Redeeming bad theories

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ABSTRACT: We give a Seiberg-like dual description of the interacting superconformal infrared fixed point of $\mathcal{N} = 4$ gauge theory in three dimensions with vanishing Chern Simons level and $N_c \leq N_f < 2N_c$ fundamental flavors. These theories are known as “bad” theories due to the existence of unitarity violating monopole operators. We show that, in a dual description, all such operators are realized by free fields and the remainder theory is the Seiberg-like dual previously identified using the type IIB brane construction.

KEYWORDS: Supersymmetry and Duality, Supersymmetric gauge theory, Extended Supersymmetry

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1 Introduction

There has lately been a resurgence of interest in gauge theories with extended supersymmetry in three dimensions. Much of the focus has been on superconformal theories in which there is a Chern-Simons kinetic term for the gauge field, instead of the usual Yang-Mills term. The search for such theories was initiated in [1] and has yielded an array of superconformal actions with $\mathcal{N} = 3, 4, 5, 6, 8$ supersymmetry in the three dimensional sense. The most thoroughly studied example is ABJM theory [2], which, for special values of the Chern-Simons level ($k$), has maximal supersymmetry. Some of these theories, and ones without a Chern-Simons term, can be constructed as low energy effective theories on a stack of D-branes in a Hanany-Witten type setup in type IIB string theory ([3]) or on M2 branes in M theory. Theories with $k = 0$ are inevitably strongly coupled in the IR because $g^2_{YM}$ has mass dimension 1, but some are believed to flow to interacting superconformal fixed points [4].

Much has been learned about the low energy dynamics of 3d gauge theories with $\mathcal{N} \geq 2$ supersymmetry by the combined use of holomorphy and localization. The most exciting discovery has been a web of dualities which relate interacting superconformal IR fixed points of different UV theories. Examples include mirror symmetry of $\mathcal{N} = 4$ theories at $k = 0$ [4], and its extensions to $\mathcal{N} = 2$ and to some theories with $k = \pm 1$ [5–9]. This was shown to descend, in some cases, from s-duality of the type IIB construction [3]. A somewhat different class of examples are known collectively as Seiberg-like duality because the rank of the dual gauge group depends on the number of flavors in the original theory as in 4d Seiberg duality [10]. These include Giveon-Kutasov duality [11] ($\mathcal{N} = 2, 3$ and...
\( k \neq 0 \) and Aharony duality [12] (\( \mathcal{N} = 2 \) and \( k = 0 \)). An \( \mathcal{N} = 4 \) Seiberg-like duality can also be argued for by considering the effect of a set of brane moves as described in [3]. It is known, however, that the dualities implied by reading off the gauge theory associated to the initial and final brane configurations are wrong. The dynamics of the low energy degrees of freedom on the Coulomb branch is incorrectly accounted for. The correct behavior in the \( \mathcal{N} = 2 \) case is given by the Aharony dual which includes extra dual fields and a somewhat complicated superpotential (the theories involved in Giveon-Kutasov duality do not have Coulomb branches and can be read off from the branes).

We will present a proposal for the case of \( \mathcal{N} = 4 \) and \( k = 0 \). These theories have a large moduli space of supersymmetric vacua where the gauge group is partially Higgsed. We will give a dual description of the interacting fixed point at the origin of the Higgs branch. Specifically, we will propose that the \( U(N_c) \) theory with \( N_c \leq N_f < 2N_c \) is dual to the \( U(N_f - N_c) \) theory with \( N_f \) flavors and \( 2N_c - N_f \) additional free (twisted) hypermultiplets. The usual meson fields of Seiberg duality are absent. The larger amount of supersymmetry present in these examples makes the analysis easier, but also implies a larger Coulomb branch [13]. The interacting part of the UV actions is the one that can be read off from the type IIB branes [3]. The decoupled fields can be argued for by considering the light fields at various point on the moduli space as was done in [14] for \( N_f = 2N_c - 1 \). The analysis is very similar to the one presented in 4d in [15].

In an IR phase with unbroken gauge symmetry, 3d gauge theories admit pointlike defects known as monopole operators [16]. In a theory with \( \mathcal{N} \geq 2 \) supersymmetry and \( k = 0 \) these can be promoted to chiral operators whose quantum numbers, including the R-charge, can be systematically deduced from the UV theory [17, 18]. The superconformal algebra implies that the dimension of these operators, or any chiral operator, in the IR theory is equal to their charge under a particular R-symmetry whose current sits in the same multiplet as the energy momentum tensor. This distinguished R-symmetry may differ from the UV R-symmetry, for example by mixing with a flavor symmetry. The IR R-charge can still, in some cases, be recovered by using \( Z \) minimization [19]. It is also possible that the distinguished R-symmetry is not part of the UV symmetry algebra (it may be accidental). For example, the R-charge implied by the UV algebra may be in conflict with unitarity. In this situation it is not in general known how to recover the correct R-charge.

Quiver theories with \( \mathcal{N} = 4 \) supersymmetry and \( k = 0 \) can be classified according to their spectrum of monopole operators [14]. Quivers with a “standard” fixed point, where all monopole operators have dimension \( \geq 1 \), are called “good”, and those with monopole operators of dimensions \( 1/2 \), which implies that they are realized by free fields, (but none with vanishing or negative dimension) are called “ugly”. Theories with monopole operators of vanishing or negative dimensions, which implies unitarity violation, are called “bad”. The distinguished IR R-symmetry for a “bad” theory is not visible in the UV [14]. It can be shown that convergence of the \( S^3 \) partition function, computed using localization, is correlated with this classification such that the partition function for “bad” theories is divergent [20]. This can be attributed to the fact that the correct evaluation of the partition function using localization is predicated on knowing the correct R-charge. This data is used to write down the action of the theory on \( S^3 \) [21]. We will see that the partition function for a “bad” theory can still be regularized and compared to its dual.
Figure 1. Brane manipulations in type IIB string theory which yield a possible Seiberg-like dual [3]. Solid vertical lines are NS5 branes. Horizontal lines are coincident D3 branes. Dashed lines are D5 branes. The legend indicates the compactification direction (t or x6) and the directions of possible triplet mass (m) terms (3,4,5), and possible triplet FI (w) terms (7 8 9). Directions (0 1 2) are common to the world volume of all branes and are suppressed. We first move $N_f$ D5 branes through the right NS5 brane, creating $N_f$ D3 branes in the process. We then exchange the two NS5 branes, changing the number of suspended D3 branes in the interval.

In section 2, we describe a proposal for the Seiberg-like duality of $\mathcal{N} = 4$ theories. The duality relates “bad” theories to “good” ones. We compare the global symmetries and moduli space of the dual pairs. In section 3 we identify a regularized version of the squashed sphere partition function of a “bad” theory (this includes the round sphere as a special case). We show that the dual pairs have the same partition function for all values of the available deformation parameters. We end with a short discussion of possible further checks of the duality and related questions.

2 Proposal and preliminary checks

We will attempt to partially characterize the IR fixed point at the origin of the Higgs branch of $\mathcal{N} = 4$ $U(N_c)$ 3d gauge theories with $N_c \leq N_f < 2N_c$ fundamental flavors and no Chern-Simons term. In the classification of [14], such theories are called “bad”. An $\mathcal{N} = 4$ theory, in the three dimensional sense, has 8 real supercharges. A fundamental flavor includes two $\mathcal{N} = 2$ chiral superfields transforming in the fundamental and anti-fundamental representations respectively. The action for the $\mathcal{N} = 4$ theories, including the real mass and Fayet-Iliopoulos deformations, can be found in [22]. The construction in terms of D-branes in type IIB string theory was described in [3]. This work also described a set of brane moves which imply a Seiberg-like duality for these theories. Figure 1 illustrates the relevant configuration. The conclusion in [3] was that the convergence of two NS5 branes in spacetime, as require by the brane moves, may create too severe a singularity which would then invalidate the duality.

2.1 A Seiberg-like duality

The simplest example of a “bad” theory, which is, however, not part of our proposal, is $U(1)$ gauge theory with $\mathcal{N} = 4$ supersymmetry and no matter. Explicit analysis, which is possible because the theory is free, shows that the low energy action is that of a free twisted hypermultiplet constructed by dualizing the gauge field $A_\mu$ into an additional scalar [14]. The R-symmetry acting on this hypermultiplet is not visible in the UV action (hence the fixed point is not “standard”). The UV action admits monopole operators which have IR
conformal dimension 0 from the point of view of the UV algebra. One can try and identify the free hypermultiplet with these operators, but the duality obscures the symmetry action.

An “ugly” theory is one with $N_f = 2N_c - 1$ fundamental flavors. There are no unitarity violating monopole operators in these theories, but there are operators of R-charge 1/2. These parametrize a decoupled free sector of the low energy theory, consisting of a single free hypermultiplet. The remainder has a dual description in terms of a $U(N_c - 1)$ theory with $2N_c - 1$ massless hypermultiplets. The remainder theory is “good” and the duality is reminiscent of Seiberg duality in that the dual theory has a gauge group $U(N_f - N_c)$ [14].

A similar attempt to describe an interacting “bad” theory fails because the theory is neither free nor has any easily identifiable free sectors although it is expected, on general grounds, that any operator which dips below the unitarity bound is actually free [10]. A general feature of these theories is that the gauge group cannot be completely Higgsed by giving vevs to the matter multiplet scalars (compatible with the scalar potential). As a result, even vacua far away on the Higgs branch contain massless vector multiplets. There could still be a singularity, and hence an interacting fixed point, at the origin of this branch, but it would be reasonable to expect that there is also a decoupled free sector.

The resolution we propose is as follows

- Theories with $N_f = N_c$ are expected to have a smooth moduli space. The IR free theory on this space can be equivalently described using $N_f$ twisted hypermultiplets corresponding to monopole operators.

- Theories with $N_c < N_f < 2N_c$ have an IR fixed point which includes an interacting sector and a decoupled free sector. The decoupled sector can be described by $2N_c - N_f$ free hypermultiplets. The interacting sector has a Seiberg-like dual description as the IR fixed point at the origin of the Higgs branch of the $\mathcal{N} = 4$ $U(N_f - N_c)$ theory with $N_f$ fundamental hypermultiplets. The Seiberg-like dual is “good” and hence has no unitarity violating monopole operators.

We will provide some standard evidence for this proposal: matching of the global symmetries, the dimension of the moduli space (actually the Higgs branch metric) and of the regularized $S^3_b$ (i.e. the squashed sphere) partition function. Note that the case $N_f = 2N_c - 1$ is well established. The equality of the $S^3_b$ partition function, which is convergent in this case, was shown in [20]. Theories with $N_f < N_c$ are also “bad”, but the duality proposal does not apply to them (the dual gauge group would have negative rank). These theories are not expected to have interacting fixed points.

2.2 Global symmetries

In three dimensions, an $\mathcal{N} = 4$ $U(N_c)$ gauge theory, with $N_f$ massless fundamental hypermultiplets and vanishing Chern-Simons level has a global bosonic symmetry group

$$SU(N_f)_{\text{flavor}} \times SO(4)_R \times U(1)_J.$$ 

The first factor is a flavor symmetry which rotates the fundamental and anti-fundamental chiral superfields in opposite directions. The second factor is an R-symmetry under which
the supercharges transform as a four-vector. $U(1)_J$ is a topological symmetry whose current is $*\text{tr}(F)$, which is divergenceless due to the Bianchi identity. Fundamental fields are not charged under $U(1)_J$, but monopole operators may be.

When written in $\mathcal{N} = 2$ language, only a $U(1)_R \times U(1)_A$ subset of the R-symmetry group is visible. The $U(1)_R$ part is an R-symmetry of the $\mathcal{N} = 2$ algebra. In a CFT with $\mathcal{N} = 2$ supersymmetry, a chiral superfield satisfies

$$D = |R| \geq \frac{1}{2}$$

where $D$ is the conformal dimension of the lowest component of the superfield and $R$ is its charge under the R-symmetry that sits in the same superconformal multiplet as the stress-energy tensor. This may or may not be identified with the charge under the $U(1)_R$ for the UV action. A chiral operator with $D = 1/2$ is realized by a free field. A gauge invariant chiral operator with $D < 1/2$ violates the unitarity bound.

Our Seiberg-like duals trivially have the same global symmetry algebra except for the “accidental” currents coming from the decoupled sector. We do not expect to be able to see these currents in the original action.

2.3 Moduli space

The classical moduli space of the original theory is a mixed Coulomb-Higgs branch where vevs for the scalars in the matter hypermultiplets partially Higgs the gauge group. At a generic point on this space, the $N_c \times N_f$ dimensional matrix of scalar vevs for the fundamental chirals has rank $r = N_f/2 < N_c$ for $N_f$ even and $r = (N_f - 1)/2$ for $N_f$ odd (the matrix of anti-fundamentals has the same rank [14]). The rank of the unbroken gauge group is $N_c - r = N_c - N_f/2$ for $N_f$ even and $N_c - r = N_c - (N_f - 1)/2$ for $N_f$ odd. The space of scalar vevs (the Higgs branch) is a hyper-Kähler manifold of dimension (we count quaternionic dimensions throughout) $N_c(N_f - N_c)$ given by the hyper-Kähler reduction of the (flat) space of hypermultiplet scalars by the gauge group $G$. The low energy action for the hypermultiplets is a hyper-Kähler sigma model [23].

There is also a Coulomb branch emanating from this space. The dimension of this branch (for $N_f$ even) is $(N_c - N_f/2) \leq f \leq N_c$ where the lower bound is the generic dimension on the bulk of the space and the upper bound occurs at the origin, where all the hypermultiplet scalars vanish. Out on this branch, the gauge group is generically further (spontaneously) broken to $U(1)^f$. Far away on the Coulomb branch, the light vector multiplets can be dualized into additional hypermultiplets. The resulting low energy theory is again a hyper-Kähler sigma model.

In the quantum theory the space above may receive corrections. It is not possible for a superpotential to be generated which would lift a part of the space, but there can be corrections to the metric [13]. $\mathcal{N} = 4$ supersymmetry guarantees that the moduli space is a hyper-Kähler manifold after all quantum corrections are taken into account. Our duality conjecture implies that the moduli space for theories with $N_c < N_f < 2N_c$ has a $2N_c - N_f$ dimensional component described by the decoupled hypermultiplets and a remainder: an $N_f - N_c$ dimensional Coulomb branch and an $(N_f - N_c)(N_f - (N_f - N_c)) = N_c(N_f - N_c)$ dimensional Higgs branch which meet at a singularity.
The equivalence of the Higgs branches for the original and dual theories, including the metric, is known from the analysis of 4d $\mathcal{N} = 2$ theories [24]. In fact, the Higgs branch has been shown to coincide with the cotangent bundle of the complex grassmanian $G_{N_c,N_f}$ and the equivalence of the Higgs branch metrics is an extension of grassmanian duality

$$G_{N_c,N_f} = G_{N_f - N_c,N_f}.$$ 

The fact that the theory does not have vacua where the gauge group is completely Higgsed provided the motivation for the Seiberg-like duality of the “ugly” theory in [14]. When $N_c < N_f \leq 2N_c - 2$, the unbroken gauge symmetry at a generic point on the mixed Higgs/Coulomb branch is $U(1)^{N_c - N_f/2}$ (for $N_f$ even). Comparing with the “ugly” case, where the unbroken gauge group is $U(1)$, we might be led to believe that the light $U(1)$ gauge multiplets can be dualized into $N_c - N_f/2$ hypermultiplets. However, the points where all gauge multiplet moduli vanish have enhanced (non-abelian) gauge symmetry, and strong gauge dynamics, which may drastically alter the action for these hypermultiplets. From the study of $\mathcal{N} = 2$ theories, it is known what such dynamics may require the introduction of additional fields even when the unbroken gauge symmetry is abelian [9] (these are the $V_\pm$ fields of [12]). It seems plausible that a similar effect requires the use of $2N_c - N_f$ hypermultiplets for $\mathcal{N} = 4$. This has the advantage of making the dimension of the branch emanating from the origin of the Higgs branch the same for a theory and its Seiberg-like dual.

2.4 Flowing from “ugly” to “bad”

A “bad” theory can be reached by an RG flow from a “good” or “ugly” theory by giving large masses to some of the hypermultiplets. Integrating out the massive matter does not induce a Chern-Simons term for the gauge field as long as the masses for the two Dirac fermions in a hypermultiplet have opposite signs, which is the case for the real mass deformations of $\mathcal{N} = 4$ [9]. Starting from the Seiberg-like duality of the “ugly” theory and giving a large mass to a single hypermultiplet on both sides we end up with a $U(N_c)$ theory with $2N_c - 2$ flavors on one side and a $U(N_c - 1)$ theory with $2N_c - 2$ flavors and a single free hypermultiplet on the other. These two theories are not dual as we will see. Instead, there is a subtlety related to following the correct vacuum across the duality. It turns out that in the vacuum dual to the origin of the $U(N_c)$ theory the dual gauge group is partially Higgsed to $U(N_c - 2) \times U(1)$ (with $2N_c - 2$ massless flavors charged under the first factor and a single massless hypermultiplet charged under the second).\(^1\) This comes about by adding a suitable vev for one of the vector multiplet moduli. The $U(1)$ factor (and the charged hypermultiplet) is then dualized into a twisted hypermultiplet. This duality is actually mirror symmetry, and is described in detail in [25]. The resulting dual theories then coincide with our proposal. In the following section, we motivate this property of the dual vacuum from the partition function of the theory on the squashed three sphere.

\(^1\)We would like to thank Ofer Aharony for suggesting this possibility.
3 The $S^3_b$ partition function

An $\mathcal{N} = 2$ theory with a conserved U(1)$_R$ symmetry can be coupled to a supergravity multiplet using the R-multiplet [21, 26, 27]. It is possible to then put the theory on a squashed three sphere while preserving some of the fermionic symmetries [28, 29]. The partition function on this compact space ($Z_b$) is finite and unambiguous, after a suitable subtraction, and can be calculated using a localization procedure similar to the ones used in [30, 31]. Specializing to the $\mathcal{N} = 4$ case with gauge group U($N_c$) and $N_f$ fundamental flavors, $Z_b$ is a function of the real mass deformations $m_a$ and the Fayet-Iliopoulos (FI) parameter $\eta$. The parameters $m_a$ can be viewed as the constant (on the squashed sphere) value of the scalar $\sigma$ in a background vector multiplet introduced by weakly gauging the SU($N_f$) flavor symmetry (note that the $m_a$ sum to 0). The FI parameter can be similarly introduced by weakly gauging the U(1)$_J$ symmetry. The partition function, deformed by the $m_a$ and $\eta$, can be computed using a matrix model derived from the localization procedure. A necessary condition for the proposed duality to hold is that $Z_b$ should agree, as a function of the deformations, for the dual theories (see [20, 22]). In this section, we test this agreement for the Seiberg-like duality introduced above.

The partition function on the round three sphere for the class of theories under consideration formally diverges. This is directly correlated with the unitarity violating dimensions of monopole operators [22]. We will see that it can still be defined by analytic continuation in certain deformation parameters. These are the would be anomalous dimensions (or corrected IR R-charges) of the various chiral multiplets [19, 32]. It should be noted that the analytic continuation presented in this section is not associated with a physical correction to the R-charges; it is merely a mathematical trick used to regulate the partition function. This is sufficient in order to match the dual partition functions as meromorphic functions of the deformation parameters [32]. As a consequence of this, some of the R-charge assignments made here will not coincide with the physical dimension of the operators. Nevertheless, the partition function with the deformations corresponding to the visible UV symmetry currents should still match.

3.1 The matrix model

Localization can be used to reduce the path integral calculation of the partition function on the squashed sphere to a matrix model. The derivation for the round sphere can be found in [30] (see [33] for a nice review) and for the squashed sphere in [28, 29]. It can be shown that the matrix integral corresponding to a “bad” $\mathcal{N} = 4$ theory diverges [22]. The physical interpretation of this fact is that the coupling to the round sphere, achieved using the UV R-multiplet, does not correctly capture the R-charges (equivalently, conformal dimensions) of all operators at the IR fixed point. The existence of monopole operators of vanishing or negative R-charge (from the UV point of view) then causes the divergence. The resulting matrix model and integral are seemingly meaningless. However, it is known from the study of $\mathcal{N} = 2$ theories that coupling the theory using the wrong R-charge merely sets the imaginary part of some deformation parameters to unphysical values (from the point of view of the flat space conformal field theory) [19]. The mixing of the flavor symmetries with
the UV R-symmetry gives an imaginary contribution to the mass parameters \(\mu_a\) and \(\nu_a\) (introduced below) and it has been argued that mixing with the \(U(1)_J\) symmetry induces a similar contribution to \(\eta\). The correct value of the partition function can be recovered by using a “trial” R-charge and extremizing the resulting integral \([19]\). The use of trial R-charges is possible when the superconformal R-symmetry gets contributions from abelian flavor symmetries or the \(U(1)_J\) current. There is no known way of incorporating the change to the R-charges of monopoles operators or from accidental symmetries.

Luckily, finding the correct R-charge is not necessary if one only wants to compare the partition function of dual theories. The comparison depends only on the correct identification of the UV symmetries and on being able to evaluate \(Z_b\) in some, not necessarily physical (in the sense of flat space), region of the parameter space. One expects that the analytic properties of \(Z_b\) ensure that its value will match at the physical points. By physical points, we mean ones where all fields are coupled to the supergravity multiplet using their IR R-charges. The \(S^3\) partition function at the physical points has an interpretation in terms of the entanglement entropy across a circle in the flat space CFT \([34]\). We will assume that the “incorrect” R-charge assignment used to put a “bad” \(\mathcal{N} = 4\) theory on the squashed sphere can be undone in this manner. We have assumed, as part of the proposal, that the dimensions of some of the monopole operators in the original theory are exactly \(1/2\). We will not be able to impose this restriction in the expression for the partition function of the original theory, but doing so in the expression for the dual is trivial (the hypermultiplets describing the monopoles are decoupled from the rest of the matrix model).

We will work with a (matrix) integral which generalizes the calculation on the squashed sphere. This is the class of hyperbolic gamma function integrals introduced in \([35]\). As explained in \([32]\), the matrix model for an \(\mathcal{N} \geq 2\) \(U(N_c)\) gauge theory with \(N_c \geq N_f\) can be written in terms of the following integral (the notation comes from \([35]\))

\[
I_{n,(2,2)}^m(\mu;\nu;\lambda) = \frac{1}{\sqrt{-\omega_1\omega_2} n!} \int_{C_n} \prod_{1 \leq j < k \leq n} \Gamma_h(\pm(x_j - x_k)) \prod_{j=1}^n \left( e^{\frac{\pi i z}{\omega_2}} \prod_{a=1}^{n+m} \Gamma_h(\mu_a - x_j) \Gamma_h(\nu_a + x_j) dx_j \right).
\]

The function \(\Gamma_h(z|\omega_1,\omega_2)\) is called a hyperbolic gamma function and satisfies (the dependence on \(\omega_{1,2}\) will often be suppressed)

\[
\begin{align*}
\Gamma_h(z + \omega_1) &= 2 \sin \left( \frac{\pi z}{\omega_2} \right) \Gamma_h(z) \\
\Gamma_h(z + \omega_2) &= 2 \sin \left( \frac{\pi z}{\omega_1} \right) \Gamma_h(z) \\
\Gamma_h(z)\Gamma_h(\omega_1 + \omega_2 - z) &= 1
\end{align*}
\]

An explanation of the relationship to the squashed sphere partition function and some identities for the hyperbolic gamma function can be found in \([36]\). The parameters for an \(\mathcal{N} = 4\) theory are identified as \([32]\)

\[
n = N_c, \quad m = N_f - N_c, \quad \mu_a = \frac{\omega}{2} - m_a, \quad \nu_a = \frac{\omega}{2} + m_a \quad \lambda = -2\eta
\]

where

\[
\omega = \frac{\omega_1 + \omega_2}{2}.
\]
Note that the factors of $\omega/2$ represent the canonical dimension of the chiral multiplets (i.e. 1/2). The elements of the integral correspond to the path integral for the theory as follows:

- The integration is over the $N_c$ “Coulomb branch” moduli (on the sphere) identified with the Cartan elements of the constant component of $\sigma$, where $\sigma$ is a scalar in the dynamical vector multiplet. 
- $\prod_{1 \leq j < k \leq n} \frac{1}{\Gamma_h(x_j-x_k)}$ represents the determinant coming from fluctuations of the fields in the dynamical vector multiplet.
- $\prod_{a=1}^{n+m} \Gamma_h(\mu_a - x_j)\Gamma_h(\nu_a + x_j)$ is the matter determinant, including the dependence on real mass parameters, of which there are two sets, $\mu_a, \nu_a$, for an $\mathcal{N} = 2$ theory with no superpotential and one set, $m_a$ for the $\mathcal{N} = 4$ theory.
- $e^{-\pi^2 x_j}$ is the contribution of an FI term.

The parameters $\omega_{1,2}$ take the values $(i, i)$ for the round sphere, and $(ib, i/b)$ for the squashed sphere. The positive number $b$ is the squashing parameter [28]. The integral $I_{n,(2,2)}^m(\mu; \nu; \lambda)$ can be extended to a meromorphic function of the deformation parameters [35]. Moreover, the points where $\Im(\rho_a) = \Im(\nu_a) = \omega / 2$ and $\Re(\nu_a) = -\Re(\mu_a) = m_a$ are regular. We will define the squashed sphere partition function of a “bad” $\mathcal{N} = 4$ theory with $N_f \geq N_c$ using the value of $I_{n,(2,2)}^m(\mu; \nu; \lambda)$

$$Z_{\mathcal{N}=4}^{N_c, N_f}((m_a); \eta) := I_{N_c,(2,2)}^{N_f-N_c}(\frac{\omega}{2} - m_a; \frac{\omega}{2} + m_a; -2\eta).$$

For a single free hypermultiplet with R-charge 1/2 and a real mass parameter $m$

$$Z_{\text{hyper}}^{\mathcal{N}=4}(m) = \Gamma_h(\frac{\omega}{2} \pm m).$$

Note that we have implicitly used the freedom to change the imaginary parts of $\mu_a$ and $\nu_a$ to achieve convergence of the integral and defined the $\mathcal{N} = 4$ partition function by analytic continuation. The value of the partition function on the round sphere matches

$$\beta^{(1)}_+ = \arg((N_f - 2N_c + 2)\omega \pm 2\eta) \neq -\pi / 2$$

and

$$\beta^{(2)}_+ = \arg((2N_c + 2 - N_f)\omega \pm 2\eta) \neq -\pi / 2$$

where we have included the conditions for having a well defined integral for the partition function of the Seiberg-like dual introduced below. For an arbitrary positive squashing parameter $b$, the phase convention being used is such that the phase is zero on the positive real line and the branch cut is along the negative imaginary axis ($\phi_+ = \phi_- = \pi / 2$). The first set of conditions are satisfied identically for any choice of $m_a$ and $\eta$. The second set is equivalent to $\beta^{(2)}_-$ can be $-\pi / 2$ or undefined. Comparison with the case of the abelian theory with no flavors suggests that there may be a delta function at $\eta = 0$. This seems likely when $N_f = 2N_c - 2$ and the asymptotic behavior is such that the integrand goes to a constant at infinity. However, when $N_f$ is even, the partition functions we find will have an ordinary pole when continued to $\eta = 0$. 

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\footnote{In the language of [35], the parameters must satisfy \(\tau = \omega\): \[(\mu, \nu) \in B_{N_f, N_f}^\omega, \quad (\omega - \mu, \omega - \nu) \in B_{N_f, N_f}^\omega\] and \[(\mu, \nu, \pm 2\eta) \in D_{N_c, N_f}^\omega, \quad (\omega - \mu, \omega - \nu, \pm 2\eta) \in D_{N_f - N_c, (N_f, N_f)}^\omega\] where we have included the conditions for having a well defined integral for the partition function of the Seiberg-like dual introduced below. For an arbitrary positive squashing parameter $b$, the phase convention being used is such that the phase is zero on the positive real line and the branch cut is along the negative imaginary axis ($\phi_+ = \phi_- = \pi / 2$). The first set of conditions are satisfied identically for any choice of $m_a$ and $\eta$. The second set is equivalent to $\beta^{(1)}_+ = \arg((N_f - 2N_c + 2)\omega \pm 2\eta) \neq -\pi / 2$ and $\beta^{(2)}_+ = \arg((2N_c + 2 - N_f)\omega \pm 2\eta) \neq -\pi / 2$. Hence the only problem that could arise is when $\eta = 0$ and $\beta^{(1,2)}_+$ can be $-\pi / 2$ or undefined. Comparison with the case of the abelian theory with no flavors suggests that there may be a delta function at $\eta = 0$. This seems likely when $N_f = 2N_c - 2$ and the asymptotic behavior is such that the integrand goes to a constant at infinity. However, when $N_f$ is even, the partition functions we find will have an ordinary pole when continued to $\eta = 0$.}
the one given in [20]. A more general class of integrals can be used to represent the $N_f < N_c$ partition function. These are the $I_{n,(s_1,s_2),t}$ type integrals of [35]. However, for the physical values $s_1 = s_2 = N_f < n = N_c$ and $t = 0$ the partition function thus defined vanishes identically (away from $\eta = 0$).

### 3.2 Seiberg-like duality

The integrals introduced above satisfy the following identity as meromorphic functions of the deformation parameters [35]

$$I_{m,n,(2,2)}(\mu;\nu;\lambda) = I_{n,m,(2,2)}(\omega - \nu;\omega - \mu; -\lambda) \prod_{a,b=1}^{n+m} \Gamma_h(\mu_a + \nu_b) \times$$

$$\times \Gamma_h \left( (m+1)\omega - \frac{1}{2} \sum_{a=1}^{n+m} (\mu_a + \nu_a) \pm \frac{1}{2} \lambda \right) c \left( \lambda \sum_{a=1}^{n+m} (\mu_a - \nu_a) \right) \right) \right)$$

(3.2)

where

$$c(x) = \exp \left( \frac{i\pi x}{2\omega_1 \omega_2} \right).$$

This identity represents the equality of partition functions in a Seiberg-like duality [32] (note that $I_{m,(2,2)}$ is 1 for $m = 0$). For “ugly” theories, the duality takes the form (we write $Z_{N_f,N_c}$)

$$Z_{N_f=4,N_c-1}^N \{m_a\}; \eta \right) = Z_{N_c-1,2N_c-1}^{N_f=4} \{m_a\}; -\eta \right) Z_{\text{hyper}}^{N_f=4} (\eta) Z_{\text{background FI}} \left( \{m_a\}; \eta \right)$$

(3.3)

which descends from the identity by way of

$$I_{N_c-1,N_c,(2,2)}^N \left( \frac{\omega}{2} - m_a; \frac{\omega}{2} + m_a; -2\eta \right) = I_{N_c-1,2N_c-2}^N \left( \frac{\omega}{2} - m_a; \frac{\omega}{2} + m_a; 2\eta \right) \Gamma_h \left( \frac{\omega}{2} \pm \eta \right) c \left( 4\eta \sum_{a=1}^{N_f} m_a \right)$$

(3.4)

and we identify

$$Z_{\text{background FI}} \left( \{m_a\}; \eta \right) = c \left( 4\eta \sum_{a=1}^{N_f} m_a \right)$$

the decoupled sector represented by $Z_{\text{hyper}}^{N_f=4} (\eta)$ is associated with the free twisted hypermultiplet which is the dual of the dimension 1/2 monopole.

In trying to identify the duality implied by the identity for “bad” theories we run into an ambiguity in the interpretation of the factor

$$\Gamma_h \left( (m+1)\omega - \frac{1}{2} \sum_{a=1}^{n+m} (\mu_a + \nu_a) \pm \frac{1}{2} \lambda \right) = \Gamma_h \left( \left( \frac{N_f}{2} - N_c + 1 \right) \omega \pm \eta \right).$$

An interpretation of this factor as arising from a single (twisted) hypermultiplet (or two chiral multiplets as in [32]) is inconsistent. In order to bypass this difficulty, we begin with the duality for the “ugly” theory and proceed by “integrating out” matter multiplets on both sides, leading to a duality for “bad” theories. We will shortly clarify what integrating out means in the context of the matrix model.
3.3 Integrating out flavors

We would like to define a procedure for integrating out flavors in the $S^3_b$ partition function, starting from the “ugly” theory with $N_f = 2N_c - 1$. In principle, this can be done by taking some of the mass parameters to infinity. Naive application of this type of limit leads to an inconsistency of the type noted in (2.4). Instead, we will try to mimic the physically acceptable picture of partial Higgsing in the matrix model. This can be done by allowing some of the integration variables to transform along with the mass parameters \[37\]. The number of such variables is a priori arbitrary. We will use the asymptotic behavior of the matrix integral, as we take the mass parameter to infinity, as a guide for identifying the correct vacuum.\(^3\) A related “degeneration” procedure is described in \[35\] (see also \[36\]).

As shown in \[35\] (5.2.6), the hyperbolic gamma function has the following asymptotic behavior

\[
\Gamma_h(x|\omega_1,\omega_2) \approx \exp\left(\pm 2\pi i \left(\frac{(x - \omega)^2}{4\omega_1\omega_2} - \frac{\omega_1^2 + \omega_2^2}{48\omega_1\omega_2}\right)\right) = \left(\zeta^{-1} e\left(\frac{(x - \omega)^2}{4}\right)\right)^{\pm 1} \quad x \to \pm \infty
\]

\[
\zeta = \exp\left(\pi i \left(\frac{\omega_1^2 + \omega_2^2}{24\omega_1\omega_2}\right)\right).
\]

We take a single mass parameter $m_1 = \xi$ to be very large and allow $\alpha$ of the integration variables to have a shift which cancels this particular parameter. The resulting integral (already at the $\mathcal{N} = 4$ values) is

\[
I_{m,n,2}^{n,2}(\mu; \nu, \lambda) = \sum_{\alpha=1}^{n} \binom{n}{\alpha} \frac{1}{\sqrt{\omega_1\omega_2}} \int_{C_{\alpha}} \prod_{1 \leq j < k \leq \alpha} \frac{1}{\Gamma_h(\pm(x_j - x_k))} \prod_{\alpha+1 \leq j \leq n} \frac{1}{\Gamma_h(\pm(x_j - x_k))} \prod_{j=\alpha+1}^{\alpha+n} e^{-\frac{\pi i \alpha(x_j - \xi)}{\omega_1\omega_2}} \Gamma_h\left(\frac{\omega}{2} - x_j + \xi\right) \Gamma_h\left(\frac{\omega}{2} + x_j + \xi\right) \prod_{a=2}^{n+m} \Gamma_h\left(\frac{\omega}{2} - m_a - x_j + \xi\right) \Gamma_h\left(\frac{\omega}{2} + m_a + x_j - \xi\right) dx_j
\]

where the first factor arises from the choice of $\alpha$ out of $n$ variables. Using the asymptotic

\[\text{– 11 –}\]
form for the hyperbolic gamma function

\[
I_{m,n(2,2)}(\mu; \nu; \lambda) \approx \sum_{\alpha=1}^{n} \left(\begin{array}{c} n \\ \alpha \end{array}\right) \frac{1}{\sqrt{-\omega_1 \omega_2} \cdot \alpha!} \int_{\mathbb{C}^n} \prod_{1 \leq j < k \leq n} \frac{\Gamma_h(\pm(x_j-x_k))}{\Gamma_h(\pm(\pm x_j-x_k))} \\
\prod_{1 \leq j \leq \alpha, \alpha+1 \leq k \leq n} c \left( (x_j - \xi - x_k - \omega)^2 - (x_j - \xi - x_k + \omega)^2 \right)
\]

\[
\prod_{j=1}^{\alpha} \left( e^{-2\pi i \eta_j (x_j - \xi)} \prod_{a=2}^{n+m} c \left( \frac{\omega}{2} - ma - x_j + \xi - \omega \right)^2 - \left( \frac{\omega}{2} + ma + x_j - \xi - \omega \right)^2 \right)
\]

\[
\prod_{j=\alpha+1}^{n} \left( c \left( \frac{\omega}{2} + \xi + x_j - \omega \right)^2 - \left( \frac{\omega}{2} - \xi - x_j - \omega \right)^2 \right) dx_j \right)
\]

(3.6)

The various factors simplify (for \( \alpha > 0 \)) as

\[
\prod_{1 \leq j, \alpha+1 \leq k \leq n} c \left( (x_j - \xi - x_k - \omega)^2 - (x_j - \xi - x_k + \omega)^2 \right) = \prod_{1 \leq j, \alpha+1 \leq k \leq n} c \left( 4\omega (x_k - x_j + \xi) \right) = c \left( 4\omega \xi (n - \alpha) \right) \prod_{1 \leq j \leq \alpha} c \left( -4\omega x_j (n - \alpha) \right) \prod_{\alpha+1 \leq k \leq n} c \left( 4\omega x_k \alpha \right)
\]

\[
\prod_{j=\alpha+1}^{n} c \left( \frac{\omega}{2} + \xi + x_j - \omega \right)^2 - \left( \frac{\omega}{2} - \xi - x_j - \omega \right)^2 = \prod_{j=\alpha+1}^{n} c \left( -2\omega (x_j + \xi) \right) = c \left( -2\omega (n - \alpha) \xi \right) \times \prod_{j=\alpha+1}^{n} c \left( -2\omega x_j \right)
\]

(3.7)

\[
\prod_{j=1}^{\alpha} \left( e^{-2\pi i \eta_j (x_j - \xi)} \prod_{a=2}^{n+m} c \left( \frac{\omega}{2} - ma - x_j + \xi - \omega \right)^2 - \left( \frac{\omega}{2} + ma + x_j - \xi - \omega \right)^2 \right)
\]

(3.8)

\[
= \prod_{j=1}^{\alpha} \left( e^{\frac{-2\pi i \eta_j (x_j - \xi)}{\omega_1 \omega_2}} \prod_{a=2}^{n+m} c \left( 2\omega (ma + x_j - \xi) \right) \right)
\]

(3.9)

\[
= c \left( 4\alpha \eta \xi \right) c \left( -2\omega \xi (n + m - 1) \alpha + 2\omega \alpha \sum_{a=2}^{n+m} ma \right) \prod_{j=1}^{\alpha} c \left( x_j (4\eta + (n + m - 1) \omega) \right).
\]
The result is then the following sum over $\alpha$ sectors
\[
I_{m,\alpha}(\xi, \omega; m_a; -2\eta) \approx I_{m-1,\alpha}(\xi, \omega; m_a; -2\eta + \omega) \cdot (-2\omega \xi n) + \sum_{\alpha=1}^{n} \left\{ I_{m-\alpha}(\xi, \omega; m_a; -2\eta) \cdot \left( -2\omega \xi (n + 2\alpha - 2) \right) + \omega \xi \sum_{a=2}^{m+n} m_a \right\}.
\]

Note that each element in the sum includes the partition function of the original theory Higgsed to $U(\alpha) \times U(n-\alpha)$ as required on physical grounds.

3.4 Matching

We now attempt to match vacua between the “ugly” theory and its dual with one flavor integrated out on both sides. That is, we will solve for the Higgsing of the dual theory in terms of the original one.

We are seeking a solution where the original gauge group is unbroken, hence $\alpha = 0$. The scaling with $\xi$ of the $\alpha = 0$ summand in the “ugly” theory ($N_f = 2N_c - 1$) is
\[
c (-2\omega \xi N_c)
\]
and an $\tilde{\alpha}$ sector in the dual ($N_c = N_c - 1, N_f = 2N_c - 1$)
\[
c (-2\omega \xi (N_c - \tilde{\alpha} - 1 + 2\tilde{\alpha}^2)).
\]

The only integral solution for which the scaling matches is $\tilde{\alpha} = 1$. Hence the gauge group of the dual is spontaneously broken in the right vacuum to $U(N_c - 2) \times U(1)$. The second factor has a single massless charged hypermultiplet. We must also check to see that the $\eta$ dependent $\xi$ scaling of the two theories matches. The “ugly” theory, having $\alpha = 0$ does not scale. For the dual, the above derivation gives a factor of (recall that the parameter $\eta$ appearing in the integral for the dual is $-\eta$ of the original integral)
\[
c (-4\tilde{\alpha} \eta \xi).
\]

There is one additional contribution coming from the duality relationship (3.2)
\[
c \left( 4\eta \sum_{a=1}^{N_f} m_a \right) = c \left( 4\eta \xi \right) c \left( 4\eta \sum_{a=2}^{N_f} m_a \right)
\]
which exactly cancels the first factor for $\tilde{\alpha} = 1$.

At this point we appeal to abelian mirror symmetry of $\mathcal{N} = 4$ theories which relates a $U(1)$ theory with a single massless charged flavor to a free twisted hypermultiplet. The real mass parameter of the twisted hypermultiplet is mapped to the FI term of the gauge theory.

\[\text{-- 13 --}\]
In terms of the hyperbolic gamma function integrals, this is nothing but the $n = 1, m = 0$ version of (3.2) which states

$$I_{1,2}(\frac{\omega}{2} - m \omega; \frac{\omega}{2} + m ; -2\eta) = \Gamma_h \left( \frac{\omega}{2} \pm \eta \right) c(4\eta m)$$

(3.11)
equivalently

$$Z_{N^4=1}^N (m; \eta) = Z_{N^4=1}^N (\eta) Z_{\text{background FI}} (m; \eta)$$

(3.12)
and we need to consider the situation where $m = 0$.

Combining (3.2), (3.10) and (3.12), and taking $\alpha = 0, \bar{\alpha} = 1$, we conclude that

$$Z_{N^4=4}^{N^4=4} (m; \eta) = Z_{N^4=4}^{N^4=4} (m; -\eta) Z_{\text{hyper}} (-\eta) \times Z_{\text{background FI}} (m; \eta)$$

(3.13)
or, defining $\eta_r = \eta + \omega/2$

$$Z_{N^4=4}^{N^4=4} (m; \eta_r) = Z_{N^4=4}^{N^4=4} (m; -\eta_r) Z_{\text{hyper}} (-\eta_r + \omega/2) \times Z_{\text{background FI}} (m; \eta_r)$$

(3.14)
which has the desired form.

What happens if we continue to integrate out flavors? Assuming that the pattern holds after integrating out $1 < k < N_c - 1$ flavors the scaling of the $U(N_c)$ theory with $2N_c - k$ flavors is

$$c (2\omega (N_c - k) \bar{\alpha} - 1 - \bar{\alpha} (N_c - 2\bar{\alpha} - 2))$$

and that of the $U(N_c - k)$ theory with $2N_c - k$ flavors

$$c (2\omega (N_c - k) \bar{\alpha} - 1 - \bar{\alpha} (N_c - 2\bar{\alpha} - 2))$$

so that again $\bar{\alpha} = 1$ and the pattern continues until $k = N_c - 1$, at which point the dual gauge group is trivial and the process terminates at the $N_f = N_c$ theory. Combining these results we conclude that for $N_c \leq N_f < 2N_c$

$$Z_{N^4=4}^{N^4=4} (m; \eta) = Z_{N^4=4}^{N^4=4} (m; -\eta) Z_{\text{background FI}} (\{m\}; \eta) Z_{\text{hyper}} (\eta) \prod_{j=1}^{2N_c - N_f - 1} Z_{\text{background FI}} (\{m\}; \eta_j).$$

(3.15)
At each step in the process of integrating out flavors, the FI parameter $\eta$ will shift by $\omega/2$ (there are $2N_c - N_f - j$ such steps). Together with the number of flavors at each step, this sets the values of the hypermultiplet deformation parameters to

$$\eta_a = \eta - (2N_c - N_f - 1) \frac{\omega}{2}, \quad \eta_j = -\eta + (2N_c - N_f + 1 - 2j) \frac{\omega}{2}.$$ 

3.5 Interpretation

We now reconsider the ambiguous factor in the duality relation (3.2)

$$\Gamma_h \left( \left( \frac{N_f}{2} - N_c + 1 \right) \omega \pm \eta \right).$$

(3.16)
First, we would like to show that indeed
\[ \Gamma_h \left( \left( \frac{N_f}{2} - N_c + 1 \right) \omega \pm \eta \right) = \prod_{i=1}^{2N_c-N_f} \Gamma_h \left( \frac{\omega}{2} \pm \eta_j \right) \] (3.17)
where we have combined the product. Recall that
\[ Z_{N=4}^{N=4} (\eta) = \Gamma_h \left( \frac{\omega}{2} \pm \eta \right) . \]
It is easily checked that the product on the r.h.s. of (3.17) is mostly telescopic with respect to the identity
\[ \Gamma_h (z) \Gamma_h (2\omega - z) = 1 \] (3.18)
and that the remainder, after cancelations, is exactly the l.h.s. We interpret this to mean that the ambiguous factor represents the contribution of \(2N_c-N_f\) hypermultiplets. The relation (3.2) then supports our duality proposal for all the relevant values of \(N_c\) and \(N_f\).

The part of the deformation parameters \(\eta_j\) which depends on \(\omega\) represents a non-canonical R-charge for the free hypermultiplets. Moreover, the R-charge is different for the two chiral fields in each hypermultiplet. Assuming the duality holds, we could set these R-charges to 1/2 and recover the partition function for the original theory, but with the correct R-charge assignment for all the fields. For instance, we could compute the entanglement entropy of the “bad” theory. There is no dual deformation in the integral expression for the original theory which would allow us to do this.

4 Discussion and conclusions

We have given some evidence in support of a proposal for a Seiberg-like duality of 3d \(\mathcal{N} = 4\) gauge theories with \(N_c \leq N_f < 2N_c\). The proposed dual has the advantage that some of the accidental symmetries present in the IR, those associated with the free hypermultiplets, are manifest. One could, in principle, search for these currents in the original theory by considering monopole operators. Such symmetry enhancement by monopole operators has been shown to exist in several examples \([14, 18, 38]\). The analysis in “bad” theories is complicated by the fact that the R-charges for monopole operators are corrected by (we assume) the very accidental symmetries we are searching for. It may still be possible to describe a self consistent scenario for the monopole operator spectrum without the need to identify an explicit dual.

The existence of a dual with enough visible symmetry allows us to compute expectation values for physical observable of a “bad” theory. This includes the entanglement entropy (i.e. the \(S^3\) partition function \([34]\)) and the expectation values of various supersymmetric Wilson loops \([30]\) or defect operators \([39, 40]\). The dual operators for the latter two must still be identified as was done for Seiberg-like Chern-Simons duals in \([41]\).

We have argued that at a generic point on the mixed Higgs-Coulomb branch, strong dynamics require the introduction of additional hypermultiplets in order to describe the low energy theory, along the same lines as in \(\mathcal{N} = 2\) \([9]\). An alternative approach to
calculating the size of a possible decoupled free sector would be to examine the corrections to the metric on the Coulomb branch. The corrected metric may be quite complicated, but the splitting of the moduli space into a flat space parametrizing the free sector and the Coulomb branch of the dual theory should be visible even far out along this space where an analysis based on instantons (and one loop diagrams) along the lines of [13] is applicable. By doing this, one may be able to see the spontaneous breaking of the Weyl group of the original gauge group.

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