An approach for evaluation of observables in analytic versions of QCD

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We present two variants of an approach for evaluation of observables in analytic QCD models. The approach is motivated by the skeleton expansion in a certain class of schemes. We then evaluate the Adler function at low energies in one variant of this approach, in various analytic QCD models for the coupling parameter, and compare with perturbative QCD predictions and the experimental results. We introduce two analytic QCD models for the coupling parameter which reproduce the measured value of the semihadronic $\tau$ decay ratio. Further, we evaluate the Bjorken polarized sum rule at low energies in both variants of the evaluation approach, using for the coupling parameter the analytic QCD model of Shirkov and Solovtsov, and compare with values obtained by the evaluation approach of Milton et al. and Shirkov.

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Consider an observable $O(Q^2)$ depending on a single space-like scale $Q^2(\equiv -q^2) > 0$ and assume that the skeleton expansion for this observable exists:

$$O_{\text{ske}}(Q^2) = \int_0^\infty \frac{dt}{t} F_\alpha^A(t) a_{pt}(te^C Q^2) + \sum_{n=2}^\infty s_n^O \left[ \prod_{j=1}^{n} \int_0^\infty \frac{dt_j}{t_j} a_{pt}(t_j e^C Q^2) \right] F_\alpha^A(t_1, \ldots, t_n).$$  (1)

The observable is normalized such that $O(Q^2) = a_{pt}$ at first order in perturbation theory. The characteristic functions $F_\alpha^A$ are symmetric functions and have the following normalization:

$$\int_0^\infty \frac{dt}{t} F_\alpha^A(t) = 1, \int \frac{dt_1}{t_1} \frac{dt_2}{t_2} F_\alpha^A(t_1, t_2) = 1, \ldots, $$  (2)

and $s_n^O$ are the skeleton coefficients. The perturbative running coupling $a_{pt}(Q^2) \equiv \alpha(Q^2)/\pi$ obeys the renormalization group (RG) equation:

$$\frac{\partial a_{pt}(Q^2)}{\partial \log Q^2} = -[\beta_0 a_{pt}(Q^2) + \beta_1 a_{pt}^3(Q^2) + \ldots].$$  (3)

In QCD, the first two coefficients $\beta_0 = (1/4)(11 - 2n_f/3)$ and $\beta_1 = (1/16)(102 - 38n_f/3)$ are scheme-independent in mass-independent schemes; $n_f$ is the number of active quarks flavors. The value of $C$ depends on the value of the scale $\Lambda$ in $a_{pt}(\Lambda^2_i C) = \Lambda^2_0 C^C)$. In $\overline{\text{MS}}$ scheme $C = \overline{C} \equiv -5/3$. The skeleton integrands and integrals are independent of $C$. Expansion (1) exists in QED if one excludes light-by-light subdiagrams [2, 2]. In the QCD case, the leading skeleton part was investigated in Refs. [1, 4]. We will assume that expansion (1) exists in a certain class of schemes.

On the other hand, the RG-improved perturbation expansion for the observable $O(Q^2)$ is given by

$$O_{\text{pt}}(Q^2) = a_{pt}(Q^2) + \sum_{n=2}^\infty c_{n-1} a_{pt}^n(Q^2).$$  (4)

Expanding $a_{pt}(te^C Q^2)$ around $t = e^{-C}$ inside the integrals in Eq. (1) must give Eq. (4).

The skeleton expansion is a reorganization of the perturbation series such that each term in (1) corresponds to the sum of an infinite number of Feynman diagrams. These sums, however, do not converge. Although it is possible to assign a value to these sums, this value is not unique, a renormalon ambiguity is present. In formulation (1) the ambiguities arise from the (nonphysical) Landau singularities of $a_{pt}(te^C Q^2)$ in the non-perturbative space-like region.
\(0 < Q^2 \leq \Lambda^2\). The difference between possible integration paths (prescriptions) is a measure of the size of the renormalon ambiguity.

The perturbative coupling is a solution of the \(n\)-loop RG equation \(3\). It can be found iteratively for \(Q^2 \gg \Lambda^2\)

\[
a_{\text{pt}}(Q^2) = \sum_{i=1}^{n} \sum_{j=0}^{i-1} k_{ij} \frac{(\log L)^j}{L^i},
\]

where \(L = \log(Q^2/\Lambda^2)\) and \(k_{ij}\) are constants depending on the \(\beta\)-function coefficients. At energies \(Q < 1 \text{ GeV}\) the perturbative result \(4\), \(6\) is not reliable. At these energies \(a_{\text{pt}}\) starts being dominated by the Landau singularities at \(0 < Q^2 \leq \Lambda^2\). These singularities are a consequence of the perturbative RG Eq. \(3\) and are located in the region where this equation is not valid. Furthermore, from general arguments (causality) \(2\) one concludes that the coupling parameter must be analytic in the whole \(Q^2\)-plane excluding the time-like (Minkowskian) semiaxis. On this semiaxis, singularities associated with asymptotic states appear.

With this motivation it seems reasonable to replace the perturbative coupling by a new coupling \(A_1(Q^2)\) differing from \(a_{\text{pt}}(Q^2)\) significantly only in the non-perturbative region and having the required analyticity properties. This replacement is not unique and should be considered as a phenomenological model. Using a dispersion relation, Shirkov and Solovtsov \(7\) proposed the following replacement:

\[
a_{\text{pt}}(Q^2) \leftrightarrow A_{1}(Q^2) = \frac{1}{\pi} \int_{-\Lambda^2}^{\infty} \frac{d\sigma}{\sigma + Q^2} \rho_1(\sigma),
\]

where \(\rho_1 = \text{Im}[a_{\text{pt}}(-\sigma - i\epsilon)]\), and \(a_{\text{pt}}\) is, e.g., given by Eq. \(6\).\(^1\) We will refer to this as the minimal analytic (MA) procedure. In MA the discontinuity of the analytic coupling along the Minkowskian semiaxis is by construction the same as the one of \(a_{\text{pt}}\).\(^2\) Below we shall consider generalizations of this analytization procedure.

Once one analytizes \(a_{\text{pt}}(Q^2)\) using Eq. \(6\) or other procedure, the question arises how to treat a known truncated perturbation series (TPS). For perturbation series \(4\) there is no unique way to analyze higher powers of \(a_{\text{pt}}\). One possibility is to apply the MA procedure to each power of \(a_{\text{pt}}\):\(^\text{10}\)

\[
a_{\text{pt}}^k(Q^2) \leftrightarrow A_{1}^k(Q^2) = \frac{1}{\pi} \int_{0}^{\infty} \frac{d\sigma}{\sigma + Q^2} \rho_k(\sigma) \quad (k = 1, 2, \ldots),
\]

where \(\rho_k = \text{Im}[a_{\text{pt}}^k(-\sigma - i\epsilon)]\) and \(a_{\text{pt}}^k\), e.g., is given by Eq. \(9\).\(^3\) Other choices could be, e.g., \(a_{\text{pt}}^1 \mapsto A_1^1, A_1^{k-2} A_2, \text{etc.}\) In this paper we propose a method to analytize a TPS based on the skeleton expansion \(\text{11}\), in any chosen version of QCD with analytic \(A_1(Q^2)\). In Eq. \(10\) the replacement \(a_{\text{pt}} \mapsto A_1\) is made, making the skeleton expansion terms well-defined integrals. Next, we Taylor-expand each \(A_1(t_\epsilon e^C Q^2)\) there around a specific \(\ln Q^2_0 = \ln(t_\epsilon e^C Q^2)\). In these expansions, we denote

\[
\tilde{A}_n(Q^2) = \frac{(-1)^{n-1}}{\beta_0^{n-1} (n-1)!} \frac{\partial^{n-1} A_1(Q^2)}{\partial (\ln Q^2)^{n-1}}, \quad (n = 2, 3, \ldots)
\]

If we know in expansion \(4\) TPS with \(n_{\text{max}} = 3\) (i.e., \(c_1\) and \(c_2\)), it is convenient to introduce the analytic couplings \(A_2\) and \(A_3\) according to

\[
\tilde{A}_2(Q^2) = A_2(Q^2) + \frac{\beta_1}{\beta_0} A_3(Q^2),
\]

\[
\tilde{A}_3(Q^2) = A_3(Q^2).
\]

When replacing here \(A_k \mapsto A_k^*\), we obtain the corresponding truncated RG equations of perturbative QCD (pQCD). Thus, once one chooses a particular analytic coupling \(A_1\), the functions \(A_k\) with \(k \geq 2\) are defined by Eqs. \(\text{8}\), or by a higher-\(n_{\text{max}}\) version thereof. In MA model \(\text{11}\), the results \(\text{11}\) and \(\text{11}\) merge when \(n_{\text{max}}\) increases \(\text{12}\) (cf. also \(\text{7}\)). In our approach, the basic set of functions is: \(A_1\) and its derivatives \(\tilde{A}_2, \tilde{A}_3, \ldots\). The set \((A_1, A_2, A_3, \ldots)\) was introduced for the convenience of comparison with pQCD.

\(^{1}\) An efficient method for evaluation of \(A_1^{(\text{MA})}\) was developed in Ref. \(\text{3}\). A different evaluation of \(a_{\text{pt}}\) and \(\rho_1\) was presented in Ref. \(\text{5}\).

\(^{2}\) Other analytization procedures of \(a_{\text{pt}}\) focus on the analyticity properties of the beta function \(\text{8, 9}\).

\(^{3}\) An extension of Eq. \(\text{7}\) to noninteger powers was developed in Refs. \(\text{11}\).
The afore-mentioned Taylor-expansion around $\ln Q_s^2 = \ln (t_s e^C Q^2)$ for the order $n_{\max} = 3$ gives

$$A_1 (te^C Q^2) \approx A_1 (Q_s^2) - \beta_0 \ln(t/t_s) A_2 (Q_s^2) + \beta_0^2 \ln^2(t/t_s) A_3 (Q_s^2)$$

$$= A_1 (Q_s^2) - \beta_0 \ln(t/t_s) A_2 (Q_s^2) + [\beta_0^2 \ln^2(t/t_s) - \beta_1 \ln(t/t_s)] A_3 (Q_s^2).$$

(10)

(11)

Keeping terms corresponding to the third-order approximation we obtain a truncated analytic version of $O(Q^2)$:

$$O_{tr}^{(an)} (Q^2) = A_1 (Q_s^2) + [\beta_0 f_1^O (t_s) A_2 (Q_s^2) + s_1^O A_1^2 (Q_s^2)]$$

$$+ [\beta_0^2 f_2^O (t_s) + \beta_1 f_1^O (t_s)] A_3 (Q_s^2) + 2 s_1^O \beta_0 f_1^O (t_s) A_1 (Q_s^2) A_2 (Q_s^2) + s_2^O A_1^3 (Q_s^2),$$

(12)

where $Q_s^2 \equiv t_s e^C Q^2$ and the momenta are

$$f_1^O (t_s) = \int_0^\infty dt \frac{d}{t} F_1^O (t) (- \log t/t_s)^i,$$

$$f_2^O (t_s) = \int \frac{dt_1}{t_1} \frac{dt_2}{t_2} F_2^O (t_1, t_2) (- \log t_1/t_s)^i (- \log t_2/t_s)^j.$$ 

(13)

Equation (12) is the result of the proposed analyticization procedure for the series (11) truncated at $\sim a_{pt}^3$. In the perturbative region one has $A_k \approx a_{pt}^k$ and Eqs. (12) and (11) merge. If $A_1 (Q^2)$ is well behaved at the origin then all $A_k (Q^2)$ ($k \geq 2$) vanish at this point. This follows from the (truncated) RG-like Eqs. (9). Comparison between Eqs. (12) and (11) gives

$$c_1 = \beta_0 f_1^O (e^{-C}) + s_1^O,$$

$$c_2 = \beta_0^2 f_2^O (e^{-C}) + \beta_1 f_1^O (e^{-C}) + 2 \beta_0 s_1^O f_1^O (e^{-C}) + s_2^O.$$ 

(14)

We assume that the skeleton expansion coefficients $s_1^O$ and characteristic functions $F_1^O (t_1, \ldots, t_j)$ are $n_f$-independent when $C$ is $n_f$-independent. Consequently, in the class of schemes where the coefficients $c_j$ of expansion (11) are polynomials in $n_f$ ($\leftrightarrow \beta_0$) of order $j$, relations (14) give us coefficients $s_1^O$ and $s_2^O$ and momenta $f_1^O$, $f_2^O$, and $f_1^O (0)$. We shall consider observables for which $c_1$ and $c_2$ are known. This approach can be continued to higher orders. The afore-mentioned class of schemes is parametrized by the RG $\beta_j$ coefficients ($j \geq 2$) which are polynomials in $n_f$ ($\leftrightarrow \beta_0$) of order $j$ such that $\beta_j = b_{j_0} + b_{j_1} \beta_0 + \cdots b_{j_k} \beta_0^k$, where $b_{j_0} = b_{j_0}$ of the MS scheme and $b_{j_k}$ ($k \geq 1$) are the free scheme parameters.

However, the knowledge of the perturbation coefficients $c_j$ by itself is not enough to obtain the higher-order coefficients which are not included in Eq. (12). At fourth-order, the coefficients at $A_2^3$ and $A_1 A_3$ cannot be obtained without certain assumptions for the characteristic function $F_1^O (t_1, t_2)$ [12].

The leading characteristic function $F_1^O (t)$ is known for many observables on the basis of their all-order large-$n_f$ ($\leftrightarrow$ large-$\beta_0$) perturbation expansion [11, 13]. Therefore, we propose to keep the leading skeleton term unexpanded, but to expand the other terms as in Eq. (12).

$$O_{shel}^{(an)} (Q^2) = \int_0^\infty dt \frac{d}{t} F_1^O (t) A_1 (te^C Q^2) + s_1^O A_1^2 (Q_s^2) + [s_2^O A_1^2 (Q_s^2) + 2 s_1^O \beta_0 f_1^O (e^{-C}) + s_2^O],$$

(15)

where we used two different expansion scales for the NL and NNL skeleton terms: $Q_s^2 \equiv t^{(2)} e^C Q^2$ and $Q_s^2 \equiv t^{(3)} e^C Q^2$, respectively. Since $f_1^O (t_1, t_2)$ = $f_1^O (e^{-C}) + \ln t_2 (e^{-C})$ and $f_1^O (e^{-C})$ is known, it is convenient to use a scale of the BLM type [12]: $\tau_2 = t^{(2)} \equiv \exp(-C - f_1^O (e^{-C}))$ such that $f_1^O (t^{(2)}) = 0$. Consequently, the $A_1 A_2$ term in Eq. (15) disappears. Further, the scheme dependence at this level shows up as the dependence of $s_2^O$ and $t^{(2)}$ on $b_{21}$ and $b_{22}$, respectively [12]. This allows us to fix the last two coefficients, for each specific observable, in such a way that $s_2^O = 0$ and, e.g., $t^{(2)} = 1$. For example, if the starting scheme is MS ($b_{21}$, $b_{22}$, and $\bar{C} \equiv -5/3$), the new scheme coefficients $b_{2j}$ are

$$b_{21} = \tau_{21} + \tau_{20} + \frac{107}{16} \tau_{11},$$

$$b_{22} = \tau_{22} + \tau_{21} - \frac{19}{4} \tau_{11} + 2 \bar{C} \tau_{10},$$

(16)

where $\tau_{jk}$ are expansion coefficients of the perturbation coefficient $c_j$, Eq. (14), in powers of $\beta_0$, in MS scheme: $\tau_j = \sum_j \tau_{jk} \beta_0^k$. In the scheme (15), the skeleton-based expansion (12) reduces to

$$O_{v1}^{(an)} (Q^2) = \int_0^\infty dt \frac{d}{t} F_1^O (t) A_1 (te^C Q^2) + s_1^O A_1^2 (e^C Q^2) + O_n,$$

(17)
where $O_n = O_4$ are now formally terms of fourth order ($\sim A_1^4, A_2^2 A_2, \ldots$). We will call formula (17) the first variant (“v1”) of our skeleton-motivated evaluation approach. Adopting the scheme (14), or higher order generalizations of it, higher order contributions are absorbed in the two terms of Eq. (14). This scheme-fixing method is particularly useful at low energies where scheme dependence is important.

The skeleton QCD expansion, if it exists, is probably valid only in a specific (yet unknown) “skeleton” scheme. A possible difference between the latter and the schemes used here will result in a difference in the evaluation of the observable $O(Q^2)$. This difference, when re-expanded in $a_{\text{pt}}$, is at most $a_{\text{pt}}^4$ subleading-$\beta_0$ (i.e., $\sim \beta_0^2 a_{\text{pt}}^4$).

The derivation up until now allows us to present yet another variant (“v2”) of the evaluation approach, by keeping the scheme (16) and simply replacing $A_1(e^C Q^2)$ by $A_2(e^C Q^2)$ in Eq. (17)

$$O_{v2}^{(3n)}(Q^2) = \int_0^\infty \frac{dt}{t} F_0^a(t) A_1(te^C Q^2) + s O A_2(e^C Q^2) + O_n ,$$

This formula can be obtained by repeating the previous derivation, but starting with the skeleton expansion (1) without the analytization replacements $a_{\text{pt}} \mapsto A_1$ there. All the expansions are then obtained as previously, but with $a_{\text{pt}}^n$ instead of $A_n, A_{n-1} A_1$, etc. In this variant, the analytization is performed at the end, by replacing $a_{\text{pt}} \mapsto A_1$ in the leading-skeleton integral, and replacing $a_{\text{pt}}^n \mapsto A_2$ in the term proportional to $s O$, leading to Eq. (18).

We wish to stress that neither variant of the evaluation approach relies on the existence of the skeleton expansion.

Our derivation can be interpreted in the following alternative way: The formal skeleton expansion (1) provides us with the tools to separate the perturbation series of the observable into several perturbation subseries. The first subseries (from the leading term) includes all the leading-$\beta_0$ terms, the second subseries (from the subleading skeleton term) includes all the leading-$\beta_0$ terms of the rest, etc. Each of these perturbation subseries is renormalization scale invariant. A specific renormalization scheme ($\beta_2, \beta_3, \ldots$) is then found such that all the perturbation subseries vanish, except the first two. In the end, the analytization of the two surviving subseries is performed.

If the perturbation coefficient $c_3$ is known, then the entire described procedure can be carried out to one higher order, i.e., the $\beta_3$-coefficients $b_{3j}$ ($j = 1, 2, 3$) can be determined so that in Eqs. (17)–(18), $O_n = O_3 = (\sim A_1^3, A_2^2 A_2, \ldots)$, under certain assumptions for the function $F_0^a(t_1, t_2)$ (12). For example, for the massless Adler function, the $\tau_3$ coefficient has been estimated as a polynomial in $r_f$ to a high degree of accuracy (15), and the scheme can be found such that in Eqs. (17)–(18) $O_n = O_5$. Of course, the higher order analytic couplings $A_n (n \geq 2)$ are defined in this case by the $n_{\text{max}} = 4$ extension of Eqs. (3)

$$\tilde{A}_2(Q^2) = A_2(Q^2) + \frac{\beta_1}{\beta_0} A_3(Q^2) + \frac{\beta_2}{\beta_0} A_4(Q^2) ,$$

$$\tilde{A}_3(Q^2) = A_3(Q^2) + \frac{5 \beta_1}{2 \beta_0} A_4(Q^2) ,$$

$$\tilde{A}_4(Q^2) = A_4(Q^2) ,$$

and expansion of $A_1(te^C Q^2)$ is now performed up to and including $\tilde{A}_4(Q^2)$, in contrast to Eq. (10).

In practical evaluations, the form of the analytic coupling parameter $A_1(Q^2)$ has to be specified. The most straightforward is the minimal analytic (MA) coupling (14). The latter model gives the value $X = A_{1(C=-5/3)} \approx 0.4$ GeV (in $\overline{\text{MS}}$ and with $n_f = 3$) from fitting high energy QCD observables (16). However, in order to reproduce the measured value of the semihadronic tau decay ratio $r_\tau$, it requires introduction of heavy first generation quark masses $m_u \approx m_d \approx 0.25$ GeV (10). Another possibility would be to modify the MA-coupling at low energies, e.g., in the following manner

$$A_1^{(M1)}(Q^2) = c_f M_0^2 \frac{Q^2}{(Q^2 + M_0^2)^2} + k_0 \frac{M_0^2}{(Q^2 + M_0^2)} + \frac{M_0^2}{(Q^2 + M_0^2)} \frac{1}{\pi} \int_{\sigma=M_0^2}^{\infty} \frac{d\sigma}{\sigma} \frac{\rho_1(\sigma) (\rho_2(\sigma) - M_0^2)}{\rho_2(\sigma + Q^2) - \rho_2(\sigma)} .$$

In this “M1” model, $k_0, c_f, c_0 = M_0^2 / \Lambda^2$, and $c_r = M_r^2 / \Lambda^2$ are four dimensionless and $C$-independent parameters which determine the low energy modification of the coupling (a special case, $k_0 = -1$, of M1 was presented in Ref. (17)).

In general, at high energies, this coupling differs from the MA-coupling by $\sim X^2 / Q^2$. However, requiring that the difference be only $\sim X^2 / Q^4$ fixes parameter $k_0$ in terms of the other three. Consequently, $X \approx 0.4$ GeV from fitting to high energy QCD observables, as in the MA case. The remaining three parameters can be fixed by requiring that the experimental value of $r_\tau$ and some other low energy observable, e.g., Bjorken polarized sum rule $d_b(Q^2)$, be reproduced by the aforementioned procedure. The experimental values of these two observables are $r_\tau = 0.196 \pm 0.010$ (18) and $d_b(Q^2) = 0.16 \pm 0.11$ at $Q^2 = 2$ GeV$^2$ (19), where the normalization was chosen such that $r_\tau = a_{\text{pt}} + O(a_{\text{pt}}^2)$ and similarly for $d_b$. The quark mass effects are subtracted (not contained) here.
The use of the MA-coupling, in our approach [15-17] and with massless first three quarks and $\bar{\Lambda} = 0.4$ GeV, gives $d_b(Q^2 = 2) \approx 0.13$ in $v_1$ (0.14 in $v_2$) which is acceptable, and $r_\tau = 0.140$ in $v_1$ (0.139 in $v_2$) which is not acceptable.\textsuperscript{4}

If we require, in model M1, the reproduction of $r_\tau \approx 0.196$ and $d_b(Q^2 = 2) \approx 0.13 - 0.14$, in the evaluation approach $v_1$, with massless quarks, and $\bar{\Lambda} = 0.4$ GeV, we obtain for the choice $c_0 = 2$ the values of $c_r \approx 0.5$ and $c_f \approx 1.7$.

Changing $c_0$ while keeping it $\sim 1$ gives us by the same procedure different values of $c_r$ and $c_f$ to reproduce the afore-mentioned values of $r_\tau$ and $d_b$. In such cases, yet another low energy observable, the (massless) Adler function, remains quite stable under the variation of $c_0$. In M1 we will take $c_0 = 2$, $c_r = 0.5$, $c_f = 1.7$ (and $\bar{\Lambda} = 0.4$ GeV).

With these parameter values: $v_1$ approach [17] gives $r_\tau = 0.197$ (0.202 when $O_n = O_4$) for $d(m^2_{eW})$ in Eq. (17) and $d_b(Q^2 = 2) \approx 0.14$; $v_2$ approach [15] gives $r_\tau = 0.210$ (0.201 when $O_n = O_4$) and $d_b(Q^2 = 2) \approx 0.14$.

Yet another, simpler, modification of the MA-model is

$$A_1^{(M2)}(Q^2) = A_1^{(MA)}(Q^2) + \tilde{c}_r \frac{M_0^2}{Q^2 + M_0^2}. \quad (21)$$

In this “M2” model,\textsuperscript{5} we will take the parameter values $\tilde{c}_r = 0.2$, $\tilde{c}_0 = \frac{M_0^2}{\Lambda^2} \approx 0.56$ and $\bar{\Lambda} = 0.4$ GeV. With these values, $v_1$ approach [17] gives $r_\tau = 0.188$ (0.194 when $O_n = O_4$) and $d_b(Q^2 = 2) \approx 0.18$; $v_2$ approach [15] gives $r_\tau = 0.188$ (0.193 when $O_n = O_4$) and $d_b(Q^2 = 2) \approx 0.19$.

Having fixed the parameters in the afore-mentioned models M1 and M2, we present in Fig. 1 low energy results of the Adler function $d(Q^2)$ associated with the hadronic part of the electromagnetic current, in the models MA, M1 and M2 (for a different evaluation of $d(Q^2)$, cf. [23]). The normalization was taken again such that in the massless quark limit for $n_f = 3$: $d(Q^2) = a_{pt} + O(a_{pt}^2)$. The lower curves in Fig. 1 represent the results of the $v_1$-evaluation [17] with three massless quarks, in the scheme where $O_n = O_5$. The higher curves represent the full quantity, i.e., the effects of the massive $c$ and $b$ quarks are added, with the coefficients as given in Ref. [24] ($d(Q^2) = (1/2)D(Q^2) - 1$, where $D$ is defined in [24]). In the contributions of $c$ and $b$, we simply replaced $a_{pt}(Q^2)$ and $a_{bt}(Q^2)$ by $A_1(Q^2)$ and $A_2(Q^2)$, and used $\Lambda = \bar{\Lambda}$ ($C = -5/3$). $A_2(Q^2)$ was constructed by the fourth-order relations [15]. The indicated $\pm$ uncertainties in these curves are those charm contributions which are $\propto A_2(Q^2)$. In contrast to the massless contributions, we do not have yet a systematic way to analyze the massive quark contributions. The experimental results [24] and the truncated pQCD series were included for comparison. Figure 1 shows that analytic versions of QCD (MA, M1, M2) in conjunction with the skeleton-motivated approach [17] give results that at low energies $Q \sim 1$ GeV behave much better than pQCD.

Applying of variant 2 of our approach, Eq. (16), gives for the Adler function results which are very close to those of variant 1, Eq. (17), because the coefficient $s_{10}^\beta$ is small; $s_{10}^\beta = 1/12$. For the Bjorken polarized sum rule $d_b(Q^2)$, this is not so, because $s_{10}^\beta = -11/12$ is appreciable. Therefore, we will take the Bjorken sum rule as a case to look at numerical differences between evaluations in our approaches $v_1$ and $v_2$, Eqs. (17)-(18). In addition, we will compare with the approach of Milton et al. and Shirkov [10, 16] (MSSSh)

$$O_{\text{MSSSh}}(Q^2) = A_1(Q^2) + c_1 A_2(Q^2) + c_2 A_3(Q^2). \quad (22)$$

In principle, the comparison between the approaches can be carried out in any analytic QCD model for the coupling parameter. However, the MSSSh approach has been applied in the literature in the MA model. Therefore, we carry out the comparison in this model. In Fig. 2 we present the MA-model predictions for $d_b(Q^2)$, with various evaluations, at third order ($O_n = O_4$ in Eqs. (17)-(18)). The results of the MSSSh approach are presented in two schemes: MS (“bMS”); and in the “b2Sk” scheme where $\beta_2$ is determined by Eqs. (16) for $d_b$. The MSSSh approach uses the three-loop MA-expressions $A_2(Q^2)$ and $A_3(Q^2)$ of Eq. (7) (Refs. [11, 16]), and variant 2 of our approach, Eq. (18), uses $A_2(Q^2)$ from Eqs. (10). Figure 2 shows that the evaluation of $d_b(Q^2)$ with variant 1 of our approach Eq. (17) (“Sk v1 b2Sk”) gives, at low energies, results which differ significantly from the MSSSh approach. On the other hand, variant 2 of our approach (“Sk v2 b2Sk”), i.e., Eq. (18), gives results which are, apparently accidentally, very close to those of the MSSSh approach in the MS scheme.

In summary, we presented two variants of skeleton-expansion-motivated evaluation of observables in analytic versions of QCD, Eqs. (17)-(18). The first variant follows more closely the skeleton expansion, in the sense that the analytization

\textsuperscript{4} For the corresponding massless Adler function $d(Q^2)$ we use the scheme where in Eqs. (14) $O_n = O_5$. If taking a scheme with by one order lower precision ($O_n = O_4$, $\beta_3 \rightarrow 0$), the value changes to $r_\tau = 0.146$ in $v_1$ (0.145 in $v_2$). Observable $r_\tau$ is evaluated by first evaluating the Adler function $d(Q^2)$ for complex values $Q^2 = m^2_{eW}$ and then applying the standard contour integration in the $Q^2$-plane. The function $F_2^D(t)$ for $d(Q^2)$ was obtained in Ref. [1]. $F_2^D(t)$ for $d_b(Q^2)$ can be obtained from the known large-$n_f$ expansion of $d_b(Q^2)$ [20], using the technique of Ref. [1], and the full perturbation coefficients $c_1$ and $c_2$ from [21].

\textsuperscript{5} In Ref. [24], power correction terms $1/(Q^2)^n$ were added to $A_1^{(MA)}(Q^2)$, but with a somewhat different motivation.
is performed at the beginning, in the skeleton expansion \((a_p \rightarrow A_1)\). In the second variant, the analytization is performed at the end, in the form \(a_k \rightarrow A_k\). The second variant can be regarded as a generalization of the evaluation approach of Milton et al. and Shirkov (MSSSh), now including the leading-\(\beta_0\) terms to all orders in the coupling parameter. Both variants use the formal structure of the skeleton expansion in order to divide the original perturbation expansion into a sum of subseries (skeleton terms), each of them renormalization scale invariant, and then using a scheme where only the first two subseries survive. Further, we introduced two alternative models (M1, M2) of analytic QCD for the coupling parameter which, for certain values of the model parameters, reproduce the measured values of the semihadronic \(\tau\) decay ratio.

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FIG. 1: Adler function as predicted by pQCD, and by our approach in several analytic QCD models (see the text).
FIG. 2: Bjorken polarized sum rule in MA as predicted by two variants of the approach [10, 16] (MSSSh), and by our approach.