PERIODIC SURFACE HOMEOMORPHISMS AND CONTACT STRUCTURES

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Abstract. We study the contact structures coming from a natural class of rational open books, in the sense of Baker-Etnyre-Morris, that correspond to the conjugacy classes of periodic surface diffeomorphisms. By considering the contact structures associated to such rational open books, we prove some fillability results for such contact structures. We also prove an analogue of Mori’s construction of explicit symplectic filling for rational open books. We also prove a sufficient condition for Stein fillability of rational open books analogous to the positivity of monodromy in honest open books as in the result of Giroux and Loi-Piergallini.

1. Introduction

The open book decomposition of a manifold has proved to be a fundamental tool for development of contact topology. Roughly speaking, an open book is a decomposition of a manifold into co-dimension 1-submanifolds fitting nicely together to form a mapping torus (see Section 2 for details) and a tubular neighborhood of a codimension 2 submanifold, called the binding. The fiber of the mapping torus is called the page. The return map or monodromy of the fibration over $S^1$ is a diffeomorphism of the page relative to its boundary. It is known [14] that all odd-dimensional manifolds admit open book decompositions.

Thurston-Winkelnkemper [12] showed how to associate a contact structure to an open book with a symplectic page and with a relative symplectomorphism as monodromy. Subsequently, a remarkable result of Giroux [5] in dimension three showed that every contact structures arises in this way. Moreover, for 3-manifolds, Giroux established a correspondence between contact structures up to isotopy and open books up to positive stabilization. Baker-Etnyre-Morris [1] defined a generalization of 3-dimensional open books called rational open books. While the monodromy of a honest open book is identity near the boundary, a rational open book may have rotations along boundary components of the page. It was shown in [1] that one can extend the Thurston-Winkelnkemper construction to the setting of rational open books and associate a unique contact structure to a rational open book.

It is an interesting pursuit to analyze the contact topological properties of rational open books in relation to their monodromies. In this paper, we discuss a natural family of rational open books arising from periodic...
surface homeomorphisms and study the fillability properties of the corresponding contact structures. The set of periodic surface homeomorphisms is an important class in the scheme of Nielsen-Thurston classification theory of surface homeomorphisms. The conjugacy class of a periodic surface homeomorphism can be efficiently encoded \([10, 11]\) by a combinatorial tuple of integers called a data set. More specifically, consider a homeomorphism \(h\) of order \(n\) on a closed orientable surface \(\Sigma\) that generates an action with \(\ell\) distinct nontrivial orbits of sizes \(n/n_i\) where for \(1 \leq i \leq \ell\), the induced local rotation angle is \(2\pi c_i/n_i\) with \(\gcd(c_i, n_i) = 1\), and whose quotient orbifold has genus \(g_0\). Then \(f\) has an associated data set given by \(D_h = (n, g_0, r; (c_1, n_1), \ldots, (c_\ell, n_\ell))\), where \(n\) is called the degree of \(D_h\). The parameter \(r\) comes into play only when \(f\) is a free rotation of the surface by \(2\pi r/n\). For example, the hyperelliptic involution on the torus, which has 4 fixed points (as illustrated in Figure 1 below) with an induced local rotation of \(\pi\) around each point, is encoded by the data set \((2, 0; (1, 2), (1, 2), (1, 2), (1, 2))\).

\[\text{Figure 1. The hyperelliptic involution on a torus.}\]

By removing (cyclically permuted) \(h\)-invariant open disks around (points in) a set \(S\) of nontrivial orbits of a periodic homeomorphism, we obtain a periodic map \(\hat{h}\) on a subsurface \((\hat{\Sigma}, \partial\hat{\Sigma})\) with corresponding boundary. One can then take the rational open book which has \(\hat{\Sigma}\) as page and \(\hat{h}\) as monodromy, and consider the contact structure associated to this rational open book, as described in \([1]\). Thus, we see that by removing rotating open disks around the orbits of a cyclic action (encoded by \(D_h\)), we can get a contact rational open book. In order to make such an association well-defined, the \(D_h\) should also include the information on the specific orbits in \(S\) around which disks were removed. Note that each pair \((c_j, n_j)\) in the multiset \(\{(c_1, n_1), \ldots, (c_\ell, n_\ell)\}\) corresponds to a distinguished orbit of the \(\langle f \rangle\)-action. Thus, we modify \(D_h\) to a marked data set, which is a tuple of the form

\[\hat{D}_h = (n \pm, g_0, r; (c_1, n_1), \ldots, (c_\ell, n_\ell), [j_1, \ldots, j_k]),\]

where \([j_1, \ldots, j_k] \subset \{1, \ldots, \ell\}\) and the (disjoint) union of the orbits that the \((c_{j_i}, n_{j_i})\) correspond to, equals \(S\). For example, assuming that the four fixed points hyperelliptic involution (above) are marked 1-4, and say we remove three disks around points 1, 2 and 4, then the marked data set is given by: \((n, 0; (1, 2), (1, 2), [1, 2, 4])\). The suffix \(\pm\) for the parameter \(n\) is added to distinguish between a positive (i.e. clockwise) and a negative (i.e. counterclockwise) local rotation around a distinguished fixed point (or orbit). We show the following result in Section 3.
Proposition 1.1. Given a marked data set \( \hat{D} \), one can associate a unique contact structure to it.

The above proposition opens the door to the possibility of describing a class of contact structures combinatorially in terms of marked data sets. Moreover, the properties of contact structures can be studied through marked data sets. Guided by this philosophy, our main results are formulated in terms of conditions on marked data sets.

In [10], Prasad-Rajeevsarathy-Sanki described a procedure for decomposing an arbitrary periodic map \( h \) of order \( n \) on \( \Sigma \) (that is not realizable as a rotation of \( \Sigma \)) into irreducible (periodic) components \( h_i \) (of order \( n \)) on surfaces \( \Sigma_i \) that have that at least one fixed point, called irreducible Type 1 maps. It is known [4] that an irreducible Type 1 map \( f_0 \) is encoded by a data set of the form \( D_{f_0} = (n, 0, (c_1, n_1), (c_2, n_2), (c_3, n)) \). Further, it was shown that a map such as \( f_0 \) is realized as a rotation of a special hyperbolic polygon (with side-pairing) by \( 2\pi c_i^{-1}/n \). The decomposition of \( h \) into irreducibles induces a decomposition of \( D_h \) into simpler data sets \( D_{h_i} \) of the form \( D_{f_0} \). This, in turn, extends to a decomposition of \( D_h \) into marked irreducible Type 1 data sets \( D_{h_i} \). This process (of decomposing \( h \) into irreducibles) can be completely reversed (to recover the original action \( h \)) by gluing pairs of such irreducible components along compatible orbits, where the induced local rotation angles are equal. More precisely, when the actions of a pair \( (h_1, h_2) \) of irreducible Type 1 components induce compatible orbits of size \( k \) (i.e., have orbits where the local rotation angles add up to \( 0 \) modulo \( 2\pi \)), we can remove (cyclically permuted) invariant disks around points in the orbits and then identify the resultant boundary components, thereby realizing a new action \( (h_1, h_2) \). We call this process of constructing \( (h_1, h_2) \) as a \( k \)-compatibility. For example, the order 6 action \( f \) on the torus with data set \( D_f = (6, 0; (1, 2), (1, 3), (1, 6)) \) is compatible with the order 6-action \( f^5 \) on the torus (represented by \( D_{f^5} = (6, 0; (1, 2), (2, 3), (5, 6)) \)) along the fixed points at the center of the realizing hexagons, resulting in the compatible pair \( (f, f^5) \) represented by \( (D_f, D_{f^5}) = (6, 0; (1, 2), (1, 2), (1, 3), (2, 3)) \). A similar compatibility of gluing can be also defined for marked data sets.

In Section [4] we investigate the fillability properties of the contact structures associated to marked data sets. The following result describes a class of Stein fillable contact structures in terms of their marked data sets.

Theorem 1.2. Let \( \hat{D} \) be a marked data set representing an action of degree \( n \) on \( \Sigma \) that decomposes into a collection of marked irreducible Type 1 data sets

\[
\hat{D}_i = (n_i, 0, (c_{i1}, n_{i1}), (c_{i2}, n_{i2}), (c_{i3}, n), [j_{i1}, \ldots, j_{ik}])
\]

representing actions on surfaces \( \Sigma_{g_i} \), for \( 1 \leq i \leq m \), such that the following conditions hold:

(i) If \( s \in \{j_{i1}, \ldots, j_{ik}\} \), then \( n_s = n \).
(ii) If \( (D_i, D_j) \) forms a compatible pair, then both \( \partial(\Sigma_{g_i} \cup \Sigma_{g_j}) \cap \partial \Sigma_{g_i} \) and \( \partial(\Sigma_{g_i} \cup \Sigma_{g_j}) \cap \partial \Sigma_{g_j} \) are non-empty.

Then the contact structure associated to \( \hat{D} \) is Stein fillable.
For example, the marked data set $\hat{D}_f = (6,0; (1,2), (1,3), (1,3), (5,6), (4))$ that is realized as a compatible pair $(D_f, D_{f^\varphi})$ of irreducible marked data sets satisfies the hypotheses of Theorem 1.2. Therefore, the contact structure associated to $\hat{D}_f$ is Stein fillable. Theorem 1.2 can be viewed as an analog of [3, Theorem 4.2] in the setting of rational open books. The result of Colin-Honda for honest open book says that a contact structure supported by an open book with a periodic monodromy (i.e. having a periodic map as its Thurston representative) is Stein fillable if the monodromy is right-veering.

Our next result pertains to producing explicit symplectic fillings of some rational open books. In [9], Mori constructed such symplectic fillings for the contact structures coming from an explicit construction of Thurston and Winkelnkemper. We give a generalization of this construction for rational open books. For the notations regarding rational open books and its monodromy we refer to section 2.5. A rational open book is characterized by a surface $\Sigma$ with boundary and a homeomorphism $\phi$ of $\Sigma$. We denote such a rational open book by $\text{ROB}(\Sigma, h)$. Let $\text{Mod}(\Sigma)$ denote the mapping class group of $\Sigma$. If $\phi$ does not permute boundaries, then $\phi$ can be written as a composition of some element $h \in \text{Mod}(\Sigma)$ and fractional rotations of the boundary components [11, Section 5]. Let $\text{Dehn}^+(\Sigma, \partial \Sigma)$ consist of relative isotopy classes of diffeomorphisms which are products of positive Dehn twists. The following result is stated for surfaces with connected boundary, but the proof holds true for surfaces with multiple boundary components.

**Theorem 1.3.** Consider a rational open book $\text{ROB}(\Sigma, \phi)$ with $\partial \Sigma$ connected. Say, $\phi$ can be written as $h \circ \partial^{q_p}_p$, where $h \in \text{Dehn}^+(\Sigma, \partial \Sigma)$ and $\partial^{q_p}_p$ is a $2\pi q_p$-rotation along $\partial \Sigma$. Then $\text{ROB}(\Sigma, \phi)$ admits a strong symplectic filling for $p > q > 0$ and $q < 0 < p$.

On similar lines, a result by Giroux [5] and Loi-Piergallini [8] for Stein-fillability of open book can be generalized to the setting of rational open books as stated follows.

**Theorem 1.4.** Let $\text{ROB}(\Sigma, \phi)$ be a rational open book with $\Sigma$ connected. Suppose $\phi$ can be written as $h \circ \partial^{q_i}_{p_i}$, where $h \in \text{Dehn}^+(\Sigma, \partial \Sigma)$ and $\partial^{q_i}_{p_i}$ is a $2\pi q_i$-rotation along $i$-th boundary component in $\partial \Sigma$. Then $\text{ROB}(\Sigma, \phi)$ admits a Stein filling if for all $i$, $p_i > q_i > 0$.

Using Theorems 1.2-1.3 one can produce many examples of symplectically fillable rational open books. The above mentioned results begin the exploration of similar results concerning fillability for the case of pseudo-periodic surface homeomorphisms, which the authors are currently pursuing.

2. Preliminaries

2.1. Periodic maps on surfaces. Let $\Sigma$ be an oriented compact surface, and let $\text{Mod}(\Sigma)$ denote the mapping class group of $\Sigma$. We recall the Neilsen-Thurston classification [13] of surface diffeomorphisms.

**Theorem 2.1.** An $f \in \text{Mod}(\Sigma)$ is represented by a homeomorphism $h$ such that at least one of the following holds.

1. $h$ is a periodic.
(2) \( h \) is a reducible (i.e. \( h \) preserves a multicurve in \( \Sigma \)).
(3) \( h \) is a pseudo-Anosov

Suppose that \( \Sigma_g \) is a closed oriented surface with genus \( g \geq 1 \). By the Nielsen realization theorem \([7]\), an \( f \in \text{Mod}(\Sigma) \) of order \( n \) is represented by a homeomorphism \( h \) of the same order. Note that we will refer to both \( h \) and \( \langle h \rangle \), interchangeably, as \( \mathbb{Z}_n \)-actions on \( \Sigma_g \). Suppose that the quotient orbifold \( \mathcal{O}_h := \Sigma_g / \langle h \rangle \) has \( \ell \) cone points \( x_i, 1 \leq i \leq \ell \). Then each \( x_i \) lifts to an orbit of size \( n/n_i \) on \( \Sigma \) (of the \( \langle h \rangle \)-action), and the local rotation induced by \( h \) around the points in each orbit is given by \( 2\pi c_i \mod n_i \), where \( c_i \equiv 1 \mod n_i \).

We will now formalize the notion of data set introduced in Section 1.

**Definition 2.2.** A data set of degree \( n \) is a tuple

\[
D = (n, g_0, r; (c_1, n_1), (c_2, n_2), \ldots, (c_\ell, n_\ell)),
\]

where \( n \geq 1, g_0 \geq 0, \) and \( 0 \leq r \leq n - 1 \) are integers, and each \( c_i \) is a residue class modulo \( n_i \) such that:

(i) \( r = 0 \) if, and only if \( \ell = 0 \), and when \( r > 0 \), we have \( \gcd(r, n) = 1 \),
(ii) each \( n_i \mid n \),
(iii) for each \( i \), \( \gcd(c_i, n_i) = 1 \),
(iv) for each \( i \), \( \text{lcm}(n_1, \ldots, \hat{n}_i, \ldots, n_\ell) = \text{lcm}(n_1, \ldots, n_\ell) \), and \( \text{lcm}(n_1, \ldots, n_\ell) = n \), if \( g_0 = 0 \), and
(v) \( \sum_{j=1}^{\ell} \frac{n}{n_j} c_j \equiv 0 \) (mod \( n \)).

The number \( g \) determined by the equation

\[
\frac{2 - 2g}{n} = 2 - 2g_0 + \sum_{j=1}^{\ell} \left( \frac{1}{n_j} - 1 \right)
\]

is called the genus of the data set.

The following proposition (see \([11, \text{Theorem 3.9}]\)) allows us to represent the conjugacy of a cyclic action by a data set.

**Proposition 2.3.** Data sets of degree \( n \) and genus \( g \) correspond to conjugacy classes of \( C_n \)-actions on \( \Sigma_g \).

We will denote the data set associated with a periodic map \( h \) by \( D_h \). For a \( D_h \) as in Definition 2.2, the integer \( r \) will be non-zero only when \( h \) is a free rotation of \( \Sigma_g \) by \( 2\pi r/n \), in which case \( D_h \) would take the form \( (n, g_0, r; s, n) \). Equation \( \text{R-H} \) in Definition 2.2 is the Riemann-Hurwitz equation associated with the branched covering \( \Sigma_g \to \mathcal{O}_h \). Before diving into the geometric realizations of cyclic actions, we recall from \([10]\) the classification of \( C_n \)-actions on \( \Sigma_g \) into three broad categories.

**Definition 2.4.** Let \( h \) be a \( C_n \)-action on \( \Sigma_g \) with \( D_h \) as in Definition 2.2. Then \( h \) is said to be a:

(i) rotational action, if either \( r \neq 0 \), or \( D \) is of the form

\[
(n, g_0; (s, n), (n - s, n), \ldots, (s, n), (n - s, n)),
\]

\( k \) pairs
for integers $k \geq 1$ and $0 < s \leq n - 1$ with $\gcd(s, n) = 1$, and $k = 1$ if, and only if $n > 2$.

(ii) Type 1 action, if $\ell = 3$ and $n_3 = n$.

(iii) Type 2 action, if $h$ is neither a rotational nor a Type 1 action.

It is apparent that rotational actions can be realized as the rotations of $\Sigma_g$ through an axis under a suitable isometric embedding $\Sigma_g \to \mathbb{R}^3$. Moreover, Gilman [4] showed that a Type 1 $C_n$-action $h$ on $\Sigma_g$ is irreducible if and only if $O_h$ is a sphere with three cone points, which in the language of data sets means that

$$D_h = (n, 0; (c_1, n_1), (c_2, n_2), (c_3, n)).$$

In [10], it was shown that every irreducible Type 1 action is as a rotation of a unique polygon with side-pairing.

**Theorem 2.5.** For $g \geq 2$, consider a irreducible Type 1 action $f$ on $\Sigma_g$ with

$$D_f = (n, 0; (c_1, n_1), (c_2, n_2), (c_3, n)).$$

Then $f$ can be realized explicitly as the rotation $\theta_f$ of a unique hyperbolic polygon $P_f$ with a suitable side-pairing $W(P_f)$, where $P_f$ is a hyperbolic $k(f)$-gon with

$$k(f) := \begin{cases} 2n, & \text{if } n_1, n_2 \neq 2, \\
            n, & \text{otherwise}, \end{cases}$$

and for $0 \leq m \leq n - 1$,

$$W(P_f) = \prod_{i=1}^{n} a_{2i}^{-1} a_{2i+1} \text{ with } a_{2m+1}^{-1} a_{2i+1} \sim a_2, \quad \text{if } k(h) = 2n, \text{ and}$$

$$\prod_{i=1}^{n} a_i \text{ with } a_{m+1}^{-1} a_i \sim a_z, \quad \text{otherwise},$$

where $z \equiv m + qj \pmod{n}$ and $q = (n/n_2)c_3^{-1}, j = n_2 - c_2$.

Further, it was shown that an arbitrary non-rotational cyclic action $h$ admits a decomposition into irreducible Type 1 components. Conversely, given such a decomposition $h$ can be recovered by piecing together the irreducible components using the following two types of constructions.

(a) *$k$-compatibility.* This construction involves the removal of cyclically permuted invariant disks around points in a pair of compatible orbits (where the induced local rotation angles are the same) by a pair $h_i$ of irreducible Type 1 action on $\Sigma_{g_i}$, then identifying the resulting boundary components thereby obtaining an action $h = (h_1, h_2)$ on $\Sigma_{g_1 + g_2 + k - 1}$. This process of constructing $h$ is called a $k$-compatibility. If this construction is performed with a pair of compatible orbits induced by a single action $h$ on $\Sigma_g$, resulting in an action on $\Sigma_{g+k}$, then the process is called a self $k$-compatibility.

(b) *Permutation additions and deletions.* In the process of permutation addition we remove (cyclically permuted) invariant disks around points in an orbit of an action size $n$ induced by the action $h$ and then paste $n$ copies of $\Sigma_g$ (i.e. $\Sigma_{g'}$ with one boundary component) to the resultant
boundary components. This results in an action on $\Sigma_{g+nq'}$ with the same fixed point and orbit data as $h$. The reverse of this process, which involves the removal of such an added permutation component, is called a permutation deletion.

The upshot of the preceding discussion is the following.

**Theorem 2.6.** For $g \geq 2$, an arbitrary non-rotational action on $\Sigma_g$ can be constructed through finitely many $k$-compatibilities, permutation additions, and permutation deletions on irreducible Type 1 actions.

We will now describe the conjugacy classes of a $k$-compatible pair $(f_1, f_2)$ of actions in terms of $D_{f_1}$ and $D_{f_2}$.

**Definition 2.7.** A pair of data sets $D = (n, g_0; (c_1, n_1), \ldots, (c_i, n_i))$ and $\tilde{D} = (\tilde{n}, \tilde{g}_0; (\tilde{c}_1, \tilde{n}_1), \ldots, (\tilde{c}_i, \tilde{n}_i))$ are said to be $(i, j)$-compatible if there exist $i, j$ such that

(i) $n_i = \tilde{n}_j = m$,

(ii) $c_i + \tilde{c}_j \equiv 0 \pmod{m}$.

Given a pair of $(i, j)$-compatible data sets $D$ and $\tilde{D}$ as above, we define

$$(D, \tilde{D}) := (m, g_0 + \tilde{g}_0; (c_1, n_1), \ldots, (c_i, n_i), \ldots, (c_l, n_l), (\tilde{c}_1, \tilde{n}_1), \ldots, (\tilde{c}_i, \tilde{n}_i), \ldots, (\tilde{c}_l, \tilde{n}_l))$$

**Example 2.8.** The 1-compatible pair $(f, f^5)$ from Section 1 is represented by a $(3, 3)$-compatible $(D_f, D_{f^5})$ of data sets, where $D_f = (6, 0; (1, 2), (1, 3), (1, 6))$ and $D_{f^5} = (6, 0; (1, 2), (2, 3), (5, 6))$. This construction is illustrated in Figure 2 below.

**Definition 2.9.** For $\ell \geq 4$, let

$D = (n, g_0; (c_1, n_1), (c_2, n_2), \ldots, (c_\ell, n_\ell))$,

be a $C_n$-action. Then $D$ is said to be $(r, s)$-self compatible, if there exist $1 \leq r < s \leq \ell$ such that

(i) $n_r = n_s = m$, and

(ii) $c_r + c_s \equiv 0 \pmod{m}$.

**Example 2.10.** Consider a Type 2 action given by the data set $D = (3, 1; (1, 3), (2, 3))$. Consider the irreducible Type 1 data sets: $D_1 = (3, 0; (1, 3), (1, 3), (1, 3))$ and $D_2 = (3, 0; (2, 3), (2, 3), (2, 3))$. $D_1$ can be realized as a $\frac{2\pi}{3}$-rotation on a hexagon with opposite sides identified (i.e. a torus). $D_2$ can
Figure 3. Obtaining \((3, 1; (1, 3), (2, 3))\) from \((3, 0; (1, 3), (1, 3), (2, 3), (2, 3))\) by a \((2, 3)\)-self compatible gluing.

be realized as a \(\frac{2\pi}{3}\)-rotation on a similar hexagon. These two tori can be glued by removing a \((1, 2)\)-compatible pair of disks and will give a genus 2 surface with the following data set: \((3, 0; (1, 3), (1, 3), (2, 3), (2, 3))\). Now we can remove another \((2, 3)\)-self compatible pair of disk on that genus 2 surface and attach a tube. This will give us a genus 3 surface with data set \(D = (3, 1; (1, 3), (2, 3))\), as shown in Figure 3.

2.2. Fractional Dehn twist coefficient and right veering homeomorphisms. We now only consider elements in \(\text{Mod}(\Sigma, \partial \Sigma)\) which are freely isotopic to a periodic or pseudo-periodic homeomorphism. Let \(\text{Homeo}^+(\Sigma)\) denote the set of orientation preserving homeomorphisms of \(\Sigma\).

Definition 2.11. A map \(\phi \in \text{Homeo}^+(\Sigma)\), freely isotopic to \(h \in \text{Mod}(\Sigma, \partial \Sigma)\), is called a Thurston representative of \(h\), if it is periodic.

Unlike \(h\), \(\phi\) may rotate the boundary components of \(\Sigma\). If \(C \subset \partial \Sigma\) is a boundary component, \(\phi|_C\) is given by a \(\frac{2\pi}{q}\)-rotation for some \(p \in \mathbb{Z}_{>0}\) and \(q \in \mathbb{Z}\) such that \(|q| \leq p\). The rational number \(c(h) = \frac{q}{p}\) is called the fractional Dehn twist coefficient (FDTC) of \(h\) with respect to \(C\). For example, the hyperelliptic involution on \(\Sigma_1\) has FDTC equal to \(\frac{1}{2}\).

Consider the mapping torus determined by \(\phi\). The induced flow, when restricted to a boundary component \(C\) of the page has periodic orbits. Let \(\gamma\) be one such orbit. Then one can write \(\gamma\), in terms of the meridian \(\mu\) and longitude \(\lambda = C\), as \(\gamma = p\lambda + q\mu\), where \(p, q\) are relatively prime integers.

Definition 2.12. The fractional Dehn twist coefficient (FDTC) of \(h\) with respect \(\partial \Sigma\) is given by \(c(h) = \frac{q}{p}\).

Note that here we follow the slope convention in [1] which interchanges \(p\) and \(q\) in the definition of \(c(h)\) in [6]. The definition is analogous in the case of multiple boundary components, one can similarly define a fractional Dehn twist coefficient \(c_i\) for the \(i\)th boundary component.

Following the notation in [3], let \(H : \Sigma \times [0, 1] \to \Sigma\) be the free isotopy from \(h(x) = H(x, 0)\) to its periodic representative \(\phi(x) = H(x, 1)\). Define \(\beta : \partial \Sigma \times [0, 1] \to \partial \Sigma \times [0, 1]\) by sending \((x, t) \to (H(x, t), t)\). We form the union of \(\partial \Sigma \times [0, 1]\) and \(\Sigma\) by gluing \(\partial \Sigma \times \{1\}\) and \(\partial \Sigma\). We then identify this union with \(\Sigma\) to construct the homeomorphism \(\beta \cup \phi\) on \(\Sigma\) which is isotopic to \(h\) relative to \(\partial \Sigma\). Here, \(\beta\) denotes the reverse isotopy from \(t = 1\) to \(t = 0\).
We will assume that $h = \beta \cup \phi$. In terms of this description of $h$, the FDTC is given by the rotation induced by $\beta(x_0, 1) - \beta(x_0, 0)$.

We will briefly recall the notion of right-veering homeomorphisms from [3]. Let $\alpha$ and $\beta$ be isotopy classes, relative to end points, of properly embedded oriented arcs $[0,1] \to \Sigma$ with a common initial point $\alpha(0) = \beta(0) = x \in \partial \Sigma$. Choose representatives $a, b$ of $\alpha, \beta$ ($\alpha \neq \beta$), respectively, so that they intersect transversely (including endpoints) and with minimal number of intersections. Then we say $\beta$ is strictly to the right of $\alpha$ if the tangent vectors $(b'(0), a'(0))$ define the orientation on $\Sigma$ at $x$. A monodromy map $h$ is right-veering if for every choice of base point $x \in \partial \Sigma$ and every choice of arc $\alpha$ based at $x$, either $h(\alpha) = \alpha$ or $h(\alpha)$ is strictly to the right of $\alpha$. The following proposition (from [6]) relates right-veering maps to the FDTC.

**Proposition 2.13.** Let $h \in \text{Mod}(\Sigma, \partial \Sigma)$. Then $h$ is right-veering if and only if $c(h) \geq 0$ for every component of $\partial \Sigma$. Similarly, $h$ is left-veering if and only if $c(h) < 0$ for every component of $\partial \Sigma$.

### 2.3. Rational open book decomposition

A rational open book decomposition for a manifold $M$ is a pair $(L, \pi)$ consisting of an oriented link $L \subset M$ and a fibration $\pi : M \setminus L \to S^1$ such that, if $N$ is a small tubular neighborhood of $L$, then no component of $\partial N \cap \pi^{-1}(\theta)$ is a meridian of a component of $L$. As mentioned in [3], a rational open book may differ from an honest open book in the following two ways.

- **(a)** A component of $\partial N \cap \pi^{-1}(\theta)$ does not have to be a longitude to a component of $L$.
- **(b)** A component of $\partial N$ intersected with $\pi^{-1}(\theta)$ does not have to be connected.

As in the case of an honest open book, $\Sigma = \pi^{-1}(\theta)$ is called a page of the rational open book for any $\theta \in S^1$ and $L$ is called the binding. Similar to the honest open book, one can describe a rational open book using the monodromy map $\phi_0 : \Sigma \to \Sigma$ of the fibration $\pi$. In case of honest open books the monodromy map is assumed to be identity near the boundary of a page. For rational open books we require that near boundary $\phi_0^m = \text{id}$ for some integer $m$.

### 2.4. Contact structures on rational open books

We say a rational open book $(L, \pi)$ for $M$ supports a contact structure $\xi$ if there is a contact form $\alpha$ for $\xi$ such that the following conditions hold.

1. $\alpha(v) > 0$ for all positively pointing tangent vectors $v \in TL$.
2. $d\alpha$ is a volume form when restricted to each page of the open book.

We recall the Thurston-Winkelnkemper construction of contact structure on a rational open book from [3].

Let $(\Sigma, \lambda)$ be the page of a rational open book, where $\lambda$ is a one form such that $d\lambda$ is an area form on $\Sigma$ and $\lambda = rd\theta$ near $\partial \Sigma \subset (\partial \Sigma \times [-1, -1 + \epsilon], (\theta, r))$ for some sufficiently small $\epsilon > 0$. Let $\lambda_{(t, x)} = t\lambda_x + (1 - t)(\phi^* \lambda)_x$ be a 1-form on $(\Sigma \times [0,1], (x, t))$. Then the 1-form $\alpha_K = \lambda_{(t, x)} + Kdt$ is contact for $K$ large enough. Thus, $\alpha_K$ defines a contact form on the mapping torus $MT(\Sigma, \phi)$. Next, we extend this 1-form over the solid tori neighborhood of the binding. Here we describe the extension for a single binding component.
Let \((S^1 \times D^2, (\theta, r, \phi))\) be a neighborhood of a binding component. Consider the following gluing map between a boundary neighborhood \(S^1 \times N(\partial D^2)\) of \(S^1 \times D^2\) and a boundary neighborhood \(N(\partial \mathcal{M}T)\) of the mapping torus.

\[
\psi : S^1 \times N(\partial D^2) \longrightarrow N(\partial \mathcal{M}T)
\]

\[
(\theta, r, \phi) \mapsto (-r, p\theta + q\phi, -q\theta + p\phi)
\]

An example of this gluing map, which is defined for \(r \in [1-\epsilon, 1]\), is illustrated in Figure 4 below. Figure 5 represents the gluing region for an orbit of 3 cone points constituting 3 permuting boundary components.

**Figure 4.** The meridian on the right is sent to the \((3, 2)\)-curve (in red) on the left. For a honest/integral open book, the meridian is sent to the longitude (in green).

**Figure 5.** Gluing of solid torus near permuting boundaries

Using this gluing map we define the total manifold of the corresponding rational book as \(M_\phi = \mathcal{M}T(\Sigma, \phi) \cup \phi(S^1 \times D^2)\). The pullback form is given by \(\psi^*\alpha_K = (-rp - Kq)d\theta + (-rq + pK)d\phi\). We now extend this form using a form \(f_0(r)d\theta + g_0(r)d\phi\). This form will be contact if and only if \(f_0(r)g_0'(r) - f_0'(r)g_0(r) > 0\). Near \(S^1 \times D^2\) (i.e. \(r = 1\)), \(f_0(r) = -rp - qK\)
and \( g_0(r) = -rq + pK \). Near the core of \( S^1 \times D^2 \) (i.e. \( r = 0 \)), \( f_0(r) = 2 - r^2 \) and \( g_0(r) = r^2 \). We can then extend the functions \( f_0 \) and \( g_0 \) to define a contact form on whole of \( S^1 \times D^2 \). Moreover, this extension is unique. For more details on rational open books, we refer to [1].

2.5. From integral open books to rational open books. One can see a rational open book of \( M \) more explicitly as an abstract integral open book \( \text{OB}(\Sigma, \phi) \) with some modification in a neighborhood of the binding \( \partial \Sigma \). Let \( L \) be a connected component of \( \partial \Sigma \). We take a solid torus neighborhood \( U_L \) of \( L \) and consider the identification of \( \partial U_L \) with \( \partial M\Sigma(\Sigma, \phi) \). Let \( \lambda \) and \( \mu \) denote the reference longitude and meridian on \( M\Sigma(\Sigma, \phi) \). Then \( \mu \) approaches \( L \) along the \((1,0)\)-curve on \( \partial U_L \). We cut out \( U_L \) from \( \text{OB}(\Sigma, \phi) \) and glue in a solid torus \( U_0 \), with longitude \( \lambda_0 \) and meridian \( \mu_0 \), by sending \( \mu_0 \) to \( p\lambda + q\mu \). This is same as a topological \( \frac{p}{q} \)-surgery on \( \text{OB}(\Sigma, \phi) \) along \( L \). Let \( L_0 \) denote the core of \( U_0 \). Note that \( \mu \) now approaches \( L_0 \) via the \( (p,q)\)-curve on \( \partial U_0 \). The surgered manifold \( M_{\frac{p}{q}}(L) \) admits an abstract open book decomposition with page \( \Sigma \) and monodromy \( \phi \circ \partial_{\frac{2\pi}{p}} \). Here, \( \partial_{\frac{2\pi}{p}} \) denotes the \( \frac{2\pi}{p} \)-rotation on the boundary component of \( \Sigma \) aproaching \( L_0 \). We will denote this rational abstract open book by \( \text{ROB}(\Sigma, \phi \circ \partial_{\frac{2\pi}{p}}) \). For \( |p| = \pm 1 \), this rational open book is an integral open book.

3. Associating contact structures to data sets

Let \( h \) be a pseudo-periodic homeomorphism on a closed surface \( \Sigma \), such that \( \Sigma \) can be decomposed into disjoint connected sub-surfaces \( \Sigma_g \)'s so that \( f_i = h|_{\Sigma_g} \) is an irreducible periodic map. In other words, \( h \) can be decomposed into finitely many compatible irreducible Type 1 actions.

3.1. Associating contact structures to Type 1 data sets. Let us first associate a contact structure to the data set corresponding to an irreducible Type 1 action. The idea is to somehow associate a page and a monodromy of a rational open book. Since a \( C_n \)-action realized by a data set lives on a closed surface, to obtain a page with periodic homeomorphism one needs to remove disks around some of the non-trivial orbit points of that action. Therefore, we define something called a marked data set. Let us first discuss the notion for a specific example.

Example 3.1. Consider the data set \( D_{\phi_0} = (5, 0; (1, 5), (3, 5), (1, 5)) \) that represents the homeomorphism \( \phi_0 \) of a genus-2 closed surface \( \Sigma_2 \), induced by a \( \frac{2\pi}{5} \)-rotation of a 10-gon with opposite sides identified as in Figure 6. This is an irreducible Type 1 periodic homeomorphism. This map has three fixed points, two of which have \( \frac{2\pi}{5} \)-rotation in their disk neighborhoods, while the third has a \( \frac{4\pi}{5} \)-rotation.

Now consider the surface \( \Sigma_2 \), obtained from \( \Sigma_2 \) by removing two disk neighborhoods, both with rotation \( \frac{2\pi}{5} \). Let \( \phi_0 \) denote the restriction homeomorphism \( \phi_0|_{\Sigma_2} \). We fix an ordering of the cone points in the data set and let the last cone point denote the center of rotation in the covering polygon. We can index the ordering as \{1, 2, 3\}, and we mention the indexing numbers of the cone points around which we are removing the disks. In our
case, if we fix the order of the cone points as written in $D$, then inclusion of $[1, 3]$ in the data set will say that disks around the first and third cone points are removed. Another aspect we need to take into consideration is the direction of rotation. Note that the data set $D$ can be realized either by a $+\frac{2\pi}{5}$-rotation or by a $-\frac{6\pi}{5}$-rotation.

![Order 5 rotation on a 10-gon](image)

**Figure 6.** Order 5 rotation on a 10-gon. The orbits among cone points are denoted by similar colours.

The two cases will make a huge difference in terms of contact structures. So, we simply add a plus or minus sign with the first entry of the data set. In particular, a marked data set like $\hat{D}_1 = (5, 0; (1, 5), (3, 5), (1, 5), [1, 3])$ will represent the map $\phi_0$ on $\Sigma_2$. Similarly, $\hat{D}_2 = (5, 0; (1, 5), (1, 5), (3, 5), [1, 2, 3])$ will represent a homeomorphism on a surface, obtained from $\Sigma_2$ by removing three disks around three fixed points, and the homeomorphism will be restriction of a homeomorphism on $\Sigma_2$ induced by a $-\frac{6\pi}{5}$-rotation of the 10-gon.

In general, let $D_{h_0} = (n_3, 0; (c_1, n_1), (c_2, n_2), (c_3, n_3))$ be a data set representing an irreducible periodic homeomorphism $h_0$ on a closed surface $\Sigma_{g_0}$. By Theorem 2.5, the conjugacy class of $h_0$ is realized by a $2\pi c_i^{-1}/n_i$-rotation on an even sided polygon with appropriate side-pairing. This means there is $g \in \text{Homeo}^+(\Sigma_{g_0})$ such that $ghg^{-1}$ is isotopic to $R_0$, where $R_0$ is the map induced on $\Sigma_0$ from the rotation on the polygon. Assume that $n_1, n_2 < n_3$. Thus, $h = R_0$ has a fixed point $x_{31}$ on $\Sigma_{g_0}$, around which $h$-induces a $\frac{2\pi c_i^{-1}}{n_i}$-rotation. Moreover, there are orbits with $\frac{n}{n_i}$ points such that $h$ cyclically permutes the orbit points, and $h^{\frac{n}{n_i}}$ induces a $\frac{2\pi c_i^{-1}}{n_i}$-rotation around each orbit point, for $i = 1, 2$ (i.e. $h$ induces $\frac{2\pi c_i^{-1}}{n_i}$-rotation). We have $N = (1 + \frac{n_1}{n_i} + \frac{n_2}{n_i})$ many cone points. Here also, the given data set can be realized either by a $+\frac{2\pi c_i^{-1}}{n_i}$-rotation or by a $-\frac{2\pi (n_3 - c_i^{-1})}{n_3}$-rotation.

**Definition 3.2 (Type 1 marked data set).** A modified data set of the form

$\hat{D}_{\pm} = (n_3 \pm, 0; (c_1, n_1), (c_2, n_2), (c_3, n_3), [j_1, \ldots, j_k])$

will be called a Type 1 marked data set.
Thus, a Type 1 marked data set \( \hat{D}_\pm = (n_{3,\pm}, 0; (c_1, n_1), (c_2, n_2), (c_3, n_3), [j_1, \cdots, j_k]) \) represents a surface \( \hat{\Sigma}_{g_0} \), obtained from \( \Sigma_{g_0} \) by removing some \( l \)-many open disks \( 1 \leq l \leq N \), and the restriction homeomorphism \( \hat{R}_0|_{\hat{\Sigma}_{g_0}} \).

We associate the induced contact structure on ROB(\( \bar{\Sigma}_0, \hat{R}_0 \)) to the marked data set \( \hat{D}_\pm \). Note that the association of a contact structure to a marked data set as above is well-defined.

### 3.2. Associating contact structures to Type 2 data sets and their relation to Type 1 data sets

Similar to the previous section, one can define marked data sets of Type 2. These are simply collection of compatible Type 1 data sets with extra marking of points. We describe the association for some specific examples.

**Example 3.3.** Consider the data set \( (6, 0; (1, 2), (1, 2), (1, 3), (2, 3)) \) from Example 2.8. This data set can be realized by combining two \((1, 5)\)-compatible Type 1 data sets:

\[
D_1 = (6, 0; (1, 2), (1, 3), (1, 6)) \text{ and } D_2 = (6, 0; (1, 2), (2, 3), (5, 6)).
\]

Now consider two marked data sets coming from these Type 1 data sets:

\[
\hat{D}_{\varphi_1} = ((6_+, 0; (1, 2), (1, 3), (1, 6), [1, 3]) \text{ and } \\
\hat{D}_{\varphi_2} = ((6_-, 0; (1, 2), (2, 3), (5, 6), [1, 2, 3]).
\]

Let \((\Sigma_{g_1}, \varphi_1)\) and \((\Sigma_{g_2}, \varphi_2)\) be the representative surfaces and monodromies corresponding to \(\hat{D}_{\varphi_1}\) and \(\hat{D}_{\varphi_2}\) respectively. We denote a compatible gluing by the following notation. A gluing represented by \((2 : 1) \sim (3 : 2)\) will mean gluing the second boundary component of the first surface to the third boundary component of the second surface. The notation \( (\hat{D}_{\varphi_1}, \hat{D}_{\varphi_2}), (3 : 1) \sim (3 : 2)\) will then represent the surface \(\Sigma_{g_{12}}\), obtained by gluing an annulus between \(\Sigma_{g_1}\) and \(\Sigma_{g_2}\) along their boundary components with rotations \(\frac{2\pi}{6}\) and \(-\frac{4\pi}{6}\) respectively, and the homeomorphism \(\varphi_{12}\), obtained as a union of \(\varphi_1\) and \(\varphi_2\) and then extended by the identity map on the connecting annulus. In other words we have a \((2, 3)\)-compatible gluing of \(\hat{D}_{\varphi_1}\) and \(\hat{D}_{\varphi_2}\).

Now, \(\varphi_i\) can be isotoped, relative to boundary, to \(h_i \circ \delta_i\), where \(h_i \in \text{Mod}(\Sigma_{g_i}, \partial \Sigma_{g_i})\) and \(\partial_i\) denotes the rotations on boundary components, for \(i = 1, 2\). More precisely, \(\partial_1\) is the union of rotations on the four boundary components of \(\Sigma_{g_1}\) – one invariant boundary component with \(\frac{2\pi}{6}\)-rotation (corresponding to \((1, 6)\) in \(\hat{D}_{\varphi_1}\)) and three permuting boundaries with \(\frac{2\pi}{6}\)-rotation (corresponding to \((1, 2)\) in \(\hat{D}_{\varphi_1}\)) on each of them. Similarly, \(\partial_2\) consists of three permuting boundaries with \(-\frac{2\pi}{6}\)-rotations, two permuting boundaries with \(-\frac{4\pi}{6}\)-rotations and one boundary with \(-\frac{2\pi}{6}\)-rotation. Let us rewrite \(h_i\) as \(h_i \circ \beta_i\), for \(i = 1, 2\). Here, \(\beta_i\) is the restriction of the free isotopy between \(\varphi_i\) and \(h_i\) on the non-permuting boundary components (which are to be glued together) and \(\bar{h}_i = \varphi_i \cup \beta_i\). It can be seen from Figure 7 that the resulting homeomorphism \(\varphi_{12}\) on \(\Sigma_{g_{12}}\) is given by \(\bar{h}_1 \cup id \cup \bar{h}_2\).
\[ \hat{h}_1 = \phi_1 \cup \beta_1 \]

\[ \hat{h}_2 = \phi_2 \cup \beta_2 \]

Figure 7. Monodromy after gluing along compatible boundaries with rotations of different signs.

**Example 3.4.** If we had the following Type 2 collection of marked data sets:
\[ \hat{D}_{\phi_1} = (6_+, 0; (1, 2), (1, 3), (1, 6), [1, 3]) \] and \[ \hat{D}_{\phi_2} = (6_+, 0; (1, 2), (2, 3), (5, 6), [1, 2, 3]) \], then a similar analysis as in Example 3.3 will show that \( \left[ \hat{D}_{\phi_1}, \hat{D}_{\phi_2} \right] \sim (3 : 1) \) represents the surface \( \Sigma_{g_12} \) with homeomorphism of the form \( \hat{h}_1' \cup T_{c_0} \cup \hat{h}_2' \), as shown in Figure 8 below.

\[ \hat{h}_1 = \phi_1 \cup \beta_1 \]

\[ \hat{h}_2 = \phi_2 \cup \beta_2 \]

Figure 8. Monodromy after gluing along compatible boundaries with rotations of same sign.

Here, \( T_{c_0} \) denotes positive Dehn twist along the curve \( c_0 \). Moreover, \( \hat{D}_1 = (6_-, 0; (1, 2), (1, 3), (1, 6), [1, 3]) \) and \( \hat{D}_2 = (6_-, 0; (1, 2), (2, 3), (5, 6), [1, 2, 3]) \) will have a homeomorphism of the form \( \hat{h}_1' \cup T_{c_0}^{-1} \cup \hat{h}_2' \). One can similarly look at the resulting homeomorphisms for self compatible data sets.

Note that we order the cone points of a marked data sets that has been obtained by compatible gluing of two other marked data sets by writing the cone point entries of \( \hat{D}_1 \) followed by those of \( \hat{D}_2 \) and then removing the cone points which are killed in compatible or self-compatible gluing. Thus in general we can define a Type 2 marked data set as follows.

**Definition 3.5.** A modified data set of the form
\[ \hat{D} = (n_{l \pm}, g_0; (c_1, n_1), (c_2, n_2), \ldots, (c_l, n_l), [j_1, \ldots, j_k]) \]

is called a Type 2 marked data set.
or self-compatible irreducible Type 1 data sets. Thus, we first consider the finitely many rational open books associated to those compatible Type 1 marked data sets and then inductively glue the pages and homeomorphisms as in Example 3.3 and Example 3.4 to give the resultant rational open book. The contact structure compatible with this resultant rational open book gives the contact structure associated to $\hat{D}$.

4. Symplectic fillability of rational open books

In this section we prove our main results. Before going into the proofs, we briefly review the notion of admissible transverse surgery on a contact manifold from [1].

4.1. Admissible transverse surgery. Let $K \subset (M, \xi)$ be a transverse knot with a fixed framing $F$.

Definition 4.1. A $\frac{q}{p}$-surgery on $K$ is called admissible if there exists a neighborhood $N \subset M$ of $K$ that is contactomorphic to a neighborhood $N_{r_0} = \{(r, \theta, z) | r \leq \sqrt{r_0}\}$ of the $z$-axis in $\mathbb{R}^3/(z \equiv z + 1)$ with the contact structure $\xi_0 = \ker(dz + r^2d\theta)$ such that $F$ goes to the product framing on $N_{r_0}$ and $-\infty < \frac{q}{p} < -\frac{1}{r_0}$.

If $M_K(\frac{q}{p})$ is obtained from $M$ by an admissible transverse surgery, then $M_K(\frac{q}{p})$ admits a natural contact structure $\xi_K(\frac{q}{p})$ on it. This contact structure is defined by a contact cut or reduction process. One takes a neighborhood of $K$ as in definition 4.1 and considers the characteristic foliations on the tori at different radii from the central knot. Topologically, a $\frac{q}{p}$-surgery sends the $(p, q)$-curve on a torus to the boundary of a disk. For more on contact cuts we refer to [2].

In some cases, one can recover a contact rational open book via admissible transverse surgery on an honest contact open book. In particular, recall the rational contact open book $\text{ROB}(\Sigma, \phi \circ R_{\frac{q}{p}})$ and the honest contact open book $\text{OB}(\Sigma, \phi)$ as described in section 2.5 with $L = \partial \Sigma$. We assume that both $p$ and $q$ are positive.

Lemma 4.2. $\text{ROB}(\Sigma, \phi \circ R_{\frac{q}{p}})$ is obtained from $\text{OB}(\Sigma, \phi)$ via a $-\frac{q}{p}$-transverse surgery on $L$.

Proof. Recall from section 2.4 that we glued a solid torus $S^1 \times D^2$ to one of the boundary components of the mapping torus $\mathcal{M}T(\Sigma, \phi)$ by sending a meridian $\{x\} \times \partial D^2$ to the $(p, q)$-curve (i.e. representing $p[l] + q[m]$ homology class) with slope $\frac{q}{p}$. In case of an honest open book, this attaching homeomorphism interchanges the meridian and longitude of the solid torus with that of the mapping torus. More precisely, $\{x\} \times \partial D^2$ goes to $(1, 0)$ and $S^1 \times \{y\}$ goes to $(0, -1)$. Therefore, a $(p, q)$-curve on $\partial \mathcal{M}T(\Sigma, \phi)$ will be identified with a $(q, -p)$-curve on $\partial(S^1 \times D^2)$. Now, there exists a neighborhood $N_L$ of the binding $L$ in $\text{OB}(\Sigma, \phi)$ such that $N_L$ is contactomorphic to $N_{r_0}$ as in Definition 4.1. So we can do a transverse $-\frac{p}{q}$ surgery along $L$. This amounts to attaching a disk along the $(q, -p)$-curve on $\partial N_{-\frac{q}{p}}$, which
is same as attaching a solid torus to $MT(\Sigma, \phi)$ along the $(p, q)$-curve on its boundary.

Note that Lemma 4.2 is also true for rational open books with multiple rotating boundary components.

**Lemma 4.3.** Consider the contact structure $(M^3, \xi_0) = \text{ROB}(\Sigma, R_0)$ associated to an irreducible marked data set

$$D = (n_+, 0; (c_1, n_1), (c_2, n_2), (c_3, n), [j_1, \cdots, j_k]),$$

where $[j_1, \cdots, j_k]$ is as mentioned in Theorem 1.2. Then, $(M^3, \xi_0)$ is Stein fillable.

Note that by Equation R-H in Definition 2.2, $\Sigma$ has genus $\frac{n-1}{2}$. Therefore, $n$ has to be odd. Suppose the contact $(M^3, \xi)$ admits an open book decomposition $\text{OB}(S, h)$ with $h$ freely isotopic to a periodic monodromy. Let $r_i$ be the FDTC of the $i$th boundary component of $S$ and $\psi$ be the periodic representative of $h$. The proof of Lemma 4.3 will require the following result [3, Theorems 4.1-4.2].

**Theorem 4.4** (Colin–Honda, [3]). If all the $r_i$ are positive, then $(M, \xi)$ is a uniquely Stein fillable $S^1$-invariant contact structure which is transverse to the $S^1$-fibers.

The next result that we need is due to Baldwin-Etnyre [2, Theorem 3.2].

**Theorem 4.5** (Baldwin–Etnyre, [2]). Let $K$ be a transverse knot in some contact manifold. Suppose $N$ is a standard neighborhood of $K$ such that the characteristic foliation on $\partial N$ is linear with slope $a$, where $n < a < n + 1$ for some integer $n$. Then, for any rational number $s < n$, admissible transverse $s$-surgery on $K$ can also be achieved by Legendrian surgery on some Legendrian link in $N$.

We now prove Lemma 4.3.

**Proof of Lemma 4.3.** Let us consider the marked data set

$$D = (n_+, 0; (c_1, n_1), (c_2, n_2), (c_3, n), [1, 2, 3]).$$

We see that $\Sigma$ is a genus-$(\frac{n-1}{2})$ surface with three boundary components. Each of these three boundary components are invariant under $R_0$ and the $i$th boundary component rotates by an angle of $\frac{2(n-1)}{n}$ for $i = 1, 2, 3$. In other words, $R_0$ is isotopic, relative to boundary, to a homeomorphism $h \circ \partial_1 \circ \partial_2 \circ \partial_3$, where $h$ is an element of Mod$(\Sigma, \partial \Sigma)$ and $\partial_i$ represents the boundary rotation with FDTC $= \frac{c_i - 1}{n}$. It is clear that $h$ is freely isotopic to $h \circ \partial_1 \circ \partial_2 \circ \partial_3$.

Since $R_0$ comes from a positive marked data set, $\frac{c_i - 1}{n} > 0$ for all $i$. Therefore, Theorem 4.4 implies that the contact open book $\text{OB}(\Sigma, h)$ is uniquely Stein fillable. Let $(W^3, d\lambda)$ denote this Stein filling.

Now by Lemma 2.5, $\text{ROB}(\Sigma, R_0) = \text{ROB}(\Sigma, h \circ \partial_1 \circ \partial_2 \circ \partial_3)$ can be obtained from $\text{OB}(\Sigma, h)$ by doing $-\frac{c_i}{n}$-transverse surgery on the $i$th boundary component for $i = 1, 2$ and 3. Let $L_i$ be the $i$th binding component in $\text{OB}(\Sigma, h)$ and let $N_i$ denote a standard contact neighborhood of $L_i$ such that the
characteristic foliation on ∂N_i has slope a such that −1 < a < 0. Note that near the binding of an honest open book, the contact plane rotates so that the slope of the characteristic foliation goes from 0 to −∞. Therefore, one can always find such an N_i. Applying Theorem 4.5 for n = −1, we get that ROB(Σ, h ∘ ∂1 ◦ ∂2 ◦ ∂3) can be obtained from OB(Σ, h) by Legendrian surgery on some link in N_1 ∪ N_2 ∪ N_3. Each of these Legendrian surgeries along a knot in (M, ξ) amounts to attaching a Stein 2-handle to (W^4, λ). Hence, ROB(Σ, R_0) is Stein fillable.

It is easy to see that the proof in the general case is exactly the same with fewer number of binding components to do surgery on. □

The final ingredient we need is the following result, which is a straightforward corollary of [1, Theorem 1.3].

**Theorem 4.6** (Baldwin–Etnyre–Morris, [1]). If OB(Σ, φ_1) and OB(Σ, φ_2) is Stein fillable, then OB(Σ, φ_1 ◦ φ_2) is Stein fillable.

We are now ready to prove Theorem 1.2.

**Proof of Theorem 1.2.** Here also we consider data sets with 3-disk neighborhoods of cone points removed, i.e., [n, n, n]. The proof is similar for the remaining cases.

Let ¯D_i = (n_i, 0; (c_{i1}, n_i), (c_{i2}, n_i), (c_{i3}, n_i), [1, 2, 3]) for i = 1, 2 be two compatible irreducible marked data sets as in the hypothesis of Theorem 1.2. Let ROB(Σ_{g_i}, R_i) be the rational open book associated to ¯D_i. Assume that ¯D_1 and ¯D_2 are (2, 3)-compatible. Which means that c_{i2} + c_{i3} ≡ 0 (mod n). According to our notation used in Example 3.3–3.4, the resultant marked data set is represented by ([ ¯D_1, ¯D_2], (2 : 1) ∼ (3 : 2)).

As in Lemma 4.3 we write the monodromy homeomorphism R_i as a composition of an element of the relative mapping class group with fractional rotations near boundary. Let R_i = h_i ◦ ∂_{i1} ◦ ∂_{i2} ◦ ∂_{i3} for i = 1, 2. As described in Lemma 4.3, both OB(Σ_{g_2}, h_1) and OB(Σ_{g_3}, h_2) are Stein fillable by Theorem 4.6.

Now recall condition (2) in Theorem 1.2: both ∂(Σ_{g_1} ∪ Σ_{g_2}) and ∂(Σ_{g_1} ∪ Σ_{g_2}) are non-empty. This implies that we can build Σ_{g_1} ∪ Σ_{g_2} by attaching 1-handles to either Σ_{g_1} or Σ_{g_2}. The reason behind this is the following. Let C_b be a boundary component of Σ_{g_1} ∪ Σ_{g_2} \ Σ_{g_1}. Let C_0 be the curve along which Σ_{g_1} and Σ_{g_2} are glued together, as shown in Figures 9. We deformation retract C_b to bring it near C_0 so that the resulting manifold is homeomorphic to the boundary connected sum of Σ_{g_1} and a 1-handlebody H_1. One can then extend h_1 to a homeomorphism \hat{h}_1 on Σ_{g_1} ∪ Σ_{g_2} by identity on H_1. Similarly we get \hat{h}_2 on Σ_{g_1} ∪ Σ_{g_2}.

Since \hat{Σ} = Σ_{g_1} ∪ Σ_{g_2} is obtained by attaching 1 handles to either Σ_{g_1} or Σ_{g_2}, OB(Σ, \hat{h}_i) is Stein fillable for i = 1, 2. Therefore, by Theorem 4.6, OB(Σ, \hat{h}_1 ∪ \hat{h}_2) is Stein fillable.

We now recall from Example 3.4 the description of the resulting homeomorphism after compatible gluing of two marked data sets. According to that description the resulting homeomorphism on \hat{Σ} is given by \hat{h}_1 ∪ T_{C_0} ∪ h_2 ∪ ∂_1 ∪ ∂_2, which is the same as the homeomorphism \hat{h}_1 ◦ T_{C_0} ◦ \hat{h}_2 ◦ ∂_1 ◦ ∂_2. Here, ∂_i denotes the union of rotations on the boundary components of Σ_i.
except the ones that are used in the compatible gluing. In particular, the contact manifold associated to \((\hat{D}_1, D_2, (c_{12}, n), (c_{12}, n))\) is given by

\[
\text{ROB}(\tilde{\Sigma}, h_1 \circ T_{C_0} \circ h_2 \circ \partial_1 \circ \partial_2),
\]

as shown in Figure 10 below.

\[
\text{ROB}(\tilde{\Sigma}, \tilde{h}_1 \circ T_{C_0} \circ \tilde{h}_2),
\]

as shown in Figure 10 below.

As an application of Theorem 1.2, we discuss two examples of marked data sets.

**Example 4.7.** Consider the following marked data set.

\[
\hat{D} = (5, 0; (3, 5), (3, 5), (1, 5), (2, 5), [1, 2, 3]).
\]

Note that \(\hat{D}\) can be realized by two \((1, 2)\)-compatible irreducible Type 1 marked data sets: \(\hat{D}_1 = (5, 0; (3, 5), (1, 5), (3, 5), [1, 2, 3])\) and \(\hat{D}_2 = (5, 0; (1, 5), (2, 5), (1, 5), [1, 3])\). This example is described in Figure 10 for the values: \(n = 5, c_1^{-1} = 2, c_{13}^{-1} = 2\) and \(c_{21}^{-1} = 1\). The resulting surface is of genus 4 with 2 genera coming from each of its irreducible components. Theorem 1.2 then says that the contact structure associated to \(\hat{D}\) is Stein fillable.
Example 4.8. Consider the following marked data set.

\[ \hat{D} = (6_+; (1, 2), (1, 3), (2, 3), (4, 5), [4]). \]

This \( \hat{D} \) can be realized by two \((3, 3)\)-compatible irreducible marked data sets: 
\[ \hat{D}_1 = (6_+; (1, 2), (1, 3), (2, 3), [3]) \] and 
\[ \hat{D}_2 = (6_+; (1, 3), (2, 3), (4, 5), [2, 3]). \]

Therefore, by Theorem 1.2, the contact structure associated to \( \hat{D} \) is Stein fillable.

4.2. The case of self compatible gluing in marked data sets. The total surface after self compatible gluing within a marked data set can be seen in two steps. First one attaches a 1-handle between the two compatible boundary components. Then one attaches a disk along the resulting connected boundary component. This is known as capping off a boundary component. In the proof of Theorem 1.2 we saw how compatible gluing between distinct surfaces preserves Stein fillability. The approach taken there breaks down for self-compatible gluing because the 1-handles are attached within the same connected component. Moreover, the change in contact structure due to capping off a boundary in the page of a contact open book is not very clear. Note that Baldwin and Etnyre have proved an interesting result that shows that capping off operation on certain universally tight contact structures may lead to overtwisted contact structures.

5. Explicit symplectic fillings of rational open books

5.1. Mori’s construction of symplectic filling. Mori \cite{Mori} constructed explicit strong symplectic filling of open books whose monodromy is composition of positive Dehn twists along disjoint curves. The part of his construction that we are interested in is the filling near binding of an open book. In particular, we will first look at the filling of an open book with identity monodromy. Note that \( \partial(S \times D^2) = OB(S, id) \). Let us consider the symplectic form \( d\alpha \oplus 2sd\phi \wedge d\phi \) on \( \Sigma \times D^2 \), where \( d\alpha \) is an exact symplectic form on \( \Sigma \) and \((s, \phi)\) are radial co-ordinates on \( D^2 \). We then attach a region, diffeomorphic to \( S^1 \times D^2 \), to \( \Sigma \times D^2 \) and extends the symplectic structure to all of \( S^1 \times D^2 \), so that the resulting manifold has a boundary contactomorphic to \( OB(S, id) \). Below we describe the procedure in more detail.

We can consider \( \Sigma \times S^1 \) sitting inside \( (\Sigma \times \partial D^2 \times (0, 1], (x, \phi, s)) \subset \Sigma \times D^2 \). This induces a symplectic structure \( \omega_0 = d(\alpha_K + s^2d\phi) \) on \( \Sigma \times \partial D^2 \times (0, 1] \). First, we embed \( \{\theta\} \times D^2 \subset S^1 \times D^2 \) into \( \mathbb{R}^3 \) by the map \((r, \phi) \mapsto (h_1(r)\cos \phi, h_1(r)\sin \phi, h_0(r)) \). Let \( w = x + iy \) and \( z = h_0(r) \). Here \( h_0, h_1 \) are smooth increasing functions defined on \([0, 1]\) such that near \( r = 1 \), \( h_0(r) = r - \frac{1}{2} \) and \( h_1(r) = 1 \), and near \( r = 0 \), \( h_0(r) = \frac{r^2}{2} \) and \( h_1(r) = r \). Thus, any point on the region \( R_0 = \{(w, z) | h_0 \circ h_1^{-1}(|w|) \leq \zeta \leq \frac{1}{2} \} \) can be represented by \( z = h_0(r), s h_1(r) \) and \( \arg(w) = \phi \), where \( s \in [0, 1] \) is determined by each point but \((0, 0)\). By description, \( R_0 \) is diffeomorphic to \( \text{int}(D^3) \). See Figure 11 below for a description of this embedding.

Using the gluing map between the boundary of a tubular neighborhood of binding and the boundary of the mapping torus, we can pull back the symplectic structure \( \omega_0 = d(\alpha_K + s^2d\phi) \) on \( \Sigma \times \partial D^2 \times (0, 1] \) to \( S^1 \times N(\partial D^2) \times (0, 1] \). One can then extend this symplectic structure by using the form
\( \omega = d(f(r)d\theta + s^2 g(r)d\phi) \), where \( f \) and \( g \) are real valued functions that interpolates between the contact structures on the mapping torus and on the solid torus neighborhood of the binding.

5.2. Modification of Mori’s construction for rational open books. Following Mori’s construction in [9], our plan is to pullback the symplectic form \( \omega_0 \) over \( S^1 \times N(\partial D^2) \times [1 - \epsilon, 1] \subset S^1 \times D^3 \) via some gluing map between \( S^1 \times N(\partial D^2) \times (0, 1] \) and \( \partial \Sigma \times (1 - \epsilon, 1] \times S^1 \times (0, 1] \subset F \times D^2 \) and then extend it to all of \( S^1 \times R_0 \) to produce a symplectic filling of ROB(\( \Sigma, \partial \Sigma \)).

We define the gluing map \( \Psi \) as follows.

\[
\Psi : S^1 \times N(\partial D^2) \times (0, 1] \rightarrow \partial \Sigma \times (1 - \epsilon, 1] \times S^1 \times (0, 1]
\]

\[
(\theta, r, \phi, s) \mapsto (p\theta + q\phi, -r, -q\theta + p\phi, s)
\]

Then \( \omega = \Psi^* \omega = d((-r p - K q - q s^2) d\theta + (-r q + p K + p s^2) d\phi) \) and \( \omega \wedge \omega = 2s(p^2 + q^2) d\theta \wedge dr \wedge ds \wedge d\phi > 0 \) for all \( s \in (0, 1] \). We want to extend this symplectic form to a 2-form \( \Omega = d(f(r,s)d\theta + g(r,s)d\phi) \), where \( f(r,s) = f_0(r) - q s^2 \) and \( g(r,s) = g_0(r) + p s^2 \), such that \( \Omega \wedge \Omega > 0 \) on \( S^1 \times D^3 \).

A simple computation shows that \( \Omega \wedge \Omega = -2s[p f_0'(r) + q g_0'(r)] d\theta \wedge dr \wedge ds \wedge d\phi \). So, together with the contact condition, we want \( f_0(r), g_0(r), p \) and \( q \) satisfying the following conditions.

1. Contact condition: \( f_0(r)g_0'(r) - f_0'(r)g_0(r) > 0 \) for all \( r \in [0, 1] \).
2. Symplectic condition: \( p f_0'(r) + q g_0'(r) < 0 \) for all \( r \in [0, 1] \).
3. Near \( r = 0 \), \( f_0(r) = 2H - r^2 \) and \( g_0(r) = r^2 \). Here \( H \) is a positive number.

Note that \( f_0(1) = -p - q K, g_0(1) = -q + p K \) and \( f_0'(r) = -2r, g_0'(r) = 2r \) near \( r = 0 \). Thus, the symplectic condition implies that \(-2r(p - q) < 0 \implies p > q \).

We will investigate the above condition by cases.

Case 1 \( (p > 0, q < 0) \): In this case we want to connect \((f_0(1), g_0(1)) = (-p - q K, -q + p K) \) and \((2H, 0) \) so that conditions (2) and (3) are satisfied. By choosing \( K \) large enough, we can always make sure that \((f_0(1), g_0(1)) \) lies in the first quadrant. Moreover, we choose...
Given this, we can always connect \((f_0(1), g_0(1))\) and \((2H, 0)\) via \((f_0(r), g_0(r))\) such that \(f'_0(r) < 0\) and \(g'_0(r) > 0\) for all \(r \in [0, 1]\) such that condition (1) and (3) is satisfied. Note that such \(f_0(r)\) and \(g_0(r)\) will also satisfy condition (2), as shown in Figure 12.

**Case 2** \((p > 0, q > 0)\): Taking \(K\) large enough we can assume \(f_0(1) < 0\) and \(g_0(1) > 0\). Moreover, \((f'_0(1), g'_0(1)) = (-p, -q)\). Again, taking \(H\) large enough we can connect \((f_0(1), g_0(1))\) and \((2H, 0)\) satisfying conditions (1) to (3), as shown in Figure 13.

**Case 3** \(q < p < 0\): For \(K\) large enough, \(f_0(1) > 0, g_0(1) < 0\ \ f'_0(1) > 0, g'_0(1) > 0\). Then, any curve joining \((2H, 0)\) and \((f_0(r), g_0(r))\) that satisfies conditions (1) and (3), must have a point \((f_0(r_0), g_0(r_0))\) such that both \(f'_0(r_0)\) and \(g'_0(r_0)\) are negative (see Figure 14). Thus, \(pf'_0(r_0) + qg'_0(r_0) > 0\) and condition (2) is violated. So, here we can not extend the symplectic form to the filling.
Thus, we have proved the following.

**Lemma 5.1.** \( \text{ROB}(\Sigma, \partial_p) \) is symplectically fillable for \( p > q > 0 \) and for \( p > 0 > q \).

The above construction also takes care of the main step in the proof of Theorem 1.3.

**Proof of Theorem 1.3.** Say, \( h = T_{c_1}^{k_1} \circ T_{c_2}^{k_2} \circ \cdots \circ T_{c_l}^{k_l} \). Let \( m = \sum_{i=1}^{l} c_i \). Identify the \( \mathbb{S}^1 \)-interval of the mapping torus of \( \text{ROB}(\Sigma, \partial_p) \) with \([0, 1] / 0 \sim 1\). Divide \([0, 1]\) into sub-intervals \( I_j = \left[ \frac{j}{m}, \frac{j+1}{m} \right] \) for \( j \in \{0, 1, \ldots, m - 1\} \). By Lemma 5.1, we can construct a symplectic filling of \( \text{ROB}(\Sigma, \partial_p) \). Let \( W_0^4 \) denote the filling. We attach \( m \) Weinstein 2-handles to \( W_0^4 \) along the appropriate \( c_i \)'s, one each in the interval \( I_j \) of the mapping torus of \( \text{ROB}(\Sigma, \partial_p) \), to obtain a symplectic filling \( W^4 \) of \( \text{ROB}(\Sigma, h \circ \partial_p) \).

**Proof of Theorem 1.4.** We will assume that the boundary \( \partial \Sigma \) is connected for simplicity of the argument. For multiple boundary components, our argument works near each boundary component. The boundary rotation, by hypothesis, is given by \( \partial \Sigma \).

We consider the honest open book \( (\Sigma, h) \). By Giroux’s result, for \( h \in \text{Dehn}^+ (\Sigma, \partial \Sigma) \) the supported contact structure \( \xi \) admits a Stein-filling say \( X \). We may chose a positive number \( R \) such that \(-\frac{q}{p} < -\frac{1}{R}\).

We briefly recall the construction of a contact form on the open book due to Thurston-Winkelnkemper. Let \((r, \theta)\) denote coordinates in a neighborhood of the binding \( \partial \Sigma \). Let \( \varphi \) denote the \( \mathbb{S}^1 \)-direction in the mapping torus. Thurston-Winkelnkemper construct a contact form having the expression \( \lambda + K d\varphi \) where \( K \) is a large constant and \( \lambda \) is 1-form appropriately chosen so that \( d\lambda > 0 \) on each \( \Sigma \times \varphi \). Further, \( \lambda = (1 + r)d\theta \) in a collar neighborhood of the boundary.
We then glue in a solid torus $S^1 \times D^2_1$ to obtain the manifold. The contact form $\lambda$ is extended on the glued solid torus as follows. We consider a 1-form of the type $\alpha = f(r)d\theta + g(r)d\phi$. We demand that $\alpha = d\theta + r^2d\phi$ near $r = 0$ and $\alpha = (1 + r)d\theta + Kd\phi$ near $r = 1$. The contact condition then translates to finding a parametrized curve $(f(r), g(r))$ in the plane such that the vectors $(f(r), g(r))$ and $(f'(r), g'(r))$ do not point in the same direction.

We refine the above construction by taking disk of radius $R + 1$ denote by $D_{R+1}$. While extending the contact form we demand that $\alpha = d\theta + r^2d\phi$ for $0 \leq r \leq R$ and $\alpha = (1 + r)d\theta + Kd\phi$ near $r = R + 1$. To find a pair $(f(r), g(r))$ satisfying contact condition we may chose to modify the constant $K$ if necessary.

Therefore, we can assume that the binding has a neighborhood that is contactomorphic to the standard neighborhood $N_R$ in $\mathbb{R}^3/(z \equiv z + 1)$ with the contact form $dz + r^2d\varphi$.

By Lemma 4.2, the contact structure $\xi$ supported by $ROB\left(\Sigma, h \circ \partial_p \partial_q\right)$ is obtained by performing a $-\frac{p}{q}$ admissible transverse surgery on the binding of $OB(\Sigma, h)$. Since $-\frac{p}{q} < -1$, by Theorem 4.5 we can realize this admissible transverse surgery by a sequence of Legendrian surgeries along some Legendrian link in the neighborhood of the binding. Thus, we can add Stein 2-handles to $X$ corresponding to each Legendrian surgery in the above sequence to get a Stein filling of $\xi$. 

\[\square\]

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