An investigation on the nonclassical and quantum phase properties of a family of engineered quantum states

A Thesis submitted by

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in partial fulfillment of the requirements for the award of the degree of

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Certificate

This is to certify that the thesis titled An investigation on the nonclassical and quantum phase properties of a family of engineered quantum states, submitted by Priya Malpani (P14EN001) to the Indian Institute of Technology Jodhpur for the award of the degree of Doctor of Philosophy, is a bonafide record of the research work done by her under my supervision. To the best of my knowledge, the contents of this report, in full or in parts, have not been submitted to any other Institute or University for the award of any degree or diploma.

Dr. V. Narayanan
Ph.D. Thesis Supervisor
## Acronyms

| Acronym | Description                                      |
|---------|--------------------------------------------------|
| BS      | Binomial state                                   |
| DFS     | Displaced Fock state                             |
| ECS     | Even coherent state                              |
| HOA     | Higher-order antibunching                        |
| HOSPS   | Higher-order sub-Poissonian photon statistics    |
| HOS     | Higher-order squeezing                            |
| KS      | Kerr state                                       |
| LE      | Linear entropy                                   |
| PADFS   | Photon added displaced Fock state                |
| PSDFS   | Photon subtracted displaced Fock state           |
| PABS    | Photon added binomial state                      |
| PAECS   | Photon added even coherent state                 |
| PAKS    | Photon added Kerr state                          |
| PASDFS  | Photon added then subtracted displaced Fock state|
| VFBS    | Vacuum filtered binomial state                   |
| VFECOS  | Vacuum filtered even coherent state              |
| VFKS    | Vacuum filtered Kerr state                       |
Declaration

I hereby declare that the work presented in this thesis entitled *An investigation on the nonclassical and quantum phase properties of a family of engineered quantum states* submitted to the Indian Institute of Technology Jodhpur in partial fulfillment of the requirements for the award of the degree of Doctor of Philosophy, is a bonafide record of the research work carried out under the supervision of Dr. V. Narayanan. The contents of this thesis in full or in parts, have not been submitted to, and will not be submitted by me to, any other Institute or University in India or abroad for the award of any degree or diploma.

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Abstract

The main focus of this thesis is to study the nonclassical and phase properties of a family of engineered quantum states, most of which show various nonclassical features. The beauty of these states is that these states can be used to establish quantum supremacy. Earlier, a considerable amount of works has been reported on various types of quantum states and their nonclassical properties. Here, complementing the earlier works, the effect of non-Gaussianity inducing operators on the nonclassical and phase properties of displaced Fock states have been studied. This thesis includes 6 chapters. In Chapter 1, motivation behind performing the present work is stated explicitly, also the basic concepts of quantum optics are discussed with a specific attention on the witnesses and measures of nonclassicality. In Chapter 2, nonclassical properties of photon added and subtracted displaced Fock states have been studied using various witnesses of lower- and higher-order nonclassicality which are introduced in Chapter 1. In Chapter 3 we have continued our investigation on photon added and subtracted displaced Fock states (and their limiting cases). In this chapter, quantum phase properties of these states are investigated from a number of perspectives, and it is shown that the quantum phase properties are dependent on the quantum state engineering operations performed. In Chapter 4, we have continued our investigation on the impact of non-Gaussianity inducing operators on the nonclassical and phase properties of the displaced Fock states. In Chapter 5, we have performed a comparison between to process that are used in quantum state engineering to induce nonclassical features. Finally, this thesis is concluded in Chapter 6, where we have summarized the findings of this thesis and have also described scope of the future works.
Dedicated to my parents for their unconditional love and support.
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Chapter 1

Introduction

1.1 Introduction

As the title of the thesis suggests, in this thesis, we aim to study the nonclassical and phase properties of a family of engineered quantum states. Before, we introduce such states and properties, it would be apt to lucidly introduce the notion of nonclassical and engineered quantum states. By nonclassical state we refer to a quantum state having no classical analogue. Such states are characterized by the negative values of Glauber-Sudarshan $P$-function or $P$-function more singular than Dirac delta function Sudarshan [1963], Glauber [1963a] and witnessed by various operational criteria (to be described in Section 1.5.1). To visualize the relevance of nonclassical states we may first note that quantum supremacy refers to the ability of performing a task using quantum resources in such a manner that either the task itself cannot be performed using classical resources or the speed/efficiency achieved using quantum resources cannot be achieved in the classical world Grover [1997]. A recent experiment performed by Google aimed at establishing quantum supremacy has drawn much of public attention Courtland [2017]. The relevance of the present study lies in the fact that to establish quantum supremacy or to perform a fundamental test of quantum mechanics, we would require a state having some features that would not be present in any classical state. As we have already mentioned, such a state having no classical analogue is referred to as the nonclassical state. Frequently used examples of nonclassical states include squeezed, antibunched, entangled, steered, and Bell nonlocal states. The relevance of the states having nonclassical features has already been established in the various domains of physics. For example, we may mention, teleportation of coherent states Furusawa et al. [1998], continuous variable quantum cryptography Hillery [2000], quantum radar Lanzagorta [2011], and many more. Further, we may note that the art of generating and manipulating quantum states as per need is referred to as the “quantum state engineering” Dakna et al. [1998], Sperling et al. [2014], Vogel et al. [1993], Miranowicz and Leonski [2004], DellAnno et al. [2006]. Particularly interesting examples of such engineered quantum states are Fock state, photon added/subtracted coherent state Agarwal and Tara [1991], displaced Fock state (DFS) which is also referred to as generalized coherent state and displaced number state Satyanarayana [1985], Wunsche [1991], Ziesel et al. [2013], Zavatta et al. [2004], De Oliveira et al. [1990], Malpani et al. [2019a], photon added DFS (PADFS) Malpani et al. [2019a], and photon subtracted DFS (PSDFS) Malpani et al. [2019a]. In what follows, we will state the relevance of such engineered quantum states in the implementation of different tasks exploiting their nonclassical and phase properties. The relatively new area of research on quantum state engineering has drawn much attention of the scientific community because of its success in experimentally producing various quantum states Zavatta et al. [2004], Torres et al. [2003], Rauschenbeutel et al. [2000], Gao et al. [2010], Lu et al. [2007] having nonclassical properties and applications in realizing quantum information processing tasks, like quantum key distribution Bennett and Brassard [1984] and quantum teleportation Brassard et al. [1998], Chen [2015]. Engineered quantum states, such as cat states, Fock state and superposition of Fock states, are known to play a crucial role in performing fundamental tests of quantum mechanics and in establishing quantum supremacy.
in the context of quantum computation and communication (Kues et al. [2017] and references therein).

As mentioned in the previous paragraph, with the advent of quantum state engineering Vogel et al. [1993]; Sperling et al. [2014]; Miranowicz and Leonski [2004]; Marchiolli and José [2004] and quantum information processing (Pathak [2013] and references therein), the study of nonclassical properties of engineered quantum states have become a very important field. This is so because only the presence of nonclassical features in a quantum state can provide quantum supremacy Grover [1997]. In the recent past, various techniques for quantum state engineering have been developed Vogel et al. [1993]; Sperling et al. [2014]; Miranowicz and Leonski [2004]; Agarwal and Tara [1991]; Lee and Nha [2010]; Marchiolli and José [2004]. If we restrict ourselves to optics, these techniques are primarily based on the clever use of beam splitters, detectors, and measurements with post selection, etc. Such techniques are useful in creating holes in the photon number distribution Escher et al. [2004] and in generating finite dimensional quantum states Miranowicz and Leonski [2004], both of which are nonclassical Pathak and Ghatak [2018]. The above said techniques are also useful in realizing non-Gaussianity inducing operations, like photon addition and subtraction Zavatta et al. [2004]; Podoshvedov [2014]. Motivated by the above, in this thesis, we aim to study the nonclassical properties of a set of engineered quantum states such as photon added, photon subtracted, and photon added then subtracted displaced Fock states which can be produced by using the above mentioned techniques. In the present thesis, we also wish to investigate the phase properties of the above mentioned engineered quantum states for the reasons explained below.

The impossibility of writing a Hermitian operator for quantum phase is a longstanding problem (see Perinová et al. [1998]; Carruthers and Nieto [1968]; Lynch [1987] for review). Early efforts of Dirac Dirac [1927] to introduce a Hermitian quantum phase operator were not successful, but led to many interesting proposals Susskind and Glogower [1964]; Pegg and Barnett [1989]; Barnett and Pegg [1986]. Specifically, Susskind-Glogower Susskind and Glogower [1964], Pegg-Barnett Pegg and Barnett [1988, 1989]; Barnett and Pegg [1990], and Barnett-Pegg Barnett and Pegg [1986] formalisms played very important role in the studies of phase properties and the phase fluctuation Imry [1971]. Thereafter, phase properties of various quantum states have been reported using these formalisms Sanders et al. [1986]; Gerry [1987]; Yao [1987]; Carruthers and Nieto [1968]; Vaccaro and Pegg [1989]; Pathak and Mandal [2000]; Alam and Mandal [2016b]; Alam et al. [2017a]; Verma and Pathak [2009]. Other approaches have also been used for the study of the phase properties. For example, quantum phase distribution is defined using phase states Agarwal et al. [1992], while Wigner Garraway and Knight [1992] and Q-Leonhardt and Paul [1993]; Leonhardt et al. [1995] phase distributions are obtained by integrating over radial parameter of the corresponding quasidistribution function. In experiments, the phase measurement is performed by averaging the field amplitudes of the Q function Noh et al. [1991, 1992]; Pegg-Barnett and Wigner phase distributions are also reported with the help of reconstructed density matrix Smithey et al. [1993]. Further, quantum phase distribution under the effect of the environment was also studied in the past leading to phase diffusion Banerjee and Srikanth [2007]; Banerjee et al. [2007]; Abdel-Aty et al. [2010]; Banerjee [2018]. A measure of phase fluctuation named phase dispersion using quantum phase distribution has also been proposed in the past Perinová et al. [1998]; Banerjee and Srikanth [2007]. Recently, quantum phase fluctuation Zheng-Feng [1992] and Pancharatnam phase Mendos and Popovic [1993] have been studied for DFS. The quantum phase fluctuation in parametric down-conversion Gantsog et al. [1991] and its revival Gantsog [1992] are also reported. Experiments on phase super-resolution without using entanglement Resch et al. [2007] and role of photon subtraction in concentration of phase information Usuga et al. [2010] are also performed. Optimal phase estimation Sanders and Milburn [1995] using different quantum states Higgins et al. [2007] (including NOON and other entangled states and unentangled single-photon states) has long been the focus of quantum metrology Giovannetti et al. [2006, 2011]. Nonclassicality measure based on the shortening of the regular distribution defined on phase difference interval broadbands due to nonclassicality is also proposed in the recent past Perina and Křepelka [2019]; Thapliyal and Perina [2019]. In brief, quantum phase properties are of intense interest of the community since long (see Pathak [2002]; Perinová et al. [1998] and references therein), and the interest in it has been further enlightened in the recent past as many new applications of quantum phase distribution and quantum phase fluctuation have been realized.
To be specific, this work is also motivated by the fact that recently several applications of nonclassical states and quantum phase properties have been reported. Specifically, squeezed states have played an important role in the studies related to phase diffusion Banerjee and Srikanth [2007]; Banerjee et al. [2007], the detection of gravitational waves in LIGO experiments Abbasi and Golshan [2013]; Abbott et al. [2016a,b]. The rising demand for a single photon source can be fulfilled by an antibunched light source Yuan et al. [2002]. The study of quantum correlations is important both from the perspective of pure and mixed states Chakrabarty et al. [2011]; Dhar et al. [2013]; Banerjee et al. [2010a,b]. Entangled states are found to be useful in both secure Ekert [1991] and insecure Bennett and Wiesner [1992]; Bennett et al. [1993] quantum communication schemes. Stronger quantum correlation present in the steerable states are used to ensure the security against all the side-channel attacks on devices used in one-side (i.e., either preparation or detector side) for quantum cryptography Branciard et al. [2012]. Quantum supremacy in computation is established due to quantum algorithms for unsorted database search Grover [1997], factorization and discrete logarithm problems Short [1999], and machine learning Biamonte et al. [2017] using essentially nonclassical states. We may further stress on the recently reported applications of quantum phase distribution and quantum phase fluctuation by noting that these have applications in quantum random number generation Xu et al. [2012]; Raffaelli et al. [2018], cryptanalysis of squeezed state based continuous variable quantum cryptography Horal [2004], generation of solitons in a Bose-Einstein condensate Denschlag et al. [2000], storage and retrieval of information from Rydberg atom Ahn et al. [2000], in phase encoding quantum cryptography Gisin et al. [2002], phase imaging of cells and tissues for biomedical application Park et al. [2018]; as well as have importance in determining the value of transition temperature for superconductors Emery and Kivelson [1995].

Now to achieve the above advantages of the nonclassical states, we need to produce these states via the schemes of quantum state engineering. For the same, there are some distinct theoretical tools, like quantum scissoring Miranowicz et al. [2001], hole-burning Escher et al. [2004]; Gerry and Benmoussa [2002]; Malpani et al. [2020a] or filtering out a particular Fock state from the photon number distribution Meher and Sivakumar [2018], applying non-Gaussianity inducing operations Agarwal [2013]. However, these distinct mechanisms are experimentally realized primarily by appropriately using beam splitters, mirrors, and single photon detectors or single photon counting module. Without going into finer details of the optical realization of quantum state engineering tools, we may note that these tools can be used to generate various nonclassical states, e.g., DFS De Oliveira et al. [1990], PADFS Malpani et al. [2019a], PSDFS Malpani et al. [2019a], photon added squeezed coherent state Thapliyal et al. [2017a], photon subtracted squeezed coherent state Thapliyal et al. [2017b], number state filtered coherent state Meher and Sivakumar [2018]. Some of these states, like photon added coherent state, have already been realized experimentally Zavatia et al. [2004].

Many of the above mentioned engineered quantum states have already been studied in detail. Primarily, three types of investigations have been performed on the engineered quantum states- (i) study of various nonclassical features of these states (and their variation with the state parameters) as reflected through different witnesses of nonclassicality. Initially, such studies were restricted to the lower-order nonclassical features. In the recent past, various higher-order nonclassical features have been predicted theoretically Alam et al. [2018a,b]; Pathak and Garcia [2006]; Pathak and Verma [2010]; Verma et al. [2008]; Thapliyal et al. [2017b] and confirmed experimentally Hamar et al. [2014]; Péflina Jr et al. [2017] and references therein in quantum states generated in nonlinear optical processes. (ii) Phase properties of the nonclassical states have been studied El-Orany et al. [2000] by computing quantum phase fluctuations, phase dispersion, phase distribution functions, etc., under various formalisms, like Susskind and Glogower Susskind and Glogower [1964]. Pegg-Barnett Pegg and Barnett [1989] and Barnett-Pegg Barnett and Pegg [1986] formalisms. (iii) Various applications of the engineered quantum states have been designed. Some of them have already been mentioned.

Motivated by the above observations, in this thesis, we would like to perform an investigation on nonclassical and phase properties of a particularly interesting set of engineered quantum states which will have the flavor of the first two facets of the studies mentioned above. Applications of the engineered quantum states will also be discussed briefly, but will not be investigated in detail. To begin with we would like to briefly describe the physical
and mathematical concepts used in this thesis, and that will be the focus of the rest of this chapter.

The rest of this chapter is organized as follows. In Section 1.2, we will briefly discuss quantum theory of radiation and introduce annihilation, creation, and number operators as well as Fock and coherent states. In Section 1.3, this will be followed by an introduction to a set of quantum states which will be studied in this thesis. After this, the notion of nonclassicality will be introduced mathematically in Section 1.4 and a set of operational criteria for observing nonclassical properties will be introduced in Section 1.5. Subsequently, the parameters used for the study of quantum phase properties will be introduced in Section 1.6. These witnesses of nonclassicality and the parameters for the study of phase properties will be used in the subsequent chapters, to investigate the nonclassical and phase properties of the quantum states discussed in Section 1.3. Finally, the structure of the rest of the thesis will be provided in Section 1.7.

1.2 Quantum theory of radiation field

Historically, quantum physics started with the ideas related to quanta of radiation. To be precise, Planck’s work on black body radiation [Planck 1901] and Einstein’s explanation of photoelectric effect [Einstein 1905] involved a notion of quantized radiation field. In Planck’s work, light was considered to be emitted from and absorbed by a black body in quanta; and in Einstein’s work, it was also considered that the radiation field propagates from one point to another as quanta. These initial works contributed a lot in the development of quantum mechanics, but after the introduction of quantum mechanics, in 1920s, in the initial days, most of the attention was given to the quantization of matter. A quantum theory of radiation was introduced by Dirac in 1927 [Dirac 1927]. In what follows, we will describe it briefly as this would form the backbone of the present thesis.

1.2.1 Creation and annihilation operator

Maxwell gave a classical description of electromagnetic field. But here the objective is to study light and its properties apart from Maxwell’s equations. So, to begin with, it would be reasonable to write Maxwell’s equations in free space:

\[ \nabla \cdot E = 0, \]
\[ \nabla \cdot B = 0, \]
\[ \nabla \times E = \frac{1}{c} \frac{\partial B}{\partial t}, \]
\[ \nabla \times B = \frac{1}{c^2} \frac{\partial E}{\partial t}, \]

where \( c = \frac{1}{\sqrt{\mu_0 \epsilon_0}} \) is the speed of light without any medium (i.e. vacuum). Using the set of above equations one can express magnetic and electric fields in the form of the solution of wave equations, like

\[ \nabla^2 E - \frac{1}{c^2} \frac{\partial^2 E}{\partial t^2} = 0. \]

The quantization of radiation field can be done by assuming a cavity of length \( L \) having a linearly polarized electric field whose direction of propagation is in \( z \) direction. Because of the linearity of the wave equation (1.5), we are
allowed to write the electric field in the form of linear combination of all the normal modes as

\[ E_x(z,t) = \sum_n A_n q_n(t) \sin(k_n z), \]  

(1.6)

where \( q_n \) being the amplitude of the \( n \)th normal mode with \( k_n = \frac{\pi n}{L} \), \( V \) is the volume of the resonator, \( A_n = \frac{2m_n c^2}{\varepsilon_0 V} \), with \( \nu_n = c k_n \), and \( m_n \) is a constant (in the units of mass). With the help of this, we try to form an analogy of radiation with mechanical oscillator. In analogy to Eq. (1.6) we are able to write the corresponding magnetic field equation in cavity as

\[ B_y(z,t) = \sum_n A_n \left( \frac{q_n}{c^2 k_n} \right) \cos(k_n z). \]  

(1.7)

So, the total energy of the field can be written as a classical Hamiltonian

\[ H = \frac{1}{2} \int_V d\tau \left( \varepsilon_0 E_x^2 + \frac{1}{\mu_0} B_y^2 \right), \]  

(1.8)

\[ H = \frac{1}{2} \sum_n \left( m_n \nu_n^2 q_n^2 + m_n p_n^2 \right) = \frac{1}{2} \sum_n \left( m_n \nu_n^2 q_n^2 + \frac{p_n^2}{m_n} \right), \]  

(1.9)

Substituting position and momentum variables by corresponding operators to obtain the quantum mechanical Hamiltonian, where \( p_n = m_n \dot{q}_n \).

The position \((q_n)\) and momentum \((p_n)\) operators follow the commutation relations

\[ [q_n, p_m] = \hbar \delta_{nm}, \quad [q_n, q_m] = [p_n, p_m] = 0, \]

where \( \hbar \) is the reduced Planck’s constant. Using these one may define a new set of operators which can be analytically written as

\[ \hat{a}_n \exp \left[ -i \nu_n t \right] = \frac{1}{\sqrt{2\hbar m_n \nu_n}} \left( m_n \nu_n q_n + i p_n \right) \]  

(1.10)

and

\[ \hat{a}^\dagger_n \exp \left[ i \nu_n t \right] = \frac{1}{\sqrt{2\hbar m_n \nu_n}} \left( m_n \nu_n q_n - i p_n \right). \]  

(1.11)

Thus, the Hamiltonian can be written as

\[ H = \sum_n \hbar \nu_n \left( \hat{a}_n \hat{a}_n^\dagger + \frac{1}{2} \right), \]  

(1.12)

and the commutation relations

\[ [\hat{a}_n, \hat{a}_m^\dagger] = \delta_{nm}, \quad [\hat{a}_n, \hat{a}_m] = [\hat{a}_n^\dagger, \hat{a}_m^\dagger] = 0, \]

with corresponding electric and magnetic fields, as given by Eq. 1.1.27 in [Scully and Zubairy, 1997]

\[ E(\vec{r}, t) = \sum_k \xi_k \hat{\xi}_k \hat{a}_k \exp \left[ -i \nu_k t + i k \cdot \vec{r} \right] + \text{H.c.} \]

and

\[ B(\vec{r}, t) = \sum_k \frac{k \times \hat{\xi}_k}{\nu_k} \xi_k \hat{a}_k \exp \left[ -i \nu_k t + i k \cdot \vec{r} \right] + \text{H.c.}, \]

where \( \xi_k = \left( \frac{\nu_k}{m_n} \right)^{1/2} \) is a constant, and \( \hat{\xi}_k \) is a unit polarization vector with the wave vector \( k \).
The above analysis shows that a single-mode field is identical to harmonic oscillator. So, in the domain of quantum optics, harmonic oscillator system plays an important role.

Notice that the quantum treatment of electromagnetic radiation hinges on annihilation $\hat{a}$ (depletes photon) and creation $\hat{a}^\dagger$ (creates photon) operators. The annihilation operator $\hat{a}$ depletes one quantum of energy and thus lowers down the system from harmonic oscillator level $|n\rangle$ to $|n - 1\rangle$, given by

$$\hat{a}|n\rangle = \sqrt{n}|n - 1\rangle.$$  \hspace{1cm} (1.13)

Here, $|n\rangle$ is called Fock or number state. Further, an application of annihilation operator on vacuum leads to 0, i.e., $\hat{a}|0\rangle = 0$. The creation operator $\hat{a}^\dagger$ creates one quantum of energy by raising the state from $|n\rangle$ to $|n + 1\rangle$. Therefore, the creation operator in the number state can be represented as

$$\hat{a}^\dagger|n\rangle = \sqrt{n + 1}|n + 1\rangle.$$  \hspace{1cm} (1.14)

If creation operator is applied to vacuum it creates a photon so these operators enables one to write a Fock state ($|n\rangle$) in terms of the vacuum state as

$$|n\rangle = \left( \hat{a}^\dagger \right)^n \frac{1}{\sqrt{n!}} |0\rangle.$$  \hspace{1cm}

In the above, we have seen that annihilation and creation operators are the important field operators and are required for the quantum description of radiation. These operators can induce nonclassicality and non-Gaussianity when applied on classical states Zavatta et al. [2004]; Agarwal [2013]. In the present thesis, we study the role of these non-Gaussianity inducing operations in controlling the nonclassicality of the quantum states which are often already nonclassical. For instance, enhancement in squeezing in a nonclassical state does not ensure advantage with respect to use as a single photon source and vice-versa. In the following subsection, we will introduce a set of other operators which can be expressed in terms of annihilation and creation operators and which play a crucial role in our understanding of the quantum states of radiation field.

### 1.2.2 Some more quantum operators of relevance

So far we have introduced some non-unitary operations (operations $\hat{O}$ which are not norm preserving and do not satisfy $\hat{O}^\dagger = \hat{O}^{-1}$, where $\hat{O}^\dagger$ and $\hat{O}^{-1}$ are the Hermitian conjugate and inverse of $\hat{O}$, respectively), namely photon addition and subtraction. We now aim to introduce some more unitary operations important in the domain of quantum state engineering in general, and in this thesis in particular. To begin with let us describe displacement operator.

#### 1.2.2.1 Displacement operator

Displacement operator is a unitary operator. The mathematical form of displacement operator is given as

$$\hat{D}(\alpha) = \exp \left( \alpha \hat{a}^\dagger - \alpha^* \hat{a} \right).$$  \hspace{1cm} (1.15)

This operator can be used as a tool to generate coherent state from vacuum. Specifically, a coherent state $|\alpha\rangle$ is defined as $|\alpha\rangle = \hat{D}(\alpha)|0\rangle$.

#### 1.2.2.2 Squeezing operator

The squeezing operator for a single mode of electromagnetic field is

$$\hat{S}(z) = \exp \left( \frac{1}{2} (z^* \hat{a}^2 - z \hat{a}^2) \right).$$  \hspace{1cm} (1.16)
The description of light is given by two quadratures namely phase \((X_1)\) and amplitude \((X_2)\) in the domain of quantum optics, mathematically defined as

\[
\hat{X}_\theta = \frac{1}{\sqrt{2}} \left( i \hat{a}^\dagger \exp[i\theta] - i \hat{a} \exp[-i\theta] \right),
\]

The corresponding uncertainty of these two quadratures is observed by relation \(\Delta X_1 \Delta X_2 \geq \hbar / 2\), where \(\Delta X_1 (\Delta X_2)\) is variance in the measured values of quadrature \(\hat{X}_1 = \hat{X}_1 (\theta = 0) \) \(\hat{X}_2 = \hat{X}_2 (\theta = \pi / 2)\). Specifically, \(\Delta X_i = \sqrt{\langle X_i^2 \rangle - \langle X_i \rangle^2}\), where \(\langle \hat{A} \rangle = \langle \psi | \hat{A} | \psi \rangle\) is the expectation value of the operator \(\hat{A}\) with respect to the quantum state \(|\psi\rangle\). Coherent state has an equal uncertainty in both quadratures so they form a circle in the phase picture (shown in Fig. 1.1). Least value of the variance for suitable \(\theta\) is studied as principle squeezing. With the advent of nonlinear optics, a very special branch of optics, the uncertainty in one of the quadratures can be reduced at the cost of increment in other quadrature’s uncertainty, which means that the circle can be squeezed.

1.2.3 Eigen states of the field operators

Here, we will discuss eigen states of some of the operators we have introduced. The eigenvalue equation can be defined as \(\hat{A} \lambda = a \lambda\) with eigen operator \(\hat{A}\), eigenvalue \(a\), and eigen function \(\lambda\). For example, Schrodinger equation \(H \psi = E \psi\) has Hamiltonian \(H\) as eigen operator with eigen functions \(\psi\) and eigenvalues as allowed energy levels.

1.2.3.1 Fock state: Eigen state of the number operator

In case of quantum optics or quantized light picture, photon number state is known as number state. The single-mode photon number states are known Fock states, and its ground state is defined as vacuum state. As the set of number states are a full set of orthonormal basis so any quantum state can be written in terms of these basis. The method of representing a quantum state as superposition of number states is known as number state representation. Now using Eq. (1.13) and (1.14), we can introduce an operator \(\hat{N} = \hat{a}^\dagger \hat{a}\).

which would satisfy the following eigen value equation

\[
\hat{N} |n\rangle = n |n\rangle.
\]

Clearly Fock states are the eigen states of the number operators and in consistency with what has already been told, a Fock state \(|n\rangle\) represent a \(n\) photon state.

1.2.3.2 Coherent state: Eigen state of the annihilation operator

Coherent state \cite{2006} is considered as a state of the quantized electromagnetic field which shows classical behaviour (specifically, behavior closest to classical states). According to Erwin Schrodinger it is a minimum uncertainty state, having same uncertainty in position and momentum \cite{1926}. According to Glauber, any of three mathematical definitions described below can define coherent state:

(i) Eigen vectors of annihilation operator \(\hat{a} |\alpha\rangle = \alpha |\alpha\rangle\), \(\alpha\) being a complex number.

(ii) Quantum states having minimum uncertainty \(\Delta X_1 = \Delta X_2 = 1 / \sqrt{2}\), with \(X_2\) and \(X_1\) as momentum and position operators.

(iii) States realized by the application of the displacement operator \(D(\alpha)\) on the vacuum state. Thus, is also known as displaced vacuum state and can be expressed as

\[
|\alpha\rangle = D(\alpha) |0\rangle.
\]
Figure 1.1: Phase picture for coherent state and squeezed state.

In Fock basis, it is expressed as infinite superposition of Fock state as

$$|\alpha\rangle = \exp \left[ -\frac{|\alpha|^2}{2} \right] \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle,$$  \hspace{1cm} (1.17)

where $\alpha$ is a complex number. Experimentally established state very close to this coherent state was possible only after the successful development of laser. Finally, one can easily see that $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$ implies $\langle \alpha|\hat{a}^\dagger = \langle \alpha|\alpha^*\rangle$ and consequently $\langle \alpha|\hat{a}\hat{a}^\dagger|\alpha\rangle = \langle \alpha|\hat{N}|\alpha\rangle = N = |\alpha|^2$ or average photon number in a coherent state is $|\alpha|^2$.

1.3 Quantum states of our interest

In this section, we provide basic mathematical details of the set of engineered quantum states studied in the present thesis.

1.3.1 Displaced Fock state

Displaced Fock state \cite{Satyanarayana1985} are formed by applying displacement operator on Fock state and thus a DFS is defined as

$$|\phi\rangle = D(\alpha)|n\rangle.$$  

Analytically it is given as

$$|\phi(n,\alpha)\rangle = \frac{1}{\sqrt{n!}} \sum_{p=0}^{n} \frac{n^p}{p!} (-\alpha^*)^{n-p} \exp \left( -\frac{|\alpha|^2}{2} \right) \sum_{m=0}^{\infty} \frac{\alpha^m}{m!} \sqrt{(m+p)!m+p} |m+p\rangle.$$

Various nonclassical properties of DFS are reported in literature \cite{DeOliveiraEtAl1990,ElOranyEtAl2000,LvovskyAndBabichev2002,MendasAndPopovic1993}.

1.3.2 Photon added and photon subtracted displaced Fock state

Using DFS, we can define a $u$ photon added DFS (i.e., a PADFS) as

$$|\psi_{+}(u,n,\alpha)\rangle = N_{+}\hat{a}^u|\phi(n,\alpha)\rangle = \frac{N_{+}}{\sqrt{n!}} \sum_{p=0}^{n} \frac{n^p}{p!} (-\alpha^*)^{n-p} \exp \left( -\frac{|\alpha|^2}{2} \right) \sum_{m=0}^{\infty} \frac{\alpha^m}{m!} \sqrt{(m+p+u)!m+p+u}.$$  \hspace{1cm} (1.19)
Similarly, a $v$ photon subtracted DFS (i.e., a PSDFS) can be expressed as

$$|\psi_{-}(v,n,\alpha)\rangle = \frac{N_{-}}{\sqrt{n!}} \sum_{p=0}^{n} \binom{n}{p} (-\alpha^{*})^{(n-p)} \exp\left(-\frac{|\alpha|^{2}}{2}\right) \sum_{m=0}^{\infty} \frac{\alpha^{m}}{m!} \left(m+p-v\right)!$$

where $m$ and $p$ are the real integers. Here,

$$N_{+} = \left[ \frac{1}{n!} \sum_{p,p'=0}^{n} \binom{n}{p} \binom{n}{p'} (-\alpha^{*})^{(n-p)} (-\alpha^{*})^{(n-p')} \exp\left[-\frac{|\alpha|^{2}}{2}\right] \sum_{m=0}^{\infty} \frac{\alpha^{m}(\alpha^{*})^{m+p-p'} (m+p+u)!}{m!(m+p-p')!} \right]^{-\frac{1}{2}}$$

and

$$N_{-} = \left[ \frac{1}{n!} \sum_{p,p'=0}^{n} \binom{n}{p} \binom{n}{p'} (\alpha^{*})^{(n-p)} (-\alpha^{*})^{(n-p')} \exp\left[-\frac{|\alpha|^{2}}{2}\right] \sum_{m=0}^{\infty} \frac{\alpha^{m}(\alpha^{*})^{m+p-p'} (m+p+u)!}{m!(m+p-p')! (m+p-v)!} \right]^{-\frac{1}{2}}$$

are the normalization constants, and subscripts $+$ and $-$ correspond to photon addition and subtraction. Thus, $|\psi_{+}(u,n,\alpha)\rangle$ and $|\psi_{-}(v,n,\alpha)\rangle$ represent $u$ photon added DFS and $v$ photon subtracted DFS, respectively, for the DFS which has been produced by displacing the Fock state $|n\rangle$ by a displacement operator $D(\alpha)$ characterized by the complex parameter $\alpha$. Clearly, the addition and the subtraction of photons on the DFS can be mathematically viewed as application of the creation and annihilation operators from the left on the Eq. (1.18). Here, it may be noted that different well-known states can be obtained as special cases of these two states. For example, using the notation introduced above to define PADFS and PSDFS, we can describe a coherent state $|\alpha\rangle$ as $|\alpha\rangle = |\psi_{+}(0,\alpha,0)\rangle = |\psi_{-}(0,\alpha,0)\rangle$, naturally, coherent state can be viewed as a special case of both PADFS and PSDFS. Similarly, we can describe a single photon added coherent state as $|\psi_{+1}\rangle = |\psi_{+}(1,\alpha,0)\rangle$, a Fock state as $|n\rangle = |\psi_{+}(0,0,n)\rangle = |\psi_{-}(0,0,n)\rangle$ and a DFS as $|\psi_{DFS}\rangle = |\psi_{+}(0,\alpha,n)\rangle = |\psi_{-}(0,\alpha,n)\rangle$.

### 1.3.3 Photon added then subtracted displaced Fock state

A PASDFS can be obtained by sequentially applying appropriate number of annihilation (photon subtraction) and creation (photon addition) operators on a DFS. Analytical expression for PASDFS (specifically, a $k$ photon added and then $q$ photon subtracted DFS) in Fock basis can be shown to be

$$|\psi(k,q,n,\alpha)\rangle = N\hat{a}^{k}\hat{a}^{q}\langle n,\alpha| = \frac{N}{\sqrt{n!}} \sum_{p=0}^{n} \binom{n}{p} (-\alpha^{*})^{(n-p)} \exp\left(-\frac{|\alpha|^{2}}{2}\right) \times \sum_{m=0}^{\infty} \frac{\alpha^{m}}{m!} \left(m+p+k\right)! \left(m+p+k-q\right)!$$

where

$$N = \left[ \frac{1}{n!} \sum_{p,p'=0}^{n} \binom{n}{p} \binom{n}{p'} (-\alpha^{*})^{(n-p)} (-\alpha^{*})^{(n-p')} \exp\left[-\frac{|\alpha|^{2}}{2}\right] \right]^{-\frac{1}{2}}$$

is the normalization factor.
1.3.4 Even coherent state and states generated by holeburning on it

Even coherent state can be defined as the superposition of two coherent states having opposite phase \(|\phi(\alpha)| \propto |\alpha| + |\alpha|\). The analytical expression for ECS in number basis can be written as

\[
|\phi(\alpha)| = \frac{\exp\left[ -|\alpha|^2 \right]}{\sqrt{2(1+\exp[-2|\alpha|^2])}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} (1 + (-1)^n) |n\rangle.
\]  

(1.24)

The parameter \(\alpha = |\alpha| \exp(i\theta)\), in Eq. (1.24), is complex in general and \(\theta\) is phase angle in the complex plane. Various schemes to generate ECS are reported in Brune et al. [1992]; Ourjoumtsev et al. [2007]; Gerry [1993]. The nonclassical properties (witnessed through the antibunching and squeezing criteria, \(Q\) function, Wigner function, and photon number distribution, etc.) of ECS have been studied in the recent past Gerry [1993].

1.3.4.1 Vacuum filtered even coherent state

As mentioned above, experimentally, an ECS or a cat state can be generated in various ways, and the same can be further engineered to produce a hole at vacuum in its photon number distribution. Specifically, filtration of vacuum will burn a hole at \(n = 0\) and produce VFECS, which can be described in Fock basis as

\[
|\phi_1(\alpha)| = N_{\text{VFECS}} \sum_{n=0, n\neq 0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} (1 + (-1)^n) |n\rangle,
\]  

(1.25)

where

\[
N_{\text{VFECS}} = \frac{4 \cosh(|\alpha|^2) - 1}{2^{1/2}}
\]  

(1.26)

is the normalization constant. For simplicity, we may expand Eq. (1.25) as a superposition of a standard ECS and a vacuum state as follows

\[
|\phi_1(\alpha)| = N_{\text{VFECS}} \left( \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} (1 + (-1)^n) |n\rangle - 2 |0\rangle \right).
\]  

(1.27)

In what follows, Eq. (1.27) will be used to explore various nonclassical features that may exist in VFECS.

1.3.4.2 Photon added even coherent state

One can define a single photon added ECS as

\[
|\phi_2(\alpha)| = N_{\text{PAECS}} a^\dagger |\phi(\alpha)| = N_{\text{PAECS}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} (1 + (-1)^n) \sqrt{n+1} |n+1\rangle,
\]  

(1.28)

where

\[
N_{\text{PAECS}} = \{ \cosh(|\alpha|^2) + |\alpha|^2 \sinh(|\alpha|^2) \}^{-1/2}/2
\]  

(1.29)

is the normalization constant for PAECS.

1.3.5 Binomial state and the states generated by holeburning on it

Binomial state is a finite superposition of Fock states having binomial photon number distribution. It is quite similar to the coherent state which is the linear combination of Fock states having the Poissonian photon number distribution Stoler et al. [1985]. BS can be defined as

\[
|p, M\rangle = \sum_{n=0}^{M} \frac{M!}{n!(M-n)!} p^n (1-p)^{M-n} \frac{1}{\sqrt{n!}} |n\rangle.
\]  

(1.30)
The binomial coefficient describes the presence of \( n \) photons with probability \( p \) in \( M \) number of ways. Recently, the study of nonclassical properties of BS, specifically, antibunching, squeezing, HOSPS Verma et al. [2008], Verma and Pathak [2010], Bazrafkan and Man’ko [2004], etc., have been studied very extensively. However, no effort has yet been made to study the nonclassical properties of VFBS and PABS.

### 1.3.5.1 Vacuum filtered binomial state

The vacuum filtration of BS can be obtained by simply eliminating vacuum state from the BS as

\[
|p, M\rangle_1 = N_{\text{VFBS}} \sum_{n=0}^{M} \left[ \frac{M!}{n!(M-n)!} p^n (1-p)^{M-n} \right]^{1/2} |n\rangle - N_{\text{VFBS}} \left[ (1-p)^M \right]^{1/2} |0\rangle,
\]

where

\[
N_{\text{VFBS}} = \left\{ 1 - (1-p)^M \right\}^{-1/2}
\]

is the normalization constant for the VFBS.

### 1.3.5.2 Photon added binomial state

A hole at \( n = 0 \) at a BS can also be introduced by the addition of a single photon on the BS. A few steps of computation yield the desired expression for PABS as

\[
|p, M\rangle_2 = N_{\text{PABS}} \sum_{n=0}^{M} \left[ \frac{M!(n+1)!}{n!(M-n)!} p^n (1-p)^{M-n} \right]^{1/2} |n+1\rangle,
\]

where

\[
N_{\text{PABS}} = (1 + Mp)^{-1/2}
\]

is the normalization constant for single photon added BS.

### 1.3.6 Kerr state and the states generated by holeburning on it

A KS can be obtained when electromagnetic field in a coherent state interacts with nonlinear medium with Kerr type nonlinearity Gerry and Grobe [1994]. This interaction generates phase shifts which depend on the intensity. The Hamiltonian involved in this process is given as

\[
H = \hbar \omega \hat{a}^\dagger \hat{a} + \hbar \chi (\hat{a}^\dagger \hat{a})^2 (\hat{a})^2,
\]

where \( \chi \) depends on the third-order susceptibility of Kerr medium. Explicit contribution of \( H \) is \( \exp \left[ -i \chi n (n-1) \right] \). Thus, the compact analytic form of the KS in the Fock basis can be given as

\[
|\psi_K (n)\rangle = \sum_{n=0}^{\infty} \frac{|\alpha|^n}{\sqrt{n!}} \exp \left( -\frac{|\alpha|^2}{2} \right) \exp \left( -i \chi n (n-1) \right) |n\rangle.
\]

### 1.3.6.1 Vacuum filtered Kerr state

Similarly, a VFKS, which can be obtained using the same quantum state engineering process that leads to VFECS and VFBS, is given by

\[
|\psi_K (n)\rangle_1 = N_{\text{VFKS}} \sum_{n=0}^{\infty} \frac{|\alpha|^n}{\sqrt{n!}} \exp \left( -i \chi n (n-1) \right) |n\rangle - |0\rangle,
\]

where

\[
N_{\text{VFKS}} = \left( \exp \left[ \left\| \alpha \right\|^2 \right] - 1 \right)^{-1/2}
\]
is the normalization constant for the VFKS.

### 1.3.6.2 Photon added Kerr state

An addition of a photon to Kerr state would yield PAKS which can be expanded in Fock basis as

\[
|\psi_K(n)\rangle_2 = N_{\text{PAKS}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \exp(-i\chi n(n-1)) \sqrt{(n+1)}|n+1\rangle,
\]

where

\[
N_{\text{PAKS}} = \left( \exp \left[ |\alpha|^2 \right] (1 + |\alpha|^2) \right)^{-1/2}
\]

is the normalization constant for the PAKS.

### 1.4 The notion of nonclassical states

Quantum states which do not have any classical analogue have been referred to as nonclassical states [Agarwal 2013]. In other words, states having their \(P\)-distribution more singular than delta function or having negative values are referred to as nonclassical states [Dodonov and Renò 2003]. This idea was possible only when Glauber and Sudarshan published papers in 1963 [Sudarshan 1963; Glauber 1963b,a]. Sudarshan found a mathematical form to represent any state in the coherent basis, mathematically given as

\[
\rho = \int P(\alpha)|\alpha\rangle \langle \alpha| d^2\alpha,
\]

where \(P(\alpha)\) is known as Glauber-Sudarshan \(P\)-function, which follows normalization condition as \(\int dP(\alpha) = 1\), but it may have negative values. Thus, it is defined as quasidistribution function or quasiprobability distribution. When \(P(\alpha)\) attains a positive probability density function, immediately it indicates that the state is classical. This leads to the definition of nonclassicality. If an arbitrary quantum state is failed to represent as mixture of coherent states, that is known as nonclassical state. To establish quantum supremacy, these nonclassical states play very essential role, for instance in theses states are useful in establishing quantum supremacy of quantum information processing, quantum communication, etc. Although \(P\)-function is not reconstructable for any arbitrary state yet it has been of major interest as it provides an important signature of nonclassicality. The negativity (positivity or non-negativity) of the \(P\)-function essentially provides the nonclassical (classical) behavior of the state under consideration. The experimental difficulty associated with the easurement of \(P\)-function in its reconstruction led to various feasible substitutes as nonclassicality witnesses. Here, we list some of those nonclassicality witnesses. These witnesses can be viewed as operational criterion of nonclassicality.

### 1.5 Nonclassical states: witnesses and measures

Using nonclassical states, the essence of quantum theory of light can be understood. There are various tools for characterization of nonclassical states. In this section, some tools are described which are used to characterize such states. If historically seen, the first such approach was aimed to check the deviation from Poissonian photon statistics, the second is to evaluate the volume of the negative part of the quasiprobability distribution in the phase space, etc. An infinite set of moments based criteria is available in literature which is used as witness of nonclassicality equivalent to \(P\)-function [Shchukin and Vogel 2005b]. Any subset of this infinite set may detect nonclassicality or fail to do so. Example of these witnesses are lower- and higher-order antibunching, sub-Poissonian photon statistics, squeezing as well as Mandel \(Q_M\) parameter, Klyshko’s, Vogel’s, and Agarwal-Tara’s criteria, \(Q\) function, etc. To quantify the amount of nonclassicality, a number of measures have been proposed, like linear entropy, Wigner volume, concurrence and many more. A small description of these criteria is given here.
CHAPTER 1. INTRODUCTION

1.5.1 Witnesses of nonclassicality

1.5.1.1 Lower- and higher-order antibunching

In this section, we study lower- and higher-order antibunching. To do so, we use the following criterion of \((l-1)\)th order antibunching [Pathak and Garcia 2006 and references therein] in terms of nonclassicality witness \(d(l-1)\) as

\[
d(l-1) = \langle \hat{a}^\dagger \hat{a}^l \rangle - \langle \hat{a}^\dagger \hat{a} \rangle^l < 0. \tag{1.41}
\]

This nonclassical feature characterizes suitability of the quantum state to be used as single photon source as the negative values of \(d(l-1)\) parameter show that the probability of photons coming bunched is less compared to that of coming independently. The signature of lower-order antibunching can be obtained as a special case of Eq. (1.41) for \(l = 2\), and that for \(l \geq 3\), the negative values of \(d(l-1)\) correspond to higher-order antibunching of \((l-1)\)th order. Figure 1.2 illustrates the scheme for studying antibunching experimentally (corresponds to \(l = 2\)). For higher values of \(l\), we require more beamsplitters and APDs. On these cascaded beamsplitters signal is mixed with vacuum and measured higher-order correlation [Avenhaus et al. 2010].

1.5.1.2 Lower- and higher-order sub-Poissonian photon statistics

The lower-order counterparts of antibunching and sub-Poissonian photon statistics are closely associated as the presence of latter ensures the possibility of observing former (see Thapliyal et al. [2014a, 2017b] for a detailed discussion). However, these two nonclassical features were shown to be independent phenomena in the past (Thapliyal et al. [2014a, 2017b] and references therein). Higher-order counterpart of sub-Poissonian photon statistics can be introduced as

\[
\mathcal{B}_h(l-1) = \sum_{e=0}^{l} \sum_{f=1}^{e} S_2(e, f) \cdot \left( \begin{array}{c} l \end{array} \right) \cdot (-1)^e d(f-1) \langle N \rangle^{l-e} < 0, \tag{1.42}
\]

where \(S_2(e, f)\) is the Stirling number of second kind, and \(\left( \begin{array}{c} l \end{array} \right)\) is the usual binomial coefficient.
1.5.1.3 Higher-order squeezing

As mentioned beforehand, the squeezing of quadrature is defined in terms of variance in the measured values of the quadrature (say, position or momentum) below the corresponding value for the coherent state, i.e., minimum uncertainty state. The higher-order counterpart of squeezing is studied in two ways, namely Hong-Mandel and Hillery-type squeezing [Hong and Mandel [1985b,a]; Hillery [1987a]]. Specifically, the idea of the higher-order squeezing originated from the pioneering work of Hong and Mandel [Hong and Mandel [1985b,a]], who generalized the lower-order squeezing using the higher-order moments of field quadrature. According to the Hong-Mandel criterion, the $l$th order squeezing can be observed if the $l$th moment (for even values of $l > 2$) of a field quadrature operator is less than the corresponding coherent state value. The condition of Hong-Mandel type higher-order squeezing is given as follows [Hong and Mandel [1985b,a]]

$$S(l) = \left\langle (\Delta X)^l \right\rangle - \left(\frac{1}{2}\right)^{\frac{l}{2}} < 0,$$

(1.43)

Here, $S(l)$ is higher-order squeezing, $\Delta X$ is the quadrature ($\Delta X_1$) as defined in Section 1.2.2. Further, $(\frac{1}{2})^{\frac{l}{2}}$ is a conventional Pochhammer symbol. The inequality in Eq. (1.43) can also be rewritten as

$$\left\langle (\Delta X)^l \right\rangle < \left(\frac{1}{2}\right)^{\frac{l}{2}} (l - 1)!!$$

(1.44)

with

$$\left\langle (\Delta X)^l \right\rangle = \sum_{r=0}^{l} \sum_{i=0}^{r-2} (-1)^r \frac{1}{2^r} (2i-1)!^2 C_i^1 C_r^1 C_2^2 (\hat{a}^\dagger + \hat{a})^{l-r} (\hat{a}^k \hat{a}^{2l-2i-k}).$$

(1.45)

1.5.1.4 Klyshko’s criterion

This criterion is relatively simpler as to calculate this witness of nonclassicality, only three consecutive probability terms are required rather than all the terms. Negative values of $B(m)$ are symbol of nonclassicality present in the state. Klyshko introduced this criterion [Klyshko [1996]] to investigate the nonclassical property using only three successive photon-number probabilities. In terms of the photon-number probability $p_m = \langle m | \rho | m \rangle$ of the state with density matrix $\rho$, the Klyshko’s criterion in the form of an inequality can be written as

$$B(m) = (m + 2)p_m p_{m+2} - (m + 1)(p_{m+1})^2 < 0.$$  

(1.46)

1.5.1.5 Vogel’s criterion

The moments-based nonclassicality criterion of the previous subsection was later extended to Vogel’s nonclassicality criterion [Shchukin and Vogel [2005b]] in terms of matrix of moments as

$$v = \begin{bmatrix}
1 & \langle \hat{a} \rangle & \langle \hat{a}^2 \rangle \\
\langle \hat{a}^\dagger \rangle & \langle \hat{a}^\dagger \hat{a} \rangle & \langle \hat{a}^\dagger (\hat{a}^\dagger)^2 \rangle \\
\langle \hat{a} \rangle & \langle \hat{a}^2 \rangle & \langle \hat{a}^2 \hat{a} \rangle 
\end{bmatrix}.$$  

(1.47)

The negative value of the determinant $dv$ of matrix $v$ in Eq. (1.47) is signature of nonclassicality.

1.5.1.6 Agarwal-Tara’s criterion

There were certain quantum states having negative $P$-function yet showing no squeezing and sub-Poissonian behavior, to witness the nonclassicality residing in those particular types of states Agarwal and Tara [Agarwal and Tara [1992]] introduced this criterion which is again a moments based criterion. This can be written in a matrix
form and expressed as
\[ A_3 = \frac{\det m^{(3)}}{\det \mu^{(3)} - \det m^{(3)}} < 0, \]  
(1.48)
where
\[ m^{(3)} = \begin{bmatrix} 1 & m_1 & m_2 \\ m_1 & m_2 & m_3 \\ m_2 & m_3 & m_4 \end{bmatrix} \]
and
\[ \mu^{(3)} = \begin{bmatrix} 1 & \mu_1 & \mu_2 \\ \mu_1 & \mu_2 & \mu_3 \\ \mu_2 & \mu_3 & \mu_4 \end{bmatrix}. \]
The matrix elements are defined as \( m_i = \langle \hat{a}^\dagger \hat{a}^i \rangle \) and \( \mu_j = (\langle \hat{a}^\dagger \hat{a}^j \rangle)^j = (m_1)^j \).

### 1.5.1.7 Mandel \( Q_M \) parameter

The Mandel \( Q_M \) parameter [Mandel 1979] illustrates the nonclassicality through photon number distribution in a quantum state. The Mandel \( Q_M \) parameter is defined as
\[ Q_M = \frac{\langle (\hat{a}^\dagger \hat{a})^2 \rangle - \langle \hat{a}^\dagger \hat{a} \rangle^2 - \langle \hat{a}^\dagger \hat{a} \rangle}{\langle \hat{a}^\dagger \hat{a} \rangle}. \]  
(1.49)
The negative values of \( Q_M \) parameter essentially indicate the negativity for \( P \)-function and so it gives a witness for nonclassicality. For the Poissonian statistics it becomes 0, while for the sub-Poissonian (super-Poissonian) photon statistics it has negative (positive) values.

### 1.5.2 Other quasiprobability distributions

Inability to give a phase space description of quantum mechanics is exploited in terms of quasiprobability distributions [Thapliyal et al. 2015] and references therein). Later, it was found that they are useful as witnesses of nonclassicality. These real and normalized quasiprobability distributions allow to calculate the expectation value of an operator as any classical probability distribution. One such quasiprobability distributions is \( Q \) function [Husimi 1940], and zeros of this function are signature of nonclassicality. Another example is Wigner [Wigner 1932] function whose negative values corresponds to the nonclassicality.

#### 1.5.2.1 \( Q \) function

\( Q \) function [Husimi 1940] is defined as
\[ Q = \frac{1}{\pi} \langle \beta | p | \beta \rangle, \]  
(1.50)
where \( |\beta\rangle \) is the coherent state \( |1.17\rangle \).

#### 1.5.2.2 Wigner function

Another quasiprobability distribution is Wigner function formulated by Wigner in 1932 [Wigner 1932] in the early stage of quantum mechanics, the motive was to connect the wavefunction approach to a probability distribution in phase space. Negativity of Wigner function represents the nonclassicality present in an arbitrary quantum state. Also the ability to reconstruct the Wigner function experimentally makes this approach more impactful than any other approach. Specifically, Wigner function obtained through optical tomography can be used to obtain other quasidistributions, however, Wigner function is stronger witness of nonclassicality than \( Q \) function while is not
singular like $P$-function. Mathematically, it is expressed as

$$W(\gamma, \gamma^*) = A \exp \left[ -2 |\gamma|^2 \right] \int d^2 \lambda \langle -\lambda | \rho | \lambda \rangle \exp \left[ -2(\gamma^* \lambda - \gamma \lambda^*) \right].$$

(1.51)

The zeros of $Q$ function while the negativity of $P$-function and Wigner function correspond to the nonclassical behavior of any arbitrary quantum state. It is worth stressing here that only $P$-function is both necessary and sufficient criterion of nonclassicality, while rest of the quasidistribution functions are only sufficient.

### 1.5.3 Measures of nonclassicality

In the above section, we have seen that there exist numerous criteria of nonclassicality. However, most of these criteria only witness the nonclassicality. They do not provide any quantification of the nonclassicality. Except $P$-function and infinite set of vogel’s criteria, all other criteria are sufficient but not necessary. However, many efforts have been made for the quantification of nonclassicality, e.g., in 1987, a distance-based measure of nonclassicality was introduced by Hillery [1987]. A trace norm based measure [Mari et al. 2011] was introduced by Mari et al., for the set of all states having the positive Wigner function. In 1991, Lee gave a measure of nonclassicality known as nonclassical depth [Lee 1991]. However, in this work, we will not study these measures. There are certain measures those can be exploited in terms of entanglement, like linear entropy [Wei et al. 2003], which we will use for our calculations and the same is described below.

#### 1.5.3.1 Linear entropy

In 2005, a measure of nonclassicality was proposed as entanglement potential, which is the amount of entanglement in two output ports of a beam splitter with the quantum state $\rho_{in}$ and vacuum $|0\rangle\langle 0|$ sent through two input ports [Asbóth et al. 2005]. The amount of entanglement quantifies the amount of nonclassicality in the input quantum state as classical state can not generate entanglement in the output. The post beam splitter state can be obtained as $\rho_{out} = U (\rho_{in} \otimes |0\rangle\langle 0|) U^\dagger$ with $U = \exp[-iH\theta]$, where $H = (\hat{a}^\dagger \hat{b} + \hat{a} \hat{b}^\dagger)/2$, and $\hat{a}^\dagger (\hat{a})$, $\hat{b}^\dagger (\hat{b})$ are the creation (annihilation) operators of the input modes. For example, considering quantum state ($|\psi\rangle = \sum_{n=0}^{\infty} c_n |n\rangle$) and a vacuum state $|0\rangle$ as input states, we can write the analytic expression of the two-mode output state as

$$|\phi\rangle = U (|\psi\rangle \otimes |0\rangle) \equiv U |\psi, 0\rangle = \sum_{n=0}^{\infty} \frac{c_n}{2^{n/2}} \sum_{j=0}^{n} \sqrt{C_j} |j, n-j\rangle.$$  

(1.52)

We can measure the amount of entanglement in the output state to quantify the amount of input nonclassicality in $|\psi\rangle$. Here, we use linear entropy of single mode subsystem (obtained after tracing over the other subsystem) as entanglement potential. The linear entropy for an arbitrary bipartite state $\rho_{AB}$ is defined as [Wei et al. 2003]

$$\mathcal{L} = 1 - \text{Tr} (\rho_B^2),$$

(1.53)

where $\rho_B$ is obtained by tracing over subsystem $A$.

### 1.6 Analytic tools for the study of phase properties of nonclassical states

In this section, we aim to introduce the parameters that are used to study phase properties of a given quantum state under consideration in this section.
1.6.1 Phase distribution function

A distribution function allows us to calculate expectation values of an operator analogous to that from the corresponding density matrix. Phase distribution function for a given density operator \cite{Banerjee and Srikanth 2007; Agarwal et al. 1992} can be defined as

\[ P_\theta = \frac{1}{2\pi} \langle \theta | \rho | \theta \rangle, \]  

(1.54)

where the phase state \(|\theta\rangle\), complementary to the number state \(|n\rangle\), is defined \cite{Agarwal et al. 1992} as

\[ |\theta\rangle = \sum_{n=0}^{\infty} e^{i\pi n} |n\rangle. \]  

(1.55)

1.6.2 Phase dispersion

A known application of phase distribution function (1.54) is that it can be used to quantify the quantum phase fluctuation. Although the variance is also used occasionally as a measure of phase fluctuation, it has a drawback that it depends on the origin of phase integration \cite{Banerjee and Srikanth 2007}. A measure of phase fluctuation, free from this problem, is phase dispersion \cite{Peˇrinová et al. 1998} defined as

\[ D = 1 - \left| \int_{-\pi}^{\pi} d\theta \exp [-t\theta] P_\theta \right|^2. \]  

(1.56)

1.6.3 Angular Q function

Analogous to the phase distribution \(P_\theta\), phase distributions are also defined as radius integrated quasidistribution functions which are used as the witnesses for quantumness \cite{Thapliyal et al. 2015}. One such phase distribution function based on the angular part of the \(Q\) function is studied in \cite{Leonhardt and Paul 1993; Leonhardt et al. 1995}. Specifically, the angular \(Q\) function is defined as

\[ Q_\theta = \int_{0}^{\infty} Q(\beta, \beta^*) |\beta| d|\beta|, \]  

(1.57)

where the \(Q\) function \cite{Husimi 1940} is defined in Eq. (1.50).

1.6.4 Phase fluctuation

In attempts to get rid of the limitations of the Hermitian phase operator of Dirac \cite{Dirac 1927}, Louisell \cite{Louisell 1963} first mentioned that bare phase operator should be replaced by periodic functions. As a consequence, sine (\(\hat{S}\)) and cosine (\(\hat{C}\)) operators appeared, explicit forms of these operators were given by Susskind and Glogower \cite{Susskind and Glogower 1964}, and further modified by Barnett and Pegg \cite{Barnett and Pegg 1986} as

\[ \hat{S} = \frac{\hat{a} - \hat{a}^\dagger}{2t (\bar{N} + \frac{1}{2})^{\frac{1}{2}}} \]  

(1.58)

and

\[ \hat{C} = \frac{\hat{a} + \hat{a}^\dagger}{2 (\bar{N} + \frac{1}{2})^{\frac{1}{2}}} \]  

(1.59)

Here, \(\bar{N}\) is the average number of photons in the measured field, and here we refrain our discussion to Barnett and Pegg sine and cosine operators \cite{Barnett and Pegg 1986}. Carruthers and Nieto \cite{Carruthers and Nieto 1968} have
introduced three quantum phase fluctuation parameters in terms of sine and cosine operators

\[ U = (\Delta N)^2 \left[ (\Delta S)^2 + (\Delta C)^2 \right] / \left[ \langle \hat{S} \rangle^2 + \langle \hat{C} \rangle^2 \right], \quad (1.60) \]

\[ S = (\Delta N)^2 (\Delta S)^2, \quad (1.61) \]

and

\[ Q = S / \langle \hat{C} \rangle^2. \quad (1.62) \]

These three phase fluctuation parameters \( U \), \( S \) and \( Q \) show phase properties of PADFS and PSDFS, while \( U \) parameter is shown relevant as a witness of nonclassicality (antibunching).

1.6.5 Quantum phase estimation parameter

Quantum phase estimation is performed by sending the input state through a Mach-Zehnder interferometer and applying the phase to be determined (\( \phi \)) on one of the arms of the interferometer. To study the phase estimation using Mach-Zehnder interferometer, angular momentum operators \( \text{Sanders and Milburn}\ [1995], \text{Demkowicz-Dobrzański et al.}\ [2015] \), defined as

\[ \hat{J}_x = \frac{1}{2} (\hat{a}^{\dagger} \hat{b} + \hat{b}^{\dagger} \hat{a}), \quad (1.63) \]

\[ \hat{J}_y = \frac{i}{2} (\hat{b}^{\dagger} \hat{a} - \hat{a}^{\dagger} \hat{b}), \quad (1.64) \]

and

\[ \hat{J}_z = \frac{1}{2} (\hat{a}^{\dagger} \hat{a} - \hat{b}^{\dagger} \hat{b}), \quad (1.65) \]

are used. Here, \( \hat{a} \) and \( \hat{b} \) are the annihilation operators for the modes corresponding to two input ports of the Mach-Zehnder interferometer. The average value of \( \hat{J}_z \) operator in the output of the Mach-Zehnder interferometer, which is one-half of the difference of photon numbers in the two output ports (1.65), can be written as

\[ \langle \hat{J}_z \rangle = \cos \phi \langle \hat{J}_z \rangle_{\text{in}} - \sin \phi \langle \hat{J}_x \rangle_{\text{in}}. \quad (1.66) \]

Therefore, variance in the measured value of operator \( \hat{J}_z \) can be computed as

\[ (\Delta J_z)^2 = \cos^2 \phi (\Delta J_z)_{\text{in}}^2 + \sin^2 \phi (\Delta J_x)_{\text{in}}^2 - 2 \sin \phi \cos \phi \text{cov} (\hat{J}_x, \hat{J}_z)_{\text{in}}, \quad (1.67) \]

where covariance of the two observables is defined as

\[ \text{cov} (\hat{J}_x, \hat{J}_z) = \frac{1}{2} (\hat{J}_x \hat{J}_z + \hat{J}_z \hat{J}_x) - \langle \hat{J}_x \rangle \langle \hat{J}_z \rangle. \quad (1.68) \]

This allows us to quantify precision in phase estimation \( \text{Demkowicz-Dobrzański et al.}\ [2015] \) as

\[ \Delta \phi = \frac{\Delta J_z}{\left| \frac{d \langle \hat{J}_z \rangle}{d \phi} \right|}. \quad (1.69) \]

Before we proceed further and conclude this chapter by noting the structure of the rest of the thesis, it would be apt to note that there exist various methods of quantum state engineering (some of which have already been mentioned) and photon addition, subtraction, filtration, punching etc., which can be viewed as examples of quantum state engineering processes. In rest of the thesis these processes will be studied with detail.
1.7 Structure of the rest of the thesis

This thesis has 6 chapters. The next 4 chapters are focused on the study of nonclassical and phase properties of the engineered quantum states and the last chapter is dedicated to conclusion. These chapters and thus the rest of this thesis is organized as follows.

In Chapter 1, in Section 1.3, quantum states of our interest (i.e., PADFS and PSDFS) have been introduced in detail. In Chapter 2, in Section 2.3, the analytical expressions of various witnesses of nonclassicality are reported. Further, the existence of various lower- and higher-order nonclassical features in PADFS and PSDFS are shown through a set of plots. Finally, we conclude in Section 2.4.

In Chapter 3, in Section 3.3, we investigate the phase properties of PADFS and PSDFS from a number of perspectives. Finally, the chapter is concluded in Section 3.4.

In Chapter 4, we describe the quantum state of interest (i.e., PASDFS) in Fock basis and calculate the analytic expressions for the higher-order moments of the relevant field operators for this state. In Section 4.3, we investigate the possibilities of witnessing various nonclassical features in PASDFS and its limiting cases by using a set of moments-based criteria for nonclassicality. Variations of nonclassical features (witnessed through different criteria) with various physical parameters are also discussed here. In Section 4.4, phase properties of PASDFS are studied. Q function for PASDFS is obtained in Section 4.5. Finally, we conclude in Section 4.6.

In Chapter 5, in Section 5.2, we have introduced the quantum states of our interest which include ECS, BS, KS, VFEC, VFBS, VFKS, PAECS, PABS, and PAKS. In Section 5.3, we have investigated the nonclassical properties of these states using various witnesses of lower- and higher-order noncassicality as well as a measure of nonclassicality. Specifically, in this section, we have compared nonclassicality features found in vacuum filtered and single photon added versions of the states of our interest using the witnesses of Higher-order antibunching (HOA), Higher-order squeezing (HOS) and Higher-order sub-Poissonian photon statistics (HOSPS). Finally, in Section 5.5, the results are analyzed, and the chapter is concluded.

Finally, the thesis is concluded in Chapter 6, where we have summarized the findings reported in Chapter 2-5 and have emphasized on the main conclusion of the present thesis. We have also discussed the scopes of future work.
Chapter 2

Lower-and higher-order nonclassical properties of photon added and subtracted displaced Fock state

In this chapter, which is based on Malpani et al. [2019a], we aim to study the nonclassical properties of the PADFS and PSDFS (which are already introduced in Section 1.3.2) using the witnesses of nonclassicality introduced in Section 1.5.1.

2.1 Introduction

As we have mentioned in Chapter 1, with the advent of quantum state engineering Vogel et al. [1993], Sperling et al. [2013], Miranowicz and Leonski [2004], Marchiolli and José [2004] and quantum information processing Pathak [2013] and references therein), the study of nonclassical properties of engineered quantum states have become a very important field. Quantum state engineering is helpful in realizing non-Gaussianity inducing operations, like photon addition and subtraction Zavatta et al. [2004], Podoshvedov [2014]. Keeping this in mind, in what follows, in this chapter, we aim to study the nonclassical properties of a set of engineered quantum states (both photon added and subtracted) which can be produced by using the above mentioned techniques.

It is already mentioned in Chapter 1 that a state having a negative $P$-function is referred to as a nonclassical state. Such a state cannot be expressed as a mixture of coherent states and does not possess a classical analogue. In contrast to these states, coherent states are classical, but neither their finite dimensional versions Miranowicz and Leonski [2004], Alam et al. [2017c] nor their generalized versions are classical Satyanarayana [1985], Thapliyal et al. [2016, 2015], Banerjee and Srikanth [2007]. Here, we would like to focus on photon added and subtracted versions of a particular type of generalized coherent state, which is also referred to as the displaced Fock state (DFS). To be precise, state of the form $|\psi\rangle = D(\alpha)|n\rangle$, where $D(\alpha)$ is the displacement operator with Fock state $|n\rangle$, is referred to as generalized coherent state (see Section 1.3 of Chapter 1), as this state is reduced to a coherent state in the limit $n = 0$. However, from the structure of the state it seems more appropriate to call this state as the DFS, and this seems to be the nomenclature usually adopted in the literature Keil et al. [2011], Wunschel [1991], Blažek [1994], Moya-Cessa [1995]. In some other works, it is referred to as displaced number state Ziesel et al. [2013], De Oliveira et al. [2006], Dodonov and De Souza [2005], but all these names are equivalent; and in what follows, we will refer to it as DFS. This is an extremely interesting quantum state for various reasons. Specifically, its relevance in various areas of quantum optics is known. For example, in the context of cavity QED, it constitutes the eigenstates of the Jaynes-Cummings systems with coherently driven atoms Alsing et al. [1992]. Naturally, various lower-order nonclassical properties and a set of other quantum features of DFS have already been studied.
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Specifically, quasiprobability distributions of DFS were studied in Wunsche [1991], phase fluctuations of DFS was investigated in Zheng-Feng [1992], decoherence of superposition of DFS was discussed in Dodonov and De Souza [2005], $Q$ function, Wigner function and probability distribution of DFS were studied in De Oliveira et al. [1990], Pancharatnam phase of DFS has been studied in Mendas and Popovic [1993]. Further, in the context of non-optical DFS, various possibilities of generating DFS from Fock states by using a general driven time-dependent oscillator has been discussed in Lo [1991], and in the trapped-ion system, quantum interference effects have been studied for the superposition of DFS Marchiolli and José [2004]. Thus, DFS seems to be a well studied quantum state, but it may be noted that a little effort has yet been made to study higher-order nonclassical properties of DFS. It is a relevant observation as in the recent past, it has been observed that higher-order nonclassicality has various applications Hillery et al. [1999], Banerjee et al. [2018], Sharma et al. [2017], Thapliyal et al. [2017c], and it can be used to detect the existence of weak nonclassical characters Thapliyal et al. [2014b], Verma and Pathak [2010], Thapliyal et al. [2017b], Alam and Manda [2016a], Alam et al. [2015], Thapliyal et al. [2017a], Prakash and Mishra [2006], Prakash et al. [2010], Das et al. [2018]. Further, various higher-order nonclassical features have been experimentally detected Allevi et al. [2012b], Avenhaus et al. [2010], Pelina Jr et al. [2017]. However, we do not want to restrict to DFS, rather we wish to focus on lower- and higher-order nonclassical properties of a set of even more general states, namely photon added DFS (PADFS) and photon subtracted DFS (PSDFS). The general nature of these states can be visualized easily, as in the special case that no photon is added (subtracted) PADFS (PSDFS) would reduce to DFS. Further, for $n = 0$, PADFS would reduce to photon added coherent state which has been widely studied Agarwal and Tara [1991], Verma et al. [2008], Thapliyal et al. [2017b] and references therein) and experimentally realized Zavatta et al. [2004, 2005]. Here it is worth noting that DFS has also been generated experimentally by superposing a Fock state with a coherent state on a beam splitter Lvovsky and Babichev [2002]. Further, an alternative method for the generation of DFS has been proposed by Oliveira et al. de Oliveira et al. [2005]. From the fact that photon added coherent state and DFS have already been generated experimentally, and the fact that the photon added states can be prepared via conditional measurement on a beam splitter, it appears that PADFS and PSDFS can also be built in the lab. In fact, inspired by these experiments, we have proposed a schematic diagram for generating the PADFS and PSDFS in Figure 2.1 using single-mode and two-mode squeezed vacuum states. Specifically, using three (two) highly transmitting beam splitters, a conditional measurement of single photons at both detectors D1 and D2 in Figure 2.1(a) (Figure 2.1(b))

Figure 2.1: A schematic diagram for the generation of PSDFS (in (a) and (b)) and PADFS (in (c) and (d)). In (a) and (c) ((b) and (d)), single-mode (two-mode) squeezed vacuum is used for generation of the desired state. Here, NLC corresponds to nonlinear crystal and D1 and D2 are photon number resolving detectors.
chapter. The obtained expressions for the above mentioned moments for PADFS and PSDFS are

\[
\langle \hat{a}^q \hat{a}^r \rangle \text{PADFS} = \sum_{m=0}^{\infty} \frac{a^m(a^*)^m}{m!(m+p-q)(m+p+r-q)} (m+p+u)! (m+p+u-r+q)!
\]

and

\[
\langle \hat{a}^q \hat{a}^r \rangle \text{PSDFS} = \sum_{m=0}^{\infty} \frac{a^m(a^*)^m}{m!(m+p-q)(m+p+r-q)} (m+p+u)! (m+p+r-q)!
\]

respectively. The values of normalization constants for PADFS and PSDFS are already given in Eqs. \ref{PADFS} and \ref{PSDFS}, respectively. In the following section, we shall investigate the possibilities of observing various types lower- and higher-order nonclassical features in PADFS and PSDFS by using Eqs. \ref{PADFS} and \ref{PSDFS}. 

Motivated by the above facts, in what follows, we investigate the possibilities of observing lower- and higher-order sub-Poissonian photon statistics, antibunching and squeezing in PADFS and PSDFS. We have studied nonclassical properties of these states through a set of other witnesses of nonclassicality, e.g., zeros of Q function, Mandel Q, parameter, Klyshko’s criterion, and Agarwal-Tara’s criterion. These witnesses of nonclassicality successfully establish that both PADFS and PSDFS (along with most of the states to which these two states reduce at different limits) are highly nonclassical. Thus, making use of the analytical expressions of moments of creation and annihilation operators, discussed below, facilitates an analytical understanding for most of the nonclassical witnesses.
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2.3 Nonclassical features of PADFS and PSDFS

The moments of number operators for PADFS and PSDFS states obtained in the previous section enable us to study nonclassical properties of these states using a set of moments-based criteria of nonclassicality Miranowicz et al. [2010] Naikoo et al. [2018]. In the recent past, an infinite set of these moments-based criteria is shown to be equivalent to the $P$-function-based criterion, i.e., it becomes both necessary and sufficient Richter and Vogel [2002]; Shchukin and Vogel [2005b]. However, in this section, we will use a subset of this infinite set as witnesses of nonclassicality to investigate various nonclassical properties of the PADFS and PSDFS. Specifically, nonclassicality will be witnessed through Mandel $Q_M$ parameter, criteria of lower- and higher-order antibunching, Agarwal-Tara’s criterion, Klyshko’s criterion, criteria of higher-order sub-Poissonian photon statistics, zeros of $Q$ function, etc. As all these criteria are already introduced in Section 2.3 here we may discuss the plots and results.

2.3.1 Mandel $Q_M$ Parameter

Negativity of this parameter indicates nonclassicality which can be calculated using Eqs. (2.1) and (2.2). In Figure 2.2 the dependence of $Q_M$ on the state parameter $\alpha$ and non-Gaussianity inducing parameters (i.e., photon addition, subtraction, and Fock parameters as they can induce non-Gaussianity in a quantum state) is shown. Specifically, variation of $Q_M$ parameter for PADFS and PSDFS is shown with state parameter $\alpha$, where the effect of the number of photons added/subtracted and the initial Fock state is also established. For $\alpha = 0$, the PADFS with an arbitrary number of photon addition has $Q_M$ parameter -1, which can be attributed to the fact that final state, which reduces to the Fock state ($|1\rangle$ chosen to be displaced in this case) is the most nonclassical state (cf. Figure 2.2 (a)). With increase in the number of photons added to the DFS, the depth of nonclassicality witness $Q_M$ increases. However, the witness of nonclassicality becomes less negative for higher values of the displacement parameter. In contrast to the photon addition, with the subtraction of photons from the DFS the $Q_M$ parameter becomes almost zero for the smaller values of displacement parameter $\alpha$ in DFS as shown in Figure 2.2 (c). This behavior can be attributed to the fact that photon subtraction from $D(\alpha)|1\rangle$ for small values of $\alpha$ will most likely yield vacuum state. Also, with the increase in the displacement parameter the witness of nonclassicality becomes more negative as with a higher average number of photons in DFS photon subtraction becomes more effective. However, for the larger values of displacement parameter the nonclassicality disappears analogous to the PADFS. For large values of $\alpha$, this parameter dominates in the behavior of the state and thus it behaves analogous to coherent state.

As Fock states are known to be nonclassical, and photon addition and subtraction are established as nonclassicality inducing operations, it would be worth comparing the effect of these two independent factors responsible for the observed nonclassical features in the present case. To perform this study, we have shown the variation of the single photon added (subtracted) DFS with different initial Fock states in Figure 2.2 (b) (Figure 2.2 (d)). Specifically, the nonclassicality present in PADFS decays faster for the higher values of the Fock states with increasing displacement parameter (cf. Figure 2.2 (b)). However, such nature was not present in PSDFS shown in Figure 2.2 (d). Note that variation of $Q_M$ parameter with $\alpha$ starts from 0 (-1) iff $u \leq n$ ($u > n$). For instance, if $u = n = 1$, i.e., corresponding to state $\hat{a}D(\alpha)|1\rangle$, nonclassicality witness is zero for $\alpha = 0$ as it corresponds to vacuum state. Therefore, the present study reveals that photon addition is a stronger factor for the nonclassicality present in the state when compared to the initial Fock state chosen to be displaced. Whereas photon subtraction is a preferred choice for large values of displacement parameter in contrast to the higher values of Fock states to displace with small $\alpha$. Among photon addition and subtraction, addition is a preferred choice for the smaller values of displacement parameter, while the choice between addition and subtraction becomes immaterial for large $\alpha$. 
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Figure 2.2: Variation of Mandel $Q_M$ parameter for PADFS (in (a) and (b)) and PSDFS (in (c) and (d)) is shown with displacement parameter $\alpha$. In (a) and (c), the value of number of photons added/subtracted (i.e., $u$ or $v$) is changed for the same initial Fock state $|1\rangle$. Different initial Fock states $|n\rangle$ are chosen to be displaced in (b) and (d) for the single photon addition/subtraction. The blue curve corresponds to vacuum for $\alpha = 0$ and thus starts from 0 unlike rest of the states which are Fock state ($n \neq 0$) in the limiting case. Therefore, nonclassicality increases first with increasing $\alpha$ before decreasing as in the rest of the cases.

2.3.2 Lower- and higher-order antibunching

The nonclassicality reflected by the lower-order antibunching criterion obtained here is the same as Mandel $Q_M$ parameter $\left(Q_M = \frac{d(1)}{d^2}\right)$ illustrated in Figure 2.2. Therefore we will rather discuss here the possibility of observing higher-order antibunching in the quantum state of our interest using Eqs. (2.1) and (2.2) in Eq. (1.41). Specifically, the depth of nonclassicality witness can be observed to increase with order for both PADFS and PSDFS as depicted in Figure 2.3 (a) and (d). This fact is consistent with the earlier observations (Thapliyal et al. [2014a,b, 2017b,a]; Alam et al. [2017b] and references therein) that higher-order nonclassicality criteria are useful in detecting weaker nonclassicality. On top of that the higher-order antibunching can be observed for larger values of displacement parameter $\alpha$, when lower-order antibunching is not present. The presence of higher-order nonclassicality in the absence of its lower-order counterpart establishes the relevance of the present study.

The depth of nonclassicality parameter $\left(d(l - 1)\right)$ was observed to decrease with an increase in the number of photons subtracted from DFS for small values of $\alpha$ in Figure 2.2 (c). A similar nature is observed in Figure 2.3 (e), which shows that for the higher values of displacement parameter, the depth of higher-order nonclassicality witness increases with the number of photon subtraction. Therefore, not only the depth of nonclassicality but the range of displacement parameter for the presence of higher-order antibunching also increases with photon addition/subtraction (cf. Figure 2.3 (b) and (e)). With the increase in the value of Fock state parameter $n$, the depth of higher-order nonclassicality witness increases (decreases) for smaller (larger) values of displacement parameter in both PADFS and PSDFS as shown in Figure 2.3 (c) and (f), respectively. Thus, we have observed that the range of $\alpha$ with the presence of nonclassicality increases (decreases) with photon addition/subtraction (Fock state) in DFS.
Figure 2.3: The presence of higher-order antibunching is shown as a function of $\alpha$ for PADFS (in (a)-(c)) and PADFS ((d)-(f)). Specifically, (a) and (d) illustrate comparison between lower- and higher-order antibunching. It should be noted that some of the curves are multiplied by a scaling factor in order to present them in one figure. Figures (b) and (e) show the effect of photon addition/subtraction, and (c) and (f) establish the effect of Fock state chosen to displace in PADFS and PSDFS, respectively. Here, without the loss of generality, we have used the notation $\psi_+^{(u,n,l-1)} (\psi_-^{(u,v,l-1)})$ for nonclassicality in photon added (subtracted) scenarios, and will follow this notation in subsequent figures of this chapter.
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2.3.3 Higher-order sub-Poissonian photon statistics

This allows us to study higher-order sub-Poissonian photon statistics using Eqs. (2.1) and (2.2) in Eq. (1.42). The presence of higher-order sub-Poissonian photon statistics (as can be seen in Figure 2.4 (a) and (d) for PADFS and PSDFS, respectively) is dependent on the order of nonclassicality unlike higher-order antibunching, which is observed for all orders. Specifically, the nonclassical feature was observed only for odd orders, which is consistent with some of the earlier observations [Thapliyal et al. 2017b], where nonclassicality in those cases could be induced due to squeezing. Along the same line, we expect to observe the nonclassicality in such cases with appropriate use of squeezing as a useful quantum resource, which will be discussed elsewhere. In case of photon addition/subtraction in DFS, a behavior analogous to that observed for higher-order antibunching is observed, i.e., the depth of nonclassicality increases with the photon addition while it decreases (increases) for small (large) values of \( \alpha \) (cf. Figure 2.4 (b) and (e)). Similar to the previous case, nonclassicality can be observed to be present for larger values of displacement parameter with photon addition/subtraction, while increase in the value of Fock parameter has an opposite effect.

Figure 2.4: Dependence of higher-order sub-Poissonian photon statistics on \( \alpha \) for PADFS (in (a)-(c)) and PSDFS ((d)-(f)) is illustrated here. Specifically, (a) and (d) show the increase in depth of nonclassicality witness with order, (b) and (e) depict the effect of photon addition and subtraction, respectively, and (c) and (f) establish the effect of choice of Fock state to be displaced in PADFS and PSDFS, respectively.
2.3.4 Higher-order squeezing

The analytical expressions of the nonclassicality witness of the Hong-Mandel type higher-order squeezing criterion for PADFS and PSDFS can be obtained with the help of Eqs. (2.1), (2.2), and (1.43)-(1.45). We have investigated the higher-order squeezing and depict the result in Figure 2.5 assuming $\alpha$ to be real. Incidentally, we could not establish the presence of higher-order squeezing phenomena in PADFS (Figure 2.5 (a)-(c)). However, the depth of the higher-order squeezing witness increases with order for small values of $\alpha$ as shown in Figure 2.5 (a), while for the higher values of the displacement parameter, higher-order squeezing disappear much quicker (cf. Figure 2.5 (d)). With increase in the number of photons subtracted, the presence of this nonclassicality feature can be maintained for the higher values of displacement parameter as well (cf. Figure 2.5 (e)). In general, photon subtraction is a preferred mode for nonclassicality enhancement as far as this nonclassicality feature is concerned. The choice of the initial Fock state is also observed to be relevant as the depth of squeezing parameter can be seen increasing with value of the Fock parameter for PSDFS in the small displacement parameter region (shown in Figure 2.5 (f)), where this nonclassical behavior is also shown to succumb to the higher values of Fock and displacement parameters. Unlike the other nonclassicalities discussed so far, the observed squeezing also depends on phase $\theta$ of the displacement parameter $\alpha = |\alpha| \exp(i\theta)$ due to the second last term in Eq. (1.45). We failed to observe this nonclassicality behavior in PADFS even by controlling the value of the phase parameter (also shown in Figure 2.6 (a)). For PSDFS, the squeezing disappears for some particular values of the phase parameter, while the observed squeezing is maximum for $\theta = n\pi$ (see Figure 2.6 (b)). It thus establishes the phase parameter of the displacement operator as one more controlling factor for nonclassicality in these engineered quantum states.
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2.3.5 Q function

Using Eqs. (1.19) and (1.20) in (1.50), we obtain the analytic expressions for the Husimi $Q$ function for PADFS and PSDFS as

$$Q_+ = \frac{N^2}{\pi} \exp \left[ -|\beta|^2 \right] \sum_{p,p'=0}^{n} \binom{n}{p} \binom{n}{p'} (-\alpha^*)^{(n-p)}(-\alpha)^{(n-p')} \exp \left[ -|\alpha|^2 \right] \times \sum_{m,m'=0}^{\infty} \frac{\alpha^m (\alpha^*)^{m'} g^{(m^d+p^d+u)}(\beta^*)^{(m^d+p^d+u)}}{m!m'!} \tag{2.3}$$

and

$$Q_- = \frac{N^2}{\pi} \exp \left[ -|\beta|^2 \right] \sum_{p,p'=0}^{n} \binom{n}{p} \binom{n}{p'} (-\alpha^*)^{(n-p)}(-\alpha)^{(n-p')} \exp \left[ -|\alpha|^2 \right] \times \sum_{m,m'=0}^{\infty} \frac{\alpha^m (\alpha^*)^{m'} g^{(m^d+p^d-v)}(\beta^*)^{(m^d+p^d-v)}}{m!m'!(m^d+p^d-v)!} , \tag{2.4}$$

respectively. We failed to observe the nonclassical features reflected beyond moments based nonclassicality criteria through a quasiprobability distribution, i.e., zeros of the $Q$ function. We have shown the $Q$ function in Figure 2.7 where the effect of photon addition/subtraction and the value of Fock parameter on the phase space distribution is shown. Specifically, it is observed that the value of Fock parameter affects the quasidistribution function more compared to the photon addition/subtraction.

2.3.6 Agarwal-Tara’s criterion

The analytic expression of $A_3$ parameter defined in Eq. (1.48) can be obtained for PADFS and PSDFS using Eqs. (2.1) and (2.2). The nonclassical properties of the PADFS and PSDFS using Agarwal-Tara’s criterion are investigated, and the corresponding results are depicted in Figure 2.8 which shows highly nonclassical behavior of the states generated by engineering. Specifically, the negative part of the curves, which is bounded by -1, ensures the existence of the nonclassicality. From Figure 2.8 it is clear that $A_3$ is 0 (-1) for the displacement parameter $\alpha = 0$ because then DFS, PADFS, and PSDFS reduce to Fock state, and $A_3 = 0 (-1)$ for the Fock state parameter $n = 0, 1 \ (n > 1)$. Nonclassicality reflected through $A_3$ parameter increases (decreases) with photon addition (subtraction) (shown in Figure 2.8 (a) and (c)). In contrast, the Fock parameter has a completely opposite effect that it leads to decrease (increase) in the observed nonclassicality for PADFS (PSDFS), which can be seen in Figure 2.8 (b) and (d). However, for larger values of displacement parameter, the depth of nonclassicality illustrated through this parameter can again be seen to increase (cf. Figure 2.8 (b)).

![Figure 2.6: Hong-Mandel type higher-order squeezing for PADFS and PSDFS is shown dependent on the phase of the displacement parameter $\alpha = |\alpha| \exp (i\theta)$ in (a) and (b), respectively.](image-url)
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Figure 2.7: Contour plots of the $Q$ function for (a) single photon added displaced Fock $|1\rangle$ state, (b) two photon added displaced Fock $|1\rangle$ state, (c) single photon added displaced Fock $|2\rangle$ state, (d) single photon subtracted displaced Fock $|1\rangle$ state, (e) two photon subtracted subtracted displaced Fock $|1\rangle$ state, (f) single photon subtracted displaced Fock $|2\rangle$ state. In all cases, $\alpha = \sqrt{2}\exp\left(\frac{i\pi}{4}\right)$ is chosen.

Figure 2.8: Variation of Agarwal-Tara’s parameter with $\alpha$ for PADFS and PSDFS is shown in (a)-(b) and (c)-(d), respectively. Specifically, the effect of photon addition/subtraction (in (a) and (c)) and the choice of Fock state ((b) and (d)) on the presence of nonclassicality in PADFS and PSDFS is illustrated.
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2.3.7 Klyshko’s Criterion

The analytic expression for the $m$th photon-number distribution for PADFS and PSDFS can be calculated (using $q = r = 1$) from Eqs. (2.1) and (2.2), respectively.

The advantage of the Klyshko’s criterion over any other existing moments based criteria is that a very small amount of information is required. Specifically, probability of only three successive photon numbers is sufficient to investigate the nonclassical property. The negative values of $B(m)$ serve as the witness of nonclassicality. Klyshko’s criterion in Eq. (1.46) is derived analytically and the corresponding nonclassical properties for both PADFS and PSDFS are investigated (cf. Figure 2.9). Specifically, the negative values of $B(m)$ are observed for different values of $m$ in case of photon addition and subtraction (cf. Figure 2.9 (a) and (c)) being the signature of nonclassicality induced via independent operations. Additionally, one can also visualize that due to the photon addition (subtraction) the negative peaks in the values of $B(m)$ shift to higher (lower) photon number regime.

A similar observation is obtained for different values of the Fock state parameter for the PADFS and PSDFS, where the negative values of the witness of nonclassicality get amplified towards higher photon number regime and becomes more negative, and the corresponding results are shown in Figure 2.9 (b) and (d), respectively. This further establishes the relevance of operations, like photon addition, subtraction, and starting with Fock states in inducing nonclassicality in the engineered quantum states.

2.4 Conclusions

The only Fock state that does not show any nonclassical feature is the vacuum state [Miranowicz et al. 2015], and displacement operator preserves its classicality. All the rest of the Fock states are maximally nonclassical and they are shown to remain nonclassical even after application of a displacement operator. Here, we set ourselves a
task: What happens when the displacement operator applied on a Fock state is followed by addition or subtraction of photon(s)? Independently, photon addition and subtraction are already established as nonclassicality inducing operations in case of displaced vacuum state (i.e., coherent state). In this chapter, we have established that photon addition/subtraction is not only nonclassicality inducing operation, it can also enhance the nonclassicality present in the DFS. It’s expected that these operations would increase the amount of nonclassicality (more precisely, the depth of the nonclassicality witnessing parameter) present in other nonclassical states, too. There is one more advantage of studying the nonclassical features of PADFS and PSDFS. These states can be reduced to a class of quantum states, most of which are already experimentally realized and found useful in several applications. Inspired by the available experimental results, we have also proposed optical designs for generation of PADFS and PSDFS from the squeezed vacuum state.

To analyze the nonclassical features of the engineered final states, i.e., PADFS and PSDFS, we have used a set of moments based criteria, namely Mandel $Q_M$ parameter, Agarwal-Tara’s $A_3$ parameter, criteria for higher-order antibunching, sub-Poissonian photon statistics, and squeezing. In addition, the nonclassical features have been investigated through Klyshko’s criterion and a quasiprobability distribution—$Q$ function. The states of interest are found to show a large variety of nonclassical features as all the nonclassicality witnesses used here (except $Q$ function) are found to detect the nonclassicality. They show that state is useful as squeezed, antibunched, as well as generation of entangled state.

This study has revealed that the amount of nonclassicality in PADFS and PSDFS can be controlled by the Fock state parameter, displacement parameter, the number of photons added or subtracted. In general, the amount of nonclassicality with respect to the witness used here is found to increase with the number of photons added/subtracted, while smaller values of Fock state and displacement parameters are observed to be preferable for the presence of various nonclassical features. On some occasions, nonclassicality has also been observed to increase with the Fock state parameter, while larger values of displacement parameter always affect the nonclassicality adversely. Most of the nonclassicality criteria used here, being moments-based criteria, could not demonstrate the effect of phase parameter of the displacement parameter. Here, higher-order squeezing witness and $Q$ function are found to be dependent on the phase of the displacement parameter. However, only higher-order squeezing criterion was able to detect nonclassicality, and thus established that this phase parameter can also be used to control the amount of nonclassicality.

Further, in the past, it has been established that higher-order nonclassicality criteria have an advantage in detecting weaker nonclassicality. We have also shown that the depth of nonclassicality witness increases with order of nonclassicality thus providing an advantage in the experimental characterization of the observed nonclassical behavior.
Chapter 3

Quantum phase properties of photon added and subtracted displaced Fock states

In this chapter, the motive is to observe the phase properties of PADFS and PSDFS. The main findings of this chapter are published in Malpani et al. [2019b]

3.1 Introduction

In the previous chapter, the nonclassical properties of PADFS and PSDFS were studied. Here, our specific interest is to study the phase properties of PADFS and PSDFS and their limiting cases. In the recent past, the nonclassical properties of this set of engineered quantum states, many of which have been experimentally generated Lvovsky et al. [2001]; Lvovsky and Babichev [2002]; Zavatta et al. [2004, 2005, 2008], were focus of various studies (see Malpani et al. [2019a] and references therein). In Section 1.3.2 we have already expressed PADFS and PSDFS as superposition of Fock states. Further, in Section 1.6 we have described the parameters used for the study of phase properties of a quantum state. In that context we have already mentioned several applications of quantum phase distribution and quantum phase fluctuation.

To stress on the recently reported applications of quantum phase distribution and quantum phase fluctuation, we note that these have applications in quantum random number generation Xu et al. [2012]; Raffaelli et al. [2018], cryptanalysis of squeezed state based continuous variable quantum cryptography Horak [2004], generation of solitons in a Bose-Einstein condensate Denschlag et al. [2000], in phase encoding quantum cryptography Gisin et al. [2002], phase imaging of cells and tissues for biomedical application Park et al. [2018]; as well as have importance in determining the value of transition temperature for superconductors Emery and Kivelson [1995].

Keeping these applications and the general nature of engineered quantum states PADFS and PSDFS in mind, in what follows, we aim to study phase distribution, $Q$ phase, phase fluctuation measures, phase dispersion, and quantum phase estimation using the concerned states and the states obtained in the limiting cases. As PADFS and PSDFS are already described, we may begin this study by describing limiting cases of these states as our states of interest.

We have already mentioned that our focus would be on PADFS and PSDFS. Due to the general form of PADFS and PSDFS, a large number of states can be obtained in the limiting cases. Some of the important limiting cases of PADFS and PSDFS in the present notation are summarized in Table 3.1. This table clearly establishes that the applicability of the results obtained in the present study is not restricted to PADFS and PSDFS; rather an investigation of the phase properties of PADFS and PSDFS would also reveal phase properties of many other
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### Table 3.1: Various states that can be obtained as the limiting cases of the PADFS and PSDFS.

| Reduction of state | Name of the state        | Reduction of state | Name of the state        |
|-------------------|--------------------------|-------------------|--------------------------|
| $|\psi_+ (u, n, \alpha)\rangle$ | $u$-PADFS | $|\psi_- (v, n, \alpha)\rangle$ | $v$-PSDFS |
| $|\psi_+ (0, n, \alpha)\rangle$ | DFS | $|\psi_- (0, n, \alpha)\rangle$ | DFS |
| $|\psi_+ (0, 0, \alpha)\rangle$ | Coherent state | $|\psi_- (0, 0, \alpha)\rangle$ | Coherent state |
| $|\psi_+ (0, 0)\rangle$ | Fock state | $|\psi_- (0, n, 0)\rangle$ | Fock state |
| $|\psi_+ (u, 0, \alpha)\rangle$ | $u$-Photon added coherent state | $|\psi_- (v, 0, \alpha)\rangle$ | $v$-Photon subtracted coherent state |

quantum states of particular interest.

### 3.2 Quantum phase distribution and other phase properties

Quantum phase operator $\hat{\phi}$ was introduced by Dirac based on his assumption that the annihilation operator $\hat{a}$ can be factored out into a Hermitian function $f(\hat{N})$ of the number operator $\hat{N} = \hat{a}^\dagger \hat{a}$ and a unitary operator $\hat{U}_{\text{Dirac}}$ as

$$\hat{a} = \hat{U} f(\hat{N}), \quad (3.1)$$

where

$$\hat{U} = e^{i\hat{\phi}}. \quad (3.2)$$

However, there was a problem with the Dirac formalism of phase operator as it failed to provide a meaning to the corresponding uncertainty relation. Specifically, in the Dirac formalism, the creation ($\hat{a}^\dagger$) and annihilation ($\hat{a}$) operators satisfy the bosonic commutation relation, $[\hat{a}, \hat{a}^\dagger] = 1$, iff $[\hat{N}, \hat{\phi}] = 1$, which leads to the number phase uncertainty relation $\Delta N \Delta \phi \geq 1$. Therefore, in order to satisfy the bosonic commutation relation under Dirac formalism, the phase uncertainty should be greater than $2\pi$ for $\Delta N < \frac{1}{2\pi}$ which lacks a physical description. Subsequently, Louisell [1963] proposed some periodic phase based method, which was followed by Susskind and Glogower formalism based on Sine and Cosine operators [Susskind and Glogower, 1964]. An important contribution to this problem is the Barnett-Pegg formalism [Barnett and Pegg, 1986] which is used in this thesis. In what follows, we will also briefly introduce notions, such as quantum phase distribution, angular $\mathcal{Q}$ phase function, phase fluctuation parameters, phase dispersion, quantum phase estimation to study the phase properties of the quantum states of our interest.

### 3.3 Phase properties of PADFS and PSDFS

The description of the states of our interest given in the previous section can be used to study different phase properties and quantify phase fluctuation in the set of quantum states listed in Table 3.1. Specifically, with the help of the quantum states defined in Eqs. (1.19)-(1.20), we have obtained the analytic expressions of phase distribution and other phase parameters defined in Section 3.2.

#### 3.3.1 Phase distribution function

From the definition of the phase distribution [1.54], it can be observed that for a Fock state, $P_\theta = \frac{1}{2\pi}$, implying it has a uniform distribution of phase. Interestingly, the states of our interest, PADFS and PSDFS, are obtained by displacing the Fock state followed by photon addition/subtraction. Therefore, we will study here what is the effect of application of displacement operator on a uniformly phase distributed (Fock) state and how subsequent
CHAPTER 3. QUANTUM PHASE PROPERTIES OF PHOTON ADDED AND SUBTRACTED DISPLACED FOCK STATES

Photon addition/subtraction further alters the phase distribution. Using phase distribution function, the information regarding uncertainty in phase and phase fluctuation can also be obtained. To begin with, we compute the analytic expressions of \( P_\theta \) for the PADFS and PSDFS, using Eq. (1.54) as

\[
P_\theta (u, n) = \frac{1}{\pi} \frac{|N_u|}{n!} \sum_{p, p' = 0}^{\infty} \binom{n}{p} \binom{n}{p'} \exp \left[ - |\alpha|^2 \right] |\alpha|^{2n-p-p'} \times \sum_{m, m' = 0}^{\infty} \frac{(-|\alpha|)^{m+m'} \sqrt{(m+p+u)(m'+p'+u)}}{m!m'!} \exp[i (\theta - \theta_2) (m' + p' - m - p)],
\]

(3.3)

and

\[
P_\theta (v, n) = \frac{1}{\pi} \frac{|N_v|}{n!} \sum_{p, p' = 0}^{\infty} \binom{n}{p} \binom{n}{p'} \exp \left[ - |\alpha|^2 \right] |\alpha|^{2n-p-p'} \times \sum_{m, m' = 0}^{\infty} \frac{(-|\alpha|)^{m+m'} \sqrt{(m+p-v)(m'+p'-v)}}{m!m'!} \exp[i (\theta - \theta_2) (m' + p' - m - p)],
\]

(3.4)

respectively. Here, \( \theta_2 \) is the phase associated with the displacement parameter \( \alpha (\alpha = |\alpha|e^{i\theta}) \). Since the obtained expressions in Eqs. (3.3) and (3.4) are complex in nature, we depict numerical (graphical) analysis of the obtained results in Figs. [3.1] and [3.2] for PADFS and PSDFS, respectively. Specifically, in Figure [3.1](a), we have shown the variation of phase distribution with phase parameter \( \theta \) for different number of photon added in the displaced single photon Fock state (\( D(\alpha |1\rangle \)) for \( \theta_2 = 0 \). A uniform phase distribution for Fock state (with a constant value of \( \frac{\pi}{2} \)) is found to transform to one that decreases for higher values of phase and possess a dip in the phase distribution for \( \theta = 0 \), which can be thought of as an approach to the Fock state. In fact, in case of classical states, \( P_\theta \) has a peak at zero phase difference \( \theta - \theta_2 \), and therefore, this contrasting behavior can be viewed as signature of quantuness of DFS. However, with the increase in the number of photons added to the DFS, the phase distribution of the PADFS is observed to become narrower. In fact, a similar behavior with increase in the mean photon number of coherent state was observed previously [Agarwal et al., 1992]. It is imperative to state that \( P_\theta \) in case of higher number of photon added to DFS has similar but narrower distribution than that of coherent state. In contrast, with increase in the Fock parameter, the phase distribution is observed to become broader (cf. Figure [3.1](b)). Thus, the increase in the number of photons added and the increase in Fock parameter have opposite effects on the phase distribution. The same is also illustrated through the polar plots in Figure [3.1](c)-(d), which not only reestablish the same fact, but also illustrate the dependence of \( P_\theta \) on the phase of the displacement parameter. Specifically, the obtained phase distribution remains symmetric along the value of phase \( \theta_2 \) (i.e., \( P_\theta \) is observed to have a mirror symmetry along \( \theta = \theta_2 \)) of the displacement parameter. The phase distribution of Fock state is shown by a black circle in the polar plot.

Instead of photon addition, if we subtract photons from the DFS, a similar effect on the phase distribution to that of photon addition is observed. Further, a comparison between photon addition and subtraction on the phase distribution establishes that a single photon subtraction has a prominent impact on phase distribution when compared to that of single photon addition, i.e., the distribution can be observed to be narrower than that of coherent state in most of the cases for \( u = v \). For instance, single photon added (subtracted) DFS is broader (narrower) than corresponding coherent state. Similarly, with the increase in the value of Fock parameter, we can observe more changes on PSDFS than what was observed in PADFS, i.e., the phase distribution broadens more with Fock parameter for PSDFS. Note that \( P_\theta \) has a peak at \( \theta = \theta_2 \) only for photon addition \( u > n \), while in case of photon subtraction it can be observed for \( v \geq n \). With the increase in the amplitude of displacement parameter (\( |\alpha| \)) initially the phase distribution becomes narrower, which is further supported by both addition and subtraction of photons, but it becomes broader again for very high \( |\alpha| \) (figure is not shown here).
Figure 3.1: Variation of phase distribution function with phase parameter for PADFS with displacement parameter $|\alpha| = 1$ for different values of photon addition ((a) and (c)) and Fock parameters ((b) and (d)). The phase distribution is shown using both two-dimensional ((a) and (b) with $\theta_2 = 0$) and polar ((c) and (d)) plots. In (c) and (d), $\theta_2 = \frac{n\pi}{2}$ with integer $n \in [0, 3]$, and the legends are same as in (a) and (b), respectively.
Figure 3.2: Variation of phase distribution function with phase parameter for PSDFS with displacement parameter $|\alpha| = 1$ for different values of photon subtraction ((a) and (c)) and Fock parameters ((b) and (d)). The phase distribution is shown using both two-dimensional ((a) and (b) with $\theta_2 = 0$) and polar ((c) and (d)) plots. In (c) and (d), $\theta_2 = \frac{n\pi}{2}$ with integer $n \in [0, 3]$, and the legends are same as in (a) and (b), respectively.
3.3.2 Angular $Q$ function of PADFS and PSDFS

The relevance of the $Q$ function as a witness of nonclassicality [Thapliyal et al., 2015] and in state tomography [Thapliyal et al., 2016] is well studied. On top of that, non-Gaussianity of the PADFS and PSDFS using $Q$ function was recently reported by us [Malpani et al., 2019a]. We further discuss a phase distribution based on $Q$ function using Eq. (157). In this particular case, we have obtained the angular $Q$ function from the $Q$ functions of the PADFS and PSDFS reported as Eqs. (15)-(16) in [Malpani et al., 2019a]. Specifically, we have shown the effect of photon addition on the DFS ($D(\alpha)|1\rangle$) for a specific value of the displacement parameter in Figure 3.3(a) for angular $Q$ function. One can clearly see that the polar plots show an increase in the peak (located at $\theta_1 = \theta_2$) of the distribution with photon addition. Further, one can compare the behavior of $Q_\theta$ with $P_\theta$ in Figure 3.1 and observe that they behave quite differently (as reported in [Agarwal et al., 1992] for the coherent states), other than increasing with the displacement parameter for small (large) value of $S$.

The parameter $S$ values of displacement parameter $S$ in the quantum phase fluctuation parameters, namely $Q$, is always peaked at the phase of the displacement parameter which also becomes a line of symmetry. Interestingly, the effect of increase in the Fock parameter of PADFS on $Q_\theta$ is similar but less prominent in comparison to photon addition. This is in quite contrast of that observed for $P_\theta$ (in Figs. 3.1 and 3.3(b)). In case of PSDFS, both photon subtraction and Fock parameter have completely different effects on $Q_\theta$ (cf. Figure 3.3(c)-(d)) which is also in contrast to that on corresponding $P_\theta$ (shown in Figure 3.2).

Specifically, with increase in photon subtraction the angular $Q$ function becomes narrower peaked at $\theta = \theta_2$, but for larger number of photon subtraction the peak value decreases quickly. However, with increasing Fock parameter (cf. Figure 3.3(d)), $Q_\theta$ behaves much like photon addition on DFS (shown in Figure 3.3(a)). The observed behavior shows the relevance of studying both these phase distributions due to their independent characteristics.

3.3.3 Quantum phase fluctuation of PADFS and PSDFS

Note that Carruthers and Nieto [Carruthers and Nieto, 1968] had introduced these parameters in terms of Susskind and Glogower operators [Susskind and Glogower, 1964]. Here, we use them in Barnett-Pegg formalism to remain consistent with [Gupta and Pathak, 2007], where $U$ parameter is shown relevant as a witness of nonclassicality [Gupta and Pathak, 2007]. Specifically, $U$ is 0.5 for coherent state, and reduction of $U$ parameter below the value for coherent state can be interpreted as the presence of nonclassical behavior [Gupta and Pathak, 2007]. In what follows, we will study quantum phase fluctuations for PADFS and PSDFS by computing analytic expressions of $U, S$ and $Q$ parameters in Barnett-Pegg formalism, with a specific focus on the possibility of witnessing nonclassical properties of these states via the reduction of $U$ parameter below the coherent state limit. Carruthers and Nieto [Carruthers and Nieto, 1968] introduced three parameters to study quantum phase fluctuation (1.60)-(1.62). It was only in the recent past that Gupta and Pathak provided a physical meaning to one of these parameters by establishing its relation with antibunching and sub-Poissonian photon statistics [Gupta and Pathak, 2007]. Thus, the quantum phase fluctuation studied here using three parameters will also be used to witness the nonclassical nature of the quantum states under consideration. Here, the effect of photon addition/subtraction and displacement parameters on these fluctuation parameters is also studied (shown in Fig. 3.4). Specifically, Figure 3.4(a)-(c) show variation of the three parameters of quantum phase fluctuation for different values of the number of photons added in the displaced Fock state ($D(\alpha)|1\rangle$) with displacement parameter $|\alpha|$. It may be clearly observed that two of the quantum phase fluctuation parameters, namely $U(u,n)$ and $Q(u,n)$ decrease with the value of displacement parameter, while $S(u,n)$ increases with $|\alpha|$. Interestingly, the photon addition and increase in the displacement parameter exhibit the same effect on all three quantum phase fluctuation parameters for PADFS, while for higher values of displacement parameter $S(u,n)$ show completely opposite effect of photon addition. In contrast, $U(v,n)$ for $v$ subtracted photons from $D(\alpha)|1\rangle$ is found to increase (decrease) with photon subtraction while decrease (increase) with the displacement parameter for small (large) value of $|\alpha|$ (cf. Figure 3.4(d)). On the other hand, parameter $S(v,n)$ is also observed to increase (decrease) with $|\alpha|$ and $v$ (cf. Figure 3.4(e)). The third parameter $Q(v,n)$ shows slightly complex behavior for PSDFS with both $|\alpha|$ and $v$ (cf. Figure 3.4(f)) as it behaves...
Figure 3.3: The polar plots for angular $Q$ function for PADFS (in (a) and (b)) and PSDFS (in (c) and (d)) for displacement parameter $|\alpha|=1$ and $\theta_2 = \frac{n\pi}{2}$ with integer $n \in [0, 3]$ for different values of photon addition/subtraction and Fock parameters. In (a) and (c), for $n = 1$, the smooth (blue), dashed (red), dot-dashed (magenta), and dotted (brown) lines correspond to photon addition/subtraction 0, 1, 2, and 3, respectively. In (b) and (d), for the single photon added/subtracted displaced Fock state, the smooth (blue), dashed (red), dot-dashed (magenta), and dotted (brown) lines correspond to Fock parameter 1, 2, 3, and 4, respectively.
analogous to PADFS for each subtracted photon for both small and large values of the displacement parameter (when it increases with \(|\alpha|\)), but for intermediate values the behavior is found to be completely opposite.

As mentioned previously, \( U(i, n) \) \( \forall i \in \{u, v\} \) has a physical significance as a witness of antibunching for values of this parameter less than \( \frac{1}{2} \). Figure 3.4(a) and (d) can be used to perform similar studies for PADFS and PSDFS, respectively. In case of PADFS, we can observe this relevant parameter to become less than \( \frac{1}{2} \), and thus to illustrate the presence of antibunching, only at higher values of the displacement parameter and photon added to the displaced Fock state. In contrast, PSDFS shows the presence of this nonclassical feature in all cases. Thus, occurrence of antibunching in PADFS and PSDFS is established here through this phase fluctuation parameter. Interestingly, a similar dependence of antibunching in PADFS and PSDFS Eq. (1.41) has been recently reported by us [Malpani et al., 2019a] using a different criterion. Further, one can observe from the expression of \( U \) in Eq. (1.60) that it is expected to be independent of the phase of the displacement parameter, which can also be understood from the use of this parameter as a witness for an intensity moments based nonclassical feature. In contrast, \( S \) and \( Q \) in Eqs. (1.61)-(1.62) show dependence on the phase of displacement parameter. Here, we have not discussed the effect of Fock parameter in detail, but in case of photon addition, \( u \) and \( n \) have same (opposite) effects on \( S \) (\( U \) and \( Q \)) parameter(s). Fock parameter has always shown opposite effect of photon subtraction on all three phase fluctuation parameters, and thus nonclassicality revealed by \( U \) can be enhanced with Fock parameter. The relevance of Fock parameter can also be visualized by observing the fact that the single photon subtracted coherent state has \( U = 0.5 \) (which is consistent with the value zero of the antibunching witness reported in Thapliyal et al. [2017b]). Thus, in this case, the origin of the induced antibunching can be attributed to the non-zero value of Fock parameter.

3.3.4 Phase Dispersion

Here, it is worth stressing that both Carruthers-Nieto parameters and phase dispersion \( D \) correspond to phase fluctuation, Our primary focus is to study phase fluctuation and further to check the correlation between these measures of phase fluctuation. Thus, it would be interesting to study phase fluctuation from the two perspectives. We compute a measure of quantum phase fluctuation based on quantum phase distribution, the phase dispersion (1.56), for both PADFS and PSDFS to perform a comparative study between them. Specifically, the maximum value of dispersion is 1 which corresponds to the uniform phase distribution, i.e., \( P_\theta = \frac{1}{2\pi} \). Both PADFS and PSDFS show a uniform distribution for the displacement parameter \( \alpha = 0 \) (cf. Figure 3.5). It is a justified result as both the states reduce to the Fock state in this case. However, with the increase in the value of displacement parameter quantum phase dispersion is found to decrease. This may be attributed to the number-phase complimentarity Banerjee and Srikanth [2010]; Srikanth and Banerjee [2009, 2010], which leads to smaller phase fluctuation with increasing variance in the number operator at higher values of displacement parameter. Thus, with an increase in the average photon number by increasing the displacement parameter, phase dispersion decreases for both PADFS and PSDFS. Addition of photons in DFS leads to decrease in the value of phase dispersion, while subtraction of photons has more complex effect on phase dispersion (cf. Figure 3.5 (a) and (c)). Specifically, for the smaller values of the displacement parameter (\(|\alpha| \leq 1\)), the phase dispersion parameter behaves differently for \( v \leq n \) and \( v > n \). This can be attributed to the sub-Poissonian photon statistics for \( v \leq n \) with \(|\alpha| < 1\) as well as the small value of average photon number (Figure 3.5 (d)). However, at the higher values of the displacement parameter \( D \) for the PSDFS behaves in a manner analogous to the PADFS. Interestingly, increase in the Fock parameter shows similar effect on PADFS and PSDFS in Figure 3.5(b) and (d), respectively.

3.3.5 Phase sensing uncertainty for PADFS and PSDFS

We finally discuss quantum phase estimation using Eq. (1.69), assuming the two mode input state in the Mach-Zehnder interferometer as \( |\psi(j, n, \alpha)\rangle \otimes |0\rangle \). The expressions for the variance of the difference in the photon numbers in the two output modes of the Mach-Zehnder interferometer for input PADFS and PSDFS and the rest of the parameters required to study phase sensing are reported in Appendix.
Figure 3.4: Variation of three phase fluctuation parameters introduced by Carruthers and Nieto with the displacement parameter with $\theta_2 = 0$. The values of photon addition ($u$), subtraction ($v$), and Fock parameter $n = 1$ are given in the legends. Parameter $U(i,n) \forall i \in \{u, v\}$ also illustrates antibunching in the states for values less than $\frac{1}{2}$. 
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Figure 3.5: Variation of phase dispersion for PADFS (in (a) and (b)) and PSDFS (in (c) and (d)) with displacement parameter for an arbitrary $\theta_2$. Dependence on different values of photon added/subtracted and the initial Fock state $|1\rangle$ (in (a) and (c)), while on different values of Fock parameter for single photon added/subtracted state (in (b) and (d)).

The obtained expressions allow us to study the optimum choice of state parameters for quantum phase estimation using PADFS and PSDFS. The variation of these parameters is shown in Figure 3.6. Specifically, we have shown that PSDFS is preferable over coherent state for phase estimation (cf. Figure 3.6 (b)). However, with the increase in the photon subtraction this phase uncertainty parameter is found to increase although remaining less than corresponding coherent state value. In contrast, with photon addition, advantage in phase estimation can be attained as the reduction of the phase uncertainty parameter allows one to perform more precise measurement. This advantage can be enhanced further by choosing large values of photon addition and Fock parameter (cf. Figure 3.6 (a) and (c)). In a similar sense, appropriate choice of Fock parameter would also be advantageous in phase estimation with PSDFS as it decreases the phase uncertainty parameter, but still PADFS remains preferable over PSDFS. This can further be controlled by an increase in $|\alpha|$ which decreases (increases) phase uncertainty parameter for PADFS (PSDFS).

3.4 Conclusions

A set of engineered quantum states can be obtained as the limiting cases from the PADFS and PSDFS, e.g., DFS, coherent state, photon added/subtracted coherent state, and Fock state. Specifically, PADFS/PSDFS are obtained by application of unitary (displacement) and non-unitary (addition and subtraction of photons) operations on Fock state. In view of the fact that the Fock states have uniform phase distribution, the set of unitary and non-unitary quantum state engineering operations are expected to affect the phase properties of the generated state. Therefore, here we have calculated quantum phase distribution, which further helped in quantifying phase fluctuation as phase dispersion. We have also computed the phase distribution as the angular $Q$ function. We have further studied phase fluctuation using three Carruthers and Nieto parameters, and have used one of them to reveal the existence of antibunching in the quantum states of our interest.
CHAPTER 3. QUANTUM PHASE PROPERTIES OF PHOTON ADDED AND SUBTRACTED DISPLACED FOCK STATES

Figure 3.6: Phase sensing uncertainty for (a) PADFS and (b) PSDFS as a function of phase to be estimated $\phi$ for different number of photon addition/subtraction with $n = 1$. The dependence for (c) PADFS and (d) PSDFS is also shown for different values of Fock parameters with $u = 1$ and $v = 1$, respectively. In all cases, we have chosen $\alpha = 0.1$.

Both the phase distribution and angular $Q$ functions are found to be symmetric along the value of the phase of the displacement parameter. The phase distribution is observed to become narrow and peak(s) to increase with the amplitude of the displacement parameter ($|\alpha|$), which further becomes broader for higher values of $|\alpha|$. Further, photon addition/subtraction and Fock parameters are observed to have opposite effects on phase distribution, i.e., distribution function becomes narrower (broaden) with photon addition/subtraction (Fock parameter). Among photon addition and subtraction operations, subtracting a photon alters the phase properties more than that of photon addition. Specifically, at the small values of the displacement parameter ($|\alpha| < 1$), the phase properties of PSDFS for $v \leq n$ and $v > n$ behave differently. This can be attributed to the fact that for $v \leq n$ with $|\alpha| < 1$, the average photon number becomes very small. Further, the peak of the phase distribution remains at the phase of displacement parameter only when the number of photons added/subtracted is more than that of the Fock parameter. However, in case the number of photons subtracted (added) is same as the Fock parameter, the peak of the phase distribution is observed (not observed) at the phase of displacement parameter. The angular $Q$ function can be observed to show similar dependence on various parameters, but the peak of the distribution remains located at the value of phase of the displacement parameter. The three phase fluctuation parameters introduced by Carruthers and Nieto [1968] show phase properties of PADFS and PSDFS, while one of them, $U$ parameter also reveals antibunching in both PADFS and PSDFS. In this case, the role of Fock parameter as antibunching inducing operation in PSDFS is also discussed. Phase dispersion quantifying phase fluctuation remains unity for Fock state reflecting uniform distribution, which can be observed to decrease with increasing displacement parameter. This may be attributed to the number-phase complementarity as the higher values of variance with increasing displacement parameter lead to smaller phase fluctuation. Fock parameter and photon addition/subtraction show opposite effects on the phase dispersion as it increases (decreases) with $n (u/v)$.

Finally, we have also discussed the advantage of the PADFS and PSDFS in quantum phase estimation and obtained the set of optimized parameters in the PADFS/PSDFS. Both photon addition and Fock parameter decrease
the uncertainty in phase estimation, while photon subtraction, though performs better than coherent state is not as advantageous as \( u \) or \( n \). In [Ou (1997)], it was established that signal-to-noise ratio is significant only when the phase shift to measure is of the same order as multiplicative inverse of the average photon number. Therefore, in case of PADFS this limitation of quantum measurement is expected to play an important role. Thus, we have shown here that state engineering tools can be used efficiently to control the phase properties of the designed quantum states for suitable applications. The study performed in this chapter can be extended for other such operations, like squeezing, photon addition followed by subtraction or vice versa.
Chapter 4

Impact of photon addition and subtraction on nonclassical and phase properties of a displaced Fock state

In this chapter, we aim to observe the nonclassical and phase properties of a PASDFS. The work done in this chapter is published in Malpani et al. [2020b].

4.1 Introduction

In Chapter 2, we have already studied nonclassical properties of PADFS and PSDFS. In Chapter 3, we have investigated phase properties of the same set state. In both chapters, we have obtained various exciting observations such as photon addition and subtraction enhance nonclassical properties of non-Gaussian DFS. Motivated by these, here we aim to study both nonclassical and phase properties for a more general quantum state. To be specific, in this chapter, we aim to study the nonclassical (both lower- and higher-order) and phase properties of a PASDFS. The reason behind selecting this particular state lies in the fact that this is a general state in the sense that in the limiting cases, this state reduces to different quantum states having known applications in continuous variable quantum cryptography (this point will be further elaborated in the next section).

As it appears from the above discussion, this investigation has two facets. Firstly, we wish to study nonclassical features of PASDFS, namely Klyshko’s Klyshko [1996], Agarwal-Tara’s Agarwal and Tara [1992], Vogel’s Shchukin and Vogel [2005b] criteria, lower- and higher-order antibunching Pathak and Garcia [2006], squeezing Hillery [1987a]; Hong and Mandel [1985a,b], and sub-Poissonian photon statistics (HOSPS) Zou and Mandel [1990]. We subsequently study the phase properties of PASDFS by computing phase distribution function Agarwal and Singh [1996]; Beck et al. [1993], phase fluctuation parameters Carruthers and Nieto [1968]; Barnett and Pegg [1986], and phase dispersion Perinová et al. [1998]. A detailed analysis of the obtained results will also be performed to reveal the usefulness of the obtained results.

4.2 Moments of the field operators for the quantum states of our interest

As mentioned in the previous section, this work is focused on PASDFS. A PASDFS as a Fock superposition state has already been expressed in Eq.(1.23). To study nonclassical and phase properties of this state, we have used nonclassicality witnesses introduced in Section 2.3 and phase parameters in Section 1.6; we would require analytic expression for moments of the field operators. A bit of computation yields the expression for higher-order moment
of annihilation and creation operator as

\[
\langle \hat{a}^\dagger \hat{a} \rangle = \langle \psi(k,q,n,\alpha) | \hat{a}^\dagger \hat{a} | \psi(k,q,n,\alpha) \rangle = \frac{N^2}{n!} \sum_{p,p'=0}^{n} \left(\frac{n}{p}\right) \left(\frac{n}{p'}\right)(-\alpha^n)(-\alpha^{n-p'})(-\alpha^{n-p}) \times \exp\left[ -|\alpha|^2 \right] \sum_{m=0}^{\infty} \frac{\alpha^m (\alpha^*)^{m+p-p'+j+t}(m+p+k)!}{m!(m+p-p'-j+t)!}(m+p+k-j+t)!.
\]

(4.1)

For different values of \(t\) and \(j\), moments of any order can be obtained, and the same may be used to investigate the nonclassical properties of PASDFS and its limiting cases by using various moments-based criteria of nonclassicality. The same will be performed in the following section, but before proceeding, it would be apt to briefly state our motivation behind the selection of this particular state for the present study (or why do we find this state as interesting?).

Due to the difficulty in realizing single photon on demand sources, the unconditional security promised by various QKD schemes, like BB84 [Bennett and Brassard 1984] and B92 [Bennett et al. 1992], does not remain unconditional in the practical situations. This is where continuous variable QKD (CVQKD) becomes relevant as they do not require single photon sources. Special cases of PASDFS has already been found useful in the realization of CVQKD. For example, protocols for CVQKD have been proposed using photon added coherent state \((k = 1, q = 0, n = 0)\) [Pinheiro and Ramos 2013; Wang et al. 2014], photon added then subtracted coherent states \((k = 1, q = 1, n = 0)\) [Borelli et al. 2016; Srikara et al. 2019], and coherent state \((k = 0, q = 0, n = 0)\) [Grosshans and Grangier 2002; Hirano et al. 2017; Huang et al. 2016; Ma et al. 2018]. Further, boson sampling with displaced single photon Fock states and single photon added coherent state [Seshadreesan et al. 2015] has been reported, and an \(m\) photon added coherent state \((k = m, q = 0, n = 0)\) has been used for quantum teleportation [Pinheiro and Ramos 2013]. Apart from these schemes of CVQKD, which can be realized by using PASDFS or its limiting cases, the fact that the photon addition and/or subtraction operation from a classical or nonclassical state can be performed experimentally using the existing technology [Parigi et al. 2007; Zavatta et al. 2004] has enhanced the importance of PASDFS.

### 4.3 Nonclassicality witnesses and the nonclassical features of PASDFS witnessed through those criteria

The negative value of the Glauber-Sudarshan \(P\)-function characterizes nonclassicality of an arbitrary state [Glauber 1963a; Sudarshan 1963]. As \(P\)-function is not directly measurable in experiments, many witnesses of nonclassicality have been proposed, such as, negative values of Wigner function [Wigner 1932; Kenfack and Zyczkowski 2004], zeroes of \(Q\) function [Husimi 1940; Lütkenhaus and Barnett 1995], several moments-based criteria [Miranowicz et al. 2010; Naikoo et al. 2018]. An infinite set of such moments-based criteria of nonclassicality is equivalent to \(P\)-function in terms of necessary and sufficient conditions to detect nonclassicality [Richter and Vogel 2002]. Here, we would discuss some of these moments-based criteria of nonclassicality and \(Q\) function (in Section 4.5) to study nonclassical properties of the state of our interest.

#### 4.3.1 Lower- and higher-order antibunching

The relevance of photon addition, photon subtraction, Fock, and displacement parameters in the nonclassical properties of the class of PASDFSs is studied here rigorously. Specifically, using Eq. (4.1) with the criterion of antibunching \([1.41]\) we can study the possibilities of observing lower- and higher-order antibunching in the quantum states of PASDFS class, where the class of PASDFSs refers to all the states that can be reduced from state \([1.23]\) in the limiting cases. The outcome of such a study is illustrated in Figure 4.1. It is observed that the depth
CHAPTER 4. IMPACT OF PHOTON ADDITION AND SUBTRACTION ON NONCLASSICAL AND PHASE PROPERTIES OF A DISPLACED FOCK STATE

(a) (b)

Figure 4.1: For PASDFS the lower-and higher-order antibunching is given as a function of displacement parameter $\alpha$. (a) Lower-order antibunching for different values of parameters of the state. (b) Higher-order antibunching for particular state. HOSPS for PASDFS for different values of (c) state parameters and (d) order of nonclassicality.

of lower- and higher-order nonclassicality witnesses can be increased by increasing the value of the displacement parameter, but large values of $\alpha$ deteriorate the observed nonclassicality (cf. Figure 4.1 (a)-(b)). The nonclassicality for higher-values of displacement parameter $\alpha$ can be induced by subtracting photons at the cost of reduction in the depth of nonclassicality witnessed for smaller $\alpha$, as shown in Figure 4.1 (a). However, photon addition is always more advantageous than subtraction. Therefore, both addition and subtraction of photons illustrate these collective effects by showing nonclassicality for even higher values of $\alpha$ at the cost of that observed for the small values of displacement parameter. Fock parameter has completely opposite effect of photon subtraction as it shows the advantage (disadvantage) for small (large) values of displacement parameter. Figure 4.1 (b) shows benefit of studying higher-order nonclassicality as depth of corresponding witness of nonclassicality can be observed to increase with the order. The higher-order nonclassicality criterion is also able to detect nonclassicality for certain values of displacement parameter for which the corresponding lower-order criterion failed to do so.

4.3.2 Higher-order sub-Poissonian photon statistics

Variation of HOSPS nonclassicality witness for class of PASDFSs obtained by different nonclassicality inducing operations show the same effect as that of antibunching witness for all the odd orders of HOSPS, and as depicted in Figure 4.1 (c). However, this nonclassical feature disappears for even orders of HOSPS (cf. Figure 4.1 (d)). In case of the odd orders of HOSPS, though the depth of nonclassicality witness increases with the order, higher-order criterion is found to fail to detect nonclassicality for certain values of $\alpha$ when corresponding HOSPS criterion for smaller values of orders shown the nonclassicality.

4.3.3 Lower- and higher-order squeezing

Out of all the nonclassicality inducing operations used in PASDFS only photon subtraction is squeezing inducing operation as shown in Figure 4.2, which is consistent with some of our recent observations Malpani et al. [2019a].
CHAPTER 4. IMPACT OF PHOTON ADDITION AND SUBTRACTION ON NONCLASSICAL AND PHASE PROPERTIES OF A DISPLACED FOCK STATE

4.3.4 Klyshko’s Criterion

For PASDFS \( p_z \) can be obtained from Eq. (4.1). Nonclassicality reflected through Klyshko’s criterion can be controlled by all the state engineering operations used here as shown in Figure 4.3. The depth of this nonclassicality witness increases at higher values of photon numbers \( z \) due to increase in photon addition and/or Fock parameter. In contrast, depth of witness increases at smaller photon numbers \( z \) due to photon subtraction. The Klyshko’s nonclassicality witness is positive for some photon numbers only if \( k + n > q \). Additionally, with increase in displacement parameter the depth of nonclassicality witness decreases, and the weight of the distribution of witness shift to higher values of \( z \).

4.3.5 Agarwal-Tara’s criterion

This nonclassicality witness is able to detect nonclassicality in all the quantum states in the class of PASDFSs (cf. Figure 4.4(a)). Note that for \( |\psi(1,2,1,\alpha)\rangle \) with small \( \alpha \), \( A_3 \) parameter is close to zero, which is due to very high probability for zero photon states.
Figure 4.3: Illustration of Klyshko’s parameter $B(z)$ with respect to the photon number $z$ for different values of state parameters with (a) $\alpha = 0.5$ and (b) $\alpha = 1$. 
Figure 4.4: Nonclassicality reflected through the negative values of (a) Agarwal-Tara’s and (b) Vogel’s criteria as a function of \( \alpha \) or different state parameters.
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Figure 4.5: Polar plot of phase distribution function for PASDFS $|\psi(k, q, n, \alpha)\rangle$ with respect to variation in displacement parameter for (a) $n = 1$, $k = 2$ and $q = 1, 2, 3$ represented by the smooth (cyan), dashed (magenta), and dot-dashed (purple) lines, respectively; (b) $n = 2$, $q = 2$ and $k = 1, 2$, and 3 illustrated by the smooth (cyan), dashed (magenta), and dot-dashed (purple) lines, respectively; and (c) $n = 1$ with $k = q = 1, 2$, and 3 shown by the smooth (cyan), dashed (magenta), and dot-dashed (purple) lines, respectively.

4.3.6 Vogel’s criterion

The negative value of the determinant $dv$ of matrix $v$ in Eq. (1.47) is signature of nonclassicality. Fock parameter has adverse effect on the nonclassicality in PASDFS detected by this criterion. This adverse effect can be compensated by photon subtraction and can be further controlled by photon addition (as shown in Figure 4.4 (b)). Notice that the nonclassical behavior illustrated by Agarwal-Tara’s (Vogel’s) criterion is related to higher-order antibunching (squeezing) criterion. However, nonclassicality witness of Vogel’s criterion is a phase independent property unlike squeezing.

4.4 Phase properties of PASDFS

The nonclassicality inducing operations are also expected to impact the phase properties of a quantum state [Banerjee and Srikanth 2007]. Recently, we have reported an extensive study on the role that such quantum state engineering tools can play in application oriented studies on quantum phase [Malpani et al. 2019b]. Specifically, relevance in quantum phase estimation, phase fluctuation, and phase distribution were discussed which can play an important role in quantum metrology [Giovannetti et al. 2011]. Here, we briefly discuss some of the phase properties of the class of PASDFSs.

4.4.1 Phase distribution function

The analytical expression for phase distribution function for PASDFS can be computed as

$$P(\theta) = \frac{1}{2\pi} \frac{N^2}{n!} \sum_{p, p'=0}^{n} \binom{n}{p} \binom{n}{p'} (-\alpha^*)^{n-p} (-\alpha)^{n-p'} \exp \left[ -|\alpha|^2 \right] \times \sum_{m, m'=0}^{\infty} \frac{\alpha^m(\alpha)^m'}{m! m!' \sqrt{(m+p+k)(m'+p'+k-q)}} \exp \left[ i \theta (m' + p' - m - p) \right].$$

(4.2)

Photon subtraction can be observed to be a more effective tool to alter phase properties of PASDFS than photon addition, as shown in Figure 4.5. Interestingly, photon addition shows similar behavior, though less prominent, as photon subtraction, Fock parameter has opposite effect.
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4.4.2 Phase Fluctuation

Here, we focus only on the first phase fluctuation parameter $U$, which is related to antibunching if $U$ is below its value for coherent state (i.e., 0.5), remaining consistent with Barnett-Pegg formalism Gupta and Pathak (2007); Pathak and Mandal (2000). One can observe that the phase fluctuation parameter is able to detect nonclassicality (specifically antibunching) only in three cases where the role of the photon subtraction is relevant (cf. Figure 4.6). The observation can be seen analogous to that observed for Vogel’s nonclassicality criterion.

4.5 Quasidistribution function: $Q$ function

Here, we will establish non-Gaussianity inducing behavior of photon addition and Fock parameter (cf. Figure 4.7), which are so far illustrated as nonclassicality inducing and phase altering operations. Clearly, with photon addition tendency of quasidistribution away from Gaussian behavior is visible, while with photon subtraction squeezing along particular phase angle chosen by displacement parameter can be observed. This squeezing can be noticed to be more appreciable for higher values of displacement parameter (cf. Figure 4.7 (c)-(d)). From Figure 4.7 (e)-(f), it can be observed that Fock parameter and photon addition have a similar effect in the phase space. As zeros of $Q$ function are signature of nonclassicality, PASDFS shows nonclassicality in Figure 4.7 (b), (e), and (f). This establishes that use of more than one state engineering tool may be helpful in generation of nonclassical states. It would be interesting to verify whether one more tools (say squeezing or photon catalysis) may further enhance the nonclassical properties.

4.6 Conclusions

In this chapter, we have investigated the nonclassical behavior of PASDFS using different witnesses of lower- and higher-order nonclassicality. The significance of this choice of state underlies the fact that a class of engineered quantum states can be achieved as the reduced case of PASDFS $|\psi(k,q,n,\alpha)\rangle$, like photon added DFS ($q=0$), photon subtracted DFS ($k=0$), DFS ($q=k=0$), photon added coherent state ($n=q=0$), photon subtracted coherent state ($n=k=0$), coherent state ($n=k=q=0$), and Fock state ($n=k=q=\alpha=0$). Some of the reduced states have been experimentally realized and in some cases optical schemes for generation have been proposed, so this family of states is apt for various challenging tasks to establish quantum dominance. The state under consideration requires various non-Gaussianity inducing quantum engineering operations and thus our focus here was to analyze the relevance of each operation independently in the nonclassical features (listed in Table 4.1) observed in PASDFS. To study the nonclassical properties of PASDFS, a set of moments-based criteria for Klyshko’s, Agrwal-Tara’s, and Vogel’s criteria, as well as lower- and higher-order antibunching, HOSPS, and
Figure 4.7: $Q$ function for PASDFS $|\psi(k, q, n, \alpha)\rangle$ with (a) $k = q = n = 1$, (b) $k = 2, q = n = 1$, and (c) $q = 2, k = n = 1$ with $\alpha = \frac{1}{\sqrt{2}} \exp(i\pi/4)$. (d) Similarly, $Q$ function of PASDFS with $q = 2, k = n = 1$ and $\alpha = \sqrt{2}\exp(i\pi/4)$. $Q$ function for $|\psi(k, q, n, \alpha)\rangle$ with (e) $k = q = 1, n = 2$ and (f) $q = 1, k = n = 2$ for $\alpha = \frac{1}{\sqrt{2}} \exp(i\pi/4)$.

The present study reveals that with an increase in the order of nonclassicality the depth of nonclassicality witnesses increase. Additionally, higher-order nonclassicality criteria were able to detect nonclassicality in the cases when corresponding lower-order criteria failed to do so. Different nonclassical features are observed for smaller values of displacement parameter, which can be sustained for higher values by increasing the number of subtracted photon. Photon addition generally improves nonclassicality, and this advantage can be further enhanced for the higher (smaller) values of displacement parameter using photon subtraction (Fock parameter). The HOSPS nonclassical feature is only observed for the odd orders. As far as squeezing is concerned, only photon subtraction could induce this nonclassicality. Large number of photon addition can be used to observe squeezing at higher values of displacement parameter at the cost of that present for smaller $\alpha$. Photon subtraction alters the phase prop-

| S. No. | Nonclassical Properties                      | Observed in PASDFS |
|--------|---------------------------------------------|---------------------|
| 1      | Lower-order and higher-order Antibunching   | yes                 |
| 2      | Higher-order sub Poissionian photon statistics | yes                 |
| 3      | Lower-order and higher-order squeezing      | yes                 |
| 4      | Klyshko’s criterion                         | yes                 |
| 5      | Agarwal-Tara’s criterion                    | yes                 |
| 6      | Vogel’s criterion                           | yes                 |
| 7      | Phase distribution function                  | -                   |
| 8      | Phase fluctuation                           | yes                 |
| 9      | $Q$ function                                | yes                 |

Table 4.1: Summary of the nonclassical properties of PASDFS.
properties more than photon addition, while Fock parameter has an opposite effect of the photon addition/subtraction. The nonclassicality revealed through phase fluctuation parameter shows similar behavior as Vogel’s criterion. Finally, we have shown the nonclassicality and non-Gaussianity of PASDFS with the help of a quasidistribution function, namely $Q$ function.
Chapter 5

Manipulating nonclassicality via quantum state engineering processes: Vacuum filtration and single photon addition

In this chapter, the objective is to study nonclassical properties associated with two different quantum state engineering processes with a specific focus on nonclassicality witnesses and measures. The work done in this chapter is published in Malpani et al. [2020a].

5.1 Introduction

So far, we have discussed nonclassical properties of engineered quantum states in detail. Here, we wish to extend the discussion and investigate the possibilities of manipulating or controlling the nonclassicality present in the system by using two specific processes of quantum state engineering. Precisely, by using vacuum filtration and single photon addition processes. To introduce the idea of these quantum state engineering operations, we can write the photon number distribution of an arbitrary quantum state in terms of Glauber-Sudarshan $P(\alpha)$ function as

$$p_n = \int |P(\alpha)|^2 |\langle n | \alpha \rangle|^2 d^2 \alpha.$$ (5.1)

If $p_n$ vanishes for a particular value of Fock state parameter $n$, we refer to that as a “hole” or a hole in the photon number distribution at position $n$ [Escher et al. 2004]. Notice that $p_n = 0$ reveals that $P(\alpha) < 0$ for some $\alpha$, which is the signature of nonclassicality. Thus, the existence of a hole in the photon number distribution implies that the corresponding state is nonclassical, and corresponding technique of quantum state engineering to create hole is called hole burning [Gerry and Benmoussa 2002]. Interestingly, this result also implies that qudits which are $d$-dimensional (finite dimensional) quantum states are always nonclassical as we can see that in such a state $p_d = p_{d+1} = \ldots = 0$. In principle, the hole can be created for an arbitrary $n$, but here for the sake of a comparative study, we restrict ourselves to the situation where the hole is created at $n = 0$, i.e., the desired engineered state has zero probability of getting vacuum state on measurement in Fock basis (in other words, $p_0 = 0$). In fact, Lee [Lee 1995] had shown that a state with $p_n = 0$ is a maximally nonclassical as long as the nonclassicality is quantitatively measured using nonclassical depth. Such a state can be constructed in various ways. To elaborate on this, we describe an arbitrary pure quantum state as a superposition of the Fock states

$$|\psi\rangle = \sum_{n=0}^{\infty} c_n |n\rangle.$$ (5.2)
where \( c_n \) is the probability amplitude of state \( |n\rangle \). A hole can be created at \( n = 0 \) by adding a single photon to obtain
\[
|\psi_1\rangle = N_1 a^\dagger |\psi\rangle,
\]
(5.3)
where \( N_1 = (\langle \psi|aa^\dagger|\psi\rangle)^{-1/2} \) is the normalization constant. If we consider the initial quantum state \( |\psi\rangle \) as a coherent state, the addition of a single photon would lead to a photon added coherent state which has been experimentally realized \cite{Zavatta2004} and extensively studied \cite{Hong1999} because of its interesting nonclassical properties and potential applications. Thus, in quantum state engineering, techniques for photon addition are known \cite{Thapliyal2017b, Malpani2019a} and experimentally realized.

An alternative technique to create a hole at vacuum is vacuum filtration. The detailed procedure of this technique is recently discussed in \cite{Meher2018}. Vacuum filtration implies removal of the coefficient of the vacuum state, \( c_0 \), in Eq. (5.2) and subsequent normalization. Clearly this procedure would yield
\[
|\psi_2\rangle = N_2 \sum_{n=1}^{\infty} c_n'|n\rangle,
\]
(5.4)
where the normalization constant \( N_2 = \left( 1 - |c_0|^2 \right)^{-1/2} \). Both these states (i.e., \( |\psi_1\rangle \), and \( |\psi_2\rangle \)) are maximally nonclassical as far as Lee’s result related to nonclassical depth is concerned \cite{Lee1991}. However, recently lower-order nonclassical properties of \( |\psi_1\rangle \) and \( |\psi_2\rangle \) in (5.3)-(5.4) are reported to be different for \( |\psi\rangle \) chosen as coherent state \cite{Meher2018}. This led to several interesting questions, like- What happens if the initial state on which addition of photon or vacuum filtration process is to be applied is already nonclassical (specifically, pure state other than coherent state \cite{Hillery1985})? How do these processes affect the higher-order nonclassical properties of the quantum states? How does the depth of nonclassicality corresponding to a particular witness of nonclassicality changes with the parameters of the quantum state for these processes? The present chapter aims to answer these questions through a comparative study using a set of interesting quantum states \( |\psi\rangle \) (and the corresponding single photon added \( |\psi_1\rangle \) and vacuum filtered \( |\psi_2\rangle \) states), each of which can be reduced to many more states. Specifically, in what follows, we would study the lower- and higher-order nonclassical properties of single photon addition and vacuum filtration of even coherent state (ECS), binomial state (BS) and Kerr state (KS). In fact, the quantum state engineering processes described mathematically in Eqs. (5.3)-(5.4) can be used to prepare a set of engineered quantum states, namely vacuum filtered ECS (VFECS), vacuum filtered BS (VFBS), vacuum filtered KS (VKFS), photon added ECS (PAECS), photon added BS (PABS) and photon added (PAKS).

We aim to look at the nonclassical properties of these states with a focus on higher-order nonclassical properties and subsequently quantify the amount of nonclassicality in all these states. In what follows, the higher-order nonclassical properties are illustrated through the criteria of HOA, HOS and HOSPS with brief discussion of lower-order antibunching and squeezing.

### 5.2 Quantum states of interest

In this chapter, we have selected a set of three widely studied and important quantum states- (i) ECS, (ii) BS and (iii) KS. We subsequently noted that these states can further be engineered to generate corresponding vacuum filtered states and single photon added states. For example, one can generate VFBS and PABS from BS by using vacuum filtration \cite{Meher2018} and photon addition \cite{Zavatta2004} processes, respectively. In a similar manner, these processes can also generate VFECS and PAECS from ECS, and VKFS and PAKS from KS. In this section, we briefly describe ECS, BS, KS, VFBS, PABS, VFECs, PAECS, VFKS and PAKS. Specifically, we describe three parent states as Fock superposition states. Similarly, the six engineered states are also expressed as Fock superposition states for the convenience of identifying the corresponding photon number distributions (each of which essentially contains a hole at the vacuum). In the rest of the study, we wish to compare the impact of these two quantum state engineering processes (i.e., vacuum filtration and photon addition processes) on the
nonclassical properties of the engineered states. In the above, we have described six (three) quantum states of our interest as Fock superposition states having (without) holes at vacuum. In what follows, these expressions will be used to study the nonclassical properties of these states using a set of witnesses of nonclassicality. Specifically, we will use a set of witnesses of nonclassicality which are based on moments of annihilation and creation operators. Keeping this in mind, in the following subsection, we report the general form of such moments for all the six engineered states of our interest and the corresponding three parent states (thus overall nine states).

5.2.1 Expressions for moments of annihilation and creation operators

In 1992, Agarwal and Tara [Agarwal and Tara 1992] introduced a criterion of nonclassicality in the form of a matrix of moments of creation and annihilation operators. This criterion was further modified to propose a moment-based criteria of entanglement Shchukin and Vogel [2005a] and nonclassicality Miranowicz et al. [2010, 2009]. Therefore, it is convenient to find out the expectation value of the most general term describing higher-order moment \( \langle \hat{a}^\dagger \hat{a}^k \rangle \) for a given state to investigate the nonclassicality using the set of moment-based criteria.

5.2.1.1 Expectation values for even coherent states and the corresponding engineered states

The analytic expression of \( \langle \hat{a}^\dagger \hat{a}^k \rangle \) is obtained for the quantum states \( i \in \{ \text{ECS}, \text{VFECS}, \text{PAECS} \} \) using Eqs. (1.27) and (1.28). For ECS and VFECS, expressions of the moments can be written in a compact form as

\[
\langle \hat{a}^\dagger \hat{a}^k \rangle_{\text{ECS}} = \frac{\exp[-(\alpha^2)]}{2(1+\exp[-2(\alpha^2)])} \sum_{n=0}^{\infty} \frac{\alpha^n (\alpha^*)^{n-k+j}}{(n-k)!} \mathcal{G}_{n,j,k}, \tag{5.5}
\]

and

\[
\langle \hat{a}^\dagger \hat{a}^k \rangle_{\text{VFECS}} = \begin{cases} 
N_{\text{VFECS}}^2 \sum_{n=1}^{\infty} \frac{\alpha^n (\alpha^*)^{n-k+j}}{(n-k)!} \mathcal{G}_{n,j,k} & \text{for } k \leq j, \\
N_{\text{VFECS}}^2 \sum_{n=1}^{\infty} \frac{\alpha^n \alpha^{n-k-j}}{(n-j)!} \mathcal{G}_{n,j,k} & \text{for } k > j,
\end{cases} \tag{5.6}
\]

respectively. Similarly, we obtained analytic expression for \( \langle \hat{a}^\dagger \hat{a}^k \rangle_{\text{PAECS}} \) for PAECS as

\[
\langle \hat{a}^\dagger \hat{a}^k \rangle_{\text{PAECS}} = N_{\text{PAECS}}^2 \sum_{n=0}^{\infty} \frac{\alpha^n (\alpha^*)^{n-k+j}(n+1)(n-k+j+1)}{(n+1-k)!} \mathcal{G}_{n,j,k}. \tag{5.7}
\]

Here, \( \mathcal{G}_{n,j,k} = (1 + (-1)^n) \left( 1 + (-1)^{n-k+j} \right) \). The above mentioned quantities are also functions of displacement parameter of ECS used to generate the engineered states, which will be used as a control parameter while discussion of nonclassicality induced due to engineering operations.

5.2.1.2 Expectation values for binomial state and the corresponding engineered states

Similarly, the compact analytic form of \( \langle \hat{a}^\dagger \hat{a}^r \rangle \) can be written as

\[
\langle \hat{a}^\dagger \hat{a}^r \rangle_{\text{BS}} = \sum_{n=0}^{M} \mathcal{A}_{p,M,n,r} \frac{M!}{(n-r)!}. \tag{5.8}
\]

In case of VFBS and PABS, the analytic form of \( \langle \hat{a}^\dagger \hat{a}^r \rangle_i \) is obtained as

\[
\langle \hat{a}^\dagger \hat{a}^r \rangle_{\text{VFBS}} = \begin{cases} 
N_{\text{VFBS}}^2 \sum_{n=1}^{M} \mathcal{A}_{p,M,n,r} \frac{M!}{(n-r)!} & \text{for } r \leq t, \\
N_{\text{VFBS}}^2 \sum_{n=1}^{M} \mathcal{A}_{p,M,n,r} \frac{M!}{(n-r)!} & \text{for } r > t,
\end{cases} \tag{5.9}
\]

and

\[
\langle \hat{a}^\dagger \hat{a}^r \rangle_{\text{PABS}} = N_{\text{PABS}}^2 \sum_{n=0}^{M} \mathcal{A}_{p,M,n,r} \frac{M![(n+1)^{r}(n+1-r+t)!]}{M!(n+1-r)(n-r+t)!}, \tag{5.10}
\]
respectively, with $I_{p,M,n,r,t} = \left[ \frac{\rho_{n-r,t}(t)p_{M-n+r,t}(M-n+r,t)}{\rho_{M-n+r,t}(M-n+r,t)} \right]^{1/2}$. Here, the obtained average values of moments are also dependent on BS parameters, which will be used to enhance/control the nonclassicality features in the generated states.

5.2.1.3 Expectation values for Kerr state and the corresponding engineered states

For Kerr state, Vacuum filtered Kerr state and Photon added Kerr state, we use the same approach to obtain a compact generalized forms of $\langle \hat{a}^n \hat{a}^\dagger \rangle_i$; and our computation yielded

$$\langle \hat{a}^n \hat{a}^\dagger \rangle_{KS} = \exp \left[ -|\alpha|^2 \right] \sum_{n=0}^{\infty} \frac{a^n(a^*)^{n-s+q}}{(n-q)!} F_{n,s,q}, \quad (5.11)$$

$$\langle \hat{a}^n \hat{a}^\dagger \rangle_{VFKS} = \left\{ \begin{array}{ll} N^2_{VFKS} \sum_{n=1}^{\infty} \frac{a^n(a^*)^{n-s+q}}{(n-q)!} F_{n,s,q}, & \text{for } s \leq q, \\ N^2_{VFKS} \sum_{n=1}^{\infty} \frac{a^s(a^*)^{n-s-q}}{(n-q)!} F_{n,s,q}, & \text{for } s > q, \end{array} \right. \quad (5.12)$$

and

$$\langle \hat{a}^n \hat{a}^\dagger \rangle_{PAKS} = N^2_{PAKS} \sum_{n=0}^{\infty} \frac{a^n(a^*)^{n+q(n+1)!}|n-s+q+1)!}{n!(n-s+q)!|n+1-s)!} F_{n,s,q}. \quad (5.13)$$

Here, $F_{n,s,q} = \exp \left[ \chi (q-s) \left( 2n+q-s-1 \right) \right]$. From the above expressions, it is clear that when $q = s$, there is no role of $\chi$ and the behavior of KS is similar to that of a coherent state. So the effect of this parameter $\langle \chi \rangle$ can be observed only in HOS which also depends on the higher-order moments other than moments of number operator, i.e., $\langle \hat{a}^n \hat{a}^\dagger \rangle_i : q \neq s$. In what follows, we use the expressions of moments given in Eqs. (5.6)-(5.13) to study various lower- and higher-order nonclassicality witnesses.

5.3 Nonclassicality witnesses

There are various criteria of nonclassicality, most of them are sufficient but not necessary in the sense that satisfaction of such a criterion can identify a nonclassical feature, but failure does not ensure that the state is classical. Further, most of the criteria (specially, all the criteria studied here) do not provide any quantitative measure of nonclassicality present in a state, and so they are referred to as witnesses of nonclassicality. These witnesses are based on either quasiprobability distribution or moments of annihilation and creation operators. In the present work, we have used a set of moment-based criteria to investigate nonclassical properties of our desired engineered quantum states. Specifically, we have investigated the possibilities of observing lower-order squeezing and antibunching as well as HOA, HOSPS, and HOS for all the states of our interest. To begin the investigation and the comparison process, let us start with the study of antibunching.

5.3.1 Lower- and higher-order antibunching

The phenomenon of lower-order antibunching is closely associated with the lower-order sub-Poissonian photon statistics [Brown and Twiss 1956]. However, they are not equivalent [Zou and Mandel 1990]. The concept of HOA also plays an important role in identifying the presence of weaker nonclassicality [Allevi et al. 2012b; Hamar et al. 2014]. It was first introduced in 1990 based on majorization technique [Lee 1990] followed by some of its modifications [An 2002; Pathak and Garcia 2006]. In this section, we study the generalized HOA criterion introduced by Pathak and Garcia [Pathak and Garcia 2006] to investigate lower-order antibunching and HOA. The analytic expressions of moments (5.6)-(5.13) can be used to investigate the nonclassicality using inequality (1.41) for the set of states. The obtained results are illustrated in Figure 5.1 where we have compared the results between the vacuum filtered and single photon added states. During this attempt, we also discuss the nonclassicality present in the quantum states used for the preparation of the engineered quantum states (cf. Figs. 5.1(a)-(c)). In Figs.
CHAPTER 5. MANIPULATING NONCLASSICALITY VIA QUANTUM STATE ENGINEERING PROCESSES: VACUUM FILTRATION AND SINGLE PHOTON ADDITION

(a) (b)

Figure 5.1: Lower- and higher-order antibunching witnesses as functions of displacement parameter $\alpha$ (for ECS and KS) and probability $p$ (for BS with parameter $M = 10$) for (a) ECS, PAECS and VFECs, (b) BS, PABS and VFBS, and (c) KS, PAKS and VFKS. The quantities shown in all the plots are dimensionless.

5.1 (b)-(c), we have shown the result for photon added and vacuum filtered BS and KS, where it can be observed that the depths of both lower- and higher-order witnesses in the negative region are larger for photon added BS and KS in comparison with the vacuum filtered BS and KS, respectively. However, an opposite nature is observed for ECS where the depth of lower- and higher-order witnesses is more for the vacuum filtration in comparison with the photon addition if the values of $\alpha$ remain below certain values; whereas for the photon addition the depth of lower- and higher-order antibunching witnesses is found to be greater than that for vacuum filtration for the higher values of $\alpha$ (cf. Figure 5.1 (a)). However, HOA is not observed for the ECS and KS and thus both operations can be ascribed as nonclassicality inducing operations as far as this nonclassical feature is concerned.

5.3.2 Lower- and higher-order squeezing

The concept of squeezing originates from the uncertainty relation. There is a minimum value of an uncertainty relation involving quadrature operators where the variance of two non-commuting quadratures (say position and momentum) are equal and their product satisfies minimum uncertainty relation. Such a situation is closest to the classical scenario, in the sense that there is no uncertainty in the classical picture and this is the closest point that one can approach remaining within the framework of quantum mechanics. Coherent state satisfies this minimum uncertainty relation and is referred to as a classical (or more precisely closest to classical state). If any of the quadrature variances reduces below the corresponding value for a minimum uncertainty (coherent) state (at the cost of increase in the fluctuations in the other quadrature) then the corresponding state is called squeezed state.

The higher-order nonclassical properties can be investigated by studying HOS. There are two different criteria for HOS [Hong and Mandel 1985a,b; Hillery 1987a]: Hong-Mandel criterion [Hong and Mandel 1985b] and Hillery criterion [Hillery 1987a]. The concept of the HOS was first introduced by Hong and Mandel using higher-order moments of the quadrature operators [Hong and Mandel 1985b]. According to this criterion, it is observed if the higher-order moment for a quadrature operator for a quantum state is observed to be less than the corresponding coherent state value. Another type of HOS was introduced by Hillery who introduced amplitude
5.3.3 Higher-order sub-Poissonian photon statistics

The higher-order moments in Eqs. (5.6)-(5.13) are used to calculate the above inequality (1.42) with the help of (1.41) for states obtained after vacuum filtration and photon addition in ECS, BS and KS as well as the parent states, powered quadrature and used variance of this quadrature to define HOS [Hillery, 1987a]. Here, we aim to analyze the possibility of HOS using Hong-Mandel criterion for $l$th order squeezing. We have investigated the possibility of observing HOS analytically using Eqs. (5.6)-(5.13) and inequality (1.44) for all engineered quantum states of our interest and have shown the corresponding results in Figs. 5.2 (a)-(c) where we have compared the HOS in the set of quantum states and the states obtained by photon addition and vacuum filtration. These operations fail to induce this nonclassical feature in the engineered states prepared from ECS, which also did not show signatures of squeezing. In Figure 5.2 (a), we illustrate Hong-Mandel type HOS with respect to parameter $p$ where we have shown the existence HOS for BS, VFBS and PABS. It can be observed that the state engineering operations fail to increase this particular feature of nonclassicality in BS. Additionally, higher-order nonclassicality is absent for higher values of $p$ when corresponding lower-order squeezing is present. In case of KS, PAKS and VFKS, we have observed that HOS is observed when the values of $\alpha$ are greater than certain values for the individual curves of the corresponding states (cf. Figure 5.2 (b)). Note that photon addition may provide some advantage in this case, but vacuum filtration would not as for the same value of displacement parameter KS and PAKS (VFKS) have (has not) shown squeezing. Interestingly, the presence of squeezing also depends upon the Kerr nonlinearity parameter $\chi$, which is shown in Figure 5.2 (c). Similar to Figure 5.2 (b) photon addition shows advantage over KS which disappears for larger values of $\chi$, while vacuum filtering is not beneficial.

In Figure 5.3 we have shown using the dark (blue) color in the contour plots of the HOS witness for PAKS that squeezing can be observed for higher values of $|\alpha|$ and smaller values of $\chi$. Additionally, the phase parameter $\theta$ of $\alpha$ is also relevant for observing the nonclassicality as squeezing occurs in the vicinity of $\theta = m\pi$, while disappears for $\theta = \frac{m\pi}{2}$ with integer $m$. Similar behavior is observed in KS and VFKS (not shown here).
and the corresponding results are depicted in Figure 5.4. Nonclassicality is not revealed by HOSPS criteria of even orders in case of ECS, while corresponding engineered states show nonclassicality. Additionally, nonclassicality is induced by vacuum filtration for odd orders while it was not observed in the parent state (cf. Figure 5.4 (a)). This clearly shows the role of hole burning operations in inducing nonclassicality for odd orders. However, in case of even orders, the same operations are also observed to destroy the nonclassicality in the parent state. From Figs. 5.4 (b) and (c), it is observed that BS and KS do not show HOSPS for the odd values of \( l \) even after application of state engineering operations. Additionally, HOSPS is not observed for the KS for even values of \( l \), too. Consequently, the nonclassical feature witnessed through the HOSPS criterion in PAKS can be attributed solely to the hole burning process.

Nonclassicality in the engineered quantum states can also be studied using quasidistribution functions [Thapliyal et al. 2015] but here we are going to quantify the amount of nonclassicality in these states. Further, the effect of decoherence on the observed nonclassicality [Banerjee and Ghosh 2007; Banerjee et al. 2010a,b; Naikoo et al. 2018] and phase diffusion [Banerjee et al. 2007; Banerjee and Srikanth 2007] can be studied.
5.4 Nonclassicality measure

We have obtained the analytic expressions of linear entropy for ECS, KS, BS and corresponding engineered states which are given as

\[
\mathcal{L}_{\text{ECS}} = 1 - \frac{\exp[-2|\alpha|^2]}{4(1+\exp[-2|\alpha|^2])} \sum_{n,m,r=0}^{\infty} f_{n,m,r} \sum_{k_1=0}^{n} \frac{n_{C_{k_1}} C_{r+k_1-m}}{2^{n+r}},
\]

for VFECS

\[
\mathcal{L}_{\text{VFECS}} = 1 - (N_{\text{VFECS}})^4 \sum_{n,m,r=1}^{\infty} f_{n,m,r} \sum_{k_1=0}^{n} \frac{n_{C_{k_1}} C_{r+k_1-m}}{2^{n+r}}
\]

and for PAECS

\[
\mathcal{L}_{\text{PAECS}} = 1 - (N_{\text{PAECS}})^4 \sum_{n,m,r=0}^{\infty} f_{n,m,r} (m+1)(n-m+r+1) \times \sum_{k_1=0}^{n+1} \frac{n_{C_{k_1}} C_{r+k_1-m}}{2^{n+r+2}}
\]

where \( f_{n,m,r} = \frac{|\alpha|^{2n+2(1+(-1)^m)(1+(-1)^n)(1+(-1)^r)(1+(-1)^{n+r}}{M^{n+r+2}} \). 

Similarly, analytical expression for linear entropy of BS

\[
\mathcal{L}_{\text{BS}} = 1 - \sum_{n,m,r=0}^{M} g_{p,M,n,m,r} \sum_{k_1=0}^{n} \frac{n_{C_{k_1}} C_{r+k_1-m}}{2^{n+r}}
\]

for VFBS

\[
\mathcal{L}_{\text{VFBs}} = 1 - (N_{\text{VFBs}})^4 \sum_{n,m,r=1}^{M} g_{p,M,n,m,r} \sum_{k_1=0}^{n} \frac{n_{C_{k_1}} C_{r+k_1-m}}{2^{n+r}}
\]
5.5 Conclusion

In summary, this chapter is focused on the comparison of the effects of two processes (vacuum state filtration and single photon addition) used in quantum state engineering to burn hole at vacuum as far as the higher-order non-
classical properties of the quantum states prepared using these two processes are concerned. Specifically, various quantum state engineering processes for burning holes at vacuum lead to different $\sum_{m=1}^{c_m} |m\rangle$ as far as the values of $c_m$s are concerned (even when the parent state is the same). To study its significance in nonclassical properties of the engineered states, we considered a small set of finite and infinite dimensional quantum states (namely, ECS, BS, and KS). This provided us a set of six engineered quantum states, namely VFECS, PAECS, VFBS, PABS, VFKS, and PAKS and three parent states for our analysis. This set of engineered quantum states can have a great importance in quantum information processing and quantum optics as they are found to be highly nonclassical. Especially when some exciting applications of their parent states are already investigated with relevance to continuous variable quantum information processing and/or quantum optics. The present study also addresses the significance of these hole burning processes in inducing (enhancing) particular nonclassical features in the large set of engineered and parent quantum states.

The general expressions for moments of the set of states are reported in the compact analytic form, which are used here to investigate nonclassical features of these states using a set of criteria of higher-order nonclassicality (e.g., criteria of HOA, HOS and HOSPS). The obtained expressions can be further used to study other moment-based criteria of nonclassicality. The hole burning operations are found to be extremely relevant as the states studied here are found to be highly nonclassical when quantified through a measure of nonclassicality (entanglement potential). In brief, both the vacuum filtration and photon addition operations can be ascribed as antibunching inducing operations in KS and ECS while antibunching enhancing operations for BS. As far as HOS is concerned no such advantage of these operations is visible as these operations fail to induce squeezing in ECS and often decrease the amount of squeezing present in the parent state. Additionally, the operations are successful in inducing HOSPS in KS and enhances this feature in the rest of the parent states. The relevance of higher-order nonclassicality in the context of the present study can be understood from the fact that these hole burning operations show an increase in the depth of HOA witness and decrease in the amount of HOS with order. While in case of HOSPS even orders show nonclassicality whereas odd orders fail to detect it. Finally, the measure of nonclassicality reveals vacuum filtration as a more powerful tool than photon addition for enhancing nonclassicality in the parent state, but photon addition is observed to be advantageous in some specific cases.
Chapter 6

Conclusions And Scope For Future Work

This concluding chapter aims to briefly summarize the results obtained in this thesis work, and it also aims to provide some insights into the scope of future works. To begin with, we may note that this thesis is a theoretical work focused on nonclassical and phase properties of some of the engineered quantum states of radiation field. Here, lower- and higher-order nonclassical properties of PADFS, PSDFS, PASDFS, ECS, VFECS, PAECS, BS, VFBS, PABS, KS, VFKS and PAKS have been witnessed through lower- and higher-order antibunching, higher-order sub-Poissonian photon statistics, higher-order squeezing, Klyshko’s criterion, Vogel’s criterion, Agarwal-Tara’s criterion, $Q$ function, Mandel $Q_M$ parameter, etc. Further, phase properties of these states have been investigated with the help of phase distribution function, phase dispersion, phase fluctuation, phase uncertainty parameter and angular $Q$ function. These investigations have revealed that the state engineering processes may help us to introduce and manipulate the nature and amount of nonclassicality present in a quantum state. Keeping this in mind, at the end of the thesis two quantum state engineering processes, which can be used to generate holes at vacuum in photon number distribution, have been compared. This systematic and rigorous study of the nonclassical and phase properties of the above mentioned engineered quantum states have led to many new findings, some of them are already mentioned in the end of the individual chapters. In what follows, we list the major findings of the present thesis.

6.1 Conclusion

The main observations of the present thesis may be summarized as follows:

1. It is observed that photon addition and subtraction are not only non-gaussianity and nonclassicality inducing operations but they can also boost the nonclassicality present in the DFS.

2. The results indicate that the amount of nonclassicality in PADFS and PSDFS can be controlled by the Fock state parameter, displacement parameter, the number of photon addition and/or subtraction.

3. Higher-order squeezing witness and $Q$ function are observed to be dependent on the phase of the displacement parameter. However, only higher-order squeezing criterion was found to be able to detect nonclassicality, and thus established that this phase parameter can also be used to control the amount of nonclassicality.

4. It is observed that the depth of nonclassicality witnesses increases with order of nonclassicality.

5. The phase distribution and angular $Q$ functions are found to be symmetric along the value of the phase of the displacement parameter.

6. Photon addition/subtraction and Fock parameters are found to induce opposite effects on phase distribution. Between photon addition and subtraction operations, subtracting a photon modifies the phase properties.
more than photon addition. Interestingly, phase properties are associated with average photon number of the state as well. Photon subtraction increases the average photon number as photon addition does. However, photon addition creates a hole at vacuum unlike photon subtraction.

7. The three phase fluctuation parameters given by Carruthers and Nieto reveal phase properties of PADFS and PSDFS, although one of them, $U$ parameter indicates antibunching in both PADFS and PSDFS.

8. Phase dispersion quantifying phase fluctuation remains unity for Fock state reflecting uniform distribution, which can be observed to decrease with increasing displacement parameter. This may be attributed to the number-phase complementarity as the higher values of variance with increasing displacement parameter lead to smaller phase fluctuation.

9. The present investigation has revealed the advantage of the PADFS and PSDFS in quantum phase estimation and has obtained the set of optimized parameters in the PADFS/PSDFS.

10. The nonclassicality and non-Gaussianity of PASDFS viewed with the help of a quasidistribution function, namely $Q$ function is shown in the present thesis.

11. The present study also provides a flavor of the significance of the hole burning processes in inducing particular nonclassical features in the family of engineered and parent quantum states. The hole burning operations are observed to be potentially relevant as the quantum states studied in this work are observed to be highly nonclassical when quantification is done through a measure of nonclassicality.

### 6.2 Scope for future work

The works reported in the present thesis give us a general idea for the investigation of phase properties and nonclassical features present in a family of engineered quantum states. This work can be further extended in various ways. Some of the possible extension of the present work are listed below with a focus on the possibilities that may be realized in the near future.

1. The work may be continued to find out the non-Gaussianity of the studied states. Subsequently, the nonclassicality and non-Gaussianity observed in these states can be used to realize various applications in quantum information processing tasks. Therefore, specific applications of the non-classical properties of the aforesaid states which (the applications) are otherwise impossible to achieve using other types of states (classical/nonclassical).

2. A major part of the results presented here can be experimentally verified using the available technology. Along this line, it would be interesting to perform resource comparison (e.g., total number of beam splitters, photo detectors, nonlinear gadgets, etc.) in generation of the aforesaid nonclassical states using quantum state engineering methods.

3. The work can be extended to quantify the amount of nonclassicality present in quantum states using different nonclassicality measures.

4. The methods adopted here and the results obtained here can be helpful in further theoretical studies on nonclassical and phase properties of other engineered quantum states (both finite as well as infinite dimensional). There could be many such states using several other quantum state engineering tools, for instance, squeezing, photon catalysis, etc.

5. Attempts can be made to observe the effect of noise in these states. Specifically, further study of the robustness of observed nonclassical properties of PADFS, PSDFS, PASDFS, BS, VFBS, PABS, KS, VFKS, PAKS under photon loss as well as inefficiency of photo-detectors.
We expect that theoretical work done in this thesis will be performed experimentally and that will lead to some important applications. We also hope this work will be very useful in quantum optics. With these hopes this thesis is concluded.
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