THE RELATION BETWEEN GROTHENDIECK DUALITY AND
HOCHSCHILD HOMOLOGY

AMNON NEEMAN

Abstract. Grothendieck duality goes back to 1958, to the talk at the ICM in Edinburgh [13] announcing the result. Hochschild homology is even older, its roots can be traced back to the 1945 article [16]. The fact that the two might be related is relatively recent. The first hint of a relationship came in 1987 in Lipman [20], and another was found in 1997 in Van den Bergh [28]. Each of these discoveries was interesting and had an impact, Lipman’s mostly by giving another approach to the computations and Van den Bergh’s especially on the development of non-commutative versions of the subject. However in this survey we will almost entirely focus on a third, much more recent connection, discovered in 2008 by Avramov and Iyengar [2] and later developed and extended in several papers, see for example [3, 19].

There are two classical paths to the foundations of Grothendieck duality, one following Grothendieck and Hartshorne [15] and (much later) Conrad [10], and the other following Deligne [11], Verdier [29] and (much later) Lipman [21]. The accepted view is that each of these has its drawbacks: the first approach (of Grothendieck, Hartshorne and Conrad) is complicated and messy to set up, while the second (of Deligne, Verdier and Lipman) might be cleaner to present but leads to a theory where it’s not obvious how to compute anything.

The point of this article is that the recently-discovered connection with Hochschild homology and cohomology (the one due to Avramov and Iyengar) changes this. It renders clearly superior the highbrow approach to the subject, the one due to Deligne, Verdier and Lipman. Not only is it (relatively) easy to set up the machinery, the computations also become transparent. And in the process we learn that Grothendieck duality is not really about residues of meromorphic differential forms, it is about the local cohomology of the Hochschild homology. By a fortuitous accident, if $f : X \rightarrow Y$ is a smooth map then the top Hochschild homology happens to be isomorphic to the relative canonical bundle, and its top local cohomology is represented by meromorphic differential forms. This is the reason that, as long as we stick to smooth maps, what comes up is residues of meromorphic forms. For non-smooth, flat maps it’s Hochschild homology and maps from it that we need to study.

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2000 Mathematics Subject Classification. Primary 14F05, secondary 13D09, 18G10.
Key words and phrases. Derived categories, Grothendieck duality.
The research was partly supported by the Australian Research Council.
0. Introduction

Let $f : X \rightarrow Y$ be a morphism of noetherian schemes. At the level of derived categories there exist natural functors $Lf^* : \mathcal{D}_{qc}(Y) \rightarrow \mathcal{D}_{qc}(X)$, its right adjoint $Rf_* : \mathcal{D}_{qc}(X) \rightarrow \mathcal{D}_{qc}(Y)$, as well as a right adjoint for $Rf_*$, nowadays (following Lipman) denoted $f^\times : \mathcal{D}_{qc}(Y) \rightarrow \mathcal{D}_{qc}(X)$. For general $f$ the functor $f^\times$ can be dreadful—it can take a bounded complex of coherent sheaves, that is an object in $\mathcal{D}^{b}_{coh}(Y) \subset \mathcal{D}_{qc}(Y)$, to a truly enormous object in $\mathcal{D}_{qc}(X)$. This functor $f^\times$ only behaves well under strong restrictions, the usual being that $f$ be proper.

To remedy this one introduces a better-behaved functor $f^!$. If $f$ is proper then $f^! = f^\times$, but for general $f$ one traditionally does some finicky manipulations to arrive at $f^!$. And, until very recently, the recipe worked only for cohomologically bounded-below complexes. That is $f^!$ has always been viewed as a functor $f^! : \mathcal{D}^{+}_{qc}(Y) \rightarrow \mathcal{D}^{+}_{qc}(X)$.

Against this background came the striking work of Avramov, Iyengar, Lipman and Nayak, see [2, 3], relating Grothendieck duality with Hochschild homology and cohomology. To give the flavor of the results let me present just one formula, and for simplicity let me give only the affine version. Suppose therefore that $X = \text{Spec}(S)$, $Y = \text{Spec}(R)$, assume that $R$ and $S$ are noetherian, and that $f : X \rightarrow Y$ is a flat, finite-type map. In an abuse of notation we will write $f^! : \mathcal{D}^{+}_{qc}(Y) \rightarrow \mathcal{D}^{+}_{qc}(X)$.

In this formula the tensor products and the Hom are all derived. The reader might find it interesting to note that, in the special case where $f : R \rightarrow S$ is finite and étale, we recover the classical formula

$$f^! N \cong S \otimes_{S^e} \text{Hom}_R(S, S \otimes_R N).$$

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$$f^! N \cong \text{Hom}_R(S, N) \cong S \otimes_{S^e} \text{Hom}_R(S, S \otimes_R N).$$

Of course for finite, étale maps the Hom and tensors are underived. We will revisit étale maps (not necessarily finite) in Remark 2.8.
Perhaps one needs some familiarity with the classical literature to appreciate how
striking this is—assuming only that \( f \) is flat we have produced a formula for \( f^! \), which
took a mere paragraph to state, and is clearly free of auxiliary choices and functorial. And
although the left-hand-side was defined on the assumption that \( N \) is bounded below—
after all we only knew \( f^! \) on the bounded-below derived category—the right-hand-side
makes sense for any \( N \). In fact the formula tells us the surprising fact that if \( N \) is an
object in \( D^+(R) \) then \( S \otimes_R \text{Hom}_R(S, S \otimes_R N) \) must belong to \( D^+(S) \). We know, from
the complicated classical construction, that \( f^! \) takes \( D^+(R) \) to \( D^+(S) \), but \( S \) is not of
finite Tor-dimension over \( S \) and we have no reason to expect an expression of the form
\( S \otimes S \text{^e} M \) to be bounded below. The derived tensor product tends to introduce lots of
negative cohomology.

In joint work with Iyengar and Lipman we revisited these results, and along the way
developed a useful new natural transformation \( \psi(f) : f^* \rightarrow f^! \), see [19]. Hints of \( \psi 
may be found in Lipman [21, Exercise 4.2.3(d)], but without the naturality properties
that make it so valuable. With all these unexpected new tools it was becoming clear
that the time may have come to revisit the foundations of Grothendieck duality. In this
article we sketch what has come out of this.

Finally we should tell the reader the structure of this survey. The early sections, §2
and §3, survey recent results that can be found elsewhere in the literature. The results
are new, meaning new in this generality—there are older avatars, what’s unusual here
is that the theory is developed in the unbounded derived category. The results might
be innovative but we still omit the proofs. With the exception of Proposition 3.3, where
the argument is included, the proofs are all to be found in recent preprints available
electronically.

In §4 and §5 this changes. Special cases of the results are known, with what turn
out to be artificial boundedness restrictions. We give a general treatment—both to
show that the results are true more generally, and to illustrate the power of the new
techniques. Because of this our treatment is complete, with proofs. The reader interested
in the highlights is advised to read the statement (not proof) of Lemma 4.3 as well as
Corollary 5.7 and Example 5.8.

The final sections, §6 and §7, are again “soft”, with no proofs presented. They review
the history and suggest open problems.

1. Conventions

In this article we consider schemes \( X \) and the corresponding derived categories \( D_{\text{qc}}(X) \),
whose objects are complexes of sheaves of \( \mathcal{O}_X \)-modules with quasicoherent cohomology.
Since abelian categories never come up, whenever there is a possible ambiguity our func-
tors should be assumed derived—thus we will write \( f^\cdot \) for \( Lf^* \), \( f_* \) for \( Rf_* \), \( \text{Hom} \)
for \( R\text{Hom} \) and \( \otimes \) for the derived tensor product \( \otimes^L \). For simplicity, in §2 §3 §4 §5 and §6
we will assume that our schemes are noetherian and morphisms of schemes are separated
and of finite type—occasionally, but not always, we will explicitly remind the reader of
these standing assumptions. Unless we specifically say otherwise all derived categories will be unbounded. For a morphism of schemes \( f : X \to Y \) we let \( f^* \dashv f_* \dashv f^\times \) be the adjoint functors which, back in \( \text{[1]} \) we referred to as \( \mathbf{L}f^* \dashv \mathbf{R}f_* \dashv f^\times \).

2. The formal theory

In this section and the next we sketch the current state of the formal theory, without worrying about who proved what and when.

Let \( f : X \to Y \) be a morphism of schemes. The functor \( f^* : \mathbf{D}_{\text{qc}}(Y) \to \mathbf{D}_{\text{qc}}(X) \) is a strict monoidal functor, meaning it respects the tensor product. Therefore for any pair of objects \( E \in \mathbf{D}_{\text{qc}}(Y) \) and \( F \in \mathbf{D}_{\text{qc}}(X) \) we have a natural map

\[
f^*[E \otimes f_* F] \xrightarrow{\sim} f^* E \otimes f^* f_* F \xrightarrow{id \otimes \varepsilon} f^* E \otimes F
\]

where the first map is the natural isomorphism, and \( \varepsilon : f^* f_* \to \text{id} \) is the counit of the adjunction \( f^* \dashv f_* \). By adjunction we obtain a natural map \( p(E,F) : E \otimes f^* F \to f_* (f^* E \otimes F) \). The map \( p(E,F) \) is known to be an isomorphism, usually called the projection formula.

This leads us to

**Definition 2.1.** Let \( f : X \to Y \) be a morphism of schemes and let \( E,F \) be objects in \( \mathbf{D}_{\text{qc}}(Y) \). The map \( \chi(f,E,F) : f^* E \otimes f^\times F \to f^\times (E \otimes F) \) is defined by applying the adjunction \( f_* \dashv f^\times \) to the composite

\[
f_* (f^* E \otimes f^\times F) \xrightarrow{p(E,F)^{-1}} E \otimes f_* f^\times F \xrightarrow{id \otimes \varepsilon'} E \otimes F,
\]

where the first map is the inverse of the isomorphism in the projection formula, while \( \varepsilon' : f_* f^\times \to \text{id} \) is the counit of the adjunction \( f_* \dashv f^\times \).

The first result in the theory is

**Theorem 2.2.** The map \( \chi(f,E,F) \) is an isomorphism whenever

(i) \( f \) is arbitrary, but \( E \) is a perfect complex.

(ii) \( E \) and \( F \) are arbitrary, but \( f \) is proper and of finite Tor-dimension.

Next recall the base-change maps. Given a commutative square of morphisms of schemes

\[
\begin{array}{ccc}
W & \xrightarrow{u} & X \\
\downarrow f & & \downarrow g \\
Y & \xrightarrow{v} & Z
\end{array}
\]

there is a canonical isomorphism of functors \( \alpha : f^* v^* \to u^* g^* \). Consider the composite

\[
f^* v^* g_* \xrightarrow{\alpha g_*} u^* g^* g_* \xrightarrow{u^* \varepsilon} u^* ,
\]

where \( \varepsilon : g^* g_* \to \text{id} \) is the counit of the adjunction \( g^* \dashv g_* \). Adjunction gives us a base-change map \( \beta : v^* g_* \to f_* u^* \); this map is not always an isomorphism, but there are important situations in which it is. This leads us to
Definition 2.3. Assume we are in a situation where the base-change map $\beta : v^*g_* \to f_*u^*$ is an isomorphism; for this article the important case where this happens is when the square

$$
\begin{array}{ccc}
W & \xrightarrow{u} & X \\
\downarrow^f & & \downarrow^g \\
Y & \xrightarrow{v} & Z
\end{array}
$$

is cartesian and the map $v$ is flat. In this scenario consider the composite

$$
f_*u^*g^\times \xrightarrow{\beta^{-1}g^\times} v^*g_*g^\times \xrightarrow{v^*\varepsilon'} v^*
$$

where the first map is the inverse of the isomorphism $\beta$ while $\varepsilon' : g_*g^\times \to \text{id}$ is the counit of the adjunction $g_* \dashv g^\times$. The (second) base change map $\Phi : u^*g^\times \to f^\times v^*$ corresponds to this composite under the adjunction $f^\times \dashv f^!$.

One can wonder when the base-change map $\Phi$ is an isomorphism. The best result to date says

Theorem 2.4. Let the notation be as in the case of Definition 2.3 which interests us in this article—that is we assume the square cartesian and $v$ flat. Let $E$ be an object in $D\text{qc}(Z)$. Then the base-change map $\Phi(E) : u^*g^\times(E) \to f^\times v^*(E)$ is an isomorphism provided $g$ is proper and one of the conditions below holds:

(i) $E$ belongs to $D^+(\text{qc})(Z) \subset D\text{qc}(Z)$.

(ii) $E \in D\text{qc}(Z)$ is arbitrary, but the map $f : W \to Y$ is of finite Tor-dimension.

Now one proceeds as follows: given any morphism $f : X \to Y$ we factor it as $X \xrightarrow{u} X \xrightarrow{p} Y$ with $u$ an open immersion and $p$ proper, and then define $f^! : D\text{qc}(Y) \to D\text{qc}(X)$ by the formula $f^! = u^*p^\times$. One of the consequences of Theorem 2.4 is that $f^!$ is well-defined, meaning that it is canonically independent of the choice of factorization. And we have the following theorem.

Theorem 2.5. The assignment, taking a morphism of schemes $f : X \to Y$ to the functor $f^! : D\text{qc}(Y) \to D\text{qc}(X)$, satisfies a long list of compatibility properties. We list some highlights.

2.5.1. Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be composable morphisms of schemes. There is a map $\rho(f,g) : (gf)^! \to f^!g^!$, which has the property that the two ways of using $\rho$ to go from $(hgf)^!$ to $f^!g^!h^!$ are equal.

2.5.2. The two functors $f^\times, f^! : D\text{qc}(Y) \to D\text{qc}(X)$ are related by a natural transformation $\psi(f) : f^\times \to f^!$. The $\psi$ is compatible with composition, in the obvious sense.
that the square below commutes

\[
\begin{array}{ccc}
(gf)^\times & \delta(f,g) & f^\times g^\times \\
\downarrow \psi(gf) & & \downarrow \psi(f) \psi(g) \\
(gf)^! & \rho(f,g) & f^! g^!
\end{array}
\]

where \( \rho(f,g) \) is the map of \textit{2.5.1} while \( \delta(f,g) : (gf)^\times \to f^\times g^\times \) is the canonical isomorphism.

2.5.3. The map \( \rho(f,g) \) is an isomorphism if \( f \) is of finite Tor-dimension or if either \( gf \) or \( g \) is proper. The map \( \psi(f) \) is an isomorphism whenever \( f \) is proper.

2.5.4. Given a pair of objects \( E, F \in Dqc(Y) \) then there is a way to mimic the construction in Definition \textit{2.1} with \( f^! \) in place of \( f^\times \). More precisely: there is a map \( \sigma(f, E, F) : f^* E \otimes f^! F \to f^!(E \otimes F) \) so that the natural square commutes

\[
\begin{array}{ccc}
f^* E \otimes f^\times F & \chi(f, E, F) & f^\times (E \otimes F) \\
\downarrow \text{id} \otimes \psi(f) & & \downarrow \psi(f) \\
f^* E \otimes f^! F & \sigma(f, E, F) & f^!(E \otimes F)
\end{array}
\]

Furthermore we have the analog of Theorem \textit{2.2} that is \( \sigma(f, E, F) \) is an isomorphism if one of the conditions below holds

(i) \( f \) is arbitrary, but \( E \) is a perfect complex.
(ii) \( E \) and \( F \) are arbitrary, but \( f \) is of finite Tor-dimension.

2.5.5. The base-change map \( \Phi \) of Definition \textit{2.4} also has an (\(-\))^! analog. Precisely: given a cartesian square as in Definition \textit{2.3}, there is a base-change map \( \theta : u^* g^! \to f^! v^* \).

2.5.6. There is an analog of Theorem \textit{2.4} for (\(-\))^! in place of (\(-\))^\times. Precisely: the map \( \theta(E) : u^* g^!(E) \to f^! v^*(E) \) is an isomorphism as long as one of the following holds

(i) \( E \) belongs to \( Dqc(Z) \subset Dqc(Z) \).
(ii) \( E \in Dqc(Z) \) is arbitrary, but the map \( f : W \to Y \) is of finite Tor-dimension.

The full list of compatibility properties is quite long, and in any case it is clearer and more compact to present it in a 2-category formulation. For this paper we content ourselves with what’s in Theorem \textit{2.5}.

Remark 2.6. In the introduction we mentioned that people have traditionally preferred \( f^! \) to \( f^\times \) because it is “better behaved”. Theorem \textit{2.5} allows us to make this more precise. If we compare \textit{2.5.4} with Theorem \textit{2.2} we see that

(i) If \( f \) is proper then \( \sigma(f, E, F) \) and \( \chi(f, E, F) \) agree up to canonical isomorphism. To see this observe that, when \( f \) is proper, then the vertical maps in the commutative square of \textit{2.5.4} are isomorphisms by \textit{2.5.3}. 

(ii) The maps $\sigma(f, E, F)$ and $\chi(f, E, F)$ are defined for every triple $f, E, F$, but $\sigma(f, E, F)$ is an isomorphism more often. If $E$ is perfect then both are isomorphisms. But for non-perfect $E$ the result 2.5.4(ii) says that $\sigma(f, E, F)$ is an isomorphism whenever $f$ is of finite Tor-dimension, whereas Theorem 2.2(ii) guarantees that $\chi(f, E, F)$ is an isomorphism only if $f$ is proper as well as of finite Tor dimension.

The same pattern repeats itself for the base-change maps $\Phi$ and $\theta$. They are defined for every cartesian square with flat horizontal morphisms, and coincide if the vertical maps are proper—if our Theorem 2.5 were less pared down this could be shown to follow from the general structure, the reader can see the introduction to [23] for the fullblown formalism. But if we ask ourselves when $\Phi$ and $\theta$ induce isomorphisms, the conditions on $\theta$ are less restrictive than on $\Phi$. Precisely: when we compare Theorem 2.4 with 2.5.6 we discover

(iii) Assume $E$ is bounded below. Then $\Phi(E) : u^*g^\times E \to f^\times v^*E$ is an isomorphism if $g$ is proper, while $\theta(E) : u^*g^!E \to f^!v^*E$ is an isomorphism unconditionally.

(iv) Let $E$ be arbitrary. Then $\Phi(E)$ is an isomorphism as long as $g$ is proper and $f$ is of finite Tor-dimension, while $\theta(E)$ is an isomorphism whenever $f$ is of finite Tor-dimension (no need for any properness).

Remark 2.7. If $f : X \to Y$ is an open immersion then the square

$$
\begin{array}{ccc}
X & \xrightarrow{id} & X \\
\downarrow{id} & & \downarrow{f} \\
X & \xrightarrow{f} & Y
\end{array}
$$

is cartesian. By 2.5.6(ii) we have that $f^! = id^*f^! \xrightarrow{\theta} id^!f^* = f^*$ is an isomorphism.

Given any morphism $g : Y \to Z$, the map $\rho(f, g) : (gf)^! \to f^!g^!$ is an isomorphism by 2.5.3—after all the open immersion $f$ is of finite Tor-dimension. Combining these isomorphisms gives

(i) If $X \xrightarrow{f} Y \xrightarrow{g} Z$ are composable morphisms of schemes, with $f$ an open immersion, then we have a canonical isomorphism $(gf)^! \xrightarrow{\rho} f^!g^! \xrightarrow{\theta} f^*g^!$.

If $g : Y \to Z$ happens to be a proper morphism then $\psi(g) : g^\times \to g^!$ is an isomorphism by 2.5.3 which we may combine with the isomorphism of (i) to deduce a canonical isomorphism $(gf)^! \cong f^*g^! \cong f^*g^\times$. The compatibilities of Theorem 2.5 force upon us the formula for $(gf)^!$. That is: any time we can factor a map $X \to Z$ as a composite $X \xrightarrow{f} Y \xrightarrow{g} Z$, with $f$ an open immersion and $g$ proper, then $(gf)^!$ must be given by the formula $(gf)^! \cong f^*g^\times$.

Remark 2.8. In passing we observe that the formula of Remark 2.7(i) generalizes, we need only assume $f$ étale. Suppose $f : X \to Y$ is an étale morphism of noetherian
schemes. Consider the following diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\Delta} & X \times_Y X \\
\downarrow^{\pi_2} & & \downarrow^{f} \\
X & \xrightarrow{f} & Y
\end{array}
\]

where the square is cartesian and \( \Delta \) is the diagonal map. We are given that \( f \) is étale, meaning that the diagonal map \( \Delta \) is an open immersion. This gives a series of isomorphisms

\[
f^! \cong \Delta^* \pi_1^! \cong \Delta^* \pi_2^! f^* \cong (\pi_2 \Delta)^! f^* \cong f^*.
\]

The first isomorphism is \( \Delta^* \pi_1^! \cong \text{id}^* = \text{id} \), the second isomorphism is the map \( \pi_1^! f^! \to \pi_2^! f^* \) of (2.5.5), which is an isomorphism by (2.5.6) ii), the third isomorphism is Remark 2.7(i) applied to the composable maps \( X \xrightarrow{\Delta} X \times_Y X \xrightarrow{\pi_2} X \) with \( \Delta \) an open immersion, and the last isomorphism is because \( \pi_2 \Delta \) is the identity. Remark 2.7(i) therefore also generalizes, we have

(i) If \( X \xrightarrow{f} Y \xrightarrow{g} Z \) are composable morphisms with \( f \) étale, then \((gf)^! \xrightarrow{\rho} f^! g^! \cong f^! \) combines to an isomorphism. The map \( \rho(f, g) \) is an isomorphism because \( f \) is of finite Tor-dimension, while \( f^! \cong f^* \) is the isomorphism above.

3. Still formal, but less familiar—the way the abstract theory is related to explicit computations

Our first aim is to obtain a formula for \( f^! \) free of auxiliary choices—that is, one that does not involve factoring \( f \) as \( X \xrightarrow{u} X \xrightarrow{p} Y \) with \( u \) an open immersion and \( p \) proper. We begin with a little Lemma.

**Lemma 3.1.** Let \( U \xrightarrow{\alpha} V \xrightarrow{\beta} W \) be finite-type morphisms of noetherian schemes so that \( \alpha \) is a closed immersion and \( \beta \alpha \) is proper. Then the maps \( \alpha^\times \psi(\beta) : \alpha^\times \beta^\times \to \alpha^\times \beta^! \) and \( \alpha^* \psi(\beta) : \alpha^* \beta^\times \to \alpha^* \beta^! \) are both isomorphisms.

**Proof.** The proof is an easy consequence of Theorem 2.5 coupled with standard facts from support theory. First 2.5.2 gives us the commutative square

\[
\begin{array}{ccc}
(\beta \alpha)^\times & \xrightarrow{\delta(\alpha, \beta)} & \alpha^\times \beta^\times \\
\downarrow^{\psi(\beta \alpha)} & & \downarrow^{\psi(\alpha) \psi(\beta)} \\
(\beta \alpha)^! & \xrightarrow{\rho(\alpha, \beta)} & \alpha^! \beta^!
\end{array}
\]

Recalling that \( \alpha \) and \( \beta \alpha \) are proper maps, we conclude from 2.5.3 that \( \rho(\alpha, \beta) : (\beta \alpha)^! \to \alpha^! \beta^! \) is an isomorphism, as are \( \psi(\alpha) : \alpha^\times \to \alpha^! \) and \( \psi(\beta \alpha) : (\beta \alpha)^\times \to (\beta \alpha)^! \). Now
\[\delta(\alpha, \beta) \text{ is trivially an isomorphism, hence in the square above the indicated maps are all isomorphisms}\]

\[
\begin{array}{c}
(\beta \alpha)^X \xrightarrow{\sim} \alpha^X \beta^X \\
\downarrow i \quad \downarrow \psi(\alpha) \psi(\beta) \\
(\beta \alpha)^! \xrightarrow{\sim} \alpha^! \beta^!
\end{array}
\]

The commutativity implies that the vertical map \(\psi(\alpha) \psi(\beta) : \alpha^X \beta^X \to \alpha^! \beta^!\) must be an isomorphism. This isomorphism can be written as the composite

\[
\alpha^X \beta^X \xrightarrow{\alpha^X \psi(\beta)} \alpha^X \beta^! \xrightarrow{\psi(\alpha) \beta} \alpha^! \beta^!
\]

but, as \(\psi(\alpha)\) is an isomorphism, we conclude that \(\alpha^X \psi(\beta)\) is also an isomorphism. Support theory, more precisely \([19\text{, Proposition A.3(ii)}]\), tells us that \(\alpha^X \psi(\beta)\) is also an isomorphism.

We will apply this little lemma in the following situation.

**Construction 3.2.** Let \(f : X \to Y\) be a finite-type, flat morphism of noetherian schemes. We may form the diagram

\[
\begin{array}{c}
X \\
\Delta \downarrow \\
X \times_Y X \\
\pi_1 \downarrow \circlearrowleft (\circlearrowright) \Downarrow f \\
X \\
f \downarrow \Downarrow Y
\end{array}
\]

where the square is cartesian, \(\pi_1\) and \(\pi_2\) are the first and second projections, and \(\Delta : X \to X \times_Y X\) is the diagonal inclusion. We assert

**Proposition 3.3.** With the notation as in Construction 3.2 there is a canonical isomorphism \(\Delta^* \pi_2^X f^* \to f^!\).

**Proof.** We apply Lemma 3.1 to the composable maps \(X \xrightarrow{\Delta} X \times_Y X \xrightarrow{\pi_2} X\); the map \(\Delta\) is a closed immersion and the composite \(\text{id} = \pi_2 \Delta\) is proper, and Lemma 3.1 tells us that \(\Delta^* \psi(\pi_2) : \Delta^* \pi_2^X \to \Delta^* \pi_1^X\) is an isomorphism.

On the other hand all the maps in the cartesian square (\(\circlearrowright\)) are flat, and [2,5.6 ii) gives that \(\theta : \pi_1^* f^! \to \pi_2^! f^*\) is an isomorphism. Consider therefore the composite

\[

\begin{array}{c}
\Delta^* \pi_2^X f^* \\
\downarrow \Delta^* \psi(\pi_2) f^* \\
\Delta^* \pi_1^X f^* \\
\downarrow \Delta^* \theta^{-1} \\
\Delta^* \pi_1^1 f^! \\
\downarrow \sim \\
\text{id}^* f^! \\
\downarrow f^!
\end{array}
\]

The first and second maps are isomorphisms by the discussion above, and the third map comes from applying the functor \((-)^*\) to the equality \(\text{id} = \pi_1 \Delta\). The composite is therefore an isomorphism \(\Delta^* \pi_2^X f^* \to f^!\). \(\square\)
Construction 3.4. Assume \( f : X \to Y \) is flat and let the notation be as above. If \( \varepsilon : \Delta_2 \Delta^\times \to \text{id} \) is the counit of adjunction \( \Delta_2 \vdash \Delta^\times \), let \( \varphi : \Delta_2 \to \pi_2^X \) be the composite

\[
\Delta_2 \xrightarrow{\varepsilon \pi_2^X} \Delta_2 \Delta^\times \pi_2^X \xrightarrow{\pi_2^X} \pi_2^X
\]

where the equality is the observation that \( \text{id} = \text{id}^X = (\pi_2 \Delta)^{\times} \cong \Delta^\times \pi_2^X \). Define \( c_f : \Delta^* \Delta^*_f - \to \Delta^* \pi_2^X f^{\ast} \) to be \( \Delta^* \varphi f^{\ast} \); combining with the isomorphism \( \Delta^* \pi_2^X f^{\ast} \to f^! \) of Proposition 3.3 we arrive at a map which, in an abuse of notation, we will also denote \( c_f : \Delta^* \Delta^*_f - \to f^! \). If we apply this map to the object \( O_Y \in D_{\text{qc}}(Y) \) and note that \( f^! O_Y = O_X \), we obtain a map \( c_f(O_Y) : \Delta^* \Delta^*_f - \to f^! O_Y \). For any integer \( d \in \mathbb{Z} \) let \( \gamma_f(d) \) be the composite

\[
(\Delta^* \Delta^*_f \mathcal{O}_X)_{{\leq}d} \xrightarrow{c_f(O_Y)} \Delta^* \Delta^*_f \mathcal{O}_X \xrightarrow{f^! \mathcal{O}_Y} f^! \mathcal{O}_Y
\]

where the first map is given by the standard \( t \)-structure truncation.

Note that we have defined the maps in great generality, globally and without auxiliary choices of coordinates. What is known so far is

**Theorem 3.5.** Suppose the map \( f : X \to Y \) is smooth and of relative dimension \( d \). Then the map \( \gamma_f(d) \) is an isomorphism.

**Remark 3.6.** The object \( f^! \mathcal{O}_Y \) might appear mysterious but \( \Delta^* \Delta^*_f \mathcal{O}_X \) isn’t, it is just the obvious object in the derived category whose cohomology is the Hochschild homology of \( \mathcal{O}_X \). If \( f : X \to Y \) is smooth and of relative dimension \( d \) then \( (\Delta^* \Delta^*_f \mathcal{O}_X)_{\leq -d} \) is nothing other than \( HH^d(\mathcal{O}_X) \), which is the relative canonical bundle in degree \( -d \). In symbols

\[
\Omega^d_{X/Y}[-d] = (\Delta^* \Delta^*_f \mathcal{O}_X)_{\leq -d}.
\]

Note that the maps \( c_f \) and \( \gamma_f(d) \) are defined for any flat \( f \) and any integer \( d \), and might contain interesting information for \( f \) which aren’t smooth. I don’t believe anyone has computed examples yet.

**Remark 3.7.** The reader can find (different) proofs of Theorem 3.5 in either Alonso, Jeremías and Lipman [1, Proposition 2.4.2] or else in [25, §1]. The point we want to make here is that the proof can’t be hard: the map is defined globally, but proving it an isomorphism is local in \( Y \) in the flat topology—hence we may assume \( Y \) an affine scheme—and local in \( X \) in the étale topology. And étale-locally any smooth map of degree \( d \) is of the form \( \text{Spec}(S) \to \text{Spec}(R) \), where \( f : R \to S \) identifies \( S \) as the polynomial ring \( S = R[x_1, x_2, \ldots, x_d] \). Let \( S^e = S \otimes_R S \); since \( S \) is flat over \( R \) it makes no difference whether we view this particular tensor product as ordinary or derived. The expression \( \Delta^* \Delta^*_f \mathcal{O}_X \) is nothing other than the derived tensor product \( S \otimes_{S^e} S \), while

\(^1\text{To see that it suffices to prove the map an isomorphism flat-locally in } Y \text{ and étale-locally in } X \text{ one needs the full strength of Theorem 2.5, the extract we presented in this survey doesn’t suffice. The reader can find the complete statement in 23.}\)
\( f^! \mathcal{O}_Y \cong \Delta^* \pi^*_2 f^* \mathcal{O}_Y \) comes down to \( S \otimes_{\mathcal{S}^*} \text{Hom}_R(S, S) \), where the tensor and the Hom are both derived. And the map \( c_f : S \otimes_{\mathcal{S}^*} S \to S \otimes_{\mathcal{S}^*} \text{Hom}_R(S, S) \) is the tensor product over \( S^* \) of the identity map \( \text{id} : S \to S \) and the obvious inclusion \( S \to \text{Hom}_R(S, S) \).

OK: a little computation is necessary to finish off the proof, the details may be found in [25, §1].

**Construction 3.8.** Now let \( W \subset X \) be the union of closed subsets \( W_i \subset X \), such that the restriction of \( f \) to each \( W_i \) is proper. For every point \( x \in X \) write \( k(x) \) for its residue field; then the full subcategory \( \mathcal{D}_{\text{qc}, W}(X) \subset \mathcal{D}_{\text{qc}}(X) \) has objects given by the formula

\[
\mathcal{D}_{\text{qc}, W}(X) = \{ E \in \mathcal{D}_{\text{qc}}(X) \mid E \otimes k(x) = 0 \text{ for all } x \notin W \}.
\]

The inclusion \( I : \mathcal{D}_{\text{qc}, W}(X) \to \mathcal{D}_{\text{qc}}(X) \) is well-known to admit a Bousfield localization, meaning it has a right adjoint \( R : \mathcal{D}_{\text{qc}}(X) \to \mathcal{D}_{\text{qc}, W}(X) \). The composite functor \( \mathcal{D}_{\text{qc}}(X) \xrightarrow{R} \mathcal{D}_{\text{qc}, W}(X) \xrightarrow{I} \mathcal{D}_{\text{qc}}(X) \) is nowadays denoted \( \Gamma_W : \mathcal{D}_{\text{qc}}(X) \to \mathcal{D}_{\text{qc}}(X) \)

and this Bousfield localization is even smashing, meaning there is a natural isomorphism \( E \otimes \Gamma_W F \to \Gamma_W(E \otimes F) \). We assumed that, for each \( W_i \), the composite \( W_i \xrightarrow{\alpha_i} X \xrightarrow{f} Y \) is proper, and a slight refinement of Lemma [3.1] tells us that \( \alpha_i^* \psi(f) \) is an isomorphism for each \( \alpha_i \). Support theory, more precisely [19, Proposition A.3(ii)], allows us to deduce that the map \( \Gamma_W \psi(f) \) is also an isomorphism.

Let \( \varepsilon : \Gamma_W = IR \to \text{id} \) be the counit of the adjunction \( I \dashv R \) and let \( \varepsilon' : f_* f^\times \to \text{id} \) be the counit of the adjunction \( f_* \dashv f^\times \). We define the map \( \int_W : f_* \Gamma_W f^! \to \text{id} \) to be the composite

\[
\int_W : f_* \Gamma_W f^! \xrightarrow{f_* \Gamma_W \psi(f)} f_* \Gamma_W f^\times \xrightarrow{f_* \varepsilon f^\times} f_* f^\times \xrightarrow{\varepsilon'} \text{id}
\]

If we apply this natural transformation to the object \( \mathcal{O}_Y \in \mathcal{D}_{\text{qc}}(Y) \), and combine with the map \( \gamma_f(d) \), we obtain a composite

\[
f_* \Gamma_W [\Delta^* \Delta_* \mathcal{O}_X] \xrightarrow{\leq -d} f_* \Gamma_W \gamma_f(d) \xrightarrow{f_* \Gamma_W f^! \mathcal{O}_Y} f_* \Gamma_W f^! \mathcal{O}_Y \xrightarrow{\int_W} \mathcal{O}_Y
\]

Now assume \( f : X \to Y \) is smooth of relative dimension \( d \). Theorem [3.5] tells us that \( \gamma_f(d) \) is an isomorphism, and in Remark [3.6] we observed that \( [\Delta^* \Delta_* \mathcal{O}_X] \leq -d \) is nothing

\[2\text{Classically it was denoted } \text{R} \Gamma W \text{—it is the right derived functor of some functor on abelian categories. But we have been suppressing all the notation that usually reminds us of the functors of abelian categories that we are deriving, and in this case it brings our notation into concert with that of Benson, Iyengar and Krause [5, 7, 8, 9]. Their choice of the letter } \Gamma \text{ was quite unrelated to Grothendieck’s, see the comment at the top of } [5 \text{ page 582}]. \text{ By a fortuitous accident the notations coincide (once the } \text{R} \text{ is eliminated in } \text{R} \Gamma).} \]
more than the relative canonical bundle in degree $-d$, that is $\Omega^d_{X/Y}[-d] = [\Delta^* \Delta_* \mathcal{O}_X] \leq -d$.

We are assuming $W = \bigcup W_i$ is the union of closed subschemes $W_i \subset X$ so that, for each $i$, the composite $W_i \to X \xrightarrow{f} Y$ is proper. Let us make the stronger assumption that $f_{\alpha i}$ is a finite morphism for each $i$. Then the functor $f_* \Gamma_W$ takes an object $E \in D_{qc}(X)$ to its local cohomology at $W$ and, in the particular case of $E = \Omega^d_{X/Y}[-d]$, this comes down (locally and modulo boundaries) to the relative meromorphic $d$–forms $\omega/s_1 s_2 \cdots s_d$ on $X$ with $(s_1, s_2, \ldots, s_d)$ a relative system of parameters. And now we are ready for the next computation.

**Theorem 3.9.** Assume $f : X \to Y$ is smooth of relative dimension $d$. Then the composite

$$f_* \Gamma_W \Omega^d_{X/Y}[-d] \to f_* \Gamma_W [\Delta^* \Delta_* \mathcal{O}_X] \leq -d \quad \to \quad f_* \Gamma_W f^! \mathcal{O}_Y \to \mathcal{O}_Y$$

is just the map taking a relative meromorphic $d$–form to its residue.

**Remark 3.10.** Early incarnations of Theorem 3.9 may be found in Verdier [29, pp. 398-400] and Hübli and Sastry [17, Residue Theorem, p. 752]. The proof of Theorem 3.9, as stated here, may be found in [25, §2]. Once again we note that while the global definition might be slightly subtle, the local computation can easily be reduced to the simple, affine case of Remark 3.7.

We have mentioned, several times, that the functor $f^!$ is “better behaved” than its cousin $f^\wedge$. One facet is that it is amenable to local computations. To illustrate this we end the section with a couple of little lemmas. The first of the lemmas just records, for a morphism $f : X \to Y$, the open subsets $U \subset X$ which we will find useful for these local computations.

**Lemma 3.11.** Suppose $f : X \to Y$ is a finite-type morphism of noetherian schemes. Suppose $U \subset X$ is an open affine subset so that the composite $U \xrightarrow{u} X \xrightarrow{f} Y$ can be factored through an open affine subset $V \subset Y$. Then

(i) If $f$ is of finite Tor-dimension then $fu$ admits a factorization $fu = hg$, where $g$ is a closed immersion of finite Tor-dimension and $h$ is smooth.

(ii) For arbitrary $f$, the map $fu$ may be factored as $fu = khg$, where $g$ is an open immersion, $h$ is a closed immersion and $k$ is smooth and proper.

**Proof.** By hypothesis the composite $U \xrightarrow{u} X \xrightarrow{f} Y$ is equal to the composite $U \xrightarrow{\alpha} V \xrightarrow{\beta} Y$ with $V$ affine and $\beta$ an open immersion. Because $\alpha : U \to V$ is a finite-type morphism of affine schemes we may factor it as $U \xrightarrow{j'} \mathbb{P}_V^n \xrightarrow{\pi'} V$ where $j'$ is a locally closed immersion, and this allows us to factor $fu$ as $U \xrightarrow{j} \mathbb{P}_Y^n \xrightarrow{k} Y$ where $j$ is also a locally closed immersion.
Now we separate the treatment into two cases. If \( f \) is of finite Tor-dimension we factor
\[ j \text{ as } U \to W \to \mathbb{P}^n_W \] with \( g \) a closed immersion and \( \delta \) an open immersion. This gives us a factorization of \( fu \) as \( U \to W \to \mathbb{P}^n_W \). Since \( k \) and \( \delta \) are both smooth so is \( h = k\delta \).

And the fact that \( fu = hg \) is of finite Tor-dimension and \( h \) is smooth means that \( g \) must be of finite Tor-dimension. We have found a factorization satisfying (i) of the Lemma.

It remains to treat the case where \( f \) is arbitrary. In this case we factor \( j : U \to \mathbb{P}^n_W \)
as \( U \to W \to h \rightarrow \mathbb{P}^n_Y \), with \( g \) an open immersion and \( h \) a closed immersion. In total this factors \( fu \) as \( khg \), with \( k : \mathbb{P}^n_Y \to Y \) smooth and proper, \( h : W \to \mathbb{P}^n_Y \) a closed immersion and \( g : U \to W \) an open immersion. \( \square \)

**Lemma 3.12.** Let \( f : X \to Y \) be a finite-type, separated morphism of noetherian schemes. We record the following boundedness and coherence statements:

(i) There exists an integer \( n \) so that \( f^! \mathcal{D}_{qc}(Y)^{\geq 0} \subset \mathcal{D}_{qc}(X)^{\geq n} \).

(ii) If \( E \in \mathcal{D}_{qc}(Y) \) is bounded below and has coherent cohomology, then the same is true for \( f^! E \).

For the next few assertions assume furthermore that \( f \) is of finite Tor-dimension.

(iii) For any object \( E \in \mathcal{D}_{qc}(Y) \), the support of \( f^! E \) is equal to the support of \( f^* E \).

(iv) If \( E \in \mathcal{D}_{qc}(Y) \) has coherent cohomology then so has \( f^! E \) [no need to assume \( E \) bounded below].

(v) There exists an integer \( n \) so that \( f^! \mathcal{D}_{qc}(Y)^{\leq 0} \subset \mathcal{D}_{qc}(X)^{\leq n} \).

(vi) There exists an integer \( n \) so that, if the Tor-amplitude of \( E \in \mathcal{D}_{qc}(Y) \) is contained in the interval \([0, \infty)\), then the Tor-amplitude of \( f_* f^! E \) is contained in the interval \([n, \infty)\).

**Proof.** First we show that (vi) follows from (i) and Theorem 2.5. Suppose we know (i); we may choose an integer \( n \) with \( f^! \mathcal{D}_{qc}(Y)^{\geq 0} \subset \mathcal{D}_{qc}(X)^{\geq n} \). Let \( E \in \mathcal{D}_{qc}(Y) \) be an object with Tor-amplitude contained in \([0, \infty)\). Then \( \mathcal{D}_{qc}(Y)^{\geq 0} \otimes E \subset \mathcal{D}_{qc}(Y)^{\geq 0} \), and
\[ \mathcal{D}_{qc}(Y)^{\geq 0} \otimes f^! E = f^! [\mathcal{D}_{qc}(Y)^{\geq 0} \otimes E] \subset \mathcal{D}_{qc}(X)^{\geq n} \]
where the equality is by 2.5.4(ii). Applying \( f_* \) we deduce
\[ \mathcal{D}_{qc}(Y)^{\geq 0} \otimes f_* f^! E = f_* [f^* \mathcal{D}_{qc}(Y)^{\geq 0} \otimes f^! E] \subset f_* \mathcal{D}_{qc}(X)^{\geq n} \subset \mathcal{D}_{qc}(Y)^{\geq n} \]
where the equality is by the projection formula. We conclude that \( f_* f^! E \) has Tor-amplitude in the interval \([n, \infty)\).

It remains to prove (i)–(v), which are all local in \( X \). This means the following: cover \( X \) by open immersions \( u_i : U_i \subset X \). Remark 2.7(i) tells us that \( (fu_i)^! \cong u_i^* f^! \). If we had a cover so that we could prove (i)–(v) for all the \( (fu_i)^! \cong u_i^* f^! \), then the statement would follow for \( f \). Cover \( Y \) by finitely many open affine subsets \( V_i \), then cover each \( f^{-1}V_i \) by finitely many open affine subsets \( U_i \), and we have covered \( X \) by open subsets as in Lemma 3.11. We are therefore reduced to proving the Lemma under the assumption that \( f = fu \) has a factorization as in Lemma 3.11(i) or (ii).

We first prove (i), (ii), (iv) and (v), all of which respect composition: this means
(vii) If $X \xrightarrow{f} Y \xrightarrow{g} Z$ are morphisms so that the map $\rho(f, g) : (gf)^! \to f^!g^!$ is an isomorphism and (i) and (ii) hold for each of $f$ and $g$, then it formally follows that (i) and (ii) hold for $gf$.

(viii) If we furthermore assume that $f$ and $g$ are of finite Tor-dimension and (iv) and (v) hold for each of $f$ and $g$, then (iv) and (v) hold for $gf$.

Next we observe that the factorizations of Lemma 3.11 behave well with respect to the functor $(-)^!$, meaning

(ix) If $f = hg$ is a factorization as in Lemma 3.11(i) then $\rho(g, h) : (hg)^! \to g^!h^!$ is an isomorphism. This is because $g$ is of finite Tor-dimension, see 2.5.3.

(x) If $f = khg$ is a factorization as in Lemma 3.11(ii) then the map $(khg)^! \to g^!h^!k^!$ is an isomorphism. The map $\rho(hg, k) : (khg)^! \to (hg)^!k^!$ is an isomorphism because $k$ is proper, while the map $\rho(g, h) : (hg)^! \to g^!h^!$ is an isomorphism because $g$ is of finite Tor-dimension. See 2.5.3.

This means that it suffices to prove (i), (ii), (iv) and (v) in the special cases where $f$ is either smooth or a closed immersion, and for the proof of (iv) and (v) we may assume that the closed immersion is of finite Tor-dimension.

If $f : X \to Y$ is a smooth map then 2.5.3(ii) tells us that $f^!(-) \cong f^*(-) \otimes f^!O_X$, while from Theorem 3.5 we learn that $f^!O_Y$ is just a shift of the relative canonical bundle. Hence (i), (ii), (iv) and (v) are all true for smooth maps $f$.

Next we prove (i), (ii), (iv) and (v) for closed immersions $f$. By 2.5.3 we know that $\psi(f) : f^* \to f^!$ is an isomorphism. We need to prove the coherence and/or vanishing of the cohomology sheaves of $f^!E \cong f^*E$, and as $f$ is a closed immersion these are equivalent to the coherence and/or vanishing of the cohomology sheaves of $f_*f^*E$. Now recall the isomorphisms $f_*f^*(-) \cong f_*\mathcal{Hom}(\mathcal{O}_X, f^*(-)) \cong \mathcal{Hom}(f_*\mathcal{O}_X, -)$. If $f$ is of finite Tor-dimension then $f_*\mathcal{O}_X \in D_{qc}(Y)$ is a perfect complex, hence (iv) and (v) are clear. It remains to prove (i) and (ii), which become assertions about the vanishing and coherence of $\mathcal{Hom}(f_*\mathcal{O}_X, E)$ where $E$ is bounded below. Illusie [18, Proposition 3.7] tells us that this may be computed locally in $Y$, and if $Y$ is affine the assertions are obvious.

It remains to prove (iii), the assertion about the supports. The map $f$ is assumed of finite Tor-dimension and 2.5.3(ii) gives an isomorphism $f^!(E) \cong f^*(E) \otimes f^!\mathcal{O}_Y$. Support theory tells us that the support of the tensor product $f^*(E) \otimes f^!\mathcal{O}_Y$ is the intersection of the support of $f^*E$ and the support of $f^!\mathcal{O}_Y$, hence it suffices to show that the support of $f^!\mathcal{O}_Y$ is all of $X$. But we have reduced to the case where $f$ has a factorization $X \xrightarrow{g} W \xrightarrow{h} Y$ as in Lemma 3.11(i), and in (ix) we noted that $\rho(g, h) : (hg)^! \to g^!h^!$ is an isomorphism. Thus

$$(hg)^!\mathcal{O}_Y \cong g^![h^!\mathcal{O}_Y] \cong [g^*h^!\mathcal{O}_Y] \otimes g^!\mathcal{O}_W$$

where the last isomorphism is by 2.5.3(ii). We wish to show that the support of $(hg)^!\mathcal{O}_Y$ is all of $X$, and it suffices to prove that the support of $h^!\mathcal{O}_Y$ is all of $W$ and the support of $g^!\mathcal{O}_W$ is all of $X$. 

Theorem 3.5 tells us that \( h^! \mathcal{O}_Y \) is just a shift of the relative canonical bundle—its support is all of \( W \). Since \( g \) is a closed immersion the support of \( g^! \mathcal{O}_W \cong g^* \mathcal{O}_W \) is equal to the support of \( g_* g^* \mathcal{O}_W \cong \mathcal{H}om(g_* \mathcal{O}_X, \mathcal{O}_W) \). But as \( g \) has finite Tor-dimension the object \( g_* \mathcal{O}_X \) is a perfect complex on \( W \), and its support (all of \( X \)) is equal to the support of the dual \( [g_* \mathcal{O}_X]^\vee = \mathcal{H}om(g_* \mathcal{O}_X, \mathcal{O}_W) \). This completes the proof of (iii). \( \square \)

4. SOME BASIC ISOMORPHISMS

In this section we prove some formal corollaries of the theory presented so far, establishing that certain natural maps are isomorphisms. None of the results is hard, but they are a little technical—their value will only become apparent when we see the applications later on. The one result we will need, in \( \S 5 \) is Lemma 4.3. At a first reading we recommend that the reader study the statement of Lemma 4.3 and skip the rest of this section.

Lemma 4.1. Let \( f : X \to Y \) be a finite-type morphism of noetherian schemes, of finite Tor-dimension. Let \( E \in D_{qc}(Y) \) be a perfect complex, and let \( \{ A_\lambda, \lambda \in \Lambda \} \) be a set of objects of \( D_{qc}(Y) \). Then the natural map \( \prod_{\lambda \in \Lambda} f^* A_\lambda \otimes f^! E \to \prod_{\lambda \in \Lambda} [f^* A_\lambda \otimes f^! E] \) is an isomorphism.

Proof. We begin by proving a special case: we show that the Lemma is true \( f \) is proper and of finite Tor-dimension. Assume therefore that \( f \) is proper and of finite Tor-dimension. Then \( f_* \) takes perfect complexes to perfect complexes, and \( 2.5.4(\text{GN2}) \) tells us that \( f^* \) has a left adjoint and respects products. If we contemplate the commutative diagram

\[
\begin{array}{cccc}
\prod_{\lambda \in \Lambda} f^* A_\lambda & \otimes f^! E & \xrightarrow{(1)} & \prod_{\lambda \in \Lambda} f^* A_\lambda \otimes f^! E \\
\sigma & & & \sigma \\
\prod_{\lambda \in \Lambda} (A_\lambda \otimes E) & \xrightarrow{(3)} & \prod_{\lambda \in \Lambda} (A_\lambda \otimes E) & \xrightarrow{(4)} \prod_{\lambda \in \Lambda} f^! [A_\lambda \otimes E]
\end{array}
\]

the vertical maps \( \sigma \) are isomorphisms by \( 2.5.4(\text{ii}) \), the map (1) is an isomorphism because \( f^* \) respects products, the map (3) is an isomorphism because \( E \) is perfect and hence \( (\cdot) \otimes E \cong \mathcal{H}om(E^\vee, \cdot) \) respects products, and the map (4) is an isomorphism because, for the proper morphism \( f \), the functor \( f^! \cong f^* \) has a left adjoint and respects products. Hence the map (2) must be an isomorphism, as we asserted.

We have proved the Lemma if \( f \) is proper and of finite Tor-dimension. Now suppose \( f \) may be factored as \( X \xrightarrow{g} W \xrightarrow{h} Y \) with \( h \) smooth and \( g \) proper and of finite Tor-dimension. Because \( h \) is of finite Tor-dimension \( 2.5.4 \) gives an isomorphism \( h^* E \otimes h^! \mathcal{O}_Y \to h^! E \), while Theorem 3.5 tells us that \( h^! \mathcal{O}_Y \) is a shift of the relative canonical
bundle. Since $E$ is assumed perfect it follows that $h^i E$, being the tensor product of two perfect complexes $h^* E$ and $h^i \mathcal{O}_Y$, must be perfect. But then we may apply the special case of the Lemma to the map $g : X \to W$, the perfect complex $h^i E \in D_{\text{qc}}(W)$ and the set of objects \{ $h^* A_\lambda, \lambda \in \Lambda$ \} of $D_{\text{qc}}(W)$. We deduce that the map (2) in the commutative square below is an isomorphism

$$
\begin{array}{ccc}
\prod_{\lambda \in \Lambda} [(hg)^* A_\lambda] \otimes (hg)^i E & \xrightarrow{(1)} & \prod_{\lambda \in \Lambda} [(h^* A_\lambda) \otimes (h^i E) \\
\tau(g,h) \otimes \rho(g,h) & & \tau(g,h) \otimes \rho(g,h)
\end{array}
$$

The vertical maps are induced by the isomorphism $\tau(g,h) : (hg)^* \to h^* h^i$ and the map $\rho(g,h) : (hg)^i \to g^i h^*$ of [2.5.1]. Because $g$ is of finite Tor-dimension [2.5.3] informs us that $\rho(g,h)$ is an isomorphism, hence both vertical maps are isomorphisms. From the commutativity we deduce that (1) is an isomorphism. In other words: the Lemma is true for any $f$ which admits a factorization as $X \xrightarrow{g} W \xrightarrow{h} Y$ with $h$ smooth and $g$ proper and of finite Tor-dimension.

Now let $f : X \to Y$ be arbitrary, fix a perfect complex $E \in D_{\text{qc}}(Y)$, as well as a set of objects \{ $A_\lambda, \lambda \in \Lambda$ \} of $D_{\text{qc}}(Y)$. If $u : U \to X$ is an open immersion with $U$ affine, and assuming that $fu : U \to Y$ can be factored through an open affine subset $V \subset Y$, then Lemma [3.11](i) guarantees that $fu$ may be factored as $hg$ with $h$ smooth and $g$ proper and of finite Tor-dimension. By the above the natural map $\prod_{\lambda \in \Lambda} (fu)^* A_\lambda \otimes (fu)^i E \to \prod_{\lambda \in \Lambda} ((fu)^* A_\lambda \otimes (fu)^i E)$ is an isomorphism. Remark [2.7](i) gives an isomorphism $(fu)^i \cong u^* f^i$, allowing us to rewrite the isomorphism above as $\prod_{\lambda \in \Lambda} (fu)^* A_\lambda \otimes u^* f^i E \to \prod_{\lambda \in \Lambda} ((fu)^* A_\lambda \otimes u^* f^i E)$. If we apply the functor $u_*$ and use the projection formula and the fact that $u_*$ has a left adjoint and hence respects products, we obtain that the natural map is an isomorphism $\prod_{\lambda \in \Lambda} u_* (u^* f^* A_\lambda) \otimes f^i E \to \prod_{\lambda \in \Lambda} [u_* u^* f^* A_\lambda \otimes f^i E].$

Now glue these isomorphisms: any time we have open immersions $u : U \to X$, $v : V \to X$, $j : U \cap V \to X$ and $w : U \cup V \to X$, then for each $\lambda$ we obtain a triangle

$$
w_* w^* f^* A_\lambda \longrightarrow u_* u^* f^* A_\lambda \otimes v_* v^* f^* A_\lambda \longrightarrow j_* j^* f^* A_\lambda
$$

Taking the product over $\lambda$ and tensoring with $f^i E$ gives a triangle, as does tensoring with $f^i E$ and then forming the product over $\lambda$. There is a map between the triangles: if two of the morphisms are isomorphisms then so is the third. Starting with the fact that we know the map to be an isomorphism if $U \subset X$ is a sufficiently small open affine, we glue to discover that it is an isomorphism for every open immersion $u : U \to X$. The case of the identity map $\text{id} : X \to X$ gives the Lemma. \qed
Lemma 4.2. As in Lemma 3.1 let $U \xrightarrow{\alpha} V \xrightarrow{\beta} W$ be finite-type morphisms of noetherian schemes so that $\alpha$ is a closed immersion and $\beta \alpha$ proper. Given a set of objects $\{A_\lambda, \lambda \in \Lambda\}$ in the category $D_{qc}(W)$, then the functors $\alpha^*$ and $\alpha^\times$ take the natural morphism

$$\beta! \left[ \prod_{\lambda \in \Lambda} A_\lambda \right] \xrightarrow{J} \prod_{\lambda \in \Lambda} \beta^! A_\lambda$$

to isomorphisms. If $\beta$ is of finite Tor-dimension the functors $\alpha^*$ and $\alpha^\times$ also take the natural morphism

$$\beta^* \left[ \prod_{\lambda \in \Lambda} A_\lambda \right] \xrightarrow{K} \prod_{\lambda \in \Lambda} \beta^* A_\lambda$$

to isomorphisms.

Proof. Consider the commutative square

$$\beta^\times \left[ \prod_{\lambda \in \Lambda} A_\lambda \right] \xrightarrow{I} \prod_{\lambda \in \Lambda} \beta^\times A_\lambda$$

The map $I$ is an isomorphism, after all $\beta^\times$ is a right adjoint and respects products. If we apply the functor $\alpha^\times$ and recall (i) that it respects products and (ii) that $\alpha^\times \psi(\beta)$ is an isomorphism by Lemma 3.1, then we deduce the commutative diagram where the indicated maps are isomorphisms

$$\alpha^\times \beta^\times \left[ \prod_{\lambda \in \Lambda} A_\lambda \right] \xrightarrow{\sim} \alpha^\times \left[ \prod_{\lambda \in \Lambda} \beta^\times A_\lambda \right]$$

Hence $\alpha^\times J$ must be an isomorphism. Support theory, more concretely [19, Proposition A.3(ii)], tells us that $\alpha^* J$ is also an isomorphism.

Now assume that $\beta$ is of finite Tor-dimension. We wish to show that the maps $\alpha^* K$ and $\alpha^\times K$ are isomorphisms, with $K$ as in the Lemma, and by [19, Proposition A.3(ii)] it suffices to consider $\alpha^* K$. 
From \[2.5.4\](ii) we have a natural isomorphism \(\beta^*(-) \otimes \beta^! \mathcal{O}_W \rightarrow \beta^!(-)\), hence the map \(J\) of the Lemma is isomorphic to the composite

\[
\beta^* \left[ \prod_{\lambda \in \Lambda} A_\lambda \right] \otimes \beta^! \mathcal{O}_W \xrightarrow{K \otimes \beta^! \mathcal{O}_W} \left[ \prod_{\lambda \in \Lambda} \beta^* A_\lambda \right] \otimes \beta^! \mathcal{O}_W \xrightarrow{H} \left[ \prod_{\lambda \in \Lambda} \beta^* A_\lambda \otimes \beta^! \mathcal{O}_W \right]
\]

Lemma \[4.1\] tells us that the map \(H\) is an isomorphism, but by the first part of the Lemma, which we already proved, we know that \(\alpha^*\) takes the composite to an isomorphism. Hence \(\alpha^*\) must take the map \(K \otimes \beta^! \mathcal{O}_W\) to an isomorphism, or to put it differently \((-) \otimes \alpha^* \beta^! \mathcal{O}_W\) takes \(\alpha^* K\) to an isomorphism. We wish to show that \(\alpha^* K\) is an isomorphism and support theory, more precisely \[19\] Proposition A.3(ii)], tells us that is suffices to show that every point of \(U\) lies in the support of \(\alpha^* \beta^! \mathcal{O}_W\). Since the support of \(\alpha^* E\) is \(\alpha^{-1} \text{supp}(E)\) it certainly suffices to show that \(\text{supp}(\beta^! \mathcal{O}_W) = V\). But \(\beta\) is of finite Tor-dimension, and Lemma \[4.3\](ii) tells us that the support of \(\beta^! \mathcal{O}_W\) is equal to the support of \(\beta^* \mathcal{O}_W = \mathcal{O}_V\), which is all of \(V\).

**Lemma 4.3.** Let \(U \xrightarrow{\alpha} V \xrightarrow{\beta} W\) be finite-type morphisms of noetherian schemes so that \(\alpha\) is a closed immersion, \(\beta\) is of finite Tor-dimension, and \(\beta \alpha\) proper. Suppose \(E, F \in \mathbf{D}_{\text{qc}}(W)\) are any objects. Then \(\alpha^*\) and \(\alpha^\times\) take the natural map \(\beta^* \text{Hom}(E, F) \rightarrow \text{Hom}(\beta^* E, \beta^* F)\) to an isomorphism.

**Proof.** Support theory, more concretely \[19\] Proposition A.3(ii)], tells us that \(\alpha^*\) will take the map to an isomorphism if and only if \(\alpha^\times\) does. It suffices to prove the assertion for \(\alpha^\times\).

Fix an object \(F \in \mathbf{D}_{\text{qc}}(W)\) and let \(\mathcal{L}\) be the full subcategory of all objects \(E \in \mathbf{D}_{\text{qc}}(W)\) such that \(\alpha^\times\) takes the map \(p_E : \beta^* \text{Hom}(E, F) \rightarrow \text{Hom}(\beta^* E, \beta^* F)\) to an isomorphism. If \(E\) is a perfect complex then the map \(p_E\) is an isomorphism as it stands, hence all perfect complexes belong to \(\mathcal{L}\). Also \(\mathcal{L}\) is obviously triangulated. Because \(\mathbf{D}_{\text{qc}}(W)\) is compactly generated it suffices to prove that \(\mathcal{L}\) is localizing; we need to show that the coproduct of any set of objects in \(\mathcal{L}\) lies in \(\mathcal{L}\).

Assume therefore that \(\{E_\lambda, \lambda \in \Lambda\}\) is a set of objects of \(\mathcal{L}\). We wish to study the map

\[
\beta^* \text{Hom} \left( \left( \prod_{\lambda \in \Lambda} E_\lambda \right), F \right) \xrightarrow{p_{E|E}} \text{Hom} \left( \beta^* \left( \prod_{\lambda \in \Lambda} E_\lambda \right), \beta^* F \right)
\]

which, up to isomorphism, rewrites as the composite

\[
\beta^* \prod_{\lambda \in \Lambda} \text{Hom}(E_\lambda, F) \xrightarrow{K} \prod_{\lambda \in \Lambda} \beta^* \text{Hom}(E_\lambda, F) \xrightarrow{\prod_{\lambda \in \Lambda} p_{E|E}} \prod_{\lambda \in \Lambda} \text{Hom}(\beta^* E_\lambda, \beta^* F)
\]

The fact that \(\alpha^\times K\) is an isomorphism is by Lemma \[4.2\]. The fact that \(\alpha^\times \left[ \prod_{\lambda} p_{E|\lambda} \right]\) is an isomorphism is because \(\alpha^\times\) respects products, and takes each \(p_{E|\lambda}\) to an isomorphism. \(\square\)
5. Some more Hochschild-style formulas

To the extent that the results surveyed in [2] and [3] are new, they arose out of trying to understand the magical formulas first discovered by Avramov and Iyengar [2], and developed further in several papers, starting with Avramov, Iyengar, Lipman and Nayak [3]. In this section we prove some of these formulas—what is novel is that they are stated and proved in the unbounded derived category. Since the results in the section are new—at least new in this generality—we present complete proofs. The reader interested in the highlights can skip ahead to Corollary 5.7 and Example 5.8.

Reminder 5.1. Let \( f : X \to Y \) be a morphism of noetherian schemes. An object \( M \in \mathbf{D}_{\mathrm{qc}}(X) \) is called \( f \)-perfect if

(i) \( M \) belongs to \( \mathbf{D}^b_{\mathrm{coh}}(X) \); that is all but finitely many of the cohomology sheaves vanish, and the non-vanishing ones are coherent.

(ii) There exists an affine scheme \( W \) and a faithfully flat morphism \( \rho : W \to X \) so that \((f\rho)_* \rho^* M \) is of finite Tor-amplitude.

Lemma 5.2. Let \( Z \to X \to Y \) be composable morphisms, with \( i \) a closed immersion and \( f \) proper. Suppose that \( M \) is an \( f \)-perfect object in \( \mathbf{D}_{\mathrm{qc}}(X) \) and \( C \in \mathbf{D}_{\mathrm{qc}}(X) \) is a perfect complex supported on the image of \( i \). Then \( f_* (M \otimes C) \) is a perfect complex in \( \mathbf{D}_{\mathrm{qc}}(Y) \).

Proof. We need to show that \( f_* (M \otimes C) \) has coherent cohomology and is of finite Tor-amplitude. The coherence of the cohomology is clear: possibly after replacing \( Z \) by an infinitesimal thickening we can assume that the complex \( C \in \mathbf{D}^b_{\mathrm{coh}}(X) \), whose cohomology is supported is on \( Z \), is of the form \( i_* \tilde{C} \) for some \( \tilde{C} \in \mathbf{D}^b_{\mathrm{coh}}(Z) \). But then \( M \otimes C = M \otimes i_* \tilde{C} \cong i_* (i^* M \otimes \tilde{C}) \), and hence \( f_* (M \otimes C) \cong f_* i_* (i^* M \otimes \tilde{C}) \). But the complex \( i^* M \otimes \tilde{C} \in \mathbf{D}_{\mathrm{qc}}(Z) \) has coherent cohomology, and as \( fi : Z \to Y \) is assumed proper so does \( f_* i_* (i^* M \otimes \tilde{C}) \cong f_* (M \otimes C) \).

Now for the Tor-amplitude. Choose a faithfully flat map \( \rho : W \to X \) as in Reminder 5.1. We know that there exists an integer \( n \) so that the Tor-amplitude of \((f\rho)_* \rho^* M \) lies in the interval \([-n, \infty)\), and therefore

\[
(f\rho)_* \left( (f\rho)^* \mathbf{D}_{\mathrm{qc}}(Y)^{\geq 0} \otimes \rho^* M \right) = \mathbf{D}_{\mathrm{qc}}(Y)^{\geq 0} \otimes (f\rho)_* \rho^* M
\]

is contained in \( \mathbf{D}_{\mathrm{qc}}(Y)^{\geq -n} \). The map \( f\rho \) is a morphism from an affine scheme to \( Y \), therefore \( f\rho \) is quasi-affine. From the proof [not the statement] of [14] Corollary 2.8] we deduce that \( (f\rho)^* \mathbf{D}_{\mathrm{qc}}(Y)^{\geq 0} \otimes \rho^* M = \rho^* \left[ f^* \mathbf{D}_{\mathrm{qc}}(Y)^{\geq 0} \otimes M \right] \) is contained in \( \mathbf{D}_{\mathrm{qc}}(W)^{\geq -n} \). Since \( \rho^* \) is faithfully flat it follows that \( f^* \mathbf{D}_{\mathrm{qc}}(Y)^{\geq 0} \otimes M \) is contained in \( \mathbf{D}_{\mathrm{qc}}(X)^{\geq -n} \). As \( C \) is assumed to be a perfect complex on \( X \) its Tor-amplitude is contained in the interval \([-m, m]\) for some \( m > 0 \), and hence \( f^* \mathbf{D}_{\mathrm{qc}}(Y)^{\geq 0} \otimes M \otimes C \) is contained in \( \mathbf{D}_{\mathrm{qc}}(X)^{\leq -m-n} \). But then

\[
f_* \left[ f^* \mathbf{D}_{\mathrm{qc}}(Y)^{\geq 0} \otimes M \otimes C \right] = \mathbf{D}_{\mathrm{qc}}(Y)^{\geq 0} \otimes f_* (M \otimes C)
\]
is contained in $f_*\mathbf{D}_{qc}(X)^{\geq -m-n} \subset \mathbf{D}_{qc}(Y)^{\geq -m-n}$, and we deduce that the Tor-amplitude of $f_*(M \otimes C)$ is contained in $[-m-n, \infty)$. \hfill \square

**Reminder 5.3.** Given any closed symmetric monoidal category $M$, that is a category $M$ with a symmetric tensor product and an internal Hom satisfying the usual adjunction, there is a canonical evaluation map $ev_{A,B} : A \otimes \mathsf{Hom}(A,B) \to B$. As is customary we deduce the following two canonical maps

$$\mathsf{Hom}(A,B) \otimes C \xrightarrow{\alpha} \mathsf{Hom}(A,B \otimes C)$$

$$A \otimes \mathsf{Hom}(B,C) \xrightarrow{\beta} \mathsf{Hom}[\mathsf{Hom}(A,B), C]$$

where $\alpha = \alpha(A,B,C)$ corresponds to the map

$$A \otimes \mathsf{Hom}(A,B) \otimes C \xrightarrow{ev_{A,B}} B \otimes C$$

and $\beta = \beta(A,B,C)$ corresponds to the composite

$$\mathsf{Hom}(A,B) \otimes A \otimes \mathsf{Hom}(B,C) \xrightarrow{ev_{A,B}} B \otimes \mathsf{Hom}(B,C) \xrightarrow{ev_{B,C}} C$$

An easy formal fact is that $\alpha(A,B,C)$ and $\beta(A,B,C)$ are isomorphisms as long as $A$ is strongly dualizable.

**Construction 5.4.** We can of course use the basic maps of Reminder 5.3 to construct variants. The situation that interests us is where $f : X \to Y$ is a morphism of schemes and $f^* \dashv f_* \dashv f^\times$ are the usual adjoint functors between $\mathbf{D}_{qc}(X)$ and $\mathbf{D}_{qc}(Y)$. If $B,C \in \mathbf{D}_{qc}(Y)$ are any objects, then the composite

$$f^*\mathsf{Hom}(B,C) \otimes f^\times B \xrightarrow{\chi(\mathsf{Hom}(B,C),B)} f^\times [\mathsf{Hom}(B,C) \otimes B] \xrightarrow{f^*ev_{B,C}} f^\times C$$

induces by adjunction a map we will denote $\gamma : f^*\mathsf{Hom}(B,C) \to \mathsf{Hom}(f^\times B, f^\times C)$. If $A \in \mathbf{D}_{qc}(X)$ and $B,C \in \mathbf{D}_{qc}(Y)$ are objects, we can consider the composites

$$\mathsf{Hom}(A,f^\times B) \otimes f^\times C \xrightarrow{\alpha} \mathsf{Hom}(A,f^\times B \otimes f^\times C) \xrightarrow{\mathsf{Hom}(A,\chi)} \mathsf{Hom}[A,f^\times (B \otimes C)]$$

$$A \otimes f^*\mathsf{Hom}(B,C) \xrightarrow{id \otimes \gamma} A \otimes \mathsf{Hom}(f^\times B, f^\times C) \xrightarrow{\beta} \mathsf{Hom}[\mathsf{Hom}(A,f^\times B), f^\times C]$$

**Lemma 5.5.** Let $Z \xrightarrow{i} X \xrightarrow{f} Y$ be composable morphism of schemes with $i$ a closed immersion and $f$ a quasi-affine map. Let $A \in \mathbf{D}_{qc}(X)$ be an object, and let $K \in \mathbf{D}_{qc}(X)$ be a perfect complex supported on the closed subset $Z$, and so that $f_*(A \otimes K)$ and $f_*(A \otimes K^\vee)$ are perfect in $\mathbf{D}_{qc}(Y)$, where $K^\vee = \mathsf{Hom}(K,0)$ is the dual of $K$. Then, with $B,C \in \mathbf{D}_{qc}(Y)$ arbitrary, the functor $K \otimes (-)$ takes the two composites at the end of Construction 5.4 to isomorphisms.
Proof. Because $K$ is perfect and supported on $Z$ it is isomorphic to a bounded complex of coherent sheaves supported on $Z$. Up to replacing $Z$ by an infinitesimal thickening we may assume there exists an object $\tilde{K} \in D^b_{\text{coh}}(Z)$ with $K \cong i_* \tilde{K}$. But then we have isomorphisms of functors $K \otimes (-) \cong i_* \tilde{K} \otimes (-) \cong i_* [\tilde{K} \otimes i^*(-)]$. It certainly suffices to prove that $\tilde{K} \otimes i^*(-)$ takes the maps at the end of Construction 5.4 to isomorphisms.

The morphism $fi$ is quasi-affine and \cite{[14], Corollary 2.8} tells us that $(fi)_* = f_* i_*$ is conservative, hence we are reduced to showing that $f_* i_* [\tilde{K} \otimes i^*(-)] \cong f_* [K \otimes (-)]$ takes these two morphisms of Construction 5.4 to isomorphisms.

But now we are in business: tensoring these two maps with the strongly dualizable $K$ has the effect of replacing $A$ with $A \otimes K^\vee$ in the first map and with $A \otimes K$ in the second. Hence we are reduced to showing that, under the assumption that $f_* A$ is a perfect complex in $D_{\text{qc}}(Y)$, the functor $f_*$ takes both of the maps of Construction 5.4 to isomorphisms. An easy exercise with the standard isomorphisms $f_* (R \otimes f^* S) \cong f_* R \otimes S$ and $f_* \text{Hom}(R, f^* S) \cong \text{Hom}(f_* R, S)$ allows us to show that the functor $f_*$ takes the two maps of Construction 5.4 to isomorphisms of Reminder 5.3, which are isomorphisms because $f_* A$ is perfect. \hfill $\Box$

Now we will apply our lemmas to the situation of Construction 3.2. We remind the reader: $f : X \to Y$ is a finite-type, flat morphism of noetherian schemes, and we formed the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\Delta} & X \\
\downarrow \pi_2 & & \downarrow f \\
X \times_Y X & \xrightarrow{\pi_1} & X \\
\end{array}
\]

where the square is cartesian, $\pi_1$ and $\pi_2$ are the first and second projections, and $\Delta : X \to X \times_Y X$ is the diagonal inclusion. We assert:

**Proposition 5.6.** Let $A, B, C \in D_{\text{qc}}(X)$ be objects and assume that $A$ is $f$–perfect. Applying Construction 5.4 to the morphism $\pi_2 : X \times_Y X \to X$ and the objects $\pi_1^* A \in D_{\text{qc}}(X \times_Y X)$ and $B, C \in D_{\text{qc}}(X)$, we obtain morphisms

\[
\begin{array}{c}
\text{Hom}(\pi_1^* A, \pi_2^* B) \otimes \pi_2 C \xrightarrow{\text{Hom}(\pi_1^* A, B \otimes C)} \text{Hom} \left[ \pi_1^* A, \pi_2^* (B \otimes C) \right] \\
\pi_1^* A \otimes \pi_2 ^* \text{Hom}(B, C) \xrightarrow{\beta \circ [\text{id} \otimes \gamma]} \text{Hom} \left[ \text{Hom}(\pi_1^* A, \pi_2^* B), \pi_2^* C \right]
\end{array}
\]
We assert that, if \( L \in D_{qc}(X \times_Y X) \) is any object supported on the diagonal, then the functors \( L \otimes (-) \) and \( \mathcal{H}om(L, -) \) take both maps to isomorphisms, as do the functors \( \Delta^* \) and \( \Delta^X \).

**Proof.** Let us first observe that the statement about \( \Delta^* \) and \( \Delta^X \) follows from the assertion about \( L \otimes (-) \) and \( \mathcal{H}om(L, -) \). Since \( \Delta \) is a closed immersion the functor \( \Delta_* \) is conservative, and to show that \( \Delta^* \) and \( \Delta^X \) take the two maps to isomorphisms is equivalent to showing that the composites \( \Delta_* \Delta^* \) and \( \Delta_* \Delta^X \) take them to isomorphisms. But it is standard that \( \Delta_* \Delta^* (-) \cong \Delta_* O_X \otimes (-) \) and \( \Delta_* \Delta^X (-) \cong \mathcal{H}om(\Delta_* O_X, -) \). Since \( \Delta_* O_X \) is supported on the diagonal this reduces us to the statements about \( L \otimes (-) \) and \( \mathcal{H}om(L, -) \).

Now let \( \mathcal{L} \subset D_{qc}(X \times_Y X) \) be the full subcategory of all objects \( L \) so that \( L \otimes (-) \) and \( \mathcal{H}om(L, -) \) take both maps of the Proposition to isomorphisms. Clearly \( \mathcal{L} \) is a localizing subcategory. We wish to show that \( \mathcal{L} \) contains the category \( D_{qc,\Delta}(X \times_Y X) \), that is the full subcategory of \( D_{qc}(X \times_Y X) \) of objects supported on the diagonal. But the subcategory \( D_{qc,\Delta}(X \times_Y X) \) is generated by the objects inside it which are compact in the larger \( D_{qc}(X \times_Y X) \); this theorem was first proved in Thomason and Trobaugh [27], and for a more general, modern proof which works for sufficiently nice algebraic stacks the reader can see Hall and Rydh [14, Theorems A, B and 4.10(2)]. This means that any localizing subcategory, containing the compact objects \( K \) supported on the diagonal, will contain all of \( D_{qc,\Delta}(X \times_Y X) \). It therefore suffices to show that every compact \( K \), supported on the diagonal, belongs to \( \mathcal{L} \). Hence we let \( K \) be a compact object supported on the diagonal, and wish to show that \( K \otimes (-) \) and \( \mathcal{H}om(K, -) \cong K^\vee \otimes (-) \) take both maps in the Proposition to isomorphisms.

Now the object \( A \in D_{qc}(X) \) is assumed \( f \)-perfect, and flat base-change tells us that \( \pi_1^* A \) is \( \pi_2 \)-perfect. Consider the composable morphisms \( X \xrightarrow{\Delta} X \times_Y X \xrightarrow{\pi_2} X \); because the composite id = \( \pi_2 \Delta \) is proper we may apply Lemma 5.2 and because it is quasi-affine Lemma 5.6 also applies. More precisely: with this pair of composable morphisms apply Lemma 5.2 to the \( \pi_2 \)-perfect object \( \pi_1^* A \in D_{qc}(X \times_Y X) \) and to the perfect complexes \( K, K^\vee \in D_{qc}(X \times_Y X) \) supported on the image of \( \Delta \), and we learn that \( \pi_2_* (\pi_1^* A \otimes K) \) and \( \pi_2_* (\pi_1^* A \otimes K^\vee) \) are perfect in \( D_{qc}(X) \). But then Lemma 5.5 allows us to conclude that \( K \otimes (-) \) and \( \mathcal{H}om(K, -) \cong K^\vee \otimes (-) \) take both maps of the Proposition to isomorphisms.

**Corollary 5.7.** Let \( f : X \to Y \) be a finite-type, flat map of noetherian schemes, and let the notation be as in Construction 3.2. For objects \( A, C \in D_{qc}(X) \) and \( B \in D_{qc}(Y) \), where \( A \) is \( f \)-perfect, we have isomorphisms

\[
\mathcal{H}om(A, f^! B) \otimes C \cong \Delta^* \mathcal{H}om \left[ \pi_1^* A, \pi_2^* (f^* B \otimes C) \right]
\]

\[
\Delta^X \left[ \pi_1^* A \otimes \pi_2^* \mathcal{H}om (f^* B, C) \right] \cong \mathcal{H}om \left[ \mathcal{H}om(A, f^! B), C \right]
\]
Proof. The classical isomorphism \( \Delta^* \mathcal{H} \text{om}(E, F) \cong \mathcal{H} \text{om}(\Delta^* E, \Delta^* F) \), coupled with the fact that \( \Delta^* \psi(\pi_2) : \Delta^* \pi_2^X \rightarrow \Delta^* \pi_2^1 \) is an isomorphism by Lemma 3.1, tell us that \( \Delta^* \) isomorphism. The isomorphism of \( \bullet \) first step we prove that for any \( \Delta \times \pi_2^X G \rangle \) isomsiphm. The isomorphism of \( \bullet \) first step we prove that for any \( \Delta \times \pi_2^X G \rangle \) isomorphism. The isomorphism of \( \bullet \) first step we prove that for any \( \Delta \times \pi_2^X G \rangle \) isomorphism. By [19, Proposition A.3(ii)] we deduce that \( \Delta^* \mathcal{H} \text{om}(E, \psi(\pi_2)) : \Delta^* \mathcal{H} \text{om}(E, \pi_2^X G) \rightarrow \Delta^* \mathcal{H} \text{om}(E, \pi_2^1 G) \) is an isomorphism. By \[ \pi_2^X G \rangle \) isomorphism in \( \bullet \) first step we prove that for any \( \Delta \times \pi_2^X G \rangle \) isomorphism. The isomorphism of \( \bullet \) first step we prove that for any \( \Delta \times \pi_2^X G \rangle \) isomorphism. By [19, Proposition A.3(ii)] we deduce that \( \Delta^* \mathcal{H} \text{om}(E, \psi(\pi_2)) : \Delta^* \mathcal{H} \text{om}(E, \pi_2^X G) \rightarrow \Delta^* \mathcal{H} \text{om}(E, \pi_2^1 G) \) is also an isomorphism.

Now we turn to the proof of the Corollary. With the notation of the Corollary, as a first step we prove

- There is a natural isomorphism \( \Delta^* \left[ \mathcal{H} \text{om}(\pi_1^* A, \pi_2^X f^* B) \right] \cong \mathcal{H} \text{om}(A, f^j B) \).

The isomorphism of \( \bullet \) comes from the following string of isomorphisms

\[
\Delta^* \left[ \mathcal{H} \text{om}(\pi_1^* A, \pi_2^X f^* B) \right] \cong \Delta^* \left[ \mathcal{H} \text{om}(\pi_1^* A, \pi_2^1 f^* B) \right] \\
\cong \Delta^* \mathcal{H} \text{om}(\pi_1^* A, \pi_2^1 f^* B) \\
\cong \Delta^* \pi_2^1 \mathcal{H} \text{om}(A, f^j B) \\
\cong \mathcal{H} \text{om}(A, f^j B)
\]

The first isomorphism is because the functor \( \Delta^* \mathcal{H} \text{om}(\pi_1^* A, -) \) takes the map \( \psi(\pi_2) : \pi_2^X f^* B \rightarrow \pi_2^1 f^* B \) to an isomorphism. The second isomorphism is because \( \theta(\Omega) : \pi_1^i f^j \rightarrow \pi_2^1 f^* \) is an isomorphism, see [25.6(ii)]. The third isomorphism in by Lemma 1.3 and the last isomorphism is because \( \Delta^* \pi_2^1 \cong \text{id}^* \).

With the preliminaries out of the way, apply Proposition 5.6 to the objects \( A, f^* B, C \in \mathcal{D}_{qc}(X) \), where \( A \) is given to be \( f \)-perfect. The Proposition tells us that the functors \( \Delta^* \) and \( \Delta^X \) take the maps below to isomorphisms

\[
\begin{align*}
\mathcal{H} \text{om}(\pi_1^* A, \pi_2^X f^* B) \otimes \pi_2^1 C & \xrightarrow{(1)} \mathcal{H} \text{om} \left[ \pi_1^1 A, \pi_2^X (f^* B \otimes C) \right] \\
\pi_1^1 A \otimes \pi_2^1 \mathcal{H} \text{om}(f^* B, C) & \xrightarrow{(2)} \mathcal{H} \text{om} \left[ \mathcal{H} \text{om}(\pi_1^* A, \pi_2^X f^* B), \pi_2^1 C \right]
\end{align*}
\]

And the first isomorphism of the Corollary is by applying \( \Delta^* \) to the map (1) while the second isomorphism is by applying \( \Delta^X \) to the map (2). Let us take these one step at a time, we begin be applying \( \Delta^* \) to (1). We obtain isomorphisms

\[
\begin{align*}
\Delta^* \mathcal{H} \text{om} \left[ \pi_1^* A, \pi_2^X (f^* B \otimes C) \right] & \cong \Delta^* \left[ \mathcal{H} \text{om}(\pi_1^* A, \pi_2^X f^* B) \otimes \pi_2^1 C \right] \\
& \cong \Delta^* \left[ \mathcal{H} \text{om}(\pi_1^* A, \pi_2^X f^* B) \right] \otimes [\Delta^* \pi_2^1 C] \\
& \cong \mathcal{H} \text{om}(A, f^j B) \otimes C
\end{align*}
\]

The first isomorphism is just \( \Delta^* \) applied to (1). The second isomorphism is because \( \Delta^* \) respects the tensor product. The third isomorphism is the tensor product of the isomorphism in \( \bullet \) with the isomorphism \( \Delta^* \pi_2^1 C \cong \text{id}^* C = C \).
A similar analysis works for $\Delta^\times$ applied to the map (2), which gives us the first isomorphism below

$$\Delta^\times\left[\pi_1^*A \otimes \pi_2^*\Hom(f^*B, C)\right] \cong \Delta^\times\Hom\left(\Hom(\pi_1^*A, \pi_2^*f^*B), \pi_2^*C\right)$$

$$\cong \Hom\left(\Delta^\times\Hom(\pi_1^*A, \pi_2^*f^*B), \Delta^\times\pi_2^*C\right)$$

$$\cong \Hom\left(\Hom(A, f^!B), C\right)$$

The second isomorphism comes from the formula $\Delta^\times\Hom(E, F) \cong \Hom(\Delta^*E, \Delta^*F)$.

The third isomorphism is the functor $\Hom(-, -)$ applied to the isomorphism in $\bullet$ and the isomorphism $\Delta^\times\pi_2^*C \cong \id^\times C = C$. 

Example 5.8. Let us work out what Corollary 5.7 says in the affine case: that is $f : R \to S$ will be a finite-type, flat homomorphism of noetherian rings and, by abuse of notation, we will also write $f : \Spec(S) \to \Spec(R)$ for the induced map of noetherian schemes. We have the usual equivalences $D(R) \cong D_{\text{qc}}(\Spec(R))$ and $D(S) \cong D_{\text{qc}}(\Spec(S))$, and $f^* : D(R) \to D(S)$, $f_* : D(S) \to D(R)$, $f^\times : D(R) \to D(S)$ and $f^! : D(R) \to D(S)$ are the affine versions of the standard functors of Grothendieck duality. Put $S^e = S \otimes_R S$. In this affine case, to say that an object $A \in D(S)$ is $f$–perfect means that $A$ must have bounded cohomology which is finite as $S$–modules, and $f_*A \in D(R)$ has finite Tor-dimension. Let $A \in D(S)$ be an $f$–perfect complex, and let $B \in D(R)$ and $C \in D(S)$ be arbitrary. Then the formulas of Corollary 5.7 come down to

$$\Hom_S(A, f^1B) \otimes C \cong S \otimes_{S^e} \Hom_R(A, B \otimes_R C),$$

$$\Hom_{S^e}[S, A \otimes_R \Hom_R(B, C)] \cong \Hom_S[\Hom_S(A, f^1B), C].$$

where the Hom and tensors are all derived. The reader can find special cases of these formulas in Avramov, Iyengar, Lipman and Nayak. 

The reader might note that we have already met special cases of the first of these formulas. If we put $A = C = S$ then the formula specializes to

$$f^1B \cong \Hom_S(S, f^1B) \otimes_S S \cong S \otimes_{S^e} \Hom_R(S, S \otimes_R B)$$

of the Introduction, and if we further specialize to $B = R$ we recover the formula $f^1R \cong S \otimes_{S^e} \Hom_R(S, S)$ of Remark 3.7.

6. A historical review

Grothendieck first mentioned that he knew how to prove a relative version of the Serre duality theorem in his ICM talk in Edinburgh in 1958, see [13]. The first published version was Hartshorne [15]; roughly speaking the construction of $f^!$ given in [15] is by gluing local data, not an easy thing to do in the derived category. Three and a half decades later Conrad [11] expanded and filled in details missing in [15]. The presentation of the subject given here is entirely different in spirit—it is based on early observations by Deligne [11] and Verdier [20], filled in and expanded greatly in Lipman [21].
second construction is much more global and functorial, the usual objection to it is that it’s difficult to compute anything.

Now it is time to say what’s different here from the classical literature. Let us begin with the observation that, until the late 1980s, no one really understood how to handle unbounded derived categories. For the first two decades of the subject the functor $f^*$, which involves a derived tensor product, was treated as a functor $f^* : D_{qc}(Y) \to D_{qc}(X)$, while the functor $f_*$, which involves injective resolutions, was classically viewed as a functor $f_* : D_{qc}(X) \to D_{qc}(Y)$. A careful reader will note that, being defined on different categories, these functors are not honest adjoints—there is no counit of adjunction $f^*f_* \to \text{id}$, and a classical version of the treatment of §2 would have had to be more delicate. Luckily for us we live in modern times and can give the clean presentation of the projection formula and the base-change maps of §2.

The article that brought modernity to this discipline was Spaltenstein [26], it taught us how to take injective and flat resolutions of unbounded complexes. Spaltenstein’s article made it clear how to define the adjoint functors $f^* : D_{qc}(Y) \to D_{qc}(X)$ and $f_* : D_{qc}(X) \to D_{qc}(Y)$. The natural question to arise was how much of Grothendieck duality could be developed in the unbounded derived category. The existence of a right adjoint $f^\times : D_{qc}(Y) \to D_{qc}(X)$ for $f_*$ was discovered soon after, the author even showed in [24] that it is possible to obtain this adjoint easily and very formally using Brown representability. At the time the author was promoting the point of view that the right way to approach all these classical results was to employ systematically the techniques of homotopy theory, like Brown representability—at the time this was still a novel idea. So Lipman challenged the author to try to use the techniques of homotopy theory to extend Verdier’s base-change theorem [29] to the unbounded derived category. Instead of a proof the author found a counterexample, see [24, Example 6.5]. There exists a cartesian square of noetherian schemes

\[
\begin{array}{ccc}
W & \xrightarrow{u} & X \\
\downarrow f & & \downarrow g \\
Y & \xleftarrow{v} & Z
\end{array}
\]

with $v$ flat (even an open immersion) and $g$ proper (even a closed immersion), and such that the base-change map $\Phi(\diamond) : u^*g^\times \to f^\times v^*$ is not an isomorphism. As an aside we note that the schemes in question are all affine.

This counterexample had the unfortunate effect of stifling the theory, for the next twenty years it put people off trying to develop the functor $f^!$ in the unbounded derived category. For example see Lipman’s book [21]—Lipman makes a real effort to give the results in the greatest generality in which they were known at the time, and for the functor $f^!$ he works almost entirely with bounded-below complexes. Drinfeld and Gaitsgory [12] generalized a version of the theory to DG schemes, and if the structure sheaf has negative
cohomology then the category $D_{qc}^+(X)$ does not make much sense. To finesse the issue they work in the category of Ind-coherent sheaves instead of the derived category.

In early 2013 I happened to run into Lipman at MSRI and he told me about exciting recent work, joint with Avramov, Iyengar and Nayak, which found a strange connection between Grothendieck duality and Hochschild homology and cohomology. In this survey we have already met this connection in Theorem 3.5 and Example 5.8 see also Remarks 3.6 and 3.7. Theorem 3.5 taught us about this bizarre new map from Hochschild homology to the dualizing complex $f^!\mathcal{O}_Y$ , and when $f$ is smooth and of relative dimension $d$ this map happens to give an isomorphism of $f^!\mathcal{O}_Y$ with a shift of the relative canonical bundle. And in §5 we saw that the formulas of §3 are only the tip of the iceberg, there and many more weird and wonderful ones—we presented two of them, together with proofs, in Example 5.8. The formulas of Example 5.8 are not new, special cases may be found in [2, 3]. What was new in §5 is that we gave them as special cases of results that hold in the unbounded derived category. Back in 2013, when Lipman told me about the work, no one knew how to define $f^!$ on the unbounded derived category.

Let us observe more carefully the second formula of Example 5.8 and for simplicity let’s put $B = R$. We remind the reader, the formula is

$$\text{Hom}_S(S, A \otimes_R C) \cong \text{Hom}_S[\text{Hom}_S(A, f^!R), C].$$

If we fix $A$ and consider the expression on the right as a functor in $C$ then it is clearly representable—the right hand side has the form $\text{Hom}_S(P, -)$, where $P$ happens to be the expression $\text{Hom}_S(A, f^!R)$. The isomorphism means that, as a functor in $C$, the expression $\text{Hom}_S(S, A \otimes_R C)$ is also representable, in particular it commutes with products—which is far from obvious. The challenge Lipman gave me was to try to use Brown representability to prove these formulas.

There is such a proof, and Iyengar and I are working on writing it up. But this survey is about another direction our research took: in trying to understand better these mysterious formulas we developed the natural transformation $\psi(f) : f^! \to f^!$—early hints of it may be found in Lipman [21, Exercise 4.2.3(d)]. What was new were the naturality and functoriality properties of $\psi$, see [19] for some illustrations of their value. Because at the time $f^!$ was defined only on the bounded-below derived category our results imposed artificial boundedness restrictions, and it was a natural challenge to try to remove them. Working in the category of Ind-coherent sheaves, as in Drinfeld and Gaitsgory, is clearly wrong for this problem—the formulas of [3] live in the derived category. The article [23] was written to address this problem, in it Grothendieck duality is developed in the unbounded derived category, and we gave a brief summary of some of the results in §2. In §3 and §5 we gave illustrations of how one can approach the unbounded versions of the formulas of [2, 3, 19] using the techniques surveyed in this paper—we proved the formula $f^! = \Delta^x \pi_2^x f^*$ for unbounded complexes in Proposition 5.3, while Example 5.8 showed us how to derive the reduction formulas of Avramov and Iyengar. These formulas occur in [2, 3, 19], but with unnatural boundedness hypotheses.
The reader might be puzzled. We mentioned that, twenty years ago, I produced a counterexample [24, Example 6.5] to the unbounded version of Verdier’s base-change theorem. There exists a cartesian square of schemes

\[
\begin{array}{ccc}
W & \xrightarrow{u} & X \\
\downarrow{f} & & \downarrow{g} \\
Y & \xrightarrow{v} & Z
\end{array}
\]

with \(v\) an open immersion and \(g\) proper, and such that the base-change map \(\Phi(\diamond) : u^*g^\times \to f^\times v^*\) is not an isomorphism. So what has changed in two decades? What’s new is Theorem 2.4(ii): it tells us that, as long as we further assume that \(f\) is of finite Tor-dimension, the problem goes away and \(\Phi(\diamond)\) is an isomorphism. When we compare two compactifications of \(X\) we end up with cartesian squares of the form

\[
\begin{array}{ccc}
X & \xrightarrow{u} & X \\
\downarrow{id} & & \downarrow{g} \\
X & \xrightarrow{v} & X
\end{array}
\]

and the identity \(id : X \to X\) is of finite Tor-dimension. Thus the cartesian squares that come up in the proof that \(f^! = u^*p^\times\) is independent of the factorization of \(f : X \to Y\) as \(X \xrightarrow{u} X \xrightarrow{p} Z\) all have base-change maps which are isomorphisms.

The place where the old counterexample rears its ugly head is when it comes to composition. The counterexample gave a commutative square (actually, even cartesian) and hence we have \(gu = vf\). Now \(u\) and \(v\) are open immersions while \(f\) and \(g\) are proper, hence \(u' = u^*, v' = v^*, f' = f^\times\) and \(g' = g^\times\). On the other hand \(u'g' = u^*g^\times\) is not isomorphic to \(f'v' = f^\times v^*\). We have already mentioned that, in the old counterexample, the base-change map \(u^*g^\times \to f^\times v^*\) is not an isomorphism, but even more is true, the functors are not isomorphic. The 2–functor \((-)^!\) is genuinely only oplax—there are natural maps \(\rho(f, g) : (vf)^! \to f^! v^!\) and \(\rho(u, g) : (gu)^! \to u^! g^!\), but clearly they cannot both be isomorphisms. As it happens, in this particular example \(\rho(u, g)\) is an isomorphism while \(\rho(f, v)\) isn’t.

The situation is not hopeless: 2.5.3 gives useful criteria for \(\rho(f, g)\) to be an isomorphism, and in [13, 14] and [15] we illustrated how this can be applied to obtain unbounded versions of the results of [2, 3, 19]. The illustrations of [13] also showed how, with all these new methods, the abstract nonsense approach to the subject pioneered by Deligne, Verdier and Lipman can produce explicit computational formulas simply and easily. The technicalities are different: the “residual complexes” of Grothendieck are replaced by the more standard tools of Hochschild homology.

In passing let me note that Hochschild homology is a \(K\)–theoretic invariant, and its appearance raises the question whether more sophisticated \(K\)–theoretic invariants might
shed even more light on the subject of Grothendieck duality. This is a volume on $K$-theory and its applications to algebraic geometry, and it seems the appropriate place to raise this question.

7. Generalizations

In March 2016 I received from the journal four referees’ reports on my article [23]. Mostly the referees’ comments were simple enough to address, but there were two difficult issues. One referee wondered what happens if we relax the noetherian hypothesis, while another suggested that I develop the entire theory in the generality of stacks. This led me to think more carefully about these points. The noetherian hypothesis seems indispensable, at least for this approach to the theory—some of the lemmas have non-noetherian versions, but there is a point at which the argument runs into a brick wall without the noetherian assumption. But I’m happy to report that, under relatively mild hypotheses, everything generalizes to noetherian algebraic stacks.

In fact the stacky version is cleaner to state. Algebraic stacks naturally form a 2-category, as do triangulated categories. The clean way to think about the theory is to view $(-)^*$, $(-)^\times$ and $(-)^!$ as 2-functors from [suitably restricted] algebraic stacks to triangulated categories, with some relations among them. These relations can be phrased in terms of the existence of certain natural transformations relating these functors, and the assertion that certain pairs of composites of natural transformations agree. For example: it turns out that $(-)^*$ has the structure of a monoid, meaning there is a pseudonatural transformation $(-)^* \times (-)^* \rightarrow (-)^*$, and $(-)^\times$ and $(-)^!$ are oplax modules over it. The map $\psi$ turns out to be an oplax natural transformation $\psi : (-)^\times \rightarrow (-)^!$ which is a module homomorphism. Anyway: the reader can find a thorough discussion in the introduction to [23].

This led to an expository conundrum in writing the current survey—it was unclear what was the right generality for the results. Avramov, Iyengar, Lipman and Nayak work with noetherian schemes, but allow the morphisms to be essentially of finite type (rather than the more restrictive finite type), and sometimes of finite Tor-dimension (rather than flat). But the methods they use don’t work for noetherian stacks—at least not yet—because one doesn’t yet know that a morphism of noetherian stacks which is essentially of finite type has a Nagata compactification. Nayak [22] proved the existence of Nagata compactifications for morphisms of noetherian schemes essentially of finite type, but so far no one has generalized this even to algebraic spaces. In other words: the results in this paper generalize in more than one direction, and at present I do not know a common generalization that covers everything that can be proved by the methods.

The compromise I made was to present the arguments in the intersection of the known cases, that is finite-type, flat maps of noetherian schemes, and leave to the reader the various generalizations. But I did make an effort to give proofs that are easily adaptable, so that the extension to (for example) algebraic stacks is straightforward.
When I gave the talk at TIFR, which amounted to a brief summary of this survey, Geisser, Kahn, Saito and Weibel raised the question of what portion of the ideas might be transferrable to the six-functor formalism. Because shortly after giving the talk I received the referees’ reports, with the questions about the non-noetherian and stacky versions of Grothendieck duality, I haven’t yet had the opportunity to think about this other question. The six-functor situation is another place where one defines functors like $f^!$ using good factorizations of $f$, and it is eminently sensible to ask if there might be an analog of the fancy, unbounded version of the base-change theorem and of its consequences.

The question is natural enough and I would like to come back to it when I have more time. In the interim I record it for others to study.

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Centre for Mathematics and its Applications, Mathematical Sciences Institute, John Dedman Building, The Australian National University, Canberra, ACT 0200, AUSTRALIA

E-mail address: Amnon.Neeman@anu.edu.au