On the randomized Euler schemes for ODEs under inexact information

Tomasz Bochacik¹ · Paweł Przybyłowicz¹

Received: 4 May 2021 / Accepted: 9 March 2022 / Published online: 11 April 2022
© The Author(s), under exclusive licence to Springer Science+Business Media, LLC, part of Springer Nature 2022

Abstract
We analyse errors of randomized explicit and implicit Euler schemes for approximate solving of ordinary differential equations (ODEs). We consider classes of ODEs for which the right-hand side functions satisfy Lipschitz condition globally or only locally. Moreover, we assume that only inexact discrete information, corrupted by some noise, about the right-hand side function is available. Optimality and stability of explicit and implicit randomized Euler algorithms are also investigated. Finally, we report the results of numerical experiments which support our theoretical conclusions.

Keywords Noisy information · Randomized Euler algorithms · Explicit and implicit schemes · nth minimal error · Optimality · Stability

Mathematics Subject Classification (2010) 65C05 · 65C20 · 65L05 · 65L06 · 65L20

In this paper we consider ordinary differential equations (ODEs) of the following form:

\[
\begin{align*}
  z'(t) &= f(t, z(t)), \quad t \in [a, b], \\
  z(a) &= \eta,
\end{align*}
\]  

¹ Faculty of Applied Mathematics, AGH University of Science and Technology, Al. A. Mickiewicza 30, 30-059 Kraków, Poland
where $-\infty < a < b < \infty$, $\eta \in \mathbb{R}^d$, $f : [a, b] \times \mathbb{R}^d \to \mathbb{R}^d$, $d \in \mathbb{Z}_+$. We will consider the class of randomized algorithms and investigate the main properties of the randomized Euler schemes under inexact information, such as error bounds, optimality and stability.

Randomized algorithms for the approximate solving of ODEs have attracted attention in the recent years, see for example [2, 4–7, 9, 16, 17]. Nevertheless, there is still a large space for further research on this topic in the setting of inexact information. This paper is an attempt in this direction. It is worth noting that related numerical problems — such as function integration and approximation, approximate solving of PDEs, stochastic integration and SDEs — are already extensively studied in the noisy information framework, see for instance [8, 10–13, 19, 20].

The noise may be caused by rounding and observational errors, as well as by previous calculations and approximations made at the problem modelling stage. Moreover, in applications such as deep learning algorithms, computations are often performed in lower precision in order to boost efficiency. Thus, the framework of inexact information considered in this paper has a strong link to practice.

Randomized implicit and explicit Euler schemes have been investigated under exact information in the articles [3, 6, 9, 16, 17]. In this paper we allow noisy information about the initial value $\eta$ and the right-hand side function $f$. We will use similar assumptions as in [1], among which the key one is the (local or global) Lipschitz condition with respect to the state variable of $f$. In some applications (e.g. switching systems in automation) irregular right-hand side functions satisfying only the local Lipschitz condition are common. Hence, local assumptions are important not only from the theoretical perspective. They have been considered also by other authors, cf. [3, 6].

The structure of this paper and considered problems resemble those from [1] but here we consider other algorithms; thus the current research can be viewed as a continuation of our previous work. We start from $L^p(\Omega)$-error analysis of randomized Euler schemes under inexact information. Error bound for the explicit scheme (Theorem 1) has been established under very mild assumptions, as it requires only local Lipschitz continuity of $f$ and linear growth of the noise function. Analogous result for the implicit scheme (Theorem 2) has been proven under global Lipschitz condition for $f$. Furthermore, we establish lower error bounds for all algorithms based on randomized inexact information in a certain class of right-hand side functions (Theorem 3) and we provide condition for optimality of the randomized Euler schemes (Proposition 1). Another novelty of our paper is stability analysis for the aforementioned algorithms. We characterize stability regions (mean-square, asymptotic and in probability) using a test problem designed to capture randomization in the time variable of $f$ and we show that stability regions are empty for the explicit scheme, whereas for the implicit scheme they cover almost entire complex plane, see Proposition 2 and Remark 3.

This paper is organized as follows. Section 1 of this paper contains basic notation, assumptions about (1), outline of the model of computation (including specification of the noise which corrupts values of the function $f$) and a definition of the $n$th minimal error. In Sections 2 and 3 we establish upper bounds of the $L^p(\Omega)$-error of the randomized Euler schemes under inexact information (explicit and implicit,
respectively). Lower bounds and condition for optimality of the randomized Euler schemes are discussed in Section 4. Section 5 is devoted to numerical experiments. In Section 6 we propose a non-classical test problem to assess stability of the investigated algorithms and we show superiority of the implicit scheme in this aspect. Main conclusions of the paper are discussed in Section 7. Finally, in Appendix we gather some auxiliary results.

1 Preliminaries

Let $\| \cdot \|$ be the one norm in $\mathbb{R}^d$, i.e. $\| x \| = \sum_{k=1}^d |x_k|$ for $x \in \mathbb{R}^d$. For $x \in \mathbb{R}^d$ and $r \in [0, \infty)$ we denote by $B(x, r) = \{ y \in \mathbb{R}^d : \| y - x \| \leq r \}$ the closed ball in $\mathbb{R}^d$ with centre $x$ and radius $r$. Moreover, $B(x, \infty) = \mathbb{R}^d$ for all $x \in \mathbb{R}^d$.

Let $(\Omega, \Sigma, \mathbb{P})$ be a complete probability space and let $\mathcal{N} = \{ A \in \Sigma : \mathbb{P}(A) = 0 \}$. For a random variable $X : \Omega \to \mathbb{R}$, defined on $(\Omega, \Sigma, \mathbb{P})$, we denote its $L^p(\Omega)$ norm by $X^p = (E|X|^p)^{1/p}$, $p \in [2, \infty]$. For a Polish space $E$ by $B(E)$ we denote the Borel $\sigma$-field on $E$. Let $\in (0, 1], K, L \in (0, \infty)$ and $R \in [0, \infty]$. As in [1], we consider a class $F_R^\infty = F_R^\infty(a, b, d, K, L)$ of pairs $(\eta, f)$ satisfying the following conditions:

(A0) $\| \eta \| \leq K$,

(A1) $f \in C ([a, b] \times \mathbb{R}^d),$

(A2) $\| f(t, x) \| \leq K (1 + \| x \|)$ for all $(t, x) \in [a, b] \times \mathbb{R}^d$,

(A3) $\| f(t, x) - f(s, x) \| \leq L|t - s|^\alpha$ for all $t, s \in [a, b], x \in B(\eta, R)$,

(A4) $\| f(t, x) - f(t, y) \| \leq L\| x - y \|$ for all $t \in [a, b], x, y \in B(\eta, R)$.

Note that $F^\infty_R$ consists of globally Lipschitz continuous functions, whereas functions from $F^\infty_0$ may not satisfy Lipschitz condition even locally. Moreover, $F^\infty_\infty \subset F^\infty_R \subset F^\infty_R$ for $\infty \geq R \geq R' \geq 0$. Parameters of the class $F^\infty_R$ are: $a, b, d, K \in (0, \infty), L \in (0, \infty)$ and $R \in [0, \infty]$. These parameters, excluding $a, b$, and $d$, are usually not known in practical applications. Thus, they will be not used as an input of algorithms presented later in the paper.

To approximate the solution of (1) for $(\eta, f) \in F^\infty_R$, we will consider randomized algorithms based on inexact information about $f$. Now we will introduce the model of computation. Let us define the following two classes of noise functions:

$$
\mathcal{K}_1(\delta) = \left\{ \tilde{\delta} : [a, b] \times \mathbb{R}^d \to \mathbb{R}^d : \tilde{\delta} \text{ is Borel measurable,} \right. \\
\left. \| \tilde{\delta}(t, y) \| \leq \delta (1 + \| y \|) \text{ for all } t \in [a, b], y \in \mathbb{R}^d \right\} \quad (2)
$$

and

$$
\mathcal{K}_2(\delta) = \left\{ \tilde{\delta} \in \mathcal{K}_1(\delta) : \| \tilde{\delta}(t, x) - \tilde{\delta}(t, y) \| \leq \delta \| x - y \| \text{ for all } t \in [a, b], x, y \in \mathbb{R}^d \right\}, \quad (3)
$$
where $\delta \in [0, 1]$ is called the precision parameter. Note that $\mathcal{K}_2(\delta) \subset \mathcal{K}_1(\delta)$ for each $\delta \in [0, 1]$ and there is no direct inclusion between these classes and the class $\mathcal{K}(\delta)$ considered in [1].

We assume that an algorithm may use only noisy evaluations of the function $f$. Specifically, for each point $(t, y) \in [a, b] \times \mathbb{R}^d$ we have

$$\tilde{f}(t, y) = f(t, y) + \tilde{\delta}_f(t, y),$$

where $\tilde{\delta}_f$ is an element of the class $\mathcal{K}_i(\delta)$ ($i \in \{1, 2\}$ depending on which framework we choose) and $\tilde{\delta}_f(t, y)$ is an error corrupting the exact value $f(t, y)$. We allow randomized choice of the evaluation points $(t, y)$. Let

$$V^i_f(\delta) = \{\tilde{f}: \exists \delta f \in \mathcal{K}_i(\delta) \tilde{f} = f + \tilde{\delta}_f\}$$

and

$$V^i_{(\eta, f)}(\delta) = B(\eta, \delta) \times V^i_f(\delta)$$

for $(\eta, f) \in F^\infty_R$, $\delta \in [0, 1]$ and $i \in \{1, 2\}$. Let us note that $V^i_{(\eta, f)}(\delta) \subset V^i_{(\eta, f)}(\delta')$ for $0 \leq \delta \leq \delta' \leq 1$ and $V^i_{(\eta, f)}(0) = \{(\eta, f)\}$. Moreover, it holds that $V^1_{(\eta, f)}(\delta) \subset V^1_{(\eta, f)}(\delta)$. An additional regularity condition has been imposed on noise functions in class $\mathcal{K}_2(\delta)$ in order to establish convergence of the implicit Euler scheme under noisy information, cf. Theorem 2.

Let $(\eta, f) \in F^\infty_R$ and $(\tilde{\eta}, \tilde{f}) \in V^1_{(\eta, f)}(\delta)$. A vector of noisy information about $(\eta, f)$ takes the following form:

$$N(\tilde{\eta}, \tilde{f}) = [\tilde{f}(t_0, y_0), \ldots, \tilde{f}(t_{i-1}, y_{i-1}), \tilde{f}(\theta_0, z_0), \ldots, \tilde{f}(\theta_{i-1}, z_{i-1}), \tilde{\eta}],$$

where $i \in \mathbb{Z}_+$ and $(\theta_0, \theta_1, \ldots, \theta_{i-1})$ is a random vector on $(\Omega, \Sigma, \mathbb{P})$. Furthermore,

$$(y_0, z_0) = \psi_0(\tilde{\eta}),$$

and

$$(y_j, z_j) = \psi_j(\tilde{f}(t_0, y_0), \ldots, \tilde{f}(t_{j-1}, y_{j-1}), \tilde{f}(\theta_0, z_0), \ldots, \tilde{f}(\theta_{j-1}, z_{j-1}), \tilde{\eta})$$

for Borel measurable mappings $\psi_j : \mathbb{R}^{(2j+1)d} \to \mathbb{R}^d \times \mathbb{R}^d$, $j \in \{0, \ldots, i - 1\}$. In particular, this implies that $N(\tilde{\eta}, \tilde{f}) : \Omega \to \mathbb{R}^{(2i+1)d}$ is a random vector. The total number of noisy evaluations of $f$ is $l = 2i$.

We consider the class $\Phi$ of algorithms $A$ which aim to compute the approximate solution $z$ of (1) using $N(\tilde{\eta}, \tilde{f})$. Such algorithms have the following form:

$$A(\tilde{\eta}, \tilde{f}, \delta) = \varphi(N(\tilde{\eta}, \tilde{f})),$$

where

$$\varphi : \mathbb{R}^{(2i+1)d} \to D([a, b]; \mathbb{R}^d)$$

is a Borel measurable function — in the Skorokhod space $D([a, b]; \mathbb{R}^d)$, endowed with the Skorokhod topology, we consider the Borel $\sigma$-field $\mathcal{B}(D([a, b]; \mathbb{R}^d))$. Therefore $A(\tilde{\eta}, \tilde{f}, \delta) : \Omega \to D([a, b]; \mathbb{R}^d)$ is $\Sigma$-to-$\mathcal{B}(D([a, b]; \mathbb{R}^d))$ measurable. Moreover, by Theorem 7.1 in [15] the $\sigma$-field $\mathcal{B}(D([a, b]; \mathbb{R}^d))$ coincides with the $\sigma$-field generated by coordinate mappings. Hence, for all $t \in [a, b]$ the mapping

$$\Omega \ni \omega \mapsto A(\tilde{\eta}, \tilde{f}, \delta)(\omega)(t) \in \mathbb{R}^d$$

(4)
is $\Sigma$-to-$\mathcal{B}(\mathbb{R}^d)$-measurable. For a given $n \in \mathbb{Z}_+$ we denote by $\Phi_n$ a class of all algorithms from $\Phi$ requiring at most $n$ noisy evaluations of $f$.

Let $p \in [2, \infty)$. For a fixed $(\eta, f) \in F_0^\delta$, the error of $A \in \Phi_n$ is given as
\[
e^{(p)}(A, \eta, f, V^i, \delta) = \sup_{(\tilde{\eta}, \tilde{f}) \in V_0^i(\eta, f)(\delta)} \left\| z(\eta, f)(t) - A(\tilde{\eta}, \tilde{f}, \delta)(t) \right\|_p,
\]
for $i \in \{1, 2\}$. (The error is well defined, see [1], Remark 2.) The worst-case error of the algorithm $A$ is defined by
\[
e^{(p)}(A, G, V^i, \delta) = \sup_{(\eta, f) \in G} e^{(p)}(A, \eta, f, V^i, \delta),
\]
where $i \in \{1, 2\}$ and $G$ is a subclass of $F_0^\delta$, see [18]. Finally, we consider the $n$th minimal error defined as
\[
e^{(p)}_n(G, V^i, \delta) = \inf_{A \in \Phi_n} e^{(p)}(A, G, V^i, \delta), \quad i \in \{1, 2\}.
\]
Of course, we have that $e^{(p)}_n(G, V^2, \delta) \leq e^{(p)}_n(G, V^1, \delta)$.

The proposed framework of inexact information is useful in the context of efficient computations on CPUs and GPUs. For example, in deep learning half precision is often used for performance reasons. For details, see Remark 1 in [1] and [8, 12, 13].

**Remark 1** We note that the assumption $\delta \leq 1$ is not necessary; however, it slightly simplifies calculations, see for example Fact 1 and Lemma 1. Furthermore, the main result (Theorem 3) is of asymptotic nature. Finally, if $\delta$ is big, the structure of the whole approximation is blurred and we cannot obtain any reliable result. Thus, the case of big $\delta$ is out of interest.

### 2 Error analysis of the randomized explicit Euler scheme under inexact information

In this section we provide upper bound for $L^p(\Omega)$-error of the randomized explicit Euler scheme when the radius $R$ appearing in assumptions (A3) and (A4) is sufficiently large. The only parameters necessary to specify $R$ are $a, b, K$ (thus, $R$ does not depend on a particular IVP).

The randomized explicit Euler method under inexact information is given by the following recurrence relation:
\[
\tilde{V}^0 = \tilde{\eta}, \quad \tilde{V}^j = \tilde{V}^{j-1} + h \cdot \tilde{f} \left( \theta_j, \tilde{V}^{j-1} \right), \quad j \in \{1, \ldots, n\},
\]
where $(\eta, f) \in F_0^\delta_R$ for some $R > 0$, $(\tilde{\eta}, \tilde{f}) \in V_0^1(\eta, f)(\delta)$, $n \in \mathbb{Z}_+$, $h = \frac{b-a}{n}$, $\tau_j = a + jh$ for $j \in \{0, 1, \ldots, n\}$, $\theta_j = \tau_{j-1} + \tau_j h$ and $\tau_j \sim U(0, 1)$ for $j \in \{1, \ldots, n\}$. We assume that $\{\tau_1, \ldots, \tau_n\}$ is an independent family of random variables. Note that
\[
\tilde{f} \left( \theta_j, \tilde{V}^{j-1} \right) = f \left( \theta_j, \tilde{V}^{j-1} \right) + \tilde{\delta}_f \left( \theta_j, \tilde{V}^{j-1} \right).
\]
for some \( \tilde{\delta}_f \in \mathcal{K}_1(\delta) \). The solution of (1) is approximated by a piecewise linear function \( \tilde{l}^{EE} : [a, b] \to \mathbb{R}^d \) given by

\[
\tilde{l}^{EE}(t) = \tilde{l}^{EE}_j(t) \text{ for } t \in [t_{j-1}, t_j]. \quad \tilde{l}^{EE}_j(t) = \frac{\tilde{V}^j - \tilde{V}^{j-1}}{h}(t - t_{j-1}) + \tilde{V}^{j-1}, \quad j \in \{1, \ldots, n\}. \quad (9)
\]

For \( \delta = 0 \) (i.e. in case of exact information) we use notation without bars: \( V^j, l^{EE} \) and \( l^{EE}_j \) instead of \( \tilde{V}^j, \tilde{l}^{EE} \) and \( \tilde{l}^{EE}_j \), respectively.

The main result of this section (Theorem 1) will be preceded by two auxiliary facts. In Fact 1 we show that the sequence generated by the randomized explicit Euler scheme (under certain assumptions) falls inside the ball \( B(\eta, R) \) for suitably chosen \( R \). Note that this is the case also for the exact solution \( t \mapsto z(t) \) of (1), as stated in Lemma 2(i) in Appendix. Fact 2 in turn provides upper bound of the difference between sequences generated by the algorithm under exact (\( \delta = 0 \)) and inexact information. Hence, we adapt the proof technique from [1] in order to cover the case considered in this paper.

**Fact 1** Let

\[
R_1 = (K + 2)e^{(K+1)(b-a)} + K - 1. \quad (10)
\]

Then for all \( (\eta, f) \in F^{g}_{R_1}, \ (\tilde{\eta}, \tilde{f}) \in V^1_{(\eta, f)}(\delta), n \in \mathbb{Z}_+, \delta \in [0, 1], \) and \( j \in \{0, 1, \ldots, n\} \)

\[
V^j, \tilde{V}^j \in B(\eta, R_1)
\]

almost surely.

**Proof** Let us note that by considering any \( \delta \in [0, 1] \) we cover both cases \( V^j \) and \( \tilde{V}^j \).

By assumption (A0) and since \( \tilde{\eta} \in B(\eta, \delta) \subset B(\eta, 1) \),

\[
\|\tilde{V}^0\| \leq \|\eta\| + \|\tilde{\eta} - \eta\| \leq K + 1.
\]

Let \( j \in \{1, \ldots, n\} \). Assumption (A2) and definition (3) imply that the following inequality holds with probability 1:

\[
\|\tilde{V}^j\| \leq \|\tilde{V}^{j-1}\| + h\|f(\theta_j, \tilde{V}^{j-1})\| + h\|\tilde{\delta}_f(\theta_j, \tilde{V}^{j-1})\|
\leq \|\tilde{V}^{j-1}\|(1 + h(K + 1)) + h(K + 1).
\]

By discrete Gronwall’s inequality:

\[
\|\tilde{V}^j\| \leq \|\tilde{V}^0\|(1 + h(K + 1))^n + (1 + h(K + 1))^n - 1 \leq (K + 2)e^{(K+1)(b-a)} - 1 = R_1 - K. \quad (11)
\]

Note that (11) holds also for \( j = 0 \) since \( R_1 - K \geq K + 1 \geq \|\tilde{V}^0\| \). By (11) and (A0),

\[
\|\tilde{V}^j - \eta\| \leq \|\tilde{V}^j\| + \|\eta\| \leq R_1
\]

for all \( j \in \{0, 1, \ldots, n\} \) and the proof is completed. \( \square \)

**Fact 2** Let \( R_1 \) be defined as in Fact 1. Then there exists a constant \( C = C(a, b, K, L) > 0 \) such that for all \( (\eta, f) \in F^{g}_{R_1}, \ (\tilde{\eta}, \tilde{f}) \in V^1_{(\eta, f)}(\delta), n \in \mathbb{Z}_+, \delta \in [0, 1] \) it holds

\[
\max_{0 \leq j \leq n} \|V^j - \tilde{V}^j\| \leq C\delta \quad (12)
\]
with probability 1.

Proof Firstly, let us note that \( \|\tilde{V}^0 - V^0\| = \|\tilde{\eta} - \eta\| \leq \delta \). For \( j \in \{1, \ldots, n\} \), by (A4), (3) and (11), we obtain

\[
\|\tilde{V}^j - V^j\| \leq \|\tilde{V}^{j-1} - V^{j-1}\| + h\|f(\theta_j, \tilde{V}^{j-1}) - f(\theta_j, V^{j-1})\| + h\|\tilde{f}(\theta_j, \tilde{V}^{j-1})\|
\]
\[
\leq (1 + hL)\|\tilde{V}^{j-1} - V^{j-1}\| + h\delta(1 + R_1 - K).
\]

By discrete Gronwall’s inequality:

\[
\|\tilde{V}^j - V^j\| \leq (1 + hL)^n \|\tilde{V}^0 - V^0\| + \delta(1 + R_1 - K)(1 + hL)^n - 1.
\]

which leads to (12).

Theorem 1 Let \( p \in [2, \infty) \). There exists a constant \( C = C(a, b, d, K, L, \varrho, p) > 0 \) such that for all \( n \geq [b - a] + 1, \delta \in [0, 1], (\eta, f) \in F_{R_0}^g, (\tilde{\eta}, \tilde{f}) \in V_{(\eta, f)}^1(\delta) \) it holds

\[
\sup_{a \leq t \leq b} \left\| z(\eta, f)(t) - \tilde{\eta}^{EE}(\tilde{\eta}, \tilde{f}, \delta)(t) \right\|_p \leq C \left( h^{\min\{p+1, 1\}} + \delta \right),
\]

where \( R_0 = \max \{R_1, R_2\} \), \( R_1 \) is given by (10) and \( R_2 \) — by (41).

Proof Let \( \Delta_j = [t_{j-1}, t_j] \) and

\[
\tilde{z}_j(t) = \frac{z(t_j) - z(t_{j-1})}{h}(t - t_{j-1}) + z(t_{j-1})
\]

for \( t \in \Delta_j, \ j \in \{1, \ldots, n\} \). In this proof \( C \) may denote different constants but always depending only on the following parameters: \( a, b, d, K, L, \varrho, p \).

Let us observe that

\[
\left\| \sup_{a \leq t \leq b} \left\| z(t) - \tilde{\eta}^{EE}(t) \right\|_p \right\| \leq \max_{1 \leq j \leq n} \sup_{t \in \Delta_j} \|z(t) - \tilde{z}_j(t)\| + \left\| \sup_{1 \leq j \leq n, t \in \Delta_j} \left\| \tilde{z}_j(t) - \tilde{\eta}^{EE}(t) \right\|_p \right\|.
\]

We will find an upper bound for the first term in the right-hand side of (14). By the Lagrange mean value theorem for \( t \in \Delta_j \) we get

\[
z(t) = (z_1(t), \ldots, z_d(t)) = (z_1(t_{j-1}) + z_1'(\alpha_{1j})(t - t_{j-1}), \ldots, z_d(t_{j-1}) + z_d'(\alpha_{dj})(t - t_{j-1})),
\]

where \( \alpha_{lj} \in [t_{j-1}, t] \subset \Delta_j, j \in \{1, \ldots, n\}, l \in \{1, \ldots, d\} \). Moreover, \( \tilde{z}_j(t_j) = z(t_j) \) for \( j \in \{0, 1, \ldots, n\} \) and

\[
\frac{z_1(t_j) - z_1(t_{j-1})}{h} = z_1'(\beta_{l_1})
\]
for some $\beta_{lj} \in (t_{j-1}, t_j) \subset \Delta_j, j \in \{1, \ldots, n\}, l \in \{1, \ldots, d\}$. As a result, by (43) in Lemma 2(ii), we obtain

$$
\|z(t) - \bar{z}_j(t)\| = \sum_{l=1}^{d} \left| z'_l(\alpha_{lj}) (t - t_{j-1}) - z'_l(\beta_{lj}) (t - t_{j-1}) \right|
$$

$$
\leq h \sum_{l=1}^{d} \left| z'_l(\alpha_{lj}) - z'_l(\beta_{lj}) \right| \leq C h \sum_{l=1}^{d} |\alpha_{lj} - \beta_{lj}|^\theta
$$

$$
\leq C d \cdot h^{\theta+1}
$$

for all $t \in \Delta_j, j \in \{1, \ldots, n\}$, which leads to

$$
\max_{1 \leq j \leq n} \sup_{t \in \Delta_j} \|z(t) - \bar{z}_j(t)\| \leq C h^{\theta+1}.
$$

Now we will analyse the second term in the right-hand side of (14). Consider $t \mapsto \alpha t + \beta \in \mathbb{R}^d$, where $\alpha, \beta \in \mathbb{R}^d$. Let $c_1, c_2 \in \mathbb{R}$ and $c_1 < c_2$. Then for every $t \in [c_1, c_2]$,

$$
\|\alpha t + \beta\| \leq \frac{c_2 - t}{c_2 - c_1} \|\alpha c_1 + \beta\| + \left(1 - \frac{c_2 - t}{c_2 - c_1}\right) \|\alpha c_2 + \beta\| \leq \max\{\|\alpha c_1 + \beta\|, \|\alpha c_2 + \beta\|\}.
$$

Note that for every $\omega \in \Omega$ for which $\tilde{V}^0, \tilde{V}^1, \ldots, \tilde{V}^n$ are well defined, $\tilde{z}_j(\cdot) - \tilde{l}_{j}^{EE}(\omega)(\cdot)$ is linear in $\Delta_j$, cf. (9) and (13). Hence, the above observation implies that

$$
\sup_{t \in \Delta_j} \|\tilde{z}_j(t) - \tilde{l}_{j}^{EE}(t)\| = \max \left\{\|z(t_{j-1}) - \tilde{V}^{j-1}\|, \|z(t_j) - \tilde{V}^j\|\right\}
$$

with probability 1. As a result

$$
\|\max_{1 \leq j \leq n} \sup_{t \in \Delta_j} \|\tilde{z}_j(t) - \tilde{l}_{j}^{EE}(t)\|\|_p \leq \|\max_{0 \leq j \leq n} \|z(t_j) - V^j\|\|_p + \|\max_{0 \leq j \leq n} \|V^j - \tilde{V}^j\|\|_p.
$$

(16)

For $k \in \{1, \ldots, n\}$:

$$
z(t_k) - V^k = \sum_{j=1}^{k} (z(t_j) - z(t_{j-1})) - \sum_{j=1}^{k} (V^j - V^{j-1})
$$

$$
= \sum_{j=1}^{k} \int_{t_{j-1}}^{t_j} z'(s) \, ds - h \sum_{j=1}^{k} f(\theta_j, V^{j-1})
$$

$$
= S_{1}^k + S_{2}^k + S_{3}^k.
$$

(17)
where

\[ S_k^1 = \sum_{j=1}^{k} \left( \int_{t_{j-1}}^{t_j} z'(s) \, ds - h z'(\theta_j) \right), \]

\[ S_k^2 = h \sum_{j=1}^{k} \left( f(\theta_j, z(\theta_j)) - f(\theta_j, z(t_{j-1})) \right), \]

\[ S_k^3 = h \sum_{j=1}^{k} \left( f(\theta_j, z(t_{j-1})) - f(\theta_j, V_{j-1}) \right). \]

Now we will show that

\[ \max_{0 \leq j \leq n} \| z(t_j) - V_j \|_p \leq e^{L(b-a)} \left( \max_{1 \leq k \leq n} \| S_k^1 \|_p + \max_{1 \leq k \leq n} \| S_k^2 \|_p \right). \tag{18} \]

Let us define \( u_0 = 0 \) and

\[ u_k = \max_{0 \leq j \leq k} \| z(t_j) - V_j \|_p = \max_{1 \leq j \leq k} \| z(t_j) - V_j \|_p, \]

for \( k \in \{1, \ldots, n\} \). Then by (A4) we get

\[ \left\| \max_{1 \leq j \leq k} \| S_j^1 \|_p \right\|_p \leq hL \sum_{j=1}^{k} \| z(t_{j-1}) - V^{j-1} \|_p \leq hL \sum_{j=0}^{k-1} u_j. \]

From the above and (17) we obtain

\[ u_k \leq \max_{1 \leq j \leq k} \| S_j^1 \|_p + \max_{1 \leq j \leq k} \| S_j^2 \|_p + hL \sum_{j=0}^{k-1} u_j. \]

Inequality (18) follows from discrete Gronwall’s inequality.

Since \( h \sum_{j=1}^{k} z'(<\theta_j) \) is the randomized Riemann sum of \( \int_{t_{j-1}}^{t_j} z'(s) \, ds \) and \( z' \) is \( \varphi \)-Hölder continuous (as stated in Lemma 2(ii)), we can use Theorem 3.1 from [9] to show that

\[ \left\| \max_{1 \leq k \leq n} \| S_k^1 \|_p \right\|_p \leq Ch^{q+\frac{1}{2}}. \tag{19} \]

In fact, by (3.3) in Theorem 3.1:

\[ \left\| \max_{1 \leq k \leq n} \| S_k^1 \|_p \right\|_p \leq \sqrt{d} \cdot \left\| \max_{1 \leq k \leq n} \| S_k^1 \|_E \right\|_p \leq Ch^{q+\frac{1}{2}} \cdot \| z' \|_{C^0([a,b])}, \]

where \( \| \cdot \|_E \) is the Euclidean norm in \( \mathbb{R}^d \) and \( \| g \|_{C^0([a,b])} \) is the Hölder norm for each \( \varphi \)-Hölder continuous function \( g \).

Furthermore,

\[ \left\| \max_{1 \leq k \leq n} \| S_k^2 \|_p \right\|_p \leq Ch \tag{20} \]

because by (A4) and (42) in Lemma 2(ii) we have

\[ \max_{1 \leq k \leq n} \| S_k^2 \|_p \leq h \sum_{j=1}^{n} \| f(\theta_j, z(\theta_j)) - f(\theta_j, z(t_{j-1})) \|_p \leq hL \sum_{j=1}^{n} C |\theta_j - t_{j-1}| \leq hLC(b-a). \]
From (18), (19) and (20) it follows that
\[
\max_{0 \leq j \leq n} \| z(t_j) - V_j \|_p \leq C \left( h^{\frac{\alpha}{2} + \frac{1}{2}} + h \right) \leq C h^{\min\left\{\frac{\alpha}{2} + \frac{1}{2}, 1\right\}} \left( 1 + (b - a)^{1 - \frac{\alpha}{2}} \right).
\] (21)

From (16), (21) and Fact 2 we get
\[
\max_{1 \leq j \leq n} \sup_{t \in [t_{j-1}, t_j]} \| \bar{z}_j(t) - \bar{l}_{IE}^j(t) \|_p \leq C h^{\min\left\{\frac{\alpha}{2} + \frac{1}{2}, 1\right\}} + \delta. \quad (22)
\]

By (14), (15) and (22), we obtain the desired claim. \( \square \)

### 3 Error analysis of the randomized implicit Euler scheme under inexact information

In this section we consider the class \( F^C_I \) of IVPs having the form (1) for which the right-hand side function satisfies the global Lipschitz condition.

The randomized implicit Euler method under inexact information is defined as follows. Let \( n \in \mathbb{Z}_+, h = \frac{b - a}{n}, t_j = a + jh \) for \( j \in \{0, 1, \ldots, n\} \) and \( \theta_j = t_{j-1} + \tau_j h \) for \( j \in \{1, \ldots, n\} \), where \( \tau_j \sim U(0, 1) \) for \( j \in \{1, \ldots, n\} \) are independent random variables on \( (\Omega, \Sigma, \mathbb{P}) \). Let \( (\eta, f) \in F^g_I \) and \( (\tilde{\eta}, \tilde{f}) \in V^2_{(\eta, f)}(\delta) \). Iterations of the algorithm are given as follows:

\[
\bar{U}^0 = \tilde{\eta}, \quad \bar{U}^j = \bar{U}^{j-1} + h \cdot \tilde{f}(\theta_j, \bar{U}^j), \quad j \in \{1, \ldots, n\}.
\] (23)

Note that
\[
\tilde{f}(\theta_j, \bar{U}^j) = f(\theta_j, \bar{U}^j) + \tilde{\delta}_f(\theta_j, \bar{U}^j)
\]
for some \( \tilde{\delta}_f \in K_2(\delta) \). The solution to (1) is approximated by \( \bar{l}_{IE}^j : [a, b] \to \mathbb{R}^d \) given by

\[
\bar{l}_{IE}^j(t) = \bar{l}_{IE}^j(t) \text{ for } t \in [t_{j-1}, t_j], \quad \bar{l}_{IE}^j(t) = \frac{\bar{U}^j - \bar{U}^{j-1}}{h}(t - t_{j-1}) + \bar{U}^{j-1}, \quad j \in \{1, \ldots, n\}.
\]

By \( U^j, l^{IE}^j \) and \( l_{IE}^j \) we denote counterparts of \( \bar{U}^j, \bar{l}_{IE}^j \) and \( \bar{l}_{IE}^j \) under exact information.

The first step in our analysis is to show that for sufficiently small \( h \) the algorithm has the solution, i.e. at each iteration there exists \( \bar{U}^j \) satisfying (23). To prove this fact we introduce the following filtration: \( \mathcal{F}_0 = \sigma(\bar{N}) \) and \( \mathcal{F}_j = \sigma(\sigma(\tau_1, \ldots, \tau_j) \cup \bar{N}) \) for \( j \in \{1, \ldots, n\} \).

**Lemma 1** Let \( \delta \in [0, 1], (\eta, f) \in F^g_I, (\tilde{\eta}, \tilde{f}) \in V^2_{(\eta, f)}(\delta), n \in \mathbb{Z}_+, h(K + 1) \leq \frac{1}{2} \) and \( h(L + 1) < 1 \). Then there exists a unique solution \( (\bar{U}^j)_{j=0}^n \) to the randomized implicit Euler scheme (23) under inexact information such that \( \sigma(\bar{U}^j) \subset \mathcal{F}_j \) for \( j \in \{0, 1, \ldots, n\} \) and

\[
\max_{0 \leq j \leq n} \| \bar{U}^j \| \leq (K + 2)e^{2(K+1)(b-a)} - 1
\] (24)

with probability 1.
Proof We will proceed by induction. Of course $\bar{U}^0 = \bar{\eta}$ is deterministic and hence $\mathcal{F}_0$-measurable. Let us assume that there exists $\mathcal{F}_{j-1}$-measurable solution $\bar{U}^{j-1}$ to (23) for some $j \in \{1, \ldots, n\}$.

We define a mapping $p_j : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ by the following formula:

$$p_j(\omega, x) = \bar{U}^{j-1}(\omega) + h\tilde{f}(\theta_j(\omega), x), \quad (\omega, x) \in \Omega \times \mathbb{R}^d$$

(25)

and a mapping $h_j : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ by taking

$$h_j(\omega, x) = p_j(\omega, x) - x, \quad (\omega, x) \in \Omega \times \mathbb{R}^d$$

(26)

For each $\omega \in \Omega$ the function $x \to h_j(\omega, x)$ is continuous, while for every $x \in \mathbb{R}^d$ the function $\omega \mapsto h_j(\omega, x)$ is $\mathcal{F}_j$-measurable since $\sigma(\bar{U}^{j-1}) \cup \sigma(\theta_j) \subset \mathcal{F}_j$. By (A4) and (3) we have for all $\omega \in \Omega, x, y \in \mathbb{R}^d$ that

$$\|p_j(\omega, x) - p_j(\omega, y)\| \leq h\|f(\theta_j(\omega), x) - f(\theta_j(\omega), y)\| + h\|\tilde{\delta}_f(\theta_j(\omega), x) - \tilde{\delta}_f(\theta_j(\omega), y)\|$$

$$\leq h(L+1)\|x - y\|.$$  

Since $h(L+1) < 1$, the function $x \mapsto p_j(\omega, x)$ is a contraction mapping for every $\omega \in \Omega$. From the Banach fixed-point theorem it follows that for every $\omega \in \Omega$ there exists a unique root $\bar{U}^j(\omega) \in \mathbb{R}^d$ of the function $x \mapsto h_j(\omega, x)$. Hence, according to Lemma 3 the mapping $\omega \mapsto \bar{U}^j(\omega)$ is $\mathcal{F}_j$-measurable and such that

$$\bar{U}^j = \bar{U}^{j-1} + h \cdot \tilde{f}(\theta_j, \bar{U}^j),$$

with probability 1.

Let us note that $\|\bar{U}^0\| \leq \|\bar{\eta}\| + \tilde{\delta} \leq K + 1$. Moreover, from (A4), (3) and (3), we obtain

$$\|\bar{U}^j\| \leq \|\bar{U}^{j-1}\| + h\left(\|f(\theta_j, \bar{U}^j)\| + \|\tilde{\delta}_f(\theta_j, \bar{U}^j)\|\right) \leq \|\bar{U}^{j-1}\| + h(K+1)(1 + \|\bar{U}^j\|).$$

for $j \in \{1, \ldots, n\}$. Since

$$0 < \frac{1}{1 - h(K+1)} \leq 1 + 2h(K+1) \leq 2$$

for $0 < h(K+1) \leq \frac{1}{2}$, we obtain

$$\|\bar{U}^j\| \leq \frac{1}{1 - h(K+1)}\|\bar{U}^{j-1}\| + \frac{h(K+1)}{1 - h(K+1)} \leq \left(1 + 2h(K+1)\right)\|\bar{U}^{j-1}\| + 2h(K+1).$$

By discrete Gronwall’s inequality we obtain for $j \in \{0, 1, \ldots, n\}$ the following inequality:

$$\|\bar{U}^j\| \leq \left(1 + 2h(K+1)\right)^n\|\bar{U}^0\| + \left(1 + 2h(K+1)\right)^n - 1 \leq (K + 2)e^{2(K+1)(b-a)} - 1.$$  

This concludes the proof.

Now we will establish a result similar to Fact 2.

**Fact 3** There exists a constant $C = C(a, b, K, L) > 0$ such that

$$\max_{0 \leq j \leq n}\|U^j - \bar{U}^j\| \leq C\delta$$
with probability 1 for all \( \delta \in [0, 1] \), \((\eta, f) \in F_0^\infty\), \((\tilde{\eta}, \tilde{f}) \in V^2(\eta, f)(\delta)\) and \( n \in \mathbb{Z}_+ \) such that \( h(K + 1) \leq \frac{1}{2} \) and \( hL \leq \frac{1}{2} \).

**Proof** Let us note that \( \bar{U}_0 - U_0 = \tilde{\eta} - \eta \leq \delta \). For \( j \in \{1, \ldots, n\} \) we obtain
\[
\|\hat{U}^j - U^j\| \leq \|\hat{U}^{j-1} - U^{j-1}\| + h f(\theta_j, \hat{U}^j) - f(\theta_j, U^j) + h \|\tilde{f}(\theta_j, \hat{U}^j)\|
\]
where \( C \) is a bound for \( \|\hat{U}^j\| \) given by (24). Thus,
\[
\|\hat{U}^j - U^j\| \leq \frac{1}{1 - hL} \|\hat{U}^{j-1} - U^{j-1}\| + \frac{h(1 + C)}{1 - hL} \delta
\]
and by discrete Gronwall’s inequality
\[
\|\hat{U}^j - U^j\| \leq (1 + 2hL)^n \|\hat{U}^0 - U^0\| + \frac{(1 + C)\delta}{L} \cdot ((1 + 2hL)^n - 1)
\]
which completes the proof.

The following theorem is an analogue of Theorem 1 for the implicit version of randomized Euler scheme. We will use similar error decomposition as in (17) but with an extra term representing a shift from \( f(\theta_j, U^{j-1}) \) to \( f(\theta_j, U^j) \).

**Theorem 2** Let \( p \in [2, \infty) \). There exists a constant \( C = C(a, b, d, K, L, \varrho, p) > 0 \) such that
\[
\left\| \sup_{a \leq t \leq b} \left| z(\eta, f)(t) - \bar{l}^E(\tilde{\eta}, \tilde{f}, \delta)(t) \right| \right\|_p \leq C \left( h^{\min\{1, \frac{3}{2}, 1\}} + \delta \right)
\]
for all \( \delta \in [0, 1] \), \((\eta, f) \in F_0^\infty\), \((\tilde{\eta}, \tilde{f}) \in V^2(\eta, f)(\delta)\) and \( n \in \mathbb{Z}_+ \) such that \( h(K + 1) \leq \frac{1}{2} \) and \( hL \leq \frac{1}{2} \).

**Proof** Proceeding analogously as in the proof of Theorem 1, we obtain
\[
\left\| \sup_{a \leq t \leq b} \left| z(t) - \bar{l}^E(t) \right| \right\|_p \leq C h^{\min\{1, \frac{3}{2}, 1\}} + \max_{0 \leq j \leq n} \|z(t_j) - U^j\|_p + \max_{0 \leq j \leq n} \|U^j - \hat{U}^j\|_p
\]
– cf. formulas (14), (15) and (16). By Fact 3,
\[
\left\| \sup_{a \leq t \leq b} \left| z(t) - \bar{l}^E(t) \right| \right\|_p \leq C \left( h^{\min\{1, \frac{3}{2}, 1\}} + \delta \right) + \max_{0 \leq j \leq n} \|z(t_j) - U^j\|_p
\]
(27)
For \( k \in \{1, \ldots, n\} \) it holds that
\[
z(t_k) - U^k = S^k_1 + S^k_2 + S^k_3 + S^k_4.
\]
where

\[ S_k^1 = \sum_{j=1}^{t_j} \left( \int_{t_{j-1}}^{t_j} z'(s) \, ds - h z'(\theta_j) \right), \]

\[ S_k^2 = h \sum_{j=1}^{k} \left[ z'(\theta_j) - f(\theta_j, z(t_{j-1})) \right], \]

\[ S_k^3 = h \sum_{j=1}^{k} \left[ f(\theta_j, z(t_{j-1})) - f(\theta_j, U^{j-1}) \right], \]

\[ S_k^4 = h \sum_{j=1}^{k} \left[ f(\theta_j, U^{j-1}) - f(\theta_j, U^j) \right]. \]

By similar arguments as in the proof of Theorem 1, we show that

\[ \max_{0 \leq j \leq n} \| z(t_j) - U^j \|_p \leq e^{L(b-a)} \cdot \left( \max_{1 \leq j \leq n} \| S_1^j \|_p + \max_{1 \leq j \leq n} \| S_2^j \|_p + \max_{1 \leq j \leq n} \| S_3^j \|_p \right), \]

cf. (18). Furthermore, by analogy to (19) and (20),

\[ \max_{0 \leq j \leq n} \| z(t_j) - U^j \|_p \leq C \cdot \left( h^\min \left\{ 1, \frac{1}{2} \right\} + \max_{1 \leq j \leq n} \| S_4^j \|_p \right). \quad (28) \]

Using assumptions (A2) and (A4), we obtain inequality

\[ \| S_k^4 \| \leq h \sum_{j=1}^{k} \left\| f(\theta_j, U^{j-1}) - f(\theta_j, U^j) \right\| \leq h^2 L \sum_{j=1}^{k} \| U^{j-1} - U^j \| \]

\[ = h^2 L \sum_{j=1}^{k} \left\| f(\theta_j, U^j) \right\| \leq h^2 L K \sum_{j=1}^{k} \left( 1 + \| U^j \| \right) \]

\[ \leq h L K (b-a) \cdot (1 + C) \]

for all \( k \in \{1, \ldots, n\} \) with probability 1, where \( C \) is a bound from (24). This combined with (28) leads to

\[ \max_{0 \leq j \leq n} \| z(t_j) - U^j \|_p \leq Ch^\min \left\{ 1, \frac{1}{2} \right\}. \quad (29) \]

The thesis follows from (27) and (29).

4 Lower bounds and optimality of randomized Euler schemes

In this section we prove the following main result of the paper.
Theorem 3 Let $p \in [2, +\infty)$, $\varrho \in (0, 1]$. There exist $C_1, C_2 > 0$, $n_0 \in \mathbb{Z}_+, \delta_0 \in [0, 1]$ such that for all $n \geq n_0$, $\delta \leq \delta_0$ the following holds

$$C_1(n^{-\varrho + \frac{1}{2}} + \delta) \leq e_n^{(p)}(F_\infty^\varrho, V^2, \delta) \leq e_n^{(p)}(F_{R_0}^\varrho, V^1, \delta) \leq C_2(n^{-\min\{1, \varrho + \frac{1}{2}\}} + \delta),$$

(30)

where $R_0$ is defined in Fact 2.

Proof To show the first inequality let us note that

$$e_n^{(p)}(F_\infty^\varrho, V^2, \delta) \geq e_n^{(p)}(F_\infty^\varrho, V^2, 0) = \Omega(n^{-(\varrho + 1/2)})$$

(31)
as $n \to \infty$, where $\Omega$ is a Landau symbol. This follows from lower bounds for an integration problem of Hölder continuous functions, see [5] and [14] for the details. Furthermore, for any algorithm $\mathcal{A} \in \Phi_n$ and any $(\eta_1, f_1), (\eta_2, f_2) \in F_\infty$ such that $V^2_{(\eta_1, f_1)}(\delta) \cap V^2_{(\eta_2, f_2)}(\delta) \neq \emptyset$ we have

$$e^{(p)}(\mathcal{A}, F_\infty^\varrho, V^2, \delta) \geq \frac{1}{2} \sup_{a \leq t \leq b} \| z(\eta_1, f_1)(t) - z(\eta_2, f_2)(t) \|$$

because for any $(\tilde{\eta}, \tilde{f}) \in V^2_{(\eta_1, f_1)}(\delta) \cap V^2_{(\eta_2, f_2)}(\delta)$ and for any $\mathcal{A} \in \Phi_n$ the following holds:

$$\sup_{a \leq t \leq b} \| z(\eta_1, f_1)(t) - z(\eta_2, f_2)(t) \| \leq \| z(\eta_1, f_1)(t) - \mathcal{A}(\tilde{\eta}, \tilde{f}, \delta)(t) \| + \| z(\eta_2, f_2)(t) - \mathcal{A}(\tilde{\eta}, \tilde{f}, \delta)(t) \| \leq e^{(p)}(\mathcal{A}, \eta_1, f_1, V^2, \delta) + e^{(p)}(\mathcal{A}, \eta_2, f_2, V^2, \delta) \leq 2e^{(p)}(\mathcal{A}, F_\infty^\varrho, V^2, \delta).$$

Let us take $(\eta_1, f_1) = (0e_1, \delta e_1), (\eta_2, f_2) = (0e_1, -\delta e_1)$, where $e_1 = (1, 0, \ldots, 0).$ These pairs belong to $F_\infty^\varrho$ if $\delta \leq \min\{K, 1\}$. Then $(0, 0) \in V^2_{(\eta_1, f_1)}(\delta) \cap V^2_{(\eta_2, f_2)}(\delta)$, $z(\eta_1, f_1)(t) = \delta(t - a)e_1$ and $z(\eta_2, f_2)(t) = -\delta(t - a)e_1.$ Thus,

$$e^{(p)}(\mathcal{A}, F_\infty^\varrho, V^2, \delta) \geq \frac{1}{2} \sup_{a \leq t \leq b} \| 2\delta(t - a)e_1 \| = (b - a)\delta.$$ 

(32)

By (31) and (32) we obtain the first inequality in (30). The second inequality follows from the fact that $F_\infty^\varrho \subset F_{R_0}^\varrho$ and $V^2_{(\eta, f)}(\delta) \subset V^1_{(\eta, f)}(\delta)$. Indeed,

$$e_n^{(p)}(F_\infty^\varrho, V^2, \delta) = \inf_{\mathcal{A} \in \Phi_n} \sup_{(\eta, f) \in F_\infty^\varrho} e^{(p)}(\mathcal{A}, \eta, f, V^2, \delta) \leq \inf_{\mathcal{A} \in \Phi_n} \sup_{(\eta, f) \in F_{R_0}^\varrho} e^{(p)}(\mathcal{A}, \eta, f, V^2, \delta) \leq \inf_{\mathcal{A} \in \Phi_n} \sup_{(\eta, f) \in F_{R_0}^\varrho} e^{(p)}(\mathcal{A}, \eta, f, V^1, \delta) = e_n^{(p)}(F_{R_0}^\varrho, V^1, \delta).$$

Finally, the last inequality in (30) is a consequence of Theorem 1. \qed

The result above implies that when $\varrho \in (0, 1/2]$, both randomized Euler schemes are optimal — implicit version in the class of globally Lipschitz right-hand side functions $F_\infty^\varrho$, whereas explicit version in a broader class $F_{R_0}^\varrho$. In Proposition 1 below we will show that this is not the case for $\varrho \in (1/2, 1]$.
Proposition 1 Let \( q \in \left( \frac{1}{2}, 1 \right] \). Then
\[
\begin{align*}
e^{(p)}(l^{EE}, F_{R_0}, V^1, \delta) &= \Theta(n^{-1} + \delta), \quad (33) \\
e^{(p)}(l^{IE}, F_{\infty}^g, V^2, \delta) &= \Theta(n^{-1} + \delta), \quad (34)
\end{align*}
\]
as \( n \to \infty \) and \( \delta \to 0^+ \).

Proof Let \( A = \min\{K, L\} \), \( \eta = A \) and \( f(t, y) = Ay \). Then \((\eta, f) \in F_{\infty}^g \) and the exact solution to (1) is given by \( z(t) = Ae^{A(t-a)}, t \in [a, b] \). For each \( n \in \mathbb{Z}_+ \) such that \( Ah \neq 1 \) we obtain the following sequences of approximated values of function \( z \) produced by explicit and implicit randomized Euler schemes, respectively, under exact information:
\[
V^j = A(1 + Ah)^j, \quad U^j = A(1 - Ah)^{-j} \quad \text{for} \quad j \in \{0, 1, \ldots, n\},
\]
where \( h = \frac{b-a}{n} \). With the help of de L’Hôpital’s rule we get that
\[
\lim_{x \to \pm\infty} \left[ x \cdot \left( e^x - (1 + \frac{x}{y})^x \right) \right] = \frac{y^2 e^y}{2}
\]
for all \( y > 0 \). Hence,
\[
\sup_{a \leq t \leq b} \left| z(t) - l^{EE}(t) \right| \geq |z(b) - V^n| = A \left( e^{A(b-a)} - \left( 1 + \frac{A(b-a)}{n} \right)^n \right) = \Omega \left( \frac{1}{n} \right),
\]
and
\[
\sup_{a \leq t \leq b} \left| z(t) - l^{IE}(t) \right| \geq |z(b) - U^n| = A \left( e^{A(b-a)} - \left( 1 - \frac{A(b-a)}{n} \right)^{-n} \right) = \Omega \left( \frac{1}{n} \right),
\]
when \( n \to \infty \). This combined with Theorem 1 and Theorem 2 completes the proof. \( \square \)

Remark 2 In [1] we have shown that the randomized two-stage Runge-Kutta scheme is optimal in the class \( F_{R}^g \) for all \( q \in (0, 1] \) and suitably chosen \( R \). However, the class of corrupting function functions \( \tilde{\delta} \) considered in [1] was smaller than \( V^1 \) — corrupting functions were assumed to be bounded. We conjecture that the optimality of the randomized R-K method is preserved when the corrupting functions belong to the class \( V^1 \).

5 Numerical experiments

In this section, we analyze the empirical rate of convergence of randomized Euler schemes obtained for two sample IVPs and we check whether numerical results are in line with the findings from previous sections. Furthermore, we compare performance of these algorithms with their deterministic counterparts and with the randomized two-stage Runge-Kutta scheme (RK2) investigated in [1].
We consider the following test problems:

\[ \begin{cases} 
  z'(t) = t^r \cdot z(t) \cdot \sin(z(t)^2), & t \in [0, 1], \\
  z(0) = 1 
\end{cases} \quad (E1) \]

and

\[ \begin{cases} 
  z'(t) = |\sin(100t)|^r \cdot z(t) \cdot \sin(z(t)^2), & t \in [-1, 2], \\
  z(-1) = 1. 
\end{cases} \quad (E2) \]

If \( r \in (0, 1] \), both IVPs satisfy the assumptions (A0)–(A4) locally with \( \varrho = r \) and \( K = 1 \). In (10), we obtain \( R_{E1}^E = 3e^3 + 2 \) for (E1) and \( R_{E2}^E = 3e^9 + 2 \) for (E2). Assumptions (A3) and (A4) hold for (E1) with \( L_{E1} = 1 + 2(R_{E1}^E + 1)^2 \), whereas for (E2) — with \( L_{E2} = 1 + 2(R_{E2}^E + 1)^2 \). Moreover, (E2) is an example of a switching system with the Hölder continuous switching function \( |\sin(100t)|^r \) that varies rapidly between 0 and 1.

In the charts in this section, by error we denote the estimate of \( L^2(\Omega) \) norm of \( \|z(t) - \tilde{z}_{\text{scheme}}(t)\| \) for the relevant randomized scheme \( \in \{EE, IE, RK2\} \), obtained using \( M \) Monte Carlo simulations (\( M = 1000 \) for explicit schemes, \( M = 100 \) for the implicit scheme). For the purpose of error estimation, we take the approximation obtained by the (deterministic) two-stage Runge-Kutta scheme with a dense grid (\( n = 10^6 \)) as the exact solution. By \( \lg \) we denote the logarithm with base 10. In the case of the implicit scheme, we use the Python function scipy.optimize.fsolve to solve a non-linear equation at each step of the algorithm.

In Figs. 1 and 2, we consider the case of exact information (charts in the left column) and the case of inexact information (charts in the right column) with the precision parameter \( \delta \) chosen as \( n^{-\min\{1, e^{+1/2}\}} \) for Euler schemes or \( n^{-(e+1/2)} \) for the randomized RK2 scheme, cf. Theorem 1, Theorem 2 and Theorem 1 in [1]. Using the OLS method, we fit the line to each set of observations and find the corresponding slope (see Table 1). In Fig. 2, i.e. for the test problem (E2), we fit the line only to observations with \( \lg(n) > 2 \). Empirical rates of convergence are generally in line with expectations. Detailed comments are provided in the next two paragraphs.

Firstly we analyze the results for (E1) displayed in Fig. 1 and in the middle column of Table 1. For randomized Euler schemes in the case of exact information, when \( r = 0.25 \), we observe a faster convergence than expected based on the theoretical upper bound. Similar remark can be made about the randomized two-stage Runge-Kutta scheme in the case of exact information for all tested values of \( r \). We do not observe any significant differences in terms of error between deterministic and randomized Euler schemes. All this might be due to the higher regularity of the right-hand side function of (E1) than assumed in (A0)–(A4). We note that if \( \delta = 0 \), the randomized RK2 scheme significantly outperforms randomized Euler schemes for all tested values of \( r \). In the inexact information setting, the randomized RK2 scheme proves to be more accurate only for \( r = 0.75 \) (among tested values of \( r \)) which is due to the difference in upper bounds for \( \varrho > 0.5 \). Moreover, under inexact information, randomized algorithms under investigation seem to achieve the theoretical upper bounds when applied to (E1).

Regarding equation (E2), we note that for \( n \leq 10^2 \) the error of deterministic schemes behaves quite irregularly — it is even not monotonic with respect to \( n \).
Fig. 1  $\lg(error)$ vs $\lg(n)$ for investigated numerical schemes applied to the test problem (E1) with $r \in \{0.25, 0.5, 0.75\}$. A Deterministic explicit Euler scheme with $\delta = 0$ (exact information). B Deterministic explicit Euler scheme with $\delta = n^{-\min\{1, \epsilon+1/2\}}$. C Randomized explicit Euler scheme with $\delta = 0$ (exact information). D Randomized explicit Euler scheme with $\delta = n^{-\min\{1, \epsilon+1/2\}}$. E Deterministic implicit Euler scheme with $\delta = 0$ (exact information). F Deterministic implicit Euler scheme with $\delta = n^{-\min\{1, \epsilon+1/2\}}$. G Randomized implicit Euler scheme with $\delta = 0$ (exact information). H Randomized implicit Euler scheme with $\delta = n^{-\min\{1, \epsilon+1/2\}}$. I Randomized two-stage Runge-Kutta scheme with $\delta = 0$ (exact information). J Randomized two-stage Runge-Kutta scheme with $\delta = n^{-\epsilon+1/2}$.
Fig. 2  $\lg(error)$ vs $\lg(n)$ for investigated numerical schemes applied to the test problem (E2) with $r \in \{0.25, 0.5, 0.75\}$. A Deterministic explicit Euler scheme with $\delta = 0$ (exact information). B Deterministic explicit Euler scheme with $\delta = n^{-\min(1, \epsilon+1/2)}$. C Randomized explicit Euler scheme with $\delta = 0$ (exact information). D Randomized explicit Euler scheme with $\delta = n^{-\min(1, \epsilon+1/2)}$. E Deterministic implicit Euler scheme with $\delta = 0$ (exact information). F Deterministic implicit Euler scheme with $\delta = n^{-\min(1, \epsilon+1/2)}$. G Randomized implicit Euler scheme with $\delta = 0$ (exact information). H Randomized implicit Euler scheme with $\delta = n^{-\min(1, \epsilon+1/2)}$. I Randomized two-stage Runge-Kutta scheme with $\delta = 0$ (exact information). J Randomized two-stage Runge-Kutta scheme with $\delta = n^{-\epsilon+1/2}$.
This issue is resolved in randomized schemes, although the rate of convergence is slower for $n \leq 10^2$ than for bigger values of $n$. For $n > 10^2$, there is a noticeable improvement in the rate of convergence when moving from deterministic to randomized schemes. Differences between the three considered randomized schemes are not material, although RK2 performs slightly better, cf. Fig. 2 and the right column of Table 1. This may indicate that assumptions of Theorem 2 can be relaxed and also in case of implicit scheme local Lipschitz and Hölder conditions may be sufficient.

In Figs. 3 and 4, we study the error of randomized Euler schemes for different values of the noise parameter $\delta$. This time, unlike in the right columns of Figs. 1 and 2, $\delta$ does not adapt to $n$. We generate the noise under two scenarios:

- **constant worst-case noise** — Figs. 3A, 3C, 4A, and 4C — in each step of the algorithm, $\tilde{f}(t, y)$ is simulated as $f(t, y) + \delta(1 + |y|)$ or $f(t, y) - \delta(1 + |y|)$, depending on which possibility leads to a bigger error;
- **random noise** — Figs. 3B, 3D, 4B, and 4D — $\tilde{f}(t, y)$ is simulated as $f(t, y) + d(1 + |y|)$, where $d$ is a random number from the uniform distribution on $[-\delta, \delta]$ (chosen independently for each noisy evaluation of $f$).

For both test problems and both randomized Euler schemes, the error in the noisy setting behaves similarly as in the case of exact information for relatively small values of $n$. For sufficiently big $n$, the noise becomes the dominant component in the scheme’s error. Thus, as $n$ increases, improvement in the error slows down from some point and finally stabilizes. Such a behaviour was expected. Of course, it is more evident in the constant worst-case noise scenario.

We remark that in the right columns of Figs. 1 and 2, the noise was simulated using the constant worst-case approach.
6 Stability of randomized Euler schemes

In this section we investigate the stability issues of the randomized Euler schemes in the case of exact information, i.e. \( \delta = 0 \). Typically the following test problem is used to analyze stability of numerical methods for ODEs:

\[
\begin{align*}
\frac{dz}{dt} &= \lambda z(t), \quad t \geq 0, \\
z(0) &= \eta 
\end{align*}
\]  

(35)

with \( \lambda \in \mathbb{C} \), \( \eta \neq 0 \). The exact solution of (35) is \( z(t) = \eta \exp(\lambda t) \). Since in this problem the right-hand side function \( f(t, z) = \lambda z \) does not depend on the time variable, (35) does not allow to capture randomization in Euler methods considered in this paper. Hence, we propose the following alternative test problem

\[
\begin{align*}
\frac{dz}{dt} &= 2\lambda t z(t), \quad t \geq 0, \\
z(0) &= \eta.
\end{align*}
\]  

(36)
Fig. 4 lg(error) vs lg(n) for randomized Euler schemes applied to the test problem (E2) with \( r = 0.5 \) for different values of the precision parameter \( \delta \). 

A Randomized explicit Euler scheme with the constant worst-case noise.

B Randomized explicit Euler scheme with the random noise.

C Randomized implicit Euler scheme with the constant worst-case noise.

D Randomized implicit Euler scheme with the random noise.

with \( \lambda \in \mathbb{C}, \eta \neq 0 \). The exact solution of (36) is \( z(t) = \eta \exp(\lambda t^2) \) and

\[
\lim_{t \to \infty} z(t) = 0 \text{ iff } \Re(\lambda) < 0. \tag{37}
\]

Similarly as in [1], we consider three stability regions:

\[
\mathcal{R}_{\text{scheme}}^{MS} = \{ h\lambda \in \mathbb{C} : W^k \to 0 \text{ in } L^2(\Omega) \text{ as } k \to \infty \},
\]

\[
\mathcal{R}_{\text{scheme}}^{AS} = \{ h\lambda \in \mathbb{C} : W^k \to 0 \text{ almost surely as } k \to \infty \},
\]

\[
\mathcal{R}_{\text{scheme}}^{SP} = \{ h\lambda \in \mathbb{C} : W^k \to 0 \text{ in probability as } k \to \infty \}, \tag{38}
\]

where (scheme, \( W \)) \( \in \{ (EE, V), (IE, U) \} \) and the sequences \( (V^k)_{k=0}^\infty \) and \( (U^k)_{k=0}^\infty \) are generated respectively by schemes (8) and (23) applied to the test problem (36) with \( \delta = 0 \) (i.e. under exact information). Regions (38) are called the region of mean-square stability, the region of asymptotic stability, and the region of stability in probability, respectively.
Proposition 2 Stability regions of randomized Euler schemes have the following properties:

(i) \( \mathcal{R}_{EE}^{MS} = \mathcal{R}_{EE}^{AS} = \mathcal{R}_{EE}^{SP} = \emptyset \),

(ii) \( \mathbb{C} \setminus (\mathbb{R}_+ \cup \{0\}) \subset \mathcal{R}_{IE}^{MS} \cap \mathcal{R}_{IE}^{AS} \cap \mathcal{R}_{IE}^{SP} \).

Proof Let \( (V^k)_{k=0}^\infty \) and \( (U^k)_{k=0}^\infty \) be the sequences generated respectively by schemes (8) and (23) applied to the test problem (36) with \( \delta = 0 \). In both cases \( \theta_j = h(j - 1 + \tau_j) \) for \( j \in \mathbb{Z}_+ \).

For the explicit scheme, let us observe that

\[
V^k = V^{k-1} \cdot (1 + 2\lambda h \theta_k) = \cdots = \eta \cdot \prod_{j=1}^{k} (1 + 2\lambda h \theta_j)
\]

for \( k \in \mathbb{Z}_+ \). As a result,

\[
|V^k|^2 = |\eta|^2 \cdot \prod_{j=1}^{k} \varphi_{h,\lambda}(\theta_j),
\]

where \( \varphi_{h,\lambda}: \mathbb{R} \ni x \mapsto |1 + 2\lambda hx|^2 \in \mathbb{R} \). Let us consider any \( h > 0 \) and any \( \lambda \in \mathbb{C} \setminus \{0\} \). Note that \( \varphi_{h,\lambda}(x) = 0 \) for \( x = -1/(2\lambda h) \), and \( \varphi_{h,\lambda}(x) > 0 \) otherwise. Thus, \( \varphi_{h,\lambda}(\theta_j) > 0 \) a.s. for every \( j \in \mathbb{Z}_+ \). Since \( \varphi_{h,\lambda}(x) = 1 + 4\Re(\lambda) h x + 4 |\lambda|^2 h^2 x^2 \) is a quadratic function with a positive coefficient at \( x^2 \), it increases for \( x \in (x_0, \infty) \), where \( x_0 = -\Re(\lambda)/(2|\lambda|^2 h) \), and tends to infinity as \( x \to \infty \). Thus, for all \( h > 0, \lambda \in \mathbb{C} \setminus \{0\} \) and for sufficiently big \( j \in \mathbb{Z}_+ \),

\[
\varphi_{h,\lambda}(\theta_j) \geq \varphi_{h,\lambda}(h(j - 1)) > 2
\]

with probability 1. As a result, \( V^k \to \infty \) as \( k \to \infty \) for all convergence types defined in (38). For completeness, let us note that for \( h > 0 \) and \( \lambda = 0 \) we have \( |V^k| = |\eta| > 0 \) for all \( k \in \mathbb{Z}_+ \). This leads to (i).

For the implicit scheme we have

\[
U^k = U^{k-1} + h \cdot 2\lambda \theta_k U^k
\]

and thus

\[
U^k = \frac{1}{1 - 2h\lambda \theta_k} U^{k-1} = \cdots = \eta \cdot \prod_{j=1}^{k} \frac{1}{1 - 2h\lambda \theta_j}
\]

for \( k \in \mathbb{Z}_+ \). Let us take \( \lambda \in \mathbb{C} \setminus (\mathbb{R}_+ \cup \{0\}) \) and \( h > 0 \). Then we have \( 1 - 2\lambda ht \neq 0 \) for all \( t \geq 0 \). Furthermore,

\[
\left| \frac{1}{1 - 2\lambda h \theta_j} \right|^2 \leq \frac{1}{1 - 4\Re(\lambda) h^2 (j - 1) + 4 |\lambda|^2 h^4 (j - 1)^2} < \frac{1}{2}
\]

with probability 1 for sufficiently big \( j \in \mathbb{Z}_+ \). Hence,

\[
|U^k| = |\eta| \cdot \prod_{j=1}^{k} \left| \frac{1}{1 - 2\lambda h \theta_j} \right| \to 0 \text{ as } k \to \infty
\]
for $\lambda \in \mathbb{C} \setminus (\mathbb{R}_+ \cup \{0\})$, $h > 0$, and for all convergence types considered in (38). This completes the proof. $\square$

**Remark 3** In order to obtain classical (deterministic) versions of Euler schemes, it suffices to set $\theta_j = t_{j-1}$ for all $j \in \mathbb{Z}_+ \cup \{0\}$ (in case of explicit method) or $\theta_j = t_j$ for all $j \in \mathbb{Z}_+ \cup \{0\}$ (in case of implicit method). Hence, inequalities (39) and (40) imply that $R_{EE} = \emptyset$ and $\mathbb{C} \setminus (\mathbb{R}_+ \cup \{0\}) \subset R_{IE}$, where $R_{EE}$ and $R_{IE}$ are the absolute stability regions for the deterministic explicit and implicit Euler schemes. This leads to the conclusion that randomization has no impact on the stability of Euler methods for both test problems (35) and (36), although for the latter the right-hand side function depends on the time variable (which is randomized). Both problems show however significant advantage of the implicit Euler scheme over the explicit one in terms of stability. This is particularly apparent for (36), where stability regions of explicit and implicit methods represent two extreme cases.

### 7 Conclusions

We have established error bounds for randomized Euler schemes under mild assumptions about the right-hand side function. Our analysis has been performed in the setting of inexact information. It turns out that the randomized Euler schemes are optimal only for values of Hölder exponent $\varrho$ not greater than $\frac{1}{2}$. Results for the explicit randomized Euler scheme are proven for broader classes of the right-hand side functions and noise functions than analogous results for the implicit scheme. On the other hand, the implicit scheme turned out to have a significant advantage in terms of stability.

### Appendix

The following lemma can be proven in the same fashion as Lemma 1(i) in [1].

**Lemma 2** Let $(\eta, f) \in F^\varnothing_{R_2}$, where

$$R_2 = K (1 + b - a)e^{K(b-a)} + K. \tag{41}$$

Then

(i) $(1)$ has a unique solution $z = z(\eta, f)$ such that $z \in C^1([a, b] \times \mathbb{R}^d)$ and $z(t) \in B(\eta, R_2)$ for all $t \in [a, b]$;

(ii) there exist $C_1 = C_1(a, b, K) \in (0, \infty)$ and $C_2 = C_2(a, b, K, L) \in (0, \infty)$ such that for all $t \in [a, b]$

$$\|z(t) - z(s)\| \leq C_1|t - s|, \tag{42}$$

$$\|z'(t) - z'(s)\| \leq C_2|t - s|^\varrho. \tag{43}$$

Next lemma is used to show existence and uniqueness of a measurable solution to the implicit randomized Euler scheme. Its proof can be found in [3] (Lemma 4.3).
Lemma 3 Let $\tilde{F}$ be a complete sub $\sigma$-algebra of the $\sigma$-algebra $\Sigma$, $M \in \tilde{F}$ with $\mathbb{P}(M) = 1$ and $h: \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ such that the following conditions are fulfilled.

(i) The mapping $x \mapsto h(\omega, x)$ is continuous for every $\omega \in M$.
(ii) The mapping $\omega \mapsto h(\omega, x)$ is $\tilde{F}$-measurable for every $x \in \mathbb{R}^d$.
(iii) For every $\omega \in M$ there exists a unique root of the function $h(\omega, \cdot)$.

Define the mapping

$$Q: \Omega \ni \omega \mapsto Q(\omega) \in \mathbb{R}^d,$$

where $Q(\omega)$ is the unique root of $h(\omega, \cdot)$ for $\omega \in M$ and $Q(\omega)$ is arbitrary for $\omega \in \Omega \setminus M$.

Then $Q$ is $\tilde{F}$-measurable.

Acknowledgements We would like to thank anonymous reviewers for many valuable comments and suggestions that allowed us to improve the quality of the paper.

Funding This research was partly supported by the National Science Centre, Poland, under project 2017/25/B/ST1/00945.

Data availability The datasets generated and analysed during this study are available from the corresponding author on a reasonable request.

Declarations

Competing interests The authors declare no competing interests.

References

1. Bochacik, T., Goćwin, M., Morkisz, P.M., Przybyłowicz, P.: Randomized Runge-Kutta method – Stability and convergence under inexact information. J. Complex. 65, 101554 (2021)
2. Daun, T.: On the randomized solution of initial value problems. J. Complex. 27, 300–311 (2011)
3. Eisenmann, M., Kovács, M., Kruse, R., Larsson, S.: On a randomized backward Euler method for nonlinear evolution equations with time-irregular coefficients. Found Comp. Math. 19, 1387–1430 (2019)
4. Heinrich, S.: Complexity of initial value problems in Banach spaces. Zh. Mat. Fiz. Anal. Geom. 9, 73–101 (2013)
5. Heinrich, S., Milla, B.: The randomized complexity of initial value problems. J. Complex. 24, 77–88 (2008)
6. Jentzen, A., Neuenkirch, A.: A random Euler scheme for carathéodory differential equations. J. Comp. Appl. Math. 224, 346–359 (2009)
7. Kacewicz, B.: Almost optimal solution of initial-value problems by randomized and quantum algorithms. J. Complex. 22, 676–690 (2006)
8. Kalużna, A., Morkisz, P.M., Przybyłowicz, P.: Optimal approximation of stochastic integrals in analytic noise model. Appl. Math Comput. 356, 74–91 (2019)
9. Kruse, R., Wu, Y.: Error analysis of randomized Runge–Kutta methods for differential equations with time-irregular coefficients. Comput. Methods Appl. Math. 17, 479–498 (2017)
10. Morkisz, P.M., Plaskota, L.: Approximation of piecewise hölder functions from inexact information. J. Complex. 32, 122–136 (2016)
11. Morkisz, P.M., Plaskota, L.: Complexity of approximating Hölder classes from information with varying Gaussian noise. J. Complex. 60, 101497 (2020)
12. Morkisz, P.M., Przybyłowicz, P.: Optimal pointwise approximation of SDE’s from inexact information. J. Comp. Appl. Math. 324, 85–100 (2017)
13. Morkisz, P.M., Przybyłowicz, P.: Randomized derivative-free Milstein algorithm for efficient approximation of solutions of SDEs under noisy information. J. Comp. Appl. Math. 383, 113112 (2021)
14. Novak, E.: Deterministic and Stochastic Error Bounds in Numerical Analysis, Lecture Notes in Mathematics, vol New York, Springer–Verlag (1349)
15. Parthasarathy, K.R.: Probability Measures on Metric Spaces. AMS Chelsea Publishing (2005)
16. Stengle, G.: Numerical methods for systems with measurable coefficients. Appl. Math. Lett. 3, 25–29 (1990)
17. Stengle, G.: Error analysis of a randomized numerical method. Numer. Math. 70, 119–128 (1995)
18. Traub, J.F., Wasilkowski, G.W., Woźniakowski, H.: Information-Based Complexity. Academic Press, New York (1988)
19. Werschulz, A.G.: The complexity of definite elliptic problems with noisy data. J. Complex. 12, 440–473 (1996)
20. Werschulz, A.G.: The complexity of indefinite elliptic problems with noisy data. J. Complex. 13, 457–479 (1997)

**Publisher's note** Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.