A note on mixed super quasi Einstein manifold

Ananta Patra¹ and Akshoy Patra²*

Abstract
Mixed super quasi Einstein manifold (MS(QE)ₙ) is a generalization of Einstein manifold. In this paper we have studied some geometric properties of MS(QE)ₙ. Also we have studied MS(QE)ₙ satisfying some curvature restriction and obtained the form of Riemannian curvature tensor. We have studied conformally flat and conformally conservative MS(QE)ₙ. We have deduced a necessary condition for a MS(QE)ₙ to be conformally conservative. Some basic properties of MS(QE)ₙ on viscous fluid MS(QE)ₙ spacetimes are discussed. We have proved that if a viscous fluid MS(QE)ₙ spacetime admitting heat flux obeys Einstein equation with a cosmological constant then none of the energy density and isotropic pressure of the fluid can be a constant.

Keywords
Mixed super quasi-Einstein manifold, conformally flat, conformally conservative, viscous fluid, heat flux, cosmological constant, energy density, isotropic pressure.

AMS Subject Classification
Primary 53C50, 53C25, 53B30; Secondary 53C80, 53B50.

1 Department of Mathematics, Kandi Raj College, Kandi-742137, Murshidabad, West Bengal, India.
2 Department of Mathematics, Govt.College of Engineering and Textile Technology, Berhampore-742101, Murshidabad, West Bengal, India.
*Corresponding author: akshopy@gmail.com

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1. Introduction
Let \( U_r = \{ x \in M : S \neq \frac{r}{n} g, atx \} \), where \( S \) and \( r \) are respectively the Ricci tensor and scalar curvature of a Riemannian manifold \((M^n, g), (n \geq 3)\). Then the manifold is said to be a quasi Einstein [4] manifold if on \( U_r \), we have

\[
S - ag = baA \otimes A,
\]

where \( A \) is a 1-form on \( U_r \) and \( a, b \) are some functions on \( U_r \). It is clear that the 1-form \( A \) as well as the function \( b \) are non zero at every point on \( U_r \). From the above definition it follows that every Einstein manifold is quasi- Einstein. The scalars \( a, b \) are known as the associated scalars of the manifold. Also the \( 1 \)-form \( A \) is called the associated 1-form of the manifold defined by \( g(X, U) = A(X) \) for any vector field \( X \); \( U \) being a unit vector field, called the generator of the manifold. Such an \( n \)-dimensional quasi Einstein manifold is denoted by \( (QE)_n \).

There are many generalization of \( (QE)_n \) in literature([1], [2], [3], [4], [5], [7]). One of them is mixed super quasi-Einstein manifold introduced by A. Bhattacharya, M. Tarafdar and D. Debnath [2]. According to them a non flat Riemannian manifold is said to be mixed super quasi-Einstein manifold if it satisfies the condition

\[
S(X, Y) = ag(X, Y) + baA(Y)A(Y) + cbX(B(Y) + d[A(X)B(Y) + A(Y)B(X)] + eD(X, Y),
\]

where \( a, b, c, d, e \) are real valued functions on \((M^n, g)\) of which \( b \neq 0, c \neq 0, d \neq 0, e \neq 0 \) and \( A, B \) are two non zero 1-forms such that \( g(X, U) = A(X), g(X, V) = B(X), g(U, U) = 1, g(V, V) = 1, g(U, V) = 0 \). \( D \) is a symmetric tensor of type \((0, 2)\) with zero trace such that \( D(X, U) = 0 \forall X \in \mathcal{X}(M) \). Here
a, b, c, d, e are called the associated scalars, A, B are called the main and the auxiliary generators and D is called the structure tensor. Such a space is denoted by MS(QE)\(_n\). The paper is organized as follows. Section 2 is concerned with preliminaries. In section 3 we have obtained some geometric properties of a MS(QE)\(_n\). Section 4 deals with conformally flat and conservative MS(QE)\(_n\). In section 5 we have studied some properties of pseudo Ricci symmetric MS(QE)\(_n\). In the last section 6 we studied viscous fluid (MSQE)\(_n\) spacetimes.

2. Preliminaries

Putting \(X = Y = e_i\) where \(\{e_i : 1 \leq i \leq n\}\) is an orthonormal basis of the tangent space of the manifold in (1.1) and summing from 1 to \(n\) we get,

\[
r = na + b + c. \tag{2.1}
\]

Putting \(X = Y = U\) in (1.1)

\[
S(U, U) = a + b. \tag{2.2}
\]

Setting \(X = Y = V\) in (1.1) we get,

\[
S(V, V) = a + c + eD(V, V). \tag{2.3}
\]

Again putting \(X = U, Y = V\) in (1.1) we get,

\[
S(U, V) = d. \tag{2.4}
\]

From above we ca state the following

**Theorem 2.1.** In MS(QE)\(_n\), the scalars \(a+b\) and \(a+c+eD(V, V)\) are the Ricci curvatures along the generators \(U\) and \(V\) respectively.

Suppose \(S(X, Y) = g(QX, Y), D(X, Y) = g(LX, Y)\), \(s^2 = \sum_i S(Qe_i, e_i), \) \(j^2 = \sum_i D(Le_i, e_i)\)

From (1.1) we get

\[
\sum_{i=1}^{n} S(Qe_i, e_i) = a(an + b + c) \tag{2.5}
\]

\[+ b(a + b) + c(a + c + eD(V, V))
\]

\[+ d(d + d) + e \sum_{i=1}^{n} D(Qe_i, e_i)
\]

\[= (n - 2)a^2 + (a + b)^2 + (a + c)^2 + 2d^2 + 2eD(V, V) + e \sum_{i=1}^{n} S(Le_i, e_i).
\]

Again from (1.1)

\[
\sum_{i=1}^{n} S(Le_i, e_i) = cD(V, V) + e \sum_{i=1}^{n} D(Le_i, e_i) \tag{2.6}
\]

\[= cD(V, V) + eJ^2.
\]

Using (2.5) and (2.6) we get

\[
s^2 = (n - 2)a^2 + (a + b)^2 + (a + c)^2 + 2d^2 + 2eD(V, V) + eJ^2. \tag{2.7}
\]

From (2.3) it is clear that

\[
s^2 = na^2 + b^2 + c^2 + 2ab + 2ac + 2eD(V, V) + eJ^2 + 2d^2
\]

\[= na^2 + b^2 + c^2 + 2ab + 2ac + 2eS(V, V) + eJ^2 + 2d^2 + 2cS(V, V).
\]

Now, \(e > \frac{2}{3}\) (res \(< 0\ or \,= 0\)) according as \(na^2 + b^2 - c^2 + 2d^2 + 2cS(V, V) < 0\ (res \,> \,= 0\). Hence we can state the following

**Theorem 2.2.** In a MS(QE)\(_n\) (\(n > 2\)) the associated scalar \(e\) is less than or equal to or greater than the ratio which the length of the Ricci tensor \(S\) bears to the length of the structure tensor \(D\) according as \(na^2 + b^2 - c^2 + 2d^2 + 2cS(V, V) > 0\ (res \,= 0 or < 0\).

3. Some geometric properties

Let us suppose that in a MS(QE)\(_n\) the generator \(U\) is parallel vector field. Then \(\nabla_X U = 0\ \forall X\). So \(R(X, Y)U = 0\) and \(S(X, U) = 0\ \forall X\)

From (1.1), \(0 = (a + b)A(X) + dB(X)\ \forall X\)

Putting \(X = V\) we obtain \(d = 0\). Again putting \(X = U\) we obtain \(a + b = 0\). Hence we have the following

**Theorem 3.1.** If the generator \(U\) of a MS(QE)\(_n\) is a parallel vector field then either \(d = 0\) or \(a + b = 0\).

**Theorem 3.2.** In a MS(QE)\(_n\) \(QU, V\) are orthogonal iff \(d = 0\).

**Proof.** \(S(U, V) = d\) i.e., \(g(QU, V) = d\), which is 0 if and only if \(d = 0\). Hence the theorem.

**Theorem 3.3.** In a MS(QE)\(_n\) \(QV, V\) are orthogonal iff \(a + c + eD(V, V) = 0\).

**Proof.**

\[S(V, V) = a + c + eD(V, V)\]

\[g(QV, V) = a + c + eD(V, V).
\]

So \(g(QV, V) = 0\) iff \(a + c + eD(V, V) = 0\).

Hence the theorem.

**Theorem 3.4.** An MS(QE)\(_n\) is a P(QGQE)\(_n\) if either of the vector field is a parallel vector field.

**Proof.** If the vector field \(U\) is a parallel vector field, then we have \(\nabla_X U = 0\ \forall X\). So \(R(X, Y)U = 0\) and eventually \(S(X, U) = 0\ \forall X\)

From (1.1), \(0 = (a + b)A(X) + dB(X), \ \forall X\)

Putting \(X = V\) we obtain \(d = 0\), i.e the manifold is P(QGQE)\(_n\) [6].

Again if the vector field \(V\) is parallel then \(R(X, Y)V = 0\), consequently \(S(Y, V) = 0\), i.e \(abY + cB(Y) + d[A(Y)] + eD(Y, V) = 0\).

Putting \(Y = U\) we get \(d = 0\), i.e the manifold is P(QGQE)\(_n\). Hence the theorem.
Theorem 3.5. In a $\text{MS}(\text{QE})_n$ $0$ is an eigen value of $L$ in the
direction of the eigen vector $U$, i.e. $LU = 0$, where $L$ is the
symmetric endomorphism of the tangent space at any point of
the manifold corresponding to the structure tensor $D$.

Proof. We have $g(LX,Y) = D(X,Y) \forall X,Y \in \chi(M)$. Putting
$X = U$, we get, $g(LU,Y) = D(U,Y) = 0 \forall Y$. So $LU = 0$, i.e $0$
is an eigen value of $L$ in the direction of $U$. □

We now consider a compact orientable $\text{MS}(\text{QE})_n (n > 2)$
without boundary. From (1.1) we get,

$$S(X,X) = ag(X,X) + bA(X)A(X) + cB(X)B(X) + d[A(X)B(X) + A(X)B(X)] + eD(X,X).$$

(3.1)

Let us assume that $\theta_u$ be the angle between $U$ and any vector $X$,
then we have $\cos \theta_u \geq \cos \theta_u$, i.e., $g(X,U) \geq g(X,V)$. Therefore,

$$S(X,X) \geq |a + b + c + 2d|[g(X,U)]^2,$$

when $a, b, c, d, e, D(X,X)$ are positive.

Definition 3.6. A vector field $H$ in a Riemannian manifold
(M$^n$, $\text{g}$) $(n > 2)$ is said to be harmonic [8] if $d\tau = 0$ and
$i\tau = 0$ where $\tau(X) = g(X,H) \forall X$.

It is known from a compact orientable Riemannian manifold
the following relations holds $\int_M[S(X,X) - \frac{1}{2}(d\tau)^2 + \langle \nabla X \rangle^2 - (\delta \tau)^2]dv = 0$, for any vector field $X$ where $dv$
notates the volume element of $M$. Now let $X \in \chi(M)$ be
harmonic vector field then $\int_M[S(X,X) + \langle \nabla X \rangle^2]dv = 0$ for
any $X$. Hence if each $a,b,c,d,e,D(X,X)$ is positive then $\int_M[(a + b + c + 2d)g(X,U)^2 + \langle \nabla X \rangle^2]dv \geq 0$, by virtue of
$a + b + c + 2d > 0$, $g(X,U) = 0$ and $\nabla X = 0$ for any vector field $X$. This follows that $X$ is orthogonal to $U$ and $X$ is a parallel vector field. Similarly if $\theta_u \geq \theta_u$, assuming as before
it can be shown $g(X,V) = 0$ and $\nabla X = 0$ for any vector field $X$. Thus we have the following theorem

Theorem 3.7. In a compact orientable $\text{MS}(\text{QE})_n (n > 2)$
without boundary any harmonic vector field $X$ is parallel
and orthogonal to one of the generators of the manifold which
makes greatest angle with vector $X$ provided $a,b,c,d,e,D(X,X)$
are positive scalars.

Let us now investigate whether a $\text{MS}(\text{QE})_n (n > 2)$ is
projectively flat or not.

Theorem 3.8. A $\text{MS}(\text{QE})_n (n > 2)$ can not be projectively
flat.

Proof. Let if possible a $\text{MS}(\text{QE})_n (n > 2)$ is projectively flat.
Then the Riemannian curvature tensor is given by

$$R(X,Y,Z,W) = \frac{1}{n-1}[S(Y,Z)g(X,W) - S(Y,Z)g(X,W)].$$

Contrasting $Y$ and $Z$ and putting $W = U$ we get

$$S(X,U) = \frac{1}{n-1}[rA(X) - S(X,U)].$$

Or,

$$S(X,U) = \frac{r}{n}A(X).$$

Putting $X = V$, in above we get $d = 0$, a contradiction. Hence
the theorem. □

4. Conformally flat and Conformally conservative $\text{MS}(\text{QE})_n$

Theorem 4.1. If the main generator of a conformally flat
$\text{MS}(\text{QE})_n$ is parallel vector field then it is a ($\text{GQE})_n$.

Proof. We recall that in a $\text{MS}(\text{QE})_n$ the scalar curvature is
given by $r = an + b + c$. Now if the manifold is conformally
flat then its Riemannian curvature tensor is given by

$$R(X,Y,Z,W) = \frac{1}{n-1}[S(Y,Z)g(X,W) - S(Y,Z)g(X,W)].$$

(4.1)

$$S(X,Z)g(Y,W) + S(X,W)g(Y,Z)$$

$$- S(Y,W)g(X,Z) - \frac{r}{(n-1)(n-2)}[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)].$$

Now using definition of $\text{MS}(\text{QE})_n$ and using $r = an + b + c$ and putting $Z = U$ we get

$$R(X,Y)U = \frac{a+b}{n-1}[A(Y)X - A(X)Y]$$

(4.2)

$$- \frac{c}{n-2}[VB(Y) + \frac{1}{(n-1)}][UA(Y) - Y]$$

$$+ \frac{d}{n-2}[B(Y)X - B(X)Y + B(X)A(Y)U - B(Y)A(X)U]$$

$$+ \frac{e}{n-2}[A(Y)LX - A(X)LY],$$

where $g(LX,Y) = D(X,Y)$. If $U$ is a parallel vector field then
$R(X,Y)U = 0$, $a+b = d = 0$, so the last equation becomes

$$\frac{c}{n-2}[VB(Y) + \frac{1}{(n-1)}][UA(Y) - Y]$$

(4.3)

$$+ \frac{e}{n-2}[A(Y)LX - A(X)LY].$$

Putting $Y = U$ we get

$$e[LX - A(X)LU] = 0.$$  

(4.4)

But $LU = 0$, so we have $eLX = 0 \forall X$, Hence $e = 0$. So, if $U$
is parallel vector field in a conformally flat $\text{MS}(\text{QE})_n$, then
$a + b = d = e = 0$, i.e the manifold reduces to ($\text{GQE})_n$. □
Theorem 4.2. A necessary condition for a MS(QE)$_n$ to be conformally conservative is 
\[ (d((n-2)a + (2n-3)b + c)(V) = 2(n-1))(dd)(U) \]

Proof. A Riemannian manifold is said to be conformally conservative if the divergence of its conformal curvature tensor is zero, i.e.
\[ (\nabla_X S)(Y, Z) - (\nabla_Y S)(X, Z) = \frac{1}{2(n-1)}[\langle r(X) g(Y, Z) - (dr)(Z)g(X, Y) \rangle]. \] 
Now putting \( X = Y = U \) and \( Z = V \), we get,
\[ (\nabla_U S)(U, U) = \frac{1}{2(n-1)}[\langle d(r)g(U, U) - (dr)(V)g(U, U) \rangle]. \] 
Now using the relations \( S(U, V) = d, S(U, U) = a + b \) and \( r = an + b + c \) in above we get
\[ (dd)(U) - d(a + b)(V) = \frac{1}{2(n-1)}[n(da)(V) + (db)(V) + (dc)(V)]. \] 
On simplification
\[ (dd)(U) - d(a + b)(V) = \frac{1}{2(n-1)}[n(da)(V) + (db)(V) + (dc)(V)], \] 
or
\[ 2(n-1)(dd)(U) - 2(n-1)d(a + b)(V) = -[n(da)(V) + (db)(V) + (dc)(V)], \] 
or
\[ 2(n-1)(dd)(U) = (dd)(U) = d((n-2)a + (2n-3)b + c)(V). \]
Hence the theorem.

5. Ricci-pseudosymmetric MS(QE)$_n$

An n-dimensional Riemannian manifold \((M^n, g)\) is called Ricci-pseudosymmetric if,
\[ (R(X, Y)S)(Z, W) = L_s g(X, Z)S(Y, W) \] 
holds on \( U_s = \{x \in M : S \neq \frac{n}{2}g, atx\} \) and \( L_s \) is a certain function on \( U_s \). Then we have,
\[ L_s[g(Y, Z)S(X, W) - g(X, Z)S(Y, W)] + g(Y, W)S(Z, X) - g(X, W)S(Y, Z) \]
holds.

Theorem 5.1. In a Ricci-pseudosymmetric MS(QE)$_n$ \( n \geq 3 \) the following results hold.
\[ R(V, U, U, V) = L_s, \] 
\[ D(R(U, U)V, V) = 0, \] 
\[ L_s = \frac{D(R(U, U)V, V)}{D(V, V)}, \] 
provided \( D(V, V) \neq 0 \).

Proof. We consider Ricci-pseudosymmetric MS(QE)$_n$. Then we have
\[ S(R(X, Y)Z, W) + S(Z, R(X, Y)W) = L_s g(Y, Z)S(X, W) - g(X, Z)S(Y, W) + g(Y, W)S(Z, X) - g(X, W)S(Y, Z), \]
or
\[ b[A(R(X, Y)Z)A(W) + A(Z)A(R(X, Y)W)] + c[B(R(X, Y)Z)B(W) + B(Z)B(R(X, Y)W)] + d[A(R(X, Y)Z)B(W) + A(W)B(R(X, Y)Z)] + e[D(R(X, Y)Z, W) + D(Z, R(X, Y)W)] + g(Y, W)A[Z(A)A(W) - g(X, Z)A(Y)A(W)] + g(Y, W)[A(Z)B(X) + A(X)B(Z)] + d[g(Y, Z)[A(X)]B(W) + A(W)B(X)] - g(X, Z)[A(Y)]B(W) + A(W)B(Y)] + e[g(Y, Z)D(X, W) - g(X, Z)D(Y, W) + g(Y, W)D(X, Z) - g(X, W)D(Y, Z)]. \]
Putting \( Z = U \) and \( W = V \) in (5.7), we get
\[ b[R(X, Y, V, U)] + c[R(X, Y, U, V)] - L_s[A(X)B(Y) - A(Y)B(X)] + e[D(R(X, Y)U, V) - L_s[A(Y)D(X, V] - A(X)D(Y, V)] - A(X)D(Y, V)] = 0. \] 
Putting \( Z = W \) in (5.7) we get
\[ d[R(X, Y)U, V) - L_s[A(Y)B(X) - A(X)B(Y)] = 0. \] 
Since, \( d \neq 0 \) we get
\[ R(X, Y)U, V) - L_s[A(Y)B(X) - A(X)B(Y)] = 0. \]
Similarly, if we take \( Z = W = V \) in (5.7) we get,
\[
d[R(X,Y)V,V] - L_s\{A(Y)B(X)\} = e[D(R(X,Y)V,V) - L_s\{B(Y)D(X,V) - B(X)D(Y,V)\}] = 0.
\]
(5.11)
Using (5.9) we get
\[
e[D(R(X,Y)V,V) - L_s\{B(Y)D(X,V) - B(X)D(Y,V)\}] = 0.
\]
Since \( e \neq 0 \), we have
\[
D(R(X,Y)V,V) = 0.
\]
(5.12)
Putting \( X = Y = U \) in (5.10) we get (5.3). Again putting \( X = V, Y = U \) in (5.12) we get (5.4). Using (5.12) in (5.11) we get
\[
D(R(X,Y)U,V) - L_s\{A(Y)D(X,V) - A(X)D(Y,V)\} = 0.
\]
(5.13)
Putting \( X = U, Y = V \) in above we get (5.5).

6. General relativistic viscous fluid spacetime admitting heat flux [6]

Let \( (M^n,g) \) be a connected semi-Riemannian viscous fluid spacetime admitting heat flux and satisfying Einstein’s equation with a cosmological constant \( \lambda \). Also let \( U \) be the unit timelike velocity vector field, \( V \) be the unit heat flux vector and \( D \) be the anisotropic pressure tensor of the fluid. The we have
\[
g(U,U) = -1, g(V,V) = 1, g(U,V) = 0 \quad (6.1)
\]
\[
D(X,Y) = D(Y,X), Tr.D = 0, D(X,U) = 0 \forall X. \quad (6.2)
\]
Let
\[
g(X,U) = A(X), g(X,V) = B(X) \forall X. \quad (6.3)
\]
Also let \( T \) be the energy-momentum tensor of type (0,2) describing the matter distribution of such fluid and it be of the following form
\[
T(X,Y) = pg(X,Y) + (\sigma + p)A(Y) + B(Y) + \{A(X)B(Y) + A(Y)B(X)\} + D(X,Y),
\]
(6.4)
where \( \sigma, p \) are the energy density and isotropic pressure respectively. General relativity flows from Einstein equation given by
\[
S(X,Y) = -\frac{r}{2}g(X,Y) + \lambda g(X,Y) = kT(X,Y), \quad (6.5)
\]
for all vector fields \( X, Y \). \( S \) is the Ricci tensor of type of type (0,2) and \( r \) is the scalar curvature, \( \lambda \) is a cosmological constant. Thus by virtue of (6.4) above equation can be written as
\[
S(X,Y) - \frac{r}{2}g(X,Y) + \lambda g(X,Y) = k\{[p(g(X,Y) + (\sigma + p)A(Y) + B(Y) + \{A(X)B(Y) + A(Y)B(X)\} + D(X,Y)].
\]
Putting this in (1.1) we get
\[
\frac{2kp - 2\lambda + 2a + b + c}{2}g(X,Y)
\]
(6.6)
and
\[
\begin{align*}
&= [b - k(\sigma + p)]A(X)A(Y) + (c - k)B(X)B(Y) + \\
&\quad + (d - k)[A(X)B(Y) + A(Y)B(X)] + (e - k)D(X,Y).
\end{align*}
\]
Putting \( X = U, Y = U \) in above we get \( 2 = d \) \( 6 \) Putting \( X = U, Y = U \) we get
\[
\sigma = \frac{2a + 3b + c - 2\lambda}{2k},
\]
(6.8)
or,
\[
\sigma = \frac{2a + 3b + c - 2\lambda}{2d}.
\]
(6.9)
Again contracting (6.6) we get
\[
r - 2r + 4\lambda = k[3p - \sigma + 1],
\]
(6.10)
or,
\[
p = \frac{6\lambda - 6a + b + c - 2d}{6d}.
\]
(6.11)
Hence we can state the following

**Theorem 6.1.** If a viscous fluid MS(QE)\( \lambda \) spacetime admitting heat flux obeys Einstein equation with cosmological constant then none of the energy density and isotropic pressure of the fluid can be a constant.

Now if the associated scalars \( a, b, c, d \) are constants with \( d > 0 \), then from (6.8) and (6.9) \( \sigma, p \) are constants. Since \( \sigma > 0, p > 0 \) we have from (6.8) and (6.9) we get \( \lambda < \frac{2a + 3b + c}{2} \) and \( \lambda > \frac{6a - b + c - 2d}{6} \). And hence
\[
\frac{6a - b + c - 2d}{6} < \lambda < \frac{2a + 3b + c}{2}. \quad (6.12)
\]
Thus we have the following

**Theorem 6.2.** If a viscous fluid MS(QE)\( \lambda \) spacetime admitting heat flux obeys Einstein equation with cosmological constant \( \lambda \), then \( \lambda \) obeys the above inequality.
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