Theory of Timeon

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Abstract

It is proposed that $T$ violation in hadronic physics, as well as the masses of $u$, $d$ quarks, arise from a pseudoscalar interaction with a new spin 0 field $\tau(x)$, odd in $P$ and $T$, but even in $C$. This interaction contains a factor $i\gamma_5$ in the quark Dirac algebra, so that the full Hamiltonian is $P$, $T$ conserving; but by spontaneous symmetry breaking, the new field $\tau(x)$ has a nonzero expectation value $<\tau> = \tau_0$ that breaks $P$ and $T$ symmetry.

Oscillations of $\tau(x)$ about its expectation value $\tau_0$ produce a new particle, the "timeon", whose mass is independent of any known quantities. If the timeon mass is within the range of present accelerators, observation of the particle can be helped with a search of $T$-violating events.

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1. Introduction

We assume that the standard model of weak interaction[1] is $CP$ and $T$ conserving. In this paper we propose that the observed $CP$ and $T$ violations are due to a new $P$ odd and $T$ odd spin zero field $\tau(x)$, called the timeon field; the same field is also responsible for the small masses of $u$, $d$ quarks, as well as for those of $\nu_1$ and the electron. In the following, our discussions will be restricted only to quarks.

Let $q_i(\uparrow)$ and $q_i(\downarrow)$ be the quark states "diagonal" in $W^\pm$ transitions[2]:

$$q_i(\downarrow) \Leftrightarrow q_i(\uparrow) + W^-$$

and

$$q_i(\uparrow) \Leftrightarrow q_i(\downarrow) + W^+$$

with $i = 1$, 2 and 3. The electric charges in units of $e$ are $+\frac{2}{3}$ for $q_i(\uparrow)$ and $-\frac{1}{3}$ for $q_i(\downarrow)$. These quark states $q_i(\uparrow)$ and $q_i(\downarrow)$ are, however, not the mass eigenstates $d$, $s$, $b$ and $u$, $c$, $t$. We assume that the mass Hamiltonians $H_\uparrow$ for $q_i(\uparrow)$ and $H_\downarrow$ for $q_i(\downarrow)$ are given by

$$H_{\uparrow/\downarrow} = \begin{pmatrix} q_1^\dagger, q_2^\dagger, q_3^\dagger \end{pmatrix}_{\uparrow/\downarrow} (G_{\gamma 4} + iF_{\gamma 4}\gamma_5)_{\uparrow/\downarrow} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}$$

with the $3 \times 3$ matrices $G_{\uparrow/\downarrow}$ and $F_{\uparrow/\downarrow}$ both real and hermitian. The mass matrix $G_{\uparrow/\downarrow}$ is the same zeroth order mass matrix as $M_0(q_i(\downarrow))$ of Ref. 2, given by

$$G_{\uparrow/\downarrow} = \begin{pmatrix} \beta\eta^2(1 + \xi^2) & -\beta\eta & -\beta\xi\eta \\ -\beta\eta & \beta + \alpha\xi^2 & -\alpha\xi \\ -\beta\xi\eta & -\alpha\xi & \alpha + \beta \end{pmatrix}_{\uparrow/\downarrow}$$

with $\alpha_\uparrow$, $\beta_\uparrow$, $\xi_\uparrow$, $\eta_\uparrow$ and $\alpha_\downarrow$, $\beta_\downarrow$, $\xi_\downarrow$, $\eta_\downarrow$ all real parameters. It can be readily verified that the determinants

$$|G_\uparrow| = |G_\downarrow| = 0.$$  

We assume $\alpha_{\uparrow/\downarrow}$ and $\beta_{\uparrow/\downarrow}$ to be all positive. The lowest eigenvalues of $G_\uparrow$ and $G_\downarrow$ are then both zero. These two real symmetric matrices can be readily
diagonalized by real, orthogonal matrices \((U^\uparrow)_0\) and \((U^\downarrow)_0\), with

\[
(U^\uparrow)_0 G^\uparrow (U^\uparrow)_0 = \begin{pmatrix}
0 & 0 & 0 \\
0 & m_0(c) & 0 \\
0 & 0 & m_0(t)
\end{pmatrix}
\] (1.6)

and

\[
(U^\downarrow)_0 G^\downarrow (U^\downarrow)_0 = \begin{pmatrix}
0 & 0 & 0 \\
0 & m_0(s) & 0 \\
0 & 0 & m_0(b)
\end{pmatrix}
\] (1.7)

where the nonzero eigenvalues are the zeroth order masses of \(c\), \(t\) and \(s\), \(b\) quarks, with

\[
m_0(c) = \beta^\uparrow [1 + \eta^\uparrow (1 + \xi^\uparrow)], \quad (1.8)
\]

\[
m_0(t) = \alpha^\uparrow (1 + \xi^\uparrow) + \beta^\uparrow, \quad (1.9)
\]

\[
m_0(s) = \beta^\downarrow [1 + \eta^\downarrow (1 + \xi^\downarrow)] \quad (1.10)
\]

and

\[
m_0(b) = \alpha^\downarrow (1 + \xi^\downarrow) + \beta^\downarrow. \quad (1.11)
\]

Thus, \(G^\uparrow\) and \(G^\downarrow\) can be each represented by an ellipse of minor and major axes given by \(m_0(c)\) and \(m_0(t)\) for \(\uparrow\) and likewise \(m_0(s)\) and \(m_0(b)\) for \(\downarrow\).

The orientations of these two elliptic plates are determined by their eigenstates. As in Ref. 2, we define four real angular variables \(\theta^\downarrow\), \(\phi^\downarrow\) and \(\theta^\uparrow\), \(\phi^\uparrow\) by

\[
\xi^\downarrow = \tan \phi^\downarrow, \quad \xi^\uparrow = \tan \phi^\uparrow
\]

\[
\eta^\downarrow = \tan \theta^\downarrow \cos \phi^\downarrow \quad \text{and} \quad \eta^\uparrow = \tan \theta^\uparrow \cos \phi^\uparrow. \quad (1.12)
\]

The eigenstates of \(G^\uparrow\) are

\[
\epsilon^\uparrow = \begin{pmatrix}
\cos \theta^\uparrow \\
\sin \theta^\uparrow \cos \phi^\uparrow \\
\sin \theta^\uparrow \sin \phi^\uparrow
\end{pmatrix} \quad \text{with eigenvalue} \ 0, \quad (1.13)
\]

\[
p^\uparrow = \begin{pmatrix}
-\sin \theta^\uparrow \\
\cos \theta^\uparrow \cos \phi^\uparrow \\
\cos \theta^\uparrow \sin \phi^\uparrow
\end{pmatrix} \quad \text{with eigenvalue} \ 0(c) \quad (1.14)
\]
and
\[
P_\uparrow = \begin{pmatrix} 0 \\ -\sin \phi_\uparrow \\ \cos \phi_\uparrow \end{pmatrix}
\] with eigenvalue \(m_0(t)\). \hspace{1cm} (1.15)

Correspondingly, the eigenstates of \(G_\downarrow\) are
\[
\epsilon_\downarrow = \begin{pmatrix} \cos \theta_\downarrow \\ -\sin \theta_\downarrow \cos \phi_\downarrow \\ -\sin \theta_\downarrow \sin \phi_\downarrow \end{pmatrix}
\] with eigenvalue 0, \hspace{1cm} (1.16)
\[
p_\downarrow = \begin{pmatrix} \sin \theta_\downarrow \\ \cos \theta_\downarrow \cos \phi_\downarrow \\ \cos \theta_\downarrow \sin \phi_\downarrow \end{pmatrix}
\] with eigenvalue \(m_0(s)\), \hspace{1cm} (1.17)
and
\[
P_\downarrow = \begin{pmatrix} 0 \\ -\sin \phi_\downarrow \\ \cos \phi_\downarrow \end{pmatrix}
\] with eigenvalue \(m_0(b)\). \hspace{1cm} (1.18)

We note that by changing \(\theta_\uparrow, \phi_\uparrow\) to \(-\theta_\downarrow, \phi_\downarrow\) the unit vectors \(\epsilon_\uparrow, p_\uparrow, P_\uparrow\) of (1.13)-(1.15) become \(\epsilon_\downarrow, p_\downarrow\) and \(P_\downarrow\) of (1.16)-(1.18). Here the signs of \(\theta_\uparrow\) and \(\theta_\downarrow\) are chosen so that the sign convention of the particle data group’s CKM matrix agrees with both \(\theta_\uparrow\) and \(\theta_\downarrow\) being positive, as we shall see. In terms of these eigenstates, the \(3 \times 3\) real unitary matrices \((U_\uparrow)_0\) and \((U_\downarrow)_0\) of (1.6)-(1.7) are given by
\[
(U_\downarrow)_0 = (\epsilon_\downarrow, p_\downarrow, P_\downarrow)
\] \hspace{1cm} (1.19)
and
\[
(U_\uparrow)_0 = (\epsilon_\uparrow, p_\uparrow, P_\uparrow).
\] \hspace{1cm} (1.20)

Thus, in the absence of the \(iF_{\uparrow/\downarrow} \gamma_4 \gamma_5\) term in (1.3), the corresponding CKM matrix in this approximation is given by
\[
(U_{\text{CKM}})_0 = (U_\downarrow)_0^\dagger (U_\uparrow)_0 =
\]
\[
\begin{pmatrix}
\cos \theta \downarrow \cos \theta \uparrow & \sin \theta \downarrow \cos \theta \uparrow & \sin \theta \uparrow \sin \phi \\
-\sin \theta \downarrow \sin \theta \uparrow \cos \phi & +\cos \theta \downarrow \sin \theta \uparrow \cos \phi \\
-\cos \theta \downarrow \sin \theta \uparrow & -\sin \theta \downarrow \sin \theta \uparrow \cos \theta \uparrow \sin \phi \\
-\sin \theta \downarrow \cos \theta \uparrow \cos \phi & +\cos \theta \downarrow \cos \theta \uparrow \cos \phi \\
\sin \theta \downarrow \sin \phi & -\cos \theta \downarrow \sin \phi & \cos \phi \\
\end{pmatrix}, \quad (1.21)
\]

in which

\[\phi = \phi \uparrow - \phi \downarrow. \quad (1.22)\]

The new hypothesis of this paper is to assume that \( T \) violation and the small masses of \( u, d \) quarks are due to the new

\[iF\gamma_4\gamma_5\]

term in (1.3), with

\[F \uparrow = F \downarrow = F = \tau_0 f \tilde{f}\]

in which \( \tau_0 \) is a constant and \( f \) a 3 dimensional unit vector represented by its \( 3 \times 1 \) real column matrix. Graphically, we can visualize \( G \uparrow \) and \( G \downarrow \) as two elliptic plates mentioned above, and \( F \) as a single needle of length \( \tau_0 \) and direction \( f \), as shown in Figure 1.

As we shall discuss, the \( F \)-term (1.24) is due to the spontaneous symmetry breaking of a new \( T \) odd, \( P \) odd and \( CP \) odd, spin 0 field \( \tau(x) \), which has a vacuum expectation value given by

\[<\tau(x)>_{vac} = \tau_0 \neq 0. \quad (1.25)\]

While the general characteristics of spontaneous time reversal symmetry breaking models have been discussed in the literature[3], the special new feature of the present model is to connect such symmetry breaking with the smallness of up, down quark masses.

In Section 2, we begin with a general \( T, P \) and \( CP \) violating mass matrix of the form

\[G\gamma_4 + iF\gamma_4\gamma_5\]

(1.26)
given by (1.3) with $G$ and $F$ both real and hermitian matrices, then compare it with an alternative form of a single term, but with $M$ complex and hermitian. As will be shown, these two different forms of mass matrices are in fact equivalent. Differences appear when one goes beyond the mass matrices. In Sections 3 and 4, we summarize the analysis of how in (1.24), the length $\tau_0$ and the direction $f$ of the needle are related to the light quark masses and the Jarlskog invariant $[4] J$. As we shall see, this leads to

$$\tau_0 \cong 33 \text{MeV}, \quad (1.28)$$
$$m_u \cong \tau_0 (\tilde{f} \epsilon_\uparrow)^2 \quad (1.29)$$
and

$$m_d \cong \tau_0 (\tilde{f} \epsilon_\downarrow)^2 \quad (1.30)$$

with $\epsilon_\uparrow$ and $\epsilon_\downarrow$ given by (1.13) and (1.16).

An important feature of the model is to assume that the constant $\tau_0$ is due to the spontaneous symmetry breaking of a new $T$ odd and $CP$ odd, spin 0 field $\tau(x)$, which has a vacuum expectation value given by (1.25). We may assume that the part of Lagrangian density that contains only $\tau(x)$ is given by

$$-\frac{1}{2} \left( \frac{\partial \tau}{\partial x_\mu} \right)^2 - V(\tau) \quad (1.31)$$

with

$$V(\tau) = -\frac{1}{2} \lambda \tau^2 (\tau_0^2 - \frac{1}{2} \tau^2) \quad (1.32)$$
in which the (renormalized) value of $\lambda$ is positive. This then yields (1.25). Expanding $V(\tau)$ around its equilibrium value $\tau = \tau_0$, we have

$$V(\tau) = -\frac{\lambda}{4} \tau_0^4 + \frac{1}{2} m_\tau^2 (\tau - \tau_0)^2 + O[(\tau - \tau_0)^3] \quad (1.33)$$

with

$$m_\tau = (2\lambda)^{\frac{1}{2}} \tau_0, \quad (1.34)$$
the mass of this new $T$ violating, $C$ violating and $CP$ violating quantum, called timeon. The interaction between $\tau(x)$ and the quark field can be readily obtained by replacing the $F = \tau_0 f \bar{f}$ factor of (1.24) with

$$F = \tau(x) f \bar{f}. \quad (1.35)$$

In this new theory, the well studied mass limit of the standard ($T$ and $CP$ conserving) Higgs boson applies only to the vibrational modes of the $\gamma_4 G_{\uparrow}$ and $\gamma_4 G_{\downarrow}$ terms in (1.3). Thus, the timeon mass $m_\tau$ would be different, and could be much lower.
2. Two Equivalent Forms of Mass Matrix

2.1 General Formulation

The "quark" mass matrix $\mathcal{M}$ and its related Hamiltonian $\mathcal{H}$ is usually written as

$$\mathcal{H} = \Psi^\dagger \mathcal{M} \gamma_4 \Psi$$

(2.1)

with

$$\mathcal{M} = \mathcal{M}^\dagger,$$

(2.2)

denoting a hermitian matrix. Throughout this section, we assume the Dirac field operator $\Psi$ to have $n$ generation-components, with $n = 3$ for quarks. Decompose $\Psi$ into a sum of left-handed and right-handed parts:

$$\Psi = \mathcal{L} + \mathcal{R}$$

(2.3)

with

$$\mathcal{L} = \frac{1}{2}(1 + \gamma_5)\Psi \quad \text{and} \quad \mathcal{R} = \frac{1}{2}(1 - \gamma_5)\Psi.$$  

(2.4)

Correspondingly, (2.1) becomes

$$\mathcal{H} = \mathcal{L}^\dagger \mathcal{M} \gamma_4 \mathcal{R} + \mathcal{R}^\dagger \mathcal{M} \gamma_4 \mathcal{L}.$$  

(2.5)

Assume $n \geq 3$ and $\mathcal{M}$ to have an imaginary part so that $\mathcal{H}$ is $T$, $C$ and $CP$ violating.

A different form of an $n$-generation $T$ and $CP$ violating mass Hamiltonian can be written in the form similar to (1.3), also with a Dirac operator $\psi$ of $n$ components:

$$H = \psi^\dagger (\mathcal{G} \gamma_4 + i\mathcal{F} \gamma_4 \gamma_5) \psi,$$

(2.6)

where $\mathcal{G}$ and $\mathcal{F}$ are both $n$-dimensional hermitian matrices,

$$\mathcal{G} = \mathcal{G}^\dagger \quad \text{and} \quad \mathcal{F} = \mathcal{F}^\dagger.$$  

(2.7)

For $n \geq 3$, $\mathcal{G}$ and $\mathcal{F}$ both nonzero, the Hamiltonian $H$ is $T$, $P$ and $CP$ violating. As in (2.3)-(2.4), we resolve $\psi$ in a similar form:

$$\psi = L + R$$

(2.8)
with
\[ L = \frac{1}{2}(1 + \gamma_5)\psi \quad \text{and} \quad R = \frac{1}{2}(1 - \gamma_5)\psi. \]  
(2.9)

Thus, (2.6) becomes
\[ H = L^\dagger(G - iF)\gamma_4R + R^\dagger(G + iF)\gamma_4L, \]  
(2.10)
different from (2.5).

In the standard model, excluding the mass Hamiltonian, only the left hand components of \( \uparrow \) and \( \downarrow \) quarks are linked by their \( W \)-interaction. Hence, the right-hand component \( R \) or \( R \) can undergo an independent arbitrary unitary transformation. Because of this freedom, we can bring (2.10) into the form (2.5), or vice versa, as we shall see.

To show this, we begin with the form (2.10). Define
\[ M = G - iF \]  
(2.11)
and assume it to be nonsingular (i.e., the eigenvalues of \( MM^\dagger \) are all nonzero.) On account of (2.7), the hermitian conjugate of \( M \) is
\[ M^\dagger = G + iF. \]  
(2.12)

Since \( MM^\dagger \) is hermitian, there exists a unitary matrix \( V_L \) that can diagonalize \( MM^\dagger \), with
\[ V_L^\dagger MM^\dagger V_L = m_D^2 = \text{Diagonal}. \]  
(2.13)
For every eigenvector \( \phi \) of \( MM^\dagger \) with eigenvalue \( \lambda \), the corresponding vector \( M^\dagger \phi \) is an eigenvector of \( M^\dagger M \) with the same eigenvalue \( \lambda \). Thus, \( M^\dagger M \) can also be diagonalized by another unitary matrix \( V_R \) as
\[ V_R^\dagger M^\dagger MV_R = m_D^2, \]  
(2.14)
with \( m_D^2 \) the same diagonal matrix of (2.13).

Multiply (2.13) on the right by \( m_D^{-1} \), it follows that
\[ V_L^\dagger MV_R = m_D, \]  
(2.15)
provided that we define
\[ V_R = M^\dagger V_L m_D^{-1}. \]  
(2.16)
One can readily see that $V_L$ and $V_R$ thus defined satisfies $V_L^\dagger V_L = 1$, $V_R^\dagger V_R = 1$ as well as (2.13) and (2.14). Since $R$ can be transformed independently from $L$, we can transform the $\psi$ field by

$$L \to V_L L \quad \text{(2.17)}$$

and

$$R \to V_R R. \quad \text{(2.18)}$$

Next let us examine the mass matrix $\mathcal{M}$ of (2.1)-(2.2). Because $\mathcal{M}$ is hermitian, it can be diagonalized by a single unitary transformation $V$, with the left-handed and right-handed components of the field operator $\Psi$ undergoing the same transformation; i.e., in contract to (2.17)-(2.18), we have

$$\mathcal{L} \to V \mathcal{L}, \quad \text{(2.19)}$$

$$\mathcal{R} \to V \mathcal{R} \quad \text{(2.20)}$$

and correspondingly

$$\mathcal{H} \to \Psi^\dagger m_D \gamma_4 \Psi \quad \text{(2.21)}$$

with $m_D$ being the corresponding diagonal matrix. So far as the mass matrices are concerned, we regard these two mass Hamiltonians $\mathcal{H}$ and $\mathcal{H}$ as equivalent, if the diagonal matrix $m_D$ of (2.21) has the same set of eigenvalues as those in (2.15). In this case, we can without loss of generality set

$$\mathcal{M}^2 = (\mathcal{G} - i\mathcal{F})(\mathcal{G} + i\mathcal{F}) \quad \text{(2.22)}$$

and

$$V = V_L; \quad \text{(2.23)}$$

hence (2.13) becomes

$$V^\dagger \mathcal{M}^2 V = m_D^2 \quad \text{(2.24)}$$

and therefore

$$V^\dagger \mathcal{M} V = m_D. \quad \text{(2.25)}$$

(Note that $\mathcal{M} \neq \mathcal{M}$ or $\mathcal{M}^\dagger$, even though $\mathcal{M}^2 = \mathcal{M} \mathcal{M}^\dagger$.)

We will now discuss theories in which both matrices $\mathcal{G}$ and $\mathcal{F}$ are real; i.e.,

$$\mathcal{G} = \mathcal{G}^* \quad \text{and} \quad \mathcal{F} = \mathcal{F}^*. \quad \text{(2.26)}$$
Since $G$ and $F$ are also hermitian; they must both be symmetric matrices. In $n$-dimension, each of these matrices can carry \( \frac{1}{2}n(n+1) \) independent real parameters, giving a total of \( n(n+1) \) real parameters. On the other hand, $M$ being a single hermitian matrix consists of only $n^2$ real parameters. Thus, knowing $G$ and $F$, by using (2.22), we can always determine uniquely the corresponding $M$, but not the converse, by expanding in power series as follows.

Decompose the hermitian $M$ into its real and imaginary parts:

$$M = R + iI.$$  

(2.27)

with $R$ and $I$ both real; hence, $R$ is symmetric and $I$ antisymmetric. On account of (2.26), the real part of (2.22) is

$$R^2 - I^2 = G^2 + F^2,$$

(2.28)

and the imaginary part is

$$\{R, I\} = [G, F].$$

(2.29)

In what follows, we assume that $G$ and $F$ are both known, as in the case when the mass matrix is given by (1.3). In addition, $F$ can be regarded as small compared to $G$. Hence, we can expend $R$ and $I$ in powers of $F$. Write

$$R = G + R_2 + R_4 + R_6 + \cdots$$

(2.30)

and

$$I = I_1 + I_3 + I_5 + \cdots,$$

(2.31)

with $R_n$ and $I_m$ to be of the order of $F^n$ and $F^m$ respectively. Eqs.(2.28) and (2.29) give

$$\{G, I_1\} = [G, F],$$

$$\{G, R_2\} = F^2 + I_1^2,$$

$$\{G, I_3\} = -\{R_2, I_1\},$$

$$\{G, R_4\} = \{I_1, I_3\} - R_2^2,$$

etc.

(2.32)

As noted before, so far as these mass matrices are concerned, the two formalisms (2.1) and (2.6) are regarded as equivalent to each other, provided
that (2.28) and (2.29) hold. Then (2.32) gives the conditions determining the series expansions (2.30)-(2.31) of $R$ and $I$ in terms of $G$ and $\mathcal{F}$.

2.2 Application

Next, we apply the above analysis to the special case when

$$\mathcal{G} = G_\uparrow \quad \text{or} \quad G_\downarrow$$

(2.33)

and

$$\mathcal{F} = F = F_\uparrow = F_\downarrow$$

(2.34)

with $G_{\uparrow/\downarrow}$ and $F$ given by (1.4) and (1.24) respectively. For clarity, we shall suppress the subscript $\uparrow$ or $\downarrow$ in this section and write (1.13)-(1.18) as

$$\epsilon = \begin{pmatrix} \cos \theta \\ \sin \theta \cos \phi \\ \sin \theta \sin \phi \end{pmatrix}, \quad p = \begin{pmatrix} -\sin \theta \\ \cos \theta \cos \phi \\ \cos \theta \sin \phi \end{pmatrix}$$

and

$$P = \begin{pmatrix} 0 \\ -\sin \phi \\ \cos \phi \end{pmatrix}.$$ 

(2.35)

In order to conform to the sign convention used in (1.13)-(1.18), we have

$$\theta = \theta_\uparrow \quad \text{in the} \quad \uparrow \text{ sector}$$

(2.36)

but

$$\theta = -\theta_\downarrow \quad \text{in the} \quad \downarrow \text{ sector}.$$ 

(2.37)

Likewise, the unit column matrix $f$ in (1.24) is

$$f = \begin{pmatrix} \cos a \\ \sin a \cos b \\ \sin a \sin b \end{pmatrix}.$$ 

(2.38)

Define matrix

$$\mathcal{G} = \nu \epsilon \tilde{\epsilon} + \mu p \tilde{p} + mP \tilde{P}$$ 

(2.39)
and
\[ F = \tau f \tilde{f} \quad (2.40) \]
with \( \nu, \mu, m, \tau \) all real constants. When \( \tau = \tau_0 \) and \( \nu = 0 \), \( F \) becomes \( F \) of (1.24) and \( G \) can be either \( G_\uparrow \) or \( G_\downarrow \) of (1.4). In this section, we retain the eigenvalue \( \nu \) in (2.39) for the formal symmetry of some of the mathematical expressions, even though \( \nu = 0 \) when we discuss physical applications of our model in other sections of the paper.

Let \( \vec{A} \) be a vector whose \( k^{th} \) component is given by the \((i, j)\)th component of the commutator between \( G \) and \( F \):
\[ [G, F]_{ij} = \epsilon_{ijk} A_k \quad (2.41) \]
with \( \epsilon_{ijk} = \pm 1 \) depending on \((ijk)\) being an even or odd permutation of \((1, 2, 3)\), and 0 otherwise. From (2.35) and (2.38), we can readily verify that
\[ A_k = \tau \left( \nu (\hat{\epsilon} \cdot \hat{f})(\hat{\epsilon} \times \hat{f}) + \mu (\hat{\rho} \cdot \hat{f})(\hat{\rho} \times \hat{f}) + m (\hat{P} \cdot \hat{f})(\hat{P} \times \hat{f}) \right)_k. \quad (2.42) \]

Next, define a real antisymmetric matrix \( I \) whose anti-commutator with \( G \) is given by
\[ \{G, I\} = [G, F]. \quad (2.43) \]
Let its \((i,j)\)th matrix element be written as
\[ I_{ij} = \epsilon_{ijk} J_k. \quad (2.44) \]
Thus, \( I \) is the same antisymmetric matrix \( I_1 \) of (2.31), i.e.,
\[ I = I_1 \quad (2.45) \]
and (2.43) is the same as the first equation in (2.32).

The corresponding vector \( \vec{J} \) is related to \( \vec{A} \) by
\[ \vec{J} \cdot \hat{\epsilon} = (\mu + m)^{-1} \vec{A} \cdot \hat{\epsilon} \]
\[ \vec{J} \cdot \hat{\rho} = (m + \nu)^{-1} \vec{A} \cdot \hat{\rho} \]
and
\[ \vec{J} \cdot \hat{P} = (\nu + \mu)^{-1} \vec{A} \cdot \hat{P}; \]

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these relations can be readily derived from (2.41)-(2.44) by writing $G$ as diagonal in the basis ($\hat{\epsilon}, \hat{p}, \hat{P}$).

As in (1.19) and (1.20), we define a real unitary matrix $U_0$ whose columns are $\epsilon, p$ and $P$ of (2.35); i.e.,

$$U_0 = \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta \cos \phi & \cos \theta \cos \phi & -\sin \phi \\ \sin \theta \sin \phi & \cos \theta \sin \phi & \cos \phi \end{pmatrix}.$$  

(2.47)

The matrix $U_0$ diagonalizes $G$, with

$$G' \equiv \tilde{U}_0GU_0 = \begin{pmatrix} \nu & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & m \end{pmatrix}.$$  

(2.48)

It also transforms $F$ into

$$F' \equiv \tilde{U}_0FU_0 = \tau f' \tilde{f}'$$  

(2.49)

where

$$f' = \begin{pmatrix} f_\epsilon \\ f_p \\ f_P \end{pmatrix}.$$  

(2.50)

with

$$f_\epsilon = \tilde{\epsilon} f, \quad f_p = \tilde{p} f, \quad f_P = \tilde{P} f.$$  

(2.51)

As before,

$$f_\epsilon^2 + f_p^2 + f_P^2 = 1.$$  

(2.52)

By using (2.48)-(2.50), we find that the same $U_0$ also transforms the matrix $\mathcal{M}^2$ into

$$(\mathcal{M}')^2 = \tilde{U}_0 \mathcal{M}^2 U_0 = (G')^2 + (F')^2 + i[G', F'],$$

(2.53)

which is given by

$$\begin{pmatrix} \nu^2 + \tau^2 f_\epsilon^2 & \tau [\tau - i(\mu - \nu)] f_\epsilon f_p & \tau [\tau - i(m - \nu)] f_P f_\epsilon \\ \tau [\tau + i(\mu - \nu)] f_\epsilon f_P & \mu^2 + \tau^2 f_p^2 & \tau [\tau - i(m - \mu)] f_p f_P \\ \tau [\tau + i(m - \nu)] f_P f_\epsilon & \tau [\tau + i(m - \mu)] f_P f_P & m^2 + \tau^2 f_P^2 \end{pmatrix}.$$  

(2.54)
For most of our applications, we are only interested in the case $\nu = 0$. Define
\[
\mathcal{M}'_0 \equiv \lim_{\nu \to 0} \mathcal{M}'
\]
and let $\lambda_1^2$, $\lambda_2^2$, $\lambda_3^2$ be the eigenvalues of $(\mathcal{M}'_0)^2$. From (2.54) and (2.55) we have
\[
\lambda_1^2 + \lambda_2^2 + \lambda_3^2 = m^2 + \mu^2 + \tau^2
\]
and
\[
\lambda_1^2 \lambda_2^2 \lambda_3^2 = |(\mathcal{M}'_0)^2| = \tau^2 f_\epsilon^4 \mu^2 m^2,
\]
These eigenvalues are also the solution $\lambda^2$ of the cubic equation
\[
|(\mathcal{M}'_0)^2 - \lambda^2| = |(\mathcal{M}'_0)^2| + A\lambda^2 + B\lambda^4 - \lambda^6 = 0
\]
where
\[
A = -\mu^2 m^2 - \tau^2 [m^2(1 - f_p^2)^2 + \mu^2(1 - f_p^2)^2 + 2m\mu f_p^2 f_P^2]
\]
and
\[
B = m^2 + \mu^2 + \tau^2.
\]
In the limit $\tau \to 0$, we see from (2.57)-(2.60) that the two heavier masses become $\mu$ and $m$, while the lightest mass is proportional to $\tau$. We shall explore this limit further in the next section.
3. Perturbative Solution and Jarlskog Invariant

3.1 Perturbation Series

In this section we return to the mass Hamiltonian (1.3) and calculate its
eigenstates by using $G\gamma_4$ as the zeroth order Hamiltonian and $iF\gamma_4\gamma_5$ as the
perturbation. From the discussions given in the last section, we see that this
is identical to the problem of finding the eigenstates of $(\mathcal{M}_0')^2$ regarding $\tau$ as
the small parameter. Using (2.54)-(2.55), we may write

\begin{equation}
(\mathcal{M}_0')^2 = \begin{pmatrix}
0 & 0 & 0 \\
0 & \mu^2 & 0 \\
0 & 0 & m^2
\end{pmatrix} + h + O(\tau^2) \quad (3.1)
\end{equation}

with

\begin{equation}
h = \tau \begin{pmatrix}
0 & -i\mu f_{\epsilon} f_p & -im f_{P} f_{\epsilon} \\
i\mu f_{\epsilon} f_p & 0 & -i(m - \mu) f_p f_P \\
im f_P f_{\epsilon} & i(m - \mu) f_p f_P & 0
\end{pmatrix}. \quad (3.2)
\end{equation}

To first order in $h$, the eigenstates of $(\mathcal{M}_0')^2$ can be readily obtained. For
applications to physical quarks, we need only to identify that $f_{\epsilon}$, $f_p$ and $f_P$ are
replaced by

\begin{equation}
(f_{\epsilon})_{\uparrow/\downarrow} = \tilde{f}_{\epsilon_{\uparrow/\downarrow}}, \quad (f_p)_{\uparrow/\downarrow} = \tilde{f}_{P_{\uparrow/\downarrow}} \quad (3.3)
\end{equation}

and

\begin{equation}
(f_P)_{\uparrow/\downarrow} = \tilde{f}_{P_{\uparrow/\downarrow}}.
\end{equation}

Likewise, neglecting $O(\tau^2)$ corrections, we can relate $\mu^2$ and $m^2$ to (quark mass)$^2$
by

\begin{align*}
\mu^2_{\uparrow} &= m^2_c, & m^2_{\uparrow} &= m^2_t \\
\mu^2_{\downarrow} &= m^2_s, & m^2_{\downarrow} &= m^2_b \quad (3.4)
\end{align*}

and set

\begin{equation}
\tau = \tau_0. \quad (3.5)
\end{equation}
Thus, to $O(\tau_0)$, in the ↑ sector the state vectors of $u, c, t$ are related to those of $\epsilon_\uparrow, p_\uparrow$ and $P_\uparrow$ by

$$
\begin{pmatrix}
u \\
c \\
t
\end{pmatrix} =
\begin{pmatrix}
(\epsilon_\uparrow|u) \\
(\epsilon_\uparrow|c) \\
(\epsilon_\uparrow|t)
\end{pmatrix}
\begin{pmatrix}
p_\uparrow \\
(c_\uparrow|c) \\
(p_\uparrow|t)
\end{pmatrix}
\begin{pmatrix}
(\epsilon_\downarrow|d) \\
(\epsilon_\downarrow|s) \\
(\epsilon_\downarrow|b)
\end{pmatrix}
\begin{pmatrix}
P_\downarrow \\
P_\uparrow \\
P_\uparrow
\end{pmatrix}
$$

with

$$
(\epsilon_\uparrow|u) = 1 + O(\tau_0^2),
(\epsilon_\uparrow|c) = -i\frac{\tau_0}{m_c}(f_\epsilon f_p)_\uparrow + O(\tau_0^2),
(\epsilon_\uparrow|t) = -i\frac{\tau_0}{m_t}(f_\epsilon f_P)_\uparrow + O(\tau_0^2),
$$

$$
(p_\uparrow|u) = -i\frac{\tau_0}{m_c}(f_\epsilon f_p)_\uparrow + O(\tau_0^2),
(p_\uparrow|c) = 1 + O(\tau_0^2),
(p_\uparrow|t) = -i\frac{\tau_0}{m_t + m_c}(f_p f_P)_\uparrow + O(\tau_0^2),
$$

$$
(P_\uparrow|u) = -i\frac{\tau_0}{m_t}(f_\epsilon f_P)_\uparrow + O(\tau_0^2),
(P_\uparrow|c) = -i\frac{\tau_0}{m_t + m_c}(f_p f_P)_\uparrow + O(\tau_0^2)
$$

and

$$
(P_\uparrow|t) = 1 + O(\tau_0^2).
$$

Likewise, for the ↓ sector, we may write

$$
\begin{pmatrix}
d \\
s \\
b
\end{pmatrix} =
\begin{pmatrix}
(\epsilon_\downarrow|d) \\
(\epsilon_\downarrow|s) \\
(\epsilon_\downarrow|b)
\end{pmatrix}
\begin{pmatrix}
p_\downarrow \\
(c_\downarrow|s) \\
(p_\downarrow|b)
\end{pmatrix}
\begin{pmatrix}
(\epsilon_\uparrow|d) \\
(\epsilon_\uparrow|s) \\
(\epsilon_\uparrow|b)
\end{pmatrix}
\begin{pmatrix}
P_\downarrow \\
P_\uparrow \\
P_\uparrow
\end{pmatrix}
$$

with

$$
(\epsilon_\downarrow|d) = 1 + O(\tau_0^2),
(\epsilon_\downarrow|s) = -i\frac{\tau_0}{m_s}(f_\epsilon f_p)_\downarrow + O(\tau_0^2),
$$
\[(\epsilon_\downarrow|b) = -i \frac{\tau_0}{m_b} (f_\epsilon f_P)_\downarrow + O(\tau_0^2), \quad (3.19)\]
\[(p_\downarrow|d) = -i \frac{\tau_0}{m_s} (f_\epsilon f_P)_\downarrow + O(\tau_0^2), \quad (3.20)\]
\[(p_\downarrow|s) = 1 + O(\tau_0^2), \quad (3.21)\]
\[(p_\downarrow|b) = -i \frac{\tau_0}{m_b + m_s} (f_p f_P)_\downarrow + O(\tau_0^2), \quad (3.22)\]
\[(P_\downarrow|d) = -i \frac{\tau_0}{m_b} (f_\epsilon f_P)_\downarrow + O(\tau_0^2), \quad (3.23)\]
\[(P_\downarrow|s) = -i \frac{\tau_0}{m_b + m_s} (f_p f_P)_\downarrow + O(\tau_0^2) \quad (3.24)\]

and
\[(P_\downarrow|b) = 1 + O(\tau_0^2). \quad (3.25)\]

### 3.2 Jarlskog Invariant

Write the CKM matrix as
\[U_{CKM} = \begin{pmatrix} U_{11} & U_{12} & U_{13} \\ U_{21} & U_{22} & U_{23} \\ U_{31} & U_{32} & U_{33} \end{pmatrix}. \quad (3.26)\]

Following Jarlskog[4], we introduce
\[S_1 = U_{11}^* U_{12}, \quad S_2 = U_{21}^* U_{22}, \quad S_3 = U_{31}^* U_{32} \quad (3.27)\]

and define
\[\mathcal{J} = \text{Im} S_1^* S_2. \quad (3.28)\]

By using (3.27) we see that
\[\mathcal{J} = \text{Im} \left[ (U_{11} U_{22}) (U_{12}^* U_{22}^*) \right]. \quad (3.29)\]

Because of unitarity of the CKM matrix,
\[S_1 + S_2 + S_3 = 0. \quad (3.30)\]
Therefore, \( J \) is equal to twice the area of the triangle whose sides are \( S_1, S_2 \) and \( S_3 \). Furthermore, from the explicit form of \( J \) given by (3.29), we see that \( J \) is symmetric with respect to the interchange between the row and column indices of the CKM matrix. It follows then in deriving \( J \), we may use the elements of either any two columns or any two rows of the CKM matrix.

It is convenient to denote \((U_{CKM})_0\) of (1.21) simply as \( V \), with

\[
V \equiv (U_{CKM})_0 = \begin{pmatrix} V_{11} & V_{12} & V_{13} \\ V_{21} & V_{22} & V_{23} \\ V_{31} & V_{32} & V_{33} \end{pmatrix}.
\]

(3.31)

In terms of the state vectors \( \epsilon_\uparrow, p_\uparrow, P_\uparrow \) and \( \epsilon_\downarrow, p_\downarrow, P_\downarrow \) of (1.13)-(1.18), we can also write \( V \) as

\[
V = \begin{pmatrix} (\epsilon_\uparrow|\epsilon_\uparrow) & (\epsilon_\uparrow|p_\uparrow) & (\epsilon_\uparrow|P_\uparrow) \\ (p_\uparrow|\epsilon_\uparrow) & (p_\uparrow|p_\uparrow) & (p_\uparrow|P_\uparrow) \\ (P_\uparrow|\epsilon_\uparrow) & (P_\uparrow|p_\uparrow) & (P_\uparrow|P_\uparrow) \end{pmatrix}.
\]

(3.32)

Likewise, the CKM matrix is given by

\[
U_{CKM} = \begin{pmatrix} (u|d) & (u|s) & (u|b) \\ (c|d) & (c|s) & (c|b) \\ (t|d) & (t|s) & (t|b) \end{pmatrix}
\]

\[= V + i\tau_0 W + O(\tau_0^2)\]

(3.33)

where the matrix elements of \( W \) are derived from (3.7)-(3.15) and (3.17)-(3.25). Using the perturbative solution of Sec. 3.1, we can readily express the matrix elements of \( U_{CKM} \) in terms of the corresponding ones of \( V \). The Jarlskog invariant can then be evaluated by using (3.29).

As will be shown in the Appendix, the result, accurate to the first power of \( \tau_0 \), is

\[
J = \tau_0 \left[ (f_{\epsilon F})(f_{\epsilon P})_{\downarrow} A_s + (f_{\epsilon F})(f_{p F})_{\downarrow} A_b + (f_{p F})(f_{p F})_{\downarrow} B_{\downarrow} \\ + (f_{\epsilon F})(f_{p F})_{\downarrow} A_c + (f_{\epsilon F})(f_{p F})_{\downarrow} A_t + (f_{p F})(f_{p F})_{\downarrow} B_{\downarrow} \right]
\]

(3.34)

where

\[
A_s = -V_{13}V_{23}V_{33} \approx -2 \cdot 10^{-4},
\]

(3.35)
\[ A_b = -V_{12}V_{22}V_{32} \approx 8.8 \cdot 10^{-3}, \]  
\[ B_\perp = -V_{11}V_{21}V_{31} \approx 1.10 \cdot 10^{-3}, \]  
\[ A_c = V_{31}V_{32}V_{33} \approx -2 \cdot 10^{-4}, \]  
\[ A_t = V_{21}V_{22}V_{23} \approx -8.8 \cdot 10^{-3} \]  
\[ B_\uparrow = V_{11}V_{12}V_{13} \approx 1.10 \cdot 10^{-3}. \]

From the definition (3.29) and (3.34), these coefficients \( A_s, \cdots, B_\uparrow \) are all products of four factors of \( V_{ij} \). As will be shown in the Appendix, because \( V \) is a real orthogonal matrix, these quartic products can all be reduced to triple products given by (3.35)-(3.40). Since \( m_c >> m_s \) and \( m_t >> m_b \), we can, as an approximation, neglect the terms related to the up sector in (3.34).
4. Determination of $\tau_0$ and $f$

4.1 A Special coordinate system

For the $\uparrow$ quarks, the parameters $\lambda_1$, $\lambda_2$ and $\lambda_3$ in (2.56)-(2.60) are related to the quark masses by

$$\lambda_1 = m_u, \quad \lambda_2 = m_c \quad \text{and} \quad \lambda_3 = m_t. \quad (4.1)$$

Likewise, $f_\epsilon$ is

$$(f_\epsilon)_{\uparrow} = \tilde{f} \epsilon_{\uparrow} \quad (4.2)$$

with $\epsilon_{\uparrow}$ given by (1.13) and $f$ the unit directional vector of (1.24). We work to leading order in $\tau_0$. From (2.57), setting $\lambda_2 = \mu$ and $\lambda_3 = m$ we have

$$m_u = \tau_0 (\tilde{f} \epsilon_{\uparrow})^2. \quad (4.3)$$

Likewise, for the $\downarrow$ quarks

$$m_d = \tau_0 (\tilde{f} \epsilon_{\downarrow})^2. \quad (4.4)$$

It is convenient to introduce a special coordinate system in which

$$\epsilon_{\downarrow} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \epsilon_{\uparrow} = \begin{pmatrix} \cos \theta_c \\ -\sin \theta_c \\ 0 \end{pmatrix}. \quad (4.5)$$

Since $p_{\downarrow}$ and $P_{\downarrow}$ are both $\perp \epsilon_{\downarrow}$, we may write

$$p_{\downarrow} = \begin{pmatrix} 0 \\ -\cos \gamma \\ \sin \gamma \end{pmatrix} \quad \text{and} \quad P_{\downarrow} = \begin{pmatrix} 0 \\ -\sin \gamma \\ -\cos \gamma \end{pmatrix}. \quad (4.6)$$

Furthermore, we shall set the zeroth order CKM matrix $(U_{CKM})_0$ of (1.21) to be

$$(U_{CKM})_0 = \begin{pmatrix} (\epsilon_{\uparrow}|\epsilon_{\downarrow}) & (\epsilon_{\uparrow}|p_{\downarrow}) & (\epsilon_{\uparrow}|P_{\downarrow}) \\ (p_{\uparrow}|\epsilon_{\downarrow}) & (p_{\uparrow}|p_{\downarrow}) & (p_{\uparrow}|P_{\downarrow}) \\ (P_{\uparrow}|\epsilon_{\downarrow}) & (P_{\uparrow}|p_{\downarrow}) & (P_{\uparrow}|P_{\downarrow}) \end{pmatrix}$$

$$= \begin{pmatrix} .974 & .227 & 5 \cdot 10^{-3} \\ -.227 & .973 & .04 \\ 5 \cdot 10^{-3} & -.04 & .999 \end{pmatrix} + O(1 \cdot 10^{-3}). \quad (4.7)$$

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Thus, with the same accuracy of $O(10^{-3})$,
\[
\cos \theta_c = .974 \quad \text{and} \quad \sin \theta_c = .227. \quad (4.8)
\]
Likewise, from $(\epsilon_\uparrow|P_\downarrow) = 5 \cdot 10^{-3}$ in (4.7), in accordance with (4.5), (4.6) and
\[
(\epsilon_\uparrow|P_\downarrow) = \sin \theta_c \sin \gamma, \quad (4.9)
\]
we find
\[
\sin \gamma = 2.2 \cdot 10^{-2}, \quad (4.10)
\]
which together with (4.5) and (4.6) give the coordinate system defined by $(\epsilon_\downarrow, p_\downarrow, P_\downarrow)$. Eq.(4.7) then, in turn, determines the corresponding coordinate system $(\epsilon_\uparrow, p_\uparrow, P_\uparrow)$.

Next, we shall determine the parameters $\tau_0$ and the directional angles $\alpha$ and $\beta$ of the unit vector
\[
f = \begin{pmatrix}
\sin \alpha \cos \beta \\
\sin \alpha \sin \beta \\
\cos \alpha
\end{pmatrix} \quad (4.11)
\]
in the coordinate system defined by (4.5)-(4.6).

### 4.2 Determination of $\beta$

From (4.5) and (4.11), we have
\[
\tilde{f}_\epsilon \epsilon_\downarrow = \sin \alpha \cos \beta \quad (4.12)
\]
and
\[
\tilde{f}_\epsilon \epsilon_\uparrow = \sin \alpha \cos(\beta + \theta_c). \quad (4.13)
\]
Thus, on account of (4.3) and (4.4),
\[
\frac{\cos^2(\beta + \theta_c)}{\cos^2 \beta} = \frac{m_u}{m_d} \quad (4.14)
\]
and therefore
\[
\frac{\cos(\beta + \theta_c)}{\cos \beta} = \pm \left(\frac{m_u}{m_d}\right)^{\frac{1}{2}}. \quad (4.15)
\]
assuming
\[ \frac{m_u}{m_d} \approx \frac{1}{2}. \] (4.16)
we find two solutions for \( \beta \):
\[ \beta \approx 48^0 \ 50' \] (4.17)
or
\[ \beta \approx 82^0 \ 20'. \] (4.18)

4.3 Determination of \( \alpha \) and \( \tau_0 \)

We shall first determine the parameter \( \alpha \) by using the Jarlskog invariant
\[ J = 3.08 \cdot 10^{-5}. \] (4.19)
Define
\[ F = 10^2 J m_b / \tau_0. \] (4.20)
From (4.4), (4.5) and (4.11), we have
\[ m_d = \tau_0 \sin^2 \alpha \cos^2 \beta \] (4.21)
and therefore
\[ F = 10^2 J (m_b / m_d) \sin^2 \alpha \cos^2 \beta. \] (4.22)
For definiteness, we shall set the various quark masses as
\[ m_d \approx 5MeV, \quad m_u \approx 2.5MeV \]
\[ m_s \approx 95MeV, \quad m_c \approx 1.25GeV \]
\[ m_b \approx 4.2GeV \quad \text{and} \quad m_t \approx 175GeV, \] (4.23)
consistent with the Particle Data Group values[5]. Thus, (4.22) becomes
\[ F(\alpha, \beta) \approx 2.6 \sin^2 \alpha \cos^2 \beta, \] (4.24)

On the other hand, from (3.34) and by using the numerical values for \( A_s, \ A_b, \ \cdots, \ B_\uparrow \) of (3.35)-(3.40) together with the various quark masses given above, the same function \( F(\alpha, \beta) \) is also
\[ F(\alpha, \beta) \approx -0.88 (f_c f_p)_\downarrow - 0.067 (f_c f_p)_\uparrow \]
+ 0.88(f_{\epsilon}f_{P})_{\downarrow} - 0.021 (f_{\epsilon}f_{P})_{\uparrow}
+ 0.11(f_{p}f_{P})_{\downarrow} + 0.0026(f_{p}f_{P})_{\uparrow}.

(4.25)

As an approximation, we may neglect the contributions of the $\uparrow$ sector. Combining (4.24) with (4.25), we find

$$2.6 \sin^{2} \alpha \cos^{2} \beta \approx 0.88 f_{\epsilon_{\downarrow}}[f_{P_{\downarrow}} - f_{p_{\downarrow}}] + 0.11 (f_{p}f_{P})_{\downarrow}$$

(4.26)

with

$$f_{\epsilon_{\downarrow}} = \tilde{f}_{\epsilon_{\downarrow}}$$

(4.27)

given by (4.12),

$$f_{p_{\downarrow}} = \tilde{f}_{p_{\downarrow}} = - \sin \alpha \sin \beta \cos \gamma + \cos \alpha \sin \gamma$$

(4.28)

and

$$f_{P_{\downarrow}} = \tilde{f}_{P_{\downarrow}} = - \sin \alpha \sin \beta \sin \gamma - \cos \alpha \cos \gamma.$$  

(4.29)

By using $\gamma$ from (4.10),

$$\beta \approx 48^{0} 50'$$

from (4.17), we find

$$\alpha \approx -36^{0} 10'.$$  

(4.30)

From (4.4) and (4.12), we have

$$\tau_{0} = m_{d}/(\sin^{2} \alpha \cos^{2} \beta).$$

(4.31)

The above values of $\alpha$, $\beta$ and $m_{d} \approx 5 MeV$ give

$$\tau_{0} \approx 33 MeV.$$  

(4.32)

On the other hand, the alternative solution $\beta \approx 82^{0} 20'$ of (4.18) leads to a much larger value $\tau_{0} \sim 5.5 GeV$. Such a large value invalidates the small $\tau_{0}$ approximation. Thus, we focus only on the solution (4.32) in this paper.
5. Remarks

While the discovery\cite{6} of $T$ and $CP$ violation was made 44 years ago, at present very little is known about its origin. The totality of our knowledge can be summarized by a single dimensionless small number, the Jarlskog invariant of the CKM matrix. In this paper, we present a simple dynamic model. The vibration of the spin 0 field $\tau(x)$ around its equilibrium value $\tau_0$ gives a new quantum, the timeon. Since $\tau(x)$ is $T$ odd and $CP$ odd, viewed from the context of the Standard Model\cite{1}, the timeon field violates maximally its $T$ conservation and $CP$ conservation. As mentioned in the Introduction, the timeon mass $m_\tau$ would be different and might be much lower than the mass limit of the ($T$ and $CP$ conserving) Higgs boson in the standard model.

In order to detect timeon, a useful signal might be the observation in any reaction, such as

$$p + p \rightarrow N + N' + \tau + \cdots,$$  \hspace{1cm} (5.1)

"$T$ violating" kinematic variables, like

$$\left(\vec{k}_N \times \vec{k}_N'\right) \cdot \vec{k}_\tau.$$  \hspace{1cm} (5.2)

Because of the final state interaction, a mere observation of such variables would not be decisive; its amplitude must be above a certain kinematic limit\cite{7}. Nevertheless these could be useful signals.
Appendix

A.1 Definitions, Corollaries and Conventions

Let \( V = (V_{i\alpha}) \) be a \( 3 \times 3 \) real orthogonal matrix with positive determinant, and indices \( i \) and \( \alpha = 1, 2 \) and \( 3 \); hence,

\[
V = V^*, \quad V^{-1} = \bar{V} \quad \text{and} \quad |V| = 1. \tag{A.1}
\]

Given any value of \( i \), define \( i', i'' \) by

\[
\epsilon_{ii'i''} = 1, \tag{A.2}
\]

so that \((i, i', i'')\) is a cyclic permutation of \((1, 2, 3)\). We shall use the same definition for each of such similar indices \( j, k, l, \alpha, \beta, \gamma \). Thus, for a given \( j \), the corresponding \( j', j'' \) satisfy

\[
\epsilon_{jj'j''} = 1, \tag{A.3}
\]

and likewise

\[
\epsilon_{kk'k''} = \epsilon_{ll'l''} = \epsilon_{\alpha\alpha'\alpha''} = \epsilon_{\beta\beta'\beta''} = \epsilon_{\gamma\gamma'\gamma''} = 1.
\]

Furthermore, for any pair \( i, \alpha \), we have by the expression for \( V^{-1} \)

\[
\begin{vmatrix}
V_{i\alpha'} & V_{i\alpha''} \\
V_{i''\alpha'} & V_{i''\alpha''}
\end{vmatrix} = |V|(V^{-1})_{\alpha i} = V_{i\alpha} \tag{A.4}
\]
on account of (A.1). This identity will enable us to reduce certain quartic products of \( V_{i\alpha} \) to triple products of \( V_{i\alpha} \), as we shall see.

A.2 Jarlskog Invariant

Similar to the relation between \( V \) of (3.32) and

\[
U = U_{CKM} \tag{A.5}
\]
of (3.33), we define \( W = (W_{i\alpha}) \) through

\[
U = V + i\tau_0 W \tag{A.6}
\]
where $W$ has the form

$$W_{i\alpha} = \sum_{j \neq i} F_{ij} V_{j\alpha} - \sum_{\beta \neq \alpha} F_{i\beta} V_{i\beta}. \quad (A.7)$$

For our purpose here, it is necessary to specify only that $F$ and $\mathcal{F}$ are real and symmetric; i.e.,

$$F_{ij} = F_{ji} = F_{ij}^* \quad (A.8)$$

and

$$\mathcal{F}_{\alpha\beta} = \mathcal{F}_{\beta\alpha} = \mathcal{F}_{\alpha\beta}^*. \quad (A.9)$$

We also define for any particular pair of indices $k$ and $\gamma$,

$$J = U_{k\gamma} U_{k'\gamma} U_{k'\gamma}^* U_{k\gamma}^* \quad (A.10)$$

and

$$J_0 = V_{k\gamma} V_{k'\gamma} V_{k'\gamma}^* V_{k\gamma}^*. \quad (A.11)$$

Thus, substituting (A.6) into (A.10), we find, to first order in $\tau_0$,

$$J - J_0 = i\tau_0 \Delta_{k\gamma} \quad (A.12)$$

where

$$\Delta_{k\gamma} = J_0 \left( \frac{W_{k\gamma}}{V_{k\gamma}} + \frac{W_{k'\gamma'}}{V_{k'\gamma'}} - \frac{W_{k'\gamma}}{V_{k'\gamma}} - \frac{W_{k\gamma'}}{V_{k\gamma'}} \right). \quad (A.13)$$

Note that $k, \gamma$ are subject to the cyclic convention typified by (A.2). [It will turn out that $\Delta_{k\gamma}$ is independent of the choice of $k$ and $\gamma$, even though this is not assumed.]

By substituting (A.7) into (A.13), we must obtain an expression of the form

$$\Delta_{k\gamma} = \sum_l A_{ll'} F_{ll'} + \sum_{\lambda} A_{\lambda\lambda'} \mathcal{F}_{\lambda\lambda'} \quad (A.14)$$

where the $A$'s and $A'$'s are to be determined. From (A.7), (A.13) and (A.14), we see that each $A$ is made of terms having the form $J_0 V_{j\alpha}/V_{i\alpha}$. Consider $A_k$: in (A.14) we must put $l'' = k$; hence $l, l'$ are $k, k'$ in some order. Thus in (A.7), $i$ is either $k'$ or $k''$. But the index $k''$ does not occur in (A.13). Therefore, we have $i = k', j = k''$ and $\alpha = \gamma$ or $\gamma'$. Therefore,

$$A_k = J_0 \left( 0 + \frac{V_{k''\gamma'}}{V_{k'\gamma'}} - 0 - \frac{V_{k''\gamma'}}{V_{k'\gamma'}} \right)$$

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\[ = V_{k\gamma} V_{k\gamma'} \begin{vmatrix} V_{k'\gamma} & V_{k'\gamma'} \\ V_{k''\gamma} & V_{k''\gamma'} \end{vmatrix} = V_{k\gamma} V_{k\gamma'} V_{k\gamma''} \]  

(A.15) on account of (A.14). Likewise,
\[ A_{k'} = V_{k'\gamma} V_{k'\gamma'} V_{k'\gamma''}. \]  

(A.16)

For \( k'' \), the calculation is different, even though the result will be similar. Note that in (A.14) \( l, l' \) can be \( k, k' \) in either order, so that in (A.7) \( i \) and \( j \) can also be \( k, k' \) in either order. Thus, we now have four terms instead of two:
\[ A_{k''} = J_0 \left( \frac{V_{k'\gamma}}{V_{k\gamma}} + \frac{V_{k\gamma'}}{V_{k'\gamma'}} - \frac{V_{k'\gamma'}}{V_{k\gamma'}} - \frac{V_{k\gamma}}{V_{k'\gamma}} \right) \]
\[ = V_{k'\gamma} V_{k'\gamma'} \begin{vmatrix} V_{k\gamma} & V_{k\gamma'} \\ V_{k'\gamma} & V_{k'\gamma'} \end{vmatrix} + V_{k\gamma} V_{k\gamma'} \begin{vmatrix} V_{k\gamma'} & V_{k\gamma} \\ V_{k'\gamma'} & V_{k'\gamma} \end{vmatrix} \]
\[ = -V_{k'\gamma} V_{k'\gamma'} V_{k''\gamma''} - V_{k\gamma} V_{k\gamma'} V_{k\gamma''} = +V_{k''\gamma} V_{k''\gamma'} V_{k''\gamma''} \]  

(A.17) since \( V \) is orthogonal.

Similar formulas are obtained for \( A_\gamma, A'_\gamma, A''_\gamma \) with a change of sign. Since (A.15)-(A.17) all have the same form, the choice of \( k, \gamma \) in (A.10) and (A.11) is immaterial and we have
\[ J = J_0 + i\tau_0 \Delta \]  

(A.18)

where
\[ \Delta = \sum_l F_{ll'} \prod_{\alpha} V_{l'\alpha} - \sum_{\lambda} \mathcal{F}_{\lambda\lambda'} \prod_i V_{i\lambda''}. \]  

(A.19)

### A.3 Applications to quarks

By consulting (3.7)-(3.15) and (3.17)-(3.25), we find that
\[ F_{ij} = \frac{f_i f_j}{m_i + m_j}, \quad \mathcal{F}_{\alpha\beta} = \frac{\bar{f}_\alpha \bar{f}_\beta}{\mu_\alpha + \mu_\beta} \]  

(A.20)

where
\[ f_1 = (f_e)_\uparrow, \quad f_2 = (f_p)_\uparrow, \quad f_3 = (f_P)_\uparrow, \quad \bar{f}_1 = (f_e)_\downarrow, \quad \bar{f}_2 = (f_p)_\downarrow, \quad \bar{f}_3 = (f_P)_\downarrow \]  

(A.21)

with
\[ m_1 = m_u = 0, \quad m_2 = m_c, \quad m_3 = m_t, \]
\[ \mu_1 = m_d = 0, \quad \mu_2 = m_s, \quad \mu_3 = m_b. \]  \hspace{1cm} (A.22)

Then (3.34) is seen to be (A.14) if we identify

\[ A_c = A_3, \quad A_t = A_2, \quad B_{\uparrow} = A_1, \]
\[ A_s = A_3, \quad A_b = A_2 \quad \text{and} \quad B_{\downarrow} = A_1. \]  \hspace{1cm} (A.23)

With these identifications, (3.38)-(3.40) are (A.15)-(A.17) and (3.35)-(3.37) are the corresponding formulas for \( A_1, \ A_2 \) and \( A_3 \).

Figure 1. A schematic drawing of the mass matrix \( M_{\uparrow/\downarrow} = G_{\uparrow/\downarrow} \gamma_4 + iF \gamma_4 \gamma_5 \).

The vibration of \( \tau(x) \) is timeon
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