ADJOINTABLE MONOIDAL FUNCTORS AND QUANTUM
GROUPOIDS

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Abstract. Every monoidal functor \( G : \mathcal{C} \to \mathcal{M} \) has a canonical factorization through the category \( R^e \mathcal{M}_R \) of bimodules in \( \mathcal{M} \) over some monoid \( R \) in \( \mathcal{M} \) in which the factor \( U : \mathcal{C} \to R^e \mathcal{M}_R \) is strongly unital. Using this result and the characterization of the forgetful functors \( \mathcal{M}_A \to R^e \mathcal{M}_R \) of bialgebroids \( A \) over \( R \) given by Schauenburg \[15\] together with their bimonad description given by the author in \[18\] here we characterize the "long" forgetful functors \( \mathcal{M}_A \to R^e \mathcal{M}_R \to \mathcal{M} \) of both bialgebroids and weak bialgebras.

1. Introduction

Takeuchi’s \( \times_R \)-bialgebras \[20\] or, what is the same \[3\], Lu’s bialgebroids \[9\] provide far reaching generalizations of the notion of bialgebra. A bialgebroid \( A \) is, roughly speaking, a bialgebra over some non-commutative \( k \)-algebra \( R \). With noncommutativity of \( R \), however, a new phenomenon appears: the separation of algebra and coalgebra structures into two different categories. While bialgebras are monoids and comonoids in the same category \( \mathcal{M}_k \), bialgebroids are monoids in \( \mathcal{M}_k \) (or in \( R^e \mathcal{M}_R \)) but comonoids in \( R^e \mathcal{M}_R \). This makes the compatibility conditions rather difficult to formulate.

Simplification, if at all, is expected on passing to the level of categories and functors. Let us take, for example, a \( k \)-algebra \( A \) and associate to it the monad \( T = \_ \otimes A \) on \( \mathcal{M}_k \). The monads obtained that way are precisely the monads which have right adjoints. For this characterization of algebras the closed monoidal structure of \( \mathcal{M}_k \) is essential. The Eilenberg-Moore category of "\( T \)-algebras" \[11, 2\], or perhaps better to say, "\( T \)-modules" is nothing but the category \( \mathcal{M}_A \) of right \( A \)-modules equipped with the forgetful functor \( \mathcal{M}_A \to \mathcal{M}_k \). Similarly, we can consider monads on the closed monoidal category \( R^e \mathcal{M}_R \equiv \mathcal{M}_{R^e} \), where \( R^e = R^{op} \otimes R \), with right adjoints. It turns out that these are, up to isomorphisms, precisely the monads \( \_ \otimes R^e \ A \) associated to monoids \( A \) in \( R^e \mathcal{M}_{R^e} \), also called \( R^e \)-rings. The forgetful functor \( U^A : \mathcal{M}_A \to R^e \mathcal{M}_R \) is monadic, has a left adjoint but has no monoidal structure. At this point bialgebroids enter naturally via Schauenburg’s theorem \[15\]: the monoidal structures on \( \mathcal{M}_A \) such that \( U^A \) is strict monoidal are in one-to-one correspondence with (right) bialgebroid structures on the \( R^e \)-ring \( A \).

Monoidal structures on \( \mathcal{M}_A \) can also be described by opmonoidal structures on the monad \( T = \_ \otimes R^e \ A \). In a recent paper \[18\] bialgebroids have been characterized as the bimonads on \( R^e \mathcal{M}_R \) the underlying functors of which have right adjoints. A bimonad, or opmonoidal monad \[12, 10\], is a monad in the 2-category \( \text{Mon}_{op} \text{Cat} \) of...
monoidal categories, opmonoidal functors and opmonoidal natural transformations. More explicitly, a bimonad \( \langle T, \gamma, \pi, \mu, \eta \rangle \) on a monoidal category \( \langle M, \otimes, i \rangle \) consists of

1. an endofunctor \( T : M \to M \)
2. a natural transformation \( \gamma_{x,y} : T(x \otimes y) \to Tx \otimes Ty \)
3. an arrow \( \pi : Ti \to i \)
4. a natural transformation \( \mu_x : T^2 x \to Tx \)
5. and a natural transformation \( \eta_x : x \to Tx \)

such that \( \langle T, \mu, \eta \rangle \) is a monad, \( \langle T, \gamma, \pi \rangle \) is an opmonoidal functor, i.e., a monoidal functor in \( \langle M^{op}, \otimes, i \rangle \), and \( \mu \) and \( \eta \) are opmonoidal natural transformations in the obvious sense. Bimonads, and therefore bialgebroids, too, form a 2-category \( \text{Bmd} \) and \( \text{Bgd} \subset \text{Bmd} \), respectively [18].

The forgetful functors \( U^A : M_A \to R M_R \) of bialgebroids over \( R \) can be characterized as the strong monoidal monadic functors to \( R M_R \) that have right adjoints [18, Corollary 4.16]. In this paper we will study analogue characterizations of the long forgetful functors \( G^A : M_A \to M_k \). This is motivated by situations where the base algebra \( R \) is not given a priori. It is also closer to the classical Tannaka-Krein situation where one reconstructs the "grouplike" object \( A \) as the set of natural transformations \( G^A \to G^A \). Apart from set theoretical controversies (\( M_A \) is not small, which will be compensated by assuming the existence of left adjoints for our functors) this reconstruction is possible for the long forgetful functor \( G^A \) but not for the short forgetful functor \( U^A \). Of course, the novelty is that now the long forgetful functor is not strong monoidal.

To recover \( R \) from \( G^A \) is in fact very easy. One takes the image of the unit object (the trivial \( A \)-module) under \( G^A \). Since the unit object is always a monoid, it is mapped by the monoidal forgetful functor to a monoid in \( M_k \). This gives us the algebra \( R \). This construction is possible for any monoidal functor and a closer look will show in Section 2 that, under mild assumptions on \( M \), every monoidal functor \( G : C \to M \) can be factorized as \( C \overset{U}{\longrightarrow} R M_R \to M \) with \( U \) monoidal but strictly unital.

The monoidal functors \( G \) for which \( U \) is strong monoidal will be called essentially strong monoidal. Clearly, the \( G^A \) of a bialgebroid is an example of such functors. Finding the extra conditions on an essentially strong monoidal functor \( G : C \to M \) that makes it (1) either factorize through the long forgetful functor \( G^A \) of a unique bialgebroid (2) or become isomorphic to such a \( G^A \) is part of a Tannaka duality program for bialgebroids. This has been carried out for "short" forgetful functors in [18]. This type of duality theory uses monad theory to characterize the large module categories of quantum groupoids together with their forgetful functors. With the results of the present paper we make some small steps in the direction of extending Tannaka theory from strong monoidal to monoidal functors. As for the state of the art of the traditional method we have to mention the recent papers by Phùng Hô Hai [14] and another one by Hayashi [6] which prove Tannaka duality theorems for Hopf algebroids and for face algebras, respectively. In their approach, as in that of Saavedra-Rivano, Deligne, Ulbrich and others (see [13, 7]) small categories are equipped with strong monoidal functors to a (sometimes rigid) category of bimodules and the task is to find a universal factorization through the comodule category of a quantum groupoid.
The organization of the paper is as follows. In Section 2 we prove the canonical factorization of general monoidal functors through a bimodule category. After touching the general case of long forgetful functors of bimonads in Section 3 we determine a class of essentially strong monoidal functors in Section 4 which factorize through the $G^A$ of a bialgebroid. Although we present the proof over the base category $\mathcal{M}_k$, $\text{Ab}$ or $\text{Set}$, it is often indicated how it could be extended to a general base category $\mathcal{V}$. In this way we intend to make playground for exotic examples of bialgebroids. Then in Section 5 the long forgetful functors of bialgebroids are characterized up to equivalence. Finally, in Section 6 the special case of weak bialgebroids. Then in Section 5 the long forgetful functors of bialgebroids are characterized up to equivalence. Finally, in Section 6 the special case of weak bialgebras are considered, now over $\mathcal{M}_k$, the long forgetful functors of which can be recognized as those that have both monoidal and opmonoidal structures and these two obey compatibility conditions that can be called a separable Frobenius structure on the forgetful functor. This characterization of weak bialgebra forgetful functors was already sketched in [17] calling them "split monoidal" functors.

2. The Canonical Factorization of Monoidal Functors

Let $\mathcal{C}$ be a monoidal category with monoidal product $\square : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and unit object $e \in \mathcal{C}$. Then we have coherent natural isomorphisms $a_{a,b,c} : a \square (b \square c) \sim (a \square b) \square c$, $l_c : e \square c \sim c$ and $r_c : c \square e \sim c$ satisfying $l_e = r_e$, the triangle and the pentagon identity. A monoid $\langle m, \mu, \eta \rangle$ in $\mathcal{C}$ is an object $m$ together with arrows $\mu : m \square m \to m$, $\eta : e \to m$ satisfying associativity and unit axioms. There is always a canonical monoid: the unit object $e$ equipped with multiplication $l_e : e \square e \to e$ and unit the identity arrow $e : e \to e$. Moreover, every object $c$ of $\mathcal{C}$ is a bimodule over this canonical monoid via the actions $l_c : e \square c \to c$ and $r_c : c \square e \to c$. The bimodule axioms follow simply from recognizing that the three associativity axioms

\begin{align*}
(2.1) \quad & l_c \circ (e \square l_c) = l_c \circ (l_c \square c) \circ a_{e,e,c} \\
(2.2) \quad & l_c \circ (e \square r_c) = r_c \circ (l_c \square e) \circ a_{e,e,e} \\
(2.3) \quad & r_c \circ (r_c \square e) = r_c \circ (c \square l_c) \circ a^{-1}_{e,e} 
\end{align*}

are consequences of special cases of the triangle diagrams valid in any monoidal category while the unit axioms become identities. In this sense every monoidal category is a category of bimodules. The more precise statement will be clear after applying the Theorem below to the identity functor of $\mathcal{C}$. Let us recall an important property of monoidal functors: They map monoids to monoids and (bi)modules to (bi)modules.

**Lemma 2.1.** Let $\langle G, G_2, G_0 \rangle : \langle \mathcal{C}, \square, e \rangle \to \langle \mathcal{M}, \otimes, i \rangle$ be a monoidal functor.

1. If $\langle m, \mu, \eta \rangle$ in $\mathcal{C}$ is a monoid in $\mathcal{C}$ then $\langle Gm, G\mu \circ G_{m,m}, G\eta \circ G_0 \rangle$ is a monoid in $\mathcal{M}$.
2. Let $m$ and $n$ be monoids in $\mathcal{C}$. If $\langle b, \lambda, \rho \rangle$ is an $m$-$n$ bimodule in $\mathcal{C}$ then the triple $\langle Gb, G\lambda \circ G_{m,b}, G\rho \circ G_{b,n} \rangle$ is a $Gm$-$Gn$ bimodule in $\mathcal{M}$.

**Proof.** (1) is well-known and can be found e.g. in [10]. (2) is also known to many authors although an explicit proof is difficult to find. Just to advertise the statement.
we compute here commutativity of the left and right actions:

\[
\lambda^\prime \circ (Gm \otimes \rho^\prime) = G\lambda \circ G_{m,b} \circ (Gm \otimes G\rho) \circ (Gm \otimes G_{b,n}) \\
= G\lambda \circ G(m \square \rho) \circ G_{m,b \square n} \circ (Gm \otimes G_{b,n}) \\
= G\rho \circ G(\lambda \square n) \circ G_{a_m,b,n} \circ G_{m,b} \otimes G(n) \circ a_{Gm,Gb,Gn} \\
= G\rho \circ G_{b,n} \circ (G\lambda \otimes Gn) \circ (G_{m,b} \otimes Gn) \circ a_{Gm,Gb,Gn} \\
= \rho^\prime \circ (\lambda^\prime \otimes Gn) \circ a_{Gm,Gb,Gn}
\]

\[\square\]

Dually, opmonoidal functors map comonoids to comonoids and (bi)comodules to (bi)comodules.

So far \(\mathcal{M}\) was an arbitrary monoidal category. In order for the category \(m,\mathcal{M}_m\) of bimodules in \(\mathcal{M}\) over a monoid \(m\) in \(\mathcal{M}\) to have a monoidal structure we need the assumptions that \(\mathcal{M}\) has coequalizers and the tensor product \(\otimes\) preserves coequalizers in both arguments. This is because the tensor product over \(m\) of a right \(m\)-module \(\rho_x : x \otimes m \to x\) with a left \(m\)-module \(\lambda_y : m \otimes y \to y\) is a coequalizer

\[
\begin{array}{ccc}
x \otimes (m \otimes y) & \xymatrix{\ar@{.>}[r]|-!{[ur];[dr]}
\ar^-{\otimes \lambda_y} [rrr] & x \otimes y & \ar^{m} \to & x \otimes_m y
}
x \otimes (m \otimes y) & \xymatrix{\ar@{.>}[r]|-!{[ur];[dr]}
\ar^-{\otimes \rho_x} [rrr] & x \otimes y & \ar^{m} \to & x \otimes_m y
}
\end{array}
\]

The construction of the monoidal product \(\otimes_m\) on \(m,\mathcal{M}_m\) together with coherence isomorphisms is a long but standard procedure. At the end one obtains a monoidal category \(\langle m,\mathcal{M}_m, \otimes_m, m \rangle\) together with a monoidal forgetful functor \(\Gamma_m : m,\mathcal{M}_m \to \mathcal{M}\) sending the bimodule \(\langle x, \lambda, \rho \rangle\) to its underlying object \(x\). The monoidal structure

\[
\begin{align}
\Gamma_{x,y}^m : \Gamma_m^m(x) \otimes \Gamma_m^m(y) & \to \Gamma_m^m(x \otimes_m y) \\
\Gamma_0^m : i & \to \Gamma_m^m(m)
\end{align}
\]

is provided by the chosen coequalizers and by the unit \(i \to m\) of the monoid \(m\), respectively.

If \(\sigma : m \to n\) is a monoid morphism then there is a functor \(\Gamma^\sigma : n,\mathcal{M}_n \to m,\mathcal{M}_m\) mapping the bimodule \(\langle x, \lambda, \rho \rangle\) to the bimodule \(\langle x, \lambda \circ (\sigma \otimes x), \rho \circ (x \otimes \sigma) \rangle\). So, \(\Gamma^\sigma \Gamma^m = \Gamma^n\). This defines a functor \(\Gamma : \text{Mon} \mathcal{M} \to \text{MonCat}/\mathcal{M}\).

**Theorem 2.2.** Let \(\langle \mathcal{M}, \otimes, i \rangle\) be a monoidal category with coequalizers and such that \(\otimes\) preserves coequalizers in both arguments. If \(\langle \mathcal{C}, \square, e \rangle\) is a monoidal category and \(G : \mathcal{C} \to \mathcal{M}\) is a monoidal functor then there is a monoid \(\langle R, \mu^R, \eta^R \rangle\) in \(\mathcal{M}\) and a strictly unital monoidal functor \(U : \mathcal{C} \to \mathcal{R} \mathcal{M}_R\) such that

1. \(G\), as a monoidal functor, can be factorized as \(\Gamma^R U\), i.e.,

\[
\begin{align}
G &= \Gamma^R U \\
G_{a,b} &= \Gamma^R U_{a,b} \circ \Gamma^R_{U_{a,b}} \\
G_0 &= \Gamma^R U_0 \circ \Gamma^R_0
\end{align}
\]

2. \(\Gamma R \Gamma^m = \Gamma^n\). This defines a functor \(\Gamma : \text{Mon} \mathcal{M} \to \text{MonCat}/\mathcal{M}\).
(2) If $S$ is a monoid in $\mathcal{M}$ and $V : \mathcal{C} \to \mathcal{M}_S$ is a monoidal functor such that $G = \Gamma^S V$, as monoidal functors, then there exists a unique monoid morphism $\sigma : S \to R$ such that $V = \Gamma^\sigma U$.

**Proof.** By Lemma 2.1 the image under $G$ of the unit monoid $e$ is a monoid $R = \langle Ge, \mu^R, \eta^R \rangle$ with underlying object $Ge$. Also by the Lemma, every object $c$ in $\mathcal{C}$, as an $e$-$e$-bimodule, is mapped by $G$ to the $R$-$R$-bimodule

$$Uc := \langle Gc, \lambda_{Uc}, \rho_{Uc} \rangle$$

where

$$\lambda_{Uc} := Gc \otimes Gc \xrightarrow{G_e \otimes G_c} G(e \square c) \xrightarrow{G_{\square}} Gc$$

$$\rho_{Uc} := Gc \otimes Ge \xrightarrow{G_e \otimes G_c} G(c \square e) \xrightarrow{G_{\square}} Gc$$

Since these actions are natural in $c$, the $G\tau$ of every arrow $\tau : c \to d$ lifts to a bimodule morphism $U\tau : Uc \to Ud$. This defines the functor $U$ which obviously satisfies (2.7). In order to define a monoidal structure for $U$ notice that

$$Ga \otimes \rho_{Gb} \circ a_{Ga,Gc,Gb} = Ga \otimes (\lambda_{Ub})$$

holds true as a consequence of the hexagon of $G_2$. Therefore, universality of the coequalizer implies the existence of a unique arrow $Ga \otimes R Gb \to G(a \square b)$ such that

$$Ga \otimes \rho_{Gb} \circ a_{Ga,Gc,Gb} = Ga \otimes (\lambda_{Ub})$$

commutes. Moreover, this new arrow lifts to a bimodule morphism $U_{a,b}$ since the other two in this diagram also lift to $\mathcal{M}_R$ and $\otimes$ preserves coequalizers. Setting $U_0 : \mathcal{M}_R \to Uc$ to be the identity arrow we obtain a monoidal functor $U : \mathcal{C} \to \mathcal{M}_R$ for which (2.8) and (2.9) hold and the unit of which, $U_0$, is an identity arrow, i.e., it is strictly unital. This proves property (1). To prove the universal property (2) we start with uniqueness. If $\sigma$ exists such that $\Gamma^\sigma U = V$ is a factorization in $\text{MonCat}$ then, in particular, $V_0 = \Gamma^\sigma U_0 \circ \Gamma_0^\sigma$. Since $U$ is strict unital, $\Gamma^S V_0 = \Gamma^S \Gamma_0^S$. This means that the underlying arrow in $\mathcal{M}$ of the bimodule morphism $\Gamma_0^S : S \to \text{coequalizer of } R\sigma(S)$ is uniquely determined by that of $V_0$. But this bimodule morphism is the same as the monoid morphism $\sigma : S \to R$ (as arrows of $\mathcal{M}$). To prove existence we therefore define $\sigma := \Gamma^S V_0 : \Gamma^S S \to \Gamma^R R$, for the time being as an arrow in $\mathcal{M}$. It preserves the unit due to the monoidal factorization,

$$\sigma \circ \eta^S = \Gamma^S V_0 \circ \Gamma_0^S = G_0 = \Gamma_0^R = \eta^R$$
and it is multiplicative because
\[ \mu^R \circ (\sigma \otimes \sigma) = G_\epsilon \circ G_{\epsilon,c} \circ (\Gamma^S V_0 \otimes \Gamma^S V_0) \]
\[ = \Gamma^S \lambda \circ \Gamma^S \nu \circ (\Gamma^S V_0 \otimes \Gamma^S V_0) \]
\[ = \Gamma^S \lambda \circ \Gamma^S \nu \circ \Gamma^S (V_0 \otimes S V_0) \circ \Gamma^S S \]
\[ = \Gamma^S \lambda \circ \Gamma^S (S \otimes S V_0) \circ \Gamma^S S = \Gamma^S V_0 \circ \Gamma^S \lambda \circ \Gamma^S S \]
\[ = \sigma \circ \mu^S . \]

So \( \sigma \) lifts to a monoid morphism \( S \to R \). The proof of \( \Gamma^\sigma U = V \) requires to show that \( \lambda_{Uc} \circ (\sigma \otimes Gc) = \lambda_{Vc} \) and \( \rho_{Uc} \circ (Gc \otimes \sigma) = \rho_{Vc} \) for all object \( c \) in \( C \). E.g.,
\[ \lambda_{Vc} = \Gamma^S \lambda \circ \Gamma^S \nu \circ \Gamma^S S \]
\[ = \Gamma^S \lambda \circ \Gamma^S \nu \circ \Gamma^S (V_0 \otimes S V_0) \circ \Gamma^S S \]
\[ = G_\epsilon \circ \Gamma^S \nu \circ \Gamma^S (V_0 \otimes Gc) \]
\[ = G_\epsilon \circ G_{\epsilon,c} \circ (\Gamma^S V_0 \otimes Gc) \]
\[ = \lambda_{Uc} \circ (\sigma \otimes Gc) . \]

In order to have \( \Gamma^\sigma U = V \) in \textbf{MonCat} we still have to show \( V_{a,b} = \Gamma^\sigma U_{a,b} \circ \Gamma^S_{U_{a,b}} \) and \( V_0 = \Gamma^\sigma U_0 \circ \Gamma^S_0 \). The latter follows directly from the definition of \( \sigma \) while the former can be shown by applying the faithful \( \Gamma^S \) on both hand sides and then composing with the coequalizer \( \Gamma^S_{V_{a,b}} \) from the right. Since a coequalizer is epi, it is right cancellable and the statement follows from [21.3].

We call \( G = \Gamma U \) the canonical factorization of the monoidal functor \( G \).

**Definition 2.3.** A monoidal functor \( G \) is called essentially strong monoidal if the \( U \) in its canonical factorization \( G = \Gamma U \) is strong monoidal.

Equivalently, \( G \) is essentially strong monoidal if for all objects \( a, b \) of \( C \) the arrow \( G_{a,b} \) is a coequalizer in the diagram
\[
\begin{array}{ccc}
Ga \otimes (Ge \otimes Gb) & \overset{Ga \otimes \lambda_{Ub}}{\rightarrow} & Ga \otimes Gb \\
\downarrow t & \quad & \downarrow G_{a,b} \\
(Ga \otimes Ge) \otimes Gb & \overset{\rho_{Ua} \otimes Gb}{\rightarrow} & Ga \otimes Gb \\
\end{array}
\]

where \( \lambda_{Uc}, \rho_{Uc} \) denote the left and right actions \( [21.1] \) and \( [21.2] \) on \( Gc \).

3. THE RELATION WITH THE EILENBERG-MOORE CONSTRUCTION

Let us recall some basic facts about monad theory [11][2]. Any functor \( G : C \to \mathcal{M} \) with a left adjoint \( F \) determines a monad \( \mathbb{M} = \langle GF, G\varepsilon F, \eta \rangle \) on \( \mathcal{M} \) where \( \varepsilon : FG \to C \) is the counit and \( \eta : M \to GF \) is the unit of the adjunction. The Eilenberg-Moore construction associates to any monad \( \mathbb{M} \) on \( \mathcal{M} \) a category \( \mathcal{M}^\mathbb{M} \) the objects of which are the \( \mathbb{M} \)-algebras \( \langle x, \alpha \rangle \) where \( x \) is an object of \( \mathcal{M} \) and \( \alpha : \mathbb{M} x \to x \) is an arrow of \( \mathcal{M} \) satisfying \( \alpha \circ \mu_x = \alpha \circ \mu_x \) and \( \alpha \circ \eta_x = x \). The arrows of \( \mathcal{M}^\mathbb{M} \) from \( \langle x, \alpha \rangle \) to \( \langle y, \beta \rangle \) are the arrows \( \tau : x \to y \) in \( \mathcal{M} \) for which \( \tau \circ \alpha = \beta \circ M\tau \).

The forgetful functor \( U^\mathbb{M} : \mathcal{M}^\mathbb{M} \to \mathcal{M} \) sending \( \langle x, \alpha \rangle \) to \( x \) has a left adjoint such that the monad associated to this adjunction is precisely the original \( \mathbb{M} \).
Any functor $G: C \to M$ with a left adjoint can be factorized as $G = U^R K$, where $M$ is the monad of the adjunction, with a unique $K: C \to M^R$, called the comparison functor. Explicitly,

(3.16) $\psi K (\langle Gc, G\varepsilon_c \rangle) \mapsto Gc.$

If $K$ is an equivalence of categories the $G$ is called monadic.

Now assume that we have a monoidal functor $G: C \to M$ with the underlying ordinary functor having a left adjoint $F$. We briefly say that $G$ is a right adjoint monoidal functor. Then we have two factorizations of $G$, the $G = \Gamma U$ provided by Theorem 2.2 and $G = U^R K$ provided by the Eilenberg-Moore construction. In this situation one expects a relation between $M$-algebras and $R$-$R$-bimodules.

Every monoid in $M$, therefore $R$ too, provides two monads $L = R \otimes -$ and $R = - \otimes R$. We can define two monad morphisms

(3.17) $\lambda^R: L \to M \quad \lambda^R_x = G(1_{Fx}) \circ G\varepsilon_{Fx} \circ (R \otimes \eta_x)$

(3.18) $\rho^R: R \to M \quad \rho^R_x = G(\eta_{Fx}) \circ GF_{x,e} \circ (\eta_x \otimes R)$

They commute in the sense of the diagram

\[
\begin{array}{ccc}
LR & \xrightarrow{\lambda^R} & M^2 \\
\downarrow & & \downarrow \\
RL & \xrightarrow{\rho^R} & M^2
\end{array}
\]

where the vertical isomorphism can be obtained from the associator as $a_{R,-,R}$.

Lemma 3.1. If $\langle x, \alpha \rangle$ is a $R$-algebra then $\langle x, \alpha \circ \lambda^R_x, \alpha \circ \rho^R_x \rangle$ is an $R$-$R$-bimodule.

This provides the object map of a functor $\Psi: M^R \to R M_R$. It is easy to show that if $\tau: \langle x, \alpha \rangle \to \langle y, \beta \rangle$ is a $R$-algebra morphism then it is also an $R$-$R$-bimodule morphism. This defines a forgetful (monadic) functor $\Psi: M^R \to R M_R$. Moreover, the Eilenberg-Moore comparison functor $K: C \to M^R$ satisfies $\Psi K = U$. Thus we can factorize the original functor $G$ in 3 steps

(3.20) $C \xrightarrow{K} M^R \xrightarrow{\Psi} R M_R \xrightarrow{\Gamma} M$

Unfortunately, only the third functor is monoidal. In order to make it a diagram in $\text{MonCat}$ we need at least a monoidal structure on the Eilenberg-Moore category $M^R$. This could be achieved under the stronger assumption that $U$ is strong monoidal and has a left adjoint. Namely, [13, Theorem 2.8] yields the following result:

Let $U: C \to T$ be a strong monoidal right adjoint functor. Then

- its monad $T$ is opmonoidal, i.e., it is a monad on $T$ in the 2-category $\text{Mon}_{\text{op}} \text{Cat}$;
- the category $T^T$ of $T$-algebras has a unique monoidal structure such that the Eilenberg-Moore forgetful functor $U^T: T^T \to T$ is strict monoidal;
- $U = U^T K$ with a strong monoidal comparison functor $K: C \to T^T$.

Theorem 3.2. Assume that

1. $\langle C, \Box, e \rangle$ is a monoidal category where $C$ is complete, well powered and has a small cogenerating set;
(2) \( (\mathcal{M}, \otimes, i) \) is a monoidal category where \( \mathcal{M} \) has coequalizers and \( \otimes \) preserves them;
(3) \( (G, G_2, G_0) \) is an essentially strong monoidal functor \( C \to \mathcal{M} \) with \( G \) having a left adjoint.

Then there is a monoid \( R \) in \( \mathcal{M} \) and an opmonoidal monad \( T \) on \( R \mathcal{M} R \) such that

\[
\begin{array}{ccc}
C & \xrightarrow{K} & (R \mathcal{M} R)^T \\
\downarrow{G} & & \downarrow{U^T} \\
\mathcal{M} & \xleftarrow{\Gamma} & R \mathcal{M} R
\end{array}
\]

is commutative in \( \text{MonCat} \) where \( K \) is the strong monoidal comparison functor, \( U^T \) is the strict monoidal forgetful functor of \( T \)-algebras and \( \Gamma \) is the monoidal forgetful functor of \( R \)-\( R \)-bimodules.

**Proof.** By assumption (2) the canonical factorization \( G = \Gamma U \) through \( R \mathcal{M} R \) exists. Since \( G \) is right adjoint, it preserves limits. But \( \Gamma \), being the forgetful functor of the monad \( R \otimes - \otimes R \) on \( \mathcal{M} \), is monadic therefore it creates limits. Therefore \( U \) preserves limits, too. By the Special Adjoint Functor Theorem [11, Corollary V.8] assumption (1) ensures that \( U \) has a left adjoint. Now using assumption (3) the \( U \) is strong monoidal right adjoint, therefore, it factorizes monoidally through the Eilenberg-Moore category of its bimonad \( T \) by the above quoted [18, Theorem 2.8].

The 3 step factorization found in the above Theorem describes what can be expected in general for continuous monoidal functors. Of course, it would be more interesting to replace the "abstract quantum groupoid" \( T \) with a concrete bialgebroid, let us say. If \( \mathcal{M} \) is the category of \( k \)-modules over a commutative ring then a necessary and sufficient condition for \( (R \mathcal{M} R)^T \) to be the monoidal category \( A \mathcal{M} \) of modules over a bialgebroid \( A \) was given in [18, Theorem 4.5]. The condition is very simple: the underlying functor \( T \) of the bimonad should have a right adjoint. Unfortunately, it is difficult to find a condition on \( G \) that guarantees a right adjoint for \( U \). In the next Section we will study a special case which allows to do so.

We note that even under the conditions of the Theorem the underlying monad of \( T \) may not be \( \mathcal{M} \), so the factorizations \( 3 \) and \( 2 \) may be different.

4. Monoidal \( V \)-functors to \( V \)

Now we turn to replace bimonads with bialgebroids. The key observation is that every monoid \( A \) in \( \mathcal{M} \) determines a monad \( A \otimes - \) on \( \mathcal{M} \) and - under certain conditions on \( \mathcal{M} \) - these monads are precisely the monads the underlying endofunctor of which has a right adjoint. The Eilenberg-Moore categories \( A \mathcal{M} \) of such monads are precisely the categories \( A \mathcal{M} \) of modules over \( A \). This can be generalized to \( (k\text{-linear}) \) bimonads as follows.

**Theorem 4.1.** [18, Thm. 4.5] Let \( \mathcal{M} = \mathcal{M}_k \) be the category of modules over a commutative ring \( k \), \( R \) be a \( k \)-algebra and \( T \) be a \( k \)-linear bimonad on \( R \mathcal{M} R \). Then there exists a bialgebroid \( A \) in \( \mathcal{M} \) over \( R \) and an isomorphism \( T \cong A \otimes - \) if and only if \( T \colon \mathcal{M} \to \mathcal{M} \) has a right adjoint. Moreover, \( T \) has a right adjoint if and only if \( U^T \colon \mathcal{M}^T \to \mathcal{M} \) has a right adjoint.
We want to give an analogue characterization of the long forgetful functors $G: \mathcal{A}\mathcal{M} \to \mathcal{M}$ of bialgebroids. Clearly, these functors have both left and right adjoints - the induction and coinduction functors - and the factorization through $R\mathcal{M}_R$ shows that they are essentially strong monoidal. Thus we expect that these properties of a functor leads to the construction of a bialgebroid with Tannaka duality.

We let $\mathcal{V}$ denote either $k\mathcal{M}$, $\mathbb{A}b$ or $\text{Set}$ and work with $\mathcal{V}$-monoidal $\mathcal{V}$-categories and $\mathcal{V}$-functors between them that have $\mathcal{V}$-adjoints [3]. Due to the fact that $\mathcal{V}$ is not only symmetric monoidal closed but its monoidal unit is a generator, we can work with the underlying ordinary categories and consider $\mathcal{V}$-functors as a special class of ordinary functors that preserve some extra structure that is encoded in the choice of $\mathcal{V}$. The set of $\mathcal{V}$-natural transformations between $\mathcal{V}$-functors is the same (under the 2-functor $\text{Hom}_\mathcal{V}(i, \_)$) as the set of ordinary natural transformations. So the complicated formalism of enriched categories can be avoided and proofs of commutativity of diagrams in $\mathcal{V}$ can be done by elements. (An element of an object $v \in \mathcal{V}$ is an arrow $i \to v$ in $\mathcal{V}$.) In a $\mathcal{V}$-category $\mathcal{C}$ we denote by $\mathcal{C}(a,b) \in \mathcal{V}$ its hom-objects and by $\text{Hom}_{\mathcal{C}}(a,b) := \text{Hom}_\mathcal{V}(i, \mathcal{C}(a,b))$ its hom-sets. Otherwise it should be clear from the context whether we speak about the $\mathcal{V}$-category $\mathcal{C}$ or its underlying ordinary category. $\mathcal{V}$ itself is a $\mathcal{V}$-category with $\mathcal{V}(x,y)$ being the internal hom object $\text{hom}(x,y)$ which is defined by $\text{Hom}_\mathcal{V}(v \otimes x,y) \cong \text{Hom}_\mathcal{V}(v, \text{hom}(x,y))$.

Our base category $\mathcal{V}$ is also complete which allows to define the Takeuchi $\times_R$-product as a pullback of equalizers. Existence of coequalizers is also needed in $\mathcal{V}$ if we want to apply Theorem 2.2. Readers interested in more general enrichments than the three cases mentioned above can take for $\mathcal{V}$ any complete category with coequalizers which is endowed with a symmetric closed monoidal structure $\langle \mathcal{V}, \otimes, i \rangle$ such that $i$ a projective generator.

We restrict ourselves to study functors with target category being the base category, i.e., $G$ is a $\mathcal{V}$-functor to $\mathcal{V}$. In this situation the existence of a left $\mathcal{V}$-adjoint $F \dashv G$ implies that $G : \mathcal{C} \to \mathcal{V}$ is representable:

$$Ga \xrightarrow{\sim} \text{hom}(i,Ga) \xrightarrow{\sim} \mathcal{C}(Fi,a)$$

for $a \in \mathcal{C}$. Thus without loss of generality we may assume that $G$ is a hom-functor $G = \mathcal{C}(g,\_)$.

If such a $G$ has a monoidal structure then - by the Yoneda Lemma - a comonoid structure $\langle g, \gamma, \pi \rangle$ on $g$ arises via

$$G : \mathcal{C} \to \mathcal{V} \quad \quad G = \mathcal{C}(g,\_)$$

$$G_{a,b} : Ga \otimes Gb \to G(a \square b) \quad \quad G_{a,b}(\alpha \otimes \beta) = (\alpha \square \beta) \circ \gamma$$

$$G_0 : i \to Ge \quad \quad G_0 = \pi$$

where in the last equation one can recognize the usual identification of elements of a hom-object of a $\mathcal{V}$-category with arrows in $\mathcal{V}$.

The monoid $R$ in the canonical factorization $G = \Gamma U$ through $R\mathcal{V}_R$ is nothing but the convolution monoid $\mathcal{C}(g,e)$ with multiplication

$$\rho \ast \rho' = L_e \circ (\rho \otimes \rho') \circ \gamma, \quad \rho, \rho' \in R$$

and unit $\pi$.

We note also that every object $Ga = \mathcal{C}(g,a) \in \mathcal{V}$ has a natural right $A$-module structure where $A := \mathcal{C}(g,g)$ is the endomorphism monoid in $\mathcal{V}$. So $G$ factorizes
through the forgetful functor $\mathcal{V}_A \to \mathcal{V}$. This is compatible with the canonical factorization $G = \Gamma U$ because we have monoid morphisms

\begin{align}
(4.27) & \quad s: R \to A \quad s(\rho) := r_g \circ (g \sqcap \rho) \circ \gamma \\
(4.28) & \quad t: R^{op} \to A \quad t(\rho) := 1_g \circ (\rho \sqcup g) \circ \gamma
\end{align}

Corresponding to the diagram $i \to R^{op} \otimes R \xrightarrow{\delta} A$ in $\text{Mon}\mathcal{V}$ there is the diagram $\mathcal{V}_A \to R\mathcal{V}_R \to \mathcal{V}$ in $\mathcal{V}\text{-Cat}$.

**Theorem 4.2.** Denoting by $\mathcal{V}$ either $\mathcal{M}_k$, or $\text{Ab}$ or $\text{Set}$ let $\mathcal{C}$ be a $\mathcal{V}$-monoidal $\mathcal{V}$-category and $G: \mathcal{C} \to \mathcal{V}$ be a representable essentially strong monoidal $\mathcal{V}$-functor. Then there exists a monoid $R$ in $\mathcal{V}$, a right bialgebroid $A$ in $\mathcal{V}$ over $R$, a strong monoidal $K: \mathcal{C} \to \mathcal{V}_A$ and a monoidal natural isomorphism $G \xrightarrow{\sim} \Gamma R \Phi^A K$,

\begin{align}
(4.29) & \quad \mathcal{C} \xrightarrow{K} \mathcal{V}_A \\
& \quad G \downarrow \cong \downarrow \Phi^A \\
& \quad \mathcal{V} \xleftarrow{\Gamma R} R\mathcal{V}_R
\end{align}

where $\Phi^A$ denotes the strict monoidal forgetful functor of the bialgebroid $A$.

**Proof.** As we have explained above, there exists a representing comonoid $\langle g, \gamma, \pi \rangle$ in $\mathcal{C}$ and a monoidal natural isomorphism $G \xrightarrow{\sim} C(g, \_)$ in $\mathcal{C}$. As a $\mathcal{V}$-functor, $C(g, \_)$ factorizes as

\begin{align}
(4.30) & \quad \mathcal{C} \xrightarrow{K} \mathcal{V}_A \\
& \quad C \xrightarrow{\Phi} R\mathcal{V}_R \xrightarrow{\Gamma} \mathcal{V}
\end{align}

where $A = C(g, g)$ is the endomorphism monoid, $R = C(g, e)$ is the convolution monoid and identifying $R\mathcal{V}_R$ with $\mathcal{V} R e$ where $R e = R^{op} \otimes R$ - $\Phi$ is the $\mathcal{V}$-functor that forgets along $t \otimes s: R e \to A$. By the very definition of $R$ we see that the canonical factorization $\Gamma U$ of $C(g, \_)$ leads to $U = \Phi K$ which is strong monoidal, hence opmonoidal. Since opmonoidal functors map comonoids to comonoids, the triple

\begin{align}
(4.31) & \quad C = U g \equiv \langle A, R \otimes A \xrightarrow{\sim} A \otimes R \xrightarrow{\Delta^\ast} A \otimes A, A \otimes R \xrightarrow{\mu} A \otimes A \xrightarrow{\mu} A \rangle \\
(4.32) & \quad \delta = \left( C \xrightarrow{U \gamma} U(g \sqcap g) \xrightarrow{U \delta^\ast} C \otimes R C \right) \\
(4.33) & \quad \varepsilon = \left( C \xrightarrow{U \pi} U e \xrightarrow{U \varepsilon^\ast} R R e \right)
\end{align}

is a comonoid in $R\mathcal{V}_R$. This comonoid allows to define a monoidal structure on $\mathcal{V}_A$ such that $\Phi$ becomes strict monoidal. As a matter of fact, let $X$ and $Y$ be right $A$-modules in $\mathcal{V}$ and define the $A$-module $X \square A Y$ as the object $X \otimes_R Y$ with $A$-action

\begin{align}
(4.34) & \quad (x \otimes_R y) \triangleleft \alpha := (x \triangleleft \alpha^{(1)}) \otimes_R (y \triangleleft \alpha^{(2)}), \quad x \in X, y \in Y, \alpha \in A.
\end{align}

But in order for this action to be well-defined we have to show that $\delta(\alpha), \alpha \in A$, belong to the subbimodule $C \times_R C \subset C \otimes_R C$ defined as the intersection of the
small set of equalizers \{e_ρ \mid ρ \in R\} where

\[
(4.35) \quad U_s(ρ) \otimes_R C \xrightarrow{ε} C \otimes_R C \xrightarrow{ρ} C \otimes_R C
\]

Here \(U_s(ρ) = s(ρ) \circ \bullet\) and \(Ut(ρ) = t(ρ) \circ \bullet\) so they are \(R\)-\(R\)-endomorphisms of \(C\).

Naturality of \(U_2\), the definition of \(δ\) and some elementary monoidal calculus yields the following identity in \(R\)\(\mathcal{V}\),

\[
U_{g,g} \circ (U_s(ρ) \otimes_R Ug) \circ δ = U(s(ρ) \Box g) \circ U_{g,g} \circ δ
\]

\[
= U(s(ρ) \Box g) \circ Uγ
\]

\[
= U((r_g \Box g) \circ ((g \Box ρ) \Box g) \circ (γ \Box g) \circ γ)
\]

\[
= U((g \Box t(ρ)) \circ (g \Box ρ \Box g)) \circ (g \Box γ) \circ γ)
\]

\[
= U(g \Box t(ρ)) \circ Uγ
\]

\[
= U_{g,g} \circ (Ug \otimes_R Ut(ρ)) \circ δ.
\]

Since \(U_{g,g}\) is invertible, \(δ\) restricts to a map \(A \to A \times_R A\). Notice that \(A \times_R A\) - as an object in \(\mathcal{V}\) - inherits a monoid structure from that of \(A \otimes A\) and then \(δ\) becomes a morphism of monoids. As a matter of fact,

\[
U_{g,g}(δ(α)δ(β)) = (α_1 \Box α_2) \circ γ \circ β
\]

\[
= γ \circ α \circ β
\]

\[
= U_{g,g}(δ(α \circ β))
\]

for \(α, β \in A\). Unitality of \(δ\) is obvious. This finishes the definition of \(\Box_A\) and the associativity coherence isomorphism \(a\) of \(\mathcal{V}_A\) as a lift of \(a\) of \(R\)\(\mathcal{V}_R\). Whether it has a unit object depends on the counit properties.

\[
ε(sε(α)) \circ β = π \circ sε(α) \circ β
\]

\[
= π \circ r_g \circ (g \Box ε(α)) \circ γ \circ β
\]

\[
= r_ε \circ (π \Box ε(α)) \circ γ \circ β = ε(α \circ β) = π \circ α \circ β
\]

and similarly, \(ε(tε(α)) \circ β = ε(α \circ β)\) holds for all \(α, β \in A\). Unitality of \(ε\) is obvious since \(ε(g) = π\) is the unit element of \(R\). Then the monoidal unit \(E\) of \(\mathcal{V}_A\) becomes \(\langle R, R \otimes A \xrightarrow{s \otimes A} A \otimes A \xrightarrow{μ} A \xrightarrow{ε} R\rangle\) and the unit coherence isomorphisms \(I\) and \(r\) of \(R\)\(\mathcal{V}_R\) lift to become the \(I\) and \(r\) of \(\mathcal{V}_A\), respectively.

This finishes the proof of that \(A\) is a bialgebroid together with the construction of a monoidal structure on \(\mathcal{V}_A\) such that \(Φ\) is strict monoidal. Since \(U\) is strong and \(Φ\) reflects isomorphisms, it follows that \(K\) is strong monoidal.

\[\square\]

5. Characterizing long forgetful functors of bialgebroids

Recall that for every closed monoidal category \(\mathcal{V}\) there is a monoidal equivalence (of ordinary categories)

\[
(5.36) \quad \mathcal{V} \simeq \text{L-}\mathcal{V}\text{-Fun}(\mathcal{V}, \mathcal{V}), \quad ν \mapsto \_ \otimes ν
\]

of \(\mathcal{V}\) with the strict monoidal category of left adjoint \(\mathcal{V}\)-functors \(\mathcal{V} \to \mathcal{V}\).
Since the $\mathcal{V}$-monads on $\mathcal{V}$ the underlying $\mathcal{V}$-functors of which have right $\mathcal{V}$-adjoints are precisely the monoids in $L:\mathcal{V}-\text{Fun}(\mathcal{V},\mathcal{V})$, they are mapped by the above equivalence into the monoids in $\mathcal{V}$.

Let $\mathcal{V}\text{-Cat}/\mathcal{V}$ denote the 2-category with objects the $\mathcal{V}$-functors to $\mathcal{V}$ and with arrows $F \to G$ the $\mathcal{V}$-functors $K$ with an isomorphism $F \cong GK$. One defines $\mathcal{V}\text{-MonCat}/\mathcal{V}$ similarly.

For a right bialgebroid $A$ in $\mathcal{V}$ we denote by $G^A$ the essentially strong monoidal forgetting $\mathcal{V}$-functor $\mathcal{V}_A \to \mathcal{V}$. $G^A$ is called the long forgetful functor of $A$.

**Theorem 5.1.** A $\mathcal{V}$-functor $G: \mathcal{C} \to \mathcal{V}$ is equivalent in $\mathcal{V}\text{-Cat}/\mathcal{V}$ to the long forgetful functor $G^A$ of a bialgebroid in $\mathcal{V}$ iff

- $G$ is monadic,
- $G$ has a right adjoint and
- there is an essentially strong monoidal structure $(G, G_2, G_0)$ on $G$.

In this case $G \simeq G^A$ in $\mathcal{V}\text{-MonCat}/\mathcal{V}$.

**Proof.** Necessity: Since the object in $\mathcal{V}$ underlying the monoidal unit of $\mathcal{V}_A$ is just the object underlying the base monoid $R$, the canonical factorization of $G^A$ is $G^A = \Gamma R \Phi^A$, in the notation of Theorem 4.2. Thus $G^A$ is essentially strong. As every forgetful functor of a monoid, $G^A$ is monadic. It has a right adjoint, namely, the coinduction functor sending the object $x$ of $\mathcal{V}$ to the object $\text{hom}(A, x)$ endowed with right $A$-action

$$(5.37) \quad \varepsilon_x: \text{hom}(A, x) \otimes A \to x$$

provided by the $x$-component of the counit of the adjunction $\_ \otimes A \dashv \text{hom}(A, \_)$.

Sufficiency: Since $G$ is monadic, it has a left adjoint $F$ and $G = U^g J$ where $\mathcal{M}$ is the monad with underlying functor $GF$, $U^g$ is its forgetful functor and $J: \mathcal{C} \to \mathcal{V}^\mathcal{M}$ is an equivalence. Let $H$ be the right adjoint of $G$. Then $GH$ is a right adjoint of $GF$, therefore, by the above remark, it is isomorphic to the monad $\_ \otimes B$ associated to a monoid $B$ in $\mathcal{V}$. Therefore $\mathcal{V}^\mathcal{M}$ is isomorphic to the category $\mathcal{V}_B$ of right $B$-modules and the decomposition $G = U^g J$ can be replaced with $G = G^B L$ where $L: \mathcal{C} \to \mathcal{V}_B$ is an equivalence. Using the latter equivalence $G^B$ is given an essentially strong monoidal structure and it has a left adjoint. Therefore Theorem 4.2 provides a monoidal isomorphism $G^B \cong \Gamma \Phi K = G^A K$ where the right bialgebroid $A$ has underlying monoid the endomorphism monoid of the representing object $g = B_0$ of $G^B$. Clearly, $A = \text{End}(B_0) \cong B$. This proves that $K: \mathcal{V}_B \to \mathcal{V}_A$, $X_B \mapsto \text{Hom}_B(AB, X_B)$, is an isomorphism of (monoidal) categories. Since in the monoidal isomorphism $G \cong G^A KL$ the functor $KL: \mathcal{C} \to \mathcal{V}_A$ is a monoidal equivalence, it defines an equivalence $\mathcal{V} \sim G^A$ in $\mathcal{V}\text{-MonCat}/\mathcal{V}$. \qed

6. The forgetful functors of weak bialgebras

In this section the base category $\mathcal{V}$ is the category $\mathcal{M}_k$ of modules over a commutative ring. So we switch to the convention of writing capital Roman letters for objects and corresponding small case letters for their elements. Weak bialgebras over $k$ are not only special bialgebroids in $\mathcal{M}_k$ but are equipped with some more structure, as well. The extra structure can be recognized in two places: (1) in the difference between a weak bialgebra counit $\epsilon: B \to k$ and a bialgebroid counit $\varepsilon: A \to R$ and (2) in the nontrivial element $\Delta(1) \in A \otimes A$ which is closely related to a separability idempotent of the separable algebra $R$. In other words, a weak
bialgebra $A$ is a bialgebroid over a separable Frobenius algebra $R$ together with a Frobenius structure $(\epsilon, \sum_i e_i \otimes f_i)$ in which $\sum_i e_i f_i = 1$ (see [19] for more details). We call $(R, \epsilon, \sum_i e_i \otimes f_i)$ a separable Frobenius structure.

Accordingly, the forgetful functor $G^A: \mathcal{M}_A \to \mathcal{M}_k$ of a bialgebroid has more structure than just a monadic essentially strong monoidal functor with right adjoint. It is equipped also with an opmonoidal structure. For concreteness let $(A, \Delta, \epsilon)$ be a weak bialgebra [4] and let $R$ be identified with the canonical right subalgebra $A^R = \{1(a)\epsilon(a1(2)) | a \in A\}$. Then $A$ becomes a right bialgebroid over $R$ with

\begin{align*}
(6.38) & \quad s: R \to A, \quad r \mapsto r \\
(6.39) & \quad t: R^op \to A, \quad r \mapsto \epsilon(r1(1))1(2) \\
(6.40) & \quad \delta: A \to A \otimes_R A, \quad \delta := \tau \circ \Delta \\
(6.41) & \quad \varepsilon: A \to R, \quad a \mapsto 1(a)\epsilon(a1(2))
\end{align*}

where $\tau: A \otimes A \to A \otimes_R A$ is the canonical epimorphism (cf. [19] Lemma 1.1).

Let $G$ denote the long forgetful functor $G^A$ of the bialgebroid $A$ (see Section 4). Then the monoidal structure

\begin{align*}
(6.42) & \quad G_{X,Y}: GX \otimes GY \to GX \square_A Y, \quad x \otimes y \mapsto x \otimes_R y \\
(6.43) & \quad G_0: k \to G\mathcal{E}, \quad \kappa \mapsto \kappa \cdot 1_R 
\end{align*}

is built of the canonical epimorphisms $X \otimes Y \to X \otimes_R Y$ for $R$-bimodules and of the unit $k \to R$ of the algebra $R$. Due to the presence of the separable Frobenius structure it has an opmonoidal counterpart

\begin{align*}
(6.44) & \quad G^{X,Y}: G(X \square_A Y) \to GX \otimes GY, \quad x \otimes_R y \mapsto \sum_i x \cdot e_i \otimes f_i \cdot y \\
(6.45) & \quad G^0: GE \to k, \quad r \mapsto \epsilon(r)
\end{align*}

So we have a monoidal $(G, G_2, G_0)$ and an opmonoidal functor $(G, G^2, G^0)$ with the same underlying functor $G$. This involves that $G_2$ with $G_0$ obey one hexagonal and two square diagrams and $G^2$ together with $G^0$ obey three analogous, but oppositely oriented, diagrams. In addition, there are compatibility conditions between the monoidal and opmonoidal structures that are not of the bialgebra type but rather of the Frobenius algebra type [4]. Namely,

\begin{align*}
(6.46) & \quad (G_{X,Y} \otimes GZ) \circ a_{G_{X,GY}GZ} \circ (GX \otimes G^{Y,Z}) = G^{X \square_A Y, Z} \circ Ga_{X,Y,Z} \circ G_{X,Y \square Z} \\
(6.47) & \quad (GX \otimes G_{Y,Z}) \circ a_{G_{X,GY}GZ}^{-1} \circ (G^{X,Y} \otimes GZ) = G^{X,Y \square Z} \circ Ga_{X,Y,Z}^{-1} \circ G_{X \square Y,Z}
\end{align*}

for all $A$-modules $X, Y, Z$ and where $\square$ stands for the monoidal product $\square_A$ of $A$-modules. At last but not least, $G_2$ is split epi and the splitting map is just $G^2$, i.e.,

\begin{equation}
(6.48) \quad G_{X,Y} \circ G^{X,Y} = G(X \square Y), \quad X, Y \in \mathcal{M}_A.
\end{equation}

Now we summarize these experiences in a

**Definition 6.1.** Let $(\mathcal{C}, \square, E)$ and $(\mathcal{M}, \otimes, I)$ be monoidal categories. A separable Frobenius structure on a functor $G: \mathcal{C} \to \mathcal{M}$ consists of natural transformations $G_2, G^2$ and arrows $G_0, G^0$ such that

1. $(G, G_2, G_0)$ is a monoidal functor,
(2) \( (G, G^2, G^0) \) is an opmonoidal functor,
(3) the Frobenius conditions \((6.46), (6.47)\) and the separability condition \((6.48)\) hold.

Of course, the functor itself may be neither separable nor Frobenius. Only its monoidal-opmonoidal structure is restricted in the above Definition.

**Lemma 6.2.** If \( (G, G_2, G_0, G^2, G^0) \) is a separable Frobenius structure on the functor \( G : \mathcal{C} \to \mathcal{M} \) then \((2.15)\) is a split coequalizer in \( \mathcal{M} \) for all pairs of objects in \( \mathcal{C} \). In particular, \( (G, G_2, G_0) \) is essentially strong monoidal.

**Proof.** For each pair \( X, Y \) of objects in \( \mathcal{C} \) we can define arrows in \( \mathcal{M} \) by

\[
\partial_0 := (GX \otimes G_{lY}) \circ (GX \otimes G_{E,Y}) \\
\partial_1 := (GrX \otimes GY) \circ (G_{X,E} \otimes GY) \circ a_{G_{X,E},GY} \\
\gamma := G_{X,Y} \\
\sigma := G_{X,Y}^{-1} \\
\tau := (GX \otimes G_{E,Y}^{-1}) \circ (GX \otimes G_{lY})
\]

Then an elementary calculation gives

\[
\begin{align*}
\text{hexagon for } G_2 & \quad \Rightarrow \quad \gamma \circ \partial_0 = \gamma \circ \partial_1 \\
\text{separability } (6.48) & \quad \Rightarrow \quad \gamma \circ \sigma = 1 \\
\text{separability } (6.48) & \quad \Rightarrow \quad \partial_0 \circ \tau = 1 \\
\text{Frobenius } (6.46) & \quad \Rightarrow \quad \partial_1 \circ \tau = \sigma \circ \gamma
\end{align*}
\]

which means precisely that \( \gamma \) is a split coequalizer, hence a coequalizer, of \( \partial_0 \) and \( \partial_1 \). Thus \( G \) is essentially strong monoidal. \( \square \)

**Lemma 6.3.** Let \( (G, G_2, G_0, G^2, G^0) \) be a separable Frobenius structure on the functor \( G : \mathcal{C} \to \mathcal{M} \) between monoidal categories. Then the image \( GE \) of the unit object of \( \mathcal{C} \) is equipped with a separable Frobenius structure in \( \mathcal{M} \).

**Proof.** \( GE \) gets a monoid structure as the image of \( E \) by \( (G, G_2, G_0) \) as we explained in Lemma 2.1. Dually, the \( (G, G^2, G^0) \) maps the comonoid \( E \) into a comonoid with underlying object \( GE \). So we obtain

\[
\begin{align*}
R &= GE \\
\mu &= G_{E,E} \circ G_{E,E} : R \otimes R \to R \\
\eta &= G_0 : I \to R \\
\sigma &= G_{E,E} \circ G_{E,E}^{-1} : R \to R \otimes R \\
\psi &= G^0 : R \to I
\end{align*}
\]

such that \( \langle R, \mu, \eta \rangle \) is a monoid and \( \langle R, \sigma, \psi \rangle \) is a comonoid in \( \mathcal{M} \). Now \((6.46), (6.47)\) and \((6.48)\) imply that these two structures on \( GE \) are compatible in the sense of satisfying

\[
\begin{align*}
(\mu \otimes R) \circ a_{R,R,R} \circ (R \otimes \sigma) &= \sigma \circ \mu \\
(R \otimes \mu) \circ a_{R,R,R}^{-1} \circ (\sigma \otimes R) &= \sigma \circ \mu \\
\mu \circ \sigma &= R
\end{align*}
\]

which are the defining relations of a separable Frobenius structure on \( R \). \( \square \)
We note that for the familiar categories, $\mathcal{M} = \mathcal{M}_k$, the $\sigma$ is uniquely determined by $\sigma(1_R)$ which in turn is uniquely determined by $\psi$. If, moreover, $k$ is a field then a separable Frobenius algebra, i.e., an algebra having a separable Frobenius structure (also called an index one Frobenius algebra), is nothing but a separable $k$-algebra.

**Lemma 6.4.** Let $(R, \mu, \eta, \sigma, \psi)$ be a separable Frobenius structure in $\mathcal{M}_k$. Then the monoidal forgetful functor of bimodules $\Gamma = \Gamma^R: R\mathcal{M}_R \to \mathcal{M}$ has the following extension to a separable Frobenius structure:

$$\Gamma^{X,Y}: X \otimes_R Y \to X \otimes Y, \quad x \otimes_R y \mapsto \sum_i x \cdot e_i \otimes f_i \cdot y_i$$

$$\Gamma^0: R \to k, \quad r \mapsto \psi(r)$$

where $\sum_i e_i \otimes f_i = \sigma(1_R)$.

**Proof.** This is left for an exercise. □

After the above preparations Theorem 5.1 has the following

**Corollary 6.5.** A $k$-linear functor $G: \mathcal{C} \to \mathcal{M}_k$ - as an object in $k\text{-Cat}/\mathcal{M}_k$ - is equivalent to the long forgetful functor of a weak bialgebra iff

- $G$ is monadic,
- $G$ has a right adjoint and
- there is a separable Frobenius structure $(G, G_2, G_0, G^0)$ on $G$.

**Proof.** $G$ is essentially strong monoidal by Lemma 6.2 therefore Theorem 5.1 provides a right bialgebroid $(A, R, s, t, \delta, \varepsilon)$ and a monoidal equivalence $G \simeq G^A$. It remains to show that the data on $A$ can be extended to the data of a weak bialgebra. (This extra structure is encoded neither in the monoidal category $\mathcal{M}_A$ nor in the monoidal functor $G^A$.) A monoidal equivalence is the same thing as an op-monoidal equivalence, so we can use $G \simeq G^A$ to pass the whole separable Frobenius structure of $G$ to $G^A$. Now Lemma 6.3 implies that $R = G^A E$, the base of $A$, is given a separable Frobenius algebra structure $(R, \mu, \eta, \sigma, \psi)$. Then weak bialgebra comultiplication and counit can be introduced by

$$\Delta := \Gamma^A A \circ \delta \quad (a) = \sum_i a_{(1)} s(e_i) \otimes a_{(2)} t(f_i),$$

$$\epsilon := \Gamma^0 \circ \varepsilon \quad (\epsilon(a) = \psi(\varepsilon(a))).$$

For the proof of that $(A, \Delta, \epsilon)$ is a weak bialgebra we refer to the proof of Proposition 7.4 where this has been done for $R$ a separable $k$-algebra over a field. However, after providing a separable Frobenius structure on $R$ that proof applies here. □

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