EXISTENCE OF NON-ABELIAN LOCAL CONSTANTS, AND THEIR PROPERTIES

SAZZAD ALI BISWAS

Abstract. In his Ph.D. thesis, John Tate attached the (abelian) local constants to the characters of a non-Archimedean local field of characteristic zero. Robert Langlands proved the existence theorem of a non-abelian local constant of a higher-dimensional complex local Galois representation. In 1990, Helmut Koch summarized Langlands’ strategy for the existence of a non-abelian local constant (group theoretically). The Brauer induction formula plays a crucial role in Langlands’ proof. Robert Boltje gives a canonical version of the Brauer induction formula. In this paper, we review Langlands’ strategy using Boltje’s canonical Brauer induction formula. We then review various properties of local constants, some applications, and open problems.

1. Introduction

Let $F$ be a non-Archimedean local field of characteristic zero, i.e., $F$ is a finite extension of $\mathbb{Q}_p$, where $p$ is a prime. Let $\overline{F}$ be an algebraic closure of $F$, and $G_F$ be the absolute Galois group of $F$. It can be proved that $G_F$ is a solvable group. One of the aims of this paper is to understand the answer to the following question:

How to attach local constants (or local root numbers, i.e., a special value (when $s = 1/2$) of the epsilon factors) with finite-dimensional complex representations of $G_F$?

The local constant is an invariant of a local Galois representation that preserves under the local Langlands correspondence, therefore, it is an important object in the Langlands program. In the Langlands program, the local constant plays an important role as part of the detection machinery in the local Langlands conjecture. Let $\chi : F^\times \to \mathbb{C}^\times$ be a nontrivial character of $F^\times$ and $\pi_F$ be a uniformizer of $F$. When the conductor of $\chi$ is zero, we call $\chi$ is unramified, and otherwise $\chi$ is ramified. With any character $\chi$ of $F^\times$, we can attach an $L$-function as follows:

$$L(\chi) = \begin{cases} 1 & \text{if } \chi \text{ is ramified}, \\ \frac{1}{(1 - \chi(\pi_F))^{-1}} & \text{if } \chi \text{ is unramified}. \end{cases}$$

John Tate showed that this $L(\chi)$ satisfies a functional equation (cf. Equation (4.4)), where the local constant $W(\chi, \psi) \in \mathbb{C}^\times$ appears, here $\psi$ is a nontrivial additive character of $F$. Now the question is:
Can we extend the notion of the local constant $W(\chi, \psi)$ for higher-dimensional local Galois representation such that the extended definition of local constants behaves well with respect to short exact sequences, change of multiplicative measure, induction (in dimension zero), and agree with Tate’s definition in the one-dimensional case?

The answer is YES (see Theorem A of \cite{35}). This was first proved by Robert Langlands in his unpublished article \cite{35} by the local method, and the paper is about 300 pages. Later, P. Deligne gave a smaller proof of the existence of a non-abelian local constant using the global method (cf. \cite{16}). In \cite{30}, Helmut Koch described the strategy of Langlands’ proof in the language of the extendible function. In Langlands’ proof, the Brauer induction formula plays a crucial role. In \cite{8}, Robert Boltje introduces a canonical Brauer induction formula. In Section 2, and Section 3, we review Koch’s paper but using Boltje’s canonical induction formula (see \cite{8}, \cite{9}).

Before going to explain, here I mention the main reasons for writing this paper:

1. Because of the uniqueness of Boltje’s canonical Brauer induction formula, we can reduce many computations of Koch\cite{30} to simple computations,
2. To show the connections between the three different conventions of local constants (due to Langlands, Deligne, and Bushnell & Henniart, see Subsection 4.4),
3. To review various properties of local constants, applications, and open problems.

Let $G$ be a finite group, and $R(G)$ be its Grothendieck group (tensor product induces multiplication on it makes it an abelian ring, and even it is called Grothendieck Λ-ring, that is, the free abelian group with basis the finite-dimensional complex representations of $G$, quotient out by short exact sequences in the usual way). Let $R_+(G)$ denote the free abelian group with basis the set of isomorphism classes of pairs $(H, \varphi)$, where $H$ is a subgroup of $G$, and $\varphi$ is a one-dimensional representation of $H$. Define a group homomorphism $b_G : R_+(G) \to R(G)$ by sending $(H, \chi) \mapsto \text{Ind}_H^G(\chi)$. By the Brauer induction theorem this a surjective group homomorphism. Because $b_G$ is surjective group homomorphism, then from the first isomorphism theorem we have

\begin{equation}
R_+\left( \frac{G}{\text{Ker}(b_G)} \right) \cong R(G).
\end{equation}

Our aim is to define local constants for every element of $R(G)$. For this, it is equivalent to studying the behavior of local root numbers on $\text{Ker}(b_G)$. Every element $\sigma \in \text{Ker}(b_G)$, we have

$$\dim(b_G(\sigma)) = \dim(0) = 0.$$  

Therefore, working on $\text{Ker}(b_G)$ is equivalent to working with virtual representations of dimension zero. Local constants for dimension zero virtual representations are invariant under induction, and it is easy to attach the non-abelian local constants for dimension zero virtual representations (cf. the definition of extendible function \cite{2.3}). Therefore, studying $\text{Ker}(b_G)$ is important for the existence proof of non-abelian local constants (cf. Remark \cite{3.1}). In \cite{35}, Langlands worked on the generator of $\text{Ker}(b_G)$ (an abelian group), and proved that the existence of a local constant for every element in $R(G)$ is equivalent to checking that certain functions vanish on $\text{Ker}(b_G)$.

\footnote{Recently, Helmut Koch and E.-W. Zink (cf. \cite{31}) revisited Langlands’ paper, \cite{35}.}
However, P. Deligne’s approach (cf. [16]) is different from that of Langlands. In Deligne’s approach (global method), \( \text{Ker}(b_G) \) is used but not its generators. Deligne (following Langlands) wrote down three types of elements in \( \text{Ker}(b_G) \) explicitly. They are called elements of type I, type II, and type III. (cf. Definition 3.2), and here is the existence theorem of non-abelian local constants:

**Theorem 1.1** (Langlands-Deligne, [16]). i. If \( G \) is abelian, the kernel \( \text{Ker}(b_G) \) is generated as an abelian group by the relation of type I.

ii. If \( G \) is nilpotent, the kernel \( \text{Ker}(b_G) \) is generated as an abelian group by the relations of type I, and type II.

iii. If \( G \) is solvable, the kernel \( \text{Ker}(b_G) \) is generated by relations of type I, type II, and type III.

**Remark 1.2.** Actually, Langlands did much more than this. He analyzed the generators of kernel \( \text{Ker}(b_G) \) for an arbitrary finite group. However, because our local Galois groups are solvable, for existence proof, we only need to study finite solvable groups. To analyze any functional properties of solvable groups, one should follow a standard pattern as follows:

first, abelian groups, then nilpotent groups, and finally solvable groups.

In this paper, we also follow this pattern to prove Theorems 1.1 and 1.3.

Article [35] is about the existence of the non-abelian local constants. These local constants are certain functions that can be extended from one-dimensional representations to higher-dimensional representations nicely. After proving Theorem 1.1, he actually proved the following theorem. This is the criterion of being an extendible function.

**Theorem 1.3** (Langlands, Theorem 3.1 of [30]). Let \( G \) be a solvable group, and let \( \mathcal{F} \) be a function on \( \mathcal{M}_G := \{(H, \varphi)|H \leq G, \varphi \in \hat{H} := \{\chi : H \rightarrow \mathbb{C}^\times\}\} \) with values in a multiplicative abelian group \( A \) satisfying certain properties (cf. Equations (2.14), and (2.15)). Let \( K \) be a normal subgroup of \( B \leq G \), and \( A := B/K \). Let \( \chi \) be a character of \( B \). Then \( \mathcal{F} \) is (weakly) extendible to \( \mathcal{M}'_G := \{(H, \rho)|H \leq G, \rho \text{ is a virtual representation of } H\} \) if and only if the following three conditions are fulfilled for all subgroups \( B \) of \( G \).

**I.** In the situation of type I (i.e., when \( A \) is a cyclic group of order a prime \( \ell \)), we have

\[
\mathcal{F}((K, \chi)) \prod_{\mu \in \hat{A}} \mathcal{F}((A, \mu)) = \prod_{\mu \in \hat{A}} \mathcal{F}((B, \mu\chi)).
\]

**II.** In the situation of type II (i.e., when \( A \) is a central extension of an abelian group of order \( \ell^2: \mathbb{Z}/(\mathbb{Z} \times \mathbb{Z})/\ell \mathbb{Z} \cong X_1 \times X_2 \xhookrightarrow{} A \xhookrightarrow{} \mathbb{Z} \)), we have

\[
\mathcal{F}((G_1, \chi_1\chi)) \prod_{\mu \in \hat{X}_1} \mathcal{F}((X_1, \mu)) = \mathcal{F}((G_2, \chi_2\chi)) \prod_{\mu \in \hat{X}_2} \mathcal{F}((X_2, \mu)),
\]

where \( G_i \) is the inverse image of \( H_i \) in \( B \) (cf. Definition 3.2 for type II).

**III.** In the situation of type III (i.e., when \( A = H \times C \) is a semidirect product where \( C \neq \{1\} \) is an abelian normal subgroup contained in all nontrivial abelian normal subgroups of \( A \) (cf. Definition 3.2 for Type III), we have

\[
\mathcal{F}((G', \chi)) \prod_{\mu \in \mathcal{T}} \mathcal{F}((A'_\mu, \mu)) = \prod_{\mu \in \mathcal{T}} \mathcal{F}((B'_\mu, \mu\chi)),
\]
where $B'_{\mu}$ is the inverse image of $A'_{\mu}$ in $B$, and $G'$ is the inverse image of $H$ in $B$.

**Remark 1.4.** For one-dimensional Galois representation of $G_F$, hence using class field theory for the character of $F^\times$, we have the notion of local constant. Properties of the local constants of one-dimensional representations are the same as the initial conditions for defining extendible function (cf. Subsection 2.3). Therefore, proving the above theorem is equivalent to proving the existence of non-abelian local constants, and Langlands explicitly did the same.

**Organization of the paper:** Including the Introduction section, there are six sections of this paper. In Section 2, we first define Robert Boltje’s canonical Brauer induction formula, and mention all the necessary properties of canonical Brauer induction. Then, for any finite group, we give the definition of the extendible function. Then, we study all the necessary conditions of a function to be extendible.

In Section 3, we first study Kernel $Ker(b_G)$. Next, we prove the Theorem 1.3. In Section 4, we review all known properties of local constants. In Section 5, we mention some applications of root numbers, and open problems. In Section 6 (Appendix), we make some remarks on the global constants (also known as Artin’s root numbers or global epsilon factors).

**2. Boltje’s canonical Brauer induction formula and extendible functions**

2.1. **Brauer Canonical induction formula.** Let $G$ be a finite group. In this subsection, we define a canonical Brauer induction formula, which is due to Robert Boltje (see [8], [9]).

A finite-dimensional complex representation $\rho$ of $G$ is called **monomial** if we have the following decomposition:

$$\rho = \rho_1 \oplus \rho_2 \oplus \cdots \oplus \rho_n,$$

where $\rho_1, \cdots, \rho_n$ are one-dimensional sub-representation of $\rho$ which are called the lines of $\rho$, and $\mathbb{C}$-linear action of $G$ on $\rho$, such that the group elements permute the lines of $\rho$.

We call a representation $\rho$ is **simple** if this action of $G$ on the lines is transitive. In this article, we need the following setting for $G$:

- $\mathbb{C}[G] :=$ the group ring of $G$ over $\mathbb{C}$.
- $R(G) :=$ the character ring or equivalently the Grothendieck group of $\mathbb{C}[G]$-modules provided with the tensor product as the multiplication, and unit representation as the unit element.
- $\mathcal{M}_G := \{(H, \varphi) \mid H \leq G, \varphi : H \to \mathbb{C}^\times, \text{linear character of } H\}$, the set of pairs consisting a subgroup and a linear character of that subgroup. Group $G$ acts on $\mathcal{M}_G$ by conjugation, i.e.,

$$G \times \mathcal{M}_G \to \mathcal{M}_G, \ G \text{ acts from the left, } g \times (H, \varphi) := (H^g, \varphi^g), \ \text{where } H^g := gHg^{-1},$$

and $\varphi^g := \varphi(g^{-1}hg)$ for all $g \in G$, and $h \in H$. In particular, this covers the whole $G$-conjugacy classes of group $H$. Only for the **normal subgroups** $H$, $H$ will be fixed. We denote the $G$-orbit of $(H, \varphi)$ by $[H, \varphi]_G = [H, \varphi] := (H, \varphi)^G$. Boltje’s notation), and we denote the set of $G$-orbit by $\mathcal{M}_G/G$.

**Poset structure.** We define natural poset (partially ordered set) structures on $\mathcal{M}_G$, and $\mathcal{M}_G/G$ by
Infima exists in \( \mathcal{M}_{G} \) but in general not in \( \mathcal{M}_{G/G} \).

- \( R_{+}(G) := \) the Free abelian group whose basis consists of isomorphic classes of irreducible monomial representations of \( G \). In our setting, \( \mathcal{M}_{G/G} \) is the basis of \( R_{+}(G) \). \( R_{+}(G) \) is also the Grothendieck group of monomial representations of \( G \). \( R_{+}(G) \) again can be provided with an abelian ring structure with the identity element \( [G, 1_{G}] \). We can write each element of \( R_{+}(G) \) as an integral linear combination of the basis elements \( [H, \varphi] \). Because \( G \) acts trivially on the pairs \( (G, \varphi), \varphi \in \hat{G} \), we can consider \( \mathbb{Z}[\hat{G}] \) as a subgroup of \( R_{+}(G) \).

- \( b_{G} : R_{+}(G) \to R(G), [H, \varphi] \mapsto \text{Ind}_{H}^{G}\varphi \) the induction map from equivalence classes of monomial pairs to the monomial representations of \( G \). This \( b_{G} \) is a surjective ring homomorphism by the Brauer induction theorem (cf. [9]).

- **Multiplication in** \( R_{+}(G) \):

  Natural multiplication in \( R_{+}(G) \) corresponds to the tensor product in \( R(G) \):

\[
[H_{1}, \chi_{1}] \cdot [H_{2}, \chi_{2}] = \sum_{(g_{1}, g_{2}) \in H_{1} \cap G \times H_{2} \cap G} [H_{1}^{g_{1}} \cap H_{2}^{g_{2}}, \chi_{1}^{g_{1}} \chi_{2}^{g_{2}}] = \sum_{g \in H_{1} \cap G/H_{2}} [H_{1}^{g} \cap H_{2}, \chi_{1}^{g} \chi_{2}],
\]

where the brackets on the right side indicate the pairs which will occur, and which have still put together into equivalence classes, such that

\[
[H_{1}^{g_{1}} \cap H_{2}^{g_{2}}, \chi_{1}^{g_{1}} \chi_{2}^{g_{2}}] = [H_{1}^{g_{1}g_{2}^{-1}} \cap H_{2}, \chi_{1}^{g_{1}} \chi_{2}^{g_{2}^{-1}}] = [H_{1} \cap H_{2}^{g_{2}g_{1}^{-1}}, \chi_{1}^{g_{1}} \chi_{2}^{g_{2}}, \chi_{2}^{g_{2}^{-1}}],
\]

because \([...,]\) denotes \( G \)-equivalence classes. Therefore, writing also the right side of equation (2.1) as a sum of equivalence classes, it is then only over a subset of pairs \( (g_{1}, g_{2}) \) representing the different equivalence classes, which means that

\[
g_{1}g_{2}^{-1} \in H_{1} \cap G/H_{2} \quad \text{or} \quad g_{2}g_{1}^{-1} \in H_{2} \cap G/H_{1}
\]

cover the different double cosets. Then

\[
b_{G}([H_{1}, \chi_{1}] \cdot [H_{2}, \chi_{2}]) = b_{G}([H_{1}, \chi_{1}]) \otimes b_{G}([H_{2}, \chi_{2}]) = \text{Ind}_{H_{1}}^{G} \chi_{1} \otimes \text{Ind}_{H_{2}}^{G} \chi_{2},
\]

because

\[
\text{Ind}_{H_{1}}^{G} \chi_{1} \otimes \text{Ind}_{H_{2}}^{G} \chi_{2} = \text{Ind}_{H_{1}}^{G} \chi_{1} \otimes \text{Res}_{H_{1}}^{G} \left( \text{Ind}_{H_{2}}^{G} \chi_{2} \right)) = \text{Ind}_{H_{1}}^{G} \chi_{1} \otimes \sum_{g \in H_{2} \cap G/H_{1}} \text{Ind}_{H_{2} \cap H_{1}}^{G} \chi_{2} \big|_{H_{2} \cap H_{1}},
\]

Similarly,

\[
\text{Ind}_{H_{1}}^{G} \chi_{1} \otimes \text{Ind}_{H_{2}}^{G} \chi_{2} = \text{Ind}_{H_{2}}^{G} \left( \text{Res}_{H_{2}}^{G} \left( \text{Ind}_{H_{1}}^{G} \chi_{1} \right) \right) \otimes \chi_{1}) = \sum_{g \in H_{1} \cap G/H_{2}} \text{Ind}_{H_{2} \cap H_{1}}^{G} \chi_{2} \big|_{H_{2} \cap H_{1}} \otimes \chi_{2} \big|_{H_{1} \cap G/H_{2}}.
\]
Remark 2.1. 1. **Twisting in** $R_+ (G)$: If $H_2 = G$, hence $[H_2, \chi_2] = (G, \chi_2)$, then (2.1) comes down to

$$[H_1, \chi_1] \cdot (G, \chi_2) = [H_1, \chi_1 \cdot \text{Res}^G_{H_1} (\chi_2)]$$

corresponding to

$$\text{Ind}^G_{H_1} (\chi_1) \otimes \chi_2 = \text{Ind}^G_{H_1} (\chi_1 \otimes \text{Res}^G_{H_1} (\chi_2)).$$

We call this twist of $[H_1, \chi_1]$ by $\chi_2$.

The twisting formula in $R_+ (G)$ shows that the pairs $[G, \varphi]$ are multiplied with each other as in $\mathbb{Z} [\hat{G}]$.

2. We also see that $R_+ (G)$ contains $\mathbb{Z} [\hat{G}]$ as a subring, and is therefore a $\mathbb{Z} [\hat{G}]$-algebra. We also see $b_G ([G, \varphi]) = \text{Ind}^G_{H_1} (\varphi) = \varphi$, then we can consider $b_G$ as a $\mathbb{Z} [\hat{G}]$-algebra homomorphism. We also have the following $\mathbb{Z} [\hat{G}]$-linear ring homomorphism:

$$\pi_G : R_+ (G) \rightarrow \mathbb{Z} [\hat{G}], \quad [H, \varphi] \rightarrow \begin{cases} \varphi & \text{if } H = G, \\ 0 & \text{if } H < G. \end{cases}$$

(2.2)

3. Using commutative multiplication (2.1), the map $b_G$ turns into a surjective homomorphism of commutative rings, and $\text{Ker} (b_G) \subset R_+ (G)$ becomes an ideal.

We also need to mention some **functionality properties of** $b_G$:

- **Restriction map** $\text{Res}^G_{+H}$. If $H \leq G$, we define a restriction map that is also a ring homomorphism

$$\text{Res}^G_{+H} : R_+ (G) \rightarrow R_+ (H)$$

by the double coset formula

$$\text{Res}^G_{+H} ([B, \mu]_G) = \sum_{s \in H \cap B^{*}} [H \cap B^{*}, \mu^{*}]_H$$

(2.3)

where $\mu^{*}$ means $\mu^{*}|_{H \cap B^{*}}$.

- **Induction** $\text{Ind}^G_{+H}$. For $H \leq G$, we define the map $\text{Ind}^G_{+H} : R_+ (H) \rightarrow R_+ (G)$ is an inclusion, i.e., $[\Gamma, \psi]_H \rightarrow [\Gamma, \psi]_G$. This map is well-defined by the following commutative diagram

$$
\begin{array}{ccc}
R_+ (H) & \xrightarrow{\text{Ind}^G_{+H}} & R_+ (G) \\
\downarrow b_H & & \downarrow b_G \\
R (H) & \xrightarrow{\text{Ind}^G_{+H}} & R (G)
\end{array}
$$

From this above diagram, we have $b_G \circ \text{Ind}^G_{+H} = \text{Ind}^G_{H} \circ b_H$.

- **Inflation** $\text{Inf}^{G}_{+G/N}$: If $f : G \rightarrow G/N =: \overline{G}$ is the canonical surjection for a normal subgroup $N$ of $G$, we obtain

$$\text{Inf}^{G}_{+G/N} ([H/N, \varphi]_{\overline{G}}) = [H, \varphi],$$

(2.4)

where $N \leq H \leq G$, and $\varphi \in \hat{H}$ vanishes on $N$. Thus $\text{Inf}^{G}_{+G/N}$ maps the basis $\mathcal{M}_{\overline{G}/G}$ injectively into the basis $\mathcal{M}_G /G$. 
Proposition 2.2. If \( \rho \in R_+(G) \) is any element, then the product with \([H, \chi] \in R_+(G)\) can be rewritten as:

\[
\rho \cdot [H, \chi] = \text{Ind}_H^G(\text{Res}_H^G(\rho) \cdot [H, \chi]),
\]

where \( \text{Res}_H^G(\rho) \cdot [H, \chi] \) is the product in \( R_+(H) \).

Proof. We may assume that \( \rho = [H_1, \chi_1] \) is one of the generators of \( R_+(G) \). Then, we obtain:

\[
\text{Res}_H^G([H_1, \chi_1]) = \sum_{g \in H_1 \backslash G/H} [H_1^g \cap H, \chi_1^g|_{H_1^g \cap H}],
\]

\[
\text{Res}_H^G([H_1, \chi_1] \cdot [H, \chi]) = \sum_{g \in H_1 \backslash G/H} [H_1^g \cap H, \chi_1^g \chi|_{H_1^g \cap H}],
\]

where for the second equality, we have used the torsion formula in the case \( G = H \). However, then applying \( \text{Ind}_H^G \) to the second equality, we see from (2.1) that the right side turns into \([H_1, \chi_1] \cdot [H, \chi]\).

\( \square \)

Remark 2.3. If \( \rho = \sum_{[H, \chi] \in M_{G/G}} \alpha_{[H, \chi]}[H, \chi] \in R_+(G) \), then \( \rho \in \text{Ker}(b_G) \) means a relation

\[
\sum_{[H, \chi] \in M_{G/G}} \alpha_{[H, \chi]} \text{Ind}_H^G(\chi) \cong 0 \in R(G).
\]

If \( G' \) is a subquotient of \( G \), which means \( G' \hookrightarrow G'' \leq G \), then our functoriality properties fit together with a commutative diagram

\[
\begin{array}{ccc}
R_+(G') & \xrightarrow{\text{Inf}_{G'}} & R_+(G'') & \xrightarrow{\text{Ind}_{G'}} & R_+(G) \\
\text{b}_{G'} \downarrow & & \text{b}_{G'} \downarrow & & \text{b}_{G} \downarrow \\
R(G') & \xrightarrow{\text{Inf}} & R(G'') & \xrightarrow{\text{Ind}} & R(G)
\end{array}
\]

Moreover, because \( \text{Ker}(b_{G'}) \) is always an ideal, and taking into account the torsion operation, we see from the diagram that

\[
(2.5) \quad \sigma \in R_+(G') \mapsto \text{Ind}_{G''_{G'}}(\chi \cdot \text{Inf}_{G'}^G(\sigma)), \quad \text{where} \ \chi \in \widehat{G''},
\]

will take \( \text{Ker}(b_{G'}) \) into \( \text{Ker}(b_G) \). This gives an answer to the following question:

Why is the kernel of \( b_{G'} \) (hence the kernel of \( b_G \)) important for studying extendible functions?

2.2. The Map \( a_G \). Now we are in a position to define the section map \( a_G \) of \( b_G \) from \( R(G) \) to \( R_+(G) \), that is, here we define the function

\[
a_G : R(G) \rightarrow R_+(G)
\]

such that \( b_G \circ a_G \) is the identity map on \( R(G) \).

Axioms of \( a_G \):

For a family of maps \( a_G : R(G) \rightarrow R_+(G) \), we consider the following two conditions:

\[
(1) \quad \begin{array}{ccc}
R(G) & \xrightarrow{a_G} & R_+(G) \\
\text{Res}_H^G \downarrow & & \downarrow \text{Res}_H^G \\
R(H) & \xrightarrow{a_H} & R_+(H)
\end{array}
\]

This diagram commutes for all subgroups relation \( H \leq G \).
(2) The following diagram is commutative for all groups $G$,

\[
\begin{array}{c}
R_+(G) \\
\uparrow_{\rho_G} \\
R(G) \\
\downarrow_{\rho_G}
\end{array}
\xrightarrow{\pi_G} Z[\hat{G}]
\]

Here $\rho_G : R(G) \to Z[\hat{G}]$ is defined by

\[
\rho_G(\chi) = \sum_{\varphi \in \hat{G}} (\varphi, \chi) \cdot \varphi.
\]

**Theorem 2.4** ([9], Theorem 2.1). There is a unique family of maps $a_G : R(G) \to R_+(G)$ satisfying conditions (1), and (2) (from above axioms), such that

\[
a_G(\chi) = \sum_{[H, \varphi] \in M_G/G} \alpha_{[H, \varphi]}(\chi) \cdot [H, \varphi] \in R_+(G).
\]

This family has the following properties (for details, cf. [9], pp. 38-40):

(a) **Description of $a_G$:**

(i) The coefficients $\alpha_{[H, \varphi]}(\chi)$ are unique for a given $\chi$.

(ii) $a_G$ is the adjoint map of $b_G$ with respect to $[-, -]$, and $(-, -)$.

(iii) We have the explicit formula

\[
a_G(\chi) = \frac{1}{|G|} \sum_{[H, \varphi] \in M_G/G} [H|\mu_{M_G}^{M_G}((H, \varphi), (H', \varphi'))(\varphi', \chi|H) \cdot [H, \varphi]]
\]

where $\mu_{M_G}^{M_G}((H, \varphi), (H', \varphi'))$ is the Möbius function of $M_G$.

(b) $b_G \circ a_G = id_{R(G)}$ ($a_G$ is a Brauer induction formula for $G$).

(c) $a_G$ is $Z[\hat{G}]$-linear ($a_G$ commutes with twists by linear characters), and for each $\varphi \in \hat{G}$ we have $a_G(\varphi) = [G, \varphi]$.

(d) For each $\chi \in R(G)$, we have

\[
\chi(1) \cdot 1_G = \sum_{[H, \varphi] \in M_G/G} \alpha_{[H, \varphi]}(\chi) Ind_{H}^{G}(1_H)
\]

and for each $\chi \in R(G)$ with $\chi(1) = 0$, we have

\[
\chi = \sum_{[H, \varphi] \in M_G/G} \alpha_{[H, \varphi]}(\chi) Ind_{H}^{G}(\varphi - 1_H).
\]

**Remark 2.5. i).** In 1947, Richard Brauer (cf. [10]) proved that any virtual representation $\rho \in R(G)$ can be expressed as follows:

\[
\rho = \sum_i n_i Ind_{H_i}^{G}(\varphi_i), \quad n_i \in \mathbb{Z}, H_i \leq G, \varphi_i \in \hat{H}_i.
\]

In [8], Robert Boltje gives an explicit, and **canonical** formula for Brauer’s induction theorem by algebraic, and combinatorial methods. Under certain functorial properties of $a_G$, it can be proved that the expression (2.6) is **unique** among all the expressions for $\rho$ as above (2.10). Because of its **uniqueness**, the expression (2.6) is called **canonical**.
ii). Here, we also mention the modified canonical Brauer induction formula (cf. Property (h) on p. 40 of [9]): for all $\rho \in R(G)$ with $\rho(1) = 0$, we have

\begin{equation}
\rho = \sum_{[H, \varphi] \in \mathcal{M}_G/G} \alpha_{[H, \varphi]}(\rho) \text{Ind}_H^G(\varphi - 1_H),
\end{equation}

where $1_H$ is the trivial character of $H$.

iii). For all $\rho \in R(G)$, we have the expression (2.6). Here, we should also mention two formulas regarding the relationship between the coefficients $\alpha_{[H, \varphi]}(\rho)$ and the dimension of $\rho = \rho(1)$, and they are (cf. Property (j) of $a_G$ on p. 40 of [9]):

\begin{equation}
\sum_{[H, \varphi] \in \mathcal{M}_G/G} \alpha_{[H, \varphi]}(\rho) = \rho(1),
\end{equation}

\begin{equation}
\sum_{[H, \varphi] \in \mathcal{M}_G/G} [G : H] \cdot \alpha_{[H, \varphi]}(\rho) = \rho(1).
\end{equation}

2.3. Extendible functions. Let $G$ be any finite group. We denote $\mathcal{M}_G^\circ$ as the set of all pairs $(H, \rho)$, where $H$ is a subgroup of $G$, and $\rho$ is a virtual representation of $H$. The group $G$ acts on $\mathcal{M}_G^\circ$ as follows:

$$(H, \rho)^g = (H^g, \rho^g), \quad g \in G,$$

$$\rho^g(x) = \rho(gxg^{-1}), \quad x \in H^g := g^{-1}Hg.$$  

Furthermore, we denote by $\hat{H}$ the set of all one-dimensional representations of $H$, and by $\mathcal{M}_G$ the subset of $\mathcal{M}_G^\circ$ of pairs $(H, \chi)$ with $\chi \in \hat{H}$. Here character $\chi$ of $H$ is always a linear character, i.e., $\chi: H \to \mathbb{C}^\times$.

Now, we define a function $\mathcal{F}: \mathcal{M}_G \to \mathcal{A}$, where $\mathcal{A}$ is a multiplicative abelian group with

\begin{equation}
\mathcal{F}((H, 1_H)) = 1,
\end{equation}

and

\begin{equation}
\mathcal{F}((H^g, \chi^g)) = \mathcal{F}((H, \chi))
\end{equation}

for all $(H, \chi)$, where $1_H$ denotes the trivial representation of $H$.

Here, a function $\mathcal{F}$ on $\mathcal{M}_G$, we mean a function which satisfies Equations (2.14) and (2.15).

A function $\mathcal{F}$ is said to be extendible if $\mathcal{F}$ can be extended to an $\mathcal{A}$-valued function on $\mathcal{M}_G^\circ$ satisfying:

\begin{equation}
\mathcal{F}((H, \rho_1 + \rho_2)) = \mathcal{F}((H, \rho_1))\mathcal{F}((H, \rho_2))
\end{equation}

for all $(H, \rho_i) \in \mathcal{M}_G^\circ$, $i = 1, 2$, and if $(H, \rho) \in \mathcal{M}_G^\circ$ with $\dim \rho = 0$, and $\Delta$ is a subgroup of $G$ containing $H$, then

\begin{equation}
\mathcal{F}((\Delta, \text{Ind}_H^\Delta \rho)) = \mathcal{F}((H, \rho)),
\end{equation}

where $\text{Ind}_H^\Delta \rho$ is the virtual representation of $\Delta$ induced from $\rho$.

Remark 2.6 (Weakly Extendible Functions, and Langlands $\lambda$-functions). We can always construct a zero-dimensional representation from a finite-dimensional representation. Let $\rho$ be a representation of $H$ with $\dim(\rho) \neq 0$. We can define a zero-dimensional representation of $H$ by $\rho$, and which is:

$$\rho_0 := \rho - \dim(\rho) \cdot 1_H.$$
So \( \dim \rho_0 \) is zero, then we use the Equation (2.17) for \( \rho_0 \), and we obtain

\[
\mathcal{F}(\Delta, \text{Ind}_H^A(\rho_0)) = \mathcal{F}(H, \rho_0). \tag{2.18}
\]

Now replace \( \rho_0 \) by \( \rho - \dim \rho \cdot 1_H \) in the above Equation (2.18), and we have

\[
\mathcal{F}(\Delta, \text{Ind}_H^A(\rho - \dim \rho \cdot 1_H)) = \mathcal{F}(H, \rho - \dim \rho \cdot 1_H)) \implies \frac{\mathcal{F}(\Delta, \text{Ind}_H^A(\rho))}{\mathcal{F}(\Delta, \text{Ind}_H^A(1_H))^{\dim \rho}} = \frac{\mathcal{F}(H, \rho)}{\mathcal{F}(H, 1_H)^{\dim \rho}}.
\]

Therefore,

\[
\mathcal{F}(\Delta, \text{Ind}_H^A(\rho)) = \left\{ \frac{\mathcal{F}(\Delta, \text{Ind}_H^A(1_H))}{\mathcal{F}(H, 1_H)} \right\}^{\dim \rho} \cdot \mathcal{F}(H, \rho)) \tag{2.19}
\]

where

\[
\lambda_H^A(\mathcal{F}) := \frac{\mathcal{F}(\Delta, \text{Ind}_H^A(1_H))}{\mathcal{F}(H, 1_H)}.
\]

However, by the definition of \( \mathcal{F} \), we have \( \mathcal{F}(H, 1_H) = 1 \), so we can write

\[
\lambda_H^A(\mathcal{F}) = \mathcal{F}(\Delta, \text{Ind}_H^A(1_H)). \tag{2.20}
\]

This \( \lambda_H^A(\mathcal{F}) \) is called Langlands \( \lambda \)-function (or simply \( \lambda \)-function) which is independent of \( \rho \). A extendible function \( \mathcal{F} \) is called strongly extendible if it satisfies equation (2.16), and fulfills Equation (2.17) for all \((H, \rho) \in M_G^v\), and if Equation (2.17) is fulfilled only when \( \dim \rho = 0 \), then \( \mathcal{F} \) is called a weakly extendible function.

**Lemma 2.7** ([13], p. 103). If extendible functions exist, they are unique.

**Proof.** Let \( \mu \) be a function on \( M_G \) and satisfy Equations (2.14), (2.15). Let \( \mu_1, \mu_2 \) be two extendible functions of \( \mu \) on \( M_G^v \). Now we have to show \( \mu_1 = \mu_2 \). By definition for one dimensional representation \( \mu_1 = \mu_2 \), and \( \mu_i((H, 1_H)) = 1, i = 1, 2 \). Again because \( \mu_1, \mu_2 \) are extendible functions of \( \mu \), they satisfy equations (2.16), and (2.17).

Let \( H \leq G \) be a subgroup of \( G \), and \( \rho \in R(H) \).

\[
\mu_1((H, \rho)) = \mu_1((H, \rho - \dim(\rho) \cdot 1_H)) = \mu_1((H, \sum_{[U_i, \chi_i] \in M_H/H} \alpha_{[U_i, \chi_i]} \text{Ind}_{U_i}^H(\chi_i - 1_{U_i}))) =
\]

\[
\prod_{[U_i, \chi_i] \in M_H/H} \mu_1((H, \text{Ind}_{U_i}^H(\chi_i - 1_{U_i})))^{\alpha_{[U_i, \chi_i]}} = \prod_{[U_i, \chi_i] \in M_H/H} \mu_1((U_i, \chi_i - 1_{U_i}))^{\alpha_{[U_i, \chi_i]}} = \prod_{[U_i, \chi_i] \in M_H/H} \mu_1((U_i, \chi_i))^{\alpha_{[U_i, \chi_i]}} \text{ because } \mu_1((U_i, 1_{U_i})) = 1
\]

\[
= \prod_{[U_i, \chi_i] \in M_H/H} \mu_2((U_i, \chi_i))^{\alpha_{[U_i, \chi_i]}} \text{ because } \mu_1 = \mu_2 \text{ on } M_G
\]

\[
= \mu_2((H, \rho)).
\]

This implies that if extendible functions exist, they are unique. \( \square \)
In the following, we mention three functoriality properties of the $\lambda$-functions which are attached to an extendible function (for arithmetic proof, cf. [35] when extendible function is local abelian root number/epsilon factor). Group theoretically, it is not hard to see its proof. They follow from the definition.

**Lemma 2.8.** Let $H$ be a subgroup of a group $G$, and $F$ be an extendible function on $\mathcal{M}_G$. Then we have the following properties of $\lambda$-factor.

1. If $g \in G$, then $\lambda^G_{g^{-1}Hg}(F) = \lambda^H_H(F)$, where $H \subseteq G$.
2. If $H'$ is a subgroup of $H$ then $\lambda^G_{H'}(F) = \lambda^H_H(F)\lambda^G_{H'}(F)^{[H:H']}$, where $[H:H']$ is the index of $H'$ in $H$.
3. If $H'$ is a normal subgroup of $G$ contained in $H$, then $\lambda^G_{H}(F) = \lambda^G_{H/H'}(F)$.

**Remark 2.9.** For explicit computation for $\lambda$-functions, see Proposition 2 on p. 124 of [28], Saito’s Theorem on p. 508 of [39], Theorems 1.1, and 1.2 on p. 182 of [4], and Theorem 1.1 of [5]. If $F$ is a non-Archimedean local field, and $K$ is a wildly ramified quadratic extension of $F$, then computation of $\lambda^{\text{Gal}(K/F)}_{\{1\}}$ is still open. When $F = \mathbb{Q}_2$, for explicit computation see pp. 60-64 of [6].

**Example 2.10.** In [35], Langlands proves that local constants are weakly extendible functions. This is the main theme of this paper. In Section 3, we prove this. The Artin root numbers (also known as global constants) are strongly extendible functions (cf. Appendix, Subsection 6.1 of this paper).

In the following theorem, the criterion for the extendability of a function using Langlands $\lambda$-function is shown. The existence of a $\lambda$-function with appropriate behavior can be transformed into a criterion for extendibility. This is a crucial idea of Langlands’ proof of the existence of nonabelian local constants.

**Theorem 2.11** (Theorem 2.1 of [35], Lemma 3.2 of [30]). Let $G$ be a finite group, and $F$ be a function on $\mathcal{M}_G$ with values in the abelian group $\mathcal{A}$. Then, $F$ is extendible to $\mathcal{M}_G^\infty$ if and only if for all subgroups $H$ of $G$, there is a function

$$U \in \mathcal{U}(H) \mapsto \lambda^H_H(F) \in \mathcal{A}$$

which is defined on the set $\mathcal{U}(H)$ of subgroups of $H$ such that:

$$\lambda^H_H(F) = 1,$$

and if there are $[U, \chi_U] \in \mathcal{M}_H/H$ such that

$$\sum_{[U, \chi_U] \in \mathcal{M}_H/H} \alpha_{[U, \chi_U]}(0) \text{Ind}_U^H(\chi_U) = 0,$$

then

$$\prod_{[U, \chi_U] \in \mathcal{M}_H/H} F([U, \chi_U])^{\alpha_{[U, \chi_U]}(0)} \lambda^H_H(F)^{\alpha_{[U, \chi_U]}(0)} = 1.$$
Proof. If \( \mathcal{F} \) is extendible, then by the definition of Langlands’ \( \lambda \)-function, and the Equations (2.16), and (2.18), conditions (2.22), and (2.23) hold.

Conversely, suppose that \( \lambda^H_H(\mathcal{F}) \) exists with (2.22), (2.23). Now for any \( H \subseteq G \), we must define a function \( \mathcal{F} : \mathcal{M}_H \rightarrow A \) such that \( \mathcal{F} \) is extendible to \( \mathcal{M}_G \). Before this, we need these two following properties of \( \lambda \)-functions:

1. For a fix \( H \subseteq G \), the function \( U \in \mathcal{U}(H) \mapsto \lambda^H_H(\mathcal{F}) \) is uniquely determined by (2.22), (2.23). Using Equation (2.11), we can write

\[
\text{Ind}^H_U(1_U) - [H : U] \cdot 1_H = \sum_{[U', \varphi_{U'}] \in \mathcal{M}_H / H} \alpha_{[U', \varphi_{U'}]} \text{Ind}^H_U(\varphi_{U'} - 1_{U'}) .
\]

This implies that

\[
\lambda^H_H(\mathcal{F}) = \prod_{[U', \varphi_{U'}] \in \mathcal{M}_H / H} \mathcal{F}((U', \varphi_{U'}))^{\alpha_{[U', \varphi_{U'}]}}.
\]

2. If \( H \subseteq G' \subseteq G \), then consider a function on \( \mathcal{U}(H) \) as follows:

\[
U \in \mathcal{U}(H) \mapsto \lambda^G_{G'}(\mathcal{F})\lambda^G_{G'}(\mathcal{F})^{-[H : U]},
\]

It turns out to be a function satisfying Equations (2.22), (2.23). Again for

\[
\sum_{[U, \chi_i] \in \mathcal{M}_H / H} \alpha_{[U, \chi_i]} \text{Ind}^H_U(\chi_i) \equiv 0,
\]

from Equation (2.13)

\[
\sum_{[U, \chi_i] \in \mathcal{M}_H / H} \alpha_{[U, \chi_i]} \cdot [H : U_i] = 0,
\]

and

\[
\sum_{[U, \chi_i] \in \mathcal{M}_H / H} \alpha_{[U, \chi_i]} \text{Ind}^G_{U_i}(\chi_i) \equiv 0.
\]

Therefore,

\[
\prod_{[U, \chi_i] \in \mathcal{M}_H / H} \mathcal{F}((U_i, \chi_i))^{\alpha_{[U, \chi_i]}}(\lambda^G_{U_i}(\mathcal{F})\lambda^G_{U_i}^{-[H : U_i]})^{\alpha_{[U, \chi_i]}} = \prod_{[U, \chi_i] \in \mathcal{M}_H / H} \mathcal{F}((U_i, \chi_i))^{\alpha_{[U, \chi_i]}}(\lambda^G_{U_i})^{\alpha_{[U, \chi_i]}} = 1,
\]

where the second equality is Condition (2.23) for our original \( \lambda \). Hence, uniqueness implies

\[
\lambda^H_H(\mathcal{F}) = \lambda^G_{G'}(\mathcal{F})\lambda^G_{G'}(\mathcal{F})^{-[H : U]}.
\]

Now, we come to the proof.

Let \( \rho \) be a virtual representation of \( H \). Then, we can uniquely write:

\[
\rho = \sum_{[U', \varphi_{U'}] \in \mathcal{M}_H / H} \alpha_{[U', \varphi_{U'}]}(\rho) \text{Ind}^H_U(\varphi_{U'}).
\]

Then, from this above equation, we define

\[
\mathcal{F}((H, \rho)) = \prod_{[U', \varphi_{U'}] \in \mathcal{M}_H / H} \mathcal{F}((U', \varphi_{U'}))^{\alpha_{[U', \varphi_{U'}]}(\rho)} \lambda^H_H(\mathcal{F})^{\alpha_{[U', \varphi_{U'}]}(\rho)}.
\]

Furthermore, it is clear that Equation (2.16) is satisfied. It remains to show property (2.18).
Let $G'$ be a subgroup of $G$ containing $H$. Then
\[ \text{Ind}_{H}^{G'}(\rho) = \sum_{[U', \varphi U'] \in \mathcal{M}_{H}/H} \alpha_{[U', \varphi U'](\rho)} \text{Ind}_{U'}^{G'}(\varphi U'). \]

Hence, by definition
\[ \mathcal{F}((G', \text{Ind}_{H}^{G'}(\rho))) = \prod_{[U', \varphi U'] \in \mathcal{M}_{H}/H} \mathcal{F}((U', \varphi U'))^{\alpha_{[U', \varphi U'](\rho)}} \lambda_{U'}^{G'}(\mathcal{F})^{\alpha_{[U', \varphi U'](\rho)}}. \]

Therefore, it is sufficient to show
\[ \chi_{U'}^{G'}(\mathcal{F}) = \chi_{U'}^{H}(\mathcal{F}) \cdot \lambda_{U'}^{G'}(\mathcal{F})^{[H:U]} \]
and it follows from equation [2.25].

This completes the proof of the theorem. \(\square\)

**Remark 2.12.** Let $u : G \rightarrow G'$ be a surjective group homomorphism. Then, we have
\[ u^* : \mathcal{M}_{G'} \rightarrow \mathcal{M}_{G}, \quad (H', \chi') \mapsto (u^{-1}(H'), \chi' \circ u), \]
and a function $\mathcal{F}$ on $\mathcal{M}_{G}$ induces a function $\mathcal{F}_{G'}$ on $\mathcal{M}_{G'}$ via
\[ \mathcal{F} \mapsto \mathcal{F}_{G'} = \mathcal{F} \circ u^*, \quad \mathcal{F}_{G'}((H', \chi')) := \mathcal{F}((u^{-1}(H'), \chi' \circ u)). \]

Alternatively, let $G' \subseteq G$ be a subgroup. Then, we have a natural injection
\[ \iota : \mathcal{M}_{G'} \hookrightarrow \mathcal{M}_{G}, \]
and a function $\mathcal{F}$ on $\mathcal{M}_{G}$ has a well-defined restriction $\mathcal{F}_{G'} = \mathcal{F} \circ \iota$ down to $\mathcal{M}_{G'}$:
\[ \mathcal{F}_{G'}((H', \chi')) := \mathcal{F}((H', \chi')), \quad \text{for } (H', \chi') \in \mathcal{M}_{G'}. \]

**Proposition 2.13.** In both cases the maps $u^*$, and $\iota$ naturally extend to **additive maps**:
\[ u^* : \mathcal{M}_{G'}^a \rightarrow \mathcal{M}_{G}^a, \quad \text{and } \iota : \mathcal{M}_{G'}^a \hookrightarrow \mathcal{M}_{G}^a, \]
and if $\mathcal{F}$ is extendible onto $\mathcal{M}_{G}^a$ then $\mathcal{F}_{G'}$ extends from $\mathcal{M}_{G'}$ onto $\mathcal{M}_{G'}^a$, using $\mathcal{F} \circ u^*$, and $\mathcal{F} \circ \iota$ resp.

**Proof.** As to check the property [2.17] for $\mathcal{F}_{G'} = \mathcal{F} \circ u^*$, we note that for $u$ we have by definition
\[ \mathcal{F}_{G'}^a((H', \rho')) = \mathcal{F}((u^{-1}(H'), \rho' \circ u)), \]
\[ \mathcal{F}_{G'}((\Delta', \text{Ind}_{H}^{G'}(\rho'))) = \mathcal{F}((u^{-1}(\Delta'), \text{Ind}_{H}^{G'}(\rho')) \circ u). \]

We must see that the left sides are equal if $\dim(\rho') = 0$. Thus, we turn to the right sides where we may use that
\[ \dim(\rho' \circ u) = \dim(\rho') = 0, \quad \text{and} \quad \text{Ind}_{H}^{G'}(\rho') \circ u = \text{Ind}_{u^{-1}(H')}^{\Delta'}(\rho' \circ u), \]
hence, the right sides are equal because $\mathcal{F}$ is extendible. \(\square\)

Before giving a list of generating relations for $\text{Ker}(b_G)$ in Section 3, we turn to one particular example which will be of crucial importance.

Let $G$ be a finite solvable group. We consider $(H, \chi) \in \mathcal{M}_{G}$. Furthermore, let $C \leq G$ be any abelian normal subgroup. Then, we consider the set of characters
\[ S := S(C, \chi) := \{ \mu \in \hat{C} | \mu|_{H \cap C} = \chi|_{H \cap C} \} \subset \hat{C}. \]

Here, $H$ acts by conjugation on $C$, and on $\hat{C}$ which induces an action $\mu \mapsto \mu^h = h^{-1}\mu h$ on the subset $S$. 
For $\mu \in S$, we let 
\[ H_\mu := \{ h \in H \mid h\mu h^{-1} = \mu \} \]
be the isotropy group. Furthermore, we put 
\[ G'_\mu = H_\mu \cdot C, \quad \mu' \in \widehat{G'_\mu} \]
the extension of $\mu \in S \subset \hat{C}$ by means of $\chi$, more precisely:
\[ \mu'(hc) = \chi(h)\mu(c) \quad \text{for} \ h \in H_\mu, c \in C. \]

Further, let $T \subset S$ be a system of representatives for the orbit $S/H$ (action of $H$ on $S$ from the right).

**Remark 2.14.** In the following lemma, we see how the monomial representation $\text{Ind}^G_H(\chi)$ splits over the system of representative $T$ for $S/H$. In the remaining parts of Sections 2, and 3, this lemma will be used many times because we have to deal with monomial representations. For every element $(H, \chi) \in \mathcal{M}_G$, we will have $T, S, G'_\mu$, therefore the following lemma is important.

**Lemma 2.15** (Lemma 15.1 of [35], Lemma 2.1 of [30]). With notation, and assumptions as we have fixed above, we have
\[ \text{Ind}^G_H(\chi) = \sum_{\mu \in T} \text{Ind}^G_{G'_\mu}(\mu'). \]

**Proof.** To prove the above assertion, we only need to show 
\[ [G : H] = \sum_{\mu \in T} [G : G'_\mu], \]
because here $\chi$, and $\mu'$ are characters of $H$, and $G'_\mu$, respectively.

Here $G'_\mu = H_\mu C$, then we can see that $G'_\mu \subseteq HC \subseteq G$. For any subgroup $K$ of $G$ containing $H$, we can write
\[ \text{Ind}^G_H(\chi) \equiv \text{Ind}^G_K(\text{Ind}^H_K(\chi)). \]

Since here we have $G'_\mu \subseteq HC \subseteq G$, we can rewrite our assertion as
\[ \text{Ind}^G_{HC}(\text{Ind}^H_{HC}(\chi)) = \text{Ind}^G_{HC}(\sum_{\mu \in T} \text{Ind}^H_{G'_\mu}(\mu')). \]

This above relation implies that it is sufficient to prove the lemma in the case $G = HC$.

Let $V$ be a complex-vector space of all functions $f : G \to \mathbb{C}$ with
\[ f(hx) = \chi(h)f(x), \quad \text{for all} \ h \in H, x \in G. \]

$G$ acts on $V$ as follows
\[ (gf)(x) := f(xg) \quad \text{for all} \ g \in G. \]

Because of $G = HC$ we have $|S| = [C : H \cap C] = [HC : H] = [G : H]$. For every $\mu \in S$, we define a function $f_\mu : G \to \mathbb{C}$ as follows:
\[ \mu \in S \mapsto f_\mu, \quad f_\mu(hc) := \chi(h)\mu(c), \quad \text{for all} \ h \in H, c \in C. \]

It can be proved (cf. Lemma 2.2 on p. 14 of [31]) that the set $\{f_\mu\}_{\mu \in S}$ is a basis of the space $V$. 


On the other hand, for every \( \mu \in S \), we can construct the subspace denoted by \( V_\mu \) of \( V \) generated by the set \( \{ f_\mu g := f_{g \mu^{-1}} \} \) for all \( g \in G/G_\mu' \). Furthermore, we can show (cf. [31]) that the subspace \( V_\mu \) is \( G \)-isomorphic to \( \text{Ind}_{G_\mu}^G(\mu') \).

And the dimension of \( \text{Ind}_{G_\mu}^G(\mu') \) is \( [G : G_\mu'] = [H : H_\mu] \) if we assume \( G = HC \), because

\[
G_\mu' \cap H = H_\mu C \cap H = H_\mu (C \cap H) = H_\mu,
\]
hence

\[
G/G_\mu' = HC/H_\mu C \sim H/H_\mu.
\]

Hence, \( [G : H] = \sum_{\mu \in T} [G : G_\mu'] \). This completes the proof. \( \square \)

**Remark 2.16.** If \( hc = 1 \), which means \( h = c^{-1} \in H \cap C \), then \( \mu \in S \) means \( \mu(c^{-1}) = \chi(h) \), and therefore

\[
\mu'(hc) = \chi(h)\mu(c) = \mu(c^{-1})\mu(c) = 1, \quad \text{as it should be.}
\]

If in particular \( H \cap C = \{1\} \), and \( H \cdot C = G \), then we have \( S = \hat{C} \), and \( T = \hat{C}/H \), hence

\[
\text{Ind}_{G}^H(\chi) = \sum_{\mu \in \hat{C}/H} \text{Ind}_{G_\mu}^G(\chi|_{H_\mu} \cdot \mu), \quad \bar{\chi} \otimes \text{Ind}_{H}^G(1_H) = \bar{\chi} \otimes \sum_{\mu \in \hat{C}/H} \text{Ind}_{H_\mu}^G(1_{H_\mu} \cdot \mu),
\]

where the second equality is only a reformation of the first one because using \( G/C \cong H \), we can extend \( \chi \) to the character \( \bar{\chi} \) of \( G \) which is trivial on \( C \). According to Subsection 8.2 of [31], this is the decomposition of \( \text{Ind}_{H}^G(\chi) \) into irreducible components.

**Lemma 2.17** (Lemma 2.4 of [31]). Let \( C \) be a abelian normal subgroup of \( G \), let \( T \) be a set of representatives of \( \hat{C}/G \), and let \( G_\mu \) be the isotropy group of \( \mu \in T \) in \( G \). Furthermore, let \( \{ H_i : i \in I \} \) be a family of subgroup of \( G \) containing \( C \), and let \( \chi_i \) be a character of \( H_i \) such that \( \chi_i|_C \in T \). If we assume a relation

\[
\sum_{i \in I} n_i \text{Ind}_{H_i}^G(\chi_i) = 0, \quad \sum_{i \in I} n_i [H_i, \chi_i]_G \in \text{Ker}(b_G),
\]

then for any fixed \( \mu \in T \), we have

\[
\sum_{\chi_i : \chi_i|_C = \mu} n_i \text{Ind}_{H_i}^G(\chi_i) = 0, \quad \sum_{\chi_i : \chi_i|_C = \mu} n_i [H_i, \chi_i]_G \in \text{Ker}(b_G).\]

**Proof.** Because \( C \subseteq H_i \subseteq G \), and \( C \) is normal we have \( C \cap G/H_i = G/H_i \), and

\[
\text{Ind}_{H_i}^G(\chi_i)|_C = \sum_{g \in G/H_i} g^\mu \chi_i|_C.
\]

Thus, we meet here only one orbit of \( \hat{C}/G \) hence, we meet only one \( \mu \). Therefore, if \( \text{Ind}_{H_i}^G(\chi_i) \) and \( \text{Ind}_{H_i}^G(\chi_j) \) lead to different representatives in \( T \) then their restrictions to \( C \) are disjoint, and therefore the representations itself must be disjoint. Thus, if we fix one \( \mu \), the assumption \[(2.27)\]
will imply that the corresponding partial sum must vanish:

\[
\sum_{\chi_i : \chi_i|_C = \mu} n_i \text{Ind}_{H_i}^G(\chi_i) = 0,
\]

hence the assumption implies separate relations \[(2.28)\] for each \( \mu \in T \). Furthermore, \( \chi_i|_C = \mu \) implies \( H_i \leq G_\mu \), and:

\[
\text{Ind}_{H_i}^G(\chi_i)|_{G_\mu} = \sum_{H_i \cap G_\mu} \text{Ind}_{H_i \cap G_\mu}^G(\chi_i^\mu).
\]
The direct sum on the right contains the term $\text{Ind}_{H_i}^{G_i}(\chi)$ for $s \in H_iG_i$, and additional components for $H_i s G_i \neq H_i G_i$. And restricting further to $C \subseteq H_i \cap G_i$ we see that those other components are disjoint from $\text{Ind}_{H_i}^{G_i}(\chi_i)$ because $s \notin G_i$. Therefore, the separate relations (2.28) will imply the assertion of our Lemma for any fixed $\mu \in T$.

\[ \]

\[ \]

Remark 2.18 (Definition of Lambda-factors). To prove Theorem 1.3, we will use Theorem 2.11. To use Theorem 2.11 we have to define

\[ U \in \mathcal{U}(H) \mapsto \lambda_H^U(\mathcal{F}) \]

with properties (2.22), (2.23), where $U$, and $H$ are two arbitrary subgroups of $G$ with $U \subseteq H$. Note that any definition of $U \mapsto \lambda_H^U(\mathcal{F})$ such that $\lambda_H^U(\mathcal{F}) = \lambda_H^H(\mathcal{F})$ allows a linear extension of $\mathcal{F}$ onto the free abelian group $\mathbb{F}$ using

\[ \mathcal{F}([U, \chi]_H) := \mathcal{F}((U, \chi)) \cdot \lambda_H^U(\mathcal{F}), \]

because $\mathcal{F}$ has the property (2.15), but we need this extension to be trivial on $\text{Ker}(b_H)$.

We fix a nontrivial minimal abelian normal subgroup $C \leq G$ which exists because $G$ is solvable. We proceed by induction over $|G|$. If $|G| = 1$, we put $\lambda_\mathbb{F} = 1$. In Lemma 2.15 put $\chi \equiv 1$ the trivial character of $U$. Thus $S = C/U \cap C$, and $\mu' : G_\mu' = U_\mu C \rightarrow \mathbb{C}^\times$ is the extension of $\mu \in S \subset \tilde{C}$ by 1, thus we can write

\[ \text{Ind}_{U}^{G}(1) \cong \sum_{\mu \in S/U} \text{Ind}_{G_{\mu'}}^{G}(\mu'). \]

If $\mathcal{F}$ is extendible, then this implies

\[ \lambda_\mathbb{F}(\mathcal{F}) = \prod_{\mu \in T} \mathcal{F}((G_{\mu}', \mu')) \lambda_{G_{\mu'}}^{\mathbb{F}}(\mathcal{F}). \]

Now we turn (2.30) into a definition of $\lambda_\mathbb{F}(\mathcal{F})$. Indeed, we can define

\[ \lambda_{G_{\mu'}}^{\mathbb{F}}(\mathcal{F}) := \lambda_{G_{\mu'}/C}^{G_{\mu'}}(\mathcal{F}), \]

because $G_{\mu'} = U_\mu C \supseteq C$, which brings us down to groups of smaller order where we may use the induction hypothesis. And $\mathcal{F}((G_{\mu}', \mu'))$ is defined anyway because $\mu'$ is one-dimensional. Therefore, we may consider Equation (2.30) as a definition of $\lambda_\mathbb{F}(\mathcal{F})$.

Using a fixed nontrivial minimal abelian normal subgroup $C \leq G$, and the isomorphism (2.29) we define:

\[ \lambda_{\mathbb{F}}^{G}(\mathcal{F}) = \mathcal{F}(\text{Ind}_{\mathbb{F}}^{G}(1)) := \prod_{\mu \in T} \mathcal{F}((G_{\mu}', \mu')) \lambda_{G_{\mu'}/C}^{G_{\mu'}}(\mathcal{F}), \]

where $\lambda$ on the right side has already been defined by the induction hypothesis.

Note here that the choice of $C$ is part of the definition because a priori we do not know that $\lambda_{\mathbb{F}}^{G}(\mathcal{F})$ will be unique. In the particular case where $U = \{1\}$ we have $G_{\mu'} = C$, and our Definitions (2.30), (2.31) turn into

\[ \lambda_{\mathbb{F}}^{G}(\mathcal{F}) := \prod_{\mu \in \tilde{C}} \mathcal{F}((C, \mu)) \lambda_{\{1\}}^{G/C}(\mathcal{F}) = \prod_{\mu \in \tilde{C}} \mathcal{F}((C, \mu)) \cdot \lambda_{\{1\}}^{G/C}(\mathcal{F})^{\text{ker}(C)}. \]
If $G$ is of prime order, then the only choice is $C = G$, and therefore the definition comes down to

$$\lambda^G_{U}(\mathcal{F}) := \prod_{\mu \in G} \mathcal{F}((G, \mu)).$$

In the particular case, where $U \supseteq C$, the definition turns into

$$\lambda^G_U(\mathcal{F}) := \lambda^{G/C}_{U/C}(\mathcal{F}).$$

This includes our original Definition (2.31).

As to the definition of $\lambda^H_U(\mathcal{F})$ for proper subgroups $H < G$, we may assume that is defined by the induction hypothesis. Now, we have the following lemma.

**Lemma 2.19.** (i). The above definition of $\lambda^G_U(\mathcal{F})$ is independent of the choice of the set $T$ of representatives for $S/U$.

(ii). 

$$\lambda^G_{U^{g}}(\mathcal{F}) = \lambda^G_U(\mathcal{F}) \quad \text{for all } g \in G.$$ 

(iii). If $N \subseteq U$ is a non-trivial normal subgroup of $G$ which sits in $U$, then $\lambda^G_U(\mathcal{F}) = \lambda^{G/N}_U(\mathcal{F})$ where the right side is already given by the induction hypothesis. In particular,

$$\lambda^G_U(\mathcal{F}) = \lambda^{e/U}_U(\mathcal{F}), \quad \text{if } U \text{ is a normal subgroup of } G.$$

**Proof.** (i). If $u \in U$, then

$$G^{\mu, u}_{u} = u^{-1}G^{\mu}_{u}, \quad (\mu^u)' = (\mu')^u.$$  

Hence, by the condition (2.15) of the extendible function,

$$\mathcal{F}((G^{\mu, u}_{u}, (\mu^u)')) = \mathcal{F}((u^{-1}G^{\mu}_{u}, (\mu')^u)) = \mathcal{F}((G^{\mu}_{u}, (\mu')))$$

and

$$\lambda^G_{G^{\mu, u}_{u}}(\mathcal{F}) = \lambda^G_{u^{-1}G^{\mu}_{u}}(\mathcal{F}) = \lambda^G_{G^{\mu}_{u}}(\mathcal{F})$$

by Lemma 2.8(1).

(ii). We must compare Equation (2.29) with the corresponding formula for $U^g = g^{-1}Ug$, instead of $U$. Then, we obtain

$$S^g := C/U^{g} \cap C, \quad \mu \in S/U \mapsto \mu^g \in S^g/U^g$$

and $\text{Stab}_{U^g}(\mu^g) = g^{-1}\text{Stab}_U(\mu)g$, hence $G^{\mu, g}_{U^g} = \text{Stab}_{U^g}(\mu^g) \cdot C = g^{-1}G^{\mu}_{g}g$, and $(\mu')^g = (\mu')^g$ which implies

$$\lambda^G_{U^g}(\mathcal{F}) = \prod_{\mu} \mathcal{F}((g^{-1}G^{\mu}_{g}, (\mu')^g))\lambda^{G/C}_{C\mu}_{U^g}\mathcal{F}(\mathcal{F}) = \lambda^G_U(\mathcal{F})$$

because $\mathcal{F}$ has the property (2.15), and $(G^{\mu, g}_{U^g}/C = (G^{\mu}_{g}/C)^g$ such that $\lambda^{G/C}_{C\mu}_{U^g}(\mathcal{F}) = \lambda^{G/C}_{G^{\mu}_{g}}(\mathcal{F})$ by the induction hypothesis.

(iii). If $U$ contains a normal subgroup $N$ of $G$, then using the notation of (2.30), we have $N \subseteq U_\mu$; indeed $\mu$ is a character of $C/U \cap C$, and $[N, C] \subseteq N \cap C \subseteq U \cap C$, hence

$$n \cdot x^{-1} \cdot n^{-1} \in N \cap C \subseteq U \cap C \quad \text{for } x \in C, n \in N,$$

and therefore

$$\frac{\mu^n}{\mu}(x) = \mu^n \mu^{-1}(x) = \mu^n(x) \mu^{-1}(x) = \mu(n \cdot x^{-1} \cdot n^{-1}) = 1.$$
Thus, we obtain $NC \leq U \mu C = G'_\mu$, and $\mu'$ (which is the extension of $\mu$ by 1) is trivial on $N$, hence (2.29) is rewritten as

$$\text{Ind}_{U}^{G/N}(1) = \sum_{\mu \in T} \text{Ind}_{G'_\mu/N}^{G/N}(\mu'), \quad \lambda_{U/N}^{G/N}(\mathcal{F}) = \prod_{\mu \in T} \mathcal{F}((G'_\mu/N, \mu')) \cdot \lambda_{G'_\mu/N}^{G/N}(\mathcal{F}),$$

because for the group $G/N$ of a smaller order, we have all properties available by the induction hypothesis. Finally:

$$\lambda_{U/N}^{G/N}(\mathcal{F}) = \lambda_{G/N}^{G}((\mathcal{F}) = \lambda_{G/C}^{G}((\mathcal{F}),$$

because all groups occurring here are of a smaller order. Therefore, we obtain

$$\lambda_{U/N}^{G/N}(\mathcal{F}) = \lambda_{U}^{G}(\mathcal{F})$$

as we have defined it using (2.31).

This completes the proof.

\[\square\]

**Remark 2.20.** In general, definition (2.30) is true if $U$ contains a normal subgroup $N$ of $G$. In this case $N \subseteq G'_\mu$ because $N \cap C \subseteq U \cap C$. Hence, $n x n^{-1} x^{-1} =: [n, x] \in U \cap C$, and $\mu([n, x]) = 1$ for $x \in C, n \in N$. Therefore, (2.30) is the inflation of the corresponding equation for $G/N, U/N, CN/N$, which is valid by the induction assumption.

The definition of $\lambda_{G}^{G}(\mathcal{F})$ has been completed, and we have proved that the definition (2.30) implies Lemma (2.8(1)), and (2.8(3)). Because of Lemma 2.19(ii), we have now a well defined $\mathbb{Z}$-linear map

$$R_{+}(G) \to \mathcal{A}, \quad [U, \chi] \mapsto \mathcal{F}([U, \chi]) := \mathcal{F}((U, \chi)) \lambda_{G}^{G}(\mathcal{F}).$$

Now, we need to check that the definition of $\lambda_{G}^{G}(\mathcal{F})$ satisfies property Lemma 2.8 (2). And we check it in the following lemma.

**Lemma 2.21** (Lemma 3.4 of [30]). If $G'$ is a subgroup of $G$ containing $U$, then

$$\lambda_{G}^{G}(\mathcal{F}) = \lambda_{G'}^{G'}(\mathcal{F}) \lambda_{G}^{G}(\mathcal{F})^{[G':U]}.$$  

**Proof.** By the definition $\lambda_{G}^{G}(\mathcal{F}) = 1$. Therefore, if $G' = G$, then the assertion is trivial.

Let $G' \neq G$, then $\lambda_{G}^{G}(\mathcal{F}_{G'})$ is defined by the induction assumption. Using $G'_\mu = U \mu C \subseteq G'C \subseteq G$, and the induction hypothesis, we obtain:

$$\lambda_{G'_\mu}^{G}(\mathcal{F}) := \lambda_{G'_\mu/C}^{G/C}(\mathcal{F})$$

$$= \lambda_{G'_\mu/C}(\mathcal{F}_{G'C}) \cdot \lambda_{G/C}(\mathcal{F})^{[G'C:G'_\mu]}$$

$$= \lambda_{G'_\mu/C}(\mathcal{F}_{G'C}) \cdot \lambda_{G/C}(\mathcal{F})^{[G'C:G'_\mu]}.$$  

This together with the relation

$$[G'C : G'] = \sum_{\mu \in T} [G'C : G'_\mu] \quad \text{(because } \text{Ind}_{G'_\mu}^{G/C}(1) \cong \sum_{\mu \in T} \text{Ind}_{G'_\mu}^{G/C}(\mu'))$$
implies
\begin{equation}
(2.32) \quad \lambda^G_{\Omega}(\mathcal{F}) := \prod_{\mu \in T} \mathcal{F}((G_{\mu}, \mu')) \lambda^G_{\Omega}(\mathcal{F}) = \lambda^G_{\Omega'}(\mathcal{F})^{|G'_{\Omega'}:G'} \prod_{\mu \in T} \mathcal{F}((G_{\mu}, \mu')) \lambda^G_{\Omega}(\mathcal{F}) = \lambda^G_{\Omega'}(\mathcal{F}) \lambda^G_{\Omega'}(\mathcal{F})^{|G'_{\Omega'}:G'|}.
\end{equation}

Similarly, we can see
\begin{equation}
(2.33) \quad \lambda^G_{U}(\mathcal{F}) = \lambda^G_{U}(\mathcal{F}) \lambda^G_{U}(\mathcal{F})^{|G_{U}:G'|}.
\end{equation}

Furthermore, if \( G' \neq G \), we have
\[ \lambda^G_{U}(\mathcal{F}) \lambda^G_{U}(\mathcal{F}) = \lambda^G_{U}(\mathcal{F}) \lambda^G_{U}(\mathcal{F}) \]
by the induction assumption. We multiply the last equation by \( \lambda^G_{U}(\mathcal{F})^{|G_{U}:G'|} \):
\[ \lambda^G_{U}(\mathcal{F}) \lambda^G_{U}(\mathcal{F})^{|G_{U}:G'|} = \lambda^G_{U}(\mathcal{F}) \lambda^G_{U}(\mathcal{F}) \lambda^G_{U}(\mathcal{F})^{|G_{U}:G'|} \]

This together with Equation (2.32), and Equation (2.33), proves the lemma in the case \( G \neq G' \).

To complete the proof, we are left with the case: when \( G = G' \). To prove this, we need the following lemma.

**Lemma 2.22** (Lemma 3.5 of [30]). For \( U \leq G' \), let \( \Omega = U C \leq G = G' C \) semidirect. Let \( \chi : \Omega \to \mathbb{C}^\times \) be a character of \( \Omega \). Then, there are subgroups \( \Omega_i \) of \( \Omega \) with \( C \subseteq \Omega_i \), and characters \( \mu_i \in \hat{\Omega}_i \) such that
\begin{equation}
(2.34) \quad \text{Ind}_{\Omega_i}^\Omega(\chi) \cong \sum_i \text{Ind}_{\hat{\Omega}_i}^{\Omega_i}(\mu_i \chi).
\end{equation}

\begin{equation}
(2.35) \quad \mathcal{F}((U, \chi)) \lambda^G_{\Omega}(\mathcal{F}) = \prod_i \mathcal{F}((\Omega_i, \mu_i \chi)) \lambda^G_{\Omega_i}(\mathcal{F}).
\end{equation}

**Proof.** The existence of (2.34) is proved in Lemma 2.15. It remains to show that Equation (2.34) implies Equation (2.35).

If \( U \neq G' \), equivalently \( G \neq \Omega \), the induction assumption applies to
\[ \text{Ind}_{\Omega_i}^\Omega(\chi) \cong \sum_i \text{Ind}_{\hat{\Omega}_i}^{\Omega_i}(\mu_i \chi) \]
hence
\begin{equation}
(2.36) \quad \mathcal{F}((U, \chi)) \lambda^G_{\Omega}(\mathcal{F}) = \prod_i \mathcal{F}((\Omega_i, \mu_i \chi)) \lambda^G_{\Omega_i}(\mathcal{F}).
\end{equation}

On the other hand, we have already proved Lemma 2.21 if \( G' C \neq G \), and therefore we may use it for \( G' = \Omega \) because \( G' C = G' \neq G \), hence we have:
\begin{equation}
(2.37) \quad \lambda^G_{U}(\mathcal{F}) \lambda^G_{\Omega}(\mathcal{F})^{|\Omega:U|} = \lambda^G_{U}(\mathcal{F}), \quad \lambda^G_{\Omega_i}(\mathcal{F}) \lambda^G_{\Omega_i}(\mathcal{F})^{|\Omega_i:U|} = \lambda^G_{\Omega_i}(\mathcal{F}).
\end{equation}
And the identity $[\Omega : U] = |C| = \sum_i [\Omega : \Omega_i]$ where the index stands for the different orbits of $U \backslash \hat{C}$ yields

$$(2.38) \quad \chi_H^G(\mathcal{F})^{[\Omega : U]} = \prod_i \chi_H^G(\mathcal{F})^{[\Omega : \Omega_i]}.$$  

Now, multiplying Equation (2.36) by Equation (2.38), then using Equalities (2.37), we obtain (2.35).

Now, assume $U = G'$, equivalently $\Omega = G$, hence $\chi$ is a character of $G = UC$, and in (2.34), we have now

$$\text{Ind}_G^U(\chi_U) = \chi \otimes \text{Ind}_U^G(1_U),$$

where $1_U$ is the trivial character of $U$. Therefore, Equation (2.34) is now the same as Equation (2.29) tensored by the character $\chi$ of $G$. Because $U \cap C = \{1\}$ we have now $S = \hat{C}$, $T = \hat{C}/U$.

If $U$ contains a nontrivial subgroup $N$ which is normal in $G$, then as we have seen in the proof of Lemma 2.19, Equation (2.29) rewrites as

$$\text{Ind}_{G/N}^G(1_U) = \sum_{\mu \in T} \text{Ind}_{G/N}^G(\mu')$$

and therefore, Equation (2.34) turns into

$$\chi \otimes \text{Ind}_{U/N}^{G/N}(1_U) = \sum_{\mu \in T} \chi \otimes \text{Ind}_{G/N}^G(\mu'), \quad \text{for} \, \mu \in \hat{G}.$$  

Because $\chi$ is trivial on $[G, G]$, and the other tensor factors are trivial on $N$, we are dealing here with a representation of $G/(N \cap [G, G])$. Thus, for $N \cap [G, G] \neq \{1\}, G/(N \cap [G, G])$ is a group of smaller order, and we can deduce Equation (2.35) by the induction hypothesis.

On the other hand, if $N \cap [G, G] = \{1\}$, then $N \subseteq N \cap [G, G]$ implies that $N$ must be contained in the center $Z = Z(G)$, hence $N \leq Z(G) \cap U \neq \{1\}$. Then, $N = Z(G) \cap U$ is a nontrivial subgroup of $U$ which is normal in $G$; and then by the induction hypothesis for $G/N$, we have Equation (2.35).

The other way around: If $Z(G) \cap U = \{1\}$, and $N \leq U$ is normal in $G$, then $N \cap [G, G] \neq \{1\}$, and we come down to the induction hypothesis.

To complete the proof of Lemma 2.22, we have to check the case: $G = U \ltimes C$, and $U$ does not contain subgroups $N$ which are normal in $G$. Again,

$$(2.39) \quad \text{Ind}_U^G(\chi_U) \cong \sum_{\mu \in T} \text{Ind}_{G/N}^G(\mu' \chi)$$

implies

$$(2.40) \quad \mathcal{F}((U, \chi_U)) \prod_{\mu \in T} \mathcal{F}((G'_\mu, \mu')) = \prod_{\mu \in T} \mathcal{F}((G'_\mu, \mu' \chi))$$

because we are assuming $\mathcal{F}$ has the property (1.4). On the other hand, by definition in Equation (2.30) we have

$$(2.41) \quad \lambda_U^G(\mathcal{F}) = \prod_{\mu \in T} \mathcal{F}((G'_\mu, \mu')) \lambda_{G/N}^G(\mathcal{F}),$$

where $\lambda_U^G(\mathcal{F})$ is the character of the representation $\mathcal{F}$ induced from $U$ to $G$.
where the last factor is defined using \( \lambda_{G/C}^{G/C}(\mathcal{F}) \). Now, multiply Equation (2.41) by \( \mathcal{F}((U, \chi_U)) \), and then using Equation (2.40), we obtain

\[
\mathcal{F}((U, \chi_U)) \lambda_{G}^{G} (\mathcal{F}) = \prod_{\mu \in T} \mathcal{F}((G_{\mu}, \mu') \chi) \lambda_{G_{\mu}}^{G_{\mu}} (\mathcal{F}).
\]

This ends the proof of Lemma 2.22 because Equations (2.39), and (2.42) are the desired Equations (2.34), and (2.35), respectively.

Now we come back to the proof of Lemma 2.21, where we are left to show that when \( U \subseteq G' \leq G = G'C \), we must have

\[
\lambda_{G}^{G} (\mathcal{F}) = \lambda_{G}^{G'} (\mathcal{F}) \lambda_{G'}^{G'} (\mathcal{F})^{[G':U]}.
\]

From the proof of Theorem 2.11 it is sufficient to show that

\[
U \in \mathcal{U}(G') \mapsto \lambda_{U}^{G'} := \lambda_{U}^{G}(\mathcal{F}) \lambda_{G'}^{G'} (\mathcal{F})^{-[G':U]}
\]

satisfies the conditions (2.22), (2.23) for \( H = G' \). The uniqueness of such a function shows then

\[
\lambda_{U}^{G}(\mathcal{F}) \lambda_{G'}^{G'} (\mathcal{F})^{-[G':U]} = \lambda_{U}^{G'} (\mathcal{F}).
\]

Using Boltje’s canonical Brauer induction formula, we can write 0 \( \in R(G') \) as follows

\[
\sum_{[U, \chi] \in \mathcal{M}_{G'/G'}} \alpha_{[U, \chi]} \text{Ind}_{U}^{G'} (\chi) = 0.
\]

Now, we must show that Equation (2.43) implies

\[
\prod_{[U, \chi] \in \mathcal{M}_{G'/G'}} \{ \mathcal{F}((U, \chi_U)) \lambda_{U}^{G'} (\mathcal{F}) \}^{\alpha_{[U, \chi]}} = 1,
\]

where \( \lambda_{U}^{G'} (\mathcal{F}) = \lambda_{U}^{G}(\mathcal{F}) \lambda_{G'}^{G'} (\mathcal{F})^{-[G':U]} \).

First, from Equation (2.43) we see:

\[
\sum_{[U, \chi] \in \mathcal{M}_{G'/G'}} \alpha_{[U, \chi]} [G':U] = 0,
\]

hence

\[
\prod_{[U, \chi] \in \mathcal{M}_{G'/G'}} \lambda_{G'}^{G'} (\mathcal{F})^{-\alpha_{[U, \chi]}[G':U]} = 1.
\]

Thus, to verify Equation (2.44), it is enough to see that Equation (2.43) will imply

\[
\prod_{[U, \chi] \in \mathcal{M}_{G'/G'}} \{ \mathcal{F}((U, \chi_U)) \lambda_{U}^{G'} (\mathcal{F}) \}^{\alpha_{[U, \chi]}} = 1.
\]

For each pair \( (U, \chi_U) \), we may consider the subgroup \( \Omega = U \cdot C \leq G = G' \cdot C \), and the character \( \chi_U \) of \( U \cdot C \) which is trivial on \( C \), and to this situation apply Lemma 2.22. Then we obtain relations of types (2.34), and (2.35), respectively, which we will write as:

\[
\text{Ind}_{U}^{G'} (\chi_U) \cong \sum_{j=1}^{r_1} \text{Ind}_{U_{ij}}^{G'} (\mu_{ij} \chi_U),
\]

where \( \{ \mu_{ij} \}_{j} \subset \hat{C} \) are representatives of the cosets \( U_{i} \backslash \hat{C} \), and where \( U_{ij} := \text{Stab}_{U_{i}}(\mu_{ij})C \), and \( \chi_U \in \hat{U}_i \) is understood to be trivial on \( C \), and \( \mu_{ij} \) extends \( \mu_{ij} \) hence is non-trivial on \( C \). And Equation (2.35) applied to \( U = U_i \) reads as follows:

\[
\mathcal{F}((U_i, \chi_U)) \lambda_{U_i}^{G'} (\mathcal{F}) = \prod_{j=1}^{r_i} \mathcal{F}((U_{ij}, \mu_{ij} \chi_U)) \lambda_{U_{ij}}^{G'} (\mathcal{F}).
\]
Implementing this into equation (2.45), the assertion becomes

\[
(2.48) \quad \prod_{[U_i, \chi_i] \in \mathcal{M}_{G_G}/G'} \prod_{j=1}^{r_G} \mathcal{F}((U_{ij}, \mu_{ij}' \chi_i)) \lambda_{U_{ij}, G_G}^{ij}(\mathcal{F}) \eta_{[U_i, \chi_i]}^{\alpha} = 1.
\]

On the other hand, applying $\text{Ind}^{G_G}_{G}$ to equation (2.43), and using equation (2.46), we obtain:

\[
(2.49) \quad \sum_{[U_i, \chi_i] \in \mathcal{M}_{G_G}/G'} \alpha_{[U_i, \chi_i]}(\sum_{j=1}^{r_G} \text{Ind}^{G_G}_{U_{ij}}(\mu_{ij}' \chi_i)) \cong 0.
\]

Again here all $U_{ij} \supseteq C$, and therefore $\lambda_{U_{ij}, G_G}^{ij}(\mathcal{F}) = \lambda_{U_{ij}/C}^{G_G/G}(\mathcal{F})$; and $\mathcal{F}((U_{ij}, \mu_{ij}' \chi_i))$ is always well defined because $\mu_{ij}' \chi_i$ is 1-dimensional. Therefore, we can write

$$
\mathcal{F}((U_{ij}, \mu_{ij}' \chi_i)) \cdot \lambda_{U_{ij}/C}^{G_G/G}(\mathcal{F}) = \mathcal{F}((U_{ij}, \mu_{ij}' \chi_i)) \cdot \lambda_{U_{ij}, G_G}^{ij}(\mathcal{F}) = \mathcal{F}(\text{Ind}^{G_G}_{U_{ij}}(\mu_{ij}' \chi_i)).
$$

Furthermore, here $\mathcal{F}$ is an extendible function. Applying $\mathcal{F}$ to Equation (2.49), we obtain the above assertion (2.48).

This completes the proof.

\[\square\]

3. Existence proof of non-abelian local constants

**Remark 3.1 (Why Kernel of $b_G$?).** In this subsection, we first discuss why we should study the kernel $\text{Ker}(b_G)$, and the reasons are as follows:

1. It can be completely described the generating set of $\text{Ker}(b_A)$ for $A := B/K$ with $G \supseteq B \supseteq K$, and $K$ normal in $G$, then inductively we will have all information about the generating set of $\text{Ker}(b_G)$. Similar to any arithmetic invariants (e.g., $L$-functions, $\gamma$-factors etc.), to study the properties of the local constants, we have to know how they behave under induction, and inflation.

2. Again, $A \leftarrow B \leq G$, and $\sigma \in \text{Ker}(b_A) \subset R_+(A)$ implies

$$
\rho = \text{Ind}^{G_G}_{B}(\chi \cdot \text{Inf}^B_+(\sigma)) \in \text{Ker}(b_G) \subset R_+(G) \quad \text{for all } \chi \in \hat{B}.
$$

For this, we have to repeat how inflation, and induction are defined for the rings $R_+$. If $[H, \psi] \in R_+(A)$, and $A \leftarrow B$ then we consider $H_B$ the full preimage of $H$ in $B$, and $\psi_H$ the lift of $\psi$ from $H$ to $H_B$. Then, we have

$$
\text{Inf}^B_+( [H, \psi] ) = [H_B, \psi_B] \in R_+(B),
$$

and this inflation map is compatible with the usual inflation $\text{Inf}^B_A : R(A) \to R(B)$. Therefore, it takes $\text{Ker}(b_A)$ to $\text{Ker}(b_B)$.

Next, if $(H_B, \psi_B) \in \mathcal{M}_B$, and $\chi \in \hat{B}$, and put

$$
\chi \cdot (H_B, \psi_B) = (H_B, \text{Res}_{H_B}^B(\chi) \cdot \psi_B),
$$

where $\text{Res}_{H_B}^B$ is the restriction from $B$ to the subgroup $H_B$. Under $b_B : R_+(B) \to R(B)$ this is compatible with the usual $\chi$-twist for virtual representations of $B$; therefore again the $\chi$-twist on $R_+(B)$ takes $\text{Ker}(b_B)$ into itself.

As to the induction, the map $\text{Ind}^{G_G}_{B} : R_+(B) \to R_+(G)$, it is induced by the identity

$$
[H_B, \psi_B] \mapsto [H_B, \psi_B]
$$
due to the fact that any subgroup $H_B$ of $B$ may also be considered as a subgroup of $G$. This map is compatible with the usual induction $Ind^G_B : R(B) \rightarrow R(G)$, and therefore again the map $Ind^G_B$ takes $Ker(b_B)$ to $Ker(b_G)$.

(3) For any $\rho \in Ker(b_G) \subset R_+(G)$,

we always have $\deg(\rho) = \deg(b_G(\rho)) = \deg(0) = 0$. This implies that for any $\rho \in Ker(b_G)$, $b_G(\rho)$ is a virtual representation of dimension zero. Defining local constants for dimension zero representations is simple because it remains invariant under induction. Because the strong extendability implies weak extendability, proving the existence of non-abelian local constants, is equivalent to prove that the extendible is strong on the virtual representations of dimension zero.

**Definition 3.2 (Type I, Type II, and Type III).** An element $\rho \in Ker(b_G)$ is called of **type I**, **type II**, and **type III**, if there is a subquotient $A \hookrightarrow B \leq G$, such that $\rho = Ind^G_B(\chi \cdot Ind^A_B(\sigma))$, where $\chi \in \hat{B}, \sigma \in Ker(b_A)$, with the following requirements on $A$, and $\sigma$:

**Type I.** Let $\ell$ be a prime, and $A$ be the cyclic group of order $\ell$, then denoting $e := \{1\} < A$ we have

$$\sigma = [e, 1_e] - \sum [A, \mu] \in Ker(b_A) \subset R_+(A),$$

because

$$Ind^A_e(1_e) = \sum \mu.$$  

**Type II.** $A$ is a central extension of an abelian group of type $\mathbb{Z}/\ell \mathbb{Z} \times \mathbb{Z}/\ell \mathbb{Z}$ where again $\ell$ is a prime. Thus, we have $Z$ central in $A$, and $A/Z = X_1 \times X_2$ is abelian, where $X_i \cong \mathbb{Z}/\ell \mathbb{Z}$, hence $A$ is two-step nilpotent, and the commutator in $A$ induces a bilinear alternating map:

$$[\cdot, \cdot] : A/Z \wedge A/Z \rightarrow Z,$$

$$\bar{a}_1 \wedge \bar{a}_2 \mapsto [a_1, a_2],$$

which must be trivial if we restrict to the cyclic subgroups $X_i \subset A/Z$. We assume that $A$ is non-abelian which means the commutator map is non-trivial. Then obviously

- the subgroups $H_i < A$ which are the full preimages of $X_i \subset A/Z$ must be maximal abelian subgroups such that $H_1 \cap H_2 = Z$, and
- the commutator subgroup is $[A, A] = [H_1, H_2] \cong \mathbb{Z}/\ell \mathbb{Z}$.

We fix characters $\chi_i$ of $H_i$ which agree on $Z : \chi_1|_Z = \chi_2|_Z$, and which are nontrivial on the commutator subgroup $[A, A]$. Then from Lemma 2.15 with $G, H, C$ replaced by $A, H_1, H_2$ we obtain

$$Ind^A_{H_1}(\chi_1) \cong Ind^A_{H_2}(\chi_2) \in R(A)$$

and we take

$$\sigma := [H_1, \chi_1] - [H_2, \chi_2] \in Ker(b_A)$$

as our relation of **type II**.

**Type III.** Based on Lemma 2.15 we now come to the **third type** of requirements on $A$, and $\sigma$:
$A = H \rtimes C$ is a semidirect product, where $C \neq \{1\}$ is an abelian normal subgroup contained in all nontrivial abelian normal subgroups of $A$. (In particular this implies that $C \neq \{1\}$ is minimal.) Making use of Lemma 2.15 with $G = A$, and $(H, 1_H) \in \mathcal{M}_A$, we have

\[ \text{Ind}^A_H(1_H) = \sum_{\mu \in T} \text{Ind}^A_{A_\mu}(\mu'), \]

and therefore

\[ \sigma := [H, 1_H] - \sum_{\mu \in T} [A_\mu', \mu'] \in \text{Ker}(b_A) \subseteq R_+(A). \]

**Remark 3.3.** For the three relations, we have indeed $b_A(\sigma) = 0$, as follows from Lemma 2.15: For type I set $H = \{e\}, C = A$, for type II set $H = H_1, C = H_2$, for type III set $H = H, C = C$.

We denote the $\mathbb{Z}$-module generated by relations of type I by $\mathcal{R}(I, G)$, for type I, and type II by $\mathcal{R}(II, G)$, and by type I, type II, and type III by $\mathcal{R}(III, G)$. This means $\mathcal{R}(-, G) \subseteq \text{Ker}(b_G) \subseteq R_+(G)$, and $\mathcal{R}(-, G)$ consists of all elements $\rho \in R_+(G)$ which are given as

\[ \rho = \sum_{i=1}^r n_i \cdot \rho_i = \sum_{i=1}^r n_i \cdot \text{Ind}^{B_i}_{+ B_i}(\chi_i \cdot \text{Inf}^{B_i}_{+ A_i}(\sigma_i)), \quad n_i \in \mathbb{Z}, \]

where $A_i = B_i/K_i$ is a subquotient of $G$, $\chi_i \in \widehat{B_i}$, and $\sigma_i \in \text{Ker}(b_{A_i})$ as specified above in the type I, type II, and type III.

We also can see that $\mathcal{R}(-, G)$ is stable under torsion because

\[ (G, \chi_G) \cdot \text{Ind}^{G}_{+ B}(\chi_B \cdot \text{Inf}^{B}_{+ A}(\sigma)) = \text{Ind}^{G}_{+ B}(\text{Res}^{G}_{B}(\chi_G)\chi_B \cdot \text{Inf}^{B}_{+ A}(\sigma)). \]

**Proof of Theorem 1.1:**

Now we are ready to prove Theorem 1.1. And we prove into three cases, and they are:

**Case-1:** When $G$ is an abelian group.

**Lemma 3.4** (Lemma 2.3 of [30]). If $G$ is abelian, then $\text{Ker}(b_G) = \mathcal{R}(I, G)$.

**Proof.** If $\rho \in \mathcal{R}(I, G)$, then from Remark 3.1(2) it is simple to see that $\rho \in \text{Ker}(b_G)$. Now to complete the proof, we have to show that $\text{Ker}(b_G) \subseteq \mathcal{R}(I, G)$.

If $G$ is commutative then the action by conjugation is always trivial, hence $[H, \chi] = (H, \chi)$, and therefore

\[ R_+(G) = \text{free abelian group over } \mathcal{M}_G = \bigoplus_{H \in \mathcal{M}_G} R(H) \]

where the second equality holds because all irreducible representations of $H$ are one-dimensional. Now assume that

\[ \rho = \sum_{[H_i, \chi_i] \in \mathcal{M}_G/G} \alpha_{[H_i, \chi_i]} \cdot (H_i, \chi_i) \in \text{Ker}(b_G) \subseteq R_+(G). \]

Thus we have:

\[ \sum_{[H_i, \chi_i] \in \mathcal{M}_G/G} \alpha_{[H_i, \chi_i]} \text{Ind}^G_{H_i}(\chi_i) \cong 0 \in R(G). \]
Moreover, because we are in the abelian case: $\text{Ind}_{\tilde{H}}^G(\chi_i) = \sum_j \tilde{\chi}_{ij}$, where $\tilde{\chi}_{ij}$ are the characters of $G$ which extend $\chi_i$. Therefore, our relation rewrites as

$$\sum_{[H,\chi_i] \in \mathcal{M}_G/G} \alpha_{[H,\chi_i]}(\sum_j \tilde{\chi}_{ij}) \cong 0 \in R(G).$$

However, $R(G)$ naturally embeds into $R_+(G)$, and therefore this is the same as

$$\sum_{[H,\chi_i] \in \mathcal{M}_G/G} \alpha_{[H,\chi_i]}(\sum_j (G, \tilde{\chi}_{ij})) \cong 0 \in R_+(G),$$

such that our original $\rho \in \text{Ker}(b_G)$ rewrites as

$$\rho = \sum_{[H,\chi_i] \in \mathcal{M}_G/G} \alpha_{[H,\chi_i]} \cdot (H, \chi_i) = \sum_{[H,\chi_i] \in \mathcal{M}_G/G} \alpha_{[H,\chi_i]} \cdot ((H, \chi_i) - \sum_j (G, \tilde{\chi}_{ij})),
$$

hence $\text{Ker}(b_G)$ is generated by the elements

$$(H, \chi) - \sum_{\tilde{\chi}} (G, \tilde{\chi})$$

where $\tilde{\chi} \in \tilde{G}$ runs over the extensions of $\chi \in \tilde{H}$. Finally, to prove our Lemma we are left to show that such elements can always be rewritten as

$$(3.1) \quad (H, \chi) - \sum_{\tilde{\chi}} (G, \tilde{\chi}) = \sum_{i} \text{Ind}_{+B_i}^G(\chi_i \cdot \text{Inf}_{+A_i}^G(\sigma_i)),
$$

where $A_i$ is a subquotient of $G$, and $\sigma_i \in \text{Ker}(A_i)$ is of type I.

If $H$ is of index $[G : H] = p$ a prime then we may take $B = G, A = G/H$, and $(3.1)$ rewrites

$$(H, \chi) - \sum_{\tilde{\chi}} (G, \tilde{\chi}) = \tilde{\chi} \cdot \text{Inf}_{+A}^G((e, 1_e) - \sum_{\mu \in \tilde{A}} (A, \mu)),
$$

where $\tilde{\chi}$ is one fixed extension of $\chi$, hence all other extensions are given as $\tilde{\chi} \cdot \mu$ for characters $\mu$ of $A = G/H$.

If $[G : H]$ is not a prime then we argue by induction on $[G : H]$ where we choose a subgroup $H'$ such that $H < H' < G$, and $[H' : H] = p$ a prime. Then, we have:

$$(3.2) \quad (H, \chi) - \sum_{\chi'} (G, \chi') = ((H, \chi) - \sum_{\chi'} (H', \chi')) + \sum_{\chi'} ((H', \chi') - \sum_{\tilde{\chi} | \chi'} (G, \tilde{\chi})),
$$

where $\chi'$ runs over the extensions of $\chi$ onto $H'$. Now the first term on right rewrites as

$$(H, \chi) - \sum_{\chi'} (H', \chi') = \text{Ind}_{+H'}^G(\chi_0' \cdot \text{Inf}_{+A}^{H'}((e, 1_e) - \sum_{\mu \in \tilde{A}} (A, \mu))),
$$

where $\chi_0' \in \tilde{H'}$ is a fixed $\chi'$, and where $A = H'/H, e = H/H$, and $H'$ plays the role of $B < G$. Then under $u : H' \to A = H'/H$ we have $u^{-1}(e) = H, u^{-1}(A) = H'$ such that

$$\chi_0' \cdot \text{Inf}_{+A}^{H'}((e, 1_e) - \sum_{\mu \in \tilde{A}} (A, \mu)) = (H, \chi) - \sum_{\chi'} (H', \chi') \in R_+(H'),
$$

and $\text{Ind}_{+H'}^G : R_+(H') \to R_+(G)$ is the identity. We are left with the other terms on the right of $(3.2)$, and here we can use the induction hypothesis because $[G : H'] < [G : H]$.

Case-2: When $G$ is a nilpotent group.
Theorem 3.5 (Langlands, Section 18 of [35]). If \( G \) is nilpotent, then \( \text{Ker}(b_G) = \mathcal{R}(II, G) \).

Proof. From Remark 3.1(2), it can be seen that \( \mathcal{R}(II, G) \subset \text{Ker}(b_G) \). Now we have to prove \( \text{Ker}(b_G) \subset \mathcal{R}(II, G) \). When \( G \) is abelian, in Lemma 3.4 it is proved. So to complete the proof, we have to prove this when \( G \) is nonabelian nilpotent group.

Because abelian groups are nilpotent, and in Lemma 3.4 it is studied, here we consider \( G \) as a nonabelian nilpotent group. Let \( Z \) be the center of \( G \). We prove Theorem 3.5 by induction on \(|G/Z|\). When \(|G/Z| = 1\), that is, \( G \) is abelian, and it is proved above. Let \(|G/Z| \neq 1\). Take an arbitrary element

\[(3.3) \quad \rho = \sum_{[H_i, \chi_i] \in \mathcal{M}_{G/G}} \alpha_{[H_i, \chi_i]} [H_i, \chi_i] \in \text{Ker}(b_G).\]

Now we have to show that \( \rho \in \mathcal{R}(II, G) \).

We can express \( b_G([H_i, \chi_i]) = \text{Ind}^G_{H_i} (\chi_i) \) as follows:

\[\text{Ind}^G_{H_i} (\chi_i) = \sum_{\chi_i' \mid u_i = \chi_i} \text{Ind}^G_{H_i, Z} (\chi_i') + \text{Ind}^G_{H_i, Z} (\text{Ind}^{H_i, Z}_{H_i} (\chi_i)) - \sum_{\chi_i' \mid u_i = \chi_i} \chi_i',\]

and

\[\text{Ind}^{H_i, Z}_{H_i} (\chi_i) - \sum_{\chi_i' \mid u_i = \chi_i} \chi_i' = b_{H_i, Z}(\rho_1),\]

where

\[\rho_1 := [H_i, \chi_i] - \sum_{\chi_i' \mid u_i = \chi_i} [H_i, Z, \chi_i']\]

Again \( \rho_1 \) can be expressed as follows:

\[\rho_1 = \text{Ind}^{H_i, Z}_{H_i, Z/\text{Ker}(\chi_i)} ([H_i/\text{Ker}(\chi_i), \chi_i] - \sum_{\chi_i' \mid u_i = \chi_i} [H_i, Z/Ker(\chi_i), \chi_i'])\]

and by Lemma 3.4

\[[H_i/\text{Ker}(\chi_i), \chi_i] - \sum_{\chi_i' \mid u_i = \chi_i} [H_i, Z/Ker(\chi_i), \chi_i'] \in \mathcal{R}(I, H_i, Z/Ker(\chi_i)),\]

hence \( \rho_1 \in \mathcal{R}(I, H_i) \). This shows that when \( Z \not\subset H_i \), \( \rho \in \mathcal{R}(II, G) \).

To complete the proof, we have to show this when \( Z \subset H_i \).

Let \( C \) be a abelian normal subgroup of \( G \) containing \( Z \) such that \(|C : Z| = \ell\) is a prime, and \( C/Z \) is in the center of \( G/Z \). With the notation of Lemma 2.15 for \( H = H_i, \chi = \chi_i \), and \( G = G \) we have

\[[H_i, \chi_i] = \sum_{\mu \in T} [C_{\mu}, \mu'] + \text{Ind}^G_{H, C} (\rho_2)\]

with

\[\rho_2 = [H_i, \chi_i]_{H, C} - \sum_{\mu \in T} [C_{\mu}, \mu']_{H, C}.\]

Therefore, a relation (3.3) with \( C \subset H_i \) are in \( \mathcal{R}(II, G) \).
Lemma 2.17 shows that a relation of type (3.3) with $C \subseteq H_i$ is a linear combination of relations induced from relations of the form

$$\sum_{[H_i, \chi_i] \in \mathcal{M}_{G_i}/G_i} \alpha_{[H_i, \chi_i]} [H_i, \chi_i] \in Ker(b_{G_i})$$

with $H_i \supseteq C$, and $\chi_i|_C = \mu$, where $G_i$ denotes the isotropy group of $\mu \in \hat{C}$ in $G$.

Furthermore

$$\sum_{[H_i, \chi_i] \in \mathcal{M}_{G_i}/G_i} \alpha_{[H_i, \chi_i]} [H_i, \chi_i] = \text{Inf}_{G_j/Ker(\mu)}(\rho_3)$$

with

$$\rho_3 = \sum u_i [H_i/Ker(\mu), \chi_i] \in Ker(\varphi_{G_i/Ker(\mu)}).$$

Because $C/Ker(\mu)$ is central in $G_i/Ker(\mu)$ by definition of $G_i$, we can apply the induction assumption to $\rho_3$.

It remains to show $\rho_2 \in \mathcal{R}(II, H_i C)$. We omit the index $i$. If $HC \neq G$, this follows from the induction assumption applied to $HC$. Hence we assume $HC = G$. Then $H$ is a normal subgroup of $G$ because $Z \subseteq H$, and $C/Z$ is central in $G/Z$. We set

$$X := \cap_{g \in G} Ker(\chi)^g.$$ 

Then $\rho_2$ is the inflation of the corresponding relation in $Ker(b_{G/X})$. Hence it is sufficient to prove:

if $X = \{1\}$, the relation $\rho_2$ is of type II.

$H$ is abelian because $[H, H] \subseteq X = \{1\}$, $Z$ is cyclic because $Ker(\chi|_Z) \subseteq X = \{1\}$. Because $C/Z$ is central in $G/Z$, the commutator $[c, h] := c^{-1} h^{-1} c h \in H$, lies in $Z$, and depends bi-additively on $c$, and $h$. We define a bilinear form $u : C/Z \otimes H/Z \rightarrow Z$ by

$$u(c, h) = [c, h].$$

Let $\bar{c}$ be a generator of $C/Z$. Because $Z$ is the center of $G$, and $G \neq Z$, the map $u(\bar{c}, \cdot) : H/Z \rightarrow Z$ is injective, and image is not trivial. Since

$$[c, h^\ell] = [c, h^\ell] = [\ell, h] = 1 \quad \text{for } h \in H,$$

we have $[H : Z] = [C : Z] = \ell$. Because $H$ is not central, and $X = \{1\}$, $\chi$ is distinct from some of its conjugates, hence $\chi([h, h]) \neq \{1\}$, and we are in the situation of type II.

\[\square\]

Case-3: When $G$ is a solvable group.

**Theorem 3.6 (Theorem 2.6 of [30]).** If $G$ is solvable, $Ker(b_G) = \mathcal{R}(III, G)$.

**Proof.** As before from Remark 3.1(2), it is simple to see that $\mathcal{R}(III, G) \subset Ker(b_G)$. Now to complete this proof, we have to show the inclusion: $Ker(b_G) \subset \mathcal{R}(III, G)$. When $G$ is nilpotent, we have proved this above. So we only need to show that for non-nilpotent solvable group $G$, we have $Ker(b_G) \subset \mathcal{R}(III, G)$.

We will prove this by induction on the order of $G$. If $|G| = 1$, then the assertion is trivial. Now let $H$ be a proper subgroup of $G$, and for $H$ the assertion is true, that is,

$$Ker(b_H) = \mathcal{R}(III, H) \quad \text{for all proper subgroups } H \text{ of } G.$$
Step 1: By using this above assumption, we can see that $R(III, G)$ is an ideal of the ring $R_+(G)$.

Let $\rho \in R(III, G)$, and $[H, \chi] \in R_+(G)$. By definition (see torsion in $R_+(G)$ on p. 5), if $H = G$, then $\rho \cdot [G, \chi]$ belongs to $R(III, G)$.

If $H \neq G$, we can see that for $\rho \in R(III, G)$, hence $\rho \in \text{Ker}(b_G)$ we have

$$b_H(\text{Res}^G_{+H}(\rho)) = \text{Res}^G_{+H}(b_G(\rho)) = \text{Res}^G_{+H}(0) = 0.$$  

This implies $\text{Res}^G_{+H}(\rho) \in \text{Ker}(b_H) = R(III, H)$, hence $\text{Res}^G_{+H}(\rho) \cdot [H, \chi] \in R(III, H)$.

Now from Proposition \ref{2.2} we can write

$$\rho \cdot [H, \chi] = \text{Ind}^G_{+H}(\text{Res}^G_{+H}(\rho) \cdot [H, \chi]) \in R(III, G).$$

This implies that $R(III, G)$ is an ideal of $R_+(G)$.

Let $Z$ be the center of $G$.

Step 2: Suppose that $Z$ be nontrivial. We use the Brauer Induction theorem for $G/Z$ (see \cite{41} Section 10.5, Theorems 19, and 20). There are nilpotent subgroups $H_i \supseteq Z$, and characters $\chi_i$ of $H_i/Z$ such that

$$1_{G/Z} = \sum_i n_i \cdot \text{Ind}^G_{H_i/Z}(\chi_i) \quad \text{with} \quad n_i \in \mathbb{Z}.\tag{3.4}$$

Here $G$, and $G/Z$ are not nilpotent, but the $H_i/Z$ are nilpotent, hence they are proper subgroups of $G/Z$ which ensures that we obtain a nontrivial relation.

Because we do induction over the order of $G$, we may assume

$$\text{Ker}(b_{G/Z}) = R(III, G/Z),$$

and by inflation we obtain

$$\sigma := [G, 1_G] - \sum_i n_i \cdot [H_i, \chi_i] \in R(III, G).\tag{3.5}$$

Given $\rho \in \text{Ker}(b_G)$, by using Proposition \ref{2.2} we have

$$\rho \sigma = \rho - \sum_i n_i \cdot (\rho \cdot [H_i, \chi_i]) = \rho - \sum_i n_i \text{Ind}^G_{H_i}(\text{Res}^G_{+H_i}(\rho) \cdot [H_i, \chi_i]).\tag{3.6}$$

However, by assumption $R(III, G)$ is an ideal in $R_+(G)$, hence $\rho \sigma \in R(III, G)$, and we have seen $\text{Res}^G_{+H_i}(\rho) \in \text{Ker}(b_{H_i}) = R(III, H_i)$ because $H_i$ are proper subgroups of $G$. Therefore, it implies

$$\rho = \rho \sigma + \sum_i n_i \text{Ind}^G_{H_i}(\text{Res}^G_{+H_i}(\rho) \cdot [H_i, \chi_i]) \in R(III, G)\tag{3.7}$$

which proves our assertion in the case $Z \neq \{1\}$.

Step 3: When $Z = \{1\}$.

Because $G$ is a solvable group, let $C$ be a minimal abelian normal subgroup of $G$, which exists because $G$ is solvable. We proceed as in the proof of Theorem \ref{3.5}. Every relation $\rho \in \text{Ker}(b_G)$ is a linear combination of the form $\rho_2$ (of the proof of Theorem \ref{3.5}) for $HC$, where $H$ is a subgroup of $G$ which does not contain $C$, and $\chi$ a character of $H$, and of relations induced from relations of the form

$$\sum_{[H_i, \chi_i] \in \mathcal{M}_{G_{\rho}}/G_{\rho}} \alpha_{[H_i, \chi_i]}[H_i, \chi_i] \in \text{Ker}(b_{G_{\rho}}).\tag{3.8}$$
with $G_\mu \supseteq H_1 \supseteq C$, and $\chi_1|_C = \mu \in \hat{C}$. Since $|G_\mu : \text{Ker}(\mu)| < |G|$, we can apply the induction assumption to (3.8). It remains to consider a relation of the form $\rho_2$.

If $HC \neq G$ we can apply the induction assumption. If $HC = G$, then $H \cap C$ is normal in $H$ and $C$, hence normal in $G$. The minimality of $C$ implies $H \cap C = \{1\}$. Therefore, we have $G = H \ltimes C$ is semidirect and using $G/C \cong H$ we can extend $\chi \in \hat{H}$ by setting $\chi(c) = 1$ for $c \in C$ to a character $\bar{\chi}$ of $G$. Then our relation is the torsion of

$$[H, 1_H] - \sum_{\mu \in T} [G'_\mu, \mu']$$

with $[G, \bar{\chi}]$, where $\mu'$ denotes the extension of $\mu$ to $G'_\mu$ which is trivial on $G'_\mu \cap H$.

Suppose there is a nontrivial normal subgroup $H_1$ of $H$ lying in the centralizer of $C$. Then $H_1$ is normal in $G$ because $G = HC$. Then $[H_1, C] \subseteq H_1 \cap C \subset H \cap C = \{1\}$. Therefore, the group $H_1$ is contained in $G'_\mu$ for all $\mu$ hence (3.9) is the inflation of

$$[H/H_1, 1_{H/H_1}] - \sum_{\mu \in T} [G'_\mu/H_1, \mu],$$

which is contained in $\mathcal{R}(III, G/H_1)$ by induction assumption, and then the inflation, and its torsion by $[G, \bar{\chi}]$ must be in $\mathcal{R}(III, G)$.

Now we assume that $G$ is the semidirect product of $H$, and $C$, that $C$ is a minimal nontrivial abelian normal subgroup of $G$, and that $H$ contains no normal subgroup lying in the centralizer $Z$ of $C$. Then $Z = (Z \cap H)C$, and $Z \cap H$ is a normal subgroup of $H$. Hence $Z = C$. If $D$ is a normal subgroup of $G$, and $D$ does not contain $C$, then, by assumption, $D \cap C = \{1\}$. The commutator of $C$, and $D$ is contained in $D \cap C$, therefore $D$ is contained in $Z = C$, and $D = \{1\}$. This means that

$$[G, \bar{\chi}] \cdot ([H, 1_H]_G - \sum_{\mu \in T} [G'_\mu, \mu']_G)$$

is of type III.

Now we are prepared to prove Theorem 1.3.

**Proof of Theorem 1.3**: Let $\mathcal{F}$ be extendible. Then in the situation of type I we have

$$\text{Ind}^B_K(\chi) = \sum_{\mu \in \hat{A}} \mu \chi.$$

Hence by Equation (2.16), and Equation (2.18)

$$\mathcal{F}((B, \text{Ind}^B_K(\chi))) = \lambda^B_K(\mathcal{F}) \mathcal{F}((K, \chi)) = \prod_{\mu \in \hat{A}} \mathcal{F}((B, \mu \chi)).$$

Furthermore,

$$\lambda^B_K(\mathcal{F}) = \mathcal{F}((B, \text{Ind}^B_K(1_K))) = \prod_{\mu \in \hat{A}} \mathcal{F}((B, \mu)) = \prod_{\mu \in \hat{A}} \mathcal{F}((A, \mu)).$$

This implies Equation (1.2).

In the situation of type II, we have

$$\text{Ind}^B_{G_1}(\chi_1 \chi) = \text{Ind}^B_{G_2}(\chi_2 \chi).$$
Hence by Equation (2.18)

$$
\lambda_{G_1}^B(F)((G_1, \chi_1)) = \lambda_{G_2}^B(F)((G_2, \chi_2)),
$$

$$
\lambda_{G_i}^B(F) = F((B, Ind_{G_i}^B(1, \mu)), \ i = 1, 2,
$$

which implies Equation (1.3).

In the situation of type III, by Lemma 2.15 we have

$$
Ind_{G_i}^B(\chi) = \sum_{\mu \in T} Ind_{B_i}^B(\mu \chi).
$$

Hence

(3.10)

$$
\lambda_{G_i}^B(F)F((G', \chi)) = \prod_{\mu \in T} \lambda_{B_i}^B(F)F((B_i', \mu \chi)).
$$

In particular for $\chi = 1_B$

(3.11)

$$
\lambda_{G_i}^B(F) = \prod_{\mu \in T} \lambda_{B_i}^B(F)F((A_i', \mu)).
$$

Equations (3.10), and (3.11) imply Equation 1.4.

Conversely, we assume that $F$ is a function which satisfies (1.2)-(1.4). We want to show that $F$ is (weakly) extendible.

Now we are able to show that $U \rightarrow \lambda_{G_i}^B(F)$ satisfies equation (2.22) for $H = G$. Theorem 2.11 shows that it is sufficient to prove that

(3.12)

$$
\sum_{[U, \chi_1] \in M_{G_i}/G_i} \alpha_{[U_1, \chi_i]}(0) Ind_{G_i}^B(\chi_i) \cong 0
$$

implies

(3.13)

$$
\prod_{[U_i, \chi_i] \in M_{G_i}/G_i} F((U_i, \chi_i))^{\alpha_{[U_i, \chi_i]}(0)} \lambda_{U_i}^G(F) = 1
$$

if (3.12) is of the form

$$
Ind_{B_i}^G(\chi \cdot Ind_{A_i}^B(\sigma')) \cong 0,
$$

where $\sigma'$ is of the form type I, type II, or type III:

**type I.**

$$
\sigma' = Ind_{e}^A(1_e) - \sum_{\mu \in \tilde{A}} \mu.
$$

Then

$$
Ind_{B_i}^G(\chi \cdot Ind_{A_i}^B(\sigma')) = Ind_{K_i}^G(\chi) - \sum_{\mu \in \tilde{A}} Ind_{B_i}^G(\mu \chi).
$$

Hence we have to show

$$
F((K, \chi)) \lambda_{K}^G(F) = \prod_{\mu \in \tilde{A}} F((B, \mu \chi)) \lambda_{B}^G(F).
$$

By Lemma 2.21 this follows from

(3.14)

$$
F((K, \chi)) \lambda_{K}^G(F) = \prod_{\mu \in \tilde{A}} F((B, \mu \chi)).$$
Therefore, Equation (1.2) implies Equation (3.14).

By definition of $\lambda^B_K(F)$,

$$\lambda^B_K(F) = \lambda^A_K(F) = \prod_{\mu \in \mathcal{A}} \mathcal{F}((A, \mu)).$$

Therefore, Equation (1.2) implies Equation (3.14).

In the case type II,

$$\sigma = \text{Ind}^A_{H_1}(\chi_1) - \text{Ind}^A_{H_2}(\chi_2),$$

and type III,

$$\sigma = \text{Ind}^A_H(1_H) - \sum_{\mu \in \mathcal{T}} \text{Ind}^A_{H_\mu}(\mu'),$$

one proceeds in a similar way as in the case type I. This concludes the proof of the existence of $U \rightarrow \lambda^B_U(F)$ with properties (2.22), (2.23), and by Theorem 2.11 the proof of Theorem 1.3.

4. Local root numbers and their properties

4.1. Local Fields and their finite extensions. Let $F$ be a non-archimedean local field, i.e., a finite extension of the field $\mathbb{Q}_p$ (field of $p$-adic numbers), where $p$ is a prime. Let $K/F$ be a finite extension of the field $F$. Let $e_{K/F}$ be the ramification index for the extension $K/F$, and $f_{K/F}$ be the residue degree of the extension $K/F$. The extension $K/F$ is called unramified if $e_{K/F} = 1$; equivalently $f_{K/F} = [K : F]$. The extension $K/F$ is totally ramified if $e_{K/F} = [K : F]$; equivalently $f_{K/F} = 1$. Let $q_F$ be the cardinality of the residue field $k_F$ of $F$. If $\gcd(p, [K : F]) = 1$, then the extension $K/F$ is called tamely ramified, otherwise wildly ramified. The extension $K/F$ is totally tamely ramified if it is both totally ramified, and tamely ramified.

For a tower of local fields $K/L/F$, we have (cf. [19], p. 39, Lemma 2.1)

$$e_{K/F}(\nu_K) = e_{K/L}(\nu_K) \cdot e_{L/F}(\nu_L),$$

where $\nu_K$ is a valuation on $K$, and $\nu_L$ is the induced valuation on $L$, i.e., $\nu_L = \nu_K|L$. For the tower of fields $K/L/F$ we simply write $e_{K/F} = e_{K/L} \cdot e_{L/F}$. Let $O_F$ be the ring of integers in the local field $F$, and $P_F = \pi_F O_F$ is the unique prime ideal in $O_F$ and $\pi_F$ is a uniformizer, i.e., an element in $P_F$ whose valuation is one, i.e., $\nu_F(\pi_F) = 1$. Let $U_F = O_F - P_F$ be the group of units in $O_F$. Let $P_F^i = \{x \in F : \nu_F(x) \geq i\}$, and for $i \geq 0$ define $U_F^i = 1 + P_F^i$ (with proviso $U_F^0 = U_F = O_F^\times$). We also consider that $a(\chi)$ is the conductor of nontrivial character $\chi : F^\times \rightarrow \mathbb{C}^\times$, i.e., $a(\chi)$ is the smallest integer $\geq 0$ such that $\chi$ is trivial on $U_F^{a(\chi)}$. We say $\chi$ is unramified if the conductor of $\chi$ is zero, and otherwise ramified. Now onwards when $K/F$ is unramified we choose uniformizers $\pi_K = \pi_F$. When $K/F$ is totally ramified (both tame, and wild) we choose uniformizers $\pi_F = N_{K/F}(\pi_K)$, where $N_{K/F}$ is the norm map from $K^\times$ to $F^\times$.

**Definition 4.1 (Different, and Discriminant).** Let $K/F$ be a finite separable extension of non-archimedean local field $F$. We define the **inverse different (or codifferent)** $\mathcal{D}^{-1}_{K/F}$ of $K$ over $F$ to be $\pi_F^{-d_{K/F}} O_K$, where $d_{K/F}$ is the largest integer (this is the exponent of the different $\mathcal{D}_{K/F}$) such that

$$\text{Tr}_{K/F}(\pi_K^{-d_{K/F}} O_K) \subseteq O_F,$$

where $\text{Tr}_{K/F}$ is the trace map from $K$ to $F$. Then the **different** is defined by:
\[ D_{K/F} = \pi_K^{d_{K/F}} O_K \]

and the **discriminant** \( D_{K/F} \) is

\[ D_{K/F} = N_{K/F}(\pi_K^{d_{K/F}})O_F. \]

Thus it is simple to see that \( D_{K/F} \) is an **ideal of** \( O_F \).

We know that if \( K/F \) is unramified, then \( D_{K/F} \) is a unit in \( O_F \). If \( K/F \) is tamely ramified, then

\[ (4.2) \quad \nu_K(D_{K/F}) = d_{K/F} = e_{K/F} - 1. \]

(see [40], Chapter III, for details about different, and discriminant of the extension \( K/F \).)

We need to mention a very important result of J-P. Serre for our purposes.

**Lemma 4.2** ([40], p. 50, Proposition 7). Let \( K/F \) be a finite separable extension of the field \( F \). Let \( I_F \) (resp. \( I_K \)) be a fractional ideal of \( F \) (resp. \( K \)) relative to \( O_F \) (resp. \( O_K \)). Then the following two properties are equivalent:

(1) \( \text{Tr}_{K/F}(I_K) \subseteq I_F \).

(2) \( I_K \subseteq I_F : D_{K/F}^{-1} \).

**Definition 4.3 (Canonical additive character).** We define the non trivial additive character of \( F \), \( \psi_F : F \to \mathbb{C}^\times \) as the composition of the following four maps:

\[ F \xrightarrow{\text{Tr}_{F/Q_p}} \mathbb{Q}_p \xrightarrow{\alpha} \mathbb{Q}_p / \mathbb{Z}_p \xrightarrow{\beta} \mathbb{Q} / \mathbb{Z} \xrightarrow{\gamma} \mathbb{C}^\times, \]

where

1. \( \text{Tr}_{F/Q_p} \) is the trace from \( F \) to \( \mathbb{Q}_p \),
2. \( \alpha \) is the canonical surjection map,
3. \( \beta \) is the canonical injection which maps \( \mathbb{Q}_p / \mathbb{Z}_p \) onto the \( p \)-component of the divisible group \( \mathbb{Q} / \mathbb{Z} \), and
4. \( \gamma \) is the exponential map \( x \mapsto e^{2\pi i x} \), where \( i = \sqrt{-1} \).

For every \( x \in \mathbb{Q}_p \), there is a rational \( r \), uniquely determined modulo 1, such that \( x - r \in \mathbb{Z}_p \).

Then \( \psi_{\mathbb{Q}_p}(x) = \psi_{\mathbb{Q}_p}(r) = e^{2\pi i r} \). The nontrivial additive character \( \psi_F = \psi_{\mathbb{Q}_p} \circ \text{Tr}_{F/Q_p} \) of \( F \) is called the **canonical additive character** (cf. [43], p. 92).

The **conductor** of any nontrivial additive character \( \psi \) of the field \( F \) is an integer \( n(\psi) \) if \( \psi \) is trivial on \( P_F^{-n(\psi)} \), but nontrivial on \( P_F^{-n(\psi)-1} \). So, from Lemma 4.2 we can observe that

\[ n(\psi_F) = n(\psi_{\mathbb{Q}_p} \circ \text{Tr}_{F/Q_p}) = \nu_F(D_{F/Q_p}), \]

because \( d_{\mathbb{Q}_p/Q_p} = 0 \), and hence \( n(\psi_{\mathbb{Q}_p}) = 0 \).

**4.2. Local Constants (or local root numbers/local epsilon factors).** Let \( F \) be a non-archimedean local field, and \( \chi \) be a character of \( F^\times \). The \( L(\chi) \)-functions are defined as follows:

\[ L(\chi) = \begin{cases} 
(1 - \chi(\pi_F))^{-1} & \text{if } \chi \text{ is unramified}, \\
1 & \text{if } \chi \text{ is ramified}.
\end{cases} \]

We denote by \( dx \) a Haar measure on \( F \), by \( d^\times x \) a Haar measure on \( F^\times \), and the relation between these two Haar measure is:

\[ d^\times x = \frac{dx}{|x|}, \]
for arbitrary Haar measure $dx$ on $F$. For a given additive character $\psi$ of $F$, and Haar measure $dx$ on $F$, we have a **Fourier transform** as:

$$ (4.3) \quad \hat{f}(y) = \int f(x)\psi(xy)dx. $$

where $f \in L^1(F^\times)$ (that is, $|f|$ is integrable), and the Haar measure is normalized such that $\hat{f}(y) = f(-y)$, i.e., $dx$ is self-dual with respect to $\psi$. By Tate (cf. [15], p. 13), for any character $\chi$ of $F^\times$, there exists a number $W(\chi, \psi, dx) \in \mathbb{C}^\times$ such that it satisfies the following local functional equation:

$$ (4.4) \quad \frac{\int \hat{f}(x)w_1x^{-1}(x)d\sigma x}{L(w_1x^{-1})} = W(\chi, \psi, dx) \int f(x)\chi(x)d\sigma x \frac{L(\chi)}{L(w_1x^{-1})}, $$

for any such function $f$ for which the both sides make sense. Here $w_s(x) = |x|^s_F = q_F^{-s\nu_F(x)}$ is unramified character of $F^\times$. The number $W(\chi, \psi, dx)$ is called the **local epsilon factor or local constant** of $\chi$.

For a nontrivial multiplicative character $\chi$ of $F^\times$, and nontrivial additive character $\psi$ of $F$, we have (cf. [35], p. 4)

$$ (4.5) \quad W(\chi, \psi, c) = \chi(c) \int_{U_F} x^{-1}(x)\psi(x/c)dx \frac{\int f(x)\chi(x)d\sigma x}{|\int f(x)\chi(x)\psi(x/c)dx|}, $$

where the Haar measure $dx$ is normalized such that the measure of $O_F$ is 1, and where $c \in F^\times$ with valuation $n(\psi) + a(\chi)$. The **modified** formula of local constant (cf. [43], p. 94) is:

$$ (4.6) \quad W(\chi, \psi, c) = \chi(c)q^{-a(\chi)/2} \sum_{x \in \frac{v_F}{v_F^{(1)}}} \chi^{-1}(x)\psi(x/c). $$

where $c = \pi_F^{a(\chi)+n(\psi)}$. Now if $u \in U_F$ is unit, and replace $c = cu$, then we have

$$ (4.7) \quad W(\chi, \psi, cu) = \chi(c)q^{-\frac{\nu_F(c)}{2}} \sum_{x \in \frac{v_F}{v_F^{(1)}}} \chi^{-1}(x/u)\psi(x/cu) = W(\chi, \psi, c). $$

Therefore, $W(\chi, \psi, c)$ depends only on the exponent $\nu_F(c) = a(\chi) + n(\psi)$. Therefore, we can simply write $W(\chi, \psi, c) = W(\chi, \psi)$, because $c$ is determined by $\nu_F(c) = a(\chi) + n(\psi)$ up to a unit $u$ which has no influence on $W(\chi, \psi, c)$. If $\chi$ is unramified, i.e., $a(\chi) = 0$, therefore $\nu_F(c) = n(\psi)$. Then from the formula of $W(\chi, \psi, c)$, we can write

$$ (4.8) \quad W(\chi, \psi, c) = \chi(c), $$

and therefore $W(1, \psi, c) = 1$ if $\chi = 1$ is the trivial character.

**Theorem 4.4 (Lamprecht-Tate formula, Proposition 1 of [13]).** \(^2\) Let $F$ be a non-Archimedean local field. Let $\chi$ be a character of $F^\times$ of conductor $a(\chi)$, and let $m$ be a natural number such that $2m \leq a(\chi)$. Let $\psi$ be a nontrivial additive character of $F$. Then there exists $c \in F^\times$, $\nu_F(c) = a(\chi) + n(\psi)$ such that

$$ (4.9) \quad \chi(1+y) = \psi(c^{-1}y) \quad \text{for all } y \in P_F^{a(\chi)-m}, $$

\(^2\)Before Tate, in 1953 in [34] Erich Lamprecht gave a formula for local epsilon factors of linear characters. Then Tate generalizes the formula for epsilon factors.
and for such a $c$ we have:

\begin{equation}
W(\chi, \psi) = \chi(c) \cdot q_F^{-\frac{a(\chi) - 2m}{2}} \sum_{x \in U_F^m / U_F^{m(x)-m}} \chi^{-1}(x) \psi(c^{-1}x).
\end{equation}

**Remark:** Note that the assumption (4.9) is obviously fulfilled for $m = 0$ because then both sides are $= 1$, and the resulting formula for $m = 0$ is the Tate formula (4.6).

For the proof we refer [3].

4.3. Some properties of $W(\chi, \psi)$.

(1) Let $b \in F^\times$ be the uniquely determined element such that $\psi' = b \psi$. Then

\begin{equation}
W(\chi, \psi', c') = \chi(b) \cdot W(\chi, \psi, c).
\end{equation}

**Proof.** Here $\psi'(x) = (b \psi)(x) := \psi(bx)$ for all $x \in F$. It is an additive character of $F$. The existence, and uniqueness of $b$ is clear. From the definition of conductor of an additive character we have

$$n(\psi') = n(b \psi) = n(\psi) + \nu_F(b).$$

Here $c' \in F^\times$ is of valuation

$$\nu_F(c') = a(\chi) + n(\psi') = a(\chi) + \nu_F(b) + n(\psi) = \nu_F(b) + \nu_F(c) = \nu_F(bc).$$

Therefore, $c' = bcu$ where $u \in U_F$ is some unit. Now

\begin{align*}
W(\chi, \psi', c') &= W(\chi, b \psi, bcu) \\
&= W(\chi, b \psi, bc) \\
&= \chi(bc)q_F^{-\frac{a(\chi)}{2}} \sum_{x \in U_F^m / U_F^{m(x)}} \chi^{-1}(x)((bc)^{-1}(b \psi))(x) \\
&= \chi(b) \cdot \chi(c)q_F^{-\frac{a(\chi)}{2}} \sum_{x \in U_F^m / U_F^{m(x)}} \chi^{-1}(x)\psi(xc^{-1}) \\
&= \chi(b) \cdot W(\chi, \psi, c).
\end{align*}

\(\square\)

(2) Let $F/\mathbb{Q}_p$ be a local field inside $\overline{\mathbb{Q}_p}$. Let $\chi$, and $\psi$ be a character of $F^\times$, and $F^+$ respectively, and $c \in F^\times$ with valuation $\nu_F(c) = a(\chi) + n(\psi)$. If $\sigma \in \text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ is an automorphism, then:

$$W_F(\chi, \psi, c) = W_{\sigma^{-1}(F)}(\chi^\sigma, \psi^\sigma, \sigma^{-1}(c)),$$

where $\chi^\sigma(y) := \chi(\sigma(y))$, $\psi^\sigma(y) := \psi(\sigma(y))$, for all $y \in \sigma^{-1}(F)$.

**Proof.** Let $L := \sigma^{-1}(F)$. Because $\sigma$ is an automorphism of $\overline{\mathbb{Q}_p}$, then we have $O_F/P_F \cong O_L/P_L$, hence $q_F = q_L$. We also can see that $a(\chi^\sigma) = a(\chi)$, and $n(\psi^\sigma) = n(\psi)$. Then
from the formula of local constant we have

\[ W_{\sigma^{-1}(F)}(\chi^\sigma, \psi^\sigma, \sigma^{-1}(c)) = W_L(\chi^\sigma, \psi^\sigma, \sigma^{-1}(c)) \]

\[ = \chi^\sigma(\sigma^{-1}(c))q_L^{-\frac{a(\chi)}{2}} \sum_{y \in \mathfrak{u}_F} (\chi^\sigma)^{-1}(y) \cdot ((\sigma^{-1}(c))^{-1}\psi^\sigma(y)) \]

\[ = \chi(c)q_F^{-\frac{a(\chi)}{2}} \sum_{x \in \mathfrak{u}_F} \chi^{-1}(x)\psi\left(\frac{x}{c}\right) \]

\[ = W_F(\chi, \psi, c). \]

Here we put \( y = \sigma^{-1}(x) \), and use \( (\sigma^{-1}(c))^{-1}\psi^\sigma = (c^{-1}\psi)^{\sigma} \).

Remark 4.5. We can simply write as before \( W_F(\chi, \psi) = W_{\sigma^{-1}(F)}(\chi^\sigma, \psi^\sigma) \). Tate in his paper [43] on local constants defines the local root number as:

\[ W_F(\chi) := W_F(\chi, \psi_F) = W_F(\chi, \psi_F, d), \]

where \( \psi_F \) is the canonical character of \( F^\times \), and \( d \in F^\times \) with \( \nu_F(d) = a(\chi) + n(\psi_F) \). Therefore, after fixing canonical additive character \( \psi = \psi_F \), we can rewrite

\[ W_F(\chi) = \chi(d(\psi_F)), \text{ if } \chi \text{ is unramified}, \]

\[ W_F(\chi) = W_{\sigma^{-1}(F)}(\chi^\sigma). \]

The last equality follows because the canonical character \( \psi_{\sigma^{-1}(F)} \) is related to the canonical character \( \psi_F \) as: \( \psi_{\sigma^{-1}(F)} = \psi_F^\sigma. \)

So we see that

\[ (F, \chi) \to W_F(\chi) \in \mathbb{C}^\times \]

is a function with properties (2.14), (2.15) of extendible functions.

(3) If \( \chi \in \hat{F}^\times \), and \( \psi \in \hat{F} \), then

\[ W(\chi, \psi) \cdot W(\chi^{-1}, \psi) = \chi(-1). \]

Furthermore if the character \( \chi : F^\times \to \mathbb{C}^\times \) is unitary (in particular, if \( \chi \) is of finite order), then

\[ |W(\chi, \psi)|^2 = 1. \]

Proof. We prove this properties by using equation [4.6]. We know that the additive characters are always unitary, hence

\[ \psi(-x) = \psi(x)^{-1} = \overline{\psi}(x). \]
On the other hand we write \( \psi(-x) = ((-1)\psi)(x) \), where \(-1 \in F^\times\). Therefore, \( \bar{\psi} = (-1)\psi \). We also have \( a(\chi) = a(\chi^{-1}) \). Therefore, by using equation (4.6) we have

\[
W(\chi, \psi) \cdot W(\chi^{-1}, \psi) = \chi(-1) \cdot q_F^{-a(\chi)} \sum_{x,y \in \frac{U_F}{\nu_F^{a(x)}}} \chi^{-1}(x)\psi(-\frac{x-y}{c})
\]

\[
= \chi(-1) \cdot q_F^{-a(\chi)} \sum_{x,y \in \frac{U_F}{\nu_F^{a(x)}}} \chi^{-1}(x)\psi(-\frac{xy - y}{c}), \text{ replacing } x \text{ by } xy
\]

\[
= \chi(-1) \cdot q_F^{-a(\chi)} \sum_{x \in \frac{U_F}{\nu_F^{a(x)}}} \chi^{-1}(x)\varphi(x),
\]

where

\[
\varphi(x) = \sum_{y \in \frac{U_F}{\nu_F^{a(x)}}} \psi(\frac{x-1}{c}).
\]

Because \( \frac{U_F}{\nu_F^{a(x)}} = \left( \frac{P_F}{P_F} \right)^x = \frac{P_F^{\nu_F^{a(x)}}}{P_F^{\nu_F^{a(x)}}} \), therefore \( \varphi(x) \) can be written as the difference

\[
\varphi(x) = \sum_{y \in \frac{U_F}{\nu_F^{a(x)}}} \psi(\frac{x-1}{c})
\]

\[
= \sum_{y \in \frac{P_F}{P_F^{\nu_F^{a(x)}}}} \psi(\frac{x-1}{c}) - \sum_{y \in \frac{P_F}{P_F^{\nu_F^{a(x)}}}} \psi(\frac{x-1}{c})
\]

\[
= \sum_{y \in \frac{P_F}{P_F^{\nu_F^{a(x)}}}} \psi(\frac{x-1}{c}) - \sum_{y \in \frac{P_F}{P_F^{\nu_F^{a(x)-1}}} \pi_F} \psi(\frac{(x-1)\pi_F}{c})
\]

\[
= A - B,
\]

where \( A = \sum_{y \in \frac{P_F}{P_F^{\nu_F^{a(x)}}}} \psi(\frac{x-1}{c}) \) and \( B = \sum_{y \in \frac{P_F}{P_F^{\nu_F^{a(x)-1}}} \pi_F} \psi(\frac{(x-1)\pi_F}{c}) \). It is simple to see that (cf. [37], p. 28, Lemma 2.1)

\[
\sum_{y \in \frac{P_F}{P_F^{\nu_F^{a(x)}}}} \psi(\frac{x-1}{c}) = \begin{cases} 
q_F^{a(\chi)} & \text{when } \alpha \in P_F^{-n(\psi)} \\
0 & \text{otherwise}
\end{cases}
\]

Therefore, \( A = q_F^{a(\chi)} \) when \( x \in U_F^{a(x)} \), and \( A = 0 \) otherwise. Similarly \( B = q_F^{a(\chi)-1} \) when \( x \in U_F^{a(x)-1} \), and \( B = 0 \) otherwise. Therefore, we have

\[
W(\chi, \psi) \cdot W(\chi^{-1}, \psi) = \chi(-1) \cdot q_F^{-a(\chi)} \cdot \{ q_F^{a(\chi)} - q_F^{a(\chi)-1} \sum_{x \in \frac{U_F^{a(x)-1}}{\nu_F^{a(x)}}} \chi^{-1}(x) \}
\]

\[
= \chi(-1) - \chi(-1) \cdot q_F^{-1} \sum_{x \in \frac{U_F^{a(x)-1}}{\nu_F^{a(x)}}} \chi^{-1}(x).
\]
Because the conductor of $\chi$ is $a(\chi)$, then it can be proved that $\sum_{x \in \mathbb{Z} / a(\chi)} \chi^{-1}(x) = 0$.

Thus we obtain
\begin{equation}
W(\chi, \psi) \cdot W(\chi^{-1}, \psi) = \chi(-1). \tag{4.13}
\end{equation}

The right side of equation (4.13) is a sign, hence we may rewrite (4.13) as
\[ W(\chi, \psi) \cdot \chi(-1)W(\chi^{-1}, \psi) = 1. \]

However, we also know from our earlier property that
\[ \chi(-1)W(\chi^{-1}, \psi) = W(\chi^{-1}, (-1)\psi) = W(\chi^{-1}, \overline{\psi}). \]

So the identity (4.13) rewrites as
\[ W(\chi, \psi) \cdot W(\chi^{-1}, \overline{\psi}) = 1. \]

Now we assume that $\chi$ is unitary, hence
\[ W(\chi^{-1}, \overline{\psi}) = W(\overline{\chi}, \overline{\psi}) = W(\chi, \psi) \]
where the last equality is obvious. Now we see that for unitary $\chi$ the identity (4.13) rewrites as
\[ |W(\chi, \psi)|^2 = 1. \]

**Remark 4.6.** From the functional equation (4.4), we can directly see the first part of the above property of local constant. Denote
\begin{equation}
\zeta(f, \chi) = \int f(x)\chi(x)d^\times x. \tag{4.14}
\end{equation}

Now replacing $f$ by $\hat{f}$ in equation (4.14), and we get
\begin{equation}
\zeta(\hat{f}, \chi) = \int \hat{f}(x)\chi(x)d^\times x = \chi(-1) \cdot \zeta(f, \chi), \tag{4.15}
\end{equation}

because $dx$ is self-dual with respect to $\psi$, hence $\hat{f}(x) = f(-x)$ for all $x \in F^+$. Again the functional equation (4.4) can be written as follows:
\begin{equation}
\zeta(\hat{f}, w_1\chi^{-1}) = W(\chi, \psi, dx) \cdot \frac{L(w_1\chi^{-1})}{L(\chi)} \cdot \zeta(f, \chi). \tag{4.16}
\end{equation}

Now we replace $f$ by $\hat{f}$, and $\chi$ by $w_1\chi^{-1}$ in equation (4.16), and we obtain
\begin{equation}
\zeta(\hat{f}, \chi) = W(w_1\chi^{-1}, \psi, dx) \cdot \frac{L(\chi)}{L(w_1\chi^{-1})} \cdot \zeta(f, w_1\chi^{-1}). \tag{4.17}
\end{equation}

Then by using equations (4.15), (4.16), from the above equation (4.17) we obtain:
\begin{equation}
W(\chi, \psi, dx) \cdot W(w_1\chi^{-1}, \psi, dx) = \chi(-1). \tag{4.18}
\end{equation}

Moreover, the convention $W(\chi, \psi)$ is actually as follows (cf. [45], p. 17, equation (3.6.4)):
\[ W(\chi w_{\frac{1}{2}}, \psi) = W(\chi w, \psi, dx). \]
By using this relation from equation (4.18) we can conclude
\[ W(\chi, \psi) \cdot W(\chi^{-1}, \psi) = \chi(-1). \]

(4) **Twisting formula of abelian local constants:**

(a) If \( \chi_1 \) and \( \chi_2 \) are two unramified characters of \( \mathbb{F}^{\times} \), and \( \psi \) is a nontrivial additive character of \( \mathbb{F} \), then from equation (4.8) we have

\[ W(\chi_1 \chi_2, \psi) = W(\chi_1, \psi)W(\chi_2, \psi). \]  

(4.19)

(b) Let \( \chi_1 \) be ramified, and \( \chi_2 \) unramified then (cf. [45], (3.2.6.3))

\[ W(\chi_1 \chi_2, \psi) = \chi_2(\pi_F)^{a(\chi_1)+n(\psi)} \cdot W(\chi_1, \psi). \]  

(4.20)

*Proof.* By the given condition \( a(\chi_1) > a(\chi_2) = 0 \). Therefore, \( a(\chi_1 \chi_2) = a(\chi_1) \). Then we have

\[ W(\chi_1 \chi_2, \psi) = \chi_1 \chi_2(c)q_F^{-a(\chi_1)/2} \sum_{x \in \mathbb{F}^{\times}} (\chi_1 \chi_2)^{-1}(x)\psi(x/c) \]

\[ = \chi_1(c)\chi_2(c)q_F^{-a(\chi_1)/2} \sum_{x \in \mathbb{F}^{\times}} \chi_1^{-1}(x)\chi_2^{-1}(x)\psi(x/c) \]

\[ = \chi_2(c)\chi_1(c)q_F^{-a(\chi_1)/2} \sum_{x \in \mathbb{F}^{\times}} \chi_1^{-1}(x)\psi(x/c), \text{ because } \chi_2 \text{ unramified} \]

\[ = \chi_2(c)W(\chi_1, \psi) \]

\[ = \chi_2(\pi_F)^{a(\chi_1)+n(\psi)} \cdot W(\chi_1, \psi). \]

□

(c) We also have a twisting formula of local constants by Deligne (cf. [16], Lemma 4.16) under some special condition, and which is as follows (for proof, cf. Corollary 3.2 (2) of [21]):

Let \( \alpha \) and \( \beta \) be two multiplicative characters of a local field \( \mathbb{F} \) such that \( a(\alpha) \geq 2 \cdot a(\beta) \). Let \( \psi \) be an additive character of \( \mathbb{F} \). Let \( y_{\alpha,\psi} \) be an element of \( \mathbb{F}^{\times} \) such that

\[ \alpha(1+x) = \psi(y_{\alpha,\psi}x) \]

for all \( x \in \mathbb{F} \) with valuation \( \nu_F(x) \geq \frac{a(\alpha)}{2} \) (if \( a(\alpha) = 0 \), \( y_{\alpha,\psi} = \pi_F^{-n(\psi)} \)). Then

\[ W(\alpha \beta, \psi) = \beta^{-1}(y_{\alpha,\psi}) \cdot W(\alpha, \psi). \]  

(4.21)

(d) **General twisting formula for characters:** In the following theorem, one can see a generalized twisting formula of local constants using local Jacobi sums.

**Theorem 4.7** (Theorem 3.5 on p. 592 of [2]). Let \( \mathbb{F} \) be a non-Archimedean local field with \( q \) as the cardinality of the residue field of \( \mathbb{F} \). Let \( \psi \) be a nontrivial additive
character of $F$. Let $\chi_1$, and $\chi_2$ be two ramified characters of $F^\times$ with conductors $n$, and $m$ respectively. Let $r$ be the conductor of character $\chi_1\chi_2$. Then

\begin{equation}
W(\chi_1\chi_2, \psi) = \begin{cases} 
q^{2}W(\chi_1, \psi)W(\chi_2, \psi) & \text{when } n = m = r, \\
q^{2}\chi_1\chi_2(x^{r-1})W(\chi_1, \psi)W(\chi_2, \psi) & \text{when } n = m > r, \\
q^{n-2}W(\chi_1, \psi)W(\chi_2, \psi) & \text{when } n > r > m,
\end{cases}
\end{equation}

Here the local Jacobi sum is:

\begin{equation}
J_l(\chi_1, \chi_2, n) = \sum_{x \in \mathbb{F}_F^\times} \chi_1^{-1}(x)\chi_2^{-1}(t - x).
\end{equation}

4.4. Connection of different conventions for local constants. Mainly there are two conventions for local constants. They are due to Langlands (35), and Deligne (16). Recently Bushnell, and Henniart (12) also give a convention of local constants. In this subsection we shall show the connection among all three conventions for local constants.\footnote{The convention $W(\chi, \psi)$ is actually due to Langlands [35], and it is:
\[
\epsilon_L(\chi, \frac{1}{2}, \psi) = W(\chi, \psi).
\]
See equation (3.6.4) on p. 17 of [15] for $V = \chi$.} We denote $\epsilon_{BH}$ as local constant of Bushnell-Henniart (introduced in Bushnell-Henniart, [12], Chapter 6).

On page 142 of [12], the authors define a rational function $\epsilon_{BH}(\chi, \psi, s) \in \mathbb{C}(q_F^{-s})$. From Theorem 23.5 on p. 144 of [12] for ramified character $\chi \in \widetilde{F}^\times$, and conductor\footnote{The definition of level of an additive character $\psi \in \widetilde{F}$ in [12] on p. 11 is the negative sign with our conductor $n(\psi)$, i.e., level of $\psi = -n(\psi)$.} $n(\psi) = -1$ we have

\begin{equation}
\epsilon_{BH}(\chi, s, \psi) = q_F^{\frac{n(\chi, s)}{2}} \sum_{x \in \mathbb{F}_F^\times} \chi(\alpha x)^{-1}\psi(\alpha x)/q_F^{\frac{n(\psi)}{2}},
\end{equation}

where $n = a(\chi) - 1$, and $\alpha \in F^\times$ with $\nu_F(\alpha) = -n$.

Also from the Proposition 23.5 of [12] on p. 143 for unramified character $\chi \in \widetilde{F}^\times$, and $n(\psi) = -1$ we have

\begin{equation}
\epsilon_{BH}(\chi, s, \psi) = q_F^{s^{-1}}\chi(\pi_F)^{-1}.
\end{equation}

(1) Connection between $\epsilon_{BH}$, and $W(\chi, \psi)$.

\begin{equation}
W(\chi, \psi) = \epsilon_{BH}(\chi, \frac{1}{2}, \psi).
\end{equation}

Proof. From [12], p. 143, Lemma 1 we see:

\begin{equation}
\epsilon_{BH}(\chi, \frac{1}{2}, b\psi) = \chi(b)\epsilon_{BH}(\chi, \frac{1}{2}, \psi)
\end{equation}

for any $b \in F^\times$. However, we have seen already that $W(\chi, b\psi) = \chi(b)W(\chi, \psi)$ has the same transformation rule. If we fix one nontrivial $\psi$ then all other nontrivial $\psi'$ are uniquely given as $\psi' = b\psi$ for some $b \in F^\times$. Because of the parallel transformation
rules it is now enough to verify our assertion for a single $\psi$. Now we take $\psi \in \mathcal{F}$ with $n(\psi) = -1$, hence $\nu_F(c) = a(\chi) - 1$. Then we obtain

$$W(\chi, \psi) = W(\chi, \psi, c) = \chi(c)q_F^{\frac{a(\chi)}{2}} \sum_{x \in \mathcal{O}_F} \chi^{-1}(x)\psi(c^{-1}x).$$

We compare this to the equation (4.24). There the notation is $n = a(\chi) - 1$, and the assumption is $n \geq 0$. This means we have $\nu_F(c) = n$, hence we may take $\alpha = c^{-1}$, and then comparing our formula with equation (4.24), we see that

$$W(\chi, \psi) = \epsilon_{BH}(\chi, \frac{1}{2}, \psi)$$
in the case when $n(\psi) = -1$.

We are still left to prove our assertion if $\chi$ is unramified, i.e., $a(\chi) = 0$. Again we can reduce to the case where $n(\psi) = -1$. Then our assertion follows from equation 4.25.

Remark 4.8. From Corollary 23.4.2 of [12], on p. 142, for $s \in \mathbb{C}$, we can write

$$(4.26) \epsilon_{BH}(\chi, s, \psi) = q_F^{\frac{1}{2} - s}n(\chi, \psi) \cdot \epsilon_{BH}(\chi, \frac{1}{2}, \psi),$$

for some $n(\chi, \psi) \in \mathbb{Z}$. In fact here $n(\chi, \psi) = a(\chi) + n(\psi)$. From above connection, we only see $W(\chi, \psi) = \epsilon_{BH}(\chi, \frac{1}{2}, \psi)$. Thus for arbitrary $s \in \mathbb{C}$, we obtain

$$(4.27) \epsilon_{BH}(\chi, s, \psi) = q_F^{\frac{1}{2} - s(a(\chi) + n(\psi))} \cdot W(\chi, \psi).$$

This equation (4.27) is very important for us. We shall use this to connect with Deligne’s convention.

In [45] there is defined a number $\epsilon_D(\chi, \psi, dx)$ depending on $\chi, \psi$, and a Haar measure $dx$ on $F$. This notion is due to Deligne [16]. We write $\epsilon_D$ for Deligne’s convention in order to distinguish it from the $\epsilon_{BH}(\chi, \frac{1}{2}, \psi)$ introduced in Bushnell-Henniart [12].

In the next Lemma we give the connection between Bushnell-Henniart, and Deligne conventions for local constants.

(2) The connection between $\epsilon_D$, and $\epsilon_{BH}$:

Lemma 4.9. We have the relation

$$\epsilon_{BH}(\chi, s, \psi) = \epsilon_D(\chi \cdot \omega_s, \psi, dx\psi),$$

where $\omega_s(x) = |x|_F^{\frac{s}{2}} = q^{-s\nu_F(x)}$ is unramified character of $F^\times$ corresponding to complex number $s$, and where $dx_\psi$ is the self-dual Haar measure corresponding to the additive character $\psi$.

Proof. From equation equation (3.6.4) of [45], we know that

$$(4.28) \epsilon_L(\chi, s, \psi) := \epsilon_L(\chi \omega_{s, \frac{1}{2}}, \psi) = \epsilon_D(\chi \omega_s, \psi, dx_\psi).$$

We prove this connection by using the relations (4.27), and (4.28). From equation (4.28) we can write our $W(\chi, \psi) = \epsilon_D(\chi \omega_{\frac{1}{2}}, \psi, dx_\psi)$. Therefore, when $s = \frac{1}{2}$, we have the relation:

$$(4.29) \epsilon_{BH}(\chi, \frac{1}{2}, \psi) = \epsilon_D(\chi \omega_{\frac{1}{2}}, \psi, dx_\psi),$$
because \( W(\chi, \psi) = \epsilon_{BH}(\chi, \frac{1}{2}, \psi) \).

We know that \( \omega_s(x) = q_F^{-sr_F(x)} \) is an unramified character of \( F^\times \). So when \( \chi \) is also unramified, we can write

\[
W(\chi, \psi) = \omega_s(\frac{1}{2}) \cdot \chi(\frac{1}{2}) = q_F^{(\frac{1}{2} - s)n(\psi)} \epsilon_{BH}(\chi, \frac{1}{2}, \psi) = \epsilon_{BH}(\chi, s, \psi).
\]

And when \( \chi \) is ramified character, i.e., conductor \( a(\chi) > 0 \), from Tate’s theorem for unramified twist (see property (4.20)), we can write

\[
W(\chi, \psi) = \omega_s(\frac{1}{2}) \cdot \chi(\frac{1}{2}) = q_F^{-n_F(a(\chi) + n(\psi))} \epsilon_{BH}(\chi, \frac{1}{2}, \psi).
\]

Furthermore from equation (4.28), we have

\[
W(\chi, \psi) = \epsilon_D(\chi, \psi, dx) = \epsilon_{BH}(\chi, s, \psi).
\]

Therefore, finally we can write

\[
(4.31) \quad W(\chi, \psi) = \epsilon_{BH}(\chi, s, \psi) = \epsilon_D(\chi, \psi, dx).
\]

\[□\]

**Corollary 4.10.** For our \( W \) we have :

\[
W(\chi, \psi) = \epsilon_{BH}(\chi, s, \psi),
\]

\[
W(\chi, \psi) = \epsilon_D(\chi, \psi, dx).
\]

**Proof.** From the equations (3.6.1), and (3.6.4) of [15] for \( \chi \), and above two connections the assertions follow. \[□\]

### 4.5. Local constants for virtual representations.

(1) To extend the concept of local constant, we need to go from one-dimensional to other virtual representations \( \rho \) of the Weil groups \( W_F \) of non-archimedean local field \( F \). According to Tate [43], the root number \( W(\chi) := W(\chi, \psi_F) \) extensions to \( W(\rho) \), where \( \psi_F \) is the canonical additive character of \( F \). More generally, \( W(\chi, \psi) \) extends to \( W(\rho, \psi) \), and if \( E/F \) is a finite separable extension then one has to take \( \psi_E = \psi_F \circ Tr_{E/F} \) for the extension field \( E \).

According to Bushnell-Henniart [12], Theorem on p. 189, the functions \( \epsilon_{BH}(\chi, s, \psi) \) extend to \( \epsilon_{BH}(\rho, s, \psi_E) \), where \( \psi_E = \psi_F \circ Tr_{E/F} \). According to Tate [15], Theorem (3.4.1) the functions \( \epsilon_D(\chi, \psi, dx) \) extends to \( \epsilon_D(\rho, \psi, dx) \). In order to get weak inductivity we have again to use \( \psi_E = \psi_F \circ Tr_{E/F} \) if we consider extensions. Then according to Tate [15] (3.6.4) the Corollary 4.10 turns into

**Corollary 4.11.** For the virtual representations of the Weil groups we have

\[\text{Note that they fix a base field } F, \text{ and a nontrivial } \psi = \psi_F \text{ (which not to be the canonical character used in Tate [43]) but then if } E/F \text{ is an extension they always use } \psi_E = \psi_F \circ Tr_{E/F}.\]
\[ W(\rho \omega_{E,s}^{-1}, \psi_E) = \epsilon_{BH}(\rho, s, \psi_E) = \epsilon_D(\rho \omega_{E,s}, \psi_E, dx_{\psi_E}). \]

Note that on the level of field extension \( E/F \) we have to use \( \omega_{E,s} \) which is defined as
\[ \omega_{E,s}(x) = |x|^s_E = q_E^{s\nu(x)}. \]

We also know that \( q_E = q_{F/F}, \) and \( \nu_E = \frac{1}{s} \cdot \nu_F(N_{E/F}) \) (cf. [19], p. 41, Theorem 2.5), therefore we can easily see that
\[ \omega_{E,s} = \omega_{F,s} \circ N_{E/F}. \]

Because the norm map \( N_{E/F} : E^\times \rightarrow F^\times \) corresponds using class field theory to the injection map \( G_E \hookrightarrow G_F, \) Tate [15] beginning from (1.4.6), simply writes \( \omega_s = ||^s, \) and consider \( \omega_s \) as an unramified character of the Galois group (or of the Weil group) instead as a character on the field. Then Corollary [41] turns into

\[ W(\rho \omega_s^{-1}, \psi_E) = \epsilon_{BH}(\rho, s, \psi_E) = \epsilon_D(\rho \omega_s, \psi_E, dx_{\psi_E}), \]

for all field extensions, where \( \omega_s \) is to be considered as one-dimensional representation of the Weil group \( W_E \subset G_E \) if we are on the \( E \)-level. The left side equation (4.33) is the \( \epsilon \)-factor of Langlands (see [15], (3.6.4)).

(2) The functional equation (4.13) extends to

\[ W(\rho, \psi) \cdot W(\rho^\vee, \psi) = \det_{\rho}(-1), \]

where \( \rho \) is any virtual representation of the Weil group \( W_F, \rho^\vee \) is the contragredient, and \( \psi \) is any nontrivial additive character of \( F. \) This is formula (3) on p. 190 of [12] for \( s = \frac{1}{2}. \)

(3) Moreover, the transformation law [15] (3.4.5) can (on the \( F \)-level) be written as unramified character twist

\[ \epsilon_D(\rho \omega_s, \psi, dx) = \epsilon_D(\rho, \psi, dx) \cdot \omega_{F,s}(c_{\rho,\psi}) \]

for any \( c = c_{\rho,\psi} \) such that \( \nu_F(c) = a(\rho) + n(\psi)\dim(\rho). \) It implies that also for the root number on the \( F \)-level we have

\[ W(\rho \omega_s, \psi) = W(\rho, \psi) \cdot \omega_{F,s}(c_{\rho,\psi}). \]

(4) Deligne-Henniart’s twisting formula: Let \( \rho_1, \) and \( \rho_2 \) be two finite-dimensional representations of \( W_F. \) Now the question is

Is there any explicit formula for \( W(\rho_1 \otimes \rho_2, \psi) ? \)

The answer to this question is not yet known. However, under some special conditions, when any of \( \rho_1 \) and \( \rho_2 \) is one dimensional, then Deligne gives an explicit formula for \( W(\rho_1 \otimes \rho_2, \psi) \) (cf. [17]), and which is as follows:

Let \( \rho_1 = \rho \) be a finite-dimensional representation of \( W_F, \) and let \( \rho_2 = \chi \) be any nontrivial character of \( F^\times. \) For each \( \chi \) there exists an element \( c \in F^\times \) such that

\[ \chi(1 + y) = \psi(cy) \]

for sufficiently small \( y. \)

For all \( \chi \) with sufficiently large conductor, we have the following formula:

\[ W(\rho \otimes \chi, \psi) = W(\chi, \psi)^{\dim(\rho)} \cdot \det(\rho)(c^{-1}). \]
Now if we define a virtual representation \( \rho_0 := \rho - \dim(\rho) \cdot 1_{W_F} \), where \( 1_{W_F} \) is the trivial representation of \( W_F \), then from above equation (4.37) we have

\[
W(\rho_0 \otimes \chi, \psi) = \det(\rho_0)(c^{-1}).
\]

In [18], Deligne, and Henniart generalize the above result (see Section 4 of [18]), for for virtual representations \( \rho \) of dimension 0, in which \( \chi \) is replaced by a representation \( \rho' \).

5. Applications, and Open Problems

In this section, we study some applications of local constants, and some open problems regarding local constants. Because local constants can be attached to every finite-dimensional complex representation of a local Galois group, in the Langlands program, the local constants play a very important role. These local constants appear in various places in modern number theory (in general, in algebraic curves, local/global Galois representations etc.). In fact, it is believed that if a mathematical object which has \( L \)-function, then we can attach this constant with it.

However, the construction of these root numbers for every mathematical object which has \( L \)-function is a difficult problem in number theory. For instance, so far, we are not able to bring root numbers in the mod-\( p \) Langlands correspondence (cf. [11]), and the geometric Langlands correspondence (cf. [36], [33]).

The explicit computation of these root numbers has many applications in modern number theory. For instance, in [46], Taylor showed that in the theory of the structure of ring of algebraic integers as a Galois module, the local root numbers determine whether or not this projective module is free, in the case of a tame extension. Furthermore, because we know that the local Langlands correspondence preserves the local root number, in the Langlands program, the local root number plays an important role for checking the local Langlands conjecture.

Besides this, on the automorphic side, by extensive study of root numbers, we can classify the automorphic representations (cf. [29], [13], [14], [15]). And it is also expected that on the Galois side, we also can do the same. Although, so far on the Galois side, we do not have any such complete result except Heiermann’s result [26].

5.1. Applications. As to application, in this article, we will only review converse theorems, and Taylor’s result [46].

Classical Converse Theorem in number theory

How to construct a modular form from a given Dirichlet series with ‘nice’ properties (e.g., analytic continuation, moderate growth, functional equation), i.e., starting with the series

\[
L(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}
\]

under what conditions is the function

\[
f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z}
\]

a modular form for some Fuchsian group? The answer to this question is known as the classical converse theorem in number theory (cf.
The classical converse theorems establish a one-to-one correspondence between “nice” Dirichlet series, and automorphic functions. Traditionally, the converse theorems have provided a way to characterize Dirichlet series associated to modular forms in terms of their analytic properties.

The modern version of classical converse theorems are stated in terms of automorphic representations instead of modular forms. Again, we know that using Langlands local correspondence that automorphic representations are associated with Galois representations. Therefore, one can ask the following questions:

1. Are there any converse theorems for automorphic representations (automorphic side of the converse theorem)?
2. Similarly, are there any converse theorems for Galois representations (Galois side of the converse theorem)?

Note: Because local root numbers for complex Galois representations are the main theme of the paper, here we only discuss converse theorems on the Galois side. For local converse theorems on the automorphic side, one can follow Dihua Jiang’s paper on local $\gamma$-factors [29].

1. Converse theorem on the Galois side:

   So far on the Galois side, we do not have any converse theorem similar to $GL_n$ side except Volker Heiermann’s [26] work. Here, we summarize his work.

   Let $k$ be a field, and $G$ be a group, on which a decreasing filtration by normal subgroups $G^\theta$ with $\theta \in [-1, +\infty)$ is defined. Furthermore, assume $G = G^{-1}$, and $G/G^0$ is cyclic. Volker Heiermann considers representations of $G$ in $k$-vector spaces of finite dimension which are trivial on one $G^\theta$. For such an indecomposable representation $\sigma$, define

   $$S(\sigma) := \sup\{\epsilon : \sigma|_{W^\epsilon} \not\supset 1\}, 0\} \cdot \deg(\sigma),$$

   (for arbitrary $\sigma$, define $S(\sigma)$ by additivity). Then he proves the following inequality:

   $$\frac{S(\sigma \otimes \tau^\vee)}{\deg(\sigma) \deg(\tau)} \leq \max\left\{\frac{S(\sigma \otimes \rho^\vee)}{\deg(\sigma) \deg(\rho)}, \frac{S(\rho \otimes \tau^\vee)}{\deg(\rho) \deg(\tau)}\right\},$$

   where $\sigma, \tau, \rho$ are three indecomposable $k$-representations of $G$ of finite dimension which are trivial on one $G^\theta$, and where it is assumed that these restrictions to $G^\theta$ are semisimple for every $\theta > 0$.

   Finally, the author also proves, under some of conditions on the residue field of $F$, that an irreducible representation $\sigma$ on $W_F$ is determined by the set of $\epsilon$-factors of the form $W(\sigma \otimes \tau)$ with $\deg(\sigma) \leq \deg(\tau)$, up to equivalence.

**Theorem 5.1** (Lemma 1, and Proposition on p. 4 of [26]). Suppose $\sigma$, and $\sigma'$ are two irreducible representations of $W_F$. Suppose that we have

$$W(\sigma \otimes \tau, s) = W(\sigma' \otimes \tau, s)$$

for all irreducible representations $\tau$ with

$$\deg(\tau) \leq \sup\{\deg(\sigma), \deg(\sigma')\}$$

of $W_F$, and for all $s \in \mathbb{C}$. Then there exists an unramified character $\mu$ of $F^\times$ for which

$$\sigma' = \sigma \otimes \mu.$$

Moreover, if the residue field of $F$ contains more than 2 elements or if $\sigma$ is induced from a character of a field of residue field of $> 2$ elements, then $\sigma$ and $\sigma'$ are isomorphic.

Remark 5.2. In Theorem 1.5 in [7], one can see a converse theorem on the Galois side regarding local Heisenberg Galois representations.

2. Taylor’s result regarding the Galois module:

Let $F$ be a number field, and $K$ be a tame Galois extension with Galois group $\Gamma := Gal(K/F)$. By Noether, it was known that the ring of integers $O_K$ of $K$ is locally free $\mathbb{Z}\Gamma$-module. For a character $\chi \in \widehat{\Gamma}$, we can associate the Artin’s root number (see Appendix 6.1 below) $W(\chi)$. Given a locally free $\mathbb{Z}\Gamma$-module $X$, $(X)$ denotes the class of $X$. Let $Cl(\mathbb{Z}\Gamma)$ denote the class group of locally free $\mathbb{Z}\Gamma$-modules, and $t(W)$ denotes a class in $Cl(\mathbb{Z}\Gamma)$ which is defined in terms of the values of Artin’s root numbers of symplectic type characters.

In [20], A. Fröhlich showed that the class of $O_K$ (when $F = \mathbb{Q}$, and $\Gamma$ is a quaternion group of order 8), is determined by the sign of the Artin root number of the irreducible symplectic character of $\Gamma$. In [46], Taylor proved the following Theorem:

Theorem 5.3 (Taylor, Theorem 1 of [46]). $(O_K) = t(W)$, and so in particular, (a) as the Artin root numbers of symplectic type characters are $\pm 1$, $(O_K)^2 = 1$. (b) the only obstructions to the vanishing of the class of $O_K$, are the signs of the Artin root numbers of symplectic characters.

5.2. Open Problems. (1) Local constants under restriction:

Let $F$ be a non-Archimedean local field of characteristic zero. Let $K/L/F$ be a tower of field extensions, where $K/F$ is Galois, and $L/F$ is finite (need not be Galois). Let $G = Gal(K/F)$, and $H = Gal(K/L)$ which is a subgroup of $G$ with finite index in $G$. Let $\rho$ be finite-dimensional complex representation of $G$, then

What is the relationship between the local constants $W(\rho, \psi)$, and $W(Res^G_H(\rho), \psi_K)$, where $\psi$ is a nontrivial additive character of $F$, $\psi_K = \psi \circ Tr_{K/F}$, and $Res^G_H(\rho)$ is the restriction of $\rho$ to $H$?

Remark 5.4. (1) One can think this restriction problem using Robert Boltje’s canonical Brauer extension (cf. [8]), because canonical Brauer induction commutes with restriction map. If we have a formula for $W(Res^G_H(\rho), \psi \circ Tr_{L/F})$, then it can be used in the Gan-Gross-Prasad conjecture (cf. [23, 21, 22]).

(2) Let $K/F$ be a finite extension of $F$. Let $\chi_K$ be a nontrivial multiplicative character of $K$. Denote

$\chi_F := \chi_K|_{F^\times}$, i.e., restriction of $\chi_K$ to $F^\times$.

Now the question is: Is there any relation between $W(\chi_F, \psi_F)$ and $W(\chi_K, \psi_K)$, where $\psi_K = \psi_F \circ Tr_{K/F}$?

If we can answer this question, it will be useful in the Langlands program. But, so far we do not have any explicit answer to this question. In [1], one can see some results in this direction.

(2). General Twisting formula:

Let $\rho_1, \rho_2$ be two complex irreducible representations of $G_F$, and let $\psi$ be a nontrivial additive character of $F$. Now the question is:
What is the explicit formula for $W(\rho_1 \otimes \rho_2, \psi)$?

When both $\rho_1, \rho_2$ are one-dimensional, we can see the property of root numbers in Section 4. When one of the $\rho_1, \rho_2$ is one dimensional, we have the Deligne-Henniart’s twisting Formula [4]. For a particular Heisenberg representation, there is an extension of Deligne-Henniart’s twisting formula (cf. [7]). Besides these, so far, we do not have any known explicit formula for $W(\rho_1 \otimes \rho_2, \psi)$ in literature. When $\rho_1$ and $\rho_2$ are one-dimensional, then via local Jacobi sum, we can give explicit formula for $W(\chi_1 \chi_2, \psi)$ (cf. [2]).

(3). Geometric analog, and arithmetic connection of local constants (or epsilon factors):

Motivation: The grand unification between various mathematical objects is the main theme of the modern mathematical research. From André Weil’s philosophy, we (conjecturally) know that there are bijective correspondences between number theory, and geometry over finite fields (using zeta function, and local information of number theory), number theory, and complex geometry (using geometric Langlands), and geometry over finite fields, and complex geometry (using cohomology). And these correspondences preserve some analytic objects (e.g., $L$-functions, $\varepsilon$-factors) which contain many arithmetic information. Further, we also know that the local root numbers remain invariant under local Langlands correspondence. Then it is a natural question to ask:

Question (a): What is the geometric interpretation/connection of the local constants? Do they have any role in the geometric Langlands program?

Because $L$-functions are very important in modern number theory, and arithmetic geometry, hence, local root numbers are. For instance, let $E$ be an elliptic curve over $\mathbb{Q}$, and $N_{E/\mathbb{Q}}$ be the conductor of $E$. We know that the $L$-function $L(E, s)$ has an analytic continuation to the entire complex plane, and it satisfies the following functional equation:

$$\Lambda(E, s) = \omega(E/\mathbb{Q}) \cdot \Lambda(E, 2 - s), \quad \text{with } \omega(E/\mathbb{Q}) = \pm 1,$$

where

$$\Lambda(E, s) := N_{E/\mathbb{Q}}^2 (2\pi)^{-s} \Gamma(s) L(E, s), \quad \Gamma(s) = \int_0^\infty t^{s-1} e^{-t} dt.$$

The number $\omega(E/\mathbb{Q})$ in the functional equation (5.2), is called the root number of $E$, and has a very important conjectural meaning in the Birch-Swinnerton-Dyer (BSD) conjecture. The Parity conjecture (a weaker version of the BSD conjecture) claims that

$$\omega(E/\mathbb{Q}) = (-1)^{\text{Rank}(E/\mathbb{Q})}.$$

If we notice, we can see that the exact sign of $\omega(E/\mathbb{Q})$ will tell us whether the rank of $E/\mathbb{Q}$ is even or odd, hence (partially) the structure of Weil-Mordell group of $E/\mathbb{Q}$, and for elliptic curves over $\mathbb{Q}$ this is the ultimate goal of number theorists.

Question (b): (regarding relationship between $\omega(E/\mathbb{Q})$, and $W(\chi, \psi)$):

Let $F$ be a local field, and $\chi$ be a multiplicative character of $F$. Let $\psi$ be a nontrivial additive character of $F$. 
(i). Is there any relation between $W(\chi, \psi)$, and $\omega(E/\mathbb{Q})$?

(ii). Because $\omega(E/\mathbb{Q})$ is a sign, so for a particular choice of elliptic curve with some admissible conductor, can we give any relation between $\omega(E/\mathbb{Q})$, and the classical Gauss sum?

(iii). Suppose that for a 1-dimensional representation $\chi$, we are able to find a connection between $W(\chi, \psi)$, and $\omega(E/\mathbb{Q})$. Then can we extend that result for any finite-dimensional Galois representations?

Remark 5.5. In [32], the authors provide some results regarding elliptic curves, and local root numbers. One also can see [38] for more information about the geometric analog of root numbers.

6. Appendix

6.1. Global constants (or global epsilon factors/Artin root numbers). Let $F$ be a global field, and

$$
\psi : \mathbb{A}_F / F \rightarrow \mathbb{C}^\times
$$

be a nontrivial additive character, and $dx$ the Haar measure on $\mathbb{A}_F$ such that

$$
\int_{\mathbb{A}_F / F} dx = 1.
$$

Here $\mathbb{A}_F$ is the adele of $F$. This is called the Tamagawa measure. Call $\psi_v$ the local component of $\psi$ at a place $v$, and let

$$
dx = \prod_v dx_v
$$

be any factorization of $dx$ into a product of local measures such that the ring $O_v$ of integers in $F_v$ gets measure 1 for almost all $v$.

Let $\rho$ be a representation of the global Weil group $W_F$, and put

$$
(6.1) \quad L(\rho, s) := \prod_v L(\rho_v \otimes \omega_s)
$$

where $s \in \mathbb{C}$, and $\omega_s : F_v^\times \rightarrow \mathbb{C}^\times$ is the unramified character such that

$$
\omega_s(\pi_v) = q_v^{-s}
$$

for $\pi_v$ a local prime element, and $q_v$ the order of the corresponding residue field.

Moreover, put

$$
(6.2) \quad W(\rho, s) := \prod_v W(\rho_v \otimes \omega_s, \psi_v, dx_v),
$$

where the existence of the non-abelian local constant on the right side is assumed. Then

**Theorem 6.1.** The product (6.1) converges for $s$ in some right half-plane, and defines a function

$L(\rho, s)$ which is meromorphic in the whole $s$-plane, and satisfies the functional equation

$$
(6.3) \quad L(\rho, s) = W(\rho, s) \cdot L(\rho^\vee, 1 - s)
$$

where $\rho^\vee$ is the dual of $\rho$. 
Remark 6.2. The global functional equation (6.3) does not come from local functional equation (local version (6.3) is wrong):

\[ L(\rho_v \otimes \omega_s) \neq W(\rho_v \otimes \omega_s, \psi_v, dx_v) \cdot L(\rho_v^* \otimes \omega_{1-s}). \]

A local functional equation is known only for \( \dim(\rho) = 1 \) (cf. \[44\], \[42\]) but not for higher dimensions, and the local functional equation for dimension 1 looks different (cf. Equation (4.4))

\[ (6.4) \]

\[ \int \hat{f}(x)w_1\chi^{-1}(x)d^\chi x \]

\[ = \frac{W(\chi, \psi, dx)\int f(x)\chi(x)d^\chi x}{L(\chi)}, \]

where \( \chi \) stands for the 1-dimensional representation \( \rho_v \otimes \omega_s \), and \( \chi^{-1} \) stands for \( \rho_v^* \otimes \omega_{1-s} \).

One obtains (6.3) by taking the product of the local functional equation (6.4) over all places \( v \), and then verifying that the product of the numerators on the left (over all places \( v \)) is the same as the product of the numerators on the right. Therefore, in the global version of the numerators can be canceled, and the (6.3) follows. For higher dimensional \( \rho \) we have no local functional equation at all. Deligne’s proof for existence, and for product formula (6.2) is different.

Remark 6.3. Let \( F \) be an algebraic number field, and \( K \) be a finite normal extension of \( F \) with Galois group \( Gal(K/F) =: G \).

This global constant satisfies the following properties:

1. **Additivity:**

   \[ W(\rho_1 \oplus \rho_2) = W(\rho_1)W(\rho_2) \]

   for representations \( \rho_1, \rho_2 \in \mathbb{R}(G) \).

2. **Invariant under inflation:** Let \( E/F \) be a finite normal extension with \( K \subseteq E \). Then

   \[ W(\text{Inf}_G^{G/N}(\rho)) = W(\rho) \]

   where \( N = Gal(K/E) \), and \( \text{Inf}_G^{G/N}(\rho) \) is the representation of \( N \) defined by inflation from \( \rho \).

3. **Invariant under induction:** Let \( L/F \) be a subextension in \( K/F \), and \( \rho_L \) be a representation of \( Gal(K/L) \). Then

   \[ W(\text{Ind}_G^{Gal(K/F)}(\rho_L)) = W(\rho_L). \]

6.2. **The canonical Brauer induction formula on the Galois side.**

Let \( F \) be an arbitrary field, and \( \bar{F} \) its separable algebraic closure. In this paper, we denote \( K/F \) for the tower \( F \subset K \subset \bar{K} \), and \( K \) is a finite Galois extension of \( F \) whose Galois group is denoted by \( G_{K/F} = Gal(K/F) \). Let \( G_F \) be the absolute Galois group, which is a profinite group with \( G_F = \lim_{\leftarrow} K/F G_{K/F} \).

We denote \( \mathbb{R}(G_F) \) the Grothendieck ring of finite-dimensional complex representations of \( G_F \) with kernel of finite index, and let \( \hat{G}_F \) be the group of linear complex representations of \( G_F \) with kernel of finite index, then we have direct limits

- \( \mathbb{R}(G_F) = \lim_{\rightarrow} \mathbb{R}(G_{K/F}) \).
- \( \hat{G}_F = \lim_{\rightarrow} \hat{G}_{K/F} \) the group of linear characters.
- \( \mathcal{R}_+(G_F) = \lim_{\rightarrow} \mathcal{R}_+(G_{K/F}) \), free abelian group with basis \( \mathcal{M}_G/G_F \) where \( \mathcal{M}_G = \lim_{\rightarrow} \mathcal{M}_{G_{K/F}} \).
• $b_{G_F} = \lim_{K/F} b_{G_{K/F}} : R_+(G_F) \to R(G_F)$, which is well defined because $b_G$ commutes with inflations.

• The section $a_{G_F}$ is $R(\hat{G_F}) = \mathbb{Z}[\hat{G_F}]$-linear, and commutes with restrictions to subgroups.

**Lemma 6.4** ([3], p. 27, Corollary 2.21). Let $N$ be a normal subgroup of $G$, and $\hat{G} = G/N$. Let $\text{Inf}_{\hat{G}}^G(\chi) \in R(G)$ come by inflation along $G \to \hat{G}$ for some $\chi \in R(\hat{G})$. Then we have

\[
a_G(\text{Inf}_{\hat{G}}^G(\chi)) = \sum_{[H, \varphi]} \alpha_{[H, \varphi]}(\chi)[H, \varphi].
\]

This means $\alpha_{[H, \varphi]}(\text{Inf}_{\hat{G}}^G(\chi)) = 0$ unless $N \leq H \leq G$ and $\varphi|_N = 1_N$, and in this case $\alpha_{[H, \varphi]}(\text{Inf}_{\hat{G}}^G(\chi)) = \alpha_{[H, \varphi]}(\chi)$.

According to Lemma 6.4, the section $a_G$ commutes with inflation, and if $\chi \in R(\hat{G})$ lives on a factor group $G \to \hat{G}$, and where $H \subset G$ denotes the full preimage of $\overline{H} \subset \overline{G}$, and $[H, \varphi]$ is always inflated from $[\overline{H}, \overline{\varphi}]$. Therefore, we can go to the profinite Galois group $G_F$, and obtain a Brauer induction formula for $G_F$ by

\[
a_{G_F} = \lim_{K/F} a_{G_{K/F}} : R(G_F) \to R_+(G_F).
\]

**6.3. Extendible function on the Galois side.** Let $K/F$ be a finite Galois extension of the field $F$. Let $\omega = (\omega_K)_{K/F}$ be an abelian invariant for $F$. Then,

• An extension $W = (W_K)_{K/F}$ of $\omega$ is a family of maps

\[W_K : R(G_K) \to A,\]

where $W_K$ extends $\omega_K : \overline{G_K} \to A$, and such that $W_{sK}(\star \chi) = W_K(\chi)$.

• A strong (resp. weak) extension of $\omega$ is an extension $W = (W_K)_{K/F}$ of $\omega$ such that $W$ is invariant under induction (resp. induction in dimension zero), i.e., for all finite field extensions $F \subset K \subset L$, and $\chi \in R(G_L)$ (plus $\chi(1) = 0$ in the weak case), we have:

\[
W_K(\text{Ind}_{G_L}^{G_K}(\chi)) = W_L(\chi).
\]

Suppose the dim $\chi \neq 0$, let $\chi_0 = \chi - \dim \chi \cdot 1_{G_L}$ which a virtual representation of $G_L$ whose dimension is zero. Then now we use equation (6.6) for $\chi_0$, and have:

\[
W_K(\text{Ind}_{G_L}^{G_K}(\chi)) = (\lambda_{G_K}^{G_L})^{\dim \chi} W_L(\chi).
\]

where

\[
\lambda_{G_K}^{G_L} := \frac{W_K(\text{Ind}_{G_L}^{G_K}1_{G_L})}{W_L(1_{G_L})}.
\]

• $\omega$ is called strongly (resp. weakly) extendible, if there exists a strong (resp. weak) extension of $\omega$.

**Remark 6.5.** By definition, strong extendibility implies weak extendibility. For finite groups $G$, and $\chi \in R(G)$, we have

\[
\chi - \chi(1) \cdot 1_G = \sum_{[H, \varphi] \in \mathcal{M}_G/G} \alpha_{[H, \varphi]}(\chi) \text{Ind}_H^G(\varphi - 1_H).
\]
Now then if $\chi(1) = 0$, we have from equation (6.9) 
\begin{equation}
\chi = \sum_{[H, \varphi] \in \mathcal{M}_{G/G}} \alpha_{[H, \varphi]}(\chi) \text{Ind}_H^G(\varphi - 1_H).
\end{equation}
Moreover,
\[W(\text{Ind}_H^G(\varphi - 1_H)) = W(\varphi - 1_H) = w(\varphi - 1_H),\]
because $\varphi - 1_H$ is of dimension 0, and then $W$ can be replaced by $w$ because $\varphi$, and $1_H$ are linear characters. Therefore, by Equation (6.9), we see that $W(\chi)$ is uniquely determined for characters $\chi$ of finite factor group $G \leftarrow G$, and as we know each $\chi \in R(\tilde{G})$ lives on a finite factor group. $W$ is uniquely determined, if it exists. This also proves that the weak extensions $W$, if existing, are unique.

6.4. Galois invariant, and abelian invariant. Assume now that for any finite field extension $K/F$ we have $G_K = \text{Gal}(\bar{F}/K)$, and map 
\[\omega_K : \tilde{G}_K \rightarrow A, \varphi \mapsto \omega_K(\varphi)\]
with values in an abelian group, which is Galois invariant:
\[\omega_{sK}([s\varphi]) = \omega_K(\varphi)\]
for all $s \in G_F$. We call such a family $\omega = (\omega_K)_{K/F}$ an abelian invariant for $F$. The abelian invariants for $F$ are in $1-1$ correspondence with the maps
\[\tilde{\omega}_F : R_+(G_F) \rightarrow A, [G_K, \varphi] \mapsto \omega_K(\varphi).\]
Because of the induction maps $\text{Ind}_{G_K}^{G_F} : R_+(G_K) \rightarrow R_+(G_F)$ the map $\tilde{\omega}_F$ will induce the maps
\begin{equation}
\tilde{\omega}_K := \tilde{\omega}_F \circ \text{Ind}_{G_K}^{G_F} : R_+(G_K) \rightarrow A
\end{equation}
such that 
\[\tilde{\omega}_{sK}([s(H, \varphi)]) = \tilde{\omega}_K([H, \varphi]),\]
for all $s \in G_F$, and $[H, \varphi] \in R_+(G_K)$, and $(H, \varphi) \in \mathcal{M}_{G_K}$.

6.5. The canonical extension. If $G_F$ is the absolute Galois group over $F$, then by definition, we have
\[a_{G_F} = \lim_{\substack{\rightarrow \ \text{K/F}}} a_{G_{K/F}} : R(G_F) \rightarrow R_+(G_F)\]
the direct limit over the inflation maps $R(G_K) \rightarrow R(G_F)$ for finite factor groups $G_K \leftarrow G_F$. Therefore, beginning from an abelian invariant $\omega = (\omega_K)_{K/F}$ over the base field $F$, we can use the compatible system $a_{G_K} : R(G_K) \rightarrow R_+(G_K)$ for all finite $K/F$ to define
\begin{equation}
W_K : \tilde{\omega}_K \circ a_{G_K}
\end{equation}
where $\tilde{\omega}_K$ comes from $\tilde{\omega}_F$ as in (6.11). We call $W = (W_K)_{K/F}$ the canonical extension of $\omega$. The Galois invariance of $\omega_K$, and $\tilde{\omega}_K$ resp.(see equation (6.11)) implies that the system $(W_K)_{K/F}$ is also Galois invariant. Therefore, the definition of $W$ gives an extension of $\omega$. Here, the canonical extension is defined for any arbitrary extendible function.
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Department of Mathematics, SRM University AP, Guntur, 522502, India

Email address: sazzadali.b@srmap.edu.in, sazzad.jumath@gmail.com