Entanglement entropy of multipartite pure states

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Consider a system consisting of \( n \) \( d \)-dimensional quantum particles and an arbitrary pure state \( |\Psi\rangle \) of the whole system. Suppose we simultaneously perform complete von Neumann measurements on each particle. One can ask: what is the minimal possible value \( S[|\Psi\rangle] \) of the entropy of outcomes joint probability distribution? We show that \( S[|\Psi\rangle] \) coincides with entanglement entropy for bipartite states. We compute \( S[|\Psi\rangle] \) for two sample multipartite states: the hexacode state \( |\overline{S}\rangle \) for \( n = 6, d = 2 \) and determinant states \( |\text{Det}_n\rangle \) for \( d = n \). The result is \( S[H] = 4 \log 2 \) and \( S[\text{Det}_n] = \log(n!) \). The generalization of determinant states to the case \( d < n \) is considered.

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I. INTRODUCTION AND MAIN RESULTS

Quantum information theory has many interesting features which have no classical analogue. One of them is entanglement or quantum correlations. It has been the object of intensive study for last years because it is the entanglement that makes possible to develop effective algorithms solving many tasks in computing, communication, and cryptography, see for example [1] and references therein.

However at the present moment the canonical definition of entanglement is missing and the question how to quantify the degree of entanglement in a given multipartite quantum state remains open. The only reasonable constraint on a functional which pretends to be an entanglement measure is the monotonicity under certain class of local quantum operations [3].

An important example of such functional is entanglement entropy [1]. For a pure state \( |\Psi\rangle \) of bipartite system, its entanglement entropy \( E[|\Psi\rangle] \) is defined as

\[
E[|\Psi\rangle] = - \sum_i p_i \log_2 p_i, \tag{1}
\]

where \( p_i \)'s denote the eigenvalues of reduced density matrices \( \rho_1 = \text{tr}_2(|\Psi\rangle\langle\Psi|) \) and \( \rho_2 = \text{tr}_1(|\Psi\rangle\langle\Psi|) \) (they have the same spectrum). This particular measure is distinguished because in the asymptotic limit (i.e. when one takes a large number of copies of a given shared state) any monotonic functional of bipartite state up to trivial rescaling coincides with entanglement entropy, see [1]. Unfortunately, when the system is divided to three or more local parts, entanglement entropy is not defined.

The functional \( E[|\Psi\rangle] \) has very simple physical sense. Suppose that each of two parties, between which the state \( |\Psi\rangle \) is distributed, performs complete von Neumann measurement on his part of the system. Such joint measurement is a complete measurement on the whole system. Its outcome is a random variable whose probability distribution depends upon the pair of complete von Neumann measurements chosen by each of two parties. This choice is equivalent to the choice of orthonormal basis in each party Hilbert space of states. One can ask: what bases should be chosen by each party to minimize the entropy of the outcomes joint probability distribution and what is the minimal value of this entropy? If the state \( |\Psi\rangle \) is factorizable, i.e. \( |\Psi\rangle = |\Psi_1\rangle \otimes |\Psi_2\rangle \), then the answer is trivial: the \( i \)-th party should complement \( |\Psi_i\rangle \) to a complete basis by any way and this choice yields zero entropy because the measurement outcome will be \( "(\Psi_1,\Psi_2)" \) with the probability one. If \( |\Psi\rangle \) is entangled, one can easily show (see Section III) that the \( i \)-th party should perform measurement in the basis where its density matrix \( \rho_i \) is diagonal (the Schmidt basis) and the minimal entropy of the outcomes coincides with \( E[|\Psi\rangle] \). Thus we can interpret entanglement entropy \( E[|\Psi\rangle] \) as the minimum of outcomes entropy over all choices of local complete von Neumann measurements.

If we consider entanglement entropy from this point of view, it can be naturally defined for arbitrary multipartite states. Suppose a system consists of \( n \) \( d \)-dimensional quantum particles distributed between \( n \) remote parties and let \( |\Psi\rangle \in (\mathbb{C}^d)^\otimes n \) be arbitrary pure state of the whole system. Denote \( B_i \) the orthonormal basis in \( \mathbb{C}^d \) chosen by \( i \)-th party and let \( |B_i(j)\rangle \in \mathbb{C}^d \) be the \( j \)-th basis vector in the basis \( B_i \), where \( i \in [1,n] \), \( j \in [1,d] \). Orthonormality condition implies that \( \langle B_i(j)|B_i(j')\rangle = \delta_{jj'} \). Now let us define the functional \( S[|\Psi\rangle] \) according to:

\[
S[|\Psi\rangle, B_1, \ldots, B_n] = S[|\Psi\rangle] = \inf_{B_1, \ldots, B_n} S[|\Psi, B_1, \ldots, B_n\rangle], \tag{2}
\]

\[
S[|\Psi, B_1, \ldots, B_n\rangle] = - \sum_j p(j) \log_2 p(j),
\]

\[
p(j) \equiv p(j_1, \ldots, j_n) = |\langle \Psi|B_1(j_1), \ldots, B_n(j_n)\rangle|^2,
\]

(the sum over multi-index \( j \) is the sum over all possible \( j_1, \ldots, j_n \in [1,d] \)). It tells us to what extent the parties may decrease the entropy of the outcomes distribution by varying the bases in which they perform the measure-
ments. As was said above, \( S[\Psi] = E[\Psi] \) for bipartite states, so we will call \( S[\Psi] \) as entanglement entropy. Its properties immediately following from the definition are \( 0 \leq S[\Psi] < n \log_2 d \); \( S[\Psi] = 0 \) iff the state \( |\Psi\rangle \) is factorizable; \( S[\Theta \otimes \Phi] = S[\Theta] + S[\Phi] \) (here we mean that \( |\Psi\rangle \) and \( |\Phi\rangle \) are the states shared by \( n \) and \( m \) parties, while \( |\Theta \otimes \Phi\rangle \) is shared by \( n + m \) parties); \( S[\Phi] \) is continuous functional of the state \( |\Psi\rangle \).

Computation of entanglement entropy in the multipartite case \( n > 2 \) is a difficult task. Probably, for generic quantum state it can be solved only numerically (note that the number of parameters to be optimized in the definition grows as \( O(n d^2) \)). It is relatively easy to get the upper bound on \( S[\Psi] \) — one just needs to choose tentatively some basis for each party. As the lower bound on \( S[\Psi] \) one can take von Neumann entropy of the mixed state of any group of parties (see Section III). However for generic state this lower bound is too weak. It can be improved if the state has some symmetry.

In this paper we consider two types of symmetry on example of determinant states \( |\text{Det}_n\rangle \in (C^n)^{\otimes n} \), see [2], and six qubit hexacode state \( |H\rangle \), see [3]. The determinant state \( |\text{Det}_n\rangle \) is invariant under unitary transformations \( U^{\otimes n} \) where \( U \in SU(n) \) is arbitrary one-party unitary operator. The hexacode state is in some sense ‘maximally uniform’ pure state — if we divide six qubits into two equal groups by arbitrary way then the mixed state of each group will be absolutely uniform. Due to these special properties, entanglement entropy of the states \( |\text{Det}_n\rangle \) and \( |H\rangle \) can be exactly computed.

Another reason for our interest to these particular states is that their entanglement entropy is rather close to the upper bound \( n \log_2 d \), so they are near-maximally entangled states. For the hexacode state \( (n = 6, d = 2) \) the computation yields \( S[H] = 4 \), see Section VII while for determinant state \( (n = d) \) one gets \( S[\text{Det}_n] = \log_2 (n!) \), see Section VII. For large \( n \) we can approximately write \( S[\text{Det}_n] \approx [1 - 1/(\ln(n))] n \log_2 n \). It means that determinant states asymptotically saturate the upper bound for normalized entropy: \( \lim_{n \to \infty} S[\text{Det}_n]/n \log_2 n = 1 \). In Section XI we construct the generalized determinant states \( |\text{Det}_{n,d}\rangle \in (C^d)^{\otimes n} \) defined if \( n = pd^p \), where \( p \) is arbitrary integer, such that \( S[\text{Det}_{n,d}] = \log_2 ((d^p)!)/d \). For fixed \( d \) and large \( n \) (i.e. large \( p \)) we can write: \( S[\text{Det}_{n,d}] \approx [1 - 1/(\ln(n))] n \log_2 d \) which again saturates the upper bound for normalized entropy. Note that the factor \( [1 - 1/(\ln(n))] \) grows sufficiently slow, e.g. it is equal 0.9 for \( n = e^{10} \approx 2 \cdot 10^4 \). We will see that for qubits \( (d = 2) \) the tensor powers of hexacode state \( |H^{\otimes n}\rangle \) have entanglement entropy greater than determinant states \( |\text{Det}_{n,2}\rangle \) if the number of qubits \( n \gtrsim 60 \).

We make concluding remarks and discuss some open questions concerning the generalized entanglement entropy in Section VII. Most interesting question concerns the stability of the definition (2) under the extension of each party space of states.

**II. BIPARTITE SYSTEM**

It is known that up to local unitary operators, any state \( |\Psi\rangle \in C^d \otimes C^d \) of bipartite system is specified by its Schmidt coefficients \( \{p_i\}_{i=1...d} \), \( p_i \geq 0 \), \( \sum_{i=1}^d p_i = 1 \). Being invariant under local unitaries \( S[\Psi] \) is a functional of the Schmidt coefficients only. Thus one suffices to compute \( S[\Psi] \) only for the special states

\[
|\Psi\rangle = \sum_{i=1}^d \sqrt{p_i} |i, i\rangle,
\]

where \( \{ |i\rangle \in C^d \}_{i=1...d} \) is the standard basis of \( C^d \). In the definition (2) we tentatively choose the standard basis for both parties, i.e. \( |B_1(i)\rangle = |B_2(i)\rangle = |i\rangle \), \( i \in [1, d] \), then \( S[\Psi, B_1, B_2] = -\sum_{i=1}^d p_i \log_2 p_i \) and thus get \( S[\Psi] \leq E[\Psi] \), see [1]. We can also prove that \( E[\Psi] \) is simultaneously the lower bound for \( S[\Psi] \). Indeed, let \( B_1^*, B_2^* \) be the optimal choice of the bases (i.e. such that \( S[\Psi] = S[\Psi, B_1^*, B_2^*] \)). Consider the density matrix \( p_i \) of the first party only: \( p_i = \text{tr}_2(|\Psi\rangle \langle \Psi|) = \sum_{i=1}^d p_i |i\rangle \langle i| \). Denote \( p_i = (B_1^*(i)|\rho_1|B_1^*(i)) = \sum_{j=1}^d p_j (B_1^*(i)|j\rangle \langle j|B_1^*(i)) \) the distribution of the first party outcomes in the optimal basis. Because the entropy of partial distribution can not exceed the entropy of joint distribution, we have:

\[
S[\Psi] \geq -\sum_{i=1}^d p_i (i) \log_2 [p_i (i)].
\]

Using the concavity of the function \( -x \log_2 x \) and normalization \( \sum_{j=1}^d (B_1^*(i)|j\rangle \langle j|B_1^*(i)) = 1 \), we get the next estimate:

\[
-p_i (i) \log_2 p_i (i) \geq -\sum_{j=1}^d (B_1^*(i)|j\rangle \langle j|B_1^*(i)) \log_2 (p_j \log_2 p_j).
\]

The summation over \( i \) can be carried out taking into account the normalization \( \sum_{i=1}^d (B_1^*(i)|i\rangle \langle i|B_1^*(i)) = 1 \). Thus from (4) we can infer \( S[\Psi] \geq E[\Psi] \) and consequently

\[
S[\Psi] = E[\Psi].
\]

**III. CONNECTION WITH VON NEUMANN ENTROPY**

Suppose the system consists of \( n \) \( d \)-dimensional quantum particles distributed between \( n \) remote parties and let \( |\Psi\rangle \in (C^d)^{\otimes n} \) be arbitrary pure state of the whole system. Let us choose a group \( X \) of \( k \) parties, for example \( X = \{1, 2, \ldots, k\} \). The chosen group of parties shares the mixed state \( \rho_x = \text{tr}_{\bar{X}} (|\Psi\rangle \langle \Psi|) \). Suppose \( E_i^* \) is the optimal basis for the \( i \)-th party. Denote \( p_x (i_1, \ldots, i_k) \equiv p_x (i) \) the optimal outcomes distribution for the parties from \( X \) only:
\( p_x(i) = \langle B^*_1(i_1), \ldots , B^*_k(i_k) | \rho_x | B^*_1(i_1), \ldots , B^*_k(i_k) \rangle \). (7)

Let \( S[p_x(i)] \) be the entropy of distribution \( p_x \). By repeating the arguments presented in Section [1] we can show that

\[ - \text{tr} (\rho_x \log_2 \rho_x) \leq S[p_x(i)] \leq S[\Psi]. \] (8)

Thus von Neumann entropy of the mixed state \( \rho_x \) can serve as the lower bound on entanglement entropy. Of course the group of parties \( X \) can be chosen by arbitrary way.

Note that the density matrix of the parties which were not selected to \( X \) has the same (positive) spectrum as \( \rho_x \). It means that one has same freedom to consider the groups of \( k \leq n/2 \) parties and the best lower estimate on \( S[\Psi] \) which we can hope to achieve is \((n/2) \log_2 d \).

As a good example, consider three qubit GHZ state \( |\text{GHZ}\rangle = 2^{-\frac{3}{2}}(|000\rangle + |111\rangle) \), \( d=2 \), \( n=3 \). The density matrix of the first qubit is absolutely uniform: \( \rho_1 = (1/2)1 \). Thus \( S[\text{GHZ}] \geq -\text{tr}(\rho_1 \log_2 \rho_1) = 1 \). On the other hand we can choose tentative bases \( B_1 = B_2 = B_3 = \{0,1\} \) which provide us with upper estimate \( S[\text{GHZ}] \leq S[\text{GHZ}, B_1, B_2, B_2] = 1 \), so that \( S[\text{GHZ}] = 1 \).

### IV. DETERMINANT STATE

Let us consider the multipartite system with \( d = n \). Choose the standard basis \( \{|i\rangle \in \mathbb{C}^n\}_{i=1,\ldots,n} \) in each copy of \( \mathbb{C}^n \) and consider the state \( |\text{Det} n\rangle \in (\mathbb{C}^n)^\otimes n \) defined as

\[ |\text{Det} n\rangle = (n!)^{-\frac{1}{2}} \sum_{i_1,\ldots,i_n} \epsilon_{i_1,\ldots,i_n} |i_1,\ldots,i_n\rangle, \] (9)

where \( \epsilon_{i_1,\ldots,i_n} \) is completely antisymmetric tensor of the rank \( n \) and the sum is over all \( i_1,\ldots,i_n \in [1,n] \). This state was used in Ref. [2] to study the limitations on the pairwise entanglement in multipartite systems. We will call the family \( \{|\text{Det} n\rangle\}_{n} \) the determinant states. Note that \( |\text{Det} 2\rangle \) is EPR singlet state \( 2^{-\frac{3}{2}}(|12\rangle - |21\rangle) \). The purpose of this section is to prove the formula

\[ S[|\text{Det} n\rangle] = \log_2 (n!) \] (10)

This result is immediate consequence of the following property of determinant states:

\[ \sup_{|\phi\rangle \in \mathbb{C}^n} |\text{Det} n\rangle |\phi_1,\ldots,\phi_n\rangle|^2 = (n!)^{-1}, \] (11)

(we employ standard designation \( |\phi_1,\ldots,\phi_n\rangle \equiv |\phi_1\rangle \otimes \cdots \otimes |\phi_n\rangle \)). It tells us that the projection of \( |\text{Det} n\rangle \) on any factorizable state has the norm at most \( \frac{1}{n!} \). To prove (11), we first note that the state \( |\text{Det} n\rangle \) is SU\( (n) \) singlet, i.e. for any \( V \in \text{SU}(n) \) we have \( V^\otimes n |\text{Det} n\rangle = |\text{Det} n\rangle \). Therefore, if the projection of \( |\text{Det} n\rangle \) on the state \( |\phi_1,\ldots,\phi_n\rangle \) is the highest one, then the projection of \( |\text{Det} n\rangle \) on the state \( |V \phi_1,\ldots,V \phi_n\rangle \) is also the highest one. So while looking for the maximum in (11) we can fix one of \( |\phi_i\rangle \), e.g. put \(|\phi_n\rangle = |n\rangle \). But according to definition (3) we have:

\[ |\langle \text{Det} n|\phi_1,\ldots,\phi_{n-1}, n\rangle|^2 = \frac{1}{n} |\langle \text{Det} n-1|\phi_1,\ldots,\phi_{n-1}\rangle|^2. \] (12)

Here by abuse of notations we denote \( |\text{Det} n-1\rangle \) the embedding of determinant state \( |\text{Det} n-1\rangle \in (\mathbb{C}^{n-1})^\otimes (n-1) \) into the space \( (\mathbb{C}^n)^\otimes n \) (the space \( \mathbb{C}^{n-1} \) is embedded into \( \mathbb{C}^n \) by adding zero \( n-1 \)-th component to all vectors). Although in (12) \(|\phi_i\rangle \in \mathbb{C}^n \), the right-hand side of (12) achieves the maximum when all states \( |\phi_1\rangle,\ldots,|\phi_{n-1}\rangle \) have zero \( n-1 \)-th component. It implies that

\[ \sup_{|\phi_i\rangle \in \mathbb{C}^n} |\langle \text{Det} n|\phi_1,\ldots,\phi_n\rangle|^2 = \frac{1}{n} \sup_{|\phi_{n-1}\rangle \in \mathbb{C}^{n-1}} |\langle \text{Det} n-1|\phi_1,\ldots,\phi_{n-1}\rangle|^2, \]

which by induction leads to equality (11).

Now let us explain why (11) implies (10). Suppose \( B^*_i \) is the optimal basis for the \( i \)-th copy of \( \mathbb{C}^n \), \( i \in [1,n] \), i.e. \( S[|\text{Det} n\rangle] = S[|\text{Det} n|B_1,\ldots,B_n\rangle] \). Let \( p^*(i_1,\ldots,i_n) \equiv |\langle B^*_1(i_1),\ldots,B^*_n(i_i) | \text{Det} n\rangle|^2 \) be the optimal distribution. According to (11), \( p^*(i_1,\ldots,i_n) \leq (n!)^{-1} \) for any outcomes \( i_1,\ldots,i_n \) and thus \( S[|\text{Det} n\rangle] \geq \log_2 (n!) \). On the other hand, we can tentatively suggest all parties to perform the measurements in the standard basis, i.e. \( |\text{B}_i(j)\rangle = |j\rangle \), \( i,j \in [1,n] \). Then \( S[|\text{Det} n,B_1,\ldots,B_n\rangle] = \log_2 (n!) \) which tells us that \( S[|\text{Det} n\rangle] \leq \log_2 (n!) \) and thus that \( S[|\text{Det} n\rangle] = \log_2 (n!) \).

### V. GENERALIZED DETERMINANT STATE

Suppose now that the dimension \( d \) of each particle is fixed. If the number of particles is \( n = pd^p \) for some integer \( p \), the space \( (\mathbb{C}^d)^\otimes n \) can be identified with \( (\mathbb{C}^{d^p})^\otimes d^p \) and thus determinant state \( |\text{Det}_{d^p}\rangle \) has its counterpart in \( (\mathbb{C}^d)^\otimes n \). This simple observation allows to construct the state \( |\text{Det}_{n,d}\rangle \in (\mathbb{C}^d)^\otimes n \) such that

\[ S[|\text{Det}_{n,d}\rangle] = \log_2 ((d^p)!), \quad n = pd^p. \] (13)

Note that although \(|\text{Det}_{d^p}\rangle \) and \(|\text{Det}_{d^p}\rangle \) represent one and the same state, \( S[|\text{Det}_{d^p}\rangle] \) might be greater than \( S[|\text{Det}_{d^p}\rangle] \) because in the first case we have less freedom in the choice of bases in (3).
Let us explain the construction of the state $|\text{Det}_{n,d}\rangle$ and derive (13) on example of the qubits, i.e. $d = 2$. Consider any one-to-one map $\varphi$ which maps the integers on the interval $[1, 2^p]$ to binary strings of the length $p$, e.g.

\begin{align}
\varphi(1) &= (0, 0, \ldots, 0, 0), \\
\varphi(2) &= (0, 0, \ldots, 0, 1), \\
& \quad \vdots \\
\varphi(2^p - 1) &= (1, 1, \ldots, 1, 0), \\
\varphi(2^p) &= (1, 1, \ldots, 1, 1). 
\end{align}  

(14)

Define the state $|\text{Det}_{n,2}\rangle \in (C^2)^\otimes n$, $n = p2^p$ as

\[
|\text{Det}_{n,2}\rangle \sim \sum_{i_1, \ldots, i_{2p}} \epsilon_{i_1, \ldots, i_{2p}} |\varphi(i_1), \ldots, \varphi(i_{2p})\rangle,
\]

where the sum is over all $i_1, \ldots, i_{2p} \in [1, 2^p]$ and we omit the normalizing factor $[(2^p)!]^{-\frac{1}{2}}$. It is written here in the standard qubit basis $\{|0\rangle, |1\rangle\}$. While computing $S[\text{Det}_{n,2}]$ we minimize over the choice of $p2^p$ one-qubit bases, see (13). But any such choice is also the choice of $2^p$ copies of $C^2$ tensors. Therefore we can say that

\[
S[\text{Det}_{n,2}] \geq S[\text{Det}_{2^p}] = \log_2[(2^p)!].
\]

(16)

On the other hand, if we will tentatively measure each of $p2^p$ qubits in the basis $\{|0\rangle, |1\rangle\}$, the entropy of the outcomes distribution will be exactly $\log_2[(2^p)!]$, see (13). Therefore $S[\text{Det}_{n,2}] \leq \log_2[(2^p)!]$ and thus the equality (13) is proven (the proof for arbitrary $d$ copies the proof for $d = 2$).

As was already mentioned in Section I the determinant states are near-maximally entangled states if the number of parties $n$ is sufficiently large. As an illustration let us consider determinant states $|\text{Det}_{n,2}\rangle$ corresponding to $n$ qubit system. In Table I we present the normalized entanglement entropy $S[\text{Det}_{n,2}]$ for the number of qubits $n$ for $p = 1, \ldots, 5$, and $p = 10$.

**Table I.** Normalized entanglement entropy of determinant states $|\text{Det}_{n,2}\rangle$. Here $n = p2^p$ is the number of qubits.

| $p$ | $n$ | $S[\text{Det}_{n,2}]$ |
|-----|-----|------------------|
| 1   | 2   | 0.50             |
| 2   | 8   | 0.57             |
| 3   | 24  | 0.64             |
| 4   | 64  | 0.66             |
| 5   | 160 | 0.74             |
| 10  | 10240 | 0.86             |

Table I suggests that we could try to find some state of $O(1)$ qubits which has entanglement entropy greater than the determinant states. This is the purpose of the next section.

**VI. HEXACODE STATE**

The hexacode state was originally defined in the context of quantum error correcting codes. It was associated with certain maximal self-dual linear subspace of $GF(2)^6$, see [1], p.30 for details. In this Section we present the alternative and more explicit definition of this state only briefly discussing its connection with quantum codes. We also prove the equality

\[
S[H] = 4,
\]

(17)

announced in Section I.

Consider graph $G = (V, E)$ shown on Fig. 1 with the set of vertices $V = \{1, 2, 3, 4, 5, 6\}$ and the set of edges $E = \{(12), (13), (14), \ldots, (56)\}$.

![Fig. 1. Graph G used in the definition of the hexacode state.](image)

We associate a qubit with each vertex $i \in V$. Let $A_{ij}$ be 6x6 adjacency matrix of $G$, i.e. $A_{ij} = 1$ if $(ij) \in E$ and $A_{ij} = 0$ if $(ij) \notin E$, $A_{ij} = A_{ji}$. The diagonal elements $A_{ii}$ will not appear anywhere. Then six qubit hexacode state $|H\rangle \in (C^2)^\otimes 6$ is defined as follows:

\[
|H\rangle = (2^6)^{-\frac{1}{2}} \sum_{x \in B^6} (-1)^{a(x)} |x\rangle,
\]

(18)

where $B^6$ denotes the set of all binary strings of the length 6 and $|x\rangle = |x_1, \ldots, x_6\rangle$.

Note that $|H\rangle$ can also be defined in terms of stabilizers operators. Denote $\sigma^x_i$ the Pauli matrix $\sigma^x$ acting on the $i$-th qubit and assign to each vertex of the graph the operator

\[
X_i = \sigma^z_i \prod_{j : (ij) \in E} \sigma^z_j, \quad i \in V.
\]

(19)

They commute with each other and stabilize the state $|H\rangle$, i.e. $X_i|H\rangle = |H\rangle$. Six operators $X_i$ generate the group $S$ of all stabilizers, $|S| = 2^6$. Each stabilizer from $S$ is a tensor product of several Pauli matrices (probably with ‘-‘ sign). As was shown in [1], any nontrivial stabilizer from $S$ is tensor product of at least four Pauli matrices, so that $|H\rangle$ is an additive quantum code coding 0 qubits into 6 qubits with the minimal stabilizer.
weight 4. Of course, such quantum code can not be used to protect quantum information from the errors. It is just symplectic state with some special properties. Note however that if some symplectic state has the minimal stabilizer weight \( d \) and if any \( [(d - 1)/2] \) or less qubits were decohered then the syndrome measurement allows to determine the positions of the decohered qubits.

The proof of the equality (17) consists of three steps. The first step is to prove that \( S[H] \leq 4 \). On the second step we establish remarkable symmetry of the state \( |H \rangle \) which is used on the third step to prove that \( S[H] \geq 4 \).

1) Let us tentatively choose the bases \( B_1 = \{ + \}, \{ - \} \) for the qubits \( i = 2, 3, 4, 5 \) and \( B_1 = B_6 = \{ + \}, \{ - \} \) where \( |\pm \rangle = 2^{-1/2}(|0 \rangle \pm |1 \rangle) \). Simple calculations show that the distribution of the outcomes \( p(x) \equiv p(x_1, \ldots, x_6) = \langle B_1(x_1), \ldots, B_6(x_6) | H \rangle^2 \) measured in these bases is following:

\[
p(x) = \begin{cases} \frac{1}{16}, & \text{if } x_1 = x_2 + x_3 + x_4 \mod 2, \\ 0, & \text{otherwise}. \end{cases}
\]

(20)

Thus \( S[H, B_1, \ldots, B_6] = 4 \) and consequently \( S[H] \leq 4 \).

2) Suppose the vertices of \( G \) are colored by black and white colors such that there are three black and three white vertices. Denote \( B \) and \( W \) the subsets of black and white vertices, \( B \cup W = V \). Consider three-qubit density matrix \( \rho_w \) describing the state of white qubits only: \( \rho_w = \text{tr}_B(H|H) \). We claim that for any partition \( V = B \cup W \), \( \rho_w = 2^{-3} \mathbb{1} \) where \( \mathbb{1} \) is the unital matrix. In other words, the mixed state of any triple of the qubits is absolutely uniform. To verify this property, fix some partition and renumber the vertices of the graph to make \( W = \{1, 2, 3\} \) and \( B = \{4, 5, 6\} \). The adjacency matrix \( A \) then can be split to four 3x3 blocks:

\[
A = \begin{pmatrix} A_{ww} & A_{wb} \\ A_{bw} & A_{bb} \end{pmatrix}
\]

(21)

which are adjacency matrices between white and white, white and black, black and black, white and black and black vertices \( A_{wb} = (A_{bw})^T \). The density matrix \( \rho_w \) depends only upon \( A_{ww} \) and \( A_{wb} \). Simple calculations yield:

\[
\langle x|\rho_w|y \rangle = \langle -1 \rangle^x A_{ww} x + y A_{wb} y . \begin{cases} 2^2 & \text{if } A_{ww} (x + y) = (0, 0, 0), \\ 0 & \text{otherwise}, \end{cases}
\]

(22)

where \( x \equiv (x_1, x_2, x_3), \ y \equiv (y_1, y_2, y_3), \ z \equiv (z_1, z_2, z_3) \) and we treat \( A_{ww}, \ A_{bw} \) as 3x3 matrices over binary field acting on binary vectors. Observe that the sum over \( z \) is zero if \( A_{bw} (x + y) \neq (0, 0, 0) \) and is equal to \( 2^3 \) if \( A_{bw} (x + y) = (0, 0, 0) \). One can explicitly verify that the matrix \( A_{bw} \) is nondegenerate over binary field \( \mathbb{F}_2 \) for any partition \( V = B \cup W \) (note that due to the symmetry of the graph \( G \), there are only three nonequivalent partitions, so this verification is very simple). It means that \( A_{bw} (x + y) = (0, 0, 0) \) only if \( x = y \). Therefore \( \langle x|\rho_w|y \rangle = 0 \) if \( x \neq y \) and \( \langle x|\rho_w|x \rangle = 2^{-3}, \) i.e. \( \rho_w = 2^{-3} \mathbb{1} \), see also (8).

3) Suppose the optimal basis for the \( i \)-th qubit is \( B_i^* \), i.e. \( S[H] = S[H, B_1^*, \ldots, B_6^*] \). Let \( p^*(x_1, \ldots, x_6) = \langle B_1^*(x_1), \ldots, B_6^*(x_6) | H \rangle^2 \) be the optimal probability distribution. Consider any partition \( V = B \cup W \). The probability distribution of three outcomes \( \{x_i : i \in W\} \) measured at white vertices only is

\[
\sum_{x_1, \ldots, x_6} p^*(x_1, \ldots, x_6) = \langle \otimes_{i \in W} B_i^*(x_i) | \rho_w | \otimes_{i \in W} B_i^*(x_i) \rangle = \frac{1}{8},
\]

(23)

regardless of configuration \( \{x_i : i \in W\} \). In other words, the optimal distribution has a nice property: the partial distribution of any triple of bits is absolutely uniform, see (3). Call such property of the distribution as 3-uniformity. An example of 3-uniform distribution is absolutely uniform distribution. There are also 3-uniform distributions which are not absolutely uniform, e.g. the distribution (24). Denote \( P^3 \) the set of all 3-uniform distributions of six bits. In Appendix A we show that

\[
\inf_{p(x) \in P^3} S[p(x)] = 4,
\]

(24)

where \( S[p(x)] \) is the entropy of probability distribution \( p(x) \). We know that \( p^*(x_1, \ldots, x_6) \in P^3 \) and thus we have \( S[H] = S[p^*(x_1, \ldots, x_6)] \geq 4 \). This completes the proof of (17).

The definition like (18) can be used to assign a state \( |G \rangle \in (\mathbb{C}^2)^{\otimes |V|} \) to any unoriented graph \( G = (V, E) \). Reasoning as above, one can show that if \( |V| = 2m \) and the adjacency matrix \( A_{bw} \) is nondegenerate over binary field for any partition \( V = B \cup W \), \( |B| = |W| = m \), then in the state \( |G \rangle \) any \( m \) qubits have absolutely uniform density matrix \( 2^{-m} \mathbb{1} \). Such state \( |G \rangle \) can be called maximally uniform pure state, because any subset of qubits which is not forbidden by Schmidt constraint to have absolutely uniform density matrix do have absolutely uniform density matrix. This property of the quantum state is interesting by itself. Surprisingly, for \( m \leq 15 \) (i.e. for \( |V| \leq 30 \)) appropriate graphs exist only for \( m = 3 \) (e.g. the graph shown on Fig. 1) and for \( m = 1 \) (e.g. \( V = \{1, 2\} \) and \( E = \{(1, 2)\} \)). It follows from the bounds on additive quantum codes. Indeed, consider a state \( |G \rangle \) assigned to such graph. Like as hexadec state, we can specify \( |G \rangle \) by stabilizer operators (19) which generate the group \( S \) of stabilizers of order \( 2^m \). Any stabilizer in \( S \) is a tensor product of several Pauli matrices (possibly with a sign ‘-’) and \( X |G \rangle = |G \rangle \). But for any operator \( Y \) acting on \( m \) or less qubits we have \( \langle Y |G \rangle |Y \rangle = 2^{-m} \text{tr}(Y) \) because the density matrix of any \( m \) qubits is proportional to \( \mathbb{1} \). Thus any stabilizer in \( S \) acts on at least \( m + 1 \) qubit. It means that \( |G \rangle \) is an additive quantum code coding 0 qubits into 2\( m \) qubits

5
with the minimal stabilizer weight $m + 1$ or greater. The results of the work [1] imply that for $m \leq 15$ such codes exist only for $m = 1, 3$.

The tensor powers of hexadec state $|H^\otimes k\rangle$ have the highest (to our knowledge) entanglement entropy for sufficiently small $k$. For example, the state $|H^{\otimes 4}\rangle$ has entanglement entropy greater than determinant state $|\text{Det}_{24,2}\rangle$, if one takes twenty-four qubits, see Table I. However we do not expect that $|H\rangle$ has the maximal entanglement entropy if all six qubit states could be considered.

VII. CONCLUSION

Although the generalization of entanglement entropy to multipartite case suggested in the present work looks rather natural, one faces a lot of difficulties while trying to compute entanglement entropy of some particular state. A progress can be achieved only if the state has some special properties or symmetry.

For fixed $n$ and $d$ the maximal entanglement entropy $S^*(n, d)$ is rather close to the upper bound $n \log_2 d$, such that the ratio $S^*(n, d)/n \log_2 d$ approaches one for fixed $d$ and sufficiently large $n$.

There is also one subtle point in the definition (2) concerning its stability under the extension of each party space of states. Suppose that each of the parties sharing the state $|\Psi\rangle \in (C^d)^\otimes n$ adds new local degrees of freedom to his part of the system thus extending his space to $C^D$, $D > d$. The original space $C^d$ is somehow embedded to extended space $C^D$ and the original state $|\Psi\rangle$ now is a vector from $(C^D)^\otimes n$. There are two ways to compute entanglement entropy of $|\Psi\rangle$: the parties may perform complete von Neumann measurements either in the original space $C^d$ or in the extended space $C^D$. Being the functional of the state $|\Psi\rangle$ only, entanglement entropy should be the same in both cases. If $S[\Psi]$ is indeed invariant under such extensions, call $|\Psi\rangle$ the stable state. In the bipartite case any state is stable, because $S[\Psi]$ is invariant functional of single party density matrix, see (13). The determinant state $|\text{Det}_n\rangle$ is also stable for any $n$. Indeed, the formula (13) which guarantees the equality $S[\text{Det}_n] = \log_2(n!)$ remains valid even if we allow the states $|\phi_i\rangle$ to be chosen from the extended space: the maximum is obviously achieved when all $|\phi_i\rangle$'s belong to original space $C^n$. However we cannot prove that arbitrary multipartite state is stable, so this question is open.

Also it is interesting to check whether generalized entanglement entropy is monotonic under local quantum operations for $n > 2$.

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APPENDIX A:

The purpose of this section is to prove the equality (24). We will consider probability distributions $p(x)$ of $n$ classical bits, i.e. $x = (x_1, \ldots, x_n)$, $x_i = 0, 1$.

By analogy with the set $P^3_n$, we will consider the sets $P^k_n$ of $k$-uniform distributions of $n$ bits. By definition, $p(x) \in P^k_n$ iff any $k$ of $n$ bits have absolutely uniform distribution (i.e. if the sum of $p(x)$ over any $n - k$ bits is equal to $2^{-k}$). For example, the set $P^0_n$ consists of just one point — absolutely uniform distribution of $n$ bits $p(x) \equiv 2^{-n}$ while the set $P^6_n$ includes all possible $n$-bit distributions. If $p(x) \in P^k_n$, consider its binary Fourier transform $q(y)$:

$$
\begin{cases}
q(y) = \sum_{x \in B^n} (-1)^{x \cdot y} p(x), \\
p(x) = 2^{-n} \sum_{y \in B^n} (-1)^{x \cdot y} q(y).
\end{cases}
$$

(A1)

Here $B^n$ denotes the set of all length $n$ binary strings and $x \cdot y = \sum_{i=1}^n x_i y_i \pmod{2}$. If $y \in B^n$, denote $\text{wt}(y) \in [0, n]$ the number of $1$ in the binary string $y$. The definition of $k$-uniformity can be rephrased in terms of Fourier components as:

$$p(x) \in P^k_n \iff \begin{cases}
q(y) = 0 \quad \text{if} \quad 1 \leq \text{wt}(y) \leq k, \\
q(0, \ldots, 0) = 1, \\
\text{for all} \quad x \in B^n \\
\sum_{y \in B^n} (-1)^{x \cdot y} q(y) \geq 0.
\end{cases}
$$

(A2)

Note also that $P^k_n$ is a convex set: if $p'(x), p''(x) \in P^k_n$ then for any $\alpha \in [0, 1]$ we have $p'(x) + (1 - \alpha)p''(x) \in P^k_n$.

It is known that the entropy is a concave functional, i.e. $S[\alpha p'(x) + (1 - \alpha)p''(x)] \geq \alpha S[p'(x)] + (1 - \alpha)S[p''(x)]$.

It means that the minimum of $S[p(x)]$ over all $p(x) \in P^k_n$ is achieved when $p(x)$ is an extremal point of $P^k_n$. Because $P^k_n$ is specified by a finite number of linear equalities and inequalities, it has a finite number of extremal points. In principle we could find all them, compute $S[p(x)]$ for each point and choose the minimal value. However the set $P^6_6$, which we are interested in, has too many extremal points and such method doesn’t work in practice.

Instead we will proceed as follows. Consider six bit probability distribution $p(x)$ defined by (20). An explicit verification shows that $p(x) \in P^3_6$ (as it should be, because any measurement on the state $|H\rangle$ produces probability distribution from $P^3_6$) and that $S[p(x)] = 4$. It tells us that
Indeed, take any \( p(x) \in \mathbb{P}_3^3 \) and average out the sixth bit. Then, by definition, the distribution of the bits 1, 2, …, 5 is 3-uniform: \( \sum x_i p(x) \in \mathbb{P}_3^3 \). The entropy of the partial distribution can not exceed the entropy of the joint distribution, so that \( S[\sum x_i p(x)] \leq S[p(x)] \), which implies (A4). Now if we will manage to prove that
\[
\inf_{p(x) \in \mathbb{P}_3^3} S[p(x)] = 4,
\]
then the work is done because (A4) implies (A5). The nonzero components of \( q(y) \) are listed below:
\[
q(1, 1, 1, 1, 1) = q, \quad q(1, 1, 0, 1, 1) = q_3,
q(0, 1, 1, 1, 1) = q_1, \quad q(1, 1, 1, 0, 1) = q_4,
q(1, 0, 1, 1, 1) = q_2, \quad q(1, 1, 1, 1, 0) = q_5.
\]

Then \( p(x) \) can be written as:
\[
p(x) = (1/32) \{ 1 + (-1)^{w(x)} [q + \sum_{i=1}^5 q_i (-1)^x_i] \}. \tag{A7}
\]
The positivity constraint \( p(x) \geq 0, x \in \mathbb{B}^3 \) specifies the convex set \( \mathbb{P}_3^3 \) in the space of \( q, q_1, \ldots, q_5 \). If \( p(x) \) is an extremal point of \( \mathbb{P}_3^3 \) then \( p(y) = 0 \) for at least one \( y \in \mathbb{B}^3 \). By the symmetry, we can assume that \( p(0, 0, 0, 0, 0) = 0 \). So to find all extremal points of \( \mathbb{P}_3^3 \) one suffices to find all extremal points of the convex set
\[
\bar{\mathbb{P}}_3^3 = \{ p(x) \in \mathbb{P}_3^3 : p(0, 0, 0, 0, 0) = 0 \}. \tag{A8}
\]
In terms of variables \( q, q_1, \ldots, q_5 \) the set \( \bar{\mathbb{P}}_3^3 \) is described by the following linear constraints:
\[
\bar{\mathbb{P}}_3^3 = \left\{ q = -1 - \sum_{i=1}^5 q_i, \quad \sum_{i=1}^5 q_i \geq -1, \quad q_i + q_j \leq 0, \quad 1 \leq i < j \leq 5 \right\}. \tag{A9}
\]
(fortunately, it appears that only part of inequalities \( p(x) \geq 0 \) is independent). It is convenient to introduce one more auxiliary set \( Q \) defined as
\[
Q = \{ (q_1, \ldots, q_5) : q_i + q_j \leq 0, \quad 1 \leq i < j \leq 5 \}. \tag{A10}
\]
It is also a convex set. One can easily show that \( Q \) has only one extremal point \( q_1 = \cdots = q_5 = 0 \) and two types of one-dimensional edges with five edges of each type coming out from this extremal point:
\[
e^{(1)}_i = \{ q_i \leq 0, \quad q_j = 0 \text{ if } j \neq i \}, \quad i \in [1, 5],
\]
\[
e^{(2)}_i = \{ q_i \geq 0, \quad q_j = -q_i \text{ if } j \neq i \}, \quad i \in [1, 5]. \tag{A11}
\]
The extremal points of \( \bar{\mathbb{P}}_3^3 \) are those extremal points of \( Q \) for which \( \sum_{i=1}^5 q_i \geq -1 \) and also the intersections of one-dimensional edges of \( Q \) with the hyperplane \( \sum_{i=1}^5 q_i = -1 \). Summarizing, there are only eleven extremal points of \( \bar{\mathbb{P}}_3^3 \):
1) \( q = -1, \quad q_1 = \ldots = q_5 = 0 \),
2) \( q = 0, \quad q_i = -1, \quad q_j = 0 \text{ if } j \neq i \); \( i \in [1, 5] \),
3) \( q = 0, \quad q_i = 1/3, \quad q_j = -1/3 \text{ if } j \neq i \); \( i \in [1, 5] \).
Here extremal points 2) and 3) represent the intersections of \( e^{(1)}_i \) and \( e^{(2)}_i \) correspondingly with the hyperplane \( \sum_{i=1}^5 q_i = -1 \) while 1) is the extremal point of \( Q \). Substituting them into (A7) one can find the corresponding distributions \( p(x) \). One can check that for extremal points 1) and 2) the probability \( p(x) \) takes only values 0 and 1/16 thus having entropy \( S[p(x)] = 4 \). For extremal points 3) the probability takes the values 0, 1/12, and 1/24. The entropy appears to be \( S[p(x)] = 17/6 + \log_2 3 \approx 4.4 \). Thus for all extremal points of \( \bar{\mathbb{P}}_3^3 \) we have \( S[p(x)] \geq 4 \) and for some extremal points \( S[p(x)] = 4 \) which implies the equality (A3).

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[7] Binary matrix \( M \) is non-degenerate if for any binary vector \( x \neq 0 \) we have \( Mx \neq 0 \) over binary field. Equivalently, \( M \) is nondegenerate if \( \det(M) = 1 \mod 2 \).
[8] According to Section \( \Pi \) it tells us that \( S[H] \geq 3 \). This estimate however do not take into account that \( \rho_w \) is absolutely uniform for any partition \( V = B \cup W \).
[9] Clearly, the outcomes distribution has the same property for all choices of the bases \( B_i \), not only for the optimal one.