Poisson Reduction

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Abstract

In this paper we exploit Lu’s momentum map to construct a theory of reduction for Poisson actions. Indeed, the local description of Poisson manifolds and the properties of Lu’s momentum map provide an explicit description of the infinitesimal generator of a Poisson action, which allows us to define a Poisson reduced space.

1 Introduction

In this paper we prove a generalization of the Marsden-Weinstein reduction to the general case of an arbitrary Poisson Lie group action on a Poisson manifold.

The results that can be found in the literature are restricted to the case when the Poisson manifold in question is in fact a symplectic manifold. In particular, Lu [8] shows that every Poisson action on a simply connected symplectic manifold has a momentum map and that for Poisson actions on symplectic manifolds with momentum mappings, the Marsden-Weinstein symplectic reduction can be carried out. We are interested in studying the reduced space associated to Poisson actions on Poisson manifolds with momentum maps and proving that it is a Poisson manifold. Basically, given a Poisson Lie group $G$ acting on a Poisson manifold $M$, we define a foliation of the orbit space $M/G$ by means of the momentum map and we show that it inherits a Poisson structure from $M$.

The paper is organized as follows. In Section 2 we recall some basic elements of Poisson geometry: Poisson manifolds and their local description, Lie bialgebroids and Poisson Lie groups. A nice review of these results can be found in [10] and [14]. The Section 3 is devoted to Poisson actions and associated momentum maps and we discuss the symplectic foliation of a Poisson Lie group introducing
the dressing actions. In Section 4 we present the main result of this paper, the Poisson reduction, and we discuss an example.

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2 Poisson manifolds, Poisson Lie groups and Lie bialgebras

In this section we discuss briefly the notion of Poisson manifolds and we give some background about Poisson Lie groups and Lie bialgebras which will be used in the paper.

2.1 Poisson manifolds and symplectic foliation

Let $\pi$ be a bivector on a manifold $M$, i.e. a skew-symmetric, contravariant 2-tensor. At each point $m$, $\pi(m)$ can be viewed as a skew-symmetric bilinear form on $T^*_m M$, or as the skew-symmetric linear map $\pi^\#: T^*_m M \to T_m M$, such that

$$\pi(m)(\alpha_m, \beta_m) = \pi^\#(\alpha_m)(\beta_m), \quad \alpha_m, \beta_m \in T^*_m M.$$  

(1)

If $\alpha, \beta$ are 1-forms on $M$, we define $\pi(\alpha, \beta)$ to be the function in $C^\infty(M)$ whose value at $m$ is $\pi(m)(\alpha_m, \beta_m)$. Given $f, g \in C^\infty(M)$ we set

$$\pi(m)(df, dg) = \{f, g\}(m).$$  

(2)

It is clear that the bracket induced by $\pi$ satisfies the Leibniz rule.

**Definition 2.1.** A **Poisson manifold** $(M, \pi)$ is a manifold $M$ with a Poisson bivector $\pi$ such that the bracket defined in eq. (2) satisfies the Jacobi identity.

**Definition 2.2.** A mapping $\phi : (M_1, \pi_1) \to (M_2, \pi_2)$ between two Poisson manifolds is called a **Poisson mapping** if $\forall f, g \in C^\infty(M_2)$ one has

$$\{f \circ \phi, g \circ \phi\}_1 = \{f, g\}_2 \circ \phi$$  

(3)

Locally, the structure of a Poisson manifold at $m \in M$ is described by the splitting theorem [17]:

**Theorem 2.1** (Weinstein). On a Poisson manifold $(M, \pi)$, any point $m \in M$ has a coordinate neighborhood with coordinates $(q_1, \ldots, q_k, p_1, \ldots, p_k, y_1, \ldots, y_l)$
centered at \( m \), such that

\[
\pi = \sum_i \frac{\partial}{\partial q_i} \wedge \frac{\partial}{\partial p_i} + \frac{1}{2} \sum_{i,j} \phi_{ij}(y) \frac{\partial}{\partial y_i} \wedge \frac{\partial}{\partial y_j} \phi_{ij}(0) = 0. \tag{4}
\]

The rank of \( \pi \) at \( m \) is \( 2k \). Since \( \phi \) depends only on the \( y_i \)s, this theorem gives a decomposition of the neighborhood of \( m \) as a product of two Poisson manifolds: one with rank \( 2k \), and the other with rank 0 at \( m \).

The equation \( y = 0 \) determines the symplectic leaf through \( m \), for any \( m \in M \). Hence, for any point \( m \in M \), we have a symplectic leaf through it. Locally, this leaf has canonical coordinates \((q_1, \ldots, q_k, p_1, \ldots, p_k)\), where the bracket is given by canonical symplectic relations. Notice that each choice of coordinates \((y_1, \ldots, y_l)\) in Theorem 2.1 gives rise to a different term

\[
\frac{1}{2} \sum_{i,j} \phi_{ij}(y) \frac{\partial}{\partial y_i} \wedge \frac{\partial}{\partial y_j} \tag{5}
\]

called the transverse Poisson structure of dimension \( l \). The transverse structures are not uniquely defined, but they are all isomorphic.

2.2 Lie bialgebras and Poisson Lie groups

Let \( \mathfrak{g} \) be a finite dimensional Lie algebra and \( \delta \) a linear map from \( \mathfrak{g} \) to \( \mathfrak{g} \otimes \mathfrak{g} \) with transpose \( ^t\delta : \mathfrak{g}^* \otimes \mathfrak{g}^* \to \mathfrak{g}^* \). Recall that a linear map on \( \mathfrak{g}^* \otimes \mathfrak{g}^* \) can be identified with a bilinear map on \( \mathfrak{g} \).

**Definition 2.3.** A Lie bialgebra is a Lie algebra \( \mathfrak{g} \) with a linear map \( \delta : \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g} \) such that

1. \( ^t\delta : \mathfrak{g}^* \otimes \mathfrak{g}^* \to \mathfrak{g}^* \) defines a Lie bracket on \( \mathfrak{g}^* \), and
2. \( \delta \) is a 1-cocycle on \( \mathfrak{g} \) relative to the adjoint representation of \( \mathfrak{g} \) on \( \mathfrak{g} \otimes \mathfrak{g} \)

There is a symmetry between \( \mathfrak{g} \) with its bracket \([\cdot, \cdot]\) and \( \mathfrak{g}^* \) with bracket \([\cdot, \cdot]_{\mathfrak{g}^*} \) defined by \( \delta \), in particular (see [14])

**Proposition 1.** If \( (\mathfrak{g}, \delta) \) is a Lie bialgebra, and \([\cdot, \cdot] \) is a Lie bracket on \( \mathfrak{g} \), then \( (\mathfrak{g}^*, \hat{\delta}) \) is a Lie bialgebra, where \( ^t[\cdot, \cdot] \) defines a Lie bracket on \( \mathfrak{g}^* \).

By definition, \( (\mathfrak{g}^*, \hat{\delta}) \) is the dual of the Lie bialgebra \( (\mathfrak{g}, \delta) \). It is easy to see that the dual of \( (\mathfrak{g}^*, \hat{\delta}) \) coincides with \( (\mathfrak{g}, \delta) \).

**Definition 2.4.** A Poisson Lie group \((G, \pi_G)\) is a Lie group equipped with a multiplicative Poisson structure \( \pi_G \).
The Poisson structure $\pi_G$ vanishes at the identity $e \in G$, and its linearization at $e$ is given by $d_e \pi_G : \mathfrak{g} \rightarrow \mathfrak{g} \wedge \mathfrak{g}$. The map $d_e \pi_G$ is a derivative and its dual map $\langle [\cdot, \cdot], \cdot \rangle : \mathfrak{g}^* \wedge \mathfrak{g}^* \rightarrow \mathfrak{g}^*$ is given by $\langle [x, y], \alpha \rangle = d_e (\pi_G (\tilde{x}, \tilde{y})))$, where $x, y \in \mathfrak{g}^*$ and $\tilde{x}$ and $\tilde{y}$ can be any 1-forms on $G$ with $\tilde{x}(e) = x$ and $\tilde{y}(e) = y$.

When $\pi_G$ is a Poisson bivector, $\langle [\cdot, \cdot], \cdot \rangle$ satisfies the Jacobi identity, so it makes $\mathfrak{g}^*$ into a Lie algebra. The Lie algebra $\langle \mathfrak{g}^*, [\cdot, \cdot]_\ast \rangle$ is just the linearization of the Poisson structure at $e$.

**Theorem 2.2.** A multiplicative bivector field $\pi_G$ on a connected Lie group $G$ is Poisson if and only if its linearization at $e$ defines a Lie bracket on $\mathfrak{g}^*$.

The proof can be found in [8]. The relation between Poisson Lie groups and Lie bialgebras is given by the following theorem:

**Theorem 2.3** (Drinfeld [4]). If $(G, \pi_G)$ is a Poisson Lie group, then the linearization of $\pi_G$ at $e$ defines a Lie algebra structure on $\mathfrak{g}^*$ such that $(\mathfrak{g}, \mathfrak{g}^*)$ form a Lie bialgebra over $\mathfrak{g}$, called the tangent Lie bialgebra to $(G, \pi_G)$. Conversely, if $G$ is connected and simply connected, then every Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$ over $\mathfrak{g}$ defines a unique multiplicative Poisson structure $\pi_G$ on $G$ such that $(\mathfrak{g}, \mathfrak{g}^*)$ is the tangent Lie bialgebra to the Poisson Lie group $(G, \pi_G)$.

Given a Poisson Lie group $(G, \pi_G)$, we consider its Lie bialgebra $\mathfrak{g}$ whose 1-cocycle is $\delta = d_e \pi_G$. Let us denote the dual Lie bialgebra by $(\mathfrak{g}^*, \delta)$. By Theorem 2.3 we know that there is a unique connected and simply connected Poisson Lie group $(G^*, \pi_{G^*})$, called the dual of $(G, \pi_G)$, associated to the Lie bialgebra $(\mathfrak{g}^*, \delta)$. If $G$ is connected and simply connected, then the dual of $G^*$ is $G$.

### 3 Poisson actions and Momentum maps

In this section we first introduce the concept of Poisson action of a Poisson Lie group on a Poisson manifold, which generalizes the canonical action of a Lie group on a symplectic manifold. We define momentum maps for Poisson actions and finally we consider the particular case of a Poisson Lie group $G$ acting on its dual $G^*$ by dressing transformations. This allows us to study the symplectic leaves of $G$ that are exactly the orbits of the dressing action. These topics can be found in [8].

Recall that a canonical action of a Lie group $G$ on a Poisson manifold $M$ is defined as a group action which preserves the Poisson structure. On the other side, a Poisson action is an action of a Poisson Lie group on a Poisson manifold satisfying a different property of compatibility between the Poisson bivectors of both manifolds. When the Poisson structure on the Lie group is trivial we recover the canonical actions.

In the following we always assume that $G$ is connected and simply connected; this brings that Theorem 2.3 holds.
**Definition 3.1.** The action of \((G, \pi_G)\) on \((M, \pi)\) is called **Poisson action** if the map \(\Phi : G \times M \to M\) is Poisson, where \(G \times M\) is a Poisson manifold with structure \(\pi_G \oplus \pi\).

If \((G, \pi_G)\) is a Poisson Lie group, the left and right actions of \(G\) on itself are Poisson actions. Note that if \(G\) carries the zero Poisson structure \(\pi_G = 0\), the action is Poisson if and only if it preserves \(\pi\). In general, when \(\pi_G \neq 0\), the structure \(\pi\) is not invariant with respect to the action.

**Proposition 2** ([10]). Assume that \((G, \pi_G)\) is a connected Poisson Lie group with associate 1-cocycle of \(g\)

\[
\delta = d_e \pi_G : g \to \wedge^2 g, \tag{6}
\]

and let \((M, \pi)\) be a Poisson manifold. The action \(\Phi : G \times M \to M\) is a Poisson action if and only if

\[
\mathcal{L}_{\xi} (\pi) = -(\delta(\xi))_M \tag{7}
\]

for any \(\xi \in g\), where \(\mathcal{L}\) denotes the Lie derivative.

**Definition 3.2.** A Lie algebra action \(\xi \mapsto \xi_M\) is called an **infinitesimal Poisson action** of the Lie bialgebra \((g, \delta)\) on \((M, \pi)\) if it satisfies eq. (7).

In this formalism the definition of momentum map reads (Lu, [8], [9]):

**Definition 3.3.** A **momentum map** for the Poisson action \(\Phi : G \times M \to M\) is a map \(\mu : M \to g^*\) such that

\[
\xi_M = \pi^*(\mu^*(\theta_{\xi}))) \tag{8}
\]

where \(\theta_{\xi}\) is the left invariant 1-form on \(g^*\) defined by the element \(\xi \in g = (T_e G^*)^*\) and \(\mu^*\) is the cotangent lift \(T^*G^* \to T^*M\).

If \(G\) has trivial Poisson structure, then \(G^* = g^*\), the differential 1-form \(\theta_{\xi}\) is the constant 1-form \(\xi\) on \(g^*\), and

\[
\mu^*(\theta_{\xi}) = d(\mu^\xi), \quad \text{where} \quad \mu^\xi(m) = \langle \mu(m), \xi \rangle. \tag{9}
\]

Thus, in this case, we recover the usual definition of a momentum map for a canonical action \(\mu : M \to g^*\), that is

\[
\xi_M = \pi^*(d(\mu^\xi)). \tag{10}
\]

In other words, \(\xi_M\) is the Hamiltonian vector field with Hamiltonian \(\mu^\xi \in C^\infty(M)\). When \(\pi_G\) is not trivial, \(\theta_{\xi}\) is a Maurer-Cartan form, hence \(\mu^*(\theta_{\xi})\) cannot be written as a differential of a Hamiltonian function. In the following we give an explicit formulation for the infinitesimal generator in this general case.
3.1 Dressing Transformations

One of the most important example of Poisson action is the dressing action of $G$ on $G^*$. Consider a Poisson Lie group $(G, \pi_G)$, its dual $(G^*, \pi_{G^*})$ and its double $D$, with Lie algebras $g$, $g^*$ and $d$, respectively.

For $\xi \in g$, let $\theta_\xi$ be the left invariant 1-form on $G^*$ with value $\xi$ at $e$. Let’s define

$$l(\xi) = \pi^{g_{\mathbb{R}}}_G(\theta_\xi). \quad (11)$$

The map $\xi \mapsto l(\xi)$ is an action of $g$ on $G^*$, whose linearization at $e$ is the coadjoint action of $g$ on $g^*$ and it is an infinitesimal Poisson action of the Lie bialgebra $g$ on the Poisson Lie group $G^*$. Similarly, the right infinitesimal dressing action of $g$ on $G^*$ is defined by

$$r(\xi) = -\pi^{g_{\mathbb{R}}}_{G^*}(\theta_\xi) \quad (12)$$

where $\theta_\xi$ is the right invariant 1-form on $G^*$. The action defined by (11) is generally called left infinitesimal dressing action of $g$ on $G^*$.

Let $l(\xi)$ (resp. $r(\xi)$) a left (resp. right) dressing vector field on $G^*$. If all the dressing vector fields are complete, we can integrate the $g$-action into a Poisson $G$-action on $G^*$ called the dressing action and we say that the dressing actions consist of dressing transformations. The following results can be found in [15].

Proposition 3. The symplectic leaves of $G$ (resp. $G^*$) are the connected components of the orbits of the right or left dressing action of $G^*$ (resp. $G$).

The momentum map for the dressing action of $G$ on $G^*$ is the opposite of the identity map from $G^*$ to itself.

Definition 3.4. A multiplicative Poisson tensor $\pi_G$ on $G$ is complete if each left (equiv. right) dressing vector field is complete on $G$.

Proposition 4. A Poisson Lie group is complete if and only if its dual Poisson Lie group is complete.

Assume that $G$ is a complete Poisson Lie group. We denote respectively the left (resp. right) dressing action of $G$ on its dual $G^*$ by $g \mapsto l_g$ (resp. $g \mapsto r_g$).

Definition 3.5. A momentum map $\mu : M \to G^*$ for a left (resp. right) Poisson action $\Phi$ is called $G$-equivariant if it is such with respect to the left dressing action of $G$ on $G^*$, that is, $\mu \circ \Phi_g = \lambda_g \circ \mu$ (resp. $\mu \circ \Phi_g = \rho_g \circ \mu$)

A momentum map is $G$-equivariant if and only if it is a Poisson map, i.e. $\mu_* \pi = \pi_{G^*}$. From now on we call Hamiltonian action a Poisson action induced by an equivariant momentum map.
4 Poisson Reduction

Now we present the main result of this paper. We show that, given a Poisson action we can define a reduced manifold in terms of momentum map. It has been proved in [9] that given a Poisson Lie group acting on a symplectic manifold $M$, the symplectic structure on $M$ induces a symplectic structure on the leaves of $M/G$ generated by the momentum map. Here we define a reduced space associated to a Poisson action on a Poisson manifold.

Given a Poisson action $\Phi : G \times M \to M$ with momentum map $\mu : M \to G^*$, we define a $G$-invariant foliation $F$ of $M$. The leaves are not Poisson manifolds, but considering the action of $G$ on the space of leaves, we prove that, for each leaf $L$, the Poisson structure on $M$ induces a Poisson structure on the orbit space $L/G_L$, where $G_L$ is the isotropic group at any point of $L$. This shows that we can reduce $M$ to another Poisson manifold $L/G_L$ that we call the Poisson reduced space.

4.1 Poisson structure on $M/G$

In this section we prove that, given a Hamiltonian action, the orbit space inherits a Poisson structure from $M$. From now on we further assume that the Poisson Lie group $G$ is complete.

In [15] Semenov-Tian-Shansky showed that, given a Poisson action, if the orbit space is a smooth manifold, it carries a Poisson structure such that the natural projection $pr : M \to M/G$ is a Poisson mapping. More precisely, given $f, h \in C^\infty(M)$ with the definitions

$$\hat{f}(m, g) := f(g \cdot m), \quad \hat{h}(m, g) := h(g \cdot m), \quad (13)$$

for any $f, h$ one finds

$$\{f, h\}_M(g \cdot m) = \{\hat{f}(m, \cdot), \hat{h}(m, \cdot)\}_G(g) + \{\hat{f}(\cdot, g), \hat{h}(\cdot, g)\}_M(m) \quad (14)$$

Then $M/G$ inherits a Poisson structure from the Poisson structure on $M$:

$$f, h \in C^\infty(M)^G \implies \{f, h\}_M \in C^\infty(M)^G. \quad (15)$$

Consider a Poisson action $\Phi : G \times M \to M$ with equivariant momentum map $\mu : M \to G^*$ and assume that the orbit space $M/G$ is a smooth manifold. Since in this general case the infinitesimal Poisson action $\xi_M$ is not a Hamiltonian vector field, the first goal of this section is to provide an explicit formulation for $\xi_M$, in terms of local coordinates. We use the properties of the momentum map and dressing action to obtain such a formulation.

We observe that the Poisson Lie group $G^*$ can be described locally in terms of coordinates $(q, p, y)$ such that $\pi_{G^*}$ is given by the splitting theorem [2.1]. In particular, the transverse structure is determined by the structure functions of the Poisson structure on $M$.
\[ \pi^*_G(y) = \{y_i, y_j\}, \] which vanishes on the symplectic leaves. As discussed in Section 3.1, the Poisson Lie group \( G \) acts on \( G^* \) by dressing action and the dressing orbits are the same as the symplectic leaves. Hence, the generic orbit \( O_x \) through \( x \in G^* \) is a closed submanifold of \( G^* \) and \( y_i \) are transversal coordinates such that \( \{y_i, y_j\} = 0 \). Note that \( \mu \) is a submersion, hence it has open image. In particular, the image of \( M \) is an open neighborhood of a generic orbit of \( G \) on \( G^* \).

Define the functions \( H_i \) as the pullbacks by \( \mu \) of the transversal coordinates \( y_i \) to the orbit on \( G^* \):

\[ H_i := y_i \circ \mu. \tag{16} \]

\( H_i \) are defined locally in a \( G \)-invariant open neighborhood \( U \) of the preimage \( N = \mu^{-1}(O_x) \). We can assume that \( x \) is a regular value of \( \mu \), hence \( N \) is a closed \( G \)-invariant submanifold of \( M \). Since \( \{y_i, y_j\} \) vanishes on the orbit \( O_x \), \( \{H_i, H_j\} \) vanishes on the preimage \( N \). The 1-forms \( \alpha_\xi = \mu^*(\theta_\xi) \) are in the span of the \( dH_i \)'s. Since the left invariant 1-form \( \theta_\xi \) in local coordinates can be expressed as a linear combination of \( dy_i \), using the definition \( \text{(16)} \) we have

\[ \alpha_\xi = \mu^*(\theta_\xi) = \sum_i c_i(\xi) dH_i \tag{17} \]

for any \( \xi \in \mathfrak{g} \). As a consequence, the infinitesimal generators \( \xi_M \) of the Hamiltonian action \( \Phi \), induced by \( \mu \), can be written as a linear combination of Hamiltonian vector fields:

\[ \xi_M = \pi^*(\alpha_\xi) = \sum_i c_i(\xi) \{H_j, \cdot\}. \tag{18} \]

We use this explicit formulation to prove that \( M/G \) inherits a Poisson structure from \( M \):

**Theorem 4.1.** Let \( \Phi : G \times M \to M \) be a Poisson action with equivariant momentum map \( \mu \). The algebra \( C^\infty(M)^G \) of \( G \)-invariant functions on \( M \) is a Lie subalgebra of \( C^\infty(M) \).

**Proof.** Let \( f, g \in C^\infty(M)^G \), then \( \xi_M[f] = \xi_M[g] = 0 \) for any \( \xi \in \mathfrak{g} \). Applying the relation \( \text{(18)} \) we have that

\[ \sum_i c_i(\xi) \{H_i, f\} = \sum_i c_i(\xi) \{H_i, g\} = 0 \tag{19} \]

that implies \( \{H_i, f\} = \{H_i, g\} = 0 \) for any \( i \). Then, using the Jacobi identity we get \( \{H_i, \{f, g\}\} = 0 \). Since \( G \) is connected we proved that

\[ \xi_M[\{f, g\}] = 0. \tag{20} \]

Hence \( \{f, g\} \) is \( G \)-invariant and the theorem is proved. \( \square \)
4.2 Poisson structure on $L/G_L$

Consider the $g^*$-valued 1-forms $\alpha_\xi$ defined by $\mu$ by eq. (17). The distribution \( \{ \alpha_\xi \mid \xi \in g \} \) defines a $G$-invariant foliation $\mathcal{F}$ on $M_{\text{reg}}$, the open submanifold of regular values of $\mu$ of $M$ by

\[
T_m L = \ker \alpha_\xi(m) = \bigcap_i \ker dH_i(m)
\]

(21)

for any leaf $L$, which is of the form $L = \mu^{-1}(x)$. The leaf $L$ is not a Poisson submanifold but we prove that, considering the action of $G$ on the space of leaves, the quotient $L/G_L$ inherits a Poisson structure by $M$, where

\[
G_L = \{ g \in G \mid g \cdot L = L \}
\]

(22)

is the stabilizer of the action of $G$ on $L$.

In order to prove this statement we observe that, since $\pi_G^*$ restricted to $O_x$ does not depend on the transversal coordinates $y_i$’s, the Poisson structure $\pi$ on $M$ depends on the coordinates $H_i$ defined in (16) only in terms $\partial_{x_i} \wedge \partial_{H_i}$. This is evident because the differential $d\mu$ between $TM|_N/TN$ and $TG^*/TO_x$ is a bijective map, so using the definition (16) the claim is proved.

Now consider the ideal $I$ generated by $H_i$. The coordinates $H_i$ are locally defined but we can show that $I$ is globally defined. Considering a different neighborhood on the orbit of $G^*$ we have transversal coordinates $y'_i$ and their pullback to $M$ will be $H'_i = y'_i \circ \mu$. The coordinates $H'_i$ are defined in a different open neighborhood $V$ of $N$, but we can see that the ideal $I$ generated by $H_i$ coincides with $I'$ generated by $H'_i$ on the intersection of $U$ and $V$, then it is globally defined. Moreover, since $\mu$ is a Poisson map we have:

\[
\{ H_i, H_j \} = \{ y_i \circ \mu, y_j \circ \mu \} = \{ y_i, y_j \} \circ \mu.
\]

(23)

Hence the ideal $I$ is closed under Poisson brackets.

**Lemma 4.1.** Suppose that $N/G$ is an embedded submanifold of the smooth manifold $M/G$, then

\[
(C^\infty(M)/I)^G = (C^\infty(M)^G + I)/I
\]

(24)

**Proof.** Let $f$ be a smooth function on $M$ satisfying $[f] \in (C^\infty(M)/I)^G$. If the equivalence class $[f]$ is $G$-invariant, we have

\[
f(G \cdot m) = f(m) + i(m),
\]

(25)

where $i \in I = \{ f \in C^\infty(M) : f|_N = 0 \}$. It is clear that $f|_N$ is $G$-invariant and hence it defines a smooth function $\tilde{f} \in C^\infty(N/G)$. Since $N/G$ is a $k$-dimensional embedded submanifold of the $n$-dimensional smooth manifold $M/G$, the inclusion map $\iota : N/G \to M/G$ has local coordinates representation:

\[
(x_1, \ldots, x_k) \mapsto (x_1, \ldots, x_k, c_{k+1}, \ldots, c_n)
\]

(26)
where \( c_i \) are constants. Hence we can extend \( \tilde{f} \) to a smooth function \( \phi \) on \( M/G \) by setting \( \tilde{f}(x_1, \ldots, x_k) = \phi(x_1, \ldots, x_k, 0, \ldots, 0) \). The pullback \( \tilde{f} \) of \( \phi \) by \( \text{pr} : M \to M/G \) is \( G \)-invariant and satisfies
\[
\tilde{f} - f|_N = 0, \tag{27}
\]
hence \( \tilde{f} - f \in \mathcal{I} \).

**Theorem 4.2.** Let \( \Phi : G \times M \to M \) be a Poisson action of \( (G, \pi_G) \) on a Poisson manifold \( (M, \pi) \) with equivariant momentum map \( \mu : M \to G^* \). For each leaf, the orbit space \( \mathcal{L}/G_L \) has a Poisson structure induced by \( \pi \).

**Proof.** First we prove that the Poisson bracket of \( M \) induces a well defined Poisson bracket on \( (C^\infty(U)^G + I)/\mathcal{I} \). In fact, for any \( f + i \in C^\infty(U)^G / \mathcal{I} \) and \( j \in \mathcal{I} \) the Poisson bracket \( \{ f + i, j \} \) still belongs to the ideal \( \mathcal{I} \). Since the ideal \( \mathcal{I} \) is closed under Poisson brackets, \( \{ i, j \} \) belongs to \( \mathcal{I} \). The function \( j \), by definition on the ideal \( \mathcal{I} \), can be written as a linear combination of \( H_i \), so \( \{ f, j \} = \sum a_i \{ f, H_i \} \). By the Theorem 4.1 we have \( \{ f, H_i \} = 0 \), hence \( \{ f + i, j \} \in \mathcal{I} \) as stated. Finally, using the isomorphism proved in the Lemma 4.1 and the identifications
\[
C^\infty(\mathcal{L}/G_L) \simeq C^\infty(N/G) \simeq (C^\infty(U)/\mathcal{I})^G, \tag{28}
\]
the theorem is proved. \( \square \)

We refer to \( \mathcal{L}/G_L \) as the **Poisson reduced space**.

**Remark 4.3.** Theorem 4.2 can be reformulated in terms of Dirac structure in order to give it a more general perspective:

Consider the Poisson action \( G \times M \to M \) with equivariant momentum map \( \mu : M \to G^* \). Let \( x \in G^* \) be a regular value of \( \mu \) and assume that the action is proper and free on \( \mu^{-1}(x) \). Then one has a natural isomorphism
\[
\mu^{-1}(x)/G_x \simeq \mu^{-1}(O_x)/G, \tag{29}
\]
where \( O_x \subset G^* \) denotes the dressing orbit of \( G \) through \( x \) and \( G_x \) denotes the isotropy group of \( x \). This isomorphism is a Poisson diffeomorphism for the unique Poisson structures on these quotients which arise from the diagram

\[
\begin{array}{ccc}
\mu^{-1}(x) & \xrightarrow{\mu^{-1}} & M \\
\downarrow & & \downarrow \\
\mu^{-1}(O_x) & \xrightarrow{\mu^{-1}} & \mu^{-1}(x)/G_x \simeq \mu^{-1}(O_x)/G \\
\end{array}
\]

where the inclusions are backward Dirac maps and the projections are forward Dirac maps.
5 An example: $\mathbb{R}^2$ action

Here we want to discuss a concrete example for the Poisson reduction. Consider the Lie bialgebra $\mathfrak{g} = \mathbb{R}^2$ with generators $\xi$ and $\eta$ such that

$$[\xi, \eta] = \eta$$  \hfill (30)

and cobracket given by

$$\delta(\xi) = 0 \quad \delta(\eta) = \xi \wedge \eta.$$  \hfill (31)

The matrix representation of $\mathfrak{g}$ is the Lie algebra $\mathfrak{gl}(2, \mathbb{R})$ and the subgroups $G$ and $G^*$ of $GL(2, \mathbb{R})$ of matrices with positive determinant are given by

$$G = \left\{ \begin{pmatrix} 1 & 0 \\ x & y \end{pmatrix} : y > 0 \right\} \quad G^* = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a > 0 \right\}$$  \hfill (32)

and we remark that the Poisson bivector on $G^*$ is

$$\pi_{G^*} = ab \partial_a \wedge \partial_b.$$  \hfill (33)

In this simple case it is clear that $\{a, b\}$ are global coordinates on $G^*$. We analyze the orbits of the dressing action of $G$ on $G^*$ for this example.

Remember that the dressing orbits $O_x$ through a point $x \in G^*$ are the same as the symplectic leaves, hence it is clear that they are generated by the equation $b = 0$. The symplectic foliation of the manifold $G^*$ is now given by two open orbit, determined by the conditions $b > 0$ and $b < 0$ respectively, and a closed orbit given by $b = 0$ and $a \in \mathbb{R}$.

Consider a Poisson action of $G$ on a generic Poisson manifold $M$ induced by the equivariant momentum map $\mu : M \to G^*$. Its pullback

$$\mu^* : C^\infty(G^*) \to C^\infty(M)$$  \hfill (34)

maps the coordinates $a$ and $b$ on $G^*$ to $\hat{a}(x) = a(\mu(x))$ and $\hat{b}(x) = b(\mu(x))$ resp. In order to simplify the notation we denote the coordinates on $M$ only with $a$ and $b$. It is important to underline that we have no information on the dimension of $M$, so $a$ and $b$ are just a couple of the possible coordinates. Nevertheless, since $\mu$ is a Poisson map, we have

$$\{a, b\} = ab$$  \hfill (35)

on $M$. The infinitesimal action of $\mathfrak{g} = \mathbb{R}^2$ on $M$ that we consider can be written in terms of these coordinates $a, b$ as

$$\Phi(\xi) = a\{b, \cdot\} \quad \Phi(\eta) = a\{a^{-1}, \cdot\}.$$  \hfill (36)

In the previous section we proved that the Poisson reduction is given equivalently either as the Poisson algebra $C^\infty(N/G)$ on the quotient $N/G$, with $N = \mu^{-1}(O_x)$ or as $(C^\infty(M)/\mathcal{I})^G$. In the following, we discuss 3 different cases of dressing orbit.
Case 1: \( b > 0 \). Consider the dressing orbit \( \mathcal{O}_x \) generated by the condition \( b > 0 \). Since \( a \) and \( b \) are both positive, we can put

\[
a = e^p, \quad b = e^q
\]

and we have

\[
\{p, q\} = 1
\]

(38)
since \( \{a, b\} = ab \). For this reason we can claim that the preimage of the dressing orbit can be split as \( N = \mathbb{R}^2 \times M_1 \) and \( C^\infty(N) \) is given explicitly by the set of functions generated by \( b^{-1} \). The infinitesimal action is given by

\[
\Phi(\xi) = e^p\{e^q, \cdot\} \quad \Phi(\eta) = e^p\{e^{-p}, \cdot\}
\]

(39)

which is just the action of \( G \) on the plane. Hence the Poisson reduction in this case is given by

\[
(C^\infty(M)[b^{-1}])^G.
\]

(40)

Case 2: \( b < 0 \). This case is similar, with the only difference that \( b = -e^q \).

Case 3: \( b = 0 \). This case is slightly different. The orbit \( \mathcal{O}_x \) is given by fixed points on the line \( b = 0 \), then we choose the point \( a = 1 \). Clearly, in this case we can not define \( b = e^p \).

Consider the ideal \( \mathcal{I} = \langle a - 1, b \rangle \) of functions vanishing on \( N \). It is easy to check that it is \( G \)-invariant, hence the Poisson reduction in this case is:

\[
(C^\infty(M)/\mathcal{I})^G.
\]

(41)

5.1 Questions and future directions

This paper generalizes the Marsden-Weinstein reduction to the formalism of the Poisson geometry. The reformulation in terms of Dirac structures suggests the idea of a generalization of the Poisson reduction to actions of Dirac Lie groups [12] on Dirac manifolds [2]. A possible development of this theory is its integration to symplectic groupoids by means of the theories on the integrability of Poisson brackets [3] and Poisson Lie group actions [7]. On the other hand, in [5], [6] a quantum momentum map has been defined. The Poisson reduction might be quantized using the quantum groups theory [1] as a quantization of the Poisson Lie groups and Kontsevich’s theory for the quantization of Poisson manifolds [13].

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