Transient analysis of power management in wireless sensor network with start-up times and threshold policy

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Abstract
Queueing models play a significant role in analysing the performance of power management systems in various electronic devices and communication systems. This paper adopts a multiple vacation queueing model with a threshold policy to analyse the power-saving mechanisms of the wireless sensor network (WSN) using the dynamic power management technique. The proposed system consists of a busy state (transmit state), wake-up state, shutdown state and inactive state. In this model, the server switches to a shutdown state for a random duration of time after serving all the events (data packets) in the busy state. Events that arrive during the shutdown period cannot be served until the system size reaches the predetermined threshold value of \( k \) and further it requires start-up time and a change of state to resume service. At the end of the shutdown period, if the system size is less than \( k \), then the server begins the inactive period; otherwise, the server switches to the wake-up state. For this system, an explicit expression for the transient and steady-state solution is computed in a closed form. Furthermore, performance indices such as mean, variance, probability that the server is in various stages of power management modes and mean power consumption are computed. Finally, graphical illustrations are made to understand the effect of the parameters on the performance of the system.

Keywords Single server · Transient probabilities · Steady-state probabilities · Threshold policy · Start-up times · Mean power consumption

1 Introduction
WSN is a smart technology used to collect information about the neighbouring environment by sensing and share the information with the user or with the base station. WSN uses smart sensors for sensing, computing and collecting information. These sensors are tiny, low-cost and battery-powered. The WSNs have a far-reaching application that has transformed human lives in many aspects and paved the way for intelligent living technology. It is applied in agriculture, defence, traffic, surveillance, natural disaster, etc. for tracking and monitoring. Although WSN is popular technology, it has got some limitations, particularly in battery life. The battery cannot be recharged during the surveillance because the nodes of WSNs may be deployed in an unfriendly environment. The lifetime of the WSN depends on the power consumption executed at each sensor node. Hence, the energy consumption in the WSNs is considered a serious issue that has attracted many researchers in recent times. To control the power consumption in sensor nodes, a variety of Dynamic Power Management (DPM) techniques have been proposed by researchers in recent times [15]. It protocols the WSN to switch power-saving modes and selectively shut down the components during the idle time for saving power. It is also applied in many portable devices for power management. The DPM policies can be classified as predictive and stochastic [1]. In this paper, a stochastic-based DPM approach is adopted. To achieve energy efficiency at a lower cost, a suitable stochastic modelling approach is implemented at the Operating System level. This paper presents a transient and steady-state analysis of the DPM using an M/M/1 queuing system with a vacation and threshold policy.

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Queueing systems with vacation play a vital role in analysing the power-saving mechanism (PSM) of computer and communication systems. Several authors have applied various vacation policies to study the performance of the PSM in various communication systems. See Dimitriou [4], Misra and Goswami [13], Sampath et al. [17] and Ren et al. [16] and references therein. In these models, the server switches to busy and vacation states frequently. To avoid frequent switching, the system designers prefer an N-policy scheme which will be more effective than other vacation schemes. In the N-policy queueing systems, the server is turned OFF or it stays idle when there is no job (data packet) in the system and the server is turned ON when the system size reaches a threshold value N. The notion of an N-policy was first studied by Yadiv and Naor [19]. Later, many researchers studied various queueing systems with N-policy in a different context. See Wang and Ke [18], Parthasarathy and Sudhesh [14] and references therein. In the multiple vacation queuing system with N-policy, the server switches to a busy state from vacation only if the system accumulates N jobs.

1.1 DPM protocol in WSN

The functions of the DPM are as follows. The DPM in WSN consists of a service requestor (SR), a service provider (SP), a power manager (PM) and an event queue (EQ). The activity of different devices in a WSN can be discrete or continuous or event-driven all of which have an impact on battery life. The PM can be a software or hardware module component. The SR and the SP in the DPM are considered as incoming events and servers respectively. The incoming event joins the EQ and waits for service. After getting the service from the SP, the jobs are removed from the EQ. The PM in the DPM monitors the states of the SR, the SP and the EQ and controls the power management in the system. The power management modes of the DPM are shutdown state, wake-up state and inactive state. At the end of a busy period, the SP switches to the shutdown state during which the jobs can join the EQ. At this stage, the server cannot be activated until the EQ reaches a threshold value \( k \). Once the EQ reaches a threshold value \( k \), the SP enters into a wake-up stage. From the wake-up state, the SP switches to the active mode. If no events arrive during the shutdown state, the SP will switch to an inactive state and wait for jobs to accumulate \( k \) jobs. The DPM with the threshold policy has many advantages over the other vacation schemes. It reduces the frequent switching of the wake-up and shutdown states, thereby increasing the life of sensor nodes and controlling the power consumption. It accumulates \( k \) events and then processes the job to make the system more energy efficient. The key feature of this work is to achieve minimum power consumption for each arrival rate using a threshold policy.

1.2 Related work in energy saving in WSN

Several analytical studies have been conducted in the literature regarding the PSM of WSN. Many researchers have analysed the PSM using queueing theory and proposed various sleep-wake schedules. The N threshold policy has been adopted by researchers to reduce the collision probability such that the total transmitting power can be reduced. To turn on the radio server, a queue threshold of N has been chosen as a counter. Using the N threshold policy as a counter basis, the minimum transmitting power consumption can be obtained.

Jiang et al. [7] applied an M/M/1 queueing model with N-policy to study the energy hole problem in WSN and discussed the steady-state results. In this paper, two states are considered namely busy and idle. Further, the power consumption function is provided and concluded that there is an optimal queue number (N) for reducing sensor node power consumption. Huang and Lee [5] studied the PSM of WSN using an N-policy M/G/1/K queueing model and presented the steady-state results. In this paper, three states are considered in this paper namely busy, sleep and idle. Blondia [2] used a finite capacity queueing model with multiple server vacations to study energy harvesting WSN. This model consists of the scheduling and vacation state. Lee and Yang [10] analysed the PSM of WSN using an N-policy Geo/G/1 queueing model. This model consists of a transmitting state and an idle state. Chen et al. [3] proposed an improved stochastic model for the WSN. This model consists of three power-saving states namely shutdown, wakeup and inactive. In this paper, the author proposed that power saving can be achieved by decreasing the number of shutdown and wake-up processes. Jayarajan et al. [6] proposed a Modified Energy Minimization Scheme to reduce the mean delay and average energy consumption of individual nodes of WSN. The authors applied the M/D/1 priority queueing model with N-policy to analyse their proposed model. Ma et al. [12] studied the PSM of WSN using a repairable M/M/2 queueing model with threshold policy and presented the steady-state results using the matrix-geometry method.

From the literature survey, it is observed that most research on the WSN has focused mainly on the steady-state analysis of the system. Surprisingly the transient analysis of the system has not received as much attention. In many real-time applications, the system experiences a change and such changes can be measured by the transient analysis. The steady-state results cannot be used to determine the number of data packets waiting in the queue during the transmit state or inactive state or shutdown state at some time instant \( t \). Hence steady-state simulation findings do not adequately depict system behaviour since it relies on the system’s long-term performance to negate the initial conditions effect. Thus, the transient analysis will be more realistic to analyse the
performance of DPM. This motivates us to study the transient results of the investigated model.

1.3 Our contribution

In this work, we investigate the performance of DPM using an M/M/1 queueing model with vacation and threshold policy. The vacation policy captures the functions of power-saving modes such as wake-up state and shutdown state. The threshold policy captures the functions of the inactive state. We present the theoretic analysis to evaluate the probability for busy, wake-up, inactive and shutdown states in both transient and steady-state. The significance and the advantages of the model are as follows. In the previous section, it is observed that there exists a threshold value for each arrival of the model are as follows. In the previous section, it is observed that there exits a threshold value for each different pair of \((\lambda, k)\) called dynamic threshold policy. According to this method, each sensor node requires a different threshold value to ensure a minimum amount of energy waste. We study a dynamic threshold policy to solve this problem, in which the method determines the threshold value for each different \(\lambda\). The key factors that affect the WSN during their operation are power consumption in each state, switching cost and duration of time spent in each state. To optimize this system, mean power consumption is discussed in the paper and a threshold value \(k\) has been derived to acquire minimum power consumption for the WSN while considering each different arrival rate. The transient analysis made in the paper enables the system analyst to understand the status of the system at any time \(t\).

The remaining part of the paper is structured as follows. The description of the model is presented in Sect. 2. The transient probabilities of the wake-up, the shutdown, the inactive and busy states are presented in Sect. 3. The mean, variance and the probability that the system is in power-saving modes are presented in Sect. 4. The steady-state probabilities are derived explicitly in Sect. 5. The performance indices of the system in the steady-state are presented in Sect. 6. The results obtained in Sects. 3–6 are graphically illustrated in Sect. 7. The Conclusion and future work are presented in Sect. 8.

2 Description of the model

The model description of the WSN with start-up times and threshold policy is presented in this section.

1. The events join the queue according to a Poisson process with a rate of \(\lambda\) and the events receive service with a rate of \(\mu\) which follows an exponential distribution.

2. After serving all the events in the busy state, the server switches to the shutdown mode of random duration \(V\). The service is only provided in the busy state. During the shutdown period, events are allowed to join the queue, but the server will not resume service until the system accumulates \(k\) events.

3. At the end of period \(V\), if the system reaches the threshold value \(k\), then the system requires a start-up time which is exponentially distributed with the rate \(\theta_1\) to begin the service. To start up, the system requires a change of state. The server switches from a shutdown state to a wake-up state which is exponentially distributed with a rate \(\theta_2\).

4. At the end of the shutdown period \(V\), if the system size is less than \(k\), then the server switches to an inactive mode with the rate \(\theta_3\) which follows an exponential distribution. Events can enter the system during the inactive period. At this epoch when the system size reaches the threshold value \(k\), the system switches to the wake-up state.

Let \(\{M(t), t \geq 0\}\) represent the status of the system at any time \(t\) and let \(Q(t)\) denotes the number of events in the system at any time \(t\).

\[
M(t) = \begin{cases} 
0 & \text{the server is in busy mode and operates with the rate } \mu \\
1 & \text{the server is in wake-up mode} \\
2 & \text{the server is in shutdown mode} \\
3 & \text{the server is in inactive mode}
\end{cases}
\]

Then, \(X(t) = \{M(t), Q(t), t \geq 0\}\) represents a continuous time Markov chain with state space \(S = \{(0, n) : n = 1, 2, 3, \ldots\} \cup \{(1, n) : n = k, k+1, k+2, \ldots\} \times \{(2, n) : n = 0, 1, 2, \ldots\} \cup \{(3, n) : n = 0, 1, 2, \ldots k-1\}\).

Let

\[
P_{0,n}(t) = P[M(t) = i, Q(t) = n], i = 0; n = 1, 2, 3, \ldots
\]

\[
P_{1,n}(t) = P[M(t) = i, Q(t) = n], i = 1; n = k, k+1, k+2, \ldots
\]

\[
P_{2,n}(t) = P[M(t) = i, Q(t) = n], i = 2; n = 0, 1, 2, \ldots
\]

\[
P_{3,n}(t) = P[M(t) = i, Q(t) = n], i = 3; n = 0, 1, 2, \ldots k-1.
\]

Then \(P_{t,n}(t)\) satisfies the following forward Kolmogorov equation. Figure 1 presents the state transition of the model.

\[
P'_{0,1}(t) = - (\lambda + \mu) P_{0,1}(t) + \mu P_{0,2}(t),
\]

\[
P'_{0,n}(t) = - (\lambda + \mu) P_{0,n}(t) + \lambda P_{0,n-1}(t)
\]

\[+ \mu P_{0,n+1}(t), n = 2, 3, 4, \ldots, k-1.\]
3 Transient analysis

This section presents the time-dependent probabilities of the wake-up state $P_{1,n}(t)$, the shutdown state $P_{2,n}(t)$, the inactive state $P_{3,n}(t)$ and the busy state $P_{0,n}(t)$.

### 3.1 Evaluation of $P_{1,n}(t)$, $P_{2,n}(t)$ and $P_{3,n}(t)$

Let $\hat{P}_{1,n}(s)$ denote the Laplace transform of $P_{1,n}(t)$ for $i = 1, 2, 3; n = 0, 1, 2, \ldots$.

Taking Laplace transform on Eqs. (2.4)–(2.9), we get

$$s\hat{P}_{1,n}(s) = -(\lambda + \theta_1)\hat{P}_{1,n}(s) + \theta_2\hat{P}_{2,n}(s) + \lambda\hat{P}_{3,n-1}(s),$$

$$s\hat{P}_{2,n}(s) = -(\lambda + \theta_2)\hat{P}_{2,n}(s) + \mu\hat{P}_{0,1}(s) + \lambda\hat{P}_{3,n-1}(s),$$

$$s\hat{P}_{3,n}(s) = -(\lambda + \theta_2)\hat{P}_{3,n}(s) + \lambda\hat{P}_{3,n-1}(s) + \lambda\hat{P}_{3,n-1}(s),$$

for $n = 1, 2, \ldots, k - 1$.
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Let $\beta_1 = \lambda + \theta_1$ and $\beta_2 = \lambda + \theta_2$. Using Eqs. (3.3) and (3.4), we get

$$
\hat{P}_{2,n}(s) = \frac{\lambda^n \mu}{(s + \beta_2)^{n+1}} \hat{P}_{0,1}(s), \quad n = 0, 1, 2, \ldots
$$

(3.7)

Substituting Eq. (3.3) in (3.5) and further using it in Eq. (3.6) after some algebraic manipulation, we obtain

$$
\hat{P}_{3,n}(s) = \frac{\mu \theta_2 \lambda^n}{(s + \lambda)(s + \beta_2)} \times \left[ \left( \frac{1}{s + \lambda} \right)^n + \sum_{i=1}^{n} \left( \frac{1}{s + \lambda} \right)^{n-i} \times \hat{P}_{0,1}(s) \right] \\
+ \frac{\lambda^n}{(s + \lambda)^{n+1}}, \quad n = 0, 1, 2, \ldots, k - 1.
$$

(3.8)

Using Eqs. (3.7) and (3.8) in Eq. (3.1) after some manipulation, we get

$$
\hat{P}_{1,k}(s) = \frac{\theta_2 \mu \lambda^k}{(s + \beta_1)(s + \beta_2)} \times \left[ \left( \frac{1}{s + \beta_2} \right)^k + \frac{1}{(s + \beta_2)^k} \times \hat{P}_{0,1}(s) \right] \\
+ \left( \frac{1}{s + \lambda} \right)^{k+1}.
$$

(3.9)

Using Eqs. (3.7) and (3.9) in Eq. (3.2), we get

$$
\hat{P}_{1,n}(s) = \frac{\mu \lambda^n \theta_2}{(s + \beta_1)(s + \beta_2)} \times \left[ \left( \frac{1}{s + \beta_1} \right)^n \times \hat{P}_{0,1}(s) \right] \\
+ \left( \frac{1}{s + \beta_1} \right)^{n-k} \times \left[ \left( \frac{1}{s + \beta_2} \right)^k + \frac{1}{(s + \beta_2)^k} \times \hat{P}_{0,1}(s) \right] \\
+ \frac{\lambda^n}{(s + \lambda)^{n+1}}, \quad n = k, k+1, k+2, \ldots
$$

(3.10)

Inversion on Eqs. (3.10), (3.7) and (3.8) respectively yields

$$
P_{1,n}(t) = \lambda^n \mu \theta_2 \exp\left[ - (\beta_1) t \right] \star \exp\left[ - (\beta_2) t \right] \times \sum_{j=k+1}^{n} \frac{\exp\left[ - (\beta_1) t \right] t^{n-j-1}}{(n-j-1)!}
$$

$$
\star \left[ \sum_{j=k+1}^{n} \frac{\exp\left[ - (\beta_1) t \right] t^{n-j-1}}{(n-j-1)!} \right] \\
\star \left[ \sum_{j=0}^{k} \frac{\exp\left[ - (\beta_1) t \right] t^{n-j-1}}{(n-j-1)!} \right] \\
\star \left[ \sum_{j=0}^{k} \frac{\exp\left[ - (\beta_2) t \right] t^{n-j-1}}{(n-j-1)!} \right]
$$

(3.11)

$$
P_{2,n}(t) = \frac{\lambda^n \mu \exp\left[ - (\beta_2) t \right] t^n}{n!} \star P_{0,1}(t), \quad n = 0, 1, 2, \ldots
$$

(3.12)

$$
P_{3,n}(t) = \frac{\lambda^n \mu \exp\left[ - (\beta_2) t \right] t^n}{n!} \star P_{0,1}(t), \quad n = 0, 1, 2, \ldots, k - 1
$$

(3.13)

where $*$ denotes convolution. The time-dependent probabilities $P_{1,n}(t)$, $P_{2,n}(t)$ and $P_{3,n}(t)$ are expressed in terms of $P_{0,1}(t)$ and an explicit expression for $P_{0,1}(t)$ is given in Eq. (3.24).

### 3.2 Evaluation of $P_{0,n}(t)$

The busy-state probability $P_{0,n}(t)$; $n=1,2,3,\ldots$ is obtained using Eqs. (2.1)–(2.3) by applying generating function.

Let

$$
H(z, t) = \sum_{n=1}^{\infty} P_{0,n}(t) z^n, \quad H(z, 0) = 0.
$$

Using Eqs. (2.1)–(2.3), we get

$$
\frac{\partial}{\partial t} H(z, t) = H(z, t) \left[ (\lambda + \mu) z^{-1} + \lambda z \right] \\
\theta_1 \sum_{n=k}^{\infty} P_{1,n}(t) z^n - \mu P_{0,1}(t).
$$

On solving,

$$
H(z, t) = \theta_1 \int_{0}^{t} \sum_{m=k}^{\infty} P_{1,m}(w) z^m \\
\times \exp\left[ - \beta + (\lambda z + \mu z^{-1}) \right] (t-w) dw.
$$
where $\beta = \lambda + \mu$. Let $\kappa = 2\sqrt{\lambda\mu}$ and $\nu = \sqrt{\lambda\mu^{-1}}$, then
\[
\exp\left(\frac{\lambda z + \mu}{z}\right) (t - w) = \sum_{n=-\infty}^{\infty} (vz)^n I_n (\kappa (t - w)).
\]
(3.15)
where $I_n(t)$ represents the modified Bessel function of the first kind of order $m$. Applying Eq. (3.15) in Eq. (3.14) and equating the coefficient of $z^n$ on both sides for $n = k + 1, k + 2, k + 3 \ldots$

\[
P_{0,n} (t) = \theta_1 \int_0^t \sum_{m=k}^{\infty} P_{1,m} (w) \exp \{-\beta (t - w)\} v^{n-m} I_{n-m} (\cdot) \, dw
\]
\[- \mu \int_0^t P_{0,1} (w) \exp \{-\beta (t - w)\} v^n I_n (\cdot) \, dw.
\]
(3.16)
Equating the coefficients of $z^{-n}$ on both sides of Eq. (3.14) for $n = k + 1, k + 2, k + 3 \ldots$ and using $I_{-n} (\cdot) = I_n (\cdot)$, we get
\[
0 = \theta_1 \int_0^t \sum_{m=k}^{\infty} P_{1,m} (w) \exp \{-\beta (t - w)\} v^{-n-m} I_{n+m} (\cdot) \, dw
\]
\[- \mu \int_0^t P_{0,1} (w) \exp \{-\beta (t - w)\} v^{-n} I_n (\cdot) \, dw.\]
(3.17)
Multiplying $v^{2n}$ on both sides of Eq. (3.17) and subtracting it from Eq. (3.16) for $n = 1, 2, 3 \ldots$, we arrive
\[
P_{0,n} (t) = \theta_1 \int_0^t \sum_{m=k}^{\infty} P_{1,m} (w) \exp \{-\beta (t - w)\} v^{n-m}
\]
\[
[I_{n-m} (\cdot) - I_{n+m} (\cdot)] \, dw.
\]
(3.18)
Taking Laplace transform on Eq. (3.18), we obtain
\[
\hat{P}_{0,n} (s) = \frac{\theta_1}{\sqrt{d^2 - \kappa^2}}
\]
\[
\sum_{m=1}^{\infty} \hat{P}_{1,m} (s) v^{n-m} \left( \hat{\zeta}(s)^{n-m} - \hat{\zeta}(s)^{n+m} \right)
\]
(3.19)
where
\[
\hat{\zeta}(s) = \frac{d - \sqrt{d^2 - \kappa^2}}{\kappa}\text{ and } d = s + \lambda + \mu.
\]
Using Eq. (3.10) in Eq. (3.19), we obtain
\[
\hat{P}_{0,n} (s) = \frac{\theta_1}{\sqrt{d^2 - \kappa^2}} \int_0^\infty \left[ \sum_{m=k}^{\infty} \left( \frac{1}{s + \beta_1} \right)^{m-j} \left( \frac{1}{s + \beta_2} \right)^{j} + \left( \frac{1}{s + \lambda} \right) \right]
\]
\[
\times \left[ \sum_{j=k+1}^{\infty} \left( \frac{1}{s + \beta_1} \right)^{j} \left( \frac{1}{s + \beta_2} \right)^{m-k} \right]
\]
\[
\times \left[ \sum_{i=1}^{k-1} \left( \frac{1}{s + \beta_1} \right)^{i} \left( \frac{1}{s + \beta_2} \right)^{m-k+i} \right] P_{0,1} (t)
\]
\[+ \lambda^m \left( \frac{1}{s + \beta_1} \right)^{m-k+1} \left( \frac{1}{s + \lambda} \right) \hat{\zeta}(s)^{n-m} - \hat{\zeta}(s)^{n+m} \right].
\]
On inversion,
\[
P_{0,n} (t) = \theta_1 \int_0^\infty \left[ \sum_{m=k}^{\infty} v^{n-m} \right]
\]
\[
\left[ \mu \theta_2 \lambda^m \exp \{- (\beta_1) t \} \ast \exp \{- (\beta_2) t \}
\right]
\[
\ast \left[ \sum_{j=k+1}^{\infty} \left( \frac{1}{s + \beta_1} \right)^{j} \left( \frac{1}{s + \beta_2} \right)^{m-k} \right]
\]
\[
\times \left[ \sum_{i=1}^{k-1} \left( \frac{1}{s + \beta_1} \right)^{i} \left( \frac{1}{s + \beta_2} \right)^{m-k+i} \right]
\]
\[
\times \left[ \left( \frac{1}{s + \lambda} \right) \hat{\zeta}(s) \right]
\]
\[\ast \left[ \left( \frac{1}{s + \beta_1} \right)^{m-k+1} \right]
\]
\[\ast \left[ \left( \frac{1}{s + \lambda} \right) \hat{\zeta}(s) \right]
\]
\[\left[ \left( \frac{1}{s + \beta_1} \right)^{m-k} \right]
\]
\[\left[ \left( \frac{1}{s + \lambda} \right) \hat{\zeta}(s) \right]
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\[\left[ \left( \frac{1}{s + \beta_1} \right)^{m-k} \right]
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\[\left[ \left( \frac{1}{s + \lambda} \right) \hat{\zeta}(s) \right]
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\[\left[ \left( \frac{1}{s + \beta_1} \right)^{m-k} \right]
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\[\left[ \left( \frac{1}{s + \beta_1} \right)^{m-k} \right]
\]
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\]
\[\left[ \left( \frac{1}{s + \beta_1} \right)^{m-k} \right]
\]
\[\left[ \left( \frac{1}{s + \lambda} \right) \hat{\zeta}(s) \right]
\]
\[\left[ \left( \frac{1}{s + \beta_1} \right)^{m-k} \right]
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\]
\[\left[ \left( \frac{1}{s + \lambda} \right) \hat{\zeta}(s) \right]
\]
\[\left[ \left( \frac{1}{s + \beta_1} \right)^{m-k} \right]
\]
\[\left[ \left( \frac{1}{s + \lambda} \right) \hat{\zeta}(s) \right]
\]
\[\left[ \left( \frac{1}{s + \beta_1} \right)^{m-k} \right]
\]
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\]
\[\left[ \left( \frac{1}{s + \beta_1} \right)^{m-k} \right]
\]
\[\left[ \left( \frac{1}{s + \lambda} \right) \hat{\zeta}(s) \right]
\]
\[\left[ \left( \frac{1}{s + \beta_1} \right)^{m-k} \right]
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\]
\[\left[ \left( \frac{1}{s + \beta_1} \right)^{m-k} \right]
\]
\[\left[ \left( \frac{1}{s + \lambda} \right) \hat{\zeta}(s) \right]
\]
\[\left[ \left( \frac{1}{s + \beta_1} \right)^{m-k} \right]
\]
\[\left[ \left( \frac{1}{s + \lambda} \right) \hat{\zeta}(s) \right]
\]
\[\left[ \left( \frac{1}{s + \beta_1} \right)^{m-k} \right]
\]
\[\left[ \left( \frac{1}{s + \lambda} \right) \hat{\zeta}(s) \right]
\]
\[\left[ \left( \frac{1}{s + \beta_1} \right)^{m-k} \right]
\]
\[\left[ \left( \frac{1}{s + \lambda} \right) \hat{\zeta}(s) \right]
\]
\[\left[ \left( \frac{1}{s + \beta_1} \right)^{m-k} \right]
\]
On inversion
\[ \hat{P}_{0,1}(s) = 2\theta_1 \sum_{m=k}^{\infty} v^{1-m} \hat{P}_{1,m}(s) \frac{\kappa^{m-1}}{(d + \sqrt{d^2 - \kappa^2})^m}. \] (3.21)

where \( d = s + \lambda + \mu \). Substituting Eq. (3.15) in Eq. (3.19) after some manipulation, we get
\[ \hat{P}_{0,1}(s) = \theta_1 \left( \frac{1}{s + \lambda} \right)^k \sum_{m=k}^{\infty} v^{1-m} \lambda^m \left( \frac{1}{s + \beta_1} \right)^{m-k+1} \]
\[ \times \frac{2\kappa^{m-1}}{(d + \sqrt{d^2 - \kappa^2})^m} \sum_{h=0}^{\infty} \left( \hat{G}_1(s) \right)^h. \] (3.22)

where
\[ \hat{G}_1(s) = \frac{\mu \theta_1 \theta_2}{(s + \beta_1)(s + \beta_2)} \sum_{m=k}^{\infty} v^{1-m} \lambda^m \]
\[ \times \left[ \sum_{j=k+1}^{m} \left( \frac{1}{s + \beta_1} \right)^{m-j} \left( \frac{1}{s + \beta_2} \right)^{j} \left( \frac{1}{s + \beta_1} \right)^{m-k} \right] \]
\[ \times \frac{2\kappa^{m-1}}{(d + \sqrt{d^2 - \kappa^2})^m}. \] (3.23)

Inversion on (3.20) gives
\[ P_{0,1}(t) = \frac{\theta_1 \exp(-\lambda t)}{(k-1)!} \sum_{m=k}^{\infty} \lambda^m \exp\{-\beta_1 t\} \frac{t^{m-k}}{(m-k)!} \]
\[ \times v^{1-m} \left[ I_{1-m}(\kappa t) - I_{1+m}(\kappa t) \right] \]
\[ \times \sum_{h=0}^{\infty} (G_1(t))^{*h}, \] (3.24)

where
\[ G_1(t) = \theta_1 \theta_2 \mu \exp\{-\beta_1 t\} \exp\{-\beta_2 t\} \]
\[ \times \sum_{m=k}^{\infty} \lambda^m \sum_{j=k+1}^{m} \exp\{-\beta_1 t\} \frac{t^{m-j-1}}{(m-j-1)!} \]
\[ \times \exp\{-\beta_2 t\} \frac{t^{j-1}}{(j-1)!} \exp\{-\beta_2 t\} \frac{t^{m-k-1}}{(m-k-1)!} \]
\[ \times v^{1-m} \left[ I_{1-m}(\kappa t) - I_{1+m}(\kappa t) \right] \]
\[ \times \sum_{h=0}^{\infty} (G_1(t))^{*h}. \] (3.25)

and ‘*h’ denotes h-fold convolution. Thus we have obtained an explicit expression for \( P_{0,1}(t) \).

4 Performance measures

The mean and variance of the system size are presented in this section.

4.1 Mean

Let \( \Omega(t) \) denote the expected system size at time \( t \). For \( t > 0 \), we have
\[ \Omega(t) = E[X(t)] = \sum_{n=1}^{\infty} n P_{0,n}(t) + \sum_{n=k}^{\infty} n P_{1,n}(t) \]
\[ + \sum_{n=1}^{\infty} n P_{2,n}(t) + \sum_{n=0}^{k-1} n P_{3,n}(t). \]

Then using Eqs. (2.1)–(2.9), we get
\[ \frac{d}{dt} \Omega(t) = \lambda - \mu \sum_{n=1}^{\infty} P_{0,n}(t) + \lambda \sum_{n=k}^{\infty} P_{1,n}(t) + \lambda \sum_{n=0}^{k-1} P_{3,n}(t). \]

The above equation gives
\[ \Omega(t) = \lambda t - \mu \sum_{n=1}^{\infty} \int_{0}^{t} P_{0,n}(y) dy + \lambda \sum_{n=k}^{\infty} \int_{0}^{t} P_{1,n}(y) dy \]
\[ + \lambda \sum_{n=0}^{k-1} \int_{0}^{t} P_{3,n}(y) dy. \]

4.2 Variance

Let \( V(t) \) denote the variance system size at time \( t \). For \( t > 0 \),
\[ V(t) = E[X^2(t)] - (E[X(t)])^2, \]

where
Using Eqs. (2.1)–(2.9), we obtain

\[
\frac{d}{dt} E \left[ X^2 (t) \right] = \lambda - \mu \sum_{n=0}^{\infty} (2n + 1) P_{0,n+1} (t) + 2\lambda \Omega (t).
\]

Then

\[
E \left[ X^2 (t) \right] = \lambda t - \mu \sum_{n=0}^{\infty} (2n + 1) \int_{0}^{t} P_{0,n+1} (y) \ dy
+ 2\lambda \int_{0}^{t} \Omega (y) \ dy.
\]

4.3 Probability that the server is in power-saving modes

Let \( P_{n,i} (t) : i = 1, 2, 3 \) denote the probability that the server is on wake-up state, shutdown state and inactive state respectively, then

\[
P_{1,i} (s) = \sum_{n=k}^{\infty} \hat{P}_{1,n} (s),
\]

\[
P_{2,i} (s) = \sum_{n=0}^{k-1} \hat{P}_{2,n} (s),
\]

\[
P_{3,i} (s) = \sum_{n=0}^{k-1} \hat{P}_{3,n} (s).
\]

Using Eqs. (3.10), (3.7) and (3.8) in the above expression and taking inversion, respectively yield

\[
P_{1,i} (t) = \mu \theta_{2} \exp \left\{ - (\beta_{1}) t \right\} \exp \left\{ - (\beta_{2}) t \right\} \]
\[
\times \left\{ \frac{\lambda^{k+1} \left\{ \delta^{'} (t) + (\beta_{1}) \delta (t) \right\}}{\theta_{2} - \theta_{1}} \right. \]
\[
\exp \left\{ - (\beta_{2}) t \right\} t^{k-1} \frac{1}{(k-1)!} \right\} \]
\[
+ \left\{ \frac{\lambda^{k+1} \exp \left\{ - (\theta_{1}) t \right\} + \lambda^{k} \delta (t)}{\theta_{2}} \right\} \]
\[
\exp \left\{ - (\beta_{2}) t \right\} t^{k-1} \frac{1}{(k-1)!} \right\} \]
\[
+ \left\{ \frac{\exp \left\{ - (\beta_{2}) t \right\} t^{k-1}}{(k-1)!} \exp \left\{ - (\lambda t) t \right\} \right\} \]
\[
+ \sum_{i=1}^{k-1} \exp \left\{ - (\lambda t) t^{i-1} \right\} \exp \left\{ - (\beta_{2}) t \right\} t^{i-1} \frac{1}{(i-1)!} \right\}.
\]

4.4 Probability that the server is in busy state

Let \( P_{b,i} (t) \) denote the probability that the server is on busy state, then

\[
\hat{P}_{b,i} (s) = \sum_{n=1}^{\infty} \hat{P}_{b,n} (s).
\]

Using Eq. (3.19) and taking inversion, we get

\[
P_{b,i} (t) = \theta_{1} \sum_{n=1}^{\infty} \sum_{m=k}^{\infty} P_{1,m} (t) u^{n-m} \]
\[
\times \left[ I_{n-m} (\lambda t) - I_{n+m} (\lambda t) \right] \exp \left\{ - (\beta_{2}) t \right\} + \left[ I_{n-m} (\lambda t) - I_{n+m} (\lambda t) \right] \exp \left\{ - (\beta_{2}) t \right\} \]

4.5 Probability that the server is either in busy state or wake-up state or shutdown state or inactive state

Let \( P (t) \) denote the probability that the server is either in busy state or in wake-up state or in shutdown state or in inactive state, then

\[
P (t) = \sum_{n=1}^{\infty} P_{0,n} (t) + \sum_{n=k}^{\infty} P_{1,n} (t)
+ \sum_{n=0}^{k-1} P_{2,n} (t) + \sum_{n=0}^{k-1} P_{3,n} (t).
\]

5 Steady-state probabilities

The system size probabilities of the wake-up state, the shutdown state and the inactive state are presented in this section.
Let \( \{ \pi_{k,n}; k = 0, 1, 2, 3; n = 1, 2, 3, \ldots \} \) represent the steady-state probability distributions for the model considered. Applying the steady-state condition \( \lim_{s \to 0} s \hat{P}_{i,n} = \pi_{i,n} \) on Eq. (2.1)–(2.9), we get

\[
\begin{align*}
(\lambda + \mu) \pi_{0,1} & = \mu \pi_{0,2}, \quad (5.1) \\
(\lambda + \mu) \pi_{n,0} & = 2 \pi_{n-1,0} + \mu \pi_{n+1,0}, \quad n = 2, 3, 4, \ldots, k - 1, \quad (5.2) \\
(\lambda + \mu) \pi_{n,1} & = \lambda \pi_{n-1,1} + \mu \pi_{n+1,1} + \theta_1 \pi_{n,1}, \quad n = k, k + 1, k + 2, \ldots, \quad (5.3) \\
(\lambda + \theta_1) \pi_{1,1} & = \theta_2 \pi_{2,1} + \lambda \pi_{3,1}, \quad (5.4) \\
(\lambda + \theta_1) \pi_{1,2} & = \theta_2 \pi_{2,2} + \lambda \pi_{3,2}, \quad (5.5) \\
(\lambda + \theta_2) \pi_{2,0} & = \mu \pi_{0,1}, \quad (5.6) \\
(\lambda + \theta_2) \pi_{2,1} & = \lambda \pi_{2,2} - 1, \quad n = 1, 2, 3, \ldots, \quad (5.7) \\
\lambda \pi_{3,0} & = \theta_2 \pi_{2,0}, \quad (5.8) \\
\lambda \pi_{3,1} & = \lambda \pi_{3,1} + \theta_2 \pi_{2,1}, \quad n = 1, 2, \ldots, k - 1. \quad (5.9)
\end{align*}
\]

Using Eqs. (5.6) and (5.7), we get

\[
\pi_{2,n} = \left( \frac{\lambda}{\beta_2} \right)^n \left( \frac{1}{\beta_2} \right) \mu \pi_{0,1}, \quad n = 0, 1, 2, \ldots. \quad (5.10)
\]

where \( \beta_2 = \lambda + \theta_2 \).

Substituting Eq. (5.10) in Eq. (5.9), we get

\[
\pi_{3,n} = \pi_{3,n-1} + \frac{\theta_2}{\lambda \beta_2} \left( \frac{\lambda}{\beta_2} \right)^n \mu \pi_{0,1}, \quad n = 1, 2, \ldots, k - 1. \quad (5.11)
\]

Equation (5.11) recursively yields

\[
\pi_{3,n} = \pi_{3,0} + \frac{\theta_2}{\lambda \beta_2} \sum_{i=1}^{n} \left( \frac{\lambda}{\beta_2} \right)^i \mu \pi_{0,1}. \quad (5.12)
\]

Using Eq. (5.8) in (5.12), we get

\[
\pi_{3,n} = \frac{\theta_2}{\lambda \beta_2} \left[ 1 + \sum_{i=1}^{n} \left( \frac{\lambda}{\beta_2} \right)^i \right] \mu \pi_{0,1}, \quad n = 0, 1, 2, \ldots, k - 1. \quad (5.13)
\]

Using Eqs. (5.10) and (5.13) in Eq. (5.4), we get

\[
\pi_{1,k} = \frac{\theta_2}{\beta_1 \beta_2} \sum_{i=0}^{k} \left( \frac{\lambda}{\beta_2} \right)^i \mu \pi_{0,1}, \quad (5.14)
\]

where \( \beta_1 = \lambda + \theta_1 \).

Equation (5.5) recursively yields

\[
\pi_{1,n} = \frac{\theta_2 \lambda^n}{\beta_1 \beta_2} \sum_{i=k+1}^{n} \left( \frac{1}{\beta_1} \right)^{n-i} \left( \frac{1}{\beta_2} \right)^i \mu \pi_{0,1} + \left( \frac{\lambda}{\beta_1} \right)^{n-k} \pi_{1,k}. \quad (5.15)
\]

Substituting Eq. (5.14) in Eq. (5.15), we get

\[
\pi_{1,n} = \frac{\theta_2}{\beta_1 \beta_2} \left[ \lambda^n \sum_{i=k+1}^{n} \left( \frac{1}{\beta_1} \right)^{n-i} \left( \frac{1}{\beta_2} \right)^i \right.
\]

\[
\left. + \left( \frac{\lambda}{\beta_1} \right)^{n-k} \left( \frac{\beta_2}{\theta_2} \right) \left( 1 - \frac{1}{\beta_2} \right)^{k+1} \right] \mu \pi_{0,1}. \quad (5.16)
\]

The steady-state probabilities of the wake-up mode, the shutdown mode and the inactive mode are expressed in terms of \( \pi_{0,1} \). To get an explicit expression for \( \pi_{0,1} \), we define a generating function as follows:

\[
G_0 (z) = \sum_{n=1}^{\infty} \pi_{0,n} z^n, \quad (5.17)
\]

\[
G_1 (z) = \sum_{n=k}^{\infty} \pi_{1,n} z^n, \quad (5.18)
\]

\[
G_1' (z) = \sum_{n=k}^{\infty} n \pi_{1,n} z^{n-1}. \quad (5.19)
\]

Using Eqs. (5.1)–(5.3), we obtain

\[
G_0 (z) \left( \lambda + \mu - \lambda z - \frac{\mu}{z} \right) = -\mu \pi_{0,1} + \theta_1 G_1 (z). \quad (5.20)
\]

After some algebra, we get

\[
G_0 (z) = \frac{\lambda \theta_1 \left[ G_1 (z) - G_1 (1) \right]}{(z \lambda - \mu) (1 - z)}. \quad (5.20)
\]

Setting \( z = 1 \), we get

\[
G_0 (1) = \frac{\lambda \theta_1 G_1' (1)}{\mu - \lambda}. \quad (5.20)
\]

Using Eq. (5.19) for \( z = 1 \) and the result (5.16) in the above equation, we get

\[
G_0 (1) = \sum_{n=k}^{\infty} \left[ \lambda^n \sum_{i=k+1}^{n} \left( \frac{1}{\beta_1} \right)^{n-i} \left( \frac{1}{\beta_2} \right)^i \right.
\]

\[
\left. + \left( \frac{\lambda}{\beta_1} \right)^{n-k} \left( \frac{\beta_2}{\theta_2} \right) \right] \mu \pi_{0,1}. \quad (5.20)
\]
where $\rho = \frac{\lambda}{\mu} < 1$. The normalization condition is given by

$$
\sum_{n=1}^{\infty} \pi_{0,n} + \sum_{n=k}^{\infty} \pi_{1,n} + \sum_{n=0}^{\infty} \pi_{2,n} + \sum_{n=0}^{k-1} \pi_{3,n} = 1.
$$

Substituting the results (5.10), (5.13), (5.16) and (5.21) in the above condition, we obtain

$$
\pi_{0,1} = \left[ \theta_1 \theta_2 \mu \frac{1}{\beta_1 \beta_2} \sum_{n=k}^{\infty} \left( n \rho \frac{1}{1 - \rho} + 1 \right) \right] \theta_2 \lambda \pi \left( \sum_{k=0}^{n} \left( \frac{\lambda}{\beta_1} \right)^{n-i} \left( \frac{1}{\beta_2} \right)^{i} + \left( \frac{\lambda}{\beta_1} \right)^{n-k} \left( \frac{\beta_2}{\theta_2} \right) \left( 1 - \frac{\lambda}{\beta_2} \right)^{k+1} \right) + \frac{1}{\theta_2} + \frac{\theta_2}{\lambda \beta_2}
+ \frac{1 + \left( \frac{\lambda}{\beta_2} \right) \left( 1 - \left( \frac{\lambda}{\beta_2} \right) \right)}{1 \theta_2} \lambda \pi \left( \sum_{n=k}^{\infty} \left( \frac{\lambda}{\beta_1} \right)^{n-i} \left( \frac{1}{\beta_2} \right)^{i} \right) \right]^{-1}.
$$

(5.22)

**Remark 1**

In the absence of wake-up state, shutdown state and if the server switches to the inactive state at the end busy period and if the server switches to busy state after the accumulation of $k$ events, Eqs. (5.1)–(5.9) reduce to

$$
(\lambda + \mu) \pi_{0,1} = \mu \pi_{0,2},
$$

$$
(\lambda + \mu) \pi_{0,n} = \lambda \pi_{0,n-1} + \mu \pi_{0,n+1},
$$

$$
\pi_{0,k} = \lambda \pi_{3,k-1} + \lambda \pi_{0,k-1} + \mu \pi_{0,k+1},
$$

$$
(\lambda + \mu) \pi_{3,0} = \mu \pi_{0,1},
$$

$$
\pi_{3,n} = \lambda \pi_{3,n-1}, n = 1, 2, 3, \ldots, k - 1.
$$

Applying the generating function (5.17) on the above equation, we get

$$
G_0(z) = \frac{z \lambda (1 - z^k)}{(z \lambda - \mu)(1 - z)} \pi_{0,1},
$$

$$
G_1(z) = \frac{1 - z^k}{1 - z} \pi_{0,1}.
$$

Setting $z = 1$ in the above equation and using the normalization condition, we get

$$
\pi_{0,1} = \frac{1 - \rho}{k}.
$$

The above equation coincide with the result (10) of Jiang et al. [7].

**Remark 2**

The Eqs. (3.10), (3.7), (3.8) coincide with the results (5.16), (5.10) and (5.13) respectively on applying $\lim_{s \to 0} \pi_{1,n} = \pi_{i,n}, i = 1, 2, 3$.

### 5.1 Probability that the server is on wake-up state, shutdown state and inactive state

Let $\pi_{i,*}; i = 1, 2, 3$ denote the probability that the server is on wake-up state, shutdown state and inactive state respectively, then

$$
\pi_{1,*} = \sum_{n=k}^{\infty} \pi_{1,n},
$$

$$
\pi_{2,*} = \sum_{n=0}^{k-1} \pi_{2,n},
$$

$$
\pi_{3,*} = \sum_{n=0}^{k} \pi_{3,n}.
$$

Using the results (5.16), (5.10) and (5.13) in the above equation, we obtain

$$
\pi_{1,*} = \frac{\theta_2}{\beta_1 \beta_2} \sum_{n=k}^{\infty} \left( \frac{\lambda}{\beta_1} \right)^{n-i} \left( \frac{1}{\beta_2} \right)^{i} + \left( \frac{\lambda}{\beta_1} \right)^{n-k} \left( \frac{\beta_2}{\theta_2} \right) \left( 1 - \left( \frac{\lambda}{\beta_2} \right) \right)^{k+1} \right) \mu \pi_{0,1},
$$

(5.23)

$$
\pi_{2,*} = \frac{\mu}{\theta_2} \pi_{0,1},
$$

(5.24)

$$
\pi_{3,*} = \frac{1}{\lambda} \left[ \sum_{n=0}^{k} \left( 1 - \left( \frac{\lambda}{\beta_2} \right)^{k} \right) \right] \mu \pi_{0,1}.
$$

(5.25)

### 6 Performance measures

This section presents expected system size in the steady-state. Let $E[N_0], E[N_1], E[N_2]$ and $E[N_3]$ be the mean number of events in the busy, wake-up, shutdown and inactive states.
respectively and let $E[N_s]$ denote the expected system size. Then,

$$E[N_s] = E[N_0] + E[N_1] + E[N_2] + E[N_3].$$

### 6.1 Expected system size in the busy state

The mean number of events in the busy state is given by

$$E(N_0) = \lim_{z \to 1} G_0'(z).$$

Differentiating Eq. (5.20) and setting $z = 1$, after some algebraic manipulation, we get

$$E(N_0) = \frac{\rho \theta_1 \theta_2}{1 - \rho \beta_1 \beta_2} \sum_{n=0}^{\infty} \left[ \lambda^n \sum_{i=k+1}^{n} \left( \frac{1}{\beta_1} \right)^{n-i} \left( \frac{1}{\beta_2} \right)^i \right] + \left( \frac{\lambda}{\beta_1} \right)^{n-k} \frac{\beta_2}{\theta_2} \left\{ 1 - \left( \frac{\lambda}{\beta_2} \right)^{k+1} \right\} \times \left[ \frac{n}{1 - \rho} \right] \mu \pi_{0,1},$$

(6.1)

where $\rho < 1$.

### 6.2 Expected system size in the wake-up state

Let $E[N_1]$ be the mean number of events in the system during wake-up mode. Then

$$E[N_1] = \sum_{n=k}^{\infty} n \pi_{1,n}.$$ 

Using the result (5.16), we obtain

$$E[N_1] = \frac{\theta_2}{\beta_1 \beta_2} \sum_{n=k}^{\infty} n \left[ \lambda^n \sum_{i=k+1}^{n} \left( \frac{1}{\beta_1} \right)^{n-i} \left( \frac{1}{\beta_2} \right)^i \right] + \left( \frac{\lambda}{\beta_1} \right)^{n-k} \frac{\beta_2}{\theta_2} \left\{ 1 - \left( \frac{\lambda}{\beta_2} \right)^{k+1} \right\} \mu \pi_{0,1}. \quad (6.2)$$

### 6.3 Expected system size in the shutdown state

Let $E[N_2]$ be the mean number of events in the system during shutdown mode. Then,

$$E[N_2] = \sum_{n=0}^{\infty} n \pi_{2,n}.$$ 

Using the result (5.10), we obtain

$$E[N_2] = \frac{\lambda}{\beta_2^2} \mu \pi_{0,1}. \quad (6.3)$$

### 6.4 Expected system size in the inactive state

Let $E[N_3]$ be the mean number of events in the system during inactive mode. Then,

$$E[N_3] = \sum_{n=0}^{k-1} n \pi_{3,n}.$$ 

Using the result (5.13), we obtain

$$E[N_3] = \frac{1}{\beta_2} \sum_{n=0}^{k-1} \left[ 1 - \left( \frac{\lambda}{\beta_2} \right)^{n+1} \right] \mu \pi_{0,1}. \quad (6.4)$$

### 6.5 Expected number of events waiting in the system

Let $E[W_s]$ denote the mean number of events waiting in the system. Then,

$$E[W_s] = \frac{E[N_s]}{\lambda}.$$ 

The mean number of events waiting in the queue is given by

$$E[W_q] = \sum_{n=k}^{\infty} n \pi_{1,n} + \sum_{n=0}^{\infty} n \pi_{2,n} + \sum_{n=0}^{k-1} n \pi_{3,n} + \sum_{n=1}^{\infty} (n-1) \pi_{0,n}.$$ 

### 6.6 Mean power consumption

This section presents the mean power consumed by the sensor node in each cycle. To minimise the mean power consumption, we define a function

$$F(k) = C_h E(N_s) + C_0 \pi_{0,1} + C_1 \pi_{1,1} + C_2 \pi_{2,1} + C_3 \pi_{3,1} + C_\mu \mu + C_{\theta_1} \theta_1 + C_{\theta_2} \theta_2.$$ 

Using the results (6.1)–(6.4) in the above expression, we get

$$F(k) = \frac{\theta_2}{\beta_1 \beta_2} \sum_{n=k}^{\infty} \left[ \lambda^n \sum_{i=k+1}^{n} \left( \frac{1}{\beta_1} \right)^{n-i} \left( \frac{1}{\beta_2} \right)^i \right] + \left( \frac{\lambda}{\beta_1} \right)^{n-k} \frac{\beta_2}{\theta_2} \left\{ 1 - \left( \frac{\lambda}{\beta_2} \right)^{k+1} \right\} \mu \pi_{0,1}.$$
The mean power consumption for all cycles can be formulated as
\[
\sum_{k} P_k = \frac{R}{\mu} \left( \theta_1 C_h \theta_0 C_0 + \frac{\rho}{1-\rho} \left( \frac{n^2-n}{2} \right) + n \theta_1 C_0 \frac{\rho}{1-\rho} + C_1 \right) + \frac{C_h \lambda}{\theta_2^2} + \frac{C_2 \lambda}{\theta_2} + \frac{C_3}{\lambda} \sum_{n=0}^{k-1} \left( 1 - \left( \frac{\lambda}{\beta_2} \right)^{n+1} \right) + \frac{C_4}{\lambda} \left( k - \frac{1}{\theta_2} \left( 1 - \left( \frac{\lambda}{\beta_2} \right)^k \right) \right) + C_5 \mu + C_6 \theta_1 + C_7 \theta_2 \right] \mu \pi_{1,1}.
\]

The parameter values are chosen as follows:
- \( \lambda = 1 \), \( \mu = 1.25 \), \( \theta_1 = 0.3 \), \( \theta_2 = 0.5 \), and \( k = 5 \).

Figure 2 illustrates the behaviour of the busy state \( P_{0,n}(t) \). From this graph, it is observed that all the probability curves start at 0 and increases to a certain extent as \( \lambda \) increases and attains a steady-state. Figure 3 portrays the behaviour of the wake-up state \( P_{1,n}(t) \). The curves of \( P_{1,n}(t) \) increase to a certain extent as \( t \) increases and then the curves decrease and further, it attains the steady-state. We also notice that the probability values of \( P_{1,5}(t) \) are greater than \( P_{1,n}(t) \), \( n = 6, 7, 8 \). This is because during the inactive state if the system reaches \( k = 5 \), the system switches to a wake-up state. As a result, the curves of \( P_{1,n}(t) \) start decreasing. Figure 4 depicts the behaviour of the shutdown state \( P_{2,n}(t) \). All the probability curves of \( P_{2,n}(t) \) start at zero, increase as \( t \) increases and attain the steady-state. Figure 5 demonstrates the behaviour of the inactive state \( P_{3,n}(t) \). The probability curves of \( P_{3,0}(t) \) start at 1 and decrease as \( t \) increases and attains the steady-state. The renaming curves of \( P_{3,n}(t) \) increase to certain extent as \( t \) increases and attains the steady-state. From Figs. 6 and 7, it is evident that as the arrival rate \( \lambda \) increases, the mean system size decreases.

\section{Numerical illustrations of transient solutions}

The system size probabilities and the expected system probabilities in the transient state are computed and their behaviour in the system is analysed in this section. The parameter values are chosen as follows: \( \lambda = 1 \), \( \mu = 1.25 \), \( \theta_1 = 0.3 \), \( \theta_2 = 0.5 \), and \( k = 5 \).

Figure 2 illustrates the behaviour of the busy state \( P_{0,n}(t) \). From this graph, it is observed that all the probability curves start at 0 and increases to a certain extent as \( \lambda \) increases and attains a steady-state. Figure 3 portrays the behaviour of the wake-up state \( P_{1,n}(t) \). The curves of \( P_{1,n}(t) \) increase to a certain extent as \( t \) increases and then the curves decrease and further, it attains the steady-state. We also notice that the probability values of \( P_{1,5}(t) \) are greater than \( P_{1,n}(t) \), \( n = 6, 7, 8 \). This is because during the inactive state if the system reaches \( k = 5 \), the system switches to a wake-up state. As a result, the curves of \( P_{1,n}(t) \) start decreasing. Figure 4 depicts the behaviour of the shutdown state \( P_{2,n}(t) \). All the probability curves of \( P_{2,n}(t) \) start at zero, increase as \( t \) increases and attain the steady-state. Figure 5 demonstrates the behaviour of the inactive state \( P_{3,n}(t) \). The probability curves of \( P_{3,0}(t) \) start at 1 and decrease as \( t \) increases and attains the steady-state. The renaming curves of \( P_{3,n}(t) \) increase to certain extent as \( t \) increases and attains the steady-state. From Figs. 6 and 7, it is evident that as the arrival rate \( \lambda \) increases, the mean system size decreases.

\section{Numerical illustrations of steady-state solutions}

The system size probabilities and the expected system probabilities in the steady-state are computed using MATLAB, and their behaviour in the system is analysed in this section. The parameter values are chosen as follows: \( \lambda = 1 \), \( \mu = 2 \), \( k = 10 \), \( \theta_1 = 0.3 \), \( \theta_2 = 0.4 \).

Figures 8, 9, and 10 demonstrate the probability of \( n \) events in the wake-up state \( \pi_{1,n} \), the shutdown state \( \pi_{2,n} \), and the inactive state \( \pi_{3,n} \) respectively. These graphs are plotted against \( n \) for varying arrival rate \( \lambda \). In Fig. 8, we notice that for fixed \( \lambda \) as \( n \) increases, the curves of \( \pi_{1,n} \) decrease. This is because the server switches to a busy state from an wake-up state if the system size reaches the threshold of \( k = 10 \). In Fig. 9, the curves of \( \pi_{2,n} \) decrease as \( n \) increases for fixed \( \lambda \). This is because the server immediately switches to an inactive state after the expiry of the shutdown period. In Fig. 10
we notice that for fixed $\lambda$ as $n$ increases, the curves of $\pi_{3,n}$ also increase. As the server turns to this state from the shutdown state, the system size probability of this state increases as $n$ increases.

Figure 11 delineates the mean number of events in the functional state $E(N_b)$. From this figure, we notice that for a fixed value of $\theta_1$, the system size increases as the arrival rate $\lambda$ increases. It is also observed that for fixed $\lambda$, the value of $E(N_b)$ decreases as $\theta_1$ increases. This is because when $\theta_1$ increases the server will quickly turn to the busy state from the wake-up state and hence the system size decreases.
Fig. 10 Probability of n events in the inactive state

Fig. 11 Mean number of events in the busy state $E(N_0)$ versus arrival rate $\lambda$

Figure 12 explains the mean number of events in the wake-up state $E(N_1)$ versus arrival rate $\lambda$. This graph is plotted against $\lambda$ for fixed $\theta_2 = 0.4$ and varying values of $\theta_1$. As the server switches to the busy state from the wake-up state, the curves of $E(N_1)$ decrease as the arrival rate $\lambda$ increases.

Figures 13 and 14 present the mean number of events in the shutdown state $E(N_2)$ and the inactive state $E(N_3)$ respectively. The graphs are plotted against $\lambda$ for varying values of $\theta_2$. As the server turns to an inactive state from the shutdown state, the curves of $E(N_2)$ decrease as the arrival rate $\lambda$ increases.

7.3 Numerical illustrations of the mean power consumption

Without loss of generality, we assume the power consumption for each state as $C_b = 3$, $C_p = 100$, $C_1 = 5$, $C_2 = 4$, $C_3 = 1$, $C_{\theta_1} = 100$, $C_{\theta_1} = 6$ and $C_{\theta_2} = 5$. Tables 1, 2, 3 and 4 presents the minimum power consumed by the system for a fixed arrival rate $\lambda$ and varying threshold value $k$. In Table 1, we observe that the mean power consumption $F(k)$

starts decreasing initially as we increase the threshold value $k$ and it starts increasing at $k = 3$. Further, we notice that the minimum power consumed at $k = 2$. Table 2, 3 and 4 present minimum power is consumed by the system for $\lambda = 1.5$, 3.5
Table 1 Mean power consumption for $\lambda = 1$

| $k$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|-----|---|---|---|---|---|---|---|---|
| $F(k)$ | 534.623 | 534.473 | 534.586 | 534.989 | 535.652 | 536.524 | 537.559 | 538.717 |

Table 2 Mean power consumption for $\lambda = 1.5$

| $k$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|-----|---|---|---|---|---|---|---|---|
| $F(k)$ | 552.291 | 552.129 | 550.276 | 552.380 | 552.621 | 553.608 | 554.552 | 555.677 |

Table 3 Mean power consumption for $\lambda = 3.5$

| $k$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|-----|---|---|---|---|---|---|---|---|
| $F(k)$ | 862.095 | 861.659 | 861.281 | 861.120 | 861.233 | 861.749 | 862.704 | 864.129 |

Table 4 Mean power consumption for $\lambda = 4.5$

| $k$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|-----|---|---|---|---|---|---|---|---|
| $F(k)$ | 1897.93 | 1896.95 | 1895.93 | 1895.13 | 1894.77 | 1895.00 | 1895.98 | 1897.79 |

and 4.5 respectively. In Table 2, the minimum power consumed at $k = 3$, in Table 3, the minimum power consumed at $k = 4$ and in Table 4, the minimum power consumed at $k = 5$. We also note that as the arrival rate increases, the threshold value $k$ for which the power consumption is minimum also increases. Hence, we conclude that there exists a pair $(k, F(k))$ for each arrival rate $\lambda$ for which the power consumption is minimum.

### 8 Conclusion and future work

The performance of the DPM in the WSNs with threshold policy is discussed in the paper. The transient and steady-state system size probabilities of the system are obtained in a closed form. The transient results presented in the paper enables us to understand the status of the system at any time $t$. To extend the sensor node’s lifetime, a dynamic optimum threshold policy is applied to ensure minimum power consumption that could be obtained while considering different data arrival rates. It is observed that there exists a threshold value for each arrival rate to minimise the power consumption. The performance indices such as mean, variance, probability that the system is in wake-up state, shut-down state and inactive state, and mean power consumption are obtained. This work may be extended to an M/M/C queueing model with working vacation and close-down times.

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