Keywords: relative entropy of entanglement, generalized W states, monogamous property

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Entanglement has played a significant role in the field of quantum information and quantum computation \cite{1}. This attracts an increasing interest in the study of quantification of entanglement for any quantum state. Lots of entanglement measures have been proposed for last two decades, e.g. entanglement of formation (EoF), concurrence, distillable entanglement and relative entropy of entanglement (REE). Among them, one of the important entanglement measures is the distillable entanglement as it is the optimal rate at which one can extract maximally entangled states out of the given state. However, it is not a trivial task to give the analytical expression of distillable entanglement for general quantum states. Fortunately, the REE has been shown to provide the tight upper bound on distillable entanglement \cite{2, 3}.

The REE can be used as a distance function on the set of density operators. It is defined as

$$E_r(\rho) = \min_{\sigma \in \mathcal{D}} S(\rho\|\sigma) = \min_{\sigma \in \mathcal{D}} \text{tr}(\rho \ln \rho - \rho \ln \sigma),$$

where $\mathcal{D}$ is the set of separable states. Although the analytical solutions have been given for the states of some special sets with highly symmetry \cite{2, 8, 10, 11}, it is still an open fundamental problem \cite{12} to obtain a closed formula for a two-qubit state due to including the convex optimization problem in terms of REE. Recently, for a given closest separable state (CSS) $\sigma$ in two-qubit system, Miranowicz and Ishizaka have given the REE of all entangled states $\rho$ which have $\sigma$ as their CSS and obtained the analytical solution of REE in some special cases \cite{8}. In fact, Friedland and Gour have found that there exists the closed formula of all entangled states for a given CSS on the boundary of separable states in \cite{13}. In \cite{8, 9}, Kim et. al. studied the inverse process and found that it is still an unsolved problem.

On the other hand, multipartite entanglement plays an important role in condensed matter physics. The seminal work of Coffman, Kundu and Wootters (CKW)\cite{14} provided a way to quantify the three-party entanglement by introducing the three-tangle. Since then, many researchers began to address the monogamy relations of entanglement \cite{15–20}. The monogamy inequality in terms of concurrence in three-qubit systems introduced by CKW:

$$C^2_{AB} + C^2_{AC} \leq C^2_{A:BC},$$

where $C^2_{AB}$ ($C^2_{AC}$) is the concurrence between $A$ and $B$ ($C$), while $C^2_{A:BC}$ — between system $A$ and $BC$. Recently, Osborne and Verstraete \cite{21} have generalized the CKW inequality to
$n$-qubit systems. In addition, the monogamy inequality holds in terms of distillable entanglement [22], negativity [23] and squashed entanglement [24], but fails for other definitions such as the concurrence itself and the EoF [22, 25] for a three-qubit state. This indicates that it is of crucial important for the proper choice of the entanglement measure to capture the monogamous nature of quantum entanglement. Now one may ask whether the monogamy inequality holds in terms of REE for a three-qubit pure state. As it is unsolved to find the analytical expression for a two-qubit state in terms of REE [8, 9], the monogamy relations of REE for three-qubit states become very troublesome problem.

It is known that there are two inequivalent classes of genuine tripartite entangled states characterized by means of local operations and classical communication (LOCC) for three-qubit states [26]. One is the Greenberger-Horne-Zeilinger (GHZ) class [27], and the other is the W class [26]. The feature of GHZ-class states is that when any one of the three qubits is traced out, the remaining two are separable. Hence, all GHZ-class states are monogamous for any entanglement measure. The W-class states have the property that their entanglement has the highest degree of endurance against loss of one of the three qubits. Thus it is worth investigating the monogamous property of W-class states. In this paper, we only investigate the monogamy relations of generalized W states

$$|\psi_W\rangle = \alpha|001\rangle + \beta|010\rangle + \gamma|100\rangle,$$

where $\alpha, \beta, \gamma \in \mathbb{R}$ and $\alpha^2 + \beta^2 + \gamma^2 = 1$.

It is found that the reduced states of the states (3) have the following form:

$$\rho = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & \sqrt{bc} & 0 \\ 0 & \sqrt{bc} & c & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where $a+b+c=1$. Moreover, the states with the form (4) are $U_x \otimes U_x$-invariant states [28].

Recall that the most general two-qubit state which is $U_x \otimes U_x$-invariant has the following form [28]:

$$\rho = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & f & 0 \\ 0 & f & c & 0 \\ 0 & 0 & d & 0 \end{pmatrix}.$$
where \(a + b + c + d = 1\) and \(bc \geq f^2\). Accordingly, the most general state which is \(U_x \otimes U_x^*\)-invariant state has the form:

\[
\rho' = \begin{pmatrix}
 b & 0 & 0 & f \\
 0 & a & 0 & 0 \\
 0 & 0 & d & 0 \\
 f & 0 & 0 & c \\
\end{pmatrix},
\]

where \(a + b + c + d = 1\) and \(bc \geq f^2\).

At first, we derive the REE of reduced states of generalized W states in three-qubit systems by use of their symmetric property.

**Theorem** For \(\rho\) with the form (4), we have

\[
E_r(\rho) = a \log a + 2(1-a) \log (1-a) + \log[(1+M)(1+N)] - (b+c) \log(bM + 2\sqrt{bcMN} + cN),
\]

where

\[
M = \frac{\sqrt{\Delta} + b - c - 2a^2b}{2ab(1 + a)},
\]

\[
N = \frac{\sqrt{\Delta - b + c - 2a^2c}}{2ac(1 + a)},
\]

with \(\Delta = (b - c)^2 + 4a^2bc\).

**Proof.** In order to prove the theorem we utilize the fact that

\[
E_r(\rho) = \inf \{ S(\rho||\sigma) | \sigma \in \mathcal{P} \mathcal{D} \}
\]

when \(\rho = P\rho\), where \(P\) denotes a projector projecting an arbitrary state onto the class (4). Moreover, a state from (5) is separable iff it has the form:

\[
\sigma^* = \begin{pmatrix}
 x & 0 & 0 & 0 \\
 0 & u & re^{i\theta} & 0 \\
 0 & re^{-i\theta} & v & 0 \\
 0 & 0 & 0 & y \\
\end{pmatrix}
\]

with \(u \geq 0, v \geq 0, x \geq 0, y \geq 0, uv - r^2 \geq 0, x + u + v + y = 1\) and \(xy - r^2 \geq 0\) by use of PPT criterion [29]. Because the entangled state \(\rho\) is not full rank, \(\sigma^*\) must be on the boundary of \(\mathcal{D}\) [13]. Therefore, either \(\sigma^*\) or its partial transposition \((\sigma^*)^{T_A}\) has at least one zero eigenvalue.
Assume \((\sigma^*)^T A\) is singular. The eigenvalues \(u, v, x, y, \sqrt{(x-y)^2 + 4r^2}\) of \((\sigma^*)^T A\) show that \((\sigma^*)^T A\) is rank deficient when \(r^2 = xy\). Thus the CSS for \(\rho\) is the state (9) with \(r^2 = xy\). Suppose \(uv - r^2 = \epsilon\), where \(\epsilon \geq 0\), we have \(xy = uv - \epsilon\). Thus

\[
E_r(\rho) = a \log a + (b + c) \log(b + c) + \min \{-\text{tr}(\rho \log \sigma^*)\}
\]

\[
\geq a \log a + (1 - a) \log(1 - a) + \min \{-a \log(0|\sigma^*|00) - (b + c) \log(\psi|\sigma^*|\psi)\}
\]

\[
= a \log a + (1 - a) \log(1 - a)
\]

\[
+ \min \{-a \log x - (b + c) \log\left[\frac{b}{b + c} u + 2\sqrt{bcx} \cos \theta + \frac{c}{b + c} v\right]\}
\]

\[
= a \log a + 2(1 - a) \log(1 - a)
\]

\[
+ \min \{-a \log x - (1 - a) \log[bu + 2\sqrt{bcx} \cos \theta + cv]\},
\]

where \(|\psi\rangle = (0, \sqrt{\frac{b}{b+c}}, \sqrt{\frac{c}{b+c}}, 0)^T\) and we have used the fact that \(f(x) = -\log x\) is convex. Now we have to minimize the function

\[
f(\theta, \epsilon) = -a \log x - (1 - a) \log[bu + 2\sqrt{bcx} \cos \theta + cv].
\]

It is easy to show that \(f(\theta, \epsilon)\) is an increasing function on \(\epsilon\) and have the minimization when \(\cos \theta = 1\). Thus it is found that

\[
f(\theta, \epsilon)_{\text{min}} = -a \log x - (1 - a) \log[bu + 2\sqrt{bcuv} + cv]
\]

and \(xy = uv\). Let

\[
x = \cos^2 \theta_1 \cos^2 \theta_2, \quad y = \sin^2 \theta_1 \sin^2 \theta_2, \quad u = \sin^2 \theta_1 \cos^2 \theta_2, \quad v = \cos^2 \theta_1 \sin^2 \theta_2.
\]

The minimization problem of \(E_r(\rho)\) has been translated into solving the minimum value of

\[
g(\theta_1, \theta_2) = a \log a + 2(1 - a) \log(1 - a) - a \log \cos^2 \theta_1 \cos^2 \theta_2
\]

\[
-(1 - a) \log[\sqrt{b} \sin \theta_1 \cos \theta_2 + \sqrt{c} \cos \theta_1 \sin \theta_2]^2.
\]

The explicit calculation of the minimization of \(E_r(\rho)\) is now a tedious but straightforward exercise, whose result is quoted in the right side of the Eq. (7). Therefore the lower limit of \(E_r(\rho)\) can be reached. In addition, if \(\sigma^*\) is singular, one can obtain the same result by use of a similar analysis, which proves the Eq. (7). \(\Box\)
Example  Set \( a = 1 - \lambda, b = c = \frac{\lambda}{2} \) when \( b = c \). The state (4) becomes the most general example of Vedral-Plenio states

\[
\rho_{vp} = (1 - \lambda)|00\rangle\langle00| + \lambda|\Psi^+\rangle\langle\Psi^+|,
\]
where \( |\Psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \). By a calculation, it is found that \( M = N = \frac{\lambda}{2 - \lambda} \) and the known result

\[
E_r(\rho_{vp}) = (1 - \lambda) \log(1 - \lambda) + (\lambda - 2) \log(1 - \frac{\lambda}{2})
\]
derived in [2, 3].

Remark  It is easy to show that the REE of the states (5) and (6) are the same because they are local unitary equivalent. Thus one can obtain the corresponding REE of \( U_x \otimes U_x^* \)-invariant states

\[
\rho' = \begin{pmatrix}
    b & 0 & 0 & f \\
    0 & a & 0 & 0 \\
    0 & 0 & 0 & 0 \\
    f & 0 & 0 & c
\end{pmatrix}.
\]

(11)

In the following we discuss the monogamy relations of the REE for the generalized W states (3). Clearly, the reduced density matrices of a generalized W state (3) can be written as:

\[
\rho_{AB} = \text{tr}_C(|\psi_W\rangle\langle\psi_W|) = \begin{pmatrix}
    \alpha^2 & 0 & 0 & 0 \\
    0 & \beta^2 & \beta \gamma & 0 \\
    0 & \beta \gamma & \gamma^2 & 0 \\
    0 & 0 & 0 & 0
\end{pmatrix},
\]

(12)

and

\[
\rho_{AC} = \text{tr}_B(|\psi_W\rangle\langle\psi_W|) = \begin{pmatrix}
    \beta^2 & 0 & 0 & 0 \\
    0 & \alpha^2 & \alpha \gamma & 0 \\
    0 & \alpha \gamma & \gamma^2 & 0 \\
    0 & 0 & 0 & 0
\end{pmatrix}.
\]

(13)

For convenience, let \( E_{AB}^r, E_{AC}^r \) and \( E_{A:BC}^r(|\psi_W\rangle) \) represent the REE of \( \rho_{AB}, \rho_{AC} \) and a bipartite state of (3) on subsystems \( A \) and \( BC \), respectively. Moreover, denote

\[
\delta = E_{A:BC}^r(|\psi_W\rangle) - E_{AB}^r - E_{AC}^r,
\]

(14)

where

\[
E_{A:BC}^r(|\psi_W\rangle) = -\gamma^2 \log \gamma^2 - (1 - \gamma^2) \log(1 - \gamma^2).
\]
In order to study the monogamy relations of generalized W states by REE, we consider the image of $\delta$ presented in FIG. 1.

The FIG. 1 shows that all generalized W states in three-qubit systems satisfy the monogamy relations of REE. Thus the entanglement $A-BC$, measured by REE, is not completely determined by its partial entanglements, $A-B$ and $A-C$. It shows the fact that the entanglement of these states is very robust against particle loss. Recently, the monogamy property of the quantum correlations in three qubit pure states is considered. Prabhu et.al. considered the monogamy property of quantum correlations by use of the quantum discord in [31]. It is found that generalized W states do not follow monogamy relation in terms of quantum discord. In addition, monogamy of quantum correlations for generalized W states using Rajagopal-Rendell quantum deficit is also addressed in [32]. It is showed that generalized W states exhibit both mono and polygamous nature unlike that of the W state. It is different from the monogamy property for generalized W states in terms of REE.

In conclusion, it is found that all generalized W states have the monogamous property in terms of REE. The result could shed new lights on the study of genuine multipartite entanglement. In addition, we have obtained the analytic formula for the reduced states of generalized W states by their symmetric property. In fact, these states are rank-2 two-qubit $U_x \otimes U_x$-invariant states. The investigation of monogamous property of generalized W states is an application of expressions of their REE. In [26], it is shown that there are six different classes for three-qubit pure states under LOCC with nonzero probability. Although it is still
not known if the monogamy relations in terms of REE hold for all three-qubit states, they are true for all generalized W states and other five classes except W-class. After all, as it is shown that a closed formula of REE for two qubits can be given only in some special cases in [8, 9], it is difficult to examine the monogamous property of all three-qubit states.

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