An algebraic approach to the completions of elementary doctrines

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Abstract

We provide a thorough algebraic analysis of three known completions having a central role in the exact completions of Lawvere’s doctrines: the one adding comprehensive diagonals (i.e. forcing equality on terms to coincide with the equality predicate), the one adding full comprehensions and the one adding quotients. We show that all these 2-adjunctions are 2-monadic and that the 2-monads arising from these adjunctions are all property-like. This entails that comprehensive diagonals, full comprehensions and quotients are algebraic properties of an elementary doctrine. Finally, we discuss and present the distributive laws between these 2-monads.

Keywords: Elementary doctrines, property-like monads, quotient completion

1. Introduction

The topic of completing a given structure with quotients to get a richer one has been widely employed in categorical logic to obtain relative consistency results and its categorical aspects have been studied extensively. The calculus of Partial Equivalence Relations has many applications in the semantics of programming languages. In Type Theory, models of abstract quotients, known as setoid models, are very useful to formalize mathematical proofs.

Over the years, several constructions and notions of completing a category to an exact category have been introduced both in category theory and categorical logic.

Freyd introduced the notion of exact completion of a regular category in [1], Carboni presented the exact completion of a lex category [2, 3, 4] and in recent works [5, 6, 7, 8], Maietti and Rosolini began to study a categorical structure involved with quotient completions, relativizing the basic concept to a doctrine equipped with a logical structure sufficient to describe the notion of an equivalence relation and quotient.

To this purpose, they considered a generalization of the notion of Lawvere’s hyper-doctrine [9, 10, 11], namely elementary doctrine, and they extend the notion of exact completion to elementary doctrines.

In [6, 8] they proved that this exact completion of elementary doctrines can be obtained as the composite of four minor constructions: the comprehension completion, the
quotient completion, the extensional collapse of an elementary doctrine, and finally the last step is the construction of the category of entire functional relations. See [8, Thm. 4.7].

As pointed out in [8] the last construction is a reformulation in the language of doctrines of that introduced by Kelly in [12].

In this work, we focus the attention on the first three free constructions involved in the exact completion of a doctrine, since they are those adding new logical structures to a given elementary doctrine.

Our main purpose is to provide a complete algebraic account, employing well-known instruments from the formal theory of monads [13, 14, 16, 14, 15], to the three constructions previously mentioned: the comprehension completion, the quotient completion and the comprehensive diagonal completion.

Notice that this approach to completions of doctrines has been applied recently for the case of the existential completion of an elementary doctrine [17] and the elementary completion of a primary doctrine [18]. Our work carries on this line and it is part of a more long-term goal whose purpose is to develop a complete description of the main logical constructions and structures in terms of 2-monads and algebras for 2-monads, in order to study logic using formal category theory and universal algebra.

For example, one of the main advantages of these methods is that, using the theory of 2-monads, we can formally distinguish properties from structures of doctrines and understand how such properties (or structures) can be combined in terms of distributive laws.

Recall that 2-monads can express uniformly and elegantly many algebraic structures, and, in particular, that an action of a 2-monad on a given object encodes a structure on that object. When the structure is uniquely determined to within unique isomorphism, to give an object with such a structure is just to give an object with a certain property. Those 2-monads for which the algebra structure is essentially unique, if it exists, are called property-like [16].

Therefore, we start by giving a detailed description of the 2-functors and the 2-adjunctions obtained from these completions, and we start our analysis of the 2-monads

$$T_c, T_d, T_q: \mathbf{ED} \longrightarrow \mathbf{ED}$$

where \(\mathbf{ED}\) denotes the 2-category of elementary doctrines, and the 2-monads are, respectively, the 2-monad \(T_c\) of comprehension completion, the 2-monad \(T_d\) of comprehensive diagonal completion, and finally the 2-monad \(T\) of quotient completion. Then, we study the 2-monadicity of the previous 2-adjunctions. In particular, we prove that the following equivalences of 2-categories hold

\[
\begin{align*}
\text{CE} & \equiv T_c\text{-Alg} \\
\text{CED} & \equiv T_d\text{-Alg} \\
\text{QED} & \equiv T_q\text{-Alg}
\end{align*}
\]

where \(\text{CE}\) is the 2-category of elementary doctrines with full comprehensions, \(\text{CED}\) is the 2-category of elementary doctrines with comprehensive diagonals, and \(\text{QED}\) is the 2-category of elementary doctrines with stable quotients.
Moreover, we show that $T_c$ is colax-idempotent, $T_d$ is pseudo-idempotent and that $T_q$ is lax-idempotent. In particular, this implies that all these 2-monads are property-like, and then we can conclude that having comprehensions, quotients or comprehensive diagonals is a property of a doctrine, and not only a structure.

Finally, we conclude by showing that the 2-monad $T_q$ can be lifted to a 2-monad on the 2-category $T_c$-$\text{Alg}$, and hence that there exists a distributive law of 2-monads $\delta: T_c T_q \to T_q T_c$, while 2-monad $T_d$ cannot be lifted either on $T_q$-$\text{Alg}$ or $T_c$-$\text{Alg}$.

In the sections 2 and 3 we recall definitions and results on 2-monads and doctrines as needed for the rest of the paper.

In section 4 we recall the notion of doctrine with full comprehensions and we construct the 2-functor and the 2-monad $T_c$ coming from comprehension completion, showing that it is colax-idempotent and that $\text{CE} \equiv T_c$-$\text{Alg}$.

In section 5 we recall the notion of doctrine with comprehensive diagonals and we construct the 2-functor and the 2-monad $T_q$ coming from comprehensive diagonal completion, and we show that it is pseudo-idempotent and that $\text{CED} \equiv T_q$-$\text{Alg}$.

In section 6 we recall the notion of doctrine with quotients and we construct the 2-functor and the 2-monad $T_d$ coming from quotient completion, showing that it is lax-idempotent and that $\text{QED} \equiv T_q$-$\text{Alg}$.

Finally, in section 7 we discuss and present the distributive laws between the 2-monads.

2. Two-dimensional monads

This section is devoted to recall some notions and results regarding the formal theory of monads, and to fix the notation. We mainly follow the usual conventions as in [13, 14, 15], and we refer the reader to the works of Kelly and Lack [16], Tanaka and Power [14, 15], and for a more general and complete description of these topics one can see the Ph.D. thesis of Tanaka [19], the articles of Marmolejo [20, 21] and the work of Kelly [22].

Recall that a 2-monad $(T, \mu, \eta)$ on a 2-category $A$ is a 2-functor $T: A \to A$ together 2-natural transformations $\mu: T^2 \to T$ and $\eta: 1 \to T$ such that the following diagrams commute

\[
\begin{array}{ccc}
T^3 & \xrightarrow{T \mu} & T^2 \\
\mu T & \downarrow & \mu \\
T^2 & \xrightarrow{\mu} & T \\
\end{array}
\quad \quad \quad
\begin{array}{ccc}
T & \xrightarrow{\eta T} & T^2 \\
\mu & \downarrow & \mu \\
T & \xrightarrow{id} & T \\
\end{array}
\]

A $T$-algebra is a pair $(A, a)$ where $A$ is an object of $A$ and $a: TA \to A$ is a 1-cell such that the diagrams

\[
\begin{array}{ccc}
T^2 A & \xrightarrow{T a} & TA \\
\mu A & \downarrow & \downarrow a \\
TA & \xrightarrow{a} & A
\end{array}
\quad \quad \quad
\begin{array}{ccc}
A & \xrightarrow{\eta A} & TA \\
1_A & \downarrow & \downarrow a \\
A & \xrightarrow{a} & A
\end{array}
\]
commute. A strict T-morphism from a T-algebra \((A, a)\) to a T-algebra \((B, b)\) is a 1-cell \(f: A \to B\) such that the following diagram commutes:

\[
\begin{array}{c}
T^0A \\
\downarrow a \\
A
\end{array}
\begin{array}{c}
\to
\end{array}
\begin{array}{c}
T^0B \\
\downarrow b \\
B
\end{array}
\]

while a lax T-morphism from a T-algebra \((A, a)\) to a T-algebra \((B, b)\) is a pair \((f, \eta)\) where \(f\) is a 1-cell \(f: A \to B\) and \(\eta\) is a 2-cell

\[
\begin{array}{c}
T^0A \\
\downarrow a \\
A
\end{array}
\begin{array}{c}
\to
\end{array}
\begin{array}{c}
\eta \\
\downarrow \\
\eta
\end{array}
\begin{array}{c}
\to
\end{array}
\begin{array}{c}
T^0B \\
\downarrow b \\
B
\end{array}
\]

which satisfies the following coherence conditions:

\[
\begin{array}{c}
T^2A \\
\downarrow \mu_A \\
T^0A
\end{array}
\begin{array}{c}
\to
\end{array}
\begin{array}{c}
T^2f \\
\downarrow \mu_B \\
T^0B
\end{array} =
\begin{array}{c}
T^2A \\
\downarrow \eta_A \\
T^0A
\end{array}
\begin{array}{c}
\to
\end{array}
\begin{array}{c}
T^2\eta \\
\downarrow \eta_B \\
T^0B
\end{array}
\]

and

\[
\begin{array}{c}
A \\
\downarrow \eta_A \\
T^0A
\end{array}
\begin{array}{c}
\to
\end{array}
\begin{array}{c}
Tf \\
\downarrow \eta_B \\
T^0B
\end{array} =
\begin{array}{c}
A \\
\downarrow 1_A \\
A
\end{array}
\begin{array}{c}
\to
\end{array}
\begin{array}{c}
A \\
\downarrow 1_B \\
B
\end{array}
\]

Observe that regions in which no 2-cell is written commute, so they are deemed to contain the identity 2-cell.
A lax morphism \((f, \overline{f})\) in which \(\overline{f}\) is invertible is said \textbf{T-morphism}. Hence, a strict T-morphism is a T-morphism where \(\overline{f}\) is the identity 2-cell.

The category of T-algebras and lax T-morphisms becomes a 2-category introducing the T-transformations as 2-cells: a T-

\[\begin{align*}
\text{T-}\text{transformation} & \quad \text{from} \\
(f, \overline{f}) & \quad \text{to} \quad (g, \overline{g})
\end{align*}\]

is a 2-cell \(\alpha: f \Rightarrow g\) in \(\mathcal{A}\) which satisfies the following coherence condition

\[
\begin{array}{c}
\begin{array}{c}
\xymatrix{
TA \ar[rr]^{\overline{T}f} & & TB \\
A \ar[rr]^a \ar[u]^\alpha & & B \ar[u]_b
}
\end{array}
\end{array} =
\begin{array}{c}
\begin{array}{c}
\xymatrix{
TA \ar[rr]^{\overline{T}g} & & TB \\
A \ar[rr]^a \ar[u]^\alpha & & B \ar[u]_b
}
\end{array}
\end{array}
\]

expressing compatibility of \(\alpha\) with \(\overline{f}\) and \(\overline{g}\).

Using the notion of T-morphism, one can express in precise mathematical terms what it means that an action of a monad \(T\) on an object \(A\) is \textit{unique to within a unique isomorphism}. In \(\mathcal{C}\) a T-algebra structure is essentially unique if, given two actions \(a, a': TA \xrightarrow{\alpha} A\), there is a unique invertible 2-cell \(\alpha: a \Rightarrow a'\) such that \((1_A, \alpha): (A, a) \Rightarrow (A, a')\) is a morphism of T-algebras. This is fixed by the following definition of property-like 2-monad.

A 2-monad \((T, \mu, \eta)\) is said \textbf{property-like} if it satisfies the following conditions:

- for every T-algebras \((A, a)\) and \((B, b)\), and for every invertible 1-cell \(f: A \xrightarrow{} B\) there exists a unique invertible 2-cell \(\overline{f}\) such that \((f, \overline{f}): (A, a) \Rightarrow (B, b)\) is a morphism of T-algebras;

\[
\begin{array}{c}
\begin{array}{c}
\xymatrix{
TA \ar[rr]^{\overline{T}f} & & TB \\
A \ar[u]^a \ar[rr]^b & & B \ar[u]_f
}
\end{array}
\end{array}
\]

- for every T-algebras \((A, a)\) and \((B, b)\), and for every 1-cell \(f: A \xrightarrow{} B\) if there exists a 2-cell \(\overline{f}\)

\[
\begin{array}{c}
\begin{array}{c}
\xymatrix{
TA \ar[rr]^{\overline{T}f} & & TB \\
A \ar[u]^a \ar[rr]^b & & B \ar[u]_f
}
\end{array}
\end{array}
\]
such that $(f, \overline{f}) : (A, a) \to (B, b)$ is a lax morphism of T-algebras, then it is the unique 2-cell with such property.

We say that a 2-monad $(T, \mu, \eta)$ is **lax-idempotent** when, for every T-algebras $(A, a)$ and $(B, b)$, and for every 1-cell $f : A \to B$, there exists a unique 2-cell $\overline{f}$

\[
\begin{align*}
TA & \xrightarrow{Tf} TB \\
\downarrow a & \quad \downarrow b \\
A & \xrightarrow{f} B
\end{align*}
\]

such that $(f, \overline{f}) : (A, a) \to (B, b)$ is a lax morphism of T-algebras. In particular every lax-idempotent 2-monad is property-like.

We conclude this section recalling the notion of distributive law between 2-monads, and we refer to [19, 14, 15] for a complete exposition of these notions in the general context of pseudo-monads.

Since the notion of 2-monad represents an elegant way to describe a structure on a category, the notion of distributive laws express how two or more such structures on a category can be combined.

Since in our work all the monads will be simply 2-monads, to simplify the reading, the results are presented in **strict** version.

In particular, given two 2-monads $(S, \mu^S, \eta^S)$ and $(T, \mu^T, \eta^T)$ on a 2-category $A$, a **distributive law** $\delta$ of $S$ over $T$ is a natural transformation $\delta : ST \to TS$ such that

\[
\begin{align*}
S^2T & \xrightarrow{S\delta} STS \xrightarrow{\delta S} TS^2 \\
\mu^ST & \xrightarrow{} ST \xrightarrow{\delta} TS
\end{align*}
\]

and

\[
\begin{align*}
ST^2 & \xrightarrow{S\delta} TST \xrightarrow{T\delta} T^2S \\
S\mu^T & \xrightarrow{} ST \xrightarrow{\delta} TS \xrightarrow{} T^2S
\end{align*}
\]

By a **lifting** of a 2-monad $T$ to the 2-category $S$-Alg of S-algebras we mean a 2-monad $\tilde{T}$ on the 2-category $S$-Alg such that $U_S \tilde{T} = TU_S$ where $U_S$ is the forgetful 2-functor for the 2-monad $S$.

**Theorem 2.1.** To give a distributive law $\delta : ST \to TS$ of 2-monad is equivalent to give a lifting of the 2-monad $T$ to a 2-monad $\tilde{T}$ on $S$-Alg.
Theorem 2.2. Given 2-monads \((S, \mu^S, \eta^S)\) and \((T, \mu^T, \eta^T)\) on a 2-category \(A\) and a distributive law \(\delta: ST \rightarrow TS\), the composite 2-functor \(TS\) acquires the structure for a 2-monad on \(A\), with multiplication given by

\[
TSTS \xrightarrow{TSS} TTSS \xrightarrow{\mu^T \mu^S} TS
\]

and \(TS\text{-Alg}\) is canonically isomorphic to \(\tilde{T}\text{-Alg}\).

3. The notion of elementary doctrine

F.W. Lawvere introduced the notion of hyperdoctrine in a series of seminal papers [9, 10, 11] to synthesize the structural properties of logical systems. Lawvere’s crucial intuition was to consider logical languages and theories as hyperdoctrines to study their 2-categorical properties.

In recent years, the notion of hyperdoctrine has been both specialized and generalized in several contexts. In this work we use the notion of elementary doctrine introduced in [5, 6, 7] in order to generalize the completion of a categorical structure with quotients.

The main idea was to relativize the concept of quotient completion to a many sorted logic, represented categorically by a doctrine validating the logical structure needed to express the notion of equivalence relations.

For the rest of the section \(C\) is assumed to be a category with binary products, and we denote by \(\text{InfSL}\) the category of inf-semilattices, i.e. the objects of \(\text{InfSL}\) are posets with finite meets, and morphisms are functions between them which preserve finite meets.

An elementary doctrine on the category \(C\) is an indexed inf-semilattice \(P: C^{\text{op}} \rightarrow \text{InfSL}\) such that for every \(A\) in \(C\) there exists an object \(\delta_A\) in \(P(A \times A)\) such that:

1. the assignment \(\exists_{(\text{id}_A, \text{id}_A)}(\alpha) := P_{\text{pr}_1}(\alpha) \land \delta_A\)

   for \(\alpha\) in \(PA\) determines a left adjoint to \(P_{(\text{id}_A, \text{id}_A)}: P(A \times A) \rightarrow PA\);

2. for every morphism \(e\) of the form \((\text{pr}_1, \text{pr}_2, \text{pr}_2): X \times A \rightarrow X \times A \times A\) in \(C\), the assignment

   \[
   \exists_e(\alpha) := P_{(\text{pr}_1, \text{pr}_2)}(\alpha) \land P_{(\text{pr}_2, \text{pr}_2)}(\delta_A)
   \]

   for \(\alpha\) in \(P(X \times A)\) determines a left adjoint to \(P_e: P(X \times A \times A) \rightarrow P(X \times A)\).

Examples 3.1. The following examples of elementary doctrine are discussed in [4, 6].

1. Let \(C\) be a category with finite limits. The functor \(\text{Sub}_C: C^{\text{op}} \rightarrow \text{InfSL}\)

   is an elementary doctrine, where \(\text{Sub}_C\) is the functor assigning to an object \(A\) of \(C\)

   the poset \(\text{Sub}_C(A)\) of subobjects of \(A\) and, for an arrow \(B \xrightarrow{f} A\) the morphism

   \(\text{Sub}_C(f): \text{Sub}_C(A) \rightarrow \text{Sub}_C(B)\) is given by pulling a subobject back along \(f\).

   The fibered equalities are the diagonal arrows.
2. Consider a category $\mathcal{D}$ with finite products and weak pullbacks. The elementary doctrine of weak subobjects is given by the functor

$$\Psi_{\mathcal{D}}: \mathcal{D}^{\text{op}} \to \text{InfSL}$$

where $\Psi_{\mathcal{D}}(A)$ is the poset reflection of the slice category $\mathcal{D}/A$, and for an arrow $B \xrightarrow{f} A$, the homomorphism $\Psi_{\mathcal{D}}(f): \Psi_{\mathcal{D}}(A) \to \Psi_{\mathcal{D}}(B)$ is given by a weak pullback of an arrow $X \xrightarrow{g} A$ with $f$.

3. Let $\mathcal{T}$ be a theory in a first order language $\mathcal{L}$. We define a primary doctrine

$$LT: \mathcal{C}_T^{\text{op}} \to \text{InfSL}$$

where $\mathcal{C}_T$ is the category of lists of variables and term substitutions:

- objects of $\mathcal{C}_T$ are finite lists of variables $\vec{x} := (x_1, \ldots, x_n)$, and we include the empty list $()$;
- a morphisms from $(x_1, \ldots, x_n)$ into $(y_1, \ldots, y_m)$ is a substitution $[t_1/y_1, \ldots, t_m/y_m]$ where the terms $t_i$ are built in $\text{Sg}$ on the variable $x_1, \ldots, x_n$;
- the composition of two morphisms $[\vec{t}/\vec{y}]: \vec{x} \to \vec{y}$ and $[\vec{s}/\vec{z}]: \vec{y} \to \vec{z}$ is given by the substitution $[s_1[\vec{t}/\vec{y}]_{/z_k}, \ldots, s_k[\vec{t}/\vec{y}]_{/z_k}]: \vec{x} \to \vec{z}$.

The functor $LT: \mathcal{C}_T^{\text{op}} \to \text{InfSL}$ sends a list $(x_1, \ldots, x_n)$ in the class $LT(x_1, \ldots, x_n)$ of all well-formed formulas in the context $(x_1, \ldots, x_n)$. We say that $\psi \leq \phi$ where $\phi, \psi \in LT(x_1, \ldots, x_n)$ if $\psi \vdash_T \phi$, and then we quotient in the usual way to obtain a partial order on $LT(x_1, \ldots, x_n)$. Given a morphism of $\mathcal{C}_T$

$$[t_1/y_1, \ldots, t_m/y_m]: (x_1, \ldots, x_n) \to (y_1, \ldots, y_m)$$

the functor $LT[\vec{t}/\vec{y}]$ acts as the substitution $LT[\vec{t}/\vec{y}](\psi(y_1, \ldots, y_m)) = \psi[\vec{t}/\vec{y}]$.

The doctrine $LT: \mathcal{C}_T^{\text{op}} \to \text{InfSL}$ is elementary exactly when $\mathcal{T}$ has an equality predicate. For all the detail we refer to [5], and for the case of a many sorted first order theory we refer to [23].

Elementary doctrines form a 2-category denoted by $\mathbf{ED}$ where

- 0-cells are elementary doctrines;
- a 1-cell is a pair $(F, b)$

![Diagram](image-url)
such that $F: \mathcal{C} \longrightarrow \mathcal{D}$ is a functor preserving products, and $b: P \longrightarrow R \circ F^{\text{op}}$ is a natural transformation preserving the structures. More explicitly, for every object $A$ in $\mathcal{C}$, the function $b_A$ preserves finite infima and

$$b_{A \times A}(\delta_A) = R_{(F \text{pr}_1, F \text{pr}_2)}(\delta_{FA}).$$

- a 2-cell is a natural transformation $\theta: F \longrightarrow G$ such that for every object $A$ in $\mathcal{C}$ and every element $\alpha$ in the fibre $PA$, we have

$$b_{A}(\alpha) \leq R_{\theta_A}(c_{A}(\alpha)).$$

### 4. Elementary doctrines with comprehensions

In [5, 6, 7] the authors intend to develop doctrines that may interpret constructive theories for mathematics. They observe that a crucial property an elementary doctrine should verify in order to sustain such interpretation relates to the axiom of comprehension.

**Definition 4.1.** Let $P: \mathcal{C}^{\text{op}} \longrightarrow \text{InfSL}$ be an elementary doctrine and let $\alpha$ be an element of $P(A)$. A comprehension of $\alpha$ is an arrow $\{ | \alpha \}: X \longrightarrow A$ of $\mathcal{C}$ such that $P\{ | \alpha \}(\alpha) = \top_X$ and, for every $f: Z \longrightarrow A$ such that $P_f(\alpha) = \top_Z$, there exists a unique arrow $g: Z \longrightarrow X$ such that $f = \{ | \alpha \} \circ g$.

One says that $P$ has comprehensions if every $\alpha$ has a comprehension, and that $P$ has full comprehensions if, moreover, $\alpha \leq \beta$ in $P(A)$ whenever $\{ \beta \}$ factors through $\{ \alpha \}$.

**Examples 4.2.** Let us consider the sub-objects doctrine $\text{Sub}_\mathcal{C}: \mathcal{C}^{\text{op}} \longrightarrow \text{InfSL}$ defined in Example 3.1. In this case, for every object $A$ and every $\alpha = [B \alpha A]$ in $\text{Sub}_\mathcal{C}(A)$, the comprehension $\{ \alpha \}$ is the arrow $B \alpha A$ in $\mathcal{C}$. Moreover, the doctrine $\text{Sub}_\mathcal{C}$ has full comprehensions.

The intuition is that a comprehension morphism represents the subset of elements of the object $A$ obtained by comprehension with the predicate $\alpha$.

In the internal language of a doctrine $P: \mathcal{C}^{\text{op}} \longrightarrow \text{InfSL}$, a comprehension of a formula $[a : A] | \phi(a)$ is a term $[x : X] | \{ [a : A] | \phi(a) \}(x) : A$ such that

$$\top \vdash \phi(\{ [a : A] | \phi(a) \}(x)) [x : X]$$

and any other term which this property can be obtained from $\{ [a : A] | \phi(a) \}(x)$ by an unique substitution.
Remark 4.3. For every $f : A' \to A$ in $C$ then the mediating arrow between the comprehensions $\{\alpha\} : X \to A$ and $\{P_f(\alpha)\} : X' \to A'$ produces a pullback

Thus comprehensions are stable under pullbacks. Moreover it is straightforward to verify that if $\{\alpha\} : B \to A$ is a comprehension of $\alpha$, then $\{\alpha\}$ is monic.

Observe the stability under pullbacks of comprehensions implies that if $\alpha \leq \beta$, where $\alpha, \beta \in P(A)$, then the unique arrow $a$ such that the following diagram commutes

is a comprehension. In particular it is the comprehension $a = \{P_\beta(\alpha)\}$, because we have that the following is a pullback

and since $\top_{\alpha'} = P_{\{\alpha\}}(\alpha) \leq P_{\{\beta\}}(\beta)$, we have $\{P_{\{\beta\}}(\beta)\} = \text{id}$ and then $a = \{P_{\{\beta\}}(\alpha)\}$.

As observed in [24], to view comprehensions as logical constructors [25] we need to assume that a choice of comprehensions is available in the doctrine.

In details, an elementary doctrine $P : C^{op} \to \text{InfSL}$ has a choice of comprehensions if there is a function $\{ - \}$ assigning a comprehension $\{\alpha\} : A_\alpha \to A$ to every object $\alpha$ of $P(A)$. Similarly, we say that $P$ has a choice of full comprehensions if $P$ has a choice of comprehensions and these are full.

Notation: since in the rest of this work we will always use doctrines with a choice of comprehensions, from now on, when we say that an elementary doctrine $P$ has comprehensions, or full comprehensions, we assume that it has a choice of comprehensions, or full comprehensions.
Remark 4.4. In many senses it is more general to treat the abstract theory of the relevant structures for the present paper in terms of fibrations. In fact, a doctrine $P: \mathcal{C}^{\text{op}} \to \text{InfSL}$ determines a faithful fibration

$$p_P: \mathcal{G}_P \to \mathcal{C}$$

by a well-known, general construction due to Grothendieck, see [25, 5]. We recall very briefly that construction in the present situation.

The data for the total category $\mathcal{G}_P$ are:

- **an object** is a pair $(A, \alpha)$, where $A$ is in $\mathcal{C}$ and $\alpha$ is in $P(A)$

- **an arrow** $f: (A, \alpha) \to (B, \beta)$ is an arrow $f: A \to B$ of $\mathcal{C}$ such that $\alpha \leq P(f)(\beta)$.

The projection on the first component extends to a functor $p_P: \mathcal{G}_P \to \mathcal{C}$ which is faithful, with a right inverse right adjoint. Setting up an appropriate 2-category for each structure (one for primary doctrines, one for faithful fibrations as above), it is easy to see that the two constructions extend to an equivalence between those 2-categories. The notions of comprehensions, full comprehensions and the requirement that comprehensions compose can be translated using the previous construction in the language of fibrations.

In this case they are called respectively fibration with *subset types*, *full subset types* and *strong coproducts*. In particular the terminology strong coproducts comes from dependent type theory, see [25, Chapter 10] and [26].

We denote by $\text{CE}$ the 2-category of elementary doctrines with full comprehensions, and the 1-cells are those $(F, b)$ such that the functor $F$ preserves comprehensions, i.e. $F([a]) = [b, A(a)]$. The 2-cells remain the same.

We recall the construction used in [6] to freely add comprehensions to a given elementary doctrine. Given an elementary doctrine $P: \mathcal{C}^{\text{op}} \to \text{InfSL}$ we define the category $\mathcal{G}_P$ as in Remark 4.4:

- **an object** of $\mathcal{G}_P$ is a pair $(A, \alpha)$, where $A$ is in $\mathcal{C}$ and $\alpha$ is in $P(A)$;

- **a morphism** $f: (A, \alpha) \to (B, \beta)$ is a morphism $f: A \to B$ in $\mathcal{C}$ such that $\alpha \leq P(f)(\beta)$;

The functor $P$ extends to functor $P_\varepsilon: \mathcal{G}_P^{\text{op}} \to \text{InfSL}$ by setting

- $P_\varepsilon(A, \alpha) = \{ \gamma \in P(A) \mid \gamma \leq \alpha \}$;

- $P_\varepsilon(f): P_\varepsilon(B, \beta) \to P_\varepsilon(A, \alpha)$ sends $\gamma \leq \beta$ into $P(f)(\gamma) \land \alpha$.

With these previous assignments, it is direct to check that the functor $P_\varepsilon: \mathcal{G}_P^{\text{op}} \to \text{InfSL}$ is an elementary doctrine and it has full comprehensions. In particular, one can observe that for every object $(A, \alpha)$ of $\mathcal{G}_P$ we can define

$$\delta_{(A, \alpha)} := \delta_A \land \alpha \otimes \alpha$$
where $\alpha \vDash \alpha := P_{\text{pr}_1}(\alpha) \land P_{\text{pr}_2}(\alpha)$, while the comprehension of an element $\alpha \in P_c(A, \beta)$ is given by the arrow $\{\alpha\} := \text{id}_A : (A, \alpha) \rightarrow (A, \beta)$.

Now we prove that the assignment $P \mapsto P_c$ can be extended to 2-functor $C : \text{ED} \rightarrow \text{CE}$ and we start defining how it acts on the 1-cells and 2-cells in $\text{ED}$.

Therefore, let us consider two elementary doctrines $P : \mathcal{C}^{\text{op}} \rightarrow \text{InfSL}$ and $R : \mathcal{D}^{\text{op}} \rightarrow \text{InfSL}$, and consider a 1-cell $(F, b)$ of $\text{ED}$:

\[
\begin{array}{c}
\mathcal{C}^{\text{op}} \\
\downarrow P \\
\mathcal{D}^{\text{op}} \\
\downarrow R
\end{array}
\rightarrow
\begin{array}{c}
\text{InfSL} \\
\downarrow F^{\text{op}} \\
\text{InfSL}
\end{array}
\]

We want to prove that the pair $(\hat{F}, \hat{b})$ where:

- $\hat{F}(A, \alpha)$ is $(FA, b_A(\alpha))$ for every $(A, \alpha) \in \mathcal{G}_P$;
- $\hat{F}(f)$ is $F(f)$ for every $f : (A, \alpha) \rightarrow (B, \beta)$;
- $\hat{b}$ is the restriction of $b$ on $P_c$,

is a 2-cell in $\text{CE}$:

\[
\begin{array}{c}
\mathcal{G}_P^{\text{op}} \\
\downarrow P_c \\
\mathcal{G}_R^{\text{op}} \\
\downarrow R_c
\end{array}
\rightarrow
\begin{array}{c}
\text{InfSL} \\
\downarrow F^{\text{op}} \\
\text{InfSL}
\end{array}
\]

**Lemma 4.5.** $(\hat{F}, \hat{b})$ is a 1-cell in $\text{CE}$.

*Proof.* It is direct to show that $\hat{F} : \mathcal{G}_P \rightarrow \mathcal{G}_R$ is a preserving products functor and that $\hat{b}$ is a natural transformation. First we show that $(\hat{F}, \hat{b})$ is a 1-cell of elementary doctrines, and then we show that it preserves comprehensions. Hence, we start observing that

\[
(R_c)(F(\text{pr}_1), F(\text{pr}_2))(\delta_{(F_A b_A(\alpha))}) = R(F(\text{pr}_1), F(\text{pr}_2))(b_A(\alpha) \boxtimes b_A(\alpha) \land \delta_{FA}) \land b_{A \times A}(\alpha \boxtimes \alpha)
\]

which is equal to

\[
R(F(\text{pr}_1), F(\text{pr}_2))(R_{\text{pr}_1}(b_A(\alpha)) \land R_{\text{pr}_2}(b_A(\alpha))) \land b_{A \times A}(\delta_A) \land b_{A \times A}(\alpha \boxtimes \alpha)
\]

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where \( pr'_i : FA \times FA \to FA \). Moreover, since \( b \) is a natural transformation, the diagram

\[
\begin{array}{ccc}
PA & \xrightarrow{pr_i} & P(A \times A) \\
\downarrow b_A & & \downarrow b_{A \times A} \\
RFA & \xrightarrow{R(pr'_i)} & RF(A \times A).
\end{array}
\]

commutes. This implies that

\[
(R_c)(F(pr_1),F(pr_2))(\delta_{FA,b_A(\alpha)}) = b_{A \times A}(P_{pr_1}(\alpha) \land P_{pr_2}(\alpha)) \land b_{A \times A}(\delta_A) \land b_{A \times A}(\alpha \boxtimes \alpha)
\]

and then

\[
b_{A \times A}(P_{pr_1}(\alpha) \land P_{pr_2}(\alpha)) = b_{A \times A}(\alpha \boxtimes \alpha).
\]

Therefore, we conclude that \((\hat{F},\hat{b})\) is a 1-cell of elementary doctrines since

\[
\hat{b}_{(A,\alpha) \times (A,\alpha)}(\delta_{(A,\alpha)}) = b_{A \times A}(\delta_A \land \alpha \boxtimes \alpha) = (R_c)(F(pr_1),F(pr_2))(\delta_{\hat{F}(A,\alpha)}).
\]

Finally, it is easy to see that \((\hat{F},\hat{b})\) preserves comprehensions since every comprehension in \( G_P \) is of the form

\[
\{\{\gamma\}\} : (A,\gamma) \to (A,\alpha)
\]

where \( \gamma \in P_c(A,\alpha) \), and \( \{\{\gamma\}\} \) is the identity on \( A \). Then the arrow

\[
F(\{\{\gamma\}\}) : (FA,b_A(\gamma)) \to (FA,b_A(\alpha))
\]

is \( \text{id}_{FA} \) by definition of \( \hat{F} \), so it is the comprehension of \( b_A(\gamma) \).

\( \square \)

**Lemma 4.6.** Let \((F,b)\) and \((G,c)\) be two objects in \( \text{ED}(P,R) \) and let \( \theta : (F,b) \to (G,c) \) be a 2-cell in \( \text{ED} \). We define

\[
\hat{\theta} : (\hat{F},\hat{b}) \to (\hat{G},\hat{c})
\]

where

\[
\hat{\theta}_{(A,\alpha)} : (FA,b_A(\alpha)) \to (GA,c_A(\alpha))
\]

is \( \theta_A \). Then it is a 2-cell in \( \text{CE} \).

**Proof.** Let \( (A,\alpha) \) be an object of \( G_P \). First, recall that we have that \( b_A(\alpha) \leq R\theta_A(c_A(\alpha)) \) because \( \theta \) is a 2-morphism. Therefore

\[
\theta_A : (FA,b_A(\alpha)) \to (GA,c_A(\alpha))
\]

is a morphism in \( G_R \). Now, consider an element \( \gamma \) of \( P_c(A,\alpha) \). Then

\[
(R_c)_{\theta_A}(c_A(\gamma)) = R\theta_A(c_A(\gamma)) \land b_A(\alpha)
\]
by definition of the functor $R_c$. Finally, observe that $b_A(\gamma) \leq b_A(\alpha)$ since $\gamma \in P_c(A, \alpha)$, and $b_A(\gamma) \leq R_{\theta_A}(c_A(\gamma))$, and then we can conclude that
$$\widehat{b}_A(\gamma) = b_A(\gamma) \leq R_{\theta_A}(c_A(\gamma)) \wedge b_A(\alpha) = (R_c)_{\theta_A}(c_A(\gamma)).$$

The previous results allow to conclude the following proposition.

**Proposition 4.7.** The assignment

$$C_{P,R}: \mathbf{ED}(P, R) \rightarrow \mathbf{CE}(P_c, R_c)$$

which maps $(F, b)$ into $(\hat{F}, \hat{b})$ and a 2-cell $\theta: (F, b)$ into $\hat{\theta}: (\hat{F}, \hat{b})$ is a functor and

$$C: \mathbf{ED} \rightarrow \mathbf{CE}$$

is a 2-functor with the assignment $C(P) = P_c$.

Now we prove that the 2-functor $C: \mathbf{ED} \rightarrow \mathbf{CE}$ is 2-left adjoint to the forgetful functor $U: \mathbf{CE} \rightarrow \mathbf{ED}$. So, we start by defining the unit and counit morphisms.

First, observe that for every elementary doctrine $P$ there is a natural embedding

$$I_P: C \rightarrow \mathcal{G}_P$$

acts as $A \mapsto (A, \top_A)$, and the morphism

$$i_P: P(A) \rightarrow P_c(A, \top_A)$$

sends $\alpha \mapsto \alpha$. These 1-cells will give the unit of the 2-

adjunction.

In order to define the counit, let us consider an elementary doctrine $P$ with full comprehensions. We can define a morphism

$$j_P: \mathcal{G}_P \rightarrow C$$

in $\mathbf{CE}$ as follow: the functor $j_P: \mathcal{G}_P \rightarrow C$ sends an object $(A, \alpha)$ of $\mathcal{G}_P$ to the object $A_\alpha$, where $A_\alpha$ is the domain of the comprehension $\{\alpha\}: A_\alpha \rightarrow A$. Given an arrow
\[ f: (A, \alpha) \rightarrow (B, \beta) \] in \( \mathcal{G}_P \), the arrow \( J_P(f): A_\alpha \rightarrow B_\beta \) is given by the vertical arrow \( (\beta \ast f) \) of the following diagram:

\[
\begin{array}{ccc}
A_\alpha & \xrightarrow{\{\alpha\}} & A \\
\downarrow a & & \downarrow f \\
A_{P(f)\beta} & \xrightarrow{\{P(f)\beta\}} & A \\
\downarrow \beta \ast f & & \downarrow f \\
B_\beta & \xrightarrow{\{\beta\}} & B \\
\end{array}
\]

(1)

where \( a: A_\alpha \rightarrow D \) exists because \( \alpha \leq P(f)\beta \). Observe that by Remark 4.3 we have that \( a = \{P(f)\beta\}(\alpha) \). The natural transformation \( j_P \) is defined by the following components: for every \((A, \alpha)\) of \( \mathcal{G}_P \) the arrow \( j_P(A, \alpha) \) acts as \( \gamma \mapsto (P(\beta)\gamma) \).

**Lemma 4.8.** With the previous assignments, \((J_P, j_P): P_c \rightarrow P\) is a 1-cell of \( \mathbf{CE} \).

**Proof.** First we prove that \( J_P: \mathcal{G}_P \rightarrow \mathcal{C} \) is a functor. Let us consider two arrows \((A, \alpha) \xrightarrow{f} (B, \beta)\) and \((B, \beta) \xrightarrow{g} (C, \gamma)\) of \( \mathcal{G}_P \). We need to prove that \( J_P(gf) = J_P(g)J_P(f) \). Let us consider the following diagrams

\[
\begin{array}{ccc}
A_\alpha & \xrightarrow{\{\alpha\}} & A \\
\downarrow a & & \downarrow f \\
A_{P(f)\beta} & \xrightarrow{\{P(f)\beta\}} & A \\
\downarrow \beta \ast f & & \downarrow f \\
B_\beta & \xrightarrow{\{\beta\}} & B \\
\end{array}
\]

which are, \( J_P(f) \), \( J_P(g) \) and \( J_P(gf) \) respectively. We have that \( J_P(g)J_P(f) = (\{\gamma\} \ast g)(\beta \ast f) \) and \( J_P(gf) = c((\gamma) \ast (gf)) \). Now notice that the following diagram
commutes because
\[ \{ \gamma \} J_P(g) J_P(f) = \{ \gamma \} (\{ \gamma \}^* g)(\{ \beta \}^* f)a = g \{ P_{\beta}(\gamma) \} b(\{ \beta \}^* f)a = g \{ \beta \} (\{ \beta \}^* f)a \]
and then
\[ \{ \gamma \} J_P(g) J_P(f) = gf \{ P_{\beta}(\gamma) \} a = gf \{ \alpha \}. \]

Then we have that
\[ J_P(g) J_P(f) = J_P(gf). \]
Moreover, it is direct to see that \( J_P(\text{id}_{(A,\alpha)}) = \text{id}_{A,\alpha} \). Hence \( J_P : \mathcal{G}_P \to \mathcal{C} \) is a functor, and it is direct to show that it preserves finite products. Now show the naturality of this assignment \( j_P \).

Let \( f : (A,\alpha) \to (B,\beta) \) be an arrow of \( \mathcal{G}_P \), we have that the diagram
\[
\begin{array}{ccc}
P_c(B,\beta) & \xrightarrow{P_c(f)} & P_c(A,\alpha) \\
P_c(f) & \downarrow{j(A,\alpha)} & \downarrow{j(B,\beta)} \\
P_c(f) & \xrightarrow{j(A,\alpha)} & P(A,\alpha) \\
\end{array}
\]
commutes because if \( \gamma \in P_c(B,\beta) \), we have that
\[ P_{P_c(f)}(j_{(B,\beta)}(\gamma)) = P_{\{\beta\}^* \gamma}(P_{\beta}(\gamma)) \]
and by definition, this is equal to \( P_{j_{A,\alpha}}(\gamma) \), which is exactly \( j_{(A,\alpha)}(P_c(f)(\gamma)) \). Therefore \( j_P : P_c \to PJ_{\text{op}} \) is a natural transformation. Finally, we have to prove that the functor \( J_P \) preserves comprehensions. Observe that every comprehension of an element \( \alpha \in P_c(A,\beta) \) is of the form \( \{ \alpha \} = \text{id}_A : (A,\alpha) \to (A,\beta) \), and then \( J_P(\{ \alpha \}) = \alpha \).
where \( a \) is the arrow such that the diagram

\[
\begin{array}{ccc}
A_\alpha & \xrightarrow{a} & \{\alpha\} \\
\downarrow & & \downarrow \\
A_\beta & \xrightarrow{\beta} & A
\end{array}
\]

commutes. By Remark 4.3 we have that \( J_P(\{\alpha\}) = a = \{P_{\beta}\}(\alpha) \), and this is exactly the comprehension of \( j_{(A,\beta)}(\alpha) = P_{\beta}(\alpha) \). Therefore we have proved that \( (J_P,j_P) \) is an 1-cell of \( \text{CE} \).

**Theorem 4.9.** The 2-functor \( C: \text{ED} \to \text{CE} \) is 2-left adjoint to the forgetful functor \( U: \text{CE} \to \text{ED} \). The unit of this 2-adjunction \( \eta: \text{id}_{\text{ED}} \to UC \) is given by \( \eta_P = (I_P,i_P) \) and the counit \( \varepsilon: \text{CU} \to \text{id}_{\text{CE}} \) is given by \( \varepsilon_P = (J_P,j_P) \).

**Proof.** It is direct to verify that \( \varepsilon \) and \( \eta \) are 2-natural transformations and that \( \text{id}_C = \varepsilon C \circ C\eta \) and that \( \text{id}_U = U\varepsilon \circ \eta U \).

The 2-adjunction of Theorem 4.9 induces a 2-monad \( T_c: \text{ED} \to \text{ED} \), whose unit is given by the unit of the 2-adjunction, and whose multiplication is defined by \( \mu = \varepsilon C \), as in 1-dimensional case. Now we show that 2-monad \( T_c \) is \emph{colax-idempotent}, and that we have the equivalence of 2-categories

\[ T_c\text{-Alg} \equiv \text{CE}. \]

This means that for an elementary doctrine, the structure of \emph{doctrine with full comprehensions} is a more than a structure: it is a \emph{property} in the sense of [16].

We start by showing that every elementary doctrine with full comprehensions is a \( T_c \)-algebra.

**Proposition 4.10.** Let \( P: \text{C}^{\text{op}} \to \text{InfSL} \) be an elementary doctrine of \( \text{CE} \), then \( (P,\varepsilon_P) \) is a \( T_c \)-algebra.

**Proof.** The diagram

\[
\begin{array}{ccc}
T^2P & \xrightarrow{T_c\varepsilon_P} & T_cP \\
\downarrow^{\mu_P} & & \downarrow^{\varepsilon_P} \\
T_cP & \xrightarrow{\varepsilon_P} & P
\end{array}
\]

commutes because \( \mu_P = \varepsilon_{\text{CP}} \) and \( \varepsilon: T_c \to \text{id}_{\text{CE}} \) is a 2-natural transformation. Similarly we have that the unit axiom for strict algebras is satisfied.

**Proposition 4.11.** Let \( (P,\langle F,b \rangle) \) be a \( T_c \)-Alg. Then the doctrine \( P \) has full comprehensions. Moreover \( (F,b) = \varepsilon_P \).
Proof. It is direct to check that given $\alpha \in P(A)$, then $F(\text{id}_A) : F(A, \alpha) \rightarrow F(A, \top_A)$ is a full comprehension of $\alpha$, where $\text{id}_A : (A, \alpha) \rightarrow (A, \top_A)$ is the comprehension of $\alpha \in P_c(A, \top_A)$. Then the doctrine $P$ has full comprehensions and the action $(F, b)$ preserves them. Now we show that $(F, b) = \varepsilon_P$. By the unit axiom of algebras, we have that $(F, b) \eta_P = \text{id}_P$, but since $P$ has comprehensions, we also have $\varepsilon_P \eta_P = \text{id}_P$. Therefore we have that $(F, b) \eta_P = \varepsilon_P \eta_P$ implies that $(F, b) = \varepsilon_P$, because $F(A, \alpha) = F((A, \top_A)\alpha) = (F(A, \top_A)\delta(A, \top)_{(\alpha)}) = (\varepsilon_P \eta_P(A))\alpha = \varepsilon_P(A, \alpha)$. Similarly one can prove that $F(f) = \varepsilon_P(f)$, because every arrow $f : (A, \alpha) \rightarrow (B, \beta)$ is the unique arrow such that the following diagram commutes

\[
\begin{array}{c}
(A, \alpha) \xrightarrow{f} (A, \top_A) \\
\downarrow \quad \downarrow f \\
(B, \beta) \xrightarrow{f} (B, \top_B)
\end{array}
\]

since $(P_c)f_{(\alpha)}(\beta) = \top(A, \alpha)$, since $\alpha \leq P_f(\beta)$. Since both $F$ and $\varepsilon_P$ preserve comprehensions, and since $F(f_{(\alpha)}) = \varepsilon_P(f_{(\alpha)})$ and $F((A, \top_A) \rightarrow (B, \top_B)) = \varepsilon_P((A, \top_A) \rightarrow (B, \top_B))$, then $F((A, \alpha) \rightarrow (B, \beta))$ must be equal to the arrow $\varepsilon_P((A, \alpha) \rightarrow (B, \beta))$ (by the unicity of the mediating arrow in the universal property of comprehensions). Hence $F = \varepsilon_P$. Finally it is direct to check that $b = j_P$. 

**Proposition 4.12.** Let $P$ and $R$ be two doctrines of $\text{CE}$, and let $(F, b) : P \rightarrow R$ be a $1$-cell of $\text{ED}$, then there exists a unique $2$-cell $\tau$ such that $((F, b), \tau) : (P, \varepsilon_P) \rightarrow (R, \varepsilon_R)$ is a colax morphism of $\text{Tc-Alg}$.

Proof. Consider the square

\[
\begin{array}{c}
P \xrightarrow{\varepsilon_P} R \\
\downarrow \quad \downarrow \varepsilon_R \\
(F, b) \xrightarrow{\varepsilon_R} R
\end{array}
\]

Let $(A, \alpha)$ be an object of $G_P$. Then we have that

$\varepsilon_R T_{(F, b)}(A, \alpha) = (FA)\delta_{(\alpha)}$

and

$(F, b)\varepsilon_P(A, \alpha) = F(A, \alpha)$. 

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We define \( \tau_{(A,\alpha)} \) as the morphism

\[
\begin{array}{c}
(FA)_{b_A(\alpha)} \sum_{b(\alpha)} \rightarrow FA \\
\downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \\
\tau_{(A,\alpha)} \sum_{b(\alpha)} \rightarrow F(A\alpha) \\
\end{array}
\]

which exists by the universal property of comprehensions, because

\[
R_{F(\{\alpha\})}(b_A(\alpha)) = b_{A\alpha} P_{\{\alpha\}}(\alpha) = \top.
\]

Now we show that the \( \tau \) is a natural transformation \( \tau: FJ \Rightarrow JR^\hat{F} \). Let us consider an arrow \( f: (A,\alpha) \rightarrow (B,\beta) \) of the category \( \mathcal{G}_P \). Then the diagram

\[
\begin{array}{c}
F(A\alpha) \rightarrow F\{\alpha\} \rightarrow FA \\
\downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \\
\tau_{(A,\alpha)} \sum_{b(\alpha)} \rightarrow F(A\alpha) \\
\end{array}
\]

commutes, because every triangle commutes and the right and back squares commute, hence, using the fact that comprehensions are mono, we can show that the left square

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commutes. Therefore, \( \tau \) is a natural transformation. Moreover we have that

\[
P(A_\alpha) \xrightarrow{b_{A_\alpha}} R(FA_\alpha) \\
\begin{array}{c}
P_c(A, \alpha) \\
\downarrow \delta_{(A, \alpha)} \\
R_c(FA, b_A(\alpha)) \xrightarrow{j_R} R((FA)b_{A}(\alpha))
\end{array}
\]

commutes, and hence we can conclude that \( \tau \) is a 2-cell of \( CE \).

Finally, it is direct to show that \( ((F, b), \tau) \) satisfies the coherence axioms of colax morphisms of algebras. For example, we have that the following axiom is satisfied

\[
P \xrightarrow{(F, b)} R \\
P_c((\hat{F}, \hat{b})) \xrightarrow{\tau \uparrow} P_c \xrightarrow{\varepsilon_R} R = 1_P \\
P \xrightarrow{(F, b)} R
\]

because, when \( \alpha = \top_A \), then we have that \( \tau_{(A, \top_A)} = \text{id}_{FA} \). Now we show that this \( \tau \) is unique. Let us consider another 2-cell \( \theta: (F, b) \varepsilon_P \Rightarrow \varepsilon_R T_c(F, b) \) such that \( ((F, b), \theta) \) is a colax-morphism

\[
P_c((\hat{F}, \hat{b})) \xrightarrow{\tau \uparrow} P_c \xrightarrow{\varepsilon_R} R
\]
of \( T_c \) algebras. Then it must satisfy the following condition

\[
\begin{array}{c}
P \xrightarrow{(F,b)} R \\
\eta_A \\ P \xrightarrow{\varepsilon_P} R \\
\eta_B \\
\end{array}
\]

\[
\begin{array}{c}
\varepsilon_R \\
\phi \parallel \\
\varepsilon_R \\
\end{array}
\]

and this means that \( \theta_{(A,\tau_A)} = \text{id}_{FA} \). Therefore, since \( \theta \) is a natural transformation from \( FJ_P \) to \( J_RF \), then the following diagram

\[
\begin{array}{c}
F(A_\alpha) \xrightarrow{F(\{\alpha\})} FA \\
\theta_{(A,\alpha)} \\
(FA)_{b_A(\alpha)} \xrightarrow{\{b_A(\alpha)\}} FA \\
\end{array}
\]

commutes and since \( \theta_{(A,\tau_A)} = \text{id}_{FA} \), then we have that \( \theta_{(A,\alpha)} \) must be \( \tau_{(A,\alpha)} \) because, by definition, \( \tau_{(A,\alpha)} \) is the unique arrow such that the diagram

\[
\begin{array}{c}
(FA)_{b_A(\alpha)} \xrightarrow{\{b(\alpha)\}} FA \\
\tau_{(A,\alpha)} \\
F(\{\alpha\}) \\
F(A_\alpha) \\
\end{array}
\]

commutes. Hence \( \theta = \tau \).

\[\square\]

**Corollary 4.13.** Let \( P \) and \( R \) be two doctrines of \( \text{CE} \), and let \( (F,f) \): \( P \xrightarrow{R} \) be an invertible 1-cell of \( \text{CE} \), then \( \tau \) is the identity.

\[\text{Proof.}\] By Proposition 6.15, \( \tau \) exists and it is unique, and since \( \varepsilon \) is 2-natural, it must be the identity. \[\square\]
Remark 4.14. Observe that if we have a $T_c$-morphism $((F, b), \gamma)$

then, since $\gamma$ is invertible, we have that $(F, b)$ is a 1-cell of $\text{CE}$, so by the previous corollary, $\gamma$ must be the identity.

Combining previous results, we directly show that the comprehensions completions is 2-monadic.

Theorem 4.15. The 2-category $T_c\text{-Alg}$ of strict algebras, algebras morphisms and $T_c$-transformation is 2-equivalent to the 2-category $\text{CE}$.

Finally, again by directly applying the previous results, we can prove the following theorem.

Theorem 4.16. The 2-monad $T_c: \text{ED} \longrightarrow \text{ED}$ is colax-idempotent.

Proof. It follows by Proposition 6.15 and by Proposition 6.14.

Notice that, in particular, the previous theorem implies that the 2-monad $T_c$ is property-like, and so we can conclude that having full comprehensions is not only a structure, but it is a property of an elementary doctrine.

Remark 4.17. Observe that having a choice of comprehensions in the doctrine plays a fundamental role in the development of the results of this section. For example, in the definition of the counit of the 2-adjunction. However, all the results we presented can be generalized if we do not assume any choice of comprehensions. In this case, the functor $\mathcal{C}$ remains a 2-functor, the unit is a 2-natural transformation, but the counit becomes a pseudo-natural transformation, and then the monad $T_c$ is just a pseudo-monad, and not a 2-monad. Then, all the results can be reformulated in terms of pseudo-monads.

5. Elementary doctrines with comprehensive diagonals

The notion of comprehensive diagonals, together with the construction called extensional collapse of an elementary doctrine, was employed by Maietti and Rosolini in \cite{5} to obtain "extensional" models of constructive theories.

We start this section by recalling this notion, which is a special case of comprehension, see Definition \ref{def:comprehensive-diagonals} and then presenting the free construction that forces the equalities of an elementary doctrine to be "extensional".

Definition 5.1. An elementary doctrine $P: C^{\text{op}} \longrightarrow \text{InfSL}$ has comprehensive diagonals if every diagonal arrow $\langle \text{id}_A, \text{id}_A \rangle: A \longrightarrow A \times A$ is the comprehension of $\delta_A$. 

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The denote by $\textsf{CED}$ the 2-category whose objects are elementary doctrines with comprehensive diagonals, and whose 1-cells and 2-cells are the same of $\textsf{ED}$.

Now we recall the free construction which freely adds comprehensive diagonals to an elementary doctrine

Let $P : C^{\text{op}} \rightarrow \text{InfSL}$ be an elementary doctrine, we define $X_P$ the extensional collapse of $P$:

- the objects of $X_P$ are the objects of $C$;
- a morphism $[f] : A \rightarrow B$ is an equivalence class of morphisms $f : A \rightarrow B$ such that $\delta_A \leq_{A \times A} P_f \times f(\delta_B)$ with respect to the equivalence $f \sim f'$ when $\delta_A \leq_{A \times A} P_f \times f'(\delta_B)$.

The indexed inf-semilattice $P^\text{op}_X : X^{\text{op}}_P \rightarrow \text{InfSL}$ will be given by $P$ itself: indeed for every $A$ in $C$, $P_A = P(A)$ and for every $[f] : A \rightarrow B$, $P_A([f]) = P(f)$ as one shows that $P(f) = P(f')$ when $f \sim f'$. See [6, Lemma 5.5].

With the previous assignments the functor $P^\text{op}_X : X^{\text{op}}_P \rightarrow \text{InfSL}$ is an elementary doctrine with comprehensive diagonals. Now we show that, as for the case of ordinary comprehensions, the assignment $P \mapsto P^\text{op}_X$ can be extended to 2-functor

$$\text{D} : \text{ED} \rightarrow \text{CED}$$

and we start defining how it acts on the 1-cells and 2-cells in $\text{ED}$.

Let $P : C^{\text{op}} \rightarrow \text{InfSL}$ and $R : D^{\text{op}} \rightarrow \text{InfSL}$ be elementary doctrines, and consider a 1-cell $(F, b)$:

$$
\begin{array}{ccc}
C^{\text{op}} & \xrightarrow{P} & \text{InfSL} \\
\downarrow F^\text{op} & & \downarrow R \\
D^{\text{op}} & \xleftarrow{b} & \\
\end{array}
$$

Let $(\tilde{F}, b)$ be the pair where

- $\tilde{F}(A)$ is $F(A)$ for every $A \in X_P$;
- $\tilde{F}([f])$ is $[F(f)]$ for every $[f] : A \rightarrow B$. and $b$ remains the same.

**Lemma 5.2.** $\tilde{F}, b$ is a 1-morphism in $\text{CED}$.

**Proof.** First we prove that $\tilde{F} : X_P \rightarrow X_R$ is a well-defined functor. If $f : A \rightarrow B$ and $g : A \rightarrow B$ are a morphism in $C$, such that $\delta_A \leq_{A \times A} P_g \times f(\delta_B)$, then we have

$$b_{A \times A}(\delta_A) \leq b_{A \times A}(P_g \times f(\delta_B))$$

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Since $b$ is a natural transformation, the following diagram commutes

$$
\begin{array}{ccc}
P(B \times B) & \xrightarrow{P_{g \times f}} & P(A \times A) \\
\downarrow^{b_{B \times B}} & & \downarrow^{b_{A \times A}} \\
RF(B \times B) & \xrightarrow{RF(g \times f)} & RF(A \times A).
\end{array}
$$

Hence we have

$$b_{A \times A}(\delta_A) \leq R_{F}(g \times f)(b_{B \times B}(\delta_B)).$$

By definition, $b_{A \times A}(\delta_A) = R_{(F(pr_1),F(pr_2))}(\delta_{F(B)})$, thus

$$R_{(F(pr_1),F(pr_2))}(\delta_{F(A)}) \leq R_{(F(pr_1),F(pr_2))}(R_{F(g \times f)}(\delta_{F(B)}))$$

where $pr_1: A \times A \to A$ and $pr_2: B \times B \to B$ are the projections. Finally

$$F(g \times f) \circ (F(pr_1),F(pr_2))^{-1} = (F(pr_1),F(pr_2)) \circ F(g) \times F(f),$$

so

$$\delta_A \leq R_{F(g \times f)}(\delta_B).$$

It is now easy to check that $\tilde{F}$ is a functor from $X_P$ to $X_R$. Then, we have that $(\tilde{F}, b)$ is a 1-cell observing that

$$b_{A \times A}(\delta_A) = (R_x)_{(\tilde{F}(pr_1),\tilde{F}(pr_2))}(\delta_{\tilde{F}(B)})$$

because $\tilde{F}(pr_1) = [F(pr_1)], \tilde{F}(B) = F(B)$ by definition of $\tilde{F}$, and

$$\langle \tilde{F}(pr_1)\rangle, \tilde{F}(pr_2) \rangle = [(F(pr_1),F(pr_2))]$$

by [6, Lemma 5.4], and

$$(R_x)_{(\tilde{F}(pr_1),\tilde{F}(pr_2))} = (R_x)_{[F(pr_1),F(pr_2)]} = R_{(pr_1,F(pr_2))}.$$
Proposition 5.4. Let \( P: C^{\text{op}} \rightarrow \text{InfSL} \) and \( R: D^{\text{op}} \rightarrow \text{InfSL} \) be elementary doctrines. The map

\[
D_{P,R}: ED(P,R) \rightarrow CED(P_x,R_x)
\]

such that \( D_{P,R}(F,b) = (\tilde{F},b) \) and \( D_{P,R}(\theta) = \tilde{\theta} \) is a functor and

\[
D: ED \rightarrow CED
\]
is a 2-functor with the assignment \( D(P) = P_x \).

We prove that the 2-functor \( D: ED \rightarrow CED \) is left adjoint to the forgetful 2-functor \( U: CED \rightarrow ED \).

First, observe that for every elementary doctrine \( P \) there is a natural embedding

\[
K_P: C \rightarrow X_P \rightarrow \text{InfSL}
\]
of elementary doctrines, where \( K_P: C \rightarrow X_P \) is the quotient functor, and \( k \) is the identity.

Similarly, if \( P \) is an elementary doctrine with comprehensive diagonals, we can define a 1-cell

\[
T_P: X_P \rightarrow C \rightarrow \text{InfSL}
\]
where \( T_P: X_P \rightarrow C \) is the identity on the objects, and it sends \([f] \mapsto f\), and \( t_P \) is the identity. Notice that \( T_P \) is a well defined functor, because if \( P: C^{\text{op}} \rightarrow \text{InfSL} \) has comprehensive diagonals, then \( f \sim g \) implies \( f = g \). In details, let \( f: A \rightarrow B \) and \( g: A \rightarrow B \) be morphisms such that \( \delta_A \leq P_{f \times g}(\delta_B) \). Then we have that

\[
\top_A \leq P_{\Delta_A}(P_{f \times g}(\delta_B)) = P_{(f,g)}(\delta_B).
\]

Thus, there exists a unique morphism \( h: A \rightarrow B \) such that the following diagram

\[
B \xrightarrow{\Delta_B} B \times B
\]

\[
A \xrightarrow{(f,g)} B
\]

\[
\xrightarrow{h}
\]

\[
\xrightarrow{\Delta_B}
\]

\[
\xrightarrow{h}
\]
commutes. Hence, if \( P \in \text{CED} \) then we have that \( f \sim g \) if and only if \( f = g \). Thus, we can define the 1-cell \((T_P,t_P): P_2 \longrightarrow P\) as before.

**Remark 5.5.** Notice that if \( P \) has comprehensive diagonal, then \( \varepsilon_P \) and \( \eta_P \) are isomorphism.

**Theorem 5.6.** The 2-functor \( D: \text{ED} \longrightarrow \text{CED} \) is 2-left adjoint to the forgetful functor \( U: \text{CED} \longrightarrow \text{ED} \). The unit of this 2-adjunction \( \eta: \text{id}_{\text{ED}} \longrightarrow UD \) is given by \( \eta_P = (K_P,k_P) \) and the counit \( \varepsilon: DU \longrightarrow \text{id}_{\text{CED}} \) is given by \( \varepsilon_P = (T_P,t_P) \).

The 2-adjunction of Theorem 5.6 induces a 2-monad \( T_d: \text{ED} \longrightarrow \text{ED} \), whose unit is given by the unit of the 2-adjunction, and whose multiplication is defined by \( \mu = \varepsilon_D \), as in 1-dimensional case.

As for the case of the comprehension completion, we show that 2-monad \( T_d \) is *pseudo-idempotent*, and that we have the equivalence of 2-categories

\[
T_d\text{-Alg} \equiv \text{CED}.
\]

However, in this case, it is immediate to show that the 2, because by Remark 5.5 we have that the multiplication of the 2-monad \( T \) is invertible.

**Theorem 5.7.** The 2-monad \( T_d: \text{ED} \longrightarrow \text{ED} \) is pseudo-idempotent.

**Proof.** It follow from Remark 5.5.

The previous result means that also having comprehensive diagonals is a property of an elementary doctrine.

**Proposition 5.8.** Let \( P: C^{op} \longrightarrow \text{InfSL} \) be an elementary doctrine of \( \text{CED} \), then \((P,\varepsilon_P)\) is a \( T_d\)-algebra.

**Proof.** The diagram

\[
\begin{array}{ccc}
T_d^2P & \xrightarrow{T_d\varepsilon_P} & T_dP \\
\mu_P & \downarrow & \varepsilon_P \\
T_dP & \xrightarrow{\varepsilon_P} & P
\end{array}
\]

commutes because \( \mu_P = \varepsilon_{DP} \) and \( \varepsilon: T_d \longrightarrow \text{id}_{\text{CED}} \) is a 2-natural transformation. Similarly we have that the unit axiom for strict algebras is satisfied.

**Proposition 5.9.** Let \( (P,(F,b)) \) be a \( T_d\text{-Alg} \). Then the doctrine \( P \) has comprehensive diagonals and \( (F,b) = \varepsilon_P \).

**Proof.** Since \( (P,(F,b)) \) is a \( T_d\)-algebra, the following diagram, i.e. the identity axiom holds
commutes. Since $P_x$ has comprehensive diagonals and it acts on morphisms as $P$, i.e. $P_x([f]) = P_f$ it is direct to show that $P$ has comprehensive diagonals, which are given by morphisms of the form $F([\Delta_A])$. Moreover, by unit axiom, $F$ must be the identity on the objects of $X_P$, and $F([f]) = F\eta F(f) = f$, i.e. $F = \varepsilon P$. So, let $f: C \rightarrow A \times A$ be a morphism of $C$ such that $\top_C \leq P_f(\Delta_A)$. \hfill \Box

**Theorem 5.10.** We have the following equivalence of categories

$$T_d\text{-Alg} \cong CED$$

Therefore, again as in the case of the comprehension completion, we have that the comprehensive diagonal completion is 2-monadic.

6. Elementary doctrines with quotients

In this section we consider the completion with quotients of an elementary doctrine introduced in [5, 6], and we show that this construction is 2-monadic and that its 2-monad is lax-idempotent.

Given an elementary doctrine $P: C^{op} \rightarrow \text{InfSL}$, an object $A$ in $C$, and an object $\rho$ in $P(A \times A)$, we say that $\rho$ is a $P$-equivalence relation on $A$ if it satisfies

- **reflexivity:** $\delta_A \leq \rho$;
- **symmetry:** $\rho \leq P_{(pr_2,pr_1)}(\rho)$, for $pr_1, pr_2: A \times A \rightarrow A$ the first and second projection, respectively;
- **transitivity:** $P_{(pr_1,pr_2)}(\rho) \land P_{(pr_2,pr_1)}(\rho) \leq P_{(pr_1,pr_2)}(\rho)$ for $pr_1, pr_2, pr_3: A \times A \times A \rightarrow A$ the first, second, and third projection, respectively.

For an elementary doctrine $P: C^{op} \rightarrow \text{InfSL}$, the object $\delta_A$ is a $P$-equivalence relation, and for every morphism $f: A \rightarrow B$, the functor

$$P_{f \times f}: P(B \times B) \rightarrow P(A \times A)$$

takes a $P$-equivalence relation $\sigma$ on $B$ to a $P$-equivalence relation on $A$. The $P$-kernel equivalence relation of $f: A \rightarrow B$ is the object $P_{f \times f}(\delta_B)$, which is a $P$-equivalence relation on $A$.

**Definition 6.1.** Let $P: C^{op} \rightarrow \text{InfSL}$ be an elementary doctrine, and let $\rho$ be an $P$-equivalence relation on $A$. A $P$-quotient of $\rho$ is a morphism $q: A \rightarrow C$ in $C$ such that $P_q(\delta_C) \geq \rho$ and for every morphism $f: A \rightarrow Z$ such that $P_f(\delta_Z) \geq \rho$, there exists a unique morphism $g: C \rightarrow Z$ such that $g \circ q = f$. 

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A quotient $q: A \rightarrow C$ of $\rho$ is said effective if $P_{q \times q}(\delta_D) = \rho$. We say that such a $P$-quotient is stable if in every pullback

$$
\begin{array}{c}
A' \xrightarrow{q'} C' \\
\downarrow f' \quad \downarrow f \\
A \xrightarrow{q} C
\end{array}
$$

in $C$, the morphism $q': A' \rightarrow C'$ is a $P$-quotient.

**Definition 6.2.** Given an elementary doctrine $P: C^{\text{op}} \rightarrow \text{InfSL}$ and a $P$-equivalence relation $\rho$ on an object $A$ in $C$, the partial order of descent data $\text{Des}_\rho$ is the suborder of $P(A)$ of those $\alpha$ such that

$$
P_{\text{pr}_1}(\alpha) \wedge \rho \leq P_{\text{pr}_2}(\alpha)
$$

where $\text{pr}_1, \text{pr}_2: A \times A \rightarrow A$ are projections.

**Remark 6.3.** Let $P: C^{\text{op}} \rightarrow \text{InfSL}$ be an elementary doctrine and consider the $P$-kernel $\rho = P_f \times f(\delta_B)$, for $f: A \rightarrow B$. The functor $P_f: P(B) \rightarrow P(A)$ takes values in $\text{Des}_\rho \subseteq P(A)$.

**Definition 6.4.** Let $P: C^{\text{op}} \rightarrow \text{InfSL}$ be an elementary doctrine. An arrow $f: A \rightarrow B$ is called of effective descent if the functor $P_f: P(B) \rightarrow \text{Des}_\rho$, where $\rho = P_f \times f(\delta_B)$ is an isomorphism.

Consider the 2-full 2-subcategory $\text{QED}$ of $\text{ED}$ whose objects are the elementary doctrines $P: C^{\text{op}} \rightarrow \text{InfSL}$ with stable effective quotients of $P$-equivalence relations and of effective descent. 1-cells of $\text{QED}$ are those 1-cells of $\text{ED}$ which preserve quotients, and the 2-cells of $\text{QED}$ are the same of $\text{ED}$.

As in the case of comprehensions, from now on we consider doctrines with a choice of quotients.

Let $P: C^{\text{op}} \rightarrow \text{InfSL}$ be an elementary doctrine, and consider the category $\mathcal{R}_P$ of $P$-equivalence relation:

- **an object** of $\mathcal{R}_P$ is a pair $(A, \rho)$ such that $\rho$ is a $P$-equivalence relation on $A$;
- **a morphism** $f: (A, \rho) \rightarrow (B, \sigma)$ is a morphism $f: A \rightarrow B$ such that $\rho \leq P_f(\sigma)$.

The indexed poset $P_q: \mathcal{R}_P^{\text{op}} \rightarrow \text{InfSL}$ will be given by the categories of descent data:

$$
P_q(A, \rho) = \text{Des}_\rho
$$

and for every morphism $f: (A, \rho) \rightarrow (B, \sigma)$ we define

$$
P_q(f) = P(f)
$$
This is a well defined elementary doctrine, see [6, Lemma 4.2], and it has descent quotients of $P$-equivalence relations, see [6, Lemma 4.4].

Following the structure of Sections 5 and 4 we prove that the assignment $Q(P) = P_q$ can be extended to 2-functor

$$ Q: \text{ED} \longrightarrow \text{QED} $$

and we start defining how it acts on the 1-cells and 2-cells in ED.

Let $P: C^{\text{op}} \longrightarrow \text{InfSL}$ and $R: D^{\text{op}} \longrightarrow \text{InfSL}$ be elementary doctrines, and consider a 1-cell $(F, b)$:

$$ \begin{tikzcd}
C^{\text{op}} \arrow{r}{p} \arrow{d}{F^{\text{op}}} & \text{InfSL} \arrow{d}{R} \\
D^{\text{op}} \arrow{r}{b} & \text{InfSL}
\end{tikzcd} $$

We want to prove that the pair $(\overline{F}, \overline{b})$ where:

- $\overline{F}(A, \rho)$ is $(FA, R_{(F(pr_1), F(pr_2))}^{-1}(b_{A \times A}(\rho)))$ for every $A \in \mathcal{R}_P$;
- $\overline{F}(f)$ is $F(f)$ for every $f: (A, \rho) \longrightarrow (B, \sigma)$;
- $\overline{b}$ is $b$ restricted to the categories of descent data;

is a 2-morphism in $\text{QED}$:

$$ \begin{tikzcd}
\mathcal{R}_{P}^{\text{op}} \arrow{r}{P_q} \arrow{d}{F^{\text{op}}} & \text{InfSL} \\
\mathcal{R}_{R}^{\text{op}} \arrow{r}{R_q} & \text{InfSL}
\end{tikzcd} $$

**Lemma 6.5.** Let $(A, \rho)$ be an object in $\mathcal{R}_P$ and let $pr_1, pr_2: A \times A \longrightarrow A$ be the two projections. Then $R_{(F(pr_1), F(pr_2))}^{-1}(b_{A \times A}(\rho))$ is a $P$-equivalence relation on $FA$.

**Proof.** Reflexivity: $\rho$ is an equivalence relation on $A$ implies $b_{A \times A}(\delta_A) \leq b_{A \times A}(\rho)$ and by definition of $b_{A \times A}$ we have $R_{(F(pr_1), F(pr_2))}(\delta_{FA}) \leq b_{A \times A}(\rho)$ . Since $F$ preserves products $(F(pr_1), F(pr_2))$ is an isomorphism. So

$$ \delta_{FA} \leq R_{(F(pr_1), F(pr_2))}^{-1}(b_{A \times A}(\rho)). $$

Symmetry and transitivity are proved similarly. \qed
Lemma 6.6. Let \( f : (A, \rho) \rightarrow (B, \sigma) \) be a morphism in \( \mathcal{R}_P \), and let \( \text{pr}_i : A \times A \rightarrow A \) and \( \text{pr}'_i : B \times B \rightarrow B \), \( i = 1, 2 \) be the projections. Then

\[
F(f) : (FA, R_{(F(\text{pr}_1), F(\text{pr}_2))}(b_{A \times A}(\rho))) \rightarrow (FB, R_{(F(\text{pr}'_1), F(\text{pr}'_2))}(b_{B \times B}(\sigma)))
\]

is a morphism in \( \mathcal{R}_R \).

Proof. Since \( f : (A, \rho) \rightarrow (B, \sigma) \) is a 1-cell, \( \rho \leq P_f \times f(\sigma) \). Thus

\[
b_{A \times A}(\rho) \leq b_{A \times A}(P_f \times f(\sigma)) = R_{F(f \times f)}(b_{B \times B}(\sigma)).
\]

Hence

\[
R_{(F(\text{pr}_1), F(\text{pr}_2))}(b_{A \times A}(\rho)) \leq R_{(F(\text{pr}_1), F(\text{pr}_2))}(R_{F(f \times f)}(b_{B \times B}(\sigma))).
\]

Since \( F(f \times f) \circ (F(\text{pr}_1), F(\text{pr}_2))^{-1} = (F(\text{pr}'_1), F(\text{pr}'_2))^{-1} \circ F(f) \times F(f) \)

it is

\[
R_{(F(\text{pr}_1), F(\text{pr}_2))}(b_{A \times A}(\rho)) \leq R_{F(f \times f)}(R_{(F(\text{pr}'_1), F(\text{pr}'_2))}(b_{B \times B}(\sigma))).
\]

\[\square\]

Remark 6.7. Consider \( (A, \rho) \in \mathcal{R}_P \), if \( \alpha \in \text{Des}_P \) then

\[
b_{A}(\alpha) \in \text{Des}_{R_{(F(\text{pr}_1), F(\text{pr}_2))}(b_{A \times A}(\rho))}.
\]

Corollary 6.8. Given \( (F, b) \in \text{ED}(P, R) \) then \( (\overline{F}, \overline{b}) \in \text{QED}(P_q, R_q) \).

Proof. By Remark 6.7 and Lemma 4.2

\[
b_{A \times A}(\rho) = R_{(F(\text{pr}_1), F(\text{pr}_2))}(R_{(F(\text{pr}_1), F(\text{pr}_2))}(b_{A \times A}(\rho)))
\]

So

\[
\overline{b}_{(A, \rho) \times (A, \rho)}(\delta(A, \rho)) = (R_{\overline{F}(\text{pr}_1), \overline{F}(\text{pr}_2)})(\delta_{\overline{F}}(A, \rho)).
\]

By Lemma 6.6 and Lemma 5.5 we can conclude that \( (\overline{F}, \overline{b}) \in \text{ED}(P_q, R_q) \). It remains to verify that \( \overline{F} \) preserves all the quotients.

Consider a \( P_q \)-equivalence relation \( \tau \) on \( (A, \rho) \). A \( P_q \)-quotient of \( \tau \) is

\[
id_A : (A, \rho) \rightarrow (A, \tau)
\]

and

\[
id_{FA} : (FA, R_{(F(\text{pr}_1), F(\text{pr}_2))}(b_{A \times A}(\rho))) \rightarrow (FA, R_{(F(\text{pr}_1), F(\text{pr}_2))}(b_{A \times A}(\tau)))
\]

is a \( R_q \)-quotient of \( R_{(F(\text{pr}_1), F(\text{pr}_2))}(b_{A \times A}(\tau)) \). So \( \overline{F} \) preserves quotients, and \( (\overline{F}, \overline{b}) \) is a 1-cell in \( \text{QED} \). \[\square\]
Proposition 6.9. Let \( \theta \) be a morphism in \( \text{ED}(P, R) \)

\[
\theta : (F, b) \longrightarrow (G, c).
\]

Then \( \theta \) is also a morphism in \( \text{QED}(P_q, R_q) \)

\[
\theta : (\overline{F}, \overline{b}) \longrightarrow (\overline{G}, \overline{c}).
\]

Proof. We must prove that for every \((A, \rho) \in R_P \)

\[
\theta_A : (FA, R(F(pr_1), F(pr_2))-1(A \times A(\rho))) \longrightarrow (GA, R(G(pr_1), G(pr_2))-1(A \times A(\rho)))
\]

is a morphism in \( R_R \). Indeed, by definition of 2-morphism we have \( b_{A \times A}(\rho) \leq R_{\theta A \times A}(c_{A \times A}(\rho)) \)

and, since \( \theta \) is a natural transformation,

\[
R(F(pr_1), F(pr_2))-1(A \times A(\rho)) \leq R_{\theta A \times A}(R(G(pr_1), G(pr_2))-1((c_{A \times A}(\rho))))
\]

Finally for every \( \alpha \in \text{Des}_{R(F(pr_1), F(pr_2))-1(A \times A(\rho))} \) we have

\[
\overline{b_A}(\alpha) \leq (R_q)_{\theta A}(\overline{c_A}(\alpha))
\]

because \( \overline{b_A}(\alpha) = b_A(\alpha), \overline{c_A}(\alpha) = c_A(\alpha) \) and \( R_q(\theta A) = R(\theta A) \).

Proposition 6.10. The assignment

\[
Q_{P,R} : \text{ED}(P, R) \longrightarrow \text{QED}(P_q, R_q)
\]

which maps \((F, b)\) into \((\overline{F}, \overline{b})\) and a 2-cell \( \theta : (F, b) \longrightarrow (G, c) \) into \( \theta : (\overline{F}, \overline{b}) \longrightarrow (\overline{G}, \overline{c}) \)

is a functor and

\[
Q : \text{ED} \longrightarrow \text{QED}
\]

is a 2-functor with the assignment \( Q(P) = P_q \).

Now we prove that the 2-functor \( Q : \text{ED} \longrightarrow \text{QED} \) is left adjoint to the forgetful 2-functor.

To simplify the notation, given an object \( A \) and a \( P \)-equivalence relation \( \rho \), the quotient of \( \rho \) is denoted by \( q_\rho : A \longrightarrow A_{\rho} \). Observe that, as in the case of comprehensions, we assume that an elementary doctrine with quotients is equipped with a choice of quotients.

First, observe that for every elementary doctrine \( P \) there is a natural embedding
of elementary doctrines, where $L_P: C \to \mathcal{R}_P$ acts as $A \mapsto (A, \delta_A)$, and $(l_P)_A: P(A) \to P_q(A, \delta_A)$ sends $\alpha \mapsto \alpha$. It is direct to check that $(L_P, l_P)$ is a 1-cell of elementary doctrines.

Given an elementary doctrine $P$ of QED, we can define a morphism

\[
\begin{array}{ccc}
\mathcal{R}_P^{op} & \to & \mathcal{C}^{op} \\
P_q & \downarrow & \& \downarrow P
\end{array}
\]

in QED as follow: the functor $V_P: \mathcal{R}_P \to \mathcal{C}$ sends an object $(A, \rho)$ of $\mathcal{R}_P$ to the object $A/\rho$ of, and an arrow $f: (A, \rho) \to (B, \sigma)$ in $\mathcal{R}_P$ is sent to the arrow $V_P(f): A/\rho \to B/\sigma$ defined as the vertical arrow $a$ of the following diagram

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow q_\rho & & \downarrow q_{\sigma} \\
\rightsquigarrow A/\rho & \xrightarrow{a} & B/\sigma
\end{array}
\]

The arrow $V_P(f) = a$ exists by the universal property of quotients, because $\rho \leq P_f \times f(\sigma) \leq P_f \times (P_{q_{\sigma}} \times q_{\sigma}(\delta_{B/\sigma}))$. The natural transformation $v_P$ is defined by the following components: for every object $(A, \rho)$ of $\mathcal{R}_P$, we have that $(v_P)_{(A, \rho)}: P_q(A, \rho) \to P(A/\rho)$ acts as $\alpha \mapsto (P_{q_\rho})^{-1}(\alpha)$. Notice that $P_{q_\rho}: P(A/\rho) \to \text{Des}_\rho$ is invertible because $P$ is a doctrine of QED, and then quotients are effective and also effective descent, i.e. $P(A/\rho) \cong \text{Des}_\rho$.

**Lemma 6.11.** With the previous assignments $(V_P, v_P): P_q \to P$ is a 1-cell of QED.

**Proof.** First we show that $V_P: \mathcal{R}_P \to \mathcal{C}$ is a functor. Let us consider the arrows $(A, \rho) \to (B, \sigma)$ and $(B, \sigma) \to (C, \gamma)$ of $\mathcal{R}_P$. Now we show that $V_P(gf) = \ldots$
Observe that the diagram

\[
\begin{array}{ccc}
A & \xrightarrow{q_\rho} & A_{/\rho} \\
\downarrow f & & \downarrow V_P(f) \\
B & \xrightarrow{q_\sigma} & B_{/\sigma} \\
\downarrow g & & \downarrow V_P(g) \\
C & \xrightarrow{q_\gamma} & C_{/\gamma}
\end{array}
\]

commutes, and then we have that \(V_P(g)V_P(f)\) is the unique arrow such that \(V_P(g)V_P(f)q_\rho = q_\gamma gf\), hence it is exactly \(V_P(gf)\). Moreover \(V_P(id) = id\) and then \(V_P: R_P \rightarrow C\) is a functor, and it is direct to check that it preserves finite products. Therefore, we can conclude that \((V_P, V_P)\) is a 1-cell of \(ED\), because \(V_P\) is a preserving product functor, and \(V_P\) is a natural transformation, whose naturality follows from the fact that every components \((V_P)(A_\rho)\) is an iso, and for every 1-cell \((F, b)\), we have that \(\overline{F}(f) = F(f)\) and \(\overline{b}\) acts as \(b\).

One can show that \((V_P, V_P)\) is also a morphism of \(QED\), i.e. \(V_P\) preserves quotients, by using the same idea of Lemma 4.6. Since every \(P_\tau\)-equivalence relation \(\tau\) on \((A, \rho)\) is also a \(P\) equivalence relation on \(A\), it is easy to see that the \(P_\tau\)-quotient of \(\tau\) is \(\overline{q_\tau} = [id_A]: (A, \rho) \rightarrow (A, \tau)\). Hence \(V_P(q_\tau)\) is the unique arrow

\[
\begin{array}{ccc}
A & \xrightarrow{q_\rho} & A_{/\rho} \\
\downarrow q_\tau & & \downarrow V_P(q_\tau) \\
A_{/\tau} & & 
\end{array}
\]

which is exactly a quotient map of \(\tau \in P(A_{/\rho} \times A_{/\rho})\). Hence we have proved that \((V_P, V_P)\) is a 1-cell of \(QED\). \(\square\)

**Theorem 6.12.** The 2-functor \(Q: ED \rightarrow QED\) is 2-left adjoint to the forgetful functor \(Q: QED \rightarrow ED\). The unit of this 2-adjunction \(\eta: id_{ED} \rightarrow UQ\) is given by \(\eta_P = (L_P, l_P)\) the counit \(\varepsilon: QU \rightarrow id_{QED}\) is given by \(\varepsilon_P = (V_P, v_P)\).

As in Sections 4 and 5 consider the following 2-monad, given 2-adjunction of Theorem 6.12, this theorem induces a 2-monad \(T_q: ED \rightarrow ED\), whose unit is given by the unit of the 2-adjunction, and whose multiplication is defined by \(\mu = \varepsilon_Q\), as in 1-dimensional case. Again, we study that 2-monad \(T_q\) and we show that we have the equivalence of 2-categories

\[T_q\text{-Alg} \equiv QED.\]
We start by showing that every elementary doctrine of QED is a $T_q$-algebra.

**Proposition 6.13.** Let $P: C^{\text{op}} \to \text{InfSL}$ be an elementary doctrine of CE, then $(P, \varepsilon_P)$ is a $T_c$-algebra.

**Proof.** The diagram

$$
\begin{array}{ccc}
T_qP & \xrightarrow{T_q\varepsilon_P} & T_qP \\
\mu_P & \downarrow & \downarrow \varepsilon_P \\
T_qP & \xrightarrow{\varepsilon_P} & P
\end{array}
$$

commutes because $\mu_P = \varepsilon_QP$ and $\varepsilon: T_q \to \text{id}_{\text{QED}}$ is a 2-natural transformation. Similarly we have that the unit axiom for strict algebras is satisfied. □

**Proposition 6.14.** Let $(P, (F, b))$ be a $T_q$-$\text{Alg}$. Then the doctrine $P$ is a doctrine of QED. Moreover $(F, b) = \varepsilon_P$.

**Proof.** It is direct to check that given an equivalence relation $\rho \in P(A \times A)$, then $F(\text{id}_A): F(A, \rho) \to F(A, \delta_A)$ is a quotient of $\rho$, where $\text{id}_A: (A, \rho) \to (A, \delta_A)$ is the quotient of $\rho \in P_q(A, \delta_A)$. Moreover, it is direct to show that quotients are stable, and effective. Now we show that $(F, b) = \varepsilon_P$. By the unit axiom of algebras, we have that $(F, b) \eta_P = \text{id}_P$, but since $P$ is a doctrine of QED, we also have $\varepsilon_P \eta_P = \text{id}_P$. Therefore we have that $(F, b) \eta_P = \varepsilon_P \eta_P$ implies that $(F, b) = \varepsilon_P$, because $F(A, \rho) = F(\{(A, \delta)\}_\rho) = (F(A, \delta_A))_{\rho(\rho)} = (\varepsilon_P \eta_P(A))_{\rho(\rho)} = \varepsilon_P(A, \rho)$. Similarly one can prove that $F(f) = \varepsilon_P(f)$, because every arrow $f: (A, \rho) \to (B, \sigma)$ is the unique arrow such that the following diagram commutes

$$
\begin{array}{ccc}
(A, \delta_A) & \xrightarrow{\rho} & (A, \rho) \\
\downarrow f & & \downarrow f \\
(B, \delta_B) & \xrightarrow{\sigma} & (B, \sigma)
\end{array}
$$

because $\rho \leq P_{f \times f}(\sigma)$. Since both $F$ and $\varepsilon_P$ preserve quotients, and since $F(q_{\rho}) = \varepsilon_P(q_{\rho})$ and $F(\{(A, \delta)\}_\rho) = \varepsilon_P(\{(A, \delta_A)\}_\rho)$, then $F(\{(A, \rho)\}_\rho) = \{(B, \sigma)\}_\rho$ (by the unicity of the mediating arrow in the universal property of quotients). Hence $F = \varepsilon_P$. Finally it is direct to check that $b = j_P$. □

**Proposition 6.15.** Let $P$ and $R$ be two doctrines of QED, and let $(F, b): P \to R$ be a 1-cell of ED, then there exists a unique 2-cell $\omega$ such that $((F, b), \tau): (P, \varepsilon_P) \to (R, \varepsilon_R)$ is a lax morphism of $T_q$-$\text{Alg}$. 34
Proof. Consider the square

\[
\begin{array}{ccc}
P & \xrightarrow{T_q(F,b)} & R_q \\
\varepsilon_P & \downarrow & \varepsilon_R \\
\downarrow & & \downarrow \\
(F,b) & \rightarrow & \rightarrow \\
P & \xleftarrow{(F,b)} & R.
\end{array}
\]

Let \((A,\rho)\) be an object of \(\mathcal{R}_P\). Then we have that

\[
\varepsilon_R T_q(F,b)(A,\rho) = (FA)_{/b(\rho)}
\]

where, to simplify the notation, we denote \(b(\rho) = R_{(Fpr_1,Fpr_2)}^{-1} b_{A \times A}(\rho)\), and

\[
(F,b)\varepsilon_P(A,\rho) = F(A_{/\rho}).
\]

We define \(\omega_{(A,\rho)}\) as the morphism

\[
\begin{array}{ccc}
F(A) & \xrightarrow{q_b(\rho)} & (FA)_{/b(\rho)} \\
\downarrow & & \downarrow \\
F(q_b) & & \omega_{(A,\rho)} \\
\downarrow & & \downarrow \\
F(A_{/\rho}) & & \\
\end{array}
\]

which exists by the universal property of quotients, because

\[
R_{F(q_b) \times F(q_b)}(\delta_{F(A_{/\rho})}) = R_{(Fpr_1,Fpr_2)}^{-1} b_{A \times A} P_{q_b \times q_b}(\delta_{A_{/\rho}}) \leq R_{(Fpr_1,Fpr_2)}^{-1} b_{A \times A}(\rho) = b(\rho).
\]

Now we show that the \(\omega\) is a natural transformation \(\omega : V_R T \Rightarrow F V_P\). Let us consider an arrow \(f : (A,\rho) \rightarrow (B,\sigma)\) of the category \(\mathcal{R}_P\). Then the diagram

\[
\begin{array}{ccc}
F(A) & \xrightarrow{q_b(\rho)} & (FA)_{/b(\rho)} \\
\downarrow & & \downarrow \\
F(q_b) & & \omega_{(A,\rho)} \\
\downarrow & & \downarrow \\
F(A_{/\rho}) & & \\
\end{array}
\]

\[
\begin{array}{ccc}
F(B) & \xrightarrow{q_b(\sigma)} & (FB)_{/b(\sigma)} \\
\downarrow & & \downarrow \\
F(q_b) & & \omega_{(B,\sigma)} \\
\downarrow & & \downarrow \\
F(B_{/\sigma}) & & \\
\end{array}
\]
commutes, because every triangle commutes and the left and the back squares commute, hence, using the fact that quotients are epi, we can show that the right square commutes. Therefore, \( \omega \) is a natural transformation. Moreover using the same argument of Proposition 6.15 we can conclude that \( \omega \) is a 2-cell of QED.

It is direct to show that \((F,b), \omega)\) satisfies the coherence axioms of lax morphisms of algebras. Again, following the idea of Proposition 6.15 we have that the following axiom is satisfied:

\[
\begin{array}{c}
P \xrightarrow{(F,b)} R \\
\eta_P \\ P_q \xrightarrow{(F,q)} R_q \\
\varepsilon_P \\
\end{array}
\]

because, when \( \rho = \delta_A \), then we have that \( \omega_{(A,\delta)} = \text{id}_{F_A} \). Now we show that this \( \omega \) is unique. Let us consider another 2-cell \( \theta : T_q(F,b)\varepsilon_R \Rightarrow \varepsilon_P(F,b) \) such that \((F,b), \theta)\) is a lax-morphism of \( T_q \) algebras. Then it must satisfy the following condition:

\[
\begin{array}{c}
P \xrightarrow{(F,b)} R_q \\
\varepsilon_P \\ P \xrightarrow{(F,b)} R \\
\end{array}
\]

and this means that \( \theta_{(A,\delta_A)} = \text{id}_{F_A} \). Therefore, since \( \theta \) is a natural transformation from
Combining previous results we can prove the following theorem, showing the 2-monadicity of the elementary quotient completion.

**Theorem 6.16.** The 2-category $T_q$-$\text{Alg}$ of strict algebras, algebras morphisms and $T_q$-transformation is 2-equivalent to the 2-category $QED$.

Finally, we can conclude, again by directly applying the previous results, with the following result.

**Theorem 6.17.** The 2-monad $T_q : ED \longrightarrow ED$ is lax-idempotent.

So, as in the cases of the comprehension completion and the comprehensive diagonal completion, we have that having quotients is a property of an elementary doctrine.

**Remark 6.18.** Observe that, as in the case of comprehension completion, the results of this section can be generalized in the case we do not assume any choice of quotients. Again, in this case, the counit of the elementary quotient completion becomes a pseudo-natural transformation, and then the monad $T_q$ is just a pseudo-monad, and not a 2-monad.

7. Distributive laws

In the previous sections we provided an algebraic framework to deal with the quotient completion and the comprehension completion of elementary doctrines. Taking the advantage of this presentation, we show how these free constructions interact, i.e. we provide a distributive law between the 2-monads $T_c$ and $T_q$. 
Recall from Section 2, in particular Theorem 2.1, that showing the existence of a distributive law between the 2-monads \( T_q \) and \( T_c \) is equivalent to provide a lifting of \( T_q \) on the 2-category \( T_c\text{-Alg} \).

Notice that combining the equivalence of 2-categories \( T_c\text{-Alg} \equiv CE \) we provide in Theorem 4.15 together with the result \([5, \text{Lemma 5.3}]\), which states that if \( P \) has comprehensions then the quotient completion \( P_q \) has comprehensions as well, the construction of a lifting of \( T_q \) on the 2-category \( T_c\text{-Alg} \) is quite direct.

Lemma 7.1. The assignment

\[
\overline{T}_q(P,a)(R,c) : T_c\text{-Alg}((P,a),(R,c)) \longrightarrow T_c\text{-Alg}((P_q,\varepsilon_{P_q}),(R_q,\varepsilon_{R_q}))
\]

mapping a 1-cell \((F,b) : (P,a) \rightarrow (R,c)\) and a 2-cell \((F,b) \theta : (F,b) \rightarrow (F,b)\) is a functor.

Proof. Since \((P,a)\) is a \( T_c \)-algebra then, by Theorem 4.15, we have that the doctrine \( P \) has comprehensions, and by \([5, \text{Lemma 5.3}]\), we can conclude that \( P_q \) has comprehensions as well. Similarly, one can directly check that if \((F,b)\) is a 1-cell of \( CE \) and \( \theta \) is a 2-cell then \( T_q(F,b) \) is again a 1-cell of \( CE \) and \( T_q \theta \) is a 2-cell of \( CE \). Therefore we conclude that \( \overline{T}_q(P,a)(R,c) \) is a functor.

Lemma 7.2. The functor defined in 7.1 can be extended to a 2-functor

\[
\tilde{T}_q : T_c\text{-Alg} \longrightarrow T_c\text{-Alg}
\]

where \( \tilde{T}_q(P,a) := (P_q,\varepsilon_{P_q}) \). Moreover it is a 2-monad, whose identity and multiplication are those induced by \( T_q \).

Theorem 7.3. There exists a distributive law \( \delta : T_c T_q \longrightarrow T_q T_c \).

Proof. If we consider the forgetful 2-functor \( U_{T_c} : T_c\text{-Alg} \longrightarrow ED \), we have the equality \( T_q U_{T_c} = U_{T_c} \overline{T}_q \). Then \( \tilde{T}_q \) is a lifting of \( T_q \). Thus, by applying Theorem 1.1 to conclude that there exists a distributive law \( \delta : T_c T_q \longrightarrow T_q T_c \).

Corollary 7.4. The 2-functor \( T_q T_c \) is a 2-monad.

Proof. It follows by Theorem 7.3 and Theorem 2.2.

Remark 7.5. As observed in \([6]\), the 2-monad \( T_d \) which freely adds comprehensive diagonals in general does not preserve full comprehensions and quotients. In particular, if \( P \) has full comprehensions, then \( T_d(P) \) has only weak comprehensions, and similarly, if \( P \) has effective quotients, the doctrine \( T_d(P) \) has only a weak form of quotients. Therefore this 2-monad \( T_d \) cannot be lifted to the 2-categories of \( T_c\text{-Alg} \) and \( T_q\text{-Alg} \).

Remark 7.6. Recall from \([3, 6]\) that all the 2-monads \( T_c, T_q \) and \( T_d \) preserve the existential structure of a doctrine, i.e. if \( P \) is an elementary existential then all the doctrines \( T_c(P) \) \( T_q(P) \) and \( T_d(P) \) are elementary and existential. Therefore, if we consider the 2-monad \( T_c \) of the existential completion introduced in \([14, \text{Theorem 1.1}]\), we have that all these 2-monad can be lifted to 2-monads on \( T_c\text{-Alg} \), i.e. the 2-category of elementary and existential doctrines. Hence we can conclude that there are the following distributive laws:
• \( \delta_1 : T_e T_c \rightarrow T_c T_e \);

• \( \delta_2 : T_e T_q \rightarrow T_q T_e \);

• \( \delta_3 : T_c T_d \rightarrow T_d T_e \).

Again this is given by the correspondence between lifting of 2-monads and distributive laws.

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