Abstract. We show that a weighted homogeneous complex surface singularity is metrically conical (i.e., bi-Lipschitz equivalent to a metric cone) only if its two lowest weights are equal. We also give an example of a pair of weighted homogeneous complex surface singularities that are topologically equivalent but not bi-Lipschitz equivalent.

1. Introduction and main results

A natural question of metric theory of singularities is the existence of a metrically conical structure near a singular point of an algebraic set. For example, complex algebraic curves, equipped with the inner metric induced from an embedding in $\mathbb{C}^N$, always have metrically conical singularities. It was discovered recently (see [1]) that weighted homogeneous complex surface singularities are not necessarily metrically conical. In this paper we show that they are rarely metrically conical.

Let $(V, p)$ be a normal complex surface singularity germ. Any set $z_1, \ldots, z_N$ of generators for $\mathcal{O}_{(V,p)}$ induces an embedding of germs $(V, p) \rightarrow (\mathbb{C}^N, 0)$. The Riemannian metric on $V - \{p\}$ induced by the standard metric on $\mathbb{C}^N$ then gives a metric space structure on the germ $(V, p)$. This metric space structure, in which distance is given by arclength within $V$, is called the inner metric (as opposed to outer metric in which distance between points of $V$ is distance in $\mathbb{C}^N$).

It is easy to see that, up to bi-Lipschitz equivalence, this inner metric is independent of choices. It depends strongly on the analytic structure, however, and may not be what one first expects. For example, we shall see that if $(V, p)$ is a quotient singularity $(V, p) = (\mathbb{C}^2/G, 0)$, with $G \subset U(2)$ finite acting freely, then this metric is usually not bi-Lipschitz equivalent to the conical metric induced by the standard metric on $\mathbb{C}^2$.

If $M$ is a smooth compact manifold then a cone on $M$ will mean the cone on $M$ with a standard Riemannian metric off the cone point. This is the metric completion of the Riemannian manifold $\mathbb{R}_+ \times M$ with metric given (in terms of an element of arc length) by $ds^2 = dt^2 + t^2 ds^2_M$ where $t$ is the coordinate on $\mathbb{R}_+$ and $ds^2_M$ is given by any Riemannian metric on $M$. It is easy to see that this metric completion simply adds a single point at $t = 0$ and, up to bi-Lipschitz equivalence, the metric on the cone is independent of choice of metric on $M$.

If $M$ is the link of an isolated complex singularity $(V, p)$ then the germ $(V, p)$ is homeomorphic to the germ of the cone point in a cone $CM$. If this homeomorphism

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can be chosen to be bi-Lipschitz we say, following \cite{Birbrair2011}, that the germ \((V, p)\) is \textit{metrically conical}. In \cite{Birbrair2011} the approach taken is to consider a semialgebraic triangulation of \(V\) and consider the star of \(p\) according to this triangulation. The point \(p\) is metrically conical if the intersection \(V \cap B_r[p]\) is bi-Lipschitz homeomorphic to the star of \(p\), considered with the standard metric of the simplicial complex.

Suppose now that \((V, p)\) is weighted homogeneous. That is, \(V\) admits a good \(\mathbb{C}^*\)-action (a holomorphic action with positive weights: each orbit \(\{\lambda x \mid \lambda \in \mathbb{C}^*\}\) approaches zero as \(\lambda \to 0\)). The weights \(v_1, \ldots, v_r\) of a minimal set of homogeneous generators of the graded ring of \(V\) are called the \textit{weights of} \(V\). We shall order them by size, \(v_1 \geq \cdots \geq v_r\), so \(v_{r-1}\) and \(v_r\) are the two lowest weights.

If \((V, p)\) is a cyclic quotient singularity \(V = \mathbb{C}^2/\mu_n\) (where \(\mu_n\) denotes the \(n\)-th roots of unity) then it does not have a unique \(\mathbb{C}^*\)-action. In this case we use the \(\mathbb{C}^*\)-action induced by the diagonal action on \(\mathbb{C}^2\).

If \((V, p)\) is homogeneous, that is, the weights \(v_1, \ldots, v_r\) are all equal, then it is easy to see that \((V, p)\) is metrically conical.

\textbf{Theorem 1.} \textit{If the two lowest weights of} \(V\) \textit{are unequal then \((V, p)\) is not metrically conical.}

For example, the Kleinian singularities \(A_k, k \geq 1, D_k, k \geq 4, E_6, E_7, E_8\) are the quotient singularities \(\mathbb{C}^2/G\) with \(G \subset SU(2)\) finite. The diagonal action of \(\mathbb{C}^*\) on \(\mathbb{C}^2\) induces an action on \(\mathbb{C}^2/G\), so they are weighted homogeneous. They are the weighted homogeneous hypersurface singularities:

- \(A_k: x^2 + y^2 + z^{k+1} = 0\) \((k + 1, k + 1, 2)\) or \((k + 1, \frac{k+1}{2}, 1)\)
- \(D_k: x^2 + y^2 z + z^{k-1} = 0\) \((k - 1, k - 2, 2), k \geq 4\)
- \(E_6: x^2 + y^3 + z^4 = 0\) \((6, 4, 3)\)
- \(E_7: x^2 + y^3 + y^3 z = 0\) \((9, 6, 4)\)
- \(E_8: x^2 + y^3 + z^5 = 0\) \((15, 10, 6)\)

By the theorem, none of them is metrically conical except for the quadric \(A_1\) and possibly \(E_7\) the quaternion group quotient \(D_4\).

The general cyclic quotient singularity is of the form \(V = \mathbb{C}^2/\mu_n\) where the \(n\)-th roots of unity act on \(\mathbb{C}^2\) by \(\xi(u_1, u_2) = (\xi^q u_1, \xi u_2)\) for some \(q\) prime to \(n\) with \(0 < q < n\); the link of this singularity is the lens space \(L(n, q)\). It is homogeneous if and only if \(q = 1\).

\textbf{Theorem 2.} \textit{A cyclic quotient singularity is metrically conical if and only if it is homogeneous.}

Many non-homogeneous cyclic quotient singularities have their two lowest weights equal, so the converse to Theorem 1 is not generally true.

We can also sometimes distinguish weighted homogeneous singularities with the same topology from each other.

\textbf{Theorem 3.} Let \((V, p)\) and \((W, q)\) be two weighted homogeneous normal surface singularities, with weights \(v_1 \geq v_2 \geq \cdots \geq v_r\) and \(w_1 \geq w_2 \geq \cdots \geq w_s\) respectively. If either \(\frac{v_1}{v_r} > \frac{w_1}{w_s}\) or \(\frac{w_1}{w_s} > \frac{v_1}{v_r}\) then \((V, p)\) and \((W, q)\) are not bi-Lipschitz homeomorphic.

\footnote{We have a tentative proof that the quaternion quotient is metrically conical, see \cite{Birbrair2011}.}
Corollary 4. Let \( V, W \subset \mathbb{C}^3 \) be defined by \( V = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : z_1^2 + z_2^{12} + z_3^{102} = 0\} \) and \( W = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : z_1^{12} + z_2^{35} + z_3^{20} = 0\} \). Then, the germs \( (V, 0) \) and \( (W, 0) \) are homeomorphic, but they are not bi-Lipschitz homeomorphic.

The corollary follows because in both cases the link of the singularity is an \( S^1 \) bundle of Euler class \(-1\) over a curve of genus 26; the weights are \((51, 2, 1)\) and \((5, 4, 3)\) respectively and Theorem 2 applies since \( \frac{2}{5} > \frac{2}{3} \).

The idea of the proof of Theorem 3 is to find an essential closed curve in \( V - \{p\} \) with the property that as we shrink it towards \( p \) using the \( \mathbb{R}^* \) action, its diameter shrinks faster than it could if \( V \) were bi-Lipschitz equivalent to a cone. Any essential closed curve in \( V - \{p\} \) that lies in the hyperplane section \( z_r = 1 \) will have this property, so we must show that the hyperplane section contains such curves. The proofs of Theorems 2 and 3 are similar.

2. Proofs

Let \( z_1, \ldots, z_r \) be a minimal set of homogeneous generators of the graded ring of \( V \), with \( z_i \) of weight \( v_i \) and \( v_1 \geq v_2 \geq \cdots \geq v_{r-1} \geq v_r \). Then \( x \mapsto (z_1(x), \ldots, z_r(x)) \) embeds \( V \) in \( C^r \). This is a \( C^* \)-equivariant embedding for the \( C^* \)-action on \( C^r \) given by \( z(z_1, \ldots, z_r) = (z^{v_1} z_1, \ldots, z^{v_r} z_r) \).

Consider the subset \( V_0 := \{x \in V \mid z_r(x) = 1\} \) of \( V \). This is a nonsingular complex curve.

Lemma 2.1. Suppose \((V, p)\) is not a homogeneous cyclic quotient singularity. Then for any component \( V'_0 \) of \( V_0 \) the map \( \pi_1(V'_0) \rightarrow \pi_1(V - \{p\}) \) is non-trivial.

Proof. Denote \( v = lcm(v_1, \ldots, v_r) \). A convenient version of the link of the singularity is given by

\[
M = S \cap V \quad \text{with} \quad S = \{z \in C^r \mid |z_1|^{2v/v_1} + \cdots + |z_r|^{2v/v_r} = 1\}.
\]

The action of \( S^1 \subset C^* \) restricts to a fixed-point free action on \( M \). If we denote the quotient \( M/S^1 = (V - \{p\})/C^* \) by \( P \) then the orbit map \( M \rightarrow P \) is a Seifert fibration, so \( P \) has the structure of an orbifold. The orbit map induces a surjection of \( \pi_1(V - \{p\}) = \pi_1(M) \) to the orbifold fundamental group \( \pi_1^{orb}(P) \) (see eg \( \mathbb{R}^\infty \)) so the lemma will follow if we show the image of \( \pi_1(V'_0) \) in \( \pi_1^{orb}(P) \) is nontrivial.

Denote \( V_r := \{z \in V \mid z_r \neq 0\} \) and \( P_r := \{z \in P \mid z_r \neq 0\} \) and \( \pi : V \rightarrow P \) the projection. Each generic orbit of the \( C^* \)-action on \( V_r \) meets \( V_0 \) in \( v_r \) points; in fact the \( C^* \)-action on \( V_r \) restricts to an action of \( \mu_{v_r} \) (the \( v_r \)-th roots of unity) on \( V_0 \), and \( V_0/\mu_{v_r} = V_r/C^* \) is cyclic, so in particular the image of any component \( V'_0 \) of \( V_0 \) is a sub orbifold with cyclic quotient. On the other hand the map \( \pi_1^{orb}(P_r) \rightarrow \pi_1^{orb}(P) \) is surjective, since \( P_r \) is the complement of a finite set of points in \( P \). Hence, the image of \( \pi_1(V'_0) \) in \( \pi_1^{orb}(P) \) is a normal subgroup with cyclic quotient. Thus the lemma follows if \( \pi_1^{orb}(P) \) is not cyclic.

If \( \pi_1^{orb}(P) \) is cyclic then \( P \) is a 2–sphere with at most two orbifold points, so the link \( M \) must be a lens space, so \((V, p)\) is a cyclic quotient singularity, say \( V = C^2/\mu_n \). Here \( \mu_n \) acts on \( C^2 \) by \( \xi(u_1, u_2) = (\xi^q u_1, \xi u_2) \) with \( \xi = e^{2\pi i/n} \), for some \( 0 < q < n \) with \( q \) prime to \( n \).

Recall that we are using the diagonal \( C^* \)-action. The base orbifold is then \( (C^2/\mu_n)/C^* = (C^2/C^*)/\mu_n = P^1/C/\mu_n \). Note that \( \mu_n \) may not act effectively on
which all have equal weight, and \((\mathbb{Z}/n')\), which is a 2-sphere with two degree \(n'/d\) cone points.

So the actual action is by a cyclic group of order \(n' := n/d\) and the orbifold \(P\) is \(P^1\mathbb{C}/(\mathbb{Z}/n')\), which is a 2-sphere with two degree \(n'/d\) cone points.

Proof of Theorem 2. Assume \(v_{r-1}/v_r > 1\). By Lemma 2.1 we can find a closed curve \(\gamma\) in \(V_0\) which represents a non-trivial element of \(\pi_1(V - \{p\})\). Suppose we have a bi-Lipschitz homeomorphism \(h\) from a neighborhood of \(p\) in \(V\) to a neighborhood in the cone \(CM\). Using the \(\mathbb{R}_+\)–action on \(V\), choose \(\epsilon > 0\) small enough that \(t\gamma\) is in the given neighborhood of \(p\) for \(0 < t \leq \epsilon\).

Consider the map \(H\) of \([0,1]\times[0,\epsilon]\) to \(V\) given by \(H(s,t) = t^{-\epsilon}h(t\gamma(s))\). Here \(t\gamma(s)\) refers to \(t\) \(\mathbb{R}_+\)–action on \(V\), and \(t^{-\epsilon}h\) refers to \(\mathbb{R}_+\)–action on \(CM\). Note that the coordinate \(z_\epsilon\) is constant equal to \(t^\epsilon\) on each \(t\gamma\) and the other coordinates have been multiplied by at most \(t^{\epsilon-1}\). Hence, for each \(t\) the curve \(t\gamma\) is a closed curve of length of order bounded by \(t^{\epsilon-1}\), so \(h(t\gamma)\) has length of the same order, so \(t^{-\epsilon}h(t\gamma)\) has length of order \(t^{\epsilon-1-\epsilon}\). This length approaches zero as \(t \to 0\), so \(H\) extends to a continuous map \(H'\) of \([0,1]\times[0,\epsilon]\) to \(V\) for which \(H([0,1]\times\{0\})\) is a point. Note that \(t^{-\epsilon}h(t\gamma)\) is never closer to \(p\) than distance \(1/K\), where \(K\) is the bi-Lipschitz constant of \(h\), so the same is true for the image of \(H'\). Thus \(H'\) is a null-homotopy of \(\epsilon\gamma\) in \(V - \{p\}\), contradicting the fact that \(\gamma\) was homotopically nontrivial.

Proof of Theorem 3. Suppose \((V,p)\) is a non-homogeneous cyclic quotient singularity, as in the proof of Lemma 2.1 and suppose Theorem 3 does not apply, so the two lowest weights are equal (in the notation of that proof this happens, for example, if \(n = 4k\) and \(q = 2k + 1\) for some \(k > 1\): the generators of the ring of functions of lowest weight are \(u_1u_2^{2k-1}, u_1^2u_2^{2k-3}, \ldots\), of weight \(2k\)). Let \(u_1^2u_2^k\) be the generator of lowest weight that has smallest \(u_1\)–exponent and choose this one to be the coordinate \(z_\epsilon\) in the notation of Lemma 2.1. Consider now the \(\mathbb{R}_+\)–action induced by the action \(t(u_1,u_2) = (t^\epsilon u_1,t^\beta u_2)\) on \(\mathbb{C}^2\) for some pairwise prime pair of positive
Let us suppose that $v \geq 1$. With respect to this $\mathbb{C}^*$-action the weight $\alpha a' + \beta b'$ of any generator $u_1^{a'}u_2^{b'}$ with $a' > 1$ will be greater than the weight $\alpha a + \beta b$ of $z_r$ (since $a' + b' \geq a + b$, which implies $\alpha a' + \beta b' = \alpha a + \alpha(a' - a) + \beta b' > \alpha a + \beta (a' - a) + \beta b' \geq \alpha a + \beta b$). On the other hand, any generator $u_1^{a'}u_2^{b'}$ with $a' < a$ had $a' + b' > a + b$ by our choice of $z_r$, and if $\alpha/\beta$ is chosen close enough to 1 we will still have $\alpha a' + \beta b' > \alpha a + \beta b$, so it will still have larger weight than $z_r$. Thus $z_r$ is then the unique generator of lowest weight, so we can carry out the proof of Theorem 1 using this $\mathbb{C}^*$-action to prove non-conicalness of the singularity.

**Proof Theorem 1.** Let $h: (V, p) \to (W, q)$ be a $K$-bi-Lipschitz homeomorphism. Let us suppose that $\frac{\alpha}{\beta} > \frac{w_1}{w_2}$. Let $\gamma$ be a loop in $V_0$ representing a non-trivial element of $\pi_1(V - \{p\})$ (see Lemma 2.2). We choose $\epsilon$ as in the previous proof. For $t \in (0, \epsilon]$ consider the curve $t\gamma$, where $t\gamma$ refers to $\mathbb{R}_+^*\gamma$-action on $V$. Its length $l(t\gamma)$, considered as a function of $t$, has the order bounded by $t^{w_r-1}$. The distance of the curve $t\gamma$ from $p$ is of order $t^{v_r}$. Since $h$ is a bi-Lipschitz map, we obtain the same estimates for $h(t\gamma)$. Since the smallest weight for $W$ is $w_s$, the curve $t^{-w_r/w_s}h(t\gamma)$ will be distance at least $1/K$ from $p$. Moreover its length will be of order at most $t^{-w_1/v_r}/w_s l(t\gamma)$ which is of order $t^{w_r-1-w_1/v_r}/w_s$. This approaches zero as $t \to 0$ so, as in the previous proof, we get a contradiction to the non-triviality of $[\gamma] \in \pi_1(V - \{p\}) = \pi_1(W - \{q\})$. By exchanging the roles of $V$ and $W$ we see that $\frac{\alpha}{\beta} > \frac{w_1}{w_2}$ also leads to a contradiction. \qed

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