OPERADS AND COHOMOLOGY

L. Kluge and E. Paal

Department of Mathematics, Tallinn Technical University
Ehitajate tee 5, 19086 Tallinn, Estonia

Abstract

It is clarified how cohomologies and Gerstenhaber algebras can be associated with linear pre-operads (comp algebras). Their relation to mechanics and operadic physics is concisely discussed.

1 Introduction and outline of the paper

Operads, in essence, were invented by Gerstenhaber [2, 3] and Stasheff [16]. The notion of an operad was formalized by May [12] as a tool for iterated loop spaces. In 1994/95 [5, 17], Gerstenhaber and Voronov published main principles of the operad calculus. Quite a remarkable research activity on operad theory and its applications can be observed in the last decade (e. g. [11, 15]). It may be said that operads are also becoming an interesting and important mathematical tool for QFT and deformation quantization.

In this paper, the essential parts of the operad algebra are presented, which are relevant to understand how the cohomology and Gerstenhaber algebra can be associated with a pre-operad. We start from simple axioms. Basic algebraic constructions associated with a linear pre-operad are introduced. Their properties and the first derivation deviations of the pre-coboundary operator are explicitly given. Under certain condition (formal associativity constraint), the Gerstenhaber algebra structure appears in the associated cohomology. At last, it is also concisely discussed how operads and Gerstenhaber algebras are related to mechanics and operadic physics.

2 Pre-operad (composition system)

Let $K$ be a unital associative commutative ring, and let $C^n$ ($n \in \mathbb{N}$) be unital $K$-modules. For homogeneous $f \in C^n$, we refer to $n$ as the degree of $f$ and often write (when it does not cause confusion) $f$ instead of $\deg f$. For example, $(-1)^f := (-1)^n$, $C^f := C^n$ and $\phi_f := \phi_n$. Also, it is convenient to use the reduced degree $|f| := n - 1$. Throughout this paper, we assume that $\otimes := \otimes_K$. 
Definition 2.1. A linear pre-operad (composition system) with coefficients in $K$ is a sequence $C := \{C^n\}_{n \in \mathbb{N}}$ of unital $K$-modules (an $\mathbb{N}$-graded $K$-module), such that the following conditions hold.

1. For $0 \leq i \leq m - 1$ there exist partial compositions $\varnothing_i \in \text{Hom}(C^m \otimes C^n, C^{m+n-1})$, $|\varnothing_i| = 0$.
2. For all $h \otimes f \otimes g \in C^h \otimes C^f \otimes C^g$, the composition (associativity) relations hold,
   $$
   (h \varnothing_i f) \varnothing_j g = \begin{cases} 
   (-1)^{|f||g|}(h \varnothing_i g) \varnothing_{i+|g|} f & \text{if } 0 \leq j \leq i - 1, \\
   h \varnothing_i (f \varnothing_{j-1} g) & \text{if } i \leq j \leq i + |f|, \\
   (-1)^{|f||g|}(h \varnothing_{j-|f|} g) \varnothing_{i} f & \text{if } i + f \leq j \leq |h| + |f|.
   \end{cases}
   $$
3. There exists a unit $I \in C^1$ such that $I \varnothing_0 f = f = f \varnothing_i I$, $0 \leq i \leq |f|$.

In the 2nd item, the first and third parts of the defining relations turn out to be equivalent.

Example 2.2 (endomorphism pre-operad $[2, 3]$). Let $A$ be a unital $K$-module and $E^n_A := \text{End}_n^A := \text{Hom}(A^\otimes n, A)$. Define the partial compositions for $f \otimes g \in E^f_A \otimes E^g_A$ as
   $$
   f \varnothing_i g := (-1)^{|i| |g|} f \varnothing(i \varnothing_A^i \otimes g \otimes \text{id}_A^\otimes(|f|-i)), \quad 0 \leq i \leq |f|.
   $$
Then $E_A := \{E^n_A\}_{n \in \mathbb{N}}$ is a pre-operad (with the unit $\text{id}_A \in E^1_A$) called the endomorphism pre-operad of $A$.

3 Associated operations

Throughout this paper fix $\mu \in C^2$.

Definition 3.1. The cup-multiplication $\cup : C^f \otimes C^g \to C^{f+g}$ is defined by
   $$
   f \cup g := (-1)^f (\mu \varnothing_0 f) \varnothing f g \in C^{f+g}, \quad |\cup| = 1.
   $$
The pair Cup $C := \{C, \cup\}$ is called a $\cup$-algebra (cup-algebra) of $C$.

Example 3.2. For the endomorphism pre-operad (Example 2.2) $E_A$ one has
   $$
   f \cup g = (-1)^f \mu \varnothing(f \otimes g), \quad \mu \otimes f \otimes g \in E^f_A \otimes E^f_A \otimes E^g_A.
   $$

Definition 3.3. The total composition $\bullet : C^f \otimes C^g \to C^{f+|g|}$ is defined by
   $$
   f \bullet g := \sum_{i=0}^{\lfloor f \rfloor} f \varnothing_i g \in C^{f+|g|}, \quad |\bullet| = 0.
   $$
The pair Com $C := \{C, \bullet\}$ is called the composition algebra of $C$.  

Definition 3.4 (tribraces and tetrabraces). Define the Gerstenhaber *tribraces* \{·, ·, ·\} as a double sum

\[
\{h, f, g\} := \sum_{i=0}^{\infty} \sum_{i+f} \sum ((h \circ_i f) \circ_j g) \in C^{h+|f|+|g|}, \quad |\{·, ·, ·\}| = 0.
\]

The *tetrabraces* \{·, ·, ·, ·\} are defined by

\[
\{h, f, g, b\} := \sum_{i=0}^{\infty} \sum_{i+f} \sum ((h \circ_i f) \circ_j g) \circ_k b \in C^{h+|f|+|g|+|b|}.
\]

It turns out that \(f \sim g = (-1)^{|f|}\{\mu, f, g\}\). In general, \(\text{Cup} \ \mathcal{C}\) is a non-associative algebra. By denoting \(\mu^2 := \mu \ast \mu\) it turns out that the associator in \(\text{Cup} \ \mathcal{C}\) reads

\[
(f \sim g) \circ h - f \sim (g \circ h) = \{\mu^2, f, g, h\}.
\]

Thus the *formal associator* \(\mu^2\) is an obstruction to associativity of \(\text{Cup} \ \mathcal{C}\). For the endomorphism pre-operad \(\mathcal{E}_A\), \(\mu^2\) reads as an associator as well:

\[
\mu^2 = \mu \circ (\mu \otimes \text{id}_A - \text{id}_A \otimes \mu), \quad \mu \in \mathcal{E}_A^2.
\]

4 Identities

In a pre-operad \(\mathcal{C}\), the Getzler identity

\[
(h, f, g) := (h \ast f) \ast g - h \ast (f \ast g) = \{h, f, g\} + (-1)^{|f||g|}\{h, g, f\}
\]

holds, which easily implies the Gerstenhaber identity

\[
(h, f, g) = (-1)^{|f||g|}(h, g, f).
\]

The *commutator* \([·, ·]\) is defined in \(\text{Com} \ \mathcal{C}\) by

\[
[f, g] := f \ast g - (-1)^{|f||g|} g \ast f = (-1)^{|f||g|}[g, f], \quad |[·, ·]| = 0. \quad (G1)
\]

The *commutator algebra* of \(\text{Com} \ \mathcal{C}\) is denoted as \(\text{Com}^{-} \mathcal{C} := \{C, [·, ·]\}\). By using the Gerstenhaber identity, one can prove that \(\text{Com}^{-} \mathcal{C}\) is a *graded Lie algebra*. The Jacobi identity reads

\[
(-1)^{|f||h|}[f, [g, h]] + (-1)^{|g||f|}[[g, h], f] + (-1)^{|h||f|}[[h, f], g] = 0. \quad (G2)
\]

5 Pre-coboundary operator

In a pre-operad \(\mathcal{C}\), define a *pre-coboundary* operator \(\delta := \delta_{\mu}\) by

\[
-\delta f := \text{ad}^\text{right}_{\mu} f := [f, \mu] := f \ast \mu - (-1)^{|f|}\mu \ast f
\]

\[
= f \sim 1 + f \ast \mu + (-1)^{|f|} 1 \sim f, \quad \text{deg} \delta = +1 = |\delta|.
\]
It turns out that $\delta^2_\mu = -\delta_\mu^2$. It follows from the Jacobi identity in $\text{Com}^{-C}$ that $\delta$ is a (right) derivation of $\text{Com}^{-C}$,

$$\delta[f, g] = (-1)^{|g|}[\delta f, g] + [f, \delta g].$$

But $\delta$ need not be a derivation of $\text{Cup}^C$, and $\mu^2$ again appears as an obstruction:

$$\delta(f \circ g) - f \circ \delta g - (-1)^{|g|}\delta f \circ g = (-1)^{|g|}\{\mu^2, f, g\}.$$

6 Derivation deviations

The derivation deviation of $\delta$ over $\bullet$ is defined by

$$\text{dev}_\bullet \delta(f \otimes g) := \delta(f \bullet g) - f \bullet \delta g - (-1)^{|g|}\delta f \bullet g.$$

**Theorem 6.1 (**[5, 8])**. In a pre-operad $C$, one has**

$$(-1)^{|g|}\text{dev}_\bullet \delta(f \otimes g) = f \circ g - (-1)^{|g|}g \circ f.$$

The derivation deviation of $\delta$ over $\{\cdot, \cdot, \cdot\}$ is defined by

$$\text{dev}_{\{\cdot, \cdot, \cdot\}} \delta(h \otimes f \otimes g) := \delta\{h, f, g\} - \{h, f, \delta g\}$$

$$- (-1)^{|g|}\{h, \delta f, g\} - (-1)^{|g|+|f|}\{\delta h, f, g\}.$$

**Theorem 6.2 (**[5, 8])**. In a pre-operad $C$, one has**

$$(-1)^{|g|}\text{dev}_{\{\cdot, \cdot, \cdot\}} \delta(h \otimes f \otimes g) = (h \bullet f) \circ g + (-1)^{|h|f}f \circ (h \bullet g) - h \bullet (f \circ g).$$

Thus the left translations in $\text{Com}^C$ are not derivations of $\text{Cup}^C$, the corresponding deviations are related to $\text{dev}_{\{\cdot, \cdot, \cdot\}} \delta$. It turns out that the right translations in $\text{Com}^C$ are derivations of $\text{Cup}^C$,

$$(f \circ g) \bullet h = f \circ (g \bullet h) + (-1)^{|h|g}(f \bullet h) \circ g.$$

By combining this formula with the one from Theorem 6.2 we obtain

**Theorem 6.3.** In a pre-operad $C$, one has

$$(-1)^{|g|}\text{dev}_{\{\cdot, \cdot, \cdot\}} \delta(h \otimes f \otimes g) = [h, f] \circ g + (-1)^{|h|f}[f \circ [h, g] - [h, f \circ g].$$

7 Associated cohomology and Gerstenhaber algebra

Now, it can be clarified how the Gerstenhaber algebra can be associated with a linear pre-operad. If (formal associativity) $\mu^2 = 0$ holds, then $\delta^2 = 0$, which in turn implies $\text{Im} \delta \subseteq \text{Ker} \delta$. Then one can form an associated cohomology ($\mathbb{N}$-graded module) $H(C) := \text{Ker} \delta / \text{Im} \delta$ with homogeneous components

$$H^n(C) := \text{Ker}(C^n \xrightarrow{\delta} C^{n+1}) / \text{Im}(C^{n-1} \xrightarrow{\delta} C^n),$$
where, by convention, \( \text{Im}(C^{-1} \xrightarrow{\delta} C^0) := 0 \). Also, in this \((\mu^2 = 0)\) case, \( \text{Cup} C \) is associative,

\[
(f \sim g) \sim h = f \sim (g \sim h),
\]

and \( \delta \) is a derivation of \( \text{Cup} C \). Recall from above that \( \text{Com}^{-1} C \) is a graded Lie algebra and \( \delta \) is a derivation of \( \text{Com}^{-1} C \). Due to the derivation properties of \( \delta \), the multiplications \([\cdot, \cdot]\) and \( \sim \) induce corresponding (factor) multiplications on \( H(C) \), which we denote by the same symbols. Then \( \{H(C), [\cdot, \cdot]\} \) is a graded Lie algebra. It follows from Theorem \(6.1\) that the induced \( \sim\)-multiplication on \( H(C) \) is graded commutative,

\[
f \sim g = (-1)^{|f||g|} g \sim f
\]

for all \( f \otimes g \in H^f(C) \otimes H^g(C) \), hence \( \{H(C), \sim\} \) is an associative graded commutative algebra. It follows from Theorem \(6.3\) that the graded Leibniz rule holds,

\[
[h, f \sim g] = [h, f] \sim g + (-1)^{|h||f|} f \sim [h, g]
\]

for all \( h \otimes f \otimes g \in H^h(C) \otimes H^f(C) \otimes H^g(C) \). At last, it is also relevant to note that

\[
0 = ||\cdot, \cdot|| \neq |\sim| = 1.
\]

In this way, the triple \( \{H(C), \sim, [\cdot, \cdot]\} \) turns out to be a Gerstenhaber algebra \(3\). The defining identities of the Gerstenhaber algebra are (G1)-(G6).

In the case of an endomorphism pre-operad, the Gerstenhaber algebra structure appears on the Hochschild cohomology of an associative algebra \(2\).

8 Discussion: x-mechanics

Some people like commutative diagrams. Consider the following one:

\[
\text{Poisson algebras} \xleftarrow{\text{algebraic abstraction}} \text{mechanics} \xrightarrow{\text{algebraic abstraction}} \text{x-mechanics}
\]

\[
\text{Gerstenhaber algebras} \xleftarrow{\text{algebraic abstraction}} \text{x-mechanics}
\]

Poisson algebras can be seen as an algebraic abstraction of mechanics. Here \(\sim\) means similarity: Gerstenhaber algebras are graded analogs of the Poisson algebras.

It may be expected that there exists a kind of mechanics (x-mechanics) associated with operads and Gerstenhaber algebras. According to the diagram, x-mechanics is a graded analogue of mechanics and observables of an x-mechanical model must satisfy the (homotopy \(5\)) Gerstenhaber algebra identities.

Cohomologies and Gerstenhaber algebras associated with pre-operads are natural objects for modelling x-mechanical systems. Physically relevant examples of the Gerstenhaber algebras and odd symplectic structures are provided by the Batalin-Vilkovisky algebras \(6\). Relevance of the operad structure in handling the renormalization problems in QFT were recently stressed in \(9\).
Acknowledgement

Research was in part supported by the Estonian Science Foundation Grant 5634.

References

[1] V. Coll, M. Gerstenhaber, and S.D. Schack. Universal deformation formulas and breaking symmetry. J. Pure Appl. Algebra, 90 (1993), 201-219.
[2] M. Gerstenhaber. The cohomology structure of an associative ring. Ann. of Math. 78 (1963), 267-288.
[3] M. Gerstenhaber. On the deformation theory of rings and algebras: III. Ann. of Math. 88 (1968), 1-34.
[4] M. Gerstenhaber, A. Giaquinto, and S.D. Schack. Algebras, bialgebras, quantum groups, and algebraic deformations. Contemp. Math. 134 (1992), 51-92.
[5] M. Gerstenhaber and A.A. Voronov. Homotopy G-algebras and moduli space operad. Intern. Math. Res. Notices, 1995, No. 3, 141-153; hep-th/9409063.
[6] E. Getzler. Batalin-Vilkovisky algebras and two-dimensional topological field theories. Comm. Math. Phys. 159 (1994), 265-285; hep-th/9212043.
[7] N. Ikeda. Topological field theories and geometry of Batalin-Vilkovisky algebras. JHEP 0210 (2002), 076; hep-th/0209042.
[8] L. Kluge and E. Paal. On derivation deviations in an abstract pre-operad. Comm. Algebra, 29 (2001), 1609-1626; math.QA/0105229.
[9] D. Kreimer. Combinatorics of (perturbative) quantum field theory. Phys. Rept. 363 (2002), 387-424; hep-th/0010059.
[10] P. van der Laan and I. Moerdijk. The renormalisation bialgebra and operads; hep-th/0210226.
[11] J.-L. Loday, J. Stasheff, and A.A. Voronov, Eds. Operads, Proceedings of Renaissance Conferences. Contemp. Math. 202 (1997).
[12] J.P. May. The Geometry of Iterated Loop Spaces. Lecture Notes in Math. 271, 1972.
[13] M. Markl, S. Schnider, and J. Stasheff. Operads in Algebra, topology and Physics. AMS, 2002
[14] A. Schwarz. Geometry of Batalin-Vilkovisky quantization. Comm. Math. Phys. 155 (1993), 249-260; hep-th/9205088.
[15] V.A. Smirnov. Simplicial and Operad Methods in Algebraic Topology. AMS, Transl. Math. Monogr. 198, 2001.
[16] J.D. Stasheff. Homotopy associativity of H-spaces, I, II. Trans. Amer. Math. Soc. 108 (1963), 275-292, 293-312.
[17] A. A. Voronov and M. Gerstenhaber. Higher-order operations on the Hochschild complex. Funktsional. Anal. i Prilozhen. 29 (1995), 1-6 (in Russian).