**On the Riemann zeta-function, Parts IV-V**

Part IV: On the Riemann zeta-function and meromorphic characteristic functions.

Part V: A relation of its nonreal zeros and first derivatives thereat to its values on \( \frac{1}{2} + 4N \).

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**Abstract**

In Part I an odd meromorphic function \( f(s) \) has been constructed from the Riemann zeta-function evaluated at one-half plus \( s \). The conjunction of the Riemann hypothesis and hypotheses advanced by the author in Part I is assumed. In Part IV we derive the two-sided Laplace transform representation of \( f(s) \) on the open vertical strip \( V \) of all \( s \) with real part between zero and four. An additional hypothesis is used to prove that the Laplace density of \( f(s) \) on the strip \( V \) is positive. Let \( z(n) \) be the \( n \)th critical zero of the Riemann zeta-function of positive imaginary part in order of magnitude thereof. In Part V an expression is derived for \( z(1) \). A relation is obtained of the pair \( z(n) \) and the first derivative thereat of the zeta-function to the preceding such pairs.

**Keywords** Riemann zeta-function; Critical roots; Riemann hypothesis; Simple zeros conjecture; Laplace transform; Analytic / entire / meromorphic / function; Mittag-Leffler partial fraction expansion; Positive definite function; Analytic / meromorphic characteristic function.

**MSC (Mathematics Subject Classification).** 11Mxx Zeta and \( L \)-functions: analytic theory. 11M06 \( \zeta(s) \) and \( L(s, \chi) \). 11M26 Nonreal zeros of \( \zeta(s) \) and \( L(s, \chi) \); Riemann and other hypotheses. 30xx Functions of a complex variable. 44A10 Laplace transform. 42A82 Positive definite functions. 60E10 Characteristic functions; other transforms.

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§1 The conditional Laplace representation of $f(s)$ on $V_0$.

Review Part I, §4, (4.3); §5, Introduction, (5.3) and (5.4).

Conditional theorem 1.1 Assume $C^\wedge$. On $V_0$: $f(s) = \int_{\mathbb{R}} d(y)e^{s\gamma}g_0(y)$.

Proof The Conditional theorem 2.2 proven in Part III, §2, is stated as Conditional theorem 5.1 in Part I, §5, Introduction.(see A. Csizmazia [3, 5]). It established that $C^\wedge$ implies that $f(s)$ is represented on $C - Z^0$ by its formal partial fraction expansion $p(s)$. Thus $\hat{f}(s) = p(s)$ on $V_0$. $C^\wedge$ implies $A$ is finite. Then the Conditional corollary 5.1 of Part I, §5, (see A. Csizmazia [3]), gives the asserted Laplace representation.

§2 Proof of the Main conditional theorem (1).
Review Part I, §5, (5.4).

Main conditional theorem (1) Assume $C^\wedge$.

(i) The equality of the conditional and unconditional Laplace densities.
If $y$ is real, then:

$$\lambda(y) + c(0) - P_0(\pi e^{2y}) = P_0(\pi e^{-2y}).$$

Eq. (*)

(i′) The boundedness of the density.
$P_0(y)$ is bounded on the real axis.

(ii) The conditional extension of the unconditional Laplace representation of $f(s)$ on $V_0'$ to $V_0$.
On the strip $V_0$: $f(s) = \int d(y) e^{sy} P_0(\pi e^{-2y}).$

Proof of (i). $g_0(y) = P_0(\pi e^{-2y})$, for $y > 0$. Say $s = x + it$, with $\frac{1}{2} < x < 4$ and $t$ real.
Consider the previous Conditional theorem 1.1 (see also Part I, §5, (5.4)) and the Main unconditional theorem (stated in Part I, §3, and) proven in Part II, §6.
(See A. Csizmazia [3-4].) Together they yield:

$$\int_{y < 0} d(y) e^{sy}(e^{y\theta(y)}) = 0, \text{ with } \theta(y) := g_0(y) - P_0(\pi e^{-2y}).$$

$C^\wedge$ implies $A$ is finite.

Conditional claim (1°) Assume $A$ is finite. Fix $x > \frac{1}{2}$. $e^{xy}|\lambda(y)|$ vanishes with exponential rapidity as $y$ recedes to $-\infty$, with $y < 0$.

Proof of Conditional claim (1°)

Review Theorem 3.2 and its consequence (') of Part I, §3. The Unconditional theorem 6.1 (4) of Part II.

$e^{xy}|P_0(\pi e^{2y})|$ vanishes with exponential rapidity as $y$ recedes to $-\infty$. $e^{xy}|g_0(y)|$
behaves likewise when $x > 0$, since $|\lambda(y)| \leq A$.

Conditional claim (2°) Assume $A < \infty$. $e^{xy}\theta(y)$ is continuous in $y$.

Proof of Conditional claim (2°) $P_0(z)$ is entire. Hence $P_0(\pi e^{2y})$ is continuous. $A$
is finite. So $\lambda(y)$ is defined and continuous, for real $y$. Thus $g_0(y)$ is continuous.
Hence so is $e^{xy}\theta(y)$.

The vanishing of the Fourier transform in (') and the Conditional claims (1°),
(2°) together imply $g_0(y) = P_0(\pi e^{-2y})$, for $y < 0$. 

Proof of (ii). In the previous Conditional theorem 1.1 apply \(g_0(y) = P_0(\pi e^{-2y})\) of (i) of this Main conditional theorem (1).

**Conditional continuity criterion** Assume \(C^\wedge\).

\[
\lim_{y \to 0^-} g_0(y) = \lim_{y \to 0^+} g_0(y).
\]

Thus

\[
\sum_{k \geq 1} c(i\gamma_k) = -(c(0)/2 + \sum_{k \geq 1} c(4k)).
\]

The previous criterion and the Conditional theorem 2.1 stated next follow from the Main conditional theorem (1), as observed in Part I, §5, (5.4). Therein Conditional theorem 2.1 is stated as Conditional theorem 5.3 (1).

**Conditional theorem 2.1**

Assume \(C^\wedge\). \(g_0(z) = P_0(\pi e^{-2z})\), respectively

\[
\lambda(z) = -c(0) + P_0(\pi e^{2z}) + P_0(\pi e^{-2z}) = -(c(0) + 2\sum_{k \geq 1} c(4k)cosh(4ky)),
\]

holds on the real line and so extends \(g_0\), respectively \(\lambda\), to an entire function on \(C\) of period \(i\pi/2\).

**§3 Proofs of the Main conditional theorems (2)-(3).**

Assume \(C^\wedge\). Apply the Main conditional theorem (1), proven in §1. Also assume \(C^5\). Then (iii) Synthesis, Positivity of \(P_0(v)\), Conditional Lemma 7.2 of §7, Part I, gives: \(P_0(v) > 0\), for all \(v > 0\); and \(\inf_{v > \varepsilon} P_0(v) > 0\), when \(\varepsilon > 0\). So the following Main conditional theorem (2), stated in Part I, §7, is attained.

**Main conditional theorem**

(2) Assume \(C^\wedge\) and \(C^5\). \(f(s)\) is an analytic characteristic function on \(V_0^\wedge:\)

\[
f(s) := 1/n(s) = \int_R d(y)e^{sy}P_0(\pi e^{-2y}),
\]

with \(P_0(v)\) positive for \(v > 0\). Also \(\inf_{v > \varepsilon} P_0(v) > 0\), for any \(\varepsilon > 0\).

Apply the previous Main conditional theorem (2) together with the Main unconditional theorem (4), stated in Part I, §3, and proven in Part II, §6, The Mellin transform representation of \(f(s, \beta)\), Results when \(\beta = 1/4\). As observed in Part I, §7, one obtains the Main conditional theorem (3) restated next.

**Main conditional theorem**
(3) Assume $C^\wedge$ and $C5$. $f(s)$ is a meromorphic characteristic function on the complex plane: When $w$ is an integer and $s$ is in $V_{4w}$,

$$(-1)^w f(s) = \int_R d(y) e^{\pi \beta y} P_{4w}(\pi e^{\pi y}),$$

with $P_{4w}(z)$ entire in $z$ and $P_{4w}(v)$ positive for $v > 0$.

Applying the relation $f(-s) = -f(s)$ one obtains the counterparts of the above results for the negative half-plane of $s$ with $\Re(s) < 0$. When $w \leq -1$, set $P_{4w}(\pi v) := P_{-4(w + 1)}(\pi / v)$, for $v > 0$.

§4 Metric norms and analytic characteristic functions.

**Review** Part I, §3, A geometric consequence of the Main unconditional theorem (4), and §7, Metric norms and analytic characteristic functions. Part II, §6, Metric result when $\beta = \frac{1}{4}$.

The next result emanates from the association of Corollary 2.2 of Part VI, Corollary 6.2 with $\beta = \frac{1}{4}$ of Part II and the Main conditional theorem (1) (ii) of §2 above.

**Conditional corollary 4.1** Say Corollary 2.2 of Part VI holds. Assume $C^\wedge$ and $C5$. Let $x, t$ be real with $x$ not a multiple of four.

$m_x(t)$ is a metric norm in $t$ on the real line.

$d_x(t_1, t_2)$ is a (finite-valued) translation invariant metric in $t_1, t_2$ on the real line.

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On the Riemann zeta-function: Part V.
A relation of its nonreal zeros and first derivatives thereat to its values on $\frac{1}{2} + 4N$.

Index of symbols

$h^\#(z), L, D(r, \omega), D_1(r, \omega), S(z, \omega), S_\omega(z, \omega), \Delta(z, \omega), c(h, k, \omega)$ and $c(h, k)$.

§1 Proof of the conditional relations of $\gamma_n, \zeta'(\frac{1}{2} + iy)$ to their predecessors and the $\zeta(\frac{1}{2} + 4k)$.

Review Part I §6, §3, Theorem 3.2 (i). Main conditional theorem (1) (i), §5, (5.4), and Part IV, §2.

In Part I, §6, the Conditional corollary 6.1 is stated and its proof is deferred to this Part V. The Conditional claim 6.1 (1) was proven assuming (*) thereof and using Conditional corollary 6.1 (1). It was observed that C^\omega and the Main conditional theorem (1) (i), with $y < 0$, together imply (*). The Conditional claim 6.1 (2) resulted from the Conditional claim 6.1 (1) and Conditional corollary 6.1 (2).

Conditional corollary 6.1 of Part I, §6, is proven as Conditional corollary 1 using Lemma 1 below.

Definition of the Poisson transform $h^\#(z)$ of $h(y)$. See (1), (2) of the next lemma.
**Lemma 1** Assume (‘): \( h(u) \) is an even entire function of \( u \), \( \theta < 1 \) and \( |h(y)| \leq K(y^\theta) \), for large positive \( y \).

(1) \( z \int_{y \geq 0} d(y)(1/(y^2 + z^2))h(y) \) converges absolutely to an analytic function, \( h^*(z) \), on the half-plane \( \text{Re}(z) > 0 \).

(2) \( h^*(z) \) extends to an entire function on \( \mathbb{C} \).

**Proof** The proof of the Lemma 1 is achieved by establishing a series of claims.

**Definitions of \( L, D(r, \omega) \), \( D_1(r, \omega) \).** Set \( L := [-\infty, -1] \cup [1, \infty] \). Fix \( \omega > 0 \). Say \( r > 0 \). Let \( D(r, \omega), D_1(r, \omega) \) be the set of all \( u \) distant from \( S \) by at least \( r \), with \( S \) respectively \( i\omega \), \( i\omega L, i\omega [-1, 1] \).

**Claim 1** Say \( \omega > 0 \) and \( y \geq \omega \). Assume (\( * \)): \( h(y) \) is a complex-valued measurable function, \( \theta < 1 \) and \( |h(y)| \leq K(y^\theta) \), for \( y \geq \omega \).

(1) \( \int_{y \geq \omega} d(y)(1/(y^2 + z^2))h(y) \) converges uniformly absolutely in \( z \) on \( D(r, \omega) \).

(2) \( \int_{y \geq \omega} d(y)(1/(y^2 + z^2))h(y) \) is analytic in \( z \) on \( \mathbb{C} - i\omega L \).

**Definition of \( S(z, \omega) \).** Let \( S(z, \omega) \) be the integral in (2) of Claim 1.

**Proof of Claim 1.** \( 1/|y^2 + z^2| = y^{-2} \cdot |1 + (z/y)^2|^{-1} \). Now \( 1/|1 + (z/y)^2| \) is bounded for \( z, y \) with \( z \) on \( D(r, \omega) \) and \( y \geq \omega \). Apply (\( * \)) of Claim 1 to obtain \( \int_{y \geq \omega} d(y)|h(y)/(y^2 + z^2)| \leq K \int_{y \geq \omega} d(y)y^{-2 + \theta} \), for some \( K \). Thus (1) and (2) hold.

**Claim 2. Taylor series for \( S(z, \omega) \).** Assume (\( * \), of Claim 1, and \( |z| < \omega \). \( S(z, \omega) = \sum_{k \geq 0} z^{2k}((-1)^k \cdot \omega^{-2k-1} \int_{y \geq \omega} d(y)y^{2k+1}h(\omega y)) \) with each integral and the series converging absolutely.

**Proof of Claim 2.** \( S(z, \omega) = \omega^{-1} \cdot \int_{y \geq 1} d(y)(1/(y^2 + u^2))h(\omega y) \), with \( u := z/\omega \), \( |u| < 1 \). \( 1/(y^2 + u^2) = \sum_{k \geq 0} (-1)^k \cdot u^{2k} \cdot y^{-2(k + 1)} \). Consider \( \sum_{k \geq 0} |u^{2k} \cdot t(k, \omega)| \), with \( t(k, \omega) := \int_{y \geq 1} d(y)y^{-2k+1}h(\omega y) \). Apply (\( * \)) to obtain \( t(k, \omega) \leq K \omega^{\theta}/(2k + 1 - \theta) \). So \( \sum_{k \geq 0} |u^{2k} \cdot t(k, \omega)| \leq K \omega^{\theta} \cdot E(|u|) \), with \( E(r) := \sum_{k \geq 0} r^{2k}/(2k + 1 - \theta) \) when \( |r| < 1 \). Then \( E(r) \leq 1/(1 - \theta) - \frac{1}{2}\log(1 - r^2) \).

**Claim 3.** Say \( \omega > 0 \). Assume \( \int_{y \geq \omega} d(y)|h(y)| \) converges.

(1) \( \int_{y \geq \omega} d(y)h(y)/(y^2 + z^2) \) converges absolutely in \( z \) on \( D_1(r, \omega) \).

(2) \( \int_{y \geq \omega} d(y)(1/(y^2 + z^2))h(y) \) is analytic in \( z \) on \( \mathbb{C} - i\omega [-1, 1] \).

**Definition of \( S_\omega(z, \omega) \).** Let \( S_\omega(z, \omega) \) be the integral in (2).

**Proof of Claim 3 (1).** \( 1/|z - iy|^2 \cdot |z + iy| \) leads to \( \int_{y \geq \omega} d(y)|h(y)/(y^2 + z^2)| \leq r^2 \cdot \int_{y \geq \omega} d(y)|h(y)| \).
Say \( \text{Re}(u) > 0 \). Set \( q(u) := u \int_{y \leq 1} d(y)/(y^2 + u^2) \). If \( y \neq \pm iz \), then \( z/(y^2 + z^2) = (2i)^{-1}(1/(y - iz) - 1/(y + iz)) \). Thus ('): \( q(u) = (2i)^{-1} \sum_{\sigma = \pm 1} \sigma \int_{y \leq 1} d(y)/(y^2 + u^2) \). If \( y \neq \pm iz \), then \( z/(y^2 + z^2) = (2i)^{-1} \frac{1}{(y - iz) - 1/(y + iz)} \).

Thus ('): \( q(u) = (2i)^{-1} \sum_{\sigma = \pm 1} \sigma \int_{y \leq 1} d(y)/(y^2 + u^2) \).

Say \( u \) is in \( C - (-\infty, 0] \).

Take \( \log(u) \) to be the principal branch \( \log(|u|) + i\theta \), with \( u = |u| e^{i\theta} \) and \(-\pi < \theta < \pi\).

Take \( \arctan(s) \) to be the principal branch that for real \( x \) has \(-\pi/2 < \arctan(x) < \pi/2\) and has \( iL \) as branch cut. \( \arctan(s) = (2i)^{-1} \sum_{\sigma = \pm 1} \sigma \log(1 + \sigma u) \).

Let \( \arccot(s) = \pi/2 - \arctan(s) \).

Say \( \text{Re}(u) > 0 \). Evaluate \( q(u) \) using ('') above. \( \sigma \int_{y \leq 1} d(y)/(y^2 + u^2) = (2i)^{-1} \sum_{\sigma = \pm 1} \sigma \log(1 - \sigma u) \).

Note the singular behavior of \( \log(u) \) as \( u \) approaches 0.

Fortunately in \( q(u) \) the \( \log(u) \) terms cancel one another, since the coefficient of their total contribution \((2i)^{-1}(-1)(\sum_{\sigma = \pm 1} \sigma)\) is 0.

Thus \( q(u) = \pi/2 + (2i)^{-1} \sum_{\sigma = \pm 1} \sigma \log(1 - \sigma u) = \arccot(u) \).

Hence \( q(u) \) has an analytic extension from the half-plane \( \text{Re}(u) > 0 \) to \( C - iL \).

Claim 4. Assume \( \omega > 0 \) and \( h(u) \) is an even entire function of \( u \). \( zS_{\omega}(z, \omega) \) extends from the half-plane \( \text{Re}(z) > 0 \) to an analytic function on \( C - iL \).

Proof of Claim 4.

Assume \( \omega > 0 \) and \( \text{Re}(z) > 0 \). \( \int_{y \leq \omega} d(y)|(j(y^2) - j(-(z^2)))/(y^2 + z^2)| \) converges.

**Definition of \( \Delta(z, \omega) \).** Set \( \Delta(z, \omega) := \int_{y \leq \omega} d(y)|(j(y^2) - j(-(z^2)))/(y^2 + z^2)| \).

\( zS_{\omega}(z, \omega) = z \cdot \Delta(z, \omega) + h(iz) \cdot \arccot(z/\omega) \).

\( h(s) = j(s^2) \), with \( j(u) \) an entire function of \( u \). Set \( j_k = j^{(k)}(0)/(k!) \).

Subclaim Say \( \omega \geq 0 \).

(1) Assume that \( z \) is on \( C - i\omega[-1, 1] \).

\[ \Delta(z, \omega) = \sum_{w \geq 0} (-z^2)^w \sum_{k \geq w + 1} (j_k/(2(k - w) - 1)) \omega^{2(k - w) - 1}, \]

with each of the series absolutely convergent.

(2) \( \Delta(z, \omega) \) is extended to an entire function of \( z, \omega \) by the latter series.

Proof of subclaim Say \( u \neq v \). \( (j(u) - j(v))/(u - v) = \sum_{k \geq 1} j_k (u^k - v^k)/(u - v) \). Also \( (u^k - v^k)/(u - v) = \sum_{0 \leq w \leq k - 1} u^{k-1-w} \cdot v^w \). Assume \( |u|, |v| \leq B \). Then \( \sum_{0 \leq w \leq k - 1} |u|^{k-1-w} \cdot |v|^w \leq k \cdot B^{k-1} \). Let \( m(s) := \sum_{k \geq 1} |j_k| \cdot s^k \). One has \( \sum_{k \geq 1} |j_k| \cdot k \cdot B^{k-1} = m'(B) \). Take \( v \).
\[-(z^2).\) Assume \(\omega \leq B^{1/2}\) and \(|z| \leq B^{1/2}\). Then \(\int_{y \geq 0} d(y) \sum_{k \geq 1} |k| \sum_{w \leq k-1} |y|^{2(k-1-w)} |v| w \leq m'(B) B^{1/2} < \infty.\) Thus the subclaim is valid.

\[zS_s(z, \omega) = z \cdot \Delta(z, \omega) + h(iz) \cdot \arccot(z/\omega),\] for \(\text{Re}(z) > 0.\) That equality and Subclaim (2) yield Claim 4.

The Maclaurin expansion, for \(z\) with \(|z| < \omega\), of \(zS_s(z, \omega)\) is determined by the latter equality from the expansions of \(\Delta(z, \omega)\) and \(h(iz) \cdot \arccot(z/\omega).\)

**Corollary** Assume (') of Lemma 1. Let \(\omega > 0.\) \(h^\#(z)\) has an analytic continuation from the half-plane \(\text{Re}(z) > 0\) to \(C - i\omega L.\)

Proof of Corollary. If \(\text{Re}(z) > 0\), then \(h^\#(z) = zS(z, \omega) + zS_s(z, \omega).\) That identity provides the analytic continuation of \(h^\#(z)\) to \(C - i\omega L,\) since each of \(S(z, \omega)\) and \(zS_s(z, \omega)\) is analytic there. So the corollary holds.

We now complete the proof of Lemma 1.

\[h^\#(z) = z(S(z, \omega) + \Delta(z, \omega)) + h(iz) \cdot \arccot(z/\omega),\]

for \(z\) on \(C - i\omega L.\) So on \(C - i\omega L\) the odd part \(h_1(z)\) of \(h^\#(z)\) is given by

\[h_1(z) := \frac{1}{2}(h^\#(z) - h^\#(-z)) = z(S(z, \omega) + \Delta(z, \omega)) - h(iz) \arctan(z/\omega).\]

Apply Claim 2 together with the Subclaim to obtain the following. Say \(\omega > 0.\) Each of \(\int_{y \geq 0} d(y)y^{-2(k+1)}h(y)\) and \(\sum_{n \geq -k} (j_n/k/(2n - 1)) \cdot \omega^{2n-1}\) converges absolutely.

**Definition of \(c(h, k, \omega)\).**

Set \(c(h, k, \omega) := (\int_{y \geq 0} d(y)y^{-2(k+1)}h(y)) + \sum_{n \geq -k} (j_n/k/(2n - 1)) \cdot \omega^{2n-1}.\)

Say \(|z| < \omega.\) \(\sum_{k \geq 0} (z^{2k+1})(-1)^k \cdot c(h, k, \omega)\) converges absolutely. \(h_1(z)\) is analytic for \(z\) with \(|z| < \omega: h_1(z) = \sum_{k \geq 0} (z^{2k+1})(-1)^k \cdot c(h, k, \omega).\)

\(c(h, k, \omega) = (-1)^k \cdot (h_0(z))^{(2k+1)}(0)/(2k + 1)!.\) Therefore \(c(h, k, \omega)\) is constant in \(\omega:\)

\[\partial_{\omega}(c(h, k, \omega)) = 0.\]

**Definition of \(c(h, k)\).** Let \(c(h, k)\) be the common value of the \(c(h, k, \omega)\) with \(\omega\) positive.

\[c(h, k) := (\int_{y \geq 0} d(y)y^{-2(k+1)}h(y)) + \sum_{n \geq -k} (j_n/k/(2n - 1)) \cdot \omega^{2n-1}.\]

\(h_1(z)\) is an entire function of \(z\) on \(C: h_1(z) = \sum_{k \geq 0} (z^{2k+1})(-1)^k \cdot c(h, k).\) Therefore \(h^\#(z) = h_1(z) + (\pi/2)h(iz)\) yields the analytic extension of \(h(z)\) from \(C - i\omega L\) to \(C.\)
\( h^\#(z) = \left( \sum_{k \geq 0} (z^{2k} + 1)^{(-1)^k} \cdot c(h, k) \right) + (\pi/2)h(iz). \)

Thus **Lemma 1** holds.

Note that \( c(h, k) = \lim_{\omega \to 0, \omega \to \infty} \sum_{n \geq 1} (j_n + k/(2n - 1)) \cdot \omega^{2n - 1}. \)

The following **Conditional corollary 1** is obtained from **Lemma 1**.

**Review Part I, §6**, definitions of \( j(u) \) and \( \upsilon(z) \).

**Conditional corollary 1** Assume (') \( \theta < 1 \) and for \( v \geq 0 \), \( |P_\theta(v)| \sim O((\log(v))^\theta) \), as \( v \to \infty \).

1. \( (z/\pi) \int_{y \geq 0} d(y)(1/(y^2 + z^2)) j(y) \) converges absolutely to an analytic function on the half-plane \( \text{Re}(z) > 0. \)

2. **Definition of \( \upsilon(z) \).** Set \( \upsilon(z) := (1/\pi)j^\#(z) \), when \( \text{Re}(z) > 0. \)

**Proof** Apply **Lemma 1** with \( h(y) = j(y) \).

**Review Part I, §5**, Introduction, **Definition of \( e(z) \).**

**Conditional corollary 2** Assume (*) \( A \) is finite and \( \lambda(y) = j(y) \), for \( y > 0. \)

1. \( e(-z) = \upsilon(z) \) on the half-plane \( \text{Re}(z) > 0: \)

\[ \sum_{k \geq 1} c(i \gamma_k) \exp(-\gamma_k z) = (z/\pi) \int_{y > 0} d(y)(1/(z^2 + y^2))(-c(0) + P_\theta(\pi e^y) + P_\theta(\pi e^{-y})). \]

2. \( e(s) \) has an analytic extension from the half-plane \( \text{Re}(s) < 0 \) to the entire complex plane.

**Proof** Assume \( A \) is finite. Say \( y \geq 0 \), \( \lambda(y) \) is bounded. Assume \( \lambda(y) = j(y) \). Then (') of **Lemma 1** holds with \( h(y) = j(y) \). One obtains (1), (2) of the previous **Conditional corollary 1**. \( A < \infty \) together with the **Conditional lemma 6.1** established in **Part I, §6**, gives \( e(-z) = (1/\pi)\lambda^\#(z) \), when \( \text{Re}(z) > 0. \) \( \lambda(y) = j(y) \) yields \( e(-z) = \upsilon(z) \). **Conditional corollary 1 (2)** now reveals that \( \upsilon(-s) \) is the analytic extension of \( e(s) \) to all of \( C \).

The proof of the **Main conditional theorem (1) (i), (i')**, was completed in **Part IV**. That now sparks the genesis of the following **Conditional corollaries 3-4**.

Each \( \gamma_n, \zeta'(1/2 + i \gamma_n) \) of the sequence \( \gamma_1, \zeta'(1/2 + i \gamma_1); \ldots \gamma_n, \zeta'(1/2 + i \gamma_n); \ldots \) can successively be expressed in terms of the predecessors \( \gamma_k, \zeta'(1/2 + i \gamma_k), \) with \( 1 \leq k \)
\(\leq n - 1\), and, in the case of \(\zeta'(\frac{1}{2} + i\gamma_n)\), also \(\gamma_n\).

(See Part I, §6, Conditional corollary 6.3)

**Conditional corollary 3** Assume \(C^\wedge\).

1. \(e(-z) = v(z)\), provided \(\text{Re}(z) > 0\). \(v(z)\) extends \(e(-z)\) to an entire function on \(C\).

2. Relations of \(\gamma_n, \zeta'(\frac{1}{2} + i\gamma_n)\) to their predecessors and the \(\zeta(\frac{1}{2} + 4k)\).

\[
\gamma_1 = \lim_{x > 0, x \to \infty} (\frac{1}{x}) \log((-1)^n v(x)).
\]

\[
\zeta'(\frac{1}{2} + i\gamma_1) = \frac{1}{(b(i\gamma_1)) \lim_{\text{Re}(z) \to \infty} \exp(\gamma_1 z) v(z)).
\]

\[
\gamma_n = \lim_{x > 0, x \to \infty} (\frac{1}{x}) \log((-1)^n (v(x) - e(-x, n - 1))).
\]

\[
\zeta'(\frac{1}{2} + i\gamma_n) = \frac{1}{(b(i\gamma_n)) \lim_{\text{Re}(z) \to \infty} \exp(\gamma_n z)(v(z) - e(-z, n - 1))).
\]

**§2 Representation of \(p_{i,+(z)}\) via \(j(y)\).**

**Review** Part I, §6. Definitions of \(p_{i,+(z)}, \Theta(\theta, z)\). Conditional corollaries 6.4-6.5. Conditional corollary 6.5 therein states without proof the following.

**Conditional corollary 4** Representation of \(p_{i,+(z)}\) via \(j(y)\).

1. Assume \(A\) is finite and for \(y > 0\), \(\lambda(y) = j(y)\). Say \(\text{Im}(z) < 0\). Then

\[
p_{i,+(z)} = \int_{\theta > 0} d(\theta) j(\theta) \Theta(\theta, z).
\]

2. Assume \(C^\wedge\). Then the previous representation of \(p_{i,+(z)}\) holds on the lower half-plane of \(z\) with \(\text{Im}(z) < 0\).

**Proof of (1).** Assume \(A\) is finite and \(\text{Re}(s) < \gamma_1\). Then Conditional corollary 6.4, proven in §6 of Part I, gives

\[
\int_{y > 0} d(y) e^{\gamma y}(-e(-y)).
\]

Also assume \(\lambda(y) = j(y)\), for \(y > 0\). Then \(|j(y)| \leq A\). Conditional corollary 2 of §1 yields

\[
e(-u) = (u/\pi) \int_{\theta > 0} d(\theta) (1/(\theta^2 + u^2)) j(\theta),
\]

when \(\text{Re}(u) > 0\). Then
\[ \text{ip}_{\nu}(\nu) = \int_{y > 0} d(y)e^{\nu y} \int_{\theta \geq 0} d(\theta) \frac{1}{\theta^2 + y^2} j(\theta). \]

Set \( x := \text{Re}(s) \). Say \( x < 0 \). Then

\[ \int_{y > 0} d(y) \int_{\theta \geq 0} d(\theta) |e^{\nu y}(-y/\pi)\frac{1}{\theta^2 + y^2})j(\theta)| \leq \alpha'\alpha A. \]

Here \( \alpha := (y/\pi)\int_{\theta \geq 0} d(\theta) \frac{1}{\theta^2 + y^2} = \frac{1}{2} \) and \( \alpha' := \int_{y > 0} d(y)e^{\nu y} = -1/x \). Apply the interchange \( \int_{y > 0} d(y) \int_{\theta \geq 0} d(\theta) = \int_{\theta \geq 0} d(\theta) \int_{y > 0} d(y) \).

**Proof of (2).** Assume \( C^\wedge \). Then \( A \) is finite. Next apply the Main conditional theorem (1) (i), stated in Part I, §5, (5.4), and proven in §2 of Part IV. \( \lambda(y) = j(y) \) results. Thus (2) of Conditional corollary 4 follows from (1) thereof.

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