INDUCED MODULES FOR MODULAR LIE ALGEBRAS

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Abstract. Let $L$ be a finite dimensional Lie algebra over a field of non-zero characteristic and let $S$ be a subalgebra. Suppose that $X$ is a finite set of finite dimensional $L$-modules and that $Y$ is a finite set of finite dimensional $S$-modules. Then there exists a category $C$ of finite dimensional $L$-modules containing the modules in $X$ and a category $D$ of $S$-modules containing the modules in $Y$ and those in $X$ regarded as $S$-modules, such that the restriction functor $\text{Res} : C \to D$ has a left adjoint $\text{Ind}^C_D : D \to C$.

1. Introduction

In the theory of finite groups, much use is made of induced representations. For an account of this theory and some examples of its use, see Curtis and Reiner [2, Chapter VIII]. If $G$ is a finite group, $H$ a subgroup and $W$ is an $FH$-module, then the induced module $\text{Ind}^G_H(W)$ is defined to be $FG \otimes_{FH} W$. Here, $FG$ denotes the group algebra of $G$ over the field $F$. An important property of this construction is Frobenius reciprocity. This, in modern terminology, essentially is that induction is a left adjoint (see Mac Lane [4] or the anonymous article [8]) to the restriction functor $\text{Res}^G_H$, the functor which converts $FG$-modules into $FH$-modules by forgetting the action of elements of $G$ which are not in $H$, that is, that there exists a natural isomorphism $\text{Hom}(\text{Ind}^G_H(W), V) \cong \text{Hom}(W, \text{Res}^G_H(V))$. If in this isomorphism, we take $V = \text{Ind}^G_H(W)$, then corresponding to the identity $\text{Ind}^G_H(W) \to \text{Ind}^G_H(W)$, we get a monomorphism $W \to \text{Ind}^G_H(W)$, known as the unit of the adjunction.

For a Lie algebra $L$ and subalgebra $S$, using the universal enveloping algebras $U(L), U(S)$, for an $S$-module $W$ we can construct an induced $L$-module $U(L) \otimes_{U(S)} W$. This construction gives a left adjoint to the restriction functor, but is of little use in the theory of finite dimensional Lie algebras since, apart from trivial cases, the induced module so constructed is infinite dimensional. There have been several attempts to define induced modules which are finite dimensional. Of necessity, these involve either restrictions on the modules covered by the theory or a weakening of the requirement that the induction functor be a left adjoint to the restriction functor.

For Lie algebras $L$ over a field $F$ of characteristic 0, in Zassenhaus [9] and in Hochschild and Mostow [3], a construction is given for the special case where $S$ is an ideal of $L$ and $L$ is the vector space direct sum of $S$ and a subalgebra $H$ of $L$. This construction is not a left adjoint to the restriction functor, but it does have the property of the unit of an adjunction mentioned above. The module constructed from $W$ contains an $S$-submodule isomorphic to $W$. This construction is used in [3] to prove a strengthening of Ado’s Theorem.

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In [6, 7], Wallach gives a construction essentially limited to the case where \( F \) is algebraically closed of characteristic 0, \( L \) is semisimple and \( S \) is a Borel subalgebra. Wallach’s construction is based on the coinduced construction, \( \text{Hom}_{U(S)}(UL, W) \) rather than the tensor product. Again, the construction has a weakened form of some of the properties of a left adjoint. Using it, Wallach gave a construction for all irreducible modules of a semisimple Lie algebra.

Over a field \( F \) of characteristic \( p \), properties of restricted Lie algebras and of \( p \)-envelopes open up possibilities not available in characteristic 0. For a restricted Lie algebra \((L, [p])\), \([p]\)-subalgebra \( S \) and character \( c : L \to F \), we can use the \( c \)-reduced enveloping algebra \( u(L, c) \) to define, for an \( S \) module \( W \) with character \( c|S \), the \( c \)-induced module \( u(L, c) \otimes u(S, c|S) W \). See Strade and Farnsteiner [5, Sections 5.3, 5.6]. This does give a left adjoint to \( \text{Res} : C \to D \) where \( C \) is the category of \( L \)-modules with character \( c \) and \( D \) the category of \( S \)-modules with category \( c|S \). Not every \( S \)-module has a character. Only in the case where the field \( F \) is algebraically closed and the module \( W \) is irreducible is the existence of a character guaranteed. Despite this limitation, this construction, combined with the use of \( p \)-envelopes is used to prove that, for a soluble Lie algebra over an algebraically closed field of characteristic \( p \), every irreducible module has dimension a power of \( p \).

In Barnes [1], the concept of a simple character cluster was introduced and used to define cluster-induced modules. Three conditions were required for this construction to work: the cluster \( C \) had to restrict simply to \( S \), that is, distinct characters in \( C \) had to have distinct restrictions to \( S \), the field \( F \) had to be perfect and the \( S \)-module \( W \) had to be amenable, that is, the module \( \bar{V} \) obtained by extending the field to the algebraic closure had to be a direct sum of modules with character in \( C|S \). In this paper, constructions are given which avoid these requirements.

If \( C \) is a category of \( L \)-modules, we have the restriction functor \( \text{Res} : C \to D \) where \( D \) is a category of \( S \)-modules containing at least those obtained from modules in \( C \) by restriction of the action. In this paper, we construct an induction functor left adjoint to \( \text{Res} \) for some categories \( C, D \) of finite dimensional modules. We shall see that for any given finite sets \( X, Y \) of \( L \)- and \( S \)-modules, we can choose \( C, D \) containing the finitely many given modules. The dimension of the induced module \( \text{Ind}_D^C(W) \) constructed depends on the choice of the categories.

In Section 2, we give the basic construction for induced modules for restricted Lie algebras. In Section 3, we give some illustrative examples and see some ways in which in some cases, the construction may be modified to reduce the dimension of the constructed modules. In Section 4, we extend the construction to non-restrictable Lie algebras.

2. Generalised induction for restricted algebras.

In the following, \((L, [p])\) is a restricted Lie algebra over the field \( F \), \( \bar{F} \) is the algebraic closure of \( F \) and \( \bar{L} = \bar{F} \otimes_F L \) is the algebra obtained by extension of the field. As we will have \( L \) embedded in various associative algebras, we will denote the Lie algebra product of \( x, y \) by \([x, y]\), reserving the notation \( xy \) for the product in the associative algebra under consideration.

By a character of \( L \) we shall understand an \( F \)-linear map \( c : L \to \bar{F} \). If \( \{e_1, \ldots, e_n\} \) is a basis of \( L \), then \( c \) can be expressed as a linear form \( c(x) = \sum a_i x_i \)
for \( x = \sum x_i e_i \), where \( a_i \in \hat{F} \). If \( \alpha \) is an automorphism of \( \hat{F}/F \), that is, an automorphism of \( \hat{F} \) which fixes all elements of \( F \), then \( e^\alpha \) is the character \( e^\alpha(x) = \sum a_i^\alpha x_i \) and is called a conjugate of \( c \). We do not distinguish in notation between \( c : L \to \hat{F} \) and its linear extension \( \bar{L} \to \hat{F} \). A character cluster is a set \( C \) of characters which contains, along with any character \( c \), all conjugates \( e^\alpha c \) of \( c \).

If \( V \) is an \( L \)-module, then \( \bar{V} \) is the \( L \)-module \( \bar{F} \otimes_F V \). The action of \( x \in L \) on \( V \) is denoted by \( \rho(x) \). The module \( \bar{V} \) has character \( c \) if \( (\rho(x)^p - \rho(x^{[p]}))v = c(x)^p v \) for all \( x \in L \) and all \( v \in V \). The cluster \( Cl(V) \) of an \( L \)-module \( V \) is the set of all characters of composition factors of \( V \).

In the universal enveloping algebra \( U(L) \), the element \( z_x = x^p - x^{[p]} \) is central. (See Strade and Farnsteiner [5, p.203].) For the module \( V \) giving the representation \( \rho \), we put \( \phi_x = \rho(x)^p - \rho(x^{[p]}) \). We then have \([\phi_x, \rho(y)] = 0\) for all \( x,y \in L \). By [1, Lemma 2.1], the map \( \phi : L \to \text{End}(V) \) defined by \( \phi_x(v) = (\rho(x^p) - \rho(x^{[p]}))v \) is \( p \)-semilinear.

Let \( W \) be an \( S \)-module. To use the standard tensor product construction of induced modules, we need enveloping algebras \( u(L), u(S) \) which are finite dimensional quotients of the universal enveloping algebras \( U(L), U(S) \) such that

1. \( W \) is a \( u(S) \)-module, and
2. \( u(S) \) is a subalgebra of \( u(L) \).

Condition (1) is easily satisfied. As \( S \) is a \( U(S) \)-module, we can take \( K \) the kernel of the representation of \( U(L) \) on \( W \) and put \( u(S) = U(S)/K \). It is in order to make choices which also satisfy condition (2) that we need the extra structure of restricted Lie algebras. We modify the construction of character reduced enveloping algebras given in Strade and Farnsteiner [5, Section 5.3].

Let \( \{e_i \mid i \in I\} \) where \( I \) is a finite ordered index set, be an ordered basis of \( L \).

We use the multi-index notation of [5, p. 51]. A product of 0 factors is interpreted as the element \( 1 \in U(L) \). The elements \( z_i = e_i^p - e_i^{[p]} \) are in the centre of \( U(L) \).

We use the standard filtration of \( U(L) \), \( U_k = \langle x_1 \ldots x_r \mid x_j \in L, r \leq k \rangle \). Let \( f \) be a family of polynomials \( f_i(t) \in F[t] \) of degrees \( d_i \geq 1 \). Putting \( \zeta_i = f_i(z_i) \) and \( v_i = e_i^{pd_i} - \zeta_i \), we have that \( \zeta_i \) is in the centre of \( U(L) \) and that \( v_i \in U_{pd_i - 1} \).

By [5, Theorem 1.9.7], \( \{e^\alpha c^\beta \mid \alpha_i < pd_i\} \) is a basis of \( U(L) \). Let \( K \) be the ideal of \( U(L) \) generated by the \( e^\alpha c^\beta \) with \( \beta \neq 0 \) and let \( u(L,f) = U(L)/K \). It follows that \( \{e^\alpha + K \mid \alpha_i < pd_i\} \) is a basis of \( u(L,f) \). We call \( u(L,f) \) the \( f \)-reduced enveloping algebra of \( L \).

We now choose the basis and the \( f_i \) to construct algebras \( u(L), u(S) \) satisfying the conditions (1), (2) above. We partition \( I = I_1 \cup I_2 \) with the elements of \( I_2 \) coming after those of \( I_1 \) and choose a basis such that \( \{e_i \mid i \in I_2\} \) is a basis of \( S \). We choose the polynomials \( f_i \) for \( i \in I_2 \) such that \( f_i(z_i)W = 0 \). It follows immediately that \( \{e^\alpha + (K \cap U(S)) \mid \alpha_i = 0 \text{ for } i \in I_1, \alpha_i < pd_i \text{ for } i \in I_2\} \) is a basis of \( u(S) = U(S)/K \cap U(S) \), the \( f \)-\( I_2 \)-reduced enveloping algebra of \( S \). By our choice of the \( f_i \) for \( i \in I_2, K \cap U(S) \) is in the kernel of the representation of \( U(S) \) on \( W \), so \( W \) is a \( u(S) \)-module. Further, \( u(S) \) is a subalgebra of \( u(L) = u(L,f) \).

**Theorem 2.1.** There exist categories \( C, D \) of finite dimensional \( L \)- and \( S \)-modules respectively, such that \( W \in D \) and the restriction functor \( \text{Res} : C \to D \) has a left adjoint \( \text{Ind}_C^D \).
Proof: We construct \( f \)-reduced enveloping algebras \( u(L), u(S) \) as above and take \( C, D \) the categories of \( u(L) \)- and \( u(S) \)-modules respectively. We have the left adjoint of \( \text{Res} \) defined by \( \text{Ind}_S^L(W, f) = u(L) \otimes_{u(S)} W \).

The dimension of the induced module is \( \dim(W)\Pi_{i \in I_1} (pd_i) \). Note that the induced module \( \text{Ind}_S^L(W, f) \) is not affected by the choice of the \( f_i \) for \( i \in I_2 \), subject to the requirement that \( f_i(z_i)W = 0 \). We do not need to choose \( f_i \) to be the minimum polynomial. If several modules are under consideration, we can choose any \( f_i \) such that \( f_i(z_i) \) annihilates all those modules. Thus we may include any given finite set \( X, Y \) of finite dimensional \( L \)- and \( S \)-modules in the categories \( C, D \). Clearly, if \( f_i^* \) are divisors of the \( f_i \), then the \( f^* \)-reduced enveloping algebra \( u^*(L) \) is a quotient of \( u(L) \) and we have a natural epimorphism \( u(L) \otimes_{u(S)} W \to u^*(L) \otimes_{u^*(S)} W \).

3. Illustrative examples

We begin by applying the above construction to the module of Example 6.6 of [1] and comparing the result with the module constructed in that paper.

Example 3.1. Let \( F \) be the field of three elements, \( L = \langle x, y \mid [x, y] = y, x^{[p]} = x, y^{[p]} = 0 \rangle \), \( S = \langle x \rangle \) and \( W = \langle b, b^2 \rangle \) with \( xb^1 = b^2 \) and \( xb^2 = -b^1 \). The cluster used was \( C = \{c_1, c_2\} \) with \( c_1(x) = i, c_2(x) = -i \) and \( c_1(y) = \alpha + i, c_2(y) = \alpha - i \) where \( \alpha, \beta \in F \) and \( i^2 = -1 \). The \( C \)-induced module \( V = \text{Ind}_S^L(W, C) \) constructed in [1] has basis the six elements \( v^r_j = y^r \otimes b^j \) for \( r = 0, 1, 2 \) and \( j = 1, 2 \). The action on these elements is:

\[
\begin{align*}
xx^0 &= x(1 \otimes b^1) = v^0_0, & xx^0 &= x(1 \otimes b^2) = -v^0_1, \\
xx^1 &= (y + yx) \otimes b^1 = v^1_1 + v^2_1, & xx^1 &= (y + yx) \otimes b^2 = v^2_1 - v^1_1, \\
xx^2 &= (-y^2 + y^2x) \otimes b^1 = -v^1_2 + v^2_2, & xx^2 &= (-y^2 + y^2x) \otimes b^2 = -v^2_2 - v^1_1, \\
yv^1 &= v^1_1, & yv^2 &= v^2_1, \\
yv^1 &= v^1_2, & yv^2 &= v^2_2, \\
yv^2 &= \alpha v^0_1 + \beta v^0_2, & yv^2 &= -\beta v^0_1 + \alpha v^0_2.
\end{align*}
\]

To apply the construction of Section 2, we take the ordered basis \( e_1 = y, e_2 = x \) and use the minimal polynomial \( f_2(t) = (t^2 + 1) \) of the action of \( (x^p - x^{[p]}) \) on \( W \). In order that the induced module shall have the characters \( c_1, c_2 \) above in its cluster, we take \( f_1(t) = t^2 + \alpha t + \alpha^2 + \beta^2 \) to produce the following example which we compare to Example 3.1.

Example 3.2. Consider the module \( M = \text{Ind}_S^L(W, f) \). This has basis the twelve elements \( m^r_j = y^r \otimes b^j \) for \( r = 0, 1, \ldots, 5 \) and \( j = 1, 2 \). In the algebra \( u(L) \), \( f_1(y^3 - y^{[p]}) = 0 \). Since \( y^{[p]} = 0 \), this gives \( y^5 + \alpha y^3 + \alpha^2 + \beta^2 = 0 \). Also in \( u(L) \), we have \( xy = y + yx, xy^2 = -y^2 + y^2x, xy^3 = y^3x, xy^4 = y^4 + y^4x, xy^5 = -y^5 + y^5x \). Using these equations we can now calculate the action of \( L \) on \( M \). The calculations involving only low powers of \( y \) go exactly as above.

\[
\begin{align*}
xx^0 &= x(1 \otimes b^1) = m^0_0, & xx^0 &= x(1 \otimes b^2) = -m^0_1, \\
xx^1 &= (y + yx) \otimes b^1 = m^1_1 + m^2_1, & xx^1 &= (y + yx) \otimes b^2 = m^1_2 - m^1_1, \\
xx^2 &= (-y^2 + y^2x) \otimes b^1 = -m^2_1 + m^2_2, & xx^2 &= (-y^2 + y^2x) \otimes b^2 = -m^2_2 - m^2_1,
\end{align*}
\]
\begin{align*}
mx_1 &= y^3 x \otimes b^1 = m_3^2, \quad & mx_2 &= -m_1^3, \\
mx_1 &= (y^4 + y^4 x) \otimes b^1 = m_4^2 + m_4^4, \quad & mx_4 &= -m_1^4 + m_2^4, \\
mx_5 &= (-y^5 + y^5 x) \otimes b^1 = -m_5^1 + m_5^2, \quad & mx_5 &= -m_5^5 - m_5^2, \\
ym_0 &= m_1^1, \quad & ym_0 &= m_2^1, \\
ym_1 &= m_1^2, \quad & ym_1 &= m_2^2, \\
ym_2 &= m_1^3, \quad & ym_2 &= m_2^3, \\
ym_3 &= m_1^4, \quad & ym_3 &= m_2^4, \\
ym_4 &= m_1^5, \quad & ym_4 &= m_2^5, \\
ym_5 &= -(\alpha y^3 + \alpha^2 + \beta^2) \otimes b^1 \quad & ym_5 &= -(\alpha y^3 + \alpha^2 + \beta^2) \otimes b^2 \\
&= -(\alpha^2 + \beta^2) m_0^0 - \alpha m_1^1, \quad & &= -(\alpha^2 + \beta^2) m_0^0 - \alpha m_2^1.
\end{align*}

Since \( \text{Ind}^E_V \) is a left adjoint to \( \text{Res}^F_E \), there is a homomorphism \( \omega : M \to V \)
mapping \( 1 \otimes w \in M \) to \( 1 \otimes w \in V \). Since \( m_j^{r+1} = ym_j^r \), in terms of the bases used above, \( \omega \) is given by:
\begin{align*}
\omega(m_0^0) &= v_1^0, \quad & \omega(m_2^0) &= v_2^0, \\
\omega(m_1^1) &= v_1^1, \quad & \omega(m_1^2) &= v_1^2, \\
\omega(m_2^2) &= v_2^0, \quad & \omega(m_2^2) &= v_2^2, \\
\omega(m_1^3) &= y v_1^2 = \alpha v_1^0 + \beta v_2^0, \quad & \omega(m_2^3) &= y v_2^2 = -\beta v_1^0 + \alpha v_2^0, \\
\omega(m_1^4) &= y \omega(m_1^3) = \alpha v_1^1 + \beta v_2^1, \quad & \omega(m_2^4) &= y \omega(m_2^3) = -\beta v_1^1 + \alpha v_2^1, \\
\omega(m_5^5) &= y \omega(m_4^1) = \alpha v_1^2 + \beta v_2^2, \quad & \omega(m_2^5) &= y \omega(m_2^4) = -\beta v_1^2 + \alpha v_2^2. 
\end{align*}
It follows that \( K = \ker(\omega) \) is spanned by the elements \( m_1^{3+r} - \alpha m_1^r - \beta m_2^r \) and \( m_2^{3+r} + \beta m_1^r - \alpha m_2^r \) for \( r = 0, 1, 2 \). Clearly, there cannot be a homomorphism \( \theta : V \to M \) mapping \( 1 \otimes w \in V \) to \( 1 \otimes w \in M \). But the construction of \( V \) involved the standard induced module construction applied to the \( c_1, c_2 \)-components \( \tilde{W}_{c_1}, \tilde{W}_{c_2} \) of \( W \). To see how this fails to give a homomorphism \( \theta \), we need to examine the character decomposition of \( M \). We have \( \phi_x = \rho(x^3 - x) \) and \( \phi_y = \rho(y^3) \). Now
\[ \tilde{M}_{c_1} = \{ m \in M \mid (\phi_x - c_1(x^3)) m = 0 = (\phi_y - c_1(y^3)m) \}. \]
\begin{align*}
\phi_x m_1^0 &= m_2^0, \quad & \phi_y m_1^0 &= m_1^1, \quad & \phi_x m_1^1 &= m_2^1, \quad & \phi_y m_1^1 &= m_1^2, \quad & \phi_x m_1^2 &= m_2^2, \quad & \phi_y m_1^2 &= m_1^3, \\
\phi_x m_2^0 &= m_1^3, \quad & \phi_y m_2^0 &= m_1^4, \quad & \phi_x m_2^1 &= m_2^1, \quad & \phi_y m_2^1 &= m_1^4, \quad & \phi_x m_2^2 &= m_2^2, \quad & \phi_y m_2^2 &= m_1^3, \\
\phi_y m_3^0 &= m_1^4, \quad & \phi_y m_3^0 &= m_1^5, \quad & \phi_y m_3^1 &= m_2^1, \quad & \phi_y m_3^1 &= m_2^2, \quad & \phi_y m_3^2 &= m_1^3, \quad & \phi_y m_3^2 &= m_1^4, \\
\phi_y m_4^0 &= m_2^3, \quad & \phi_y m_4^0 &= m_2^4, \quad & \phi_y m_4^1 &= m_2^2, \quad & \phi_y m_4^1 &= m_2^3, \quad & \phi_y m_4^2 &= m_1^3, \quad & \phi_y m_4^2 &= m_1^4, \\
\phi_y m_5^0 &= m_2^4, \quad & \phi_y m_5^0 &= m_2^5, \quad & \phi_y m_5^1 &= m_2^3, \quad & \phi_y m_5^1 &= m_2^4, \quad & \phi_y m_5^2 &= m_1^3, \quad & \phi_y m_5^2 &= m_1^4, \\
\phi_y m_5^3 &= (\alpha^2 + \beta^2)m_1^0 - \alpha m_1^3, \quad & \phi_y m_5^3 &= (\alpha^2 + \beta^2)m_1^0 - \alpha m_2^3, \\
\phi_y m_5^4 &= (\alpha^2 + \beta^2)m_1^1 - \alpha m_1^4, \quad & \phi_y m_5^4 &= (\alpha^2 + \beta^2)m_1^1 - \alpha m_2^4, \\
\phi_y m_5^5 &= (\alpha^2 + \beta^2)m_1^2 - \alpha m_1^5, \quad & \phi_y m_5^5 &= (\alpha^2 + \beta^2)m_1^2 - \alpha m_2^5. 
\end{align*}
We rewrite this table in terms of eigenvectors. For $\phi_x$, we have:

$$\phi_x(m_1^r + im_2^r) = -im_1^r + m_2^r = c_1(x)^3(m_1^r + im_2^r),$$

$$\phi_x(m_1^r - im_2^r) = im_1^r + m_2^r = c_2(x)^3(m_1^r - im_2^r)$$

for $r = 0, 1, \ldots, 5$. For $\phi_y$, we have:

$$\phi_y(m_j^{r+3} - (\alpha + i\beta)m_j^r) = -(\alpha^2 + \beta^2)m_j^r - \alpha m_j^{r+3} - (\alpha + i\beta)m_j^{r+3}$$

$$= (\alpha - i\beta)(m_j^{r+3} - (\alpha + i\beta)m_j^r)$$

$$= c_1(y)^3(m_j^{r+3} - (\alpha + i\beta)m_j^r),$$

$$\phi_y(m_j^{r+3} - (\alpha - i\beta)m_j^r) = c_2(y)^3(m_j^{r+3} - (\alpha - i\beta)m_j^r)$$

for $j = 1, 2$ and $r = 1, 2, 3$.

Consider the elements $k_r = m_1^{r+3} - \alpha m_1^r - \beta m_2^r + i(m_2^{r+3} + \beta m_1^r - \alpha m_2^r) = (m_1^{r+3} + im_2^{r+3}) - (\alpha - i\beta)(m_1^r + im_2^r)$.

These are in $\ker(\omega)$ and $\phi_y k_r = c_1(x)^3 k_r$. But $\phi_y k_r = c_2(y)^3 k_r$. Thus the $k_r$ span a submodule with character $c_3$ where $c_3(x) = c_1(x) = i$ and $c_3(y) = c_2(y) = \alpha - i\beta$.

Let $c_4$ be the conjugate of $c_3$, thus $c_4(x) = -i$ and $c_4(y) = \alpha + i\beta$.

Suppose that $\beta \neq 0$. Then $c_1, c_2, c_3, c_4$ are distinct. We have

$$M_{c_1} = \langle m_1^{r+3} + im_2^{r+3} - (\alpha + i\beta)(m_1^r + im_2^r) \mid r = 0, 1, 2 \rangle,$$

$$M_{c_2} = \langle m_1^{r+3} - im_2^{r+3} - (\alpha - i\beta)(m_1^r - im_2^r) \mid r = 0, 1, 2 \rangle,$$

$$M_{c_3} = \langle m_1^{r+3} + im_2^{r+3} - (\alpha - i\beta)(m_1^r + im_2^r) \mid r = 0, 1, 2 \rangle,$$

$$M_{c_4} = \langle m_1^{r+3} - im_2^{r+3} - (\alpha + i\beta)(m_1^r - im_2^r) \mid r = 0, 1, 2 \rangle.$$

Putting $C' = \{c_3, c_4\}$, the cluster of $M$ is $C \cup C'$ and $M$ has the cluster decomposition (see [1, Theorem 4.1]) $M = M_C \oplus M_{C'}$ where

$$M_C = \langle m_1^{1+r} - \alpha m_1^r + \beta m_2^r, m_2^{3+r} - \beta m_1^r - \alpha m_2^r \mid r = 0, 1, 2 \rangle$$

and

$$M_{C'} = \langle m_1^{3+r} - \alpha m_1^r - \beta m_2^r, m_2^{3+r} + \beta m_1^r - \alpha m_2^r \mid r = 0, 1, 2 \rangle = \ker(\omega).$$

The construction of [1, Section 6] gives, for each $W$ in $D$, an induced module $\text{Ind}_W^D(W, C) \in \mathcal{C}^1$, the category of $u(L)$-modules with cluster $C$. We can recover this from our construction of $\text{Ind}_W^D(\cdot, f) : D \to \mathcal{C}$ by combining $\text{Ind}_W^D(\cdot, f)$ with the functor $\pi : \mathcal{C} \to \mathcal{C}^1$ sending each $V \in \mathcal{C}$ to the $C$-component of its cluster decomposition.

Now consider the case in the above example where $\beta = 0$. We then have $c_3 = c_1$ and $c_4 = c_2$. The above expressions give only the parts $K_{c_i}$ of $M_{c_i}$ contained in $K$.

We have

$$M_{c_1} = \langle m_1^r + im_2^r \mid r = 0, 1, 2 \rangle + K_{c_1},$$

$$M_{c_2} = \langle m_1^r - im_2^r \mid r = 0, 1, 2 \rangle + K_{c_2}.$$

Note that $\phi_y - \alpha$ acts nilpotently on $M_{c_1}$ but is not zero as

$$(\phi_y - \alpha)(m_1^r + im_2^r) = m_1^{r+3} + im_2^{r+3} - \alpha(m_1^r + im_2^r) \in K_{c_1}.$$
If we are given an $S$-module $W$, we can always extend the characters in $\text{Cl}(W)$ to $L$ so that the extended cluster restricts simply to $S$. If however, we seek information on a given $L$-module $V$ by representing it as an induced module, we may not have $\text{Cl}(V)$ restricting simply to $S$. We modify Example 3.1 to illustrate this.

**Example 3.3.** Let $(L, [p])$ and $V$ be the algebra and $L$-module of Example 3.1 with $\alpha = 1$ and $\beta = 0$. We take $S = \langle y \rangle$ and $W = \langle w \rangle$ with $yw = w$. We identify $w$ with $v_1^0 + v_1^1 + v_1^2$. The characters $c_1(x) = i$ and $c_2(x) = -i$ require that we use $f_x(t) = t^2 + 1$ in the construction of $\text{Ind}_S^L(W, f)$. This gives $\text{Ind}_S^L(W, f) = \langle x^r \otimes w \mid r = 0, 1, \ldots 5 \rangle$ with $x^0 = x - 1$ in $u(L)$. The action of $y$ on $\text{Ind}_S^L(W, f)$ may be calculated using the commutation rule $yx = xy - [x, y] = xy - y$. Thus $y(x \otimes w) = x \otimes w - 1 \otimes w$. The homomorphism $\theta : \text{Ind}_S^L(W, f) \rightarrow V$ mapping $1 \otimes w$ to $v_1^0 + v_1^1 + v_1^2$ is an isomorphism.

In the character decomposition $\tilde{V} = \tilde{V}_1 \oplus \tilde{V}_2$, we have

$$\tilde{V}_1 = \langle (v_1^0 - iv_2^0), (v_1^1 - iv_2^1), (v_1^2 - iv_2^2) \rangle$$

and

$$\tilde{V}_2 = \langle (v_1^0 + iv_2^0), (v_1^1 + iv_2^1), (v_1^2 + iv_2^2) \rangle.$$ 

Embedding $W$ in $V$ effectively gives a character decomposition of $\tilde{W} = \tilde{W}_1 \otimes \tilde{W}_2$ with $W_1 = \langle (v_1^0 + v_1^1 + v_1^2) - i(v_1^0 + v_1^1 + v_1^2) \rangle$ and $W_2 = \langle (v_1^0 + v_1^1 + v_1^2) + i(v_1^0 + v_1^1 + v_1^2) \rangle$. The construction of $\text{Ind}_S^L(W, f)$ applies the character induction construction to $W_1$ and $W_2$ and selects an $F$-subspace of the direct sum of the induced modules.

Finally, we give an example illustrating the construction when the field is not perfect.

**Example 3.4.** Let $F = \mathbb{F}_3(\tau)$ be the field of rational functions in the indeterminate $\tau$ over the field $\mathbb{F}_3$ of 3 elements. Again we take $L = \langle x, y \rangle$ with $[x, y] = y$ and $x^{[p]} = x$, $S = \langle y \rangle$ and $W = \langle w \rangle$ with $yw = w$. We have to choose a character $c : L \rightarrow F$. Only the value of $c(x)$ affects the induced module constructed. The minimal polynomial of $x^3 - x^{[p]}$ over $F$ is $t - c(x)^3$, so to get a different result from the perfect field case, we need to choose $c(x)^3 \notin F$. So take $c(x) = \tau^{1/3}$. We then take $f_x(t) = (t - \tau^{1/3})^3$. Then in $u(L, f)$, we have $(x^3 - x - \tau^{1/3})^3 = 0$, thus $x^9 = x^3 + \tau$. Let $V = u(L, f) \otimes_{u(S, f)} W$. Then $V$ has basis $\{v_0, \ldots, v_8\}$ where $v_r = x^r \otimes w$. The action of $x$ on $V$ is given by $xv_r = v_{r+1}$ for $r = 0, \ldots, 7$ and $xv_8 = \tau v_0 + v_3$. To calculate the action of $y$, we use the commutation rule $yx^r = (x - 1)^r y$. We obtain

\[
\begin{align*}
yv_0 &= v_0, & yv_1 &= -v_0 + v_1, \\
yv_2 &= v_0 + v_1 + v_2, & yv_3 &= -v_0 + v_3, \\
yv_4 &= v_0 - v_1 - v_3 + v_4, & yv_5 &= -v_0 - v_1 - v_2 + v_3 + v_4 + v_5, \\
yv_6 &= v_0 + v_3 + v_6, & yv_7 &= -v_0 + v_1 - v_3 + v_4 - v_6, \\
yv_8 &= v_0 + v_1 + v_2 + v_3 + v_4 + v_5 + v_6 + v_7.
\end{align*}
\]

In the above example, $(t - \tau^{1/3})^3$ is the minimal polynomial of the action of $x^3 - x^{[p]}$ on $\tilde{V}$. It follows that the solution spaces of $(x^3 - x - \tau^{1/3})v = 0$ for $r = 1, 2, 3$ are distinct. It follows that $V$ is irreducible and that $\tilde{V}$ has a unique composition series with three isomorphic 3-dimensional factors.
4. USE OF $p$-ENVELOPES.

In this section, we drop the assumption that $L$ is a restricted Lie algebra with $S$ a $[p]$-subalgebra and investigate the extent to which the use of $p$-envelopes can provide the needed extra structure. Let $(L^*, [p])$ be a $[p]$-envelope of $L$ and let $S^{[p]}$ be the $[p]$-closure of $S$.

Let $\hat{S} \subset U(S)$ be the universal $p$-envelope of $S$. We choose a homomorphism $j : S^{[p]} \to \hat{S}$ such that, for $s \in S$, we have $j(s) = s \in U(S)$. (See [5, Proposition 2.5.6 and Theorem 5.1.1].) Any $S$-module $W$ is also a $U(S)$-module and the map $j$ makes it into an $S^{[p]}$-module. Thus $j$ defines a functor $J$ from any category $D$ of $S$-modules into a category $JD$ of $S^{[p]}$-modules.

Starting from an $S$-module $W$, we form $JW$, choose a family $f$ of polynomials as above and form the algebras $u(L^*), u(S^{[p]})$. With $D$ the category of $u(S^{[p]})$-modules regarded as $S$-modules, we now have a functor $\text{Ind}_{S^{[p]}}^{L^*}(\cdot, f) \circ J : D \to C^*$. Combining this with the forgetful functor $\text{Res}_L^L : C^* \to C$ which forgets the action of elements of $L^*$ not in $L$, we get a functor $T = \text{Res}_L^L \circ \text{Ind}_{S^{[p]}}^{L^*}(\cdot, f) \circ J : D \to C$ which we might consider as induction from $D$ to $C$.

Lemma 4.1. Let $V \in C$ and $W \in D$ and let $\theta : W \to \text{Res}_L^L(V)$ be an $S$-module homomorphism. Then there exists an $L$-module homomorphism $\psi : T(W) \to V$ such that $\psi(1 \otimes w) = \theta(w)$ for all $w \in W$.

Proof. We can make $V$ into a $L^*$-module $V^*$ by extending the action. There exists a unique $L^*$-module homomorphism $\psi^* : \text{Ind}_{S^{[p]}}^{L^*}(JW) \to V^*$ such that $\psi^*(1 \otimes w) = \theta(w)$. Forgetting the action of elements not in $L$ gives the required homomorphism $\psi$. \hfill $\Box$

This does not establish $T$ as a left adjoint to $\text{Res}_L^L$ as the map $\psi$ might not be unique. Suppose that $V$ is irreducible and that $\rho_1 \neq \rho_2$ are extensions of the action on $V$ giving $L^*$-modules $V_1, V_2$. Suppose that $\theta \neq 0$. We show that the resulting homomorphisms $\psi_1, \psi_2$ are not equal. For some $v \in V$ and some $x \in L^*$, $\rho_1(x)v \neq \rho_2(x)v$. Our assumptions imply that $\psi_1^*, \psi_2^*$ are surjective, so there exists $k \in T(W)$ such that $\psi_1^*(k) = v$. If $\psi_2^*(k) \neq v$, then $\psi_1 \neq \psi_2$, so we may suppose that $\psi_2(k) = v$. But then $\psi_1(xk) = \rho_1(x)v \neq \rho_2(x)v = \psi_2(xk)$.

This situation can arise. The following example develops [5, Exercise 2.3.9].

Example 4.2. Let $L = \langle x, a, b \rangle$ with $[a, b] = 0, [x, a] = a, [x, b] = \lambda b$ where $\lambda \in F$, $\lambda^p \neq \lambda$. Then $L$ is not restrictable. We put $L^* = \langle d, L \rangle$ with $[d, x] = [d, a] = 0$ and $[d, b] = b$. Setting $a^p = b^p = 0, d^p = d$, and $x^p = x + (\lambda^p - \lambda)d$ makes $(L^*, [p])$ a restricted Lie algebra. Let $H = \langle h_0, \ldots, h_{p-1} \rangle$ with $xh_r = rh_r, ah_r = h_{r+1}$ and $bh_r = 0$. Let $K = \langle k_0, \ldots, k_{p-1} \rangle$ with $aK = 0, xk_r = \lambda rk_r$ and $bk_r = k_{r+1}$. Then $V = H \otimes K$ is an irreducible $L$-module.

To see that $V$ is irreducible, take $v_0 = \sum \mu_{ij} h_i \otimes k_j$ any non-zero element of $V$ and let $V_0$ be the submodule generated by $v_0$. Suppose that $n > 1$ of the $\mu_{ij}$ are non-zero. Acting on $v_0$ by $a$ or $b$ does not change the number of non-zero coefficients, so we may suppose that $\mu_{00} \neq 0$. The coefficient of $h_i \otimes k_j$ in $xv_0$ is $(i + \lambda)\mu_{ij}$. Since $\lambda$ is not in the prime field, $(i + \lambda)j \neq 0$ unless $i = j = 0$. Thus $xv_0$ has $n-1$ non-zero coefficients. It follows that there exists $v_1 \in V_0$ with exactly 1 non-zero coefficients, and so that every $h_i \otimes k_j \in V_0$. Since $(a)$ and $(b)$ are the only minimal ideals of $L$, it follows that $V$ is also faithful.
We now extend the action to make $V$ into an $L^*$-module. Since $[d, x] = [d, a] = 0$, we may set $d h_r = \mu h_r$ for any $\mu \in F$ to get an action of $L^*$ on $H$. To get an action on $K$, we must set $d h_r = (\nu + r)k_r$ for some $\nu \in F$. Different choices of $\nu, \mu$ give us different actions of $L^*$ on $V$. Now take $S = \langle a \rangle$ and $W = \langle w_0 \rangle$ with $a w_0 = w_0$. Then $S^{[p]} = S$. We have a homomorphism $\theta : W \to V$ defined by $\theta (w_0) = (h_0 + \ldots + h_{p-1}) \otimes k_0$.

Now consider the actions of $\phi_a, \phi_b, \phi_x, \phi_d$ on $V^*$ given by $\mu, \nu$. We have $\phi_a (h_i \otimes k_j) = h_i \otimes k_j, \phi_b (h_i \otimes k_j) = h_i \otimes k_j, \phi_d (h_i \otimes k_j) = ((\mu + \nu)^p - (\mu + \nu)) (h_i \otimes k_j)$ and $\phi_d (h_i \otimes k_j) = -(\lambda^p - \lambda) (\mu + \nu) (h_i \otimes k_j)$. If we construct the algebras $u(L^*), u(S)$ using the polynomials $f_\lambda(t) = (t - 1), f_\nu(t) = (t - 1), f_\epsilon(t) = (t - \epsilon), f_d(t) = (t - \eta)$, we have $V^*$ an $L^*$-module provided $(\mu + \nu)^p - (\mu + \nu) = \eta$ and $(\lambda^p - \lambda) (\mu + \nu) = -\epsilon$. The choice of $\epsilon$ determines $(\mu + \nu)$ and $\eta$. There are then multiple solutions for $\mu$ and $\nu$ satisfying all the assumptions of the above discussion.

The failure of the map $\psi$ of Lemma 4.1 to be unique is easily repaired. It arises when $L^* \neq L$ because the elements $1 \otimes w$ do not generate $u(L^*, f) \otimes_{u(S^{[p]}, f)} W$ as $L$-module. For these same categories $\mathcal{C}, \mathcal{D}$, if we define $\text{Ind}_{\mathcal{D}}^\mathcal{C}(W)$ to be the $L$-submodule of $T(W)$ generated by the elements $1 \otimes w$, then we have a unique map $\psi|\text{Ind}_{\mathcal{D}}^\mathcal{C}(W) \to V$ and $\text{Ind}_{\mathcal{D}}^\mathcal{C} : \mathcal{D} \to \mathcal{C}$ is a left adjoint to $\text{Res} : \mathcal{C} \to \mathcal{D}$. Summarising the above discussions, we have

**Theorem 4.3.** Let $L$ be a Lie algebra over the field $F$ of characteristic $p \neq 0$ and let $S$ be a subalgebra of $L$. Let $X$ be a finite set of finite dimensional $L$-modules and let $Y$ be a finite set of finite dimensional $S$-modules. Then there exists a category $\mathcal{C}$ of finite dimensional $L$-modules containing the modules in $X$, and a category $\mathcal{D}$ of finite dimensional $S$-modules containing the modules in $\mathcal{C}$ regarded as $S$-modules and the modules in $Y$ such that the restriction functor $\text{Res} : \mathcal{C} \to \mathcal{D}$ has a left adjoint $\text{Ind}_{\mathcal{D}}^\mathcal{C} : \mathcal{D} \to \mathcal{C}$.

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