New examples of Anosov flows on higher dimensional manifolds
which fibre over 3-dimensional Anosov flows
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Abstract. We construct non-algebraic Anosov flows in dimension $3+2n$, $n \geq 2$, by suspending the action of the fundamental group of a finite cover of the Bonatti-Langevin flow.

1 Introduction

We say that a flow $\psi^t : E \to E$ is Anosov if the tangent bundle $TE$ splits into invariant subbundles $TE = E^0 \oplus E^s \oplus E^u$ under $d\psi^t$ such that there exist constants $C > 0, \lambda < 0$, so for $t > 0$,

$$\|d\psi^t(v)\| \leq Ce^{\lambda t}\|v\|, \text{ if } v \in E^s; \quad \|d\psi^{-t}(v)\| \leq Ce^{-\lambda t}\|v\|, \text{ if } v \in E^u,$$

and $E^0$ denotes the flow direction.

Definition 1 (Algebraic in the narrow sense). We say a flow on $M$ is algebraic in the narrow sense if it is $(G, \Gamma)$-induced in the sense of [Tom70]. That is, $M = G\backslash \Gamma$ (note that we switched the left and right actions to match our definition of the suspension space in Construction 6 below), where $G$ is a connected Lie group, $K$ a compact subgroup, $\Gamma$ a uniform lattice, and the flow is generated by left multiplication of a one parameter subgroup $\exp(t\alpha)$ which normalizes $K : Kg\Gamma \to \exp(t\alpha)Kg\Gamma$.

We often denote the flow as the system $(G, K, \Gamma, \alpha)$.

Definition 2. We say two flows $\varphi^t : E \to E$ and $\psi^t : E' \to E'$ are orbit equivalent if there exists a homeomorphism $h : E \to E'$ and $\alpha : E \times \mathbb{R} \to \mathbb{R}$ with $\alpha(t, x) \geq 0$ if $t \geq 0$ such that

$$h \circ \varphi^{\alpha(x, t)}(x) = \psi^{t} \circ h(x).$$

Tomter [Tom70] classified algebraic Anosov flows in the narrow sense and it was elaborated by Barbot and Maquera [BM13] as the following theorem.

Theorem 3 (Theorem 4 of [BM13]). Let $(G, K, \Gamma, \alpha)$ be an algebraic Anosov flow. Then $(G, K, \Gamma, \alpha)$ is commensurable (i.e., up to a finite cover or topological conjugacy; for a complete list see Section 2.2 of [BM13]) to either the suspension of an Anosov automorphism of a nilmanifold, or to a hyperbolic nil-suspension over a geodesic flow of a locally symmetric space of real rank one that is covered by a noncompact symmetric space.

Definition 4 (Algebraic in the broad sense). We say an Anosov flow is algebraic in the broad sense if it is orbit equivalent to an algebraic Anosov flow in the narrow sense of Definition 1.

Remark 5. There is a complete list of noncompact symmetric spaces of real rank 1, by É. Cartan (see for example [Pet06]). They are: real hyperbolic space $SO(n, 1)$, the complex hyperbolic space $SU(n, 1)$, the quaternionic hyperbolic space $Sp(n, 1)$, and the Cayley hyperbolic plane $F_4$ (which has dimension 16).

Therefore Theorem 3 tells us that an algebraic Anosov flow (of the broad sense) lives only on either a bundle of nilmanifold fibres over a circle base, or a bundle of nilmanifold fibres over a quotient of one of the above 4 types of symmetric spaces.

In the rest of the paper, we consider the flows being algebraic or non-algebraic in the broad sense, i.e., up to orbit equivalence.

There are many non-algebraic examples of Anosov flows constructed in dimension 3, for example, [FW80], [HT80], [BL94], etc. Examples in dimension $> 3$ are rare. Barthelmé, Bonatti, Gogolev, and Rodríguez Hertz [BBGRH21] pointed out the deficiency of Franks and Williams’ construction [FW80] in higher dimensions and gave an alternative construction in the similar spirit, surging the geodesic flows on unit tangent bundles of hyperbolic manifolds in higher dimensions. The same paper also discussed fibrewise Anosov flows, which we will define below, but there were no concrete examples given.
The algebraic examples suggested us to suspend the actions of certain groups. Such suspension spaces are also commonly used for proofs of results in group actions on manifolds. In this paper, we construct non-algebraic Anosov flows by suspending the actions of (or maybe finite index subgroups of) the fundamental group of a non-algebraic 3-dimensional flow. This is also the standard construction of flat bundles.

**Construction 6** (The suspension space). Suppose $\varphi^t : X \to X$ is a flow and let $\widetilde{X}$ denote a normal covering space of $X$. Denote $\Gamma := \pi_1 X / \pi_1 \widetilde{X}$, the deck group. Suppose $\varphi^t$ lifts to a flow $\widetilde{\varphi}^t : \widetilde{X} \to \widetilde{X}$.

We form the product $\tilde{E} = \widetilde{X} \times \mathbb{T}^d$. Let $\psi^t : \tilde{E} \to \tilde{E}$ be a flow given by $\psi^t(x, y) = (\tilde{\varphi}^t(x), y)$.

Suppose we have a monodromy representation $\rho : \Gamma \to SL(d, \mathbb{Z})$. With $\rho$, we can build a bundle. Let $\Gamma$ act on $\tilde{E}$ by the diagonal action, i.e., for $g \in \Gamma$

$$(x, y)g = (xg, \rho(g)^{-1}y).$$

We then get a flow on the quotient $E := \tilde{E}/\Gamma$, denoted by $\psi^t : E \to E$.

**Definition 7.** We call an Anosov flow $\psi^t : E \to E$ a fibrewise Anosov flow if $E$ is a fibre bundle with fibre $F$ and base $B$, such that the tangent bundle $TE = H \oplus V$ splits into the horizontal and vertical bundle where $TF = V = E^s \oplus E^u$ and $E^s, E^u$ are tangent to the fibres and invariant under $d\psi^t$, and there exist constants $C > 0, \lambda \in (0, 1)$, for $t > 0$,

$$\|d\psi^t(v)\| \leq Ce^{\lambda t}\|v\|, \text{ if } v \in E^s; \quad \|d\psi^{-t}(v)\| \leq Ce^{\lambda t}\|v\|, \text{ if } v \in E^u.$$ 

Moreover, $\psi^t$ covers an Anosov flow $\varphi^t : B \to B$ on the base, i.e., for the bundle projection $p : E \to B$, we have $p \circ \psi^t = \varphi^t \circ p$.

Note that flows constructed as in 6 are fibrewise. If it is fibrewise Anosov, but not a suspension flow, the dimension of the stable distribution and the dimension of the unstable distribution have to equal, and both $\geq 3$ (Theorem 1.2 of [BBGRH21]), because of the existence of a periodic orbit that is freely homotopic to its own inverse (Corollary E of [Fen98]).

It is then natural to ask: *Is there an Anosov flow which is not a suspension flow, but has stable and unstable distributions with different dimensions?*

Unfortunately, because of the above remarks, our examples constructed here will still have equal stable and unstable dimensions.

We will work out a class of examples over the fourth cover of the Bonatti-Langevin’s example in Section 2 and prove that this class of examples is not algebraic in Section 3.

**Acknowledgements.** I would like to thank my advisor Andrey Gogolev for suggesting the problem, discussions, and reading my writing; Zihao Fang, Zhining Wei, and Shifan Zhao for conversations while I was working out the linear algebra. I would also like to thank Christian Bonatti for questions on the draft which have cleared my thoughts.

## 2 The fibrewise flow over the Bonatti-Langevin’s example

The structure of this section: We first recall the Bonatti-Langevin construction. Then we give a construction of our bundle by specifying a representation. We prove that the flow we constructed is Anosov by finding a Poincaré section, defining a metric on the section, and showing that its return map is Anosov.

### 2.1 The construction of the 3-dimensional flow and the 4-sheeted cover

For completeness we first give a recap of the Bonatti-Langevin’s example. We take $\overline{N} = \mathbb{R} \times [-1, 1] - \bigcup_{i \in \mathbb{Z}} \mathbb{D}((2i, 0), \frac{1}{4})$, the infinite band with a disk of radius $\frac{1}{4}$ taken off at each even integer on the $y$-axis. Form the product $\overline{M} := \overline{N} \times \mathbb{S}^1$.

Consider a vector field on the base, $\overline{X} = -\phi(x)\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$ where $\phi : \mathbb{R} \to [-1, 1]$ is a $C^\infty$ odd function which satisfies $\phi(x + 1) = -\phi(x)$ and generates the integral curve shown in Figure 1. For
an example, \( \phi(x) = \frac{1}{2}x \) if \( x \in [-\frac{1}{3}, \frac{1}{3}] \) and is smoothed out otherwise will do the trick. Note it follows that \( \phi \) is 2-periodic.

It is shown in Lemma 2.1 of [BL94] that there is a function \( \omega : \overline{\mathcal{N}} \to \mathbb{R}/4\mathbb{Z} \) which is constant on each integral curve of \( \overline{\mathcal{X}} \) such that

\[
\omega(x, y) = \begin{cases} 
\omega(x, 1) = x & \text{if } y > 0, \\
\omega(x, -1) = 2 - x & \text{if } y < 0, \\
1 & \text{if } x \in (4i, 4i + 2), i \in \mathbb{Z} \text{ and } y = 0, \\
-1 & \text{if } x \in (4i + 2, 4i + 4), i \in \mathbb{Z} \text{ and } y = 0.
\end{cases}
\]

We use \( \omega \) as the new coordinate instead of \( x \), also for \( \partial \mathbb{D}((2i, 0), \frac{1}{4}), i \in \mathbb{Z} \).

With the new coordinate, we start to add a component to the vector field in the \( \theta \)-direction. Let

\[
\overline{\mathcal{Y}}_t := -\left( \alpha(x, y) + t \beta(|y|) \sin \left( \frac{\pi}{2} \omega(x, y) \right) \right) \frac{\partial}{\partial \theta},
\]

where

- \( \alpha : \mathbb{R}^2 \to \mathbb{R} \) is a \( C^\infty \) radial function, such that it is only nonzero on \( \mathbb{D}((2i + 1, 0), \frac{1}{3}), i \in \mathbb{Z} \), equal to 1 on a small open disk at \((1, 0)\), vanishing outside, and \( \alpha(x + 2, y) = -\alpha(x, y) \).
- \( \beta : [0, 1] \to [0, 1] \) is a \( C^\infty \) function that is strictly positive in \( (\frac{1}{3}, \frac{2}{3}) \) and vanishes outside.

Let \( \overline{\mathcal{Z}}_t = \overline{\mathcal{X}} + \overline{\mathcal{Y}}_t \). The holonomy along the \( \theta \)-axis when the orbits cross the supports of \( \alpha \) and \( \beta \) is now rather easy to compute.

Suppose the orbit is denoted as \( r(s) = (x(s), y(s)) \) entering from \((x, 1)\). Then the orbit deviates, when passing through a neighborhood of \((x, y) = (2k + 1, 0), k \in \mathbb{Z} \) by

\[
-\int_{x^2 + y^2 < \frac{1}{9}} \alpha(r(s)) \, ds,
\]

and by (note that with the above \( \overline{\mathcal{X}} \), the \( y \)-coordinate is \( y = e^{-s} \) if we solve the ODE)

\[
-\int_{y^2 < \frac{1}{3}} y^{-\frac{1}{2}} t \beta(|y(s)|) \sin \left( \frac{\pi}{2} \omega(r(s)) \right) \, ds = -t \sin \left( \frac{\pi}{2} x \right) \int_{-\infty}^{\infty} \beta(e^s) \, ds,
\]

when passing through \( y \in (\frac{1}{2}, \frac{3}{2}) \). Note that \( \int_{-\infty}^{\infty} \beta(e^s) \, ds \) is a fixed amount. Thus how much the orbit deviates only depends on where it starts, i.e., the \( \omega \)-coordinate.

**Notation 8.** Let us denote the amount of the deviation in the \( \theta \) direction of an orbit starting from \( \omega \) as \( f(\omega) \), where \( f : \mathbb{R} \to \mathbb{R} \).

Now we take two identifications. First let \( \varphi : \overline{\mathcal{M}} \to \overline{\mathcal{M}} \) be such that \( \varphi(x, y, \theta) = (x + 2, -y, -\theta) \).

Note that \( \varphi^2(x, y, \theta) = (x + 4, y, \theta) \), so \( M_0 := \overline{\mathcal{M}}/\varphi^2 \) is a trivial circle bundle over an annulus with two discs removed. Let \( M_1 := M_0/\varphi \). We denote the torus boundaries as \( \mathbb{T}_i, i = 1, 2, 3, 4 \) as shown in Figure 1, with induced coordinates from \( \omega \), and the two torus boundaries of \( M_1 \) by \( \mathbb{T}_{1,4} \) and \( \mathbb{T}_{2,3} \). The second identification \( A : \mathbb{T}_{1,4} \to \mathbb{T}_{2,3} \) is given by \( A(\omega, \theta) = (\theta, -\omega) \), so \( A^{-1}(\omega, \theta) = (-\theta, \omega) \) (useful for our later calculations). Denote \( M_2 := M_1/A \).
Now consider the fourth cover $M_3$ that we obtained by gluing 4 copies of $M_1$, after the first identification $\varphi$ from the above (see Figure 2).

Each copy before we glue is a nonorientable circle bundle over a möbius band with a hole, and has the fundamental group

$$\langle a_i, b_i, \theta_i \mid \theta_i = a_i \theta_i^{-1} a_i^{-1} = b_i \theta_i^{-1} b_i^{-1} \rangle.$$

We are able to compute the fundamental group of $M_3$ with the Seifert-Van Kampan’s Theorem (or say the amalgamation of groups), which is

$$\begin{align*}
  &a_1, b_1, \theta_1, a_2, b_2, \theta_2 \mid \\
  &a_3, b_3, \theta_3, a_4, b_4, \theta_4 \mid \\
  &c^4 \\
  &\theta_1 = a_1 \theta_1^{-1} a_1^{-1} = b_1 \theta_1^{-1} b_1^{-1} = (a_2 b_2)^{-1} = c^4 a_4 b_4^{-1} c^{-4} , \\
  &\theta_2 = a_2 \theta_2^{-1} a_2^{-1} = b_2 \theta_2^{-1} b_2^{-1} = a_1 b_1^{-1} = (a_3 b_3)^{-1} , \\
  &\theta_3 = a_3 \theta_3^{-1} a_3^{-1} = b_3 \theta_3^{-1} b_3^{-1} = a_2 b_2^{-1} = (a_4 b_4)^{-1} , \\
  &\theta_4 = a_4 \theta_4^{-1} a_4^{-1} = b_4 \theta_4^{-1} b_4^{-1} = a_3 b_3^{-1} = c^{-4} (a_1 b_1)^{-1} c^4.
\end{align*}$$

See Figure 2 for what the generators are: For $i = 1, 2, 3, 4$, $a_i, b_i$ are loops based at $\times$ which we take as the basepoint and oriented from the upper left corner to the lower right corner, and $\theta_i$’s are the
generators of the circle fibres in the virtual direction. We denote $c$ as $1/4$ (just for convenience of understanding) of the loop coming out from $T_1$ and going back, in the cover, and note that $c$ would be a loop in $M_2$ itself.

2.2 The representation

According to [BL94], we can fix a large enough parameter $t$ so the vector field $Z_t$ induces an Anosov flow $f^t : M_3 \rightarrow M_3$. Consider a fixed metric on $M_3$ and lift it to $\tilde{M}_3$, the universal cover of $M_3$.

We want to follow Construction 6 to construct a bundle.

First form the product $\tilde{M}_3 \times \mathbb{T}^d$. Then we specify a representation $\rho : \pi_1 M_3 \rightarrow \text{GL}(d, \mathbb{Z})$. We give a way to find such a representation.

Let $d = 4$. Take a 2-by-2 hyperbolic matrix, for example, $C_1 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ and its inverse $C_2 = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$.

Let $D_1 = C_2^2$ and $D_2 = C_1^2$, and

$$C = \begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix}, \quad \text{and} \quad D = \begin{pmatrix} 0 & D_1 \\ D_2 & 0 \end{pmatrix}.$$ 

One can check that $D^{-1} C D = C^{-1}$. Now let

$$\rho(\theta_1) = C^{t\theta_1}, \quad \rho(\theta_2) = C^{t\theta_2}, \quad \rho(\theta_3) = C^{t\theta_3}, \quad \rho(\theta_4) = C^{t\theta_4}, \quad \rho(c) = C^{t\theta}.$$ 

$$\rho(a_1) = C^{m_1} D, \quad \rho(a_2) = C^{m_2} D, \quad \rho(a_3) = C^{m_3} D, \quad \rho(a_4) = C^{m_4} D,$$

$$\rho(b_1) = C^{m_1} D, \quad \rho(b_2) = C^{m_2} D, \quad \rho(b_3) = C^{m_3} D, \quad \rho(b_4) = C^{m_4} D,$$

where $t_0, t_i, m_i, n_i \in \mathbb{Z}, i = 1, \ldots, 4$. Note that $D^2 = \text{id}$, so

$$\rho(a_i b_i) = C^{m_i} D C^{m_i} D = C^{m_i} D^2 D^{-1} C^{m_i} D = C^{m_i - n_i}, \quad \text{and} \quad \rho(a_i b_i^{-1}) = C^{m_i - n_i}.$$

The relators in the fundamental group then give

$$C^{t\theta_1} = C^{m_2 - m_2} = C^{m_4 - n_4},$$

$$C^{t\theta_2} = C^{m_3 - m_3} = C^{m_1 - n_1},$$

$$C^{t\theta_3} = C^{m_4 - m_4} = C^{m_2 - n_2},$$

$$C^{t\theta_4} = C^{m_1 - m_1} = C^{m_3 - n_3}.$$ 

Take $t_0 = 1, m_1 = 1, n_1 = 2, m_2 = 3, n_2 = 4, m_3 = 5, n_3 = 4, m_4 = 6, n_4 = 5$. Check that

$$\rho(\theta_i), \quad \rho(a_i b_i), \quad \rho(a_i b_i^{-1}), \quad \rho(c)$$

have no eigenvalues on the unit circle and the eigenspaces coincide.

Let $E := \tilde{M}_3 \times \mathbb{T}^d / \pi_1 M_3$. Note that $d$ does not have to be $= 4$.

**Notation 9.** For the rest of the paper, we denote $C^t := \exp(t\alpha)$ if $C = \exp(\alpha)$.

2.3 The holonomies on the torus boundaries in the base

We first compute the holonomy maps in the base 3-manifold. Note that by our construction the holonomy in the fibre is always constant, so the hyperbolicity would come from the metric which we are going to define, as for the usual suspension flow. See Figure 2 to recall what $T_i, \ i = 1, 2, 3, 4, 5$ are.

We now consider the lift of the torus boundaries in Figure 2 to $\mathbb{R}^2$, and the $\mathbb{T}^d$ fibres when the product bundle $\tilde{M} \times \mathbb{T}^d$ are restricted to the $\mathbb{R}^2$ that covers the torus boundaries. We compute the holonomy maps in the universal cover, as to write the maps in the product coordinates. Denote the holonomies as

$$h_i : \mathbb{R}^2 \times \mathbb{T}^d \rightarrow \mathbb{R}^2 \times \mathbb{T}^d, \ i = 1, 2, 3, 4,$$

where the domain $\mathbb{R}^2 \times \mathbb{T}^d$ is a lift of the restriction of the bundle to $T_i$, and the image $\mathbb{R}^2 \times \mathbb{T}^d$ is a lift of the restriction of the bundle to $T_i+1$.

For simplicity, assume we start from a point with $\theta = 0$. Here we denote $\tilde{f}$ as $f$ restricted to $\mathbb{R}\setminus \{2k+1\}$ mod 2, $k \in \mathbb{Z}$. Note that $\tilde{f}(-\omega) = -\tilde{f}(\omega)$ and $\tilde{f}(0) = 0$. Then we have
The metric $(1)$ should be more or less implicit in the original B-L paper, but we state for completeness.

2.4 The Poincaré section

Let $D_i, i = 1, 2, 3, 4$ denote small 2-dimensional discs with $\varepsilon$ radius centered at $(\omega, \theta) = (1, 0)$ in each copy of Figure 2. We fix an arbitrary but very small $\varepsilon$ now.

Now we want to consider the Poincaré section. Let us first consider how the orbits go in the base 3-manifold. Thinking of starting from points in $T_1$ (see figure 2), the orbits can go along any path in the following diagram.

$$
\begin{array}{c}
T_1 \rightarrow T_2 \rightarrow T_3 \rightarrow T_4 \rightarrow T_5 \\
D_1 \cup \bullet \quad D_2 \cup \bullet \quad D_3 \cup \bullet \quad D_4 \cup \bullet
\end{array}
$$

(1)

Let us now fix the time $T_0$ which is the time when the orbit at $(1,0)$ in each copy returns to each disc $D_i$.

The Poincaré section is a union of 5 pieces, which are, the $\mathbb{T}^d$-bundle restricted to the torus boundary $T_1$ (after we have already glued to get the fourth cover), and the torus bundle restricted to $D_i, i = 1, 2, 3, 4$. We denote the former as $S_0$ and the latter as $S_i, i = 1, 2, 3, 4$. The following should be more or less implicit in the original B-L paper, but we state for completeness.

Claim: There is a time $T > 0$ such that all the orbits going out from $S := \cup_{i=0}^{d} S_i$ return (at least once). We prove the claim by the following observations. Recall that we have fixed the radius $\varepsilon$ of the discs. The return times completely rely on the return times in the base, so we only need to consider the base.

1. Fix a large $T_1 >> T_0$. Because $(1,0)$ is a hyperbolic fixed point, if we think of the base of (we say the “base” of the circle bundle before we glue, since after the gluing the mainifold is not a circle bundle anymore) the B-L 3-manifold, there is another smaller open neighborhood $U_{T_1} \subset D_i$ such that all the points of $U_{T_1}$ return to $D_i$ after $T_1$.

2. Consider any orbits in one copy of B-L that do not intersect $U_{T_1}$ at all. Say the orbit comes out from $x \in T_i$. There is a time $T_2$ after which all such orbits reach $T_{i+1}$. This is because the manifold without an open disc is compact so the maximum is achieved.

3. $T_2$ is also an upperbound of the time needed for the orbits coming out from $T_i$ to reach $D_i$, and the orbits coming out from $D_i$ to reach $T_{i+1}$.

4. If an orbit going out from $T_i$ hits $D_i$ in time $T_2$, but does not return to $D_i$ in time $T_1$, it would hit $T_{i+1}$ in another $T_2$. So the time this orbit takes to go from $T_i$ to $T_{i+1}$ is bounded above by $T_1 + 2T_2$.

Then there is a uniform bound for the time $4(T_1 + 2T_2)$, when every orbit returns at least once.

2.5 The metric

We first carefully define a metric on the bundle restricted to $T_1$. Afterwards we define the metric on the bundles restricted to the discs. Then we are able to extend the metric to a metric on the entire manifold with a partition of unity.
2.6 The invariant subbundles

Observation 10. The orbits starting from and returning to the $\mathbb{T}^d$ bundle restricted to $T_1$ are homotopic to multiples of $\theta_i$’s and concatenating with the $c$’s.

Remark 11. The orbit of the B-L example that is not transverse to a torus is freely homotopic to its own inverse via a Klein bottle. This isotopy lifts to the bundle.

Also note that when chasing along an orbit of the B-L example below, we are not very careful with if the orbit is freely homotopic to $\theta$ or $\theta^{-1}$ (considering it in the covering space). This might cause confusion, but we want to point out that since $\theta$ and the conjugations of $\theta^{-1}$ involved produce the same matrix under the representation, it does not matter which representative we use. Hence
in addition, for a better picture in our minds, it is helpful to think of the \( \omega \in [0,2] \) part of the B-L 3-manifold, where \( f \geq 0 \) (also easier for computations).

We only need to check for the \( T^d \) coordinate that the flow is Anosov. Since in the covering space \( \hat{M} \times T^d \) we have constant cones tangent to the fibres along the orbits, we only need to check that the vectors are contracted or expanded as we want.

Now we write the return map \( R : S \to S \). Recall that it is the map that we computed in the last section in the base on \( S_0 \), on \( S_i, i = 1,2,3,4 \) the return map in the base is irrelevant. It is always the identity in the fibres.

With the following lemma we will finish the proof.

**Lemma 12.** The tangent space of the \( T^d \) fibres in \( S \) splits into invariant subspaces under \( R : S \to S \), i.e., \( T T^d = E^s \oplus E^u \) where \( E^s \) and \( E^u \) are the unstable and stable eigenspaces of \( C \) respectively (note that the swap of the stable and unstable is intentional), and there exists \( \lambda \in (0,1) \) such that

\[
\|dR(v)\| \leq \lambda \|v\|, \text{ if } v \in E^s, \text{ and}
\]

\[
\|dR^{-1}(v)\| \leq \lambda \|v\|, \text{ if } v \in E^u.
\]

**Proof.** Looking at diagram 1, there are all possibilities under \( R: S_0 \to S_0, S_0 \to S_i, S_i \to S_{i+1}, S_i \to S_0, i = 1,2,3,4 \).

If the orbit comes out from \( S_i \) and first returns to \( S_i \), we see that for \( R(x,y) = (x',y) \in U_{T_1} \times T^d \), from the metric we defined above,

\[
\|dR(v)\|_{(x',y)} = \|C^{-T_{t_0}}v\|_y, \quad v \in T^d, \quad T \text{ is the return time.}
\]

This shows that the orbits that travel from \( S_i \) to \( S_i \) are hyperbolic.

In the case where the orbit comes out from \( S_0 \) and first returns to \( S_0 \), let \( v \) be a vector tangent to the \( T^d \) fibre at \( (\omega,0) \in T_1 \). We have \( dR(v) = v \). Compute

\[
\|v\|_{(\omega,0,z)} = \|C^{-(t_2\omega)}v\|_{0,z}
\]

and

\[
\|dR(v)\|_{R(\omega,0,z)} = \|C^{-(t_2\omega-\bar{f}(\omega-\bar{f}(\omega)+\bar{f}(\omega-\bar{f}(\omega)))\cdot t_1}v\|_{0,z}, \text{ where } t_1 = -t_2 = 1
\]

\[
= \|C^{-(\bar{f}(\omega)-\bar{f}(\omega)+\bar{f}(\omega-\bar{f}(\omega)))\cdot(-\bar{f}(\omega)+\bar{f}(\omega-\bar{f}(\omega)))\cdot t_1}v\|_{0,z}
\]

Note that here \( t_1 = -t_2 \) we picked matches the negative sign coming out from the gluing \( A^{-1}(\omega,\theta) = (-\theta,\omega) \) in the first coordinate, which makes the stable and unstable subspaces match each other after we glue.

Recall we have picked \( \bar{f} \geq 0 \). If \( v \) lies in the eigenspace of \( C \) with eigenvalue \( > 1 \), we take \( \lambda \) a negative power \( < -4 \) of this eigenvalue. Then

\[
\|dR(v)\|_{R(\omega,0,z)} \leq \lambda \|v\|_{(\omega,0,z)},
\]

and \( v \) is a vector in the stable subspace of the return map. Similarly for the unstable.

The cases where we have \( S_0 \to S_i \) and \( S_i \to S_0 \) are symmetric, so we only consider \( S_0 \to S_i \). The orbit is still a concatenation of \( \theta_i \)'s and \( c_i \)'s, and the computation is exactly the same as the last case with less terms in the exponent of \( C \).

\[ \blacksquare \]

Note that in particular if \( v \) is a vector tangent to the fibre at \( (0,0) \in T_1 \), we have

\[
\|dR(v)\|_{(0,0,z)} = \|C^{-4}v\|_{0,z},
\]

which corresponds to the action along exactly the closed orbit homotopic to \( c^4 \).
3 The examples are non-algebraic

Proposition 13. The flow we constructed above is non-algebraic.

Proof. Let \( \Lambda \) denote the fundamental group of a nilmanifold. The fundamental group of a suspension of a nilmanifold automorphism is \( \Lambda \rtimes \mathbb{Z} \). The fundamental group of a nil-suspension is \( \Lambda \rtimes \rho \Gamma \) where \( \Gamma \) denotes the fundamental group of the unit tangent bundle of a hyperbolic surface, or a cocompact lattice in a rank 1 symmetric space of dimension \( \geq 3 \) (the fundamental group of whose unit tangent bundle is the same as its own), and \( \rho \) is a representation of \( \Gamma \) acting on \( \Lambda \).

We will always denote our base 3-manifold as \( M \) and the total space as \( E \) in this proof. We have the short exact sequence

\[
0 \rightarrow \mathbb{Z}^d \rightarrow \pi_1(E) \rightarrow \pi_1(M) \rightarrow 0,
\]

where \( d \geq 4 \).

Assume \( \pi_1(E) = \Lambda \rtimes \mathbb{Z} \) or \( \Lambda \rtimes \rho \Gamma \).

If \( \pi_1(E) \) is solvable, \( \pi_1(M) = \pi_1(E)/\mathbb{Z}^d \) is also solvable. But the fundamental group of the B-L example as a 3-manifold group satisfies the Tits alternative (as a consequence of the Geometrisation; see for example [BBB*10]) and contains a free group of rank 2 (in particular \( a_i, b_i \) generate \( F_2 \)), which cannot be virtually solvable. This is essentially the same argument as in [HT80].

We know that \( \mathbb{Z}^d \) is injective into \( \pi_1(E) \). Neither of the fundamental group of the unit tangent bundle of a surface or a hyperbolic space contains a rank 4 free abelian subgroup. Thus we also know that \( \mathbb{Z}^d \triangleleft \pi_1(E) \) implies \( \mathbb{Z}^d \triangleleft \Lambda \).

Claim: The universal cover \( \tilde{N} \) of the nilmanifold has an \( \mathbb{R}^d \) subgroup, so \( N \) has dimension \( \geq d \).

If the dimension of \( N \) is exactly \( d \), \( N \) has to be a torus.

To see this, note that the exponential map for a connected and simply connected nilpotent Lie group is a diffeomorphism. We lift basis elements \( \gamma_1, \gamma_2, \ldots, \gamma_d \) of \( \mathbb{Z}^d \) to elements \( X_1, X_2, \ldots, X_d \) of the Lie algebra \( \mathfrak{n} \) of \( \tilde{N} \). Because \( \exp([X_i, X_j]) = [\gamma_i, \gamma_j] = \text{id}, [X_i, X_j] = 0 \). Then \( \mathfrak{n} \) has a subalgebra \( \mathbb{R}^d \), and \( \tilde{N} \) has a subgroup \( \mathbb{R}^d \). Since \( \tilde{N} \) is connected, if \( \dim \tilde{N} = d \), then \( N = \mathbb{T}^d \).

Now we show that the fundamental group of the B-L example cannot be \( \Lambda \rtimes \rho \Gamma \).

The bundle from our construction is of dimension \( 3 + 2n \), \( n \geq 2 \) where \( d = 2n \). From the dimension counting, the nilmanifold fibres should have dimension \( \geq d \) and thus must be exactly a torus of dimension \( d = 2n \) because the dimension of the base is \( \geq 3 \). The base then has to be the unit tangent bundle of a real hyperbolic surface and \( \Gamma \) is its fundamental group.

From the short exact sequences,

\[
\begin{array}{ccc}
0 & \longrightarrow & \mathbb{Z}^d \\
\downarrow f & & \downarrow g \\
\Lambda & \longrightarrow & \Lambda \rtimes \rho \Gamma \\
\downarrow k & & \downarrow s \\
0 & \longrightarrow & \Gamma
\end{array}
\]

The first and second vertical arrows \( f \) and \( g \) are isomorphisms and \( f \) is induced by \( g \). Consider the section \( s \) of \( \Gamma \) into \( \Lambda \rtimes \rho \Gamma \), which is injective. Then there is a map \( h \circ g^{-1} \circ s \) from \( \Gamma \) onto \( \pi_1(M) \). We want to show that this is 1-to-1.

Suppose \( \gamma_1, \gamma_2 \in \Gamma \) and \( h \circ g^{-1} \circ s(\gamma_1) = h \circ g^{-1} \circ s(\gamma_2) \). Then \( (g^{-1} \circ s(\gamma_1))^{-1}(g^{-1} \circ s(\gamma_2)) = g^{-1} \circ s(\gamma_1 \gamma_2) \in i(\mathbb{Z}^d) \). But now \( g \circ g^{-1} \circ s(\gamma_1 \gamma_2) \in \Lambda \). Thus \( s(\gamma_1 \gamma_2) = \text{id} \) and \( \gamma_1 = \gamma_2 \).

Therefore \( k = h \circ g^{-1} \circ s \) is an isomorphism. This gives a contradiction.

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