Potentials for transverse trace-free tensors

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Abstract

In constructing and understanding initial conditions in the 3 + 1 formalism for numerical relativity, the transverse and trace-free (TT) part of the extrinsic curvature plays a key role. We know that TT tensors possess two degrees of freedom per space point. However, finding an expression for a TT tensor depending on only two scalar functions is a non-trivial task. Assuming either axial or translational symmetry, expressions depending on two scalar potentials alone are derived here for all TT tensors in flat 3-space. In a more general spatial slice, only one of these potentials is found, the same potential given in (Baker and Puzio 1999 Phys. Rev. D 59 044030) and (Dain 2001 Phys. Rev. D 64 124002), with the remaining equations reduced to a partial differential equation, depending on boundary conditions for a solution. As an exercise, we also derive the potentials which give the Bowen-York curvature tensor in flat space.

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1. Introduction

This paper is devoted to transverse and trace-free (TT) tensors in 3-space. These are symmetric 2-index tensors which are both transverse (divergence-free) and trace free. They play several key roles in General Relativity (GR). In the analysis of GR as a Hamiltonian system, initiated by ADM [3] the ‘true gravitational degrees of freedom’ were identified as a pair of TT tensors. In particular, the momentum constraint reduces to $\nabla_a (K^{ab} - Kg^{ab}) = 0$ where $K^{ab}$ is the extrinsic curvature and $K$ is its trace. If we add the condition that $K = 0$, we have that $K^{ab}$ must be a TT tensor. In the conformal method of solving the constraints, TT tensors form a key part of the gravitational momentum (see [4–6]). Therefore any understanding we can gain of the space of TT tensors will add to our ability to understand the space of solutions of the Einstein equations.

A symmetric 2-tensor in 3-space contains six independent components. For a TT tensor there are four conditions on these six components, and therefore one expects TT tensors to have two degrees of freedom per space point. Can one explicitly identify these two degrees of freedom? Further, can one construct general TT tensors starting from these variables? We
have failed to do this in general, but succeeded in several special cases by adding two extra conditions. We restrict our attention to TT tensors on flat space and further assume that the tensor has an extra symmetry, either translational or rotational. Under these assumptions we can express ALL such TT tensors in terms of two free scalar potentials. This gives the desired representation of the two degrees of freedom and shows how to explicitly construct all TT tensors of these special kinds. However since TT tensors are conformally covariant, if \( A^{ij} \) is TT with respect to a metric \( g_{ij} \), then \( \bar{A}^{ij} = \phi^{-10}A^{ij} \) is TT with respect to \( \bar{g}_{ij} = \phi^4g_{ij} \) for any positive function \( \phi \). Therefore once we find all axi-symmetric (or translationally symmetric) TT tensors on flat space, we can also find all such tensors when the metric is conformally flat rather than flat. In the axisymmetric case, with appropriate boundary conditions, we can even conformally compactify and find TT tensors on a compact manifold without boundary.

The manifolds considered will also be assumed to be contractible, i.e. that any closed loop can be continuously contracted to a point. By Poincaré’s lemma, this means that every closed differential form is exact. Specifically, in three-dimensions, if the curl of a vector field vanishes, the vector field can be given by the gradient of a scalar potential, and if a vector field has vanishing divergence, it can be given by the curl of some other vector field. This assumption only rules out manifolds with non-trivial homotopy.

In the 3+1 formalism for numerical relativity, the initial conditions are given by a space-like hypersurface, with a 3-space metric \( g_{ab} \) and an extrinsic curvature tensor \( K_{ab} \), giving the embedding of the hypersurface in the space-time manifold. The decomposition of the Einstein equations however, gives a set of constraint equations for the metric and curvature, restricting the choices for 4 of the 12 components (both \( g_{ab} \) and \( K_{ab} \) being symmetric 3-space tensors). In the weak-field regime, [3], it was realised early on that the true degrees of freedom consisted of a pair of TT tensors on flat space. Thus we have the desired four degrees of freedom, once the four degrees associated with the choice of coordinates are removed. This four-fold freedom exactly matches up to the two degrees of freedom in the radiation field. One can see an immediate parallel with Maxwell’s equations, where the degrees of freedom consists of two divergence-free vectors.

The most successful technique for constructing solutions to the constraints, i.e., initial data for the gravitational field, has been the conformal transverse trace-free (CTT) decomposition, originally developed in [4–6] (see also the text books [7, 8]). In the simplest version of the CTT decomposition, the extrinsic curvature is assumed to be trace-free, and thus TT. The way a TT tensor remains TT under a conformal transformation is used to solve the Hamiltonian constraint. Further, in the weak-field regime, the other two wave degrees of freedom are represented by the TT part of the metric. The ability to completely represent these wave degrees of freedom explicitly means that one can add arbitrary gravitational waves to any fixed background (assuming it to be conformally flat) without ever having to perform tensor decompositions.

Unfortunately, explicit expressions for a TT tensor, depending on two scalar functions alone, have not been found in the general case. However, assuming the existence of a surface-orthogonal Killing field, a single scalar potential is found in [1], which, in both cylindrical and spherical-polar type coordinates, generates two of the six tensor components, leaving the remaining four unspecified. This result is formulated in a coordinate-independent manner by Dain in [2], using a ‘time/Killing-coordinate symmetry’ from [9], which implies that the four unspecified components vanish. In this paper we find the ‘other’ potential, assuming the metric to be flat, which coincides with the part that Dain had to set to zero. Our technique works also in the case of translational symmetry. Thus we have a complete explicit representation of all axi-symmetric (and translationally symmetric) TT tensors on flat space in terms of two scalar potentials.
It is obviously useful to assume that the coordinates we use explicitly represent the desired symmetry. Therefore, in section 3 using both Cartesian and cylindrical coordinates, we find the two potentials with a linear symmetry condition, and in section 4, in cylindrical and spherical coordinates, we find the two potentials with axial symmetry. For a more general space, the scalar potential found in both [1, 2] is derived in section 5 in cylindrical coordinates, for axially-symmetric tensors. Though a second potential cannot be found explicitly, the remaining equations reduce to a second order partial differential equation in two of the remaining components.

For both the translationally symmetric and axially-symmetric tensors, the specific choice of potentials which give the Bowen-York curvature tensors [10] are computed in section 6. These are a very special sub-class of TT tensors on flat space. In the tensor decomposition language, the Bowen-York tensors are those TT tensors which are simultaneously pure conformal Killing forms.

The momentum constraint is very similar to the Maxwell constraint $\nabla_a D^a = 0$. The traditional way of solving this, is to take advantage of the fact that a divergence-free vector must be the curl of another vector, from Poincaré’s lemma. If however, the vector is either translationally symmetric or axially symmetric, we can express the vector in terms of two scalar potentials. Clearly this is closely related to finding potentials for TT tensors. We will lay out the vector calculation in the next section.

The main result of the paper can be seen immediately from (52), an expression in cylindrical coordinates depending solely on the two scalar potentials $R_A(\rho, z)$ and $Y_A(\rho, z)$, giving all axially symmetric TT tensors in flat space. Without going through the derivation, one can directly check that this is an axi-symmetric TT tensor from (37a)–(37d). An equivalent expression in spherical polar coordinates is given by (56), which can be checked from (44a)–(44d). Showing that all TT tensors can be expressed in these forms makes use of Stokes’ theorem, and in particular the fact that a curl-free vector is the gradient of a scalar. Again, this holds as long as the manifold is contractible.

2. Potentials for divergence-free vectors

Let us assume that we have a divergence-free vector $B^a$ on flat space which we assume to be translationally invariant in one direction and we pick this direction as the $z$ direction of a Cartesian coordinate system on the flat space. Therefore we assume $B^a \equiv B^a(x, y)$. The divergence equation now becomes

$$\partial_x B^x + \partial_y B^y + \partial_z B^z = 0.$$  \hfill (1)

We now can solve equation (1) by picking any function $F(x, y)$ and choosing

$$B^x = \partial_y F, \quad B^y = -\partial_x F.$$  \hfill (2)

Therefore we can construct any translationally invariant divergence-free vector on flat space from a pair of scalars $[F(x, y), B^x(x, y)]$.

This argument extends to the situation where the metric is not flat, but is also translationally invariant in the same direction, which we again pick as the $z$ direction. We fix the coordinates so that the metric is $z$ independent. We continue to call the other two coordinates $(x, y)$ but we do not assume any connection between them and any Cartesian system. We continue to have $B^a \equiv B^a(x, y)$ and also $g \equiv g(x, y)$, where $g$ is the determinant of the metric.

The divergence equation becomes

$$g^{-1/2} \partial_x (\sqrt{g} B^x) + g^{-1/2} \partial_y (\sqrt{g} B^y) + g^{-1/2} \partial_z (\sqrt{g} B^z) = 0.$$  \hfill (3)
We now can solve equation (3) by again picking any function $F(x, y)$ and choosing
\[
\sqrt{g^B} = \partial_y F, \quad \sqrt{g^B} = -\partial_x F.
\]
Therefore we can construct any translationally invariant divergence-free vector on a translationally symmetric space from a pair of scalars $[F(x, y), B^F(x, y)]$.

This works equally well when the divergence-free vector is axially symmetric. In flat space we choose the symmetry direction to be the $\phi$ coordinate and construct cylindrical polar coordinates using the $\phi$ as the angular coordinate. Therefore we assume $B^\rho \equiv B^\rho(\rho, z)$. The divergence equation now becomes
\[
\rho - 1/2 \partial_\rho (\rho B^\rho) + \rho - 1/2 \partial_z (\rho B^z) + \rho - 1/2 \partial_\phi (\rho B^\phi) = 0.
\]
We can now solve (5) by picking any function $F(\rho, z)$ and choosing
\[
\rho B^\rho = \partial_z F, \quad \rho B^z = -\partial_\rho F.
\]
Therefore we can construct any axially-symmetric divergence-free vector on flat space from a pair of scalars $[F(\rho, z), B^\phi(\rho, z)]$.

This extends to the case where the space has the same axial symmetry as the vector, again call it $\phi$. We pick coordinates so that the metric is $\phi$-independent. We continue to call the other two coordinates $(\rho, z)$ but we do not assume any connection between them and any standard coordinates. We continue to have $B^\rho \equiv B^\rho(\rho, z)$ and also $g \equiv g(\rho, z)$, where $g$ is the determinant of the metric.

The divergence equation becomes
\[
g^{-1/2} \partial_\rho (\sqrt{g} B^\rho) + g^{-1/2} \partial_z (\sqrt{g} B^z) + g^{-1/2} \partial_\phi (\sqrt{g} B^\phi) = 0.
\]
We can now solve (7) by again picking any function $F(\rho, z)$ and choosing
\[
\sqrt{g} B^\rho = \partial_z F, \quad \sqrt{g} B^z = -\partial_\rho F.
\]
Therefore we can construct any axially-symmetric divergence-free vector on an axially-symmetric space from a pair of scalars $[F(\rho, z), B^\phi(\rho, z)]$.

It will emerge that the TT tensor case is not quite as nice as the divergence-free vector case. Everything works when we assume that the space is flat. Both translationally symmetric and axially-symmetric TT tensors can be derived from two scalar potentials. However, if the space is axially symmetric (but not flat) only one of the two potentials emerges naturally. We will show this in the following sections.

3. Flat space TT tensors with linear symmetry

We begin with linear symmetry. Of course, this is a much less interesting case than axial symmetry, because we lose the possibility of asymptotic flatness. One application however, would be a situation where one would wish to add gravitational waves to a standard cosmological model. In the linear symmetry case the calculations are both easier and also help us to understand what needs be done in the axial symmetry case.

The analysis obviously simplifies if one uses coordinates which reflect the underlying symmetry, and with translational symmetry, the natural coordinates are Cartesian and cylindrical-polars. TT tensor expressions are thus derived in both Cartesian and cylindrical coordinates. As desired, the Cartesian expression is dependent on two scalar potentials alone. However, the cylindrical expression appears to be more complicated. It contains an integral, as well as a function of integration. Knowing the simpler expression in Cartesian coordinates, a coordinate transformation is performed to relate the potentials in the two coordinate systems. This, in turn, shows us how to write an expression in cylindrical coordinates which has no integral and no function of integration. This trick turns out to be extremely useful in the axi-symmetric case.
3.1. Cartesian coordinates

In Cartesian coordinates \((x, y, z)\), the flat 3-space line element is given simply by:

\[
dl^2 = dx^2 + dy^2 + dz^2,
\]

with completely vanishing connection coefficients. Now, a symmetric two-index tensor \(T^{ab}\) is invariant along a Killing vector field \(\eta^a\), if and only if its Lie derivative with respect to that vector field is zero:

\[
0 = \mathcal{L}_\eta T^{ab} = \eta^c \partial_c T^{ab} - T^{cb} \partial_c \eta^b - T^{ac} \partial_c \eta^b.
\]

(10)

Taking the Killing vector to coincide with the coordinate vector \(z\), (10) reduces to:

\[
0 = \partial_z T^{ab} = \partial_c T^{ab} - \partial_c \eta^b = \partial_c T^{ab},
\]

(11)

giving a simple condition for \(T^{ab}\) to be linearly-symmetric. Combining this condition with the equations for \(T^{ab}\) to be TT:

\[
0 = D_y T^{ab} = \partial_x T^{xy} + \partial_y T^{yx} + \partial_z T^{xz},
\]

(12a)

\[
0 = D_y T^{ab} = \partial_x T^{xy} + \partial_y T^{yx} + \partial_z T^{xz},
\]

(12b)

\[
0 = D_y T^{ab} = \partial_x T^{xy} + \partial_y T^{yx} + \partial_z T^{xz},
\]

(12c)

\[
0 = T^{xx} + T^{yy} + T^{zz}.
\]

(12d)

By the equivalence of mixed partial derivatives, (12a), (12b) and (12c) then imply the existence of scalar potentials \(P, Q, S\), with:

\[
-\partial_x T^{xx} = \partial_y T^{xy} \iff T^{xx} = \partial_y P, \quad T^{yy} = \partial_x P,
\]

(13)

\[
-\partial_x T^{xx} = \partial_y T^{xy} \iff T^{xx} = \partial_y Q, \quad T^{yy} = \partial_x Q,
\]

(14)

\[
-\partial_x T^{xx} = \partial_y T^{xy} \iff T^{xx} = -\partial_y S, \quad T^{yy} = \partial_x S.
\]

(15)

Let us go through this more carefully, in particular let us focus on (13). Let us consider a vector \(v^i\) in flat 3-space with \(v^i = (T^{xy}, -T^{xx}, 0)\) where \(T^{xy}\) and \(T^{xx}\) depend on \((x, y)\) only. Now compute the curl of \(v^i\) to get \(\nabla \times v^i = (0, 0, \partial_z T^{xx} - \partial_y T^{yx}) = (0, 0, 0)\), by equation (12a). As long as the space is contractible, Poincaré’s lemma implies that \(v^i\) is the gradient of a scalar. This breaks down only if the base manifold has a non-trivial homotopy. Therefore the existence of unique potentials \(P, Q, S\), up to additive constants, is guaranteed.

Also, since \(T^{ab}\) is symmetric, the expressions for \(T^{xy}\) in (13) and \(T^{yz}\) in (14) can be equated, implying the existence of another potential \(R\), with:

\[
\partial_x P = -\partial_y Q \iff P = \partial_x R, \quad Q = -\partial_y R.
\]

(16)

Substituting (16) into (13) and (14), and using (12d) to find \(T^{zz}\), the tensor \(T^{ab}\) is given in matrix form by:

\[
T^{ab} = \begin{pmatrix}
-\partial_y R & \partial_x R & -\partial_y S \\
\partial_x R & -\partial_x R & \partial_x S \\
-\partial_y S & \partial_x S & \partial_x R + \partial_y R
\end{pmatrix},
\]

(17)

depending on the choice of the two scalar potentials \(R\) and \(S\) alone. Of course, it is assumed that \(R\) and \(S\) are independent of \(z\). This expression is then transverse, trace-free, and linearly-symmetric along the \(z\) coordinate.
3.2. Cylindrical coordinates

In cylindrical coordinates $(\rho, \phi, z)$, the flat 3-space line element is given by:

$$d\ell^2 = d\rho^2 + \rho^2 d\phi^2 + dz^2,$$

with non-zero connection coefficients:

$$\Gamma^\rho_{\phi\phi} = -\rho, \quad \Gamma^\rho_{\rho\phi} = \Gamma^\phi_{\rho\rho} = \frac{1}{\rho}.\quad (19)$$

With the divergence of a tensor $T^{ab}$ given by:

$$D_b T^{ab} = \partial_b T^{ab} + \Gamma^a_{bc} T^{cb} + \Gamma^b_{bc} T^{ac},$$

the TT conditions for a symmetric tensor $T^{ab}$, along with the condition of linear symmetry along $z$ from (11), are given by the equations:

$$0 = D_b T^{rb} = \partial_b T^{r\rho} + \partial_\rho T^{rb} - \rho T^{\phi\phi} + \frac{1}{\rho} T^{r\rho}.\quad (21a)$$
$$0 = D_b T^{rb} = \partial_b T^{\phi\rho} + \partial_\rho T^{rb} + \frac{3}{\rho} T^{\phi\phi},\quad (21b)$$
$$0 = D_b T^{rb} = \partial_b T^{c\phi} + \partial_\phi T^{rb} + \frac{1}{\rho} T^{c\phi},\quad (21c)$$
$$0 = T^{r\rho} + T^{c\phi} + \rho^2 T^{\phi\phi}.\quad (21d)$$

Firstly, (21c) is manipulated:

$$0 = \frac{1}{\rho} \partial_\rho (\rho T^{c\phi}) + \partial_\phi (T^{c\phi})$$

and by the equivalence of mixed partial derivatives, there must exist a scalar potential $Y_L$, such that:

$$\partial_\phi Y_L = -\rho T^{c\phi}, \quad \partial_\rho Y_L = \rho T^{c\phi},$$

$$\Leftrightarrow T^{c\phi} = -\frac{1}{\rho} \partial_\phi Y_L, \quad T^{c\phi} = \frac{1}{\rho} \partial_\rho Y_L.\quad (23)$$

Similarly with (21b):

$$0 = \frac{1}{\rho^3} \partial_\rho (\rho^3 T^{\phi\phi}) + \partial_\phi (T^{\phi\phi})$$

and again, by the equivalence of mixed partial derivatives there must exist a scalar potential $X_L$, such that:

$$\partial_\phi X_L = \rho^3 T^{\phi\phi}, \quad \partial_\rho X_L = -\rho^3 T^{\phi\phi},$$

$$\Leftrightarrow T^{\phi\phi} = \frac{1}{\rho^3} \partial_\rho X_L, \quad T^{\phi\phi} = -\frac{1}{\rho^3} \partial_\phi X_L.\quad (25)$$
Finally, beginning with (21a) and substituting from (25):

\[ 0 = \partial_\rho T^{\rho\rho} + \partial_\phi T^{\rho\phi} - \rho T^{\phi\phi} + \frac{1}{\rho} T^{\rho\rho} \]

\[ = \frac{1}{\rho} \partial_\rho (\rho T^{\rho\rho}) + \frac{1}{\rho^2} \partial_\phi \phi X_L + \frac{1}{\rho^2} \partial_\rho X_L, \]

\[ \Leftrightarrow \partial_\rho (\rho T^{\rho\rho}) = -\frac{1}{\rho} \partial_\rho X_L - \frac{1}{\rho^2} \partial_\phi \phi X_L, \]

\[ \Leftrightarrow T^{\rho\rho} = -\frac{1}{\rho} \int \left[ \frac{1}{\rho} \partial_\rho X_L + \frac{1}{\rho^2} \partial_\phi \phi X_L \right] d\rho - \frac{1}{\rho} f_L(\phi), \]  

(26)

noting the addition of a function of integration, depending on \( \phi \), but not \( \rho \) (or \( z \)).

Transverse, trace-free and linearly-symmetric tensors in flat space can then be given, in cylindrical coordinates, by the matrix expression:

\[
T^{ab} = \begin{pmatrix}
-\frac{1}{\rho} \int \left[ \frac{1}{\rho} \partial_\rho X_L + \frac{1}{\rho^2} \partial_\phi \phi X_L \right] d\rho & \frac{1}{\rho} \partial_\rho X_L & -\frac{1}{\rho} \partial_\rho Y_L \\
\frac{1}{\rho} \partial_\phi X_L & -\frac{1}{\rho} \partial_\rho X_L & \frac{1}{\rho} \partial_\rho Y_L \\
-\frac{1}{\rho} \partial_\phi Y_L & \frac{1}{\rho} \partial_\rho Y_L & -\frac{1}{\rho} \partial_\rho X_L + \frac{1}{\rho} f_L(\phi) \\
\end{pmatrix},
\]

(27)

with two scalar potentials \( X_L \) and \( Y_L \), and a function of integration \( f_L(\phi) \).

### 3.3. Coordinate transformations

Since both (17) and (27) give expressions for the same type of tensors, but in different coordinate systems, the potentials from each should be related. A coordinate transformation is therefore performed, and since the Cartesian expression depends on the desired two potentials alone, we transform (17) into cylindrical coordinates.

To begin, the derivatives of \( R \) and \( S \) with respect \( x \) and \( y \) are first found in terms of cylindrical coordinates, by use of the chain rule, giving:

\[
\partial_\rho R = \cos^2 \phi \partial_\rho R + \frac{1}{\rho} \sin^2 \phi \partial_\rho R - \frac{2}{\rho} \cos \phi \sin \phi \partial_\rho \phi R
\]

\[ + \frac{2}{\rho^2} \cos \phi \sin \phi \partial_\phi R + \frac{1}{\rho^2} \sin^2 \phi \partial_\phi \phi R, \]

\[
\partial_\phi R = \cos \phi \sin \phi \partial_\rho R - \frac{1}{\rho} \cos \phi \sin \phi \partial_\rho R + \frac{1}{\rho} (\cos^2 \phi - \sin^2 \phi) \partial_\phi \phi R
\]

\[ - \frac{1}{\rho^2} (\cos^2 \phi - \sin^2 \phi) \partial_\phi R - \frac{1}{\rho^2} \cos \phi \sin \phi \partial_\phi \phi R, \]

\[
\partial_\rho S = \sin^2 \phi \partial_\rho R + \frac{1}{\rho} \cos^2 \phi \partial_\rho R + \frac{2}{\rho} \cos \phi \sin \phi \partial_\phi \phi R
\]

\[ - \frac{2}{\rho^2} \cos \phi \sin \phi \partial_\phi R + \frac{1}{\rho^2} \cos^2 \phi \partial_\phi \phi R, \]

\[
\partial_\phi S = \cos \phi \partial_\rho S - \frac{1}{\rho} \sin \phi \partial_\phi S,
\]

\[
\partial_\rho S = \sin \phi \partial_\rho S + \frac{1}{\rho} \cos \phi \partial_\phi S. \]

(28)
These expressions are then substituted into the transformation of (17) from Cartesian into cylindrical coordinates:

\[
T_{\rho \rho} = -\frac{1}{\rho} \partial_\rho R - \frac{1}{\rho^2} \partial_\phi R,
\]

\[
T_{\phi \phi} = -\frac{\sin^2 \phi}{\rho^2} \partial_\rho R - \frac{2 \cos \phi \sin \phi}{\rho^2} \partial_\phi R - \frac{\cos^2 \phi}{\rho^2} \partial_\rho R = -\frac{1}{\rho^2} \partial_{\rho \rho} R,
\]

\[
T_{\rho \phi} = \frac{\cos \phi \sin \phi}{\rho} \partial_\rho R + \left( \frac{\cos^2 \phi}{\rho} - \frac{\sin^2 \phi}{\rho} \right) \partial_\phi R = -\frac{1}{\rho} \partial_{\rho \phi} R,
\]

\[
T_{zz} = \partial_{xx} R + \partial_{yy} R = \partial_{\rho \rho} R + \frac{1}{\rho} \partial_\rho R + \frac{1}{\rho^2} \partial_\phi R,
\]

\[
T_{\rho z} = -\frac{\cos \phi}{\rho} \partial_y R + \frac{\sin \phi}{\rho} \partial_x R = -\frac{1}{\rho} \partial_\phi S
\]

\[
T_{\phi z} = \frac{\sin \phi}{\rho} \partial_y R + \frac{\cos \phi}{\rho} \partial_x R = \frac{1}{\rho} \partial_\rho S,
\]

(29)

giving a new expression for a linearly symmetric TT tensor in cylindrical coordinates. In matrix form:

\[
T^{ab} = \begin{pmatrix}
-\frac{1}{\rho} \partial_\rho R & -\frac{1}{\rho^2} \partial_\phi R & \frac{1}{\rho} \partial_\rho R & -\frac{1}{\rho} \partial_\phi R \\
\frac{1}{\rho} \partial_\phi R & -\frac{1}{\rho^2} \partial_\phi R & \frac{1}{\rho} \partial_\rho R & \frac{1}{\rho} \partial_\phi R \\
\frac{1}{\rho} \partial_\rho S & \frac{1}{\rho} \partial_\phi S & \partial_\rho R + \frac{1}{\rho} \partial_\rho R + \frac{1}{\rho^2} \partial_\phi R & \partial_\phi R
\end{pmatrix},
\]

(30)

depending only on the two scalar potentials \( R \) and \( S \), with no function of integration.

Comparing the \( T^{\rho z} \) and \( T^{\phi z} \) terms of (27) with (30), it can easily be seen that the potentials \( Y_L \) and \( S \) are the same. Equating each of the remaining terms, and using integration by parts, an expression can be found for the potential \( X_L \) in terms of \( R \):

\[
X_L = \rho \partial_\rho R - R,
\]

(31)

with \( X_L \) invariant under the gauge change \( R \rightarrow R + \rho h_L(\phi) \) for any function \( h_L(\phi) \). There is also a general solution for the ordinary differential equation (31):

\[
R = \rho \int \frac{1}{\rho^2} X_L \, d\rho + \rho h_L(\phi),
\]

(32)

with the function of integration \( h_L(\phi) \) accounting for the gauge invariance in \( X_L \). Finally, (27) also contains a function of integration \( f_L(\phi) \) which can be found from the function \( h_L \), giving the full set of relations:

\[
Y_L = S, \quad X_L = \rho \partial_\rho R - R, \quad f_L = h_L + \partial_\phi h_L,
\]

(33)

with both \( X_L \) and \( f_L \) contained in the potential \( R \).

We have gone through this long and fairly messy calculation entirely to find (31). In the next section, the axi-symmetric case, in both the cylindrical and spherical polar cases, we find an integral equivalent to the one in (27). A transformation of variables similar to (31) can again be used to eliminate the integral. Guessing this, without having the guidance of translational case, would be extremely difficult.
4. Flat space TT tensors with axial symmetry

The techniques of section 3 are used in this section to derive expressions for an axially-symmetric TT tensor, in both cylindrical and spherical coordinates. A similar relation to (33) is also found, transforming the tensor in cylindrical coordinates into an integral-free expression. A coordinate transformation is then carried out on this expression, to allow us do the same in spherical coordinates.

4.1. Cylindrical coordinates

In this section, the cylindrical coordinates are given in the order \((\rho, z, \phi)\), for easy comparison with the spherical coordinates \((r, \theta, \phi)\). However it must be noted that this order produces a ‘left hand orthogonality’, reversing the sign of the Levi-Civita tensor. The flat 3-space line element, in these coordinates, is given by:

\[
dl^2 = d\rho^2 + dz^2 + \rho^2 d\phi^2,
\]

with its non-zero connection coefficients, as with (19):

\[
\Gamma_\phi^{\rho\phi} = -\rho, \quad \Gamma_\rho^{\phi\rho} = \frac{1}{\rho}.
\]

For a symmetric two-index tensor \(T^a_b\) to be axially symmetric, the Killing vector field \(\eta^a\) from (10) can be taken to coincide with the azimuthal coordinate vector \(\phi\), giving the condition:

\[
0 = \phi \partial_c T^a_b - T^c_b \partial_c \phi^a - T^a_c \partial_c \phi^b \equiv \partial_\phi T^a_b.
\]

The TT conditions for a symmetric tensor \(T^a_b\), along with the condition of axial symmetry, are then given by the equations:

\[
\begin{align*}
0 &= D_\rho T^\rho\rho = \partial_\rho T^\rho\rho + \partial_\rho T^\rho z + \partial_\rho T^\rho\phi - \rho T^\phi\phi - \frac{1}{\rho^2} T^\rho\rho, \\
0 &= D_\rho T^\rho z = \partial_\rho T^\rho z + \partial_\rho T^\rho\phi + \frac{1}{\rho} T^\rho\phi, \\
0 &= D_\rho T^\rho\phi = \partial_\rho T^\rho\phi + \frac{3}{\rho} T^\rho\phi, \\
0 &= T^\rho\rho + T^\rho z + \rho^2 T^\rho\phi.
\end{align*}
\]

By the equivalence of mixed partial derivatives, \((37b)\) and \((37c)\) respectively imply the existence of scalar functions \(X_A\) and \(Y_A\), with:

\[
\partial_\rho (\rho T^\rho\rho) = \partial_z (-\rho T^\rho z) \Leftrightarrow \partial_\rho X_A = \rho T^\rho z, \quad \partial_z X_A = -\rho T^\rho z,
\]

\[
\begin{align*}
\partial_\rho (\rho^3 T^\rho\phi) &= \partial_z (-\rho^3 T^\rho\phi) \Leftrightarrow \partial_\rho Y_A = \rho^3 T^\rho\phi, \quad \partial_z Y_A = -\rho^3 T^\rho\phi,
\end{align*}
\]

Then using \((37d)\) and substituting from \((38)\) above, \((37a)\) becomes:

\[
0 = \partial_\rho T^\rho\rho + \partial_\rho T^\rho z + \frac{1}{\rho} (T^\rho\rho + T^\rho z) + \frac{1}{\rho^2} T^\rho\rho
\]

\[
\Rightarrow \partial_\rho (\rho^2 T^\rho\rho) = \partial_\rho X_A - \rho \partial_z X_A,
\]

\[
\Rightarrow T^\rho\rho = \frac{1}{\rho^2} \int [\partial_\rho X_A - \rho \partial_z X_A] d\rho + \frac{1}{\rho^2} f_A(z),
\]

again noting the function of integration, depending here on \(z\), but not \(\rho\) (or \(\phi\)).
Transverse, trace-free and axially-symmetric tensors in flat space can then be given, in cylindrical coordinates, by the matrix expression:

\[
T_{ab} = \begin{pmatrix}
\frac{1}{\rho} \int [\rho \partial_\rho X_A + \partial_\rho X_A] d\rho \\
\frac{1}{\rho} \partial_\rho X_A & -\frac{1}{\rho^2} \rho \partial_\rho X_A - \frac{1}{\rho} \partial_\rho Y_A \\
\frac{1}{\rho^2} \rho \partial_\rho Y_A & -\frac{1}{\rho^3} \partial_\rho Y_A - \frac{1}{\rho^2} f_A(z) \\
& -\frac{1}{\rho^4} \int [-\rho \partial_\rho X_A + \partial_\rho X_A] d\rho
\end{pmatrix},
\]

depending on the two scalar potentials \(X_A\) and \(Y_A\), and the function of integration \(f_A(z)\).

### 4.2. Spherical coordinates

The flat 3-space line element, in spherical-polar coordinates \((r, \theta, \phi)\), is given by:

\[
\text{d}l^2 = \text{d}r^2 + r^2 \text{d}\theta^2 + r^2 \sin^2 \theta \text{ d}\phi^2,
\]

and the non-zero connection coefficients by:

\[
\Gamma^r_{\theta \theta} = -r, \quad \Gamma^\theta_{r \theta} = \frac{1}{r}, \quad \Gamma^\phi_{r \phi} = \Gamma^\phi_{\phi r} = \frac{1}{r},
\]

\[
\Gamma^r_{\phi \phi} = -r \sin^2 \theta, \quad \Gamma^\theta_{\phi \phi} = -\cos \theta \sin \theta, \quad \Gamma^\phi_{\theta \phi} = \frac{\cos \theta}{\sin \theta}.
\]

Imposing axial symmetry from (36), the conditions for the symmetric tensor \(T_{ab}\) to be both \(TT\) are given by the equations:

\[
0 = D_b T^{\theta \theta} = \partial_\theta T^{\theta \theta} + \partial_\phi T^{\theta \phi} + \frac{2}{r} T^{\theta \phi} - r T^{\theta \theta} - r \sin^2 \theta \ T^{\phi \phi} + \frac{\cos \theta}{\sin \theta} T^{r \theta},
\]

\[
0 = D_b T^{\phi \theta} = \partial_\theta T^{\phi \theta} + \partial_\phi T^{\theta \phi} + \frac{\cos \theta}{\sin \theta} T^{\theta \theta} - \cos \theta \sin \theta T^{\phi \phi} + \frac{4}{r} T^{\phi \phi},
\]

\[
0 = T^{rr} + r^2 T^{\theta \theta} + r^2 \sin^2 \theta T^{\phi \phi}.
\]

Beginning with (44a), and removing the \(T^{\theta \theta}\) and \(T^{\phi \phi}\) terms with (44d):

\[
0 = \frac{3}{r} T^{rr} + \partial_\rho T^{rr} + \frac{\cos \theta}{\sin \theta} T^{r \theta} + \partial_\rho T^{\theta \theta}
\]

\[
= \frac{1}{r^3} \partial_\rho (r^3 T^{rr}) + \frac{1}{\sin \theta} \partial_\rho (\sin \theta T^{r \theta}),
\]

\[
\Leftrightarrow \quad \partial_\rho (r^3 \sin \theta T^{rr}) = \partial_\rho (r^3 \sin \theta T^{r \theta}),
\]

and by the equivalence of mixed partial derivatives, there must exist a scalar potential \(V\) such that:

\[
\partial_\rho V = r^3 \sin \theta T^{rr}, \quad \partial_\theta V = -r^3 \sin \theta T^{r \theta},
\]

\[
\Leftrightarrow T^{rr} = \frac{\partial_\rho V}{r^3 \sin \theta}, \quad T^{r \theta} = -\frac{\partial_\theta V}{r^3 \sin \theta}.
\]
Now taking (44c):
\[
0 = \frac{4}{r} \mathcal{T}_{\phi \phi} + \alpha_r \mathcal{T}_{r \phi} + \frac{3 \cos \theta}{\sin \theta} \mathcal{T}_{\theta \phi} + \partial_\rho \mathcal{T}_{r \phi}
\]
\[
= \frac{1}{r^4} \partial_r (r^4 \mathcal{T}_{r \phi}) + \frac{1}{\sin^3 \theta} \partial_\theta (\sin^3 \theta \mathcal{T}_{r \phi}),
\]
\[\Leftrightarrow \partial_r (-r^4 \sin^3 \theta \mathcal{T}_{r \phi}) = \partial_\theta (r^4 \sin^3 \theta \mathcal{T}_{\theta \phi}), \tag{47}\]
and again, by the equivalence of mixed partial derivatives, there must exist a scalar potential \( W \) such that:
\[
\partial_\theta W = -r^4 \sin^3 \theta \mathcal{T}_{r \phi}, \quad \partial_r W = r^4 \sin^3 \theta \mathcal{T}_{\theta \phi}, \tag{48}\]
\[\Leftrightarrow \partial_r (\mathcal{T}_{r \phi}) = \frac{1}{r^4} \partial_r (r^4 \sin^3 \theta \mathcal{T}_{r \phi}) = \partial_\theta (r^4 \sin^3 \theta \mathcal{T}_{\theta \phi}), \quad (49)\]
noting a function of integration, similar to the cylindrical coordinates, depending here on \( r \) but not \( \theta \) (or \( \phi \)).

Hence, transverse, trace-free and axially-symmetric tensors in flat space can be given, in spherical coordinates, by the matrix expression:
\[
\mathcal{T}^{ab} = \begin{pmatrix}
\frac{1}{r^3 \sin \theta} \partial_\theta V & -\frac{1}{r^3 \sin \theta} \partial_r V & -\frac{1}{r^3 \sin \theta} \partial_\phi W \\
-\frac{1}{r^3 \sin \theta} \partial_r V & \frac{1}{r^3 \sin \theta} \partial_r V & \frac{1}{r^3 \sin \theta} \partial_\phi V \\
-\frac{1}{r^3 \sin \theta} \partial_\phi W & \frac{1}{r^3 \sin \theta} \partial_\phi W & \frac{1}{r^3 \sin \theta} \partial_\phi V + \frac{\sin \theta}{r^2} g(r) \\
\end{pmatrix} , \tag{50}\]
depending on the two scalar potentials \( V \) and \( W \), and the function of integration \( g(r) \).

### 4.3. Removing integrals

Since the tensor in cylindrical coordinates has a similar form to the linear symmetry case, we are looking for a relation based on (31) between the scalar function \( X_A \) and some scalar potential \( R_A \), to eliminate the integral. We find that essentially the same function works. Using
\[
X_A = \partial_\rho R_A + \frac{1}{\rho} R_A , \tag{51}\]
with \( X_A \) invariant under the gauge change \( R_A \to R_A + h_A(z)/\rho \), we get a new form for an axially-symmetric TT tensor in cylindrical coordinates:
noting the function of integration in the spherical components in terms of $RA$

The components of

$$\partial_\rho R_A = \frac{1}{\rho} \partial_\rho R_A + \frac{1}{\rho} \partial_\rho R_A + \frac{1}{\rho} \partial_\rho R_A + \frac{1}{\rho} \partial_\rho R_A$$

\[ \begin{pmatrix}
-\frac{1}{\rho} & \frac{1}{\rho} & \frac{1}{\rho} & \frac{1}{\rho} \\
\frac{1}{\rho} & \frac{1}{\rho} & \frac{1}{\rho} & \frac{1}{\rho} \\
\frac{1}{\rho} & \frac{1}{\rho} & \frac{1}{\rho} & \frac{1}{\rho} \\
\frac{1}{\rho} & \frac{1}{\rho} & \frac{1}{\rho} & \frac{1}{\rho}
\end{pmatrix} \cdot \frac{1}{\rho} \partial_\rho R_A + \frac{1}{\rho} \partial_\rho R_A + \frac{1}{\rho} \partial_\rho R_A + \frac{1}{\rho} \partial_\rho R_A, \tag{52} \]

with no integrals, and as a result, no function of integration. An expression for the potential $RA$ in terms of $XA$ can also be found from the general ordinary differential equation solution for (51):

$$RA = \frac{1}{\rho} \int \rho X_A \, d\rho + \frac{1}{\rho} h_A(z), \tag{53}$$

noting the function of integration $h_A(z)$, similar to (32), which accounts for the gauge invariance in $XA$. As with the linear case, the function of integration $f_A(z)$ in (41) can be given in terms of $h_A$ as $f_A = \partial_z h_A$. Thus both $X_A$ and $f_A$ are contained in $R_A$, reducing the determination of ALL axi-symmetric TT tensors in cylindrical coordinates to the choice of the two scalar potentials $RA$ and $YA$.

To find an axially-symmetric TT tensor in spherical coordinates, without the presence of an integral, a coordinate transformation can be carried out on (52). As with the linear case, the derivatives of the cylindrical potentials $RA$ and $YA$ are found with respect to $r$ and $\theta$ using the chain rule:

$$\partial_\rho R_A = \frac{1}{\rho} \partial_\rho R_A + \frac{1}{\rho} \cos \theta \partial_\rho R_A.$$

$$\partial_r R_A = \cos \theta \partial_r R_A + \frac{1}{r} \cos \theta \partial_r R_A.$$

$$\partial_\theta R_A = \sin \theta \partial_\theta R_A + \frac{1}{r} \sin \theta \partial_\theta R_A.$$

$$\partial_\phi R_A = \frac{1}{r} \cos \theta \partial_\phi R_A + \frac{1}{r^2} \cos \theta \partial_\phi R_A.$$

$$\partial_\rho Y_A = \frac{1}{r} \sin \theta \partial_\rho Y_A + \frac{1}{r} \cos \theta \partial_\rho Y_A.$$

$$\partial_\theta Y_A = \cos \theta \partial_\theta Y_A - \frac{1}{r} \sin \theta \partial_\theta Y_A.$$

$$\partial_\phi Y_A = \cos \theta \partial_\phi Y_A + \frac{1}{r} \sin \theta \partial_\phi Y_A.$$

The components of $T^{ab}$ are then transformed from cylindrical to spherical coordinates, giving the spherical components in terms of $RA$ and $YA$:

$$T^{rr} = -\frac{1}{r^2 \sin \theta} \partial_\rho R_A + \frac{1}{r^2 \sin ^2 \theta} \cos \theta \partial_\rho R_A + \frac{1}{r^2 \sin \theta} \partial_\rho R_A,$$

$$T^{66} = -\frac{1}{r^3 \sin \theta} \partial_\rho R_A + \frac{1}{r^4 \sin \theta} \cos \theta \partial_\rho R_A + \frac{1}{r^3 \sin \theta} \partial_\rho R_A + \frac{cos \theta}{r^3 \sin ^2 \theta} \partial_\phi R_A + \frac{1}{r^4 \sin \theta} \partial_\phi R_A.$$
These then give an integral-free expression for an axially-symmetric TT tensor in spherical coordinates:

\[
T_{\theta\theta} = -\frac{1}{r^3 \sin^2 \theta} \partial_\theta R_A + \frac{1}{r^3 \sin^2 \theta} \partial_\theta R_A + \frac{1}{r^5 \sin^2 \theta} \partial_\theta R_A - \frac{1}{r^5 \sin^2 \theta} R_A,
\]

\[
T_{\phi\phi} = +\frac{1}{r^3 \sin^2 \theta} \partial_\phi R_A + \frac{1}{r^3 \sin^2 \theta} \partial_\phi R_A + \frac{1}{r^5 \sin^2 \theta} \partial_\phi R_A - \frac{1}{r^5 \sin^2 \theta} R_A,
\]

\[
T_{r\phi} = -\frac{1}{r^4 \sin^3 \theta} \partial_\phi R_A - \frac{1}{r^4 \sin^3 \theta} \partial_\phi R_A - \frac{1}{r^5 \sin^2 \theta} \partial_\phi R_A - \frac{1}{r^5 \sin^2 \theta} R_A,
\]

\[
T_{r\theta} = +\frac{1}{r^4 \sin^3 \theta} \partial_\theta R_A + \frac{1}{r^4 \sin^3 \theta} \partial_\theta R_A + \frac{1}{r^5 \sin^2 \theta} \partial_\theta R_A + \frac{1}{r^5 \sin^2 \theta} R_A.
\]

(55)

These then give an integral-free expression for an axially-symmetric TT tensor in spherical coordinates:

\[
T_{ab} = \begin{pmatrix}
-\frac{1}{r^3 \sin^2 \theta} \partial_\theta R_A + \frac{1}{r^3 \sin^2 \theta} \partial_\theta R_A + \frac{1}{r^5 \sin^2 \theta} \partial_\theta R_A & -\frac{1}{r^4 \sin^3 \theta} \partial_\phi R_A - \frac{1}{r^4 \sin^3 \theta} \partial_\phi R_A - \frac{1}{r^5 \sin^2 \theta} \partial_\phi R_A - \frac{1}{r^5 \sin^2 \theta} R_A

\end{pmatrix},
\]

(56)

depending on $R_A$ and $Y_A$ alone, with no additional functions. This can be compared with (50) to show an equivalence between the cylindrical potential $Y_A$ and the spherical potential $W$. It proves a little more difficult however, to find a direct relation between $V$ and either $R_A$ or $X_A$.

**5. General space TT tensors**

In this section, the flat space metric is replaced by a more general spatial metric with an axial symmetry. The conditions for an axially-symmetric 2-tensor to be TT are given, but only one scalar potential can be found using the techniques of the previous two sections, coinciding with that given in [1, 2]. The remaining equations are reduced to a second order partial differential equation in two of the components.

**5.1. Metric and tensors with axial symmetry**

A general axially-symmetric 3-space metric can be given by the Brill wave metric (3) of [11], which was credited there to H Bondi. This is geometrically equivalent to the metric used in [1], with a related set of functions. In cylindrical-polar type coordinates $(\rho, z, \phi)$, the conformal part of this metric can be expressed as:

\[
dl^2 = e^{2q} \, d\rho^2 + e^{2q} \, dz^2 + \rho^2 \, d\phi^2.
\]

(57)

for a differential function $q(\rho, z)$ such that:

\[
q|_{\rho=0} = \partial_\rho q|_{\rho=0} = 0,
\]

(58)

and that $q$ decays faster than $1/r$ at infinity, and is reasonably differential. The non-zero connection coefficients for this metric are then given by:

\[
\Gamma^\rho_{\rho\rho} = \partial_\rho q, \quad \Gamma^\rho_{\rho\phi} = \partial_\rho q, \quad \Gamma^\rho_{\rho\phi} = -\partial_\rho q,
\]

\[
\Gamma^\rho_{\phi\phi} = -\rho \, e^{-2q}, \quad \Gamma^\rho_{\phi\phi} = \partial_\rho q,
\]

(59)
and the conditions for a symmetric tensor $T^{ab}$ to be $TT$ with respect to (57), along with the condition of axial-symmetry (36), are given by the equations:

$$0 = D_b T^{ab} = \partial_a T^{\rho \rho} + \partial_\rho T^{\rho \phi} + \partial_\phi T^{\phi \phi} + \left( 3 \frac{\partial_\rho q + \frac{1}{\rho} }{\rho} \right) T^{\rho \rho} + 4 \frac{\partial_\phi T^{\rho \phi} }{\rho} - \partial_\phi q T^{\phi \phi}, \quad (60a)$$

$$0 = D_b T^{\phi \phi} = \partial_\rho T^{\rho \rho} + \partial_\phi T^{\rho \rho} + \partial_\rho T^{\phi \phi} - \partial_\phi q T^{\rho \rho} + 3 \frac{\partial_\phi q + \frac{1}{\rho} }{\rho} T^{\rho \rho} + \left( 4 \frac{\partial_\rho q + \frac{3}{\rho} }{\rho} \right) T^{\rho \rho} + 2 \frac{\partial_\phi T^{\rho \rho} }{\rho}, \quad (60b)$$

$$0 = D_b T^{\rho \phi} = \partial_\rho T^{\rho \rho} + \partial_\phi T^{\rho \rho} + \partial_\rho T^{\phi \phi} + \partial_\phi T^{\phi \phi} + \left( 2 \frac{\partial_\rho q + \frac{3}{\rho} }{\rho} \right) T^{\rho \rho} + 2 \frac{\partial_\phi T^{\rho \rho} }{\rho}, \quad (60c)$$

$$0 = e^{2q} T^{\rho \rho} + e^{2q} T^{\phi \phi} + \rho^2 T^{\phi \phi}. \quad (60d)$$

Working from (60c):

$$0 = \frac{1}{\rho^3} \partial_\rho (\rho^3 e^{2q} T^{\rho \phi}) + \frac{1}{\rho^3} \partial_\phi (\rho^3 e^{2q} T^{\phi \phi})$$

$$\Leftrightarrow \quad \partial_\rho (\rho^3 e^{2q} T^{\rho \phi}) = \partial_\phi (-\rho^3 e^{2q} T^{\phi \phi}), \quad (61)$$

and the equivalence of mixed partial derivatives implies the existence of a scalar potential $w$ such that:

$$\partial_\rho w = \rho^3 e^{2q} T^{\rho \phi}, \quad \partial_\phi w = -\rho^3 e^{2q} T^{\phi \phi},$$

$$\Leftrightarrow \quad T^{\rho \phi} = \rho^{-3} e^{-2q} \partial_\rho w, \quad T^{\phi \phi} = -\rho^{-3} e^{-2q} \partial_\phi w. \quad (62)$$

With the space-time assumed to have a ‘time-rotation’ symmetry, these components are shown in [9] to give the only non-zero components of the extrinsic curvature. In this case, the tensor agrees exactly with the curvature given by (62). The components given by (62) can also be seen to agree with the corresponding curvature components of [1], and if $q = 0$, i.e., the metric is reduced to a flat space metric, (62) is also equivalent to (39) from section 4.1.

### 5.2. Equations for remaining tensor components

Manipulating both (60a) and (60b) similar to section (4.1), using (60d) to remove the $T^{\phi \phi}$ component from (60a):

$$0 = \frac{1}{\rho^2} \partial_\rho (\rho^2 e^{2q} T^{\rho \rho}) + \frac{1}{\rho^2} \partial_\phi (\rho^2 e^{2q} T^{\rho \rho}) + \frac{1}{2} T^{\rho \rho} \partial_\rho e^{2q}$$

$$+ T^{\rho \rho} \partial_\phi e^{2q} - \frac{1}{2} \rho^2 T^{\rho \rho} \partial_\rho (\rho^2 e^{2q}) \partial_\phi e^{2q}, \quad (63)$$

$$0 = \frac{1}{\rho^2} \partial_\rho (\rho^2 e^{2q} T^{\rho \rho}) + \frac{1}{\rho^2} \partial_\phi (\rho^2 e^{2q} T^{\rho \rho}) + T^{\rho \rho} \partial_\rho e^{2q}$$

$$- \frac{1}{2} \rho^2 e^{2q} T^{\rho \rho} \partial_\rho \rho^2 + \frac{1}{2} T^{\rho \rho} \partial_\phi e^{2q} - \frac{1}{2} T^{\rho \rho} \partial_\phi \rho^2, \quad (64)$$

where there remains three tensor components in each equation, rather than the two required for the techniques used previously. However both equations do contain similar terms for the $T^{\rho \rho}$ component. Taking (64) first, and bringing all of the $T^{\rho \rho}$ terms to one side:

$$\partial_\rho (\rho^2 e^{2q} T^{\rho \rho}) + \rho^2 T^{\rho \rho} \partial_\rho e^{2q} - \frac{1}{2} \rho^2 e^{2q} T^{\rho \rho} \partial_\rho \rho^2$$

$$= -\partial_\rho (\rho^2 e^{2q} T^{\rho \rho}) - \frac{1}{2} \rho^2 T^{\rho \rho} \partial_\rho e^{2q} + \frac{1}{2} \rho^2 T^{\rho \rho} \partial_\rho \rho^2, \quad (65)$$
Since the $T^{\mu\nu}$ terms have derivatives with respect to $\rho$ here, and $z$ in (63), both sides are integrated with respect to $\rho$:

$$
\rho^2 e^{2q} T^{\rho\zeta} + \int (\rho^2 T^{\rho\zeta}) \, d\rho = \frac{1}{2} \int (e^{2q} T^{\rho\zeta}) \, d\rho^2
$$

$$
= -\int \rho (\rho^2 e^{2q} T^{\rho\zeta}) \, d\rho - \frac{1}{2} \int e^{2q} T^{\rho\zeta} \, d\rho
$$

$$
+ \frac{1}{2} \int (\rho^2 T^{\rho\rho} \partial_{\rho} e^{2q}) \, d\rho + f_I(z).
$$

(66)

Taking now (63), and again bringing the $T^{\rho\zeta}$ terms to one side:

$$
\frac{1}{\rho^2} \partial_{\rho}(\rho^2 e^{2q} T^{\rho\zeta}) + T^{\rho\zeta} \partial_{\rho} e^{2q}
$$

$$
= -\frac{1}{\rho^2} \partial_{\rho}(\rho^2 T^{\rho\rho}) - \frac{1}{2} T^{\rho\rho} \partial_{\rho} e^{2q} + \frac{1}{2} \rho^2 T^{\zeta\zeta} \partial_{\rho} (\rho^{-2} e^{2q}).
$$

(67)

With both sides of (67) integrated with respect to $z$, there is still a difference with (66), however the missing term can be given by first adding a term involving the derivative $\partial_{\rho} e^{2q}$, which itself evaluates to zero. Hence (67) is equivalent to:

$$
\rho^2 e^{2q} T^{\rho\zeta} + \int (\rho^2 T^{\rho\zeta}) \, d\rho = \frac{1}{2} \int (e^{2q} T^{\rho\zeta}) \, d\rho^2
$$

$$
= -\int \rho (\rho^2 T^{\rho\rho}) \, d\rho - \frac{1}{2} \int (\rho^2 T^{\rho\rho} \partial_{\rho} e^{2q}) \, d\rho + f_I(z).
$$

(68)

Since the left hand sides of both (66) and (68) are equivalent, the right hand sides can be equated to give a single equation, depending on $T^{\rho\rho}$ and $T^{\zeta\zeta}$ alone:

$$
\int \rho (\rho^2 T^{\rho\rho}) \, d\rho - \frac{1}{2} \int (\rho^2 T^{\rho\rho} \partial_{\rho} e^{2q}) \, d\rho + f_I(z)
$$

$$
= -\int \rho (\rho^2 T^{\rho\rho}) \, d\rho - \frac{1}{2} \int (\rho^2 T^{\rho\rho} \partial_{\rho} e^{2q}) \, d\rho + f_I(z).
$$

(69)

Differentiating both sides with respect to both $\rho$ and $z$, gives a second order partial differential equation in $T^{\rho\rho}$ and $T^{\zeta\zeta}$:

$$
\partial_{\rho} \partial_{\zeta}(\rho^2 T^{\rho\zeta}) + \frac{1}{2} \partial_{\rho} (\rho^2 T^{\rho\rho} \partial_{\zeta} e^{2q}) - \frac{1}{2} \partial_{\rho} (\rho^2 T^{\rho\rho} \partial_{\rho} e^{2q})
$$

$$
= \partial_{\rho} \partial_{\rho} (\rho^2 T^{\rho\rho}) + \frac{1}{2} \partial_{\rho} (\rho^2 T^{\rho\rho} \partial_{\rho} e^{2q}) - \frac{1}{2} \partial_{\rho} (\rho^4 T^{\zeta\zeta} \partial_{\rho} (\rho^{-2} e^{2q})).
$$

(70)

Unfortunately, finding a relation between the components $T^{\rho\rho}$ and $T^{\zeta\zeta}$ involves solving this equation, requiring two separate boundary conditions to give a unique solution. Assuming enough information is available to solve (70), the relation between $T^{\rho\rho}$ and $T^{\zeta\zeta}$ can be used, along with any boundary conditions, to solve either (63) or (64) for a relation involving $T^{\rho\zeta}$. The full tensor can then be found using (60a), depending on the potential $w$, and a potential obtained from the relations between $T^{\rho\rho}$, $T^{\zeta\zeta}$ and $T^{\rho\zeta}$.

6. Potentials for Bowen-York curvature

Since the Bowen-York conformal extrinsic curvature [10] is transverse, trace-free and axially symmetric in a conformally flat space, it should be given by an appropriate choice of potentials for the tensors derived in section 4.
6.1. Angular momentum part

Taking first the case of zero linear momentum, the conformal Bowen-York curvature is given by:
\[
\tilde{K}_{ab} = \frac{3}{r^3} (\epsilon_{acd} q_b + \epsilon_{bcd} q_a) q^c J^d,  \tag{71}
\]
depending on the angular momentum \( J^a \) alone, with \( q^a \) the unit normal to a sphere of constant radius, and \( \epsilon_{abc} \) the Levi-Civita alternating tensor. In cylindrical coordinates \((\rho, z, \phi)\), with the angular momentum directed in the axial direction, i.e., along the positive \( z \) coordinate, the vectors \( q^a \) and \( J^a \) are given by:
\[
q^a = \left( \frac{\rho, z, 0}{\sqrt{\rho^2 + z^2}} \right), \quad J^a = (0, J, 0). \tag{72}
\]

Recalling from section 4.1, that the Levi-Civita tensor has a reversed sign for the coordinates given in this order, the non-zero terms of the conformal Bowen-York curvature are given, in cylindrical coordinates, by:
\[
\tilde{K}_{\rho \phi} = \tilde{K}_{\phi \rho} = \frac{3J\rho^3}{r^5}, \quad \tilde{K}_{\rho z} = \tilde{K}_{z \rho} = \frac{3J\rho^2 z}{r^5}. \tag{73}
\]

To find the necessary choice of potentials, the components of the tensor expressions (41) and (52) are set equal to those of the Bowen-York curvature (71) with indices raised, both giving:
\[
\partial_z Y_A = \frac{3J\rho^4}{r^5}, \quad \partial_\rho Y_A = -\frac{3J\rho^3 z}{r^5}, \tag{74}
\]
with the remaining tensor components equal to zero, implying the potentials \( X_A, R_A \) must be constants. Integrating the two equations in (74), with respect to \( z \) and \( \rho \) respectively, leads to the solution:
\[
Y_A = J \frac{3\rho^2 z + 2z^3}{\rho^3}, \tag{75}
\]
plus a constant of integration, which is differentiated out when \( Y_A \) is used to produce a TT tensor. Since \( Y_A \) is equivalent to the spherical potential \( W \), and the remaining potential must be a constant, (75) can easily be translated into spherical coordinates, giving:
\[
W = -J(3 \sin^2 \theta \cos \theta + 2 \cos^3 \theta) = J(\cos^3 \theta - 3 \cos \theta), \tag{76}
\]
which agrees with (21) of [12], for the curvature tensor derived in [2].

6.2. Linear momentum part

Taking now, the angular momentum to be zero, the conformal Bowen-York extrinsic curvature is given by:
\[
\tilde{K}^\pm_{ab} = \frac{3}{2r^2} \left[ P_a q_b + P_b q_a - (\tilde{\gamma}_{ab} - q_a q_b) P^c q_c \right] \nonumber + \frac{3\omega^2}{2r^2} \left[ P_a q_b + P_b q_a + (\tilde{\gamma}_{ab} - 5q_a q_b) P^c q_c \right], \tag{77}
\]
where \( P^a \) denotes the linear momentum of a single source, \( q^a \) the unit normal to a sphere of constant radius and \( a \) an arbitrary constant. This can only give an axially-symmetric tensor, if the momentum is directed along the axis. Hence, in cylindrical type coordinates \((\rho, z, \phi)\), the linear momentum vector and unit space-like normal \( q^a \) are given by:
\[
P^a = (0, P, 0), \quad q^a = \left( \frac{\rho, z, 0}{\sqrt{\rho^2 + z^2}} \right). \tag{78}
\]
giving the non-zero components of the conformal Bowen-York curvature as:

\[
\tilde{K}^{\pm}_{\rho \rho} = \frac{3Pz}{2 r^3} (-r^2 + \rho^2) + \frac{3a^2 Pz}{2 r^3} (r^2 - 5 \rho^2), \quad (79a)
\]

\[
\tilde{K}^{\pm}_{\rho z} = \frac{3P\rho}{2 r^3} (r^2 + z^2) \mp \frac{3a^2 P\rho}{2 r^3} (r^2 - 5 z^2), \quad (79b)
\]

\[
\tilde{K}^{\pm}_{zz} = \frac{3Pz}{2 r^3} (r^2 + z^2) \mp \frac{3a^2 Pz}{2 r^3} (3 r^2 - 5 z^2), \quad (79c)
\]

\[
\tilde{K}^{\pm}_{\phi \phi} = -\frac{3P\rho^2 z}{2 r^3} \mp \frac{3a^2 P\rho^2 z}{2 r^3}, \quad (79d)
\]

with \( r = \sqrt{\rho^2 + z^2} \), and the vanishing components showing the potential \( Y_A \) to be a constant.

Working firstly with the potential \( X_A \) and the expression (41), the \( \rho z \) components are equated with those of (79a) with indices raised:

\[
X_A = \frac{3}{2} P\rho^2 \int \frac{1}{r^3} (\rho^2 + z^2) \, dz \mp \frac{3}{2} a^2 P\rho^2 \int \frac{1}{r^3} (\rho^2 - 4z^2) \, dz, \quad (80)
\]

and equating next the \( z^2 \) components:

\[
X_A = -\frac{3}{2} Pz \int \frac{\rho}{r^2} (\rho^2 + z^2) \, d\rho \mp \frac{3}{2} a^2 Pz \int \frac{\rho}{r^2} (3\rho^2 - 2z^2) \, d\rho. \quad (81)
\]

Carrying out the two sets of integrals, noting that \( r = \sqrt{\rho^2 + z^2} \), gives:

\[
X_A = P \frac{3\rho^2 z + 4z^3}{2r^3} \mp Pa^2 \frac{3\rho^2 z}{2r^3}. \quad (82)
\]

A constant of integration can also be added, but as with (75), it does not have an influence on the tensor. To find the corresponding expression for the potential \( R_A \) of (52), the relation (53) between \( R_A \) and \( X_A \) is used:

\[
R_A = \frac{1}{\rho} \int \rho X_A \, d\rho = \frac{1}{\rho} \int P \frac{3\rho^2 z + 4z^3}{2r^3} \, d\rho \mp \frac{1}{\rho} \int Pa^2 \frac{3\rho^2 z}{2r^3} \, d\rho. \quad (83)
\]

Although functions of integration of the form:

\[
\frac{1}{\rho} \int \rho \, k \, d\rho + \frac{1}{\rho} \, h_A(z), \quad (84)
\]

can also be added, where \( k \) is the constant of integration from (82), these will all be cancelled or differentiated out when forming a tensor using (52).

The two solutions for the Bowen-York curvature can be combined, with both momenta directed along the \( z \)-axis, and the choice of scalar potentials for (41) and (52) given by:

\[
X_A = P \frac{3\rho^2 z + 4z^3}{2r^3} \mp Pa^2 \frac{3\rho^2 z}{2r^3}, \quad Y_A = J \frac{3\rho^2 z + 2z^3}{r^3},
\]

\[
R_A = \frac{1}{\rho} \int P \frac{3\rho^2 z + 4z^3}{2r^3} \, d\rho \mp \frac{1}{\rho} \int Pa^2 \frac{3\rho^2 z}{2r^3} \, d\rho. \quad (85)
\]

Due to the relation with the Bowen-York curvature, in general \( X_A, R_A \) can be considered to represent a linear momentum and \( Y_A \) an angular momentum, when used with (41), (52) for an axially-symmetric TT tensor.
7. Conclusion

In flat 3-space, expressions have successfully been given for transverse trace-free tensors in different coordinate systems, with both linear and axial symmetries, depending on only two scalar potentials. In a more general axially-symmetric 3-space, a single potential has been derived, equivalent to [1, 2], for two of the components. For the remaining components, the TT conditions have been reduced to a second order partial differential equation, requiring boundary conditions to be solved. The axially-symmetric flat space tensors have also been compared with the Bowen-York curvature, and specific choices of the potentials shown to give the Bowen-York curvature. There is also a distinct relationship between each of the potentials, and either the angular or linear momentum of the Bowen-York space. The expressions derived could however benefit from a coordinate independent form, in line with that given in [2]. An expanded version of some of the content of this paper can be found in [13].

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