Emergent geometry and path integral optimization for a Lifshitz action

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Extending the background metric optimization procedure for Euclidean path integrals of two-dimensional conformal field theories, first introduced by Caputa et al. [1, 2], to a $z = 2$ anisotropically scale-invariant (2 + 1)-dimensional Lifshitz field theory of a free massless scalar field, we find optimal geometries for static and dynamic correlation functions. For the static correlation functions, the optimal background metric is equivalent to an AdS metric on a Poincare patch, while for dynamical correlation functions, we find Lifshitz like metric. This results suggest that a MERA-like tensor network, perhaps without unitarity, would still be considered an optimal background spacetime configuration for the numerical description of this system, even though the classical action we start with is not a conformal field theory.

An important quest of many body physics is the search for efficient variational characterizations of correlated quantum systems. (for a review see, e.g., [3]). A class of tensor network states, particularly geared towards the description of scale-invariant systems, are called the multi-scale entanglement renormalization ansatz (MERA) [4, 5]. MERA is used to represent approximate ground states of 1D quantum spin chains at criticality described by 2D conformal field theory (CFT)[6]. The scale-invariance of the MERA network turned out to also play a special role in connecting it to holographic duals in the sense of the AdS/CFT correspondence [7]. Here, the bulk of a MERA network can be understood as a discrete realization of 3D anti-de Sitter space ($AdS_3$), identifying the extra holographic direction with the renormalization group (RG) flow in the MERA [7].

Motivated by the procedure of tensor network renormalization in [8], where the path integral is first discretized into a lattice and then mapped into a tensor network which turns out to be a MERA, Caputa et. al, in a recent series of works [1, 2], took a step further in studying this relationship from the viewpoint of optimizing Euclidean path integrals that represent the ground state wave functional of two-dimensional CFT. Starting with flat Euclidean metric with a UV cutoff, they argued that their optimization procedure amounts to minimizing the Jacobian of the scale transformation for the path integral measure. In the conformally flat gauge, this translates to solving the equation of motion of the Liouville effective action from which they find that the $AdS_3$ metric a Poincare patch $H_2$ naturally emerges. This new approach is very appealing, as it shows that the procedure can be successfully applied in systems of interest beyond a CFT. We show how natural geometries arise from the path integral optimization procedure. Our results are illustrated in Fig. 1.

The quantum Lifshitz model is a canonical example of a (2+1)-dimensional Lifshitz field theory known [10]. This model describes a free massless scalar field with dynamical scaling exponent $z = 2$ and represents an important example of a conformal quantum critical point. Different aspects of this theory have been studied and analyzed in [10][12]. For example, it emerges as the scaling limit of the square lattice quantum dimer model [12].

The quantum Lifshitz Hamiltonian [10] of a $z = 2$ theory of a massless scalar field $\phi(t, x)$ in 2+1 dimensions is given by

$$H = \int d^2x \left\{ \pi_\phi^2 + (\Delta_s \phi)^2 \right\}.$$  

The Euclidean action of the field $\phi(t, x)$ coupled to a background metric $g_{ij}$ is given by

$$S = \int d^2x dt N \sqrt{h} \left( N^{-2} (\partial_i \phi)^2 + \lambda (\Delta_s \phi)^2 \right),$$

where $\Delta_s$ is the spatial Laplace-Beltrami operator

$$\Delta_s = \frac{1}{\sqrt{h}} \partial_i h^{ij} \sqrt{h} \partial_j ,$$

FIG. 1. The two geometries emerging for the quantum Lifshitz model. (a) An $AdS_3$-like geometry arises when considering equal time correlation functions and (b) A Lifshitz metric that is optimal for computing correlation functions with a temporal component.
and $h_{ij}$ is the spatial component of the background metric $\tilde{g}$.

$$ds^2 = N^2 dt^2 + h_{ij} dx^i dx^j .$$

(4)

where $N$ is called the lapse function. The action in $[2]$ is invariant under the following foliation-preserving diffeomorphism transformations

$$t \rightarrow \tilde{t}(t), \quad x^i \rightarrow \tilde{x}^i(\tilde{x})$$

(5)

and anisotropic Weyl scaling transformations

$$N \rightarrow e^{\sigma} N ; \quad h_{ij} \rightarrow e^{2\sigma} h_{ij} , \quad i,j \neq t .$$

(6)

As stated before, in $[1]$, such a starting point led, via path integral optimization, to an AdS metric. The path integral optimization suggested in $[1]$ looks for the extremal measure over all choices of the gauge $\sigma$, due to the Weyl anomaly in the model. Here we use the same structure, though with the anisotropic Weyl scaling appropriate.

Here we ask the following question: what is the optimal geometry associated with a path integral computation of correlation functions in the quantum Lifshitz model? In contrast to the CFT case, due to the non-relativistic nature of the model, equal time correlation functions and dynamical correlation functions should be treated differently. Indeed, we find two separate geometries associated with the optimal calculation, described in Fig. $[1]$ a). For equal-time correlation functions, we consider Weyl transformations which are translationally invariant in space, but not in time, Fig. $[1]$ b), covered by case (1) below.

Consider dynamical correlation functions on the other hand. To find the optimized geometry to describe two point functions, such as, say, $\langle \phi(t,r)\phi(r',t) \rangle$, we can choose the spatial axis $r - r'$ to be in the $y$ direction, due to spatial rotational invariance of the model. We concentrate therefore on the computation of the description of the state in the $t,y$ plane, and thus choose a Weyl scaling which is homogeneous in $t,y$, but can depend on the third coordinate $x$, Fig. $[1]$ b) as explained in case (2) below.

Of particular interest to us in this paper, is the Weyl anomaly of this model which has first been computed holographically in $[14]$ and by Baggio et al in $[15]$ using heat kernel expansion and the holographic renormalization methods in $[16]$. In $[17]$, Lifshitz Weyl anomalies have been computed cohomologically in different dimensions and for different values of the dynamical scaling exponent $z$. In $[19]$, the heat kernel expansion has been generalized to calculate effective actions and Weyl anomalies for Lifshitz field theories. A general framework for computing one loop effective action for Lifshitz theory via heat kernel coefficients has been presented in several places, see e.g. $[19]$ $[20]$.

We note that in contrast with $[1]$, here, We do not start from the quantum effective action and then derive the equation of motion as they do but rather directly compute the variation in the Lifshitz effective action due to an infinitesimal transformation of the Weyl transformation parameter $\sigma$. Our starting point is a flat metric, deformed by a Weyl scaling, therefore $\sigma$ carries the entire information on the metric in the space of metrics we explore. We compute the variation of the effective action explicitly utilizing the particular structure of our metric and finally obtain differential equations for the scaling factor $\sigma$. Concretely, we compute the variation of the one loop effective action under $\sigma \rightarrow \sigma + \delta \sigma$. In this case,

$$\delta W[\sigma] = \frac{1}{2} \int d\bf{r} \delta \sigma(\bf{r}) e^{\rho D} [\bf{r}] ,$$

(7)

where $\bf{r} = (x,t)$, $\rho(\bf{r}) = \frac{1}{\sqrt{\phi(\bf{r})}}$ and $D = -\frac{1}{N \sqrt{h}} \frac{1}{2} \partial_t N^{-1} \sqrt{h} \partial_t + \frac{1}{N} \frac{1}{2} \Delta_x N \Delta_x [15]$. In our system we fix our gauge so that $N = e^{2\sigma}$, $h_{ij} = \lambda N \delta_{ij}$. In this case we have:

$$D = (-\partial_t^2 + \lambda^2 (\partial_x^2 + \partial_y^2))^2 .$$

(8)

We note that upon varying $\sigma$ we have $\delta D = \delta \sigma$. The $\epsilon \rightarrow 0$ behavior of (7) is dominated by the short distance behavior of the heat kernel $|\bf{r}| e^{\rho D} [\bf{r}]$.

Now, as promised, we specialize on cases where $\delta \sigma$ depends either on the time coordinate $t$ alone, or on one of the spatial coordinates, say $x$. Denoting $\rho = e^{-\sigma_0}$, we expand $\rho$ close to a given point $\bf{r}_0$,

$$\rho(\delta \bf{r} + \bf{r}_0) = \rho_0 + \delta \rho ,$$

(9)

where $\rho_0 = \rho(\bf{r}_0) = \frac{1}{\sqrt{\phi(\bf{r}_0)}}|\bf{r} = \bf{r}_0|$.

To obtain the variation we carry out a second order perturbation calculation of the heat kernel, using :

$$e^{-\epsilon(\rho_0 + \delta \rho)D} = e^{-\epsilon D} - \frac{1}{\rho_0} \int_0^\epsilon e^{-(\epsilon-s)D} \delta \rho D e^{-sD} D d\epsilon \delta \rho$$

$$+ \frac{1}{\rho_0^2} \int_0^\epsilon ds \int_0^s ds_1 e^{-(\epsilon-s)D} \delta \rho D e^{-s_1 D} \delta \rho D e^{-s_1 D}$$

where $\epsilon = \rho_0 \epsilon$. We assume that the operator $D$ is diagonal in momentum, and that $\delta \rho$ depends on a single coordinate such as $x$ or $t$ and has an expansion:

$$\delta \rho = \Sigma_{m=1}^\infty c_m (x-x_0)^m$$

(11)

Explicitly evaluating the heat kernel through second order perturbation series in $\delta \rho$, we find that the leading (in $\epsilon$) contributions to $\delta W$ up to two derivatives are given as

$$(1) \quad \sigma = \sigma(t).$$

In this case:

$$\delta W = \frac{1}{2} \int dt d^2x \delta \sigma \left( \frac{e^{4\sigma}}{16 \pi^2 \lambda} - \frac{1}{24 \pi \lambda} \frac{d^2 \sigma}{dt^2} \right),$$

(12)
In this case, the leading in \( \epsilon \) contributions, read:
\[
\delta W = \frac{1}{2} \int dt d^2 x \delta \sigma \left( \frac{e^{4\sigma}}{16\pi \epsilon \lambda} - \frac{e^{2\sigma} (\frac{d^2 \sigma}{dx^2} + \frac{d^2 \sigma}{dy^2})}{12\pi^{3/2} \sqrt{\epsilon}} \right). \tag{13}
\]

**Optimized geometry for equal time correlation functions.** Following [1], we search for a profile \( \rho(t) \) to minimize the effective action by solving for \( \delta W = 0 \). Eq. (12) implies that the optimal \( \sigma(t) \) obeys the Liouville equation:
\[
\frac{e^{4\sigma}}{\epsilon} - \frac{2}{3} \frac{d^2 \sigma}{dt^2} = 0 \quad (14)
\]

This surprising result suggests that indeed a some type of a hierarchical tensor network would still be the optimal discrete spacetime configuration even if the field theory we started with is only anisotropically scale invariant. It is interesting note how the combination \( \sigma^4 / \epsilon \) arises naturally in (14). This is a natural scaling: If we consider our path integral with action (2) as describing, e.g. the ground state of the quantum Lifshitz Hamiltonian, and considering the gap scaling as \( \sigma^4 / \epsilon \), we find that the correlation functions match quite well with the holographic two-point function. The authors consider in this work have been studied in [10] and more recently in [22] where they have been compared with the holographic two-point function. The authors find that the correlation functions match quite well with the scaling obtained from a holographic calculation with a Lifshitz geometry, thereby strengthening our expectation that a tensor network description of the system will inherit the features of a Lifshitz geometry.

We find it quite striking that a semi-classical description for the quantum Lifshitz model that we consider in this work have been studied in [10] and more recently in [22] where they have been compared with the holographic two-point function. The authors find that the correlation functions match quite well with the scaling obtained from a holographic calculation with a Lifshitz geometry, thereby strengthening our expectation that a tensor network description of the system will inherit the features of a Lifshitz geometry. We find it quite striking that a semi-classical description for the quantum Lifshitz model that we consider in this work have been studied in [10] and more recently in [22] where they have been compared with the holographic two-point function. The authors find that the correlation functions match quite well with the scaling obtained from a holographic calculation with a Lifshitz geometry, thereby strengthening our expectation that a tensor network description of the system will inherit the features of a Lifshitz geometry.

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To obtain our equations we carry out a second order perturbation calculation of the heat kernel, using:

\[ e^{-\epsilon(\rho_0 + \delta \rho)D} = e^{-\epsilon D} - \frac{1}{\rho_0} \int_0^{\epsilon} e^{-(\tilde{\epsilon} - s)D} \delta \rho D e^{-sD} \, ds + \frac{1}{\rho_0^2} \int_0^{\epsilon} \int_0^{\epsilon - s} e^{-sD} \delta \rho D e^{-sD} \, ds \]

where \( \tilde{\epsilon} = \rho_0 \epsilon \). For convenience, set \( \rho_0 = 0 \) throughout the calculation, and reinstate its value in the end. We assume that the operator \( D \) is diagonal in momentum, and that \( \delta \rho \) depends on a single coordinate \( x \), and has an expansion:

\[ \delta \rho = \sum_{m=1} c_m x^m \]

Taking \( q \) to be the momentum in the \( x \) direction and \( K \) to be the momentum vector in all other directions, the zeroth order contribution to the heat kernel reads:

\[ A_0 = \langle 0 \vert e^{-\epsilon D} \vert 0 \rangle = \frac{1}{(2\pi)^{d+1}} \int d^d K \, dq \, e^{-\epsilon D(K,q)} \]

The contribution from the first order term in (23) is

\[ A_1 = -\frac{1}{\rho_0} \langle 0 \vert e^{-\epsilon D} \vert 0 \rangle \delta \rho D e^{-sD} \, ds = \]

\[ \frac{1}{\rho_0} \int_0^{\epsilon} \int_0^{\epsilon - s} e^{-sD} \delta \rho D e^{-sD} \, ds \]

which can also be expressed in the form:

\[ A_1 = \frac{1}{\rho_0} \int_0^{\epsilon} ds \int d^d K \, dq \, e^{-\epsilon D(K,q)} (\delta \rho D) \sum_{m=1} c_m (\bar{\epsilon}D'(K,q) + \tilde{\epsilon}D(K,q)) D(K,q) e^{-sD(K,q)} \]

where \( B_{h,m} \) are Bell polynomials. In the case we are interested in, due to the time reversal/space inversion symmetry the first non zero contribution comes from \( c_2 = \frac{1}{2} \partial^2 x^2 \delta \rho \):

\[ A_1 \approx \frac{1}{\rho_0} \int_0^{\epsilon} ds \int d^d K \, dq \, e^{-\epsilon D(K,q)} \left( -\frac{1}{2} \partial^2 D(K,q) \bar{\epsilon}^2 + \frac{1}{3} (D'(K,q))^2 \tilde{\epsilon}^2 \right) \]

The second order contribution is given by:

\[ A_2 = \frac{1}{\rho_0^2} \int_0^{\epsilon} ds \int_0^{\epsilon - s} ds_1 e^{-sD} \delta \rho D e^{-sD} \delta \rho D e^{-s_1D} \, ds \]

\[ \frac{1}{\rho_0^2} \int_0^{\epsilon} ds \int d^d K \, dq \, \left( \sum_{m=1} c_m (\bar{\epsilon}D'(K,q) + \tilde{\epsilon}D(K,q) + \bar{\epsilon}D'(K,q) + \tilde{\epsilon}D(K,q)) \right) D(K,q) e^{-sD(K,q)} \left( -\frac{1}{2} \partial^2 D(K,q) \bar{\epsilon}^2 + \frac{1}{3} (D'(K,q))^2 \tilde{\epsilon}^2 \right) \]