Para-Hermitian Geometry and Doubled Aspects of Vaisman Algebroid

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Abstract. The geometry of double field theory (DFT) is described by a para-Hermitian manifold $M$. A tangent bundle of the para-Hermitian manifold $T M$ is decomposed into two eigenbundles $L$ and $\tilde{L}$ associated with the eigenvalues of the para-complex structure $K$. We define a Lie algebroid structure on the eigenbundles $L$, $\tilde{L}$. The gauge symmetry algebra of DFT is governed by the $C$-bracket. The algebraic structure based on the $C$-bracket is not a Courant algebroid, but a metric algebroid proposed by Vaisman (the Vaisman algebroid). We show that the Vaisman algebroid in DFT is naturally defined on $T M$ by an analogue of the Drinfel’d double of $L$, $\tilde{L}$. We also find that an algebraic origin of the strong constraint is the condition for $(L, \tilde{L})$ to become a Lie bialgebroid.

1. Introduction

String theory is studied as a candidate of quantum gravity. This theory has fruitful mathematical structures. In particular, duality, which relates various different theories, is an important concept to uncover the whole picture of the theory. For example, we consider a string propagating in a compact space. The strings can wrap around a compactified direction and have energy proportional to the number of windings (the winding mode $w \in \mathbb{Z}$). The Hamiltonian of the string is characterized by the quantized momentum along the compactified direction (the Kaluza-Klein (KK) mode $n \in \mathbb{Z}$) and the winding mode. At this time, the Hamiltonian $H$ is invariant under the interchange of the KK-mode and the winding mode (at the same time, the radius of the compactified direction becomes an inverse number): $H(n, w) \leftrightarrow H(w, n)$. This invariance is called the target space duality, or T-duality [1]. When the compact space is a $D$-dimensional torus $T^D$, T-duality is expressed as a symmetry by $O(D, D)$ group. In supergravity, which is a low energy effective theory of strings, T-duality is hidden symmetry.

One of the approaches to deal geometrically with T-duality is the Hitchin’s generalized geometry (GG) [2]. This is defined on an extended bundle that combines a tangent bundle and a cotangent bundle of the target space: $TM = TM \oplus T^*M$. The gauge symmetry with respect to the target space of non-linear sigma models of strings is described by a Courant algebroid appearing in GG. The Courant algebroid accommodates the diffeomorphism of the spacetime and the NSNS $B$-field gauge symmetry algebra. In contrast to doubling the tangent bundle in GG, there is a formalism that doubles the base space, which is called double field theory (DFT) [3]. The action of DFT can be written in the Einstein-Hilbert-like form, i.e., DFT is Einstein’s gravity theory with T-duality as a manifest symmetry. DFT is defined in the doubled spacetime $M^{2D}$, which is locally characterized by the doubled coordinates $X^M = (x^\mu, \tilde{x}_\mu)$. Since
DFT includes extra degrees of freedom, it is necessary to impose the *physical section condition* in order to obtain a true physical theory. There are two section conditions: the *weak constraint* derived from the level matching condition of closed strings and the *strong constraint* derived from a condition of the gauge symmetry of DFT. The gauge symmetry algebra of DFT is governed by the *C-bracket* [4]. The C-bracket reduces to the Courant bracket in GG when the strong constraint is imposed.

It is interesting to study the mathematical aspects of DFT because the construction of DFT was done physically rather than mathematically. The *metric algebroid* (we call this the Vaisman algebroid) is proposed in [5] as the structure of symmetric algebras based on the C-bracket. The doubled spacetime of DFT is given by a para-Hermitian manifold, and the Vaisman algebroid is an algebraic structure defined by the C-bracket on its tangent bundle [6]. Recently, the Vaisman algebroid was rediscovered as the pre-DFT algebroid in the study of membrane sigma models that corresponds to DFT [7].

It is known that a Courant algebroid is obtained by an analogue of the Drinfel’d double of a Lie bialgebroid [8]. Therefore the Courant algebroid has the doubled structure. Some aspects of the doubled structure of the Vaisman algebroid were studied [9]. In particular, the Vaisman algebroid is obtained as a double of a pair of Lie algebroids [10]. At this time, the key is the derivation condition of Lie bialgebroids.

In this contribution, we show that the Vaisman algebroid in DFT is given by an analogue of the Drinfel’d double of a Lie algebroid pair naturally defined on a para-Hermitian manifold. In section 2, we give a lightning review of DFT for grasping the physical aspects of DFT. We introduce important notions in DFT for the following discussion. In section 3, we review a para-Hermitian manifold. DFT is described as a flat para-Hermitian manifold. In section 4, we calculate the “double” for the Vaisman algebroid in DFT setup. In section 5, we give the conclusion and the future directions. We also discuss the gauge symmetry in DFT. The main results in this contribution are based on the original work [10].

### 2. A Lightning Review of Double Field Theory

We give a quick introduction to DFT in this section. The T-duality transformation in DFT is expressed as an $O(D, D)$ group action in the doubled spacetime. The doubled spacetime is given by a $2D$-dimensional differential manifold $\mathcal{M}^{2D}$. The coordinates of $\mathcal{M}^{2D}$ is given by

$$X^M = \left( \tilde{x}_\mu, x^\mu \right) \quad (M = 1, \ldots, 2D; \, \mu = 1, \ldots, D).$$

(1)

Here $x^\mu$ is the coordinate of a conventional physical spacetime. $\tilde{x}_\mu$ is the Fourier dual of the winding modes of strings and called the winding coordinate. The doubled index is raised and lowered by the $O(D, D)$ invariant structure (the $\eta$-metric):

$$\eta_{MN} = \begin{pmatrix} 0 & \delta^\mu_\nu \\ \delta_\mu^\nu & 0 \end{pmatrix}, \quad \eta^{MN} = \begin{pmatrix} 0 & \delta_\mu^\nu \\ \delta^\mu_\nu & 0 \end{pmatrix}. \quad (2)$$

The dynamical fields in DFT are the generalized metric $\mathcal{H}(g, B)$ and the DFT dilaton $d(g, \phi)$. Here, $g$ is a conventional $D$-dimensional spacetime metric, $B$ is the NSNS 2-form and $\phi$ is the dilaton in type II supergravities. The action of DFT is given by the Einstein-Hilbert-like form

$$S_{\text{DFT}} = \int d^{2D} X \, e^{-2d} R(\mathcal{H}, d),$$

(3)

where $R(\mathcal{H}, d)$ is called the generalized Ricci scalar. Detailed expressions for $\mathcal{H}$, $d$ and $R$ is found in [11]. This action is invariant by the $O(D, D)$ group action. Moreover, this action is invariant under the DFT gauge transformation.
The infinitesimal DFT gauge transformation of $O(D, D)$ tensors is given by the generalized Lie derivative $\hat{\mathcal{L}}_\Xi$. The generalized Lie derivative on a doubled vector $V^M$ is defined by

$$\hat{\mathcal{L}}_\Xi V^M = \Xi^K \partial_K V^M + (\partial^M \Xi_K - \partial_K \Xi^M) V^K. \tag{4}$$

Here $\Xi$ is a gauge parameter and is a doubled vector. The generalized Lie derivative is also called the D-bracket. The commutator of the generalized Lie derivatives is calculated as

$$[\hat{\mathcal{L}}_{\Xi_1}, \hat{\mathcal{L}}_{\Xi_2}] = \hat{\mathcal{L}}_{[\Xi_1, \Xi_2]}. \tag{5}$$

This means that the DFT gauge algebra is closed by the C-bracket. The symbol “$\approx$” in (5) means equality when imposing the strong constraint. The C-bracket is defined by

$$[\Xi_1, \Xi_2]_C = \Xi^K \partial_K \Xi_2^M - \Xi^K \partial_K \Xi_1^M - \frac{1}{2} \eta_{KL}(\Xi^K \partial^M \Xi_L^K - \Xi^K \partial^M \Xi_K^M). \tag{6}$$

The strong constraint is a condition for reducing extra degrees of freedom in DFT and obtaining a true physical theory. The strong constraint is given by

$$\eta^{MN} \partial_M * \partial_N * = 0. \tag{7}$$

Here $*$ denotes arbitrary DFT fields and gauge parameters. One of a trivial solution to this constraint is that the derivative of the winding coordinates vanishes $\partial^\mu * = 0$. This solution is called the supergravity frame because it shows that fields in the theory depend only on the coordinate $x^\mu$ that are the Fourier dual to the KK-modes. In the supergravity frame $\partial^\mu * = 0$, the DFT action (3) is reduced to the NSNS sector of type II supergravity:

$$S_{\text{DFT}} \overset{\partial^\mu *=0}{\longrightarrow} S_{\text{NSNS}} = \int d^D x \sqrt{-g} e^{-2\phi} \left[ R + 4(\partial \phi)^2 - \frac{1}{12} (H_\beta)^2 \right], \tag{8}$$

where $R$ is the Ricci scalar with the metric $g$, and $H_\beta = dB$ is the field strength of the $B$-field. Similarly, the C-bracket is reduced to the Courant bracket in GG:

$$[\Xi_1, \Xi_2]_C \overset{\partial^\mu *=0}{\longrightarrow} [A + \alpha, B + \beta]_C = [A, B]_L + \mathcal{L}_A \beta - \mathcal{L}_B \alpha - \frac{1}{2} d(\iota_A \beta - \iota_B \alpha). \tag{9}$$

Then, $\Xi_1 = A + \alpha$, $\Xi_2 = B + \beta$ are denoted as a formal sum of vector fields and 1-forms.

We focus on the doubled structure of the C-bracket. The component of the gauge parameters $\Xi^M$ and the basis of the doubled vector fields are given by

$$\Xi_1^M = \begin{pmatrix} \alpha_\mu \\ A^\mu \end{pmatrix}, \quad \Xi_2^M = \begin{pmatrix} \beta_\mu \\ B^\mu \end{pmatrix}, \quad \partial_M = \begin{pmatrix} \partial^\mu \\ \partial^\mu \end{pmatrix}. \tag{10}$$

Here the doubled vector field is given by $\Xi^M \partial_M$. We call $\alpha_\mu \partial^\mu$ appearing as a component of the doubled vector field as the winding vector field. The Lie bracket and the dual “winding” Lie bracket are defined by

$$[A, B]_L = (A^\nu \partial_\nu B^\mu - B^\nu \partial_\nu A^\mu) \partial_\mu, \quad [\alpha, \beta]_L = (\alpha_\nu \partial^\nu \beta_\mu - \beta_\nu \partial^\nu \alpha_\mu) \partial^\mu. \tag{11}$$

We define naively the exterior derivative $d$ acting on the winding vector fields, the winding exterior derivative $d$ acting on the vector fields and the interior products $\iota_A, \iota_\alpha$:

$$d\alpha = \partial_\mu \alpha_\nu \partial^\nu \land \partial^\mu, \quad dA = \partial_\mu A^\nu \partial^\mu \land \partial_\nu, \quad \iota_A \alpha = \iota_\alpha A = A^\mu \alpha_\mu. \tag{12}$$
The Lie derivative $\mathcal{L}_A$ acting on the winding vector fields and the winding Lie derivative $\tilde{\mathcal{L}}_\alpha$ acting on the vector fields are given by analogous formulæ of the Cartan’s homotopy formula:

$$\mathcal{L}_A \alpha = (d\iota_A + \iota_A d)\alpha, \quad \tilde{\mathcal{L}}_\alpha A = (\tilde{d}\tilde{\iota}_\alpha + \tilde{\iota}_\alpha \tilde{d})A. \quad (13)$$

Then we can rewrite the $C$-bracket (6) into a non-$O(D, D)$ covariant form:

$$\{\Xi_1, \Xi_2\}_C = [A, B]_L + \mathcal{L}_A \beta - \mathcal{L}_B \alpha - \frac{1}{2} d(\iota_A \beta - \iota_B \alpha) + [\alpha, \beta]_L + \tilde{\mathcal{L}}_\alpha B - \tilde{\mathcal{L}}_\beta A + \frac{1}{2} d(\iota_A \beta - \iota_B \alpha). \quad (14)$$

### 3. Para-Hermitian Geometry

In this section, we introduce a para-Hermitian manifold as the doubled spacetime of DFT [5, 6, 12, 13]. Vaisman [5] first mentioned that the doubled geometry of DFT is described as a para-Hermitian geometry.

#### 3.1. Almost Para-Hermitian Manifold

An almost para-complex manifold is defined by $(\mathcal{M}, K)$ where $\mathcal{M}$ is a differential manifold and $K$ is an endomorphism $K : T\mathcal{M} \rightarrow T\mathcal{M}$ satisfying $K^2 = 1$. $K$ is called the almost para-complex structure. The prefix “para” means a real analogue of the (almost) complex structure $J^2 = -1$. By using a para-complex structure $K$, the tangent bundle is decomposed into $T\mathcal{M} = L \oplus \tilde{L}$. $L$ and $\tilde{L}$ are the eigenbundles associated with the eigenvalues $K = +1, -1$. This decomposition is performed via the projection operators:

$$P = \frac{1}{2}(1 + K), \quad \tilde{P} = \frac{1}{2}(1 - K). \quad (15)$$

These can also be viewed as maps $P : T\mathcal{M} \rightarrow L$ and $\tilde{P} : T\mathcal{M} \rightarrow \tilde{L}$. The eigenbundles $L, \tilde{L}$ are distributions in $T\mathcal{M}$.

An almost para-Hermitian manifold is an almost para-complex manifold $\mathcal{M}$ equipped with a neutral metric $\eta : T\mathcal{M} \times T\mathcal{M} \rightarrow \mathbb{R}$. $\eta$ satisfies the compatibility condition $\eta(K\cdot, K\cdot) = -\eta(\cdot, \cdot)$.

#### 3.2. Integrability

The integrability condition of the para-complex structure is that a real analogue of the Nijenhuis tensor $N_K$ vanishes. Here $N_K$ is given by

$$N_K(X, Y) = \frac{1}{4} \left\{ [K(X), K(Y)] + [X, Y] - K([K(X), Y] + [X, K(Y)]) \right\}. \quad (16)$$

where $X, Y \in T\mathcal{M}$. When $\mathcal{M}$ is an almost para-complex manifold, if the integrability condition is imposed on the para-complex structure, $\mathcal{M}$ becomes a para-complex manifold. Similarly, an almost para-Hermitian manifold becomes a para-Hermitian manifold.

On the other hand, the integrability of a distribution is represented by the Frobenius theorem. For any vector field $A, B$ belonging to a distribution $L$, if the bracket $[A, B]$ also belongs to $L$, then the distribution $L$ is called involutive. That is, $L$ and $\tilde{L}$ being involutive means that the following tensors are zero:

$$N_P(X, Y) := \tilde{P}[P(X), P(Y)], \quad N_{\tilde{P}}(X, Y) := P[\tilde{P}(X), \tilde{P}(Y)]. \quad (17)$$

The Frobenius theorem states that a distribution $L$ is integrable if and only if $L$ is involutive. Adding two tensors in (17) gives the Nijenhuis tensor of $K$: $N_K(X, Y) = N_P(X, Y) + N_{\tilde{P}}(X, Y)$. In contrast to the usual complex case, the integrability of the two distributions $L, \tilde{L}$ is independent each other in the para-complex case.
3.3. Relation between DFT and Generalized Geometry
An alternative representation of the Frobenius theorem states that a subbundle \( E \subset TM \) is integrable if and only if it is defined by a foliation of \( \mathcal{M} \). Therefore \( L \) and \( \tilde{L} \) have foliation structures when they are integrable:

\[
L = Tf, \quad \tilde{L} = T\tilde{f}.
\]

Here the foliation \( f \) is given by the union of leaves \( \bigsqcup_{[p]} \mathcal{M}_{[p]} \). A leaf \( M_p \) is a subspace of \( f \) that pass through a point \( p \in \mathcal{M} \). The index space in the union is the leaf space \( \mathcal{M}/f \). For \( f \), the local coordinates \( x^\mu \) is given along a leaf \( M_p \). The one for the transverse directions to leaves is \( \tilde{x}_\mu \). Therefore \( \tilde{x}_\mu \) is a constant on a leaf \( M_p \) in \( f \). The same discussion for \( \tilde{L} \) is true. This discussion means that a leaf \( M_p \) is a physical spacetime. In the supergravity frame \( \tilde{\partial}^*_\psi = 0 \), the field dependence is fixed on the leaf.

The metric \( \eta \) over \( \mathcal{M} \) can be seen as a map \( \eta : TM = L \oplus \tilde{L} \to T^*\mathcal{M} = L^* \oplus \tilde{L}^* \). By using \( \eta \), the following two isomorphisms \( \phi^+ : L \to L^* \) and \( \phi^- : L \to \tilde{L}^* \) are defined. They map vectors in \( L (L) \) to forms in \( L^* (L^*) \). Given these maps, the following new isomorphisms are defined:

\[
\Phi^+ : TM \to L \oplus L^* \quad \text{and} \quad \Phi^- : TM \to \tilde{L} \oplus \tilde{L}^*.
\]

These maps are called the natural isomorphisms. In particular, \( \Phi^+ \) is utilized to relate DFT and generalized geometry.

3.4. DFT as A Flat Para-Hermitian Manifold
In the following, we consider DFT in a torus background (we call this the toroidal DFT) without fluxes. The toroidal DFT is given by a flat para-Hermitian manifold \( \mathcal{M} \). \( K \) and \( \eta \) are given by

\[
K = \begin{pmatrix}
-1 & 0 \\
0 & +1
\end{pmatrix}, \quad \eta = \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}.
\]

The integrability condition \( N_K = 0 \) is satisfied because \( N_P, N_\beta \) are naturally zero. The tangent space \( TM \) is spanned by \( \partial_M \). Vector fields on \( TM \) are decomposed by the projection operators \( P, \tilde{P} \). Namely, we have

\[
\Xi^M \partial_M = A^\mu(x, \tilde{x}) \partial_\mu + \alpha_\mu(x, \tilde{x}) \tilde{\partial}^\mu,
\]

where \( \Xi \in TM, A \in L, \alpha \in \tilde{L} \). Here \( X^M = (x^\mu, \tilde{x}_\mu) \) is the decomposition induced by \( P, \tilde{P} \) of the local coordinates on the base space \( \mathcal{M} \). \( L \) is spanned by \( \partial_\mu \) while \( \tilde{L} \) is spanned by \( \tilde{\partial}^\mu \).

4. Doubled Structure of Vaisman Algebroid in DFT Setup
4.1. A Lie Algebroid in DFT
A Lie algebroid is a generalization of a Lie algebra defined over a manifold. Lie algebras are defined by using left invariant vector fields and determined by structure constants. On the other hand, Lie algebroids are defined by using general vector fields and determined by structure functions. We use the following theorem to define the Lie algebroid structures on the para-Hermitian manifold. The theorem by Vaintrob [14] states that a Lie algebroid over a vector bundle \( V \to M \) and a Gerstenhaber algebra over multi-vectors \( \Lambda^* V \) are equivalent. A Gerstenhaber algebra is defined by the multi-vector fields on a manifold through the Schouten-Nijenhuis bracket [15]. Therefore a Lie algebroid structure in DFT is defined by the exterior algebra of multi-vectors with the Schouten-Nijenhuis bracket on the para-Hermitian manifold.

We define a natural exterior algebra of multi-vectors on \( TM, L \) and \( \tilde{L} \). We introduce a set of the doubled multi-vectors \( \mathcal{A}^k(\mathcal{M}) = \Gamma(\Lambda^k TM) \). Here \( \Gamma \) means the section of a vector
bundle. If we define $\mathcal{A}^{r,s}(\mathcal{M})$ as the section of $(\wedge^r L) \wedge (\wedge^s \tilde{L})$, then we obtain the decomposition $\tilde{\mathcal{A}}^k(\mathcal{M}) = \bigoplus_{k=r+s} \mathcal{A}^{r,s}(\mathcal{M})$. This decomposition is given by $\pi^{r,s} : \mathcal{A}^{r+s}(\mathcal{M}) \rightarrow \mathcal{A}^{r,s}(\mathcal{M})$ induced by $P, \tilde{P}$. The exterior derivatives acting on $L$ and $\tilde{L}$ are defined by

$$
\tilde{d} : \mathcal{A}^{r,s}(\mathcal{M}) \rightarrow \mathcal{A}^{r+1,s}(\mathcal{M}) \quad (\text{i.e. } \wedge^r L \rightarrow \wedge^{r+1} L),
$$

(22)

$$
d : \mathcal{A}^{r,s}(\mathcal{M}) \rightarrow \mathcal{A}^{r,s+1}(\mathcal{M}) \quad (\text{i.e. } \wedge^s \tilde{L} \rightarrow \wedge^{s+1} \tilde{L}).
$$

(23)

\(\tilde{d}\) and \(d\) have the following properties: $\tilde{d}^2 = 0$, $d^2 = 0$, $\tilde{d}d + d\tilde{d} = 0$. They are called the para-Dolbeault operators. We also define the interior products and the Lie derivatives (see [10] for details).

We give an explicit form of the Schouten-Nijenhuis bracket and the exterior derivatives. We introduce the “odd coordinates” $\zeta_\mu := \partial_\mu$, then an $r$-vector is

$$
A = \frac{1}{r!} A^{\mu_1 \cdots \mu_r} \partial_{\mu_1} \wedge \cdots \wedge \partial_{\mu_r} = \frac{1}{r!} A^{\mu_1 \cdots \mu_r} \zeta_{\mu_1} \cdots \zeta_{\mu_r}.
$$

(24)

Here $\zeta_\mu$ can be treated as a Grassmann number, so the derivative $\partial/\partial \zeta_\mu$ is defined by the right derivative. By using $\partial/\partial \zeta_\mu$, the Schouten-Nijenhuis bracket is explicitly given by

$$
[A, B]_s = \left( \frac{\partial}{\partial \zeta_\mu} A \right) \partial_\mu B - (-1)^{(p-1)(q-1)} \left( \frac{\partial}{\partial \zeta_\mu} B \right) \partial_\mu A,
$$

(25)

where $A \in \wedge^p L$, $B \in \wedge^q L$. The same discussion holds for $[\cdot, \cdot]_s$ on $\tilde{L}$ where $\zeta_\mu = \partial_\mu$ is replaced by $\zeta^\mu = \tilde{\partial}^\mu$. By using the local coordinate, we find that the action of $\tilde{d}$ on an $r$-vector $A \in \wedge^r L$ is explicitly given by

$$
\tilde{d}A = \frac{1}{r!} \tilde{\partial}^\nu A^{\mu_1 \cdots \mu_r} \partial_\nu \wedge \partial_{\mu_1} \wedge \cdots \wedge \partial_{\mu_r}.
$$

(26)

This definition is compatible with the dual Schouten-Nijenhuis bracket $[\cdot, \cdot]_s$ (see [10] for details). $d$ on $\tilde{L}$ has the same definition with $\partial_\mu$ and $\partial^\mu$ interchanged. $\tilde{d}$ and $d$ defined here are exactly equivalent to the exterior derivative $\tilde{d}$ and $d$ defined naively (12).

The exterior algebras of multi-vectors are consistently defined. Therefore, we have defined the Lie algebroid $(\wedge^r L, [\cdot, \cdot]_s, d)$ and its dual Lie algebroid $(\wedge^r \tilde{L}, [\cdot, \cdot]_s, \tilde{d})$ in DFT. In the DFT setup, $L$ and $\tilde{L}$ are integrable, so they are tangent Lie algebroids. The anchor maps $\rho_L$ and $\rho_{\tilde{L}}$ are defined by the identity map.

4.2. The Double of Lie Algebroid Pair in DFT

A Lie bialgebroid is defined by the Lie algebroid $(L, [\cdot, \cdot]_L, \rho_L, d)$ and its dual Lie algebroid $(L^*, [\cdot, \cdot]_{L^*}, \rho_{L^*}, d_*)$ with a compatibility condition between them called the derivation condition:

$$
d_*[A, B]_s = [d_* A, B]_s + [A, d_* B]_s \quad (A, B \in L),
$$

(27)

where $d : \wedge^k L^* \rightarrow \wedge^{k+1} L^*$ and $d_* : \wedge^k L \rightarrow \wedge^{k+1} L$ are the exterior derivatives. It is known that the Drinfel’d double of a Lie bialgebroid is the Courant algebroid [8].

A double of a pair of Lie algebroids $(L, \tilde{L})$ is given by a direct sum $L \oplus \tilde{L}$ of a vector space with a bilinear form and a skew-symmetric bracket. Now, the term “double” means an analogous notion of the Drinfel’d double. A double of a pair of Lie algebroids does not satisfy some axioms of the Courant algebroid because the derivation condition of Lie bialgebroids is not imposed [10].

In the following, we discuss a double of a pair of Lie algebroids $(L, \tilde{L})$ in DFT setup. We examine the derivation condition in DFT by explicit calculation. We obtain

$$
\tilde{d}[A, B]_s = [\tilde{d}A, B]_s + [A, \tilde{d}B]_s + (\partial_\mu A^\nu \tilde{\partial}^\rho B^\nu + \tilde{\partial}^\rho A^\mu \partial_\mu B^\nu) \partial_\nu \wedge \partial_\nu.
$$

(28)
The last contribution is representing the violation of the derivation condition (27). Therefore, it is shown in DFT that \( L \) and \( \tilde{L} \) pairs generally do not form a Lie bialgebroid. We define the two structures required for a double of \((L, \tilde{L})\). The bilinear form \((\cdot, \cdot)\) is given by

\[
(A + \alpha, B + \beta) = \frac{1}{2} \left\{ \langle\langle \alpha, B \rangle \rangle + \langle\langle \beta, A \rangle \rangle \right\},
\]

(29)

where \(\langle\langle \cdot, \cdot \rangle \rangle\) is the symmetric pairing and is defined by \(\langle\langle \alpha, A \rangle \rangle = A^\mu \alpha_\mu\). The symmetric pairing is regarded as the inner product when \(\tilde{L}\) and \(L^*\) are identified by \(\Phi^+\). The skew-symmetric bracket called the Vaisman bracket is given in the same form as the C-bracket (14). The anchor map is defined as \(\rho_v = \rho_L + \rho_{\tilde{L}}\) and the differential operator is defined as \(D = d + \tilde{d}\). Then, the quadruple \((L \oplus \tilde{L}, [\cdot, \cdot]_C, \rho_v, (\cdot, \cdot))\) defines a Vaisman algebroid.

We note that the last contribution in (28) is rewritten as

\[
\partial\rho A^\mu \tilde{\rho} B^\nu + \tilde{\partial}\rho A^\mu \partial\rho B^\nu = \eta^{MN} \partial_M A^\mu \partial_N B^\nu.
\]

(30)

This becomes zero by imposing the strong constraint. Therefore, when the strong constraint is imposed, the derivation condition is satisfied and \((L, \tilde{L})\) becomes a Lie bialgebroid. Then, the double \(L \oplus \tilde{L}\) defines a Courant algebroid. This completely agrees with the analysis in [7]. We ascertained that an algebraic origin of the strong constraint in DFT is the derivation condition of Lie bialgebroids.

5. Conclusion and Discussion

In this contribution, we discussed a doubled structure of the Vaisman algebroid in the para-Hermitian geometry. We defined Lie algebroid structures on the two eigenbundles of the para-Hermitian manifolds and examined their doubles. The derivation condition for a pair of Lie algebroids to become a Lie bialgebroid is violated in the form of the strong constraint. Hence, we found an algebraic origin of the strong constraint in DFT.

Based on the doubled structure of the Vaisman bracket in the para-Hermitian manifold, we discuss the physical aspects of the gauge symmetry generated by the C-bracket. Winding vectors in \(\tilde{L}\) are identified with 1-forms in \(L^*\) by \(\Phi^+\), and the Lie bracket \([\cdot, \cdot]_{\tilde{L}}\) in \(\tilde{L}\) represents the \(B\)-field gauge symmetry algebra. However, \([\cdot, \cdot]_{\tilde{L}}\) in the Vaisman bracket is generally non-zero. Therefore, in DFT, the \(B\)-field gauge symmetry should be effectively non-Abelian.

The strong constraint is necessary for the DFT gauge algebra to close. However, in more general setups (e.g. gauged DFT [16]), there is no need to impose the strong constraint. The Vaisman algebroid would play even more important roles in applications of DFT. It is interesting in the case of DFT on group manifolds used in the study of the Poisson-Lie T-duality [17]. The Drinfel’d double is also an important notion to understand the Poisson-Lie T-duality [18–20].

In this discussion, we used a para-Hermitian geometry, but in order to introduce the generalized metric in DFT, it is necessary to consider in the framework of Born geometry [21]. A conventional generalized metric can be obtained by performing the \(B\)-transformation \(e^B\) to a Riemannian metric in Born geometry. This transformation \(e^B\) may lead to a twisted Vaisman bracket.

The Vaisman algebroid is an algebra of the infinitesimal gauge transformation in DFT. The finite gauge transformation in DFT is governed by an “integration” of the Vaisman algebroid. We are working on the problem of finding the groupoid-like structure corresponding to the Vaisman algebroid [22]. We believe that the geometric structure of the DFT gauge symmetry leads to an understanding of the string winding effects to spacetimes [23–26].

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