Quantum probes of timelike naked singularity with scalar hair

O. Svítek\textsuperscript{a}\textsuperscript{,1}, T. Tahamtan\textsuperscript{b}\textsuperscript{,1,2}, Adamantia Zampeli\textsuperscript{c}\textsuperscript{,1}

\textsuperscript{1}Institute of Theoretical Physics, Faculty of Mathematics and Physics, Charles University, V Holešovičkách 2, 180 00 Prague 8, Czech Republic
\textsuperscript{2}Astronomical Institute, Czech Academy of Sciences, Boční II 1401, Prague, CZ-141 31, Czech Republic

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Abstract We study a curvature singularity resolution via relativistic quantum mechanics on a fixed background based on the Klein–Gordon and the Dirac equations for a static spacetime with a scalar field producing a timelike naked singularity. We show that both the Klein–Gordon and the Dirac particles see this singularity. For comparison with previous method we study the Canonical Quantization via conditional symmetries. Subsequently we check the results by applying a maximal acceleration existence in the Covariant Loop Quantum Gravity described recently and obtain a resolution of singularity. In the process we study radial geodesics and their congruences in the spacetime.

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1 Introduction

Curvature singularities that generally appear in solutions of General Relativity show the limits of validity of this theory. If they are hidden beneath the horizon they are not influencing external observers and the situation is at least practically less serious (although the problem for the theory itself is not diminished). However, naked singularities represent a rather undesirable feature (motivating the Cosmic Censorship Hypothesis) of a solution to the Einstein equations especially if the associated matter content seems quite ordinary. One expects that the singularities might be removed once a quantum gravity theory is established in some form. In this work, we probe the singularity using quantum mechanics to determine whether the quantum matter can experience its presence at all. Then, we move a step further and use two approaches of quantum gravity to determine whether the singularity might survive this specific form of spacetime quantization. Using these theories allows us to investigate a singularity resolution on the level of a full quantum gravity picture (however this is just one of the candidate theories) or in other words on the level of a spacetime quantization itself. For this reason it should be considered as more fundamental than the first method where the spacetime is treated as fixed. i) the canonical quantization via conditional symmetries and ii) recent results in the Covariant Loop Quantum Gravity.

Our first approach to the problem of naked singularity presence is based on the pioneering work of Wald\textsuperscript{[1]}, which was further developed by Horowitz and Marolf (HM)\textsuperscript{[2]}. The main idea is to probe a classical timelike curvature singularity in static spacetime with quantum test particles obeying the Klein–Gordon equation. Later the method was applied in many specific geometries containing singularity\textsuperscript{[3–11]}. In this approach the singular character of the spacetime geometry is determined based on the number of self-adjoint extensions of an evolution operator. The evolution operator is extracted from the field equation selected for the analysis – originally it was the Klein–Gordon equation, but the approach might be straightforwardly extended to other field equations. The extended operator is then defined on a Hilbert space (usually an $L_2$ space over a domain) covering the singularity position as well. If the self-adjoint extension is unique (so called essentially self-adjoint operator), it is said that the spacetime is quantum mechanically regular. This is connected to the

\textsuperscript{a}e-mail: ota@matfyz.cz
\textsuperscript{b}e-mail: tahamtan@utf.mff.cuni.cz
\textsuperscript{c}e-mail: azampeli@phys.uoa.gr
fact that one can in general select a self-adjoint extension by demanding a specific boundary conditions for the eigenfunctions of the operator. However this cannot be applied in the singularity where we do not have any control over physics and therefore the extension should be unique automatically. This subsequently ensures a uniquely defined evolution for the wave-function thus mimicking a globally hyperbolic spacetime.

In the canonical quantization via conditional symmetries, the starting point is a 4-dimensional spacetime written in a minisuperspace form. In these classical spacetimes, the lapse function is not gauge fixed. This allows for the presence of extra symmetries in the configuration space called conditional symmetries (see e.g. [12] [17]). In [18], the relation between the Lie point symmetries and the conditional symmetries of the minisuperspace was established which in the constant potential lapse parametrization coincide with the conditional symmetries in the phase space. For more details and applications see e.g. [19]–[21] and for a recent review [22]. The system of the Einstein’s field equations is solved by using the first integrals of motion associated to the conditional symmetries; this method facilitates the solution of these equations. For the quantization of the system, we first promote the first-class constraints to operators annihilating the wave function, according to Dirac [23]. Then, the generators of the conditional symmetries are also promoted to operators, thus providing a system of quantum constraints with additional eigenvalue equations on the wave function. The outcome is a unique wave function, not containing arbitrary functions, which is used to find a semiclassical spacetime. This is done by writing it in polar form and setting up the corresponding semiclassical equations following Bohm’s approach to quantum theory. This semiclassical solution is studied to see whether the singularity exists or not in the new solution [19]–[22],[24].

The third approach utilizes recent developments in the Covariant Loop Quantum Gravity (CLQG) [25]. This relatively recent development in quantum gravity research builds on the previous results of the Spinfoams approach [26] (derived using a Feynman-style “sum over geometries”) and the Loop Quantum Gravity [27] (canonical GR quantization leading to spin network states). The method is based on the observation that there is a maximal acceleration in this theory [28]. This upper bound appears in an analogous way to the minimal area in the original canonical Loop Quantum Gravity [29]. We derive a characteristic measure of acceleration in our spacetime and apply the upper bound yielding a resolution of our singularity.

As a spacetime for studying the curvature singularity presence on the quantum level we will use a specific subcase of the recently derived Robinson–Trautman solution minimally coupled to a free massless scalar field [30] (a broader overview of the standard Robinson–Trautman solution with many references can be found there). This solution contains a relatively wide range of special cases [31] with some peculiar properties.

The Robinson–Trautman geometry is defined by the presence of a nonshearing, nontwisting and expanding null geodesic congruence. This family of spacetimes contains the Schwarzschild or the Vaidya solutions but general members are nonsymmetric and dynamical. These spacetimes generically contain (exact) gravitational waves that carry away the asymmetries and a large class of these spacetimes settle down to the Schwarzschild solution (or its simple generalizations) asymptotically.

One of the important cases studied in [31] and obtained from the general Robinson–Trautman solution with a scalar field is a static spherically symmetric solution with a static scalar field which represents a parametric limit of the Janis–Newman–Winicour scalar field spacetime [32,33]. This spacetime is asymptotically flat and the scalar field is vanishing at infinite (retarded) time. As observed in the Chase theorem [34] (see [35] for a recent generalization to a large class of potentials) the static configurations with a scalar field do not possess a regular horizon which is the case for the considered solution as well. This special solution contains a naked singularity which is moreover timelike (unlike for the generic Janis–Newman–Winicour spacetime). It is sourced by quite an ordinary scalar field which seems to be physically realistic source satisfying standard energy conditions. It is important to understand how the singularity created solely by a scalar field behaves when analyzed by the above mentioned two quantum approaches.

2 The spacetime with a scalar field

One of the subcases of the Robinson–Trautman scalar field solution analyzed in [31] is a static algebraic type D spacetime with a scalar field. The metric corresponding to this solution is (we are using the (+, −, −, −) signature convention to retain the standard Newman–Penrose formalism choice)

$$\begin{align*}
\frac{d^2 s}{d \tau^2} &= -\left( r^2 - \chi_0^2 \right) d\Omega^2, \\
\frac{d\Omega^2}{d \tau} &= \sin^2 \theta \, d\varphi^2.
\end{align*}$$

The geometry is obviously spherically symmetric and the static scalar field is given by

$$\begin{align*}
\Phi(r) = \frac{1}{\sqrt{2}} \ln \left\{ \frac{r - \chi_0}{r + \chi_0} \right\}.
\end{align*}$$
The Ricci scalar and the Kretschmann invariant have the following form
\[
\text{RicciSc} = -\frac{2\chi_0^2}{(r^2 - \chi_0^2)^2},
\]
\[
\text{Kretschmann} = 3(\text{RicciSc})^2
\]
and they give the positions of curvature singularities as points where they diverge.

One can easily observe that the singularity at \( r = \chi_0 \) (we consider only this one) is naked, either directly from the metric or by looking for marginally trapped surfaces. The singularity is pointlike and timelike. When \( r \to \infty \), the scalar field vanishes and the metric is asymptotically flat. The area of spherical surfaces \( r = \chi_0 \) it grows only linearly.

The newly introduced coordinate \( \rho \) is a correct areal radius.

For the subsequent calculations we retain the original form \[\text{1}\], since it leads to easier and more familiar expressions in both techniques analyzing the quantum aspects of the naked singularity at \( r = \chi_0 \).

### 3 Quantum Fields

#### 3.1 Self-adjoint extension method

First, we present a method for probing singularities with quantum mechanics used in \[\text{2}\] in order to use it in the specific case of a massless scalar particle and a Dirac particle on the background spacetime described by \[\text{1}\]. Consider a static spacetime \((\mathcal{M}, g_{\mu\nu})\) with a timelike Killing vector field \(\xi^\mu\) \[\text{2}\]. Let \( t \) denote the affine parameter along the Killing field and \( \Sigma \) denote a static spatial slice (with singular points removed).

The Klein-Gordon equation can then be written in this form
\[
\frac{\partial^2 \psi}{\partial t^2} = \sqrt{f} \nabla^i \left( \sqrt{f} D_i \psi \right) - f M^2 \psi = -A \psi,
\]
in which \( f = \xi^\mu \xi_\mu \) (using the selected signature of a spacetime metric) and \( D_i \) is the spatial covariant derivative on \( \Sigma \) induced from the full spacetime covariant derivative. The Hilbert space \( \mathcal{H} = L^2(\Sigma) \) is a space of square integrable functions on \( \Sigma \). The operator \( A \) is evidently real, positive and symmetric and therefore its self-adjoint extensions (covering the extension of Hilbert space to encompass the singular point) always exist. If this extension is unique then \( A \) is called essentially self-adjoint.

For analyzing the essential self-adjointness we use the following procedure. Consider the eigenfunction equation
\[
A \psi \pm i \psi = 0.
\]

Then the operator \( A \) (coming from the equation \[\text{9}\]) will be essentially self-adjoint if one of the two solutions of this equation (for each sign of the imaginary term) fails to be square integrable near the singularity. This means that the operator can be unambiguously extended to the singularity and the corresponding wave functions are part of the Hilbert space. Such a system is then considered quantum mechanically regular. If \( A \) is essentially self-adjoint for \( M = 0 \), it is essentially self-adjoint for all \( M > 0 \) as well \[\text{36}\]. For simplicity, we consider only massless scalar and Dirac particles.

#### 3.1.1 Klein–Gordon particle

The Klein–Gordon equation for a massless scalar particle is given by
\[
\square \tilde{\psi} = g^{-1/2} \partial_\mu \left[ g^{1/2} g^{\mu\nu} \partial_\nu \right] \tilde{\psi} = 0.
\]

For the metric \[\text{1}\], the Klein–Gordon equation becomes
\[
\frac{\partial^2 \tilde{\psi}}{\partial t^2} = \left\{ \frac{\partial^2}{\partial r^2} + \frac{2r}{r^2 - \chi_0^2} \frac{\partial}{\partial r} + \frac{1}{r^2 - \chi_0^2} \left( \frac{\partial^2}{\partial \theta^2} + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \varphi^2} + \frac{\cos \theta}{\sin \theta} \frac{\partial}{\partial \theta} \right) \right\} \tilde{\psi}.
\]

In analogy with the equation \[\text{6}\], the spatial operator \( A \) has the following form
\[
A = - \left\{ \frac{\partial^2}{\partial r^2} + \frac{2r}{r^2 - \chi_0^2} \frac{\partial}{\partial r} + \frac{1}{r^2 - \chi_0^2} \left( \frac{\partial^2}{\partial \theta^2} + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \varphi^2} + \frac{\cos \theta}{\sin \theta} \frac{\partial}{\partial \theta} \right) \right\}.
\]

Using a separation of variables, \( \psi = e^{i \omega t} R(r) Y_{l m}^m(\theta, \varphi) \), we obtain an equation for the radial function from equation \[\text{9}\]. Its left-hand side is the radial part (the most important one for our
analysis since the remaining coordinates have compact ranges) of the operator $A$

\[
\frac{d^2 R}{dt^2} + \frac{2r}{r^2 - \chi_0^2} \frac{dR}{dr} - \left( \frac{l(l+1)}{r^2 - \chi_0^2} \right) R = -\omega^2 R . \tag{11}
\]

The equation that we have to analyze in order to judge about the essential self-adjointness is (7). So we have to deal with an ODE

\[
\frac{d^2 \psi_{\pm}}{dr^2} + \frac{2r}{r^2 - \chi_0^2} \frac{d\psi_{\pm}}{dr} - \left( \frac{l(l+1)}{r^2 - \chi_0^2} \pm i \right) \psi_{\pm} = 0. \tag{12}
\]

This is a Heun (singly) Confluent equation which is obtained from the general Heun equation containing four regular singularities through a confluence process, that is, a process where two singularities coalesce. This confluence procedure is performed by redefining parameters and taking limits resulting in a single (typically irregular) singularity \[37\]. In our case (12) we have two regular singularities at $r = \pm \chi_0$ and an irregular one at infinity. The solution for the above equation is expressed using Heun Confluent functions

\[
\psi_{\pm}(r) = C_1 \text{HeunC} \left( 0, -\frac{1}{2}, 0, \mp i \chi_0^2, \eta, \frac{r^2}{\chi_0^2} \right) + \quad C_2 r \text{HeunC} \left( 0, \frac{1}{2}, 0, \mp i \chi_0^2, \eta, \frac{r^2}{\chi_0^2} \right) , \tag{13}
\]

where

\[
\eta = \frac{1}{4} (\pm i \chi_0^2 - l(l+1) + 1) .
\]

If we do not consider the subdominant $\pm i$ term in (12), the Confluent Heun functions simplify and the solution can be expressed in the following form

\[
\psi(r) = C_1 P_l \left( \frac{r}{\chi_0} \right) + C_2 Q_l \left( \frac{r}{\chi_0} \right) , \tag{14}
\]

where $P, Q$ are the Legendre functions of the first and second kind respectively.

For analyzing the square-integrability it is worth to know the asymptotic behaviors of the above functions around the singular point $r = \chi_0$. The Legendre function $P_l \left( \frac{r}{\chi_0} \right)$ at $r = \chi_0$ is regular

$P_l(1) = 1$

and the Legendre function of the second kind, $Q_l \left( \frac{r}{\chi_0} \right)$, can be written as

\[
Q_l \left( \frac{r}{\chi_0} \right) = \frac{1}{2} P_l \left( \frac{r}{\chi_0} \right) \ln \left[ \frac{r + \chi_0}{r - \chi_0} \right] - \frac{2l-1}{l} P_{l-1} \left( \frac{r}{\chi_0} \right) - \cdots \tag{15}
\]

The square integrability of the solution (14) is checked by calculating a squared norm in a proper functional space on each $t = \text{const}$ hypersurface $\Sigma$. We consider the Hilbert space $\mathcal{H} = L^2(\Sigma, \mu)$, where $\mu$ is a measure given by the spatial metric volume element. It is straightforward to show that both solutions are square integrable at $r = \chi_0$ since the logarithmic divergence in (13) is compensated by the volume form $(r^2 - \chi_0^2)dr$ to give a finite limit at $r = \chi_0$ for the integrand. One might be worried that by removing the complex term from the equation we have changed its nature too much. However, as shown in [37] one of the solutions of Confluent Heun equation has (for the specific values of our parameters) logarithmic divergence — as in the case of $Q_l$ — and the other one is regular.

3.1.2 Dirac particle

The Newman-Penrose (NP) formalism [38] will be used here to analyze the properties of operator governing the massless Dirac particles (fermions). The Chandrasekhar-Dirac (CD) [39] equations (representing a reformulation into the Newman-Penrose formalism) are suitable for this task and are given by

\[
(D + \epsilon - \rho) F_1 + (\delta + \pi - \alpha) F_2 = 0 ,
\]

\[
(\nabla + \mu - \gamma) F_2 + (\delta + \beta - \tau) F_1 = 0 ,
\]

\[
(D + \bar{\epsilon} - \bar{\rho}) G_2 - (\bar{\delta} + \bar{\pi} - \bar{\alpha}) G_1 = 0 ,
\]

\[
(\nabla + \bar{\mu} - \bar{\gamma}) G_1 - (\bar{\delta} + \bar{\beta} - \bar{\tau}) G_2 = 0 ,
\]

where $F_1, F_2, G_1$ and $G_2$ are the components of the Dirac wave function (bispinor), $\epsilon, \rho, \pi, \alpha, \mu, \gamma, \beta$ and $\tau$ are the NP spin coefficients and the "bar" denotes a complex conjugation. The null tetrad vectors for the metric [1] are defined by

\[
l^a = (1, 1, 0, 0) , \tag{17}
\]

\[
n^a = \left( \frac{1}{2} - \frac{1}{2}, 0, 0, 0 \right) ,
\]

\[
m^a = \frac{1}{\sqrt{2(r^2 - \chi_0^2)}} \left( 0, 0, 1, -i \frac{\cot \theta}{\sin \theta} \right) .
\]

The directional derivatives in the CD equations are given by $D = n^a \partial_a, \nabla = n^a \partial_a$ and $\delta = m^a \partial_a$. To simplify the analysis we define auxiliary differential operators

$D_0 = D,$

$D_0^* = -2\nabla,$

$L_0 = \sqrt{2(r^2 - \chi_0^2)} \delta$ and $L_0^* = L_0 + \frac{\cot \theta}{2},$

$L_0 = \sqrt{2(r^2 - \chi_0^2)} \delta$ and $L_1 = L_0 + \frac{\cot \theta}{2}.$


Evidently, the spatial parts of $D_0$ and $D_1^0$ are purely radial operators, while $L_{0,1}$ and $L_{1,1}^0$ are purely angular operators.

The nonzero spin coefficients for the metric \( [1] \) are given by

$$\rho = 2\mu = \frac{r}{r^2 - \lambda_0^2}, \quad \beta = -\alpha = \frac{1}{2\sqrt{2}} \frac{\cot \theta}{\sqrt{r^2 - \lambda_0^2}}.$$  \hspace{1cm} (19)

Substituting these nonzero spin coefficients and the definitions of the operators \([18]\) given above into the CD equations \([16]\) leads to

$$\begin{align*}
(D_0 + \frac{r}{r^2 - \lambda_0^2}) F_1 + \frac{1}{\sqrt{2(r^2 - \lambda_0^2)}} L_1 F_2 &= 0, \\
-\frac{1}{2} (D_0 + \frac{r}{r^2 - \lambda_0^2}) F_2 + \frac{1}{\sqrt{2(r^2 - \lambda_0^2)}} L_1^* F_1 &= 0, \\
(D_0 + \frac{r}{r^2 - \lambda_0^2}) G_1 - \frac{1}{\sqrt{2(r^2 - \lambda_0^2)}} L_1^* G_1 &= 0, \\
\frac{1}{2} (D_0^1 + \frac{r}{r^2 - \lambda_0^2}) G_1 + \frac{1}{\sqrt{2(r^2 - \lambda_0^2)}} L_1^0 G_2 &= 0.  \hspace{1cm} (20)
\end{align*}$$

For solving these CD equations, we assume a separable form of a solution

$$\begin{align*}
F_1 &= f_1(r) Y_1(\theta)e^{ik(r^2 + m\phi)}, \\
F_2 &= f_2(r) Y_2(\theta)e^{ik(r^2 + m\phi)}, \\
G_1 &= g_1(r) Y_3(\theta)e^{ik(r^2 + m\phi)}, \\
G_2 &= g_2(r) Y_4(\theta)e^{ik(r^2 + m\phi)}.  \hspace{1cm} (21)
\end{align*}$$

Here \( \{f_1, f_2, g_1, g_2\} \) and \( \{Y_1, Y_2, Y_3, Y_4\} \) are functions of \( r \) and \( \theta \) respectively. Additionally, \( m \) is the azimuthal quantum number and \( k \) is the frequency of the Dirac wave function, both are assumed to be real and positive. By substituting \([21]\) into \([20]\) and using these assumptions

$$\begin{align*}
f_1(r) &= g_2(r) \quad \text{and} \quad f_2(r) = g_1(r),  \hspace{1cm} (22) \\
Y_1(\theta) &= Y_3(\theta) \quad \text{and} \quad Y_2(\theta) = Y_4(\theta),  \hspace{1cm} (23)
\end{align*}$$

we can reduce the system \([20]\) into just two equations. The important radial parts of these two remaining Chandrasekhar – Dirac equations become

$$\begin{align*}
(D_0 + \frac{r}{r^2 - \lambda_0^2}) f_1(r) &= \frac{\lambda}{\sqrt{2(r^2 - \lambda_0^2)}} f_2(r), \\
\frac{1}{2} (D_0^1 + \frac{r}{r^2 - \lambda_0^2}) f_2(r) &= \frac{\lambda}{\sqrt{2(r^2 - \lambda_0^2)}} f_1(r),  \hspace{1cm} (24)
\end{align*}$$

where \( \lambda \) is a separation constant. For further simplification we introduce a new functions

$$\begin{align*}
f_1(r) &= \frac{\zeta_1(r)}{\sqrt{r^2 - \lambda_0^2}}, \\
f_2(r) &= \frac{\sqrt{2}\zeta_2(r)}{\sqrt{r^2 - \lambda_0^2}},
\end{align*}$$

and the equations \([24]\) transform into the following coupled system

$$\begin{align*}
D_0^1 \zeta_1(r) &= \frac{\lambda}{\sqrt{r^2 - \lambda_0^2}} \zeta_2(r),  \hspace{1cm} (25) \\
D_0 \zeta_2(r) &= \frac{\lambda}{\sqrt{r^2 - \lambda_0^2}} \zeta_1(r).
\end{align*}$$

or explicitly

$$\begin{align*}
\left( \frac{d}{dr} + ik \right) \zeta_1(r) &= \frac{\lambda}{\sqrt{r^2 - \lambda_0^2}} \zeta_2(r),  \hspace{1cm} (26) \\
\left( \frac{d}{dr} - ik \right) \zeta_2(r) &= \frac{\lambda}{\sqrt{r^2 - \lambda_0^2}} \zeta_1(r).
\end{align*}$$

In order to write the above equation in a more compact form we combine the solutions in the following way, \( \Xi_+ = \zeta_1 + \zeta_2 \), \( \Xi_- = \zeta_2 - \zeta_1 \), and square the operators to end up with a pair of one-dimensional Schrödinger-like stationary equations with effective potentials,

$$\left( \frac{d^2}{dr^2} + k^2 \right) \Xi_\pm = V_\pm \Xi_\pm,  \hspace{1cm} (27)$$

$$V_\pm = \frac{\lambda^2}{r^2 - \lambda_0^2} + \frac{r\lambda}{(r^2 - \lambda_0^2)^{3/2}}.  \hspace{1cm} (28)$$

In analogy with the equation \([6]\), the spatial operator \( A \) for the massless case is

$$A = -\frac{d^2}{dr^2} + V_\pm,$$

so from the self-adjoint extension method \([7]\) we have to analyze the solutions of

$$\left( -\frac{d^2}{dr^2} + \left[ \frac{\lambda^2}{r^2 - \lambda_0^2} \mp \frac{r\lambda}{(r^2 - \lambda_0^2)^{3/2}} \right] \right) \pm i \psi_\pm = 0.  \hspace{1cm} (29)$$

For finding the solutions of the above equation, we ignore the subdominant \( \pm i \) part and obtain

$$\psi_\pm = C_1 \left( \pm 2\sqrt{r^2 - \lambda_0^2} + r \right) \left( \sqrt{r^2 - \lambda_0^2} + r \right)^{\mp \lambda} + C_2 \left( \sqrt{r^2 - \lambda_0^2} + r \right)^{\pm \lambda}  \hspace{1cm} (30)$$
in which $\lambda$ should be an integer. Obviously, when $r \to \chi_0$ (which is the singular point in our spacetime) the above two solutions are both finite and their Hilbert space norms near the singular point as well.

Thus we have seen that both the Klein–Gordon and the Dirac particles see the singularity because in both cases all the solutions are square-integrable and therefore the system is quantum mechanically singular according to [2].

### 4 Canonical Quantization via conditional symmetries

We next move on to study the resolution of the singularity by canonically quantizing the system via the conditional symmetries method [16]. The initial point is the general form of the spacetime metric

$$ds^2 = a^2(r)dt^2 - \frac{\gamma(r)}{4a^2(r)}dr^2 - b^2(r)d\Omega^2$$  \hspace{1cm} (31)

where $a(r), b(r)$ are scale factors and $N(r)$ is the lapse function, i.e. no choice of gauge has been done. We will see that the spacetime metric \([1]\) coupled to the scalar field \([2]\) pops up as a solution of the equations of motion for a specific gauge choice. The Lagrangian of the geometry \([31]\) coupled to a massless scalar field $\phi(r)$ in the constant potential parametrization is

$$L = -N - \frac{8ab^2}{N} - \frac{4a^2b^2}{N} + \frac{2a^2b^2\phi^2}{N}$$  \hspace{1cm} (32)

where $' \equiv \frac{d}{dr}$. The metric on the configuration space of the variables $a, b, \phi$ can be read off from the kinetic part of \([32]\),

$$G_{\alpha\beta} = \begin{pmatrix} 0 & -8ab & 0 \\ -8ab & -8a^2 & 0 \\ 0 & 0 & 4a^2b^2 \end{pmatrix}$$  \hspace{1cm} (33)

We are interested in its Killing symmetries which are

$$\xi_1 = -a\partial_a + b\partial_b, \quad \xi_2 = -a\phi\partial_a + b\phi\partial_b + 2\ln a\partial_b, \quad \xi_3 = \partial_b, \quad \xi_h = \frac{a}{2}\partial_a.$$  \hspace{1cm} (34)

where $\xi_h$ denotes the homothetic vector field. We can construct the first integrals on the phase space from the relations $Q_i = \xi_i^\alpha p_\alpha$, $i = 1, 2, 3$ and $Q_h = \xi_h^\alpha p_\alpha + \int dt \ n(t)$. Then, in our variables they become

$$Q_1 = -\frac{8ab^2}{N} = \kappa_1, \quad \kappa_1 = 2\kappa_2,$$  \hspace{1cm} (35a)

$$Q_2 = -\frac{8ab^2}{n} = \kappa_3,$$  \hspace{1cm} (35b)

$$Q_3 = 4a^2b^2\phi = \kappa_3,$$  \hspace{1cm} (35c)

$$Q_h = 4a^2b^2\phi = \kappa_h - \int dt \ N(t).$$  \hspace{1cm} (35d)

If we replace to the system \([35]\), the values for the variables from the metric and field configurations \([1], [2]\), i.e. $a = 1, N = 1, b = r^2 - \chi_0^2, \phi = \frac{1}{2}\ln(r^2 + \chi_0^2)$, we find that it will be satisfied under the choice for the constants

$$\kappa_1 = \kappa_2 = \kappa_h = 0, \quad \kappa_3 = 2\sqrt{2}\chi_0.$$  \hspace{1cm} (36)

Turning at the quantum level, we promote the conserved charges $Q_i$ together with the constraints to operators and impose them as conditions on the wave function. The new eigen-equations cannot be imposed simultaneously on the wave function because of the conditions \([35a], [35c]\), i.e. $a_1, N_1, b_1 = r^2 - \chi_0^2, \phi = \frac{1}{2}\ln(r^2 + \chi_0^2)$, we find that it will be satisfied under the choice for the constants

$$\kappa_1 = \kappa_2 = \kappa_h = 0, \quad \kappa_3 = 2\sqrt{2}\chi_0.$$  \hspace{1cm} (36)

In the following, we consider the case of the two dimensional subalgebra \{Q_1, Q_3\} and solve the following equations

$$\hat{Q}_1\Psi = i(-b\partial_b + a\partial_a)\Psi = \kappa_1\Psi,$$  \hspace{1cm} (37)

$$\hat{Q}_3\Psi = -i\partial_\phi\Psi = \kappa_3\Psi,$$  \hspace{1cm} (38)

together with the constraint equation (Wheeler-DeWitt)

$$\hat{H}\Psi = \frac{1}{2}\frac{\partial^2}{\partial \phi^2} \left( (-1 + 32a^2b^2) \Psi - 2(2\partial_\phi - b\partial_b + a(\partial_a - 2b\partial_b + a\partial_\phi))) \Psi = 0 \right.$$

The solution of this system is

$$\Psi = e^{i2\chi_0} (A_1J_1(4ab) + A_2Y_2(4ab)),$$  \hspace{1cm} (40)

$$\lambda = \frac{1}{2}(-1 + \sqrt{3 - 64\chi_0^2}).$$  \hspace{1cm} (39)

To get a rough idea about the consequences the above wave function has for the fate of the singularity one can consider probability distribution on the superspace. Specifically one shall consider probability density including the correct measure coming from the metric \([33]\),

$$p = |\Psi|^2 \cdot 16a^2b^2$$  \hspace{1cm} (41)

As the plots of the above quantity for two sets of pa-
parameters — first one for $A_1 = 1, A_2 = 0$ corresponding to BesselJ function only (figure 1) and the second one for $A_1 = 0, A_2 = 1$ corresponding to BesselY function only (figure 2) — in the plane of variables $a, b$ show the probability of the system to be in the state corresponding to either $a = 0$ or $b = 0$ is highly suppressed. Since the physical singularity for the metric (31) appears only for $b = 0$ (as one can easily derive from corresponding Kretschmann scalar) on the quantum level this situation seems to be avoided.

Another possible approach to derive physical consequences from the wave function considers Bohmian interpretation. This suits well with quantum cosmology since it makes possible the definition of quantum paths on the configuration space through the guidance equations

$$\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial S}{\partial q_i}.$$ (42)

These are defined through the identification $p_\alpha \equiv \partial_\alpha S$ and $S(q)$ is the function in the polar form expression of the wave function, $\Psi = \Omega e^{iS}$. As in the case of the Schrödinger equation, inserting it in the Wheeler-DeWitt equation

$$\hat{H}\Psi = \left(-\frac{1}{2\sqrt{G}}\partial_\mu G^{\mu\nu}\partial_\nu - \frac{d-2}{8(d-1)}R + 1\right)\Psi = 0$$ (43)

we obtain a modified Hamilton-Jacobi

$$\frac{1}{2}G^{\alpha\beta}\partial_\alpha S\partial_\beta S = \frac{1}{2}\Box\Omega + 1 = 0.$$ (44)

which contains an additional potential term. In the case this term is nonzero, the solution of the (42) will be different from the classical one while when it vanishes we should recover the classical spacetime.

For our particular case, we will consider the approximation $A_1 \to 0$ for small and large arguments of the spherical Bessel functions to bring the wave function in polar form, thus obtaining

$$\Psi_{sm} = C_1(ab)^{-\frac{1}{4} - \frac{1}{8}\sqrt{3 - 64\chi_0^2}}e^{i\chi_0\phi},$$ (45a)

$$\Psi_{la} = C_2\sin(-4ab + \frac{\pi}{4}(-1 + \sqrt{3 - 64\chi_0^2}))e^{i\chi_0\phi}.$$ (45b)

The above cases give us two different solutions all of which differ from the classical solution of the system (35), since the quantum potential does not vanish. We assume two subcases for the small arguments, corresponding to negative or positive value of the quantity under the square root (since the function $S(a, b, \phi)$ differs). Then the spacetime elements we obtain are

$$ds^2 = dt^2 - dr^2 - \lambda_1 d\theta^2 - \sin^2 \theta d\phi^2,$$ (46a)

$$ds^2 = \lambda_2 dt^2 - \lambda_2 r^2 dr^2 - r^2(d\theta^2 + \sin^2 \theta d\phi^2)$$ (46b)

where $\lambda_1, \lambda_2$ are essential constants which characterize the geometry of the spacetimes.

For the large arguments, the solution is again (46a). The scalar functions of these line elements inform us that it is only for the range $-\sqrt{\frac{3}{8}} < \chi_0 < \sqrt{\frac{3}{8}}$ of the constant $\chi_0$ for the small arguments and any range for the large arguments that the singularity vanishes from the semiclassical line element.
5 Covariant Loop Quantum Gravity

5.1 Geodesic equations

Before going in the direction of Quantum Gravity investigation we need to understand the nature of the singularity more at the classical level. In this section we want to study the trajectory for a test particle moving on a timelike geodesic. The simplest approach is to use the variational principle or Euler–Lagrange equations for timelike geodesics. The Lagrangian has the following form

\[ 2L = g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = \dot{t}^2 - \dot{r}^2 - (r^2 - \chi_0^2) \left( \dot{\theta}^2 + \sin^2 \theta \dot{\varphi}^2 \right), \]  

where the dot denotes a derivative with respect to the proper time \( \tau \). The Euler–Lagrange equations

\[ \frac{\partial L}{\partial x^\mu} - \frac{d}{d\tau} \left( \frac{\partial L}{\partial \dot{x}^\mu} \right) = 0 \]  

give us two conserved quantities, namely the energy \( (E) \) and the angular momentum \( (L) \)

\[ \frac{d}{d\tau} \left( \frac{\partial L}{\partial \dot{t}} \right) = 0 \Rightarrow \dot{t} = E, \]  

\[ \frac{d}{d\tau} \left( \frac{\partial L}{\partial \dot{\varphi}} \right) = 0 \Rightarrow \dot{\varphi} = \frac{L}{(r^2 - \chi_0^2) \sin^2 \theta}. \]  

We consider motion in the equatorial plane \( \theta = \frac{\pi}{2} \). Substituting (49b) in (47), we obtain

\[ E^2 - \dot{r}^2 - \frac{L^2}{(r^2 - \chi_0^2)} = 1 \]  

and for a qualitative analysis of geodesics we employ the standard effective potential method. We can write the equation for radial velocity in the following form

\[ \dot{r}^2 + V_{eff} = E^2, \]  

\[ V_{eff} = 1 + \frac{L^2}{(r^2 - \chi_0^2)}. \]  

The effective potential is plotted in Figure 3. It is evidently repulsive and acting similarly to a centrifugal barrier (in a flat space) and in fact its origin is similar. Compared to centrifugal barrier it allows particle to travel closer to the origin (at \( r = \chi_0 \)). Note that for vanishing angular momentum \( l \) the radial velocity is constant so the radial particles (or observers) are traveling like in an empty flat space with a constant velocity.

5.2 Covariant Loop Quantization

So far we have probed the singularity just by quantum particles on the fixed background spacetime so the results might not be convincing or even correct from the nonperturbative point of view. To proceed further we should consider the quantum gravity picture. When the static spacetime possesses a horizon covering the central spacelike singularity one can use the Loop Quantum Cosmology method since the spacetime below the horizon (which is no longer static) can often be mapped onto some symmetric cosmological model whose singularities are generally resolved. Our spacetime however contains a naked timelike singularity so we cannot use this trick. Instead we can apply the recent discovery on the level of the Covariant Loop Quantum Gravity [28] that quite generally the singularities are resolved due to the upper bound for the acceleration of observers arising on the quantum level. The derivation is based on considering the Rindler observers but is later applied to cosmology with the characteristic acceleration being the mutual acceleration of nearby comoving observers. For the spacetime in question, we consider essentially the same quantity, a relative acceleration with respect to a radial geodesic as given by the geodesic deviation equation. This also measures the tidal forces acting upon an object approaching the singularity.
The four velocity of a radial geodesic (considered in the equatorial plane for simplicity) is described by
\[ u^\alpha \partial_{\alpha} = \sqrt{(u^r)^2 + 1} \partial_t + u^r \partial_r, \tag{52} \]
with the radial velocity \( u^r \) being a constant. The deviation vector is considered in the form \( \delta = \delta^\alpha \partial_{\alpha} \). The geodesic deviation equation then assumes the following form
\[ \frac{D^2 \delta^\alpha}{d\tau^2} = -R^\alpha{}_{\beta\gamma\sigma} u^\beta \delta^\gamma u^\sigma = - \frac{\chi_0^2 (u^r)^2}{(r^2 - \chi_0^2)^2} \delta^\alpha. \tag{53} \]
Evidently, the tidal force grows unbounded when approaching the singularity even though the radial geodesic observer is not accelerated with respect to the asymptotic observer (see the end of section 5.1). As a measure of the acceleration, we will use the invariant norm of (53) with respect to a unit separation
\[ a = \frac{\chi_0^2 (u^r)^2}{(r^2 - \chi_0^2)^{3/2}}. \tag{54} \]
According to \[28\], the acceleration is bounded by a maximum value \( a_{\text{max}} \approx \sqrt{\frac{1}{8 \pi G \hbar c}} \) (in nongeometric units). This result is moreover derived in a fully covariant theory unlike previous upper bounds to the acceleration \[41\]. Inspecting (54), one immediately sees that the upper bound to the acceleration means that the divergent factor \( (r^2 - \chi_0^2)^{-1} \) appearing in the curvature scalars \[3\] is also bounded and therefore the singularity is resolved at the level of Covariant Loop Quantum Gravity. Accordingly, the tidal forces are bounded as well and an object can in principle survive the fall into the singularity (or to the region where the curvature singularity appears classically). However, the bound is extremely large so it is hard to imagine any realistic object that would not be crushed.

So one can conclude that the critical behavior of General Relativity (its breakdown at the position of singularity) is cured at the quantum level (infinites are cured), but the practical result of approaching the singularity (the destruction of an extended object) remains effectively the same.

6 Conclusion and final remarks

We have shown that for both the Klein–Gordon particle and the Dirac particle all solutions of (7) are square integrable which means that the corresponding operators in both cases are not essentially self-adjoint and therefore the problem is quantum mechanically singular. So the quantum probes still see the singularity in this case.

In the canonical quantization approach there are clear indications that the singularity is resolved using two alternative interpretations of the wave equation.

In the case of the Covariant Loop Quantum Gravity the maximal acceleration existence provides the means to effectively remove the singularity as demonstrated above. On the other hand we have not performed complete quantization of the spacetime here and one should still view this result as an indication of singularity resolution in this theory rather than a complete proof. At the same time the results of an observer approaching the previous position of a singularity seem catastrophic even in this quantum picture because the upper bound on the tidal forces is extremely large.

Evidently, the spacetime quantization approaches yield results contradicting the quantum particle approach. Since these method are based on quantum description of spacetime one should give them preference over the quantum particle approach where the spacetime itself is classical only the probes are quantum.

One obvious direction of a possible future investigation concerns the use of the Quantum Field Theory on a curved background, ideally including semi-classical backreaction effects. This would fit in-between the approaches presented here. Using quantum probe field is certainly closer to a realistic scenario than relying only on the quantum mechanical particles. On the other hand this approach should be superseded by a full spacetime quantization, e.g. using Covariant Loop Quantum Gravity to perform full spacetime quantization going beyond spherically symmetric model.

Recently, we have studied the Janis–Newman–Winicour solution \[42\] which can serve very well as another special case for the investigation using all these approaches.

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