UNITARY PERTURBATIONS OF COMPRESSED N-DIMENSIONAL SHIFTS

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ABSTRACT. Given a purely contractive matrix-valued analytic function \( \Theta \) on the unit disc \( \mathbb{D} \), we study the \( U(n) \)-parameter family of unitary perturbations of the operator \( Z_\Theta \) of multiplication by \( z \) in the Hilbert space \( L^2_\Theta \) of \( n \)-component vector-valued functions on the unit circle \( \overline{\mathbb{T}} \) which are square integrable with respect to the matrix-valued measure \( \Omega_\Theta \) determined uniquely by \( \Theta \) and the matrix-valued Herglotz representation theorem.

In the case where \( \Theta \) is an extreme point of the unit ball of bounded \( \mathbb{M}_n \)-valued functions we verify that the \( U(n) \)-parameter family of unitary perturbations of \( Z_\Theta^* \) is unitarily equivalent to a \( U(n) \)-parameter family of unitary perturbations of \( X_\Theta \), the restriction of the backwards shift in \( H^2_n(\mathbb{D}) \), the Hardy space of \( C^n \) valued functions on the unit disc, to \( K^2_\Theta \), the de Branges-Rovnyak space constructed using \( \Theta \). These perturbations are higher dimensional analogues of the unitary perturbations introduced by D.N. Clark in the case where \( \Theta \) is a scalar-valued \((n = 1)\) inner function, and studied by E. Fricain in the case where \( \Theta \) is scalar-valued and an extreme point of the unit ball of \( H^\infty(\mathbb{D}) \).

A matrix-valued disintegration theorem for the Aleksandrov-Clark measures associated with matrix-valued contractive analytic functions \( \Theta \) is obtained as a consequence of the Weyl integration formula for \( U(n) \) applied to the family of unitary perturbations of \( Z_\Theta \). This disintegration formula generalizes a recent result of S. Elliott to arbitrary matrix-valued contractive analytic functions. Following results of Clark and Fricain in the scalar case, a necessary and sufficient condition on \( \Theta \) for \( K^2_\Theta \) to contain a total orthogonal set of point evaluation or reproducing kernel vectors is provided.

Key words and phrases: Hardy space, model subspaces, Aleksandrov disintegration theorem, Clark's unitary perturbations, Aleksandrov-Clark measures, matrix-analytic functions, symmetric/isometric linear transformations

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1. Introduction

Let \( \Theta \) be an \( \mathbb{M}_n \)-valued contractive analytic function on \( \mathbb{D} \), the unit disc in the complex plane \( \mathbb{C} \). Here \( \mathbb{M}_n \) denotes the \( n \times n \) matrices with entries in \( \mathbb{C} \). Recall cf. [1, Proposition V.2.1], that \( \Theta \) can be block-diagonalized as \( \Theta = \Theta_0 \oplus \Theta_1 \) where \( \Theta_0 \) is a unitary constant and \( \Theta_1 \) is purely contractive, i.e. \( \|\Theta_1(0)\| < 1 \). We will assume throughout that \( \Theta \) is purely contractive. For such a function it follows easily that \( \|\Theta(z)\| < 1 \) for all \( z \in \mathbb{D} \). Recall that the function \( \Theta \) is said to be inner if \( \Theta(\zeta) \), \( \zeta \in \overline{\mathbb{T}} \), is unitary a.e. with respect to Lebesgue measure on the unit circle \( \mathbb{T} \) (here \( \Theta(\zeta) \) is the non-tangential limit of \( \Theta(z) \) for \( z \) approaching \( \zeta \) non-tangentially in \( \mathbb{D} \)).

Given any \( A \in \mathbb{M}_n \), the closed unit ball of \( \mathbb{M}_n \), let \( \Theta_A := \Theta A^* \) and define

\[
B_{\Theta_A}(z) := \frac{1 + \Theta(z)A^*}{1 - \Theta(z)A^*}
\]
This is clearly analytic on \( \mathbb{D} \) since \( \|A\| \leq 1 \) and \( \|\Theta(z)\| < 1 \) for all \( z \in \mathbb{D} \). It is straightforward to calculate that

\[
(1.2) \quad \text{Re} (B_{\Theta_A}(z)) := \frac{1}{2} (B_{\Theta_A}(z) + B_{\Theta_A}(z)^*) = (1 - \Theta(z)A^*)^{-1}(1 - \Theta(z)A^*A\Theta(z)^*)(1 - A\Theta(z)^*)^{-1}.
\]

This is clearly positive so that by the matrix-valued Herglotz theorem [2, Theorem 3], it follows that for each such \( A \) there is a unique positive \( \mathbb{M}_n \) valued measure \( \Omega_{\Theta_A} \) on \( \mathbb{T} \) such that

\[
(1.3) \quad \text{Re} (B_{\Theta_A}(z)) = \text{Re} \left( \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} \Omega_{\Theta_A}(d\zeta) \right).
\]

The imaginary part of \( B_{\Theta_A}(z) \) is

\[
(1.4) \quad \text{Im} (B_{\Theta_A}(z)) := \frac{1}{2i} (B_{\Theta_A}(z) + B_{\Theta_A}(z)^*) = -i(1 - \Theta(z)A^*)^{-1}(\Theta(z)A^* - A\Theta(z)^*)(1 - A\Theta(z)^*)^{-1}.
\]

It is then straightforward to calculate that

\[
(1.5) \quad B_{\Theta_A}(z) = \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} \Omega_{\Theta_A}(d\zeta) + i\text{Im} (B_{\Theta_A}(0)).
\]

In the case where \( A \) is unitary, the measures \( \Omega_{\Theta_A} \) are the matrix-valued Aleksandrov-Clark measures introduced in [2]. In the case where \( \Theta \) is a scalar-valued and \( A \in \mathbb{T} \), these are the usual Aleksandrov-Clark measures, first introduced in [3], and studied since by many authors. We will sometimes write \( \Omega_A \) and \( B_A \) in place of \( \Omega_{\Theta_A} \) and \( B_{\Theta_A} \) respectively when there is no chance of confusion.

Given a contractive analytic function \( \Theta \), let \( L^2_\Theta(\mathbb{T}) \) or simply \( L^2_\Theta \) denote the Hilbert space of \( \mathbb{C}^n \)-valued functions on \( \mathbb{T} \) which are square integrable with respect to the matrix-valued measure \( \Omega_\Theta := \Omega_{\Theta_A} \). Explicitly, let \( \{e_i\}_{i=1}^n, n = \text{rank}(\Theta) \) be a fixed orthonormal basis for \( \mathbb{C}^n \), \( \Omega_\Theta(I)_{ij} \) the matrix entries of \( \Omega_\Theta(I) \) with respect to this basis \( (I \subset \mathbb{T} \) is some fixed Borel set). The Hilbert space \( L^2_\Theta \) contains a copy of \( \mathbb{C}^n \). It will be convenient to denote the embedding of \( \mathbb{C}^n \) into \( L^2_\Theta \) by \( V_n \). Let \( b^i_\zeta := V_ne_i \), the \( b^i_\zeta \) are the constant functions \( b^i_\zeta(\zeta) = e_i, \zeta \in \mathbb{T} \). Elements \( f,g \in L^2_\Theta \), will be viewed as column vectors of functions with entries \( f_i(\zeta) := (f(\zeta),b^i_\zeta(\zeta)), (\cdot,\cdot) \) denotes the inner product in \( \mathbb{C}^n \). Then the inner product in \( L^2_\Theta \) is given by the formula

\[
(1.6) \quad (f,g)_\Theta := \int_{\mathbb{T}} (\Omega_\Theta(d\zeta)f(\zeta),g(\zeta)) = \sum_{i,j=1}^n \int_{\mathbb{T}} \overline{g_i(\zeta)}\Omega_\Theta(d\zeta)_{ij}f_j(\zeta).
\]

Let \( Z_\Theta \) denote the operator of multiplication by the independent variable \( \zeta \) in this space. Let \( \mathfrak{D}_+ := \mathbb{C}\{b^+\}_{i=1}^n \), the subspace spanned by the vectors \( b^+_{\zeta}(\zeta) := 1/\zeta b^i_\zeta(\zeta) \). Here \( \mathbb{C}\{b^+\} \) denotes the linear span of the set \( \{b^+\} \). Then let \( \mathfrak{D}_- := Z_\Theta\mathfrak{D}_+ = \mathbb{C}\{b^-\} \). Let \( P_\pm \) denote the projectors onto \( \mathfrak{D}_\pm \).

In what follows, we assume that \( \Theta(0) = 0 \) so that \( \Omega_\Theta(\mathbb{T}) = \mathbb{1}_n \) and the \( b^+ \) are orthonormal basis vectors for \( \mathfrak{D}_\pm \). Given any \( A \in (\mathbb{M}_n)_{1} \), we will identify \( A \) with the operator \( \hat{A} \in \mathcal{B}(L^2_\Theta) \) defined by

\[
(1.7) \quad \hat{A} := \sum_{i,j=1}^n (\cdot,b^-_{\zeta})_\Theta A_{ij}b^-_{\zeta} = ((\cdot,b^-_{\zeta})_\Theta,\ldots,(\cdot,b^-_{\zeta})_\Theta) \left( \begin{array}{c} b^-_1 \\ \vdots \\ b^-_n \end{array} \right).
\]

We will identify \( \hat{A} \) with \( A \) and simply write \( A \) for \( \hat{A} \) from now on. For each such \( A \) define \( Z_\Theta(A) := Z_\Theta + P_-(A - \mathbb{1}_n)P_-Z_\Theta \), a perturbation of \( Z_\Theta \). To simplify notation, we will sometimes write \( \hat{Z}(A) \) in place of \( Z_\Theta(A) \) when the choice of \( \Theta \) is clear. If \( A \in \mathcal{U}(n) \) then \( Z_\Theta(A) \) is unitary, and \( Z_\Theta(\mathbb{1}) = Z_\Theta \). Here \( \mathcal{U}(n) \) denotes the group of unitary \( n \times n \) matrices. The family of unitary
operators $Z_{\Theta}(U)$: $U \in \mathcal{U}(n)$ can be seen as the family of unitary extensions of the simple isometric linear transformation $Z'_\Theta := Z_{\Theta}(0)|_{B_2^\perp \otimes \mathbb{D}}$. This will be discussed in greater detail in Section \[5\].

This paper will now proceed as follows. Consider $\Lambda_{\Theta(U)}(I) := \chi_I(Z_{\Theta}(U))$, where $U \in \mathcal{U}(n)$, $I$ is a Borel subset of $\mathbb{T}$, $\chi_I$ is the characteristic function of $I \subset \mathbb{T}$, and $\chi_I(Z_{\Theta}(U))$ is a spectral projection defined using the Borel functional calculus for the unitary operator $Z_{\Theta}(U)$. In the next section we will prove that $\Omega_{\Theta_{\Theta}}(I) = [(\Lambda_{\Theta_{\Theta}})(I)_{ij}] = [(\Lambda_{\Theta_{\Theta}})(I)b_{ij} b_{ij}^\perp)_{\Theta}]$. With this identification and a straightforward application of the Weyl integration formula for the Lie group $\mathcal{U}(n)$, a matrix-version of Aleksandrov’s disintegration theorem for arbitrary $\mathcal{M}_n$-valued purely contractive analytic functions on $\mathbb{D}$ satisfying $\Theta(0) = 0$ will be established. This will extend the main result of Elliott \[2\] (Theorem 15) which establishes the disintegration theorem for $\Theta$ which are the product of a scalar function in $(H^\infty(\mathbb{D}))(U)$.

In Section 3, the Cauchy integral representation for the de Branges-Rovnyak space $K^2_{\Theta}$, associated with $\Theta$ as presented in \[3\] (Chapter III), is adapted to the case where $\Theta$ is $\mathcal{M}_n$-valued (we refer the reader to this section for the formal definition of $K^2_{\Theta}$). In direct analogy with the scalar case it is shown that there is a unitary transformation $V_{\Theta}$ of $H^2_{\Theta}$, the closure of the polynomials in $L^2_{\Theta}$ onto the de Branges-Rovnyak space $K^2_{\Theta}$ which takes $Z_{\Theta}$ onto a rank $n$ perturbation of $X_{\Theta} := S^*|_{K^2_{\Theta}}$, the restriction of the backwards shift $S^*$ to $K^2_{\Theta}$. We will then verify that, as in the case where $\Theta$ is scalar, $H^2_{\Theta} = L^2_{\Theta}$ if and only if $\Theta$ is an extreme point of the unit ball of $H^\infty_{\mathcal{M}_n}(\mathbb{D})$, the Hardy space of $\mathcal{M}_n$-valued analytic functions on $\mathbb{D}$ whose supremum norms on circles of radius $0 \leq r < 1$ are uniformly bounded. We will further check that $\Theta$ is an extreme point if and only if the trace of $\ln(1-|\Theta|)$ fails to be Lebesgue integrable on $\mathbb{T}$. In the case that $\Theta$ is a Borel subset of $\mathbb{T}$, we determine when the inverse Cayley transform of $X_{\Theta}$ is $M_{\Phi}$, where $\Phi := \Theta^{-1}$ is a contractive analytic function on $\mathbb{D}$ and $K^2_{\Theta} = H^2_{\mathbb{D}} \otimes \mathcal{M}_n \otimes \Phi H^2_{\mathbb{D}}(U)$, maps $M_{\Phi}$ onto $\mathcal{M}_n$, the symmetric operator of multiplication by $z$ in $K^2_{\Theta}$. Finally in Section 5, we consider the isometric linear transformation $Z'_{\Theta} := Z_{\Theta}(0)|_{B_2^\perp \otimes \mathbb{D}}$. This is a simple isometric linear transformation with deficiency indices $(n, n)$ and Lifschitz characteristic function equal to $\Theta$ \[3\]. Let $\mu(z) := \frac{z-1}{z+1}$; $\mu : \mathbb{D} \to \mathbb{D}$ where $\mathbb{D}$ denotes the open unit upper half-plane. Then $\mu^{-1}(z) = \frac{1-z}{1+z}$. Using the theory of Lifschitz we determine when the inverse Cayley transform $\mu^{-1}(Z'_{\Theta})$ of $Z'_{\Theta}$ is a densely defined symmetric operator. If $\Theta$ is inner, the canonical unitary transformation that takes $L^2_{\mathbb{D}} = H^2_{\mathbb{D}} \otimes M_{\Phi}$ where $\Phi := \Theta^{-1}$ is a contractive analytic function on $\mathbb{D}$ and $K^2_{\Theta} = H^2_{\mathbb{D}} \otimes \Phi H^2_{\mathbb{D}}(U)$, maps $\mu^{-1}(Z'_{\Theta})$ onto $M_{\Phi}$, the symmetric operator of multiplication by $z$ in $K^2_{\Theta}$. We verify that, as in the scalar $(n=1)$ case, $K^2_{\Theta}$ has a $\mathcal{U}(n)$-parameter family of total orthogonal sets of point evaluation vectors $\{v_{\lambda_j(U)} \mid j \in \mathbb{Z}, U \in \mathcal{U}(n)\}$, such that the sequences $(\lambda_j(U)) \subset \mathbb{R}$.
have no finite accumulation point (It will be shown in Section 3 that $(\lambda_j(U))_{j \in \mathbb{Z}}$ is necessarily a sequence of real values) if and only if $\Theta$ is analytic on some open neighbourhood of any given $x \in \mathbb{R}$. Here $K^2_0 \subset H^2_n(U) \subset L^2_n(R)$, where $L^2_n(R)$ is the Hilbert space of $\mathbb{C}^n$ valued functions on $\mathbb{R}$ which are square integrable with respect to Lebesgue measure. This provides a class of vector-valued reproducing kernel Hilbert spaces of functions on $\mathbb{R}$ which have total orthogonal sets of point evaluation vectors.

Such reproducing kernel Hilbert spaces have the special property that their elements are perfectly reconstructible from the values they take on certain discrete sets of points. Indeed, suppose that $\mathcal{H}$ is a RKHS of $\mathbb{C}^n$-valued functions on a set $X \subset \mathbb{C}$, i.e. for any $\vec{y} \in \mathbb{C}^n$ and any $x \in X$, the linear functional which evaluates an element $f \in \mathcal{H}$ at $x$ and takes its inner product with $\vec{y}$ is bounded. By the Riesz representation theorem, for each $\vec{y} \in \mathbb{C}^n$ and $x \in X$, there is then a ‘point evaluation vector’ $\delta^{\vec{y}}_x \in \mathcal{H}$ such that $(f, \delta^{\vec{y}}_x) = (f(x), \vec{y})$. Here $\langle \cdot, \cdot \rangle$ is the inner product on $\mathcal{H}$ and $(\cdot, \cdot)$ is, as before, the inner product in $\mathbb{C}^n$. If $\mathcal{H}$ has a total orthogonal set of point evaluation vectors $\{\delta^{\vec{y}}_x\}$, then it follows that for any $f \in \mathcal{H}$,

$$f = \sum_j (f, \delta^{\vec{y}}_x) \frac{\delta^{\vec{y}}_{x_j}}{||\delta^{\vec{y}}_{x_j}||^2} = \sum_j (f(x_j), \vec{y}_j) \frac{\delta^{\vec{y}}_{x_j}}{\langle \delta^{\vec{y}}_{x_j}(x_j), \vec{y}_j \rangle}.$$  

This shows that any element $f \in \mathcal{H}$ can be perfectly reconstructed from the values $\{(f(x_j), \vec{y}_j)\}$, the $\mathbb{C}^n$-inner products of its values taken on the set of points $\{x_j\} \subset X$ with the vectors $\vec{y}_j \in \mathbb{C}^n$.

2. The Matrix-valued Aleksandrov disintegration theorem

Throughout this section we will assume that $\Theta(0) = 0$.

2.1. Identification of the matrix-valued Aleksandrov-Clark measures. The purpose of this subsection is to establish two key facts needed for the proof of the disintegration theorem. First it will be shown that the Aleksandrov-Clark measures $\Omega_{\Theta_U}$ associated with $\Theta$ and unitary $U \in \mathcal{U}(n)$ are such that $(\Omega_{\Theta_U}(I))_{ij} = (\chi_I(Z_{\Theta_U}(U))b_i^*, b_j^T)_{\Theta}$ where $b_i^T(z) = 1/zb_i^-$ for $z \in \mathbb{T}$, and $b_i^-$ are the constant co-ordinate functions. Recall here that $\chi_I$ is the characteristic function of the Borel subset $I \subset \mathbb{T}$. In fact, we will establish something stronger than this. Given any $A \in (\mathcal{M}_n)_1$ the operator $Z_{\Theta}(A)$ is a completely non-unitary contraction since if $\|A\| < 1$, any unitary restriction of $Z_{\Theta}(A)$ would have to be a unitary restriction of $Z_{\Theta}$ to a subspace orthogonal to $\mathcal{D}_+ = \mathbb{C}\{1/zb_i^-\}$. This is not possible. If $Z_{\Theta}$ has a unitary restriction to a subspace $S$ which is orthogonal to $\mathcal{D}_+$, then $S$ is reducing for $Z_{\Theta}$ and $Z_{\Theta}S = S$ for all $k \in \mathbb{Z}$. Since $Z_{\Theta}$ is unitary it would then follow that $S$ is orthogonal to $\bigvee_{k \in \mathbb{Z}}(z^k b_i^-)_{i=1}^n$ which is dense in $L^2_\Theta$ so that $S = \{0\}$. Let $U_A$ acting on $\mathcal{K}_A \supset L^2_\Theta$ be the minimal unitary dilation of $Z_{\Theta}(A)$, and let $P_A$ be the orthogonal projection of $\mathcal{K}_A$ onto $L^2_\Theta$. We will show that the positive matrix-valued measure $\Lambda_A(I) := P_A\chi_I(U_A)P_A$, $I \in \text{Bor}(\mathbb{T})$ is such that $\Lambda_A(I)_{ij} = (\Lambda_A(I)e_i, e_j) = (\Omega_{\Theta_A}(I)b_{i}^+, b_{j}^T)_{\Theta}$. Here $\text{Bor}(\mathbb{T})$ denotes the Borel subsets of $\mathbb{T}$.

Secondly we will show that $\Omega_{\Theta} = m$ where $m$ denotes $\mathcal{M}_n$-valued normalized Lebesgue measure on $\mathbb{T}$, i.e. $m(I)_{ij} = \mu(I)\delta_{ij}$, and $\mu$ is normalized Lebesgue measure on $\mathbb{T}$, and $\delta_{ij}$ is the Kronecker delta.

To identify the matrix valued Aleksandrov-Clark measures $\Omega_{\Theta_A}$ with the spectral measures associated with the perturbations $Z_{\Theta}(A)$, for any $A \in (\mathcal{M}_n)_1$, we will apply the following Proposition taken from [2 Proposition 14]. Although the statement of the proposition in [2] assumes that $A$ is unitary, the proof for general $A$ is identical.
Proposition 2.1.1. (S. Elliott) Let $\Theta : \mathbb{D} \to \mathbb{M}_n$ be analytic and purely contractive with $\Theta(0) = 0$. Then for any $A \in (\mathbb{M}_n)$,
\begin{equation}
\int_\mathbb{T} \zeta^n \Omega_A(d\zeta) = \begin{cases} 
\sum_{k=1}^{\infty} \int_\mathbb{T} \zeta^{-n}(A\Theta(\zeta^*)^k m(d\zeta) & n \geq 1 \\
\sum_{k=1}^{\infty} \int_\mathbb{T} \zeta^n (\Theta(\zeta^*)^k m(d\zeta) & n \leq -1 \\
1 & n = 0
\end{cases}
\end{equation}

Proposition 2.1.2. Let $\Theta \in (H^{\infty}_2(\mathbb{D}))$ be a purely contractive analytic function with $\Theta(0) = 0$. Then $(\Omega_A(\cdot)e_i, e_j) = (\Lambda_A(b_i, b_j))^\Theta$ where $\Lambda_A$ is the positive $\mathbb{M}_n$-valued measure associated with $Z_\Theta(A)$.

2.1.3. Notation. Here $(e_i)_{i=1}^{\infty}$ will be an orthonormal basis of $\mathbb{C}^n$ that is fixed throughout this paper. As in the introduction $V_n : \mathbb{C}^n \to L^2_{\mathbb{D}}$ is an isometry defined by $V_n e_i = b_i^-$, where the $b_i^-$ form an orthonormal basis for $\mathbb{D}$, the copy of $\mathbb{C}^n$ in $L^2_{\mathbb{D}}$. The above proposition can be stated more succinctly as $V_n^* P \Lambda_A P V_n = \Omega_{\Theta}(A)$.

Recall that if $A = U$ is unitary then $\Lambda_U(I) = \chi_I(Z_{\Theta}(A))$ is a projection for any $I \subset \text{Bor}(\mathbb{T})$, the Borel subsets of $\mathbb{T}$. To simplify the presentation of the proof, we first establish a few lemmas. Consider the power series for $\Theta$, $\Theta(z) := \sum_{k=1}^{\infty} c_k z^k$, $c_k \in \mathbb{M}_n$ (recall we assume that $\Theta(0) = 0$). Let $l_j(A)$ denote the $j^{th}$ coefficient in the power series of $\sum_{k=1}^{\infty} (\Theta(z)A^*)^k =: \Phi(z)$, and observe that by Proposition 2.1.2, $l_j(A) = \int_\mathbb{T} \zeta^{-j} \Omega_A(d\zeta)$, and that $l_j(1) = V_n^* P \sum_{k=1}^{\infty} \zeta^{-j} \sum_{k=1}^{\infty} \zeta^{-j} = \int_\mathbb{T} \zeta^{-j} \Omega_{\Theta}(d\zeta) e_i, e_j)$ for all $j \in \mathbb{N} \cup \{0\}$.

Lemma 2.1.4. The coefficients $l_k(A)$ obey the recurrence relations $l_k(A) = c_k A^* + \sum_{j=1}^{k-1} c_j A^* l_{k-j}(A) = c_k A^* + \sum_{j=1}^{k-1} l_j(A) c_k A^* c_{k-j} A^*$.

Proof. By definition $c_k$ is the $k^{th}$ coefficient of $\Theta(z)$ and $l_k(A)$ is the $k^{th}$ coefficient of $\Theta(z) A^* + \ldots + (\Theta(z) A^*)^k$. Let $\Gamma_k$ denote the linear functional which picks out the $k^{th}$ coefficient of a power series. Then clearly
\begin{equation}
l_k(A) = c_k A^* + \Gamma_k[(\Theta(z) A^*)^2 + \ldots + (\Theta(z) A^*)^k] = c_k A^* + \Gamma_k[(\Theta(z) A^*) (\Theta(z) A^*)^2 + \ldots + (\Theta(z) A^*)^{k-1}].
\end{equation}
Let $b_j$ denote the coefficients in the power series of $\Phi(z) := \Theta(z) A^* + \ldots + (\Theta(z) A^*)^{k-1}$. Since $\Theta(0) = 0 = c_0$, it is easy to see that $b_j = l_j(A)$ for all $1 \leq j \leq k - 1$. Hence it follows that $\Gamma_k[\Theta(z) A^* \Phi(z)] = c_1 A^* l_{k-1}(A) + c_2 A^* l_{k-2}(A) + \ldots + c_{k-1} A^* l_1(A) = \sum_{j=1}^{k-1} c_j A^* l_{k-j}(A)$. Also since $\Theta(z) A^*$ commutes with $\Phi(z)$ it follows that $\Gamma_k[\Theta(z) A^* \Phi(z)] = \Gamma_k[\Phi(z) \Theta(z) A^*] = \sum_{j=1}^{k-1} l_{k-j}(A) c_j A^* = \sum_{j=1}^{k-1} l_j(A) c_k A^*$. □

The following combinatorial fact will be needed:

Lemma 2.1.5. Let $(a_i)$, $(b_i)$, and $(c_i)$, $i = 1, \ldots, n$ be arbitrary sequences of (in general non-commuting) variables. Then the sum $\sum_{i=1}^{n-1} \sum_{j=1}^{i-1} a_i b_j c_{n-i-j}$ is a rearrangement of the sum $\sum_{i=1}^{n-1} \sum_{j=1}^{i-1} a_i b_j c_{n-i-j}$.

The above lemma can be established with a straightforward proof by induction. We omit the proof. Let $l_j := l_j(1)$.
Lemma 2.1.6. The $l_j$ and $l_j(A)$ obey the recurrence relation

\begin{equation}
(2.3) \quad l_k(A) = l_k A^* + \sum_{j=1}^{k-1} l_j(\mathbb{1})[A^* - \mathbb{1}] l_{k-j}(A).
\end{equation}

Proof. For convenience, let $q_n := l_n(A)$ for the remainder of the proof. By Lemma 2.1.4, we have that

\begin{equation}
(2.4) \quad q_n = c_n A^* + \sum_{i=1}^{n-1} c_i A^* q_{n-i} \text{ and } c_n = l_n - \sum_{i=1}^{n-1} l_i c_{n-i}.
\end{equation}

Substituting the second equation into the first yields

\begin{equation}
(2.5) \quad q_n = l_n A^* + \sum_{i=1}^{n-1} l_i A^* q_{n-i} - \left( \sum_{i=1}^{n-1} l_i c_{n-i} A^* + \sum_{i=1}^{n-1} \sum_{j=1}^{i} l_j c_{i-j} A^* q_{n-i} \right).
\end{equation}

To prove the lemma, we need to show that

\begin{equation}
(2.6) \quad \sum_{i=1}^{n-1} l_i q_{n-i} = \sum_{i=1}^{n-1} l_i c_{n-i} A^* + \sum_{i=1}^{n-1} \sum_{j=1}^{i} l_j c_{i-j} A^* q_{n-i}.
\end{equation}

Substituting equation (2.4) into the left hand side of this expression gives:

\begin{equation}
(2.7) \quad \sum_{i=1}^{n-1} l_i \left( c_{n-i} A^* + \sum_{j=1}^{n-i-1} c_j A^* q_{n-i-j} \right) = \sum_{i=1}^{n-1} \left( l_i c_{n-i} A^* + \sum_{j=1}^{n-i-1} l_j c_{i-j} A^* q_{n-i} \right).
\end{equation}

Canceling like terms and applying the identity from Lemma 2.1.5 proves the claim. \qed

Proof. (Proposition 2.1.2)

Proposition 2.1.1 shows that $l_j(A) = \int_{\mathbb{R}} \mathbb{P} \Omega_{\theta_j} (dz)$ for all $j \in \mathbb{N}$. Hence to prove this proposition, it suffices to show that $d_k := V_n P_\neg \neg Z(A)^{-k} P_\neg \neg V_n = \int_{\mathbb{R}} \mathbb{P} V_n P_\neg \neg \Lambda A \neg \neg (dz) P_\neg \neg V_n = l_k(A)$ for all $k \in \mathbb{N}$. The fact that $d_k = l_k(A)$ for $k \in \mathbb{N}$ will follow from taking adjoints, and since $\Theta(0) = 0$, it follows that $\Omega_{\theta}(\mathbb{R}) = \mathbb{P}$ so that $d_0 = \mathbb{P}_n = l_0(A)$. Thus if we can prove that $d_k = l_k(A)$ for all $k \in \mathbb{R}$, then all moments of the measures $V_n P_\neg \neg \Lambda A \neg \neg P_\neg \neg V_n$ and $\Omega_{\theta A}$ agree so that they must be equal.

This will be accomplished by proving that the $d_k$ obey the same recurrence formula as the $l_k(A)$ given in the previous lemma. For simplicity identify the standard basis $\{e_i\}$ of $\mathbb{C}^n$ with the basis $\{b_i\}$ of $\mathcal{D}_1 \subset L^2_{\omega}$, and let $P := P_\neg \neg$ so that we can write $V_n P_\neg \neg (Z(A))^k P_\neg \neg V_n$ as $P(Z(A))^k P_\neg \neg$. The calculation proceeds as follows

\begin{align*}
P_Z(A)^k P_\neg \neg &= P \left( Z^{-1} + Z^{-1} P(A^* - \mathbb{1}) P \right) \left( Z^{-1} + Z^{-1} P(A^* - \mathbb{1}) P \right)^{k-1} P_\neg \neg \\
&= P Z^{-1} P(A^* - \mathbb{1}) P \left( Z^{-1} + Z^{-1} P(A^* - \mathbb{1}) P \right)^{k-1} P_\neg \neg + P Z^{-1} \left( Z^{-1} + Z^{-1} P(A^* - \mathbb{1}) P \right)^{k-2} P_\neg \neg \\
&= l_1(A^* - \mathbb{1}) d_{k-1} + P Z^{-2} P(A^* - \mathbb{1}) P \left( Z^{-1} + Z^{-1} P(A^* - \mathbb{1}) P \right)^{k-2} P_\neg \neg \\
&= l_1(A^* - \mathbb{1}) d_{k-1} + P Z^{-2} P(A^* - \mathbb{1}) P \left( Z^{-1} + Z^{-1} P(A^* - \mathbb{1}) P \right)^{k-2} P_\neg \neg \\
&= l_1(A^* - \mathbb{1}) d_{k-1} + l_2(A^* - \mathbb{1}) d_{k-2} + ... + l_k(A^* - \mathbb{1}) d_1 \\
&= l_k A^* + \sum_{j=1}^{k-1} l_j(A^* - \mathbb{1}) d_{k-j}.
\end{align*}

(2.8)
This is the same formula as in Lemma 2.1.6. We conclude that $d_k = l_k(A)$, and hence that $\Omega_{\Theta_A} = \Lambda_A$. □

2.2. The Weyl integral formula and proof of the disintegration theorem. Let $T$ be a contraction on a separable Hilbert space $\mathcal{H}$. The defect operators $D_T, D_{T^*}$ are defined by $D_T := \sqrt{1 - T^* T}$, the defect subspaces $\mathcal{D}_T, \mathcal{D}_{T^*}$ by $\mathcal{D}_T := \text{Ran}(D_T)$ and the defect indices by $\mathfrak{d}_T := \dim(\mathcal{D}_T)$. We say a contraction $T$ has defect indices $(n, n)$ if $\mathfrak{d}_T = n = \mathfrak{d}_{T^*}$. Let $P_T$ denote the projection onto $\mathcal{D}_T$. Then $T_0 := T - TP_T$ is a partial isometry with kernel $\mathcal{D}_T$ and with range the orthogonal complement of $\mathcal{D}_{T^*}$.

The Nagy-Foias characteristic function of a contraction $T$ is defined as

$$\Theta_T(z) = (-T + zD_{T^*}(1 - zT^*)^{-1}D_T)|_{\mathcal{D}_T},$$

and is a contractive analytic function with domain $\mathcal{D}_T$ and range $\mathcal{D}_{T^*}$. Two contractions $T, T'$ are unitarily equivalent if and only if their characteristic functions coincide, i.e. if and only if there are isometries $U, V$ such that $U \Theta_T = \Theta_T V$. It is straightforward to check that $T$ is a partial isometry if and only if $\Theta_T(0) = 0$. It follows that if $T$ is a contraction with defect indices $(n, n)$, then $\Theta_T(0) = 0$. In Section 5.3 we will show that given any partial isometry $V$ with defect indices $(n, n)$, that $\Theta_V$ coincides with $\Theta_{Z_{\Theta_V}(0)}$, so that $V$ is unitarily equivalent to $Z_{\Theta_V}(0)$. It will follow that any contraction $T$ with defect indices $(n, n)$ is unitarily equivalent to some extension of the partial isometry $Z_{\Theta_{T_0}}(0)$.

Given $T$ and $T_0$, let $\mathcal{D}_+ := \mathcal{D}_T$ and $\mathcal{D}_- := \mathcal{D}_{T^*}$, and let $\{\psi_i^+\}_{i=1}^n, \{\psi_i^-\}$ be orthonormal bases for $\mathcal{D}_\pm$. Fix an isometry $W$ of $\mathcal{D}_+$ onto $\mathcal{D}_-$ by $W \psi_i^+ = \psi_i^-$. Now define for any $U \in \mathcal{U}(n)$, $T(U) := T_0 + WU$, where $\hat{U} : \mathcal{D}_+ \to \mathcal{D}_+$ is the bijection isometry defined by

$$\langle \cdot, \psi_i^+ \rangle, \ldots, \langle \cdot, \psi_n^+ \rangle \right] [U_{ij}] = \sum_{i,j=1}^n U_{ij} \langle \cdot, \psi_i^+ \rangle \psi_j^+.$$

If $T = Z_{\Theta}$, this notation agrees with that of the previous section if we choose $\psi_i^+ = b_i^+$ and $\psi_i^- = b_i^-$. Now any $U \in \mathcal{U}(n)$ can be written as $U = V^* DV$ where $V \in \mathcal{U}(n)$ and $D \in \mathbb{T}^n$, i.e. $D = \text{diag}(z_1, \ldots, z_n)$ with $z_i \in \mathbb{T}$. Hence $U_{ij} = \sum_k z_k V_{ki} V_{kj}$ and we can write

$$\hat{U} = \sum_{i,j,k=1}^n z_k V_{ki} V_{kj} \langle \cdot, \psi_i^+ \rangle \psi_j^+ =: z_1 R_1 + z_2 R_2 + \ldots + z_n R_n,$$

and

$$T(U) = R_0 + z_1 R_1 + \ldots z_n R_n.$$

Here $R_0 := T(0) = T_0$ and for $i \geq 1$ the $R_i$ are all finite rank operators depending on $V$ and not on $D$, i.e. the $R_i = R_i(V)$ are independent of the $z_i \in \mathbb{T}$.

Hence for any polynomial $p(z) = \sum_{k=0}^i p_k z^k$, it follows that

$$p(T(U)) = p(T(V^* DV)) = \sum_{i_1, \ldots, i_n=0}^k z_1^{i_1} \ldots z_n^{i_n} A_{i_1, \ldots, i_n}(V),$$

where the coefficient operators $A_{i_1, \ldots, i_n}(V)$ depend only on $V$, and so are constant if $V$ is fixed.
Weyl’s integration formula for $\mathcal{U}(n)$ (see e.g. [1]) states that if $H$ is Haar measure on $\mathcal{U}(n)$, $\mathbb{T}^n$ denotes the subgroup of diagonal unitary matrices, $G := \mathcal{U}(n)/\mathbb{T}^n$, and $H_G$ Haar measure on $G$ then:

**Theorem 2.2.1.** (Weyl Integration Formula) If $f$ is a continuous function on $\mathcal{U}(n)$, then,

$$\int_{\mathcal{U}(n)} f(U)dH(U) = \frac{1}{n!} \int_G \left( \int_{\mathbb{T}^n} f(VDV^*) \Delta(D) \Delta(D)dD \right) dH_G(V\mathbb{T}^n). \tag{2.14}$$

In the above if $D = \text{diag}(z_1, \ldots, z_n)$ then $dD := dz_1 \ldots dz_n$, and $\Delta(D) := \prod_{j<k}(z_j - z_k)$. The following fact is a straightforward consequence of Weyl’s integration formula

**Proposition 2.2.2.** If $T$ is a completely non-unitary contraction with defect indices $(n,n)$, and $f = h + g$ for $h, g \in H^\infty(\mathbb{T})$, then

$$\int_{\mathcal{U}(n)} f(T(U))dH(U) = f(T(0)). \tag{2.15}$$

Here if $h \in H^\infty$, and $T$ is a completely non-unitary contraction, then $h(T)$ is defined as $h^*(T^*)$, where $h^*(z) = h(\mathbb{T}) \in H^\infty$.

**Proof.** It suffices to establish the formula in the case where $f = p$ is a polynomial. The more general formula follows by taking adjoints, and limits with the aid of the $H^\infty$ functional calculus for completely non-unitary contractions (see e.g. [1]). For fixed $V \in \mathcal{U}(n)$, equation (2.13) implies that

$$p(T(U)) = p(T(0)) + \sum^t_{i} z_1^{i_1} \cdots z_n^{i_n} A_{i_1, \ldots, i_n}(V), \tag{2.16}$$

where the prime denotes that the sum is taken over all values of the $i_1, \ldots, i_n$ where at least one of the $i_j; 1 \leq j \leq n$ is non-zero.

By Weyl’s integration formula,

$$\int_{\mathcal{U}(n)} p(T(U))dH(U) = p(T(0)) + \sum^t_{A_{i_1, \ldots, i_n}} \int_G \left( \int_{\mathbb{T}^n} \prod_{j<k} (z_j - z_k)^2 dz_1 \ldots dz_n \right) dH_G(V\mathbb{T}^n). \tag{2.17}$$

Hence, to prove the proposition, it suffices to show that provided at least one of the $i_1, \ldots, i_n$ is non-zero, that

$$0 = \int_{\mathbb{T}^n} \prod_{j<k} (z_j - z_k)^2 dz_1 \ldots dz_n. \tag{2.18}$$

This is straightforward to show. If one expands out $\prod_{j<k} (z_j - z_k)^2$ one obtains a sum of terms of the form $z_1^{j_1} \cdots z_n^{j_n}$ where $j_1 + j_2 + \ldots + j_n = 0$. It follows that the product $z_1^{j_1} \cdots z_n^{j_n} \left| \prod_{j<k} (z_j - z_k) \right|^2$ is the sum of terms of the form $z_1^{k_1} \cdots z_n^{k_n}$ where $k_1 + \ldots + k_n \geq 1$, and therefore there is at least one $k_i \geq 1$ in each such term. But then it is clear that the above integral in (2.13) vanishes, since if $k_i \geq 1$, then

$$\int_{\mathbb{T}^n} \prod_{j<k} (z_j - z_k)^2 dz_1 \ldots dz_n = 0. \tag{2.19}$$
There is one final observation to make before presenting the matrix-valued disintegration theorem. If $\Omega_{\Theta_A}$ are the Aleksandrov-Clark measures discussed in the previous section, recall that,

$$B_A(z) = \frac{1 - \Theta(z)A^*}{1 - \Theta(z)} = \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} \Omega_{\Theta_A}(d\zeta).$$

Taking $A = 0$ shows that $\mathbb{1}_{\mathbb{M}_n} = \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} d\Omega_0(\zeta)$. Letting $m$ denote the diagonal positive matrix valued measure given by $n$ copies of Lebesgue measure on the diagonal, then

$$\int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} m(d\zeta) = \sum_{k=0}^{\infty} \int_{\mathbb{T}} ((\zeta^*)^k + (\zeta)^{k+1}) m(d\zeta) = \mathbb{1}_n.$$

By the uniqueness of the representing measure in the matrix-valued Herglotz theorem [2, Theorem 3], it follows that $\Omega_0 = m$.

**Theorem 2.2.3.** Let $\Theta$ be a $\mathbb{M}_n$-valued contractive analytic function on $\mathbb{D}$, and $\Omega_{\Theta_A}$ the AC measures associated with $\Theta$ for any $A \in (\mathbb{M}_n)_1$. Then $\Omega_0 = m$ and for any continuous function $f$ on $\mathbb{T}$,

$$\int_{U(n)} \int_{\mathbb{T}} f(\zeta) \Omega_{\Theta_U}(d\zeta) dH(U) = \int_{\mathbb{T}} f(\zeta) m(d\zeta).$$

**Proof.** Recall that if $A \in (\mathbb{M}_n)_1$, $\Lambda_A$ denotes the positive operator valued measure obtained as the compression of the projection valued measure of the unitary dilation of $Z(A)$ to $L^2_{\Theta}(\mathbb{T})$. In the case where $A = U$ is unitary, $\Lambda_U$ is the projection-valued measure obtained from $Z_0(U)$.

Now by the previous proposition, Proposition 2.2.2 if $f = \overline{q} + p$ where $p, q$ are polynomials, then

$$\int_{U(n)} \int_{\mathbb{T}} f(\zeta) \Lambda_U(d\zeta) dH(U) = \int_{U(n)} f(Z(U)) dH(U) = q^*(Z(0)^*) + p(Z(0)).$$

By Proposition 2.1.2 if we again identify $\mathbb{C}^n$ with $\mathcal{D} \subset L^2_\Theta$ and let $P := P_-$, the projector of $L^2_\Theta$ onto $\mathcal{D}_-$, then $PA_A P = \Omega_{\Theta_A}$, so that

$$\int_{U(n)} \int_{\mathbb{T}} f(\zeta) \Omega_U(d\zeta) dH(U) = \int_{U(n)} \int_{\mathbb{T}} f(\zeta) P \Lambda_U(d\zeta) P dH(U)$$

$$= P \int_{U(n)} \int_{\mathbb{T}} f(\zeta) \Lambda_U(d\zeta) dH(U) P$$

$$= P \int_{U(n)} f(Z_\Theta(U)) dH(U) P$$

$$= P(q^*(Z(0)^*) + p(Z(0))) P$$

$$= \int_{\mathbb{T}} f(\zeta) \Omega_0(d\zeta)$$

$$= \int_{\mathbb{T}} f(\zeta) m(d\zeta).$$

(2.24)
For \( f \) an arbitrary continuous function, the statement follows by approximating \( f \) by functions \( f_n \) of the form \( f_n = \eta_n + p_n \) since such functions are dense in the Banach space of continuous functions on \( \mathbb{T} \).

\[ \square \]

3. The Cauchy Integral Representation of \( K_\Theta^2 \)

Let \( H_\Theta^2 \) be the closure of the polynomials in \( L_\Theta^2 \), i.e. the closed subspace of \( L_\Theta^2 \) generated by \( Z_\Theta \) and \( \mathbb{D} - \), the constant functions. In this section we construct an isometry \( V_\Theta : H_\Theta^2 \rightarrow K_\Theta^2 \), where \( K_\Theta^2 \) is the de Branges-Rovnyak space associated with \( \Theta \) and show that the image of \( Z_\Theta^* \) under this transformation is a rank-\( n \) perturbation of \( X_\Theta \), the restriction of the backwards shift from \( H_\Theta^2(\mathbb{D}) \) to \( K_\Theta^2 \). In this section we do not assume that \( \Theta(0) = 0 \) in general.

3.1. de Branges-Rovnyak spaces. Let \( L_n^2(\mathbb{T}) \) denote the Hilbert space of \( \mathbb{C}^n \)-valued functions which are square integrable with respect to normalized matrix-valued Lebesgue measure \( m \) on \( \mathbb{T} \), and recall that \( H_n^2(\mathbb{D}) \subset L_n^2(\mathbb{T}) \) is the subspace of \( \mathbb{C}^n \) valued functions which are analytic in \( \mathbb{D} \) and whose \( L^2 \) norm on circles of radii \( r < 1 \) remains bounded as \( r \to 1 \).

Given \( \Theta \in (H_{\infty,0}(\mathbb{D}))_1 \), the de Branges-Rovnyak space \( K_\Theta^2 \) is defined as follows. Let \( P_{H^2} \) denote the projection of \( L_n^2(\mathbb{T}) \) onto \( H_n^2(\mathbb{D}) \), and let \( T_\Theta \) denote the operator of multiplication by \( \Theta \) on \( H_n^2(\mathbb{D}) \). The de Branges-Rovnyak space \( K_\Theta^2 \) is defined as the range of \( R_\Theta := \sqrt{1 - T_\Theta T_\Theta^*} \) endowed with the inner product that makes \( H_n^2(\mathbb{D}) \) onto its range. Hence if \( f, g \in H_n^2(\mathbb{D}) \) and at least one of \( f, g \) is orthogonal to the kernel of \( R_\Theta \), then \( \langle R_\Theta f, R_\Theta g \rangle_\Theta = \langle f, g \rangle \), see [4] for more details. We will denote the inner product in \( K_\Theta^2 \) by \( \langle \cdot, \cdot \rangle_\Theta \) to distinguish it from the inner product of \( H_n^2(\mathbb{D}) \) which is denoted by \( \langle \cdot, \cdot \rangle_{\mathbb{D}} \). For \( z, w \in \mathbb{D} \), let

\[
\Delta_w(z) := \frac{1 - \Theta(z)\Theta(w)^*}{1 - zw},
\]

be the matrix kernel function at \( w \). The Hilbert space \( K_\Theta^2 \) is the closed linear span of the point evaluation functions

\[
\delta_x^z := \Delta_x \mathbb{1}^z,
\]

for \( x \in \mathbb{C}^n \) and \( z \in \mathbb{D} \). The notation \( \delta_x^z := \delta_x^{z^*} \) where \( \{e_j\} \) as before is an ON basis of \( \mathbb{C}^n \) will sometimes be used. Inner products with \( \delta_x^z \) gives point evaluations at \( z \in \mathbb{D} \):

\[
\langle f, \delta_x^z \rangle_\Theta = \langle f(z), x \rangle_{\mathbb{C}^n},
\]

for any \( f \in K_\Theta^2 \).

We will now discuss the Cauchy integral representation for vector-valued de Branges-Rovnyak spaces \( K_\Theta^2 \). This will be a straightforward generalization of the methods of [4] Chapter III]. Since most of the arguments generalize with only trivial modifications, many of the results will be stated without proof.

3.2. The Cauchy Integral Representation of \( K_\Theta^2 \). Recall that \( \Omega_\Theta \) is the unique positive \( \mathbb{M}^n \)-valued measure on \( \mathbb{T} \) associated with the purely contractive \( \Theta \) by the Herglotz theorem.

One defines the Cauchy integral of \( \Omega_\Theta \) by

\[
C\Omega_\Theta(z) := \int_{\mathbb{T}} \frac{1}{1 - z\zeta} \Omega_\Theta(d\zeta).
\]
This is clearly a \( M_n \)-valued function which is analytic in \( \mathbb{D} \). Next for any \( f \in L^2_{\Theta}(\mathbb{T}) \) define the Cauchy integral of \( f \) by
\[
C_\Theta f(z) := \int_\mathbb{T} \frac{1}{1 - \bar{\zeta} z} \Omega_\Theta(d\zeta)f(\zeta).
\]
For each such \( f \) this is an analytic \( \mathbb{C}^n \)-valued function on \( \mathbb{D} \).

By definition \( C_\Theta f(z) = (f, k_z)_\Theta \) where
\[
k_z(\zeta) := (1 - \bar{\zeta} \zeta)^{-1},
\]
kernel \( z \in L^2_{\Theta} \) for \( z \in \mathbb{D} \). Hence the kernel of the map \( C_\Theta \) is the orthogonal complement of the span of the kernel functions \( k_z, z \in \mathbb{D} \) in \( L^2_{\Theta} \). The closed linear span of the kernel functions \( k_z := k_z \bar{z} \) where \( \bar{z} \in \mathbb{C}^n \) is easily seen to be the span of the polynomials in \( L^2_{\Theta} \) which we defined previously to be \( H^2_{\Theta} \).

The following Lemma is easy to verify and its proof is omitted:

**Lemma 3.2.1.** The following identity holds:
\[
\int_\mathbb{T} \frac{1}{1 - \bar{a} z} \frac{1}{1 - \bar{b} z} \Omega_\Theta(dz) = (\mathbb{1} - \Theta(b))^{-1}\Delta_n(b)(\mathbb{1} - \Theta(a)^*)^{-1}.
\]

Given \( f \in H^2_{\Theta} \), define \( V_\Theta f(z) := (1 - \Theta(z))C_\Theta f(z) \). We will write \( k^i_z \) for \( k^i_z \) where \( \{e_i\} \) is the canonical ON basis for \( \mathbb{C}^n \). Then observe that by applying the above lemma,
\[
V_\Theta k^i_z(z) = (1 - \Theta(z)) \int_\mathbb{T} \frac{1}{1 - \bar{a} w} \frac{1}{1 - \bar{z} w} \Omega_\Theta(dw)e_i
\]
\[
= \Delta_n(z)(\mathbb{1} - \Theta(a)^*)^{-1}e_i.
\]

This shows that \( V_\Theta k^i_z \) is a linear combination of the point evaluation functions \( \{\delta^j_a\}_{j=1}^n \subset K^2_\Theta \) so that \( V_\Theta \) is a linear map from \( H^2_{\Theta} \) into \( K^2_{\Theta} \). Here, as above \( \delta^j_a = \delta^j_a = \Delta_n e_j \).

**Proposition 3.2.2.** The linear map \( V_\Theta : H^2_{\Theta} \rightarrow K^2_{\Theta} \) is an isometry of \( H^2_{\Theta} \) onto \( K^2_{\Theta} \).

**Proof.** For any \( a \in \mathbb{D} \), \( (\mathbb{1} - \Theta(a)^*) \) is invertible so that \( \{(\mathbb{1} - \Theta(a)^*)e_i\}_{i=1}^n \) is a basis for \( \mathbb{C}^n \). It follows that the span of the set of functions \( S := \{k^i_a \mid a \in \mathbb{D}, 1 \leq j \leq n\} \subset H^2_{\Theta} \) where \( k^i_a(z) = \frac{1}{1 - \bar{a} z} (\mathbb{1} - \Theta(a)^*)e_j \) is equal to the span of the \( \{k^i_a \mid a \in \mathbb{D}, 1 \leq j \leq n\} \). The span of the last set is dense in \( H^2_{\Theta} \), and hence so is the span of \( S \). Hence to prove that \( V_\Theta \) can be uniquely extended to an isometry of \( H^2_{\Theta} \) onto \( K^2_{\Theta} \), it suffices to show that
\[
\left( k^i_a, k^j_b \right)_\Theta = (V_\Theta k^i_a, V_\Theta k^j_b)_\Theta.
\]

The left hand side of the above equation is equal to
\[
\left( \int_\mathbb{T} \frac{1}{1 - \bar{a} z} \frac{1}{1 - \bar{b} z} \Omega_\Theta(dz)(\mathbb{1} - \Theta(a)^*)e_i, (\mathbb{1} - \Theta(b)^*)e_j \right) = (\Delta_n(b)e_i, e_j),
\]
by the previous lemma.

By the calculation preceding this proposition, \( V_\Theta k^i_a(z) = \delta^j_a(z) \) so that the right hand side of equation (3.9) is equal to \( (\delta^j_a, \delta^j_b)_\Theta \). Since these are the point evaluation functions in \( K^2_\Theta \), this is equal to \( (\delta^j_a, e_j) = (\Delta_n(b)e_i, e_j) \). Hence both sides are equal and \( V_\Theta \) is an isometry. 

\( \square \)
Let $Y_\Theta(A) := Z_\Theta(A)|_{H^2_\Theta}$ (it is clear that $H^2_\Theta$ is invariant for each $Z_\Theta(A)$, $A \in (\mathcal{M}_n)_1$, and let $X_\Theta := S^*|_{K^2_\Theta}$ where $S$ is the shift (multiplication by $z$) in $H^2_n(\mathbb{T})$.

**Proposition 3.2.3.** The compressed backwards shift $X_\Theta$ and $Y^*_\Theta$ are related as follows:

\[(3.11)\]

\[X_\Theta V_\Theta = V_\Theta Y^*_\Theta \left(1 - (1_n - \Theta(0)) \sum_{i=1}^{n} \cdot \Theta_i b_i^{-}\right).\]

In the above statement, $Y_\Theta := Y_\Theta(1)$. If we let $P = P_- = \sum_{i=1}^{n} \cdot \Theta_i b_i^{-}$, the projector onto the constant functions in $H^2_\Theta$, then the claim can be written

\[(3.12)\]

\[V^*_\Theta X_\Theta V_\Theta = Y^*_\Theta (1 - P(1 - \Theta(0)) P).\]

This proof of this proposition is an obvious $n$-dimensional generalization of calculations in [4].

**Proof.** For simplicity identify the fixed basis of $C^n$, $\{e_i\}$ with the basis $\{b_i^{-}\}$ for $\mathcal{D}_-$, the constant functions in $L^2_\Theta$. This basis is orthonormal if $\Theta(0) = 0$. Given any $f \in H^2_\Theta$, consider $V_\Theta Y^*_\Theta f(z) = (1 - \Theta(z)) C_\Theta Y^*_\Theta f(z)$. First as in [4] it is easy to calculate that

\[(3.13)\]

\[
(C_\Theta Y^*_\Theta f(z), e_i)_{C^n} = \int_{\mathbb{T}} \frac{1}{1 - \overline{w} z} (\Omega_\Theta dw) Y^*_\Theta f(w, e_i) = \frac{1}{\overline{z}} (C_\Theta f(z) - C_\Theta f(0)).
\]

It follows that

\[(3.14)\]

\[V_\Theta Y^*_\Theta f = S^* V_\Theta f + S^* \Theta(C_\Theta f(0)).\]

Applying this formula to the case where $f = e_i$, and using equation (3.8) yields

\[(3.15)\]

\[V_\Theta Y^*_\Theta e_i = S^* \Delta_{0}(1 - \Theta(0)^*)^{-1} e_i - S^* \Theta \int_{\mathbb{T}} \Omega_\Theta (dw) e_i.\]

Short calculations show that $S^* \Delta_{0} = -(S^* \Theta)\Theta(0)^*$ while Lemma (3.2.1) implies that $\int_{\mathbb{T}} \Omega_\Theta (dw) = (1 - \Theta(0)^{-1}(1 - \Theta(0))\Theta(0)^*(-1)^{-1}$. Substituting these formulas into equation (3.15) and simplifying leads to

\[(3.16)\]

\[V_\Theta Y^*_\Theta e_i = S^* \Theta(1 - \Theta(0))^{-1} e_i,
\]

or equivalently that

\[(3.17)\]

\[S^* \Theta e_i = V_\Theta Y^*_\Theta (1 - \Theta(0)) e_i.
\]

Since

\[(3.18)\]

\[C_\Theta f(0) = \sum_{i=1}^{n} (C_\Theta f(0), e_i) e_i = \sum_{i=1}^{n} \int_{\mathbb{T}} (\Omega_\Theta (dw) f, e_i) e_i = \sum_{i=1}^{n} (f, e_i)_{\Theta} e_i,
\]
it follows that
\[
V_\Theta Y_\Theta^* f = S^* V_\Theta f + S^* \Theta(C_\Theta f)(0) \\
= S^* V_\Theta f + \sum_{i=1}^{n} (f, e_i)_{\Theta} V_\Theta Y_\Theta^*(\mathbb{1} - \Theta(0))e_i \\
= X_\Theta V_\Theta (\mathbb{1} - \sum_{i=1}^{n} (\cdot, e_i)_{\Theta} (\mathbb{1} - \Theta(0))e_i) f.
\] (3.19)

\[
3.3. \textbf{Extreme points.} \text{ In the case where } \Theta \text{ is scalar-valued, it is well known that } \Theta \text{ is an extreme point of the unit ball of } H^\infty \text{ if and only if } 1 - |\Theta| \text{ fails to be log-integrable } \mathbb{R} \text{ pgs. 138-139}, \text{ and that this happens if and only if } H^2_\Theta = L^2_{\Theta}. \text{ These facts follow easily from Szegö's theorem } \mathbb{R} \text{ pgs. 49-50} \text{ and the fact that the derivative of the absolutely continuous part of } \Omega_\Theta \text{ with respect to Lebesgue measure is } \frac{1 - |\Theta|^2}{|1 - \Theta|^2}.
\]

These facts generalize almost verbatim to the case where } \Theta \text{ is } \mathbb{M}_n \text{-valued and purely contractive. First, it is easy to check } \mathbb{R} \text{ Theorem 9} \text{ that the derivative of the absolutely continuous part of } \Omega_\Theta \text{ with respect to Lebesgue measure is}

\[
W_\Theta(\zeta) = (\mathbb{1} - \Theta(\zeta))^{-1} (\mathbb{1} - \Theta(\zeta)\Theta(\zeta)^*) (\mathbb{1} - \Theta(\zeta)^*)^{-1}.
\] (3.20)

By the Helson-Lowdenslager generalization of Szegö's Theorem,

\[
\exp \left( \int_{\mathbb{D}} \text{tr} \left( \text{ln}(W_\Theta(\zeta))m(d\zeta) \right) \right) = \inf_{A_0, P} \int_{\mathbb{D}} \text{tr} \left( (A_0 + P(\zeta))^*(A_0 + P(\zeta))\Omega_\Theta(d\zeta) \right).
\] (3.21)

Here the infimum is taken over all } n \times n \text{ matrices } A_0 \text{ of determinant one, and all polynomial matrix functions } P(z) = \sum_{j=1}^{n} A_j z^j, A_j \in \mathbb{M}_n \text{ for } z \in \mathbb{D} \text{ which vanish at the origin } \mathbb{R} \text{ Theorem 8}.

With this fact in hand, and the fact that if } A, B \text{ are positive definite matrices the identity } \text{tr}(\text{ln}(AB)) = \text{tr}(\text{ln}(A)) + \text{tr}(\text{ln}(B)) \text{ holds (this follows from the multiplicative property of the determinant), one can show as in the scalar case that } H^2_\Theta = L^2_{\Theta} \text{ if and only if } \int_{\mathbb{D}} \text{tr} \left( (\mathbb{1} - |\Theta(z)|)m(dz) \right) = -\infty \text{. Indeed, in this case the left hand side of equation } \mathbb{R} \text{ vanishes, and this implies that if } A_0^* := \bigvee_{k \in \mathbb{N}} Z_{\Theta}^k \mathcal{D}_- \text{ that } \mathcal{D}_- \subset A_0^* \text{ and hence that } \mathcal{D}_- \subset A_0 \text{ where } A_0 := \bigvee_{k \in \mathbb{N}} Z_{\Theta}^k \mathcal{D}_-. \text{ This readily leads to the conclusion that } H^2_\Theta = L^2_{\Theta}. \text{ Moreover, using the fact that by } \mathbb{R} \text{ Theorem 9,}

\[
\int_{\mathbb{D}} \text{tr} \left( |\Theta(z)|^2 m(dz) \right) > -\infty, \text{ it is easy to generalize the proof characterizing extreme points of the unit ball of } H^\infty(\mathbb{D}) \mathbb{R} \text{ pgs. 138-139} \text{ to obtain an analogous characterization of extreme points of the unit ball of } H^\infty_{\mathbb{D}}(\mathbb{D}). \text{ In summary one can establish the following without difficulty:}
\]

\[
\textbf{Theorem 3.3.1.} \text{ Given } \Theta \in (H^\infty_{\mathbb{D}}(\mathbb{D}))_1, \text{ the following are equivalent:}
\]

(i) \( \Theta \) is an extreme point.

(ii) \( \int_{\mathbb{D}} \text{tr} \left( (\mathbb{1} - |\Theta(z)|)m(dz) \right) = -\infty \)

(iii) \( L^2_{\Theta} = H^2_{\Theta} \)

\[
3.3.2. \textbf{Remark.} \text{ For brevity we will say that } \Theta \text{ is extreme if it is an extreme point of the unit ball of } H^\infty_{\mathbb{D}}(\mathbb{D}). \text{ In this case since } H^2_{\Theta} = L^2_{\Theta} \text{ we have that } Y_\Theta = Z_\Theta|_{H^2_{\Theta}} = Z_\Theta \text{ in Proposition 3.2.3}.
\]

\[
3.4. \textbf{Determination of AC measures.} \text{ The Cauchy integral representation of } K^2_{\Theta} \text{ provides another way of proving that } \Omega_{\Theta U} = \Lambda_U \text{ for } U \text{ unitary that is independent of the methods used in Section 2.1. In this subsection we do this and prove that } Z_\Theta(U) \text{ is unitarily equivalent to } Z_\Theta(U^*).
\]

Recall that } \Theta_U = \Theta U^*.
Suppose that $\Theta(0) = 0$. Consider the subspace $K_0$ of $K^2_0$ spanned by the point evaluation functions at $z = 0$, $\delta_j^0$, $1 \leq j \leq n$ where $\delta_j^0(z) = \Delta_0(z)e_j = (1 - \Theta(z)\Theta(0))^*e_j = e_j$ since $\Theta(0) = 0$. Then from earlier calculations we see that if $P = P_-$ denotes the projection onto the constant functions in $H^2_0$ and $Q$ the projector onto $K_0$, the constant functions in $K^2_0$ then $V_0P = QV_0$. Let $R_0$ denote the projection of $L^2_0$ onto $H^2_0$. Before we defined $Z_0(A) = Z_0 + P(A - \mathbb{1})PZ_0$. Since $H^2_0$ is invariant for $Z_0(A)$ it follows that

$$Y_0(A)^* = R_0Z_0(A)^*R_0 = R_0Z^*_0R_0 + R_0Z^*_0P(A^* - \mathbb{1})PR_0$$

(3.23)

$$= Y_0^* + Y_0^*P(A^* - \mathbb{1})P.$$  

Now by the intertwining relation of Proposition 3.2.3, $V_0Y_0V_0^* = X_0 + V_0Y_0^*PQV_0^*$. As calculated previously in equation 3.17 $V_0Y_0^*e_i = S^*\Theta e_i$. Hence we get that $V_0Y_0^*PQV_0^* = S^*\Theta V_0PQV_0^* = S^*\Theta Q$. This shows that

$$V_0Y_0^*V_0^* = X_0 + S^*\Theta Q,$$

and hence that

$$V_0Y_0^*(A)V_0^* = (X_0 + S^*\Theta Q)(\mathbb{1} + Q(A^* - \mathbb{1})Q) = X_0 + S^*\Theta QA^*Q.$$  

(3.24)

Note that here the operator $QAQ$ denotes the operator $\sum^n_{i=1}\{\delta_i\Theta A_{ij}\delta_i^0}$ where $\{\delta_i^0 = e_j\}$ is an ON basis for the constant functions $K_0 \subset K^2_0$ and $A \in (\mathbb{H}_n)$. In particular we conclude that $Y_0(0)^*$ is unitarily equivalent to $X_0$ (under our assumption that $\Theta(0) = 0$). If $\Theta$ is extreme then also $Y_0(A) = Z_0(A)$.

Now recall that the de Branges-Rovnyak spaces $K^2_\Theta$ are the ranges of $R_\Theta = \sqrt{1 - T_\Theta T^*_\Theta}$. If we define $\Theta_A := \Theta A^*$ for $A \in (\mathbb{H}_n)$, then it follows that $K^2_\Theta = K^2_{\Theta A}$ for unitary $U$.

**Lemma 3.4.1.** Given any $U \in U(n)$, let $W_U := V_0^*V_0U$. Then $W_U Y_0^* = Y_0(U)^*W_U$.

**Proof.** By previous calculations,

$$W_U V_0^* V_0^* = X + S^*\Theta U Q = X + S^*\Theta U Q.$$  

(3.25)

Here $X := X_0 = X_{\Theta U}$ acts on $K^2_\Theta = K^2_{\Theta U}$. But by equation (3.24) this agrees with $V_0Y_0(U)^*V_0^*$.

**Proposition 3.4.2.** Suppose that $\Theta(0) = 0$. For any $U \in U(n)$, let $\Omega_{\Theta U}$ be the measure associated with $\Theta U$ by the Herglotz theorem, and let $\Lambda_U$ denote the $\mathbb{V}_n$ valued positive measure on $\mathbb{V}$ defined by $\Lambda_U(I) := |\chi_I(\Theta_U(U))(e_i, e_j)\Theta\rangle$. Then $\Omega_{\Theta U} = \Lambda_U$.

**Proof.** Clearly the claim holds for $U = \mathbb{1}$. Now suppose $U \neq \mathbb{1}$. Recall that $H^2_\Theta$ is invariant for $Z_0(U)$ and that $Y_0(U) := Z_0(U)|H^2_\Theta$. By the previous lemma, there is a unitary operator $W_U$ which intertwines $Y_0^* := Y_0^*U$ and $Y(U)^* := Y_0(U)^*$. Since $H^2_\Theta$ is invariant for $Y(U)$, it is semi-invariant for $Y^*(U)$, for any $U \in U(n)$. Recall here that a subspace $S$ of a Hilbert space $\mathcal{H}$ is said to be semi-invariant for a semigroup of operators $\mathcal{G}$ if $S = S_1 \Theta S_2$ where $S_1 \supset S_2$ are invariant subspaces for $\mathcal{G}$. If $S$ is semi-invariant for the semigroup $\mathcal{G}$, then the compression of $\mathcal{G}$ to $S$ is a semigroup of operators on $S$ [10].

Moreover it is not hard to show that $Z_\Theta(U)$ is the minimal unitary dilation of $Y^*(U)$. To prove this, it suffices to show that the linear span of $Z(U)^{-k}H^2_\Theta$, for $k \in \mathbb{Z}$ is dense in $L^2_\Theta$. Recall that $P$ projects onto the constant functions in $H^2_\Theta$. Now $\text{Ran}(Z(U)^{-1}P = Z^{-1}U^*P) \supset \text{Ran}(Z^{-1}P)$, and $Z(U)^{-2}P = Z^{-2}P + Z^{-1}P(U^* - \mathbb{1})PZ^{-1}P$. Since the range of the second term is contained in $\text{Ran}(Z^{-1}P) \subset \text{Ran}(Z^{-1}(U)P)$, it follows that the range of $Z^{-2}P$ is contained in the closed linear span of the ranges of $Z^{-2}(U)P$ and $Z^{-1}(U)P$. Continuing in this fashion we get that
for any $A$ and $Y$ of $Z$ this case.

Remark. 3.4.3. Let

\[ L_\Theta U \subset \bigvee_{k \in \mathbb{Z}} \text{Ran}(Z_k P) \subset \bigvee_{k \in \mathbb{Z}} \text{Ran}(Z_k(U)). \]

Since the first set is dense in $L_\Theta^2$, so is the second so that $Z^*(U)$ acting on $L_\Theta^2$ is indeed the minimal unitary dilation of $Y^*(U)$ acting on $H_\Theta^2$. The same argument shows that $Z^*_U$ is the minimal unitary dilation of $Y^*_U$.

Since there is a unitary $W_U$ intertwining $Y^*(U)$ and $Y^*_U$, the intertwiner version of the commutant lifting theorem \( \text{Lift} \) \( \text{pg. 66} \) implies that there is a unitary $\hat{W}_U : L^2_{\Theta_U} \to L^2_{\Theta}$ such that $\hat{W}_U|_{H^2_{\Theta_U}} = W_U$ and such that $\hat{W}_U Z_U = Z^*_U \hat{W}_U$. If $P_U$ denotes the projector onto the constant functions in $H^2_{\Theta_U}$, then, by construction $W_U P_U = PW_U$ since $W_U = V_{\Theta} V_{\Theta_U}$, and it is clear that $\hat{W}_U$ obeys the same formula, $\hat{W}_U P_U = PW_U$. In particular if $\{b_i^-\}$ is the canonical ON basis of $D_-$ in $H^2_{\Theta_U}$ and $\{\beta_j^-\}$ is the corresponding basis in $H^2_{\Theta_U}$, then $\hat{W}_U \beta_j^- = b_j^-$. It follows that for any Borel set $I \subset \mathbb{T}$, $[\Lambda_U(I)]_{ij} := [(\chi_I(Z(U)) b_i^-, b_j^-)]_{\Theta_U} = [(\chi_I(Z_U) \hat{W}_U b_i^-, \hat{W}_U b_j^-)]_{\Theta_U} = [\Omega_U]_{ij}$, where the last equality follows from the fact that $Z_U$ is multiplication by the independent variable in $L^2_{\Theta_U}$.

Note that if $\Theta$ is extreme so that $L^2_{\Theta} = H^2_{\Theta}$, then the above argument simplifies. In particular in this case $Z^*_U = Y^*_U$ and we have no need to use dilation theory.

\[ \square \]

3.4.3. Remark. By the proof of the above proposition, $Z_{\Theta_U}$, $Z_{\Theta}(U)$ are the minimal unitary dilations of $Y_{\Theta_U}$ and $Y_{\Theta}(U)$, respectively. By Lemma 3.4.4 there is a unitary operator $W_U$ intertwining $Y_{\Theta_U}$ and $Y_{\Theta}(U)$. The above proof shows that there is a unitary $\hat{W}_U : L^2_{\Theta_U} \to L^2_{\Theta_U}$ which intertwines $Z\Theta(U)$ and $Z\Theta$, and satisfies $W_U|_{H^2_{\Theta_U}} = W_U$.

Recall that the earlier Proposition 2.1.2 established the more general statement that $\Omega_{\Theta_A} = \Lambda_A$ for any $A \in \mathcal{L}_{\mathbb{A}}$. This more general fact is not needed to prove the disintegration theorem. In the proof of the disintegration theorem, one simply needs to show that $\Omega_{\Theta_U} = \Lambda_U$ for $U \in \mathcal{U}(n)$, as shown in the above proposition, Proposition 3.4.4 as well as the fact that $\Omega_0 = \Lambda_0 = m$. Below we provide a proof of this fact which does not rely on the methods of Subsection 2.1 so that the disintegration theorem, Theorem 2.2.3 as given in Subsection 2.2 can be proven completely using the results of this section instead of those of Subsection 2.1.

Lemma 3.4.4. $\Omega_0 = \Lambda_0 = m$.

Proof. That $\Omega_0 = m$ follows from the uniqueness of the Herglotz representation as described before the statement of Theorem 2.2.3.

By definition $\int T \zeta_k [\Lambda_0(d\zeta)]_{ij}$ evaluates to $(Z(0)^k b_i^-, b_j^-)$ if $k \geq 0$ and to $(Z(0)^*)^k b_i^-, b_j^-)$ if $k \leq 0$. The only non-vanishing moment occurs when $k = 0$ in which case this evaluates to $(b_i^-, b_j^-) = \delta_{ij}$. This proves that $m = \Lambda_0$ since they have the same moments.

\[ \square \]

4. Total orthogonal sets of point evaluation vectors

If $\Theta$ is scalar-valued, necessary and sufficient conditions for the point evaluation vectors $\delta_\zeta(z) := \frac{1 - \Theta(z)}{1 - \Theta(z)}$ to belong to $K^2_{\Theta}$ in the case where $\zeta \in \mathbb{T}$ can be given in terms of the existence of the Carathéodory angular derivative (CAD) of $\Theta$ at $\zeta$ [4, VI-4]. In $\text{[3]}$ (for inner $\Theta$) and $\text{[5]}$, it is shown that $K^2_{\Theta}$ has a total orthogonal set of point evaluation vectors if and only if there is a $\zeta \in \mathbb{T}$ for which the measure $\Omega_{\Theta_{\zeta}}$ is purely atomic. It is easy to show that if $\{\delta_{\lambda_n}\}_{n \in \mathbb{Z}}$ is a total orthogonal set in $K^2_{\Theta}$, then $\{\lambda_n\} \subset \mathbb{T}$.  

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This section will verify that these results generalize straightforwardly to the case where \( \Theta \) is matrix-valued. To accomplish this, it will first be useful to show how the theorems of \cite{4} Chapter VI on angular derivatives extend to the matrix-valued case.

4.1. Caratheodory angular derivatives. Let \( \Theta \) be purely contractive. There is no need to assume that \( \Theta(0) = 0 \) in this subsection. The analytic function \( \Theta \) is said to have a Carathéodory angular derivative (CAD) at \( \zeta \in \mathbb{T} \) if \( \Theta \) has a non-tangential limit \( \Theta(\zeta) \) at \( \zeta \), \( |\Theta(\zeta)| = 1 \), and the non-tangential limit of \( \Theta' \) at \( \zeta \) exists. In this case the CAD of \( \Theta \) at \( \zeta \) is defined as the limit of \( \Theta'(z) \) as \( z \to \zeta \) non-tangentially, and is denoted by \( \Theta'(\zeta) \).

It is fairly easy to generalize \cite{4} VI-4 to prove the following:

**Theorem 4.1.1.** If \( \Theta \in (H_\infty^\infty(D))_1 \) and \( \zeta \in \mathbb{T} \), the following are equivalent:

1. \( \Theta \) has a CAD at \( \zeta \in \mathbb{T} \).
2. \( c_\zeta := \lim Inf_{z \to \zeta} \|\frac{1-\Theta(z)\Theta'(z)}{1-|z|^2}\| < \infty \).
3. There is a \( U \in \mathcal{U}(n) \) such that \( \frac{\Theta(z) - U}{z - \zeta} \to K_2^n \) for all \( z \in \mathbb{C}^n \).
4. Every element of \( K_2^n \) has a non-tangential limit at \( \zeta \).

If the above conditions hold then \( \delta_\zeta^x \in K_2^n \) for any \( z \in \mathbb{C}^n \), if \( f \in K_2^n \) then \( \langle f, \delta_\zeta^x \rangle = (f(z), \bar{x}) \) and \( \delta_\zeta^x \) is the norm limit of \( \delta_\zeta^{x_n} \) as \( z \to \zeta \). Moreover, \( \Theta'(\zeta) = \frac{\bar{A}\Theta(\zeta)}{\bar{A}^2} \) where \( A > 0 \) (so that \( \Theta'(\zeta) \) is invertible) and \( \frac{1-\Theta(z)\Theta'(z)}{1-|z|^2} \) converges to \( A \) as \( z \) approaches \( \zeta \) non-tangentially.

In the above \( z \to \zeta \) denotes the non-tangential convergence of \( z \in \mathbb{D} \) to \( \zeta \in \mathbb{T} \). The above theorem can be proven by following the proof for the case of scalar \( \Theta \). The only part of the proof which could be considered slightly more complicated is the proof that if (3) holds, then \( \delta_\zeta^x \) converges to \( \delta_\zeta^x \) weakly, which is used in the proof that (3) \( \Rightarrow \) (4). We will show how this is accomplished and omit the rest of the proof.

As in the proof of \cite{4} VI-4, to show that \( \delta_\zeta^x \) converges weakly to \( \delta_\zeta^x \) it suffices to show that the functions \( \delta_\zeta^x \) are bounded in norm as \( z \) approaches \( \zeta \) non-tangentially. To show this it suffices to show that \( \Delta_z \) is bounded in norm in this limit where \( \Delta_z : \mathbb{C}^n \to K_2^n \) is the linear map defined by \( \Delta_z(x) := \Delta(x \xi) = \delta_\zeta^x(z) \). This follows from an argument that can be found in the proof of \cite{12} Lemma 8.3: Consider

\[
0 \leq \langle (1-\Theta(z)\Theta(z)^*), \bar{x} \rangle + \langle (\Theta(\zeta)^* - \Theta(z)^*), (\Theta(\zeta)^* - \Theta(z)^*)\bar{x} \rangle \\
\leq \langle (1-\Theta(\zeta)^*), \bar{x} \rangle + \langle \bar{x}, (1-\Theta(z))\Theta(z)^* \rangle, 
\]

and observe that both terms on the right hand side of the inequality on the first line are positive. Recall that \( \Delta_z(w) = \frac{\bar{\Theta}(\zeta) - \bar{\Theta}(z)}{1 - |\zeta|^2} \). It follows that

\[
\|\Delta_z(x)\|^2 \leq \frac{1 - |\zeta|^2}{1 - |z|^2} \langle \Delta(\Theta^*, \Delta, \zeta)^2 \rangle + \frac{1 - |\zeta|^2}{1 - |z|^2} \langle \Delta(z, \zeta)^2 \rangle \\
\leq \frac{2(1 - |\zeta|^2)}{1 - |z|^2} \langle \Delta(\Theta^*, \Delta(z, \zeta))^2 \rangle.
\]

This inequality shows that \( \Delta_z \) is bounded in norm as \( z \) approaches \( \zeta \) non-tangentially.

4.1.2. *Remark.* More generally, given \( \bar{x} \in \mathbb{C}^n \) we will say that \( \Theta \bar{x} \) has a CAD at \( \zeta \in \mathbb{T} \) if \( \Theta \bar{x} \) has a non-tangential limit \( \Theta(\zeta)\bar{x} \) at \( \zeta \), \( \|\Theta(\zeta)\bar{x}\| = \|\bar{x}\| \), and \( \Theta' \bar{x} \) has a non-tangential limit at \( \zeta \). One can prove a version of the above theorem for such vector functions. We will not write this result down.
here, but we note that one can show that \( \delta^2 \in K^2_\Theta \) if and only if there is a unitary \( U \) such that \( \Theta U^* \bar{x} \) has a CAD at \( \zeta \).

4.2. Spectra of the unitary perturbations \( Z_\Theta(U) \). Earlier we defined \( Z'_\Theta := Z_\Theta|_{L^2_\Theta \otimes \mathcal{D}_+} \). This is clearly an isometric linear transformation from \( L^2_\Theta \otimes \mathcal{D}_+ \) onto \( L^2_\Theta \otimes \mathcal{D}_- \). The deficiency indices of an isometric linear transformation \( V \) are defined as \( (n_+, n_-) \) where \( n_+ := \dim(\operatorname{Dom}(V)^\perp) \) and \( n_- := \dim(\operatorname{Ran}(V)^\perp) \). If \( \Theta \) has rank \( n \), it follows that the deficiency indices of \( Z'_\Theta \) are \( (n, n) \). An isometric linear transformation is called simple if it has no unitary restriction to a proper subspace. It is easy to see that \( Z'_\Theta \) is simple, as if it were not, then \( Z'_\Theta \) would have a reducing subspace orthogonal to \( \mathcal{D}_- = \{1/\zeta e_i\} \), which, as discussed at the beginning of Section 2.4, is not possible. A point \( \lambda \in \mathbb{C} \) is called regular for an isometric linear transformation \( V \) if \( V - \lambda \) is bounded below. \( V \) is called regular if every \( \lambda \in \mathbb{C} \setminus \{1\} \) is regular for \( V \) (i.e. if every \( \lambda \in \mathbb{C} \) is regular for the symmetric linear transformation \( S = \mu^{-1}(V) \) defined on \( \operatorname{Ran}(V - 1) \) where \( \mu(z) = \frac{1}{z-1} \)).

As proven by Lifschitz in [3], any simple isometric linear transformation \( V \) with indices \( (n, n) \) is unitarily equivalent to \( Z'_\Theta \) for some purely contractive \( \Theta \) with \( \Theta(0) = 0 \). The following theorem characterizes the essential spectrum of \( Z'_\Theta \) (and hence of \( Z_\Theta \)) [6, Theorem 4]

**Theorem 4.2.1.** (Lifschitz) A point \( \zeta \in \mathbb{T} \) is a regular point of \( Z'_\Theta \) if and only if both of the following conditions are satisfied:

1. \( \Theta \) is analytic on some open neighbourhood of \( \zeta \).
2. There is a neighbourhood \( N_\zeta \) of \( \zeta \) such that \( \Theta(\lambda) \) is unitary for all \( \lambda \in N_\zeta \cap \mathbb{T} \).

By the above theorem the essential spectrum, \( \sigma_e(Z_\Theta(U)) \) of any of the unitary perturbations \( Z_\Theta(U) \) is the set of all \( \zeta \in \mathbb{T} \) which fail to satisfy at least one of the above conditions in the theorem. We will denote this set by \( \operatorname{sp}(\Theta) \). Assume that \( \zeta \in \mathbb{T} \setminus \operatorname{sp}(\Theta) \). Then \( Z_\Theta(U) - \zeta \) is a finite rank perturbation of \( Z_\Theta(0) - \zeta \) which has Fredholm index 0 since both \( Z_\Theta(0) \) and its adjoint are simple. It follows that \( \sigma(Z_\Theta(U)) = \sigma(\Theta) \cup \sigma_p(Z_\Theta(U)) \), where \( \sigma_p(Z_\Theta(U)) \) is the set of eigenvalues of \( Z_\Theta(U) \).

To determine the spectrum of \( Z_\Theta(U) \) it remains to determine its eigenvalues.

It is worth noting that one can show using the basic theory of isometric/symmetric linear transformations that given a simple isometric linear transformation \( V \) with deficiency indices \( (n, n) \), any eigenvalue of any unitary extension \( U \) of \( V \) has multiplicity not exceeding \( n \), and if \( \bar{x} \) is any point in \( \mathbb{T}^n \) consisting of regular points for \( V \), there is a unitary extension \( U \) of \( V \) which has the entries of \( \bar{x} \) as eigenvalues. Moreover each distinct pair of unitary extensions \( U(V) \) and \( V(U') \) can share no more than \( n - 1 \) eigenvectors. See for example [13, Section 83]

**Proposition 4.2.2.** Suppose that \( \Theta(0) = 0 \) and \( \lambda \in \mathbb{T} \).

1. \( \lambda \in \sigma_p(Z_\Theta(U)) \setminus \sigma(\Theta) \) if and only if \( \operatorname{Ker}(\Theta(\lambda)^* - U^*) \neq \emptyset \). A vector \( \bar{x} \in \mathbb{C}^n \) belongs to \( \operatorname{Ker}(\Theta(\lambda)^* - U^*) \) if and only if \( \delta_{\lambda}(\bar{x}) \) is an eigenvector of \( Z_{\Theta_U} \) to eigenvalue \( \lambda \).
2. \( \lambda \) is not an eigenvalue of any \( Z_\Theta(U) \) if and only if \( \lim_{z \to \lambda} \frac{1}{z-\lambda} U(U - \Theta(z))^{-1} = 0 \). This happens if and only if the angular derivative of \( \Theta \bar{x} \) at \( \lambda \) does not exist for any \( \bar{x} \in \mathbb{C}^n \).

In the above \( \delta_{\lambda}(\bar{x}) \in L^2_\Theta \) is the point mass function which takes the value \( \bar{x} \) at \( \lambda \in \mathbb{T} \) and vanishes elsewhere on \( \mathbb{T} \).

**Proof.** By Remark 3.4.3, \( Z_\Theta(U) \) is unitarily equivalent to \( Z_{\Theta_U} \) which acts as multiplication by \( z \) in \( L^2_{\Theta_U} \). It follows that \( \lambda \in \mathbb{T} \) is an eigenvalue of \( Z_\Theta(U) \) if and only if \( \Theta_U \) has a point mass at \( \lambda \), i.e. if and only if \( \Omega_{\Theta_U}(\{\lambda\}) \neq 0 \).
Now by the Herglotz theorem
\begin{equation}
2(1 - \Theta(z)U^*)^{-1} = B_{\Theta_U}(z) + 1 = 2 \int_1 \frac{1}{1 - z}\Omega_{\Theta_U}(d\zeta),
\end{equation}
and note that \((1 - \Theta(z)U^*)^{-1} = U(U - \Theta(z))^{-1}\). It follows easily from this that
\begin{equation}
\Omega_{\Theta_U}(\lambda) = \lim_{z \to \lambda} (1 - z\lambda)U(U - \Theta(z))^{-1}.
\end{equation}
In the above limit, we assume \(z\) converges to \(\lambda\) non-tangentially. Hence \(\lambda\) is not an eigenvalue of \(Z_\Theta(U)\) if and only if this limit is identically 0. This happens if and only if
\begin{equation}
\lim_{z \to \lambda} \frac{\|\Theta(z) - U\|}{\|z - \lambda\|} = \infty,
\end{equation}
for every \(\vec{x} \in \mathbb{C}^n\). This shows that \(\lambda\) is not an eigenvalue of any \(Z_\Theta(U), U \in \mathcal{U}(n)\) if and only if the angular derivative of \(\Theta(z)\vec{x}\) at \(\lambda\) does not exist for any \(\vec{x} \in \mathbb{C}^n\) (see Remark 4.1.2).

Since \(Z_{\Theta_U}\) acts as multiplication by \(z\), clearly \(\lambda\) is an eigenvalue of \(Z_{\Theta_U}\) if and only if there is a \(\vec{x} \in \mathbb{C}^n\) such that \(\delta_{(\lambda)}\vec{x}\) is an eigenvector of \(Z_{\Theta_U}\). If \(\delta_{(\lambda)}\vec{x}\) is such an eigenvector, then \(\vec{x} \in \mathbb{C}^n\) must be in the range of the non-zero projection \(\Omega_{\Theta_U}(\lambda)\) \(\in \mathcal{M}_n(\mathbb{C})\).
\begin{equation}
\vec{x} = \Omega_{\Theta_U}(\lambda)\vec{x} = \lim_{z \to \lambda} (1 - z\lambda)(1 - \Theta(z)U^*)^{-1}\vec{x}.
\end{equation}
This in turn implies that \(\lim_{z \to \lambda} (1 - \Theta(z)U^*)\vec{x} = 0\) so that \((\Theta(\lambda)^* - U^*)\vec{x} = 0\).

Conversely suppose that \(\vec{x} \in \text{Ker}(\Theta(\lambda)^* - U^*)\). If \(\lambda \notin \text{sp}(\Theta)\), it follows from Theorem 4.2.1 that \(\Theta\) is analytic in a neighbourhood of \(\lambda\), so that in particular the angular derivative of \(\Theta\) exists at \(\lambda\). By Theorem 4.1.1 the angular derivative \(\Theta'(\lambda)\) is invertible, and it is the limit of the invertible matrices \(A(z) := \frac{\Theta(z) - \Theta(\lambda)}{z - \lambda}\) as \(z \to \lambda\) non-tangentially.

Recall the matrix analytic function \(\Delta_{\lambda}(z) := \frac{z - \Theta(z)\Theta(\lambda)^*}{1 - z\lambda}\). By Theorem 4.1.1 \(\Delta_{\lambda}(z)\) converges to \(\Delta_{\lambda}(\lambda) := \lambda\Theta(\lambda)^*\Theta'(\lambda)\) as \(z \to \lambda\) non-tangentially, and this limit is an invertible operator. The non-tangential limit of \(\Delta_{\lambda}(z)^{-1}\) at \(\lambda\) is equal to the projection \(\Omega_{\Theta_{\Theta(\lambda)}}(\lambda)\) by equation (4.3), so that \(\Delta_{\lambda}(z)^{-1}\) is norm bounded in this limit and the non-tangential limit of \(\Delta_{\lambda}(z)^{-1}\) is equal to \(\Delta_{\lambda}(\lambda)^{-1}\). Since this is an invertible projection, \(\Delta_{\lambda}(\lambda) = \Delta_{\lambda}(\lambda)^{-1} = 1\).

Let \(B(z) := \frac{z - \Theta(z)U^*}{1 - z\lambda}\). Previous calculations in this proof have shown that \(B(z)^{-1} \to \Omega_{\Theta_U}(\lambda)\) \(\vec{x}\). To show that \(\delta_{(\lambda)}^{(\lambda)}\vec{x}\) is an eigenvector of \(\Theta_{\Theta_U}\), we need to show that \(B(z)^{-1}\vec{x}\) converges to \(\vec{x}\) as \(z \to \lambda\) non-tangentially. This is easily accomplished by observing that \(\|B(z)^{-1}\vec{x} - \vec{x}\| \leq \|B(z)^{-1}\||\vec{x} - B(z)^{-1}\vec{x}\| = \|B(z)^{-1}\||\vec{x} - \Delta_{\lambda}(z)\vec{x}\|\). The last equality follows from the fact that \(\vec{x} \in \text{Ker}(\Theta(\lambda)^* - U^*)\).

Since \(\|B(z)^{-1}\|\) is bounded as \(z \to \lambda\) non-tangentially, and \(\Delta_{\lambda}(z)\) converges to \(1\), the proof is complete.

4.3. Total orthogonal sets of point evaluation vectors. Recall the matrix kernel functions \(\Delta_w(z) := \frac{z - \Theta(w)\Theta(\lambda)^*}{1 - z\lambda}\), and the point evaluation functions \(\delta_{\vec{x}}^w := \Delta_w\vec{x} \in K_{\Theta}^2\) which satisfy \((f, \delta_{\vec{x}}^w)_{\Theta} = (f(w), \vec{x})\) for all \(f \in K_{\Theta}^2\), all \(w \in \mathbb{D}\), and all \(\vec{x} \in \mathbb{T}\) for which the angular derivative of \(\Theta\) at \(w\) exists.

In this section we determine necessary and sufficient conditions for \(K_{\Theta}^2\) to have a total orthogonal set of point evaluation functions. Suppose that \(\Lambda := \{\delta_{\vec{x}}^\lambda\}_{\vec{x} \in \mathbb{T}} \subset K_{\Theta}^2\) is such a set. For convenience define \(\delta_{\vec{x}} := \delta_{\vec{x}}^\lambda\). Let \(N_{\Lambda} : K_{\Theta}^2 \to K_{\Theta}^2\) be the normal operator \(N_{\Lambda} := \sum_{\vec{x} \in \mathbb{T}} \lambda_{\vec{x}} (\delta_{\vec{x}}\delta_{\vec{x}}^\lambda)^{\frac{1}{2}}\).

Proposition 4.3.1. If \(\Lambda := \{\delta_{\vec{x}}\}_{\vec{x} \in \mathbb{T}} \subset K_{\Theta}^2\) is a total orthogonal set, then \(\Theta\) is extreme and \(N_{\Lambda}\) is unitarily equivalent to \(Z_\Theta(U)\) for some \(U \in \mathcal{U}(n)\). Hence \(N_{\Lambda}\) is unitary and \(\{\lambda_{\vec{x}}\}_{\vec{x} \in \mathbb{T}} \subset \mathbb{T}\).

The following simple fact will be used in the proof of the above proposition.
Lemma 4.3.2. Let $V$ be an isometric linear transformation with deficiency indices $(n, n)$, and let $P, Q$ be the projectors onto $\text{Dom}(V)$ and $\text{Ran}(V)$ respectively. If $(P \lor Q)^\perp = 0$ then any normal extension of $V$ must be unitary.

Proof. Given any $\phi \in \mathcal{H}$ there exist $\phi_1 \in Q\mathcal{H}$ and $\phi_2 \in P\mathcal{H}$ such that $\phi = \phi_1 + \phi_2$. Any extension of $V$ can be written as $V(A) = V \oplus A$ on $\mathcal{H} = \text{Dom}(V) \oplus \text{Dom}(V)^\perp$ where $A: \text{Dom}(V)^\perp \to \text{Ran}(V)^\perp$.

If $V(A)$ is a normal extension of $V$ then $V(A)^\star V(A) = P + A^\star A = V(A)V(A)^\star = Q + A A^\star$ where $A^\star A$ vanishes on $\text{Dom}(V)$ and $AA^\star$ vanishes on $\text{Ran}(V)$.

It follows that $V(A)^\star V(A)\phi = V(A)V(A)\phi_1 + V(A)V(A)^\star \phi_2 = P\phi_1 + Q\phi_2 = \phi_1 + \phi_2 = \phi$. Hence $V(A)$ is unitary.

Lemma 4.3.3. Let $P$ be the projector onto $D_- \subset H^2_\Theta$ and let $Q := V\Theta PV\Theta^\star$. The restriction $\Theta(0) = Z_\Theta(0)|_{\mu^\Theta_n}$ and $X_\Theta$, the restriction of the backwards shift to $K^2_\Theta$ are related by the following formula:

\[ V_\Theta \Theta(0)^\star V^\Theta_\Theta = X_\Theta(\mathbb{1} - Q). \]

Proof. Let $R_\Theta$ be the projection of $L^2_\Theta$ onto $H^2_\Theta$ so that $\Theta(0)^\star = R_\Theta Z_\Theta(0)^\star R_\Theta = Y_\Theta(\mathbb{1} - P)$. Then

\[ V_\Theta \Theta(0)^\star V^\Theta_\Theta = V_\Theta Y_\Theta(0)^\star V_\Theta^\star = V_\Theta Y_\Theta(0)^\star V_\Theta(\mathbb{1} - P)V_\Theta^\star = V_\Theta Y_\Theta(0)^\star V_\Theta(\mathbb{1} - Q). \]

By equations (4.12) and (3.15),

\[ V_\Theta Y_\Theta(0)^\star V^\Theta_\Theta = X_\Theta + V_\Theta Y_\Theta^\star (\mathbb{1} - \Theta(0)) PV^\Theta_\Theta \]

\[ = X_\Theta + (S^\star \Theta) P(\mathbb{1} - \Theta(0))^{-1} (\mathbb{1} - \Theta(0)) PV^\Theta_\Theta \]

\[ = X_\Theta + (S^\star \Theta) PV^\Theta_\Theta. \]

In the above note that any $A \in M_n(\mathbb{C})$ is viewed as the operator $\sum_{ij}(\cdot, b_i^\perp) \Theta A_{ij} b_j^\perp$ where $\{b_i^\perp\}$ is the fixed ON basis of $D_-$ so that in particular $P \Theta(0) P = \Theta(0) P = \Theta(0)$. Equation (4.8) becomes

\[ V_\Theta \Theta(0)^\star V^\Theta_\Theta = (X_\Theta + (S^\star \Theta) PV^\Theta_\Theta)(\mathbb{1} - Q) = X_\Theta(\mathbb{1} - Q). \]

Proof. (of Proposition 4.3.3) Recall the canonical unitary transformation $V_\Theta : H^2_{\Theta} \to K^2_{\Theta}$ from the Cauchy integral representation of $K^2_{\Theta}$. Let $P = P_{\mathbb{1}}$ be the projector onto the constant functions in $H^2_{\Theta}$ spanned by the basis $\{e_i\}_{i=1}^n$ and let $Q = V_\Theta PV_\Theta^\star$ be the projector in $K^2_{\Theta}$ onto the span of the vectors $\delta^\Theta_{i} = \Delta_0 e_i$ for $1 \leq i \leq n$ (if $\Theta(0) = 0$ these are constant functions). If $A \in M_n(\mathbb{C})$ then $Z_{\Theta}(A)^\star = Z_{\Theta}^\perp(1 + (A - \mathbb{1}))$. Let $P_{\Theta}$ denote the projector onto $H^2_{\Theta}$. Then $\Theta(0)^\star = P_{\Theta} Z_{\Theta}(A)^\star P_{\Theta} = P_{\Theta} Z_{\Theta}^\perp P_{\Theta}(\mathbb{1} + (A - \mathbb{1}))$.

By Lemma 4.3.3 $V_\Theta Y_\Theta^\star(0) V_{\Theta}^\star = X_\Theta(\mathbb{1} - Q)$. Now if $f \in K^2_{\Theta} \ominus Q K^2_{\Theta}$, then since $0 = \langle f, \delta^\Theta_i \rangle_\Theta$ for all $1 \leq i \leq n$, it follows that $f(0) = 0$. Hence $f(z) = z g(z)$ for some $g \in H^2_{\Theta}(\mathbb{D})$. Moreover since $K^2_{\Theta}$ is invariant for $S^\star$, it follows that $S^\star f = g \in K^2_{\Theta}$. Hence for any $x^\perp \in \mathbb{C}^n$ and any $\lambda \in \mathbb{D}$ or $\lambda \in \mathbb{T}$ for which the angular derivative of $\Theta$ at $\lambda$ exists,

\[ \langle X_\Theta f, \delta^\perp_{\lambda} \rangle_\Theta = (g(\lambda), \overline{x^\perp}) = \overline{\lambda}(f, \delta^\perp_{\lambda})_\Theta. \]

It follows that

\[ X_\Theta(\mathbb{1} - Q)f = X_\Theta f = \sum_{n \in \mathbb{Z}} \overline{\lambda}_n \frac{\langle f, \delta^\Theta_n \rangle_\Theta \delta^\Theta_n}{||\delta^\Theta_n||^2} = N^\star f. \]
It can be concluded that $N^*$ is a normal extension of $X_\Theta|_{\mathcal{Q}^+}$, so that $\tilde{N} := V_{\Theta}^* N^* V_{\Theta}$ is a normal and contractive extension of $Y_\Theta(0)^*|_{P^+ L_\Theta^2}$.

Now $Y_\Theta^*$ is a co-isometry, and by the Wold decomposition it can be decomposed into the direct sum of a unitary operator, and a purely co-isometric operator (an operator isomorphic to the direct sum of copies of the adjoint of the unilateral shift). If $Y_\Theta^*$ had a non-zero purely co-isometric part, then it would have non-zero Fredholm index. However, $\tilde{N}^*$ is a normal finite rank perturbation of $Y_\Theta^*$, and hence is Fredholm. Any normal Fredholm operator must have index zero. Since the index is invariant under compact perturbations, $Y_\Theta^*$ also has index 0 and hence $Y_\Theta^*$ is unitary. It follows that $H_\Theta^2 = L_\Theta^2$, that $Y_\Theta = Z_\Theta$ and that $\Theta$ is extreme.

In conclusion $\tilde{N}^*$ is a normal extension $Z_\Theta(0)^*$ which is a partial isometry with deficiency indices $(n, n)$. By Lemma 4.3.2, $N^*$ and hence $\tilde{N}$ must be unitary so that $\tilde{N} = Z_\Theta(U)$ for some $U \in \mathcal{U}(n)$. Since $\tilde{N}$ is unitary its spectrum is contained in the unit circle so that $\{\lambda_n\} \subset \mathbb{T}$. \hfill $\Box$

**Theorem 4.3.4.** $K_\Theta^2$ has a total orthogonal set of point evaluation vectors if and only if there is a $U \in \mathcal{U}(n)$ such that the measure $\Omega_U$ is purely atomic. If $K_\Theta^2$ has such a set then $\Theta$ is inner.

**Proof.** If $K_\Theta^2$ has a total orthogonal set of point evaluation vectors $\{\delta_i\}$, where $\delta_i = \delta_{\lambda_i}$, then by the previous proposition, there is a $U \in \mathcal{U}(n)$ such that $Z_\Theta(U)$ has a total orthogonal set of eigenfunctions. Therefore $Z_{\Theta_U}$ which acts as multiplication by $z$ in $H_{\Theta_U}^2 = L_{\Theta_U}^2$ has $\{\delta_{\lambda_1} \vec{x}_i\}$ as a total orthogonal set of eigenfunctions, and the measure $\Omega_{\Theta_U} = \Omega_U = \sum_{n \in \mathbb{Z}} \Omega_U(\{\lambda_i\}) \delta_{\lambda_i}$ is purely atomic.

Conversely if $\Omega_U = \sum_{n \in \mathbb{Z}} \Omega_U(\{\lambda_i\}) \delta_{\lambda_i}$ is purely atomic then $\{\delta_{\lambda_i} \vec{x}_i\} \downarrow_{j \in \mathbb{Z}} \vec{x}_i \in E_{\lambda_i}$ is an ON basis for $\Omega_U(\{\lambda_i\}) \mathbb{C}^n$, and $k_i \leq n$, is a total orthogonal set of eigenvectors to $Z_{\Theta_U}$. Note here that each $\Omega_U(\{\lambda_i\})$ is a projection. Under the canonical unitary transformation $V_{\Theta_U} : H_{\Theta_U}^2 \to K_{\Theta}^2,$

\begin{equation}
V_{\Theta_U} \delta_{\lambda_i} \vec{x}_i(z) = (1 - \Theta_U(z)) \int_{\mathbb{T}} \frac{\delta_{\lambda_i}(w)}{1 - wz} \Omega_{\Theta_U}(dw) \cdot \vec{x}_i = \frac{1 - \Theta(z) U^*}{1 - z \lambda_i} \vec{x}_i,
\end{equation}

By Proposition 4.2.2 $\vec{x}_i \in \text{Ker}(\Theta(z) - U^*)$, so that $V_{\Theta_U} \delta_{\lambda_i} \vec{x}_i = \delta_{\lambda_i} \vec{x}_i$. We conclude that $\{\delta_i\}$ where $\delta_i = \delta_{\lambda_i} \vec{x}_i$ is a total orthogonal set of point evaluation vectors in $K_\Theta^2$.

If $\Theta$ is not inner, then there is a set $I \subset \text{Bor}(\mathbb{T})$ with $m(I) > 0$ such that $\Theta(z)$ is not unitary for $z \in I$. Let $\Omega_{\Theta_U}$ denote the absolutely continuous part of $\Omega_{\Theta_U}$ with respect to $m$. Then

\begin{equation}
(1 - \Theta(z) U^*)^{-1} (1 - \Theta(z) \Theta(z)^* - U \Theta(z)^*)^{-1} = \int_{\mathbb{T}} \frac{1 - |z|^2}{1 - z \zeta^2} \Omega_{\Theta_U}(d\zeta),
\end{equation}

for $z \in \mathbb{D}$, and

\begin{equation}
\frac{d\Omega_{\Theta_U}(\zeta)}{dm}(\zeta) = (1 - \Theta(\zeta) U^*)^{-1} (1 - \Theta(\zeta) \Theta(\zeta)^* - U \Theta(\zeta)^*)^{-1},
\end{equation}

almost everywhere $\zeta \in \mathbb{T}$ with respect to Lebesgue measure. For a proof of this fact in the matrix setting, see [1] Theorem 9. Hence if $\Theta$ is not inner, $\Omega_{\Theta_U}$ cannot be purely atomic for any $U \in \mathcal{U}(n)$ so that $K_\Theta^2$ cannot have a total orthogonal set of point evaluation vectors. \hfill $\Box$

5. Representation of simple symmetric operators with deficiency indices $(n, n)$

In this final section, we wish to point out that any simple symmetric operator with deficiency indices $(n, n)$ is unitarily equivalent to the symmetric operator of multiplication by the independent
variable in a model subspace $K^2_{\Phi}$ where $\Phi \in H^\infty_{\mathcal{M}_n}(U)$ is inner, $\Phi(0) = 0$ and $\Phi$ is analytic on some open neighbourhood of any given point $x \in \mathbb{R}$. We will see that such $K^2_{\Phi}$ have a $U(n)$-parameter family of total orthogonal sets of point evaluation vectors. Recall that $U$ denotes the open upper half plane, and $H^\infty_n(U)$ is the Hardy space of bounded analytic $\mathbb{M}_n$-valued functions on $U$.

There is a bijective correspondence between $\Phi \in H^\infty_{\mathcal{M}_n}(U)$ and $\Theta \in H^\infty_n(\mathbb{D})$ given by $\Phi = \Theta \circ \mu$ and $\Theta = \Phi \circ \mu^{-1}$ where $\mu(z) = \frac{z+i}{z-i}$ and $\mu^{-1}(z) = i\frac{1+z}{1-z}$. Further recall that there is a canonical unitary transformation $U : H^\infty_n(\mathbb{D}) \rightarrow H^\infty_n(U)$ given by

$$Uf(z) = \frac{1 - \mu(z)}{\sqrt{\pi}} f \circ \mu(z),$$

and that $U$ takes $K^2_{\Theta}$ onto $K^2_{\Phi}$.

5.1. Representation of simple symmetric linear transformations with deficiency indices $(n,n)$. Recall the Lifschitz characteristic function of a simple isometric linear transformation $V$. Here $\text{Dom}(V)$ and $\text{Ran}(V)$ are contained in a separable Hilbert space $\mathcal{H}$. Let $\mathcal{D}_+ : = \text{Dom}(V)^+$ and $\mathcal{D}_- : = \text{Ran}(V)^-$, fix a unitary extension $U$ of $V$ and let $(\psi_i^\pm)_{i=1}^n$ be orthonormal bases of $\mathcal{D}_\pm$ such that $U\psi_i^+ = \psi_i^-$. Given $W \in U(n)$, we define $V(W) := V \oplus \sum_i (\psi_i^+, \psi_i^-) W_{ij} \psi_j^-$ on $\mathcal{H} := \text{Dom}(V) \oplus \mathcal{D}_+$, so that $\{V(W)\}_{W \in U(n)}$ is the $U(n)$-parameter family of unitary extensions of $V$.

5.1.1. Definition. Fix $U \in U(n)$. For $1 \leq i,k \leq n$, let $A,B$ be matrix valued functions on $\mathbb{D}$ with entries $A_{ik}(z) = z((V(U) - z)^{-1} \psi_i^+, \psi_k^+)$, $B_{ik}(z) := ((V(U) - z)^{-1} V(U) \psi_i^+, \psi_k^+)$. The Lifschitz characteristic function of the simple isometric linear transformation $V$ is defined as $\Theta_V(z) := A(z)B(z)^{-1}$.

One can show that $\Theta_V(z)$ is always a purely contractive matrix analytic function on $\mathbb{D}$ with $\Theta_V(0) = 0$. Two contractive matrix analytic functions on $\mathbb{D}$, $\Theta_1$ and $\Theta_2$ are said to coincide if there are fixed unitaries $U, V$ in $U(n)$ such that $U\Theta_1 = \Theta_2 V$. In [6], it is shown that two simple isometric linear transformations $V_1, V_2$ are unitarily equivalent if and only if their characteristic functions coincide. Moreover one can show that choosing a different $U$ in the definition of $\Theta_V$ yields another purely contractive function which coincides with the original so that $\Theta_V$ is unique up to such coincidence. Now given $\Theta_V$, consider the operator $Z_{\Theta_V}$ of multiplication by the independent variable in $L^2_{\Theta_V}$. As discussed in the beginning of Section 4.2 the transformation $Z_{\Theta_V} = Z_{\Theta_V}|_{L^2_{\Theta_V} \ominus \mathcal{D}_+}$ is a simple isometric linear transformation with deficiency indices $(n,n)$. It is not difficult to show that the characteristic function of $Z_{\Theta_V}$ is $\Theta_V$ so that $V$ is always unitarily equivalent to $Z_{\Theta_V}$.

**Theorem 5.1.2. (Lifschitz) Any simple isometric linear transformation $V$ with deficiency indices $(n,n)$ is unitarily equivalent to $Z_{\Theta_V}$, which acts as multiplication by the independent variable on $\text{Dom}(Z_{\Theta_V}) = L^2_{\Theta_V} \ominus \mathcal{D}_+$.**

**Proof.** Let $\Theta : = \Theta_V$. As in the proof of Proposition 4.2.2 it is easy to check that

$$((B_\Theta - 1)e_i, e_k) = 2z \int_\pi 1_{\zeta - z} (\Omega_\Theta(d\zeta) e_i, e_k) = 2z ((Z_\Theta - z)^{-1} b_i^+, b_k^+)_{\Theta},$$

and similarly that

$$((B_\Theta + 1)e_i, e_k) = 2((Z_\Theta(Z_\Theta - z)^{-1} b_i^+, b_k^+)_{\Theta}.$$}

This shows that the Lifschitz characteristic function of $Z_{\Theta}$ coincides with $\Theta$ since $\Theta = (B_\Theta - 1)(B_\Theta + 1)^{-1}$. \qed
There is a bijective correspondence between simple isometric linear transformations $V$ and simple symmetric linear transformations $B$ given by $B = \mu^{-1}(V)$ with $\text{Dom}(B) := (V - 1)\text{Dom}(V)$ and $V = \mu(B)$ with $\text{Dom}(V) = \text{Ran}(B + i)$. Recall here that a symmetric linear transformation is called simple if it has no self-adjoint restriction to a proper subspace.

The following provides necessary and sufficient conditions on $\Theta$ for the symmetric linear transformation $\mu^{-1}(Z_{\Theta})$ to be a densely defined symmetric operator. This is a straightforward generalization of a result of Lifschitz for the case $n = 1$, and the proof is virtually identical.

**Lemma 5.1.3.** Let $V$ be a simple isometric linear transformation with deficiency indices $(n, n)$. Then $B = \mu^{-1}(V)$ is a densely defined symmetric operator if and only if $z = 1$ is not an eigenvalue of any unitary extension of $V$.

**Proof.** If $\mu^{-1}(V)$ is densely defined, then $\text{Ran}(V - 1)$ is dense so that if $U$ is any unitary extension of $V$, then $\text{Ran}(U - 1) \supset \text{Ran}(V - 1)$ is also dense. This can only happen if $z = 1$ is not an eigenvalue of any unitary extension $U$.

Conversely if $\mu^{-1}(V)$ is not densely defined then there is a $\xi \in \mathcal{H}$ such that $\langle (V - 1)\psi, \xi \rangle = 0$ for all $\psi \in \text{Dom}(V)$. Let $\{\psi_{i}^{+}\}$ and $\{\psi_{i}^{-}\}$ be ON bases for $\text{Dom}(V)^{\perp}$ and $\text{Ran}(V)^{\perp}$ respectively. For $A \in M_{n}$ define $V(A) := V \oplus \hat{A}$ on $\mathcal{H} = \text{Dom}(V) \oplus \text{Dom}(V)^{\perp}$ where $\hat{A} : \text{Dom}(V)^{\perp} \to \text{Ran}(V)^{\perp}$ is given by $\hat{A} = \sum_{i,j=1}^{n} A_{ij} \langle \cdot, \psi_{i}^{+} \rangle \psi_{j}^{-}$.

Given any $\psi \in \mathcal{H} = \text{Dom}(V) \oplus \text{Dom}(V)^{\perp}$, $\psi = \psi_{V} + \sum_{i=1}^{n} c_{i} \psi_{i}^{+}$ where $\psi_{V} \in \text{Dom}(V)$. Hence,

$$\langle (V(A) - 1)\psi, \xi \rangle = \langle (V - 1)\psi, \xi \rangle + \sum_{i=1}^{n} c_{i} \langle (\hat{A} - 1)\psi_{i}^{+}, \xi \rangle. \quad (5.4)$$

Now $\xi$ is not orthogonal to $\text{Ran}(V)^{\perp}$, as otherwise there would exist a $\xi' \in \text{Dom}(V)$ such that $V\xi' = \xi$. This would imply that

$$0 = \langle (V - 1)\psi, V\xi' \rangle = \langle \psi, (1 - V)\xi' \rangle, \quad (5.5)$$

for all $\psi \in \text{Dom}(V)$ so that $(V - 1)\xi' \in \text{Dom}(V)^{\perp}$. The fact that $\xi'$ is orthogonal to $\text{Dom}(V)^{\perp}$ and that $\|V\xi'\| = \|\xi\|$ would then imply that $V\xi' = \xi'$, contradicting the simplicity of $V$. We conclude that $\xi$ is not orthogonal to $\text{Ran}(V)^{\perp}$, so that we can choose $A$ so that $\langle (\hat{A} - 1)\psi_{i}^{+}, \xi \rangle = 0$.

It follows that for this choice of $A$, $\langle (V(A) - 1)\psi, \xi \rangle = \langle (V - 1)\psi, \xi \rangle = 0$ for all $\psi \in \mathcal{H}$ so that $V(A)^{*}\xi = \xi$. Now $\xi = \xi_{V}^{+} + \psi^{-}$ where $\xi_{V}^{+} \in \text{Ran}(V)$ and $\psi^{-} \in \text{Ran}(V)^{\perp}$, and $V(A)^{*} = V^{*} \oplus \hat{A}^{*}$ on $\mathcal{H} = \text{Ran}(V) \oplus \text{Ran}(V)^{\perp}$. A simple calculation shows

$$\|\xi_{V}^{+}\|^{2} + \|\psi^{-}\|^{2} = \|\xi\|^{2} = \|V(A)^{*}\xi\|^{2} = \|V^{*}\xi_{V}^{+}\|^{2} + \|\hat{A}^{*}\psi^{-}\|^{2} = \|\xi_{V}^{+}\|^{2} + \|\hat{A}^{*}\psi^{-}\|^{2}, \quad (5.6)$$

so that $\|\hat{A}^{*}\psi^{-}\| = \|\psi^{-}\|$. It follows that we can choose $U \in \mathcal{U}(n)$ such that $\hat{U}^{*}\psi^{-} = \hat{A}^{*}\psi^{-}$, and that with this choice of $U$, $V(U)$ is a unitary extension of $V$ with $z = 1$ as an eigenvalue. \hfill \Box

**Theorem 5.1.4.** The simple symmetric linear transformation $\mu^{-1}(Z_{\Theta})$ will be a densely defined simple symmetric operator if and only if the limit of $(1 - z)U(U - \Theta(z))^{-1}$ as $z$ approaches 1 non-tangentially vanishes for all $U \in \mathcal{U}(n)$. This happens if and only if the angular derivative of $\Theta \hat{x}$ at $z = 1$ does not exist for any $\hat{x} \in \mathbb{C}^{n}$.

**Proof.** This is an immediate consequence of the previous lemma and Proposition 4.2.2. \hfill \Box
5.2. Regular simple symmetric operators with deficiency indices \((n, n)\). Now suppose that \(B\) is a simple symmetric linear transformation on \(\mathcal{H}\) with deficiency indices \((n, n)\). Such a linear transformation is called regular if \(B - z\) is bounded below for all \(z \in \mathbb{C}\). The isometric linear transformation \(V = \mu(B)\) is called the Cayley transform of \(B\). This \(V\) is a simple regular isometric linear transformation with deficiency indices \((n, n)\). Here an isometric linear transformation is called regular if \(V - z\) is bounded below for all \(z \in \mathbb{C} \setminus \{1\}\). The fact that \(V\) is regular, and the results of Section 5.2.2 show that \(\Theta_V\) is inner and analytic on some neighbourhood of any given point \(z \in \mathbb{T} \setminus \{1\}\).

Let \(\Theta := \Theta_V\). Since \(\Theta_V(0) = 0\), \((Z_\Theta)^*\) is unitarily equivalent to \((X_\Theta)'\), the isometric linear transformation which acts as multiplication by \(1/z\) on the orthogonal complement of the \(n\)-dimensional subspace spanned by the vectors \(\{\delta_i^n \mid x \in \mathbb{C}^n\} \subset K_{\Theta}^2\) of point evaluations at zero. Since \(\Theta_V(0) = 0\), these are the constant functions in \(K_{\Theta}^2\). Let \(\Phi = \Theta \circ \mu\), and let \(M\) be the self-adjoint operator of multiplication by the independent variable in \(L_2^n(\mathbb{R})\). Then the image of \((X_\Theta)'\) under the canonical unitary map \(\mathcal{U}\) of \(K_{\Theta}^2\) onto \(K_{\Theta}^2 \subset H_2^2(\mathbb{U})\) is the isometric linear transformation \(\mu^*(M)_\Phi\) which acts as multiplication by \(\mu^*(z) = \mu(\overline{z}) = i \frac{1}{z - i}\) on the domain of all functions in \(K_{\Theta}^2\) which vanish at \(z = i\).

Let \(M_\Phi := (\mu^*)^{-1}(\mu^*(M)_\Phi)\). Then \(M_\Phi\) is a simple symmetric linear transformation which acts as multiplication by the independent variable on its domain \(\text{Dom}(M_\Phi) = \text{Ran}(\mu^*(M)_\Phi - 1)\).

We will say that the inner function \(\Phi_B = \Theta \circ \mu\) is the Lifschitz characteristic function of \(B\). Note that since \(\Theta_V(0) = 0\), \(\Phi_B(i) = 0\). Combining these observations with Lifschitz’ result, Theorem 5.1.2 yields the following:

**Theorem 5.2.1.** A simple symmetric linear transformation \(B\) with deficiency indices \((n, n)\) and characteristic function \(\Phi_B\) is regular if and only if \(\Phi_B \in H_{\infty}^\infty(\mathbb{U})\) is an inner function which has an analytic extension to an open neighbourhood of any fixed \(x \in \mathbb{R}\). In this case \(B\) is unitarily equivalent to \(M_{\Phi_B}\) which acts as multiplication by the independent variable on \(\text{Dom}(M_{\Phi_B}) \subset K_{\Theta}^2\).

5.2.2. **Remark.** The result stated above can be generalized to any simple symmetric linear transformation whose characteristic function \(\Phi_B\) is an extreme point, for in this case \(H_{\Theta}^2 = L_{\Theta}^2\) (where \(\Theta = \Phi \circ \mu^{-1}\)), and the canonical unitary transformations from \(H_{\Theta}^2\) onto \(K_{\Theta}^2\) and \(K_{\Theta}^2\) onto \(K_{\Theta}^2\) take \(\mu^{-1}(Z_\Theta)\) onto \(M_\Phi\).

5.2.3. **Remark.** Theorem 5.1.4 provides necessary and sufficient conditions on \(\Phi\) for \(M_\Phi\) to be a densely defined symmetric operator.

Now suppose that \(\Phi = \Theta \circ \mu\) satisfies the conditions of Theorem 5.1.4 so that \(M_\Phi\) is densely defined, and that \(\text{sp}(\Phi) := \mu^{-1}(\text{sp}(\Theta)) \subset (\infty, \{\infty\}\), so that \(M_\Phi\) is regular. Let \(M_\Phi(U)\) be the image of \(\mu^{-1}(Z_\Theta(U))\) under the canonical unitary transformation of \(H_{\Theta}^2\) onto \(K_{\Theta}^2\). Then the regularity of \(M_\Phi\) implies that the spectrum of each \(M_\Phi(U)\) is purely discrete with no finite accumulation point. Hence if \(\sigma(M_\Phi(U)) = \{\lambda_i(U)\} \subset \mathbb{R}\), it follows that there are vectors \(\beta_i(U) \in \mathbb{C}^n\) such that the point evaluation vectors \(\{\delta_{\lambda_i(U)}\}\) form a total orthogonal set of eigenvectors to \(M_\Phi(U)\) for each \(U \in \mathcal{U}(n)\),

\[
M_\Phi(U) \delta_{\lambda_i(U)} = \lambda_i(U) \delta_{\lambda_i(U)}.
\]

Here the point evaluation vectors in \(K_{\Theta}^2\) have the form

\[
\delta_{\lambda}^u(z) = \frac{i}{2\pi} \frac{1 - \Phi(z) \Phi^*(\lambda)}{z - \lambda} \beta.
\]

Moreover if \(M_\Phi\) is densely defined then each point evaluation vector \(\delta_{\lambda}^u\) is an eigenvector to \(M_\Phi^*\). Indeed, given any \(f \in \text{Dom}(M_\Phi)\),

\[
(M_\Phi f, \delta_{\lambda}^u)_\Phi = \lambda(f(\lambda), \overline{\beta}) = (f, \overline{\lambda} \delta_{\lambda}^u)_\Phi,
\]

which shows that \(\delta_{\lambda}^u \in \text{Dom}(M^*_\Phi)\) and that \(M^*_\Phi \delta_{\lambda}^u = \overline{\lambda} \delta_{\lambda}^u\). Here, \((\cdot, \cdot)_\Phi\) denotes the inner product in \(K_{\Theta}^2\) (which is the usual \(L^2\) inner product since we are assuming \(\Phi\) is inner).
In summary any regular simple symmetric linear transformation $B$ with deficiency indices $(n, n)$ is unitarily equivalent to multiplication by the independent variable, $M_\lambda$ in a model subspace $K_\delta^2 \subset H^2_\mathbb{C}(\mathbb{U})$, where $\Phi = \Phi_B$ is the Liouville characteristic function of $B$. $\Phi \in H^\infty_\mathbb{C}(\mathbb{U})$ is inner, and the fact that $B$ is regular implies that $\Phi$ has an analytic extension to some open neighbourhood of each $x \in \mathbb{R}$. The transformation $B$ is densely defined if and only if $\Theta := \Phi \circ \mu^{-1}$ is such that the angular derivative of $\Theta \bar{x}$ at $z = 1$ does not exist for any $\bar{x} \in \mathbb{C}^n$. In this case $K_\delta^2$ has a $\mathcal{U}(n)$-parameter family of total orthogonal sets of point evaluation vectors $\{\delta^2_{\lambda_i}(U)\}$ which are eigenvectors to self-adjoint extensions $M_\Phi(U)$ of $M_\Phi$ with eigenvalues $\lambda_i(U)$. The spectra $\sigma(M_\Phi(U)) = \{\lambda_i(U)\}$ are purely discrete with no finite accumulation points. Moreover each $\delta^2_{\lambda_i}$ is an eigenvector of $M_\Phi$ to eigenvalue $\lambda_i$.

5.2.4. Remark. The above representation results for densely defined regular simple symmetric linear operators with deficiency indices $(n, n)$ apply in particular to regular symmetric differential operators of any finite order, and to their self-adjoint extensions.

5.3. A model for c.n.u. contractions with defect indices $(n, n)$. In this subsection we show that if $V$ is any partial isometry with finite and equal defect indices $(n, n)$, then $V$ is unitarily equivalent to the partial isometry $Z_{\Theta_V}(0)$ acting in $L^2_{\Theta_V}(\mathbb{T})$. If $V' := V|_{\ker(V)^\perp}$, an isometric linear transformation with deficiency indices $(n, n)$, then as shown in [3] (and reproduced in Theorem 5.1.2 above), $V'$ is unitarily equivalent to $Z_{\Theta_V}' := Z_{\Theta_V}(0)|_{\ker(Z_{\Theta_V}(0))^\perp}$. This establishes that the Liouville characteristic function of any isometric linear transformation $V'$ with indices $(n, n)$ is equal to the Nagy-Foias characteristic function of the partial isometry $V'$ to the entire Hilbert space. While natural, and known in the case where $\Theta_V$ is inner $\mathbb{I}$, for non-inner $\Theta_V$, this is not immediately obvious from the definitions of these two different characteristic functions.

To prove this, recall that $\Theta$, a contractive $\mathbb{M}_n$-valued analytic function on $\mathbb{D}$ is the characteristic function of a partial isometry $V$ if and only if $\Theta(0) = 0$. So to prove $V$ is isomorphic to $Z_{\Theta_V}(0)$ it suffices to show that the characteristic function of $Z_{\Theta_V}(0)$ coincides with $\Theta_V$.

Let $T := Z_{\Theta_V}(0)$, $Z := Z_{\Theta_V}$ and let $P_+, P_-$ be the projectors onto $\mathcal{D}_T$ and $\mathcal{D}_T^\perp$ respectively. Then the Nagy-Foias characteristic function of $T$ is

$$\Theta_T(z) = z P_-(1 - z T^*)^{-1} P_+ = P_- \sum_{m=0}^{\infty} z^{m+1} (T^*)^m P_+ \tag{5.9}$$

We will show that this coincides with $\Theta(z)$ by showing that if $\Theta(z) = \sum_{k=1}^{\infty} c_k z^k$ that $c_k = P_- (Z^{-1} - Z^{-1} P_-)^k Z^{-1} P_-$ $=: d_k$. As in Section 2.1 let $l_k := P_- Z^{-k} P_-$, and let $P = P_-$. Then,

$$d_k = P(Z^{-1} - Z^{-1} P)^{k-2} (Z^{-1} - Z^{-1} P) Z^{-1} P$$
$$= P(Z^{-1} - Z^{-1} P)^{k-3} Z^{-2} P - P(Z^{-1} - Z^{-1} P)^{k-2} P Z^{-1} P$$
$$P(Z^{-1} - Z^{-1} P)^{k-3} (Z^{-1} - Z^{-1} P) Z^{-2} P - d_{k-l_1}$$
$$= P(Z^{-1} - Z^{-1} P)^{k-3} Z^{-2} P - d_{k-2} - d_{k-l_1}$$
$$= P(Z^{-1} - Z^{-1} P)^{k-4} Z^{-2} P - d_{k-l_1}$$
$$P(Z^{-1} - Z^{-1} P)^{k-3} Z^{-2} P - d_{k-l_1}$$
$$= l_k - d_{k-1} - d_{k-1} - d_{k-1} l_1 \tag{5.10}$$
The matrix function

Proposition 5.3.3. With the Nagy-Foias characteristic function of \( \Theta \) (5.13)
using the same methods as in Lemma 2.1.4 it is easy to calculate that

\[ U_d = \sum_{j=1}^c d_j L_j. \]

Finally, using these two formulas and the one relating the \( l_k \) and \( c_k \), one can use a combinatorial identity, as in the proof of Proposition 2.1.2 to show that \( d_j = b_j \). 

5.3.2. Remark. It follows that any completely non-unitary contraction \( T \) with defect indices \((n, n)\) is unitarily equivalent to some extension of the partial isometry \( Z_\Theta(0) \) from Section 2.2 that if \( \Theta(0) = 0 \). The characteristic function \( \Theta(0) \) coincides with the Nagy-Foias characteristic function of \( \Theta \) (5.13). Hence if \( V \) is any partial isometry with finite defect indices \((n, n)\), \( V \) is unitarily equivalent to \( Z_\Theta(0) \).

Proposition 5.3.3. The matrix function \( \Gamma(z) = \Theta(z)(1 - A^*\Theta(z))^{-1} \).

Proof. (Sketch) This proof is very similar to previous calculations in Section 2.1. As before let \( \Theta(z) := \sum_{k=1}^\infty c_k z^k \), and let \( \Gamma(z) := \sum_{k=1}^\infty d_k z^k \).

(5.12) \[ \Gamma(z) = \sum_{m=0}^{\infty} z^{m+1} P_-(Z_\Theta(A^*)^m Z^{-1} P_-. \]

Let \( b_k, k \in \mathbb{N} \) be the coefficients of \( \Theta(z)(1 - A^*\Theta(z))^{-1} \) and \( d_k \) be the coefficients of \( \Gamma(z) \). Now using the same methods as in Lemma 2.1.4 it is easy to calculate that

(5.13) \[ b_m = l_m + \sum_{j=1}^{m-1} l_j (A^* - 1)b_{m-j}. \]

By the definition of the \( d_j \) and Elliott’s formula, Proposition 2.1.1 one can show, as in the proof of Proposition 2.1.2 that

(5.14) \[ d_j = c_j + \sum_{k=1}^{j-1} c_k A^* d_{j-k}. \]

Finally, using these two formulas and the one relating the \( l_k \) and \( c_k \), one can use a combinatorial identity, as in the proof of Proposition 2.1.2 to show that \( d_j = b_j \). 

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