TOPOLOGICAL ENTROPY AND BURAU REPRESENTATION

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Abstract. Let $f$ be an orientation-preserving homeomorphism of the disk $D^2$, $P$ a finite invariant subset and $[f_P]$ the isotopy class of $f$ in $D^2 \setminus P$. We give a non trivial lower bound of the topological entropy for maps in $[f_P]$, using the spectral radius of some specializations in $GL(n, \mathbb{C})$ of the Burau matrix associated with $[f_P]$ and we discuss some examples.

1. Introduction

Let $f$ be an orientation-preserving homeomorphism of the disk $D^2$ and $P \subset \text{int} \ (D^2)$, a finite invariant subset of $n$ points. $f$ induces a homeomorphism $f_P$ on $D_P = D^2 \setminus P$ and $f_P$ induces an automorphism $f_P^#$ on $\pi_1(D_P) \cong F_n$, the free group with $n$ generators. If $h(f)$ is the topological entropy $[1]$ of $f$ and $\text{GR}(f_P^#)$ is the growth rate of $f_P^#$ on $\pi_1(D_P)$, Bowen $[3]$ has shown that

$$ h(f) \geq \ln(\text{GR}(f_P^#)). $$

The problem is that in general, it is very hard to compute $\text{GR}(f_P^#)$.

By the way, let $f_{P^*1}$ be the isomorphism induced by $f_P$ on the first homology group $H_1(D_P; \mathbb{C})$, then

$$ \text{GR}(f_P^#) \geq R(f_{P^*1}) $$

where $R(f_{P^*1})$ is the spectral radius of the isomorphism $f_{P^*1}$. But $f_{P^*1}$ is just a permutation and so inequality $[2]$ is always trivial. Therefore, we cannot by this way detect a positive topological entropy.

It is a well-known fact $[2]$ that the subgroup of automorphism of $F_n$ induced by homeomorphisms of $D_P$ is isomorphic to Artin's Braid group $B_n$. $^1$

There exists a famous representation $[2]$ of $B_n$ in $GL(n, \mathbb{Z}[t, t^{-1}])$, the Burau representation. In this paper, we prove the following:

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$^1$ To be perfectly exact, this result supposed that we have fixed the star point of $\pi_1(D_P)$ on the boundary of $D_P$ !
Theorem 1. Let $\alpha \in B_n \subset \text{Aut}(F_n)$ and $B_\alpha$ the Burau matrix of $\alpha$; we have

$$GR(\alpha) \geq \sup \{R(B_\alpha(t)); t \in \mathbb{C}, |t| = 1\},$$

$R(B_\alpha(t))$ being the spectral radius of the matrix $B_\alpha(t) \in GL(n, \mathbb{C})$ when evaluating at a complex number $t \in \mathbb{C}$.

Remark 2. This result is not really new since it appears not explicitly as a corollary of results from David Fried in [5]. Nevertheless, we give here a direct proof of it without any reference to “twisted” cohomology, using only the definition of the Burau representation and Fox free differential calculus [4].

2. The Braid group and the Burau representation

2.1. Artin braid group. There are several ways to introduce the Braid group $B_n$. All this paragraph is just a short summary of what can be found in [2]. We define $B_n$ in a purely algebraic way as the subgroup of right automorphisms $\alpha$ of the free group

$$F_n = \langle x_1, x_2, \ldots, x_n \rangle$$

which verify:

$$\begin{align*}
(x_i)\alpha &= A_i x_\mu_i A_i^{-1}, & 1 \leq i \leq n, \\
(x_1x_2 \cdots x_n)\alpha &= x_1x_2 \cdots x_n
\end{align*}$$

where $(\mu_1, \mu_2, \ldots, \mu_n)$ is a permutation of $(1, 2, \ldots, n)$ and $A_i$ is an element of $F_n$.

This group is of finite type with generators

$$\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$$

where:

$$\begin{align*}
(x_i)\sigma_i &= x_ix_{i+1}x_i^{-1}, \\
(x_{i+1})\sigma_i &= x_i, \\
(x_j)\sigma_i &= x_j, & j \neq i, i + 1.
\end{align*}$$

and relations

$$\begin{align*}
\sigma_i\sigma_{i+1}\sigma_i &= \sigma_{i+1}\sigma_i\sigma_{i+1}, & 1 \leq i \leq n - 2, \\
\sigma_i\sigma_j &= \sigma_j\sigma_i, & |i - j| \geq 2
\end{align*}$$

2.2. Free differential calculus. Let $\mathbb{Z}F_n$ be the group ring of $F_n$ with integer coefficients. For $j = 1, 2, \ldots, n$, there exists a well-defined map:

$$\frac{\partial}{\partial x_j} : \mathbb{Z}F_n \rightarrow \mathbb{Z}F_n$$
called free derivative and verifying:

\[
\frac{\partial}{\partial x_j}(c_1 + c_2) = \frac{\partial}{\partial x_j}(c_1) + \frac{\partial}{\partial x_j}(c_2), \quad c_1, c_2 \in \mathbb{Z}F_n; \tag{12}
\]

\[
\frac{\partial}{\partial x_j}(g_1 g_2) = \frac{\partial}{\partial x_j}(g_1) + g_1 \frac{\partial}{\partial x_j}(g_2), \quad g_1, g_2 \in F_n; \tag{13}
\]

\[
\frac{\partial}{\partial x_j}(x_i) = \delta_{i,j} \quad 1 \leq i \leq n. \tag{14}
\]

Finally, let \( \langle t \rangle \) be the free group with one generator generated by \( t \) and

\[
\varphi \left\{ \begin{array}{l}
F_n \rightarrow \langle t \rangle \\
x_i \rightarrow t, \quad 1 \leq i \leq n
\end{array} \right.
\]

\( \varphi \) extends into a morphism \( \mathbb{Z}F_n \rightarrow \mathbb{Z} \langle t \rangle = \mathbb{Z}[t, t^{-1}] \) that we will continue to call \( \varphi \).

2.3. Burau representation. Consider an element \( \alpha \) in \( B_n \) and define the \( n \)-square matrix \( B_\alpha = (b_{ij}) \) by:

\[
b_{ij} = \varphi \left( \frac{\partial}{\partial x_j}((x_i)\alpha) \right) \in \mathbb{Z}[t, t^{-1}] \quad (1 \leq i, j \leq n) \tag{16}
\]

It is easy to show [4] the following relation:

\[
B_\alpha \beta = B_\alpha B_\beta. \tag{17}
\]

The maps \( \alpha \mapsto B_\alpha \) from \( B_n \) to \( GL(n, \mathbb{Z}[t, t^{-1}]) \) is the Burau representation of the Braid group \( B_n \).

Let \( M = (\mathbb{Z}[t, t^{-1}])^n \) and \( V_1, V_2, \ldots, V_n \) denote the canonical basis of the free module \( M \). A Burau matrix \( B \) acts (on the right) on raw vectors of \( M \).

Any Burau matrix \( B = (b_{ij}) \) satisfies

\[
\sum_k t^{k-1}b_{kj} = t^{j-1}, \quad (\forall j). \tag{18}
\]

Hence, the submodule \( H \) of \( M \) defined by

\[
X_1 + X_2 + \cdots + X_n = 0 \tag{19}
\]

is invariant under \( B \). The Burau representation is reducible into an \((n - 1)\)-dimensional representation \( B^r \), called the reduced Burau representation.

Remark 3. There is an interesting relation between the Burau representation of a braid \( \alpha \) and the Alexander polynomial, a famous invariant of links. If we close the braid \( \alpha \) to obtain a link \( \tilde{\alpha} \) and we let

\[
L = \tilde{\alpha} \cup m
\]

where \( m \) is the braid axis (that is a circle which surrounded all the stings of the braid), \( L \) characterizes the link \( \tilde{\alpha} \) in the 3-sphere \( S^3 \) [2] and the Alexander polynomial of \( L \) is given by (cf. [7]):

\[
\Delta(x, t) = \det(B^r_\alpha(t) - xI_{n-1}) \tag{20}
\]
2.4. Spectral properties of Burau matrices. A Burau matrix $B = (b_{ij})$ satisfies the following relations:

$$\sum_k b_{ik} = 1, \quad (\forall i).$$

Hence, the raw vector $(1, t, \ldots, t^{n-1})$ is an eigenvector for $B$ with eigenvalue $1$. If $P_B$ is the characteristic polynomial of $B$, we have thus:

$$P_B(X) = \det(XI_n - B) = (X - 1)P_{B^r}(X).$$

Squier has shown that the reduced Burau representation is unitary, in the sense that there exits a non-singular matrix $J \in GL(n - 1, \mathbb{Z}[t, t^{-1}])$ such that:

$$B^r J B^r = J$$

where $B^r$ is the transpose of the matrix obtained from $B$ by exchanging $t$ and $t^{-1}$.

From equation (23) we deduce that

$$P_{B^r}(1/X) = (-1/X)^{n-1}(-t)^e P_{B^r}(X),$$

where $e$ is the algebraic sum of exponents of $\sigma_i$ in expression of $\alpha$.

Therefore, if we let $t \in \mathbb{C}, \ |t| = 1$ then this relation becomes:

$$P_{B^r}(1/X) = (-1/X)^{n-1}(-t)^e \overline{P_{B^r}(X)}.$$ 

Hence, if $\lambda$ is an eigenvalue of $B^r_\alpha$ then $1/\lambda$ is also an eigenvalue of $B^r$ with the same multiplicity.

3. Growth rate of a finite type group automorphism

Let $G$ be a finite type group, $S = \{x_1, x_2, \ldots, x_n\}$ a family of generators for $G$ and $L_S(g)$ be the minimal length of $g \in G$ relatively to $S$, that is the minimum number of letters $x_i$ or $x_i^{-1}$ to needed to express $g$. The growth rate of an automorphism $\alpha : G \to G$ is defined to be

$$GR(\alpha) = \sup_{g \in G} \left\{ \limsup_{p \to +\infty} \left( L_S((g\alpha^p))^{1/p} \right) \right\}.$$

It is not hard to see that $GR(\alpha)$ is independent of the generating system $S$, that $1 \leq GR(\alpha) < +\infty$ and that

$$GR(\alpha) = \limsup_{p \to +\infty} \left\{ \max_{1 \leq i \leq n} \left( L_S((x_i\alpha^p))^{1/p} \right) \right\}.$$ 

Introducing the occurrence matrix $A_\alpha = (a_{ij})$ where $a_{ij}$ is the number of occurrence of $x^{\pm 1}$ in the reduced word $(x_i)\alpha$ and norm $\|A_\alpha\| = \max_i(\sum_j a_{ij})$, equation (27) gives:

$$GR(\alpha) = \lim_{p \to +\infty} \|A_\alpha^p\|^{1/p}.$$
Remark 4. If $G = F_n$ is the free group, $S = \{x_1, x_2, \ldots, x_n\}$ a system of free generators, $\alpha \in \text{Aut}(F_n)$ and if moreover there is are no cancellations when we iterate $\alpha$, then $L_S((x_i x_j)\alpha) = L_S((x_i)\alpha) + L_S((x_j)\alpha)$, for $1 \leq i, j \leq n$. Hence, $A_{\alpha^p} = A_\alpha^p$, and $GR(\alpha)$ is just the spectral radius of matrix $A_\alpha$. In the general case, we do not know any recurrence formula for $A_{\alpha^p}$.

4. Proof of main theorem

Let’s come back to the case where $G = F_n$ and $\alpha \in B_n$. Theorem 1 is a consequence of preceding considerations as we shall see now.

Let $\omega = x_1^{\varepsilon_1} x_2^{\varepsilon_2} \cdots x_n^{\varepsilon_n} \in F_n$ be a reduced word with $\varepsilon_k = \pm 1$. We have

$$\frac{\partial}{\partial x_j}(\omega) = \sum_{k=1}^{r} \varepsilon_k \delta_{\mu_k,j} x_1^{\varepsilon_1} x_2^{\varepsilon_2} \cdots x_{\mu_k}^{(\varepsilon_k-1)/2}.$$  (29)

Therefore, the number of occurrence of $x_j^{\pm 1}$ in $\omega$ is equal to the number of monomials in $(\frac{\partial}{\partial x_j})(\omega)$. Moreover, if we let

$$\varphi \left( \frac{\partial}{\partial x_j}(\omega) \right) = \sum a_nt^n,$$

then the number of monomials in $(\frac{\partial}{\partial x_j})(\omega)$ is greater than $\sum |a_n|$. Now, taking $t \in \mathbb{C}$, $|t| = 1$, we have

$$\left| \sum a_nt^n \right| \leq \left| \sum a_n \right|.$$

Hence, if $\alpha \in B_n$, $A_\alpha$ is the occurrence matrix and $B_\alpha$ is the Burau matrix, we get for all $t \in \mathbb{C}$, $|t| = 1$:

$$a_{ij} \geq |b_{ij}(t)|.$$

This shows that

$$\|A_{\alpha^p}\| \geq \|B_{\alpha^p}(t)\| = \|B_\alpha^p(t)\|,$$

which concluded the proof.

Remark 5. For $t = 1$, $B_\alpha(1)$ is precisely the matrix of the map induced by $\alpha$ on the homology group $H^1(D_P; \mathbb{C})$, which is just a permutation matrix. Hence $t = 1$ will always give a trivial result.

5. Examples

5.1. Example 1. $\alpha = \sigma_1 \sigma_2^{-1} \in B_3$.

The action on $F_3$ is given by

$$\begin{align*}
(x_1)\alpha &= x_1 x_3 x_1^{-1} \\
(x_2)\alpha &= x_1 \\
(x_3)\alpha &= x_3^{-1} x_2 x_3
\end{align*}$$  (30)
In this example, there are no cancellations when we iterate $\alpha$ on each generator $x_i$. Hence, if $A_{\alpha^n}$ is the occurrence matrix of $\alpha^n$, we have

$$A_{\alpha^n} = A^n_{\alpha}.$$ 

Therefore, we can compute explicitly $GR(\alpha)$ as the spectral radius of $A$ and we find $(3 + \sqrt{5})/2$. By the way, the characteristic polynomial of $B_{\alpha}$ is given by

$$P_{B_{\alpha}}(X) = (X - 1)(X^3 - (1 - t - t^{-1})X + 1)$$

For $t = -1$, the root of biggest modulus of $P_{B_{\alpha}}$ is precisely $(3 + \sqrt{5})/2$. In this case inequality (3) is an equality.

This is however not surprising; $GR(\alpha)$ is always equal to the spectral radius of $B_{\alpha}(-1)$ if $\alpha \in B_3$. Let $\overline{D}_P$ be the blow-up of the 3-punctured disc (where each puncture is replaced by a circle $\mathbb{S}^1$). The isotopy class represented by $\alpha$ in $\overline{D}_P$ is either periodic or pseudo-Anosov (in the Thurston sense $\mathbb{S}^1$). In the first case, $GR(\alpha) = R(B_{\alpha}) = 1$. In the second case, the foliation of a pseudo-Anosov map $\phi$ in this class has only singularities on the boundary of $\overline{D}_P$. The two-fold cover of $\overline{D}_P$ that we shall denote $\overline{D}_P^2$, known as the hyperbolic involution, is a torus with 4 holes (each corresponding to each boundary component of $\overline{D}_P$). The lift $\tilde{\phi}$ of $\phi$ induces an Anosov diffeomorphism on the torus obtained by replacing each boundary curve by a point. The matrix $B_{\alpha}(-1) \in SL(2, \mathbb{Z})$ is exactly the matrix of a linear Anosov diffeomorphism $A$ in this class. Hence, the entropy of $\phi$ which is equal to the one of $A$ is equal to $R(B_{\alpha}(-1))$.

5.2. Example 2. $\alpha = \sigma_1\sigma_2^{-1}\sigma_3^{-1} \in B_4$.

The action on $F_4$ is given by

$$\left\{ \begin{array}{l}
(x_1)\alpha = x_1x_4x_1^{-1} \\
(x_2)\alpha = x_1 \\
(x_3)\alpha = x_4^{-1}x_2x_4 \\
(x_4)\alpha = x_4^{-1}x_3x_4 \\
\end{array} \right.$$ 

(31)

In this example, there are cancellations when we iterate $\alpha$ and so we are not able to compute straightforwardly $GR(\alpha)$. The characteristic polynomial of the Burau matrix is:

$$P_{B_{\alpha}}(X) = (X - 1)(X^3 - (1 - t - t^{-1})X^2 + (t - t^{-1} + 1)X + t^{-1}).$$

For $t = -1$, we get:

$$P_{B_{\alpha}}(X) = (X - 1)^3,$$

which gives nothing. But for $t = j = e^{2\pi i}/3$ we obtain

$$P_{B_{\alpha}}(X) = X^3 - 2X^2 - 2jX + j$$

which has a root of modulus strictly greater than 1. We can therefore conclude that $h(f) > 0$ for all maps in the corresponding isotopy class. In fact, in this case this isotopy class is of pseudo-Anosov type. The growth rate $GR(\alpha)$ is the greatest positive root of the polynomial $X^4 - 2X^3 - 2X + 1$.
Lemma 6. In Example 2, we have:

\[ \lambda > \sup_{|t|=1} \{ R(B_\alpha(t)) \} . \]

Proof. Since \( t \mapsto R(B_\alpha(t)) \) is a continuous function on the compact set \( |t|=1 \), if there is equality in (3) for this \( \alpha \), there must exist a complex number \( t \ (|t|=1) \) for which this bound is obtained. Hence there must exist \( t \ (|t|=1) \) such that the polynomial

\[ \lambda^3 X^3 - (1 - t - t^{-1}) \lambda^2 X^2 + (t^{-2} - t^{-1} + 1) \lambda X + t^{-1} \]

has a root on the unit circle \( |X|=1 \).

But a necessary condition for a polynomial \( P = a_n X^n + \cdots + a_0 \ (a_n \neq 0) \) to have a root on the unit circle is that \( P \) and \( Q = \overline{a_0} X^n + \cdots + \overline{a_n} \) have a common root. To check this we compute the resultant of the two polynomials

\[ \lambda^3 X^3 - (1 - t - t^{-1}) \lambda^2 X^2 + (t^{-2} - t^{-1} + 1) \lambda X + t^{-1} \]
\[ tX^3 + (1 - t + t^2) \lambda X^2 - (1 + t - t^{-1}) \lambda^2 X + \lambda^3 \]

and we verify that there is no value of \( t \ (|t|=1) \) for which the resultant vanishes.

5.3. Example 3. \( \alpha = \sigma_4 \sigma_3 \sigma_2 \sigma_1 \sigma_4 \sigma_3 \in B_5 \).

The action on \( F_5 \) is given by

\[
\begin{cases}
(x_1)\alpha = x_1 x_2 x_1^{-1} \\
(x_2)\alpha = x_1 x_3 x_4 x_3^{-1} x_1^{-1} \\
(x_3)\alpha = x_1 x_3 x_5 x_3^{-1} x_1^{-1} \\
(x_4)\alpha = x_1 x_3 x_1^{-1} \\
(x_5)\alpha = x_1
\end{cases}
\]

(32)

This is another interesting example. Here again \( \alpha \) represent a pseudo-Anosov class. The growth rate \( GR(\alpha) \) is the greatest positive root of \( X^4 - X^3 - X^2 - X + 1 \) that we shall call \( \lambda \) (cf. [6, appendix A]).

By the way, we have

\[ P_{B_\alpha(-1)}(X) = (X - 1)(X^4 + X^3 - X^2 + X + 1) \]

which has \(-\lambda \) as a root. Hence, in this case (3) is an equality. As in example 1, the invariant foliation of a pseudo-Anosov representative \( \phi \) in this class has only singularities on the boundary of \( \overline{D}_p \). Since \( \text{card}(P) = 5 \) is odd, it can be shown that the lift \( \tilde{\phi} \) of \( \phi \) on the two-fold cover \( \overline{D}_p^2 \) has only singularities on the boundary and there are of even order [3]. Hence the invariant foliation of \( \tilde{\phi} \) is transversally orientable and, \( h(\tilde{\phi}) = R(\tilde{\phi}_{s1}) \).

But \( B_\alpha^r(-1) \) is the homology matrix of \( \tilde{\phi} \) and so

\[ \lambda = h(\tilde{\phi}) = R(\tilde{\phi}_{s1}) = R(B_\alpha^r(-1)). \]
This case of equality will happen each time \( \text{card}(P) \) is odd and \( \alpha \) represent a pseudo-Anosov class with singularities only on the boundary of \( \overline{D}_P \).

**References**

1. R. L. Adler, A. G. Konheim, and M. H. McAndrew, *Topological entropy*, Trans. Amer. Math. Soc. 114 (1965), 309–319. MR 30 #5291
2. Joan S. Birman, *Braids, links, and mapping class groups*, Princeton University Press, Princeton, N.J., 1974, Annals of Mathematics Studies, No. 82. MR 51 #11477
3. Rufus Bowen, *Entropy and the fundamental group*, The structure of attractors in dynamical systems (Proc. Conf., North Dakota State Univ., Fargo, N.D., 1977), Lecture Notes in Math., vol. 668, Springer, Berlin, 1978, pp. 21–29. MR 80d:58049
4. Richard H. Crowell and Ralph H. Fox, *Introduction to knot theory*, Springer-Verlag, New York, 1977, Reprint of the 1963 original, Graduate Texts in Mathematics, No. 57. MR 56 #3829
5. David Fried, *Entropy and twisted cohomology*, Topology 25 (1986), no. 4, 455–470. MR 88k:58074
6. Boris Kolev, *Dynamique topologique en dimension 2: Orbites périodiques et entropie topologique*, Ph.D. thesis, Université de Nice, France, 1991.
7. H. R. Morton, *Exchangeable braids*, Low-dimensional topology (Chelwood Gate, 1982), London Math. Soc. Lecture Note Ser., vol. 95, Cambridge Univ. Press, Cambridge, 1985, pp. 86–105. MR 87e:57010
8. Craig C. Squier, *The Burau representation is unitary*, Proc. Amer. Math. Soc. 90 (1984), no. 2, 199–202. MR 85b:20056
9. William P. Thurston, *On the geometry and dynamics of diffeomorphisms of surfaces*, Bull. Amer. Math. Soc. (N.S.) 19 (1988), no. 2, 417–431. MR 89k:57023

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