Advantages for controls imposed in a proper subset∗

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Abstract

In this paper, we study time optimal control problems for heat equations on $\Omega \times \mathbb{R}^+$. Two properties under consideration are the existence and the bang-bang properties of time optimal controls. It is proved that those two properties hold when controls are imposed on some proper subsets of $\Omega$; while they do not stand when controls are active on the whole $\Omega$. Besides, a new property for eigenfunctions of $-\Delta$ with Dirichlet boundary condition is revealed.

Keywords. time optimal control, heat equations, bang-bang property, property of eigenfunctions of the Laplacian

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1 Introduction.

Let $\Omega$ be a bounded domain in $\mathbb{R}^n$. Let $\omega$ be a non empty and open subset of $\Omega$. Write $\chi_\omega$ for its characteristic function. Consider the following controlled heat equation:

$$\begin{cases}
\partial_t y(x,t) - \Delta y(x,t) = \chi_\omega(x)u(x,t) & \text{in } \Omega \times \mathbb{R}^+, \\
y(x,t) = 0 & \text{on } \partial\Omega \times \mathbb{R}^+, \\
y(x,0) = y_0(x) & \text{in } \Omega,
\end{cases}$$

(1.1)

where $u$ is a control function taken from a control constraint set and $y_0$ is an initial state taken from $L^2(\Omega)$. The solution of (1.1) corresponding to $u$ and $y_0$ will be treated as a function from $\mathbb{R}^+$ to $L^2(\Omega)$ and denoted by $y(\cdot; u, y_0)$.

The purpose of this study is to reveal the following fact: Some properties hold for some time optimal control problems of (1.1) when $\omega$ is a proper subset of $\Omega$, but do not stand when $\omega = \Omega$. Consequently, the local control may be more effective than the global control for heat equations in some cases.

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We begin with introducing time optimal control problems. Let \( \{\xi_i\}_{i=1}^{\infty} \) be a complete set of eigenfunctions for \(-\Delta\) with Dirichlet boundary condition such that it serves as a normalized orthonormal basis of \(L^2(\Omega)\). Write \( \{\lambda_i\}_{i=1}^{\infty} \), with \( 0 < \lambda_1 < \lambda_2 \leq \cdots < +\infty \), for the corresponding set of eigenvalues. Then, we take the following target set:

\[
\mathcal{S}_m = \text{span}\{\xi_{m+1}, \xi_{m+2}, \cdots\}, \text{ where } m \geq 2 \text{ is arbitrarily fixed.}
\]

Next, we define, for each natural number \(k\) and each finite sequence of positive numbers \(\{\bar{a}_i\}_{i=1}^k\), the following control constraint set:

\[
U_{\{\bar{a}_i\}_{i=1}^k} = \left\{ \sum_{i=1}^k a_i(\cdot)\xi_i \mid \text{each } a_i(\cdot) \text{ is measurable from } \mathbb{R}^+ \text{ to } [-\bar{a}_i, \bar{a}_i] \right\}.
\]

Consider the following time optimal control problem:

\[
(P) \quad \inf \left\{ t \geq 0 \mid y(t; u, y_0) \in \mathcal{S}_m \right\}, \text{ where the infimum is taken over all } u \in U_{\{\bar{a}_i\}_{i=1}^k}.
\]

Two properties of Problem \((P)\) under consideration are as follows: \((i)\) The existence of time optimal controls; \((ii)\) The bang-bang property: any optimal control \(u^* = \sum_{i=1}^k a_i^*\xi_i\) satisfies that for each \(i\), \(|a_i^*(t)| = \bar{a}_i\) for almost every \(t \in (0, t^*)\), where \(t^*\) is the optimal time. In the case that \(\Omega, \omega, k\) and \(y_0 \notin \mathcal{S}\) are fixed, we say Problem \((P)\) has optimal controls if for any finite sequence of positive numbers \(\{\bar{a}_i\}_{i=1}^k\), it has optimal controls. When \(\omega = \Omega\), \(y_0 \notin \mathcal{S}_m\) and \(k\) are given, Problem \((P)\) has optimal controls if and only if there is a finite sequence of positive numbers \(\{b_i\}_{i=1}^k\) such that the problem \((P)\), with \(\{b_i\}_{i=1}^k\), has optimal controls (see Remark 2.3).

The main results of this paper are broadly stated as follows: \((a)\) Suppose that \(\omega = \Omega\) and \(y_0 \notin \mathcal{S}_m\). Then, \(k\) and \(y_0\) are such that Problem \((P)\) has no optimal control if and only if \(k < m\) and \(y_0\) satisfies

\[
\begin{align*}
\langle (y_0, \xi_{k+1}), (y_0, \xi_{k+2}), \cdots, (y_0, \xi_m) \rangle^T &\neq 0; \quad (1.2)
\end{align*}
\]

\((b)\) Suppose that \(\omega = \Omega\) and \(y_0 \notin \mathcal{S}_m\). Assume that either \(k \geq m\) or \(k < m\) and \(y_0\) does not satisfy \((1.2)\). Then, in general, Problem \((P)\) does not hold the bang-bang property; \((c)\) Suppose that \(\Omega\) and \(\omega\) satisfy accordingly the following conditions:

- **(D1)** The eigenvalues \(\lambda_1 \cdots \lambda_m\) are simple, i.e., \(\lambda_1 < \lambda_2 < \cdots < \lambda_m\), and

- **(D2)** \(\langle \chi_\omega \xi_i, \xi_j \rangle \neq 0\) for all \(i \in \{1, 2, \cdots, m\}\) and \(j \in \{1, 2, \cdots, k\}\).

Then, for each \(k \geq 1\), each \(y_0 \notin \mathcal{S}_m\) and each finite sequence of positive numbers \(\{\bar{a}_i\}_{i=1}^k\), Problem \((P)\) has optimal controls and holds the bang-bang property.

It is worth mentioning that for any fixed bounded domain \(\Omega\), there are a lot of open subsets \(\omega\) in \(\Omega\) such that \(\langle \chi_\omega \xi_i, \xi_j \rangle \neq 0\) for all \(i, j = 1, 2, \cdots\) (see Theorem 1.2 for a new property of the eigenfunctions \(\{\xi_i\}_{i=1}^{\infty}\)); while there are a lot of bounded domains \(\Omega\) such that the property \((D1)\) holds (see Remark 4.1).
2 Studies of Problem (P) where $\Omega = \omega$

The following result is another version of Theorem 2.5 in [1]. It will be used later.

**Lemma 2.1.** Let $\hat{A} \in \mathbb{R}^{d \times d}$ and $\hat{B} \in \mathbb{R}^{d \times l}$, where $d$ and $l$ are natural numbers. Suppose that
\[
\operatorname{rank} \left( \hat{B}, \hat{A}\hat{B}, \hat{A}^2\hat{B}, \ldots, \hat{A}^{d-1}\hat{B} \right) = d,
\]
and the spectrum of $\hat{A}$ belongs to the left half plane of $\mathbb{C}$. Then, for each finite sequence of positive numbers $\{b_i\}_{i=1}^{l}$ and each $w_0$ in $\mathbb{R}^d$, there are a $\hat{t} \geq 0$ and a control $\hat{\beta}$ in the set:
\[
\bar{V} \equiv \left\{ \beta = (\beta_1, \ldots, \beta_l)^T \mid \text{each } \beta_i \text{ is measurable from } \mathbb{R}^+ \text{ to } [-b_i, b_i] \right\},
\]
such that the solution $\hat{w}(\cdot; \hat{\beta}, w_0)$ to the equation:
\[
\begin{align*}
\dot{\hat{w}}(t) &= \hat{A}\hat{w}(t) + \hat{B}\hat{\beta}(t), \quad t \in \mathbb{R}^+, \\
\hat{w}(0) &= w_0,
\end{align*}
\]
reaches zero at $\hat{t}$.

**Theorem 2.2.** Suppose $\omega = \Omega$ and $y_0 \notin S_m$. Then, $k$ and $y_0$ are such that Problem (P) has no optimal control if and only if $k < m$ and $y_0$ satisfies (1.2).

**Proof.** The proof will be organized in three steps as follows:

**Step 1.** Suppose that $k < m$ and $y_0$ satisfies (1.2). Then, for any finite sequence of positive numbers $\{\bar{a}_i\}_{i=1}^{k}$, Problem (P) has no optimal control.

Let $\{\bar{a}_i\}_{i=1}^{k}$ be a finite sequence of positive numbers. Then each $u(\cdot) \in U(\bar{a}_i)_{i=1}^{k}$ can be expressed as $u(t) = \sum_{i=1}^{k} \alpha_i(t)\xi_i$. Write $y(t; u, y_0) = \sum_{i=1}^{\infty} y_i(t)\xi_i$. Clearly, the controlled equation (1.1) is equivalent to the following system:
\[
\begin{align*}
\dot{y}_i(t) + \lambda_i y_i(t) &= \sum_{j=1}^{k} \alpha_j(t) \langle \chi_\omega \xi_i, \xi_j \rangle, \quad y_i(0) = \langle y_0, \xi_i \rangle, \quad i = 1, 2, \ldots.
\end{align*}
\]

Write
\[
\begin{align*}
z(t) &= \begin{pmatrix} y_1(t) \\
y_2(t) \\
\vdots \\
y_m(t) \end{pmatrix}, \\
A &= \begin{pmatrix} \lambda_1 \\
\lambda_2 \\
\vdots \\
\lambda_m \end{pmatrix}, \\
\alpha(t) &= \begin{pmatrix} \alpha_1(t) \\
\alpha_2(t) \\
\vdots \\
\alpha_k(t) \end{pmatrix},
\end{align*}
\]
and
\[
B = \left( \langle \chi_\omega \xi_i, \xi_j \rangle \right)_{i,j} \in \mathbb{R}^{m \times k}.
\]
Let

\[ U_{\{\bar{a}_i\}_{i=1}^k} = [-\bar{a}_1, \bar{a}_1] \times [-\bar{a}_2, \bar{a}_2] \times \cdots \times [-\bar{a}_k, \bar{a}_k]. \]

Consider the following time optimal control problem:

\[
(P) \quad \inf \{ t \geq 0 \mid z(t;\alpha, z_0) = 0 \},
\]

where the infimum is taken over all \( \alpha \) from the control constraint set:

\[
\mathcal{V}_{\{\bar{a}_i\}_{i=1}^k} \triangleq \{ \alpha = (\alpha_1, \cdots, \alpha_k)^T \mid \text{each } \alpha_i \text{ is measurable from } \mathbb{R}^+ \text{ to } [-\bar{a}_i, \bar{a}_i] \},
\]

and \( z(\cdot; \alpha, z_0) \) is the solution to the following equation:

\[
\begin{cases}
\dot{z}(t) + Az(t) = B\alpha(t), & t \in \mathbb{R}^+,

z(0) = (\langle y_0, \xi_1 \rangle, \cdots, \langle y_0, \xi_m \rangle)^T.
\end{cases} \tag{2.5}
\]

Clearly, Problems \((P)\) and \((\tilde{P})\) are equivalent, i.e., \( t^* \) and \( u^* = \sum_{i=1}^k \alpha_i^* \xi_i \) are accordingly the optimal time and an optimal control to Problem \((P)\) if and only if \( t^* \) and \( (\alpha_1^*, \cdots, \alpha_k^*)^T \) are the optimal time and an optimal control to Problem \((\tilde{P})\) respectively.

Since \( \omega = \Omega \) and \( k < m \), it follows from (2.4) that \( B = \begin{pmatrix} I_{k \times k} & 0 \end{pmatrix} \) in this case. Let \( z_1(t) = (y_1(t), \cdots, y_k(t))^T \) and \( z_2(t) = (y_{k+1}(t), \cdots, y_m(t))^T \). Write

\[
A_1 = \begin{pmatrix} \lambda_1 & \cdots & \lambda_k \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} \lambda_{k+1} & \cdots & \lambda_m \end{pmatrix}.
\]

Then, Equation (2.5) can be written as

\[
\frac{d}{dt} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}(t) + \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}(t) = \begin{pmatrix} I_{k \times k} \\ 0 \end{pmatrix} \alpha(t), \tag{2.6}
\]

together with the initial condition:

\[
(z_1(0), z_2(0))^T = (\langle y_0, \xi_1 \rangle, \cdots, \langle y_0, \xi_k \rangle)^T, (\langle y_0, \xi_{k+1} \rangle, \cdots, \langle y_0, \xi_m \rangle)^T)^T
\]

This, along with the condition (1.2), indicates that \( z_2(t) \neq 0 \), for each \( t > 0 \) and each control \( \alpha \) in \( \mathcal{V}_{\{\bar{a}_i\}_{i=1}^k} \). Consequently, Problem \((P)\) has no time optimal control.

Step 2. Suppose that \( k < m \) and \( y_0 \) does not satisfy (1.2). Then, Problem \((P)\) has optimal controls.

Let \( \{\bar{a}_i\}_{i=1}^k \) be a finite sequence of positive numbers. Since \( y_0 \) does not satisfy (1.2), it holds that \( z_2(0) = 0 \). Thus, it follows from (2.6) that \( z_2(t) = 0 \) for all \( t \geq 0 \). Hence, Problem
(\tilde{P}) shares the same optimal time and optimal controls with the following time optimal control problem:

\[ (\tilde{P}_1) : \quad \inf \{ t \geq 0 \mid z_1(t; \alpha) = 0 \} , \]

where the infimum is taken over all \( \alpha \) from \( V_{\bar{a}_1}^{m-1} \), and \( z_1(\cdot; \alpha) \) is the solution to the equation:

\[
\dot{z}_1(t) + A_1 z_1(t) = I_{k \times k} \alpha(t), \quad t \in \mathbb{R}^+, \quad z_1(0) = (\langle y_0, \xi_1 \rangle, \ldots, \langle y_0, \xi_k \rangle)^T .
\]

According to Lemma 2.1, Problem \((\tilde{P}_1)\) has admissible controls. Then, by the standard argument (see either Theorem 13 and the note after it in Chapter III on Page 130 in [13] or Theorem 3.1 on Page 31 in [1]), one can easily verify that Problem \((\tilde{P}_1)\) has optimal controls. Consequently, Problem \((\tilde{P})\) has optimal controls.

**Step 3. Suppose that** \( k \geq m \). Then, Problem \((\tilde{P})\) admits optimal controls.

Let \( \{ \bar{a}_i \}_{i=1}^{k} \) be a finite sequence of positive numbers. Since \( B = (I_{m \times m}, 0_{m \times (k-m)}) \) in the case that \( k \geq m \), control variables \( \alpha_{m+1}(\cdot), \ldots, \alpha_k(\cdot) \) play no role in Equation (2.5) when \( k > m \). Hence, in the case that \( k \geq m \), the effective controls in Problem \((\tilde{P})\) have the form: \( \tilde{\alpha} = (\alpha_1(\cdot), \ldots, \alpha_m(\cdot))^T \). Therefore, Problem \((\tilde{P})\) shares the same optimal time and optimal controls with the following time optimal control problem:

\[ (\tilde{P}_2) : \quad \inf \{ t \geq 0 \mid z(t; \tilde{\alpha}) = 0 \} , \]

where the infimum is taken over all \( \tilde{\alpha} \triangleq (\alpha_1, \ldots, \alpha_m)^T \) from the control constraint set:

\[
V_{\bar{a}_i}^{m} \triangleq \{ \alpha = (\alpha_1, \cdots, \alpha_m)^T \mid \text{each } \alpha_i \text{ is measurable from } \mathbb{R}^+ \text{ to } [-\bar{a}_i, \bar{a}_i] \},
\]

and \( z(\cdot; \alpha) \) is the solution of the following equation:

\[
\begin{align*}
\dot{z}(t) + A z(t) &= I_{m \times m} \tilde{\alpha}(t), \quad t \in \mathbb{R}^+, \\
z(0) &= z_0 \triangleq (\langle y_0, \xi_1 \rangle, \langle y_0, \xi_2 \rangle, \ldots, \langle y_0, \xi_m \rangle)^T .
\end{align*}
\]

(2.7)

Then, by Lemma 2.1 using the same argument as that in Step 2, one can prove that Problem \((\tilde{P})\) has optimal controls.

In summary, we complete the proof. \( \square \)

**Remark 2.3.** From the proof of Theorem 2.2, one can easily verify the following: (a) Suppose that \( \omega = \Omega \). Let \( y_0 \notin S_m \) and \( k \) be given. Then, Problem \((\tilde{P})\) has optimal controls if and only if there is a finite sequence of positive numbers \( \{ \bar{b}_i \}_{i=1}^{k} \) such that the problem \((\tilde{P})\), with \( \{ \bar{b}_i \}_{i=1}^{k} \), has optimal controls. (b) In the case that \( \omega = \Omega \) and \( y_0 \notin S_m \), Problem \((\tilde{P})\) has optimal controls, provided either \( k \geq m \) or \( k < m \) and \( y_0 \) does not satisfy (1.2).
Theorem 2.4. Let \( \omega = \Omega \) and \( y_0 \notin S_m \). Let \( \{\bar{a}_i\}_{i=1}^k \) be a finite sequence of positive numbers. For each \( i \in \{1, \cdots, k\} \), write

\[
T_i = \frac{1}{\lambda_i} \ln \left( 1 + \frac{\lambda_i}{\bar{a}_i} \langle y_0, \xi_i \rangle \right).
\] (2.8)

Then Problem (P), with \( \{\bar{a}_i\}_{i=1}^k \), does not have the bang-bang property, if either of the following conditions stands: (i) \( k \geq m \) and the numbers \( T_1, \cdots, T_m \) are not the same; (ii) \( k < m \), \( y_0 \) does not satisfy (1.2) and the numbers \( T_1, \cdots, T_k \) are not the same.

Proof. Simply write (P) for the problem (P), with \( \{\bar{a}_i\}_{i=1}^k \). For each \( i \in \{1, \cdots, k\} \), define

\[
\bar{\alpha}_i(\cdot) = -\chi_{[0,T_i]}(\cdot)\text{sgn} \left( \langle y_0, \xi_i \rangle \right) \bar{a}_i \triangleq \begin{cases} 
\chi_{[0,T_i]}(\cdot)\bar{a}_i, & \text{if } \langle y_0, \xi_i \rangle < 0, \\
0, & \text{if } \langle y_0, \xi_i \rangle = 0, \\
-\chi_{[0,T_i]}(\cdot)\bar{a}_i, & \text{if } \langle y_0, \xi_i \rangle > 0.
\end{cases}
\] (2.9)

We first prove the following property \((H_1)\): When \( k \geq m \), \( \bar{T} \) and \( \bar{u} \) are the optimal time and an optimal control to Problem (P) respectively, where

\[
\bar{T} \triangleq \max \{T_1, T_2, \cdots, T_m\} \quad \text{and} \quad \bar{u} \triangleq \sum_{i=1}^m \bar{\alpha}_i \xi_i.
\]

By the equivalence of Problems (P) and \((\bar{P}_2)\) (see Step 3 in the proof of Theorem 2.2), we need only to verify that \( \bar{T} \) and \( \bar{\alpha} \) are the optimal time and an optimal control to Problem \((\bar{P}_2)\) respectively, where \( \bar{\alpha} \triangleq (\bar{\alpha}_1, \cdots, \bar{\alpha}_m)^T \).

For this purpose, we observe from direct computation that for each \( i \in \{1, \cdots, m\} \), \( T_i \) and \( \bar{\alpha}_i(\cdot) \) are the optimal time and the optimal control to the following time optimal control problem:

\[
(P_i) : \quad \inf \left\{ t \geq 0 \mid z_i(t; \alpha_i) = 0 \right\},
\]

where the infimum is taken over all \( \alpha_i(\cdot) \) from the set of all measurable functions from \( R^+ \) to \([-\bar{a}_i, \bar{a}_i]\), and \( z_i(\cdot; \alpha_i) \) solves the following equation:

\[
\dot{z}_i(t) + \lambda_i(t)z_i(t) = \alpha_i(t), \quad z_i(0) = \langle y_0, \xi_i \rangle.
\]

Clearly, \( \bar{\alpha} \in V_{\{\bar{a}_i\}_{i=1}^m} \) and \((z_1(\cdot; \bar{\alpha}_1), \cdots, z_m(\cdot; \bar{\alpha}_m))^T\) is the solution \( z(\cdot; \bar{\alpha}) \) to Equation (2.7) with \( \dot{\alpha} = \bar{\alpha} \). Since \( z_i(T_i; \bar{\alpha}_i) = 0 \), it holds that

\[
z_i(\bar{T}; \bar{\alpha}_i) = 0 \quad \text{for all } i \in \{1, 2, \cdots, m\}, \quad \text{i.e., } \quad z(\bar{T}; \bar{u}) = 0.
\] (2.10)

Hence, the optimal time to Problem \((\bar{P}_2)\) is not bigger than \( \bar{T} \). On the other hand, if \( \dot{\alpha} \triangleq (\dot{\alpha}_1 \cdots, \dot{\alpha}_m)^T \in V_{\{\bar{a}_i\}_{i=1}^m} \) and \( \bar{T} > 0 \) are such that \( z(\bar{T}; \dot{\alpha}) = 0 \), then it stands that

\[
z_i(\bar{T}; \dot{\alpha}_i) = 0 \quad \text{for all } i \in \{1, \cdots, m\}.
\]
By the optimality of \( T_i \) to Problem \((P_i)\), we see that \( \hat{T} \geq T_i \) for all \( i \in \{1, \cdots, m\} \), from which, it follows that \( \hat{T} \geq \bar{T} \). Therefore, \( \hat{T} \) is the optimal time to Problem \((P_2)\). Along with (2.10), this yields that \( \tilde{\alpha} \) is an optimal control to this problem. Hence, the property \((\mathcal{H}_1)\) stands.

Since \( y_0 \notin S_m \), it holds that \( \hat{T} > 0 \). Because \( T_1, \cdots, T_m \) are not the same, there is an \( i_0 \in \{1, 2, \cdots, m\} \) such that \( T_{i_0} < \hat{T} \). Then, it follows from (2.9) that \( \tilde{\alpha}_{i_0}(t) = 0 \) for all \( t \in (T_{i_0}, \hat{T}] \). Thus, the optimal control \( \tilde{u} \) does not satisfy the bang-bang property.

Using the very similar argument to that in the proof of the property \((\mathcal{H}_1)\), one can easily show the following property \((\mathcal{H}_2)\): When \( k < m \), \( y_0 \) does not satisfy \((1.2)\), \( \hat{T} \) and \( \tilde{u} \) are the optimal time and an optimal control to Problem \((\mathcal{P})\), where

\[
\hat{T} \triangleq \max\{T_1, T_2, \cdots, T_k\} \quad \text{and} \quad \tilde{u} \triangleq \sum_{i=1}^{k} \tilde{\alpha}_i \xi_i.
\]

Then, by the property \((\mathcal{H}_2)\), (2.9) and the assumptions that \( y_0 \notin S_m \) and the numbers \( T_1, \cdots, T_k \) are not the same, one can easily show that the optimal control \( \tilde{u} \) does not satisfy the bang-bang property. This completes the proof.

### 3 Studies of Problem \((\mathcal{P})\) where \( \omega \) is a proper subset of \( \Omega \)

**Theorem 3.1.** Let \( \Omega \) satisfy the condition \((D1)\). Suppose that \( \omega \) holds the condition \((D2)\). Then, for each \( k \geq 1 \), each \( y_0 \notin S_m \) and each finite sequence of positive numbers \( \{\tilde{\alpha}_i\}_{i=1}^{k} \), Problem \((\mathcal{P})\) has optimal controls.

**Proof.** By the same way as that in Step 1 of the proof of Theorem \((2.2)\) we define the matrices \( A \) and \( B \), and the problem \((\mathcal{P})\). Write \( B_{ij} \) for the element in \( i \)-th row and \( j \)-th column of \( B \), namely, \( B_{ij} = \langle \chi_\omega \xi_i, \xi_j \rangle \). Let \( B_1 = (B_{11}, \cdots, B_{m1})^T \). We first claim that

\[
\text{rank}(B_1, AB_1, A^2B_1, \cdots, A^{m-1}B_1) = m. \tag{3.1}
\]

In fact, since

\[
A^j B_1 = \begin{pmatrix}
\lambda_1 & \cdots & \\
& \ddots & \\
& & \lambda_m
\end{pmatrix}
\begin{pmatrix}
B_{11} \\
\vdots \\
B_{m1}
\end{pmatrix}
= \begin{pmatrix}
\lambda_1^j B_{11} \\
\vdots \\
\lambda_m^j B_{m1}
\end{pmatrix},
\]

it holds that

\[
\begin{vmatrix}
B_{11} & \cdots & \lambda_1^{m-1} B_{11} \\
B_{21} & \cdots & \lambda_2^{m-1} B_{21} \\
\vdots & \cdots & \vdots \\
B_{m1} & \cdots & \lambda_m^{m-1} B_{m1}
\end{vmatrix},
\]

which is a determinant of Vandermonde’ type and equals to \( \prod_{i=1}^{m} B_{ii} \prod_{i<j}(\lambda_i - \lambda_j) \). Because of conditions \((D1)\) and \((D2)\), this determinant is not zero, which implies (3.1).
Now, according to Lemma 2.1, Problem (\(\tilde{P}\)) has admissible controls. Then, by the standard argument (see either Theorem 13 and the note after it in Chapter III on Page 130 in [13] or Theorem 3.1 on Page 31 in [1]), one can easily show that Problem (\(\tilde{P}\)) admits optimal controls. This, along with the equivalence of Problems (\(P\)) and (\(\tilde{P}\)), completes the proof. 

Remark 3.2. From the proof of the above theorem, it follows that Theorem 3.1 still stands when the condition \((D2)\) is replaced by the following condition:

- \((\tilde{D}2)\) \(\langle \chi_\omega \xi_i , \xi_1 \rangle \neq 0 \) for all \(i \in \{1,2,\cdots,m\}\).

Before studying the bang-bang property for Problem (\(\tilde{P}\)) where \(\omega\) is a proper open subset of \(\Omega\), we recall the general position condition which plays an important role in the studies of the bang-bang property for linear controlled ordinary differential equations. Let \(\hat{A}\) and \(\hat{B}\) be \(m \times m\) and \(m \times k\) matrices respectively. Let \(\hat{V}\) be a closed polyhedron in \(\mathbb{R}^k\). We say that \(\hat{V}\) satisfies the general position condition with respect to \((\hat{A},\hat{B})\), if for each nonzero vector \(v\), which is parallel to one of the edges of \(\hat{V}\), the vectors \(\hat{B}v, \hat{A}\hat{B}v, \cdots \hat{A}^{m-1}\hat{B}v\)

are linearly independent. Consider the following time optimal control problem:

\[(\hat{P}) : \inf \{t : z(t;v,z_0) = 0\},\]

where the infimum is taken over all measurable functions \(v\) from \(\mathbb{R}^+\) to the polyhedron \(\hat{V}\), and \(z(\cdot;v,z_0)\) is the solution to the following equation:

\[\dot{z}(t) + \hat{A}z(t) = \hat{B}v(t), \quad t > 0; \quad z(0) = z_0,\]

with \(z_0\) a non-zero vector in \(\mathbb{R}^m\).

Lemma 3.3. (see [13], [4]) Suppose that the closed polyhedron \(\hat{V}\) satisfies the general position condition with respect to \((\hat{A},\hat{B})\). Then any optimal control \(\bar{u}(t)\) to Problem (\(\hat{P}\)), if exists, takes values on the vertices of \(\hat{V}\) and has a finite number of switchings.

Theorem 3.4. Let \(\Omega\) satisfy the condition \((D1)\). Suppose that \(\omega\) satisfies the condition \((D2)\). Then, for each \(k \geq 1\), each \(y_0 \notin S_m\) and each finite sequence of positive numbers \(\{\bar{a}_i\}_{i=1}^k\), Problem (\(P\)) holds the bang-bang property.

Proof. By the same way as that in Step 1 of the proof of Theorem 2.2, we define the matrices \(A\) and \(B\), and the problem (\(\hat{P}\)). According to Lemma 3.3, Theorem 3.1 and the equivalence of Problems (\(P\)) and (\(\hat{P}\)), it suffices to prove the general condition of \(U_{\{\bar{a}_i\}_{i=1}^k}\) with respect to \((A,B)\). Clearly, the later is equivalent to the statement that for each \(j \in \{1,\cdots,k\}\), the vectors \(Be_j, AB\bar{e}_j, \cdots, A^{m-1}Be_j\) are linearly independent, where \(\{e_1,\cdots,e_k\}\) is the standard basis of \(\mathbb{R}^k\).
Let $F_j = (Be_j, ABe_j, \cdots, A^{m-1}Be_j)$. It is clear that
\[
|F_j| = |(Be_j, ABe_j, \cdots, A^{m-1}Be_j)| = \left| \begin{array}{ccc}
B_{1j} & \lambda_1 B_{1j} & \cdots & \lambda_1^{m-1} B_{1j} \\
B_{2j} & \lambda_2 B_{2j} & \cdots & \lambda_2^{m-1} B_{2j} \\
\vdots & \vdots & \ddots & \vdots \\
B_{mj} & \lambda_m B_{mj} & \cdots & \lambda_m^{m-1} B_{mj}
\end{array} \right| = \prod_{i=1}^m B_{ij} \prod_{k>l}(\lambda_k - \lambda_l).
\]
This, together with the conditions (D1) and (D2), yields that $|F_j| \neq 0$ for each $j \in \{1, \cdots, k\}$. Hence, $U_{\{a_i\}_{i=1}^k}$ satisfies the general position condition with respect to $(A, B)$. This completes the proof.

\section{Further studies on the conditions (D1) and (D2)}

In this section, we first give a remark and a theorem, which reveal accordingly some properties for eigenvalues and eigenfunctions of $-\Delta$ with Dirichlet boundary condition. From the remark, it follows that there are a lot of $\Omega$ satisfying the property (D1). From the theorem, it follows that for any bounded domain $\Omega$ in $\mathbb{R}^n$, there are a lot of $\omega \subset \Omega$ where the property (D2) holds. We end this section with another remark which provides an open problem.

\textbf{Remark 4.1.} It is presented in [9] (see also [15], [11]) that there are a lot of $\Omega$ of class $C^3$ satisfies the condition (D1) in the following sense: Let $\Omega$ be a bounded open set of class $C^3$ in $\mathbb{R}^n$. For each $\varepsilon \in (0, 1)$, an $\varepsilon-$neighborhood of $\Omega$ is defined to be the image $(I+\psi)(\Omega)$, where $I$ is the identity map over $\mathbb{R}^n$ and $\psi \in C^3(\mathbb{R}^n; \mathbb{R}^n)$, with the $C^3-$norm less than $\varepsilon$. For each bounded open set $\Omega$ of class $C^3$ in $\mathbb{R}^n$, Write $\Delta_\Omega$ for the self-adjoint operator in $L^2(\Omega)$ generated by the Laplacian on $\Omega$ with the homogeneous Dirichlet boundary condition. Then, for each $\varepsilon \in (0, 1)$, there is an $\varepsilon-$neighborhood of $\Omega^\varepsilon$ such that $-\Delta_{\Omega^\varepsilon}$ has only simple eigenvalues.

Before presenting the theorem, we introduce the following notations: for each $x \in \mathbb{R}^n$ and each $\rho > 0$, $B_\rho(x)$ stands for the open ball in $\mathbb{R}^n$, centered at $x$ and of radius $\rho$; $\overline{B_\rho(x)}$ denotes the closure of the ball $B_\rho(x)$; for each $\rho > 0$,
\[
\Omega^\rho \triangleq \left\{ x \in \Omega \setminus \overline{\omega} \mid \text{dist} (\partial B_\rho(x), \partial \Omega) > 0, \text{dist} (\partial B_\rho(x), \partial \omega) > 0 \right\},
\]
where $\text{dist} (E_1, E_2) \triangleq \inf_{x_1 \in E_1, x_2 \in E_2} \|x_1 - x_2\|_{\mathbb{R}^n}$ for any subsets $E_1$ and $E_2$ in $\mathbb{R}^n$.

\textbf{Theorem 4.2.} Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ and $\omega$ be an open subset of $\Omega$ such that $\Omega \setminus \overline{\omega} \neq \emptyset$. Then, for any $\varepsilon > 0$, there exists an $\varepsilon_0 \in (0, \varepsilon)$ such that $\Omega^{\varepsilon_0} \neq \emptyset$ and for almost every $\tilde{x} \in \Omega^{\varepsilon_0}$,
\[
\left\langle \chi_{\omega \cup \overline{B_{\varepsilon_0}}(\tilde{x})} \xi_i, \xi_j \right\rangle \neq 0 \quad \text{for all } i, j \in \mathbb{N}. \quad (4.1)
\]
Proof. We recall that each eigenfunction \( \xi_i \) belongs to \( C^\infty(\Omega) \) (see Page 335 in [2]). Let
\[
\varphi(x, \tau) = \xi_i(x)e^{\sqrt{\lambda_i}\tau}, \quad (x, \tau) \in \Omega \times \mathbb{R}.
\]
It is obvious that
\[
\triangle_x \varphi(x, \tau) + \partial^2_\tau \varphi(x, \tau) = 0, \quad (x, \tau) \in \Omega \times \mathbb{R}.
\]
By the property of harmonic functions (see Page 6 in [5]), the function \( \varphi(\cdot, \cdot) \) is real analytic over \( \Omega \times \mathbb{R} \). Thus, each eigenfunction \( \xi_i \) is real analytic over \( \Omega \). Write
\[
D = \{ (x, \rho) \in \Omega \times \mathbb{R}^+ \mid B_\rho(x) \subset \Omega \}.
\]
Then, for each pair \((i, j)\), we define a function \( F_{i,j}(\cdot, \cdot) \) from \( D \) to \( \mathbb{R} \) by setting:
\[
F_{i,j}(x, \rho) = \int_{B_1(0)} \xi_i(x + \rho \eta) \xi_j(x + \rho \eta) d\eta, \quad (x, \rho) \in D. \tag{4.2}
\]
Clearly, it is well defined. The rest of the proof will be carried out by the following three steps:

**Step 1.** Suppose that \( f \) is a real analytic function over \( \Omega \). Define the function \( F : D \to \mathbb{R} \) by
\[
F(x, \rho) = \int_{B_1(0)} f(x + \rho \eta) d\eta, \quad (x, \rho) \in D. \tag{4.3}
\]
Then \( F \) is real analytic over \( D \).

We need only to explain that \( F \) is real analytic in a small neighborhood of \( (x_0, \rho_0) \) for any point \((x_0, \rho_0) \in D \). First, there is a neighborhood \( U \) of \( (x_0, \rho_0) \) in \( \mathbb{R}^n \times \mathbb{R}^+ \) such that \( B_\rho(x) \subset \Omega \) for any \((x, \rho) \in U \). Hence, the function \( f(x + \rho \eta) \) is real analytic in \((x, \rho, \eta)\) over \( U \times B_1(0) \). Extend \( f \) to a complex-valued function in \((z, w, \eta)\) over a small neighborhood \( U_c \times B_1(0) \) of \( U \times B_1(0) \) in \( \mathbb{C}^n \times \mathbb{C} \times B_1(0) \) by making use of the power series expansion. We then get \( f_c(z + w\eta) \), which is real analytic over \((z, w, \eta) \in U_c \times B_1(0) \) and holomorphic in \((z, w) \in U_c \) for each fixed \( \eta \in B_1(0) \). Clearly, it holds that
\[
f_c(x + \rho \eta) = f(x + \rho \eta) \quad \text{for all } (x, \rho, \eta) \in U \times B_1(0).
\]

Now we define a function \( F_c : U_c \to \mathbb{C} \) by setting:
\[
F_c(z, w) = \int_{B_1(0)} f(z + w \eta) d\eta, \quad (z, w) \in U_c,
\]
and define the operator \( \bar{\partial} \) in the standard way:
\[
\bar{\partial} u(z, w) = \sum_{j=1}^n \frac{\partial u}{\partial \bar{z}_j} d\bar{z}_j + \frac{\partial u}{\partial w} d\bar{w},
\]
where
\[
\frac{\partial}{\partial \bar{z}_j} = \frac{1}{2} \left( \frac{\partial}{\partial (\text{Re}(z_j))} + \sqrt{-1} \frac{\partial}{\partial (\text{Im}(z_j))} \right), \quad \frac{\partial}{\partial w} = \frac{1}{2} \left( \frac{\partial}{\partial (\text{Re}(w))} + \sqrt{-1} \frac{\partial}{\partial (\text{Im}(w))} \right)
\]
are the standard Cauchy-Riemann operators (see [6]). It follows from the holomorphic property of \( f_c \) in \((z, w)\) that
\[
\bar{\partial} F_c(z, w) = \int_{B_1(0)} \bar{\partial} f_c(z + w\eta) d\eta = \int_{B_1(0)} 0 \, d\eta = 0.
\]
Hence, \( F_c \) is holomorphic in \( U_c \). In particular, the function \( F_c(\cdot, \cdot) \) is real analytic, i.e. \( F(\cdot, \cdot) \) is analytic in \( U \). Thus, \( F(\cdot, \cdot) \) is real analytic over \( D \). Consequently, for each pair \((i, j)\), the function \( F_{i, j}(\cdot, \cdot) \) is real analytic over \( D \).

**Step 2.** For each pair \((i, j)\), \( F_{i, j}(\cdot, \cdot) \) is not identically a constant over \( D \).

By the unique continuation property of the eigenfunctions (see [7]), we see that for each \( i \in \{1, 2, \cdots \} \),
\[
\xi_i(x) \neq 0 \quad \text{for almost every} \quad x \in \Omega.
\]
Thus, it holds that for each pair \((i, j)\),
\[
(\xi_i \xi_j)(x) \neq 0 \quad \text{for almost every} \quad x \in \Omega.
\]
Since the function \((\xi_i \xi_j)(\cdot)\) is continuous in \( \Omega \) and \( \Omega \setminus \mathcal{W} \neq \emptyset \), there is an \( \hat{x} \in \Omega \setminus \mathcal{W} \) such that \((\xi_i \xi_j)(\hat{x}) \neq 0\). Hence, when \( \delta > 0 \) is small enough, the function \((\xi_i \xi_j)(\cdot)\) is either positive or negative over \( B_\delta(\hat{x}) \), and \( B_\delta(\hat{x}) \subset \Omega \). Now, it follows from the definition of the function \( F_{i, j}(\cdot, \cdot) \) that
\[
F_{i, j}(\hat{x}, \delta_1) \neq F_{i, j}(\hat{x}, \delta_2), \quad \text{when} \ \delta_1 \text{ and } \delta_2 \text{ are different numbers in } (0, \delta).
\]
Since \((\hat{x}, \delta_1)\) and \((\hat{x}, \delta_2)\) belong to \( D \), \( F_{i, j} \) is not identically zero over \( D \) for each pair \((i, j)\).

**Step 3.** To prove (4.1).

Since each \( F_{i, j} \) is real analytic and is not identically a constant over \( D \), the set
\[
W_{i, j} \triangleq \{(x, \rho) \in D \mid F_{i, j}(x, \rho) + \langle \chi_\omega \xi_i, \xi_j \rangle = 0 \}
\]
is a real analytic subvariety with dimension at most \( n \). Thus, the \( \mathbb{R}^{n+1} \)-Lebesgue measure of the set
\[
W \triangleq \bigcup_{i, j} W_{i, j}
\]

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is zero. Write \( W^\rho = \{ x \in \Omega \mid (x, \rho) \in W \} \). Denote by \( m(W^\rho) \) the \( \mathbb{R}^n \)-Lebesgue measure of \( W^\rho \). According to Fubini’s Theorem,

\[
0 = \int_{\Omega \times (0, \infty)} \chi_{W}(x, \rho) dx d\rho = \int_0^\infty \int_\Omega \chi_{W^\rho}(x) dx d\rho = \int_0^\infty m(W^\rho) d\rho.
\]

Thus, there is a subset \( E \subset (0, \infty) \) of zero measure such that \( m(W^\rho) = 0 \) for each \( \rho \in (0, \infty) \setminus E \). On the other hand, since \( \Omega \setminus \mathcal{U} \neq \emptyset \), there is \( \bar{\rho} > 0 \) such that \( \Omega^\rho \neq \emptyset \) for all \( \rho \in (0, \bar{\rho}] \).

Now, for each \( \varepsilon > 0 \), we arbitrarily take an \( \varepsilon_0 \) from the set \( (0, \min\{\bar{\rho}, \varepsilon\}) \setminus E \). Then, it stands that \( m(W^{\varepsilon_0}) = 0 \) and \( \Omega^{\varepsilon_0} \neq \emptyset \). Hence,

\[
m(\Omega^{\varepsilon_0} \setminus W^{\varepsilon_0}) = m(\Omega^{\varepsilon_0}). \tag{4.4}
\]

Clearly, the statement that \( x \in \Omega^{\varepsilon_0} \setminus W^{\varepsilon_0} \) is equivalent to the statement that

\[
x \in \Omega^{\varepsilon_0} \text{ and } F_{i,j}(x, \varepsilon_0) + \langle \chi_{\omega} \xi_i, \xi_j \rangle \neq 0 \text{ for all } i, j \in \mathbb{N}.
\]

This, together with (4.4), yields that

\[
F_{i,j}(x, \varepsilon_0) + \langle \chi_{\omega} \xi_i, \xi_j \rangle \neq 0 \text{ for all } i, j \in \mathbb{N} \text{ and for almost every } x \in \Omega^{\varepsilon_0}. \tag{4.5}
\]

Finally, by the definition of \( \Omega^{\varepsilon_0} \), we see that

\[
B_{\varepsilon_0}(x) \subset \Omega \text{ and } B_{\varepsilon_0}(x) \bigcap \omega = \emptyset \text{ for all } x \in \Omega^{\varepsilon_0}.
\]

Along with (4.5), these indicate (4.1). This completes the proof. \( \square \)

**Remark 4.3.** Let \( \{a_i\}_{i=1}^\infty \in l^2_+ \triangleq \{b_i\}_{i=1}^\infty \in l^2 \mid b_i > 0 \text{ for all } i \}. \) Consider the following time optimal control problem \((P)\): \( \inf \{ t : y(t; u, y_0) = 0 \} \), where the infimum is taken over all \( u \) from the set:

\[
U_{ad} = \left\{ u = \sum_{i=1}^\infty u_i(t) \xi_i \mid \text{each } u_i(\cdot) \text{ is measurable from } \mathbb{R}^+ \text{ to } [-a_i, a_i] \right\},
\]

and \( y(\cdot; u, y_0) \) is the solution to Equation (1.1). The set \( U_{ad} \) is called a control constraint set of the rectangular type. We say Problem \((P)\) has the bang-bang property if any optimal control \( u^* = \sum_{i=1}^\infty u_i^*(t) \xi_i \) satisfies that for each \( i \), \( u_i^*(t) = a_i \) for a.e. \( t \in (0, t^*) \), where \( t^* \) is the optimal time.

It is not clear to us what conditions are needed to obtain the bang-bang property for Problem \((P)\). With regard to this question, we would like to mention the following: (i) It is necessary to impose certain conditions on \( \{a_i\}_{i=1}^\infty \in l^2_+ \) to ensure the existence of optimal controls for Problem \((P)\) (see [8]); (ii) When \( U_{ad} \) is replaced by the following control constraint sets of the ball type:

\[
\tilde{U}_{ad} \triangleq \left\{ u(\cdot) \in L^\infty(\mathbb{R}^+; L^2(\Omega)) \mid u(t) \in \bar{B}(0, r) \right\},
\]

where \( \bar{B}(0, r) \) is the ball in \( L^2(\Omega) \), centered at the origin and of radius \( r > 0 \), the bang-bang property for the corresponding time optimal control problem \((P)\) has been studied (see [3], [10], [16] and [12]).
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