QUASI-SYMMETRIC DESIGNS ON 56 POINTS

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ABSTRACT. Computational techniques for the construction of quasi-symmetric block designs are explored and applied to the case with 56 points. One new (56, 16, 18) and many new (56, 16, 6) designs are discovered, and non-existence of (56, 12, 9) and (56, 20, 19) designs with certain automorphism groups is proved. The number of known symmetric (78, 22, 6) designs is also significantly increased.

1. Introduction

A block design with parameters \((v, k, \lambda)\) consists of a set of \(v\) points and a family of \(k\)-element subsets called blocks such that every pair of points is contained in \(\lambda\) blocks. The number of blocks through a single point \(r = \lambda \cdot \frac{v-1}{k-1}\) and the total number of blocks \(b = \lambda \cdot \frac{v(v-1)}{k(k-1)}\) are also determined by the parameters. The degree of a design is the number of cardinalities \(|B_1 \cap B_2|\) occurring as intersections of two blocks \(B_1\) and \(B_2\). Symmetric designs are designs with \(v = b\) or, equivalently, designs of degree 1. A design is called quasi-symmetric if it is of degree 2, i.e. if any pair of blocks intersects in \(x\) or in \(y\) points, for some integers \(x < y\). We refer to [1] for results about block designs and generalisations such as \(t\)-designs and partially balanced designs, and to [22, 23] for results about quasi-symmetric designs (QSDs).

Neumaier [17] published a table of admissible parameters of QSDs with \(v \leq 40\). It contains 15 parameter sets for which the existence of QSDs was unknown at the time. Although all but 3 of these parameter sets have been eliminated in the meantime, the recent construction of (56, 16, 18) QSDs [13] gives hope that there may be other parameters for which QSDs exist, but have not been discovered yet. In this paper we focus on quasi-symmetric designs on 56 points and computational techniques for their construction. We explore known methods relying
on assumed automorphism groups and enhance them sufficiently to be able to thoroughly examine the case \( v = 56 \). One new \((56, 16, 18)\) and many new \((56, 16, 6)\) QSDs are constructed, and non-existence of \((56, 12, 9)\) and \((56, 20, 19)\) QSDs with certain automorphism groups is proved.

The layout of our paper is as follows. In Section 2, we survey known results about QSDs on 56 points and eliminate two parameter sets by a theorem of Calderbank [5]. Section 3 is devoted to \((56, 16, 18)\) QSDs. An algorithm for generating good orbits of \(k\)-element subsets is developed. Together with an approach based on clique search, it is used to classify \((56, 16, 18)\) QSDs with a permutation group \( G_{48} \) of order 48, resulting in a new design. In Section 4, the group \( G_{48} \) is used to construct 876 new \((56, 16, 6)\) QSDs. A complete classification of these designs with the Frobenius group \( Frob_{21} \) of order 21 is carried out by computations based on orbit matrices. This technique is also used to construct 303 new \((56, 16, 6)\) QSDs with automorphism groups isomorphic to the alternating group \( A_4 \). More examples are constructed from the binary linear codes associated with the constructed designs. The new \((56, 16, 6)\) QSDs significantly increase the number of known symmetric \((78, 22, 6)\) designs, in which they can be embedded as residual designs. In Section 5, the developed construction techniques are applied to \((56, 12, 9)\) and \((56, 20, 19)\) QSDs with automorphism groups from the previous sections. It is shown that \( G_{48} \) cannot be an automorphism group of these QSDs, and \((56, 12, 9)\) QSDs with \( Frob_{21} \) and some subgroups of \( G_{48} \) are also eliminated. The results of our computations are summarised in the final Section 6.

The paper relies heavily on computer calculations. We use GAP [10] for group calculations, and nauty [15] to check isomorphism and compute full automorphism groups of designs. Cliquer [18, 19] is used to search for cliques in weighted graphs, and Magma [2] for calculations with codes associated with the designs. The most time-critical calculations are performed with our own programs written in the C language.

2. The known QSDs on 56 points

M. S. Shrikhande’s survey in the Handbook of Combinatorial Designs [22] includes a table of admissible parameters of quasi-symmetric designs with \( v \leq 70 \) (Table 48.25). We reproduce the six rows relating to designs on 56 points in Table 1 with updated information on the number of designs in column NQSD.

The first QSD on 56 points was constructed by Tonchev [24] by embedding the \( 3-(22, 6, 1) \) design in a symmetric \((78, 22, 6)\) design.
Table 1. The known QSDs with $v = 56$.

The corresponding residual design is quasi-symmetric with parameters $(56, 16, 6)$, $x = 4$, $y = 6$, appearing in row no. 52. Another QSD with these parameters was constructed in [16] from words of weight 16 in the binary linear code associated with the first design.

Three QSDs with parameters $(56, 16, 18)$, $x = 4$, $y = 8$ from row 47 were constructed in [13] by assuming a suitable automorphism group. The question marks in rows 48 and 50 of [22, Table 48.25] can be eliminated by Calderbank’s Theorem 2 from [5]. A version of the theorem specialised to quasi-symmetric designs is reproduced here for the convenience of the reader.

**Theorem 2.1** (Calderbank [5]). Let $p$ be an odd prime and let $D$ be a $(v, k, \lambda)$ QSD with block intersection numbers $x \equiv y \equiv s \pmod{p}$. Then either

1. $r \equiv \lambda \pmod{p^2}$,
2. $v \equiv 0 \pmod{2}$, $v \equiv k \equiv s \equiv 0 \pmod{p}$, $(-1)^{v/2}$ is a square in $GF(p)$,
3. $v \equiv 1 \pmod{2}$, $v \equiv k \equiv s \not\equiv 0 \pmod{p}$, $(-1)^{(v-1)/2} s$ is a square in $GF(p)$,
4. $r \equiv \lambda \equiv 0 \pmod{p}$ and either
   a. $v \equiv 0 \pmod{2}$, $v \equiv k \equiv s \not\equiv 0 \pmod{p}$,
   b. $v \equiv 0 \pmod{2}$, $k \equiv s \not\equiv 0 \pmod{p}$, $v/s$ is a nonsquare in $GF(p)$,
   c. $v \equiv 1 \pmod{2p}$, $r \equiv 0 \pmod{p^2}$, $k \equiv s \not\equiv 0 \pmod{p}$,
   d. $v \equiv p \pmod{2p}$, $k \equiv s \equiv 0 \pmod{p}$,
   e. $v \equiv 1 \pmod{2}$, $k \equiv s \equiv 0 \pmod{p}$, $v$ is a nonsquare in $GF(p)$,
   f. $v \equiv 1 \pmod{2}$, $k \equiv s \equiv 0 \pmod{p}$, $v$ and $(-1)^{(v-1)/2}$ are squares in $GF(p)$.

For $p = 3$, the parameters $(56, 15, 42)$, $x = 3$, $y = 6$ and $(56, 21, 24)$, $x = 6$, $y = 9$ do not satisfy any of the conditions of Theorem 2.1, so
these designs don’t exist. This seems to have been overlooked in Table I of Calderbank’s paper [5] and the omission was copied in later editions of the table, including [22, Table 48.25]. Designs with parameters from rows 49 and 51 cannot be eliminated by this theorem. We shall attempt to construct them by computational techniques relying on assumed automorphism groups and to increase the number of known designs in rows 47 and 52.

Tables of admissible parameters of QSDs also appear in [20], organised by the associated strongly regular graphs (SRGs). Parameters from rows 47 and 52 appear in [20, Table 1], where the SRGs are known to exist. However, parameters from rows 48 to 51 are missing from [20, Table 3], where existence of the SRGs is unknown.

3. A new (56, 16, 18) QSD

Let $G$ be a permutation group on a $v$-element set, say $V = \{1, \ldots, v\}$. Finding $(v, k, \lambda)$ designs with $V$ as the set of points and $G$ as an automorphism group is done in two steps:

1. generate the orbits of $G$ on $k$-element subsets of $V$,
2. select orbits comprising blocks of the design.

For quasi-symmetric designs, only good orbits need to be considered, i.e. orbits containing $k$-element sets intersecting in $x$ or $y$ points.

In [13], a group $H$ isomorphic to $(\mathbb{Z}_2)^4 \rtimes A_5$ was used to find three (56, 16, 18) designs with intersection numbers $x = 4$, $y = 8$. Here and in the sequel, $N \rtimes M$ denotes a semidirect product of groups $N$ and $M$, where $N$ is normal in the product. The computation in [13] was fairly small because the order of the group $|H| = 960$ exceeds the number of blocks of the design, $b = 231$. Orbits of size greater than $b$ can be omitted, and the “short orbits” can be generated efficiently by an algorithm based on stabilisers, as described in [13].

Let us now consider the subgroup $G_{48} = \langle \alpha \beta \alpha^{-1}, \alpha^{-1} \beta \alpha^2, \beta \alpha \beta \alpha^{-1} \beta \rangle \cong (\mathbb{Z}_2)^4 \rtimes \mathbb{Z}_3$, where $\alpha$ and $\beta$ are the generators of $H$ given in [13]. Now we need to find orbits of all sizes, up to $|G_{48}| = 48$. We use an orderly algorithm of Read-Faradžev type [9, 21] (see also [14]). Subsets of $V$ are compared lexicographically. Suppose $U = \{u_1, \ldots, u_k\}$ and $W = \{w_1, \ldots, w_k\}$, for some $u_1 < \ldots < u_k$ and $w_1 < \ldots < w_k$. Then $U < W$ provided there is an index $i$ such that $u_i < w_i$ and $u_j = w_j$, for all $j < i$. We denote by $m(U)$ the minimal element of the orbit $\{gU \mid g \in G\}$ with respect to this total order. The call \texttt{GoodOrbits(0)} of the following recursive algorithm will output the minimal representative from each good orbit of $k$-element subsets of $V$.

1: \textbf{procedure} \texttt{GoodOrbits}(U: subset of $V$)
The correctness of the algorithm is based on the following observation. If \( U' \) is minimal, i.e. \( m(U') = U' \), and \( U \) is obtained by removing the largest element of \( U' \), then \( U \) is also minimal (\( m(U) = U \)). A program written in C based on this algorithm needs about 6 days of CPU time on a 2.66 GHz processor to generate all \( G_{48} \)-orbits of 16-element subsets of \( V = \{1, \ldots, 56\} \). The total number of orbits is \( 867,693,085,859 \) and there are 301,080 good orbits (with intersection numbers \( x = 4, y = 8 \)) among them.

The second step of the computation is usually more difficult. In [13], we used the Kramer-Mesner approach based on solving systems of linear equations over \( \{0, 1\} \), a known NP complete problem. The equations correspond to the requirement that 2-element subsets of \( V \) are covered exactly \( \lambda \) times by blocks of the design (balancedness). Together with information on compatibility of the orbits, corresponding to the requirement that blocks intersect in \( x \) or \( y \) points, we could handle problems of about 1000 orbits in [13].

Now we have significantly more orbits and use a different approach based on clique search, another NP complete problem. We define a graph with the good orbits as vertices and edges between compatible orbits. This is the compatibility graph of the orbits. To each vertex we assign a weight equal to the size of the orbit, and use the program Clique [18, 19] to find all cliques of weight \( b \) in this graph. The cliques correspond to families of \( b \) subsets of size \( k \), intersecting in \( x \) or \( y \) points. In the end we check which of the families are balanced, i.e. designs, and eliminate isomorphic copies. A similar approach was used in [7] and [16]. The calculation is summarised by the following flowchart.
Compute the good $G$-orbits of $k$-subsets of $V$ and define the compatibility graph.

Find cliques of weight $b$ in the compatibility graph [Cliquer]. For each clique, check if the corresponding family of $k$-subsets is balanced.

Eliminate isomorphic copies among the balanced families [nauty].

For the group $G_{48}$, the compatibility graph has 301,080 vertices and 21,193,946 edges (density $4.676 \cdot 10^{-4}$). Cliquer needs about 2 hours of CPU time to find all 1,049,792 cliques of weight 231. Among them there are 1,216 cliques corresponding to designs, and four designs are non-isomorphic. The result is stated in the next theorem (nauty [15] was also used to compute the full automorphism group, and GAP [10] to analyse its structure).

**Theorem 3.1.** There are four $(56, 16, 18)$ QSDs with $G_{48}$ as automorphism group. Three of them are the designs $D_1$, $D_2$, $D_3$ of [13, Theorem 4.1], and the fourth is a new design $D_4$ with full automorphism group of order 192 isomorphic to $(((\mathbb{Z}_2)^4 \rtimes \mathbb{Z}_2) \rtimes \mathbb{Z}_2) \rtimes \mathbb{Z}_3$.

The block graphs of the four designs are isomorphic to the strongly regular Cameron graph [4] with parameters $(231, 30, 9, 3)$. We also classified $(56, 16, 18)$ QSDs with other subgroups of $H$ of orders 48 and 32, but only found these four designs.

Let $C_i$ be the binary code spanned by block incidence vectors of the design $D_i$, for $i = 1, 2, 3, 4$. The codes are self-orthogonal, because the intersection numbers $x = 4$, $y = 8$ are even. The codes $C_1$ and $C_2$ are equivalent, of dimension 23. The code $C_3$ is a subcode of dimension 19. The new code $C_4$ is also of dimension 23, but can be distinguished by the weight enumerator $W(x) = \sum_{i=0}^{56} a_i x^i$. Coefficients are given in

| dim | $a_0$ | $a_8$ | $a_{12}$ | $a_{16}$ | $a_{20}$ | $a_{24}$ | $a_{28}$ |
|-----|-------|-------|----------|----------|----------|----------|----------|
| $C_{1,2}$ | 23    | 1     | 75       | 0        | 21,657   | 353,536  | 2,059,035 | 3,520,000 |
| $C_3$     | 19    | 1     | 0        | 0        | 1,722    | 19,936   | 134,085   | 212,800   |
| $C_4$     | 23    | 1     | 15       | 216      | 20,493   | 359,200  | 2,044,899 | 3,538,960 |

**Table 2.** Dimensions and weight distributions of self-orthogonal binary codes spanned by $(56, 16, 18)$ QSDs.
Table 2 with $a_{56-i} = a_i$ because the codes contain the all-one vector. The codes were analysed with Magma [2].

We tried to construct other $(56, 16, 18)$ QSDs from words of weight 16 using clique search in the associated compatibility graphs, as in [16]. The 1 722 codewords of $C_3$ support only the design $D_3$. We could not perform a complete search for the 21 657 codewords of $C_{1,2}$ and the 20 493 codewords of $C_4$. The compatibility graphs have respective densities 0.5650 and 0.5497, and Cliquer needs much more time than for the previous graph with 301 080 vertices. We did partial searches by identifying codewords into orbits under various automorphism groups, but only found the four designs of Theorem 3.1. By using a dihedral group of order 10, we got $D_1$, $D_2$ and $D_3$ from the codewords of $C_{1,2}$. From the codewords of $C_4$ we only got the design $D_4$, by using automorphism groups of orders as small as 3.

4. New $(56, 16, 6)$ QSDs

The group $G_{48}$ can also be used to construct new quasi-symmetric $(56, 16, 6)$ designs. Among the 867 693 085 859 orbits of 16-element subsets, there are 5352 good orbits with intersection numbers $x = 4, y = 6$. The compatibility graph has 5352 vertices and 379 369 edges (density 0.02649). Cliquer quickly found 224 256 cliques of weight $b = 77$; all of them correspond to QSDs. Using nauty, we computed the number of non-isomorphic designs and their full automorphism groups.

**Theorem 4.1.** There are 876 quasi-symmetric $(56, 16, 6)$ designs with $G_{48}$ as automorphism group. They have $G_{48}$ as their full automorphism group and are not isomorphic to the two known designs from [16, 24].

Next, we want to classify $(56, 16, 6)$ QSDs with the Frobenius group $Frob_{21} \cong \mathbb{Z}_7 \times \mathbb{Z}_3$ of order 21. The full automorphism group of the QSD from [24] is of order 168 and has a subgroup isomorphic to $Frob_{21}$, but we want to proceed systematically and consider all possible actions.

**Lemma 4.2.** An automorphism of order 7 of a $(56, 16, 6)$ QSD does not fix any points and blocks.

**Proof.** The automorphism $\alpha$ maps a block through two fixed points $F_1$ and $F_2$ on a block through $F_1$ and $F_2$. Since there are $\lambda = 6$ such blocks, and $\alpha$ is of order 7, the blocks through $F_1$ and $F_2$ are fixed by $\alpha$. Dually, a point on the intersection of two fixed blocks is fixed, because the intersection numbers are $x = 4, y = 6$. Any fixed block contains 9 or 16 fixed points and through any fixed point there are 8, 15 or 22 fixed blocks. The set of fixed points and blocks of $\alpha$ is a partially balanced design (PBD) with parameters $(7m, \{9, 16\}, 6)$ and
points of degrees 8, 15, and 22. Let \( v_8, v_{15}, v_{22} \) be the number of points of the respective degrees in this PBD, and \( b_9, b_{16} \) the number of blocks of degrees 9 and 16. Clearly

\[ v_8 + v_{15} + v_{22} = 7m, \]  
and by double counting incident point-line pairs we get

\[ 8v_8 + 15v_{15} + 22v_{22} = 9b_9 + 16b_{16}. \]

By double counting triples \((P, Q, B)\) of two points \(P\) and \(Q\) incident with a block \(B\) we get

\[ \binom{9}{2} b_9 + \binom{16}{2} b_{16} = \binom{7m}{2} \cdot 6. \]

Finally, through each of the \(v_8\) points incident with 8 fixed blocks there are 14 non-fixed blocks, and through each of the \(v_{15}\) points incident with 15 fixed blocks there are 7 non-fixed blocks. Together with the fixed blocks, this number cannot be greater than the total number of blocks:

\[ 14v_8 + 7v_{15} + b_9 + b_{16} \leq 77. \]

The system of (in)equalities (1)-(4) is inconsistent for \(m = 1, \ldots, 7\). Therefore, \(m = 0\) and \(\alpha\) does not fix any points and blocks. \(\Box\)

Thus, the group \(Frob_{21}\) acts in orbits of length 7 and 21 on the points and blocks of a \((56, 16, 6)\) QSD. The action on each orbit is unique up to permutational isomorphism. This allows three possible actions on the 56 points, with orbit size distributions \(\nu^{(1)} = (7, 7, 7, 7, 7, 7, 7), \nu^{(2)} = (7, 7, 7, 7, 7, 21), \text{and} \nu^{(3)} = (7, 7, 21, 21)\). Our orderly algorithm needs about 7 CPU days to generate the orbits of 16-element subsets:

- \(\nu^{(1)}\): 107 602 880 good orbits (1 983 283 532 181 total orbits),
- \(\nu^{(2)}\): 98 909 810 good orbits (1 983 283 449 525 total orbits),
- \(\nu^{(3)}\): 584 272 493 good orbits (1 983 283 432 389 total orbits).

There are far too many good orbits to search for cliques in the compatibility graphs. We can reduce the number of orbits that need to be considered without losing generality by using orbit matrices. Let \(O_1, \ldots, O_m\) be the point-orbits and \(B_1, \ldots, B_n\) the block-orbits of a group \(G\) acting on a \((v, k, \lambda)\) design. Denote the orbit sizes by \(\nu_i = |O_i|\) and \(\beta_j = |B_j|\); then \(\sum_{i=1}^m \nu_i = v\) and \(\sum_{j=1}^n \beta_j = b\). For \(G \cong Frob_{21}\) and our \((56, 16, 6)\) QSD, the possible point-orbit size distributions are \(\nu^{(1)}, \nu^{(2)}, \nu^{(3)}\) given above, and the block-orbit size distributions are \(\beta^{(1)} = (7, 7, 7, 7, 7, 7, 7, 7, 7), \beta^{(2)} = (7, 7, 7, 7, 7, 7, 7, 21), \beta^{(3)} = (7, 7, 7, 7, 21, 21), \text{and} \beta^{(4)} = (7, 7, 21, 21, 21)\).

Let \(a_{ij} = |\{ P \in O_i \mid P \in B \}|\), for some \(B \in B_j\). This number does not depend on the choice of \(B\), because the orbits form a tactical
decomposition of the design. The matrix $A = [a_{ij}]$ has the following properties:

1. $\sum_{i=1}^{m} a_{ij} = k$,
2. $\sum_{j=1}^{n} \beta_j a_{ij} = r$,
3. $\sum_{j=1}^{n} \frac{\beta_j}{\nu} a_{ij} a_{ij'} = \begin{cases} \lambda \nu_i, & \text{for } i \neq i', \\ \lambda (\nu_i - 1) + r, & \text{for } i = i'. \end{cases}$

A matrix with these properties is called an orbit matrix for $(v, k, \lambda)$ and $G$. Orbit matrices were used for the construction of block designs with prescribed automorphisms in many papers, e.g. [8, 11, 12].

The entries of an orbit matrix are bounded by $0 \leq a_{ij} \leq \nu_i$. In our case we can also exclude entries $a_{ij} = 2$ and 5 whenever $\nu_i = \beta_j = 7$.

An orbit $B_j$ of size 7 is stabilised by a subgroup of order 3 of $Frob_{21}$, which has a fixed point and two orbits of size 3 on the 7 points of $O_i$. Thus, $a_{ij}$ must be a sum of 1, 3 and 3. Furthermore, for a quasi-symmetric design with intersection numbers $x$ and $y$, the matrix $A$ has the additional properties

4. $\sum_{i=1}^{m} \frac{\beta_i}{\nu_i} a_{ij} a_{ij'} = \begin{cases} sx + (\beta_j - s)y, & \text{for } j \neq j', 0 \leq s \leq \beta_j, \\ sx + (\beta_j - 1 - s)y + k, & \text{for } j = j', 0 \leq s < \beta_j. \end{cases}$

An orbit matrix satisfying these equations is called good. In [8], equations 4. were used for the classification of $(28, 12, 11)$ QSDs with $x = 4$, $y = 6$ and an automorphism of order 7.

If the number of columns $n$ is not too large, we can classify all orbit matrices up to rearrangements of rows and columns by an orderly Read-Faradžev type algorithm described in [12]. We then check equations 4. and the requirement that $a_{ij} \neq 2, 5$ whenever $\nu_i = \beta_j = 7$.

Matrices exist in 4 of the 12 combinations of point- and block-orbit size distributions for $Frob_{21}$ on a $(56, 16, 6)$ QSD:

1. $\nu^{(1)}, \beta^{(2)} \leadsto 2$ orbit matrices,
2. $\nu^{(2)}, \beta^{(3)} \leadsto 6$ orbit matrices,
3. $\nu^{(3)}, \beta^{(3)} \leadsto 4$ orbit matrices,
4. $\nu^{(3)}, \beta^{(4)} \leadsto 1$ orbit matrix.

In case (4), the orbit matrix is

$$A = \begin{bmatrix} 4 & 0 & 3 & 2 & 1 \\ 0 & 4 & 3 & 2 & 1 \\ 9 & 9 & 4 & 6 & 6 \\ 3 & 3 & 6 & 6 & 8 \end{bmatrix}.$$

The $j$-th column of this matrix tells us how the points on a block of $B_j$ are distributed among the point-orbits $O_1, \ldots, O_m$. We can adapt the algorithm from Section 3 to search for orbits compatible with a column of $A$: simply add the conditions $|(U \cup \{e\}) \cap O_i| \leq a_{ij}$, $i = 1, \ldots, m$ to the if statement in line 8. In our case, good orbits with the required intersection pattern exist only for the fourth column of $A$. For the other columns, there are no compatible good orbits and therefore QSDs corresponding to this orbit matrix do not exist.

Similarly, for the 8 orbit matrices of cases (1) and (2), compatible good orbits do not exist for at least one column. Only the orbit matrices of case (3) allow good orbits for every column. Here are two of the four matrices, transposed, with numbers of compatible good orbits:

\[
A^\tau = \begin{bmatrix}
4 & 0 & 6 & 6 \\
4 & 3 & 6 & 3 \\
3 & 1 & 3 & 9 \\
1 & 3 & 3 & 9 \\
1 & 0 & 9 & 6 \\
2 & 3 & 6 & 5 \\
1 & 2 & 7 & 6 \\
\end{bmatrix}, \quad \begin{bmatrix}
4 & 0 & 6 & 6 \\
3 & 4 & 6 & 3 \\
3 & 1 & 9 & 3 \\
3 & 1 & 3 & 9 \\
0 & 1 & 9 & 6 \\
2 & 3 & 5 & 6 \\
1 & 2 & 6 & 7 \\
\end{bmatrix} \rightarrow 882 \text{ orbits},
\]

\[
\rightarrow 3674412 \text{ orbits},
\]

\[
\rightarrow 3628548 \text{ orbits}.
\]

We still get millions of good orbits of size $\beta_6 = \beta_7 = 21$ from the last two columns, but now we have information on how they must be chosen. The design is comprised of one orbit from each set compatible with a column of the orbit matrix. A backtracking program written in C can complete the search. The left matrix gives rise to the known $(56, 16, 6)$ QSD, and the right matrix gives a new design. The remaining two orbit matrices of case (3) do not yield designs. Thus, we can conclude

**Theorem 4.3.** There are two $(56, 16, 6)$ QSDs with an automorphism group isomorphic to $\text{Frob}_{21}$. One is the known design of [24] with full automorphism group of order 168, and the other one is a new design with full automorphism group of order 21.

The computational proof of Theorem 4.3 is summarised by the following flowchart.
Classify all good orbit matrices up to rearrangements of rows and columns.

For every orbit matrix, generate the $G$-orbits of $k$-subsets of $V$ compatible with each column.

Pick an orbit for each column so that the chosen orbits are mutually compatible [backtracking]. Check if the corresponding families of $k$-subsets are balanced.

Eliminate isomorphic copies among the balanced families [nauty].

We tried to construct more $(56, 16, 6)$ QSDs with the alternating group $A_4$ of order 12. The two designs of [16, 24] and the 876 designs of Theorem 4.1 allow two actions of $A_4$, with orbit size distributions $\nu^{(1)} = (4, 4, 6, 6, 6, 12, 12)$, $\beta^{(1)} = (1, 1, 1, 3, 3, 4, 4, 6, 6, 12, 12, 12)$ and $\nu^{(2)} = (1, 3, 4, 6, 6, 12, 12, 12)$, $\beta^{(2)} = (1, 1, 3, 4, 4, 4, 6, 6, 6, 6, 12, 12, 12)$. There are 3 good orbit matrices in the first case and 13 in the second case. Now our backtracking search for designs from orbits compatible with an orbit matrix takes considerably more time, and we could not complete the search. However, we did find new designs with $A_4$ as their full automorphism group: 67 in the first case and 236 in the second case. Since we performed an incomplete search, there are probably more designs in both cases.

We can further increase the number of known $(56, 16, 6)$ QSDs by considering the associated self-orthogonal binary codes, in the spirit of [16]. The designs of [16, 24] span a code $C_1$ of dimension 26. The design of Theorem 4.3 with full automorphism group $Frob_{21}$ spans an inequivalent code $C_2$, also of dimension 26. The 876 designs of Theorem 4.1 span 23 inequivalent codes $C_3, \ldots, C_{25}$ of dimensions 22–25. Finally, the 303 designs with full automorphism group $A_4$ span 16 inequivalent codes. Two of them are equivalent to the previous codes $C_3$ and $C_8$, and the others are new codes $C_{26}, \ldots, C_{39}$ of dimensions 26 and 27. Weight enumerators of the codes are given in Table 3. Some inequivalent codes have equal weight distributions; in total 23 different weight enumerators occur. The computation was done in Magma.

We managed to find 228 more $(56, 16, 6)$ QSDs with full automorphism groups of order 16 by searching among words of weight 16 of
the codes $C_3, \ldots, C_{25}$ with Cliquer. By using nauty once more, we can conclude

**Theorem 4.4.** There are at least 1410 quasi-symmetric $(56, 16, 6)$ designs. Their distribution by order of full automorphism group is given in Table 4.

The block graphs of the QSDs are strongly regular with parameters $(77, 16, 0, 4)$. Such a graph is unique [3] and this means that all the QSDs can be embedded as residuals of symmetric $(78, 22, 6)$ designs, as noted by Tonchev [24]. The first example of symmetric $(78, 22, 6)$ designs was constructed in [11] and it is self-dual. Two more dual pairs were obtained by embedding QSDs in [16, 24]. More examples were constructed in [6] by assuming an automorphism of order 6, bringing

| dim | $a_0$ | $a_8$ | $a_{12}$ | $a_{16}$ | $a_{20}$ | $a_{24}$ | $a_{28}$ |
|-----|-------|-------|----------|----------|----------|----------|----------|
| $C_1$ | 26 | 1 | 91 | 2016 | 15245 | 2939776 | 16194619 | 28531008 |
| $C_2$ | 26 | 1 | 2016 | 155365 | 2926336 | 16224019 | 28493376 |
| $C_3$ | 24 | 1 | 75 | 0 | 40089 | 730368 | 4055835 | 7124480 |
| $C_{4,6,9,10}$ | 22 | 1 | 15 | 0 | 9933 | 183168 | 1012515 | 1783040 |
| $C_{7,11,13}$ | 25 | 1 | 75 | 672 | 77721 | 1465984 | 8103963 | 14257600 |
| $C_8$ | 25 | 1 | 75 | 960 | 75417 | 1474048 | 8087835 | 14277670 |
| $C_{14}$ | 22 | 1 | 15 | 0 | 10701 | 178560 | 1024035 | 1767680 |
| $C_{15}$ | 23 | 1 | 15 | 288 | 19917 | 361216 | 2040867 | 3544000 |
| $C_{16}$ | 23 | 1 | 15 | 96 | 19917 | 365056 | 2028579 | 3561280 |
| $C_{17}$ | 24 | 1 | 75 | 160 | 39833 | 728704 | 4062235 | 7115200 |
| $C_{18}$ | 22 | 1 | 15 | 64 | 9677 | 183424 | 1012771 | 1782390 |
| $C_{19,21,24}$ | 22 | 1 | 15 | 64 | 10445 | 178816 | 1024291 | 1767040 |
| $C_{20,22}$ | 22 | 1 | 15 | 16 | 10061 | 182080 | 1015459 | 1779040 |
| $C_{23}$ | 25 | 1 | 75 | 1280 | 74905 | 1470720 | 8100635 | 14259200 |
| $C_{25}$ | 25 | 1 | 75 | 992 | 77209 | 1462656 | 8116763 | 14239400 |
| $C_{26}$ | 27 | 1 | 139 | 4992 | 307161 | 5848832 | 32477083 | 56941312 |
| $C_{27}$ | 27 | 1 | 99 | 4304 | 305873 | 5872320 | 32406731 | 57039072 |
| $C_{28,29}$ | 27 | 1 | 99 | 4112 | 307409 | 5866944 | 32417483 | 57025632 |
| $C_{30}$ | 26 | 1 | 147 | 1008 | 158529 | 2920512 | 16231467 | 28485536 |
| $C_{31,32,34,35}$ | 27 | 1 | 147 | 3696 | 309057 | 5820976 | 32423979 | 57018016 |
| $C_{33,39}$ | 27 | 1 | 147 | 4976 | 307009 | 5849664 | 32475179 | 56943776 |
| $C_{36}$ | 26 | 1 | 75 | 2240 | 153241 | 2931200 | 16218395 | 28498560 |
| $C_{37,38}$ | 27 | 1 | 75 | 4416 | 305817 | 5871616 | 32408859 | 57036160 |

*Table 3. Dimensions and weight distributions of self-orthogonal binary codes spanned by $(56, 16, 6)$ QSDs.*
The number of known \((78, 22, 6)\) designs up to 413. This number is now further increased by embedding the new QSDs from Theorem 4.4.

**Theorem 4.5.** There are at least 3141 symmetric \((78, 22, 6)\) designs. Their distribution by order of full automorphism group is given in Table 4.

The design from [11] is still the only known self-dual \((78, 22, 6)\) design, and the other examples form 1570 dual pairs.

5. **Nonexistence of \((56, 12, 9)\) and \((56, 20, 19)\) QSDs with certain automorphism groups**

In this section, we consider designs with parameters from rows 49 and 51 of [22, Table 48.25] and groups that yielded new designs in the previous sections. Acting on 12-element subsets, the group \(G_{48}\) has only 12 good orbits with intersection numbers \(x = 0, y = 3\), of sizes 1 and 3. A \((56, 12, 9)\) QSD would have \(b = 210\) blocks, and thus clearly does not allow \(G_{48}\) as automorphism group. The same holds for subgroups of \(G_{48}\) of orders 16 and 8. They have only small numbers of good orbits (180 and 740, respectively), of sizes 1 and 2, which cannot be used to build \((56, 12, 9)\) QSDs.

Subgroups of order 12 are isomorphic to \(A_4\) and have significantly more good orbits, including “long orbits” of size 12. Two actions on the 56 points occur, with point-orbit size distributions \(\nu^{(1)}\) and \(\nu^{(2)}\) given earlier. There are 3148236 good orbits of 12-element subsets for \(\nu^{(1)}\), among them 16588 “short orbits” of size less than 12, and 5664770 for \(\nu^{(2)}\), among them 53954 short orbits. The compatibility graphs are to large to invoke Cliquer directly, but we can proceed in

| \(|\text{Aut}|\) | \(#(56, 16, 6)\) | \(#(78, 22, 6)\) |
|-----------------|-----------------|-----------------|
| 168             | 1               | 2               |
| 78              | 0               | 1               |
| 48              | 876             | 1664            |
| 24              | 1               | 378             |
| 21              | 1               | 2               |
| 16              | 228             | 456             |
| 12              | 303             | 606             |
| 6               | 0               | 32               |

Table 4. Distribution of the known \((56, 16, 6)\) QSDs and symmetric \((78, 22, 6)\) designs by full automorphism group order.
the following way. Since $b = 210$ is not divisible by 12, the design must have at least one short block-orbit. For every short orbit $B$ of 12-element subsets, we consider all orbits compatible with $B$ and search for cliques in the corresponding subgraph of the compatibility graph. Cliquer could eliminate cliques of weight $b - |B|$ in all ensuing subgraphs in a few days of CPU time. Since $G_{48}$ has no subgroups of order 24, we can conclude

**Theorem 5.1.** Let $G$ be a subgroup of order at least 8 of the permutation group $G_{48}$. Then, quasi-symmetric $(56, 12, 9)$ designs with $G$ as automorphism group do not exist.

Next, we consider $Frob_{21}$ as an automorphism group of $(56, 12, 9)$ QSDs. Again, we want to consider all possible actions on the 56 points. To reduce the number of possibilities, we need the following

**Lemma 5.2.** An automorphism of order 7 of a $(56, 12, 9)$ QSD does not fix any points and blocks.

**Proof.** Let $\alpha$ be an automorphism of order 7 fixing $7m$ points. A fixed block contains 5 or 12 fixed points; denote the number of such blocks by $b_5$ and $b_{12}$. If $m = 1$, then $b_{12} = 0$, $b_5$ is at least 7, and any pair of fixed blocks intersects in $y = 3$ fixed points. This would yield a family of 5-subsets of the 7 fixed points pairwise intersecting in 3 points. Such a family can have at most 3 subsets, and thus $m \geq 2$. The number of point-orbits of size 7 is $8 - m$, and since every $b_5$-block contains such an orbit, we have $b_5 \leq 6$.

Consider the $\lambda = 9$ blocks through two fixed points $F_1$ and $F_2$. Either two of them are fixed and 7 are in a block-orbit $B$, or all 9 are fixed blocks. A pair of blocks from $B$ intersects in the fixed points $F_1$ and $F_2$, and therefore must have a third intersection point $T$. If $T$ is fixed, the remaining $k - 3 = 9$ points on a block $B \in B$ belong to different point-orbits of size 7. This is not possible because we have at most 6 point-orbits of size 7. If $T$ is not fixed, a block $B \in B$ contains at most 3 points from each orbit $O$ of size 7. If $B$ contains 3 points from one orbit $O$, then $O$ and $B$ form a Fano plane and the remaining $k - 5 = 7$ points on $B$ belong to different point-orbits of size 7. The only other possibility is for $B$ to contain two points from 3 orbits of size 7, and one point from further $k - 8 = 4$ orbits of size 7. Both possibilities would require more than 6 point-orbits of size 7.

Hence, there are 9 fixed blocks through $F_1$ and $F_2$ and the set of fixed points and blocks forms a PBD with parameters $(7m, \{5, 12\}, 9)$. Double counting triples $(P, Q, B)$ of two fixed points incident with a
fixed block yields

\[ \binom{5}{2} b_5 + \binom{12}{2} b_{12} = \binom{7m}{2} \cdot 9. \quad (5) \]

From (5), \( b_{12} = 63m(7m-1) - 20b_5 \) and this expression is not an integer for \( 2 \leq m \leq 7 \) and \( 0 \leq b_5 \leq 6 \). Therefore, \( m = 0 \) and \( \alpha \) has no fixed points and blocks. \( \square \)

We need to consider three possible actions of \( \text{Frob}_{21} \) on the 56 points, with orbit size distributions \( \nu^{(1)}, \nu^{(2)}, \nu^{(3)} \) as in Section 4. We computed the orbits of 12-element subsets with the algorithm from Section 3:

- \( \nu^{(1)} \): 5,824 good orbits (26,589,705,660 total orbits),
- \( \nu^{(2)} \): 459,550 good orbits (26,589,687,420 total orbits),
- \( \nu^{(3)} \): 53,578 good orbits (26,589,683,340 total orbits).

The compatibility graphs have densities \( 2.173 \cdot 10^{-2}, 2.328 \cdot 10^{-5}, \) and \( 3.612 \cdot 10^{-5} \), respectively. We used Cliquer to establish that the maximum weight of a clique in the graphs are 21, 77, and 35. This is less than the required number of blocks \( b = 210 \).

**Theorem 5.3.** Quasi-symmetric \((56, 12, 9)\) designs with an automorphism group isomorphic to \( \text{Frob}_{21} \) do not exist.

Finally, we turn to \((56, 20, 19)\) QSDs with intersection numbers \( x = 5, y = 8 \) and \( G_{48} \) as automorphism group. Generating orbits of 20-element subsets with the algorithm from Section 3 would take about \( \binom{56}{20}/\binom{56}{16} \approx 19 \) times longer than for the designs with \( k = 16 \), and only a tiny fraction of the orbits are expected to be good. We can speed up the computation by adding the condition \( |U \cap g(U)| \leq y \), for \( g \in G \setminus \{1\} \), to the if statement in line 8. A set \( U \) intersecting its image \( g(U) \) in more than \( y \) points cannot be extended to a \( k \)-element representative of a good orbit, unless the extension \( U' \supset U \) belongs to a short orbit. The new condition causes the algorithm to miss the short good orbits, but we can generate them quickly by the algorithm from [13]. It took about 2 CPU days to generate the 384 long good orbits (of size 48) and a few more minutes for the 3851 short good orbits (of size less than 48). The compatibility graph has 4235 vertices and 163,766 edges (density 0.01827). The maximum weight of a clique is 142, less than the required number of blocks \( b = 154 \).

**Theorem 5.4.** Quasi-symmetric \((56, 20, 19)\) designs with \( G_{48} \) as automorphism group do not exist.

Assuming a smaller automorphism group, e.g. a subgroup of \( G_{48} \) or some permutation representation of \( \text{Frob}_{21} \) on 56 points, increases the number of good orbits quite dramatically. We can no longer perform
the second step of the computation, neither by clique search nor by using orbit matrices.

6. Conclusion

New bounds on numbers of non-isomorphic quasi-symmetric designs on 56 points are given in Table 5. The incidence matrices of the constructed designs can be downloaded from our web page

https://web.math.pmf.unizg.hr/~krcko/results/quasisym.html

Previously, only two (56, 16, 6) QSDs [16] and three (56, 16, 18) QSDs [13] were known. Almost any approach we tried for (56, 16, 6) QSDs increased the number of known designs. On the other hand, only one new (56, 16, 18) QSD was found, and the existence of (56, 12, 9) and (56, 20, 19) QSDs remains open.

| No. | v   | k   | λ   | r   | b  | x  | y  | NQSD |
|-----|-----|-----|-----|-----|----|----|----|------|
| 47  | 56  | 16  | 18  | 66  | 231| 4  | 8  | ≥4   |
| 48  | 56  | 15  | 42  | 165 | 616| 3  | 6  | 0    |
| 49  | 56  | 12  | 9   | 45  | 210| 0  | 3  | ?    |
| 50  | 56  | 21  | 24  | 66  | 176| 6  | 9  | 0    |
| 51  | 56  | 20  | 19  | 55  | 154| 5  | 8  | ?    |
| 52  | 56  | 16  | 6   | 22  | 77 | 4  | 6  | ≥1410|

Table 5. An updated table of QSDs with \( v = 56 \).

Regarding computational techniques, the algorithm from Section 3 solves the problem of generating orbit representatives of \( k \)-element subsets satisfactorily. It can be adapted to generate orbits compatible with an orbit matrix (see Section 4), and speed-up is possible when the intersection number \( y \) is comparatively small (see the proof of Theorem 5.4). The main computational problem remains putting the orbits or individual \( k \)-subsets together to form QSDs.

The requirement that every pair of \( k \)-subsets intersects in \( x \) or \( y \) points proved stronger than the requirement that they cover every 2-subset exactly \( λ \) times. We could simply ignore the second requirement until the end of the computation. In some cases all the constructed structures were balanced (Theorem 3.1), and in others the number of non-balanced structures was not too large (Theorem 3.1). The opposite approach would give many more block designs that are not quasi-symmetric.

The critical factor for the computation is the number of candidates for blocks of the design (good orbits, or individual \( k \)-subsets obtained
from codes). The difficulty of the problem also depends on the number of compatible candidates, i.e. candidates intersecting in $x$ or $y$ points, measured by the density of the compatibility graph. For low densities we could handle problems with hundreds of thousands or even millions of candidates (Theorem 5.1), and for higher densities problems with only a few thousand candidates. The number of candidates can sometimes be reduced by considering orbit matrices, as in the proof of Theorem 4.3. For this approach the number of blocks of the design must be small enough to allow complete classification of the orbit matrices.

Despite our efforts in Section 5, we did not find any $(56, 12, 9)$ and $(56, 20, 19)$ QSDs. If these designs exist, we believe some new insight or completely different computational approach will be necessary for their construction.

References

[1] T. Beth, D. Jungnickel, H. Lenz, Design theory, second edition, Cambridge University Press, 1999.
[2] W. Bosma, J. Cannon, C. Playoust, The Magma algebra system. I. The user language, J. Symbolic Comput. 24 (1997), no. 3–4, 235–265.
[3] A. E. Brouwer, The uniqueness of the strongly regular graph on 77 points, J. Graph Theory 7 (1983), 455–461.
[4] A. E. Brouwer, Uniqueness and nonexistence of some graphs related to $M_{22}$, Graphs Combin. 2 (1986), no. 1, 21-29.
[5] A. R. Calderbank, Geometric invariants for quasisymmetric designs, J. Combin. Theory Ser. A 47 (1988), no. 1, 101–110.
[6] D. Crnković, D. Đumić Danilović, S. Rukavina, On symmetric $(78, 22, 6)$ designs and related self-orthogonal codes, Util. Math. 109 (2018), 227–253.
[7] D. Crnković, B. G. Rodrigues, S. Rukavina, V. D. Tonchev, Quasi-symmetric 2-(64, 24, 46) designs derived from $AG(3, 4)$, Discrete Math. 340 (2017), no. 10, 2472–2478.
[8] Y. Ding, S. Houghten, C. Lam, S. Smith, L. Thiel, V. D. Tonchev, Quasi-symmetric 2-(28, 12, 11) designs with an automorphism of order 7, J. Combin. Des. 6 (1998), no. 3, 213-223.
[9] I. A. Faradžev, Constructive enumeration of combinatorial objects. Problemes combinatoires et théorie des graphes (Colloq. Internat. CNRS, Univ. Orsay, Orsay, 1976), pp. 131–135, Colloq. Internat. CNRS, 260, CNRS, Paris, 1978.
[10] The GAP Group, GAP – Groups, Algorithms, and Programming, Version 4.8.10, 2018, http://www.gap-system.org.
[11] Z. Janko, T. van Trung, Construction of a new symmetric block design for $(78, 22, 6)$ with the help of tactical decompositions, J. Combin. Theory Ser. A 40 (1985), 451–455.
[12] V. Krčadinac, Steiner 2-designs $S(2, 4, 28)$ with nontrivial automorphisms, Glas. Mat. Ser. III 37(57) (2002), 259–268.
[13] V. Krčadinac, R. Vlahović, New quasi-symmetric designs by the Kramer-Mesner method, Discrete Math. 339 (2016), no. 12, 2884–2890.
[14] B. D. McKay, *Isomorph-free exhaustive generation*, J. Algorithms **26** (1998), no. 2, 306–324.

[15] B. D. McKay, A. Piperno, *Practical graph isomorphism, II*, J. Symbolic Comput. **60** (2014), 94-112.

[16] A. Munemasa, V. D. Tonchev, *A new quasi-symmetric 2-(56,16,6) design obtained from codes*, Discrete Math. **284** (2004), no. 1-3, 231–234.

[17] A. Neumaier, *Regular sets and quasisymmetric 2-designs*, in: *Combinatorial theory (Schloss Rauischholzhausen, 1982)*, Lecture Notes in Math., **969**, Springer, 1982, pp. 258-275.

[18] S. Niskanen, P. R. J. Östergård, *Cliquer user’s guide, version 1.0*, Communications Laboratory, Helsinki University of Technology, Espoo, Finland, Tech. Rep. T48, 2003.

[19] P. R. J. Östergård, *A fast algorithm for the maximum clique problem*, Discrete Appl. Math. **120** (2002), no. 1–3, 197–207.

[20] R. M. Pawale, M. S. Shrikhande, S. M. Nyayate, *Conditions for the parameters of the block graph of quasi-symmetric designs*, Electron. J. Combin. **22** (2015), no. 1, Paper 1.36, 30 pp.

[21] R. C. Read, *Every one a winner or how to avoid isomorphism search when cataloguing combinatorial configurations*. Algorithmic aspects of combinatorics (Conf., Vancouver Island, B.C., 1976). Ann. Discrete Math. **2** (1978), 107–120.

[22] M. S. Shrikhande, *Quasi-symmetric designs*, in: *The Handbook of Combinatorial Designs, Second Edition* (eds. C. J. Colbourn and J. H. Dinitz), CRC Press, 2007, pp. 578–582.

[23] M. S. Shrikhande, S. S. Sane, *Quasi-symmetric designs*, Cambridge University Press, 1991.

[24] V. D. Tonchev, *Embedding of the Witt-Mathieu system $S(3,6,22)$ in a symmetric 2-(78,22,6) design*, Geom. Dedicata **22** (1987), no. 1, 49–75.

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