STRONG UNIQUENESS OF THE RICCI FLOW ON THE EUCLIDEAN SPACE
IN HIGHER DIMENSIONS

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Abstract. In this paper, we prove a strong uniqueness theorem for the Ricci flow on the Euclidean space under certain conditions. Let \((\mathbb{R}^n, g_t)_{t \in [0,T]}\), where \(n \geq 3\), be a Ricci flow solution with \(g_0 = g_E\), where \(g_E\) is the standard Euclidean metric. Assume that either (1) \(g_t\) is uniformly lower equivalent to a fixed complete metric, or (2) \(g_t\) satisfies a logarithmic lower bound for the Ricci curvature. Then we have that \(g_t \equiv g_E\) for all \(t \in [0,T]\). This partially answers a question proposed by B-L. Chen [1].

1. Introduction

The Ricci flow is a geometric evolution equation introduced by Hamilton [8], which deforms a Riemannian manifold by the Ricci curvature

\[
\frac{\partial}{\partial t} g_t = -2 \text{Ric}_{g_t}.
\]

On the one hand, the Ricci flow equation is nonlinear, which implies that almost certainly a Ricci flow develops singularity and does not exist for all time; see [10]. By a thorough analysis of the singularities in dimension three [13], Perelman successfully overcame the obstacles when solving the Poincaré and geometrization conjectures [13, 14, 15] using Hamilton’s program [10]. On the other hand, the Ricci flow equation is parabolic, and consequently, many problems related to parabolic partial differential equations arise in this field. For instance, the short-time existence problem (c.f. [6, 9, 17]), the long-time existence problem (c.f. [4, 10, 16, 18, 19]), and the uniqueness problem (c.f. [2, 11, 12]).

Another interesting problem in the study of parabolic equations is that of strong uniqueness. In general, the solution to the initial value problem of a parabolic equation on an unbounded region is not unique. For instance, the 1-dimensional linear heat equation \(u_t - u_{xx} = 0\) on \(\mathbb{R}^1\) admits a nontrivial solution with the initial value \(u(\cdot, 0) \equiv 0\), namely, the well-known Tychonoff’s example

\[
u(x, t) = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!} \frac{d^k}{dt^k} \exp(-\frac{1}{t}).
\]

In this respect, B-L. Chen [1] proved a strong uniqueness result in dimension three, namely, starting from a 3-dimensional complete Riemannian manifold with bounded and nonnegative sectional curvature, two Ricci flows must be identical to each other for a short time. As a consequence, a Ricci flow starting from the standard 3-dimensional Euclidean space must remain Euclidean as long as it exists. It is therefore interesting to ask whether

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such property is true for the Euclidean space in higher dimensions. We shall give an affirmative answer to this question with some additional assumptions. First of all, we show that the strong uniqueness is true provided that the solution is uniformly lower equivalent to a fixed and complete metric.

**Definition 1.1.** A complete Ricci flow $(M^n, g_t)_{t \in [0, T]}$ is called uniformly lower equivalent to $\bar{g}$, where $\bar{g}$ is a complete smooth Riemannian metric on $M$, if there exists a positive constant $c$, such that $g_t \geq c \bar{g}$ for all $t \in [0, T]$.

**Theorem 1.2.** Let $(\mathbb{R}^n, g_t)_{t \in [0, T]}$ be a complete and smooth solution to the Ricci flow with $g_0 = g_E$, where $g_E$ is the standard Euclidean metric. Furthermore, assume that $g_t$ is uniformly lower equivalent to some complete smooth metric $\bar{g}$ on $\mathbb{R}^n$. Then we have $g_t \equiv g_E$ for all $t \in [0, T]$.

Another condition under which we can prove the strong uniqueness theorem is a logarithmic lower bound for the Ricci curvature. Note that the conditions in Definition 1.1 and in Definition 1.3 do not imply each other.

**Definition 1.3.** A complete Ricci flow $(M^n, g_t)_{t \in [0, T]}$ has a logarithmic lower bound for the Ricci curvature if there exist constants $\alpha > 0, \beta > 0, \gamma \in (0, 1)$, and a fixed point $x_0 \in M$, such that

\[
\text{Ric}_{g_t}(x) \geq -\alpha \tau^{-\gamma} \cdot \ln \left( \text{dist}_{g_t}(x_0, x) + \beta \right) g_t \quad \text{for all} \quad (x, t) \in M \times [0, T].
\]

**Theorem 1.4.** Let $(\mathbb{R}^n, g_t)_{t \in [0, T]}$ be a complete and smooth solution to the Ricci flow with $g_0 = g_E$, where $g_E$ is the standard Euclidean metric. Furthermore, assume that $g_t$ satisfies Definition 1.3. Then we have $g_t \equiv g_E$ for all $t \in [0, T]$.

**Remark 1.5.** In general, if a complete Ricci flow $(M^n, g_t)_{t \in [0, T]}$ starts from a $C^2$ Riemannian metric $g_0$, then Shi’s local estimates [17] imply that $g_t$ is smooth on $M \times (0, T]$ and continuous on $M \times [0, T]$. However, if $g_0$ is a $C^\infty$ Riemannian metric, then $g_t$ is smooth on $M \times [0, T]$. In particular, $g_t$ is smooth at $t = 0$. To see that this is true, we fix a point $p \in M$, and apply Theorem 11 to conclude that for any $r > 0$ and $k \in \mathbb{N}$, there are positive constants $C_{k,r} < \infty$ and $\bar{r}_{k,r} > 0$, such that

\[
|\nabla^k Rm| \leq C_{k,r} \quad \text{on} \quad B_{g_0}(p, r) \times [0, \bar{r}_{k,r}].
\]

Then, by the evolution equation of the Ricci flow, $g_t$ is smooth at $t = 0$.

The idea of the proof is to apply Perelman’s $L$-geometry. The initial condition $g_0 = g_E$ implies that the reduced volume evaluated at $t = 0$ must be equal to 1. Since the reduced volume is monotone, it follows that it is equal to 1 for all the time. Perelman’s monotonicity formula then implies that the Ricci flow in question must be the Gaussian shrinker, namely, the static Euclidean space. In fact, the reason why we make the above assumptions is because under these assumptions alone is Perelman’s reduced distance known to bear many nice properties.

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2. Preliminaries

In this section, we collect some well-known results applied in the proof of our main theorem. These results are chiefly about Perelman’s reduced distance and reduced volume, and most of them can be found in [13] [20]. Let $(M^n, g(t))_{t \in [0, T]}$ be a solution to the
backward Ricci flow equation

\[ \frac{\partial}{\partial \tau} g(\tau) = 2Ric_{g(\tau)} \]

where \( \tau \) stands for the backward time. In fact, if \((M^n, g_t)_{t \in [0,T]} \) is a Ricci flow, then \( g(\tau) := g_{T-\tau} \) solves the backward Ricci flow equation.

Let \( \gamma(s) : [0, \tau] \to M \), where \( \tau \in (0, T] \), be a piecewise \( C^1 \) curve, then Perelman’s \( \mathcal{L} \)-energy is defined as

\[ \mathcal{L}(\gamma) = \int_0^\tau \sqrt{s} \left( R_{g(s)}(\gamma(s)) + |\gamma'(s)|^2_{g(s)} \right) ds. \]

One may view \( \gamma \) as a curve in the Ricci flow space-time from \( (\gamma(0), 0) \) to \( (\gamma(\tau), \tau) \in M \times [0, T] \), satisfying \( (\gamma(s), s) \in M \times \{s\} \) for all \( s \in [0, \tau] \).

Let \( x_0 \in M \) be a fixed base point. For any \( (x, \tau) \in M \times (0, T] \), we define

\[ L(x, \tau) = \inf_{\gamma} \mathcal{L}(\gamma(s)), \quad \ell(x, \tau) = \frac{L(x, \tau)}{2 \sqrt{\tau}}, \]

where the infimum is taken over all piecewise \( C^1 \) curves \( \gamma(s) : [0, \tau] \to M \) satisfying \( \gamma(0) = x_0 \) and \( \gamma(\tau) = x \). The minimizer is called a minimal \( \mathcal{L} \)-geodesic, and \( \ell(\cdot, \cdot) : M \times (0, T] \to \mathbb{R} \) is called Perelman’s reduced distance function or the \( \ell \)-function based at \( (x_0, 0) \). Furthermore, we defined Perelman’s reduced volume based at \( (x_0, 0) \) as

\[ \mathcal{V}(\tau) = \int_M (4\pi \tau)^{-\frac{n}{2}} e^{-\ell(x, \tau)} dg(\tau). \]

In Perelman’s study \cite{Perelman} of the \( \mathcal{L} \)-geometry, a general (although implicit) assumption is bounded sectional curvature. However, Ye \cite{Ye} studied the properties of the \( \ell \)-function and the reduced volume assuming only a lower bound for the Ricci curvature. In the next section, we will show that Perelman’s theory of \( \mathcal{L} \)-geometry can be generalized to a setting where the condition in either Definition \( \ref{def:bounded sectional} \) or Definition \( \ref{def:lower bound} \) is satisfied.

We also need the following logarithmic Sobolev inequality of Gross \cite{Gross}. This inequality is important in the proof of our main theorem, since it is strongly related to Perelman’s monotonicity formulas. The particular form of the following logarithmic Sobolev inequality can be found in \cite{Perelman} Theorem 22.5.

**Theorem 2.1** (Logarithmic Sobolev inequality in the Euclidean space). For any \( W^{1,2} \) function \( \varphi \) on \( \mathbb{R}^n \), we have

\[ \int_{\mathbb{R}^n} \left( 2|\nabla \varphi|^2 - \varphi^2 \log \varphi^2 \right) dg_E + \log \left( \int_{\mathbb{R}^n} \varphi^2 dg_E \right) \int_{\mathbb{R}^n} \varphi^2 dg_E \geq \frac{n}{2} \log(2\pi e) + n \int_{\mathbb{R}^n} \varphi^2 dg_E. \]

### 3. Perelman’s \( \mathcal{L} \)-geometry under certain conditions

In this section, let us consider a complete backward Ricci flow \((M^n, g(\tau))_{\tau \in [0,T]} \). We shall show that Perelman’s theory of \( \mathcal{L} \)-geometry can be extended to the cases under the assumptions made in Theorem \( \ref{thm:bounded sectional} \) and Theorem \( \ref{thm:lower bound} \) respectively. For the technical convenience, we assume

\[ R_{g(\tau)}(x) \geq 0 \quad \text{for all} \quad (x, \tau) \in M \times [0, T] \]

throughout the whole section. In fact, this assumption can be replaced by a lower bound of the scalar curvature. In the following two subsections, our goal is to show the existence of minimal \( \mathcal{L} \)-geodesics from the base point \((x_0, 0)\) to any point \((x, \tau) \in M \times (0, T] \).
3.1. Uniform lower equivalence. Let us assume that there exists a positive number c and a complete smooth Riemannian metric $g$ on $M$, such that

$$g(\tau) \geq c \tilde{g} \quad \text{for all} \quad \tau \in [0, T].$$

To prove the existence of minimal $L$-geodesics, we first show that any minimizing sequence of the $L$-energy must lie within some compact set in space-time.

**Lemma 3.1.** For any $L > 0$, there exists $A = A(L, c, T)$, such that the following holds. Let $x_0$ be a fixed point and let $(x, \tau) \in M \times (0, T]$ be a point in space-time. Then for any piecewise $C^1$ curve $\gamma : [0, \tau] \to M$ satisfying $\gamma(0) = x_0$, $\gamma(\tau) = x$, and $\mathcal{L}(\gamma) \leq L$, we have

$$\gamma(s) \in B_{\tilde{g}}(x_0, A) \quad \text{for all} \quad s \in [0, \tau].$$

**Proof.** This is a straightforward application of the definition of the $L$-energy. Let $\gamma$ be a curve satisfying all the assumptions in the statement of the lemma. Then, by (3.1) and (3.2), we may estimate

$$L \geq L(\gamma) = \int_0^\tau \sqrt{s} \left( R_{\tilde{g}(\tau)}(\gamma(s)) + |\gamma'(s)|^2_{\tilde{g}(s)} \right) ds \geq \int_0^\tau \sqrt{s} |\gamma'(s)|^2_{\tilde{g}} ds \geq c \int_0^\tau \sqrt{s} |\gamma'(s)|^2_{\tilde{g}} ds$$

$$= \frac{c}{2} \int_0^\tau |\zeta'(s)|^2_{\tilde{g}} ds,$$

where we have applied the change of variable $\zeta(\sigma) := \gamma(\sigma^2)$. Then, by the Cauchy-Schwarz inequality, we have

$$2c^{-1}L \geq \int_0^\tau |\zeta'(s)|^2_{\tilde{g}} ds \geq \int_0^\tau \sqrt{s} |\zeta'(s)|^2_{\tilde{g}} ds \geq \frac{\left( \text{dist}_{\tilde{g}}(\gamma(\sigma), x_0)^2 \right)^2}{\sqrt{s}} \geq \frac{\left( \text{dist}_{\tilde{g}}(\gamma(s), x_0)^2 \right)^2}{\sqrt{T}}$$

for all $s \in [0, \tau]$. The lemma follows immediately.

**Proposition 3.2** (Existence of minimal $L$-geodesic). Under the assumptions (3.1) and (3.2), the following holds. For any $(x_0, 0)$ and $(x, \tau) \in M \times (0, T]$, there exists a minimal $L$-geodesic $\gamma : [0, \tau] \to M$ satisfying $\gamma(0) = x_0$ and $\gamma(\tau) = x$.

**Proof.** Let us fix an arbitrary piecewise $C^1$ curve $\beta : [0, \tau] \to M$ satisfying $\beta(0) = x_0$ and $\beta(\tau) = x$. Let $L = \mathcal{L}(\beta)$. Then, letting $A = A(L, c, T)$ be the constant in Lemma 3.1, we have that every minimizing sequence $\{\gamma_i\}_{i=1}^\infty$ of the $L$-energy with $\mathcal{L}(\gamma_i) \leq L$ from $(x_0, 0)$ to $(x, \tau)$ is contained in $\mathcal{B}_{\tilde{g}(0)}(x_0, A) \times [0, \tau]$, which is a compact set in space-time. The conclusion then follows from the standard theory of variation.

3.2. Logarithmic lower bound for the Ricci curvature. In this subsection, we assume that the backward Ricci flow in question has a logarithmic lower bound for the Ricci curvature. In particular, we let $\alpha > 0, \beta > 0, \gamma \in (0, 1)$ be constants and $x_0$ be a fixed point, such that

$$\text{Ric}_{\tilde{g}(\tau)}(x) \geq -\alpha(T - \tau)^{-\gamma} \cdot \ln \left( \text{dist}_{\tilde{g}(\tau)}(x_0, x) + \beta \right) g(\tau) \quad \text{for all} \quad (x, \tau) \in M \times [0, T].$$

As in the previous subsection, we show the existence of a minimal $L$-geodesic from $(x_0, 0)$ to $(x, \tau) \in M \times (0, T]$, provided that $T$ is small enough.
Lemma 3.3 (Distance distortion). For any $\tau_1$ and $\tau_2 \in [0, T]$ with $\tau_1 \leq \tau_2$ and for any $x \in M$, we have

$$\text{dist}_{g(\tau)}(x_0, x) + \beta \leq \left(\text{dist}_{g(\tau)}(x_0, x) + \beta\right)^{\exp\left(\frac{\tau}{\tau^{1-\gamma}}\right)}.$$  

Proof. For any $\tau \in [0, T]$, let us define $r(\tau) := \text{dist}_{g(\tau)}(x_0, x)$ and let $\gamma : [0, r(\tau)] \to M$ be a $g(\tau)$-geodesic connecting $x_0$ and $x$ with unit speed. We may apply (3.3) to compute

$$\frac{dr}{d\tau} = \int_0^\tau \text{Ric}(\gamma'(s), \gamma'(s))ds \geq -\alpha(T - \tau)^{-\gamma} \cdot r \ln(r + \beta) \geq -\alpha(T - \tau)^{-\gamma} \cdot (r + \beta) \ln(r + \beta),$$

and equivalently

$$\frac{d}{d\tau} \ln (r + \beta) \geq -\alpha(T - \tau)^{-\gamma}.$$  

Integrating the above inequality from $\tau_1$ to $\tau_2$, we obtain the lemma. \qed

Next, we prove that a minimizing sequence of the $\mathcal{L}$-energy is always contained in a compact set in space-time.

Lemma 3.4. There exists a positive number $\overline{T} = \overline{T}(\alpha, \gamma) > 0$ with the following property. If $T < \overline{T}$, then for any $r > 0$ and $L > 0$, there is a positive number $A = A(r, L, T, \alpha, \beta, \gamma)$, such that the following holds. For any $(x, \tau) \in M \times (0, T]$ with

$$\text{dist}_{g(\tau)}(x_0, x) \leq r,$$

if $\gamma : [0, \tau] \to M$ is a piecewise $C^1$ curve satisfying $\gamma(0) = x_0$, $\gamma(\tau) = x$, and

$$\mathcal{L}(\gamma) \leq L$$

Then we have

$$\gamma(s) \in \overline{B}_{g(\tau)}(x_0, A) \quad \text{for all} \quad s \in [0, \tau].$$

Proof. We shall argue by contradiction. Let $A \gg r$ be a large number to be fixed. Define

$$\overline{\tau} := \inf \left\{ \tau' \mid \gamma(s) \in B_{g(\tau)}(x_0, A) \text{ for all } s \in [\tau', \tau] \right\} \in [0, \tau).$$

We assume that $\overline{\tau} > 0$ and show that there is a contradiction when $A$ is large enough. By this contradictory assumption, we have

$$\text{dist}_{g(\tau)}(x_0, \gamma(\overline{\tau})) = A,$$

$$\text{dist}_{g(\tau)}(x_0, \gamma(s)) < A \quad \text{for all} \quad s \in [\overline{\tau}, \tau].$$

By Lemma 3.3, we have

$$\text{dist}_{g(\tau)}(x_0, \gamma(\tau_2)) \leq \left(\text{dist}_{g(\tau)}(x_0, \gamma(\tau_1)) + \beta\right)^{\exp\left(\frac{\tau}{\tau^{1-\gamma}}\right)} \quad \text{for all} \quad 0 \leq \tau_1 \leq \tau_2 \leq \tau,$$

and consequently

$$\text{dist}_{g(\tau)}(x_0, \gamma(\tau_2)) \leq (A + \beta)^{\exp\left(\frac{\tau}{\tau^{1-\gamma}}\right)} \quad \text{for all} \quad \overline{\tau} \leq \tau_1 \leq \tau_2 \leq \tau$$

(3.5)

$$\text{dist}_{g(\tau)}(x_0, \gamma(\tau_2)) \leq (r + \beta)^{\exp\left(\frac{\tau}{\tau^{1-\gamma}}\right)}.$$  

(3.6)
By (3.3) and (3.5), we may estimate
\[ \frac{d}{d\tau_1} |y'(\tau_2)|_{g(\tau_1)}^2 = 2Ric_{g(\tau_1)}(y'(\tau_2), y'(\tau_2)) \]
\[ \geq -2\alpha(T - \tau)^{-\gamma} \cdot \ln \left( (A + \beta) \exp \left( \frac{2}{\gamma} \frac{1}{T - \tau} \right) + \beta \right) |y'(\tau_2)|_{g(\tau_1)}^2, \]
for all \( \hat{\tau} \leq \tau_1 \leq \tau_2 \leq \tau \). Integrating over \( \tau_1 \), we have
\[ |y'(\tau_2)|_{g(\tau)}^2 \geq \left( (A + \beta) \exp \left( \frac{2}{\gamma} \frac{1}{T - \tau} \right) + \beta \right) |y'(\tau_2)|_{g(\tau)}^2 \]
\[ \geq (2A) \cdot \frac{2\gamma - 1}{\gamma} \cdot \exp \left( \frac{2}{\gamma} \frac{1}{T - \tau} \right) \]
\[ \geq (2A) \cdot \frac{2\gamma - 1}{\gamma} \cdot \exp \left( \frac{2}{\gamma} \frac{1}{T - \tau} \right) \]
\[ \geq (2A) \cdot \frac{2\gamma - 1}{\gamma} \cdot \exp \left( \frac{2}{\gamma} \frac{1}{T - \tau} \right), \]
for all \( \hat{\tau} \leq \tau_2 \leq \tau \), if \( A \geq \Delta(\alpha, \beta, \gamma, T) \) for some large positive constant \( \Delta(\alpha, \beta, \gamma, T) \). By the definition of the \( \mathcal{L} \)-energy and the assumption (3.1), we have
\[ L \geq \mathcal{L}(\gamma) \geq \int_{\hat{T}}^{T} \sqrt{g(s)} |y'(s)|_{g(s)}^2 ds \]
\[ \geq (2A) \cdot \frac{2\gamma - 1}{\gamma} \cdot \exp \left( \frac{2}{\gamma} \frac{1}{T - \tau} \right) \]
\[ \geq (2A) \cdot \frac{2\gamma - 1}{\gamma} \cdot \exp \left( \frac{2}{\gamma} \frac{1}{T - \tau} \right), \]
where we have applied (3.4) and (3.6). Finally, if we take \( T < \mathcal{T}(\alpha, \gamma) \) such that \( \frac{2\gamma - 1}{\gamma} \cdot \exp \left( \frac{2}{\gamma} \frac{1}{T - \tau} \right) \leq 1 \), then we have
\[ L \geq \frac{(A - (r + \beta) \exp \left( \frac{2}{\gamma} \frac{1}{T - \tau} \right))^2}{4A \sqrt{T}}, \]
and this obviously is a contradiction if \( A > \Delta(r, L, T, \alpha, \beta, \gamma) \). \( \square \)

**Proposition 3.5** (Existence of minimal \( \mathcal{L} \)-geodesic). If \( T < \mathcal{T} \), where \( \mathcal{T} \) is the constant in the statement of Lemma 3.2, then for any \( (x, \tau) \in M \times (0, T) \), there is a minimal \( \mathcal{L} \)-geodesic from \( (x_0, 0) \) to \( (x, \tau) \).

**Proof.** We fix an arbitrary \( (x, \tau) \in M \times (0, T) \) and a piecewise \( C^1 \) curve \( \gamma : [0, \tau] \to M \) with \( \gamma(0) = x_0 \) and \( \gamma(\tau) = x \). Let
\[ r := \text{dist}_{g(\tau)}(x_0, x), \]
\[ L := \mathcal{L}(\gamma) \]
and let \( A = A(r, L, T, \alpha, \beta, \gamma) \) be the positive constant given by Lemma 3.4. Then we have that, by Lemma 3.4, any minimizing sequence \( \{\gamma_i\}_{i=1}^{\infty} \) of the \( \mathcal{L} \)-energy with \( \mathcal{L}(\gamma_i) \leq L \) from \( (x_0, 0) \) to \( (x, \tau) \) lies in the space-time compact set
\[ \bigcup_{s \in [0, T]} \overline{B}_{g(s)}(x_0, A) \times \{s\}, \]
and the conclusion follows from the standard theory of variation. \( \square \)
3.3. Conclusion. Once we have established the above propositions, the other results of Perelman’s $L^2$-geometry can be built on it. We leave it to the reader to check that all proofs in [20] are valid, and we summarize some important results below.

**Theorem 3.6** ([13], see also [20] Proposition 2.14, Lemma 2.22, Theorem 2.23). Let $(M^n, g(t))_{t \in [0,T]}$ be a backward Ricci flow and $x_0 \in M$ be a point such that either one of the following is true.

1. $(3.7)$ and $(3.2)$ both hold; $x_0$ is arbitrarily fixed.
2. $(3.7)$ and $(3.3)$ both hold; $T < T(\alpha, \gamma)$, where $T$ is the constant in the statement of Lemma 3.4; $x_0$ is the point in $(3.3)$.

Let $\ell$ be the reduced distance function based at $(x_0,0)$. Then $\ell$ is locally Lipschitz on $M \times (0,T]$ and the following equation and inequalities hold almost everywhere in the smooth sense on $M \times (0,T]$.

\begin{align*}
(3.8) & \quad 2 \frac{\partial \ell}{\partial \tau} + |\nabla \ell|^2 - R + \frac{\ell}{\tau} = 0, \\
(3.9) & \quad \frac{\partial}{\partial \tau} \ell - \Delta \ell + |\nabla \ell|^2 - R + \frac{n}{2 \tau} \geq 0, \\
(3.10) & \quad 2 \Delta \ell - |\nabla \ell|^2 + R + \frac{\ell - n}{\tau} \leq 0.
\end{align*}

Furthermore, $(3.9)$ and $(3.10)$ both hold on $M \times (0,T)$ in the sense of distribution. That is to say, for any $0 < \tau_1 < \tau_2 < T$ and for any nonnegative Lipschitz function $\phi$ compactly supported on $M \times [\tau_1, \tau_2]$, it holds that

\begin{align*}
(3.11) & \quad \int_{\tau_1}^{\tau_2} \int_M \left( \nabla \ell \cdot \nabla \phi + \left( \frac{\partial}{\partial \tau} \ell + |\nabla \ell|^2 - R + \frac{n}{2 \tau} \right) \phi \right) dg(\tau) d\tau \geq 0,
\end{align*}

and, for any $\tau \in (0,T)$ and any nonnegative Lipschitz function $\phi$ compactly supported on $M$, it holds that

\begin{align*}
(3.12) & \quad \int_M \left( -2 \nabla \ell \cdot \nabla \phi + \left( -|\nabla \ell|^2 + R + \frac{\ell - n}{\tau} \right) \phi \right) dg(\tau) \leq 0.
\end{align*}

**Theorem 3.7** ([13], see also [20] Theorem 4.3, Theorem 4.5). Let $\mathcal{V}(\tau)$ be the reduced volume based at $(x_0,0)$. Under the same assumptions as in the above theorem, we have

1. $0 < \mathcal{V}(\tau) \leq 1$ for all $\tau \in (0,T]$;
2. $\mathcal{V}(\tau)$ is non-increasing in $\tau$;
3. if $\mathcal{V}(\tau) = 1$ for some $\tau \in (0,T)$, then $M^n = \mathbb{R}^n$ and $g(s) \equiv g_E$ for all $s \in [0,\tau]$, where $g_E$ is the standard Euclidean metric.

4. The proof of the main theorem

Let $(\mathbb{R}^n, g_0)_{t \in [0,T]}$ be the complete Ricci flow in the statement of Theorem 1.2 or Theorem 1.4. In the case of Theorem 1.4 we also would like to assume that $T < T(\alpha, \gamma)$, so that we may apply Theorem 3.7. Suppose that this is not the case, we may first prove the strong uniqueness on the time interval $[0, \frac{1}{2} T(\alpha, \gamma)]$, and then on the interval $[\frac{1}{2} T(\alpha, \gamma), T(\alpha, \gamma)], [T(\alpha, \gamma), 2 T(\alpha, \gamma) - \ldots]$

First of all, we observe that, in general cases, $(3.11)$ and $(3.12)$ may not be valid up to $\tau = T$, which is the initial time-slice of the Ricci flow. However, because the Ricci flow in the statement of Theorem 1.4 is smooth at $t = 0$, and because the initial metric is $g_E$, we may extend it backwards by letting $g_t$ be the static Euclidean metric when $t \leq 0$. 
Lemma 4.1. The Ricci flow \((\mathbb{R}^n, g_t)_{t \in [0,T]}\) in the statement of Theorem 1.4 can be extended to a complete and smooth ancient solution by letting \(g_t \equiv g_E\) for all \(t \in (-\infty, 0]\). Consequently, we have
\[
R_{g_t} \geq 0 \quad \text{for all} \quad t \in [0,T].
\]

Proof. The second statement is a consequence of the first one and [11]. To see that the first statement is true, one needs only to observe that, by our assumption on \(g_t\) or by Remark 1.5 if we define \(g_t \equiv g_E\) for all \(t \leq 0\), then we have
\[
\left. \frac{\partial_t g_t}{\partial t} \right|_{t=0} = -2\text{Ric}_{g_E} = \left. \frac{\partial g_t}{\partial t} \right|_{t=0},
\]
and the same condition for higher derivatives can be similarly verified. Hence the conclusion of the lemma follows. \(\quad \Box\)

Henceforth we shall consider the backward Ricci flow \(g(\tau) := g_{T-\tau}\). Let \(x_0 \in \mathbb{R}^n\) be a fixed point. In the case of Theorem 1.4 we also assume \(x_0\) to be the point in Definition 1.5. Let \(\ell : \mathbb{R}^n \times (0, \infty) \to \mathbb{R}\) be the reduced distance based at \((x_0, 0)\).

Lemma 4.2. Theorem 3.7 is valid for \(\ell\) on \(M \times (0, \infty)\), and in particular, at \(\tau = T\).

Proof. We consider only the case of Theorem 1.4. The case of Theorem 1.2 is easier and is left to the reader. The idea of the proof, as in Section 2, is to show the existence of minimal \(L\)-geodesic from \((x_0, 0)\) to any point in \(M \times (0, \infty)\). Let \((x, \tau)\) be an arbitrary point with \(\tau > T\), and let \(\gamma : [0, \tau] \to M\) be a curve satisfying \(\gamma(0) = x_0, \gamma(\tau) = x\), and \(L(\gamma) \leq L\), where \(L\) is a positive number. Then we may estimate
\[
L = L(\gamma) \geq \int_T^\tau \sqrt{T \left( R_{g_\gamma}(\gamma(s)) + |\gamma'(s)|^2_{g_\gamma} \right)} \, ds \\
\geq \int_T^\tau \sqrt{T |\gamma'(s)|^2_{g_\gamma}} \, ds \geq \sqrt{T} \frac{\text{dist}_{g_{T}(x, \gamma(T))}^2}{\tau - T}.
\]

Note that we have used the fact \(g(\tau) \equiv g_E\) for all \(\tau \geq T\). Hence we have
\[
\text{dist}_{g_{T}(x_0, \gamma(T))} \leq \sqrt{LT^{-\frac{1}{2}}(\tau - T) + \text{dist}_{g_x}(x_0, x)} =: r.
\]

By Lemma 3.4 if we let \(A = A(r, L, T, \alpha, \beta, \gamma)\) be the constant in its statement, then we have
\[
\gamma(s) \in \overline{B_{g_\gamma}(x_0, A)} \quad \text{for all} \quad s \in [0, T].
\]

This obviously shows that every minimizing sequence of \(L\)-energy from \((x_0, 0)\) to \((x, \tau)\) lies in a space-time compact set. Hence there exists a minimal \(L\)-geodesic from \((x_0, 0)\) to any point in \(M \times (0, \infty)\). One may then apply the same method as in [20] to prove the lemma. \(\quad \Box\)

In the following, we shall focus on the time-slice of the \(\ell\)-function
\[
\ell_T := \ell(\cdot, T) : \mathbb{R}^n \to \mathbb{R}.
\]

Lemma 4.3. We have
\[
(4.1) \quad \int_{\mathbb{R}^n} |\nabla \ell_T|_{g_E}^2 (4\pi T)^{-\frac{n}{2}} e^{-\ell_T} dE \leq \frac{2n}{T} < \infty.
\]

Consequently, \((4\pi T)^{-n/4} e^{-\ell_T/2}\) is a \(W^{1.2}\) function on \(\mathbb{R}^n\).
Proof. This lemma is the same as [3 Lemma 6.6]. We shall include its proof for the convenience of the reader. Rewriting (3.12) at \( \tau = T \), we have

\[
\int_{\mathbb{R}^n} \left( -2 \nabla \ell_T \cdot \nabla \phi - |\nabla \ell_T|^2_\mathbb{R} \phi \right) d\mu_E \leq -\int_{\mathbb{R}^n} \left( R_{\mathbb{R}^n} + \frac{\ell_T - n}{T} \right) \phi d\mu_E
\]

where we have implemented the facts that \( \phi \) and subsequently \( \ell \) are valid test functions of (3.12), since we obtain (4.1). The boundedness of the \( L^2 \)-norm of \( \varphi \) that \( R \) is a true, one needs only to recall the definition (2.1), (2.2) of the reduced distance and the fact that \( R_{\mathbb{R}^n} \geq 0 \) for all \( R \geq 0 \) (c.f. Lemma 4.1).

Next, we take

\[
\phi := \varphi_A(4\pi T)^{-\frac{n}{2}} e^{-\ell_T},
\]

where

\[
\varphi_A(x) = \eta \left( \frac{|x - x_0|}{A} \right).
\]

\( x_0 \) is the fixed base point, and \( \eta : [0, \infty) \to [0, 1] \) is a smooth and decreasing function satisfying \( \eta(0) = 1, \eta(1) = 0, \) and \(-2 \leq \eta'(s) \leq 0 \) for all \( s \in [0, \infty) \). Obviously, \( \phi \) is a valid test function of (3.12), since \( \varphi_A \) is smooth and compactly supported and \( \ell_T \) is locally Lipschitz. Then, (4.2) becomes

\[
\int_{\mathbb{R}^n} \left( \varphi_A^2 |\nabla \ell_T|^2_\mathbb{R} - 4 \langle \nabla \ell_T, \nabla \varphi_A \rangle_{\mathbb{R}^n} \varphi_A \right) (4\pi T)^{-\frac{n}{2}} e^{-\ell_T} d\mu_E \leq \frac{n}{T} \int_{\mathbb{R}^n} \varphi_A^2 (4\pi T)^{-\frac{n}{2}} e^{-\ell_T} d\mu_E.
\]

Hence we have

\[
\int_{\mathbb{R}^n} \varphi_A^2 |\nabla \ell_T|^2_\mathbb{R} (4\pi T)^{-\frac{n}{2}} e^{-\ell_T} d\mu_E \leq \frac{n}{T} \int_{\mathbb{R}^n} \varphi_A^2 (4\pi T)^{-\frac{n}{2}} e^{-\ell_T} d\mu_E + 4 \int_{\mathbb{R}^n} \langle \nabla \ell_T, \nabla \varphi_A \rangle_{\mathbb{R}^n} \varphi_A (4\pi T)^{-\frac{n}{2}} e^{-\ell_T} d\mu_E \leq \frac{n}{T} \int_{\mathbb{R}^n} \varphi_A^2 (4\pi T)^{-\frac{n}{2}} e^{-\ell_T} d\mu_E + \frac{1}{2} \int_{\mathbb{R}^n} \varphi_A^2 |\nabla \ell_T|^2_\mathbb{R} (4\pi T)^{-\frac{n}{2}} e^{-\ell_T} d\mu_E + 8 \int_{\mathbb{R}^n} \nabla \varphi_A^2 (4\pi T)^{-\frac{n}{2}} e^{-\ell_T} d\mu_E,
\]

and subsequently

\[
\int_M \varphi_A^2 |\nabla \ell_T|^2_\mathbb{R} (4\pi T)^{-\frac{n}{2}} e^{-\ell_T} d\mu_E \leq \left( \frac{64}{A^2} + \frac{2n}{T} \right) \int_M (4\pi T)^{-\frac{n}{2}} e^{-\ell_T} d\mu_E = \left( \frac{64}{A^2} + \frac{2n}{T} \right) \mathcal{V}(T) \leq \frac{64}{(A^2 + \frac{2n}{T})},
\]

where we have applied the fact that \( \mathcal{V}(T) \leq 1 \) due to Theorem 3.7(1). Taking \( A \to \infty \), we obtain (4.1). The boundedness of the \( L^2 \)-norm of \( (4\pi T)^{-n/4} e^{-\ell_T/2} \) is but a consequence of the fact that \( \mathcal{V}(T) \leq 1 \).

Finally, Theorem 1.2 and Theorem 1.4 follows from the lemma below and Theorem 3.7(3).
Lemma 4.4. We have \( V(T) = 1 \).

Proof. Rewriting (4.2) using the same cut-off function as defined in (4.3) as well as the fact that \( R_{gx} = 0 \), we have

\[
\int_{\mathbb{R}^n} \left( \nabla \ell_T \right)^2_{gx} + \frac{\ell_T - n}{T} \phi_A^2 (4\pi T)^{-\frac{n}{2}} e^{-\ell_T} \, dg_E \\
\leq 4 \int_{\mathbb{R}^n} \langle \nabla \ell_T, \nabla \phi_A \rangle_{gx} \phi_A (4\pi T)^{-\frac{n}{2}} e^{-\ell_T} \, dg_E \\
\leq 4 \left( \int_{\mathbb{R}^n} \left| \nabla \ell_T \right|^2_{gx} (4\pi T)^{-\frac{n}{2}} e^{-\ell_T} \, dg_E \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^n} \left| \nabla \phi_A \right|^2_{gx} (4\pi T)^{-\frac{n}{2}} e^{-\ell_T} \, dg_E \right)^{\frac{1}{2}} \\
\leq 4 \left( \frac{2n}{T} \right)^{\frac{1}{2}} \left( \frac{4}{A} V(T) \right)^{\frac{1}{2}},
\]

where we have applied Lemma 4.3. Taking \( A \to \infty \), we have

(4.4) \[
\int_{\mathbb{R}^n} \left( T|\nabla \ell_T|_{gx}^2 + \ell_T - n \right) (4\pi T)^{-\frac{n}{2}} e^{-\ell_T} \, dg_E \leq 0.
\]

Henceforth, we shall assume \( T = \frac{1}{2} \) for the sake of convenience. In fact, one may always make such assumption by a parabolic scaling. By Lemma 4.3, we have

\[
\phi(x) := (2\pi)^{-\frac{n}{2}} e^{-\frac{1}{4}\ell_T(x)} \in W^{1,2}(\mathbb{R}^n).
\]

We can rewrite (4.4) as

\[
\int_{\mathbb{R}^n} \left( 2|\nabla \phi_A|_{gx}^2 - \phi^2 \log \phi^2 \right) \, dg_E \\
\leq \left( n - \frac{n}{2} \log(2\pi) \right) \int_{\mathbb{R}^n} \phi^2 \, dg_E \\
\leq \int_{\mathbb{R}^n} \left( 2|\nabla \phi|_{gx}^2 - \phi^2 \log \phi^2 \right) \, dg_E + \log \left( \int_{\mathbb{R}^n} \phi^2 \, dg_E \right) \int_{\mathbb{R}^n} \phi^2 \, dg_E,
\]

where in the second inequality we have applied Theorem 2.1. Consequently, we have

\[
\log \left( \int_{\mathbb{R}^n} \phi^2 \, dg_E \right) \int_{\mathbb{R}^n} \phi^2 \, dg_E \geq 0.
\]

Since, by Theorem 3.7

\[
\int_{\mathbb{R}^n} \phi^2 \, dg_E = V\left( \frac{1}{2} \right) \in (0, 1],
\]

we have

\[
0 \leq V\left( \frac{1}{2} \right) \log \left( V\left( \frac{1}{2} \right) \right) = \log \left( \int_{\mathbb{R}^n} \phi^2 \, dg_E \right) \int_{\mathbb{R}^n} \phi^2 \, dg_E \leq 0.
\]

Therefore, it must hold that \( V\left( \frac{1}{2} \right) = 1 \), and the lemma follows. \( \square \)
REFERENCES

[1] Chen, Bing-Long. Strong uniqueness of the Ricci flow. Journal of Differential Geometry, 2009, 82(2):363-382.
[2] Chen, Bing-Long, and Xi-Ping Zhu. Uniqueness of the Ricci flow on complete noncompact manifolds. Journal of Differential Geometry, 2006, 74(1): 119-154.
[3] Cheng, Liang, and Yongjiu Zhang. Perelman-type no breather theorem for noncompact Ricci flows. Transactions of the American Mathematical Society 374.11 (2021): 7991-8012.
[4] Chen, Xiuxiong, and Bing Wang. On the conditions to extend Ricci flow (III). International Mathematics Research Notices 2013.10 (2013): 2349-2367.
[5] Chow, B.; Chu, S.; Glickenstein, D.; Guenther, C.; Isenberg, J.; Ivey, T.; Knopf, D.; Lu, P.; Luo, F.; Ni, L. The Ricci flow: techniques and applications. Part III. Geometric-Analytic Aspects, Mathematical Surveys and Monographs, vol.163, AMS, Providence, RI, 2010.
[6] DeTurck, Dennis M. Deforming metrics in the direction of their Ricci tensors. Journal of Differential Geometry, 1983, 18(1): 157-162.
[7] Gross, Leonard. Logarithmic sobolev inequalities. American Journal of Mathematics 97.4 (1975): 1061-1083.
[8] Hamilton, Richard S. Three-manifolds with positive Ricci curvature. Journal of Differential geometry, 1982 17(2): 255-306.
[9] Hamilton, Richard S. The inverse function theorem of Nash and Moser. Bulletin of the American Mathematical Society, 1982, 17(1): 65-222.
[10] Hamilton, Richard S. The formation of singularities in the Ricci flow. Surveys in Differential Geometry, 1995(2):7-136.
[11] Kotschwar, Brett L. Backwards uniqueness for the Ricci flow. International Mathematics Research Notices 2010, 21: 4064-4097.
[12] Peng Lu, Gang Tian. Uniqueness of standard solutions in the work of Perelman. https://math.berkeley.edu/~lott/ricciflow/StanUniqWork2.pdf
[13] G.Perelman, The entropy formula for the Ricci flow and its geometric applications. http://arxiv.org/abs/math/0211159
[14] G.Perelman, Ricci flow with surgery on three-manifolds. http://arxiv.org/abs/math/0303109v1
[15] G.Perelman, Finite time extinction for the solutions to the Ricci flow on certain three-manifold. http://arxiv.org/abs/math/0307245
[16] Sesum, Natasa. Limiting behavior of Ricci flows. Massachusetts Institute of Technology, doctoral thesis.
[17] Shi, Wan-Xiong. Deforming the metric on complete Riemannian manifolds. J. Differential Geom. 1989, 30(1):223-301.
[18] Wang, Bing. On the conditions to extend Ricci flow. International Mathematics Research Notices 2008.9 (2008): rnn012-rnn012.
[19] Wang, Bing. On the conditions to extend Ricci flow (II). International Mathematics Research Notices 2012.14 (2012): 3192-3223.
[20] Ye, Rugang. On the L-Function and the Reduced Volume of Perelman I. Transactions of the American Mathematical Society, 2008, 360(1):507-531.

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