Alpha-Viscosity Effects in Slender Tori

Jin Horák,1 Marek A. Abramowicz,2,4,5 Lina Levin,2,3 Rikard Slapak,2 and Odele Straub4

1 Astronomical Institute, Academy of Sciences, Boční II, CZ-141 31 Prague, Czech Republic
2 Department Physics, Göteborg University, S-412 96 Göteborg, Sweden
3 Swinburne University of Technology, Centre for Astrophysics and Supercomputing, Mail H30, PO Box 218, VIC 3122, Australia
4 Copernicus Astronomical Centre PAN, Bartyska 18, 00-716 Warszawa, Poland
5 Institute of Physics, Faculty of Philosophy and Science, Silesian University in Opava, Bezuřovo nám. 13, CZ-746-01 Opava, Czech Republic

(Received 2011 November 8; accepted 2012 January 24)

Abstract

We explore effects of the Shakura–Sunyaev α-viscosity on the dynamics and oscillations of slender tori. We start with a slow secular evolution of the torus. We show that the angular-momentum profile approaches the Keplerian one on a timescale longer than a dynamical one by a factor on the order of 1/α. We then focus our attention on the oscillations of the torus. We discuss the effects of various angular-momentum distributions. Using a perturbation theory, we have found a rather general result that the high-order acoustic modes are damped by the viscosity, while the high-order inertial modes are enhanced. We calculate viscous growth rates for the lowest-order modes, and show that already the lowest-order inertial mode is unstable for less-steep angular-momentum profiles, or very close to the central gravitating object.

Key words: accretion disks — X-rays: binaries

1. Introduction

The stability and propagation of waves in viscous accretion disks have been studied extensively for more than thirty years, since the pioneering work of Kato (1978). Adopting the local approximation, he has shown that the horizontal p-mode oscillations of the Keplerian isothermal disk become over-stable when a small viscosity of the fluid is introduced. This instability arises because the azimuthal component of the viscous force varies in phase with the azimuthal velocity during oscillations, and thus positive work on the oscillations is done. The source of the energy for the oscillations is the angular momentum flowing radially in the background flow. Under suitable conditions, given mainly by the temperature and density dependence of the viscosity, and also by the mode eigenfrequencies and the shape of the modal eigenfunctions, this process may provide energy sufficient to amplify the oscillations against other normal viscous damping processes. The instability of the p-modes is even stronger in the relativistic disk, where the gradient of the angular velocity is steeper.

Later on, this problem was reconsidered by other authors, and the analysis has been expanded also to other types of oscillation modes. For example, Blumenthal, Lin, and Yang (1984) studied the stability of p-modes in disks with different ratios of the gas to the radiation pressure; Kato (1991) discussed the possibility of viscous excitation of the one-armed corrugation mode in relativistic disks. A local approximation was released in the work of Nowak and Wagoner (1992), who estimated the growth rates of p-modes trapped near the inner edge of the disk, and g-modes trapped near the radius of the maximum of the radial epicyclic frequency.

Some of the analysis go even beyond the standard Shakura and Sunyaev (1973) prescription, which is based on mixing-length theory. Nowak and Wagoner (1992) considered the effects of an anisotropic viscosity, and showed that the growth of the trapped g-modes is enhanced, while that of the p-modes is suppressed. Perhaps even more advanced is the analysis of c-mode oscillations in relativistic disks by Kato (1994), who uses the transport equations to calculate the stress tensor perturbations based on his second-order closure scheme.

In this note we consider the effects of viscosity on the oscillation modes of a slender torus. The main differences of this configuration from the thin disks studied in the work cited above are the angular-momentum profiles, which can be significantly different from the Keplerian distribution, and the following substantial pressure and density gradient in the radial direction (i.e., in the same direction as the angular velocity is changing). The plan of the paper is as follows. In section 2 a basic theory of inviscid tori and the limit of slender tori are reminded. Section 3 deals with the secular evolution of the torus that arises when a small viscosity is introduced to the flow. Section 4 is devoted to a general discussion of the oscillation modes, and the effects of viscosity on the oscillation frequencies. There, we derive a general formula for the eigenfrequency shifts, which we subsequently apply in sections 5 and 6. Section 5 contains results for the modes of high order and for Keplerian tori, and the section 6 focuses on the lowest-order modes. Finally, section 7 is devoted to discussion and conclusions.

2. Inviscid Slender Tori

The construction of an inviscid axisymmetric polytropic slender torus is described by Blaes (1985) in the Newtonian
gravitational field, and by Blaes et al. (2007) in the case of a general axisymmetric gravitational potential. The fluid of the torus is in pure rotation, e.g., its velocity in polar coordinates \( \{r, \phi, z\} \) can be expressed as \( v^i = \Omega \delta^i_\phi \), where \( \Omega \) is the angular velocity. The shape of the equi-pressure and equi-
density surfaces coincide because of the polytropic equation of state, \( p \propto \rho^{\gamma} \), and can be uniformly described by the Lane–Emden function, \( f(x) \). The density and pressure are then given by \( \rho = \rho_0 f^n(x) \) and \( p = p_0 f^{n+1}(x) \) (the subscript zero indicates the evaluation at the center of the torus, where the density and pressure are maximal). Another consequence of using the polytropic equation of state is that the angular velocity is a function of the radial coordinate only, \( \Omega = \Omega(r) \). In addition, throughout this paper we assume that this function is slowly varying, i.e., \( \partial t \Omega \approx \Omega / r \). The structure of these surfaces follows from the poloidal part of the Euler equation,

\[
\nabla^i \Phi - r \Omega^2 \delta^i_\phi = -\frac{1}{\rho} \nabla^i \nabla^j f,
\]

where \( \beta^2 = 2(n + 1) p_0 / (\rho_0^{n+1} \Omega_0^2) \) is a slenderness parameter, \( \beta \), that gives the thickness of the torus, both the radial and vertical extent of the torus are of the order of \( \beta r_0 \).

The limit \( \beta \to 0 \) describes slender tori. The surfaces of constant pressure and density have elliptic cross-sections, whose centers correspond to the maximal-pressure circle at \( r = r_0 \) in the equatorial plane. It is convenient to introduce dimensionless coordinates, \( \tilde{x} \), and \( \tilde{y} \), contracting with the torus as \( \beta \to 0 \),

\[
\tilde{x} = \frac{r - r_0}{\beta r_0}, \quad \tilde{y} = \frac{z}{\beta r_0}.
\]

The shapes of the equi-pressure surfaces are then given by

\[
f(\tilde{x}, \tilde{y}) = 1 - (\tilde{\alpha}_2^x - \tilde{\alpha}_2^z)^2 - \tilde{\alpha}_2^y \tilde{z}^2,
\]

where \( \tilde{\alpha}_2^x \) and \( \tilde{\alpha}_2^z \) are the radial and vertical epicyclic frequencies normalized by \( \Omega_0 \) and \( \tilde{\kappa}^2 = (d \ln \kappa^2 / d \ln r)_0 \) describes distribution of the angular momentum \( \ell = r^2 \Omega(r) \), which is in the slender-torus limit. It follows from equation (1) that \( \Omega_0 = \Omega_{\kappa}(r_0) \). The outermost ellipse given by \( f = 0 \) describes the surface of the torus. On the other hand, the torus center at \( (\tilde{x}, \tilde{y}) = (0, 0) \) corresponds to \( f = 1 \).

### 3. Secular Evolution of Viscous Tori

The introduction of a viscosity causes a slow secular evolution of the torus governed by the Navier–Stokes equation,

\[
\frac{\partial v^i}{\partial t} + v^k \nabla_k v^i + \frac{1}{\rho} \nabla^i p + \nabla^i \Phi = \frac{1}{\rho} \nabla^k g^{hi} F_{\text{visc}}^i,
\]

The stress tensor of the viscous flow is given by

\[
\sigma^{ik} = \eta (\nabla^k v^i + \nabla^i v^k) + \left( \xi - \frac{2}{3} \eta \right) \left( \nabla^k v^i \right) g^{ik},
\]

where \( \eta \) and \( \xi \) are coefficients of the dynamic and bulk viscosity, respectively, and \( g^{ik} \) is a metric tensor. We extend the standard ‘\( \alpha \rho \)’-parameterization of the shear viscosity, \( \eta \), traditional in the accretion theory to also include the bulk viscosity, \( \xi \), by introducing a coefficient, \( a \),

\[
\eta = \frac{\rho_0 \xi}{\Omega_0} \left( \xi - \frac{2}{3} \right), \quad a = \frac{\rho_0 \xi}{\Omega_0}, \quad a \ll 1.
\]

The velocity of the fluid is still dominated by the azimuthal component, but also small poloidal components appear due to the viscosity. In our analysis, we take \( \alpha \) as a small parameter with respect to which both the equations and their solutions are expanded. For the poloidal velocity we assume that \( v^i / (r \Omega) = O(\alpha), i = r, z \), the azimuthal velocity is of the order of the Keplerian orbital velocity at the center of the torus. From a balance among the leading-order terms in the poloidal part \( (i = r, z) \) of the Navier–Stokes equation we obtain again

\[
\frac{1}{\rho} \nabla^i p + \nabla^i \Phi - r \Omega^2 \delta^i_\phi = 0,
\]

while the toroidal part and the continuity equation gives

\[
\frac{\partial \ell}{\partial t} = -\frac{\partial \ell}{\partial r} / (r \rho v^r) - \frac{\partial}{\partial z} (\rho v^z).
\]

The right-hand sides of the latter two equations are of the first order in \( \alpha \); these equations therefore describe a slow evolution of the angular momentum and the density on the time scale \( \sim 1 / (\alpha \Omega) \). On the other hand, equation (7) states the equilibrium among the pressure, gravitational and centrifugal forces that is reached on a much shorter timescale of \( \sim 1 / \Omega \). This equation determines the structure of the equipressure surfaces in the torus, and is identical to the case of the inviscid flow. The overall evolution can therefore be regarded as a sequence of inviscid tori, whose shapes are given by the Lane–Emden function, \( f \), defined in equation (3), but whose main determining parameters (\( \tilde{\kappa}, \beta, \) and \( r_0 \)) are slowly changing in time.

Calculating the total mass, \( M \), and the angular momentum, \( \mathcal{L} \), of an infinitely slender torus by integrating \( \rho \) and \( \rho \ell \) over the torus volume, we obtain the relation

\[
\mathcal{L} = \ell_{\kappa}(r_0) M.
\]

Since both quantities must be conserved during the process, we may conclude that the position of the torus center at \( r = r_0 \) remains unaffected. This is an artefact of the reflection symmetry with respect to \( \tilde{x} = 0 \) that appears in the limit \( \beta \to 0 \). The centers of thicker tori that do not have this symmetry will be, in general, slowly drifting in time toward the central objects.

Substituting the approximate expressions (3) for \( f \) with \( \tilde{\kappa} = \tilde{\kappa}(t) \) into equations (8) and (9) and assuming a linear profile of \( \ell(r) \), we obtain

\[
\frac{d \ln \beta^2}{d t} + \frac{d \tilde{\kappa}^2}{d t} \tilde{x}^2 + \frac{f_{n-1}}{2 \rho_0} \left( \frac{\partial}{\partial \tilde{x}} (f^n v^r) + \frac{\partial}{\partial \tilde{y}} (f^n v^z) \right) = 0
\]

and

\[
\frac{d \tilde{\kappa}^2}{d t} - \frac{\ell_{\kappa}}{r_0} (4 - \tilde{\kappa}^2) (\tilde{\alpha}_2^2 - \tilde{\kappa}^2) \beta \tilde{x} + \frac{1}{r_0} \tilde{\kappa}^2 v^r = 0.
\]
Now, making an ansatz,
\[ \nu'(t, \bar{x}, \bar{y}) = \beta \bar{x} V'(t), \quad \nu'(t, \bar{x}, \bar{y}) = \beta \bar{y} V'(t), \]  
we obtain polynomial equations in \( \bar{x}, \bar{y} \). Comparing appropriate coefficients, we finally arrive at a set of four equations:
\begin{align*}
    \frac{d\ln k^2}{dt} &= -\frac{1}{r_0} (V' + V^2), \\
    \frac{d\bar{k}^2}{dt} &= \frac{1}{r_0} \left[ (2n+1) V' + V^2 \right] (\tilde{\omega}_r^2 - \bar{k}^2), \\
    0 &= \frac{1}{r_0} [V' + (2n+1) V^2] \tilde{\omega}_r^2, \\
    \frac{d\bar{k}^2}{dt} &= \frac{\alpha \xi_0}{r_0} \left( 4 - \bar{k}^2 \right) (\tilde{\omega}_r^2 - \bar{k}^2) - \frac{1}{r_0} \bar{k}^2 V',
\end{align*}
for unknowns \( \bar{k}(t) \), \( \beta(t) \), \( V'(t) \), and \( V^2(t) \). The time of the secular evolution from the constant-angular momentum torus to the configuration, characterized by \( \bar{k} \), is
\[ t = \frac{1}{\alpha \Omega_0} \int_{0}^{\bar{k}(t)} \frac{\tilde{\omega}_r^2 - q \bar{k}}{(4 - k)(\tilde{\omega}_r^2 - \bar{k})^2} dk = \frac{1}{\alpha \Omega_0 (4 - \tilde{\omega}_r^2)^2} \left[ (4q \tilde{\omega}_r^2 (4 - \bar{k}^2) / 4(\tilde{\omega}_r^2 - \bar{k}^2) \right] + (1-q) \tilde{\omega}_r^2 (4 - \bar{k}^2), \]
where \( q = (2n + 3)/(4n + 4) \).

The solution, \( \bar{k}(t) \), is shown in the top panel of figure 1.

The angular momentum from the inner part of the torus is transported outward; its distribution becomes steeper when approaching the Keplerian one (corresponding to \( k = \tilde{\omega}_r \)) at long times. A characteristic timescale of the viscous diffusion is
\[ t_{\text{visc}} = \alpha^{-1} \Omega_0^{-1}. \]  

The bottom panel shows the shape of the torus at several moments during the secular evolution. The torus becomes wider in a radial direction and shrinks in the vertical, as the angular momentum distribution approaches the Keplerian one.

In the next sections we calculate the damping rates of the lowest-order oscillation modes due to viscosity in tori with arbitrary angular-momentum distributions. Since the oscillations occur on timescales that are much shorter than \( t_{\text{visc}} \), we ignore viscous diffusion in the rest of the paper.

4. Perturbations

The unperturbed state is axisymmetric, and can be regarded as being stationary on timescales shorter than \( t_{\text{visc}} \). Therefore, the \( t \) and \( \phi \)-dependence of the perturbation is \( \propto \exp[-i(\omega t - m\phi)] \), where \( \omega \) is the eigenfrequency of the oscillation mode, and \( m \) its azimuthal wavenumber. Papaloizou and Pringle (1984) showed that it is convenient to express all perturbations in terms of a single quantity, \( W = -\delta p/(\rho \sigma) \), where \( \sigma = \omega - m\Omega \). A linear Euler perturbation of the continuity equation gives
\begin{equation}
\frac{1}{r} \frac{\partial}{\partial r} (rf^n \delta v^r) + \frac{\partial}{\partial z} (fn^2 \delta v^z) + \frac{\partial}{\partial \phi} \left[ \frac{m}{r^2} \delta v^\phi \right] = 0,
\end{equation}
where \( \delta = \sigma / \Omega_0 \). The perturbation of the velocity \( \delta v^i \) can be calculated from \( W \) using perturbed Navier–Stokes equations, which take the following forms:
\begin{align*}
    i \delta v^r + \frac{2r \Omega}{\sigma} \delta v^\phi + \frac{\partial W}{\partial r} - \frac{m}{\sigma} d\Omega/d\Omega W &= -\frac{1}{\sigma} \delta F_r, \\
    i \delta v^\phi - \frac{\sigma}{2r \Omega} \delta v^r + \frac{im}{r^2} W &= -\frac{1}{\sigma} \delta F_\phi, \\
    i \delta v^z + \frac{\partial W}{\partial z} &= -\frac{1}{\sigma} \delta F_z.
\end{align*}

The perturbation of the viscous force in terms of the velocity perturbation \( W \) is:
\begin{align*}
    \delta F_r &= -\frac{\alpha \Omega_0 f^{-n}}{2(n+1)} \left[ \frac{\partial}{\partial y} \left( f^{n+1} \left( \frac{\partial \delta v^r}{\partial y} + \frac{\partial \delta v^z}{\partial x} \right) \right) \right], \\
    \delta F_\phi &= -\frac{\alpha \Omega_0 f^{-n}}{2(n+1)} \left[ \frac{\partial}{\partial x} \left( f^{n+1} \left( \frac{\partial \delta v^\phi}{\partial x} + \frac{\partial \delta v^z}{\partial y} \right) \right) \right] + \frac{\partial}{\partial y} \left( f^{n+1} \left( \frac{\partial \delta v^\phi}{\partial y} + \frac{im}{r^2} \delta v^r \right) \right), \\
    \delta F_z &= -\frac{\alpha \Omega_0 f^{-n}}{2(n+1)} \left[ \frac{\partial}{\partial x} \left( f^{n+1} \left( \frac{\partial \delta v^z}{\partial x} + \frac{im}{r^2} \delta v^r \right) \right) \right].
\end{align*}
where we keep only leading-order terms in the limit $\beta \to 0$ (i.e., we neglect effects of the azimuthal curvature of the torus). Equations (21)–(23) together with (24)–(26) are then solved for the poloidal velocity in terms of $W$. This procedure is easier if we assume that $\partial W/\partial \tilde{x} \gtrsim 1$, i.e., that the wavelength of a perturbation is at least comparable to the size of the torus. Then, up to the first order in $\alpha$, and in the lowest order in $\beta$, the solutions are

$$
\begin{align*}
\delta v' &= \frac{i}{\beta \rho_0} \frac{\sigma^2}{\sigma^2 - \kappa^2} \frac{\partial W}{\partial \tilde{x}} - \alpha \frac{\sigma^2}{\sigma^2 - \kappa^2} \frac{f^{-n}}{2(n+1)\tilde{x} \beta_0 \rho_0} \frac{\partial}{\partial \tilde{x}} \left[ f^{n+1} \frac{\partial^2 W}{\partial \tilde{x} \partial \tilde{y}} \right] + \frac{\partial}{\partial \tilde{y}} \left[ f^{n+1} \left( 2a + \frac{\partial^2 W}{\partial \tilde{x}^2} + a \frac{\partial W}{\partial \tilde{y}^2} \right) \right] - 8(n+1)f^n \left( \frac{1}{1 - \frac{\sigma^2}{2n}} \frac{\partial W}{\partial \tilde{x}} \right) \\
&\quad + \frac{\kappa^2}{\sigma^2 - \kappa^2} \left[ 3 + a \right] \frac{\partial}{\partial \tilde{x}} \left( f^{n+1} \frac{\partial^2 W}{\partial \tilde{x}^2} \right) + 2 \frac{\partial}{\partial \tilde{y}} \left( f^{n+1} \frac{\partial^2 W}{\partial \tilde{x} \partial \tilde{y}} \right) \right] \right),
\end{align*}
$$

and

$$
\begin{align*}
\delta v'' &= \frac{i}{\beta \rho_0} \frac{\partial W}{\partial \tilde{y}} - \alpha \frac{\sigma^2}{\sigma^2 - \kappa^2} \frac{f^{-n}}{2(n+1)\tilde{x} \beta_0 \rho_0} \frac{\partial}{\partial \tilde{x}} \left[ f^{n+1} \frac{\partial^2 W}{\partial \tilde{x} \partial \tilde{y}} \right] + \frac{\partial}{\partial \tilde{y}} \left[ f^{n+1} \left( a \frac{\partial^2 W}{\partial \tilde{x}^2} + [2 + a] \frac{\partial W}{\partial \tilde{y}^2} \right) \right] + \frac{\kappa^2}{\sigma^2 - \kappa^2} \left[ \frac{\partial}{\partial \tilde{x}} \left( f^{n+1} \frac{\partial^2 W}{\partial \tilde{x} \partial \tilde{y}} \right) + a \frac{\partial}{\partial \tilde{y}} \left( f^{n+1} \frac{\partial^2 W}{\partial \tilde{x}^2} \right) \right].
\end{align*}
$$

Substituting equations (27) and (28) into the continuity equation (20), and keeping only dominant terms when $\beta \to 0$, we obtain a single operator equation for $W$,

$$
\tilde{\sigma}^4 W + \tilde{\sigma}^2 \tilde{B} W + \tilde{C} W = i \alpha \tilde{F}(\tilde{\sigma}) W
$$

with

$$
\begin{align*}
\tilde{B} &= \frac{f^{1-n}}{2n} \left[ \frac{\partial}{\partial \tilde{x}} \left( f^n \frac{\partial W}{\partial \tilde{x}} \right) + \frac{\partial}{\partial \tilde{y}} \left( f^n \frac{\partial W}{\partial \tilde{y}} \right) \right] - \kappa^2, \\
\tilde{C} &= -\kappa^2 \frac{f^{1-n}}{2n} \frac{\partial}{\partial \tilde{y}} \left( f^n \frac{\partial W}{\partial \tilde{y}} \right),
\end{align*}
$$

and

$$
\tilde{F}(\tilde{\sigma}) = -\frac{\tilde{\sigma}^2 - \kappa^2}{4n(n+1)\tilde{x} f^{-n-1}} \left[ a \left( \frac{\partial^2}{\partial \tilde{x}^2} \left( f^{n+1} \frac{\partial^2 W}{\partial \tilde{x}^2} \right) \right) + \frac{\partial^2}{\partial \tilde{x} \partial \tilde{y}} \left( f^{n+1} \frac{\partial^2 W}{\partial \tilde{x} \partial \tilde{y}} \right) + (2 + a) \frac{\partial^2}{\partial \tilde{x}^2} \left( f^{n+1} \frac{\partial^2 W}{\partial \tilde{x}^2} \right) \right].
$$

When $\alpha = 0$, equation (29) is a second-order eigenvalue problem, and describes oscillation modes of the inviscid slender torus. If, in addition, $\tilde{\kappa} = 0$ (constant angular momentum tori), equation (29) is reduced to the first-order eigenvalue problem for the operator $\tilde{B}$. In that case the operator $\tilde{B}$ is simply related to the operator $\tilde{L}^{(0)}$ introduced in Blaes, Arras, and Fragile (2006) by $\tilde{B} = 2n\tilde{L}^{(0)}$. Blaes (1985) and Blaes, Arras, and Fragile (2006); Blaes et al. (2007) showed that this operator is Hermitian with respect to the scalar product,

$$
\langle U, V \rangle = \int U^* V f^{-n-1} d \tilde{x} d \tilde{y},
$$

and demonstrated how this fact can be used in a perturbation approach to slightly non-slender tori. Blaes (1985) used this theory to calculate the growth-rate of the Papaloizou–Pringle instability and, more recently, the same approach has been employed in calculating the eigenfrequency corrections to the epicyclic modes (Blaes et al. 2007).

A small viscous damping of the oscillation modes can be treated in a similar way. In addition, in this paper we extend the perturbation approach of Blaes et al. (2007) to the case of a general angular-momentum distribution. In appendix 1 we formulate first-order perturbation theory for the eigenvalue problem (29). Due to the viscosity term on the right-hand side of equation (29), the eigenfrequency of the $v$-th mode is slightly changed from the value $\sigma_v^{(0)}$, which corresponds to the inviscid flow to

$$
\sigma_v = \sigma_v^{(0)} + \alpha \sigma_v^{(1)} + \cdots.
$$

The first-order correction is given by

$$
\sigma_v^{(1)} = \frac{i}{2\sigma_v^{(0)}} \left\langle W_v^{(0)}, \tilde{F} W_v^{(0)} \right\rangle \left( \tilde{B} + 2\sigma_v^{(0)2} W_v^{(0)} \right)
$$

with $W_v^{(0)}$ being the eigenfunction corresponding to the $v$-th mode of the inviscid torus.

Since the oscillations of the inviscid tori are not damped, the growth-rate of the viscous torus is given by

$$
\gamma_v = \alpha \Omega_0 \text{Im}(\sigma_v^{(1)}).
$$

The damping corresponds to the negative values of $\gamma$. 

Downloaded from https://academic.oup.com/pasj/article-abstract/64/4/76/2898275 by guest on 28 July 2018
5. Special Cases

5.1. The WKBJ Limit

In the limit of short wavelengths of the perturbation, the eigenfunctions can be approximated using the WKBJ ansatz,

\[
W \propto \exp \left(i \int k_x dx + i \int k_y dy \right),
\]

(37)

where \(k_x\) and \(k_y\) are the horizontal and vertical components of the wavevector. Blaes, Arras, and Fragile (2006) identified two distinct classes of modes, acoustic and inertial, that obey the dispersion relations:

\[
\delta \omega = \frac{f}{2n} k^2 \quad \text{and} \quad \delta \omega = \kappa^2 \frac{k_y^2}{k_x^2},
\]

(38)

respectively. While the frequency of the acoustic modes grows with increasing \(k = (k_x^2 + k_y^2)^{1/2}\), that of the inertial modes is always smaller, or at most equal to \(\kappa\).

After calculations briefly summarized in appendix 2, we find that the approximate expression for the growth rates of the acoustic modes is

\[
\gamma = -\frac{n}{n + 1} \left(1 + \frac{a}{2} \right) \delta \omega \Omega_0,
\]

(39)

while that of the inertial modes is proportional to

\[
\gamma \sim \left(1 - \frac{\delta \omega}{\kappa^2}\right) \frac{\alpha \Omega_0}{(n + 1)\Omega_2^2},
\]

(40)

where \(\lambda_s\) is a typical wavelength of the mode. Therefore, while the high-frequency acoustic modes are always damped by the viscosity, the low-frequency inertial modes become unstable when the \(\alpha\)-viscosity is introduced.

5.2. Keplerian Angular Momentum Distribution

In the limit of the Keplerian angular-momentum distribution, the characteristic frequency of the inertial oscillations is equal to the radial epicyclic frequency, \(\kappa = \omega_r\). The Lane–Emden function does not depend on \(\tilde{x}\) and is a function of \(\tilde{y}\) only,

\[
f = 1 - \tilde{\omega}_r^2 \tilde{y}^2.
\]

(41)

One set of the oscillation modes that depends linearly on \(\tilde{x}\) have eigenfunctions given by the Gegenbauer polynomials, \(C^1_j\):

\[
W_j = C^{1-1/2}_j \tilde{\omega}_r \tilde{y}, \quad \text{or} \quad W_j = \tilde{x} C^{1-1/2}_j \tilde{\omega}_r \tilde{y},
\]

(42)

where \(j \geq 1\) denotes a number of vertical nodes (see Blaes et al. 2006). The corresponding eigenfrequencies are

\[
\delta \omega_j = \frac{1}{2n} j(j + 2n - 1) \tilde{\omega}_r^2.
\]

(43)

The vertical epicyclic mode corresponds to \(j = 1\). Applying our theory, we find that the growth rate of the \(j\)-th mode is

\[
\gamma_j = -\frac{1 + a}{2} \left(\frac{j + 2n - 1}{2(n + 1)} \right) \tilde{\omega}_r^2 \Omega_0
\]

(44)

(see appendix 3). Hence, all of the modes are damped in the Keplerian limit. In addition, for high-order modes with \(j \gg 1\) the damping rate agrees with that predicted by the WKBJ approximation.

6. Results for the Lowest-Order Modes

Blaes, Arras, and Fragile (2006) derived eigenfunctions and eigenfrequencies of the lowest-order modes of slender tori with an arbitrary angular momentum distribution. The eigenfunctions of these modes are given by polynomials of low-order in \(\tilde{x}\) and \(\tilde{y}\). The most simple is the corotation mode corresponding to \(\delta = 0\), and a zero-order polynomial, \(W = 1\). It may become unstable when \(\beta > 0\); in that case it corresponds to the principal mode of the Papaloizou–Pringle instability. The eigenfunctions linear in \(\tilde{x}\) and \(\tilde{y}\) characterize two epicyclic modes: in slender tori they correspond to a rigid motion in radial or vertical direction with associated epicyclic frequencies. Four second-order modes (the X-mode, inertial mode, breathing mode and plus-mode) have eigenfunctions quadratic in \(\tilde{x}\) and \(\tilde{y}\). Their velocity patterns have a single node at the center of the torus. Blaes, Arras, and Fragile (2006) show how to construct modes of arbitrary order by solving appropriate sets of linear equations. In this section, we calculate the viscous growth-rates of the lowest order modes (up to the third order) using the equations (35) and (36). For this purpose we use formulae for the eigenfunctions and eigenfrequencies derived by Blaes, Arras, and Fragile (2006).

6.1. The Epicyclic Modes

The radial and vertical epicyclic modes correspond to linear eigenfunctions, \(W_{\text{rad}} = \tilde{x} \) or \(W_{\text{vert}} = \tilde{y}\). In the inviscid slender torus, they describe uniform displacements from the equilibrium positions with frequencies equal to epicyclic frequencies of a freely moving particle. Therefore, the shear of the corresponding velocity field vanishes, and so may vanish the viscous damping. In fact, this is the case of the vertical epicyclic mode for which \(\hat{F}W_{\text{vert}} = 0\), and therefore \(\gamma_{\text{vert}} = 0\). In the case of the radial epicyclic modes, however, we find that

\[
\hat{F}W_{\text{rad}} = \frac{1}{\omega_r} (\tilde{\omega}_r^2 - \kappa^2)(4 - \kappa^2) \tilde{x},
\]

(45)

and therefore the viscous growth-rate is

\[
\gamma_{\text{rad}} = -\frac{1}{2} \left(1 - \frac{\tilde{\omega}_r^2}{\kappa^2} \right)^2 (4 - \kappa^2) \alpha \Omega_0.
\]

(46)

This is because during the radial oscillations, the viscous force does not vanish, even though the shear of the perturbed velocity field does. Because of variations in the kinematic viscosity coefficient, \(\eta\) (that arise due to a pressure variations), a nonzero azimuthal viscous force appears,

\[
\delta F^\phi_{\text{visc}} \propto -\frac{\omega_r \kappa^2}{2} \exp[-i(\omega t - m\phi)].
\]

(47)

This force is in the anti-phase with the velocity perturbations

\[
\delta v^\phi \propto \frac{\omega_r \kappa^2}{2r} \exp[-i(\omega t - m\phi)],
\]

(48)

and therefore causes the damping of oscillations. A similar effect is absent in the case of the vertical oscillations, because the \(\phi\)-component of the shear of the equilibrium velocity vanishes.

Equation (46) shows that the growth-rate vanishes in the limit of the Keplerian angular momentum distribution. This is
because the amplitude of the pressure variations decays when approaching that limit [note that \( \delta p/p \sim (1 - \omega^2/k^2)(\delta v^2/\Omega) \) for the radial epicyclic mode]. This is consistent with the behavior of \( p \)-modes in geometrically thin accretion disks. Although their amplitudes grow, the growth-rates are inversely proportional to the squared wavelengths of the modes (e.g., Kato 1978; Kato et al. 1998). The \( p \)-modes with infinite wavelength (that corresponds to our radial epicyclic mode) are not affected by the viscosity.

6.2. Constant-Angular-Momentum Tori

Figure 2 shows a radial dependence of the damping rates (in units of \( \alpha \Omega_0 \)) of the second-order (solid line) and third order modes (dashed lines). The third-order modes have been labeled according to the convention of Blaes, Arras, and Fragile (2006). The coefficient, \( \alpha \), was set to \( \alpha = -2/3 \), which corresponds to the vanishing bulk viscosity coefficient, \( \xi = 0 \). One may notice that the gravitational field enters into all formulae only through the expressions for the orbital and epicyclic frequencies \( (\Omega_0, \dot{\omega}_r, \text{and } \dot{\omega}_z) \). Similarly to other authors, we use the Schwarzschild expressions for these frequencies, despite the fact that the formulae have been derived in the framework of Newtonian physics. We believe that this inconsistency will not affect the qualitative discussion of the paper significantly.

The radial coordinate of the torus center is shown in units of the gravitational radii, \( GM/c^2 \).

The analytic formulae for damping rates are rather complicated. The exception is the X-mode, whose growth/damping rate is

\[
\gamma_x = -\frac{2 \bar{\omega}_r^2 (1 + \bar{\omega}_z^2)}{\bar{\omega}_r^2 + \bar{\omega}_z^2} \alpha \Omega_0.
\]

The damping rate does not depend on the bulk viscosity coefficient, \( \alpha \), because the X-mode oscillations are incompressible. On the other hand, both the plus and breathing modes weakly depend on the value of \( \alpha \), as it is shown in figure 3.

Figure 2 suggests that the ratios \( \gamma_r/(\alpha \Omega_0) \) tend to the constant for \( r \gg GM/c^2 \). This behavior nicely illustrates the scaling invariance of Newtonian gravity. The Newtonian value of the damping rates of the plus and the breathing modes are

\[
\gamma_r = 2 \alpha \Omega_0 \text{ and } \gamma_{\text{breath}} = (2 + a) \alpha \Omega_0.
\]

6.3. Dependence on the Angular-Momentum Profile

The behavior of the damping rates of the four lowest order modes with changing angular momentum distribution is shown in figure 4. The top panel is devoted to the situation when torus is placed close to the central object at radius \( r_0 = 10GM/c^2 \). The lower panel shows the same for the torus at large radii and corresponds to the Newtonian limit. The damping rates of all the modes, except the breathing one, are significantly reduced for the Keplerian angular-momentum distribution. Although our analysis predicts that the damping of all three modes vanishes completely, it is still possible that the actual damping rate is of a higher order in \( \beta \), say on a timescale of \( \sim (\beta \alpha \Omega)^{-1} \).

The growth-rate of the breathing mode in the Keplerian limit (\( \bar{k} \to 1 \)) is

\[
\gamma_{\text{breath}} = -\bar{\omega}_r^2 \left( 1 + \frac{a}{2} \right) \alpha \Omega_0.
\]

Our results also suggest that the torus is unstable with respect to the inertial mode. In the relativistic case (top panel) this mode grows in time independently of \( \bar{k} \), while in the Newtonian limit this growth is suppressed for less-steep angular momentum profiles. Figure 5 shows regions of stable and unstable configurations in the \( (r, \bar{k}) \)-plane.

7. Discussion and Conclusions

In this paper we have considered viscosity effects in slender tori. Due to angular-momentum transport, the torus becomes wider in the horizontal direction, and thinner in the vertical as the angular momentum distribution approaches the Keplerian one. We have shown that this process occurs on a timescale of \( \sim \alpha^{-1} \Omega_0^{-1} \), and can be regarded as being quasi-steady, i.e., as a series of steady inviscid tori configurations, because the
poloidal velocity due to the viscosity is negligible. We have also found that because the angular-momentum transport is symmetric, the location of the center of the torus remains unchanged. This result is related to the reflection symmetry around the \( \hat{x} = 0 \) surface, which appears in the slender torus limit. Including the higher-order terms in \( \beta \) that break this symmetry, we would likely find a shift of the torus center towards the central object.

In the rest of the paper we dealt with the oscillations of viscous tori. For simplicity, we neglected the secular evolution, which allowed us to assume a harmonic time dependence of the perturbations. We derived an inhomogeneous master equation (29) with the left-hand side giving the normal modes of the inviscid tori, and right-hand side describing the effects of viscosity on the modes. We formulated a general first-order perturbation theory for the case when the right-hand side is small; also, with the aid of this theory we obtained a general formula for the mode growth rates. Applying this formula in the short-wave limit, we found that the viscosity acts to damp the acoustic modes. On the other-hand, viscous tori are unstable with respect to the inertial oscillations. The growth-rate of the instability increases with decreasing wavelength. We also calculated the damping rates of the acoustic oscillations in the Keplerian limit, and showed that they agree with results based on the WKBJ theory in the limit of high nodal numbers. Finally, we calculated the growth rates of the lowest-order modes, and explored their behavior with changing various parameters. We found that the damping of three of four modes is significantly reduced as the angular momentum distribution approaches the Keplerian one.

We gratefully acknowledge the kind hospitality of Mrs. Malwina from Stary Gierałtów, where the work on this project was initiated. JH also acknowledges many helpful discussions with his colleagues from Prague Astronomical Institute and support of the Grant Agency of the Czech Republic (project no. P209/11/2004). MAA acknowledges the support of the Czech CZ. 1.07/2.300/20.0071 “Synergy” grant supporting international collaboration of IF Opava, and the Polish NCN grant UMO-2011/01/B/ST9/05439.

**Appendix 1. First-Order Perturbation Theory for Tori with Arbitrary Angular-Momentum Distribution**

Here, we deal with a perturbation theory for equation (29) with the assumption that \( \alpha \) is small. The right-hand side is therefore treated as a perturbation. We start with the observation that both, the operators \( \hat{B} \) and \( \hat{C} \) are Hermitian with respect to the scalar product (33), i.e.,

\[
\left\{ \hat{B} W^{(0)}_\mu, W^{(0)}_\nu \right\} = \left\{ W^{(0)}_\mu, \hat{B} W^{(0)}_\nu \right\},
\]

\[
\left\{ \hat{C} W^{(0)}_\mu, W^{(0)}_\nu \right\} = \left\{ W^{(0)}_\mu, \hat{C} W^{(0)}_\nu \right\}.
\]

The eigenfunctions \( W^{(0)}_\nu \) that solve equation (29) with \( \alpha = 0 \) form a complete set, and can therefore be used as a basis with respect to which solutions of the perturbed equation will be expanded. However, contrary to the first-order eigenvalue problems, the eigenfunctions \( W^{(0)}_\nu \) are not orthogonal. Instead, they satisfy the pseudo-orthogonality relation,

\[
-\left( \sigma^{(0)2}_\mu + \bar{\sigma}^{(0)2}_v \right) \left\{ W^{(0)}_\mu, W^{(0)}_v \right\} + \left\{ W^{(0)}_\mu, \hat{B} W^{(0)}_v \right\} = 0.
\]
for $\mu \neq \nu$.

The solutions of the perturbed problem are then expanded via $\alpha$ as

$$\tilde{\sigma}_v = \tilde{\sigma}_v^{(0)} + \alpha \tilde{\sigma}_v^{(1)} + \ldots,$$

and

$$W_v = W_v^{(0)} + \alpha \sum_{\mu} c_{\mu} W_{\mu}^{(0)} + \ldots,$$

(A4)

and substituted into equation (29). Comparing the coefficient of the same powers of $\alpha$, we obtain in the first order

$$-\sum_{\mu} c_{\mu} \left[ (\tilde{\sigma}_\mu^{(02)} - \tilde{\sigma}_v^{(02)}) \right] \left[ (\tilde{\sigma}_\mu^{(02)} + \tilde{\sigma}_v^{(02)}) + \tilde{B} \right] W_\mu^{(0)}$$

$$+ 2\tilde{\sigma}_v^{(0)} \tilde{\sigma}_\nu^{(1)} (2\tilde{\sigma}_v^{(02)} + \tilde{B}) W_\nu^{(0)} = i\dot{W}_v^{(0)}.$$

(A6)

If we now perform a scalar product with $W_\nu^{(0)}$, and use the pseudo-orthogonality relation, we obtain

$$2\tilde{\sigma}_v^{(0)} \tilde{\sigma}_\nu^{(1)} \left[ \dot{W}_\nu^{(0)} \left( \tilde{B} + 2\tilde{\sigma}_v^{(02)} \right) W_\nu^{(0)} \right] = i \langle W_\nu^{(0)}, \dot{W}_\nu^{(0)} \rangle,$$

(A7)

from which we immediately recover relation (35).

**Appendix 2. The WKBJ Approximation**

In the WKBJ limit, $k \to \infty$, we find that

$$\dot{\tilde{W}} = - \frac{(\tilde{\sigma}^2 - \tilde{k}^2) f^2 k^2}{4n(n+1)} \times \left[ (2 + a)k^2 + \frac{3 + a}{2} \tilde{k}^2 \right] W,$$

and

$$\left( \hat{B} + 2\tilde{\sigma}^2 \right) W = \left( -\frac{1}{2n} f k^2 + 2\tilde{\sigma}^2 - \tilde{k}^2 \right) W.$$

(A9)

For the acoustic modes, the second term in the square bracket of equation (A8) vanishes because $\tilde{\sigma} \gg \tilde{k}$, and using the corresponding dispersion relation we obtain

$$\langle \dot{W}, \tilde{W} \rangle = -\frac{(2 + a)n\tilde{\sigma}^5}{n+1} \langle W, W \rangle.$$

(A10)

Similarly, we find that

$$\langle W, (\hat{B} + 2\tilde{\sigma}^2) W \rangle = \tilde{\sigma}^2 \langle W, W \rangle.$$

(A11)

With aid of equations (35) and (36), we therefore find that the growth/damping rate is

$$\gamma = -\frac{a}{n+1} \left( 1 + \frac{a}{2} \right) \tilde{\sigma}^2 \alpha \Omega_0.$$

(A12)

In the case of the inertial modes, we find that

$$\langle \dot{W}, \tilde{W} \rangle = -\frac{\tilde{k}^2 - \tilde{\sigma}^2}{4n(n+1)\tilde{\sigma}} \langle W, k^4 f^2 W \rangle$$

(A13)

and

$$\langle W, (\hat{B} + 2\tilde{\sigma}^2) W \rangle = -\frac{1}{2n} \langle W, k^2 f^2 W \rangle.$$

(A14)

The growth rate of these modes is therefore

$$\gamma = \frac{\tilde{k}^2 - \tilde{\sigma}^2}{2(n+1)\tilde{\sigma}^2} \langle W, k^4 f^2 W \rangle \alpha \Omega_0.$$

(A15)

Both scalar products on the right-hand side are positive, since their are integrals of positive quantities. Therefore, they cannot change the sign of $\gamma$, and we can conclude that the high-order inertial modes grow. Moreover, introducing a typical wavelength of the mode, $\lambda_*$, the upper and lower scalar products can be estimated as $\sim \lambda_*^{-8}(W, W)$ and $\sim \lambda_*^{-2}(W, W)$, respectively. Hence, the growth rate can be estimated as

$$\gamma \sim \left( 1 - \frac{\tilde{\sigma}^2}{\tilde{k}^2} \right) \frac{\alpha \Omega_0}{(n+1)\lambda_*^2}.$$

(A16)

**Appendix 3. Damping of the Oscillations in the Keplerian Limit**

In the Keplerian limit, the Lane–Emden function $f$, independent of $\check{x}$, and the eigenfunctions, $W_j$, depend on $\check{x}$ at most linearly. Applying the operator $\hat{F}$ given by relation (32) on $W$ we obtain

$$\langle W_j, \hat{F} \rangle = -\frac{(\tilde{\sigma}^2 - \tilde{\omega}_j^2)(2 + a)}{4n(n+1)\tilde{\sigma}} f \frac{\partial^2 W_j}{\partial \tilde{y}^2} \left( f^{n+1} \frac{\partial^2 W_j}{\partial \tilde{y}^2} \right).$$

(A17)

Performing twice the integration by parts in $\tilde{y}$, we find that the scalar product, $\langle W_j, \hat{F} W_j \rangle$, is given by

$$\langle W_j, \hat{F} W_j \rangle \sim -\frac{(\tilde{\sigma}^2 - \tilde{\omega}_j^2)(2 + a)}{4n(n+1)\tilde{\sigma}} f I^2,$$

(A18)

where

$$I^2 = \int \left( \frac{\partial W_j}{\partial \tilde{y}} \right)^2 f^{n+1} \tilde{y} d \tilde{y},$$

(A19)

and we omitted the factor coming from the integration over $\check{x}$. Similarly,

$$\langle W_j, (\hat{B} + 2\tilde{\sigma}^2) W_j \rangle = -\frac{1}{2n} I^1 + (2\tilde{\sigma}^2 - \tilde{k}^2) I^0,$$

(A20)

where

$$I^0 = \int |W_j|^2 f^{n-1} \tilde{y} d \tilde{y},$$

(A21)

and

$$I^1 = \int \left| \frac{\partial W_j}{\partial \tilde{y}} \right|^2 f^n \tilde{y} d \tilde{y}.$$

(A22)

Using the substitution $\zeta = \tilde{\omega}_j \tilde{y}$ and with the aid of standard rules for the derivatives of Gegenbauer polynomials and their normalization, the integrals $I^0$, $I^1$, and $I^2$ can be evaluated as:

$$I^0 = \frac{1}{\tilde{\omega}_j} \int_1^{z_{j-1/2}} \frac{(C_{n-1/2}^j)^2}{\left( 1 - \zeta^2 \right)^{n-1}} d \zeta = \frac{\pi \Gamma(j + 2n - 1)}{\tilde{\omega}_j 4^{n-1}(j + n + 1/2) \Gamma^2(n-1/2) \Gamma(j + 1)}.$$

(A23)

$$I^1 = \tilde{\omega}_j^2 \int_1^{z_{j-1/2}} \frac{d}{d \zeta} \left( C_{n-1/2}^j \right)^2 \left( 1 - \zeta^2 \right)^n d \zeta = (2n-1)^2 \tilde{\omega}_j \int_1^{z_{j-1/2}} \left( C_{n-1/2}^j \right)^2 \left( 1 - \zeta^2 \right)^n d \zeta = -\frac{\pi(2n-1)^2 \tilde{\omega}_j \Gamma(2n + j)}{4^n(j + n - 1/2) \Gamma^2(n + 1/2) \Gamma(j + 1)}.$$

(A24)
and

\[ I^2 = \bar{\omega}_z^3 \int_{-1}^{1} \left( \frac{d^2}{d\zeta^2} C_j \right)^2 \left( 1 - \zeta^2 \right)^{n+1} \, d\zeta \]

\[ = (2n-1)^2 (2n+1)^2 \bar{\omega}_z^3 \int_{-1}^{1} \left( C_{j-2}^{n+3/2} \right)^2 \left( 1 - \zeta^2 \right)^{n+1} \, d\zeta \]

\[ = -\frac{\pi (2n-1)^2 (2n+1)^2 \bar{\omega}_z^3 \Gamma(j+2n+1)}{4^{n+1} \Gamma(n+1/2) \Gamma(j+1)}. \]  \hspace{1cm} (A25)

Then, substituting into equations (A18) and (A20), and finally into equations (35) and (36), we obtain

\[ \gamma_j = -\frac{\left( 1 + \frac{a}{2} \right) (j+2n)(j-1)}{2(n+1)} \bar{\omega}_z^2 \Omega_0. \]  \hspace{1cm} (A26)

References

Blaes, O. M. 1985, MNRAS, 216, 553
Blaes, O. M., Arras, P., & Fragile, P. C. 2006, MNRAS, 369, 1235
Blaes, O. M., Šrámková, E., Abramowicz, M. A., Kluzniak, W., & Torkelsson, U. 2007, ApJ, 665, 642
Blumenthal, G. R., Yang, L. T., & Lin, D. N. C. 1984, ApJ, 287, 774
Kato, S. 1978, MNRAS, 185, 629
Kato, S. 1991, PASJ, 43, 557
Kato, S. 1994, PASJ, 46, 415
Kato, S., Fukue, J., & Mineshige, S. 1998, Black-Hole Accretion Disks (Kyoto: Kyoto University Press)
Nowak, M. A., & Wagoner, R. V. 1992, ApJ, 393, 697
Papaloizou, J. C. B., & Pringle, J. E. 1984, MNRAS, 208, 721
Shakura, N. I., & Sunyaev, R. A. 1973, A&A, 24, 337