Painlevé Field Theory

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Abstract

We propose multidimensional versions of the Painlevé VI equation and its degenerations. These field theories are related to the isomonodromy problems on flat holomorphic infinite rank bundles over elliptic curves and take the form of non-autonomous Hamiltonian equations. The modular parameter of curves plays the role of “time”. Reduction of the field equations to the zero modes leads to SL(N, C) monodromy preserving equations. In particular, the latter coincide with the Painlevé VI equation for N=2. We consider two types of the bundles. In the first one the group of automorphisms is the centrally and cocentrally extended loop group L(SL(N, C)) or some multiloop group. In the case of the Painlevé VI field theory in D=1+1 four constants of the Painlevé VI equation become dynamical fields. The bundles of the second type are defined by the group of automorphisms of the noncommutative torus. They lead to the equations in dimension 2+1. In both cases we consider trigonometric, rational and scaling limits of the theories. Generically (except some degenerate cases) the derived equations are nonlocal. We consider Whitham quasiclassical limit to integrable systems. In this way we derive two and three dimensional integrable nonlocal versions of the integrable Euler-Arnold tops.

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1 Introduction
1.1 Painlevé VI and non-autonomous elliptic tops

The six Painlevé equations were discovered in the 1900-1910 period [66, 26, 27] as the second order differential equations that have only poles in the complex plane as movable singularities [34, 37, 38, 9]. The Painlevé equations have a lot of applications in the contemporary mathematical and theoretical physics [1, 19, 2]. The most general equation – the Painlevé VI (PVI) has the following form:

$$\frac{d^2X}{dt^2} = \frac{1}{2} \left( \frac{1}{X} + \frac{1}{X-1} + \frac{1}{X-t} \right) \left( \frac{dX}{dt} \right)^2 - \left( \frac{1}{t} + \frac{1}{t-1} + \frac{1}{X-t} \right) \frac{dX}{dt} + \frac{X(X-1)(X-t)}{t^2(t-1)^2} \left( \alpha \frac{t}{X^2} + \beta \frac{t-1}{(X-1)^2} + \delta \frac{t(t-1)}{(X-t)^2} \right), \quad (1.1)$$

where $(\alpha, \beta, \gamma, \delta)$ are arbitrary complex constants. One of the main goal of this paper is to construct analog of this equation in dimensions 1+1. It takes the form (1.32). We consider also some multi-component generalizations of (1.1) and construct their field theory analogs in higher dimensions. These
generalizations share two main properties of PVI: they are monodromy preserving equations for some linear problems and they have Hamiltonian form.

Before going to the general case we analyze structures behind PVI. The PVI can be represented in the elliptic form [67, 60]

\[
\frac{d^2 u}{dt^2} = \sum_{a=0}^{3} \nu_a^2 \psi'(u + \omega_a) \tag{1.2}
\]

via the change of variables \( X = \frac{\psi(u) - e_1}{e_2 - e_1}, t = \frac{e_1 - e_4}{e_2 - e_1} \) with \( e_k = \varphi(\omega_k) \) and identification of constants \( (\nu_0^2, \nu_1^2, \nu_2^2, \nu_3^2) = \frac{1}{(2\pi i)^2}(\alpha, -\beta, \gamma, \frac{1}{2} - \delta). \) Here

\[
(\omega_0, \omega_1, \omega_2, \omega_3) = \left(0, \frac{1}{2}, \frac{\tau + 1}{2}, \frac{\tau}{2}\right) \tag{1.3}
\]

are the half-periods of the elliptic curve \( \Sigma_\tau = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z}), \) \( \text{Im} \tau > 0, \) \( \varphi(x) \) is the Weierstrass \( \varphi \)-function (see Appendix A). The equation (1.2) is described as the Hamiltonian system with the Hamiltonian function

\[
H = \frac{1}{2} \rho^2 - \sum_{a=0}^{3} \nu_a^2 \varphi(u + \omega_a) \tag{1.4}
\]

and canonical Poisson bracket \( \{p, u\} = 1. \) The system is non-autonomous since the potential explicitly depends on the moduli \( \tau \) (of \( \Sigma_\tau \)) which is the "time" variable. It is non-autonomous version of the Calogero-Inozemtsev system [35]. In the case when \( \nu_a = \frac{1}{2} \nu \) for all \( a \) we come to the elliptic two-particle non-autonomous Calogero-Moser model

\[
\frac{d^2 u}{d\tau^2} = \nu^2 \varphi'(2u). \tag{1.5}
\]

In this paper we deal with another (also elliptic) form of the PVI. It is the non-autonomous version of the Zhukovsky-Volterra gyrostat [90] (NAZVG). Namely, it was shown in [52] that (1.2) can be written as dynamics of three-dimensional complex-valued vector \( \vec{S} = (S_1, S_2, S_3): \)

\[
\partial_t \vec{S} = \vec{J}(\vec{S}) + \vec{S} \times \vec{\nu}', \tag{1.6}
\]

where \( \vec{J}(\vec{S}) = (J_1S_1, J_2S_2, J_3S_3), \) \( J_k = \varphi(\omega_k) \) and \( \vec{\nu}' = (\nu'_1, \nu'_2, \nu'_3) \) - vector of linear combinations of constants \( \nu_a \) from (1.2) multiplied by some ratios of theta-constants (see [52]). The fourth independent linear combination of the constants \( \nu'_0 = \frac{1}{2} \sum_0^{3} \nu_c \) appears to be the length of \( \vec{S}: \) \( \nu'^2_0 = \sum_{a=1}^{3} S'^2_\alpha. \) Equation (1.6) can be rewritten in terms of \( \text{sl}(2, \mathbb{C}) \)-valued \( \mathbf{S} = \sum_{\gamma=1}^{3} \sigma_\gamma S_\gamma, \) where \( \sigma_\gamma \) are the Pauli matrices, as

\[
\partial_t \mathbf{S} = [\mathbf{S}, \mathbf{J}(\mathbf{S})] + [\mathbf{S}, \nu'], \tag{1.7}
\]

where \( \nu' = \sum_{\gamma=1}^{3} \sigma_\gamma \nu'_\gamma. \) It is generated by the quadratic Hamiltonian

\[
H = \frac{1}{2} \text{tr} (\mathbf{S} \mathbf{J}(\mathbf{S})) + \text{tr}(\mathbf{S} \nu'). \tag{1.8}
\]

and the linear Poisson-Lie brackets on \( \text{sl}^*(2, \mathbb{C}): \) \( \{S_\alpha, S_\beta\} = \epsilon_{\alpha\beta\gamma} S_\gamma \) Explicit change of variables \( S_\alpha(v, u) \) can be found in [52]. The equation (1.7) reduces to the non-autonomous elliptic \( \text{sl}(2, \mathbb{C}) \)-valued system when \( \nu'_{1,2,3} = 0. \)

In general case we deal with the non-autonomous Euler-Arnold tops on corresponding groups. They have the following description [7]. Let \( \mathfrak{g} \) be the corresponding Lie algebra, \( \mathfrak{g}^* \) its Lie coalgebra, and \( \mathbf{J} \) is a map \( \mathfrak{g}^* \to \mathfrak{g}. \) The conjugate operator \( \mathbf{J}^* : \mathfrak{g} \to \mathfrak{g}^* \) is called the inertia tensor of the top. The elements

\[\text{It is also shown in [52] that (1.8) can be described by the quadratic Poisson brackets generalizing Sklyanin’s ones [77].}\]
of $S \in g^*$ are called the angular momenta, while the elements of $F = J(S) \in g$ are the angular velocities. The equations assume the form
\[
\partial_t S = \text{ad}^*_g J(S) = [S, J(S)].
\] (1.9)
The phase space of the non-autonomous tops are the coadjoint orbits
\[
\mathcal{O} = \{S \in g^* | S = \text{Ad}^*_g(S_0), \; g \in G, \; S_0 \text{ is a fixed element of } g^* \}.
\] (1.10)
The form (1.9) is convenient for different generalizations in the sense that $S$ can be generalized to some field $S(x)$ while $J$ becomes a differential (or pseudo-differential) operator in $x$. Let us recall that some important examples of integrable equations in dimensions 1+1 and 2+1 were interpreted as integrable field $S$ equations $\partial_x^2 S = 0$. We obtain
\[
J^* \partial_x F = (\text{ad}^*_S J^*(F)) - (\partial_x J^*)(F) \tag{1.11}
\]
The multi-component generalization of PVI (related to the group $SL(N,\mathbb{C})$) is defined by the following form of the operator $J$ in the basis $\{\alpha\}
\[
S = \sum_{\alpha \in \mathbb{Z}^N_2} S_\alpha T_\alpha, \quad J : S_\alpha \to J_\alpha S_\alpha, \quad J_\alpha = \psi_{1\alpha} (\alpha) := \psi(\frac{\alpha_1 + \alpha_2}{\sqrt{N}} | \tau).
\] (1.12)
In the case when the orbit $\mathcal{O}$ (1.10) has the minimal dimension $\dim(\mathcal{O}) = 2N - 2$
\[
S_0 = \nu \text{diag}(N - 1, \ldots, -1)
\] (1.13)
the corresponding non-autonomous top is equivalent to the non-autonomous version of the elliptic $SL(N,\mathbb{C})$ Calogero-Moser model \[52\] (for $sl_2$ case see (1.5))
\[
\frac{d^2 u_i}{d\tau^2} = \nu^2 \sum_{k \neq i} \psi'(u_i - u_k), \quad i = 1, \ldots, N.
\] (1.14)
similarly to the equivalence between (1.2) and (1.6). Here $\nu$ is the single coupling constant. Its square is proportional to the value of the quadratic Casimir function of $S$. A general orbit (1.10) corresponds to the "spin" generalizations of Calogero model \[39\] \[50\].
Another type of the "multi-component" generalization (which is better to refer as "multi-color") comes from the Schlesinger systems \[63\]. In this case the phase space consists of $n$ orbits (1.10) $S_a$, $a = 1, \ldots, n$. The equations of motion can be written in the following general form:
\[
\partial_t S_a = [S_a, J_a^I(S_a)] + \sum_{c \neq a} [S_a, J_a^{II}(S_c)].
\] (1.15)
Here we have two types of the (inverse) inertia tensors $J_a^I$ and $J_a^{II}$. The model is described by quadratic Hamiltonian
\[
H_\tau = \sum_{a=1}^n \frac{1}{2} \text{tr}(S_a J_a^I(S_a)) + \sum_{b \neq c} \text{tr}(S_b J_b^{II}(S_c))
\] (1.16)
and direct sum of the linear Poisson-Lie brackets generated by the structure constants \[13\] for each $S_a$.$^4$ Originally, the Schlesinger model is given by the Hamiltonians corresponding to $n$ marked points while $H_\tau$ (1.10) appears in the elliptic case and corresponds to the moduli $\tau$ of $\Sigma_\tau$. In the special $sl_2$-case
$^4$For a group $G = SDiff(M)$ of the volume preserving diffeomorphisms of a Riemann manifold $M$ the equation takes the form of the Euler-Bernoulli equation (see \[7\]). In this case $J^*$ is the Laplace-Beltrami operator and the last term in (1.11) is absent.
$^5$Similarly to the case of PVI in the form of NAVZG there exists the quadratic Poisson structure describing the same equations \[22\] \[14\].
when the marked points on \( \Sigma \) are the half-periods \( \omega_\alpha \) \((1.3)\) there exists the (reflection) symmetry which generates constraints

\[
S_a^\beta = \exp(4\pi i(\omega_\alpha \partial_\tau \omega_\beta - \omega_\beta \partial_\tau \omega_\alpha))S_a^\beta, \quad a = 0, \ldots, 3, \quad S_a = \sum_{\gamma=1}^{3} S_\gamma^a \sigma_\gamma. \tag{1.17}
\]

The constraints save \( S_0 \) and reduce all other orbits to non-dynamical constant matrices

\[
S_a^\beta = \delta_{a\beta} \bar{\nu}_\beta \sigma_\beta, \quad a = 1, 2, 3, \tag{1.18}
\]

where \( \bar{\nu}_\beta \) differ from \( \nu_\beta \) by the ratios of theta-constants factors \([52]\). In this way we reproduce the PVI in the NAVZG form \((1.7)\). Explicit equations for the elliptic Schlesinger system can be found in \([12]\).

### 1.2 Linear problems and monodromy preserving equations

The described non-autonomous models share another common property. The non-autonomous elliptic equations described above can be considered as a monodromy preserving conditions for some linear problems. This approach was suggested in \([27, 28, 73]\) and then developed in \([25, 40]\) (see also \([38]\)). Different linear problems are known for Painlevé equations. The scalar examples were found in \([26, 28]\). The matrix-valued linear problems were also obtained in many different variations. See, for instance, \([40]\) for \( \text{sl}_2 \)-valued examples and \([41]\) for \( \text{sl}_3 \)-valued ones. Our approach is based on consideration of the monodromy preserving equations as non-autonomous version \([49]\) of the Hitchin systems \([32]\). From the computational viewpoint it is based on Krichever’s anzats for the Lax pairs of elliptic integrable systems \([46]\) and the classical Painlevé-Calogero correspondence \([50]\) (see also \([87, 71]\)). The correspondence claims, in particular, that the Lax pair of the elliptic Calogero model also satisfies the monodromy preserving condition.

The linear problem for the PVI equation in the form of NAZVG has the following description. Let \( L = L(S, \nu, w, \tau) \) and \( M = M(S, \nu, w, \tau) \) be \( 2 \times 2 \) matrices depending on the spectral parameter \( w \in \mathbb{C} \). The linear system assumes the form

\[
\begin{align*}
\partial_w + L \Psi &= 0, \\
\partial_\tau \Psi &= 0, \\
\partial_\tau + M \Psi &= 0,
\end{align*}
\tag{1.19}
\]

where \( \Psi = \Psi(S, \nu, w, \tau) \) - \( 2 \times 2 \) matrix-valued function of solutions. The first and the second equations in \((1.19)\) mean that \( \partial_w + L \) is the component of the flat connection of the \( \text{SL}(2, \mathbb{C}) \)-bundle over a complex curve with local coordinates \((w, \bar{w})\). The third equation means that the monodromies of \( \Psi(w, \tau) \) are independent of \( \tau \). The consistency condition of the first and the last equations is the monodromy preserving equation (or zero-curvature equation \([3]\) for NAZVG

\[
\partial_\tau L - \partial_w M = [L, M]. \tag{1.20}
\]

In \([52]\) the explicit expressions of \( L \) and \( M \) were obtained:

\[
L^{\text{NAZVG}} = \sum_{\alpha=1}^{3} \left( S_0^\alpha \varphi_\alpha(w) + \bar{\nu}_\alpha \varphi_\alpha(w) - \omega_\alpha \right) \sigma_\alpha = \sum_{\alpha=1}^{3} \left( S_0^\alpha \varphi_\alpha(w) + \nu_\alpha \frac{1}{\varphi_\alpha(w)} \right) \sigma_\alpha, \tag{1.21}
\]

\[
M^{\text{NAZVG}} = \sum_{\alpha=1}^{3} S_3^\alpha \varphi_\alpha(w) \varphi_\alpha(w) \varphi_\alpha(w) \sigma_\alpha + E_1(w)L^{\text{NAZVG}}(w),
\]

where \( \varphi_\alpha(w) \) and \( E_1(w) \) are defined in the Appendix A. In this paper we obtain the field-theoretical generalizations of the linear problems \((1.19)\) with \((1.21)\). Let us also mention that the L-M pair \((1.21)\) was obtained from the another one describing PVI \((1.2)\) suggested in \([93]\) by a singular gauge transformation. The gauge transformation is the Hecke operator (or modification of bundle \([6, 51]\) ). Another L-M pair can be derived from the rational Schlesinger system by the change of variables in the linear problem \([53]\).

\( ^{6} \text{Sometimes this equation is also referred to as the Lax equation since it appears from \((1.23)\) by } L \to \partial_w + L. \)
In the general case $L$ and $M$ are elements of a Lie algebra acting on a finite-dimensional module. Their matrix elements depend on the local coordinates on the phase space $\mathcal{M}$

$$\xi = (\xi_1, \ldots, \xi_{\dim \mathcal{M}})$$

and the spectral parameter $w \in \Sigma$, where $\Sigma$ is $\mathbb{C}P^1$, or an elliptic curve, or more generally, any complex curve. Examples of systems of this type are the Schlesinger system \([73]\), the Painlevé equations and their generalizations.

There exists a special limit of (1.20) to the classical integrable systems. Naively, we replace the derivative $\partial_w$ with $\kappa \partial_w$, where $\kappa$ is a small parameter. In the limit $\kappa \to 0$ we come from (1.20) to the isospectral problem:

$$\partial_t L = [L, M].$$

(1.23)

It is the Lax equation of a finite-dimensional classical autonomous completely integrable system. In particular, from PVI we come in this way to the BC$_1$ Calogero-Inozemtsev model \([35]\) and from the Schlesinger system to the Gaudin model \([29]\).

In the autonomous case the wide class of elliptic integrable systems (Gaudin models) was obtained in \([68, 64]\). More general models classified by characteristic classes were described on \([54, 55]\) (see also \([56]\)). Their isomonodromy version was described recently as well \([57]\).

Different types of degenerations of the elliptic autonomous and non-autonomous models to the trigonometric and rational case including scaling limits of the Inozemtsev type \([36]\) can be found in \([14, 3, 4, 78]\). We consider particular cases of these reductions in the field theories.

### 1.3 1+1 field theories

Let us start with 1+1 generalizations of the autonomous mechanical finite-dimensional integrable systems. The later systems are described by the Lax equations (1.23). First of all, any 1+1 generalization implies that the variables (1.22) of the phase space $\mathcal{M}$ become fields on a real line or a circle $S^1$:

$$\xi \to \xi(x) = (\xi_1(x), \ldots, \xi_{\dim \mathcal{M}}(x)), \ x \in S^1.$$  

(1.24)

Instead of the Lax equation (1.23) the equations of motion in 1+1 case acquire the form of the Zakharov-Shabat equation \([89]\)

$$\partial_t L - \partial_x M = [L, M].$$

(1.25)

In general the matrices $L$ and $M$ in this equation cannot be obtained from the mechanical one by a direct substitution (1.24). However, in some cases this approach does work. For example, it happens for the sl$_2$ elliptic top and its 1+1 field version - Landau-Lifshitz model \([10, 76]\): $L_{LL}(z) = \sum_{\alpha=1}^{3} S^\alpha(x) \varphi_\alpha(z) \sigma_\alpha$. The equations of motion

$$S_t = [S, J(S)] + [S, S_{xx}]$$

(1.26)

show that in this case the field generalization is achieved by replacing the conjugate inertia tensor $J$ in (1.9) with the (local) differential operator

$$J \xrightarrow{1.26} \partial_x^2 + J.$$  

(1.27)

Another two-dimensional version of the integrable Euler-Arnold tops \([7]\) are the $N$-wave equations \([88]\). The integrable field versions of the interacting particles models is the Toda field theory \([58, 91, 62]\) and the Calogero-Moser field theory \([44, 51]\). It turns out that the two-particle case of the Calogero-Moser field theory is equivalent to the Landau-Lifshitz equation similarly to relation between PVI and NAVZG. They are examples of the soliton equations which can be solved by the famous Inverse Scattering Method \([59, 21, 17, 80]\).
More generally, the field-theoretical generalization of the Hitchin systems \[32\] was suggested in \[51\] (see also \[91, 92, 79\]). Namely, it was shown that the 1+1 version of integrable systems appears by changing the structure group \(G\) of the underlying Higgs bundle by the centrally extended loop group \(\hat{L}(G)\).

To construct multi-dimensional version of the monodromy preserving equations we apply proposed in \[49\] approach to standard isomonodromy problems. It is similar to the Hitchin derivation of classical integrable systems \[32\], but the Higgs \(G\)-bundles over Riemann surfaces are replaced by flat bundles. It should be noted that the quasi-classical (Whitham) limit leads to the Hitchin integrable systems related to the Higgs \(G\)-bundles. To pass to field theory analogs we replace a simple finite-dimensional Lie groups \(G\) by infinite groups, and in this way obtain field-theoretical generalizations of the isomonodromy problems and the Hitchin systems \[8\]. The latter field theories are different from described above \[51\].

In this paper we consider two types of infinite-dimensional groups. One of them is related to the cocentrally extended loop group \(\hat{\mathcal{L}}(G)\) of the centrally extended loop group \(\hat{L}(G)\) and described in detail in Section 3. Another field theories are related to the group of noncommutative torus (NCT) (Section 5), and its dispersionless limit to the area preserving diffeomorphisms of the two-dimensional torus \(SDiff(T^2)\). The local integrable hierarchies related to this group were considered in \[83\]. We come to this point below.

Before going to the isomonodromic field theory let us comment on the difference between the construction used in this paper (let us call it the "\(k\)-case") and the one described in \[51\] (the "\(\tilde{k}\)-case"). The latter leads to integrable hierarchies related to the Inverse Scattering Problem and, in particular, to (1.26), while the \(\tilde{k}\)-case provides different type of equations (see below). Comments on NCT-based generalization is given below for the isomonodromic version.

It will be explained in Section 3, the field generalizations, arising (via reduction) from the double extended (centrally and cocentrally) loop groups in \(k\)-case, have the following property: the L-operator depends on two scalar parameters \(k\) (dual to the central extension) and \(\tilde{k}\):

\[
L = L(k, \tilde{k})
\]

In fact, the variable \(k\) appears as the coefficient in front of \(\partial_x\) and exists in the described above \(k\)-case as well. In the same time, the dependence on the another variable \(\tilde{k}\) is non-trivial (\(\tilde{k}\) is absent in \(k\)-case approach). Moreover, these variables satisfy equations of motion

\[
\partial_t \tilde{k} = k, \quad \partial_t k = 0.
\]

Therefore, the Lax operator depends non-trivially on linear function of time and the field theory is non-autonomous even in the isospectral deformations. Summarizing, the filed theories of \(\tilde{k}\)-case are originally non-autonomous. As we will see below the isomonodromic version adds another type of the non-autonomous dependence. Similarly to mechanics, the dependence on \(\tau\) appears from the corresponding definitions of the elliptic functions as the module of the underlying elliptic curve (say, \(\vartheta(u, \tau)\) for the theta-function), while the dependence on \(\tilde{k}\) enters in another way – as \(\vartheta(u + k, \tau)\).

There is one more special property of the \(\tilde{k}\) case. In contrast to the \(k\)-case (LL-like soliton equations (1.26)) the term \(\partial_x M\) does not provide any input into equations of motion from (1.25). In this case

\[
\partial_t L = \partial_t L|_{\tilde{k}=\text{const}} + \partial_{\tilde{k}}L|_{\tilde{k}=\text{const}} + k\partial_{\tilde{k}}L = \partial_t L|_{\tilde{k}=\text{const}} + \frac{1}{2}k\partial_{\tilde{k}}M.
\]

As we will see the last term \(k\partial_{\tilde{k}}M\) cancels exactly \(k\partial_x M\)-term. It happens because the Hamiltonian density can be computed in this case likewise in finite-dimensional mechanics:

\[
H = \frac{1}{2} \oint tr L^2.
\]

(see more precise expression (1.28)). This Hamiltonian is not conserved on its own dynamics due to explicit dependence on the time-variable. This situation is different from \(k\)-case, where there is a whole

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8 A single example related to the local 1+1 integrable hierarchy – the isomonodromy version of the Heisenberg model, which is not based on this approach is given in Section 3.7.
hierarchy of conservation laws generated by monodromy operator $\text{Pexp}\begin{array}{c} f \end{array} L(x)$. The $\bar{k}$-case is more like the case of monodromy preserving equations \|[1.20]\|, \text{ where } \partial_w M \text{ is canceled by } \partial_t L\big|_{t=\tau}$. The reason of the difference the $k$-case from the $\bar{k}$ case can be explained in the following way. In derivation of the both types systems we use the symplectic reduction procedure. While in derivation of the $k$-systems we demand that symplectic structure $\omega$ and Hamiltonians $H_s$ are invariant, in the $\bar{k}$ case we assume that invariant only their special combination - the Poincaré-Cartan form \|[7]\|.

$$\omega^{PC} = \omega - \sum \delta H_s \wedge \delta t_s,$$

where $t_s$ are the times related to the Hamiltonians. It turns out that derived in this way $\bar{k}$ systems are nonlocal. The obtained field theories can be considered as $\mathfrak{gl}_\infty$ Gaudin systems. As it is known from \|[63]\| the $\mathfrak{gl}_N$ Gaudin models are spectrally dual to magnetics of Landau-Lifshitz type on $N$ cites in finite-dimensional case. Therefore, we can await that the spectral duality may be hold on the level of $\mathfrak{gl}_\infty$.

1.4 Painlevé-Schlesinger field-theoretical generalizations

Our goal is to find a field-theoretical generalization of the monodromy preserving equations. It means that the mixture of \|[1.25]\| and \|[1.20]\| should appear as the zero-curvature equation \|[1.31]\|, i.e.

$$\partial_t L - \partial_x M - \partial_w M = [L, M] \quad (1.31)$$

In other words we replace $L \rightarrow \partial_x + L$, where $L$ and $M$ are still finite-dimensional matrices. Due to the term $\partial_w M$ in \||(1.20)\| this generalization is highly non-trivial.

More generally, we allow $L$ in \||(1.20)\| to be an integro-differential operator acting in some functional space. In this way the field-theoretical generalizations of the monodromy preserving equations we consider here are still the monodromy preserving conditions for the connection $\partial_w + L$ acting on sections of a bundle of infinite rank. To write down the systems we assume that:

- some reductions of \||(1.20)\|, (for example, taking $x$-independent solutions) leads to PVI \||(1.1)\| or to its finite-dimensional generalizations;
- \||(1.31)\| are equations of motion for some non-autonomous Hamiltonian system.

Let us start from concrete example. Consider the space of twelve fields parameterized by the three-vectors

$$S_\alpha^a(x), \quad \alpha \in (\mathbb{Z}_2 \times \mathbb{Z}_2) \setminus (0,0), \quad b \in \mathbb{Z}_2 \times \mathbb{Z}_2.$$ and let $J^I(\beta, \partial_x, \tau), J^{II}(\beta, b, c, \partial_x, \tau)$ be the pseudo-differential operators defined in \||(4.104)\|. Then $\text{PVI}^{FT}$ has the form of four interacting non-autonomous Euler-Arnold tops \|[7]\| \|[8]\|, related to the loop group $L(\text{SL}(2, \mathbb{C}))$

$$\frac{\partial}{\partial \tau} S_\alpha^a(x) = \sum_{\beta \neq \alpha} \left( S_{\beta}^{\alpha - \beta} (x) J^I(\alpha, \beta, \partial_x, \tau) S_\beta^\beta (x) + \sum_{c \neq b} S_{\beta}^{\alpha - \beta} (x) J^{II}(\alpha, \beta, b, c, \partial_x, \tau) S_{\beta}^c (x) \right) \quad (1.32)$$

and subjected to the constraints

$$S_\alpha^a (x) = (-1)^{b \times a} S_\alpha^a (-x). \quad (1.33)$$

The equation \||(1.32)\| itself can be considered as a field generalization of the elliptic $\mathfrak{sl}_2$ Schlesinger system \|[12]\| with four marked points. There are analog of constraints \||(1.33)\| in the finite-dimensional case that leads to the vanishing of Hamiltonians related to the marked points located at the half-periods of elliptic \footnote{There is also a different problem, where the both equations \||(1.25)\| and \||(1.20)\| are used \|[39]\|.}
curves. The remaining nontrivial Hamiltonian defines an evolution with respect to the modular parameter \( \tau \) of the elliptic curves. This evolution is the equation for the non-autonomous Zhukovsky-Volterra gyrostat (NAZVG) \((1.33)\). We show that the zero modes of the field model \((1.32)\) along with the constraints \((1.33)\) satisfies the equation for the non-autonomous Zhukovsky-Volterra gyrostat (NAZVG) \((1.3)\) for three variables \( S^\alpha \) – zero modes of \( S^0 \).

In terms of the Lax operators corresponding to \((1.32)\) we replace \( L \) and \( M \) from \((1.19)\) with operators, which act on the space of functions on additional variable \( x \in S^1 \). In this way we come from the ODE PVI to pseudo-differential equations for functions on the spaces of dimension 1+1. Namely, we define two operators \( L(S^b_0(x), x, w, \tau) \) and \( M(S^b_0(x), x, w, \tau) \) to be two by two matrices which matrix elements are pseudodifferential operators in \( x \), such that the linear system

\[
\begin{aligned}
(\partial_w + \partial_x + L(S^b_0(x), x, w, \tau))\Psi(x, w, \tau) &= 0, \\
\partial_w \Psi(x, w, \tau) &= 0, \\
(\partial_x + M(S^b_0(x), x, w, \tau))\Psi(x, w, \tau) &= 0
\end{aligned}
\]  

(1.34)

is consistent. Then \((1.34)\) is equivalent to \((1.32)\). The general isomonodromy problems we are going to consider have the form \((1.34)\), where \( L, M \) are operators related to infinite-dimensional Lie algebras. We will exploit also another important property of PVI. It is a non-autonomous Hamiltonian system, where the Hamiltonian is defined by \( L \). In this way all equations we consider here are flows of infinite-dimensional Hamiltonian systems.

In general case the monodromy preserving equations we derived here are nonlocal and described by pseudodifferential operators. But some their degenerations lead to PDE. For example, for the Lie algebra \( \text{SL}(2, \mathbb{C}) \) the dispersionless limit of the nonlocal equations leads to the equation in the 2+1 space \((1.35)\).

\[
\partial_x \partial^2 Z F(x, \tau) - \{\partial^2 Z F(x, \tau), F(x, \tau)\} + \epsilon_2 \partial_{x_2} \bar{\partial}_Z F(x, \tau) = 0,
\]

(1.35)

We use the approach which was earlier applied for derivation of the classical integrable systems \((1.23)\). It was based on the symplectic reduction of the Higgs bundles over \( \Sigma \) \((32)\). To come to \((1.20)\) one should replace the Higgs bundles with the space of smooth connections \((49)\). In this way starting from \( \text{SL}(N, \mathbb{C}) \)-bundles over elliptic curves we derived earlier the Painlevé VI equation, its multi-component generalizations and the elliptic Schlesinger system \((12, 39, 52)\).

The field theory \((1.25)\) can be derived similarly to \((1.24)\), but in this case the Higgs bundle has an infinite rank, and the group of automorphisms of the bundle is an infinite-dimensional group. Following this idea we construct monodromy preserving equations in higher dimension using the symplectic reduction procedure applied to the space of smooth connections on the infinite rank bundles.

In this paper we use a few types of infinite-dimensional groups. First, we consider the bundles with the loop group \( L(\text{SL}(N, \mathbb{C})) \) as the group of its automorphisms. In this case we come to 1+1 theory. In particular, the equations \((1.32)\) are related the \( \text{SL}(2, \mathbb{C}) \) theory. In general case they have the form of the so-called non-autonomous elliptic hydrodynamics \((43)\). To derive the trigonometric and the rational systems we should extend the loop-group \( L(\text{SL}(N, \mathbb{C})) \) to the two-loop and three-loop groups. In this way we come to models in two-dimensional and three-dimensional spaces.

For the next type of bundles the group of its automorphisms is a specially defined group \( \text{GL}(\infty) \), or more exactly, the group \( \text{SIN}_\theta \) of the noncommutative torus (NCT) \( T_\theta \), where \( \theta \) is the parameter of non-commutativity. In this case the field theories are defined on the two-dimensional noncommutative space \( T_\theta \). The group \( \text{SIN}_\theta \) for \( \theta = \frac{1}{b} \) is isomorphic to the two-loop group \( LL(\text{GL}(2, \mathbb{C})) \) while \( \text{GL}(2, \mathbb{C}) \) is a subgroup of \( LL(\text{GL}(2, \mathbb{C})) \). In this sense we obtain the field-theoretical generalization of the particular case of PVI \((1.1)\). We consider also the classical limit \((\theta \to 0)\) of \( \text{SIN}_\theta \) to the group of the volume
preserving diffeomorphisms $SDiff(T^2)$ of the two-dimensional torus $T^2$ (see Appendix B). The equation (1.35) is one of the equations related to this group.

We also consider the Whitham quasiclassical limit which reduces the monodromy preserving equations to integrable systems. In this way we come to integrable nonlocal field theories. Notice that this construction leads to a new class of integrable field models. For example, starting with $SL(2,\mathbb{C})$ Euler-Arnold top we come to the nonlocal equation (4.50), while the old construction leads to the Landau-Lifshitz equation [51]. On the other hand, many interesting integrable equations, related to noncommutative space were analyzed (see, for example [16, 47, 48, 31]). It would be also interesting to compare results with the classification suggested in [24]. In particular, for the case of NCT the integrable Toda field theories [33, 72] and the so-called elliptic hydrodynamic [43, 65] were obtained.

As it was already mentioned generally we get some nonlinear integro-differential equations. In some approximation they can be described as a tower of differential equations. For the loop group there exists a parameter $k$. It is the cocentral charge corresponding to the term $k\hat{\partial}_x$. Taking perturbations of the equations for small $k$ we come to a tower of equations related to degrees $k^j$. For $j = 0$ the equation is just the original one-dimensional system parameterized by the space coordinate $x$. On the level $j > 0$ the equations are linear differential equations of order $j$ with respect to $S_j$, where $S = \sum_{j\geq 0} S_j k^j$. Its coefficients have quadratic dependence on $S_{j-1}, \ldots, S_0$.

In all instances the derived equations take the form of non-autonomous Euler-Arnold tops on corresponding groups. In a general case the operator $J$ is a pseudodifferential operator and in this way we come to nonlocal equations. There are two ways to come to local equations. Generally, we consider degenerations of the systems related to degenerations of the underlying elliptic curves. Namely, we consider trigonometric, rational and scaling limits. Some of these equations can be considered as the field-theoretical versions of the PI-PV equations. Notice that the trigonometric degenerations of the Euler-Arnold tops are rather subtle. We use here the approach from [78] that allows us to come to non-trivial equations. Another way, that works in some cases is to rewrite (1.9) in terms of the angular velocities $F$ (see (1.11)). Equations (1.35) are of this type.

The paper is organized as follows. In Section 2 we propose the general construction of the monodromy preserving equations from the flat bundles over Riemann surfaces. The case of elliptic curves is considered in details. In Section 3 we define the bundles over curves with marked points with loop groups as the groups of their automorphisms. Using these bundles we derive related monodromy preserving equations, their degenerations and corresponding integrable field theories. Section 4 is devoted to derivations of the trigonometric and rational equations based on the two and three-loop algebras. In Section 5 we derive and analyze field theories related to the noncommutative torus. We consider also their classical limits wherein the equations are simplified.

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2 Monodromy Preserving Equations as Non-autonomous Version of Hitchin Systems

Here we describe a general approach to the monodromy preserving equations, based on the symplectic reduction of the space of smooth connection on a bundle of any rank (finite or infinite) over a Riemann surface with marked points. The connection takes value in a complex Lie algebra $\mathfrak{g}$ (finite-dimensional or infinite-dimensional). The only restriction is that $\mathfrak{g}$ has an invariant bilinear form that allows us to identify $\mathfrak{g}$ and its dual. After the symplectic reduction procedure one comes to the monodromy preserving equations. The finite-dimensional algebras lead to ODE. This construction was proposed in
It is easily generalized to the infinite-dimensional algebras, such as affine algebras and algebras of noncommutative torus. Our main goal is to consider equations related to these two types of algebras in detail.

### 2.1 Flat bundles

Let $\Sigma_g$ be a smooth oriented compact surface of genus $g$. For a complex Lie group $G$ consider a principal $G$-bundle $P$ and the associated bundle $E_G = P \times_G V$, where $V$ is a representation space of $G$. Let $FConn_{\Sigma_g, G}$ be the space of flat connections on $E$

$$FConn_{\Sigma_g, G} = \{d + A | dA + \frac{1}{2} A \wedge A = 0\}. \quad (2.1)$$

The group of automorphisms $\mathcal{G} = Aut(FConn_{\Sigma_g, G})$ (the gauge group) is the group of smooth maps $\mathcal{G} = Map_{C^\infty} : \Sigma_g \rightarrow G$ acting as $A \rightarrow f^{-1}df + f^{-1}Af$. The moduli space of the flat connections $FBun_{\Sigma_g, G}$ is the quotient $FConn_{\Sigma_g, G}/\mathcal{G}$. This space can be also described as a result of the symplectic reduction of the space of all smooth connections $Conn_{\Sigma_g, G}$ equipped with the symplectic form

$$\omega = \frac{1}{2} \int_{\Sigma_g} (\delta A \wedge \delta A) \quad (2.2)$$

invariant under the action of the gauge group. Here the brackets means the Killing form in the Lie algebra $\mathfrak{g} = Lie(G)$. The moduli space of flat bundles is the coset space

$$FBun_{\Sigma_g, G} := FConn_{\Sigma_g, G}/\mathcal{G} = Conn_{\Sigma_g, G}/\mathcal{G}, \quad (2.3)$$

where the double slash is the symplectic action of $\mathcal{G}$. In this way $Conn_{\Sigma_g, G}$ will be considered as an unrestricted phase space equipped with the symplectic form $\omega$ (2.2). The flatness condition (2.1) plays the role of the moment constraint generating the automorphisms $\mathcal{G}$.

### 2.2 Deformation of complex structures

Introduce a complex structure on $\Sigma_g$. The choice of the complex structure defines the polarization of $Conn_{\Sigma_g, G}$. Then the connection is decomposed in $(1, 0)$ and $(0, 1)$ parts $A = (A, \bar{A})^{10}$. We can write $(1, 0)$ and $(0, 1)$ components of the connection in local coordinates $(z, \bar{z})$

$$(\kappa \partial + A) \otimes dz, \quad (\bar{\partial} + \bar{A}) \otimes d\bar{z}, \quad (\partial = \partial_z, \quad \bar{\partial} = \partial_{\bar{z}})\quad (2.4)$$

The holomorphic sections of bundles are those which are annihilated by $\bar{\partial} + \bar{A}$. The moduli space of holomorphic bundles is the quotient

$$Bun_{\Sigma_g, G} = \{\bar{\partial} + \bar{A}'\}/\mathcal{G}. \quad (2.5)$$

In terms of $(A, \bar{A}')$ (2.2) assumes the form

$$\omega = \int_{\Sigma_g} (\delta A \wedge \delta \bar{A}'). \quad (2.6)$$

Let $E_G$ be a holomorphic bundle corresponding to a point $b \in Bun_{\Sigma_g, G}$ and

$$Flat_b = \{\partial + A | F(A, \bar{A}) = \partial \bar{A} - \bar{\partial}A + [A, \bar{A}] = 0\} \quad (2.6)$$

is the space of flat holomorphic connections on $E_G$. There is a map of the moduli space $FBun_{\Sigma_g, G}$ (2.4) to $Bun_{\Sigma_g, G}$ (2.4). The fiber of this projection at the point $b$ is isomorphic to $Flat_b$. These fibers are Lagrangian with respect to $\omega$ (2.5).

---

10 The notation $\bar{A}'$ will be justify below.

11 We introduce here a new parameter $\kappa$ (\kappa-connection \) in order to pass later from the monodromy preserving equations to integrable systems in the limit $\kappa \rightarrow 0$. 

12
The tangent space to the moduli $\text{Bun}(\Sigma_g, G)$ at some point $b$ is isomorphic to the first cohomology group $H^1(\Sigma_g, \text{End}E_b)$. The dual vector space is $H^1(\Sigma_g, \text{End}E_b \otimes \Omega^1(\Sigma_g))$. The space $\text{Flat}_b$ [2.16] is isomorphic to the principal homogeneous space over $H^0(\Sigma_g, \text{End}E_b \otimes \Omega^1(\Sigma_g))$.

The vector fields generated by $\text{Lie}(G) = \{\epsilon\}$ act on $\text{Conn}(\Sigma_g, G)$. They have the form $\delta_\epsilon A = \partial \epsilon + [A, \epsilon]$, $\delta_\epsilon \tilde{A}' = \partial \epsilon + [\tilde{A}', \epsilon]$, where $\delta_\epsilon = di_\epsilon + i_\epsilon d$ is the Lie derivative. The corresponding moment map $F : \text{Conn}(\Sigma_g, G) \rightarrow \text{Lie}^*(G)$ is defined as $i_\epsilon \omega = \int_{\Sigma_g} (\epsilon, \delta F)$. The Hamiltonian of the gauge transformations $\delta_\epsilon$ assumes the form

$$h^\text{gauge} = \int_{\Sigma_g} (\epsilon, F), \quad F = \bar{\partial} A - \partial \tilde{A}' + [\tilde{A}', A].$$

Consider small deformations the complex structure on $\Sigma_g$. The complex structure on $\Sigma_g$ is defined by the $\bar{\partial}$ operator. Consider the change of variables

$$w = z - \epsilon(z, \bar{z}), \quad \bar{w} = \bar{z} - \bar{\epsilon}(z, \bar{z}),$$

where $\epsilon(z, \bar{z})$ is small. Up to a common multiplier the partial derivatives assumes the form

$$\begin{aligned}
\theta_w &= \partial_z + \bar{\mu} \partial_{\bar{z}}, \\
\bar{\theta}_w &= \partial_{\bar{z}} + \mu \partial_z,
\end{aligned}$$

where

$$\mu = \frac{\bar{\partial} \epsilon}{1 - \epsilon} \sim \bar{\partial} \epsilon$$

is the Beltrami differential $\mu \in \Omega^{(-1,1)}(\Sigma_g)$. We pass from $(w, \bar{w})$ to the chiral coordinates $(w, \bar{\epsilon} = \bar{z})$ because the dependence on $\bar{\mu}$ is nonessential in our construction

$$\begin{aligned}
\theta_w &= \partial_z, \\
\bar{\theta}_w &= \partial_{\bar{z}} + \mu \partial_z,
\end{aligned}$$

Note that $\bar{w}$ does not mean the complex conjugation. The pair $(w, \bar{w})$ is just a pair of local coordinate on $\Sigma_g$. What is important is that $\bar{\theta}_w$ annihilates holomorphic functions $\partial_w f(w) = 0$. We assume that $\mu(w, \bar{w})$ is equivalent to $\mu(z, \bar{z})$, if $w(z, \bar{z})$ is a global diffeomorphism. The equivalence relations in $\Omega^{(-1,1)}(\Sigma_g)$ under the action of $D\text{iff}_C(\Sigma_g)$, is the moduli space $\mathcal{M}(\Sigma_g)$ of complex structures on $\Sigma_g$. The tangent space to the moduli space is the Teichmüller space $\mathcal{T}_g \sim H^1(\Sigma_g, \Gamma)$, where $\Gamma \in T\Sigma_g$. From the Riemann-Roch theorem one has

$$\dim(\mathcal{M}(\Sigma_g)) = 3(g - 1).$$

Let $(\mu_1^0, \ldots, \mu_l^0)$ be a basis in the vector space $H^1(\Sigma_g, \Gamma)$. Then

$$\mu = \sum_{l=1}^{3g-3} \tau_l \mu_l^0,$$

where the local coordinates $\tau_l$ will play the role of times in the isomonodromic deformation problem.

The monodromy preserving equations can be derived from the classical integrable systems (the Hitchin systems) by the so-called Whitham quantization. Here we will consider the inverse procedure - the quasiclassical limit of the monodromy preserving equations to integrable systems. The introduced above constant $\kappa$ plays the role of the Planck constant. Deform $(0,1)$ component of connection $\tilde{A}'$ as

$$\tilde{A}' = \tilde{A} - \frac{1}{\kappa} \mu A, \quad \partial + \mu \partial + \tilde{A} = \partial_{\bar{w}} + \tilde{A}.$$

In other words, $\tilde{A}$ is the $(0,1)$ component of the connection in the deformed structure. The form $\omega$ becomes

$$\omega = \int_{\Sigma_g} (\delta A \wedge \delta \tilde{A}) - \frac{1}{\kappa} \int_{\Sigma_g} (A, \delta A) \delta \mu.$$
The connections \((A, \bar{A})\) play the role of the canonical coordinates in \(\text{Conn}(\Sigma, G)\). Taking into account (2.11) we rewrite \(\omega\) as the differential of the Poincaré-Cartan one-form \(\vartheta\) \((\omega = \delta \vartheta)\) [7]

\[
\omega = \omega_0 - \frac{1}{\kappa} \sum_{l=1}^{3g-3} \delta H_l \delta \eta_l, \quad H_l = \frac{1}{2} \int_{\Sigma_g} (A, A) \mu^0_l, \quad \omega_0 = \int_{\Sigma_g} (\delta A \wedge \delta A). \tag{2.13}
\]

We will discuss this form below. The Poincaré-Cartan form gives rise to the action functional

\[
S = \sum_{l=1}^{3g-3} \int_0^\infty \left( \int_{\Sigma_g} (A, \partial_t A) - \frac{1}{2\kappa} (A, A) \mu^0_l \right) d\eta_l, \quad (\partial_t = \partial_{\eta_l}).
\]

The equations of motion following from the action (or from the Hamiltonians) are

\[
\partial_t A = \frac{1}{\kappa} A \mu^0_l, \quad \partial_t A = 0. \tag{2.14}
\]

These equation are the compatibility conditions for the following linear system

\[
\begin{align*}
1. \quad (\kappa \partial + A) \psi &= 0, \\
2. \quad (\bar{\partial} + \sum_{l=1}^{3g-3} \tau_l \mu^0_l \partial + \bar{A}) \psi &= 0, \\
3. \quad \kappa \partial \psi &= 0,
\end{align*} \tag{2.15}
\]

where \(\psi \in \Omega^{(0)}(\Sigma_g, \text{Aut } E_G)\). The equations of motion (2.14) for \(A\) and \(\bar{A}\) are the consistency conditions of (1.&3.) and (2.&3.) in (2.15). The monodromy of \(\psi\) is the transformation

\[
\psi \rightarrow \psi Y, \quad Y \in \text{Rep}(\pi_1(\Sigma_{g,n}) \rightarrow G).
\]

The equation (3.2.15) means that the monodromy is independent on the times. The consistency condition of 1. and 2. is the flatness constraint (2.1).

\[
\partial_{\bar{a}} A - \kappa \partial \bar{A} + [A, \bar{A}] = 0. \tag{2.16}
\]

The linear equations (2.14), defined on \(\text{Conn}(\Sigma, G)\) become nontrivial on \(FBun_{\Sigma,n} G\). Before consideration of (2.14) we extend the phase space by including the quasi-parabolic structures at the marked points on the surfaces [75].

2.3 Quasi-parabolic structures and deformation of elliptic curve

Gauge groups and flag varieties

Assume that the algebra \(g\) has the decomposition on subalgebras

\[
g = n^- \oplus \mathfrak{h} \oplus n^+, \quad [\mathfrak{h}, n^+] = n^+.
\]

For complex simple \(g\) it is the decomposition on the positive and negative nilpotent subalgebras and the Cartan subalgebra. The subalgebra \(\mathfrak{h} \oplus n^+ = \mathfrak{b}\) is the Borel subalgebra. Let \(B\) be the corresponding Borel subgroup. The quotient \(G/B\) is the variety of the \(FL(G)\).

Denote by \(\Sigma_{g,n}\) the surface \(\Sigma_g\) with \(n\) marked points \(\vec{x} = (x_1, \ldots, x_n)\). We fix \(G\)-flags at these points and assume that the gauge group preserves \(FL_a\). It means that \(G\) is reduced to the Borel subgroup \(B \subset G\) at the marked points. In other words, for \(f \in G\) and \(t_a = z - x_a, \ f(t_a), f_a |_{t_a=0} \in B\). We denote this group \(G_B\). The quotient \(\{\bar{\partial} + A\}/G_B\) is moduli space \(Bun(\Sigma_{g,n}, G)\) (see (2.3)) of the holomorphic bundles with the quasi-parabolic structures at the marked points.

Consider the coadjoint \(G\)-orbits in the Lie coalgebra \(g^*\) located at the marked points

\[
\mathcal{O}_a = \{ S_a = Ad_g S^0_a, \ g \in G, \ S^0_a \in g^* \}, \quad (a = 1, \ldots, n). \tag{2.17}
\]

13
\(O_a\) is the affine space over the cotangent bundle of the flag variety \(T^*F_{a}\). It is a symplectic variety with the Kirillov-Kostant symplectic forms \(\omega^{KK} = \delta(S^0, \delta g g^{-1}) = \langle S_g^{-1} \delta g \wedge g^{-1} \delta g \rangle\). The space of connections \(Conn(\Sigma_{g,n}, G)\) is equipped with the symplectic form
\[
\omega + \sum_{a=1}^{n} \omega^{KK}_a, \quad \omega^{KK}_a = (S_a g^{-1} \delta g \wedge g^{-1} \delta g),
\]
where \(\omega\) is \(\text{(2.2)}\). We assume now that the flat connections have logarithmic singularities at the marked points with the residues taking values in \(O_a\) \(\text{(2.17)}\). In this way the flatness condition \(\text{(2.1)}\) is replaced with
\[
FConn_{\Sigma_{g,n}, G} = \{ d + A \mid d A + \frac{1}{2} A \wedge A = \sum_{a=1}^{n} S_a \delta(x_a) \}. \tag{2.19}
\]
The gauge group \(G_B\) is the group of symplectic automorphisms of the symplectic space \(Conn(\Sigma_{g,n}, G)\) and \(\text{(2.19)}\) is the moment constraint condition generated by this action. The moduli space of flat connections \(FBun_{\Sigma_{g}, G}\) is the result of symplectic reduction
\[
FBun_{\Sigma_{g}, G} = FConn_{\Sigma_{g,n}, G}/G_B = Conn(\Sigma_{g,n}, G)/\{G_B\}.
\]

**Moving points**

Consider the moduli space \(\mathfrak{M}(\Sigma_{g,n})\) of complex structures of \(\Sigma_{g,n}\). This space is foliated over the moduli space of complex structures of compact curves \(\mathfrak{M}(\Sigma_{g})\) with fibers \(U \subset \mathbb{C}^n\) corresponding to the moving marked points. The moduli space \(\mathfrak{M}(\Sigma_{g,n})\) is the classes of equivalence relation in the space of differentials \(\Omega_{C}^{1(-1,1)}(\Sigma_{g,n})\) under the action of the group of diffeomorphisms \(Diff_{C}^{\infty}(\Sigma_{g,n})\) vanishing at the marked points.

Consider local coordinates in a fiber. The basis is defined as follows. Let \((z, \bar{z})\) be the local coordinates in a neighborhood of the marked point \(x_a^0\) and \(U_a\) is a neighborhood of \(x_a^0\) such that \(x_b^0 \notin U_a\) if \(x_b^0 \neq x_a^0\). Define the \(C^\infty\) function
\[
\chi(t_a, \bar{t}_a) = \begin{cases} 
1, & z \in U_a' \subset U_a \\
0, & z \not\in U_a.
\end{cases} \tag{2.20}
\]
The moving points \((x^0_a \to x_a)\) correspond to the following local deformation
\[
w = z - \sum_{a=1}^{n} \epsilon_a(z, \bar{z}), \quad \epsilon_a(z, \bar{z}) = -(t_a + \sum_{j=1}^{n} t_a^{(j)} \frac{1}{j!}(z - x_a^0)^{j})\chi(x_a^0, \bar{x_a^0}).
\]
In particular,
\[
t_a = x_a - x_a^0, \quad t_a^{(1)} = \partial_{z} w|_{z=x_a^0} - 1, \quad t_a^{(j)} = \partial_{z}^{j} w|_{z=x_a^0}, \quad (j > 1). \tag{2.21}
\]
Under the action of \(Diff_{C}^{\infty}(\Sigma_{g,n})\) we can kill all the terms except the first. Thus, in general we have only time \(t_a\). The part of the Beltrami differential related to the marked points assumes the form
\[
\mu = \sum_{a=1}^{n} t_a \mu_a^{(0)}, \quad \mu_a^{(0)} = \bar{\partial} \chi_a(z, \bar{z}). \tag{2.22}
\]

**Deformation of elliptic curve**

Let \(T^2 = \{ (x, y) \in \mathbb{R} \mid x, y \in \mathbb{R}/\mathbb{Z} \}\) be a torus. Complex structure on \(T^2\) is defined by the complex coordinate \(z = x + \tau_0 y, \Im \tau_0 > 0\). In this way we define the elliptic curve \(\Sigma_{\tau_0} \sim \mathbb{C}/(\mathbb{Z} + \tau_0 \mathbb{Z})\). It follows from \(\text{(2.23)}\) that \(T^2\) should have at least one moving point. Only this case \((n = 1)\) will be considered here. Since there is \(\mathbb{C}\) action on \(\Sigma_{\tau_0}\) \(z \to z + c\), it is possible to put the one marked point at \(z = 0\) and we leave with one module related to the deformation \(\Sigma_{\tau_0} \to \Sigma_{\tau} \sim \mathbb{C}/(\mathbb{Z} + \tau \mathbb{Z})\). We assume that \(\tau - \tau_0\)
we use the coordinates

\( P \) bundle

The choice of the complex structure defines the polarization (2.4 Equations of motion and the isomonodromy problem)

\( \chi \) where

\( \tau \)

Taking into account (2.22) define

\( \mu \)

Notice that

\( a \)

From (2.8) and (2.24) we find the form of

\( \mu_a^{(0)} \)

\( \mu_a^{(0)} = \bar{\partial}_\tau \chi_a(\bar{z}, \bar{z}) \), \( t_a = x_a - x_a^0 \).

The dual to the Beltrami-differentials basis \( \mu_a^{(0)} \), \( \{ \mu_a^{(0)}, a = 1, \ldots n \} \) with respect to the integration over \( \Sigma \) is the first Eisenstein functions (A.2) \( E_1(z - x_a) \) and 1. There is only one time \( t_\tau \) for the one marked point case (see (2.23)).

2.4 Equations of motion and the isomonodromy problem

The choice of the complex structure defines the polarization \( (A, \bar{A}) \) of \( Conn(\Sigma_{g,n}, G) \). We define the bundle \( P(G) \) over the Teichmüller space \( T_{g,n} \) with the local coordinates

\( (A, \bar{A}, S, t) \), \( ResA_{z=x_a} = S_a \), \( S = (S_1, \ldots, S_n) \), \( t = (\tau_1, \ldots, \tau_{3g-3}, t_1, \ldots, t_n) \),

\[ P(G) \]

\[ \downarrow \]

\[ Conn(\Sigma_{g,n}, G) \]

The bundle \( P(G) \) plays the role of the extended phase space while \( Conn(\Sigma_{g,n}, G) \) is the standard phase space with degenerate form (2.18) and \( t \) is the set of times. The form (2.13) acquires additional terms

\[ \omega = \omega_0 + \sum_{a=1}^n \omega_a - \frac{1}{\kappa} \left( \sum_{l=1}^{3g-3} \delta H_l \delta \tau_l + \sum_{a=1}^n \delta H_a \delta t_a \right), \quad H_a = \frac{1}{2} \int_{U_a} (A, \bar{A}) \bar{\partial}_\tau \chi_a(\bar{z}, \bar{z}). \]
The symplectic form $\omega$ is defined on the total space of $\mathcal{P}(G)$. It is degenerate on $3g - 3 + n$ vector fields $D_{s}$: $\omega(D_{s}, \cdot) = 0$, where

$$D_{l} = \partial_{\tau_{l}} + \frac{1}{\kappa}\{H_{l}, \cdot\}_{\omega_{0}}, \quad (l = 1, \ldots, 3g - 3), \quad D_{a} = \partial_{t_{a}} + \frac{1}{\kappa}\{H_{a}, \cdot\}_{\omega_{0}}, \quad (a = 1, \ldots, n).$$

The Poisson brackets corresponding to $\omega_{0}$ are the Darboux brackets, and those corresponding to $\omega_{a}$ are the Lie brackets. They are non-degenerate on the fibers. The vector fields $D_{s}$ define the equations of motion for any function $f$ on $\mathcal{P}(G)$.

$$\frac{df}{dr_{s}} = \partial_{r_{s}}f + \frac{1}{\kappa}\{H_{s}, f\}, \quad r_{s} = \tau_{s}, \text{ or } r_{s} = t_{s}.$$

The compatibility conditions are the so-called the Whitham equations [44]:

$$\kappa \partial_{s}H_{r} - \kappa \partial_{r}H_{s} + \{H_{r}, H_{s}\} = 0. \quad (2.28)$$

The Hamiltonians are the Poisson commuting quadratic Hitchin Hamiltonians. It means that there exists the generating function (the tau-function)

$$H_{s} = \frac{\partial}{\partial t_{s}} \log \tau, \quad \tau = \exp \frac{1}{2} \sum_{a=1}^{l} \int_{\Sigma_{g,n}} A_{a} \delta_{m}^{(2)}.$$

### 2.5 Symplectic reduction

Up to now the equations of motion, the linear problem, and the tau-function are trivial. The meaningful equations arise after imposing the corresponding constraints (2.19) and the gauge fixing. Here we explain in detail the structure of the moduli space of the flat bundles upon the choice of the polarization. The form $\omega$ (2.27) is invariant under the action of the gauge group $\mathcal{G}_{B}$.

$$A \rightarrow f^{-1}\kappa \partial f + f^{-1}Af, \quad \bar{A} \rightarrow f^{-1}\bar{\partial}f + f^{-1}\bar{A}f.$$

Let us fix $\bar{A}$ such that $\bar{\bar{A}}$ parameterized orbits of the $\mathcal{G}_{B}$ action

$$\bar{A} = f(\bar{\partial} + \mu \partial)f^{-1} + f\bar{L}f^{-1}. \quad (2.29)$$

Then the dual field is obtained from $A$ by the same element $f$

$$L = f^{-1}\kappa \partial f + f^{-1}Af, \quad (2.30)$$

Thus in local coordinates the moment equation takes the form (see 2.19)

$$(\bar{\partial} + \partial \mu)\bar{L} - \kappa \bar{\partial} \bar{L} + [\bar{L}, L] = 2\pi i \sum_{a=1}^{n} S_{a} \delta(x_{a}) \quad (2.31)$$

Let $U_{a}$ be a small neighborhood of the marked point $x_{a}$, where $f(x_{a}, \bar{x}_{a}) \in B$. we fix the gauge as $\bar{\bar{L}} = 0$ locally on $U_{a}$. Then the Hamiltonian $h^{\text{gauge}}$ (2.7) takes the form

$$h^{\text{gauge}} = \int_{U_{a}} (\epsilon, \bar{\partial}L), \quad \epsilon|_{t_{a}=0} = b + O(|z|), \quad b \in \text{Lie}(B).$$

The zero moment map condition with respect to the $\mathcal{G}_{B}$ action is

$$\bar{\partial}L|_{U_{a}} = \sum_{a=1}^{n} S_{a} \delta^{(2)}(t_{a}, \bar{t}_{a}), \quad S_{a} \in O_{a}, \quad \text{Res}_{x_{a}} L = S_{a}. \quad (2.32)$$
If $G$ is a finite-dimensional group the gauge fixing (2.29) and the moment constraint (2.32) kill almost all degrees of freedom. The fibers $FBun(\Sigma_{g,n}, G) = \{L, \bar{L}, \mathbf{S}\}$ become finite-dimensional as well as the bundle $\mathcal{P}(G)$:

$$\dim (FBun(\Sigma_{g,n}, G)) = 2 \dim (G)(g - 1) + n \dim \mathcal{O},$$

$$\dim \mathcal{P}(G) = \dim (FBun(\Sigma_{g,n}, G)) + 3g - 3 + n.$$  

Due to the invariance of $\omega$ (2.21) it preserves its form on $FBun(\Sigma_{g,n}, G)$:

$$\omega_0 = \int_{\Sigma_{g,n}} (dL, d\bar{L}) + \sum_{a=1}^{n} \omega_{a}^{K K}, \quad H_s(L) = \frac{1}{2} \int_{\Sigma_{g,n}} (L^2)|^0_s.$$  

(2.33)

But now, due to (2.31), the system is no longer free because $L$ depends on $\bar{L}$ and $\mathbf{S}$. Moreover, since $L$ depends explicitly on $t$ the system (2.33) is non-autonomous. Let $M_s = \partial_s f f^{-1}$. It follows from (2.29) and (2.30) that the equations of motion (2.14) on the space $FBun(\Sigma_{g,n}, G)$ take the form

$$\left\{ \begin{array}{l}
1. \quad \kappa \partial_t L - \kappa \partial M_s + [M_s, L] = 0, \quad s = 1, \ldots, l, \\
2. \quad \kappa \partial_t L - (\bar{\partial} + \partial \mu) M_s + [M_s, \bar{L}] = L \mu_0^0.
\end{array} \right.$$  

(2.34)

The equations 1. (2.34) are the Lax equations. The essential difference with the integrable systems is the differentiation $\bar{\partial}$ with respect to the spectral parameter. On $FBun(\Sigma_{g,n}, G)$ the linear system (2.15) assumes the form

$$\left\{ \begin{array}{l}
1. \quad (\kappa \bar{\partial} + L) \psi = 0, \\
2. \quad (\partial + \sum_{s=1}^{l} t_s \mu_0^0 \partial L + \bar{L}) \psi = 0, \\
3. \quad (\kappa \partial_s + M_s) \psi = 0, \quad (s = 1, \ldots, l). 
\end{array} \right.$$  

(2.35)

The equations 1. and 2. (2.34) are consistency conditions for (1. 3.) and (2. 3.), and (1. 2.) is the flatness condition (2.31). The equations 3. (2.35) provides the isomonodromy property of the system 1.,2.(2.34), with respect to variations of the times $t_s$. For this we refer to the nonlinear equations (2.34) as the Hierarchy of the Isomonodromic Deformations.

### 2.6 Isomonodromic deformations and integrable systems

We can consider the isomonodromy preserving equations as a deformation (Whitham quantization) of integrable equations [22, 19]. The level $\kappa$ plays the role of the deformation parameter. Introduce the independent times $t_s = (\tau_1^0, x_a^0)$ as $\tau_i = \tau_i^0 + \kappa t_i, \quad t_a = \kappa t_a$ for $\kappa \to 0$. It means that $t_s = (t_s, \kappa t_a)$ play the role of a local coordinates in a neighborhood of the point $(t_s^0, x_a^0)$ on the moduli space $\mathcal{M}(\Sigma_{g,n})$. In this limit the equations of motion 1. (2.34) are the standard Lax equation (the Zakharov-Shabat equation in the 1+1 case)

$$\partial_s L^{(0)} + [M_s^{(0)}, L^{(0)}] = 0, \quad s = 1, \ldots, l,$$  

(2.36)

where $L^{(0)} = L(t_s^0), \quad (M_s^{(0)} = M_s(t_s^0))$. The linear problem for this system is obtained from the linear problem for the isomonodromy problem (2.35) by the analog of the quasiclassical limit in quantum mechanics. Represent the Baker-Akhiezer function in the WKB form

$$\psi = \Phi \exp \left( \frac{S^{(0)}}{\kappa} + S^{(1)} \right).$$  

(2.37)

and substitute (2.37) into the linear system (2.35). If $\partial_2 S^{(0)} = 0$ and $\partial_1 S^{(0)} = 0$, then the terms of order $\kappa^{-1}$ vanish. In the quasiclassical limit we put $\partial S^{(0)} = \lambda$. In the zero order approximation we come to the linear system

$$\left\{ \begin{array}{l}
i. \quad (\lambda + L^{(0)}(z, \tau_0))Y = 0, \\
ii. \quad \partial_2 Y = 0, \\
iii. \quad (\partial_{\kappa} + M_s^{(0)}(z, \tau_0))Y = 0.
\end{array} \right.$$  

(2.38)

The Baker-Akhiezer function $Y$ takes the form

$$Y = \Phi e^{\sum_{s=1}^{l} t_s \frac{\partial_s}{\partial_2}} S^{(0)}.$$

The consistency condition of $i$ and $ii$ is the Lax equation (2.36).
3 Affine Algebras and Isomonodromic Deformations

We apply the general scheme to the bundles related to affine algebras, and briefly describe here their central and the cocentral extensions [12].

3.1 Affine Lie algebras

Let \( \mathfrak{g} \) be a simple complex Lie algebra and \( L(\mathfrak{g}) = \mathfrak{g} \otimes \mathbb{C}[x, x^{-1}], \) \( x \in \mathbb{C}^* \) is the loop algebra of Laurent polynomials. Consider its central extension \( \hat{L}(\mathfrak{g}) = \{ (\mathcal{A}(x), \lambda) \} \). The commutator in \( \hat{L}(\mathfrak{g}) \) assumes the form

\[
[(\mathcal{X}_1, \lambda_1), (\mathcal{X}_2, \lambda_2)] = \left( [(\mathcal{X}_1, \mathcal{X}_2)]_0, \oint (\mathcal{X}_1, \partial_x \mathcal{X}_2) \right),
\]

where \( [(\mathcal{X}_1, \mathcal{X}_2)]_0 \) is a commutator in \( \mathfrak{g} \), and \( \langle , \rangle \) is the Killing metric on \( \mathfrak{g} \). The cocentral extension \( \hat{L}(\mathfrak{g}) \) of \( \hat{L}(\mathfrak{g}) \) is the algebra

\[
\hat{L} = \{ \mathcal{W} = (k, Y, \lambda) = (k \partial_x + Y, \lambda), \ Y \in L(\mathfrak{g}), \ k \in \mathbb{C}, \ \lambda \in \mathbb{C} \},
\]

with the commutator

\[
[\mathcal{W}_1, \mathcal{W}_2] = [(\mathcal{Y}_1, k_1, \lambda_1), (\mathcal{Y}_2, k_2, \lambda_2)] = (k_1 \partial_x \mathcal{Y}_2 - k_2 \partial_x \mathcal{Y}_1 + [\mathcal{Y}_1, \mathcal{Y}_2]_0, 0, \oint (\mathcal{Y}_1, \partial_x \mathcal{Y}_2))
\]

(3.2)

and the invariant non-degenerate form

\[
(\mathcal{W}_1, \mathcal{W}_2) = \frac{1}{2} \left( \oint (\mathcal{Y}_1, \mathcal{Y}_2) + k_1 \lambda_2 + k_2 \lambda_1 \right).
\]

(3.3)

Let \( L(G) \) be the loop group corresponding to the Lie algebra \( L(\mathfrak{g}) \). The following statement can be checked directly.

Lemma 3.1 The group \( L(G) \) acts as \( \text{Aut}(\hat{L}(\mathfrak{g})) \) preserving the metric (3.3)

\[
\mathcal{W} \to (kf^{-1} \partial_x f + f^{-1} \mathcal{Y} f, k, \lambda - 2 \oint \langle \mathcal{Y}, \partial_x f f^{-1} \rangle - k \oint \langle \partial_x f f^{-1}, \partial_x f f^{-1} \rangle).
\]

(3.4)

The Lie algebra \( \text{End}(\hat{L}(\mathfrak{g})) = L(\mathfrak{g}) \) is generated by the vector fields \( \epsilon \in L(\mathfrak{g}) \)

\[
\delta_\epsilon \mathcal{Y} = k \partial_x \epsilon + [\mathcal{Y}, \epsilon], \ \delta_\epsilon k = 0, \ \delta_\epsilon \lambda = - \oint \langle \mathcal{Y}, \partial_x \epsilon \rangle.
\]

(3.5)

3.2 Vector bundles of infinite rank

Let \( \mathcal{P} \) be a principal \( L(G) \) bundle over the complex curve \( \Sigma_g \). Consider the adjoint bundle \( E = \mathcal{P} \otimes \hat{L}(\mathfrak{g}) \). Its local sections are the maps \( \Sigma_g \to \mathcal{W} \) \( \text{(3.1)} \). The space of connections on \( E \) is

\[
\text{Conn}(\Sigma_g, L(G)) = \{ d_A \}, \ d_A : \Gamma(E) \to \Gamma(E) \otimes \Omega^1(\Sigma), \ d_A = d + \mathcal{A},
\]

where \( d_A \) has a component description (3.1), and locally

\[
d_A = d + \mathcal{A} = (d + k \partial_x + \mathcal{Y}, d + \lambda).
\]

(3.6)

The component \( d + \lambda \) is a connection in a trivial line bundle over \( \Sigma_g \).

The gauge group of the bundle \( \mathcal{G} \) is the map \( \text{Map}_{C^\infty}(\Sigma_g \to L(G)) \). It acts on sections as (3.4).

Using the form (3.3) we introduce the symplectic structure in the space \( \text{Conn}(\Sigma_g, L(G)) \)

\[
\omega = \frac{1}{2} \int_{\Sigma_g,n} \oint \langle \delta \mathcal{A} \wedge \delta \mathcal{A} \rangle = \frac{1}{2} \int_{\Sigma_g,n} \oint \langle \delta \mathcal{Y} \wedge \delta \mathcal{Y} \rangle + \delta k \wedge \delta \lambda.
\]

(3.7)

The form is invariant under the gauge action.
3.3 The gauge group

In what follows we use the central extension $\hat{G}$ of the gauge group $G$. It is based on the central extension of the loop group

$$1 \rightarrow \mathbb{C}^* \rightarrow \hat{L}(G) \rightarrow L(G) \rightarrow 1$$

and defined as

$$\hat{G} = \text{Map}_{C^\infty}(\Sigma_g \rightarrow \hat{L}(G)) = \{ f(z, \bar{z}, x), s(z, \bar{z}) \},$$

where $f(z, \bar{z}, x)$ takes values in $G$, and $s$ is the map $\Sigma_g$ to the central element of $\hat{L}(G)$. The multiplication is pointwise with respect to $\Sigma_g$

$$(f_1, s_1) \times (f_2, s_2) = (f_1f_2, s_1s_2 \exp C(f_1, f_2)).$$

Here $\exp C(f_1, f_2)$ is a map from $\Sigma_g$ to the central element of $\hat{L}(G)$ defined by the Wess-Zumino-Witten action [23 69].

Consider the Lie algebra $\text{Lie}(\hat{G}) = \{(\epsilon, \bar{\epsilon})\}$, where $\epsilon$ is a smooth map of $\Sigma_g$ to $g$ and $\bar{\epsilon}$ is a smooth map of $\Sigma_g$ to $\mathbb{C}$. $\text{Lie}(\hat{G})$ is represented by the Hamiltonian vector fields: (see (3.5))

$$\delta_\epsilon \mathcal{Y} = (d + k \partial_x)\epsilon + [\mathcal{Y}, \epsilon], \quad \delta_{\bar{\epsilon}} k = 0, \quad \delta_{\bar{\epsilon}} \lambda = -\bar{\epsilon} [\mathcal{Y}, \partial_x \epsilon], \quad \delta_{\epsilon} k = 0, \quad \delta_{\epsilon} \lambda = d\bar{\epsilon}.$$  \hspace{1cm} (3.9)

It defines the moment map $F : \text{Conn}(\Sigma_g, L(G)) \rightarrow \text{Lie}^*(\hat{G})$, with

$$\mathcal{F} = (F, dk), \quad F = (d + k \partial_x)\mathcal{Y} + \frac{1}{2}[\mathcal{Y}, \mathcal{Y}].$$

Rewrite $\mathcal{A}$ in terms of a complex structure on $\Sigma_g$. Let $(z, \bar{z})$ be local coordinates on $\Sigma_g$. Then

$$\mathcal{A} = (A, \bar{A}') = ((\kappa \partial_z + k \partial_x + \mathcal{Y}, \partial_z + \lambda) \otimes dz, (\partial_z + \bar{k}' \partial_x + \bar{\mathcal{Y}}', \partial_z + \bar{\lambda}') \otimes d\bar{z}),$$

In these terms $\omega$ (3.7) and the action of $\text{Lie}(\hat{G})$ (3.9) assumes the form

$$\omega = \oint_{\Sigma_g} \langle \delta \mathcal{A} \wedge \delta \bar{A}' \rangle = \oint_{\Sigma_g,n} \langle \delta \mathcal{Y} \wedge \delta \bar{\mathcal{Y}}' \rangle + \delta k \wedge \delta \bar{k}' + \delta \lambda \wedge \delta \bar{\lambda}'$$ \hspace{1cm} (3.10)

$$\delta_\epsilon \mathcal{Y} = (\kappa \partial_z + k \partial_x)\epsilon + [\mathcal{Y}, \epsilon], \quad \delta_{\bar{\epsilon}} \lambda = -\bar{\epsilon} [\mathcal{Y}, \partial_x \epsilon], \quad \delta_{\epsilon} k = 0,$$

$$\delta_{\bar{\epsilon}} \bar{\mathcal{Y}}' = (\partial_z + \bar{k}' \partial_x)\epsilon + [\bar{\mathcal{Y}}', \epsilon], \quad \delta_{\bar{\epsilon}} \bar{k}' = 0, \quad \delta_{\epsilon} \bar{\lambda} = \kappa \partial_z \epsilon, \quad \delta_{\bar{\epsilon}} \bar{\lambda}' = \partial_z \bar{\epsilon}.$$ \hspace{1cm} (3.11)

Then we come to the following expression for the moment

$$\mathcal{F} = \left( (\kappa \partial_z + k \partial_x)\bar{\mathcal{Y}}' - (\partial_z + \bar{k}' \partial_x)\mathcal{Y} + [\mathcal{Y}, \bar{\mathcal{Y}}'], (\kappa \partial_z \bar{k}' - \partial_z k) \right).$$ \hspace{1cm} (3.12)

The flatness condition $\mathcal{F} = 0$ is the moment constraint.

As in (2.12) we pass to the new components

$$\bar{Y} = \bar{Y}' + \frac{1}{\kappa} \bar{\mu} Y, \quad \bar{k}' = \bar{k}' + \frac{1}{\kappa} \bar{\mu} k, \quad \bar{\lambda}' = \bar{\lambda}' + \frac{1}{\kappa} \bar{\mu} \lambda.$$  \hspace{1cm} (3.13)

In this way using (2.9) we pass to the flat connection in the coordinates $(w, \bar{w})$:

$$\left\{ \begin{array}{l}
((\kappa \partial_w + k \partial_x + Y), (\partial_w + \lambda)) \otimes dw, \\
((\partial_{\bar{w}} + k \partial_x + \bar{Y}), (\partial_{\bar{w}} + \bar{\lambda})) \otimes d\bar{w}.
\end{array} \right.$$ \hspace{1cm} (3.13)

Recall that $\mu = \sum_t t s t s^0_\epsilon$. Then $\omega$ (3.10) becomes

$$\omega = \omega_0 - \frac{1}{\kappa} \delta H_s \delta t_s,$$ \hspace{1cm} (3.14)
\[ \omega_0 = \oint \int_{\Sigma_{g,n}} (\delta Y \wedge \delta \bar{Y}) + \sum_{\gamma} (\delta \kappa \wedge \delta \bar{\kappa} + \delta \lambda \wedge \delta \bar{\lambda}) , \]

\[ H_s = \oint \int_{\Sigma_{g,n}} \left( \oint \langle Y, Y \rangle + k \lambda \right) \mu^0_s. \] (3.15)

The equations of motion corresponding to these Hamiltonians take the form:

\[ \partial_s Y = 0, \quad \partial_s \lambda = 0, \quad \partial_s k = 0, \quad \kappa \partial_s \bar{Y} = Y \mu^0_s, \quad \kappa \partial_s \bar{k} = \bar{k} \mu^0_s, \quad \kappa \partial_s \bar{\lambda} = \lambda \mu^0_s. \] (3.16)

### 3.4 Quasi-parabolic structure

Let \( B \subset G \) be a Borel subgroup of \( G \). Define \( L^+(G) = (B + G \otimes t\mathbb{C}[x]) \) the Borel subgroup of \( L(G) \). The quotient space \( L(G)/L^+(G) \) is the affine flag variety \( FL^a_{Aff} \). It is the loop analog of the finite-dimensional flag variety. Assume that \( \mathcal{G} \) is restricted to \( L^+(G) \) at the marked points on \( \Sigma_{g,n} \). As in the general case we denote this subgroup \( \mathcal{G}_B \). The existence of the quasi-parabolic structure for these bundles means that we fix affine flags \( Gr^a_{Aff} \) at the marked points \( x_a (a = 1, \ldots, n) \). To define the moduli space of flat connections we attach the coadjoint orbits \( \mathcal{O}(\mathcal{S}^{(0)}_{0}, r_a) \) of \( \hat{L}(\text{SL}(N, \mathbb{C})) \) to the marked points

\[ \mathcal{O}(\mathcal{S}^{(0)}_{a}, r_a) = \{ \mathcal{S}_a = -g^{-1} \mathcal{S}^{(0)}_{a} g - r_a g^{-1} \partial_x g, \ r_a, \ (\mathcal{S}^{(0)}_{a}, r_a) \in \hat{L}^*(g) \}. \] (3.17)

The coadjoint orbits are infinite-dimensional symplectic manifold with the Kirillov-Kostant form

\[ \omega^K_a = \oint \langle \delta(\mathcal{S}_a g^{-1}) \wedge \delta g \rangle + \oint \langle g^{-1} \delta g \wedge \delta(g^{-1} \partial_x g) \rangle. \] (3.18)

Notice that due to the residue theorem

\[ \sum_{a=1}^n r_a = 0. \] (3.19)

The orbits are the affine spaces over the cotangent bundle \( T^*FL^a_{Aff} \) of the affine flag varieties \( FL^a_{Aff} \). For the bundles with the quasi-parabolic structure we replace \( \omega_0 \) in (3.11) with \( \omega_0 + \sum_{a=1}^n \omega^K_a \). The flatness condition for these bundles takes the form (see (3.12)):

\[ (\kappa \partial_z + k \partial_x) \bar{Y} - (\partial_z + \bar{k} \partial_x) Y + [Y, \bar{Y}] = -\sum_{a=1}^n S_a \delta(x_a), \]

\[ \kappa \partial_{\bar{z}} \bar{Y} - \partial_{\bar{z}} k = -\sum_{a=1}^n r_a \delta(x_a) , \]

or from (3.13) we get:

\[ (\kappa \partial_{\bar{w}} + k \partial_{\bar{x}}) \bar{Y} - (\partial_{\bar{w}} + \bar{k} \partial_{\bar{x}}) Y + [Y, \bar{Y}] = -\sum_{a=1}^n S_a \delta(x_a), \] (3.20)

\[ \kappa \partial_{\bar{w}} \bar{k} - \partial_{\bar{w}} k = -\sum_{a=1}^n r_a \delta(x_a). \]

### 4 Penlevé type field theory related to \( L(\text{SL}(N, \mathbb{C})) \) bundles

Here we consider a particular example of general theory and assume that \( G = \text{SL}(N, \mathbb{C}) \) and the base of the loop group bundle is taken over elliptic curve \( \Sigma_{\tau} = \mathbb{C}/(\mathbb{Z} \oplus \tau \mathbb{Z}) \) with coordinates \((z, \bar{z})\). We consider the case of one marked point \( z = 0 \).
4.1 Bundles over $\Sigma$  

Let $V$ be an integrable representation of $\hat{L}(\mathfrak{sl}(N,\mathbb{C}))$ of level $l$ [42]. It means that $\lambda$ acts on $V$ as a scalar. Consider the adjoint bundle $E = \mathcal{P} \times_{L(\mathfrak{sl}(N,\mathbb{C}))} V$. The sections $s(w,\bar{w},x) \in \Gamma(E)$ satisfies the quasi-periodicity conditions

\[
\begin{aligned}
1. \quad s(w + 1, \bar{w} + 1, x) &= Q^{-1}s(w, \bar{w}, x), \quad (\bar{\Lambda} = -e^{-w-\tau/2})\Lambda, \quad (B.22) \\
2. \quad s(w + \tau, \bar{w} + \tau_0, x) &= \Lambda^{-1}e(-\frac{k}{2\pi i}\partial_x)s(w, \bar{w}, x) = \Lambda^{-1}s(w, \bar{w}, x - k),
\end{aligned}
\]

(4.21)

To describe the sections of the bundles of this type it is convenient to work with the sin-basis $T^\alpha$ in $\mathfrak{sl}(N,\mathbb{C})$ [B.21] $s(w) = \sum_\alpha s^\alpha(w)T^\alpha$. The sections $s^\alpha(w)$ are defined by the quasi-periodicities (4.21). It follows from (3.13) and (4.21) that we pass to the connections

\[
\begin{aligned}
k\partial_w + (k\partial_x + Y, \lambda), \\
\bar{\partial}_\bar{w} + (\bar{Y}, \bar{\lambda}).
\end{aligned}
\]

(4.22)

It follows from (3.11) that $k, \bar{k}, \lambda$ and $\bar{\lambda}$ can be chosen to be double-periodic, while

\[
\begin{aligned}
Y(w + \tau) &= -2\pi i Id_N + \Lambda Y(w)\Lambda^{-1}e(-\frac{k}{2\pi i}\partial_x), \quad Y(w + 1) = QY(w)Q^\dagger, \\
\bar{Y}(w + \tau) &= \Lambda\bar{Y}(w)\Lambda^{-1}e(-\frac{k}{2\pi i}\partial_x), \quad \bar{Y}(w + 1) = \bar{Q}\bar{Y}(w)\bar{Q}^\dagger.
\end{aligned}
\]

(4.23)

Let $\hat{G} = C^\infty(\text{Map}) : \Sigma_r \to \hat{L}(\text{GL}(N,\mathbb{C}))$ be the gauge group [3.8]. Its elements satisfy the quasi-periodicity conditions (4.21). The gauge transformation (3.11) allows one to choose $k$ and $\bar{k}$ to be $(z, \bar{z})$-independent, $\lambda$ to be antiholomorphic, and $\bar{\lambda}$ holomorphic. Since $\lambda$ and $\bar{\lambda}$ are double-periodic, then they are also $(z, \bar{z})$-independent. For the bundles (4.21) almost all $\bar{Y}$ can be represented as a pure gauge $\bar{Y} = -f^{-1}(\partial_\bar{w})f$. Then the gauge transformed connections (4.22) assume the form

\[
\begin{aligned}
k\partial_w + (k\partial_x + L, \lambda), \\
\bar{\partial}_\bar{w} + (0, \bar{\lambda}),
\end{aligned}
\]

(4.24)

where $k, \lambda, \bar{k}, \bar{\lambda}$ are constants. Let us consider $\Sigma_r$ with a single marked point $z = 0$. Then $L$ is fixed by the following conditions:

1. **The flatness** of the connections (4.21)

   \[\partial_\bar{w}L = 0.\]

(4.25)

2. **The quasi-periodicity conditions**

   \[L(x, w + \tau) = -2\pi i Id_N + \Lambda L(w)\Lambda^{-1}e(-\frac{k}{2\pi i}\partial_x), \quad L(w + 1) = QL(w)Q^{-1},\]

(4.26)

3. **The quasi-parabolic structure**

   \[\text{Res}_{w=0}L = S - 2\pi i Id_N, \quad S \in \mathcal{O}(S(0), r = 0)\]

(4.27)

Since $\det(\hat{\Lambda}(z, \tau)) = (-1)^N e^{(z - \frac{1}{2}\tau)}$ we see that $\hat{\Lambda}$ takes value in $\text{GL}(N, \mathbb{C})$, but not in $\text{SL}(N, \mathbb{C})$. Thus, for fixed $x$ (4.26) defines $\text{GL}(N, \mathbb{C})$ bundles. These $\text{GL}(N, \mathbb{C})$-bundles have degree one. In this way, the $\hat{L}(\text{SL}(N, \mathbb{C}))$-bundles which we consider here are the analogues of the $\text{GL}(N, \mathbb{C})$-bundles of degree one (see [53] [55] for the bundles of arbitrary characteristic class). We rewrite the Hamiltonians (3.15) in terms of $L$

\[H_\tau = \int_{\Sigma_r} \left( \oint (L, L) + k\lambda \right) \mu_\tau^0, \quad \mu_\tau^0 = \frac{1}{\rho_0},\]

(4.28)

where we preserve the notion $\lambda$ after the gauge transformation.

\[\text{Res}_{w=0}L = S - 2\pi i Id_N, \quad S \in \mathcal{O}(S(0), r = 0)\]

(4.27)
4.2 Spin variables

We refer to the elements \( S = \sum_{\alpha \in \mathbb{Z}^2} \alpha(x) T^\alpha \) of the orbit \( \mathcal{O}(S(0), r = 0) \subset L^*(\mathfrak{g}) \) as the spin variables. To be more precise, we consider the loop algebra \( L(\mathfrak{g}) \) of trigonometric polynomials in the basis \( T_n = e(nx) T^\alpha \). In other words,

\[
L(\mathfrak{g}) = \{ \mathbf{W}(x) = \sum_{n \in \mathbb{Z}} \sum_{\alpha \in \mathbb{Z}^2} W_n^\alpha T_n^\alpha, \; W_n^\alpha = 0 \text{ for almost all } n \}.
\]

The dual space \( L^*(\mathfrak{g}) \) is the space of distributions on \( L(\mathfrak{g}) \). For the functionals \( S^\alpha(x) = \langle S(x), T^{-\alpha} \rangle \) on \( L^*(\mathfrak{g}) \) we have the Poisson-Lie brackets

\[
\{ S^\alpha(x), S^\beta(y) \} = S^{\alpha+\beta}(x) C(\alpha, \beta) \delta(x - y),
\]

or, in terms of the Fourier expansion

\[
\{ S_n^\alpha, S_{m}^\gamma \} = S_{n+m}^{\alpha+\gamma} C(\alpha, \gamma).
\]

The brackets are non-degenerate on the orbits \( \mathcal{O} \). We can identify the algebra \( L(\mathfrak{g}) \) of the trigonometric polynomials with the subspace in \( L^*(\mathfrak{g}) \) by means of the contour via the contour integral and the Killing form on \( \mathfrak{g} \). The element

\[
C_2 = \sum_{n \in \mathbb{Z}} \sum_{\alpha \in \mathbb{Z}^2} T_n^\alpha T_{-n}^{-\alpha}
\]

is the central element of the algebra \( [\mathfrak{g}, \mathfrak{g}] \). Though it is not an element of the universal enveloping algebra \( \mathcal{U}(L(\mathfrak{g})) \) it is well defined on some representations of \( L(\mathfrak{g}) \). The Casimir function of the brackets \( (4.29), (4.30) \) is given by

\[
H_2 = \sum_{n \in \mathbb{Z}} \sum_{\alpha \in \mathbb{Z}^2} S_n^\alpha S_{-n}^{-\alpha} = \oint \langle S(x), S(x) \rangle.
\]

In addition to \( (3.11) \) the gauge transformation of \( S \) takes the form:

\[
\delta S = [S, \epsilon^0], \; \epsilon^0 = \epsilon(x, z, \bar{z})|_{z=0}.
\]

4.3 Lax operator and equations of motion

Represent the Lax operator in the form of the Fourier expansion

\[
L(x, w) = \frac{k}{N} E_1(w|\tau) I_d N + \sum_{n \in \mathbb{Z}} \sum_{\alpha \in \mathbb{Z}^2} L_n^\alpha(w) T_n^\alpha, \; T_n^\alpha = e(nx) T^\alpha, \; \epsilon( ) = \exp 2\pi i( )
\]

Taking into account the flatness condition \( (4.25) \) and \( (4.27) \) \( L_n^\alpha(w) \) is \( \bar{\omega} \) independent and

\[
L_n^\alpha(w)|_{w \to 0} \sim \frac{S_n^\alpha}{\omega}.
\]

From the quasi-periodicities of \( E_1 \) \( (4.18) \) and \( \phi(-\alpha_r + \bar{\kappa} n, w) \) \( (4.20) \) we find that

\[
L_n^\alpha(w) = S_n^\alpha X_n^\alpha, \; X_n^\alpha = e\left(-\frac{w \alpha_2}{N}\right) \phi(-\alpha_r + \bar{\kappa} n, w)
\]

with \( \alpha_r = \frac{\alpha_1 + \bar{\kappa} \alpha_2}{\kappa} \) has the quasi-periodicity conditions \( (4.26) \). In addition, from \( (4.8) \) we come to \( (4.34) \). Therefore, in terms of the functions \( S_n^\alpha(x) = \sum_{\alpha \in \mathbb{Z}^2} S_n^\alpha e(nx) \)

\[
L(x, w) = \frac{k}{N} E_1(w|\tau) I_d N + \sum_{\alpha \in \mathbb{Z}^2} S^\alpha(x) e\left(-\frac{w \alpha_2}{N}\right) \phi(-\alpha_r + \bar{\kappa} \frac{\partial_x}{2\pi i} w) T^\alpha.
\]

22
Hamiltonian, phase space and equations of motion

Recall that for the one marked point case we have only one time \( t_\tau \). In order to calculate the Hamiltonian we consider the integral

\[
\int \text{tr} L^2(x, w) \mu_\tau^{(0)} = \left( \frac{\kappa^2}{N} E_1^2(w) + \sum_{n \in \mathbb{Z}} \sum_{\alpha \in \mathbb{Z}^2} L_n^\alpha, L_{-n}^{-\alpha} \right) \mu_\tau^{(0)}
\]

\[
= \left( \frac{\kappa^2}{N} E_1^2(w) + \sum_{n \in \mathbb{Z}} \sum_{\alpha \in \mathbb{Z}^2} S_n^\alpha S_{-n}^{-\alpha} \phi(-\alpha + \bar{k}n, w) \phi(\alpha - \bar{k}n, w) \right) \mu_\tau^{(0)}
\]

\[
\tag{4.37}
= \left( \frac{\kappa^2}{N} E_1^2(w) + \sum_{\alpha \in \mathbb{Z}^2} \sum_{n \in \mathbb{Z}} S_n^\alpha S_{-n}^{-\alpha} (E_2(w) - E_2(\alpha + \bar{k}n)) \right) \mu_\tau^{(0)}.
\]

Notice that the first term of this expression is not double periodic (see (4.13)). It reflects the fact that \( L \) is not the Higgs field but a component of the connection of the flat bundle over \( \Sigma \). It follows from the definition of \( \mu_\tau^{(0)} = \mu_\tau^{(0)} \) (2.20) that

\[
\int_{\Sigma} E_2(w) \mu_\tau^{(0)} = -2\pi i, \quad \int_{\Sigma} \mu_\tau^{(0)} = 1, \quad \int_{\Sigma} E_1^2(w) \mu_\tau^{(0)} = \text{const}.
\]

Then integrating (4.37) over \( \Sigma_\tau \) and using (4.38) we obtain the Casimir function (4.31)

\[
\int_{\Sigma_\tau} E_2(w) \sum_{n \in \mathbb{Z}} S_n^\alpha S_{-n}^{-\alpha} \mu_\tau^{(0)} = H_2,
\]

the Hamiltonian \( H_\tau \)

\[
H_\tau = k\lambda + \sum_{\alpha \in \mathbb{Z}^2} \sum_{n \in \mathbb{Z}} S_n^\alpha S_{-n}^{-\alpha} E_2(-\alpha + \rho_0 n)
\]

\[
\tag{4.39}
and the constant term proportional to \( \frac{\kappa^2}{N} \) coming from the last integral in (4.38). It does not depend on the dynamical variables \( (S_n^\alpha, k, \lambda) \) and we ignore it. In terms of fields the Hamiltonian takes the form

\[
H_\tau = k\lambda + \sum_{\alpha \in \mathbb{Z}^2} \int_{\Sigma_\tau} \left( E_2(-\alpha + \frac{\rho_0}{\bar{k}} \partial_x) S^\alpha(x) \right)
\]

\[
\tag{4.40}
\]

The Hamiltonian can be written in the form of the Euler-Arnold top on \( \tilde{L}(g) \)

\[
H_\tau = k\lambda + \int_{\Sigma_\tau} \left( \mathbf{J}(x) \right),
\]

where \( \mathbf{J} : L^*(g) \to L(g) \) is the conjugate inertia tensor

\[
\mathbf{J} : S^\alpha(x) \to E_2(-\alpha + \frac{\bar{k}}{2\pi i} \partial_x) S^\alpha(x).
\]

The phase space \( \mathcal{M} \) of the dynamical system is defined by means of the functionals (4.32), (4.40)

\[
\mathcal{M} = \{ S^\alpha(x) \mid H_2(S) < \infty, \quad H_\tau(S) < \infty \}
\]

Taking into account the Poisson brackets (4.29) we find that the evolution on \( \mathcal{M} \) generated by \( H_\tau \) is given as follows:

\[
\kappa \partial_\tau S^\alpha(x) = \sum_{\gamma \in \mathbb{Z}^2, \gamma \neq \alpha} \left( E_2(-\gamma + \frac{\bar{k}}{2\pi i} \partial_x) S^{-\gamma}(x) \right) S^{\alpha+\gamma}(x) C(\alpha, \gamma).
\]

\[
\tag{4.41}
\]
In terms of the Fourier modes the equations of motion assume the form

\[
\kappa \partial_{\tau} S^\alpha_n = \sum_{j} \sum_{m} S^{\gamma}_m S^{\alpha+\gamma}_{n+m} E_2(-\gamma_{\tau} + km|\tau) C(\alpha, \gamma)
\] (4.42)

The equation (4.41) can written in the top like form (1.9)

\[
\kappa \partial_{\tau} S(x) = ad^*_{J_1} S(x) , \quad \kappa \partial_{\tau} S(x) = [J \cdot S(x), S(x)] .
\]

From (3.16) we find

\[
\kappa \partial_{\tau} k = \kappa , \quad \kappa \partial_{\tau} k = 0 .
\] (4.43)

**Remark 4.1** Consider the subalgebra \( g \subset L(g) \) of the \( x \)-independent loops. Then (4.42) coincides with the monodromy preserving equation related to Sl\((N,\mathbb{C})\) for \( g = \text{sl}(N, \mathbb{C}) \). For \( \text{sl}(2, \mathbb{C}) \) (4.42) is the particular case of the Painlevé VI (1.1) (or, to be exact (1.3)). In this way (4.42) is the field-theoretical generalization of the Painlevé VI equation.

It is useful to rewrite (4.41) in terms of velocities:

\[
F^\alpha(x) = E_2(-\alpha_{\tau} + \frac{\tau k}{2\pi i}) S^\alpha(x) .
\]

They are the functionals on \( L(g) \). Then (4.41) takes the form:

\[
\kappa \partial_{\tau} \varphi^{-1}(\alpha_{\tau} + \frac{\tau k}{2\pi i}) F^\alpha(x) = \sum_{\gamma \in \pi_2(2), \gamma \neq \alpha} F^{-\gamma}(\tau) \varphi^{-1}(\alpha_{\tau} + \gamma + \frac{\tau k}{2\pi i}) F^{\alpha+\gamma}(x) .
\] (4.44)

In some limiting cases considered below (4.41) is simplified to some local equation.

**Perturbation with respect to \( k \)**

Assume that \( k \) is a small parameter and consider expansion of the equation of motion in degrees \( k^j \). Using (4.43) we can take \( k = \tau k \). Let

\[
S^\alpha(x, k) = \sum_{j \geq 0} S^\alpha_j k^j , \quad J(k) = E_2(-\gamma_{\tau} + \frac{\tau k}{2\pi i}) = \sum_{j \geq 0} J_j \partial_x^j , \quad J_j = \frac{\partial_j E_2(-\gamma_{\tau})(\tau k)^j}{(2\pi i)^j j!} .
\]

Then we come to the system of equations

\[
0) \quad \kappa \partial_{\tau} S_0(x) = [J_0 S(x), S_0(x)] ,
1) \quad \kappa \partial_{\tau} S_1(x) = [J_1 \partial_x S_0(x), S_0(x)] + [J_0 S_1(x), S_0(x)] + [J_0 S_0(x), S_1(x)] ,
\]

\[
\ldots
\]

\[
j) \quad \kappa \partial_{\tau} S_j(x) = [J_j \partial_x^j S_0(x), S_0(x)] + [J_{j-1} \partial_x^{j-1} S_1(x), S_0(x)] + \ldots + [J_0 S_j(x), S_0(x)] .
\] (4.45)

The equation 0) describes non-autonomous \( \text{L}(\text{SL}(N, \mathbb{C})) \) Euler-Arnold top. The next equation is linear on the background of solutions \( S_0(x) \). In this way we come to the tower of linear equation. The equation on level \( j \) has as background solutions of the lower equations 0),1),...,\((j-1)\) \( S_j = S_j(S_0, S_1, \ldots, S_{j-1}) \).

For \( \text{SL}(2, \mathbb{C}) \) \( J_{2i+1} = 0 \) and \( \partial_x \) appears first time on the second level

\[
\kappa \partial_{\tau} S_0(x) = [J_0 S_0(x), S_0(x)] , \quad J_0 = (E_2(\tau/2), E_2((1 + \tau)/2), E_2(1/2)) ,
\]

\[
\kappa \partial_{\tau} S_1(x) = [J_0 S_1(x), S_0(x)] + [J_0 S_0(x), S_1(x)] ,
\]

\[
\kappa \partial_{\tau} S_2(x) = [J_2 \partial_x^2 S_0(x), S_0(x)] + [J_0 S_2(x), S_0(x)] + [J_0 S_0(x), S_2(x)] + [J_0 S_1(x), S_1(x)] .
\] (4.46)
Lax equations

In addition to the connection $A$ we define the connection $\kappa \partial_x + M$. Thus, we consider the pair of operators

\[
\begin{cases}
\kappa \partial_w + k \partial_x + L, \\
\kappa \partial_x + M.
\end{cases}
\]

Proposition 4.1 The Lax equation

\[
\left[ \kappa \partial_x + M, \kappa \partial_w + k \partial_x + L \right] = 0 \quad (4.47)
\]

is equivalent to the equations of motion (4.41)-(4.43) for

\[
M(x, w) = \frac{\kappa}{N} \partial_x \partial(w|\tau) + \sum_{n \in \mathbb{Z}} \sum_{\alpha \in \tilde{\mathbb{Z}}(2)} M_n^\alpha(w) T_n^\alpha, \quad (4.48)
\]

where

\[
M_n^\alpha = S_n^\alpha Y_n^\alpha, \quad Y_n^\alpha = \frac{1}{2\pi i} e\left(-\frac{w\alpha}{N}\right) f\left(-\alpha + \bar{k}n, w\right),
\]

where $f(A.9)$.

Proof.

From (4.47) we have

\[
\begin{align*}
\alpha &= \kappa \partial_x L_n^\alpha - (k \partial_w + kn) M_n^\alpha + \frac{\kappa}{N} \partial_x \partial(w|\tau) - \frac{\kappa}{N} \partial_x \partial w \partial(w|\tau) + \\
&+ \sum_\gamma \sum_m M_n^\gamma L_n^{-\gamma - m} C(\gamma, \alpha) = 0, \quad (4.49)
\end{align*}
\]

where $L_n^\alpha = S_n^\alpha X_n^\alpha$ is defined in (4.33). First of all, notice that $c = 0$. Let us substitute $L_n^\alpha(w)$ and $M_n^\alpha(w)$ in (4.49). Taking into account the equation of motion (4.43) we find

\[
\begin{align*}
\alpha &= \kappa \partial_x L_n^\alpha = \kappa \partial_x S_n^\alpha X_n^\alpha(w, \bar{w}) + \kappa S_n^\alpha \partial_x X_n^\alpha(w, \bar{w}) = \\
&= \frac{1}{2} \kappa \partial_x S_n^\alpha X_n^\alpha(w, \bar{w}) + \left(-\frac{\alpha_2}{N} + \frac{kn}{2}\right) S_n^\alpha Y_n^\alpha(w, \bar{w}) + \\
&+ \frac{1}{2} S_n^\alpha e\left(-\frac{w\alpha_2}{N}\right) \kappa \partial_x \phi\left(-\alpha + \bar{k}n, w\right).
\end{align*}
\]

On the other hand

\[
\begin{align*}
b &= (k \partial_w + kn) M_n^\alpha = \\
&= \frac{1}{2} S_n^\alpha \left(-\frac{\alpha_2}{N} Y_n^\alpha(w, \bar{w}) + \frac{\kappa}{2\pi i} e\left(-\frac{w\alpha_2}{N}\right) \partial_w f\left(-\alpha + \bar{k}n, w\right)\right) + \\
&+ \frac{1}{2} kn S_n^\alpha Y_n^\alpha(w, \bar{w}).
\end{align*}
\]

Finally,

\[
\begin{align*}
\epsilon &= \sum_\gamma \sum_m M_n^\gamma L_n^{-\gamma - m} C(\gamma, \alpha) = \sum_\gamma \sum_m S_n^{-\gamma} S_{n+m}^{\alpha+\gamma} X_n^{-\gamma} Y_{n+m}^{\alpha+\gamma} C(\alpha, \gamma) = 
\end{align*}
\]
\[ e \left( - \frac{\mu_0 q}{N} \right) \sum_{\gamma} \sum_{m} S_{m}^{-\gamma} S_{n+m}^{\alpha+\gamma} \phi (\gamma \tau - \hat{k} m, w) f \left( - (\gamma + \alpha) \tau + \hat{k} (n + m), w \right) C (\alpha, \gamma) \]

where \[ J \]

is the famous Landau-Lifshitz equation [76].

The Heisenberg model (4.89). We define below the isomonodromic version of the Landau-Lifshitz equation is the existence of the isomonodromic version (4.46) of the former system while for the latter system is unknown. The rational limit of (4.51) in compare with the Landau-Lifshitz equation is the existence of the isomonodromic version (4.46) of the former system while for the latter system is unknown. The rational limit to integrable systems

In this way we come to the equations of motion (4.42).

Comparing the terms in these expressions find

\[ \frac{1 + 8}{2 + 5} = 0, \quad \frac{4 + 6}{2 + 7} = 0. \]

In this way we come to the equations of motion (4.42).

**Limit to integrable systems**

Applying the general construction we obtain the integrable equations from (4.41)

\[ \partial_t S^\alpha (x) = \sum_{\gamma} \left( E_2 (\tau, m, n) \partial_x \right) S^{-\gamma} S^{\alpha+\gamma} C (\alpha, \gamma), \quad \partial_t \hat{k} = k / \rho_0, \]

or in terms of the Fourier modes

\[ \partial_t S_n^\alpha = \sum_{\gamma} \sum_{m} S_{m}^{-\gamma} S_{n+m}^{\alpha+\gamma} E_2 (\gamma \tau + \hat{k} m) C (\alpha, \gamma). \]

These equations are the two-dimensional (nonlocal) version of the integrable Euler-Arnold SL(N, C)-top [68, 43]. As above consider the perturbation of these equation for small \( k \). We come to the system (4.51), where \( \kappa \partial_x \) is replaced with \( \partial_t \). Again for sl(2, C) we obtained

1) \( \partial_t S_0 (x) = [J_0 S_0 (x), S_0 (x)], \quad J_0 = (E_2 (\tau/2), E_2 ((1 + \tau)/2), E_2 (1/2)), \)
2) \( \partial_t S_1 (x) = [J_0 S_1 (x), S_0 (x)] + [J_0 S_0 (x), S_1 (x)], \)
3) \( \partial_t S_2 (x) = [J_2 \partial_x S_0 (x), S_0 (x)] + [J_0 S_2 (x), S_0 (x)] + [J_0 S_0 (x), S_2 (x)] + [J_0 S_1 (x), S_1 (x)], \)
4) \( \partial_t S_j (x) = [J_j \partial_x S_0 (x), S_0 (x)] + [J_{j-1} \partial_x S_1 (x), S_0 (x)] + \ldots + [J_0 S_0 (x), S_j (x)], \)

where \( J_j = 0 \) for \( j \) odd. There exists a local two-dimensional version of the SL(C, 2) Euler top. It is the famous Landau-Lifshitz equation [76]

\[ \partial_t S_0 (x) = [J_0 S_0 (x), S_0 (x)] + [S_0 (x), \partial^2 S_0], \]

The advantage of (4.51) in compare with the Landau-Lifshitz equation is the existence of the isomonodromic version [40] of the former system while for the latter system is unknown. The rational limit of the Landau-Lifshitz equation is the Heisenberg model. We define below the isomonodromic version of the Heisenberg model (1.80).

In the integrable case we come to the Lax equation (4.46). The linear system (2.28) assumes the form

\[ \begin{cases} (k \partial_x + L) Y = 0, \\ (\partial_t + M) Y = 0, \end{cases} \]

where \( L (x, z, \bar{z}) = \sum_{n \in \mathbb{Z}} \sum_{\alpha \in \mathbb{Z} (2)} L_\alpha^n (x, \bar{z}) T_\alpha^n, \)

\[ L_\alpha^n (x, \bar{z}) = S_\alpha^n X_\alpha^n, \quad X_\alpha^n = e \left( - \frac{\alpha z}{N} \right) \phi \left( - \alpha + t k n, x \right), \]

26
\( M(x, z, \bar{z}) = \sum_{n \in \mathbb{Z}} \sum_{\alpha \in \mathbb{Z}(n)} M^\alpha_n T^\alpha_n, \)

\[
M^\alpha_n = S^\alpha_n Y^\alpha_n, \quad Y^\alpha_n = \frac{1}{2\pi i} e\left( -\frac{2\alpha_2}{N} \right) f\left( -\alpha_\tau + t k n, z \right).
\]

The Lax equation assumes the form:

\[
\partial_t L - k \partial_z M + [M, L] = 0.
\]

It is a simplified version of Proposition 4.1.

### 4.4 Trigonometric and rational limits

Let \( \tau = \tau_1 + \tau_2 \) and \( \Im m \tau_2 \to +\infty \). Then using (A.27) we derive the trigonometric degeneration of the equations of motion (4.42).

\[
\kappa \partial_x S^\alpha(x) = \sum_{\gamma \in \mathbb{Z}(2)} \sum_{\gamma \neq \alpha} \left( \pi^2 \sin^{-2}\pi \left( -\gamma_\tau + \frac{k}{2\pi i} \partial_x \right) S^{-\gamma}(x) \right) S^{\alpha+\gamma}(x) C(\alpha, \gamma), \quad (4.52)
\]

where we preserved notation \( \tau \) for \( \tau_1 \). This equation is the compatibility condition for the isomonodromy problem

\[
\begin{aligned}
& (\kappa \partial_w + k \partial_x + L^{tr}(S, w)) \Psi(w) = 0, \\
& (\kappa \partial_\tau + M^{tr}(S, w)) \Psi(w) = 0,
\end{aligned}
\]

Let

\[
T^\alpha_{n,j} = e(nx + jy)T^\alpha,
\]

where \( T^\alpha \) (B.21). Then the trigonometric operators take the form

\[
L^{tr} = \sum_{n \in \mathbb{Z}} \sum_{\alpha \in \mathbb{Z}(n), j \in \mathbb{Z}} L^{tr,\alpha}_{n,j} T^\alpha_n, \quad M^{tr} = \sum_{n \in \mathbb{Z}} \sum_{\alpha \in \mathbb{Z}(n), j \in \mathbb{Z}} M^{tr,\alpha}_{n,j} T^\alpha_n,
\]

\[
L^{tr,\alpha}_{n,j} = S^\alpha_{n,j} X^\alpha_{n,j} X^\alpha_{n,j} = e\left( -\alpha_\tau \right) \pi \left( \cot \pi (-\alpha_\tau - j\tau + \bar{k}n) + \cot (\pi w) \right),
\]

\[
M^{tr,\alpha}_{n,j} = S^\alpha_{n,j} Y^\alpha_{n,j} Y^\alpha_{n,j} = -e\left( -\alpha_\tau \right) \pi^2 \sin^{-2}\pi \left( -\alpha_\tau - j\tau + \bar{k}n \right),
\]

where we use the forms of \( \partial^{tr} \) and \( f^{tr} \) (A.25), (A.26). The compatibility of (4.53) can be checked directly. Rewrite (4.52) in the form (4.44). Define the second order difference operator

\[
D^{tr}_x(\tau) F^\alpha(x) = \frac{1}{4} \left( e(-\alpha_\tau) F^\alpha(x + \bar{k}) + e(\alpha_\tau) F^\alpha(x - \bar{k}) - F^\alpha(x) \right).
\]

It plays the role of the inertia operator. Then (4.52) assumes the form of the difference in \( x \) equation

\[
K \partial_\tau D^{tr,\alpha}_x(\tau) F^\alpha(x) = \sum_{\gamma \in \mathbb{Z}(2), \gamma \neq \alpha} F^{-\gamma}(x) D^{tr,\alpha}_x(\tau) F^{\alpha+\gamma}(x) C(\alpha, \gamma).
\]

For the \( \text{sl}(2, \mathbb{C}) \) case we have three fields (see (B.25)) arranged in the three vector

\[
\vec{F} = (F^1 = F^{(1,0)}, F^1 = F^{(1,1)}, F^3 = F^{(0,1)}),
\]

and the vector \( \vec{D^{tr}} \vec{F} = (D^{tr,\alpha}_x(\tau) F^\alpha) \). Then the \( \text{sl}(2, \mathbb{C}) \) commutator becomes the wedge product and (4.55) is simplified to

\[
K \partial_\tau (\vec{D^{tr}} \vec{F})(x, \tau) = \vec{F}(x, \tau) \wedge (\vec{D^{tr}} \vec{F})(x, \tau)
\]

(4.56)
In the rational case we come to the equations

$$\kappa \partial_t S^\alpha(x) = \sum_{\gamma \in \mathbb{Z}^{(2)}, \gamma \neq \alpha} \left( -\gamma + \frac{k}{2 \pi i} \partial_x \right) \left( -2 S^{-\gamma}(x) \right) S^{\alpha+\gamma}(x) C(\alpha, \gamma),$$

(4.57)

The corresponding isomonodromy problem is defined by the operators

$$L_n^{\alpha} = S_n^\alpha X_n^\alpha, \quad X_n^\alpha = e\left( -\frac{\omega_\alpha}{N} \right) \left( \frac{1}{\alpha - \alpha + k_n + \frac{1}{w}} \right),$$

$$M_n^{\alpha} = S_n^\alpha Y_n^\alpha, \quad Y_n^\alpha = -e\left( -\frac{\omega_\alpha}{N} \right) \left( -\alpha + k_n - 2 \right),$$

The rational inertia operator takes the form:

$$D_x^{\alpha,\alpha}(x) F^\alpha(x) = \left( -\alpha + k \right)^2 F^\alpha(x).$$

(4.58)

Then (4.57) can be rewritten in terms of the angular velocities

$$\kappa \partial_t D_x^{\alpha,\alpha}(\tau) F^\alpha(x) = \sum_{\gamma \in \mathbb{Z}^{(2)}, \gamma \neq \alpha} F^{-\gamma}(x) D_x^{\alpha,\alpha}(\tau) F^{\alpha+\gamma}(x) C(\alpha, \gamma).$$

(4.59)

For sl(2, C) case we introduce as above two three-dimensional vectors \( \vec{F} = (F^\alpha) \) and \( \vec{D} \). Then as in the trigonometric case we have

$$\kappa \partial_t \vec{D}^{-1}(x, \tau) = \vec{F}(x) \wedge \vec{D}^{-1}(x, \tau)$$

(4.60)

### 4.4.1 Special trigonometric and rational limits for the sl(2, C) case

In the sl(2, C) case another interesting limit can be described explicitly using the results of [78]. We consider sl(2, C) Lax operators in the basis of the sigma matrices

$$L = \sum_{n \in \mathbb{Z}} e(n x) \left( \begin{array}{ccc}
S_1^3 \varphi_{10} (k w) & S_1^1 \varphi_{01} (\bar{k} w) & S_1^1 \varphi_{01} (\bar{k} w) - i S_1^2 \varphi_{11} (\bar{k} w) \\
S_1^1 \varphi_{01} (k w) + i S_1^2 \varphi_{10} (k w) & -S_1^3 \varphi_{10} (k w) \\
& & \end{array} \right) + \frac{\kappa}{2} E_1 (w | r) I d,$$

(4.61)

$$M = \frac{1}{2 \pi i} \sum_{n \in \mathbb{Z}} e(n x) \left( \begin{array}{ccc}
S_1^3 f_{10} (k w) & S_1^1 f_{10} (\bar{k} w) & S_1^1 f_{10} (\bar{k} w) - i S_1^2 f_{11} (k w) \\
S_1^1 f_{10} (\bar{k} w) + i S_1^2 f_{10} (k w) & -S_1^3 f_{10} (k w) \\
& & \end{array} \right) + \frac{\kappa}{2} \partial_r (w | r) I d,$$

(4.62)

where

$$\varphi_{10} (k w) = \varphi \left( -\frac{1}{2} + \bar{k} w \right), \quad \varphi_{01} (k w) = \varphi \left( -\frac{\tau}{2} + k w \right),$$

$$\varphi_{11} (k w) = \varphi \left( -\frac{1 + \tau}{2} + \bar{k} w \right),$$

and functions \( f_{ij} \) are related to the function \( f \) in the same way. Hamiltonian for the sl(2, C) case can be written as

$$H = k \lambda - \frac{1}{4 \pi i} \sum_{\alpha = 1, 2, 3} \sum_{n \in \mathbb{Z}} S_1^\alpha S_2^\alpha E_2 \left( -\alpha r + \bar{k} n \right).$$

(4.63)

The Poisson brackets are of the form

$$\{ S^i_n, S^j_m \} = 2 \epsilon_{ijk} S^k_{n+m}, \quad \{ \lambda, \bar{k} \} = 1.$$  

(4.64)
In what follows we are going to use the Chevalley basis:

\[ S_n^+ = \frac{1}{2} \left( S_n^1 + i S_n^2 \right), \quad S_n^- = \frac{1}{2} \left( S_n^1 - i S_n^2 \right), \]

\[ \{ S_n^3, S_m^\pm \} = \pm 2 S_{n+m}^\pm, \quad \{ S_n^+, S_m^- \} = S_{n+m}^3. \]  

(4.65)

One can also write down the Poisson brackets in terms of the field variables:

\[ \{ S^3(x), S^\pm(y) \} = \pm 2 S^\pm(y) \delta(x-y), \quad \{ S^+(x), S^-(y) \} = S^3(y) \delta(x-y). \]

To perform the non-autonomous trigonometric limit, we decompose the parameter of the elliptic curve as

\[ \tau = \tau_1 + \tau_2, \]

where \( \tau_1 \) plays the role of time of the limiting system and \( \tau_2 \) gives the trigonometric limit \( \Im m \tau_2 \to +\infty \). Then we introduce time-independent change of variables

\[ S_n^3 \to S_n^3, \quad S_n^+ \to q_2^{1/4} S_n^+, \quad S_n^- \to q_2^{-1/4} S_n^-, \]

where \( q_2 \equiv e(\tau_2) \). Notice that the Poisson structure (4.65) is preserved under this transformation.

The trigonometric Lax operator can be derived as the limit of the gauge transformed elliptic Lax operator

\[ L^T = \lim_{q_2 \to 0} A^T(q_2) L A^T(q_2)^{-1}, \]

where

\[ A^T(q_2) = \begin{pmatrix} q_2^{1/8} & 0 \\ 0 & q_2^{-1/8} \end{pmatrix}. \]

This gives

\[ L^T = \pi \sum_{n \in \mathbb{Z}} \left( \frac{(\cot(\pi w) - \tan(\pi \bar{k} n)) S_n^3}{\sin(\pi w)} S_n^+ + 8 \sqrt{q_1} \sin \left( \pi w + 2 \pi \bar{k} n \right) S_n^- - \left( \cot(\pi w) - \tan(\pi \bar{k} n) \right) S_n^3 \right) e(n x) + \frac{\pi \kappa}{2} \cot(\pi w) \text{Id}, \]

where \( q_1 \equiv e(\tau_1) \). Trigonometric Hamiltonian is defined via

\[ H^T = \lim_{q_2 \to 0} H = k \lambda + \frac{i \pi}{4} \sum_{n \in \mathbb{Z}} \left( \frac{S_n^3 S_{-n}^3}{\cos^2 \left( \pi \bar{k} n \right)} - 16 \sqrt{q_1} \cos \left( 2 \pi \bar{k} n \right) S_n^- S_{-n}^- \right), \]

or in terms of the field variables

\[ H^T = k \lambda + \frac{i \pi}{4} \oint \left( S^3(x) \cos^2 \left( \frac{\bar{k}}{2} \partial_x \right) S^3(x) - 16 \sqrt{q_1} S^- S^-(x) \cos \left( \bar{k} \partial_x \right) S^-(x) \right). \]

Applying the same gauge transformation as for the \( L \)-operator, we come to the trigonometric \( M \)-operator

\[ M^T = \lim_{q_2 \to 0} A^T(q_2) M A^T(q_2)^{-1} = \frac{\pi}{2 \Delta} \sum_{n \in \mathbb{Z}} \left( \frac{-1}{\cos^2 \left( \pi \bar{k} n \right)} S_n^3 \frac{1}{16 \sqrt{q_1} \cos \left( \pi w + 2 \pi \bar{k} n \right) S_n^-} \right) e(n x). \]

The Lax equation

\[ \kappa \partial_{\tau_1} L^T - \kappa \partial_w M^T - k \partial_x M^T = [L^T, M^T]. \]
in the trigonometric case is equivalent to the Hamiltonian equations of motion

$$\kappa \partial_{\tau_1} \bar{k} = k, \quad \kappa \partial_{\tau_1} S^3_n = \frac{16\pi}{i} \sqrt{q_1} \sum_{m \in \mathbb{Z}} \cos (2\pi km) S^-_{n-m} S^+_m,$$

$$\kappa \partial_{\tau_1} S^+_n = \pi i \sum_{m \in \mathbb{Z}} \left( \frac{S^+_m S^3_{n-m}}{\cos^2 (\pi km)} - 8\sqrt{q_1} \cos (2\pi km) S^3_{n-m} S^-_m \right),$$

$$\kappa \partial_{\tau_1} S^-_n = \frac{\pi}{i} \sum_{m \in \mathbb{Z}} \frac{S^-_{n-m} S^3_m}{\cos^2 (\pi km)}.$$

Using the field variables, one can rewrite the equations of motion as follows:

$$\kappa \partial_{\tau_1} \bar{k} = k, \quad \kappa \partial_{\tau_1} S^3(x) = \frac{16\pi}{i} \sqrt{q_1} S^-(x) \cos (i\bar{k} \partial_x) S^-(x),$$

$$\kappa \partial_{\tau_1} S^+(x) = \pi i \left( \left( \frac{S^+(x) \cos^2 \left( \frac{\nu}{2} \partial_x \right) S^3(x) - 8\sqrt{q_1} S^3(x) \cos \left( i\bar{k} \partial_x \right) S^-(x) \right) \right),$$

$$\kappa \partial_{\tau_1} S^-(x) = \frac{\pi}{i} S^-(x) \cos^2 \left( \frac{i\bar{k}}{2} \partial_x \right) S^3(x).$$

The zero Fourier modes are related to the canonical coordinates $u, v$ by the so-called bosonization formulas:

$$S^0_0 = -\frac{v}{\pi \tan (2\pi u)} - \frac{\nu}{4\pi \sin^2 (2\pi u)}, \quad S^-_0 = -\frac{v}{4\pi \sin (2\pi u)} - \frac{\nu \cos (2\pi u)}{4 \sin^2 (2\pi u)},$$

$$S^+_0 = \frac{v \cos^2 (2\pi u)}{\pi \sin (2\pi u)} + \frac{\nu \cos (2\pi u) (1 + \sin^2 (2\pi u))}{\sin^2 (2\pi u)}.$$

Thus, in the zero modes we obtain a non-autonomous version of the trigonometric Calogero-Moser system with the following Hamiltonian:

$$H^T_0 = k\lambda + \frac{i}{4\pi} \left( \frac{v^2}{\sin^2 (2\pi u)} (\cos^2 (2\pi u) - \sqrt{q_1}) + \frac{\pi^2 \nu^2}{\sin^4 (2\pi u)} (1 - \sqrt{q_1} \cos^2 (2\pi u)) \right) +$$

$$+ \frac{\nu}{2} \frac{v \cos (2\pi u)}{\sin^2 (2\pi u)} \cot (2\pi u) \left( 1 - \sqrt{q_1} \right).$$

The equations of motion for this system are equivalent to the following second-order differential equation:

$$\frac{d^2 u}{d\tau_1^2} (\cos^2 (2\pi u) - \sqrt{q_1}) + 2\pi \left( \frac{d u}{d\tau_1} \right)^2 \cot (2\pi u) (1 - \sqrt{q_1}) +$$

$$+ \pi i \frac{d u}{d\tau_1} \sqrt{q_1} + \frac{\pi}{2} \nu (v + 1) \cot (2\pi u) \sqrt{q_1} = 0.$$

The rational limit can be described in terms of the small parameter $y \to 0$ and the following transformations:

$$S^3_n \to S^3_n + \frac{4}{y^2} S^-_n, \quad S^-_n \to \frac{S^-_n}{y^2}, \quad S^+_n \to y S^+_n - 2S^3_n - \frac{4}{y} S^-_n,$$

$$w \to \frac{y}{\pi} w, \quad \bar{k} \to \frac{y}{\pi} \bar{k}, \quad \tau_1 \to \frac{y^2}{\pi} t_1, \quad k \to \frac{y}{\pi} k, \quad \lambda \to \frac{\pi}{y} \lambda.$$

Again, the Poisson structure (4.60) remains unchanged along with the canonical bracket

$$\{ \lambda, \bar{k} \} = 1.$$

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The Hamiltonian for the zero Fourier modes can be written in canonical coordinates
once the solution of the form

\[ H = \left( \begin{array}{cc} 1 - \frac{2}{w}S_n^3 - 2(w + 2kn)S_n^- & \frac{2}{w}S_n^- \\ \frac{2}{w}S_n^+ - 2(w + 2kn)S_n^+ + \varphi_R(kn, t_1)S_n^- - \frac{1}{w}S_n^3 + 2(w + 2kn)S_n^- & \end{array} \right) e(nx) + \kappa \frac{\kappa}{2w} Id, \]

where

\[ \varphi_R(kn, t_1) = 8(w + 2kn)t_1 - 2(8k^3n^3 + 8k^2n^2w + 4knw^2 + w^3). \]

The rational Hamiltonian assumes the following form:

\[ H^R = \lim_{y \to 0} \frac{y^2}{1 + \pi} H^T = -\pi i \kappa \lambda + \frac{2}{w} \sum_{n \in \mathbb{Z}} \left( S_n^3 S_n^- + (5k^2n^2 - 2t_1) S_n^- S_n^- \right), \]

or in terms of the field variables

\[ H^R = -\pi i \kappa \lambda + 2 \int \left( S^3(x) S^-(x) - S^-(x) \right) \left( 2t_1 + \frac{5k^2}{4\pi^2} \partial_x^2 \right) \]

The rational \( M \) operator is related to the trigonometric one as

\[ M^R = \lim_{y \to 0} \frac{y^2}{1 + \pi} A^R(y) M^T A^R(y)^{-1}, \]

which gives

\[ M^R = 2 \sum_{n \in \mathbb{Z}} \left( \begin{array}{cc} S_n^3 - 2(2t_1 - w^2 - 4wk\nu - 6k^2n^2) S_n^- & 0 \\ 0 & -S_n^3 \end{array} \right) e(nx). \]

The Lax equation

\[ \kappa \partial_t L^R - \kappa \partial_u M^R - k \partial_x M^R = [L^R, M^R] \]

is equivalent to the equations of motion

\[ \begin{cases} \kappa \partial_t S^3(x) = 4S^-(x)S^3(x) - 8S^-(x) \left( 2t_1 + \frac{5k^2}{4\pi^2} \partial_x^2 \right) S^-(x), \\ \kappa \partial_t S^+(x) = 4S^+(x)S^+(x) - 2(S^3(x))^2 + 4S^3(x) \left( 2t_1 + \frac{5k^2}{4\pi^2} \partial_x^2 \right) S^-(x), \\ \kappa \partial_t S^-(x) = -4(S^-(x))^2, \quad \kappa \partial_t k = -\pi i \kappa. \end{cases} \]

The Hamiltonian for the zero Fourier modes can be written in canonical coordinates \( u, v \) as follows:

\[ H^0_1 = v^2 - \frac{t_1}{w^3} (v + uw) v - \frac{\nu^2}{4u^2} \left( 1 + \frac{t_1}{u^2} \right), \]

where \( \nu = const \) and we use the following bosonization formulas:

\[ S_0^3 = uw - \frac{\nu}{2}, \quad S_0^+ = -\frac{w^3v}{2} + \frac{3\nu u^2}{4}, \quad S_0^- = \frac{v}{2u} + \frac{\nu}{4u^2}. \]

The equations of motion can be represented as the second-order differential equation

\[ \frac{d^2u}{dt_1^2} (u^2 - t_1) + \frac{du}{dt_1} - \left( \frac{du}{dt_1} \right)^2 \frac{t_1}{u} + \frac{\nu}{u} (\nu + 1) = 0 \]

with the solution of the form

\[ 4u^2 = -C_1 - 4C_1 \nu (\nu + 1) + 16C_1^2 + 4t_1 + 16C_1 t_1 + 4 \frac{t_1^2}{C_1}. \]
4.4.2 Scaling limit

In this subsection we construct a limiting procedure based on generalizations [3, 4] of the Inozemtsev limit [36]. To define the procedure we decompose the parameter $\tau$ of the elliptic curve:

$$\tau = \tau_1 + \tau_2,$$

where $\tau_1$ plays the role of time of the limiting system and $\tau_2$ gives the trigonometric limit $\Im m\tau_2 \to +\infty$. Before taking the trigonometric limit we shift the spectral parameter and coordinate on $S^1$

$$w = \tilde{w} + \frac{\tau}{2}, \quad x = \tilde{x} + \frac{\bar{\kappa}}{2}$$  \hspace{1cm} \text{(4.67)}

and scale the coordinates in the following way:

$$S_n^{\alpha_1, \alpha_2} = \tilde{S}_n^{\alpha_1, \alpha_2} q_2^{g(\alpha_2)}, \quad q_2 \equiv e(\tau_2), \quad g(\alpha_2) = \frac{1 - \delta_{\alpha_2, 0}}{2N}. \quad \text{(4.68)}$$

After scalings \textbf{(4.68)} we obtain the contraction of the Poisson algebra in the limit $\Im m\tau_2 \to +\infty$

$$\left\{ \tilde{S}_n^{\alpha_1, \alpha_2}, \tilde{S}_m^{\beta_1, \beta_2} \right\} = \frac{N}{\pi} \sin \left( \frac{\pi}{N} (\alpha_1 \beta_2 - \alpha_2 \beta_1) \right) \tilde{S}_{n+m}^{\alpha_1 + \beta_1, \alpha_2 + \beta_2} q_2^{g(\alpha_2) + g(\beta_2) - g(\alpha_2 + \beta_2)},$$

where $\tilde{S}_\alpha \equiv \tilde{S}_{\alpha_1, \alpha_2}, \alpha \in \tilde{\mathbb{Z}}^{(2)}$. Scaled coordinates $\tilde{S}_\alpha$ with the Poisson brackets form the algebra in the limit of $\Im m\tau_2 \to +\infty$ provided that

$$\forall \alpha_2, \beta_2 \in \mathbb{Z} : \quad g(\alpha_2) + g(\beta_2) - g(\alpha_2 + \beta_2) \geq 0. \quad \text{(4.69)}$$

For $g(\alpha_2) = (1 - \delta_{\alpha_2, 0})/(2N)$ the condition \textbf{(4.69)} is trivial and we can write down all nonzero brackets corresponding to the equality in \textbf{(4.69)}

$$\left\{ \tilde{S}_n^{\alpha_1, \beta}, \tilde{S}_m^{\beta_1, \beta_2} \right\} = \frac{N}{\pi} \sin \left( \frac{\pi}{N} \alpha_1 \beta_2 \right) \tilde{S}_{n+m}^{\alpha_1 + \beta_1, \beta_2}. \quad \text{(4.70)}$$

We start from the $\text{sl}(2, \mathbb{C})$ case. The scalings of coordinates in the basis of the Pauli matrices can be written as follows:

$$S_n^1 \to S_n^1 q_2^{-1/4}, \quad S_n^2 \to S_n^2 q_2^{-1/4}, \quad S_n^3 \to S_n^3. \quad \text{(4.71)}$$

Then the Lax operator \textbf{(4.61)} acquires the following form in the limit $\Im m\tau_2 \to +\infty$:

$$L' = 4\pi \sum_{n \in \mathbb{Z}} L_n' e(n\tilde{x}),$$

where $L_n'$ is $2 \times 2$ matrix of the form

$$L_n' = \begin{pmatrix} -\frac{t S_n^3}{4 \cos(\pi k N)} & \left( \sin(\pi \tilde{w} + \pi \tilde{k} n) S_n^1 - \cos(\pi \tilde{w} + \pi \tilde{k} n) S_n^2 \right) q_1^{1/4} \\ \left( \sin(\pi \tilde{w} + \pi \tilde{k} n) S_n^1 + \cos(\pi \tilde{w} + \pi \tilde{k} n) S_n^2 \right) q_1^{1/4} & \frac{t S_n^3}{4 \cos(\pi k N)} \end{pmatrix}$$

and $q_1 \equiv e(\tau_1)$. For the limit of the Hamiltonian \textbf{(4.63)} we have

$$H' = k \lambda + \frac{\sqrt{8}}{4} \sum_{n \in \mathbb{Z}} \left( \frac{S_n^3 S_{-n}^3}{\cos^2(\pi k N)} - 8 \sqrt{q_1} \cos(2\pi \tilde{k} n) \left( S_n^1 S_{-n}^1 - S_n^2 S_{-n}^2 \right) \right). \quad \text{(4.72)}$$

The Poisson structure \textbf{(4.64)} transforms as well. In addition to the other zero brackets we get

$$\{ S_n^1, S_m^2 \} = 0, \quad \{ S_n^1, S_m^3 \} = -2s S_n^2 S_{n+m}^2, \quad \{ S_n^2, S_m^3 \} = 2s S_n^1 S_{n+m}^2, \quad \{ \lambda, \tilde{k} \} = 1,$$
or, in terms of the field variables
\[
\{S^1(x), S^2(y)\} = 0, \quad \{S^1(x), S^3(y)\} = -2iS^2(y)\delta(x - y),
\]
\[
\{S^2(x), S^3(y)\} = 2iS^1(y)\delta(x - y).
\]
The equations of motion
\[
\kappa_2 \dot{k} = k, \quad \kappa_3 \dot{S} = -8\pi\sqrt{q_1} \sum_{m \in \mathbb{Z}} \cos (2\pi km) \left(S^1_{n-m} S^2_m + S^1_m S^2_{n-m}\right), \quad (4.73)
\]
\[
\kappa_3 \dot{S} = -\pi \sum_{m \in \mathbb{Z}} \frac{S^2_{n-m} S^3_m}{\cos^2 (\pi km)}, \quad \kappa_2 \dot{S} = \pi \sum_{m \in \mathbb{Z}} \frac{S^1_{n-m} S^3_m}{\cos^2 (\pi km)} \quad (4.74)
\]
can be represented in the form of the Lax equation
\[
\kappa \partial_x L^I - \kappa \partial_{\bar{w}} M^I - k \partial_{\bar{z}} M^I = [L^I, M^I],
\]
with
\[
M^I = \pi^2 \sum_{n \in \mathbb{Z}} \left(-\frac{S^3_n}{\cos^2 (\pi kn)} + 4e \left(-\frac{\bar{w}}{2} - \frac{\bar{k}}{2}\right) \left(S^1_n + iS^2_n\right) q_1^{1/4} \frac{S^3_n}{\cos^2 (\pi kn)}\right) e(n\bar{z}).
\]
Equations (4.74) can be simplified by the following change of coordinate:
\[
S^3_n = \cos^2 (\pi kn) \tilde{S}^3_n.
\]
Then in terms of the field variables the Hamiltonian (4.72) and equations of motion (4.73), (4.74) assume the following form:
\[
H^I = k\lambda - 2i\pi\sqrt{q_1} \int \left(S^1(x) \cos (i\bar{k}\partial_{\bar{z}}) S^1(x) - S^2(x) \cos (i\bar{k}\partial_{\bar{z}}) S^2(x)\right) +
\]
\[
+ \frac{\pi}{4} \int \tilde{S}^3(x) \cos^2 \left(\frac{i\bar{k}}{2}\partial_{\bar{z}}\right) \tilde{S}^3(x),
\]
\[
\kappa_2 \dot{S} = -\pi S^2_{n-m} \tilde{S}^3_m, \quad \kappa_3 \dot{S} = \pi S^1_{n-m} \tilde{S}^3_m.
\]
The Hamiltonian for the zero Fourier modes can be written in canonical coordinates \(u, v\):
\[
H^I_0 = k\lambda - \frac{i}{2\pi} \left(\frac{\bar{u}^2}{2} + 4M^2\pi^2 \sqrt{q_1} \cos (4\pi u)\right),
\]
where we use the bosonization formulas of the form
\[
S^1_0 = M \cos (2\pi u), \quad S^2_0 = -M \sin (2\pi u), \quad S^3_0 = \frac{\bar{u}}{\pi}.
\]
Equations of motion for the coordinates \(u, v\) in the form of the second-order differential equation
\[
\frac{d^2u}{dt^2} = -4M^2\pi \sqrt{q_1} \sin (4\pi u)
\]
are equivalent to a particular case of the Painlevé III equation:
\[
\frac{1}{t} \frac{d}{dt} \frac{d(4\pi u)}{dt} = \sin (4\pi u),
\]

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where \( t = 8Me(\tau_i/4) \). The Hamiltonian of the limiting system in the case of \( \text{sl}(N > 2, C) \) has the following form:

\[
H_{\tau_1} = k\lambda + \pi^2 \sum_{\alpha_1 \neq 0, n} \frac{\widetilde{S}^{\alpha_1,0}_{n}}{\sin^2 \left( \pi \left( \frac{\alpha_1}{N} - kn \right) \right)} - 8\pi^2 q_1^\frac{1}{2} \sum_{\alpha_1, n} e \left( \frac{\alpha_1}{N} - \tilde{k}n \right) \widetilde{S}^{\alpha_1,1}_{n-1} \widetilde{S}^{-\alpha_1,1}_{n-1}. \tag{4.75}
\]

It’s worth noting that the Hamiltonian \((4.75)\) contains only coordinates of the form \( S^{\alpha_1,0}_{n} \), \( \alpha_2 = 0, \pm 1 \). Other elements have simple linear dynamics which can be integrated once we know the dynamics of the coordinates included in the Hamiltonian. The equations of motion of the limiting system can be presented as the Hamiltonian equations with respect to the brackets \((4.70)\):

\[
\kappa \partial_\tau \widetilde{S}^{\gamma_1,0}_m = 8\pi N q_1^\frac{1}{2} \sum_{\alpha_1, n} e \left( \frac{\alpha_1}{N} - \tilde{k}n \right) \sin \left( \frac{\pi \gamma_1}{N} \right) \left( \widetilde{S}^{\alpha_1+\gamma_1,1}_{n+m} \widetilde{S}^{-\alpha_1,1}_{n} - \widetilde{S}^{\alpha_1,1}_{n} \widetilde{S}^{-\alpha_1+\gamma_1,1}_{n-m} \right), \tag{4.76a}
\]

\[
\kappa \partial_\tau \widetilde{S}^{\gamma_1,1}_m = 2\pi N \sum_{\alpha_1, n} \sin \left( \frac{\pi \alpha_1}{N} \right) \sin^2 \left( \pi \left( \frac{\alpha_1}{N} - kn \right) \right) \widetilde{S}^{\alpha_1+\gamma_1,1}_{n+m} \widetilde{S}^{-\alpha_1,0}_{n}, \tag{4.76b}
\]

\[
\kappa \partial_\tau \widetilde{S}^{-\gamma_1,1}_m = -2\pi N \sum_{\alpha_1, n} \sin \left( \frac{\pi \alpha_1}{N} \right) \sin^2 \left( \pi \left( \frac{\alpha_1}{N} - kn \right) \right) \widetilde{S}^{\alpha_1,1}_{n+m} \widetilde{S}^{-\alpha_1+\gamma_1,1}_{n-m}. \tag{4.76c}
\]

The equations of motion of the original system admit the Lax representation \((4.47)\) which is equivalent to

\[
\kappa \partial_\tau L - \kappa \partial_x M - \kappa \partial_z M = [L, M]. \tag{4.77}
\]

Since substitutions \((4.67)\) are time-dependent, equation \((4.77)\) transforms as follows

\[
\kappa \partial_\tau L - \kappa \partial_x (M + \frac{1}{2} L) - \kappa \partial_z (M + \frac{1}{2} L) = [L, M],
\]

where \( L = L(S(\bar{x} + \bar{k}/2), \bar{w} + \tau/2, \tau), M = M(S(\bar{x} + \bar{k}/2), \bar{w} + \tau/2, \tau) \), and we use the following properties:

\[
(\partial_x M)_{\tau,\bar{w}} = (\partial_x L)_{\tau,\bar{w}}, \quad (\partial_x L)_{\tau,\bar{w}} = (\partial_x L)_{\tau,\bar{w}},
\]

\[
(\partial_x L)_{\bar{w},\bar{x}} = (\partial_x L)_{w,\bar{x}} + \frac{1}{2} (\partial_x L)_{\tau,\bar{w}} + \frac{1}{4\pi i} (\partial_x L)_{\tau,\bar{w}} \partial_\tau \bar{k}.
\]

Thus, the Lax pair of the limiting system is defined via

\[
\tilde{L} = \lim_{\Im \tau_2 \to +\infty} L, \quad \tilde{M} = \lim_{\Im \tau_2 \to +\infty} \left( M + \frac{1}{2} L \right) \tag{4.79}
\]

and the Lax equation assumes the form

\[
\kappa \partial_\tau \tilde{L} - \kappa \partial_\bar{w} \tilde{M} - \kappa \partial_\bar{z} \tilde{M} = [\tilde{L}, \tilde{M}],
\]

where

\[
\tilde{L} = -\pi \sum_{n, \in \mathbb{Z}} \sum_{\alpha_1 = 1}^{N-1} \frac{\widetilde{S}^{\alpha_1,0}_{n}}{\sin \left( \pi \left( \frac{\alpha_1}{N} - kn \right) \right)} e(\bar{n} \bar{x})T^{\alpha_1,0} + 
\]

\[
+ 2\pi i \sum_{n, \in \mathbb{Z}} \sum_{\alpha_1 = 1}^{N-1} \frac{\widetilde{S}^{\alpha_1,1}_{n}}{\sin \left( \pi \left( \frac{\alpha_1}{N} - \frac{\bar{w}}{N} - \frac{\bar{k}n}{2} \right) \right)} \frac{q_1^\frac{1}{2}}{q_1^\frac{1}{2}} e(\bar{n} \bar{x})T^{\alpha_1,1} - 
\]

\[
- 2\pi i \sum_{n, \in \mathbb{Z}} \sum_{\alpha_1 = 1}^{N-1} \frac{\widetilde{S}^{\alpha_1,1}_{n}}{\sin \left( \pi \left( \frac{\alpha_1}{N} - \frac{\bar{w}}{N} + \frac{\bar{k}n}{2} \right) \right)} \frac{q_1^\frac{1}{2}}{q_1^\frac{1}{2}} e(\bar{n} \bar{x})T^{\alpha_1,1},
\]

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\[ \tilde{M} = \frac{\pi}{4} \sum_{n \in \mathbb{Z}} \sum_{n_{a_1}=1}^{N-1} \widetilde{S}^{0}_{n_{a_1}} e^{\left( \frac{kn}{2} \right)} \frac{e^{\left( \frac{\alpha_{N} - \frac{kn}{2}}{N} \right)} - e^{\left( \frac{\bar{\alpha} - \frac{kn}{2}}{N} \right)}}{\sin^{2} \left( \pi \left( \frac{\alpha_{N}}{N} - \frac{kn}{2} \right) \right)} e^{(n \bar{x}) T^{0_{a_1}}, -} \]

\[ -4 \pi \sum_{n \in \mathbb{Z}} \sum_{n_{a_1}=0}^{N-1} \widetilde{S}^{1}_{n_{a_1}} e^{\left( \frac{\alpha_{N}}{N} - \frac{\bar{w}}{N} - e^{\left( \frac{\bar{k}n}{2} \right)} \right)} q_1^{\frac{1}{4}} e^{(n \bar{x}) T^{a_{1}, 1}} - \]

\[ -4 \pi \sum_{n \in \mathbb{Z}} \sum_{n_{a_1}=0}^{N-1} \widetilde{S}^{1,-1}_{n_{a_1}} e^{\left( \frac{\bar{k}n}{2} + \frac{\bar{w}}{N} \right)} q_1^{\frac{1}{4}} e^{(n \bar{x}) T^{a_{1}, -1}}. \]

### 4.4.3 1+1 Heisenberg model and its isomonodromic version

Here we consider the Lax equation for the non-autonomous 1+1 Heisenberg model that was derived by direct calculations without applying the symplectic reduction method used in general cases. It will be interesting to derived this system within our framework.

The complex version of 1+1 Heisenberg model

\[ \partial_{t} S + \frac{k^2}{4 \lambda^2} [S, S_{xx}] = 0, \quad S_{xx} = \partial_{x}^{2} S \] (4.80)

is written in terms of the matrix-valued periodic function taking values in the loop coalgebra \( S(x) \in \hat{\mathfrak{sl}}(2, \mathbb{C}), \quad S(x + 2\pi) = S(x), \quad x \in \mathbb{S}^1 \)

with \( x \)-independent eigenvalues

\[ S^2 = \lambda^2 1, \quad \text{Spec}(S) = (\lambda, -\lambda), \quad \partial_{x} \lambda = 0. \] (4.81)

The later also means that

\[ SS_x + S_x S = 0. \] (4.82)

This property allows to represent (4.80) in the form of the zero-curvature equation

\[ \partial_{t} L^H - k \partial_{x} M^H = [L^H, M^H] \] (4.83)

with rational L-A pair depending on the spectral parameter \( z \) - local coordinate on \( \mathbb{C}P^1 \):

\[ \left\{ \begin{array}{ll} L^H = \frac{1}{z} S, \\ M^H = \frac{1}{z} h^H + \frac{1}{z^2} S, \quad h^H = -\frac{k}{4 \lambda^2} [S, S_x]. \end{array} \right. \] (4.84)

The local isomonodromic version of the Heisenberg model appears from the natural generalization of (4.83)

\[ \partial_{t} L - k_1 \partial_{x} M - k_2 \partial_{x} M = [L, M]. \] (4.85)

The later equation can be treated as the compatibility condition for the linear problem:

\[ \left\{ \begin{array}{ll} (k_1 \partial_{x} + k_2 \partial_{x}) L = L \Psi, \\ (\partial_{t} + M) \Psi = 0. \end{array} \right. \] (4.86)

**Proposition 4.2** The zero-curvature equation (4.85) with rational L-A pair

\[ \left\{ \begin{array}{ll} L = \frac{1}{z-1} S, \\ M = \frac{1}{z^2} h, \quad h = c k_2 S_x + c [S, S_x] - \frac{1}{k_2} S. \end{array} \right. \] (4.87)
where \( c \) is a constant and condition

\[ k_2 = \pm 2\lambda \]  

provides the following "isomonodromic version" of the Heisenberg model:

\[ \frac{1}{ck_1} \partial_t S = k_2 S_{xx} + [S, S_{xx}] - \frac{1}{ck_2} S_x \]  

(4.89)

The proof is based on the anzats \( L = \frac{1}{z-\tau} S \), \( M = \frac{1}{z-\tau} h \) which leads to the following system

\[
\begin{cases}
\partial_t S = k_1 \partial_x h, \\
S + k_2 h = [S, h].
\end{cases}
\]  

(4.90)

The later can be resolved as given in (4.87) with condition (4.88). Notice that this condition does not allow to make a limit to the initial "autonomous" 1+1 Heisenberg model (4.80) because \( \lambda \to 0 \) degenerates the dynamical variables. Written as \( \lambda = \pm \frac{1}{2} k_2 \) (4.88) can be also considered as "quantization" of the eigenvalue of \( S \). Unfortunately, the Hamiltonian description is unknown. In this sense the suggested equation looks exotic. In the same time let us mention that one can obtain multi-point generalization (of Schlesinger type). The corresponding autonomous system is the rational 1+1 Gaudin model. See, for example [91].

### 4.5 Non-autonomous Zhukovsky-Volterra gyrostat field theory

Up to now we considered the Painlevé VI field theory depending essentially on one constant corresponding to the finite-dimensional case (1.5) and its SL(\( N, \mathbb{C} \)) generalization (1.9), (1.14). Here we give the field-theoretical generalization of the complete Painlevé VI equation (1.2) in the form (1.6). We consider the bundle \( E = \mathcal{P} \otimes g \) with \( g = \text{sl}(2, \mathbb{C}) \) over elliptic curves with four marked points \( \Sigma_{r,4} \) identified with the half-periods \( \omega_a \) (3.30). Similarly to the finite-dimensional case [52] consider the involution \( \zeta \) of \( E \) (\( \zeta^2 = 1 \)). It acts on the basic spectral curve \( \Sigma_r \) and on the fields as

\[ \zeta : (w, x) = (-w, -x). \]  

(4.91)

The crucial point is that the half-periods \( w = \omega_a \) and \( w = 0 \) are the fixed points of \( \zeta \) under the action of it on \( \Sigma_r \). There are two eigenspaces of the operator \( \zeta \) acting on the space of sections

\[ \Gamma(\text{End}E) = \Gamma^+(\text{End}E) \oplus \Gamma^-(\text{End}E), \]  

(4.92)

where \( \zeta \Gamma^+ = \Gamma^+ \) and \( \zeta \Gamma^+ = -\Gamma^+ \). The Lax operators corresponding to the bundle over \( \Sigma_{r,4} \) can be decomposed in the basis of the Pauli matrices (3.28)

\[
L(k, x, w) = \frac{\kappa}{2} E_1(w|\tau) \sigma_0 + \sum_{\alpha=1}^{3} \sum_{b=0}^{3} S^\alpha_{b,n} \varphi_\alpha (\bar{k}_n, w - \omega_b) e \left( nx - \frac{n \bar{k}}{2} b_2 \right) \sigma_\alpha,
\]

\[
M(k, x, w) = \frac{\kappa}{2} \partial_\tau E_1(w|\tau) \sigma_0 + \frac{1}{2\pi i} \sum_{\alpha=1}^{3} \sum_{b=0}^{3} S^\alpha_{b,n} f_\alpha (\bar{k}_n, w - \omega_b) e \left( nx - \frac{n \bar{k}}{2} b_2 \right) \sigma_\alpha - \]

\[ - \sum_{\alpha=1}^{3} \sum_{b=0}^{3} S^\alpha_{b,n} \frac{\partial \omega_b}{\partial \tau} \varphi_\alpha (\bar{k}_n, w - \omega_b) e \left( nx - \frac{n \bar{k}}{2} b_2 \right) \sigma_\alpha,
\]

where \( \varphi_\alpha (\bar{k}_n, w) \) and \( f_\alpha (\bar{k}_n, w) \) are defined in (3.31) and (3.32) and \( \sigma_\alpha \) are the Pauli matrices (3.28). Coordinates \( S^\alpha_{b,n} \) are the Fourier modes of the field variables \( S^\alpha_b \):

\[ S^\alpha_b (x) = \sum_{n \in \mathbb{Z}} S^\alpha_{b,n} e(nx). \]
The Lax pair can be rewritten in terms of these field variables as
\[
L(\bar{k}, x, w) = \frac{k}{2} E_1(w|\tau)\sigma_0 + \sum_{b=0}^{3} \sum_{\alpha=1}^{3} \varphi_{\alpha} \left( \frac{\bar{k}}{2\pi i} \partial_x, w - \omega_b \right) S^\alpha_b \left( x - \frac{k b_2}{2} \right) \sigma_\alpha,
\]
\[
M(\bar{k}, x, w) = \frac{k}{2} \partial_x \vartheta(w|\tau)\sigma_0 + \frac{1}{2\pi i} \sum_{b=0}^{3} \sum_{\alpha=1}^{3} f_{\alpha} \left( \frac{\bar{k}}{2\pi i} \partial_x, w - \omega_b \right) S^\alpha_b \left( x - \frac{k b_2}{2} \right) \sigma_\alpha - \sum_{b=0}^{3} \sum_{\alpha=1}^{3} \frac{\partial \omega_b}{\partial \tau} \varphi_{\alpha} \left( \frac{\bar{k}}{2\pi i} \partial_x, w - \omega_b \right) S^\alpha_b \left( x - \frac{k b_2}{2} \right) \sigma_\alpha.
\]
The latter operators give the Lax equation
\[
\kappa \partial_{\tau} L - \kappa \partial_{\omega} M - \frac{k}{\rho_0} \partial_x M = [L, M],
\]
which is equivalent to the equation of the elliptic top
\[
\frac{\partial S^\gamma_{b,n}}{\partial \tau} = \sum_{m=0}^{3} \sum_{\alpha,\beta=1}^{3} \epsilon_{\alpha,\beta}\gamma S^{\alpha-\beta}_{b,n-m} S^\beta_{b,c,m} J^{\beta}_{b,c,m}, \tag{4.93}
\]
where
\[
J^{\beta}_{b,c,m} = \frac{1}{\pi} \left( 1 - \delta_{bc} \right) \left( f_{\beta} (k m, \omega_b - \omega_c) + 2\pi i \frac{\partial (\omega_b - \omega_c)}{\partial \tau} \varphi_{\beta} (k m, \omega_b - \omega_c) \right) \text{e} \left( \frac{mk}{2} (b_2 - c_2) \right) - \frac{1}{\pi} \delta_{bc} \text{E}_2 \left( -\omega_3 + \bar{k} m \right).
\]
In terms of the field variables equation (4.93) acquires the following form:
\[
\frac{\partial S^\beta_b(x)}{\partial \tau} = \sum_{c=0}^{3} \sum_{\alpha,\beta=1}^{3} \epsilon_{\alpha,\beta} S^\alpha_b(x) J^{\beta}_{b,c}(\partial_x) S^\beta_c(x), \tag{4.94}
\]
where the conjugate inertia tensor is \( \Psi \) DO
\[
J^{\beta}_{b,c}(\partial_x) = \frac{1}{\pi} \left( 1 - \delta_{bc} \right) f_{\beta} \left( \frac{k}{2\pi i} \partial_x, \omega_b - \omega_c \right) \text{e} \left( \frac{k (b_2 - c_2)}{4\pi i} \partial_x \right) - \frac{1}{\pi} \delta_{bc} \text{E}_2 \left( -\omega_3 + \frac{k}{2\pi i} \partial_x \right) + 2\pi i \frac{\partial (\omega_b - \omega_c)}{\partial \tau} \varphi_{\beta} \left( \frac{k}{2\pi i} \partial_x, \omega_b - \omega_c \right) \text{e} \left( \frac{k (b_2 - c_2)}{4\pi i} \partial_x \right).
\]
Now, in accordance with (4.91) we take
\[
\varsigma L(x, w) = -L(-x, -w), \quad \varsigma M(x, w) = M(-x, -w), \tag{4.95}
\]
and define
\[
L^\pm = \frac{1}{2} \left( L \pm \varsigma(L) \right), \quad L^\pm \in \Gamma^\pm(\text{End}E),
\]
\[
M^\pm = \frac{1}{2} \left( M \pm \varsigma(M) \right), \quad M^\pm \in \Gamma^\pm(\text{End}E).
\]
The constraint
\[
L^- = 0 \tag{4.96}
\]
is consistent with the involutions (4.91), defined by \( M^+ \)
\[
\kappa \partial_{\tau} L^+ - \kappa \partial_{\omega} M^+ - \frac{k}{\rho_0} \partial_x M^+ = [L^+, M^+]. \tag{4.97}
\]
Consider the action of \( \varsigma \) on the arguments of \( \varphi_{\alpha}(\bar{k}n, w) \):

\[
\varsigma \varphi_{\alpha}(\bar{k}n, w - \omega_b) = -\varsigma \left( e^{-(w - \omega_b \frac{\alpha_{\gamma}}{2})} \phi(-\alpha_{\gamma} + \bar{k}n, w - \omega_b) \right) = -e^{(w + \omega_b \frac{\alpha_{\gamma}}{2})} \phi(-\alpha_{\gamma} + \bar{k}n, w - \omega_b).
\]

we remind that \( \omega_b = \frac{b_1 + b_2}{2} \) (\( b_1, b_2 = 0, 1 \)), and \( \alpha_{\gamma} = \frac{\alpha_1 + \alpha_2}{2} \) (\( \alpha_1, \alpha_2 = 0, 1 \)). Then from (A.14) and (A.20) we find

\[
\varphi_{\alpha}(\bar{k}n, w - \omega_b) = -(1) \vec{b} \times \vec{a} e(\bar{b} \bar{k}n) \varphi_{\alpha}(\bar{k}n, w - \omega_b), \tag{4.98}
\]

where \( \vec{b} \times \vec{a} = b_1 \alpha_2 - b_2 \alpha_1 \). In this way following (4.95) we come to the expression

\[
\varsigma L(\bar{k}, x, w) = \frac{\kappa}{2} E_1(w|\tau) \sigma_0 + \sum_{n \in \mathbb{Z}} \sum_{b = 0}^{3} \sum_{\alpha_1 = 1}^{3} \left( \frac{1}{2} S_{\alpha_1}^a \varphi_{\alpha}(w - \omega_b)(1 + (-1) \vec{b} \times \vec{a}) + \right.
\]

\[
\left. \sum_{n > 0} S_{b,n}^a \left( \varphi_{\alpha}(\bar{k}n, w - \omega_b)e \left( nx - \frac{\bar{k}n}{2} b_2 \right) + (-1) \vec{b} \times \vec{a} \varphi_{\alpha}(\bar{k}n, w - \omega_b)e \left( \frac{\bar{k}n}{2} b_2 - nx \right) \right) \right) \sigma_\alpha.
\]

Then the constraints (4.90) and (4.98) imply

\[
S_{b,n}^\alpha = (-1) \vec{b} \times \vec{a} S_{b,-n}^\alpha, \tag{4.99}
\]

which in terms of the field variables can be written as

\[
S_b^\alpha(x) = (-1) \vec{b} \times \vec{a} S_b^\alpha(-x).
\]

Using the relation (4.99), we get that the constraint (4.90) also implies

\[
M^- = 0, \tag{4.100}
\]

or, equivalently,

\[
M(x, w) = M(-x, -w).
\]

Thus, the equations of motion of the Zhukovsky-Volterra gyrostat are defined by the Lax equation (4.97).

From (4.99) we find the invariant part of the Lax operator

\[
L^{+}(x, w) = \frac{\kappa}{2} E_1(w|\tau) \sigma_0 + \sum_{n \in \mathbb{Z}} \sum_{b = 0}^{3} \sum_{\alpha_1 = 1}^{3} \left( \frac{1}{2} S_{\alpha_1}^a \varphi_{\alpha}(w - \omega_b)(1 + (-1) \vec{b} \times \vec{a}) + \right.
\]

\[
\left. \sum_{n > 0} S_{b,n}^a \left( \varphi_{\alpha}(\bar{k}n, w - \omega_b)e \left( nx - \frac{\bar{k}n}{2} b_2 \right) + (-1) \vec{b} \times \vec{a} \varphi_{\alpha}(\bar{k}n, w - \omega_b)e \left( \frac{\bar{k}n}{2} b_2 - nx \right) \right) \right) \sigma_\alpha.
\]

Now we can write down the equations of motion of the Zhukovsky-Volterra gyrostat, using the Lax equation (4.97). For the positive Fourier modes we obtain:

\[
n > 0 : \quad \frac{\partial S_{b,n}^\gamma}{\partial \tau} = -\frac{1}{\pi} \sum_{m \in \mathbb{Z}} \sum_{\alpha, \beta = 1}^{3} \epsilon_{\alpha \beta \gamma} S_{b,n-m}^\alpha S_{b,m}^\beta J^I(\beta, m) + \]

\[
\frac{1}{\pi} \sum_{m \in \mathbb{Z}} \sum_{c \neq b} \sum_{\alpha, \beta = 1}^{3} \epsilon_{\alpha \beta \gamma} S_{b,n-m}^\alpha S_{c,m}^\beta J^{II}(b, c, \beta, m), \tag{4.101}
\]

where \( J^I(\beta, m) = E_2(-\omega_{\beta} + \bar{k}m) \) and

\[
J^{II}(b, c, \beta, m) = \left( f_\beta(\bar{k}m, \omega_{b} - \omega_{c}) + 2i \frac{\partial (\omega_{b} - \omega_{c})}{\partial \tau} \varphi_{\beta}(\bar{k}m, \omega_{b} - \omega_{c}) \right) e \left( \frac{m \bar{k}}{2} (b_2 - c_2) \right).
\]

If we put \( n \) to be zero, the last term in the right hand side of (4.101) vanishes on the constraint (4.99) and we get the equations for the zero Fourier modes in the form:

\[
\vec{b} \times \vec{a} = 0 : \quad \frac{\partial S_{b,0}^\gamma}{\partial \tau} = -\frac{1}{\pi} \sum_{m \in \mathbb{Z}} \sum_{\alpha, \beta = 1}^{3} \epsilon_{\alpha \beta \gamma} S_{b,-m}^\alpha S_{b,m}^\beta E_2(-\omega_{\beta} + \bar{k}m) +
\]
the Eisenstein functions:

In order to compute the Hamiltonians of the system we consider the following expansion in the basis of the Eisenstein functions:

\[ \text{const } \frac{1}{\pi} \sum_{n=1}^{3} \epsilon_{\alpha\beta\gamma} S_{b_{-m}}^{\alpha} S_{c_{-n}}^{\beta} S_{d_{0}}^{\gamma} (\tilde{k}m, \omega_{b} - \omega_{c}) e \left( \frac{m\tilde{k}}{2} (b_{2} - c_{2}) \right). \quad (4.102) \]

In order to compute the Hamiltonians of the system we consider the following expansion in the basis of the Eisenstein functions:

\[ \text{const } \frac{1}{\pi} \sum_{n=1}^{3} \epsilon_{\alpha\beta\gamma} S_{b_{-m}}^{\alpha} S_{c_{-n}}^{\beta} S_{d_{0}}^{\gamma} (\tilde{k}m, \omega_{b} - \omega_{c}) e \left( \frac{m\tilde{k}}{2} (b_{2} - c_{2}) \right). \quad (4.103) \]

where

\[ H_{2,b} = \frac{1}{4\pi i} \sum_{n=1}^{3} S_{b,n}^{\alpha} S_{b_{-n}}^{\alpha}, \]

\[ H_{1,b} = -\frac{1}{2\pi i} \sum_{n=1}^{3} S_{b,n}^{\alpha} S_{b_{-n}}^{\alpha} \varphi_{\alpha} (\tilde{k}n, \omega_{c} - \omega_{b}) e \left( \frac{m\tilde{k}}{2} (c_{2} - b_{2}) \right), \]

\[ H_{\tau} = \frac{1}{4\pi i} \sum_{n=1}^{3} S_{b,n}^{\alpha} S_{b_{-n}}^{\alpha} f_{\alpha} (\tilde{k}n, \omega_{c} - \omega_{b}) e \left( \frac{m\tilde{k}}{2} (c_{2} - b_{2}) \right) - \frac{1}{2\pi i} \sum_{c=1}^{3} S_{b,n}^{\alpha} S_{b_{-n}}^{\alpha} E_{2} (\tilde{k}n - \omega_{a}). \]

In terms of the field variables \( H_{\tau} \) assumes the following form:

\[ H_{\tau} = \frac{1}{4\pi i} \sum_{c=1}^{3} S_{b}^{\alpha} (x - \tilde{k}c_{2}/2) J^{I} (\alpha, b, c, \partial_{x}, \tau) S_{b}^{\alpha} (x - \tilde{k}b_{2}/2) - \frac{1}{4\pi i} \sum_{b=0}^{3} S_{b}^{\alpha} (x) J^{I} (\alpha, b, c, \partial_{x}, \tau) S_{b}^{\alpha} (x), \]

where

\[ J^{I} (\alpha, b, c, \partial_{x}, \tau) = E_{2} \left( \omega_{a} - \frac{\tilde{k}}{2\pi i} \partial_{x}, \tau \right), \quad J^{II} (\alpha, b, c, \partial_{x}, \tau) = f_{\alpha} \left( \frac{\tilde{k}}{2\pi i} \partial_{x}, \omega_{c} - \omega_{b} \right). \quad (4.104) \]

Then we impose the constraint (4.99), which yields

\[ H_{1,b} = 0. \]

Other functions from the expansion (4.103) can be represented as

\[ H_{2,b} = \frac{1}{8\pi i} \sum_{n=0}^{3} S_{b,n}^{\alpha} S_{b_{-n}}^{\alpha} \left( 1 + (-1)^{b \times a} \right), \]

\[ H_{\tau} = \frac{1}{8\pi i} \sum_{n=0}^{3} S_{b,n}^{\alpha} S_{b_{-n}}^{\alpha} \left( 1 + (-1)^{b \times a} \right) E_{2} (\omega_{a}) - \frac{1}{2\pi i} \sum_{n=0}^{3} S_{b,n}^{\alpha} S_{b_{-n}}^{\alpha} E_{2} (\tilde{k}n - \omega_{a}) + \frac{1}{2\pi i} \sum_{n=0}^{3} S_{b,n}^{\alpha} S_{b_{-n}}^{\alpha} E_{2} (\tilde{k}n - \omega_{a}) + \frac{1}{16\pi} \sum_{c=1}^{3} S_{b,n}^{\alpha} S_{b_{-n}}^{\alpha} \left( 1 + (-1)^{b \times a} \right) \left( 1 + (-1)^{c \times a} \right) f_{\alpha} (\omega_{c} - \omega_{b}) + \frac{1}{2\pi i} \sum_{n=0}^{3} S_{b,n}^{\alpha} S_{b_{-n}}^{\alpha} f_{\alpha} (\tilde{k}n, \omega_{c} - \omega_{b}) e \left( \frac{m\tilde{k}}{2} (c_{2} - b_{2}) \right) + \frac{1}{2\pi i} \sum_{n=0}^{3} S_{b,n}^{\alpha} S_{b_{-n}}^{\alpha} f_{\alpha} (\tilde{k}n, \omega_{c} - \omega_{b}) e \left( \frac{m\tilde{k}}{2} (c_{2} - b_{2}) \right) + \]

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\[
+ \sum_{n>0} \sum_{c \neq b} \sum_{a=1}^{3} S_{b,n}^a S_{c,-n}^a \frac{\partial (\omega_c - \omega_b)}{\partial \tau} \phi_a (k n, \omega_c - \omega_b) e^{\left( \frac{nK}{2} (c_2 - b_2) \right)}.
\] (4.105)

The latter function gives the equations of motion (4.101), (4.102) with respect to the Poisson structure
\[
\left\{ S_{b,n}^\alpha, S_{c,m}^\beta \right\} = i \delta_{bc} \sum_{\gamma=1}^{3} \epsilon_{\alpha \beta \gamma} \left( S_{b,n+m}^\gamma + (-1)^{\delta x \delta y} S_{b,n-m}^\gamma \right).
\] (4.106)

Thus, taking into account (3.15) we get the Hamiltonian of the Zhukovsky-Volterra gyrostat field theory
\[
H_{ZVFT} = H_x + k \lambda.
\] (4.107)

Notice that the other hypothetical Hamiltonians \(H_{2,b}\) are the Casimir function with respect to the Poisson brackets (4.106). Also, the Poisson structure (4.106) can be reformulated in terms of the field variables as follows:
\[
\left\{ S^\alpha_0 (x), S^\beta_0 (y) \right\} = i \delta_{bc} \sum_{\gamma=1}^{3} \epsilon_{\alpha \beta \gamma} S^\gamma_b (y) \left( \delta (x - y) + (-1)^{\delta x \delta y} \delta (x + y) \right).
\] (4.108)

This gives the Hamilton equations of motion (4.94) for the Zhukovsky-Volterra gyrostat field variables.

5 Trigonometric and Rational Systems and Multiloop Algebras

Our goal here is to derive trigonometric and rational versions of the monodromy preserving field theories based on the centrally and cocentrally extended Lie algebra \(\hat{L}(sl(N, \mathbb{C}))\). To this end we replace the elliptic curve \(\Sigma = \mathbb{C} / (\mathbb{Z} + \tau \mathbb{Z})\) with the cylinder \(Cyl = \mathbb{C} / (\mathbb{Z} + 0)\) (trigonometric case) and the complex curve \(\mathbb{C}\) (rational case). In the elliptic case we used the sine-basis \(\{T^\alpha, \alpha = (\alpha_1, \alpha_2)\}\) of the Lie algebra \(sl(N, \mathbb{C})\) and put in correspondence to it a finite lattice
\[
L^{ell} = \left\{ \alpha_\tau = \frac{\alpha_1 + \tau \alpha_2}{N} \right\} = (\mathbb{Z}_N \oplus \tau \mathbb{Z}_N) \langle 0, 0 \rangle \subset \Sigma_\tau.
\] (5.1)

The element \(\alpha_\tau\) are arguments of elliptic functions \(\phi\) and \(f\), which are matrix elements of the Lax operators. In the trigonometric and rational cases we consider their degenerated versions (A.23)-(A.27) and (A.28)-(A.27). The trigonometric and rational functions are well defined on \(Cyl\) and \(\mathbb{C}\). Their arguments can be elements of the lattices
\[
L^{rat} = \frac{(\mathbb{Z} + \tau \mathbb{Z}) \langle 0, 0 \rangle}{N} \subset \mathbb{C}, \quad L^{tr} = \frac{(\mathbb{Z}_N \oplus \tau \mathbb{Z}_N) \langle 0, 0 \rangle}{N} \subset Cyl = \mathbb{C} / (\mathbb{Z} + 0).
\] (5.2)

Moreover, they satisfy the addition theorems (A.10) and (A.11). In the previous Section we consider bundles with connections taking values in the Lie algebra \(\mathfrak{a}^{ell} = L(sl(N, \mathbb{C}))\) (3.1). For trigonometric and rational systems we will extend this algebra as
\[
\mathfrak{a}^{tr} = \mathfrak{a}^{ell} \otimes P(y), \quad \mathfrak{a}^{rat} = \mathfrak{a}^{ell} \otimes P(y_1, y_2),
\]
where \(P(y)\) and \(P(y_1, y_2)\) are trigonometric polynomials in one and two variables. In this way we will establish interrelations between the following classes of objects

| | I elliptic | II trigonometric | III rational |
|----------------|----------------|-----------------|-------------|
| Dimension      | 2              | 3               | 4           |
| Surfaces       | \(\Sigma_\tau = \mathbb{C} / (\mathbb{Z} + \tau \mathbb{Z})\) | \(Cyl = \mathbb{C} / (\mathbb{Z} + 0)\) | \(\mathbb{C}\) |
| Lattices       | \(L^{ell}\)    | \(L^{tr}\)      | \(L^{rat}\)  |
| Algebras       | \(\mathfrak{a}^{ell} = L(sl(N, \mathbb{C}))\) | \(\mathfrak{a}^{tr}\) | \(\mathfrak{a}^{rat}\) |
5.1 Loop algebras $L(\mathfrak{gl}(N, \mathbb{C}))$ and $LL(\mathfrak{gl}(N, \mathbb{C}))$

$L(\mathfrak{gl}(N, \mathbb{C}))$

Introduce the following notations. Let

$$Z_N^{tr,(2)} = (\mathbb{Z} \oplus \mathbb{Z}) \setminus (0, 0) = \{ (\hat{\alpha}_1, \alpha_2) \}, \quad \alpha_2 = \hat{\alpha}_2 + mN, \ 0 \leq \hat{\alpha}_2 < N. \quad \text{(5.3)}$$

For the loop algebra $L(\mathfrak{gl}(N, \mathbb{C}))$ we introduce the basis corresponding to $Z_N^{tr,(2)}$ (and to the factor-lattice $L^{tr} (5.2)$):

$$T_N^\alpha = T_N^{\hat{\alpha}_1, \alpha_2} \sim T_N^{\hat{\alpha}_1, \hat{\alpha}_2} e(\alpha_2 y), \ y \in \mathbb{C}, \ (\hat{\alpha}_1, \hat{\alpha}_2) \in Z_N^{(2)} \quad \text{(5.4)}$$

with the commutator

$$[T_N^{\hat{\alpha}_1, \hat{\alpha}_2} e(\alpha_2 y), T_N^{\tilde{\beta}_1, \tilde{\beta}_2} e(\beta_2 y)] = C_{1/N}(\tilde{\alpha}, \tilde{\beta}) T_N^{\hat{\alpha}_1 + \tilde{\beta}_1, \hat{\alpha}_2 + \tilde{\beta}_2} e((\alpha_2 + \beta_2)y). \quad \text{(5.5)}$$

The notion $\hat{\alpha}_1 + \tilde{\beta}_1$ means that the sum is taken $mod \ N$. The algebra has the representation by the basis in $\mathfrak{gl}(N, \mathbb{C})$

$$T_N^{\hat{\alpha}_1, \hat{\alpha}_2} = e_N(\hat{\alpha}_1 \alpha_2) Q^{\hat{\alpha}_1} A_{tr}^{\hat{\alpha}_2}, \quad Q = \text{diag}(1, e_N(1), \ldots, e_N(N - 1)), \ A_{tr}^{\hat{\alpha}_2} = \{E_{j,j+\hat{\alpha}_2}\}. \quad \text{(5.6)}$$

This parametrization means that we use the principle gradation of $L(\mathfrak{sl}(N, \mathbb{C}))$ [12] (chapter 14) defined on the generators as

$$yE_{j,j+1} (j = 1, \ldots, N - 1), \ y^{-1}E_{j+1,j} (j = 2, \ldots, N). \quad \text{(5.4)}$$

To pass to the standard gradation $L(\mathfrak{gl}(N, \mathbb{C})) = \{T_N^{\hat{\alpha}_1, \hat{\alpha}_2} e(m\xi)\} (\alpha_2 = \hat{\alpha}_2 + mN)$ one should conjugate the basis [5,4] by the diagonal matrix $h = \text{diag}(y, y^2, \ldots, y^N)$

$$T_N^{\hat{\alpha}_1, \hat{\alpha}_2} e(m\xi) = hT_N^{\hat{\alpha}_1, \hat{\alpha}_2} e(\alpha_2 y) h^{-1}, \ \xi = y^N. \quad \text{(5.5)}$$

The Poisson algebra of function on $L^*(\mathfrak{sl}(N, \mathbb{C}))$ has the following generators:

$$S^{\hat{\alpha}_1, \alpha_2} = \sum_{(\hat{\beta}_1, \beta_2) \in Z_N^{tr,(2)}} \text{tr} (S^{-\hat{\beta}_1, -\beta_2} T_N^{-\hat{\beta}_1, -\beta_2} T_N^{\hat{\alpha}_1, \alpha_2}) \quad \text{(5.7)}$$

with the brackets

$$\{S^{\hat{\alpha}_1, \alpha_2}, S^{\hat{\beta}_1, \beta_2}\} = C_{1/N}(\alpha, \beta) S^{\hat{\alpha}_1 + \hat{\beta}_1, \alpha_2 + \beta_2}. \quad \text{(5.8)}$$

Introduce the currents

$$S^{\hat{\alpha}_1}(y) = \sum_{\alpha_2 \in \mathbb{Z}} S^{\hat{\alpha}_1, \alpha_2} e(\alpha_2 y). \quad \text{(5.9)}$$

In terms of the currents the Poisson structure [4,8] takes the form

$$\{S^{\hat{\alpha}_1}(y), S^{\hat{\beta}_1}(y')\} = \frac{1}{2} N \left( \delta_{y+y', \hat{\alpha}_1 \hat{\beta}_1} S^{\hat{\alpha}_1 + \hat{\beta}_1}(y + \frac{\hat{\beta}_1}{2N}) - \delta_{y-y', \hat{\alpha}_1 \hat{\beta}_1} S^{\hat{\alpha}_1 + \hat{\beta}_1}(y - \frac{\hat{\beta}_1}{2N}) \right). \quad \text{(5.10)}$$

$LL(\mathfrak{gl}(N, \mathbb{C}))$

Consider generators $T^\alpha$ of $sin\theta$ algebra for a rational $\theta = M/N$. The algebra is isomorphic to the two-loop algebra $LL(\mathfrak{gl}(N, \mathbb{C}))$ with the basis [3,26] and the commutator [3,27]. For the dual variables we have the Poisson brackets

$$\{S^{\alpha_1, \alpha_2}, S^{\beta_1, \beta_2}\} = C_{1/N}(\alpha, \beta) S^{\alpha_1 + \beta_1, \alpha_2 + \beta_2}. \quad \text{(5.11)}$$

In terms of the currents

$$S(y_1, y_2) = \sum_{\alpha \in \mathbb{Z}^{(2)}} S^{\alpha_1, \alpha_2} e(\alpha_1 y_1 + \alpha_2 y_2)$$

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the Poisson brackets are the Moyal brackets

\[
\{ S(y_1, y_2), F(y_1, y_2) \}^{\text{int}} = [ S(y_1, y_2), F(y_1, y_2) ]_{1/N} = N \left( S(y_1, y_2) \star F(y_1, y_2) - F(y_1, y_2) \star S(y_1, y_2) \right) = S(y_1 + \frac{1}{2N} \partial_{y_2}, y_2 - \frac{1}{2N} \partial_{y_1}) F(y_1, y_2) - S(y_1 - \frac{1}{2N} \partial_{y_2}, y_2 + \frac{1}{2N} \partial_{y_1}) F(y_1, y_2).
\]

(5.12)

In this way we come to the nonlocal Poisson brackets (as well as in the trigonometric case (5.3)). Nonlocal Poisson brackets in integrable hierarchies were considered recently [15, 81].

5.2 Equations of motion

5.2.1 Trigonometric systems

We consider the Poisson algebra \( \mathfrak{g}[G] \) in the principle gradation \( S^{\hat{\alpha}_1, \alpha_2} \) and take into account the \( x \)-dependence of \( \mathfrak{g}[G] \)

\[
S^{\hat{\alpha}_1}(x, y) = \sum_{\alpha_2 \in \mathbb{Z}} S^{\hat{\alpha}_1, \alpha_2} e(nx + \alpha_2 y).
\]

(5.1)

In this way we deal with the two-loop algebra with the central extension in the \( x \) direction

\[
\mathbb{C}^* \times \mathbb{C}^* \rightarrow \text{sl}(N, \mathbb{C}) = \{ S^{\hat{\alpha}_1}(x, y) \}.
\]

The Poisson brackets for the Fourier modes assume the form

\[
\{ S^{\hat{\alpha}_1, \alpha_2}, S^{\hat{\beta}_1, \beta_2} \} = S^{\hat{\alpha}_1 + \hat{\beta}_1, \alpha_2 + \beta_2} C_N(\alpha \times \beta).
\]

(5.2)

The conjugate inertia tensor

\[
(J^{-\text{tr}})^{\hat{\alpha}_1, \alpha_2}(n) = E_2^{\text{tr}} (-\frac{\hat{\alpha}_1 + \tau \alpha_2}{N} + \bar{k} n) = \pi^2 \sin^{-2} \pi(\frac{-\hat{\alpha}_1 + \tau \alpha_2}{N} + \bar{k} n), \quad \alpha_2 \in \mathbb{Z}.
\]

defines the trigonometric Euler-Arnold Hamiltonian

\[
H^\text{tr} = k \lambda + \sum_{\hat{\alpha}_1 \in \mathbb{Z}, \alpha_2, n \in \mathbb{Z}} S^{\hat{\alpha}_1, \alpha_2} (J^{-\text{tr}})^{\hat{\alpha}_1, \alpha_2} (-n) (S^{-\hat{\alpha}_1, \alpha_2}),
\]

(5.3)

or in term of velocities \( F^{\hat{\alpha}_1, \alpha_2}_n = (J^{-\text{tr}})^{\hat{\alpha}_1, \alpha_2}(n) S^{\hat{\alpha}_1, \alpha_2} \)

\[
H^\text{tr} = k \lambda + \sum_{\hat{\alpha}_1 \in \mathbb{Z}, \alpha_2, n \in \mathbb{Z}} S^{\hat{\alpha}_1, \alpha_2} F^{-\hat{\alpha}_1, \alpha_2}.
\]

Then we come to the equations of motion

\[
\kappa \partial_{\tau} S^{\hat{\alpha}_1, \alpha_2} = \sum_{\gamma_1 \in \mathbb{Z}, \gamma_2, n' \in \mathbb{Z}} F^{-\hat{\gamma}_1, \gamma_2}_n S^{\hat{\alpha}_1 + \hat{\gamma}_1, \alpha_2 + \gamma_2}_n C_{1/N}(\gamma, \alpha).
\]

(5.4)

Rewrite the equations in terms of fields. Then the conjugate inertia tensor

\[
J^{\text{tr}}_{\hat{\alpha}_1, \alpha_2}(\partial_{\tau}) = \pi^2 \sin^{-2} \pi(\frac{-\hat{\alpha}_1 + \tau \alpha_2}{N} + \frac{\bar{k}}{2\pi \hat{\alpha}_1} \partial_{\tau}), \quad \alpha_2 \in \mathbb{Z}
\]

(5.5)

acts on the \( y \) Fourier components

\[
S^{\hat{\alpha}_1, \alpha_2}(x) \rightarrow (J^{\text{tr}}_{\hat{\alpha}_1, \alpha_2}(\partial_{\tau}) S^{\hat{\alpha}_1, \alpha_2})(x).
\]

The Hamiltonian is the integral

\[
H^\text{tr} = k \lambda + \sum_{\hat{\alpha}_1 \in \mathbb{Z}, \alpha_2 \in \mathbb{Z}} \oint_x S^{\hat{\alpha}_1, \alpha_2}(x) J^{\text{tr}}_{\hat{\alpha}_1, \alpha_2}(\partial_{\tau})(S^{-\hat{\alpha}_1, \alpha_2})(x),
\]

(5.6)
or in term of velocities $F_{n}^{\tilde{\alpha}_{1},\alpha_{2}}(x) = J_{\tilde{\alpha}_{1},\alpha_{2}}^{tr}(\partial_{x})S^{\tilde{\alpha}_{1},\alpha_{2}}(x)$

$$H_{tr}^{\tau} = k\lambda + \sum_{\tilde{\alpha}_{1}\in\mathbb{Z}_{N},\alpha_{2}\in\mathbb{Z}} \int_{x} S^{\tilde{\alpha}_{1},\alpha_{2}}(x)F^{-\tilde{\alpha}_{1},-\alpha_{2}}(x).$$

The equations of motion for the fields

$$\kappa \partial_{t} S^{\tilde{\alpha}_{1}\alpha_{2}}(x) = \sum_{\gamma_{1}\in\mathbb{Z}_{N},\gamma_{2}\in\mathbb{Z}} F^{-\gamma_{1},-\gamma_{2}}(x)S^{\tilde{\alpha}_{1}+\gamma_{1},\alpha_{2}+\gamma_{2}}(x)C_{1/N}(\gamma,\alpha). \quad (5.7)$$

Taking into account the equation in motion $\tilde{k} = \frac{1}{N}k$ rewrite the conjugate inertia operator in terms of the currents $(5.9)$

$$(J_{\alpha_{1}}^{tr})(\partial_{x},\partial_{y}) = \pi^{2} \sin^{-2} \pi \left( -\frac{\alpha_{1}}{N} + \frac{\tau}{2\pi t} \left( \frac{1}{N} \partial_{y} + \frac{1}{k} \partial_{x} \right) \right).$$

Thus the inertia operator becomes the difference operator of the second order

$$(J_{\alpha_{1}}^{tr})^{*}(f)(x,y) = \frac{2f(x,y) - e(-2\alpha_{1}/N)f(x+2\tau k/k,y+2\tau/N) - e(2\alpha_{1}/N)f(x-2\tau k/k,y-2\tau/N)}{\pi^{2}/4}. \quad (5.8)$$

Then we come to the Hamiltonian

$$H_{tr}^{\tau} = k\lambda + \sum_{\alpha_{1}\in\mathbb{Z}_{N}} \int_{T^{2}} S^{\alpha_{1}}(x,y)(J_{\alpha_{1}}^{tr})^{*}(S^{-\alpha_{1}})(x,y), \quad \left( \int_{T^{2}} = \int_{x} \int_{y} \right). \quad (5.9)$$

Define the dual fields

$$F^{-\alpha_{1}}(x,y) = (J_{\alpha_{1}}^{tr})^{*}(\partial_{x},\partial_{y})(S^{-\alpha_{1}})(x,y)$$

Then

$$H_{tr}^{\tau} = k\lambda + \sum_{\alpha_{1}\in\mathbb{Z}_{N}} \int_{T^{2}} S^{\alpha_{1}}(x,y)F^{-\alpha_{1}}(x,y)$$

The equations of motion assume the form

$$\kappa \partial_{t} S^{\alpha_{1}}(x,y) = \frac{1}{N} \sum_{\gamma_{1}\in\mathbb{Z}_{N}} \left( F^{-\gamma_{1}}(x,y + \frac{\alpha_{1} + \gamma_{1}}{2})S^{\alpha_{1}+\gamma_{1}}(x,y - \frac{\gamma_{1}}{2}) - 

-F^{-\gamma_{1}}(x,y - \frac{\alpha_{1} + \gamma_{1}}{2})S^{\alpha_{1}+\gamma_{1}}(x,y + \frac{\gamma_{1}}{2}) \right),$$

or in term of velocities

$$\kappa \partial_{t} J_{\alpha_{1}}^{tr} F^{\alpha_{1}}(x,y) = \frac{1}{N} \sum_{\gamma_{1}\in\mathbb{Z}_{N}} \left( F^{-\gamma_{1}}(x,y + \frac{\alpha_{1} + \gamma_{1}}{2})J_{\alpha_{1}+\gamma_{1}}^{tr} F^{\alpha_{1}+\gamma_{1}}(x,y - \frac{\gamma_{1}}{2}) - 

-F^{-\gamma_{1}}(x,y - \frac{\alpha_{1} + \gamma_{1}}{2})J_{\alpha_{1}+\gamma_{1}}^{tr} F^{\alpha_{1}+\gamma_{1}}(x,y + \frac{\gamma_{1}}{2}) \right),$$

Lax representation

Recall that (see(5.4)) $T_{N,n}^{\alpha} = e(nx + \alpha_{2}y)T_{N,1}^{\tilde{\alpha}_{1},\tilde{\alpha}_{2}}$, where

$$\alpha \in \mathbb{Z}_{N}^{tr(2)}, \quad \alpha_{2} = \tilde{\alpha}_{2} + mN \in \mathbb{Z}, \quad 0 \leq \tilde{\alpha}_{2} < N \quad \alpha_{r} = \frac{\tilde{\alpha}_{1} + \alpha_{2}}{N}.$$

The Lax pair assumes the following form.

$$L_{n}^{\tau} = \sum_{\alpha_{1}\in\mathbb{Z}_{N}^{tr}(2)} L_{n}^{\tau,\alpha_{1}} T_{N,n}^{\alpha_{1}}, \quad M_{n}^{\tau} = \sum_{m,n\in\mathbb{Z}} \sum_{\alpha_{2}\in\mathbb{Z}_{N}^{tr(2)}} M_{n,\alpha_{2}}^{\tau,\alpha_{2}} T_{N,n}^{\alpha_{2}},$$

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Then we come to the equations of motion

\[ L_{n,\alpha}^{tr} = S_n^\alpha X_{n}^\alpha, \quad X_{n}^\alpha = e\left(-w_1^\alpha N\right)\phi_{\alpha,\bar{\lambda}}^{tr}(\overline{k}n, w), \]
\[ M_{n,\alpha}^{tr} = \frac{1}{2\pi n^\alpha} Y_n^\alpha, \quad Y_n^\alpha = \frac{1}{2\pi n^\alpha} e\left(-\frac{w_2^\alpha}{N}\right) f_{\alpha,\bar{\lambda}}^{tr}(\overline{k}n), \]

\[ \phi_{\alpha,\bar{\lambda}}^{tr}(\overline{k}n, w) = \pi \cot \pi (-\alpha + \overline{k}n + \pi \cot (\pi w), \quad f_{\alpha,\bar{\lambda}}^{tr}(\overline{k}n) = -\pi^2 \sin^2 \pi (-\alpha + \overline{k}n), \]

\[ S^\alpha(x, y) = \sum_{\alpha,\bar{\lambda},n} S_{n,\alpha}^{\bar{\alpha},\bar{\lambda}}(nx + \alpha y), \quad S_n^\alpha = S_n^{\bar{\alpha},\bar{\lambda}}. \]

The both functions \( X_n^\alpha, Y_n^\alpha \) are well defined on lattice \( L_2^{tr} \). In terms of fields we have

\[ L_{n,\alpha}^{tr} = S_n^\alpha \pi e\left(-\frac{w_2^\alpha}{N}\right) \phi_{\alpha,\bar{\lambda}}^{tr}(\overline{k}n, w) T_{n,n}^\alpha, \quad M_{n,\alpha}^{tr} = -S_n^\alpha \pi^2 e\left(-\frac{w_2^\alpha}{N}\right) f_{\alpha,\bar{\lambda}}^{tr}(\overline{k}n) T_{n,n}^\alpha. \]

**Proposition 5.1** The Lax equation

\[ [\kappa \partial_\tau + M^{tr}, \kappa \partial_\tau + k \partial_x + L^{tr}] = 0 \]  

is equivalent to the equations of motion \( (5.7) \).

**Proof.**

The proof is the same as in the elliptic case (Proposition 4.1). It is based on the addition formulas for \( \phi_{\alpha,\bar{\lambda}}^{tr} \) and \( f_{\alpha,\bar{\lambda}}^{tr}. \)

**5.2.2 Rational systems**

Let us introduce the \( x \)-dependence in the Poisson algebra \( S^\alpha, (\alpha \in \mathbb{Z}^{(2)}) \)

\[ S(x, y_1, y_2) = \sum_{\alpha,\bar{\lambda},n} S_{n,\alpha,\bar{\lambda}}^{\bar{\alpha},\bar{\lambda}}(nx + \alpha y_1 + \bar{\alpha} y_2) \]  

\[ \{S_{n,\alpha}^{\bar{\alpha},\bar{\lambda}}, S_{m,\beta}^{\bar{\alpha},\bar{\beta}}\} = C_{1/N}(\alpha, \beta) S_{n+m,\alpha}^{\bar{\alpha},\bar{\beta}}. \]

The fields \( S(x, y_1, y_2) \) represent three-loop coalgebra (with the cocentral extension in the \( x \)-direction). Define the conjugate inertia tensor

\[ J_{n,\alpha}^{rat}(n) = E_{2n} \left(-\frac{\alpha + \tau \bar{\alpha}}{N} + \overline{k}n\right) = \left(-\frac{\alpha + \tau \bar{\alpha}}{N} + \overline{k}n\right)^2, \]

and the Hamiltonian

\[ H_{\tau}^{rat} = k\lambda + \sum_{\alpha,\bar{\lambda},n} S_{n,\alpha,\bar{\lambda}}^{\bar{\alpha},\bar{\lambda}} J_{n-\alpha,\bar{\lambda}}^{rat}(-n)(S_{-n}^{-\alpha,\bar{\lambda}}), \]

or in term of velocities \( F_{n,\alpha}^{\bar{\alpha},\bar{\lambda}} = J_{n,\alpha,\bar{\lambda}}^{rat}(n) \)

\[ H_{\tau}^{rat} = k\lambda + \sum_{\alpha,\bar{\lambda},n} S_{n,\alpha,\bar{\lambda}}^{\bar{\alpha},\bar{\lambda}} F_{n}^{\alpha,\bar{\lambda}}. \]

Then we come to the equations of motion

\[ \kappa \partial_\tau S_{n,\alpha}^{\bar{\alpha},\bar{\lambda}} = \sum_{\gamma,\alpha,\bar{\lambda}} C_{1/N}(\gamma, \alpha) F_{n-n'}^{\gamma,\bar{\gamma}} S_{n'}^{\gamma,\bar{\gamma}}^{\alpha,\bar{\lambda}} \]

**The Lax representation** for these equations is based on the addition theorems \( (A.10), (A.11) \) and similar to the trigonometric case.
6 Noncommutative Torus and Isomonodromic Deformations

6.1 Non-autonomous top on NCT

Let \( g = \sin \theta \) be the sine-algebra and \( g^* = \sin^*_\theta \) is its Lie co-algebra. Consider corresponding to the basis \( \{ T^\beta \} \) [B.3], \( \{ B.15 \) in \( \sin \theta \) the basis \( S^\alpha \) in the Poisson algebra \( \sin^*_\theta \). Then the Lie Poisson-Lie structure on \( \sin^*_\theta \) assumes the form

\[
\{ S^\alpha, S^\beta \} = C_\theta(\alpha, \beta) S^\alpha + S^\beta, \quad \alpha, \beta \in \hat{\mathbb{Z}}^{(2)}. \tag{6.1}
\]

The function \( C_2 = \sum_{\alpha \in \mathbb{Z}^{(2)}} S^\alpha S^{-\alpha} \) is the Casimir function with respect to (6.1). Consider the representation (B.11)

\[
S(x) = \sum_{\gamma \in \mathbb{Z}^{(2)}} S^\gamma T^\gamma(x) \tag{6.3}
\]

as a function on the noncommutative torus. Then \( C_2 \) has the integral form (B.13)

\[
C_2 = -\frac{1}{4\pi^2} \int_{T^2_\theta} S^2(x). \tag{6.4}
\]

Let \( \epsilon_1, \epsilon_2 \) be real numbers such that \( 0 < \theta \epsilon_j < 1 \) and \( \theta \epsilon_j \) are irrational. Define the Weierstrass function on \( \Sigma_\tau \) [B.30]

\[
\wp_\theta(\alpha) = \wp(\theta(\epsilon_1 \alpha_1 + \epsilon_2 \alpha_2 \tau)|\tau). \tag{6.5}
\]

It determines the conjugate inertia operator

\[
J : \sin^*_\theta \rightarrow \sin_\theta, \quad S_\alpha \rightarrow J_\alpha S_\alpha, \quad J_\alpha = \wp_\theta(\alpha). \tag{6.6}
\]

We rewrite this map in terms of basis [B.2] of the NCT \( T^2_\theta \). For this purpose introduce a complex structure on \( T^2_\theta \) and define the operator \( \bar{\partial} = \partial \bar{Z} \). In the basis [B.2]

\[
\partial \bar{Z}(U_1^{\alpha_1}U_2^{\alpha_2}) = \theta(\epsilon_1 \alpha_1 + \epsilon_2 \alpha_2 \tau)(U_1^{\alpha_1}U_2^{\alpha_2}). \tag{6.7}
\]

In these terms the conjugate inertia operator \( J \) becomes the pseudodifferential operator

\[
J(S(x)) = \wp(\partial \bar{Z})S(x). \tag{6.8}
\]

Define the Hamiltonian

\[
H = -\frac{1}{2} \int_{T^2_\theta} S(x) \wp_\theta(\partial \bar{Z})S(x) = -\frac{1}{2} \sum_{\gamma \in \mathbb{Z}^{(2)}} S_\gamma \wp_\theta(\gamma) S^{-\gamma}. \tag{6.9}
\]

The phase space of the system is defined by the conditions

\[
\mathcal{R}^* = \{ S | C_2(S) < \infty, \ H(S) < \infty \}. \tag{6.10}
\]

The equations of motion for the non-autonomous Euler-Arnold top \( \kappa \partial_\tau S = ad^*_J(S) \) for the group \( SIN_\theta \) with inertia operator [6.6] take the form

\[
\kappa \partial_\tau S_\alpha = \sum_{\gamma \in \mathbb{Z}^{(2)}} S_{\alpha-\gamma} \wp_\theta(\gamma) C_\theta(\gamma, \alpha) \tag{6.11}
\]

or in terms of the noncommuting fields

\[
\kappa \partial_\tau S(x) = [S(x), \wp(\partial \bar{Z})S(x)]_\theta \tag{6.12}
\]

In this section we work with the Weierstrass function instead of \( E_2 \) (see [A.6]).
6.2 Lax representation

The goal of this subsection is the Lax representation of the equation of motion (6.11). Consider an infinite rank principal bundle \( P \) with the structure group \( \text{SIN}_\theta \) over the deformed elliptic curve \( \Sigma_{\tau} \) (2.24). As in subsection 3.2 define the adjoint bundle \( E_\theta = P \otimes \text{SIN}_\theta V \), where \( V \) is a vector representation of \( \text{SIN}_\theta \).

We don’t need exact form of \( V \), because we will work with connections on \( E_\theta \). Sections \( s \in \Gamma(E_\theta) \) are transformed by the transition operators

\[
\begin{align*}
    s(w + 1) &= \text{Ad}_{U^{-\epsilon_2}} s(w) \\
    s(w + \tau) &= \text{Ad}_{-e_\theta(-w - \frac{\tau}{2}) U^{-\epsilon_1}} s(w),
\end{align*}
\]

\( U^{-\epsilon_2}, U^{-\epsilon_1} \) are well defined in the representation (B.8), or in the Moyal representation (B.11). In the former case \( s(w) = s(x, w) \), \( x \in \mathbb{R} \). Consider connections on \( E_\theta \)

\[
\begin{align*}
    \kappa \partial_w + L(w, \tilde{w}, \tau) \\
    \partial_{\tilde{w}} + \tilde{L}(w, \tilde{w}, \tau),
\end{align*}
\]

where \( L \) will play the role of the Lax operator. For the almost all bundles (6.12) \( \tilde{L} \) can be gauged away: \( f^{-1} \partial_{\tilde{w}} f + f^{-1} L f = 0 \). As a result of the symplectic reduction we assume that the bundle is flat (2.16).

As in the case of affine algebras (4.25), (4.26), (4.27) the Lax operator can be fixed by the following conditions:

1. The flatness

\[
\partial_{\tilde{w}} L = 0. \tag{6.13}
\]

2. The quasi-periodicity conditions:

\[
\begin{align*}
    L(w + 1) &= U^{-\epsilon_2}_1 L^{(1)}(w) U^{-\epsilon_2}_1, \\
    L(w + \tau) &= \tilde{\Lambda} L(w) \tilde{\Lambda}^{-1} + 2\pi i \theta, \\
    \tilde{\Lambda}(w, \tau) &= -e_\theta(-w - \frac{\tau}{2}) U^{-\epsilon_1}_2.
\end{align*}
\]

3. The quasi-parabolic structure: \( L \) has a simple pole at \( w = 0 \) and

\[
\text{Res}_{w=0} L(w) = S - \kappa \theta \cdot \text{Id}, \tag{6.15}
\]

where \( S \) is defined by (6.9). All degrees of freedom will come from the residue

Lemma 6.1 The Lax operator assumes the form

\[
L(w) = -\kappa \theta E_1(w|\tau) \text{Id} + \sum_{\alpha \in \mathbb{Z}^2} S_\alpha \varphi_{\theta, \alpha}(w) T^\alpha, \tag{6.16}
\]

where \( E_1(w|\tau) \) (A.2) and \( \varphi_{\theta, \alpha}(z) \) (B.37).)

Proof

It follows from (B.4) that

\[
U^{-1}_1 T^\alpha U_1 = e_\theta(-1) T^\alpha, \quad U^{-1}_2 T^\alpha U_2 = e_\theta(1) T^\alpha, \quad e_\theta(1) = \exp 2\pi i \theta.
\]

Then (6.14) follows from (B.39), while (6.15) from (A.2) and (A.8). \( \blacksquare \)

Proposition 6.1 The equation of motion of the non-autonomous top (6.10) is the monodromy preserving equation for the linear system

\[
\begin{align*}
    (\kappa \partial_w + L(w, \tilde{w}, \tau)) \psi &= 0, \\
    \partial_{\tilde{w}} \psi &= 0.
\end{align*}
\]

They have the Lax representation

\[
\kappa \partial_v L - \kappa \partial_w M + [M, L] = 0, \tag{6.17}
\]

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where $L$ is defined by (6.10),

$$M = -\frac{\kappa}{N} \partial_x \ln \phi(w; \tau) Id + \frac{1}{2\pi i} \sum_{\gamma \in \mathbb{Z}^2} S_{\gamma} f_{\theta \gamma}(w) T^\gamma,$$

and $f_{\theta \gamma}(w)$ is defined by (6.39).

Proof.

The proof of the equivalence of (6.10) and (6.17) is based on the addition formula (A.10) and the heat equation (A.13). Let us substitute $L$ (6.16) and $M$ in (6.17). Then in the basis $T^\alpha$ we come to the equation

$$\kappa \partial_x (S_{\alpha} \varphi_{\theta \alpha}(w)) - \kappa \partial_w (S_{\alpha} f_{\theta \alpha}(w)) = \sum_{\gamma} S_{\alpha-\gamma} S_{\gamma} C_{\theta}(\gamma, \alpha) \varphi_{\theta \gamma-\gamma}(w) f_{\theta \gamma}(w).$$

(6.19)

The l.h.s. equals

$$\kappa \partial_x (S_{\alpha}) \varphi_{\theta \alpha}(w) + S_{\alpha} \left( \kappa \partial_x \varphi_{\theta \alpha}(w) - \frac{1}{2\pi i} \kappa \partial_w f_{\theta \alpha}(w) \right).$$

The expression in the brackets $\kappa \partial_x \varphi_{\theta \alpha}(w) - \frac{1}{2\pi i} \kappa \partial_w f_{\theta \alpha}(w)$ vanishes due to the heat equation (A.13).

The r.h.s. of (6.19) gives

$$\sum_{\gamma} S_{\alpha-\gamma} S_{\gamma} C_{\theta}(\gamma, \alpha) f_{\alpha-\gamma}(w) \varphi_{\gamma}(w) \stackrel{(A.10)}{=} \sum_{\gamma} S_{\alpha-\gamma} S_{\gamma} C_{\theta}(\gamma, \alpha) \varphi_{\theta \gamma}(\gamma) - \varphi_{\theta}(\alpha-\gamma) \varphi_{\theta \alpha}(w)$$

Comparing the l.h.s. and the r.h.s. we come to the equation of motion. □

### 6.3 Trigonometric limit

The trigonometric limit is associated with $Im(\tau) \to +\infty$, where $\tau$ is the modulus of torus $\tau = \omega_2/\omega_1$. In this limit the elliptic curve degenerates into pinched torus. For our purposes it is natural to consider it as an infinite complex cylinder with one additional point at infinity. In our systems $\tau$ also plays the role of time and we can no longer consider it as a free parameter of the limit. To avoid this difficulty we introduce the scaling of $\tau$ in the following way

$$\tau = \tau^{tr} \tau_2,$$

where $\tau^{tr}$ plays the role of time and $\tau_2$ is a free parameter. With this substitution the equations of motion (6.10) are modified in a simple way. The parameter of the elliptic curve $\tau$ is originally constrained to $Im(\tau) > 0$. In what follows we consider $\tau^{tr}$ as real and $\tau^{tr} > c > 0$ for some fixed $c$, while $Im(\tau_2) > 0$. Then we take $Im(\tau_2) \to +\infty$ and simultaneously put $\epsilon_2 \to 0$, $\kappa \to \infty$ in such a way that combinations

$$\theta \epsilon_2 \tau_2 = \tilde{\epsilon}_2, \quad \frac{\kappa}{\tau_2} = \tilde{\kappa}, \quad \frac{w}{\tau_2} = \tilde{w}$$

remain fixed. One can note that with this substitution the corresponding Lax equation preserves the form (6.17) with $\tilde{\kappa}$ and $\tilde{w}$ instead of $\kappa$ and $w$. Using expansion (A.32) we get the limiting Hamiltonian

$$H^{tr} = \frac{\pi^2}{2} \sum_{\alpha \in \mathbb{Z}^2} S_{\alpha} S_{-\alpha} \sin^{-2} \left( \pi (\tilde{\epsilon}_1 \alpha_1 + \tilde{\epsilon}_2 \alpha_2 \tau^{tr}) \right),$$

(6.20)

where $\tilde{\epsilon}_1 = \theta \epsilon_1$. The Hamilton equations of motion

$$\tilde{\kappa} \partial_x \tau^{tr} S_{\alpha} = \pi^2 \sum_{\gamma \in \mathbb{Z}^2} \frac{C_{\theta}(\gamma, \alpha) S_{\alpha+\gamma} S_{-\gamma}}{\sin^2 \left( \pi (\tilde{\epsilon}_1 \gamma_1 + \tilde{\epsilon}_2 \gamma_2 \tau^{tr}) \right)}$$

(6.21)
This equations of motion have the form (1.9) with \( \tilde{\kappa} \) instead of \( \kappa \) and
\[
J = \{ \pi^2 \sin^{-2}(\tilde{\epsilon}_1 \alpha_1 + \tilde{\epsilon}_2 \alpha_2 \tau^{fr}) \}.
\]

If we restrict ourselves to \( 0 < \Re(\tilde{w}) < 1 \) the Hamilton equations of motion can be presented as an equation of the zero curvature
\[
\tilde{\kappa} \partial_{\tau^r} L^{tr} - \tilde{\kappa} \partial_{\tilde{w}} M^{tr} + [M^{tr}, L^{tr}] = 0,
\]
with the Lax matrices \( L^{tr}, M^{tr} \) constructed as the limit of the elliptic ones
\[
L^{tr} = \lim_{q_2 \to 0} L, \quad M^{tr} = \lim_{q_2 \to 0} M,
\]
where \( q_2 \equiv e(\tau_2) \). Using (A.36) and (A.37) we get the following limit of the Lax pair:
\[
L^{tr}(w) = \pi \sum_{\alpha \in \mathbb{Z}(2)} S_{\alpha} e(\tilde{\epsilon}_2 \alpha_2 \tilde{w}) \left( \cot \left( \pi(\tilde{\epsilon}_1 \alpha_1 + \tilde{\epsilon}_2 \alpha_2 \tau^{fr}) \right) + i \right) T^\alpha, \tag{6.23}
\]
\[
M^{tr}(w) = \pi \frac{1}{2} \sum_{\alpha \in \mathbb{Z}(2)} S_{\alpha} e(\tilde{\epsilon}_2 \alpha_2 \tilde{w}) \frac{\sin^2(\pi(\tilde{\epsilon}_1 \alpha_1 + \tilde{\epsilon}_2 \alpha_2 \tau^{fr}))}{\sin^2(\pi(\tilde{\epsilon}_1 \alpha_1 + \tilde{\epsilon}_2 \alpha_2 \tau^{fr}))} T^\alpha, \tag{6.24}
\]

### 6.4 Rational limit

Here we reduce the system constructed in previous subsection further with a limit which we call rational. This limit is motivated by the analogy with rational limit of the finite-dimensional monodromy-preserving systems. First, we make the following substitution:
\[
\tau^{tr} = \tau^r a^2, \quad \tilde{w} = w^r a, \quad \tilde{\epsilon}_1 = \epsilon_1^r a, \quad \tilde{\epsilon}_2 = \frac{\epsilon_2^r}{a},
\]
where \( a \) in the finite-dimensional case had the meaning of the inverse first period of the original curve \( a \propto 1/\omega_1 \). Using this substitution we adjust the scaling of the Hamiltonian and the Lax pair in such a way that equations of motion and the equation of zero curvature preserve the form after this substitution with new \( \tau^r, w^r \). Then we take the limit \( a \to 0 \) using the above scalings
\[
H^r = \lim_{a \to 0} a^2 H^{tr}, \quad L^r = \lim_{a \to 0} a L^{tr}, \quad M^r = \lim_{a \to 0} a^2 M^{tr}.
\]
Thus, for the Hamiltonian we have:
\[
H^r = \frac{1}{2} \sum_{\alpha \in \mathbb{Z}(2)} \frac{S_{\alpha} S_{-\alpha}}{(\epsilon_1^r \alpha_1 + \epsilon_2^r \alpha_2 \tau^r)^2}. \tag{6.25}
\]

The corresponding Hamilton equations has form (6.11)
\[
\kappa^r \partial_{\tau^r} S(x) = [S(x), (\partial_{\tilde{w}})^{-2} S(x)]_\theta \tag{6.26}
\]
Equation (6.26) can be presented as an equation of the zero curvature with limiting Lax matrices \( L^r \) and \( M^r \) which have the following form:
\[
L^r(w) = \sum_{\alpha \in \mathbb{Z}(2)} S_{\alpha} \frac{e(\epsilon_2^r \alpha_2 w^r)}{\epsilon_1^r \alpha_1 + \epsilon_2^r \alpha_2 \tau^r} T^\alpha, \tag{6.27}
\]
\[
M^r(w) = \frac{1}{2\pi i} \sum_{\alpha \in \mathbb{Z}(2)} S_{\alpha} \frac{e(\epsilon_2^r \alpha_2 w^r)}{(\epsilon_1^r \alpha_1 + \epsilon_2^r \alpha_2 \tau^r)^2} T^\alpha, \tag{6.28}
\]
6.5 Scaling limit

To define the limit we decompose the parameter $\tau$ of the elliptic curve as

$$\tau = \tau_1 + \tau_2,$$  \hspace{1cm} (6.29)

where $\tau_1$ plays the role of time of the limiting system and $\tau_2$ gives the trigonometric limit $\Im m \tau_2 \to +\infty$. The limiting procedure under consideration consists of the shift of the spectral parameter

$$w = \tilde{w} + \tau/2,$$  \hspace{1cm} (6.30)

the scalings of the coordinates

$$S_\alpha = \tilde{S}_\alpha q_2^{-g(\alpha_2)}, \quad \text{where} \quad q_2 \equiv e(\tau_2), \quad g(\alpha_2) = \theta \epsilon_2 \frac{1 - \delta_{\alpha_2,0}}{2},$$  \hspace{1cm} (6.31)

and the trigonometric limit $\Im m \tau_2 \to +\infty$. After scalings (6.31), in the limit $\Im m \tau_2 \to +\infty$ we obtain the contraction of the Poisson algebra (6.1)

$$\left\{ \tilde{S}_\alpha, \tilde{S}_\beta \right\} = C_{\theta}(\alpha, \beta)\tilde{S}_{\alpha + \beta} q_2^{g(\alpha_2) + g(\beta_2) - g(\alpha_2 + \beta_2)},$$  \hspace{1cm} (6.32)

where $\tilde{S}_\alpha \equiv \tilde{S}_{\alpha_1,\alpha_2}$, $\alpha \in \mathbb{Z}^{(2)}$. Scaled coordinates $\tilde{S}_\alpha$ with the Poisson brackets form an algebra provided that

$$\forall \alpha_2, \beta_2 \in \mathbb{Z} : \quad g(\alpha_2) + g(\beta_2) - g(\alpha_2 + \beta_2) \geq 0.$$  \hspace{1cm} (6.33)

For $g(\alpha_2) = \theta \epsilon_2 (1 - \delta_{\alpha_2,0})/2$ condition (6.33) is trivial and we can write down all nonzero brackets corresponding to the equality in (6.33)

$$\left\{ \tilde{S}_{\alpha_1,0}, \tilde{S}_{\beta_1,\beta_2} \right\} = \frac{1}{\pi \theta} \sin (\pi \theta \alpha_1 \beta_2) \tilde{S}_{\alpha_1 + \beta_1,\beta_2}.$$  \hspace{1cm} (6.34)

An important component of the procedure is that we consider the subsystem with arbitrary large but finite number of nonzero coordinates $S_\alpha$. To compute the limits of the Hamiltonian (6.8) and the Lax operators (6.16), (6.18) we consider those values of constants $\theta, \epsilon_2$ that for all nonzero coordinates $S_\alpha$ the following condition is true:

$$|\theta \epsilon_2 \alpha_2| < 1.$$

Then for the Hamiltonian of the limiting system we have

$$H = -\frac{\pi^2}{2} \sum_{\alpha_1 \in \mathbb{Z} \setminus \{0\}} \tilde{S}_{\alpha_1,0} \tilde{g}_{-\alpha_1,0} \sin^2 (\pi \theta \epsilon_1 \alpha_1) + 4\pi^2 \frac{\theta}{q_1} \sum_{\alpha_2 \in \mathbb{Z}} e(\theta \epsilon_1 \alpha_1) \tilde{S}_{\alpha_1,1} \tilde{S}_{-\alpha_1,-1},$$  \hspace{1cm} (6.36)

where $q_1 \equiv e(\tau_1)$. Note that coordinates included in the Hamiltonian form a subalgebra of the limiting Poisson algebra (6.34). Thus, the Hamilton equations of motion for these coordinates do not depend on the coordinates which are not included in the Hamiltonian. Equations of motion with respect to the brackets (6.34)

$$\partial_{\tau_1} \tilde{S}_\alpha = \left\{ H, \tilde{S}_\alpha \right\}$$  \hspace{1cm} (6.37)

can be also obtained as a limit of (6.10). For coordinates included in the Hamiltonian the equations of motion are of the form

\begin{align*}
\partial_{\tau_1} \tilde{S}_{\gamma_1,0} &= -\frac{4\pi}{\theta} \frac{\theta_2}{q_1} \sum_{\alpha_1 \in \mathbb{Z}} e(\theta \epsilon_1 \alpha_1) \sin (\pi \theta \gamma_1) \left( \tilde{S}_{\gamma_1 + \alpha_1,1} \tilde{S}_{-\alpha_1,-1} - \tilde{S}_{\gamma_1 - \alpha_1,1} \tilde{S}_{-\alpha_1,1} \right), \\
\partial_{\tau_1} \tilde{S}_{\gamma_1,1} &= -\frac{4\pi}{\theta} \sum_{\alpha_1 \in \mathbb{Z} \setminus \{0\}} \tilde{S}_{\alpha_1 + \gamma_1,1} \sin^2 (\pi \theta \epsilon_1 \alpha_1) \tilde{S}_{-\alpha_1,0}, \\
\partial_{\tau_1} \tilde{S}_{\gamma_1,-1} &= \frac{4\pi}{\theta} \sum_{\alpha_1 \in \mathbb{Z} \setminus \{0\}} \tilde{S}_{\alpha_1 + \gamma_1,-1} \sin^2 (\pi \theta \epsilon_1 \alpha_1) \tilde{S}_{-\alpha_1,0}.
\end{align*}

(6.38)
To construct the Lax representation for the equations (6.38) we consider the limit of the Lax operators (6.10), (6.18) and the equation (6.17). Since the shift of the spectral parameter (6.30) in the limiting procedure under consideration is time-dependent, equation (6.17) turns into

\[\partial_t \tilde{L} - \frac{1}{2\pi i} \partial_{\tilde{w}} (M + \pi t L) = [L, M], \]  

(6.39)

where \( L = L(S, \tilde{w} + \tau/2, \tau) \), \( M = M(S, \tilde{w} + \tau/2, \tau) \). Thus, the Lax pair of the limiting system is defined via

\[\tilde{L} = \lim_{\Im \tau_2 \to +\infty} L, \quad \tilde{M} = \lim_{\Im \tau_2 \to +\infty} (M + \pi t L) \]  

(6.40)

and the Lax equation assumes the form

\[\partial_t \tilde{L} - \frac{1}{2\pi i} \partial_{\tilde{w}} \tilde{M} = \left[ \tilde{L}, \tilde{M} \right], \]  

(6.41)

where

\[
\tilde{L} = \pi \sum_{\alpha_1 \in \mathbb{Z} \setminus \{0\}} \frac{\mathbf{e}(-\theta_1 \alpha_1/2)}{\sin (\pi \theta_1 \alpha_1)} \tilde{S}_{\alpha_1,0} T^{\alpha_1,0} - 2\pi i q_1^{\theta_2/2} \sum_{\alpha_1 \in \mathbb{Z}} \mathbf{e}(\theta_2 \tilde{w}) \tilde{S}_{\alpha_1,1} T^{\alpha_1,1} +
+ 2\pi i q_1^{\theta_2/2} \sum_{\alpha_1 \in \mathbb{Z}} \mathbf{e}(\theta_1 \alpha_1 - \theta_2 \tilde{w}) \tilde{S}_{\alpha_1,-1} T^{\alpha_1,-1},
\]  

(6.42)

\[
\tilde{M} = -\frac{\pi^2}{2} \sum_{\alpha_1 \in \mathbb{Z} \setminus \{0\}} \frac{1 + \mathbf{e}(-\theta_1 \alpha_1)}{\sin^2 (\pi \theta_1 \alpha_1)} \tilde{S}_{\alpha_1,0} T^{\alpha_1,0} + 2\pi^2 q_1^{\theta_2/2} \sum_{\alpha_1 \in \mathbb{Z}} \mathbf{e}(\theta_2 \tilde{w}) \tilde{S}_{\alpha_1,1} T^{\alpha_1,1} +
+ 2\pi^2 q_1^{\theta_2/2} \sum_{\alpha_1 \in \mathbb{Z}} \mathbf{e}(\theta_1 \alpha_1 - \theta_2 \tilde{w}) \tilde{S}_{\alpha_1,-1} T^{\alpha_1,-1}.
\]  

(6.43)

As one can see, coordinates \( \tilde{S}_{\alpha}, |\alpha_2| > 1 \) are not present in Lax operators. Therefore, we have the Lax representation (6.41) for equations of motion (6.38). Since the Hamiltonian (6.36) and Lax operators (6.42), (6.43) depend only on coordinates of the form \( S_{\alpha}, |\alpha_2| \leq 1 \), we can pass to the following three field variables:

\[ h = h(x_1) = \sum_{\alpha_1 \in \mathbb{Z}} \frac{1}{\sin^2 (\pi \theta_1 \alpha_1)} \tilde{S}_{\alpha_1,0} \mathbf{e}(\alpha_1 x_1), \]  

(6.44)

\[ f = f(x_1) = \sum_{\alpha_1 \in \mathbb{Z}} \tilde{S}_{\alpha_1,1} \mathbf{e}(\alpha_1 x_1), \quad g = g(x_1) = \sum_{\alpha_1 \in \mathbb{Z}} \tilde{S}_{\alpha_1,-1} \mathbf{e}(\alpha_1 x_1). \]  

(6.45)

Then the Hamiltonian (6.36) can be rewritten as follows

\[ H = \pi^2 \int_{S^1} \left( \frac{1}{2} h(x_1) \sinh^2 \left( \frac{\theta_1 \partial_{x_1}}{2} \right) h(x_1) \right) + 4q_1^{\theta_2} g(x_1) e^{-\theta_1 \partial_{x_1}} f(x_1) \right) dx_1. \]  

(6.46)

The equations of motion in terms of field variables \( f, g, h \) acquire the following form:

\[ \sinh^2 \left( \frac{\theta_1 \partial_{x_1}}{2} \right) \partial_{\tau_1} h = -\frac{4\pi i}{\theta} q_1^{\theta_2} \sinh \left( \frac{\theta \partial_{x_1}}{2} \right) \left( f e^{-\theta_1 \partial_{x_1}} g - g e^{\theta_1 \partial_{x_1}} f \right), \]  

(6.47)

\[ \partial_{\tau_1} f = -\frac{\pi i}{\theta} f \sinh \left( \frac{\theta \partial_{x_1}}{2} \right) h, \]  

(6.48)

\[ \partial_{\tau_1} g = \frac{\pi i}{\theta} g \sinh \left( \frac{\theta \partial_{x_1}}{2} \right) h. \]  

(6.49)
Lax operators in terms of field variables can be defined via the shift operators $e^{\pm \theta \partial_x}$:

$$\tilde{L} = \frac{1}{4\theta} (h(x_1) - h(x_1 - \theta \epsilon_1)) + \frac{i}{\theta} \theta_1^{\theta_2/2} e^{(\theta \epsilon_2 \tilde{w})} f(x_1 + \frac{\theta}{2}) e^{\theta \partial_x} - \frac{2}{\theta} \frac{\theta_1^{\theta_2/2}}{\epsilon_1} e^{(-\theta \epsilon_2 \tilde{w})} g(x_1 - \theta \epsilon_1 - \frac{\theta}{2}) e^{-\theta \partial_x},$$

$$\tilde{M} = -\frac{\pi i}{4\theta} (h(x_1) + h(x_1 - \theta \epsilon_1)) - \frac{\pi i}{\theta} \theta_1^{\theta_2/2} e^{(\theta \epsilon_2 \tilde{w})} f(x_1 + \frac{\theta}{2}) e^{\theta \partial_x} - \frac{\pi i}{\theta} \frac{\theta_1^{\theta_2/2}}{\epsilon_1} e^{(-\theta \epsilon_2 \tilde{w})} g(x_1 - \theta \epsilon_1 - \frac{\theta}{2}) e^{-\theta \partial_x}. \quad (6.50)$$

6.6 Dispersionless limit

6.6.1 General case

In the "dispersionless" limit ($\theta \to 0$) the Lie algebra $\text{sing}_\theta$ becomes the Poisson-Lie algebra $\text{Ham}(T^2)$ of Hamiltonians on the two-dimensional torus, see (B.40). It can be represented by the Lie algebra of the corresponding divergence-free vector fields $\text{SVect}(T^2)$. More precisely, to pass from $\text{Ham}(T^2)$ to $\text{SVect}(T^2)$ one has to discard the constant Hamiltonians, but add the "flux" vector fields $\partial/\partial x_1$ and $\partial/\partial x_2$ corresponding to multivalued Hamiltonian functions $x_1$ and $x_2$ on the torus. We define the non-autonomous top related to the Lie group $\text{SDiff}(T^2)$ by considering the limit $\theta \to 0$ of the described above systems. Let

$$\theta \to 0, \; \epsilon_{1,2} \to \infty, \; \text{such that} \lim_{\theta \to 0} (\theta \epsilon_{1,2}) = \epsilon'_{1,2} < 1, \; \epsilon'_{1,2} \text{ are irrational}. \quad (6.52)$$

In what follows we drop the superscript $'$. Let $S = \sum_{\alpha} S^\alpha e(\alpha \cdot x) \in \text{Ham}^*(T^2)$, where $e(\alpha \cdot x)$ is the Fourier basis (B.41) of $\text{Ham}^*(T^2)$. In the Fourier basis, the linear Poisson bracket assumes the form (B.42)

$$\{S^\alpha, S^\beta\}_1 = (\alpha \times \beta) S^{\alpha + \beta}, \; \alpha \times \beta = \alpha_1 \beta_2 - \alpha_2 \beta_1. \quad (6.53)$$

The conjugate inertia operator $J : \text{Ham}^*(T^2) \to \text{Ham}(T^2)$ becomes

$$J : S^\alpha \to \varphi(\epsilon \cdot \alpha) S^\alpha, \; \varphi(\epsilon \cdot \alpha) = \varphi(\epsilon_1 \alpha_1 + \epsilon_2 \alpha_2 \tau; \tau),$$

where $\alpha \in \mathbb{Z}^2$. The operator is well defined since $\epsilon_j$ are irrational. In terms fields

$$J : S(x) \to \varphi(\tilde{\partial}_Z)S(x) \quad (6.54)$$

where

$$\tilde{\partial}_Z = \frac{1}{2\pi i} ((\epsilon_1 \partial_1 + \epsilon_2 \tau \partial_2)$$

is the operator of the complex structure on the commutative torus $T^2$. In fact, the complex structure depends on the ratio $\tau \epsilon_2/\epsilon_1$. The quadratic Hamiltonian of the system is

$$H = -\frac{1}{2} \sum_\gamma S^\gamma \varphi(\epsilon \cdot \gamma) S^{-\gamma} = 2\pi^2 \int_{T^2} S(\varphi(\tilde{\partial}_Z)S), \quad (6.55)$$

(see (B.46)), and the corresponding equations of motion are (see (B.43))

$$\partial_\tau S = \{S, \varphi(\tilde{\partial}_Z)\} \quad (6.56)$$

For the velocities the equation takes the form

$$\partial_\tau (\varphi^{-1}(\tilde{\partial}_Z)F) = \{\varphi^{-1}(\tilde{\partial}_Z)F, F\}. \quad (6.57)$$

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Define the Lax operator
\[ L(x; w) = \sum_{\alpha \in \mathbb{Z}^2} S^\alpha \varphi(\epsilon \cdot \alpha, w)e(\alpha \cdot x) \tag{6.58} \]

Notice that
\[ \int_{T^2} L(x, w) = 0. \]

In this way \( L \) defines divergence-free vector field \( \partial_1 L \partial_2 - \partial_2 L \partial_1 \).

**Proposition 6.2** The equations of motion \((6.50)\) have the dispersionless Lax representation
\[ \partial_r L - \partial_w M = \{L, M\} \]
with \( L \) given by \((6.58)\) and
\[ M(x; w) = \sum_{\alpha \in \mathbb{Z}^2} S^\alpha f(\epsilon \cdot \alpha, w)e(\alpha \cdot x) \tag{6.59} \]
where
\[ f(\epsilon \cdot \alpha, w) = e(\epsilon_2 \alpha_2 w)\partial_\alpha \phi(u, w)|_{u = \epsilon_1 \alpha_1 + \epsilon_2 \alpha_2}. \]

The proof of Proposition \(6.2\) is the same as Proposition \(6.1\). Notice that operators \( L \) and \( M \) satisfy the quasi-periodicity properties
\[
\begin{align*}
L(x_1, x_2; w) &= L(x_1 + \epsilon_2, x_2; w + 1), \\
L(x_1, x_2; w) &= L(x_1, x_2 - \epsilon_1; w + \tau), \\
M(x_1, x_2; w) &= M(x_1 + \epsilon_2, x_2; w + 1), \\
M(x_1, x_2; w) - M(x_1, x_2 - \epsilon_1; w + \tau) &= 2\pi i L(x_1, x_2, w). 
\end{align*}
\]

**6.6.2 Trigonometric limit**

In the trigonometric case the equations of motion \((6.11)\) assume the form
\[
\partial_r S(x) = \{S(x), \left( \pi^2/\kappa^2 \right) \sin^{-2}(\pi \partial Z) S(x) \} \\
\partial_Z = \frac{1}{2\pi i} (\partial_1 + \partial_2 \tau r \partial_2). \tag{6.60}
\]

In terms of the velocities \( F(x) \) \((1.11)\)
\[
F(x) = (J^\tau)^{-1} S(x), \quad J^\tau = (\kappa^2/\pi^2) \sin^2(\pi \partial Z)
\]
\[
((J^\tau)^{-1} F)(x) = F(x_1 + 2\pi \tilde{\epsilon}_1, x_2 + 2\pi \tilde{\epsilon}_2 \tau r) + F(x_1 - 2\pi \tilde{\epsilon}_1, x_2 - 2\pi \tilde{\epsilon}_2 \tau r) - 2F(x)
\]
it can be rewritten as
\[
\sin^2(\pi \partial Z) \partial_r F(x) = \{\sin^2(\pi \partial Z) F(x), F(x)\} - \pi \epsilon_2 \sin(\pi \partial Z) \cos(\pi \partial Z) \partial_2 F(x) \tag{6.61}
\]
or
\[
\partial_r \left( F(x_1 + 2\pi \tilde{\epsilon}_1, x_2 + 2\pi \tilde{\epsilon}_2 \tau r) + F(x_1 - 2\pi \tilde{\epsilon}_1, x_2 - 2\pi \tilde{\epsilon}_2 \tau r) - 2F(x) \right) = 0 \tag{6.62}
\]
\[
= \left\{ \left( F(x_1 + 2\pi \tilde{\epsilon}_1, x_2 + 2\pi \tilde{\epsilon}_2 \tau r) + F(x_1 - 2\pi \tilde{\epsilon}_1, x_2 - 2\pi \tilde{\epsilon}_2 \tau r) \right), F(x) \right\} - \pi \epsilon_2 \partial_2 \left( F(x_1 + 2\pi \tilde{\epsilon}_1, x_2 + 2\pi \tilde{\epsilon}_2 \tau r) - F(x_1 - 2\pi \tilde{\epsilon}_1, x_2 - 2\pi \tilde{\epsilon}_2 \tau r) \right).
\]

This equation can be presented as an equation of the zero curvature with Lax matrices from \((6.23)\) and \((6.24)\). In dispersionless limit the decomposition with respect to generators \( T^\alpha \) turns into Fourier decomposition:
\[
L^\tau(x_1, x_2, \tilde{w}) = \pi \sum_{\alpha \in \mathbb{Z}^2_f} S^\alpha e(\tilde{\epsilon}_2 \alpha_2 \tilde{w}) (\cot (\pi (\tilde{\epsilon}_1 \alpha_1 + \tilde{\epsilon}_2 \alpha_2 \tau r)) + i) e(\alpha_1 x_1 + \alpha_2 x_2) \tag{6.63}
\]
\[ M^r(x_1, x_2, \tilde{w}) = \frac{\pi}{2t} \sum_{\alpha \in \mathbb{Z}^2} S^\alpha \frac{e^{(\epsilon_2 \alpha_2 \tilde{w})}}{\sin^2((\pi \alpha_1 + \epsilon_2 \alpha_2 \tau^r))} e^{(\alpha_1 x_1 + \alpha_2 x_2)}, \quad (6.64) \]

Moreover, (6.61) (or (6.62)) are the monodromy preserving condition for linear partial differential equation

\[ \partial_z \Psi(x_1, x_2, \tilde{w}) + \{ L(x_1, x_2, \tilde{w}), \Psi(x_1, x_2, \tilde{w}) \} = 0. \quad (6.65) \]

### 6.6.3 Rational case

Here we apply dispersionless limit to the rational system constructed in subsection 6.6.1. In this way we come to the equation of motion for the Fourier modes

\[ \partial_{\tau^r} S^\alpha = \sum_{\gamma \in \mathbb{Z}^2} S^\gamma \frac{1}{(\epsilon^r \cdot \gamma)^2} S^{\alpha - \gamma}. \quad (6.66) \]

where in comparison with (6.20) we omitted \( \kappa^r \). This can be done since in rational case it is nothing else than scale of free parameters \( \epsilon_1^r \) and \( \epsilon_2^r \). In terms of fields \( S(x) \in \text{Ham}(T^2) \) equation of motion (6.66) turns into:

\[ \partial_{\tau^r} S = \{ S, \tilde{\partial}_Z^{-2} S \}. \quad (6.67) \]

The equations of motion are defined by the corresponding quadratic Hamiltonian

\[ H = \frac{1}{2} \int_{T^2} S \tilde{\partial}_Z^{-2} S = \frac{1}{2} \sum_{\alpha \in \mathbb{Z}^2} \frac{1}{(\epsilon^r \cdot \alpha)^2} S^\alpha S^{-\alpha}. \quad (6.68) \]

with respect to the linear Poisson bracket on \( \text{Ham}(T^2) \). It is worth noting that equation of motion (6.67) is equivalent to some partial differential equation. To show that we rewrite it in terms of the angular velocities \( F(x) \)

\[ \partial_{\tau^r} \tilde{\partial}_Z F(x, \tau^r) - \{ \tilde{\partial}_Z^2 F(x, \tau^r), F(x, \tau^r) \} + \epsilon_2^r \partial_2 \tilde{\partial}_Z F(x, \tau^r) \quad (6.69) \]

where \( \tilde{\partial}_Z = \frac{\epsilon_1^r \partial_1 - \epsilon_2^r \tau^r \partial_2}{2\pi i} \). This equation can be presented as an equation of the zero curvature. Using (6.27) and (6.28) we get

\[ L^r(x_1, x_2, w^r) = \sum_{\alpha \in \mathbb{Z}^2} S^\alpha \frac{e^{(\epsilon_2^r \alpha_2 w^r)}}{\epsilon_1^r \alpha_1 + \epsilon_2^r \alpha_2 \tau^r} e^{(\alpha_1 x_1 + \alpha_2 x_2)} = (\tilde{\partial}_Z)^{-1} S(x_1, x_2 + \epsilon_2^r w^r), \quad (6.70) \]

\[ M^r(x_1, x_2, w^r) = \frac{1}{2\pi i} \sum_{\alpha \in \mathbb{Z}^2} S^\alpha \frac{e^{(\epsilon_2^r \alpha_2 w^r)}}{\epsilon_1^r \alpha_1 + \epsilon_2^r \alpha_2 \tau^r} e^{(\alpha_1 x_1 + \alpha_2 x_2)} \quad (6.71) \]

\[ = \frac{1}{2\pi i} (\tilde{\partial}_Z)^{-2} S(x_1, x_2 + \epsilon_2^r w^r). \]

The equation (6.69) is the monodromy preserving condition for linear partial differential equation

\[ \partial_{w^r} \Psi(x_1, x_2, \tau^r, w^r) + \{ L^r(x_1, x_2, \tau^r, w^r), \Psi(x_1, x_2, \tau^r, w^r) \} = 0, \quad (6.72) \]

because the Baker-Akhiezer function satisfies the equation

\[ \partial_{\tau^r} \Psi(x_1, x_2, \tau^r, w^r) + \{ M^r(x_1, x_2, \tau^r, w^r), \Psi(x_1, x_2, \tau^r, w^r) \} = 0, \]
6.6.4 Scaling limit

In this subsection we construct the limiting procedure analogous to the one proposed in section 6.5. We use the same decomposition (6.29) of the parameter $\tau$ of the elliptic curve

$$
\tau = \tau_1 + \tau_2,
$$

(6.73)

where $\tau_1$ plays the role of time of the limiting system and $\tau_2$ gives the trigonometric limit $\Im m \tau_2 \to +\infty$. The limiting procedure consists of the shift of the spectral parameter

$$
w = \tilde{w} + \tau/2,
$$

(6.74)

the scalings of the coordinates

$$
S^\alpha = \tilde{S}^\alpha q_2^{-g(\alpha_2)}, \quad q_2 \equiv e(\tau_2), \quad g(\alpha_2) = c_2 \frac{1 - \delta_{\alpha_2,0}}{2},
$$

(6.75)

and the trigonometric limit $\Im m \tau_2 \to +\infty$. After scalings (6.31), we obtain the contraction of Poisson algebra (6.53) in the limit $\Im m \tau_2 \to +\infty$. All nonzero brackets for coordinates $\tilde{S}^\alpha$ of the limiting system have the form

$$
\{ \tilde{S}^{\alpha_1,0}, \tilde{S}^{\beta_1,\beta_2} \} = \alpha_1 \beta_2 \tilde{S}^{\alpha_1+\beta_1,\beta_2},
$$

(6.76)

where $\tilde{S}^{\alpha_1,\alpha_2} \equiv \tilde{S}^\alpha$, $\alpha \in \mathbb{Z}^2$. Again, it is important that we consider the subsystem with arbitrary large but finite number of nonzero coordinates $S^\alpha$. Though this subsystem is open in the initial system, it turns out to be closed in the limiting one. To compute the limits of the Hamiltonian (6.55) and the Lax operators (6.58), (6.59) we consider those values of constant $\epsilon_2$ such that for all nonzero coordinates $S^\alpha$ the following condition is true:

$$
|\epsilon_2 \alpha_2| < 1.
$$

(6.77)

Then for the Hamiltonian of the limiting system we have

$$
H = -\frac{\pi^2}{2} \sum_{\alpha_1 \in \mathbb{Z}^2 \setminus \{0\}} \frac{\tilde{S}^{\alpha_1,0} \tilde{S}^{\alpha_1,0}}{\sin^2 (\pi \epsilon_1 \alpha_1)} + 4\pi^2 q_1^2 \sum_{\alpha_1 \in \mathbb{Z}} e (\epsilon_1 \alpha_1) \tilde{S}^{\alpha_1,1} \tilde{S}^{-\alpha_1,1},
$$

(6.78)

where $q_1 \equiv e(\tau_1)$. Note that coordinates included in the Hamiltonian form a subalgebra of the limiting Poisson algebra (6.76). Thus, the Hamilton equations of motion for these coordinates

$$
\partial_{\tau_1} \tilde{S}^\alpha = \left\{ H, \tilde{S}^\alpha \right\}
$$

(6.79)

do not depend on the coordinates which are not included in the Hamiltonian. For the coordinates included in the Hamiltonian the equations of motion are of the form

$$
\partial_{\tau_1} \tilde{S}^{\gamma_1,0} = -4\pi^2 q_1^2 \sum_{\alpha_1 \in \mathbb{Z}} \gamma_1 \left( \tilde{S}^{\alpha_1+\gamma_1,1} e (\epsilon_1 \alpha_1) \tilde{S}^{-\alpha_1,-1} - \tilde{S}^{\alpha_1+\gamma_1,-1} e (-\epsilon_1 \alpha_1) \tilde{S}^{-\alpha_1,1} \right),
$$

(6.80)

$$
\partial_{\tau_1} \tilde{S}^{\gamma_1,1} = -\pi^2 \sum_{\alpha_1 \in \mathbb{Z} \setminus \{0\}} \tilde{S}^{\alpha_1+\gamma_1,1} \frac{\alpha_1}{\sin^2 (\pi \epsilon_1 \alpha_1)} \tilde{S}^{-\alpha_1,0},
$$

(6.81)

$$
\partial_{\tau_1} \tilde{S}^{\gamma_1,-1} = \pi^2 \sum_{\alpha_1 \in \mathbb{Z} \setminus \{0\}} \tilde{S}^{\alpha_1+\gamma_1,-1} \frac{\alpha_1}{\sin^2 (\pi \epsilon_1 \alpha_1)} \tilde{S}^{-\alpha_1,0}.
$$

(6.82)

These equations of motion also can be obtained by applying the dispersionless limit (6.32). We define Lax pair of the limiting system as limits of operators (6.58) and (6.59):

$$
\tilde{L} = \lim_{q_2 \to 0} L, \quad \tilde{M} = \lim_{q_2 \to 0} (M + \pi i L).
$$

(6.83)
This pair admits Lax equation
\[ \partial_{\tau_1} \tilde{L} - \frac{1}{2\pi i} \partial_{\omega} \tilde{M} = [\tilde{L}, \tilde{M}], \] (6.84)

where
\[
\tilde{L} = \pi \sum_{\alpha_1 \in \mathbb{Z} \setminus \{0\}} \frac{e^{-\epsilon_1 \alpha_1/2}}{\sin(\pi \epsilon_1 \alpha_1)} \tilde{S}^{\alpha_1,0} e(\alpha_1 x_1) - 2\pi i q_1^{\epsilon_2/2} \sum_{\alpha_1 \in \mathbb{Z}} e(\epsilon_2 \tilde{w}) \tilde{S}^{\alpha_1,1} e(\alpha_1 x_1 + x_2) + 2\pi i q_1^{\epsilon_2/2} \sum_{\alpha_1 \in \mathbb{Z}} e(\epsilon_1 \alpha_1 - \epsilon_2 \tilde{w}) \tilde{S}^{\alpha_1,-1} e(\alpha_1 x_1 - x_2),
\] (6.85)

\[
\tilde{M} = -\frac{\pi^2}{2} \sum_{\alpha_1 \in \mathbb{Z} \setminus \{0\}} \frac{1 + e^{-\epsilon_1 \alpha_1}}{\sin^2(\pi \epsilon_1 \alpha_1)} \tilde{S}^{\alpha_1,0} e(\alpha_1 x_1) + 2\pi^2 q_1^{\epsilon_2/2} \sum_{\alpha_1 \in \mathbb{Z}} e(\epsilon_2 \tilde{w}) \tilde{S}^{\alpha_1,1} e(\alpha_1 x_1 + x_2) + 2\pi^2 q_1^{\epsilon_2/2} \sum_{\alpha_1 \in \mathbb{Z}} e(\epsilon_1 \alpha_1 - \epsilon_2 \tilde{w}) \tilde{S}^{\alpha_1,-1} e(\alpha_1 x_1 - x_2).
\] (6.86)

Equation (6.84) is equivalent to equations of motion (6.80)–(6.82). Thus, in the limiting system Lax representation describes the equations of motion only for the coordinates included in the Hamiltonian (6.78). Since the Hamiltonian (6.78) depends only on coordinates of the form \( S^\alpha, |\alpha_2| \leq 1 \), we can pass to the following three field variables:
\[
h = h(x_1) = \sum_{\alpha_1 \in \mathbb{Z} \setminus \{0\}} \tilde{S}^{\alpha_1,0} e(\alpha_1 x_1), \tag{6.87}
\]

\[
f = f(x_1) = \sum_{\alpha_1 \in \mathbb{Z}} \tilde{S}^{\alpha_1,1} e(\alpha_1 x_1), \quad g = g(x_1) = \sum_{\alpha_1 \in \mathbb{Z}} \tilde{S}^{\alpha_1,-1} e(\alpha_1 x_1). \tag{6.88}
\]

Then the Hamiltonian (6.78) can be rewritten as follows
\[
H = \pi^2 \int_{S^1} \left( 2h(x_1) \sinh^{-2} \left( \frac{\epsilon_1 \partial_{x_1}}{2} \right) h(x_1) + 4q_1^{\epsilon_2} g(x_1) e^{-\epsilon_1 \partial_{x_1}} f(x_1) \right) dx_1. \tag{6.89}
\]

Equations of motion (6.80)–(6.82) in terms of field variables \( f, g, \) and \( h \) acquire the following form:
\[
\partial_{\tau_1} h = 2\pi i q_1^{\epsilon_2} \partial_{x_1} \left( f e^{-\epsilon_1 \partial_{x_1}} g - g e^{\epsilon_1 \partial_{x_1}} f \right), \tag{6.90}
\]

\[
\partial_{\tau_1} f = f \frac{\pi i}{2 \sinh^2(\epsilon_1 \partial_{x_1}/2)} \partial_{x_1} h, \tag{6.91}
\]

\[
\partial_{\tau_1} g = -g \frac{\pi i}{2 \sinh^2(\epsilon_1 \partial_{x_1}/2)} \partial_{x_1} h. \tag{6.92}
\]

These equations are again Hamiltonian
\[
\partial_{\tau_1} h = \{H, h\}, \quad \partial_{\tau_1} f = \{H, f\}, \quad \partial_{\tau_1} g = \{H, g\} \tag{6.93}
\]

with respect to the following "Poisson" brackets:
\[
\{h(x), f(y)\} = \frac{f(y)}{2\pi i} \delta'(x - y), \quad \{h(x), g(y)\} = -\frac{g(y)}{2\pi i} \delta'(x - y),
\]
\[
\{f(x), g(y)\} = 0, \tag{6.94}
\]

where \( \delta'(x - y) \) is the first derivative of the Dirac delta function. Notice that equations (6.91) and (6.92) imply
\[
\partial_{\tau_1} (f(x_1, \tau_1) g(x_1, \tau_1)) = 0. \tag{6.95}
\]
7 Appendix

7.1 List of abbreviations

- PI,...,PVI – Painlevé I,...,VI equations
- PVIFT – Painlevé VI field theory
- NCT – noncommutative torus
- NAVZG – non-autonomous Zhukovsky-Volterra gyrostat
- sinθ – noncommutative analogue of sin-algebra
- SDiff(\(M\)) – group of volume-preserving diffeomorphisms of \(M\)
- \(FConn\) – space of flat connections
- \(FBun\) – moduli space of flat connections
- \(F\operatorname{Conn}\) – space of smooth connections
- \(\omega^KK\) – Kirillov-Kostant symplectic form
- \(F\operatorname{T}\) – Painlevé VI field theory
- \(\operatorname{Conn}\) – space of smooth connections
- \(\operatorname{NAVZG}\) – non-autonomous Zhukovsky-Volterra gyrostat
- \(\sin\theta\) – noncommutative analogue of \(\sin\)-algebra
- \(\omega_1, \omega_2\) – fundamental half-periods, \(\tau = \omega_2/\omega_1\).
- \(\eta(\tau) = q^{\frac{1}{24}} \prod_{n>0} (1 - q^n)\).

7.2 Appendix A: Elliptic functions

Notations.

\( (e(x) = \exp 2\pi i(x), \quad q = e^{\frac{1}{2}\tau}) \).

\(\omega_1, \omega_2\) – fundamental half-periods, \(\tau = \omega_2/\omega_1\).

The theta function:

\[
\vartheta(z|\tau) = \sum_{n\in\mathbb{Z}} (-1)^n e^{\frac{1}{2}((n + \frac{1}{2})^2\tau + (2n + 1)z)} = 2q^{\frac{3}{8}} \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)} \sin \pi z \tag{A.1}
\]

The Eisenstein functions

\[
E_1(z|\tau) = \partial_z \log \vartheta(z|\tau), \quad E_1(z|\tau)|_{z\to 0} \sim \frac{1}{z} - 2\eta_1(z), \tag{A.2}
\]

\[
E_2(z|\tau) = -\partial_z E_1(z|\tau) = \partial_z^2 \log \vartheta(z|\tau), \quad E_2(z|\tau)|_{z\to 0} \sim \frac{1}{z^2} + 2\eta_1(z). \tag{A.3}
\]

Here

\[
\eta_1(\tau) = \frac{3}{\pi^2} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{1}{(m\tau + n)^2} = \frac{24 \eta'(\tau)}{2\pi i \eta(\tau)}, \tag{A.4}
\]

where

\[
\eta(\tau) = q^{\frac{1}{24}} \prod_{n>0} (1 - q^n). \]

is the Dedekind function.

Relation to the Weierstrass functions

\[
\zeta(z, \tau) = E_1(z, \tau) + 2\eta_1(\tau)z, \tag{A.5}
\]

\[
\wp(z, \tau) = E_2(z, \tau) - 2\eta_1(\tau). \tag{A.6}
\]

Important functions:

\[
\phi(u, z) = \frac{\vartheta(u + z)\vartheta'(0)}{\vartheta(u)\vartheta(z)}. \tag{A.7}
\]
It has a pole at \( z = 0 \) and
\[
\phi(u, z) = \frac{1}{z} + E_1(u) + \frac{z}{2}(E_1^2(u) - \varphi(u)) + \ldots \tag{A.8}
\]

Let
\[
f(u, z) = \partial_u \phi(u, z) = \phi(u, z)(E_1(u + z) - E_1(u)). \tag{A.9}
\]

**Addition theorems.**
\[
\phi(u, z)f(v, z) - \phi(v, z)f(u, z) = (E_2(v) - E_2(z))\phi(u + v, z). \tag{A.10}
\]
\[
\phi(u, z)\phi(-u, z) = (E_2(z) - E_2(u)). \tag{A.11}
\]
\[
\phi(u, z)\phi(-u, w) = \phi(u, z - w)[E_1(u + z - w) - E_1(u) + E_1(w) - E_1(z)]. \tag{A.12}
\]

**Heat equation**
\[
\partial_z \phi(u, w) - \frac{1}{2\pi i} \partial_u \partial_w \phi(u, w) = 0. \tag{A.13}
\]

**Parity**
\[
\phi(u, z) = \phi(z, u), \quad \phi(-u, -z) = -\phi(u, z). \tag{A.14}
\]
\[
E_1(-z) = -E_1(z), \quad E_2(-z) = E_2(z). \tag{A.15}
\]
\[
f(-u, -z) = f(u, z). \tag{A.16}
\]

**Quasi-periodicity**
\[
\vartheta(z + 1) = -\vartheta(z), \quad \vartheta(z + \tau) = -q^{-\frac{1}{2}} e^{-2\pi i \vartheta(z)}, \tag{A.17}
\]
\[
E_1(z + 1) = E_1(z), \quad E_1(z + \tau) = E_1(z) - 2\pi i, \tag{A.18}
\]
\[
E_2(z + 1) = E_2(z), \quad E_2(z + \tau) = E_2(z), \tag{A.19}
\]
\[
\phi(u, z + 1) = \phi(u, z), \quad \phi(u, z + \tau) = e^{-2\pi i u} \phi(u, z). \tag{A.20}
\]
\[
f(u, z + 1) = f(u, z), \quad f(u, z + \tau) = e^{-2\pi i u} f(u, z) - 2\pi i \phi(u, z). \tag{A.21}
\]

**Particular values**
\[
E_2^{(2j + 1)}(\tau/2) = E_2^{(2j + 1)}(1/2 + \tau/2) = E_2^{(2j + 1)}(1/2) = 0, \quad \langle E_2^{(j)}(u) = \partial_u \partial_j E_2(u) \rangle. \tag{A.22}
\]

**Degenerations I.**
Let \( \Im \tau \to +\infty \). Then
\[
\vartheta(z|\tau) \sim \sin(\pi z), \tag{A.23}
\]
\[
E_1^r(z) = \pi \cot(\pi z), \tag{A.24}
\]
\[
\varphi^{tr}(u, z) = \pi(\cot \pi u + \cot \pi z), \tag{A.25}
\]
\[
f(u, z)^{tr} = -\pi^2 \sin^{-2}\pi u, \tag{A.26}
\]
\[
E_2^{tr}(z) = \frac{\pi^2}{\sin^2(\pi z)}. \tag{A.27}
\]

The rational limit of the trigonometric functions assumes the form
\[
E_1^r(z) = \frac{1}{z}. \tag{A.28}
\]
\[ \phi'(u, z) = \frac{1}{u} + \frac{1}{z}, \]  
(A.29)

\[ f'(u, z) = \frac{1}{u^2}, \]  
(A.30)

\[ E_2(z) = \frac{1}{z^2}. \]  
(A.31)

The both types of functions satisfy the addition formulae (A.10), (A.11).

**Degenerations II.**

To evaluate limits of the Hamiltonian we need the decomposition of the second Eisenstein function with shifted argument in trigonometric limit \( \Im(\tau) \to +\infty \), or equivalently \( q = e^{2\pi i} \to 0 \). Directly from definition we get:

\[
E_2(u - g\tau) = \begin{cases} 
-4\pi^2 \frac{e(u)}{(1 - e(u))^2}, & \{g\} = 0, \\
-4\pi^2 e(-u) q^{\{g\}}, & 0 < \{g\} < \frac{1}{2}, \\
-4\pi^2 q^{1/2} (e(-u) + e(u)), & \{g\} = \frac{1}{2}, \\
-4\pi^2 e(u) q^{1-\{g\}}, & \frac{1}{2} < \{g\} < 1.
\end{cases} 
\]  
(A.32)

To evaluate various limits of Lax pair we need the decomposition of \( \sigma \) function with shifted arguments in trigonometric limit. Using definition (A.27) we reduce the expansion of \( \phi(u - \sigma, z - \varsigma \tau) \) to the expansion of theta functions:

\[
\phi(u - \sigma, z - \varsigma \tau) = \frac{\vartheta(u + z - (\sigma + \varsigma)\tau)\vartheta'(0)}{\vartheta(u - \sigma\tau)\vartheta(z - \varsigma \tau)},
\]  
(A.33)

and for the expansion of theta function we have:

\[
\vartheta(z + \sigma \tau) = [1 + o(1)] q^{\left(\sigma z - \frac{1}{2} \frac{\sigma}{\varsigma} \right)} \left[1 - \frac{1}{2\pi} \ln \left| \sin \left(\frac{\pi u}{\pi u + z} \right) \right| \right] \times
\]  
(A.34)

\[
\left[1 - \frac{1}{2\pi} \ln \left| \sin \left(\frac{\pi u}{\pi u + z} \right) \right| \right] \times
\]  
(A.35)

where \( \lfloor \sigma \rfloor \) is the integer part of \( \sigma \) and \( \{\sigma\} \) is the fractional part of \( \sigma \). This gives the following answer:

\[
\phi(u + \sigma, z + \varsigma \tau) = (1 + o(1)) \times
\]  
(A.36)
To evaluate the limits of \( f_\alpha (u + \omega \beta, z) \) we use the identity (A.9) and the expansion of \( E_1(u - \sigma \tau) \):

\[
E_1(u - \sigma \tau) = 2\pi i |\sigma| + \begin{cases} 
\pi \cot(\pi u) + o(1), & \{\sigma\} = 0, \\
\pi i + 2\pi i q^{(\sigma)} e(-u) + o(q^{(\sigma)}), & 0 < \{\sigma\} < \frac{1}{2}, \\
\pi i + 2\pi i^q (e(-u) - e(u)) + o(q^2), & \{\sigma\} = \frac{1}{2}, \\
\pi i - 2\pi i q^{1-(\sigma)} e(u) + o(q^{1-(\sigma)}), & \frac{1}{2} < \{\sigma\} < 1.
\end{cases}
\]

(37)

7.3 Appendix B: Noncommutative torus

In this Appendix we use [21, 70].

1. Definition and representation. The noncommutative torus \( T^2_\theta \) is a unital algebra with the two generators \( (U_1, U_2) \) that satisfy the relation

\[
U_1 U_2 = e^{-\frac{1}{\theta} \pi i \theta} U_2 U_1,
\]

(1)

Elements of \( T^2_\theta \) are the double sums

\[
T^2_\theta = \left\{ X = \sum_{a_1,a_2 \in \mathbb{Z}} c_{a_1,a_2} U_1^{a_1} U_2^{a_2} | \ c_{a_1,a_2} \in \mathbb{C} \right\}.
\]

(2)

It is convenient to introduce the following basis in \( T^2_\theta \)

\[
T^a = \frac{i}{2\pi \theta} e^{(-\frac{a_1 a_2}{2})} U_1^{a_1} U_2^{a_2} \quad \alpha \in \mathbb{Z}^{(2)} = \mathbb{Z} \oplus \mathbb{Z}.
\]

(3)

It follows from (B.1) that

\[
T^a T^b = -2\pi \theta e^{(a \times b) \frac{\pi i}{2} \theta} T^{a+b}.
\]

(4)

Therefore,

\[
T^a T^b = e^{(a \times b) \frac{\pi i}{2} \theta} T^{b} T^a.
\]

(5)

We can identify \( U_1, U_2 \) with matrices from \( \text{GL}(\infty) \). Define \( \text{GL}(\infty) \) as the associative algebra of infinite matrices \( c_{jk} E_{jk} \), where \( E_{jk} = |\delta_{jk}| \), such that

\[
\sup_{j,k \in \mathbb{Z}}|c_{jk}|^2 |j - k|^n < \infty \quad \text{for} \quad n \in \mathbb{N}.
\]

Consider the following two matrices from \( \text{GL}(\infty) \):

\[
Q = \text{diag}(e(j\theta)) \quad \text{and} \quad \Lambda = ||\delta_{j,j+1}||, \ j \in \mathbb{Z}.
\]

(6)

We have the following identification

\[
U_1 \rightarrow Q, \ U_2 \rightarrow \Lambda.
\]

(7)

Another useful realization of \( T^2_\theta \) in the Schwartz space on \( \mathbb{R} \) by the operators

\[
U_1 f(x) = f(x - \theta), \quad U_2 f(x) = \exp(2\pi i x) f(x).
\]

(8)

The trace functional on \( T^2_\theta \) is defined as

\[
\langle X \rangle = \text{tr}(X) = c_{00}.
\]

(9)

It satisfies the evident identities

\[
\langle 1 \rangle = 1, \quad \langle XY \rangle = \langle Y X \rangle.
\]

(10)
The relation of $T^2$ with the commutative algebra of smooth functions on the two-dimensional torus $T^2 = \mathbb{R}^2 / \mathbb{Z} \oplus \mathbb{Z}$ comes from the identification

$U_1 \rightarrow e(x_1), \ U_2 \rightarrow e(x_2), \ e(x_1) \ast e(x_2) = e^{-2\pi i\theta} e(x_2) \ast e(x_1),$

$$f(x) = \sum_{a \in \mathbb{Z}^{(2)}} f_a T_a(x), \ T^\alpha = T^\alpha(x) = \frac{i}{2\pi \theta} e \left( \frac{a_1 a_2}{2} \theta \right) e(a_1 x_1) e(a_2 x_2). \quad (B.11)$$

The multiplication on $T^2$ becomes the Moyal multiplication:

$$(f \ast g)(x) := fg + \sum_{n=1}^{\infty} \frac{(\pi \theta/n)!}{n!} \varepsilon_{r_1 S_1} \ldots \varepsilon_{r_n S_n} \hat{A}_{r_1 \ldots r_n}^n (\hat{A}_{S_1 \ldots S_n}^n f)(\hat{A}_{S_1 \ldots S_n}^n g), \quad \hat{\partial}_j = \frac{1}{2\pi i} \partial_j. \quad (B.12)$$

The trace functional (B.9) in the Moyal identification is the integral

$$\text{tr}_f = -\frac{1}{4\pi^2} \int_{\mathcal{A}_\theta} f dx_1 dx_2 = f_{00}. \quad (B.13)$$

2. sin-algebra.

We denote by $\mathfrak{g} = sin_\theta$ the Lie algebra with the generators $T^\alpha$ ($\alpha \in \mathbb{Z}^{(2)}$) over the ring $\mathbb{S}$

$$sin_\theta = \{ \psi = \sum_{\alpha \in \mathbb{Z}^{(2)}} \psi_\alpha T^\alpha, \ \psi_\alpha \in \mathbb{S} \}, \ \mathbb{S} = \{ \psi_\alpha = 0 \ \text{almost for all}, \ \alpha \in \mathbb{Z}^{(2)} \}. \quad (B.14)$$

In other words, $sin_\theta$ is a noncommutative analog of the algebra of the trigonometric polynomials. It follows from (B.11) that the commutator has the form

$$[T^\alpha, T^\beta] = C_\theta(\alpha, \beta) T^{\alpha + \beta}, \quad (B.15)$$

where

$$C_\theta(\alpha, \beta) = \frac{1}{\pi \theta} \sin \pi \theta (\alpha \times \beta), \ \alpha \times \beta = \alpha_1 \beta_2 - \alpha_2 \beta_1. \quad (B.16)$$

In the Moyal representation (B.12) the commutator of $sin_\theta$ has the form

$$[f(x_1, x_2), g(x_1, x_2)]_\theta := \frac{1}{\theta} (f \ast g - g \ast f) \quad (B.17)$$

The trace functional (B.9) allows one to define the coalgebra $\mathfrak{g}^* = sin_\theta^*$. It is defined as the linear space of distributions

$$sin_\theta^* = \left\{ \mathcal{S} \mid \int_{\mathcal{A}_\theta} \mathcal{S} \cdot \psi < \infty, \ \text{for} \ \psi \in sin_\theta \right\} \quad (B.18)$$

The group $SIN_\theta$ is the group of invertible elements from $\mathcal{A}_\theta$.

$$SIN_\theta = \{ \Psi = \sum_{\alpha \in \mathbb{Z}^{(2)}} \Psi_\alpha T^\alpha \}. \quad (B.19)$$

In this group $\frac{1}{2\pi \theta} T^0$ play the role of the identity element and $(T^\alpha)^{-1} = T^{-\alpha}$.

3. sine basis in $\mathfrak{gl}(N, \mathbb{C})$. This basis is a finite-dimensional version of (B.6). Let

$$\mathbb{Z}_N^{(2)} = (\mathbb{Z}_N \oplus \mathbb{Z}_N) \setminus \{0, 0\}. \quad (B.20)$$

Then

$$T^\alpha = e^{2\pi i a_1 a_2 Q^\alpha \Lambda^\alpha}, \ \alpha \in \mathbb{Z}_N^{(2)} \quad (B.21)$$
where $Q$ and $\Lambda$ are the t’Hooft matrices

$$Q = \text{diag}(e_N(1), e_N(2), \ldots, 1), \quad e_N(x) = \exp \frac{2\pi i}{N} x,$$

$$\Lambda = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$ \hfill (B.22)

$$[T^\alpha, T^\beta] = C(\alpha, \beta) T^{\alpha+\beta}, \quad C_N(\alpha, \beta) = \frac{N}{\pi} \sin \frac{\pi}{N}(\alpha \times \beta)$$ \hfill (B.23)

The action of the transition matrices is diagonalized in this basis

$$QT\alpha Q^{-1} = e(-\alpha_2/N)T^\alpha, \quad \Lambda T^\alpha \Lambda^{-1} = e(\alpha_1/N)T^\alpha.$$ \hfill (B.24)

If $a \in Z_{N}^2 \cup (0, 0)$ the $\{T^a\}$ forms a basis $\text{gl}(N, C)$.

4. NCT and two-loop algebras.

Assume that $\theta$ is a rational number $\theta = M/N$, $M < N$ and $(M, N)$ are coprime numbers. In this case $\sin_{\theta=M/N}$ is isomorphic to the two-loop algebra

$$LL(\text{gl}(N, C)) : C^* \times C^* \to \text{gl}(N, C).$$ \hfill (B.25)

This algebra has the basis

$$T_N^a = T^{a_1, a_2} e(a_1y_1 + a_2y_2), \quad a_j = a_j + m_jN, \quad 0 \leq a_j < N, \quad m_j \in Z$$ \hfill (B.26)

with commutation relations

$$[T_N^a, T_N^b] = C_N(\tilde{a}, \tilde{b}) T_N^{a+b}.$$ \hfill (B.27)

where $C_N(\tilde{a}, \tilde{b})$ is defined by (B.29). The isomorphisms is provided by the map of the basis $T^a \to T_N^a$.

For $\theta = M/N$ the commutator (B.27) coincides with (B.15).

5. $\text{SL}(2, C)$ case.

For $\text{SL}(2, C)$ the basis $T_\alpha$ is the basis of sigma-matrices

$$\sigma_0 = \text{Id}, \quad \sigma_1 = i\pi T^{0,1}, \quad \sigma_2 = i\pi T^{1,1}, \quad \sigma_3 = -i\pi T^{1,0},$$

$$\{\sigma_a\} = \{\sigma_0, \sigma_a\}, \quad (a = 0, \alpha), \quad (\alpha = 1, 2, 3), \quad [\sigma_\alpha, \sigma_\beta] = \epsilon_{\alpha, \beta, \gamma} \sigma_\gamma,$$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$ \hfill (B.28)

The standard theta-functions with the characteristics are

$$\theta_{0,0} = \theta_3, \quad \theta_{1,0} = \theta_2, \quad \theta_{0,1} = \theta_4, \quad \theta_{1,1} = \theta_1.$$ \hfill (B.29)

Half-periods:

$$\omega_a = (\omega_0, \omega_\alpha), \quad \omega_a = \frac{\alpha_1 + \alpha_2 \tau}{2}, \quad \alpha = (\alpha_1, \alpha_2), \quad \alpha_j = 0, 1.$$ \hfill (B.30)

| $\alpha$ | $(1,0)$ | $(0,1)$ | $(1,1)$ |
|----------|---------|---------|---------|
| $\sigma_\alpha$ | $\sigma_3$ | $\sigma_1$ | $\sigma_2$ |
| half-periods | $\omega_1 = 1$ | $\omega_2 = \frac{1}{2}$ | $\omega_3 = \frac{3+i}{2}$ |
| $\phi_\alpha(z)$ | $\varphi(z) \psi(0)$ | $\varphi(z) \psi(1)$ | $\varphi(z) \psi(2)$ |
| $\varphi(z) \psi(0)$ | $\varphi(z) \psi(1)$ | $\varphi(z) \psi(2)$ | $\varphi(z) \psi(3)$ |
Let
\[
\varphi_{\alpha}(u, z) = e(-\alpha z/2)\phi(-\omega_1 + u, z),
\]
(\ref{A.31})
\[
f_{\alpha}(u, z) = e(-\alpha z/2)f(-\omega_1 + u, z).
\]
(\ref{A.32})

Then from \((A.14), (A.20)\)
\[
\varphi(-u, z - \omega_b) = \begin{cases} 
-\varphi_0(u, z) & b = 0, \\
-e(-u_2\omega_b)\varphi_0(u, z - \omega_b) & b = \alpha, \\
e(-u_2\omega_b)\varphi_0(u, z - \omega_b) & b \neq \alpha,
\end{cases}
\]
(\ref{A.33})
\[
\varphi_{-\alpha}(u, z - \omega_b) = \varphi_0(u, z - \omega_b).
\]
(\ref{A.34})

6. Elliptic constants related to NCT \(A_\theta\).
Introduce two numbers \(\epsilon = (\epsilon_1, \epsilon_2)\) such that \(\epsilon_1 \theta < 1\) and \(\epsilon_2 \theta\) are irrational. Consider the dense set \(\mathbb{Z}_{\theta, \epsilon}(\tau)\) in \(E_\tau\):
\[
\mathbb{Z}_{\theta, \epsilon}(\tau) = \{((\epsilon_1 \gamma_1 + \tau \epsilon_2 \gamma_2)\theta = \theta \epsilon \cdot \gamma \in E_\tau \mid (\gamma_1, \gamma_2) \in \mathbb{Z}^2(\downarrow)\}
\]
(\ref{B.35})
The corresponding elliptic functions with the arguments from \(\mathbb{Z}_{\theta, \epsilon}(\tau)\) are as follows:
\[
\vartheta(\theta \epsilon \cdot \gamma), \quad \varphi(\theta \epsilon \cdot \gamma), \quad E_1(\theta \epsilon \cdot \gamma), \quad E_2(\theta \epsilon \cdot \gamma),
\]
(\ref{B.36})
\[
\varphi_{\theta \epsilon \cdot \gamma}(z) = e_\theta(\epsilon_2 \gamma_2 \omega_1)\phi(\theta \epsilon \cdot \gamma, z).
\]
(\ref{B.37})
\[
f_{\theta \epsilon \cdot \gamma}(z) = e_\theta(\epsilon_2 \gamma_2 \omega_1)\partial_x \phi(u, z)|_{u = \theta \epsilon \cdot \gamma}.
\]
(\ref{B.38})

It follows from \((A.20)\) that
\[
\varphi_{\theta \epsilon \cdot \gamma}(z + 1) = e_\theta(\epsilon_2 \gamma_2 \omega_1)\varphi_{\theta \epsilon \cdot \gamma}(z), \quad \varphi_{\theta \epsilon \cdot \gamma}(z + \tau) = e_\theta(-\epsilon_1 \gamma_1)\varphi_{\theta \epsilon \cdot \gamma}(z).
\]
(\ref{B.39})

7. Dispersionless limit.
In the limit \(\theta \to 0\) the Lie group \(SIN_\theta\) becomes the group of the volume preserving diffeomorphisms \(SDiff(T^2)\) of the two-dimensional torus \(T^2\) and the algebra \(sin_\theta\) becomes the Lie algebra of Hamiltonian functions
\[
Ham(T^2) \sim C^\infty(T^2)/\mathbb{C}
\]
(\ref{B.40})
equipped with the canonical Poisson brackets. In \(Ham(T^2)\) we have the Fourier basis
\[
e(\alpha \cdot x) = \exp(2\pi i(\alpha_1 x_1 + \alpha_2 x_2))
\]
(\ref{B.41})
instead of the basis \((B.33)\). The commutator \((B.15)\) becomes
\[
[e(\alpha x), e(\beta x)] = (\alpha \times \beta)e((\alpha + \beta) \cdot x),
\]
(\ref{B.42})
or in terms of functions \(f, g \in Ham(T^2)\)
\[
[f, g] := \{f, g\} = \partial_1 f \partial_2 g - \partial_2 f \partial_1 g, \quad \partial_j = \partial_{x_j}.
\]
(\ref{B.43})
The algebra \(Ham(T^2)\) (without constant Hamiltonians) is isomorphic to Lie algebra \(SVect_0(T^2)\) of the divergence-free zero-flux vector fields on \(T^2\) equipped with the area form \(-4\pi^2 dx_1 dx_2\). Let \(h(x_1, x_2) \in Ham(T^2)\). Then the Hamiltonian field \(V_h\) corresponding to the Hamiltonian function \(h\) is
\[
V_h = -\frac{1}{4\pi^2}((\partial_2 h)\partial_1 - (\partial_1 h)\partial_2),
\]
(\ref{B.44})
while
\[
[V_h, V_{h'}] = V_{\{h, h'\}}.
\]
(\ref{B.45})
For \(f(x) = \sum_\alpha f_\alpha e(\alpha \cdot x)\)
\[
\int_{T^2} f = -\frac{1}{4\pi^2}f_0.
\]
(\ref{B.46})
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