GSI Oscillations as Laboratory for Testing of New Physics

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We analyse recent experimental data on the GSI oscillations of the hydrogen–like heavy $^{142}$Pm$^{60+}$ ions that is the time modulation of the K–shell electron capture (EC) decay rate. We follow the mechanism of the GSI oscillations, caused by the interference of the neutrino flavour mass–eigenstates in the content of the electron neutrino. We give arguments that these experimental data show i) an existence of sterile neutrinos that is necessary for an explanation of a phase–shift, ii) an observation of CP violation, related to a phase–shift, and iii) an influence of the Quantum Zeno Effect, explaining different values of the amplitude and phase–shift for two runs of measurements with different time resolutions and different numbers of consecutive measurements. For new runs of experiments on the GSI oscillations we propose to measure the EC and bound–state β–decay rates of the H–like heavy ions $^{108}$Ag$^{60+}$. These measurements should verify the $\Delta$-scaling of the periods of the time modulation, where $\Delta$ is the mass number of the parent ion, and give an important information on masses of neutrino (antineutrino) mass–eigenstates.

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I. INTRODUCTION

The measurements of the K–shell electron capture (EC) decays of the hydrogen–like (H–like) heavy ions $p \rightarrow d + \nu_e$, where $p$ and $d$ are the parent and daughter ions in their ground states and $\nu_e$ is the electron neutrino, at GSI \cite{1–4} in Darmstadt showed the time modulation of the rates of the number $N_d(t)$ of daughter ions. The experimental data on the time modulation \cite{1,2} have been fitted by the equation

$$\frac{dN_d(t)}{dt} = \lambda_{EC}(t)N_p(t),$$  \hspace{1cm} (1)

where $dN_d(t)/dt$ is the rate of the number of daughter ions $d$, $N_p(t)$ is the number of the parent H–like heavy ions $p$ in the ground hyperfine state $(1s)_{\text{F} = \pm \frac{1}{2}}$, $M_F = \pm \frac{1}{2}$, and $\lambda_{EC}(t)$ is the time–dependent EC decay rate in the laboratory frame, given by

$$\lambda_{EC}(t) = \lambda_{EC} (1 + a \cos(\omega t + \phi)).$$ \hspace{1cm} (2)

Here $t = 0$ corresponds to the moment of the injection of parent ions into the Experimental Storage Ring (ESR), $\lambda_{EC}$ is the EC decay constant and $a$, $T = 2\pi/\omega$ and $\phi$ are the amplitude, the period, and the phase–shift of the time modulation. The time modulation of the EC decay rates of the H–like heavy ions has been dubbed the “GSI oscillations” \cite{3}.

As theoretical explanation of the GSI oscillations we have proposed the time modulation mechanism, caused by the interference of the neutrino flavour mass–eigenstates in the final state of the EC decays \cite{5,6}. It is well–known from theoretical and experimental investigations of the neutrino lepton flavour oscillations \cite{7} that the electron neutrino $|\nu_e\rangle$ is a superposition $|\nu_e\rangle = \sum_{j=1}^{N_e} U_{ej}^* |\nu_j\rangle$ of the neutrino flavour mass–eigenstates $|\nu_j\rangle$ with masses $m_j$, where $U_{ej}$ are the matrix elements of a $N_e \times N_e$ unitary mixing matrix $U$ of the $N_e$ neutrino flavour mass–eigenstates \cite{8}, which are treated as Dirac particles. As has been shown in \cite{5,6} the time modulation frequency of the EC decay rates of the H–like heavy ions, calculated in the rest frame of parent ions, is related to the masses of neutrino flavour mass–eigenstates and the mass of parent ions $M_p$ by $\omega_{ij} = \Delta m_{ij}^2/2M_p$, where $\Delta m_{ij}^2 = m_i^2 - m_j^2$ and $i > j$. Since the time modulation periods, $T \approx (7–6) s$, measured for the EC decay rates of the H–like heavy ions $^{140}$Pd$^{58+}$, $^{142}$Pm$^{60+}$, and $^{122}$I$^{52+}$ \cite{1,2,3}, impose the constraint $\Delta m_{ij}^2 \sim 10^{-4} eV^2$, the interferences between the neutrino flavour mass–eigenstates with $i > j$

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and \(i \geq 3\), having \(\Delta m_{ij}^2\) larger compared with \(10^{-3} \text{eV}^2\), should lead to a time modulation with periods by orders of magnitude smaller compared with the experimental values \(T \approx (7 - 6) \text{s}\). This implies that the time modulation of the EC decays of the H–like heavy ions \(^{140}\text{Pr}^{58+}, \quad ^{142}\text{Pr}^{60+}\) and \(^{122}\text{P}^{62+}\) may be induced by the interference of the neutrino flavour mass–eigenstates \(|\nu_i\rangle\) and \(|\nu_j\rangle\) with \(\Delta m_{ij}^2\), which we call as \((\Delta m_{21}^2)_{\text{GSI}}\). The combined value \((\Delta m_{21}^2)_{\text{GSI}} = 2.19(3) \times 10^{-5} \text{eV}^2\), obtained from the experimental data on the periods of the time modulation of the EC decays of the H–like heavy ions \(^{140}\text{Pr}^{58+}, \quad ^{142}\text{Pr}^{60+}\) and \(^{122}\text{P}^{62+}\) by a factor of 2.9 exceeds the experimental value \((\Delta m_{21}^2)_{\text{K,L}} = 7.59(21) \times 10^{-5} \text{eV}^2\), determined by the KamLAND Collaboration from the electron antineutrino oscillations \(\nu_e \leftrightarrow \bar{\nu}_e\). However, as has been shown in \([3]\), the neutrino flavour mass–eigenstates can acquire mass–corrections \(\delta m_{ij}\) in the Strong Coulomb fields of the heavy ions. These mass–corrections change the masses of the neutrino flavour mass–eigenstates in the strong Coulomb fields of the daughter ions, i.e. \(m_j \to \tilde{m}_j = m_j + \delta m_{ij}\). This gives \((\Delta m_{21}^2)_{\text{GSI}} = \tilde{m}_1^2 - \tilde{m}_2^2\) and allows to explain an increase of \((\Delta m_{21}^2)_{\text{GSI}}\) with respect to \((\Delta m_{21}^2)_{\text{K,L}}\). As has been also discussed in \([13]\), the corresponding mass–corrections to the neutrino flavour mass–eigenstates should be taken into account for a correct elaboration of the experimental data by the KamLAND Collaboration.

The time modulation mechanism of the EC decays of the H–like heavy ions, proposed in \([3, 5]\), explains i) the suppression of the time modulation of the rates of the \(\beta^+\) decays, i.e. \(p \to d' + e^+ + \bar{\nu}_e\), of the H–like heavy ions \([12]\) and ii) the proportionality of the time modulation period of the EC decay rates to the mass number \(A\) of the parent ions \(T = 4\pi M_p/(\Delta m_{21}^2)_{\text{GSI}} \sim A\). The suppression of the time modulation of the \(\beta^+\) decay rates of the H–like heavy ions has been observed experimentally in \([3, 13]\) and reported as a preliminary result. As has been found in \([3, 5]\), the amplitude of the time modulation of the \(\beta^+\)–decays \(a = 0.03(3)\) is commensurable with zero in agreement with the results, obtained in \([12]\).

Recently the new experimental data on the rates of the EC decays and on the three–body \(\beta^+\) decays of the H–like heavy ions \(^{142}\text{Pr}^{60+}\) have been reported in \([13]\). The suppression of the time modulation of the \(\beta^+\) decay rates of the H–like heavy ions \([12]\) has been observed in recent high resolution measurements of the EC and \(\beta^+\) decays of the H–like heavy ions \(^{142}\text{Pr}^{60+}\) and \(^{122}\text{P}^{62+}\) with the amplitude of the time modulation \(a = 0.027(27)\), agreeing well with the preliminary result \(a = 0.03(3)\) \([3, 4]\).

An other mechanism of the GSI oscillations, proposed by Giunti \([14]\) and Kienert et al. \([15]\), is based on the assumption of the interference of two closely spaced energy levels of the daughter ions in the ground state. The main problem of this mechanism is the prediction of the time modulation for the \(\beta^+\) decay rates of the H–like heavy ions with the same modulation period as the EC decay rates \([5, 16]\). The explanation of the GSI oscillations by means of a neutrino magnetic moment, proposed by Gal \([17]\), suffers from the same problem as that by Giunti \([14]\) and Kienert et al. \([15]\). In addition these two mechanisms as well as the mechanisms, proposed by Pavlichenkov \([18, 19]\), Krainov \([20]\), Lambiase et al. \([21, 22]\) and Giacosa and Pagliara \([23]\), do not predict the \(A\)–scaling \([5]\). The suppression of the time modulation of the \(\beta^+\)–decays \(a = 0.03(3)\) may be induced by the interference of the neutrino flavour mass–corrections \(\delta m_{ij}\) in the strong Coulomb fields of the heavy ions.

Finally we would like to notice that a one–dimensional model of the GSI oscillations, proposed by Lipkin \([24, 25]\), being extended to 3–dimensions, leads to the EC decay rate in the rest frame of parent ions, given by \([7]\)

\[
\lambda_{\text{EC}}(t) = \lambda_{\text{EC}}(r.f.) \left(1 + \sin 2\theta \left(P_{\text{sup}} - P_{\text{sub}}\right) \frac{\sin(\Omega_L t)}{\Omega_L t}\right) \left(1 + \sum_{i > j} 2\text{Re}[U_{ei}^* U_{ej}] \cos(\omega_{ij} t)\right),
\]

where \(\lambda_{\text{EC}}(r.f.)\) is the EC decay constant in the rest frame (r.f.) of parent ions, \(\omega_{ij} = \Delta m_{ij}^2/2M_p\), \(\Omega_L = 2Q_{\text{EC}}|\delta\tilde{p}|/M_p\) and \(M_p\) is a mass of parent ions. According to Lipkin \([24, 26]\), the GS oscillations are caused by the incoherent contributions of the EC decays from a superradiant \(|p|_{\text{sup}} = |\cos \theta|p(\tilde{p} + \delta\tilde{p}) + |\sin \theta|p(\tilde{p} - \delta\tilde{p})\rangle\) and |subradiant\rangle\)$p_{\text{sub}} = |\cos \theta|p(\tilde{p} - \delta\tilde{p}) - |\sin \theta|p(\tilde{p} + \delta\tilde{p})\rangle$ states of a parent ion, where \(\theta\) is a mixing angle. The superradiant and subradiant states are formed with probabilities \(P_{\text{sup}}\) and \(P_{\text{sub}}\) respectively, obeying the condition \(P_{\text{sup}} + P_{\text{sub}} = 1\). If \(P_{\text{sup}} = P_{\text{sub}}\), the EC decay rate Eq. \([3]\) reduces to ours \([3]\). For \(P_{\text{sup}} \neq P_{\text{sub}}\) Lipkin’s model predicts unobservable oscillations with a frequency \(\Omega_L = 2Q_{\text{EC}}|\delta\tilde{p}|/M_p\). One can show \([3]\) that the term, oscillating with the frequency \(\Omega_L\), appears also in the \(\beta^+\) decay rates of the H–like heavy ions \([8]\). This contradicts to both the experimental data \([3, 4, 13]\) and our theoretical analysis \([12]\).

In spite of a certain success of the description of the time modulation in the EC and \(\beta^+\) decays of the H–like heavy ions by the interference of the neutrino flavour mass–eigenstates, such a mechanism of the GSI oscillations has been criticised in publications \([27, 28]\), where i) the parent and daughter ions have been treated as the nuclei, unaffected by the measurements, and ii) energy–momentum conservation has been accepted for all decay channels \(p \to d + \nu_j\) \((j = 1, 2, \ldots, N_d)\) in the EC decay \(p \to d + \nu_e\). However, as has been pointed out in \([3, 5]\) and shown recently in \([24]\), energy and 3–momentum in the GSI experiments on the EC decays of the H–like heavy ions are not conserved in the decay channels \(p \to d + \nu_j\) \((j = 1, 2, \ldots, N_d)\) due to interactions of parent and daughter ions with the resonant and capacitive pickups in the ESR \([31]\). Qualitatively the result of such interactions can be described by the smearing of energy and momentum of daughter ions over the regions \(\delta E_d \sim 2\pi/\delta t_d \sim 10^{-13} \text{eV}\).
and $|\delta q_i| \sim (M_d/Q_{EC})\delta E_d \sim 10^{-9}$ eV, calculated in the rest frame of parent ions (see section V). Since in the rest frame of parent ions the differences of energies and 3–momenta of particles in the decay channels $p \to d + \nu_i$ and $p \to d + \nu_j$ are of order $\omega_{ij} \sim 2\pi/T \sim 10^{-15}$ eV and $|k_i - k_j| = |q_i - q_j| \sim (M_d/Q_{EC})\omega_{ij} \sim 10^{-11}$ eV, such a violation of the energy–momentum conservation provides i) an overlap of the wave functions of the daughter ions in the decay channels $p \to d + \nu_i$ and $p \to d + \nu_j$, causing an overlap of these decay channels, and, as a result, ii) indistinguishability of the decay channels $p \to d + \nu_j$ ($j = 1, 2, \ldots, N_\nu$) in the EC decay $p \to d + \nu_e$, which is necessary for the interference of the neutrino flavour mass–eigenstates.

In this paper we propose a theoretical analysis of recent experimental data on a time modulation of the EC decay rates of the H–like heavy ions $^{142}$Pm$^{60+}$. In section II we present shortly the experimental data, given in Table 1 of Ref. [13]. In section III we show that in the mechanism of the interference of the neutrino flavour mass–eigenstates a phase–shift of the GSI oscillations can be explained by extending the number of neutrino flavours from $N_\nu = 3$ to $N_\nu \geq 4$. For an illustration we set $N_\nu = 4$ and show that the neutrino flavour mass–eigenstates $|\nu_1\rangle$ and $|\nu_2\rangle$, the interference of which is responsible for the observed time modulation, acquire a relative phase $\delta_{12}$, violating CP invariance. In section IV we show that different values of the amplitude and phase–shift of the time modulation, measured with time resolutions $\delta t_d = 32$ ms and $\delta t_d = 64$ ms and the number of consecutive measurements $N = 1688$ and $N = 844$, can be explained by means of an influence of the Quantum Zeno Effect (QZE). In section VI we discuss the obtained results and analyse the periods of the time modulation of the EC and bound–state $\beta^–$–decay rates of the H–like $^{108}$Ag$^{46+}$ heavy ions. These results can be used for new runs of experiments on the GSI oscillations for the verification of the $A$–scaling of the periods of the time modulation. They should be also of great deal of importance for the estimate of the masses of neutrino (antineutrino) mass–eigenstates.

II. RECENT EXPERIMENTAL DATA ON GSI OSCILLATIONS [13]

Recent experimental data on the GSI oscillations, reported in [13] (see Table 1 of Ref.[13]), show the time modulation of the EC decay rates of the H–like heavy ions $^{142}$Pm$^{60+}$ with i) the period $T = 7.11(11)$ s, the amplitude $a = 0.107(24)$ and the phase–shift $\phi = 2.35(48)$ rad and ii) the period $T = 7.12(11)$ s, the amplitude $a = 0.134(27)$ and the phase–shift $\phi = 1.78(44)$ rad, observed with time resolutions $\delta t = 32$ ms and $\delta t = 64$ ms and the number of consecutive measurements $N = 1688$ and $N = 844$, respectively.

III. PHASE–SHIFT OF GSI OSCILLATIONS AS A SIGNAL FOR STERILE NEUTRINOS AND CP VIOLATION

In the mechanism of the GSI oscillations, caused by the interference of the neutrino flavour mass–eigenstates [3]–[5], a phase–shift can appear because of an extension of the 3–flavour structure of neutrinos with a certain leptonic charge $|\nu_s\rangle = \sum_{j=1}^{3} U_{s\alpha}^\dagger |\nu_j\rangle$, where $U_{s\alpha}$ are the matrix elements of the 3 × 3 mixing matrix $U$ [9] and $\alpha = e, \mu, \tau$, to $N_\nu$–flavour structure $|\nu_s\rangle = \sum_{j=1}^{N_\nu} U_{s\alpha}^\dagger |\nu_j\rangle$ with $N_\nu \geq 4$, $j = 1, 2, 3, \ldots, N_\nu$ and $\alpha = e, \mu, \tau, \ldots$. New neutrinos $|\nu_s\rangle = \sum_{j=1}^{N_\nu} U_{s\alpha}^\dagger |\nu_j\rangle$ with $s \neq e, \mu, \tau$ are named sterile neutrinos. In order to illustrate an appearance of a phase–shift in the time modulated term of the EC decay rates we consider below some extensions with $N_\nu = 4$.

According to [3]–[5], a 4 × 4 mixing matrix of the neutrino flavour mass–eigenstates can be defined in two schemes of an inclusion of sterile neutrino lepton flavours, which are 3 + 1 and 2 + 2 four–family neutrino flavour mixing, respectively. The 4 × 4 mixing matrices are given by [31]

$$U = U(\theta_{14}, 0)U(\theta_{24}, 0)U(\theta_{34}, 0)U(\theta_{13}, \delta_{13})U(\theta_{12}, \delta_{12}),$$
$$U = U(\theta_{14}, 0)U(\theta_{13}, 0)U(\theta_{24}, 0)U(\theta_{23}, \delta_{23})U(\theta_{34}, \delta_{34})U(\theta_{12}, \delta_{12}),$$

for the 3 + 1 and 2 + 2 four–family neutrino flavour mixing, respectively, and

$$U = U(\theta_{34}, 0)U(\theta_{24}, 0)U(\theta_{23}, \delta_{23})U(\theta_{14}, 0)U(\theta_{13}, \delta_{13})U(\theta_{12}, \delta_{12})$$
$$U = U(\theta_{34}, 0)U(\theta_{24}, 0)U(\theta_{23}, \delta_{23})U(\theta_{14}, 0)U(\theta_{13}, \delta_{13})U(\theta_{12}, \delta_{12})$$

for the 3 + 1 four–family neutrino flavour mixing. The 4 × 4 matrices $U(\theta_{ij}, \delta_k)$ are constructed in analogy with 3 × 3 matrices of the three–family neutrino flavour mixing [31]–[52], where the phases $\delta_{ij}$ are responsible for CP violation. Indeed, it is well–known that for neutrino flavour mass–eigenstates, treated as Dirac particles, a $N_\nu \times N_\nu$ neutrino
flavour mixing matrix is defined by $N_\nu(N_\nu - 1)/2$ independent mixing angles $\theta_{ij}$ and $(N_\nu - 1)(N_\nu - 2)/2$ independent phases, violating CP invariance [33].

Since the time modulated term of the EC decay rate is defined by the interference of the neutrino flavour mass–eigenstates $|\nu_1\rangle$ and $|\nu_2\rangle$, neglecting the contributions of the mixing angles $\theta_{13}$ and $\theta_{12}$ for $i = 1, 2, 3$ and $j = 4$ [34], the matrix elements $U_{ij}$ of the mixing matrices for the schemes $3+1$ and $2+2$ are equal to $U_{ij} = (\cos \theta_{ij}, \sin \theta_{ij} e^{-i\phi_{ij}}, 0, 0)$, where $\theta_{12}$ is a mixing angle between the neutrino flavour mass–eigenstates $|\nu_1\rangle$ and $|\nu_2\rangle$ and $\phi_{12}$ is a CP violating phase of the wave function of the neutrino flavour mass–eigenstate $|\nu_2\rangle$. It is defined relative to the phases of the wave functions of the neutrino flavour mass–eigenstates $|\nu_{ij}\rangle$ ($j = 1, 3, 4$). Following then the procedure, expounded in [5–7], we obtain the EC decay rate of the H–like heavy ions as a function of time, defined in the laboratory frame (see also section V)

$$\lambda_{EC}(t) = \lambda_{EC}(1 + a \cos(\omega t - \delta_{12})), \quad (6)$$

where $\lambda_{EC}$ and $\omega$ are the EC decay constant and frequency of the time modulation in the laboratory frame, related to the EC decay constant $\lambda_{EC}(r,f.)$ and frequency $\omega_{21}$ of the time modulation in the rest frame of parent ions as $\lambda_{EC} = \lambda_{EC}(r,f.)/\gamma$ and $\omega = \omega_{21}/\gamma$, and $\gamma = 1.43$ is the Lorentz factor [13]. From the comparison of Eq.(6) with Eq.(2) we define the phase–shift of a time modulation in terms of $\delta_{12}$. This gives $\phi = -\delta_{12}$.

It may be interesting to notice that one more possible signal for an existence of sterile neutrinos is a deficit of reactor antineutrinos at distances smaller than 100 m, observed in [35]. Recently such a deficit of reactor antineutrinos has been confirmed in [36] for the new world average value $\tau_r = 880.1(1.1)\text{s}$ of the neutron lifetime [37]. The theoretical lifetime of the neutron $\tau_n = 879.6(1.1)\text{s}$, agreeing well with the world average one $\tau_n = 880.1(1.1)\text{s}$, has been recently calculated in [37].

For further analysis of the experimental data on the GSI oscillations we propose to transcribe Eq.(4), defined in the laboratory frame, into the form

$$\lambda_{EC}(\tau) = \lambda_{EC}(1 + a \cos(\omega \tau - \Delta \phi)), \quad (7)$$

where $\tau = t - \pi/\omega = t - T/2$ and $\Delta \phi = \pi - \phi = \pi + \delta_{12}$. A change of a time dependence from $t$ to $\tau = t - T/2$, where $T/2 = 3.56(6)$s, takes into account that a velocity spread of the injected parent ions has been reduced from 1% up to $\Delta v/v \approx 5 \times 10^{-7}$ during first 3.5s after the injection into the ESR [13]. For the experimental values of the phase–shift $\Delta \phi$, i.e. $\Delta \phi = 0.79(16)$ rad and $\Delta \phi = 1.36(34)$ rad, we have retained 20 % and 25 % uncertainties of the phase–shifts $\phi = 2.35(48)$ rad and $\phi = 1.78(44)$ rad, measured with the number of consecutive measurements $N = 1688$ and $N = 844$, respectively. Since a phase $\delta_{12}$ is not measurable in the neutrino (antineutrino) lepton flavour oscillation experiments [31, 32], the GSI oscillations give an unprecedented possibility for an observation of such a phase in the EC decays of the H–like heavy ions.

IV. QUANTUM ZENO EFFECT AND DEPENDENCE OF PARAMETERS OF GSI OSCILLATIONS ON THE NUMBER OF CONSECUTIVE MEASUREMENTS

The experimental frequencies of the time modulation $\omega = 0.882(14)\text{s}$ and $\omega = 0.884(14)\text{s}$, measured during the same observation time $54$ s with the time resolutions $\delta t_{ad} = 32$ ms and $\delta t_{ad} = 64$ ms and $N = 1688$ and $N = 844$ consecutive measurements, respectively, do not show a dependence on the number of consecutive measurements $N$. In turn, the experimental data on the amplitude and phase–shift of the time modulation show, in principle, such a dependence, which we explain below as an influence of the Quantum Zeno Effect (QZE) [35, 38].

For a quantum field theoretic analysis of the QZE one has to investigate a behaviour of a survival probability of an unstable quantum state during short time intervals, which are much smaller than a total observation time. Since a short time evolution of unstable quantum systems differs from the exponential decay law [33, 34], according to the QZE, frequent consecutive measurements with short time intervals can prevent from an evolution of an unstable quantum state. Indeed, let $\Delta \tau$ be a time interval between two consecutive measurements such as $\tau = N \Delta \tau$, where $\tau$ is a total observation time and $N$ is the number of consecutive measurements. Suppose that a time interval between two consecutive measurements $\Delta \tau$ is small enough. In this case a survival probability of an evolution of an unstable quantum state during a time interval $\Delta \tau$ may be given by the form $\lambda_{EC}(1 + a \cos(\omega \tau - \Delta \phi))$.

$$P(\Delta \tau) = 1 - (\Delta \tau)^2/\tau_2^2, \quad (8)$$

where $\tau_2$ is the Zeno time [35, 38]. If a time interval $\Delta \tau$ is smaller compared with the Zeno time, i.e. $\tau_2 \gg \Delta \tau$, an evolution of an unstable quantum state can be significantly slowed down [17, 48]. In terms of a survival probability
this can be illustrated as follows. After \( N \) consecutive measurements a survival probability of an unstable quantum state can be defined by \[ P_N(\Delta \tau) = (1 - (\Delta \tau)^2/\tau_Z)^N = e^{-\tau^2/N\tau_Z^2}. \] In the limit \( N \to \infty \) a survival probability tends to unity, i.e. an unstable quantum state is left stable after continuous consecutive measurements, carried out during an observation time \( \tau \). This is the QZE.\[ \square \]

In the case of the GSI experiments on the EC decays of the H–like heavy ions the QZE can, in principle, i) lead to a delay of the EC decays, i.e. to a decrease of the EC decay constants, and ii) prevent from a time modulation of the EC decay rates, leading to a decrease of the amplitudes and a change of the phase–shifts.

For this aim we calculate the Zeno time \( \tau_Z \) or a time scale of an influence of frequent consecutive measurements on the lifetime of the H–like heavy ions, caused by the EC decays. Skipping intermediate calculations we give the result. For example, for the H–like heavy ions \(^{142}\)Pm\(^{60+}\) the Zeno time is equal to \( \tau_Z = \sqrt{3\pi/\lambda_{EC}Q_{EC}} = \sqrt{3\pi\gamma/\lambda_{EC}(r.f.)Q_{EC}} = 6.01 \times 10^{-10} \text{s} \), where \( Q_{EC} = 4.817 \text{MeV} \) and \( \lambda_{EC}(r.f.) = 0.0051(1) \text{s}^{-1} \) are the \( Q \)–value and the EC constant of the H–like heavy ions \(^{142}\)Pm\(^{60+}\). Thus, in order to observe a delay of the EC decays of the H–like heavy ions \(^{142}\)Pm\(^{60+}\), i.e. a decrease of the EC decay constant \( \lambda_{EC} \) as a consequence of frequent consecutive measurements, a time interval \( \Delta \tau \) between two consecutive measurements should be smaller compared to the Zeno time, i.e. \( \tau_Z \gg \Delta \tau \). Since measurements of the EC decays of the H–like heavy ions with time intervals \( \Delta \tau \ll \tau_Z = 6.01 \times 10^{-10} \text{s} \) are unreal, an influence of frequent consecutive measurements on the value of the EC decay constant of the H–like heavy ions \(^{142}\)Pm\(^{60+}\) can be neglected.

For time intervals \( \Delta \tau \), which are commensurable with time resolutions of the GSI experiments, we define a survival probability of parent ions during a time interval \( \Delta \tau = \tau/N \) as follows:\[ \square \]

\[
P(\Delta \tau) = 1 - \lambda(\Delta \tau)\Delta \tau.
\]

After \( N \) consecutive measurements a survival probability is given by

\[
P_N(\Delta \tau) = (1 - \lambda(\Delta \tau)\Delta \tau)^N = e^{-\lambda(\tau/N)\tau}.
\]

In the limit \( N \to \infty \) we arrive at the exponential decay law

\[
P(\tau) = \lim_{N \to \infty} P_N(\Delta \tau) = \lim_{N \to \infty} e^{-\lambda(\tau/N)\tau} = e^{-\lambda(0)\tau},
\]

where \( \lambda(0) = \lim_{N \to \infty} \lambda(\tau/N) \). In order to confirm our assertion that frequent consecutive measurements do not delay the EC decays of the H–like heavy ions we have to show that \( \lambda(0) = \lambda_{EC} \). However, it is important to notice that a linear dependence of the exponent of a survival probability Eq.\( 12 \) on the observation time \( \tau \), obtained in the limit of infinite number of consecutive measurements, implies a suppression of a time modulation.

In order to obtain a relation between \( \lambda(0) \) and the EC decay constant \( \lambda_{EC} \) we use Eq.\( 7 \). This gives

\[
\lambda(\Delta \tau) = \frac{1}{\Delta \tau} \int_0^{\Delta \tau} \lambda_{EC}(r.f.) dr = \lambda_{EC}(1 + a \cos(\Delta \phi)) + \Delta \tau \frac{1}{2} a \lambda_{EC} \omega \sin(\Delta \phi) = \\
\lambda_{EC}(1 + a \cos(\Delta \phi)) + \frac{\tau}{N} \frac{1}{2} a \lambda_{EC} \omega \sin(\Delta \phi). 
\]

Taking the limit \( N \to \infty \) we arrive at the relation

\[
\lambda(0) = \lambda_{EC}(1 + a \cos(\Delta \phi)).
\]

It is seen that the constant \( \lambda(0) \) coincides with the EC decay constant \( \lambda_{EC} \) if the amplitude of a time modulation \( a \) vanishes in the limit \( N \to \infty \).

In order to find the amplitude and phase–shift of a time modulation as functions of a number of consecutive measurements \( N \) we have to calculate the Zeno time or a time scale of an influence of frequent consecutive measurements on the amplitude and phase–shift of the GSI oscillations. For this aim we follow \[ \square \] and define the required Zeno time as follows

\[
\tau^a_Z = \sqrt{1/N'(\Delta \tau)}|_{\Delta \tau=0} = \sqrt{2/a} \lambda_{EC} \omega \sin(\Delta \phi) = \sqrt{2\gamma/a} \lambda_{EC}(r.f.) \omega \sin(\Delta \phi),
\]

where \( \lambda'(\Delta \tau) \) is a derivative of \( \lambda(\Delta \tau) \) with respect to \( \Delta \tau \). Using the experimental data \( \lambda_{EC}(r.f.) = 0.0051(1) \text{s}^{-1} \)\[ \square \] and \( \omega = 0.884(14) \text{s}^{-1} \) (or \( \omega = 0.882(14) \text{s}^{-1} \)) and \( \gamma = 1.43 \) we calculate the Zeno time

\[
\tau^a_Z = 25.20(32)/\sqrt{a} \sin(\Delta \phi) s.
\]
This is a time scale of an influence of frequent consecutive measurements on the amplitude and phase–shift of the GSI oscillations. For the experimental values of the amplitudes $a_1 = 0.107(24)$ and $a_2 = 0.134(27)$ and the phase–shifts $\Delta \phi_1 = 0.79(16) \text{ rad}$ and $\Delta \phi_2 = 1.36(34) \text{ rad}$, measured during the same observation time 54 s with the time resolutions $\delta t_d = 32 \text{ ms}$ and $\delta t_d = 64 \text{ ms}$, respectively, we calculate the Zeno times $\tau_Z^{(a_1)} = 82(10) \text{ s}$ and $\tau_Z^{(a_2)} = 78(10) \text{ s}$, respectively. One may see that the Zeno times are much larger than the time resolutions $\delta t_d$, and commensurable with the observation time 54 s. This implies a strong influence of the QZE on the values of the amplitude and phase–shift of a time modulation.

As a first step to the definition of the Zeno time, the amplitude and the phase–shift of the GSI oscillations as a function of $N$ we set

$$\tau_Z^{(a)} = 13.6(1.1) N^{1/4} \text{ s}. \quad (17)$$

For $N = 1688$ and $N = 844$ we obtain $\tau_Z^{(a)} = 87(7) \text{ s}$ and $\tau_Z^{(a)} = 73(6) \text{ s}$, which agree well with the values $\tau_Z^{(a_1)} = 82(10) \text{ s}$ and $\tau_Z^{(a_2)} = 78(10) \text{ s}$.

Substituting Eq. (17) into Eq. (16) we define the product $a \sin(\Delta \phi)$ as a function of the number of consecutive measurements

$$a \sin(\Delta \phi) = 3.43(56)/\sqrt{N}. \quad (18)$$

For $N = 1688$ and $N = 844$ we get $a \sin(\Delta \phi) = 0.083(14)$ and $a \sin(\Delta \phi) = 0.118(19)$, which agree well with the experimental values $a \sin(\Delta \phi) = 0.076(21)$ and $a \sin(\Delta \phi) = 0.131(28)$, respectively. Using the experimental data on the phase–shifts and averaging over two experimental values we obtain the amplitude of a time modulation as a function of $N$

$$a = 4.17(64)/\sqrt{N}. \quad (19)$$

The amplitudes of a time modulation $a = 0.102(16)$ and $a = 0.144(22)$, calculated for $N = 1688$ and $N = 844$, agree well with the experimental values $a = 0.107(24)$ and $a = 0.134(27)$, respectively. For the phase–shift $\Delta \phi$ we find the following dependence on the number of consecutive measurements

$$\Delta \phi = 2\pi \times 47.5(7.7)/N^{4/5}. \quad (20)$$

For $N = 1688$ and $N = 844$ the function Eq. (20) gives $\Delta \phi = 0.78(13) \text{ rad}$ and $\Delta \phi = 1.36(22) \text{ rad}$, which fit well the experimental data.

In the limit $N \to \infty$ the amplitude $a$ and phase–shift $\Delta \phi$ vanish, giving $\phi \to \pi$. This suppresses a time modulation of the EC decay rates and corroborates the equality $\lambda(0) = \lambda_{\text{EC}}$, implying that the QZE does not affect the lifetime of the $\text{H}$–like heavy ions.

Using the functions Eq. (19) and Eq. (20) we may correct the dependence of the Zeno time $\tau_Z^{(a)}$ on the number of consecutive measurements. For sufficiently large $N$ in comparison to $N = 1688$ and $N = 844$ we obtain

$$\tau_Z^{(a)} = 0.72(9) N^{0.65} \text{ s}. \quad (21)$$

In turn, for $N = 1688$ and $N = 844$ we get $\tau_Z^{(a_1)} = 90(10) \text{ s}$ and $\tau_Z^{(a_2)} = 60(8) \text{ s}$, which do not contradict to the values $\tau_Z^{(a_1)} = 82(10) \text{ s}$ and $\tau_Z^{(a_2)} = 78(10) \text{ s}$, respectively.

Finally we would like to notice that the amplitude and phase–shift of a time modulation, obtained with the number of consecutive measurements $N = 1688$ and $N = 844$, agree within one standard deviation. In spite of this fact we assume that they are affected by the QZE and define them as some functions of the number of consecutive measurements $N$, vanishing in the limit of the infinite number of consecutive measurements and suppressing a time modulation as it is required by the QZE.

V. QUANTUM FIELD THEORETIC ANALYSIS OF GSI OSCILLATIONS

A description of the electron neutrino $|\nu_e\rangle$ as a superposition of the neutrino flavour mass–eigenstates $|\nu_j\rangle$ with masses $m_j$, i.e. $|\nu_e\rangle = \sum_{j=1}^{N_{\nu}} U^*_{ej} |\nu_j\rangle$, where $N_{\nu}$ is the number of neutrino flavours, implies an existence of $N_{\nu}$ decay channels $p \to d + \nu_j$ in the EC decay $p \to d + \nu_e$. An indistinguishability of the decay channels $p \to d + \nu_j$ ($j = 1, 2, \ldots, N_{\nu}$) in the EC decay $p \to d + \nu_e$, which is the necessary condition of an interference of the neutrino flavour mass–eigenstates $|\nu_j\rangle$, requires an overlap of them. An overlap between the decay channels $p \to d + \nu_j$
(j = 1, 2, ..., Nν) may occur only if energies and 3–momenta of the daughter ions are smeared with energy δEδ and 3–momentum |δqδ| uncertainties, which are larger compared with the differences of energies and 3–momenta of the neutrino flavour mass–eigenstates and daughter ions, produced in two decay channels p → d + νj and p → d + νj. Since energy δEδ and 3–momentum |δqδ| uncertainties lead to a violation of energy–momentum conservation in the decay channels p → d + νj (j = 1, 2, ..., Nν) with accuracies δEδ and |δqδ|, a non–conservation of energy and 3–momentum in the decay channels p → d + νj (j = 1, 2, ..., Nν) is the sufficient condition of the appearance of an interference between the neutrino flavour mass–eigenstates |νj⟩ in the rate of the EC decay p → d + νe. In other words, a violation of energy–momentum conservation should guarantee that an overlap of the decay channels p → d + νj (j = 1, 2, ..., Nν), as the necessary condition of an interference of the neutrino flavour mass–eigenstates, may be fulfilled. Of course, energy δEδ and 3–momentum |δqδ| uncertainties should not violate the Fermi Golden Rule. Such a requirement is fulfilled if δEδ and |δqδ| obey the constraints δEδ ≪ Td = QEC/2Mδ and |δqδ| ≪ QEC, where QEC is the Q–value of the EC decay and Td = QEC/2Mqδ is a kinetic energy of the daughter ions δqδ, as has been shown in [8]. A required violation of energy and momentum in the EC decays of the GSI experiments occurs due to interactions of ions with the measuring apparatus, i.e. the resonant and capacitive pickups in the ESR.

Since the origin of energy δEδ and 3–momentum |δqδ| uncertainties in the GSI experiments is the detection of the daughter ions with a time resolution δtd, we have to check that the uncertainties δEδ and |δqδ| satisfy the constraints necessary for the appearance of the interference of the neutrino flavour mass–eigenstates. For quantitative analysis of the required overlap of the decay channels p → d + νj (j = 1, 2, ..., Nν) and violation of energy–momentum conservation we use i) energy and 3–momentum differences of the neutrino flavour mass–eigenstates and daughter ions from two decay channels p → d + νj and p → d + νj, defined by ωij = Eν(δj) − Eν(δj) − Eδ(δj) − Eδ(δj) and |δ⃗q| = |q̃j − q̃j|, where (Eν(δj), q̃j) and (Eδ(δj), q̃j) are the energies and 3–momenta of the neutrino flavour mass–eigenstate |νj⟩ and the daughter ion j in the decay channel p → d + νj, and ii) energy δEδ and 3–momentum |δqδ| uncertainties, induced by the detection of the daughter ions with a time resolution δtd. If δEδ ≫ ωij = Eν(δj) − Eν(δj) − Eδ(δj) − Eδ(δj) and |δqδ| ≫ |δ⃗q| = |q̃j − q̃j| the decay channels p → d + νj (j = 1, 2, ..., Nν) in the EC decay p → d + νj should be indistinguishable. Such an indistinguishability leads to the interference between the decay channels without violation of the Fermi Golden Rule if δEδ ≪ Td = QEC/2Mδ and |δqδ| ≪ QEC. This is the basis of the mechanism of the GSI oscillations, caused by an interference of the neutrino flavour mass–eigenstates [8, 9, 10].

In the rest frame of parent ions the kinematics of particles in the EC decay channels p → d + νj (j = 1, 2, ..., Nν) is given by the relations: kν(Mν, 0) = kν(Eν(δj), 0) + kν(j, 0), and νj are not detected, they move away from the origin of energy and 3–momentum conservation of the decay channels p → d + νj. Since the neutrino flavour mass–eigenstates |νj⟩ are not detected, they move away from the origin of energy and 3–momentum conservation of the decay channels p → d + νj. In principle, because of energy and 3–momentum conservation the daughter ions, produced in the decay channel p → d + νj, should go away with energies Eν(δj) and 3–momenta q̃j = −⃗k. If it is so, one is able to distinguish a decay channel p → d + νj from a decay channel p → d + νj with i ≠ j. In this case there are no interferences between decay channels p → d + νj and p → d + νj and, correspondingly, a time modulation of the EC decay rate or of the rate of the number of daughter ions, caused by the EC decays. However, as has been pointed out in [5] and shown in [8] such a kinematics is not the case for the GSI experiments.

Indeed, because of the detection with a time resolution δtd energies and 3–momenta of daughter ions, produced in the decay channels p → d + νj (j = 1, 2, ..., Nν), are smeared with δEδ ∼ 2π/δtd and |δqδ| ∼ (Mδ/QEC)δEδ [8] [10]. For example, for the EC decays of the H–like heavy ions 142Pm60+ we get δEδ ∼ 2π/δtd ∼ 10−13 eV and |δqδ| ∼ (Mδ/QEC)δEδ ∼ 10−9 eV, where QEC = 4.817 MeV is the Q–value of the EC decay 142Pm60+ → 142Nd60++νe. Mδ is the daughter ion mass and δtd = 32 ms and δtd = 64 ms are the time resolutions of the GSI experiments [12].

For the experimental value of the time modulation period T ≃ 7 s the differences of energies of the neutrino mass–eigenstates and of the daughter ions of the decay channels p → d + νj and p → d + νj are of order ωij ∼ 2π/T ∼ 10−15 eV. In turn, the differences of 3–momenta of the neutrino flavour mass–eigenstates and of the daughter ions of the decay channels p → d + νj and p → d + νj are of order |q̃j − q̃j| ∼ (Mδ/QEC)(2π/T) ∼ 10−11 eV. Since δEδ ≫ ωij and |δqδ| ≫ |q̃j − q̃j| in the GSI experiments on the EC decays of the H–like heavy ions i) energy–momentum conservation is violated with accuracies δEδ ∼ 10−13 eV and |δqδ| ∼ 10−9 eV, respectively, and ii) the decay channels
with

\[ \vec{q} \]

\[ e \]

\[ p \to d + \nu_1 \] and \[ p \to d + \nu_2 \] are indistinguishable. This means that the daughter ions, produced in the decay channels \[ p \to d + \nu_j \ (j = 1, 2, \ldots, N_p) \], are not detected with 3–momenta \[ \vec{q}_j \] and energies \[ E_d(\vec{q}_j) \] but they are detected with a 3–momentum \[ \vec{q} \] and an energy \[ E_d(\vec{q}) \], obeying the constraints \[ |\delta \vec{q}_j| \gg |\vec{q} - \vec{q}_j| \] and \[ |\delta E_d| \gg |E_d(\vec{q}_j) - E_d(\vec{q})| \]. Thus, in the GSI experiments on the EC decays of the H–like heavy ions the necessary and sufficient conditions for the interference of the neutrino flavour mass–eigenstates are fulfilled and one may expect the appearance of the time modulation of the EC decay rates [34].

As has been shown in [33], the amplitude of the EC decay \[ p \to d + \nu_2 \] calculated with the \( \varepsilon \)–regularisation in the rest frame of parent ions within a time dependent perturbation theory [51], takes the form

\[
A(p \to d \nu_2)(t) = -\delta M_p, -\frac{1}{2} 2 \sqrt{3} \sqrt{M_p E_d(\vec{q})} |\mathcal{M}_{GT}| \sum_j U_{ej} \sqrt{E_j(\vec{k}_j)} e^{i(\Delta E_j - i\varepsilon)t} \frac{2\pi}{\Delta E_j - i\varepsilon} \Phi_d(\vec{k}_j + \vec{q}),
\]

(24)

where \( \Delta E_j = E_d(\vec{q}) + E_j(\vec{k}_j) - M_p \) is the difference of energies of the final and initial state in the decay channel \( p \to d + \nu_j \ (j = 1, 2, \ldots, N_p) \) with a daughter ion, detected with a time resolution \( \delta t \) and described by the wave function \( \Phi_d(\vec{k}_j + \vec{q}) \), taken in the form of the wave packet and localised around \( \vec{k}_j + \vec{q} \approx 0 \) with an accuracy of about \( |\delta \vec{q}_j| \sim 10^{-9} \text{eV} \). Then, \( |\mathcal{M}_{GT}| \) is a nuclear matrix element of the Gamow–Teller transition \( ^{142}\text{Pm}^{60+} \to ^{142}\text{Nd}^{60+} \) and \( \langle \psi(\vec{k}_j) \rangle \) is an average value of the Dirac wave function of the electron in the ground state of the H–like parent ion \( ^{142}\text{Pm}^{60+} \) [49, 52]. The Kronecker symbol \( \delta M_p, -\frac{1}{2} \) implies that in the rest frame and with the spin quantisation axis anti–parallel to the neutrino 3–momentum parent ions are unstable under the EC decay in the hyperfine state \( 1s_{F,M_F} \) with \( F = 1/2 \) and \( M_F = -1/2 \) only. The EC decay probability per unit time is [6]

\[
P(p \to d \nu_2)(t) = \frac{d}{dt} |A(p \to d \nu_2)(t)|^2,
\]

(25)

where we have denoted [5]

\[
d \frac{d}{dt} |A(p \to d \nu_2)(t)|^2 = \lim_{\varepsilon \to 0} \frac{d}{dt} \frac{1}{2} \sum_{M_F} |A(p \to d \nu_2)(t)|^2 = 6M_p E_d(\vec{q}) |\mathcal{M}_{GT}|^2 \left\{ \sum_j |U_{ej}|^2 E_j(\vec{k}_j) 2\pi \delta(\Delta E_j) \times |\Phi_d(\vec{k}_j + \vec{q})|^2 + \sum_{j' > j} |U_{ej}||U_{ej'}| \sqrt{E_j(\vec{k}_j)E_{j'}(\vec{k}_{j'})} |\Phi_d(\vec{k}_j + \vec{q})||\Phi_d(\vec{k}_{j'} + \vec{q})| \left( 2\pi \delta(\Delta E_{j'}) + 2\pi \delta(\Delta E_{j}) \right) \cos(\omega_{j'j} t + \phi_{j'j}) \right\}
\]

(26)

with \( \omega_{j'j} = \Delta m_{2j'}^2 / 2M_p \) and \( \phi_{j'j} = \arg U_{ej'} - \arg U_{ej} \). We would like to note that before we take the limit \( \varepsilon \to 0 \) we obtain the time modulated terms proportional to \( \cos((\Delta E_j - \Delta E_{j'})t + \phi_{j'j}) = \cos((E_j(\vec{k}_j) - E_{j'}(\vec{k}_{j'}))t + \phi_{j'j}) \). After the use of Eq. (26) the time modulated terms become proportional to \( \cos(\omega_{j'j} t + \phi_{j'j}) \). Then, taking the limit \( \varepsilon \to 0 \) we arrive at Eq. (26).

It is well–known that the matrix elements of the mixing matrix may undergo phase transformations \( U_{ej} \to e^{-i\beta_j} U_{ej} e^{+i\alpha_j} \). It is obvious that the observables should be invariant under such transformations [53]. Since the decay rate Eq. (26) is an observable quantity, it should be invariant under phase transformations of the matrix elements of the mixing matrix. One may see that the time independent term of Eq. (26) is invariant under the phase transformations \( U_{ej} \to e^{-i\beta_j} U_{ej} e^{+i\alpha_j} \). In order to show that the time modulated term is also invariant quantity we propose to rewrite it as follows

\[
\sum_{j' > j} |U_{ej}||U_{ej'}| \sqrt{E_j(\vec{k}_j)E_{j'}(\vec{k}_{j'})} |\Phi_d(\vec{k}_j + \vec{q})||\Phi_d(\vec{k}_{j'} + \vec{q})| \left( 2\pi \delta(\Delta E_{j'}) + 2\pi \delta(\Delta E_{j}) \right) \cos(\omega_{j'j} t + \phi_{j'j}) = \sum_{j' > j} \text{Re} \left( U_{ej} U^*_{ej'} \Phi_d^*(\vec{k}_j + \vec{q'}) \Phi_d^*(\vec{k}_{j'} + \vec{q'}') e^{i\omega_{j'j} t} \right) \left( 2\pi \delta(\Delta E_{j'}) + 2\pi \delta(\Delta E_{j}) \right).
\]

(27)

Making the phase transformations of the matrix elements of the mixing matrix \( U_{ej} \to e^{-i\beta_j} U_{ej} e^{+i\alpha_j} \) and \( U_{ej} \to e^{+i\beta_j} U^*_j e^{-i\alpha_j} \) and the phase transformations of the wave functions of the daughter ions \( \Phi_d(\vec{k}_j + \vec{q}) \to \Phi_d(\vec{k}_j + \vec{q}) \) and \( \Phi_d^*(\vec{k}_{j'} + \vec{q'}) \to \Phi_d^*(\vec{k}_{j'} + \vec{q'}) \) we leave the time modulated term unchanged.

For the calculation of the EC decay rate we have to integrate \( P(p \to d \nu_2) \) over the phase–volume of the final states of the decays \( p \to d + \nu_j \ (j = 1, 2, \ldots, N_p) \). For this aim we set zero masses of the neutrino flavour mass–eigenstates everywhere in comparison to the \( Q \)–value of the EC decay, i.e. \( E_i(\vec{k}_i) \approx |\vec{k}_i| \) and \( E_j(\vec{k}_j) \approx |\vec{k}_j| \). Then, due to the
the influence of the QZE on the delay of the EC decays (or on the EC decay constant 
Such a dependence we have explained assuming an influence of the Quantu
m Zeno Effect (QZE). We have shown that
the time modulation of the rates of the number of daughter ions from the
EC decays as the interference of the neutrino
The mechanism of GSI oscillations, caused by the interference of the neutrino flavour mass–
eigenstates $|\nu_j\rangle$, allows to
explain the phase–shift of the time modulation by means of an extension of the number of neutrino flavours from
$N_{\nu} = 3$ to $N_{\nu} \geq 4$, assuming an existence of so–called sterile neutrinos. For an illustration we have considered the
case with $N_{\nu} = 4$. We have shown that the known schemes of an inclusion of sterile neutrinos, i.e. (3 + 1) and
(2 + 2) schemes, lead to an appearance of a phase–shift $\phi = -\delta t_d$ of a phase–shift of the time modulation of the EC decays of the H–like heavy
ions, where $\delta t_d$ is a phase of the wave function of the neutrino flavour mass–eigenstate $|\nu_d\rangle$, defined relative to the phases of the wave functions of other neutrino flavour mass–eigenstates $|\nu_j\rangle$ ($j = 1, 2, \ldots, N_{\nu}$). The important
property of $\delta t_d$ is to violate CP invariance. It is remarkable that in the considered schemes of an inclusion of sterile neutrinos a phase $\delta t_d$ does not appear in the probabilities of the neutrino lepton flavour oscillations $\nu_0 \leftrightarrow \nu_3$. This means that the experiments on the GSI oscillations give unprecedented possibilities for an observation of such a phase, implying also an extension of the number of neutrino flavour mass–eigenstates from $N_{\nu} = 3$ to $N_{\nu} \geq 4$. We would like also to note that a so–called reactor antineutrino flux anomaly [5] (see also [5]) may serve as one more hint on a low–energy confirmation of an existence of sterile neutrinos. For recent analysis of sterile neutrinos and estimates of $\Delta m^2_{ij}$ we refer to the paper by Martini et al. [5].
We have given a quantum field theoretic derivation of the EC decay rate of the H–like heavy ions with the time modulation, published in [32]. We have discussed in more detail the necessary and sufficient conditions for the appearance of the interference of the neutrino flavour mass–eigenstates. We have accentuated that the necessary and sufficient conditions of this effect are related to i) an overlap of the decay channels \( p \to d + \nu_{j} (j = 1, 2, \ldots, N_{\nu}) \) in the EC decay \( p \to d + \nu_{e} \) and ii) a violation of energy–momentum conservation in the decay channels \( p \to d + \nu_{j} (j = 1, 2, \ldots, N_{\nu}) \), caused by a detection of the daughter ions with a time resolution \( \delta t_{\text{d}} \). Such a detection introduces energy \( \delta E_{\text{d}} \sim 2\pi/\delta t_{\text{d}} \sim 10^{-13} \text{eV} \) and 3–momentum \( |\delta \vec{q}_{\nu}| \sim (M_{d}/E_{\text{d}}) \delta E_{\text{d}} \sim 10^{-9} \text{eV} \) uncertainties, which are larger compared with the differences of energies \( \delta E_{\text{d}} \gg \omega_{j} \sim 2\pi/T \sim 10^{-15} \text{eV} \) and of 3–momenta \( |\delta \vec{q}_{\nu}| \gg |\vec{k}_{j} - \vec{k}_{l}| \gg |\vec{q}_{e} - \vec{q}_{\nu}| \sim 10^{-11} \text{eV} \) of the neutrino flavour mass–eigenstates and of the daughter ions, produced in the decay channels \( p \to d + \nu_{j} \) and \( p \to d + \nu_{e} \). As result, the daughter ions are detected with an average 3–momentum \( \vec{q}_{\nu} \) and an energy \( E_{\text{d}}(\vec{q}_{\nu}) \). This makes the decay channels \( p \to d + \nu_{j} \) and \( p \to d + \nu_{e} \) experimentally indistinguishable in the EC decay \( p \to d + \nu_{e} \).

According to quantum mechanical principle of superposition [51], indistinguishability of the decay channels \( p \to d + \nu_{j} (j = 1, 2, \ldots, N_{\nu}) \) allows to calculate the amplitude of the EC decay \( p \to d + \nu_{e} \) as a superposition of the amplitudes of the decays \( p \to d + \nu_{j} (j = 1, 2, \ldots, N_{\nu}) \), multiplied by the matrix elements \( U_{\nu_{j}} \) of the mixed matrix \( \mathcal{M}_{\nu} \), where \( \sum_{j} U_{\nu_{j}}^{*} U_{\nu_{j}} = 1 \). The time modulated EC decay rates can be calculated setting \( E_{\nu_{j}}(\vec{k}_{j}) \approx E_{\nu_{e}}(\vec{k}_{e}) \) as dynamical masses \( m_{\nu_{j}} \) of the neutrino mass–eigenstates are much smaller than the \( Q \)–values of the EC decays, i.e. \( Q_{\text{EC}} \gg m_{\nu_{j}} \). Because of fulfilment of inequalities \( |\delta \vec{q}_{\nu}| \gg |\vec{k}_{j} - \vec{k}_{l}| \gg |\vec{q}_{e} - \vec{q}_{\nu}| \), \( \delta E_{\text{d}} \gg |E_{\nu_{j}}(\vec{k}_{j}) - E_{\nu_{e}}| \) and \( \delta E_{\text{d}} \gg |M_{\nu_{j}} - M_{\nu_{e}}(\vec{q}_{\nu}) - E_{\nu_{e}}| \) any deviations from the Fermi Golden Rule cannot be practically observable experimentally.

We would like to emphasise that the authors [27–29], criticising the mechanism of the time modulation of the EC decay rates, caused by the interference of the neutrino flavour mass–eigenstates, did not take into account that fact that parent and daughter ions interact with the measuring apparatus, i.e. the resonant and capacitive pickups in the ESR, and such interactions lead to violation of energy and momentum in the EC decays in the GSI experiments. Assuming energy–momentum conservation in the decay channels \( p \to d + \nu_{j} (j = 1, 2, \ldots, N_{\nu}) \) they came to the conclusion that the mechanism of the interference of the neutrino flavour mass–eigenstates cannot be used for the explanation of the GSI oscillations. This is not a surprise, since dealing with kinematics \( \vec{q}_{\nu} = -\vec{k}_{e} \) and \( M_{\nu_{j}} = E_{\nu_{e}}(\vec{k}_{e}) + E_{\nu_{j}}(\vec{k}_{j}) \) for the decay channels \( p \to d + \nu_{j} (j = 1, 2, \ldots, N_{\nu}) \), one is doomed to show the absence of the time modulation, caused by the interference of the neutrino flavour mass–eigenstates \( |\nu_{j}\rangle \) for the EC decay \( p \to d + \nu_{e} \).

We would like also to mention our attention to the orthogonality of the wave functions of the final states \( |d\nu_{e}| |d\nu_{j}\rangle \sim \delta\nu_{ij} \) in the decay channels \( p \to d + \nu_{j} \) and \( p \to d + \nu_{e} \), has no influence on the suppression of the time modulation as the interference of neutrino flavour mass–eigenstates \( |\nu_{j}\rangle \) for the EC decay \( p \to d + \nu_{e} \).

For the further verification of the \( A \)–scaling of the periods of the time modulation of the H–like heavy ions in the GSI experiments and the estimate of masses of neutrino (antineutrino) mass–eigenstates we propose to measure the weak decays of the H–like \(^{108}\text{Ag}\) \(^{46+}\). In the ground state the odd–odd nucleus \(^{108}\text{Ag}\) \(^{47+}\) with quantum numbers \( I^{\pi} = 1^{-} \) has the unique feature of decaying as neutral atom with a half-life time of \( T_{1/2} = 2.37(1) \text{m} \) [52], with the three–body and two–body decay channels 97.15 % and 2.85 % [52], respectively. The H–like ions \(^{108}\text{Ag}\) \(^{46+}\) are unstable under i) the EC decay \(^{108}\text{Ag}\) \(^{46+}\) \( \to^{108}\text{Pd}\) \(^{46+}\) \( + \nu_{e} \), ii) the bound–state \( \beta^{-} \)–decay \(^{108}\text{Ag}\) \(^{46+}\) \( \to^{108}\text{Cd}\) \(^{46+}\) \( + \bar{\nu}_{e} \) and iii) the \( \beta^{-} \)–decay \(^{108}\text{Ag}\) \(^{46+}\) \( \to^{108}\text{Cd}\) \(^{47+}\) \( + e^{-} + \bar{\nu}_{e} \). Storing H–like \(^{108}\text{Ag}\) \(^{46+}\) ions in the ESR of heavy ions one may study then a time modulation of the \( EC \) and bound-state \( \beta^{-} \)–decays from the same parent H–like ion and thus compare the properties of mixed massive neutrino and antineutrino flavour mass–eigenstates, emitted in these decays, respectively, directly in a CPT type of test.

Following [12] we predict that the rates of the \( \beta^{-} \)–decay \(^{108}\text{Ag}\) \(^{46+}\) \( \to^{108}\text{Cd}\) \(^{47+}\) \( + e^{-} + \bar{\nu}_{e} \) should not show a time modulation, whereas the EC and bound-state \( \beta^{-} \)–decay rates should have a periodic time dependence with periods \( T_{\text{EC}} \) and \( T_{\beta_{sb}} \), respectively. Following [5, 8, 11] we predict that for H–like ions \(^{108}\text{Ag}\) \(^{46+}\), moving with the Lorentz factor \( \gamma = 1.43 \), the period of the time modulation of the EC decay rate \(^{108}\text{Ag}\) \(^{46+}\) \( \to^{108}\text{Pd}\) \(^{46+}\) \( + \nu_{e} \) should be equal to

\[
T_{\text{EC}} = \frac{4\pi\gamma M_{p}}{(\Delta m_{21})_{\text{GSI}}} \approx \frac{A}{20} = 5.4 \text{s}, \tag{29}
\]

where \( (\Delta m_{21})_{\text{GSI}} = (m_{2} + \delta m_{2})^{2} - (m_{1} + \delta m_{1})^{2} \). Here \( m_{2} \) and \( m_{1} \) are bare neutrino masses and \( \delta m_{2} \) and \( \delta m_{1} \) are the mass–corrections, caused by polarisation \( \nu_{ij} \to \sum_{j} \epsilon^{-}\vec{W}^{+} \to \nu_{j} \) of the neutrino flavour mass–eigenstates \( \nu_{j} \) in the strong Coulomb field of the daughter ion \(^{108}\text{Pd}\) \(^{46+}\) (see Table I) [5, 8, 11]. For the bound–state \( \beta^{-} \)–decay \(^{108}\text{Ag}\) \(^{46+}\) \( \to^{108}\text{Cd}\) \(^{46+}\) \( + \bar{\nu}_{e} \) we predict the period of the decay rate modulation equal to

\[
T_{\beta_{sb}} = \frac{4\pi\gamma M_{p}}{(\Delta m_{21})_{\text{GSI}}} = \frac{(\Delta m_{21})_{\text{GSI}}}{\langle \Delta m_{21}^2 \rangle_{\text{GSI}}} T_{\text{EC}}, \tag{30}
\]
where \((\Delta m^2_{21})_{\text{GSI}} = (m_2 + \delta m_2)^2 - (m_1 + \delta m_1)^2\) is the difference of the squared dynamical masses of the antineutrino flavour mass–eigenstates, emitted in the bound–state \(\beta^–\text{decay} \; ^{108}\text{Ag}^{46+} \rightarrow ^{108}\text{Cd}^{47+} + e^– + \bar{\nu}_e\) as constituents of the electron antineutrino \(\bar{\nu}_e = \sum_{j=1} U_{ej} |\bar{\nu}_j\rangle\). Then, \(m_j\) for \(j = 1, 2\) are bare masses of the antineutrino flavour mass–eigenstates \(|\bar{\nu}_j\rangle\). In case of CPT invariance we may set \(m_j = m_j\), for \(j = 1, 2\). The mass–corrections \(\delta m_j\) for \(j = 1, 2\), caused by polarisations of the antineutrino flavour mass–eigenstates \(\bar{\nu}_j \rightarrow \sum_{l} f W^– \rightarrow \bar{\nu}_j\) in the strong Coulomb field of the daughter ions \(^{108}\text{Cd}^{47+}\), are given in Table I.

| \(10^7 \Delta m_2/\delta m_2\) | \(10^7 \Delta m_2/\delta m_2\) | \(10^7 \Delta m_2/\delta m_2\) |
|-----------------|-----------------|-----------------|
| \(^{108}\text{Cd}^{47+}/10^8\text{Pd}^{46+}\) | +8.055/ -7.253 | +4.589/ -4.136 |

TABLE I: Numerical values of mass–corrections for the antineutrino and neutrino flavour masses–eigenstates in the strong Coulomb field of daughter nuclei \(^{108}\text{Cd}^{47+}\) and \(^{108}\text{Pd}^{46+}\), calculated for \(R = 1.1 \times A^{1/3}\). [5, 11].

Using the masses \(m_1 = 0.01345\) eV and \(m_2 = 0.01991\) eV of the neutrino flavour mass–eigenstates, calculated from the experimental data on the GSI oscillations of the H–like \(^{142}\text{Pr}^{60+}\), \(^{140}\text{Pt}^{59+}\) and \(^{122}\text{I}^{52+}\) heavy ions, for \((\Delta m^2_{21})_{\text{GSI}}\) in the period of the EC decay rate of \(^{108}\text{Ag}^{46+}\) we obtain \((\Delta m^2_{21})_{\text{GSI}} = 2.18 \times 10^{-4}\) eV\(^2\). This agrees well with the values \((\Delta m^2_{21})_{\text{GSI}}\), extracted from the periods of the EC decays of \(^{142}\text{Pr}^{60+}\), \(^{140}\text{Pt}^{59+}\) and \(^{122}\text{I}^{52+}\) heavy ions [5]. In turn, for \((\Delta m^2_{21})_{\text{GSI}}\) we get \((\Delta m^2_{21})_{\text{GSI}} = 2.12 \times 10^{-4}\) eV\(^2\). This gives the modulation period of the bound–state \(\beta^–\text{decay}\) rate equal to \(T_{\beta_h} \approx 5.6\) s. A deviation from \(T_{\beta_h} \approx 5.6\) s should testify a violation of CPT invariance. The frequencies of the time modulation of the EC and bound–state \(\beta^–\text{decay}\) rates are \(\omega_{\text{EC}} = 2\pi/T_{\text{EC}} = 1.164\) rad/s and \(\omega_{\beta_h} = 2\pi/T_{\beta_h} = 1.122\) rad/s, respectively.

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