Upper bounds on the $Q$-spectral radius of book-free and/or $K_{s,t}$-free graphs

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Abstract
In this paper, we prove two results about the signless Laplacian spectral radius $q(G)$ of a graph $G$ of order $n$ with maximum degree $\Delta$. Let $B_n = K_2 + K_n$ denote a book, i.e., the graph $B_n$ consists of $n$ triangles sharing an edge.

(1) Let $1 < k \leq l < \Delta < n$ and $G$ be a connected $\{B_{k+1}, K_{2,l+1}\}$-free graph of order $n$ with maximum degree $\Delta$. Then

$$q(G) \leq \frac{1}{4}[3\Delta + k - 2l + 1 + \sqrt{(3\Delta + k - 2l + 1)^2 + 16l(\Delta - n - 1)}].$$

with equality holds if and only if $G$ is a strongly regular graph with parameters $(\Delta, k, l)$.

(2) Let $s \geq t \geq 3$, and let $G$ be a connected $K_{s,t}$-free graph of order $n$ ($n \geq s+t$). Then

$$q(G) \leq n + (s - t + 1)^{1/t}n^{1-1/t} + (t - 1)(n - 1)^{1-3/t} + t - 3.$$ 

Key Words: complete bipartite subgraph, Zarankiewicz problem, signless Laplacian spectral radius.

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1 Introduction

Our graph notation follows Bollobás [1]. In particular, let $G = (V(G), E(G))$ be a simple graph. Denote by $v(G)$ the order of $G$ and $e(G)$ the size of $G$, that is to say, $v(G) = |V(G)|$, and $e(G) = |E(G)|$. Set $\Gamma_G(u) = \{v | uv \in E(G)\}$, and $d_G(u) = |\Gamma_G(u)|$, or simply $\Gamma(u)$ and $d(u)$, respectively. Let $\delta = \delta(G)$ and $\Delta = \Delta(G)$ denote the minimal degree and maximal degree of graph $G$, respectively.

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For a simple graph $G$ of order $n$, let the matrix $D(G) = \text{diag}(d_1, d_2, \ldots, d_n)$, and $A(G) = (a_{ij})_{n \times n}$ be the adjacency matrix of $G$ with $a_{ij} = 1$ if $v_i$ is adjacent to $v_j$, and $a_{ij} = 0$ otherwise. The matrix $Q(G) = D(G) + A(G)$ is called the signless Laplacian matrix of $G$. The largest eigenvalue of $A(G)$ and $Q(G)$ are called spectral radius and signless Laplacian spectral radius (or simply Q-spectral radius) of $G$, respectively, and marked $\rho(G)$ and $q(G)$, respectively.

Let $X$ be a set of vertices of $G$, $G[X]$ is the graph induced by $X$, and $e(X) = e(G[X])$. Let $P_k$, $C_k$ and $K_k$ be the path, cycle, and complete graph of order $k$, respectively. If all vertices of $G$ have the same degree $k$, then $G$ is $k$-regular. A $k$-regular graph is called strongly regular with parameters $(k, a, c)$ whenever each pair of adjacent vertices have $a \geq 0$ common neighbors, and each pair of non-adjacent vertices have $c \geq 1$ common neighbors.

The main results of this paper are in the spirit of the trend in the famous Zarankiewicz problem [5]:

**Problem A** How many edges can have a graph of order $n$ if it does not contain a complete bipartite subgraph $K_{s,t}$?

In 1996, Füredi [4] gave an upper bound on the above Zarankiewicz problem. In 2010, Nikiforov [6] improved his result. That is, if $G$ is a $K_{s,t}$-free graph of order $n$, then

$$e(G) \leq \frac{1}{2}(s - t + 1)^{1/t}n^{2-1/t} + \frac{1}{2}(t - 1)n^{2-2/t} + \frac{1}{2}(t - 2)n.$$ 

The spectral version of the Zarankiewicz problem is the following one:

**Problem B** How large can be the spectral radius $\rho(G)$ of a graph $G$ of order $n$ that does not contain $K_{s,t}$?

There are some results for some value of $s$ and $t$.

In 2007, the upper bound on the signless Laplacian spectral radius of $K_{2,l+1}$-free graph as the corollary of the following Lemma [1.1] was proved in [9] by Shi and Song.

**Lemma 1.1.** $0 \leq k \leq l \leq \Delta < n$ and $G$ be a connected $\{B_{k+1},K_{2,l+1}\}$-free graph of order $n$ with maximum degree $\Delta$. Then

$$\rho(G) \leq \lfloor k - l + \sqrt{(k-l)^2 + 4\Delta + 4l(n-l)} \rfloor / 2,$$

with equality if and only if $G$ is a strongly regular with parameters $(\Delta, k, l)$.

In 2007, Nikiforov [7] improved the above bound showing that

**Lemma 1.2.** Let $l \geq k \geq 0$. If $G$ is a $\{B_{k+1},K_{2,l+1}\}$-free graph of order $n$ with maximum degree $\Delta$. Then

$$\rho(G) \leq \min\{\Delta, \frac{1}{2}[k - 1 + 1 + \sqrt{(k - l + 1)^2 + 4l(n - 1)}]\}.$$

If $G$ is connected, equality holds if and only if one of the following conditions holds:

1. $\Delta^2 - \Delta(k-l+1) \leq l(n-1)$ and $G$ is $\Delta$-regular;
2. $\Delta^2 - \Delta(k-l+1) > l(n-1)$ and every two vertices of $G$ have $k$ common neighbors if they are adjacent, and $l$ common neighbors otherwise.

Setting $l = \Delta$ or $k = l$, Lemma [1.2] implies assertions that strengthen Corollaries 1 and 2 of [9].

In 2010, Nikiforov [6] also gave a bound as the following lemma.
Lemma 1.3. Let $s \geq t \geq 2$, and let $G$ be a $K_{s,t}$-free graph of order $n$. If $t = 2$, then
$$\rho(G) \leq \frac{1}{2} + \sqrt{(s - 1)(n - 1) + 1/4}.$$  
If $t \geq 3$, then
$$\rho(G) \leq (s - t + 1)^{1/t}n^{1-1/t} + (t - 1)n^{1-2/t} + t - 2.$$  
and
$$e(G) < \frac{1}{2}(s - t + 1)^{1/t}n^{2-1/t} + \frac{1}{2}(t - 1)n^{2-2/t} + \frac{1}{2}(t - 2)n.$$  

A newer trend in extremal graph theory is the Zarankiewicz problem for signless Laplacian spectral radius of graphs:

**Problem C** How large can be the signless Laplacian spectral radius $q(G)$ of a graph $G$ of order $n$ that does not contain subgraph $K_{s,t}$?

When $s = t = 2$, we notice that the $K_{2,2}$-free graph is the same as $C_4$-free graph. Also in 2013, de Freitus [2] has proved that if $G$ contains no $C_4$, then
$$q(G) < q(F_n),$$

unless $G = F_n$, where $F_n$ is the friendship graph of order $n$. For $n$ odd, $F_n$ is a union of $\lfloor n/2 \rfloor$ triangles sharing a single common vertex, and for $n$ even, $F_n$ is obtained by hanging an edge to the common vertex of $F_{n-1}$.

In this paper, we discuss upper bounds on the signless Laplacian spectral radius of Book-free and/or $K_{2,l+1}$-free ($l > 1$) graphs of order $n$ with maximum degree $\Delta$.

**Theorem 1.4.** Let $1 < k \leq l < \Delta < n$ and $G$ be a connected $\{B_{k+1}, K_{2,l+1}\}$-free graph of order $n$ with maximum degree $\Delta$. Then
$$q(G) \leq \frac{1}{4}[3\Delta + k - 2l + 1 + \sqrt{(3\Delta + k - 2l + 1)^2 + 16l(\Delta + n - 1)}].$$ (1)

with equality holds if and only if $G$ is a strongly regular graph with parameters $(\Delta, k, l)$.  

Because every graph is obviously $K_{2,\Delta+1}$-free, Theorem 1.4 readily implies a sharp upper bound for book-free graph.

**Corollary 1.5.** Let $1 < k < \Delta < n$ and $G$ be a connected $B_{k+1}$-free graph of order $n$ with maximum degree $\Delta$. Then
$$q(G) \leq \frac{1}{4}[\Delta + k + 1 + \sqrt{(\Delta + k + 1)^2 + 32\Delta(n - 1)}].$$

with equality if and only if $G$ is a strongly regular graph with parameters $(\Delta, k, \Delta)$.  

Because a $K_{2,l}$-free graph is also $B_l$-free. Theorem 1.4 with $k = l$ also implies a sharp upper bound for $K_{2,l}$-free graphs.

**Corollary 1.6.** Let $1 < l < \Delta$ and $G$ be a connected $K_{2,l+1}$-free graph of order $n$ with maximum degree $\Delta$. Then
$$q(G) \leq \frac{1}{4}[3\Delta - l + 1 + \sqrt{(3\Delta - l + 1)^2 + 32l(n - 1)}].$$

with equality if and only if $G$ is a strongly regular graph with parameters $(\Delta, l, l)$.  


Furthermore we will discuss $s \geq t \geq 3$, let $G$ be a connected graph of order $n$, when $n < s + t$, then $G$ must contain no $K_{s,t}$, so we only discuss $n \geq s + t$.

**Theorem 1.7.** Let $s \geq t \geq 3$, and let $G$ be a connected $K_{s,t}$-free graph of order $n$ ($n \geq s + t$). Then

$$q(G) \leq n + (s - t + 1)^{1/t}n^{1-1/t} + (t - 1)(n - 1)^{1-3/t} + t - 3.$$ 

2 Main Lemmas

In this section, we state some well-know results which will be used in this paper.

**Lemma 2.1.** Let $s \geq 2$, $t \geq 2$, $0 \leq k \leq s - 2$, and let $G(A, B)$ be a bipartite graph with parts $A$ and $B$. Suppose that $G$ contains no copy of $K_{s,t}$ with a vertex class of size $s$ in $A$ and a vertex class of size $t$ in $B$. Then $G(A, B)$ has at most 

$$(s - k - 1)^{1/t}|B||A|^{1-1/t} + (t - 1)|A|^{1+k/t} + k|B|$$

edges.

**Lemma 2.2.** ([3], [8]) For every graph $G$, we have

$$q(G) \leq \max_{u \in V(G)} \{d(u) + \frac{1}{d(u)} \sum_{v \in \Gamma(u)} d(v)\}.$$ 

3 Proofs

**Proof of Theorem 1.4.** Let $Q_i$ denote the $i$th row vector of $Q := Q(G)$ and let $x = (x_1, x_2, \ldots, x_n)^T$ be the Perron-eigenvector of $Q$ corresponding to $q(G)$. Then $x_i > 0$ for $1 \leq i \leq n$. Since $G$ is $\{B_{k+1}, K_{2,t+1}\}$-free, each pair of adjacent vertices has at most $k$ common neighbors and each pair of non-adjacent vertices has at most $l$ common neighbors. Thus

$$\sum_{i=1}^{n} \sum_{v_p,v_q \in \Gamma(v_i)} x_p x_q \leq k \sum_{v_p,v_q \in E(G)} x_p x_q + l \sum_{v_p,v_q \notin E(G)} x_p x_q. \quad (2)$$

Then by virtue of $x^T A(K_n)x \leq \rho(K_n) = n - 1$. Thus

$$q(G) = x^T Q x = x^T D x + x^T A x = \sum_{i=1}^{n} d_i x_i^2 + 2 \sum_{v_p,v_q \in E(G)} x_i x_p$$

$$\leq \Delta + x^T A(K_n)x - 2 \sum_{v_p,v_q \notin E(G)} x_i x_p$$

$$\leq \Delta + n - 1 - 2 \sum_{v_p,v_q \notin E(G)} x_i x_p.$$
Also we can obtain
\[ q(G) = x^T Q x = \sum_{i=1}^{n} \sum_{j=1, i<j}^{n} 2q_{i,j}x_ix_j + \sum_{i=1}^{n} d_i x_i^2 \]
\[ \leq \sum_{i=1}^{n} \sum_{j=1, i<j}^{n} q_{i,j} (x_i^2 + x_j^2) + \sum_{i=1}^{n} d_i x_i^2 \]
\[ = \sum_{i=1}^{n} \sum_{j=1, i<j}^{n} q_{i,j} x_i^2 + \sum_{i=1}^{n} d_i x_i^2 \]
\[ = 2 \sum_{i=1}^{n} d_i x_i^2. \]

So
\[ \sum_{i=1}^{n} d_i x_i^2 \geq \frac{q}{2}. \]

Then
\[ q^2(G) = \|Qx\|^2 = \sum_{i=1}^{n} (Q_i x)^2 = \sum_{i=1}^{n} (d_i x_i + \sum_{v_i v_p \in E(G)} x_p)^2 \]
\[ = \sum_{i=1}^{n} [d_i^2 x_i^2 + 2d_i x_i \sum_{v_i v_p \in E(G)} x_p + (\sum_{v_i v_p \in E(G)} x_p)^2] \]
\[ = \sum_{i=1}^{n} d_i x_i^2 + 2 \sum_{i=1}^{n} d_i \sum_{v_i v_p \in E(G)} x_i x_p + \sum_{i=1}^{n} d_i x_i^2 + 2 \sum_{i=1}^{n} \sum_{v_i v_p \in \Gamma(v_i)} x_i x_q \]
\[ \leq (\Delta + 1) \sum_{i=1}^{n} d_i x_i^2 + 2\Delta \sum_{i=1}^{n} \sum_{v_i v_p \in E(G)} x_i x_p \]
\[ + 2k \sum_{v_i v_p \in E(G)} x_p x_q + 2l \sum_{v_i v_p \notin E(G)} x_p x_q \]
\[ = (\Delta + 1) \sum_{i=1}^{n} d_i x_i^2 + (4\Delta + 2k) \sum_{v_i v_p \in E(G)} x_i x_p + 2l \sum_{v_i v_p \notin E(G)} x_p x_q \]
\[ \leq (2\Delta + k) (\sum_{i=1}^{n} d_i x_i^2 + 2 \sum_{v_i v_p \in E(G)} x_i x_p) \]
\[ - (\Delta + k - 1) \sum_{i=1}^{n} d_i x_i^2 + 2l \sum_{v_i v_p \notin E(G)} x_p x_q \]
\[ \leq (2\Delta + k) q - \frac{\Delta + k - 1}{2} q + l(\Delta + n - 1 - q) \]
\[ = \frac{1}{2} (3\Delta + k - 2l + 1) q + l(\Delta + n - 1). \]

Solving the inequality gives the upper bound
\[ q(G) \leq \frac{1}{4} [3\Delta + k - 2l + 1 + \sqrt{(3\Delta + k - 2l + 1)^2 + 16l(\Delta + n - 1)}]. \]
If the upper bound of (1) is attained then all inequalities in the above argument must be equalities. In particular, from (2) and \(x_i > 0\) for \(1 \leq i \leq n\), we have that each pair of adjacent vertices in \(G\) has exactly \(k\) common neighbors and each pair of non-adjacent vertices in \(G\) has exactly \(l\) common neighbors. Moreover, by (3), \(G\) must be \(\Delta\)-regular. Thus \(G\) must be a strongly regular graph with parameters \((\Delta, k, l)\). □

**Proof of Theorem 1.7.** By Lemma 2.2 let \(w\) be a vertex of \(G\) such that

\[
d(w) + \frac{1}{d(w)} \sum_{i \in \Gamma(w)} d(i) = \max_{u \in V(G)} \{d(u) + \frac{1}{d(u)} \sum_{v \in \Gamma(u)} d(v)\}.
\]

Then

\[
q(G) \leq d(w) + \frac{1}{d(w)} \sum_{i \in \Gamma(w)} d(i).
\]

Note first that if \(d(w) \leq s + t - 1\), then

\[
q(G) \leq d(w) + \frac{1}{d(w)} \sum_{i \in \Gamma(w)} d(i) \leq d(w) + \Delta(G)
\]

\[
\leq s + t - 1 + n - 1 = s + t + n - 2
\]

\[
\leq n + (s - t + 1)^{1/x} n^{1-1/t} + (t - 1)(n - 1)^{1-3/t} + t - 3.
\]

Therefore we shall assume that \(s + t - 1 \leq d(w) \leq n - 1\). Let \(U\) and \(W\) be disjoint sets satisfying \(|U| = d(w)\) and \(|W| = n - 1\), and let \(\varphi_U\) and \(\varphi_W\) be bijections

\[
\varphi_U : U \to \Gamma(w), \varphi_W : W \to V(G) \setminus \{w\}.
\]

Define a bipartite graph \(H\) with vertex classes \(U\) and \(W\) by joining \(u \in U\) and \(v \in W\) whenever \(\{\varphi_U(u), \varphi_W(v)\} \in E(G)\).

Then we can get that \(H\) does not contain a copy of \(K_{s-1,t}\) with \(s - 1\) vertices in \(W\) and \(t\) vertices in \(U\). Indeed, the map \(\psi : V(H) \to V(G)\) defined as

\[
\psi(x) = \begin{cases} 
\varphi_U(x), & \text{if } x \in U, \\
\varphi_W(x), & \text{if } x \in W.
\end{cases}
\]

is a homomorphism of \(H\) into \(G - w\). Assume for a contradiction that \(F \subset H\) is a copy of \(K_{s-1,t}\) with a set of \(S\) of \(s - 1\) vertices in \(W\) and a set of \(T\) of \(t\) vertices in \(U\). Clearly \(S\) and \(T\) are the vertex classes of \(F\). Note that \(\psi(F)\) is a copy of \(K_{s-1,t}\) in \(G - w\), and \(\psi(S) = \varphi_W(S) \subset V(G) \setminus \{w\}\) and \(\psi(T) = \varphi_U(T) \subset \Gamma_G(w)\) are the vertex classes of \(\psi(F)\) of size \(s - 1\) and size \(t\), respectively. Now, adding \(w\) to \(\psi(F)\), we see that \(G\) contains a \(K_{s,t}\), a contradiction proving the claim.

Suppose that \(0 \leq k \leq \min\{s, t\} - 2\). Setting \(k' = k - 1, s' = s - 1, t' = t, A = W, B = U\), then from Lemma 2.1 we have

\[
e(H) \leq (s - k - 1)^{1/t} |U||W|^{1-1/t} + (k - 1)|U| + |T| \leq (k - 1)|W|^{1+(k-1)/t}
\]

\[
= (s - k - 1)^{1/t} d(w)n^{1-1/t} + (k - 1)d(w) + (t - 1)(n - 1)^{1+(k-1)/t}.
\]

On the other hand, we have

\[
e(H) = \sum_{v \in \Gamma(w)} d(v) - d(w),
\]
and so,
\[
\sum_{v \in \Gamma(w)} d(v) \leq ((s - k - 1)^{1/t} n^{-1/t} + k) d(w) + (t - 1)(n - 1)^{1+(k-1)/t}.
\]

And then from Lemma 2.2 we have
\[
q(G) \leq d(w) + \frac{1}{d(w)} \sum_{i \in \Gamma(w)} d(i)
\leq d(w) + \frac{(t - 1)(n - 1)^{1+(k-1)/t}}{d(w)} + (s - k - 1)^{1/t} n^{-1/t} + k.
\]

Since the function
\[
f(x) = x + \frac{(t - 1)(n - 1)^{1+(k-1)/t}}{x}
\]

is convex for \(x > 0\), its maximum in any closed interval is attained at one of the ends of this interval. In the case \(s + t - 1 \leq d(w) \leq n - 1\), then,
\[
q(G) \leq d(w) + \frac{1}{d(w)} \sum_{i \in \Gamma(w)} d(i)
\leq \max\{s + t - 1 + \frac{(t - 1)(n - 1)^{1+(k-1)/t}}{s + t - 1}, n - 1 + \frac{(t - 1)(n - 1)^{1+(k-1)/t}}{n - 1}\}
+ (s - k - 1)^{1/t} n^{-1/t} + k
\leq (s - k - 1)^{1/t} n^{-1/t} + k + \frac{(t - 1)(n - 1)^{1+(k-1)/t}}{n - 1} + n - 1
= (s - k - 1)^{1/t} n^{-1/t} + k + (t - 1)(n - 1)^{(k-1)/t} + n - 1.
\]

Now, if \(s + t \geq 3\), setting \(k = t - 2\), we obtain
\[
q(G) \leq n + (s - t + 1)^{1/t} n^{-1/t} + (t - 1)(n - 1)^{1-3/t} + t - 3.
\]

So, the proof is complete. \(\square\)

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