THE EIGENVALUE APPROXIMATIONS OF THE LAPLACE OPERATOR DEFINED ON A DOMAIN WITH STRONGLY DEFORMED BOUNDARY

In this paper the eigenvalue approximations of the Laplace operator defined on a domain with strongly deformed boundary are presented. Because the exact eigenfunctions exhibit complicated behaviour in the vicinity of singular points of the used conformal mapping, the B-spline trial functions are used in order to improve the quality of the eigenfunction approximations near the singular points.

1. Introduction

The eigenvalue problem for the two-dimensional Laplace operator defined on domains with complicated boundary shape arises in many practical situations, for example in mechanical engineering, microwave theory and techniques and biomechanics [3, 6]. The complicated shape form of domains is of interest in practice when the Laplace operator defined on the standard domains as a circle and square does not offer the optimum eigenvalue distribution needed for meeting the design requirements. Standard methods and their combinations with various special techniques have achieved the solution of such problems. The author of this paper [5] has recently presented the eigenvalue computations using this technique based on the sine trial functions. However, because of the presence of geometrical singularities of the exact eigenfunctions, the convergence of the Ritz eigenvalue approximations for the large deformation of domain under consideration is very slow.

On the other hand, these singularities are of local character and in this case more precise approximations can be obtained using a local approximation, for example the spline approximation and finite element method.

In this paper the eigenvalue approximations of the Laplace operator defined on a domain with strongly deformed boundary are presented. Because of the presence of shape singularities of the exact eigenfunctions the B-spline trial functions are used in order to improve the quality of the eigenfunction approximations near the singular points.

2. Formulation of the problem

The eigenvalue problem for the Laplace operator, known also as the homogeneous Helmholtz equation, is given by

![Fig. 1 Conformal mapping w = tg(z/2) maps the region Ωs onto the region Ωe](image-url)
\[- \frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial y^2} = \lambda \psi \quad \text{in } \Omega \]  
with the Dirichlet boundary condition
\[ \psi = 0 \quad \text{on } \partial \Omega, \]  
where \( \Omega \) is a bounded two-dimensional domain with a piecewise smooth boundary.

We deal with the problem in which the domain \( \Omega \) in the \( w \)-plane \((w = x + iy)\) is generated by conformal mapping \( w = f(z) \) of a rectangle in the \( z \)-plane \((z = x + iy)\). The conformal mapping \( w = ig(z)/2 \) maps the region \( S_i \) in the \( z \)-plane onto the region \( \Omega_i \) in the \( w \)-plane bounded by arcs of orthogonal circles, see Fig. 1.

Using the conformal mapping \( w = f(z) \) the eigenvalue problem (1), (2) is transformed to the equation
\[ -\Delta U(x, y) = \sigma(x, y) \lambda U(x, y) \quad \text{in } \Omega \]  
with the Dirichlet boundary condition
\[ U = 0 \quad \text{on } \partial \Omega, \]  
Here the function \( \sigma(x, y) = \left| \frac{d(f(z))}{dz} \right| \) is defined as follows
\[ \sigma(x, y) = \frac{1}{(\cos x + \cosh y)^2}. \]  
The nearest singular points of conformal mapping \( w = ig(z)/2 \) to the regions \( S_i \) are \( T_i = [-\pi, 0], T_2 = [\pi, 0] \). The shapes of the function \( \sigma(x, y) \) corresponding to the weakly deformed domain \( \Omega_1 \), and to the strongly deformed domains \( \Omega_2 \) are plotted in the left and right in Figure 2, respectively.

The domains \( \Omega_1 \) and \( \Omega_2 \) are created by the conformal mapping \( w = ig(z)/2 \) of the square \( S_1 = (-\pi/2, \pi/2) \times (-\pi/2, \pi/2) \) and the rectangle \( S_2 = (-1.9\pi/2, 1.9\pi/2) \times (-\pi/2, \pi/2) \), respectively.

3. Spline approximation

Definition 1. Let \( t_i, i = 1, 2, \ldots, n \) be an increasing sequence of points of the real axis. The function \( B^k_i(t) \) with \( i + k \leq n \) is called \( k \)-th algebraic \( B \)-spline of order \( k \) (see Fig. 3), if the following properties are satisfied:

(a) \( B^k_i(t) \neq 0 \) only for \( t \in (t_i, t_{i+k}) \).
(b) \( B^k_i(t) \) is algebraic polynomial of order \( (k - 1) \) on the each interval \( (t_i, t_{i+k}) \) \( i \leq i + k - 1 \).
(c) \( B^k_i(t) \) is continuous function with continuous derivatives up to the order \( (k - 2) \) on the whole real axis.

![Fig. 3 Shape of the i-th algebraic B-spline of order 6](image)

For the calculation of the \( B \)-splines and their derivatives the following numerically stable recurrence relations \([2]\) are used
\[ B^k_i(t) = t - t_{i+k-1} - t_i B_{i+k-1}^k(t) + \frac{t_{i+k} - t}{t_{i+k} - t_{i+1}} B_{i+1}^{k-1}(t) \quad (6) \]  
if \( B^k_i(t) = \begin{cases} 1 & t \in (t_i, t_{i+1}) \\ 0 & t \not\in (t_i, t_{i+1}) \end{cases} \quad (7) \]  
and \( B^k_i(t)^m = (k - 1) \left[ \frac{B_{i+k-1}^{k-1}(t)^{m-1}}{t_{i+k} - t_i} - \frac{B_{i+1}^{k-1}(t)^{m-1}}{t_{i+k} - t_{i+1}} \right]. \) (8)

4. Numerical experiments

The numerical experiments presented in this article are based on the Rayleigh - Ritz method applied on the equation (3) using the \( B \)-spline trial functions of the form

![Fig. 2 Shapes of the function \( \sigma(x, y) \) corresponding to the domain \( \Omega_1 \) (left) and \( \Omega_2 \) (right)](image)
\[ \psi_{kl}(x, y) = (\cos x + \cosh y) \tilde{\psi}_{kl}(x, y), \]  
\[ \tilde{\psi}_{kl}(x, y) = (x - a)(b - x) \left( \frac{\pi}{2} y + \frac{\pi}{2} \right) B_i(x) B_j(y) \]  
\[ (l, k = 1, 2, \ldots, n, \, l = 1, 2, \ldots, n) \]  
\[ n = m/2 \]  
\[ \text{for } r = 8, k = 1, 2, \ldots, n, \, l = 1, 2, \ldots, n, \text{ and } a = b = -\pi/2 \]  
\[ \Omega_1 \]  
\[ \Omega_\delta \]  
\[ \text{Table 1 and Table 3 correspond to the case of weakly deformed domain } \Omega_\delta, \text{ while the results reported in Table 2 and Table 4 correspond to the case of strongly deformed domain } \Omega_1. \]

The Rayleigh-Ritz eigenvalue approximations of the selected eigenvalues using \( n \) trial functions are presented in Table 1 – 4. For the sake of convergence comparisons the eigenvalue approximations shown in Table 1 and Table 2 are taken from the author’s previous article [5] and correspond to the sine trial functions. The eigenvalue approximations shown in Table 3 and Table 4 are computed using the B-spline trial functions (9). The results reported in Table 1 and Table 3 correspond to the case of weakly deformed domain \( \Omega_\delta \), while the results reported in Table 2 and Table 4 correspond to the case of strongly deformed domain \( \Omega_1 \).
5. Concluding remarks

The presented numerical results indicate that the B-spline trial functions offer more precise eigenvalue approximations than the sine trial functions. This difference in the eigenvalue convergence is caused by the presence of shape singularities of the exact eigenfunctions which are the consequence of very steep gradient of the function $f(x, y)$ in the vicinity of the points $[-\pi, 0]$, $[\pi, 0]$ as seen in Figure 2 (on the right). This case corresponds to the strongly deformed domain $\Omega$. Because the B-spline trial functions are able to match singular behaviour of functions more precisely than approximations based on the sine trial functions, the corresponding eigenvalue approximations exhibit essentially better convergence. Finally the B-spline trial functions are recommended for use at least in the cases when the domain with complicated boundary shape is generated by a conformal mapping of square or rectangle.

References

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