MODULI SPACES OF D-CONNECTIONS AND DIFFERENCE PAINLEVÉ EQUATIONS

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Abstract. We show that difference Painlevé equations can be interpreted as isomorphisms of moduli spaces of d-connections on \( \mathbb{P}^1 \) with given singularity structure. In particular, we derive a difference equation that lifts to an isomorphism between \( A_2(1) \)-surfaces in Sakai’s classification \[29\]; it degenerates to both difference Painlevé V and classical (differential) Painlevé VI equations. This difference equation has been known before under the name of asymmetric discrete Painlevé IV equation.

1. Introduction

This paper is about difference Painlevé equations and their geometric properties. The term ‘discrete (difference, q-difference, or elliptic) Painlevé equation’ is rather vague: there exist different ways of discretizing the classical (2nd order differential) Painlevé equations, see e.g. \[13\], \[23\], \[22\], \[19\], \[29\]. We will consider the equations that fit into Sakai’s classification described in \[29\].

Any equation of Sakai’s hierarchy, by definition, originates from a birational automorphism of \( \mathbb{C}^2 \) that lifts to a regular isomorphism between the blow-ups of \( \mathbb{P}^2 \) at 9 points. This geometric property allows to classify the equations according to the type of the resulting surface. The hierarchy also includes the classical Painlevé equations for which the surfaces are viewed as spaces of initial conditions, see \[24\], \[25\].

In the last few years several researchers computed the so-called ‘gap probabilities’ in various discrete probabilistic models of random matrix type, see \[4\], \[6\], \[7\], \[1\], \[8\]-\[12\]. Surprisingly, these quantities were often expressible in terms of certain specific solutions of equations from Sakai’s hierarchy. Later it was demonstrated that the equations arising in probabilistic models can be viewed as reductions of isomonodromy transformations of systems of linear difference equations with rational coefficients \[5\]. Further discussion of monodromy for difference equations can be found in \[20\].

The goal of this paper is two-fold. First, we show how the geometric approach to isomonodromy transformations implies that the transformations lift to isomorphisms between suitable surfaces. This provides a conceptual explanation of the above coincidence. The surfaces are geometrically interpreted as suitable moduli spaces of d-connections (short for “difference connections”) on the Riemann sphere. Second, we derive an equation of Sakai’s hierarchy that lifts to an isomorphism between \( A_2(1) \)-surfaces in Sakai’s classification \[29\]. We call this equation the difference Painlevé VI, or dPVI.

Let us briefly describe our results.
Consider a matrix linear difference equation
\begin{equation}
(1.1) \quad y(z+1) = A(z)y(z), \quad A(z) = A_0 z^n + \cdots + A_{n-1} z + A_n, \quad A_i \in \text{Mat}(m, \mathbb{C}).
\end{equation}
We will always assume that $A_0$ is invertible. According to [5], isomonodromy transformations of this equation consist of maps of the form
\begin{equation}
(1.2) \quad A(z) \mapsto A'(z) = R(z+1) A(z) R(z)^{-1}
\end{equation}
for suitable rational matrix-valued functions $R(z)$. For generic $A(z)$, these transformations are parameterized by integral shifts of the zeros of $A(z)$ and of certain exponents at $z = \infty$ with total sum of shifts equal to 0, see [5][Theorem 2.1]. We can then express the matrix elements of $A'(z)$ as functions of the matrix elements of $A(z)$; in special cases, the expressions give rise to the difference Painlevé equations.

However, the isomonodromy transformation is defined only when $A(z)$ is generic enough. Therefore, the resulting maps are rational rather than regular, that is, the formulas for matrix elements of $A'(z)$ have singularities. In order to resolve these singularities it is convenient to use the geometric approach.

Let $\mathcal{L}$ be a vector bundle on $\mathbb{P}^1$ of rank $m$. Assume that we are given a $d$-connection on $\mathcal{L}$ which is, by definition, a linear operator $A(z) : \mathcal{L}_z \to \mathcal{L}_{z+1}$ that depends on $z$ polynomially. (Here $\mathcal{L}_z$ is the fiber of $\mathcal{L}$ over $z$.) If $\mathcal{L}$ is the trivial vector bundle, $A(z)$ is a matrix difference equation [14].

There is a natural operation on vector bundles with $d$-connection called modification: it is induced by a rational isomorphism $R : \mathcal{L} \to \mathcal{L}'$ between two vector bundles. A $d$-connection $A$ on $\mathcal{L}$ then induces a $d$-connection $A'$ on $\mathcal{L}'$ and vice versa. Isomonodromy transformations described above can be viewed as special cases of such modifications.

Let us consider the example that leads to the difference Painlevé V equation (dPV). Take $m = \text{rank of } \mathcal{L} = 2$; assume that $A(z)$ has four simple zeros $a_1, a_2, a_3, a_4 \in \mathbb{C}$, and that there exists a trivialization of $\mathcal{L}$ in a neighborhood of $z = \infty$ with respect to which the matrix of $A(z)$ has the form
\begin{equation*}
A(z) = \begin{bmatrix} \rho_1 & 0 \\ 0 & \rho_2 \end{bmatrix} z^2 + \begin{bmatrix} \rho_1 d_1 & 0 \\ 0 & \rho_2 d_2 \end{bmatrix} z + O(1), \quad z \to \infty.
\end{equation*}

**Proposition (isomonodromy transformation).** Under certain non-degeneracy conditions on the parameters $(a_1, \ldots, a_4, \rho_1, \rho_2, d_1, d_2)$, for any vector bundle $\mathcal{L}$ with $d$-connection $A$ as above and any integral shifts of the parameters $a_1, \ldots, a_4, d_1, d_2$, there exists a unique vector bundle $\mathcal{L}'$ with $d$-connection $A'$ related to $(\mathcal{L}, A)$ by a modification, and such that it satisfies the above assumptions with shifted values of parameters.

Note that we do not need to assume that $(\mathcal{L}, A)$ is generic. This means that the modifications of this proposition give (regular, not birational) isomorphisms of the moduli spaces of vector bundles with $d$-connections with given singularity structure, provided the parameters are generic.

From now on, let us also assume that
\[ \text{deg}(\mathcal{L}) = -(a_1 + \cdots + a_4 + d_1 + d_2) = -1. \]
This condition implies that $\mathcal{L}$ is always isomorphic to $\mathcal{O} \oplus \mathcal{O}(-1)$. (Notice that an isomonodromy transformation fixes $\text{deg}(\mathcal{L})$ if and only if the corresponding shifts of the parameters $a_1, \ldots, a_4, d_1, d_2$ add up to zero.) By a choice of basis in $\mathcal{L}$, the
moduli space of d-connections can be identified with equivalence classes of $2 \times 2$ matrices with polynomial entries satisfying

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad \text{deg } a_{11} \leq 2, \quad \text{deg } a_{22} \leq 2, \quad \text{deg } a_{21} \leq 1, \quad \text{deg } a_{12} \leq 3,$$

$$\det A(z) = \text{const}(z - a_1)(z - a_2)(z - a_3)(z - a_4),$$

$$a_{11} + a_{22}(1 + z^{-1}) = (\rho_1 + \rho_2)z^2 + (d_1 \rho_1 + d_2 \rho_2)z + O(1),$$

modulo the gauge transformations of the form \([1,2]\) with polynomial

$$R = \begin{bmatrix} r_{11} & r_{12} \\ 0 & r_{22} \end{bmatrix}, \quad r_{11} = \text{const}, \quad r_{22} = \text{const}, \quad \text{deg } r_{12} \leq 1.$$

It is not hard to see that this moduli space is two-dimensional. We show that its smallest compactification is a surface of the Sakai type $D_4^{(1)}$; in particular, it is a blow-up of $\mathbb{P}^2$ at 9 points (we use a different description as a blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$). The moduli space itself is the complement of 5 curves (the support of the unique effective anti-canonical divisor) inside this surface.

In order to connect this picture to dPV, we introduce coordinates on the moduli spaces.

**Theorem (dPV).** Take the zero of the linear polynomial $a_{21}$ as the first coordinate, denote it by $q$, and take the value of the matrix element $a_{11}$ at $q$ divided by $(q - a_3)(q - a_4)$ as the second coordinate, denote it by $p$. Consider the modification of $(\mathcal{L}, \mathcal{A})$ to $(\mathcal{L}', \mathcal{A}')$ that shifts

$$a_1 \mapsto a_1 - 1, \quad a_2 \mapsto a_2 - 1, \quad d_1 \mapsto d_1 + 1, \quad d_2 \mapsto d_2 + 1.$$

Then the coordinates $(p', q')$ on the moduli space of $(\mathcal{L}', \mathcal{A}')$ are related to $(p, q)$ by

$$\begin{align*}
q' + q &= a_3 + a_4 + \frac{\rho_1(d_1 + a_3 + a_4)}{p - \rho_1} + \frac{\rho_2(d_2 + a_3 + a_4)}{p - \rho_2}, \\
p'p &= \frac{(q' - a_1 + 1)(q' - a_2 + 1)}{(q' - a_3)(q' - a_4)} \cdot \rho_1 \rho_2.
\end{align*}$$

This is exactly the dPV equation of [13], [29].

**Remark.** The idea of using $(q, p)$ as coordinates on the moduli space is by no means new. For Painlevé equations it has been used, for example, in [18] in the continuous situation, and in [19] in the discrete situation.

Another example that we consider in detail deals with rank 2 vector bundles $\mathcal{L}$ with d-connection $\mathcal{A}(z)$ which has 6 simple zeros $a_1, \ldots, a_6 \in \mathbb{C}$ and whose behavior near $z = \infty$ in a suitable trivialization is given by

$$A(z) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} z^{3} + \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix} z^{2} + O(z).$$

Quite similarly to the case of dPV discussed above, there is an action of $\mathbb{Z}^8$ which is parametrized by integral shifts of $a_i$'s and $d_j$'s. The group acts by isomorphisms of moduli spaces. Let us again assume that $\deg(\mathcal{L}) = -(a_1 + \cdots + a_6 + d_1 + d_2) = -1$.

\[1\] A different reduction of an isomonodromy transformation to dPV can be found in [6].
The difference Painlevé VI equation above is equivalent to the asymmetric Painlevé VI equation as well. This also simplifies various degenerations to other Painlevé equations.

It should be noted that formulas for all isomorphisms of Sakai surfaces in principle can be written using coordinates of [29]. The computation, however, can be rather tedious.
There are simple degenerations that turn dPVI into dPV and the classical PVI equations. In a sense, this can be done simultaneously. Let us consider, in addition to the flow given by the shift (1.3), the flow generated by the shift

\[ a_3 \mapsto a_3 - 1, \quad a_4 \mapsto a_4 - 1, \quad d_1 \mapsto d_1 + 1, \quad d_2 \mapsto d_2 + 1. \]

Clearly, the flow given by the shift (1.4) is also described by dPVI with a slightly different p-coordinate. Now let \( a_1, a_2, d_1, \) and \( d_2 \) go to infinity at speeds \(-\rho_1, -\rho_2, \rho_1,\) and \( \rho_2, \) respectively. In the limit, the dPVI equation corresponding to (1.3) converges to a continuous vector field which is equivalent to the classical PVI. At the same time, the flow corresponding to (1.4) converges to a discrete flow described by dPV. As the result, we get two commuting flows on the same surface (of the Sakai type \( D_4^{(1)} \)): a vector field given by dPVI and a discrete dynamics given by dPV.

This limiting picture can be seen from two points of view. First, the classical PVI possesses the so-called Bäcklund transformations which can be described via dPV, see [11]. Second, there is a natural continuous isomonodromy deformation that moves the parameters \( \rho_1, \rho_2 \) in the dPV setting; it can be reduced to the classical PVI. Finally, the geometric Mellin transform (a version of the Fourier transform) relates the two approaches. These interrelations (except for the Bäcklund transformations) are discussed in detail in the body of the paper.

The paper is organized as follows. In Section 1 we state our main results. In Section 2 we study general properties of d-connections and discuss various operations on them. Section 3 is dedicated to dPV and the corresponding moduli space. In Section 4 we describe the relations between dPV and PVI. Finally, in Section 5 we deal with dPVI, the associated moduli space, and degenerations of dPVI to dPV and PVI.

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1.1. Notation. In this paper, the ground field is \( \mathbb{C} \), so ‘variety’ means ‘variety over \( \mathbb{C} \)’, \( \mathbb{P}^1 \) means ‘\( \mathbb{P}^1_\mathbb{C} \)’ and so on. \( z \) stands for the coordinate on the projective line \( \mathbb{P}^1 \). For a vector bundle \( \mathcal{L} \) on \( \mathbb{P}^1 \), the fiber of \( \mathcal{L} \) over \( z \in \mathbb{P}^1 \) is denoted by \( \mathcal{L}_z \) and the space of global sections of \( \mathcal{L} \) is denoted by \( \Gamma(\mathbb{P}^1, \mathcal{L}) \). \( \mathcal{O}(k) \) stands for the line bundle (vector bundle of rank 1) on \( \mathbb{P}^1 \) whose sections are functions on \( \mathbb{P}^1 \) with a pole of order at most \( k \) (or zero of order at least \( -k \), if \( k < 0 \)) at \( \infty \in \mathbb{P}^1 \).

\( \text{diag}(\alpha_1, \ldots, \alpha_m) \) stands for the diagonal \( m \times m \) matrix with entries \( \alpha_1, \ldots, \alpha_m \).

2. Main Results

2.1. d-connections and their moduli. Let \( \mathcal{L} \) be a vector bundle on \( \mathbb{P}^1 \) of rank \( m \).[2]

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[2] The classical PVI was also obtained as a limit of other discrete Painlevé equations in [19, 20].
Definition 2.1. A (rational) $d$-connection on $\mathcal{L}$ is a linear operator

$$\mathcal{A}(z) : \mathcal{L}_z \to \mathcal{L}_{z+1}$$

that depends on a point $z \in \mathbb{P}^1 - \{\infty\}$ in a rational way (in particular, $\mathcal{A}(z)$ is defined for all $z \in \mathbb{C}$ outside of a finite set); here $\mathcal{L}_z$ is the fiber of $\mathcal{L}$ over $z \in \mathbb{P}^1$. In other words, $\mathcal{A}$ is a rational map between the vector bundle $\mathcal{L}$ and its pullback $s^*(\mathcal{L})$ via the automorphism $s : \mathbb{P}^1 \to \mathbb{P}^1 : z \mapsto z + 1$.

Remark 2.2. Essentially, a $d$-connection is a system of (rational) linear difference equations $y(z + 1) = \mathcal{A}(z)y(z)$ on a section $y(z)$ of the vector bundle $\mathcal{L}$. Notice that any vector bundle $\mathcal{L}$ is trivial when restricted to $\mathbb{A}^1 = \mathbb{P}^1 - \{\infty\}$. If we pick a trivialization $\mathcal{S}(z) : \mathbb{C}^n \to \mathcal{L}_z$, $z \in \mathbb{A}^1$ of this restriction (a basis of $\mathcal{L}$), $\mathcal{A}$ can be written in coordinates as the matrix-valued function $A(z) = \mathcal{S}(z + 1)^{-1}\mathcal{A}(z)\mathcal{S}(z)$ (the matrix of the $d$-connection). For two trivializations $\mathcal{S}_i(z) : \mathbb{C}^n \to \mathcal{L}_z$ ($i = 1, 2$), the corresponding matrices $A_i = \mathcal{S}_i(z + 1)^{-1}\mathcal{A}(z)\mathcal{S}_i(z)$ differ by a $d$-gauge transformation:

$$A_2(z) = R(z + 1)^{-1}A_1(z)R(z),$$

for the ‘$d$-gauge matrix’ $R := S_1^{-1}S_2$. Thus, classification of $d$-connections is equivalent to the classification of their matrices up to the $d$-gauge transformation.

We work with $d$-connections that have simple zeroes on $\mathbb{A}^1$ and whose behavior at infinity is ‘simple’ in the sense of the following definition:

Definition 2.3. Let $\mathcal{L}$ be a rank 2 vector bundle on $\mathbb{P}^1$ and $\mathcal{A}(z)$ be a $d$-connection on $\mathcal{L}$. Suppose $\mathcal{A}(z)$ satisfies the following conditions:

1. The only zeroes and poles of $\mathcal{A}(z)$ are as follows: a pole of order $n$ at infinity, and simple zeroes at $k$ distinct points $a_1, \ldots, a_k \in \mathbb{A}^1$. Here we say that $a_i$ is a simple zero of $\mathcal{A}(z)$ if, at $a_i$, $\mathcal{A}(z)$ is regular and $\det(\mathcal{A}(z))$ has zero of order 1.

2. On the formal neighborhood of $\infty \in \mathbb{P}^1$, there exist a trivialization $\mathcal{R}(z) : \mathbb{C}^2 \to \mathcal{L}_z$ ($\mathcal{R}(z)$ is essentially a matrix-valued Taylor series in $z^{-1}$) such that the matrix of $\mathcal{A}$ with respect to $\mathcal{R}$ satisfies

$$\mathcal{R}(z + 1)^{-1}\mathcal{A}(z)\mathcal{R}(z) = \begin{bmatrix} \rho_1(z^n + d_1z^{n-1}) & 0 \\ 0 & \rho_2(z^n + d_2z^{n-1}) \end{bmatrix}$$

for some numbers $\rho_1, \rho_2, d_1, d_2 \in \mathbb{C}$.

We call such a $d$-connection $\mathcal{A}(z)$ (or, more precisely, the pair $(\mathcal{L}, \mathcal{A})$) a $d$-connection of type $\theta = (a_1, \ldots, a_k; \rho_1, \rho_2, d_1, d_2; n)$.

Remark 2.4. One can also consider $d$-connections that have simple poles besides simple zeroes. As it turns out, addition of poles does not lead to a significantly different object: in Section 3.2, we discuss an operation (‘multiplication by a scalar’) that turns a pole of a $d$-connection into a zero and vice versa.

Remark 2.5. The second condition of Definition 2.3 might seem unnatural, however, Corollary 3.4 shows that a generic $d$-connection satisfies it. See also Remark 3.2 for a reformulation of this condition in terms of formal solutions to a difference equation.

Denote by $M_\theta$ the moduli space of $d$-connections of type $\theta$. One can think of $M_\theta$ in several different ways: as a set (the set of isomorphism classes of connections...
of given type), a category (the category of such connections), a scheme (the corresponding coarse moduli space), or an algebraic stack (the fine moduli stack). In this paper, we work with the coarse moduli space, although some results also hold for other ‘incarnations’ of $M_\theta$. (Note that we need to impose some conditions on $\theta$ to make sure that the coarse moduli space of d-connections of type $\theta$ is a scheme.)

It is easy to see (Corollary 3.11) that $M_\theta$ is empty unless

$$k = 2n;$$

$$\deg(\theta) := -d_1 - d_2 - \sum_{i=1}^{k} a_i$$

Let us also consider the following non-degeneracy assumptions on $\theta$:

$$-d_j - \sum_{i \in I} a_i \notin \mathbb{Z} \text{ for any } I \subset \{1, \ldots, k\}, j = 1, 2;$$

$$a_i - a_j \notin \mathbb{Z} \text{ for any } i \neq j;$$

$$\rho_1, \rho_2 \neq 0; \rho_1 \neq \rho_2.$$

Let $\Theta_{2n}$ be the set of all collections $\theta = (a_1, \ldots, a_{2n}; \rho_1, \rho_2, d_1, d_2; n)$, and let $\Theta_{2n}^2 \subset \Theta_{2n}$ be the set of $\theta$ that satisfy (2.2)–(2.6). Set $\Theta = \bigsqcup_n \Theta_{2n}$, $\Theta^2 = \bigsqcup_n \Theta_{2n}^2$.

**Remark 2.6.** Informally speaking, we impose the conditions (2.4)–(2.6) for the following reasons: (2.5), (2.6) simplify modifications of d-connections (Section 3.2), while (2.4) implies that d-connections of type $\theta$ are irreducible (Lemma 3.12). Irreducibility can be used to prove that the moduli space $M_\theta$ is ‘nice’: for example, one can show (using the same ideas as in [3]) that $M_\theta$ is a smooth variety of dimension $2n - 2$ for any $\theta \in \Theta_{2n}^2$.

### 2.2. Difference $PV$. We want to study the moduli space $M_\theta$ for $\theta \in \Theta_{2n}^2$. As Remark 2.6 shows, the first interesting case is when $2n = 4$: then $M_\theta$ is a smooth algebraic surface. We will also assume that $\deg(\theta) = -1$ (the degree is defined in (2.3)).

**Remark 2.7.** The assumption is not too restrictive: using ‘modifications’ of d-connections (described in Section 3.2), we can construct for any $\theta$ an isomorphism $M_{\tilde{\theta}} \cong M_{\theta}$, where $\deg(\tilde{\theta}) = -1$.

We describe the surface $M_{\theta}$ by introducing ‘coordinates’ $(q, p) \in (\mathbb{P}^1)^2$; more precisely, $M_{\theta}$ is described as an open subset in a blow-up of $(\mathbb{P}^1)^2$. The construction imitates the description of the moduli space of connections ([3], [10]) which goes back to Okamoto ([24], [26]).

**Theorem A.** Suppose

$$\theta = (a_1, a_2, a_3, a_4; \rho_1, \rho_2, d_1, d_2; 2) \in \Theta_{4}^2$$

has $\deg(\theta) = -1$. Let $\sigma_1 : K_1 \to (\mathbb{P}^1)^2$ be the blow-up of $(\mathbb{P}^1)^2$ at the following 6 points: $(q, p) = (a_1, 0), (a_2, 0), (a_3, \infty), (a_4, \infty), (\infty, \rho_1), \text{ and } (\infty, \rho_2)$ (here $q$ and $p$ are the projections $(\mathbb{P}^1)^2 \to \mathbb{P}^1$). Consider the two exceptional curves $E_j = \sigma_1^{-1}(\infty, \rho_j) \subset K_1$, $j = 1, 2$; homogeneous coordinates on $E_j$ are given by $(1/q : p - \rho_j)$. Let $\sigma_2 : K_2 \to K_1$ be the blow-up of $K_1$ at the two points $(1/q : p - \rho_j) = (1 : \rho_1(d_1 + a_3 + a_4)), j = 1, 2$ (one point on each exceptional curve).

1. There exists an open embedding $P_2 : M_{\tilde{\theta}} \hookrightarrow K_2$. 

Remark 2.8. \(K_2\) is the smallest smooth compactification of \(M_\theta\) (cf. [3] Corollary 5)): any open embedding \(M_\theta \hookrightarrow \bar{M}\) with smooth projective \(\bar{M}\) induces a regular morphism \(\bar{M} \to K_2\). Note also that \((K_2, K_2 - M_\theta)\) is an Okamoto-Painlevé pair (of type \(\tilde{D}_4\)) in the sense of [27], [28]: in particular, \(K_2\) is a surface of the Sakai type \(D_4^{(1)}\).

In particular, the composition \(P : M_\theta \hookrightarrow K_2 \to (\mathbb{P}^1)^2\) is birational. Therefore, one can view the components of \(P\) as a kind of ‘rational coordinates’ on \(M_\theta\). We denote the components by \(q\) and \(p\), so that \(P = (q, p)\).

The natural operations on \(d\)-connections (modifications and multiplications by scalar) define isomorphisms between the spaces \(M_\theta\) for different \(\theta\) (see Section 3.2). Our next result describes such an isomorphism for one of the simplest modifications of \(d\)-connections. The description can be viewed as a non-linear difference equation in ‘coordinates’ \((p, q)\) (the difference \(PV\)).

As before, suppose

\[
\theta = (a_1, a_2, a_3, a_4; \rho_1, \rho_2, d_1, d_2; 2) \in \Theta_4^d
\]

has \(\deg(\theta) = -1\). Set

\[
\theta' = (a_1 - 1, a_2 - 1, a_3, a_4; \rho_1, \rho_2, d_1 + 1, d_2 + 1; 2) \in \Theta_4^d,
\]

Modification of \(d\)-connections defines an isomorphism \(dPV : M_\theta \to M_{\theta'}\). Explicitly, for every \((\mathcal{L}, \mathcal{A}) \in M_\theta\), the image \(dPV(\mathcal{L}, \mathcal{A}) = (\mathcal{L}', \mathcal{A}')\) is the only \(d\)-connection of type \(\theta'\) that admits a rational isomorphism \(R : \mathcal{L} \to \mathcal{L}'\) that agrees with the \(d\)-connections: \(R(z + 1)A'(z) = A(z)/R(z)\).

**Theorem B.** Set \(p' := p \circ dPV, q' := q \circ dPV : M_\theta \to \mathbb{P}^1\). Then

\[
\begin{align*}
q' + q &= a_3 + a_4 + \frac{\rho_1(d_1 + a_3 + a_4) + \rho_2(d_2 + a_3 + a_4)}{p - \rho_1} \\
p' \cdot p &= \frac{(q' - a_3 + 1)(q' - a_3)}{p - a_3}(q' - a_4) \cdot \rho_1 \rho_2
\end{align*}
\]

(2.7)

2.3. Difference \(PV\) and classical \(PVI\). As we mentioned above, \(d\)-connections and ordinary connections have many common properties. Let us consider the following class of (ordinary) connections:

Denote by \(\Lambda \subset \mathbb{C}^4\) the set of all collections \(\lambda = (\lambda_1^-, \lambda_1^+, \ldots, \lambda_4^-, \lambda_4^+)\) such that

\[
\sum_{i=1}^4 (\lambda_i^- + \lambda_i^+) \in \mathbb{Z}, \quad \lambda_i^+ - \lambda_i^- \notin \mathbb{Z}, \quad \sum_{i=1}^4 \lambda_i^{\epsilon_i} \notin \mathbb{Z}
\]

for any choice of upper indexes \(\epsilon_i \in \{+,-\}\). Let \(X \subset (\mathbb{P}^1)^4\) be the set of all collections \(x = (x_1, \ldots, x_4)\) of four distinct points of \(\mathbb{P}^1\):

\(X := \{(x_1, \ldots, x_4)| x_i \neq x_j \text{ for } i \neq j \} \subset (\mathbb{P}^1)^4\).

**Definition 2.9.** Suppose \((x, \lambda) \in X \times \Lambda\). A connection of type \((x, \lambda)\) is a pair \((\mathcal{L}, \nabla)\) such that \(\mathcal{L}\) is a rank 2 vector bundle on \(\mathbb{P}^1\), \(\nabla : \mathcal{L} \to \mathcal{L} \otimes \Omega_{\mathbb{P}^1}(x_1 + \cdots + x_4)\) is a connection with simple poles at \(x_i\)’s, and the residue of \(\nabla\) at \(x_i\) has eigenvalues \(\{\lambda_i^-, \lambda_i^+\}\).
For \((x, \lambda) \in X \times \Lambda\), we denote the coarse moduli space of connections of type \((x, \lambda)\) by \(M_{(x, \lambda)}\). It can be thought of as the space of initial conditions of the Painlevé equation \(PVI\). The space \(M_{(x, \lambda)}\) has a geometric description, which goes back to K. Okamoto; we remind the description in Proposition 5.1. It is easy to see from the description that \(M_0\) and \(M_{(x, \lambda)}\) are isomorphic for a suitable choice of parameters (they are both surfaces of type \(D_4^{(1)}\)):

**Theorem C.** Suppose

\[
\theta = (a_1, a_2, a_3, a_4; \rho_1, \rho_2, d_1, d_2; 2) \in \Theta_4^2
\]

has \(\text{deg}(\theta) = -1\). Set

\[x = (x_1, x_2, x_3, x_4) := (0, \rho_1, \rho_2, \infty) \in X,\]

\[\lambda = (\lambda_1^-, \lambda_1^+, \ldots, \lambda_n^-; \lambda_n^+) := (a_1, a_2, 0, d_1 + a_3 + a_4, 0, d_2 + a_3 + a_4, -a_3, -a_4) \in \Lambda.\]

Then \(M_0 \simeq M_{(x, \lambda)}\).

**Remark 2.10.** Theorem C can be proved by direct calculations, but it can also be explained in terms of moduli spaces. In Section 5.6 we describe a one-to-one correspondence between d-connections of type \(\theta\) and connections of type \((x, \lambda)\). Up to small ‘twists’, the correspondence is the geometric Mellin transform of \([21]\); it is constructed using de Rham cohomology and equivariant cohomology groups. The Mellin transform is a particular case of the duality for generalized one-motives (also defined in \([21]\)).

Now let us fix \(\lambda \in \Lambda\) and consider surfaces \(M_{(x, \lambda)}\) for all \(x \in X\). They can be viewed as fibers of an algebraic family \(M_{\lambda} \to X\). The sixth Painlevé equation \(PVI\) is an algebraic connection on this family; the (analytic) integral curves of \(PVI\) correspond to isomonodromy deformation of connections.

By Theorem C the sixth Painlevé equation \(PVI\) induces a connection on a family of moduli spaces of d-connections. It turns out that this connection can be defined for arbitrary \(\theta \in \Theta_2^n\) (not necessarily when \(2n = 4\)). More precisely:

**Theorem D.** Let \(n\) be a positive integer. Fix \(a_1, \ldots, a_{2n}, d_1, d_2 \in C\) that satisfy (2.3)–(2.5), and set \(P = \{(\rho_1, \rho_2) \in C^2 : \rho_1, \rho_2 \neq 0, \rho_1 \neq \rho_2\}\). For all \(\rho := (\rho_1, \rho_2) \in P\), set \(\theta(\rho) = (a_1, \ldots, a_{2n}; \rho_1, \rho_2, d_1, d_2; n) \in \Theta_2^{2n}\), and consider the coarse moduli spaces \(M_{\theta(\rho)}\). Clearly, they form a family \(M \to P\).

1. The family \(M \to P\) carries a natural algebraic connection (defined in Section 5.5).
2. In the case \(2n = 4\), this connection coincides with the \(PVI\) connection under the isomorphism of Theorem C.

**Remark 2.11.** The connection of Theorem D can be thought of as a ‘continuous’ isomonodromy deformation of d-connections.

### 2.4. Difference \(PVI\).

So far, we have worked with d-connections of type \(\theta\), where \(\theta\) is non-degenerate in the sense of (2.4)–(2.6). It turns out that a different class of d-connections enjoys similar properties. Namely, let us replace (2.6) with the following condition:

\[
\rho_1 = \rho_2 \neq 0; d_1 \neq d_2.
\]
Let \( \Theta^\flat_{2n} \subset \Theta_{2n} \) be the set of all \( \theta \) that satisfy (2.2), (2.3), and (2.5), and set \( \Theta^\flat = \bigsqcup_n \Theta^\flat_{2n} \). It can be shown that for \( \theta \in \Theta^\flat_{2n} \), the coarse moduli space \( M_\theta \) is a smooth variety of dimension \( 2n - 4 \) (recall that for \( \theta \in \Theta^\flat_{2n} \), we have \( \dim(M_\theta) = 2n - 2 \)). Therefore, the first ‘interesting’ case is \( \theta \in \Theta^\flat_6 \); then \( M_\theta \) is an algebraic surface. As before, we assume \( \deg(\theta) = -1 \).

Similarly to Theorem [A] we can describe the moduli space \( M_\theta \) using ‘coordinates’ \( (q,p) \in (\mathbb{P}^1)^2 \).

**Theorem E.** Suppose

\[
\theta = (a_1, a_2, a_3, a_4, a_5, a_6; \rho, \rho, d_1, d_2; 3) \in \Theta^\flat_6
\]

has \( \deg(\theta) = -1 \). Let \( \sigma_1 : K_1 \to (\mathbb{P}^1)^2 \) be the blow-up of \((\mathbb{P}^1)^2\) at the following 7 points: \((q,p) = (a_1,0), (a_2,0), (a_3,0), (a_4,\infty), (a_5,\infty), (a_6,\infty), (\infty,\rho)\) (here \( q \) and \( p \) are the projections \((\mathbb{P}^1)^2 \to \mathbb{P}^1 \)). Consider the exceptional curve \( E = \sigma^{-1}_1(\infty,\rho) \subset K_1 \); a homogeneous coordinate on \( E \) is given by \( (1/q : p - \rho) \).

Let \( \sigma_2 : K_2 \to K_1 \) be the blow-up of \( K_1 \) at the two points \((1/q : p - \rho) = (1 : \rho(d_1 + a_4 + a_5 + a_6))\), \( j = 1, 2 \).

1. There exists an open embedding \( P_2 : M_\theta \to K_2 \).
2. The complement to \( P_2(M_\theta) \) in \( K_2 \) is the union of the proper preimages of the following curves: \( \mathbb{P}^1 \times \{0\}, \mathbb{P}^1 \times \{\infty\}, \{\infty\} \times \mathbb{P}^1 \subset (\mathbb{P}^1)^2 \), and the exceptional curve \( E \subset K_1 \), \( j = 1, 2 \).

**Remark 2.12.** Using multiplication by scalar, it is easy to see that the moduli space \( M_\theta \) for \( \theta = (a_1, \ldots, a_{2n}; \rho, \rho, d_1, d_2; n) \) does not depend on \( \rho \). Therefore, we can assume that \( \rho = 1 \) without loss of generality.

**Remark 2.13.** \( K_2 \) is not the smallest smooth compactification of \( M_\theta \) (unlike the case when \( \theta \in \Theta^\flat_4 \), see Remark 2.3). Indeed, the proper preimage of \( \{\infty\} \times \mathbb{P}^1 \subset (\mathbb{P}^1)^2 \) is an exceptional curve in \( K_2 - M_\theta \). Contracting the exceptional curve, we obtain the smallest smooth compactification of \( M_\theta \), which is a surface of the Sakai type \( A_2^{(1)+} \).

Modifications of \( d \)-connections define natural isomorphisms between spaces \( M_\theta \). Similarly to Theorem [H] we describe a simple isomorphism of this kind explicitly. We call the resulting difference equation ‘the difference PV’ - as we will see, it degenerates into both the difference PV (Section 6.3) and the usual PV (Section 6.4).

Suppose

\[
\theta = (a_1, a_2, a_3, a_4, a_5, a_6; 1, 1, d_1, d_2; 3) \in \Theta^\flat_6
\]

has \( \deg(\theta) = -1 \). Set

\[
\theta' = (a_1 - 1, a_2 - 1, a_3, a_4, a_5, a_6; 1, 1, d_1 + 1, d_2 + 1; 3) \in \Theta^\flat_6.
\]

Modification of \( d \)-connections induces an isomorphism \( dPV : M_\theta \to M_{\theta'} \). Explicitly, for every \((L,A) \in M_\theta\), the image \( dPV(L,A) = (L',A')\) is the only \( d \)-connection of type \( \theta' \) that admits a rational isomorphism \( R : L' \to A' \) that agrees with the \( d \)-connections: \( R(z + 1)A'(z) = A(z)R(z) \).

**Theorem F.** Set \( p' := p \circ dPV, q' := q \circ dPV : M_\theta \to \mathbb{P}^1 \). For \( j = 1, 2 \), set

\[
c_j := \frac{(d_j + a_1 + a_2 + a_4 - 1)(d_j + a_1 + a_2 + a_5 - 1)(d_j + a_1 + a_2 + a_6 - 1)}{(d_j - d_{3-j})}
\]
(the denominator is $\pm(d_1 - d_2)$). Then

\[
q' = (p - 1)(q + 1 - a_1 - a_2) + pa_3 + \sum_{j=1,2} \frac{c_{jp}}{p - \frac{1}{p}(1-a_1-a_2-d_j-a_1)} \\
q' = (q' - a_1 + 1)(q' - a_2 + 1) \cdot ((p - 1)(q' - q) + q' - a_3)
\]

(2.9)

\[
p' \cdot p = \frac{(q' - a_1 + 1)(q' - a_2 + 1)}{(q' - a_3)(q' - a_6)} \cdot ((p - 1)(q' - q) + q' - a_3)
\]

Remark 2.14. Theorem [C] identifies $M_\theta$ for $\theta \in \Theta^2_A$ with a moduli space of connections of certain kind. A similar statement holds for $\theta = (a_1, \ldots, a_6; \rho, \rho, d_1, d_2; 3) \in \Theta^3_6$. In this case, $M_\theta$ is isomorphic to the moduli space of pairs $(\mathcal{L}, \nabla)$, where $\mathcal{L}$ is a rank 3 bundle on $\mathbb{P}^1$ and $\nabla$ is a connection on $\mathcal{L}$ with first order poles at $\rho, 0,$ and $\infty$ (and no other poles); the residues at the poles have eigenvalues $\{0, d_1 + a_4 + a_6, d_2 + a_4 + a_5 + a_6\}, \{a_1, a_2, a_3\}, \{-a_4, -a_5, -a_6\}$, respectively. The isomorphism can be constructed using the Mellin transform (similarly to Section 5.6).

Notice that if we interpret $M_\theta$ as a moduli space of rank 3 bundles with connections on $\mathbb{P}^1$, then dPVI becomes an isomorphism between such moduli spaces (a Bäcklund transformation) which corresponds to a modification of such bundles.

3. General $d$-connections

3.1. Formal behavior at infinity. Let $\mathcal{L}$ be a vector bundle on $\mathbb{P}^1$ and $\mathcal{A}(z) : \mathcal{L}_z \to \mathcal{L}_{z+1}$ be a rational $d$-connection on $\mathcal{L}$. Since $\infty \in \mathbb{P}^1$ is the only fixed point of the transformation $z \mapsto z + 1$, it is natural to study the restriction of $\mathcal{A}$ to a neighborhood of infinity. Here the word ‘neighborhood’ can be understood either analytically (a small disk) or formally (the formal disk). In this section, we work with the formal neighborhood: the corresponding classification problem is significantly easier. The situation is somewhat similar to classification of irregular singularities for ordinary differential equations: the formal classification is much simpler than the analytic one (because of the Stokes’ phenomenon).

In the language of difference equations, the problem is to classify matrices $A(z)$ over the ring of formal Laurent series $\mathbb{C}((z^{-1}))$ modulo $d$-gauge transformations

\[A(z) \mapsto R(z + 1)^{-1}A(z)R(z),\]

where the gauge matrix $R(z)$ is an invertible matrix over the ring of formal Taylor series $\mathbb{C}[[z^{-1}]]$.

If $A$ is generic, the answer is given by the following easy statement (see for instance [B], Proposition 1.1):

Proposition 3.1. Suppose that the $m \times m$ matrix $A(z) = \sum_{i \leq n} A_i z^i$ over $\mathbb{C}((z^{-1}))$ satisfies the following condition:

\[
\text{All eigenvalues of the leading term } A_n \text{ are distinct and non-zero (in other words, } A_n \text{ is invertible, regular, and semisimple).}
\]

Then there exists a gauge matrix $R(z) = \sum_{i \leq 0} R_i z^i$ with invertible $R_0$ such that

\[
R(z + 1)^{-1}A(z)R(z) = A'_n z^n + A'_{n-1} z^{n-1},
\]

where $A'_n$ and $A'_{n-1}$ are diagonal matrices. The matrix $R(z)$ is uniquely determined up to right multiplication by a permutation matrix and a constant diagonal matrix. \qed
Denote the diagonal entries of $A'_n$ by $\rho_1, \ldots, \rho_m$; notice that $\rho_i$’s are the eigenvalues of $A_n$, in particular, all $\rho_i$ are distinct and non-zero. Denote the corresponding diagonal entries of $A'_n-1$ by $c_1, \ldots, c_m$. Set $d_i := c_i/\rho_i$; we work with $d_i$ rather than $c_i$ because it simplifies formulas (2.3), (2.4). We call the collection $(\rho_1, \ldots, \rho_m, d_1, \ldots, d_m; n)$ the formal type of $A(z)$ at infinity. Proposition 3.1 implies that the formal type is determined by $A(z)$ up to a simultaneous permutation of $\rho_i$’s and $d_i$’s, that is, up to the action of the symmetric group $S_m$.

Remark 3.2. Proposition 3.1 is sometimes (for instance, in [5]) formulated in terms of formal solutions to the difference equation: the claim is that the equation $Y(z + 1) = A(z)Y(z)$ has a formal solution of the form

$$Y(z) = (\Gamma(z))^n \left( \sum_{i \leq 0} \hat{Y}_i z^i \right) \text{diag}(\rho_1^i z^{d_1}, \ldots, \rho_m^i z^{d_m}),$$

where $\hat{Y}_i$ are $m \times m$ matrices, $\hat{Y}_0$ is invertible, and $\rho_1, \ldots, \rho_m, d_1, \ldots, d_m \in \mathbb{C}$.

Note that $\sum_{i \leq 0} \hat{Y}_i z^i$ does not coincide with $R(z)$ of Proposition 3.1.

Remark 3.3. The formal type of $A(z)$ can be determined directly, without diagonalizing $A(z)$. Indeed, denote by $\sigma_i(z)$ and $\sigma'_i(z)$ ($i = 1, \ldots, m$) the coefficients of the characteristic polynomials of $A(z)$ and $R(z+1)^{-1}A(z)R(z)$ respectively, so that $\sigma_i(z) = -\text{tr} A(z)$ and $\sigma'_m(z) = (-1)^m \det A(z)$. Clearly, $\sigma_i(z)$ and $\sigma'_i(z)$ have pole of order $i \cdot n$ at infinity. One can easily check that the order of pole of $\sigma_i(z) - \sigma'_i(z)$ is at most $i \cdot n - 2$. Thus, the two leading terms of $\sigma_i(z)$ and $\sigma'_i(z)$ coincide. It is now easy to see that the formal type of $A(z)$ is determined (up to the $S_m$-action) by the pairs of leading terms of $\sigma_i(z)$, $i = 1, \ldots, m$.

In particular, if we assume $A_n$ is diagonal, then its diagonal entries are the $\rho_i$’s, and the diagonal entries of $A_{n-1}$ equal $\rho_i d_i$, even if $A_{n-1}$ is not diagonal.

Let us now translate Proposition 3.1 into the language of d-connections. For simplicity, we only consider vector bundles of rank 2.

Corollary 3.4. Let $\mathcal{A}(z)$ be a d-connection on a rank 2 vector bundle $\mathcal{L}$. Denote by $n$ the order of pole of $\mathcal{A}$ at infinity and by $\mathcal{A}_n : \mathcal{L} \rightarrow \mathcal{L}$ the leading term of $\mathcal{A}$ (that is, $n$ is the smallest number such that the limit

$$\mathcal{A}_n := \lim_{z \rightarrow \infty} \mathcal{A}(z)z^{-n}$$

exists). Suppose all eigenvalues of $\mathcal{A}_n$ are distinct and non-zero. Then $\mathcal{A}(z)$ satisfies the second condition of Definition 2.3 (for some $\rho_1, \rho_2, d_1, d_2 \in \mathbb{C}$).

We call the collection $(\rho_1, \rho_2, d_1, d_2; n)$ the formal type of the d-connection $\mathcal{A}(z)$. It is determined by $\mathcal{A}(z)$ up to the action of $S_2$. Notice also that in the situation of Corollary 3.4 the condition 2.6 holds automatically.

3.2. Operations on d-connections. Let us now discuss some natural operations on d-connections. As we will see, the operations allow us to identify the moduli spaces (or moduli stacks, or sets of isomorphism classes, or categories) of d-connections of type $\theta$ for different $\theta$. As a trivial example, notice that $M_{\theta'} = M_{\theta}$ if $\theta'$ is obtained from $\theta$ by a permutation of $a_i$’s or a simultaneous permutation of $\rho_i$’s and $d_i$’s.
Multiplication by a scalar: Let \( f(z) \neq 0 \) be a rational function on \( \mathbb{P}^1 \), and let \( \mathcal{A}(z) \) be a d-connection on a vector bundle \( \mathcal{L} \). Clearly, the product \( f(z)\mathcal{A}(z) \) is again a d-connection on \( \mathcal{L} \).

In the language of difference equations, this operation corresponds to multiplication of solutions by \( \Gamma \)-functions. Indeed, let us write \( f(z) \) again a d-connection on \( \mathcal{O} \) one bundle \( f \) product of d-connections. We can view

\[
\text{Suppose } \mathcal{L} \text{ is reduced to classification of d-connections with simple zeroes only.}
\]

Then \( f(z)\mathcal{A}(z) \) becomes the natural d-connection on the tensor product \( \mathcal{L} = \mathcal{L} \otimes \mathcal{O}_{\mathbb{P}^1} \); then \( f(z)\mathcal{A}(z) \) becomes the natural d-connection on the tensor product \( \mathcal{L} = \mathcal{L} \otimes \mathcal{O}_{\mathbb{P}^1} \) of two vector bundles with d-connections.

**Remark 3.5.** For any d-connection \( \mathcal{A}(z) \), we can pick a function \( f(z) \) so that the only pole of the product \( f(z)\mathcal{A}(z) \) is at infinity. For instance, suppose \( \mathcal{L} \) has rank 2, and the d-connection \( \mathcal{A}(z) \) has a simple pole at \( z = z_0 \); this means that all matrix elements of \( \mathcal{A}(z) \) (in some basis) have at most a simple pole and \( \det(\mathcal{A}(z)) \) has a simple pole at \( z = z_0 \). Then \( (z - z_0)\mathcal{A}(z) \) has a simple zero at \( z_0 \). In this way, classification of rank 2 d-connections with simple poles and simple zeroes on \( \mathbb{P}^1 - \{ \infty \} \) is reduced to classification of d-connections with simple zeroes only.

Now suppose \( (\mathcal{L}, \mathcal{A}(z)) \in \mathcal{M}_0 \) for \( \theta \in \Theta \). Let \( f(z) \) be a rational function; clearly, the product \( (\mathcal{L}, f(z)\mathcal{A}(z)) \) is a d-connection of type \( \theta' \) (for some \( \theta' \in \Theta \)) if and only if the function \( f(z) = c \) is a non-zero constant. If \( f(z) = c \in \mathbb{C} - \{0\} \), then

\[
(\mathcal{L}, c\mathcal{A}) \in \mathcal{M}_0, \quad \theta' = (a_1, \ldots, a_k; cp_1, cp_2, d_1, d_2; n).
\]

Clearly, the correspondence \( (\mathcal{L}, \mathcal{A}) \mapsto (\mathcal{L}, c\mathcal{A}) \) gives an isomorphism \( \mu = \mu_c : \mathcal{M}_0 \rightarrow \mathcal{M}_0 \); the inverse map is \( \mu_{c^{-1}} \).

**Modification:** Suppose \( \mathcal{R} : \mathcal{L} \rightarrow \mathcal{L}' \) is a rational isomorphism between two vector bundles \( \mathcal{L} \) and \( \mathcal{L}' \) on \( \mathbb{P}^1 \). Then a d-connection \( \mathcal{A}(z) \) on \( \mathcal{L} \) induces a d-connection \( \mathcal{A}' \) on \( \mathcal{L}' \) (and vice versa).

In the language of difference equation, this operation is the d-gauge transformation

\[
(3.3) \quad \mathcal{A}'(z) = R(z + 1)^{-1}A(z)R(z),
\]

where \( R \), \( A \), and \( A' \) are the matrices of \( \mathcal{R} \), \( \mathcal{A} \), and \( \mathcal{A}' \) respectively (corresponding to some choice of bases). We call \( \mathcal{A}' \) a modification of \( \mathcal{A} \) (of course, \( \mathcal{A} \) is also a modification of \( \mathcal{A}' \)).

**Remark 3.6.** Modifications can also be viewed as an isomonodromy deformation in the sense of [5]. Indeed, the monodromies of \( \mathcal{A} \) and \( \mathcal{A}' \) coincide (for the monodromies to exist, \( \mathcal{A} \) and \( \mathcal{A}' \) have to satisfy the assumptions of Corollary [3.4]).

The simplest class of modifications is the so-called elementary modifications:

**Definition 3.7.** Suppose the rational isomorphism \( \mathcal{R} : \mathcal{L} \rightarrow \mathcal{L}' \) is regular and has exactly one simple zero. In this case, \( \mathcal{A}' \) is an elementary upper modification of \( \mathcal{A} \), and \( \mathcal{A} \) is an elementary lower modification of \( \mathcal{A}' \).

Note that an upper elementary modification \( \mathcal{R} : \mathcal{L} \rightarrow \mathcal{L}' \) is uniquely determined by the pair \((x, l)\), where \( x \in \mathbb{P}^1 \) is the only zero of \( \mathcal{R} \) and the one-dimensional subspace \( l \subset \mathcal{L}_x \) is given by \( l = \ker(\mathcal{R}(x) : \mathcal{L}_x \rightarrow \mathcal{L}_x') \subset \mathcal{L}_x \). Conversely, any pair \((x \in \mathbb{P}^1, l \subset \mathcal{L}_x)\) defines an elementary upper modification. Similarly, elementary...
lower modifications of $\mathcal{L}'$ are in on-to-one correspondence with pairs $(x, l')$ where $x \in \mathbb{P}^1$, $l' \subset L'_x$ is a subspace of codimension one (for $R : \mathcal{L} \to \mathcal{L}'$, $x$ is the only zero of $R$ and $l' = \text{im}(R(x) : \mathcal{L}_x \to \mathcal{L}'_x)$).

**Proposition 3.8.** Suppose $(\mathcal{L}, \mathcal{A}) \in M_\theta$ for $\theta = (a_1, \ldots, a_k; \rho_1, \rho_2, d_1, d_2; n)$, and $\rho_1 \neq \rho_2$. Let $(\mathcal{L}', \mathcal{A}')$ be an elementary upper modification of $\mathcal{L}$ given by $(x \in \mathbb{P}^1; l \subset L_x)$. Then the only cases when $(\mathcal{L}', \mathcal{A}')$ belongs to $M_{\theta'}$ for some $\theta' \in \Theta$ are as follows:

1. If $x = \infty$, then $l$ must be an eigenspace of $\mathcal{A}_n : L_\infty \to L_\infty$ (the leading term of $\mathcal{A} = \mathcal{A}_n z^n + \text{lower order terms}$). If, for instance, $l = \ker(\mathcal{A}_n - \rho_1) \subset L_\infty$, then $\theta' = (a_1, \ldots, a_k; \rho_1, \rho_2, d_1 - 1, d_2; n)$, and an analogous formula holds when $l = \ker(\mathcal{A}_n - \rho_2)$.

2. If $x = a_i$ is a zero of $\mathcal{A}$ and $x - 1 \neq a_i$ is not, then $l$ must be the kernel of $\mathcal{A}(x) : \mathcal{L}_x \to \mathcal{L}_{x + 1}$; in this case, $\theta' = (a_1, \ldots, a_{i - 1}, \ldots, a_k; \rho_1, \rho_2, d_1, d_2; n)$.

In either case, the elementary modifications define an isomorphisms $M_\theta \sim M_{\theta'}$. 

**Remark 3.9.** Sometimes an elementary modification of a $d$-connection of type $\theta$ has simple poles, which can be turned into simple zeroes using multiplication by a scalar (for example, this happens if neither $x$ has simple poles, which can be turned into simple zeroes using multiplication by a scalar). However, this procedure does not lead to an isomorphism between the moduli spaces $M_\theta$ (at least assuming $(2.2)-(2.6)$ hold), because the corresponding spaces have different dimensions.

Thus, elementary modifications (upper or lower) allow to identify $M_{\theta'}$ and $M_\theta$ if $\theta'$ is obtained from $\theta$ by adding or subtracting 1 to one of $a_i$’s or $d_i$’s, provided certain conditions hold. Composing such identifications, we get other isomorphisms between $M_\theta$ for different $\theta \in \Theta_k$.

The situation is particularly simple if $\theta$ satisfies the conditions $(2.4), (2.6)$. Then $M_\theta$ and $M_{\theta'}$ are naturally isomorphic if $\theta'$ is obtained from $\theta$ by adding integers to $a_i$’s and $d_i$’s. In other words, we have a natural action of the group $G = (\mathbb{Z})^k \times (\mathbb{Z})^2$ on $\Theta_k$, and for any $\theta \in \Theta_k$ satisfying $(2.4), (2.6)$ (in particular, for any $\theta \in \Theta_k^1$), we get isomorphisms $M_\theta \sim M_{g\theta}$ for all $g \in G$.

### 3.3. Irreducibility of $d$-connections

Let $\mathcal{A}(z)$ be a $d$-connection on a vector bundle $\mathcal{L}$ on $\mathbb{P}^1$. Assume that $\mathcal{A}(z)$ is non-degenerate at infinity in the sense that $(3.1)$ holds. Denote by $(p_1, \ldots, p_m, d_1, \ldots, d_m; n)$ the formal type of $\mathcal{A}(z)$ at infinity.

For the morphism $\mathcal{A}(z) : \mathcal{L}_z \to \mathcal{L}_{z + 1}$, its determinant is a map $\det \mathcal{A}(z) : \bigwedge^m \mathcal{L}_z \to \bigwedge^m \mathcal{L}_{z + 1}$; in other words, $\det \mathcal{A}(z)$ is a $d$-connection on the line bundle $\det \mathcal{L} := \bigwedge^m \mathcal{L}$. It is easy to see that $\det \mathcal{L}$ has formal type $(p_1 p_2 \cdots p_m, d_1 + \cdots + d_m; mn)$ at infinity. Let $a_1, \ldots, a_k \in \mathbb{A}^1$ and $b_1, \ldots, b_l \in \mathbb{A}^1$ be zeroes and poles (counted with multiplicity), respectively, of $\det \mathcal{A}(z)$ on $\mathbb{A}^1$.

**Lemma 3.10.** The collection $(a_1, \ldots, a_k; b_1, \ldots, b_l; \rho_1, \ldots, \rho_m, d_1, \ldots, d_m; n)$ satisfies the following equalities:

\[
\begin{align*}
mn &= k - l \\
\deg(\mathcal{L}) &= - \sum_{i=1}^m d_i - \sum_{i=1}^k a_i + \sum_{i=1}^l b_i.
\end{align*}
\]
Corollary 3.11. Let \((\mathcal{L}, \mathcal{A})\) be a d-connection of type
\[
\theta = (a_1, \ldots, a_k; \rho_1, \rho_2, d_1, d_2; n) \in \Theta.
\]
Then \(k = 2n\) and \(\deg(\theta) = \deg(\mathcal{L})\) (see (2.3) for the definition of \(\deg(\theta)\)); in particular, \(\deg(\theta)\) is an integer.

Lemma 3.12. Suppose \(\theta \in \Theta\) satisfies (2.4). Then any \((\mathcal{L}, \mathcal{A}) \in M_\theta\) is irreducible: there is no rank 1 subbundle \(\ell \subset \mathcal{L}\) such that \(\mathcal{A}(\ell_z) \subset \ell_{z+1}\) for all \(z\).

Proof. (Both the statement and its proof are completely analogous to Proposition 1.) Suppose \(\ell \subset \mathcal{L}\) is an invariant subbundle of rank 1, so that \(\mathcal{A}\) induces a d-connection \(\mathcal{A}|_\ell\) on \(\ell\). All zeroes of \(\mathcal{A}|_\ell\) belong to \(\{a_1, \ldots, a_k\}\); besides, the formal type of \(\mathcal{A}|_\ell\) at infinity is either \((\rho_1, d_1; n)\) or \((\rho_2, d_2; n)\). Now Lemma 3.10 leads to a contradiction.

Corollary 3.13. Suppose \((\mathcal{L}, \mathcal{A}) \in M_\theta\) and suppose that \(\theta \in \Theta_{2n}\) satisfies (2.4). If \(\mathcal{L} \cong \mathcal{O}(n_1) \oplus \mathcal{O}(n_2)\), then \(|n_1 - n_2| \leq n\).

Proof. Without loss of generality, we can assume \(n_1 \geq n_2\). Let \(\ell \subset \mathcal{L}\) be a rank 1 subbundle of degree \(n_1\). Since \((\mathcal{L}, \mathcal{A})\) is irreducible, \(\ell\) is not \(\mathcal{A}\)-invariant, and so the rational map \(\alpha : \ell \to \mathcal{L} \to s^*\mathcal{L} \to s^*(\mathcal{L}/\ell)\) is not identically zero. Notice that \(\alpha\) can have at most a pole of order \(n\) at \(\infty\) (and no other poles); thus, \(n_1 = \deg(\ell) \leq n + \deg(\mathcal{L}/\ell) = n + n_2\).

4. Difference PV

In this section, we study \(M_\theta\) for
\[
\theta = (a_1, a_2, a_3, a_4; \rho_1, \rho_2, d_1, d_2; 2) \in \Theta_4^2.
\]
We assume \(\deg(\theta) = -1\) (that is, \(-d_1 - d_2 - \sum_{i=1}^4 a_i = -1\)). Using modifications, we can make this assumption without loss of generality.

4.1. \(M_\theta\) as a quotient. Let \((\mathcal{L}, \mathcal{A}) \in M_\theta\). By Corollary 3.13, \(\mathcal{L}\) is isomorphic to \(\mathcal{O} \oplus \mathcal{O}(-1)\). Let us choose an isomorphism \(\mathcal{S} : \mathcal{O} \oplus \mathcal{O}(-1) \cong \mathcal{L}\); then \(\mathcal{A}\) induces the d-connection \(\mathcal{S}(z + 1)^{-1} \mathcal{A}(z) \mathcal{S}(z)\) of type \(\theta\) on \(\mathcal{O} \oplus \mathcal{O}(-1)\). Such a d-connection can be written as a matrix

\[
A = \begin{pmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{pmatrix}, \quad \begin{array}{c}
a_{11}, a_{22} \in \Gamma(\mathbb{P}^1, \mathcal{O}(2)) \\
a_{12} \in \Gamma(\mathbb{P}^1, \mathcal{O}(3)) \\
a_{21} \in \Gamma(\mathbb{P}^1, \mathcal{O}(1)).
\end{array}
\]

Of course, \(\mathcal{S}\) is not unique: it can be composed with an automorphism of \(\mathcal{O} \oplus \mathcal{O}(-1)\). Such an automorphism can be written as a matrix

\[
R = \begin{pmatrix}
r_{11} & r_{12} \\
0 & r_{22}
\end{pmatrix}, \quad \begin{array}{c}
r_{11}, r_{22} \in \mathbb{C} - \{0\} \\
r_{12} \in \Gamma(\mathbb{P}^1, \mathcal{O}(1)).
\end{array}
\]

If we replace \(\mathcal{S}\) with \(\mathcal{S} \circ R\), then \(A\) is replaced with its d-gauge transform

\[
R(z + 1)^{-1} A(z) R(z).
\]
Lemma 4.1. Let $A$ be a $d$-connection on $\mathcal{O} \oplus \mathcal{O}(-1)$; its matrix $A$ is of the form (4.1). We claim that $A$ is of type $\theta$ if and only if $A$ satisfies the following conditions:
\begin{align}
\det(A) &= (z - a_1)(z - a_2)(z - a_3)(z - a_4)\rho_1\rho_2 \quad \text{(4.4)} \\
a_{11} + a_{22}(1 + z^{-1}) &= (\rho_1 + \rho_2)z^2 + (d_1\rho_1 + d_2\rho_2)z + t(z^{-1}) \quad \text{(4.5)}
\end{align}
where $t(z^{-1}) \in \mathbb{C}[[z^{-1}]]$ is a Taylor series in $z^{-1}$.

Proof. $A$ is of type $\theta$ if and only if it satisfies the two conditions of Definition 2.3. Let us reformulate the conditions in terms of $A$.

Definition 2.3(1) is equivalent to the condition that
\begin{equation}
\det(A) = c(z - a_1)(z - a_2)(z - a_3)(z - a_4) \quad \text{for some } c \in \mathbb{C} - \{0\} \tag{4.6}
\end{equation}
(here we use that $\det(A)$ is a polynomial of degree 4 in $z$). Now set
\[S(z) := \begin{bmatrix} 1 & 0 \\ 0 & z^{-1} \end{bmatrix}\]
(S is a essentially a basis of $\mathcal{O} \oplus \mathcal{O}(-1)$ in a neighborhood of $\infty \in \mathbb{P}^1$). By Remark 3.3, Definition 2.3(2) is equivalent to the following two conditions:
\begin{align}
&\det(S(z + 1)^{-1}A(z)S(z)) = \rho_1\rho_2z^4 + \rho_1\rho_2(d_1 + d_2)z^3 + t_1(z^{-1})z^2 \quad \text{(4.7)} \\
&\operatorname{tr}(S(z + 1)^{-1}A(z)S(z)) = (\rho_1 + \rho_2)z^2 + (d_1\rho_1 + d_2\rho_2)z + t_2(z^{-1}). \quad \text{(4.8)}
\end{align}
Here $t_1, t_2$ are Taylor series in $z^{-1}$.

It is easy to see that (4.4) is equivalent to the combination of (4.6) and (4.7) (here we use that $\deg(\theta) = -1$), and (4.5) is equivalent to (4.8). \hfill \square

Corollary 4.2. Denote by $X_\theta$ the space of matrices $A$ of the form (4.1) that satisfy (4.4) and (4.5); denote by $G$ be the group of matrices $R$ of the form (4.2). Let $G$ act on $X_\theta$ via $d$-gauge transformations (1.3). Then the quotient $X_\theta/G$ is canonically isomorphic to $M_\theta$. \hfill \square

4.2. Geometric description of $M_\theta$. In this section, we will derive Theorem $A$ from another geometric description of $M_\theta$ (Theorem 4.4). Recall that Theorem $A$ realizes $M_\theta$ as an open subset of a blow-up of $(\mathbb{P}^1)^2$; in Theorem 4.4 we use a different rational surface in place of $(\mathbb{P}^1)^2$. Of the two descriptions, Theorem 4.4 uses somewhat more natural constructions (however, see Remark 4.5; for instance, all four points $a_1, \ldots, a_4$ appear in a symmetric manner. On the other hand, the advantage of Theorem $A$ is that $(\mathbb{P}^1)^2$ has natural coordinates $(q,p)$, which can then be viewed as ‘rational coordinates’ $q,p : M_\theta \to \mathbb{P}^1$. This makes Theorem $A$ more suitable for writing formulas.

As before, $(\mathcal{L},A) \in M_\theta$, $S : \mathcal{O} \oplus \mathcal{O}(-1) \to \mathcal{L}$, and $A$ is the matrix of $A$ relative to $S$. Notice that the matrix element $a_{21} \in \Gamma(\mathbb{P}^1, \mathcal{O}(1))$ is not identically zero, because $(\mathcal{L},A)$ is irreducible. Therefore, $a_{21}$ has a single zero on $\mathbb{P}^1$; let us denote it by $q \in \mathbb{P}^1$. Set $\tilde{p} := a_{11}(q) \in (\mathcal{O}(2))_q$.

Proposition 4.3. $\tilde{p}$ and $q$ depend only on $(\mathcal{L},A) \in M_\theta$ and not on $S$.

Proof. This statement can be easily checked directly by calculating the $d$-gauge transformation (4.3) with the gauge matrix (1.2). It is also possible to provide a geometric explanation in the spirit of [3, Section 4.1]. \hfill \square
(q, 3) can be viewed as a map \( \tilde{P} : M_\theta \to \tilde{K} \), where \( \tilde{K} := \mathcal{V}(\mathcal{O}(2)^\vee) \) is the total space of the line bundle \( \mathcal{O}(2) \). As we will see, the map \( \tilde{P} : M_\theta \to \tilde{K} \) is a regular birational morphism. Since \( M_\theta \) is a smooth algebraic surface, \( \tilde{P} \) identifies \( M_\theta \) with an open subset of a blow-up of \( \tilde{K} \). Let us describe the blow-up.

Let us start with some general remarks about the geometry of \( \tilde{K} \). Clearly, \( \tilde{K} \) is fibered over \( \mathbb{P}^1 \) so that the fiber fiber over \( z \in \mathbb{P}^1 \) is \( \mathcal{O}(2)_z \). If \( f \) is a (rational) section of \( \mathcal{O}(2) \) that is regular at \( z \), then its value \( f(z) \in \mathcal{O}(2)_z \) can be viewed as a point of \( \tilde{K} \); we will denote this point by \((z, f(z))\). For example, \((z, 0(z))\) is the zero element in the fiber of \( \tilde{K} \) over \( z \in \mathbb{P}^1 \).

Now let \( \tilde{\sigma}_c : \tilde{K}_c \to \tilde{K} \) be the blow-up of \( \tilde{K} \) at \( c := (z, f(z)) \). Then the exceptional divisor \( \tilde{\sigma}_c^{-1}(c) \subset \tilde{K}_c \) is isomorphic to the projective line \( \mathbb{P}(T_c \tilde{K}) \); that is, points of \( \tilde{\sigma}_c^{-1}(c) \) correspond to lines in the tangent space to \( \tilde{K} \) at \( c \). Any smooth curve \( \tilde{C} \subset \tilde{K} \) that passes through \( c \) defines such a line (the tangent line to \( \mathcal{C} \) at \( c \)). In particular, we can take \( C \) to be the graph \( \{(x, f(x)) : x \in \mathbb{P}^1\} \) of \( f \); denote the corresponding point of \( \tilde{K}_c \) by \((z, f'(z))\). Any other rational section \( g \) of \( \mathcal{O}(2) \) defines a point \((z, g'(z)) \in \tilde{K}_c \) provided \( g \) is regular at \( z \) and \( g(z) = f(z) \).

**Theorem 4.4.**

1. The map \( \tilde{P} : M_\theta \to \tilde{K} \) is a regular birational morphism of smooth algebraic surfaces.

2. Let \( \tilde{\sigma}_1 : \tilde{K}_1 \to \tilde{K} \) be the blow-up of \( \tilde{K} \) at the following 6 points: \((a_1, 0(a_1)) \) \( (i = 1, \ldots, 4) \) and \( (\infty, (\rho_j z^2)(\infty)) \) \( (j = 1, 2) \). Let \( \tilde{\sigma}_2 : \tilde{K}_2 \to \tilde{K}_1 \) be the blow-up of \( \tilde{K}_1 \) at the two points \((\infty, (\rho_j z^2 + p_j d_3 z')(\infty)), j = 1, 2 \) (these points belong to the preimages of \((\infty, (\rho_j z^2)(\infty)), j = 1, 2 \)). Then the map \( \tilde{P} \) induces an open embedding \( \tilde{P}_2 : M_\theta \to \tilde{K}_2 \).

3. The complement to \( \tilde{P}_2(M_\theta) \) in \( \tilde{K}_2 \) is the union of the proper preimages of the following curves: the zero section \( \{(z, 0(z)) : z \in \mathbb{P}^1\} \subset \tilde{K} \), the fiber at infinity \( \{(\infty, az^2(\infty)) : a \in \mathbb{C}\} \subset \tilde{K}, \) and two exceptional curves \( \tilde{\sigma}_1^{-1}(\infty, (\rho_j z^2)(\infty)) \subset \tilde{K}_1 \).

The proof of Theorem 4.4 is given in Section 4.3. Let us now derive Theorem A from Theorem 4.4.

**Proof of Theorem A.** For \((\mathcal{L}, \mathcal{A}) \in M_\theta \), consider the expression

\[
(4.9) \quad p := \frac{\tilde{p}}{(q - a_3)(q - a_4)}.
\]

Here the denominator is the value of the section \((z - a_3)(z - a_4) \in \Gamma(\mathbb{P}^1, \mathcal{O}(2)) \) at \( z = q \in \mathbb{P}^1 \). Both the numerator and the denominator are elements of \( \mathcal{O}(2)_q \); therefore, \( p \in \mathbb{C} \) provided the denominator does not vanish. We can view \( p \) as a rational mapping \( p : M_\theta \to \mathbb{P}^1 \). Actually, Theorem 4.4 implies that \( p : M_\theta \to \mathbb{P}^1 \) is regular: the corresponding rational mapping \( \tilde{K} \to \mathbb{P}^1 \) has singularities at \( (a_3, 0(a_3)), (a_4, 0(a_4)) \in \tilde{K} \), but the blow-up \( \tilde{K}_1 \to \tilde{K} \) resolves the singularities. We therefore obtain a regular mapping \( P := (q, p) : M_\theta \to (\mathbb{P}^1)^2 \). We claim that \( P \) induces an embedding \( P_2 : M_\theta \to K_2 \), where \( K_2 \) is the blow-up of \( (\mathbb{P}^1)^2 \) described in Theorem A.

Let us consider the birational mapping \( \Phi : (q, \tilde{p}) \to (q, p) : \tilde{K} \to (\mathbb{P}^1)^2 \). It is easy to see that \( \Phi \) induces an open embedding \( \Phi_1 : \tilde{K}_1 \to \tilde{K}_1 \), and the complement \( K_1 - \Phi(\tilde{K}_1) \) is the proper preimage of \( \mathbb{P}^1 \times \{\infty\} \subset (\mathbb{P}^1)^2 \) under the blow-up \( K_1 \to \mathbb{P}^1 \).
To complete the proof, we should now check that \( \Phi_1 \) maps the centers of the blow-up \( \widetilde{K}_2 \to \widetilde{K}_1 \) to the centers of the blow-up \( K_2 \to K_1 \). This also follows from the formulas. \( \square \)

**Remark 4.5.** Geometrically, formula (4.3) can be explained as a multiplication of a d-connection by a scalar. For \((L, A) \in M_\theta\), consider the d-connection

\[
\tilde{A} := \frac{1}{(z - a_3)(z - a_4)} A
\]
on \( L \). Then \( \tilde{A} \) has simple zeroes at \( a_1, a_2 \), simple poles at \( a_3, a_4 \), and its formal type at infinity is \((p_1, p_2; a_1^2 + a_2, a_3 + a_4; 0)\). Moreover, we can then view \( M_\theta \) as the moduli space of d-connections of this kind (as in Remark 3.5). For d-connections of this kind, \( p \) plays the role of \( \tilde{p} \), and Theorem A plays the role of Theorem 4.3.

**4.3. Proof of Theorem 4.4** The most direct way to prove Theorem 4.4 is by bringing matrices (4.1) to some ‘normal form’. We will not reproduce all calculations here; the idea of the proof is as follows:

Denote by \( \tilde{M}_\theta \) the open subset of \( \tilde{K}_2 \) described in Theorem 4.4(3) (that is, the complement of proper preimages of the zero section, the fiber at infinity, and two exceptional curves). We need to show that the map \( \tilde{P} : \tilde{M}_\theta \to \tilde{K} \) lifts to an isomorphism \( M_\theta \to \tilde{M}_\theta \). Let us consider open sets

\[
U_0 := q^{-1}(\mathbb{P}^1 - \{\infty\}) \subset M_\theta, \quad U_\infty := q^{-1}(\mathbb{P}^1 - \{0\}) \subset M_\theta
\]

\[
\tilde{U}_0 := q^{-1}(\mathbb{P}^1 - \{\infty\}) \subset \tilde{M}_\theta, \quad \tilde{U}_\infty := q^{-1}(\mathbb{P}^1 - \{\infty\}) \subset \tilde{M}_\theta.
\]

It suffices to show that \( \tilde{P} \) lifts to isomorphisms \( U_0 \to \tilde{U}_0, U_\infty \to \tilde{U}_\infty \). We will show this by writing \( U_0, U_\infty \) explicitly as zero loci of polynomial equations.

Let \((L, A)\) be a point of \( U_0 \). Then \( q = q(L, A) \in \mathbb{C} \) and \( \tilde{p} = \tilde{p}(L, A) \in (\mathcal{O}(2))_q = \mathbb{C} \). It is easy to see that there exists an isomorphism \( \mathcal{S} : \mathcal{O} \oplus \mathcal{O}(-1) \to L \), unique up to a multiplicative constant, such that the matrix of the d-connection \( A \) relative to \( \mathcal{S} \) is

\[
(4.10) \quad A = \begin{bmatrix} a_{11} &=& \tilde{p} & a_{12} \\ a_{21} &=& z - q & a_{22} \end{bmatrix}, \quad a_{22} \in \Gamma(\mathbb{P}^1, \mathcal{O}(2)) \quad a_{12} \in \Gamma(\mathbb{P}^1, \mathcal{O}(3)).
\]

Essentially, (4.10) serves as a normal form of d-connections \((L, A)\) (provided \( q \neq \infty \)). The conditions (4.4), (4.5) now become equations on \( a_{12}, a_{22} \). Explicitly, \( a_{12} \) and \( a_{22} \) are determined by their coefficients

\[
a_{12} = a_{12,3}z^3 + a_{12,2}z^2 + a_{12,1}z + a_{12,0}
\]

\[
a_{22} = a_{22,2}z^2 + a_{22,1}z + a_{22,0},
\]

and (4.4), (4.5) is a system of polynomial equations on \( a_{12,j}, \tilde{p}, \) and \( q \). Solving these equations, we find polynomial (in \( \tilde{p} \) and \( q \)) formulas for all \( a_{12,j} \), except for \( r := a_{22,0} \). The equation on \( r \) looks as follows:

\[
(4.11) \quad \tilde{p}r = F(\tilde{p}, q),
\]

where \( F(\tilde{p}, q) \) is a polynomial. Thus, \( U_0 \) is identified with the zero locus of the equation (4.11) in the three-dimensional space with coordinates \( \tilde{p}, q, r \).

Besides, \( F(0, q) = c(q - a_1)(q - a_2)(q - a_3)(q - a_4) \) for some \( c \in \mathbb{C} - \{0\} \). Therefore, the map \((\tilde{p}, q) : U_0 \to \mathbb{A}^2\) identifies \( U_0 \) with the complement to the
proper preimage of ‘the q-axis’ \{ (0, q) \} in the blow-up of \( \mathbb{A}^2 \) at the four points \( (\tilde{p}, q) = (0, a_i), \) \( i = 1, \ldots, 4. \) This complement is exactly \( \tilde{U}_0. \)

Similar approach works for \( U_\infty. \) For \( (L, A) \in U_\infty, \) set \( \omega := (q(L, A))^{-1} \in \mathbb{C}, \) \( \pi := \frac{\bar{p}(L, A)}{q(L, A)} \in \mathbb{C}, \) where the denominator is understood as the value of \( z^2 \in \Gamma(\mathbb{P}^1, \mathcal{O}(2)) \) at \( z = q. \) One can think of \( \omega \) and \( \pi \) as the coordinates on the complement to the zero locus of \( q \) in \( \tilde{K}. \) Then there is a unique up to a multiplicative constant choice of \( \mathcal{S} : \mathcal{O} \oplus \mathcal{O}(-1) \rightarrow \mathcal{L} \) such that the matrix of \( A \) is

\[
A = \begin{bmatrix}
\pi z^2 & a_{12} \\
1 - \omega z & a_{22}
\end{bmatrix}.
\]

Again, we get a system of polynomial equations on the coefficients of \( a_{i2}. \) Solving the equations, we find polynomial (in \( \pi \) and \( \omega \)) formulas for all \( a_{i2}, \) except for \( r = a_{22}. \) In this case, the equation on \( r \) is

\[
(\pi \omega^2 r = G(\pi, \omega),
\]

where \( G(\pi, \omega) \) is a polynomial. Therefore, \( U_\infty \) is the zero locus of the equation \( (4.12) \) in the three-dimensional space with coordinates \( \pi, \omega, r. \) Again, from the formula for \( G(\pi, \omega), \) one easily sees the isomorphism \( U_\infty \cong \tilde{U}_\infty. \)

For instance, let us consider the neighborhood of \( \omega = 0 \) (the complement of \( \omega = 0 \) is covered by \( U_0). \) One can check that \( G(\pi, 0) = (\pi - \rho_1)(\pi - \rho_2), \) so when \( \omega = 0, \) either \( \pi = \rho_1, \) or \( \pi = \rho_2. \) Consider the neighborhood of the set \( \omega = 0, \pi = \rho_1 \) in \( U_\infty. \) It follows that \( \pi_1 := (\pi - \rho_1)/\omega \) is a regular function on the neighborhood \( (\pi_1 \) is the coordinate on the blow-up of the \( \omega - \pi \) plane at \( (\omega, \pi) = (0, \rho_1). \) We can then rewrite \( (4.12) \) in variables \( \pi_1, \omega, \) and \( r: \)

\[
(\omega \pi_1 + \rho_1) r \omega = H(\pi_1, \omega),
\]

where \( H(\pi_1, \omega) \) is a polynomial such that \( H(\pi_1, 0) = (\rho_2 - \rho_1)(\pi_1 - \rho_1 d_1); \) therefore, \( r \) is essentially the coordinate on the blow-up of the \( \omega - \pi \) plane at \( (\omega, \pi) = (0, \rho_1 d_1). \) Of course, the neighborhood of the set \( \omega = 0, \pi = \rho_2 \) in \( U_\infty \) has a similar description.

**Remark 4.6.** Theorem \([\mathbf{A}] \) can be also proved in a more geometric way, in the spirit of \([\mathbf{3}], \) Theorem 3].

### 4.4. Proof of Theorem \([\mathbf{3}]. \) The proof of Theorem \([\mathbf{3}]. \) is also based on calculations. The calculations are simplified by the observation that it suffices to check the formulas \( (2.7) \) on a dense subset of \( M_0; \) we can therefore assume that \( q, q' \neq \infty. \)

Take \( (L, A) \in M_0 \) and set \( (L', A') := dPV(L, A). \) Let us assume \( q(L, A) \neq \infty \) (that is, \( (L, A) \in U_0), \) then there is an isomorphism \( \mathcal{S} : \mathcal{O} \oplus \mathcal{O}(-1) \rightarrow \mathcal{L} \) such that the matrix of \( A \) relative to \( \mathcal{S} \) is of the form \( (4.10). \) Using the formula \( \tilde{p} = p(q - a_3)(q - a_4), \) we can write the matrix as

\[
(4.13) \quad A = \begin{bmatrix}
 p(q - a_3)(q - a_4) & a_{12} \\
z - q & a_{22}
\end{bmatrix}, \quad a_{22} \in \Gamma(\mathbb{P}^1, \mathcal{O}(2)),
\]

Recall also that \( a_{12}, a_{22} \) are polynomials of \( z \) whose coefficients are rational functions of \( p, q. \)

Similarly, if we assume \( q(L', A') \neq 0, \) there exists an isomorphism \( \mathcal{S}' : \mathcal{O} \oplus \mathcal{O}(-1) \rightarrow \mathcal{L}' \) such that the matrix of \( A' \) relative to \( \mathcal{S}' \) is of the form

\[
(4.14) \quad A' = \begin{bmatrix}
 p'(q' - a_3)(q' - a_4) & a'_{12} \\
z - q' & a'_{22}
\end{bmatrix}, \quad a'_{22} \in \Gamma(\mathbb{P}^1, \mathcal{O}(2)).
\]
By the definition of $dPV$, the matrix $A'$ is the d-gauge transformation of $A$:

$$A'(z) = R(z + 1)^{-1} A(z) R(z),$$

where $R$ is the matrix of the rational map $R : L' \to L$ (from the definition of $dPV$) with respect to the bases $S$, $S'$. It follows from the properties of modifications (Section 3.2) that $R$ induces a regular map $L' \to L \otimes \mathcal{O}(1)$ whose determinant has simple zeroes at $a_1, a_2$ and no other zeroes. In other words, $R$ is of the form

$$R = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix}, \quad r_{11}, r_{22} \in \Gamma(\mathbb{P}^4, \mathcal{O}(1))$$

$$r_{21} \in \mathbb{C}, r_{12} \in \Gamma(\mathbb{P}^4, \mathcal{O}(2)),$$

such that

$$\det(R) = c(z - a_1)(z - a_2) \quad (c \in \mathbb{C} - \{0\}).$$

(4.16) yields polynomial equations on the coefficients of $r_{11}, r_{12}, r_{21}, r_{22}$; the condition that (4.15) gives a matrix $A'$ of the form (4.14) also gives such equations. The resulting system determines $R$ up to a multiplicative constant. From (4.15), we now obtain a formula for the matrix $A'$ in terms of $p$ and $q$; in particular, we can derive (2.7). \hfill \square

5. DIFFERENCE PV AND CLASSICAL PVI

5.1. Geometry of PVI. Let us recall the description of the surface $M_{(x, \lambda)}$. We will suppose that

$$\sum_{i=1}^{4} (\lambda_i^- + \lambda_i^+) = 1.$$  

(5.1)

It is easy to see that $M_{(x, \lambda)}$ only depends on the classes of $\lambda_i^\pm$ in $\mathbb{C}/\mathbb{Z}$ (because of modifications of bundles with connections), so our assumption does not restrict the generality.

Suppose $x \in X$, $\lambda \in \Lambda$, and let $K_x$ be the total space of the line bundle $\Omega_{\mathbb{P}^4}(x_1 + \cdots + x_4)$. Let $b_i \subset K_x$ be the fiber over $x_i \in \mathbb{P}^4$. Notice that the residue of 1-forms identifies the fiber of $\Omega(x_1 + \cdots + x_4)$ over $x_i$ with $\mathbb{C}$, so we get a canonical isomorphism $\text{res}_i : b_i \cong \mathbb{A}$. Denote by $\tilde{M}_{(x, \lambda)}$ the blow-up of $K_x$ at the eight points $(\text{res}_i)^{-1}(\lambda_i^\pm)$, $i = 1, \ldots, 4$, and let $M'_{(x, \lambda)} \subset \tilde{M}_{(x, \lambda)}$ be the complement to the proper preimages of $b_i \subset K_x$.

Proposition 5.1. There exists an isomorphism $M_{(x, \lambda)} \cong M'_{(x, \lambda)}$. \hfill \square

Proposition 5.1 is a slight generalization of [3, Theorem 3] (see also [16, Theorem 2.2]): [3] works only with $SL(2)$-bundles, which corresponds to assuming $\lambda_i^- + \lambda_i^+ = 0$, ($i = 2, 3, 4$). However, the general case is easily reduced to this special case. Let us sketch the construction of the map $M_{(x, \lambda)} \cong M'_{(x, \lambda)}$.

Given $(\mathcal{L}, \nabla) \in \tilde{M}_{(x, \lambda)}$, one can show that $\mathcal{L} \cong \mathcal{O} \oplus \mathcal{O}(-1)$ (this is similar to Corollary 3.13). If we fix an isomorphism $\mathcal{O} \oplus \mathcal{O}(-1) \cong \mathcal{L}$, the connection $\nabla$ is determined by its matrix $M(z) = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$. $m_{11}, m_{22} \in \Gamma(\mathbb{P}^4, \Omega_{\mathbb{P}^4}(x_1 + \cdots + x_4))$ $m_{12} \in \Gamma(\mathbb{P}^4, \Omega_{\mathbb{P}^4}(x_1 + \cdots + x_4) \otimes \mathcal{O}(1))$ $m_{21} \in \Gamma(\mathbb{P}^4, \Omega_{\mathbb{P}^4}(x_1 + \cdots + x_4) \otimes \mathcal{O}(-1))$.

It can be proved that $m_{21}$ is not identically zero (because $(\mathcal{L}, \nabla)$ is irreducible; this is similar to Lemma 3.12). Therefore, $m_{21}$ has a single zero on $\mathbb{P}^1$; denote it by $q_{PV\text{I}}$. 
Set \( p^{PV I} := m_{11}(q^{PV I}) \). Note that \( p^{PV I} \) belongs to the fiber of \( \Omega_{21}(x_1 + \cdots + x_4) \) over \( q^{PV I} \in \mathbb{P}^1 \). In other words, \( p^{PV I} \) is a point of the total space \( K_x \) (in the notation of Section 4.2) the point is \((q^{PV I}, p^{PV I}) \in K_x \). One can check that \( q^{PV I} \) and \( p^{PV I} \) depend only on \((L, \nabla)\), not on the choice of \( \mathcal{O} \oplus \mathcal{O}(-1) \rightarrow L \). Therefore, we obtain a regular map \( M_{(x, \lambda)} \rightarrow K_x \). Proposition 5.1 claims the map induces an isomorphism \( M_{(x, \lambda)} \rightarrow M'_{(x, \lambda)} \).

**Proof of Theorem A.** Let \( \theta \in \Theta_4^x, x \in X, \) and \( \lambda \in \Lambda \) be as in Theorem A we will define the isomorphism \( M_{\theta} \rightarrow M_{(x, \lambda)} \) by explicit formulas. Let \( q, p : M_{\theta} \rightarrow \mathbb{P}^1 \) be the ‘coordinates’ from Theorem A Consider the expression

\[
\tilde{p}^{PV I} := (z^{-1}dz)_{z=p},
\]

where \((z^{-1}dz)_{z=p} \in (\Omega_{21}(x_1 + \cdots + x_4))_p \) is the value of \( z^{-1}dz \in \Gamma(\mathbb{P}^1, \Omega_{21}(x_1 + \cdots + x_4)) \) at \( z = p \). Then \( p^{PV I} \in (\Omega_{21}(x_1 + \cdots + x_4))_p \), provided \( q \neq \infty \). Let us also set \( q^{PV I} := p \).

If \( q \neq \infty \), we have \((q^{PV I}, p^{PV I}) \in K_x \); in this manner, we get a rational map

\[
M_{\theta} \rightarrow K_x : (q, p) \mapsto (q^{PV I}, p^{PV I}).
\]

Using Theorem A and Proposition 5.1, it is easy to see that the map is actually regular, and that it lifts to an isomorphism \( M_{\theta} \rightarrow M_{(x, \lambda)} \).

5.2. **Classical PV I.** The isomonodromy deformation of bundles with connections gives a system of differential equations on the ‘coordinates’ \( q^{PV I}, p^{PV I} \) (the ‘usual’ PV I). Here \( q^{PV I}, p^{PV I} \) are viewed as functions of \( x_1, \ldots, x_4, \) while \( \lambda^\pm_i \) are fixed parameters. Let us recall the explicit formulas (which we adapted from [17]).

For simplicity, we assume, in addition to (5.1), that \( x_4 = \infty \). Define the new parameters by \( \kappa_i := \lambda^+_i - \lambda^-_i, \) \( i = 1, \ldots, 4, \) and let us replace the variable \( p^{PV I} \) with

\[
\tilde{p}^{PV I} := (p^{PV I}/dz) - \sum_{i=1}^{\lambda^-} \frac{\lambda^-}{z - x_i}.
\]

Since \( p^{PV I} \in (\Omega_{21}(x_1 + \cdots + x_4))_x^{PV I} \), the ratio \( p^{PV I}/dz \) (if it is defined) is a number. The advantage of \( \tilde{p}^{PV I} \) is that the differential equations for \( q^{PV I}, \tilde{p}^{PV I} \) involve fewer parameters: \( \kappa_i \)'s, rather than \( \lambda^\pm_i \)'s.

Set also

\[
\kappa_0 := \frac{1}{2} \left( 1 - \sum_{i=1}^{4} \kappa_i \right),
\]

and \( q_i := q^{PV I} - x_i, \) \( i = 1, 2, 3. \) Define the Hamiltonians \( h_i, i = 1, 2, 3 \) by

\[
h_i := (q_1q_2q_3)(\tilde{p}^{PV I})^2 - ((\kappa_i - 1)q_i q_k + \kappa_j q_i q_k + \kappa_k q_i q_j) \tilde{p}^{PV I} + \kappa_0 (\kappa_0 + \kappa_4).
\]

The equations can then be written in the Hamiltonian form as

\[
\frac{\partial q^{PV I}}{\partial x_i} = \frac{\partial h_i}{\partial \tilde{p}^{PV I}}, \quad \frac{\partial \tilde{p}^{PV I}}{\partial x_i} = \frac{\partial h_i}{\partial q^{PV I}} \quad (i = 1, 2, 3).
\]

The system (5.2) can be reduced to the usual form of PV I as follows: set

\[
y := \frac{q^{PV I} - x_1}{x_2 - x_1}, \quad x := \frac{x_3 - x_1}{x_2 - x_1},
\]

and define

\[
n_i := q_i - x_i, \quad i = 1, 2, 3.
\]

The Hamiltonians in this new system can be written as

\[
h_i := (n_1n_2n_3)(\tilde{p}^{PV I})^2 + \kappa_0 (\kappa_0 + \kappa_4).
\]

The system (5.2) can then be reduced to the usual form of PV I as follows: set

\[
y := \frac{q^{PV I} - x_1}{x_2 - x_1}, \quad x := \frac{x_3 - x_1}{x_2 - x_1},
\]

and define

\[
n_i := q_i - x_i, \quad i = 1, 2, 3.
\]

The Hamiltonians in this new system can be written as

\[
h_i := (n_1n_2n_3)(\tilde{p}^{PV I})^2 + \kappa_0 (\kappa_0 + \kappa_4).
\]
Then \( \text{(5.2)} \) implies that \( y \) depends only on \( x \), not on \( x_1, x_2, x_3 \), and that \( y \) satisfies the PVI equation

\[
\frac{d^2 y}{dx^2} = \frac{1}{2} \left( \frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-x} \right) \left( \frac{dy}{dx} \right)^2 - \left( \frac{1}{x} + \frac{1}{x-1} + \frac{1}{y-x} \right) \frac{dy}{dx} + \frac{y(y-1)(y-x)}{x^2(x-1)^2} \left( \kappa_1^2 - \frac{x-1}{y} + \kappa_2^2 - \frac{1}{(y-1)^2} + (1 + \kappa_2^2)(x-1) \right).
\]

### 5.3. Isomonodromy deformation of d-connections

Let us prove Theorem 11). Informally, we need to show that, given \( \theta \in \Theta_{2n}^\circ \) and \( \rho_1, \rho_2 \in \mathbb{C} \), any d-connection of type \( \theta \) has a natural ‘first order deformation’ that is ‘of type’

\[
\theta^\epsilon := (a_1, \ldots, a_{2n}; \rho_1 + \epsilon \rho_1, \rho_2 + \epsilon \rho_2, d_1, d_2; n).
\]

Here \( \epsilon \) is the parameter of the deformation, and all calculations are done modulo \( \epsilon^2 \), that is, over the ring of dual numbers \( \mathbb{C}^\epsilon := \mathbb{C}[\epsilon]/(\epsilon^2) \). First, let us prove a ‘formal’ statement:

**Proposition 5.2.** Suppose the matrix \( A(z) = \sum_{i \leq n} A_i z^i \) over \( \mathbb{C}((z^{-1})) \) has formal type \( (\rho_1, \ldots, \rho_m; d_1, \ldots, d_m; n) \) at infinity (see Proposition 44). For any collection \( \rho_1', \ldots, \rho_m' \in \mathbb{C} \), there exists a gauge matrix \( R(\epsilon)^* = R(z) + \epsilon R'(z) \), where \( R(z) \) is as in Proposition 44 (that is, \( R(z) \) is an invertible \( m \times m \) matrix over \( \mathbb{C}[[z^{-1}]] \)), and \( R'(z) \) is an \( m \times m \) matrix over the ring of formal Laurent series \( \mathbb{C}((z^{-1})) \) such that

\[
R(z+1)^{-1}A(z)R(\epsilon)^* = \text{diag}((\rho_1 + \rho_1\epsilon),(\rho_2 + \rho_2\epsilon), \ldots, (\rho_m + \rho_m\epsilon)).
\]

The matrix \( R(\epsilon)^* \) is unique up to right multiplication by a diagonal matrix with entries in \( \mathbb{C}^\epsilon \).

**Proof.** \( \text{(5.4)} \) is equivalent to the following two conditions:

\[
\text{(5.5)} \quad R(z+1)^{-1}A(z)R(z) = \text{diag}(\rho_1 z^n + \rho_1 d_1 z^{n-1}, \ldots, \rho_m z^n + \rho_m d_m z^{n-1}),
\]

\[
\text{(5.6)} \quad R(z+1)^{-1}A(z)R'(z) - R(z+1)^{-1}R'(z+1)R(z+1)^{-1}A(z)R(z) = \text{diag}(\rho_1' z^n + \rho_1' d_1 z^{n-1}, \ldots, \rho_m' z^n + \rho_m' d_m z^{n-1}).
\]

As \( A(z) \) has formal type \( (\rho_1, \ldots, \rho_m; d_1, \ldots, d_m; n) \) at infinity, there exists a matrix \( R(z) \) satisfying \( \text{(5.5)} \); moreover, \( R(z) \) is unique up to right multiplication by a constant diagonal matrix (Proposition 31). Once \( \text{(5.5)} \) is satisfied, \( \text{(5.6)} \) can be rewritten as

\[
\text{(5.7)} \quad B(z)S(z) - S(z+1)B(z) = \text{diag}(\rho_1' z^n + \rho_1' d_1 z^{n-1}, \ldots, \rho_m' z^n + \rho_m' d_m z^{n-1}),
\]

where we set \( B(z) \) := \( \text{diag}(\rho_1 z^n + \rho_1 d_1 z^{n-1}, \ldots, \rho_m z^n + \rho_m d_m z^{n-1}) \), and \( S(z) := R(z)^{-1}R'(z) \). One can view \( \text{(5.4)} \) as a difference equation on the matrix \( S(z) \); it is easy to see that the only solutions whose matrix elements are Laurent series are given by \( S(z) = \text{diag}((\rho_1'/\rho_1) z + c_1, \ldots, (\rho_m'/\rho_m) z + c_m) \), where \( c_i \)’s are arbitrary constants. This implies the statement.

Proposition 5.2 allows us to construct the natural first order deformation, thus proving Theorem 11). The construction is most easily described using the following well-known statement.
Lemma 5.3. Let \( L \) be a vector bundle on \( \mathbb{P}^1 \) and let \( S(z) : \mathbb{C}^2 \to L_z \) be a trivialization of \( L \) in the punctured formal neighborhood of \( \infty \) (so \( S(z) \) is essentially a matrix whose entries belong to \( \mathbb{C}((z^{-1})) \)). Then there exists a unique vector bundle \( L^S \) such that \( L \) and \( L^S \) have equal restrictions to \( \mathbb{P}^1 - \{ \infty \} \) and that the map

\[
S(z) : \mathbb{C}^2 \to L_z = (L^S)_z
\]

extends to a trivialization of \( L^S \) in the formal neighborhood of \( \infty \). \( \square \)

Notice that Lemma 5.3 still works when \( S \) depends on parameters. In this case, the modification \( L^S \) will also depend on the parameters.

Proof of Theorem 5.4.1. Take \( \rho = (\rho_1, \rho_2) \in P, (L, A) \in M_0(\rho) \). Take a tangent vector \( \tau = \rho_1 \frac{\partial}{\partial z_1} + \rho_2 \frac{\partial}{\partial z_2} \) to \( P \) at \( \rho \). Let us construct a natural lifting of \( \tau \) to a tangent vector a tangent vector \( \tau_M \) to \( M \) at \( (L, A) \in M \).

Choose a trivialization \( S(z) : \mathbb{C}^2 \to L_z \) on the neighborhood of \( \infty \in \mathbb{P}^1 \). The matrix

\[
A(z) := S^{-1}(z+1)A(z)S(z)
\]

of \( A \) relative to \( S \) satisfies the assumption of Proposition 5.2. Let us set \( S'(z) := S(z)R^\epsilon(z) \), where the matrix \( R^\epsilon(z) \) is given by Proposition 5.2. We can view \( S'(z) \) as a trivialization of \( L \) in the punctured formal neighborhood of \( \infty \in \mathbb{P}^1 \) that depends on \( \epsilon \in \mathbb{C}^\ast \). Lemma 5.3 defines a vector bundle \( L' := L^{S'} \) that depends on \( \epsilon \).

\( L' \) and \( L \) coincide on \( \mathbb{P}^1 - \{ \infty \} \) (for any value of the parameter \( \epsilon \)), thus the d-connection \( A' \) on \( L' \) induces a d-connection \( A' \) on \( L' \). Notice also that when \( \epsilon = 0 \), we have \( L' = L, A' = A \). The pair \( (L', A') \) define a tangent vector \( \tau_M \) to \( M \) at \( (L, A) \). The vector \( \tau_M \) does not depend on the choice of \( R^\epsilon \). It is easy to see that as \( \tau \) and \( (L, A) \) vary, the lifting \( \tau_M \) defines a flat algebraic connection on \( M \to P \). \( \square \)

5.4. Isomonodromy deformation for \( 2n = 4 \). Suppose now that \( 2n = 4 \), deg(\( \theta \)) = -1. Then the construction of the previous section can be reformulated more explicitly. Instead of working with d-connections, let us consider their matrices (that is, we think of \( M_0 \) as a quotient \( X_0/G \), see Corollary 5.2).

Let \( (L, A) \) and \( (L', A') \) be as above. Choose a trivialization \( S' : \mathcal{O} \oplus \mathcal{O}(-1) \to L' \) (depending on \( \epsilon \)). When \( \epsilon = 0 \), \( S' \) becomes a trivialization \( S : \mathcal{O} \oplus \mathcal{O}(-1) \to L \). Let \( A \) be the matrix of \( A \) relative to \( S \), and \( A' \) be the matrix of \( A' \) relative to \( S' \). Let us summarize the properties of \( A' \):

Proposition 5.4. The matrix \( A'(z) = A(z) + \epsilon A'(z) \), where

\[
A' = \begin{bmatrix} a'_{11} & a'_{12} \\ a'_{21} & a'_{22} \end{bmatrix}, \quad a'_{11}, a'_{22} \in \Gamma(\mathbb{P}^1, \mathcal{O}(2)) \\
\quad a'_{12} \in \Gamma(\mathbb{P}^1, \mathcal{O}(3)) \\
\quad a'_{21} \in \Gamma(\mathbb{P}^1, \mathcal{O}(1))
\]

satisfies the following conditions:

1. For some \( 2 \times 2 \) matrix \( S'(z) = 1 + \epsilon S'(z) \), where the entries of \( S'(z) \) are polynomials in \( z \) (of arbitrary degree), we have \( A'(z) = S'(z+1)^{-1} A(z) S'(z) \).
2. For some \( 2 \times 2 \) matrix

\[
R'(z) = \begin{bmatrix} 1 & 0 \\ 0 & z^{-1} \end{bmatrix}(T(z^{-1}) + \epsilon T'(z^{-1})),
\]
where $T, T'$ are $2 \times 2$ matrices over $\mathbb{C}[[z^{-1}]]$ and $T$ is invertible (that is, $\det(T|_{z^{-1}=0}) \neq 0$), we have

$$R'(z+1)^{-1} A'(z) R'(z) = \text{diag}((\rho_1 + \epsilon \rho'_1)(z^2 + d_1 z), (\rho_2 + \epsilon \rho'_2)(z^2 + d_2 z)).$$

Conversely, a matrix $A'$ with such properties corresponds to the 'continuous isomonodromy deformation' of Theorem (D1). Actually, we can reformulate Theorem (D1) (for $2n = 4$, $\deg(\theta) = -1$) as the following statement:

**Proposition 5.5.** Let $A(z) \in X_\theta$, $\theta \in \Theta^1_\infty$, $\deg(\theta) = -1$.

1. There is a deformation $A'(z)$ that satisfies the conditions of Proposition 5.4.

2. $A'(z)$ is unique up to a $d$-gauge transformation

$$A'(z) \mapsto R'(z+1)^{-1} A'(z) R'(z)$$

for a gauge matrix $R'(z) = 1 + \epsilon R'(z)$, where $R'(z)$ is of the form

$$R'(z) = \begin{bmatrix} r'_{11} & r'_{12} \\ 0 & r'_{22} \end{bmatrix}, \quad r'_{11}, r'_{22} \in \mathbb{C} - \{0\}, \quad r'_{12} \in \Gamma(\mathbb{P}^1, \mathcal{O}(1)).$$

5.5. Isomonodromy deformation of $d$-connections as $PVI$. Let us now use coordinates $q, p$ on $M_\theta$ to write the connection of Theorem (D1) as a system of differential equations on $p$ and $q$. Suppose $(\mathcal{L}, A) \in M_\theta$ and let $A \in X_\theta$ be the matrix of $A$ relative to some trivialization $\mathcal{S} : \mathcal{O} \oplus \mathcal{O}(-1) \to \mathcal{L}$. We need to find $A'$ that satisfies the conditions of Proposition 5.4. As in Section 4.4, it suffices to do so when $(\mathcal{L}, A)$ belong to a dense subset of $M_\theta$; we can thus assume that $q(\mathcal{L}, A) \neq \infty$.

We will look for $A'$ in the form

$$A'(z) = \begin{bmatrix} a'_{11} & a'_{12} \\ a'_{21} & a'_{22} \end{bmatrix} = S'(z+1)^{-1} A(z) S'(z)$$

for the gauge matrix

$$S'(z) = 1 + \begin{bmatrix} s'_{11} & s'_{12} \\ s'_{21} & s'_{22} \end{bmatrix} \epsilon \quad s'_{11}, s'_{22} \in \Gamma(\mathbb{P}^1, \mathcal{O}(2)), \quad s'_{12} \in \Gamma(\mathbb{P}^1, \mathcal{O}(1)), \quad s'_{21} \in \mathbb{C}.$$

(Actually, the proof of Proposition 5.4 shows that $A'$ is necessarily of this form.) Then $A'$ automatically satisfies Proposition 5.4.1, so we only need to make sure that Proposition 5.4.2 is satisfied. From Lemma 4.1 (which still holds for $d$-connections that depend on $\epsilon$), we see that Proposition 5.4.2 is equivalent to the following equations:

\begin{align*}
\det(A') &= (z-a_1)(z-a_2)(z-a_3)(z-a_4)(\rho_1^r \rho_2^r) \\
(\rho_1^r + \rho_2^r)(z+z^{-1}) &= (\rho_1^r + \rho_2^r)(z + d_1 \rho_1^r + d_2 \rho_2^r) + t(z^{-1}),
\end{align*}

where $\rho_1^r = \rho_1 + \rho_1^r \epsilon$, and $t(z^{-1}) \in \mathbb{C}[[z^{-1}]]$ is a Taylor series in $z^{-1}$ with coefficients in $\mathbb{C}^\epsilon$. Solving these equations, we can find formulas for $q^r$, $p^r$ in terms of $\rho_1^r$, $\rho_2^r$, and $\theta$; here $q^r$ and $p^r$ are determined by the condition

$$a'_{21}(q + cq') = 0 \in \mathbb{C}^\epsilon, \quad a'_{11}(q + cq') = (p + cp')(q + cq' - a_3)(q + cq' - a_4).$$
The formulas for \( q' \) and \( p' \) can then be viewed as a system on differential equations on \( q \) and \( p \) (considered as functions of \( \rho_i \)):

\[
dq = \frac{\rho_1 d\rho_2 - \rho_2 d\rho_1}{\rho_1 - \rho_2} \left( \frac{p(q-a_3)(q-a_4)}{\rho_1 \rho_2} - \frac{(q-a_1)(q-a_2)}{p} \right)
\]

\[
dp = \frac{\rho_1 d\rho_2 - \rho_2 d\rho_1}{\rho_1 - \rho_2} \left( \frac{\rho_1 d\rho_2 - \rho_2 d\rho_1}{\rho_1 - \rho_2} \left( -\frac{\rho_1 \rho_2}{\rho_1 \rho_2} \right) \right)
\]

(5.9)

\[
\begin{align*}
\frac{p}{\rho_1 \rho_2} &\left( d_1 \rho_1 + d_2 \rho_2 + 2q(\rho_1 + \rho_2) \right) + \\
&\frac{p}{\rho_1 \rho_2} \left( d_1 \rho_1 + d_2 \rho_2 + 2q(\rho_1 + \rho_2) \right)
\end{align*}
\]

Proof of Theorem D(2). We need to verify that (5.9) is obtained from the PVI [2] by plugging in the formulas for \( \rho PV I \), \( q PV I \), \( \zeta \)'s, and \( \lambda \)'s (from Theorem C and Section 5.1). This is a straightforward calculation. □

Remark 5.6. Theorem D(2) can also be proved by an indirect argument. Indeed, both \( PV I \) and (5.9) define algebraic connections on the family \( M \rightarrow P \) from Theorem D. The difference between two such connections is a vector field on the moduli space \( M_\delta \); on the other hand, it is known that \( M_\delta \) has no non-zero global vector fields ([2, Theorem 3, Lemma 3], [28, Proposition 2.1]).

Still another, more geometric, proof of Theorem D(2) uses the Mellin transform described in Section 5.6. It is easy to see that under the transform, the continuous isomonodromy deformation of d-connections (from Theorem D(1)) corresponds to the isomonodromy deformation of ordinary connections, which is described by the sixth Painlevé equation.

5.6. Mellin transform. In this section (which is completely independent from the rest of the paper), we sketch the geometric construction underlying Theorem C. Fix \( \theta \in \Theta_3, x \in X \), and \( \lambda \in \Lambda \) as in Theorem C.

Take \( (\hat{\mathcal{L}}, \nabla) \in M_{(x, \lambda)} \). For any \( z \in \mathbb{C} \), consider the connection

\[ \nabla_z := \nabla - z\zeta^{-1}d\zeta : \hat{\mathcal{L}} \rightarrow \hat{\mathcal{L}} \times \Omega_{\mathcal{P}_3}(x_1 + x_2 + x_3 + x_4), \]

where we denote by \( \zeta \) the coordinate on \( \mathbb{P}^1 \). Recall that \( x_1 = 0, x_4 = \infty \), so subtraction of \( z\zeta^{-1}d\zeta \) from \( \nabla \) does not introduce new poles. Denote by \( \hat{\mathcal{L}}_z \supset \hat{\mathcal{L}} \) the smallest quasi-coherent sheaf that contains \( \hat{\mathcal{L}} \) and such that \( \nabla_z(\hat{\mathcal{L}}_z) \subset \hat{\mathcal{L}}_z \) for all \( z \in \mathbb{C} \). (In terms of D-modules, \( \hat{\mathcal{L}}_z \) can be constructed by taking the intermediate extension of \( \nabla_z \) from \( \mathbb{P}^1 - \{x_1, x_2, x_3, x_4\} \) to \( \mathbb{P}^1 - \{0, \infty\} \) and then extending to \( \mathbb{P}^1 \).)

Consider the first de Rham cohomology group \( H^1_{DR}(\hat{\mathcal{L}}_z, \nabla_z) \). Since \( \hat{\mathcal{L}}_z \) and \( \hat{\mathcal{L}}_z \otimes \Omega_{\mathcal{P}_3} \) have no higher cohomologies, it can be computed by the formula

\[ H^1_{DR}(\hat{\mathcal{L}}_z, \nabla_z) = \text{coker}(\nabla_z : \Gamma(\mathbb{P}^1, \hat{\mathcal{L}}_z) \rightarrow \Gamma(\mathbb{P}^1, \hat{\mathcal{L}}_z \otimes \Omega_{\mathcal{P}_3})). \]

\( H^1_{DR}(\hat{\mathcal{L}}_z, \nabla_z) \) depends on \( z \) in an algebraic way; more precisely, it is the fiber over \( z \in \mathbb{C} \) of a natural quasi-coherent sheaf \( \mathcal{L}_z \) on \( \mathbb{P}^1 - \{0, \infty\} \). The sheaf \( \mathcal{L}_z \) is the Mellin transform of \( \hat{\mathcal{L}}_z \) in terms of \( \mathbb{C} \).

Consider now the rational map \( a : \hat{\mathcal{L}} \rightarrow \hat{\mathcal{L}} : s \mapsto \zeta \). Note that \( a \) satisfies the relation \( a \circ \nabla_z = \nabla_{z+1} \circ a \). It is also easy to see that \( a \) induces an automorphism of \( \hat{\mathcal{L}}_z \); therefore, it becomes an isomorphism of D-modules (that is, quasi-coherent sheaves with connections) \( (\hat{\mathcal{L}}_z, \nabla_z) \cong (\hat{\mathcal{L}}_z, \nabla_{z+1}) \). Hence \( a \) yields an identification.

\[
\tilde{A}(z) : H^1_{DR}(\hat{\mathcal{L}}_z, \nabla_z) \cong H^1_{DR}(\hat{\mathcal{L}}_z, \nabla_{z+1}).
\]
As \( z \in \mathbb{C} \) varies, we can view \( \tilde{A}(z) \) as a d-connection on the quasicoherent sheaf \( \mathcal{L}_s \). One can check that \( \mathcal{L}_s \) contains a unique coherent locally free subsheaf of rank 2 (that is, a rank 2 vector bundle) \( \mathcal{L} \subset \mathcal{L}_s \) such that

\[
\mathcal{A}(z) := (z - a_3)(z - a_4)\tilde{A}(z)
\]

is a d-connection of type \( \theta \) on \( \mathcal{L} \). The correspondence

\[
(\hat{\mathcal{L}}, \nabla) \mapsto (\mathcal{L}, \mathcal{A})
\]

gives a map \( M_{(x, \lambda)} \rightarrow M_{\theta} \). Note that the scalar multiple \( (z - a_3)(z - a_4) \) also appears in Remark 1.3.

To describe the inverse map \( M_{\theta} \rightarrow M_{(x, \lambda)} \), let us reconstruct \((\hat{\mathcal{L}}, \nabla)\) from \((\mathcal{L}, \mathcal{A})\). For any \( \zeta \in \mathbb{C} - \{0\} \), consider the d-connection

\[
\tilde{A}_\zeta := \zeta^{-1} \frac{\mathcal{A}}{(z - a_3)(z - a_4)}
\]
on \( \mathcal{L} \). Let \( \mathcal{L}_{s!} \) be the smallest quasicoherent sheaf on \( \mathbb{P}^1 \) that contains \( \mathcal{L} \) and such that \( \tilde{A}_\zeta \) induces an isomorphism \((\mathcal{L}_{s!})_z \rightarrow (\mathcal{L}_{s!})_{z+1}\) for all \( z \) and \( \zeta \) (the quotient \( \mathcal{L}_{s!}/\mathcal{L} \) is the direct sum of length 1 skyscraper sheaves supported at points \( a_1, a_1 - 1, a_1 - 2, \ldots; a_2, a_2 - 1, \ldots; a_3 + 1, a_3 + 2, \ldots; a_4 + 1, a_4 + 2, \ldots \)). For any \( \zeta \in \mathbb{C} - \{0\} \), we obtain a structure of a \( \mathbb{Z} \)-equivariant sheaf on \( \mathcal{L}_{s!} \), where \( 1 \in \mathbb{Z} \) acts on \( \mathbb{P}^1 \) by \( z \mapsto z + 1 \) and on \( \mathcal{L}_{s!} \) by \( \tilde{A}_\zeta \) (in some sense, \( \mathcal{L}_{s!} \) is obtained from \( \mathcal{L} \) by an ‘intermediate extension’ for \( \mathbb{Z} \)-equivariant sheaves). Consider the corresponding equivariant cohomology group \( H^2_\mathbb{Z}(\mathcal{L}_{s!}, \tilde{A}_\zeta) \), which can be computed by the formula

\[
H^2_\mathbb{Z}(\mathcal{L}_{s!}, \tilde{A}_\zeta) = \text{coker}(\tilde{A}_\zeta - 1 : \Gamma(\mathbb{P}^1, \mathcal{L}_{s!}) \rightarrow \Gamma(\mathbb{P}^1, \mathcal{L}_{s!})).
\]

\( H^2_\mathbb{Z}(\mathcal{L}_{s!}, \tilde{A}_\zeta) \) is the fiber over \( \zeta \in \mathbb{C} - \{0\} \) of the quasicoherent sheaf \( \hat{\mathcal{L}}_{s!} \) on \( \mathbb{P}^1 - \{\infty, 0\} \).

For every \( \zeta \in \mathbb{C} - \{0\} \), consider the rational map

\[
\delta(\zeta) : \mathcal{L} \rightarrow \mathcal{L} : s \mapsto z\zeta^{-1}s.
\]

\( \delta(\zeta) \) induces a regular map \( \mathcal{L}_s \rightarrow \mathcal{L}_{s!} \), and, therefore, a map

\[
\delta_*(\zeta) : \Gamma(\mathbb{P}^1, \mathcal{L}_s) \rightarrow \Gamma(\mathbb{P}^1, \mathcal{L}_{s!}).
\]
The map \( \delta_*(\zeta) \) satisfies the following commutativity relation

\[
\delta_*(\zeta)\tilde{A}_\zeta = \tilde{A}_\zeta\delta_*(\zeta) + \frac{d\tilde{A}_\zeta}{d\zeta}.
\]

Now let us consider the trivial quasicoherent sheaf over \( \mathbb{P}^1 - \{0, \infty\} \) whose fiber over every point \( \zeta \in \mathbb{P}^1 - \{0, \infty\} \) equals \( \Gamma(\mathbb{P}^1, \mathcal{L}_s) \). The formula \( \tilde{A}_\zeta - 1 \) gives an endomorphism of this sheaf; the cokernel of the endomorphism is \( \hat{\mathcal{L}}_{s!} \). Notice now that \( \tilde{A}_\zeta - 1 \) is horizontal with respect to the connection \( \nabla = d + \delta_*(\zeta)d\zeta \) on the sheaf. Therefore, \( \nabla \) induces a connection \( \hat{\mathcal{L}}_{s!} \rightarrow \hat{\mathcal{L}}_{s!} \otimes \Omega_{\mathbb{P}^1} \) (which we will also denote by \( \nabla \)). Finally, \( \hat{\mathcal{L}} \subset \hat{\mathcal{L}}_{s!} \) can be reconstructed as the only coherent locally free subsheaf of rank 2 such that \( \nabla \) is a connection of type \( (x, \lambda) \) on \( \hat{\mathcal{L}} \).

6. Difference \( PVI \)

In this section, we study \( M_{\theta} \) for \( \theta \in \Theta^0_k \). We will need suitable versions of several statements from Section 3.
6.1.

**Proposition 6.1** (cf. Proposition 3.1). Suppose that the matrix $A(z) = \sum_{i \leq n} A_i z^i$ over $\mathbb{C}((z^{-1}))$ satisfies the following condition:

\begin{equation}
\text{(6.1) The leading term } A_n \text{ is a non-zero scalar matrix while all eigenvalues of the next term } A_{n-1} \text{ are distinct.}
\end{equation}

Then there exists a gauge matrix $R(z) = \sum_{i \leq 0} R_i z^i$ with invertible $R_0$ such that\(^{(6.2)}\)

\[ R(z+1)^{-1}A(z)R(z) = A'_n z^n + A'_{n-1} z^{n-1}, \]

where $A'_n$ and $A'_{n-1}$ is diagonal. $R(z)$ is uniquely determined up to right multiplication by a permutation matrix and a constant diagonal matrix. □

As before, we will denote the only eigenvalue of $A'_n$ by $\rho = \rho_1 = \cdots = \rho_n$, and the eigenvalues of $A'_{n-1}$ by $\rho d_1, \ldots, \rho d_n$. It is easy to see that $A_n = A'_n$ (so $\rho$ is also the eigenvalue of $A_n$) and $A_{n-1}$ is conjugate to $A'_{n-1}$ (so $\rho d_1, \ldots, \rho d_n$ are also eigenvalues of $A'_{n-1}$): this can be thought of as a version of Remark 3.3.

**Proposition 6.2** (cf. Proposition 3.8). Suppose $\theta = (a_1, \ldots, a_k; \rho, \rho, d_1, d_2; n)$, and $d_1 \neq d_2$. Let $(\mathcal{L}', \mathcal{A}')$ be an elementary upper modification of $(\mathcal{L}, \mathcal{A}) \in \mathcal{M}_\theta$ given by $(x \in \mathbb{P}^1; l \subset \mathcal{L}_x)$. Then the only cases when $(\mathcal{L}', \mathcal{A}')$ belongs to $\mathcal{M}_{\theta'}$ for some $\theta' \in \Theta$ are as follows:

1. If $x = \infty$, then $l$ must be an eigenspace of $A_{n-1} : \mathcal{L}_\infty \to \mathcal{L}_\infty$ (the second term of $A = \rho z^n + A_{n-1} z^{n-1} + \text{lower order terms}$). If, for instance, $l = \ker(A_{n-1} - \rho d_1) \subset \mathcal{L}_\infty$, then $\theta' = (a_1, \ldots, a_k; \rho_1, \rho_2, d_1 - 1, d_2; n)$, and an analogous formula holds when $l = \ker(A_{n-1} - \rho d_2)$.

2. If $x = a_i$ is a zero of $A$ and $x - 1 \neq a_j$ is not, then $l$ must be the kernel of $A(x) : \mathcal{L}_x \to \mathcal{L}_{x+1}$; in this case, $\theta' = (a_1, \ldots, a_{i-1}, a_j; \rho_1, \rho_2, d_1, d_2; n)$.

In either case, the elementary modifications define an isomorphisms $\mathcal{M}_\theta \cong \mathcal{M}_{\theta'}$.

**Corollary 6.3.** Suppose $\theta \in \Theta_k$ satisfies (1A), (2A). Then $\mathcal{M}_\theta$ is naturally isomorphic to $\mathcal{M}_{\theta'}$ whenever $\theta'$ is obtained from $\theta$ by adding integers to $a_i$'s and $d_i$'s.

□

**Lemma 6.4** (cf. Corollary 5.13). Suppose $(\mathcal{L}, \mathcal{A}) \in \mathcal{M}_\theta$ and suppose that $\theta \in \Theta_{2n}$ satisfies (1A), (2A). If $\mathcal{L} \simeq \mathcal{O}(n_1) \oplus \mathcal{O}(n_2)$, then $|n_1 - n_2| \leq n - 1$.

Proof. The proof repeats that of Corollary 5.13: the only difference is that the order of pole of $\alpha$ at $\infty$ cannot exceed $n - 1$ (because the coefficient of $z^n$ in $\alpha$ is an off-diagonal element of a scalar matrix; that is, zero). □

6.2. **Proof of Theorems 12, 13**. The proof of Theorem 12 follows the same ideas as that of Theorem 1. Fix $\theta \in \Theta_0$, $\deg(\theta) = -1$. For any $(\mathcal{L}, \mathcal{A}) \in \mathcal{M}_\theta$, Lemma 6.4 implies $\mathcal{L} \cong \mathcal{O} \oplus \mathcal{O}(-1)$. Choosing an isomorphism $\mathcal{S} : \mathcal{O} \oplus \mathcal{O}(-1) \to \mathcal{L}$, we can write $\mathcal{A}$ as a matrix

\[ A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad a_{11}, a_{22} \in \Gamma(\mathbb{P}^1, \mathcal{O}(3)) \]

\[ a_{12} \in \Gamma(\mathbb{P}^1, \mathcal{O}(4)) \]

\[ a_{21} \in \Gamma(\mathbb{P}^1, \mathcal{O}(2)) \]

Choosing a different isomorphism $\mathcal{S}$ replaces $A$ with its $d$-gauge transformation (4.5), where the gauge matrix $R$ is given by (4.2).
Lemma 6.5 (cf. Lemma 4.1). Let $\mathcal{A}$ be a $d$-connection on $\mathcal{O} \oplus \mathcal{O}(-1)$; its matrix $A$ is of the form (6.3). We claim that $\mathcal{A}$ is of type $\theta$ if and only if $A$ satisfies the following conditions:

\[
a_{12} \in \Gamma(\mathbb{P}^1, \mathcal{O}(3)); \quad a_{21} \in \Gamma(\mathbb{P}^1, \mathcal{O}(1)); \quad a_{11} - \rho z^3, a_{22} - \rho z^3 \in \Gamma(\mathbb{P}^1, \mathcal{O}(2))
\]

\[
det(A) = (z - a_1)(z - a_2)(z - a_3)(z - a_4)(z - a_5)(z - a_6) \rho^2
\]

\[
(a_{11} - \rho z^3)(a_{22}(1 + z^{-1}) - \rho z^3) - a_{12} a_{21} = d_1 d_2 \rho^2 z^4 + \text{lower order terms.}
\]

$\square$

Remark. The last condition of the lemma can be more naturally written as

\[
det(R(z + 1)^{-1} AR(z) - \rho z^3) = d_1 d_2 \rho^2 z^4 + \text{lower order terms,}
\]

where $R(z) := \text{diag}(1, z^{-1})$ is a trivialization of $\mathcal{O} \oplus \mathcal{O}(-1)$ near $\infty \in \mathbb{P}^1$.

We can now think of $M_\theta$ as the quotient of the space of all matrices (6.3) that satisfy Lemma 6.5 modulo $d$-gauge transformations with gauge matrices (4.2) (cf. Corollary 4.2). For any matrix $A$ (6.3) that satisfies Lemma 6.5 denote by $q \in \mathbb{P}^1$ the only zero of $a_{21}$, and set $\tilde{p} \in (\mathcal{O}(3))_q$. It is easy to see that $q$ and $\tilde{p}$ do not change under $d$-gauge transformations with gauge matrices (4.2); therefore, $\tilde{P} := (q, \tilde{p})$ can be viewed as a map $M_\theta \to \tilde{K}$, where $\tilde{K} := \mathbb{V}(\mathcal{O}(3)^\vee)$ is the total space of the line bundle $\mathcal{O}(3)$. We can now use the map $\tilde{P}$ for a geometric description of $M_\theta$ (we are using the notation of Theorem 4.4):

**Theorem 6.6.**

1. The map $\tilde{P} : M_\theta \to \tilde{K}$ is a regular birational morphism of smooth algebraic surfaces.

2. Let $\tilde{\sigma}_1 : \tilde{K}_1 \to \tilde{K}$ be the blow-up of $\tilde{K}$ at the following 7 points: $(a_i, 0(a_i))$ $(i = 1, \ldots, 6)$ and $(\infty, (\rho z^3)(\infty))$. Let $\sigma_2 : \tilde{K}_2 \to \tilde{K}_1$ be the blow-up of $\tilde{K}_1$ at the two points $(\infty, (\rho z^3 + \rho \tilde{p} z^2)(\infty))$, $j = 1, 2$ (these points belong to the preimage $\tilde{\sigma}_1^{-1}(\infty, (\rho z^3)(\infty)) \subset \tilde{K}_1$). Then the map $\tilde{P}$ induces an open embedding $\bar{P}_2 : M_\theta \hookrightarrow \bar{K}_2$.

3. The complement to $\bar{P}_2(M_\theta)$ in $\bar{K}_2$ is the union of the proper preimages of the following curves: the zero section $\{(z, 0(z)) : z \in \mathbb{P}^1\} \subset \bar{K}$, the fiber at infinity $\{((\infty, az^3(\infty)) : a \in \mathbb{C}) \subset \bar{K}$, and the exceptional curve $\tilde{\sigma}_1^{-1}(\infty, (\rho z^3)(\infty)) \subset \bar{K}_1$.

$\square$

The proof of Theorem 6.6 is completely analogous to that of Theorem 4.4 (Section 4.3). Now Theorem 6.4 easily follows: we set

\[
p := \frac{\tilde{p}}{(q - a_1)(q - a_5)(q - a_6)},
\]

and it is not hard to check that the map $P := (q, p) : M_\theta \to (\mathbb{P}^1)^2$ (which is birational by Theorem 6.6) is regular and induces an embedding $M_\theta \hookrightarrow K_2$ with the required properties.

**Proof of Theorem 6.7.** The proof repeats the proof of Theorem 4.4 (given in Section 4.4) almost word-for-word (of course, the calculations involved are somewhat more complicated). The only real difference is formulas (4.13), (4.14); the corresponding
formulas in our case are
\[ A = \left[ z^3 - q^3 + p(q - a_4)(q - a_5)(q - a_6) \over z - q \right] - a_{12} \over a_{22}, \quad a_{22} \in \Gamma (\mathbb{P}^1, \mathcal{O}(3)); \]
\[ A' = \left[ z^3 - (q')^3 + p'(q' - a_3)(q' - a_4)(q' - a_6) \over z - q' \right] - a'_{12} \over a'_{22}, \quad a'_{22} \in \Gamma (\mathbb{P}^1, \mathcal{O}(3)). \]

6.3. Degeneration to difference PV. Given
\[ \tilde{\theta} = (\tilde{a}_1, \tilde{a}_2, \tilde{a}_3, \tilde{a}_4; \tilde{\rho}_1, \tilde{\rho}_2, \tilde{d}_1, \tilde{d}_2; 2) \in \Theta^d_4, \]
let us define \( \theta(t) \) for \( t \in \mathbb{C} - \{0\} \) by
\[ \theta(t) = (\tilde{a}_1, \tilde{a}_2, -\tilde{\rho}_1/t, -\tilde{\rho}_2/t, \tilde{a}_3, 1, 1, \tilde{d}_1 + (\tilde{\rho}_1/t), \tilde{d}_2 + (\tilde{\rho}_2/t); 3); \]
clearly, \( \theta(t) \in \Theta^d_6 \) for all but countably many \( t \). Denote the components of \( \theta = \theta(t) \) by \( a_i = a_i(t), d_j = d_j(t) \). Formulas (2.9) define a family of equations depending on parameter \( t \in \mathbb{C} - \{0\} \). Let us show that the difference PV (2.7) is the limit of this family as \( t \to 0 \).

Replace \( p \) with a new variable \( \tilde{\rho} := (\tilde{\rho}_2 + qt)p \); accordingly, set \( \tilde{p}' := (\tilde{\rho}_2 + qt)p' \). After we plug the formulas for \( \theta(t), \tilde{\rho}, \) and \( \tilde{p}' \) into (2.9), it becomes the following system:
\[
\begin{align*}
q + q' &= \tilde{a}_3 + \tilde{a}_4 + \tilde{p}_1(\tilde{d}_1 + \tilde{a}_4 + \tilde{a}_5) + \tilde{p}_2(\tilde{d}_2 + \tilde{a}_4 + \tilde{a}_5) + O(t) \\
\tilde{p}\tilde{p}' &= \frac{(q' - \tilde{a}_3 + 1)(q' - \tilde{a}_2 + 1)}{(q' - \tilde{a}_3)(q' - \tilde{a}_4)} \cdot \tilde{p}_1\tilde{p}_2 + O(t),
\end{align*}
\]
where \( O(t) \) stands for a Taylor series in \( t \) with no constant term. This is exactly the difference PV equation (2.7).

Remark 6.7. The degeneration of (2.9) to (2.7) has a clear geometric meaning; let us sketch it. It is easy to construct a family of moduli spaces \( \nu : N \to \mathbb{A}^1 \) such that the fiber \( \nu^{-1}(t) \) over \( t \in \mathbb{A}^1 - \{0\} \) equals \( M_{\theta(t)} \) whenever \( \theta(t) \in \Theta^d_6 \), while \( \nu^{-1}(0) = M_\tilde{\rho} \). Similarly, one can define a family \( \nu' : N' \to \mathbb{A}^1 \) such that \( (\nu')^{-1}(t) = M_{\theta'(t)} \) if \( t \neq 0, \theta(t) \in \Theta^d_6 \) and that \( (\nu')^{-1}(0) = M_\tilde{\rho}' \). Here
\[ \theta' = (\tilde{a}_1 + 1, \tilde{a}_2 + 1, -\tilde{\rho}_1/t, -\tilde{\rho}_2/t, \tilde{a}_3, 1, 1, \tilde{d}_1 + (\tilde{\rho}_1/t) - 1, \tilde{d}_2 + (\tilde{\rho}_2/t) - 1; 3). \]
The modification of d-connections defines a rational isomorphism \( N \to N' \) that is regular over a neighborhood of \( 0 \in \mathbb{A}^1 \); this isomorphism is given by (2.9) if \( t \neq 0 \) and \( \theta(t) \in \Theta^d_6 \) and by (2.7) if \( t = 0 \).

6.4. Degeneration to classical PVI. Let us now show how difference PVI (2.9) degenerates into the classical PVI. Fix
\[ \tilde{\theta} = (\tilde{a}_1, \tilde{a}_2, \tilde{a}_3, \tilde{a}_4; \tilde{\rho}_1, \tilde{\rho}_2, \tilde{d}_1, \tilde{d}_2; 2) \in \Theta^d_4, \]
and set
\[ \theta(t) := (-\tilde{\rho}_1/t, -\tilde{\rho}_2/t, \tilde{a}_1, \tilde{a}_2, \tilde{a}_3, 1, 1, \tilde{d}_1 + (\tilde{\rho}_1/t), \tilde{d}_2 + (\tilde{\rho}_2/t); 3) \quad (t \in \mathbb{C} - \{0\}); \]
again, \( \theta(t) \in \Theta^d_6 \) for all but countably many \( t \). Let us also set
\[ \theta'(t) := (-\tilde{\rho}_1/t - 1, -\tilde{\rho}_2/t - 1, \tilde{a}_1, \tilde{a}_2, \tilde{a}_3, 1, 1, \tilde{d}_1 + (\tilde{\rho}_1/t) + 1, \tilde{d}_2 + (\tilde{\rho}_2/t) + 1; 3), \]
so that $dPVI$ is an isomorphism $M_{\theta(t)} \rightarrow M_{\theta'(t)}$. Note that the formula for $\theta'(t)$ is obtained from the formula for $\theta(t)$ if we substitute

\[(6.5) \quad \hat{\rho}_i' := \hat{\rho}_i + t,\]

for $\hat{\rho}_i$, $i = 1, 2$.

Let us replace $p$ with $\hat{p} := (q - \hat{a}_2)tp$; accordingly, set $\hat{p}' := (q' - \hat{a}_2)tp'$. Then (2.9) can be written as

\[(6.6) \quad \begin{cases} 
\frac{q' - q}{t} = \frac{(q - \hat{a}_3)(q - \hat{a}_4)}{\hat{\rho}_1\hat{\rho}_2} \hat{p} - \frac{(q - \hat{a}_1)(q - \hat{a}_2)}{\hat{\rho}} + O(t) \\
\frac{\hat{p}' - \hat{p}}{t} = \hat{a}_1 + \hat{a}_2 - 2q + 2(\hat{\rho}_1 + \hat{\rho}_2)q + \hat{d}_1\hat{\rho}_1 + \hat{d}_2\hat{\rho}_2 \hat{p} + \hat{a}_3 + \hat{a}_4 - 2q\hat{p}^2 + O(t),
\end{cases}\]

where $(q,p)$ are the coordinates on $M_{\theta(t)}$ and $(q',p')$ are the coordinates on $M_{\theta'(t)}$. As $t \to 0$, the left hand sides tend to derivatives of $q$ and $p$ with respect to $t$.

Similarly, (6.5) becomes the expression

\[\frac{d\hat{\rho}_i}{dt} = 1, \quad (i = 1, 2);\]

all other parameters $\hat{a}_1, \ldots, \hat{a}_4, \hat{d}_1, \hat{d}_2$ do not depend on $t$. Now it is easy to see that (6.6) is obtained from (5.9) (which is equivalent to the sixth Painlevé equation) by changing variables from $\hat{\rho}_1, \hat{\rho}_2$ to $t$.

The degeneration of (2.9) to (6.6) has a geometric interpretation similar to that given for the degeneration to (2.7) (Remark 6.7). The details are left to the reader.

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