An Algebra of Fault Tolerance

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Abstract

Every system of any significant size is created by composition from smaller sub-systems or components. It is thus fruitful to analyze the fault-tolerance of a system as a function of its composition. In this paper, two basic types of system composition are described, and an algebra to describe fault tolerance of composed systems is derived. The set of systems forms monoids under the two composition operators, and a semiring when both are concerned. A partial ordering relation between systems is used to compare their fault-tolerance behaviors.

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1 Introduction

Fault tolerance is a subject of immense importance in the design and analysis of many kinds of systems. It has been studied in distributed computing [11, 10, 6] and elsewhere in computer science [3, 4], in the context of safety-critical systems [17, 15], and in many other places. Authors such as Perrow [14] and Neumann [13] note many instances where standard assumptions made in designing fault-tolerant systems seem not to hold, and draw conclusions from these deviations.

In this paper, we take a different, algebraic view of fault tolerance, based on the composition of a system. The basic notion is that every system of any significant size is created by composition from smaller sub-systems or components. The fault-tolerance of the overall system is influenced by that of the components that underlie it. Conversely, a system may itself be a module or part of a larger system, so that its fault-tolerance affects that of the whole of which it is a part.

It is thus fruitful to analyze the fault tolerance of a system as a function of its composition. In the preliminary analysis, it is assumed that

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component failures occur independently—failure of one component does not automatically imply the failure of another.

Composition of components to create a larger system is considered in Section 2 to happen in two ways: direct sum, denoted $+$, and direct product, denoted $\times$. This is then used to describe an arithmetic on systems. The fault tolerance of systems is formally described in Section 3 using functions from the set of systems to the set of natural numbers, whose basic properties are obtained. In Section 4, the arithmetic on systems is extended to define system monoids by direct sum and direct product, and later, a system semiring considering both. Using this as a basis, a partial ordering of systems by fault tolerance is given in Section 5 and basic properties are derived. In Section 6, it is shown that monoids can be defined on the set of equivalence classes under fault tolerance, of systems (rather than on the set of systems as done in Section 4). Section 7 briefly indicates how this could be used to create a semiring on the set of equivalence classes of systems under worst-case fault tolerance, essentially allowing for the development of results analogous to those of Section 5.

The mathematics used is standard algebra as might be learned in a first-level graduate or upper-level undergraduate course, and is mostly self-explanatory. Standard works in the field such as Hungerford [9] or Lang [12], and monographs on semiring theory [7, 8] may be useful references.

2 Notation and Preliminaries

We use lowercase $p$, $q$, etc., to denote individual components that are assumed to be atomic at the level of discussion, i.e., they have no components or sub-systems of their own. Systems that are not necessarily atomic are denoted by $A, B$, etc., with or without subscripts. Where particular clarity is required, component will be used to refer to an atomic component, while subsystem will be used to refer to a component that is not atomic. Sets of systems—atomic or not—are denoted by $\mathcal{P}, \mathcal{Q}$, etc., and in particular $\mathcal{U}$ is the universal set of all systems in the domain of discourse. The set of natural numbers is denoted by $\mathbb{N}$, and the set of integers by $\mathbb{Z}$. Other mathematical terms and notation are defined in the sections where they are first found.

In the beginning, we assume that our components and subsystems are disjoint, i.e., that they do not share any components among themselves, and that they fail, if they do, independently of one another.

A fault is a failure of a subsystem or component, while a failure applies to some system as a whole, but the latter term will also be applied in the context of events that are faults from a larger system perspective but may be considered failures from a component or subsystem perspective.

Let $A$ and $B$ be two components that can form a system.
**Definition 2.1.** The $+$ operator is considered to apply for systems consisting of two components when the failure of either would cause the system as a whole to fail. Equivalently, we may say that the direct sum $A + B$ of components $A$ and $B$ is a system where the failure of either $A$ or $B$ will cause overall system failure.

One example is a computer program that needs two resources, such as two files, in order to perform, with lack of functionality resulting from the lack of availability of either.

A more basic example of such a system is two light-bulbs connected in series, with a voltage applied to cause them to glow. If either bulb burns out, the connection is broken and system failure occurs.

In colloquial terms, a direct sum is a situation where the “weakest link” component needs to fail, for the system to fail.

**Definition 2.2.** The $\times$ operator applies for systems consisting of two components when the failure of both is necessary to cause the system as a whole to fail. Equivalently, we may say that the direct product $A \times B$ of components $p$ and $q$ is a system where the failure of both is necessary to cause overall system failure.

One example is a computer program that needs a file that is available in two replicas; the program would only fail to perform if both replicas were to be unavailable.

A more basic example of such a system is one consisting of two light-bulbs connected in parallel across a voltage source, with the system considered functional if at least one bulb glows. System failure occurs only when both bulbs burn out.

In colloquial terms, a direct product is a situation where the “strongest link” component needs to fail, for the system to fail.

Note that these composition operators are different from the way composition is described elsewhere in the literature [1, 2], and are also not significantly related to the theory of “fault tolerance components” proposed by Arora and Kulkarni [5].

Given the previous definitions of the direct sum and direct product operators, we obtain an “arithmetic” on systems as follows [16], where $\cong$ stands for isomorphism.

**Proposition 2.3.** Given disjoint components $A$, $B$, and $C$, we have the following.

(i) $+$ and $\times$ are commutative and associative: $A \times B = B \times A$; $A + C = C + A$; $A + (B + C) = (A + B) + C$; and $A \times (B \times C) = (A \times B) \times C$.

(ii) $\times$ distributes over $+$: $A \times (B + C) = (A \times B) + (A \times C)$. 

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Proof. The properties given in (i) are obvious from the Definitions 2.1 and 2.2.

For the distributive property of (ii), consider that \( A \times (B + C) \) will fail when both \( A \) and \( B + C \) fail, and that this happens precisely when \( (A \times B) + (A \times C) \) fails.

Therefore, systems are described by polynomial expressions on variables denoting their components, such as \( (P^3 + 3Q) \times 5R^2 \).

In addition, by analogy with regular arithmetic, we can say the following.

**Proposition 2.4.** Any system described by an expression of direct sum and product operations on components \( A_{i,j} \) can be expressed in a canonical sum of products form such as \( \{A_0,0 \times A_0,1 \times \ldots \times A_0,n_0-1\} + \{A_1,0 \times A_1,1 \times \ldots \times A_1,n_1-1\} + \{A_{m-1,0} \times A_{m-1,1} \times \ldots \times A_{m-1,n_{m-1}-1}\} \).

In Proposition 2.4, there are \( m \) product terms to be added together, and a product term \( i \) consists of the direct product of \( n_i \) components.

### 3 Fault Tolerance of Composed Systems

About systems that are members of \( U \), we can state the following results from [16], with \( A^n \) describing the direct product of \( A \in U \) with itself \( n \) times, and \( nA \) describing the direct sum of \( A \in U \) with itself \( n \) times.

**Lemma 3.1.** A system \( A^n \) can tolerate faults in \( n - 1 \) of the subsystems, where \( n \in \mathbb{Z}_+ \), and a system \( nA \) can tolerate zero faults.

**Proof.** This is immediate, given the definitions of the direct sum and product. A system composed of the direct sum of \( A \) with itself \( n \) times will fail should any one of the components fail, whilst one composed of the direct product of \( A \) with itself \( n \) times will only fail if all of them fail (i.e., it can tolerate faults in \( n - 1 \) of them).

As a consequence, we have the following.

**Corollary 3.2.** A system \( mA^n \) consists of \( m \times n \) subsystems, where \( m, n \in \mathbb{Z}_+ \), but can only tolerate faults in \( n - 1 \) of them in the worst case.

Notice, however, that the following holds.

**Corollary 3.3.** In the best case a system \( mA^n \) can tolerate faults in \( m \times (n - 1) \) components.

**Proof.** In the best case, a system \( mA^n = A^n + \ldots + A^n \) would have \( (n - 1) \) failures in each of the \( A^n \) and still be functional, for a tolerance of \( m \times (n - 1) \) faults.
In the previous results, we have dealt with just one type of component. However, the same idea can be generalized to multiple types of components, as follows. It is assumed here that a failure of any component, regardless of its type, is of equal value and adds 1 to the number of failures in the overall system.

The idea is to specify the fault-tolerance (in the best case and in the worst case) recursively, in terms of sub-systems.

**Definition 3.4.** If, as stated, previously, \( \mathcal{U} \) is the set of all systems, then \( \Phi_{\text{best}} : \mathcal{U} \rightarrow \mathbb{N} \) and \( \Phi_{\text{worst}} : \mathcal{U} \rightarrow \mathbb{N} \) are functions giving the best- and worst-case fault-tolerances of a system, such that:

(i) \( \Phi_{\text{best}}(A) = i \) for \( A \in \mathcal{U} \), if \( i \) is the cardinality of the largest set of components of \( A \) that can fail without causing \( A \) to fail.

(ii) \( \Phi_{\text{worst}}(B) = j \) for \( B \in \mathcal{U} \), if \( j + 1 \) is the cardinality of the smallest set of components in \( B \) whose failure causes \( B \) to fail.

Quite obviously, we have \( \Phi_{\text{best}}(A) \geq \Phi_{\text{worst}}(A), \forall A \in \mathcal{U} \). The values of these functions may be computed recursively as indicated in the following theorem.

**Theorem 3.5.** Given systems \( A, B \in \mathcal{U} \), if \( |A| \) and \( |B| \) are the numbers of components therein, the following hold:

(i) \( \Phi_{\text{best}}(A \times B) = \max\{\Phi_{\text{best}}(A) + |B|, |A| + \Phi_{\text{best}}(B)\} \);

(ii) \( \Phi_{\text{worst}}(A \times B) = \Phi_{\text{worst}}(A) + \Phi_{\text{worst}}(B) + 1 \);

(iii) \( \Phi_{\text{best}}(A + B) = \Phi_{\text{best}}(A) + \Phi_{\text{best}}(B) \);

(iv) \( \Phi_{\text{worst}}(A + B) = \min\{\Phi_{\text{worst}}(A), \Phi_{\text{worst}}(B)\} \).

**Proof.** For parts (i) and (ii), we know that \( A \times B \) will fail only when both \( A \) and \( B \) fail.

Therefore, \( \Phi_{\text{best}}(A \times B) \) reflects the case where one of \( A \) or \( B \) fails fully (every component in it failing), and the other has its best-case limit of component failures. Therefore, the total number is \( \max\{\Phi_{\text{best}}(A) + |B|, |A| + \Phi_{\text{best}}(B)\} \) failures. Likewise, \( \Phi_{\text{worst}}(A \times B) = \Phi_{\text{worst}}(A) + \Phi_{\text{worst}}(B) + 1 \).

For parts (iii) and (iv), we know that a system \( A + B \) will fail should either \( A \) or \( B \) fail.

Therefore, \( \Phi_{\text{best}}(A + B) = \Phi_{\text{best}}(A) + \Phi_{\text{best}}(B) \), because in the best case \( A \) and \( B \) will sustain their limit of failures and yet not fail.

Likewise, \( \Phi_{\text{worst}}(A + B) = \min\{\Phi_{\text{worst}}(A), \Phi_{\text{worst}}(B)\} \), as in the worst case the minimum of \( \min\{\Phi_{\text{worst}}(A), \Phi_{\text{worst}}(B)\} \) is all that is necessary to cause one of the two, \( A \) or \( B \), to fail. \( \square \)
The theorem just stated can also be extended in the obvious way to systems of $n > 2$ components.

**Theorem 3.6.** Given subsystems $A_i \in \mathcal{U}, 0 \leq i \leq n - 1$, we have the following, with $\Pi$ and $\Sigma$ denoting, respectively, the direct product and direct sum of multiple subsystems, and $|A_i|$ denoting the number of components in subsystem $A_i$:

(i) For the best-case tolerance of a product of $n$ subsystems:

$$\Phi_{\text{best}}(\prod_{i=0}^{n-1} A_i) = \sum_{i=0}^{n-1} |A_i| - |A_k| + \Phi_{\text{best}}(A_k),$$

where $A_k$ is a subsystem with the lowest difference between its own number of components and its own best-case fault tolerance.

(ii) For the worst-case tolerance of a product of $n$ subsystems:

$$\Phi_{\text{worst}}(\prod_{i=0}^{n-1} A_i) = \sum_{i=0}^{n-1} \Phi_{\text{worst}}(A_i) + n - 1$$

(iii) For the best-case tolerance of a sum of $n$ subsystems:

$$\Phi_{\text{best}}(\sum_{i=0}^{n-1} A_i) = \sum_{i=0}^{n-1} \Phi_{\text{best}}(A_i)$$

(iv) For the worst-case tolerance of a sum of $n$ subsystems:

$$\Phi_{\text{worst}}(\sum_{i=0}^{n-1} A_i) = \min\{\Phi_{\text{worst}}(A_i)\}$$

**Proof.** The proof is exactly similar to that of Theorem 3.5, though a little more involved.

For parts (i) and (ii), we know that the system will fail when all of the $A_i$ subsystems fail.

Therefore, the best-case fault tolerance of the product of the $A_i$ is when all the subsystems save one fail, and collectively sustain the maximum possible number of component failures. If the one that does not fail also sustains component failures, then the best-case fault tolerance is obtained when the non-failing subsystem has the lowest difference, among all subsystems $A_i$, between its own number of components and its best-case fault tolerance.

Likewise, the worst-case fault tolerance of the product of the $A_i$ is when all subsystems but one fail, but sustain the least possible number of component failures in doing so. This happens when each $A_i$ sustains its worst-case
number of failures, and then further when \( n - 1 \) of them sustain one more failure each, meaning that all but one subsystem have failed.

For parts (iii) and (iv), we know that the system will fail when any one subsystem \( A_i \) fails.

Therefore, the best-case fault tolerance of the sum of the \( A_i \) is when each \( A_i \) sustains its best-case limit of failures. Likewise, the worst-case fault tolerance of the sum of the \( A_i \) is when just one \( A_i \) sustains its worst-case limit of failures.

\[\square\]

4 System Monoids and Semirings

Given the system arithmetic previously defined, we can posit the existence of two identity operators, one each for the direct sum and direct product. Informally, an identity element is one which leaves the system unchanged, under the relevant operator.

**Definition 4.1.** The multiplicative and additive identities are defined as follows.

(i) The additive identity \( 0 \) is the system such that for any system \( A \),
\[
A + 0 = 0 + A = A.
\]

(ii) The multiplicative identity \( 1 \) is the system such that for any system \( A \),
\[
A \times 1 = 1 \times A = A.
\]

By the commutativity of the + and \( \times \) operators, we observe that the identity elements are two-sided.

Informally, we may describe these elements as follows:

(i) The additive identity \( 0 \) is a system “that is always up.” The direct sum of such a system and \( A \) is obviously \( A \) itself.

(ii) The multiplicative identity \( 1 \) is a system “that is always down.” The direct product of such a system and \( A \) is likewise \( A \) itself.

Then \( \mathcal{U} \), combined with the + operator, is a **monoid** (a set with an associative operator and a two-sided identity element) \[9\]. Similarly, \( \mathcal{U} \) is also a monoid when considering the \( \times \) operator. For notational convenience, we denote these monoids as \((\mathcal{U}, +)\) and \((\mathcal{U}, \times)\).

It is further clear that the set \( \mathcal{U} \) is a **semiring** when taken with the operations + and \( \times \) because the following conditions \[7\] for being a semiring are satisfied:

(i) \((\mathcal{U}, +)\) is a commutative monoid with identity element \( 0 \);

(ii) \((\mathcal{U}, \times)\) is a monoid with identity element \( 1 \);
(iii) \( \times \) distributes over + from either side;

(iv) \( 0 \times A = 0 = A \times 0 \) for all \( A \in U \).

This system semiring will be denoted by \((U, +, \times)\), and its properties are as indicated in the following.

**Remark 4.2.** The semiring \((U, +, \times)\) is zerosumfree, because \( A + B = 0 \) implies, for all \( A, B \in U \), that \( A = B = 0 \).

This condition shows \([7]\) that the monoid \((U, +)\) is completely removed from being a group, because no non-trivial element in it has an inverse.

**Remark 4.3.** \((U, +, \times)\) is entire, because there are no non-zero elements \( A, B \in U \) such that \( A \times B = 0 \).

This likewise shows that the monoid \((U, \times)\) is completely removed from being a group, as there is no non-trivial multiplicative inverse.

**Remark 4.4.** \((U, +, \times)\) is simple, because \( 1 \) is infinite, i.e., \( A + 1 = 1, \forall A \in U \).

Given that the semiring \((U, +, \times)\) is both zerosumfree and entire, we can call it an information algebra \([7]\).

We may state another important definition \([7]\) about semirings, and observe a property of \((U, +, \times)\).

**Definition 4.5.** The center \( C(U) \) of \( U \) is the set \( \{ A \in U \mid A \times B = B \times A, \text{ for all } B \in U \} \).

**Remark 4.6.** The semiring \((U, +, \times)\) is commutative because \( C(U) = U \).

5 Fault-Tolerance Partial Ordering

Consider a partial ordering relation \( \preceq \) on \( U \), the set of all systems, such that \((U, \preceq)\) is a poset. This is a fault-tolerance partial ordering where \( A \preceq B \) means that \( A \) has a lower measure of some fault metric than \( B \) (e.g., \( A \) has fewer failures per hour than \( B \), or has a better fault tolerance than \( B \)).

Formally, \( \preceq \) is a partial ordering on the semiring \((U, +, \times)\) if the following conditions are satisfied \([8]\).

**Definition 5.1.** If \((U, +, \times)\) is a semiring and \((U, \preceq)\) is a poset, then \((U, +, \times, \preceq)\) is a partially ordered semiring if the following conditions are satisfied for all \( A, B, \) and \( C \) in \( U \).

(i) The monotony law of addition:

\[
A \preceq B \implies A + C \preceq B + C
\]
(ii) The monotony law of multiplication:

\[ A \preceq B \rightarrow A \times C \preceq B \times C. \]

It is assumed that \( 0 \preceq A, \forall A \in U \), on the reasoning that \( 0 \), being a system that never fails, must have the least possible measure of any fault metric. Similarly, \( A \preceq 1 \), because \( 1 \), being a system that is always down, must have the greatest possible measure of any fault metric.

Given Definition 5.1, it is instructive to consider the behavior of the partial order under composition. We begin with a couple of simple results.

**Lemma 5.2.** If \( \preceq \) is a fault-tolerance partial order as defined, then \( \forall A, B \in U \):

(i) \( A \preceq A + B \), and

(ii) \( A \times B \preceq A \).

**Proof.** For (i), consider that \( 0 \preceq B \). Using the monotony law of addition, we get \( 0 + A \preceq B + A \). Considering that \( 0 \) is the additive identity element and that addition is commutative, we get \( A \preceq A + B \).

For (ii), consider that \( B \preceq 1 \). Using the monotony law of multiplication, we get \( B \times A \preceq 1 \times A \). Considering that \( 1 \) is the multiplicative identity element and that multiplication is commutative, we get \( A \times B \preceq A \). \( \Box \)

A property of \( U \) in consideration of the + operator can now be noted.

**Remark 5.3.** The positive cone of \((U, +, \preceq)\), which is the set of elements \( A \in U \) for which \( A \preceq A + B, \forall B \in U \), is the set \( U \) itself. The negative cone is empty.

This is a direct consequence of Lemma 5.2 (i), and it also follows that the set of elements \( \{A \mid A + B \preceq A\} = \emptyset \), showing that the negative cone is empty.

The analogous property of \( U \) in consideration of the \( \times \) operator can also be noted.

**Remark 5.4.** The negative cone of \((U, \times, \preceq)\), which is the set of elements \( A \in U \) for which \( A \times B \preceq A, \forall B \in U \), is the set \( U \) itself. The positive cone is empty.

As before, this is a direct consequence of Lemma 5.2 (ii), and it likewise also follows that the set of elements \( \{A \mid A \preceq A \times B\} = \emptyset \), showing that the positive cone is empty.

**Theorem 5.5.** If \( A + B \preceq C \), then \( A \preceq C \) and \( B \preceq C \).
Proof. The proof is by contradiction. Assume the contrary. Then \( A + B \preceq C \), and at least one of \( A \preceq C \) or \( B \preceq C \) is false.

Without loss of generality, assume that \( C \preceq A \). Using the monotony law of addition and the commutativity of the + operator, \( B + C \preceq A + B \).

Now, by Lemma 5.2 (i), \( C \preceq B + C \). Given the transitivity of \( \preceq \), we get \( C \preceq A + B \), which is a contradiction. \( \square \)

An analogous result can also be stated in terms of the product, as follows.

**Theorem 5.6.** If \( A \preceq B \times C \), then \( A \preceq B \) and \( A \preceq C \).

**Proof.** The proof is again by contradiction. Assume the contrary. Then \( A \preceq B \times C \) and at least one of \( A \preceq B \) and \( A \preceq C \) is false.

Without loss of generality, assume that \( B \preceq A \). Using the monotony law of multiplication and the commutativity of the \( \times \) operator, we get \( B \times C \preceq A \times C \).

Now, by Lemma 5.2 (ii), \( A \times C \preceq A \). Given the transitivity of \( \preceq \), we get \( B \times C \preceq A \), which is a contradiction. \( \square \)

The following generalization of Lemma 5.2 can be made.

**Lemma 5.7.** If \( \preceq \) is a fault-tolerance partial order, then \( \forall n \in \mathbb{Z}_+ \) and \( \forall A \in \mathcal{U} \),

(i) \( A \preceq nA \), and

(ii) \( A^n \preceq A \).

**Proof.** For (i), consider that by Lemma 5.2 (i) we have \( A \preceq 2A \) (just set \( B \) to be \( A \) in the Lemma). Likewise, \( kA \preceq (k + 1)A \), for any \( k \geq 2 \). By the transitivity of \( \preceq \), we therefore have the result.

For (ii), the reasoning is very similar, using Lemma 5.2 (ii) and the transitivity of \( \preceq \). \( \square \)

The following is an obvious corollary.

**Corollary 5.8.** If \( \preceq \) is a fault-tolerance partial order, then \( \forall m, n \in \mathbb{Z}_+ \) and \( \forall A \in \mathcal{U} \), if \( m < n \), we have:

(i) \( mA \preceq nA \), and

(ii) \( A^n \preceq A^m \).

Similarly, the following corollary generalizing Theorems 5.5 and 5.6 applies. The proof is omitted as obvious.

**Corollary 5.9.** The following hold for all \( A, B \in \mathcal{U} \) and all \( n \in \mathbb{Z}_+ \):
In a system semiring, the following result concerning preservation of fault-tolerance behavior under composition also applies.

**Theorem 5.10.** If $\preceq$ is a partial order as described, and if $A \preceq B$ and $C \preceq D$, then,

(i) $A + C \preceq B + D$, and  
(ii) $A \times C \preceq B \times D$.

**Proof.** These results can be proven directly. Only (i) is proved, the proof of (ii) being very similar. 

We know the following:

$$A \preceq B \quad (1)$$

and:

$$C \preceq D \quad (2)$$

From (1) and the monotony law of addition (considering the direct sum of $D$ and both sides), we have:

$$A + D \preceq B + D. \quad (3)$$

Similarly, from (2) and the monotony law (considering the direct sum of $A$ and both sides), we have:

$$A + C \preceq A + D. \quad (4)$$

By considering transitivity in respect of (1) and (3), we get:

$$A + C \preceq B + D.$$

QED.

The following corollary can be stated.

**Corollary 5.11.** If $\preceq$ is a fault-tolerance partial order and if $A \preceq B$, then $\forall n \in \mathbb{Z}_+$,

(i) $nA \preceq nB$, and  
(ii) $A^n \preceq B^n$. 

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Proof. These are straight-forward iterated consequences of Theorem 5.10 and the transitivity of \( \preceq \).

In consideration of the transitivity of the partial order \( \preceq \), the following generalization of Theorem 5.10 applies. The proof is immediate.

**Theorem 5.12.** If \( A_i \preceq B_i \), with \( 0 \leq i \leq n-1 \) and \( A_i, B_i \in U \), then:

\[
\sum_{i=0}^{n-1} A_i \preceq \sum_{i=0}^{n-1} B_i,
\]

and

\[
\prod_{i=0}^{n-1} A_i \preceq \prod_{i=0}^{n-1} B_i.
\]

The following is a result along the same lines as the previous two.

**Theorem 5.13.** If \( A \preceq B \) and \( m \leq n \), then:

(i) \( mA \preceq nB \); and

(ii) \( A^n \preceq B^m \).

Note that \( A^m \) and \( B^n \) cannot be compared just based on the data given.

**Proof.** For (i), we have that \( mA \preceq nB \) by Corollary 5.11 (i), because \( A \preceq B \). We further have that \( nB \preceq nB \) by Corollary 5.8 (i), because \( m \leq n \). By the transitivity of \( \preceq \), we have the desired result.

For (ii), we have that \( A^n \preceq A^m \) by Corollary 5.8 (ii), because \( m \leq n \). We further have that \( A^m \preceq B^m \) by Corollary 5.11 (ii), because \( A \preceq B \). As before, by the transitivity of \( \preceq \), we have the desired result.

These results can be generalized in the obvious way, as follows.

**Theorem 5.14.** If \( A_i \preceq A_{i+1}, n_i \leq n_{i+1} \), and \( m_i \geq m_{i+1} \), for some range of values \( i \), then we have:

\[
n_iA_i^{m_i} \preceq n_{i+1}A_{i+1}^{m_{i+1}}
\]

**Proof.** It is clear that \( A_i^{m_i} \preceq A_{i+1}^{m_{i+1}} \), noting that \( A_i \preceq A_{i+1}, m_i \geq m_{i+1} \), and applying Theorem 5.13 (ii). Now, given that \( A_i^{m_i} \preceq A_{i+1}^{m_{i+1}} \), the fact that \( n_i \leq n_{i+1} \), in light of Theorem 5.13 (i), gives the result.

This theorem leads to the following obvious corollary.

**Corollary 5.15.** If \( m \leq n \), then \( mA^m \preceq nA^m \).
6 Monoids of Fault-Tolerance Equivalence Classes

It has previously been noted in Section 4 that the set of all systems, along with the direct sum (respectively: direct product) and the additive (respectively: multiplicative) identity defines a monoid. Another kind of monoid may be defined on classes of systems, as follows.

We define an equivalence relation on fault tolerance, as follows.

**Definition 6.1.** A fault-tolerance equivalence relation by worst-case fault tolerance, $\mathcal{R}(\sim)$, is given by $A \sim B \rightarrow \Phi_{\text{worst}}(A) = \Phi_{\text{worst}}(B)$ for all $A, B \in \mathcal{U}$.

Based on this, we note a simple theorem, given in [16].

**Theorem 6.2.** $A_1 \sim B_1$ and $A_2 \sim B_2$ together imply:

(i) $A_1 + A_2 \sim B_1 + B_2$; and

(ii) $A_1 \times A_2 \sim B_1 \times B_2$.

**Proof.** For part (i), we note from Theorem 3.5 (iv) that $\Phi_{\text{worst}}(A_1 + A_2) = \min\{\Phi_{\text{worst}}(A_1) + \Phi_{\text{worst}}(A_2)\}$. Let this be $\Phi_{\text{worst}}(A_1)$. Similarly, then, $\Phi_{\text{worst}}(B_1 + B_2) = \Phi_{\text{worst}}(B_1)$, and the result obtains.

For part (ii), we note from Theorem 3.5 (ii) that $\Phi_{\text{worst}}(A_1 \times A_2) = \Phi_{\text{worst}}(A_1) + \Phi_{\text{worst}}(A_2) + 1$. This is also equal to $\Phi_{\text{worst}}(B_1) + \Phi_{\text{worst}}(B_2) + 1$ by the definition of $\mathcal{R}$, which in its turn is equal to $\Phi_{\text{worst}}(B_1 \times B_2)$. \[\square\]

Now we can state the major result (given for monoids by Hungerford [9]), on constructing a monoid on the equivalence classes of systems. The proof given here is as given by Hungerford.

**Theorem 6.3.** Let $\mathcal{R}(\sim)$ be defined as in Definition 6.1. Then the set $\mathcal{U}/\mathcal{R}$ of all equivalence classes of $\mathcal{U}$ under $\mathcal{R}$ is a monoid under the binary operation defined by $\overline{A} \ominus \overline{B} = \overline{A + B}$, where $\overline{A}$ denotes the equivalence class of $A \in \mathcal{U}$.

**Proof.** If $\overline{A_1} = \overline{A_2}$ and $\overline{B_1} = \overline{B_2}$, where $A_i, B_i \in U$, then $A_1 \sim A_2$ and $B_1 \sim B_2$. By Theorem 6.2 part (i), $A_1 + B_1 \sim A_2 + B_2$, so that $\overline{A_1 + B_1} = \overline{A_2 + B_2}$. Therefore, the binary operation $\ominus$ in $\mathcal{U}/\mathcal{R}$ is well-defined (i.e., it is independent of the choice of equivalence-class representatives). It is associative since $\overline{A + (B + C)} = \overline{A} \ominus (\overline{B} \ominus \overline{C}) = \overline{A} \ominus \overline{B + C} = \overline{(A + B) + C} = \overline{A + B} \ominus \overline{C} = \overline{(A \ominus B) \ominus C}$.

The identity element is $\overline{0}$, the equivalence class of all systems that are always up, since $\overline{A} \ominus \overline{0} = \overline{A + 0} = \overline{A}$. Therefore, $(\mathcal{U}/\mathcal{R}, \ominus)$ is a monoid. \[\square\]

An analogous theorem can also be stated in respect of the $\times$ operator, with an identity element $\overline{1}$, the equivalence class of systems that are always up. The proof is exactly similar and is thus omitted.
Theorem 6.4. Let $\mathcal{U}$ be the set of all systems, and the relation $\mathcal{R}(\sim)$ be defined as in Theorem 6.2. Then the set $\mathcal{U}/\mathcal{R}$ of all equivalence classes of $\mathcal{U}$ under $\mathcal{R}$ is a monoid under the binary operation defined by $(\overline{A}) \boxtimes (\overline{B}) = A \times B$, where $\overline{A} \subseteq \mathcal{U}$ denotes the equivalence class of $A$.

Therefore, we have a second monoid, $(\mathcal{U}/\mathcal{R}, \boxtimes)$.

Remark 6.5. Note that $\mathcal{U}/\mathcal{R}$ is a set of classes of systems, rather than of systems themselves. Therefore, $\mathcal{U}/\mathcal{R} \subseteq 2^{\mathcal{U}}$, and any $A \subseteq \mathcal{U}$ (or: $\overline{A} \in 2^{\mathcal{U}}$), where $\overline{A}$ denotes the equivalence class of $A$ by worst-case fault tolerance.

It is also possible to re-work Theorem 6.3, considering the best-case fault tolerance. To show how, we first state the analogue of Theorem 6.2.

Theorem 6.6. If we define an equivalence relation $\mathcal{R}(\sim)$ on systems by best-case fault tolerance, such that $A \sim B \rightarrow \Phi_{\text{best}}(A) = \Phi_{\text{best}}(B)$, then $A_1 \sim B_1$ and $A_2 \sim B_2$ imply $A_1 + A_2 \sim B_1 + B_2$.

Proof. We note from Theorem 3.5 (iii) that $\Phi_{\text{best}}(A_1 + A_2) = \Phi_{\text{best}}(A_1) + \Phi_{\text{best}}(A_2)$. If $A_1 \sim B_1$ and $A_2 \sim B_2$, then $\Phi_{\text{best}}(A_1) + \Phi_{\text{best}}(A_2) = \Phi_{\text{best}}(B_1) + \Phi_{\text{best}}(B_2)$. This latter expression is $\Phi_{\text{best}}(B_1 + B_2)$, giving us the result. \hfill $\square$

Note that the analogue of Theorem 6.2(ii) is not generally true—if $\mathcal{R}(\sim)$ denotes an equivalence relation by best-case fault-tolerance, then $A_1 \sim B_1$ and $A_2 \sim B_2$ do not imply $A_1 \times A_2 \sim B_1 \times B_2$.

Therefore, we can re-state Theorem 6.3 (but not Theorem 6.4) using the new definition of $\mathcal{R}$. The statement and proof run just as previously, however, so we do not belabor the point.

We may summarize the results of this section, however, to note that there are three types of equivalence-class monoids so obtained:

- the monoid of worst-case equivalence classes under direct sums;
- the monoid of worst-case equivalence classes under direct products; and
- the monoid of best-case equivalence classes under direct sums.

7 The Semiring of Fault-Tolerance Equivalence Classes

It has been shown previously in Theorems 6.3 and 6.4 that there exist monoids $(\mathcal{U}/\mathcal{R}, \boxtimes)$ and $(\mathcal{U}/\mathcal{R}, \boxtimes)$ on the equivalence classes of systems by worst-case fault tolerances, and that these are commutative monoids.

It is easily seen that the other two conditions for a semiring [7] (see Section 4) are also satisfied:
- ⊠ distributes over ⊕—for any $\mathcal{A}, \mathcal{B}, \mathcal{C} \in 2^U$, we have:

$$
(\mathcal{A} \otimes \mathcal{B}) \oplus (\mathcal{A} \otimes \mathcal{C}) = (\mathcal{A} \times \mathcal{B}) \oplus (\mathcal{A} \times \mathcal{C}) = (\mathcal{A} \times \mathcal{B}) + (\mathcal{A} \times \mathcal{C}) = \mathcal{A} \times (B + C) = \mathcal{A} \otimes (\mathcal{B} \oplus \mathcal{C})
$$

- $\mathcal{0} \otimes \mathcal{A} = \mathcal{0} \times \mathcal{A} = \mathcal{0} = \mathcal{A} \otimes \mathcal{0}$, for all $\mathcal{A} \subseteq U$

Therefore, we can consider a different semiring, $(2^U, \oplus, \otimes)$, and it turns out that as with $(U, +, \times)$, this too is zerosumfree, entire, simple, and commutative (compare with the corresponding Remarks 4.2, 4.3, 4.4, and 4.6).

It is further possible to define a partial ordering relation (denoted by the symbol $\succeq$, for example) comparing the fault tolerances of different classes of systems. All of Section 5 can thus be repeated with $2^U$ in place of $U$, $\mathcal{A}$ and such in place of $A$, and $\succeq$ in place of $\succeq$.

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