On \((p, q)\)-analogue of Bernstein Operators

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Abstract

In this paper, we introduce a new analogue of Bernstein operators and we call it as \((p, q)\)-Bernstein operators which is a generalization of \(q\)-Bernstein operators. We also study approximation properties based on Korovkin’s type approximation theorem of \((p, q)\)-Bernstein operators and establish some direct theorems. Furthermore, we show comparisons and some illustrative graphics for the convergence of operators to a function.

Keywords and phrases: \((p, q)\)-Bernstein operator; modulus of continuity; positive linear operator; Korovkin type approximation theorem.

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1 Introduction and preliminaries

In 1912, S.N. Bernstein [5] introduced the following sequence of operators \(B_n : C[0, 1] \to C[0, 1]\) defined for any \(n \in \mathbb{N}\) and for any function \(f \in C[0, 1]\)

\[
B_n(f; x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right), \quad x \in [0, 1].
\]  (1.1)

Later it was found that Bernstein polynomials possess many remarkable properties, so new applications and generalizations are being discovered of it. The aim of these generalizations is to provide appropriate and powerful tools to application areas such as numerical analysis, computer-aided geometric design, and solutions of differential equations. The rapid development of \(q\)-calculus has led to the discovery of new generalizations of Bernstein polynomials involving \(q\)-integers. Lupas [15] was the first who introduced the \(q\)-analogue of the well known Bernstein polynomials and investigated its approximating and shape-preserving properties. Let \(f \in C[0, 1]\). The linear operator \(L_{n,q} : C[0, 1] \to C[0, 1]\), defined by

\[
L_{n,q}(f; x) = \sum_{k=0}^{n} f\left(\left[\frac{k}{n}\right]_q\right) b_{n,k}^q(x),
\]  (1.2)

where

\[
b_{n,k}^q(x) = \frac{n!_q}{k!(n-k)!_q} q^{k(k-1)/2} x^k (1-x)^{n-k} \prod_{j=0}^{n-1} (1-x + q^j x)
\]}
is called Lupaş $q$-analogue of Bernstein polynomials.

Another $q$-generalization of the classical Bernstein polynomials is due to Phillips [24]. After that many generalizations of well-known positive linear operators, based on $q$-integers were introduced and studied by several authors. Recently the approximation properties have also been investigated for $q$-analogue polynomials. For instance, $q$-analogues of Bernstein–Kantorovich operators in [15]; $q$-Baskakov–Kantorovich operators in [8]; generalized integral Bernstein operators based on $q$-integers in [18]; $q$-Szász–Mirakjan operators in [23]; $q$-Bleimann, Butzer and Hahn operators in [3] and [7]; $q$-analogue of Baskakov and Baskakov-Kantorovich operators in [16]; $q$-analogue of Szász–Kantorovich operators in [17]; $q$-analogue of Stancu-Beta operators in [4] and [22]; and $q$-Lagrange polynomials in [21] were defined and their approximation properties were investigated.

Recently, Mursaleen et al introduced and studied approximation properties for new positive linear operators of Lagrange type in [19] and also studied approximation properties of the $q$-analogue of generalized Berstein–Shurer operators in [20].

In this paper, we introduce a new generalization ($(p, q)$)-analogue) of Bernstein operators as $(p, q)$-Bernstein operators. We study the approximation properties based on Korovkin’s type approximation theorem and also establish some direct theorems. We also show comparisons and some illustrative graphics for the convergence of operators to a function.

Details on the $q$-calculus can be found in [14] and for the applications of $q$-calculus in approximation theory, one can refer [1].

Now we recall certain notations of $(p, q)$-calculus.

The $(p, q)$ integer $[n]_{p,q}$ is defined by

$$[n]_{p,q} := \frac{p^n - q^n}{p - q}, \quad n = 0, 1, 2, \ldots, \quad 0 < q < p \leq 1.$$  

The $(p, q)$-binomial expansion is given as

$$(ax + by)^n_{p,q} := \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right]_{p,q} a^{n-k} b^k x^{n-k} y^k$$

$$(x + y)^n_{p,q} := (x + y)(px + qy)(p^2 x + q^2 y) \cdots (p^{n-1} x + q^{n-1} y),$$

and the $(p, q)$-binomial coefficients are defined by

$$\left[ \begin{array}{c} n \\ k \end{array} \right]_{p,q} := \frac{[n]_{p,q}!}{[k]_{p,q}![n-k]_{p,q}!}.$$  

Details on $(p, q)$-calculus can be found in [10, 12, 13, 26].
2 Construction of Operators

Now, we introduce \((p, q)\)-analogue of Bernstein operators as

\[
B_{n,p,q}(f; x) = \sum_{k=0}^{n} \binom{n}{k}_{p,q} x^k \prod_{s=0}^{n-k-1} (p^s - q^s x) \quad f \left( \frac{[k]_{p,q}}{[n]_{p,q}} \right), \quad x \in [0, 1]. \tag{2.1}
\]

Note that for \(p = 1\), \((p, q)\)- Bernstein operators given by (2.1) turn out to be \(q\)-Bernstein operators.

We have the following basic result:

Lemma 2.1. For \(x \in [0, 1]\), \(0 < q < p \leq 1\), we have

(i) \(B_{n,p,q}(1; x) = 1\);
(ii) \(B_{n,p,q}(t; x) = x\);
(iii) \(B_{n,p,q}(t^2; x) = \left( \frac{p^2 x + 1 - x}{n} \right)^{n-1} - \frac{q[n-1]_{p,q}}{[n]_{p,q}} x^2 ;
\)
(iv) \(B_{n,p,q}(t^3; x) = \left( \frac{p^3 x + 1 - x}{n} \right)^{n-1} + \frac{2p+q}{[n]_{p,q}} \left( \frac{[n]_{p,q}}{[n]_{p,q}} \right)^{n-2} - \frac{q^2[n-1]_{p,q}[n-2]_{p,q}}{[n]_{p,q}} x^3 ;
\)
(v) \(B_{n,p,q}(t^4; x) = \left( \frac{p^4 x + 1 - x}{n} \right)^{n-1} + \frac{3p+2q}{[n]_{p,q}} \left( \frac{[n]_{p,q}}{[n]_{p,q}} \right)^{n-2} + \frac{3pq(n-1)_{p,q}}{[n]_{p,q}} \left( \frac{[n]_{p,q}}{[n]_{p,q}} \right)^{n-3} - \frac{q^3[n-1]_{p,q}[n-2]_{p,q}[n-3]_{p,q}}{[n]_{p,q}} x^4 .
\)

Proof. (i)

\[
B_{n,p,q}(1; x) = \sum_{k=0}^{n} \binom{n}{k}_{p,q} x^k \prod_{s=0}^{n-k-1} (p^s - q^s x) = 1.
\]

(ii)

\[
B_{n,p,q}(t; x) = \sum_{k=0}^{n} \binom{n}{k}_{p,q} x^k \prod_{s=0}^{n-k-1} (p^s - q^s x) \frac{[k]_{p,q}}{[n]_{p,q}}
\]

\[
= \sum_{k=1}^{n} \frac{[n]_{p,q}}{[k]_{p,q}} \binom{n-1}{k-1}_{p,q} x^k \prod_{s=0}^{n-k-2} (p^s - q^s x) \frac{[k]_{p,q}}{[n]_{p,q}}
\]

\[
= \sum_{k=0}^{n-1} \binom{n-1}{k}_{p,q} x^k \prod_{s=0}^{n-k-2} (p^s - q^s x) = x.
\]

3
(iii)

\[ B_{n,p,q}(t^2; x) = \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right]_{p,q} x^k \prod_{s=0}^{n-k-1} (p^s - q^s x) \frac{[k]_{p,q}^2}{[n]_{p,q}^2} \]

\[ = \sum_{k=1}^{n} \left[ \begin{array}{c} n-1 \\ k-1 \end{array} \right]_{p,q} x^k \prod_{s=0}^{n-k-1} (p^s - q^s x) \frac{[k]_{p,q}}{[n]_{p,q}} \]

\[ = \sum_{k=0}^{n-1} \left[ \begin{array}{c} n-1 \\ k \end{array} \right]_{p,q} x^{k+1} \prod_{s=0}^{n-k-2} (p^s - q^s x) \frac{[k+1]_{p,q}}{[n]_{p,q}}. \]

Using \([k+1]_{p,q} = p^k + q[k]_{p,q}\), we have

\[ B_{n,p,q}(t^2; x) = \sum_{k=0}^{n-1} \left[ \begin{array}{c} n-1 \\ k \end{array} \right]_{p,q} x^{k+1} \prod_{s=0}^{n-k-2} (p^s - q^s x) \frac{p^k + q[k]_{p,q}}{[n]_{p,q}} \]

\[ = \frac{x}{[n]_{p,q}} \sum_{k=0}^{n-1} \left[ \begin{array}{c} n-1 \\ k \end{array} \right]_{p,q} (xp)^k \prod_{s=0}^{n-k-2} (p^s - q^s x) \]

\[ + q \sum_{k=1}^{n-1} [n-1]_{p,q} \left[ \begin{array}{c} n-2 \\ k-1 \end{array} \right]_{p,q} x^{k+1} \prod_{s=0}^{n-k-2} (p^s - q^s x)[k]_{p,q} \]

\[ = \frac{(px + 1 - x)^{n-1} x}{[n]_{p,q}} + q[n-1]_{p,q} x^2. \]

(iv)

\[ B_{n,p,q}(t^3; x) = \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right]_{p,q} x^k \prod_{s=0}^{n-k-1} (p^s - q^s x) \frac{[k]_{p,q}^3}{[n]_{p,q}^3} \]

\[ = \sum_{k=0}^{n-1} \left[ \begin{array}{c} n-1 \\ k \end{array} \right]_{p,q} x^{k+1} \prod_{s=0}^{n-k-2} (p^s - q^s x) \frac{[k+1]_{p,q}^2}{[n]_{p,q}^2} \]

\[ = \sum_{k=0}^{n-1} \left[ \begin{array}{c} n-1 \\ k \end{array} \right]_{p,q} x^{k+1} \prod_{s=0}^{n-k-2} (p^s - q^s x) \frac{(p^k + q[k]_{p,q})^2}{[n]_{p,q}^2} \]

\[ = \frac{(p^2 x + 1 - x)^{n-1} x}{[n]_{p,q}^2} + \frac{(2p + q)[n-1]_{p,q}(px + 1 - x)^{n-2} x^2}{[n]_{p,q}^2} \]

\[ + q^3[n-1]_{p,q}[n-2]_{p,q} x^3. \]
Using Lemma 2.1 and by linearity of the operators we can find our desired result.

\[
B_{n,p,q}(t^4; x) = \sum_{k=0}^{n-1} \binom{n}{k}_{p,q} x^k \prod_{s=0}^{n-k-1} (p^s - q^s x) \frac{k!_{p,q}}{n!_{p,q}}
\]

\[
= \sum_{k=0}^{n-1} \binom{n-1}{k}_{p,q} x^{k+1} \prod_{s=0}^{n-k-2} (p^s - q^s x) \frac{(k+1)!_{p,q}}{n!_{p,q}}
\]

\[
= \sum_{k=0}^{n-1} \binom{n-1}{k}_{p,q} x^{k+1} \prod_{s=0}^{n-k-2} (p^s - q^s x) \frac{(p+k+q[k]_{p,q})^3}{n!_{p,q}}
\]

\[
= \frac{(p^2x + 1 - x)^{n-1}x}{[n]_{p,q}^3} + \frac{3pq[n-1]_{p,q}(p^2x + 1 - x)^{n-2}x^2}{[n]_{p,q}^3} + \frac{q^3[n-1]_{p,q}(p^2x + 1 - x)^{n-2}x^2}{[n]_{p,q}^3}
\]

\[
+ \frac{(3p+2pq+q^2)q^3[n-1]_{p,q}[n-2]_{p,q}(px + 1 - x)^{n-3}x^3}{[n]_{p,q}^3}
\]

\[
+ \frac{q^6[n-1]_{p,q}[n-2]_{p,q}[n-3]_{p,q}x^4}{[n]_{p,q}^3}.
\]

**Lemma 2.2.** For \( x \in [0, 1] \), \( 0 < q < p \leq 1 \), we have

(i) \( B_{n,p,q}\left((t - x); x\right) = 0; \)

(ii) \( B_{n,p,q}\left((t - x)^2; x\right) = \frac{(px + 1 - x)^{n-1}x}{[n]_{p,q}} + \left(\frac{q[n-1]_{p,q}}{[n]_{p,q}} - 1\right)x^2; \)

(iii) \( B_{n,p,q}\left((t - x)^4; x\right) = \frac{(p^2x + 1 - x)^{n-1}x}{[n]_{p,q}} + \left(\frac{3pq[n-1]_{p,q}(p^2x + 1 - x)^{n-2}x^2}{[n]_{p,q}} - 4(p^2x + 1 - x)^{n-1}\right)x^2
\]

\[
+ \left\{\frac{(3p+2pq+q^2)q^3[n-1]_{p,q}[n-2]_{p,q}(px + 1 - x)^{n-3}x^3}{[n]_{p,q}} - \frac{4q^2[n-1]_{p,q}[n-2]_{p,q}}{[n]_{p,q}} - 3\right\}x^4.
\]

**Proof.** (ii) By linearity of the operator we have

\[
B_{n,p,q}\left((t - x)^2; x\right) = B_{n,p,q}(t^2; x) - 2x B_{n,p,q}(t; x) + x^2 B_{n,p,q}(1; x)
\]

\[
= \frac{x(px + 1 - x)^{n-1}}{[n]_{p,q}} + \left(\frac{q[n-1]_{p,q}}{[n]_{p,q}} - 1\right)x^2
\]

\[
= \frac{x(px + 1 - x)^{n-1}x^2}{[n]_{p,q}} + \left(\frac{q[n-1]_{p,q}}{[n]_{p,q}} - 1\right)x^2.
\]

(iii) Using Lemma 2.1 and by linearity of the operators we can find our desired result.
3 Korovkin type approximation theorem

We know that the space \( C[0, 1] \) of all continuous functions on \([0, 1]\) is a Banach space with sup-norm
\[
\|f\|_{C[0, 1]} := \sup_{x \in [0, 1]} |f(x)|, \quad f \in C[0, 1].
\]

For typographical convenience, we will write \( \| \cdot \| \) in place of \( \| \cdot \|_{C[0, 1]} \) if no confusion arises.

Let \( C[a, b] \) be the linear space of all real valued continuous functions \( f \) on \([a, b]\) and let \( T \) be a linear operator which maps \( C[a, b] \) into itself. We say that \( T \) is positive if for every non-negative \( f \in C[a, b] \), we have \( T(f, x) \geq 0 \) for all \( x \in [a, b] \).

The classical Korovkin approximation theorem \([2, 11, 27]\) states as follows:

Let \((T_n)\) be a sequence of positive linear operators from \( C[0, 1] \) into \( C[0, 1] \). Then \( \lim_n \|T_n(f, x) - f(x)\|_{C[0, 1]} = 0 \), for all \( f \in C[0, 1] \) if and only if \( \lim_n \|T_n(f_i, x) - f_i(x)\|_{C[0, 1]} = 0 \), for \( i = 0, 1, 2 \), where \( f_0(x) = 1 \), \( f_1(x) = x \) and \( f_2(x) = x^2 \).

One can find more convergence results mainly for positive linear operators in \([9]\).

**Theorem 3.1.** Let \( 0 < q_n < p_n \leq 1 \) such that \( \lim_{n \to \infty} p_n = 1 \) and \( \lim_{n \to \infty} q_n = 1 \). Then for each \( f \in C[0, 1] \), \( B_{n,p_n,q_n}(f; x) \) converges uniformly to \( f \) on \([0, 1]\).

**Proof.** By the Korovkin’s Theorem it suffices to show that
\[
\lim_{n \to \infty} \|B_{n,p_n,q_n}(t^m; x) - x^m\|_{C[0,1]} = 0, \quad m = 0, 1, 2.
\]

By Lemma 2.1(i)–(ii), it is clear that
\[
\lim_{n \to \infty} \|B_{n,p_n,q_n}(1; x) - 1\|_{C[0,1]} = 0;
\]
\[
\lim_{n \to \infty} \|B_{n,p_n,q_n}(t; x) - x\|_{C[0,1]} = 0.
\]

Using \( q_n[n - 1]_{p_n,q_n} = [n]_{p_n,q_n} - p_n^{n-1} \) and by Lemma 2.1(iii), we have
\[
|B_{n,p_n,q_n}(t^2; x) - x^2|_{C[0,1]} = \left| \frac{(xp_n + 1 - x)^{n-1}x}{[n]_{p_n,q_n}} + \frac{q_n[n - 1]_{p_n,q_n}}{[n]_{p_n,q_n}} - 1 \right| x^2 \leq \frac{(xp_n + 1 - x)^{n-1}x}{[n]_{p_n,q_n}} + \frac{p_n^{n-1}}{[n]_{p_n,q_n}} x^2.
\]

Taking maximum of both sides of the above inequality, we get
\[
\|B_{n,p_n,q_n}(t^2; x) - x^2\|_{C[0,1]} \leq \frac{2p_n^{n-1}}{[n]_{p_n,q_n}}
\]
which yields
\[
\lim_{n \to \infty} \|B_{n,p_n,q_n}(t^2; x) - x^2\|_{C[0,1]} = 0.
\]
Thus the proof is completed.

**Remark 3.1.** If we choose \( q_n = 1 - \frac{1}{n} \), and \( p_n = \frac{n}{n+1} \) such that \( 0 < q_n < p_n \leq 1 \), it is easily seen that \( \lim_{n \to \infty} p_n = 1 \), \( \lim_{n \to \infty} q_n = 1 \) and \( \lim_{n \to \infty} p_n^n = e^{-1} \), \( \lim_{n \to \infty} q_n^n = 0 \). Hence we guarantee that \( \lim_{n \to \infty} [n]_{p_n,q_n} = \infty \). Since \( [n+1]_{p_n,q_n} = p_n^n + q_n^n[n]_{p_n,q_n} \) and \( \frac{[n]_{p_n,q_n}}{[n+1]_{p_n,q_n}} = \frac{1}{q_n + \frac{2}{n} p_n q_n} \), we have \( \lim_{n \to \infty} \frac{[n]_{p_n,q_n}}{[n+1]_{p_n,q_n}} = 0 \) and \( \lim_{n \to \infty} \frac{[n]_{p_n,q_n}}{[n+1]_{p_n,q_n}} = 1 \).

For \( q \in (0,1) \) and \( p \in (q,1) \) it is obvious that \( \lim_{n \to \infty} [n]_{p,q} = \frac{1}{p-q} \). In order to reach to convergence results of the operator \( B_{n,p,q} \) we take a sequence \( q_n \in (0,1) \) and \( p_n \in (q_n,1) \) such that \( \lim_{n \to \infty} p_n = 1 \), \( \lim_{n \to \infty} q_n = 1 \). So we get that \( \lim_{n \to \infty} [n]_{p_n,q_n} = \infty \).

Thus Remark 3.1 provides an example that such a sequence can always be constructed.

### 4 Direct Theorems

The Peetre’s \( K \)-functional is defined by

\[
K_2(f, \delta) = \inf \{ \| f - g \| + \delta \| g'' \| : g \in W^2 \},
\]

where

\[
W^2 = \{ g \in C[0,1] : g', g'' \in C[0,1] \}.
\]

By [6], there exists a positive constant \( C > 0 \) such that \( K_2(f, \delta) \leq C\omega_2(f, \delta^+) \), \( \delta > 0 \); where the second order modulus of continuity is given by

\[
\omega_2(f, \delta^+) = \sup_{0 < h \leq \delta^+} \sup_{x \in [0,1]} | f(x + 2h) - 2f(x + h) + f(x) |
\]

Also for \( f \in C[0,1] \) the usual modulus of continuity is given by

\[
\omega(f, \delta) = \sup_{0 < h \leq \delta} \sup_{x \in [0,1]} | f(x + h) - f(x) |
\]

**Theorem 4.1.** Let \( f \in [0,1] \) and \( 0 < q < p \leq 1 \). Then for all \( n \in \mathbb{N} \), there exists an absolute constant \( C > 0 \) such that

\[
| B_{n,p,q}(f;x) - f(x) | \leq C\omega_2(f, \delta_n(x)),
\]

where

\[
\delta_n^2(x) = \left( \frac{q[n-1]_{p,q}}{[n]_{p,q}} - 1 \right) x^2 + \frac{(xp + 1 - x)^{n-1}}{[n]_{p,q}} x.
\]

**Proof.** Let \( g \in W_2 \). From Taylor’s expansion, we get

\[
g(t) = g(x) + g'(x)(t-x) + \int_x^t (t-u) g''(u) \, du, \quad t \in [0, A], \ A > 0,
\]

\[
\lim_{n \to \infty} \frac{[n]_{p_n,q_n}}{[n+1]_{p_n,q_n}} = 0, \quad \lim_{n \to \infty} \frac{[n]_{p_n,q_n}}{[n+1]_{p_n,q_n}} = 1.
\]

Thus the proof is completed.
and by Lemma 2.1, we get
\[ B_{n,p,q}(g; x) = g(x) + B_{n,p,q}\left(\int_x^t (t-u) g''(u) \, du; x\right) \]
\[ | B_{n,p,q}(g; x) - g(x) | \leq B_{n,p,q}\left(\int_x^t (t-u) g''(u) \, du; x\right) \]
\[ \leq B_{n,p,q}\left(\int_x^t |t-u| \, |g''(u)| \, du; x\right) \]
\[ \leq B_{n,p,q}\left((t-x)^2; x\right) \|g''\|. \]

Using Lemma 2.2(ii), we obtain
\[ | B_{n,p,q}(g; x) - g(x) | \leq x^2 \left(\frac{q[n-1]}{n} - 1\right) \|g''\| + x \frac{(xp + 1 - x)^{n-1}}{[n]_{p,q}} \|g''\|. \]

On the other hand, by the definition of \( B_{n,p,q}(f; x) \), we have
\[ | B_{n,p,q}(f; x) | \leq \|f\|. \]

Now
\[ | B_{n,p,q}(f; x) - f(x) | \leq | B_{n,p,q}((f-g); x) - (f-g)(x) | + | B_{n,p,q}(g; x) - g(x) | \]
\[ \leq \|f-g\| + x^2 \left(\frac{q[n-1]}{n} - 1\right) \|g''\| + x \frac{(xp + 1 - x)^{n-1}}{[n]_{p,q}} \|g''\|. \]

Hence taking infimum on the right hand side over all \( g \in W^2 \), we get
\[ | B_{n,p,q}(f; x) - f(x) | \leq CK_2(f, \delta_n^2(x)) \]

In view of the property of \( K \)-functional, we get
\[ | B_{n,p,q}(f; x) - f(x) | \leq C\omega_2(f, \delta_n(x)). \]

This completes the proof of the theorem.

**Theorem 4.2.** Let \( f \in C[0,1] \) be such that \( f', f'' \in C[0,1] \), and the sequences \( (p_n), (q_n) \) satisfying \( 0 < q_n < p_n \leq 1 \) such that \( p_n \to 1, q_n \to 1 \) and \( p_n^a \to a, q_n^b \to b \) as \( n \to \infty \) where \( 0 \leq a, b < 1 \). Then the following equality holds
\[ \lim_{n \to \infty} [n]_{p_n,q_n} (B_{n,p_n,q_n}(f; x) - f(x)) = \frac{x(\lambda - ax)}{2} f''(x) \]
uniformly on $[0, 1]$ where $0 < \lambda \leq 1$.

**Proof.** By the Taylor’s formula we may write

$$f(t) = f(x) + f'(x)(t - x) + \frac{1}{2}f''(x)(t - x)^2 + r(t, x)(t - x)^2$$

(4.1)

where $r(t, x)$ is the remainder term and $\lim_{t \to x} r(t, x) = 0$. Applying $B_{n,p,q_n}(f; x)$ to (3.1), we obtain

$$[n]_{p_n,q_n} (B_{n,p,q_n}(f; x) - f(x)) = [n]_{p_n,q_n} B_{n,p,q_n}((t - x); x)f'(x)$$

$$+ [n]_{p_n,q_n} B_{n,p,q_n}((t - x)^2; x)\frac{f''(x)}{2}$$

$$+ [n]_{p_n,q_n} B_{n,p,q_n}(r(t, x)(t - x)^2; x).$$

By the Cauchy-Schwartz inequality, we have

$$B_{n,p,q_n} (r(t, x)(t - x)^2; x) \leq \sqrt{B_{n,p,q_n} (r^2(t, x); x)} \times \sqrt{B_{n,p,q_n} ((t - x)^4; x)}$$

(4.2)

Observe that $r^2(x, x) = 0$ and $r^2(., x) \in C[0, 1]$. Then it follows from Theorem 3.1 that

$$B_{n,p,q_n} (r^2(t, x); x) = r^2(x, x) = 0$$

(4.3)

uniformly with respect to $x \in [0, 1]$, in view of the fact that $B_{n,p,q_n}((t - x)^4; x) = O\left(\frac{1}{n}\right)$. Now from (4.2), (4.3) and Lemma 2.2 (ii), we get

$$\lim_{n \to \infty} [n]_{p_n,q_n} B_{n,p,q_n} (r(t, x)(t - x)^2; x) = 0.$$

(4.4)

Now we compute the followings:

$$\lim_{n \to \infty} [n]_{p_n,q_n} B_{n,p,q_n}((t - x); x) = 0;$$

(4.5)

$$\lim_{n \to \infty} [n]_{p_n,q_n} B_{n,p,q_n}((t - x)^2; x) = x \lim_{n \to \infty} (p_n x + 1 - x)^{n-1} + \lim_{n \to \infty} (q_n[n - 1]_{p_n,q_n} - [n]_{p_n,q_n}) x^2$$

$$= \lambda x + \lim_{n \to \infty} \left( q_n p_n^{n-1} - q_n^{n-1} - \frac{p_n^n - q_n^n}{p_n - q_n} \right) x^2$$

$$= \lambda x - ax^2;$$

(4.6)

where $\lambda \in (0, 1]$ depending on the sequence $(p_n)$. Finally from (4.4), (4.5) and (4.6), we get the required result.

This completes the proof of the theorem.
Theorem 4.3. If \( f \in C[0, 1] \), then
\[
|B_{n,p,q}(f; x) - f(x)| \leq 2\omega(f, \sqrt{\frac{p^n - 1}{[n]_{p,q}}})
\]
holds.

Proof. Since \( B_{n,p,q}(1, x) = 1 \), we have
\[
|B_{n,p,q}(f; x) - f(x)| \leq \sum_{k=0}^{n} \left[ \frac{n}{k} \right]_{p,q} x^k \prod_{s=0}^{n-k-1} (p^s - q^s x) \left| f\left( \frac{[k]_{p,q}}{[n]_{p,q}} \right) - f(x) \right|
\]
In view of (4.4), we get
\[
|B_{n,p,q}(f; x) - f(x)| \leq \left\{ \sum_{k=0}^{n} \left[ \frac{n}{k} \right]_{p,q} x^k \prod_{s=0}^{n-k-1} (p^s - q^s x) \left( \frac{[k]_{p,q}}{[n]_{p,q}} - x \right)^2 + 1 \right\} \omega(f, \delta)
\]
\[
= \left\{ \frac{1}{\delta^2} \left( B_{n,p,q}(t^2; x) - 2xB_{n,p,q}(t; x) + x^2 B_{n,p,q}(1; x) \right) + 1 \right\} \omega(f, \delta)
\]
\[
= \left\{ \frac{1}{\delta^2} \left( \frac{x(xp + 1 - x)^{n-1}}{[n]_{p,q}} + \frac{q[n - 1]_{p,q} - 1}{[n]_{p,q}} x^2 \right) + 1 \right\} \omega(f, \delta)
\]
\[
= \left\{ \frac{1}{\delta^2} \left( \frac{x(xp + 1 - x)^{n-1}}{[n]_{p,q}} - \frac{p^n - 1}{[n]_{p,q}} x^2 \right) \right\} \omega(f, \delta)
\]
\[
\leq \left\{ \frac{1}{\delta^2} \left( \frac{x(xp + 1 - x)^{n-1}}{[n]_{p,q}} \right) + 1 \right\} w_f(\delta) \leq \left\{ \frac{1}{\delta^2} \left( \frac{p^n - 1}{[n]_{p,q}} \right) + 1 \right\} \omega(f, \delta).
\]
Choosing \( \delta = \delta_n = \sqrt{\frac{p^n - 1}{[n]_{p,q}}} \), we have
\[
|B_{n,p,q}(f; x) - f(x)| \leq 2\omega(f, \delta_n).
\]
This completes the proof of the theorem.

Now we give the rate of convergence of the operators \( B_{n,p,q} \) in terms of the elements of the usual Lipschitz class \( Lip_M(\alpha) \).

Let \( f \in C[0, 1], \ M > 0 \) and \( 0 < \alpha \leq 1 \). We recall that \( f \) belongs to the class \( Lip_M(\alpha) \) if the inequality
\[
|f(t) - f(x)| \leq M |t - x|^{\alpha} \quad (t, x \in [0, 1])
\]
Theorem 4.4. Let \(0 < q < p \leq 1\). Then for each \(f \in Lip_{M}(\alpha)\) we have
\[
|B_{n,p,q}(f; x) - f(x)| \leq M\delta_{n}(x)^{\alpha}
\]
where
\[
\delta_{n}^{2}(x) = \left(\frac{q[n - 1]_{p,q} - 1}{n_{p,q}}\right)x^{2} + \frac{(xp + 1 - x)^{n-1}}{n_{p,q}}x.
\]

Proof. Let us denote \(P_{n,k}(x) = \left[\begin{array}{c} n \\ k \end{array}\right]_{p,q} x^{k} \prod_{s=0}^{n-k-1} (p^{s} - q^{s}x)\). Then by the monotonicity of the operators \(B_{n,p,q}\), we can write
\[
|B_{n,p,q}(f; x) - f(x)| \leq B_{n,p,q}(|f(t) - f(x)|; x)
\]
\[
\leq \sum_{k=0}^{n} P_{n,k}(x) \left| f\left(\frac{[k]_{p,q}}{n_{p,q}}\right) - f(x)\right|
\]
\[
\leq M \sum_{k=0}^{n} P_{n,k}(x) \left| \frac{[k]_{p,q}}{n_{p,q}} - x\right|^{\alpha}
\]
\[
= M \sum_{k=0}^{n} \left( P_{n,k}(x) \left( \frac{[k]_{p,q}}{n_{p,q}} - x\right) \right)^{2} \frac{\alpha}{2} P_{n,k}^{2-\alpha}(x).
\]
Now applying the Hölder’s inequality for the sum with \(p = \frac{2}{\alpha}\) and \(q = \frac{2}{2-\alpha}\) and taking into consideration Lemma 2.1(i) and Lemma 2.2(ii), we have
\[
|B_{n,p,q}(f; x) - f(x)| \leq M \left( \sum_{k=0}^{n} P_{n,k}(x) \left( \frac{[k]_{p,q}}{n_{p,q}} - x\right) \right)^{\alpha/2} \left( \sum_{k=0}^{n} P_{n,k}(x) \right)^{\frac{2-\alpha}{2}}
\]
\[
= M \left\{ B_{n,p,q}(t - x)^{2}; x \right\}^{\frac{\alpha}{2}}.
\]
Choosing \(\delta : \delta_{n}(x) = \sqrt{B_{n,p,q}(t - x)^{2}; x}\), we obtain
\[
|B_{n,p,q}(f; x) - f(x)| \leq M\delta_{n}(x)^{\alpha}
\]
Hence, the desired result is obtained.

5 Example

With the help of Matlab, we show comparisons and some illustrative graphics for the convergence of operators (2.1) to the function \(f(x) = (x - \frac{1}{3})(x - \frac{1}{2})(x - \frac{3}{4})\) under different parameters.
From figure 1, we can observe that as the value the \(q\) increases, \((p, q)\)-Bernstein operators given by (1.2) converges towards the function.
Convergence of \((p,q)\)-Bernstein operators to function
For \(q=.1, p=1\)
For \(q=.6, p=1\)
For \(q=.9, p=1\)

Figure 1:

In comparison to figure 1, as the value the \(n\) increases, operators given by (2.1) converges towards the function which is shown in figure 2.

Convergence of \((p,q)\)-Bernstein operators to function
For \(q=.1, p=1\)
For \(q=.6, p=1\)
For \(q=.9, p=1\)

Figure 2:

Similarly for different values of parameters \(p, q\) and \(n\) convergence of operators to the function is shown in figure 3,4.
Convergence of $(p,q)$–Bernstein operators to function

For $q=0.1$, $p=0.15$

For $q=0.6$, $p=0.65$

For $q=0.9$, $p=1$

Figure 3:

Convergence of $(p,q)$–Bernstein operators to function

For $q=0.1$, $p=0.15$

For $q=0.6$, $p=0.65$

For $q=0.9$, $p=1$

Figure 4:

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