Solution of quantum eigenvalue problems by means of algebraic consistency conditions

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Abstract
We present a simple algebraic procedure that can be applied to solve a range of quantum eigenvalue problems without the need to know the solution of the Schrödinger equation. The procedure, presented with a pedagogical purpose, is based on algebraic consistency conditions that must be satisfied by the eigenvalues of a couple of operators proper of the problem. These operators can be either bilinear forms of the raising and lowering operators appropriate to the problem, or else auxiliary operators constructed by resorting to the factorization of the Hamiltonian. Different examples of important quantum-mechanical textbook problems are worked out to exhibit the clarity and simplicity of the algebraic procedure for determining the spectrum of eigenvalues without knowing the eigenfunctions. For this reason the material presented may be particularly useful for undergraduate students or young physicists.

Keywords: quantum eigenvalues, algebraic method, ladder operators

1. Introduction
One may find in the literature several approaches to the quantum eigenvalue problem based on operator methods, the most well known ones being the method of factorization of the Hamiltonian and the supersymmetric approach (susy, see e.g. references [1–4]). Reviews of susy quantum mechanics from the point of view of a general factorization method can be found in [5, 6]; further, references [7–9] contain examples of skillful combinations of both approaches that are of relevance to the present work.

On the other hand, since the early introduction of raising and lowering operators for the harmonic oscillator (HO) and the angular momentum, significant effort has been invested to study the relationship between ladder operators and the factorization method. All in all, this relationship has proved to be far from trivial. Specifically, it has become clear that knowing

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the ladder operators does not mean that we know how to factorize the Hamiltonian [10], and conversely, knowing how to factorize the Hamiltonian does not in general lead to the construction of differential ladder operators. The work by Infeld and Hull [1], followed by that of Mielnik and collaborators (see [3] and references therein), has led first to a family of solvable potentials isospectral to the HO, and thereafter to more general solvable potentials with an arbitrary factorization energy. However, as mentioned in [11], the factorization method is by now well developed, and the set of problems that can be solved through this technique can hardly be expanded. The procedure has also become quite elaborate and difficult to follow for somebody not involved in this area. A valuable general reference is the book by de Lange and Raab [12], which contains an accessible introduction to operator methods for the solution of quantum-mechanical problems.

Here we present an alternative method aimed at solving the quantum eigenvalue problem without solving the Schrödinger equation, which is characterized by its simplicity, generality and directness. The proposed procedure, based on a few algebraic consistency conditions, was first developed a number of years ago in reference [13] and applied to typical elementary textbook problems in which the ladder operators do not depend on the state on which they operate, thereby solving the eigenvalue problem without the need to know the solution of the Schrödinger equation. Well-known instances are precisely the HO and the angular momentum, as well as several other textbook examples. A few years thereafter the algebraic technique was successfully applied to solve the radial component of most popular central-force quantum problems [14]. Our purpose here is to show how the proposed procedure can be generalized and applied to a wider range of quantum-mechanical problems. Due to its transparency and directness, the method of the consistency conditions could result of particular importance or interest to undergraduate or early graduate physics students.

For clarity in the presentation, the paper starts with an introduction of the basic properties of ladder operators, which allows us to readily derive some general algebraic consistency conditions that must be satisfied by a couple of bilinear combinations of these operators. The distinction is made between ladder operators with individual components that either depend or do not depend on the state to which they are applied. A couple of textbook examples illustrate the method for state-independent ladder operators. It is further shown that the consistency conditions can be applied also to solve quantum eigenvalue problems that lend themselves to the factorization of the Hamiltonian, typically central-force problems, including Dirac’s H-atom. Further application to a simple one-dimensional problem serves to stress the advantage of state-independent operators as a tool to solve the eigenvalue problem by algebraic means. Finally, an elementary example is presented that illustrates how the algebraic method can also be used in connection with perturbative calculations.

2. State-dependent and state-independent ladder operators

Let us consider the eigenvector basis \{\ket{k}\} of a certain Hermitian operator \(P\) of interest with discrete spectrum; we use this basis to construct a pair of operators \(\eta^\dagger, \eta\) as follows,

\[
\eta^\dagger = \sum_i C_i \ket{i + 1} \bra{i} = \sum_i C_i \eta_i^\dagger, \tag{1a}
\]

\[
\eta = \sum_i C_i^* \ket{i} \bra{i + 1} = \sum_i C_i^* \eta_i, \tag{1b}
\]
with
\[ \eta_i = |i+1\rangle\langle i|, \quad \eta_i = |i\rangle\langle i+1| \] (1c)
and with the constants \( C_i \) to be determined according to the needs. Note from (1c) that
\[ \eta_i^\dagger |n\rangle = |n+1\rangle \delta_{in}, \quad \eta_i |n\rangle = |n-1\rangle \delta_{in-1}. \] (2)
The individual operators \( \eta_i^\dagger, \eta_i \) may depend explicitly or not on the state \( i \) on which they operate, in which case we speak of state-dependent or state-independent (individual) operators, respectively.
The action of \( \eta_i^\dagger \) and \( \eta_i \) given by (1) over a ket \( |k\rangle \) of the basis is
\[ \eta_i^\dagger |k\rangle = \sum_i C_i |i+1\rangle \langle i| |k\rangle = C_k |k+1\rangle, \] (3a)
\[ \eta_i |k\rangle = \sum_i C_i^* |i-1\rangle \langle i| |k\rangle = C_{k-1} |k-1\rangle, \] (3b)
so that \( \eta_i^\dagger \) and \( \eta_i \) are, respectively, raising and lowering operators in the Hilbert space of \( \{|k\rangle\} \), for any selection of the coefficients \( C_k \). It also follows that
\[ \eta_i \eta_i^\dagger |k\rangle = |C_k|^2 |k\rangle, \] (4a)
\[ \eta_i^\dagger \eta_i |k\rangle = |C_{k-1}|^2 |k\rangle. \] (4b)
We see that the basis \( \{|k\rangle\} \) corresponds to the eigenvectors of the operators \( \eta_i \eta_i^\dagger \) and \( \eta_i^\dagger \eta_i \), and the quantities \( |C_k|^2, |C_{k-1}|^2 \) are the corresponding eigenvalues. Therefore, the operators
\[ S \equiv \eta_i \eta_i^\dagger + \eta_i^\dagger \eta_i = \{\eta, \eta^\dagger\}, \quad A \equiv \eta_i \eta_i^\dagger - \eta_i^\dagger \eta_i = \{\eta, \eta\}, \] (5)
which are respectively the anticommutator and commutator of \( \eta \) and \( \eta^\dagger \), satisfy the eigenvalue equations
\[ S |k\rangle = s_k |k\rangle, \quad A |k\rangle = a_k |k\rangle, \] (6)
with
\[ s_k = |C_k|^2 + |C_{k-1}|^2, \] (7a)
\[ a_k = |C_k|^2 - |C_{k-1}|^2. \] (7b)
whence
\[ 2|C_k|^2 = s_k + a_k = s_{k+1} - a_{k+1} \] (8)
and
\[ s_{k+1} - s_k = a_{k+1} + a_k. \] (9)
These basic consistency conditions between the eigenvalues of \( S \) and \( A \) corresponding to adjacent states will play an important role in what follows. In particular, it is clear from equations (7) that
\[ s_k \geq |a_k|. \] (10)
We will consider that the spectrum of interest is bounded from below. As is customary, we shall denote the ground state with $|k_{\text{min}}\rangle = |0\rangle$ unless otherwise specified, so that, from equation (3b),
\[ C_{-1} = 0, \tag{11} \]
and from equations (7),
\[ s_0 = a_0. \tag{12} \]
Successive application of equation (9), together with (12), leads to the recurrence relation
\[ s_k = a_k + 2 \sum_{i=0}^{k-1} (-1)^{i-(k-1)} s_i, \tag{13} \]
establishing the relationship between the eigenvalues of $A$ and those of $S$.

In the case of a finite Hilbert space, with $\{ |k\rangle \} = \{ |0\rangle, \ldots, |k_{\text{max}}\rangle \}$, the upper bound imposes the condition $\eta_k^\dagger |k_{\text{max}}\rangle = 0$, so that $C_{k_{\text{max}}} = 0$ and
\[ s_{k_{\text{max}}} = -a_{k_{\text{max}}}. \tag{14} \]

As we shall see below, for some applications it proves convenient to introduce a couple of auxiliary operators $\alpha, \beta$, indirectly defined through the relations
\[ \eta^\dagger = \frac{1}{\sqrt{2}} (\alpha - i \beta), \tag{15a} \]
\[ \eta = \frac{1}{\sqrt{2}} (\alpha + i \beta). \tag{15b} \]

From equations (5) and (15) it follows that
\[ A = i[\beta, \alpha], \tag{16a} \]
\[ S = \alpha^2 + \beta^2. \tag{16b} \]

In this form, when the ladder operators are known, the consistency conditions allow us to determine the spectrum of $A$ in terms of the spectrum of $S$ or vice versa. This is particularly useful when either one of these operators is identified with a dynamical variable of interest, as is the case in the examples discussed in section 3.1.

It may happen that the operator corresponding to the relevant dynamical variable—the one whose eigenvalues are looked for—is not of the form of $A$ or $S$, but a linear combination of these (with $p_0, p_S$ and $p_A$ real parameters) [8, 13],
\[ P = p_0 I + p_S S + p_A A. \tag{17} \]

As exemplified in section 3.2, the algebraic method may be applied in some cases even without resorting to the explicit form of the ladder operators, by establishing a functional relationship between the operators $A$ and $S$ through an auxiliary operator $B$. 

Before closing this section, we note that the definition of ladder operators given in equations (1) can be readily extended to operators connecting states that are not adjacent but separated a distance $T$, i.e.,

$$\eta^{(T)} |k\rangle = C^{(T)}_{k-T} |k-T\rangle.$$

The corresponding operators $S^{(T)}, A^{(T)}$ satisfy the eigenvalue equations

$$S^{(T)} |k\rangle = s^{(T)}_k |k\rangle, \quad A^{(T)} |k\rangle = a^{(T)}_k |k\rangle,$$

leading to

$$2 \left| C^{(T)}_k \right|^2 = s^{(T)}_k + a^{(T)}_k = s^{(T)}_{k+T} - a^{(T)}_{k+T}$$

and

$$s^{(T)}_{k+T} - s^{(T)}_k = a^{(T)}_{k+T} - a^{(T)}_k,$$

as immediate generalizations of equations (8) and (9), respectively, with $C^{(T)}_k = 0$. It is important to notice, however, that although $T$ subsequent operations with single-step raising operators take the system from state $|k\rangle$ to state $|k+T\rangle$, the combined coefficient resulting from such operations is not equal to $C^{(T)}_k$ in general, as the reader may confirm from the examples presented in the following sections.

3. Applications

3.1. Textbook examples

3.1.1. 1D harmonic oscillator. As a first and most useful example of state-independent ladder operators we recall the well-known case of the one-dimensional HO of mass $m$ and frequency $\omega$, with raising and lowering operators given respectively by

$$\eta^+ = \frac{1}{2\sqrt{m}} (m\omega x - ip), \quad \eta = \frac{1}{2\sqrt{m}} (m\omega x + ip),$$

with $p = -i\hbar(d/dx)$. Notice that the ladder operators $\eta^+, \eta$ differ from the standard ones $a^+, a$ by a factor $\sqrt{\hbar \omega}/2$. From equation (15) we get

$$\alpha = \sqrt{\frac{m}{2}} \omega x, \quad \beta = \frac{1}{\sqrt{2m}} p,$$

which introduced in equations (16) gives

$$A = -\frac{i\hbar}{2} [x, p] = \frac{1}{2} \hbar \omega, \quad S = \frac{1}{2m} (m^2 \omega^2 x^2 + p^2) = H.$$

The corresponding eigenvalues are therefore

$$a_k = \frac{1}{2} \hbar \omega, \quad s_k = E_k.$$

Equation (10) and the consistency condition (9) give

$$E_k \geq \frac{1}{2} \hbar \omega, \quad E_{k+1} - E_k = \hbar \omega.$$
Iterating the last expression gives
\[ E_k = k\hbar\omega + E_0, \quad k = 0, 1, \ldots, \quad (22) \]
where \( E_0 \) corresponds to the ground state, \( s_0 = a_0 = \frac{1}{2}\hbar\omega \). For a detailed history of the theory of the quantum HO, see reference [15].

3.1.2. Angular momentum. This is another important example of state-independent ladder operators that is directly amenable to the above procedure. The angular-momentum operators \( J = (J_1, J_2, J_3) \) are defined as usual through the eigenvalue equations
\[ J^2|\lambda, \mu\rangle = \lambda|\lambda, \mu\rangle, \]
\[ J_3|\lambda, \mu\rangle = \mu|\lambda, \mu\rangle, \]
and the commutation relation (in units of \( \hbar \)),
\[ J_3 = -i[J_1, J_2]. \]
From these, it is a simple exercise to construct the raising and lowering operators
\[ \eta^+ = \frac{1}{\sqrt{2}}(J_1 + ij_2) = J_+, \quad \eta = \frac{1}{\sqrt{2}}(J_1 - ij_2) = J_. \quad (23) \]

Since \( J_+ \), \( J_- \), and \( J_3 \) commute with \( J^2 \), they leave the value of \( \lambda \) unchanged and operate on \( \mu \) only. This means that we look for the eigenvalues of \( J_3 \) for a given magnitude of the total angular momentum (i.e., for fixed \( \lambda \)). The construction of ladder operators for orbital angular momentum that modify the value of \( \lambda \) can be seen in [16].

Equations (23) suggest introducing
\[ \alpha = J_1, \quad \beta = -J_2, \]
so that from equations (16)
\[ A = i[J_1, J_2] = -J_3, \quad (24) \]
\[ S = J_1^2 + J_2^2 = J^2 - J_3^2, \quad (25) \]
whence
\[ a_k = -\mu_k, \quad s_k = \lambda - \mu_k^2. \quad (26) \]
From equations (9) and (10) one obtains
\[ \lambda \geq \mu_k^2 + |\mu_k| \quad (27) \]
and
\[ |\mu_k + \frac{1}{2}| = |\mu_{k+1} - \frac{1}{2}|, \]
whence
\[ \mu_{k+1} = \mu_k + 1. \quad (28) \]
which gives after iteration
\[ \mu_k = \mu_0 + k, \quad k = 0, 1, 2, \ldots \] (29)
where \( k \) is the number of unit steps required to go from \( \mu_0 \) to \( \mu_k \), and \( \mu_0 \) denotes the minimum eigenvalue in \( \{ \mu_k \} \). Introducing \( \lambda = j(j + 1) \), with \( j > 0 \), equation (27) rewrites as
\[
j + \frac{1}{2} \geq \begin{cases} \mu_k + \frac{1}{2}, & \mu_k \geq 0 \\ \mu_k - \frac{1}{2}, & \mu_k \leq 0 \end{cases}
\] (30)
which implies
\[-j \leq \mu_k \leq j.
Since the number of steps required to go from \( \mu_0 = -j \) to \( \mu_M = j \) must be an integer, the value of \( j \) must be an integer or half integer, and from (29) one gets
\[ \mu_k = -j + k, \quad k = 0, 1, \ldots, 2j. \] (31)
The values obtained from (8) and (26) for the coefficients, namely
\[ |C_k|^2 = \frac{1}{2} \{ j(j + 1) - \mu_k(\mu_k + 1) \} = \frac{1}{2} \{ (j + \mu_k + 1)(j - \mu_k) \}, \]
reproduce well-known results for the (squared modulus of the) matrix elements of the ladder operators in the basis \( \{ k \} \). Note that although the coefficients depend on both indices \( j \) and \( k \), the dependence on the former is passive, so to say, meaning that \( j \) appears as a fixed parameter whilst \( k \) refers to the state on which the ladder operators act.

3.2. The N-dimensional radial problem
There exists a family of problems for which a factorization method allows to construct appropriate state-independent operators \( A \) and \( S \) that can be used to solve the eigenvalue problem with the help of the consistency relations. These are \( N \)-dimensional radial problems, i.e., central-force problems in \( N \) dimensions (\( N = 1, 2, 3 \)) that have been reduced to the radial variable only. Although the present procedure has been applied to these problems in reference [14], we shall briefly present it here, along with some examples by way of illustration.
To investigate the spectrum associated with the radial factor of such problems, we introduce a couple of operators [7, 14] that depend on a continuous, dimensionless variable \( q \) as follows (here \( \Lambda \) is a real parameter)
\[ B_{n \pm} = q^n \pm \left( \frac{-\Lambda}{q^n} + \frac{1}{q^{n-2}} \frac{d^2}{dq^2} \right), \] (32)
with the properties
\[
\left[ q \frac{d}{dq}, B_{n \pm} \right] = n B_{n \pm}, \quad (33a)
\]
\[ B_{n+} = 2q^n - B_{n-}. \] (33b)
This procedure is justified by the fact that for certain values of the exponent $n$, $B_n$ happens to have the form of an important quantum-mechanical radial Hamiltonian, as will be seen below.

Careful inspection of equations (33) suggests to select

$$\alpha = B_{n+},$$

(34a)

$$i\beta = q \frac{d}{dq} - \left(q \frac{d}{dq}\right)^\dagger = 2q \frac{d}{dq} + (n - 1)I,$$

(34b)

so that after a few simple transformations one obtains

$$S = \alpha^2 + \beta^2 = B_n^2 + (n^2 - 1 - 4\Lambda)I,$$

(35a)

$$A = i[\beta, \alpha] = 2nB_n^-.$$  

(35b)

This couple of equations indicates that the eigenvalues of the operators $A$ and $S$ are related through the eigenvalues of the operator $B_n^-$, which are the ones of interest (in this case the operator $P$ of equation (17) is just proportional to $A$). Denoting these eigenvalues with $b_k$, we obtain from the consistency condition (9)

$$b_{k+1}^2 - b_k^2 = 2n(b_{k+1} + b_k).$$

(36)

Upon elimination of the common factor $b_{k+1} + b_k$ (assuming it is different from zero, which is true in the cases of interest), this equation simplifies into

$$b_{k+1} - b_k = 2n,$$

with solution

$$b_k = b_0 + 2nk.$$  

(37)

For $n > 0$, which is the only case we shall study here, there is no upper bound. Equation (13) gives $2nb_0 = b_0^2 + n^2 - 1 - 4\Lambda$, whence

$$b_0 = n \pm \sqrt{1 + 4\Lambda} \geq 0.$$  

(38)

We see that the proposed selection of the operators $\alpha, \beta$ has led to the general solution expressed in equation (37). The gist of the method was to obtain a nonlinear relation between $s_k$ and $a_k$, with $a_k \propto b_k$, and use the consistency condition to determine $b_k$.

We shall now apply this procedure to a couple of examples.

3.2.1. The isotropic $N$-dimensional harmonic oscillator. The radial equation for the $N$-dimensional isotropic HO is

$$\left(-\frac{\hbar^2}{2m} \frac{d^2}{dr^2} + \frac{g}{2r^2} + \frac{1}{2}m\omega^2 r^2\right)u = Eu,$$

with $u = rR$, $R(r)$ being the radial part of the wave function, and

$$g = \left[\frac{1}{4}(N - 1)(N - 3) + l(l + N - 2)\right] \frac{\hbar^2}{m}.$$  

(39)

Notice that the angular momentum $l$ enters here as a fixed parameter. The case $N = 1$ has already been dealt with in section 3.1.1, so we shall consider here $N = 2, 3$. In terms of the
dimensionless variable $q = r/\alpha_0$ with $\alpha_0 = \sqrt{\hbar/m\omega}$, the energy becomes expressed in units of $\frac{1}{2}\hbar\omega$ and the Hamiltonian coincides with $B_n^{-}$ for $n = 2$ [14].

$$H = -\frac{d^2}{dq^2} + \frac{\Lambda}{q^2} + q^2 = B_{2^{-}},$$

(40)

where

$$\Lambda = mg/h^2 = \frac{1}{4}(N-1)(N-3) + l(l + N - 2).$$

(41)

Equation (38) with $n = 2$ gives for $E_0$

$$E_0 = \frac{1}{2}\hbar\omega b_0 = \frac{1}{2}\hbar\omega \left(2 \pm \sqrt{(N + 2l - 2)^2}\right).$$

(42)

For $N = 2, 3$ and arbitrary $l$ the condition $E_0 \geq 0$ precludes the negative sign, so that

$$E_0 = \frac{1}{2}\hbar\omega(N + 2l); \quad N = 2, 3.$$  

Combining this result with equation (37) for $n = 2$, we obtain for the energy corresponding to the radial state $k$ and a given angular momentum $l$,

$$E_{kl} = \hbar\omega(2k + l + N/2).$$

(43)

The subindex $l$ has been added to the energy eigenvalues to remind us that the angular momentum eigenvalue enters as a parameter.

3.2.2. The hydrogen-like atom. The radial Schrödinger equation for the hydrogen-like atom in $N$ dimensions is

$$\left[ -\frac{\hbar^2}{2m} \frac{d^2}{dr^2} + \frac{r}{2l^2} - \frac{Ze^2}{r} \right] u = Eu.$$  

(44)

Multiplying by $r$, in terms of the variable $q = \beta\varphi r$ with $\beta\varphi = \sqrt{2m|E|/\hbar}$, the radial Hamiltonian reduces after some simple rearrangements to the operator $B_{n^{-}}$ for $n = 1$ [14],

$$\left( -q\frac{d^2}{dq^2} + \frac{\Lambda}{q} + q \right) u = B_{1^{-}}u = \frac{Ze^2}{\hbar} \sqrt{\frac{2m}{|E|}} u,$$

with $\Lambda$ given by (41). To determine $b_0 = (Ze^2/\hbar)\sqrt{2m/|E_0|}$ we proceed as above, using (38) now with $n = 1$, thus obtaining $b_0 = N - 1 + 2l$ (the minus sign in equation (38) is ruled out since $b_0 \geq 0$ for any $N$ and $l$). Using equation (37) we get $b_k = N - 1 + 2(l + k)$, which combined with $b_k = (Ze^2/\hbar)\sqrt{2m/|E_k|}$ gives for the energy eigenvalues

$$E_{kl} = -\frac{Z^2 e^4 m}{2\hbar^2 (k + l + \frac{N-1}{2})^2} = -\frac{Z^2 e^4 m}{2\hbar^2 n^2}.$$  

(45)

Note that the principal quantum number, denoted by $n$ (roman), $n = k + l + \frac{N-1}{2}$, is an integer in the 1D and 3D cases, and a half-integer in the 2D case.

In reference [17], the H atom problem is solved by means of a similar procedure involving factorization of the Hamiltonian. Additional examples of the central problem in several
dimensions can be seen in references [8, 12–14]. All these examples show that even when central-force problems can be reduced for their solution to one-dimensional (radial) problems regardless of their physical dimension $N$, the energy spectrum does depend explicitly on the number of dimensions.

3.2.3. The Dirac H-atom. The consistency conditions apply whenever the operators $S$ and $A$ can be written in the form of equations (35), where $B_{n-}$ represents the Hamiltonian operator expressed in the appropriate dynamical variable. It is therefore possible to apply them also to a relativistic problem. Let us select for this purpose the radial component of the hydrogen atom as described by the Dirac theory.

For a central problem, a standard—though elaborate—procedure (see, e.g., references [18, 19]) allows to transform the Dirac equation of first order for a four-component bi-spinor into a couple of first-order equations, each for a two-component spinor and the corresponding amplitudes. A further separation of the angular and radial variables leads to a second-order equation corresponding to the Klein–Gordon equation plus additional terms that represent corrections generated by the coupling of the spin of the particle to the external field. Finally, the angular and radial functions can be factored out.

For the present purpose let us consider the stationary case. The transformation just described leads to the stationary radial equation for the hydrogen-like atom,

$$\left(\nabla^2_r + \frac{\Lambda_D}{r^2} + \frac{2c_E}{r} - d_E\right) R = 0,$$

with

$$\nabla^2_r = \frac{1}{r} \frac{\partial^2}{\partial r^2} r, \quad \Lambda_D = l(l + 1) - \alpha^2 Z^2,$$

$$d_E = \frac{m^2 c^2}{\hbar^2} \left(1 - c^2\right), \quad c_E = \frac{Z m e^2}{\hbar^2} \mathcal{E},$$

where

$$\mathcal{E} = \frac{mc^2 + E}{mc^2}$$

stands for the energy of the atom, including the rest energy, expressed in units of the rest energy, and $\alpha = e^2 / (\hbar c)$ is the fine-structure constant, $\alpha \simeq 1/137$. In what follows we shall limit the calculation to the hydrogen atom by making the atomic number $Z = 1$.

In terms of the function $u = r R$, equation (46) acquires the same structure as (44). They both correspond to the operator $B_{1-}$, of course with different sets of parameters, and (46) is therefore equivalent to

$$\left(-q \frac{d^2}{dq^2} + \frac{\Lambda_D}{q} + q\right) u = B_{1-} u = \frac{2c_E}{d_E^2} u,$$

$$\Lambda_D = l(l + 1) - \alpha^2, \quad q = d_E^{1/2} r.$$ 

Hence the solution for the Schrödinger atom can be translated to the Dirac atom,

$$b_k = b_0 + 2k,$$

$$\text{(51)}$$
with $b_0$ given according to equation (38) by $b_0 = 1 + \sqrt{1 + 4\Lambda D}$. Here again the minus sign is ruled out since $b_0 \geq 0$ for any $l$. From equations (48) and (51) we obtain after some algebra

$$\mathcal{E}_k = \left[ 1 + \frac{4\alpha^2}{(b_0 + 2k)^2} \right]^{-1/2},$$

which is the exact result for the energy of the Dirac H-atom in units of $mc^2$ [18].

3.3. State-dependent individual operators; a simple example

The HO, the angular momentum and the radial problems just presented have an important feature in common: in all these cases, the successive eigenvalues of the relevant operator $P$ (with $P = H, J_3, B_n$, respectively) are equally spaced,

$$p_k = p_0 + \gamma k,$$

whence $p_{k+1} - p_k = \gamma$, where the value of the parameter $\gamma$ depends on the specific problem ($\gamma = \hbar \omega, 1, 2n$, respectively).

For the HO and the angular momentum, the individual ladder operators $\eta^\dagger_k, \eta_k$ do not depend explicitly on the state $k$. For state-dependent operators $\eta^\dagger_k, \eta_k$ by contrast, using the algebraic method may not be the most convenient procedure. This is clearly seen in reference [20], where the eigenvalue problem of the H atom is solved by constructing a pair of radial ladder operators, which turn out to contain the angular momentum $l$ and thus give rise to $l$-dependent commutators, such as $[\eta_l, \eta^{\dagger}_l] = (l + l' + 2)/r^2$. Here we illustrate the point by resorting to another well-known elementary textbook problem, namely the 1D infinite square well, with Hamiltonian given by

$$H = \frac{p^2}{2m} = \frac{\hbar^2}{2m} \frac{d^2}{dx^2}; \quad 0 \leq x \leq L.$$  

This problem has received considerable attention along the years, and ladder operators have been obtained with the help of the susy technique [4] and by means of alternative algebraic methods (see [9] and references therein). In this case the individual operators $\eta^\dagger_k, \eta_k$ turn out to be explicitly state-dependent. In terms of the dimensionless variable $y = (\pi/L)x$, their expression [21], when acting on $\varphi_k(y) = (y/k)\varphi$, is

$$\eta^\dagger_k \varphi_k = \left( \cos \frac{1}{k} \sin \frac{d}{dy} \right) \varphi_k \delta_{ik},$$

$$\eta_k \varphi_k = \left( \cos \frac{1}{k} \sin \frac{d}{dy} \right) \varphi_k \delta_{ik-1}. \quad (55)$$

It can be readily confirmed that these operators satisfy equations (2) by applying them to the (well-known) solutions for the problem, $\varphi_k(y) = \sqrt{2/L} \sin ky$. In fact, given the trigonometric properties of the sine function it is an easy matter to infer the structure of $\eta^\dagger_k, \eta_k$. With the selection of the coefficients appearing in equations (1) as

$$C_k = \sqrt{k(k + 1)/2},$$

equations (3) become

$$\eta^\dagger_k \varphi_k = \sqrt{k(k + 1)/2} \varphi_{k+1}, \quad \eta_k \varphi_k = \sqrt{k(k - 1)/2} \varphi_{k-1},$$

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and equations (7) give
\[ s_k = k^2, \quad a_k = k. \] 
(56)

Thus the effect of operator \( S \) is proportional to the effect of the Hamiltonian, \( H \varphi_k = E_k S \varphi_k \), with \( E_1 = \hbar^2 \pi^2 / 2mL^2 \), as follows from equation (54) applied to \( \varphi_k(x) \). The operator \( A \), in turn, turns out to be isospectral to the HO Hamiltonian, equation (22).

Clearly, when the eigenfunctions for a specific problem are known in advance, as in the example just shown, the algebraic procedure becomes unnecessary. However, in cases where the eigenfunctions are not known or the Schrödinger equation is not so easily solved, construction of the ladder operators, or else direct construction of a relevant operator, may be the pathway to solve the eigenvalue problem with the help of the algebraic consistency conditions, as in the examples discussed above.

3.4. An example from perturbation theory

As a demonstration of extra possibilities of the algebraic method presented here, let us apply it to a traditional and simple exercise of perturbation theory. We consider the 1D HO perturbed by an interaction Hamiltonian of the form \( \delta H = \epsilon x^4 \), and calculate the energy spectrum to first order in \( \epsilon \). For this purpose it is convenient to first establish some fundamental relations.

We write the perturbed Hamiltonian as \( H = H^0 + \delta H \), and similarly \( \alpha = \alpha^0 + \delta \alpha \), \( \beta = \beta^0 + \delta \beta \), \( S = S^0 + \delta S \), \( A = A^0 + \delta A \). Then to first order in \( \epsilon \),

\[ \delta S = \alpha^2 + \beta^2 - S^0 = \alpha^0 \delta \alpha + \delta \alpha \alpha^0 + \beta^0 \delta \beta + \delta \beta \beta^0, \] 
(57a)
\[ \delta A = -i [\alpha^0, \delta \beta] - i [\delta \alpha, \beta^0]. \] 
(57b)

We take \( S = H \); since \( S^0 = H^0 \), the eigenvalues of \( S \) are, to first order, \( s_k = E_k = E_k^0 + \delta E_k \), and from the consistency condition (9) it follows, after cancelling out the zero-order terms, that

\[ \delta E_{k+1} - \delta E_k = \delta \alpha_{k+1} + \delta \alpha_k, \] 
(58)

with \( \{ \delta \alpha_k \} \) the eigenvalues of \( \delta A \). This is sufficient for our present needs.

From the selection (see equations (19))

\[ \alpha^0 = \sqrt{\frac{m}{2 \omega x}}, \quad \beta^0 = \frac{1}{\sqrt{2m}} \]

it follows that \( \delta \beta = 0 \) and \( \delta \alpha \) becomes a function of \( \delta H \), which being a function of \( x \) commutes with \( \alpha^0 \). Therefore,

\[ \delta S = \delta H = 2 \alpha^0 \delta \alpha = \sqrt{2m} \omega x \delta \alpha, \]

whence

\[ \delta \alpha = \left( \frac{\epsilon}{\sqrt{2m} \omega} \right) x^3. \]
With these values, equation (57b) becomes
\[ \delta A = -\frac{i\epsilon}{2m\omega} [x^3, p] = \frac{3\hbar}{2m\omega^2} x^2, \]
which introduced in equation (58) gives
\[ \delta E_{k+1} - \delta E_k = \frac{3\hbar}{2m\omega} \left[ \langle k + 1 | x^2 | k + 1 \rangle + \langle k | x^2 | k \rangle \right] \]
\[ = \frac{3\hbar}{2m\omega} \left[ \frac{E_{k+1}^0}{m\omega^2} + \frac{E_k^0}{m\omega^2} \right] = \frac{3\hbar^2}{m^2\omega^2} (k + 1). \] (59)

An iteration gives
\[ \delta E_k = \delta E_0 + \frac{3\hbar^2}{m^2\omega^2} \sum_{i=1}^{k} = \delta E_0 + \frac{3\hbar^2}{2m^2\omega^2} k(k + 1). \]

To fix the value of \( \delta E_0 \) we recall that \( s_0 = a_0 \), which gives
\[ s_0 = \delta E_0 = \delta a_0 = \frac{3\hbar}{2m\omega} \langle 0 | x^2 | 0 \rangle = \frac{3\hbar^2}{4m^2\omega^2}. \]

The first-order correction to the energy eigenvalue is therefore
\[ \delta E_k = \frac{3\hbar^2}{2m^2\omega^2} \left( k^2 + k + \frac{1}{2} \right). \] (60)

For a more general study of the oscillator subject to a polynomial perturbation in \( x \) and \( p \), see reference [22].

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