Using Aitken method to solve Volterra-Fredholm integral equations of the second kind with Homotopy perturbation method

Talaat I. Hasan¹, Nejmaddin A. Sulaiman¹, Shaharuddin Salleh²

1- Department of Mathematic, College of Education, Salahaddin University, Erbil, Kurdistan Region, Iraq.
2- Department of Mathematical Sciences, Universiti Technologi Malaysia, Johor, Malaysia

ARTICLE INFO

Article History:
Received: 01/06/2017
Accepted: 05/08/2017
Published: 20/12/2017

Keywords:
Aitken method, Volterra-Fredholm integral equations of the second kind and Homotopy perturbation method.

*Corresponding Author:
Talaat I. Hasan
Talhat.hasan@su.edu.krd

ABSTRACT

In this paper, we consider Volterra-Fredholm integral equations of the second kind (VFIE-2), we investigated Homotopy perturbation method (HPM) to solve VFIE-2 and proposed Aitken method on Homotopy perturbation method (AM on HPM) for solving the same problem. In addition, a theorem and two new algorithms are introduced. They are illustrated by numerical examples and simulations using Matlab.

1. INTRODUCTION

The theory of integral equations has been an active research field for many years and it is based on analysis, function theory and functional analysis, also the theory and application of a system integral equation play an important role in applied mathematical modelling of many fields [Chniti 2002, Saadati 2009], the integral equation method is widely used for solving many problem in mathematic physics and engineering [Shidfar 2011, Waz Waz 2004]. These problems can be constructed from differential equations of type’s initial value problem and boundary value problem. The Volterra-Fredholm integral equations arise from parabolic boundary value problems, the mathematical modelling of the spatial – temporal development of an epidemic, also arises from various science as physical and biological problems in addition, in different problems in electrical engineering in modelling.
of dynamic impulse system [Rahmeny 2007, Merzaee 2013]. Volterra-Fredholm integral equations are usually difficult to solve analytically and so the numerical approaches are created to overcome the complexities of analytical methods. A numerical approaches for solving integral equations are an essential work in scientific research. In 1997 Waz waz used HPM for solving linear Fredholm integral equation of the second the kind. Rahmani used HPM to find approximate solution of Fredholm integral equation of the second kind. In 2008 Saeed used HPM to solve system of nonlinear Fredholm integral equation of the second kind.

We extend the work further for linear Volterra-Fredholm integral equations of the second kind by using HPM and AM on HPM.

**Definition 1** The integral equation

$$y(s) = f(s) + \lambda \int_{a}^{b} k(s,t)y(t)dt + \lambda \int_{a}^{b} g(s,t)y(t)dt, \quad (1)$$

is called linear Volterra-Fredholm integral equation of the second kind where the functions $k(s,t)$ and $g(s,t)$ are called kernels of integral equation, such that $f(s), k(s,t)$ and $g(s,t)$ are known functions on $R = \{(s,t) | a < t < s < b\}$, such that $\lambda, \lambda'$ are constants and $y(s)$ is unknown function.

### 2- Homotopy perturbation method

Until recently, the application of the Homotopy perturbation method (HPM) in nonlinear problems has been developed by scientists and engineers, because this method deforms the difficult problems under study into a simple problem which is easy to solve [Waz waz 2004]. In discussing the Homotopy perturbation method (HPM), consider the general equation of the form:

$$L(y) = 0, \quad (2)$$

where $L$ is the integral or differential operator. Convex Homotopy $H(y, p)$ is defined as

$$H(y, p) = (1 - p)F(y) + pL(y) = 0, \quad (3)$$

such that $F(y)$ is a function operator with known solutions $y_0$, which can be obtain easily. It is clear that

$$H(y, p) = 0, \quad (4)$$

from which we have $H(y, 0) = F(y)$ and $H(y, 1) = L(y)$. This shows that $H(y, p)$ continuously traces an implicitly defined curve from a starting point $H(y_0, 0)$ to the solution $H(y, 1)$. The embedding parameter $p$ monotonously increases from zero to a unit as the trivial problem $F(y) = 0$ continuously deforms to the original problem $L(y) = 0$. The embedding parameter $p \in [0,1]$ is the expanding parameter expressed as follows:

$$y = \sum_{i=0}^{\infty} p^i y^i = y^0 + py^1 + p^2 y^2 + p^3 y^3 + ... \quad (5)$$

When $p$ approach to 1, equation (5) becomes the approximate solution of FIE-2

$$y = y^0 + y^1 + y^2 + y^3 + ...$$

This series is convergent for most cases, and the rate of convergence depends on $L(y) = 0$, [He 1999, Jafari 2010], Substituting equation (5) in (3), using $F(s) = y(s) - f(s)$, we obtain

$$\sum_{i=0}^{\infty} y^i(s) = f(s) + \lambda \int_{a}^{b} k(s,t)y^i(t)dt$$

$$y^0(s) + y^1(s) + ... = f(s) + \lambda \int_{a}^{b} k(s,t)y^0(t)dt + \lambda \int_{a}^{b} k(s,t)y^1(t)dt + ...$$
Equating the terms of same powers of the embedding parameter \( p \) we get the following recurrence relation

\[ p^0 : y^0(s) = f(s) \]

\[ p^{i+1} : y^{i+1}(s) = \lambda \int_{a}^{b} k(s,t)y'(t)dt \]  \hspace{1cm} (6)

**3. Homotopy perturbation method for solving single VFIE-2**

In this section, we propose a new iterative method to solve VFIE-2. We have observed that the new technique is successful in the iterative process. Consider the VFIE-2, which is given in Equation (1), In view of Equation \((l-p)[y(s) - f(s)] + p[y(s) - f(s)]\)

\[ -\lambda \int_{a}^{b} k(s,t)y(t)dt - \lambda \int_{a}^{b} g(s,t)y(t)dt = 0, \]  \hspace{1cm} (7)

Where \( p \) is the parameter defined in equation (2) or

\[ y(s) = f(s) + p\lambda \int_{a}^{b} k(s,t)y(t)dt + p\lambda \int_{a}^{b} g(s,t)y(t)dt \]  \hspace{1cm} (8)

Let \( y(s) = \sum_{i=0}^{n} p^i y'(s) \) be the approximate solution of VFIE-2. Substituting it into Equation (3) we obtain

\[ \sum_{i=0}^{n} p^i y_i(s) = f(s) + \lambda \int_{a}^{b} k(s,t)\sum_{i=0}^{n} p^i y_i(t)dt + \lambda \int_{a}^{b} g(s,t)\sum_{i=0}^{n} p^i y_i(t)dt, \]  \hspace{1cm} (9)

Equating the terms with identical powers of \( p \) we get

\[ p^0 : y^0(s) = f(s) \]

\[ p^1 : y^1(s) = \lambda \int_{a}^{b} k(s,t)y'(t)dt + \lambda \int_{a}^{b} g(s,t)y'(t)dt \]

\[ p^2 : y^2(s) = \lambda \int_{a}^{b} k(s,t)y'(t)dt + \lambda \int_{a}^{b} g(s,t)y'(t)dt \]

\[ p^3 : y^3(s) = \lambda \int_{a}^{b} k(s,t)y'(t)dt + \lambda \int_{a}^{b} g(s,t)y'(t)dt \]

It follows that the general form is

\[ p^n : y^n(s) = \lambda \int_{a}^{b} k(s,t)y^{n-1}(t)dt + \lambda \int_{a}^{b} g(s,t)y^{n-1}(t)dt \]  \hspace{1cm}, \( n = 0,1,2,... \) \hspace{1cm} (10)

In addition, according to Equation (5) the partial sums are obtained as follows:

\[ S_0(s) = f(s) \]

\[ S_n(s) = \sum_{i=0}^{n} y_i(s) \]  \hspace{1cm} (11)

From Equations (6) and (11) we obtain

\[ S_0(s) = f(s) \]

\[ S_n(s) = f(s) + \lambda \int_{a}^{b} k(s,t)S_{n-1}(t)dt + \lambda \int_{a}^{b} g(s,t)S_{n-1}(t)dt \]  \hspace{1cm} (12)

The series in Equation (5) converges to the exact solution \( y(s) \) if the partial sum in Equation (11) converges to \( y(s) \). This proves the following theorem.

**Theorem 1.** Consider the partial sum of the iterations:

\[ S_n(s) = f(s) \]
\[
S_n(s) = f(s) + \lambda \int_a^b k(s,t)S_{n-1}(t)dt + \lambda \int_a^b g(s,t)S_{n-1}(t)dt
\]

for \( n = 1, 2, 3, \ldots, \text{max} \) and \( \{y'(s)\} \) be a sequence generate by the iteration scheme
\[
y^0(s) = f(s),
\]
\[
y^{r+1}(s) = T(y'(s)) = f(s) + \lambda \int_a^b k(s,t)y'(t)dt + \lambda \int_a^b g(s,t)y'(t)dt,
\]

for \( r = 0, 1, 2, 3, \ldots, \text{max} \), where \( \text{max} \) is the maximum number of iteration

\[
M_1^2 = \int_a^b \int_a^b k^2(s,t)dtds < \infty \quad \text{and}
\]
\[
M_2^2 = \int_a^b \int_a^b g^2(s,t)dtds < \infty,
\]

If \( f(s) \in L^2(a,b) \), \( |\lambda| < \frac{1}{M_1 + M_2} \)

The series is convergent to the exact solution of VFIE-2, in the norm \( L^2(a,b) \).

**Proof.** We want to prove that the sequence generate by the iteration scheme in Equation (2.65) is convergence to the exact solution of VFIE-2. For each \( m, n \in N \), we have

\[
\|T(y^m(s)) - T(y^n(s))\| = \|y^{m+1}(s) - y^{n+1}(s)\|
\]
\[
= \|f(s) + \lambda \int_a^b k(s,t)y^m(t)dt + \lambda \int_a^b g(s,t)y^m(t)dt - \lambda \int_a^b k(s,t)y^n(t)dt - \lambda \int_a^b g(s,t)y^n(t)dt\|
\]
\[
= \|\lambda \int_a^b k(s,t)[y^m(t) - y^n(t)]dt + \lambda \int_a^b g(s,t)[y^m(t) - y^n(t)]dt\|
\]
\[
\leq |\lambda| \int_a^b \int_a^b k^2(s,t)dtds \|y^m(s) - y^n(s)\| + |\lambda| \int_a^b \int_a^b g^2(s,t)dtds \|y^m(s) - y^n(s)\|
\]

Therefore \( \|T(y^m(s)) - T(y^n(s))\| \)

\[
\leq |\lambda| \int_a^b \int_a^b k^2(s,t)dtds \|y^m(s) - y^n(s)\| + |\lambda| \int_a^b \int_a^b g^2(s,t)dtds \|y^m(s) - y^n(s)\|
\]

by suppose we have

\[
|\lambda| < \frac{1}{M_1 + M_2} = \left[ M_1 + M_2 \right]^{-\frac{1}{2}},
\]

since \( \int_a^b \int_a^b k^2(s,t)dtds = M_1^2 < \infty \) and
\( \int_a^b \int_a^b g^2(s,t)dtds = M_2^2 < \infty \),

then
\[
\int_a^b \int_a^b k^2(s,t)dtds \frac{1}{2} = M_1 \quad \text{and}
\]
\[
\int_a^b \int_a^b g^2(s,t)dtds \frac{1}{2} = M_2,
\]

we get

\[
|\lambda| < \left[ \int_a^b \int_a^b k^2(s,t)dtds \frac{1}{2} + \int_a^b \int_a^b g^2(s,t)dtds \frac{1}{2} \right]^{-1},
\]

hence we obtain

\[
\|T(y^m(s)) - T(y^n(s))\| \leq |\lambda| \|y^m(s) - y^n(s)\|.
\]

Then the sequence generate by the iteration scheme in Equation (2.65) is convergence to the exact solution of VFIE-2.
HPM Algorithm:-

Input: $a,b,n,s,Tol$

Step 1: Suppose that $y(s) = \sum_{i=0}^{n} p^i y^i(s)$ be the numerical solution of VFIE-2.

$$W = 0$$

For $i = 1$ to $n$

Step 2: Using equation (10) to obtain the approximation solution $y^i(s)$.

Step 3: find $W = W + y^i(s)$

Step 4: Calculate the absolute error by

$$e_i = |y(s) - W|.$$

If $e_i > Tol$

Go to out put

End if

End for

Output: the results of approximate solutions and $e_i$.

Step 5: we see that $\sum_{i=0}^{n} p^i y^i(s) \rightarrow y(s)$ as $n$ increase.

4. Numerical Examples:

In this section, we illustrate the numerical method HPM through an example. From table comparison between the exact solution and numerical solution, illustrates the results with value of $s$ and different iterations $n$. These methods are useful for finding the numerical solutions of VFIE-2. The computations associated with the examples were performed using MALAB version 12.

**Example 1.** Find approximate solution of a VFIE-2, which is of the form

$$y(s) = 1 + s + \int_0^s y(t)\,dt - \int_0^s (st)y(t)\,dt$$

where the exact solutions $y(s) = e^s$.

Solution. Applying HPM and its Matlab program, obtained the approximate solutions of $y(s)$ as follows and shown in table 1.

$$y^0(s) = 1 + s, \quad y^1(s) = s + \frac{s^2}{2} - \frac{5s}{6}$$

$$y^2(s) = \frac{s^2}{12} + \frac{s^3}{6} - \frac{13s}{72}, \quad y^3(s) = \frac{13s}{2160} - \frac{13s^2}{144} + \frac{s^3}{36} + \frac{s^4}{24}$$

| $i$  | $s$ | Exact values | HPM       |
|------|-----|--------------|-----------|
|      |     | $y^i(s)$     | $e_i$     |
| 2    | 0.2 | 1.22140275   | 1.25333333 | 3.193 × 10^{-4} |
| 4    | 0.2 | 1.22140275   | 1.21977033 | 1.633 × 10^{-3} |
| 6    | 0.2 | 1.22140275   | 1.22148033 | 6.757 × 10^{-5}  |
| 8    | 0.2 | 1.22140275   | 1.22139951 | 3.244 × 10^{-6}  |
| 10   | 0.2 | 1.22140275   | 1.22140286 | 1.089 × 10^{-7}  |
| 12   | 0.2 | 1.22140275   | 1.22140275 | 1.866 × 10^{-8}  |
| 14   | 0.2 | 1.22140275   | 1.22140275 | 1.193 × 10^{-9}  |
| 16   | 0.2 | 1.22140275   | 1.22140275 | 1.727 × 10^{-10} |
| 18   | 0.2 | 1.22140275   | 1.22140275 | 2.214 × 10^{-11} |
| 20   | 0.2 | 1.22140275   | 1.22140275 | 9.931 × 10^{-12} |
5. Aitkens formula for iterative methods

In this section, we propose a new technique to solve VFIE-2 based on the previous iterative methods to accelerate the iterative numerical methods for faster convergence to any sequence that is linearly convergent. This convergence of the iterative methods can be improve with the help of the Aitkin method.

Theorem 2 (Convergence of Aitkens method)

Let \( \{y_i(s)\}_{i=0}^{\infty} \) be a sequence in the set real number that converges to the limit \( p \). Then the new sequence

\[
\hat{y}_i(s) = \frac{y_{i+2}y_i - y^2_{i+1}}{y_{i+2} - 2y_{i+1} + y_i}, \quad \text{for } i \geq 1
\]

converges to \( p \) faster than \( \{y_i(s)\}_{i=0}^{\infty} \) if

\[
y_{i+1} - p = (\gamma - \delta_i)(y_i - p)
\]

where \( |\gamma| < 1 \) and \( \lim_{i \to \infty} \delta_i = 0 \). This implies that \( \frac{\hat{y}_i - p}{y_i - p} \to 0 \) as \( i \to \infty \).

Proof: See [Kincaid, 2002]

Let \( \lim_{n \to \infty} \{y_i(s)\}_{i=0}^{n} \) be a linear sequence with convergence \( y_i(s) \to p \) and the error in the \( i^{th} \) sequence given by \( e_i = y_{i+1} - p \). The convergence to a small number \( a \) can be formulated as

\[
\lim_{i \to \infty} \left| \frac{e_{i+1}}{e_i} \right| = a \quad \text{for } 0 < a < 1.
\] (13)

To investigate the construction of the sequence \( \{y_i(s)\}_{i=0}^{\infty} \) for convergence to \( p \) where \( e_{i+1} = y_{i+1} - p \) let sufficiently large that the ratio can be used to approximate the limit. In addition, supposing they have the same sign, we get

\[
e_{i+1} = ae_i \quad \text{and} \quad e_{i+2} = ae_{i+1}
\]

The error in two successive approximations for linear convergence can be written as:

\[
e_{i+1} = y_{i+1} - p \quad \text{and} \quad e_{i+2} = y_{i+2} - p \text{ then}
\]

\[
y_{i+2} \approx e_{i+2} + p \approx ae_{i+1} + p
\]

\[
y_{i+2} \approx a(y_{i+1} - p) + p \quad \text{(14)}
\]

Replacing \( i+1 \) by \( i \) Equation (14) we get

\[
y_{i+1} \approx a(y_i - p) + p \quad \text{(15)}
\]

For finding value \( a \), subtracting Equation (15) from Equation (14), gives

\[
y_{i+2} - y_{i+1} \approx a(y_{i+1} - p) + p - [a(y_i - p) + p]
\]

\[
y_{i+2} - y_{i+1} \approx ay_{i+1} - ap + p - ay_i + ap - p
\]

\[
y_{i+2} - y_{i+1} \approx ay_{i+1} - ay_i
\]

\[
a = \frac{y_{i+2} - y_{i+1}}{y_{i+1} - y_i}
\] (16)
Substituting (14) into (12) yields
\[ y_{i+2} = \frac{(y_{i+2} - y_{i+1})}{y_{i+1} + y_i} (y_{i+1} - p) + p \]

\[ (y_{i+2} - y_{i+1})y_{i+1} - (y_{i+2} - y_{i+1})p + (y_{i+1} - y_i)p = y_{i+2}(y_{i+1} - y_i) \]

\[ y_{i+2}y_{i+1} - y_{i+2}p + y_{i+1}p + y_{i+1}p - y_i = y_{i+2}y_{i+1} - y_{i+2}y_i \]

\[ (-y_{i+2} + 2y_{i+1} - y_i)p = y_{i+2}^2 - y_{i+2}y_i \]

Hence,
\[ p = \frac{(y_{i+2}^2 - y_{i+2}y_i)}{-y_{i+2} + 2y_{i+1} - y_i} = \frac{(-y_{i+2}^2 - y_{i+2} + y_i)}{-y_{i+2}^2 - y_{i+1} + y_i} \]

\[ = \frac{y_{i+2}y_{i+1} - y_{i+2}^2}{y_{i+2} - 2y_{i+1} + y_i} \]

The computed value \( \hat{y}_{i+1}(s) = p \) for \( y_i(s) \) in the Aitken method is based on the assumption on the sequence \( \{y_i(s)\}_{i=0}^{\infty} \). We get
\[ \hat{y}_{i+1}(s) = \frac{y_{i+2}y_{i+1} - y_{i+2}^2}{y_{i+2} - 2y_{i+1} + y_i} \]

for \( i = 1, 2, ..., \text{max} \). This formula converges faster to \( p \) than \( \{y_i(s)\}_{i=0}^{\infty} \) where \( p \) is the exact value at \( s = s_0 \). Therefore, we can construct an iterative sequence \( \{\hat{y}_i(s)\}_{i=0}^{\infty} \).

6. Aitken Method on HPM for solving VFIE-2.

In this sub-section, we have used a new technique to accelerate the MHPM, which is AM on HPM for solving the VFIE-2. We have observed that the new technique is successful and better than HPM in the iterative process.

First, we find the sequence \( \{y_i(s)\}_{i=0}^{\infty} \) by using HPM then we apply Aitken method on HPM by using the formula in equation (6). We obtain new numerical solution \( \hat{y}_i(s) \) for \( i = 1, 2, 3, ..., \text{max} \).
\[ \hat{y}_i(s) = \frac{y_{i+2}y_{i+1} - y_{i+2}^2}{y_{i+2} - 2y_{i+1} + y_i} \]

for \( i = 0, 1, 2, ..., \text{max} \). This formula converges faster to \( p \) than \( \{y_i(s)\}_{i=0}^{\infty} \).

7. Numerical Example

We discuss numerical example for solving VFIE-2 by using AM on HPM.

Test Example 2. Solve example (1) for \( y(s) = e^s \) using AM on HPM.

Table 2- Comparison of results for \( y(s) = e^s \) at \( s = 0.2 \), \( y(0.2) = 1.221402758 \) with \( \hat{y}_i(s) \) and absolute error \( e_i = |\hat{y}_i(s) - y(s)| \).

| \( i \) | \( s \) | Exact values | AM on HPM |
|------|------|-------------|-----------|
| 2    | 0.2  | 1.22140275  | 1.23355176 | 1.214×10^{-2} |
| 4    | 0.2  | 1.22140275  | 1.22064324 | 7.521×10^{-4} |
| 6    | 0.2  | 1.22140275  | 1.22139476 | 2.804×10^{-5} |
| 8    | 0.2  | 1.22140275  | 1.22140156 | 1.193×10^{-6} |
| 10   | 0.2  | 1.22140275  | 1.22140285 | 5.731×10^{-8} |
| 12   | 0.2  | 1.22140275  | 1.22140275 | 3.049×10^{-9} |
| 14   | 0.2  | 1.22140275  | 1.22140275 | 1.798×10^{-11} |
| 16   | 0.2  | 1.22140275  | 1.22140275 | 1.207×10^{-13} |
| 18   | 0.2  | 1.22140275  | 1.22140275 | 2.005×10^{-15} |
| 20   | 0.2  | 1.22140275  | 1.22140275 | 9.931×10^{-17} |

Tables 3- Give a comparison between the exact solution and the approximated solution of examples (1) by using HPM and AM on HPM with two, four, six and eight iterations depending on the least square error and running times.
8. Conclusion

In this work, we propose two methods called HPM and AM on HPM for solving the problem. Several numerical examples were tested on applied algorithm of HPM and AM on HPM for solving SVFIE-2. From the results given in Tables 1, 2 and 3, indicate clearly that both methods achieve good convergence as a number of iteration increases when the error decreases. The mentioned examples demonstrated the validity and applicability of the techniques. Finally, we concluded that AM on HPM converges faster than HPM as shown in the above tables for the number of iterations, LSE and RT.

Acknowledgments

The authors would like to extend their sincere thanks to the referees for their helpful remarks and suggestions. Many thanks are due to the Salahaddin University and Universiti Teknologi Malaysia (UTM) for providing the financial support for this research.

References

ABDOU, M. 2012. Volterra- Fredholm integral equation of first kind and contact Problem. J Applied Mathematics, 25, 177-193.

CHNITI, C. and HOU, J. 2002. Numerical method for solving non Volterra integral equations with convolution kernel. International J of Applied Mathematics, 43, 43-48.

DASTJERDI, H. L. and MAALEK, F. M. 2012. Numerical solution of Volterra-Fredholm integral equations by moving least square method and Chebyshev polynomials, J Applied Mathematics and computation, 8, 69-84.

HENDI, F. A. 2010. Numerical solution for Volterra Fredholm integral equations of the second kind by using collection and Galerkin methods, J King Saud university. 22, 37-40.

HE, J. H. 2000. Homotopy perturbation technique, computer methods in applied mechanics and engineering, International J nonlinear mechanics,35, 37-43.

HIDFAR, A. and MOLAHRAMI, A. 2011. Solving a system of integral equations by an analytic method. J Mathematical and computer Modeling. 54, 828-835.

KINCAID, D. and CHEN, W. 2002. Numerical and Mathematics of scientific computing third edition. Thomson Learning. Brook/Cole.

MIRZAEI, F. and HOSEINI, A. 2013. Numerical solution of nonlinear Volterra-Fredholm integral equations using hybrid of block-pulse functions and Taylor series, Alexandria Engineering, 3, 551-555.

RAHMAN, M. 2007.Integral equations and the application. Printed in Great Britain Press Ltd.

SAADATI, R. and et al. 2009. T- Stability approach to variation iteration method for solving integral equations, J Hindawi publishing Corporation. 209, 1-9.

SULAIMAN, N. A. HASAN, T. I. and SALLAH S. 2014. Solving a System of Volterra-Fredholm integral equation of the second kind via series solution method, J Applied Mathematic Sciences. Theory and Application. 8, 2181-2194.

WAZ WAZ, A. M. 1997. A first course of integral equation. Scientific Publishing Co. Pte. Ltd.

WAZ WAZ, A. M. 2004. Linear and non linear linear integral equations methods and applications. Springer-Verlag Berlin Heidelberg.