Kloosterman Sums with Multiplicative Coefficients

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Abstract. Let $f(n)$ be a multiplicative function satisfying $|f(n)| \leq 1$, $q \leq N^2$ be a positive integer and $a$ be an integer with $(a, q) = 1$. In this paper, we shall prove that

$$\sum_{n \leq N, (n, q) = 1} f(n)e(\frac{\bar{n}a}{q}) \ll \sqrt{\frac{\tau(q)}{q}} N \log \log (6N) + q^{\frac{1}{2} + \frac{1}{2} + N^3} \frac{\log (6N)}{N} + \frac{N}{\sqrt{\log \log (6N)}}$$

where $\bar{n}$ is the multiplicative inverse of $n$ such that $\bar{n}n \equiv 1 \pmod{q}$, $e(x) = \exp(2\pi ix)$, $\tau(q)$ is the divisor function.

1. Introduction

Let $\mu(n)$ be the Möbius function, $q$ be a positive integer and $a$ be an integer with $(a, q) = 1$. In 1988, D. Hajela, A. Pollington and B. Smith [8] proved that

$$\sum_{n \leq N, (n, q) = 1} \mu(n)e(\frac{\bar{n}a}{q}) \ll \varepsilon Nq^{\frac{1}{2} + N^3} \frac{\log (6N)}{N^\frac{1}{2}}$$

where $\bar{n}$ is the multiplicative inverse of $n$ such that $\bar{n}n \equiv 1 \pmod{q}$, $e(x) = \exp(2\pi ix)$ and $\varepsilon$ is a sufficiently small positive constant. This estimate is nontrivial for $(\log N)^{\frac{5}{2} + 10\varepsilon} \ll q \ll N^{\frac{1}{2} - 3\varepsilon}$.

Later, P. Deng [4], G. Wang and Z. Zheng [9] independently improved the above estimate to

$$\sum_{n \leq N, (n, q) = 1} \mu(n)e(\frac{\bar{n}a}{q}) \ll \varepsilon N\tau(q) \frac{\log (6N)}{q^\frac{1}{2}} + \frac{q^{\frac{1}{2}} (\log N)^{\frac{13}{2}}}{N^\frac{1}{2}}$$

where $\tau(q)$ is the divisor function, which is nontrivial for $(\log N)^{\frac{5}{2} + \varepsilon} \ll q \ll N^{1 - \varepsilon}$. It was stated in [4] that under the Generalized Riemann Hypothesis,
one can get
\[
\sum_{n \leq N \atop (n, q) = 1} \mu(n)e\left(\frac{an}{q}\right) \ll \varepsilon q^{\frac{1}{2}N^{\frac{1}{3}} + \varepsilon}.
\]

We also mention some progress on the relative topic. In 1998, E. Fouvry and P. Michel [6] proved that if \( q \) is a prime number, \( g(x) = \frac{P(x)}{Q(x)} \) is any rational function with \( P(x) \) and \( Q(x) \) relatively prime monic polynomials in \( \mathbb{Z}[x] \), then for \( 1 \leq N \leq q \), one has
\[
\sum_{\substack{p \leq N \atop (p, q) = 1}} e\left(\frac{g(p)}{q}\right) \ll \varepsilon q^{\frac{1}{2}N^{\frac{1}{3}} + \varepsilon} N^{\frac{29}{48}},
\]
where \( p \) runs through prime numbers, the implied constant also depends on the degrees of \( P \) and \( Q \). This estimate is nontrivial for \( N \leq q \ll N^{\frac{2}{3} - \varepsilon} \).

It was stated in [6] that the same method can produce
\[
\sum_{n \leq N \atop (n, q) = 1} \mu(n)e\left(\frac{g(n)}{q}\right) \ll \varepsilon q^{\frac{1}{16} + \varepsilon} N^{\frac{29}{48}}
\]
for the prime number \( q \) and \( 1 \leq N \leq q \). Some further results can be found in [5].

In 2011, E. Fouvry and I. E. Shparlinski [7] proved that for \( (a, q) = 1 \) and \( N^{\frac{2}{3}} \leq q \leq N^{\frac{4}{5}} \), one has
\[
\sum_{\substack{N < p \leq 2N \atop (p, q) = 1}} e\left(\frac{ap}{q}\right) \ll \varepsilon q^{\frac{1}{4}N^{\frac{4}{5}} + N^{\frac{4}{15}}},
\]
which is nontrivial for \( N^{\frac{2}{3}} \leq q \ll N^{\frac{4}{5} - 6\varepsilon} \). They also proved that if \( (a, q) = 1 \), then
\[
\sum_{\substack{N < p \leq 2N \atop (p, q) = 1}} e\left(\frac{ap}{q}\right) \ll N \left(\tau\left(\frac{\log N}{2}\right) \frac{1}{q^{\frac{1}{2}}} + \tau(q) q^{\frac{1}{2}} \frac{1}{N^{\frac{1}{2}}}ight),
\]
which is nontrivial for \( \log N \ll q \ll N^{\frac{4}{5} - \varepsilon} \). In 2012, R. C. Baker [1] gave improvement under some conditions.

When the first author visited the University of Montreal, Professor A. Granville suggested him to study the general sum
\[
\sum_{n \leq N \atop (n, q) = 1} f(n)e\left(\frac{an}{q}\right),
\]
(1.1)
where $f(n)$ is a multiplicative function satisfying $|f(n)| \leq 1$.

In this paper, we shall apply the method in Section 2 of [3], which is called as the finite version of Vinogradov’s inequality, to give a nontrivial estimate for the sum in (1.1) when $q$ is in a suitable range.

**Theorem.** Let $f(n)$ be a multiplicative function satisfying $|f(n)| \leq 1$, $q \leq N^2$ be a positive integer and $a$ be an integer with $(a, q) = 1$. Then we have

$$\sum_{\substack{n \leq N \atop (n, q) = 1}} f(n)e\left(\frac{an}{q}\right) \ll \sqrt{\frac{\tau(q)}{q}} N \log \log(6N)$$

$$+ q^{\frac{1}{4} + \frac{\varepsilon}{2}} N^{\frac{1}{2}} (\log(6N))^{\frac{1}{2}} + \frac{N}{\sqrt{\log \log(6N)}}$$

The estimate in (1.2) is nontrivial for

$$(\log \log(6N))^{2 + \varepsilon} \ll q \ll N^{2 - 5\varepsilon}.$$

In a private communication, Ping Xi remarked that when $q$ is a prime number, if Lemma 2 below is replaced by Theorem 16 in [2], then the upper bound in the above nontrivial range can be extended to $q \ll N^A$, where $A$ is any given large constant.

Throughout this paper, we assume that $N$ is sufficiently large and set

$$d_0 = \sqrt{\log \log(6N)}, \quad D_0 = e^{d_0} = \exp(\sqrt{\log \log(6N)}),$$

$$d_1 = d_0^2 = \log \log(6N), \quad D_1 = e^{d_1} = \log(6N).$$

Let $p$ denote a prime number, $\tau(q)$ denote the divisor function, $\varepsilon$ be a sufficiently small positive constant.

### 2. Some preliminaries

Write

$$S = \{n : 1 \leq n \leq N, n \text{ has a prime factor in } [D_0, D_1]\},$$

$$T = \{n : 1 \leq n \leq N, n \text{ has no prime factor in } [D_0, D_1]\}.$$

**Lemma 1.** We have

$$|T| \ll \frac{N}{\sqrt{\log \log(6N)}}.$$
Proof. Let

\[ P(N) = \prod_{D_0 \leq p < D_1} p. \]

We have

\[ |T| = \sum_{n \leq N} 1 = \sum_{n \leq N} \sum_{d \mid n} \mu(d) \sum_{d \mid P(N)} 1 \]

\[ = \sum_{d \mid P(N)} \mu(d) \left( \frac{N}{d} + O(1) \right) \]

\[ = N \sum_{d \mid P(N)} \frac{\mu(d)}{d} + O\left(2^{\pi(D_1)}\right) \]

\[ = N \prod_{D_0 \leq p < D_1} \left( 1 - \frac{1}{p} \right) + O\left(\frac{2D_1}{\log D_1} \right) \]

\[ \ll N \frac{\log D_0}{\log D_1} + O\left(\frac{2 \log(6N)}{\log(6N)}\right) \]

\[ \ll \frac{N}{\sqrt{\log \log(6N)}}. \]

Hence, Lemma 1 holds true.

By Lemma 1, we have

\[ \sum_{n \leq N} f(n)e^{\frac{a \bar{n}}{q}} = \sum_{n \leq N} f(n)e^{\frac{\bar{n}}{q}} + O\left(\frac{N}{\sqrt{\log \log(6N)}}\right). \] (2.2)

Let

\[ P_r = \{ p : e^r \leq p < e^{r+1}, \quad \text{if} \quad [d_0] \leq r \leq [d_1]. \} \] (2.3)

Then

\[ \bigcup_{r = [d_0] + 1}^{[d_1] - 1} P_r \subseteq \{ p : D_0 \leq p < D_1 \} \subseteq \bigcup_{r = [d_0]}^{[d_1]} P_r. \]

The prime number theorem yields

\[ |P_r| \ll \frac{e^r}{r}. \] (2.4)
Write

\[ S' = \{ n : 1 \leq n \leq N, \ n \text{ has a prime factor in } \bigcup_{r = [d_0]}^{[d_1]} P_r \}, \]

\[ S'' = \{ n : 1 \leq n \leq N, \ n \text{ has a prime factor in } \bigcup_{r = [d_0]+1}^{[d_1]-1} P_r \}. \]

Then

\[ S'' \subseteq S \subseteq S'. \]

Hence,

\[ |S\setminus S''| \leq |S'\setminus S''| \ll \sum_{p \in P_0} \frac{N}{p} + \sum_{p \in P_{1]} \frac{N}{p} \]

\[ \ll N \left( \frac{|P_{[d_0]}|}{e^{[d_0]}} + \frac{|P_{[d_1]}|}{e^{[d_1]}} \right) \]

\[ \ll \frac{N}{d_0} = \frac{N}{\sqrt{\log \log (6N)}}. \]

We note that

\[ |\{ n : 1 \leq n \leq N, \ n \text{ has at least two prime factors in the same one of } P_r' ((d_0] + 1 \leq r \leq [d_1] - 1)\}| \]

\[ \ll \sum_{r = [d_0]+1}^{[d_1]-1} \sum_{p \in P_r} \sum_{p' \in P_r} \frac{N}{pp'} \]

\[ \ll N \sum_{r = [d_0]+1}^{[d_1]-1} \left( \frac{|P_r|}{e^{r}} \right)^2 \]

\[ \ll N \sum_{r = [d_0]+1}^{[d_1]-1} \frac{1}{r^2} \]

\[ \ll \frac{N}{d_0} = \frac{N}{\sqrt{\log \log (6N)}}. \]

Therefore for

\[ S''' = \{ n : 1 \leq n \leq N, \ n \text{ has exact one prime factor in one of } P_r' ((d_0] + 1 \leq r \leq [d_1] - 1)\}, \]

we have

\[ S''' \subseteq S''. \]
and 
\[ |S'' \setminus S'''| \ll \frac{N}{\sqrt{\log \log(6N)}}. \]

The set \( S''' \) can be decomposed as 
\[ S''' = \bigcup_{r = [d_0] + 1}^{[d_1]-1} S_r, \tag{2.5} \]
where
\[ S_r = \{ n : 1 \leq n \leq N, n \text{ has exact one prime factor in } P_r \text{ and has no prime factor in } \bigcup_{i < r} P_i \}. \tag{2.6} \]

By the prime number theorem, it is easy to see that each \( S_r \) \((r = [d_0] + 1, \ldots, [d_1] - 1)\) is not empty. The sets \( S_r \) are disjoint from each other.

Every element \( n \in S_r \) can be written in exact one way as 
\[ n = py, \tag{2.7} \]
where \( p \in P_r \), \( y \) has no prime factor in \( \bigcup_{i \leq r} P_i \), \( py \leq N \).

From the above discussion, we get
\[
\sum_{n \leq N} f(n)e\left(\frac{an}{q}\right) = \sum_{n \leq N, n \in S'''} f(n)e\left(\frac{an}{q}\right) + O\left(\frac{N}{\sqrt{\log \log(6N)}}\right)
\]
\[
= \sum_{r = [d_0] + 1}^{[d_1]-1} \sum_{n \leq N, n \in S_r} f(n)e\left(\frac{an}{q}\right) + O\left(\frac{N}{\sqrt{\log \log(6N)}}\right)
\]
\[
= \sum_{r = [d_0] + 1}^{[d_1]-1} \sum_{e^r \leq p < e^{r+1}} \sum_{y \leq N \atop (p, y) = 1} f(py)e\left(\frac{apy}{q}\right) + O\left(\frac{N}{\sqrt{\log \log(6N)}}\right) \quad \tag{2.8}
\]

\[ + O\left(\frac{N}{\sqrt{\log \log(6N)}}\right) \]
$$\sum_{r=|d_0|+1}^{[d_1]-1} \sum_{y \leq \frac{N}{p^e}} f(y) \sum_{e^r \leq p < e^{r+1}} f(p) e\left(\frac{ap\overline{y}}{q}\right) + O\left(\frac{N}{\sqrt{\log \log(6N)}}\right)$$

$$\ll \sum_{r=|d_0|+1}^{[d_1]-1} \sum_{y \leq \frac{N}{p^e}} f(y) \sum_{e^r \leq p < e^{r+1}} f(p) e\left(\frac{ap\overline{y}}{q}\right) + \frac{N}{\sqrt{\log \log(6N)}}.$$

Let

$$Y = \frac{N}{e^r}. \quad (2.9)$$

We shall estimate the sum

$$\sum_1 = \sum_{y \leq Y} \left| \sum_{e^r \leq p < e^{r+1}} f(p) e\left(\frac{ap\overline{y}}{q}\right) \right|. \quad (2.10)$$

**Lemma 2.** For the positive integer $q$ and the integer $b$, we have

$$\sum_{X < n \leq Z \atop (n, q) = 1} e\left(\frac{bn\overline{q}}{q}\right) \ll \left(\frac{Z - X}{q} + 1\right)(b, q) + q^{1/2} + \varepsilon. \quad (2.11)$$

**Proof.** Lemma 2.1 in [7] states that

$$\sum_{X < n \leq Z \atop (n, q) = 1} e\left(\frac{bn\overline{q}}{q}\right) \ll \mu^2\left(\frac{q}{(b, q)}\right)\left(\frac{Z - X}{q} + 1\right) \cdot \frac{\varphi(q)}{\varphi\left(\frac{q}{(b, q)}\right)}$$

$$+ \tau(q) \tau((b, q)) \log(2q)q^{1/2}.$$ 

Then the bounds

$$\frac{\varphi(q)}{\varphi\left(\frac{q}{(b, q)}\right)} = q \prod_{p|q} \left(1 - \frac{1}{p}\right) \cdot \left(\frac{q}{(b, q)} \prod_{p|\varphi(q)} \left(1 - \frac{1}{p}\right)^{-1}\right.\left.\right)$$

$$= (b, q) \prod_{p|q} \left(1 - \frac{1}{p}\right) \leq (b, q)$$

and

$$\tau(q) \ll q^{\frac{6}{7}}$$

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produce the conclusion in Lemma 2.

3. The proof of Theorem

By Cauchy’s inequality,

$$
\sum_1 \leq \frac{Y^\frac{1}{2}}{2} \left( \sum_{y \leq Y} \sum_{y' \leq p < e^{r+1}} f(p) e\left(\frac{ap_1}{q} \right)^2 \right)^{\frac{1}{2}}.
$$

(3.1)

An application of Lemma 2 to

$$
\sum_2 = \sum_{y \leq Y} \sum_{y' \leq p < e^{r+1}} f(p) e\left(\frac{ap_1}{q} \right)^2
$$

produces

$$
\sum_2 = \sum_{y \leq Y} \sum_{y' \leq p < e^{r+1}} f(p_1) f(p_2) e\left(\frac{a(p_1 - p_2)y}{q} \right)
$$

$$
\leq \sum_{e^{r} \leq p_1 < e^{r+1}} \sum_{(p_1, q) = 1} e\left(\frac{a(p_1 - p_2)y}{q} \right)
$$

$$
\leq \sum_{e^{r} \leq p_1 < e^{r+1}} \sum_{(p_1, q) = 1} e\left(\frac{a(p_1 - p_2)y}{q} \right)
$$

$$
\leq Ye^r + \sum_{e^{r} \leq p_1 < e^{r+1}} \sum_{(p_1, q) = 1} \left( \frac{Y}{q} + 1 \right) (a(p_1 - p_2), q) + q^{\frac{1}{2} + \epsilon}
$$

$$
\leq Ye^r + \left( \frac{Y}{q} + 1 \right) \sum_{e^{r} \leq p_1 < e^{r+1}} \sum_{(p_1, q) = 1} (p_1 - p_2, q) + q^{\frac{1}{2} + \epsilon} e^{2r}.
$$
We have
\[ \sum_{\substack{\ell \leq p_1 < \ell + 1 \\ (p_1, q) = 1}} \sum_{\substack{\ell \leq p_2 < \ell + 1 \\ p_2 \neq p_1 \atop (p_2, q) = 1}} (\bar{p}_1 - \bar{p}_2, q) \]
\[ = \sum_{k \mid q} k \sum_{\ell \leq p_1 < \ell + 1} \sum_{\substack{\ell \leq p_2 < \ell + 1 \\ p_2 \neq p_1 \atop (p_2, q) = 1}} 1 \]
\[ \leq \sum_{k \mid q} k \sum_{\ell \leq p_1 < \ell + 1} \sum_{\substack{\ell \leq p_2 < \ell + 1 \\ p_2 \neq p_1 \atop (p_2, q) = 1}} 1. \]

(3.3)

In the above sum, if \( k \geq e^{r + 1} \), then \( p_2 \equiv p_1 \) (mod \( k \)) and \( p_1, p_2 < e^{r + 1} \implies p_2 = p_1 \), which contradicts the fact \( p_2 \neq p_1 \). Hence, it follows that
\[ \sum_{k \mid q} k \sum_{\ell \leq p_1 < \ell + 1} \sum_{\substack{\ell \leq p_2 < \ell + 1 \\ p_2 \neq p_1 \atop (p_2, q) = 1}} 1 \]
\[ = \sum_{k \mid q} k \sum_{\ell \leq p_1 < \ell + 1} \sum_{\substack{\ell \leq p_2 < \ell + 1 \\ p_2 \neq p_1 \atop (p_2, q) = 1}} 1 \]
\[ \leq \sum_{k \mid q} k \sum_{\ell \leq p_1 < \ell + 1} \sum_{\substack{n_1 < \ell + 1 \\ n_2 = n_1 \, (\text{mod} \, k)}} 1 \]
\[ \ll \sum_{k \mid q} k \cdot e^{r + 1} \cdot \frac{e^{r + 1}}{k} \]
\[ \ll \tau(q) e^{2r}. \]

Thus we get the estimate
\[ \sum_{\substack{\ell \leq p_1 < \ell + 1 \\ (p_1, q) = 1}} \sum_{\substack{\ell \leq p_2 < \ell + 1 \\ p_2 \neq p_1 \atop (p_2, q) = 1}} (\bar{p}_1 - \bar{p}_2, q) \ll \tau(q) e^{2r}. \]

(3.4)
By the above discussion, we have

\[ \sum_2 \ll Y e^r + \left( \frac{Y}{q} + 1 \right) \tau(q) e^{2r} + q^{\frac{1}{2} + \varepsilon} e^{2r} \]
\[ \ll \frac{\tau(q)}{q} Y e^{2r} + Y e^r + q^{\frac{1}{2} + \varepsilon} e^{2r}. \]

It follows that

\[ \sum_1 \ll Y^{\frac{1}{2}} \left( \frac{\tau(q)}{q} Y e^{2r} + Y e^r + q^{\frac{1}{2} + \varepsilon} e^{2r} \right)^{\frac{1}{2}} \]
\[ \ll \sqrt{\frac{\tau(q)}{q}} Y e^r + Y e^{\frac{r}{2}} + Y^{\frac{1}{2}} q^{\frac{1}{2} + \varepsilon} e^{r} \]
\[ \ll \sqrt{\frac{\tau(q)}{q}} N + \frac{N}{e^{\frac{r}{2}}} + q^{\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \varepsilon} e^{\frac{r}{2}}. \]

Applying this estimate to (2.8), we get

\[
\sum_{\substack{n \leq N \\ (n, q) = 1}} f(n) e\left(\frac{an}{q}\right) \\
\ll \left[ \sum_{r = [d_0] + 1}^{[d_1] - 1} \left( \sqrt{\frac{\tau(q)}{q}} N + \frac{N}{e^{\frac{r}{2}}} + q^{\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \varepsilon} e^{\frac{r}{2}} \right) \right] + \frac{N}{\sqrt{\log \log (6N)}} \]
\ll \sqrt{\frac{\tau(q)}{q}} N \log \log (6N) + \frac{N}{\exp\left(\frac{1}{2} \sqrt{\log \log (6N)}\right)} \]
\[ + q^{\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \varepsilon} N^{\frac{1}{2}} (\log (6N))^{\frac{1}{2}} + \frac{N}{\sqrt{\log \log (6N)}} \]
\ll \sqrt{\frac{\tau(q)}{q}} N \log \log (6N) + q^{\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \varepsilon} N^{\frac{1}{2}} (\log (6N))^{\frac{1}{2}} + \frac{N}{\sqrt{\log \log (6N)}}. \]

So far the proof of Theorem is complete.

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