CONVERGENCE OF T MEANS WITH RESPECT TO VILENKIN SYSTEMS OF INTEGRABLE FUNCTIONS

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Abstract. In this paper we derive converge of T means of Vilenkin-Fourier series with monotone coefficients of integrable functions in Lebesgue and Vilenkin-Lebesgue points. Moreover, we discuss pointwise and norm convergence in $L_p$ norms of such T means.

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1. Introduction

The definitions and notations used in this introduction can be found in our next Section. It is well-known (see e.g. the book [35]) that there exists an absolute constant $c_p$, depending only on $p$, such that

$$\|S_n f\|_p \leq c_p \|f\|_p, \quad \text{when } p > 1.$$ 

On the other hand, (for details see [7, 8, 30, 41, 44]) boundedness does not hold for $p = 1$. The analogue of Carleson’s theorem for Walsh system was proved by Billard [3] for $p = 2$ and by Sjölin [36] for $1 < p < \infty$, while for bounded Vilenkin systems by Gosselin [15]. For Walsh-Fourier series, Schipp [31, 32, 35] gave a proof by using methods of martingale theory. A similar proof for Vilenkin-Fourier series can be found in Schipp and Weisz [33, 48]. In each proof, they show that the maximal operator of the partial sums is bounded on $L_p$, i.e. there exists an absolute constant $c_p$ such that

$$\|S^* f\|_p \leq c_p \|f\|_p, \quad \text{when } f \in L_p, \quad p > 1.$$ 

Hence, if $f \in L_p(G_m)$, where $p > 1$, then $S_n f \to f$, a.e. on $G_m$. Stein [37] constructed the integrable function whose Vilenkin-Fourier (Walsh-Fourier) series diverges almost everywhere. In [35] was proved that there exists an integrable function whose Walsh-Fourier series diverges everywhere. a.e convergence of subsequences of Vilenkin-Fourier series was considered in [6], where was used methods of martingale Hardy spaces.

If we consider the following restricted maximal operator $\tilde{S}_n^* f := \sup_{n \in \mathbb{N}} |S_n f|$, we have weak $(1, 1)$ type inequality for $f \in L_1(G_m)$. Hence, if $f \in L_1(G_m)$, then $\tilde{S}_n f \to f$, a.e. on $G_m$. Moreover, for any integrable function it is known that a.e. point is Lebesgue point and for any such point $x$ of integrable function $f$ we have that

$$S_{M_n} f(x) \to f(x), \quad \text{as } n \to \infty, \quad \text{for any Lebesgue point } x \text{ of } f \in L_1(G_m).$$

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In the one-dimensional case Yano [47] proved that
\[ \| \sigma_n f - f \|_p \to 0, \quad \text{when} \quad n \to \infty, \quad (f \in L_p(G_m), \quad 1 \leq p \leq \infty). \]

If we consider the maximal operator of Féjer means
\[ \sigma^* f := \sup_{n \in \mathbb{N}} |\sigma_n f|, \]
then
\[ \lambda \mu \{ \sigma^* f > \lambda \} \leq c \| f \|_1, \quad f \in L_1(G_m), \quad \lambda > 0. \]

This result can be found in Zygmund [51] for the trigonometric series, in Schipp [34] and [12, 20, 27, 38, 42, 43] for Walsh series and in Pál, Simon [25] for bounded Vilenkin series (see also Weisz [49, 50]). The boundedness does not hold from Lebesgue space \( L_1(G_m) \) to the space \( L_1(G_m) \). The weak \( (1, 1) \) type inequality follows that for any \( f \in L_1(G_m) \),
\[ \sigma_n f(x) \to f(x), \quad \text{a.e., as} \quad n \to \infty. \]

Moreover, in [11] (see also [10]) was proved that for any integrable function it is known that a.e. point is Vilenkin-Lebesgue points and for any such point \( x \) of integrable function \( f \) we have that
\[ \sigma_n f(x) \to f(x), \quad \text{as} \quad n \to \infty. \]

Móricz and Siddiqi [18] investigate approximation properties of some special Nörlund means of Walsh-Fourier series of \( L_p \) functions in norm. Similar results for the two-dimensional case can be found in Nagy [19, 20], Nagy and Tephnadze [21, 22, 23, 24], Gogolashvili and Tephnadze [13, 14] (see also [2], [17]). Approximation properties of general summability methods can be found in [41, 55]. Fridli, Manchanda and Siddiqi [9] improved and extended results of Móricz and Siddiqi [18] to Martingale Hardy spaces. The a.e. convergence of Nörlund means of Vilenkin-Fourier series with monotone coefficients of \( f \in L_1 \) was proved in [28] (see also [29]). In [45] was proved that the maximal operators of \( T \) means \( T^* \) defined by \( T^* f := \sup_{n \in \mathbb{N}} |T_n f| \) either with non-increasing coefficients, or non-decreasing sequence satisfying condition
\[ \frac{q_{n-1}}{Q_n} = O \left( \frac{1}{n} \right), \quad \text{as} \quad n \to \infty \]
are bounded from the Hardy space \( H_{1/2} \) to the space weak \(-L_{1/2} \). Moreover, there exists a martingale and such \( T \) means for which boundedness does not hold from the Hardy space \( H_p \) to the space \( L_p \) when \( 0 < p \leq 1/2 \).

One of the most well-known mean of \( T \) means is the Riesz summability. In [39] (see also [16]) it was proved that the maximal operator of Riesz logarithmic means
\[ R^* f := \sup_{n \in \mathbb{N}} |R_n f| \]
is bounded from the Hardy space \( H_{1/2} \) to the space weak \(-L_{1/2} \) and is not bounded from \( H_p \) to the space \( L_p \), for \( 0 < p \leq 1/2 \). There was also proved that Riesz summability has better properties than Fejér means.

In this paper we derive convergence of \( T \) means of Vilenkin-Fourier series with monotone coefficients of integrable functions in Lebesgue and Vilinkin-Lebesgue points.

This paper is organized as follows: In order not to disturb our discussions later on some definitions and notations are presented in Section 2. For the proofs of the main results we need some auxiliary Lemmas, some of them are new and of independent interest. These
results are presented in Section 3. The main results and some of its consequences and
detailed proofs are given in Section 4.

2. Definitions and Notation

Denote by \( \mathbb{N}_+ \) the set of the positive integers, \( \mathbb{N} := \mathbb{N}_+ \cup \{0\} \). Let \( m := (m_0, m_1, \ldots) \) be a sequence of the positive integers not less than 2. Denote by
\[
Z_{m_k} := \{0, 1, \ldots, m_k - 1\}
\]
the additive group of integers modulo \( m_k \).

Define the Vilenkin group \( G_m \) as the complete direct product of the groups \( Z_{m_i} \) with the product of the discrete topologies of \( Z_{m_j} \)'s (for details see [46]). In this paper we discuss bounded Vilenkin groups, i.e. the case when \( \sup n_m < \infty \). The direct product \( \mu \) of the measures \( \mu_k \{\{j\}\} := \frac{1}{m_k} \) \( (j \in Z_{m_k}) \) is the Haar measure on \( G_m \) with \( \mu (G_m) = 1 \). The elements of \( G_m \) are represented by sequences
\[
x := (x_0, x_1, \ldots, x_j, \ldots), \ (x_j \in Z_{m_j}).
\]
It is easy to give a basis for the neighborhoods of \( G_m \):
\[
I_0 (x) := G_m, \ I_n (x) := \{y \in G_m \mid y_0 = x_0, \ldots, y_{n-1} = x_{n-1}\}, \quad \text{where} \ x \in G_m, \ n \in \mathbb{N}.
\]

If we define the so-called generalized number system based on \( m \) in the following way:
\[
M_0 := 1, \ M_{k+1} := m_k M_k \quad (k \in \mathbb{N}),
\]
then every \( n \in \mathbb{N} \) can be uniquely expressed as \( n = \sum_{j=0}^{\infty} n_j M_j \), where \( n_j \in Z_{m_j} \) \( (j \in \mathbb{N}_+) \) and only a finite number of \( n_j \)'s differ from zero.

We introduce on \( G_m \) an orthonormal system which is called the Vilenkin system. First, we define complex-valued function \( r_k (x) : G_m \to \mathbb{C} \), the generalized Rademacher functions, by
\[
r_k (x) := \exp \left(2\pi i x_k / m_k\right), \quad (i^2 = -1, x \in G_m, \ k \in \mathbb{N}).
\]
Next, we define the Vilenkin system \( \psi := (\psi_n : n \in \mathbb{N}) \) on \( G_m \) by:
\[
\psi_n (x) := \prod_{k=0}^{\infty} r_k^n (x), \quad (n \in \mathbb{N}).
\]
Specifically, we call this system the Walsh-Paley system when \( m \equiv 2 \).

The norms (or quasi-norms) of the spaces \( L_p (G_m) \) and weak \(-\) \( L_p (G_m) \) \( (0 < p < \infty) \) are respectively defined by
\[
\|f\|_p := \int_{G_m} |f|^p \ d\mu, \quad \|f\|_{\text{weak} - L_p} := \sup_{\lambda > 0} \lambda^p \mu (f > \lambda) < +\infty.
\]

The Vilenkin system is orthonormal and complete in \( L_2 (G_m) \) (see [46]).

Now, we introduce analogues of the usual definitions in Fourier-analysis. If \( f \in L_1 (G_m) \) we can define Fourier coefficients, partial sums and Dirichlet kernels with respect to the Vilenkin system in the usual manner:
\[
\hat{f} (n) := \int_{G_m} f \overline{\psi}_n \ d\mu, \quad S_n f := \sum_{k=0}^{n-1} \hat{f} (k) \psi_k, \quad D_n := \sum_{k=0}^{n-1} \psi_k, \quad (n \in \mathbb{N}_+).
\]
Recall that
\[
\int_{G_m} D_n(x) \, dx = 1,
\]
\[
D_{M_n-1}(x) = D_{M_n}(x) - \psi_{M_n-1}(x)D_j(x), \quad j < M_n.
\]
The convolution of two functions \(f, g \in L_1(G_m)\) is defined by
\[
(f * g)(x) := \int_{G_m} f(x-t) \, g(t) \, dt \quad (x \in G_m).
\]
It is easy to see that if \(f \in L_p(G_m), g \in L_1(G_m)\) and \(1 \leq p < \infty\). Then \(f * g \in L_p(G_m)\) and
\[
\|f * g\|_p \leq \|f\|_p \|g\|_1.
\]
Let \(\{q_k : k \geq 0\}\) be a sequence of nonnegative numbers. The \(n\)-th Nörlund mean \(t_n\) for a
Fourier series of \(f\) is defined by
\[
t_n f = \frac{1}{Q_n} \sum_{k=1}^{n} q_{n-k} S_k f, \quad \text{where} \quad Q_n := \sum_{k=0}^{n-1} q_k.
\]
It is obvious that
\[
t_n f(x) = \int_{G_m} f(t) F_n(x-t) \, d\mu(t), \quad \text{where} \quad F_n := \frac{1}{Q_n} \sum_{k=0}^{n-1} q_k D_k
\]
is called \(T\) kernel.

**Proposition 1.** Let \(\{q_k : k \in \mathbb{N}\}\) be a sequence of non-increasing numbers. Then, for any
\(n, N \in \mathbb{N}_+\),
\[
\int_{G_m} F_{M_n}(x) \, d\mu(x) = 1,
\]
\[
\sup_{n \in \mathbb{N}} \int_{G_m} |F_{M_n}(x)| \, d\mu(x) \leq c < \infty,
\]
\[
\sup_{n \in \mathbb{N}} \int_{G_m \setminus I_N} |F_{M_n}(x)| \, d\mu(x) \to 0, \quad \text{as} \quad n \to \infty,
\]
Let \(\{q_k : k \geq 0\}\) be a sequence of non-negative numbers. The \(n\)-th \(T\) means \(T_n\) for a
Fourier series of \(f\) are defined by
\[
T_n f := \frac{1}{Q_n} \sum_{k=0}^{n-1} q_k S_k f, \quad \text{where} \quad Q_n := \sum_{k=0}^{n-1} q_k.
\]
It is obvious that
\[
T_n f(x) = \int_{G_m} f(t) F^{-1}_n(x-t) \, d\mu(t), \quad \text{where} \quad F^{-1}_n := \frac{1}{Q_n} \sum_{k=0}^{n-1} q_k D_k
\]
is called the \(T\) kernel.

We always assume that \(\{q_k : k \geq 0\}\) is a sequence of non-negative numbers and \(q_0 > 0\).
Then the summability method (10) generated by \(\{q_k : k \geq 0\}\) is regular if and only if
\[
\lim_{n \to \infty} Q_n = \infty.
\]
It is easy to show that, for any real numbers \(a_1, \ldots, a_m, b_1, \ldots, b_m\) and \(a_k = A_k - A_{k-1},\) \(k = n, \ldots, m,\) we have so called Abel transformation:

\[
\sum_{k=m}^{n} a_k b_k = A_n b_n - A_{m-1} b_m + \sum_{k=m}^{n-1} A_k (b_k - b_{k+1}).
\]

For \(a_j = A_j - A_{j-1}, j = 1, \ldots, n,\) if we invoke Abel transformations

\[(11) \quad \sum_{j=1}^{n-1} a_j b_j = A_{n-1} b_{n-1} - A_0 b_1 + \sum_{j=0}^{n-2} A_j (b_j - b_{j+1}),\]
\[(12) \quad \sum_{j=MN}^{n-1} a_j b_j = A_{n-1} b_{n-1} - A_{MN-1} b_{MN} + \sum_{j=MN}^{n-2} A_j (b_j - b_{j+1}),\]

For \(b_j = q_j, a_j = 1\) and \(A_j = j\) for any \(j = 0, 1, \ldots, n\) we get the following identity:

\[(13) \quad Q_n = \sum_{j=0}^{n-1} q_j = q_0 + \sum_{j=1}^{n-1} q_j = q_0 + \sum_{j=1}^{n-2} (q_j - q_{j+1}) j + q_{n-1}(n - 1),\]
\[(14) \quad \sum_{j=MN}^{n-1} q_j = \sum_{j=MN}^{n-2} (q_j - q_{j+1}) j + q_{n-1}(n - 1) - (M_N - 1)q_{MN}.
\]

Moreover, if use \(D_0 = K_0 = 0\) for any \(x \in G_m\) and invoke Abel transformations \((11)\) and \((12)\) for \(b_j = q_j, a_j = D_j\) and \(A_j = jK_j\) for any \(j = 0, 1, \ldots, n - 1\) we get identities:

\[(15) \quad F_n^{-1} = \frac{1}{Q_n} \sum_{j=0}^{n-1} q_j D_j = \frac{1}{Q_n} \left( \sum_{j=1}^{n-2} (q_j - q_{j+1}) jK_j + q_{n-1}(n - 1)K_{n-1} \right),\]
\[(16) \quad \frac{1}{Q_n} \sum_{j=MN}^{n-1} q_j D_j = \frac{1}{Q_n} \left( \sum_{j=MN}^{n-2} (q_j - q_{j+1}) jK_j + q_{n-1}(n - 1)K_{n-1} - q_{MN}(M_N - 1)K_{MN-1} \right).
\]

Analogously, if use \(S_0 f = \sigma_0 f = 0,\) for any \(x \in G_m\) and invoke Abel transformations \((11)\) and \((12)\) for \(b_j = q_j, a_j = S_j\) and \(A_j = j\sigma_j\) for any \(j = 0, 1, \ldots, n - 1\) we get identities:

\[(17) \quad T_n f = \frac{1}{Q_n} \sum_{j=0}^{n-1} q_j S_j f = \frac{1}{Q_n} \left( \sum_{j=1}^{n-2} (q_j - q_{j+1}) j\sigma_j f + q_{n-1}(n - 1)\sigma_{n-1} f \right),\]
\[(18) \quad \frac{1}{Q_n} \sum_{j=MN}^{n-1} q_j S_j f = \frac{1}{Q_n} \left( \sum_{j=MN}^{n-2} (q_j - q_{j+1}) j\sigma_j f + q_{n-1}(n - 1)\sigma_{n-1} f - q_{MN}(M_N - 1)\sigma_{MN-1} f \right).
\]
If \( q_k \equiv 1 \) in (6) and (10) we respectively define the Fejér means \( \sigma_n \) and Fejér Kernels \( K_n \) as follows:

\[
\sigma_n f := \frac{1}{n} \sum_{k=1}^{n} S_k f, \quad K_n := \frac{1}{n} \sum_{k=1}^{n} D_k.
\]

It is well-known that (for details see [1])

\[
n |K_n| \leq c \sum_{l=0}^{\lfloor n \rfloor} M_l |K_{M_l}|
\]

and for any \( n, N \in \mathbb{N}_+ \),

\[
\int_{G_m} K_n(x) d\mu(x) = 1,
\]

\[
\sup_{n \in \mathbb{N}} \int_{G_m} |K_n(x)| d\mu(x) \leq c < \infty,
\]

\[
\sup_{n \in \mathbb{N}} \int_{G_m \setminus I_N} |K_n(x)| d\mu(x) \to 0, \text{ as } n \to \infty,
\]

The well-known example of Nörlund summability is the so-called \((C, \alpha)\) mean (Cesàro means) for \( 0 < \alpha < 1 \), which are defined by

\[
\sigma_{n}^{\alpha} f := \frac{1}{A_{n}^{\alpha}} \sum_{k=0}^{n} A_{n-k}^{\alpha-1} S_k f, \quad \text{where} \quad A_{0}^{\alpha} := 0, \quad A_{n}^{\alpha} := \frac{(\alpha + 1) \cdots (\alpha + n)}{n!}.
\]

We also consider the "inverse" \((C, \alpha)\) means, which is an example of \( T \) means:

\[
U_{n}^{\alpha} f := \frac{1}{A_{n}^{\alpha}} \sum_{k=0}^{n-1} A_{k}^{\alpha-1} S_k f, \quad 0 < \alpha < 1.
\]

Let \( V_{n}^{\alpha} \) denote the \( T \) mean, where \{\( q_0 = 0, \ q_k = k^{\alpha-1} : k \in \mathbb{N}_+ \)\}, that is

\[
V_{n}^{\alpha} f := \frac{1}{Q_{n}} \sum_{k=1}^{n-1} k^{\alpha-1} S_k f, \quad 0 < \alpha < 1.
\]

The \( n \)-th Riesz logarithmic mean \( R_n \) and the Nörlund logarithmic mean \( L_n \) are defined by

\[
R_{n} f := \frac{1}{l_n} \sum_{k=1}^{n} \frac{S_k f}{k} \quad \text{and} \quad L_{n} f := \frac{1}{l_n} \sum_{k=1}^{n-1} \frac{S_k f}{n-k}, \quad \text{where} \quad l_n := \sum_{k=1}^{n-1} 1/k.
\]

Up to now we have considered \( T \) means in the case when the sequence \{\( q_k : k \in \mathbb{N} \)\} is bounded but now we consider \( T \) summabilities with unbounded sequence \{\( q_k : k \in \mathbb{N} \)\}.

Let \( \alpha \in \mathbb{R}_+, \ \beta \in \mathbb{N}_+ \) and \( \log^{(\beta)} x := \log \ldots \log x \). If we define the sequence \{\( q_k : k \in \mathbb{N} \)\} by \{\( q_0 = 0, \ q_k = \log^{(\beta)} k^{\alpha} : k \in \mathbb{N}_+ \)\}, then we get the class \( B_{n}^{\alpha,\beta} \) of \( T \) means with non-decreasing coefficients:

\[
B_{n}^{\alpha,\beta} f := \frac{1}{Q_{n}} \sum_{k=1}^{n-1} \log^{(\beta)} k^{\alpha} S_k f.
\]
3. **Auxiliary Lemmas**

First we consider kernels of $T$ kernels with non-increasing sequences:

**Lemma 1.** Let $\{q_k : k \in \mathbb{N}\}$ be a sequence of non-increasing numbers, satisfying the condition

$$
\frac{q_0}{Q_n} = O \left( \frac{1}{n} \right), \quad \text{as} \quad n \to \infty.
$$

Then, for some constant $c$, we have that

$$
|F_n^{-1}| \leq \frac{c}{n} \left\{ \sum_{j=0}^{|n|} M_j |K_{M_j}| \right\}.
$$

**Proof.** Let sequence $\{q_k : k \in \mathbb{N}\}$ be non-increasing. Then, by using (2) we get that

$$
\frac{1}{Q_n} \left( \sum_{j=1}^{n-2} (q_j - q_{j+1} + q_{n-1}) \right) \leq \frac{1}{Q_n} \left( \sum_{j=1}^{n-2} (q_j - q_{j+1} + q_{n-1}) \right) \leq \frac{q_0}{Q_n} \leq \frac{c}{n}.
$$

Hence, if we apply (19) and use the equalities (13) and (15) we immediately get that

$$
|F_n^{-1}| \leq \left( \frac{1}{Q_n} \left( \sum_{j=1}^{|n-1|} |q_j - q_{j+1} + q_{n-1}| \right) \right) \sum_{i=0}^{|n|} M_i |K_{M_i}| \leq \frac{c}{n} \sum_{i=0}^{|n|} M_i |K_{M_i}|.
$$

The proof is complete by just combining the estimates above. $\square$

**Lemma 2.** Let $\{q_k : k \in \mathbb{N}\}$ be a sequence of non-increasing numbers. Then, for any $n, N \in \mathbb{N}_+$,

$$
\int_{G_m} F_n^{-1}(x) d\mu(x) = 1,
$$

$$
\sup_{n \in \mathbb{N}} \int_{G_m} |F_n^{-1}(x)| d\mu(x) < \infty,
$$

$$
\sup_{n \in \mathbb{N}} \int_{G_m \setminus I_N} |F_n^{-1}(x)| d\mu(x) \to 0, \quad \text{as} \quad n \to \infty.
$$

**Proof.** According to (3) we easily obtain proof of (24). By using (21) combined with (13) and (15) we get that

$$
\frac{1}{Q_n} \left( \sum_{j=0}^{n-2} (q_j - q_{j+1}) j \int_{G_m} |K_j| d\mu + q_{n-1}(n-1) \int_{G_m} |K_{n-1}| d\mu \right) 
\leq \frac{c}{Q_n} \left( \sum_{j=0}^{n-2} (q_j - q_{j+1}) j + q_{n-1}(n-1) \right) \leq c < \infty,
$$

so also (25) is proved. By using (22) and inequalities (13) and (15) we can conclude that
\[
\int_{G_m \setminus I_N} |F_n^{-1}| \, d\mu \leq \frac{1}{Q_n} \sum_{j=0}^{n-1} (q_j - q_{j+1}) \int_{G_m \setminus I_N} |K_j| \, d\mu + \frac{q_{n-1}(n-1)}{Q_n} \int_{G_m \setminus I_N} |K_{n-1}|
\]

\[
\leq \frac{1}{Q_n} \sum_{j=0}^{n-2} (q_j - q_{j+1}) j \alpha_j + \frac{q_{n-1}(n-1)\alpha_{n-1}}{Q_n} = I + II,
\]

where \(\alpha_n \to 0\), as \(n \to \infty\). Since sequence is non-increasing we can conclude that

\[
II = \frac{q_{n-1}(n-1)\alpha_{n-1}}{Q_n} \leq \alpha_{n-1} \to 0, \text{ as } n \to \infty.
\]

Moreover, for any \(\varepsilon > 0\) there exists \(N_0 \in \mathbb{N}\), such that \(\alpha_n < \varepsilon\) when \(n > N_0\). Furthermore,

\[
I = \frac{1}{Q_n} \sum_{j=0}^{n-2} (q_j - q_{j+1}) j \alpha_j = \frac{1}{Q_n} \sum_{j=0}^{N_0} (q_j - q_{j+1}) j \alpha_j + \frac{1}{Q_n} \sum_{j=N_0+1}^{n-2} (q_j - q_{j+1}) j \alpha_j := I_1 + I_2.
\]

The sequence \(\{q_k : k \in \mathbb{N}\}\) is non-increasing and therefore \(|q_j - q_{j+1}| < 2q_0,\)

\[
I_1 \leq \frac{2q_0 N_0}{Q_n} \to 0, \text{ as } n \to \infty
\]

and

\[
I_2 = \frac{1}{Q_n} \sum_{j=N_0+1}^{n-2} (q_j - q_{j+1}) j \alpha_j \leq \frac{\varepsilon}{Q_n} \sum_{j=N_0+1}^{n-2} (q_j - q_{j+1}) j \leq \frac{\varepsilon}{Q_n} \sum_{j=0}^{n-2} (q_j - q_{j+1}) j < \varepsilon.
\]

and we can conclude that \(I_2 \to 0\) so proof is complete. \(\square\)

Next we consider kernels of \(T\) means with non-decreasing sequences:

**Lemma 3.** Let \(\{q_k : k \in \mathbb{N}\}\) be a sequence of non-decreasing numbers, satisfying the condition

\[
(27) \quad \frac{q_{n-1}}{Q_n} = O \left( \frac{1}{n} \right), \text{ as } n \to \infty.
\]

Then for some constant \(c\),

\[
|F_n^{-1}| \leq \frac{c}{n} \left\{ \sum_{j=0}^{n} M_j |K_{M_j}| \right\}.
\]

**Proof.** Since the sequence \(\{q_k : k \in \mathbb{N}\}\) be non-decreasing if we apply the condition (27) we find that

\[
\frac{1}{Q_n} \left( \sum_{j=1}^{n-2} |q_j - q_{j+1}| + q_{n-1} \right) = \frac{1}{Q_n} \left( \sum_{j=1}^{n-2} (q_{j+1} - q_j) + q_{n-1} \right) \leq \frac{2q_{n-1}}{Q_n} \leq \frac{c}{n}.
\]

If we apply Abel transformation (16) combined with (19) and (28) we get that

\[
|F_n^{-1}| \leq \left( \frac{1}{Q_n} \left( \sum_{j=1}^{n-1} |q_j - q_{j+1}| + q_{n-1} + q_0 \right) \right) \sum_{i=0}^{n} M_i |K_{M_i}| \leq \frac{c}{n} \sum_{i=0}^{n} M_i |K_{M_i}|.
\]

The proof is complete. \(\square\)
Lemma 4. Let \( \{q_k : k \in \mathbb{N}\} \) be a sequence of non-decreasing numbers, satisfying the condition (27). Then, for some constant \( c \),

\[
\int_{G_m} F_n^{-1}(x) d\mu(x) = 1,
\]

\[
\sup_{n \in \mathbb{N}} \int_{G_m} |F_n^{-1}(x)| d\mu(x) \leq c < \infty,
\]

\[
\sup_{n \in \mathbb{N}} \int_{G_m \setminus I_N} |F_n^{-1}(x)| d\mu(x) \to 0, \quad \text{as} \quad n \to \infty.
\]

Proof. If we compare the estimation of \( F_n \) in Lemma 2 with the estimation of \( F_n \) in Lemma 3 we find that they are quite same. It follows that proof is analogical to Lemma 2. So, we leave out the details. \( \square \)

Finally, we study some special subsequences of kernels of \( T \) means:

Lemma 5. Let \( n \in \mathbb{N} \). Then

\[
F_{M_n}^{-1}(x) = D_{M_n}(x) - \psi_{M_n-1}(x) F_{M_n}(x).
\]

Proof. By using (4) we get that

\[
F_{M_n}^{-1}(x) = \frac{1}{Q_{M_n}} \sum_{k=0}^{M_n-1} q_k D_k(x) = \frac{1}{Q_{M_n}} \sum_{k=1}^{M_n} q_{M_n-k} D_{M_n-k}(x)
\]

\[
= \frac{1}{Q_{M_n}} \sum_{k=1}^{M_n} q_{M_n-k} \left( D_{M_n}(x) - \psi_{M_n-1}(x) D_k(x) \right) = D_{M_n}(x) - \psi_{M_n-1}(x) F_{M_n}(x).
\]

The proof is complete. \( \square \)

Corollary 1. Let \( \{q_k : k \in \mathbb{N}\} \) be a sequence of non-decreasing numbers. Then for some constant \( c \),

\[
\int_{G_m} F_{M_n}^{-1}(x) d\mu(x) = 1,
\]

\[
\sup_{n \in \mathbb{N}} \int_{G_m} |F_{M_n}^{-1}(x)| d\mu(x) \leq c < \infty,
\]

\[
\sup_{n \in \mathbb{N}} \int_{G_m \setminus I_N} |F_{M_n}^{-1}(x)| d\mu(x) \to 0, \quad \text{as} \quad n \to \infty,
\]

for any \( N \in \mathbb{N}_+ \).

Proof. The proof is direct consequence of Proposition 1 and Lemma 5. \( \square \)

4. PROOF OF MAIN RESULT

Theorem 1. Let \( p \geq 1 \) and \( \{q_k : k \in \mathbb{N}\} \) be a sequence of non-increasing numbers. Then

\[
\|T_n f - f\|_p \to 0 \quad \text{as} \quad n \to \infty
\]

for all \( f \in L_p(G_m) \).

Let function \( f \in L_1(G_m) \) is continuous at a point \( x \). Then

\[
T_n f(x) \to f(x), \quad \text{as} \quad n \to \infty.
\]
Moreover,
\[
\lim_{n \to \infty} T_n f(x) = f(x)
\]
for all Vilenkin-Lebesgue points of \( f \in L_p(G_m) \).

Let \( p \geq 1 \) and \( \{q_k : k \in \mathbb{N}\} \) be a sequence of non-decreasing numbers satisfying the condition (27). Then
\[
\|T_n f - f\|_p \to 0 \quad \text{as} \quad n \to \infty
\]
for all \( f \in L_p(G_m) \).

Let function \( f \in L_1(G_m) \) is continuous at a point \( x \). Then
\[
T_n f(x) \to f(x), \quad \text{as} \quad n \to \infty.
\]
Moreover,
\[
\lim_{n \to \infty} T_n f(x) = f(x)
\]
for all Vilenkin-Lebesgue points of \( f \in L_p(G_m) \).

**Proof.** Let \( \{q_k : k \in \mathbb{N}\} \) be a non-increasing sequence. Lemma 2 immediately follows stated norm and pointwise convergences.

Suppose that \( x \) is either point of continuity or Vilenkin-Lebesgue point of function \( f \in L_p(G_m) \). Then
\[
\lim_{n \to \infty} |\sigma_n f(x) - f(x)| = 0.
\]

Hence,
\[
|T_n f(x) - f(x)| \leq \frac{1}{Q_n} \left( \sum_{j=0}^{n-2} (q_j - q_{j+1}) j |\sigma_j f(x) - f(x)| + q_{n-1}(n - 1)|\sigma_{n-1} f(x) - f(x)| \right)
\]
\[
\leq \frac{1}{Q_n} \sum_{j=0}^{n-2} (q_j - q_{j+1}) j \alpha_j + \frac{q_{n-1}(n - 1)\alpha_{n-1}}{Q_n} := I + II, \quad \text{where} \quad \alpha_n \to 0, \quad \text{as} \quad n \to \infty.
\]

To prove \( I \to 0, \) as \( n \to \infty \) and \( II \to 0, \) as \( n \to \infty \), we just have to analogous steps of Lemma 2. It follows that Part a) is proved.

Now we assume that the sequence is non-decreasing and satisfying condition (27). According to (29) in Lemma 4 get norm and and pointwise convergence. To prove convergence in Vilenkin-Lebesgue points we use estimation
\[
|T_n f(x) - f(x)| \leq \frac{1}{Q_n} \sum_{j=0}^{n-2} (q_{j+1} - q_j) j \alpha_j + \frac{q_{n-1}(n - 1)\alpha_{n}}{Q_n} := III + IV, \quad \text{where} \quad \alpha_n \to 0, \quad \text{as} \quad n \to \infty.
\]

It is evident that
\[
IV \leq \frac{q_{n-1}(n - 1)\alpha_{n}}{Q_n} \leq \alpha_n \to 0, \quad \text{as} \quad n \to \infty.
\]

On the other hand, for any \( \varepsilon > 0 \) there exists \( N_0 \in \mathbb{N} \), such that \( \alpha_n < \varepsilon/2 \) when \( n > N_0 \). We can write that
\[
\frac{1}{Q_n} \sum_{j=1}^{n-2} (q_{j+1} - q_j) j \alpha_j = \frac{1}{Q_n} \sum_{j=1}^{N_0} (q_{j+1} - q_j) j \alpha_j + \frac{1}{Q_n} \sum_{j=N_0+1}^{n-2} (q_{j+1} - q_j) j \alpha_j = III_1 + III_2.
\]
Since the sequence \( \{q_k\} \) is non-decreasing, we obtain that \( |q_{j+1} - q_j| < 2q_{j+1} < 2q_{n-1} \). Hence,
\[
III_1 \leq \frac{2q_0N_0}{Q_n} \to 0, \quad \text{as } n \to \infty
\]
and
\[
III_2 \leq \frac{1}{Q_n} \sum_{j=N_0+1}^{n-2} (q_{n-j-1} - q_{n-j}) j \alpha_j \leq \frac{\varepsilon(n-1)}{Q_n} \sum_{j=N_0+1}^{n-2} (q_{n-j} - q_{n-j-1})
\]
\[
\leq \frac{\varepsilon(n-1)}{Q_n} (q_0 - q_{n-N_0}) \leq \frac{2q_0 \varepsilon(n-1)}{Q_n} < \varepsilon.
\]
Therefore, also \( III \to \infty \) so that the proof of part b) is also complete.

\[\square\]

**Corollary 2.** Let \( f \in L_p \), where \( p \geq 1 \). Then
\[
R_n f \to f, \quad \text{a.e., as } n \to \infty, \quad U_n^\alpha f \to f, \quad \text{a.e., as } n \to \infty,
\]
\[
V_n^\alpha f \to f, \quad \text{a.e., as } n \to \infty, \quad B_n^{\alpha,\beta} f \to f, \quad \text{a.e., as } n \to \infty.
\]

**Theorem 2.** Let \( p \geq 1 \) and \( \{q_k : k \in \mathbb{N}\} \) be a sequence of non-decreasing numbers. Then
\[
\| T_{M_n} f - f \|_p \to 0 \quad \text{as } n \to \infty
\]
for all \( f \in L_p(G_m) \).

Let function \( f \in L_1(G_m) \) is continuous at a point \( x \). Then
\[
T_{M_n} f(x) \to f(x), \quad \text{as } n \to \infty.
\]
Moreover,
\[
\lim_{n \to \infty} T_{M_n} f(x) = f(x), \quad \text{for all Lebesgue points of } f \in L_p(G_m).
\]

**Proof.** Corollary 1 immediately follows norm and and pointwise convergence. To prove a.e convergence we use first identity in Lemma 5 to write
\[
T_{M_n} f(x) = \int_{G_m} f(t) F_n^{-1}(x-t) \, d\mu(t)
\]
\[
= \int_{G_m} f(t) D_{M_n}(x-t) \, d\mu(t) - \int_{G_m} f(t) \psi_{M_n-1}(x-t) F_{M_n}(x-t) = I - II.
\]
By applying (1) we can conclude that \( I = S_{M_n} f(x) \to f(x) \) for all Lebesgue points of \( f \in L_p(G_m) \). By using \( \psi_{M_n-1}(x-t) = \psi_{M_n-1}(x) \psi_{M_n-1}(t) \) we can conclude that
\[
II = \psi_{M_n-1}(x) \int_{G_m} f(t) \overline{F_{M_n}(x-t)} \overline{\psi_{M_n-1}(t)} d(t)
\]
By combining (5) and Proposition 1 we find that function
\[
f(t) \overline{F_{M_n}(x-t)} \in L_p \quad \text{where } p \geq 1 \text{ for any } x \in G_m,
\]
and \( II \) is Fourier coefficients of integrable function. According to Riemann-Lebesgue Lemma we get that \( II \to 0 \) for any \( x \in G_m \). The proof is complete.

\[\square\]
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