Spin precession in the Schwarzschild spacetime: circular orbits

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Abstract

We study the behaviour of nonzero rest mass spinning test particles moving along circular orbits in the Schwarzschild spacetime in the case in which the components of the spin tensor are allowed to vary along the orbit, generalizing some previous work.

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1. Introduction

The equations of motion for a spinning test particle in a given gravitational background were deduced by Mathisson and Papapetrou \cite{1, 2} and read

\[
\frac{DP^\mu}{d\tau_U} = -\frac{1}{2} R^\mu_{\nu\alpha\beta} U^\nu S^\alpha^\beta \equiv F^{(\omega)\mu},
\]
\[
\frac{DS^{\mu\nu}}{d\tau_U} = P^\mu U^\nu - P^\nu U^\mu,
\]

where $P^\mu$ is the total 4-momentum of the particle, $S^{\mu\nu}$ is the antisymmetric spin tensor and $U^\mu$ is the 4-velocity of the timelike ‘centre of mass line’ used to make the multipole reduction. In order to have a closed set of equations, equations (1.1) and (1.2) must be completed by adding supplementary conditions (SC) whose standard choices in the literature are the

1. Corinaldesi–Papapetrou \cite{3} conditions (CP): $S^{\nu\nu} = 0$, where the index $\nu$ corresponds to a coordinate component and $\tau$ is a timelike slicing coordinate;
2. Pirani \cite{4} conditions (P): $S^{\mu\nu} U_\nu = 0$;
3. Tulczyjew \cite{5} conditions (T): $S^{\mu\nu} P_\nu = 0$. 

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Only solutions of the combined equations for which both $U$ and $P$ are timelike vectors are considered, in order to have a meaningful interpretation describing a spinning test particle with nonzero rest mass and physical momentum.

Not much is known about actual solutions of these equations in explicit spacetimes which satisfy the Einstein equations. In a previous article [6], we considered the simplest special case of a spinning test particle moving uniformly along a circular orbit in the static spherically symmetric Schwarzschild spacetime, but because these equations are still complicated, we looked for solutions with constant frame components of the spin tensor in the natural symmetry adapted static frame, i.e., coinciding with a static tensor field along the path. Such a static spin tensor is a very strong restriction on the solutions of these equations of motion, leading to special solutions in which the spin vector is perpendicular to the plane of the orbit, and contributes to an adjustment in the acceleration of the orbit.

Here we consider the slightly less restrictive case where the spin components are not constant, but the motion is still circular. However, in this case it is clear that if the spin tensor has time-dependent components, its feedback into the acceleration of the test particle path will break the static symmetry of that path unless the spin precession is very closely tied to the natural Frenet–Serret rotational properties of the path itself. Indeed, we find that only the Pirani supplementary conditions permit such specialized solutions since they allow the spin tensor to be described completely by a spatial spin vector in the local rest space of the path itself. By locking spin vector precession to the Frenet–Serret rotational velocity of the path, solutions are found with a spin vector Fermi–Walker transported along an accelerated centre of mass world line. The remaining choices for the supplementary conditions have no natural relationship to the Frenet–Serret properties of the particle path and do not admit such specialized solutions.

With the assumption of circular motion, one can solve the equations of motion explicitly up to constants of integration. By a process of elimination, one can express them entirely in terms of the spin components and particle mass as a constant coefficient linear system of first- and second-order differential equations. By systematic solving and backsubstitution, one gets decoupled linear second-order constant coefficient equations for certain spin components, which are easily solved to yield exponential, sinusoidal or quadratic solutions as functions of the proper time, from which the remaining variables may be calculated. Imposing the choice of supplementary conditions then puts constraints on the constants of integration or leads to inconsistencies. The details of the decoupling and solution of the equations of motion are left to the appendix, leaving the imposition of the supplementary conditions to the main text.

2. Circular orbits in the Schwarzschild spacetime

Consider the case of the Schwarzschild spacetime, with the metric written in standard coordinates

$$ds^2 = - \left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2), \quad (2.1)$$

and introduce the usual orthonormal frame adapted to the static observers following the time lines

$$e_t = (1 - 2M/r)^{-1/2}\partial_t, \quad e_r = (1 - 2M/r)^{1/2}\partial_r, \quad e_\theta = \frac{1}{r}\partial_\theta, \quad e_\phi = \frac{1}{r \sin \theta}\partial_\phi, \quad (2.2)$$
with dual frame
\[ \omega^\tau = (1 - 2M/r)^{1/2} dt, \quad \omega^\phi = (1 - 2M/r)^{-1/2} dr, \]
\[ \omega^\theta = r \, d\theta, \quad \omega^\varphi = r \sin \theta \, d\phi, \]
where \{\partial_\tau, \partial_\theta, \partial_\varphi, \partial_\varphi^\perp\} and \{dt, dr, d\theta, d\phi\} are the coordinate basis and its dual, respectively.

In order to investigate the simplest special solutions of the combined equations of motion, we explore the consequences of assuming that the test particle 4-velocity \( U \) corresponds to a timelike constant speed circular orbit confined to the equatorial plane \( \theta = \pi/2 \). Then, it must have the form
\[ U = \Gamma[\partial_\tau + \zeta \partial_\varphi] = \gamma[\partial_\tau + v e_\varphi], \quad \gamma = (1 - v^2)^{-1/2}, \]
where \( \zeta \) is the angular velocity with respect to infinity, \( v \) is the azimuthal velocity as seen by the static observers, \( \gamma \) is the associated gamma factor and \( \Gamma \) is a normalization factor which assures that \( U \cdot U = -1 \). These are related by
\[ \zeta = (-g_{\tau \varphi} / g_{\varphi \varphi})^{1/2} v, \quad \Gamma = (-g_{\tau \tau} - \zeta^2 g_{\varphi \varphi})^{-1/2} = (-g_{\tau \tau})^{-1/2} \gamma, \]
so that \( \Gamma \gamma v / (g_{\varphi \varphi})^{1/2} \), which reduces to \( \gamma \nu / r \) in the equatorial plane.

Here \( \zeta \) and therefore \( \nu \) are assumed to be constant along the world line. We limit our analysis to the equatorial plane \( \theta = \pi/2 \); as a convention, the physical (orthonormal) component along \( -\partial_\theta \) which is perpendicular to the equatorial plane will be referred to as ‘along the positive z-axis’ and will be indicated by the index \( \hat{e} \) when convenient: \( e_z = -e_\theta \).

Note both \( \theta = \pi/2 \) and \( r = r_0 \) are constants along any given circular orbit, and that the azimuthal coordinate along the orbit depends on the coordinate time \( t \) or proper time \( \tau \) along that orbit according to
\[ \phi - \phi_0 = \zeta t = \Omega_U \tau_U, \quad \Omega_U = \gamma \nu / r, \]
defining the corresponding coordinate and proper time orbital angular velocities \( \zeta \) and \( \Omega_U \). These determine the rotation of the spherical frame with respect to a nonrotating frame at infinity.

Among all circular orbits, the timelike circular geodesics merit special attention, whether co-rotating (\( \zeta_+ \)) or counter-rotating (\( \zeta_- \)) with respect to increasing values of the azimuthal coordinate \( \phi \) (counter-clockwise motion). Their time coordinate angular velocities \( \zeta_\pm \equiv \pm \zeta K = \pm (M/r^3)^{1/2} \), which are identical with the Newtonian Keplerian values, lead to the expressions
\[ U_\pm = \gamma_K [e_\tau \pm \nu_K e_\varphi], \quad v_K = \left[ \frac{M}{r - 2M} \right]^{1/2}, \quad \gamma_K = \left[ \frac{r - 2M}{r - 3M} \right]^{1/2}, \]
where the timelike condition \( v_K < 1 \) is satisfied if \( r > 3M \). At \( r = 3M \), these circular geodesics go null.

It is convenient to introduce the Lie relative curvature \([7, 8]\) of each orbit
\[ k_{(\text{Lie})} = -\partial_\nu \ln \sqrt{g_{\varphi \varphi}} = -\frac{1}{r} \left( \frac{2M}{r} \right)^{1/2} = -\frac{\zeta K}{v_K}, \]
and a Frenet–Serret intrinsic frame along \( U \) \([9]\), defined by
\[ E_0 = U, \quad E_1 = e_\tau, \quad E_2 = \gamma [v e_\tau + e_\varphi], \quad E_3 = e_\varphi, \]
satisfying the following system of evolution equations along the constant radial acceleration orbit
\[ \frac{dU}{d\tau_U} = a(U) = \kappa E_1, \quad \frac{dE_1}{d\tau_U} = \kappa U + \tau_1 E_2, \quad \frac{dE_2}{d\tau_U} = -\tau_1 E_1, \quad \frac{dE_3}{d\tau_U} = 0, \]
where in this case

\[ \kappa = k_{\text{Lie}} \gamma^2 \left[ \nu^2 - \nu_{\text{K}}^2 \right] = -\frac{\gamma^2 (\nu^2 - \nu_{\text{K}}^2)}{\nu_{\text{K}}} \xi_{\text{K}}, \tag{2.11} \]

\[ \tau_1 = -\frac{1}{2\gamma^2} \frac{d\kappa}{d\nu} = -k_{\text{Lie}} \frac{\gamma^2 \nu}{\nu_{\text{K}}} \xi_{\text{K}}. \]

The projection of the spin tensor into the local rest space of the static observers defines the spin vector by spatial duality

\[ S^\alpha = \frac{1}{2} \eta^\alpha_{\beta \gamma \delta} (e^\text{t})^\beta S^{\gamma \delta}, \tag{2.12} \]

where \( \eta^\alpha_{\beta \gamma \delta} = \sqrt{-g} \epsilon^\alpha_{\beta \gamma \delta} \) is the unit volume 4-form constructed from the Levi-Civita alternating symbol \( \epsilon^\alpha_{\beta \gamma \delta} (\epsilon^\text{t} \hat{\text{r}} \hat{\text{t}} \hat{\text{r}} \hat{\text{t}} = 1) \), leading to the correspondence

\[ (S^\text{t}, S^\theta = -S^\phi, S^\phi) = (S^\text{t}_\text{h}, -S^\text{r}_\text{h}, S^\text{r}_\text{h}). \tag{2.13} \]

For the CP supplementary conditions only these components of the spin tensor remain nonzero, while in the remaining cases the other nonzero components are determined from these through the corresponding orthogonality condition. The total spin scalar is also useful

\[ s^2 = \frac{1}{2} S_{\mu \nu} S^{\mu \nu} = -S_{\text{t} \text{t}} - S_{\theta \theta}^2 - S_{\phi \phi}^2 + S_{\text{t} \phi}^2 + S_{\text{t} \text{r}}^2 + S_{\theta \phi}^2 + S_{\text{r} \phi}^2, \tag{2.14} \]

and, in general, is not constant along the trajectory of the spinning particle. In the Schwarzschild field, the total spin must be small enough compared to the mass of the test particle and of the black hole \(|s|/(mM) \ll 1\) for the approximation of the Mathisson–Papapetrou model to be valid. This inequality follows from requiring that the characteristic length scale \(|s|/m\) associated with the particle’s internal structure be small compared to the natural length scale \(M\) associated with the background field in order that the particle backreaction can be neglected, i.e., that the description of a test particle on a background field make sense [10].

3. Solving the equations of motion: preliminary steps

Consider first the evolution equation for the spin tensor (1.2). By contracting both sides of equation (1.2) with \( U_c \), one obtains the following expression for the total 4-momentum:

\[ P^\mu = -(U \cdot P) U^\mu - U^\alpha \frac{DS^{\mu \nu}}{dtU} \equiv mU^\mu + P^\mu_s, \tag{3.1} \]

which then defines the particle’s mass \( m \), which \textit{a priori} does not have to be constant along the orbit, while \( P^\mu_s = U_\alpha DS^{\mu \nu}/dtU \) is the part of the 4-momentum orthogonal to \( U \). Finally, let \( U_\rho \) denote the timelike unit vector associated with the total 4-momentum \( P = \|P\|U_\rho \equiv \mu U_\rho \).

Backsubstituting this representation, equation (3.1) of the momentum into the spin evolution equation (1.2) expressed in the static observer frame leads to

\[ 0 = \frac{dS_{t\phi}}{dtU} - v \frac{dS_{t\text{h}}}{dtU} + \gamma \frac{\xi_{\text{K}}}{\nu_{\text{K}}} (\nu^2 - \nu_{\text{K}}^2) S_{t\text{h}}, \tag{3.2} \]

\[ 0 = \frac{dS_{t\text{h}}}{dtU} - v \frac{dS_{t\text{h}}}{dtU} - \frac{\nu \gamma}{\nu_{\text{K}}} \xi_{\text{K}} S_{t\phi}, \tag{3.3} \]

\[ 0 = \frac{dS_{t\phi}}{dtU} - \gamma \frac{\nu_{\text{K}} \xi_{\text{K}}}{\nu_{\text{K}}} S_{t\text{h}} + \gamma \frac{\nu \xi_{\text{K}}}{\nu_{\text{K}}} S_{t\phi}. \tag{3.4} \]
From (3.1), using the definition of \( P_s \) and equations (3.2)–(3.4), it follows that the total 4-momentum \( P \) can be written in the form
\[
P = \gamma (m + \nu m_s) e_{\hat{t}} + \frac{1}{\gamma} \left[ \frac{dS_{\hat{t}}}{d\tau_U} - \gamma v \frac{\xi K}{v K} S_{\hat{\phi}} \right] e_{\hat{r}} + \frac{1}{\gamma} \left[ \frac{dS_{\hat{\phi}}}{d\tau_U} - \gamma v K \xi K S_{\hat{\phi}} \right] e_{\hat{\phi}} + \gamma (m v + m_s) e_{\hat{\delta}}
\]
with
\[
m_s = \frac{dS_{\hat{\delta}}}{d\tau_U} + \gamma v K \xi K S_{\hat{\phi}} - \gamma v K \xi K S_{\hat{\delta}}.
\]

Next, consider the equation of motion (1.1). The Riemann tensor spin–curvature coupling force term is
\[
F^{(sc)} = \gamma \xi K \left( v S_{\hat{\phi}} e_{\hat{t}} + [2 S_{\hat{t}} + v S_{\hat{\phi}}] e_{\hat{r}} - [S_{\hat{t}} + 2 v S_{\hat{\phi}}] e_{\hat{\delta}} - S_{\hat{t}} e_{\hat{\delta}} \right).
\]
Using (3.1), the balance condition which allows a circular orbit of this type to exist can be written as
\[
ma(U) = F^{(so)} + F^{(sc)}.
\]
where \( a(U) \) is the acceleration of the \( U \) orbit and \( F^{(so)} \equiv -\frac{dP}{d\tau_U} \) defines the spin–orbit coupling force term, which arises from the variation of the spin along the orbit.

Taking (3.5) and (3.6) into account, equation (1.1) gives rise to the following set of ordinary differential equations:
\[
0 = \frac{d^2 S_{\hat{t}}}{d\tau_U^2} - 2 \gamma v K \xi K \frac{dS_{\hat{t}}}{d\tau_U} + \frac{\gamma}{v K} \left( v^2 + v K^2 \right) \frac{dS_{\hat{t}}}{d\tau_U} - \gamma^2 v K \xi K \left( v^2 - v K^2 \right) S_{\hat{t}} + \frac{dm}{d\tau_U}, \tag{3.9}
\]
\[
0 = \frac{d^2 S_{\hat{\phi}}}{d\tau_U^2} - \frac{d^2 S_{\hat{\delta}}}{d\tau_U^2} - 2 \gamma v K \xi K \frac{dS_{\hat{\phi}}}{d\tau_U} + v v K^2 \xi K \frac{dS_{\hat{\delta}}}{d\tau_U} - \zeta K \left[ \frac{v^2}{v K} + 2 \right] S_{\hat{\phi}} - m \gamma \frac{\xi K}{v K} \left( v^2 - v K^2 \right), \tag{3.10}
\]
\[
0 = \frac{d^2 S_{\hat{\phi}}}{d\tau_U^2} - \frac{d^2 S_{\hat{\delta}}}{d\tau_U^2} + \frac{v}{v K} \left( v^2 + v K^2 \right) \frac{dS_{\hat{\phi}}}{d\tau_U} + 2 v v K \xi K S_{\hat{\phi}} + \zeta K \left[ \frac{v^2}{v K} - 1 \right] S_{\hat{\phi}} + \frac{dm}{d\tau_U}, \tag{3.11}
\]
\[
0 = \frac{d^2 S_{\hat{\delta}}}{d\tau_U^2} - \frac{d^2 S_{\hat{\phi}}}{d\tau_U^2} - \frac{d^2 S_{\hat{\delta}}}{d\tau_U^2} + \frac{\gamma}{v K} \left( v^2 - v K^2 \right) \frac{dS_{\hat{\delta}}}{d\tau_U} + 2 \gamma v K \xi K S_{\hat{\delta}} - \zeta K \left[ \frac{v^2}{v K} - 1 \right] S_{\hat{\delta}} + \frac{dm}{d\tau_U}. \tag{3.12}
\]
Note that there are two equations containing the second derivative of \( S_{\hat{t}} \); this is due to the presence of its first derivative in two different components of \( P \) (more precisely, in \( P^t \) and \( P^\phi \), see equations (3.5) and (3.6)).

Once the system of constant coefficient linear differential equations (3.2)–(3.4) and (3.9)–(3.12) is solved for \( m \) and the spin tensor components, one may then calculate \( P \). The system must be decoupled, leading to functions which are either exponentials, sinusoids or at most quadratic functions of the proper time along the particle world line. The elimination method for decoupling the equations is crucially different depending on whether \( v \) has the values 0 or \( \pm v K \) or none of these values, since one or the other or neither term drops out of the spin equation (3.2) and so must be considered separately. From the details of their derivations discussed in the appendix, one sees why there are several zones approaching the horizon where the solutions change character.
4. Particles at rest: the \( \nu = 0 \) case

When the particle is at rest, the solutions for the components of the spin tensor and the varying mass \( m \) of the spinning particle are given by

(i) \( 2M < r < 3M \):

\[
\begin{align*}
S_{00} &= c_1, \\
S_{ij} &= c_2 e^{\omega_0 t} + c_3 e^{-\omega_0 t} + \frac{v_K}{\xi_k} \frac{c_m}{2 + v_K}, \\
m &= -v_K \xi_k [c_2 e^{\omega_0 t} + c_3 e^{-\omega_0 t}] + \frac{2c_m}{2 + v_K},
\end{align*}
\]

\( (4.1) \)

(ii) \( r = 3M \):

\[
\begin{align*}
S_{00} &= c_1, \\
S_{ij} &= c_2 e^{\gamma/3M} + c_3 e^{-\gamma/3M} + \sqrt{3} M c_m, \\
m &= \frac{\sqrt{3}}{9M} [c_2 e^{\gamma/3M} + c_3 e^{-\gamma/3M}] + \frac{2c_m}{3},
\end{align*}
\]

\( (4.2) \)

(iii) \( r > 3M \):

\[
\begin{align*}
S_{00} &= c_1, \\
S_{ij} &= c_2 e^{\omega_0 t} + c_3 e^{-\omega_0 t} + \frac{v_K}{\xi_k} \frac{c_m}{2 + v_K}, \\
m &= -v_K \xi_k [c_2 e^{\omega_0 t} + c_3 e^{-\omega_0 t}] + \frac{2c_m}{2 + v_K}.
\end{align*}
\]

\( (4.3) \)

where \( c_m, c_1, \ldots, c_9 \) are the integration constants and

\[
\omega_0 = i \omega_0 = \frac{\xi_k}{\gamma_K} = \sqrt{\frac{M(r - 3M)}{r^3(r - 2M)}}, \quad \omega_1 = \xi_k \left(2 + v_K^2\right)^{1/2} = \sqrt{\frac{M(2r - 3M)}{r^3(r - 2M)}}.
\]

From equation (3.5), the total 4-momentum \( P \) then has the value

\[
P = m e_1 + \omega_0 [c_2 e^{\omega_0 t} - c_3 e^{-\omega_0 t}] e_j - \frac{\xi_k}{v_K} \left[S_{ij} - \frac{c_8}{\gamma_K}\right] e_j - \frac{\xi_k}{v_K} \left[S_{i\phi} - \frac{c_9}{\gamma_K}\right] e_{\phi}
\]

\( (4.5) \)
in cases (i) and (iii), and

$$P = me + \frac{1}{3M} [c_2 e^{\gamma/(3M)} - c_3^{-\gamma/(3M^2)}] e_\gamma - \left[ \frac{\sqrt{3}}{9M} S_{\vec{t}\theta} - c_4 \right] e_{\vec{t}} - \left[ \frac{\sqrt{3}}{9M} S_{\vec{r} \phi} - c_6 \right] e_{\vec{r}} \quad (4.6)$$

in case (ii).

At this point, the supplementary conditions impose constraints on the constants of integration which appear in the solution. For a particle at rest ($v = 0$), the CP and P conditions coincide and imply that $S_{\vec{t}\theta} = 0$, namely

$$c_2 = c_3 = c_4 = c_5 = c_6 = c_7 = 0, \quad c_m = 0, \quad (4.7)$$

leaving arbitrary values for $c_1$, $c_8$ and $c_9$. As a consequence, $m$ should be 0 as well, implying that $P$ should be spacelike and therefore physically inconsistent.

The T supplementary conditions when $v = 0$ imply instead

$$0 = S_{\vec{t}\theta} \frac{dS_{\vec{t}\theta}}{dt_U} + S_{\vec{t}\phi} \frac{dS_{\vec{t}\phi}}{dt_U} + S_{\vec{t}\gamma} \frac{dS_{\vec{t}\gamma}}{dt_U} - v_k \zeta_k [S_{\vec{t}\theta} S_{\vec{t}\phi} + S_{\vec{t}\phi} S_{\vec{t}\theta}],$$

$$0 = S_{\vec{t}\theta} \frac{dS_{\vec{t}\theta}}{dt_U} + S_{\vec{t}\phi} \frac{dS_{\vec{t}\phi}}{dt_U} - v_k \zeta_k (S_{\vec{t}\theta}^2 + S_{\vec{t}\phi}^2) - m S_{\vec{t}\gamma},$$

$$0 = S_{\vec{t}\gamma} \frac{dS_{\vec{t}\gamma}}{dt_U} - S_{\vec{t}\theta} \frac{dS_{\vec{t}\theta}}{dt_U} - v_k \zeta_k S_{\vec{t}\phi} S_{\vec{t}\phi} - m S_{\vec{t}\theta},$$

$$0 = S_{\vec{t}\phi} \frac{dS_{\vec{t}\phi}}{dt_U} + S_{\vec{t}\gamma} \frac{dS_{\vec{t}\gamma}}{dt_U} - v_k \zeta_k S_{\vec{t}\theta} S_{\vec{t}\phi} + m S_{\vec{t}\phi}. \quad (4.8)$$

By substituting the solutions given by equations (4.1)–(4.3) into these equations, one finds that all the integration constants except $c_1$ must vanish. This, in turn, implies $m = 0$, which again leads to a spacelike $P$. Thus, a spinning particle with nonzero rest mass cannot remain at rest in the given gravitational field.

5. Geodesic motion: the $\nu = \pm v_K$ case

When the test particle’s centre of mass moves along a geodesic (the orbit has zero acceleration $a(U) = 0$) with azimuthal velocity $v = \pm v_K$, the spin-curvature and the spin–orbit forces balance each other (see equation (3.8)): $F^{(\phi)} = -F^{(\omega)}$. The solution of equations (3.9)–(3.12) determines the spin which leads to this balancing. In the Schwarzschild spacetime, timelike circular geodesics only exist for $r > 3M$. We consider separately the various cases:

(i) $3M < r < 6M$:

$$S_{\vec{t}\theta} = c_1 \cos \omega_4 \tau + c_4 \sin \omega_4 \tau,$$

$$S_{\vec{t}\phi} = c_5 \cos \omega_3 \tau + c_6 \sin \omega_3 \tau \pm \frac{S_{\vec{t}\phi}^2}{v_K},$$

$$S_{\vec{t}\gamma} = \gamma_K v_K [c_5 \sin \omega_3 \tau - c_6 \cos \omega_3 \tau],$$

$$m = c_m, \quad (5.1)$$

$$S_{\vec{t}\theta} = c_7 e^{\omega_2 \tau} + c_8 e^{-\omega_2 \tau} \pm \frac{2 \gamma_K}{\xi_K} \frac{c_2}{4 - 3\gamma_K^2},$$

$$S_{\vec{t}\phi} = \pm v_K [c_7 e^{\omega_2 \tau} + c_8 e^{-\omega_2 \tau}] + c_1 + \frac{2 \gamma_K}{\xi_K} \frac{c_2}{4 - 3\gamma_K^2},$$

$$S_{\vec{t}\gamma} = \pm \frac{2}{\gamma_K} \left( 3\gamma_K^2 - 4 \right)^{-1/2} [c_7 e^{\omega_2 \tau} - c_1 e^{\omega_2 \tau}] + c_3 - 3 \gamma_K^2 \frac{c_2}{4 - 3\gamma_K^2} \tau;$$
(ii) \( r = 6M \):
\[
S_{\delta \delta} = c_3 \cos \frac{\sqrt{3} r}{18M} + c_4 \sin \frac{\sqrt{3} r}{18M},
\]
\[
S_{\delta \theta} = c_5 \cos \frac{\sqrt{6} r}{36M} + c_6 \sin \frac{\sqrt{6} r}{36M} \pm 2S_{\delta \phi},
\]
\[
S_{\delta \phi} = \frac{\sqrt{3}}{3} \left[ c_5 \sin \frac{\sqrt{6} r}{36M} - c_6 \cos \frac{\sqrt{6} r}{36M} \right],
\]
\[
m = c_m,
\]
\[
S_{\theta \theta} = c_7 \tau + c_8,
\]
\[
S_{\theta \phi} = \pm \frac{\sqrt{2} r}{12M} \left[ \frac{c_7}{2} \tau + c_8 \right] + c_9.
\]

(iii) \( r > 6M \):
\[
S_{\delta \delta} = c_3 \cos \omega_3 \tau + c_4 \sin \omega_3 \tau,
\]
\[
S_{\delta \theta} = c_5 \cos \omega_3 \tau + c_6 \sin \omega_3 \tau \pm \frac{S_{\delta \phi}}{v_K},
\]
\[
S_{\delta \phi} = \gamma_K v_K \left[ c_5 \sin \omega_3 \tau - c_6 \cos \omega_3 \tau \right],
\]
\[
m = c_m,
\]
\[
S_{\theta \theta} = c_7 \cos \omega_2 \tau + c_8 \sin \omega_2 \tau \pm \frac{\gamma_K v_M}{\zeta_K} \frac{c_2}{4 - 3\gamma_K^2},
\]
\[
S_{\phi \phi} = \pm \sqrt{2} \tau \left[ \frac{c_7}{2} \tau + c_8 \right] + c_9 - \frac{3\gamma_K^2}{4 - 3\gamma_K^2} \frac{c_2}{4 - 3\gamma_K^2} \tau,
\]

where \( c_m, c_1, \ldots, c_9 \) are the integration constants, and three real frequencies are defined for each open interval of radial values by
\[
\omega_2 = i\omega_2 = \zeta_K \left( 4 - 3\gamma_K^2 \right)^{1/2} = \sqrt{\frac{M(r - 6M)}{r^3(r - 3M)}},
\]
\[
\omega_3 = \zeta_K = \left( \frac{M}{r^3} \right)^{1/2},
\]
\[
\omega_4 = i\omega_4 = \zeta_K \left( 3\gamma_K^2 - 2 \right)^{1/2} = \frac{1}{r} \sqrt{\frac{M}{r - 3M}}.
\]

Consider first the open interval cases \( r \neq 6M \). From equation (3.5), the total 4-momentum \( P \) is given by
\[
P = \left\{ m\gamma_K + \zeta_K \left[ S_{\phi \phi} - (1 - 2\nu_K^2)\nu_K^2 c_1 - \frac{\nu_K}{\gamma_K \zeta_K} \frac{c_2}{1 - 4\nu_K^2} \right] \right\} e_\ell
\]
\[
\pm \frac{\zeta_K}{\nu_K} \left\{ (1 + 2\nu_K^2) \left[ S_{\theta \theta} + \frac{c_2}{1 - 4\nu_K^2} \tau \right] + (1 - 4\nu_K^2) c_9 \right\} e_\ell
\]
\[
+ \left\{ \frac{\zeta_K}{\nu_K} \right\} \left\{ S_{\theta \phi} + (1 - 2\nu_K^2)\nu_K^2 c_1 - \frac{\nu_K}{\gamma_K} \frac{c_2}{1 - 4\nu_K^2} \right\} e_\phi
\]
\[
\pm \left\{ m\gamma_K \right\} \left\{ S_{\phi \phi} + (1 - 2\nu_K^2)\nu_K^2 c_1 - \frac{\nu_K}{\gamma_K} \frac{c_2}{1 - 4\nu_K^2} \right\} e_\phi.
\]
We next impose the standard supplementary conditions. The CP conditions imply that
\[ S_{i\dot{0}} = 0, \]
only leading to the trivial solution
\[ S_{i\dot{0}} = \pm v_K S_{i\dot{0}} = 0, \quad S_{i\dot{0}} = v_K S_{i\dot{0}} = 0, \]
which lead only to the trivial solution
\[ c_1 = c_2 = c_3 = c_4 = c_5 = c_6 = c_7 = c_8 = c_9 = 0, \]
so that the only nonvanishing component of the spin tensor is \( S^2 = S_{i\dot{0}} = c_1 \equiv s \), leaving
arbitrary values for \( c_m \) as well. From equation (5.5), the total 4-momentum \( P \) becomes (using
\[ m_s = s y v_K \zeta_K \] which follows from equation (3.6))
\[ P = m U_\perp + s y v_K \zeta_K E_\dot{0}, \]
with \( U_\perp \) given by equation (2.7). Re-examining equation (3.7) shows that the spin-curvature
force then acts radially, balancing the radial spin–orbit force.

The P conditions imply
\[ S_{i\dot{0}} = 0, \quad S_{i\dot{0}} = v_K S_{i\dot{0}} = 0, \quad S_{i\dot{0}} = v_K S_{i\dot{0}} = 0, \]
which lead only to the trivial solution
\[ c_1 = c_2 = c_3 = c_4 = c_5 = c_6 = c_7 = c_8 = c_9 = 0, \]
with \( c_m \) arbitrary, or in other words the components of the spin tensor must all be zero, which
means that a nonzero spin is incompatible with geodesic motion for a spinning particle.

The T supplementary conditions when \( v = \pm v_K \) imply
\[ 0 = S_{i\dot{0}} \frac{dS_{i\dot{0}}}{dt} + S_{i0} \frac{dS_{i0}}{dt} + \gamma^2 S_{i\dot{0}} \frac{dS_{i\dot{0}}}{dt} \pm m y^2 v_K S_{i\dot{0}} \]
\[ = \gamma K \zeta_K [S_{i\dot{0}} \pm v_K S_{i\dot{0}}], \]
which implies that the only nonvanishing components of the spin tensor are
\[ S^2 = S_{i\dot{0}} = \mp c_1, \quad S_{i\dot{0}} = c_0, \]
and either
\[ c_1, c_9 \quad \text{arbitrary}, \quad c_m = \pm y K \zeta_K c_1, \]
which implies that spin component \( S^1 \) is proportional to the mass, locking them together by a
constant of proportionality depending on the orbit velocity, or
\[ c_1 = 0 = c_9, \quad c_m \quad \text{arbitrary}. \]
corresponding to the zero-spin case where geodesic motion is, of course, allowed. In the former case, the total spin invariant (2.14) reduces to
\[ s^2 = -c_9^2 + c_1^2, \]  
so that the condition \(|s|/(mM) \ll 1\) preserving the validity of the Mathisson–Papapetrou model reads
\[ \frac{|s|}{mM} = \frac{1}{M\gamma_K \xi_K} \left( 1 - \frac{c_9^2}{c_1^2} \right)^{1/2} \ll 1, \]  
implying either \(c_1 > c_9\) or \(r \approx 3M\) (where \(\gamma_K \to \infty\)). In the limit \(r \to 3M\) where the circular geodesics become null and require a separate treatment, one has a solution for which this spin component \(S^z\) is fixed to have a value determined by the constant mass \(m\) and the azimuthal velocity, the \(t-\phi\) component of the spin is arbitrary. If one takes the limit \(m \to 0\), then the component of the spin vector out of the orbit vanishes, leaving the spin vector locked to the direction of motion as found by Mashhoon [11] who discussed the null geodesic case using the P supplementary conditions, the latter being the only physically relevant in such a limit.

Finally, consider the remaining case \(r = 6M\). Equation (3.5) then shows that the total 4-momentum \(P\) is given by
\[
P = \left\{ \begin{array}{l}
\frac{2}{3} \sqrt{3m + \frac{\sqrt{6}}{108M} [3S_{\phi} - 2c_1]} \ e_t + \left\{ \frac{\sqrt{6}}{36M} S_{\phi} + \frac{\sqrt{3}}{2} c_1 \right\} e_\phi \\
- \left\{ \frac{\sqrt{6}}{18M} S_{\phi} \pm \frac{1}{6M} \left[ c_3 \sin \sqrt{\frac{3\tau}{18M}} - c_4 \cos \sqrt{\frac{3\tau}{18M}} \right] \right\} e_\phi \\
\pm \frac{1}{2} \left\{ \frac{2}{3} \sqrt{3m + \frac{\sqrt{6}}{108M} [3S_{\phi} - 2c_1]} \right\} e_\phi,
\end{array} \right.
\]  
Imposing the standard supplementary conditions gives rise to the same result as for the general case \(r \neq 6M\).

The CP conditions imply
\[ c_2 = c_3 = c_4 = c_5 = c_6 = c_7 = c_8 = 0, \]  
so that the only nonvanishing component of the spin tensor is \(S^z = -S_{\phi} = -c_1\), for arbitrary values of \(c_m\), leading to constant mass \(m\). The P conditions give only the trivial solution
\[ c_1 = c_2 = c_3 = c_4 = c_5 = c_6 = c_7 = c_8 = 0, \]  
with \(c_m\) arbitrary, leading to constant mass \(m\). Finally, the T conditions imply
\[ c_3 = c_4 = c_5 = c_6 = c_7 = c_8 = 0, \]  
so that the only nonvanishing components of the spin tensor are
\[ S^t = -S_{\phi} = -c_1, \quad S_{\phi} = c_2, \]  
and either
\[ c_1, c_2 \text{ arbitrary, } \quad c_m = \pm \frac{\sqrt{3}}{18M} c_1 \]  
or
\[ c_1 = 0 = c_2, \quad c_m \text{ arbitrary,} \]  
with constant mass \(m\) in both cases. In the former case, the spin invariant (2.14) reduces to
\[ s^2 = -c_1^2 + c_2^2. \]
so that the condition $|s|/(mM) \ll 1$ reads

$$\frac{|s|}{mM} = \frac{18}{\sqrt{2}} \left( 1 - \frac{c_2^2}{c_1^2} \right)^{1/2} \ll 1,$$

implying $c_1 \gg c_2$.

Thus, if the centre of mass of the test particle is constrained to be a circular geodesic, either the spin is forced to be zero or have an arbitrary constant value of the single nonzero component $S^\phi$ of the spin vector out of the plane of the orbit.

6. The general case: $\nu \neq 0$ and $\nu \neq \pm \nu_K$

For general circular orbits excluding the previous cases $v = 0$ and $v = \pm v_K$, the solutions of the equations of motion for the components of the spin tensor and the mass $m$ of the spinning particle are

$$S_{\theta \phi} = A \cos \Omega \tau + B \sin \Omega \tau,$$

$$S_{\phi \phi} = C \cos \Omega_1 \tau + D \sin \Omega_1 \tau + F S_{\theta \phi}, \quad F = \frac{3v \nu_K^2}{v^2 (1 + 2v_K^2) - v_K^2 (1 - v_K^2)},$$

$$S_{\theta \phi} = -\frac{\nu \nu_K (1 + 2v_K^2)(v^2 - v_K^2)}{\nu \nu_K (v^2 (1 + 2v_K^2) - v_K^2 (1 - v_K^2))} [A \sin \Omega \tau - B \cos \Omega \tau] + \nu_K \gamma \nu \nu_K [D \cos \Omega_1 \tau - C \sin \Omega_1 \tau],$$

$$S_{\phi \phi} = \frac{v_K \nu^2 - v_K^2}{\nu_K^2 + v_K^2 (2 + v_K^2)} \left[ \gamma \nu \nu_K - \frac{\nu^2}{\nu_K} \frac{v^2 (1 - 4v_K^2) + v_K^2 (2 + v_K^2)}{(v^2 - v_K^2)^2} c_0 \right] + c_1 e^{\Omega \tau} + c_2 e^{-\Omega \tau} + c_3 e^{\Omega_1 \tau} + c_4 e^{-\Omega_1 \tau},$$

$$S_{\phi \phi} = \frac{1}{\nu_K} \left[ (3v \nu_K + \Phi) [c_1 e^{\Omega \tau} + c_2 e^{-\Omega \tau}] + (3v \nu_K - \Phi) [c_3 e^{\Omega_1 \tau} + c_4 e^{-\Omega_1 \tau}] \right].$$

$$S_{\phi \phi} = \frac{1}{\nu_K} \left[ \nu \nu_K v^2 - v_K^2 \right] \left[ \Omega_+ - \frac{3v_K^2 + \Phi}{2v_K (1 + 2v_K^2)} - 1 \right] [c_1 e^{\Omega \tau} + c_2 e^{-\Omega \tau}] + \Omega_+ \left[ \frac{3v_K^2 - \Phi}{2v_K (1 + 2v_K^2)} - 1 \right] [c_3 e^{\Omega_1 \tau} - c_4 e^{-\Omega_1 \tau}],$$

$$m = \frac{\nu K}{\nu_K} [v S_{\theta \phi} - v_K^2 S_{\phi \phi}] + c_m,$$

where $A$, $B$, $C$, $D$, $c_m$, $c_0$, $c_1$, $c_2$, $c_3$, $c_4$ are the integration constants, and the real positive frequencies $\Omega$ and $\Omega_1$ are given by

$$\Omega = \frac{\nu K}{\nu_K} \left( 1 + 2v_K^2 \right)^{1/2} \frac{|v|}{v_K}, \quad \Omega_1 = \frac{\nu K}{\nu_K},$$

assumed to be distinct for the above equations to be valid, and the remaining abbreviations are

$$\Omega_+ = -\frac{\nu K}{\nu_K} \left[ v^2 - v_K^2 \pm \frac{\nu K^2}{2} \Phi \right]^{1/2}, \quad \Phi = 3v_K^2 \left[ 1 - \frac{v_K^2}{v^2} \right]^{1/2}. $$


Figure 1. The azimuthal velocities \( \bar{\nu} \) (\( \bar{\nu}_b \) in the plot), \( \tilde{\nu} \) (\( \tilde{\nu}_t \) in the plot) and \( \nu_K \) (dashed curve) are plotted as functions of the radial parameter \( r/M \). The dashed vertical line corresponds to the value \( r/M = 6 \), where they all coincide. There are five different regions (explicitly indicated in the plot), depending on the relative sign of the azimuthal velocity \( \nu \) with respect to \( \bar{\nu} \) and \( \tilde{\nu} \), which correspond to intervals for which \( \Omega_{\pm} \) are real, purely imaginary or neither, as explained in the text.

The behaviour of the azimuthal velocities \( \bar{\nu}, \tilde{\nu} \) and \( \nu_K \) as a function of the radial parameter \( r/M \) is compared in figure 1. They all coincide at \( r = 6M \), where \( \bar{\nu} = \tilde{\nu} = \nu_K = 1/2 \); for \( 2M < r < 6M \) one has \( \tilde{\nu} < \bar{\nu} \), while \( \tilde{\nu} > \bar{\nu} \) for \( r > 6M \).

The quantities \( \Omega_{\pm} \) also lead to angular velocities for certain intervals of values of the azimuthal velocity \( \nu \). In fact, we are interested in those values for which \( \Omega_{+} \), and/or \( \Omega_{-} \) are purely imaginary, since the imaginary parts can be interpreted as additional frequencies characterizing spin precession. One must distinguish the cases \( 2M < r < 6M \) and \( r > 6M \), referring to figure 1 and equation (6.9):

(a) \( r > 6M \):
- if \( \nu > \bar{\nu} \) (region I), the quantities \( \Omega_{\pm} \) are both complex;
- if \( \nu = \bar{\nu} \), \( \Omega_{+} = \Omega_{-} \) is purely imaginary, since \( \tilde{\nu} > \bar{\nu} \);
- if \( \tilde{\nu} < \nu < \bar{\nu} \) (region II), \( \Omega_{-} \) is purely imaginary, while \( \Omega_{+} \) can be either real (even zero) or purely imaginary;

with

\[
\bar{\nu}^2 = \frac{\nu^2 K^2}{2} \left( 1 + 2\nu^2 \right), \quad \tilde{\nu}^2 = \frac{9 \nu^2 K^2}{8 \left( 1 + 2\nu^2 \right)}. \quad (6.10)
\]
Spin precession in the Schwarzschild spacetime: circular orbits

- if \( v < \tilde{v} \) (region III), \( \Omega_+ \) is purely imaginary, while \( \Omega_- \) can be either real (even zero) or purely imaginary;
- if \( 2M < r < 6M \):
  - if \( v > \tilde{v} \) (region IV), the quantities \( \Omega_\pm \) are both complex;
  - if \( v = \tilde{v} \), \( \Omega_+ = \Omega_- \) is real, since \( \tilde{v} < \tilde{v} \);
  - if \( v < \tilde{v} \) (region V), \( \Omega_+ \) is real, since \( \tilde{v} < \tilde{v} \), while \( \Omega_- \) can be either real (even zero) or purely imaginary.

All of these remarks so far assume that the two frequencies \( \Omega \) and \( \Omega_1 \) are distinct, necessary for the decoupling procedure which leads to this solution. A different result follows in the special case \( \Omega = \Omega_1 \). This occurs for the particular value of the azimuthal velocity

\[
v_0 = \pm \frac{v_K}{\gamma_K} \left( 1 + 2v_K^2 \right)^{-1/2} = \pm \left( \frac{M(r - 3M)}{r(r - 2M)} \right)^{1/2},
\]

which vanishes at \( r = 3M \) and is real for \( r > 3M \), rising to its peak speed at \( r \approx 3.934M \) and decreasing asymptotically towards the geodesic speed from below as \( r \to \infty \). The solutions for the components \( S_{\hat{a}\hat{b}}, S_{\hat{a}\hat{d}} \) and \( S_{\hat{a}\hat{d}} \) of the spin tensor are given by

\[
S_{\hat{a}\hat{b}} = A \cos \Omega \tau + B \sin \Omega \tau,
\]
\[
S_{\hat{a}\hat{d}} = \left[ C - \frac{3\gamma_0^2}{2} \xi_K^2 (A - B \Omega \tau) \right] \cos \Omega \tau + \left[ D - \frac{3\gamma_0^2}{2} \xi_K^2 A \tau \right] \sin \Omega \tau,
\]
\[
S_{\hat{a}\hat{d}} = \left[ \frac{\gamma_0^3}{2} \xi_K^2 (B + A \Omega \tau) + \frac{2\gamma_0^2}{\gamma_0^2} (v_0B - v_K^2 D) \right] \cos \Omega \tau + \left[ 3v_0v_K^2 \xi_K^2 (2A + B \Omega \tau) - 2\frac{\gamma_0^2}{\gamma_0^2} (v_0A - v_K^2 C) \right] \sin \Omega \tau,
\]

with

\[
\Omega \equiv \Omega_1 = \frac{\xi_K}{v_K} \left[ \frac{1 + 2v_K^2}{1 + v_K^2 (1 + v_K^2)} \right]^{1/2} = \frac{\sqrt{M}}{r} \left[ \frac{r - 3M}{r(r - 3M) + 3M^2} \right]^{1/2},
\]

while that corresponding to the remaining components as well as to the varying mass \( m \) are obtained simply by evaluating the general solutions (6.4)–(6.7) at \( v = v_0 \). The reality properties of the quantities \( \Omega_\pm \) can be determined as done in the general case, noting that \( v_0 < \tilde{v} \) (corresponding to region V) always holds in the interval \( 3M < r < 6M \). For \( r > 6M \) however, we must distinguish two different regions: (a) \( 6M < r < \tilde{r}_0 \), with \( \tilde{r}_0 = 6M(1 + \sqrt{2}/2) \approx 10.24M \) such that \( v_0 = \tilde{v} \), where \( v_0 < \tilde{v} \) (corresponding to region III); and (b) \( r > \tilde{r}_0 \), where \( v_0 > \tilde{v} \) (corresponding to region II).

The behaviour of \( S, U \) and \( P \) along the world line itself is completely determined by the initial conditions

\[
S_{\hat{a}\hat{b}}(0), \quad \frac{dS_{\hat{a}\hat{b}}}{d\tau U} \bigg|_{\tau = 0}
\]

and the corresponding conditions on the mass \( m \) of the particle which follow from equation (6.7). Thus, in the special case in which the ‘centre of mass line’ is directed along a circular orbit, the completion of the scheme for the spinning test particle is equivalent to a choice of initial conditions.

In principle, the components of the spin tensor which are not constants should precess with the different frequencies which appear in equations (6.1)–(6.6), leading to non-periodic
motion, a feature that seems to characterize the general situation in the Schwarzschild [12] and Kerr [13–15] spacetimes. However, this does not occur in practice once the CP, P and T supplementary conditions are imposed, as we will see below. It turns out that the nonvanishing components of the spin tensor are all constant in the CP and T cases, while the motion is periodic with a unique frequency in the P case. As one might expect, the particle mass $m$ turns out to be constant in all three cases.

6.1. The CP supplementary conditions

The CP supplementary conditions require

$$S_{t\hat{r}} = 0, \quad S_{t\hat{\theta}} = 0, \quad S_{t\hat{\phi}} = 0.$$  \hspace{2cm} (6.17)

From equation (6.5), this forces

$$c_1 = c_2 = c_3 = c_4 = 0, \quad c_0 = \frac{\gamma}{\nu v_K \nu K} \left( v^2 - v_K^2 \right) c_m.$$  \hspace{2cm} (6.18)

Substituting these values into equation (6.4) then gives

$$c_m = -\frac{\gamma v \zeta K}{\nu \nu K} S_{\hat{r} \hat{\phi}},$$  \hspace{2cm} (6.19)

so from equation (6.7) we get

$$S_{\hat{r} \hat{\phi}} = s = \frac{m}{\nu v_K \xi K}.$$  \hspace{2cm} (6.20)

Finally, from equations (6.1) and (6.3), it follows that

$$S_{\hat{t} \hat{\theta}} = 0, \quad S_{\hat{r} \hat{\theta}} = 0.$$  \hspace{2cm} (6.21)

However, from equation (3.5), it follows that

$$P = -sv_K \zeta K e_{\hat{\phi}},$$  \hspace{2cm} (6.22)

since $m_\gamma = -s v_K \zeta K = -m/v$, a consequence of equations (3.6) and (6.20). This result is unphysical since the total 4-momentum $P$ is spacelike.

6.2. The P supplementary conditions

The P supplementary conditions require

$$S_{\hat{r} \hat{\phi}} = 0, \quad S_{\hat{t} \hat{r}} + S_{\hat{r} \hat{\phi}} v = 0, \quad S_{\hat{t} \hat{\theta}} + S_{\hat{\theta} \hat{\phi}} v = 0.$$  \hspace{2cm} (6.23)

Under these conditions, the components of the spin vector $S_U$ in the local rest space of the particle, $S_U^\beta = \frac{1}{2} \eta_\alpha^{\beta \gamma \delta} U^\alpha S_{\gamma \delta}$, expressed in the Frenet–Serret frame, are just

$$\left( S_U^1, S_U^2, S_U^3 \right) = (\gamma^{-1} S_{\hat{t} \hat{\theta}}, S_{\hat{r} \hat{\theta}}, \gamma^{-1} S_{\hat{r} \hat{\phi}}).$$  \hspace{2cm} (6.24)

Comparing the first equation (6.23) with equation (6.6) we get

$$c_1 = c_2 = c_3 = c_4 = 0,$$  \hspace{2cm} (6.25)

so that $S_{\hat{t} \hat{\theta}}$, $S_{\hat{r} \hat{\phi}}$ and the particle mass $m$ are all constant. Equations (6.4) and (6.5) together with the second of the Pirani conditions (6.23) imply

$$c_0 = \frac{\gamma v}{v_K} \frac{\left( 1 + v^2 \right) v^2}{\gamma K} \frac{(v^2 - v_K^2)^2}{v^2(2 - 5v_K^2) + v_K^2(1 + 2v_K^2)} c_m.$$  \hspace{2cm} (6.26)
hence from equation (6.7)
\[ c_m = \left[ 1 + \frac{1}{\gamma_k} \frac{v^2 - v_k^2}{v^2(1 - 4v_k^2) + v_k^2(2 + v_k^2)} \right] \, m. \] (6.27)

Next, by substituting these values of the constants \( c_0 \) and \( c_m \) into equation (6.4), we obtain
\[ \nu S^3_U = S^3 = S_{\nu \phi} = -m \frac{\nu K}{v \xi K} \frac{v^2 - v_k^2}{v^2(1 - 4v_k^2) + v_k^2(2 + v_k^2)} \cdot \] (6.28)

Finally, comparing the last of the Pirani conditions (6.23) with equations (6.1)–(6.2) and (6.12)–(6.13) leads to two possibilities: either

case P1: \( A = B = C = D = 0 \),
which places no constraint on \( \nu \) and the spin vector is constant and out of the plane of the orbit, or

case P2: \( C = 0 = D, \quad F = \nu \),
the latter of which (again from equation (6.2)) leads to the special azimuthal velocity
\[ \nu = \nu(P2) = \pm 2v_k \left( \frac{1 - v_k^2}{1 + 2v_k^2} \right)^{1/2} = \pm 2 \left( \frac{M(r - 9M/4)}{r(r - 2M)} \right)^{1/2} \cdot \] (6.31)
The case P1 has already been considered previously [6], leaving only P2 to be considered here. The corresponding azimuthal speed \( \nu(P2) \) vanishes at \( r = 9/4M \) and is real for \( r > 9/4M \), rising to a maximum speed of 1 at \( r = 3M \), corresponding to the two null circular geodesics, and decreasing asymptotically towards twice the geodesic speed from below as \( r \to \infty \).

The corresponding values of \( \gamma \) and \( \Omega \) are, respectively,
\[ \gamma(P2) = \left( \frac{1 + 2v_k^2}{1 - v_k^2} \right)^{1/2} = \frac{\sqrt{r(r - 2M)}}{r - 3M} \] (6.32)
and
\[ \Omega(P2) = \gamma_k \xi K \left[ (4 - v_k^2)(1 + 2v_k^2) \right]^{1/2} = \frac{\sqrt{M \frac{4r - 9M}{r - 3M}}} {r - 3M} \] (6.33)
using equation (2.6). To get the angular velocity of precession with respect to a frame which is nonrotating with respect to infinity, one must subtract away the precession angular velocity \( \Omega_U = \gamma \nu / r \) of the spherical frame. In the case P2, one finds \( \Omega(P2) - \Omega_U = 0 \), so the spin does not precess with respect to a frame which is nonrotating at infinity.

Substituting these values back into equation (6.28) then leads to
\[ S^3_U = \pm \frac{m}{2\Omega(P2)} \] (6.34)
The remaining nonzero spin components (6.1)–(6.3) can then be expressed in the form
\[ S^1_U = \gamma^{-1} S_{\theta \phi} = \gamma^{-1} \left[ A \cos \Omega(P2) \tau_U + B \sin \Omega(P2) \tau_U \right], \] \[ S^2_U = S_{\phi \theta} = \gamma^{-1} \left[ A \sin \Omega(P2) \tau_U - B \cos \Omega(P2) \tau_U \right], \] \[ S^1_U(0) = \gamma^{-1} A, \quad S^2_U(0) = -\gamma^{-1} B, \quad S^3_U(0) = \pm \frac{m}{2\Omega(P2)} \] (6.35) leading to
\[ \begin{pmatrix} S^1_U \\ S^2_U \\ S^3_U \end{pmatrix} = \begin{pmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} S^1_U(0) \\ S^2_U(0) \\ S^3_U(0) \end{pmatrix}. \] (6.37)
The spin invariant (2.14) becomes in this case
\[ s^2 = \frac{1}{\gamma^2} \left[ A^2 + B^2 + \frac{m^2}{4 \xi_k^2} \right]. \]  
(6.38)

The Mathisson–Papapetrou model is valid if the condition \(|s|/(mM) \ll 1\) is satisfied. From the previous equation we have that either \(\gamma \to \infty\) or the sum of the bracketed terms must be small, i.e.,
\[ \frac{A^2}{m^2 M^2} \ll 1, \quad \frac{B^2}{m^2 M^2} \ll 1, \quad \left[ 4M^2 \xi_k^2 (4 - v_k^2) \right]^{-1} \ll 1. \]  
(6.39)
The latter possibility cannot occur for any allowed values of \(r/M\), since the third term (which is dimensionless) of (6.39) is always greater than \(\approx 1.88\), as is easily verified. The former possibility is realized only in the case of ultrarelativistic motion, which equation (6.32) implies occurs only as \(r \to 3M\), where the orbits approach null geodesics and the limit \(m \to 0\) forces the component of the spin vector out of the plane of the orbit to vanish, locking the spin vector to the direction of motion exactly as discussed by Mashhoon [11].

It is well known that the spin vector \(s_U = S_\delta T_\delta\), lying in the local rest space of \(U\), is Fermi–Walker transported along \(U\) in the \(P\) case, so it must satisfy
\[ 0 = \frac{D_{(\omega)} S_U}{d\tau_U} \equiv P(U) \frac{D S_U}{d\tau_U} = \begin{bmatrix} \frac{dS_U^1}{d\tau_U} + S_U^0 \tau_1 \\ \frac{dS_U^2}{d\tau_U} + S_U^0 \tau_1 \\ \frac{dS_U^3}{d\tau_U} - S_U^0 \tau_1 \end{bmatrix} E_1 + \begin{bmatrix} \frac{dS_U^1}{d\tau_U} - S_U^0 \tau_1 \\ \frac{dS_U^2}{d\tau_U} - S_U^0 \tau_1 \end{bmatrix} E_2, \]  
(6.40)
from (2.10), where \(P(U) E_\alpha = \delta_\alpha^0 + U^\mu U_\mu\) projects into the local rest space of \(U\). To check this we must show that the following two equations are identically satisfied:
\[ \frac{dS_U^1}{d\tau_U} + \tau_1 S_U^2 = 0, \quad \frac{dS_U^2}{d\tau_U} - \tau_1 S_U^1 = 0. \]  
(6.41)
But these two equations follow immediately from (6.35), since \(\tau_1 = \Omega_{(P2)}\) results from the direct evaluation of the expression (2.11) for \(\tau_1\), with \(v\) given by (6.31).

Thus, given the rest mass \(m\) of the test particle, the constant component of the spin orthogonal to the orbit is then fixed by the orbit parameters, while the component in the plane of the orbit as seen within the local rest space of the particle itself is locked to a direction which is fixed with respect to the distant observers, since the angle of precession with respect to the spherical axes is exactly the azimuthal angle of the orbit, but in the opposite sense. In other words, the precession of the spin, which introduces a time-varying force into the mix, must be locked to the first torsion of the orbit itself in order to maintain the alignment of the 4-velocity with a static direction in the spacetime, and the spin does not precess with respect to observers at spatial infinity. Furthermore, the specific spin of the test particle cannot be made arbitrarily small except near the limiting radius where the 4-velocity of this solution goes null, and the spin vector is then locked to the direction of motion. Apparently, the imposition of a circular orbit on the centre of mass world line of the test particle is just too strong a condition to describe an interesting spin precession.

The total 4-momentum \(P\) given by equations (3.5) and (3.6) can be written in this case as
\[ P = mU + m_s E_2 + \gamma^{-1} \left[ \frac{dS_U^0}{d\tau_U} - \gamma v_k \xi_k S_{\phi} \right] e_\phi. \]  
(6.42)
with \(v\) given by (6.31) and \(m_s\) a constant
\[ m_s = \gamma \frac{\xi_k}{v_k} (v S_\phi - v_k^2 S_{\phi}) = \gamma^2 \frac{\xi_k}{v_k} (v^2 - v_k^2) S_{\phi}^3. \]  
(6.43)
but the final term in $P$ (out of the plane of the orbit) oscillates as the spin precesses in the plane of the orbit. Note that the radial component of $P$ is zero.

The spin-curvature force (3.7) simplifies to

$$F^{(sc)} = \gamma \zeta_k^2 [2S_{\theta\phi} + vS_{\phi\phi}e_\tau - [S_{\theta\phi} + 2vS_{\phi\phi}]e_\phi],$$

$$= 3\gamma^2v\zeta_k^2 (S_{\theta\tau} e_\tau - S_{\phi\phi} e_\phi), \tag{6.44}$$

while the term on the left-hand side of equation (1.1) can be written as

$$\frac{dP}{d\tau_U} = [m\kappa - m_\kappa e_\tau - \gamma \zeta_k^2 [S_{\theta\phi} + 2vS_{\phi\phi}]e_\phi]$$

$$= [m\kappa - m_\kappa e_\tau - 3\gamma^2v\zeta_k^2 S_{\theta\tau} e_\tau]. \tag{6.45}$$

The force balance equation (3.8) reduces to

$$ma(U)_\tau = F^{(sc)}_\tau,$$

$$ma(U)_\phi = 0 = F^{(so)}_\phi,$$

$$ma(U)_\theta = \kappa,$$

$$F^{(so)}_\tau = -m_s \left( \frac{DE_\phi}{d\tau_U} \right)_\tau = m_s \tau_1,$$

$$F^{(sc)}_\tau = 3\gamma^2v\zeta_k^2 S_{\theta\tau}^2,$$

$$F^{(so)}_\phi = -m \left( \frac{DE_\phi}{d\tau_U} \right)_\phi = -3\gamma^2v\zeta_k^2 S_{\theta\phi}^2. \tag{6.47}$$

### 6.3. The Tulczyjew (T) supplementary conditions

The T supplementary conditions imply from (3.1)

$$0 = \zeta_{\theta\phi} \frac{dS_{\theta\phi}}{d\tau_U} + S_{\theta\phi} \frac{dS_{\theta\phi}}{d\tau_U} + \frac{2\gamma^2 S_{\theta\phi} \frac{dS_{\theta\phi}}{d\tau_U} + m\gamma^2 vS_{\theta\phi}}$$

$$- \gamma \kappa K \left\{ [vS_{\theta\tau} + v^2 S_{\phi\phi} S_{\phi\phi}] - \gamma \zeta_k^2 S_{\theta\tau} [vS_{\theta\tau} - v^2 S_{\phi\phi} S_{\phi\phi}] \right\},$$

$$0 = \zeta_{\phi\phi} \left[ \frac{dS_{\phi\phi}}{d\tau_U} - \gamma^2 [vS_{\phi\phi} - S_{\phi\phi}] \frac{dS_{\phi\phi}}{d\tau_U} + \gamma \kappa K [S_{\phi\phi}^2 - v^2 S_{\phi\phi}^2] - \gamma \zeta_k^2 [vS_{\phi\phi} - v^2 S_{\phi\phi} S_{\phi\phi}] \right]$$

$$0 = \gamma^2 [vS_{\phi\phi} - S_{\phi\phi}] \left[ \frac{dS_{\phi\phi}}{d\tau_U} - \gamma \kappa K S_{\phi\phi} \frac{dS_{\phi\phi}}{d\tau_U} + \gamma \zeta_k^2 \right]$$

$$0 = \gamma^2 \left[ vS_{\phi\phi} - S_{\phi\phi} \right] \left[ \frac{dS_{\phi\phi}}{d\tau_U} - \gamma \kappa K S_{\phi\phi} \frac{dS_{\phi\phi}}{d\tau_U} + \gamma \zeta_k^2 \right]$$

$$0 = \gamma^2 \left[ vS_{\phi\phi} - S_{\phi\phi} \right] \left[ \frac{dS_{\phi\phi}}{d\tau_U} - \gamma \kappa K S_{\phi\phi} \frac{dS_{\phi\phi}}{d\tau_U} + \gamma \zeta_k^2 \right]$$

$$0 = \gamma^2 \left[ vS_{\phi\phi} - S_{\phi\phi} \right] \left[ \frac{dS_{\phi\phi}}{d\tau_U} - \gamma \kappa K S_{\phi\phi} \frac{dS_{\phi\phi}}{d\tau_U} + \gamma \zeta_k^2 \right]$$

By solving for the first derivatives, a straightforward calculation shows that the above set of equations simplifies to

$$0 = \frac{dS_{\theta\phi}}{d\tau_U} + m \frac{S_{\theta\phi}}{S_{\phi\phi}} - \gamma \kappa K S_{\phi\phi}, \tag{6.49}$$

$$0 = \frac{dS_{\phi\phi}}{d\tau_U} + m \gamma K \left[ vS_{\phi\phi} - v^2 S_{\phi\phi} S_{\phi\phi} \right], \tag{6.50}$$
leads to

\[ 0 = \frac{dS_{i\phi}}{dt_U} - \frac{m}{S_{i\phi}} - \gamma'vK \xi K S_{i\phi}, \quad (6.51) \]

\[ 0 = \frac{m}{\gamma' S_{i\phi}} \left[ S_{i\phi} S_{i\phi} - S_{i\phi} S_{i\phi} - S_{i\phi} S_{i\phi} \right]. \quad (6.52) \]

provided that \( S_{i\phi} \neq 0 \) is assumed. Substituting equations (A.30), (6.7) and then equations (A.34) and (A.35) into equation (6.50) (see the equations listed in appendix A.3) leads to

\[ 0 = S_{i\phi} - \frac{vK}{\xi K} \left[ \gamma'vcm - \frac{vK c_0}{\xi K \nu^2 - vK} \right]. \quad (6.53) \]

Substituting equation (6.4) into equation (6.53) leads to

\[ c_1 = c_2 = c_3 = c_4 = 0, \quad c_0 = \frac{\gamma'v \xi K}{\nu', vK} \left[ 1 + \frac{1}{\nu^2} + \frac{1}{vK^2} \right] c_m. \quad (6.54) \]

implying that \( S_{i\phi} = 0 \), and \( \theta \) and \( S_{i\phi} \) are constant, from equation (6.6), and equations (6.4) and (6.5), respectively. But from equation (6.49), it follows that \( S_{i\phi} = 0 \), or \( A = B = C = D = 0 \), from equation (6.2), so that \( S_{i\phi} = 0 \) and \( S_{i\phi} = 0 \) as well, from equations (6.1) and (6.3), respectively. This contradicts the assumption \( S_{i\phi} \neq 0 \), so only the case \( S_{i\phi} = 0 \) remains to be considered.

If \( S_{i\phi} = 0 \), the set of equations (6.48) reduces to

\[ 0 = \frac{dS_{i\phi}}{dt_U} - \left[ \frac{m}{vS_{i\phi} - S_{i\phi}} + \gamma'vK \xi K \right] S_{i\phi}, \quad (6.55) \]

\[ 0 = \frac{dS_{i\phi}}{dt_U} - \frac{vS_{i\phi} - S_{i\phi}}{vS_{i\phi} - S_{i\phi}} + \gamma'vK \xi K \left[ vS_{i\phi} - S_{i\phi} \right]. \quad (6.56) \]

provided that \( vS_{i\phi} - S_{i\phi} \neq 0 \). Substituting equations (A.30), (6.7) and then equations (A.34) and (A.35) into equation (6.56), we obtain

\[ 0 = \xi K \left( \nu^2 - vK^2 \right) \left[ vK^2 S_{i\phi}^2 - S_{i\phi}^2 \right] + vK^2 c_0 [vS_{i\phi} - S_{i\phi}] - \gamma'vK \xi K c_m (\nu^2 - vK^2) [S_{i\phi} - vS_{i\phi}]. \quad (6.57) \]

Substituting equation (6.4) into equation (6.57) then gives

\[ c_1 = c_2 = c_3 = c_4 = 0, \quad (6.58) \]

implying

\[ S_{i\phi} = 0, \quad \frac{dS_{i\phi}}{dt_U} = 0 = \frac{dS_{i\phi}}{dt_U}. \quad (6.59) \]

from equation (6.6), and equations (6.4) and (6.5), respectively. Hence, equation (6.55) is identically satisfied; moreover,

\[ c_0^{( \pm )} = c_m \frac{\nu'v \xi K}{2 \nu'vK} \left[ 2 - \nu^2 (1 - 5vK^2) \right] (\nu^2 - vK^2)^2 + 3vK^2 \left[ (3 - vK^2) \right] \]

\[ + vK^2 \left( 4 - vK^2 \right) \left[ \frac{vK^2}{\nu^2} + \frac{vK^2}{2 + vK^2} \right] \left[ \nu^2 \left( 13vK^2 + 4v^2 \right) - 8vK^2 \right] \]

\[ \times \left[ \nu^2 + vK^2 \left( 2 + vK^2 \right) \right]^{-1}. \quad (6.60) \]
Next, substituting equations (6.4) and (6.5) and then equation (6.60) into equation (6.7) leads to
\[
c_{m}^{(\pm)} = -\frac{m}{2} \left\{ \frac{v_{K}^{2} v^{4} - v^{2} [1 - v_{K}^{2} (3 - v_{K}^{2})]}{v^{2} + 2v_{K}^{2}} \right\}^{-1} \left\{ 2 \frac{v^{6}}{v_{K}^{2}} - v^{4} [1 - v_{K}^{2} (3 + v_{K}^{2})] \right. \\
+ v^{6} v_{K}^{2} \left[ 2 - v_{K}^{2} (18 - v_{K}^{2}) \right] + 4v_{K}^{4} (2 + v_{K}^{2}) \pm \frac{v}{v_{K}} \left[ v^{2} [1 - v_{K}^{2} (3 + v_{K}^{2})] \right. \\
\left. + v_{K}^{2} (2 + v_{K}^{2}) \right\} \left[ v^{2} (13v_{K}^{2} + 4v^{2}) - 8v_{K}^{2} \right]^{1/2} \right\}. \tag{6.61}
\]

Finally, substituting equations (6.60) and (6.61) into equations (6.4) and (6.5), we obtain expressions for the only nonvanishing components of the spin tensor
\[
S_{\nu \nu} = \frac{m}{2y \zeta K} v_{K}^{2} - v_{K}^{2} \left[ \frac{v^{2} (2 - 3v_{K}^{2}) + 4v_{K}^{2} \pm \nu v_{K} \left[ v^{2} (13v_{K}^{2} + 4v^{2}) - 8v_{K}^{2} \right]^{1/2} \right]}{v_{K}^{2} (v^{2} - 2) - v^{2} [1 - v_{K}^{2} (3 - v_{K}^{2})]}, \tag{6.62}
\]
\[
S_{\nu \nu} = \frac{m}{2y \zeta K} \left\{ \frac{v_{K}^{2} (v^{2} - 2) - v^{2} [1 - v_{K}^{2} (3 - v_{K}^{2})]}{v_{K}^{2} (v^{2} - 2) - v^{2} [1 - v_{K}^{2} (3 - v_{K}^{2})]} \right\}^{-1} \left\{ \nu v_{K} \left[ (v^{2} + 2v_{K}^{2}) (4v^{2} - 3 - v_{K}^{2}) \right.ight.
\left. - 2(v^{2} - v_{K}^{2}) \right\} \left[ v^{2} (1 - 3v_{K}^{2}) - 2v_{K}^{2} \right] \left[ v^{2} (13v_{K}^{2} + 4v^{2}) - 8v_{K}^{2} \right]^{1/2} \right\}.
\tag{6.63}
\]

which are in agreement with the condition \( v_{S_{\nu \nu}} = S_{\nu \nu} \neq 0 \) assumed above. This solution, having constant spin components, was already found in previous work [6].

Equation (6.59) together with the fact that \( S_{\dot{\nu} \dot{\nu}} = 0 \) and \( S_{\dot{\nu} \dot{\nu}} = 0 \) shows that the total 4-momentum \( P \) (see equation (3.5)) also lies in the cylinder of the circular orbit
\[
P = mU + m_{s}E_{\phi}, \tag{6.64}
\]
with
\[
m_{s} = y \frac{\zeta K}{v_{K}} (v_{S_{\nu \nu}} - v_{K}^{2} s_{\nu \nu}). \tag{6.65}
\]

It can therefore be written in the form \( P = \mu U_{\nu} \) with
\[
U_{\nu} = \gamma_{\nu} [e_{\nu} + v_{\nu} e_{\phi}], \quad v_{\nu} = \frac{v + m_{s}/m}{1 + vm_{s}/m}, \quad \mu = \frac{\gamma_{\nu}}{y_{\nu}} (m + vm_{s}), \tag{6.66}
\]
where \( y_{\nu} = (1 - v_{\nu}^{2})^{-1/2} \), provided that \( m + vm_{s} \neq 0 \). The T supplementary conditions can then be written as
\[
S_{\dot{i} \dot{i}} = 0, \quad S_{\dot{i} \dot{i}} + S_{\dot{\nu} \dot{\nu}} v_{\nu} = 0, \quad S_{\dot{\nu} \dot{\nu}} + S_{\dot{\nu} \dot{\nu}} v_{\nu} = 0, \tag{6.67}
\]
the last condition being identically satisfied, and with the equivalent azimuthal velocity \( v_{\nu} \) given by
\[
\nu v_{\nu} = \frac{1}{2} v_{K}^{2} + 2v_{K}^{2} \left\{ v_{K}^{2} \pm \left[ v^{2} (13v_{K}^{2} + 4v^{2}) - 8v_{K}^{2} \right]^{1/2} \right\}. \tag{6.68}
\]

from equations (6.62) and (6.63). The reality condition of (6.68) requires that \( v \) takes values outside the interval \((-\hat{v}, \hat{v})\), with \( \hat{v} = v_{K} \sqrt{2} + 3 \sqrt{33} / 4 \simeq 0.727v_{K} \); moreover, the timelike condition for \( |v_{\nu}| < 1 \) is satisfied for all values of \( v \) outside the same interval.

From (6.67), the spin vector orthogonal to \( U_{\nu} \) is just \( y_{\nu}^{-1} S_{\dot{\nu} \dot{\nu}} E_{\phi} \). The spin–curvature force (3.7) turns out to be radially directed
\[
E^{(sc)} = y \zeta K [2S_{\nu \nu} + vS_{\nu \phi}] e_{\phi}. \tag{6.69}
\]
The term on the left-hand side of equation (1.1) can be written as
\[
\frac{DP}{dt_U} = [m\kappa - m_s\tau_1]e_r, \tag{6.70}
\]
so that the balance equation (3.8) reduces to
\[
ma(U)e_r = F_{(so)}e_r + F_{(sc)}e_r, \tag{6.71}
\]
where
\[
ma(U)e_r = m\kappa, \quad F_{(so)}e_r = m_s\tau_1, \quad F_{(sc)}e_r = \gamma\zeta^2[K^2S_t e_r + \nu S_r e_{\phi}]. \tag{6.72}
\]

7. Conclusions

Spinning test particles in circular motion around a Schwarzschild black hole have been discussed in the framework of the Mathisson–Papapetrou approach supplemented by the usual standard conditions. One finds that apart from very special (and indeed artificially constrained) orbits where the spin tensor is closely matched to the curvature and torsion properties of the world line of the test particle or the static observer spin vector is constant and orthogonal to the plane of the orbit, the assumption of circular motion is not compatible with these equations. Indeed, even in the former case, the test particle assumption is then violated except in the limit of massless particles following null geodesics, where the spin vector must be aligned with the direction of motion from general considerations. The spin-curvature force generically forces the motion away from circular orbits, so one needs a much more complicated machinery to attempt to study explicit solutions of this problem, solutions which must break the stationary axisymmetry.

Appendix. Derivation of the solutions of the equations of motion

This appendix derives the solutions of the equations of motion alone (equations (3.3)–(3.4) and (3.9)–(3.12)) for the spin tensor along a circular orbit without supplementary conditions imposed. Standard elimination and differentiation techniques are used to find decoupled second-order linear constant coefficient equations for certain spin components, from which one may calculate the remaining spin components that do not already satisfy decoupled first-order such equations.

A.1. The \( \nu = 0 \) case

When the particle is at rest \( \nu = 0 \) relative to the static observers, these equations reduce to
\[
0 = \frac{dS_{\phi\phi}}{dt_U} - \nu K\zeta K S_{\phi\phi}, \tag{A.1}
\]
\[
0 = \frac{dS_{\theta\phi}}{dt_U}, \tag{A.2}
\]
\[
0 = \frac{dS_{\theta\phi}}{dt_U} - \nu K\zeta K S_{\theta\phi}, \tag{A.3}
\]
\[
0 = \frac{dm}{dt_U} + \nu K\zeta K \frac{dS_{\phi\phi}}{dt_U}, \tag{A.4}
\]
\[
0 = \frac{d^2S_{\phi\phi}}{dt_U^2} - 2\zeta^2 K S_{\phi\phi} + mv_{\phi}\zeta K. \tag{A.5}
\]
0 = \frac{d^2 S_{\theta\theta}}{d\tau_U^2} + \frac{\zeta_K}{r} S_{\theta\theta} - \zeta_K S_{\theta\theta} \frac{dS_{\theta\theta}}{d\tau_U}, \quad (A.6)

0 = \frac{d^2 S_{\phi\theta}}{d\tau_U^2} + \frac{\zeta_K}{r} S_{\phi\theta} - \zeta_K S_{\phi\theta} \frac{dS_{\phi\theta}}{d\tau_U}. \quad (A.7)

Equation (A.2) implies that \( S_{\theta\theta} = c_1 \) is constant. Solving equation (A.3) for \( dS_{\theta\theta}/d\tau_U \) and substituting the result into equation (A.6) leads to the decoupled equation

0 = \frac{d^2 S_{\theta\theta}}{d\tau_U^2} + \omega_0 S_{\theta\theta}, \quad \omega_0 = \frac{\zeta_K}{\gamma_K} = \frac{\sqrt{M(r - 3M)}}{r^3(r - 2M)}. \quad (A.8)

Similarly solving equation (A.1) for \( dS_{\phi\theta}/d\tau_U \) and substituting the result into equation (A.7) leads to an analogous decoupled equation

0 = \frac{d^2 S_{\phi\theta}}{d\tau_U^2} + \omega_0 S_{\phi\theta}. \quad (A.9)

Equation (A.4) leads immediately to \( m = -v_K \zeta_K S_{\theta\theta} + c_m \) and substituting this into (A.5) yields

0 = \frac{d^2 S_{\theta\theta}}{d\tau_U^2} + \omega_1 S_{\theta\theta} \zeta_K c_m, \quad \omega_1 = \zeta_K (2 + \nu_K^2)^{1/2} = \frac{\sqrt{M(2r - 3M)}}{r^3(r - 2M)}. \quad (A.10)

The three second-order constant coefficient equations (A.8)–(A.10) are easily integrated, from which expressions for the remaining components of the spin tensor are then obtained from equations (A.1) and (A.3). These have either oscillatory or exponential solutions depending on whether the squared frequencies in equations (A.8) and (A.10) are positive or negative, or linear solutions when zero. This distinguishes the two intervals \( 2M < r < 3M \) and \( r > 3M \), whose corresponding solutions are given by equations (4.1) and (4.3), respectively. The special case \( r = 3M \) can be easily discussed by setting \( v_K = 1 \) and \( \zeta_K = \sqrt{3}/(9M) \) in equations (A.1)–(A.7). The corresponding solution is given by equation (4.2).

A.2. The \( \nu = \pm v_K \) case

When the particle moves along a geodesic with \( \nu = \pm v_K \), equations (3.2)–(3.4) and (3.9)–(3.12) simplify to

0 = \frac{dS_{\theta\theta}}{d\tau_U} \mp v_K \frac{dS_{\theta\theta}}{d\tau_U}, \quad (A.11)

0 = \frac{dS_{\phi\theta}}{d\tau_U} \mp v_K \frac{dS_{\phi\theta}}{d\tau_U} \mp \frac{\zeta_K}{\gamma_K} S_{\phi\theta} \frac{dS_{\theta\theta}}{d\tau_U}, \quad (A.12)

0 = \frac{dS_{\phi\theta}}{d\tau_U} \pm \zeta_K \gamma_K [S_{\theta\theta} \mp v_K S_{\phi\theta}], \quad (A.13)

0 = \frac{d^2 S_{\phi\theta}}{d\tau_U^2} \pm \frac{1}{\nu_K} \frac{dm}{d\tau_U} \pm 2\zeta_K \gamma_K \left[ \frac{dS_{\theta\theta}}{d\tau_U} \mp v_K \frac{dS_{\phi\theta}}{d\tau_U} \right]. \quad (A.14)

0 = \frac{d^2 S_{\theta\theta}}{d\tau_U^2} \mp v_K \frac{d^2 S_{\phi\theta}}{d\tau_U^2} \mp 2\zeta_K \gamma_K \frac{dS_{\phi\theta}}{d\tau_U} - 3\zeta_K^2 S_{\phi\phi}, \quad (A.15)

0 = \frac{d^2 S_{\phi\theta}}{d\tau_U^2} \mp v_K \frac{d^2 S_{\phi\theta}}{d\tau_U^2} + \zeta_K [S_{\theta\theta} \pm 2v_K S_{\phi\theta}], \quad (A.16)
$0 = \frac{d^2 S_{i\phi}}{d\tau_U^2} \pm v_k \frac{dm}{d\tau_U} \mp \pm 2\chi_0 Y_k \left[ \frac{dS_{i\phi}}{d\tau_U} \mp v_k \frac{dS_{i\phi}}{d\tau_U} \right]. \quad (A.17)$

The difference of equations (A.14) and (A.17) leads to $dm/d\tau_U = 0$, so $m = c_m$ is constant, which then implies from the same equations that

$0 = \frac{d^2 S_{i\phi}}{d\tau_U^2} \pm 2\chi_0 Y_k \left[ \frac{dS_{i\phi}}{d\tau_U} \mp v_k \frac{dS_{i\phi}}{d\tau_U} \right]. \quad (A.18)$

Integration of equation (A.11) yields

$S_{i\phi} = \pm v_k S_{i\bar{r}} + c_1, \quad (A.19)$

and using this to replace $S_{i\phi}$ in equation (A.18) and then integrating gives

$\frac{dS_{i\phi}}{d\tau_U} \pm 2\chi_0 v_k S_{i\bar{r}} = c_2. \quad (A.20)$

Using equations (A.19) and (A.20) to replace $S_{i\phi}$ and $dS_{i\phi}/d\tau_U$ in equation (A.15) then leads to the decoupled second-order equation

$0 = \frac{d^2 S_{i\phi}}{d\tau_U^2} \mp 2\chi_0 Y_k c_2. \quad (A.21)$

Taking the $\tau_U$ derivative of equation (A.12) and using it and equation (A.13) in equation (A.16) leads to the second-order equation

$0 = \frac{d^2 S_{i\phi}}{d\tau_U^2} + \omega_2 = \chi_0 \left( 4 - 3\gamma_k^2 \right)^{1/2} = \sqrt{\frac{M(r - 6M)}{r^3(r - 3M)}}. \quad (A.22)$

Finally, using equations (A.22) and (A.13) in the equation obtained by taking the $\tau_U$ derivative of equation (A.12) gives a second decoupled second-order equation

$0 = \frac{d^2 S_{i\phi}}{d\tau_U^2} \mp \omega_4 = \chi_0 \left( 3\gamma_k^2 - 2 \right)^{1/2} = \sqrt{\frac{M}{r^3(r - 3M)}}. \quad (A.23)$

Integrating the two decoupled second-order equations (A.21) and (A.23), one can then integrate equation (A.22) too. The remaining components of the spin tensor are then determined by equations (A.13), (A.19) and (A.20). Note that the frequencies $\omega_3$ and $\omega_4$ agree only for $\gamma_k = 1$, or $v_k = 0$, which would imply $M = 0$: so this special case is not relevant.

Now the two intervals $3M < r < 6M$ and $r > 6M$ have differing signs for the squared angular velocities and the corresponding solutions are given by equations (5.1) and (5.3), respectively. The special case $r = 6M$ can be easily handled as well, setting $v_k = 1/2$ and $\chi_0 = \sqrt{6}/(36M)$ in equations (A.11)–(A.17). The corresponding solution is given by equation (5.2).

A.3. The general case: $v \neq 0$ and $v \neq \pm v_K$

Here we deal with the general form of equations (3.2)–(3.4) and (3.9)–(3.12). Solving equations (3.2)–(3.4) for their first terms and substituting these derivative terms into equations (3.9)–(3.12), one finds

$\frac{d^2 S_{i\phi}}{d\tau_U^2} = \gamma \frac{\chi_0}{v^2 \pm v_k} \left[ (2v_k^2 - 1)v^2 - v_k^2 \right] \frac{dS_{i\phi}}{d\tau_U} - \gamma^2 \chi_0 (v^2 - v_k^2) S_{i\phi} - \frac{1}{v} \frac{dm}{d\tau_U}. \quad (A.24)$
Spin precession in the Schwarzschild spacetime: circular orbits

\[
\frac{d^2 S_{\phi}}{dt^2} = -\gamma^3 v \frac{\xi_k}{v_k} [v^2 + v_k^2 - 2] \frac{dS_{\phi}}{dt} - \gamma^4 v \frac{\xi_k}{v_k} \left[ (3v_k^2 - 1)v^2 - 2v_k^2 \right] S_{\phi} \\
- \gamma^4 v \frac{\xi_k}{v_k} (v^2 - v_k^2) S_{\phi} + m \gamma^3 v \frac{\xi_k}{v_k} (v^2 - v_k^2).
\] (A.25)

\[
\frac{d^2 S_{\phi}}{dt^2} = \frac{\gamma^3 v \xi_k}{v_k^2} v dS_{\phi} - \frac{\gamma^4 v \xi_k}{v_k} \left[ (1 + 2v_k^2) v^2 - 3v_k^2 \right] S_{\phi}.
\] (A.26)

\[
\frac{d^2 S_{\phi}}{dt^2} = v \frac{\xi_k}{v_k} [v^2 + v_k^2 - 2] \frac{dS_{\phi}}{dt} - \gamma^2 v \frac{\xi_k}{v_k} \left[ (1 + 2v_k^2) v^2 - 1 \right] S_{\phi} - v \frac{dm}{dt}.
\] (A.27)

Solving equations (A.24) and (A.27) for \(dm/dt\) and \(d^2 S_{\phi}/dT^2\), one finds

\[
\frac{dm}{dt} = \gamma \frac{\xi_k}{v_k} (v^2 - v_k^2),
\] (A.28)

\[
\frac{d^2 S_{\phi}}{dt^2} = -2 \gamma^2 v \frac{\xi_k}{v_k} dS_{\phi} + \gamma^2 v \frac{\xi_k}{v_k} (v^2 - v_k^2) S_{\phi}.
\] (A.29)

Solving equation (3.2) for \(S_{\phi}\), leads to

\[
S_{\phi} = \frac{1}{\gamma} \frac{v_k}{v} \frac{1}{v^2 - v_k^2} \left[ \gamma \frac{dS_{\phi}}{dt} - \frac{dS_{\phi}}{dt} \right].
\] (A.30)

Using equation (A.30) in equation (A.28) leads to

\[
m = \gamma \frac{\xi_k}{v_k} [vS_{\phi} - v_k^2 S_{\phi}] + c_m.
\] (A.31)

Then, using equations (A.30) and (A.31) in equation (A.25) leads to

\[
0 = \left[ (1 - 2v_k^2) v^2 + v_k^2 \right] \frac{d^2 S_{\phi}}{dt^2} + v \left[ (v^2 + v_k^2 - 2) \frac{d^2 S_{\phi}}{dt^2} + \gamma^2 v \frac{\xi_k}{v_k} (v^2 - v_k^2) S_{\phi} \\
+ v \frac{\xi_k}{v_k} (v^2 - v_k^2) \left[ (1 - 4v_k^2) v^2 + v_k^2 (2 + v_k^2) \right] S_{\phi} + \gamma \frac{\xi_k}{v_k} (v^2 - v_k^2) c_m.
\] (A.32)

Using equation (A.30) in equation (A.29) leads to

\[
\frac{d^2 S_{\phi}}{dt^2} - v \frac{d^2 S_{\phi}}{dt^2} = \frac{\gamma^2 v \xi_k}{v_k} [v^2 - v_k^2] S_{\phi} + v S_{\phi} = 0.
\] (A.33)

Then, solving these last two equations for \(d^2 S_{\phi}/dt^2\) and \(d^2 S_{\phi}/dT^2\) gives

\[
\frac{d^2 S_{\phi}}{dt^2} = -\gamma^2 v \frac{\xi_k}{v_k} \left[ v^2 - v_k^2 \right] S_{\phi} - \gamma^2 v \frac{\xi_k}{v_k} S_{\phi} \\
+ v^2 + v_k^2 - 2 \frac{c_0}{v^2 - v_k^2} + \gamma^3 v \frac{\xi_k}{v_k} (v^2 - v_k^2) c_m.
\] (A.34)

\[
\frac{d^2 S_{\phi}}{dt^2} = -\gamma^2 v \frac{\xi_k}{v_k} \left[ v^2 + v_k^2 \right] S_{\phi} + v^2 v \frac{\xi_k}{v_k} + (1 + 2v_k^2) S_{\phi} \\
- \gamma^2 v \frac{\xi_k}{v_k} (1 - 2v_k^2) + \frac{c_0}{v^2 - v_k^2} + \gamma^3 v \frac{\xi_k}{v_k} (v^2 - v_k^2) c_m.
\] (A.35)
whose solution is given by equations (6.4) and (6.5). Then, substituting these solutions into equation (A.30), one obtains (6.6).

Substituting equations (3.3) and (3.4) into equation (3.11) gives

$$0 = \frac{d^2 \hat{S}_{\hat{\theta} \hat{\phi}}}{d\tau_U^2} + \Omega_1^2 \hat{S}_{\hat{\theta} \hat{\phi}} + 3\gamma^2 \nu^2 \frac{\zeta_2}{\nu_K} \hat{S}_{\hat{\theta} \hat{\phi}}.$$  (A.36)

Taking the $\tau_U$ derivative of equation (3.3) and using equations (A.36) and (3.4), one gets

$$0 = \frac{d^2 \hat{S}_{\hat{\theta} \hat{\phi}}}{d\tau_U^2} + \Omega^2 \hat{S}_{\hat{\theta} \hat{\phi}}, \quad \Omega^2 = \frac{\gamma^2 \nu^2 \frac{\zeta_2}{\nu_K}}{1 + 2\nu^2_K}.$$  (A.37)

This is easily integrated to give the solution (6.1). The remaining components of the spin can be obtained directly from equations (A.36) and (3.4) yielding equations (6.2) and (6.3). The frequencies $\Omega$ and $\Omega_1$ agree for the particular value of the azimuthal velocity given by

$$\nu_0 = \pm \frac{\nu_K}{\nu} \left( 1 + 2\nu^2_K \right)^{-1/2}.$$  (A.38)

The solution corresponding to this special case is given by equations (6.12)–(6.14) together with equations (6.4)–(6.7) evaluated at $\nu = \nu_0$.

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