MORE BRIEKSORN SPHERES BOUNDING RATIONAL BALLS

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Abstract. We call an integral homology sphere non-trivially bounds a rational homology ball if it is obstructed from bounding an integral homology ball. After Fintushel and Stern’s well-known example $\Sigma(2, 3, 7)$, Akbulut and Larson recently provided the first infinite families of Brieskorn spheres non-trivially bounding rational homology balls: $\Sigma(2, 4n + 1, 12n + 5)$ and $\Sigma(3, 3n + 1, 12n + 5)$ for odd $n$. Using their technique, we present new such families: $\Sigma(2, 4n + 3, 12n + 7)$ and $\Sigma(3, 3n + 2, 12n + 7)$ for even $n$. Also manipulating their main argument, we simply recover some classical results of Akbulut and Kirby, Fickle, Casson and Harer, and Stern about Brieskorn spheres bounding integral homology balls.

1. Introduction

The integral homology cobordism group $\Theta^3_Z$ has been playing a central role in both low- and high-dimensional topology [Man18]. Working with rational coefficients, the rational homology cobordism group $\Theta^3_Q$ can be also taken into account. There is a canonical homomorphism $\psi: \Theta^3_Z \rightarrow \Theta^3_Q$ induced by inclusion. One way to measure the complexity between homology cobordism groups goes through exploring the kernel of the map $\psi$, denoted by $\text{Ker}(\psi)$. For the other perspectives of studying the map $\psi$, see [KL14], [AL18a], [ACJ18] and [GL18].

The natural source for integral homology spheres is Brieskorn homology spheres $\Sigma(p,q,r) = \{x^p + y^q + z^r = 0\} \cap S^5 \subset \mathbb{C}^3$ with pairwise coprime positive integers $p,q$ and $r$. Fintushel and Stern [FS84] gave the first example, $\Sigma(2,3,7)$, non-trivially bounding a rational homology ball. This result can be interpreted as the non-triviality of $\text{Ker}(\psi)$. As the Brieskorn sphere $\Sigma(2,3,7)$ has a non-vanishing Neumann-Siebenmann invariant $\bar{\mu}$, this also shows the existence of an infinite order subgroup $\mathbb{Z}$ in $\text{Ker}(\psi)$. The knowledge about the structure of $\text{Ker}(\psi)$ is limited to those outcomes.

The Brieskorn sphere $\Sigma(2,3,7)$ has remained the single example for more than thirty years. The difficulty of finding another such examples is due to the handle decomposition of four-manifolds. If such an integral homology sphere exists, then the corresponding rational homology ball necessarily contains three-handles. This challenge gives the motivation for searching more examples. Akbulut and Larson [AL18b] made a huge progress in this area initially showing that the Brieskorn sphere $\Sigma(2,3,19)$ also non-trivially bounds a rational homology ball. Additionally, they provided first infinite families of Brieskorn spheres non-trivially bounding rational homology balls: $\Sigma(2,4n+1,12n+5)$ and $\Sigma(3,3n+1,12n+5)$ for odd $n$. Using their technique, we present new such families.

**Theorem 1.1.** The Brieskorn spheres $\Sigma(2,4n+3,12n+7)$ and $\Sigma(3,3n+2,12n+7)$ bound rational homology balls. When $n$ is even, they both have $\bar{\mu} = 1$ and thus they non-trivially bound rational homology balls.

\[\text{For } n = 1, \text{ these spheres bound even contractible four-manifolds, see [FS81] and [Fic84].}\]
It is not possible to detect linear independence of these spheres in $\Theta^3_\mathbb{Z}$ by calculating their current integral homology cobordism invariants.\footnote{If it were possible with an affirmative way, this would imply the existence of an infinitely generated subgroup $\mathbb{Z}^\infty$ in $\text{Ker}(\psi)$.}

Continuing with the manipulation of Akbulut and Larson’s main argument, we simply recover some classical results proven around nineteen eighties following the work of Kirby \cite{Kir78}. The initial two were showed in \cite{AK79} and \cite{Fic84} respectively. The following two families came from \cite{CH81} and their first elements were also appeared in \cite{AK79}. The rest were due to \cite{Ste78}.

**Theorem 1.2** (Akbulut and Kirby, Fickle, Casson and Harer, and Stern). The following Brieskorn spheres bound integral homology balls: $\Sigma(2, 3, 13)$, $\Sigma(2, 3, 25)$, $\Sigma(2, 4n + 1, 4n + 3)$, $\Sigma(3, 3n + 1, 3n + 2)$, $\Sigma(2, 4n + 1, 20n + 7)$, $\Sigma(3, 3n + 1, 21n + 8)$, $\Sigma(2, 4n + 3, 20n + 13)$ and $\Sigma(3, 3n + 2, 21n + 13)$.

**Organization.** The structure of the paper is as follows. In Section 2, we present the proof of new Brieskorn spheres bounding rational homology balls. To obstruct half of them from bounding integral balls, we compute their Neumann-Siebenmann invariants. In Section 3, we give the proof of manipulation of Akbulut and Larson’s argument. We finally recover some classical results about Brieskorn spheres bounding integral balls by using our new argument in a simple way.

## 2. Rational Homology Balls

A knot in the three-sphere is called *rationally slice* if it bounds a smoothly properly embedded disk in a rational homology ball. The basic example of rational slice knots is the figure-eight knot, see \cite[Theorem 4.16]{Cha07} and \cite[Section 3]{AL18b}.

The main argument of Akbulut and Larson relates rationally slice knots with rational homology balls via surgeries.

**Lemma 2.1** \cite[Lemma 2]{AL18b}. Let $Y$ be the three-manifold obtained by zero-surgery on a rationally slice knot in the three-sphere. Then any integral homology sphere obtained by an integral surgery on $Y$ bounds a rational homology ball.

Now we are ready to prove our main theorem.

**Proof of Theorem 1.1**. Following the recipe in \cite[Section 1.1]{Sav02}, it can be easily seen that the Brieskorn spheres $\Sigma(2, 4n + 3, 12n + 7)$ and $\Sigma(3, 3n + 2, 12n + 7)$ are respectively the boundary of the negative-definite unimodular plumbing graphs shown in Figure 1. To complete the proof via Lemma 2.1 we use the dual approach by giving integral surgeries from their plumbing graphs.

In particular, we show that $\Sigma(2, 4n + 3, 12n + 7)$ and $\Sigma(3, 3n + 2, 12n + 7)$ are both obtained by $(-1)$-surgery on $Y$ when the knot is figure-eight knot. Their surgery diagrams corresponding to their plumbing graphs appear in Figure 4 and the dark black $(-1)$-framed components give the necessary surgery to $Y$.

We first consider $\Sigma(2, 4n + 3, 12n + 7)$ and split its proof into two parts. For the base case $n = 1$, the sequence of blow downs are displayed by dark black $(-1)$-framed components and this explicit procedure can be seen in Figure 2. For the general case $n > 1$, we reduce the proof to the base case by applying $n$-times blow downs and to fulfil the rest of the proof we address to the base case, see Figure 4.

\footnote{It is also unrealizable for the infinite families of Akbulut and Larson.}

\footnote{Originally, these spheres were all shown to be bound contractible four-manifolds.}
Since their plumbing graphs are quite similar, we follow the identical flow for the proof of Brieskorn spheres $\Sigma(3, 3n+2, 12n+7)$. For details, one can see Figure 3 and Figure 4 respectively. Since the blow down operation does not change the boundary three-manifold, we end up the first part of the proof.
The computation of Neumann-Siebenmann invariant $\bar{\mu}$ ([Neu80] and [Sie80]) is well-known for the negative-definite unimodular plumbing graphs due to the combinatorial approach in [NR78]. Note that plumbings of $\Sigma(2, 4n + 3, 12n + 7)$ and $\Sigma(3, 3n + 2, 12n + 7)$ have both signature $-n - 5$, and when $n$ is even the square of their spherical Wu classes are both $-n - 13$ (otherwise they are $-n - 5$). Thus for even $n$ these Brieskorn spheres have $\bar{\mu} = 1$. Therefore, they do not bound integral homology balls, see [Sav02, Corollary 7.34].

\[ \begin{align*}
\text{Figure 4. The sequence of blow downs for } \Sigma(2, 4n + 3, 12n + 7) \text{ and } \\
\Sigma(3, 3n + 2, 12n + 7) \text{ for } n > 1
\end{align*} \]

Remark 2.2. We shall call the two-stage blow down procedure in the proof of Theorem 1.1 reduction trick and we shortly denote it by RT.
3. Integral Homology Balls

A knot in the three-sphere is called *smoothly slice* if it bounds a smoothly properly embedded disk in the four-ball. The typical examples of smoothly slice knots are the unknot and the stevedore knot.

To associate smoothly slice knot surgeries with integral homology balls, we manipulate the main argument of Akbulut and Larson. As expected, its proof is so similar with theirs and it has been known to experts in low-dimensional topology.

**Lemma 3.1.** Let $Y'$ be the three-manifold obtained by zero-surgery on a smoothly slice knot in the three-sphere. Then any integral homology sphere obtained by an integral surgery on $Y'$ bounds an integral homology ball.

**Proof.** We consider the four-manifold $X = B^4 \setminus \nu D$, where $\nu D$ denotes the tubular neighborhood of the slice disk $D$. Since $\nu D$ is diffeomorphic to $D \times B^2$, we clearly have $\partial X = Y'$. Further, $X$ has the integral homology of $S^1 \times B^3$ by a simple Mayer-Vietoris argument for the triplet $(B^4, \nu D, X)$.

An integral surgery on $Y'$ corresponds to attaching a two-handle $B^2 \times B^2$ to $X$. Thus if the resulting three-manifold is an integral homology sphere, then the four-manifold $W$ obtained by attaching the corresponding two-handle to $X$ must be an integral homology ball. This claim can be easily seen from combining the Mayer-Vietoris sequence for the triplet $(W, X, B^2 \times B^2)$, the long exact sequence for the pair $(W, \partial W)$, and the Poincaré-Lefschetz duality together. □

We finally recover the results of Akbulut and Kirby, Fickle, Casson and Harer, and Stern in the following simple way.

**Proof of Theorem 1.2.** Completing the proof by Lemma 3.1, we prove that all these spheres are all obtained by $(-1)$-surgery on $Y'$ when the knot is unknot or the stevedore knot. We shall begin with single examples $\Sigma(2, 3, 13)$ and $\Sigma(2, 3, 25)$. By [Example 1.4, Sav02], we know that $\Sigma(2, 3, 6n + 1)$ are obtained by $(+1)$-surgery on twist knot $K_n$ for $n \geq 0$, see the first picture of Figure 5. Remark that $\Sigma(2, 3, 1) = S^3$, $K_0$ is the unknot and $K_2$ is the stevedore knot. Blowing down the dark black $(-1)$-framed components in the second, third and last pictures of Figure 5 results in $(+1)$-surgery on $K_2$, $K_2$ and $K_4$, respectively.

![Figure 5](image)

**Figure 5.** The twist knot $K_n$, $\Sigma(2, 3, 13)$, $\Sigma(2, 3, 13)$, and $\Sigma(2, 3, 25)$ respectively

Recap that $\Sigma(2, 4n + 1, 4n + 3)$, $\Sigma(3, 3n + 1, 3n + 2)$, $\Sigma(2, 4n + 1, 20n + 7)$, $\Sigma(3, 3n + 1, 21n + 8)$, $\Sigma(2, 4n + 3, 20n + 13)$ and $\Sigma(3, 3n + 2, 21n + 13)$ are respectively the boundary of the negative-definite unimodular plumbing graphs shown in Figure 6.
To apply Lemma 3.1, we use the dual approach by giving integral surgeries from their plumbing graphs.

Figure 6. The plumbing graphs

Figure 7. The Akbulut-Larson trick

We initially deal with $\Sigma(2, 4n + 1, 4n + 3)$, $\Sigma(3, 3n + 1, 3n + 2)$ and $\Sigma(2, 4n + 1, 20n + 7)$, $\Sigma(3, 3n + 1, 21n + 8)$. The surgery diagrams corresponding to the first element of their plumbing graphs are shown in Figure 8 and Figure 9 respectively. Moreover, the dark black $(-1)$-framed components again give the necessary surgery to $Y'$. Following the sequence of blow downs in Figure 8 and Figure 9, we reach
zero surgery on the unknot for $\Sigma(2, 5, 7)$ and $\Sigma(3, 4, 5)$, and zero surgery on the stevedore knot for $\Sigma(2, 5, 27)$ and $\Sigma(3, 5, 29)$. Then the general families are obtained by applying the Akbulut-Larson trick\(^4\) successively, see Figure 7. Hence this part of the proof is done.

Since the plumbing graphs of $\Sigma(2, 4n + 3, 20n + 13)$ and $\Sigma(3, 3n + 2, 21n + 13)$ are quite similar with ones in Theorem 1.1, we use the reduction trick. We describe their proof in Figure 9 and Figure 10 respectively. In the base cases $n = 1$, the essential ingredient of proofs comes from the reduction trick and the remaining blow downs are clearly shown in Figure 9. Again we reduce proofs to the base cases from the general cases $n > 1$ and we complete them by using the base, see Figure 10. As blow down does not change the boundary three-manifold, this finishes the proof. \(\square\)

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{diagram.png}
\caption{The $(-1)$-surgeries from $\Sigma(2, 5, 7)$ and $\Sigma(3, 4, 5)$ to $Y'$}
\end{figure}

Remark 3.2. It will be interesting to find Brieskorn spheres bounding homology balls by using Lemma 2.1 and Lemma 3.1 but by starting with slice knots other than twist knots. Note that the rationally slice knots in the three-sphere are in abundance due to Kawauchi \cite{Kaw79} and \cite{Kaw09}, and Kim and Wu \cite{KW18}. Indeed, any hyperbolic amphichiral knot is rationally slice. Further, any fibered amphichiral knot with irreducible Alexander polynomial is rationally slice.

\(^4\)This is the critical observation of Akbulut and Larson for the proof of their main theorem, which explains the iterative procedure for passing from the surgery diagram to consecutive one.
Figure 9. The \((-1)\)-surgeries from \(\Sigma(2, 5, 27), \Sigma(3, 5, 29), \Sigma(2, 7, 44)\) and \(\Sigma(3, 5, 34)\) to \(Y'\)
Figure 10. The sequence of blow downs for $\Sigma(2, 4n + 3, 20n + 13)$ and $\Sigma(3, 3n + 2, 21n + 13)$
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