On homogeneous second order linear general quantum difference equations

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Abstract
In this paper, we prove the existence and uniqueness of solutions of the $\beta$-Cauchy problem of second order $\beta$-difference equations

$$a_0(t)D^{2}_\beta y(t) + a_1(t)D_\beta y(t) + a_2(t)y(t) = b(t), \quad t \in I,$$

$a_0(t) \neq 0$, in a neighborhood of the unique fixed point $s_0$ of the strictly increasing continuous function $\beta$, defined on an interval $I \subseteq \mathbb{R}$. These equations are based on the general quantum difference operator $D_\beta$, which is defined by

$$D_\beta f(t) = \frac{t'\beta(t) - f(t)}{\beta(t) - t}, \quad \beta(t) \neq t.$$ 

We also construct a fundamental set of solutions for the second order linear homogeneous $\beta$-difference equations when the coefficients are constants and study the different cases of the roots of their characteristic equations. Finally, we drive the Euler-Cauchy $\beta$-difference equation.

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1 Introduction
Quantum calculus allows us to deal with sets of non-differentiable functions by substituting the classical derivative by a difference operator. Non-differentiable functions are used to describe many important physical phenomena. Quantum calculus has a lot of applications in different mathematical areas such as the calculus of variations, orthogonal polynomials, basic hyper-geometric functions, economical problems with a dynamic nature, quantum mechanics and the theory of scale relativity; see, e.g., [1–9]. The general quantum difference operator $D_\beta$ is defined, in [10, p.6], by

$$D_\beta f(t) = \begin{cases} \frac{f(\beta(t)) - f(t)}{\beta(t) - t}, & t \neq s_0, \\ f'(s_0), & t = s_0, \end{cases}$$

where $f : I \to \mathbb{X}$ is a function defined on an interval $I \subseteq \mathbb{R}$, $\mathbb{X}$ is a Banach space and $\beta : I \to I$ is a strictly increasing continuous function defined on $I$, which has only one fixed point $s_0 \in I$ and satisfies the inequality: $(t - s_0)(\beta(t) - t) \leq 0$ for all $t \in I$. The function $f$ is said to be $\beta$-differentiable on $I$, if the ordinary derivative $f'$ exists at $s_0$. The $\beta$-difference operator yields the Hahn difference operator when $\beta(t) = qt + \omega$, $\omega > 0$, $q \in (0,1)$, and the Jackson

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Assume that $f$ is $\beta$-differentiable two times over $I$, then the second order derivative of $f$ is denoted by $D^j_\beta f = D^j_\beta(D^i_\beta f)$. Furthermore, $S(y_0, b) = \{y \in \mathbb{K} : \|y - y_0\| \leq b\}$ and the rectangle $R = \{(t, y) \in I \times \mathbb{K} : |t - s_0| \leq a, \|y - y_0\| \leq b\}$, where $a, b$ are fixed positive real numbers.

2 Preliminaries

In this section, we present some needed results associated with the $\beta$-calculus from [10, 17–19].

Lemma 2.1 The following statements are true:

(i) The sequence of functions $\{\beta^k(t)\}_{k=0}^\infty$ converges uniformly to the constant function $\beta(0) := s_0$ on every compact interval $V \subseteq I$ containing $s_0$.

(ii) The series $\sum_{k=0}^\infty |\beta^k(t) - \beta^{k+1}(t)|$ is uniformly convergent to $|t - s_0|$ on every compact interval $V \subseteq I$ containing $s_0$.

Lemma 2.2 If $f : I \rightarrow \mathbb{K}$ is a continuous function at $s_0$, then the sequence $\{f(\beta^k(t))\}_{k=0}^\infty$ converges uniformly to $f(s_0)$ on every compact interval $V \subseteq I$ containing $s_0$.

Theorem 2.3 If $f : I \rightarrow \mathbb{K}$ is continuous at $s_0$, then the series $\sum_{k=0}^\infty \|(\beta^k(t) - \beta^{k+1}(t)) \times f(\beta^k(t))\|$ is uniformly convergent on every compact interval $V \subseteq I$ containing $s_0$.

Lemma 2.4 Let $f : I \rightarrow \mathbb{K}$ be $\beta$-differentiable and $D^j_\beta f(t) = 0$ for all $t \in I$. Then $f(t) = f(s_0)$ for all $t \in I$.

Theorem 2.5 Assume that $f : I \rightarrow \mathbb{K}$ and $g : I \rightarrow \mathbb{R}$ are $\beta$-differentiable functions on $I$. Then:

(i) the product $fg : I \rightarrow \mathbb{K}$ is $\beta$-differentiable on $I$ and

$$D^j_\beta (fg)(t) = (D^j_\beta f(t))g(t) + f(\beta(t))D^j_\beta g(t)$$

$$= (D^j_\beta f(t))g(\beta(t)) + f(t)D^j_\beta g(t),$$
(ii) \( f/g \) is \( \beta \)-differentiable at \( t \) and
\[
D_\beta(f/g)(t) = \frac{(D_\beta f(t))g(t) - f(t)D_\beta g(t)}{g(t)g(\beta(t))},
\]
provided that \( g(t)g(\beta(t)) \neq 0 \).

**Theorem 2.6** Assume \( f : I \to \mathbb{R} \) is continuous at \( s_0 \). The function \( F \) defined by
\[
F(t) = \sum_{k=0}^{\infty} (\beta^k(t) - \beta^{k+1}(t))f(\beta^k(t)), \quad t \in I
\]
(2.1)
is a \( \beta \)-antiderivative of \( f \) with \( F(s_0) = 0 \). Conversely, a \( \beta \)-antiderivative \( F \) of \( f \) vanishing at \( s_0 \) is given by (2.1).

**Definition 2.7** Let \( f : I \to \mathbb{R} \) and \( a, b \in I \). The \( \beta \)-integral of \( f \) from \( a \) to \( b \) is
\[
\int_a^b f(t) \, d_\beta t = \int_a^b f(t) \, d_\beta t - \int_a^s f(t) \, d_\beta t,
\]
where
\[
\int_a^s f(t) \, d_\beta t = \sum_{k=0}^{\infty} (\beta^k(x) - \beta^{k+1}(x))f(\beta^k(x)), \quad x \in I,
\]
presented that the series converges at \( x = a \) and \( x = b \). \( f \) is called \( \beta \)-integrable on \( I \) if the series converges at \( a \) and \( b \) for all \( a, b \in I \). Clearly, if \( f \) is continuous at \( s_0 \in I \), then \( f \) is \( \beta \)-integrable on \( I \).

**Definition 2.8** The \( \beta \)-exponential functions \( e_{p,\beta}(t) \) and \( E_{p,\beta}(t) \) are defined by
\[
e_{p,\beta}(t) = \prod_{k=0}^{\infty} \frac{1}{1 - p(\beta^k(t))(\beta^k(t) - \beta^{k+1}(t))}, \quad (2.2)
\]
and
\[
E_{p,\beta}(t) = \prod_{k=0}^{\infty} \left[ 1 + p(\beta^k(t))(\beta^k(t) - \beta^{k+1}(t)) \right], \quad (2.3)
\]
where \( p : I \to \mathbb{C} \) is a continuous function at \( s_0 \) and both infinite products are convergent to a non-zero number for every \( t \in I \) and \( e_{p,\beta}(t) = \frac{1}{E_{p,\beta(t)}} \).

It is worth mentioning that both products in (2.2) and (2.3) are convergent since \( \sum_{k=0}^{\infty} |p(\beta^k(t))(\beta^k(t) - \beta^{k+1}(t))| \) is uniformly convergent. See [18, Definition 2.1].

**Theorem 2.9** The \( \beta \)-exponential functions \( e_{p,\beta}(t) \) and \( E_{p,\beta}(t) \) are, respectively, the unique solutions of the \( \beta \)-initial value problems:
\[
D_\beta y(t) = p(t)y(t), \quad y(s_0) = 1,
\]
\[
D_\beta y(t) = -p(t)y(\beta(t)), \quad y(s_0) = 1.
\]
Theorem 2.10 Assume that \( p, q : I \rightarrow \mathbb{C} \) are continuous functions at \( s_0 \in I \). The following properties are true:

(i) \( \frac{1}{e_p(t)} = e_{-p}(1 + \beta(t) - \beta(t)) \),
(ii) \( e_{p,q}(t)e_{q,p}(t) = e_{p+q}(1 + \beta(t) - \beta(t)) \),
(iii) \( e_{p,q}(t)/e_{q,p}(t) = e_{p-q}(1 + \beta(t) - \beta(t)) \).

Definition 2.11 The \( \beta \)-trigonometric functions are defined by

\[
\cos_{p,\beta}(t) = \frac{e_{ip,\beta}(t) + e_{-ip,\beta}(t)}{2},
\sin_{p,\beta}(t) = \frac{e_{ip,\beta}(t) - e_{-ip,\beta}(t)}{2i}.
\]

Theorem 2.12 For all \( t \in I \). The following relation holds true:

\[ e_{p,\beta}(t) = \cos_{p,\beta}(t) + i \sin_{p,\beta}(t). \]

Theorem 2.13 Assume that the function \( f : R \rightarrow \mathbb{X} \) is continuous at \( (s_0, y_0) \in R \) and satisfies the Lipschitz condition (with respect to \( y \))

\[
\| f(t, y_1) - f(t, y_2) \| \leq L \| y_1 - y_2 \|, \quad \text{for all } (t, y_1), (t, y_2) \in R.
\]

Then the \( \beta \)-initial value problem \( D_{\beta}y(t) = f(t, y), \ y(s_0) = y_0, \ t \in I \) has a unique solution on \([s_0 - \delta, s_0 + \delta]\), where \( L \) is a positive constant and \( \delta = \min\{a, \frac{b}{1+|\beta|}, \frac{\rho}{1} \} \) with \( M = \sup_{(t, y) \in R} \| f(t, y) \| < \infty, \ \rho \in (0, 1) \).

3 Main results

In this section, we prove the existence and uniqueness of solutions of the \( \beta \)-Cauchy problem of second order \( \beta \)-difference equations in a neighborhood of \( s_0 \). Furthermore, we construct a fundamental set of solutions for the second order linear homogeneous \( \beta \)-difference equations when the coefficients are constants and study the different cases of the roots of their characteristic equations. Finally, we derive the Euler-Cauchy \( \beta \)-difference equation.

3.1 Existence and uniqueness of solutions

Theorem 3.1 Let \( f_i(t, y_1, y_2) : I \times \prod_{i=1}^{2} S_i(x_i, b_i) \rightarrow \mathbb{X}, \ s_0 \in I, \) such that the following conditions are satisfied:

(i) for \( y_i \in S_i(x_i, b_i), \ i = 1, 2, f_i(t, y_1, y_2) \) are continuous at \( t = s_0 \),
(ii) there is a positive constant \( A \) such that, for \( t \in I, y_i, \tilde{y}_i \in S_i(x_i, b_i), \ i = 1, 2, \) the following Lipschitz condition is satisfied:

\[
\| f_i(t, y_1, y_2) - f_i(t, \tilde{y}_1, \tilde{y}_2) \| \leq A \sum_{i=1}^{2} \| y_i - \tilde{y}_i \|.
\]

Then there exists a unique solution of the \( \beta \)-initial value problem, \( \beta \)-IVP,

\[
D_{\beta}y(t) = f_i(t, y_1(t), y_2(t)), \quad y_i(s_0) = x_i, \ i = 1, 2, \ t \in I.
\]
Proof Let $y_0 = (x_1, x_2)^T$ and $b = (b_1, b_2)^T$, where $(\cdot, \cdot)^T$ stands for vector transpose. Define the function $f : I \times \prod_{i=1}^{2} S_i(x_i, b_i) \rightarrow \mathbb{R} \times \mathbb{R}$ by $f(t, y_1, y_2) = (f_1(t, y_1, y_2), f_2(t, y_1, y_2))^T$. It is easy to show that system (3.1) is equivalent to the $\beta$-IVP

$$D_\beta y(t) = f(t, y(t)), \quad y(s_0) = y_0. \quad (3.2)$$

Since each $f_i$ is continuous at $t = s_0$, $f$ is continuous at $t = s_0$. The function $f$ satisfies the Lipschitz condition because for $y, \tilde{y} \in \prod_{i=1}^{2} S_i(x_i, b_i)$,

$$\|f(t, y) - f(t, \tilde{y})\| = \|f(t, y_1, y_2) - f(t, \tilde{y}_1, \tilde{y}_2)\| \leq A \sum_{i=1}^{2} \|y_i - \tilde{y}_i\| = A \|y - \tilde{y}\|.$$ 

Applying Theorem 2.13, see the proof in [19], there exists $\delta > 0$ such that (3.2) has a unique solution on $[s_0, s_0 + \delta]$. Hence, the $\beta$-IVP (3.1) has a unique solution on $[s_0, s_0 + \delta]$. □

Corollary 3.2 Let $f(t, y_1, y_2)$ be a function defined on $I \times \prod_{i=1}^{2} S_i(x_i, b_i)$ such that the following conditions are satisfied:

(i) for any values of $y_i \in S_i(x_i, b_i), i = 1, 2$, $f$ is continuous at $t = s_0$,

(ii) $f$ satisfies the Lipschitz condition

$$\|f(t, y_1, y_2) - f(t, \tilde{y}_1, \tilde{y}_2)\| \leq A \sum_{i=1}^{2} \|y_i - \tilde{y}_i\|,$$

where $A > 0, y_i, \tilde{y}_i \in S_i(x_i, b_i), i = 1, 2$ and $t \in I$. Then

$$D_\beta^2 y(t) = f(t, y(t), D_\beta y(t)), \quad D_\beta^{i-1} y(s_0) = x_i, i = 1, 2 \quad (3.3)$$

has a unique solution on $[s_0, s_0 + \delta]$. Proof Consider equation (3.3). It is equivalent to (3.1), where $(\phi_i(t))_{i=1}^{2}$ is a solution of (3.1) if and only if $\phi_i(t)$ is a solution of (3.3). Here,

$$f_i(t, y_1, y_2) = \begin{cases} y_2, & i = 1, \\ f(t, y_1, y_2), & i = 2. \end{cases}$$

Hence, by Theorem 3.1, there exists $\delta > 0$ such that system (3.1) has a unique solution on $[s_0, s_0 + \delta]$. □

The following corollary gives us the sufficient conditions for the existence and uniqueness of the solutions of the $\beta$-Cauchy problem (3.3).

Corollary 3.3 Assume the functions $a_j(t) : I \rightarrow \mathbb{C}, j = 0, 1, 2$, and $b(t) : I \rightarrow \mathbb{R}$ satisfy the following conditions:
(i) \( a_j(t), j = 0, 1, 2 \) and \( b(t) \) are continuous at \( s_0 \) with \( a_0(t) \neq 0 \) for all \( t \in I \),
(ii) \( a_j(t)/a_0(t) \) is bounded on \( I, j = 1, 2 \). Then

\[
a_0(t)D_\beta^2y(t) + a_1(t)D_\beta y(t) + a_2(t)y(t) = b(t),
\]

\[
D_\beta^{s_0} y = x_i, \quad x_i \in \mathbb{X}, i = 1, 2,
\]

has a unique solution on subinterval \( I \subseteq I, s_0 \in I \).

Proof Dividing by \( a_0(t) \), we get

\[
D_\beta^2y(t) = A_1(t)D_\beta y(t) + A_2(t)y(t) + B(t),
\]

where \( A_1(t) = -a_1(t)/a_0(t) \) and \( B(t) = b(t)/a_0(t) \). Since \( A_j(t) \) and \( B(t) \) are continuous at \( t = s_0 \), the function \( f(t, y_1, y_2) \), defined by

\[
f(t, y_1, y_2) = A_1(t)y_2 + A_2(t)y_1 + B(t),
\]

is continuous at \( t = s_0 \). Furthermore, \( A_1(t) \) is bounded on \( I \). Consequently, there is \( A > 0 \) such that \( |A_1(t)| \leq A \) for all \( t \in I \). We can see that \( f \) satisfies the Lipschitz condition with Lipschitz constant \( A \). Thus, \( f(t, y_1, y_2) \) satisfies the conditions of Corollary 3.2. Hence, there exists a unique solution of (3.5) on \( I \).

\[\blacksquare\]

### 3.2 Fundamental solutions of linear homogeneous \( \beta \)-difference equations

The second order homogeneous linear \( \beta \)-difference equation has the form

\[
a_0(t)D_\beta^2y(t) + a_1(t)D_\beta y(t) + a_2(t)y(t) = 0, \quad t \in I,
\]

where the coefficients \( a_0(t) \neq 0, a_j(t), j = 1, 2 \) are assumed to satisfy the conditions of Corollary 3.3.

**Lemma 3.4** If the function \( y \) is a solution of the homogeneous equation (3.6), such that \( y(s_0) = 0 \) and \( D_\beta y(s_0) = 0, s_0 \in I, \) then \( y(t) = 0, \) for all \( t \in I \).

Proof By Corollary 3.3, if \( x_i = 0, i = 1, 2 \) in the \( \beta \)-IVP (3.4), which has a unique solution on \( I \), then \( y(t) = 0 \) for all \( t \in I \) is a unique solution of the \( \beta \)-difference equation (3.6), which satisfies the given initial conditions \( y(s_0) = 0, D_\beta y(s_0) = 0 \). Hence we have the desired result.

\[\blacksquare\]

**Theorem 3.5** The linear combination \( c_1y_1 + c_2y_2 \) of any two solutions \( y_1 \) and \( y_2 \) of the homogeneous linear \( \beta \)-difference equation (3.6) is also a solution of it in \( I \), where \( c_1 \) and \( c_2 \) are arbitrary constants.

Proof The proof is straightforward.

\[\blacksquare\]

**Theorem 3.6** Let \( y_1 \) and \( y_2 \) be any two linearly independent solutions of the \( \beta \)-difference equation (3.6) in \( I \). Then every solution \( y \) of (3.6) can be expressed as a linear combination \( y = c_1y_1 + c_2y_2 \).
Proof Let
\[ \phi = \begin{pmatrix} y \\ D_\beta y \end{pmatrix}, \quad \phi_1 = \begin{pmatrix} y_1 \\ D_\beta y_1 \end{pmatrix}, \quad \phi_2 = \begin{pmatrix} y_2 \\ D_\beta y_2 \end{pmatrix}, \]
be the solutions of the linear system \( D_\beta y_i(t) = a_i(t)y_i(t), \ i = 1, 2, \) corresponding, respectively, to the solutions \( y_1, y_2 \) of homogeneous linear \( \beta \)-difference equation (3.6). Since \( y_1, y_2 \) are linearly independent in \( J \), then \( \phi_1, \phi_2 \) are linearly independent in \( J \). Then there exist two constants \( c_1, c_2 \) such that \( \phi = c_1\phi_1 + c_2\phi_2 \). The first component of this is \( y = c_1y_1 + c_2y_2 \). Thus the results hold.

\[ \square \]

**Definition 3.7** A set of two linearly independent solutions of the second order homogeneous linear \( \beta \)-difference equation (3.6) is called a fundamental set of it.

**Theorem 3.8** There exists a fundamental set of solutions of the second order homogeneous linear \( \beta \)-difference equation (3.6).

**Proof** By Corollary 3.3, there exist unique solutions \( y_1 \) and \( y_2 \) of equation (3.6), such that \( y_1(s_0) = 1, D_\beta y_1(s_0) = 0 \) and \( y_2(s_0) = 0, D_\beta y_2(s_0) = 1 \).

Suppose that \( y_1 \) and \( y_2 \) are linear dependent, so there exist constants \( c_1 \) and \( c_2 \) not both zero, such that
\[ c_1y_1(t) + c_2y_2(t) = 0, \quad \text{for all } t \in J, \]
\[ c_1D_\beta y_1(t) + c_2D_\beta y_2(t) = 0, \quad \text{for all } t \in J. \]

We have \( c_1 = c_2 = 0 \) at \( t = s_0 \), which is a contradiction. Thus the solutions \( y_1 \) and \( y_2 \) are linearly independent in \( J \). Then there exists a fundamental set of the two solutions \( y_1 \) and \( y_2 \) of equation (3.6). \( \square \)

**Definition 3.9** Let \( y_1, y_2 \) be \( \beta \)-differentiable functions. Then we define the \( \beta \)-Wronskian of the functions \( y_1, y_2 \), defined on \( I \), by
\[ W_\beta(y_1, y_2)(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ D_\beta y_1(t) & D_\beta y_2(t) \end{vmatrix}, \quad t \in I. \]

**Lemma 3.10** Let \( y_1(t), y_2(t) \) be functions defined on \( I \). Then, for any \( t \in I, t \neq s_0 \),
\[ D_\beta W_\beta(y_1, y_2)(t) = \begin{vmatrix} y_1(\beta(t)) & y_2(\beta(t)) \\ D_\beta^2 y_1(t) & D_\beta^2 y_2(t) \end{vmatrix}. \] (3.7)

**Proof** Since \( W_\beta(y_1, y_2)(t) = y_1(t)D_\beta y_2(t) - y_2(t)D_\beta y_1(t) \), then
\[ D_\beta W_\beta(y_1, y_2)(t) = y_1(\beta(t))D_\beta^2 y_2(t) - y_2(\beta(t))D_\beta^2 y_1(t), \]
which is the desired result. \( \square \)
**Theorem 3.11** Assume that $y_1(t)$ and $y_2(t)$ are two solutions of equation (3.6). Then their $\beta$-Wronskian, $W_\beta$,

\[
W_\beta(y_1, y_2)(t) = e^{-\alpha_1(t) \beta(t) - \alpha_2(t) \beta(t)} W_\beta(y_1, y_2)(s_0), \quad t \in I,
\]

where $r_1(t) = \frac{\alpha_1(t)}{a_0(t)}$ and $r_2(t) = \frac{\alpha_2(t)}{a_0(t)}$ satisfy the conditions of Corollary 3.3.

**Proof** Since $y_1$ and $y_2$ are solutions of equation (3.6), from (3.7) we have

\[
D_\beta W_\beta(y_1, y_2)(t) = \begin{vmatrix}
\frac{y_1(\beta(t))}{\alpha_0(t)} & \frac{y_2(\beta(t))}{\alpha_0(t)} \\
\frac{y_1(\beta(t))}{\alpha_0(t)} & \frac{y_2(\beta(t))}{\alpha_0(t)}
\end{vmatrix}
+ \begin{vmatrix}
\frac{y_1(\beta(t))}{\alpha_0(t)} & \frac{y_2(\beta(t))}{\alpha_0(t)} \\
\frac{y_1(\beta(t))}{\alpha_0(t)} & \frac{y_2(\beta(t))}{\alpha_0(t)}
\end{vmatrix}
\beta(t) - t
\begin{vmatrix}
y_1(t) & y_2(t) \\
y_1(t) & y_2(t)
\end{vmatrix}
\]

\[
= \frac{\alpha_1(t)}{a_0(t)} \begin{vmatrix}
y_1(t) & y_2(t) \\
D_\beta y_1(t) & D_\beta y_2(t)
\end{vmatrix}
+ \frac{\alpha_2(t)}{a_0(t)} \begin{vmatrix}
y_1(t) & y_2(t) \\
D_\beta y_1(t) & D_\beta y_2(t)
\end{vmatrix}
\]

\[
= \left[-r_1(t) + r_2(t) (\beta(t) - t)\right] W_\beta(y_1, y_2)(t),
\]

which has the solution

\[
W_\beta(y_1, y_2)(t) = W_\beta(y_1, y_2)(s_0) e^{-r_1(t) + r_2(t) (\beta(t) - t)}, \quad t \in I.
\]

Using Theorem 3.11 and Lemma 3.4, we can prove the following corollaries.

**Corollary 3.12** Two solutions $y_1$ and $y_2$ of $\beta$-difference equation (3.6) are linearly dependent in $I$ if and only if $W_\beta(y_1, y_2)(t) = 0$, for all $t \in I$.

**Corollary 3.13** The value of $W_\beta(y_1, y_2)(t)$ of $\beta$-difference equation (3.6) either is zero or unequal to zero for all $t \in I$.

### 3.3 Homogeneous equations with constant coefficients

Equation (3.6) can be written as

\[
Ly(t) = aD_\beta^2 y(t) + bD_\beta y(t) + cy(t) = 0,
\]

where $a$, $b$, and $c$ are constants. The characteristic polynomial of equation (3.8) is

\[
P(\lambda) = a\lambda^2 + b\lambda + c = 0,
\]

where $y(t) = e_{\lambda, \beta}(t)$ is a solution of equation (3.8). Since equation (3.9) is a quadratic equation with real coefficients, it has two roots, which may be real and different, real but repeated, or complex conjugates.

**Case 1: real and different roots of the characteristic equation (3.9).**

Let $\lambda_1$ and $\lambda_2$ be real roots with $\lambda_1 \neq \lambda_2$, then $y_1(t) = e_{\lambda_1, \beta}(t)$ and $y_2(t) = e_{\lambda_2, \beta}(t)$ are two solutions of equation (3.8). Therefore,

\[
y(t) = c_1 e_{\lambda_1, \beta}(t) + c_2 e_{\lambda_2, \beta}(t)
\]
is a general solution of equation (3.8), with
\[ c_1 = \frac{D\beta y_0 - y_0\lambda_2}{\lambda_1 - \lambda_2} e^{-\lambda_1,\beta}(s_0) \quad \text{and} \quad c_2 = \frac{y_0\lambda_1 - D\beta y_0}{\lambda_1 - \lambda_2} e^{-\lambda_2,\beta}(s_0). \]

**Example 3.14** Find the solution of the \( \beta \)-initial value problem
\[ D^2\beta y(t) + 5D\beta y(t) + 6y(t) = 0, \quad y(s_0) = 2, D\beta y(s_0) = 3. \]

By assuming that \( y(t) = e_{\lambda,\beta}(t) \), we obtain the solution
\[ y(t) = 9e_{-2,\beta}(t) - 7e_{-3,\beta}(t). \]

**Case 2: complex roots of the characteristic equation (3.9).**
Let \( \lambda_1 = \nu + i\mu \) and \( \lambda_2 = \nu - i\mu \), where \( \nu \) and \( \mu \) are real numbers. Then \( y_1(t) = e_{(\nu+i\mu),\beta}(t) \) and \( y_2(t) = e_{(\nu-i\mu),\beta}(t) \) are two solutions of equation (3.8). By Theorems 2.10, 2.12, \( e_{(\nu+i\mu),\beta}(t) = e_{\nu,\beta}(t)e^{\frac{i\mu}{\Gamma(\nu/\beta)-\beta}(t)} \). So,
\[ e_{(\nu+i\mu),\beta}(t) = e_{\nu,\beta}(t)\left(\cos \frac{\mu}{\Gamma(\nu/\beta)-\beta} \beta(t) + i \sin \frac{\mu}{\Gamma(\nu/\beta)-\beta} \beta(t)\right). \]

We have
\[ y_1(t) + y_2(t) = 2e_{\nu,\beta}(t)\cos \frac{\mu}{\Gamma(\nu/\beta)-\beta} \beta(t) \]
and
\[ y_1(t) - y_2(t) = 2ie_{\nu,\beta}(t)\sin \frac{\mu}{\Gamma(\nu/\beta)-\beta} \beta(t). \]

Therefore,
\[ u(t) = e_{\nu,\beta}(t)\cos \frac{\mu}{\Gamma(\nu/\beta)-\beta} \beta(t) \quad \text{and} \quad v(t) = e_{\nu,\beta}(t)\sin \frac{\mu}{\Gamma(\nu/\beta)-\beta} \beta(t) \]
are two solutions of equation (3.8). If the \( \beta \)-Wronskian of \( u \) and \( v \) is not zero, then \( u \) and \( v \) form a fundamental set of solutions. The general solution of equation (3.8) is
\[ y(t) = c_1 e_{\nu,\beta}(t)\cos \frac{\mu}{\Gamma(\nu/\beta)-\beta} \beta(t) + c_2 e_{\nu,\beta}(t)\sin \frac{\mu}{\Gamma(\nu/\beta)-\beta} \beta(t), \]
where \( c_1 \) and \( c_2 \) are arbitrary constants.

**Example 3.15** Find the general solution of
\[ D^2\beta y(t) + D\beta y(t) + y(t) = 0. \quad (3.10) \]

The characteristic equation is \( \lambda^2 + \lambda + 1 = 0 \), and its roots are
\[ \lambda_{1,2} = \frac{-1}{2} \pm i \frac{\sqrt{3}}{2}. \]
Thus, the general solution of equation (3.10) is
\[ y(t) = c_1 e^{t \lambda_1 / t} + c_2 e^{t \lambda_2 / t}. \]

**Case 3: repeated roots.**
Consider the case that the two roots \( \lambda_1 \) and \( \lambda_2 \) are equal, so \( \lambda_1 = \lambda_2 = -b/2a. \)

Therefore, the solution \( y_1(t) = e^{t \lambda_1 / t} \) is one solution of the \( \beta \)-difference equation (3.8), and we give the second solution by the following example:

**Example 3.16** Solve the \( \beta \)-difference equation
\[ D_\beta^2 y(t) + 4D_\beta y(t) + 4y(t) = 0. \] (3.11)

The characteristic equation is \((\lambda + 2)^2 = 0\), so \( \lambda_1 = \lambda_2 = -2\). Therefore, \( y_1(t) = e^{-2 \beta(t)} \) is a solution of equation (3.11). To find the second solution, let \( y(t) = v(t)e^{-2 \beta(t)} \). Then \( D_\beta^2 v(t) = 0 \).

Therefore, \( v(t) = c_1 t + c_2 \), where \( c_1 \) and \( c_2 \) are arbitrary constants. Then the general solution is
\[ y(t) = c_1 te^{-2 \beta(t)} + c_2 e^{-2 \beta(t)}, \]
where the two solutions \( y_1(t) = e^{-2 \beta(t)} \) and \( y_2(t) = te^{-2 \beta(t)} \) form a fundamental set of solutions of equation (3.11).

### 3.4 Euler-Cauchy \( \beta \)-difference equation
The Euler-Cauchy \( \beta \)-difference equation takes the form
\[ t\beta(t)D_\beta^2 y(t) + atD_\beta y(t) + by(t) = 0, \quad t \in I, t \neq s_0, \] (3.12)
where \( a, b \) are constants. The characteristic equation of (3.12) is given by
\[ \lambda^2 + (a - 1)\lambda + b = 0. \] (3.13)

**Theorem 3.17** If the characteristic equation (3.13) has two distinct roots \( \lambda_1 \) and \( \lambda_2 \), then a fundamental set of solutions of (3.12) is given by \( e_{\lambda_1 I, \beta}(t) \) and \( e_{\lambda_2 I, \beta}(t) \).

**Proof** Let \( y(t) = e_{\lambda I, \beta}(t) \), where \( \lambda \) is a root of equation (3.13). It follows that
\[ D_\beta y(t) = \frac{\lambda}{t} y(t), \quad D_\beta^2 y(t) = \frac{\lambda^2 - \lambda}{t \beta(t)} y(t). \]

Consequently, we have
\[ t\beta(t)D_\beta^2 y(t) + atD_\beta y(t) + by(t) = (\lambda^2 + (a - 1)\lambda + b)y(t) = 0. \]
Assume that $\lambda_1$ and $\lambda_2$ are distinct roots of the characteristic equation (3.13). Then, we have

$$\lambda_1 + \lambda_2 = 1 - a, \quad \lambda_1 \lambda_2 = b.$$  

Moreover, $W_\beta(e_{\lambda_1(t/\beta), \beta}, e_{\lambda_2(t/\beta), \beta})(t) \neq 0$, since $\lambda_1 \neq \lambda_2$. Hence, $e_{\lambda_1(t/\beta), \beta}(t)$ and $e_{\lambda_2(t/\beta), \beta}(t)$ form a fundamental set of solutions of (3.12).

The following theorem gives us the general solution of the Euler-Cauchy $\beta$-difference equation in the double root case.

**Theorem 3.18** Assume that $1/\beta(t)$ is bounded on $I$ and $0 \notin I$. Then the general solution of the Euler-Cauchy $\beta$-difference equation

$$t\beta(t)D^2_\beta y(t) + (1 - 2\gamma)tD_\beta y(t) + \gamma^2 y(t) = 0, \quad t \in I,$$

is given by

$$y(t) = c_1 e_{\gamma/\beta}(t) + c_2 e_{\gamma/\beta}(t) \int_s^t \frac{e^{-r_1(\tau)}}{1 + \frac{\gamma}{\beta}(\beta(t) - \tau)} \, d_\beta \tau.$$  

**Proof** The characteristic equation of (3.14) is

$$\lambda^2 - 2\gamma \lambda + \gamma^2 = 0.$$  

Then the characteristic roots are $\lambda_1 = \lambda_2 = \gamma$. Hence one linearly independent solution of equation (3.14) is $y_1(t) = e_{\gamma/\beta}(t)$. To obtain the second linearly independent solution, we can rewrite equation (3.14) in the form

$$D^2_\beta y(t) + r_1(t)D_\beta y(t) + r_2(t)y(t) = 0,$$  

(3.15)

where $r_1(t) = \frac{1 - 2\gamma}{\beta(t)}$ and $r_2(t) = \frac{\gamma^2}{\beta(t)}$. Consequently,

$$-r_1(t) + r_2(t)\left(\beta(t) - t\right) = \frac{\gamma^2}{t} - \frac{(\gamma - 1)^2}{\beta(t)}.$$  

Let $u$ be a solution of equation (3.15) such that $u(s_0) = 0$, $D_\beta u(s_0) = 1$. Then

$$W_\beta(e_{\gamma/\beta}, u)(t) = e_{-r_1(t) + r_2(t)(\beta(t) - t)}(t) = e_{\gamma^2/\beta(t)} - \frac{(\gamma - 1)^2}{\beta(t)}.$$

By Theorem 2.5, we find that $u$ satisfies the following $\beta$-difference equation:

$$D_\beta\left(\frac{u}{e_{\gamma/\beta}(t)}\right)(t) = \frac{W_\beta(e_{\gamma/\beta}, u)(t)}{e_{\gamma/\beta}(t)e_{\gamma^2/\beta(t)}(\beta(t))}$$

$$= \frac{e_{\gamma^2/\beta(t)} - \frac{(\gamma - 1)^2}{\beta(t)}}{e_{\gamma/\beta}(t)(1 + \frac{\gamma}{\beta}(\beta(t) - t)).}$$
Then
\[ u(t) = e_{\gamma}^2(t) \int_{s_0}^{t} e_{\gamma}^{-1} \left( \frac{\gamma - 1}{\beta(\tau)} \right) \frac{e_{\gamma}^{-1} \left( \frac{\gamma - 1}{\beta(\tau)} \right) (1 + \frac{\gamma - 1}{\beta(\tau)} d_\beta \tau). \]

Also,
\[ \frac{e_{\gamma}^{-1} \left( \frac{\gamma - 1}{\beta(\tau)} \right) (1 + \frac{\gamma - 1}{\beta(\tau)} d_\beta \tau) \right). \]

Therefore,
\[ y(t) = c_1 e_{\gamma}^2(t) + c_2 e_{\gamma}^{-1}(t) \int_{s_0}^{t} \frac{e_{\gamma}^{-1}(\beta(\tau)) \left( \frac{\gamma - 1}{\beta(\tau)} \right) (1 + \frac{\gamma - 1}{\beta(\tau)} d_\beta \tau) \right). \]

is the general solution of equation (3.14). □

4 Conclusion
In this paper, the existence and uniqueness of solutions of the \( \beta \)-Cauchy problem of second order \( \beta \)-difference equations were proved. Moreover, a fundamental set of solutions for second order linear homogeneous \( \beta \)-difference equations when the coefficients are constants was constructed. Also, the different cases of the roots of the characteristic equations of these equations were studied. Finally, the Euler-Cauchy \( \beta \)-difference equation was derived.

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Authors’ contributions
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