A New Limit of the $AdS_5 \times S^5$ Sigma Model

Nathan Berkovits

Instituto de Física Teórica, State University of São Paulo
Rua Pamplona 145, 01405-900, São Paulo, SP, Brasil

Using the pure spinor formalism, a quantizable sigma model has been constructed for the superstring in an $AdS_5 \times S^5$ background with manifest $PSU(2,2|4)$ invariance. The $PSU(2,2|4)$ metric $g_{AB}$ has both vector components $g_{ab}$ and spinor components $g_{\alpha \beta}$, and in the limit where the spinor components $g_{\alpha \beta}$ are taken to infinity, the $AdS_5 \times S^5$ sigma model reduces to the worldsheet action in a flat background.

In this paper, we instead consider the limit where the vector components $g_{ab}$ are taken to infinity. In this limit, the $AdS_5 \times S^5$ sigma model simplifies to a topological A-model constructed from fermionic N=2 superfields whose bosonic components transform like twistor variables. Just as $d=3$ Chern-Simons theory can be described by the open string sector of a topological A-model, the open string sector of this topological A-model describes $d=4$ N=4 super-Yang-Mills. These results might be useful for constructing a worldsheet proof of the Maldacena conjecture analogous to the Gopakumar-Vafa-Ooguri worldsheet proof of Chern-Simons/conifold duality.

March 2007

1 e-mail: nberkovi@ift.unesp.br
1. Introduction

Maldacena’s conjecture [1] relating d=4 N=4 super-Yang-Mills and the superstring on AdS$_5 \times S^5$ has been verified in various limiting cases. However, in the limit where d=4 N=4 super-Yang-Mills is weakly coupled, it has been difficult to verify the conjecture because the AdS$_5 \times S^5$ background is highly curved. Although there exists a quantizable sigma model description of the superstring in an AdS$_5 \times S^5$ background using the pure spinor formalism [2], the sigma model naively becomes strongly coupled when the AdS$_5 \times S^5$ radius goes to zero.

In an AdS$_5 \times S^5$ background, the sigma model action using the pure spinor formalism has the form [2][3][4][5]

$$S = \frac{1}{\Lambda} \int d^2 z \left[ \frac{1}{2} \eta_{ab} J^a \mathcal{T}^b + \eta_{\alpha \beta} \left( \frac{3}{4} J^{\beta} \mathcal{T}^\alpha - \frac{1}{4} \mathcal{T}^\beta J^\alpha \right) + \text{ghost contribution} \right] \tag{1.1}$$

where $J^a$ for $a = 0$ to 9 and $(J^\alpha, J^{\widehat{\beta}})$ for $\alpha, \widehat{\beta} = 1$ to 16 are bosonic and fermionic \(PSU(2,2|4)/SO(4,1) \times SO(5)\) currents constructed from the worldsheet Green-Schwarz variables $(x, \theta, \widehat{\theta})$ as in the Metsaev-Tseytlin construction [3], $\eta_{ab}$ is the d=10 Minkowski metric and $\eta_{\alpha \widehat{\beta}} = (\gamma^{01234})_{\alpha \widehat{\beta}}$. BRST invariance together with \(PSU(2,2|4)\) invariance uniquely fixes the relative coefficients in the action, so the AdS$_5 \times S^5$ radius $r$ only appears in the action through the sigma model coupling constant $\Lambda = \alpha'/r^2$ where $\alpha'$ is the inverse string tension. So the sigma model seems to be strongly coupled when the AdS$_5 \times S^5$ radius is small. However, this conclusion may be too naive since it assumes that the \(PSU(2,2|4)\) algebra remains undeformed when the AdS$_5 \times S^5$ radius is taken to zero.

One limit of the sigma model which is well-understood is the d=10 flat space limit where the AdS$_5 \times S^5$ radius goes to infinity. Naively, one would go to the flat space limit by simply taking $\Lambda \rightarrow 0$, however, this limit would preserve \(PSU(2,2|4)\) invariance instead of the desired d=10 super-Poincaré invariance. The correct way to go to the flat space limit is to rescale the spinor component of the \(PSU(2,2|4)\) metric $g_{\alpha \widehat{\beta}} = \eta_{\alpha \widehat{\beta}}$ to

$$g_{\alpha \widehat{\beta}} = r \eta_{\alpha \widehat{\beta}} \tag{1.2}$$

in the sigma model action of (1.1), together with an appropriate rescaling of the \(PSU(2,2|4)\) structure constants. In the limit where $r$ goes to infinity, the \(PSU(2,2|4)\) algebra is deformed into the d=10 super-Poincaré algebra and the second-order kinetic term for the fermions in (1.1) blows up. Nevertheless, this limit can be taken
smoothly by writing the second-order kinetic term $r\eta_{\alpha\beta}\mathcal{J}^\beta\mathcal{J}^\alpha$ as the first-order kinetic term $\mathcal{T}^\alpha d_\alpha + J^\beta \hat{d}_\beta + r^{-1} \eta^{\alpha\beta} d_\alpha \hat{d}_\beta$ where $d_\alpha$ and $\hat{d}_\beta$ are auxiliary fermionic variables. In the limit where $r \to \infty$, one obtains a first-order action for the worldsheet fermions $(\theta^\alpha, d_\alpha)$ and $(\hat{\theta}^\beta, \hat{d}_\beta)$, which is the flat space version of the worldsheet action using the pure spinor formalism.

Since the structure constants of the algebra are related to the superspace torsions $T_{AB}{}^C$, this limiting procedure can be understood as a rescaling of the $AdS_5 \times S^5$ superspace torsions into the flat superspace torsions. In an $AdS_5 \times S^5$ background, $T_{\alpha \alpha}{}^{\hat{\beta}}$ and $T_{\alpha \beta}{}^{a}$ are non-vanishing torsions which are related by $T_{\alpha \alpha}{}^{\hat{\beta}} \eta_{\beta \hat{\beta}} = T_{\alpha \beta}{}^{b} \eta_{ab}$. On the other hand, in a flat background, $T_{\alpha \beta}{}^{a}$ is non-vanishing and $T_{\alpha \alpha}{}^{\hat{\beta}} = 0$. The rescaling of the structure constants and $g_{\alpha \beta}$ as in (1.2) rescales the torsions such that

$$\frac{T_{\alpha \beta}{}^{b} \eta_{ab}}{T_{\alpha \alpha}{}^{\hat{\beta}} \eta_{\beta \hat{\beta}}} = r. \quad (1.3)$$

So when $r \to \infty$, $T_{\alpha \alpha}{}^{\hat{\beta}} \to 0$ which corresponds to flat space.

In this paper, we will consider a different limit of the $AdS_5 \times S^5$ sigma model in which, instead of the spinor component of the $PSU(2, 2|4)$ metric $g_{\alpha \beta}$ being rescaled, the vector component $g_{ab}$ will be rescaled as

$$g_{ab} = r^{-1} \eta_{ab}. \quad (1.4)$$

Furthermore, the $PSU(2, 2|4)$ structure constants will be rescaled such that in the limit where $r \to 0$, the $PSU(2, 2|4)$ superalgebra is deformed into an $SU(2, 2) \times SU(4)$ bosonic algebra with 32 abelian fermionic symmetries. This corresponds to rescaling the torsions such that (1.3) remains satisfied when $r \to 0$, which implies that the resulting background has non-vanishing $T_{\alpha \alpha}{}^{\hat{\beta}}$ but has $T_{\alpha \beta}{}^{a} = 0$. Since the usual construction of supergravity backgrounds assumes that $T_{\alpha \beta}{}^{a} = \gamma_{\alpha \beta}^{a} \quad [4]$, this $r \to 0$ limit does not correspond to a standard supergravity background.

Nevertheless, the resulting sigma model action when $T_{\alpha \beta}{}^{a} \to 0$ is very simple and can be expressed as a linear N=2 sigma model constructed from 16 chiral and antichiral N=2 superfields denoted by $\Theta^{rj}$ and $\Theta_{jr}$, where $r = 1$ to 4 are $SU(2, 2)$ indices and $j = 1$ to 4 are $SU(4)$ indices. Unlike the bosonic superfields in standard N=2 sigma models, $\Theta^{rj}$ and $\Theta_{jr}$ are fermionic superfields. It is interesting that in the open-closed matrix model duality of [8], the matter variables are also described by fermions with a second-order kinetic action.
The lowest components of $\Theta^r_j$ and $\Theta^r_{jr}$ are linear combinations of the $\theta$ and $\hat{\theta}$ variables, and the bosonic components of $\Theta^r_j$ and $\Theta^r_{jr}$ are twistor-like combinations of the ten $x'$s and 22 pure spinor ghosts. Just as the fermionic variables had a first-order kinetic action in the flat space sigma model obtained by rescaling (1.2), the bosonic variables now have a first-order kinetic action in the N=2 sigma model obtained by rescaling (1.4).

Moreover, this N=2 sigma model is twisted as an A-model where the pure spinor BRST operator from the original $AdS_5 \times S^5$ sigma model acts in the usual topological manner as the scalar worldsheet supersymmetry generator. So the N=2 sigma model is a topological A-model with the worldsheet action

$$S = \int d^2z d^4\kappa \, \Theta^r_j \Theta^r_{jr}$$

where $(\kappa_+, \kappa_-, \kappa_+^\dagger, \kappa_-^\dagger)$ are the Grassmann parameters of the N=(2,2) superspace. This model is invariant under the bosonic isometries $SU(2,2) \times SU(4) \times U(1)$ which act on the superfields as

$$\delta \Theta^r_j = i\Lambda^r_s \Theta^s_j + i\Theta^r_{ks} \Omega^i_k \Theta^r_{ij} + i\Sigma \Theta^r_j, \quad \delta \Theta^r_{jr} = -i\Theta^r_{js} \Lambda^s_i - i\Omega^j_k \Theta^r_{kr} - i\Sigma \Theta^r_{jr},$$

where $(\Lambda^r_s, \Omega^j_k, \Sigma)$ are constant parameters satisfying $\Lambda^r_r = \Omega^j_j = 0$, and is invariant under the 32 abelian fermionic isometries

$$\delta \Theta^r_j = \alpha^r_j, \quad \delta \Theta^r_{jr} = \overline{\alpha}_{jr}$$

where $\alpha^r_j$ and $\overline{\alpha}_{jr}$ are constant Grassmann parameters. Note that the bosonic isometries of this model include a “bonus” $U(1)$ symmetry in addition to the $SU(2,2) \times SU(4)$ isometries of the original $AdS_5 \times S^5$ sigma model.

Introducing fermionic worldsheet superfields whose bosonic components are twistor-like coordinates has been useful in classical descriptions of the superstring where kappa-symmetry is replaced by worldsheet supersymmetry [10][11][12]. The N=2 model in this paper shares many features with this “super-embedding” approach, however, it has the advantage of being quantizable because of the second-order action for the fermionic superfields. Since the second-order action for fermionic superfields is generated by the Ramond-Ramond background, it might be possible to generalize the twistor-like methods of this paper to more general Ramond-Ramond backgrounds.

The abelianization of the fermionic isometries of (1.7) comes from setting $T_{\alpha\beta^a} = 0$ and means that the supersymmetry generators anticommute with each other. To relate this
model to super-Yang-Mills where supersymmetry acts in the conventional way, it is useful to interpret (1.3) as the limit of a non-linear topological A-model which is constructed such that the isometries of (1.3) and (1.4) are deformed into $SU(2,2|4)$ isometries.

The worldsheet action for this non-linear topological A-model is

$$S = \frac{1}{\Lambda} \int d^2z d^4\kappa \left[ \overline{\Theta}^{rj} \Theta^{jr} - \frac{1}{2R^2} \overline{\Theta}^{rj} \Theta^{js} \overline{\Theta}^{sk} \Theta^{kr} + \frac{1}{3R^4} \overline{\Theta}^{rj} \Theta^{js} \overline{\Theta}^{sk} \Theta^{kt} \overline{\Theta}^{tl} \Theta^{lr} + \ldots \right]$$

where $R$ is a new parameter which, in the limit $R \to \infty$, takes the non-linear sigma model into the linear sigma model of (1.3). This non-linear action will be shown to be one-loop conformally invariant, and is invariant under the same $SU(2,2) \times SU(4) \times U(1)$ transformations as (1.4). But the fermionic transformations of (1.4) are modified to

$$\delta \Theta^{rj} = \alpha^{rj} + \frac{1}{R^2} \Theta^{rk} \overline{\alpha}_{ks} \Theta^{sj}, \quad \delta \overline{\Theta}^{jr} = \overline{\alpha}^{jr} + \frac{1}{R^2} \overline{\Theta}^{js} \alpha^{sk} \overline{\Theta}^{kr},$$

which anticommute to form the superalgebra $SU(2,2|4)$.

It will be conjectured that the BRST cohomology in the closed string sector of this non-linear topological A-model is trivial, which implies that the open string physical states are independent of $R$ and $\Lambda$ in (1.8). This would be similar to the topological A-model for $d=3$ Chern-Simons which has physical states only in the open string sector [13], but would be different from the topological B-model for the twistor-string [14] which describes N=4 $d=4$ super-Yang-Mills in the open sector and N=4 $d=4$ conformal supergravity in the closed sector.

In the topological A-model for $d=3$ Chern-Simons, the open string boundary conditions are $X^I = \overline{X}_I$ where $X^I$ and $\overline{X}_I$ are chiral and anti-chiral superfields for $I = 1$ to 3. Similarly, the open string boundary conditions in the non-linear topological A-model of (1.8) are $\Theta^{rj} = \overline{\Theta}_{jr}$. These boundary conditions eliminate half of the 32 $\theta$’s and break $SU(2,2|4)$ invariance down to an $OSp(4|4)$ subgroup, which is the N=4 supersymmetry algebra on $AdS_4$. In this open topological A-model, the BRST cohomology of physical states will be shown to describe $d=4$ N=4 super-Yang-Mills, where the bosonic components of $\Theta^{rj}$ are interpreted as twistor coordinates constructed from the four $x$’s of $AdS_4$ together with an N=4 $d=4$ pure spinor.

The similarities between Chern-Simons and N=4 $d=4$ super-Yang-Mills are not surprising since, using the pure spinor formalism, the $d=10$ super-Yang-Mills action can be
written in the Chern-Simons form $S = \langle VQV + \frac{2}{3}V^3 \rangle$ where $Q$ is the pure spinor BRST operator and $V$ is the super-Yang-Mills vertex operator \cite{13} \cite{16}. Furthermore, there is a gauge/geometry correspondence relating Chern-Simons and the resolved conifold which has many features in common with the Maldacena conjecture relating N=4 d=4 super-Yang-Mills and $AdS_5 \times S^5$. The Chern-Simons/conifold correspondence was first proposed by Gopakumar and Vafa \cite{17}, and was later proven using open-closed duality arguments by Ooguri and Vafa \cite{18}.

The basic idea behind the open-closed duality proof of Gopakumar-Vafa-Ooguri is that, in a certain limit, the closed topological string theory for the resolved conifold geometry develops a new branch corresponding to “holes” on the closed worldsheet. These holes were then shown to correspond to the open string sector of the topological A-model that describes d=3 Chern-Simons.

Since the open string sector of the topological A-model in this paper describes d=4 N=4 super-Yang-Mills, and since this topological A-model is related to a certain limit of the closed superstring in an $AdS_5 \times S^5$ background, it is natural to try to construct a similar open-closed duality proof for the Maldacena conjecture. However, there are some questions that need to be answered before such a proof can be attempted.

One question is to explain the interpretation of the torsion ratio of (1.3) as the $AdS \times S^5$ radius. Although this interpretation is easily understood in the flat space limit where $r \to \infty$, it is not obvious this interpretation is correct in the limit where $r \to 0$. So it is not clear that the limit discussed in this paper corresponds to weak coupling on the super-Yang-Mills side of the duality.

A second question is to compute the complete cohomology of physical states for the topological A-model of (1.8). Although it will be shown that the cohomology in the open string sector of this A-model describes d=4 N=4 super-Yang-Mills, it remains to be shown that there are no physical states in the closed string sector of this A-model.

Finally, a third question which needs to be answered is if the open string topological A-model in this paper can be interpreted as a branch of the closed string $AdS_5 \times S^5$ sigma model which emerges in the limit where $T_{\alpha\beta}^a \to 0$. Perhaps the “bonus” $U(1)$ symmetry in (1.6) will play a role in the emergence of this branch.

In section 2 of this paper, the $AdS_5 \times S^5$ sigma model using the pure spinor formalism is reviewed and the flat space limit is discussed. In section 3, the $AdS_5 \times S^5$ sigma model is shown to reduce to a linear topological A-model in the limit where $T_{\alpha\beta}^a \to 0$. In section 4, this linear topological A-model is deformed into a non-linear topological A-model with $PSU(2,2|4)$ invariance. And in section 5, the open string sector of this non-linear topological A-model is shown to describe d=4 N=4 super-Yang-Mills.
2. Review of Pure Spinor Formalism in $AdS_5 \times S^5$ Background

Using the pure spinor formalism, the superstring can be quantized in any consistent d=10 supergravity background [19]. Unlike the Green-Schwarz formalism where the gauge-fixing procedure of kappa-symmetry is poorly understood even in a flat background, the pure spinor formalism is quantized using a BRST operator which can be defined in any consistent supergravity background. In an $AdS_5 \times S^5$ background, the BRST transformations act in a geometric manner, which has been useful for proving the quantum consistency of this background [3].

2.1. Sigma model action

The sigma model for the superstring in an $AdS_5 \times S^5$ background is manifestly $PSU(2,2|4)$-invariant and is constructed from the Metsaev-Tseytlin left-invariant currents [6]

$$J^A = (G^{-1} \partial G)^A, \quad \bar{J}^A = (\bar{G}^{-1} \partial \bar{G})^A,$$

(2.1)

where $G(x, \theta, \hat{\theta})$ takes values in the coset $PSU(2,2|4)_{SO(4,1)\times SO(5)}$, $A = ([ab], c, \alpha, \hat{\alpha})$ ranges over the 30 bosonic and 32 fermionic elements in the Lie algebra of $PSU(2,2|4)$, $[ab]$ labels the $SO(4,1) \times SO(5)$ “Lorentz” generators, $c = 0$ to 9 labels the “translation” generators, and $\alpha, \hat{\alpha} = 1$ to 16 label the fermionic “supersymmetry” generators. The action in the pure spinor formalism also involves left and right-moving bosonic ghosts, $(\lambda_\alpha, w_\alpha)$ and $(\hat{\lambda}_{\hat{\alpha}}, \hat{w}_{\hat{\alpha}})$, which satisfy the pure spinor constraints $\lambda_\alpha \gamma^c \lambda = \hat{\lambda}_{\hat{\alpha}} \gamma^c \hat{\lambda} = 0$. Because of the pure spinor constraints, $w_\alpha$ and $\hat{w}_{\hat{\alpha}}$ can only appear in combinations which are invariant under the gauge transformations

$$\delta w_\alpha = \xi^c(\gamma_c \lambda)_\alpha, \quad \delta \hat{w}_{\hat{\alpha}} = \hat{\xi}^c(\gamma_c \hat{\lambda})_{\hat{\alpha}}.$$

(2.2)

As in standard coset constructions, the $PSU(2,2|4)_{SO(4,1)\times SO(5)}$ coset $G(x, \theta, \hat{\theta})$ is defined up to right multiplication by a local $SO(4,1) \times SO(5)$ parameter $\Omega^[ab](x, \theta, \hat{\theta})$ as

$$\delta G(x, \theta, \hat{\theta}) = G(x, \theta, \hat{\theta}) (\Omega^[ab](x, \theta, \hat{\theta}) T^[ab])$$

(2.3)

where $T^[ab]$ are the $SO(4,1) \times SO(5)$ generators. Under these gauge transformations, the pure spinors are defined to transform covariantly as

$$\delta \lambda^\alpha = -\frac{1}{2} \Omega^[ab](\gamma^[ab] \lambda)^\alpha, \quad \delta w_\alpha = \frac{1}{2} \Omega^[ab](\gamma^[ab] w)_\alpha,$$

(2.4)

$$\delta \hat{\lambda}^{\hat{\alpha}} = -\frac{1}{2} \Omega^[ab](\gamma^[ab] \hat{\lambda})^{\hat{\alpha}}, \quad \delta \hat{w}_{\hat{\alpha}} = \frac{1}{2} \Omega^[ab](\gamma^[ab] \hat{w})_{\hat{\alpha}}.$$
A convenient way to write the sigma model action in a manifestly gauge-invariant manner is [20, 2]

\[
S = \frac{1}{\Lambda} \int d^2 z \left[ \frac{1}{2} \eta_{AB}(J^A - A^A)(J^B - A^B) \right.
\]
\[
+ \mathcal{B} + w_\alpha (\bar{\partial} \lambda + \frac{1}{2} A^{[ab]} \gamma_{[ab]} \lambda)^\alpha + \tilde{w}_\alpha (\bar{\partial} \tilde{\lambda} + \frac{1}{2} \tilde{A}^{[ab]} \gamma_{[ab]} \tilde{\lambda})^{\tilde{\alpha}} \right]
\]
\[
= \frac{1}{\Lambda} \int d^2 z \left[ \frac{1}{2} \eta_{[cd]}(J^{[ab]} - A^{[ab]})(J^{cd} - \tilde{A}^{[cd]}) + \frac{1}{2} \eta_{cd} J^c J^d + \frac{1}{4} \eta_{\alpha\beta}(J^{\beta} J^{\alpha} - J^\beta J^\alpha) \right.
\]
\[
+ \frac{1}{2} \eta_{\alpha\beta}(J^{\beta} J^\alpha - J^\beta J^\alpha) + w_\alpha (\bar{\partial} \lambda + \frac{1}{2} \tilde{A}^{[ab]} \gamma_{[ab]} \lambda)^\alpha + \tilde{w}_\alpha (\bar{\partial} \tilde{\lambda} + \frac{1}{2} \tilde{A}^{[ab]} \gamma_{[ab]} \tilde{\lambda})^{\tilde{\alpha}} \right],
\]

where \( \eta_{AB} \) is the \( \text{PSU}(2, 2|4) \) metric, \( \eta_{[cd]} = \eta_{[c|d]} \) when \( a, b, c, d = 0 \) to 4, \( \eta_{[cd]} = -\eta_{[c|d]} \) when \( a, b, c, d = 5 \) to 9, \( \eta_{cd} \) is the d=10 Minkowski metric, \( \eta_{\alpha\beta} = (\gamma^{01234})_{\alpha\beta} \), \( A^{[ab]} \) and \( \tilde{A}^{[ab]} \) are worldsheet \( SO(4, 1) \times SO(5) \) gauge fields, and \( \mathcal{B} \) is the Wess-Zumino term which in an \( AdS_5 \times S^5 \) background takes the simple form [20]

\[
\mathcal{B} = \frac{1}{2} \eta_{\alpha\beta}(J^{\beta} J^\alpha - J^\beta J^\alpha).
\]  

Since \( A^{[ab]} \) and \( \tilde{A}^{[ab]} \) satisfy auxiliary equations of motion, they can be integrated out to obtain the action

\[
S = \frac{1}{\Lambda} \int d^2 z \left[ \frac{1}{2} \eta_{cd} J^c J^d + \eta_{\alpha\beta}(\frac{3}{4} J^{\beta} J^\alpha - \frac{1}{4} J^\beta J^\alpha) \right.
\]
\[
\left. + w_\alpha (\bar{\nabla} \lambda)^\alpha + \tilde{w}_\alpha (\bar{\nabla} \tilde{\lambda})^{\tilde{\alpha}} - \frac{1}{2} \eta_{[cd]}(w_{\gamma}[ab] \lambda)(\tilde{w}_{\gamma}[cd] \tilde{\lambda}) \right],
\]

where \( (\bar{\nabla} \lambda)^\alpha = \bar{\partial} \lambda + \frac{1}{2} \tilde{J}^{[ab]} (\gamma_{[ab]} \lambda)^\alpha \) and \( (\bar{\nabla} \tilde{\lambda})^{\tilde{\alpha}} = \bar{\partial} \tilde{\lambda} + \frac{1}{2} J^{[ab]} (\gamma_{[ab]} \tilde{\lambda})^{\tilde{\alpha}} \). Using the Maurer-Cartan equations, the action of (2.7) can be shown to be invariant under the BRST transformation generated by [3]

\[
Q + \bar{Q} = \int dz \, \eta_{\alpha\tilde{\alpha}} \lambda^\alpha J^{\tilde{\alpha}} + \int d\bar{z} \, \eta_{\alpha\tilde{\alpha}} \tilde{\lambda}^{\tilde{\alpha}} \bar{J}^\alpha
\]

which transform the \( \text{PSU}(2, 2|4) \) coset and pure spinor ghosts as

\[
\delta G = G(\epsilon \lambda^\alpha T_\alpha + \epsilon \tilde{\lambda}^{\tilde{\alpha}} T_{\tilde{\alpha}}), \quad \delta w_\alpha = \epsilon \eta_{\alpha\beta} J^{\beta}, \quad \delta \tilde{w}_\alpha = \epsilon \eta_{\alpha\beta} \tilde{J}^\beta,
\]

where \( T_\alpha \) and \( T_{\tilde{\alpha}} \) are the 32 fermionic generators of \( \text{PSU}(2, 2|4) \) and \( \epsilon \) is a constant Grassmann parameter.
This BRST invariance, together with $PSU(2, 2|4)$ invariance, fixes the relative coefficients of the terms in the sigma model action of (2.7). So, naively, the $AdS_5 \times S^5$ radius $r$ can only appear in the action through the coupling constant $\Lambda = \alpha'/r^2$. However, if one allows the $PSU(2, 2|4)$ algebra to be deformed as the value of $r$ is changed, the $r$ dependence of the action can be more complicated and the form of the action can be modified. For example, in the flat space limit where $r \to \infty$, the $PSU(2, 2|4)$ algebra is deformed to the $N=2$ d=10 super-Poincaré algebra. As will now be discussed, this modifies the sigma model action of (2.7) to a quadratic action.

2.2. Flat space limit

Although the naive limit as $r \to \infty$ is obtained by simply taking $\Lambda \to 0$ in the sigma model action of (2.7), this limit would preserve $PSU(2, 2|4)$ invariance instead of the desired $N=2$ d=10 super-Poincaré invariance of flat Minkowski superspace. To obtain the correct flat space limit, one needs to rescale the $PSU(2, 2|4)$ structure constants such that when $r \to \infty$, the $PSU(2, 2|4)$ algebra is deformed into the $N=2$ d=10 super-Poincaré algebra.

The non-vanishing $PSU(2, 2|4)$ structure constants $f^C_{AB}$ are

\begin{equation}

f^c_{\alpha\beta} = \gamma^c_{\alpha\beta}, \quad f^{\hat{c}}_{\alpha\beta} = \gamma^{\hat{c}}_{\alpha\beta}, \quad f^{d}_{\alpha\beta} = \gamma^{d}_{\alpha\beta}, \quad f^{\hat{d}}_{\alpha\beta} = \gamma^{\hat{d}}_{\alpha\beta},
\end{equation}

(2.10)

\begin{align*}
 f^{\hat{c}}_{\alpha\hat{c}} &= -\gamma^{\hat{c}}_{\alpha\beta} \delta^{\beta\hat{c}}, \\
 f^{\hat{d}}_{\alpha\hat{d}} &= -\gamma^{\hat{d}}_{\alpha\beta} \delta^{\beta\hat{d}}, \\
 f^{[ef]}_{\alpha\beta} &= \pm (\gamma^{ef})_{\alpha\beta} \gamma^{\hat{c}}_{\beta\hat{c}}, \\
 f^{[ef]}_{\alpha\hat{c}} &= \pm \delta^{[e\delta f]}_{\alpha\hat{c}}, \\
 f^{\delta\gamma}_{[cd]}_{ef} &= \eta_{c\delta} \delta^{\delta\gamma}_{fe} - \eta_{d\delta} \delta^{\delta\gamma}_{ce} + \eta_{f\delta} \delta^{\delta\gamma}_{de} - \eta_{ef} \delta^{\delta\gamma}_{cd}, \\
 f^{\delta\gamma}_{[cd]e} &= \eta_{e\delta} \delta^{\delta\gamma}_{cd}, \\
 f^{\delta\gamma}_{[cd]e} &= 1/2 (\gamma_{cd})_{\alpha\beta}, \\
 f^{\delta\gamma}_{[cd]e} &= 1/2 (\gamma_{cd})_{\alpha\beta},
\end{align*}

where the + sign in the third line is if $(c, d, e, f) = 0$ to 4, and the − sign is if $(c, d, e, f) = 5$ to 9.

To deform these structure constants to the super-Poincaré structure constants in the $r \to \infty$ limit, one should rescale (2.10) such that

\begin{equation}

f^c_{\alpha\beta} = \gamma^c_{\alpha\beta}, \quad f^{\hat{c}}_{\alpha\beta} = \gamma^{\hat{c}}_{\alpha\beta}, \quad f^d_{\alpha\beta} = \gamma^d_{\alpha\beta}, \quad f^{\hat{d}}_{\alpha\beta} = \gamma^{\hat{d}}_{\alpha\beta},
\end{equation}

(2.11)

\begin{align*}
 f^{\hat{c}}_{\alpha\hat{c}} &= -r^{-1}\gamma^{\hat{c}}_{\alpha\beta} \delta^{\beta\hat{c}}, \\
 f^{\hat{d}}_{\alpha\hat{d}} &= -r^{-1}\gamma^{\hat{d}}_{\alpha\beta} \delta^{\beta\hat{d}}, \\
 f^{\hat{c}}_{\alpha\hat{c}} &= -r^{-1}\gamma^{\hat{c}}_{\alpha\beta} \delta^{\beta\hat{c}}, \\
 f^{\hat{d}}_{\alpha\hat{d}} &= -r^{-1}\gamma^{\hat{d}}_{\alpha\beta} \delta^{\beta\hat{d}},
\end{align*}
\[ f_{\alpha \beta}^{[ef]} = \pm r^{-2}(\gamma_{ef})_{\alpha} \gamma_{\beta}, \quad f_{cd}^{[e f]} = \pm r^{-2}\delta_{c}^{[e} \delta_{d]}^{f}, \]

\[ f_{[cd][ef]}^{[gh]} = \eta_{ce} \delta_{d}^{[g} \delta_{f}^{h]} - \eta_{cf} \delta_{d}^{[g} \delta_{e}^{h]} + \eta_{df} \delta_{c}^{[g} \delta_{e}^{h]} - \eta_{de} \delta_{c}^{[g} \delta_{f}^{h]}, \]

\[ f_{J[cd]e}^{f} = \eta_{e[c} \delta_{d]}/r, \quad f_{J[cde]}^{\alpha} = \frac{1}{2}(\gamma_{cd})_{\alpha}^{\beta}, \quad f_{J[cd]e}^{\alpha} = \frac{1}{2}(\gamma_{cd})_{\alpha}^{\beta}. \]

The metric \(g_{AB}\) should satisfy the property that \(f_{A B}^{C} g_{C D}\) is graded-antisymmetric under permutations of \([ABD]\), so the rescaling of (2.11) implies one should also rescale \(g_{\alpha \beta} \rightarrow r \eta_{\alpha \beta}\) and \(g_{[ab][cd]} = \eta_{[ab][cd]}\) to

\[ g_{\alpha \beta} \rightarrow r \eta_{\alpha \beta}, \quad g_{[ab][cd]} = r^{2} \eta_{[ab][cd]}, \quad (2.12) \]

Since the structure constants \(f_{A B}^{C}\) are proportional to the superspace torsions \(T_{A B}^{C}\), the rescaling of (2.11) implies that

\[ \frac{T_{\alpha \beta}^{b} \eta_{ab}}{T_{\alpha \alpha}^{b} \eta_{b b}} = r. \quad (2.13) \]

If \(T_{\alpha \beta}^{b}\) is fixed to satisfy \(T_{\alpha \beta}^{b} = \gamma_{\alpha \beta}^{b}\), (2.13) implies that \(T_{\alpha \beta} = r^{-1} \gamma_{\alpha \beta}^{a} \eta_{a b} \eta_{b b}\), which is the correct \(r\) dependence since the \(AdS\) curvature \(R_{aba} \beta\) goes like \(1/r^{2}\), and Bianchi identities imply that \(R_{aba} \beta\) is proportional to \(T_{aa} \alpha^{a} T_{b \gamma} \gamma^{b}\).

Since \(g_{\alpha \beta} \rightarrow r \eta_{\alpha \beta}\) blows up when \(r \rightarrow \infty\), it is convenient to write the second-order kinetic term for the fermions in (2.7) in the first-order form as

\[ \frac{1}{\Lambda} \int d^{2} z r \eta_{\alpha \beta} \left( \frac{3}{4} J^{\alpha \beta} J_{\alpha}^{\beta} - \frac{1}{4} J^{\beta} J_{\alpha}^{\alpha} \right) \]

\[ = \frac{1}{\Lambda} \int d^{2} z r \eta_{\alpha \beta} \left( \frac{1}{2} J^{\alpha \beta} J_{\alpha}^{\beta} + \frac{1}{4} J^{\beta} \wedge J_{\alpha}^{\alpha} \right) \]

\[ = \frac{1}{\Lambda} \int d^{2} z J^{\alpha} d_{\alpha} + J^{\alpha} \tilde{d}_{\alpha} + 2r^{-1} \eta^{\alpha \beta} d_{\alpha} \tilde{d}_{\beta} + \frac{1}{4} r \eta_{\alpha \beta} \int d \sigma_{3} \left( d J^{\beta} \wedge J^{\alpha} \right) \]

\[ = \frac{1}{\Lambda} \int d^{2} z J^{\alpha} d_{\alpha} + J^{\alpha} \tilde{d}_{\alpha} + 2r^{-1} \eta^{\alpha \beta} d_{\alpha} \tilde{d}_{\beta} + \frac{1}{4} \int d \sigma_{3} \left( \gamma_{\alpha \beta} J^{c} \wedge J^{\alpha} \wedge J^{\beta} - \gamma_{\alpha \beta} J^{c} \wedge J^{\hat{c}} \wedge J^{\hat{c}} \right) \]

where \(d_{\alpha}\) and \(\tilde{d}_{\alpha}\) are auxiliary variables and the two-form \(J^{\beta} \wedge J^{\alpha} \equiv J^{\beta} J^{\alpha} - J^{\alpha} J^{\beta}\) has been written as the integral of a Wess-Zumino-Witten three-form using the Maurer-Cartan equations

\[ d J^{\hat{\beta}} = f_{\alpha a}^{\hat{\beta}} J^{c} \wedge J^{a} = r^{-1} \gamma_{\alpha a} \eta^{\hat{\beta} \hat{\alpha}} J^{c} \wedge J^{\alpha}, \quad (2.15) \]
\[
d J^\beta = f^\beta_{\alpha c} J^c \wedge \hat{J}^\alpha = r^{-1} \gamma^\beta_{\alpha \beta} \gamma^{\beta \hat{\beta}} J^c \wedge \hat{J}^\hat{\alpha}.
\]

Furthermore, the BRST operator \( Q + \bar{Q} \) of (2.8) can be written as

\[
Q + \bar{Q} = \int dz \lambda^\alpha d_\alpha + \int d\hat{z} \hat{\lambda}^\hat{\alpha} \hat{d}_{\hat{\alpha}}
\]  
(2.16)

using the auxiliary equations of motion for \( d_\alpha \) and \( \hat{d}_{\hat{\alpha}} \).

When \( r = \infty \), the left-invariant currents \( (J^c, J^\alpha, J^{\hat{\beta}}, J^{[a\hat{b}]} ) \) simplify to

\[
J^c = \Pi^c = \partial x^c + \theta \gamma^c \partial \theta + \hat{\theta} \gamma^c \partial \hat{\theta}, \quad J^\alpha = \partial \theta^\alpha, \quad J^{\hat{\beta}} = \partial \hat{\theta}^{\hat{\beta}}, \quad J^{[a\hat{b}]} = 0.
\]  
(2.17)

So the action of (2.7) reduces to

\[
S = \frac{1}{\Lambda} \int d^2 z \left[ \frac{1}{2} \eta_{cd} \Pi^c \Pi^d - d_\alpha \bar{\partial} \theta^\alpha - \hat{d}_{\hat{\alpha}} \bar{\partial} \hat{\theta}^{\hat{\alpha}} + w_\alpha \bar{\partial} \lambda^\alpha + \hat{w}_{\hat{\alpha}} \bar{\partial} \hat{\lambda}^{\hat{\alpha}} \right] + \frac{1}{4} \int d\sigma_3 (\gamma_{c\alpha \beta} \Pi^c \wedge \partial \theta^\alpha \wedge \partial \theta^\beta - \gamma_{c\alpha \beta} \Pi^c \wedge \partial \hat{\theta}^{\hat{\alpha}} \wedge \partial \hat{\theta}^{\hat{\beta}}),
\]  
(2.18)

which is the worldsheet action in a flat background using the pure spinor formalism. By defining

\[
p_\alpha = d_\alpha + ..., \quad \hat{p}_{\hat{\alpha}} = \hat{d}_{\hat{\alpha}} + ...
\]  
(2.19)

where ... are functions of \((x, \theta, \hat{\theta})\), this action can be written in quadratic form as

\[
S = \frac{1}{\Lambda} \int d^2 z \left[ \frac{1}{2} \eta_{cd} \partial x^c \partial x^d - p_\alpha \bar{\partial} \theta^\alpha - \hat{p}_{\hat{\alpha}} \bar{\partial} \hat{\theta}^{\hat{\alpha}} + w_\alpha \bar{\partial} \lambda^\alpha + \hat{w}_{\hat{\alpha}} \bar{\partial} \hat{\lambda}^{\hat{\alpha}} \right].
\]  
(2.20)

3. New Limit of Sigma Model

In the previous section, we constructed the flat space limit of the \( AdS_5 \times S^5 \) sigma model in which \( T_{c\alpha \hat{\beta}} \rightarrow 0 \) and \( T_{\alpha \beta}^c = \gamma_{\alpha \beta}^c \). In this section, we shall consider a different limit of the model in which \( T_{\alpha \beta}^c \rightarrow 0 \) and \( T_{c\alpha \hat{\beta}} = \gamma_{c\alpha \beta} \eta^{\beta \hat{\beta}} \). If one defines \( r \) as in (2.13), this formally corresponds to the limit \( r \rightarrow 0 \) of the \( AdS_5 \times S^5 \) background. However, since supergravity backgrounds are usually defined such that \( T_{\alpha \beta}^c = \gamma_{\alpha \beta}^c \), this limit cannot be identified with a conventional supergravity background.
3.1. $T_{\alpha \beta} \rightarrow 0$ limit

To construct the sigma model in this new limit, one needs to rescale the $PSU(2, 2|4)$ structure constants of (2.10) as

$$f^c_{\alpha \beta} = r \gamma^c_{\alpha \beta}, \quad f^c_{\hat{\alpha} \hat{\beta}} = r \gamma^c_{\alpha \beta},$$

$$f^{[ef]}_{\alpha \beta} = \pm r (\gamma^{[ef]}_{\alpha \beta} \gamma^{\gamma \beta} - \gamma^{[ef]}_{\alpha \beta} \gamma^{\gamma \beta}), \quad f^{[ef]}_{\hat{\alpha} \hat{\beta}} = \pm \delta^{[e \delta f]}_{\alpha \beta},$$

$$f^{[gh]}_{[cd][ef]} = \eta e [\delta^g_d \delta^h_f] - \eta e [\delta^g_d \delta^h_f] + \eta e [\delta^g_d \delta^h_f] - \eta e [\delta^g_d \delta^h_f]$$

Furthermore, to preserve the graded-antisymmetry of $f^C_{AB} g_{CD}$ under permutation of $[ABD]$, one needs to also rescale $g_{ab} = \eta_{ab}$ and $g_{[ab][cd]} = \eta_{[ab][cd]}$ to

$$g_{ab} = r^{-1} \eta_{ab}, \quad g_{[ab][cd]} = r^{-1} \eta_{[ab][cd]}, \quad (3.2)$$

When $r \rightarrow 0$, the structure constants $f^A_{\alpha \beta} \rightarrow 0$ which implies that the 32 fermionic isometries become abelian. In this limit, the $PSU(2, 2|4)$ coset $G$ splits into a bosonic coset $H^r_r$ for $r, r' = 1$ to 4 which parameterizes $AdS_5 = \frac{SU(2, 2)}{SO(4, 1)}$, a bosonic coset $\tilde{H}^j_{j'}$ for $j, j' = 1$ to 4 which parameterizes $S^5 = \frac{SU(4)}{SO(5)}$, and two fermionic matrices $\theta^{r j}$ and $\bar{\theta}^{j r}$ for $r, j = 1$ to 4. The index $r = 1$ to 4 labels a fundamental representation of the global $SU(2, 2)$, and the index $j = 1$ to 4 labels a fundamental representation of the global $SU(4)$. Furthermore, the index $r' = 1$ to 4 labels a spinor representation of the local $SO(4, 1)$, and the index $j' = 1$ to 4 labels a spinor representation of the local $SO(5)$. Note that $r'$ indices can be raised and lowered with an antisymmetric $SO(4, 1)$-invariant tensor $\varepsilon^{r's'}$, and $j'$ indices can be raised and lowered with an antisymmetric $SO(5)$-invariant tensor $\varepsilon^{j'k'}$. Under the 32 global fermionic isometries,

$$\delta \theta^{r j} = \alpha^{r j}, \quad \delta \bar{\theta}^{j r} = \alpha^{j r}, \quad \delta H^r_{r'} = 0, \quad \delta \tilde{H}^j_{j'} = 0, \quad (3.3)$$

where $\alpha^{r j}$ and $\alpha^{j r}$ are constant Grassmann parameters.

Since $g_{ab} = r^{-1} \eta_{ab}$ blows up when $r \rightarrow 0$, it is convenient to write the second-order kinetic term for the bosons in the first-order form as

$$\frac{1}{2 \Lambda} \int d^2 z [r^{-1} \eta_{[ab][cd]} (J^{[ab]} - A^{[ab]})(J^{[cd]} - A^{[cd]}) + r^{-1} \eta_{cd} J^c J^d] \quad (3.4)$$
\[
S = \frac{1}{\Lambda} \int d^2 z [(J^{[ab]} - \Lambda^{[ab]} P_{[ab]} + (\bar{J}^{[ab]} - \Lambda^{[ab]} P_{[ab]} + J^c \bar{P}_c + \bar{J}^c P^c \\
+ 2r(\eta^{[ab][cd]} P_{[ab]} \bar{P}_{[cd]} + \eta^{cd} P_c \bar{P}_d)] \]
\]

where \( [P_{[ab]}, \bar{P}_{[ab]}, P_c, \bar{P}_c] \) are auxiliary fields. So the \( AdS_5 \times S^5 \) sigma model action of (2.5) reduces in this limit \( r \to 0 \) to

\[
S = \frac{1}{\Lambda} \int d^2 z [(J^{[ab]} - \Lambda^{[ab]} P_{[ab]} + (\bar{J}^{[ab]} - \Lambda^{[ab]} P_{[ab]} + J^c \bar{P}_c + \bar{J}^c P^c \\
+ \frac{1}{4} \eta_{\alpha \beta}(\bar{J}^\beta \bar{J}^\alpha + \bar{J}^{\hat{\beta}} J^\alpha) + B + w_\alpha(\partial \lambda + \frac{1}{2} \Lambda^{[ab]} \gamma_{[ab]} \lambda)^\alpha + \bar{w}_\alpha(\partial \lambda + \frac{1}{2} A^{[ab]} \gamma_{[ab]} \lambda)^\alpha] \\
\]

where \( B \) is the Wess-Zumino-Witten term of (2.6). Since \( \int d^2 z B = \frac{1}{2} \int d^2 z \int d\sigma_3 (\gamma_{c\alpha \beta} J^c \wedge J^\alpha \wedge J^\beta - \gamma_{c\alpha \beta} J^c \wedge J^{\hat{\alpha}} \wedge J^{\hat{\beta}}) \), the Wess-Zumino-Witten term can be eliminated from the action by shifting \( P_c \) and \( \bar{P}_c \).

Furthermore, when \( r \to 0 \), the currents \( J^c \) and \( J^{[cd]} \) simplify to

\[
J^c = (H^{-1} \partial H)^{s'}_{s'}(\sigma^c)^{s'}_{s'}, \quad J^{[cd]} = (H^{-1} \partial H)^{s'}_{s'}(\sigma^{[cd]})^{s'}_{s'} \quad \text{when } c, d = 0 \to 4, \\
J^c = (\tilde{H}^{-1} \partial \tilde{H})^{k'}_{k'}(\sigma^c)^{k'}_{k'}, \quad J^{[cd]} = (\tilde{H}^{-1} \partial \tilde{H})^{k'}_{k'}(\sigma^{[cd]})^{k'}_{k'} \quad \text{when } c, d = 5 \to 9,
\]

where \( \sigma^c \) and \( \sigma^{[cd]} \) are \( 4 \times 4 \) Pauli matrices which generate an \( SU(2, 2) \) algebra when \( c = 0 \) to 4, and generate an \( SU(4) \) algebra when \( c = 5 \) to 9. Expressing the \( SO(9, 1) \) spinors \( J^\alpha \) and \( \tilde{J}^{\hat{\alpha}} \) in terms of \( SO(4, 1) \times SO(5) \) spinors as \( J^\alpha = J^{r'} j' \) and \( \tilde{J}^{\hat{\alpha}} = \tilde{J}^{\hat{r}'} j' \), one finds that when \( r \to 0 \), \( J^{r'} j' \) and \( \tilde{J}^{\hat{r}'} j' \) simplify to

\[
J^{r'} j' = (H^{-1})^{r'}_r (\tilde{H}^{-1})^{j'}_j \partial \theta^{rj} + \epsilon^{r's'} \epsilon^{j'k'} H^{s'}_s \tilde{H}^{j'}_k \partial \bar{\theta}_{jr}, \\
\tilde{J}^{\hat{r}'} j' = (H^{-1})^{r'}_r (\tilde{H}^{-1})^{j'}_j \partial \theta^{rj} - \epsilon^{r's'} \epsilon^{j'k'} H^{s'}_s \tilde{H}^{j'}_k \partial \bar{\theta}_{jr}.
\]

Plugging these currents into (3.5), one finds that the action simplifies to

\[
S = \frac{1}{\Lambda} \int d^2 z [(J^{[ab]} - \Lambda^{[ab]} P_{[ab]} + (\bar{J}^{[ab]} - \Lambda^{[ab]} P_{[ab]} + J^c \bar{P}_c + \bar{J}^c P^c \\
+ \partial \bar{\theta}_{jr} \partial \theta^{rj} + w_\alpha(\partial \lambda + \frac{1}{2} \Lambda^{[ab]} \gamma_{[ab]} \lambda)^\alpha + \bar{w}_\alpha(\partial \lambda + \frac{1}{2} A^{[ab]} \gamma_{[ab]} \lambda)^\alpha].
\]

12
3.2. Twistor-like variables

The final step in simplifying this action is to express the pure spinors in $SO(4,1) \times SO(5)$ notation as $\lambda^\alpha = \lambda^{r'j'}$ and $\tilde{\lambda}^\alpha = \tilde{\lambda}^{r'j'}$ and to define the new variables $Z^{rj}$ and $\overline{Z}_{jr}$ as

$$Z^{rj} = H_r^p \tilde{H}_j^q \lambda^{r'j'}, \quad \overline{Z}_{jr} = (H^{-1})_r^p (\tilde{H}^{-1})_j^q \tilde{\lambda}_{r'q'},$$

(3.9)

where $\tilde{\lambda}_{r'q'} = \epsilon_{r'k'} \epsilon_{r's'} \tilde{\lambda}^{s'k'}$. Note that $Z^{rj}$ and $\overline{Z}_{jr}$ are twistor-like variables since they transform covariantly under the global $SU(2,2) \times SU(4)$ isometries and since they are constructed out of the pure spinors and the ten $x$'s parameterized by the cosets $H$ and $\tilde{H}$. Similarly, one can define the conjugate twistor-like variables $Y_{jr}$ and $\overline{Y}^{rj}$ as

$$Y_{jr} = (H^{-1})_r^p (\tilde{H}^{-1})_j^q w_{j'r'}, \quad \overline{Y}^{rj} = H_r^p \tilde{H}_j^q \tilde{w}_{r'q'},$$

(3.10)

where $w_{\alpha} = w_{j'r'}$ and $\tilde{w}_{\alpha} = \epsilon_{j'k'} \epsilon_{r's'} \tilde{w}^{s'k'}$ are the original conjugate pure spinor variables written in $SO(4,1) \times SO(5)$ notation.

Using

$$Y_{jr} \overline{Z}^{rj} = w_{\alpha} \partial \lambda^\alpha + (H^{-1} \partial H)^{r'})_{s'} w_{j'r'} \lambda^{s'j'} + (\tilde{H}^{-1} \partial \tilde{H})_{s'} \tilde{w}_{j'r'} \tilde{\lambda}^{s'j'},$$

(3.11)

one finds that

$$w_{\alpha} \partial \lambda^\alpha = Y_{jr} \overline{Z}^{rj} - (w_{\sigma c}) \overline{J}^c - \frac{1}{2} (w_{\sigma [cd]} \lambda) J^{[cd]}$$

(3.12)

where $(w_{\sigma c} \lambda) = w_{j'r'} \sigma^{r'} j' j$ and $(w_{\sigma [cd]} \lambda) = w_{j'r'} \sigma^{[r'} j' j)_{s'} \lambda^{s'j'}$ for $c = 0$ to $4$, and $(w_{\sigma c} \lambda) = w_{j'r'} \sigma^{r'} j' j \tilde{\lambda}^{s'k'}$ and $(w_{\sigma [cd]} \lambda) = w_{j'r'} \sigma^{[r'} j' j)_{s'} \tilde{\lambda}^{s'k'}$ for $c = 5$ to $9$. Similarly,

$$\tilde{w}_{\alpha} \partial \tilde{\lambda}^\alpha = \overline{Y}^{rj} \partial Z_{jr} - (\tilde{w}_{\sigma c} \tilde{\lambda}) J^c - \frac{1}{2} (\tilde{w}_{\sigma [cd]} \tilde{\lambda}) J^{[cd]}.$$  

(3.13)

So after defining

$$P^{[c} = P^{c} - (w_{\sigma c} \lambda), \quad \overline{P}^{[c} = \overline{P}^{c} - (\tilde{w}_{\sigma c} \tilde{\lambda}),$$

(3.14)

$$P^{[cd]} = P^{[cd]} - \frac{1}{2} (w_{\sigma [cd]} \lambda), \quad \overline{P}^{[cd]} = \overline{P}^{[cd]} - \frac{1}{2} (\tilde{w}_{\sigma [cd]} \tilde{\lambda}),$$

one can write the action of (3.8) as

$$S = \frac{1}{\Lambda} \int d^2 z [(J^{[ab]} - A^{[ab]}) \overline{P}^{[ab]} + (\overline{J}^{[ab]} - \overline{A}^{[ab]}) P^{[ab]} + J^{[c}] \overline{P}^c + \overline{J}^c P^{[c]}$$

$$+ \overline{\theta}_{jr} \partial \theta^{rj} + Y_{jr} \partial \overline{Z}^{rj} + \overline{Y}^{rj} \partial Z_{jr}].$$

(3.15)
The shift of (3.14) implies that under the gauge transformation $\delta w_\alpha = \xi^c (c_\gamma \lambda)_\alpha$ and $\delta \hat{w}_{\hat{\alpha}} = \hat{\xi}^c (\hat{c}_\gamma \hat{\lambda})_{\hat{\alpha}}$ of (2.2), $P'_c$ and $\bar{P}'_c$ must transform as

$$\delta P'_c = \xi^c \epsilon_r s' \epsilon_j t' \lambda^{r' j} \lambda^{s' k'} = \xi^c (\lambda^0 1234 \lambda),$$  
(3.16)

and

$$\delta \bar{P}'_c = \hat{\xi}^c \epsilon' r' s' \epsilon' j' \hat{\lambda}_{r' j} \hat{\lambda}_{s' k'} = \hat{\xi}^c (\hat{\lambda}^0 1234 \hat{\lambda}).$$

So assuming that $(\lambda^0 1234 \lambda)$ and $(\hat{\lambda}^0 1234 \hat{\lambda})$ are non-zero, one can use this invariance to gauge-fix $P^{nc} = \bar{P}^{nc} = 0$. Furthermore, integrating out $A^{[ab]}$ and $\bar{A}^{[ab]}$ implies that $P'^{[ab]} = \bar{P}'^{[ab]} = 0$.

So finally, one can write the action in quadratic form as

$$S = \frac{1}{\Lambda} \int d^2 z [\partial \theta_{jr} \partial \theta^{rj} + Y_{jr} \partial Z^{rj} + \bar{Y}^{rj} \partial \bar{Z}_{jr}].$$  
(3.17)

Instead of the original action containing ten $x$’s and 22 left and right-moving pure spinors, (3.17) contains 16 left-moving and 16 right-moving unconstrained bosonic spinors. So the second-order action for $x$ has been converted into a first-order action for ten left and right-moving bosons which effectively removes the constraint on the pure spinors. The removal of the pure spinor constraint is related to the fact that $T_{\alpha \beta}^c = 0$ in this background. Since the BRST operator acts as $Q = \lambda^\alpha \nabla_\alpha$, $Q^2 = \lambda^\alpha \lambda^\beta \{ \nabla_\alpha, \nabla_\beta \} = \lambda^\alpha \lambda^\beta T_{\alpha \beta} A \nabla_A$. When $T_{\alpha \beta}^c = \gamma^c_{\alpha \beta}$, the pure spinor constraint $\lambda^c \gamma^c \lambda = 0$ is required for $Q$ to be nilpotent. However, when $T_{\alpha \beta}^c = 0$, the nilpotence of $Q$ does not require $\lambda^a$ to satisfy the pure spinor constraint.

3.3. $N=2$ worldsheet supersymmetry

In terms of the variables $(\theta^{rj}, \bar{\theta}_{jr}, Z^{rj}, \bar{Z}_{jr}, Y_{jr}, \bar{Y}^{rj})$, the BRST transformations are

$$\delta \theta^{rj} = \epsilon Z^{rj}, \quad \delta \bar{\theta}_{jr} = \epsilon \bar{Z}_{jr}, \quad \delta Y_{jr} = \epsilon \partial \bar{\theta}_{jr}, \quad \delta \bar{Y}^{rj} = \epsilon \partial \theta^{rj},$$  
(3.18)

which are generated by $Q + \bar{Q}$ where

$$Q = \int dz Z^{rj} \partial \bar{\theta}_{jr}, \quad \bar{Q} = \int dz \bar{Z}_{jr} \partial \theta^{rj}.$$  
(3.19)

Unlike in a flat background where it is difficult to construct $b$ and $\bar{b}$ ghosts satisfying $\{Q, b\} = T$ and $\{\bar{Q}, \bar{b}\} = \bar{T}$, it is easy to construct $b$ and $\bar{b}$ ghosts in this background as

$$b = Y_{jr} \partial \theta^{rj}, \quad \bar{b} = \bar{Y}^{rj} \partial \bar{\theta}_{jr},$$  
(3.20)
\[ T = \partial \theta^r_j \partial \theta^r_j + Y^r_j \partial Z^r_j, \quad \overline{T} = \overline{\partial} \theta^r_j \overline{\partial} \theta^r_j + \overline{Y}^r_j \overline{\partial} Z^r_j. \]  

(3.21)

Since \( Y^r_j \) and \( \overline{Y}^r_j \) have conformal weight \((1,0)\) and \((0,1)\), the action of (3.17) has A-twisted N=(2,2) supersymmetry and can be interpreted as a topological A-model. This topological A-model can be expressed in N=(2,2) superspace by combining the component fields into the chiral and antichiral superfields

\[ \Theta^r_j = \theta^r_j + \kappa^+ Z^r_j + \kappa^- Y^r_j + \kappa^+_+ \kappa^- f^r_j, \]  

(3.22)

\[ \overline{\Theta}^r_j = \overline{\theta}^r_j + \overline{\kappa}^+ Y^r_j + \overline{\kappa}^- Z^r_j + \overline{\kappa}^+_+ \overline{\kappa}^- \overline{f}^r_j, \]

where \((\kappa^+, \overline{\kappa}^+)\) and \((\kappa^-, \overline{\kappa}^-)\) are the left and right-moving N=(2,2) Grassmann parameters, and \((f^r_j, \overline{f}^r_j)\) are auxiliary fields.

In terms of \( \Theta^r_j \) and \( \overline{\Theta}^r_j \), the action of (3.17) is

\[ S = \frac{1}{\Lambda} \int d^2 z \int d^4 \kappa \overline{\Theta}^r_j \Theta^r_j, \]  

(3.23)

and the global bosonic isometries act as

\[ \delta \Theta^r_j = i \Lambda^r_s \Theta^s_j + i \Theta^r_k \Omega^k_j + i \Sigma \Theta^r_j, \quad \delta \overline{\Theta}^r_j = -i \overline{\Theta}^r_s \Lambda^s_r - i \Omega^k_j \overline{\Theta}^r_k - i \Sigma \overline{\Theta}^r_j, \]  

(3.24)

where \((\Lambda^r_s, \Omega^k_j, \Sigma)\) are constant parameters satisfying \( \Lambda^r_r = \Omega^k_k = 0 \). Note that in addition to the \( SU(2,2) \times SU(4) \) bosonic isometries, there is an additional “bonus” \( U(1) \) symmetry parameterized by \( \Sigma \). Under the fermionic isometries of (3.3), the superfields transform as

\[ \delta \Theta^r_j = \alpha^r_j, \quad \delta \overline{\Theta}^r_j = \overline{\alpha}^r_j. \]  

(3.25)

4. Non-Linear Topological A-Model

To compute the physical states of the linear topological A-model of (3.23), it will be useful to define a non-linear topological A-model which reduces to the linear model of (3.23) in a certain large-radius limit. In the non-linear model, the \( SU(2,2) \times SU(4) \times U(1) \) bosonic isometries will combine with the 32 fermionic isometries to form an \( SU(2,2|4) \) supergroup. Since this supergroup includes the \( PSU(2,2|4) \) isometries of the \( AdS_5 \times S^5 \) background, it is tempting to try to identify this non-linear topological A-model at large but finite radius with the \( AdS_5 \times S^5 \) sigma model at small but non-zero \( T_{\alpha \beta c} \). However, this identification does not seem possible since when \( T_{\alpha \beta c} \) is non-zero, the \( AdS_5 \times S^5 \) sigma model contains a Wess-Zumino-Witten term which is antisymmetric under exchange of \( z \) and \( \overline{z} \) and which breaks \( SU(2,2|4) \) down to \( PSU(2,2|4) \). On the other hand, the non-linear topological A-model is symmetric under exchange of \( z \) and \( \overline{z} \) and preserves \( SU(2,2|4) \) invariance. So it appears that the \( AdS_5 \times S^5 \) sigma model and the non-linear topological A-model can only be identified in the limit where \( T_{\alpha \beta c} = 0 \) in the \( AdS_5 \times S^5 \) model and where the radius is infinite in the non-linear model.
4.1. Superspace action

Although the non-linear topological A-model has both N=(2,2) worldsheet supersymmetry and SU(2,2|4) invariance, both these symmetries can not be simultaneously made manifest. The worldsheet supersymmetry can be made manifest by expressing the non-linear action in superspace as

\[ S = \frac{1}{\Lambda} \int d^2 z d^4 \kappa [\Theta_{rj} \Theta^{jr} - \frac{1}{2R^2} \Theta_{rj} \Theta^{js} \Theta_{sk} \Theta^{kr} + \frac{1}{3R^4} \Theta_{rj} \Theta^{js} \Theta_{sk} \Theta^{kt} \Theta_{tl} \Theta^{lr} + \ldots] \]

\[ = \frac{R^2}{\Lambda} \int d^2 z d^4 \kappa \text{Tr} \log(1 + \frac{1}{R^2} \Theta \Theta) \]

where \( \Theta_{rj} \) and \( \Theta_{jr} \) are the same superfields as in (3.22), and \( R \) is the radius of this model which is unrelated to the AdS\(_5 \times S^5 \) radius \( r \). In the limit \( R \to \infty \), this non-linear model reduces to the linear topological A-model of (3.23). The non-linear action of (4.1) is invariant under the same SU(2,2) \( \times SU(4) \times U(1) \) transformations as (3.24), but the fermionic isometries of (3.25) are modified to

\[ \delta \Theta_{rj} = \alpha_{rj} + \frac{1}{R^2} \Theta^{rk} \Theta_{ks} \Theta_{sj}, \quad \delta \Theta_{jr} = \alpha_{jr} + \frac{1}{R^2} \Theta_{js} \Theta_{kr}, \]

(4.2)

which close with the bosonic isometries into the SU(2,2|4) supergroup.

4.2. Coset action

These SU(2,2|4) isometries can be made manifest by rescaling \( \Theta_{rj} \to R \Theta_{rj} \) and \( \Theta_{jr} \to R \Theta_{jr} \) and writing the non-linear action in terms of the component fields \( (\theta^{rj}, \theta_{jr}, Z^{rj}, \bar{Z}_{jr}, Y_{jr}, \bar{Y}_{rj}) \) using a coset space construction. The coset \( G \) will be defined to take values in \( PSU(2,2|4) / SU(2) \times SU(4) \), and since the coset has only fermionic elements, \( G \) can be gauged to the form

\[ G_{s}^{k} = \delta_{s}^{k}, \quad G_{s}^{r} = \delta_{s}^{r}, \quad G^{rj} = \Theta^{rj}, \quad G_{jr} = \Theta_{jr}. \]

(4.3)

In terms of the left-invariant currents \( J^{A} = (G^{-1} \partial G)^{A} \) and \( \bar{J}^{A} = (G^{-1} \bar{G})^{A} \) where \( A \) is an SU(2,2|4) index, the action is

\[ S = \frac{R^2}{\Lambda} \int d^2 z [(\bar{J} - \bar{A})_{s}^{r}(J - A)^{s}_{r} - (\bar{J} - \bar{A})_{j}^{k}(J - A)^{j}_{k}] \]

\[ + \bar{J}_{jr} J^{rj} + Y_{jr} (\bar{\partial} Z + \bar{A} Z)^{rj} + \bar{Y}^{rj} (\partial Z - A Z)_{jr} \]

(4.4)
\[
\frac{R^2}{\Lambda} \int d^2z [J_{jr} J^{rj} + Y_{jr} \nabla Z^{rj} + \overline{Y}^{rj} \nabla \overline{Z}_{jr} + Y_{jr} Z^{s} k^{r} \overline{Z}_{ks} + \overline{Z}^{s j} - Z^{rj} Y_{js} \overline{Z}^{ks} \overline{Z}_{kr}] \quad (4.5)
\]

where \((A^A, \overline{A}^A)\) are \(SU(2, 2) \times SU(4)\) gauge fields, \(\nabla Z^{rj} = \partial Z^{rj} + J^r s \overline{Z}^{js} + J^r j \overline{Z}^{rk}\), and \(\nabla \overline{Z}_{rj} = \partial \overline{Z}_{rj} - J^r s \overline{Z}^{sj} - J^r j \overline{Z}^{rk}\). Note that

\[
\overline{J}^{rj} J^{rj} - J_{jr} \overline{J}^{rj} = \partial \overline{J}_{U(1)} - \overline{\partial} J_{U(1)} \quad (4.6)
\]

is a total derivative where \(J_{U(1)}\) is the “bonus” \(U(1)\) current, so the term \(\int d^2z \overline{J}^{rj} J^{rj}\) is symmetric under exchange of \(z\) and \(\overline{z}\).

Although \(SU(2, 2|4)\) invariance is manifest in the action of (4.4), \(N=(2,2)\) worldsheet supersymmetry is not manifest. Nevertheless, one can easily construct the twisted \(N=(2,2)\) worldsheet supersymmetry generators as

\[
Q = \int dz Z^{rj} J_{jr}, \quad \overline{Q} = \int dz \overline{Z}^{rj} \overline{J}^{rj}, \quad b = Y_{jr} J^{rj}, \quad \overline{b} = \overline{Y}^{rj} \overline{J}^{rj}. \quad (4.7)
\]

After parameterizing \(G\) as in (4.3), the action of (4.5) coincides with the superspace action of (4.1) after integrating out the auxiliary fields \(f^{rj}\) and \(\overline{f}_{jr}\).

### 4.3. One-loop conformal invariance

To show that the non-linear topological A-model has no one-loop conformal anomaly, one can either use the superspace version of the action of (4.1) and compute \(\log \det(\partial \overline{J} K)\) where \(K\) is the Kahler potential, or one can use the coset version of the action of (4.5) and compute the anomaly with the background field method of [20] and [4]. Absence of this anomaly is necessary for the topological twisting to be consistent at the quantum level.

Using the superspace action of (4.1), \(K = Tr \log (1 + \overline{\Theta} \Theta)\) implies that

\[
\partial_{ks} \overline{J}^{rj} K = \partial_{ks} [\Theta^l [(1 + \overline{\Theta} \Theta)^{-1}]^j_l] \quad (4.8)
\]

\[
= \delta^r_s [(1 + \overline{\Theta} \Theta)^{-1}]_k^j - \Theta^l [(1 + \overline{\Theta} \Theta)^{-1}]_l^m \overline{\Theta}^m_s [(1 + \overline{\Theta} \Theta)^{-1}]_k^j
\]

\[
= [(1 + \overline{\Theta} \Theta)^{-1}]_s^j [(1 + \overline{\Theta} \Theta)^{-1}]_k^j.
\]

So there is no conformal anomaly since

\[
\log \det(\partial_{ks} \overline{J}^{rj} K) = \log \det[(1 + \Theta \overline{\Theta})^{-1}] + \log \det[(1 + \overline{\Theta} \Theta)^{-1}] \quad (4.9)
\]

\[
= -Tr \log(1+\Theta \overline{\Theta}) - Tr \log(1+\overline{\Theta} \Theta) = -Tr \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (\Theta \overline{\Theta})^n + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (\overline{\Theta} \Theta)^n = 0
\]
where we have used that $Tr[(\Theta\bar{\Theta})^n] = -Tr[(\bar{\Theta}\Theta)^n]$ for $n > 0$.

Using the background field method for the coset action of (4.5), the matter sector of $\int d^2z J_{jr} J^{jr}$ contributes no conformal anomaly since, when $G/H$ is a symmetric space, the $G/H$ coset model has the same conformal anomaly as the principal chiral model based on $G$ [20]. In this case, $PSU(2,2|4)/(SU(2,2) \times SU(4))$ is a symmetric space, and the principal chiral model based on $PSU(2,2|4)$ has no conformal anomaly [21].

Furthermore, the ghost sector of (4.5) contributes no conformal anomaly because of a cancellation between the $Y_{jr} \nabla Z^{jr} + \bar{Y}^{jr} \nabla \bar{Z}_{jr}$ contribution and the $Y_{jr} Z^{rk} \bar{Z}_{ks} \bar{Y}^{sj} - Z^{rj} Y_{js} \bar{Y}^{sk} \bar{Z}_{kr}$ contribution. As shown in [4], the $Y_{jr} \nabla Z^{jr} + \bar{Y}^{jr} \nabla \bar{Z}_{jr}$ term contributes an anomaly proportional to the dual coxeter number of the group, and $Y_{jr} Z^{rk} \bar{Z}_{ks} \bar{Y}^{sj} - Z^{rj} Y_{js} \bar{Y}^{sk} \bar{Z}_{kr}$ contributes an anomaly proportional to the level $k$ in the OPE of the Lorentz currents. In the $AdS_5 \times S^5$ case, the relevant group was $SO(4,1) \times SO(5)$ with dual coxeter number 3, which cancels the level $k = -3$ in the OPE of the Lorentz currents constructed from pure spinors [4]. In this case, the relevant group is $SU(2,2) \times SU(4)$ with dual coxeter number 4, which cancels the level $k = -4$ in the OPE of Lorentz currents constructed from unconstrained bosonic spinors.

4.4. Open string sector

Just as $d=3$ Chern-Simons theory is described by the open string sector of a topological A-model [13], it will be shown that the open string sector of the non-linear topological A-model of (4.1) describes N=4 d=4 super-Yang-Mills. The open string boundary condition for the A-model of (4.1) will be defined as

$$\overline{\Theta}_{jr} = \delta_{jk} \epsilon_{rs} \Theta^{sk}$$

(4.10)

where $\epsilon_{rs}$ is an antisymmetric tensor which breaks $SU(2,2)$ to $SO(3,2)$ and $\delta_{jk}$ is a symmetric tensor which breaks $SU(4)$ to $SO(4)$. The boundary condition of (4.10) is similar to the open string boundary condition for the Chern-Simons topological string which is $\overline{X}_I = \delta_{IJ} X^J$ for $I, J = 1$ to 3. Note that the open string boundary for the A-model is defined by

$$z = \overline{z}, \quad \kappa_+ = \overline{\kappa}_-, \quad \overline{\kappa}_+ = \kappa_-,$$

(4.11)

so (4.10) implies that

$$\overline{\theta}_{jr} = \delta_{jk} \epsilon_{rs} \theta^{sk}, \quad \overline{Z}_{jr} = \delta_{jk} \epsilon_{rs} Z^{sk}, \quad Y_{jr} = \delta_{jk} \epsilon_{rs} \overline{Y}^{sk}.$$

(4.12)
The boundary condition of (4.10) breaks half of the fermionic isometries and reduces the $SU(2,2|4)$ supergroup of isometries to the supergroup $OSp(4|4)$. This supergroup contains $SO(3,2) \times SO(4)$ bosonic isometries and 16 fermionic isometries, and is the N=4 supersymmetry algebra on $AdS_4$.

To show that the BRST cohomology of open string states in this model describes N=4 d=4 super-Yang-Mills, it will be assumed that, as in the topological A-model for Chern-Simons, the cohomology in the closed string sector is trivial. This assumption is reasonable since N=(2,2) worldsheet supersymmetric D-terms are BRST-trivial, and there are naively no global obstructions to writing supersymmetric expressions involving fermionic superfields as superspace D-terms. However, since the A-model of (4.1) is constructed from fermionic superfields in a non-conventional manner, there might be unexpected subtleties in the model which invalidate this assumption.

With this assumption, the cohomology computation in the open string sector is independent of $\Lambda$ and $R$ in (4.1), and can be performed at $\Lambda = 0$ where only the constant modes of $\Theta^{rj}$ contribute. Furthermore, if the closed string sector has no cohomology, the open string physical states should be independent of $SU(2,2|4)/OSp(4|4)$ rotations which modify the D-brane boundary conditions of (4.10). So although only $OSp(4|4)$ symmetry is manifest in the open topological A-model, the physical spectrum should be invariant under the full $SU(2,2|4)$ supergroup.

After imposing the open string boundary condition of (4.10) and restricting to constant worldsheet modes, the superspace action of (4.1) reduces to

$$S = R^2 \int d\tau d^2\kappa \left[D_+ \Theta (1 + \Theta \Theta)^{-1} D_- \Theta (1 + \Theta \Theta)^{-1}\right]$$

(4.13)

where $\Theta_{jr} = \delta_{jk} \epsilon_{rs} \Theta^{sk}$ is an N=2 superfield whose component expansion is

$$\Theta^{rj} = \theta^{rj} + \kappa_+ Y^{rj} + \kappa_- Z^{rj} + \kappa_+ \kappa_- f^{rj},$$

(4.14)

and $D_\pm = \partial / \kappa^\pm + \kappa^\mp \partial / \partial \tau$. Alternatively, using the coset construction, the action of (4.13) reduces to

$$S = R^2 \int d\tau [\epsilon_{rs} J^{rj} J^{sj} + (J - A)^{s}_{\bar{s}}(J - A)^{r}_{\bar{r}} - (J - A)^{s}_{\bar{s}}(J - A)^{r}_{\bar{r}}] + Y_{jr} \left(\partial / \partial \tau Z + AZ\right)^{rj}$$

(4.15)

$$= R^2 \int d\tau [\epsilon_{rs} J^{rj} J^{sj} + Y_{jr} (\nabla Z)^{rj} + (YZ)^{k}_{r}(YZ)^{j}_{k} - (YZ)^{s}_{r}(YZ)^{s}_{r}],$$

19
where \( J^A = (G^{-1} \frac{\partial}{\partial \tau} G)^A \) are left-invariant currents taking values in the Lie algebra of \( OSp(4|4) \), \( G(\theta) \) takes values in the coset \( \frac{OSp(4|4)}{SO(3,2) \times SO(4)} \), \( A = ([rs], [jk], jr) \) labels the \( OSp(4|4) \) generators, \( r = 1 \) to \( 4 \) labels \( Sp(4) \) indices which are raised and lowered using the antisymmetric metric \( \epsilon^{rs} \), \( j = 1 \) to \( 4 \) labels \( SO(4) \) indices which are raised and lowered using \( \delta_{jk} \), \( A^A \) is an \( Sp(4) \times SO(4) \) worldline gauge field, and \( (\nabla Z)^{rj} = \frac{\partial}{\partial \tau} Z^{rj} + J^r_s Z^{sj} + J^j_k Z^{rk} \).

The N=2 worldline supersymmetry generators for this action are

\[
Q = Z^{rj} J_{jr}, \quad b = Y_{jr} J^{rj}.
\]

(4.16)

5. Cohomology of Open Topological A-Model

Before showing that the BRST cohomology of the worldline action of (4.15) describes N=4 d=4 super-Yang-Mills, it will be useful to review the superspace description of on-shell super-Yang-Mills.

5.1. On-shell super-Yang-Mills in superspace

In ten flat dimensions, on-shell super-Yang-Mills is described by a spinor superfield \( A_\alpha(x, \theta) \) where \( \alpha = 1 \) to \( 16 \). This superfield can be understood as a spinor connection which covariantizes the superspace derivative \( D_\alpha = \frac{\partial}{\partial \theta^\alpha} + \gamma^{c}_{\alpha \beta} \frac{\partial}{\partial x^c} \) to \( \nabla_\alpha = D_\alpha - A_\alpha(x, \theta) \). Since \( \{D_\alpha, D_\beta\} = \gamma^{c}_{\alpha \beta} \frac{\partial}{\partial x^c} \), it is natural to impose that \( A_\alpha \) is defined such that

\[
\{\nabla_\alpha, \nabla_\beta\} = \gamma^{c}_{\alpha \beta} \nabla_c
\]

(5.1)

where \( \nabla_c = \frac{\partial}{\partial x^c} - A_c(x, \theta) \) and \( A_c(x, \theta) \) is a vector connection whose \( \theta = 0 \) component is the usual gauge field.

These spinor and vector superspace connections are defined up to the gauge transformation

\[
\delta A_\alpha = \nabla_\alpha \Omega, \quad \delta A_c = \nabla_c \Omega
\]

(5.2)

where \( \Omega \) is a scalar superfield, and the Bianchi identity of (5.1) implies that

\[
D_\alpha A_\beta + D_\beta A_\alpha - \{A_\alpha, A_\beta\} = \gamma^{c}_{\alpha \beta} A_c.
\]

(5.3)

Equation (5.3) implies that \( A_c \) is determined from \( A_\alpha \) and that \( A_\alpha \) must satisfy the constraint

\[
(\gamma^{abde}_{\alpha \beta}) \alpha \beta (D_\alpha A_\beta - \frac{1}{2} \{A_\alpha, A_\beta\}) = 0
\]

(5.4)
for any five-form direction $abcde$.

The constraint of \( [5,4] \) together with the gauge invariance of \( [3,2] \) implies that \( A_\alpha(x, \theta) \) can be gauged to the form

\[
A_\alpha(x, \theta) = a_e(x)(\gamma^e \theta)_\alpha + \xi^\beta(x)(\gamma^e \theta)_{\beta}(\gamma^c \theta)_\alpha + \ldots
\]

where \( a_e(x) \) and \( \xi^\alpha(x) \) are the on-shell gluon and gluino, and \( \ldots \) involves spacetime derivatives of \( a_e(x) \) and \( \xi^\alpha(x) \).

To describe N=4 d=4 super-Yang-Mills, one simply decomposes the d=10 vectors and spinors into d=4 vectors, scalars and spinors in the usual manner as

\[
\text{spinors into d=4 vectors, scalars and spinors in the usual manner as}\]

\[
\theta^\alpha \rightarrow (\theta^{\mu j}, \overrightarrow{\theta}_j), \quad A_\alpha \rightarrow (A_{\mu j}, \overrightarrow{A}_j), \quad A_e \rightarrow (A_m, A_{[jk]})
\]

where \( m = 0 \) to 3, \( \mu, \tilde{\mu} = 1 \) to 2, \( j = 1 \) to 4, and \( [jk] = 1 \) to 6. The corresponding covariant spinor and vector derivatives satisfy the Bianchi identities

\[
\{ \nabla_{\mu j}, \nabla^k \} = \delta^k_j \sigma^m_{\mu \tilde{\mu}} \nabla_m, \quad \{ \nabla_{\mu j}, \nabla_{\nu k} \} = \epsilon_{\mu \nu \lambda} A_{[jk]}, \quad \{ \nabla^j, \nabla^k \} = \frac{1}{2} \epsilon_{\mu \nu \dot{\lambda}} \epsilon^{hi\dot{j}} A_{[hi]},
\]

where \( \sigma^m_{\mu \tilde{\mu}} \) are the d=4 Pauli matrices. So the N=4 d=4 spinor connections satisfy the equations

\[
D_\mu j = \Delta_\mu j A - \{ A_{\mu j}, \overrightarrow{A}_\mu \} = \delta^k_j \sigma^m_{\mu \tilde{\mu}} A_m,
\]

\[
D_{(\mu j} A_{\nu k)} - \{ A_{\mu j}, A_{\nu k} \} = \epsilon_{\mu \nu \lambda} A_{[jk]}, \quad \overrightarrow{D}^{(\mu j} - \{ \overrightarrow{A}_{\mu j}, \overrightarrow{A}_{\nu k} \} = \frac{1}{2} \epsilon_{\mu \nu \dot{\lambda}} \epsilon^{hi\dot{j}} A_{[hi]},
\]

and the gauge transformations

\[
\delta A_{\mu j} = \nabla_{\mu j} \Omega, \quad \delta \overrightarrow{A}_\mu = \overrightarrow{\nabla}_\mu \Omega, \quad \delta A_m = \nabla_m \Omega.
\]

Since N=4 d=4 super-Yang-Mills is superconformally invariant, the Bianchi identities of \( (5,7) \) are valid both in flat d=4 Minkowski space and in AdS4 space. The only difference is that in a flat background, the superspace derivatives are

\[
D_\mu j = \frac{\partial}{\partial \theta^{\mu j}} + \overrightarrow{\sigma}_j^m \sigma^m_{\mu \tilde{\mu}} \frac{\partial}{\partial x_m}, \quad \overrightarrow{D}^i = \frac{\partial}{\partial \theta^i} + \theta^{\mu j} \sigma^m_{\mu \tilde{\mu}} \frac{\partial}{\partial x_m}, \quad D_m = \frac{\partial}{\partial x_m},
\]

whereas in an AdS4 background,

\[
D_A = E_A^M \frac{\partial}{\partial Y^M} + w_A^{[mn]} M_{[mn]} + w_A^{[jk]} M_{[jk]}
\]

where \( E_A^M \) is the AdS4 super-vierbein, \( Y^M = (y^m, \xi^{\mu j}, \overrightarrow{\xi}_j) \) are the AdS4 superspace coordinates, \( w_A \) is the AdS4 super-connection, and \( M_{[mn]} \) and \( M_{[jk]} \) are the SO(3,1) and SO(4) generators. As will be shown in subsection 5.3, the AdS4 super-vierbein and super-connection can be naturally constructed from a supercoset \( \frac{OSp(4|4)}{SO(3,1) \times SO(4)} \) in the same manner as the AdS5 × S5 super-vierbein and super-connection are constructed from the \( \frac{PSU(2,2|4)}{SO(4,1) \times SO(5)} \) supercoset.

21
5.2. First-quantized description of \( N=4 \) \( d=4 \) super-Yang-Mills

Just as \( d=3 \) Chern-Simons can be obtained by quantizing the worldline action 
\[
\int d\tau \left( \frac{1}{2} \partial x^I \partial x^I + \bar{\psi}_I \frac{\partial}{\partial \tau} \psi^I \right)
\]
with the BRST operator \( Q = \psi^I \frac{\partial}{\partial \tau} x_I \) where \( I = 1 \) to 3, 
\( d=10 \) super-Yang-Mills can be obtained by quantizing the worldline action 
\[
\int d\tau \left( \frac{1}{2} \partial x^c \partial x^c + p_\alpha \frac{\partial}{\partial \tau} \theta^\alpha + w_\alpha \frac{\partial}{\partial \tau} \lambda^\alpha \right)
\]
with the BRST operator \( Q = \lambda^\alpha d_\alpha \) where \( d_\alpha = p_\alpha + (\gamma_c \theta)_\alpha \frac{\partial}{\partial \tau} x^c \) and 
\( \lambda^\alpha \) is a pure spinor satisfying \( \lambda \gamma^c \lambda = 0 \) for \( c = 0 \) to 9 [15][23].

At ghost-number one, the states in the cohomology of \( Q = \lambda^\alpha d_\alpha \) are described by 
\( V = \lambda^\alpha A_\alpha(x, \theta) \) where \( A_\alpha(x, \theta) \) is a spinor superfield. \( QV = 0 \) implies that \( \lambda^\alpha \lambda^\beta D_\beta A_\alpha = 0 \) where 
\( D_\alpha = \frac{\partial}{\partial \theta^\alpha} + (\gamma_c \theta) \frac{\partial}{\partial x^c} \), and since \( \lambda \gamma^c \lambda = 0 \), \( \lambda^\alpha \lambda^\beta D_\beta A_\alpha = 0 \) implies that \( D_\alpha A_\beta + D_\beta A_\alpha = \gamma^c \epsilon_{\alpha\beta} A_c \) for some \( A_c \). Also, \( \delta V = Q \Omega \) implies that \( \delta A_\alpha = D_\alpha \Omega \). By comparing with (5.3) and (5.2), one sees that \( A_\alpha(x, \theta) \) describes the linearized on-shell \( d=10 \) super-Yang-Mills fields.

The structure of \( V = \lambda^\alpha A_\alpha(x, \theta) \) in \( d=10 \) super-Yang-Mills using the BRST operator \( Q = \lambda^\alpha d_\alpha \) closely resembles the structure of \( V = \psi^I A_I(x) \) in Chern-Simons theory using the BRST operator \( Q = \psi^I \frac{\partial}{\partial \tau} x_I \). In Chern-Simons theory, \( QV = 0 \) implies that \( \partial_I A_J - \partial_J A_I = 0 \) and \( \delta V = Q \Omega \) implies that \( \delta A_I = \partial_I \Omega \). Furthermore, as in Chern-Simons theory, the super-Yang-Mills ghost is described by the BRST cohomology at ghost-number zero, the super-Yang-Mills fields are described by the BRST cohomology at ghost-number one, the super-Yang-Mills antifields are described by the BRST cohomology at ghost-number two, and the super-Yang-Mills antighost is described by the BRST cohomology at ghost-number three [15]. This structure can be seen from the Batalin-Vilkovisky action for \( d=10 \) super-Yang-Mills which can be written in the Chern-Simons-like form \( S = \langle VQV + \frac{2}{3} V^3 \rangle \) using the normalization convention that \( \langle (\lambda \gamma^a \theta)(\lambda \gamma^b \theta)(\lambda \gamma^c \theta)(\theta \gamma_{abc} \theta) \rangle = 1 \).

This construction for \( d=10 \) super-Yang-Mills is easily generalized to \( N=4 \) \( d=4 \) super-Yang-Mills by eliminating six of the ten \( x \)'s and decomposing the \( d=10 \) spinors into \( N=4 \) \( d=4 \) spinors as
\[
\theta^\alpha \rightarrow (\theta^{\mu j}, \overline{\theta}^{\dot{\mu} j}), \quad p_\alpha \rightarrow (p_{\mu j}, \overline{p}^{\dot{\mu} j}), \quad \lambda^\alpha \rightarrow (\lambda^{\mu j}, \overline{\lambda}^{\dot{\mu} j}), \quad w_\alpha \rightarrow (w_{\mu j}, \overline{w}^{\dot{\mu} j}), \quad (5.12)
\]
where \( \mu, \dot{\mu} = 1 \) to 2 and \( j = 1 \) to 4. The pure spinor condition \( \lambda \gamma^c \lambda = 0 \) implies that \( \lambda^{\mu j} \) and \( \overline{\lambda}^{\dot{\mu} j} \) satisfy the constraints
\[
\lambda^{\mu j} \overline{\lambda}^{\dot{\mu} j} = 0, \quad (5.13)
\]
\[
\epsilon_{\mu \nu} \lambda^{\mu j} \lambda^{\nu k} = \frac{1}{2} \epsilon_{\mu \dot{\nu}} \epsilon^{hi j k} \lambda^{\mu} \overline{\lambda}^{\dot{\nu} i j k}. \quad (5.14)
\]
Although (5.13) and (5.14) contain ten constraints, only five of these constraints are independent. This is easy to verify since
\[
\lambda \mu j = 0 \text{ implies that } \lambda \mu j = 0, \text{ which implies that } 
\]
\[
\epsilon_{\mu \nu} \lambda^{\mu \nu} = \frac{1}{2} e^{2 \phi} e^{h i j k} \epsilon_{\mu \nu} \lambda^\mu \lambda^\nu 
\]
for some \( \phi \). So if the four constraints in (5.13) are satisfied, any one of the constraints in (5.14) imply that \( \phi = 0 \), which implies that the remaining five constraints in (5.14) are satisfied.

Since the four constraints of (5.13) are almost strong enough to define an N=4 d=4 pure spinor, it will be convenient to define a “semi-pure” spinor \((\lambda'^{\mu_j}, \lambda'^{\dot{\mu}_j})\) which is only required to satisfy the four constraints of (5.13) that
\[
\lambda'^{\mu_j} = 0. 
\]
A semi-pure spinor has 12 independent components and is related to a pure spinor \((\lambda^{\mu_j}, \lambda^\mu_j)\) by a \(U(1)\) “R-transformation” as
\[
\lambda'^{\mu_j} = e^{\phi} \lambda^{\mu_j}, \quad \lambda'^{\dot{\mu}_j} = e^{-\phi} \lambda^\dot{\mu}_j 
\]
where \( \phi \) is determined from
\[
e^{2 \phi} = \frac{\epsilon_{\mu \nu} \lambda^{\mu \nu} \lambda^{\nu k}}{\frac{1}{2} e^{h i j k} \epsilon_{\mu \nu} \lambda^\mu \lambda^\nu}. 
\]

In flat d=4 Minkowski space, the worldline action for N=4 d=4 super-Yang-Mills will be defined as
\[
S = \int d\tau \left[ \frac{1}{2} p_{\mu j} \frac{\partial}{\partial \tau} \theta^{\mu j} + \frac{1}{2} \bar{p}^\mu \frac{\partial}{\partial \tau} \bar{\theta}^{\mu j} + w'_{\mu j} \frac{\partial}{\partial \tau} \lambda'^{\mu j} + \bar{w}'_{\dot{\mu} j} \frac{\partial}{\partial \tau} \bar{\lambda}'_{\dot{\mu} j} \right] 
\]
with the BRST operator
\[
Q = \lambda^{\mu j} d_{\mu j} + \bar{\lambda}'_{\dot{\mu} j} d'_{\dot{\mu} j} 
\]
where \( d_{\mu j} = p_{\mu j} + \sigma^m_{\mu \dot{\mu}} \bar{p}^\dot{\mu} \frac{\partial}{\partial \tau}, \quad d'_{\dot{\mu} j} = \bar{p}^\dot{\mu} + \sigma^m_{\mu \dot{\mu}} \theta^{\mu j} \frac{\partial}{\partial \tau}, \) and \( \lambda^{\mu j} \) and \( \bar{\lambda}'_{\dot{\mu} j} \) are semi-pure spinors satisfying (5.16). Note that \( Q^2 = 0 \) since \( \{d_{\mu j}, \bar{d}'_{\dot{\mu} k}\} = \delta^k_j \sigma^m_{\mu \dot{\mu}} \frac{\partial}{\partial \tau}, \) and that \( w'_{\mu j} \) and \( \bar{w}'_{\dot{\mu} j} \) can only appear in combinations which are invariant under the gauge transformations
\[
\delta w'_{\mu j} = \xi_m \sigma^m_{\mu \dot{\mu}} \lambda'^{\mu j}, \quad \delta \bar{w}'_{\dot{\mu} j} = \xi_m \sigma^m_{\mu \dot{\mu}} \lambda'^{\mu j}. 
\]
The action and BRST operator of (5.19) and (5.20) are invariant under the $U(1)$ $R$-transformation

\[ \theta^{\mu j} \rightarrow e^{\theta^{\mu j}}, \quad \overline{\theta}^{\mu j} \rightarrow e^{-1}\overline{\theta}^{\mu j}, \quad p_{\mu j} \rightarrow e^{-1}p_{\mu j}, \quad \overline{p}^{\mu j} \rightarrow e^{\overline{p}^{\mu j}}, \]  

(5.22)

\[ \lambda^{\mu j} \rightarrow c\lambda^{\mu j}, \quad \overline{\lambda}^{\mu j} \rightarrow c^{-1}\overline{\lambda}^{\mu j}, \quad w'_{\mu j} \rightarrow c^{-1}w'_{\mu j}, \quad \overline{w}^{\mu j} \rightarrow e^{\overline{w}^{\mu j}}, \]

however, $N=4$ $d=4$ super-Yang-Mills does not contain such a $U(1)$ symmetry. Since the variable $\phi$ of (5.18) transforms under (5.22) as

\[ \phi \rightarrow \phi + \frac{1}{2} \log c, \]  

(5.23)

$\phi$ can be interpreted as a “compensator” for $U(1)$ $R$-transformations which cancels the $U(1)$ $R$-transformation of $\theta^{\mu j}$ and $\overline{\theta}^{\mu j}$. Physical states will therefore be defined as states of $+1$ ghost-number in the BRST cohomology which are invariant under the $R$-transformation of (5.22).

At ghost-number one, $R$-invariant states are described by

\[ V = e^{-\frac{\phi}{2}}\lambda^{\mu j} A_{\mu j}(x, \theta e^{-\frac{\phi}{2}}, \overline{\theta} e^{\frac{\phi}{2}}) + e^{\frac{\phi}{2}}\overline{\lambda}^{\mu j} \overline{A}_{\mu j}(x, \theta e^{-\frac{\phi}{2}}, \overline{\theta} e^{\frac{\phi}{2}}) \]  

(5.24)

where $\phi$ is defined in (5.18) and cancels the $R$-transformation of $\lambda'$ and $\theta$. In other words,

\[ V = \lambda'^{\mu j} A'_{\mu j}(x, \theta', \overline{\theta}') \]  

(5.25)

where $A'_{\mu j}(x, \theta', \overline{\theta}') = e^{-\frac{\phi}{2}} A_{\mu j}(x, \theta e^{-\frac{\phi}{2}}, \overline{\theta} e^{\frac{\phi}{2}})$ and $\overline{A}'^{\mu j}(x, \theta', \overline{\theta}') = e^{\frac{\phi}{2}} \overline{A}_{\mu j}(x, \theta e^{-\frac{\phi}{2}}, \overline{\theta} e^{\frac{\phi}{2}})$ are the $R$-transformed versions of $A_{\mu j}(x, \theta, \overline{\theta})$ and $\overline{A}_{\mu j}(x, \theta, \overline{\theta})$ using the $R$-parameter $c = e^{-\frac{\phi}{2}}$ in (5.22). The equation $QV = 0$ implies that

\[ e^{-\frac{\phi}{2}}\lambda^{\mu j} \chi^{\nu k} D_{\mu j} A_{\nu k} + e^{\frac{\phi}{2}}\overline{\lambda}^{\nu k} \overline{D}_{\nu j} \overline{A}_{\nu k} + \lambda'^{\mu j} \overline{\lambda}'^{\nu k} (D_{\mu j} \overline{A}_{\nu k} + \overline{D}_{\nu j} A_{\mu j}) = 0, \]  

(5.26)

which implies using the pure spinor constraints of (5.13) - (5.18) that

\[ D_{\mu j} \overline{A}_{\nu k} + \overline{D}_{\nu j} A_{\mu j} = \delta^{k}_{j} \sigma^{m}_{\nu \mu} A_{m}, \quad D_{(\mu j} A_{\nu k)} = \epsilon_{\mu \nu} A_{[j k]}, \quad \overline{D}^{(\mu j} \overline{A}^{\nu k)} = \frac{1}{2} \epsilon^{\mu \nu} \epsilon^{h i j k} A_{[h i]}, \]  

(5.27)

for some superfields $A_{m}(x, \theta, \overline{\theta})$ and $A_{[j k]}(x, \theta, \overline{\theta})$. Furthermore, the gauge transformation $\delta V = Q\Omega(x, e^{-\frac{\phi}{2}} \theta, e^{\frac{\phi}{2}} \overline{\theta})$ implies that

\[ \delta A_{\mu j} = D_{\mu j} \Omega, \quad \delta \overline{A}_{\mu j} = \overline{D}_{\mu j} \Omega, \quad \delta A_{m} = \partial_{m} \Omega. \]  

(5.28)

So when $V$ of (5.24) is in the BRST cohomology, $A_{\mu j}$ and $\overline{A}_{\mu j}$ satisfy the linearized $N=4$ $d=4$ super-Yang-Mills equations and gauge invariances of (5.8) and (5.9) in flat Minkowski space.
5.3. \(N=4\ d=4\) super-Yang-Mills in \(AdS_4\)

To generalize this construction to \(N=4\ d=4\) super-Yang-Mills in an \(AdS_4\) background, one needs to modify the worldline action and BRST operator of (5.19) and (5.20) to be \(OSp(4|4)\) invariant. This can be done using a coset construction based on \(\frac{OSp(4|4)}{SO(3,1) \times SO(4)}\) which contains four bosonic generators and sixteen fermionic generators. As in the \(AdS_5 \times S^5\) construction, it is convenient to define left-invariant currents \(J^A = (g^{-1} \frac{\partial}{\partial t}g)^A\) where \(g(x, \theta)\) takes values in the \(\frac{OSp(4|4)}{SO(3,1) \times SO(4)}\) coset, \(A = (m, [mn], [jk], rj)\) label the \(OSp(4, 4)\) generators, \(m = 0\) to \(3\) label the “translation” generators, \([mn]\) label the \(SO(3, 1)\) and \(SO(4)\) generators, and \(rj\) label the “supersymmetry” generators for \(r = 1\) to \(4\) and \(j = 1\) to \(4\). Note that the two-component \(\mu\) index corresponds to \(r = 1, 2\), the two-component \(\dot{\mu}\) index corresponds to \(r = 3, 4\), and the antisymmetric \(\epsilon_{rs}\) tensor has non-zero components \(\epsilon_{12} = -\epsilon_{21} = \epsilon_{34} = -\epsilon_{43} = 1\).

The \(OSp(4|4)\)-invariant worldline action is

\[
S = R^2 \int dt \left\{ \frac{1}{4} J^m J_m + \epsilon_{rs} J^{rj} J^{sj} + w'_r J^{rj} (\frac{\partial}{\partial t} \lambda^r + A \lambda^r)^{rj} \right. \\
+ (J^{[mn]} - A^{[mn]})(J_{[mn]} - A_{[mn]}) - (J^{[jk]} - A^{[jk]})(J_{[jk]} - A_{[jk]}) \right\}
\]

\[
= R^2 \int dt \left\{ \frac{1}{4} J^m J_m + \epsilon_{rs} J^{rj} J^{sj} + w'_r (\nabla \lambda)^{rj} + (w' \lambda')_j^k (w' \lambda')_k^j - (w' \sigma^{mn} \lambda')(w' \sigma_{mn} \lambda') \right\},
\]

where \((w' \lambda')_j^k = w'_r J^{rj} \lambda^{rk}, (w' \sigma^{mn} \lambda') = (\sigma^{mn})_s^r w'_r J^{sj} \lambda^{sk} \text{ and } (\nabla \lambda)^{rj} = \frac{\partial}{\partial t} \lambda^{rj} + \frac{1}{2} J_{[mn]} (\sigma^{[mn]})_s^r \lambda^{rsj} + J_{[jk]} \lambda^{rk}\). This action is invariant under local \(SO(3, 1) \times SO(4)\) transformations where \(\lambda^r\) and \(w^r\) transform covariantly, and is also invariant under the BRST transformations

\[
\delta g = g(\epsilon \lambda^{rj} T_{rj}), \hspace{1cm} \delta w'_r = \epsilon J_{rj},
\]

generated by the BRST operator \(Q = \lambda^{rj} J_{rj}\) where \(T_{rj}\) are the fermionic generators of \(OSp(4|4)\).

Defining the ghost-number one vertex operator as

\[
V = \lambda^{rj} A^r_{rj} = \lambda^\mu_j A^r_{\mu j} + \overline{\lambda}_{\dot{\mu} j} \overline{A}^{r j}_{\dot{\mu}},
\]

the BRST-transformation of (5.30) implies that

\[
QV = \lambda^\mu_j \lambda^{\nu k} \nabla_{\mu j} A^r_{\nu k} + \overline{\lambda}_{\dot{\mu} j} \overline{\lambda}_{\dot{\nu} k} \overline{A}^{r j}_{\dot{\mu} \dot{\nu}} + \lambda^\mu_j \overline{\lambda}^{\nu k}_{\dot{\mu} j} (\nabla_{\mu j} \overline{A}^{r k}_{\dot{\nu}} + \overline{\nabla}^{r k}_{\dot{\nu}} A^r_{\mu j}),
\]
superconformal transformations are denoted respectively by $P_{\mu j}, K_{\mu j}, D, R_j^k$, and whose fermionic generators for supersymmetry and superconformal transformations are denoted respectively by $[Q_{\mu j}, \overline{Q}_j^\mu, S_\mu, \overline{S}_{\mu j}]$. Under an N=4 superconformal transformation parameterized by the $PSU(2,2|4)$ element $\Omega$,

$$g_{AdS_4}(y, \xi, \overline{\xi}) \rightarrow g'_{AdS_4}(y', \xi', \overline{\xi}) = \Omega \ g_{AdS_4}(y, \xi, \overline{\xi})$$

where

$$h = e^{c_m K_m + w^{mn} M_{mn} + a_j^k R_j^k + b D + x^\mu S_\mu + \overline{\xi}^\mu \overline{S}_{\mu j}}$$

and the parameters $[c^m, w^{mn}, a_j^k, b, x^\mu, \overline{\xi}^\mu]$ in (5.36) are chosen such that

$$g'_{AdS_4} = e^{y'^m (P_m + K_m) + \xi'^{\mu j} (Q_{\mu j} + S_\mu \delta_{jk}) + \overline{\xi}_j^\mu (\overline{Q}_j^\mu + \overline{S}_{\mu j} \delta_{jk})}$$

for some $(y^m(y, \xi, \overline{\xi}), \xi^{\mu j}(y, \xi, \overline{\xi}), \overline{\xi}^{\mu j}(y, \xi, \overline{\xi}))$.

Similarly, a point $(x^m, \theta^{\mu j}, \overline{\theta}_j^\mu)$ in N=4 d=4 Minkowski superspace can be represented as

$$g_{Mink}(x, \theta, \overline{\theta}) = e^{x^m P_m + \theta^{\mu j} Q_{\mu j} + \overline{\theta}_j^\mu \overline{Q}_j^\mu}$$

where under an N=4 superconformal transformation parameterized by $\Omega$,

$$g_{Mink}(x, \theta, \overline{\theta}) \rightarrow g'_{Mink}(x', \theta', \overline{\theta}') = \Omega \ g_{Mink}(x, \theta, \overline{\theta}) \ h(x, \theta, \overline{\theta})$$
and the parameters \([c^m, w^{mn}, a^j_k, b, \chi^\mu_j, \xi^\mu_j]\) in h of (5.36) are now chosen such that
\[ g'_{\text{Mink}} = e^{x^m P_m + \theta^\mu j Q_{\mu j} + \overline{\theta}^\nu_j \Omega_j} \text{ for some } (x^m(x, \theta, \overline{\theta}), \theta^\mu j(x, \theta, \overline{\theta}), \overline{\theta}^\nu_j(x, \theta, \overline{\theta})). \]

To superconformally map \(N=4 \text{ AdS}_4\) superspace into \(N=4 \text{ d}=4\) Minkowski superspace, define
\[ g_{\text{Mink}}(x, \theta, \overline{\theta}) = g_{\text{AdS}_4}(y, \zeta, \overline{\zeta}) h(y, \zeta, \overline{\zeta}) \]
where the parameters \([c^m, w^{mn}, a^j_k, b, \chi^\mu_j, \xi^\mu_j]\) in h of (5.36) are chosen such that
\[ g_{\text{Mink}} = e^{x^m P_m + \theta^\mu j Q_{\mu j} + \overline{\theta}^\nu_j \Omega_j} \text{ for some functions } (x^m(y, \zeta, \overline{\zeta}), \theta^\mu j(y, \zeta, \overline{\zeta}), \overline{\theta}^\nu_j(y, \zeta, \overline{\zeta})). \]
After writing the \(\text{AdS}_4\) superspace variables \((y^m, \zeta^\mu j, \overline{\zeta}^\mu j)\) in terms of the Minkowski superspace variables \((x^m, \theta^\mu j, \overline{\theta}^\nu j)\) using this superconformal map, the superfield equations of (5.33) simplify to
\[ D_{\mu j} A^k_{\nu} + \overline{D}^k_{\nu} A^j_{\mu} = \delta^k_j \sigma^{m\nu} A_m, \quad e^\phi D_{(\mu j} A'_{\nu k)} = \epsilon_{\mu\nu} A_{[jk]}, \quad e^{-\phi} \overline{D}^{(\mu j} A^\nu k) = \frac{1}{2} \epsilon^{\mu\nu} \epsilon^{hijk} A_{[hi]j}, \]
(5.41)
where \(D_{\mu j}\) and \(\overline{D}^j_{\mu}\) are the flat superspace derivatives. So if one defines \(A'_\mu j(x, \theta', \overline{\theta}') = e^{-\phi} A_{\mu j}(x, \theta e^{-\overline{\phi}}, \overline{\theta} e^{\overline{\phi}})\) and \(A'^j_{\mu}(x, \theta', \overline{\theta}) = e^{\phi} \overline{A}^j_{\mu}(x, \theta e^\overline{\phi}, \overline{\theta} e^{\overline{\phi}})\) as in (5.25), one finds that
\[ D_{\mu j} A^k_{\nu} + \overline{D}^k_{\nu} A^j_{\mu} = \delta^k_j \sigma^{m\nu} A_m, \quad D_{(\mu j} A_{\nu k)} = \epsilon_{\mu\nu} A_{[jk]}, \quad \overline{D}^{(\mu j} A^\nu k) = \frac{1}{2} \epsilon^{\mu\nu} \epsilon^{hijk} A_{[hi]j}, \]
(5.42)
which are the same equations as (3.27). So the \(\text{OSp}(4|4)\)-invariant worldline action of (5.29) also describes \(N=4 \text{ d}=4\) super-Yang-Mills.

5.4. Equivalence with open topological A-model

It will now be shown that the worldline action of (5.29), which is based on the \(\text{OSp}(4|4)\) coset together with semi-pure spinors, is related by a field redefinition to the worldline action of (4.15), which is based on the \(\text{OSp}(4|4)\) coset together with unconstrained spinors. This field redefinition combines the four \(x\)'s of the \(\text{OSp}(4|4)\) coset with the 12 components of the semi-pure spinors to form an unconstrained 16-component spinor which transforms covariantly like a twistor variable under \(\text{SO}(3, 2)\) transformations. The construction of this \(\text{AdS}_4\) twistor variable is very similar to the construction of the \(\text{AdS}_5 \times S^5\) twistor variable of subsection 3.2 in which the ten \(x\)'s of the \(\text{PSU}(2, 2|4)\) coset were combined with the 22 components of the pure spinors to form two unconstrained 16-component spinors.

To construct the field redefinition, first decompose the \(\text{OSp}(4|4)\) coset as
\[ g(x, \theta) = e^{\theta^r T_r} e^{x^m T_m} \equiv G(\theta) H(x) \]
(5.43)
where \( G(\theta) = e^{\theta T_{rj}} \) takes values in \( \frac{OSp(4|4)}{Sp(4) \times SO(4)} \), \( H(x) = e^{x^m T_m} \) takes values in \( \frac{Sp(4)}{SO(3,1)} \), and \( T_{rj} \) and \( T_m \) are the “supersymmetry” and “translation” generators of \( \frac{OSp(4|4)}{SO(3,1) \times SO(4)} \).

Now define the twistor-like variable as
\[
Z^{rj} = H^r_s \lambda^{rsj}
\] (5.44)
which combines the four \( x \)'s in \( H^r_s \) with the 12 components of the semi-pure spinor \( \lambda' \). Similarly, define the conjugate twistor-like variable as
\[
Y_{jr} = (H^{-1})^r_s w'_js.
\] (5.45)

Using
\[
J = (g^{-1} \frac{\partial}{\partial \tau} g) = (H^{-1} \frac{\partial}{\partial \tau} H) + H^{-1} (G^{-1} \frac{\partial}{\partial \tau} G) H,
\] (5.46)
one finds that
\[
Y_{jr} \frac{\partial}{\partial \tau} Z^{rj} = w'_{rj} \frac{\partial}{\partial \tau} \lambda^{rsj} + (H^{-1} \frac{\partial}{\partial \tau} H)^s_r (w' \lambda')^r_s
\] (5.47)
\[
= w'_{rj} \frac{\partial}{\partial \tau} \lambda^{rsj} + J^m_r (w' \sigma^m \lambda') - (G^{-1} \frac{\partial}{\partial \tau} G)^s_r (Y Z)^r_s - (G^{-1} \frac{\partial}{\partial \tau} G)^j_k (Y Z)^j_k,
\]
where \((w' \lambda')^r_s = w'_{js} \lambda'^{rsj}, (w' \lambda')^j_k = Y_{kr} Z^{rj}\), \((w' \sigma^m \lambda') = (\sigma^m)^r_s w'_{rj} \lambda'^{rsj}\), and \((\nabla \lambda')^{rj} = \frac{\partial}{\partial \tau} \lambda'^{rsj} + \frac{1}{2} J^{mn}(\sigma^m \lambda')^{rj} + J^j_k \lambda'^{jrk}\). Furthermore,
\[
(w' \sigma^m \lambda')(w' \sigma^m \lambda') = (w' \lambda')^r_s (w' \lambda')^r_s - (w' \sigma^m \lambda')(w' \sigma^m \lambda')
\] (5.48)
\[
= (Y Z)^r_s (Y Z)^r_s - (w' \sigma^m \lambda')(w' \sigma^m \lambda').
\]

Plugging (5.47) and (5.48) into the action of (5.29), and introducing an auxiliary variable \( P_m \) to write the \( J_m J^m \) kinetic term in first-order form, one finds that the action of (5.29) can be written as
\[
S = \int d\tau [P_m J^m - P_m P^m + \epsilon_{rs} J^{rj} J^{sj} + Y_{jr} (\nabla Z)^{rj}
\] (5.49)
\[
+ (Y Z)^j_k (Y Z)^j_k - (Y Z)^r_s (Y Z)^r_s - J^m (w' \sigma^m \lambda') + (w' \sigma^m \lambda')(w' \sigma^m \lambda')]
= \int d\tau [P'_m (J^m - 2 w' \sigma^m \lambda') - P'_m P'^m + \epsilon_{rs} J^{rj} J^{sj} + Y_{jr} (\nabla Z)^{rj} + (Y Z)^j_k (Y Z)^j_k - (Y Z)^r_s (Y Z)^r_s],
\]
where \((\nabla Z)^{rj} = \frac{\partial}{\partial \tau} Z^{rj} + (G^{-1} \frac{\partial}{\partial \tau} G)^r_s Z^{sj} + (G^{-1} \frac{\partial}{\partial \tau} G)^j_k Z^{rk}\) and...
\[ P_m' = P_m - (w' \sigma_m \lambda'). \]  

(5.50)

Under the gauge transformation \( \delta w'_{rj} = \xi^m (\sigma_m)_{r}^{s} \lambda'_{sj} \) of (5.21), (5.50) implies that

\[ \delta P_m' = \xi^n (\sigma_{mn})_r^{s} \lambda'^{rj} \lambda'_{sj}. \]  

(5.51)

For generic values of \( \lambda'^{rj} \), \( \det(\delta P'/\delta \xi) \) is non-zero, so one can consistently gauge \( P_m' = 0 \). Moreover, it is expected that the Fadeev-Popov factor from this gauge-fixing of \( P_m' \) is cancelled by the measure factor which converts the four \( x \)'s and 12 constrained \( \lambda \)'s into the 16 unconstrained \( Z^{rj} \)'s.

In the gauge \( P_m' = 0 \), the action of (5.49) reduces to

\[ S = \int d\tau[\epsilon_{rs} J^{rj} J^{sj} + Y_{rj}(\nabla Z)^{rj} + (YZ)^k_j(YZ)^{j}_k - (YZ)^r_s(YZ)^s_r], \]  

(5.52)

where (5.40) implies that \( \epsilon_{rs} J^{rj} J^{sj} = \epsilon_{rs}(G^{-1} \frac{\partial}{\partial \tau} G)^{rj}(G^{-1} \frac{\partial}{\partial \tau} G)^{sj} \). Since \( G \) parameterizes the coset \( OSp(4|4)/SO(3,2) \times SO(4) \), the worldline action of (5.52) is equivalent to the worldline action of (4.15) coming from the open topological A-model. And since the BRST cohomology of (5.29) describes \( d=4 \ N=4 \) super-Yang-Mills, this equivalence implies that the physical states in the open sector of the topological A-model are \( d=4 \ N=4 \) super-Yang-Mills states.

6. Conclusions

In this paper, a new limit of the \( AdS_5 \times S^5 \) sigma model was considered in which the vector components of the \( PSU(2,2|4) \) metric \( g_{ab} \rightarrow \infty \) and the superspace torsion \( T_{\alpha \beta a} \rightarrow 0 \), while the spinor components of the \( PSU(2,2|4) \) metric \( g_{\hat{\alpha} \hat{\beta}} \) and the superspace torsion \( T_{\alpha a} \hat{\beta} \) are held fixed. This is the opposite procedure from the flat space limit, and if \( (T^b_{\alpha \beta} \eta_{ab})/(T^b_{\alpha a} \eta_{\hat{\alpha} \hat{\beta}}) \) is interpreted as the \( AdS_5 \times S^5 \) radius, it corresponds to taking this radius to zero.

In this limit, the \( PSU(2,2|4) \) algebra deforms into an \( SU(2,2) \times SU(4) \) bosonic algebra with 32 abelian fermionic isometries, and the \( AdS_5 \times S^5 \) sigma model reduces to a linear topological A-model constructed from fermionic N=2 superfields. The bosonic components of these fermionic superfields involve twistor-like combinations of the \( x \)'s and pure spinor ghosts, and the linear topological A-model can be interpreted as the limit of a \( PSU(2,2|4) \)-invariant non-linear topological A-model whose open string sector describes N=4 d=4 super-Yang-Mills.
These results have many parallels with the open-closed duality found by Gopakumar and Vafa which relates Chern-Simons theory and the resolved conifold [17]. In this open-closed duality, Chern-Simons theory is described by the open sector of a topological A-model [13], which is interpreted as a Coulomb branch of the closed string theory for the resolved conifold. As pointed out in [17] and [18], the Chern-Simons/conifold duality shares many features with the Yang-Mills/AdS$_5 \times S^5$ duality, suggesting that the Ooguri-Vafa worldsheet proof of Chern-Simons/conifold duality [18] might have a generalization to a worldsheet proof of the Maldacena conjecture.

However, before attempting a proof of Maldacena’s conjecture using the results of this paper, one would need to understand better both the properties of the $T_{\alpha\beta} \rightarrow 0$ limit of the $AdS_5 \times S^5$ sigma model, and the properties of the open topological A-model for N=4 d=4 super-Yang-Mills.

For example, it is not clear that the $T_{\alpha\beta} \rightarrow 0$ limit of the sigma model can be interpreted as the small $AdS_5 \times S^5$ radius limit, and that a separate Coulomb branch is developed in this limit. Furthermore, although it was shown that the physical states of the open topological A-model describes N=4 d=4 super-Yang-Mills, it was not shown how to compute perturbative super-Yang-Mills scattering amplitudes using this A-model. Hopefully, the d=10 pure spinor formalism will provide some useful clues for computing these amplitudes. For example, if the d=10 pure spinor measure factor \(\langle(\lambda\gamma^a\theta)(\lambda\gamma^b\theta)(\lambda\gamma^c\theta)(\theta\gamma_{abc}\theta)\rangle = 1\) is dimensionally reduced to four dimensions, the field theory action for the open A-model

\[
S = \langle VQV + \frac{2}{3}VVV \rangle
\]

appears to correctly reproduce the N=4 d=4 super-Yang-Mills action [15] [16]. So using the interaction vertex from (6.1), it should be possible to at least compute 3-point super-Yang-Mills tree amplitudes with the open topological A-model. A much bigger challenge would be to compute 4-point tree amplitudes using the A-model, and perhaps the twistor-string methods of [14] [24] [25] will be useful in these computations.

Acknowledgements: I would like to thank Rajesh Gopakumar, Chris Hull, Lubos Motl, Nikita Nekrasov, Hirosi Ooguri, Sasha Polyakov, Warren Siegel, Cumrun Vafa, Brenno Carlini Vallilo, Edward Witten, and especially Juan Maldacena for useful discussions, CNPq grant 305814/2006-0 and FAPESP grant 04/11426-0 for partial financial support, and the Fundação Instituto de Física Teórica for their hospitality. I would also like to thank the organizers of the Twistor String Theory
workshop in Oxford University where initial stages of this research were presented at http://www.maths.ox.ac.uk/~lmsa/Tws/programme.html.
References

[1] J. Maldacena, *The large N limit of superconformal field theories and supergravity*, Adv. Theor. Math. Phys. 2 (1998) 231, hep-th/9711200.

[2] N. Berkovits, *Super-Poincaré covariant quantization of the superstring*, JHEP 0004 (2000) 018, hep-th/0001035.

[3] N. Berkovits and O. Chandia, *Superstring vertex operators in an AdS$_5 \times S^5$ background*, Nucl. Phys. B596 (2001) 185, hep-th/0009168.

[4] B.C. Vallilo, *One-loop conformal invariance of the superstring in an AdS$_5 \times S^5$ background*, JHEP 0212 (2002) 042, hep-th/0210064.

[5] N. Berkovits, *Quantum consistency of the superstring in AdS$_5 \times S^5$ background*, JHEP 0503 (2005) 041, hep-th/0411170.

[6] R.R. Metsaev and A.A. Tseytlin *Type IIB superstring action in AdS$_5 \times S^5$ background*, Nucl. Phys. B533 (1998) 109, hep-th/9805028.

[7] P. Howe and P. West, *The complete N=2 D=10 supergravity*, Nucl. Phys. B238 (1984) 181.

[8] D. Gaiotto and L. Rastelli, *A paradigm of open/closed duality: Liouville D-branes and the Kontsevich model*, JHEP 0507 (2005) 053, hep-th/0312196.

[9] K. Intriligator, *Bonus symmetries of N=4 super-Yang-Mills correlation functions via AdS duality*, Nucl. Phys. B551 (1999) 575, hep-th/9811047.

[10] D. Sorokin, V. Tkach, D. Volkov and A. Zheltukhin, *From the superparticle Siegel symmetry to the spinning particle proper time supersymmetry*, Phys. Lett. B216 (1989) 302.

[11] M. Matone, L. Mazzucato, I. Oda, D. Sorokin and M. Tonin, *The superembedding origin of the Berkovits pure spinor covariant quantization of superstrings*, Nucl. Phys. B639 (2002) 182, hep-th/0206104.

[12] D. Sorokin, *Superbranes and superembeddings*, Phys. Rept. 329 (2000) 1, hep-th/9906142.

[13] E. Witten, *Chern-Simons gauge theory as a string theory*, Prog. Math. 133 (1995) 637, hep-th/9207094.

[14] E. Witten, *Perturbative gauge theory as a string theory in twistor space*, Comm. Math. Phys. 252 (2004) 189, hep-th/0312171.

[15] N. Berkovits, *Covariant quantization of the superparticle using pure spinors*, JHEP 0109 (2001) 016, hep-th/0105050.

[16] J.H. Schwarz and E. Witten, private communication.

[17] R. Gopakumar and C. Vafa, *On the gauge theory/geometry correspondence*, Adv. Theor. Math. Phys. 3 (1999) 1415, hep-th/9811131.

[18] H. Ooguri and C. Vafa, *Worldsheet derivation of a large N duality*, Nucl. Phys. B641 (2002) 3, hep-th/0205297.
[19] N. Berkovits and P. Howe, *Ten-dimensional supergravity constraints from the pure spinor formalism for the superstring*, Nucl. Phys. B635 (2002) 75, hep-th/0112160.

[20] N. Berkovits, M. Bershadsky, T. Hauer, S. Zhukov and B. Zwiebach, *Superstring theory on AdS$_2 \times S^2$ as a coset supermanifold*, Nucl. Phys. B567 (2000) 61, hep-th/9907200.

[21] N. Berkovits, C. Vafa and E. Witten, *Conformal field theory of AdS background with Ramond-Ramond flux*, JHEP 9903 (1999) 018, hep-th/9902098.

[22] W. Siegel, *Superfields in higher-dimensional spacetime*, Phys. Lett. B80 (1979) 220.

[23] P. Howe, *Pure spinors lines in superspace and ten-dimensional supersymmetric theories*, Phys. Lett. B258 (1991) 141.

[24] R. Roiban, M. Spradlin and A. Volovich, *A googly amplitude from the B-model in twistor space*, JHEP 0404 (2004) 012.

[25] N. Berkovits, *An alternative string theory in twistor space for N=4 super-Yang-Mills*, Phys. Rev. Lett. 93 (2004) 011601, hep-th/0402045.