Ballistic random walks in random environment as rough paths: convergence and area anomaly

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Abstract

Annealed functional CLT in the rough path topology is proved for the standard class of ballistic random walks in random environment. Moreover, the ‘area anomaly’, i.e. a deterministic linear correction for the second level iterated integral of the rescaled path, is identified in terms of a stochastic area on a regeneration interval. The main theorem is formulated in more general settings, namely for any discrete process of uniformly bounded increments which has a regenerative structure with regeneration times of finite moments. Here the largest finite moment translates into the degree of regularity of the topology. In particular, the convergence holds the $\alpha$-Hölder rough path topology for all $\alpha < \frac{1}{2}$ whenever all the moments are finite, which is the case for the class of ballistic random walks in random environment.

Key words: Levy area, rough paths, annealed invariance principle, area anomaly, random walks in random environment, ballisticity conditions, regenerative structure

1 Introduction

Introduced by Lyons in ‘98 [Lyo98], rough path theory has been extensively analyzed and developed ever since. The theory gives a framework to solutions to SDEs driven by non-regular signals (such as Brownian motions) while keeping the solution map continuous with respect to the signal. This was a solution to a long standing open problem mainly since the Ito theory of stochastic integration, being an $L^2$ theory in essence, does not allow integration path-by-path, and hence cannot give rise to solutions with that continuity property.

As it was observed by Lyons, that difficulty was indeed a real problem in the sense that in any separable Banach space $B \subset C[0,1]$ containing the sample paths of Brownian motions a.s. the map $(f,g) \rightarrow \int_0^1 f(t)g(t)dt$ defined on smooth maps cannot be extended to a continuous map on $B \times B$ (see [FH14] Proposition 1.1 and the references therein). Some additional information on the path is needed to achieve continuity, namely the so called “iterated integrals”, where the levels of iteration needed are determined by the regularity of the signal.

Fix $T > 0$ and $x : [0, T] \rightarrow \mathbb{R}^d$. The $M$-th level iterated iterated integral of $x$ is

$$ S_{s,t}^M(x) = \int_{s < u_1 < \ldots < u_N < t} dx_{u_1} \cdots dx_{u_N}, \quad s < t, s, t \in [0, T]. $$

Note that the definition of iterated integrals assumes a notion of integration with respect to $x$.

Lyons’ theory uses the information coming from the iterated integral as a postulated high level information and construct a space (called the rough path space) in which solutions to SDEs driven by Brownian motion are continuous with respect to the latter. In our case two levels of iteration are enough since the Brownian motion is $\alpha$-Hölder for some $\alpha \geq \frac{1}{2}$ (and actually for all $\alpha < \frac{1}{2}$). More generally, roughly speaking, in case the signal is $\alpha$-Hölder continuous for some $\alpha > 0$, then $M = [1/\alpha]$ levels of iteration are sufficient (and necessary).

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A related question solved by rough path theory is concerning approximations of solutions to SDEs. Let \((B^N)\) be a sequence of processes converging weakly in the uniform topology to a Brownian motion \(B\). An interesting question is to understand the approximating differential equations, where \(B^N\) is considered as noise. Assume \(X\) is a solution to a SDE with some nice (in an appropriate sense) drift and diffusion coefficients. Let \(X^N\) be the discrete solution to corresponding difference equation driven by \(B^N\). Does \(X^N\) converge to \(X\)? By the Wong-Zakai Theorem the answer to this question is negative if the convergence hold in the classical case [WZ65]. However, if the weak convergence of \((B^N)\) to \(B\) holds in the rough path space of regularity \(\alpha\), for some \(\alpha \in (\frac{1}{3}, \frac{1}{2})\), then the answer is affirmative. A natural question then arises: what can be said about convergence to a Brownian motions in the rough path topology, and moreover what is the largest possible regularity of spaces on which the convergence takes place.

For discrete processes with regenerative structure such as ballistic RWREs, invariance principles are well known. Our main result, Theorem 3.3, shows that we have as well a scaling limit in the rough path topology where the regularity is determined by the moments of the regenerations.

The application to ballistic RWREs is then immediate, see Theorem 5.3 and since regeneration times have all moments [Szn00], the convergence in spaces of regularity \(\alpha\) is taken all the way to \(\alpha < \frac{1}{2}\).

The power of rough paths became even more robust with Gubinelli’s abstract formulation [Gub04] which then inspired M. Hairer to develop the theory of regularity structures [Hai14], which is now extensively studied. In light of the intimate relation between the theories a fundamental question of relevance to both is what can we learn if instead of mollifying the noise by a smooth function we take a more complicated approximation? For example, in this paper we consider discrete processes as ballistic RWREs as the noise approximation.

With a different point of view, if a scaling limit is known for some process in the uniform topology, one might be interested to get a richer information about the limit. For inhomogeneous random walks with regenerative structure, an interesting phenomenon yields. As it turns out, unlike the “classical” invariance principles, when considering the second level iterated integral, which is related to the running signed area of the process as we later show, the local fluctuations do not disappear in the limit, and a correction has to be considered. Moreover, thanks to the i.i.d structure of the walk on regeneration intervals, the law of large numbers allows us to write the correction as a function \((\bar{\Gamma})_{0 \leq t \leq T}\), linear in time, where \(\bar{\Gamma}\) is a deterministic matrix which is expected signed area accumulated in a regeneration interval, divided by the expected length of the interval, see the main result, Theorem 3.3.

The fundamental result related to our work is the Donsker invariance principle in the rough path topology [BFH09]. An extension to random walks with general covariances were proved in [Kel16]. In [LS17a] and [LS17b] the authors have studied some discrete processes converging to Brownian motion in \(\mathbb{R}^d\) in rough path topology with area anomaly which was constructed explicitly. Our main idea of our proof is inspired by theirs, with two main differences. First, we do not use the strong Markov property for the excursions nor finitely support of the jump distribution which implies that the excursions have exponential tail. Instead, we only assume i.i.d. regenerative structure and moments of the regeneration times. Second, the discrete processes in those papers are homogeneous in space (a simple random walk on periodic graphs [LS17a], or hidden Markov walk where the jumps are independent on the current location but depends only on the increment [LS17b]). In our case we allow the process to be inhomogeneous in space.

The problem of discrete processes seen as rough paths is dealt with in other contexts as well. In [Kel16] and [KM16], and the more recent [FZ18], used the rough path framework to deal with discrete approximations of SDEs. The case of random walks on nilpotent covering graphs was considered in [IKN18a, IKN18b, Nam17] where the corresponding area anomaly is identified in terms of harmonic embeddings (see [IKN18a, equation (2.6)]). Other paper concerning discrete processes which is of relevance is [CF17]. In that paper the authors showed a robust construction of rough SDEs that can deal with rough paths with jumps (and hence not limited to linear interpolation).

### 1.1 Structure of the paper

In order to keep the paper as self-contained as possible in Chapter 2 we discuss basic notions in rough path theory and set up the framework to be used in the rest of the paper. In Chapter 3 we formulate our main
result, Theorem 3.3. In chapter 4 we give some simple examples of processes converging in the rough path topology and having or lacking area anomaly. In Chapter 5 we present some special cases of our main result. A particular case is RWREs, for which we also present an open problem. Finally, in Chapter 6 we give the proof of Theorem 3.3.

2 Basic notions in rough paths theory

The aim of this section is to introduce briefly the basic objects in our framework. We assume here no familiarity of the reader with rough path theory. For an extensive account of the theory the reader may consult the books [FV10] and [FH14].

Initially developed for solving differential equations, rough path theory is also useful in the discrete setting, and in particular when studying the convergence of discrete processes. For example, in the uniform topology a simple random walk (SRW) to which we add the same deterministic four steps loop every two steps (see figure 1) converges to the same Brownian motion as a SRW which stays still for four steps every two steps. Thus in the uniform topology the loops simply disappear at the limit.

Figure 1: A simple random walk with a deterministic loop every two steps. The double arrows (in red) are the random walk’s steps while the added loops are presented by the single arrows (in blue).

The loops certainly do not play a role if one is interested in the limit trajectory only. However, if one wishes to study more aspects of the limit, the accumulated area created by the loops could be also taken into consideration. A basic example for accumulated area in the continuous setting is provided by the “bubble areas” of Lejay [Lej03]. This weakness of the uniform topology is precisely one of the problems that rough path theory palliates.

Following [FV10] we denote by \( \otimes \) two different actions:

- for two elements of vector spaces, it is the usual tensor product: if \( V \) and \( W \) are \( d \)-dimensional, respectively \( d' \)-dimensional vector spaces, for \( v \in V \) and \( w \in W \), \( v \otimes w \) is the matrix \( (v_i w_j)_{1 \leq i \leq d, 1 \leq j \leq d'} \);

- for two elements of a group (in our case, \( G^2(\mathbb{R}^d) \), defined below), it denotes the corresponding group operation.

The continuous process obtained by linear interpolation (or any other piecewise \( C^1 \) interpolation) of a discrete process, as well as its iterated integrals can be “encoded” in a particular nilpotent Lie group (see [FH14] Section 2.3 for more details). For simplicity, and since our motivation in this paper is to prove convergence to Brownian motion, which is \( \alpha \)-Hölder for all \( \alpha < 1/2 \), we adapt the general point of view taken in the book [FH14] and consider (1) in the case \( M \leq 2 \), i.e. with only two levels of iteration. Therefore
in the rest of the paper, we write $S_{s,t}(X)$ for $S^2_{s,t}(X)$. The pairs $(X_{s,t}, S_{s,t}(X))$, $s < t$, with $X_{s,t} = X_t - X_s$, for a smooth path $X$, have a natural group structure with respect to increments concatenation. Here is the formal definition (see also the algebraic conditions in Proposition 2.4 for the corresponding formulation in terms of paths).

**Definition 2.1** (The group $G^2(\mathbb{R}^d)$). The step-2 nilpotent Lie group $G^2(\mathbb{R}^d) \subset \mathbb{R}^d \oplus (\mathbb{R}^d)^{\otimes 2}$ is defined as follows. An element can be presented by a pair $(a, b) \in \mathbb{R}^d \times \mathbb{R}^{d \times d}$ (that is $a$ is a vector and $b$ is a matrix), the group operation $\otimes$ is defined by

$$(a, b) \otimes (a', b') = (a + a', b + b' + a \otimes a'),$$

and the following condition holds

$$\forall (a, b) \in G^2(\mathbb{R}^d), \ Sym(b) = \frac{1}{2} a \otimes a,$$

where $Sym(\cdot)$ gives the symmetric part of an element, that is $Sym(b)_{i,j} = \frac{1}{2}(b_{i,j} + b_{j,i})$. (For clarity, we emphasize that above we used $a \otimes a' = (a, a')_{i,j}$ for the tensor product).

For an element $(a, b)$, $a$ and $b$ are called the first and the second level, respectively.

The topology of $G^2(\mathbb{R}^d)$ is induced by the Carnot-Caradory norm $\| \cdot \|_{G^2(\mathbb{R}^d)}$, which gives for an element $(a, b) \in G^2(\mathbb{R}^d)$ the length of the shortest path with bounded variation that can be “encoded” as $(a, b)$, i.e., whose increment is $a$ and iterated integral is $b$. In other words

$$\| (a, b) \|_{G^2(\mathbb{R}^d)} := \inf \left\{ \int_0^1 |\gamma(t)| \, dt \mid \gamma : [0, 1] \to \mathbb{R}^d \text{ of bounded variation, } (\gamma_{0,1}, S_{0,1}(\gamma)) = (a, b) \right\}.$$  

The norm defined in this fashion induces a continuous metric $d$ on $G^2(\mathbb{R}^d)$ through the application

$$d : \ G^2(\mathbb{R}^d) \times G^2(\mathbb{R}^d) \to \mathbb{R}_+ \quad (g, h) \mapsto \| g^{-1} \otimes h \|_{G^2(\mathbb{R}^d)}.$$  

($G^2(\mathbb{R}^d), d$) is then a geodesic space, i.e. for any two elements we can find another element whose length equals the distance between them.

**Definition 2.2** (Rough paths on $G^2(\mathbb{R}^d)$). Let $1/3 < \alpha < 1/2$. An $\alpha$-Hölder geometric rough path on $G^2(\mathbb{R}^d)$ is an element $X = (X, \Xi) \in C^\alpha([0, T], G^2(\mathbb{R}^d))$ which is $\alpha$-Hölder continuous with respect to the distance $d$.

Without going into details we remark that in rough path theory also deals with rough paths which are not geometric, i.e., those for which (3) does not hold.

An immediate example of such rough paths in the probabilistic setting is the Brownian motion rough path, which is also known as the enhanced Brownian motion. It is constructed using Stratonovich integration as follows:

$$B_{s,t} = (B_t - B_s, \int_{s \leq u < v \leq t} \circ dB_u \otimes \circ dB_v), \quad 0 \leq s \leq t.$$  

The group structure on $G^2(\mathbb{R}^d)$ and the Carnot-Caradory norm and distance are particularly tamed for treating path concatenations. For example the norm is sub-additive, and this can be compactly demonstrated. For a path $X = (X, \Xi)$ which takes value in $G^2(\mathbb{R}^d)$ let $X_{s,u} := X_u^{-1} \otimes X_t$. Then for every $s < u < t$

$$\| X_{s,u} \|_{G^2(\mathbb{R}^d)} = \| X_{s,u} \otimes X_{u,t} \|_{G^2(\mathbb{R}^d)} \leq \| X_{s,u} \|_{G^2(\mathbb{R}^d)} + \| X_{u,t} \|_{G^2(\mathbb{R}^d)}.$$  

The next proposition can found be useful for actual estimations. It leans on the equivalence

$$C^{-1} \leq \frac{\| (a, b) \|_{G^2(\mathbb{R}^d)}}{|a|_{\mathbb{R}^d} + |b|^{1/2}_{\mathbb{R}^d \otimes \mathbb{R}^d}} \leq C$$
for some $C \geq 1$, where $| \cdot |_{\mathbb{R}^d}$ is the Euclidean norm on $\mathbb{R}^d$ and $| \cdot |_{\mathbb{R}^d \otimes \mathbb{R}^d}$ is the matrix norm with respect to $| \cdot |_{\mathbb{R}^d}$.

Assume that $(X_s,t, X_s,t)$, $0 < s < t < T$ take value in $\mathbb{R}^d \times \mathbb{R}^d \times d$. Define

$$|||(X,X)|||_{\alpha} := ||X||_{\alpha} + ||X||_{2\alpha}^2,$$  \hspace{1cm} (7)

where

$$||X||_{\alpha} = \sup_{s < t, s, t \in [0,T]} \frac{|X_{s,t}|_{\mathbb{R}^d}}{|t-s|^\alpha}$$ and $$||X||_{2\alpha} = \sup_{s < t, s, t \in [0,T]} \frac{|X_{s,t}|_{\mathbb{R}^d \otimes \mathbb{R}^d}}{|t-s|^{2\alpha}}.$$ \hspace{1cm} (8)

Proposition 2.3. (FH14, Proposition 2.4) Let $\alpha \in (\frac{1}{3}, \frac{1}{2})$. $X_t = (X_t, X_t)$ is a geometric rough path as in Definition 2.2 if and only if the following assumptions hold:

- $|||(X,X)|||_{\alpha} < \infty$.
- $X_{s,t} = X_{s,u} + X_{u,t} + X_{s,u} \otimes X_{u,t}$ for every $s < u < t$ (Chen’s relation).
- $\text{Sym}(X_{s,t}) = \frac{1}{2} X_{s,t} \otimes X_{s,t}$ for every $s < t$ (integration by parts property).

To end this brief review we mention an alternative definition of the group which has some nice interpretation in terms of a signed area. A path $X$ considered in Definition 2.2 has increments in $G^2(\mathbb{R}^d)$. This is relevant for the notion of integration, which is based on “sewing” according to the increments. However, since the symmetric part of the second level depends entirely on the first level, to handle path increments a the following alternative group on which the rough paths are considered is sometimes more useful. The corresponding antisymmetric group operation $\wedge$ is defined by

$$(a, b) \wedge (a', b') = (a + a', b + b' + \frac{1}{2}(a \otimes a' - a' \otimes a)).$$ \hspace{1cm} (9)

In particular, unlike the case of the law $\otimes$ where an element $(a, b)$ represents a path with an increment $a$ and an iterated integral $b$, in the case of the antisymmetric product an element $(a, b)$ represents a path where $a$ is still an increment but $b$ is now the corresponding area. In other words, for $(X, X) \in G^2(\mathbb{R}^d)$ we consider $(X, A)$ instead, where $A_{s,t}^{i,j} = \frac{1}{2}(X_{s,t}^{i,j} - X_{s,t}^{j,i})$.

For example, the Brownian motion considered as a rough path in the case of the antisymmetric product $\wedge$ has the form

$$\mathbb{E}_{s,t}^{\wedge} = (B_t - B_s, A_{s,t}), \ 0 \leq s \leq t,$$ \hspace{1cm} (10)

where $A$ is the stochastic signed area of $B$, called the Stratonovich Lévy area.

The operation defined in (9) has a geometrical interpretation which shows how the group is suitable for concatenating paths. The first level translates into path concatenation, whereas the second one gives the law of the “area concatenation” (area of concatenated paths). Figure 2 demonstrates how to calculate the area of two concatenated curves. The areas of $\gamma_1$, $\gamma_2$ and that of the triangle (formed by the increments of $\gamma_1$ and $\gamma_2$) in the figure correspond respectively to $b$, $b'$ and $\frac{1}{2}(a \otimes a' - a' \otimes a)$ in formula (9). This rule for the area of concatenated paths is also commonly referred as the Chen’s relation. It plays a fundamental role in the theory of rough paths.
3 Main result

**Definition 3.1.** Let \((X, \mathbb{P}, \Omega, \mathcal{F})\) be a discrete time stochastic process on \(\mathbb{R}^d\). We say that \(X\) admits a regenerative structure in direction \(\ell \in S^{d-1}\) if there are \(\mathcal{F}\)-measurable integer valued random variables \((\tau_k)_{k \in \mathbb{N}_0}\) so that \(0 = \tau_0 < \tau_1 < \tau_2 < ... < \infty\) \(\mathbb{P}\)-a.s. and \((\tau_k - \tau_{k-1}, (X_{\tau_{k-1}}, \tau_{k-1} + m \cdot \ell), 0 \leq m \leq \tau_k - \tau_{k-1})\) are independent random variables for \(k \geq 1\), and have the same law for \(k \geq 2\).

**Definition 3.2.** Let \(X\) be a discrete time stochastic process on \(\mathbb{R}^d\). We define below the associated discrete second level iterated integral \(S_{n}(X))_{n \geq 0}\) of \(X\) and its area \(A_{n}(X))_{n \geq 0}\). Set

\[
S_{m,n}^{i,j}(X) = \sum_{m+1 \leq k \leq n} \Delta X_k^i \Delta X_k^j \quad \text{and} \quad A_{m,n}^{i,j}(X) = \frac{1}{2}(S_{m,n}^{i,j}(X) - S_{m,n}^{j,i}(X)),
\]

where \(\Delta X_k := X_k - X_{k-1} = X_{k-1,k}\) are the increments. Then, \(S_n(X) := S_{0,n}(X)\) and \(A_n(X) := A_{0,n}(X)\).

For a sequence \(x = (x_n)_{n}\) of elements of \(\mathbb{R}^d\), we introduce its continuous rescaled version \(x^{(N)}\):

\[
x^{(N)}_t = x^{\lfloor Nt \rfloor} + (Nt - \lfloor Nt \rfloor)(x^{\lfloor Nt \rfloor+1} - x^{\lfloor Nt \rfloor}).
\]

We denote the rough path corresponding to \(x^{(N)}\) by

\[
\iota^{(N)}(x)_{s,t} := \left( x^{(N)}_{s,t}, S_{s,t}(x^{(N)}) \right)
\]

where \(S_{s,t}(x^{(N)})\) denotes the second level iterated integral of \(x^{(N)}\) between \(s\) and \(t\). As usual \(\iota^{(N)}(x)_{t} := \iota^{(N)}(x)_{0,t}\).

**Theorem 3.3.** Let \(X\) be a discrete time stochastic process on \(\mathbb{R}^d\) with bounded jumps \(|X_{n+1} - X_n|_{\mathbb{R}^d} < K\) a.s. Assume that \(X\) admits a regenerative structure in direction \(\ell \in S^{d-1}\) in the sense of Definition 3.1 and let \((\tau_k)_{k \geq 1}\) be the corresponding regeneration times. Assume further that \(X\) satisfies a strong law of large numbers

\[
\mathbb{P} \left( \lim_{n \to \infty} \frac{X_n}{n} = v \right) = 1.
\]
Lemma 3.5. \[ \bar{\Gamma} = \Gamma + \text{Brownian motion} \]

Also, assume that \( \bar{X}_n = X_n - nv \) satisfies an annealed invariance principle with covariance matrix

\[ M = \frac{E[\bar{X}_{\tau_1, \tau_2} \bar{X}^T_{\tau_1, \tau_2}]}{E[\tau_2 - \tau_1]} . \]

Last assumption is the following moment condition:

\[ E[(\tau_k - \tau_{k-1})^{2p}] < \infty \]

for some \( p > 4 \). Then the rescaled lift \( l^{(N)}(\bar{X})_{t \in [0, T]} \) of the random walk \( \bar{X} \) converges weakly to

\[ (B_t, S_t + t\bar{\Gamma})_{t \in [0, T]} \text{ in } C^\alpha([0, 1], C^2(\mathbb{R}^d)) \]

for all \( \alpha \in (\frac{1}{2}, \frac{p^* - 1}{2p^*}) \), where \( p^* = \min\{[p], 2[p/2]\} \), and the couple \( (B, S) \) are the Brownian motion with covariance matrix \( M \) and its second level iterated Stratonovich integral process. Moreover, the correction is the antisymmetric matrix

\[ \bar{\Gamma} = \frac{E_0[A_{\tau_1, \tau_2}(\bar{X})]}{E_0[\tau_2 - \tau_1]} . \]

In particular, if the moment condition holds true for all (large enough) \( p < \infty \) then the convergence holds true for all \( \alpha < \frac{1}{2} \).

Remark 3.4. The corresponding result for the area with the same correction \( \bar{\Gamma} \) holds as well, that is whenever the path is considered with the antisymmetric operation, \( S(X) \) is replaced by \( A(X) \) and the enhanced Brownian motion \( S \) is replaced by the Stratonovich Levy area \( A \).

The correction matrix \( \bar{\Gamma} \) has the following decomposition.

**Lemma 3.5.** \( \bar{\Gamma} = \Gamma + R \) where

\[ \Gamma = \frac{E_0[A_{\tau_1, \tau_2}(X)]}{E_0[\tau_2 - \tau_1]} \text{ and } R^{i,j} = \frac{v_i E[S_{\tau_1, \tau_2}^{i,i}(X)] - v_j E[S_{\tau_1, \tau_2}^{j,j}(X)]}{2E[\tau_2 - \tau_1]} . \]

**Proof.** By construction \( \bar{X}_n = X_n - nv \). Therefore

\[ \Delta \bar{X}_n \otimes \Delta \bar{X}_m = \Delta X_n \otimes \Delta X_m + v \otimes v - (v \otimes \Delta X_m + \Delta X_n \otimes v) . \]

Neglecting symmetric terms we decompose \( \bar{\Gamma} \) as \( \bar{\Gamma} = \Gamma + R \) where

\[ R = \frac{1}{2E}[E \sum_{\tau_1 \leq m < n \leq \tau_2} (v \otimes \Delta X_m + \Delta X_n \otimes v) - (\Delta X_m \otimes v + v \otimes \Delta X_n)] \]

and

\[ \Gamma = \frac{1}{2E}[E \sum_{\tau_1 \leq m < n \leq \tau_2} \Delta X_n \otimes \Delta X_m - \Delta X_m \otimes \Delta X_n] \]

is the expected Levy area of \( X \) in the time interval \([\tau_1, \tau_2]\) divided by the expected interval’s size. Therefore,

\[ R^{i,j} = \frac{v_i E[S_{\tau_1, \tau_2}^{i,i}] - v_j E[S_{\tau_1, \tau_2}^{j,j}]}{2E[\tau_2 - \tau_1]} . \]

\[ \square \]
4 Simple examples for area anomaly

4.1 SRW on $\mathbb{Z}^2$ with drift in direction $e_1$

This example demonstrates that drift is not enough to create area anomaly. Let $X_{n+1} - X_n = \zeta_n$ so that $(\zeta_n)_{n}$ are i.i.d. and $\zeta_1 = e_1$ with probability $p/2$, $-e_1$ with probability $(1-p)/2$, and $\pm e_2$ with probability $1/4$. Then $X^n \to B$ in distribution in the uniform topology, $B$ is a BM with covariance matrix $\left( \begin{array}{cc} 2p(1-p) & 0 \\ 0 & 1 \end{array} \right)$, where $X^n(t) := \frac{\tilde{X}_{nt}}{\sqrt{n}} + \frac{(nt-|nt|)(\tilde{X}_{nt-1} - \tilde{X}_{nt})}{\sqrt{n}}$, $\tilde{X}_n = X_n - \frac{2p-1}{d}ne_1$.

One can check that in this case there is no area anomaly: $i^{(N)}(X)_{t\in[0,1]} \to (B_t, S_t)_{t\in[0,1]}$ in distribution in $C^\alpha([0,1], G^2(\mathbb{R}^2))$, $\alpha < 1/2$, where $(S_t)_{t\in[0,1]}$ is the Stratonovich second level iterated integral of $B$.

4.2 One-periodic conductances

This example aims to demonstrate that reversibility should create no area anomaly. Consider $\mathbb{Z}^2$, and let horizontal edges have weight 2 while vertical ones have weight 1. The walk then jumps along an edge proportionally to its weight. More formally, here $X_n - X_{n-1} \in \{e_1, -e_1\}$ uniformly with probability $2/3$ and $X_n - X_{n-1} \in \{e_2, -e_2\}$ uniformly with probability $1/3$. Then $X^n \to B$ in distribution in the uniform topology, where $B$ is a Brownian motion with covariance matrix $\left( \begin{array}{cc} 2/3 & 0 \\ 0 & 1/3 \end{array} \right)$, $X^n(t)$ as in the last example, and $\tilde{X}_n = X_n$. There is no area anomaly also in this case: $i^{(N)}(X)_{t\in[0,1]} \to (B_t, S_t + \tilde{\Gamma} t)_{t\in[0,1]}$ in distribution in $C^\alpha([0,1], G^2(\mathbb{R}^2))$, $\alpha < 1/2$, where $(S_t)_{t\in[0,1]}$ is the Stratonovich second level iterated integral of $B$.

4.3 Rotating drift [LS17b]

The following example shows that area anomaly is possible even with speed. Consider $\mathbb{Z}^2 \subset \mathbb{C}$. Let $(\zeta_n)_{n}$ be i.i.d. $P(\zeta_1 = 1) = p = 1 - P(\zeta_1 = -1)$. Define $X_n - X_{n-1} := i^n \zeta_n$, $i = e^{\pi i}$. Then $X^n \to B$ in distribution in the uniform topology, $B$ is a BM with covariance $2p(1-p)I$, where $I$ is the identity matrix, $X^n(t)$ is the linear interpolation as before, and $\tilde{X}_n = X_n$. However, after rescaling $i^{(N)}(X)_{t\in[0,1]} \to (B_t, S_t + \tilde{\Gamma} t)_{t\in[0,1]}$ in distribution in $C^\alpha([0,1], G^2(\mathbb{R}^2))$, $\alpha < 1/2$, where $(S_t)_{t\in[0,1]}$ is the Stratonovich second level iterated integral of $B$ and

$$
\tilde{\Gamma} = \frac{1}{4} \left( \begin{array}{cc} 0 & (2p-1)^2 \\ -(2p-1)^2 & 0 \end{array} \right).
$$

4.4 Non-elliptic periodic environment

The same example as the last one has the following representation. Consider $\mathbb{Z}^2$ and fix $1/2 < p < 1$. Let $\omega$ be the two-periodic environment given by: $\omega(0,1) = p = 1 - \omega(0,-1)$, $\omega(1,1+i) = p = 1 - \omega(1,1-i)$, $\omega(1+i, i) = p = 1 - \omega(1+i, 2+i)$, and $\omega(i, 0) = p = 1 - \omega(i, 2i)$. Finally, let $X$ be the random walk on $\mathbb{Z}^2$ in the deterministic environment $\omega$.

5 Applications

5.1 Random walks in random environment

We first define random walks in random environment on $\mathbb{Z}^d$. Let $\mathcal{E} := \{e_i : i = 1, \ldots, 2d\} \subset \mathbb{Z}^d$ be the set of neighbors of the origin. Let $\mathcal{P}$ be the space of probability distributions on $\mathcal{E}$. We call $\Omega = \mathcal{P}^{\mathbb{Z}^d}$ the space of environments on $\mathbb{Z}^d$. In particular, an environment $\omega \in \Omega$ is of the form $\omega = (\omega(z,e))_{z \in \mathbb{Z}^d, e \in \mathcal{E}}$ so that $\omega(z,e) \geq 0$ and $\sum_{e \in \mathcal{E}} \omega(z,e) = 1$.

For a fixed environment $\omega \in \Omega$ and a starting point $x \in \mathbb{Z}^d$ we define a nearest neighbor walk $X$ on $\mathbb{Z}^d$ to be the Markov chain with jump distributions $P_{x,\omega}(X_0 = x) = 1$ and $P_{x,\omega}(X_{n+1} = X_n + e | X_n = y) = \omega(y, y+e)$.
Given a probability distribution $\mathbb{P}$ on $\Omega$, the annealed (or averaged) law of the walk $X$ is characterized by $P_{\mathbb{P}}(\cdot) := \int P_{x,\omega}(\cdot) \mathbb{P}(d\omega)$. We also call $P_{x,\omega}$ the quenched law. We say that the environment is i.i.d. if $P_{x,\omega}$ is an i.i.d. sequence under $\mathbb{P}$. The random walk $X$ in i.i.d. random environment is called uniformly elliptic if there is some deterministic $\kappa > 0$ so that $P_{0}(X_{1} = e) \geq \kappa$ for all $e \in \mathcal{E}$.

We now define some ballisticity conditions and for that adapt the notation of [BDR14]. Let $L \geq 0$ and $\ell \in \mathbb{S}^{d-1}$ an element of the unit sphere. Then we write

$$H_{\ell}^{L} := \inf\{n \in \mathbb{N}_{0} : X_{n} \cdot \ell > L\} \quad (15)$$

for the first entrance time of $(X_{n})$ into the half-space \{ $x \in \mathbb{Z}^{d} : x \cdot \ell > L$\}, and where $\mathbb{N}_{0} = \{0, 1, 2, \ldots\}$.

**Definition 5.1** (Sznitman $(T')|\ell$ condition [Szn01]). Let $\gamma \in (0, 1]$ and $\ell \in \mathbb{S}^{d-1}$. We say that condition $(T)_{\gamma}$ is satisfied with respect to $\ell$ (written $(T)_{\gamma}|\ell$ or $(T)_{\gamma}$) if for each $\ell'$ in a neighborhood of $\ell$ and each $b > 0$ one has that

$$\limsup_{L \to \infty} L^{-\gamma} \ln P_{0}(H_{\ell'}^{L} > H_{bL}^{-\ell'}) < 0.$$ 

We say that condition $(T')$ is satisfied with respect to $\ell$ (written $(T')|\ell$ or $(T')$), if for each $\gamma \in (0, 1)$, condition $(T)_{\gamma}|\ell$ is fulfilled.

**Definition 5.2** (Berger-Drewitz-Ramirez $(P_{M}^{\ast}|\ell$) condition [BDR14]). Let $M > 0$ and $\ell \in \mathbb{S}^{d-1}$. We say that condition $P_{M}^{\ast}|\ell$ is satisfied with respect to $\ell$ if the following holds: For all $b > 0$ and all $\ell' \in \mathbb{S}^{d-1}$ in some neighborhood of $\ell$, one has that

$$\limsup_{L \to \infty} L^{M} P_{0}(H_{bL}^{-\ell'} < H_{L}^{\ell'}) = 0, \quad (16)$$

**Theorem 5.3.** Let $X$ be a random walk in i.i.d. and uniformly elliptic random environment on $\mathbb{Z}^{d}$, where $d \geq 2$. Let $\ell \in \mathbb{S}^{d-1}$ and assume that the Sznitman-type condition $P_{M}^{\ast}|\ell$ of Berger-Drewitz-Ramirez holds for some $M > 15d + 5$. Then the conditions of Theorem 3.3 are satisfied, and moreover the moment condition holds for all $1 \leq p < \infty$.

**Proof.** Berger, Drewitz, and Ramirez [BDR14, Theorem 1.6] states that in this case the stronger condition $(T')|\ell$ of Sznitman also holds. The law of large numbers, including the existence of regeneration times were proved in [SZ99] where the independence mentioned only the increments $X_{\tau_{k}} - X_{\tau_{k-1}}$. However, the proof of [SZ99] actually shows the walk on different intervals $(X_{\tau_{k-1}}, m, \tau_{k-1} < m \leq \tau_{k})$, is independent for $k \geq 1$ and identically distributed for $k \geq 2$, and appears specifically in [Ber08, Claim 3.4]. Therefore $X$ admits a regenerative structure in the sense of Definition 3.1. Annealed invariance principle was proved in [Szn00, Theorem 4.1] and [Szn01, Theorem 3.6] based on the finiteness of all moments for the regeneration time, which was proved in [Szn01, Theorem 3.4].

**Remark 5.4.** One version of Sznitman’s Conjecture states as follows: For random walk in random environment on $\mathbb{Z}^{d}$, $d \geq 2$, in i.i.d. and uniformly elliptic environment under the annealed law, a.s. directional transience in some direction $\ell$ is enough for attaining finiteness of all moments for the regeneration time. Therefore, assuming the conjecture then annealed a.s. directional transience in some direction $\ell$ is enough for annealed convergence in the rough path topology. Moreover, the convergence is in the $\alpha$-H"older topology to all $\alpha < 1/2$. In other words, there should be no example of a more singular convergence, or, more accurately no example for which there are some $\alpha < \beta \leq 1/2$ so that the convergence holds in $\alpha$-H"older but not in $\beta$-H"older).

**Remark 5.5.** Rather than using the $\alpha$-H"older norm, one can define rough paths using the $p$-variation norm (which was in fact the original definition in Lyons [Ly08]). This definition yields more robust estimates and is typically useful when the jumps have only second moment. However, in view of the last remark, there’s no advantage here for the $p$-variation version.
We close this section with an open problem. As one can notice in the examples given in Chapter \[1\] to construct a law with area anomaly it is not enough to have an asymptotic direction and non-trivial covariations. Area anomaly might hint that there is some asymmetry in the shape of the path with respect to the asymptotic direction. This leads to the following

**Problem 5.6.** Is there a RWRE satisfying the conditions of Theorem \[5.3\] for which the area anomaly \(\bar{\Gamma}\) is non-zero? Note that the question is open even for stationary and ergodic RWRE, which is weaker.

### 5.2 Periodic graphs or hidden Markov walks

Theorem 3.3 naturally generalizes the main result in [LS17b] and [LS17a].

**Theorem 5.7** ([LS17a] and [LS17b]). Let \(X\) be either an irreducible Markov chain on a periodic graph (see the definition in [LS17b]) or an irreducible hidden Markov walk driven by a finite state Markov chain (see the definition in [LS17a]), then the conditions of Theorem 3.3 are satisfied.

**Proof.** If \((Y_n)\) is an irreducible Markov chain on a periodic graph or an irreducible hidden Markov walk, it admit an underlying irreducible Markov chain \((X_n)\) on a finite state space. More precisely, for every \(n \geq 1\), the increment \(Y_{n+1} - Y_n\) depends on \(X_n\) in an appropriate way.

We can thus define a sequence on stopping times for \((X_n)\) as

\[
T_0 = 0 \text{ and } T_n = \inf\{k > T_{n-1} : X_k = X_0\}, \ n \geq 1.
\]

In particular, it is a sequence of return times to the initial position of \((X_n)\). By construction, the sequence \((T_n)\) is strictly increasing and, as \((X_n)\) is irreducible, all \(T_n\) are finite a.s. The increments \((T_{n+1} - T_n)\) are i.i.d., as well as the variables \((Y_{T_{n+1}} - Y_{T_n})\) (see proof in [LS17b]) and, more generally,

\[
(T_{n+1} - T_n, (Y_m - Y_{T_n})_{T_n \leq m < T_{n+1}})_{n \geq 0}.
\]

Consequently the process \((Y_n)\) admits a regenerative structure.

Moreover, since \((X_n)\) is irreducible and takes values in a finite state space, all moments of the increments \(T_{n+1} - T_n\) are finite (and actually have geometric tails). Concluding the law of large numbers and the invariance principle is now routine.

### 6 Proof of Theorem 3.3

We shall take the general route of [LS17a], where the authors proved first convergence for the path on a sequence of return times with exponential tails, and then moved to the full path where they identified an area correction. For both identification of the limit and tightness they used strong Markov property and the tail bounds of the stopping times. To demonstrate the idea in a rather simple way the reader is suggested to think about the case of random walks on a deterministic periodic environment where the decomposition is done according to returns of the walk to the origin *modulo the period*. In our proof, we decompose the path according to the regeneration times which are not stopping times and therefore strong Markov property does not apply. Fortunately, the i.i.d. nature of our decomposition plus finiteness of the regeneration time interval moments are enough to conclude.

**Proof of Theorem 3.3.** The proof will be divided in four steps:

- Convergence in distribution of the centered discrete process given by the sum of the \(\tilde{X}_{\tau_{k+1}} - \tilde{X}_{\tau_k}\) using the rough path version of the Donsker theorem.

- Convergence of the finite-dimensional marginals of the subsequence \(\left((\xi^{(\tau_k)}(\tilde{X}))_{t \leq T}\right)_{k \geq 0}\) where we see the area anomaly \(\bar{\Gamma}\).
• Convergence of finite-dimensional marginals of the full process \( (\ell(N)(X))_{t \leq T} \).

• Tightness of the sequence \( (\ell(N)(X))_{N \geq 0} \).

**Step 1:** Let \( Y_n =: \bar{X}_n \). We claim that \( \ell(N)(Y)_{t \leq T} \to (B', S'_t)_{t \leq T} \) in distribution in \( C^\alpha([0, T], G^2(\mathbb{R}^d)) \) for all \( \alpha \in \left( \frac{1}{3}, \frac{1}{2} - \frac{1}{2p} \right) \), where \( B' \) is a Brownian motion with covariance matrix \( \mathbb{E}[\bar{X}_{\tau_1, \tau_2} \bar{X}_{\tau_1, \tau_2}^T] \) and \( S' \) is its corresponding second level iterated Stratonovich integral.

Indeed, \( Y_n = \sum_{i=1}^n \Delta Y_i \) is a sum of i.i.d. centered random variables with values in \( \mathbb{R}^d \) and with covariance \( \mathbb{E}[\Delta Y_1 \Delta Y_1^T] = \mathbb{E}[\bar{X}_{\tau_1, \tau_2} \bar{X}_{\tau_1, \tau_2}^T] \). Moreover, since the jumps are a.s. bounded \( |\Delta X_n|_{\mathbb{R}^d} \leq K \), then \( |\Delta Y_n|_{\mathbb{R}^d} \leq Rd(\tau_n - \tau_{n-1}) \) and therefore also have finite \( 2p \) moment. Lemma 3.1 of [Kell16], implies weak convergence of \( Y(N) \) in the uniform topology and therefore convergence of the finite-dimensional marginals (Indeed, one takes \( V = 1 \) in the equation appears in that lemma and \( D \) there plays the role of our covariance matrix \( \mathbb{E}[\bar{X}_{\tau_1, \tau_2} \bar{X}_{\tau_1, \tau_2}^T] \). The tightness in \( C^\alpha([0, 1], G^2(\mathbb{R}^d)) \) for all \( \alpha \in \left( \frac{1}{3}, \frac{1}{2} - \frac{1}{2p} \right) \) is showed in the proof of that lemma using the Kolmogorov Criterion.)

**Step 2:** Denote by \( \delta \) the standard dilatation by \( \epsilon \), that is \( \delta_\epsilon(x, a) = (\epsilon x, \epsilon^2 a) \). We have the following decomposition of the rough path associated to \( \bar{X} \)

\[
\delta_{N^{1/2} \ell(N)(\bar{X})}_t = \bigotimes_{k=1}^{[Nt]} \left( \Delta \bar{X}_k, \frac{1}{2} \Delta \bar{X}_k^\otimes_2 \right) \otimes \delta_{Nt-\lfloor Nt \rfloor} \left( \Delta \bar{X}_{\lfloor Nt \rfloor+1}, \frac{1}{2} \Delta \bar{X}_{\lfloor Nt \rfloor+1}^\otimes_2 \right)
\]

since by construction the points of \((X_n)_n\) are connected by a geodesic embedding induced by the increments \( \bar{X}_{k+1} - \bar{X}_k \). Then, using the properties of integrals for piecewise linear processes, for \( r \in \mathbb{N} \), we get the decomposition

\[
\delta_{\tau_{r-1}/2 t^{r}}(\bar{X})_1 = \bigotimes_{k=1}^{r} \left( \Delta (\bar{X})_k, \frac{1}{2} \Delta (\bar{X})_k^\otimes_2 \right) \otimes \bigotimes_{k=1}^{r} (0, a_k)
\]

(17)

where \( \Delta (\bar{X})_k = \bar{X}_{\tau_k} - \bar{X}_{\tau_{k-1}} \) and

\[
ak_k = \frac{1}{2} \sum_{\tau_k+1 \leq m < n \leq \tau_{k}} \Delta \bar{X}_n \otimes \Delta \bar{X}_m - \Delta \bar{X}_m \otimes \Delta \bar{X}_n
\]

is the antisymmetric part of the process between the moments \( \tau_{k+1} \) and \( \tau_k \).

We note that the first term in the product at the right hand side of (17) corresponds to the rough path of a partial sum of our i.i.d. variables \( \bar{X}_{\tau_{r+1}} - \bar{X}_{\tau_1} \). We have seen in step 1 that the sequence of rough paths corresponding to these partial sums converges in distribution to the enhanced Brownian motion in the \( \alpha \)-Hölder topology, which implies that the corresponding finite-dimensional marginals converge in distribution to those of the Brownian motion.

On the other hand, for every fixed \( s \in \mathbb{N} \) and \( 0 < t_1 < \ldots < t_s \), using the fact that the process \( X \) admits an i.i.d. regenerative structure, we conclude that \( a_k, k \geq 2 \), are i.i.d. and thus, by the law of large numbers, we have the following convergence

\[
\left( \frac{1}{r} \sum_{k=1}^r a_{[t_1,k]}, \ldots, \frac{1}{r} \sum_{k=1}^r a_{[t,r]} \right) \to \mathbb{E}[a_2](t_1, \ldots, t_s) \text{ a.s.}
\]

The strong law of large numbers implies \( \frac{r}{n} \to \mathbb{E}[\tau_2 - \tau_1] =: \beta \) a.s. Since \( \otimes_{k=1}^{r} (0, a_k) = (0, \sum_{k=1}^{r} a_k) \), we can use Slutsky’s theorem [Shu25] as in [LS17a, Lemma 2.3.2] to conclude that we have the following convergence in distribution

\[
\left( \ell(\tau_r)(X)_{t_1}, \ldots, \ell(\tau_r)(X)_{t_s} \right) \to \left( (B_{t_1}, S_{t_1} + t_1 \Gamma), \ldots, (B_{t_s}, S_{t_s} + t_s \Gamma) \right)
\]
where $\bar{\Gamma} = \beta^{-1} \mathbb{E}[\alpha_2]$ is an antisymmetric matrix, $B = \beta^{-1/2} B'$ and $S$ is its corresponding Stratonovich iterated integral.

**Step 3:** Set $\kappa(n)$ to be the unique integer such that $\tau_{\kappa(n)} \leq n < \tau_{\kappa(n)+1}$. We use $[\bar{\Gamma}]^t$ together with the fact that $\bar{X}$ has bounded increments a.s. to deduce

$$d\left(\delta_{N-1/2}\left(\ell(1)(X)_{T_{\kappa(n)}(N)}\right), \delta_{N-1/2}\left(\ell(1)(X)_{[N]}\right)\right) = N^{-1/2}d\left(\left(\ell(1)(X)_{T_{\kappa(n)}(N)}\right), \left(\ell(1)(X)_{[N]}\right)\right) \leq dKN^{-1/2}(\lceil Nt \rceil - \tau_{\kappa(n)}(N)).$$

Applying the Markov inequality we obtain the following convergence for any $\epsilon > 0$

$$\mathbb{P}\left(d\left(\delta_{N-1/2}\left(\ell(1)(X)_{T_{\kappa(n)}(N)}\right), \delta_{N-1/2}\left(\ell(1)(X)_{[N]}\right)\right) > \epsilon\right) \leq Kd \frac{\mathbb{E}[\lceil Nt \rceil - \tau_{\kappa(n)}(N)]}{N^{1/2}\epsilon} \xrightarrow{N \to \infty} 0.$$  

Also, using the decomposition of $\tau_{\kappa(n)}$ into i.i.d. variables and the strong law of large numbers one deduces that $\kappa(n)/n$ converges a.s. to $\beta^{-1}$. Hence the conclusion of Step 2 together with Slutsky’s Theorem $[\text{Slu25}]$ imply the convergence in distribution

$$\ell(N)(X) \to (B_t, S_t + t\bar{\Gamma})$$

for any fixed $t \in [0, T]$. Extending the convergence to all finite-dimensional marginals of $\ell(N)(X)$, is done similarly using Slutsky’s Theorem on $\mathbb{R}^d$.

**Step 4:** It is left to prove the tightness of the process. In order to do this, we use the Kolmogorov tightness criterion for rough paths $[\text{FH14}, \text{Theorem 3.10}]$. That is, in order to obtain tightness for $\alpha < \frac{p^* - 1}{2p^*}$, it is enough to show that there exists a positive constant $c$ such that, for all $0 \leq s < t \leq T$,

$$\sup_N \mathbb{E}\left[\left\|\ell(N)(X)_{s,t}\right\|_{G^{2p^*}(\mathbb{R}^d)}\right] \leq c|t-s|^{p^*}. \quad (18)$$

To avoid heavy notation we write $X_{s,t} := \ell(N)(X)_{s,t}$ and assume WLOG that $\tau_1$ has the same distribution as $\tau_k - \tau_{k-1}$ for $k > 1$. From the definition of linear interpolation proving $(18)$ boils down to showing that there is a constant $c$ so that

$$\mathbb{E}\left[\left\|X_{\tau_k}\right\|_{G^{2p^*}(\mathbb{R}^d)}\right] \leq c(k-\ell)^{p^*}$$

uniformly on $0 \leq \ell < k \leq NT$. Note that by the i.i.d regenerative structure

$$\mathbb{E}\left[\left\|X_{\tau_k, \tau_\ell}\right\|_{G^{2p^*}(\mathbb{R}^d)}\right] = \mathbb{E}\left[\left\|X_{\tau_\ell - \tau_k}\right\|_{G^{2p^*}(\mathbb{R}^d)}\right].$$

The tightness argument $[\text{BFH09}, \text{Step 2 in Chapter 3}]$ then immediately implies

$$\mathbb{E}\left[\left\|X_{\tau_k}\right\|_{G^{2p^*}(\mathbb{R}^d)}\right] = O(\tau_k^{p^*}),$$

which in turn implies

$$\mathbb{E}\left[\left\|X_{\tau_k}\right\|_{G^{2p^*}(\mathbb{R}^d)}\right] = O(k^{p^*}),$$

where we used the fact that $\mathbb{E}[\tau_k^{p^*}] = O(k^{p^*})$. Next, if $k, \ell$ are in the same regeneration interval, the fact that the jumps are bounded, regeneration intervals have a finite $2p^*$ moments, and the definition $(4)$ imply

$$\mathbb{E}\left[\left\|X_{\tau_k, \tau_\ell}\right\|_{G^{2p^*}(\mathbb{R}^d)}\right] \leq C'$$

for some constant $C'$. Therefore by sub-additivity $(5)$, and using Hölder’s inequality together with $(6)$ we can find a constant $2p^*$ so that

$$\mathbb{E}\left[\left\|X_{\tau_k, \tau_\ell}\right\|_{G^{2p^*}(\mathbb{R}^d)}\right] \leq C_{2p^*}(2C' + \mathbb{E}\left[\left\|X_{\tau_k, \tau_{\ell+1}}\right\|_{G^{2p^*}(\mathbb{R}^d)}\right]) = O((k-\ell)^{p^*}).$$

We conclude that the Kolmogorov criterion is satisfied and so the sequence $(\ell(N)(X))_N$ is tight in $C^\alpha([0, T], G^2(\mathbb{R}^d), \alpha < \frac{p^* - 1}{2p^*})$. \qed
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