Discrete Randomness in Discrete Time Quantum Walk: Study via Stochastic Averaging

D. Ellinas*, A. J. Bracken# and I. Smyrnakis$

*Technical University of Crete Department of Sciences
MΦQ Research Unit GR-731 00 Chania Crete Greece
ellinas@science.tuc.gr

#Centre for Mathematical Physics, Department of Mathematics, University of Queensland, Brisbane 4072 Australia
ajb@maths.uq.edu.au

$Technological Education Institute of Crete, P.O. Box 1939, GR-71004, Heraklion, Greece
smyrnaki@tem.uoc.gr

May 2, 2014

Abstract

The role of classical noise in quantum walks (QW) on integers is investigated in the form of discrete dichotomic random variable affecting its reshuffling matrix parametrized as a SU(2)/U(1) coset element. Analysis in terms of quantum statistical moments and generating functions, derived by the completely positive trace preserving (CPTP) map governing evolution, reveals a pronounced eventual transition in walk’s diffusion mode, from a quantum ballistic regime with rate $O(t)$ to a classical diffusive regime with rate $O(\sqrt{t})$, when condition (strength of noise parameter) $^2 \times$ (number of steps) = 1, is satisfied. The role of classical randomness is studied showing that the randomized QW, when treated on the stochastic average level by means of an appropriate CPTP averaging map, turns out to be equivalent to a novel quantized classical walk without randomness. This result emphasizes the dual role of quantization/randomization in the context of classical random walk. Keywords: Quantum walk, randomness, CP map, quantization

1 Introduction

Quantum Walk on $\mathbb{Z}$: Recapitulation. The essential feature of a QW on a line is the promotion of the mathematical correspondence: left/right $\rightarrow$ head/tails,
between walker’s move directions and coin’s sides, to a dynamic interaction among two physical quantum systems\textsuperscript{11-3}. This is realized by introducing state Hilbert spaces $H_w = \text{span}(|m\rangle)_{m \in \mathbb{Z}} \approx \mathbb{C}^\infty$ and $H_c = \text{span}(|+\rangle, |−\rangle) \approx \mathbb{C}^2$, for the quantum walker and coin systems respectively. In $H_w$ the algebra Euclidean group $\text{ISO}(2)$ with generators $\{ \hat{L}, \hat{E}_+, \hat{E}_− \}$ satisfying relations $[\hat{L}, \hat{E}_\pm] = ±\hat{E}_\pm$, $[\hat{E}_+, \hat{E}_−] = 0$, is represented by the step operators $\hat{E}_\pm |m\rangle = |m ± 1\rangle$ and the position operator $\hat{L} |m\rangle = m |m\rangle$. Introduce also the coin space projection operators $P_\pm = |±\rangle\langle±|$. One step of a CRW is described by the unitary ‘walker-coin’ operator

$$\rho \rightarrow \rho \mathcal{E}_c \rho := \text{Tr}_c (\mathcal{E}_c \rho \rho \mathcal{E}_c),$$

where the partial trace $\text{Tr}_c$, i.e. $\mathcal{E}_c = \text{Tr}_c (\rho \mathcal{E}_c) = \text{Tr}_c (\mathcal{E}_c \rho)$, realizes the walker’s move. Explicitly, coin $\rho_c$ and walker $\rho_w$ density matrices, initially in a product state $\rho = \rho_c \rho_w$, interact unitarily via the map $\rho_c \rho_w \rightarrow V_{cl} \rho_c \rho_w V_{cl}^\dagger = \text{Ad} V_{cl} (\rho_c \rho_w)$ (written alternatively in terms of the adjoint action $\text{Ad} X (\rho) := X (\rho) X^\dagger$), and then de-couple by an unconditional quantum measurement of the coin subsystem, realized by the partial trace $\text{Tr}_c$. Consider the unitary ‘walker-coin’ operator $V(s) = V_{cl} U(s) \otimes 1_w$, or explicitly

$$V(s) = P_+ U(s) \otimes \hat{E}_+ + P_- U(s) \otimes \hat{E}_−,$$

where the reshuffling matrix $U(s)$, according to the Appendix, is taken to be an element of the coset $SU(2)/U(1) \approx CP$. Referring to their coordinates of the complex projective variable $\theta \in CP$, is chosen to be a dichotomous random variable (rv) $\theta = se$, $e > 0$, with values $\theta = ± e$, where $s = ± 1$, with equal probabilities. The reshuffling matrix

$$U(s) = \mathcal{N} \left( \begin{array}{cc} 1 & se \\
−se & 1 \end{array} \right),$$

with $\mathcal{N} \equiv \frac{1}{\sqrt{1+e^2}}$, provides the unitaries $V(s = ±1)$ that apply at random, with probability $q_± = 1/2$ (unbiased walk), so that the ‘walker-coin’ density operator evolves from the $N$-th to $(N+1)$-th step as

$$\rho_{N+1} = \frac{1}{2} \sum_{s=±1} V(s) \rho_N V(s)^\dagger.$$
Rewrite this as $\rho_{N+1} = \hat{\mathcal{E}}(\rho_N) \equiv \langle V(s)\rho_N V(s)\rangle = \frac{1}{2} \sum_{s = \pm} V_q(s)\rho_N V_q(s)^\dagger = \frac{1}{2} \text{Ad} V(s)\rho_N$, where $\hat{\mathcal{E}} : \text{Lin}(H_c \otimes H_w) \rightarrow \text{Lin}(H_c \otimes H_w)$ stands for a CPTP map acting on the composite coin-walker system which implements the ensemble average of the random unitary evolution. This density matrix iteration provides the sequence $\{\rho_N\}_{N=1}^\infty$, which subsequently provides, for any observable operator $A_c \otimes B_w$ on $H_c \otimes H_w$, a sequence of quantum moments $\{\mu_N\}_{N=1}^\infty$, as $\mu_{N+1} = \text{Tr} (\rho_{N+1} A_c \otimes B_w) \equiv (A_c \otimes B_w)_N$. Utilizing the state-observable (trace inner product) duality, we derive the moments

$$\mu_{N+1} = \text{Tr}(\hat{\mathcal{E}}(\rho_N) A_c \otimes B_w) = \text{Tr}(\rho_N \hat{\mathcal{E}}^*(A_c \otimes B_w)) = \text{Tr}(\rho_1 \hat{\mathcal{E}}^{N*}(A_c \otimes B_w)),$$

(4) either evolving the state $\hat{\mathcal{E}}$, or the observables by the dual CP map $\hat{\mathcal{E}}^*$. The latter has as Kraus generators, the hermitian conjugates of those of $\hat{\mathcal{E}}$. By means of the dual map $\hat{\mathcal{E}}^*$, moment iterations for various choices of observables $A_c \otimes B_w$, are investigated next.

Let $\rho_N = \sum_{\alpha, \beta} \sum_{k, l} \rho_{N, k, \alpha, l, \beta} |\alpha\rangle \langle \beta| \otimes |k\rangle \langle l|$, be the density matrix describing the state of the “coin-walker” system, where $(\rho_{N, k, \alpha, l, \beta})$ are elements labelling walker positions $k, l \in \mathbb{Z}$, and coin states $\alpha, \beta \in \{+\, -, \}$. Assume initially a factorized state $\rho_0 = \rho_0^{(c)} \otimes \rho_0^{(w)}$, $\rho_0^{(w)} = |0\rangle \langle 0|$, $\rho_0^{(c)} = \text{diag}(\cos^2 \gamma, \sin^2 \gamma)$, for $\gamma \in [0, \pi)$. The resulting matrix valued recurrence relation implies that $\rho_N$ is also diagonal. Setting $(\rho_{N, k, \alpha, l, \beta}) = \text{diag}(\alpha_{Nk}, \beta_{Nk})$, for diagonal components obtained by expectation values of the projectors,

$$\alpha_{Nk} = \text{Tr}(\rho_N P_+ \otimes P_k) \equiv (P_+ \otimes P_k)_N,$$

$$\beta_{Nk} = \text{Tr}(\rho_N P_- \otimes P_k) \equiv (P_- \otimes P_k)_N.$$

leads to recurrence relations

$$\alpha_{N+1k} = \mathcal{N}^2 (\alpha_{Nk-1} + \epsilon^2 \beta_{Nk-1}),$$

$$\beta_{N+1k} = \mathcal{N}^2 (\epsilon^2 \alpha_{Nk+1} + \beta_{Nk+1}),$$

for $N = 0, 1, 2, \ldots$, $k \in \mathbb{Z}$, with initial conditions $\alpha_{00} = \cos^2 \gamma$, $\beta_{00} = \sin^2 \gamma$, $\alpha_{0k} = \beta_{0k} = 0$ for $k \neq 0$.

### 3 Recurrence relations for moments

The recurrence relations are difficult to solve exactly, except when $\epsilon = 0$ or 1. We consider instead the quantum moments of the distribution of probability over $k$-values (‘walker positions’) after $N$ iterations. Accordingly, let $A_{N}^{(p)} = \sum_{k=-\infty}^{\infty} \alpha_{Nk} k^p$, and $B_{N}^{(p)} = \sum_{k=-\infty}^{\infty} \beta_{Nk} k^p$, for $p = 0, 1, 2, \ldots$. Then $A_{N}^{(p)}$ is the $p$-th moment of the distribution of probability over $k$ values, with coin spin ‘up,’ and $B_{N}^{(p)}$ is the $p$-th moment of the distribution of probability over $k$ values, with coin spin ‘down.’ The sum $A_{N}^{(p)} + B_{N}^{(p)}$ is the total $p$-th moment of the distribution of probability over $k$ values. The operator expression of these
moments and their combinations \( S_N^{(p)} = A_N^{(p)} + B_N^{(p)} \), and \( D_N^{(p)} = A_N^{(p)} - B_N^{(p)} \), read

\[
\begin{align*}
A_N^{(p)} &= \text{Tr}(P_+ \otimes \hat{L}^p \rho_N), \quad B_N^{(p)} = \text{Tr}(P_- \otimes \hat{L}^p \rho_N), \tag{9} \\
D_N^{(p)} &= \text{Tr}((P_+ - P_-) \otimes \hat{L}^p \rho_N), \tag{10} \\
S_N^{(p)} &= \text{Tr}((P_+ + P_-) \otimes \hat{L}^p \rho_N) = \text{Tr}(1 \otimes \hat{L}^p \rho_N) =: \text{Tr}(\hat{L}^p \rho_N^{(w)}). \tag{11}
\end{align*}
\]

The resulting recurrence relations for the moments are also difficult to solve in closed form for general \( N \) and \( p \), except small values of \( p \). In this way we find that

\[
A_N^{(0)} = \frac{1}{2} \left( 1 + r^N \cos(2\gamma) \right), \quad B_N^{(0)} = \frac{1}{2} \left( 1 - r^N \cos(2\gamma) \right). \tag{12}
\]

For the higher moments, we find

\[
D_N^{(1)} = \frac{(1 - r^N)/(1 - r)}{S_N^{(1)} = r \cos(2\gamma) [1 - r^N] / [1 - r]}, \tag{13}
\]

and

\[
D_N^{(2)} = 2r \cos(2\gamma) (1 - r^N) / (1 - r)^2 - Nr^N \cos(2\gamma) (1 + r) / (1 - r), \tag{14}
\]

\[
S_N^{(2)} = N (1 + r) / (1 - r) - 2r (1 - r^N) / (1 - r)^2. \tag{15}
\]

The second total moment i.e. the expectation value of the square of the position operator \( \hat{L} \), can also be written as

\[
S_N^{(2)} = \text{Tr}((\hat{L}^2) \rho_N^{(w)} \equiv \left\langle \hat{L}^2 \right\rangle_N \tag{16}
\]

\[
= \frac{1}{2 \epsilon^4} (2N \epsilon^2 - 1 + \epsilon^4 - (1 - \epsilon^2)^N) / (1 + \epsilon^2)^{N-1}. \tag{17}
\]

There are two regimes of interest for the second moment, corresponding to two value ranges of discrete time steps \( N \) relative to the strength of randomness coefficient \( 1/\epsilon^2 \). The first regime is when \( 1/\epsilon^2 \gg N \gg 1 \), and then

\[
(1 - \epsilon^2)^N + (1 + \epsilon^2)^N - 1 = 2N \epsilon^2 + (2N^2 - 1) \epsilon^4 + \mathcal{O}(N\epsilon^2)^3, \tag{18}
\]

so that

\[
S_N^{(2)} = N^2 + \frac{1}{2 \epsilon^4} \mathcal{O}(N\epsilon^2)^3. \tag{19}
\]

This is the ‘ballistic’ or ‘inertial’ regime, where the growth rate of the second moment with \( N \) is \( \sqrt{\left\langle \hat{L}^2 \right\rangle_N} \sim \mathcal{O}(N) \), that is characteristic of a quantum walk (QW) \cite{2}.

The second regime is when \( N \gg 1/\epsilon^2 \gg 1 \), and then

\[
S_N^{(2)} = \frac{N}{\epsilon^2} \left\{ 1 + \mathcal{O}(1/(N\epsilon^2)) \right\}. \tag{20}
\]
i.e. \( \sqrt{\langle L^2 \rangle_N} \sim O(\sqrt{N}) \), showing the typical growth rate of a CRW or diffusion process. Note that, however small the \( \epsilon^2 \), eventually \( N \) becomes so large that the second regime is reached. The changeover occurs for \( N \epsilon^2 \approx 1 \). Thus we see that for very small \( \epsilon^2 \), the process behaves — as far as the second moment is concerned — initially like a QW, but eventually like a CRW. This is precisely the behavior observed in numerical simulations of a variety of quantum walks with unitary noise \([7]\) and in previous studies e.g. \([8] - [11]\), of which the present process may be considered a simple, special case, with the advantage that it is amenable to a complete mathematical analysis.

4 Randomization and Quantization of Walks

Let us return to the evolution equation (3) for the total system density matrix. The following elaboration is possible; factoring the evolution operator as

\[ V(s) = V_{cl} U(s) \otimes 1_w \]

makes explicit that the stochastic averaging amounts to averaging with respect to the matrices \( U(s = +), U(s = -) \) with probabilities \( \rho_{\pm} = \frac{1}{2} \), which is implemented by a CPTP map \( \hat{E}_c = \frac{1}{2} \sum \epsilon_{\pm} \text{Ad}(U(s) \otimes 1_w) \), acting nontrivially only on \( H_c \). Explicitly we obtain

\[ \rho_{N+1} = \hat{E}(\rho_N) = V_{cl} \hat{E}_c(\rho_N)V_{cl}^\dagger = \text{Ad}(U_{cl} \otimes 1_w) \circ \hat{E}_c(\rho_N) \]

Noticeable, the averaging map \( \hat{E} = \text{Ad}(U_{cl} \otimes 1_w) \circ \hat{E}_c \) can be identified to be itself the CPTP evolution map of another QW. This new QW will have as its “walker space” the total Hilbert space of the initial walk i.e. \( H_t := H_c \otimes H_w \). In addition to the operator sum decomposition of the map \( \hat{E} \) defined following (3) it could also be described by a unitary dilation matrix \( Y \) in an extended space \( H_{aux} \otimes H_c \otimes H_w \), where now an auxiliary vector space \( H_{aux} \) should be attached to the initial space \( H_t \), as an additional “coin space”. Such particular QW-like unitary dilation for \( \hat{E} \) is given in terms of the unitary operator \( Y = \sum \epsilon_{\pm} P_{\epsilon} Q \otimes V(s) \), or \( Y = Y_{cl} Q \otimes 1_w \), acting on \( H_{aux} \otimes H_c \otimes H_w \). Under fairly general conditions, for CP maps with unitary Kraus generators (and \( \hat{E} \) provides an example), such a QW-like unitary dilation it is always possible (cf. \([12]\), for a proof and related discussion).

Explicitly for \( \hat{E} \) we have that

\[ \rho_{N+1} = \hat{E}(\rho_N) = \text{Tr}_{aux} \text{Ad}(Y_{aux} \otimes \rho_N) = \text{Tr}_{aux} \text{Ad}(Y_{cl} \otimes 1) \circ (\text{Ad}(Q \otimes 1_w))(\rho_{aux} \otimes \rho_N). \]

(21)

Let \( \rho_{aux} = |a\rangle \langle a| \) be auxiliary system’s state with \( |a\rangle \) a basis vector. The non-unique unitary \( Q \) is chosen so that \( q_{\pm} = \langle \pm | Q \circ Q^* | a \rangle \), i.e.

\[ Q = \left( \begin{array}{cc} \sqrt{q_+} & \sqrt{q_-} \\ -\sqrt{q_-} & \sqrt{q_+} \end{array} \right) \]

If \( q_{\pm} = \frac{1}{2} \), matrix \( Q \) is the \( \pi/4 \) rotation matrix.

In conclusion, the QW on integers with evolution map \( \hat{E} = \text{Tr}_{cl} \text{Ad}(U_{cl} \otimes 1_w) \), randomized by un-bias dichotomous noise, when treated at the stochastic average level with map \( \hat{E} = \text{Ad}(U_{cl} \otimes 1_w) \), becomes equivalent to a novel non-random QW with map \( \hat{E} = \text{Tr}_{aux} \text{Ad}(Y_{cl} \otimes 1_w) \). The latter QW, via iterative relations obeyed by quantum moments of its CP evolution map, exhibits an eventual transition from a ballistic regime of fast spreading rate to diffusive regime of slow spreading.
rate. Similar transition phenomenon occurs and quantization/randomization interplay is exhibited when randomization via continuous random variables is introduced.\cite{13}

Appendix: Lemma 1. The probabilities \( p_i \) determining the CP map of QW
\[
\mathcal{E}(\rho_w) = \sum_{i=\pm} p_i E_i \rho_w E_i^\dagger,
\]
are given by \( p_i = \langle i | W \circ W^* | c \rangle \), where \( W \) is an element of the coset \( SU(2)/U(1) \cong CP \) coset corresponding to the reshuffling matrix \( U \), and the basis vector \( | c \rangle \) is determined by the initial state of the coin.

Proof. The Hadamard or element-wise product \((A \circ B)_{ij} = A_{ij} B_{ij}\) of matrices \( A, B \in M(C^N) \), is invariant \((DA \circ (DB)^* = A \circ B^*)\) under diagonal \( D \) unimodular \(|D_{ii}| = 1\) matrices. Let \( V_q = V_c U \otimes 1_t \), be a unitary dilation of CP map \( \mathcal{E} \), viz. \( \mathcal{E}(\rho_w) = \operatorname{Tr}_c(V_q \rho_c \otimes \rho_w V_q^\dagger) = \sum_{i=\pm} p_i E_i \rho_w E_i^\dagger \), where \( p_i = \operatorname{Tr}(P_i U \rho_c U^\dagger) = \langle i | U \circ U^* | c \rangle \), for \( \rho_c = | c \rangle \langle c | \). If \( U = D W \) be the \( SU(2)/U(1) \) coset decomposition of \( U \), for \( D \in U(1) \), we obtain \( p_i = \langle i | W \circ (W)^* | c \rangle \). If \( D = \exp(i \phi \sigma_3) \),
\[
W(\theta) = \begin{pmatrix}
\sqrt{1-|\theta|^2} & z \\
-z^* & \sqrt{1-|\theta|^2}
\end{pmatrix} = \frac{1}{\sqrt{1+|\theta|^2}} \begin{pmatrix}
1 & \theta \\
-\theta^* & 1
\end{pmatrix},
\]
where \( z = \frac{\sin\theta}{\eta} = \frac{\theta}{\sqrt{1+|\theta|^2}} \in CP \). In terms of \( p := (1 + |\theta|^2)^{-1} = 1 - |z|^2 \), the bi-stochastic matrix \( W \circ (W)^* = p 1 + (1 - p) \sigma_1 \) reads
\[
W \circ (W)^* = \frac{1}{1 + |\theta|^2} \begin{pmatrix}
1 & \frac{|\theta|^2}{1} \\
\frac{|\theta|^2}{1} & 1
\end{pmatrix}.
\]

Acknowledgments. The first named author (D. E) is grateful to the Department of Mathematics, The University of Queensland, for the hospitality extended to him during a sabbatical stay in which this work was completed.

References

[1] Y. Aharonov, L. Davidovich and N. Zagury, Quantum random walks, Phys. Rev. A 48, 1687 (1993).

[2] A. Ambainis, E. Bach, A. Nayak, A. Vishwanath and J. Watrous, One dimensional quantum walks, Proceedings of the 33rd Annual ACM symposium on Theory of Computing, 2001, 37-49.

[3] E. Bach, S. Coppersmith, M.P. Goldschen, R. Joynt, J. Watrous, one-dimensional quantum walks with absorbing boundaries, quant-ph/0207008.

[4] M. A. Nielsen and I. Chuang, Quantum Computation and Quantum Information, Cambridge University Press 2000.
[5] A. Bracken, D. Ellinas and I. Tsohantjis, *Pseudo memory effects, majorization and entropy in quantum random walks*. J. Phys. A: Math. Gen. 37, L91-L97 (2004).

[6] D. Ellinas and I. Smyrnakis, *Quantum optical random walk: quantization rules and quantum simulation of asymptotics*, Phys. Rev. A 76, 022333 (2007).

[7] D. Shapira, O. Biham, A. Bracken, and M. Hackett, *One dimensional quantum walk with unitary noise*, Phys. Rev. A 68, 062315 (2003).

[8] Kosk, J., Buzek, V., Hillery, M., *Quantum walks with random phase shifts*, Phys. Rev. A 74, 022310, (2006).

[9] Y. Yin, D.E. Katsanos, S.N. Evangelou, *Quantum walks on a random environment*, Phys. Rev. A 77, 022302 (2008).

[10] N. Konno, *One-dimensional discrete-time quantum walks on random environments*, Quant. Inf. Processing, 8387 (2009).

[11] A. Schreiber, K. N. Casseiro, V. Potoček, A. Gábris, I. Jex, and Ch. Silberhorn, *Decoherence and disorder in quantum walks: from ballistic spread to localization*, Phys. Rev. Lett. 106, 180403 (2011).

[12] D. Ellinas, *Convex geometry of quantum walks*, to appear.

[13] D. Ellinas, A. J. Bracken and I. Smyrnakis, *Randomized one-dimensional quantum walk: the averaging CP map approach*, to appear.