SEMIDEFINITE PROGRAMMING VIA IMAGE SPACE ANALYSIS

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Abstract. In this paper, we investigate semidefinite programming by using the image space analysis and present some equivalence between the (regular) linear separation and the saddle points of the Lagrangian functions related to semidefinite programming. Some necessary and sufficient optimality conditions for semidefinite programming are also given under some suitable assumptions. As an application, we obtain some equivalent characterizations for necessary and sufficient optimality conditions for linear semidefinite programming under Slater assumption.

1. Introduction. As pointed by Vandenberghe and Boyd [12], there are good reasons for studying semidefinite programming. First, positive semidefinite (or definite) constraints directly arise in a number of important applications. Second, many convex optimization problems, e.g., linear programming and (convex) quadratically constrained quadratic programming, can be cast as semidefinite programs. So, semidefinite programming offers a unified way to study the properties of and derive algorithms for a wide variety of convex optimization problems. Most importantly, semidefinite programs can be solved very efficiently, both in theory and in practice. Extensive lists of applications from various areas can be found in [9, 12]. In recent years, many authors have investigated semidefinite programming (see, for example, [5, 11, 13, 15]).

The image of a constrained extremum problem was developed by Giannessi [2], by exploiting previous results on theorems of the alternative. Recently, there has been an increasing interest in the Image Space Analysis (for short, ISA) of constrained variational inequalities and constrained optimization problems (see, for example, [3, 1, 4, 6, 7, 14]).

The ISA is a powerful tool and a unifying scheme for studying both variational inequalities and optimization problems. This approach can be applied to any kind of problem that can be expressed under the form of the impossibility of a parametric system. The impossibility of such a system is reduced to the disjunction of two suitable subsets of the image space. The disjunction between the two suitable subsets is proved by showing that they lie in two disjoint level sets of a separating functional.

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The purpose of this paper is to carry on the ISA of semidefinite programming. We present some equivalence between the (regular) linear separation and the saddle points of the Lagrangian functions related to semidefinite programming. We give necessary and sufficient optimality conditions for semidefinite programming under suitable assumptions. As an application, under Slater assumption we obtain some equivalent characterizations for necessary and sufficient optimality conditions for linear semidefinite programming.

The paper is organized as follows. In Section 2, we recall some notations that will be used in the sequel. In Section 3, we define the image of semidefinite programming and the conical extension of the image, and give the equivalence between the solvability of semidefinite programming and an empty intersection of $\mathcal{H}$ and the conical extension of the image. In Section 4, we characterize the (regular) linear separation for semidefinite programming and we give the equivalence between (regular) linear separation and saddle points of Lagrangian functions for semidefinite programming. We present some equivalence between the (regular) linear separation and the saddle points of the Lagrangian functions related to semidefinite programming. We give necessary and sufficient optimality conditions for semidefinite programming under suitable assumptions. As an application, under Slater assumption we obtain some equivalent characterizations for necessary and sufficient optimality conditions for linear semidefinite programming.

2. Notations and problem formulation. Let $\mathbb{R}^k$ be the $k$ dimensional Euclidean space, where $k$ is a given positive integer. We denote by $\mathbb{R}^k_+ := \{x := (x_1, \cdots, x_k)^\top : x_i \geq 0, \ i = 1, \cdots, k\}$ and $\mathbb{R}^k_{++} := \{x := (x_1, \cdots, x_k)^\top : x_i > 0, \ i = 1, \cdots, k\}$, where the $\top$ denotes the transpose. Let $\mathbb{R}_+ := \mathbb{R}^1_+$ and $\mathbb{R}_{++} := \mathbb{R}^1_{++}$. A nonempty subset $P$ of $\mathbb{R}^k$ is said to be a cone with apex at the origin if $\lambda P \subseteq P$ for all $\lambda \geq 0$. $P$ is said to be a convex cone if $P$ is a cone and $P + P = P$. The dual cone (or positive polar cone) of a convex cone $P$ is given by

$$P^* := \{z \in \mathbb{R}^k : \langle z, x \rangle \geq 0, \ \forall x \in P\},$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product.

Denote by $\text{dom} \ h := \{x \in \mathbb{R}^k : h(x) < +\infty\}$ the effective domain of $h : \mathbb{R}^k \to \mathbb{R} \cup \{+\infty\}$. We say that $h : \mathbb{R}^k \to \mathbb{R} \cup \{+\infty\}$ is proper if $\text{dom} \ h \neq \emptyset$. We say that $h : \mathbb{R}^k \to \mathcal{R} \cup \{+\infty\}$ is convex on a convex set $K \subseteq \mathbb{R}^k$ if $h(tx + (1 - t)y) \leq th(x) + (1 - t)h(y)$ for any $x, y \in K, t \in [0, 1]$. The subdifferential of a proper convex function $h : \mathbb{R}^k \to \mathcal{R} \cup \{+\infty\}$ at $x \in \text{dom} \ h$ is given by

$$\partial h(x) := \{s \in \mathbb{R}^k : h(y) - h(x) \geq \langle s, y - x \rangle, \ \forall y \in \mathbb{R}^k\}.$$ 

It is well known that if $h$ is differential at $x$, then $\partial h(x) = \{\nabla h(x)\}$, where $\nabla$ denotes the gradient. Let $K$ be a nonempty subset of $\mathcal{R}^k$. The indicator function of $K$ is defined by

$$i_K(x) := \begin{cases} 
0, & x \in K, \\
+\infty, & x \notin K.
\end{cases}$$

The normal cone to $K$ at $x \in K$, denoted by $N_K(x)$, is defined by $N_K(x) := \{x^* \in \mathbb{R}^k : \langle x^*, y - x \rangle \leq 0, \ \forall y \in K\}$. It is well known that if $K$ is convex, so is $i_K(x)$ and so $\partial i_K(x) = N_K(x)$, where $x_0 \in K$.

Denote by $\mathcal{S}^l$ the linear subspace of the symmetric $l \times l$ matrices with real entries, i.e.,

$$\mathcal{S}^l := \{A \in \mathbb{R}^{l \times l} : A = A^\top\}.$$
Denote by $S^l_+$ the cone of the symmetric positive semidefinite $l \times l$ matrices with real entries, i.e.,

$$S^l_+ := \{ A \in S^l : x^T A x \geq 0, \forall x \in \mathbb{R}^l \},$$

and by $S^l_{++}$ the set of the symmetric positive definite $l \times l$ matrices with real entries (which actually coincides with int$S^l_+$, see [11]), i.e.,

$$S^l_{++} := \{ A \in S^l : x^T A x > 0, \forall x \in \mathbb{R}^l \text{ with } x \neq 0 \}.$$

Then the so-called Löwner partial order can be introduced as follows:

$$A \succeq B \iff A - B \in S^l_{++}, \forall A, B \in S^l,$$

and

$$A \succ B \iff A - B \in S^l_{++}, \forall A, B \in S^l.$$

The scalar product and Frobenius norm in $S^l$ are given by

$$\langle A, B \rangle := \sum_{i,j=1}^{l} A_{ij} B_{ij} = \text{trace}(A^T B), \forall A, B \in S^l,$$

and

$$\| A \| := \sqrt{\langle A, A \rangle},$$

respectively, where $A_{ij}$ and $B_{ij}$ are $(i, j)$ elements of $A$ and $B$, respectively. It is well known that the space $S^l$ is self-dual, i.e., $(S^l)^* = S^l$. It is proved in [15] that the convex cone $S^l_+$ is also self-dual, i.e., $(S^l_+)^* = S^l_+$. Also one can easily check that $S^l_+$ is a closed subset of $S^l$, i.e., cl$S^l_+ = S^l_+$, where the cl denotes the closed hull. Notice that $S^l$ can be considered as the $\frac{d(d+1)}{2}$ dimensional Euclidean space.

We say that $g : \mathbb{R}^k \to S^l$ is $S^l_+$-convex (res., $S^l_+$-convexlike) on a convex set $K \subseteq \mathbb{R}^k$ if $g(tx + (1-t)y) \preceq tg(x) + (1-t)g(y)$ for any $x, y \in K, t \in [0, 1]$ (res., $g(K) + S^l_+$ is convex). Clearly, if $g$ is $S^l_+$-convex on $K$, then $g(K) + S^l_+$ is convex. But the converse is not true in general.

In this paper, without other specifications, let $K$ be a nonempty convex subset of $\mathbb{R}^k$ and let $f : \mathbb{R}^k \to \mathbb{R}$ and $g : \mathbb{R}^k \to S^l$. We consider the following semidefinite programming (for short, SDP):

$$\min_{x \in K} f(x) \quad \text{subject to} \quad g(x) \succeq 0.$$

Denote by $F_p := \{ x \in K : g(x) \succeq 0 \}$ the feasible set of SDP.

Let $c_0 \in \mathbb{R}^k$, $A_0, A_j \in S^l(j = 1, \cdots, k)$. Let $f(x) := c_0^T x$ and $g(x) := A_0 + \sum_{j=1}^{k} x_j A_j$. Then SDP collapses to the following linear semidefinite programming (for short, LSDP):

$$\min_{x \in K} c_0^T x \quad \text{subject to} \quad A_0 + \sum_{j=1}^{k} x_j A_j \succeq 0.$$

3. Preliminaries results on ISA for SDP. In this section, we shall carry on ISA for SDP. Observe that, $\bar{x} \in F_p$ solves SDP if and only if the system (in the unknown $x$):

$$\begin{cases} f(\bar{x}) - f(x) > 0 \\ g(x) \succeq 0 \\ x \in K \end{cases} \quad (3.1)$$
is impossible. We can associate SDP with the following sets:

\[ 
H := \{ (u, A) \in \mathbb{R} \times S^l_l : u > 0, A \succeq 0 \} = \mathbb{R}^+ \times S^l_+ , \\
K(\bar{x}) := \{ (u, A) \in \mathbb{R} \times S^l_l : u = f(\bar{x}) - f(x), A = g(x), x \in K \} .
\]

We call the set \( K(\bar{x}) \) the image of SDP at \( \bar{x} \in F_p \) and \( \mathbb{R} \times S^l_+ \) image space. Define the mapping \( G_{\bar{x}} : \mathbb{R}^k \rightarrow \mathbb{R} \times S^l_l \) by

\[ 
G_{\bar{x}}(x) := (f(\bar{x}) - f(x), g(x)) .
\]

Then \( K(\bar{x}) = \{ G_{\bar{x}}(x) : x \in K \} = G_{\bar{x}}(K) . \)

Clearly, we have the following results:

**Proposition 3.1.** Let \( \bar{x} \in F_p \). System (3.1) is impossible if and only if

\[ 
H \cap K(\bar{x}) = \emptyset. \tag{3.2}
\]

Consequently, \( \bar{x} \in F_p \) is a solution of SDP if and only if (3.2) is true.

As pointed by Giannessi, to prove directly whether (3.2) holds or not is generally too difficult. The reason is that, in general, the image of SDP is not convex even when the functions involved enjoy some convexity properties. To overcome this difficulty, similar to that in [3, 4] (see also [6, 7, 14]), we introduce a regularization of the image \( K(\bar{x}) \), namely, the extension with respect to the cone \( \text{cl} H \), denoted by \( \mathcal{E} \):

\[ 
\mathcal{E} := K(\bar{x}) - \text{cl} H
\]

= \[ G_{\bar{x}}(K) - \mathbb{R}^+ \times S^l_+ \]

= \[ \{ (u, A) \in \mathbb{R} \times S^l_l : u \leq f(\bar{x}) - f(x), A \preceq g(x), x \in K \} . \]

Set \( H_u := \{ (u, A) \in \mathbb{R} \times S^l_l : u > 0, A = 0 \} = \mathbb{R}^+ \times \{ 0 \} . \)

We have the following:

**Proposition 3.2.** Let \( \bar{x} \in F_p \). The following statements are true:

1. The mapping \(-G_{\bar{x}} \) is \( \mathbb{R}^+ \times S^l_+ \)-convexlike on \( K \), i.e., \(-G_{\bar{x}}(K) + \mathbb{R}^+ \times S^l_+ \) is convex, if and only if the set \( \mathcal{E} \) is convex;
2. System (3.1) is impossible, or (3.2) holds if and only if

\[ 
H \cap \mathcal{E} = \emptyset, \tag{3.3}
\]

or equivalently,

\[ 
H_u \cap \mathcal{E} = \emptyset. \tag{3.4}
\]

**Proof.**

1. Clearly, the mapping \(-G_{\bar{x}} \) is \( \mathbb{R}^+ \times S^l_+ \)-convexlike on \( K \), i.e., \(-G_{\bar{x}}(K) + \mathbb{R}^+ \times S^l_+ \) is convex, if and only if \( G_{\bar{x}}(K) - \mathbb{R}^+ \times S^l_+ \) is convex. Since \( \mathcal{E} = G_{\bar{x}}(K) - \mathbb{R}^+ \times S^l_+ \), this is equivalent to the convexity of \( \mathcal{E} \).

2. We first show the equivalence between (3.2) and (3.3). Since \( K(\bar{x}) \subseteq \mathcal{E} \), it suffices to prove (3.2) implies (3.3). Let (3.2) hold, i.e., \( H \cap K(\bar{x}) = \emptyset \). Suppose to the contrary that (3.3) is false, i.e.,

\[ 
H \cap \mathcal{E} \neq \emptyset.
\]

Then there is \( (u^0, A^0) \in H \cap \mathcal{E} \), i.e., there exists \( x^0 \in K \) such that

\[ 
u^0 \leq f(\bar{x}) - f(x^0), A^0 \preceq g(x^0) .\]
and
\[ u^0 > 0, A^0 \succeq 0. \]
As a consequence,
\[ 0 < f(\bar{x}) - f(x^0), 0 \leq g(x^0), \]
which implies that \( \mathcal{H} \cap \mathcal{K}(\bar{x}) \neq \emptyset \), a contradiction.

We now prove (3.3) and (3.4) are equivalent. We only need to prove (3.4) implies (3.3) since \( \mathcal{H} \cup \mathcal{K}(\bar{x}) \subseteq \mathcal{H} \). Let (3.4) hold, i.e., \( (\bar{x}, A^0) \in \mathcal{H} \cap \mathcal{K}(\bar{x}) \). Then \( (u^0, A^0) \in \mathcal{K}(\bar{x}) \) and \( (0, A^0) \in \text{cl} \mathcal{H} \). Since
\[ E - \text{cl} \mathcal{H} = E - \mathbb{R}_+ \times S^t_+ = \mathcal{K}(\bar{x}) - \mathbb{R}_+ \times S^t_+ = \mathcal{K}(\bar{x}) - \text{cl} \mathcal{H} = E, \]
one has \( (u^0, A^0) - (0, A^0) = (u^0, 0) \in E \), which is a contradiction since \( (u^0, 0) \in \mathcal{H} \).

**Remark 3.3.** It is easy to verify that if \( f \) is convex on \( K \) and \( -g \) is \( S^t_+ \)-convex on \( K \) (i.e., \( tg(x) + (1-t)g(y) \preceq g(tx + (1-t)y) \) for any \( t \in [0, 1], x, y \in K \)), then the mapping \( -G_{\bar{x}} \) is \( \mathbb{R}_+ \times S^t_+ \)-convexlike on \( K \).

4. **Linear separation and saddle points of Lagrangian functions for SDP.**
In this section, we shall characterize the (regular) linear separation of SDP and investigate the saddle points of the Lagrangian function associated to SDP.

**Definition 4.1.** Let \( \bar{x} \in F_p \). The sets \( \mathcal{K}(\bar{x}) \) and \( \mathcal{H} \) are said to

(1) be linearly separable, if there exists \((\bar{\lambda}, \bar{A}) \in \mathbb{R}_+ \times S^t_+, \) with \((\bar{\lambda}, \bar{A}) \neq 0, \) such that:
\[ \bar{\lambda}u + \langle \bar{A}, A \rangle \leq 0, \forall (u, A) \in \mathcal{K}(\bar{x}), \]
or equivalently,
\[ \bar{\lambda}(f(\bar{x}) - f(x)) + \langle \bar{A}, g(x) \rangle \leq 0, \forall x \in K; \quad (4.1) \]

(2) admit a regular linear separation, if there is \((\bar{\lambda}, \bar{A}) \in \mathbb{R}_+ \times S^t_+, \) such that (4.1) holds.

Similarly, we can define the (regular) linear separation of the sets \( \mathcal{K}(\bar{x}) \) and \( \mathcal{E} \).

The following lemma is well known (see, for example, [6]).

**Lemma 4.2.** Let \( B \succ 0. \) Then \( \langle B, A \rangle > 0 \) for all \( A \in S^t_+ \setminus \{0\}. \)

**Theorem 4.3.** Let \( \bar{x} \in F_p. \) Consider the following statements:

(1) The sets \( \mathcal{K}(\bar{x}) \) and \( \mathcal{H} \) are linearly separable;
(2) The sets \( \mathcal{E} \) and \( \mathcal{H} \) are linearly separable;
(3) The sets \( \mathcal{K}(\bar{x}) \) and \( \mathcal{H} \) admit a regular linear separation;
(4) The sets \( \mathcal{E} \) and \( \mathcal{H} \) admit a regular linear separation.

Then we have (1)\( \Leftrightarrow \) (2)\( \Leftrightarrow \) (3)\( \Leftrightarrow \) (4). If the following Slater condition holds:
\[ \exists x^0 \in K \text{ such that } g(x^0) > 0, \]
then statements (1)-(4) are equivalent.
Proof. It is easy to see that (3)⇒(1) and (4)⇒(2).

We first prove (1)⇒(2). Clearly, (2)⇒(1) since \( K(\bar{x}) \subseteq \mathcal{E} \). Assume that (1) is true, i.e., the sets \( K(\bar{x}) \) and \( \mathcal{H} \) are linearly separable. Then, there exists \((\lambda, \bar{A}) \in \mathbb{R}^+_+ \times S^d_+\), with \((\lambda, \bar{A}) \neq 0\), such that (4.1) holds. Let \((u, A) \in \mathcal{E} = K(\bar{x}) - \mathbb{R}^+_+ \times S^d_+\). Then there are \( x^0 \in K, t^0 \in \mathbb{R}^+_+, A^0 \in S^d_+ \) such that \((u, A) = (f(\bar{x}) - f(x^0), g(x^0)) - (t^0, A^0)\). Therefore it follows from (4.1) that
\[
\lambda u + \langle \bar{A}, A \rangle = \lambda(f(\bar{x}) - f(x^0)) + \langle \bar{A}, g(x^0) \rangle - (\lambda t^0 + \langle \bar{A}, A^0 \rangle) \leq 0,
\]
which yields that \( K(\bar{x}) \) and \( \mathcal{E} \) are linearly separable. This proves (1)⇒(2).

Similarly, we can prove (3)⇒(4).

Now we show that if the Slater condition (4.2) holds, then (1)⇒(3). Assume that the Slater condition (4.2) holds and (1) is true, i.e., \( K(\bar{x}) \) and \( \mathcal{H} \) are linearly separable. Then there exists \((\bar{\lambda}, \bar{A}) \in \mathbb{R}^+_+ \times S^d_+\), with \((\bar{\lambda}, \bar{A}) \neq 0\), such that (4.1) holds. Suppose to the contrary that \( \bar{\lambda} = 0 \). Then \( \bar{A} \neq 0 \) and from Lemma 4.2 it follows that
\[
\bar{\lambda}(f(\bar{x}) - f(x^0)) + \langle \bar{A}, g(x^0) \rangle = \langle \bar{A}, g(x^0) \rangle > 0,
\]
which is a contradiction with (4.1). This completes the proof. \( \Box \)

Let \( \bar{x} \in F_p \). Consider the generalized Lagrangian function associated to SDP, defined by \( L : \mathbb{R}^+_+ \times S^d_+ \times K \rightarrow \mathbb{R} \),
\[
L(\bar{x}; \lambda, A, x) := \lambda(f(x) - f(\bar{x})) - \langle A, g(x) \rangle, \quad \forall(\lambda, A, x) \in \mathbb{R}^+_+ \times S^d_+ \times K.
\]

We also consider the following Lagrangian function associated to SDP, defined by \( L_0 : S^d_+ \times K \rightarrow \mathbb{R} \),
\[
L_0(A, x) := f(x) - \langle A, g(x) \rangle, \quad \forall(A, x) \in S^d_+ \times K.
\]

**Definition 4.4.** The point \((\bar{\lambda}, \bar{A}, \bar{x}) \in \mathbb{R}^+_+ \times S^d_+ \times K\) is said to be a saddle point of the generalized Lagrangian function \( L \) on \( \mathbb{R}^+_+ \times S^d_+ \times K \), if the following inequalities hold:
\[
L(\bar{x}; \lambda, A, x) \leq L(\bar{x}; \bar{\lambda}, \bar{A}, \bar{x}) \leq L(\bar{x}; \bar{\lambda}, \bar{A}, x), \quad \forall(\lambda, A, x) \in \mathbb{R}^+_+ \times S^d_+ \times K;
\]
The point \((\bar{A}, \bar{x}) \in S^d_+ \times K\) is said to be a saddle point of the Lagrangian function \( L_0 \) on \( S^d_+ \times K \), if the following inequalities hold:
\[
L_0(A, \bar{x}) \leq L_0(\bar{A}, \bar{x}) \leq L_0(\bar{A}, x), \quad \forall(A, x) \in S^d_+ \times K.
\]

It is easy to verify the following proposition:

**Proposition 4.5.** Let \( \bar{x} \in F_p \) and \((\bar{\lambda}, \bar{A}) \in \mathbb{R}^+_+ \times S^d_+\). Then the point \((\bar{\lambda}, \bar{A}, \bar{x}) \in \mathbb{R}^+_+ \times S^d_+ \times K\) is a saddle point of the generalized Lagrangian function \( L \) on \( \mathbb{R}^+_+ \times S^d_+ \times K \), if and only if \((\bar{\lambda}, \bar{A}, \bar{x}) \in S^d_+ \times K\) is a saddle point of the Lagrangian function \( L_0 \) on \( S^d_+ \times K \).

We now characterize the linear separation for SDP by using the saddle points of the generalized Lagrangian function related to SDP.

**Theorem 4.6.** Let \( \bar{x} \in F_p \). Then the sets \( K(\bar{x}) \) and \( \mathcal{H} \) are linearly separable if and only if there exists \((\bar{\lambda}, \bar{A}) \in \mathbb{R}^+_+ \times S^d_+, \) with \((\bar{\lambda}, \bar{A}) \neq 0\), such that the point \((\bar{\lambda}, \bar{A}, \bar{x})\) is a saddle point for \( L \) on \( \mathbb{R}^+_+ \times S^d_+ \times K \).
Proof. Necessity. Suppose that $\mathcal{K}(\bar{x})$ and $\mathcal{H}$ are linearly separable. Then there exists $(\bar{\lambda}, \bar{A}) \in \mathbb{R}_+ \times S_+^l$, with $(\bar{\lambda}, \bar{A}) \neq 0$, such that (4.1) holds. Letting $x := \bar{x}$ in (4.1) allows that $\langle \bar{A}, g(\bar{x}) \rangle \leq 0$. Since $\bar{x} \in F_p$, we have $g(\bar{x}) \geq 0$ and therefore $\langle \bar{A}, g(\bar{x}) \rangle \geq 0$. As a consequence, $\langle \bar{A}, g(\bar{x}) \rangle = 0$ and again from (4.1) one has

$$L(\bar{x}; \bar{\lambda}, \bar{A}, \bar{x}) = 0$$

$$= \bar{\lambda}(f(\bar{x}) - f(\bar{x})) - \langle \bar{A}, g(\bar{x}) \rangle$$

$$\leq \bar{\lambda}(f(\bar{x}) - f(\bar{x})) - \langle \bar{A}, g(x) \rangle$$

$$= L(\bar{x}; \bar{\lambda}, \bar{A}, x), \quad \forall x \in K.$$ 

Again from $g(\bar{x}) \geq 0$ we have

$$L(\bar{x}; \lambda, A, \bar{x}) = -\langle A, g(\bar{x}) \rangle \leq 0, \quad \forall (\lambda, A) \in \mathbb{R}_+ \times S_+^l.$$

This shows that the point $(\bar{\lambda}, \bar{A}, \bar{x})$ is a saddle point for $L$ on $\mathbb{R}_+ \times S_+^l \times K$.

Sufficiency. Suppose that $(\bar{\lambda}, \bar{A}, \bar{x})$ is a saddle point for $L$ on $\mathbb{R}_+ \times S_+^l \times K$. Then

$$-\langle A, g(\bar{x}) \rangle \leq -\langle \bar{A}, g(\bar{x}) \rangle$$

$$\leq \bar{\lambda}(f(x) - f(\bar{x})) - \langle \bar{A}, g(x) \rangle, \quad \forall (A, x) \in S_+^l \times K.$$ (4.3)

Letting $A = 0$ in the first inequality leads to $\langle \bar{A}, g(\bar{x}) \rangle \leq 0$. Since $\bar{x} \in F_p$, one has $g(\bar{x}) \geq 0$ and so $\langle \bar{A}, g(\bar{x}) \rangle \geq 0$ since $\bar{A} \in S_+^l$. Thus $\langle \bar{A}, g(\bar{x}) \rangle = 0$ and it follows from (4.3) that

$$0 \leq \bar{\lambda}(f(x) - f(\bar{x})) - \langle \bar{A}, g(x) \rangle, \quad \forall x \in K,$$

or equivalently,

$$\bar{\lambda}(f(x) - f(\bar{x})) + \langle \bar{A}, g(x) \rangle \leq 0, \quad \forall x \in K,$$

which yields that the sets $\mathcal{K}(\bar{x})$ and $\mathcal{H}$ are linearly separable.

Similarly, we can prove the following result:

**Theorem 4.7.** Let $\bar{x} \in F_p$. Then the sets $\mathcal{K}(\bar{x})$ and $\mathcal{H}$ admit a regular linear separation if and only if there exists $(\bar{\lambda}, \bar{A}) \in \mathbb{R}_+ \times S_+^l$, such that the point $(\bar{\lambda}, \bar{A}, \bar{x})$ is a saddle point for $L$ on $\mathbb{R}_+ \times S_+^l \times K$.

From Proposition 4.5 and Theorem 4.7 we have:

**Theorem 4.8.** Let $\bar{x} \in F_p$. Then the sets $\mathcal{K}(\bar{x})$ and $\mathcal{H}$ admit a regular linear separation if and only if there exists $(\bar{\lambda}, \bar{A}) \in \mathbb{R}_+ \times S_+^l$, such that $(\bar{A}, \bar{x}) \in S_+^l \times K$ is a saddle point of the Lagrangian function $L_0$ on $S_+^l \times K$.

Under some convexity assumptions of $f$ and $-g$ we also have the following proposition:

**Proposition 4.9.** Let $\bar{x} \in F_p$. Suppose that $f$ is convex on $K$ and $-g$ is $S_+^l$-convex on $K$. Then the point $(\bar{\lambda}, \bar{A}, \bar{x}) \in \mathbb{R}_+ \times S_+^l \times K$ is a saddle point of the generalized Lagrangian function $L$ on $\mathbb{R}_+ \times S_+^l \times K$, if and only if it is a solution of the following system:

$$\begin{cases}
0 \in \lambda \partial f(x) + \partial(-\langle A, g(x) \rangle)(x) + N_K(x), \\
\langle A, g(x) \rangle = 0, \\
x \in F_p, \ (\lambda, A) \in \mathbb{R}_+ \times S_+^l, \ with \ (\lambda, A) \neq (0, 0).
\end{cases}$$ (4.4)
Proof. Necessity. Suppose that the point \((\bar{\lambda}, \bar{A}, \bar{x}) \in \mathbb{R}_+ \times S^I_+ \times K\) is a saddle point of the generalized Lagrangian function \(L\) on \(\mathbb{R}_+ \times S^I_+ \times K\). Then from the proof of sufficiency in Theorem 4.6, we have \(\langle \bar{A}, g(\bar{x}) \rangle = 0\). It follows from (4.3) that
\[
0 \leq \bar{\lambda}(f(\bar{x}) - f(\bar{x})) - \langle \bar{A}, g(x) \rangle, \quad \forall x \in K.
\] (4.5)

Let \(f^*(x) := \bar{\lambda}(f(x) - f(\bar{x})) - \langle \bar{A}, g(x) \rangle + i_K(x)\). Then \(\bar{x}\) is the minimum point of the function \(f^*\) on \(\mathbb{R}^k\). Since \(K\) is convex, \(f^*\) is convex on \(K\) and \(-g\) is \(S^I_+\)-convex on \(K\), we have \(-\langle \bar{A}, g(\cdot) \rangle\) is convex on \(K\) (see, for example [8]) and as a consequence, \(f^*\) is convex on \(\mathbb{R}^k\). Since \(\text{dom}(\bar{\lambda}(f(\cdot) - f(\bar{x}))) \cap \text{dom}(\langle \bar{A}, g(\cdot) \rangle) \cap \text{dom}_K = K\) and \(\text{ri}K \neq \emptyset\), it follows from [10] that
\[
0 \in \partial f^*(\bar{x})
= \partial(\bar{\lambda}(f(\cdot) - f(\bar{x})) - \langle \bar{A}, g(\cdot) \rangle + i_K(\bar{x}))
= \bar{\lambda}\partial f(\bar{x}) + \partial(-\langle \bar{A}, g(\cdot) \rangle)(\bar{x}) + N_K(\bar{x}),
\]
which implies that \((\bar{\lambda}, \bar{A}, \bar{x})\) solves system (4.4).

Sufficiency. Let \((\bar{\lambda}, \bar{A}, \bar{x})\) solves system (4.4). Again since \(\text{dom}(\bar{\lambda}(f(\cdot) - f(\bar{x}))) \cap \text{dom}(\langle \bar{A}, g(\cdot) \rangle) \cap \text{dom}_K = K\) and \(\text{ri}K \neq \emptyset\), it follows from [10] that
\[
0 \in \tilde{\lambda}\partial f(\bar{x}) + \partial(-\langle \bar{A}, g(\cdot) \rangle)(\bar{x}) + N_K(\bar{x})
= \partial(\bar{\lambda}(f(\cdot) - f(\bar{x})) - \langle \bar{A}, g(\cdot) \rangle + i_K(\bar{x}))
= \partial f^*(\bar{x}),
\]
where \(f^*(x) := \bar{\lambda}(f(x) - f(\bar{x})) - \langle \bar{A}, g(x) \rangle + i_K(x)\). Thus it follows that (4.5) holds and since \(\langle \bar{A}, g(\bar{x}) \rangle = 0\), we have
\[
L(\bar{x}; \bar{\lambda}, \bar{A}, \bar{x}) = \bar{\lambda}(f(\bar{x}) - f(\bar{x})) - \langle \bar{A}, g(\bar{x}) \rangle
= 0
\leq \bar{\lambda}(f(x) - f(\bar{x})) - \langle \bar{A}, g(x) \rangle
= L(\bar{x}; \bar{\lambda}, \bar{A}, x), \quad \forall x \in K.
\]

Since \(\bar{x} \in F_p\), i.e., \(g(\bar{x}) \succeq 0\), we have \(\langle A, g(\bar{x}) \rangle \geq 0\) for any \(A \in S^I_+\). As a consequence,
\[
L(\bar{x}; \lambda, A, \bar{x}) = \lambda(f(\bar{x}) - f(\bar{x})) - \langle A, g(\bar{x}) \rangle
= -\langle A, g(\bar{x}) \rangle
\leq 0, \quad \forall (\lambda, A) \in \mathbb{R}_+ \times S^I_+.
\]

This proves that \((\bar{\lambda}, \bar{A}, \bar{x}) \in \mathbb{R}_+ \times S^I_+ \times K\) is a saddle point of \(L\) on \(\mathbb{R}_+ \times S^I_+ \times K\). \(\square\)

Similarly, we have

**Proposition 4.10.** Let \(\bar{x} \in F_p\). Suppose that \(f\) is convex on \(K\) and \(-g\) is \(S^I_+\)-convex on \(K\). Then the point \((\bar{A}, \bar{x}) \in S^I_+ \times K\) is a saddle point of the Lagrangian function \(L_0\) on \(S^I_+ \times K\), if and only if it is a solution of the following system:
\[
\begin{cases}
0 \in \partial f(x) + \partial(-\langle A, g(\cdot) \rangle)(x) + N_K(x), \\
\langle A, g(x) \rangle = 0, \\
x \in F_p, A \in S^I_+.
\end{cases}
\]

**Corollary 4.11.** Let \(\bar{x} \in F_p\). Suppose that \(K\) is open, \(f\) is convex and differentiable on \(K\) and \(-g\) is \(S^I_+\)-convex on \(K\) and \(\langle A, g(\cdot) \rangle\) is differentiable on \(K\) for each
A ∈ \mathcal{S}_+^l$. Then the point \((\bar{A}, \bar{x}) \in \mathcal{S}_+^l \times K\) is a saddle point of the Lagrangian function \(L_0\) on \(\mathcal{S}_+^l \times K\), if and only if it is a solution of the following system:

\[
\begin{cases}
0 = \nabla f(x) - \nabla \langle A, g(\cdot) \rangle(x), \\
\langle A, g(x) \rangle = 0, \\
x ∈ F_p, A ∈ \mathcal{S}_+^l.
\end{cases}
\]

**Proof.** Since \(K\) is open, \(f\) and \(\langle A, g(\cdot) \rangle\) are differentiable on \(K\), where \(A ∈ \mathcal{S}_+^l\), one has \(\partial f(x) = \{\nabla f(x)\}, \partial \langle A, g(\cdot) \rangle(x) = \{\nabla \langle A, g(\cdot) \rangle(x)\}\) and \(N_K(x) = \{0\}\) for any \(x ∈ K\). The conclusion follows immediately from Proposition 4.10. \(\square\)

5. **Optimality conditions for SDP.** In this section, we shall present the necessary and sufficient optimality conditions for SDP. First we present the following necessary optimality condition for SDP.

**Theorem 5.1.** Let \(\bar{x} ∈ F_p\). Assume that the mapping \(-G_\bar{x}\) is \(\mathbb{R}_+ × \mathcal{S}_+^l\)-convexlike on \(\mathcal{S}_+^l\). If \(\bar{x}\) is a solution of SDP, then the sets \(K(\bar{x})\) and \(\mathcal{H}\) are linearly separable.

**Proof.** Clearly, the set \(\mathcal{H}\) is convex and from (1) of Proposition 3.2, \(\mathcal{E}\) is convex. If \(\bar{x} ∈ F_p\) is a solution of SDP, then from Proposition 3.2 (2) we have

\[\mathcal{H} ∩ \mathcal{E} = ∅.\]

Therefore, by a standard separation theorem (see, for example, [10]), there exist \(\alpha ∈ \mathbb{R}\) and a vector \((\bar{\lambda}, \bar{A}) ∈ \mathbb{R} × \mathcal{S}_+^l\), with \((\bar{\lambda}, \bar{A}) ≠ 0\), such that

\[\bar{\lambda}(f(\bar{x}) - f(x)) + \langle \bar{A}, g(x) \rangle - (\bar{\lambda}t + \langle \bar{A}, Y \rangle) ≤ \alpha, \quad ∀x ∈ K, Y ∈ \mathcal{S}_+^l, t ∈ \mathbb{R}_+, \]

and

\[\bar{\lambda}u + \langle \bar{A}, A \rangle ≥ \alpha, \quad ∀(u, A) ∈ \mathcal{H} = \mathbb{R}_+ × \mathcal{S}_+^l.\] (5.2)

Letting \(A := 0\) in (5.2) we have, as \(u → 0, 0 ← \bar{\lambda}u ≥ \alpha\). This implies that \(\alpha ≤ 0\). We may assume that \(\alpha := 0\). In fact, if there is \((u^0, A^0) ∈ \mathcal{H}\) such that \(\bar{\lambda}u^0 + \langle \bar{A}, A^0 \rangle < 0\), then letting \(u := tu^0\) and \(A := tA^0\) \((t ∈ \mathbb{R}_+)\) in (5.2) allows that \(\bar{\lambda}u^0 + \langle \bar{A}, A^0 \rangle ≥ 0\), as \(t → +∞\), a contradiction. Therefore it follows immediately from (5.2) that \((\bar{\lambda}, \bar{A}) ∈ \mathbb{R}_+ × \mathcal{S}_+^l\). Setting \(Y := 0\) and \(t := 0\) in (5.1) allows that

\[\bar{\lambda}(f(\bar{x}) - f(x)) + \langle \bar{A}, g(x) \rangle ≤ 0, ∀x ∈ K.\]

This proves that the sets \(K(\bar{x})\) and \(\mathcal{H}\) are linearly separable. \(\square\)

From Theorems 5.1 and 4.6 we have the following corollary:

**Corollary 5.2.** Let \(\bar{x} ∈ F_p\). Assume that the mapping \(-G_\bar{x}\) is \(\mathbb{R}_+ × \mathcal{S}_+^l\)-convexlike on \(\mathcal{S}_+^l\). If \(\bar{x}\) is a solution of SDP, then there exists \((\bar{\lambda}, \bar{A}) ∈ \mathbb{R}_+ × \mathcal{S}_+^l\), with \((\bar{\lambda}, \bar{A}) ≠ 0\), such that the point \((\bar{\lambda}, \bar{A}, \bar{X})\) is a saddle point for \(L\) on \(\mathbb{R}_+ × \mathcal{S}_+^l × \mathcal{S}_+^l\).

Now we present the following sufficient optimality condition for SDP.

**Theorem 5.3.** Let \(\bar{x} ∈ F_p\). If the sets \(K(\bar{x})\) and \(\mathcal{H}\) admit a regular linear separation, then \(\bar{x}\) is a solution of SDP.

**Proof.** Suppose that the sets \(K(\bar{x})\) and \(\mathcal{H}\) admit a regular linear separation. Then there is \((\bar{\lambda}, \bar{A}) ∈ \mathbb{R}_+ × \mathcal{S}_+^l\) such that (4.1) holds. We declare that \(\bar{x}\) is a solution of SDP. If not, then there is \(x^0 ∈ K\) with \(g(x^0) ∈ \mathcal{S}_+^l\) such that \(f(\bar{x}) > f(x^0)\). As a consequence,

\[\bar{\lambda}(f(\bar{x}) - f(x^0)) + \langle \bar{A}, g(x^0) \rangle > 0,\]

a contradiction with (4.1). \(\square\)
From Theorems 5.3 and 4.3 we have the following corollary:

**Corollary 5.4.** Let $\bar{x} \in F_p$. If the sets $\mathcal{K}(\bar{x})$ and $\mathcal{H}$ are linearly separable and the Slater condition (4.2) holds, then $\bar{x}$ is a solution of SDP.

6. **Applications to LSDP.** In this section, we shall apply the obtained results to characterize the necessary and sufficient optimality conditions of LSDP.

Let $\bar{x} \in F_p$. Define

$$\mathcal{H} := \{(u, A) \in \mathbb{R} \times S^l : u > 0, A \succeq 0\} = \mathbb{R}^+ \times S^l_+,$$

$$\mathcal{H}_u := \{(u, A) \in \mathbb{R} \times S^l : u > 0, A = 0\} = \mathbb{R}^+ \times \{0\},$$

$$\mathcal{K}(\bar{x}) := \{(u, A) \in \mathbb{R} \times S^l : u = c_0^T (\bar{x} - x), A = A_0 + \sum_{j=1}^k x_j A_j, x \in K\},$$

$$\mathcal{E} := \{(u, A) \in \mathbb{R} \times S^l : u \leq u = c_0^T (\bar{x} - x), A \succeq A_0 + \sum_{j=1}^k x_j A_j, x \in K\}. $$

Then from Propositions 3.1 and 3.2 we have:

**Proposition 6.1.** Let $\bar{x} \in F_p$. The following statements are true:

1. The set $\mathcal{E}$ is convex;
2. $\bar{x}$ solves LSDP if and only if $\mathcal{H} \cap \mathcal{K}(\bar{x}) = \emptyset$ if and only if $\mathcal{H} \cap \mathcal{E} = \emptyset$ if and only if $\mathcal{H}_u \cap \mathcal{E} = \emptyset$.

From Theorems 4.3, 4.6, 5.1 and 5.3, Propositions 4.5 and 4.9, we have:

**Theorem 6.2.** Let $\bar{x} \in F_p$. Suppose that $K$ is open and the following Slater condition holds:

$$\exists x^0 \in K \text{ such that } A_0 + \sum_{j=1}^k x_j^0 A_j \succ 0,$$

Then the following statements are equivalent:

1. The point $\bar{x}$ is a solution of LSDP;
2. The sets $\mathcal{K}(\bar{x})$ and $\mathcal{H}$ are linearly separable;
3. The sets $\mathcal{E}$ and $\mathcal{H}$ are linearly separable;
4. The sets $\mathcal{K}(\bar{x})$ and $\mathcal{H}$ admit a regular linear separation;
5. The sets $\mathcal{E}$ and $\mathcal{H}$ admit a regular linear separation;
6. There exists $(\lambda, A) \in \mathbb{R}_+ \times S^l_+$, with $(\lambda, A) \neq 0$, such that the point $(\lambda, A, \bar{x})$ is a saddle point for $L$ on $\mathbb{R}^+ \times S^l_+ \times K$.
7. There exists $(\lambda, A) \in \mathbb{R}_+ \times S^l_+$, such that $(\frac{\lambda}{A}, \bar{x}) \in S^l_+ \times K$ is a saddle point of $L_0$ on $S^l_+ \times K$.
8. The point $(\lambda, A, \bar{x}) \in \mathbb{R}_+ \times S^l_+ \times K$ is a solution of the following system:

$$\begin{cases}
0 = \lambda c_0 - ((A, A_1), \cdots, (A, A_k))^\top, \\
(A, A_0) + \sum_{j=1}^k x_j (A, A_j) = 0, \\
x \in F_p, (\lambda, A) \in \mathbb{R}_+ \times S^l_+, \text{ with } (\lambda, A) \neq (0, 0).
\end{cases}$$

**Proof.** The conclusion follows immediately from the facts that $\partial f(\bar{x}) = \{c_0\}$ and $\partial (-\langle A, g(\cdot) \rangle)(\bar{x}) = \{- (\langle A, A_1 \rangle, \cdots, (A, A_k))\}$. \(\square\)
Example 6.3. Let $K := \{ x := (x_1, x_2)^T : x_i > -2, \ i = 1, 2 \} \subseteq \mathbb{R}^2$, $c_0 := (1, 1)^T \in \mathbb{R}^2$, $A_0 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $A_1 := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $A_2 := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

Then $f(x) := x_1 + x_2$ and $g(x) := \begin{pmatrix} 1 + x_1 & 0 \\ 0 & 1 + x_2 \end{pmatrix}$, where $x := (x_1, x_2)^T$. Simple computation leads to $F_p = \{ x := (x_1, x_2)^T : x_i \geq -1, \ i = 1, 2 \}$ and the Slater condition holds with $x^0 := (0, 0)^T$. Clearly, $\bar{x} := (-1, -1)^T$ is the solution of LSDP and Theorem 6.1 holds for $\lambda := 1$ and $\bar{A} := A_0$.

7. Conclusion. Recently, there has been an increasing interest in the ISA. Separation plays a vital role in the ISA. In this paper, the ISA was employed to investigate semidefinite programming. Some equivalence between the (regular) linear separation and the saddle points of the Lagrangian functions related to the problem were characterized. Some necessary and sufficient optimality conditions for semidefinite programming were given under some suitable assumptions. An application to linear semidefinite programming was also given to illustrate the obtained results.

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