A Wilson loop for off-shell amplitudes

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Abstract

We introduce a Wilson loop prototype that encodes off-shell gluon scattering amplitudes in $D$ space-time dimensions. We calculate it at one-loop order and provide formulas that interpolate between the off-shell 4-dimensional kinematics to on-shell dimensionally regularized case. Agreement is observed with known gluon scattering amplitudes in respective limits. We address differences between the off/on-shell Sudakov behavior of the Wilson loop and its relation to the octagon/cusp anomalous dimensions.
1 Introduction

On-shell scattering amplitudes occupy the central stage of perturbative QCD studies over several decades [1] since its inception. The past fifteen years have witnessed an explosion of activity of their studies within the framework of a distant QCD cousin, the planar maximally supersymmetric Yang-Mills theory. This is where the gauge/string correspondence [2] endows one with the power to probe all values of 't Hooft coupling ranging from weak to strong. A major breakthrough that led to this development was an observation of a duality between one-shell gluon scattering amplitudes and bosonic Wilson loops on a polygonal light-like contour, first demonstrated at strong coupling [3] and then at weak one-loop order [4, 5]. This reformulation of gluon amplitudes paved a way for the use of heavy machinery of integrable models to solve dynamics on the 2D surface stretched on the Wilson loop contour [6-8].

To date, limited attention has been paid to off-shell gauge amplitudes however. While these are not immediately gauge invariant, they play a crucial role in QCD studies. Of particular importance to phenomenological applications in hadron-hadron collisions is the so-called high-energy factorization. Here, contrary to more conventional collinear factorization, amputated legs of Green’s functions after the LSZ reduction are not set to their mass shell. Restoration of the gauge invariance can be accomplished by means of the Lipatov’s effective action [9], which was implemented in efficient codes [10]. In Ref. [11], the leading double logarithmic asymptotics of $2 \to 2$ off-shell scattering was related to their on-shell limit by studying infrared evolution equations. Yet another interest in off-shell results is driven by Berends-Giele recursion relations for currents with different number of off-shell gluon legs [12]. Gluon polarization is the above two circumstances are different, i.e., they are longitudinal versus transverse.

The goal of this note is to propose a prototypical bosonic Wilson loop that is dual to off-shell gluon scattering amplitudes in maximally supersymmetric Yang-Mills theory. The latter made their appearance in Ref. [13] which studied an intriguing equivalence of four-gluon off-shell amplitude to the light-cone limit of the so-called octagon [14], an object that corresponds to the square root of a four-point correlation functions of half-BPS operators with infinitely large $R$-charges. As we demonstrate below, the off-shellness requires the Wilson loop contour to reside in more than four dimensions. This motivates us to start from a 10$D$ holonomy and perform its dimensional reduction down to $D = 4 + 2\varepsilon$ with $\varepsilon > 0$. The four out of $D$ are the usual Minkowski coordinates $x_\mu$ while the rest will be used as auxiliary variables parametrized by $2\varepsilon$-component vectors $y^I$. Since vector spaces of non-integer dimensions do not exists, the orthogonal $2\varepsilon$ subspace is in fact has to be regarded as infinite dimensional with $I = 1, 2, \ldots, \infty$. This is akin to the use of conventional dimensional regularization for calculation of divergent field-theoretical integrals where integration rules over fractional dimensions are shown to be consistent and proven upon working in infinite dimensional linear spaces and further analytic continuation to other values of $D$, fractional or even complex (see, for instance, the standard text on renormalization [15]). This loophole will allow us to accommodate arbitrary complex vectors $y^I$ possessing zero norm $(y^I)^2 = 0$.

Since off-shell amplitudes rapidly become complicated and extremely laborious to calculate with increasing loop order, we prefer to deal with their integrands. This inevitably forces us to use the formalism of operator insertions familiar from studies of renormalization of composite operators [16], which was recently popularized within the context of correlation functions starting from [17]. The usefulness of the method of Lagrangian insertion for the Wilson loop is practically

\footnote{See also Refs. [6-8] for motivational studies.}
twofold. First, as we just said, it allows one to define the notion of the integrand for the bosonic Wilson loop starting already from the one loop order. Unfortunately, this procedure does not offer any computational simplifications in perturbative studies on par to the correlator story. Notwithstanding, its very existence is complemented by yet another feature of paramount importance: the ability to analytically continue and vary the \((2\varepsilon)\)-regulator dimensions independently of the off-shellness condition for external kinematical variables. It is this property that will let us to continuously navigate between the off-shell case in 4D and the on-shell one in dimensionally regularized theory.

2 An off-shell Wilson loop

To start with, let us provide a one-paragraph recap of the on-shell amplitudes/Wilson loop duality. It is based on the following T-dual relation between gluons’ four-momenta \(p_\ell\) and corresponding dual coordinates \(x_\ell\), i.e., \(p_\ell = x_\ell - x_{\ell+1} \equiv x_{\ell\ell+1}\), which automatically solve the energy-momentum conservation condition \(\sum_i p_i = 0\). Then, the ratio \(W_N^{\text{on-shell}}\) of the \(N\)-gluon (MHV) amplitude to its tree-level expression is determined by the expectation value of the Wilson loop in the fundamental representation of \(SU(N_c)\) gauge group \([3, 4, 5]\).

\[
W_N^{\text{on-shell}} \equiv A_N / A_N^{(0)} = \left\langle \frac{1}{N_c} \text{tr} P \exp i \oint_C dx_\mu A_\mu(x) \right\rangle , \tag{1}
\]

on a \(N\)-polygonal contour \(C = [x_1, x_2] \cup [x_2, x_3] \cup \cdots \cup [x_N, x_1]\) with each segment being light-like \(x_{\ell\ell+1}^2 = 0\) to mimic the gluon on-shellness condition \(p_\ell^2 = 0\). This equivalence withstood a number of tests for various values of \(N\) and loop orders and formed the foundation for the integrability framework of Ref. \([8]\).

Up to now, the number of attempts to devise a version of the Wilson loop that corresponds to off-shell gluon scattering amplitudes can be counted on the fingers of one hand. Namely, Ref. \([18]\) proposed to cut-off the immediate vicinity of the cusps thus breaking gauge invariance of the loop. The leading double logarithmic dependence of the four-site loop was shown to be gauge independent however. While Ref. \([19]\) proposed to modify the coordinate-space propagators “by hand” shifting the Lorentz invariant distance between two points by a mass parameter, i.e., \(x^2 \to x^2 + m^2\). While it works at one loop, this off-shell regulator was demonstrated to fail exponentiation starting from two loops. These two examples complete the list of proposals that we are aware off.

Our goal in this section will be to find a holonomy of the gauge connection whose expectation value develops the perturbative expansion which matches the one of an off-shell four-gluon amplitude ratio function devised in Ref. \([13]\). Namely, the latter was given to the lowest few orders by the expansion in ’t Hooft coupling \(g^2 = g_{\text{YM}}^2 N_c / (4\pi)^2\)

\[
W_N^{\text{off-shell}} = 1 - g^2 \left( z_{13}^2 z_{24}^2 g_{1234} + g^4 \left( z_{13}^2 z_{24}^2 h_{13;24} + z_{13}^4 z_{24}^4 h_{24;13} \right) \right) + O(g^6) , \tag{2}
\]

in terms of the standard one- and two-loop ladder integrals rewritten via the dual coordinates.

\(^2\)Strictly speaking \([13]\) did not compute any amplitudes instead they interpreted a 10D light-like limit of a generating function of four-point correlators as a scattering amplitude on restricted Coulomb branch of \(\mathcal{N} = 4\) SYM. It would definitely be important to calculate off-shell gluon amplitudes directly.

\(^3\)The one-loop off-shell amplitude in \(\mathcal{N} = 4\) SYM was already calculated in terms of the box integral back in Ref. \([20]\), see Eqs. (6.5.67) - (6.5.68) there.
as
\[ g_{1234} = \frac{1}{\pi^2} \int \frac{d^4x_0}{x_{01}^2 x_{02}^2 x_{03}^2 x_{04}^2}, \quad h_{13;24} = \frac{1}{\pi^4} \int \frac{d^4x_0 d^4x_0'}{(x_{01}^2 x_{02}^2 x_{04}^2)} x_{00'}^2 (x_{02}^2 x_{03}^2 x_{04}^2). \] (3)

In this expression, the external kinematics displays the full 10D Lorentz symmetry through the invariants \( z_{\ell \ell'} \equiv x_{\ell \ell'}^2 - y_{\ell \ell'}^2 \), while the loop integrations are performed only on its Minkowski 4D subspace. The nearest-neighbor points in 10D live on light rays obeying the null conditions \( x_{\ell \ell+1}^2 = y_{\ell \ell+1}^2 = m^2_q \) with the extra-dimensional invariants \( y_{\ell \ell+1}^2 \) playing the role of the off-shellness (or external mass). The necessity to have massless internal lines forces one to impose another null condition on all \( y \)-coordinates themselves, i.e.,
\[ y_{\ell \ell}^2 = 0. \] (4)

This equation allows one to restore the above 10D symmetry on the level of the integrand provided the integration variables are localized only in four dimensions [13].

Having this motivation in mind, we start with a 10D bosonic holonomy and dimensionally reduce it down to \( D = 4 + 2\varepsilon \) dimension with \( \varepsilon > 0 \),
\[ W_{\text{off-shell}}^4 = \left\langle \frac{1}{N_c} \text{tr} P \exp i \oint_{C} dz_M A^M (z) \right\rangle, \] (5)

We take the contour \( C \) to be a polygon with segments \( [z_\ell, z_{\ell+1}] \) starting and ending on adjacent points \( z^M_\ell = (x^\mu_\ell, y^I_\ell) \) with the four-dimensional Minkowski coordinates \( x^\mu_\ell \) and \( (2\varepsilon) \)-orthogonal fractional subspace spanned by the infinite-component vectors \( y^I_\ell, I = 1, 2, \ldots, \infty \). The latter are null and thus can be thought as having complex-valued components. The reason to stay above four-dimensions is to couple the propagating gauge fields to the auxiliary coordinates \( y^I_\ell \), which play a crucial role in the proper introduction of the off-shellness. Namely, any two adjacent points along the polygonal contour in \( D \)-dimensions (see the right panel (a) in Fig. 1) form light-like distances thus enforcing the constraint
\[ x_{\ell \ell+1}^2 = y_{\ell \ell+1}^2. \] (6)

4We use \( z, x \) and \( y \) variables for the 10D, 4D and 6D coordinates, respectively. Below, we recycle them for the \( D \)-dimensional, 4 and \( (2\varepsilon) \) spaces, accordingly.

5One explicit realization that a reader can adopt for the four cusps of the loop contour are \( y^I_\ell = \delta_{I,1} + i\delta_{I,\ell}. \)
It is interpreted in light of the T-duality relation for four-dimensional momenta as developing nonvanishing virtualities \(y_{\ell+1}^2\) for external gluons of scattering amplitudes. In what follows, we will assume for simplicity that all of them possess the same value \(y_{\ell+1}^2 = m^2\).

Next, in the spirit of the computation of correlation functions \([17, 21]\), we define the L-loop order planar corrections to the rectangular Wilson loop (3) expectation value

\[
W_{4\text{-shell}} = 1 + W_{4(1)} + W_{4(2)} + \ldots ,
\]

via the method of Lagrangian insertions as

\[
W_{4(L)} = \int d^D z_0 \ldots d^D z_{(L-1)} W_{4(L)}
\]

with the integrand being

\[
W_{4(L)} = i^L g_{YM}^{-2L} \left\langle \frac{1}{N_c} \text{tr} P \exp i \oint_C A^M (z) \mathcal{L}(z) \ldots \mathcal{L}(z_{(L-1)}) \right\rangle_{(0)},
\]

possessing the following perturbative behavior in Yang-Mills coupling \(g_{YM}\), \(W_{4(L)} = O(g_{YM}^{2L})\). Here the differential operator insertion \(\mathcal{L}\) read:\[6\]

\[
\mathcal{L} = \text{tr} \left( \frac{1}{2} F_{MN} F^{MN} + \frac{1}{8} g_{YM}^2 [\phi^{AB}, \phi^{CD}] [\bar{\phi}_{AB}, \bar{\phi}_{CD}] - \frac{1}{\sqrt{2}} g_{YM} \lambda^a A^\lambda A^a + \frac{1}{\sqrt{2}} g_{YM} \lambda^a \bar{\lambda}_a [\phi^{AB}, \bar{\lambda}_B] \right).
\]

Since the field operators in it are normal ordered, in order to avoid direct pairings among the fields of the Wilson loop itself (3), the latter will also be understood as normal ordered as well. This way radiative corrections only arise from Wick contractions of the gauge-link field to the ones in the Lagrangian insertion (10). We will see momentarily the importance of this requirement. Parametrizing each \([\ell, \ell+1]-\)segment of the D-dimensional contour by the proper time \(\tau_\ell\), the gauge field on each of them resides at the position \(z^\tau_{\ell+1} = \tau_\ell z_\ell + \tau_\ell z_{\ell+1}\) integrated over the interval \(0 \leq \tau_\ell \leq 1\). Notice that in the present note, only the first term in \(\mathcal{L}\) will be relevant for the analysis that follows. So only the gauge propagator is required and it reads

\[
\langle A^a_M (z_0) A^b_N (z^\tau_{\ell+1}) \rangle = - \frac{g_{YM}^2}{4\pi D/2} \eta_{MN} \delta^{ab} \frac{\Gamma(D/2 - 1)}{-(z_0 - z^\tau_{\ell+1})^2 + i0} D/2 - 1 ,
\]

in the Feynman gauge. We set the conventional dimensional regulator scale to unity \(\mu = 1\), so as not too pollute formulas, however, we will recover it when it becomes important for our arguments. The resulting integrand is a function of the space-time dimension \(D\) of the operator insertions and of the external kinematics, in particular, the off-shellness \(x^2_{\ell+1} = m^2\). We will consider it as an independent function of both. Since short-distance/light-cone divergences are now regularized by \(m\) and we can safely send \(\varepsilon \rightarrow 0\). Thus, at this step we project out the overlap between the internal and external variables on the \((2\varepsilon)\)-subspace. With this understanding of the integrand, the invariant distance in the Feynman propagator reduces to

\[
(x_0 - z^\tau_{\ell+1})^2 = (x_0 - x^\tau_{\ell+1})^2 - (y^\tau_{\ell+1})^2 = (x_0 - x^\tau_{\ell+1})^2 + \tau_\ell x^2_{\ell+1}.
\]

\[6\]It is obtained from the \(N = 4\) Lagrangian \(\mathcal{L}_{N=4}\) of Ref. \([22]\) by rescaling the gauge field \(g_{YM} A_M \rightarrow A_M\), differentiating the result with respect to \(g_{YM}\), i.e., \(\mathcal{L} = g_{YM}^4 \partial^2 g_{YM} (L_{N=4} g_{YM} A_M \rightarrow A_N)\). We omit the gauge fixing term \(1/(2g_{YM}) (\partial \cdot A)^2\) since we are dealing with a gauge-invariant Wilson loop.
Here the light-cone condition on the external kinematics (3) was used simultaneously with the
null conditions (1) on the individual $y_t$ variables. The resulting Wilson loop with $D > 4$ external
kinematics and $D = 4$ internal integrations is visualized in the (a)-panel of Fig. 1 before and in
(b)-panel after imposing the aforementioned light-like constraints. In the latter case, trajectories
of the probes sourcing the gauge flux cease to be straight but rather develops a pull into auxiliary
kinematical directions encoded by a unit vector $n_t$, i.e., $z_{\ell\ell+1}^r = \tau_\ell x_\ell + \tau_\ell x_{\ell+1} + m\sqrt{\tau_\ell n_\ell}$. As we
will see in the next subsection, this setup yields an exact one-loop match between the integrand of the
off-shell four-gluon scattering amplitude (2) and the integrand of the Wilson loop constructed
in this manner.

### 3 One-loop test

The first order of business is to demonstrate the equivalence of the just proposed Wilson loop
along with its calculational procedure to the result quoted in Eq. (2). The contributing one-loop
graphs are shown in Fig. 2. Their integrands possess the structure

$$
\langle \frac{1}{N_c} P \text{tr}[z_\ell, z_{\ell+1}] [z_{\ell'}, z_{\ell'+1}] L(x_0) \rangle = -\frac{N_c g^{4}}{8\pi^4} \int_0^1 d\tau_\ell \int_0^1 d\tau_{\ell'} \frac{n(x_0; z_\ell, z_{\ell+1}; z_{\ell'}, z_{\ell'+1})}{(x_0 - z_{\ell\ell+1})^4(x_0 - z_{\ell'\ell'+1})^4},
$$

(13)

where, somewhat abusing notations, we designated the gauge links by their corresponding inter-
vals $[z_\ell, z_{\ell+1}]$. The numerator, while superficially appearing to have nontrivial dependence on the
proper times $\tau_\ell, \tau_{\ell'}$, is in fact independent of them

$$
n(x_0; z_\ell, z_{\ell+1}; z_{\ell'}, z_{\ell'+1}) = z_{\ell\ell+1} \cdot z_{\ell'\ell'+1} (x_0 - z_\ell) \cdot (x_0 - z_{\ell'}) - z_{\ell\ell+1} \cdot (x_0 - z_{\ell'}) z_{\ell'\ell'+1} \cdot (x_0 - z_\ell).$$

(14)

This is the first harbinger of successful comparison to the amplitude (2): correlation functions
computed with the Lagrangian insertion procedure in the pair-wise light-cone limits possess the
very same numerators on a diagram-by-digram basis. The second ingredient is to convert and
match the proper-time integrals to the expected denominators of the cross integral $g_{1234}$ in Eq.
(3). This follows immediately upon rearranging the denominators in Eq. (13) as follows

$$
(x_0 - z_{\ell\ell+1})^2 = \tau_\ell x_{0\ell}^2 + \tau_\ell x_{0\ell+1}^2,
$$

(15)

with any explicit mentioning of the off-shellness lost in translation. The remaining $\tau$-integrals
immediately produce products of the free scalar propagators $1/x_{0\ell}^2$. Now, all that is left to do is
to show that the sum of all graphs, displayed in Fig. 2 and their inequivalent permutations, add
up to $z_{13}^2 z_{24}^2 g_{1234}$. This is almost warranted by the equivalence of the numerators between the
integrands of the Wilson loop and correlation functions alluded to above and can be verified by
rearrangement of numerators for the three topologies of graphs in Fig. 2. Namely, one finds

$$
n(x_0; z_1, z_2; z_3, z_4) = \frac{1}{4} \left[ z_{13}^2 z_{24}^2 - z_{13}^2 [x_{04}^2 + x_{02}^2] - z_{24}^2 [x_{01}^2 + x_{03}^2] - [x_{01}^2 - x_{02}^2] [x_{03}^2 - x_{04}^2] \right],
$$

(16)

$$
n(x_0; z_1, z_2; z_2, z_3) = \frac{1}{4} \left[ 2 z_{13}^2 x_{02}^2 - [x_{01}^2 - x_{02}^2] [x_{02}^2 - x_{03}^2] \right],
$$

(17)

$$
n(x_0; z_1, z_2; z_1, z_2) = -\frac{1}{2} [x_{01}^2 - x_{02}^2]^2,
$$

(18)
for the diagrams in (a), (b) and (c), respectively. The cancellation mechanism is then self-explanatory, i.e., the second and third terms in Eq. (16) (and its analogue with the gluon insertion stretching between the other opposite sides) cancel against the first term in the vertex contribution (17) and its cyclic permutations, while the fourth term in (16) together with the last one in (16) conspire to cancel the self-energy contributions in Eq. (18) and its permutations. This leaves us just the (double of the) first term in (16) and hence confirming the agreement between the one-loop off-shell Wilson loop and the one-loop gluon amplitude (2) landing the first evidence in favor of our proposal.

4 Off-shell to on-shell

In this section, we relax the condition for internal integrations to be four-dimensional. While massless one-loop calculations can be done on the back of an envelop, massive calculations in dimensionally regularized theory are far from being trivial. The motivation for this is to study the dependence of the Wilson loop expectation value both on the off-shellness and dimensional regulator parameters at the same time and thus be able to interpolate between the off-shell and on-shell regimes. When the segments’ virtualities tend to zero $x_{\ell+1}^2 \rightarrow 0$, the light-like Wilson loop suffers from well-known ultraviolet divergences. To tame them properly, we have to analytically continue $\varepsilon$ in $D = 4 + 2\varepsilon$ in Eq. (5) to negative values $\varepsilon < 0$. This is how we will understand the dimensionality of the internal integration in this section. The distances are however taken to be same as they arise from the dimensional reduction, i.e., see Eq. (12).

When the space-dimension $D$ is kept away from four, the massive cancellation observed between various graphs observed at the end of the previous section no longer hold except for the self-energy-like terms, i.e., the last contributions in Eq. (16) and (17) and Eq. (18) itself, which are total derivatives as we show in the Appendix A. So the exchange and the vertex graphs have to be computed from scratch. To cast the emerging parametric integrals in the as most eye-pleasing form as possible, we use a somewhat unorthodox version of the Feynman parametrization to join the two denominators in Eq. (13) promoted to their $D$-dimensional versions (11),

$$\frac{\Gamma(\nu_1)\Gamma(\nu_2)}{A_1^{\nu_1} A_2^{\nu_2}} = \tau_{12} \int_{\tau_2}^{\tau_1} d\sigma (\tau_1 - \sigma)^{\nu_1 - 1}(\sigma - \tau_2)^{\nu_2 - 1} \frac{\Gamma(\nu_1 + \nu_2)}{[(\tau_1 - \sigma)A_2 + (\sigma - \tau_2)A_1]^{\nu_1 + \nu_2}},$$

with the adopted short-hand notation $\tau_{12} \equiv \tau_1 - \tau_2$. The right-hand side is independent of the choice for $\tau$’s and they can be selected at will. To achieve the above simplification they are taken
to be $\tau_1 = \tau_\ell$ and $\tau_2 = \tau'_{\ell}$ for $A_1 = -(x_0 - z_{\ell\ell+1})^2$ and $A_2 = -(x_0 - z'_{\ell'\ell+1})^2$. Performing the integral over the insertion point $z_0$, we find\[ W_{\text{off-shell}}^{4(1,a)} = \frac{g^2_N N_c}{32\pi^{D/2}} \left\langle \left[ z_{13}^2 z_{24}^2 - z_{13}^2 x_2^2 \sigma + 2m^2 \bar{\sigma} - z_{24}^2 x_2^2 \sigma + 2m^2 \bar{\sigma} \right] \frac{\Gamma(\frac{D}{2})}{D_{x_{13},x_{24}}^{D/2}} \right. \\
+ \left. \frac{2 z_{13}^2 + z_{24}^2 \Gamma(\frac{D}{2} - 1)}{D_{x_{13},x_{24}}^{D/2-1}} \right\rangle, \]
\[ W_{\text{off-shell}}^{4(1,b)} = \frac{g^2_N N_c}{32\pi^{D/2}} \left\langle \left[ 2z_{13}^2 m^2 \bar{\sigma} - 2z_{13}^2 \frac{\Gamma(D/2 - 1)}{D_{x_{13},0}^{D/2-1}} \right] \right\rangle, \]
(19)
for the respective contributing graphs $(a), (b)$ to the Wilson loop. Notice that the integrand of the vertex diagram is related to the one of the exchange graph by sending $4 \rightarrow 2$ and flipping the overall sign. Here the double angle brackets stand for the three-fold parametric integral
\[ \langle\langle\ldots\rangle\rangle \equiv \int_0^1 d\tau_1 \int_0^1 d\tau_2 \int_{\tau_2}^{\tau_1} d\sigma [ (\tau_1 - \sigma)(\sigma - \tau_2)]^{D/2-1} \ldots, \]
(20)
and the denominator being
\[ D_{x_{13},x_{24}} \equiv (\tau_1 - \sigma)(\sigma - \tau_2) [-\bar{\tau}_1 \bar{\tau}_2 x_{13}^2 - \tau_1 \tau_2 x_{24}^2] - \sigma \bar{\sigma} \tau_{12}^2 m^2. \]
(21)

The above parametric integrals are not immediately expressible in terms of known special functions. However, it is relatively straightforward to derive Mellin-Barnes representations for them. Details of calculations involved are relegated to Appendix A. Here we merely cite the result for the main structure $z_{13}^2 z_{24}^2$ which survives the four-dimensional limit. We found the following three-fold contour integral for it
\[ \left\langle \left\langle \frac{\Gamma(D/2)}{D_{x_{13},x_{24}}^{D/2}} \right\rangle \right\rangle = \frac{(-m^2)^{-D/2}}{\Gamma^2 (2 - D/2)} \int \prod_{i=1}^{3} \frac{d\bar{j}_i}{2\pi i} \frac{\Gamma(-\bar{j}_i)}{\Gamma(-\bar{j}_i)} \left( \frac{x_{13}^2}{m^2} \right)^{\bar{j}_1} \left( \frac{x_{24}^2}{m^2} \right)^{\bar{j}_2} \]
\[ \times \frac{\Gamma(-j_3) \Gamma(j_1 + j_3 + 1) \Gamma(j_2 + j_3 + 1) \Gamma(1 - \frac{D}{2} - j_1 - j_3) \Gamma(1 - \frac{D}{2} - j_2 - j_3) \Gamma(\frac{D}{2} + j_1 + j_2 + j_3)}{\Gamma(\frac{D}{2} + j_1 + j_2)} \].
(22)

Other contributions to the exchange graph are unfortunately not as simple: they admit a four-fold Mellin-Barnes representation which can be found in Eq. \[33\]. Finally, the vertex integral is given in Eq. \[36\]. The complete result for the off-shell dimensionally regularized one-loop Wilson is then obtained by adding one more exchange contribution and three corrections to other cusp vertices of the rectangular Wilson loop, namely,
\[ W_{\text{off-shell}}^{4(1)} = W_{\text{off-shell}}^{4(1,a)} + W_{\text{off-shell}}^{4(1,a)}|_{1\rightarrow 4\rightarrow 3\rightarrow 2\rightarrow 1} + \sum_{\text{cyclic 1234}} W_{\text{off-shell}}^{4(1,b)}. \]
(23)
studying the following two limits: the four-dimensional vanishing off-shellness and the on-shell dimensionally regularized case. These respective asymptotic expressions read\(^8\)

\[
W_{4(1)}^{\text{off-shell}}|_{D=4,m^2 \to 0} = -\frac{g_Y^2 N_c}{4\pi^2} \left[ \log^2 \frac{m^2}{\sqrt{x_{13}^2 x_{24}^2}} + \frac{1}{2} \zeta_2 \right],
\]

(24)

\[
W_{4(1)}^{\text{off-shell}}|_{D=4+2\varepsilon,m^2=0} = -\frac{g_Y^2 N_c}{8\pi^2} \left[ \log \frac{\mu^2}{x_{13}} \log \frac{\mu^2}{x_{24}^2} - 2\zeta_2 + \text{Poles} \right].
\]

(25)

In the second formula, we restored the ‘cut-off’ dimensional scale \(\mu\) and absorbed in it certain accompanying transcendental constants along the way. We observe indeed the well-known ‘factor of two difference’ between the Sudakov asymptotics in the off-shell [23] and on-shell regimes [24], which is attributed to the non-vanishing contribution from ultra-soft integration in addition to the soft regime, present in both cases, as explained in Ref. [25] (see, in particular, Appendix A there), [26] (see section 12.1 there) and Ref. [27]. These one-loop results are in accord with considerations of the on-shell [28] and off-shell four-leg one-loop amplitude in Ref. [37]. However, the off-shell amplitude/Wilson loop (2) starts to deviate from the latter starting from two loops.

## 5 Beyond one loop

Let us briefly discuss higher loops. The consideration of Ref. [37] suggests that the all-order Sudakov behavior of off-shell gluon amplitude is driven by the very same cusp anomalous dimension as their massless counterparts albeit for an overall factor of two. Equation (2) taken as a ratio function of gluon amplitudes on the restricted Coulomb branch paints a different picture and suggests instead that the double logarithmic asymptotics accompanied by the so-called octagon instead of the cusp anomalous dimension [14]. The former is known to all orders in \(\text{'t Hooft coupling}\) \(g\) from [14, 30, 31] as \(\Gamma_{\text{oct}} = 2 \log \cosh(2\pi g^2)/\pi^2\) while \(\Gamma_{\text{cusp}}\) from [32]. Such that

\[
\ell^{-2} \log W_{4(1)}^{\text{off-shell}}|_{D=4,m^2 \to 0} = -\frac{1}{2} \Gamma_{\text{cusp}}(g) = -2g^2 + \frac{2}{3} \pi^2 g^4 - \frac{22}{45} \pi^4 g^6 + \ldots,
\]

(26)

\[
\ell^{-2} \log W_{4(1)}^{\text{off-shell}}|_{D=4+2\varepsilon,m^2=0} = -\Gamma_{\text{oct}}(g) = -4g^2 + \frac{8}{3} \pi^2 g^4 - \frac{128}{45} \pi^4 g^6 + \ldots,
\]

(27)

in perturbative series, where \(\ell = \log(m^2/Q^2)\) and we chose the values of external kinematical variables in the symmetric point \(x_{13}^2 = x_{24}^2 \equiv Q^2\) for easier comparison of the two cases, also we set \(\mu = m\).

We have not performed a full-fledged two-loop calculation of the off-shell Wilson loop and will content ourselves in this note with a heuristic, semi-quantitative argument as to why we anticipate it to reproduce the box-integral expansion (2). The one-loop case was an exemplar of what has to happen before one can hope for this to take place. Namely, a necessary condition is the independence of the integrand’s numerator of the proper times such that its structure would match the one of the light-cone limit of four-point correlators. There are only two maximally non-Abelian graphs that one has to address. They are displayed in Fig. 3 (a) and (b). The graph in (a) does not require a new analysis since it involves two independent propagator with the Lagrangian insertions whose transverse nature insures the sought after cancellation. It is the same as in Eq. (14). It appears that the three-gluon vertex diagram (b) would require a separate

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\(^8\)The pole contribution is Poles = \(-2\varepsilon^{-2} + \varepsilon^{-1} \log(\mu^4/x_{13}^2 x_{24}^2)\).
consideration since it contains two gluon propagators without insertions. However, the very fact that Lagrangian insertions in gluon propagators ‘gauge transform’ them from the Feynman to the Landau gauge, these two can be taken in the Landau gauge from the get-go to enforce the transversality condition. Having cancelled the proper-time dependence in numerators, all \( \tau \)-integrals of four-dimensional gluon propagators building up the integrand can be taken to produce products of scalar denominators, in the same manner as observed at one loop. The resulting integrands are the same on a diagram-by-diagram basis as the ones of four-point correlators of half-BPS scalar operators\(^9\). This points towards the anticipated structure of the two-loop result \(^2\) with correct numerator structure in the sum of all graphs. The explicit calculation is very tedious and will be reported in a separate publication.

\section{Discussion}

In this note, we proposed a generalization of the amplitude/Wilson loop duality to the off-shell regime by using a \( D \)-dimensional extension of the holonomy. A reader might feel uneasy about manipulations which we invoked while varying the number of space-time dimensions to equip the loop with the notion of link’s virtuality. However, since the Wilson loop is not per se a measurable observable, one had to advocate a consistent regularization procedure for it. This is what we hoped to achieve in this paper and verified it at one loop with a plausible extension to higher orders in ’t Hooft coupling. The consideration of this paper opens up several directions for further research. First, two-loop corrections to the four-site Wilson loop have to be studied in full-fledged form not just at the heuristic level as was done in Section \(^5\). Generalization to a larger number of links is begging for attention as well.

From a different perspective, as a way to understand the dynamics of the emerging two-dimensional world-sheet, the following questions have to be answered: what are the elementary excitations propagating on the off-shell world sheet, what are their dispersion relations and scattering matrices, are these factorizable? It appears that the excitations are not the same as the ones sources by the light-like flux of the cusp story \([34, 35]\). This can be concluded by looking

\(^9\)Notice the two-loop calculation in Ref. \([33]\) employed a different component of the stress-tensor supermultiplet to extract the integrands. In this case graphs with non-Abelian vertices did not contribute and the result was given by exchange-like diagrams. Since we are working with a bosonic Wilson loop such a simplification is not available to us.
at the $O(g^0)$ contribution at strong-coupling to the cusp and octagon anomalous dimensions,

$$
\Gamma_{\text{cusp}}(g) = 2g \left(1 - \frac{3 \log 2}{4\pi g} + \ldots\right), \quad \Gamma_{\text{oct}}(g) = \frac{4g}{\pi} \left(1 - \frac{\log 2}{2\pi g} + \ldots\right). \quad (28)
$$

The coefficient $3 \log 2$ in $\Gamma_{\text{cusp}}$ is a sum of integrals of the dispersion relations for the 2 gluon excitations of mass $\sqrt{2}$, 5 scalars of zero mass, one composite excitations of mass 2 and 8 fermions of unit mass \[36, 34, 35\]. What is the world-sheet origin of the same correction to $\Gamma_{\text{oct}}$?

Both of the above anomalous dimensions arise as solutions to certain integral equations. $\Gamma_{\text{cusp}}$ stems from the flux-tube equation of Ref. \[32\]. $\Gamma_{\text{oct}}$, on the other hand, was shown to obey two different equations \[30\], one is based on the BMN and the other on the GKP vacuum. It was unclear why there is a GKP description at all since $\Gamma_{\text{oct}}$ drives the dynamics of a four-point correlation function of large-$R$ charge operators. In light of the connection to off-shell scattering amplitudes pointed out in Ref. \[13\] and the Wilson loop advocated here, this link becomes more natural. Notice that both $\Gamma_{\text{cusp}}$ and $\Gamma_{\text{oct}}$ were shown in Ref. \[38\] to be solutions to a flux-tube equation depending on a parameter reducing to one or another depending on its value. While it is tempting to interpret the parameter as related in some way to the off-shellness, the precise map remain obscure. Hopefully, the study of the off-shell Wilson loop initiated here will shed some light on this interpretation.

There is another aspect that needs a thorough study and further clarification. As we have recapitulated in Section 4, there is a factor of 2 difference between the on- and off-shell amplitudes at one loop order, which repeats the Sudakov form factor story known for almost half a century. However, it appears that it was taken for granted that this doubling is the only effect that occurs at higher loop orders for the infrared asymptotics of scattering amplitudes for dimensional versus off-shell regulator, see \[27, 37\]. Results of Ref. \[13\] and this paper seem to suggest a different mechanism and point to a completely different function of the coupling that drives the infrared evolution of amplitudes in the off-shell regime, it is the octagon anomalous dimension. Last but not least, what is its analogue in QCD?

### Appendix A Mellin-Barnes representation

Let us provide details for the reduction of parametric integrals in the body of the paper to contour integral representations. We will focus on the most complicated exchange contribution in Fig. 2 (a) and merely state results for the rest. We spit it according to the structure of the numerator in terms of its $z^2$-dependence, i.e., coefficients accompanying $z_{13}^2, z_{24}^2, z_{13}^2, z_{24}^2$ and the rest. The starting point is the well-known formula \[39, 40\]

$$
\frac{1}{[A_1 + A_2 + A_3]^{\nu}} = \frac{1}{A_3^{\nu}} \int_{c-i\infty}^{c+i\infty} \frac{dj_1dj_2}{(2\pi i)^2} \left(\frac{A_1}{A_3}\right)^{j_1} \left(\frac{A_2}{A_3}\right)^{j_2} \frac{\Gamma(-j_1)\Gamma(-j_2)\Gamma(j_1 + j_2 + \nu)}{\Gamma(\nu)} F_1(\beta, -\gamma, -\delta, \alpha + \beta, -\frac{\tau_{12}}{\tau_2}, \frac{\tau_{12}}{\tau_2}). \quad (29)
$$

with implicit integration contours running along the imaginary axis and separating poles of the Euler Gamma functions with positive and negative signs of j’s. We apply it to the denominator $D$ defined in Eq. \[21\]. After this, the integral over $\sigma$ can be performed in terms of the Appell function $F_1$ \[41\] (see Eq. (3.6) there), which reads

$$
\int_{\tau_2}^{\tau_1} d\sigma (\tau_1 - \sigma)^{\alpha-1} (\sigma - \tau_2)^{\beta-1} \sigma^\gamma \bar{\sigma}^\delta = \tau_2^{\gamma} \bar{\tau}_2^{\delta} t_{12}^{\alpha+\beta-1} \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)} F_1 \left(\beta, -\gamma, -\delta, \alpha + \beta, -\frac{\tau_{12}}{\tau_2}, \frac{\tau_{12}}{\tau_2}\right).
$$
Finally, employing the Mellin-Barnes representation for such that one has to make sure that both arguments of the function have phases obeying the constraints for the Appell function itself.

Before we do it, however, we observe that for the $z_{13}^2 z_{24}^2$-structure, the Appell function actually reduces to the simpler hypergeometric function $\binom{2}{1}$ by means of the known identity

$$F_1(\alpha; \beta_1, \beta_2; \beta_1 + \beta_2; \tau_1, \tau_2) = (1 - \tau_2)^{-\alpha} \binom{2}{1}(\alpha, \beta_1, \beta_2, \frac{\tau_{12}}{\tau_2}) ,$$

such that

$$\left\langle \binom{D/2}{D/2} \right\rangle_{\mathcal{D}_{x_{13}, x_{24}}} = (-m^2)^{-D/2} \int_0^1 d\tau_1 \int_0^1 d\tau_2 \int_{\ell=1}^2 \frac{dj_1 \Gamma(-j_1)}{2\pi i} \Gamma\left( \frac{x_{13}^2}{m^2} \right) \left( \frac{x_{24}^2}{m^2} \right)^{j_2} \frac{\Gamma\left( \frac{D}{2} + j_1 + j_2 \right)^3}{\Gamma(D + 2j_1 + 2j_2)} \times \tau_1^{j_2 - l_1 - j_1} \tau_2^{j_2 - l_2 - j_1} \tau_1^{l_1} \tau_2^{l_2} \binom{2}{1} \left( \frac{D}{2} + j_1 + j_2, \frac{D}{2} + j_1 + j_2; D + 2j_1 + 2j_2; -\frac{\tau_{12}}{\tau_1 \tau_2} \right) .$$

Finally, employing the Mellin-Barnes representation for $\binom{2}{1}$, we obtain a three-fold contour integral which is quoted in Eq. (22) of the main text.

For $z_{13}^2$-coefficient, the $F_1$ does not simplify and is left as it is. However, to employ a well-known contour integral form for it \cite{22}, namely,

$$F_1(\alpha; \beta_1, \beta_2; \gamma; \tau_1, \tau_2) = \frac{\Gamma(\gamma)}{\Gamma(\gamma - \alpha) \Gamma(\gamma - \beta)} \times \int \frac{dj_1}{2\pi i} (1 - \tau)^j \Gamma(\gamma - j) \Gamma(\gamma + j) \Gamma(\gamma - \alpha - \beta - j) ,$$

we have to make sure that both arguments of the function have phases obeying the constraints $|\arg(-\tau_{1,2})| < \pi$. This is not the case in Eq. (30) since both integration regions $\tau_1 \leq \tau_2$ contribute on equal footing. The problem can be partially alleviated by reducing the region $\tau_1 < \tau_2$ to the other one $\tau_1 > \tau_2$. In fact the result for the latter just doubles. However, after this is done, while the first argument in Eq. (30) satisfies the above condition, the second one does not. To correct this, we use the transformation

$$F_1(\alpha; \beta_1, \beta_2; \gamma; \tau_1, \tau_2) = (1 - \tau_2)^{-\beta_2} (1 - \tau_1)^{-\alpha - \beta_1 + \gamma} \binom{2}{1}(\gamma - \alpha; \gamma - \beta_1 - \beta_2, \beta_2; \tau_1, \frac{\tau_{12}}{\tau_2}) .$$

Assembling everything together, we obtain the following four-fold integral

$$\left\langle \left[ x_{24}^2 + 2m^2 \sigma \right] \frac{\Gamma(D/2)}{\mathcal{D}_{x_{13}, x_{24}}^{D/2}} - 2 \frac{\Gamma(D/2 - 1)}{\mathcal{D}_{x_{13}, x_{24}}^{D/2 - 1}} \right\rangle = \frac{4}{(-m^2)^{D/2}} \int \frac{dj_1}{2\pi i} \Gamma(-j_1) \left( \frac{x_{13}^2}{m^2} \right)^{j_1} \left( \frac{x_{24}^2}{m^2} \right)^{j_2} .$$
\[
\times \Gamma(j_3 + 1)\Gamma(j_3 + j_4 + 1) \Gamma(D/2 + j_1 + j_2) \Gamma(D/2 + j_1 + j_2 + j_3) \Gamma(D/2 + j_1 + j_2 + j_3 + j_4) \\
\times \frac{\Gamma(D/2 - j_2 - j_3 + 2) \Gamma(D/2 + j_1 - j_2 + j_3 + 3) \Gamma(D/2 - j_1 - j_2 + j_3 + 1)}{\Gamma(-D/2 - j_2 + j_3 + 3) \Gamma(-D/2 + j_2 + j_2 + j_3 + j_4) \Gamma(D + 2j_2 + j_2 + j_3 + j_4)} \left[ \frac{2m^2(j_4 + 1)}{D + 2j_1 + 2j_2 + 2j_4 - 2} f_{111} \right. \\
+ \frac{m^2(D + 2j_1 + 2j_2 - 2)}{D + 2j_1 + 2j_2 + 2j_4 - 2} f_{110} + \frac{x_{24}^2(D + 2j_2 - 2j_3 - 4)(D + j_2 + j_4 - 3)}{(D + 2j_2 + 2j_4 - 2)(D - 2j_1 + 2j_2 - 2j_3 - 4)} f_{001} \right]
\]

where

\[
f_{\alpha_1\alpha_2\beta} = 3 F_2 \left( \begin{array}{c} -j_2, \alpha_2 - D/2 + 2 + j_1 - j_2 + j_3, \alpha_1 - D/2 + 1 - j_2 - j_4 \\ \alpha_2 - D/2 + 2 - j_2 + j_3, \alpha_2 - D + 3 - j_2 - j_4 \end{array} \right) \bigg| 1 \right).
\]

Finally, let us address the last term in the numerator of Eq. (16), which we referred to in the main body as to the self-energy-like contribution. Indeed in the exchange graph (a) it can be integrated over the proper times to get

\[
W_{4(1, 2)}^{\text{off-shell}} \rightarrow \frac{i\Gamma^2(D/2)}{32\pi^D} \int_0^1 d\tau_1 \int_0^1 d\tau_2 \int d^Dx_0 \left[ \frac{[x_{01}^2 - x_{02}^2][x_{03}^2 - x_{04}^2]}{[-\tau_1 x_{01}^2 - \tau_1 x_{02}^2][D/2] [-\tau_2 x_{03}^2 - \tau_2 x_{04}^2][D/2}] \right],
\]

and depends only on the boundary values of the links. The last contribution to the graph (b) as well the diagram (c) admit the same form and add up to zero.

Last but not least, the vertex graph is deduced from the exchange by taking the residue at \( j_2 = 0 \) and setting \( x_{24}^2 = 0 \). In this case, the hypergeometric function \( 3 F_2 \) reduces to products of Euler Gammas and the result reads

\[
\left\langle 2\frac{\zeta_{13}^2 m^2 \sigma}{D_{13, 0}} - 2\frac{\zeta_{13}^2 \Gamma(D - 1)}{D_{13, 0}} \right\rangle = -\frac{4}{(-m^2)^{D/2 - 1}} \int \prod_{\ell=1}^3 \frac{dj_\ell}{2\pi i} \Gamma(-j_\ell) \left( \frac{x_{13}^2}{m^2} \right)^{j_\ell}
\]

\[
\times \left( \frac{\zeta_{13}^2}{2(\frac{d}{2} - j_2 - 3)} \frac{d}{2} - j_3 - 4 \right) \Gamma(d + 2j_3 - 4) \Gamma\left( \frac{d}{2} + j_3 + 1 \right) \Gamma\left( -\frac{d}{2} - j_3 + 2 \right)
\]

\[
\times \left( \frac{\zeta_{13}^2}{2} \frac{d}{2} + j_1 + j_2 + 3 \right) \Gamma\left( \frac{d}{2} + j_1 + j_2 + j_3 - 1 \right) \Gamma\left( \frac{d}{2} - j_1 - j_2 - j_3 + 1 \right) \Gamma\left( \frac{d}{2} + j_1 + j_2 + j_3 \right)
\]

\[
\Gamma\left( -\frac{d}{2} + j_2 + 3 \right) \Gamma\left( -d - j_3 + 4 \right) \Gamma(d + 2j_1 + j_2 + j_3)
\]

It was also rechecked by an explicit calculation as well.

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