On spectral properties of one class difference operators

A G Baskakov¹, G V Garkavenko², M Yu Glazkova³ and N B Uskova⁴

¹Voronezh State University, Applied Mathematics, Informatics and Mechanics Faculty, Voronezh, Russia
²Voronezh State Pedagogical University, Physics and Mathematics Faculty, Voronezh, Russia
³Voronezh State Technical University, Mechanical Engineering and Aerospace Engineering Faculty, Voronezh, Russia
⁴Voronezh State Technical University, Information Technologies and Computer Security Faculty, Voronezh, Russia

Email: anatbaskakov@yandex.ru, g.garkavenko@mail.ru, nat-uskova@mail.ru

Abstract. In this paper, we analyse basic facts of infinite matrix theory. We construct a similarity transform which allows one to represent matrices in a certain class of 5-th diagonal matrices of a difference operator in a diagonal or block-diagonal form. For such matrices, asymptotic estimates of eigenvalues and eigenvectors are obtained. Such matrices are considered in game theory. They are also used in the study fourth-order difference operators with growing potential. The problem of invariant subspaces is also considered.

1. Introduction
Let Z be the group of all integers, N – the set of all positive integers, Zᵢ = N ∪ {0}, and C – the field of complex numbers.

Definition 1. A matrix A is a map from the Cartesian product \( \mathcal{I} \times \mathcal{I} = \mathcal{I}^2 \) to C. Elements of the matrix A will be denoted by \( A(n,m) \), \( n,m \in \mathcal{I} \). The set of all matrices \( A : \mathcal{I}^2 \to C \) naturally forms a linear space, which we will denote by \( \text{Matr}(\mathcal{I}^2, C) \).

Definition 2. The sequence \( c_k : \mathcal{I} \to C \) given by \( c_k(n) = A(n,k) \), \( n,k \in \mathcal{I} \), is called the k-th column of the matrix A. The sequence \( b_k : \mathcal{I} \to C \), \( b_k(n) = A(k,n) \), \( n,k \in \mathcal{I} \), is called the k-th row of the matrix A. The sequence \( a_k : Z \to C \), where \( a_k(n) = A(n,m) \), \( n-m = k \), \( n,m \in \mathcal{I} \), \( k \in Z \) is called the k-th diagonal of the matrix A.

If \( k = 0 \) the k-th diagonal is called the main diagonal.

If \( a_k(n) = a_k \) for all \( n \in \mathcal{I}, k \in Z \) then A is called Toeplitz matrix.

Infinite matrices play an important role in many areas of analysis and beyond, for example, in game theory. We note, that in [1], a matrix was also defined as mapping from the Cartesian product of sets of indices to \( \mathbb{R} \). Different definition of a matrix were used in [2, 3].

We consider the class of infinite matrices that are given by
and $L_t$. are Fourier coefficients of the function $f_L$. are $j \neq L_t - t_0$, where $j \neq i$ acting by formula $j \equiv L_t$ and $j \pi$ was considered. The operator $j \in L_t$ has the five-diagonal matrix with $j \neq L_t$. This matrix $\tilde{A}$ is the five-diagonal matrix.

Infinite finite-diagonal matrices arise in various mathematical models. We begin with two examples.

**Example 1.** The matrix $\tilde{A}$ obtained by discretizing the fourth-order differential operator with the growing potential. Thus, matrix $\tilde{A}$ can be considered as a matrix of fourth-order difference operator.

**Example 2.** Let $L_2 = L_2[0, 1]$ be the Hilbert space of Lebesgue measurable square integrable functions (equivalence classes) on $[0, 1]$ with values in $C$. The inner product and the norm in $L_2$ are given by

$$(x, y) = \int_0^1 x(t) \overline{y(t)} dt, \quad \|x\|_2^2 = \int_0^1 |x(t)|^2 dt, \quad x, y \in L_2.$$ 

By $W^2_2 = W^2_2[0, 1]$ we denote the Sobolev space $\{x \in L_2, \ x' \text{ is absolutely continuous}, x'' \in L_2\}$. We consider the differential operator $E : D(E) \subset L_2 \to L_2$ acting by formula

$$(Ey)(t) = -y''(t) - q(t)y(t),$$

where $D(E) = \{y \in W^2_2[0, 1], \ y'(1) = y'(0)e^{i\Theta}, \ y(1) = y(0)e^{i\Theta}\}, \ \Theta \in (0, 2\pi), \ \Theta \neq \pi$.

Such boundary conditions are called unseparated quasiperiodic boundary conditions. A review of the results concerning the operator $E$ can be found in [4, 5]. In [4] case $q \in L_2$ was considered. The similar operator method with preliminary similar transform was used, estimates of the eigenvalues are obtained. Let $q(t) = \sum_{l \in Z} \hat{q}(l)e^{i2\pi l t}$, where $\hat{q}(n), n \in Z$ are Fourier coefficients of the function $q \in L_2$. In the standard basis $\{e^{i2\pi l t}, n \in Z\}$ in $L_2$ the operator $E$ has the five-diagonal matrix with $c_i = \hat{q}(l), l = \pm 1, \pm 2$ and $e_0 = -\hat{q}(0) + (\pi(2n + \Theta))^2$. We will also use the similar operator method, without the preliminary similar transform.

And, finally, five-diagonal matrices are used in game theory.

The properties of differential and difference operator are connected (see [6]). The results of the present paper may be used in the linear differential equation theory, including the differential equation with unbounded operator coefficients [6].

In [7] there was given a complete bibliographic survey of results for the various classes of infinite tridiagonal matrices.
Usually, one of the main methods of finding the eigenvalues of infinite finite diagonal matrices is to approximate of the eigenvalues of truncated (finite) matrices (see [8-12]). We use the similar operator method for solving this problem. We also note the works [13-15], in which eigenvalues of the Toeplitz matrices of large dimensions are calculated.

In this paper, we transform the matrix \( \tilde{A} \) to a diagonal (or a block-diagonal) matrix. We use the method of similar operator [4, 7, 16-18]. This method is a very effective tool for spectral analysis of various classes of differential and difference operators. We also consider the problem of invariant subspaces.

The diagonalization of some classes of linear operators was studied in [19, 20].

2. Tools and methods

Let \( \ell_p(\mathbb{S}) \), \( 1 \leq p < \infty \), be the complex Banach space of \( p \)-summable sequences \( x: \mathbb{S} \to \mathbb{C} \) with the norm \( \|x\|_p = \left( \sum_{n=3}^{\infty} |x(n)|^p \right)^{1/p} \). For \( p = \infty \), the norm is given by \( \|x\|_\infty = \sup_{n \in \mathbb{S}} |x(n)| \), \( x: \mathbb{S} \to \mathbb{C} \). If \( p = 2 \) then \( \ell_2(\mathbb{S}) \) is a complex Hilbert. The inner product in \( \ell_2(\mathbb{S}) \) is defined by \( (x, y) = \sum_{n=3}^{\infty} x(n)y(n) \).

The vectors \( e_n, n \in \mathbb{S} \), where \( e_n(k) = \delta_{nk} \), \( k \in \mathbb{S} \), \( \delta_{nk} \) - standard Kronecker delta, form an unconditional basis in all spaces \( \ell_p(\mathbb{S}) \), \( 1 \leq p < \infty \). In space \( \ell_2(\mathbb{S}) \) this basis is orthonormal basis.

**Definition 3.** A sequence \( x: \ell_p(\mathbb{S}) \to \mathbb{C} \) is called finite if the set of indexes \( n \in \mathbb{S} \) such that the number \( x(n) \neq 0 \) is finite.

Let \( \ell_{p,0} \subset \ell_p \), \( 1 \leq p \leq \infty \), be the subset of all finite sequences. If \( p \neq \infty \) then \( \ell_{p,0} = \ell_p \). For an abstract Banach space \( \mathbb{X} \) we denote by \( \text{End} \mathbb{X} \) the Banach algebra of all bounded linear operator in \( \mathbb{X} \). The norm in \( \text{End} \mathbb{X} \) is defined by \( \|X\| = \sup_{x \in \mathbb{X}} \|Xx\|, \ X \in \text{End} \mathbb{X} \), \( x \in \mathbb{X} \).

For each matrix \( A \in \text{Matr}(\mathbb{S}^2, \mathbb{C}) \) we construct a linear operator \( A: \ell_{p,0} \to \ell_p \), \( 1 \leq p \leq \infty \), defined by \( (Ax)(k) = \sum_{j=3}^{\infty} A(k, j)x(j), \ x \in \ell_{p,0}, k \in \mathbb{S} \).

**Definition 4.** The matrix \( A \in \text{Matr}(\mathbb{S}^2, \mathbb{C}) \) is called \( p \)-correct for same \( 1 \leq p \leq \infty \) if the operator \( A \) admits an extension to some closed operator in the space \( \ell_p(\mathbb{S}) \).

We denote this operator by the same symbol \( A: D(A) \subset \ell_p(\mathbb{S}) \to \ell_p(\mathbb{S}) \).

If a matrix is \( p \)-correct for all \( 1 \leq p \leq \infty \) then it is called a correct matrix.

**Lemma 1.** Each diagonal matrix is a correct matrix.

Let \( A \in \text{Matr}(\mathbb{S}^2, \mathbb{C}) \) be a diagonal matrix. Then an operator \( A: D(A) \subset \ell_p \to \ell_p \) is defined by \( (Ax)(k) = A(k, k)x(k), \ k \in \mathbb{S} \), and \( D(A) = \{x \in \ell_p(\mathbb{S}); \sum_{k=3}^{\infty} |A(k, k)x(k)|^p < \infty \} \), \( 1 \leq p < \infty \), or \( D(A) = \{x \in \ell_\infty(\mathbb{S}); \sup_{k=3} \sum_{k=3}^{\infty} |A(k, k)x(k)| < \infty \} \). We note, that \( D(A) \) is dense in \( \ell_p(\mathbb{S}) \), \( 1 \leq p < \infty \).
Let $H$ be the complex separable Hilbert space and $e_n, n \in \mathbb{Z}$ is the fixed orthonormal basis in $H$. Then an operator that is diagonal with respect to the choosen basis $e_n, n \in \mathbb{Z}$ has the form $\tilde{A} = \sum_{i \in \mathbb{Z}} a_i(e_i \otimes e_i)$. The symbol $u \otimes v$ denote $(u \otimes v)x = (x, u)v$ [21, 22, 23].

Let $A, B \in \text{Matr}(\mathfrak{H}^2, C)$. We define the product $AB = C$ to be the matrix of the operator $C = AB$.

We consider the sequence $d_A : \mathbb{Z} \to \mathbb{R}_+$ given by $d_A(k) = \sup_{i-j=k} |\mathbf{A}(i, j)|$, $i, j \in \mathfrak{H}, k \in \mathbb{Z}$.

**Definition 5.** A matrix $A \in \text{Matr}(\mathfrak{H}^2, C)$ is said to have summable diagonals if $d(A) = \sum_{n \in \mathbb{Z}} d_A(n) < \infty$.

Let $A \in \text{End}X$ and assume that its matrix $A \in \text{Matr}(\mathfrak{H}^2, C)$ has summable diagonals. Then $A \in \text{End}_1X \subset \text{End}X$, where $\text{End}_1X$ is the space of operators having a matrix with summable diagonals. The norm $\|A\|$ in the space $\text{End}_1X$ is given by $\|A\| = d(A)$.

Throughout the rest of the paper, we denote an abstract complex Hilbert space by $H$.

**Definition 6.** [6] Two linear operators $A_i : D(A_i) \subset H \to H$, $i = 1, 2$, are called *similar*, if there exists a continuously invertible operator $U \in \text{End}H$ such that $A_2Ux = UA_2x$, $x \in D(A_2)$, $UD(A_2) = D(A_i)$.

The operator $U$ is called the *similarity transform* of $A_i$ into $A_2$.

**Definition 7.** A subspace $M$ from the Banach space $X$ is called invariant for a linear operator $A : D(A) \subset X \to X$ if $Ax \in M$ for all $x \in M \cap D(A)$.

Directly from Definition 6, we have the following result about the spectral properties of similar operators.

**Lemma 2.** [4] Let $A_i : D(A_i) \subset H \to H$, $i = 1, 2$, be two similar operators with the operator $U$ being the similarity transform of $A_i$ into $A_2$. Then the following properties hold.

1) We have $\sigma(A_i) = \sigma(A_2)$, $\sigma_p(A_i) = \sigma_p(A_2)$ and $\sigma_c(A_i) = \sigma_c(A_2)$, where $\sigma_p$ denotes the point spectrum and $\sigma_c$ denotes the continuous spectrum;

2) If $\lambda_0$ is an eigenvalue of the operator $A_2$ and $x$ is a corresponding eigenvector, then $y = Ux$ is an eigenvector of the operator $A_i$ corresponding to the same eigenvalue $\lambda_0$.

3) Assume that the operator $A_2$ admits a decomposition $A_2 = A_{21} \oplus A_{22}$ with respect to a direct sum $H = H_1 \oplus H_2$, where $A_{21} = A_2|H_1$ and $A_{22} = A_2|H_2$ are the restrictions of $A_2$ to the respective subspaces. Then the operator $A_i$ admits a decomposition $A_i = A_{i1} + A_{i2}$ with respect to a direct sum $H = \tilde{H}_1 \oplus \tilde{H}_2$, where $A_{i1} = A_i|\tilde{H}_1$ and $A_{i2} = A_i|\tilde{H}_2$ are the restrictions of $A_i$ to the respective invariant subspaces. Moreover, if $P$ is the projection onto to $H_1$ parallel to $H_2$, then $\tilde{P} = UP^{-1}$ is the projection onto $\tilde{H}_1$ parallel to $\tilde{H}_2$.

The key notion of the method of similar operators is that of an admissible triplet. Once such a triplet is constructed, achieving the goal of the method becomes a routine task.
**Definition 8.** [4] Let \( \mathcal{R} \) be a linear subspace that is continuously embedded in \( \text{End} \, \mathcal{H} \), \( \mathcal{R} \) has a norm \( \|X\| \). We also consider linear operators \( J: \mathcal{R} \rightarrow \mathcal{R} \) and \( G: \mathcal{R} \rightarrow \mathcal{R} \). The collection \( (\mathcal{R}, J, G) \) is called an admissible triplet for the operator \( A \) and the space \( \mathcal{R} \) is the space of admissible perturbations, if the following five properties hold.

1. \( J \) and \( G \) are bounded linear operators; and \( J \) is an idempotent.
2. \( (GX)D(A) \subset D(A) \) and

\[
(AGX - GXA)x = (X - JX)x, \quad x \in D(A), \quad X \in \mathcal{R},
\]

moreover \( GX \in \mathcal{R} \) is the unique solution of the equation

\[
AY - YA = X - JX,
\]

that satisfies \( JY = 0 \).
3. \( XGY \), \( (GX)Y \in \mathcal{R} \) for all \( X, Y \in \mathcal{R} \), and there is a constant \( \gamma > 0 \) such that

\[
\|G\| \leq \gamma, \quad \max \|XGY\|, \|GX\| \leq \gamma \|X\| \|Y\|.
\]
4. \( J((GX)Y)) = 0 \) for all \( X, Y \in \mathcal{R} \).
5. For every \( X \in \mathcal{R} \) and \( \epsilon > 0 \) there exists a number \( \lambda_\epsilon \in \rho(A) \), such that \( \|X(A - \lambda_\epsilon I)^{-1}\| \leq \epsilon \).

To formulate the main theorem of the method of similar operators \([4, 7, 16-18]\) for an operator \( A - B \), we use the function \( \Phi : \mathcal{R} \rightarrow \mathcal{R} \) given by

\[
\Phi(X) = BGX - (GX)(JB) - (GX)J(BGX) + B.
\]

**Theorem 1.** [4] Assume that \( (\mathcal{R}, J, G) \) is an admissible triplet for an operator \( A : D(A) \subset \mathcal{H} \rightarrow \mathcal{H} \) and \( B \in \mathcal{R} \). Assume also that

\[
4\gamma\|J\|\|B\| < 1,
\]

where \( \gamma \) comes from the Property 3 of Definition 8. Then the operator \( A - B \) is similar to the operator \( A - JX_* \), where \( X_* \in \mathcal{R} \) is the unique fixed point of the function \( \Phi \) given by equation (2), and the similarity transform of \( A - B \) into \( A - JX_* \) is given by \( I + GX_* \in \mathcal{R} \). Moreover, the map \( \Phi : \mathcal{R} \rightarrow \mathcal{R} \) is a contraction in the ball \( \{X \in \mathcal{R} : \|X - B\| \leq 3\|B\|\} \), and the fixed point \( X_* \) can be found as a limit of simple iterations: \( X_0 = 0, \ X_1 = \Phi(X_0) = B \), etc.

**Theorem 2.** Assume that \( (\mathcal{R}, J, G) \) is an admissible triplet for the operator \( A : D(A) \subset \mathcal{H} \rightarrow \mathcal{H} \), \( B \in \mathcal{R} \), \( JB = 0 \) and the condition

\[
3\gamma\|J\|\|B\| < 1
\]

hold. Then the operator \( A - B \) is similar to the operator \( A - JX_* \), where \( X_* \in \mathcal{R} \) is the unique fixed point of the function \( \Phi : \mathcal{R} \rightarrow \mathcal{R} \) given by

\[
\Phi(X) = BGX - (GX)J(BGX) + B
\]

The fixed point \( X_* \in \mathcal{R} \) can be founded as a limit of simple iteration \( X_0 = 0 \), \( X_1 = B \), \( X_2 = \Phi(X_1) \), etc.
3. Results and discussion

Definition 9. [24] Let $H$ be a Hilbert space. The operator $E : D(E) \subset H \to H$, $\overline{D(E)} = H$ is normal if the adjoint operator $E^* : D(E^*) \subset H \to H$ satisfies $D(E^*) = D(E)$ and $\|Ex\| = \|E^*x\|$ for all $x \in D(E)$.

In this part $H = \ell_2(Z)$.

We write $\widetilde{A} = A_0 - B$, where the matrix $\widetilde{A}$ of the operator $\widetilde{A} : D(\widetilde{A}) \subset H \to H$ is defined by

$$(A_i x)(n) = c_0(n)x(n), \quad n \in \mathbb{Z}, \quad \text{and} \quad D(\widetilde{A}) = D(A_0) = \{ x \in \ell_2(Z); \sum_{n \in \mathbb{Z}} |x(n)c_0(n)|^2 < \infty \}.$$

The matrices $\widetilde{A}$, $A_0$ are correct and operators $\widetilde{A}$, $A_0$ are closed linear operators. The operator $B$ is defined by formulae

$$(Bx)(n) = -c_2x(n-2) - c_1x(n-1) - c_{-1}x(n+1) - c_{-2}x(n+2), \quad n \in \mathbb{Z}.$$

The operator $B$ is bounded, $B \in \text{End} \ell_2(Z)$ or $B \in \text{End}_c \ell_2(Z)$ and

$$\|B\| = |c_1| + |c_2| + |c_{-1}| + |c_{-2}|.$$

The operator $B$ has an infinite Toeplitz five-diagonal matrix with zero main diagonal.

The operator $A_0$ is called an unperturbed operator, the operator $\widetilde{A}$ - perturbed operator, and the bounded operator $B$ is called a perturbation.

The operator $A_0$ is normal. The spectrum $\sigma(A_0)$ of the operator $A_0$ can be written as

$$\sigma(A_0) = \bigcup_{i \in \mathbb{Z}} \{ \lambda_i = c_0(i) \},$$

where $\lambda_i = c_0(i)$, $i \in \mathbb{Z}$, are simple eigenvalues. The corresponding eigenvectors are $e_i$, $i \in \mathbb{Z}$, where $\{ e_i \}$, $i \in \mathbb{Z}$ is a standard basis in $\ell_2(Z)$. The spectral projections $P_n = P(\lambda_n, A_0)$, $n \in \mathbb{Z}$, are given by $P_n = e_n \otimes e_n$, $n \in \mathbb{Z}$, and $P_{(m)} = \sum_{|i| \leq m} P_i$.

Let $R = \text{End}_c \ell_2(Z)$ and $\|X\| = \|X\|_2$ for $X \in \text{End}_c \ell_2(Z)$.

Let $X \in R$, $X = (X(i,j))$, $i, j \in \mathbb{Z}$. We define a family of operators $J_k X$ and $G_k X$, $k \in \mathbb{Z}_+$, with a help of their matrices as follows

$$(J_k X)(i,j) = \begin{cases} 
X(i,j), \quad i = j, \\
X(i,j), \quad \max \{|i|, |j|\} \leq k, \\
0, \quad \text{in other cases},
\end{cases}$$

$$(G_k X)(i,j) = \begin{cases} 
\frac{X(i,j)}{c_0(i) - c_0(j)}, \quad (J_k X)(i,j) = 0, \\
0, \quad (J_k X)(i,j) \neq 0.
\end{cases}$$

In other words,

$$J_k X = P_{(k)} X P_{(k)} + \sum_{|i|, |j| \geq k} P_i X P_j, \quad k \geq 0, \quad k \geq 0.$$
moreover, \( JX = J_0X = \sum_{i \in \mathbb{Z}} P_i XP_i \), where the series \( \sum_{i \in \mathbb{Z}} P_i XP_i \) is convergent, because \( X \in \text{End}_1 \ell_2(Z) \). The matrix of the operator \( JX \) is diagonal, the matrix of the operator \( J_kX \) is block-diagonal.

The operators \( GX = \sum_{i,j \in \mathbb{Z}, c_0(i) - c_0(j) \neq 0} X(i,j) \) and \( G_kX = \sum_{i,j \in \mathbb{Z}, c_0(i) - c_0(j) \neq 0} X(i,j) \) also belong to \( \text{End}_1 \ell_2(Z) \) and the constant \( \gamma \) (from the definition 7) is

\[
\gamma = \left( \min_{i \neq j} |c_0(i) - c_0(j)| \right)^{-1},
\]

\[
\gamma_k = d_k^{-1} = \left( \inf_{|j| \leq k, |j| > k} |c_0(i) - c_0(j)| \right)^{-1},
\]

We note, that \( \gamma_k \to 0 \) if \( k \to \infty \).

Analogously to [16] the following lemma takes place.

**Lemma 3.** For each \( k \geq 0 \) the triplet \( (\text{End}_1 \ell_2(Z), J_k, G_k) \) is admissible for the unperturbed operator \( A_0 \).

Lemma 3, Theorem 1, Theorem 2, and estimates (4) and (5) imply Theorem 3.

**Theorem 3.** If

\[
|c_1| + |c_2| + |c_{-1}| + |c_{-2}| < 3\gamma^{-1},
\]

then the operator \( \tilde{A} \) is similar to the diagonal operator \( A_0 - JX_\ast \), where \( X_\ast \in \text{End}_1 \ell_2(Z) \) is the solution of the operator equation (3),

\[
\tilde{A}(I + GX_\ast) = (I + GX_\ast)(A_0 - \sum_{i \in \mathbb{Z}} P_i XP_i).
\]

There is a number \( k \in \mathbb{Z} \), such that the operator \( \tilde{A} \) is similar to the block-diagonal operator \( A_0 - J_kX_\ast \), where \( X_\ast \in \text{End}_1 \ell_2(Z) \) is the solution of (2), with the operators \( J_k, G_k \in \text{End}(\text{End}_1 \ell_2(Z)) \), and

\[
\tilde{A}(I + G_kX_\ast) = (I + G_kX_\ast)(A_0 - \sum_{|j| \leq k} P_i XP_i - P_{(k)} X_\ast P_{(k)}).
\]

Assume that Theorem 3 holds true. Then the similarity of operators \( \tilde{A} \) and \( A_0 - J_kX_\ast, \ k \geq 0 \), yields \( \sigma(\tilde{A}) = \sigma(A_0 - J_kX_\ast) = \sigma(A_{(k)}) \cup \bigcup_{|H| > k} \sigma(A_j) \), where \( A_{(k)} = (P_{(k)} A_0 - P_{(k)} X_\ast)_{H_{(k)}} \), \( H_{(k)} = \text{Im}P_{(k)} \), \( A_j = (P_i A_0 - P_i X_\ast)_{H_j} \), \( H_j = \text{Im}P_i \), \( |i| > k \). Since the operator \( X_\ast \) is unknown and we know only its approximations, we have \( A_i = (P_i A_0 - P_i B - P_i(X_\ast - B))_{H_j} \) and \( \|P(X - B)P\| \leq \text{const} \ d_i^{-1} \) \( |i| > k \), where \( d_i = \inf_{i \neq j} |c_0(i) - c_0(j)| \).

The following theorem describes the spectral properties of the operator \( \tilde{A} \).

**Theorem 4.** We have
\[ \sigma(\tilde{A}) = \sigma_{(k)} \cup \left( \bigcup_{i=k}^{\infty} \{ \mu_i \} \right), \]

where \( \sigma_{(k)} \) consists of no more than \( 2k + 1 \) eigenvalues and

\[
\mu_i = c_0(i) + O(d_i^{-1}),
\]

\[
\mu_i = c_0(i) - c_2 \cdot c_2 \left( (c_0(i + 2) - c_0(i))^{-1} + (c_0(i - 2) - c_0(i))^{-1} \right) - c_1 \cdot c_1 \left( (c_0(i + 1) - c_0(i))^{-1} + (c_0(i - 1) - c_0(i))^{-1} + O(d_i^{-2}) \right). \]

The corresponding eigenvectors \( \tilde{\xi}_i, i \in \mathbb{Z} \) form a Riesz basis in \( \ell_2(\mathbb{Z}) \).

We go back to the example 2. From the Theorem 4 it follows

**Theorem 5.** There is a number \( k \in \mathbb{Z} \) such that

\[ \sigma(E) = \sigma_{(k)} \bigcup \left( \bigcup_{i=k}^{\infty} \mu_i \right), \]

where \( \sigma_{(k)} \) consists of no more than \( 2k + 1 \) eigenvalues, \( \{ \mu_i \} \) is one-point sets, and

\[ \mu_i = (\pi(2i + \Theta))^2 - \hat{q}(0) + O\left(\frac{1}{i}\right). \]

We don’t write out higher order approximations because of their bulkiness.

**Definition 10.** A subspace \( M \subset X \) is called bi-invariant if it is invariant, complementable, and one of its complements is invariant for the operator \( A : D(A) \subset X \rightarrow X \).

**Definition 11.** Let \( X = H \), where \( H \) is Hilbert space. \( A \) nontrivial closed subspace \( M \subset H \) is called bi-invariant if it is invariant for the operator \( F : D(F) \subset H \rightarrow H \) with its complements.

**Lemma 4.** Let an operator \( A : D(A) \subset X \rightarrow X \) commute with a projection \( P \). Then \( \text{Ker} \ P \) and \( \text{Im} \ P \) are bi-invariant subspaces for \( A \).

Let \( F : D(F) \subset H \rightarrow H \) is a normal linear operator and \( \sigma_i(F) = \bigcup_{i \in \mathbb{Z}} \sigma_i, \sigma_i \cap \sigma_j = 0, \)

\( P_i(F) = P(\{ \sigma_i \}, F), i \in \mathbb{Z}, \ H_i = \text{Im} \ P_i(F). \)

Then the Hilbert space \( H \) represented as a direct sum of orthogonal non-zero closed subspaces \( H_i, i \in \mathbb{Z}, \) that is

\[ H = \bigoplus_{i \in \mathbb{Z}} H_i, \quad (7) \]

where \( H_i \) is orthogonal to \( H_j \) as \( i \neq j, i, j \in \mathbb{Z}, \)

\[ x = \sum_{i \in \mathbb{Z}} x_i, x_i \in H_i, \| x \|^2 = \sum_{i \in \mathbb{Z}} \| x_i \|^2. \]

According to [16, Ch. 5] the system of subspace \( H_i, i \in \mathbb{Z} \) is an orthogonal basis of subspace in \( H \).

**Definition 12.** The decomposition

\[ H = \bigoplus_{i \in \mathbb{Z}} UH_i \]
of the Hilbert space $H$, where $U$ is invertible operator in $End H$ and $H$ is orthogonal direct sum \( \sum \) of subspaces $H_i, i \in \mathbb{Z}$ will be said to be quasiorthogonal decomposition or Riesz decomposition. If the operator $U$ can be represented in the form $U = I + W$, where $W \in \sigma_2(H)$ then the quasiorthogonal decomposition is called a Bari decomposition.

Further, according to [25, Ch.5] the system of subspaces $UH_i, i \in \mathbb{Z}$ is a basis of subspace equivalent to an orthogonal one, or a rectifiable basis [26].

We go back to the Definition 6. Let the operator $A_i$ have the bi-invariant subspace $M \subset H$. Then the operator $A_i$ have the bi-invariant subspace $UM$. This subspace is called the Riesz bi-invariant subspace. If the transformation operator $U$ (see Definition 6) can be represented in the form $U = I + W$, $W \in \sigma_2(H)$ then the bi-invariant subspace $UM$ is called the Bari bi-invariant subspace.

Going back to the perturbed operator $\tilde{A} : D(\tilde{A}) \subset H \to H$, let $\tilde{P}_n, n \in \mathbb{Z}$ be the spectral projections $\tilde{P}_n = P(\{\mu_n\}, \tilde{A}), n \in \mathbb{Z}$, corresponding to the one-point spectral sets $\sigma_n = \{\mu_n\}, n \in \mathbb{Z}$, of the operator $\tilde{A}$, and $\tilde{P}_{(k)} = \sum \tilde{P}_i$. From the Lemma 2, Lemma 4 and Theorem 4, we derive

**Theorem 5.** The spectral projections $\tilde{P}_i, \|i\| > k, \tilde{P}_{(k)}$ are given by

$$
\tilde{P}_i = (I + G_kX_*)P(I + G_kX_*)^{-1},
$$

$$
\tilde{P}_{(k)} = (I + G_kX_*)P_{(k)}(I + G_kX_*)^{-1},
$$

$$
\tilde{P}_i - P_i = (G_kXP_i - P_iG_kX_*)(I + G_kX_*)^{-1},
$$

$$
\tilde{P}_{(k)} - P_{(k)} = (G_kXP_{(k)} - P_{(k)}G_kX_*)(I + G_kX_*)^{-1}, \|i\| > k.
$$

The subspaces $\text{Im} \tilde{P}_i, \text{Ker} \tilde{P}_i, \|i\| > k$, $\text{Im} \tilde{P}_{(k)}, \text{Ker} \tilde{P}_{(k)}$ are bi-invariant Riesz subspaces for the operator $\tilde{A}$.

4. **Conclusion**

In this paper we discussed a few spectral properties of the operator $\tilde{A}$ given by a matrix $\tilde{A}$ of the form (1). Asymptotic estimates of eigenvalues were obtained. We also looked at an (infinite) matrix as a map and provided relevant definitions and basic facts. The main method for this exploration was the method of similar operators which allowed us to reduce the operator to one with a block-diagonal matrix. A matrix $\tilde{A}$ can be considered as a matrix of forth-order difference operator. A matrix $\tilde{A}$ also is a matrix of Hill operator with a special kind of potential. Thus, the results obtained can be applied to difference and differential equations.

**Acknowledgments**

We thank I. Krishtal for stimulating discussions that helped us improve this manuscript. We also acknowledge the following financial support. The first author was supported in part by the Ministry of Science and Higher Education of the Russian Federation in the frameworks of the project part of the state work quota (Project No 1.3464.2017/4.6). The forth author was supported in part by RFBR grant 19-01-00732.

**References**

[1] McKinsey J.C.C. 1952 *Introduction to the theory of games* (New York. Toronto. London.
McGRaw-Hill book Company inc.)

[2] Cooke R G 1950 Infinite matrices and sequence spaces (London).

[3] Golub G H, Van Loan C F 1989 Matrix Computation (Baltimore and London: The John Hopkins University Press)

[4] Baskakov A G and Polyakov D M 2017 The method of similar operators in the spectral analysis of the Hill operator with nonsmooth potential, Sb. Math. 208:1 1–43

[5] Djakov P V and Mityagin B S 2006 Instability zones of periodic one-dimensional Schrödinger and Dirac operators, Russian Math. Surveys 61:4 pp. 663–788. 61:4 pp 663–766

[6] Baskakov A G 1996 Linear differential operators with unbounded operator coefficients and semigroups of difference operators Math. Notes 56 6 pp 586–593

[7] Braeutigam I and Polyakov D M 2019 Asymptotics of eigenvalues of infinite block matrices Ufa Math. J. 11 3 pp 11–28

[8] Malejki M 2018 Eigenvalues for some complex infinite tridiagonal matrices J. Adv. Math. Comp. Sci. 26:5 pp 1–9 36781038

[9] Malejki M 2009 Asymptotics of large eigenvalues for some discrete unbounded Jacobi matrices Lin. Alg. Appl. 431:10 pp 1952–1970

[10] Malejki M 2010 Asymptotics behaviour and approximation of eigenvalues for unbounded block Jacobi matrices Opuscula Math. 30:3 pp 311–330

[11] Boutet de Monvel A. and Zielinski L 2010 Approximation of eigenvalues for unbounded block Jacobi matrices using finite submatrices Cent. Eur. J.Math. 12:3 pp 445–463

[12] Ikebe Y, Asai N, Miyazaki Y and Cai D 1996 The eigenvalue problem for infinite complex symmetric tridiagonal matrices with application Lin. Alg. Appl. 241–243 pp 599–618

[13] Batalschchikov A A, Grudsky S M and Stukopin V A 2015 Asymptotics of eigenvalues of symmetric Toeplitz band matrices Linear algebra and its application 469 pp 464–486

[14] Batalschchikov A A, Grudsky S M, Ramirez de Arellano E and Stukopin V A 2015 Asymptotics of Eigenvectors of Large Symmetric Banded Toeplitz Matrices. Integral Equations and Operator Theory 83:3 pp 301–330

[15] Zolotych S A and Stukopin V A 2017 Asymptotics of eigenvalues of simple multiloop banded Toeplitz matrices of a special type Math. Notes 102:4 pp 575–579

[16] Garkavenko G V and Uskova N B 2017 Method of similar operators in research of spectral properties of difference operators with growing potential Siber. Electr. Math. Reports 14 pp 673–689

[17] Polyakov D M 2018 A one-dimensional schrödinger operator with square-integrable potential Sib Math J. 59 16 pp 470–485

[18] Braeutigam I and Polyakov D M 2018 On the asymptotic of eigenvalues of a fourth-order differential operators with matrix coefficients Differ. Equ. 54 pp 450-467

[19] Skrynnikov A V 1983 Quasinilpotent variant of Friedrichs' method in the theory of similarity of linear operators Func. Anal. Appl. 17 pp 239–240

[20] Hinkkanen A 1985 On the diagonalization of a certain class of operators Michigan Math. J. 32

[21] Foias C, Jung I B, Ko E and Pearcy C 2007 Rank-one perturbation of normal operators J. Funct. Anal. 253 pp 628 - 642.

[22] Foias C, Jung I B, Ko E and Pearcy C 2008 Rank-one perturbation of normal operators II, Indiana Univ. Math. J. 57 pp 2745 – 2760.

[23] Foias C, Jung I B, Ko E and Pearcy C 2011 Spectral decomposability of rank-one perturbation of normal operators J. Math. Anal. Appe. 375 pp 602 – 609.

[24] Rudin W 1973 Functional Analysis (McGraw-Hill, New York)

[25] Gohberg I C and Krein M G 1969 Introduction to the Theory of Linear Nonselfadjoint Operators in Hilbert Space (American Mathematical Society. Series: Translations of mathematical monographs) p.378

[26] Fage M K 1950 The rectification of bases in Hilbert space Dokl. Akad. Nauk SSSR 74 pp 1053–1056 (in Russian)