ON THE CONNECTION BETWEEN SUPERVISED LEARNING AND LINEAR INVERSE PROBLEMS

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ABSTRACT. In this paper we investigate the connection between supervised learning and linear inverse problems. We first show that a linear inverse problem can be viewed as a function approximation problem in a reproducing kernel Hilbert space (RKHS) and then we prove that to each of these approximation problems corresponds a class of inverse problems. Analogously, we show that Tikhonov solutions of this class correspond to the Tikhonov solution of the approximation problem. Thanks to this correspondence, we show that supervised learning and linear discrete inverse problems can be thought of as two instances of the approximation problem in a RKHS. These instances are formalized by means of a sampling operator which takes into account both deterministic and random samples and leads to discretized problems. We then analyze the discretized problems and we study the convergence of their solutions to the ones of the approximation problem in a RKHS, both in the deterministic and statistical framework. Finally, we prove there exists a relation between the convergence rates computed with respect to the noise level and the ones computed with respect to the number of samples. This allows us to compare upper and lower bounds given in the statistical learning and in the deterministic infinite dimensional inverse problems theory.

1. INTRODUCTION

We consider the inverse problem of recovering a function $f$ such that

$$y = Af$$

where $f$ and $y$ are elements of Hilbert spaces and $A$ is a linear operator. For estimating $f$ one can consider to have noisy infinite dimensional data, e.g. $y^\delta$ such that $\|y^\delta - y\| \leq \delta$, or, more realistically, finite dimensional noisy samples $\{y_1, \ldots, y_n\}$ taken at points $\{x_1, \ldots, x_n\}$. Solutions of inverse problems are usually achieved by using regularization methods, i.e. methods with specific convergence properties of the $L^2$-norm of the error when $\delta$ goes to 0.

On the other hand, we consider the supervised learning problem of finding a function $g$ from a set of examples $\{(X_i, Y_i)\}_{i=1,\ldots,n}$ randomly drawn from an unknown probability distribution $\rho$. The function $g$ has to explain the relationship between input-output, i.e.

$$Y_i \sim g(X_i)$$

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for all \(i = 1, \ldots, n\) and \(g(x)\) has to be a good estimate of the output when a new input \(x\) is given. In statistical learning the emphasis is on the consistency of estimators of \(g\) and the convergence is required in expectation (or in probability) when the number of examples \(n\) goes to infinity.

One of the main evidences of the connection occurring across these two problems is that regularization methods, usually developed for inverse problems, such as the spectral regularization, can be used for solving learning problems \([22, 39]\). More in general, it is possible to apply regularization, e.g. \(\ell_1\)-penalized methods, for solving both learning and inverse problems \([17]\). Furthermore, a recent trend is to use neural networks, a common tool for learning problems, for solving inverse problems \([24, 1]\). The fact that these problems can be solved using the same methods begs the question of to what extent they are similar and which are the key points of the connection. In this paper we discuss the connection at three levels: at the infinite dimensional level where both problems can be described as function approximation problems; at the discrete level where the two problems differentiate according to the nature of the data; at the level of convergence rates which have been considered under the same source conditions but different noise hypotheses.

1.1. Our contribution. One of the main differences between supervised learning and discrete inverse problems comes from the hypothesis on available data: in the first case, data are usually assumed to be given as the result of a stochastic process whose underlying distribution is unknown. On the contrary, in the second case, data are assumed to be given according to a deterministic scheme, at least for the independent variables and even when the dependent variables are assumed to be drawn in a stochastic manner, the underlying distribution is supposed to be known. As the difference concerns hypotheses on the discrete data and our aim is to identify the key points of the connection, the starting point of this work is to consider the two problems at the infinite dimensional level. Indeed, the fact that the range of a bounded operator is provided with a Reproducing Kernel Hilbert Space (RKHS) structure in a natural way (see \([19]\)) allows us to describe the two problems as the same function approximation problem. Here, we define the function approximation in a RKHS as an optimization problem. In particular, by introducing a suitable (non-linear) generalization of the Moore-Penrose inverse, we prove that the solution of an approximation problem in a RKHS can always be associated with a solution of a certain inverse problem. Conversely, we prove that the set of solutions of a class of inverse problems corresponds to the solution of a certain approximation problem in a RKHS. This set is defined up to the action of the unitary group. Moreover, as in both frameworks Tikhonov regularization is used for obtaining stable solutions, we prove that there exists a correspondence between Tikhonov solutions of the approximation problem in a RKHS and of their corresponding inverse problems.

Then, we define a sampling operator for deriving both learning and inverse problems from the infinite dimensional formulation. The peculiarity of this sampling operator is that it can take into account both deterministic and stochastic samples and it is an extension of the sampling operator defined in \([31]\). By means of this sampling operator, supervised learning and inverse problems can be thought of as two instances of the same infinite dimensional approximation problem. We then analyze such discretized problems in both deterministic and stochastic frameworks and we study the convergence of their solutions to the ones of the original approximation problem. The study of the convergence has to be conducted under different hypotheses according to the statistical or deterministic context. In the statistical framework, the convergence is well-studied and is related
to the argmax continuous theorem [35], which guarantees the convergence of the solutions of the discretized problems to the ideal ones. In the deterministic case, we show the convergence of solutions of the discrete inverse problems to ideal ones in a general setting by means of the fundamental Gamma convergence theorem [9].

Finally, we analyze the relationship between error convergence rates arising from the analysis of inverse problems where the error is introduced as a bounded infinite dimensional perturbation of the data, say $\delta$, and (inverse) learning problems where estimates of the solution are computed with a finite number of samples, say $n$. We introduce an estimator starting from a finite number of random samples with two properties: first, under the usual source conditions, it shares the same optimal rates of the spectral regularizer for learning problems, and second its rates are related to the ones of the classical spectral regularization for deterministic inverse problems. In particular, for this estimator we prove an inequality between the infinite dimensional error given a certain degree of noise $\delta$ and the expected error given $n$ samples. Such an inequality allows us to convert upper bounds with respect to the number of samples $n$ (typically analyzed in the statistical inverse framework [37, 8]) to upper bounds with respect to the noise level $\delta$ (typically analyzed in the deterministic inverse framework [14]) and conversely lower bounds depending on $\delta$ to lower bounds depending on $n$. A further contribution of this work is then to show that the optimal rate obtained in statistical learning is worse than the optimal one obtained in the deterministic error analysis, quantifying the difference between optimal rates in the two frameworks.

1.2. Related works. In the literature several authors proposed to solve learning problems by using regularization techniques originally developed for inverse problems, offering a glimpse of the connection between supervised learning and inverse problems [15, 32, 31, 20, 11, 25, 38]. In recent years, a rigorous formalization of this connection between supervised learning and linear inverse problems have been proposed according to two strategies: the first considers the learning problem as an instance of an inverse one (see e.g. [13, 22]) whereas the second introduces a bounded operator in the model equation of the statistical learning and it is known as inverse learning (see e.g. [23, 8, 28]). The first strategy interprets a learning problem as an inverse one in which the forward operator is an inclusion and its main objective is to draw a connection between consistency in kernel learning and regularization in inverse problems, without dealing with convergence rates. On the other hand, the second strategy considers inverse problems from a statistical estimation perspective highlighting the fact that statistical inverse problems can be thought of as learning problems starting from indirect data. In this case, under appropriate probabilistic source conditions, upper and lower bounds of convergence rates are provided for predictive and estimation error of spectral methods. The common thread between these two studies is to prioritize the learning context. Indeed, in both these approaches data are samples randomly drawn from an unknown distribution, the typical assumption of the learning framework. However, as far as inverse problems are concerned, the theory is provided both in a statistical and infinite dimensional deterministic setting. In [7] a comprehensive study on the convergence rates with infinite dimensional deterministic and stochastic noise is provided. Our analysis of the convergence rates of the proposed estimator is based on the results in this paper.

The paper is organized as follows. In Section 2 we introduce the approximation problem in an infinite dimensional RKHS. Then, we show that this infinite dimensional problem serves as
an abstract prototype of linear inverse problems and supervised learning problems and we prove that such problems are equivalent up to the action of the unitary group. Finally, we prove that the Tikhonov regularized solutions in these two frameworks correspond to each other. In Section 3 we focus on discretized problems by considering both the statistical and the deterministic framework and we analyze the convergence of the solution of discretized problems to their corresponding ideal ones, showing that the argmax continuous theorem and the fundamental theorem of Gamma convergence express the conditions for the convergence of the empirical solutions. In Section 4 we give a result for converting error convergence rates with respect to the number of samples (in the statistical framework) to error convergence rates with respect to the noise level (in infinite dimensional inverse problems framework) and vice-versa. In Section 5 we present the conclusions of our analysis.

2. INFINITE DIMENSIONAL SETTING

Reproducing kernel Hilbert spaces arise in a number of areas, including statistical machine learning theory, approximation theory, generalized spline theory and inverse problems [11]. The usual definition of a Reproducing Kernel Hilbert Space (RKHS) is given for a Hilbert space of functions, as follows:

Definition 1. Let $\mathcal{H}$ be an Hilbert space of real valued functions on a non-empty set $\mathcal{X}$. $\mathcal{H}$ is said a reproducing kernel Hilbert space if for all $x \in \mathcal{X}$ the evaluation functional $L_x : f \in \mathcal{H} \rightarrow L_x(f) := f(x)$ is continuous.

An important characterization of RKHSs, which can be even considered as an alternative definition, is the following: $K : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}$ is a reproducing kernel of an Hilbert space $\mathcal{H}$ if for all $f \in \mathcal{H}$, $f(x) = \langle f, K_x \rangle_{\mathcal{H}}$, where $K_x := K(x, \cdot) \in \mathcal{H}$, $\forall x \in \mathcal{X}$. The definition of RKHS is not restricted to function spaces but allows us to consider reproducing kernels $K$ defined on $\mathcal{X} \times \mathcal{X}$, where $\mathcal{X}$ is a Borel set. For function spaces $\mathcal{X}$ shall be $\mathbb{R}$ or $\mathbb{C}$, but in general it can be a countable set or a finite set [4] (e.g. a pixel space). This perspective takes to see the reproducing kernel $K$ as function of two variables $(x, x')$, which can be continuous variables, e.g. $x, x' \in \mathbb{R}$, or can be represented by indexes $(i, j)$, e.g. countable variables $i, j \in \mathbb{N}$ or finite discrete variables $i, j \in \{1, \ldots, n\}$. In the latter case, the kernel $K$ is an infinite or finite matrix. We now define an approximation problem for functions, sequences or vectors, by requiring that the solution belongs to a suitable RKHS.

2.1. Approximation problems in RKHS. We introduce the approximation problem in a RKHS as the problem of finding the closest element of the RKHS to a given one. Let us call $y$ the element to approximate in a given Hilbert space $\mathcal{H}_2$ and let $\mathcal{H}_K \subseteq \mathcal{H}_2$ be a RKHS with reproducing kernel $K$. We define the solution of the approximation problem as the minimizer of a functional $R_y : \mathcal{H}_2 \rightarrow \mathbb{R}$ over the RKHS $\mathcal{H}_K$, i.e.

$$\begin{equation}
g_r := \arg\min_{g \in \mathcal{H}_K} R_y(g). \tag{2}
\end{equation}$$

The idea is that $R_y$ measures the approximation error. We require that $R_y(g) \geq 0$ for all $g \in \mathcal{H}_2$, and $R_y(g) = 0$ iff $g = y$. Under these hypotheses, if $y \in \mathcal{H}_2$ the existence and uniqueness are assured by
requiring that \( R_y \) is strictly convex. Otherwise, if \( y \notin \mathcal{H}_K \) the existence and uniqueness are assured either by requiring that

a) \( R_y \) is lower semicontinuous, strictly convex and coercive with respect to the norm \( \| \cdot \|_{\mathcal{H}_2} \) and \( \mathcal{H}_K \subseteq \mathcal{H}_2 \) is closed, or

b) \( R_y \) is lower semicontinuous, strictly convex and coercive with respect to the norm \( \| \cdot \|_{\mathcal{H}_K} \).

A typical example is \( R_y(g) = \| y - g \|^2_{\mathcal{H}_2} \) with \( \mathcal{H}_K \) closed in \( \mathcal{H}_2 \).

2.2. Linear inverse problems in Hilbert spaces. Let \( \mathcal{H}_1 \) be an Hilbert space (generally different from \( \mathcal{H}_2 \)) and \( A \) a bounded linear operator \( A : \mathcal{H}_1 \to \mathcal{H}_2 \). The inverse problem associated to the operator \( A \) consists in finding \( f \) satisfying equation (1) given \( y \in \mathcal{H}_2 \). The ill-posedness of inverse problems leads to the definition of the generalized solution, usually denoted by \( f^\dagger \), which, from a variational point of view, can be seen as the minimal norm solution of the least squares problem

\[
\min_{f \in \mathcal{H}_1} \| y - Af \|^2_{\mathcal{H}_2}.
\]

The variational form of the generalized inverse suggests that a strategy for approximating the solution of an inverse problem is to minimize a functional along \( f \). Then, we consider the set of \( R_y \)-minimum solutions of the problem (1) defined by

\[
\mathcal{S}_{A,R_y} := \arg \min_{f \in \mathcal{H}_1} R_y(Af)
\]

and take the minimum norm solution. When at least an \( R_y \)-minimum solution \( f_{R_y} \) exists, \( \mathcal{S}_{A,R_y} \) is the affine subspace given by \( f_{R_y} + \text{Ker}(A) \), where \( \text{Ker}(A) \) denotes the nullspace of \( A \).

**Definition 2.** \( f^\dagger_{R_y} \in \mathcal{H}_1 \) is called the \( R_y \)-generalized solution of the inverse problem (1) if it is the \( R_y \)-minimum solution of (1) with minimum norm, i.e.

\[
f^\dagger_{R_y} = \arg \min_{f \in \mathcal{S}_{A,R_y}} \| f \|_{\mathcal{H}_1}.
\]

As in section 2.1 we require that \( R_y(g) \geq 0 \) for all \( g \in \mathcal{H}_2 \), and \( R_y(g) = 0 \) iff \( g = y \). We discuss some hypotheses which assure the existence and uniqueness of the \( R_y \)-generalized solution. We denote with \( \mathcal{S}(A) \) the range of \( A \). Under these hypotheses, if \( y \in \mathcal{S}(A) \) the existence and uniqueness are assured by requiring that \( R_y \) is strictly convex. Otherwise, if \( y \notin \mathcal{S}(A) \) the existence and uniqueness are assured either by requiring that

a) \( R_y \) is lower semicontinuous, strictly convex and coercive with respect to the norm \( \| \cdot \|_{\mathcal{H}_2} \) and \( \mathcal{S}(A) \subseteq \mathcal{H}_2 \) is closed, or

b) \( f \in \mathcal{H}_1 \mapsto R_y(Af) \) is lower semicontinuous, strictly convex and coercive with respect to the norm \( \| \cdot \|_{\mathcal{H}_1} \).

When \( R_y \) is different from the least squares functional, this procedure provides a generalization of the so-called Moore Penrose generalized solution. Such a generalization is needed to develop the equivalence between approximation problems in RKHSs and classes of linear inverse problems. We introduce it in the next paragraph.

2.3. Equivalence between problems. We show the equivalence between an approximation problem in a RKHS and an inverse problem by proving that there is a natural correspondence of the solutions of the two problems. We make use of the following:
Assumption 1. Let \( \mathcal{H}_1 \) be a real separable Hilbert space and \( \mathcal{H}_2 \) be a real Hilbert space on a Borel space \( \mathcal{X} \). For all \( x \in \mathcal{X} \) and for all \( f \in \mathcal{H}_1 \) there exists a constant \( c > 0 \) such that
\[
|Af(x)| \leq c\|f\|_{\mathcal{H}_1}.
\]

The assumption 1 together with the Riesz’s representation theorem implies that for all \( x \) there exists an element \( \phi_x \in \mathcal{H}_1 \) such that
\[
(Af)(x) = \langle f, \phi_x \rangle_{\mathcal{H}_1}.
\]
Moreover, it is well known that the range of the operator \( A \) is a RKHS (e.g. see [33]). The following proposition is an adaptation of this result to our context.

**Proposition 1.** \( \mathcal{Z}(A) \) equipped with the norm
\[
\|g\|_{\mathcal{Z}_K} = \min\{\|w\|_{\mathcal{H}_1} : w \in \mathcal{H}_1 \text{ s.t } g(x) = \langle w, \phi_x \rangle_{\mathcal{H}_1}, x \in \mathcal{X} \}
\]
is a RKHS with kernel
\[
K : \mathcal{X} \times \mathcal{X} \to \mathbb{R}
\]
\[
(x, r) \mapsto K(x, r) := \langle \phi_x, \phi_r \rangle_{\mathcal{H}_1}.
\]

We remark that \( K \) by definition is a positive semi-definite kernel over \( \mathcal{X} \) and \( \phi \) represents the feature map on the feature space \( \mathcal{H}_1 \). Furthermore we have
\[
\mathcal{Z}(A) = \text{span}\{K_x, x \in \mathcal{X}\}.
\]
Moreover, it is worth observing that conditions usually required on a reproducing kernel and on its associated RKHS are satisfied: \( \mathcal{H}_K \) is separable since \( \mathcal{H}_1 \) is separable and \( A \) is a partial isometry from \( \mathcal{H}_1 \) to \( \mathcal{Z}(A) \), and for all \( x \in \mathcal{X} \) \( K(x, x) \leq c^2 \) since assumption 1 applies.

Now we introduce the restriction of \( A \) to the space orthogonal to its kernel and we prove the main result of this section which identifies the solutions of the two problems \( g_{R_y} \) and \( f^\dagger_{R_y} \) as defined in equations (2) and (4), respectively. We denote with \( \tilde{A} \) the restriction operator, i.e.
\[
\tilde{A} := A_{|\text{Ker}(A)^\perp} : \text{Ker}(A)^\perp \to \mathcal{Z}(A).
\]

By definition, \( \tilde{A} \) admits the inverse operator \( \tilde{A}^{-1} \).

**Theorem 1.** Let \( g_{R_y} \) be the solution of the approximation problem in the RKHS \( \mathcal{H}_K \) with kernel \( K \) defined in equation (2). Let \( f^\dagger_{R_y} \) be the solution of the inverse problem defined in equation (4) with the operator \( A \) defined in equation (6). If \( \forall x, x' \in \mathcal{X} \) \( K(x, x') = \langle \phi_x, \phi_{x'} \rangle_{\mathcal{H}_1} \), we have
\[
f^\dagger_{R_y} = \tilde{A}^{-1} g_{R_y}.
\]

**Proof.** By hypothesis we have the following identification \( \mathcal{Z}(A) = \mathcal{H}_K \) intended as RKHSs. Thanks to this identification the hypotheses on \( R_y \) in problems (2) and (4) are exactly the same: the hypotheses a) in section 2.1 and in section 2.2 are trivially the same hypothesis, the hypotheses in b) in section 2.1 and in section 2.2 are equivalent by noting that the coercivity of \( R_y \) with respect to the norm \( \|\cdot\|_{\mathcal{Z}_K} \) corresponds to the coercivity of \( f \mapsto R_y(Af) \) with respect to the norm \( \|\cdot\|_{\mathcal{H}_1} \). Let \( g_{R_y} \) be the solution of the problem (2) and let \( \tilde{f} := \tilde{A}^{-1} g_{R_y} \). Then for all \( f \in \mathcal{H}_1 \) we have
\[
R_y(\tilde{A} f) \geq \min_{g \in \mathcal{Z}(A)} R_y(g) = R_y(g_{R_y}) = R_y(\tilde{f})
\]
i.e. \( \tilde{f} \) is solution of problem (3). Furthermore, by definition of \( \tilde{A}^{-1}, \tilde{f} \in \text{Ker}(A) \perp \) and therefore \( \tilde{f} \) is the solution of (4), that is \( \tilde{f} = f^\dagger_{R_y} \).

**Remark 1.** Under assumption 1, given an inverse problem described by a linear operator \( A \) (characterized by a map \( \phi \)), it is always possible to associate with it an approximation problem in the RKHS \( \mathcal{H}_K \) with kernel \( K \) defined by the map \( \phi \), i.e. \( K(x,x') = \langle \phi_x, \phi_{x'} \rangle_{\mathcal{H}_1} \) for all \( x,x' \in \mathcal{X} \).

**Remark 2.** Given an approximation problem in the RKHS \( \mathcal{H}_K \) with kernel \( K \), it is always possible to associate with it a feature map \( \phi : x \in \mathcal{X} \rightarrow \phi_x \in \mathcal{H}_1 \), where \( \mathcal{H}_1 \) is an Hilbert space and such that \( K(x,x') = \langle \phi_x, \phi_{x'} \rangle_{\mathcal{H}_1} \) for all \( x,x' \in \mathcal{X} \). In such a way we define \( \mathcal{F} = \text{span}\{ \phi_x, x \in \mathcal{X} \} \), which is the feature space, and an inverse problem whose operator \( A \) is given in equation (6). By construction we have the identification between the feature space and the orthogonal of the kernel of the operator, i.e. \( \mathcal{F} = \text{Ker}(A) \perp \). In the case that \( K \) is a continuous reproducing kernel, the Mercer theorem gives us the way to describe the feature map \( \phi \) and the feature space is \( \ell_2 \), while in the general case (when \( K \) is not necessarily continuous) we can consider the canonic feature map, that is \( \phi : \mathcal{X} \rightarrow \mathcal{H}_K \) where \( \forall x \in \mathcal{X} \phi_x = K_x \).

From the second remark the feature map associated with a given kernel \( K \) is determined up to the action of unitary group on \( \mathcal{H}_1 \), i.e.

\[
\forall f, f' \in \mathcal{H}_1
\]

\[
f \sim f' \iff \exists U \in \mathcal{U} \quad | \quad f' = U f.
\]

We can also define an equivalence \( \sim^X \) between feature maps. Let \( \phi, \phi' \in \mathcal{H}_1^X \)

\[
\phi \sim^X \phi' \iff \phi_x \sim \phi'_x, \forall x \in \mathcal{X}.
\]

Then, we define the map

\[
\mathcal{K} : \mathcal{H}_1^X \rightarrow \mathbb{R}^X \times \mathcal{X}
\]

\[
\phi \mapsto K_\phi
\]

with \( K_\phi(x,x') = \langle \phi_x, \phi_{x'} \rangle_{\mathcal{H}_1} \). Therefore, from equations (10) and (12) we have a bijection

\[
\mathcal{H}_1^X / \sim^X \leftrightarrow \mathcal{S}(\mathcal{X}) \subset \mathbb{R}^X \times \mathcal{X}
\]

\[
\tilde{\phi} \leftrightarrow K_{\tilde{\phi}},
\]

where \( \tilde{\phi} \) is the class induced by the equivalence relation \( \sim^X \) in (12). We denote with \( A_{\phi} \) the operator defined in equation (6).

\[
g_{R_y} = A_{\phi} f^\dagger_{R_y} = A_{\phi} (f^\dagger_{R_y}',
\]

where \( \phi \sim^X \phi' \) and \( f^\dagger_{R_y} \sim (f^\dagger_{R_y})' \). Then we also have a bijection

\[
\mathcal{H}_1 / \sim \leftrightarrow \mathcal{H}_K
\]

\[
\tilde{f} \leftrightarrow g_{R_y},
\]
stating that, for any \( R_y \) satisfying conditions of problem (2) (or equivalently (4)) and for any \( y \in \mathcal{H}_2 \), the class of \( R_y \)-generalized solutions \( f_{R_y}^\dagger \) corresponds to the solution \( g_{R_y} \) of the approximation problem in the RKHS \( \mathcal{H}_K \) (2). Let us now fix an element \( y \in \mathcal{H}_2 \) and a functional \( R_y \). For each \( K \in \mathfrak{I}(\mathcal{H}) \) we define the function \( T_{R_y}(K) := g_{R_y} \) which maps the kernel \( K \) to the solution of the approximation problem in a RKHS (2). In the same way, for each \( \phi \in \mathcal{H}^X_1 \) we define the function \( S_{R_y}^\dagger(\phi) := f_{R_y}^\dagger \) which maps the feature map \( \phi \) to the \( R_y \)-generalized solution of the inverse problem (1). Then, for each class \( \bar{\phi} \), we can define a map \( \overline{S_{R_y}^\dagger} : \mathcal{H}^X_1 / \sim^X \to \mathcal{H}_1 / \sim \) as follows

\[
\overline{S_{R_y}^\dagger}(\bar{\phi}) := \pi(\overline{S_{R_y}^\dagger}(\phi)),
\]

where \( \phi \) is a representer of \( \bar{\phi} \) and \( \pi \) is the quotient map with respect to the equivalence relation \( \sim \) in (11). This definition is well-posed since it does not depend on the choice of the representer \( \phi \). We can summarize this discussion with the commutative diagram in fig. 1. In synthesis, when an approximation problem in a RKHS is provided with a feature map, it is equivalent to a linear inverse problem. If a feature map is not given, we can associate with the approximation problem in a RKHS as many inverse problems as feature maps (and so features spaces) which give rise to the same kernel.

### 2.4. Equivalence between methods.

When \( y \) is corrupted by noise, the inverse problem needs to be addressed in a different way as the \( R_y \)-generalized solution \( f_{R_y}^\dagger \) may not exist or it may not depend continuously on the data. A well-known strategy common to both approximation and inverse problems is Tikhonov regularization [14]. It allows us to find solutions of the problem which depend continuously on the data by re-stating the approximation problem in RKHS \( \mathcal{H}_K \) defined in equation (2) as follows

\[
\hat{g}_{R_y, \lambda} = \arg \min_{g \in \mathcal{H}_K} R_y(g) + \lambda \Omega(g),
\]
and the inverse problem associated to the operator $A$ given data $y$ defined in equation (4) as follows

$$
\hat{f}_{R,\lambda} = \arg \min_{f \in \mathcal{H}} R_y(Af) + \lambda \Omega(f).
$$

In these generalized Tikhonov regularization schemes $R_y$ is usually called the data fidelity term, $\Omega$ is the penalty term and $\lambda > 0$ is the regularization parameter. The purpose of the penalty term is to induce stability and to allow the incorporation of a priori information about the desired solution according to the magnitude of the parameter $\lambda$. In this context we assume that the penalty term has the following form

$$
\Omega(h) := \psi(\|h\|_{\mathcal{H}}),
$$

where $\psi : [0, +\infty) \to \mathbb{R}_+$ is a continuous convex and strictly monotonically increasing real-valued function, $h$ is an element of an Hilbert space $\mathcal{H}$ and $\|\cdot\|_{\mathcal{H}}$ denotes its norm. Now we show that the result of theorem 1 can be extended to the case of Tikhonov regularized solutions $\hat{f}_{R,\lambda}$ and $\hat{g}_{R,\lambda}$.

**Theorem 2.** Under the same assumptions of theorem 1 we have

$$
\hat{f}_{R,\lambda} = \tilde{A}^{-1} \hat{g}_{R,\lambda}.
$$

**Proof.** As in the proof of the theorem 1 we have the identification $\exists(A) = \mathcal{H}_K$ as RKHSs and the hypotheses on functionals to minimize in (14) and (15) are the same. Let $\tilde{f} := \tilde{A}^{-1} \hat{g}_{R,\lambda}$. By definition of $\tilde{A}^{-1}$, $\tilde{f} \in \text{Ker}(A)^\perp$ and so $\|\hat{g}_{R,\lambda}\|_{\mathcal{H}_K} = \|	ilde{f}\|_{\mathcal{H}_1}$. For all $f \in \mathcal{H}_1$ we have

$$
R_y(Af) + \lambda \psi(\|f\|_{\mathcal{H}_1}) \geq \min_{g \in \exists(A)} R_y(g) + \lambda \psi(\|g\|_{\mathcal{H}_1})
$$

$$
= R_y(\hat{g}_{R,\lambda}) + \lambda \psi(\|\hat{g}_{R,\lambda}\|_{\mathcal{H}_1})
$$

$$
= R_y(\tilde{A} \hat{g}_{R,\lambda}) + \lambda \psi(\|\tilde{A} \hat{g}_{R,\lambda}\|_{\mathcal{H}_1})
$$

i.e. $\tilde{f}$ is solution of problem (15). This concludes the proof. $\square$

As in the case of $R_y$-generalized solutions, we have a commutative diagram for Tikhonov regularized solutions. The diagram has exactly the same shape of the one shown in fig. 1 but arrows and nodes refer to the solution of problems (14) and (15). In particular, we have to replace: $T^\dagger_{R_{\lambda}}$ with the function $T_{R,\lambda}(K) := \hat{g}_{R,\lambda}$ which maps the kernel $K$ to the Tikhonov solution (14); $S^\dagger_{R_{\lambda}}$ with the function $S_{R,\lambda}(\phi) := \hat{f}_{R,\lambda}$ which maps the feature map $\phi$ to the Tikhonov solution (15); $\tilde{S}^\dagger_{R_{\lambda}}$ with the map $\tilde{S}_{R,\lambda}$ defined as in equation (13) by substituting $\tilde{S}^\dagger_{R_{\lambda}}$ with $S_{R,\lambda}$; $\tilde{f}_{R,\lambda}$ with $\hat{f}_{R,\lambda}$, which is the class of Tikhonov solutions corresponding to the Tikhonov solution of the approximation problem in the RKHS represented by $K_\phi$.

### 3. Discrete data

The purpose of this section is to show that some applied problems, namely discrete inverse problems, interpolation problems and statistical (inverse) learning, despite appearing different, can be thought of as instances of the approximation problem in a RKHS (2). To this end, we introduce a suitable discretization operator mapping the infinite dimensional data $y$ to a finite number of samples together with a specific form of the functional $R_y$. The idea of the discretization operator
is to consider, in place of the data \( y \), a set of samples \( \{(X_i, Y_i)\}_{i=1}^n \) statistically or deterministically related to \( y \). In this way we will retrieve the formulation of various applied problems by minimizing the empirical form of the ideal functional \( R_y \). To realize the discretization operator, i.e. a map from \( \mathcal{H}_2 \) to a sample space, we proceed as follows. Let us consider the set \( \mathcal{P} \) of all possible Borel probability distributions over a compact space \( \mathcal{Y} \subseteq \mathbb{R} \) and let \( F_V : \mathcal{P} \rightarrow \mathbb{R} \) be a function defined by

\[
F_V(\rho) := \arg\min_{w \in \mathbb{R}} \int_{\mathcal{Y}} V(Y, w) \, d\rho(Y),
\]

where \( V \) is called \textit{loss function} in the statistical learning terminology [29]. The function \( F_V \) is defined provided that \( V : \mathcal{Y} \times \mathbb{R} \rightarrow [0, +\infty) \) is measurable and integrable with respect to the first variable and \( V(Y, \cdot) \) is lower semicontinuous, strictly convex and coercive \( \forall \ Y \in \mathcal{Y} \). Given a function \( V \), \( F_V(\rho) \) can represent a characteristic of the distribution \( \rho \): by instance, if \( V \) is the square loss usually used in regression problems, i.e. \( V(Y, w) = (w - Y)^2 \), or \( V \) is the Kullback-Leibler divergence, then \( F_V(\rho) \) is the expected value, i.e. \( F_V(\rho) = \mathbb{E}(Z) \), where \( Z \) is a random variable with probability distribution \( \rho \); if \( V \) is the square loss usually used in classification problems, i.e. \( V(Y, w) = (1 - Y)^2 \) then \( F_V(\rho) = \mathbb{E}(Z)/\mathbb{E}(Z^2) \); if \( V \) is the absolute value loss, i.e. \( V(Y, w) = |w - Y| \) then \( V \) is the median of the distribution \( \rho \).

We now want to define a map from \( \mathbb{R} \) to \( \mathcal{P} \), roughly speaking an inverse of \( F_V \). We introduce an application

\[
\theta : \mathbb{R} \rightarrow \mathcal{P},
\]

\[
z \rightarrow \rho_z,
\]

mapping \( z \in \mathbb{R} \) in a distribution \( \rho_z \) such that \( F_V \circ \theta = \text{id} \). Given a function \( y \), \( \theta \) maps \( y(x) \) to a distribution \( \rho_{y(x)} \) such that \( y(x) \) is the characteristic of \( \rho_{y(x)} \) for each \( x \in \mathcal{X} \). Therefore we define the following sampling operator

\[
S_{\xi, \theta}^{(n)} : \mathcal{H}_2 \rightarrow \mathcal{Y}^n
\]

\[
y \rightarrow (Y_i)_{i=1, \ldots, n}
\]

where each \( Y_i \) is drawn from the distribution \( \rho_{y(x)} := \theta(y(x)) \) and the set of points \( \bar{x} = \{x_1, \ldots, x_n\} \subset \mathcal{X} \) can be either given a priori (in a deterministic manner) or drawn from a probability distribution \( \nu \) over \( \mathcal{X} \). Once \( \nu \) is fixed, for any chosen sampling \( S_{\xi, \theta}^{(n)} \) let us consider the functional defined as

\[
R_y(g) := \int_{\mathcal{X} \times \mathcal{Y}} V(Y, g(X)) \, d\bar{\rho}_{y(x)}(Y) \, d\nu(X)
\]

which depends on \( \nu \) and \( \theta \) as well as on \( y \) and on \( V \). When the set of sampling points \( \bar{x} \) is given in a deterministic manner, we assume \( d\nu(x) = dx \). From now on, we will denote the distribution \( \bar{\rho}_{y(x)} = \rho(\cdot | X = x) \) since it will represent the conditional distribution with respect to \( X = x \).

Henceforth, we consider the approximation problem in a RKHS (2) with functional \( R_y \) given in (22). By applying \( S_{\xi, \theta}^{(n)} \) to the data \( y \), we now show that we can retrieve the formulation of different applied problems according to whether \( \rho \) and \( \nu \) are known or not and, if they are known, according to their specific explicit form. In general, when just a finite number of sample values is known, all
these problems are addressed by minimizing the following empirical form of the functional, i.e.

\[ R_{\mathcal{F}}(g) := \frac{1}{n} \sum_{i=1}^{n} V((S_{x,i}^{(n)}(y))_i, g(x_i)) . \]

In particular, when \( \rho(\cdot|\cdot) \) and \( \nu \) are not known we retrieve the formulation of statistical learning problems, while if \( \rho(\cdot|\cdot) \) and \( \nu \) are given we have the following cases.

i) Stochastic case: the samples \( Y_i \) are drawn from \( \rho(Y|X = x_i) \) and \( x_i \) are the elements of \( \bar{x} \), given at random according to a distribution \( \nu \). In this case we can describe inverse regression problems with random matrix design.

ii) Semi-stochastic case: the samples \( Y_i \) are drawn from a generic probability distribution \( \rho(Y|X = x_i) \) and \( x_i \) are given not at random. In this case we assume \( d\nu(x) = dx \). This is the setting used for describing discrete inverse problems with random noise under the maximum likelihood approach or inverse regression problems with deterministic matrix design.

iii) Deterministic case: the samples \( Y_i \) are the values of the function \( y \) at the points \( x_i \), given not at random. This can be thought of as the samples \( Y_i \) are drawn from \( \rho(Y|X = x_i) = \delta(Y - y(X)|X = x_i) \). In this case we assume \( d\nu(x) = dx \) and the sampling operator can be denoted by \( S_{\bar{x}}^{(n)} \) as in [31]. This is the setting used for describing interpolation problems or discrete inverse problems with deterministic noise when we consider a noisy version \( y^{\delta} \) of \( y \).

The crucial difference between the first and the latter two cases is that in the first case discretization has to be defined according to a stochastic process while in the second and third cases at least a part of the discretization is usually defined in a deterministic manner. Incidentally, we notice that in learning problems a given point can be sampled more than once whereas in inverse problems each sample \( x \in \mathcal{X} \) is usually taken once. For example, in a machine learning problem the samples can be view as the result of a sampling process which takes place upstream of the definition of the problem itself, or in any way, independently of the will of the learner. It is indeed formalized as an empirical process in accordance with an unknown distribution. On the contrary, in an inverse problem the discretization usually takes place downstream of the problem: for example, in the case of an industrial device, it can be defined during the design phase or determined even later, after the signal acquisition, as a variable to be optimized in the inversion process. The table 1 summarizes the main sampling schemes corresponding to different applications.

| Sampling \( S_{\bar{x},\theta}^{(n)} \) | \( \rho(\cdot|\cdot) \) and \( \nu \) unknown | \( \bar{x} \) given and \( \rho(\cdot|\cdot) \) known |
|--------------------------------------|------------------------------------------|-----------------------------------------------|
| direct learning inverse learning     | discrete inverse problems                |                                               |

**Table 1.** Discretization schemes of a reproducing kernel approximation problem
We remark that for learning problems this formulation differs from the classical one where the samples are drawn without any discretization process. In the classical formulation the crucial hypothesis is that the samples are drawn independently and identically distributed according to a distribution $\rho(\cdot, \cdot)$ and there is no need to introduce from the beginning $\nu$ and $\rho(\cdot \mid \cdot)$, but these last two distributions are the result of the factorization of $\rho$. Moreover, $y$ is introduced after $\rho$, it depends on the choice of $V$ and represents the parameter of $\rho$ which one wants to learn.

3.1. Learning from examples. We introduce the supervised learning problem in the standard way to highlight the link with our formulation. We suppose to know a finite number of samples

$$ (\mathcal{X}_n := \{ (X_1, Y_1), \ldots, (X_n, Y_n) \}.$$  

Such samples are drawn independently identically distributed according to a given (but unknown) probability distribution $\rho$ on $\mathcal{X} = \mathcal{X} \times \mathcal{Y}$ where $\mathcal{X} \subseteq \mathbb{R}^d$, with $d > 0$, and $\mathcal{Y} \subseteq \mathbb{R}$. $\mathcal{X}$ and $\mathcal{Y}$ can be assumed to be compact spaces and $\rho$ admits the following factorization

$$ \rho(X, Y) = \rho(Y \mid X) \nu(X) $$

where $\nu$ is the marginal distribution on $\mathcal{X}$ and $\rho(\cdot \mid X = x)$ is the conditional distribution on $\mathcal{Y}$ for almost $x \in \mathcal{X}$. Given a measurable function $g$ the ability of $g$ to describe the distribution $\rho$ is measured by the expected risk defined as

$$ R_\rho(g) = \int_{\mathcal{X} \times \mathcal{Y}} V(Y, g(X)) \, d\rho(X, Y). $$

We remark that thanks to the hypothesis (25) $y$, defined as $y(x) = F_\nu(\rho(\cdot \mid X = x))$, is the minimizer of the expected risk (26) (over all measurable functions), i.e. it can be seen as an ideal estimator of the unknown distribution $\rho$. However only the set $\mathcal{X}_n$ is available and therefore learning is performed by minimizing over the RKHS $\mathcal{H}_k$ the empirical risk given by

$$ R_{\mathcal{X}_n}(g) = \frac{1}{n} \sum_{i=1}^{n} V(Y_i, g(X_i)). $$

Therefore the problem (2) reduces to

$$ \hat{g}_R^{(n)} := \arg \min_{g \in \mathcal{H}_k} R_{\mathcal{X}_n}(g). $$

From a numerical point of view the solution $\hat{g}_R^{(n)}$ is not stable and therefore, following the approach of Tikhonov regularization, it is useful to introduce a penalty term in order to stabilize the solution. Therefore, the regularized problem is the following:

$$ \hat{g}_{R,\lambda}^{(n)} := \arg \min_{g \in \mathcal{H}_k} R_{\mathcal{X}_n}(g) + \lambda \psi(\|g\|_{\mathcal{H}_k}), $$

where $\lambda$ is the regularization parameter. This is the classical formulation of statistical learning theory, in which $\mathcal{X}$ and $\mathcal{Y}$ represent the input and the output space, respectively, and the aim is to find a function $g$ such that $g(x)$ is a good estimate of the output when a new input $x$ is given.

The result of this construction can be obtained by (21) and (22) by taking unknown $\vartheta$ and $\nu$, i.e the samples $\mathcal{X}_n$ can be seen as the result of the action of the sampling operator $S_{\nu, \vartheta}$, where $X_1, \ldots, X_n$ are drawn from the distribution $\nu$, the samples $Y_1, \ldots, Y_n$ are drawn from $\rho(Y \mid X)$ and the
the factorization in (25) applies. Moreover, we note that this formulation takes into account the inverse statistical learning problem by considering $Af$ instead of $g$ as given in equation (1) [8, 28].

3.2. Discrete inverse problems. In this paragraph we introduce discrete inverse problems with a deterministic discretization scheme [14, 16]. We suppose to know a set of $n$ samples of the infinite dimensional data $y$ (or of a noisy version $y^\delta$) computed in the points $x_1, \ldots, x_n$. This assumption can be formalized by means of the sampling operator $S^{(n)}_x$ which yields the set of samples

$$Z_n := \{(x_1, y_1), \ldots, (x_n, y_n)\}$$

where $y_1 := y(x_1), \ldots, y_n := y(x_n)$. In this case, the functional (22) takes the form

$$R_y(Af) = \int_X V(y(x), Af(x)) dx .$$

Obviously when $V(y(x), Af(x)) = (y(x) - Af(x))^2$ the problem reduces to the least squares minimization with $R_y(Af) = \|y - Af\|^2$. When the set of samples $Z_n$ is available we minimize the functional

$$R_{Z_n}(Af) = \frac{1}{n} \sum_{i=1}^n V(y_i, (Af)(x_i)) .$$

In this way we estimate the solution of the following discretized inverse problem

$$y_i = (Af)(x_i) \quad i \in \{1, \ldots, n\} .$$

Analogously to the procedure followed in section 2, we introduce the $R_{Z_n}$-generalized solution as follows: $(\hat{f}^{(n)}_R)^+$ is the minimum norm solution of the problem

$$\arg \min_{f \in \mathcal{H}_1} R_{Z_n}(Af) .$$

If we define $\mathcal{S}_{A, R_{Z_n}}$ the set of solution of (34), we have that

$$(\hat{f}^{(n)}_R)^+ = \arg \min_{f \in \mathcal{S}_{A, R_{Z_n}}} \|f\|_{\mathcal{H}_1} .$$

As we noticed in the previous section, it is preferable to regularize the $R_{Z_n}$-generalized solution for a stability issue, being the data $y_i$ usually corrupted by noise. Therefore, in general we solve the Tikhonov regularization problem, i.e.

$$(\hat{f}^{(n)}_{R, \lambda}) := \arg \min_{f \in \mathcal{H}_1} R_{Z_n}(Af) + \lambda \psi(\|f\|_{\mathcal{H}_1}) ,$$

where $\lambda > 0$ is the regularization parameter.

Remark 3. Maximum likelihood approach [6, 37, 18, 34]. We consider the discretized inverse problem (33), where $x_1, \ldots, x_n$ are $n$ points deterministically identified, and for each $i \in \{1, \ldots, n\}$ we know the sample $Y_i$ from a given probability distribution $\rho(Y|X = x_i)$. The main difference with respect to the learning framework is that here the probability distribution $\rho(\cdot|\cdot)$ is known and the quantity to be determined is the parameter $f$ which characterizes the distribution $\rho(\cdot|\cdot)$. For this reason we denote the distribution $\rho(\cdot|\cdot)$ with $\rho_{Af}(\cdot|\cdot)$ to highlight that it depends on the parameter
This approach can be formalized by means of the sampling operator $S_{\tilde{x},\theta}^{(n)}$ which yields the sample set

$$\mathcal{Z}_n := \{(x_1,Y_1),\ldots,(x_n,Y_n)\},$$

where $\vartheta(Af(x)) := \rho_{Af}(\cdot|X=x)$. In the maximum likelihood approach the choice of $V$ is such that

$$V(Y, Af(x)) \ d\rho_{Af}(Y|X=x) = -\log \rho_{Af}(Y|X=x) \ dY.$$

With this choice equation (23) takes the form

$$R_{\mathcal{Z}}(g) = \frac{1}{n} \sum_{i=1}^n -\log(\rho_{Af}(Y_i|x_i)),$$

which corresponds to the negative-log formulation of the maximum likelihood approach.

The general discrete minimization problem (23) depends on the set of points $\mathcal{Z}_n$ but not on their statistical or deterministic origin, i.e. it does not depend on the specific choice of $\tilde{x}$ and $\vartheta$. For this reason, we use the same notation for the solutions of the discretized problems $\hat{g}_R^{(n)}$, $(\hat{f}_R^{(n)})^\dagger$, $\tilde{g}_R^{(n)}$, and $\tilde{f}_R,\lambda$ regardless the nature of samples $\mathcal{Z}_n$. In this respect, we conclude this subsection by giving the following

**Corollary 1.** Given $\mathcal{Z}_n$ a set of samples, under assumption 1 and by assuming that $\forall x,x' \in \mathcal{X}$

$$K(x,x') = \langle \phi_x, \phi_{x'} \rangle_{\mathcal{H}}$$

we have that

$$\tilde{f}_R^{(n)} = \tilde{A}^{-1} \tilde{g}_R^{(n)}.$$

Furthermore, the solutions $\tilde{g}_R^{(n)}$ and $\tilde{g}_R,\lambda^{(n)}$ of problems (28) and (29) correspond to the set of solutions

$$\{U(\hat{f}_R^{(n)})^\dagger \mid U \in \mathcal{U}\}$$

and

$$\{U \tilde{f}_R,\lambda^{(n)} \mid U \in \mathcal{U}\},$$

respectively, where we remind that $\mathcal{U}$ is the set of unitary operators on $\mathcal{H}_1$.

This result is valid for any choice of $\tilde{x}$ and $\theta$, i.e. independently of the discretization scheme. The proof is omitted since it is a straightforward application of theorem 1 and theorem 2.

### 3.3. Convergence

In this paragraph we discuss the convergence of the empirical functional (23) to the ideal one (22) and the convergence of their respective minimizers. In the case $\mathcal{Z}_n$ is randomly drawn the convergence is defined in terms of probabilities and the conditions are well established [36, 27]. However, if $\mathcal{Z}_n$ is assumed to be generated in a deterministic manner, the convergence is defined in terms of norms and the theoretical tools for proving the convergence are slightly different. Indeed, whereas in the statistical framework convergence is a consequence of a straightforward application of the argmax continuous theorem [35], we show that in the deterministic framework we need a result relying on the notion of $\Gamma$-convergence [9].

#### 3.3.1. Statistical setting

We recall a classical theorem ensuring the consistency of a sequence of argmax-estimators in an argmin version suitable for our framework [35]. Let $(H,d)$ be a metric space and $(F_n)$ be a sequence of random functions over $H$ given a probability distribution $\nu$. 
Theorem 3. (Argmax continuous theorem). Let us suppose
\begin{equation}
\sup_{h \in H} |F_n(h) - F(h)| \rightarrow^P 0,
\end{equation}
where \( F \) is a fixed function over \( H \) and for each \( \varepsilon > 0 \)
\begin{equation}
\inf_{h \in H : d(h, h^*) \geq \varepsilon} F(h) > F(h^*),
\end{equation}
where \( h^* \) is the minimizer of \( F \). Moreover, if \( F_n(h^{(n)}) \leq F_n(h^*) + o_P(1) \), we have
\begin{equation}
h^{(n)} \rightarrow^P h^*
\end{equation}
where \( h^{(n)} \) is the minimizer of \( F_n \).

Whereas the second hypothesis is a property of the limit function \( F \) at its minimum point \( h^* \), which is assured when \( F \) is strictly convex, coercive and lower semi-continuous, the first hypothesis (41) requires the uniform convergence of \( (F_n) \). When \( F_n \) takes the form of the empirical risk (equation (23)) and \( F \) is given by equation (22) the condition (41) is satisfied if \( H \) is a uniform Glivenko-Cantelli class (uGC), provided that \( V \) has some Lipschitz property [26]. Then, we have the following

Corollary 2. Let \( \mathcal{H}_K \) be uGC. Let \( R_n \) be defined in (22) and let \( V \) be a loss function as in section 3 with the additional Lipschitz property described in [26]. Assume that \( V \) satisfies the following coercivity property: for each sequence \( (g_k) \subseteq \mathcal{H}_K \) such that \( \|g_k\|_{\mathcal{H}_K} \rightarrow \infty \), as \( k \rightarrow \infty \) then \( V(Y, g_k(X)) \rightarrow \infty \), as \( k \rightarrow \infty \), for each \( Y \in \mathcal{Y} \) and \( X \in \mathcal{X} \). Then as \( n \rightarrow +\infty \),
\begin{equation}
g^{(n)}_R \rightarrow^P g_{R_n} \quad \text{and} \quad (f^{(n)}_R)^\dagger \rightarrow^P f^\dagger_{R_n},
\end{equation}
where \( g^{(n)}_R \) is defined in equation (28), \( g_{R_n} \) is the minimizer of \( R_n \) over \( \mathcal{H}_K \), \( (f^{(n)}_R)^\dagger \) is defined in equation (35) and \( f^\dagger_{R_n} \) is the \( R_n \)-generalized solution in accordance to the definition in (4), respectively.

Proof. Let us take \( F_n := R_{\mathcal{X}_n} \) (where \( R_{\mathcal{X}_n} \) is defined in (23)) and \( F := R_Y \) in theorem 3. Condition (41) is verified for the uGC hypothesis on \( \mathcal{H}_K \). Condition (42) is verified thanks to the hypothesis of uniqueness of the minimizer of \( R_n \). Moreover, the sequence \( g^{(n)}_R \) satisfies \( R_{\mathcal{X}_n}(g^{(n)}_R) \leq R_{\mathcal{X}_n}(g_{R_n}) + o_P(1) \) as \( g^{(n)}_R \) is the minimizer of \( R_{\mathcal{X}_n} \). Using the equivalence of learning and inverse problems, we have the following equalities
\begin{equation}
\|g^{(n)}_R - g_{R_n}\|_{\mathcal{H}_K} = \|A(f^{(n)}_R)^\dagger - A f^\dagger_{R_n}\|_{\mathcal{H}_K} = \|(f^{(n)}_R)^\dagger - f^\dagger_{R_n}\|_{\mathcal{H}_K}.
\end{equation}
This completes the proof. \( \square \)

Remark 4. The same convergence result of corollary 2 applies for Tikhonov type regularized solutions, i.e. fixed \( \lambda > 0 \) we have that \( \hat{g}^{(n)}_{R,\lambda} \) and \( \hat{f}^{(n)}_{R,\lambda} \) converge in probability to \( \hat{g}_{R_n,\lambda} \) and \( \hat{f}_{R_n,\lambda} \), respectively.
3.3.2. Deterministic setting. The convergence in the deterministic case needs the use of the fundamental theorem of $\Gamma$-convergence [9]. First, we recall the $\Gamma$-convergence definition for a given sequence $(F_n)$ of functions on a metric space $(H,d)$ with respect to the distance $d$.

**Definition 3.** The sequence $(F_n)$ $\Gamma$-converges in $H$ to a fixed function $F$ if for all $h \in H$ the lim inf inequality holds, i.e. for all sequence $h_n$ such that $d(h_n,h) \to 0$, as $n \to +\infty$

\[
F(h) \leq \liminf_n F_n(h_n)
\]

and the lim sup inequality holds, i.e. there exists a sequence $h_n$ such that $d(h_n,h) \to 0$, as $n \to +\infty$ such that

\[
F(h) \geq \limsup_n F_n(h_n).
\]

In order to prove the $\Gamma$-convergence of a sequence we use the following characterization of the equi-coerciveness of a sequence [12].

**Lemma 1.** $(F_n)$ is an equi-coercive sequence $\iff$ there exists a lower semicontinuous coercive function $G$ such that $F_n \geq G$ on $H$, for each $n \in \mathbb{N}$.

We also exploit the following result which is a consequence of the fundamental theorem of $\Gamma$-convergence (see [10] for details).

**Proposition 2.** Let $(F_n)$ be an equi-coercive sequence $\Gamma$-converging to $F$. Let $h_n$ be a minimizer of $F_n$, and we assume $F$ admits a unique point of minimum $h$. Then $h_n \to h$, as $n \to +\infty$, i.e. $d(h_n,h) \to 0$, as $n \to +\infty$.

We now prove the convergence of the minimizer of $R_{\mathcal{X}_n}$ to the one of $R_y$ over $\mathcal{X}_K$, where $R_y$ is defined in equation (31) and $V$ is strictly convex, Lipschitz continuous with respect to the second variable and coercive in the sense of the definition given in corollary 2.

**Proposition 3.** Let $x_1, \ldots, x_n \in \mathcal{X}$ such that the sequence of points $(x_n)$ is dense in $\mathcal{X}$. Let $R_{\mathcal{X}_y}$ be defined in equation (32) and $R_y$ be defined in equation (31), with $V$ a Lipschitz continuous with respect to the second variable and coercive function. Then the sequence $(R_{\mathcal{X}_n})$ is an equi-coercive sequence and it $\Gamma$-converges to $R_y$.

**Proof.** To prove the equi-coerciveness of the sequence $(R_{\mathcal{X}_n})$, it is sufficient to observe that $R_{\mathcal{X}_1} \geq R_{\mathcal{X}_n}$ for all $n \in \mathbb{N}$ where $R_{\mathcal{X}_1}(g) = V(y_1, g(x_1))$ and then $R_{\mathcal{X}_1}$ is coercive and continuous for the hypothesis on $V$. Now we prove that $(R_{\mathcal{X}_n})$ $\Gamma$-converges to $R_y$. Without loss of generality we assume $\mathcal{X} = [0,1]^d$. Let $g \in \mathcal{X}_K$ and let $(g_n)$ be a sequence converges to $g$, i.e. $\|g_n - g\|_{\mathcal{X}_K} \to 0$, then we have the following inequality

\[
|R_{\mathcal{X}_n}(g_n) - R_y(g)| \leq |R_{\mathcal{X}_n}(g) - R_y(g)| + |R_{\mathcal{X}_n}(g_n) - R_{\mathcal{X}_n}(g)|.
\]

The first term in the r.h.s in eq. (48) converges to 0 as $n \to +\infty$ for the definition of the Riemann integral and for the density of the points $x_i$ in $\mathcal{X}$. Now we prove that the second term in the r.h.s in eq. (48) converges to 0. Under the assumption 1 we have that $\|K_{x_i}\|_{\mathcal{X}_K} \leq c$, $\forall x_i$, where $c$ is a
fixed constant. By using the Lipschitz continuity of \(V\) and the reproducing property of \(K\) we have the following inequalities

\[
|R_{\mathcal{X}_n}(g_n) - R_{\mathcal{X}_n}(g)| \leq \frac{1}{n} \sum_{i=1}^{n} |V(g_n(x_i), y_i) - V(g(x_i), y_i)| \\
\leq \frac{1}{n} \sum_{i=1}^{n} \sigma|g_n(x_i) - g(x_i)| \leq c \sigma \|g_n - g\|_{\mathcal{H}_K},
\]

where \(\sigma\) is the Lipschitz constant of \(V\). Therefore, for each sequence \((g_n)_n\) converging to \(g\) there exists \(\lim_{n \to \infty} R_{\mathcal{X}_n}(g_n) = R(g)\). Then \((R_{\mathcal{X}_n})\) \(\Gamma\)-converges to \(R\).

**Corollary 3.** Under the assumptions of proposition 3 and requiring that \(V\) is strictly convex with respect to the second variable we consider \(\hat{g}_{R,n}, \hat{g}_R, \hat{(\hat{g}_{R,n})}^\dagger\) and \(\hat{f}_R^\dagger\), defined in equations \((28), (2), (35)\) and \((4)\), respectively. Then, as \(n \to \infty\)

\[
\hat{g}_{R,n} \to \hat{g}_R, \quad \text{and} \quad \hat{(\hat{g}_{R,n})}^\dagger \longrightarrow \hat{f}_R^\dagger,
\]

where the convergence is uniform in \(\mathcal{H}_K\) and \(\mathcal{H}_I\), respectively.

**Proof.** The convergence in \(\mathcal{H}_K\) follows from proposition 2 and proposition 3, by observing that \(R_Y\) admits a unique minimizer. The convergence in \(\mathcal{H}_I\) follows from the equality in \((45)\).

**Remark 5.** The same convergence result of corollary 3 applies for Tikhonov type regularized solutions, i.e. \(\hat{g}_{R,n,\lambda}, \hat{\hat{g}}_{R,n,\lambda}\) converge to \(\hat{g}_{R,\lambda}, \hat{\hat{g}}_{R,\lambda}\) and \(\hat{f}_{R,\lambda}, \hat{\hat{f}}_{R,\lambda}\), respectively. Such a result follows from the fact that \((R_{\mathcal{X}_n} + \lambda \psi(\|\cdot\|_{\mathcal{H}_K}))\) is equi-coercive and \(\Gamma\)-converges to \(R_Y + \lambda \psi(\|\cdot\|_{\mathcal{H}_K})\) which is a straightforward consequence of proposition 3 and the fact that \(\lambda \psi(\|\cdot\|_{\mathcal{H}_K})\) is continuous (see [9]).

Finally, as the convergence property of the \(R_{\mathcal{X}_n}\)-generalized solution holds regardless the discretization scheme we can summarize functionals, solutions, convergence and discretization with the commutative diagrams shown in fig. 2.

The vertexes of the rear side of the cube represent the four minimizing functionals and the vertexes of the front side represent the corresponding solutions. The empirical and ideal cases are shown on the left and right sides, respectively. The arrows from left to right represent the convergence, while the arrows from right to left, on the rear side, represent the discretization. The arrows from rear to front show the minimizing process. In particular, along horizontal arrows of the front side of the cube we show the convergence of the empirical solutions to the ideal ones (corollaries 2 and 3); along vertical arrows we show the correspondence between solutions of approximation problems in a RKHS and inverse problems (theorem 1 and corollary 1).

### 3.4. Application of the representer theorem.

The representer theorem and its generalizations prove that the solution of problem (29) belongs to a finite dimensional subspace of \(\mathcal{H}_K\) [30]. Under the assumption 1 on the linear operator \(A : \mathcal{H}_I \to \mathcal{H}_2\), let

\[
\mathcal{H}_K^{(n)} := \text{span}\{K_{x_1}, \ldots, K_{x_n}\}
\]

and

\[
\mathcal{H}_I^{(n)} := \text{span}\{\phi_{x_1}, \ldots, \phi_{x_n}\},
\]

where \(\phi_{x_i}\) and \(K_{x_i}\) are the \(\phi\) and \(K\) functions respectively. Under the assumption 1 on the linear operator \(A : \mathcal{H}_I \to \mathcal{H}_2\), let
be two finite dimensional subspaces $\mathcal{H}_K^{(n)} \subset \mathcal{H}_K$ and $\mathcal{H}_1^{(n)} \subset \mathcal{H}_1$, where $\phi$ and $K$ are related by the equation (7). Under the aforementioned conditions on the loss function $V$ and $\psi$ (on which depends the penalty term), in the statistical learning setting the representer theorem allows us to write

\[
\hat{g}^{(n)}_{R,\lambda} = \sum_{i=1}^{n} \beta_i K_{x_i},
\]

where $\beta_i \in \mathbb{R}$ for all $i \in \{1, \ldots, n\}$ are appropriate coefficients. Thus, the problem (29) can be re-formulated as follows

\[
\hat{g}^{(n)}_{R,\lambda} := \arg \min_{g \in \mathcal{H}_K^{(n)}} R_{\mathcal{H}_K}(g) + \lambda \psi(\|g\|_{\mathcal{H}_K}),
\]

where the optimization is performed on the finite dimensional subspace $\mathcal{H}_K^{(n)}$. Clearly, corollary 1 can be exploited to provide a representer theorem for $\hat{f}^{(n)}_{R,\lambda}$.

**Proposition 4.** The solution of the problem (36) admits the following representation

\[
\hat{f}^{(n)}_{R,\lambda} = \sum_{i=1}^{n} \beta_i \phi_{x_i},
\]

where $\beta_i \in \mathbb{R}$, for all $i \in \{1, \ldots, n\}$ are the same coefficients of equation (53). Finally the problem (36) can be re-formulated as follows

\[
\hat{f}^{(n)}_{R,\lambda} := \arg \min_{f \in \mathcal{H}_1^{(n)}} R_{\mathcal{H}_1}(Af) + \lambda \psi(\|f\|_{\mathcal{H}_1}),
\]
where $\mathcal{H}_1^{(n)}$ is defined in equation (52).

The major consequence of this result is that it is sufficient to determine coefficients $\{\beta_j\}_{j=1}^n$ in order to solve both problems (54) and (56). For the sake of completeness, we report the explicitly computation of the coefficients $\beta_j$ in the classical Tikhonov regularization case.

**Example 1.** Let us consider the Tikhonov regularization for a linear inverse problem which is known as penalized least square approach in supervised learning. Under the usual assumptions, we write the problem (54) as

$$
\hat{g}_\lambda^{(n)} = \arg\min_{g \in \mathcal{H}_1} \frac{1}{n} \sum_{i=1}^{n} (y_i - g(x_i))^2 + \lambda \|g\|_{\mathcal{H}_2}^2,
$$

and the problem (56) as

$$
\hat{f}_\lambda^{(n)} = \arg\min_{f \in \mathcal{H}_1} \frac{1}{n} \sum_{i=1}^{n} (y_i - Af(x_i))^2 + \lambda \|f\|_{\mathcal{H}_1}^2.
$$

Using the representer theorem and equation (40) the solution of the two problems (57) and (58) is given by solving the following

$$
\hat{\beta}_\lambda^{(n)} = \arg\min_{\beta \in \mathbb{R}^n} \frac{1}{n} \|y - K\beta\|_2^2 + \lambda \beta^T K \beta,
$$

where $y$ is the $n$-dimensional vector $y = (y_1, \ldots, y_n)^T$, and $K$ is the matrix with entries $K_{ij} := K(x_i, x_j)$, for each $i, j \in \{1, \ldots, n\}$. In such a case the solution $\hat{\beta}_\lambda^{(n)}$ is given by

$$
\hat{\beta}_\lambda^{(n)} = (K + \lambda nI)^{-1} y,
$$

and therefore solutions of problems (57) and (58) are given respectively by

$$
\hat{g}_\lambda^{(n)} = k^T (K + \lambda nI)^{-1} y,
$$

where $k = (K_{x_1}, \ldots, K_{x_n})^T$, and

$$
\hat{f}_\lambda^{(n)} = \Phi^T (K + \lambda nI)^{-1} y,
$$

where $\Phi = (\phi_{x_1}, \ldots, \phi_{x_n})^T$.

Analogously, the solutions $\hat{g}_R^{(n)}$ and $(\hat{f}_R^{(n)})^+$ defined in equations (28) and (35) respectively admit a finite representation. This follows from the fact that $\hat{g}_R^{(n)}$ can be seen as the minimizer of (54) with $\psi = 0$. Hence, at least a minimizer has a finite representation as $\psi$ is non-decreasing and it is unique as $R$ is strictly convex [40, 3]. In the next proposition we give a simple alternative proof of the fact that $\hat{g}_R^{(n)}$ and $(\hat{f}_R^{(n)})^+$ admit a finite representation based on $\Gamma$-convergence.

**Proposition 5.** Let $R_{x_i}$ be defined in equation (32), with $V$ strictly convex, coercive (as the definition in corollary 2) and Lipschitz continuous function. The solution $\hat{g}_R^{(n)}$ defined in equation (28) admits the following representation

$$
\hat{g}_R^{(n)} = \sum_{j=1}^{n} \alpha_j K_{x_j},
$$
where $\alpha_j \in \mathbb{R}$, for all $j \in \{1, \ldots, n\}$ are appropriate coefficients.

**Proof.** Let $\lambda > 0$ and let $\psi$ be a continuous convex and strictly increasing real-valued function. Fixed $n \in \mathbb{N}$, the sequence $(R_{\chi_i} + \lambda \psi(\| \cdot \|_{\mathcal{H}_K}))_{\lambda}$ satisfies the hypotheses of proposition 2 and it $\Gamma$-converges to $R_{\chi_i}$ as $\lambda \to 0$. This proves the convergence of minimizers, i.e. $\hat{s}_{R,\lambda}^{(n)} \to \hat{s}_R^{(n)}$ as $\lambda \to 0$, uniformly in $\| \cdot \|_{\mathcal{H}_K}$ for all $n \in \mathbb{N}$, where $\hat{s}_{R,\lambda}^{(n)}$ is defined in equation (27). Moreover, $\hat{s}_R^{(n)}$ admits the following representation

$$
\hat{s}_R^{(n)} = \sum_{j=1}^{n} \beta_j^\lambda K_{x_j},
$$

where $\beta_j^\lambda \in \mathbb{R}$ for all $j \in \{1, \ldots, n\}$. Therefore, $\sum_{j=1}^{n} \beta_j^\lambda K_{x_j}$ pointwise converges to $\hat{s}_R^{(n)} \in \mathcal{H}_K$ as $\lambda \to 0$ and each $\beta_j^\lambda$ has to converge to some value $\beta_j^0$. The limit can be written as $\sum_{j=1}^{n} \beta_j^0 K_{x_j}$, and this shows that $\hat{s}_R^{(n)} \in \mathcal{H}_K^{(n)}$. \hfill $\square$

**Corollary 4.** Under assumptions of proposition 5, consider $(\hat{f}_R^{(n)})^\dagger$ defined in equation (35). Then $(\hat{f}_R^{(n)})^\dagger$ admits the following representation

$$
(\hat{f}_R^{(n)})^\dagger = \sum_{j=1}^{n} \alpha_j \phi_{x_j},
$$

where $\alpha_j \in \mathbb{R}$, for all $j \in \{1, \ldots, n\}$ are the same coefficients in equation (63).

The proof is a straightforward consequence of proposition 5 and corollary 1.

### 4. Connection between convergence rates

The study of the convergence rates is carried out in parallel in different settings as in (inverse) learning problems [5, 8], statistical inverse problems [7], inverse problems with deterministic noise [2, 14]. In this section we focus on the convergence rates provided in the statistical learning setting and in the linear inverse problems with deterministic noise. The crucial difference between these two approaches lies in the independent variable which the error depends on. Whereas for learning problems the independent variable is the number of examples $n$, for inverse problems it is the noise level $\delta$ dealing with infinite dimensional noisy data. The relation between the optimal rates provided in these two settings under the same source condition is not straightforward [14, 5]. It is evident that there is no transformation between $n$ and $\delta$ (independently of the rate) mapping one rate to the other. The aim of this section is to find a relation between the two rates and to quantify their difference. To do this we introduce an estimator in the statistical learning setting which is different from the usual one, with the following properties.

- The expected error given by this estimator is always larger than the error given by the standard spectral regularized solution provided that a suitable relation between $n$ and $\delta$ holds true. Such an inequality allows to convert upper convergence rates depending on $n$ to upper convergence rates depending on $\delta$ and viceversa, lower convergence rates depending on $\delta$ to lower convergence rates depending on $n$ (section 4.1).
- It has the same upper rates of the spectral regularization methods [8] (section 4.2).
4.1. A link between the number of examples \( n \) and the noise level \( \delta \). Let \( L^\lambda \) be a linear regularization operator family by varying of \( \lambda \geq 0 \) from an Hilbert space \( \mathcal{H}_2 \) to another \( \mathcal{H}_1 \) [14]. Let \(( \mathcal{T}, \Theta, \mu)\) be a measure space with respect to the measure \( \mu \) on \(( \mathcal{T}, \Theta)\) where \( \mathcal{T} \) is a nonempty set and \( \Theta \) is a \( \sigma \)-algebra and \(( \mathcal{X}, \Sigma, \nu)\) be a measure space with respect to the measure \( \nu \) on \(( \mathcal{X}, \Sigma)\), where \( \mathcal{X} \) is a nonempty set and \( \Sigma \) is a \( \sigma \)-algebra. We assume that \( \nu \) is a positive and finite measure. We suppose that \( \mathcal{H}_1 \) is the \( L^2(\mathcal{T}, \nu) \) space (the Hilbert space of square integrable functions on \( \mathcal{T} \) with respect to the measure \( \nu \)) and \( \mathcal{H}_2 \) is the \( L^2(\mathcal{X}, \nu) \) space (the Hilbert space of square integrable functions on \( \mathcal{X} \) with respect to the measure \( \nu \)). We assume that \( L^\lambda \) has the following form

\[
L^\lambda y = \int_{\mathcal{X}} \ell^\lambda_x y(x) \, d\nu(x),
\]

where \( \ell^\lambda_x \in \mathcal{H}_1 \), \( \ell^\lambda_x(t) := \ell^\lambda(x,t) \) and \( \ell^\lambda(\cdot, t) \in \mathcal{H}_2 \) for each \( x \in \mathcal{X} \) and for each \( t \in \mathcal{T} \). Thanks to this last assumption the integral in equation (66) is finite. Moreover, we assume \( \sup_{t \in \mathcal{T}} \| \ell^\lambda(\cdot, t) \|_{\mathcal{H}_2} < \infty \). Such an assumption implies that \( L^\lambda \) is uniformly bounded and then for each \( y \in \mathcal{H}_2 \) \( L^\lambda y \) is bounded in supremum norm which assures that \( L^\lambda y \in \mathcal{H}_1 \). We denote with \( f^\delta \) the regularized solution given by the linear regularization operator \( L^\lambda \) applied to the noise free data \( y \), i.e.

\[
f^\lambda = L^\lambda y,
\]

and with \( f^\delta \) the regularized solution given by the noisy data \( y^\delta \), i.e.

\[
f^\delta = L^\lambda y^\delta,
\]

when \( \| y - y^\delta \| \leq \delta \). We introduce the following estimator computed from a set of discrete data as follows

\[
\hat{f}_n^\lambda = L^\lambda_x y = \frac{1}{n} \sum_{i=1}^{n} \ell^\lambda_{x_i} Y_i
\]

where \( x = (X_1, \ldots, X_n) \) and \( y = (Y_1, \ldots, Y_n) \) denote the samples.

Convergence rates are usually studied for linear regularization methods based on spectral theory. Now we introduce the standard regularization operator used in the spectral theory in our notation. We denote with \( s_\lambda \) the regularization function. Then the regularized solution \( f^\lambda \) is given by

\[
f^\lambda = s_\lambda( A^*A ) A^* y,
\]

where \( A^* \) is the adjoint operator of \( A \). With straightforward computations the solution (70) can be re-written in the form (67) by setting

\[
\ell^\lambda_x = s_\lambda( A^*A ) \phi_x,
\]

with \( x \in \mathcal{X} \). We remark that the required hypotheses on \( \ell^\lambda_x \), are satisfied under the assumption \( \sup_{t \in \mathcal{T}} \| \phi(\cdot, t) \|_{\mathcal{H}_2} < \infty \). Furthermore, the estimator defined in (69) takes the following form

\[
\hat{f}_n^\lambda = s_\lambda( A^*A ) A^*_x y,
\]

where \( A^*_x \) is the sampling operator associated to the set of samples, which is defined as \( A^*_x : \mathcal{H}_1 \to \mathbb{R}^n \)

\[
(A^*_x f)_j = \langle f, \phi_{x_j} \rangle_{\mathcal{H}_1},
\]
∀ \ j = 1, \ldots, \ n \text{ and } A_\phi^\ast \text{ is its adjoint operator given by}
\[ A_\phi^\ast y = \frac{1}{n} \sum_{j=1}^{n} \ Y_j \phi_{X_j}. \]

We want to highlight the difference with the usual statistical learning estimator and we denote the latter with
\[ \hat{f}_{\eta, \text{learn}} = \lambda (A_\phi^\ast A_\phi A_\phi^\ast)A_\phi^\ast y. \]

Along the lines of inverse learning problems, we make the following assumptions.

**Assumption 2.** The \( n \) observations \( (X_i, Y_i) \) are i.i.d. drawn from a probability distribution \( \rho \) on a Borel space \( \mathcal{X} \times \mathcal{Y} \) so that \( \nu \) is the marginal distribution of \( X \) (see section 3.1). We assume that the conditional expectation with respect to \( \rho(\cdot | \cdot) \) of \( Y \) given \( X \) is equal to
\[ E(Y|X = x) = A f^\dagger(x) = y(x) \]
for \( \nu \)-almost \( x \in \mathcal{X} \), where \( f^\dagger \in \mathcal{H}_1 \) is the generalized solution. We assume also that the variance of the conditional probability is
\[ \text{Var}(Y|X = x) = \sigma^2 \]
for \( \nu \)-almost \( x \in \mathcal{X} \), where \( \sigma \) is a constant.

Now we are in the position to prove a first inequality between the expected error provided by \( \hat{f}_{\eta}^\lambda \) as a function of \( n \) and the error provided by \( f_{\delta}^\lambda \). In what follows, to make it easier the writing, we do not write the subscript of the norms and we denote with \( E \) the mean computed with respect to the measure \( \rho^n \).

**Lemma 2.** Let \( \hat{f}_{\eta}^\lambda \) be defined in equation (69). Under assumption 2 we have
\[ E(\|\hat{f}_{\eta}^\lambda - f^\dagger\|^2) \geq \frac{\sigma^2}{n} \|L^\lambda\|^2_{HS} + \|f_{\delta}^\lambda - f^\dagger\|^2, \]
where \( \| \cdot \|_{HS} \) denotes the Hilbert Schmidt norm.

**Proof.** Denote with \( \varepsilon_n \) the difference between the estimate \( \hat{f}_{\eta}^\lambda \) obtained with \( n \) samples and the sought solution \( f^\dagger \). For any \( t \in \mathcal{T} \) we have
\[ \varepsilon_n^2(t) = \left( \frac{1}{n} \sum_{i=1}^{n} \ell_{X_i}(t)Y_i - f^\dagger(t) \right)^2 \]
\[ = \frac{1}{n^2} \sum_{i,j=1}^{n} \ell_{X_i}(t)\ell_{X_j}(t)Y_iY_j - \frac{2}{n} f^\dagger(t) \sum_{i=1}^{n} \ell_{X_i}(t)Y_i + (f^\dagger(t))^2. \]

By integrating over \( \mathcal{Y}^n \), we get
\[ \int_{\mathcal{Y}^n} \varepsilon_n^2(t) \ d\rho(\cdot | \cdot)^n = \frac{1}{n^2} \sum_{i=1}^{n} (\ell_{X_i}(t))^2 \sigma^2 + \frac{1}{n^2} \sum_{i,j=1}^{n} \ell_{X_i}(t)\ell_{X_j}(t) y(X_i)y(X_j) \]
\[ - \frac{2}{n} f^\dagger(t) \sum_{i=1}^{n} \ell_{X_i}(t)y(X_i) + (f^\dagger(t))^2, \]
\[ \int_{\mathcal{Y}^n} \varepsilon_n^2(t) \ d\rho(\cdot | \cdot)^n = \frac{1}{n^2} \sum_{i=1}^{n} (\ell_{X_i}(t))^2 \sigma^2 + \frac{1}{n^2} \sum_{i,j=1}^{n} \ell_{X_i}(t)\ell_{X_j}(t) y(X_i)y(X_j) \]
\[ - \frac{2}{n} f^\dagger(t) \sum_{i=1}^{n} \ell_{X_i}(t)y(X_i) + (f^\dagger(t))^2, \]
where $\varphi(\cdot) \in \mathcal{Y}$. Then, by integrating over $\mathcal{X}^n$ we obtain

$$
\int_{\mathcal{X}^n} \int_{\mathcal{Y}^n} \varphi^2(t) \, d\varphi(\cdot)^\otimes n \, d\varphi^n = \frac{\sigma^2}{n^2} \sum_{i=1}^n \int_{\mathcal{X}} (\varphi^2_{X_i}(t)) \, d\varphi(X_i) + \frac{1}{n^2} \sum_{i=1}^{n^2} \left( \int_{\mathcal{X}} \varphi^2_{X_i}(t) \, d\varphi(X_i) \right)^2 + \frac{1}{n^2} \sum_{i=1}^n \int_{\mathcal{X}} \varphi^2_{X_i}(t) \, d\varphi(X_i) - \frac{2}{n} \varphi^2(t) \sum_{i=1}^n \int_{\mathcal{X}} \varphi^2_{X_i}(t) \, d\varphi(X_i) + (\varphi^2(t))^2
$$

(79)

where we used that $\varphi$ is a probability measure on $\mathcal{X}$. Therefore, we have

$$
\mathbb{E} \left( \| \hat{\varphi}^2 - f^\dagger \|^2 \right) \geq \int_{\mathcal{Y}^n} \frac{\sigma^2}{n} \int_{\mathcal{X}} (\varphi^2_{X_i}(t)) \, d\varphi(X_i) + (\varphi^2(t))^2 \, d\mu(t)
$$

(80)

as required. □

**Proposition 6.** Let us consider the inverse problem (1), a family of linear regularization methods $L^\lambda$ and its discretized version (69). Then, for each $n \in \mathbb{N}$ there exists a function $\Delta(n, \lambda)$ such that for each $0 < \delta \leq \Delta(n, \lambda)$ and infinite dimensional noisy data $y^\delta$ such that $\| y^\delta - y \| \leq \delta$, the following inequality holds

$$
\| f^\delta - f^\dagger \|^2 \leq \mathbb{E} \left( \| \hat{\varphi}^2 - f^\dagger \|^2 \right).
$$

Moreover,

$$
\Delta(n, \lambda) = \frac{1}{\sqrt{\frac{\sigma^2}{n} + \varepsilon(\lambda)^2 + \varepsilon(\lambda)}} \frac{\sigma^2}{n},
$$

(81)

(82)
where $\varepsilon(\lambda) = \|f^\lambda - f^\dagger\|_{HS}$. 

Conversely, for each $\delta > 0$ there exists a function $N(\delta, \lambda)$ such that for each $n \in \mathbb{N}$ such that $n \leq N(\delta, \lambda)$ the equation (81) applies and

$$N(\delta, \lambda) = \frac{\sigma^2}{\delta^2 + 2\varepsilon(\lambda)}.$$  

Proof. We start from the result of lemma 2. Easy manipulation of formula (76) leads to

$$\sqrt{\mathbb{E} \left( \|\hat{f}_n^\lambda - f^\dagger\|^2 \right)} \geq \Delta(n, \lambda) \|L^\lambda\|_{HS} + \|f^\lambda - f^\dagger\|$$  

where $\Delta(n, \lambda)$ is defined as in equation (82). For each $\delta > 0$, let $y^\delta$ s.t. $\|y^\delta - y\| \leq \delta$, then a simple calculation gives

$$\|f^\lambda_\delta - f^\dagger\| \leq \delta \|L^\lambda\| + \|f^\lambda - f^\dagger\|.$$  

Further, for each $\delta \leq \Delta(n, \lambda)$ we have

$$\sqrt{\mathbb{E} \left( \|\hat{f}_n^\lambda - f^\dagger\|^2 \right)} \geq \delta \|L^\lambda\| + \|f^\lambda - f^\dagger\|$$  

as $\|\cdot\|_{HS} \geq \|\cdot\|$. From equations (85) and (86) we obtain $\forall \delta \leq \Delta(n, \lambda)$

$$\|f^\lambda_\delta - f^\dagger\|^2 \leq \mathbb{E} \left( \|\hat{f}_n^\lambda - f^\dagger\|^2 \right)$$  

for each $y^\delta$ for which $\|y^\delta - y\| \leq \delta$.

Conversely, let $\delta > 0$. For each $n \leq N(\delta, \lambda)$, with $N(\lambda, \delta)$ defined by equation (83) we have

$$\delta \leq \Delta(n, \lambda)$$  

and so the thesis is proved. □

Functions $\Delta(n, \lambda)$ and $N(\delta, \lambda)$ express the dependency between the noisy level $\delta$ and the number of samples $n$. To make explicit this dependency we need to specify the rate of convergence of $\lambda \to 0$ both considered as a function of $\delta$ and $n$. For the sake of convenience, we introduce the following

**Definition 4.** For any given $\lambda_n$ we define

$$\tilde{\delta}(n) := \Delta(n, \lambda_n).$$  

Conversely, for any given $\lambda_\delta$ we define

$$\tilde{n}(\delta) := \lfloor N(\delta, \lambda_\delta) \rfloor,$$

where the symbol $\lfloor \cdot \rfloor$ denotes the integer part.

From now on, in order to express asymptotic behaviors we make use of the Landau symbols $O$, $\Omega$ and $\Theta$.

**Lemma 3.** Let $\varepsilon(\lambda) \in \Theta(\lambda^\gamma)$, with $\gamma \geq 0$. If $\lambda_n \in \Theta(n^{-p})$, with $p > 0$, then

$$\tilde{\delta}(n) \in \Theta \left(n^{-\max(\frac{1}{2} - p\gamma)} \right).$$
If \( \lambda_\delta \in \Theta(\delta^{p'}) \), with \( p^* > 0 \), then
\[
\tilde{n}(\delta) \in \Theta\left(\delta^{-\min(2,p^*+1)}\right).
\]

Proof. The equation (91) follows from the definition of \( \tilde{\delta} \) and from hypotheses \( \lambda_n \in \Theta(n^{-p}) \) and \( \varepsilon(\lambda) \in \Theta(\lambda^\gamma) \). In the same way the equation (92) follows from the definition of \( \tilde{n} \) and from hypotheses \( \lambda_\delta \in \Theta(\delta^{p'}) \) and \( \varepsilon(\lambda) \in \Theta(\lambda^\gamma) \).

Lemma 4. Given \( \lambda_n \) there exists a unique \( \lambda_\delta \) such that
\[
\tilde{\delta} \circ \tilde{n} = \text{id}_{\tilde{\delta}(\delta)},
\]
where \( \text{id}_{\tilde{\delta}(\delta)} \) indicates the identity on the set \( \tilde{\delta}(\delta) = \{ \delta > 0 \mid \frac{\sigma^2}{\delta^2 + 2\delta \varepsilon(\delta)} \in \mathbb{N}\} \) and
\[
\Lambda^\delta = \Lambda^\delta \circ \tilde{\delta},
\]
where \( \Lambda^n : \mathbb{N} \to \mathbb{R} \) and \( \Lambda^\delta : \mathbb{R} \to \mathbb{R} \) are such that \( \lambda_n = \Lambda^n(n) \) and \( \lambda_\delta = \Lambda^\delta(\delta) \). Furthermore,
\[
\tilde{n} \circ \tilde{\delta} = \text{id}_\mathbb{N}.
\]

Proof. The existence and uniqueness of \( \lambda_\delta \) such that (93) and (94) are verified follow by defining \( \lambda_\delta := \Lambda^n(\tilde{n}(\delta)) \). With straightforward calculus it can be verified that (94) implies (95).

Similarly, we give the converse result.

Lemma 5. Given \( \lambda_\delta \), there exists a unique \( \lambda_n \) such that
\[
\tilde{n} \circ \tilde{\delta} = \text{id}_\mathbb{N}
\]
and
\[
\Lambda^\delta = \Lambda^n \circ \tilde{n}
\]
where we have used the same notation of lemma 4. Furthermore,
\[
\tilde{\delta} \circ \tilde{n} = \text{id}_{\tilde{\delta}(\delta)}.
\]

The proof is analogous to the one of lemma 4 by defining \( \lambda_n = \Lambda^\delta(\tilde{n}(\delta)) \).

The following result relates a given upper convergence rate computed with respect to \( n \) to the one computed with respect to \( \delta \).

Theorem 4. Let the average upper rate with respect to the number of samples identically and independently drawn according to a distribution \( \rho \) be equal to \( n^{-\alpha} \) for a given \( \alpha > 0 \), i.e.
\[
\mathbb{E}(\|\hat{f}_n^\lambda - f^\dagger\|^2) \in O\left(\frac{1}{n}\right)^\alpha,
\]
given \( \lambda = \lambda_n = \Theta(n^{-p}) \), with \( p > 0 \) and \( \varepsilon(\lambda) = \Theta(\lambda^\gamma) \), with \( \gamma > 0 \). Then the upper rate of the error with respect to the noise level \( \delta \to 0 \) is given by
\[
\|f_\delta^\lambda - f^\dagger\|^2 \in \begin{cases} 
O\left(\frac{\sigma^2}{\delta^2 + 2\delta \varepsilon(\delta)}\right) & \text{if } p\gamma \geq \frac{1}{2} \\
O\left(\delta^{-\alpha\gamma}\right) & \text{if } p\gamma < \frac{1}{2}, 
\end{cases}
\]
where \( f^o_\delta \) is defined in equation (68), \( y^\delta \) is such that \( \| y^\delta - y \| \leq \delta \), and \( \lambda = \lambda_\delta \) is defined in lemma 4 and it has the following rate

\[
\lambda_\delta \in \begin{cases} \Theta (\delta^{2p}) & \text{if } p\gamma \geq \frac{1}{2} \\ \Theta \left( \delta^{-\frac{1}{p\gamma}} \right) & \text{if } p\gamma < \frac{1}{2}. \end{cases}
\]  

Proof. Given \( \lambda_n = \Lambda^\alpha(n) \), we define \( \lambda_\delta = \Lambda^\delta(\delta) \) according to lemma 4, so that equations (93) and (94) hold. The rate of \( \lambda_\delta \) given in (100) can be found by using the hypothesis \( \lambda_n = \Theta(n^{-p}) \) and lemma 3. Now we prove (99). Thanks to proposition 6 and lemma 3, for each \( \lambda > 0 \) and \( \delta > 0 \), there exists \( \tilde{n}(\delta) \) such that for all \( n \leq \tilde{n}(\delta) \)

\[
\| f^o_\delta - f^\dagger \|^2 \leq \mathbb{E}(\| f_n^\lambda - f^\dagger \|^2). 
\]

Let \( n = \tilde{n}(\delta) \), then

\[
\| f^\lambda_\delta - f^\dagger \|^2 \leq \mathbb{E}(\| f^\lambda_{\tilde{n}(\delta)} - f^\dagger \|^2). 
\]

Let \( \lambda = \lambda_\delta \). Then there exist \( n_0 \in \mathbb{N} \) and \( M > 0 \) such that

\[
\| f^\lambda_{\tilde{n}(\delta)} - f^\dagger \|^2 \leq \mathbb{E}(\| f^\lambda_{\tilde{n}(\delta)} - f^\dagger \|^2) - \mathbb{E}(\| f^\lambda_{\tilde{n}(\delta)} - f^\dagger \|^2) \leq M \left( \frac{1}{\tilde{n}(\delta)} \right)^\alpha,
\]

for all \( \tilde{n}(\delta) > n_0 \). From equations (100) and (103), and by using lemma 3 we obtain

- if \( p\gamma \geq \frac{1}{2} \) then \( \lambda_\delta = \Theta(\delta^{2p}) \), therefore from proposition 6 we have \( \tilde{n}(\delta) \in \Theta(\delta^2) \) and from equation (103) we obtain \( \| f^\lambda_\delta - f^\dagger \|^2 \in O(\delta^{2\alpha}) \)

- if \( p\gamma < \frac{1}{2} \) then \( \lambda_\delta \in \Theta(\delta^{-\frac{1}{p\gamma}}) \), therefore from proposition 6 we have \( \tilde{n}(\delta) \in \Theta(\delta^{\frac{1}{1-p\gamma}}) \) and from equation (103) we obtain \( \| f^\lambda_\delta - f^\dagger \|^2 \in O(\delta^{\frac{1}{1-p\gamma}}) \).

This completes the proof.

Now we give the converse result regarding lower rates.

Theorem 5. Let \( f^o_\delta \) be defined in (68). Let the lower rate of the convergence error with respect to the noise level \( \delta \to 0 \) equal to \( \delta^\alpha \) for a given \( \alpha > 0 \), i.e.

\[
\| f^o_\delta - f^\dagger \|^2 \in \Omega(\delta^\alpha),
\]

where \( y^\delta \) is such that \( \| y^\delta - y \| \leq \delta \), \( \lambda = \lambda_\delta = \Theta(\delta^{p^*}) \), with \( p^* > 0 \) and \( e(\lambda) = \Theta(\lambda^\gamma) \), with \( \gamma > 0 \). Then the average lower rate with respect to the number of samples \( n \to \infty \) is given by

\[
\mathbb{E}(\| f^\lambda_n - f^\dagger \|^2) \in \begin{cases} \Omega \left( n^{-\frac{\alpha}{p^*}} \right) & \text{if } p^*\gamma \geq 1 \\ \Omega \left( n^{-\frac{\alpha}{1+p^*\gamma}} \right) & \text{if } p^*\gamma < 1 \end{cases}
\]

where \( \lambda = \lambda_n \) is defined in lemma 5 and it has the following rate

\[
\lambda_n \in \begin{cases} \Theta \left( n^{-\frac{\alpha}{p^*}} \right) & \text{if } p^*\gamma \geq 1 \\ \Theta \left( n^{-\frac{\alpha}{1+p^*\gamma}} \right) & \text{if } p^*\gamma < 1. \end{cases}
\]
Proof. The proof exploits a similar argument to the one used for theorem 4. Given \( \lambda_\delta = \Lambda^\delta(\delta) \), by defining \( \lambda_n = \Lambda^n(n) \) according to lemma 5, it can be proved that the rate of \( \lambda_n \) is given by equation (106). To prove (105) one has to reverse the role of \( n \) and \( \delta \) in the proof of theorem 4 and use proposition 6 and hypothesis (104). In such a way one obtains that for each \( n \in \mathbb{N} \), there exist \( \delta_0 > 0 \) and \( M' > 0 \) such that

\[
E(||\tilde{f}_n^\lambda - f^\dagger||^2) \geq ||f_{\delta(n)}^\Lambda(\delta(n)) - f^\dagger||^2 \geq M'(\delta(n))^{\alpha},
\]

for all \( \delta(n) < \delta_0 \). The thesis follows from (106) and (107) and lemma 3. \( \square \)

4.2. Convergence rates in the statistical setting. In the statistical learning framework a lot of research is devoted to investigate convergence rates [8, 5, 21]. In this section we show that, under the same hypotheses considered in [8], the upper rate of the estimator \( \tilde{f}_n^\lambda \) defined in (72) is of the same order of the one of the classical spectral estimator [8]. First we describe the usual assumptions considered in the study of convergence rates. Source conditions are expressed in terms of restrictions of the probability space, and they correspond to assume a certain degree of smoothness of the infinite dimensional solutions and operators. The first restriction applies to the smoothness of the sought solutions. We assume that the solution belongs to the set

\[
\omega(r,R) := \{ f \in \mathcal{H}_1 : f = B^\dagger w, \| w \|_{\mathcal{H}_1} \leq R \},
\]

where \( B := A^*A \), with \( A^* \) the adjoint operator and \( r > 0 \) and \( R > 0 \). In the statistical framework this assumption is given as a requirement on the probability \( \rho(\cdot) \). In particular, \( \rho(\cdot) \) has to be such that (74) holds and the sought solution belongs to the set \( \omega(r,R) \). The second restriction applies to the eigenvalue decay of the operator \( B \). We assume that

\[
\mu_j \leq \frac{d}{j^b}
\]

where \( \mu_j \) are the eigenvalues of \( B \) for each \( j \in \mathbb{N}, j \geq 1 \), \( d > 0 \) and \( b > 1 \). In the statistical framework this assumption is given as a requirement on the probability \( \nu \) which \( B \) depends on. These two assumptions are restrictions on \( \rho(\cdot) \) and \( \nu \) respectively and they are summarized as a single restriction on the probability space by requiring that \( \rho \) given by equation (25) belongs to a suitable subspace \( \mathcal{M}(r,R,b) \) of the probability space where \( \mathcal{M}(r,R,b) \) represents the class of models (for details see [8]).

We give an upper rate for \( E(||\tilde{f}_n^\lambda - f^\dagger||^2) \) by exploiting the upper bound given in [7], where a more general mixed type noise model is considered and the stochastic part of the noise is modeled as an Hilbert-space process. We recall the properties of the regularization function \( s_\lambda \).

Definition 5. The regularization (or filtering) functions \( s_\lambda \) for \( \lambda > 0 \) defined on the spectrum of \( A^*A \), denoted by \( \sigma(A^*A) \), have to satisfy the following properties

- there exists a constant \( D > 0 \) such that

\[
\sup_{t \in \sigma(A^*A)} |ts_\lambda(t)| \leq D \text{ uniformly in } \lambda > 0
\]

- there exists a constant \( E > 0 \) such that

\[
\sup_{\lambda > 0} \sup_{t \in \sigma(A^*A)} |\lambda s_\lambda(t)| \leq E
\]
there exists \( q > 0 \) called qualification of the method and constants \( C_q > 0 \) such that

\[
\sup_{t \in \sigma(A^*A)} |t^\nu(1 - ts_\lambda(t))| \leq C_q \lambda^\nu \forall \lambda > 0 \text{ and } 0 \leq \nu \leq q.
\]

We remark that \( \mathbb{E}(\|\hat{f}_n^\lambda - f^\dagger\|^2) \) satisfies the bias-variance decomposition as follows

\[
\mathbb{E}(\|\hat{f}_n^\lambda - f^\dagger\|^2) = B(\hat{f}_n^\lambda)^2 + \mathbb{E}(\|\hat{f}_n^\lambda - \mathbb{E}(\hat{f}_n^\lambda)\|^2),
\]

where \( B(\hat{f}_n^\lambda) := \mathbb{E}(\|\hat{f}_n^\lambda - f^\dagger\|) \) is the bias term and \( \mathbb{E}(\hat{f}_n^\lambda) = f^\lambda \). Under the source condition (108) the bias term can be bounded by

\[
B(\hat{f}_n^\lambda) \leq C_r \lambda^r R,
\]

where \( C_r \) is the constant of the property (112) of the regularization function \( s_\lambda \). Hereafter, we consider \( r \leq q \). The estimation of the variance term needs more manipulations. In the following result we show the optimal upper rate achieved by \( \hat{f}_n^\lambda \).

**Lemma 6.** Let \( \hat{f}_n^\lambda \) be defined in (72) and let the model be described by (74) and (75). Under the source conditions (108) and (109) we have

\[
\mathbb{E}(\|\hat{f}_n^\lambda - f^\dagger\|^2) \in O\left(\left(\frac{1}{n}\right)^{\frac{2r}{2r+1+b}}\right),
\]

with \( \lambda \in \Theta\left(\left(\frac{1}{n}\right)^{\frac{2r}{2r+1+b}}\right) \).

**Proof.** We follow the argument given in the section 4.3 in [7]. We define \( \tilde{\varepsilon} \) as an Hilbert-space noise process such that \( A^*\tilde{\varepsilon} = A\lambda \tilde{y} - A^*A f^\dagger \). The noise \( \varepsilon = \tilde{\sigma} \tilde{\varepsilon} \), where \( \tilde{\sigma} = \frac{\tilde{\sigma}}{\sqrt{n}} \) with \( \tilde{\sigma} \) constant depending on the variance \( \sigma^2 \), satisfies the assumption of the Theorem 3 in [7]. Then, we have the following bound

\[
\mathbb{E}(\|\hat{f}_n^\lambda - \mathbb{E}(\hat{f}_n^\lambda)\|^2) = \mathbb{E}(\|s_\lambda(A^*A)A^*\tilde{\varepsilon}\|^2) \leq \frac{C}{n} \frac{1}{\lambda^2} \int_0^\lambda \beta^{-\frac{1}{2}} d\beta = \frac{C}{n} \frac{1}{\lambda^2} L = \frac{C}{n},
\]

under assumption (109) and where \( L \) is a constant which depends on \( D \) and \( E \) (see properties (110) and (111)) and constants in the assumption (109). Therefore, under assumption (108) we obtain

\[
\mathbb{E}(\|\hat{f}_n^\lambda - f^\dagger\|^2) \leq C^2 \lambda^{2r} R^2 + \frac{C}{n} \frac{1}{\lambda^{1+b}}.
\]

By balancing terms in the r.h.s. of (117) we have the thesis. \( \square \)

Then, the upper rate given in (115) is the same of the classical spectral estimator \( \hat{f}_{n,\text{learn}}^\lambda \) defined in (73).
4.3. Conversion of convergence rates. Under assumptions (108) and (109) we use theorem 4 to transform the upper rate (115) to an upper rate for the classical spectral regularization depending on \( \delta \). Let \( f_\delta^\lambda := s_\lambda (A^*A)A^*y_\delta \) and \( \gamma \) be defined as in theorem 4 then we have two cases:

- if \( \frac{\gamma}{2r+1+b} \geq \frac{1}{2} \)

\[
(118) \quad \| f_\delta^\lambda - f^\dagger \| \in O \left( \delta^{\frac{1}{2r+1+b}} \right)
\]

where \( \lambda \in \Theta \left( \delta^{\frac{1}{2r+1+b}} \right) \).

- if \( \frac{\gamma}{2r+1+b} < \frac{1}{2} \)

\[
(119) \quad \| f_\delta^\lambda - f^\dagger \| \in O \left( \delta^{\frac{1}{2r+1+b} - \gamma} \right)
\]

where \( \lambda \in \Theta \left( \delta^{\frac{1}{2r+1+b} - \gamma} \right) \).

The first case gives a faster rate with respect to the second one. We remark that in both cases the obtained upper rate is slower than the classical optimal one, i.e. \( O \left( \delta^{\frac{2}{2r+1+b}} \right) \) [14]. The ratio between the first best case in (118) and the classical optimal one \( O \left( \delta^{\frac{2}{2r+1+b}} \right) \) is equal to \( \tau = \frac{2r+1+b}{2r+1} > 1 \). \( \tau \) represents the loss factor converting the optimal rate in the statistical setting to a rate depending on the noise level \( \delta \). We have \( \tau < 2 \) and \( \tau \) is close to 1 when \( b \) is large, which means that the eigenvalues decay (109) has to be fast.

**Example 2.** We consider the Tikhonov solution. In this case we have \( \gamma = r + \frac{1}{2} \).

- **Upper rate conversion.** Using the upper rate (115) and theorem 4 we have

\[
(120) \quad \| f_\delta^\lambda,\text{Tik} - f^\dagger \| \in O \left( \delta^{\frac{r}{2r+1+b}} \right),
\]

where \( \lambda \in \Theta \left( \delta^{\frac{r}{2r+1+b}} \right) \). By comparing with the classical optimal rate \( O \left( \delta^{\frac{r}{2r+1+b}} \right) \), we have that the loss factor is \( \tau_{\text{Tik}} = \frac{2r+1+b}{2r+1} < 3 \).

- **Lower rate conversion.** The classical lower rate established for Tikhonov solution with respect to the noise level \( \delta \), i.e. \( \| f_\delta^\lambda,\text{Tik} - f^\dagger \| \in \Omega \left( \delta^{\frac{r}{2r+1+b}} \right) \), under the source condition (108) can be converted to a lower rate with respect to the number of samples \( n \) using theorem 5, that is

\[
(121) \quad \mathbb{E}(\| f_{\delta,\text{Tik}}^\lambda - f^\dagger \|^2) \in \Omega \left( \left( \frac{1}{n} \right)^{\frac{2r+1+b}{2r+1+b}} \right),
\]

where \( \lambda \in \Theta \left( \left( \frac{1}{n} \right)^{\frac{2r+1+b}{2r+1+b}} \right) \). In [8] it is proven that \( \left( \frac{1}{n} \right)^{\frac{2r+1+b}{2r+1+b}} \) is a minimax rate for the estimator \( f_{n,\text{learn}}^\lambda \). The lower bound is given under the source condition (108) and the following
hypothesis on the eigenvalues decay of the operator $B$

$$\mu_j \geq \frac{a}{j^p}$$

where $a > 0$, $b > 1$ and $j \geq 1$. Comparing the result in (121) and the lower rate established in [8] we note that the lower rate in (121) is retrieved as $b \to \infty$. In such a case the r.h.s of the eigenvalue condition (122) vanishes for all $j \geq 2$ when $b \to \infty$. Thus, it takes the form $\mu_1 \geq a > 0$ which can be read as a condition $B \neq 0$.

5. Conclusion

In this paper we attempted to give a uniform vision of discrete inverse problems and supervised learning. We started from the infinite dimensional approximation problem in a RKHS, showing that there is a natural correspondence between its solution and the solutions of a certain class of inverse problems. Such a correspondence suggests that these problems are equivalent to some extent: we showed that as well as the data space of a linear inverse problem is a RKHS, the feature space of a learning problem can be thought of as the parameter space of a linear inverse problem. Then, we distinguished learning and discrete inverse problems according to a different discretization scheme of the same infinite dimensional problem. We analyzed the convergence of the discretized functionals and solutions to their corresponding ideal ones, and in the case of a deterministic discretization we gave some mild sufficient conditions to have the convergence relying on the $\Gamma$-convergence theory. Finally, we investigated the connection between error convergence rates in the case the error is computed as a function of the noise level $\delta$ and as a function of the number of examples $n$. We quantified the deviation between optimal rates in the two frameworks.

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