Exploiting Structure of Uncertainty for Efficient Combinatorial Semi-Bandits

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Abstract

We improve the efficiency of algorithms for stochastic combinatorial semi-bandits. In most interesting problems, state-of-the-art algorithms take advantage of structural properties of rewards, such as independence. However, while being minimax optimal in terms of regret, these algorithms are intractable. In our paper, we first reduce their implementation to a specific submodular maximization. Then, in case of matroid constraints, we design adapted approximation routines, thereby providing the first efficient algorithms that exploit the reward structure. In particular, we improve the state-of-the-art efficient gap-free regret bound by a factor \(\sqrt{k}\), where \(k\) is the maximum action size. Finally, we show how our improvement translates to more general budgeted combinatorial semi-bandits.

1. Introduction

Stochastic bandits model sequential decision-making in which an agent selects an arm (a decision) at each round and observes a realization of the corresponding unknown reward distribution. The goal is to maximize the expected cumulative reward, or equivalently, to minimize the expected regret, defined as the difference between the expected cumulative reward achieved by an oracle algorithm always selecting the optimal arm and that achieved by the agent. To accomplish this objective, the agent must trade-off between exploration (gaining information about reward distributions) and exploitation (using greedily the information collected so far) as it was already discovered by Robbins (1952). Bandits have been applied to many fields such as mechanism design (Mohri & Munoz, 2014), search advertising (Tran-Thanh et al., 2014), and personalized recommendation (Li et al., 2010). We improve the computational efficiency of their combinatorial generalization, in which the agent selects at each round a subset of arms, that we refer to as action in the rest of the paper (Cesa-Bianchi & Lugosi, 2012; Gai et al., 2012; Audibert et al., 2014; Chen et al., 2014).

Different kinds of feedback provided by the environment are possible. First, with bandit feedback (also called full bandit or opaque feedback), the agent only observes the total reward associated to the selected action. Second, with semi-bandit feedback, the agent observes the partial reward of each individual arm in the selected action. Finally, with full information feedback, the agent observes the partial reward of all arms. We give results for semi-bandit feedback.

There are two main questions that come up with combinatorial (semi)-bandits: 1° How can the stochastic structure of the reward vector be exploited to reduce the regret? and 2° Can algorithms be efficient? Combes et al. (2015) answer the first question assuming that reward distributions are mutually independent. Later, Degenne & Perchet (2016) generalize the algorithm of Combes et al. (2015) to a larger class of sub-Gaussian rewards by exploiting the covariance structure of the arms. They also show the optimality of proposed algorithms, in particular, that an upper bound on their regret matches the asymptotic lower bound of this class. The second question is studied by Kveton et al. (2015), who give an efficient algorithm based on the UCB algorithm of Auer et al. (2002). However, the algorithms of Combes et al. (2015) and Degenne & Perchet (2016) are computationally inefficient. While being efficient, the algorithm of Kveton et al. (2015) assumes the worst case class of arbitrary correlated rewards\(^1\), i.e., it does not exploit any properties of rewards, and therefore does not match the lower bound of Degenne & Perchet (2016).

Related work Efficiency in combinatorial bandits, or more generally, in linear bandits with a large set of arms is an open problem. Some methods as convex hull representation, (Koolen et al., 2010), hashing (Jun et al., 2017), or DAG-encoding (Sakaue et al., 2018) reduce the running time of algorithms. Concerning semi-bandit feedback, in the adversarial case, Neu & Bartók (2013) proposed an efficient implementation via geometric resampling. In the

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stochastic case, many efficient Bayesian algorithms exist
(see e.g., Russo & Van Roy, 2013), but are only shown to
be optimal for bayesian regret.\footnote{An easier problem
where arm distributions depend on a random parameter
drawn from a known prior, and expectation of the
regret is also taken with respect to this parameter.}
For frequentist regret, we are not aware of a work that
provides an efficient algorithm improving the regret bound of
Kveton et al. (2015) for the class of sub-Gaussian rewards.

**Our contributions** For \( k \) being the maximum action size,
we provide efficient algorithms that improve the regret
bound of Kveton et al. (2015) by a factor \( \sqrt{k} \) on the class of
sub-Gaussian rewards, when the combinatorial action set
is a matroid (Whitney, 1935). We first locate the source of
inefficiency of classical minimax optimal algorithms: At
each round, they solve a submodular maximization
problem. We then provide efficient, adapted LOCALSEARCH
and GREEDY-based algorithms that exploit the submodularity
to give approximation guarantees on the regret upper
bound. These algorithms can be of independent interest.
We also extend our approximation techniques to
more challenging budgeted combinatorial semi-bandits via
binary search methods and exhibit the same improvement
for this setting as well.

## 2. Background

We denote the set of arms by \([n] \triangleq \{1, 2, \ldots, n\}\), we type-
set vectors in bold and indicate components with indices,
i.e., \( \mathbf{a} = (a_i)_{i \in [n]} \in \mathbb{R}^n \). We let \( \mathcal{P}([n]) \triangleq \{A, A \subset [n]\} \)
be the power set of \([n]\). Let \( \mathbf{e}_i \in \mathbb{R}^n \) denote the \( i \)th
canonical unit vector. The incidence vector of any set \( A \in \mathcal{P}([n]) \)
is
\[
\mathbf{e}_A \triangleq \sum_{i \in A} \mathbf{e}_i.
\]
The above definition allows us to represent a subset of \([n]\)
as an element of \( \{0, 1\}^n \). We denote the Minkowski sum of
two sets \( Z, Z' \subset \mathbb{R}^n \) as \( Z + Z' \triangleq \{z + z', z \in Z, z' \in Z'\} \),
and \( Z + z' \triangleq Z + \{z'\} \). Let \( A \subset \mathcal{P}([n]) \) be a set of actions.
We define the maximum possible cardinality of an element of
\( A \) as \( k \triangleq \max\{|A|, A \in A\} \).

### 2.1. Stochastic Combinatorial Semi-Bandits

In combinatorial semi-bandits, an agent selects an action
\( A_t \in A \) at each round \( t \in \mathbb{N}^+ \), and receives a reward \( \mathbf{e}_{A_t}^\top \mathbf{X}_t \),
where \( \mathbf{X}_t \in \mathbb{R}^n \) is an unknown random vector of rewards.
The successive reward vectors \( \{\mathbf{X}_t\}_{t \geq 1} \) are i.i.d. with an
unknown mean \( \mathbf{\mu}^* \triangleq \mathbb{E}X_1 \in \mathbb{R}^n \), where \( \mathbf{X} = \mathbf{X}_1 \).
After selecting an action \( A_t \) in round \( t \), the agent observes
the partial reward of each individual arm in \( A_t \). The goal
of the agent is to minimize the expected regret, defined with
\[
A^* \in \arg \max_{A \in A} \mathbf{e}_A^\top \mathbf{\mu}^* \quad \text{as}
\]
\[
R_T \triangleq \mathbb{E} \left[ \sum_{t=1}^T (\mathbf{e}_{A_t}^\top - \mathbf{e}_{A_t}^\top)^\top \mathbf{X}_t \right].
\]
For any action \( A \in A \), we define its gap as the difference
\( \Delta_A \triangleq (\mathbf{e}_{A_t}^\top - \mathbf{e}_{A_t}^\top)^\top \mathbf{\mu}^* \). We then rewrite the expected
regret as \( R_T = \mathbb{E} \left[ \sum_{t=1}^T \Delta_{A_t} \right] \). We set \( \Delta \triangleq \min_{A \in A}, A \neq 0 \Delta_A \).

Semi-bandits have been introduced by
Cesa-Bianchi & Lugosi (2012). More recently, different
algorithms have been proposed (Combes et al., 2015;
Kveton et al., 2015; Degenne & Perchet, 2016), depending
whether the random vector \( X \) satisfies specific properties.
Some of these properties commonly assumed are a subset
of the following ones:

1. \( X_1, \ldots, X_n \in \mathbb{R} \) are mutually independent,
2. \( X_1, \ldots, X_n \in \mathbb{R} \) are arbitrary correlated,
3. \( X \in [−1, 1]^n \),
4. \( X \in \mathbb{R}^n \) is multivariate sub-Gaussian,
i.e., \( \mathbb{E}[e^{\lambda^\top X - \mu^\top X}] \leq e^{\lambda \lambda^\top / 2}, \forall \lambda \in \mathbb{R}^n \),
5. \( X \in \mathbb{R}^n \) is component-wise sub-Gaussian,
i.e., \( \mathbb{E}[e^{\lambda_i (X_i - \mu_i)}] \leq e^{\lambda_i^2 / 2}, \forall i \in [n], \forall \lambda \in \mathbb{R}^n \).

### 2.2. Lower Bounds

Combining some of the above properties, we consider dif-
erent classes of possible distributions for \( X \). In Table 1,
we show two existing gap-dependent lower bounds on \( R_T \)
that depend on the respective class. They are valid for at
least one distribution of \( X \) belonging to the corresponding
class, one combinatorial structure \( A \subset \mathcal{P}([n]) \), and for any
consistent algorithms (Lai & Robbins, 1985), for which
the regret on any problem verifies \( R_T = o(T^n) \) as \( T \to \infty \) for
all \( a > 0 \). Table 1 suggests that a tighter regret rate can
be reached with some prior knowledge on the random vector \( X \).

### 3. (In)efficiency of Existing Algorithms

In this section, we discuss the efficiency of algorithms
matching the lower bounds in Table 1. We consider that
an algorithm is **efficient** as soon as the time and space
complexity for each round \( t \) is polynomial in \( n \) and polylogarithmic\footnote{In streaming settings with near real-time requirements, it is
imperative to have algorithms that can run with a complexity that
stay almost constant across rounds.} in \( t \). Notice that the per-round complexity depends
substantially on \( A \). We assume \( A \) is such that linear opti-
mization problems on \( A \) — of the form \( \max_{A \in A} \mathbf{e}_A^\top \delta \) for
Exploiting Structure of Uncertainty for Efficient Combinatorial Semi-Bandits

Table 1. Gap-dependent lower bounds proved on different classes of possible distributions for $X$.

| Class of possible reward distributions | Gap-dependent lower bound |
|----------------------------------------|---------------------------|
| $(i) + (iii)$                           | $\Omega \left( \frac{n \log T}{\Delta} \right)$ Combes et al., 2015 |
| $(i) + (v)$                             |                           |
| $(i) + (iv)$                            |                           |
| $(ii) + (iii)$                          | $\Omega \left( \frac{nk \log T}{\Delta} \right)$ Kveton et al., 2015 |
| $(ii) + (v)$                            |                           |

some $\delta \in \mathbb{R}^n$ — can be solved efficiently. As a consequence, an agent knowing $\mu^*$ can efficiently compute $A^*$.

### 3.1. A Generic Algorithm

As mentioned above, the action $A_t$ is selected based on the feedback received up to round $t - 1$. A common statistic computed from this feedback is the empirical average of each arm $i \in [n]$, defined as

$$\overline{X}_{i,t-1} = \frac{\sum_{u \in [t-1]} \mathbb{1}\{i \in A_u\} X_{i,u}}{N_{i,t-1}},$$

where $N_{i,t-1} \triangleq \sum_{u \in [t-1]} \mathbb{1}\{i \in A_u\}$. Many combinatorial semi-bandit algorithms, in particular, those listed in Table 2, can be seen as a special case of Algorithm 1 for different confidence regions $C_t$ around $\overline{X}_{i,t-1}$.

**Algorithm 1** Generic confidence-region-based algorithm

At each round $t$:

- Find a confidence region $C_t \subset \mathbb{R}^n$.
- Solve the bilinear program

$$\left( \mu_t, A_t \right) \in \arg \max_{\mu \in C_t, A \in A} e_A^T \mu.$$

- Play $A_t$.

We further assume that $C_t$ is defined through some parameters $p, r \in \{1, \infty\}$, and some functions $g_{i,t}, i \in [n]$ by

$$C_t \triangleq [-r, r]^n \cap \left( \overline{X}_{i,t-1} + \left\{ \delta \in \mathbb{R}^n, \left\| (g_{i,t}(\delta_i))_i \right\|_p \leq 1 \right\} \right),$$

where $g_{i,t} = 0$ if $N_{i,t-1} = 0$ and, otherwise, is convex, strictly decreasing on $[-r - \overline{X}_{i,t-1}, 0]$ and strictly increasing on $[0, r - \overline{X}_{i,t-1}]$ such that $g_{i,t}(0) = 0$. Typically, $r = 1$ under assumption (iii) and $r = \infty$ otherwise. Table 2 lists variants of Algorithm 1, with the corresponding reward class under which they can be used. Each of these algorithms is matching the lower bound corresponding to the respective reward class considered in Table 1, i.e., $R_T$ is a ‘big $O$’ of the lower bound, up to a polylogarithmic factor in $k$ (Degenne & Perchet, 2016). To the best of our knowledge, the algorithms that are variants of Algorithm 1 are the only ones matching the corresponding lower bound. Indeed, the analysis of Thompson sampling by Wang & Chen (2018) matches the lower bound $nk \log(T)/\Delta$, but only for mutually independent rewards — in other words, the bound is not tight. The regret upper bound of algorithms in Table 1 with $p = 1$ have an additive constant term w.r.t. $T$ but exponential in $n$, which can be replaced with a different analysis to get either:

- an exponential term in $k$ plus a term of order $1/\Delta^2$,
- a term of order $1/\Delta^2$ (changing log($t$) to log($t$) + $k \log \log(t)$ in the algorithm),
- or can be removed (changing log($t$) to log($t$) + $n \log \log(t)$ in the algorithm).

On one hand, the arbitrary correlated case can be considered as solved, since the matching lower bound algorithm CUCB (Kveton et al., 2015) is efficient. On the other hand, if we tighten the reward class to target a sharper regret bound, the known algorithms are intractable. We further develop on the efficiency of Algorithm 1 in the following subsection.

### 3.2. Submodular Maximization

In Algorithm 1, only $A_t$ needs to be computed. It is a maximizer over $A$ of the set function

$$\mathcal{P}([n]) \rightarrow \mathbb{R}^n \rightarrow \max_{\mu \in C_t} e_A^T \mu.$$

We can easily evaluate the function (1) above for some set $A \in \mathcal{P}([n])$, since it only requires solving a linear optimization problem on the convex set $C_t$. In Proposition 1, we show that in some cases, the evaluation can be even simpler. However, maximizing the function (1) over a combinatorial set $A$ is not straightforward. Before studying this function more closely, Definition 1 recalls some well-known properties that can be satisfied by a set function $F : \mathcal{P}([n]) \rightarrow \mathbb{R}$.

**Definition 1**. A set function $F$ is:

- **normalized**, if $F(\emptyset) = 0$,
- **linear (or modular)** if $F(A) = e_A^T \delta + b$, for some $\delta \in \mathbb{R}^n, b \in \mathbb{R}$,
- **non-decreasing** if $F(A) \leq F(B) \forall A \subset B \subset [n]$.

4In Theorem 1, we recover that $A_t$ is computed by Algorithm 1 by solving a linear optimization problem.
5$A$ may have up to $2^n$ elements.
6$C_t$ is convex since functions $g_{i,t}$ are.
Algorithm and Example

1. **δ**

2. **ESCB-1 (Combes et al., 2015)**

3. **ESCB-2 (Combes et al., 2015)**

4. **OLS-UCB (Degene & Perchet, 2016)**

- **submodular if for all** $A, B \subseteq [n],\n
F(A \cup B) + F(A \cap B) \leq F(A) + F(B).$

Equivalently, $F$ is submodular if for all $A \subseteq B \subseteq [n], and i \notin B,$ $F(A \cup \{i\}) - F(A) \geq F(B \cup \{i\}) - F(B).$

The function (1) is clearly normalized, and it can be decomposed into two set functions in the following way,

$$\forall A \subseteq [n], \max_{\mu \in C_{t}} e_{A}^{i} \mu = e_{A}^{i} \pi_{t-1} + \max_{\delta \in \pi_{t-1}} e_{A}^{i} \delta.$$

The linear part $A \mapsto e_{A}^{i} \pi_{t-1}$ is efficiently maximized alone, we thus focus on the other part, $A \mapsto \max_{\delta \in \pi_{t-1}} e_{A}^{i} \delta,$ usually called exploration bonus. It aims to compensate for the negative selection bias of the first term. We define

$$C_{t}^{+} \triangleq [-r, r]^n \cup \{(\delta \in \mathbb{R}_+^n, \|f_i(\delta_i)\|_p \leq 1)\}$$

$$= C_{t} \cap \{\mu \in \mathbb{R}_+^n, \mu \geq \pi_{t-1}\},$$

and rewrite $A \mapsto \max_{\delta \in C_{t} \cap \pi_{t-1}} e_{A}^{i} \delta$ through Lemma 1.

**Lemma 1.** For all $A \in \mathcal{P}(\mathbb{N}), \max_{\delta \in \pi_{t-1}} e_{A}^{i} \delta = \max_{\delta \in C_{t} \cap \pi_{t-1}} e_{A}^{i} \delta.$

The proof is a consequence of $\{\{\delta_i^+\}_{i}, \delta \in C_{t} \cap \pi_{t-1}\} \subset C_{t} \cap \pi_{t-1}$. As a corollary, this set function is non-negative, and non-decreasing. It can be written in closed form under additional assumptions, see Proposition 1 and Example 1.

**Proposition 1.** Let $A \in \mathcal{P}(\mathbb{N}), t \in \mathbb{N}^*, p = 1.$ Assume that for all $i \in A, g_{i,t}$ has a strictly increasing, continuous derivative $g'_{i,t},$ defined on $[0, r - \pi_{t-1}].$ For $i \in A, let$

$$f_{i}(\lambda) \triangleq \begin{cases} g'_{i,t}(1/\lambda) & \text{if } 1/\lambda < g'_{i,t}(r - \pi_{t-1}) \\ r - \pi_{t-1} & \text{otherwise,} \end{cases}$$

defined for $\lambda \geq 0.$ Then, the smallest $\lambda^*$ satisfying

$$e_{A}^{i} (g_{i,t}(f_{i}(\lambda^*))) \leq 1$$

is such that

$$(\delta_i^*)^i \triangleq (1\{i \in A\} f_{i}(\lambda^*))^i \in \arg\max_{\delta \in C_{t}^{+} \cap \pi_{t-1}} e_{A}^{i} \delta.$$

**Proof.** It suffices to maximize on the coordinates of $\delta$ belonging to $A$ (the others being zero). For all $i \in A,$ we let

$$\eta^*_i \triangleq (1 - \lambda^* g'_{i,t}(r - \pi_{t-1}))\{\delta^*_i = r - \pi_{t-1}\}$$

$$\gamma^*_i \triangleq (\lambda^* g'_{i,t}(0) - 1)\{\delta^*_i = 0\}.$$

For all $i \in A,$ the function $f_i$ is continuous, non-increasing on $\mathbb{R}_+,$ hence so is $\lambda \mapsto e_{A}^{i} (g_{i,t}(f_{i}(\lambda))).$ If $e_{A}^{i} (g_{i,t}(f_{i}(\lambda^*)))^i < 1,$ then necessarily $\lambda^* = 0.$ Thus, the following KKT conditions are satisfied:

$$\lambda^* \{\sum_{i \in A} g_{i,t}(\delta^*_i) - 1\} = 0,$$

$$\forall i \in A, \lambda^* g'_{i,t}(\delta^*_i) + \eta^*_i - \gamma^*_i = 0,$$

$$\gamma^*_i (\delta^*_i - r + \pi_{t-1}) = 0,$$

$$-\lambda^* \delta^*_i = 0,$$

which concludes the proof by the convexity of the constraints and the objective function. \hfill \square

An important use-case example of Proposition 1 is the following

**Example 1.** Let $A \in \mathcal{P}(\mathbb{N}), t \in \mathbb{N}^*.$ If for all $i \in [n], g_{i,t} = (.)^2 \alpha_{i,t}$ for some $\alpha_{i,t} > 0,$ and $r = \infty, p = 1,$ then

$$\max_{\delta \in C_{t}^{+} \cap \pi_{t-1}} e_{A}^{i} \delta = e_{A}^{i} \left(\frac{1}{\alpha_{i,t}}\right)^i.$$
It is the square root of a non-decreasing linear set function. Such a set function is known to be submodular (Stobbe & Krause, 2010). This interesting property helps for maximizing the function \( f \). In Theorem 1, we prove that \( A \mapsto \max_{\delta \in C^+} \mu^T \delta \) is in fact always submodular.

**Theorem 1.** The following two properties hold.

- For \( p = \infty \), \( A \mapsto \max_{\delta \in C^+} \mu^T \delta \) is linear.
- For \( p = 1 \), \( A \mapsto \max_{\delta \in C^+} \mu^T \delta \) is submodular.

The proof is deferred to Appendix A and uses a result on polymatroids by He et al. (2012). Theorem 1 first implies the efficiency of any variant of Algorithm 1 with \( p = \infty \), since it reduces to optimizing a linear set function over \( A \). Theorem 1 also yields that when the reward class is strength to target the tighter lower bound \( n \log(T)/\Delta \). Algorithm 1 reduces to maximizing a submodular set function over \( A \) (the sum of a linear and a submodular function is submodular). Submodular maximization problems have been applied in machine learning before (see e.g., Krause & Golovin, 2011; Bach, 2011), however, maximizing a submodular function \( F \), even for \( A = \{A, |A| \leq k\} \) and \( F \) non-decreasing, is NP-Hard in general (Schrijver, 2008), with an approximation factor of \( 1 + 1/(e - 1) \) by the Greedy algorithm (Nemhauser et al., 1978). This is problematic as the typical analysis is based on controlling with high probability the error \( \Delta_{A_i} \) at round \( t \) using \( \max_{\delta \in C^+} \mu^T \delta \), which converges to zero as \( C_t - \mu_{t-1} \) becomes increasingly tight along axis \( i \in A_t \) (because counters \( N_{i,t-1} \) increases for \( i \in A_t \)). More precisely, since \( \mu^* \) belongs with high probability to the confidence region \( C_t - \mu_{t-1} \) belongs with high probability to \( C_t - \mu_{t-1} \). For \( \kappa \geq 1 \), a \( \kappa \)-approximation algorithm for maximizing the function (1) would only guarantee the following:

\[
\Delta_{A_i} = (e_{A_i} - e_{A_i^*})^T \mu^*
\leq \max_{\mu \in C_t} e_{A_i}^T \mu - e_{A_i^*}^T \mu^* \\
= \max_{\mu \in C_t} e_{A_i}^T \mu - e_{A_i^*}^T \mu^* \\
\leq \kappa \max_{\mu \in C_t} e_{A_i}^T \mu - e_{A_i^*}^T \mu^* \\
= \kappa \max_{\delta \in C_t} e_{A_i}^T \delta \\
+ (1 - \kappa) e_{A_i}^T (\mu_{t-1} - \mu^*) \\
\leq \kappa + 1 \max_{\delta \in C_t} e_{A_i}^T (\delta_t) + (1 - \kappa) e_{A_i}^T \mu_{t-1}.
\]

The term \( (\kappa - 1) e_{A_i}^T \mu_{t-1} \) gives linear regret bounds. In the next section, with a stronger assumption on \( A \), we show that both parts of the objective can have different approximation factors. More precisely, we show how to approximate the linear part with factor 1, and the submodular part with a constant factor \( \kappa \geq 1 \). Then, (2) can be replaced by

\[
\kappa \cdot \max_{\mu \in C_t} e_{A_i}^T \mu + 1 \cdot e_{A_i}^T \mu_{t-1}.
\]

Therefore, in (3), the extra \( (\kappa - 1) e_{A_i}^T \mu_{t-1} \) term is removed.

### 4. Efficient Algorithms for Matroid Constraints

In this section, we will consider additional structure on \( A \), using the notion of matroid, recalled below.

**Definition 2.** A matroid is a pair \((\mathcal{I}, \mathcal{C})\), where \( \mathcal{I} \) is a family of subsets of \( \{1, \ldots, n\} \), called the independent sets, with the following properties:

- The empty set is independent, i.e., \( \emptyset \in \mathcal{I} \).
- Every subset of an independent set is independent, i.e., for all \( A \in \mathcal{I} \), if \( A' \subset A \), then \( A' \in \mathcal{I} \).
- If \( A \) and \( B \) are two independent sets, and \( |A| > |B| \), then there exists \( x \in A \setminus B \) such that \( B \cup \{x\} \in \mathcal{I} \).

Matroids generalize the notion of linear independence. A maximal (for the inclusion) independent set is called basis and all bases have the same cardinality \( k \), which is called the rank of the matroid (Whitney, 1935). Many combinatorial problems such as building a spanning tree for network routing (Oliveira & Pardalos, 2005) can be expressed as a linear optimization on a matroid (see Edmonds & Fulkerson, 1965 or Perfect, 1968, for other examples).

Let \( \mathcal{I} \in \mathcal{P}(\{1, \ldots, n\}) \) be such that \((\{1, \ldots, n\}, \mathcal{I})\) forms a matroid. Let \( B \subset \mathcal{I} \) be the set of bases of the matroid \((\{1, \ldots, n\}, \mathcal{I})\). In the following, we may assume that \( A \) is either \( \mathcal{I} \) or \( B \). We also assume that an independence oracle is available, i.e., given an input \( A \subset \{1, \ldots, n\} \), it returns true if \( A \in \mathcal{I} \) and false otherwise. Maximizing a linear set function \( L \) on \( A \) is efficient, and it can be done as follows (Edmonds, 1971): Let \( \sigma \) be a permutation of \( [n] \) and \( j \) an integer such that \( j = k \) in case \( A = B \) and otherwise, \( j \) satisfies

\[
\ell_1 \geq \cdots \geq \ell_j \geq 0 \geq \ell_{j+1} \geq \cdots \geq \ell_n,
\]

where \( \ell_i = L(\{\sigma(i)\}) \) for \( i \in [n] \). The optimal \( S \) is built greedily starting from \( S = \emptyset \), and for \( i \in [j], \sigma(i) \) is added to \( S \) only if \( S \cup \{\sigma(i)\} \in \mathcal{I} \).

Matroid bandits with \( A = B \) has been studied by Kveton et al. (2014). In this case, the two lower bounds in Table 1 coincide to \( \Omega(n \log(T)/\Delta) \), and CUCB reaches it, with the following gap-free upper bound: \( R_T(\text{CUCB}) = O\left(\sqrt{n k T \log(T)}\right) \). Assuming sub-Gaussian rewards to
use any Algorithm of Table 2 with $p = 1$ would tighten
this gap-free upper bound to $O\left(n \log^2(k)T \log(T)\right)$
(Degenne & Perchet, 2016). In the rest of this section, we
provide efficient approximation routines to maximize the
function (1) on $A = \mathcal{I}$ and $B$ without having the extra undesirable term $(\kappa - 1)e_A^T \pi_{t-1}$, that a standard $\kappa$-approximation algorithm would suffer. Therefore, using these routines in Algorithm 1 do not alter its regret upper bound.

Let $L$ be a normalized, linear set function, that will
correspond to the linear part $A \mapsto e_A^T \pi_{t-1}$; and let $F$
 denote a normalized, non-decreasing, submodular function,
that will correspond to the submodular part $A \mapsto \max_{S \in \mathcal{C}_A^t - \pi_{t-1}} e_A^S$. Unless stated otherwise, we further assume that $F$ is positive (for $A \neq \emptyset$). This is a mild assumption as it holds for $A \mapsto \max_{S \in \mathcal{C}_A^t - \pi_{t-1}} e_A^S$ in the unbounded case (i.e., if $(iii)$ is not assumed and $r = \infty$).

If $(iii)$ is true, then adding an extra $e_A^T \left(\frac{1}{\sqrt{\kappa}} \pi_{t-1}\right)$ term will recover positivity and increase the regret upper bound by only an additive constant. In the following subsections, we will provide algorithms that efficiently outputs $S$ such that
\[ L(S) + \kappa F(S) \geq L(O) + F(O), \quad \forall O \in A, \] (4)
with some appropriate approximation factor $\kappa \geq 1$. It is possible to efficiently output $S_1$ and $S_2$ such that $L(S_1) \geq L(O_1)$ and $\kappa F(S_2) \geq F(O_2)$ for any $O_1, O_2 \in A$. Although we can take $O_1 = O_2, S_1$ and $S_2$ are not necessarily equal, and (4) is not straightforward. The last subsection is an application to budgeted matroid semi-bandits.

4.1. Local Search Algorithm

In this subsection, we assume that $A = \mathcal{I}$. In Algorithm 2 we provide a specific instance of LOCALSEARCH that we tailored to our needs to approximately maximize $L + F$. It starts from the greedy solution $S_{\text{init}} = \arg \max_{A \in \mathcal{I}} L(A)$. Then, Algorithm 2 repeatedly tries three basic operations in order to improve the current solution. Since every $S \in A$ can potentially be visited,
only significative improvements are considered, i.e., improvements greater than $\frac{\delta}{k} F(S)$ for some input parameter $\delta > 0$.

The smaller $\delta$ is, the higher complexity will be. Notice the improvement threshold $\frac{\delta}{k} F(S)$ does not depend on $L$. In fact, this is crucial to ensure that the approximation factor of $L$ is 1. However, this can increase the time complexity. To prevent this increase, the second essential ingredient is the initialization, where only $L$ plays a role. In Theorem 2, we state the approximation guarantees for Algorithm 2 and its time complexity. The proof of Theorem 2 can be found in Appendix B. For $C_t$ given by any algorithm of Table 2, $F(A) = \max_{S \in \mathcal{C}_A^t - \pi_{t-1}} e_A^S$, and $\varepsilon = 1$, the time complexity is bounded by $O\left(k^2 \log(kT)\right)$, and is thus tractable. Another algorithm enjoying an improved time complexity is provided in the next subsection, in the case where $A = B$.

\[ L(S) + 2(1 + \varepsilon) F(S) \geq L(O) + F(O), \quad \forall O \in \mathcal{I}. \]

Its complexity is $O\left(\kappa n \log\left(\frac{\max_{A \in \mathcal{I}} F(A)}{F(S_{\text{init}})}\right) / \log(1 + \frac{\varepsilon}{k})\right)$.

Theorem 2 gives a parameter $\kappa$ arbitrary close to 2 in (4).\footnote{We could design a different version of Algorithm 2, based on NON-OBLIVIOUSLOCALSEARCH (Filmus & Ward, 2012), in order to get $\kappa$ arbitrary close to $1 + 1/(\varepsilon - 1)$ but with a much worst time complexity. Actually, Sviridenko et al. (2013) proposed such an approach, with an approximation factor for $L$ arbitrary close to 1, but not equal, so we would get back the undesirable term, which would require a complexity polynomial in $t$ to control.}
4.2. Greedy Algorithm

In this section, we assume that $A = B$. This situation happens, for instance, under a non-negativity assumption on $L$, i.e., if we consider non-negative rewards $X_i$. We show that the standard GREEDY algorithm (Algorithm 3) improves over Algorithm 2 by exploiting this extra constraint, both in terms of the running time and the approximation factor.

We state the result in Theorem 3 and prove it in Appendix C. Notice that another advantage is that we do not need to assume $F(A) > 0$ for $A \neq \emptyset$ here.

Algorithm 3 GREEDY for maximizing $L + F$ on $B$.

Input: $L$, $F$, $I$, $k$.
Initialization: $S \leftarrow \emptyset$.

for $i \in [k]$ do
  $x \in \arg\max_{x \notin S, S \cup \{x\} \in I}(L + F)\left(S \cup \{x\}\right)$.
  $S \leftarrow S \cup \{x\}$.
end for

Output: $S$.

Theorem 3. Algorithm 3 outputs $S \in B$ such that

$$L(S) + 2F(S) \geq L(O) + F(O), \quad \forall O \in B.$$ 

Its complexity is $O(kn)$.

Combining the results before, we get the following theorem.

Theorem 4. With approximation techniques, the cumulative regret for the combinatorial semi-bandits is bounded as

$$R_T \leq O\left(\sqrt{n \log^2(k)T \log T}\right)$$

with per-round time complexity of order $O((\log(k)t)^2n)$ (resp., $O(kn)$) for $A = I$ (resp., $A = B$).

4.3. Budgeted Matroid Semi-Bandits

In this subsection, we extend results of the two previous subsections to budgeted matroid semi-bandits. In budgeted bandits with single resource and infinite horizon (Ding et al., 2013; Xia et al., 2016a), each arm is associated with both a reward and a cost. The agent aims at maximizing the cumulative reward under a budget constraint for the cumulative costs. Xia et al. (2016b) studied budgeted bandits with multiple play, where a $k$-subset $A$ of arms is selected at each round. An optimal (up to a constant term) offline algorithm chooses the same action $A^*$ within each round, where $A^*$ is the minimizer of the ratio “expected cost paid choosing $A^*$” over “expected reward gained choosing $A^*$”. In the setting of Xia et al. (2016b), the agent observes the partial random cost and reward of each arm in $A$ (i.e., semi-bandit feedback), pays the sum of partial costs of $A$ and gains the sum of partial rewards of $A$. $A^*$ can be computed efficiently, and a Xia et al. (2016b) give an algorithm based on CUBC. It minimizes the ratio where the averages are replaced by UCBs. We extend this setting to matroid constraints. We assume that total costs/rewards are non-negative linear set functions of the chosen action $A$. The objective is to minimize a ratio of linear set functions. As previously, two kinds of constraints can be considered for the minimization: either $A = I$ or $A = B$. Theorem 1 implies that an optimistic estimation of this ratio is of the form $\frac{L_2(F_2)}{L_1(F_1)}$, where $F_i$ is positive (except for $\emptyset$), normalized, non-decreasing, submodular; and $L_i$ are non-negative and linear. $L_1 - F_1$ is a high-probability lower bound on the expected cost paid, and $L_2 + F_2$ is a high-probability upper bound on the expected reward gained. Notice that the numerator, $L_1 - F_1$, can be negative, which can be an incitement to take arms with a high cost/low rewards. Therefore, we consider the minimization of the surrogate $\frac{L_1 - F_1}{L_2 + F_2}$. Indeed, $(L_1 - F_1)/L_2 + F_2$ is a high probability lower bound on the ratio of expectation, so by monotonicity of $x \mapsto x^+$ on $\mathbb{R}$, $((L_1 - F_1)/(L_2 + F_2))^+$ is also a high-probability lower bound. We assume $L_2$ is normalized, but not necessarily $L_1$. $L_1(\emptyset)$ can be seen as an entry price. When $L_1$ is normalized, we assume that $\emptyset$ is not feasible.

Remark 2. Notice, if $A = I$, and $L_1$ is normalized, then there is an optimal solution of the form $\{s\} \in I$: If $L_1 - F_1$ is negative for some $S = \{s\} \subset I$, then such $S$ is a minimizer. Otherwise, by submodularity (and thus subadditivity, since we consider normalized functions), $L_1 - F_1$ is non-negative, and we have

$$\frac{L_1(S) - F_1(S)}{L_2(S) + F_2(S)} \geq \frac{\min_{s \in S} L_1(\{s\}) - F_1(\{s\})}{\sum_{s \in S} L_2(\{s\}) + F_2(\{s\})}.$$

This minimization problem is at least as difficult as previous submodular maximization problems (taking $L_1 = 1$ and $F_1 = 0$). In order to use our approximation algorithms, we consider the Lagrangian function associated to the problem (see e.g., Fujishige, 2005):

$$\mathcal{L}(\lambda, S) \triangleq L_1(S) - F_1(S) - \lambda(L_2(S) + F_2(S)),$$

for $\lambda \geq 0$ and $S \subset [n]$. For a fixed $\lambda \geq 0$, $-\mathcal{L}(\lambda, \cdot)$ is a sum of a linear and a submodular function, and both Algorithms 2 and 3 can be used. However, the first step is to find $\lambda$ sufficiently close to the optimal ratio $\lambda^* = \min_{A \in A} \left(\frac{L_1(A) - F_1(A)}{L_2(A) + F_2(A)}\right)^+$. Remark 3. For some $\lambda \geq 0$,

$$\min_{A \in A} \mathcal{L}(\lambda, A) \geq 0 \Rightarrow \lambda \leq \lambda^*,$
min \_\_{A \in A} L(\lambda, A) \leq 0 \Rightarrow \begin{cases} \lambda \geq \lambda^*, & \text{or} \\ \min_{A \in A} L_1(A) - F_1(A) \leq 0, & \text{which further gives } \lambda^* = 0. \end{cases}

From Remark 3, if it was possible to compute \( \min_{A \in A} L(\lambda, A) \) exactly, then a binary search algorithm would find \( \lambda^* \). This dichotomy method can be extended to \( \kappa \)-approximation algorithms by defining the approximation Lagrangian as

\[
L_\kappa(S) \triangleq L_1(S) - \kappa F_1(S) - \lambda(L_2(S) + \kappa F_2(S)),
\]

for \( \lambda \geq 0 \) and \( S \subset [n] \). The idea is to use the following approximation guarantee for a \( \kappa \)-approximation algorithms outputing \( S \),

\[
\min_{A \in A} L_\kappa(\lambda, A) \leq L_\kappa(\lambda, S) \leq \min_{A \in A} L(\lambda, A).
\]

Thus, for a given \( \lambda \), either the l.h.s is strictly negative or the r.h.s is non-negative, depending on the sign of \( L_\kappa(\lambda, S) \). So a lower bound \( \lambda_1 \) on \( \lambda^* \), and an upper bound \( \lambda_2 \) on \( \min_{A \in A} \left( \frac{L_1(A) - \kappa F_1(A)}{L_2(A) + \kappa F_2(A)} \right)^+ \) can be computed. As in the proposed method, given in Algorithm 4. Notice that it takes as input some \( \text{ALGO}_\kappa \), that can be either Algorithm 2 or Algorithm 3, depending on the assumption on the constraint (either \( A = \mathcal{I} \) or \( A = \mathcal{B} \)). We denote the output as \( \text{ALGO}_\kappa(L + F) \), for some linear set function \( L \) and some submodular set function \( F \), for maximizing the objective \( L + F \) on \( A \), so that \( S = \text{ALGO}_\kappa(L + F) \) satisfies \( L(S) + \kappa F(S) \geq \max_{A \in A} L(A) + F(A) \), i.e., \( \kappa = 2(1 + \varepsilon) \) if \( \text{ALGO}_\kappa = \text{Algorithm 2}, A = \mathcal{I} \) and \( \kappa = 2 \) if \( \text{ALGO}_\kappa = \text{Algorithm 3}, A = \mathcal{B} \).

**Algorithm 4** Binary search for minimizing the ratio \( \left( \frac{(L_1 - F_1)}{L_2 + F_2} \right)^+ \).

**Input:** \( L_1, L_2, F_1, F_2, \text{ALGO}_\kappa, \eta > 0 \).

\[
\delta \leftarrow \frac{\eta \min_{A \in A} F_1(\{\star\})}{L_2(B) + \kappa F_2(B)} \quad \text{with } B = \text{ALGO}_\kappa(L_2 + \kappa F_2).
\]

\[
A \leftarrow A_0 \in A \setminus \{\emptyset\} \text{ arbitrary.}
\]

**if** \( \text{ALGO}_\kappa((0, A) > 0) \) **then**

\[
\lambda_1 \leftarrow 0, \quad \lambda_2 \leftarrow \frac{L_2(A) - F_1(A)}{L_2(A) + F_2(A)}.
\]

**while** \( \lambda_2 - \lambda_1 \geq \delta \) **do**

\[
\lambda \leftarrow \frac{\lambda_1 + \lambda_2}{2}.
\]

\[
S \leftarrow \text{ALGO}_\kappa(-L(\lambda, \cdot)).
\]

**if** \( \text{ALGO}_\kappa(S) \geq 0 \) **then**

\[
\lambda_1 \leftarrow \lambda.
\]

**else**

\[
\lambda_2 \leftarrow \lambda.
\]

\[
A \leftarrow S.
\]

**end if**

**end while**

**Output:** \( A \).

![Figure 1. Cumulative regret for the minimum spanning tree setting in up to 1000 rounds, averaged over 20 independent simulations.](image)

In Theorem 5, we state the result for the output of Algorithm 4 and prove it in Appendix D.

**Theorem 5.** Algorithm 4 outputs \( A \) such that:

\[
\left( \frac{L_1(A) - (\kappa + \eta)F_1(A)}{L_2(A) + \kappa F_2(A)} \right)^+ \leq \lambda^*,
\]

where \( \lambda^* \) is the minimum of \( \left( \frac{L_1 - F_1}{L_2 + F_2} \right)^+ \) over \( \mathcal{I} \) if \( \text{ALGO}_\kappa = \text{Algorithm 2} \), and over \( \mathcal{B} \) if \( \text{ALGO}_\kappa = \text{Algorithm 3} \). For \( C_t \) given by any algorithm of Table 2, \( F(A) = \max_{\delta \in C_t^+} e_\delta \gamma \), the complexity is of order \( \log(kt/\eta) \) times the complexity of \( \text{ALGO}_\kappa \).

**5. Experiments**

We provide experiments for graphic matroid, evaluating our algorithms on the problem of learning a routing network for an Internet service provider. We take the network 1755 (European Backbone) from the RocketFuel dataset (Spring et al., 2004), which contains 87 nodes and 161 edges weighted by their expected latencies. The oracle policy keeps selecting a spanning tree of this network that has the lowest expected latency on its edges. Using our notation, \( [n] \) are the edges, \( \mathcal{I} \) are subforests, and \( \mathcal{B} \) are the spanning trees.

For an edge \( i \), we model its random latency by \( X_i \sim \text{TruncatedExponential}((\lambda_i)) \), with \( \lambda_i \) chosen such that \( \mathbb{E}[X_i] = 1/\lambda_i + 1/(1 - e^{\lambda_i}) \) is the expected latency of arm \( i \) (that we normalize by 40 ms to lay in \([0, 1]) \). We thus have \( X_1, \ldots, X_n \) are mutually independent, in \([0, 1]) \). A state-of-the-art efficient algorithm for this setting is KL-CUCB (Talebi & Proutiere, 2016). Figure 1 illustrates the comparison between this algorithm and our Algorithm 3 used in
ESCB1 (Combes et al., 2015), showing the behavior of the regret vs. time horizon. We observe that although we are approximating the confidence region within a factor 2 (and thus force the exploration), our efficient implementation outperforms KL-CUCB. Indeed, we take advantage (gaining a factor $\sqrt{k}$) of the previously inefficient algorithm (that we made efficient) to use the reward structure, which the more conservative KL-CUCB is not able to. Regarding the running time, our efficient approximation of ESCB1 takes roughly the same time as the efficient of KL-CUCB using our implementation.

6. Discussion

In this paper we have provided an approximation schemes for the confidence regions and applied to combinatorial semi-bandits with matroid constraints and their budgeted version. We believe our approximation methods can be extended to more general non-linear objective functions, and more general constraints, adapting other approximation algorithms so that each part of the objective has a different approximation factor.

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A. Proof of Theorem 1

Proof. Let \( t \in \mathbb{N}^+ \). We consider here the restriction of \( g_{i,t} \) to \([0, r - \pi_{i,t-1}]\), that we still denote as \( g_{i,t} \). Notice that for all \( i \in [n] \), \( g_{i,t} \) is either 0 or a bijection on \([0, r - \pi_{i,t-1}]\) by assumption. For \( p = \infty \), we have that

\[
\max_{\delta \in \mathcal{E}_{C_1^+, \pi_{l-1}}} e_A^\delta = e_A^\delta \left( \min \{ g_{i,t}^{-1}(1), r - \pi_{i,t-1} \} \mathbb{I}\{ N_{i,t-1} \geq 1 \} + r \mathbb{I}\{ N_{i,t-1} = 0 \} \right),
\]

is a linear set function of \( A \). Assume now that \( p = 1 \). To show the submodularity of \( A \mapsto \max_{\delta \in \mathcal{E}_{C_1^+, \pi_{l-1}}} e_A^\delta \) in this case, we will use the notion of polymatroid.

Definition 3 (Polymatroid). A polymatroid is a polytope of the form \( \{ \delta' \in \mathbb{R}^n_+, e_A^\delta' \leq F(A), \forall A \subset [n] \} \), where \( F \) is a non-decreasing submodular function.

Fact 1 (Theorem 3 of He et al., 2012). Let \( P \) be a polymatroid, and let \( h_1, \ldots, h_n \) be concave functions. Then \( A \mapsto \max_{\delta \in P} e_A^\delta(h(\delta')) \) is submodular.

Notice that \( g_{i,t}^{-1}(\{0\}) = [0, r - \pi_{i,t-1}] \) when \( N_{i,t-1} = 0 \), and that \( g_{i,t}^{-1}(\cdot) \) is a strictly increasing concave function on \([0, g_{i,t}(r - \pi_{i,t-1})] \), as the inverse function of a strictly increasing convex function when \( N_{i,t-1} \geq 1 \). So we can rewrite \( C_1^+, \pi_{l-1} \) as an union of product sets:

\[
C_1^+, \pi_{l-1} = \left\{ \delta \in \prod_{i \in [n]} [0, r - \pi_{i,t-1}], \sum_{i \in [n]} g_{i,t}(\delta_i) \leq 1 \right\} = \bigcup_{\delta' \in \prod_{i \in [n]} \{0, g_{i,t}(r - \pi_{i,t-1})\}, \sum_{i \in [n]} \delta_i' \leq 1} \prod_{i \in [n]} g_{i,t}^{-1}(\{\delta_i'\}).
\]

We can thus rewrite our function as

\[
\max_{\delta \in \mathcal{E}_{C_1^+, \pi_{l-1}}} e_A^\delta = \max_{\sum_{i \in [n]} \delta_i' \leq 1} e_A^\delta(g_{i,t}^{-1}(\delta_i')),
\]

with the convention \( g_{i,t}^{-1}(0) = r - \pi_{i,t-1} \) when \( N_{i,t-1} = 0 \).

The constraints’ set \( \{ \delta' \in \prod_{i \in [n]} [0, g_{i,t}(r - \pi_{i,t-1})], \sum_{i \in [n]} \delta_i' \leq 1 \} \) is equal to the intersection between \( \prod_{i \in [n]} [0, g_{i,t}(r - \pi_{i,t-1})] \) and the polymatroid \( \{ \delta' \in \mathbb{R}^n_+, e_A^\delta' \leq \mathbb{I}\{ A \neq \emptyset \}, \forall A \subset [n] \} \). This intersection is itself equal to the polymatroid \( \{ \delta' \in \mathbb{R}^n_+, e_A^\delta' \leq \min_{B \subset A} \{ \mathbb{I}\{ B \neq \emptyset \} + e_B^\delta(g_{i,t}(r - \pi_{i,t-1})) \} \}, \forall A \subset [n] \} \).

Thus, \( \max_{\delta \in \mathcal{E}_{C_1^+, \pi_{l-1}}} e_A^\delta \) is the optimal objective value on a polymatroid of a separable concave function, as a function of the index set \( A \). Now, using Fact 1, it is submodular.

\[\square\]

B. Proof of Theorem 2

Before proving Theorem 2, we state some well known results about submodular optimization on a matroid.

Proposition 2. Let \( A, B \subset [n] \). If \( F \) is submodular, then

\[
\sum_{b \in B \setminus A} (F(B) - F(B \setminus \{b\})) \leq F(B) - F(A \cap B), \quad \sum_{a \in A \setminus B} (F(B \cup \{a\}) - F(B)) \geq F(A \cup B) - F(B).
\]

Proof. Let \( (b_1, \ldots, b_{|B \setminus A|}) \) be an ordering of \( B \setminus A \). Then, by submodularity of \( F \),

\[
\sum_{i=1}^{|B \setminus A|} (F(B) - F(B \setminus \{b_i\})) \leq \sum_{i=1}^{|B \setminus A|} (F(B \setminus \{b_1, \ldots, b_{i-1}\}) - F(B \setminus \{b_1, \ldots, b_i\})) = F(B) - F(A \cap B).
\]

In the same way, let \( (a_1, \ldots, a_{|A \setminus B|}) \) be an ordering of \( A \setminus B \). Then, by submodularity of \( F \),

\[
\sum_{i=1}^{|A \setminus B|} (F(B \cup \{a_i\}) - F(B)) \geq \sum_{i=1}^{|A \setminus B|} (F(B \cup \{a_1, \ldots, a_{i-1}\}) - F(B \cup \{a_1, \ldots, a_i\})) = F(A \cup B) - F(B).
\]
Fact 2 (Theorem 1 of Lee et al., 2010). Let $A, B \in \mathcal{A}$. Then, there exists a mapping $\alpha : B \setminus A \rightarrow A \setminus B \cup \{\emptyset\}$ such that

- $\forall b \in B \setminus A$, $A \setminus \{\alpha(b)\} \cup b \in \mathcal{A}$
- $\forall a \in A \setminus B$, $|\alpha^{-1}(a)| \leq 1$

Proposition 3. Let $A, B \in \mathcal{A}$. Let $F$ be a submodular function and $\alpha : B \setminus A \rightarrow A \setminus B \cup \{\emptyset\}$ be the mapping given in Fact 2. Then

$$\sum_{b \in B \setminus A} (F(A) - F(A \setminus \{\alpha(b)\} \cup \{b\})) + \sum_{a \in A \setminus B, \alpha^{-1}(a) = \emptyset} (F(A) - F(A \setminus \{a\})) \leq 2F(A) - F(A \cup B) - F(A \cap B).$$

Proof. We decompose $\sum_{b \in B \setminus A} (F(A) - F(A \setminus \{\alpha(b)\} \cup \{b\}))$ into sum of two terms:

$$\sum_{b \in B \setminus A} (F(A) - F(A \setminus \{\alpha(b)\} \cup \{b\})) + \sum_{b \in B \setminus A} (F(A \setminus \{\alpha(b)\})) - F(A \setminus \{\alpha(b)\} \cup \{b\}).$$

Remark that the first part is equal to

$$\sum_{a \in \alpha(B \setminus A)} (F(A) - F(A \setminus \{a\})) = \sum_{a \in A \setminus B, \alpha^{-1}(a) \neq \emptyset} (F(A) - F(A \setminus \{a\})).$$

Thus, together with $\sum_{a \in A \setminus B, \alpha^{-1}(a) = \emptyset} (F(A) - F(A \setminus \{a\}))$, we get that

$$\sum_{b \in B \setminus A} (F(A) - F(A \setminus \{\alpha(b)\} \cup \{b\})) + \sum_{a \in A \setminus B, \alpha^{-1}(a) = \emptyset} (F(A) - F(A \setminus \{a\}))$$

is equal to

$$\sum_{a \in A \setminus B} (F(A) - F(A \setminus \{a\})) + \sum_{b \in B \setminus A} (F(A \setminus \{\alpha(b)\})) - F(A \setminus \{\alpha(b)\} \cup \{b\}).$$

Finally, we upper bound the first term by $F(A) - F(A \cap B)$ using first inequality of Lemma 2, and the second term by $F(A) - F(A \cup B)$ using first, the submodularity of $F$ to remove $\alpha(b)$ in the summands, and then the second inequality of Lemma 2.

Proof of Theorem 2. The proof is divided into two parts:

Approximation guarantee. If Algorithm 2 outputs $\emptyset$ before entering in the while loop, then by submodularity, for any $S \in \mathcal{I}$,

$$(L + F)(S) \leq \sum_{x \in S} (L + F)(\{x\}) \leq 0.$$ 

Thus, $\emptyset$ is a maximizer of $L + F$.

Otherwise, the output $S$ of Algorithm 2 satisfies the local optimality of the while loop. We apply Proposition 3 with $A = S$ and $B = O$ for $L$ and $F$ separately,

$$\sum_{b \in O \setminus S} (L(S) - L(S \setminus \{\alpha(b)\} \cup \{b\})) + \sum_{a \in S \setminus O, \alpha^{-1}(a) = \emptyset} (L(S) - L(S \setminus \{a\})) \leq 2L(S) - L(S \cup O) - L(S \cap O),$$

$$\sum_{b \in O \setminus S} (F(S) - F(S \setminus \{\alpha(b)\} \cup \{b\})) + \sum_{a \in S \setminus O, \alpha^{-1}(a) = \emptyset} (F(S) - F(S \setminus \{a\})) \leq 2F(S) - F(S \cup O) - F(S \cap O).$$

Then, we sum these two inequalities,

$$\sum_{b \in O \setminus S} ((L + F)(S) - (L + F)(S \setminus \{\alpha(b)\} \cup \{b\})) + \sum_{a \in S \setminus O, \alpha^{-1}(a) = \emptyset} ((L + F)(S) - (L + F)(S \setminus \{a\}))$$
\[ \leq 2(L + F)(S) - (L + F)(S \cup O) - (L + F)(S \cap O) = 2F(S) - F(S \cup O) - F(S \cap O) + L(S) - L(O), \]

where the last equality uses linearity of \( L \). Since \( F \) is increasing and non-negative, \( F(S \cup O) + F(S \cap O) \geq F(O) \), and we get

\[ \sum_{b \in O \setminus S} ((L + F)(S) - (L + F)(S \setminus \{a\} \cup \{b\})) + \sum_{a \in S \setminus O, \ a^{-1}(a) = \emptyset} ((L + F)(S) - (L + F)(S \setminus \{a\})) \leq 2F(S) - F(O) + L(S) - L(O). \]

From the local optimality of \( S \), the left hand term in this inequality is lower bounded by

\[ \sum_{b \in O \setminus S} \frac{-\varepsilon}{k} F(S) + \sum_{a \in S \setminus O, \ a^{-1}(a) = \emptyset} \frac{-\varepsilon}{k} F(S) \geq -2\varepsilon F(S). \]

The last statement finishes the proof for the approximation inequality.

**Time complexity** Computing \( S_0 \) has a negligible complexity compared to the while loop. The following lemma gives a characterization of \( S_0 \).

**Lemma 2.** We have \( S_0 = \max\{L(A), \ A \in \mathcal{I}, (F + L)(A) > 0\} \).

**Proof.** If \( S_{\text{init}} \neq \emptyset \), then \( S_0 = S_{\text{init}} = \max_{A \in \mathcal{I}} L(A) \geq 0 \). Thus, \( F(S_0) > 0 \) by assumption on \( F \), giving \( (F + L)(S_0) > 0 \), which ends the proof. If \( S_{\text{init}} = \emptyset \), then \( S_0 = \max\{L(\{x\}), \ \{x\} \in \mathcal{I}, (F + L)(\{x\}) > 0\} \).

Let \( A \in \arg \max\{L(A), A \in \mathcal{I}, (F + L)(A) > 0\} \). By submodularity of \( F + L \), there exists \( x \in A \) such that \( (F + L)(\{x\}) > 0 \). Let \( L \) be non-increasing from \( S_{\text{init}} = \emptyset \), so we get \( L(\{x\}) \geq L(A) \), which finishes the proof.

From this lemma, necessarily \( L(S_0) \geq L(S_\ell) \) for every iterations \( \ell \geq 1 \), since the sequence \( \{L(S_\ell) + F(S_\ell)\} \) is increasing. At each iteration \( \ell \geq 1 \), Algorithm 2 constructs \( S_\ell \) such that

\[ F(S_\ell) > \left(1 + \frac{\varepsilon}{k}\right) F(S_{\ell-1}) + L(S_{\ell-1}) - L(S_\ell). \]

Thus, we must have

\[ F(S_\ell) - \left(1 + \frac{\varepsilon}{k}\right)^\ell F(S_0) \geq \sum_{j=1}^{\ell} \left(1 + \frac{\varepsilon}{k}\right)^{\ell-j} (L(S_{j-1}) - L(S_j)) \]

\[ = L(S_0) \left(1 + \frac{\varepsilon}{k}\right)^{\ell-1} - \frac{\varepsilon}{k} \sum_{j=1}^{\ell-1} L(S_j) \left(1 + \frac{\varepsilon}{k}\right)^{\ell-j-1} - L(S_\ell) \]

\[ \geq L(S_0) \left(1 + \frac{\varepsilon}{k}\right)^{\ell-1} - \frac{\varepsilon}{k} \sum_{j=1}^{\ell-1} L(S_0) \left(1 + \frac{\varepsilon}{k}\right)^{\ell-j-1} - L(S_0) = 0, \]

where the last inequality uses \( L(S_0) \geq L(S_\ell) \ \forall \ell \geq 1 \). This gives the following upper bound on the number of iteration \( \ell \):

\[ \ell \leq \frac{\log \left( \frac{F(S_\ell)}{F(S_0)} \right)}{\log \left(1 + \frac{\varepsilon}{k}\right)} \leq \frac{\log \left( \frac{\max_{A \in \mathcal{I}} F(A)}{F(S_0)} \right)}{\log \left(1 + \frac{\varepsilon}{k}\right)}. \]

Finally, the result follows remarking that time complexity per iteration is \( O(kn) \).
C. Proof of Theorem 3

As we did in the previous section, before starting the proof of Theorem 3, we state some useful results.

**Fact 3** (Brualdi’s lemma). Let \( A, B \in B \). Then, there exists a bijection \( \beta : A \to B \) such that
\[
\forall a \in A, \ A \setminus \{a\} \cup \{\beta(a)\} \in B.
\]
Furthermore, \( \beta \) is the identity on \( A \cap B \).

**Proof.** A proof is given by Brualdi (1969) and is also proved by Schrijver (2008), as Corollary 39.12a. \( \square \)

**Proposition 4.** Let \( A, B \in B \). Let \( F \) be a submodular function and \( \beta : A \to B \) be the mapping given in Fact 3. Let \( a_1, \ldots, a_k \) be elements of \( A \), and \( A_i = \{a_1, \ldots, a_i\} \). Then,
\[
\sum_{i \in [k]} (F(A_i) - F(A_{i-1} \cup \{\beta(a_i)\})) \leq 2F(A) - F(A \cup B) - F(\emptyset).
\]

**Proof.** We can split \( \sum_{i \in [k]} (F(A_{i-1} \cup \{a_i\}) - F(A_{i-1} \cup \{\beta(a_i)\})) \) into two terms,
\[
\sum_{i=1}^{k} (F(A_{i-1} \cup \{a_i\}) - F(A_{i-1})) + \sum_{i=1}^{k} (F(A_{i-1}) - F(A_{i-1} \cup \{\beta(a_i)\})).
\]

The first term is equal to \( F(A_k) - F(\emptyset) \). Using submodularity of \( F \), the second term is upper bounded by
\[
\sum_{i=1}^{k} (F(A_k) - F(A_k \cup \{\beta(a_i)\})),
\]
which is upper bounded by \( F(A_k) - F(A_k \cup B) \) thanks to Proposition 2 and its second inequality. \( \square \)

**Proof of Theorem 3.** The time complexity proof is trivial. Let \( S_i \triangleq \{s_1, \ldots, s_i\} \) be the set maintained in Algorithm 3 after \( i \) iterations. Instantiating Proposition 4 with \( A_i = S_i \) and \( B = O \), we have
\[
\sum_{i \in [k]} (F(S_i) - F(S_{i-1} \cup \{\beta(s_i)\})) \leq 2F(S) - F(S \cup O) - F(\emptyset).
\] (5)

Furthermore, we also have, by linearity of \( L \), and bijectivity of \( \beta \),
\[
\sum_{i \in [k]} (L(S_i) - L(S_{i-1} \cup \{\beta(s_i)\})) = \sum_{i \in [k]} (L(\{s_i\}) - L(\{\beta(s_i)\})) = L(S) - L(O).
\] (6)

Thus, we can sum up (5) and (6) to get
\[
\sum_{i \in [k]} ((L + F)(S_i) - (L + F)(S_{i-1} \cup \{\beta(s_i)\})) \leq 2F(S) - F(S \cup O) - F(\emptyset) + L(S) - L(O)
\]
\[
\leq 2F(S) - F(O) + L(S) - L(O),
\]
where the last inequality uses the fact that \( F \) is increasing and \( F(\emptyset) = 0 \). We finish the proof remarking that by definition of Algorithm 3, \( (L + F)(S_i) - (L + F)(S_{i-1} \cup \{\beta(s_i)\}) \geq 0 \). \( \square \)
D. Proof of Theorem 5

Proof. Let $A$ be the output of Algorithm 4 and let

$$\mathcal{L}_{\kappa, \kappa_2}(\lambda, S) \triangleq L_1(S) - \kappa_1 F_1(S) - \lambda(L_2(S) + \kappa_2 F_2(S)).$$

Recall that $\mathcal{L}_\kappa = \mathcal{L}_{\kappa, \kappa}$. Algorithm 4 satisfies either $\mathcal{L}_\kappa(0, A_0) \leq 0$ (in which case Theorem 5 is trivial since $\lambda^* = 0$) or $\mathcal{L}_\kappa(0, A_0) > 0$, in which case we have:

$$0 > \mathcal{L}_\kappa(\lambda_2, A) \geq \mathcal{L}_{\kappa + \eta, \kappa}(\lambda_2 - \delta, A) \geq \mathcal{L}_{\kappa + \eta, \kappa}(\lambda_1, A) \geq \mathcal{L}_{\kappa + \eta, \kappa}(\lambda^*, A). \quad (7)$$

The first inequality is comes from the update of $\lambda_2$: Notice that before the while loop, we have

$$\lambda_2 = \frac{L_1(A_0) - F_1(A_0)}{L_2(A_0) + F_2(A_0)} > \frac{L_1(A_0) - \kappa F_1(A_0)}{L_2(A_0) + \kappa F_2(A_0)} > 0,$$

since $F_2(A_0) > 0$, so $0 > \mathcal{L}_\kappa(\lambda_2, A_0)$ multiplying by $L_2(A_0) + F_2(A_0)$ on both sides. Notice that in particular, this inequality gives that $A \neq \emptyset$.

The second inequality follows from

$$\delta = \frac{\eta \min(s) \in A F_1(\{s\})}{L_2(B) + \kappa^2 F_2(B)} \leq \frac{\eta F_1(A)}{L_2(A) + \kappa F_2(A)} \quad \text{since } A \neq \emptyset \text{ and } L_2(B) + \kappa^2 F_2(B) \geq L_2(A) + \kappa F_2(A).$$

Thus, multiplying by $L_2(A) + \kappa F_2(A) > 0$, and adding $L_1(A) - \kappa F_1(A) - \lambda_2(L_2(A) + \kappa F_2(A))$ gives

$$L_1(A) - (\kappa + \eta) F_1(A) - (\lambda_2 - \delta)(L_2(A) + \kappa F_2(A)) \leq L_1(A) - \kappa F_1(A) - \lambda_2(L_2(A) + \kappa F_2(A)),$$

i.e., $\mathcal{L}_{\kappa + \eta, \kappa}(\lambda_2 - \delta, A) \leq \mathcal{L}_\kappa(\lambda_2, A)$.

The third inequality uses $\lambda_2 - \lambda_1 \leq \delta$, and the last inequality uses $\lambda_1 \leq \lambda^*$. Indeed, since $\mathcal{L}_\kappa(\lambda_1, S) \geq 0$, the approximation relation given by $\text{ALGO}_\kappa$,

$$\mathcal{L}_\kappa(\lambda_1, S) \leq \mathcal{L}(\lambda_1, O),$$

where $O$ is the minimizer of $\left(\frac{L_1 - F_1}{L_2 + F_2}\right)^+$ (for the constraints considered by $\text{ALGO}_\kappa$), gives $0 \leq \mathcal{L}(\lambda_1, O)$. Thus,

$$\mathcal{L}^+(\lambda_1, O) \triangleq (L_1(O) - F_1(O))^+ - \lambda_1(L_2(O) + F_2(O)) \geq \mathcal{L}(\lambda_1, O) \geq 0.$$

Finally, since $L_2(O) + F_2(O) > 0 (O \neq \emptyset)$, we have $\lambda_1 \leq \lambda^*$.

In (7), since $A \neq \emptyset$, we have $\frac{L_1(A) - (\kappa + \eta) F_1(A)}{L_2(A) + \kappa F_2(A)} \leq \lambda^*$ and therefore, $\left(\frac{L_1(A) - (\kappa + \eta) F_1(A)}{L_2(A) + \kappa F_2(A)}\right)^+ \leq \lambda^*$.

The time complexity for the binary search is $O(\log(1/\delta)) \leq O(\log(kt/\eta))$ for $C_t$ given by any algorithm of Table 2, and $F(A) = \max_{\delta \in C_t} -e_1^T \delta$. 

\qed