FLATNESS OVER PI COIDEAL SUBALGEBRAS

BY

SERGE SKRYABIN

Institute of Mathematics and Mechanics, Kazan Federal University
Kremlevskaya St. 18, 420008 Kazan, Russia
e-mail: Serge.Skryabin@kpfu.ru

ABSTRACT

Under the assumption that a residually finite-dimensional Hopf algebra $H$ has an Artinian ring of fractions, it is proved that $H$ is a flat module over any right coideal subalgebra satisfying a polynomial identity and is faithfully flat over any polynomial identity Hopf subalgebra. As a consequence we find a large class of Hopf algebras which are flat over all coideal subalgebras and are faithfully flat over all Hopf subalgebras.

Introduction

Construction of algebraic homogeneous spaces for group schemes of finite type over a field is linked closely with the flatness of orbit morphisms [6]. This motivated the study of the flatness property for Hopf algebras. At a certain time it seemed plausible that an arbitrary Hopf algebra over a field might be faithfully flat over any Hopf subalgebra (see [19, Question 3.5.4]). This is not true, as was shown by Schauenburg [27]. Still, faithful flatness holds under additional assumptions, although there is no any single result which would unify the known cases.
To interpret homogeneous spaces one needs not only Hopf subalgebras but the larger class of right coideal subalgebras. Over coideal subalgebras one can expect flatness in favourable cases but not necessarily faithful flatness. In the author’s preceding article [35] it was proved that a residually finite-dimensional Noetherian Hopf algebra is a flat left module over any right Noetherian right coideal subalgebra.

An obvious deficiency of that result is that nothing is said about flatness when the Noetherian condition on the subalgebra is dropped. Even a finitely generated commutative Hopf algebra may contain non-Noetherian coideal subalgebras. Such a possibility stems from the existence of algebraic groups admitting a finite-dimensional linear representation such that the respective algebra of invariant polynomial functions is not finitely generated (see [11]).

In the new paper we deal with right coideal subalgebras satisfying a polynomial identity. Polynomial identity algebras, PI algebras for short, constitute an important class which includes, in particular, all commutative algebras and all algebras module-finite over some of their commutative subalgebras.

We fix a field $k$ which is assumed to be the base field for all algebras. Recall that an algebra is said to be residually finite-dimensional if its ideals of finite codimension have zero intersection [19].

**Theorem 0.1:** A residually finite-dimensional Noetherian Hopf algebra $H$ is left and right flat over any PI right coideal subalgebra and is left and right faithfully flat over any PI Hopf subalgebra.

By a result of Schneider [28, Corollary 1.8], for any Hopf algebra with bijective antipode faithful flatness over a Hopf subalgebra is equivalent to projectivity (see also Masuoka and Wigner [16]). This applies in the situation of Theorem 0.1, so in the case of Hopf subalgebras we have an even stronger conclusion. If $H$ is additionally assumed to be itself PI, then the polynomial identity is inherited by all subalgebras of $H$, which yields

**Corollary 0.2:** A residually finite-dimensional Noetherian PI Hopf algebra $H$ is left and right flat over any right coideal subalgebra and is a projective generator as either left or right module over any Hopf subalgebra.

This conclusion extends immediately to the case when the Hopf algebra $H$ is the union of a directed family of residually finite-dimensional Noetherian PI Hopf subalgebras. In such a form our result covers two classes of Hopf algebras.
which have been previously known to be flat over all coideal subalgebras: the
commutative Hopf algebras [16] and the finite-dimensional ones [32].

Faithful flatness and projectivity over all Hopf subalgebras have been known
in the two cases just mentioned [38], [22], and in two other cases rather different
by the arguments employed: the Hopf algebras with cocommutative coradicals
[15] (including pointed Hopf algebras [24]) and the cosemisimple ones [4]. Partial
results on faithful flatness over PI Hopf subalgebras of some type can be found
in [29], [33], [39].

Over coideal subalgebras the flatness property turns out to be more elusive.
There are several results in which faithful flatness over some coideal subalgebras
has been proved, but this requires usually more severe restrictions than in the
case of Hopf subalgebras (see [15], [21], [33]).

Theorem 0.1 is another instance of the approach developed in [34]. In that
paper it was shown that a residually finite-dimensional Hopf algebra $H$ is left flat
over a right coideal subalgebra $A$ whenever both $H$ and $A$ have right Artinian
classical right quotient rings (the Ore rings of fractions). For a residually finite-
dimensional Noetherian Hopf algebra $H$ and its right Noetherian right coideal
subalgebras the existence of such quotient rings has been established in [35].
Here we deal with PI coideal subalgebras. By passage to the dual Hopf algebra
the whole problem is reformulated in the context of $H$-prime $H$-module algebras
where $H$ is now an arbitrary Hopf algebra over the base field $k$.

As in [35], we aim first to construct a generalized quotient ring $Q(A)$ and then
to verify that $Q(A)$ is the Ore localization of $A$. However, the right Gabriel
topology used in [35] is well suited only when $A$ is right Noetherian. In the
present paper we use a different Gabriel topology defined in terms of Gelfand–
Kirillov dimension. It is characterized by means of the corresponding class of
torsion modules.

Another essential difference with [35] is that we have to work with a symmet-
tric quotient ring $Q(A)$ defined with respect to a pair of right and left Gabriel
topologies. It can be compared to the symmetric Martindale quotient ring,
but in our case two filters of one-sided ideals are involved in the construction.
Although extra work is needed to verify that $Q(A)$ is semiprimary, once this
is done we gain considerable advantage since the situation becomes completely
left-right symmetric, unlike what we had in [35]. This intermediate step is im-
portant since it enables us to use known results on the structure of semiprimary
algebras.
The first two sections of the paper provide a purely ring-theoretic background. For a finitely generated PI algebra $A$ we introduce certain torsion classes $T_r$ and $T_l$ of right and left $A$-modules and prove several lemmas aimed at recognition of torsion modules. The Gabriel topologies $G_r$ and $G_l$ correspond to $T_r$ and $T_l$.

Section 2 is concerned with the $(G_l, G_r)$-symmetric quotient ring $Q(A)$. Proposition 2.7 gives a set of conditions under which $Q(A)$ is shown to be an Artinian classical quotient ring of $A$. In such a form this result is probably not very useful since the ring-theoretic assumptions are rather demanding.

However, all necessary conditions can be verified for a finitely generated $H$-prime $H$-module PI algebra $A$ satisfying the ACC (ascending chain condition) on right and left annihilators when the action of $H$ on $A$ is locally finite. This is done in section 3, and the final result is presented in Theorem 3.9.

In section 4 we apply Theorem 3.9 to coideal subalgebras. Theorem 4.1 is the main result of the paper which includes Theorem 0.1 as a special case. With the assumption that $H$ has an Artinian classical quotient ring the Noetherianness of $H$ is not needed.

Theorem 4.5 is another special case of Theorem 4.1 stated for a PI Hopf algebra $H$ which is affine, i.e., finitely generated as an ordinary algebra. Here the Noetherian condition on $H$ is relaxed to the ACC on annihilators. We are led to suggest that the conclusion might hold even in larger generality:

**Conjecture 0.3:** Every affine PI Hopf algebra $H$ is flat over all right coideal subalgebras and is faithfully flat over all Hopf subalgebras.

Among all Hopf algebras those satisfying a polynomial identity are more manageable, but even this class of Hopf algebras is not understood sufficiently well. Let us look at the following assertions:

(a) every affine PI Hopf algebra is Noetherian,
(b) every affine PI Hopf algebra is representable in the sense that it embeds in a finite-dimensional algebra over an extension field of the ground field,
(c) every affine PI Hopf algebra is residually finite-dimensional.

It is not known whether any of them is true or not. As to (a), this was asked by Brown [3, Question C]. The implications (a)$\Rightarrow$(b)$\Rightarrow$(c) follow from results of Anan’ in [1] and Malcev [14]. Knowledge of (b) would be sufficient to confirm Conjecture 0.3.
In the direction opposite to (a) Wu and Zhang asked whether every Noetherian PI Hopf algebra is affine [39, Question 5.1]. Proposition 4.8 and Corollary 4.9 in our paper are partial results on that question deduced quickly from the main result.

A recent preprint by Greenfeld, Rowen and Small [10] deals with representability of arbitrary Noetherian PI algebras. For our purposes it is important to know that $H$ is residually finite-dimensional. This suggests the question whether every Noetherian PI Hopf algebra is at least residually finite-dimensional, even if not affine.

Acknowledgement. I would like to thank the referee for helpful comments and for pointing out two related papers.

1. The torsion determined by the Gelfand–Kirillov dimension

Let $A$ be a finitely generated PI algebra. By an algebra we mean an associative unital algebra over a base field $k$. We will use standard facts from PI theory discussed thoroughly in [17], [23], [25]. Concerning the definition and properties of Gelfand–Kirillov dimension we refer to [13], [17], [25].

The algebra $A$ has finite Gelfand–Kirillov dimension $\text{GKdim } A$ [25, Th.6.3.25]. For an $A$-module $M$ we denote its Gelfand–Kirillov dimension by $\text{GKdim } M$ or by $\text{GKdim}_A M$ when there is a need to specify the algebra. We have

$$\text{GKdim } M \leq \text{GKdim } A.$$ 

If $\text{GKdim } M' = \text{GKdim } M$ for all nonzero submodules $M' \subset M$, then $M$ is called $\text{GK-pure}$ (or $\text{GK-homogeneous}$). In particular, $A$ is right $\text{GK-pure}$ if

$$\text{GKdim } I = \text{GKdim } A$$

for all nonzero right ideals $I$ of $A$. It is known that each prime right Goldie algebra is right GK-pure [13, Lemma 5.12].

If $I$ is an ideal of $A$ then the Gelfand–Kirillov dimension of $A/I$ as an algebra is the same as that of $A/I$ as either right or left $A$-module. Put

$$d(A) = \max\{\text{GKdim } A/P \mid P \in \text{Spec } A\},$$

$$\Omega(A) = \{P \in \text{Spec } A \mid \text{GKdim } A/P = d(A)\},$$

where $\text{Spec } A$ is the set of prime ideals of $A$. 
If $P' \subset P$ is a proper inclusion between two prime ideals, then

$$\text{GKdim } A/P < \text{GKdim } A/P'$$

by [13, Prop. 3.15] since the ideal $P/P'$ of the prime factor algebra $A/P'$ contains a regular element (nonzero divisor) of the latter. Therefore all ideals in the set $\Omega(A)$ are minimal primes of $A$. It is known that $A$ has finitely many minimal prime ideals [23, Ch. 5, Cor. 2.2, 2.4]. Hence $\Omega(A)$ is a finite set.

The prime radical $N$ of $A$ is the intersection of all prime ideals of $A$. By [13, Cor. 3.3] we have

$$d(A) = \text{GKdim } A/N \leq \text{GKdim } A.$$

Denote by $T_r$ the class of all right $A$-modules $M$ which admit a finite chain of submodules $M = M_0 \supset M_1 \supset \cdots \supset M_n = 0$ with all factors $M_{i-1}/M_i$ of Gelfand–Kirillov dimension less than $d(A)$.

In particular, a right $A$-module $M$ lies in $T_r$ whenever $\text{GKdim } M < d(A)$. The converse is true when $A$ is Noetherian but may be false in general since the Gelfand–Kirillov dimension is not always an exact dimension function.

A ring $Q$ is said to be a classical right quotient ring (resp., left quotient ring) of a ring $R$ if $R$ embeds in $Q$ as a subring, all regular elements (nonzero divisors) of $R$ are invertible in $Q$ and each element of $Q$ can be written as $as^{-1}$ (resp., as $s^{-1}a$) for some $a, s \in R$ with $s$ being regular. If the right and left conditions in this definition are satisfied simultaneously, then $Q$ is a classical quotient ring of $R$.

Rings admitting a semisimple Artinian classical one-sided quotient ring were characterized by Goldie. If $P \in \text{Spec } A$, then the prime factor ring $A/P$ is right and left Goldie by Posner’s Theorem. Therefore $A/P$ has a simple Artinian classical quotient ring $Q(A/P)$.

**Lemma 1.1:** For a right $A$-module $M$ annihilated by a prime ideal $P \in \Omega(A)$ the following conditions are equivalent:

(a) $M \in T_r$,

(b) $\text{GKdim } M < d(A)$,

(c) $M \otimes_A Q(A/P) = 0$.

**Proof.** If $M \otimes_A Q(A/P) = 0$, then each element of $M$ is annihilated by a regular element of $A/P$, i.e., $M$ is a torsion right $A/P$-module in the classical sense. In this case $\text{GKdim } M < \text{GKdim } A/P$ by [17, 8.3.6] since the ring $A/P$ is right Goldie. Thus (c) $\Rightarrow$ (b). The implication (b) $\Rightarrow$ (a) is obvious.
Suppose that $M \otimes_A Q(A/P) \neq 0$. Since each $Q(A/P)$-module is semisimple, and each simple right $Q(A/P)$-module is isomorphic to a right ideal of $Q(A/P)$, we have $\text{Hom}_A(M, Q(A/P)) \neq 0$. In other words, some nonzero epimorphic image $M'$ of $M$ is isomorphic to a right $A$-submodule of $Q(A/P)$. But each nonzero $A$-submodule of $Q(A/P)$ has nonzero intersection with $A/P$. Therefore there is a nonzero right ideal $I$ of $A/P$ isomorphic to a submodule of $M'$. Recalling that prime right Goldie algebras are right GK-pure [13, Lemma 5.12], we get

$$\text{GKdim } M \geq \text{GKdim } M' \geq \text{GKdim } I = \text{GKdim } A/P = d(A).$$

The preceding argument shows that (b)$\Rightarrow$(c). If

$$M = M_0 \supset M_1 \supset \cdots \supset M_n = 0$$

is a chain of submodules such that $\text{GKdim } M_{i-1}/M_i < d(A)$ for each $i = 1, \ldots, n$, then $M_{i-1}/M_i \otimes_A Q(A/P) = 0$ for each $i$, whence $M \otimes_A Q(A/P) = 0$. Thus (a)$\Rightarrow$(c) too. ■

If $M$ is annihilated by a prime ideal $P$ of $A$ such that $P \notin \Omega(A)$, then

$$\text{GKdim } M \leq \text{GKdim } A/P < d(A).$$

In this case $M \in \mathcal{T}_r$.

**Lemma 1.2:** The class $\mathcal{T}_r$ is closed under submodules, factor modules, extensions and coproducts. In other words, $\mathcal{T}_r$ is the torsion class of a hereditary torsion theory (see [37, Ch. VI, §3]).

The proof is obtained by standard verifications. We note only that one can use chains of submodules whose length $n$ does not depend on the module. Since the prime radical $N$ of $A$ is nilpotent [2], there exists a finite sequence of prime ideals $P_1, \ldots, P_n$ such that $P_1 \cdots P_n = 0$. Combined with Lemma 1.1 this implies

**Lemma 1.3:** Any $A$-module $M$ has a finite chain of submodules $M = M_0 \supset M_1 \supset \cdots \supset M_n = 0$ in which each factor is annihilated by a prime ideal of $A$. We have $M \in \mathcal{T}_r$ if and only if $\text{GKdim } M_{i-1}/M_i < d(A)$ for each $i$.

The modules in $\mathcal{T}_r$ can be characterized by means of a variation of the notion of reduced rank (see [9, Ch. 11] and [17, Ch. 4]). Let $P$ be any minimal prime ideal of $A$. Recall that a **Loewy series** for a right $A$-module $M$ is any
finite chain of submodules $M = M_0 \supset M_1 \supset \cdots \supset M_n = 0$ with all factors annihilated by the prime radical $N$ of $A$. Define the reduced rank $\rho_P(M)$ of $M$ at $P$ as follows. Choose any Loewy series for $M$ and put

$$\rho_P(M) = \sum_{i=1}^{n} \text{length } M_{i-1}/M_i \otimes_A Q(A/P)$$

where the summands are the lengths of right $Q(A/P)$-modules. It is assumed that

$$\rho_P(M) = +\infty$$

when at least one of the $Q(A/P)$-modules $M_{i-1}/M_i \otimes_A Q(A/P)$ is not finitely generated.

The number $\rho_P(M)$ does not depend on the choice of a Loewy series. If the whole $M$ is annihilated by $N$, this follows from the equality

$$\sum_{i=1}^{n} \text{length } M_{i-1}/M_i \otimes_A Q(A/P) = \text{length } M \otimes_A Q(A/P)$$

which holds because $Q(A/P)$ is flat as a left $A/N$-module. In general, independence of a choice is proved by passing to refinements.

The reduced rank $\rho(M)$ of $M$ is the sum $\sum \rho_P(M)$ over all minimal primes $P$. However, we will need the values $\rho_P(M)$ only at primes $P \in \Omega(A)$, while there could exist minimal primes outside this set.

**Lemma 1.4:** For a right $A$-module $M$ we have $M \in T_r$ if and only if $\rho_P(M) = 0$ for each $P \in \Omega(A)$.

**Proof.** Take a chain of submodules $M = M_0 \supset M_1 \supset \cdots \supset M_n = 0$ in which each factor $M_{i-1}/M_i$ is annihilated by a prime ideal, say $P_i$, of $A$. Such a chain exists by Lemma 1.3, and it is a Loewy series since $N \subset P_i$ for each $i = 1, \ldots, n$.

For $P \in \Omega(A)$ the equality $\rho_P(M) = 0$ holds if and only if

$$M_{i-1}/M_i \otimes_A Q(A/P) = 0$$

for each $i$. If $P_i \neq P$ for some $i$, then the latter equality is always true since the image of $P_i$ in the Goldie ring $A/P$ is a nonzero ideal, and therefore there exists $s \in P_i$ such that the coset $s + P$ is a regular element of $A/P$, while at the same time $s$ annihilates $M_{i-1}/M_i$. In other words, only those indices $i$ for which $P_i = P$ have to be checked.
Thus $\rho_P(M) = 0$ for each $P \in \Omega(A)$ if and only if

$$M_{i-1}/M_i \otimes_A Q(A/P_i) = 0$$

for each $i$ with $P_i \in \Omega(A)$. By Lemma 1.1 this is equivalent to the condition that $M_{i-1}/M_i \in T_r$ for all such indices $i$. We also know that $M_{i-1}/M_i \in T_r$ for all $i$ with $P_i \notin \Omega(A)$, and the desired conclusion follows from the fact that $M \in T_r$ if and only if $M_{i-1}/M_i \in T_r$ for each $i = 1, \ldots, n$.

**Lemma 1.5:** Let $M', M''$ be two submodules of a right $A$-module $M$. Suppose that $\rho_P(M) < \infty$ for each $P \in \Omega(A)$. Then:

(a) $M/M' \in T_r$ if and only if $\rho_P(M) = \rho_P(M')$ for each $P \in \Omega(A)$.

(b) If $M/M' \in T_r$ and $M'' \cong M'$, then $M/M'' \in T_r$ too.

**Proof.** Let us build a Loewy series for $M$ by first taking the preimages in $M$ of the terms of any chosen Loewy series for $M/M'$, and then adding the remaining terms from a Loewy series for $M'$. The factors $M_{i-1}/M_i$ of the Loewy series thus obtained will be those of the two Loewy series used in the construction. From this it is clear that

$$\rho_P(M) = \rho_P(M') + \rho_P(M/M')$$

for each minimal prime $P$ of $A$. Finiteness of $\rho_P(M)$ implies that $\rho_P(M') < \infty$ too and $\rho_P(M/M') = \rho_P(M) - \rho_P(M')$. Lemma 1.4 tells us that $M/M' \in T_r$ if and only if $\rho_P(M/M') = 0$ for each $P \in \Omega(A)$, which amounts to (i).

Similarly, $M/M'' \in T_r$ if and only if $\rho_P(M) = \rho_P(M'')$ for each $P \in \Omega(A)$. If $M'' \cong M'$, then $\rho_P(M'') = \rho_P(M')$ for all $P \in \Omega(A)$, and (ii) is immediate.

**Corollary 1.6:** Let $\alpha \in \text{End}_A M$. If $\text{Ker} \alpha = 0$ and $\rho_P(M) < \infty$ for each $P \in \Omega(A)$ then $M/\alpha(M) \in T_r$.

**Proof.** Take $M' = M$ and $M'' = \alpha(M) \cong M$ in Lemma 1.5.

**Lemma 1.7:** Let $M$ be a right $A$-module, $M'$ its submodule such that $M/M' \in T_r$. Then $\rho_P(M) = \rho_P(M')$ for each $P \in \Omega(A)$. In particular, $\rho_P(M) < \infty$ if and only if $\rho_P(M') < \infty$.

**Proof.** This is immediate from the equality $\rho_P(M) = \rho_P(M') + \rho_P(M/M')$ which we have observed in the proof of Lemma 1.5. Indeed, $\rho_P(M/M') = 0$ for each $P \in \Omega(A)$ by Lemma 1.4.
Lemma 1.9 below serves as a means to verify that $\rho_P(M) < \infty$. Its proof makes use of Lemma 1.8. A bimodule $M$ is said to be $k$-central if the two module structures on $M$ are extensions of the same $k$-vector space structure, i.e., $av = va$ for all $a \in k$ and $v \in M$.

**Lemma 1.8:** Let $M$ be a $k$-central $(F,E)$-bimodule where $F$ and $E$ are extension fields of the base field $k$. Suppose that $F$ is a finitely generated extension of $k$ with $\text{trdeg} \, F/k = \text{trdeg} \, E/k$. If $\dim_F M < \infty$, then $\dim_E M < \infty$.

This follows from Schofield’s result on bimodules over stratiform simple Artinian rings [30, Th. 24]. The special case that we need is much easier, but it is not clear whether it has ever been considered separately.

**Lemma 1.9:** Let $M$ be a $k$-central $(F,A)$-bimodule where $F$ is a finitely generated extension field of the base field $k$. Suppose that $\text{trdeg} \, F/k = d(A)$. If $\dim_F M < \infty$, then $\rho_P(M) < \infty$ for each $P \in \Omega(A)$.

**Proof.** There is a chain of $A$-submodules $M = M_0 \supset M_1 \supset \cdots \supset M_n = 0$ in which each factor is annihilated by a prime ideal of $A$. In the construction of such a chain we may take $M_i = M_{i-1}P_i$ with $P_i \in \text{Spec} \, A$ for each $i = 1, \ldots, n$. Then each $M_i$ is a subbimodule of $M$. Since

$$\rho_P(M) = \sum \rho_P(M_{i-1}/M_i),$$

it suffices to consider the case $n = 1$ when $M$ is annihilated by $P_1$.

Let $P \in \Omega(A)$. If $P \neq P_1$, then $M \otimes_A Q(A/P) = 0$, and $\rho_P(M) = 0$. Suppose that $P = P_1$. Then $M$ is a right $A/P$-module. Its torsion submodule $M'$ consists of all elements whose annihilator in $A/P$ contains a regular element of that ring. We have $M' \otimes_A Q(A/P) = 0$. Clearly $M'$ is a subbimodule of $M$.

The factor bimodule $M'' = M/M'$ is torsionfree as a right $A/P$-module. Each regular element of $A/P$ acts on $M''$ as an injective $F$-linear transformation. Since $\dim_F M'' < \infty$, this transformation is invertible. Hence $M''$ is a right $Q(A/P)$-module in a natural way. The center $E$ of $Q(A/P)$ is an extension field of $k$. By [13, Th. 10.5] or [25, Prop. 6.3.40]

$$\text{trdeg} \, E/k = \text{GKdim} \, A/P = d(A) = \text{trdeg} \, F/k.$$ 

Lemma 1.8 applied to the $(F,E)$-bimodule $M''$ yields $\dim_E M'' < \infty$. Hence $M \otimes_A Q(A/P) \cong M''$ is a finitely generated $Q(A/P)$-module. □
Hereditary torsion theories on the categories of modules are characterized by means of certain filters of one-sided ideals called Gabriel topologies \[37, \text{Ch. VI, Th. 5.1}\]. The filter \(G_r\) corresponding to the torsion class \(T_r\) consists of all right ideals \(I\) of the algebra \(A\) such that \(A/I \in T_r\). It is a set of right ideals satisfying the following properties of a **right Gabriel topology**:

\(T4\) if \(J \in G_r\) and \(I\) is a right ideal such that \(J \subseteq I\) then \(I \in G_r\),

\(T4\) if \(I, J \in G_r\) then \(I \cap J \in G_r\),

\(T4\) if \(I \in G_r\) then \((I : a) \in G_r\) for each \(a \in R\),

\(T4\) if \(J \in G_r\) and \(I\) is a right ideal such that \((I : a) \in G_r\) for all \(a \in J\) then \(I \in G_r\), where \((I : a) = \{x \in A \mid ax \in I\}\).

A right \(A\)-module \(M\) belongs to the torsion class \(T_r\) if and only if each element of \(M\) is annihilated by a right ideal in \(G_r\). We will use the term \(G_r\)-**torsion module** for a module in this torsion class.

An arbitrary right \(A\)-module \(M\) has a largest \(G_r\)-torsion submodule. We say that \(M\) is \(G_r\)-**torsionfree** if it contains no nonzero \(G_r\)-torsion submodules. Such modules form the torsionfree class of the torsion theory considered here. By the definition of \(T_r\) any nonzero \(G_r\)-torsion module contains a nonzero submodule of Gelfand–Kirillov dimension less than \(d(A)\). Therefore a right \(A\)-module \(M\) is \(G_r\)-torsionfree if and only if \(\text{GKdim } M' \geq d(A)\) for each nonzero submodule \(M' \subset M\).

The previous definitions were concerned with right \(A\)-modules and right ideals of \(A\). Denote by \(T_l\) the torsion class of left \(A\)-modules defined similarly to \(T_r\), and let \(G_l\) be the left Gabriel topology determined by the left ideals \(I\) of \(A\) such that \(A/I \in T_l\). We will speak in a similar way about \(G_l\)-**torsion module** and \(G_l\)-**torsionfree** left \(A\)-modules. The reduced ranks \(\rho_P(M)\) make sense also for left \(A\)-modules.

**Lemma 1.10:** For a two-sided ideal \(I\) the following conditions are equivalent:

(a) \(I \in G_l\),

(b) \(I \in G_r\),

(c) \(\text{GKdim } A/P < d(A)\) for each prime ideal \(P\) of \(A\) such that \(I \subseteq P\).

**Proof.** By Lemma 1.3 the factor algebra \(A/I\) has a finite chain of right ideals in which each factor is annihilated by a prime ideal of \(A/I\), i.e., each factor is annihilated by a prime ideal of \(A\) containing \(I\). Property (c) implies that each of these factors has Gelfand–Kirillov dimension less than \(d(A)\), and therefore \(A/I \in T_r\), i.e., \(I \in G_r\). This shows that (c)⇒(b).
Conversely, if \( A/I \in T_r \), then \( A/P \in T_r \) for each prime ideal \( P \) of \( A \) containing \( I \), but this is equivalent to (c) in view of Lemma 1.1. Thus (b) \( \iff \) (c). By symmetry we also have (a) \( \iff \) (c).

2. The symmetric quotient ring

Each Gabriel topology on a ring \( R \) gives rise to a localization of \( R \) [37, Ch. IX]. We are going to apply this construction to the Gabriel topology \( G_r \) introduced in section 1. We continue to assume that \( A \) is a finitely generated PI algebra. Furthermore, we will need several additional assumptions:

(A1) \( A \) satisfies the ACC on right and left annihilators,
(A2) \( \text{GKdim } I \geq d(A) \) for each nonzero one-sided ideal \( I \) of \( A \),
(A3) \( \rho_P(A_A) < \infty \) and \( \rho_P(A_A) < \infty \) for each \( P \in \Omega(A) \).

The subscripts in \( A_A \) and \( A A \) indicate that \( A \) is regarded as a module over itself with respect to either right or left multiplications, respectively. Condition (A2) should be understood as the bound on the Gelfand–Kirillov dimension of all right ideals regarded as right \( A \)-modules and all left ideals regarded as left \( A \)-modules. It means precisely that \( A_A \) is \( G_r \)-torsionfree and \( A A \) is \( G_l \)-torsionfree.

The **right quotient ring** of \( A \) with respect to \( G_r \) is defined as the direct limit

\[
Q_r(A) = \lim_{\longrightarrow} \text{Hom}_A(I, A).
\]

Each element of \( Q_r(A) \) is represented by a right \( A \)-linear map \( \alpha : I \to A \) with a right ideal \( I \in G_r \), and another right \( A \)-linear map \( \beta : J \to A \) with \( J \in G_r \) represents the same element of \( Q_r(A) \) if and only if \( \beta \) agrees with \( \alpha \) on a smaller right ideal from the filter \( G_r \). The multiplication in \( Q_r(A) \) is induced by the composition of maps. The algebra \( A \) is identified with the subring of \( Q_r(A) \) consisting of all elements represented by left multiplications in \( A \).

For each \( q \in Q_r(A) \) there exists \( I \in G_r \) such that \( qI \subset A \), and \( qI \neq 0 \) whenever \( q \neq 0 \). Indeed, if \( q \) is represented by \( \alpha : I \to A \) then \( qx = \alpha(x) \) for all \( x \in I \). It follows that \( Q_r(A) \) is \( G_r \)-torsionfree as a right \( A \)-module, and each nonzero right \( A \)-submodule of \( Q_r(A) \) has nonzero intersection with \( A \). (This implies that \( Q_r(A) \) is a rational extension of \( A \) in the category of right \( A \)-modules, and so \( Q_r(A) \) is a subring of the maximal right quotient ring of \( A \), as discussed in [8].)
Now we introduce the **symmetric quotient ring** of $A$. Put

$$Q(A) = \{ q \in Q_r(A) \mid \text{there exists } J \in \mathcal{G}_l \text{ such that } Jq \subset A \}.$$ 

We will write $Q_r = Q_r(A)$ and $Q = Q(A)$ for short, and the notation $Q_A$ will be used when $Q$ is regarded as a right $A$-module.

**Lemma 2.1:** The set $Q$ is a subring of $Q_r$ containing $A$. It is $\mathcal{G}_l$-torsionfree as a left $A$-module and $\mathcal{G}_r$-torsionfree as a right $A$-module.

For each $q \in Q$ there exist one-sided ideals $J \in \mathcal{G}_l$ and $I \in \mathcal{G}_r$ such that $Jq \subset A$ and $qI \subset A$. In other words, the $A$-bimodule $Q/A$ is $\mathcal{G}_l$-torsion and $\mathcal{G}_r$-torsion.

**Proof.** By the definition above $Q$ consists of all $q \in Q_r$ such that the coset $q + A$ is a $\mathcal{G}_l$-torsion element of the left $A$-module $Q_r/A$. By the axioms of Gabriel topologies the $\mathcal{G}_l$-torsion elements form a submodule. This means that $Q$ is a left $A$-submodule of $Q_r$ such that $Q/A$ is $\mathcal{G}_l$-torsion and $Q_r/Q$ is $\mathcal{G}_l$-torsionfree.

Given $x \in Q$, we thus have $Ax \subset Q$. It follows that $Qx/(Qx \cap Q)$ is an epimorphic image of the left $A$-module $Q/A$. Hence $Qx/(Qx \cap Q)$ is $\mathcal{G}_l$-torsion. On the other hand, this module is $\mathcal{G}_l$-torsionfree since it embeds in $Q_r/Q$. This is possible only when $Qx/(Qx \cap Q) = 0$, i.e., $Qx \subset Q$. Hence $Q$ is closed under the multiplication in $Q_r$, and so $Q$ is indeed a subring.

Suppose that $Jq = 0$ for some $q \in Q_r$ and $J \in \mathcal{G}_l$. There exists $I \in \mathcal{G}_r$ such that $qI \subset A$. Since $JqI = 0$ and $A$ is $\mathcal{G}_l$-torsionfree, we deduce that $qI = 0$, but then $q = 0$. This shows that $Q_r$ is not only $\mathcal{G}_r$-torsionfree, but also $\mathcal{G}_l$-torsionfree. Hence so too is $Q$.

The last assertion in the lemma is clear from the definitions of $Q_r$ and $Q$. 

**Remark 1:** Consider the left quotient ring $Q_l$ of $A$ defined with respect to the left Gabriel topology $\mathcal{G}_l$. Then $Q$ is canonically isomorphic to the subring of $Q_l$ consisting of all elements $q \in Q_l$ such that $qI \subset A$ for some $I \in \mathcal{G}_r$. In this sense the construction of $Q$ is left-right symmetric.

The initial reason which forces us to work with the subring $Q$, rather than with the whole ring $Q_r$, is that for $Q_r$ we do not have the analog of the next lemma. The ACC on annihilators will be important in proving that $Q$ is semi-primary.

For a ring $R$ and its subset $X$ we denote by $\text{rann}_RX$ and $\text{lann}_RX$ the right and left annihilators of $X$ in $R$. 
Lemma 2.2: The ring $Q$ satisfies the ACC on right and left annihilators.

Proof. Given $q \in Q$, there exists $J \in \mathcal{G}_l$ such that $Jq \subset A$. Since $Q$ is $\mathcal{G}_l$-torsionfree, we have $\text{rann}_Q q = \text{rann}_Q Jq$. It follows that for each subset $X \subset Q$ there exists $X' \subset A$ such that $\text{rann}_Q X = \text{rann}_Q X'$.

Let $p \in Q$, and let $I \in \mathcal{G}_r$ be such that $pI \subset A$. Since $Q$ is $\mathcal{G}_r$-torsionfree, the equality $Xp = 0$ is equivalent to $XpI = 0$. It follows that $p \in \text{rann}_Q X$ if and only if $pI \subset A \cap \text{rann}_Q X$.

If $K_1 \subset K_2 \subset \cdots$ is an ascending chain of annihilator right ideals of $Q$, then their intersections with $A$ form an ascending chain of annihilator right ideals of $A$. Since $A$ satisfies the ACC on right annihilators, there is an integer $n > 0$ such that $A \cap K_i = A \cap K_n$ for all $i > n$. But each annihilator right ideal of $Q$ is completely characterized by its intersection with $A$, as explained in the preceding paragraph. Therefore $K_i = K_n$ for all $i > n$. This shows that $Q$ satisfies the ACC on right annihilators, and the case of left annihilators is similar.

Lemma 2.3: Let $I = uQ + \text{rann}_Q u$ where $u \in Q$ is an element satisfying $\text{rann}_Q u = \text{rann}_Q u^2$.

Then $I$ is a right ideal of $Q$ such that $Q/I$ is $\mathcal{G}_r$-torsion as a right $A$-module. We have $A \cap I \in \mathcal{G}_r$ and, moreover, $A \cap uQ + A \cap \text{rann}_Q u \in \mathcal{G}_r$.

Proof. Put $M = Q/\text{rann}_Q u$. If $x \in Q$ is any element such that $ux \in \text{rann}_Q u$, then $u^2x = 0$, whence $x \in \text{rann}_Q u$ by the assumption about $u$. This shows that the left multiplication by $u$ induces an injective endomorphism $\alpha$ of $M$ as a right $Q$-module, and therefore also as a right $A$-module.

As we know, $Q/A$ is $\mathcal{G}_r$-torsion, i.e., $Q/A \in \mathcal{T}_r$. By Lemma 1.7 assumption (A3) implies that $\rho_P(Q_A) < \infty$ for each $P \in \Omega(A)$. Since $\rho_P(M) \leq \rho_P(Q_A)$, we have $\rho_P(M) < \infty$ for each $P \in \Omega(A)$. Now the first conclusion follows from Corollary 1.6 since $Q/I \cong M/\alpha(M)$.

The right $A$-module $A/(A \cap I)$ is $\mathcal{G}_r$-torsion since it embeds in $Q/I$. This means that $A \cap I \in \mathcal{G}_r$. Next, put

$$I_1 = A \cap uQ \quad \text{and} \quad I_2 = A \cap \text{rann}_Q u.$$  

For each right ideal $K$ of $Q$ the right $A$-module $K/(A \cap K)$ is $\mathcal{G}_r$-torsion since it embeds in $Q/A$. In particular, both $uQ/I_1$ and $(\text{rann}_Q u)/I_2$ are $\mathcal{G}_r$-torsion.
Hence so is $I/(I_1 + I_2)$, being the sum of epimorphic images of the previous two $A$-modules. Since $Q/I$ is also $G_r$-torsion, so too is $Q/(I_1 + I_2)$. It follows that $A/(I_1 + I_2) \in T_r$, which yields $I_1 + I_2 \in G_r$.

**Lemma 2.4:** If $I$ is a right ideal of $Q$ such that $A \cap I \in G_r$, then each right $A$-linear map $f : I \to Q$ is induced by a left multiplication in the larger ring $Q_r$.

**Proof.** Put $I_0 = A \cap I$ and $I' = A \cap f^{-1}(A)$. Both are right ideals of $A$ and $I' \subset I_0$. The right $A$-module $I_0/I'$ is $G_r$-torsion since it embeds in $Q/A$ by means of the map induced by $f$. Since $A/I_0$ is also $G_r$-torsion, so is $A/I'$, i.e., $I' \in G_r$.

The right $A$-linear map $f|_{I'} : I' \to A$ represents an element $q \in Q_r$ such that $qx = f(x)$ for all $x \in I'$. It remains to show that $qx = f(x)$ for all $x \in I$.

The right $A$-linear map $h : I \to Q_r$ defined by the rule $h(x) = qx - f(x)$ for $x \in I$ vanishes on $I'$. Hence $h$ factors through $I/I'$. Note that the right $A$-module $I/I'$ is a submodule of $Q/I'$, and the latter is an extension of $A/I'$ by $Q/A$. It follows that $I/I'$ is $G_r$-torsion. Since $Q_r$ is $G_r$-torsionfree, we have $\text{Hom}_A(I/I', Q_r) = 0$. Therefore $h = 0$, and we are done.

**Lemma 2.5:** Given $u \in Q$ satisfying

$$\text{rann}_Q u = \text{rann}_Q u^2 \quad \text{and} \quad \text{lann}_Q u = \text{lann}_Q u^2,$$

there exist two elements $e, v \in Q$ such that

$$eu = ue = u, \quad ev = ve = v, \quad uv = vu = e, \quad e^2 = e.$$

**Proof.** In the ring $Q$ consider its right and left ideals

$$I = uQ + \text{rann}_Q u, \quad J = Qu + \text{lann}_Q u.$$

By Lemma 2.3 $A \cap I \in G_r$.

The assumption about $u$ implies that the sum $uQ + \text{rann}_Q u$ is direct. Hence there is a right $Q$-linear map $f : I \to Q$ such that $f(x) = x$ for all $x \in uQ$ and $f(x) = 0$ for all $x \in \text{rann}_Q u$. By Lemma 2.4 $f$ is the restriction to $I$ of the left multiplication by some element $e \in Q_r$. Then

$$eu = u \quad \text{and} \quad \text{rann}_Q u \subset \text{rann}_Q e.$$

Since $(ue - u)I = 0$ and $(e^2 - e)I = 0$, it follows that $ue = u$ and $e^2 = e$, again by $G_r$-torsionfreeness of $Q_r$. Indeed, $\text{lann}_Q, I = 0$ since $A \cap I \in G_r$. 

We have to show that \( e \in Q \). For this we have to find a left ideal \( J' \) of \( A \) with the properties that \( J' \in \mathcal{G}_l \) and \( J'e \subset A \). Let us take

\[
J' = J_1 + J_2 \quad \text{where} \quad J_1 = A \cap Qu \text{ and } J_2 = A \cap \text{lann}_Q u.
\]

By the left-hand version of Lemma 2.3 applied to \( J \) we have \( J' \in \mathcal{G}_l \).

If \( y \in \text{lann}_Q u \), then \( yeu = yu = 0 \). Since \( ex = 0 \) for all \( x \in \text{rann}_Q u \), we deduce that \( yeI = 0 \), whence \( ye = 0 \) by \( \mathcal{G}_r \)-torsionfreeness of \( Q_r \). In particular, \( ye = 0 \) for all \( y \in J_2 \). For \( t \in Qu \) we have \( te = t \) since \( ue = u \).

In particular, \( te = t \in A \) for all \( t \in J_1 \). We see that \( J'e \subset A \), and the inclusion \( e \in Q \) is thus proved.

Since \( \text{rann}_Q u \subset \text{rann}_Q e \), there is a right \( Q \)-linear map \( g : I \to Q \) such that \( g(u) = e \) and \( g(x) = 0 \) for all \( x \in \text{rann}_Q u \). By Lemma 2.4 \( g \) is the restriction to \( I \) of the left multiplication by some element \( v \in Q_r \). Then

\[
vu = e \quad \text{and} \quad \text{rann}_Q u \subset \text{rann}_Q v.
\]

Observing that \( (uv - e)I = 0 \), \( (ev - v)I = 0 \), \( (ve - v)I = 0 \), we get the equalities \( uv = e \) and \( ev = ve = v \).

Consider the left ideals \( J_1, J_2, J' \) of \( A \) defined earlier, and put \( J'' = K_1 + J_2 \) where \( K_1 = A \cap J' u \subset J_1 \). The left \( A \)-module \( J_1/K_1 \) embeds in \( Qu/J'u \), which is an epimorphic image of \( Q/J' \in \mathcal{T}_l \). Hence all these modules are \( \mathcal{G}_l \)-torsion, and so too is \( J'/J'' \). Since \( J' \in \mathcal{G}_l \), we get \( J'' \in \mathcal{G}_l \).

For all \( y \in J_2 \) we have \( yvI = 0 \) since \( yvu = ye = 0 \) and \( vx = 0 \) for all \( x \in \text{rann}_Q u \). Hence \( J_2 v = 0 \) by \( \mathcal{G}_r \)-torsionfreeness of \( Q_r \). Also \( K_1 v \subset A \) since \( J'u v = J'e \subset A \). It follows that \( J''v \subset A \), and therefore \( v \in Q \).

A ring is called \textbf{semiprimary} if its Jacobson radical is nilpotent and the factor ring by the Jacobson radical is semisimple Artinian.

**Lemma 2.6:** The ring \( Q \) is semiprimary.

**Proof.** Since the ring \( Q \) satisfies the ACC on right and left annihilators by Lemma 2.2, all its nil subrings are nilpotent [12, Th. 1]. Hence \( Q \) has a nilpotent ideal \( N \) which contains every nil right ideal of \( Q \).

We claim that each right ideal \( I \) of \( Q \) has the form \( I = eQ + K \) where \( e \in Q \) is an idempotent and \( K \) a nil right ideal of \( Q \), so that \( K \subset N \). In order to prove this consider the set

\[
X = \{ u \in I \mid \text{rann}_Q u = \text{rann}_Q u^2 \text{ and } \text{lann}_Q u = \text{lann}_Q u^2 \}.
\]
The lattices of annihilator right and left ideals of $Q$ are antiisomorphic to each other. Therefore the ACC on left annihilators implies that $Q$ also satisfies the DCC on right annihilators. In particular, the set \{rann$_Q x \mid x \in X$\} has a minimal element. Pick $u \in X$ such that rann$_Q u$ is a minimal element of that set of right ideals.

By Lemma 2.5 there is an idempotent $e \in Q$ such that $uQ = eQ$ and $Qu = Qe$. The first of these two equalities shows that $e \in I$ since $u \in I$. The second equality implies that rann$_Q u = \text{rann}_Q e = (1 - e)Q$. Since $Q = eQ \oplus (1 - e)Q$, we get

$I = eQ \oplus K$ \quad \text{where} \quad K = I \cap (1 - e)Q$.

We will prove that $K$ is nil. Suppose on the contrary that $K$ contains a non-nilpotent element $y$. The right ideals rann$_Q y^i$, $i = 1, 2, \ldots$, form an ascending chain which has to stabilize by the ACC. Replacing $y$ with $y^n$ for sufficiently large $n$, we may assume therefore that rann$_Q y^2 = \text{rann}_Q y$.

Put $t = e + y \in I$. Since $y \in K$, we have $eQ \cap yQ = 0$. Hence

$$\text{rann}_Q t = \text{rann}_Q e \cap \text{rann}_Q y \subset (1 - e)Q.$$ 

Since $ey = 0$, but $ty = y^2 \neq 0$, the last inclusion is proper. Next, $t^2 = e + ye + y^2$.

If $t^2 q = 0$ for some $q \in Q$, then $eq = 0$ since $eQ \cap yQ = 0$, and we must also have $y^2 q = 0$. It follows that

$$\text{rann}_Q t^2 = \text{rann}_Q e \cap \text{rann}_Q y^2 = \text{rann}_Q t,$$

and therefore rann$_Q t^i = \text{rann}_Q t$ for all $i > 0$. Since the ascending chain of left ideals lann$_Q t^i$, $i = 1, 2, \ldots$, stabilizes, we have $t^m \in X$ for sufficiently large $m$. On the other hand, rann$_Q t^m$ is properly contained in rann$_Q u = (1 - e)Q$, which contradicts the choice of $u$.

Thus our claim about the right ideals of $Q$ has been proved. It follows that each right ideal of the factor ring $Q/N$ is generated by an idempotent. Hence $Q/N$ is semisimple Artinian, and $N$ is the Jacobson radical of $Q$.

When we apply notions of the $G_l$-torsion and $G_r$-torsion theories to $Q$-modules we always regard $Q$-modules as $A$-modules.

**Proposition 2.7:** In addition to (A1), (A2), (A3) assume another condition:

(A4) each right $Q$-module is $G_r$-torsionfree and each left $Q$-module is $G_l$-torsionfree.

Then $Q$ is a right and left Artinian classical quotient ring of $A$. 

Proof. In several arguments below we follow the proof of [35, Prop. 4.9] with some modifications. By Lemma 2.6 \( Q \) is semiprimary. Hence \( Q \) has finitely many maximal ideals, and all factor rings by those ideals are simple Artinian.

**Claim 1:** If \( M \) is any maximal ideal of \( Q \), then \( M \cap A \) is a prime ideal of \( A \).

There is an embedding of factor rings \( A/(M \cap A) \hookrightarrow Q/M \). Viewing \( Q/M \) as a right \( A/(M \cap A) \)-module, we deduce from Lemma 1.3 that \( Q/M \) has a nonzero element whose right annihilator contains a prime ideal of \( A/(M \cap A) \). In other words, there exist \( y \in Q \) and \( P \in \text{Spec} \ A \) such that \( y \in M \), \( M \cap A \subset P \) and \( yP \subset M \).

Arguing as in [35] we deduce that there is \( I \in \mathcal{G}_r \) such that the left ideal \( L(I) = \{ x \in Q \mid xIP \subset M \} \)
of \( Q \) coincides with the whole \( Q \). Then \( IP \subset M \), and therefore also \( I'P \subset M \) where \( I' = AI \) is a two-sided ideal of \( A \) containing \( I \). Since \( I' \in \mathcal{G}_l \) by Lemma 1.10. This shows that the image of \( P \) in \( Q/M \) is a \( \mathcal{G}_l \)-torsion left \( A \)-submodule of \( Q/M \). Since \( Q/M \) is \( \mathcal{G}_l \)-torsionfree by condition \((A4)\), we get \( P \subset M \). Hence \( M \cap A = P \).

**Claim 2:** Denote by \( N \) the prime radical of \( A \) and by \( J \) the Jacobson radical of \( Q \). Then \( N = J \cap A \) and \( J = NQ \).

Let \( M_1, \ldots, M_k \) be all the maximal ideals of \( Q \). Then \( J = \bigcap M_i \). By Claim 1 \( P_i = M_i \cap A \) is a prime ideal of \( A \) for each \( i \). We have \( J \cap A = N' \) where \( N' = \bigcap P_i \). Since \( J \) is nilpotent, \( N' \) is a nilpotent ideal of \( A \). Hence \( N' \) is contained in each prime ideal of \( A \), and therefore \( N' = N \).

The equality \( N = J \cap A \) implies that \( N \subset J \) and \( J/N \) is a \( \mathcal{G}_r \)-torsion right \( A \)-module. Then \( NQ \subset J \) and \( J/NQ \) is \( \mathcal{G}_r \)-torsion as well. Since \( J/NQ \) is a right \( Q \)-module, it is \( \mathcal{G}_r \)-torsionfree by \((A4)\). It follows that \( J/NQ = 0 \), i.e., \( J = NQ \).

**Claim 3:** The factor ring \( Q/J \) is a classical quotient ring of \( A/N \).

Put \( S = Q/J \) and \( R = A/N \). The ring \( S \) is semisimple Artinian, while \( R \) is a finitely generated semiprime PI algebra. By [23, Ch. V, Th. 2.5] \( R \) is semiprime right and left Goldie. Let \( \pi : Q \to S \) be the canonical homomorphism. By Claim 2 \( R \) is identified with the subring \( \pi(A) \) of \( S \).
Consider the set \( \{ \pi(I) \mid I \in \mathcal{G}_r \} \) of right ideals of \( R \). By (A4) \( S \) is \( \mathcal{G}_r \)-torsionfree as a right \( A \)-module, which means that \( \text{lam}_S \pi(I) = 0 \) for each \( I \in \mathcal{G}_r \). If \( q \in Q \), then \( qI \subset A \), and therefore \( \pi(q)\pi(I) \subset \pi(A) \), for some \( I \in \mathcal{G}_r \). Now Claim 3 follows from [35, Lemma 4.8].

Denote by \( C \) the set of all elements \( u \in A \) which are regular modulo \( N \), i.e., whose images \( \pi(u) \) in the ring \( \pi(A) \cong A/N \) are regular elements of that ring. If \( u \in C \), then \( \pi(u) \) is invertible in \( Q/J \) by Claim 3; this implies that \( u \) is invertible in \( Q \) since \( J \) is the Jacobson radical of \( Q \). Thus all elements of \( C \) are regular in \( A \).

Conversely, suppose that \( u \) is any regular element of \( A \). Then \( \text{rann}_Q u \) is a right ideal of \( Q \) which has zero intersection with \( A \). This entails \( \text{rann}_Q u = 0 \). Similarly, \( \text{lam}_Q u = 0 \), and so \( u \) is regular in \( Q \). Such an element satisfies the assumptions of Lemma 2.5. Hence \( uQ = eQ \) and \( Qu = Qe \) for some idempotent \( e \in Q \). But then \( (1 - e)u = 0 \), which forces \( e = 1 \). We conclude that \( u \) is invertible in \( Q \) (more generally, it is known that any regular element of a semiprimary ring is invertible). Since \( \pi(u) \) is then invertible in \( Q/J \), we get \( u \in C \).

It follows that \( C \) is the set of all regular elements of \( A \), and each of these elements is invertible in \( Q \). By [35, Lemma 4.8] each right ideal in the set
\[
\{ \pi(I) \mid I \in \mathcal{G}_r \}
\]
contains a regular element of the ring \( \pi(A) \). Therefore
\[
I \cap C \neq \emptyset
\]
for all \( I \in \mathcal{G}_r \). Given any \( q \in Q \), we have \( qI \subset A \) for some \( I \in \mathcal{G}_r \), whence there exists \( u \in C \) such that \( qu \in A \). This shows that \( Q \) is a classical right quotient ring of \( A \).

We have not verified yet that \( Q \) is Artinian. For the proof of this fact we will need another consequence of assumption (A4):

**Claim 4:** \( \text{GKdim} A/P = d(A) \) for each minimal prime ideal \( P \) of \( A \).

Indeed, we have observed in the proof of Claim 2 that the prime radical \( N \) of \( A \) is the intersection of the prime ideals \( P_i = M_i \cap A \). Therefore any minimal prime ideal \( P \) of \( A \) must coincide with one of these \( P_i \). In other words, \( P = M \cap A \) for some maximal ideal \( M \) of \( Q \). But then \( A/P \) embeds in \( Q/M \). Since \( Q/M \) is \( \mathcal{G}_r \)-torsionfree by (A4), so too is \( A/P \). This yields \( \text{GKdim} A/P \geq d(A) \), while the opposite inequality is clear from the definition of \( d(A) \).
We see that $\Omega(A)$ is the set of all minimal prime ideals of $A$. Denoting by $Q(A/P)$ the classical quotient ring of the prime PI algebra $A/P$, we have

$$Q/J \cong \prod_{P \in \Omega(A)} Q(A/P)$$

in view of Claim 3. Since $J$ is a two-sided ideal of $Q$ equal to $NQ$ by Claim 2, we have $J^i = N^i Q$, and since $Q$ is left flat over $A$ by a standard property of classical right quotient rings, there are isomorphisms $J^i \cong N^i \otimes_A Q$ and

$$J^i/J^{i+1} \cong N^i/N^{i+1} \otimes_A Q/J \cong \prod_{P \in \Omega(A)} N^i/N^{i+1} \otimes_A Q(A/P)$$

for all $i \geq 0$. The ideals $N^i$ form a Loewy series for the right $A$-module $A_A$. The finiteness of $\rho_P(A_A)$ implies that all right $Q/J$-modules $N^i/N^{i+1} \otimes_A Q(A/P)$ have finite length. Hence so too do the $Q/J$-modules $J^i/J^{i+1}$, and we conclude that $Q$ has finite length as a right module over itself.

Thus $Q$ is right Artinian. By the left-right symmetry $Q$ is also left Artinian and is a classical left quotient ring of $A$.

3. Finitely generated PI module algebras

Let $H$ be a Hopf algebra over the base field $k$ and $A$ an $H$-module algebra. This means that $A$ is a $k$-algebra equipped with a left $H$-module structure such that

$$h(ab) = \sum (h(1)a)(h(2)b) \quad \text{for all } h \in H \text{ and } a, b \in A,$$

and $h1_A = \varepsilon(h)1_A$ for all $h \in H$ where $\varepsilon: H \to k$ is the counit. The algebra $A$ is said to be $H$-semi-prime if $A$ has no nonzero nilpotent $H$-stable ideals, and $A$ is $H$-prime if $A \neq 0$ and $IJ \neq 0$ for each pair of nonzero $H$-stable ideals $I$ and $J$ of $A$.

Our aim in this section is to prove that an $H$-prime $H$-module algebra $A$ has an Artinian classical quotient ring in the case when $A$ is a finitely generated PI algebra with the additional assumptions that $A$ satisfies the ACC on right and left annihilators and the action of $H$ on $A$ is locally finite in the sense that $\dim Ha < \infty$ for all $a \in A$ (see Theorem 3.9). As an intermediate step we will investigate the symmetric quotient ring $Q = Q(A)$ introduced in Section 2. We will verify that $A$ satisfies all conditions needed for an application of Proposition 2.7.
The antipode $S$ of $H$ is not assumed to be bijective. Because of this we need slightly more complicated constructions in which the Hopf subalgebra $S(H)$ of $H$ is taken into account. In some places $A$ is assumed to be $S(H)$-prime, which is the stronger requirement that $IJ \neq 0$ for each pair of nonzero $S(H)$-stable ideals $I$ and $J$ of $A$. However, the difference with the $H$-primeness eventually disappears.

It has been observed in [35, Cor. 2.9] that each $S(H)$-submodule of a locally finite-dimensional $H$-module is automatically an $H$-submodule. This is an easy consequence of the fact that the antipode of the dual Hopf algebra $H^\circ$ is always injective (see [31]). In particular, we have

**Lemma 3.1:** If the action of $H$ on $A$ is locally finite, then each $S(H)$-stable ideal of $A$ is $H$-stable, and so the $H$-primeness of $A$ is equivalent to the $S(H)$-primeness.

For an arbitrary algebra $A'$ and a coalgebra $C$ denote by $[C, A']$ the vector space $\text{Hom}_k(C, A')$ equipped with the convolution multiplication. If $\dim C < \infty$, then

$$[C, A'] \cong A' \otimes C^*$$

as algebras. Let $C^{\text{cop}}$ be $C$ with the opposite comultiplication. Then $[C^{\text{cop}}, A']$ is $\text{Hom}_k(C, A')$ equipped with the multiplication

$$(\xi \times \eta)(c) = \sum \xi(c_{(2)})\eta(c_{(1)}), \quad \xi, \eta \in \text{Hom}_k(C, A'), \quad c \in C.$$

For an ideal $I$ of $A$ and a subcoalgebra $C$ of $H$ put

$$I_C = \{ a \in A \mid Ca \subset I \},$$

which is another ideal of $A$. Define a map $\theta : A \to [C^{\text{cop}}, A/I]$, $a \mapsto \theta_a$, setting

$$\theta_a(c) = S(c)a + I, \quad a \in A, \quad c \in C.$$

We have $\theta_{ab} = \theta_a \times \theta_b$ for $a, b \in A$ since

$$S(c)(ab) = \sum (S(c_{(2)})a)(S(c_{(1)})b)$$

for all $c \in C$. Note that $\theta_1$ is the identity element $c \mapsto \varepsilon(c) + I$ of the algebra $[C^{\text{cop}}, A/I]$. Thus $\theta$ is a homomorphism of unital algebras and

$$\text{Ker} \theta = I_{S(C)} = \{ a \in A \mid S(C)a \subset I \}.$$
Lemma 3.2: Let $C \in \mathcal{F}$. Then the algebra $B = [C^{\text{cop}}, A/I]$ is a finitely generated module over its subalgebra $\theta(A)$ both on the left and on the right.

Proof. We will prove that $B$ is generated as a $\theta(A)$-module by the finite-dimensional subspace $C^* \subset \text{Hom}_k(C, A/I)$. Note that $\theta$ factors as

$$A \longrightarrow [C^{\text{cop}}, A] \longrightarrow [C^{\text{cop}}, A/I]$$

where the first map is the special case of $\theta$ obtained for the zero ideal of the algebra $A$, while the second map is induced by the canonical surjection $A \rightarrow A/I$. Thus it suffices to prove the lemma assuming that $I = 0$. In this case we define a linear transformation $\Psi$ of the space $\text{Hom}_k(C, A)$ by the rule

$$(\Psi \xi)(c) = \sum S(c(2))\xi(c(1)), \quad \xi \in \text{Hom}_k(C, A), \ c \in C.$$ 

It is bijective with the inverse transformation

$$(\Psi^{-1} \xi)(c) = \sum S^2(c(2))\xi(c(1)).$$

The space $\text{Hom}_k(C, A)$ is generated by $C^*$ as a left $A$-module with respect to the plain action of $A$ such that

$$(a \xi)(c) = a \xi(c) \quad \text{for} \ a \in A, \ \xi \in \text{Hom}_k(C, A), \ c \in C.$$ 

Since

$$\Psi(a \xi)(c) = S(c(2))(a \xi(c(1))) = \sum (S(c(3))a)(S(c(2))\xi(c(1)))$$

$$= \sum \theta_a(c(2))(\Psi \xi)(c(1)) = (\theta_a \times \Psi \xi)(c)$$

for all $c$, we have

$$\Psi(a \xi) = \theta_a \times \Psi \xi$$

for all $a$ and $\xi$. Thus $\Psi$ transforms the left module structure on $\text{Hom}_k(C, A)$ given by the plain action of $A$ to the one obtained via the algebra homomorphism $\theta$. Since $\Psi \xi = \xi$ for all $\xi \in C^*$, we get $B = \theta(A)C^*$ in the algebra $B$. The other equality $B = C^* \theta(A)$ is obtained by means of similar arguments which use the bijective transformation $\Phi$ of the space $\text{Hom}_k(C, A)$ defined by the rule

$$(\Phi \xi)(c) = \sum S(c(1))\xi(c(2)).$$

Lemma 3.3: Let $A$ be a finitely generated $S(H)$-prime $H$-module PI algebra. If $P$ is a prime ideal of $A$ such that $\text{ann}_A P \neq 0$, then $P_{S(C)} = 0$ for some $C \in \mathcal{F}$. In this case $\theta : A \rightarrow [C^{\text{cop}}, A/P]$ is injective.
Proof. For each $C \in \mathcal{F}$ the left annihilator $\text{lann}_A P_{S(C)}$ is a two-sided ideal of $A$ since so is $P_{S(C)}$. Denote by $V(C)$ the intersection of all prime ideals of $A$ containing $\text{lann}_A P_{S(C)}$.

If $C, C' \in \mathcal{F}$ and $C \subset C'$, then $P_{S(C)} \supset P_{S(C')}$, whence $V(C) \subset V(C')$. Since $C_1 + C_2 \in \mathcal{F}$ for any two subcoalgebras $C_1, C_2 \in \mathcal{F}$, the set of semiprime ideals $\{V(C) \mid C \in \mathcal{F}\}$ is directed by inclusion, i.e., for any two ideals in this set there is a larger ideal in the same set. Since $A$ satisfies the ACC on semiprime ideals by [23, Ch. V, Cor. 2.2] or [25, Cor. 6.3.36], the set $\{V(C) \mid C \in \mathcal{F}\}$ has a largest element $V_0$. Thus $V(C) \subset V_0$ for all $C \in \mathcal{F}$.

Put $K = \text{lann}_A P$. If $a \in K$ and $b \in P_{S(C)}$, then

$$(ca)b = \sum (c(1)a)(c(2)S(c(3))b) = \sum c(1)(aS(c(2))b) = 0$$

for all $c \in C$ since $S(C)b \subset P$. This shows that $CK \subset \text{lann}_A P_{S(C)} \subset V(C)$. Hence $CK \subset V_0$ for each $C \in \mathcal{F}$, and therefore $HK \subset V_0$.

Denote by $I$ the ideal of $A$ generated by the $H$-submodule $HK$. Then $I$ is stable under the action of $H$ and $I \subset V_0$. Since $K \neq 0$ by the hypothesis, we also have $I \neq 0$.

Now pick $C \in \mathcal{F}$ such that $V_0 = V(C)$. Note that $V_0 / \text{lann}_A P_{S(C)}$ is the prime radical of the factor algebra $A / \text{lann}_A P_{S(C)}$. Since the prime radicals of finitely generated PI algebras are nilpotent, we have $V_0^n \subset \text{lann}_A P_{S(C)}$ for some $n > 0$. Then $I^n \subset \text{lann}_A P_{S(C)}$, and therefore $P_{S(C)} \subset \text{rann}_A I^n$.

Note that $I^n$ is an $H$-stable ideal of $A$. If $a \in I^n$ and $b \in \text{rann}_A I^n$, then

$$a(S(h)b) = \sum (S(h(2))h(3)a)(S(h(1))b) = \sum S(h(1))((h(2)a)b) = 0$$

for all $h \in H$ since $Ha \subset I^n$. This shows that the ideal $\text{rann}_A I^n$ is stable under the action of $S(H)$. Since $A$ is $S(H)$-prime and $I \neq 0$, we must have $\text{rann}_A I^n = 0$. Hence $P_{S(C)} = 0$ too, and therefore $\text{Ker} \theta = 0$.

Remark 2: The conclusion of Lemma 3.3 remains true when $A$ satisfies the ACC on left annihilators, but is not necessarily finitely generated PI. This follows from the fact that the set of annihilators $\{\text{lann}_A P_{S(C)} \mid C \in \mathcal{F}\}$ has a largest element, and this largest ideal in this set contains the $H$-stable ideal $I$ introduced in the proof of the lemma. By similar arguments it can be shown that Lemma 3.3 holds also in the case when $A$ satisfies the ACC on right annihilators and is $S^2(H)$-prime.
Lemma 3.4: Let $A$ be a finitely generated $S(H)$-prime $H$-module PI algebra. Then:

(i) $\text{GKdim } I = \text{GKdim } A = d(A)$ for each nonzero one-sided ideal $I$ of $A$,

(ii) $\rho_P(A_A) < \infty$ and $\rho_P(A_A) < \infty$ for each $P \in \Omega(A)$.

Proof. Take any $P' \in \text{Spec } A$ such that $\text{lann}_A P' \neq 0$. Such a prime ideal exists, as is seen from Lemma 1.3 applied to $M = A$. By Lemma 3.3 there is a coalgebra $C \in \mathcal{F}$ such that $A$ embeds in the algebra $B = [C^{\text{cop}}, A/P']$ by means of $\theta : A \to B$.

One immediate consequence of this embedding is that $\text{GKdim } A \leq \text{GKdim } B$. Now $B \cong A' \otimes D$ where $A' = A/P'$ and $D$ is the algebra dual to the coalgebra $C^{\text{cop}}$. Since $\dim D < \infty$, we have

$$\text{GKdim } B = \text{GKdim } A' \leq \text{GKdim } A.$$ 

Thus an equality is attained here, which shows also that $\text{GKdim } A = d(A)$.

Let $I$ be a right ideal of $A$. Since $B$ is a finitely generated left $\theta(A)$-module according to Lemma 3.2, there is an inequality

$$\text{GKdim}_B(I \otimes_A B) \leq \text{GKdim}_A I$$

obtained by viewing $B$ as an $(A, B)$-bimodule and applying [17, 8.3.14]. Since $\theta(I)B$ is an epimorphic image of the right $B$-module $I \otimes_A B$, we get

$$\text{GKdim}_B(\theta(I)B) \leq \text{GKdim}_A I.$$ 

We may identify $A'$ with the subalgebra $A' \otimes 1$ of $B \cong A' \otimes D$. It is clear from the definition of Gelfand–Kirillov dimension that $\text{GKdim}_B M \geq \text{GKdim}_{A'} M$ for each right $B$-module $M$. The prime PI algebra $A'$ is GK-pure by [13, Lemma 5.12]. It follows that $\text{GKdim}_{A'} M = \text{GKdim } A'$ whenever $M$ is a nonzero submodule of a free $A'$-module since such a module $M$ contains a submodule isomorphic to a nonzero right ideal of $A'$. In particular, this can be applied to right ideals of $B$ since $B$ is free as a right $A'$-module. If $I \neq 0$, then $\theta(I)B$ is a nonzero right ideal of $B$, and so

$$\text{GKdim}_B(\theta(I)B) \geq \text{GKdim}_{A'}(\theta(I)B) = \text{GKdim } A'.$$

Hence $\text{GKdim}_A I \geq \text{GKdim } A'$. On the other hand,

$$\text{GKdim}_A I \leq \text{GKdim } A = \text{GKdim } A'.$$

This shows that $\text{GKdim}_A I = \text{GKdim } A$. By symmetry, such an equality holds also for each nonzero left ideal of $A$. This completes the proof of (i).
The prime PI algebra \( A' \) has a simple Artinian classical quotient ring \( Q' \) which is finite-dimensional over its center \( F \). By [13, Th. 10.5] or [25, Prop. 6.3.40] \( \text{GKdim } A' = \text{trdeg } F/k \). Hence \( \text{trdeg } F/k = d(A) \). Note that \( F \) is a finitely generated extension field of \( k \). This follows from the fact that there is an element \( s \neq 0 \) in the center \( Z \) of \( A' \) such that the localization \( A'[s^{-1}] \) is an Azumaya algebra over its center \( Z[s^{-1}] \) (see [25, Cor. 6.1.36]). By the Artin–Tate Lemma \( Z[s^{-1}] \) is a finitely generated algebra over \( k \) since so is \( A'[s^{-1}] \). Therefore the quotient field of \( Z[s^{-1}] \), equal to \( F \), is a finitely generated extension of \( k \).

The algebra \( B \), and therefore also the algebra \( A \), embed in \( Q' \otimes D \). We may view \( M = Q' \otimes D \) as an \((F, A)\)-bimodule. Since \( \dim_F M < \infty \), Lemma 1.9 can be applied. It yields \( \rho_P(M) < \infty \) for each \( P \in \Omega(A) \). Since \( A \) is an \( A \)-submodule of \( M \), we get \( \rho_P(A_A) < \infty \) for each \( P \in \Omega(A) \). Viewing \( Q' \otimes D \) as an \((A, F)\)-bimodule, we deduce similarly the other part of (ii).

The action of \( H \) on \( A \) gives rise to certain tensoring operations on the categories of right and left \( A \)-modules. Our subsequent arguments will use the fact that the classes of \( G_r \)-torsion and \( G_l \)-torsion modules are stable under those operations.

If \( M \) is a right \( A \)-module and \( U \) is a left \( H \)-comodule, then the vector space \( M \otimes U \) is a right \( A \)-module with respect to the twisted action of \( A \) defined by the rule

\[
(m \otimes u)a = \sum m(u_{(-1)}a) \otimes u_{(0)}, \quad m \in M, \ u \in U, \ a \in A.
\]

For a left \( A \)-module \( M \) and a right \( H \)-comodule \( U \) there is a twisted left \( A \)-module structure on the vector space \( M \otimes U \) defined by the rule

\[
a(m \otimes u) = \sum (u_{(1)}a)m \otimes u_{(0)}.
\]

Here the symbolic notation

\[
\sum u_{(-1)} \otimes u_{(0)} \in H \otimes U \quad \text{and} \quad \sum u_{(0)} \otimes u_{(1)} \in U \otimes H
\]

is used to represent the values of the comodule structure maps

\[
U \to H \otimes U \quad \text{and} \quad U \to U \otimes H,
\]

respectively. Note that the above tensoring operations make sense for an arbitrary \( H \)-module algebra \( A \).
Lemma 3.5: Let $M$ be a right $A$-module and $U$ a left $H$-comodule. Then

$$\text{GKdim } M \otimes U \leq \text{GKdim } M.$$ 

Suppose that $A$ is a finitely generated $H$-module PI algebra. Then $M \otimes U \in T_r$ whenever $M \in T_r$.

Similar conclusions hold for a left $A$-module and a right $H$-comodule.

Proof. Let $V \subset A$ and $X \subset M \otimes U$ be finite-dimensional subspaces with $1 \in V$. We can find a subspace $Y \subset M$ and a subcomodule $U' \subset U$, both of finite dimension, such that $X \subset Y \otimes U'$. There is a coalgebra $C \in \mathcal{F}$ such that

$$\sum u_{(-1)} \otimes u_{(0)} \in C \otimes U'$$

for all $u \in U'$. Put $W = CV$, which is a finite-dimensional subspace of $A$. Note that $CV^n \subset W^n$ for each integer $n > 0$. Hence

$$XV^n \subset (Y \otimes U')V^n \subset Y(CV^n) \otimes U' \subset YW^n \otimes U',$$

which yields $\dim XV^n \leq (\dim U')(\dim YW^n)$ and

$$\limsup_{n \to \infty} \log_n (\dim XV^n) \leq \limsup_{n \to \infty} \log_n (\dim YW^n) \leq \text{GKdim } M.$$

Taking the supremum over all $V$ and $X$, we get the required inequality.

If $M \in T_r$, then $M$ has a chain of submodules $M = M_0 \supset M_1 \supset \cdots \supset M_n = 0$ such that

$$\text{GKdim } M_{i-1}/M_i < d(A)$$

for all $i$. The spaces $M_i \otimes U$ form a chain of submodules in $M \otimes U$. For each $i$ we have

$$(M_{i-1} \otimes U)/(M_i \otimes U) \cong (M_{i-1}/M_i) \otimes U,$$

which is an $A$-module of Gelfand–Kirillov dimension less than $d(A)$ by the already established inequality. It follows that $M \otimes U \in T_r$. \[\qed\]

Corollary 3.6: The action of $H$ on $A$ is $G_r$-continuous and $G_l$-continuous in the sense that for every $h \in H$, $I \in G_r$, $J \in G_l$ there exist $I' \in G_r$ and $J' \in G_l$ such that $hI' \subset I$ and $hJ' \subset J$.

Proof. The right $A$-module $M = A/I$ is $G_r$-torsion since $I \in G_r$. Denote by $v$ the coset $1+I \in M$. Consider $H$ as a left $H$-comodule with respect to the comultiplication in $H$. Then $M \otimes H$ is a right $A$-module with respect to the twisted
action of $A$. It is $\mathcal{G}_r$-torsion by Lemma 3.5. Hence the element $v \otimes h \in M \otimes H$ is annihilated by some right ideal $I' \in \mathcal{G}_r$. If $a \in I'$, then

$$\sum v(h(1)a) \otimes h(2) = (v \otimes h)a = 0 \quad \text{in } M \otimes H,$$

and it follows that

$$v(ha) = \sum \varepsilon(h(2))v(h(1)a) = 0.$$ 

Since $I$ is the annihilator of $v$ in $A$, we see that $ha \in I$ for all $a \in I'$, as required.

The other conclusion is obtained in a similar way by considering the left $A$-module $M \otimes H$ where $M = A/J$ and $H$ is regarded as a right $H$-comodule.

**Proposition 3.7:** Let $A$ be a finitely generated $S(H)$-prime $H$-module PI algebra satisfying the ACC on right and left annihilators.

Then its symmetric quotient ring $Q$ with respect to the Gabriel topologies $\mathcal{G}_l, \mathcal{G}_r$ is a semiprimary $S(H)$-prime $S(H)$-module algebra.

**Proof.** Assumptions (A2), (A3) of section 2 are satisfied by Lemma 3.4, while (A1) is included in the hypothesis. By Lemma 2.6 $Q$ is semiprimary. Since the action of $H$ on $A$ is $\mathcal{G}_r$-continuous, it extends to the right quotient ring $Q_r$ of $A$ with respect to $\mathcal{G}_r$, as explained in [20, Th. 3.13] (in a special case such an action was first introduced by Cohen [5, Th. 18]). This action makes $Q_r$ into an $H$-module algebra containing $A$ as an $H$-stable subalgebra.

If $q \in Q$, then $Jq \subset A$ for some $J \in \mathcal{G}_l$. Given $h \in H$, there is a coalgebra $C \in \mathcal{F}$ containing $h$. By Corollary 3.6 there exists $J' \in \mathcal{G}_l$ such that $cJ' \subset J$ for all $c \in C$. Then

$$a(S(h)q) = \sum (S(h(2))h(3)a)(S(h(1))q) = \sum S(h(1))((h(2)a)q) \in A$$

for all $a \in J'$. Hence $J'S(h)q \subset A$, and therefore $S(h)q \in Q$. This shows that $Q$ is an $S(H)$-stable subalgebra of $Q_r$.

If $I$ is any nonzero $S(H)$-stable ideal of $Q$, then $I \cap A$ is a nonzero $S(H)$-stable ideal of $A$. The condition that $A$ is $S(H)$-prime implies that $(I \cap A)(J \cap A) \neq 0$, and therefore $IJ \neq 0$, for each pair of nonzero $S(H)$-stable ideals of $Q$. Hence $Q$ is $S(H)$-prime.  

Proposition 3.7 does not give us all we want. It is not clear whether one can achieve more under the same assumptions about $A$. At this point we are going to employ the local finiteness of the action.
Recall that an $H$-module algebra is said to be $H$-simple if it has no $H$-stable ideals other than the zero ideal and the whole algebra. Our further arguments are based on the already known result for which we cite [35, Prop. 4.2]:

**Proposition 3.8:** Suppose that $Q$ is a semiprimary $H$-semiprime $H$-module algebra containing an $H$-stable subalgebra $A$ such that the action of $H$ on $A$ is locally finite and $I \cap A \neq 0$ for each nonzero right ideal $I$ of $Q$. Then there is an isomorphism of $H$-module algebras $Q \cong Q_1 \times \cdots \times Q_n$ where $Q_1, \ldots, Q_n$ are $H$-simple $H$-module algebras.

**Theorem 3.9:** Let $A$ be a finitely generated $H$-prime $H$-module PI algebra satisfying the ACC on right and left annihilators. Suppose that the action of $H$ on $A$ is locally finite. Then:

(i) $\text{GKdim } I = \text{GKdim } A$ for each nonzero one-sided ideal $I$ of $A$,
(ii) $\text{GKdim } A/P = \text{GKdim } A$ for each minimal prime ideal $P$ of $A$,
(iii) $A$ has a right and left Artinian classical quotient ring $Q$,
(iv) $Q$ is an $H$-simple $H$-module algebra.

**Proof.** By Lemma 3.1 $A$ is $S(H)$-prime. Hence Lemma 3.4 and Proposition 3.7 apply to $A$. In particular, assumptions (A1)–(A3) of section 2 are satisfied, and (i) holds by Lemma 3.4. Let $Q$ be the symmetric quotient ring of $A$ constructed in section 2. We will verify condition (A4) in the statement of Proposition 2.7. This will establish assertion (iii), and (ii) will follow at once from Lemma 3.4 and Claim 4 in the proof of Proposition 2.7.

By Proposition 3.7 $Q$ is a semiprimary $S(H)$-prime $S(H)$-module algebra. The algebra $A$ embeds in $Q$ as an $S(H)$-stable subalgebra. By the construction of $Q$, each nonzero right $A$-submodule of $Q$ has nonzero intersection with $A$. Hence we can apply Proposition 3.8 with $H$ replaced by $S(H)$. It follows that $Q \cong Q_1 \times \cdots \times Q_n$ where each $Q_i$ is an $S(H)$-simple $S(H)$-module algebra. Since $Q$ is $S(H)$-prime, we must have $n = 1$, and so $Q$ is $S(H)$-simple.

For a right $A$-module $M$ and a left $H$-comodule $U$ we have defined the twisted action of $A$ on the vector space $M \otimes U$. If $M$ is a right $Q$-module and $U$ is a left $S(H)$-comodule, then $M \otimes U$ is a right $Q$-module with respect to the twisted action of $Q$. Take $U = S(H)$, which is a left $S(H)$-comodule with respect to the comultiplication. If $I$ is the annihilator of $M$ in $Q$, then the annihilator of $M \otimes S(H)$ in $Q$ is the $S(H)$-stable ideal

$$I_{S(H)} = \{ q \in Q \mid S(H)q \subseteq I \}.$$
Indeed, if \( q \in Q \) annihilates \( M \otimes S(H) \), then for all \( m \in M \) and \( h \in S(H) \) we have
\[
0 = (m \otimes h)q = \sum m(h_{(1)}q) \otimes h_{(2)} \quad \text{in } M \otimes S(H),
\]
and applying the map \( \text{id} \otimes \varepsilon \), we get \( m(hq) = 0 \), which shows that \( q \in I_{S(H)} \).
Conversely, it is seen from the formula for the twisted action that all elements of \( I_{S(H)} \) annihilate \( M \otimes S(H) \).

If \( M \neq 0 \), then \( I_{S(H)} = 0 \) since \( Q \) is \( S(H) \)-simple. In this case the \( Q \)-module \( M \otimes S(H) \) is faithful. Suppose that \( M \) is \( G_r \)-torsion. Then so is \( M \otimes S(H) \) by Lemma 3.5. Since \( Q \) is semiprimary, it satisfies the DCC on finitely generated right ideals [37, Ch. VIII, Prop. 5.5]. Hence \( Q \) has a simple right ideal, say \( V \). Since \( V \) does not annihilate \( M \otimes S(H) \), it embeds in \( M \otimes S(H) \) as a \( Q \)-submodule. But then \( V \) has to be \( G_r \)-torsion, which contradicts the fact that \( Q \) is \( G_r \)-torsionfree (see Lemma 2.1).

The preceding argument shows that there exist no nonzero right \( Q \)-modules which are \( G_r \)-torsion. If \( M \) is any right \( Q \)-module, then its largest \( G_r \)-torsion \( A \)-submodule \( T \) is in fact a \( Q \)-submodule, whence \( T = 0 \). This proves that each right \( Q \)-module is \( G_r \)-torsionfree. Using tensoring operations on left \( Q \)-modules, it is proved in a similar way that each left \( Q \)-module is \( G_l \)-torsionfree.

Now we can use Proposition 2.7 and all parts in its proof. There we have seen that \( Q \) is a classical quotient ring of \( A \) and each right ideal \( I \in G_r \) contains a regular element of \( A \). This implies that \( Q = Q_r \). Indeed, for each \( q \in Q_r \) there exists a regular element \( s \in A \) such that \( qs \in A \), but then \( q \in Q \) since \( s^{-1} \in Q \).

In the proof of Proposition 3.7 we have seen that the action of \( H \) on \( A \) extends to \( Q_r \). Therefore \( Q \) is not only an \( S(H) \)-module algebra, but an \( H \)-module algebra. It is \( H \)-simple by Proposition 3.8. This proves the remaining assertion (iv).

The existence of an Artinian classical quotient ring proved in Theorem 3.9 for an \( H \)-prime algebra can be extended to the \( H \)-semiprime case by applying standard ring-theoretic methods. This result will not be needed, so we state it without proof:

**Theorem 3.10:** Let \( A \) be a finitely generated \( H \)-semiprime \( H \)-module PI algebra satisfying the ACC on right and left annihilators. Suppose that the action of \( H \) on \( A \) is locally finite. Then \( A \) has a right and left Artinian classical quotient ring.
4. Flatness over coideal subalgebras and Hopf subalgebras

We have come to the main results of the paper. Let \( H \) be a Hopf algebra over the field \( k \) with the comultiplication \( \Delta \), the counit \( \varepsilon \) and the antipode \( S \). Any subalgebra \( A \subset H \) such that
\[
\Delta(A) \subset A \otimes H
\]
is called a right coideal subalgebra.

If \( H \) has a right Artinian classical right quotient ring \( Q(H) \), then \( S \) is bijective by \([31, \text{Th. A}]\). In this case \( S \) is an antiautomorphism of \( H \), and \( S \) extends to an antiautomorphism of \( Q(H) \), which implies that \( Q(H) \) is also a left quotient ring and is left Artinian. Therefore it will not lessen generality of results if we use two-sided conditions on \( Q(H) \) instead of one-sided. By an Artinian ring we mean a ring which is left and right Artinian.

**Theorem 4.1:** Let \( A \) be a PI right coideal subalgebra of a residually finite-dimensional Hopf algebra \( H \) which has an Artinian classical quotient ring \( Q(H) \). Then:

(i) \( A \) has an Artinian classical quotient ring \( Q(A) \),

(ii) \( H \) is left and right \( A \)-flat,

(iii) if \( A \) is a Hopf subalgebra, then \( H \) is left and right faithfully \( A \)-flat.

**Proof.** Let \( A^{op} \) and \( H^{op} \) be \( A \) and \( H \) with the opposite multiplications. Note that \( H^{op} \) is a residually finite-dimensional Hopf algebra, and \( A^{op} \) a PI right coideal subalgebra of \( H^{op} \). Passage from \( A \) to \( A^{op} \) shows that the left versions of the properties in (i)–(iii) are equivalent to their right versions. It is now clear that (i) \( \Rightarrow \) (ii) by \([34, \text{Th. 1.8}]\) or \([35, \text{Th. 5.4}]\). Also, (ii) \( \Rightarrow \) (iii) by \([16, \text{Th. 2.1}]\).

Thus only (i) has to be proved. Suppose first that \( A \) is a finitely generated algebra. Consider the dual Hopf algebra \( H^\circ \) which consists of all linear functions \( H \to k \) vanishing on an ideal of finite codimension in \( H \). There is a locally finite action of \( H^\circ \) on \( A \) defined by the rule
\[
f \to a = \sum f(a_{(2)})a_{(1)}, \quad f \in H^\circ, \ a \in A.
\]
With respect to this action \( A \) is an \( H^\circ \)-prime \( H^\circ \)-module algebra (see \([35, \text{Lemma 5.2}]\) for an easy verification). Since \( A \) embeds in the Artinian ring \( Q(H) \), it satisfies the ACC on right and left annihilators. Therefore (i) follows from Theorem 3.9.

Consider now the case when \( A \) is not finitely generated. Denote by \( \mathcal{I} \) the set of all finitely generated right coideal subalgebras of \( H \) contained in \( A \). The subalgebras generated by finite-dimensional right \( H \)-subcomodules of \( A \) belong to \( \mathcal{I} \). Since each element of \( A \) is contained in a finite-dimensional right subcomodule, the set \( \mathcal{I} \) is directed by inclusion and the union of all subalgebras \( A' \in \mathcal{I} \) gives the whole \( A \). It follows that

\[
V \otimes_A H \cong \varinjlim_{A' \in \mathcal{I}} V \otimes_{A'} H
\]

for each right \( A \)-module \( V \). We know already that \( H \) is left \( A' \)-flat for each \( A' \in \mathcal{I} \). Hence the functors \( \otimes_{A'} H \) are exact, and so is \( \otimes_A H \) by exactness of inductive direct limits. In other words, \( H \) is left \( A \)-flat. By symmetry \( H \) is right \( A \)-flat. This proves (ii), and (iii) also follows.

To complete the proof of (i) we need some facts which will be stated separately. Each algebra \( A' \in \mathcal{I} \) has an Artinian quotient ring \( Q(A') \), as we have established already. By Lemma 4.3 \( Q(A') \) embeds in \( Q(H) \) as a subring. Put

\[
Q(A) = \bigcup_{A' \in \mathcal{I}} Q(A') \subset Q(H).
\]

If \( A' \in \mathcal{I} \), then each regular element of \( A' \) is invertible in \( Q(H) \), and therefore regular in \( A \). Conversely, each regular element \( s \) of \( A \) is regular in any subalgebra \( A' \) containing \( s \). Hence all regular elements of \( A \) are invertible in \( Q(H) \). If \( q \in Q(A) \), then \( q \in Q(A') \) for some \( A' \in \mathcal{I} \), whence \( qs \in A' \subset A \) for some regular element \( s \in A' \). This shows that \( Q(A) \) is a classical right quotient ring of \( A \). By symmetry \( Q(A) \) is also a classical left quotient ring of \( A \).

By Lemma 4.4 \( Q(H) \) is faithfully flat as a left \( Q(A') \)-module for any \( A' \in \mathcal{I} \). Therefore the map \( I \mapsto IQ(H) \) embeds the lattice \( \mathcal{L}(A') \) of right ideals of \( Q(A') \) into the lattice \( \mathcal{L}(H) \) of right ideals of \( Q(H) \). Denote by \( \mathcal{L}_f(A) \) the lattice of finitely generated right ideals of \( Q(A) \). Given \( I \in \mathcal{L}_f(A) \), there exists a subalgebra \( A' \in \mathcal{I} \) containing a finite set of generators of \( I \). Then

\[
I = I'Q(A) \quad \text{and} \quad IQ(H) = I'Q(H)
\]

for some \( I' \in \mathcal{L}(A') \). For two right ideals \( I, J \in \mathcal{L}_f(A) \) we can find \( A' \in \mathcal{I} \) such that both \( I \) and \( J \) are extensions of right ideals of \( Q(A') \). It follows that \( \mathcal{L}_f(A) \) embeds in \( \mathcal{L}(H) \). Since \( Q(H) \) is right Artinian, we deduce that \( \mathcal{L}_f(A) \) satisfies the ACC and the DCC.
By the ACC on finitely generated right ideals an arbitrary right ideal $I$ of $Q(A)$ must coincide with the right ideal generated by a finite subset of $I$. Thus the lattice $L_f(A)$ contains all right ideals of $Q(A)$, and the DCC for this lattice implies that $Q(A)$ is right Artinian. By symmetry $Q(A)$ is left Artinian.

We have not yet finished with part (i) of Theorem 4.1 in the case when $A$ is not finitely generated. The next three lemmas provide additions needed to complete the proof.

**Lemma 4.2:** Let $H$ be an arbitrary Hopf algebra, $D$ an $H$-module algebra, and $B$ an $H$-stable subalgebra of $D$. Suppose that $B$ is semiprimary and $S(H)$-simple. If $D$ is left $B$-flat, then $D$ is left faithfully $B$-flat.

**Proof.** Denote by $\mathcal{T}_{D/B}$ the class of all right $B$-modules $V$ such that

$$V \otimes_B D = 0.$$ 

We have to show that $\mathcal{T}_{D/B}$ contains no nonzero modules. If $V \in \mathcal{T}_{D/B}$, then $\mathcal{T}_{D/B}$ contains all submodules of $V$ by the flatness assumption. (In fact $\mathcal{T}_{D/B}$ is the torsion class of a hereditary torsion theory.) The arguments that follow are similar to the proof that there exist no nonzero $G_r$-torsion $Q$-modules in the setup of Theorem 3.9.

First we show that $\mathcal{T}_{D/B}$ is closed under tensoring with right comodules (note a difference with section 3 where right modules were tensored with left comodules). For a right $H$-comodule $U$ and a right $B$-module $V$ we will consider the vector space $U \otimes V$ as a right $B$-module with respect to the twisted action of $B$ defined as

$$(u \otimes v)b = \sum u_{(0)} \otimes v(S(u_{(1)})b), \quad u \in U, \ v \in V, \ b \in B.$$ 

Similar tensoring operations are defined for right $D$-modules. With these conventions there is an isomorphism of right $D$-modules

$$U \otimes (V \otimes_B D) \cong (U \otimes V) \otimes_B D$$

constructed as follows. We may identify $V \otimes_B D$ with the factor space of $V \otimes D$ by its subspace spanned by

$$\{v \otimes bd - vb \otimes d \mid v \in V, b \in B, d \in D\}.$$ 

Then

$$U \otimes (V \otimes_B D) \cong (U \otimes V \otimes D)/R_1, \quad (U \otimes V) \otimes_B D \cong (U \otimes V \otimes D)/R_2.$$
where $R_1$ and $R_2$ are the subspaces of $U \otimes V \otimes D$ spanned, respectively, by
\[
\{ u \otimes v \otimes bd - u \otimes vb \otimes d \mid u \in U, v \in V, b \in B, d \in D \}
\]
and
\[
\{ u \otimes v \otimes bd - \sum u(0) \otimes v(S(u(1))b) \otimes d \mid u \in U, v \in V, b \in B, d \in D \}.
\]
Define linear transformations $\Phi$ and $\Psi$ of $U \otimes V \otimes D$ by the rules
\[
\Phi(u \otimes v \otimes d) = \sum u(0) \otimes v \otimes u(1)d,
\Psi(u \otimes v \otimes d) = \sum u(0) \otimes v \otimes S(u(1))d.
\]
Then $\Psi = \Phi^{-1}$. For $u \in U, v \in V, b \in B, d \in D$ we have
\[
\Phi(u \otimes v \otimes bd) = \sum u(0) \otimes v \otimes (u(1)b)(u(2)d) \equiv \sum u(0) \otimes v(S(u(1))u(2)b) \otimes u(3)d
\]
\[
= \sum u(0) \otimes vb \otimes u(1)d = \Phi(u \otimes vb \otimes d) \mod R_2,
\]
\[
\Psi(u \otimes v \otimes bd) = \sum u(0) \otimes v \otimes (S(u(2))b)(S(u(1))d) \equiv \sum u(0) \otimes v(S(u(2))b) \otimes S(u(1))d
\]
\[
= \sum \Psi(u(0) \otimes v(S(u(1))b) \otimes d) \mod R_1.
\]
It follows that $\Phi(R_1) \subset R_2$ and $\Psi(R_2) \subset R_1$. Hence $\Phi(R_1) = R_2$. Since $\Phi$
commutes with the action of $D$ by right multiplications on the third tensorand, we see that $\Phi$
induces the required isomorphism of $D$-modules.

If $V \otimes_B D = 0$, then $(U \otimes V) \otimes_B D = 0$. In other words, $U \otimes V \in T_{D/B}$
whenever $V \in T_{D/B}$, as claimed.

Now take $U = H$ with the comodule structure given by the comultiplication.

The annihilator of the twisted right $B$-module $H \otimes V$ is the largest $S(H)$-stable
ideal of $B$ contained in the annihilator of $V$. Suppose that $V \neq 0$. Then
this ideal is zero since $B$ is $S(H)$-simple. In other words, $H \otimes V$ is a faithful
$B$-module.

Since $B$ is semiprimary, it has a simple right ideal, say $K$. This right ideal
embeds in $H \otimes V$ as a submodule. If $V \in T_{D/B}$, then $H \otimes V \in T_{D/B}$,
whence $K \in T_{D/B}$ as well. But this is impossible since $K \otimes_B D \cong KD \supset K$,
and so $K \otimes_B D \neq 0$.

It follows that $V \notin T_{D/B}$, i.e., $V \otimes_B D \neq 0$, for each nonzero right $B$-
module $V$. Thus the flat left $B$-module $D$ is faithfully flat. □
Lemma 4.3: Let $A$ be a right coideal subalgebra of a residually finite-dimensional Hopf algebra $H$. Suppose that $A$ and $H$ have right Artinian classical right quotient rings $Q(A)$ and $Q(H)$. Then the inclusion $A \hookrightarrow H$ extends to an injective homomorphism of $H^\circ$-simple $H^\circ$-module algebras $Q(A) \to Q(H)$.

Proof. First, the action of $H^\circ$ on $A$ and $H$ extends to $Q(A)$ and $Q(H)$ by [36, Th. 2.2]. The $H^\circ$-module algebras $Q(A)$ and $Q(H)$ are $H^\circ$-prime since so are $A$ and $H$, and by Proposition 3.8 they are in fact $H^\circ$-simple. The conclusion of Lemma 4.3 is now a special case of [34, Lemma 1.7].

Lemma 4.4: Under the assumptions of Lemma 4.3 $Q(H)$ is faithfully flat as a left $Q(A)$-module.

Proof. By Lemma 4.3 $Q(A)$ is identified with an $H^\circ$-stable subalgebra of $Q(H)$, and this subalgebra is $H^\circ$-simple. By [34, Th. 1.8] $H$ is left $A$-flat. Since $Q(H)$ is left $H$-flat, $Q(H)$ is also left $A$-flat by transitivity of flatness. Noting that

$$V \otimes_{Q(A)} Q(H) \cong (V \otimes_A Q(A)) \otimes_{Q(A)} Q(H) \cong V \otimes_A Q(H)$$

for each right $Q(A)$-module $V$, we deduce that $Q(H)$ is left $Q(A)$-flat. The existence of a right Artinian classical right quotient ring $Q(H)$ implies that the antipode of $H$ is bijective. Then so is the antipode of $H^\circ$. Now Lemma 4.2 applies with $B = Q(A)$, $D = Q(H)$, and $H$ replaced by $H^\circ$.

If $H$ is a residually finite-dimensional Noetherian Hopf algebra, then the existence of an Artinian classical quotient ring $Q(H)$ has been established in [35, Th. 5.5]. Thus Theorem 0.1 stated in the introduction is a special case of Theorem 4.1. And here is another case:

Theorem 4.5: Let $H$ be a residually finite-dimensional PI Hopf algebra, finitely generated as an ordinary algebra. Suppose that $H$ satisfies the ACC on right and left annihilators. Then $H$ has an Artinian classical quotient ring.

As a consequence $H$ is flat as either right or left module over any right coideal subalgebra, and faithfully flat over any Hopf subalgebra.

Proof. Since $H$ is an $H^\circ$-prime $H^\circ$-module algebra, we can apply first Theorem 3.9, and then Theorem 4.1, noting that each subalgebra of $H$ is PI.

Corollary 4.6: Suppose that $H$ is a right Noetherian PI Hopf algebra, finitely generated as an ordinary algebra. Then all conclusions of Theorem 4.5 are true. Also, each Hopf subalgebra of $H$ is Noetherian.
Proof. It was proved by Anan’in [1] that any right Noetherian finitely generated PI algebra is residually finite-dimensional and, moreover, representable. This property implies the ACC on right and left annihilators. Thus $H$ satisfies the assumptions of Theorem 4.5. Since the antipode of $H$ is bijective, the algebra $H$ is also left Noetherian. By faithful flatness the Noetherian property descends to Hopf subalgebras.

**Lemma 4.7:** Let $H$ be as in Theorem 4.1. Then the lattice of PI Hopf subalgebras of $H$ embeds into the lattice of left ideals of $H$.

**Proof.** For a PI Hopf subalgebra $A$ of $H$ put

$$A^+ = \{ a \in A \mid \varepsilon(a) = 0 \}.$$  

Then $HA^+$ is a left ideal and a coideal of $H$. Since $H$ is faithfully flat over $A$ by Theorem 4.1, we can apply Takeuchi’s result [38, Th. 1] which shows that $A$ can be reconstructed from the quotient coalgebra $H/HA^+$ as

$$A = \{ h \in H \mid (\pi \otimes \text{id})\Delta(h) = \pi(1) \otimes h \text{ in } (H/HA^+) \otimes H \}$$

where $\pi : H \to H/HA^+$ is the canonical map. In other words,

$$A = \{ h \in H \mid \Delta(h) - 1 \otimes h \in HA^+ \otimes H \}.$$  

It follows that the assignment

$$A \mapsto HA^+$$

defines an injective inclusion-preserving map from the set of PI Hopf subalgebras to the set of left ideals of $H$.

As an application of the previous results we consider the relationship between Noetherianess and finite generation. It was observed by Molnar [18] that a commutative Hopf algebra is Noetherian if and only if it is finitely generated. Motivated by this fact, Wu and Zhang asked whether every Noetherian Hopf algebra is affine, i.e., finitely generated as an ordinary algebra [39, Question 5.1]. Faithful flatness over Hopf subalgebras is an essential argument for deriving such a conclusion. We have established this property for a certain class of Hopf algebras, but this property alone is not sufficient. We are still able to state a somewhat weaker result:

**Proposition 4.8:** Let $H$ be a residually finite-dimensional Noetherian PI Hopf algebra. Then $H$ is finitely generated as a Hopf algebra.
Proof. By [35, Th. 5.5] $H$ has an Artinian classical quotient ring, i.e., $H$ satisfies the assumptions of Theorem 4.1. This enables us to use Lemma 4.7. Note that all subalgebras of $H$ are PI. Since the ACC is satisfied in the lattice of left ideals of $H$, it is also satisfied in the lattice of Hopf subalgebras of $H$. It follows that there exists a largest finitely generated Hopf subalgebra $A$. Clearly $A$ must contain all elements of $H$, i.e., $A = H$. □

If in Proposition 4.8 each element of $H$ is contained in a finite-dimensional subspace stable under the action of the antipode $S$, then $H$ will be finitely generated as an ordinary algebra. One particular case where this happens is presented below:

Corollary 4.9: If $H$ is a coquasitriangular residually finite-dimensional Noetherian PI Hopf algebra, then $H$ is finitely generated as an ordinary algebra.

Proof. It is well-known that $S^2(C) = C$ for any subcoalgebra $C \subset H$. This is a consequence of the fact that the category of right comodules for any coquasitriangular Hopf algebra is braided, which implies that $V^{**} \cong V$ for each finite-dimensional right $H$-comodule $V$. Moreover, $S^2$ is a coinher operator according to Doi [7, Th. 1.3] and Schauenburg [26, Lemma 3.3.2]. This means that there exists a convolution invertible linear map $\lambda : H \to k$ such that

$$S^2(h) = \lambda \otimes h \leftarrow \lambda^{-1}$$

for all $h \in H$.

In view of Proposition 4.8 we can find a finite-dimensional subcoalgebra $C$ which generates $H$ as a Hopf algebra. The subcoalgebra $C' = C + S(C)$ is finite dimensional and $S$-invariant. Since the subalgebra of $H$ generated by $C'$ is a Hopf subalgebra, it coincides with $H$. □

References

[1] A. Z. Anan’iu, Representability of Noetherian finitely generated algebras, Archiv der Mathematik 59 (1992), 1–5.

[2] A. Braun, The nilpotency of the radical in a finitely generated P.I. ring, Journal of Algebra 89 (1984), 375–396.

[3] K. A. Brown, Noetherian Hopf algebras, Turkish Journal of Mathematics 31 (2007), 7–23.

[4] A. Chirvasitu, Cosemisimple Hopf algebras are faithfully flat over Hopf subalgebras, Algebra & Number Theory 8 (2014), 1179–1199.
[5] M. Cohen, *Hopf algebras acting on semiprime algebras*, in *Group Actions on Rings*, Contemporary Mathematics, Vol. 43, American Mathematical Society, Providence, RI, 1985, pp. 49–61.

[6] M. Demazure and P. Gabriel, *Groupes Algébriques. I*, Masson, Paris, 1970.

[7] Y. Doi, *Braided bialgebras and quadratic bialgebras*, Communications in Algebra 21 (1993), 1731–1749.

[8] K. R. Goodearl, *Ring Theory*, Pure and Applied Mathematics, Vol. 33, Marcel Dekker, New York–Basel, 1976.

[9] K. R. Goodearl and R. B. Warfield Jr., *An Introduction to Noncommutative Noetherian Rings*, London Mathematical Society Student Texts, Vol. 61, Cambridge University Press, Cambridge, 2004.

[10] B. Greenfeld, L. Rowen and L. Small, *Noetherian PI-algebras are representable*, https://arxiv.org/abs/2008.11041.

[11] F. D. Grosshans, *Algebraic Homogeneous Spaces and Invariant Theory*, Lecture Notes in Mathematics, Vol. 1673, Springer, Berlin, 1997.

[12] I. N. Herstein and L. Small, *Nil rings satisfying certain chain conditions*, Canadian Journal of Mathematics 16 (1964), 771–776.

[13] G. R. Krause and T. H. Lenagan, *Growth of Algebras and Gelfand–Kirillov Dimension*, Graduate Studies in Mathematics, Vol. 22, American Mathematical Society, Providence, RI, 2000.

[14] A. I. Malcev, *On representations of infinite algebras*, Matematicheski˘ı Sbornik 13 (1943), 263–286.

[15] A. Masuoka, *On Hopf algebras with cocommutative coradicals*, Journal of Algebra 144 (1991), 451–466.

[16] A. Masuoka and D. Wigner, *Faithful flatness of Hopf algebras*, Journal of Algebra 170 (1994), 156–164.

[17] J. C. McConnell and J. C. Robson, *Noncommutative Noetherian Rings*, Graduate Studies in Mathematics, Vol. 30, American Mathematical Society, Providence, RI, 2001.

[18] R. K. Molnar, *A commutative Noetherian Hopf algebra over a field is finitely generated*, Proceedings of the American Mathematical Society 51 (1975), 501–502.

[19] S. Montgomery, *Hopf Algebras and their Actions on Rings*, CBMS Regional Conference Series in Mathematics, Vol. 82, American Mathematical Society, Providence, RI, 1993.

[20] S. Montgomery and H.-J. Schneider, *Hopf crossed products, rings of quotients, and prime ideals*, Advances in Mathematics 112 (1995), 1–55.

[21] E. F. Müller and H.-J. Schneider, *Quantum homogeneous spaces with faithfully flat module structures*, Israel Journal of Mathematics 111 (1999), 157–190.

[22] W. D. Nichols and M. B. Zoeller, *A Hopf algebra freeness theorem*, American Journal of Mathematics 111 (1989), 381–385.

[23] C. Procesi, *Rings with Polynomial Identities*, Pure and Applied Mathematics, Vol. 17, Marcel Dekker, New York, 1973.

[24] D. E. Radford, *Pointed Hopf algebras are free over Hopf subalgebras*, Journal of Algebra 45 (1977), 266–273.

[25] L. H. Rowen, *Ring Theory. Vols. I, II*, Pure and Applied Mathematics, Vols. 127, 128, Academic Press, Boston, MA, 1988.
[26] P. Schauenburg, On Coquasitriangular Hopf Algebras and the Quantum Yang–Baxter Equation, Algebra Berichte, Vol. 67, Reinhard Fisher, Munich, 1992.

[27] P. Schauenburg, Faithful flatness over Hopf subalgebras: counterexamples, in Interactions Between Ring Theory and Representations of Algebras, Lecture Notes in Pure and Applied Mathematics, Vol. 210, Marcel Dekker, New York, 2000, pp. 331–344.

[28] H.-J. Schneider, Normal basis and transitivity of crossed products for Hopf algebras, Journal of Algebra 152 (1992), 289–312.

[29] H.-J. Schneider, Some remarks on exact sequences of quantum groups, Communications in Algebra 21 (1993), 3337–3357.

[30] A. H. Schofield, Stratiform simple Artinian rings, Proceedings of the London Mathematical Society 53 (1986), 267–287.

[31] S. Skryabin, New results on the bijectivity of antipode of a Hopf algebra, Journal of Algebra 306 (2006), 622–633.

[32] S. Skryabin, Projectivity and freeness over comodule algebras, Transactions of the American Mathematical Society 359 (2007), 2597–2623.

[33] S. Skryabin, Projectivity of Hopf algebras over subalgebras with semilocal central localizations, Journal of K-Theory 2 (2008), 1–40.

[34] S. Skryabin, Models of quasiprojective homogeneous spaces for Hopf algebras, Journal für die Reine und Angewandte Mathematik 643 (2010), 201–236.

[35] S. Skryabin and F. Van Oystaeyen, The Goldie theorem for H-semiprime algebras, Journal of Algebra 305 (2006), 292–320.

[36] B. Stenström, Rings of Quotients, Die Grundlehren der mathematischen Wissenschaften, Vol. 217, Springer, New York–Heidelberg, 1975.

[37] M. Takeuchi, Relative Hopf modules—equivalences and freeness criteria, Journal of Algebra 60 (1979), 452–471.

[38] Q.-S. Wu and J. J. Zhang, Noetherian PI Hopf algebras are Gorenstein, Transactions of the American Mathematical Society 355 (2003), 1043–1066.