Screw Photon-like (3+1)-Solitons in Extended Electrodynamics

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Abstract

This paper aims to present explicit photon-like (3+1) spatially finite soliton solutions of screw type to the vacuum field equations of Extended Electrodynamics (EED) in relativistic formulation. We begin with emphasizing the need for spatially finite soliton modelling of microobjects. Then we briefly comment the properties of solitons and photons and recall some facts from EED. Making use of the localizing functions from differential topology (used in the partition of unity) we explicitly construct spatially finite screw solutions. Further a new description of the spin momentum inside EED, based on the notion for energy-momentum exchange between $F$ and $*F$, is introduced and used to compute the integral spin momentum of a screw soliton. The consistency between the spatial and time periodicity naturally leads to a particular relation between the longitudinal and transverse sizes of the screw solution, namely, it is equal to $\pi$. Planck’s formula $E = h\nu$ in the form of $ET = h\nu$ arises as a measure of the integral spin momentum.

1 Introduction

The very notion of really existing objects, i.e. physical objects carrying energy-momentum, necessarily implies that all such objects must have definite stability properties, as well as properties that do not change with time; otherwise everything would constantly change and we could not talk about objects and structure at all, moreover no memory and no knowledge would be possible. Through definite procedures of measurement we determine, where and when this is possible, quantitative characteristics of the physical objects. The characteristics obtained differ in: their nature and qualities; in their significance to understand the structure of the objects they characterize; in their abilities to characterize interaction among objects; and in their universality.

Natural objects may be classified according to various principles. The classical point-like objects (called usually particles) are allowed to interact continuously with each other just through exchanging (through some mediator usually called field) universal conserved quantities: energy, momentum, angular momentum, so that, the set of objects "before" interaction is the same as the set of objects "after" interaction, no objects have disapeared and no new objects have appeared, only the conserved quantities have been redistributed. This is in accordance with the assumption of point-likeness, i.e. particles are assumed to have no internal structure, so they are undestroyable. Hence, classical particles may be subject only to elastic interaction.

Turning to study the set of microobjects, called usually elementary particles: photons, electrons, etc., physicists have found out, in contrast to the case of classical particles, that a given set of microobjects may transform into another set of microobjects under definite conditions, for example, the well known anihilation process: $(e^+ , e^-) \rightarrow 2\gamma$. These transformations obey also the energy-momentum and angular+spin momentum conservation, but some features may
disappear (e.g. the electric charge) and new features (e.g. motion with the highest velocity) may appear. Hence, microobjects **allow to be destroyed**, so they have structure and, consequently, they would **NOT** admit the approximation "point-like objects". In view of this we may conclude that any theory aiming to describe their behaviour must take in view their structure. In particular, assuming that the Planck formula \( E = h \nu \) is valid for a free microobject, the only reasonable way to understand where the characteristic frequency comes from is to assume that this microobject has a periodic dynamical structure.

The idea of conservation as a dynamics generating rule was realized and implemented in physics firstly by Newton a few centuries ago in the frame of real (classical) bodies: he invented the quantity **momentum**, \( p \), for an isolated body (treated as a point, or structureless object), as its integral characteristic, postulated its time-constancy (i.e. conservation), and wrote down his famous equation \( \dot{p} = F \). This equation says, that the integral characteristic **momentum** of a given point-like object may (smoothly) increase **only** if some other object loses (smoothly) the same quantity of **momentum**. The concept of **force**, \( F \), measures the momentum transferred per unit time. The conservative quantities **energy** and **angular momentum**, consistent with the momentum conservation, were also appropriately incorporated. The most **universal** among these turned out to be the **energy**, since, as far as we know, every natural object carries energy. This property of universality of energy makes it quite distinguished among the other conservative quantities, because its change may serve as a reliable measure of **any** kind of interaction and transformation of real objects.

We especially note, that all these conservative quantities are carried by some object(s), **no energy-momentum can exist without corresponding objects-carryers**. In this sense, the usual words "energy quanta" are senseless if the corresponding carriers are not pointed out.

So, theoretical physics started with idealizing the natural objects as particles, i.e. objects **without structure**, and the real world was theoretically viewed as a collection of **interacting**, i.e. energy-momentum exchanging, particles. As far as the behaviour of the real objects as a whole is concerned, and the interactions considered do **NOT** lead to destruction of the bodies-particles, this theoretical model of the real world worked well.

The 19th century physics, due mainly to Faraday and Maxwell, created the theoretical concept of **electromagnetic field** as the **interaction carrying object**, responsible for the observed mutual influence between distant electrically charged (point-like) objects. This concept presents the electromagnetic field as an extended (in fact, infinite) continuous object, having dynamical structure, and although it influences the behaviour of the charged particles, it does **NOT** destroy them. The theory of the electromagnetic field was based also on balance relations of new kind of quantities. Actually, the new concepts of **flux of a vector field through a 2-dimensional surface** and **circulation of a vector field along a closed curve** were coined and used extensively. The Faraday-Maxwell equations in their integral form establish, in fact, where the time-changes of the fluxes of the electric and magnetic fields go to, or come from, in both cases of a **closed** 2-surface, and of a **not-closed** 2-surface with a boundary, and in this sense they introduce a kind of balance relations. We note, that these fluxes are new quantities, specific to the continuous character of the physical object under consideration; the field equations of Faraday-Maxwell do **NOT** express directly energy-momentum balance relations as the above mentioned Newton’s law \( \dot{p} = F \) does. Nevertheless, they are consistent with energy-momentum conservation, as it is well known. The corresponding local energy-momentum quantities turn out to be quadratic functions of the electric and magnetic vectors.

Although very useful for considerations in finite regions with boundary conditions, the pure field Maxwell equations have time-dependent solutions in the whole space that could hardly be considered as mathematical models of really existing fields. As a rule, if these solutions are time-stable and not static, they occupy the whole 3-space, or its infinite sub-region (e.g. plane waves), and, hence, they carry **infinite** energy and momentum (infinite objects). On the
other hand, according to Cauchy’s theorem for the D’Alembert wave equation [1], which is necessarily satisfied by any component of the vacuum field in Maxwell theory, every finite (and smooth enough) initial field configuration is strongly time-unstable: the initial condition blows up radially and goes to infinity, and its forefront and backfront propagate with the velocity of light. Hence, Faraday-Maxwell equations cannot describe finite time-stable localized field-objects.

The inconsistencies between theory and experiment appeared in a full scale at the end of the last century and it soon became clear that they were not avoidable in the frame of classical physics. After Planck and Einstein created the notion of elementary field quanta, named later by Lewis [2] photon, physicists faced the above mentioned problem: the light quanta appeared to be real objects of a new kind, namely, they did NOT admit the point-like approximation like Newton’s particles did. In fact, every photon propagates (transitionally) as a whole with a constant velocity and keeps unchanged the energy-momentum it carries, which should mean that it is a free object. On the other hand, it satisfies Planck’s relation $E = h\nu$, which means that the very existence of photons is intrinsically connected with a periodical process of frequency $\nu$, and periodical processes in classical physics are generated by external force-fields, which means that the object should not be free. The efforts to overcome this undesirable situation resulted in the appearance of quantum theory, quantum electrodynamics was built, but the assumption ”the point-like approximation works” was kept as a building stone of the new theory, and this brought the theory to some of the well known singularity problems.

Modern theory tries to pay more respect to the view that the point-like approximation does NOT work in principle in the set of microobjects satisfying the above Planck’s formula. In other words, the right theoretical notion for these objects should be that of extended continuous finite objects, or finite field objects. During the last 30 years physicists have been trying seriously to implement in theory the ”extended point of view” on microobjects mainly through the string/brane theories, but the difficulties these theories meet, generated partly by the great purposes they set before themselves, still do not allow to get satisfactory results. Of course, attempts to create extended point view on elementary particles different from the string/brane theory approach, have been made and are being made these days [3].

Anyway, we have to admit now, that after one century away from the discovery of Planck’s formula, we still do not have a complete and satisfactory self-consistent theory of single photons. So, creation of a self-consistent extended point of view and working out a corresponding theory is still a challenge, and this paper aims to consider such an extended point of view on photons as screw solitons in the frame of the newly developed EED [4]. First we summarize the main features/properties of solitons and photons.

## 2 Solitons and Photons

The concept of soliton appears in physics as a nonlinear elaboration - physical and mathematical - of the general notion for exitation in a medium. It includes the following features:

I. PHYSICAL.

1. The medium is homogeneous, isotropic and has definite properties of elasticity.
2. The exitation does not destroy the medium.
3. The exitation is finite:

   -at every moment it occupies comparatively small volume of the medium,
- it carries finite quantities of energy-momentum and angular momentum, and of any other physical quantity too,
- it may have translational-rotational (may time-periodical) dynamical structure, i.e. besides its straightline propagation as a whole it may have internal rotational degrees of freedom.

4. The excitation is time-stable, i.e. at lack of external perturbations its dynamical evolution does not lead to a self-ruin. In particular, the spatial shape of the excitation does not (significantly) change during its propagation.

The above 4 features outline the physical notion of a solitary wave. A solitary wave becomes a soliton if it has in addition the following property of stability:

5. The excitation survives when collides with another excitation of the same nature.

We make some comments on the features 1-5.

Feature 1 requires homogeneity and some elastic properties of the medium, which means that it is capable to bear the excitation, and every region of it, subject to the excitation, i.e. dragged out of its natural (equilibrium) state, is capable to recover entirely after the excitation leaves that region.

Feature 2 puts limitations on the excitations considered in view of the medium properties: they should not destroy the medium.

Feature 3 is very important, since it requires finite nature of the excitations, it enables them to represent some initial level self-organized physical objects with dynamical structure, so that these objects "feel good" in this medium. This finite nature assumption admits only such excitations which may be created and destroyed; no point like and/or infinite excitations are admitted. The excitation interacts permanently with the medium and the time periodicity available may be interpreted as a measure of this interaction.

Feature 4 guarantees the very existence of the excitation in this medium, and the shape keeping during propagation allows its recognition and identification when observed from outside. This feature 4 carries in some sense the first Newton’s principle from the mechanics of particles to the dynamics of continuous finite objects, it implies conservation of energy-momentum and of the other characteristic quantities of the excitation.

The last feature 5 is frequently not taken in view, especially when one considers single excitations. But in presence of many excitations in a given region it allows only such kind of interaction between/among the excitations, which does not destroy them, so that the excitations get out of the interaction (almost) the same. This feature is some continuous version of the elastic collisions of particles.

II. MATHEMATICAL

1. The excitation defining functions \( \Phi^a \) are components of one mathematical object (usually a section of a vector/tensor bundle) and depend on \( n \) spatial and 1 time coordinates.

2. The components \( \Phi^a \) satisfy some system of nonlinear partial differential equations (except the case of (1+1) linear wave equation), and admit some "running wave" dynamics as a whole, together with available internal dynamics.

3. There are (infinite) many conservation laws.

4. The components \( \Phi^a \) are localized (or finite) functions with respect to the spatial coordinates, and the conservative quantities are finite.

5. The multisoliton solutions, describing elastic interaction (collision), tend to many single
Comments:

1. Feature 1 introduces some notion of integrity: one excitation - one mathematical object, although having many algebraically independent but differentially interrelated (through the equations) components $\Phi^a$.

2. Usually, the system of PDE is of evolution kind, so that the excitation is modelled as a dynamical system: the initial configuration determines fully the evolution. The "running wave" dynamics as a whole introduces Galileo/Lorentz invariance and corresponds to the physical feature 4. The nonlinearity of the equations is meant to guarantee the spatially localized (finite) nature of the solutions.

3. The infinite many conservation laws frequently lead to complete integrability of the equations.

4. The spatially localized nature of $\Phi^a$ represents the finite nature of the excitation.

5. The many single soliton asymptotics at $t \to \infty$ of a multisoliton solution mathematically represents the elastic character of the interactions admitted, and so it takes care of the stability of the physical objects being modelled.

The above physical/mathematical features are not always strictly accounted for in the literature. For example, the word soliton is frequently used for a solitary wave excitation. Another example is the usage of the word soliton just when the energy density, being usually a quadratic function of the corresponding $\Phi^a$, has the above soliton properties [5]. Also, one usually meets this soliton terminology for spatially localized, i.e. going to zero at spatial infinity, but not spatially finite $\Phi^a$, i.e. when the spatial support of $\Phi^a$ is a compact set. In fact, all soliton solutions of the well known KdV, SG, NLS equations are localized and not finite. It is curious, that the linear (1+1) wave equation has spatially finite soliton solutions of arbitrary shape.

Further in this paper we shall present 1-soliton screw solutions of the vacuum EED equations, so we may, and shall, use the more attractive word soliton for solitary wave. We hope this will not bring any troubles to the reader.

The screw soliton solutions we are going to present are of photon-like character, i.e. the velocity of their translational component of propagation is equal to the velocity of light $c$, and besides of the energy-momentum, they carry also internal angular (spin, helicity) momentum accounting for the available rotational component of propagation. Therefore, this seems to be the proper place to recall some of the well known properties of photons.

First of all, as it was explained in the Introduction, photons should not be considered as point-like objects since they respect the Planck’s relation $E = h\nu$ and carry internal (spin) momentum, so they have to be considered as extended finite objects with periodical rotational-translational dynamical structure. Therefore, we assume that the concept of soliton, as described above, may serve as a good mathematical tool in trying to represent mathematically the real photons, so that, their well known integral properties to appear as determined by their dynamical structure.

We give now some of the more important for our purposes properties of photons.

1. Photons have zero proper mass and electric charge. The (straightline) translational component of their propagation velocity is constant and equal to the experimentally established velocity of light in vacuum.

2. Photons are time-stable objects. Every interaction with other objects kills them.
3. The existence of photons is generically connected with some time-periodical process of period $T$ and frequency $\nu = T^{-1}$, so that the Planck relation $E = h\nu$, or $ET = h$, where $h$ is the Planck constant, holds.

4. Every single photon carries momentum $p$ with $|p| = h\nu/c$ and spin momentum equal to the Planck constant $h$.

5. Photons are polarized objects. The polarization of every single photon is determined through the relation between the translational and rotational directions of propagation, hence, the 3-dimensionality of the real space allows just two polarizations. We call the polarization right, if when looking along its translational component of propagation, i.e. from behind, we find its rotational component of propagation to be clock-wise, and the polarization is left when under the same conditions we find anti-clock-wise rotational component of propagation.

6. Photons do not interact with each other. i.e. they pass through each other without changes.

These well known properties of photons would hardly need any comments. However, according to our opinion, these properties strongly suggest to make use of the soliton concept for working out a mathematical model of their structure and propagation. And this was one of the main reasons to develop the extension of Faraday-Maxwell theory to what we call now Extended Electrodynamics (EED). We proceed now to recall the basics of EED, in the frame of which the screw soliton model of photons will be worked out.

3 Basics of Extended Electrodynamics in Vacuum

We are going to consider just the vacuum case of EED in relativistic formulation. The signature of the space time pseudometric $\eta$ is $(−,−,−,+)$, the canonical coordinates will be denoted by $(x^1, x^2, x^3, x^4) = (x, y, z, \xi = ct)$, so the components $\eta_{\mu\nu}$ in any canonical coordinate system are $−\eta_{11} = −\eta_{22} = −\eta_{33} = \eta_{44} = 1$, and $\eta_{\mu\nu} = 0$ for $\mu \neq \nu$. The corresponding volume 4-form $\omega_o$ is given by $\omega_o = dx \wedge dy \wedge dz \wedge d\xi$ since $|\det(\eta_{\mu\nu})| = 1$. The Hodge $*$-operator is defined by $\alpha \wedge \beta = \eta^{\ast}(\alpha, \beta) \omega_o$, where $\alpha$ and $\beta$ are $p$ and $4−p$ forms respectively. In terms of $\varepsilon_{\mu\nu\rho\sigma}$ we have $(\ast F)_{\mu\nu} = −\frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}$. We have also the exterior derivative $d$ and the coderivative $\delta = \ast d\ast$.

The physical interpretation of $F_{\mu\nu}$ are: $F_{\mu\nu}$, $i = 1, 2, 3$, are the components $E^1, E^2, E^3$ of the electric vector $E$, and $(F_{23}, −F_{13}, F_{12})$ are the components $B^1, B^2, B^3$ of the magnetic vector $B$, respectively.

In terms of $\delta$ the vacuum Maxwell equations are given by

$$\delta \ast F = 0, \quad \delta F = 0. \quad (1)$$

In EED the above equations (1) are extended to

$$\delta \ast F \wedge F = 0, \quad \delta F \wedge \ast F = 0, \quad \delta F \wedge F − \delta \ast F \wedge \ast F = 0. \quad (2)$$

In components, equations (2) are respectively:

$$(\ast F)_{\mu\nu}(\delta \ast F)^\nu = 0, \quad F_{\mu\nu}(\delta F)^\nu = 0, \quad (\ast F)_{\mu\nu}(\delta F)^\nu + F_{\mu\nu}(\delta \ast F)^\nu = 0. \quad (3)$$

The Maxwell energy-momentum tensor

$$Q^\nu_{\mu} = \frac{1}{8\pi} [F_{\mu\sigma} F^{\nu\sigma} + (\ast F)_{\mu\sigma} (\ast F)^{\nu\sigma}] \quad (4)$$
is assumed as energy-momentum tensor in EED because its divergence
\[ \nabla_\nu Q'_\mu = \frac{1}{4\pi} \left[ F_{\mu\nu}(\delta F)^\nu + (F^*F)_{\mu\nu}(\delta F)^\nu \right] \]

is obviously zero on the solutions of equations (3) and no problems of Maxwell theory at this point are known. The physical sense of equations (3) is, obviously, local energy-momentum redistribution during the time-evolution: the first two equations say that \( F \) and \( *F \) keep locally their energy-momentum, and the third equation says (in correspondence with the first two), that the energy-momentum transferred from \( F \) to \( *F \) is always equal locally to the energy-momentum transferred from \( *F \) to \( F \), hence, any of the two expressions \( F_{\mu\nu}(\delta *F)^\nu \) and \( (F^*F)_{\mu\nu}(\delta F)^\nu \) may be considered as a measure of the rotational component of the energy-momentum redistribution between \( F \) and \( *F \) during propagation (recall that the spatial part of \( \delta F \) is \( \text{rot}B \) and the spatial part of \( \delta *F \) is \( \text{rot}E \)).

Obviously, all solutions of (1) are solutions to (3), but equations (3) have more solutions. In particular, those solutions of (3) which satisfy the relations
\[ \delta F \neq 0, \quad \delta *F \neq 0 \]

are called nonlinear. Further we are going to consider only the nonlinear solutions of (3).

Some of the basic results in our previous studies of the nonlinear solutions of equations (3) could be summarized in the following way: For every nonlinear solution \((F, *F)\) of (3) there exists a canonical system of coordinates \((x, y, z, \xi)\) in which the solution is fully represented by two functions \( \Phi(x, y, \xi + \varepsilon z) \), \( \varepsilon = \pm 1 \), and \( \varphi(x, y, z, \xi) \), \( |\varphi| \leq 1 \), as follows:
\[
F = \varepsilon \Phi \varphi dx \land dz + \Phi \varphi dx \land d\xi + \varepsilon \Phi \sqrt{1 - \varphi^2} dy \land dz + \Phi \sqrt{1 - \varphi^2} dy \land d\xi,
\]
\[
*F = -\Phi \sqrt{1 - \varphi^2} dx \land dz - \varepsilon \Phi \sqrt{1 - \varphi^2} dy \land d\xi + \Phi \varphi dy \land dz + \varepsilon \Phi \varphi dy \land d\xi.
\]
We call \( \Phi \) the amplitude function and \( \varphi \) the phase function of the solution. The condition \(|\varphi| \leq 1\) allows to set \( \varphi = \cos\psi \), and further we are going to work with \( \psi \), and \( \psi \) will be called phase. As we showed [4], the two functions \( \Phi \) and \( \varphi \) may be introduced in a coordinate free manner, so they have well defined invariant sense. Every nonlinear solution satisfies the following important relations:
\[
(\delta F)^2 < 0, \quad (\delta *F)^2 < 0, \quad |\delta F| = |\delta *F|, \quad (\delta F)_\sigma(\delta *F)^\sigma = 0, \quad F_{\mu\nu}(\delta *F)^\nu = F_{\mu\nu}(*F)^\nu = 0.
\]

We recall also the scale factor \( L \), defined by the relation \( L = |\Phi|/|\delta F| \). A simple calculation shows that it depends only on the derivatives of \( \psi \) in these coordinates and is given by
\[
L = \frac{1}{|\psi_\xi - \varepsilon \psi_z|}.
\]

4 Screw Soliton Solutions in Extended Electrodynamics

Note that EED considers the field as having two components: \( F \) and \( *F \). As we mentioned earlier, the third equation of (3) describes how much energy-momentum is redistributed locally with time between the two components \( F \) and \( *F \) of the field: \( F_{\mu\nu}(\delta *F)^\nu dx^\mu \) gives the transfer from \( F \) to \( *F \), and \( (*F)_{\mu\nu}\delta*F^\nu dx^\mu \) gives the transfer from \( *F \) to \( F \), thus, if there is such an energy-momentum exchange equations (3) require permanent and equal mutual energy-momentum transfers between \( F \) and \( *F \). Since \( F \) and \( *F \) are always orthogonal to each other \( [F_{\mu\nu}(*F)^{\mu\nu} = 0] \) and these two mutual transfers depend on the derivatives of the field functions
through $\delta F$ and $\delta \ast F$ (i.e. through $\text{rot} \mathbf{B}$ and $\text{rot} \mathbf{E}$, which are not equal to zero in general), we may interpret this property of the solution as a description of an \textit{internal rotation-like} component of the general dynamics of the field. Hence, any of the two expressions $F_{\mu\nu}(\delta \ast F)_{\nu}^{\mu} dx^{\mu}$ or $(\ast F)_{\mu\nu\rho} \delta F_{\nu}^{\rho} dx^{\mu}$, having the sense of local energy-momentum change, may serve as a natural measure of this rotational component of the energy-momentum redistribution during the propagation. Therefore, after some appropriate normalization, we may interpret any of the two 3-forms $(\ast F) \wedge (\delta \ast F)$ and $F \wedge \delta F$ as local \textit{spin-momentum} of the solution. Making use of the above expressions for $F$ and $\ast F$ we compute $F \wedge \delta F = (\ast F) \wedge (\delta \ast F)$:

$$F \wedge \delta F = - \varepsilon \Phi^2(\psi_{\xi} - \varepsilon \psi_z) dx \wedge dy \wedge dz - \Phi^2(\psi_{\xi} - \varepsilon \psi_z) dx \wedge dy \wedge d\xi.$$  

Since $\Phi \neq 0$, this expression says that we shall have nonzero local spin-momentum only if $\psi$ is \textit{not} a running wave along $z$. The above idea to consider the 3-form $F \wedge \delta F$ as a measure of the spin momentum of the solution suggests also some additional equation for $\psi$, because the spin momentum is a conserved quantity and its integral over the 3-space should not depend on the time variable $\xi$. This requires to have some nontrivial \textit{closed 3-form} on $\mathcal{R}^4$, such that when restricted to the 3-space and (spatially) integrated to give the integral spin momentum of the solution. Therefore we assume the additional equation

$$\mathbf{d}(F \wedge \delta F) = 0. \quad (8)$$

In our system of coordinates this equation is reduced to

$$\mathbf{d}(F \wedge \delta F) = \varepsilon \Phi^2 (\psi_{\xi\xi} + \psi_{zz} - 2\varepsilon \psi_{z\xi}) dx \wedge dy \wedge dz \wedge d\xi = 0, \quad (9)$$

i.e.

$$\psi_{\xi\xi} + \psi_{zz} - 2\varepsilon \psi_{z\xi} = (\psi_{\xi} - \varepsilon \psi_z)_{\xi} - \varepsilon (\psi_{\xi} - \varepsilon \psi_z)_{z} = 0. \quad (10)$$

Equation (10) has the following solutions:

1. Running wave solutions $\psi = \psi(x, y, \xi + \varepsilon z)$,
2. $\psi = \xi g(x, y, \xi + \varepsilon z) + b(x, y)$,
3. $\psi = \varepsilon g(x, y, \xi + \varepsilon z) + b(x, y)$,
4. Any linear combination of the above solutions with coefficients which are allowed to depend on $(x, y)$. The functions $g(x, y, \xi + \varepsilon z)$ and $b(x, y)$ are arbitrary in the above expressions.

The running wave solutions $\psi_1$, defined by 1$^a$ lead to $F \wedge \delta F = 0$ and to $|\delta F| = 0$, and by this reason they have to be ignored. The solutions $\psi_2$ and $\psi_3$, defined respectively by 2$^a$ and 3$^a$, give the same scale factor $L = 1/|g|$. Since at all spatial points where the field is different from zero we have $\xi + \varepsilon z = \text{const}$, we may choose $|g(x, y, \xi + \varepsilon z)| = 1/l(x, y) > 0$, so we obtain the following \textit{nonrunning} wave solutions of (10):

$$\psi_2 = \frac{\kappa \xi}{l(x, y)} + b(x, y); \quad \psi_3 = \frac{\kappa \varepsilon}{l(x, y)} + b(x, y), \quad (11)$$

where $\kappa = \pm 1$ accounts for the two different polarizations. Clearly, the physical dimension of $l(x, y)$ is \textit{length}, $b(x, y)$ is dimensionless and the scale factor is $L = l(x, y)$.

The corresponding electric $\mathbf{E}$ and magnetic $\mathbf{B}$ vectors for the case 2$^a$ in view of (11) are

$$\mathbf{E} = \left[ \Phi(x, y, \xi + \varepsilon z) \cos \left( \frac{\pm \xi}{l(x, y)} + b(x, y) \right); \Phi(x, y, \xi + \varepsilon z) \sin \left( \frac{\pm \xi}{l(x, y)} + b(x, y) \right) \right]; 0, \quad (12)$$

$$\mathbf{B} = \left[ \varepsilon \Phi(x, y, \xi + \varepsilon z) \sin \left( \frac{\pm \xi}{l(x, y)} + b(x, y) \right); -\varepsilon \Phi(x, y, \xi + \varepsilon z) \cos \left( \frac{\pm \xi}{l(x, y)} + b(x, y) \right) \right]; 0, \quad (13)$$
A characteristic feature of the solutions, defined by (12)-(13), is that the direction of the electric and magnetic vectors at some initial moment \( t_0 \), as seen from (12)-(13), is entirely determined by \((\psi_2)_{t_0}\) (under \( L = 1/|g| = l(x,y) \), and so, it does not depend on the coordinate \( z \). Therefore, at all points of the 3-region \( \Omega_o \), occupied by the solution at \( t_0 \), under the additional conditions \( L = l(x,y) = \text{const} \) and \( b(x,y) = \text{const} \), the directions of the representatives of \( E \) at all spatial points will be the same, independently on the spatial shape of the 3-region \( \Omega_o \). At every subsequent moment this common direction will be rotationally displaced, but will stay the same for the representatives of \( E \) at the different spatial points. The same feature will hold for the representatives of \( B \) too. Thus, the representatives of the couple \((E,B)\) will rotate in time, clockwise \((\kappa = -1)\) or anticlockwise \((\kappa = 1)\), coherently at all spatial points, so these solutions show no twist-like, or screw, propagation component even if the region \( \Omega_o \) has a screw shape.

Hence, these solutions may be soliton-like but not of screw kind.

The electric \( E \) and magnetic \( B \) vectors for the case \( 3^o \) in view of (11) are

\[
E = \left[ \Phi(x,y,\xi + \varepsilon z)\cos\left(\frac{\pm z}{l(x,y)} + b(x,y)\right); \Phi(x,y,\xi + \varepsilon z)\sin\left(\frac{\pm z}{l(x,y)} + b(x,y)\right); 0 \right], \quad (14)
\]

\[
B = \left[ \varepsilon\Phi(x,y,\xi + \varepsilon z)\sin\left(\frac{\pm z}{l(x,y)} + b(x,y)\right); -\varepsilon\Phi(x,y,\xi + \varepsilon z)\cos\left(\frac{\pm z}{l(x,y)} + b(x,y)\right); 0 \right], \quad (15)
\]

We are going to use this particular solution (14)-(15) to construct a theoretical example of a screw photon-like soliton solution. We have to give an explicit form of the amplitude function \( \Phi \). In accordance with the finite nature of the solution \( \Phi \) must have compact spatial support, and this guarantees that the solution is finite because the phase function \( \varphi = \cos \), so, the products \( \Phi.\cos(\psi), \Phi.\sin(\psi) \) are also finite.

First we outline the idea (FIG1).

We mean to choose \( \Phi = \Phi_o \) at \( \xi = 0 \) in such a way that \( \Phi_o(x,y,z) \) to be localized inside a screw cylinder of radius \( r_o \) and height \( |z_2 - z_1| = 2\pi l_o \), and we shall denote this region by \( \Omega_o \); the \( z \)-prolongation of \( \Omega_o \) will be an infinite screw cylinder denoted by \( \Omega \). Let this screw cylinder have the coordinate axis \( z \) as an outside axis, i.e. \( \Omega \) windes around the \( z \)-axis and never crosses it. The initial configuration \( \Omega_o \) shall be made propagating along \( \Omega \) through both: \( z \)-translation, and a consistent rotation around itself and around the \( z \)-axis. This kind of propagation will be achieved only if every spatial point inside \( \Omega_o \) will keep moving along its own screw line and will never cross its neighbors' screw lines.

We consider the plane \((x,y)\) at \( z = 0 \) and choose a point \( P = (a,a) \), \( a > 0 \) in the first quadrant in this plane, so that the points \((x,y,z = 0)\) where \( \Phi_o(x,y,z = 0) \neq 0 \) are inside the
circle \( x^2 + y^2 \leq r_o^2 \) centered at \( P = (a, a) \), and the distance \( a\sqrt{2} \) between \( P \) and the zero-point \((0, 0, 0)\) is much greater than \( r_o \).

Now we consider the function

\[
\frac{1}{\cosh \left[ (x - a)^2 + (y - a)^2 \right]}.
\]

It is concentrated mainly inside \( \Omega \) and goes to zero outside \( \Omega \) although it becomes zero just at infinity. Restricted to the plane \( z = 0 \) it is concentrated inside the circle \( A_{r_o} \) of radius \( r_o \) around the point \( P \) and it becomes zero at infinity \((x = \infty, y = \infty)\). We want this concentrated but still infinite object to become finite, i.e. to become zero outside \( \Omega \). In order to do this we shall make use of the so called localizing functions, which are smooth everywhere, are equal to 1 on some compact set \( A \) and quickly go to zero outside \( A \). (These functions are very important in differential topology for making partition of unity and glueing up various structures \([6]\)). We shall denote these functions by \( \theta(x, y, \ldots) \). Let \( \theta(x, y; r_o) \) be a localizing function around the point \( P \) in the plane \((x, y)\), such that it is equal to 1 inside the circle \( A_r \), centered at \( P \), of radius \( r \), and \( r \) is considered to be a bit shorter but nearly equal to \( r_o \), and \( \theta(x, y; r_o) = 0 \) outside the circle \( A_{r_o} \). We modify now the above considered function as follows:

\[
\frac{\theta(x, y; r_o)}{\cosh \left[ (x - a)^2 + (y - a)^2 \right]}.
\]

Let now \( \theta(z; l_o) \) be a localizing function with respect to the interval \((z_1, z_2)\), \( z_2 > z_1 \), where \( |z_2 - z_1| = 2\pi l_o \). Hence, the function

\[
\Phi_o(x, y, 0 + z) = \frac{\theta(x, y; r_o)\theta(z; l_o)}{\cosh \left[ (x - a)^2 + (y - a)^2 \right]} \cos \left( \frac{z}{l_o} + b(x, y) \right),
\]

is different from zero only inside \( \Omega_o \). We choose now the scale factor \( l(x, y) \) to be a constant equal to \( l_o \), and making use of the phase \( \psi_3 \) with \( \kappa = 1 \) we define the function

\[
\Phi_o\psi_3^+ = C \frac{\theta(x, y; r_o)\theta(z; l_o)}{\cosh \left[ (x - a)^2 + (y - a)^2 \right]} \cos \left( \frac{z}{l_o} + b(x, y) \right),
\]

where \( C \) is a constant with appropriate physical dimension. This function (16) represents the first component \( \mathbf{E}_1 \) of the electric vector at \( t = 0 \). Similarly, for \( \mathbf{E}_2 \) at \( t = 0 \) we write

\[
\Phi_o\psi_3^+ = C \frac{\theta(x, y; r_o)\theta(z; l_o)}{\cosh \left[ (x - a)^2 + (y - a)^2 \right]} \sin \left( \frac{z}{l_o} + b(x, y) \right),
\]

The right hand sides of (16) and (17) define the initial state of the solution, and this initial condition occupies the screw cylinder \( \Omega_o \), its internal axis is a screw line away from the \( z \)-axis at a distance of \( d = a\sqrt{2} \). Hence, the solution in terms of \((\mathbf{E}, \mathbf{B})\) in this system of coordinates will look like (we assume \( C > 0 \), \( \varepsilon = -1 \), \( b(a, a) = 3\pi \) and we write just \( b \) for \( b(x, y) \)).

\[
\mathbf{E} = \left\{ \begin{array}{l}
C \frac{\theta(x, y; r_o)\theta(\xi - z; l_o)}{\cosh \left[ (x - a)^2 + (y - a)^2 \right]} \cos \left( \frac{z}{l_o} + b \right); \\
C \frac{\theta(x, y; r_o)\theta(\xi - z; l_o)}{\cosh \left[ (x - a)^2 + (y - a)^2 \right]} \sin \left( \frac{z}{l_o} + b \right); 0
\end{array} \right\}
\]
\[
B = \left\{ -C \frac{\theta(x, y; r_o) \theta(x - z; l_o)}{\cosh [(x - a)^2 + (y - a)^2]} \sin \left( \frac{z}{l_o} + b \right); \quad \frac{\theta(x, y; r_o) \theta(x - z; l_o)}{\cosh [(x - a)^2 + (y - a)^2]} \cos \left( \frac{z}{l_o} + b \right); \ 0 \right\} \quad (19)
\]

We consider now the vectors \( E \) and \( B \) on the screw line passing through the point \((a, a, 0)\), where \( b(a, a) = 3\pi \). We choose the coordinates \( x, y, z \) as usual: \( x \) grows rightwards, \( y \) grows upwards, and then \( z \) is determined by the requirement to have a right orientation. If we count the phase anticlockwise from the \( x \)-axis, then at \( \xi = ct = 0 \), at the point \( P = (a, a, z = 0) \) the vector \( B \) is directed to the point \((0, 0, 0)\) and \( E \) is orthogonal to \( B \) in such a way that \((E \times B)\) is directed to \(+z\). When \( z \) grows from 0 to \( 2\pi l_o \) the magnetic vector \( B \) on this screw line (which is the central screw line of the screw cylinder) turns around the axis \( z \) and stays always directed to it, while the electric vector \( E \) rotates around the \( z \)-axis and it is always tangent to this rotation; at the same time the Poynting vector keeps its \(+z\)-orientation. This situation corresponds to the clockwise polarizaton (looking from behind). If \( \kappa = -1 \) then the electric vector \( E \) stays always directed to the \( z \)-axis and since we have chosen \( \varepsilon = -1 \), i.e. the solution propagates along \(+z\), the magnetic vector \( B \) rotates around the \( z \)-axis anticlockwise (looking from behind) and is always tangent to this rotation. So, this corresponds to the anticlockwise polarization.

We note once again that any actual assumption \( b(x, y) = \text{const} \) would make the representatives of the magnetic vector (when \( \kappa = 1 \)) \( B \) parallel to each other at every point of the plane \( z = z_o \), \( z_1 < z_o < z_2 \) at \( t = 0 \). This may cause some instabilities with time, so in order to have all representatives of \( B \) (or \( E \) in the other polarization) at every point of that plane to be directed to the \( z \)-axis and not to be parallel to each other, we may use \( b(x, y) \) for the necessary corrections. This remark shows the importance of the relation \( b(x, y) \neq \text{const, } b(x, y) \) must be equal to the angle between the two lines passing through the points \([[0, 0, z_o]; (a, a, z_o)]\) and \([[0, 0, z_o]; (x, y, z_o)]\), where \( x^2 + y^2 \leq r_o^2 \).

Now, every point \((x, y, z)\) inside the solution region follows its own screw line so, that the distance \( \rho = \sqrt{x^2 + y^2} \) between this screw line and the \( z \)-axis is kept the same when \( z \) grows. Since the spatial periodicity along \( z \) is equal to \( 2\pi l_o \) (note that \( l_o \) is in fact the maximum value of \( l(x, y) \)), we obtain an interpretation of the scale factor \( L = l_o \) as the distance between the screw line inside \( \Omega \) along which \( L = l_o \) and the \( z \)-axis. This, in particular, means that for \( r_o << a\sqrt{2} \), i.e. for a very thin screw cylinder, the scale factor \( L = l(x, y) \) is approximated by \( L = l_o(\theta(x, y; r_o)) \); it also follows that the relation between the longitudinal and transverse sizes of the solution is approximately equal to \( \pi \).

5 The Spin-momentum

We turn now to the integral spin-momentum (helicity) computation. According to our assumption its density is given by any of the correspondingly normalized two 3-forms \( F \wedge \delta F \), or \((*F) \wedge (\delta *F)\). In order to have the appropriate physical dimension we consider now the 3-form \( \beta \) defined by

\[
\beta = 2\pi \frac{L^2}{c} F \wedge \delta F = 2\pi \frac{L^2}{c} \left[ -\varepsilon \Phi^2(\psi_x - \varepsilon \psi_z)dx \wedge dy \wedge dz - \Phi^2(\psi_x - \varepsilon \psi_z)dx \wedge dy \wedge d\xi \right]. \quad (20)
\]

Its physical dimension is "energy-density \times time". Since \( L = L(x, y) \) at most, and in view of (8), we see that \( \beta \) is closed: \( d\beta = 0 \). The restriction of \( \beta \) to \( \mathcal{R}^3 \) (which will be denoted also by \( \beta \)) is also closed:

\[
d\beta = -2\pi d \left[ \frac{L^2(x, y)}{c} \varepsilon \Phi^2(\psi_x - \varepsilon \psi_z)dx \wedge dy \wedge dz \right] = 0,
\]
and we may use the Stokes’ theorem. We shall make use of the solutions 3° of equation (10). We have \( \psi_\xi = 0, \psi_z = \kappa/l(x, y), L = |\psi_\xi - \varepsilon \psi_z|^{-1} = l, \) so, in view of our approximating assumption \( L = l_o, \) we integrate

\[
\beta = \frac{2\pi l_o}{c} \kappa \Phi^2 dx \wedge dy \wedge dz
\]

over the 3-space and obtain

\[
\int_{\mathbb{R}^3} \beta = \kappa E \frac{2\pi l_o}{c} = \kappa ET = \pm ET,
\]

where \( E \) is the integral energy of the solution, \( T = 2\pi l_o/c \) is the intrinsically defined time-period, and \( \kappa = \pm 1 \) accounts for the two polarizations. According to our interpretation this is the integral intrinsic angular momentum, or spin-momentum, of the solution, for one period \( T. \) This intrinsically defined action \( ET \) of the solution is to be identified with the Planck’s constant \( h, h = ET, \) or \( E = h\nu, \) if we are going to interpret the solution as an extended model of a single photon.

**Remark.** For the connection of \( F \wedge \delta F \) with our earlier definition of the local intrinsic spin-momentum through the Nijenhuis torsion tensor of \( F^\nu_\mu \) one may look in our paper, cited as the last one in [5].

### 6 Conclusion

According to our view, based on the conservation properties of the energy-momentum and spin-momentum, photons are real objects and NOT theoretical imagination. This view on these microobjects as real extended objects, obeying the famous Planck relation \( E = h\nu, \) almost necessarily brings us to favor the soliton concept as the most appropriate and self-consistent working theoretical tool for now, because no point-like conception is consistent with the availability of frequency and spin-momentum of photons, as well as with the possibility photons to be destroyed. So, the dynamical structure of photons is a real thing, clearly manifesting itself through a consistent rotational-translational propagation in space, and their finite nature reveals itself through the finite 3-volumes of definite shape they occupy at every moment of their existence, and through the finite values of the universal conserved quantities they carry.

The dynamical point of view on these objects reflects theoretically in the possibility to make use of, more or less, arbitrary initial configurations, i.e. to consider them as dynamical systems. This important moment allows the localizing functions \( \theta(x,...) \) from differential topology to be used for making the spatial dimensions of the solution FINITE, and NOT smoothly vanishing just at infinity as is the case of the usual soliton solutions.

Our theoretical screw example, presenting an exact solution of the nonlinear vacuum EED equations, was meant to present a, more or less, visual image of the well known properties of photons, of their translational-rotational dynamical structure, and, especially, of the nature of their spin-momentum. Of course, we do not insist on the function \( 1/\cosh(....) \) chosen, any other localisable inside the circle \( A_{r_o} \) function would do the same job in this theoretical example. So far we have not experimental data concerning the shape of single photons, and we do not exclude various ones, so our choices for the amplitude \( \Phi \) and for the scale factor \( L(x, y) \) are rather admissible approximations than correct mathematical images. The more important moment was to recognize the dynamical sense of the quantity \( F \wedge \delta F, \) and to find that the spin-momentum conservation equation \( d(F \wedge \delta F) = 0 \) gives solutions for the phase function \( \varphi = \cos \psi, \) which helps very much to visualize the dynamical properties of photons in our aproach. So, we are able to obtain a general expression for the integral spin-momentum, which, in fact, is the Planck’s formula \( h = ET. \) Moreover, the whole solution \( (F, *F) \) describes naturally the polarization
properties of photons, it clearly differs the clockwise and anticlockwise polarizations through pointing out the different roles of $E$ and $B$ in the two cases.

An important moment in our approach is that $F$ and $*F$ are considered as two components of the same solution, so the couples $(E, B)$ and $(-B, E)$ give together the 3-dimensional picture of one solution. This, of course, would require the full energy-momentum tensor $Q^\mu_\nu$, to be two times the usual $Q^\mu_\nu$, given by (4): $Q^\mu_\nu(F, *F) = 2Q^\mu_\nu(F) = 2Q^\mu_\nu(*F)$.

Finally, we’d like to mention that, making use of the localizing functions $\theta(x, y, z; \ldots)$, we may choose an amplitude function $\Phi$ of a ”many-lump” kind, i.e. at every moment $\Phi$ to be different from zero inside many non-overlapping 3-regions, probably of the same shape, so we are able to describe a flow of consistently propagating photon-like excitations of the same polarization and of the same phase. Some of such ”many-lump” solutions (i.e. flows of many 1-soliton solutions) may give the macroimpression of, or to look like as, (parts of) plane waves.

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