On the relation between
Staruszkiewicz’s quantum theory of the
Coulomb field and the causal
perturbative approach to QED

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Abstract

At the classical level the electromagnetic field can be well identified at the spatial infinity. Staruszkiewicz pointed out that the quantization of the electromagnetic field at spatial infinity is essentially unique and follows from the two fundamental principles: 1) gauge invariance and 2) canonical commutation relations for canonically conjugated generalized coordinates, and constructed a simple and mathematically transparent quantum theory of the Coulomb field, predicting (among other things) a relation between the theory of unitary representations of the $SL(2,\mathbb{C})$ group and the fine structure constant. Until now this theory has stayed outside the main stream of the perturbative development in QED, mainly due to the unsolved infrared-type (IR) problems in the perturbative approach. Recently however there has been performed a more careful analysis of mass less free gauge fields, such as the electromagnetic potential field, their Wick and chronological products, which revealed the need for a more careful construction of these fields, and which opened a way to resolve IR problems (at least those which shows up at each order separately). In particular the more careful definition of these fields reveals the need for the analysis of the free and the interacting fields in QED at spatial infinity which have been unnoticed until now because the zero mass gauge fields have been treated with insufficient care. A need for comparison of the perturbatively constructed field at spatial infinity with the quantum phase field of the Staruszkiewicz theory arises, which leads to the proof of universality of the unit of charge. We give here a commentary on these facts.

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1 Introduction

Thirty years ago Staruszkiewicz [32] (compare also essentially the same text in [32]) noted that the quantization of the electromagnetic field at spatial infinity is essentially unique and follows from the two fundamental principles: 1) gauge invariance and 2) canonical commutation relations for canonically conjugated generalized coordinates, and constructed a simple and mathematically transparent quantum theory of the Coulomb field, predicting (among other things) a relation between the theory of unitary representations of the $SL(2, \mathbb{C})$ group and the fine structure constant. Until now this theory has stayed outside the main stream of the perturbative development in QFT, especially perturbative QED, mainly due to the unsolved infrared-type problems of the perturbative approach. By now the problem of UV-divergences has been solved in physical QFT, such as QED, within the causal perturbative method. This method involves the adiabatic switching off the interaction at infinity, as the intermediate stage, and the theory becomes complete, at least at the order-by-order level, when the adiabatic limit of restoring the interaction at infinity is performed. Now in QED we successfully control the adiabatic limit only at the cross section level in each order separately, as well as for the Green and Wightman functions (free vacuum expectation values) likewise at the order-by-order level. Restoring the interaction at infinity (in the adiabatic limit) even at each order term separately in the perturbation series for interacting fields (and not merely for numerical cross sections, or merely for Green or Wightman functions) has until now been unsuccessful. This is the unsolved infrared (IR) divergence problem. In particular any endeavour of identification of the (interacting) quantum field at spatial infinity, even at the order-by-order level, would be problematic (if possible at all) within the causal perturbative QED (not to mention the problem of convergence of the perturbation series). Perhaps the reader well trained in the causal perturbative QED will not even see any necessity in passing to spatial infinity when dealing with the causal perturbative approach. Let us explain how it arises.

2 Infrared problem and its solution

The work of Blanchard and Seneor [2] in which they discovered the existence, order-by-order, of the adiabatic limit for Wightman functions in QED has been received as a great success for the causal method, which successfully eliminates the UV divergences, just working carefully with distributions, operator-valued distributions, and their Wick products. The initial enthusiasm is easy to understand. Namely, we can in principle stop the calculation at the chosen order of the perturbative series, obtaining the Wightman functions at the chosen order. By the reconstruction theorem of Wightman we can, at least in principle, reconstruct the Hilbert space in which the Wightman operator valued fields act, i.e. initial free Fock space, together with its Fock space structure. Of course this reconstruction, at least in principle, depends on the order, and the dense domain(s) on which the operator valued Wightman fields act, may be related
to the simple finite number particle states of the Fock space in a very involved fashion. But the enthusiasm had been based on the comparison with the scalar massive theory formerly worked out by Epstein and Glaser. Indeed joining the reconstruction theorem of Wightman with the general properties of the so called “weak” and “strong” topologies used for the operator valued distributions [49] shows that the existence of the adiabatic limit for Wightman functions should force the existence of the adiabatic limit in the weak sense and then in strong sense on a suitably chosen domain. This is indeed the case for massive field theories, such as the \( \varphi^4 \): scalar massive field. The explicit construction of the relevant domain for the scalar massive field, which meets the requirements and works in each order is possible and has been given in [11]. But when passing to QED (and all the more when passing to other theories with non abelian gauge) situation seems to be substantially different and nobody (so far as the author is aware of) has been able to construct the domain on which the adiabatic limit exists in the strong sense, giving the interacting fields order-by-order. In particular it seems that even the free vacuum state cannot belong to the domain on which the interacting field is well defined as an operator valued distribution (in each order separately). Thus from the initial enthusiasm we have quickly arrived at the paradoxical situation, because the Blanchard-Seneor’s result together with the essential equivalence of both, weak and strong topologies, for the existence of limits of operator valued distributions [49], forces existence of a dense domain on which the strong adiabatic limit exists.

In order to solve this paradox we pointed out in [46] that the lack of success in the construction of the dense domain lies not in the causal construction (of Epstein and Glaser) of the chronological product of Wick polynomials of free fields but rather in the careless construction of the free local mass less gauge fields themselves, such as the electromagnetic potential field within the Gupta-Bleuler (and generally BRST) method, before passing to their Wick and chronological products. Is this a real surprise? This is (or would be) not very much surprising, at least for some of us, e.g. for Haag who points out on page 48 of his book [17] that “Reviewing our construction of free fields from the irreducible [unitary] representations of the double covering of the Poincaré group we notice that the two most important fields, the Dirac field and the electromagnetic potential are not directly obtained”. The construction of the free fields to which Haag is appealing to is just the Wightman construction of the free field as presented in [43]. Bogoliubov frequently repeated during his lectures that “the free electromagnetic field is a true mystery among other free fields” and some specialists, e.g. Schroer [27], emphasize that indefiniteness of the inner product (in the Gupta-Bleuler or BRST method) makes the mathematically controllable construction of the free mass less gauge fields difficult. On the other hand the Gupta-Bleuler or BRST method with indefinite product is unavoidable in the construction of free mass less gauge fields, such as the electromagnetic potential, with local transformation law, and this locality is crucial for the causal method avoiding UV divergences, so that the problem is real.

In order to reach the free and local electromagnetic potential field (within the Gupta-Bleuler or BRST method) at the level of rigour which is achieved in
Wightman construction we recall first his construction, [43], Ch. 3, working for massive free fields or for zero mass fields (but non gauge fields, e.g. the scalar zero mass field). Then we reformulate it, in this simplified case, in a manner general enough to be capable of generalization on the gauge fields. Then we show that Wightman’s construction can be naturally generalized on gauge fields with the same level of rigour, and thus that essentially his definition works as well for gauge fields. We give now the construction of [43], Ch. 3, already in a form, working as well for gauge fields. Namely the construction consists of two parts. The first part depends totally on the Hilbert space structure and a representation $U$ of the double (and thus universal in this case) covering $T_4 \otimes SL(2, \mathbb{C})$ of the Poincaré group in the single particle Hilbert space $H$ together with its amplification $\Gamma(U)$ acting in the Fock space $\Gamma(H)$. Presumably the representation $U$ acting in the single particle Hilbert space $H$ is an irreducible unitary representation (for massive fields or non-gauge fields it is indeed unitary, but for the electron-positron Dirac free field it is not irreducible, although in most of the cases it is likewise irreducible) and its choice depends on the specific kind of field we want to construct. By the standard classification of unitary irreducible representations of a (regular) semi-direct product of locally compact groups all irreducible representations of $T_4 \otimes SL(2, \mathbb{C})$ are classified by the specific orbit in the momentum space (under the Lorentz hyperbolic rotations) and the corresponding representation is the representation induced in the Mackey sense by a (presumably irreducible and unitary) representation of the stationary subgroup leaving fixed an arbitrary (but fixed) point of the chosen orbit. By general principles we are concerned with representations associated with orbits lying within the positive energy cone (one sheet of the two-sheeted hyperboloid – in case of massive fields, or the positive energy sheet of the light cone without the apex – in case of zero mass free fields). The stationary subgroups in these two cases are respectively the double covering $SU(2, \mathbb{C})$ of the three dimensional rotation group or the double covering of the group of motions of the Euclidean plane. There are further restrictions, and among them is the local transformation law of the free field which means that the Fourier transforms of the states of the single particle representation should have momentum-independent multiplier in the transformation law so that their inverse Fourier transform (over space-time) have local transformation law. Still at this Hilbert space level we define the families of annihilation and creation operators, $a(\tilde{\varphi}), a(\tilde{\varphi})^*, \tilde{\varphi} \in H$, densely defined in $\Gamma(H)$. This is the first part. In case of zero mass gauge fields, we have in addition to the Hilbert space structure of $H$ a unitary fundamental symmetry operator $\mathcal{J}$, i.e. involution, in $H$: $\mathcal{J}^* = \mathcal{J}, \mathcal{J}^2 = 1$, which makes the Hilbert space $H$ a Krein space. The representation $U$ in the single particle Hilbert-Krein space $H$ is no longer unitary with respect to the Hilbert space inner product $(\cdot, \cdot)$ but it preserves the Krein inner product $(\cdot, \mathcal{J} \cdot)$, but it is in general even not Krein-unitary but only Krein-isometric unbounded with respect to the Hilbert space inner product. This makes the difference in comparison to non gauge fields. But the point is that the relevant class of Krein isometric representations, including the representations needed for the construction of the free zero mass gauge fields, has the property that the Mackey theory of induced representa-
tions can be extended on them, so that the first part of Wightman construction is extendible on the free zero mass gauge fields, in particular the tensor product construction extends on them, and the second quantization functor $\Gamma$ is applicable in this class of representations. In particular we can likewise build the families of annihilation and creation operators, $a(\bar{\varphi}), a(\bar{\varphi})^+, \bar{\varphi} \in \mathcal{H}$, densely defined in the Hilbert space $\Gamma(\mathcal{H})$, which together with the fundamental symmetry (Gupta-Bleuler operator) $\eta = \Gamma(\mathfrak{3})$, is a Krein space in which the representation $\Gamma(U)$ acts as a Krein-isometric representation (with densely defined and closable representors). Now we pass to the second part of Wightman construction of free field. Namely, in order to pass to fields (say operator-valued distributions)

we need to pass to the level of a more subtle structure than just the Hilbert space $\mathcal{H}$ and $\Gamma(\mathcal{H})$, namely to the nuclear densely embedded subspaces $E$ and $(E)$ of $\mathcal{H}$ and $\Gamma(\mathcal{H})$ respectively, composing Gelfand triples $E \subset \mathcal{H} \subset E^*$ and $(E) \subset \Gamma(\mathcal{H}) \subset (E)^*$ closely related to the chosen nuclear test function space in space-time coordinates (namely $E$ is the space of restrictions $\tilde{\varphi}\vert_\mathcal{O}$ of Fourier transforms $\tilde{\varphi}$ of test functions $\varphi$ to the orbit $\mathcal{O}$ in question which defines the induced representation $U$). Each representor of the representation $U$, or respectively $\Gamma(U)$, transforms bi-uniquely and bi-continuously the nuclear space $E \subset \mathcal{H}$, or respectively the nuclear space $(E) \subset \Gamma(\mathcal{H})$, onto itself with respect to the nuclear topology. This is the second part of Wightman construction of the free field. Since Bohr and Rosenfeld analysis we know that we need a test space over the space time coordinates in $(\mathbb{R}^4, g_{\mu\nu})$ (as well as its Fourier transform image in the momentum picture). Now it is crucial to note that in the construction of the field the argument $\tilde{\varphi}$ of the creation-annihilation families $a(\bar{\varphi}), a(\bar{\varphi})^+$ of (here ordinary densely defined) operators, which in general may be equal to any element of the single particle Hilbert space, is replaced by elements of $E$, i.e. by the restriction $\tilde{\varphi}\vert_\mathcal{O}$ of the Fourier transforms of the elements $\varphi$ of space-time test function space to the orbit in question which defines the representation $U$ in the single particle space. It is essential to note that the restriction of these Fourier transforms which are functions over $\mathbb{R}^4$ to the orbit submanifold should be a continuous map in the nuclear topology in order to obtain an operator valued distribution given by the Wightman formula

$$A(\varphi) = a(\tilde{\varphi}\vert_\mathcal{O}) + \eta a(\tilde{\varphi}\vert_\mathcal{O})^+ \eta,$$

over a nuclear test functions $\varphi$ defined on the space-time (of course in case of non gauge fields $\eta = 1$). Now in case of massive fields when the corresponding orbit is just the one sheet of the two-sheeted hyperboloid everything works fine if we take as the space-time test space the ordinary Schwartz space $S(\mathbb{R}^4)$, as the orbit is a smooth submanifold of $\mathbb{R}^4$ and when using the natural spatial coordinates of the momentum as coordinates on the orbit then the restriction to the orbit understood as a map $S(\mathbb{R}^4) \to S(\mathbb{R}^3)$ is continuous in the nuclear Schwartz topology on $S(\mathbb{R}^4)$ and respectively on $S(\mathbb{R}^3)$. For the zero mass fields

\footnote{$(E) \subset \Gamma(\mathcal{H}) \subset (E)^*$ is the Gelfand triple over the Fock space obtained by the standard amplification of the Gelfand triple $E \subset \mathcal{H} \subset E^*$ over the single particle Hilbert space, compare e.g. \cite{21} or \cite{19}. $(E)$ is the Hida’s test functional space.}
the Schwartz space as the test space doesn’t work because the restriction to the orbit is no longer continuous in the nuclear topology as a consequence of the singularity of the cone at the apex, being the orbit in the zero mass case. The zero mass fields (say a zero mass gauge fields) need another test space, which in momentum picture is the closed subspace $S^0(\mathbb{R}^4)$ of the Schwartz space of all those functions which vanish at zero together with all their derivatives, and its inverse Fourier image $S^{00}(\mathbb{R}^4)$ as the nuclear test space of functions on the space-time. It seems that this important fact has not been noticed before.

Summing up we solved the problem of rigorous construction of zero mass gauge fields in the indicated manner. In particular a brief account of the free electromagnetic potential can be found in [46], a slightly more detailed account of the electromagnetic potential (especially its single particle subspace) is presented in [45], and the full theory extending Mackey theory of induced representations on Krein-isometric representations, sufficient for the zero mass gauge fields encountered in the Standard Model (with the Higgs field), can be found in [48] or [44]. Although the full and rather long proof, especially concerning the distribution theory aspect, can be found in [48]. Especially the white noise construction of the free electromagnetic potential requires a considerable work, in particular the standard construction of the family of spaces $S^0(\mathbb{R}^n)$, capable of the amplification to the Fock space, required a considerable amount of technicalities. The fermionic massive fields are easier, and in principle require only ordinary application of the general theory [30].

Note that although $S^{00}(\mathbb{R}^4)$ contains no elements of compact support it is well suited as a test space for zero mass fields. First note that $S^{00}(\mathbb{R}^4)$ is sufficiently reach to contain element for each arbitrary small open conic type-shape set $C$, with the support in $C$. In particular the causal relations may be expressed within $S^{00}(\mathbb{R}^4)$. Because the commutator functions of zero mass free fields are homogeneous, then $S^{00}(\mathbb{R}^4)$ is also sufficient to provide sufficient basis for the splitting problem of these commutator functions and their tensor products, for the proof compare Subsection 5.7, of [48].

The point is that the zero mass fields, such as the electromagnetic potential, treated with insufficient care, e.g. with incorrectly chosen test spaces, could not be used for the construction of the relevant domain on which they are indeed well defined operator valued distributions, contrary to the case of massive fields where the construction of the free field is fully correct, which partially explains the lack of success in construction of the domain working for the adiabatic limit. Epstein-Glaser method (including the adiabatic switching-on and -off of the interaction) works well for QED (and very likely for the other theories with non abelian gauge, at least the causal method may be applied with the adiabatically switched off interaction [10-12]) and in particular the Blanchard-Seneor proof with the correctly constructed free electromagnetic potential works with only minor corrections. It even becomes much simpler when the correct construction

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2 In the following text Subsections will always refer to the reference [48].

3 In case of SM with the Higgs field, with zero mass gauge fields before interaction in switched on, our correct construction of these fields clarifies and simplifies the situation, although the computations are still on the way.
of the electromagnetic potential is used, as for example the Malgrange split-
ing of a distribution becomes unique when using the correct test spaces \( S^0(\mathbb{R}^4) \) and \( \mathcal{S}^{00}(\mathbb{R}^4) \) in momentum space and over space-time respectively. Moreover the Pauli-Jordan zero mass function is not clearly defined as a continuous functional over the ordinary Schwartz test space \( \mathcal{S}(\mathbb{R}^4) \), but becomes a well defined continuous functional over \( \mathcal{S}^{00}(\mathbb{R}^4) \) whose Fourier transform is a continuous functional over \( \mathcal{S}^0(\mathbb{R}^4) \) with the support equal to the light cone in the momentum space, having all the properties discovered by Dirac and formulated formally by him in [5], Chap. XII. This would be impossible when using the ordinary Schwartz test spaces.

Thus the adiabatic limit exists (at least in QED) and the interacting field can be constructed order by order without encountering UV and IR divergences. There are however at least two important differences in comparison with the scalar massive case. First, when constructing the free zero mass gauge field \( A(x) \) and the free spinor field \( \psi(x) \) we are using the white noise calculus and Gelfand triple techniques in the construction of these fields and in particular the domain working for the adiabatic limit is the Hida test functional space \( \mathcal{E} \) of the total system of the free Dirac field and electromagnetic potential. In particular there exists a realization of the free potential field, which gives formally the same formula for the interacting field \( A_{\text{int}}(g,x) \) as that presented in [26], Ch. 4.9. In particular the first order correction to \( A_{\text{int}}(g,x) \) (before the adiabatic limit \( g \to 1 \) is performed) has the form

\[
- \frac{e}{4\pi} \int d^3x_1 \frac{g(x_0 - |x_1 - x|, x_1)}{|x_1 - x|} : \bar{\psi} \gamma^\mu \psi : (x_0 - |x_1 - x|, x_1).
\]

(1)

Its meaning cannot be clearly inferred when the free fields (especially the zero mass gauge fields) present in this expression are not carefully defined. When they are carefully defined within the white noise calculus, then each free field, e.g. \( A(x) \), at a specified space-time point \( x \) is an operator transforming continuously the nuclear Hida test space \( \mathcal{E} \) into its dual \( \mathcal{E}^* \) (which likewise can be treated as an operator valued distribution on the Hida test functional space \( \mathcal{E} \), but this is not very much helpful here). The crucial point is that we can apply the Fock expansion theorems of Hida-Obata-Saitô-Shimada, which allows to give the meaning to the integral [1] as a well defined operator \( A^{\mu(1)}(g, x) \) transforming continuously \( \mathcal{E} \) into \( \mathcal{E}^* \). Namely, the general theory of Fock expansions for integral kernel operators assures existence of such a continuous operator \( A^{\mu(1)}(g, x) : (\mathcal{E}) \to (\mathcal{E})^* \).
theory, e.g. [19], [30]. Existence of the limit $g \to 1$ of this generalized operator is equivalent to the existence in the strong sense, and defines a well defined generalized operator mapping continuously $(E)$ into $(E)^*$, and *a fortiori* a well defined distribution in the Fock space of free fields and the Hida test functional space $(E)$ as the dense domain, i.e.

$$A^{(1)}(g = 1, x) = \frac{e}{4\pi} \int d^3x \frac{1}{|x_1 - x|} : \bar{\psi} \gamma^\mu \psi : (x_0 - |x_1 - x|, x_1).$$

(2)

is the continuous operator $(E) \to (E)^*$, fulfilling the said condiutions with $g = 1$ set for the coupling function $g$. Similarly application of the Fock expansion theory of Hida-Obata-Saitô-Shimada gives the adiabatic limit order-by-order for all higher order terms in the causal perturbative expansion of the interacting field $A_{\mu}(g = 1, x)$.

Before going on we remark here that already at the level of extending the Noether theorem to the realm of free quantum fields we encounter expressions like (2). For example when considering a free field, Bogoliubov and Shirikov postulated the following: *We shall assume, as the basic postulate of quantization of wave fields, that the operators of the energy-momentum four vector $P^\mu$, the angular momentum tensor $M^{\mu\nu}$, the total charge $Q$, and so on, which are the generators of infinitesimal transformations of state vectors, can be expressed in terms of the quantum field operator functions of the fields by the same formal expressions as in classical field theory, with the operator fields, arranged in the Wick order. In particular if $T^{\mu\nu}$ is the energy-momentum tensor, then we should expect

$$\int : T^{0\mu} : (x_0, x) d^3x = P^\mu = d\Gamma(P^\mu),$$

(3)

where on the right hand side are the translation generators acting in the Fock space of the field. The crucial difficulty lies in the fact that on the left hand side we have operator-valued distributions (and not merely unbounded operators), and their integrals over the spatial coordinates, exactly as in the expression (2) or (1). Particularly hard difficulties arise in proving (3) for mass less gauge free fields (in fact the problem stayed open in this case, compare e.g. [25]). However the problem may be solved if the gauge fields are constructed as above with the help of white noise calculus. In this case the Fock space expansion theory of Hida-Obata-Saitô for integral kernel operators may be applied to give the result (3): the left hand side is well defined continous operator $(E) \to (E)^*$, which have an extension to a continous operator $(E) \to (E)$, thus to a densely defined operator on the Fock space; then the standard Riesz-Szökefalvy-Nagy criterion gives the essential self-adjointness of the operator on the nuclear space $(E)$ (alhtough the full proof of (3) is long and nontrivial, similarly assertion that (2) is well defined continous operator operator $(E) \to (E)^*$ requires a considerable amount of technicalities essentially the same as in the proof of Bogoliubov-Shirkov postulate (4), compare [48]).

Of special computational use are the *exponential vectors* contained in $(E)$, (in case of presence of the fermion fields these are the vectors residing in the
even part of the fermion Fock space). The exponential vectors are coherent-type states which do not contain states which are finite number particle states of the free Fock space. Such coherent-type states are inconvenient for the analysis of scattering processes (at least at the more naive order-by-order level where the free Fock finite number particle states serve as the asymptotic states). At the less naive level, at which one is confronted with the problem of convergence of the perturbative series, this difference is of less importance, because at this level the finite number particle states of the Fock space of free fields cannot serve as asymptotic states and the physical particle states are not eigenstates of the mass operator.

The second, and much more important difference, is the following, and this is the main theme of our comment. For fields which contain zero mass gauge fields (before the interaction is switched on) there is one nontrivial problem we are confronted with (one aspect of which shows up already at the free field level) not encountered when working with non gauge fields. In case of non gauge fields, when the representation $U$ acting in the single particle subspace $\mathcal{H}$, and thus its amplification $\Gamma(U)$ in $\Gamma(\mathcal{H})$ is unitary, the free field is essentially uniquely, i.e. up to unitary equivalence, determined by its general properties: i.e. by the transformation rule pertinent to the concrete representation $U$ which already includes the “generalized charges” pertinent to the field, for example the allowed spin of the single particle states, e.t.c.. We should expect of the correctly constructed gauge quantum free fields that they are likewise essentially uniquely determined by the corresponding “generalized charged” structure pertinent to the field. But in case of zero mass gauge fields, such as the electromagnetic potential field, the representation $U$ and its amplification $\Gamma(U)$ is unbounded and Krein-isometric. The natural equivalence for such representations is the existence of Krein isometric mapping transforming bi-uniquely and bi-continuously the nuclear space $E$, resp. $(E)$, into itself, and which intertwines the representations. Now this equivalence is weaker in comparison to the case of unitary equivalence of non-gauge fields where the continuous Hilbert space isometry defining the equivalence, and which is continuous on the respective nuclear space, can be extended to a bounded operator – in fact even to a unitary operator. This is the problem we are confronted with already at the free field level. One consequence of this weaker equivalence is the following. One can construct two equivalent local electromagnetic potential free fields based on the common nuclear spaces $E = S^0(\mathbb{R}^3; \mathbb{C}^4)$ and $(E)$ (regarded as functions on the orbit, i.e. on the positive energy cone without the apex, with the spatial components of the momentum as the natural coordinates on the cone without the apex) in the single particle spaces and in the Fock spaces respectively, which have different infrared content. Let us formulate this assertion more precisely. The different representatives of free fields of the same equivalence class are constructed by using different inner products and fundamental symmetry operators on $E$ continuous with respect to nuclear topology on $E$, which after completion with respect to the respective inner products give the respective single particle Hilbert spaces of the respective representatives of the field. In general different representatives of the same equivalence class of the free field may be constructed in this way.
The single particle representations \( U \) in case of the two representatives of the free electromagnetic potential field differ substantially; in the first case \( U \), when restricted to the \( SL(2, \mathbb{C}) \) subgroup, can be written as a direct integral (with respect to Hilbert space inner product of the single particle Hilbert space of the corresponding representative of the field) of representations (in general non unitary) acting naturally on the functions of the corresponding homogeneities on the cone, and in the second case of the restriction of \( U \) to \( SL(2, \mathbb{C}) \) corresponding to the other representative of the free field no such direct integral decomposition is possible. This possibility is no surprise as the (unbounded) equivalence operator of representations whose representors of Lorentz hyperbolic rotations are unbounded does not forces any bounded equivalence for the action of Lorentz representors. As already noted by Epstein and Glaser, the action of the Lorentz subgroup is of less importance in causal perturbative approach to QFT (in fact only translational covariance takes a material role in the perturbative series as well as the spectral behaviour of states of the relevant domains with respect to the joint spectrum of translation generators) and on the other hand the (weaker) equivalence for the translation generators which are by construction unitary and Krein-unitary reduces to ordinary unitary and Krein unitary equivalence for the action of the translation representors. Nonetheless sensitivity of the infrared asymptotic behaviour to the particular choice within one equivalence class of the free field cannot be simply ignored. This is because different asymptotic behaviour corresponding to different concrete realizations of the free field within the same equivalence class may survive when passing to interacting fields, and on the other hand the electromagnetic field has non-trivial infrared content corresponding to the Coulomb interaction, so that its asymptotic behaviour may (and in fact should) reflect important physical properties which cannot be ignored. Moreover it cannot \textit{a priori} be excluded (and even it should be expected) that this asymptotic behaviour is important in fixing the correct choice among different realizations of the free field within one and the same equivalence class. Therefore in the causal perturbative approach the condition of Lorentz covariance is not entirely optional when passing to the zero mass gauge field, such as the interacting electromagnetic potential quantum field, which shows up when we treat the field with more care. Moreover a need for the analysis of the (free and interacting) field at spatial infinity naturally arises.

3 Interacting field at spatial infinity

In order to solve this problem we recall that there exists a simple and elegant theory of the quantized homogeneous of degree \(-1\) part of the electromagnetic potential field \( A \), which resides at spatial infinity, i.e. at the three dimensional one-sheet hyperboloid, say the three dimensional de Sitter space-time, compare \cite{32} – \cite{42}. At the classical level extraction of the electromagnetic field which resides at spatial infinity is in principle unique and well defined, and it is the homogeneous of degree \(-1\)“part” of the field \( A \) which is free, \cite{16}, determined
by a scalar \( S(x) \) (of “electric type”) and a scalar \( M(x) \) (of “magnetic type”) on de Sitter 3-hyperboloid fulfilling the homogeneous wave equation on de Sitter 3-hyperboloid. As shown in [32] or [33] its quantization can be performed within a natural way with the commutation relations based essentially on the two principles: the gauge invariance and the canonical commutation relations for the conjugated generalized coordinates, [32] or [33]. As shown in [33] the phase of the wave function (of the charge carrying particle, before the second quantization is performed) is the generalized coordinate conjugated to the total charge, and at the classical level the phase has been determined in [33] as equal to the electric part \( S(x) = -e x^\mu A_\mu(x) \) of the field at infinity, with \( A_\mu \) homogeneous of degree \(-1\) (in general distributional solution of d’Alembert equation). The crucial point is that in computing the total charge we do not need the global solution of the Maxwell equations but need only to know the solution outside the light cone e.g. knowing the Dirac homogeneous solution of d’Alembert equation (distributional). [5] pp. 303-304, which coincides with the ordinary Coulomb potential field outside the light cone is pretty sufficient.

In particular the corresponding field induced on de Sitter 3-hyperboloid by the Dirac homogeneous solution corresponds to the classical Coulomb field and is determined by the homogeneous of degree \(-1\) Coulomb field solution of Maxwell equations at spatial infinity. Therefore the standard commutation rules between the phase and the total charge (so identified at spatial infinity with the respective constants in the general scalar solution of the wave equation on de Sitter 3-hyperboloid) determine uniquely the commutation rules for the scalar field on de Sitter 3-hyperboloid and include the Coulomb field. [33]. In particular it contains the total electric charge as an operator acting in the Hilbert space of the quantum phase field \( S(x) \), as a scalar field on de Sitter 3-hyperboloid, and explains discrete character of the charge. This theory is remarkable for several reasons. First it is very simple and mathematically transparent. The paper [33] does not enter mathematical analysis of the theory, but the theory of continuous functionals on \( S^0(\mathbb{R}^4) \) and \( S^0(\mathbb{R}^4) \), respectively over space-time and in momentum space, provide the distributional background for [33], compare Section 6.

In particular the homogeneous Dirac’s solution of d’Alembert equation is a well defined distribution over the test space \( S^0(\mathbb{R}^4) \) whose Fourier transform is a continuous functional over \( S^0(\mathbb{R}^4) \) with the light cone in the momentum space as the support, for the proof compare Subsection 6.1. We show in particular that the support of the Dirac solution as a distribution on \( S^0(\mathbb{R}^4) \) is equal to that part of space-time which lies outside the light cone. Similar property we have for the transversal homogeneous of degree \(-1\) electric type solutions of d’Alembert equation generated by the Lorentz transforms of the Dirac solution. This solutions extend over to the (here not the correct) test function space \( S(\mathbb{R}^4) \), but in highly non unique fashion. In general such extensions destroy their space-time support which in general cease to be confined to the outside part of the light cone. When treated as distributions on the correct test space \( S^0(\mathbb{R}^4) \) they become uniquely determined with their spacetime supports necessary lying outside the light cone, which has very important physical consequences, compare
We also show (a detailed proof can be found in Subsection 6.5) that the standard representation of the commutation relations of Staruszkiewicz theory, proposed in [33], can be characterized (among the infinite family of other possible representations) by the condition that in each reference frame the gauge group $U(1)$ can be reconstructed spectrally in the sense of spectral geometry of Connes, by the phase and the charge operators $V = e^{iS(u)}, D = (1/e)Q$ of his theory, compare Subsection 6.5. For other possible non-standard representations of the commutation relations of Staruszkiewicz this would be impossible with $V = e^{iS(u)}, D = (1/e)Q$. The standard representation of [33] is in fact the one which is actually used in the subsequent papers [32]–[42]. Second, it involves the fine structure constant and relates it nontrivially to the theory of irreducible unitary representations of $SL(2, \mathbb{C})$, mainly through the unitary representation of $SL(2, \mathbb{C})$ acting in the Hilbert space of the quantum phase field $S(x)$, a mathematical theory which have attained a mature form full of computational devices thanks mainly to Gelfand and his school, Neumark, Harish-Chandra and others. In particular (as shown in [35]) the representation acting in the eigenspace of the total charge operator corresponding to the lowest (regarding the absolute value) non-zero charge contains the supplementary series component (and if any it must enter discretely) only if the fine structure is sufficiently small. Third, this theory contains the quantized Coulomb field (at least as it concerns the asymptotic part outside the light cone). This is perhaps the most remarkable feature of the theory of Staruszkiewicz [33], at least for the perturbative causal approach to QED. Indeed so far as the gauge electromagnetic field was treated with insufficient care the existence of the adiabatic limit was unclear in QED and in particular the status of the Coulomb field so that the identification of the quantum (interacting) field $A_{\text{int}}(x)$ at spatial infinity was impossible within the causal perturbative approach due essentially to the troubles with the adiabatic limit. But with the electromagnetic potential field treated more carefully we restore the adiabatic limit and at least in principle we can compute $A_{\text{int}}(x)$ as a formal power series in which the switching off coupling $g(x)$ is moved to infinity, so that the interacting field is now a formal power series in the ordinary fine structure constant and not the function $g(x)$, with each order term equal to an operator-valued distribution acting in the Fock space of free fields. This is of capital importance because now we can compare the homogeneous of degree zero part of the field $x_{\mu}A_{\mu\text{int}}(x)$ with the quantum phase field $S(x)$ of Staruszkiewicz theory. For this plan to be realizable we have to learn how to extract a homogeneous part of a fixed homogeneity $\chi$, fulfilling d’Alembert equation, of a quantum (interacting) field. Although this task is still non trivial there are several circumstances which both allow the computation to be effective and connect this computation to important physical phenomena. Let us explain this in more details now. Concerning the extraction of the homogeneous part, fulfilling d’Alembert equation, of a given interacting field, say $x_{\mu}A_{\mu\text{int}}(x)$, we do it gradually.

First we observe that a free zero mass field, say a quantum scalar field fulfilling d’Alembert equation (or even not necessary fulfilling d’Alembert equation,
as is the case for \( x_\mu A^\mu(x) \), even when \( A^\mu(x) \) is free), when constructed with the correct test function spaces \( S^0(R^4) \) and \( S^0(R^4) \) (over space-time and in the momentum picture respectively), allows a natural construction of a homogeneous part, which is effectively a field on de Sitter 3-hyperboloid, fulfilling d’Alembert equation (or wave equation on de Sitter 3-hyperboloid which is inhomogeneous in general if the homogeneity degree \( \chi \neq 0 \)). Now when looking at the single particle state space we should construct a Hilbert space of homogeneous (of a fixed degree \( \chi \)) solutions of d’Alembert equation. In general such solutions have distributional sense and are continuous functionals on the test space \( S^0(R^4) \) (with topology inherited from the Schwartz topology on \( S(R^4) \)) and whose Fourier transforms are continuous functionals on the test space \( S^0(R^4) \) (again with the topology inherited from \( S(R^4) \)) and have the support concentrated on the (positive sheet) of the cone in the momentum space. Now the restriction to the light cone of the Fourier transforms of these functionals are continuous functionals on the nuclear test space \( E = S^0(R^3) \) of restrictions of the elements of \( S^0(R^4) \) to the light cone without the apex with the spatial momentum coordinates as the natural coordinates on the cone. This gives us a general obstruction on the homogeneous generalized single particle states of the homogeneous part of the field we are interested in: they should be the continuous functionals on the nuclear space \( E \subset H \subset E^* \) is the Gelfand triple in the single particle Hilbert space \( H \) of the initial field in question. Now let us fix a closed subspace \( E^*_\chi \) of \( E^* \) of functions (functionals) on the cone of fixed homogeneity \( \chi \). The representation \( U \) of the restriction of the double covering of the Poincaré group to the subgroup \( SL(2, \mathbb{C}) \) acting in \( H \) by the very construction of the field has the property that each representor maps continuously \( E = S^0(R^3) \) onto \( E \) with respect to the nuclear topology of \( E \) and each representor of its linear dual or transpose (let us denote it by the same sign \( U \)) transforms \( E^* \) continuously onto \( E^* \) (with its natural strong topology). In particular all elements of \( E^* \) of fixed homogeneity \( \chi \) have a fixed transformation law, let us denote them by \( E^*_\chi \). The representation \( U \) acting on \( E^*_\chi \) is uniquely determined by the action on the homogeneous regular elements of \( E^*_\chi \) i.e functions on the 2-sphere \( S^2 \) of unit rays on the cone in momentum space which are smooth on \( S^2 \). Note that \( E = S^0(R^3) \) has the structure of tensor product (for a proof compare Subsect. 5.6) and as a nuclear space is isomorphic to \( S^0(R) \otimes C^\infty(S^2) \) and similarly for its dual \( E^* = S^0(R)^* \otimes C^\infty(S^2)^* \) by the kernel theorem. Now using the results of [15] one can classify all possible Hilbert space inner products on \( E^*_\chi \) invariant under the representation \( U \) of \( SL(2, \mathbb{C}) \). If such an invariant inner product exists for a fixed homogeneity \( \chi \) (in general does not exist and if any it is essentially unique) we have to meet our obstruction mentioned to above before we use it as a single particle inner product of the homogeneous part of the field of homogeneity \( \chi \). Namely it may happen that the closure of \( E^*_\chi \) with respect to this invariant inner product leads us out of the space \( E^* \) which is impossible for a field homogeneous of a fixed degree, fulfilling d’Alembert equation, and thus inducing a field on the de Sitter 3-hyperboloid fulfilling the wave equation.

\[ \text{In general } \chi \text{ may assume complex values, although far not all of them are admitted.} \]
on the 3-hyperboloid. If the closure of $E^*_\chi$ with respect to the invariant inner product lies within the dual space $E^*$, then we obtain a well defined field when using the closure of $E^*_\chi$ with respect to the invariant inner product as the single particle Hilbert space by the application of the functor $\Gamma$. The ordinary inverse Fourier transform of the elements of a complete system in this single particle Hilbert space, which are homogeneous (distributional) solutions of d’Alembert equation, determine by restrictions to de Sitter 3-hyperboloid the fundamental modes (waves) fulfilling the wave equation on de Sitter 3-hyperboloid. By the kernel theorem for nuclear spaces $E^*=\mathcal{S}^0(\mathbb{R})^*\otimes C^\infty(\mathbb{S}^2)^*$, $(E\otimes E)^*=E^*\otimes E^*$ and the Fock structure of the Hilbert space of the homogeneous part of the field (when it exists at all) is essentially inherited form the Fock structure of the Hilbert space of the initial field itself, and in particular the creation and annihilation operators $a(\vec{\varphi})^+,a(\vec{\varphi})$ of the homogeneous part of the field are well defined, with $\vec{\varphi}$ belonging to the closure of $E^*_\chi$ with respect to the invariant Hilbert space inner product, by assumption contained in $E^*$. It frequently happens that the spherical harmonics (scalar, spinor, e.t.c, depending on the field) on the unit 2-sphere of rays on the cone in momentum space, extended by homogeneity, and regarded as elements of $E^*_\chi$ are sufficient to provide a complete system in the single particle subspace of the homogeneous part of the field.

Note in particular that the homogeneous part (fulfilling d’Alembert equation) of a free field of degree $\chi$ makes sense only for some particular values of $\chi$, which of course was to be expected.

In the next step we observe that the extraction of a homogeneous part of fixed homogeneity $\chi$, fulfilling d’Alembert equation, of a zero mass field presented above, works also for local free massive fields without any essential changes. Namely we can extract in a natural way a homogeneous part (of fixed homogeneity $\chi$) fulfilling d’Alembert equation, of a massive local free field. This is possible because the nuclear spaces $\mathcal{S}^{00}(\mathbb{R}^4)$ and $\mathcal{S}^{0}(\mathbb{R}^4)$ are closed subspaces of the Schwartz space $\mathcal{S}(\mathbb{R}^4)$ with their topologies inherited from $\mathcal{S}(\mathbb{R}^4)$ (and this holds in any dimension $n$, i.e. for $\mathcal{S}^{00}(\mathbb{R}^n)$, $\mathcal{S}^{0}(\mathbb{R}^n)$, $\mathcal{S}(\mathbb{R}^n)$). In case of massive fields the role of the space-time test space is played by functions (in general scalar valued, vector valued, spinor valued, depending on the field in question) of $\mathcal{S}(\mathbb{R}^4)$, and the role of the nuclear space $E$ is played by the Fourier transforms of (scalar valued, vector valued, e.t.c. depending on the field) functions of $\mathcal{S}(\mathbb{R}^4)$, composing likewise the space $\mathcal{S}(\mathbb{R}^4)$, restricted to the positive energy sheet $\mathcal{O}_{m,0,0,0}$ of the two-sheeted mass $m$-hyperboloid, i.e. $E=\mathcal{S}(\mathbb{R}^3)$ (with the spatial components of the momentum as the natural coordinates on $\mathcal{O}_{m,0,0,0}$—the corresponding orbit of the representation pertinent to the field in question, i.e. just the Lobachevsky space). The point is that the representation $U$ in the single particle space of the field in question, uniquely determinates the represen-

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5We are using essentially two types of linear topological spaces: the nuclear spaces and the Hilbert spaces. When writing $E\otimes E$ with nuclear spaces $E$, we mean the projective (coinciding in this case with the equicontinuous) tensor product, which is thus essentially unique, and when writing $H\otimes H$ for Hilbert spaces $H$ we mean the Hilbert space tensor product. Note however that the Hilbert space tensor product, projective tensor product and equicontinuous tensor product are all different for Hilbert spaces of infinite dimension.
representation acting on the test space \( S(\mathbb{R}^4) \), and thus on \( S^{00}(\mathbb{R}^4) \) – its closed subspace, and \textit{a fortiori} a representation acting on the Fourier transform image and its closed subspace \( S^0(\mathbb{R}^4) \), as well as on the restrictions to the cone of the elements of \( S^0(\mathbb{R}^4) \) composing \( S^0(\mathbb{R}^3) \), regarded as functions on the cone. In particular we have uniquely determined the action of the \( SL(2, \mathbb{C}) \) subgroup on the elements of \( S^0(\mathbb{R}^3)^* \), in particular homogeneous distributions in \( S^0(\mathbb{R}^3)^* \), whose ordinary inverse Fourier transforms are homogeneous solutions of d’Alembert equation. Each distribution \( S \in S^0(\mathbb{R}^3)^* \) defines a unique distribution \( F \) over \( S^0(\mathbb{R}^4) \) concentrated on the (positive sheet \( \mathcal{O} \) of the) cone, determined by the condition that \( F(\tilde{\varphi}) = S(\tilde{\varphi}_{|_{\mathcal{O}}}) \), well defined because the restriction to \( \mathcal{O} \) maps continuously \( S^0(\mathbb{R}^4) \) onto \( S^0(\mathbb{R}^3) \). The ordinary Fourier transforms of such \( F \)-s, regarded as functionals in \( S^0(\mathbb{R}^4)^* \), are homogeneous solutions of d’Alembert equation, in general distributional, i.e. belonging to \( S^{00}(\mathbb{R}^4)^* \). Thus the representation \( U \) induces a unique representation on \( S^0(\mathbb{R}^3)^* \) and \( S^{00}(\mathbb{R}^4)^* \). In particular choosing a subspace of \( S^0(\mathbb{R}^3)^* \) of fixed homogeneity \( \chi \) we can, in case of the scalar field, use the classification of invariant inner products of [15] on homogeneous functions on the cone, and construct the homogeneous part of the field of fixed homogeneity \( \chi \) as shown above.

The crucial point is that the representation \( U \) acting in the single particle subspace of the local massive quantum free field in question, determines a unique representation acting in \( S^0(\mathbb{R}^3)^* \) (regarded as a space of functions on the cone), or resp. in \( S^{00}(\mathbb{R}^4)^* \). Moreover the elements of \( S^0(\mathbb{R}^3)^* \), or \( S^{00}(\mathbb{R}^4)^* \) may in a natural manner be regarded as generalized states of the single particle subspace of the massive field in question, i.e. as elements of \( E^* = S(\mathbb{R}^3)^* \). It is instructive to look at this circumstance from the point of view of harmonic analysis on the Lobachevsky space – the orbit \( \mathcal{O}_{m,0,0,0} \) pertinent to the representation \( U \) defining the field. Namely we can decompose the restriction of \( U \) acting in the single particle space to the subgroup \( SL(2, \mathbb{C}) \). We obtain a direct integral decomposition into irreducible (this time \( U \) is unitary) sub-representations. Each of the irreducible sub-representations is canonically a representation acting on functions of fixed homogeneity \( \chi \) on the cone. In fact each of the irreducible sub-representations act on Hilbert spaces which up to a measure zero set may be regarded as ordinary functions on the unit sphere \( S^2 \) of rays on the cone, except for the supplementary series representations (if it enters the decomposition at all, which is rather exceptional), whose representation space as a complete Hilbert space contains elements which cannot be identified with ordinary functions on the cone. But in each case the elements of the irreducible representation are homogeneous distributions over \( S^0(\mathbb{R}^3) \) regarded as the space of restrictions of the elements \( S^0(\mathbb{R}^4) \) to the positive sheet of the cone. In particular for the massive scalar field we obtain this assertion without much ado using the decomposition of the representation acting on the scalar functions on the Lobachevsky space, acting in the ordinary Hilbert space of square integrable functions with respect to the invariant measure given in [15], Ch VI.4. In order to obtain this theorem in full generality we have to prove that all unitary irreducible representations of \( SL(2, \mathbb{C}) \), can be realized on (the closure with respect to an invariant inner product of) homogeneous functions on the cone. For the
spherical-type representations this is already known to be true (the case of the supplementary series representations and the spherical-type representations of the principal series has been presented in this manner in [35]-[42] and in [15]). In Subsection 6.2 we give a proof that the closure of the space of homogeneous functions of the supplementary series representation under the invariant inner product is contained within the space $S^0(\mathbb{R}^3)^*$. It can be however proven for all irreducible unitary representations (and the proof for the remaining irreducible representations easily follows from the results of Subsect. 6.1), or even for all completely irreducible, and not necessary unitary, representations of $SL(2, \mathbb{C})$.

In particular any unitary representation $(l_0, l_1 = i\nu)$, $m \in \mathbb{Z}$, $\nu \in \mathbb{R}$, of Gefand-Minlos-Shapiro [12] (not necessary spherical-type, i.e. with $l_0$ not necessary equal to zero), can be realized on the space of scalar functions on the cone, homogeneous of degree $-1 - i\nu$, on using in addition in the transformation formula a homogeneous of degree zero phase factor $e^{i\Theta}$ in the transformation formula, raised to the integer or half-integer power $\pm l_0$ depending on the representation $(l_0, l_1 = i\nu)$ we want to achieve, where the phase $e^{i\Theta}$ is the one found in [45]. The inner product is given by the ordinary $L^2(S^2)$-norm defined for the restrictions of the homogeneous functions to the unit sphere $S^2$. The decomposition of $U$, restricted to $SL(2, \mathbb{C})$, can be think of as an application of the general Gelfand-Neumark Fourier transform corresponding to the decomposition of $U$. Because the $SL(2, \mathbb{C})$ group is nonabelian, then the Fourier transform of a function (even scalar valued) on the Lobachevsky space is no longer scalar valued, but the values of the transform are homogeneous distributions over $S^0(\mathbb{R}^3)$ regarded as a space of functions on the cone. Each such distribution $S$ defines a unique distribution $F$ over $S^0(\mathbb{R}^4)$ concentrated on the (positive sheet $\mathcal{O}$ of the) cone, by the condition that $F(\tilde{\varphi}) = S(\tilde{\varphi}|_{\mathcal{O}})$, well defined because the restriction to $\mathcal{O}$ maps continuously $S^0(\mathbb{R}^4)$ onto $S^0(\mathbb{R}^3)$. The ordinary Fourier transforms of such $F$-s, regarded as functionals in $S^0(\mathbb{R}^4)^*$, are homogeneous solutions of d’Alembert equation, in general distributional, i.e. belonging to $S^{00}(\mathbb{R}^4)^*$. Now because $E = S(\mathbb{R}^3)$, the single particle Hilbert space $\mathcal{H}$ and the space of distributions $E^*$ compose a Gelfand triple $E \subset \mathcal{H} \subset E^*$ (or a rigged Hilbert space), then by [14], the elements of the Hilbert space $\mathcal{H}_\chi$ in the decomposition

$$\mathcal{H} = \oplus \int \mathcal{H}_\chi d\sigma(\chi)$$

corresponding to the decomposition of the representation $U$, restricted to $SL(2, \mathbb{C})$, belong to $E^*$ because the Casimir operators transform $E$ continuously onto itself. Moreover by the analytic continuation of a distribution, [13], also the other distributions homogeneous of degree $\chi$ over $S^0(\mathbb{R}^3)$, with $\chi$ not entering the decomposition belong to $E^* = S(\mathbb{R}^3)^*$, i.e. to the space of generalized states of the single particle space of the massive field. Although the application of the general Gelfand-Neumark Fourier transform gives a general framework working for all fields, it is not in general computationally useful, because we lose any immediate relation of the transformation formula of the field, to the respective irreducible components $(l_0, l_1)$ entering the decomposition of $U$ (of course restricted to $SL(2, \mathbb{C})$). For example the explicit realization of the ir-
reducible representation \((l_0, l_1)\) through its action on the scalar homogeneous functions on the cone (with the additional phase factor multiplier \(e^{i\Theta}\) raised to the appropriate power) is not much helpful because the phase factor \(e^{i\Theta}\) in the transformation formula depends on the momentum, which means that its inverse-Fourier-transformed image has non-local transformation law and an additional work is needed in recovering the local transformation law of the field in question. In particular taking a direct sum of such irreducible representations with the additional multiplier respectively equal \(e^{i\Theta}\) and \(e^{-i\Theta}\), acting on functions homogeneous of degree \(-1\), we obtain the representation acting in the single particle space of a homogeneous of degree \(-2\) part of the local Riemann-Silberstein quantum vector field, but this is far not obvious, compare \([45]\), \([1]\), \([46]\).

Therefore in practical computations it is much better to choose another way when computing a homogeneous part of a given local massive quantum free (scalar, spinor, e.t.c.) field. Namely we construct first the free zero mass counterpart of the (scalar, spinor, e.t.c.) field. There exists a general construction of such local zero mass fields, compare \([45]\) or the introductory part of Section 2 (and there is quite a long tradition in constructing such fields, compare e.g. \([4]\) for the zero mass Dirac field). The homogeneous of degree \(-2\) commutator functions of such fields are just the quasi asymptotic distributions of the corresponding commutator functions of the massive fields, which we encounter in computing the singularity degree when splitting the massive commutator functions due to Epstein-Glaser. It is not obvious if such counterpart zero mass fields exist, but this is indeed the case at least for fields we are interested in. Then to the single particle representation of the zero mass (scalar, spinor, e.t.c.) field, acting on the (scalar, spinor, e.t.c., respectively) functions on the cone (in the momentum picture) we apply the Gelfand-Graev-Vilenkin Fourier transform (or its immediate generalization on spinor, etc., valued, functions on the cone) in order to recover the representations acting on (spinor, tensor, e.t.c.) homogeneous functions, entering the decomposition of this representation. The point is that it is much easier to extend the Gelfand-Graev-Vilenkin Fourier theory on spinor, tensor, etc. valued functions, then to search at random among the direct sums in the general decomposition of the representation \(U\) (restricted to \(SL(2, \mathbb{C})\)) acting in the single particle subspace of the massive field, those which recover the correct local transformation formula of the homogeneous part of the field. This task however can be reduced to the results obtained by Gelfand and Neumark on the classification of unitary representations of \(SL(2, \mathbb{C})\). The case of the scalar field we have already at hand without any additional computations.

Summing up we have constructed homogeneous of degree \(\chi \in \mathbb{C}\) part of a local free field (working for massive as well as for mass less fields, for non gauge fields and for gauge fields) which is well defined only for particular values of \(\chi\). Before we extend this extraction on still more general local fields we should stop for a moment at the level of free fields. First note that the introduction of the new class of test spaces \(S^0(\mathbb{R}^n)\) and \(S^{00}(\mathbb{R}^n)\), essential for the construction of mass less fields is likewise essential as the distributional basis for \([33]\), compare Section 6. Second note that the same test spaces are essential in extraction
of homogeneous parts of the free fields. A more rigourous definition and construction of a homogeneous part of a free field the reader will find in Subsection 6.3. And finally let us go back to the comparison \( x_\mu A^\mu(x) = S(x) \), with the homogeneous of degree zero part of the scalar field \( x_\mu A^\mu(x) \) at the free field level. It turns out that at the free field level, by extracting of the homogeneous part of degree zero, fulfilling d’Alembert equation, of the field \( x_\mu A^\mu(x) \), with \( A^\mu \) the free potential, we indeed recover the degenerate case of the theory of Staruszkiewicz, with the fine structure constant put equal to zero, and with the Hilbert space which degenerates to the eigenspace of the total charge operator corresponding to the eigenvalue zero, compare Subsection 6.4 where we provide a detailed construction. Moreover this result holds true for any representative of the free electromagnetic potential. Of course this is far not obvious if this results hold true in the full interacting theory and if it is sensitive to the choice of the representative of the free potential as the building block of the causal perturbative series.

As the next step we construct the homogeneous part, in general not fulfilling d’Alembert equation, of a local field equal to a Wick polynomial of free fields. Let the homogeneity of the part to be extracted be \( \chi \). In fact we can confine attention to Wick monomials. In particular in order to extract the homogeneous part of the field \( : \psi(x)^{n_1} A(x)^{n_2} : \) we sum up

\[
\sum_{n_1 \chi_1 + n_2 \chi_2 = \chi} : \psi^{\chi_1}(x)^{n_1} A^{n_2}(x) : 
\]

over all fields

\[
: \psi^{\chi_1}(x)^{n_1} A^{n_2}(x) :
\]

where \( \psi^{\chi_1}(x) \) is the homogeneous part of degree \( \chi_1 \), fulfilling d’Alembert equation, of the Dirac field \( \psi(x) \) and similarly \( A^{\chi_2}(x) \) is the homogeneous of degree \( \chi_2 \) part of the field \( A(x) \), fulfilling d’Alembert equation;

or we put zero for this sum in case when

no \( \chi_i \) exist such that \( n_1 \chi_1 + n_2 \chi_2 = \chi \).

As the final step we would like to extract a homogeneous part of an interacting field, especially \(-x_\mu A^\mu(x)\). On the other hand the interacting field itself is beyond our reach, because, so far we have not yet given any precise meaning to the limit of the perturbative series, leaving aside such questions as its convergence. In order to pass over this difficulty we go back to the causal perturbative series for the interacting field \( A_{\text{int}}(x) \) after the adiabatic switching on the interaction at infinity is performed. Then into each order term of the causal perturbative series we “insert”, in place of each free field operator, its homogeneous part with the respective paring functions replaced by the homogeneous of degree \(-2\) zero mass counterparts. Here “insertion” means that each integration \( d^4x_i \) is replaced with integration over de Sitter hyperboloid and with the homogeneous integrand treated as operator distribution on de Sitter hyperboloid. Then we confine attention to each order term separately. Next we
extract the homogeneous part of the chronological product of Wick polynomials of free fields, similarly as for the Wick product of free fields, just summing over all summands with factors whose homogeneousities sum up to $\chi$.

For example in order to compute the first order correction to the homogeneous part of the interacting field $A_\mu^\text{int}(x)$, in case when the representative of the free potential field is used which leads to the formulas for the interacting fields which are given in [26], Ch. 4.9, we need to compute the homogeneous of degree $-1$ part of the generalized operator [2]. According to our prescription we “insert” the homogeneous of degree $-3$ part of the field $\bar{\psi}\gamma^\mu\psi : (x)$ into the formula [2], where the “insertion” means that the integral in [2] of the homogeneous integrand is replaced by the integral over the intersection of the spacelike plane $x_0 = \text{const}$ with de Sitter 3-hyperboloid and the integrand is now regarded as the field on de Sitter 3-hyperboloid, which it naturally induces as a homogeneous field in Minkowski space-time. It is important to note that quantum fields on de Sitter 3-hyperboloid space-time may be integrated over Cauchy surfaces and this integration produces well defined (densely defined) operators in their Hilbert spaces. We give a proof of it using white noise calculus in Sections 6.4 and 6.3. But the same proof can be performed by using the unitary representation of $SL(2, \mathbb{C})$ acting in the Hilbert space of the homogeneous field. This fact seems to be rather known for those who have worked with fields on de Sitter space-time [41] or on the static Einstein Universe space-time [28], [29] which are similar in this respect.

Why is this comparison

$$\left(x_\mu A_\mu^\text{int}(x)\right)_{\chi=0} = S(x), \quad (4)$$

where $S(x)$ is the quantum phase field of Staruszkiewicz theory, so interesting? First of all in extracting the homogeneous of degree $-1$ part of the interacting field $A_\mu^\text{int}(x)$ only the first and zero order contributions are non zero:

$$\left(A_\mu^\text{int}(x)\right)_{\chi=-1} = \left(A_\mu^\text{free}(x)\right)_{\chi=-1} + \left(A_\mu^{(1)}(g = 1, x)\right)_{\chi=-1},$$

where $\left(A_\mu^{(1)}(g = 1, x)\right)_{\chi=-1}$ is the homogeneous of degree $-1$ part of the generalized operator [2] defined as above. This is of capital importance. The mechanism which cuts out the higher order terms is in principle very simple: the allowed homogeneousities $\chi$ for the massive fields coupled to $A_\mu$ are restricted to relatively small set. In particular the allowed homogeneousities $\chi$ for the scalar massive field are equal: $-1 < \chi < 0$ or $\chi = -1 + i\nu$, $\nu \in \mathbb{R}$, for the proof compare Subsection 6.2, and Remark 4 of Subsection 6.2. Similar situation we have for other massive fields, e.g. for the Dirac field. On the other hand positive homogeneousities for the homogeneous parts of the free field $A_\mu$ are not allowed. In fact we have not finished yet the full classification of allowed homogeneousities in this case (in Subsection 6.3 we have reduced the classification to application of the Gelfand-Graev-Vilenkin method for classification of invariant bilinear forms on a nuclear space, and we present some partial results in Subsection 6.3). But the assumption that positive homogeneousities are impossible is physically reasonable.
On the other hand each factor coming from the retarded (resp. advanced) parts of the commutator functions contributes additional homogeneity $-2$. Because the number of these factors grows together with the order, there remains no room for keeping homogeneity $-1$ of each higher order contribution.

This in fact is what one should expect, by comparison with the scattering at the classical level in the infrared regime: the scattered charges produce infrared electromagnetic field but the infrared electromagnetic field does not scatter charges.

Moreover, $(A^\mu(1)(g = 1, x))_{\chi=-1}$ and $(A^\mu_{\text{free}}(x))_{\chi=-1}$ by construction commute. This again is of capital importance and makes (4) still more plausible with $x_\mu(A^\mu_{\text{free}}(x))_{\chi=-1}$ corresponding to

$$\sum_{l=1}^{\infty} \sum_{m=-l}^{m=l} \{c_{lm}f_{lm}^{(+)}(\psi, \theta, \phi) + \text{h.c.}\}$$

(5)

and $x_\mu(A^\mu(1)(g = 1, x))_{\chi=-1}$ corresponding to $-eQ\text{th}\psi$ in the expansion of the quantum phase operator

$$S(\psi, \theta, \phi) = S_0 - eQ\text{th}\psi + \sum_{l=1}^{\infty} \sum_{m=-l}^{m=l} \{c_{lm}f_{lm}^{(+)}(\psi, \theta, \phi) + \text{h.c.}\}$$

of the Staruszkiewicz theory (we are using the notation of [33]). The operator $S_0$ in $S(x)$ is that part which cannot be reproduced by $x_\mu(A^\mu_{\text{int}}(x))_{\chi=-1}$, which again could have been foreseen by comparison with the classical theory of infrared fields.

Now the computation of the homogeneous of degree $-3$ part of the free current field (in case of the Dirac field coupled to the potential the free current is equal : $\bar{\psi}\gamma^\mu\psi : (x)$) is not entirely trivial, even in the simpler scalar QED, because there are in general continuum-many possible homogeneity degrees $\chi$ to play with. Let us explain this in the simpler case of the scalar QED, where the spinor field $\psi(x)$ is replaced with a scalar (boson) massive complex field, let us denote it likewise by $\psi(x)$. In the scalar QED the field : $\bar{\psi}\gamma^\mu\psi : (x)$ is replaced with : $\bar{\psi}\partial^\mu\psi : (x)$. According to our definition each part $\psi_\chi(x)$ of homogeneity $\chi$ of the scalar field $\psi(x)$ contributes to the homogeneous of degree $-3$ part of the field : $\bar{\psi}\partial^\mu\psi : (x)$ if $\chi + \chi = -2$. In particular each homogeneous of degree $\chi = -2$. That only first order contribution to the interacting field at spatial infinity should survive also at the quantum field theory level has been foreseen by Schwinger, as prof. Staruszkiewicz has kindly informed me. Schwinger observes that the only charge carrier fields are massive. The infrared photons carry too small an energy to produce pairs sufficient to create massive charge carrying particle. On the other hand we should expect the first order contribution to be nonzero. That there persist a kind of “back-reaction” we should expect by comparison with the ordinary nonrelativistic charged quantum particle in the infrared Bremsstrahlung-type infrared field: a nonzero contribution to the phase shift will arise for each plane wave of the particle which produces nontrivial change of the packet-type wave function of the particle, compare e.g. [31], [18]. This is reflected by the nonzero first order contribution.

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$-1 - i \nu$, $\nu \in \mathbb{R}$, part $\psi_{x=-1-i\nu}(x)$ of the scalar field $\psi(x)$, has a contribution. Each such homogeneous of degree $\chi = -1 - i \nu$ part $\psi_{x=-1-i\nu}(x)$ of the scalar field $\psi(x)$ is nontrivial, and is constructed on the unitary irreducible representation $U^{\chi=-1+i\nu} = (l_0, l_1 = i \nu)$ of $SL(2, \mathbb{C})$ acting on the homogeneous of degree $\chi = -1 + i \nu$ scalar functions on the cone (in the momentum space) as the single particle subspace $\mathcal{H}_{\chi}$ of the field $\psi_{x=-1-i\nu}(x)$, and is a spherical-type representation of $SL(2, \mathbb{C})$ of the principal series.

Now when working with a finite set of possible homogeneities, say $\chi_1 = -1 - i \nu_1, \ldots, \chi_n = -1 - i \nu_n$, we need only to consider the finite sum $\Gamma(\mathcal{H}_{\chi_1}) \otimes \ldots \otimes \Gamma(\mathcal{H}_{\chi_n})$ of their Fock spaces, by the known property of the functor $\Gamma$:

$$\Gamma(\mathcal{H}_{\chi_1} \oplus \ldots \oplus \mathcal{H}_{\chi_n}) = \Gamma(\mathcal{H}_{\chi_1}) \otimes \ldots \otimes \Gamma(\mathcal{H}_{\chi_n}),$$

as the homogeneous of degree $-3$ part of the field $\bar{\psi} \partial^\mu \psi : (x)$.

But already passing from finite set of possible homogeneities to a denumerable set $\chi_1, \chi_2, \ldots$, there arises a subtle point of generalizing the last theorem to the following

$$\Gamma\left( \bigoplus_{n=1}^{\infty} \mathcal{H}_{\chi_n} \right) = \prod_{n \in \mathbb{N}} \otimes \Gamma(\mathcal{H}_{\chi_n}),$$

where it seems that the $\mathcal{C}$-adic infinite direct tensor product, $\prod_{n \in \mathbb{N}} \otimes$, of von Neumann [20] should work here (although, so far as the author is aware, no proof has until now been performed). And when considering the decomposition

$$U = \int_{\nu > 0} \mathcal{U}^{\chi=-1+i\nu} d\nu$$

of the restriction $U$ of the double covering of the Poincaré group to the $SL(2, \mathbb{C})$ acting in the single particle subspace $\mathcal{H}$ of the scalar field $\psi(x)$ into irreducible components $\mathcal{U}^{\chi=-1+i\nu}$ acting in $\mathcal{H}_{\chi=-1+i\nu}$, we encounter the following formula

$$\Gamma\left( \int_{\nu > 0} \mathcal{H}_{\chi=-1+i\nu} d\nu \right) = \prod_{\nu \in \mathbb{R}} \otimes \Gamma(\mathcal{H}_{\chi=-1+i\nu}),$$

but this time it is far not obvious that the $\mathcal{C}$-adic infinite direct tensor product of von Neumann is sufficient here (we expect rather a new infinite tensor product to be needed here); with (intentionally) infinite system of continuum-many independent fields of respective homogeneities $\chi = -1 - i \nu$ acting on the corresponding Fock spaces $\Gamma(\mathcal{H}_{\chi=-1+i\nu})$.

We propose not to enter these unsolved problems, and confine attention to just one part $\psi_{x_1=-1-i\nu_1}(x)$ of $\psi(x)$ of fixed homogeneity $\chi_1 = -1 - i \nu_1$, and
then investigate invariant subspaces of the field: \( \bar{\psi}_{x_1 = -1 - i\nu_1} \rho^\mu \psi_{x_1 = -1 - i\nu_1} : (x) \)

(or resp. \( \bar{\psi}_{x_1} \gamma^\mu \psi_{x_1} : (x) \)).

On the other hand one can show (compare Subsect. 6.4 and 6.6) that the Hilbert space of the quantum phase \( S(x) \) of Staruszkiewicz theory has the following structure

\[ \mathcal{H} = \mathcal{H}_0 \otimes \mathcal{H}_0. \]  

Here \( \mathcal{H}_0 \) is the closed subspace of the Hilbert space \( \mathcal{H} \) spanned by \( e^{i m S_0} |0 \rangle, m \in \mathbb{Z}. \)

Note that the direct summand with fixed \( m \) spanned by

\[ \left( e^{i m S_0} |0 \rangle \right) \otimes \mathcal{H}_0 \]

in (6) is the eigenspace of the total charge operator \( Q \) corresponding to the eigenvalue \( m \). The direct summand \( \mathbb{C} \otimes \mathcal{H}_0 = \mathcal{H}_0 \) is the eigenspace corresponding to the eigenvalue zero of \( Q \). The Hilbert space \( \mathcal{H}_0 \) is equal to the Fock space \( \mathcal{H}_0 = \Gamma(\mathcal{H}_0^1) \) over the single particle space \( \mathcal{H}_0^1 \) of “infrared transversal photons” spanned by \( c^+_{lm} |0 \rangle \).

The representation of \( SL(2, \mathbb{C}) \) acts on \( \mathcal{H}_0^1 \) through the Gelfand-Minlos-Shapiro irreducible representation \( (l_0 = 1, l_1 = 0) \) of the principal series and through its amplification \( \Gamma(l_0 = 1, l_1 = 0) \) on \( \mathcal{H}_0 = \Gamma(\mathcal{H}_0^1) \), and trivially on the factor \( \mathbb{C} \) in (6), [36], [47]. The factorization property (6) is preserved (compare Subsection 6.6) under the representation \( U \) of \( SL(2, \mathbb{C}) \) acting in \( \mathcal{H} \):

\[ U \mathcal{H} = (U \mathcal{H}_0 U^{-1}) \otimes (U \mathcal{H}_0 U^{-1}) \]

\[ = \mathcal{H}_0' \otimes \left( \Gamma(l_0 = 1, l_1 = 0) \mathcal{H}_0 \Gamma(l_0 = 1, l_1 = 0)^{-1} \right) = \mathcal{H}_0' \otimes \mathcal{H}_0. \]

But under the action of \( U \) only the second factor in (6) is invariant under \( U \) where \( U \) acts through \( \Gamma(l_0 = 1, l_1 = 0) \), as said above. The first factor in (6) is transformed under \( U \) into another subspace \( \mathcal{H}_0' \subset \mathcal{H} \) spanned by

\[ U e^{i m S_0} U^{-1} |0 \rangle, m \in \mathbb{Z}. \]

Finally to the tensor product factorization (6) of the Hilbert space of the phase field \( S(x) \) there correspond the tensor product factorization \( \mathcal{H}_0' \otimes \mathcal{H}_0' \) of the Hilbert space of the operator

\[ x_\mu \left( A_{\text{im}}^\mu (x) \right)_{x_\chi = -1} = x_\mu \left( A_{\text{free}}^\mu (x) \right)_{x_\chi = -1} + x_\mu \left( A_{(1)}^\mu (g = 1, x) \right)_{x_\chi = -1}, \]

where \( \mathcal{H}_0' \) is the Fock Hilbert space of the field \( x_\mu \left( A_{\text{im}}^\mu (x) \right)_{x_\chi = -1} \) and \( \mathcal{H}_0' \) is the Hilbert space of the field \( x_\mu \left( A_{(1)}^\mu (g = 1, x) \right)_{x_\chi = -1} \), by construction equal to an invariant subspace of the Fock space of a homogeneous of degree \(-3\) part.
The Fock space $H$ for the Hilbert space with the action of the operator (5) on $H$ gives us full equality (equivalence) and compatibility with gauge invariance, will identify the coupling constant and the various massive charged fields (say, scalar, spinor, e.t.c.) with the coupling $eQ$ and its action on this space can be naturally identified with the action of the operator (5) on $H_0$, for the proof compare Subsect. 6.4 and 6.6. Both $\chi$ and $\chi'$ act as the unit operator on the respective first factors in $\mathcal{H}'_1 \otimes \mathcal{H}'_0$ and respectively $\mathcal{H}_0 \otimes \mathcal{H}_0$. Similarly $-eQ\theta\psi$ and $\chi$ act as the unit operator on the respective second factor, for the proof compare Subsect. 6.4 and 6.6. Thus indeed the operators $x_\mu(\mathcal{A}_\text{free}(x))_{\chi=-1}$ and (5) understood as operators in the respective Hilbert spaces $\mathcal{H}'_1 \otimes \mathcal{H}'_0$ and $\mathcal{H}_0 \otimes \mathcal{H}_0$ can be equated, up to a trivial multiplicity. This in particular means that the equality (equivalence) of the operators $-eQ\theta\psi$ and $\chi$ in their action on the respective first factors would give us full equality (equivalence)

$$x_\mu(\mathcal{A}_\text{free}(x))_{\chi=-1} = \sum_{l=1}^{\infty} \sum_{m=-l}^{m+l} (c_l m f_l^\Gamma(\psi, \theta, \phi) + \text{h.c.})$$

and

$$x_\mu(\mathcal{A}(g = 1, x))_{\chi=-1} = -eQ\theta\psi.$$

Perhaps the most important reason for the comparison with Staruszkiewicz theory at spatial infinity lies in giving the realization of the proof for the universality of the unit of charge, outlined in [39]. Namely in completing the construction of the subspace invariant for the operator $x_\mu(\mathcal{A}(g = 1, x))_{\chi=-1}$ on which indeed it can be identified with $-eQ\theta\psi$ for the potential coupled with various massive charged fields (say, scalar, spinor, e.t.c.) with the coupling compatible with gauge invariance, will identify the coupling constant and the charge with the respective constant of the Staruszkiewicz theory. More precisely: if the equality of $x_\mu(\mathcal{A}(g = 1, x))_{\chi=-1}$ to the part $-eQ\theta\psi$ of phase $S(x)$ of Staruszkiewicz theory is indeed true, then in the Hilbert space of the field $x_\mu(\mathcal{A}(g = 1, x))_{\chi=-1}$ there must exist the operator $e^{iS_0}$ which together with the operator $(1/e)Q$ provides a spectral realization of the global gauge group $U(1)$. This follows from the fact that this is the case for Strausszkiewicz theory. Various contributions to $-eQ\theta\psi$ coming from various charge carrying fields coupled to the potential $A$ should give the total charge operator $Q$ which together with the corresponding phase provides a spectral contraction of the global gauge group, as in the case of the Staruszkiewicz theory, in which $V = e^{iS(x)}$, $D = (1/e)Q$ (or $V = e^{iS_0}, (1/e)Q$ define spectrally the gauge $U(1)$ group, compare Subsection 6.5). This will give us the universality of the unit of charge because the various contributions to the global charge $Q$ coming from the various charge carrying fields all should have common spectrum $e\mathbb{Z}$. Otherwise
the total charge operator could not serve as the Dirac operator for the $U(1)$ manifold, as the contributions coming from various charge carrying fields would destroy the spectrum $e\mathbb{Z}$ needed for the spectral reconstruction of the global gauge $U(1)$ group, compare Section 6.5. Thus the common scale for the electric charge comes from the condition that the infrared fields of each isolated system (involving various charge carrying fields with the couplings to $A$ preserving gauge invariance) provide a spectral description (in their total Hilbert space of infrared states) of the global gauge group $U(1)$ as in case of Stauszkiewicz theory, compare Subsect. 6.5. This mechanism forcing universality of the scale of the electric charge still works even for non standard representation of the commutation rules of the Staruszkiewicz theory. The only difference would be in changing of the spectrum of the total charge $Q$ from $e\mathbb{Z}$ into $ce\mathbb{Z}$ for some constant $c > 1$, and in changing $V = e^{iS(u)}D = (1/e)Q$ into $V = e^{icS(u)}D = (1/e)Q$ in the spectral construction of $U(1)$, compare Subsection 6.5 for the definition of the non standard representation.

Note that in this proof of universlity based on the comparison with Staruszkiewicz theory we do not need to have the operator $S_0$ as constructed in terms of the homogeneous part of the interacting field. Its computation within the homogeneous part of the interacting fields is perhaps more tricky in comparison to $Q$. We suspect that the non-perturbative construction of the causal phase (compare [26], Chap. 2.9) will be hepful here, but we had not enough time to try this way in computations.

Unfortunately we have not lead the proof that $x_\mu(A_\mu(1)(g = 1,x))\chi_{x=-1}$ is equal to the part $-eQ\theta\psi$ of $S(x)$ to an end in this work.

Perhaps we should remark that various limit operations involved in our computation were commuted rather freely. Especially we have computed first the homogeneous of degree $-1$ part $(A^\mu_{\text{int}}(x))_{x=-1}$ of the field $A^\mu_{\text{int}}(x)$ and then constructed homogeneous of degree zero part of the field $x_\mu A^\mu_{\text{int}}(x)$ by putting it equal to $x_\mu(A^\mu_{\text{int}}(x))_{x=-1}$. We did so only to simplify computations, but one should remember that the causal perturbative series for interacting fields in general does not depend on the order of the following operations: first compute $A^\mu_{\text{int}}(x)$ and then multiply by $x_\mu$: $x_\mu A^\mu_{\text{int}}(x)$ or first multiply by $x_\mu$: $x_\mu A^\mu(x)$, and then compute $(x_\mu A^\mu(x))_{x=-1}$, because $x_\mu$ is a $c$-number. This order is also unimportant at the free theory level, and it is irrelevant if we first compute the homogeneous of degree $-1$ part of the potential field, and then multiply by $x_\mu$, or first multiply by $x_\mu$ and then compute the homogeneous of degree zero part.

Moreover in extracting the homogeneous part we work effectively with fields on de Sitter 3-hyperboloid space-time. On this space-time quantum fields, including interacting fields with naturally defined interactions, behave much better than in the Minkowski spacetime. Similar fact has been discovered by mathematicians, mainly Segal, Zhou and Paneitz, [28], [29], for the $\phi^4$ theory and for QED on the static Einstein Universe space-time, with the help of the harmonic analysis on the Einstein Universe, which the authors worked out extensively in a series of papers: [22]-[24]. Nonetheless the mechanisms simplyfing matters are exactly the same for the quantum fields on de Sitter 3-hyperboloid.

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In particular the curvature of de Sitter 3-hyperboloid is crucial here (for example QED on the toral compactification of the Minkowski space-time is still very singular, although the set of modes is discrete).

Suppose the operation of multiplication by $x_\mu$ is performed at the very end of the process of computation after the operation of extraction of the homogeneous of degree $-1$ part of the interacting potential, i.e. for the homogeneous of degree zero part of the field $x_\mu A_{\text{int}}^\mu (x)$ we put $x_\mu (A_{\text{int}}^\mu (x))_{x=-1}$. It seems that in this computation the various equivalent realizations of the free potential field, which are then used in the construction of the perturbative series of the interacting fields, give the same $x_\mu (A_{\text{int}}^\mu (x))_{x=-1}$, because all of them have the same pairings.

Thus passing to infinity is not merely a way to simplify matters but possibly an indispensable step in constructing full theory.

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References

[1] Bialynicki-Birula, I.: Progress in Optics 36, 245 (1996).
[2] Blanchard, P., Seneor, R.: Annales de l’ I. H. P. A23, 147 (1975).
[3] Bohr, N., Rosenfeld, L.: Mat.-fys. Medd. Dan. Vid. Selsk. 12, no. 8 (1933).
[4] Bollini, C. G.: Nuovo Cimento 8, 39 (1958).
[5] Dirac, P. A. M.: The principles of quantum mechanics, Clarendon Press, Oxford 1947, Third ed..
[6] Dütsch, M., Krahe, F., Scharf, G.: Nuovo Cimento A 103, 871 (1990).
[7] Dütsch, M., Krahe, F., Scharf, G.: Nuovo Cimento A 1029, 871 (1993).
[8] Dütsch, M., Krahe, F., Scharf, G.: Nuovo Cimento A 107, 375 (1994).
[9] Dütsch, M., Krahe, F., Scharf, G.: Nuovo Cimento A 108, 737 (1995).
[10] Epstein, H., Glaser, V.: Ann. Inst. H. Poincaré A19, 211 (1973).
[11] Epstein, H., Glaser, V.: Contribution to the meeting on renormalization theory. C. N. R. S., Marseille, June 1971; C. E. R. N., preprint TH 1344 (1951).
[12] Gelfand, I. M., Minlos, R. A., Shapiro, Z. Ya.: Representations of the rotation and Lorentz groups and their applications. Pergamon Press Book, The Macmillan Company, New York, 1963.
[13] Gelfand, I. M. and Shilov, G. E.: Properties of Operations: Generalized functions. Vol., Acad. Press, New York, 1964.

[14] Gelfand, I. M. and Vilenkin, N. Ya.: Applications of Harmonic Analysis: Generalized functions. Vol. 4., Acad. Press, New York, 1964.

[15] Gelfand, I. M., Graev, M. I., and Vilenkin, N. Ya.: Integral Geometry and Representation Theory: Generalized functions. Vol. 4., Acad. Press, New York, 1964.

[16] Gervais, J.-L., Zwanziger, D.: Phys. Lett. B94, 389 (1980).

[17] Haag, R.: Local Quantum Physics. Springer Verlag, 1996.

[18] Herdegen, A.: “Asymptotic structure of electrodynamics revisited”, arXiv: 1604.04170v3 [hep-th].

[19] Hida, T, Obata, N., Saitô, K.: Nagoya Math. J. 128, 65 (1992).

[20] von Neumann, J.: Compositio Math. 6, 1 (1939).

[21] Obata, N.: White noise calculus and Fock space. Springer Verlag, 1994.

[22] Paneitz, S. M. and Segal, I. E., J.: Funct. Anal. 47, 78 (1982).

[23] Paneitz, S. M. and Segal, I. E., J.: Funct. Anal. 49, 335 (1982).

[24] Paneitz, S. M., I. E., J.: Funct. Anal. 54, 18 (1983).

[25] Requardt, M.: Commun. Math. Phys. 50, 259 (1976).

[26] Scharf, G: Finite Quantum electrodynamics, Dover Publications, Mineola, New York, 2014.

[27] Schroer, B.: A Course on: “Modular Localization and Nonperturbative Local Quantum Physics” CBPF, Rio de Janeiro, March 1998; arxiv: hep-th/9805093, pp. 93-94.

[28] Segal, I. E. and Zhou, Z.: Ann. Phys. 218, 279 (1992).

[29] Segal, I. E. and Zhou, Z.: Ann. Phys. 232, 279 (1994).

[30] Shimada, Y.: white noise distribution theory for the fermion system. arXiv: 0503051v3 [math-ph] (2005).

[31] Staruszkiewicz, A.: Acta Phys. Polon. B12, 327 (1981).

[32] Staruszkiewicz, A.: “Quantum mechanics of phase and charge and quantization of the Coulomb field”, Preprint TPJU-12/87, Institute of Physics, Jagellonian University, Krakow 1987.

[33] Staruszkiewicz, A.: Ann. Phys. (N.Y.) 190, 354 (1989).
[34] Staruszkiewicz, A.: Acta Phys. Polon. **B 23**, 927 (1992).

[35] Staruszkiewicz, A.: Acta Phys. Polon. **B23**, 591 (1992) and ERRATUM in Acta Phys. Pol **B23**, 959 (1992).

[36] Staruszkiewicz, A.: Acta Phys. Polon. **B26**, 1275 (1995).

[37] Staruszkiewicz, A.: Condensed Matter Phys. 1, 587 (1998).

[38] Staruszkiewicz, A.: Quantum mechanics of the electric charge in: *Developments in Quantum Field Theory*, Ed. Damgaard and Jurkiewicz, Plenum Press, New York, 1998, pp. 179-185.

[39] Staruszkiewicz, A.: “Quantum Mechanics of the Electric Charge”. In: “Quantum Coherence and Reality”. Ed.: J. S. Anandan and J. L. Safko. World Scientific, Singapore, 1994. arXiv:9810084 [hep-th].

[40] Staruszkiewicz, A.: Foundations of Physics **32**, 1863 (2002).

[41] Staruszkiewicz, A.: Acta Phys. Polon. **B35**, 2249 (2004).

[42] Staruszkiewicz, A.: Reports on Math. Phys. **64**, 293 (2009).

[43] Streater, R. F. and Wightman, A. S.: PCT, Spin and Statistics, and All That, W. A. Benjamin, Inc., New York, 1964.

[44] Wawrzycki, J.: Preprint math-ph/150402273.

[45] Wawrzycki, J.: Preprint math-ph/160400482.

[46] Wawrzycki, J.: Acta Phys. Polon. **B47**, 2163 (2016).

[47] Wawrzycki, J.: Preprint math-ph/171205306.

[48] Wawrzycki, J.: Preprint math-ph/180206719.

[49] Woronowicz, S. L.: Studia Mathematica, 39, 217, (1971).