A Kähler Einstein structure on the cotangent bundle of a Riemannian manifold

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Abstract

We use the natural lifts of the fundamental tensor field $g$ to the cotangent bundle $T^*M$ of a Riemannian manifold $(M, g)$, in order to construct an almost Hermitian structure $(G, J)$ of diagonal type on $T^*M$. The obtained almost complex structure $J$ on $T^*M$ is integrable if and only if the base manifold has constant sectional curvature and the second coefficient, involved in its definition is expressed as a rational function of the first coefficient and its first order derivative. Next one shows that the obtained almost Hermitian structure is almost Kählerian. Combining the obtained results we get a family of Kählerian structures on $T^*M$, depending on one essential parameter. Next we study the conditions under which the considered Kählerian structure is Einstein. In this case $(T^*M, G, J)$ has constant holomorphic curvature.

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Introduction

The differential geometry of the cotangent bundle $T^*M$ of a Riemannian manifold $(M, g)$ is quite similar to that of the tangent bundle $TM$. However there are some differences, due to the fact that the lifts (vertical, complete, horizontal etc.) to $T^*M$ cannot be defined just like in the case of $TM$.

In the present paper we study a family of natural Kähler Einstein structures $(G, J)$, of diagonal type induced on $T^*M$ from the Riemannian metric $g$. They are obtained in a manner quite similar to that used in [10] (see also [9]) by

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using a similar parametrization. The considered natural Riemannian metric \( G \) of diagonal type on \( T^*M \) is defined by using two parameters \( u, v \) which are smooth functions depending on the energy density \( t \) on \( T^*M \). The vertical and horizontal distributions are orthogonal to each other and the dot products induced on them from \( G \) are isomorphic (isometric) by duality.

Next, the family of the natural almost complex structures \( J \) on \( T^*M \) that interchange the vertical and horizontal distributions depends on the same two essential parameters \( u, v \). From the integrability condition for \( J \) it follows that the base manifold \( M \) must have constant sectional curvature \( c \) and the second parameter \( v \) must be expressed as a rational function depending on the first parameter \( u \) and its derivative. Of course, in the obtained formula there are involved too the constant \( c \) and the energy density \( t \).

Next it follows that \( G \) is Hermitian with respect to \( J \) and it follows that the fundamental 2-form \( \phi \), associated to the almost Hermitian structure \( (G, J) \) is the fundamental form defining the usual symplectic structure on \( T^*M \), hence it is closed. In the case where the integrability condition for \( J \) is fulfilled, we get a Kählerian structure on \( T^*M \) and this structure depends on one essential parameter \( u \).

In the case where the considered Kählerian structure is Einstein we get a second order differential equation fulfilled by the parameter \( u \) and we have been able to find the general solution of this equation. We have a generic case, when the curvature \( c \) is negative and whole \( (T^*M, G, J) \) is Kähler Einstein. Then there is another case where \( c \) is positive and \( (G, J) \) is defined on a tube around the zero section in \( T^*M \). In both cases one obtains that \( (T^*M, G, J) \) has constant holomorphic curvature. The cases where some singularities occur will be discussed in some forthcoming papers.

The manifolds, tensor fields and geometric objects we consider in this paper, are assumed to be differentiable of class \( C^\infty \) (i.e. smooth). We use the computations in local coordinates but many results from this paper may be expressed in an invariant form. The well known summation convention is used throughout this paper, the range for the indices \( h, i, j, k, l, r, s \) being always \( \{1, ..., n\} \) (see [3], [13], [14], [22]). We shall denote by \( \Gamma(T^*M) \) the module of smooth vector fields on \( T^*M \).

1. Some geometric properties of \( T^*M \)

Let \( (M, g) \) be a smooth \( n \)-dimensional Riemannian manifold and denote its cotangent bundle by \( \pi : T^*M \rightarrow M \). Recall that there is a structure of a \( 2n \)-dimensional smooth manifold on \( T^*M \), induced from the structure of smooth \( n \)-dimensional manifold of \( M \). From every local chart \( (U, \varphi) = (U, x^1, \ldots, x^n) \)
on $M$, it is induced a local chart $(\pi^{-1}(U), \Phi) = (\pi^{-1}(U), q^1, \ldots, q^n, p_1, \ldots, p_n)$ on $T^*M$ as follows. For a cotangent vector $p \in \pi^{-1}(U) \subset T^*M$, the first $n$ local coordinates $q^1, \ldots, q^n$ are the local coordinates $x^1, \ldots, x^n$ of its base point $x = \pi(p)$ in the local chart $(U, \varphi)$ (in fact we have $q^i = \pi^*x^i = x^i \circ \pi$, $i = 1, \ldots, n$).

The last $n$ local coordinates $p_1, \ldots, p_n$ of $p \in \pi^{-1}(U)$ are the vector space coordinates of $p$ with respect to the natural basis $(dx^{1}_{\pi(p)}, \ldots, dx^{n}_{\pi(p)})$, defined by the local chart $(U, \varphi)$, i.e. $p = p_1 dx^{1}_{\pi(p)}$. Due to this special structure of differentiable manifold for $T^*M$ it is possible to introduce the concept of $M$-tensor field on it. An $M$-tensor field of type $(r, s)$ on $T^*M$ is defined by sets of $n^{r+s}$ components (functions depending on $q^i$ and $p_i$), with $r$ upper indices and $s$ lower indices, assigned to induced local charts $(\pi^{-1}(U), \Phi)$ on $T^*M$, such that the local coordinate change rule is that of the local coordinate components of a tensor field of type $(r, s)$ on the base manifold $M$ (see [5] for further details in the case of the tangent bundle); e.g., the components $p_i$, $i = 1, \ldots, n$, corresponding to the last $n$ local coordinates of a cotangent vector $p$, assigned to an induced local chart $(\pi^{-1}(U), \Phi)$ define an $M$-tensor field of type $(0, 1)$ on $T^*M$. A usual tensor field of type $(r, s)$ on $M$ may be thought of as an $M$-tensor field of type $(r, s)$ on $T^*M$. If the considered tensor field on $M$ is covariant only, the corresponding $M$-tensor field on $T^*M$ may be identified with the induced (pullback by $\pi$) tensor field on $T^*M$. Some useful $M$-tensor fields on $T^*M$ may be obtained as follows. Let $u : [0, \infty) \longrightarrow \mathbb{R}$ be a smooth function and let $\|p\|^2 = g_{\pi(p)}(p, p)$ be the square of the norm of the cotangent vector $p \in \pi^{-1}(U)$ ($g^{-1}$ is the tensor field of type $(2, 0)$ on $M$ having as components the entries $g^{ij}(x)$ of the inverse of the matrix $(g_{ij}(x))$ defined by the components of $g$ in the local chart $(U, \varphi)$). If $\delta^i_j$ are the Kronecker symbols (in fact, they are the local coordinate components of the identity tensor field $I$ on $M$), then the components $u(\|p\|^2)\delta^i_j$ define an $M$-tensor field of type $(1, 1)$ on $T^*M$. Similarly, if $g_{ij}(x)$ are the local coordinate components of the metric tensor field $g$ on $M$ in the local chart $(U, \varphi)$, then the components $u(\|p\|^2)g_{ij}(\pi(p))$ define a symmetric $M$-tensor field of type $(0, 2)$ on $T^*M$. The components $g^{0i} = g^{0i}_x$, as well as $u(\|p\|^2)g^{0i}$, define $M$-tensor fields of type $(1, 0)$ on $T^*M$. Of course, all the components considered above are in the induced local chart $(\pi^{-1}(U), \Phi)$.

We shall use the horizontal distribution $HT^*M$, defined by the Levi Civita connection $\nabla$ of $g$, in order to define some first order natural lifts to $T^*M$ of the Riemannian metric $g$ on $M$. Denote by $VT^*M = \text{Ker } \pi_* \subset TT^*M$ the vertical distribution on $T^*M$. Then we have the direct sum decomposition

$$TT^*M = VT^*M \oplus HT^*M.$$
If \((\pi^{-1}(U), \Phi) = (\pi^{-1}(U), q^1, \ldots, q^n, p_1, \ldots, p_n)\) is a local chart on \(T^*M\), induced from the local chart \((U, \varphi) = (U, x^1, \ldots, x^n)\), the local vector fields \(\frac{\partial}{\partial p_1}, \ldots, \frac{\partial}{\partial p_n}\) on \(\pi^{-1}(U)\) define a local frame for \(VT^*M\) over \(\pi^{-1}(U)\) and the local vector fields \(\frac{\delta}{\delta q^1}, \ldots, \frac{\delta}{\delta q^n}\) define a local frame for \(HT^*M\) over \(\pi^{-1}(U)\), where

\[
\frac{\delta}{\delta q^i} = \frac{\partial}{\partial q^i} + \Gamma^k_{ih} \frac{\partial}{\partial p_k}, \quad \Gamma^k_{ih} = p_k \Gamma^k_{ih}
\]

and \(\Gamma^k_{ih}(\pi(p))\) are the Christoffel symbols of \(g\).

The set of vector fields \(\left(\frac{\partial}{\partial p_1}, \ldots, \frac{\partial}{\partial p_n}, \frac{\delta}{\delta q^1}, \ldots, \frac{\delta}{\delta q^n}\right)\) defines a local frame on \(T^*M\), adapted to the direct sum decomposition (1). Remark that \(\frac{\partial}{\partial p_i} = (dx^i)^V\), \(\frac{\delta}{\delta q^i} = (\frac{\partial}{\partial x^i})^H\), where \((dx^i)^V\) denotes the vertical lift to \(T^*M\) of the 1-form \(\theta\) on \(M\) and \((\frac{\partial}{\partial x^i})^H\) denotes the horizontal lift to \(T^*M\) of the vector field \(X\) on \(M\).

Now we shall present the following auxiliary result.

**Lemma 1.** If \(n > 1\) and \(u, v\) are smooth functions on \(T^*M\) such that

\[
ug_{ij} + vp_ip_j = 0, \quad p \in \pi^{-1}(U),
\]

on the domain of any induced local chart on \(T^*M\), then \(u = 0, v = 0\).

The proof is obtained easily by transvecting the given relation with components \(g^{ij}\) of the tensor field \(g^{-1}\) and \(g^{0i}\) (Recall that the functions \(g^{ij}(x)\) are the components of the inverse of the matrix \((g_{ij}(x))\), associated to \(g\) in the local chart \((U, \varphi)\) on \(M\)).

**Remark.** From the relations of the type

\[
ug^{ij} + vg^{0i}g^{0j} = 0, \quad p \in \pi^{-1}(U),
\]

\[
u\delta^i_j + vg^{0i}p_j = 0, \quad p \in \pi^{-1}(U),
\]

it is obtained, in a similar way, \(u = v = 0\). We have used the notation \(g^{0i} = p_ng^{hi}\).

Since we work in a fixed local chart \((U, \varphi)\) on \(M\) and in the corresponding induced local chart \((\pi^{-1}(U), \Phi)\) on \(T^*M\), we shall use the following simpler (but less clear) notations

\[
\frac{\partial}{\partial p_i} = \partial^i, \quad \frac{\delta}{\delta q^i} = \delta_i.
\]
Denote by
\[
(2) \quad t = \frac{1}{2} ||p||^2 = \frac{1}{2} g^{-1}_\pi(p, p) = \frac{1}{2} g^{ik}(x)p_ip_k, \quad p \in \pi^{-1}(U)
\]
the energy density defined by \( g \) in the cotangent vector \( p \). We have \( t \in [0, \infty) \) for all \( p \in T^*M \). For a vector field \( X \) on \( M \) we shall denote by \( g_X \) the 1-form on \( M \) defined by \( gX(Y) = g(X, Y) \), for all vector fields \( Y \) on \( M \). For a 1-form \( \theta \) on \( M \), we shall denote by \( \theta^\sharp = g^{-1}_\theta \theta \) the vector field on \( M \) defined by the usual musical isomorphism, i.e. \( g(\theta^\sharp, Y) = \theta(Y) \), for all vector field \( Y \) on \( M \).

Remark that, for \( p \in T^*M \), we can consider the vector \( p^\sharp \), tangent to \( M \) in \( \pi(p) \).

2. A natural almost Kähler structure of diagonal type on the cotangent bundle

From now on we shall work in a fixed local chart \((U, \varphi)\) on \( M \) and in the induced local chart \((\pi^{-1}(U), \Phi)\) on \( T^*M \). Consider two real valued smooth functions \( u, v \) defined on \([0, \infty) \subset \mathbb{R}\) and define the following \( M \)-tensor field of type \((0, 2)\) defined by the Riemannian metric \( g \) on the cotangent bundle \( T^*M \) of the \( n \)-dimensional Riemannian manifold \((M, g)\)
\[
(3) \quad G_{ij}(p) = u(t)g_{ij}(\pi(p)) + v(t)p_ip_j.
\]

It follows easily that the matrix \((G_{ij})\) is positive definite if and only if \( u > 0, \ u + 2tv > 0 \). The inverse of this matrix has the entries
\[
(4) \quad H^{kl}(p) = \frac{1}{u(t)}g^{kl}(\pi(p)) + w(t)g^{0k}g^{0l},
\]
where \( g^{0k} = p_hg^{hk} \) and
\[
(5) \quad w = -\frac{v}{u(u + 2tv)}.
\]

One shows easily that the components \( H^{kl} \) assigned to the induced local chart \((\pi^{-1}(U), q^1, \ldots, q^n, p_1, \ldots, p_n)\) on \( T^*M \) define an \( M \)-tensor field of type \((2, 0)\). The property of the matrix \((H^{kl})\) to be the inverse of the matrix \((G_{ij})\)
is expressed by the formulas

\[ G_{ik} H^{kj} = \delta^j_i. \]

**Remark.** If the matrix \((G_{ij})\) is positive definite, its inverse \((H^{kl})\) is positive definite too. This aspect can be seen directly since we have

\[ \frac{1}{u} > 0, \quad \frac{1}{u} + 2tw = \frac{1}{u + 2tv} > 0. \]

Using the \(M\)-tensor fields defined by \(G_{ij}, H^{kl}\), the following Riemannian metric may be considered on \(T^*M\)

\[ G = G_{ij} dq^i dq^j + H^{ij} Dp_i Dp_j, \]

where \(Dp_i = dp_i - \Gamma^0_{ij} dq^j\) is the absolute (covariant) differential of \(p_i\) with respect to the Levi Civita connection \(\nabla\) of \(g\) (We have used the notation \(\Gamma^0_{ij} = p_h \Gamma^h_{ij}\), where \(\Gamma^h_{ij}\) are the Christoffel symbols defined by \(g\)). Equivalently, we have

\[ G(\delta_i, \delta_j) = G_{ij}, \quad G(\partial^i, \partial^j) = H^{ij}, \quad G(\partial^i, \delta_j) = G(\delta_j, \partial^i) = 0. \]

Remark that \(HT^*M, VT^*M\) are orthogonal to each other with respect to \(G\), but the Riemannian metrics induced from \(G\) on \(HT^*M, VT^*M\) are not the same, so the considered metric \(G\) on \(T^*M\) is not a metric of Sasaki type. However, the matrix associated to the dot product induced from \(G\) on \(VT^*M\) is the inverse of the matrix associated to the dot product induced from \(G\) on \(HT^*M\), so that the metrics induced from \(G\) on \(VT^*M, HT^*M\) could be considered as being isomorphic (isometric). The \(2n \times 2n\)-matrix associated to \(G\), with respect to the adapted local frame \((\delta_1, ..., \delta_n, \partial^1, ..., \partial^n)\) has two \(n \times n\)-blocks on the first diagonal

\[ G = \begin{pmatrix} G_{ij} & 0 \\ 0 & H^{ij} \end{pmatrix}. \]

The Riemannian metric \(G\) is called a **natural lift of diagonal type** of \(g\). Remark also that the system of 1-forms \((dq^1, ..., dq^n, Dp_1, ..., Dp_n)\) defines a local frame on \(T^*T^*M\), dual to the local frame \((\delta_1, ..., \delta_n, \partial_1, ..., \partial^n)\) adapted to the direct sum decomposition (1).
Next, an almost complex structure $J$ is defined on $T^*M$ by the same $M$-tensor fields $G_{ij}$, $H^{kl}$, expressed in adapted local frames by

$$J\delta_i = G_{ik}\partial^k, \quad J\partial^i = -H^{ik}\delta_k.$$  

The matrix of $J$ with respect to the adapted local basis $(\delta_1, ..., \delta_n, \partial^1, ..., \partial^n)$ is

$$J = \begin{pmatrix} 0 & -H^{ij} \\ G_{ij} & 0 \end{pmatrix}.$$  

From the property of the $M$-tensor field $H^{kl}$ to be defined by the inverse of the matrix defined by the components of the $M$-tensor field $G_{ij}$, it follows easily that $J$ defines an almost complex structure on $T^*M$. The almost complex structure defined by $J$ is called a natural lift of diagonal type of $g$.

**Proposition 2.** The total space of the cotangent bundle $T^*M$, endowed with the Riemannian metric $G$ and the almost complex structure $J$ (both natural lifts of $g$ of diagonal type) has a structure of almost Kählerian manifold.

**Proof.** Since the matrix $(H^{kl})$ is the inverse of the matrix $(G_{ij})$, it follows easily that

$$G(J\delta_i, J\delta_j) = G(\delta_i, \delta_j) = G_{ij}, \quad G(J\partial^i, J\partial^j) = G(\partial^i, \partial^j) = H^{ij},$$

$$G(J\delta_i, J\partial^j) = G(J\partial^i, J\delta_j) = G(\delta_i, \partial^j) = G(\partial^i, \delta_j) = 0$$

Hence $G(JX, JY) = G(X, Y)$, for all vector fields $X, Y$ on $T^*M$. Thus $(T^*M, G, J)$ is an almost Hermitian manifold. The fundamental 2-form associated with this almost Hermitian structure is $\phi$, defined by

$$\phi(X, Y) = G(X, JY),$$

for all vector fields $X, Y$ on $T^*M$. By a straightforward computation we get

$$\phi(\delta_i, \delta_j) = G(\delta_i, G_{jk}\partial^k) = 0, \quad \phi(\partial^i, \partial^j) = G(\partial^i, H^{jk}\delta_k) = 0,$$

$$-\phi(\delta_j, \partial^i) = \phi(\partial^i, \delta_j) = G(\partial^i, G_{jk}\partial^k) = G_{jk}H^{ik} = \delta^i_j.$$  

It follows that

$$\phi = Dp_i \wedge dq^i = dp_i \wedge dq^i,$$

due to the symmetry of $\Gamma^0_{ij} = p_h\Gamma^h_{ij}$. It follows that $\phi$ does coincide with the fundamental 2-form defining the usual symplectic structure on $T^*M$. Of course, we have $d\phi = 0$, i.e. $\phi$ is closed. Therefore $(T^*M, G, J)$ is an almost Kähler manifold.
3. A natural Kähler structure on $T^*M$

We shall study the integrability of the almost complex structure defined by $J$ on $T^*M$. To do this we need the following well known formulas for the brackets of the vector fields $\partial^i = \frac{\partial}{\partial p_i}$, $\delta^i = \frac{\delta}{\delta q^i}$, $i = 1, \ldots, n$

\[
[\partial^i, \partial^j] = 0; \quad [\partial^i, \delta^j] = \Gamma^i_{jk} \partial^k; \quad [\delta^i, \delta^j] = R^0_{kij} \partial^k,
\]

where $R^{h}_{kij}(\pi(p))$ are the local coordinate components of the curvature tensor field of $\hat{\nabla}$ on $M$ and $R^0_{kij}(p) = p_h R^{h}_{kij}$ . Of course, the components $R^0_{kij}, R^{h}_{kij}$ define M-tensor fields of types $(0,3), (1,3)$ on $T^*M$, respectively.

**Theorem 3.** The Nijenhuis tensor field of the almost complex structure $J$ on $T^*M$ is given by

\[
N(\delta^i, \delta^j) = \{(v(2tu' - u) + uu')(\delta^i g_{jk} - \delta^j g_{ik}) - R^h_{kij}\}p_h \partial^k,
\]

\[
N(\delta^i, \partial^j) = H^{kl} H^{jr} \{(v(2tu' - u) + uu')(\delta^i g_{rl} - \delta^l g_{ir}) - R^h_{irj}\}p_h \partial^k,
\]

\[
N(\partial^i, \partial^j) = H^{ik} H^{jk} \{(v(2tu' - u) + uu')(\delta^k g_{rj} - \delta^j g_{kr}) - R^h_{jkr}\}p_h \partial^k.
\]

**Proof.** Recall that the Nijenhuis tensor field $N$ defined by $J$ is given by

\[N(X, Y) = [JX, JY] - J[JX, Y] - J[X, JY] - [X, Y], \quad \forall X, Y \in \Gamma(T^*M).\]

Then, we have $\delta_k t = 0$, $\partial^k t = g^{0k}$ and $\hat{\nabla}_i G_{jk} = 0$, $\hat{\nabla}_i H^{jk} = 0$, where

\[
 \hat{\nabla}_i G_{jk} = \delta_i G_{jk} - \Gamma^l_{ij} G_{lk} - \Gamma^l_{ik} G_{lj},
\]

\[
 \hat{\nabla}_i H^{jk} = \delta_i H^{jk} + \Gamma^l_{ij} H^{lk} + \Gamma^l_{ik} H^{lj}.
\]

The above expressions for the components of $N$ can be obtained by a quite long, straightforward computation.

**Theorem 4** The almost complex structure $J$ on $T^*M$ is integrable if and only if the base manifold $M$ has constant sectional curvature $c$ and the function $v$ is given by

\[v = \frac{c - uu'}{2tu' - u}.\]
Proof. From the condition $N = 0$ one obtains
\[
\{(v(2tu' - u) + uu') (\delta^h_i g_{jk} - \delta^h_j g_{ik}) - R^h_{kij}\} p_h = 0.
\]
Differentiating with respect to $p_l$, taking $p_h = 0 \forall h \in \{1, ..., n\}$, it follows that
the curvature tensor field of $\nabla$ has the expression
\[
R^h_{kij} = u(0)[-v(0) + u'(0)] (\delta^l_i g_{jk} - \delta^l_j g_{ik}).
\]
Using the Schur theorem (in the case where $M$ is connected and $\dim M \geq 3$), it follows that $(M, g)$ has the constant sectional curvature $c = u(0)[-v(0) + u'(0)]$. Then we obtain the expression (8) of $v$.

Conversely, if $(M, g)$ has constant sectional curvature $c$ and $v$ is given by (8), it follows in a straightforward way that $N = 0$.

Remark. In the case where $u^2 - 2ct = 0$, we have $uu' - c = 0$, $u - 2tu' = 0$ too. So, this case must be thought of as a singular case and should be considered separately. Recall that the function $u$ must fulfill the conditions
\[
u > 0, \ u + 2tv = \frac{2ct - u^2}{2tu' - u} > 0.
\]
Thus the family of natural Kählerian structures of diagonal type on $T^*M$ (when $N = 0$) depends on one essential coefficient $u$ satisfying some supplementary conditions.

4. The Levi Civita connection of the metric $G$ and its curvature tensor field

The Levi Civita connection $\tilde{\nabla}$ on a Riemannian manifold $(M, g)$ is determined by the conditions
\[
\tilde{\nabla} g = 0, \quad \tilde{T} = 0,
\]
where $\tilde{T}$ is its torsion tensor field. The explicit expression of this connection is obtained from the formula
\[
2g(\tilde{\nabla}_X Y, Z) = X(g(Y, Z)) + Y(g(X, Z)) - Z(g(X, Y)) +
+ g([X, Y], Z) - g([X, Z], Y) - g([Y, Z], X), \quad \forall X, Y, Z \in \Gamma(M).
\]

We shall use this formula in order to obtain the expression of the Levi Civita connection $\nabla$ of $G$ on $T^*M$. The final result can be stated as follows
Theorem 7. The Levi Civita connection $\nabla$ of $G$ on $T^*M$ has the following expression in the local adapted frame $(\partial^1, \ldots, \partial^n, \delta^1, \ldots, \delta^n)$

\[
\begin{align*}
\nabla_{\partial^i} \partial^j &= Q^i_{\partial} \partial_j, \\
\nabla_{\delta^i} \delta_j &= P^i \delta_j, \\
\nabla_{\partial^i} \delta_j &= \Gamma^h_{ij} \partial^h + S^h_{ij} \delta^h,
\end{align*}
\]

where $Q^i_{\partial}, P^i, S^h_{ij}$ are $M$-tensor fields on $T^*M$, defined by

\[
\begin{align*}
Q^i_{\partial} &= \frac{1}{2} G^{hk}(\partial^i H^{jk} + \partial^j H^{ik} - \partial^k H^{ij}), \\
R^i_j &= \frac{1}{2} H^{hk}(\partial^i G_{jk} - H^{il} R^i_{lk}), \\
S^h_{ij} &= \frac{1}{2} R^i_{jk} \partial^h G_{ij} + \frac{1}{2} P^i_{hk}.
\end{align*}
\]

After replacing of the expressions of the involved $M$-tensor fields and their derivatives, one obtains

\[
\begin{align*}
Q^i_{\partial} &= \frac{1}{2} G^{hk}(\partial^i H^{jk} + \partial^j H^{ik} - \partial^k H^{ij}), \\
R^i_j &= \frac{1}{2} H^{hk}(\partial^i G_{jk} - H^{il} R^i_{lk}), \\
S^h_{ij} &= \frac{1}{2} R^i_{jk} \partial^h G_{ij} + \frac{1}{2} P^i_{hk}.
\end{align*}
\]

In the case of a Kähler structure on $T^*M$, the final expressions of these $M$-tensor fields can be obtained by doing the necessary replacements of the functions $v, w$ from (8) and (5) and their derivatives. However, the final expressions are quite complicated but they may be obtained quite automatically by using the Mathematica package RICCI for doing tensor computations (see [4]).

Now we shall indicate the obtaining of the components of the curvature tensor field of the connection $\nabla$.

The curvature tensor $K$ field of the connection $\nabla$ is obtained from the well known formula

\[
K(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z.
\]

The components of $K$ with respect to the adapted local frame $(\partial^1, \ldots, \partial^n, \delta^1, \ldots, \delta_n)$ are obtained easily:
\[
K(\partial^i, \partial^j)\partial^k = PPP_h^{ijk} \partial^h = (\partial^i Q_h^{jk} - \partial^j Q_h^{ik} + Q_h^{lk} Q_h^{ij} - Q_h^{lk} Q_h^{ij}) \partial^h,
\]
\[
K(\partial^i, \partial^j)\delta_k = PPP_k^{ijh} \delta_h = (\partial^i P_k^{hj} - \partial^j P_k^{hi} + P_k^{lj} P_k^{hi} - P_k^{lj} P_k^{hi}) \delta_h,
\]
\[
K(\delta_i, \delta_j)\partial^k = QQP_{ijk}^h \delta_h = (-R_{hij}^k - R_{lij}^i P_h^{kl} + S_{hij} P_{k}^{il} - S_{lij} P_{h}^{il}) \delta_h,
\]
\[
K(\delta_i, \delta_j)\delta_k = QQP_{ijk}^h \delta_h = (R_{kij}^h - R_{lij}^i P_h^{kl} + S_{ljk} P_{h}^{il} - S_{lij} P_{k}^{il}) \delta_h,
\]
\[
K(\partial^i, \delta_j)\partial^k = PQP_{ijk}^h \delta_h = (\partial^i P_j^{hj} + P_k^{hi} P_j^{pk} - Q_l^{jk} P_j^{pk}) \delta_h.
\]

The explicit expressions of these components are obtained after some quite long and hard computations, made by using the package RICCI.

Next, the components of the Ricci tensor field are obtained as traces of \(K\):
\[
\begin{align*}
\text{Ric}(\partial^i, \partial^k) &= \text{Ric}PP^k = PPP_h^{hjk} - PQP_h^{jk}, \\
\text{Ric}(\delta_j, \delta_k) &= \text{Ric}QQ_{jk} = QQP_{h}^{jk} + PQP_{h}^{jk}, \\
\text{Ric}(\partial^i, \delta_k) &= \text{Ric}(\delta_k, \partial^i) = 0.
\end{align*}
\]

### 5 The cotangent bundle \(T^*M\) as a Kähler Einstein manifold

Doing the necessary computations, we obtain the final expressions of the components of the Ricci tensor field of \(\nabla\):
\[
\begin{align*}
\text{Ric}PP^k &= \frac{a}{2(u-2tu)^2}g^{jk} + \frac{\alpha}{2u^2(u-2tu)^2}g_{jk} P_k, \\
\text{Ric}PP^k &= \frac{a}{2u^2(u-2tu)^2}g^{jk} + \frac{\beta}{2u^2(u-2tu)^2}g_{jk} g_{0k},
\end{align*}
\]
where the coefficients \(a, \alpha, \beta\) are given by
\[
\begin{align*}
a &= n(u - 2tu'') (2cu - 2ctu' - u^2u') + 2(2ct - u^2) (tuu'' + uu' - tu') , \\
\alpha &= n(u - 2tu') (2cu - 2ctu' - u^2u') + 3cu^4u' - 12c^2t^2uu' - 3u^5u'' + 8c^2t^3u^3 - 4ctu^2u^3 + 4tu^4u^3 - 4c^2t^2u^2u'' + 4ctu^4u'' - u^6u''),
\end{align*}
\]
\[ +2(2ct - u^2)(-3cu^3u' + 7ctu^2u'' + 4u^4u'^2 - 8ct^2u'u^3 - 8tu^3u'^3 + 4ct^3u'^4 + 4t^2u^2u'^4 - \]
\[ - 7ctu^3u'' + 2u^5u'' + 6ct^2u^2u'' + 3tu^4u'u'' - 6t^2u^3u'^2u'' - 8ct^3u^2u''^2 + \]
\[ + 4t^2u'^2u''^2 - 2ct^2u^3u'(3) + tu^5u'(3) + 4ct^3u^2u'u'(3) - 2t^2u^4u'(3)), \]
\[ \beta = n(u - 2tu')(2c^2u - 2c^2tu' - 3cu^2u' + 6ctuu^2 - u^3u'^2 - 4ctu^2u'^3 + 2tu^2u'^3 + \]
\[ + 2ctu^2u'' - u^4u'' + \]
\[ + 2(2^2tu^2u' - 5ctu^2u'^2 - 2u^4u'^2 + 8ct^2u'u^3 + 4tu^3u'^3 - 4ct^3u'^4 - \]
\[ - 2t^2u^4u'^4 + 2ct^2uu'' + 5ct^3u'' - 2u^5u'' - 6ct^2u^2u'u'' - tu^4u'u'' + 4t^2u^3u'^2u'' + \]
\[ + 8ct^3u^2u''^2 - 4t^2u^4u''^2 + 2ct^2u^3u'(3) - tu^5u'(3) - 4ct^3u^2u'u'(3) + 2t^2u^4u'u'(3)). \]

In order to find the conditions under which \((T^* M, G, J)\) is Einstein, we consider the differences

\[
\begin{align*}
\text{Diff}QQ_{jk} &= \text{Ric}QQ_{jk} - \frac{\alpha}{2u(u-2tu')^2} G_{jk}; \\
\text{Diff}PP_{jk} &= \text{Ric}PP_{jk} - \frac{\alpha}{2u(u-2tu')^2} H_{jk},
\end{align*}
\]

whose explicit expressions are

\[
\begin{align*}
\text{Diff}QQ_{jk} &= \frac{u^2 - 2ct}{2u^2(u-2tu')} \gamma P_{jk}; \\
\text{Diff}PP_{jk} &= \frac{1}{2u^2(u-2ct)(u-2tu')} \gamma g^{0j}g^{0k}.
\end{align*}
\]

The expression of the factor \(\gamma\) is given by

\[
\gamma = n(u^2 - 2ct)(2tu' - u)(u^2u'' - 2tu'^3 + 2u'u^2) + \\
+ 2(2cu^3u' - 4ctu^2u'^2 - 3u^4u'^2 + 6ct^2u'^3 + 5tu^3u'^3 - 4ct^3u'^4 - 2t^2u^2u'^4 + \]
\[ + 6ctu^3u'' - 2u^5u'' - 4ct^2u^2u'u'' - 2tu^4u'u'' + 4t^2u^3u'^2u'' + 8ct^3u^2u''^2 - \]
\[ - 4t^2u'^2u''^2 + 2ct^2u^3u'(3) - tu^5u'(3) - 4ct^3u^2u'u'(3) + 2t^2u^4u'u'(3)), \]

We are interested in finding the functions \(u\) for which \(\text{Diff}QQ_{jk} = 0, \text{Diff}PP_{jk} = 0\). We should exclude the case \(u^2 = 2ct = 0\) which leads to a singularity. Thus we must see what happens in the case \(\gamma = 0, u^2 - 2ct \neq 0\). Generally, this equation is almost impossible to solve. However, it is reasonable
to ask for the solution $u$ to be independent of the dimension $n$ of $M$. In this case we must have

$$u^2 u'' - 2tu^3 + 2uu'^2 = 0 \quad (9)$$

The general solution of this equation is obtained transforming it in an equation in the inverse function $t = t(u)$ of $u$. One obtains an equation of Euler type from which we get

$$u = A \pm \sqrt{A^2 + Bt}, \quad A > 0, \ B \neq 0, \ A^2 + Bt > 0,$$

where $A, B$ are the integration constants.

**Remark.** The equation (9) has two other singular solutions

$$u = A, \ A > 0, \ A^2 - 2ct > 0,$$

$$u = At, \ A > 0, \ 2c - A^2t > 0,$$

which will be discussed in some forthcoming papers.

From now on we shall consider only the case of the function $u = A + \sqrt{A^2 + Bt}$. Asking for the found function to be a solution for the remaining part of the equation $\gamma = 0$, one finds $B = -2c$. Thus the general solution for the condition for $(T^*M, G, J)$ to be Einstein is

$$u = A + \sqrt{A^2 - 2ct}, \ A > 0, \ A^2 - 2ct > 0. \quad (10)$$

If the constant sectional curvature $c$ of $M$ is negative, the solution $u$ is defined on the whole $T^*M$.

If $c$ is positive, the solution $u$ is defined only in the tube around the zero section in $T^*M$, defined by $0 \leq \|p\|^2 < \frac{A^2}{c}$. Then we obtain easily

$$v = \frac{1}{2t}(A - \frac{4ct}{A} - \sqrt{A^2 - 2ct}), \quad w = \frac{-A^3 + 3A^2t + (A^2 - 2ct)^{3/2}}{4ct^2(A^2 - 2ct)}.$$

Remark that $v, w$ are well defined even if $t = 0$. Then we have

$$u + 2tv = \frac{2(A^2 - 2ct)}{A} > 0,$$

thus the conditions for the existence of the Kähler Einstein manifold $(T^*M, G, J)$ are fulfilled.
The components of the Ricci tensor field defined by $G$ are

$RicQQ_{jk} = \frac{(n+1)c}{A} G_{jk}$,
$RicPP_{jk} = \frac{(n+1)c}{A} H^j^k$,

$Ric(\partial^i, \delta_k) = Ric(\delta_k, \partial^i) = 0$.

Hence we may state our main result.

**Theorem 5.**
1. Assume that the Riemannian manifold $(M, g)$ has constant negative sectional curvature $c$. Then $(T^*M, G, J)$ defined by $u$ given in (10), with $A > 0$ is a Kähler Einstein manifold.
2. Assume that $(M, g)$ has constant positive sectional curvature $c$. Then the tube around the zero section in $T^*M$, defined by the condition $0 \leq ||p||^2 < \frac{A^2}{c}$ is a Kähler Einstein manifold, with $(G, J)$ defined as in the case 1.

6 The holomorphic sectional curvature of $(T^*M, G, J)$

Recall that a Kählerian manifold $(M, g, J)$ has constant holomorphic sectional curvature $k$ if its curvature tensor field $R$ is given by

$R(X, Y)Z = \frac{k}{4}(g(Z, Y)X - g(Z, X)Y + g(Z, JY)JX - g(Z, JX)JY + 2g(X, JY)JZ)$,

for all vector fields $X, Y, Z$ defined on $M$.

In the case of the Kähler Einstein structure $(G, J)$ on $T^*M$ (on a tube around zero section in $T^*M$) obtained in Theorem 5, we can check by a straightforward computation that the components of the curvature tensor field $K$ of $\nabla$ are given by

$K(\delta_i, \delta_j)\delta_k = \frac{c}{2A}(\delta^h_iG_{jk} - \delta^h_jG_{ik})\delta_h$,
$K(\delta_i, \delta_j)\partial^k = \frac{c}{2A}(\delta^h_kG_{ih} - \delta^h_iG_{jh})\partial^h$,

$K(\partial^i, \delta_j)\delta_k = \frac{c}{2A}(\delta^j_kH^{ih} - \delta^j_hH^{jk})\delta_h$,
$K(\partial^i, \partial^j)\partial^k = \frac{c}{2A}(\delta^h_kH^{ij} - \delta^h_iH^{jk})\partial^h$,

$K(\partial^i, \delta_j)\partial^k = \frac{c}{2A}(\delta^j_kG_{jk} + \delta^j_hG_{jh} + 2\delta^j_iG_{kh})\partial^h$,

$K(\delta_i, \delta_j)\partial^k = \frac{c}{2A}(\delta^h_iH^{jk} + \delta^h_jH^{ih} + 2\delta^h_iH^{jh})\delta_h$.

From these relations we get

**Theorem 6.** The Kähler Einstein structure $(G, J)$ on $T^*M$ (on a tube around zero section in $T^*M$) obtained in Theorem 5 has constant holomorphic sectional curvature $k = \frac{2c}{A}$. 

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