Curves with normal planes at constant distance 
from a fixed point

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Abstract

We consider the curves whose all normal planes are at the same dis-
tance from a fixed point and obtain some characterizations of them in the
3-dimensional Euclidean space.

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1 Introduction

Beng-Yen Chen investigated the curves whose position vectors lie in their recti-
fying planes in the 3-dimensional Euclidean space, \([1]\). These curves are called
the rectifying curves which are not plane curves. They can be equivalently de-
defined as the twisted curves whose osculating planes are at the same distance
from a fixed point. Here we consider the curves whose all normal planes are at
the same distance from a fixed point and obtain some characterizations of them
in the 3-dimensional Euclidean space.

We first review the results on the rectifying curves in the 3-dimensional
Euclidean space, \([1]\).

Let \(C\) be a unit-speed curve with the position vector \(\mathbf{r}(s)\). Therefore \(s\) is the
natural parameter of \(C\). Frenet formulas follow as

\[
\mathbf{t}' = \kappa \mathbf{n}, \quad \mathbf{n}' = -\kappa \mathbf{t} + \tau \mathbf{b}, \quad \mathbf{b}' = -\tau \mathbf{n}
\]  

(1)

where \(\mathbf{t}, \mathbf{n}\) and \(\mathbf{b}\) are Frenet vectors and \(\kappa\) the curvature and \(\tau\) the torsion of
the curve.
A rectifying curve can be defined by the equation

\[ <r, n> = 0 \] (2)

By means of (1) we can write it as

\[ <r, t'> = 0 \] (3)

Accordingly \(<r, t>' = <r, t'> + <t, t'> = 1\) and so

\[ <r, t> = \int_0^s 1 \, ds = s \] (4)

Integrating (4) we have

\[ r = r = s^2 + c \] (5)

where \(c\) is a constant. Without loss the generality we can write (5) as

\[ \rho^2 = s^2 + a^2 \] (6)

where \(a\) is a positive number.

(2), (4) and (6) imply that the position vector can be written as

\[ r = st + ab \] (7)

Differentiating (7) we have

\[ t = t + s(\kappa n) + a(-\tau n) \]

and so

\[ s\kappa = a\tau \] (8)

This implies that a rectifying curve is a twisted curve, that is \(\tau \neq 0\).

On the other hand, from (7) we have

\[ <r, b> = a \] (9)

So if a curve is a rectifying curve, then its osculating planes are at the same distance from the origin. Conversely, let us assume that osculating planes of a twisted curve are at the same distance from the origin. Differentiating (9) we get (2) and so it is a rectifying curve. Therefore we have the following property:

A twisted curve is a rectifying curve if and only if its osculating planes are at the same distance from a fixed point.
On the other hand, if a unit spherical curve \( C_1 \) is defined by the equation

\[
r_1(s) = \frac{r(s)}{\sqrt{s^2 + a^2}}
\]

where \( r(s) \) is the position vector of a rectifying curve \( C \) with the natural parameter \( s \), it can be show that the position vector of a rectifying curve \( C \) can be represented as

\[
r(s_1) = (a \sec s_1) r_1(s_1)
\]

where \( s_1 \) is the natural parameter of the spherical curve \( C_1 \). Then the tangent unit vector of the curve \( C \) can be written as

\[
t(s_1) = (\sin s_1) r_1(s_1) + (\cos s_1) t_1(s_1)
\]

where \( t_1(s_1) \) is the tangent unit vector of the spherical curve \( C_1 \). Differentiating the last equation we have

\[
\kappa \sec^2 s_1 = (\cos s_1) (r_1 + \kappa_1 n_1)
\]

\( \kappa_1 \) is the curvature of the spherical curve \( C_1 \).

From (13) we find the following relation between the curvatures of the curves \( C \) and \( C_1 \) associated with each other:

\[
a^2 \kappa^2 = (\cos s_1)^6 (\kappa_1^2 - 1)
\]

The last relation can be written, in terms of the natural parameter \( s \) of the rectifying curve, as

\[
(s^2 + a^2)^3 \kappa^2 = a^4 (\kappa_1^2 - 1)
\]

2 Curves with normal planes at the same distance from a fixed point

Let us assume that all normal planes of a curve \( C \) with the position vector \( r = r(s) \) are at the same distance from a fixed point where \( s \) is the natural parameter of \( C \). We denote Frenet vectors and the curvature and the torsion of \( C \) by \( t, n \) and \( b \) and \( \kappa \) and \( \tau \) respectively. We can choose the fixed point as the origin. Therefore our condition becomes

\[
< r, t > = c_1 = \text{constant}
\]
By the integration we find that

\[ < \mathbf{r}, \mathbf{r} > = \rho_1^2 = 2c_1 s + c_2 \]  

(17)

where \( c_2 \) is a constant.

Since the case of \( c_1 = 0 \) corresponds to a spherical curve, we can assume that \( c_1 \neq 0 \). Then, without loss the generality, we can write (17) as

\[ < \mathbf{r}, \mathbf{r} > = \rho_1^2 = 4c^2 s, \quad c = \text{sign}(s) \]

(18)

where \( c \) is a positive constant. Then (16) reduces to

\[ < \mathbf{r}, \mathbf{t} > = 2c^2 \]

(19)

(18) and (19) imply

\[ \epsilon s \geq c^2 \]

(20)

Differentiating (19) we have

\[ \kappa < \mathbf{r}, \mathbf{n} > = -1 \]

or

\[ < \mathbf{r}, \mathbf{n} > = -R \]

(22)

where \( R = 1/\kappa \) is the radius of curvature.

From (19) and (22) we can write the position vector as

\[ \mathbf{r} = 2c^2 \mathbf{t} - R \mathbf{n} + A \mathbf{b} \]

(23)

Differentiating (23) we have

\[ \mathbf{t} = \mathbf{t} + (2\epsilon c^2 \kappa - R' - A\tau) \mathbf{n} + (A' - R\tau) \mathbf{b} \]

(24)

and

\[ A\tau = 2\epsilon c^2 \kappa - R', \quad A' = R\tau \]

(25)

Therefore we first have

\[ RR' = 2\epsilon c^2 \iff \tau = 0 \]

(26)
So the curve is a plane curve if and only if
\[ RR' = 2\epsilon c^2 \] (27)

Without the generality, for case of a plane curve we can write
\[ r = 2\epsilon c^2 t + ( - R) n \] (28)

Then from (18) and (28)
\[ R^2 = 4c^2(\epsilon s - c^2) \] (29)

Accordingly the natural equations of the curve are
\[ \kappa = \frac{1}{2\epsilon c^2 \sqrt{\epsilon s - c^2}}, \quad \tau = 0 \] (30)

We can now assume that our curve is a twisted curve, that is \( \tau \neq 0 \). Then
\[ A = 2\epsilon c^2 \kappa T - R'T, \quad A' = R\tau \] (31)
where \( T = \frac{1}{\tau} \) is the radius of torsion. Hence (23) can be written as
\[ r = (2\epsilon c^2) t + ( - R) n + (2\epsilon c^2 \kappa T - R'T) b \] (32)

Therefore, according to (18) we have
\[ 4\epsilon c^2 s = 4c^4 + R^2 + (2\epsilon c^2 \kappa T - R'T)^2 \] (33)

On the other hand, from (31) we can write
\[ R\tau + (R'T)' - 2\epsilon c^2 (\kappa T)' = 0 \] (34)

We can show that (33) and (34) are equivalent equations. In fact, the equation (34) can be written as
\[ [R\tau + (R'T)' - 2\epsilon c^2 (\kappa T)'][RR' - 2\epsilon c^2] = 0 \] (35)

because of \( RR' - 2\epsilon c^2 = 0 \), corresponds to the plane curve. Then we have
\[ 2\epsilon c^2 = [R\tau + (R'T)'] - 2\epsilon c^2 (\kappa T)'R'T + 4c^4 \kappa T(\kappa T)' - 2\epsilon c^2 \kappa T(R'T)' \] (36)
and
\[ 4c^2 = 2RR' + 2(2\epsilon c^2 \kappa T - R'T)[2\epsilon c^2 (\kappa T)' - (R'T)'] \] (37)
This implies that

\[ 4\epsilon c^2 = (R^2)' + [(2\epsilon c^2 \kappa T - R'T)^2]' \]  

(38)

Therefore we have

\[ R^2 + (2\epsilon c^2 \kappa T - R'T)^2 = \int_{c^2}^{c^2} 4\epsilon^2 dt = 4\epsilon c^2 s - 4c^4 \]

It is obvious that using the equation (33) we obtain the equation (34).

3 Spherical curves associated with a curve with normal planes at constant distance from a point

Let us define a unit spherical curve \( C_1 \) by the equation

\[ r_1(s) = \frac{r(s)}{2c\sqrt{\epsilon s}} \]  

(39)

where \( r(s) \) is the position vector of a curve with normal planes at constant distance from a fixed point.

So we have

\[ r(s) = 2c\sqrt{\epsilon s}r_1(s) \]  

(40)

Differentiating we get

\[ t = \frac{c\epsilon}{\sqrt{\epsilon s}}r_1(s) + (2c\sqrt{\epsilon s})r_1'(s) \]  

(41)

From the last equation we find

\[ |r_1'(s)| = \frac{\sqrt{\epsilon s - c^2}}{2\epsilon c s} = \frac{ds_1}{ds} \]  

(42)

for the speed of the spherical curve. And so the natural parameter of the unit spherical curve \( C_1 \), from \( s_1 = \epsilon \int_{c^2}^{c^2} \sqrt{\epsilon s - c^2} dt \) is obtained as

\[ s_1 = \epsilon\left(\frac{\sqrt{\epsilon s - c^2}}{c} - \arctan\frac{\sqrt{\epsilon s - c^2}}{c}\right) \]  

(43)

Since
there exists a function \( s(s_1) \) which satisfies the equation (43). Therefore (39) can be written as

\[
\mathbf{r}_1(s_1) = \frac{\mathbf{r}(s(s_1))}{2c\sqrt{\epsilon s}}
\]  

(45)

So for a given curve \( C \) with normal planes at constant distance from a fixed point, whose position vector is \( \mathbf{r}(s_1) = \mathbf{r}(s(s_1)) \), we have a unit spherical curve \( C_1 \) whose position vector \( \mathbf{r}_1(s_1) = \mathbf{r}_1(s(s_1)) \) is defined by (45) with the natural parameter \( s_1 \). We call \( C_1 \) the unit spherical curve associated with the curve \( C \) with normal planes at constant distance from a fixed point. Now let us consider a curve \( C \) defined by

\[
\mathbf{r}(s_1) = (2c\sqrt{\epsilon s}) \mathbf{r}_1(s_1)
\]  

(46)

where \( \mathbf{r}_1(s_1) \) is the position vector of a unit spherical curve \( C_1 \) with the natural parameter \( s_1 \) and \( s(s_1) \) is defined by (43) and \( c \) is constant. Let \( \mathbf{t}_1, \mathbf{n}_1, \mathbf{b}_1 \) and \( \kappa_1 \) and \( \tau_1 \) be Frenet vectors and the curvature and the torsion of \( C_1 \) respectively. According to (43)

\[
s' = \frac{ds}{ds_1} = \frac{2c\epsilon s}{\sqrt{\epsilon s - c^2}}
\]  

(47)

Differentiating (46) with respect to \( s_1 \) we have

\[
\mathbf{r}' = \frac{d\mathbf{r}}{ds_1} = 2c^2 \frac{\sqrt{\epsilon s}}{\sqrt{\epsilon s - c^2}} \mathbf{r}_1 + (2c\sqrt{\epsilon s}) \mathbf{t}_1
\]  

(48)

Since \( \langle \mathbf{r}_1, \mathbf{r}_1 \rangle = 1 \), \( \langle \mathbf{r}_1, \mathbf{t}_1 \rangle = 0 \) and \( \langle \mathbf{t}_1, \mathbf{t}_1 \rangle = 1 \) from (48) we have

\[
|\mathbf{r}'| = \frac{2c\epsilon s}{\sqrt{\epsilon s - c^2}}
\]

So the unit tangent vector of \( C \) is found as

\[
\mathbf{t}(s_1) = \frac{\epsilon c}{\sqrt{\epsilon s}} \mathbf{r}_1(s_1) + \frac{\sqrt{\epsilon s - c^2}}{\sqrt{\epsilon s}} \mathbf{t}_1(s_1)
\]  

(49)

Since \( \langle \mathbf{r}, \mathbf{r}_1 \rangle = 2c\sqrt{\epsilon s} \) and \( \langle \mathbf{r}, \mathbf{t}_1 \rangle = 0 \)

\[
\langle \mathbf{r}, \mathbf{t} \rangle = 2c^2
\]  

(50)
This means that the curve \(C\) is a curve with normal planes at constant distance from a fixed point. We call \(C\) the curve with normal planes at constant distance from a fixed point associated with the unit spherical curve \(C_1\).

Let us now differentiate (49) with respect to \(s\). We have

\[
\kappa n = -\frac{c}{2\epsilon s \sqrt{\epsilon s}} r_1 + \frac{1}{2\sqrt{\epsilon s} \sqrt{\epsilon s - c^2}} t_1 + \frac{\epsilon s - c^2}{2\epsilon c s \sqrt{\epsilon s}} \kappa_1 n_1 \tag{51}
\]

Using \(<n, n> = 1, <r_1, r_1> = 1, <t_1, t_1> = 1, <n_1, n_1> = 1, <r_1, t_1> = 0, <t_1, n_1> = 0\) and \(\kappa_1 <r_1, n_1> = -1\), from (51) we obtain the following relation between curvatures of the curves \(C\) and \(C_1\) associated with each other:

\[
\kappa^2 = \frac{3s^2 - 3c^2\epsilon s + c^4}{4\epsilon s^3(s - c^2)} + \frac{(\epsilon s - c^2)^2}{4c^2\epsilon s^3} \kappa_1^2 \tag{52}
\]

or

\[
\kappa^2 = \frac{1}{4c^2(\epsilon s - c^2)} + \frac{(\epsilon s - c^2)^2}{4c^2\epsilon s^3}(\kappa_1^2 - 1) \tag{53}
\]

Since \(\kappa_1 <r_1, n_1> = -1\), the curvature of the unit spherical curve is not smaller than 1, that is \(\kappa_1 \geq 1\).

**Example 1**

The equation (53) implies that the curvature of the unit spherical curve associated with the plane curve given by the natural equations (30) is \(\kappa_1 = 1\). This means that the spherical curve is a great circle of the unit sphere. Hence we can obtain the cartesian equations of the plane curve associated with a great circle of the unit sphere using the equation (40). In fact, we can choose the equation of a great circle of the unit sphere as

\[
r_1 = (\cos s_1, \sin s_1, 0) \tag{54}
\]

Since \(s_1 = \epsilon(\frac{\sqrt{\epsilon s - c^2}}{c} - \arctan \frac{\sqrt{\epsilon s - c^2}}{c})\),

\[
\cos s_1 = \cos \frac{\sqrt{\epsilon s - c^2}}{c} \frac{c}{\sqrt{\epsilon s}} + \epsilon \sin \frac{\sqrt{\epsilon s - c^2}}{c} \frac{\sqrt{\epsilon s - c^2}}{\sqrt{\epsilon s}}
\]

and

\[
\sin s_1 = \epsilon \sin \frac{\sqrt{\epsilon s - c^2}}{c} \frac{c}{\sqrt{\epsilon s}} - \cos \frac{\sqrt{\epsilon s - c^2}}{c} \frac{\sqrt{\epsilon s - c^2}}{\sqrt{\epsilon s}}
\]

Then from (40) we obtain the Cartesian equations of the plane curve with normal
planes at constant distance from the origin as

\[
\begin{align*}
x &= 2c^2 \cos \frac{\sqrt{\epsilon s - c^2}}{c} + 2c \epsilon \sqrt{\epsilon s - c^2} \sin \frac{\sqrt{\epsilon s - c^2}}{c} \\
y &= 2c^2 \sin \frac{\sqrt{\epsilon s - c^2}}{c} - 2c \epsilon \sqrt{\epsilon s - c^2} \cos \frac{\sqrt{\epsilon s - c^2}}{c} \\
z &= 0
\end{align*}
\]  

(55)

This curve is a plane curve with normal planes at a distance \(2c^2\) from the origin. This means that all normal lines of the curve are at a distance \(2c^2\) from the origin. So it is an involute of the circle of radius \(2c^2\) centered at the origin.

Since it is an involute of a plane curve, it is also a plane curve \([3, p.88]\). In fact, the position vector of an involute \(C\) of the curve \(C_2\) whose position vector is \(r_2(s_2)\) can be written as

\[
r = r_2(s_2) + (c_2 - s_2)t_2(s_2)
\]  

(56)

where \(c_2\) is constant and \(t_2(s_2)\) is unit tangent vector to the curve \(C_2\), \([3, p.69]\), \([5, p.40]\), \([4, p.99]\). Then

\[
r' = (c_2 - s_2)\kappa_2n_2
\]  

(57)

where \(\kappa_2\) is the curvature of \(C_2\) and \(n_2\) the principal normal vector. Therefore the unit tangent vector \(t\) to \(C\) can be written as

\[
t = -\epsilon_2n_2, \quad -\epsilon_2 = \text{sign}(c_2 - s_2)
\]  

(58)

Since \(C_2\) is a circle of radius \(2c^2\) centered at the origin,

\[
r_2 = (2c^2 \cos \frac{s_2}{2c^2}, 2c^2 \sin \frac{s_2}{2c^2}, 0)
\]  

(59)

\[
t_2 = (-\sin \frac{s_2}{2c^2}, \cos \frac{s_2}{2c^2}, 0)
\]  

(60)

\[
n_2 = (-\cos \frac{s_2}{2c^2}, -\sin \frac{s_2}{2c^2}, 0)
\]  

(61)

So \(\langle r_2, t_2 \rangle = 0\), \(\langle r_2, n_2 \rangle = -2c^2\). Then from (56) and (58) we have

\[
\langle r, t \rangle = 2\epsilon_2c^2
\]  

(62)

This means that the involute \(C\) of the circle \(C_2\) is a plane curve with normal planes at a distance \(2c^2\) from the origin.
Figure 1

Figure 1 illustrates the plane curve \( (55) \) with normal planes at a distance \( 2c^2 = 2(1)^2 = 2 \) from the origin (the solid line for \( \epsilon = 1 \), the dashed line for \( \epsilon = -1 \)). It is an involute of the circle of radius \( 2c^2 = 2 \) centered at the origin. Let us note the equations \( (55) \) can be also obtained using the equations \( (56)-(61) \) or using \( (55) \) from the equations of a plane curve given by

\[
\mathbf{r} = \mathbf{r}(s) = (x(s), y(s), 0) = (\int \cos(\int \kappa(s)ds)ds, \int \sin(\int \kappa(s)ds)ds, 0) \quad (63)
\]

where \( s \) is the natural parameter of the curve, \([3, p.87], [2, p.28], [4, p.99]\).

In the following we give an example of a twisted curve with normal planes at constant distance from a fixed point.

Example 2

Let us choose the unit spherical curve \( C_1 \) as the circle of radius \( \frac{1}{\sqrt{2}} \) given by the equation

\[
\mathbf{r}_1 = \left( \frac{1}{\sqrt{2}} \cos(\sqrt{2}s_1), \frac{1}{\sqrt{2}} \sin(\sqrt{2}s_1), \frac{1}{\sqrt{2}} \right) \quad (64)
\]

Then the curve \( C \) associated with \( C_1 \) is given by the equation

\[
\mathbf{r}(s) = (c\sqrt{2}\sqrt{\epsilon s} \cos(\sqrt{2}s_1), c\sqrt{2}\sqrt{\epsilon s} \sin(\sqrt{2}s_1), c\sqrt{2}\sqrt{\epsilon s}) \quad (65)
\]

where \( s_1 = \epsilon \left( \frac{\epsilon s - c^2}{c} - \arctan \frac{\sqrt{2}s_1}{c} \right) \).

Since the curvature of the circle \( C_1 \) is \( \kappa_1 = \sqrt{2} \), \( (53) \) implies that

\[
\kappa^2 = \frac{1}{4c^2(\epsilon s - c^2)} + \frac{(\epsilon s - c^2)^2}{4c^2\epsilon s^3} = \frac{\epsilon s^3 - (\epsilon s - c^2)^3}{4c^2\epsilon s^3(\epsilon s - c^2)} \quad (66)
\]
Figure 2 illustrates the twisted curve with normal planes at a distance \(2c^2 = 2(1)^2 = 2\) from the origin (the solid line for \(\epsilon = 1\), the dashed line for \(\epsilon = -1\)).

4 Curves of constant curvature with normal planes at the same distance from a fixed point

If \(\kappa = \frac{1}{r}\) =constant, (32) reduce to

\[
r = (2\epsilon c^2)t + (-r)n + 2\epsilon c^2 T b \tag{67}
\]

From (67) we have \(<r, n> = -r = \text{constant}\). This means that rectifying planes of a curve of constant curvature with normal planes at constant distance from a fixed point are also at constant distance from the same point.

Conversely if normal planes and rectifying planes of a curve are at constant distance from a fixed point, it is a curve of constant curvature. In fact by above conditions we can write

\[
<r, t> = c_1 = \text{constant} \tag{68}
\]

and

\[
<r, n> = c_2 = \text{constant} \tag{69}
\]

Differentiating (68) we have

\[
1 + <r, \kappa n> = 0
\]

and so
Because of (69) and (70) the curve is a curve of constant curvature.

On the other hand for a curve of constant curvature with normal planes at constant distance from a fixed point (33) reduces to

\[ 4ec^2 s = 4c^4 + r^2 + \frac{4c^4}{r^2} T^2 \]

Therefore the natural equations of such a curve are

\[ R = r = \text{constant}, \quad T^2 = r^2 \frac{2ehs - r^2 - h^2}{h^2} \]

where \( h = 2c^2 \) is the distance of the normal planes from the origin and \( r \) is the distance of the rectifying planes from the origin.

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