Classical Chern-Simons on manifolds with spin structure

Jerome A. Jenquin

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Abstract

We construct a 2+1 dimensional classical gauge theory on manifolds with spin structure whose action is a refinement of the Atiyah-Patodi-Singer $\eta$-invariant for twisted Dirac operators. We investigate the properties of the Lagrangian field theory for closed, spun 3-manifolds and compact, spun 3-manifolds with boundary where the action is interpreted as a unitary element of a Pfaffian line of twisted Dirac operators. We then investigate the properties of the Hamiltonian field theory over 3-manifolds of the form $\mathbb{R} \times Y$, where $Y$ is a closed, spun 2-manifold. From the action we derive a unitary line bundle with connection over the moduli stack of flat gauge fields on $Y$.

By now, Chern-Simons on oriented 3-manifolds is well studied from various points of view. On the one hand, Witten shows that the quantum theory provides an example of a Topological Quantum Field Theory, or TQFT [23]. Briefly, this means that quantum Chern-Simons associates a complex number to a closed, oriented 3-manifold; a complex vector space to a closed, oriented 2-manifold; and algorithms for how to decompose these associations when cutting along codimension one submanifolds. (See [5] for a detailed exposition). His arguments, however, pivot around the Feynman Integral, which is not mathematically well-defined. Nevertheless, based on these physical arguments, he is able to show that the 3-manifold invariants of this physical TQFT correspond to certain knot invariants.

On the other hand, mathematicians (especially the authors of [8]) construct a TQFT out of Frobenius algebras that they derive from the same knot invariants. Although there is no map between the physical and mathematical TQFTs, there is substantial evidence that the two are “isomorphic”. This evidence appears at all dimensions of the theory. In certain cases, the two TQFTs have been shown to generate the same 3-manifold invariants [15]. The two TQFTs associate vector spaces of equal dimension to a given 2-manifold. And finally, the two TQFTs associate isomorphic algebras to the circle.

We see then that Chern-Simons provides a correspondence between a quantized field theory and a purely algebro-categorical construction. Historically, Atiyah deduced the axioms of the latter by abstracting certain properties of
the Feynman Integral. But soon after, the algebro-categorical TQFTs outpaced
the field theoretic TQFTs. Indeed, mathematicians soon discovered knot in-
variants corresponding to spun 3-manifolds [21], [2]. Out of these Blanchet
and Mausbaum construct a new TQFT by applying the same categorical techniques
developed in [3]. In this new TQFT one finds that the invariants, vector spaces,
and algebras all depend on the spin structures of the 3-manifolds, 2-manifolds,
and 1-manifolds, respectively. Blanchet and Mausbaum dub this new construc-
tion a “spin-TQFT”.

Chern-Simons provides a correspondence between the the original “unspun”
TQFTs and a quantized physical theory. In light of the construction of Blanchet
and Mausbaum one is naturally lead to ask, “Where is the physical correspon-
dence to the spin-TQFT?” In this paper we put forth an answer of sorts.

Chern-Simons is a field theory. In fact, its a gauge theory on oriented 3-
manifolds. Only after quantizing do physicists obtain a TQFT. What we require
is a new field theory on 3-manifolds; one that incorporates spin structure in a
(preferrably) natural way. In this paper we construct just such a field theory
and study its classical aspects. We dub this new theory “spin-Chern-Simons”.
The careful analysis of classical spin-Chern-Simons in this paper will lead to an
equally careful treatment of the quantum theory in the paper [19]. Thus, in this
paper we propose the “answer” to the question above; while in [19] we provide
evidence that this does indeed provide the missing correspondence.

While the “missing physical correspondence” question provides a lot of mo-
tivation for studying spin-Chern-Simons and especially quantum spin-Chern-
Simons, the classical theory is interesting in its own right. Its a field theory
that incorporates aspects of classical Chern-Simons with geometric index theory,
a rather novel feature. The action is a spectral invariant associated to
operators of Dirac-type; and the prequantum line bundle and its geometry is
Quillen’s determinant line with the connection constructed by Bismut and Freed
[6]. It is the author’s hope that this approach will one day lead to a solid phys-
ical argument for the work of Freed, Hopkins, and Teleman [10]. This would
require an understanding of classical spin-Chern-Simons over 1-manifolds: at
present a work in progress. For now, we content ourselves with a study of the
Lagrangian theory over spun 3-manifolds and the Hamiltonian theory over spun
2-manifolds. Let us summarize our results.

We construct a classical gauge theory on a spin manifold $M$, where $M$ is a
compact 3-manifold with spin structure $(X, \Sigma)$ for the Lagrangian theory or a
compact 2-manifold with spin structure $(Y, \sigma)$ for the Hamiltonian theory. The
sigmas label the spin structure. As with most gauge theories, we require some
initial data: a compact Lie group $G$ and a real, rank zero virtual representation,
$\rho$, that we call (for now) the “level”.

The classical fields are the category of $G$-bundles with connection over $M$.
That is, the objects are pairs $(P, A)$ where $P \to M$ is a principal $G$-bundle
and $A$ is a connection on $P$; and the morphisms – the classical symmetries
– are $G$-bundle morphisms which cover the identity map of $M$. So far, this
description also applies to standard Chern-Simons. We now come to the aspects
of spin-Chern-Simons that differentiate it from the standard theory.
By choosing a metric on $M$ one naturally has a Dirac operator. Given any real representation of $G$ one creates a Dirac-like operator by coupling a $G$-connection to the Dirac operator via the representation. On a 3-manifold with spin structure this operator is quaternionic and self-adjoint while on a 2-manifold with spin structure it is complex skew-symmetric.

Over $(X, \Sigma)$ we have a Lagrangian field theory. To construct the action, we take any two real representations, $\rho_1$ and $\rho_2$, such that $\rho_2 - \rho_1 = \rho$. A $G$-connection, $A$, induces two Dirac-like operators, $D_1(A)$ and $D_2(A)$, via $\rho_1$ and $\rho_2$ respectively. Evaluated at $A$, the classical action is

$$\frac{\xi}{2}(D_2(A)) - \frac{\xi}{2}(D_1(A)) \pmod{1}$$

where $\xi$ is the spectral invariant that appears in the Atiyah-Patodi-Singer index theorem [2].

In general $\xi \pmod{1}$ is smooth with respect to smooth parameters. Here $\xi/2 \pmod{1}$ is smooth because the operators are quaternionic. Also, note that the action depends only on the difference $\rho_2 - \rho_1 = \rho$. The APS index theorem tells us that because $\rho$ is rank zero, the action is independent of the metric and that the critical points of the action are exactly the flat $G$-connections. Lastly, the action is invariant with respect to $G$-bundle morphisms, so that it is a well defined function on the moduli space of $G$-connections.

The action depends on the spin structure $\Sigma$. Using the Atiyah-Patodi-Singer flat index theorem [3], we track this dependence to obtain some useful results. One result allows us to prove, using cobordism arguments, that the action only depends on $\rho$ up to the element it generates in $E^4(BG)$. Here $E^*(\cdot)$ is a generalized cohomology theory generated by a spectrum, each element of which is a twisted product of two Eilenberg-MacLane spaces. In fact, the degree-four element of the spectrum is the bottom of the Postnikov tower for $BSO$. For example, $E^4(BSO_3) \cong \mathbb{Z}$ and $E^4(BSU_2) \cong H^4(BSU_2) \cong \mathbb{Z}$ so that for these groups the spin-Chern-Simons theory has integer-valued levels.

For the Hamiltonian field theory we must consider $G$-connections over $(Y, \sigma)$. Given a real representation $\rho_0$, the $G$-connections over $Y$ provide a family of (complex skew-symmetric) Dirac-like operators. Over this family we can take the Pfaffian line bundle $\mathcal{L}_{Y}^{\rho_0}$ which has a natural unitary structure and connection [12]. In fact, we consider the Pfaffian line of an elliptic operator as a graded line with the grading given by the mod 2 index of the operator. One of the motivations for considering the grading is revealed when we discuss the quantum theory in [19].

If the level is represented by $\rho = \rho_2 - \rho_1$ then the (graded) line bundle we consider is $\mathcal{L}_{Y}^{\rho_1} \otimes (\mathcal{L}_{Y}^{\rho_2})^*$. $G$-bundle morphisms naturally lift to a unitary, connection-preserving action on the Pfaffian bundle so that one gets a unitary line bundle with connection on the moduli stack of flat $G$-connections $\mathcal{M}_G(Y)$. This data – a unitary line bundle with connection over the classical phase space – is required when we quantize the Hamiltonian field theory.

If we allow $X$ to have a boundary with $\partial(X, \Sigma) = (Y, \sigma)$ then by imposing boundary conditions analogous to those in [11] the action can still be defined.
In fact, we consider the exponentiation of the action given by

\[ \tau_X^{1/2}(A) = \frac{\exp \pi i \xi(D_2(A))}{\exp \pi i \xi(D_1(A))} \]

and we know that \( \tau_X^{1/2}(A) \)'s dependence on the boundary conditions naturally identifies it with a unitary element of the line \( \mathcal{L}_Y^\rho \otimes (\mathcal{L}_Y^\rho)^*|_A \).

If, more generally, \((Y, \sigma) \subset (X, \Sigma)\), then we also have a gluing law that tells us how to factorise \( \tau_X^{1/2}(A) \) into information on \( \mathcal{L}_Y^\rho \otimes (\mathcal{L}_Y^\rho)^*|_A \). We point out that the notation \( \tau_X^{1/2} \) for the exponentiated action is chosen to be consistent with the notation in [11].

The outline of this paper is as follows.

In Section 1 we define the action of the theory for \( G \)-connections over closed spin 3-manifolds and define the Lagrangian field theory. We show that the theory is gauge-invariant and independent of the metric. We compute the Euler-Lagrange equations to see that the classical solutions are (gauge group orbits of) flat \( G \)-connections. We also track the action’s dependence on the spin structure. Finally we show that the “levels” of the theory are actually elements of a certain generalized cohomology theory of the classifying space \( BG \).

In Section 2 we define Lagrangian field theory for \( G \)-connections over compact spin 3-manifolds with boundary. What we find is that, when the boundary is non-empty, the action is properly considered to be an element of the Pfaffian line for the twisted Dirac operator of the boundary. To a family of \( G \)-connections the action assigns a section of the Pfaffian line bundle. With respect to the natural covariant derivative of that bundle we determine that the section is independent of the metric and derive the same Euler-Lagrange equations that we derive in Section 1. Finally we see that the action obeys a “gluing law”.

In Section 3 we consider the Hamiltonian field theory over spin 3-manifolds that are isometric to a spin 2-manifold crossed with an interval. The Euler-Lagrange equations reveal that the space of classical solutions is equivalent to the moduli stack of flat \( G \)-connections over the 2-manifold. The action determines a (Pfaffian) line bundle over the moduli stack and determines the symplectic structure as well. We determine that the subset of smooth points of the moduli stack is the symplectic reduction of the space of \( G \)-connections with respect to the gauge group action. Finally we point out a functorial relationship between the classical Lagrangian and Hamiltonian field theories.

We include an appendix for computations that are necessary but, we feel, lie outside the narrative flow of the paper’s body.

Finally, the author warmly thanks his thesis advisor, Dan Freed, for introducing him to this project and for all of his help during its completion.
The Classical Lagrangian Theory

1.1 A review of ζ-invariants and the APS-index theorem

In this section we review some salient points from index theory for manifolds with boundary as worked out in [2, 3, 4]. For the most part – though much of this discussion applies more generally – we focus on 4-manifolds with boundary and closed 3-manifolds.

Let \( M \) be an oriented Riemannian 4-manifold with boundary \( \partial M \) and assume that \( M \) is spin; that is, assume \( M \) has given a spin structure. Let \( S^\pm \rightarrow M \) denote the chiral spinor bundles and consider the chiral Dirac operator

\[
D_M : C^\infty(S^+) \rightarrow C^\infty(S^-)
\]

between smooth sections of \( S^+ \) and \( S^- \). If \( \partial M = \emptyset \) then the local form of the Atiyah-Singer Index Theorem tells us that

\[
\text{index}(D_M) = \int_M \left[ \hat{A}(\Omega^M) \right]_{(4)}
\]

where \( \Omega^M \) is the Riemannian curvature two-form and \( \hat{A} \) is the usual \( \hat{A} \)-polynomial so that

\[
\hat{A}(\Omega) = \sqrt{\det \left( \frac{\Omega/4\pi}{\sinh \Omega/4\pi} \right)}.
\]

The notation \([\cdot]_{(4)}\) is meant to imply the taking the degree-4 component of the differential form within.

In the case where \( \partial M \) is not empty, Clifford multiplication by the (inward pointing) normal vector provides an identification \( S^+|_{\partial M} = S^-|_{\partial M} \). We can identify either of these with the spinor bundle \( S \) of \( \partial M \). To obtain something akin to the local expression in (1.1) the authors of [2] establish global boundary conditions that take into account the Dirac operator on the boundary

\[
D_{\partial M} : C^\infty(S) \rightarrow C^\infty(S).
\]

If we restrict the Dirac operator on \( M \) to the subspace of spinor sections which satisfy these global boundary conditions and the geometric data is isomorphic to \([0,1) \times \partial M \) near the boundary, then the Atiyah-Patodi-Singer (APS) Index Theorem offers the expression:

\[
\text{index}(D_M) = \int_M \hat{A}(\Omega^M) - \xi(D_{\partial M})
\]

(1.2)

where \( \xi(D_{\partial M}) \) depends only on the geometric data of the boundary. We take a moment now to discuss this boundary term as it is the key player in this paper.

The Dirac operator \( D \) on any odd-dimensional manifold (such as the operator \( D_{\partial M} \)) is self-adjoint and elliptic. Its spectrum is real and discrete so that we may define

\[
\eta_D(s) = \sum_{\lambda \neq 0} \frac{\text{sign} \lambda}{|\lambda|^s}, \quad Re(s) \gg 0,
\]
where the sum ranges over the nonzero spectrum of $D$. One may think of this as a $\zeta$-function regularization of the spectral asymmetry

$$\# \text{ of positive eigenvalues of } D - \# \text{ of negative eigenvalues of } D.$$ 

It is well established that $\eta_D(s)$ converges if $s > \dim \partial M/2$ and is analytic in $s$; and it has a meromorphic continuation to $s \in \mathbb{C}$ that is regular at $s = 0$. The value $\eta_D(0)$ is the $\eta$-invariant of $D$.

What actually appears in the APS-index theorem – as the reader has already seen – is the $\xi$-invariant:

$$\xi(D) = \frac{\eta_D(0) + \dim \ker D}{2}$$

Under a smooth change of parameters $\xi$ will experience, at worst, integer jumps so that $\xi \pmod{1}$ is smooth.

All of this also carries over for operators of Dirac-type; that is Dirac operators twisted by a unitary connection on some vector bundle $E \to M$. All that changes is the integrand that appears in (1.1) and (1.2), which becomes the differential form

$$\hat{A}(\Omega^M)ch(\Omega^E).$$

where $\Omega^E$ is the curvature two-form of the connection on $E$ and $ch$ is the Chern character polynomial so that

$$ch(\Omega) = \text{Tr} \exp(i\Omega/2\pi).$$

We now take into consideration the dimensions of $M$ and $\partial M$. From the representation theory of $Spin_4$ and $Spin_3$ one has that the bundles $S^\pm$ and $S$ all have spin-equivariant quaternionic structures $J^\pm$ and $J$ (respectively). These induce bounded operators between the $L^2$-sections of the corresponding spinor bundles, which we also denote by $J^\pm$ and $J$. The Dirac operators are quaternionic by which we mean

$$\bar{D}_{\partial M} \circ J = J \circ D_{\partial M} \quad \text{and} \quad J^- \circ D_M = \bar{D}_M \circ J^+.$$ 

There are two upshots to this extra structure: First, the kernel and cokernel of $D_M$ are each even dimensional so that index $(D_M)$ is divisible by two. Second, the eigenvalues of $D_{\partial M}$ (including zero) occur with even multiplicity so that, under smooth changes in the geometric parameters, $\xi$ will, at worst experience even valued jumps. Thus $\xi/2 \pmod{1}$ is a smooth function of the geometric parameters. Of course, this discussion more generally applies to manifolds of dimensions 4 and 3 (mod 8) due to the Bott periodicity of the real Clifford algebras.

To incorporate this refinement with operators of Dirac type we must impose some restrictions on the twisting bundle $E \to M$ and its connection. The discussion above hinged upon the existence of quaternionic structures on $S^\pm$ and $S$ and the compatibility of the respective Dirac operators with these structures.
In order for the bundles $S^\pm \otimes CE$ and $S \otimes CE|\partial M$ to be quaternionic, $E$ must be real. In order for the twisted Dirac operators to be compatible, the connection on $E$ must be orthogonal. In these cases $\xi/2 \pmod{1}$ is a smooth function of the geometric and twisting parameters.

So far we have only considered the $\xi$-invariant in the context of manifolds with boundary, but this need not always be the case. We may simply look at a closed, compact Riemannian 3-manifold with spin structure and consider the $\xi$-invariant of its Dirac operator which is clearly independent of any 4-manifold that the 3-manifold might bound. We do not, however, wish to altogether abandon $\xi$’s role as a boundary term in the APS-index theorem. It is particularly useful when taking the derivative of $\xi$ with respect to (smoothly) changing geometric data. To explain this we must consider families of Dirac operators. We follow the exposition given in [13].

A family of Riemannian 3-manifolds is defined as a fiber bundle $\pi: X \to Z$ with the following structure. First, the fibers are each diffeomorphic to some given compact 3-manifold. Second, there is a metric structure on the relative tangent bundle $T(X/Z)$. Third, there is a projection $P: TX \to T(X/Z)$, the kernel of which is a horizontal distribution on $X$. We assume that $T(X/Z)$ has an orientation and spin structure. To each $z \in Z$ we assign the Dirac operator $D_z$ of the fiber $\pi^{-1}(z)$ and this is our family of Dirac operators parametrized by $Z$.

To define a family of twisted Dirac operators parametrized by $Z$ we further require a vector bundle with connection $(E, \nabla) \to X$. Since $\nabla$ restricts to a connection $\nabla^z$ over the fiber $\pi^{-1}(z)$ we can assign to $z$ the twisted Dirac operator $D^z_{E_z}$, In the context of families $\xi \pmod{1}$ gives us a smooth function $z \mapsto \xi(D^z_{E_z}) \pmod{1}$ on $Z$.

The APS-index theorem can be used to track changes in $\xi \pmod{1}$ along any smooth path in $Z$. In the untwisted case one sees that, for infinitesimal changes,

$$d\xi = \left[ \int_{X/Z} \hat{A}(\Omega^{X/Z}) \right] \pmod{1},$$

where $\Omega^{X/Z}$ is the curvature two form of the relative Levi-Civita connection. The notation is meant to imply that we are integrating the form $\hat{A}(\Omega^{X/Z})$ over the fibers and considering only the degree-1 component on $Z$. There are, of course, corresponding expressions for the $\xi/2$-invariant and families of twisted Dirac operators given by

$$\frac{d\xi}{2} = \frac{1}{2} \left[ \int_{X/Z} \hat{A}(\Omega^{X/Z}) ch(\Omega^E) \right] \pmod{1} \quad (1.3)$$

Before we end this section we would like to point out how the $\xi$-invariants of twisted Dirac operators behave under direct sums of the twisting data. Let $(E_1, \nabla^{E_1})$ and $(E_2, \nabla^{E_2})$ be two vector bundles with unitary connections and let $D_E$ denote the Dirac operator twisted by $(E, \nabla^E)$. Then, as is easy to see
from its definition,
\[ \xi(D_{E_1 \oplus E_2}) = \xi(D_{E_1}) + \xi(D_{E_2}) \] (1.4)
so that \( \xi \) behaves additively under direct sums. This motivates an obvious definition for the \( \xi \)-invariant of a Dirac operator twisted by virtual vector bundles with connection. The “virtual” here just means formal differences. Thus, for the virtual vector bundle with connection \((E_1, \nabla^{E_1}) - (E_2, \nabla^{E_2})\), we define
\[ \xi(D_{E_1 - E_2}) = \xi(D_{E_1}) - \xi(D_{E_2}) \] (1.5)
Of course, this also applies to the refined \( \xi/2 \)-invariant and virtual real vector bundles with connection.

1.2 The data of the classical gauge theory

In this section we will describe the data that goes into defining our classical field theory.

To begin, fix a closed, compact, Riemannian 3-manifold \( X \) with spin structure. Let \( D \) denote the corresponding Dirac operator. It is well known that any oriented 3-manifold admits a spin structure so that this imposes no constraints on \( X \). Now we define our classical gauge theory over \( X \).

Any gauge theory requires some structural data. Our gauge theory requires a compact Lie group \( G \) and a virtual real representation \( \rho = \rho_2 - \rho_1 \) of rank zero. For now, we call \( \rho \) the “level” of our theory.

Given this structural data we take our “fields” to be objects in the category \( \mathcal{C}_G(X) \) of principal \( G \)-bundles with connection over \( X \). Throughout we refer to the objects in this category as \( G \)-connections on \( X \) or just connections when \( G \) or \( X \) is understood. The “symmetry” group \( G_G(X) \) of this theory is taken to be \( G \)-bundle morphisms which cover the identity of \( X \). That is, we consider any two connections to be physically equivalent when there is a morphism (covering \( id_X \) ) that maps one to the other.

Before we define the action of our field theory we say a few words about associated vector bundles and establish some notation. This will facilitate the definition of our action and much of what follows. Given a pair \( (P, A) \) – where \( P \to X \) is a \( G \)-bundle and \( A \) is a connection on \( P \) – the associated bundle construction assigns to any representation \( \rho' : G \to GL(V) \) a vector bundle with connection
\[ (\rho' P, \rho'A) = (P \times_{\rho'} V, d + \rho'(A)). \]
Here \( d \) is the exterior derivative on \( P \) and \( \rho'(A) \) is the \( gl(V) \)-valued one form obtained from \( A \) and the Lie algebra homomorphism induced by \( \rho' : g \to gl(V) \). The connection \( d + \rho'(A) \) acts on equivariant maps \( P \to V \) which are, afterall, the sections of the associated bundle \( \rho'P \). Likewise, this construction assigns to any virtual representation (like the level \( \rho \) above) a virtual vector bundle with connection, the formal difference of two representations becoming the formal difference of the two corresponding vector bundles with connection.
That being said, we define our action to be the map

$$C_G(X) \longrightarrow \mathbb{R}/\mathbb{Z}$$

$$(P, A) \longrightarrow \xi/2(D_{\rho A}) \pmod{1}$$

where $D_{\rho A}$ is the Dirac operator on $X$ twisted by the virtual vector bundle with connection $(\rho P, \rho A)$.

Let $\mathbb{T}$ denote the elements of $\mathbb{C}$ with modulus 1. As our action takes values in $\mathbb{R}/\mathbb{Z}$, it is perhaps just as natural (if not more natural as we shall see later) to take the action to be the $\mathbb{T}$-valued exponential of the $\xi/2$-invariant. In [11] Dai and Freed analyze the exponentiated $\xi$-invariant $\tau_X(D) = \exp 2\pi i \xi(D)$ and following their notation we define

$$\tau_X^{1/2}(D) = \exp 2\pi i \xi(D)/2$$

the square root being well defined for the reasons given in the previous section. From here on out our action will be the $\mathbb{T}$-valued map

$$C_G(X) \longrightarrow \mathbb{T}$$

$$(P, A) \longrightarrow \tau_X^{1/2}(D_{\rho A}).$$

Though, if the 3-manifold $X$ is understood or peripheral, it is sometimes dropped from the notation.

Given the structure and the action described above we call this classical field theory “spin-Chern-Simons” at level $\rho$. This name is purposely chosen to invoke the memory of classical Chern-Simons theory, where the action is defined to be the Chern-Simons 3-form associated to a $G$-connection [10], [12].

1.3 Functoriality of $\tau^{1/2}$

In this section we enumerate some of the features of our action.

The structure of this next proposition emulates Proposition 2.7 of [12].

**Proposition 1.1.** Let $X$ be a closed oriented 3-manifold with Riemannian metric $g$ and spin structure $\sigma$. Then the spin-Chern-Simons action

$$\tau_X^{1/2} : C_G(X) \longrightarrow \mathbb{T}$$

satisfies the following properties:

1. (Functoriality) If $\varphi : P' \rightarrow P$ and $F : S_{g',\sigma'} \rightarrow S_{g,\sigma}$ are any $G$-bundle and spinor bundle morphisms (respectively) covering an orientation and spin structure preserving isometry $f : (X',g') \rightarrow (X,g)$, and $A$ is a connection on $P$, then

$$\tau_{X'}^{1/2}(D_{\rho'(\varphi^* A)}) = \tau_X^{1/2}(D_{\rho A})$$

where $D'$ is the Dirac operator on $X'$. 

2. (Orientation) Let \(-X\) denote \(X\) with the opposite orientation. Then
\[
\tau_{\pm}^{1/2}(D_\rho A) = \tau_{\mp}^{1/2}(D_\rho A).
\]

3. (“Additivity”) If \(X = X_1 \sqcup X_2\) is a disjoint union, and \(A_j\) are connections over \(X_j\), then
\[
\tau_{X_1 \sqcup X_2}^{1/2}(A_1 \sqcup A_2) = \tau_{X_1}^{1/2}(A_1) \cdot \tau_{X_2}^{1/2}(A_2).
\]

It follows from (1) that the action is invariant under the symmetry group \(G_G(X)\) of the theory. Or what is equal, there is an induced action
\[
\tau_X^{1/2} : \mathcal{C}_{G}(X) \to \mathbb{T}
\]
where \(\mathcal{C}_{G}(X)\) is the set of equivalence classes.

**Proof.** To prove (1) we point out that
\[
D'_{\rho(\varphi^* A)} = \Phi_{F,\varphi}^{-1} \circ D_\rho A \circ \Phi_{F,\varphi}
\]
where \(\Phi_{F,\varphi}\) denotes the induced unitary map on sections given by
\[
s' \mapsto (F \otimes \rho \varphi) \circ s' \circ f^{-1}.
\]
Thus \(D'_{\rho(\varphi^* A)}\) and \(D_\rho A\) have the same eigenvalues and eigenvalue degeneracies so that their respective \(\tau^{1/2}\)-invariants are equal.

To prove (2) we point out that on oriented spin 3-manifolds – and oriented odd dimensional spin manifolds in general – the orthonormal volume form provides a covariantly constant, spin-equivariant isomorphism between spinor bundles \(\omega : S_X \to S_{-X}\). We denote the induced unitary map on sections by \(\omega\) as well. A standard argument shows that \(D_{-X} \circ \omega = -\omega \circ D_X\). Thus, if \(\lambda\) is an eigenvalue of \(D_X\), \(-\lambda\) will be an eigenvalue of \(D_{-X}\).

The proof for (3) is easy and we leave it to the reader.  

The \(\tau^{1/2}\)-invariant also behaves well under extension of the structure group. We explain what we mean. An inclusion of Lie groups \(i : G \hookrightarrow G'\) induces an inclusion of principal bundles \(i_P : P \hookrightarrow P'\), where \(P\) is any \(G\)-bundle and \(P' = P \times_i G'\). If \(A\) is a connection on \(P\) then there is a natural extension to a connection \(A'\) on \(P'\) determined by \(i(A) = i_P(A')\) (cf. [20]). This is the situation we consider in the next proposition.

**Proposition 1.2.** Let \(\rho\) be a virtual representation of \(G'\). For any \(G\) connection \(A\) over \(X\), its extension \(A'\) to a \(G'\) connection satisfies \(\tau^{1/2}(D_{(\rho \circ i)A}) = \tau^{1/2}(D_{\rho A'})\).  

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Another way to express Proposition 1.2 is to say that the extension functor \( i_X : (P, A) \mapsto (P', A') \) fits into the following commutative diagram:

\[
\begin{array}{ccc}
C_G(X) & \xrightarrow{\iota_X} & C_{G'}(X) \\
\tau^{1/2} \downarrow & & \downarrow \tau^{1/2} \\
T & \xrightarrow{id_T} & T
\end{array}
\]

Proof. There is a natural isomorphism between the associated virtual vector bundles \( \rho P' \rightarrow (\rho \circ i)P \) which sends the associated virtual connection \( \rho A' \) to \( (\rho \circ i)A \). Thus the twisted Dirac operators \( D_{\rho A'} \) and \( D_{(\rho \circ i)A} \) will have the same eigenvalues and eigenvalue degeneracies and so the same \( \tau^{1/2} \)-invariants.

1.4 Dependence of \( \tau^{1/2} \) on smooth parameters

To facilitate the discussion which follows we establish some conventions and notation. Any real representation \( \rho' \) of a compact lie group \( G \) generates an Ad-invariant bilinear form on the lie algebra \( g \):

\[
\langle \eta_1, \eta_2 \rangle_{\rho'} = -\frac{1}{8\pi^2} \text{Tr}(\rho'(\eta_1)\rho'(\eta_2)) \quad (1.6)
\]

Remark 1.3. It is easy to check that if \( [\eta_1, \eta_2] = 0 \) and \( \exp(\eta_1) = \exp(\eta_2) = 1_G \) then \( \langle \eta_1, \eta_2 \rangle_{\rho'} \) is \( \mathbb{Z} \)-valued. This provides some motivation for the normalization. Another motivation comes from Chern-Weil theory. If \( P \) is a principal \( G \)-bundle and \( \rho'P \) is the associated vector bundle then the first Pontryagin class \( p_1(\rho'P) \) can be represented in de Rham cohomology by the 4-form

\[
\langle \Omega^A \wedge \Omega^A \rangle_{\rho'},
\]

for any connection \( A \) on \( P \). Notice that if \( \rho = \rho_2 \oplus \rho_1 \) then

\[
\langle \eta_1, \eta_2 \rangle_{\rho} = \langle \eta_1, \eta_2 \rangle_{\rho_2} + \langle \eta_1, \eta_2 \rangle_{\rho_1}
\]

For this reason, if we consider a virtual representation \( \rho = \rho_2 - \rho_1 \) we define its Ad-invariant bilinear form \( \langle \cdot, \cdot \rangle_{\rho} \) by

\[
\langle \eta_1, \eta_2 \rangle_{\rho} = \langle \eta_1, \eta_2 \rangle_{\rho_2} - \langle \eta_1, \eta_2 \rangle_{\rho_1}
\]

As it is defined the action seems to depend on the Riemannian structure on \( X \). But, in fact we have the following:

Proposition 1.4. If \( \rho \) is a real rank zero virtual representation then \( \tau^{1/2}(D_{\rho A}) \) is independent of the metric.

Proof. If \( \text{Met}(X) \) denotes the (convex) space of metrics on \( X \) then we take the obvious family \( \text{Met}(X) \times X \to \text{Met}(X) \). We fix the pair \( (P, A) \) and compute the differential over \( \text{Met}(X) \) using the twisted version of (1.3):

\[
d\tau^{1/2} = \tau^{1/2} \left[ \pi i \int_X \hat{A}(\Omega) \text{ch}(\rho \Omega^A) \right]_{(1)}.
\]
where (for typographical reasons) the script-less \( \Omega \) is standing in for the Riemannian curvature of the relative tangent bundle \( T((\text{Met}(X) \times X) / \text{Met}(X)) \).

Unraveling the polynomials \( \hat{A} \) and \( ch \) we get a more explicit expression for the integrand

\[
\hat{A}(\Omega)ch(\rho\Omega^A) = \text{rank}(\rho)\hat{A}(\Omega)_{(4)} + \langle \Omega^A \wedge \Omega^A \rangle_{\rho} + \text{higher degree terms}.
\]

Since \( \rho \) is rank zero the first term goes away. The second term – since \( A \) is fixed – is the pullback (by the vertical projection \( \text{Met}(X) \times X \to X \)) of a 4-form on a 3-manifold and is thus identically zero.

Thus the Lagrangian field theory is independent of the metric on \( X \). We would now like to see how \( \tau^{1/2} \) behaves with respect to infinitesimal changes in the connection (the only remaining smooth parameter in our theory). Let \( \mathcal{C}(P) \) denote the affine space of connections on a principal \( G \)-bundle \( P \to X \). Any smooth variation of connections will occur within such a space.

**Proposition 1.5.** Let \( A_t \) be a path in \( \mathcal{C}(P) \) and let \( \Omega_t \) denote the curvature of the connection \( A_t \), \( \dot{A}_t \) the tangent vector along the path. Then

\[
\frac{d}{dt}\tau^{1/2}(A_t) = 2\pi i \cdot dt \cdot \tau^{1/2}(A_t) \int_X \left( \Omega_t \wedge \dot{A}_t \right).
\]

**Proof.** The connections \( A_t \) define a family of twisted Dirac operators parametrized by \([0,1]\). Or what is the same, they give us a single connection on the \( G \)-bundle \( P \times [0,1] \). The curvature of this single connection is \( dt \wedge \dot{A}_t + \Omega_t \). As before, we use the twisted version of (1.3) to compute

\[
\frac{d}{dt}\tau^{1/2}(A_t) = \tau^{1/2}(A_t) \left[ \pi i \int_X \langle (dt \wedge \dot{A}_t + \Omega_t) \wedge (dt \wedge \dot{A}_t + \Omega_t) \rangle \right]_{(1)}
\]

\[
= \pi i \cdot dt \cdot \tau^{1/2}(A_t) \int_X 2\langle \dot{A}_t \wedge \Omega_t \rangle
\]

proving the proposition.

**Remark 1.6.** Assuming \( \langle \cdot, \cdot \rangle_{\rho} \) is nondegenerate, this proposition implies that \( d\tau^{1/2} \bigl|_{A} = 0 \) if and only if \( \Omega^A = 0 \); that is, if and only if \( A \) is flat. This is the content of the Euler-Lagrange equation

\[
\Omega^A = 0 \quad (1.7)
\]

which is first order. Since \( \tau^{1/2}_X \) is invariant with respect to gauge transformations \( \mathcal{G}_G(X) \), so too is the space of solutions to the Euler-Lagrange equation. This is also obvious from (1.7). We let

\[
\mathcal{M}_G(X) \subset \mathcal{C}_G(X)
\]

denote the space of equivalence classes of solutions to (1.7). We will have more to say about \( \mathcal{M}_G(X) \) when we discuss the Hamiltonian field theory in Section 1.3.
1.5 Dependence of $\tau^{1/2}$ on the spin structure

We now track the dependence of the action on the spin structure $\sigma$ assigned to $X$. This piece of the data is discrete. In fact, if $\text{spin}(X)$ denotes the set of equivalence classes of spin structures on $X$, then $\text{spin}(X)$ is affine over the vector space $H^1(X; \mathbb{Z}/2\mathbb{Z})$. This vector space is in one-to-one correspondence with equivalence classes of flat, orthogonal line bundles on $X$ so that, throughout the rest of the paper, we often identify a flat line bundle with its first Stiefel-Whitney class.

If $\ell$ is a flat line bundle and $S_\sigma$ is the spinor bundle associated to the spin structure $\sigma$, then recall that we can identify $S_\sigma \otimes \ell$ with $S_\sigma \otimes \rho$ – we assume the metric is fixed. The Dirac operator associated to $\sigma \otimes \ell$ is therefore the Dirac operator associated to $\sigma$ twisted by $\ell$.

To track the dependence of the action on $\sigma$ we fix the metric and the pair $(P,A)$. From the discussion in the previous paragraph we would like to compute the ratio

$$q_\sigma(P,\ell) = \frac{\tau^{1/2}(D_\ell \otimes \rho)}{\tau^{1/2}(D_\rho)} = \frac{\tau^{1/2}(D_{(\ell-1) \otimes \rho})}{\tau^{1/2}(D_{(\ell-1) \otimes \rho})}$$

effectively comparing the $\tau^{1/2}$-invariants for the twisted Dirac operators on $S_\sigma \otimes \rho$ and $S_\sigma \otimes \ell \otimes \rho$.

We point out that, since $\Omega^{\rho A \otimes \ell} = \Omega^{\rho A}$, Proposition 1.5 implies that $q_\sigma$ is independent of the connection $A$. Along with that Proposition 1.4 implies that $q_\sigma$ depends only on the topological type of $P$ as $\tau^{1/2}$ is invariant under bundle morphisms. Furthermore, (1.4) and (1.5) imply that $q_\sigma$ depends only on the element of $KO(X)$ represented by the virtual vector bundle $\rho P$. Thus $q_\sigma$ depends only on discrete topological parameters. Indeed, it has an entirely KO-theoretic interpretation given by the Atiyah-Patodi-Singer (APS) Flat Index Theorem \[3\]. We state the nature of $q_\sigma$ in the following proposition.

**Proposition 1.7.** Let $E$ be a rank zero element of $KO(X)$.

1. If $w_1(E) = 0$ and $w_2(E) = 0$ then $q_\sigma(E,\cdot) \equiv 1$.

2. If $w_1(E) = 0$ then $q_\sigma(E,\ell) = (-1)^{w_2(E) \cdot \ell}$.

3. In general, $q_\sigma(E,\cdot)$ is a $\mathbb{Z}/4\mathbb{Z}$-valued quadratic refinement of the bilinear form

$$B_E(\ell_1,\ell_2) = \frac{q_\sigma(E,\ell_1 \otimes \ell_2)}{q_\sigma(E,\ell_1)q_\sigma(E,\ell_2)} = (-1)^{w_1(E) \cdot \ell_1 \cdot \ell_2}.$$
a cobordism invariant. (One can easily show this using the twisted version of \(1.2\)). Thus we have a homomorphism

\[
q_\sigma : \Omega^{\text{spin}}_3(BSO \times B(\mathbb{Z}/2\mathbb{Z})) \longrightarrow \mathbb{Z}/2\mathbb{Z}^\times
\]

where \(\Omega^{\text{spin}}_3(BSO \times B(\mathbb{Z}/2\mathbb{Z}))\) is the cobordism group of smooth 3-manifolds with spin structure, oriented rank zero element of \(KO\), and flat line bundle. Even better, assuming the validity of (1), it is clear that this homomorphism factors through \(\Omega^{\text{spin}}_3(BSO \wedge B(\mathbb{Z}/2\mathbb{Z}))\) which, in Proposition 4.3 of the appendix, we show is isomorphic to \(\mathbb{Z}/2\mathbb{Z}\). But in Proposition 4.5 we show that, on the 3-torus, we have exactly the formula

\[
q_\sigma(E, \ell) = (-1)^{w_2(E) - \ell}
\]

called for in (2) of this proposition. Since Stiefel-Whitney numbers are also cobordism invariants \([18]\) and the structures on the 3-torus clearly generate \(\Omega^{\text{spin}}_3(BSO \wedge B(\mathbb{Z}/2\mathbb{Z}))\) this gives us (2) for general compact 3-manifolds.

To prove (3) we decompose \(E = (\ell - 1) + F\) where \(w_1(E) = \ell\) and \(F \in KO(X)\) is such that \(w_1(F) = 0\) and \(w_2(F) = w_2(E)\). Then

\[
B_E(\ell_1, \ell_2) = B_{\ell-1}(\ell_1, \ell_2) \cdot B_F(\ell_1, \ell_2) = B_{\ell-1}(\ell_1, \ell_2)
\]

since \(B\) is linear with respect to \(E\) and is zero if \(w_1(E) = 0\). Now, rearranging some of the factors, we see that

\[
B_{\ell-1}(\ell_1, \ell_2) = \tau^{1/2}(D_{(\ell-1)\otimes(\ell_1\otimes\ell_2\oplus\ell_1-\ell_2)}) = \tau(\ell_1 \otimes \ell_2 \oplus 1 - \ell_1 - \ell_2, \ell) = (-1)^{\ell_1 \otimes \ell_2 - \ell}
\]

where the last equality follows from (2) and the fact that \(w_1(\ell_1 \otimes \ell_2 \oplus 1 - \ell_1 - \ell_2) = 0\) and \(w_2(\ell_1 \otimes \ell_2 \oplus 1 - \ell_1 - \ell_2) = \ell_1 - \ell_2\).

### 1.6 Redefining the “level”

As it is defined above the level is a real, rank zero, virtual representation \(\rho = \rho_1 - \rho_2\). Even at first glance one sees that we need not be so specific. In particular, fixing all other parameters – spin structure, connection, metric – \(\tau^{1/2}(D_{\rho A})\) only depends on the equivalence class represented by \(\rho\) in the rank zero representation ring \(\widetilde{RO}(G)\). If we look a little harder, in fact, we can see that even this is too specific and that the level need not be so refined. We elaborate on this next.

In Section A.3 of the appendix we define a generalized cohomology theory \(X \mapsto E^\bullet(X)\). As discussed in the appendix, a virtual representation \(\rho\) determines a class \(\lambda(\rho) \in E^4(BG)\). Our claim is that \(\tau^{1/2}(D_{\rho A})\) really only depends on the class \(\lambda(\rho)\). Or, what’s equivalent, that the map

\[
\widetilde{RO}(G) \longrightarrow \mathbb{T}
\]

\[
\rho \longmapsto \tau^{1/2}(D_{\rho A})
\]
factors through the map \( \lambda : RO(G) \to E^4(BG) \). This is exactly what is implied by the following proposition.

**Proposition 1.8.** If \( \lambda(\rho) = 0 \) then \( \tau^{1/2}(D_{\rho A}) = 1 \) for all \( G \)-connections \( A \).

**Proof.** Let \( \rho = \rho_1 - \rho_2 \). Without loss of generality we can assume that both \( \rho_1 \) and \( \rho_2 \) map \( G \) into \( SO(V) \). Recall – again, from the appendix – that \( E^4(BG) \) fits into an exact sequence

\[
0 \to H^4(BG) \to E^4(BG) \to H^2(BG; \mathbb{Z}/2\mathbb{Z})
\]

where the third homomorphism is induced by the 2nd Stiefel-Whitney class \( w_2 \in H^2(BSO) \). Given the hypothesis of the proposition this implies that \( w_2(\rho) = 0 \) so that, for any \( G \)-bundle \( P \), \( w_2(\rho P) = 0 \). Together with Proposition 1.7 this implies that \( \tau^{1/2}(D_{\rho}) \) is independent of the spin structure.

Yet another fact from Section A.3 is that the first Pontryagin class \( p_1 \in H^4(BSO) \) induces a homomorphism \( E^4(BG) \to H^4(BG) \). Together with the hypothesis this implies that, for any \( G \)-bundle \( P \), \( p_1(\rho P) = 0 \). Furthermore, from the Chern-Weil isomorphism

\[
H^\bullet(BG) \otimes \mathbb{R} \cong \text{Sym}^\bullet(g^* \text{AdG}),
\]

we see that \( \langle \cdot, \cdot \rangle_\rho = 0 \).

This last revelation implies that \( \tau^{1/2}(D_{\rho}) \) is a cobordism invariant. Indeed, if all of the parameters on \( X \) – the pair \( (P_X, A_X) \), metric \( g_X \), and spin structure \( \sigma_X \) – bound corresponding parameters on a 4-manifold \( M = (P_M, A_M) \), \( g_M \), \( \sigma_M \) – then according to Proposition 1.7

\[
\tau^{1/2}(D_{\rho A}) = \exp \pi i \left( \int_M \langle \Omega^{A_M}, \Omega^{A_M}_\rho \rangle \right) = 1.
\]

Since there is no obstruction to extending the smooth parameters, \( \tau^{1/2}(D_{\rho(\cdot)}) \) induces a homomorphism \( \Omega^3_{\text{spin}}(BG) \to \mathbb{T} \), where \( \Omega^3_{\text{spin}}(BG) \) is the cobordism group of 3-manifolds with spin-structure and a \( G \)-bundle. In fact, this homomorphism factors through \( \Omega^3_{\text{spin}}(BSO) \).

Now we borrow one last fact from the appendix. In Section A.1 we find that \( \Omega^3_{\text{spin}}(BSO) \cong \mathbb{Z}/2\mathbb{Z} \) and is generated by rank zero oriented vector bundles on \( S^1 \times S^2 \) and the non-bounding spin structure over the \( S^1 \) cartesian factor. However, \( \tau^{1/2}(D_{\rho(\cdot)}) \) is independent of the spin structure (as shown above) so that we can place the bounding spin structure on the \( S^1 \) cartesian factor to obtain the same element of \( \mathbb{T} \). But, that being done, all of the structure bounds so that, in fact, \( \tau^{1/2}_{S^1 \times S^2}(D_{\rho(\cdot)}) = 1 \). The cobordism invariance proves the proposition for a general compact 3-manifold. \( \square \)

### 1.7 spin-Chern-Simons and Chern-Simons

We end our consideration of the classical Lagrangian field theory by establishing its relation to Chern-Simons field theory; in particular, as it was studied in [12].
Proposition 1.9. Let \( \rho \) be a real rank zero virtual representation of a connected, simply connected, compact Lie group \( G \) and let \( \langle , \rangle_\rho \) denote the symmetric pairing defined by (1.6). If \( A \) is a \( G \)-connection over a closed, spin 3-manifold \( X \), we let \( \exp 2\pi i S_X(A) \) denote the \( T \)-valued Chern-Simons invariant determined by the pairing \( \frac{1}{2} \langle , \rangle_\rho \), as defined in [12]. Then

\[
\tau_X^{1/2}(D\rho A) = \exp 2\pi i S_X(A).
\]

Proof. The proof relies on the fact that the cobordism group \( \Omega^\text{spin}_3(BG) = 0 \) whenever \( G \) is compact and simply connected. In that case there exists a spin 4-manifold \( M \) such that \( \partial M = X \) as a spin manifold and there exists an extension \( A' \) of \( A \) over \( M \). On the one hand it is well-known that

\[
\exp 2\pi i S_X(\rho A) = \exp 2\pi i \int_M \frac{1}{2} \langle \Omega^{A'}, \Omega^{A'} \rangle_\rho.
\]

On the other hand, since \( \rho \) has rank zero the APS index theorem implies

\[
\tau_X^{1/2}(D\rho A) = \exp \pi i \int_M \langle \Omega^{A'}, \Omega^{A'} \rangle_\rho,
\]

and this proves the proposition.

To make the equality of these theories even stronger we point out the correspondence between their respective levels. Recall that for Chern-Simons the levels are elements of \( H^4(BG) \) while for spin-Chern-Simons the level are elements of \( E^4(BG) \). However, for \( G \) simply connected there is a natural isomorphism \( i : H^4(BG) \rightarrow E^4(BG) \) so that Chern-Simons theory at level \( \alpha \in H^4(BG) \) is isomorphic to spin-Chern-Simons at level \( i(\alpha) \in E^4(BG) \).

2 Spin-Chern-Simons on 3-Manifolds with Boundary

In this chapter we extend our investigation of the Lagrangian spin-Chern-Simons field theory to compact 3-manifolds with boundary. When \( X \) is without boundary our action \( \tau^{1/2} \) is a \( T \)-valued function of the smooth parameters. What we see in this chapter is that when \( X \) has non-empty boundary \( \partial X \), \( \tau^{1/2} \) is readily interpreted as a section of the Pfaffian line associated to \( \partial X \). This interpretation and many of its consequences stem from previous work done for the \( \tau \)-invariant. We review these less refined results before stating the corresponding results for \( \tau^{1/2} \).

2.1 A review of \( \tau \) and DF-boundary conditions

In this section we summarize some of the features of \( \tau \)-invariants for manifolds with boundary as worked out in [11] and [14]. For the most part – though
much of this discussion applies more generally in odd dimensions – we focus on 3-manifolds with boundary.

These results are formulated in terms of graded vector spaces and especially graded lines. For that reason we review a few salient points about graded lines from [11] and defer to the source for a more detailed account.

The general situation we consider is a line of the form $\text{Det} V = \bigwedge^{\dim V} V$ for some finite dimensional complex vector space $V$ and we assign to it the grading $|\text{Det} V| = \dim V \pmod{2}$. The gradings add or subtract when we tensor two lines or tensor one by the inverse of the other, respectively. One of the ways in which gradings affect our computations is when we consider the natural isomorphisms:

$$L^* \otimes L \to \mathbb{C} \quad \text{and} \quad L \otimes L^* \to \mathbb{C}$$

where $L$ is a graded line with grading $|L| \in \mathbb{Z}/2\mathbb{Z}$. In the graded category we take the convention that the first isomorphism $a^{-1} \otimes b \mapsto a^{-1}(b)$ does not involve any signs. However, this implies that the second isomorphism must take the form $b \otimes a^{-1} \mapsto (-1)^{|L|}a^{-1}(b)$. The second isomorphism is called the supertrace and is denoted $\text{Tr}_s$. Notice that if $L$ has an even grading it is equivalent to the normal trace and if $L$ has an odd grading then it is equivalent to minus the normal trace.

Let $X$ be a spin Riemannian 3-manifold with boundary $\partial X$. Recall that $\tau$ is a spectral invariant of the (possibly twisted) Dirac operator. When $\partial X = \emptyset$ the Dirac operator $D_X$ is self-adjoint and elliptic so that $\tau$ is well-defined. To maintain these analytic properties when $\partial X \neq \emptyset$ Dai and Freed impose elliptic boundary conditions similar to those introduced by Atiyah-Patodi-Singer but require an additional piece of data to adapt the APS boundary conditions to odd-dimensional manifolds. We explain this next.

Since $\partial X$ is even-dimensional its spinor bundle $S_{\partial X}$ splits as $S_{\partial X} = S_+^{\partial X} \oplus S_-^{\partial X}$ and the Dirac operator $D_{\partial X} : C^\infty(S_+^{\partial X}) \to C^\infty(S_-^{\partial X})$ interchanges the components. The cobordism invariance of the index implies that $\dim \text{Ker}^+ D_{\partial X} = \dim \text{Ker}^- D_{\partial X}$. The additional piece of data is an isometry

$$T : \text{Ker}^+ D_{\partial X} \to \text{Ker}^- D_{\partial X}.$$ 

These boundary conditions then ensure that $\tau(D_X)$ is well-defined. Though it does depend on the isometry $T$ its dependence can be factored out in such a way that

$$\tau_X \in \text{Det}_{\partial X}^{-1}$$

where $\text{Det}_{\partial X}$ is the determinant line of the Dirac operator $D_{\partial X}$:

$$\text{Det}_{\partial X} = (\text{DetKer}^- D_{\partial X}) \otimes (\text{DetKer}^+ D_{\partial X})^{-1}. \quad (2.1)$$

Here and in what follows $L^{-1} = L^*$ for any (abstract) line $L$. Furthermore, $|\tau_X|^2 = 1$ in the Quillen metric on $\text{Det}_{\partial X}$. From the definition we see that the determinant line $\text{Det}_Y$ associated to a closed, even dimensional Riemannian spin manifold $Y$ has the grading index $(D_Y) \pmod{2}$. 17
Remark 2.1. Above we (parenthetically) stated that all of this was possible in the context of Dirac operators twisted by virtual vector bundles. For clarity, and because it is essential to our considerations below, we make explicit what we mean. First we point out that if \((E_j, \nabla^{E_j}) \) \(j = 1, 2\) are two vector bundles with unitary connections over a spin Riemannian 3-manifold \(X\) then \((2.1)\) implies

\[
\text{Det}_{\partial X, E_1 \oplus E_2} = \text{Det}_{\partial X, E_1} \otimes \text{Det}_{\partial X, E_2}.
\]

The definition that is compatible with addition of virtual vector bundles is therefore

\[
\text{Det}_{\partial X, E_1 - E_2} = \text{Det}_{\partial X, E_1} \otimes \text{Det}_{\partial X, E_2}^{-1}
\]

and so \(\tau_X(D_{E_1 - E_2})\) is an element of \(\text{Det}_{\partial X, E_1 - E_2}^{-1}\).

If \(X \to Z\) is a family of Riemannian 3-manifolds with boundary then \(\partial X \to Z\) is a family of 2-manifolds. (Also allowed is a twisting virtual vector bundle \(E \to X\) with unitary connection \(\nabla^E\)). The lines \((2.1)\) patch together to form a smooth line bundle \(\text{Det}_{\partial X/Z} \to Z\) with the Quillen metric and a natural unitary connection \(\nabla'\), defined in \([6]\). In this context we have that

\[
\tau_{X/Z} : Z \longrightarrow \text{Det}_{\partial X/Z}^{-1}
\]

is a smooth unitary section. One of the basic results regarding this section is a variation formula which computes its covariant derivative over the family.

\textbf{Theorem 2.2} ([11], Theorem 1.9). With respect to the natural connection \(\nabla'\) on \(\text{Det}^{-1}_{\partial X/Z}\),

\[
\nabla' \tau_{X/Z} = 2\pi i \left[ \int_{X/Z} \hat{A}(\Omega^{X/Z}) \text{ch}(\Omega^E) \right] (2.2)
\]

Notice that this is just the (exponentiated) generalization of \((1.3)\).

For now, this completes our review of the results in \([11]\). We will continue this discussion later in this chapter when we review a gluing law for \(\tau_X\) and then again in the next chapter when we consider Hamiltonian spin-Chern-Simons theory over closed 2-manifolds.

\section{\(\tau^{1/2}\) and DF-boundary conditions}

We come back to \(\tau^{1/2}\) which is the spectral invariant relevant to our spin-Chern-Simons field theory. With some care but little difficulty the results summarized above for \(\tau\) can be refined for \(\tau^{1/2}\). We explain this next. Though much of what we say generalizes to \(3 \mod 8\) dimensional manifolds (thanks to Bott periodicity), we maintain our focus on 3-manifolds with boundary.

Let \(X\), once again, be a spin Riemannian 3-manifold with boundary \(\partial X\). Recall that when \(\partial X = \emptyset\) the spectral invariant \(\tau^{1/2}\) is well defined because the (possibly twisted) Dirac operator is self-adjoint, elliptic, and compatible with a quaternionic structure \(J\) on the the space \(C^\infty(S_X)\) of spinor fields on \(X\).
To maintain these analytic properties when \( \partial X \neq \emptyset \), including quaternionic compatibility, we require a refinement of the DF boundary conditions. We explain this next.

Since \( \partial X \) is a closed 2-manifold the spinor bundles \( S^+_{\partial X} \) and \( S^-_{\partial X} \) are naturally dual to each other and the Dirac operator \( D_{\partial X} : C^\infty(S^+_{\partial X}) \rightarrow C^\infty(S^-_{\partial X}) \) is complex skew-symmetric with respect to that duality. This informs our requirements on the isometry

\[
T : \text{Ker}^+ D_{\partial X} \rightarrow \text{Ker}^- D_{\partial X}.
\]

The cobordism invariance of the mod-2 index implies that \( \dim \text{Ker}^+ D_{\partial X} \) is even dimensional and so supports a skew-symmetric isometry from itself to its dual. We require that the isometry \( T \) be skew-symmetric.

These boundary conditions ensure that \( \frac{1}{\sqrt{2}} (D_X) \) is well-defined. Indeed, restricted the subspace of spinors that satisfy the boundary conditions, \( D_X \) is elliptic, self-adjoint, and quaternionic. Though it depends on \( T \) that dependence can be factored out so that one observes

\[
\tau^{1/2}_X \in \text{Pfaff}^{-1}_{\partial X}
\]

where \( \text{Pfaff}_{\partial X} \) is the Pfaffian line of the Dirac operator \( D_{\partial X} \):

\[
\text{Pfaff}_{\partial X} = \text{Det} \text{Ker}^- D_{\partial X}
\]

Notice that, because of the duality mentioned above, there is a natural identification

\[
\text{Pfaff}^\otimes_2 = \text{Det} \partial X.
\]

Under this identification we have (not surprisingly) \( \tau^{1/2}_X \otimes \tau^{1/2}_X = \tau_X \) This implies \( \tau^{1/2}_X \in \text{Pfaff}^{-1}_{\partial X} \). We also point out that the Pfaffian line \( \text{Pfaff}^Y \) associated to a closed, 2 (mod 8)-dimensional Riemannian spin manifold \( Y \) has the grading ind

\[
\text{Pfaff}^Y = \text{Det} \text{Ker}^+ D_Y
\]

(2.3) implies

\[
\text{Pfaff}_{\partial X,E_1 \oplus E_2} = \text{Pfaff}_{\partial X,E_1} \otimes \text{Pfaff}_{\partial X,E_2}.
\]

The definition that is compatible with addition of virtual vector bundles is therefore

\[
\text{Pfaff}_{\partial X,E_1 \ominus E_2} = \text{Pfaff}_{\partial X,E_1} \otimes \text{Pfaff}^-_{\partial X,E_2}.
\]

and so \( \tau^{1/2}_X (D_{E_1 - E_2}) \) is an element of \( \text{Pfaff}^-_{\partial X,E_1 - E_2} \).

Let \( X \rightarrow Z \) be a family of Riemannian 3-manifolds with boundary and \( \partial X \rightarrow Z \) the corresponding family of closed 2-manifolds. (Also allowed is a
twisting real virtual vector bundle $E \to X$ with orthogonal connection $\nabla^E$.

The lines $2.3$ patch together to form a smooth line bundle $\text{Pfaff}_{\partial X/Z} \to Z$ with the square root of the Quillen metric and a natural unitary connection $\nabla$. In this context we have that $\tau^{1/2}_{X/Z} : Z \to \text{Pfaff}_{\partial X/Z}^{-1}$ is a smooth unitary section. The identification $\tau^{1/2}_{X/Z}$ also patches together so that $\text{Pfaff}_{\partial X/Z}^{\otimes 2} = \text{Det}_{\partial X/Z}$ and $\tau^{1/2}_{X/Z} \otimes \tau^{1/2}_{X/Z} = \tau_{X/Z}$. Furthermore, we can identify the connections $\nabla^{\otimes 2} = \nabla'$. This implies a variation formula for the section $\tau^{1/2}_{X/Z}$ which follows trivially from $2.2$.

**Proposition 2.4.** With respect to the natural connection $\nabla$ on $\text{Pfaff}_{\partial X/Z}^{-1}$,

$$\nabla \tau^{1/2}_{X/Z} = \pi i \left[ \int_{X/Z} \hat{A}(\Omega_{X/Z}) ch(\Omega^E) \right] \tau^{1/2}_{X/Z}. \quad (2.5)$$

Actually, many of these considerations are too general for our needs (they are true in all dimensions $3 \pmod{8}$ for instance) and so we specialize to spin-Chern-Simons Lagrangian field theory next.

2.3 **Functoriality of** $\tau^{1/2}_{X} \in \text{Pfaff}_{\partial X}^{-1}$

Let $G$ be a compact Lie group and $\rho$ a real oriented virtual representation of $G$. Let $Y$ be a closed spin Riemannian 2-manifold with Dirac operator $D_Y : C^\infty(S^+) \to C^\infty(S^-)$. We define $\mathcal{C}_G(Y)$ to be the category of $G$-connections over $Y$. We further define a functor from $\mathcal{C}_G(Y)$ to the category of complex lines $\mathcal{L}$

$$\mathcal{L}^\rho : \mathcal{C}_G(Y) \to \mathcal{L}$$

$$(Q, B) \mapsto \text{Pfaff}_{\partial X}^{-1}(D_B) \quad (2.6)$$

where $\text{Pfaff}_{\partial X}^{-1}(D_B)$ is the inverse Pfaffian line of the twisted Dirac operator $D_B$ on $Y$, as defined in $2.3$. From the discussion above, if $B_u$ is a smooth family of $G$-connections on $Y$ varying over a smooth manifold $U$ then the inverse Pfaffian lines $\mathcal{L}^\rho(B_u)$ form a smooth hermitian line bundle with unitary connection over $U$.

We come back to a spin Riemannian 3-manifold $X$ with boundary $\partial X$ and Dirac operator $D_X : C^\infty(S) \to C^\infty(S)$. From the discussion above, if $\partial X \neq \emptyset$, and $\mathcal{C}_G(X, B)$ is the category of $G$-connections $A$ on $X$ such that $A|\partial X = B$, then we have the assignment

$$\mathcal{C}_G(X, B) \to \mathcal{L}^\rho(B)$$

$$A \mapsto \tau^{1/2}_X(D_A)$$

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From the discussion above if $A_u$ is a smooth family of $G$-connections on $X$ varying over a smooth manifold $U$ then the elements $\tau_{X}^{1/2}(D_{\rho A_u}) \in \mathcal{L}_{\rho}(A_u|_{\partial X})$ form a smooth section of line bundle formed by the lines $\mathcal{L}_{\rho}(A_u|_{\partial X})$. To allow for the case when $X$ is closed we set $\mathcal{L}^\rho = \mathbb{C}$ when $Y = \emptyset$. If $X$ itself is empty then we set $\tau_{X}^{1/2} = 1$. In light of this new interpretation for the action of our field theory, we present a properly adjusted version of Proposition 1.1.

**Proposition 2.5.** Let $G$ be a compact Lie group and $\rho$ a virtual real representation of $G$. Then the assignments

$$B \mapsto \mathcal{L}^\rho(B), \quad B \in C_G(Y)$$

$$A \mapsto \tau_{X}^{1/2}(D_{\rho A}), \quad A \in C_G(X),$$

defined above for closed spin Riemannian 2-manifolds $Y$ and compact spin Riemannian 3-manifolds $X$ are smooth and satisfy:

1. (Functoriality) If $\psi: Q' \to Q$ and $H: S_{g', \sigma'} \to S_{g, \sigma}$ are any $G$-bundle and spinor bundle morphisms (respectively) covering an orientation and spin structure preserving isometry $h: (Y', g') \to (Y, g)$, and $B$ is a connection on $Q$, then there is an induced isometry

$$\partial(F \otimes \rho \phi)^* \tau_{X'}^{1/2}(D_{\rho A}) = \tau_{X}^{1/2}(D_{\rho A})$$

(2.8)

and these compose properly. If $\psi: P \to P$ and $F: S_{g', \sigma'} \to S_{g, \sigma}$ are any $G$-bundle and spinor bundle morphisms (respectively) covering an orientation and spin structure preserving isometry $f: (X', g') \to (X, g)$, and $A$ is a connection on $P$, then

$$\partial(F \otimes \rho \phi)^* \tau_{X'}^{1/2}(D_{\rho A}) = \tau_{X}^{1/2}(D_{\rho A})$$

(2.8)

where $D'$ is the Dirac operator on $X'$ and $\partial(F \otimes \rho \phi): \partial(S_{X'} \otimes \rho P) \to \partial(S_X \otimes \rho P)$ is the induced map over the boundary.

2. (Orientation) There is a natural isometry

$$\mathcal{L}^\rho(\bar{Y}, B) \cong \mathcal{L}^{-\rho}(Y, B)$$

(2.9)

such that

$$\tau_{X}^{1/2}(D_{\rho A}) = (-1)^{\frac{\ell_{Y}}{2}} \tau_{X}^{1/2}(D_{\rho A})^{-1},$$

(2.10)

where $k$ is the number of components $Y \subset \partial X$ such that the twisted (chiral) Dirac operator on $Y$ has non-trivial mod 2 index.

3. (“Additivity”) If $Y = Y_1 \sqcup Y_2$ is a disjoint union, and $B_j$ are connections over $Y_j$, then there is a natural isometry

$$\mathcal{L}^\rho(B_1 \sqcup B_2) \cong \mathcal{L}^\rho(B_1) \otimes \mathcal{L}^\rho(B_2).$$

(2.11)

If $X = X_1 \sqcup X_2$ is a disjoint union, and $A_j$ are connections over $X_j$, then

$$\tau_{X_1 \sqcup X_2}(A_1 \sqcup A_2) = \tau_{X_1}(A_1) \otimes \tau_{X_2}(A_2).$$
Here and in what follows if \( a \in L \) is an element of a line then we denote by \( a^{-1} \in L^{-1} \) the unique dual element such that \( a^{-1}(a) = 1 \). In particular, this applies to part (2) of the proposition.

**Remark 2.6.** Let us pause to take stock of what is implied by the functorial statements in this proposition. When we defined the assignment

\[
L^\rho : C_G(Y) \rightarrow L
\]

we stated, without proof, that it was a functor. Part (1) of the proposition proves this claim. From a less lofty point of view, if \( Q \rightarrow Y \) is a \( G \)-bundle over a closed spin Riemannian 2-manifold there is an associated line bundle \( L^\rho(Q) \rightarrow C(Y) \) over the space of connections, and the action of the gauge transformations \( G(Y) \) lifts to \( L^\rho(Q) \). Also, a \( G \)-bundle \( P \rightarrow X \) over a compact spin Riemannian 3-manifold determines a restriction map \( C(P) \rightarrow C(\partial P) \) and so a pulled back line bundle \( L^\rho(P) \rightarrow C(P) \). The action of the gauge transformations \( G(P) \) lifts to \( L^\rho(P) \) and \( \tau_{1/2}^X \) is an invariant section of \( L^\rho(P) \). We will have more to say about the implications of this proposition once we have stated the gluing law (2.14) for \( \tau_{1/2}^X \).

**Proof.** To prove (1) let \( \Phi_{H,\psi} \) denote the isometry on sections induced by \( H \otimes \rho \psi \). Then \( \Phi_{H,\psi} \) restricts to an isometry \( \text{Ker}^{-D\rho B} \rightarrow \text{Ker}^{-D\rho(\psi^*B)} \), where \( D' \) is the Dirac operator on \( Y' \). We denote the restriction by \( \Phi_{H,\psi} \) as well so that \( \text{Det}(\Phi_{H,\psi}) \) is the isometry in (2.8). That \( \tau_{1/2} \) obeys (2.8) follows from the proof for part (1) of Proposition 1.1 and the fact that the induced isometry on sections \( \Phi_{F,\phi} \) clearly preserves the (refined) DF-boundary conditions.

To prove (2) recall that under change in orientation there is a switch in chirality, \( S^\pm_Y = S_Y^\mp \). The duality of \( S_Y^+ \) and \( S_Y^- \), together with the comments in Remark 2.3 imply (2.8). That \( \tau_{1/2} \) obeys (2.10) follows partly from the proof for part (2) of Proposition 1.1 and the fact that the DF boundary conditions behave appropriately upon change in orientation. For the sign factor, recall that the isomorphism is one between graded lines and so will involve signs. In particular, identifying \( L_1^{-1} \otimes \cdots \otimes L_n^{-1} \) with \( (L_1^{-1} \otimes \cdots \otimes L_n^{-1}) \) involves permuting \( \binom{n}{2} \) odd elements past each other whereby we obtain the sign in (2.10).

The proof for (3) is simple and left to the reader. \( \square \)

We offer a properly adjusted version of Proposition 1.2 for 3-manifolds with boundary.

**Proposition 2.7.** Let \( i : G \rightarrow G' \) be an inclusion of compact Lie groups. Suppose \( \rho \) is a real virtual representation of \( G' \). Then if \( (Q,B) \) is a \( G \)-connection over a closed, spin Riemannian 2-manifold \( Y \), and \( (Q',B') \) its \( G' \) extension, there is a natural isometry

\[
i_B : L^{\rho \circ i}(B) \rightarrow L^\rho(B'). \tag{2.12}
\]

If \( A \) is a \( G \)-connection over a compact, spin Riemannian 3-manifold \( X \), and \( A' \) its \( G' \) extension, then

\[
i_B \left( \tau_{1/2}^X (D_{(\rho \circ i)B}) \right) = \tau_{1/2}^X (D_{\rho B'}).\]
In categorical language, \( i \) induces a transformation from the functor \( \mathcal{C}_G(Y) \to \mathcal{L} \) to the functor \( \mathcal{C}_G(Y) \to \mathcal{C}_G(Y) \to \mathcal{L} \). For each \( X \) this induces a transformation from \( \mathcal{C}_G(X) \to \mathcal{C}_G(\partial X) \to \mathcal{L} \) to \( \mathcal{C}_G(X) \to \mathcal{C}_G(\partial X) \to \mathcal{C}_G(\partial X) \to \mathcal{L} \), that preserves the elements \( \tau_{X}^{1/2} \).

**Proof.** There is a natural isomorphism between the associated virtual vector bundles \( \rho Q' \to (\rho \circ i) Q \) which sends the associated virtual connection \( \rho B \) to \( (\rho \circ i) B \). The induced isometry between the sections restricts from the kernel of \( D_{\rho B} \) to the kernel \( D_{(\rho \circ i) B} \). If we let \( \Phi \) denotes the restriction then \( \text{Det}\Phi \) provides the isometry (2.12). The rest of the proposition follows from the proof Proposition 1.2.

### 2.4 Gluing Formulae

We return to the results of [11]. In particular, we review the gluing formula for the \( \tau \)-invariant. The context of the gluing formula is as follows. Let \( X \) be a compact, spin Riemannian 3-manifold and \( Y \to X \) a closed, spin 2-dimensional submanifold. We cut along \( Y \) to obtain a new manifold \( X_{\text{cut}} \) with \( \partial X_{\text{cut}} = \partial X \sqcup Y \sqcup -Y \). Recall that \( \tau_{X} \in \text{Det}_{\partial X}^{-1} \) and \( \tau_{X_{\text{cut}}} \in \text{Det}_{\partial X}^{-1} \otimes \text{Det}_{Y}^{-1} \otimes \text{Det}_{-Y}^{-1} \) and notice that \( \text{Det}_{-Y}^{-1} \cong (\text{Det}_{Y}^{-1})^{-1} \). Thus we can apply the supertrace to the last two factors of the tensor product to obtain an element of \( \text{Det}_{\partial X}^{-1} \). That being said we state the gluing formula for \( \tau \).

**Theorem 2.8 ([11], Theorem 2.20).** In the context described above

\[
\text{Tr}_s(\tau_{X_{\text{cut}}}) = \tau_{X}.
\]  

(2.13)

With some care but very little difficulty we can refine the discussion and results just given to Pfaffian lines and the \( \tau^{1/2} \)-invariant. Recall that the Pfaffian line \( \text{Pfaff} \) associated to a closed, \( 2 \) (mod 8)-dimensional Riemannian spin manifold \( Y \) has the grading \( \text{ind}_2(D_Y) \). With that in mind, in the context of the gluing formula we have that

\[
\tau_{X}^{1/2} \in \text{Pfaff}_{\partial X}^{-1} \quad \text{and} \quad \tau_{X_{\text{cut}}}^{1/2} \in \text{Pfaff}_{\partial X}^{-1} \otimes \text{Pfaff}_{Y}^{-1} \otimes \text{Pfaff}_{-Y}^{-1}.
\]

We apply the supertrace to the last two factors of the tensor product to obtain the gluing formula for \( \tau^{1/2} \).

**Proposition 2.9.** In this case

\[
\text{Tr}_s(\tau_{X_{\text{cut}}}^{1/2}) = \tau_{X}^{1/2}.
\]

(2.14)

This very general gluing formula, of course, applies to the action of our spin-Chern-Simons theory. Due to its importance in the upcoming chapter on the Hamiltonian field theory we restate the gluing formula for \( \tau^{1/2} \) in the notation of the previous section. The reader should consider it as an addendum to Proposition 2.8.
Corollary 2.10 (Gluing). Let $X$, $Y$, $G$, and $\rho$ be as in Proposition 2.9. Now suppose $Y \hookrightarrow X$ is a closed spin submanifold and $X^{cut}$ is the manifold obtained by cutting $X$ along $Y$. Then $\partial X^{cut} = \partial X \sqcup Y \sqcup -Y$. Suppose $A$ is a $G$-connection over $X$, with $A^{cut}$ the induced connection over $X^{cut}$, and $B = A|_Y$. Then

$$\tau_{X}^{1/2}(D_{\rho A}) = \text{Tr}_s(\tau_{X^{cut}}^{1/2}(D_{\rho A^{cut}}))$$

where $\text{Tr}_s$ is the contraction

$$\text{Tr}_s : L^\rho(A^{cut}) \cong L^\rho(\partial A) \otimes L^\rho(B) \otimes L^{-\rho}(B) \rightarrow L^\rho(\partial A)$$

taking the supertrace of the last two factors of the tensor product.

Remark 2.11. We return to the comments in Remark 2.6. Part (3) of Proposition 2.5 and the gluing formula expresses the fact that the action is a local functional of local fields. Part (1) implies that the action is invariant under the symmetries of the fields, and part (2) expresses the fact that the action is unitary.

2.5 Dependence of $\tau^{1/2}$ on smooth parameters: redux

Let $X$ be a compact, spin 3-manifold with a possibly non-empty boundary $\partial X$. In Propositions 1.4 and 1.5 we compute how the function $\tau^{1/2}$ varied with respect to infinitesimal changes in the metric and connection. This was done assuming $\partial X = \emptyset$. We now offer analogous propositions, adjusted to the case $\partial X \neq \emptyset$, so that we compute how the section $\tau^{1/2}$ varies with respect to infinitesimal changes. Of course, we make this computation using the natural connection on the Pfaffian line bundle, whose properties we reviewed above. In particular, we apply the variation formula (2.5) to the section determined by the action of our spin-Chern-Simons theory.

Let $P \rightarrow X$ be any principal $G$-bundle. Then we take the obvious family of Riemannian 3-manifolds with $G$-connections

$$\text{Met}(X) \times C(P) \times X \longrightarrow \text{Met}(X) \times C(P) = Z.$$ Given a real virtual representation $\rho$ we obtain a family of twisted Dirac operators parametrized by $Z$. From Remark 2.6 we know that there is a line bundle $L^\rho(X, P) \rightarrow Z$ such that the action is a unitary section $\tau^{1/2} : Z \rightarrow L^\rho(X, P)$. Recall that $L^\rho(X, P)$ is a bundle of Pfaffian lines and therefore has a natural connection $\nabla$. Having said that, we can state how the action behaves under infinitesimal variations. We begin with a generalization of Proposition 1.6.

**Proposition 2.12.** If $\rho$ is a real rank zero virtual representation then, with respect to $\nabla$, $\tau^{1/2}$ is covariantly constant along $\text{Met}(X)$.

**Proof.** Using the variation formula (2.5) the proof of Proposition 1.6 carries over almost word for word to this situation. \qed

Now we generalize Proposition 1.6.
Proposition 2.13. Let $A_t$ be a path in $C(P)$ and let $\Omega_t$ denote the curvature of the connection $A_t$, $\dot{A}_t$ the tangent vector along the path. Then

$$\nabla_{\dot{A}_t} \tau^{1/2}(A_t) = 2\pi i \cdot dt \cdot \tau^{1/2}(A_t) \int_X \langle \Omega_t \wedge \dot{A}_t \rangle_\rho.$$ 

Proof. Once again, using the variation formula (2.5) the proof of Proposition 1.5 carries over almost word for word to this situation.

Remark 2.14. Assuming $\langle , \rangle_\rho$ is nondegenerate, this proposition implies that $\nabla \tau^{1/2} \big|_A = 0$ if and only if $\Omega^A = 0$; that is, if and only if $A$ is flat. Since the section $\tau^{1/2}$ is invariant with respect to gauge transformations $G$, so too is the space of solutions to the Euler-Lagrange equation. This is also obvious from (1.7). We let

$$\mathcal{M}_G(X) \subset \mathcal{C}_G(X)$$

denote the space of equivalence classes of solutions to (1.7). We will have more to say about $\mathcal{M}_G(X)$ when we discuss the Hamiltonian field theory in Section 1.3.

3 The Classical Hamiltonian Theory

In the Lagrangian spin-Chern-Simons theory the “spacetimes” are compact, spin, Riemannian 3-manifolds, possibly with non-empty boundary. In the Hamiltonian field theory the spacetimes are spin, Riemannian 3-manifolds that are globally products of a closed, spin 2-manifold (“space”) and an infinite interval (“time”). The classical solutions to the Euler-Lagrange equations are flat connections and we are only interested in equivalence classes of these with respect to the gauge group. This is the classical phase space. In the Hamiltonian theory the solutions are constant with respect to time so that the classical phase space is the moduli space of flat connections on the 2-manifold. These spaces have been the topic of much study in the past twenty years. Since our action is defined using certain elements of geometric index theory, it leads quite naturally to appearance of Pfaffian line bundles with their natural geometry: a metric and a unitary connection. We have already seen foreshadowings of this in the previous section. Here the line bundles themselves will take center stage, and their geometry will inform our formulation of the classical Hamiltonian field theory.

3.1 Moduli spaces of flat connections

Recall that in the Lagrangian field theory over a compact spin 3-manifold $X$, the classical space of solutions to the Euler-Lagrange equations is the moduli space of flat connections $\mathcal{M}_G(X)$. This space appears again in the Hamiltonian field theory. For that reason we review some standard facts about moduli spaces of flat connections. They reflect on how the classical theory (both Lagrangian and Hamiltonian) probes the topology of the spacetimes.
Proposition 3.1. Let $X$ be any smooth manifold and let $\{x_i\}_{i \in \pi_0(X)}$ be a set of basepoints for each component of $X$. Then the holonomy provides a natural identification

$$\mathcal{M}_G(X) = \prod_i \frac{\text{Hom}(\pi_1(X, x_i), G)}{G},$$

(3.1)

where $G$ acts on $\text{Hom}(\pi_1(X, x_i), G)$ by conjugation. Furthermore, this identification is independent of the basepoints.

The proof is standard.

Typically the moduli space $\mathcal{M}_G(X)$ is not a manifold. However, if $Y$ is a compact, oriented 2-manifold then it is well known that $\mathcal{M}_G(Y)$ is a stratified space and that the stratum of top dimension is a smooth manifold \cite{17, 1}. To investigate this manifold structure near the equivalence class of a flat connection $A$, we consider the twisted complex

$$0 \rightarrow \Omega^0_X(\text{ad} P) \xrightarrow{d_A} \Omega^1_X(\text{ad} P) \rightarrow \ldots.$$  

(3.2)

Here $d_A$ is the usual extension of $A$ to act on differential forms with value in $\text{ad} P$; and that this is a complex follows from $d_A^2 = 0$. If we denote the cohomology groups of this complex by $H^\bullet(X; d_A)$, then one might guess that the tangent space at $A$ is

$$T_A \mathcal{M}_G(X) \cong H^1(X, d_A).$$

(3.3)

Indeed, this is certainly the case whenever $A$ represents a smooth point of $\mathcal{M}_G(X)$.

Many of the properties of the usual deRham cohomology carry over for twisted cohomology. For example, if $Y$ is a compact, oriented 2-manifold then for a flat connection $B$ there is a nondegenerate pairing

$$H^0(Y, d_B) \otimes H^2(Y, d_B) \rightarrow \mathbb{R},$$

which mimics Poincare duality in the usual deRham cohomology. The zeroth cohomology $H^0(Y, d_B)$ is the Lie algebra of the stabilizer $G(B)$. Of course, it vanishes if $B$ is irreducible since, in that case, the stabilizer of $B$ is the center of $G$ which is finite. At smooth points the index theorem for the twisted complex gives

$$\dim \mathcal{M}_G(Y) = \dim H^1(Y, d_B) = - \dim G \cdot \chi(Y) + 2 \dim G(B).$$

(3.4)

When $Y$ has a complex structure the flat $G$-connections can identified with holomorphic structures on $G$ principal bundles and this imbues $\mathcal{M}_G(Y)$ with its own complex structure \cite{22, 1}. This is manifest in the dimension (3.4) which is always even.

Remark 3.2. More properly, what we actually work with is the moduli stack of flat connections. We briefly explain what we mean. Recall that the space of fields is the category of $G$-connections $C_G(X)$. The objects are pairs $(P, A)$ –
where \( P \to X \) is a \( G \)-bundle and \( A \) is a connection on \( P \) — and the morphisms \( \mathcal{G}_G(X) \) are \( G \)-bundle isomorphisms that cover the identity of \( X \). As all of the elements of \( \mathcal{G}_G(X) \) are invertible, the category \( \mathcal{C}_G(X) \) is a groupoid. The subcategory of flat \( G \)-connections is a subgroupoid. In working with the moduli stack, instead of the moduli space, we keep track of, not just the equivalence class of \( A \), but the automorphisms of \( A \) as well. That is, if \( A \) has a non-trivial stabilizing subgroup \( \mathcal{G}(A) \subset \mathcal{G}(P) \), then the moduli stack keeps track of that information; whereas the moduli space only sees the equivalence class of \( A \). Notice that if we only work with those connections with trivial stabilizing subgroups then working with the moduli stack provides the same information as working with the moduli space.

3.2 The space of fields for the Hamiltonian theory

We consider a 3-manifold with boundary of the form \( X = [0, \infty) \times Y \) where \( Y \) is a closed spin Riemannian 2-manifold. \( X \) has a natural product spin structure determined entirely by the spin structure on \( Y \), and we impose the cylindrical metric (flat in the \([0, \infty)\) direction) on \( X \). The Dirac operator on \( X \), even with the DF-boundary conditions, is not an elliptic operator. In particular, the spectrum is not entirely discrete so that the \( \tau^{1/2} \)-invariant is not defined. Nonetheless, we can consider the critical points by taking compactly supported variations. Indeed, the log derivative \( d\tau^{1/2}X/\tau^{1/2}X \) has an expression in terms of local fields that is well defined when the variation is taken along compactly supported fields. Thus, we define a critical \( G \)-connection \( A \) to be one for which

\[
\Omega^A = 0.
\]

In the Hamiltonian formulation we may interpret the space of fields as a space of paths. To do so we require the following proposition.

**Proposition 3.3 ([12] Proposition 3.14).** Let \( \{Q\} \) be a set of representations for the equivalence classes of principal \( G \) bundles over \( Y \). Then there is an identification

\[
\mathcal{C}_G([0, \infty) \times Y) = \bigsqcup_{\{Q\}} \text{Map}([0, \infty), \mathcal{C}(Q))/\mathcal{G}(Q).
\]

The proof makes use of the fact that \([0, \infty)\) is contractible so that the topological type of a \( G \)-bundle \( P \to [0, \infty) \times Y \) is determined by \( \partial P \to Y \). It also uses the fact that every connection on \([0, \infty) \times Y \) is isomorphic to a connection that is trivial in the \([0, \infty)\) direction, which is exactly a path of connections on \( Y \). In the physics literature, this is the statement that we can always work in a “temporal gauge” in which the connection is trivial along time direction.

If \( A \) is a flat connection then \( \partial A = A|_Y \) is also flat. The next proposition implies that the equivalence class of the classical solutions on \([0, \infty) \times Y \) are completely determined by the equivalence class of their initial value.
Proposition 3.4 ([12] Proposition 3.16). The restriction to the boundary
\[ M_G(\mathbb{R} \times Y) \subset C_G(\mathbb{R} \times Y) = C_G(\{0\} \times Y) \]
is an isomorphism of \( M_G(\mathbb{R} \times Y) \) onto the moduli space \( M_G(Y) \) of flat connections over \( Y \).

In Hamiltonian classical mechanics the fields over a cylinder are paths in a symplectic manifold. We address the issue of symplectic structure next by bringing the Pfaffian lines \( \mathcal{L}^\rho \) to the forefront.

### 3.3 Geometry of the Pfaffian line bundle

To further our discussion we turn to results from [7] and [13]. Let \( Y \to Z \) be a family of closed, spin Riemannian 2-manifolds, and \( E \to Y \) a real vector bundle with orthogonal connection \( \nabla^E \). Let \( \mathcal{L} = \text{Pfaff}_{Y/Z,E} \to Z \) be the inverse Pfaffian line bundle over the family of twisted Dirac operators. Denote \( I = [0,1] \) and take a path \( \gamma : I \to Z \). We form the pullback \( \gamma^*Y \to I \) and observe that \( \gamma^*Y \) is 3-manifold with boundary \( \partial(\gamma^*Y) = Y_{\gamma(1)} \cup -Y_{\gamma(0)} \). If we place the standard metric on \( I \), then we get a metric on \( \gamma^*Y \) thanks to the metric \( g_{\gamma^*Y/I} \) and the distribution of horizontal planes. From this data we get a twisted Dirac operator \( D_E \) on \( \gamma^*Y \) and so a \( \tau_{1/2} \)-invariant
\[ \tau_{\gamma^*Y}(D_E) \in \mathcal{L}_{\gamma(1)} \otimes \mathcal{L}_{\gamma(0)}^{-1}. \tag{3.6} \]

Let us look at the algorithm we have generated. To a path in \( Z \) – using the \( \tau_{1/2} \)-invariant – we assign a linear map between the lines over the endpoint of the path. This is exactly what occurs in parallel transport. In fact, the following theorem confirms that the algorithm described above is just that.

**Theorem 3.5 ([13], Theorem 3.1).** Let \( E \to Y \) be a real rank zero virtual vector bundle with orthogonal connection over the family of closed, spin Riemannian 2-manifolds \( Y \to Z \).

1. If \( \gamma : I \to Z \) is a path then the linear map
\[ \tau_{\gamma^*Y}(D_E) : \mathcal{L}_{\gamma(0)} \to \mathcal{L}_{\gamma(1)} \tag{3.7} \]
is parallel transport over \( \gamma \) with respect to the natural connection on \( \mathcal{L} \).

2. Let \( \Omega \) denote the curvature of the natural connection on \( \mathcal{L} \to Z \). Then
\[ \Omega = -\pi i \left[ \int_{Y/Z} \hat{A}(\Omega^{Y/Z}) \text{ch}(\Omega^E) \right]_{(2)}. \tag{3.8} \]

If \( \gamma \) is closed path in \( Z \) then we can form a 3-manifold \( Y_\gamma \) fibered over the circle and \( \tau_{Y_\gamma}(D_E) \) is a unitary number. Recall that \( S^1 \) has two spin structures: the non-bounding spin structure is the trivial double cover of the circle and the bounding spin structure is the non-trivial double cover. In this case we have the following theorem.
Theorem 3.6 ([4], Theorem 3.18). Suppose \( \gamma : I \to \mathbb{Z} \) is a closed path (which is constant on \([0, \delta]\) and \([1 - \delta, 1]\) for some \( \delta \)). Then the holonomy around \( \gamma \) is given by

\[
\text{hol}_{\gamma} = \begin{cases} 
(-1)^{\text{ind}_{2}(D_{Y,E})} \tau_{Y,\gamma}^{1/2}(D_{E}), & \text{nonbounding spin structure on } S^1; \\
\tau_{Y,\gamma}^{1/2}(D_{E}), & \text{bounding spin structure on } S^1.
\end{cases}
\] (3.9)

We point out some subtleties that have been sidestepped. What we have written here is a really a specialized version of what appears in [13] which applies more generally for \( E \) of any rank (and, as usual, in dimensions 2 (mod 8)). By restricting \( E \) to have rank zero we avoid having to take an adiabatic limit of \( \tau_{1/2}^{X} \)-invariants. In general, upon taking the adiabatic limit, the linear map (3.6) becomes independent of the parametrization of the path \( \gamma \). The adiabatic limit is taken over the metric we place on \( I \); but on 3-manifolds, when \( E \) is rank zero, we have seen that the \( \tau_{1/2}^{X} \)-invariants are independent of the metric. Thus our linear map is independent of the parametrization of the path even before taking an adiabatic limit. For details we defer to the source cited.

3.4 The line bundle and symplectic structure

Though we want our physical parameters to be the \( G \)-connections over a closed spin 2-manifold \( Y \), to be completely egalitarian we allow the Riemannian metric to vary as well. Just as the metrics are “unphysical” in the Lagrangian field theory, so they are in the Hamiltonian field theory. We explain this shortly. Thus, if we fix a principal \( G \) bundle \( Q \to Y \), our in nomine parameter space is \( \text{Met}(Y) \times C(Q) \) but our de facto parameter space is \( C(Q) \).

Given a real virtual representation \( \rho \), we have a canonical family of twisted Dirac operators associated to \( \text{Met}(Y) \times C(Q) : (g, A) \mapsto D_{g,\rho A} \); and so we have a canonical inverse Pfaffian line bundle

\[
\mathcal{L}^{\rho}(Q) \to \text{Met}(Y) \times C(Q). \tag{3.10}
\]

This line bundle is the central figure in the Hamiltonian field theory over \( Y \), just as \( \tau_{X}^{1/2}(D_{\rho}) \) was in the Lagrangian field theory over the spin 3-manifold \( X \). We use the theorems above to formulate some of the geometric properties of \( \mathcal{L}^{\rho}(Q) \).

Proposition 3.7. Let \( \rho \) be a real, rank zero virtual representation of a compact Lie group \( G \) and let \( Q \to Y \) be principal \( G \)-bundle over a closed spin 2-manifold.

1. If \( \gamma : I \to \text{Met}(Y) \times C(Q) \) is a path then

\[
\text{hol}_{\gamma} = \tau_{Y \times I,\rho A,\gamma}^{1/2}(D_{g,\gamma,\rho A,\gamma}) \tag{3.11}
\]

where \( \text{hol}_{\gamma} \) is the holonomy along \( \gamma \) with respect to the natural connection on \( \mathcal{L}^{\rho}(Q) \) and \( g, A \), \( A \gamma \) are the metric and \( G \)-connection on \( Y \times I \) determined by \( \gamma \).
2. The natural connection on $L^\rho(Q)$ is flat along $\text{Met}(Y)$ and if $\Omega$ denotes the curvature of $L^\rho(Q)$ then
\[ \Omega(\dot{A}_1, \dot{A}_2) = 2\pi i \int_Y \langle \dot{A}_1 \wedge \dot{A}_2 \rangle_\rho. \] (3.12)

where $\dot{A}_j \in \Omega^1(\text{ad}Q)$, $j = 1, 2$ are tangent vectors along $C(Q)$.

Remark 3.8. We explain in a more precise way what we mean when we say that the metric is “unphysical” in the Hamiltonian theory. Since the connection is flat along the convex space $\text{Met}(Y)$ we can use parallel transport to identify the fiber $L^\rho(g_1, B)$ with the fiber $L^\rho(g_2, B)$ in a way that is independent of the path in $\text{Met}(Y) \times \{B\}$ connecting the metrics $g_1, g_2$. Also, since the curvature is flat along $\text{Met}(Y)$ and does not depend on the metric in any way, $\Omega$ must lie in the image of the pullback by the projection $\text{Met}Y \times C(Q) \to C(Q)$. In fact, assuming the pairing $\langle \cdot, \cdot \rangle_\rho$ is non-degenerate, $\omega = \Omega / 2\pi i$ defines a symplectic form $\omega$ on $C(Q)$. From now on we require that the pairing be non-degenerate, so that our phase space is properly identified as the symplectic space $(C(Q), \omega)$.

Proof. Let $g_t, A_t, t = (t_1, t_2) \in [-1, 1]^2$ be a two parameter family of metrics and $G$-connections such that $(\partial A_t / \partial t_j)|_{t=0} = \dot{A}_j$. This gives us a natural metric $g$ and connection $A$ on $Q \times [-1, 1]^2$. From (3.8) we have
\[ \Omega = -\pi i \left[ \int_Y \hat{A}(\Omega^g) \text{ch}(\Omega^\rho) \right] \] (3.13)

where the second equality follows from the unraveling of the characteristic polynomials $\hat{A}$ and $\text{ch}$ and the fact that $\rho$ is rank zero. If $\Omega^A_t$ denotes the curvature of the connection $A_t$ then,
\[ \Omega^A = dt_1 \wedge \dot{A}_1 + dt_2 \wedge \dot{A}_2 + \Omega^{A_t}. \]

Plugging this into (3.13) we obtain the result. $\square$

3.5 The line bundle and symplectic reduction

In the Lagrangian field theory over a spin 3-manifold $X$ we insisted that two $G$-connections be physically equivalent if they are isomorphic by some element of $G_C(X)$. Indeed, we saw that the action $\tau^{1/2}$ is invariant with respect to this symmetry. The next few propositions offer some consequences of this symmetry in the Hamiltonian field theory.

Proposition 3.9. The action of $G(Q)$ on $C(Q)$ lifts to $L^\rho(Q)$ and the lifted action preserves the metric and the connection. The induced moment map is
\[ \mu_\zeta(B) = \int_Y \langle \Omega^B \wedge \zeta \rangle_\rho, \] (3.14)

where $\zeta \in \Omega^0(\text{ad}Q)$ is an infinitesimal gauge transformation.
Remark 3.10. In the language of symplectic geometry, this proposition implies that the moduli space of flat $G$-connections on which $\mathcal{G}(Q)$ acts freely is the symplectic quotient $\mathcal{C}(Q)/\mathcal{G}(Q)$. As we shall explain in the proof, there is an induced line bundle $\mathcal{L}\prime(Q) \to \mathcal{M}(Q)$ with a metric and a connection whose curvature is $2\pi i$ times the symplectic form that naturally comes with the symplectic quotient of flat connections.

Proof. Recall that $\mathcal{L}$ is a lift of the $\mathcal{G}(Q)$ action on $\mathcal{C}(Q)$ to $\mathcal{L}\prime(Q)$ that preserves the metric. Given the formulation (3.11) of parallel transport, (2.8) implies that this lift preserves the connection. Thus the action of $\mathcal{G}(Q)$ preserves the curvature of $\mathcal{L}\prime(Q)$ and so the symplectic form $\omega$. We can, therefore, compute the moment map of the symplectic action of $\mathcal{G}(Q)$.

Before we compute the moment map of the $\mathcal{G}(Q)$ action we recall how this is done in the context of automorphisms on line bundles. Whenever $L \to M$ is a hermitian line bundle with unitary connection and $\beta: G \to \text{Aut}(L)$ is a $G$-action on $L$ that preserves the metric and connection, the moment map $\mu_\beta$ of $L$ to $M$ is

$$\mu_\beta(m) = \frac{\text{vert}(\beta(\zeta)u)}{2\pi i}, \quad \zeta \in \mathfrak{g},$$

(3.15)

where $u \in L_m$ is a unitary element, $\beta(\zeta)$ is the vector field on $L$ corresponding to $\zeta \in \mathfrak{g}$, and $\text{vert} (\cdot)$ is the vertical part of the vector with respect to the connection on $L$. This is precisely the obstruction to the connection descending to the quotient $L/G$. If it dissapears (as it does in our case for flat $G$-connections) then the connection descends to the quotient bundle, as claimed in the remarks following the proposition.

Now we apply this to our situation. Suppose $\zeta \in \Omega^0(\text{ad}Q)$ and $\phi_s \in \mathcal{G}(Q)$ is a path of gauge transformations with $\phi_0 = \text{id}_Q$ and $\phi_1 = \zeta$. Consider the path of $G$-connections $B_s = \phi_s^* B$ which forms a connection $A_t$ on $Q \times [0, t]$. To compute the vertical action we "divide" the automorphism $(\rho \phi_t)^*$ by the parallel transport $\tau_{Y \times [0, t]}^{1/2}(D_{\rho A_t})$. Then (2.8) and (3.11) imply that this number is the $\tau^{1/2}$-invariant of the $G$-connection $(Q_t, A_t)$ over $Y \times S^1_b(t)$ gotten by gluing together the endpoints of $(Q \times [0, t], A_t)$ with $\phi_t$. Here $S^1_b(t) = [0, t]/\{0, t\}$ denotes the circle of length $t$ with the bounding spin structure.

To ease our computation, let $\phi$ denote the automorphism of $Q \times [0, t]$ given by $(p, s) \mapsto (\phi_s^{-1} p, s)$. It descends to $Q_t$ and we denote the descendant automorphism by $\phi$, as well. A simple computation shows that

$$\phi^* A_t = B - t \zeta \cdot \theta$$

where $\theta$ is the standard unit measure on $S^1$. Since $\tau^{1/2}$ is invariant with respect to gauge transformations, we have reduced the computation of the vertical action to computing

$$\tau_{Y \times S^1}^{1/2}(D_{\rho (B - t \zeta \cdot \theta)}).$$

An easy application of (1.5) gives us the infinitesimal vertical action

$$\text{vert}((\rho \phi^*_t)|_{t=0}) = \int_Y \langle \Omega^B \wedge \zeta \rangle_{\rho}$$

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and this, together with (3.15), gives us (3.14).

The gluing law (2.14) (perhaps more appropriately called a “cutting law”) is used to good effect in the proof above. We use it again to obtain a result for the action of the stabilizer subgroup $G(B) \subset G(Q)$. In general, the gauge transformations do not act freely on $C(Q)$. The subgroup $G(B)$ are the elements of $C^\infty(\text{Ad}Q)$ that are parallel with respect to $B$. Therefore, the value of a stabilizer of $B$ at $y \in Y$ – which lies in $\text{Aut}(Q_y)$ – commutes with the holonomy group of $B$ at $y$.

**Proposition 3.11.** The action of $G(B)$ on $L^{\rho}(B)$ is constant on components of $\rho G(B)$ and so factors through an action of the finite group $\pi_0(G(B))$ on $L^{\rho}(B)$.

**Proof.** Let $\phi_t, t \in I$ be a path in $G(B)$ and let $(Q_t, B_t)$ denote the vector bundle with connection over on $S^1_b \times Y$ gotten by gluing the together the endpoints of $(Q \times I, B)$ with $\phi_t$. In this way we obtain a connection over $I \times S^1_b \times Y$ which is flat along $I \times S^1$. We use parallel transport along $I$ to obtain a morphism between the vector bundles with connection $(Q_0, B_0)$ and $(Q_1, B_1)$. Thus

$$\tau_{1/2}^{S^1_b \times Y}(D_{\rho B_0}) = \tau_{1/2}^{S^1_b \times Y}(D_{\rho B_1})$$

and so, according to (2.14), $\phi_0$ and $\phi_1$ induce the same automorphism on $L^{\rho}B$.

3.6 The functor of the line bundle

We end this chapter with an observation of how the Lagrangian and Hamiltonian field theories are related to one another. Let $X$ be any compact spin 3-manifold with boundary $\partial X$. There is a restriction functor $r_X : C_G(X) \to C_G(\partial X)$ which sends $(P, A) \mapsto (P, A)|_{\partial X}$. This functor restricts to flat connection and descends to equivalence classes so that we have the commutative diagram

$$
\begin{array}{ccc}
\mathcal{M}_G(X) & \longrightarrow & \overline{C}_G(X) \\
r_X \downarrow & & \downarrow r_X \\
\mathcal{M}_G(\partial X) & \longrightarrow & \overline{C}_G(\partial X).
\end{array}
$$

Having said that we offer the following proposition.

**Proposition 3.12.** The map $r_X : M_G(X) \to M_G(\partial X)$ is Lagrangian. In fact, the action $\tau_{1/2}^X$ is a flat section of the pullback $r_X^* L^\rho(\partial X) \to M_G(X)$ so that the induced symplectic form $r_X^* \omega$ vanishes.

This proposition implies that the holonomy on $r_X^* L^\rho(\partial X)$ is trivial. The image of $r_X$ is an example of a Bohr-Sommerfeld orbit of the line bundle $L^\rho(\partial X)$. These play a pivotal role in quantization with real polarizations. We will have more to say about this when we quantize the Hamiltonian theory over a genus one surface.
Proof. To simplify the exposition we restrict to the connections on a particular \( G \)-bundle \( P \to X \). Let \( \mathcal{C}(P)_{\text{flat}} \) denote the subset of flat connections on \( P \). Over this space the variation formula (2.13) gives

\[
\nabla_{\dot{A}} \tau_{X}^{1/2} = \pi_{i} \cdot \tau_{X}^{1/2} \int_{X} (\dot{A} \wedge \Omega^{A})_{\rho}
\]

where \( \dot{A} \) is some vector at \( T_{A} \mathcal{C}(Q) \). Since \( A \) is flat the right hand side is zero and so \( \tau_{X}^{1/2} \) is a flat section.

It remains to show that, if \( A \) and \( A|_{\partial X} \) represent smooth points of \( \mathcal{M}_{G}(X) \) and \( \mathcal{M}_{G}(\partial X) \) respectively, then

\[
2 \dim \ \text{image}(r_{X})_{*} = \dim H^{1}(X; d_{A}).
\]

As we have nothing new to add we defer to the proof that appears in Proposition 3.27 of [12].

We end by summarizing the main point of the Hamiltonian field theory, that being the assignment

\[ Y \mapsto \mathcal{L}^{\rho}(Y) \to \mathcal{M}_{G}(Y). \]

This clearly obeys the same functoriality, orientation, additivity and gluing laws that appear in Proposition 2.5. The Hamiltonian theory makes contact with the Lagrangian theory when \( Y = \partial X \). In this case we have the assignment

\[ X \mapsto (\tau_{X}^{1/2} : \mathcal{M}_{G}(X) \to r_{X}^{\rho}(\partial X)), \]

and we have seen this adheres to the same four rules mentioned above. This ends our study of the classical spin-Chern-Simons field theory.

4 Appendix

4.1 \( KO \) of a compact 3-manifold

We compute the \( KO \) group of a closed, connected 3-manifold in terms of its \( \mathbb{Z}/2\mathbb{Z} \)-valued cohomology. First of all we have that, for \( X \) connected,

\[ KO(X) \cong \mathbb{Z} \oplus \widetilde{KO}(X) \]

where \( \widetilde{KO}(X) \) is the kernel of the rank map.

Now we consider the group structure on \( H^{1}(X; \mathbb{Z}/2\mathbb{Z}) \times H^{2}(X; \mathbb{Z}/2\mathbb{Z}) \) given by the product

\[
(a_{1}, b_{1}) \cdot (a_{2}, b_{2}) = (a_{1} + a_{2}, a_{1} \sim a_{2} + b_{1} + b_{2}).
\]

We let \( H^{1}(X; \mathbb{Z}/2\mathbb{Z}) \times H^{2}(X; \mathbb{Z}/2\mathbb{Z}) \) denote the set \( H^{1}(X; \mathbb{Z}/2\mathbb{Z}) \times H^{2}(X; \mathbb{Z}/2\mathbb{Z}) \) with this group structure.
Theorem 4.1. For $X$ a closed, connected, compact 3-manifold, the map

$$\tilde{KO}(X) \longrightarrow H^1(X; \mathbb{Z}/2\mathbb{Z}) \times H^2(X; \mathbb{Z}/2\mathbb{Z})$$

$$E - F \mapsto (w_1(E) + w_1(F), w_2(E) + w_2(F) + w_1(E)w_1(F) + w_1(F)^2)$$

is an isomorphism.

To prove Theorem 4.1 we will use the following fact about oriented vector bundles over a compact 3-manifold.

Proposition 4.2. For $n \geq 3$ the 2nd Stiefel-Whitney class provides a one-to-one correspondence between topological $SO_n$ bundles over $X$ and classes in $H^2(X; \mathbb{Z}/2\mathbb{Z})$.

Proof. We first show injectivity. From the short exact sequence

$$1 \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow Spin_n \rightarrow SO_n \rightarrow 1$$

we get an exact sequence of Čech cohomology groups

$$\cdots \rightarrow H^1(X; Spin_n) \rightarrow H^1(X; SO_n) \xrightarrow{w_2} H^2(X; \mathbb{Z}/2\mathbb{Z})$$

For $n \geq 3$ $Spin_n$ is simply connected so that, according to a simple argument in obstruction theory, any $Spin_n$ principal bundle over a 3-manifold is topologically trivial. Thus the map $w_2$ is injective.

We now show $w_2$ is surjective. To see this one takes the short exact sequence

$$0 \rightarrow \mathbb{Z} \rightarrow Spin_n \rightarrow SO_n \rightarrow 1$$

and the induced long exact sequence in cohomology

$$\cdots \rightarrow H^2(X; \mathbb{Z}) \rightarrow H^2(X; \mathbb{Z}/2\mathbb{Z}) \rightarrow H^3(X; \mathbb{Z}) \rightarrow \cdots$$

The image of $\beta$ in $H^3(X; \mathbb{Z}) \cong \mathbb{Z}$ will be torsion and so must be zero. Thus the “mod 2” cohomology map is surjective. Of course, $H^2(X; \mathbb{Z})$ parametrizes the topological $SO_2$ bundles over $X$ via the 1st Chern class

$$c_1 : H^1(X; SO_2) \rightarrow H^2(X; \mathbb{Z}).$$

The standard inclusion homomorphism $SO_2 \hookrightarrow SO_n$ and the associated bundle map

$$H^1(X; SO_2) \rightarrow H^1(X; SO_n)$$

$$[P] \mapsto [P \times_{SO_2} SO_n]$$

give us the commutative diagram

$$\begin{array}{ccc}
H^1(X; SO_2) & \xrightarrow{\text{assoc. bundle}} & H^1(X; SO_n) \\
\downarrow \text{c}_1 \mod 2 & & \downarrow w_2 \\
H^2(X; \mathbb{Z}/2\mathbb{Z}) & \xrightarrow{\text{identity}} & H^2(X; \mathbb{Z}/2\mathbb{Z})
\end{array}$$

so that, since the “mod 2” map is surjective, $w_2$ is surjective. \qed
With this result at our disposal we can now prove Theorem 4.1.

Proof of Theorem 4.1. That the map is well-defined and a homomorphism easily follow from the properties of the Stiefel-Whitney classes.

To prove surjectivity, choose any \( b \in H^2(X; \mathbb{Z}/2\mathbb{Z}) \) and \( a \in H^1(X; \mathbb{Z}/2\mathbb{Z}) \). Let \( \ell \) be a real line bundle over \( X \) such that \( w_1(\ell) = a \); and, according to Lemma (A.1.2), we can choose an oriented rank 2 bundle \( E \) such that \( w_2(E) = b \). Then

\[
(w_1, w_2)(E \oplus \ell) = (a, b)
\]

proving surjectivity.

To prove injectivity we consider the kernel of the homomorphism. Indeed, suppose \( E - F \in \tilde{KO}(X) \) is such that \( w_1(E) = w_1(F) \) and \( w_2(E) = w_2(F) + w_1(F)w_1(E) \). The first equality implies \( \text{Det}E \cong \text{Det}F \) so that in \( KO(X) \)

\[
E - F = (E \oplus \text{Det}E) - (F \oplus \text{Det}F)
\]

Since the map is well defined and

\[
w_1(E \oplus \text{Det}E) = w_1(F \oplus \text{Det}F) = 0
\]

we must have that

\[
w_2(E \oplus \text{Det}E) = w_2(F \oplus \text{Det}F)
\]

From Lemma 4.2 we know this equality implies the equivalence

\[
E \oplus \text{Det}E \cong F \oplus \text{Det}F
\]

so that in \( KO(X) \)

\[
E - F = (E \oplus \text{Det}E) - (F \oplus \text{Det}F) = 0.
\]

Thus the kernel of the homomorphism is trivial.

4.2 Cobordism groups

In this section we compute certain cobordism groups that pop up Chapter 2. In particular, they appear in the proofs to Propositions 1.7 and 1.8.

Proposition 4.3. Let \( \Omega_{n}^{\text{spin}}(M) \) denote the degree-\( n \) spin cobordism group of the topological space \( M \). Then

\[
\Omega_{3}^{\text{spin}}(BSO \wedge B(\mathbb{Z}/2\mathbb{Z})) = \mathbb{Z}/2\mathbb{Z} \quad \text{and} \quad \Omega_{3}^{\text{spin}}(BSO) \cong \mathbb{Z}/2\mathbb{Z}.
\]

and the generator of \( \Omega_{3}^{\text{spin}}(BSO) \) is represented by \( S_{1b}^{1} \times F \to S_{1b}^{1} \times S^{2} \). Here \( F \to S^{2} \) is an oriented rank zero virtual vector bundle with non-trivial \( w_{2} \) and \( S_{1b}^{1} \) is the circle with the non-bounding spin structure.

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Proof. The first isomorphism can be seen as follows. Since $BSO$ is 1-connected and $B(\mathbb{Z}/2\mathbb{Z})$ is connected the wedge product $BSO \wedge B(\mathbb{Z}/2\mathbb{Z})$ is 2-connected. The Hurwitz theorem implies that the first non-trivial homology (for any coefficient group) is $H_3$ and from the $E^2$ term of the Atiyah-Hirzebruch spectral sequence it is clear we need only determine $H_3(BSO \wedge B(\mathbb{Z}/2\mathbb{Z}))$. This is easily computed by considering the long exact homology sequence induced by the maps

$$BSO \wedge B(\mathbb{Z}/2\mathbb{Z}) \hookrightarrow BSO \times B(\mathbb{Z}/2\mathbb{Z}) \to BSO \wedge B(\mathbb{Z}/2\mathbb{Z}).$$

One then sees that $H_3(BSO \wedge B(\mathbb{Z}/2\mathbb{Z})) \cong \mathbb{Z}/2\mathbb{Z}$ and this proves the first isomorphism of the proposition.

The first isomorphism is used to prove Proposition 1.7 and now we use that proposition to prove the second isomorphism. From the $E^2$ term of the Atiyah-Hirzebruch spectral sequence we can conclude that $\Omega_3^{\text{spin}}(BSO_N)$ is isomorphic to either $\mathbb{Z}/2\mathbb{Z}$ or $\{0\}$ for any $N > 2$. We will now show that certain structures on $S^1 \times S^2$ cannot be the boundary of structures on a 4-manifold so that the cobordism group must be isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

Let $(Q, B) \to S^2$ be an $SO_N$ connection such that $Q$ has non-trivial $w_2$. Contrary to the claim of the proposition, we assume that there exists $(P, A) \to M$ such that

$$\partial((P, A) \to M) = (S^1_{nb} \times (Q, B) \to S^1_{nb} \times S^2). \quad (4.1)$$

On the other hand, there is no question that there exists $(P', A') \to D^2 \times S^2$ such that

$$\partial((P', A') \to D^2 \times S^2) = (S^1_b \times (Q, B) \to S^1_b \times S^2).$$

Let $\rho = id_{SO_N} - N$ be the identity representation minus the rank $N$ trivial representation and let $\ell$ represent the non-trivial element of $H^1(S^1; \mathbb{Z}/2\mathbb{Z})$. According to Proposition 1.7

$$\frac{\tau^{1/2}_{S^1_b \times S^2}(D_{\rho B})}{\tau^{1/2}_{S^1_b \times S^2}(D_{\rho B})} = \frac{\tau^{1/2}_{S^1_b \times S^2}(D_{\rho B \otimes (\ell - 1)})}{\tau^{1/2}_{S^1_b \times S^2}(D_{\rho B})} = (-1)^{w_2(Q) - \ell} = -1.$$

However (4.1) implies

$$\frac{\tau^{1/2}_{S^1_b \times S^2}(D_{\rho B})}{\tau^{1/2}_{S^1_b \times S^2}(D_{\rho B})} = \exp \pi i \int_{M \cup -(D^2 \times S^2)} \{\Omega^{A \cup A'} \wedge \Omega^{A \cup A'}\}_{\rho} = 1$$

where the last equality follows from the fact the integral is an even integer (as it is the index of the twisted chiral Dirac operator $D_{\rho (A \cup A')}$ on $M \cup -(D^2 \times S^2)$). Thus we get a contradiction and thereby prove the second isomorphism and the claim following it. \(\square\)
4.3 Computations on the 3-Torus

Here we compute the quadratic form $q$ as defined in (1.8) when the closed 3-manifold is a 3-torus. These particular computations are used in the cobordism argument for parts (2) and (3) of Proposition 1.7.

**Proposition 4.4.** Let $E \in \widetilde{KO}(X)$ be such that $w_1(E) = 0$. Then $q_\sigma(E, \ell)$ is independent of the spin structure $\sigma$ and depends linearly on $\ell \in H^1(X; \mathbb{Z}/2\mathbb{Z})$.

**Proof.** An easy computation shows that if $\ell_0 \in H^1(X; \mathbb{Z}/2\mathbb{Z})$ then $w_1(E \otimes \ell_0) = 0$ and $w_2(E \otimes \ell_0) = w_2(E)$. Thus, by (1.8) $E \otimes \ell_0 = E$ in $KO(X)$. This implies

$$q_{\sigma+\ell_0}(E, \ell) = q_{\sigma}(E \otimes \ell_0, \ell) = q_{\sigma}(E, \ell)$$

where the first equality follows from the fact that the Dirac operator for $\sigma + \ell_0$ is just the Dirac operator for $\sigma$ twisted by $\ell_0$. This proves independence with respect to the spin structure.

Now consider $q_{\sigma}(E, \ell_1 \otimes \ell_2)$. Using the argument in the above paragraph we see that

$$q_{\sigma}(E, \ell_1 \otimes \ell_2) = q_{\sigma}(E \otimes \ell_1, \ell_1 \otimes \ell_2) = q_{\sigma}(E, \ell_1) \cdot q_{\sigma}(E, \ell_2)$$

where the second equality follows easily from the definition of $q$ and the fact that (in the case $w_1(E) = 0$) $q$ is $\mathbb{Z}/2\mathbb{Z}$-valued. \qed

We are now in a good position to make some actual computations. In particular we look at $q$ on the 3-torus $T^3$. What we find is that

**Proposition 4.5.** Let $E \in \widetilde{KO}(T^3)$ be such that $w_1(E) = 0$. Then for any $\ell \in H^1(T^3; \mathbb{Z}/2\mathbb{Z})$

$$q_\sigma(E, \ell) = (-1)^{w_2(E) - \ell}.$$

**Proof.** Take $T^3 = \mathbb{R}^3/\mathbb{Z}^3$ with the three standard projections $\pi_j : T^3 \to T$ given by $(x_1, x_2, x_3) \mapsto x_j$ for $j = 1, 2, 3$. Let $\ell_0 \in H^1(T; \mathbb{Z}/2\mathbb{Z})$ be the non-trivial element. Then the elements $\ell_j = \pi_j^* \ell_0$ generate $H^1(T^3; \mathbb{Z}/2\mathbb{Z})$ and the elements $\ell_1 \sim \ell_2$ generate $H^2(T^3; \mathbb{Z}/2\mathbb{Z})$. We prove the proposition for a set of generators and then appeal to linearity for the other cases.

Without loss of generality we assume $w_2(E) = \ell_1 \sim \ell_2$ and consider the cases $\ell = \ell_1$ or $\ell_2$. Then $w_2(E) \sim \ell = 0$. On the other hand, all of the topological data is trivial along the 1-cycle $c_3 = (0 \oplus 0 \oplus \mathbb{R})/\mathbb{Z}^3$. We are free to choose the smooth parameters to be trivial along $c_3$ as well. In fact, according to Proposition 4.4 just above, we can even assume that the spin structure $\sigma$ extends across a disk bounded by $c_3$. This allows us to implement the gluing formula (2.13) on each of the $\tau^{1/2}$ factors appearing in $q$. We find

$$\tau^{1/2}_{T^2 \times S^1_{\theta}}(D_{E \otimes \ell}) = 1$$

and

$$\tau^{1/2}_{T^2 \times S^1_{\theta}}(D_E) = 1$$

so that, in this case, $q_\sigma(E, \ell) = 1 = (-1)^{w_2(E) - \ell}$.
Now we consider the case $\ell = \ell_3$. Then $w_2(E) \sim \ell = 1$. On the other hand, we can take all of the parameters to be trivial along $c_3$ (including a bounding spin structure) except for the line bundle $\ell_3$. But we can still use the gluing formula. Indeed we see that

$$\tau^{1/2}_{T^2 \times S^1_h}(DE \otimes \ell_3) = (-1)^{\text{ind}_2(D_{T^2, E'})}$$

and $\tau^{1/2}_{T^2 \times S^1_h}(DE) = 1$

where $D_{T^2, E'}$ is the Dirac operator on $T^2$ twisted by the virtual vector bundle $E'$ which is the restriction of $E$ to $T^2 \subset T^3$. Since $\text{ind}_2(D_{T^2, E'}) = w_2(E') = 1$ we get that, in this case, $q_\sigma(E, \ell) = -1 = (-1)^{w_2(E) \sim \ell}$.

### 4.4 The “Level” of the Theory

To define the action of our spin-Chern-Simons theory we require an element of the virtual oriented representation ring, $\widehat{\text{RSO}}(G)$. In fact, to eliminate metric dependence, we require that the element have rank zero. Such elements form an ideal which we denote by $\widetilde{\text{RSO}}(G)$.

A representation, $\rho : G \to \text{SO}_N$, generates a unique homotopy class of maps, $B_\rho : BG \to \text{BSO}$, between classifying spaces. Homotopy classes of maps into $\text{BSO}$ is a bit too rich in structure for our needs. In order to obtain a leaner structure we cap-off all of the homotopy groups of $\text{BSO}$ above $\pi_4$. What is left is a space whose only nontrivial homotopy lies in $\pi_2$ and $\pi_4$. In fact, this capped-off $\text{BSO}$ is homotopic to a fibration, the total space of which we’ll denote by $E^4$. The fibration is

$$K(\mathbb{Z}, 4) \xrightarrow{\text{inclusion}} E^4 \xrightarrow{w_2} K(\mathbb{Z}/2\mathbb{Z}, 2)$$

The map $w_2$ is an extension of the usual second Steiffel-Whitney class on $\text{BSO}$. Thus $E^4$ is a twisted product of Eilenberg-MacLane spaces. Such fibrations with fiber $K(A, m)$ and base $K(B, n)$ are classified by elements of $H^{m+1}(K(B, n), A)$. The fibration seen above is determined by $\beta \circ Sq^2(\iota)$, where $\beta$ is the Bockstein homomorphism and $\iota$ is the fundamental class of $H^2(K(\mathbb{Z}/2, 2); \mathbb{Z}/2)$. The “levels” in this theory are homotopy classes of maps from $BG$ into $E^4$. This has a group structure since $E^4$ is, much like an Eilenberg-MacLane space, homotopic to a loop space. Indeed, if we take the fibration

$$K(\mathbb{Z}, 5) \xrightarrow{\text{inclusion}} E^5 \xrightarrow{w} K(\mathbb{Z}/2\mathbb{Z}, 3)$$

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which is determined by $\beta \circ Sq^2(\iota')$ — where $\iota'$ is the fundamental class of $H^3(K(\mathbb{Z}/2,3);\mathbb{Z}/2)$ — we have that $E^4 \sim \Omega E^5$.

In fact, the two previous fibrations are part of something in homotopy theory called a “spectrum”. For any counting number, $N$, we have a fibration

$$K(\mathbb{Z}, N) \xrightarrow{\text{inclusion}} E^N \xrightarrow{w} K(\mathbb{Z}/2\mathbb{Z}, N-2)$$

such that $E^{N-1} = \Omega E^N$. Each such fibration is determined by $\beta \circ Sq^2(\iota_{N-2})$, where $\iota_{N-2}$ is the fundamental class of $H^{N-2}(K(\mathbb{Z}/2, N-2);\mathbb{Z}/2)$. Such a spectrum of fibrations induces a long exact sequence of homotopy groups. In particular we have

$$\cdots \to [X, K(\mathbb{Z}, N)] \to [X, E^N] \to [X, K(\mathbb{Z}/2\mathbb{Z}, N-2)] \to [X, K(\mathbb{Z}, N+1)] \to \cdots$$

or more succinctly

$$\cdots \to H^N(X;\mathbb{Z}) \to E^N(X) \to H^{N-2}(X;\mathbb{Z}/2\mathbb{Z}) \xrightarrow{\beta \circ Sq^2} H^{N+1}(X;\mathbb{Z}) \to \cdots$$

We will be interested in the following portion of the long exact sequence for $BG$:

$$\cdots \to H^1(BG;\mathbb{Z}) \to E^4(BG) \to H^2(BG;\mathbb{Z}/2) \to \cdots$$

An easy argument shows that the first map is injective. Indeed, the map $Sq^2$ is zero on $H^1(BG;\mathbb{Z}/2\mathbb{Z})$ so that $\beta \circ Sq^2 : H^1(BG;\mathbb{Z}/2\mathbb{Z}) \to H^4(BG;\mathbb{Z})$ is zero as well. From this definition it is clear that when $G$ is simply connected $E^4(BG) \cong H^4(BG;\mathbb{Z})$ since $H^4(BG;\mathbb{Z}/2\mathbb{Z}) = 0$.

The first Pontryagin map $p_1 : BSO \to K(\mathbb{Z}, 4)$ extends to a map on $E^4$. One can show that

**Proposition 4.6.** The sequence of homomorphisms

$$H^1(BG;\mathbb{Z}) \to E^4(BG) \xrightarrow{p_1} H^4(BG;\mathbb{Z})$$

is multiplication by 2 in $H^4(BG;\mathbb{Z})$.

**Proof.** We begin by noting that the fibration

$$K(\mathbb{Z}, 4) \hookrightarrow E^4 \xrightarrow{w_2} K(\mathbb{Z}/2\mathbb{Z}, 2)$$

can be derived from the fibration

$$BSpin \hookrightarrow BSO \xrightarrow{w_2} K(\mathbb{Z}/2\mathbb{Z}, 2)$$

by capping off all the homotopy generators of $\pi_n(BSpin)$ for $n > 4$. Thus, when we consider the sequence of maps

$$K(\mathbb{Z}, 4) \hookrightarrow E^4 \xrightarrow{p_1} K(\mathbb{Z}, 4)$$

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we are effectively comparing the pullback of $p_1 \in H^4(BSO; \mathbb{Z})$ to a generator of $H^4(BSpin; \mathbb{Z})$. If we denote the pullback by $p_1$ as well (as is commonly done in the literature) then it is well known that there is a generating class “$p_1/2$” $\in H^4(BSpin; \mathbb{Z})$ such that $p_1 = 2 \cdot (p_1/2)$. This proves the proposition.

\[\Box\]

**Proposition 4.7.** If $G$ is a connected compact lie group then $H^4(BG; \mathbb{Z})$ is torsionless.

**Proof.** Let $T \subset G$ be a maximal torus. Consider the cohomology spectral sequence $E_r$ of the fibration $G/T \hookrightarrow BT \rightarrow BG$. We have two useful facts to help us along: $H^{odd}(G/T) = 0$ and $H^{even}(G/T)$ is torsionless.

Now $E_2^{4,0} = H^4(BG; \mathbb{Z})$ and it's clear from all of the zeros in the degree 3 diagonal line that $E_\infty^{4,0} = E_2^{4,0}$. Since $E_\infty^{4,0}$ is isomorphic to a subgroup of $H^4(BT)$, which is torsionless, we conclude that $H^4(BG; \mathbb{Z})$ is torsionless.  

**Proposition 4.8.** The homomorphism $(p_1, w_2) : E^4(BG) \rightarrow H^4(BG; \mathbb{Z}) \oplus H^2(BG; \mathbb{Z}/2)$ is injective up to 2-torsion elements of $H^4(BG; \mathbb{Z})$ so that, if $G$ is connected, $(p_1, w_2)$ is injective.

As an easy corollary we have

**Corollary 4.9.** Let $\lambda(\rho)$ be the class in $E^4(BG)$ induced by the representation $\rho$. Then we have $\lambda(\rho_0 \oplus \rho_1) = \lambda(\rho_0) + \lambda(\rho_1)$.

As an example – and because we will later use these results when we consider the quantum theory – we consider $E^4(BSU_2)$ and $E^4(BSO_3)$. Since $SU_2$ is simply connected we easily have

$$E^4(BSU_2) \cong H^4(BSU_2; \mathbb{Z}) = \mathbb{Z} \cdot c_2$$

where $c_2$ is the 2nd Chern class. More relevant to spin-Chern-Simons is the image of the map

$$\lambda : \widetilde{RO}(BSU_2) \rightarrow E^4(BSU_2).$$

Let $\rho : SU_2 \rightarrow SO_4$ be the realization of the standard $SU_2$ representation on $\mathbb{C}^2$. We take the rank zero representation $(\rho - 4)$; that is, $\rho$ minus the trivial 4 dimensional representation. With some foresight, we denote $1' = \lambda(\rho - 4)$ and claim that $E^4(SU_2) = \mathbb{Z} \cdot 1'$. Indeed, $p_1(1') = -2c_2$; and combined with Proposition 4.8 this proves the claim. Then, according to Corollary 4.9 to hit all of the levels of $SU_2$ spin-Chern-Simons we need only consider integer multiples of $(\rho - 4)$.

Of course, $SO_3$ is not simply connected. In fact, $H^2(BSO_3; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ is generated by the 2nd Stiefel-Whitney class. We consider the image of the map

$$\lambda : \widetilde{RO}(BSO_3) \rightarrow E^4(BSO_3).$$
In particular we consider the element $1 = \lambda(id_{SO_3} - 3)$ and claim that $E^4(BSO_3) = \mathbb{Z} \cdot 1$. Indeed, $w_2(1) \in H^2(BSO_3; \mathbb{Z}/2\mathbb{Z})$ is the non-trivial element so that $w_2$ is surjective. Thus we have a short exact sequence

$$0 \to (H^4(BSO_3; \mathbb{Z}) \cong \mathbb{Z}) \to E^4(BSO_3) \xrightarrow{w_2} (H^2(BSO_3; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}) \to 0$$

where $H^4(BSO_3; \mathbb{Z})$ is generated by the 1st Pontryagin class. This implies that $E^4(BSO_3)$ is isomorphic to either $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}$. If it is isomorphic to the former then, with respect to that isomorphism, $\text{image}(p_1 \oplus w_2) = 2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. However, it is clear that $(p_1 \oplus w_2)(1) = 1 \oplus (1 \mod 2)$, and so it must be that $E^4(BSO_3)$ is isomorphic to $\mathbb{Z}$. Now Propositions 4.6 and 4.9 imply that 1 is a generator.

We end these considerations by pointing out that the standard 2:1 covering homomorphism $\beta : SU_2 \to SO_3$ induces the homomorphism

$$B^*_\beta : E^4(BSO_3) \longrightarrow E^4(BSU_2)$$

$$k \cdot 1 \longmapsto 2k \cdot 1'.$$

Indeed, consider the commutative diagram

$$\begin{array}{cccccc}
H^4(BSO_3; \mathbb{Z}) & \longrightarrow & E^4(BSO_3) & \xrightarrow{w_2} & H^2(BSO_3; \mathbb{Z}/2\mathbb{Z}) & \\
B^*_\beta \downarrow & & B^*_\beta \downarrow & & B^*_\beta \downarrow \\
H^4(BSU_2; \mathbb{Z}) & \longrightarrow & E^4(BSU_2) & \xrightarrow{w_2} & 0 & \\
\end{array} (4.2)$$

A simple argument in Chern-Weil theory shows that $B^*_\beta$ sends the generator of $H^4(BSO_3; \mathbb{Z})$ to 4 times the generator of $H^4(BSU_2; \mathbb{Z})$. Thus, the diagram 4.2 is isomorphic to the diagram

$$\begin{array}{cccccc}
\mathbb{Z} & \xrightarrow{\times 2} & \mathbb{Z} & \xrightarrow{(\text{mod} 2)} & \mathbb{Z}/2\mathbb{Z} & \\
\downarrow \times 4 & & B^*_\beta \downarrow & & \downarrow 0 \\
\mathbb{Z} & \xrightarrow{\times 1} & \mathbb{Z} & \longrightarrow & 0 & \\
\end{array}$$

and from this diagram’s commutativity it is clear that $B^*_\beta$ is effectively multiplication by 2.

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