Direct and inverse spectral problems for a class of non-selfadjoint band matrices

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Abstract

The spectral properties of a class of band matrices are investigated. The reconstruction of matrices of this special class from given spectral data is also studied. Necessary and sufficient conditions for that reconstruction are found. The obtained results extend some results on the direct and inverse spectral problems for periodic Jacobi matrices and for some non-self-adjoint tridiagonal matrices.

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1 Introduction

Inverse eigenvalue problems arise in mathematics as well as in many areas of engineering and science such as chemistry, geology, physics etc. Often the mathematical model describing a certain physical system involves matrices whose spectral data allow the prediction of the behavior of the system. Determining the spectra of those matrices is the so-called direct problem, while the inverse problem consists in the reconstruction of the matrices from the knowledge of the behavior of the system, frequently expressed by spectral data.

Inverse eigenvalue problems, in general, and for structured matrices, in particular, have attracted attention of many researchers, some of them motivated by the numerous applications of this scientific area (see e.g. [1], [16]). The mathematical background employed in those investigations may involve rather sophisticated techniques such as algebraic curves, functional analysis, matrix theory, etc. (see [25], [1], [11], [2], [25] and the references therein).

Inverse eigenvalue problems for band matrices have been actively investigated, e.g. see [10] and their references. The inverse spectral problem for a periodic Jacobi matrix, that is, a real symmetric matrix of the form

\[
L_n = \begin{pmatrix}
a_1 & b_1 & 0 & \ldots & 0 & b_n \\
b_1 & a_2 & b_2 & \ldots & 0 & 0 \\
0 & b_2 & a_3 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & a_{n-1} & b_{n-1} \\
b_n & 0 & 0 & \ldots & b_{n-1} & a_n
\end{pmatrix}, \quad b_i > 0, \quad i = 1, \ldots, n
\]

(1.1)

deserved the attention of researchers, see [11], [13], [2], [29] and their references. These matrices appear in studies of the periodic Toda lattice, inverse eigenvalue problems for Sturm-Liouville equations and Hill’s equation [10], [13]. If \(b_n = 0\), the the matrices \(L_n\) of the form (1.1) reduce to tridiagonal symmetric matrices called the Jacobi matrices. The Jacobi matrices motivated intensive study as an useful tool in the investigation of orthogonal polynomials, in the theory of continued fractions, and in numerical analysis [27], [9], [8]. Namely, the inverse problems for Jacobi matrices have been an intensive topic of research since the seminal papers by Hochstadt and Hald [20], [19] in the seventies of the last century.
In the present work, we study spectral properties of complex matrices of the form

\[
J_n = \begin{pmatrix}
 c_1 & b_1 & 0 & \cdots & 0 & \bar{b}_n \\
 \bar{c}_1 & c_2 & b_2 & \cdots & 0 & 0 \\
 0 & \bar{b}_2 & c_3 & \cdots & 0 & 0 \\
 \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
 0 & 0 & 0 & \cdots & c_{n-1} & b_{n-1} \\
 b_n & 0 & 0 & \cdots & \bar{b}_{n-1} & a_n
\end{pmatrix},
\]

(1.2)

where \( b_1, \ldots, b_{n-1}, b_n \in \mathbb{C} \setminus \mathbb{R}, c_1, \ldots, c_{n-1} \in \mathbb{R}, \) and \( a_n \in \mathbb{C}, \) and solve the direct problem for such matrices. (Here \( \bar{z} \) means the complex conjugate of \( z. \) The matrices of the form (1.2) constitute the class \( \mathcal{J}_n. \)

We also solve the inverse spectral problem for matrices from the subclass \( \widehat{\mathcal{J}}_n \) of the class \( \mathcal{J}_n. \) This class consists of the matrices of the form

\[
\widehat{J}_n = \begin{pmatrix}
 \hat{c}_1 & \hat{b}_1 & 0 & \cdots & 0 & \bar{b}_n \\
 \hat{\bar{c}}_1 & \hat{c}_2 & \hat{b}_2 & \cdots & 0 & 0 \\
 0 & \hat{\bar{b}}_2 & \hat{c}_3 & \cdots & 0 & 0 \\
 \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
 0 & 0 & 0 & \cdots & \hat{c}_{n-1} & \hat{b}_{n-1} \\
 \hat{b}_n & 0 & 0 & \cdots & \hat{\bar{b}}_{n-1} & \hat{a}_n
\end{pmatrix},
\]

(1.3)

where \( \hat{b}_1, \ldots, \hat{b}_{n-1}, \hat{c}_1, \ldots, \hat{c}_{n-1} \in \mathbb{R}, \) and \( \hat{a}_n, \hat{b}_n \in \mathbb{C}, \) \( \hat{b}_n \neq 0. \)

Note that in [3] (see also [4]), Arlinskii and Tsekhanovskii considered the matrices of the form (1.3) with \( \hat{b}_n = 0, \) \( a_n \in \mathbb{C} \setminus \mathbb{R}, \) and solved the direct and inverse eigenvalue problems for those matrices. In [26], the direct and inverse spectral problems for the matrices the form (1.3) with \( b_n \in \mathbb{R} \setminus \{0\} \) and \( a_n \in \mathbb{R} \) (that is, for the matrices of the form (1.1)) were solved, and necessary and sufficient conditions for solvability of the inverse problem were found.

Recall that in [26], it was established that the necessary and sufficient conditions for the inverse spectral problem for the matrices of the form (1.1) to be solvable are

\[
\prod_{j=1}^{n} |\mu_k - \lambda_j| \geq (-1)^{n-k-1} \beta, \quad k = 1, \ldots, n - 1.
\]

(1.4)

Here \( \beta = b_1 \cdots b_n \) and the sets \( \{\lambda_j\}_{j=1}^{n-1} \) and \( \{\mu_k\}_{k=1}^{n-1} \) are the spectra of the matrix (1.1) and its \( (n-1) \times (n-1) \) leading principal submatrix, respectively.

In [3] (see also [4]), it was established that any matrix of the form (1.3) with \( \hat{b}_n = 0 \) and \( \text{Im} \ a_n > 0 \) has its eigenvalues in the open half-plane of the complex plane. The inverse spectral problem of such matrices was also solved in [3].

In the present work, we extend the results of the work [26] and find necessary and sufficient conditions for solvability of the inverse spectral problem for the matrices from the class \( \widehat{\mathcal{J}}_n. \) Those conditions are also necessary for the matrices from the class \( \mathcal{J}_n \) and generalize the inequality (1.4). Also we extended the results of the paper [3] to the matrices from the class \( \mathcal{J}_n. \)

The paper is organized as follows. In Section 2, we survey some general properties of the real symmetric tridiagonal matrices. Section 3 is devoted to the study of spectral properties of matrices from the class \( \mathcal{J}_n. \) We study the eigenvalue location for real and nonreal numbers \( \beta = b_1 \cdots b_n \) and \( a_n, \) and find a necessary condition for the spectra of those matrices. In Section 4 we show that for any matrix from the class \( \mathcal{J}_n, \) there exists a matrix from the class \( \widehat{\mathcal{J}}_n \) with the same spectral data. In this section, we solve the inverse spectral problem for the matrices from the class \( \widehat{\mathcal{J}}_n, \) and establish that the necessary condition obtained in Section 4 is also sufficient for the inverse spectral problem to be solvable.

In Sections 5 and 6 we follow the approach developed in [26]. However, we simplify the substantiation of their technique and extend it to a more wide class of matrices.
2 Preliminaries

Let us denote by $J_k, k = 1, \ldots, n-1$, the $k$th principal submatrix of $J_n$, and consider the matrix $J_{n-1}$

\[
J_{n-1} = \begin{pmatrix}
c_1 & b_1 & 0 & \ldots & 0 & 0 \\
b_1 & c_2 & b_2 & \ldots & 0 & 0 \\
0 & b_2 & c_3 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & c_{n-2} & b_{n-2} \\
0 & 0 & 0 & \ldots & b_{n-2} & c_{n-1}
\end{pmatrix}.
\]

(2.1)

This matrix is self-adjoint, so its characteristic polynomial

\[
\chi_{n-1}(\lambda) = \det(\lambda I_{n-1} - J_{n-1})
\]

has only real zeroes, the eigenvalues of the matrix $J_{n-1}$. Here $I_{n-1}$ is the identity matrix of size $n-1$. Note that the spectrum of $J_{n-1}$ coincides with the spectrum of the matrix (cf. [14])

\[
\tilde{J}_{n-1} = \begin{pmatrix}
c_1 & [b_1] & 0 & \ldots & 0 & 0 \\
[b_1] & c_2 & [b_2] & \ldots & 0 & 0 \\
0 & [b_2] & c_3 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & c_{n-2} & [b_{n-2}] \\
0 & 0 & 0 & \ldots & [b_{n-2}] & c_{n-1}
\end{pmatrix}.
\]

(2.2)

But this matrix is a real Jacobi matrix so its eigenvalues are simple. Thus we obtain that the spectrum of the matrix $J_{n-1}$, $\sigma(J_{n-1}) = \{\mu_1, \ldots, \mu_{n-1}\}$, is real and simple (e.g., see [14]):

\[
\mu_1 < \mu_2 < \cdots < \mu_{n-1},
\]

so

\[
\text{sign} \left( \chi_{n-1}(\mu_k) \right) = (-1)^{n-k-1},
\]

(2.3)

where $\chi_{n-1}(\lambda)$ is the derivative of the polynomial $\chi_{n-1}(\lambda)$

By $u_k = (u_{k1}, \ldots, u_{k,n-1})^T \in \mathbb{C}^{n-1}$ we denote the eigenvector of $J_{n-1}$ corresponding to the eigenvalue $\mu_k, k = 1, \ldots, n-1$ such that

\[
\tilde{u}_j^T u_k = \delta_{jk},
\]

(2.4)

where $\delta_{jk}$ is the Kronecker symbol. Then the \textit{resolvent} of the matrix $J_{n-1}$ has the form

\[
(\lambda I_{n-1} - J_{n-1})^{-1} = \sum_{k=1}^{n-1} \frac{u_k \tilde{u}_k^T}{\lambda - \mu_k}
\]

(2.5)

It is easy to see that

\[
e_1^T (\lambda I_{n-1} - J_{n-1})^{-1} e_{n-1} = \frac{b_1 b_2 \cdots b_{n-2}}{\chi_{n-1}(\lambda)},
\]

where $e_1 = (1, 0, \ldots, 0, 0)^T \in \mathbb{C}^{n-1}$ and $e_{n-1} = (0, 0, \ldots, 0, 1)^T \in \mathbb{C}^{n-1}$. The formula (2.5) gives us

\[
\begin{aligned}
\frac{b_1 b_2 \cdots b_{n-2}}{\chi_{n-1}(\lambda)} &= \sum_{k=1}^{n-1} \frac{u_k \tilde{u}_k}{\lambda - \mu_k},
\end{aligned}
\]

so we obtain

\[
b_k u_k b_{n-1} \tilde{u}_{k,n-1} = \frac{\beta}{\chi_{n-1}(\mu_k)}, \quad k = 1, \ldots, n-1,
\]

(2.6)

where

\[
\beta = b_1 b_2 \cdots b_n \neq 0.
\]

(2.7)

In particular, $u_{k1} \neq 0$ and $u_{k,n-1} \neq 0$.

The formula (2.6) implies that

\[
|b_k u_k|^2 |b_{n-1} \tilde{u}_{k,n-1}|^2 = \frac{|\beta|^2}{|\chi_{n-1}(\mu_k)|^2}, \quad k = 1, \ldots, n-1.
\]

(2.8)
3 Spectral properties of the matrices in \( J_n \)

In this section, we characterize the spectra of the matrices in \( J_n \). Note first that any matrix \( J_n \) of the form \( (1.2) \) can be represented as follows

\[
J_n = \begin{pmatrix} J_{n-1} & y \\ y^T & a_n \end{pmatrix},
\]

where \( y = (\vec{a}_n, 0, 0, \ldots, 0, b_{n-1})^T \in \mathbb{C}^{n-1} \). By the Schur determinant formula \( [29] \), we obtain (cf. \( [26] \))

\[
\frac{\chi_n(\lambda)}{\chi_{n-1}(\lambda)} = \lambda - a_n - \frac{\sum_{k=1}^{n-1} |\lambda|^2}{\lambda - \mu_k},
\]

where \( \chi_n(\lambda) \) and \( \chi_{n-1}(\lambda) \) are the characteristic polynomials of the matrices \( J_n \) and \( J_{n-1} \), respectively:

\[
\chi_n(\lambda) = \det(\lambda I_n - J_n), \quad \chi_{n-1}(\lambda) = \det(\lambda I_{n-1} - J_{n-1}).
\]

Furthermore, the formula \( [23] \) implies

\[
\left( \sum_{k=1}^{n-1} |\lambda|^2\right)^{-1} y = \sum_{k=1}^{n-1} \frac{|b_{n-1}u_k|^2}{|\lambda - \mu_k|} = \sum_{k=1}^{n-1} \frac{|b_n u_{k1} + \vec{a}_{n-1} u_{k,n-1}|^2}{|\lambda - \mu_k|}.
\]

From \( (3.1) \) and \( (3.2) \) we get

\[
\frac{\chi_n(\lambda)}{\chi_{n-1}(\lambda)} = \lambda - a_n - \sum_{k=1}^{n-1} \frac{a_k}{\lambda - \mu_k},
\]

where

\[
a_k = -\frac{\chi_n(\mu_k)}{\chi_{n-1}(\mu_k)} = |b_{n-1}u_{k1}|^2 + |\vec{a}_{n-1} u_{k,n-1}|^2 \geq 0
\]

or

\[
a_k = |b_n u_{k1}|^2 + |b_{n-1} u_{k,n-1}|^2 + 2 \text{Re}(b_n u_{k1} b_{n-1} \vec{u}_{k,n-1}) \geq 0.
\]

Note that \( \chi_n(\mu_k) \) and \( \chi_{n-1}(\mu_k) \) imply that

\[
(-1)^{n-k} \chi_n(\mu_k) \geq 0.
\]

Thus, the function

\[
R_{n-1}(\lambda) = \lambda - \sum_{k=1}^{n-1} \frac{\alpha_k}{\lambda - \mu_k}
\]

maps the upper half-plane of the complex plane to itself, since

\[
\text{sign Im} \frac{-\alpha_k}{\lambda - \mu_k} = \text{sign Im} \frac{-\alpha_k (\lambda - \mu_k)}{|\lambda - \mu_k|^2} = \text{sign Im} \lambda.
\]

Therefore, the zeroes and poles of \( R_{n-1}(\lambda) \) are real, simple and interlacing (see e.g. \( [21] \) and references therein), so \( \chi_n(\lambda) \) can be represented as follows

\[
\frac{\chi_n(\lambda)}{\chi_{n-1}(\lambda)} = R_{n-1}(\lambda) - a_n.
\]

By \( (3.3) \), \( \chi_n(\mu_j) = 0 \) for some \( j, j = 1, \ldots, n - 1 \), if and only if \( \alpha_j = 0 \), or equivalently, if and only if

\[
b_n u_{k1} + \vec{a}_{n-1} u_{k,n-1} = 0.
\]

Let \( N = \{j_1, \ldots, j_m\} \subset \{1, 2, \ldots, n - 1\} \) be the set of indices such that \( \alpha_j = 0 \) for \( j \in N \). Then the function \( R_{n-1}(\lambda) \) has the form

\[
R_{n-1}(\lambda) = \lambda - \sum_{k=1}^{n-1} \frac{\alpha_k}{\lambda - \mu_k},
\]

and the polynomial \( \chi_n(\lambda) \) has \( m \) zeroes in common with \( \chi_{n-1}(\lambda), \mu_{j_1}, \mu_{j_2}, \ldots, \mu_{j_m} \) while its other zeroes are the solutions of the equation

\[
R_{n-1}(\lambda) = a_n.
\]
If \( a_n \in \mathbb{R} \), then the function \( R_{n-1}(\lambda) - a_n \) maps the upper half-plane to itself, so the solutions of the equation (3.9) are real and simple and interlace the numbers \( \mu_k \), \( k \in \{1, 2, \ldots, n-1\} \setminus N \). However, they may coincide with some numbers \( \mu_j \), \( j \in N \). So for \( a_n \in \mathbb{R} \), the eigenvalues of \( J_n \) are real and of multiplicity at most two. Multiple eigenvalues are always eigenvalues of \( J_{n-1} \). This property of the spectrum of \( J_n \) with \( b_k \in \mathbb{C} \), \( k = 1, 2, \ldots, n \) is the same as for real \( b_k \), e.g. [20].

Suppose now that \( a_n \) is nonreal, and recall that \( R_{n-1}(\lambda) \) maps the upper (lower) half-plane to itself. Therefore, all solutions of the equation (3.9) lie in the upper (lower) half-plane of the complex plane whenever \( \text{Im} \ a_n > 0 \) (\( \text{Im} \ a_n < 0 \)), and cannot be real since the functions mapping the upper half-plane to itself are always real on the real line (see e.g. [21] and references there). Note that the famous Hermite-Biehler theorem can be proved by the same technique, see [24].

Finally, we note that the formula (3.1) and (2.6) imply that
\[
\alpha_k = \frac{[b_1 u_{k1}]^2 + \text{Re}(b_n u_{k1} b_{n-1} \bar{u}_{k,n-1})^2 + \text{Im}(b_n u_{k1} b_{n-1} \bar{u}_{k,n-1})^2}{[b_1 u_{k1}]^2} = (3.10)
\]
This formula shows that if \( \text{Im} \beta \neq 0 \), then \( \alpha_k \neq 0 \) for any \( k = 1, \ldots, n-1 \). At the same time, if \( \beta \in \mathbb{R} \setminus \{0\} \), then
\[
\alpha_k = \frac{[\chi'_{n-1}(\mu_k)] [b_1 u_{k1}]^2 + \beta^2}{[b_1 u_{k1}]^2 [\chi'_{n-1}(\mu_k)]^2}.
\]
Therefore, for real nonzero \( \beta \), the number \( \alpha_k \) equals zero if and only if \( \chi'_{n-1}(\mu_k) [b_1 u_{k1}]^2 + \beta = 0 \), that is equivalent to the following equality
\[
|\chi'_{n-1}(\mu_k)| [b_1 u_{k1}]^2 = (-1)^{n-k} \beta,
\]
by (2.3). This equality and (3.9) give us:

for \( \beta > 0 \)
\[
\begin{align*}
(-1)^{n-k} \chi_n(\mu_k) &> 0, & k = n-1, n-3, n-5, \ldots, \\
(-1)^{n-k} \chi_n(\mu_k) &\geq 0, & k = n-2, n-4, n-6, \ldots,
\end{align*}
\]
and for \( \beta < 0 \)
\[
\begin{align*}
(-1)^{n-k} \chi_n(\mu_k) &> 0, & k = n-1, n-3, n-5, \ldots, \\
(-1)^{n-k} \chi_n(\mu_k) &\geq 0, & k = n-2, n-4, n-6, \ldots,
\end{align*}
\]

If additionally \( a_n \in \mathbb{R} \), then the eigenvalues of the matrices \( J_n \) and \( J_{n-1} \) are distributed as follows:

for \( \beta < 0 \)
\[
\cdots \leq \lambda_{n-5} < \mu_{n-4} < \lambda_{n-4} \leq \mu_{n-3} \leq \lambda_{n-2} < \mu_{n-2} < \lambda_{n-1} \leq \mu_{n-1} \leq \lambda_n,
\]
and for \( \beta > 0 \)
\[
\cdots < \lambda_{n-5} \leq \mu_{n-4} < \lambda_{n-4} < \mu_{n-3} < \lambda_{n-2} \leq \mu_{n-2} \leq \lambda_{n-1} < \mu_{n-1} < \lambda_n.
\]

Thus, we come to the following statements.

**Theorem 3.1.** Let \( \beta \) be nonreal. If \( a_n \in \mathbb{R} \), then all the eigenvalues of the matrix \( J_n \) defined in (1.2) are real and simple and interlace the eigenvalues of \( J_{n-1} \), which are real and simple as well.

If \( a_n \in \mathbb{C} \setminus \mathbb{R} \), then all the eigenvalues of the matrix \( J_n \) lie in the open upper (lower) half-plane of the complex plane whenever \( \text{Im} \ a_n > 0 \) (\( \text{Im} \ a_n < 0 \)).

Moreover, for complex \( a_n \) (real or nonreal) the characteristic polynomial of the matrix \( J_n \) satisfies the inequalities
\[
(-1)^{n-k} \chi_n(\mu_k) > 0, \quad k = 1, \ldots, n-1.
\]

**Theorem 3.2.** Let \( \beta \in \mathbb{R} \). If \( a_n \in \mathbb{R} \), then all the eigenvalues of the matrix \( J_n \) given in (1.2) are real and of multiplicity at most 2. Any multiple eigenvalue of \( J_n \) is an eigenvalue of \( J_{n-1} \).

If \( a_n \in \mathbb{C} \setminus \mathbb{R} \), then all the eigenvalues of the matrix \( J_n \) lie in the closed upper (lower) half-plane of the complex plane whenever \( \text{Im} \ a_n > 0 \) (\( \text{Im} \ a_n < 0 \)). An eigenvalue of \( J_n \) is real if and only if it is an eigenvalue of \( J_{n-1} \).

Moreover, the eigenvalues of \( J_n \) and \( J_{n-1} \) are distributed as in (3.13)–(3.14) for real \( a_n \), or satisfy the inequalities (3.11)–(3.12) for any complex \( a_n \).
Remark 3.3. From (3.3) it follows that if \( a_n \) is nonreal, then
\[
\chi_n(\lambda) = \chi_{n-1}(\lambda - \beta) - \sum_{k=1}^{n} \alpha_k \prod_{j=1}^{n-1} (\lambda - \mu_j) - i \text{Im} a_n \cdot \chi_{n-1}(\lambda).
\] (3.15)
that is, \( \chi_{n-1}(\lambda) \) is the imaginary part (up to the constant factor \( \text{Im} a_n \)) of the polynomial \( \chi_n(\lambda) \), so \( \chi_n(\mu_k) \in \mathbb{R}, k = 1, \ldots, n - 1 \).

We now study a necessary condition which the spectra of the matrices in the class \( J_n \) must satisfy.

Theorem 3.4. Let \( J_n \in J_n, \chi_n(\lambda) = \det(\lambda I_n - J_n), \) and \( \sigma(J_{n-1}) = \{\mu_1, \ldots, \mu_{n-1}\} \). Then
- for nonreal \( \beta \)
  \[
  (-1)^{n-k}\chi_n(\mu_k) > 0 \quad \text{and} \quad |\chi_n(\mu_k) + 2 \Re \beta| \geq 2|\beta|, \quad k = 1, \ldots, n - 1,
  \] (3.16)
- for real nonzero \( \beta \)
  \[
  (-1)^{n-k}\chi_n(\mu_k) \geq 0 \quad \text{and} \quad |\chi_n(\mu_k)| \geq 4(-1)^{n-k-1}\beta, \quad k = 1, \ldots, n - 1,
  \] (3.17)
Here \( \beta \) is defined in (2.7).

Proof. In fact, taking into account that \( b_n u_{k1} \neq 0 \) by (2.9) and \( \chi_{n-1}(\mu_k) \neq 0 \) by (3.3), from (3.10) we obtain that \( X_k := |b_n u_{k1}|^2 \) satisfies the following equation
\[
[\chi_n(\mu_k)X_k + \Re \beta]^2 + \chi_n(\mu_k)\chi_n(\mu_k)X_k + (\Im \beta)^2 = 0,
\] (3.18)
or
\[
[\chi_n(\mu_k)X_k]^2 + (\chi_n(\mu_k) + 2 \Re \beta)\chi_n(\mu_k)X_k + (\Im \beta)^2 = 0.
\] (3.19)
The solutions of these equations have the form
\[
X_k^{(1,2)} = \frac{(-1)^{n-k}(\chi_n(\mu_k) + 2 \Re \beta) \pm \sqrt{[\chi_n(\mu_k) + 2 \Re \beta]^2 - 4|\beta|^2}}{2|\chi_n(\mu_k)|}.
\] (3.20)
Since \( b_n u_{k1} \neq 0 \), \( X_k \) is positive, so we have for any (real or nonreal) \( \beta \) and \( a_n \) that (see (3.6))
\[
[\chi_n(\mu_k) + 2 \Re \beta]^2 - 4|\beta|^2 \geq 0,
\] (3.21)
\[
(-1)^{n-k}(\chi_n(\mu_k) + 2 \Re \beta) \geq 0,
\] (3.22)
\[
(-1)^{n-k}\chi_n(\mu_k) \geq 0,
\] (3.23)
and \( |\chi_n(\mu_k) + 2 \Re \beta|^2 - 4|\beta|^2 \geq 0 \) and \( (-1)^{n-k}(\chi_n(\mu_k) + 2 \Re \beta) \geq 0 \) do not equal zero simultaneously.

We now show that the inequality in (3.22) follows from (3.21) and (3.23). Indeed, the inequality (3.21) implies
\[
(\chi_n(\mu_k) + 4 \Re \beta)\chi_n(\mu_k) \geq 4(\Im \beta)^2 \geq 0,
\]
so
\[
(-1)^{n-k}(\chi_n(\mu_k) + 4 \Re \beta) \geq 0.
\]
Therefore, if \( (-1)^{n-k} \Re \beta < 0 \), then
\[
(-1)^{n-k}(\chi_n(\mu_k) + 2 \Re \beta) > (-1)^{n-k}(\chi_n(\mu_k) + 4 \Re \beta) \geq 0,
\]
and if \( (-1)^{n-k} \Re \beta \geq 0 \), then
\[
(-1)^{n-k}(\chi_n(\mu_k) + 2 \Re \beta) \geq (-1)^{n-k}\chi_n(\mu_k) \geq 0.
\]

Thus, we obtain that for any complex (real or nonreal) \( \beta \) and \( a_n \), the eigenvalues of the matrices \( J_n \) and \( J_{n-1} \) satisfy the inequalities (3.21) and (3.22) that is equivalent to (3.16).

Let now \( \beta \in \mathbb{R}\setminus\{0\} \), so \( \Re \beta = \beta \) and \( \Im \beta = 0 \). In this case, the inequality (3.21) has the form
\[
\chi_n(\mu_k)(\chi_n(\mu_k) + 4 \beta) \geq 0,
\]
Thus, (3.20) and (3.27) imply
\[|\chi_n(\mu_k)| > 4\beta(-1)^{n-k-1}.\] (3.25)
Moreover, for real \(\beta\) the eigenvalues of \(J_n\) and \(J_{n-1}\) satisfy the inequalities (3.11)–(3.12), which include (3.24).

It is easy to see that (3.11)–(3.12) and (3.25) imply (3.17), as required.

**Remark 3.5.** By the Hermite–Biehler theorem (see e.g. [24]), the condition \((-1)^{n-k}\chi_n(\mu_k) > 0\) in (3.19) can be changed to \(\chi_n(\mu_k) \in \mathbb{R}\), since for \(\Im a_n > 0\) (\(\Im a_n < 0\)) the polynomial \(\chi_n(\lambda)\) has all roots in the open upper (lower) half-plane of the complex plane, and since \(\chi_{n-1}(\lambda)\) is its imaginary part (see Remark 3.3).

**Remark 3.6.** If \(\beta\) is a pure imaginary number, that is, \(\beta = i\gamma, \gamma \in \mathbb{R}\setminus\{0\}\), then the conditions (3.10) have the form
\[|\chi_n(\mu_k)| > 2|\gamma|, \quad k = 1, \ldots, n-1.\]

**Remark 3.7.** The inequalities (3.14) were established in [25] under the assumption of reality of all the entries of the matrix \(J_n\) and \(b_n > 0\).

**Remark 3.8.** The formulae (3.25) and (3.26) also imply that
\[\alpha_k = \frac{|b_{n-1}u_{k,n-1}|^2 + \Re(b_{n-1}u_{k,n-1})^2 + |\Im(b_{n-1}u_{k,n-1})|^2}{|b_{n-1}u_{k,n-1}|^2} = \frac{|\chi_{n-1}(\mu_k)|^2 + \Re(b_{n-1}u_{k,n-1})^2 + |\Im(b_{n-1}u_{k,n-1})|^2}{|b_{n-1}u_{k,n-1}|^2 |\chi_{n-1}(\mu_k)|^2}.\] (3.26)
Hence from (3.24) and (3.26) we get that \(X_k := |b_{n-1}u_{k,n-1}|^2\) satisfies the equation (3.18). Therefore, if \(X_k^{(1,2)}\) are solutions of (3.18), then \(X_k^{(1)} = |b_1u_{k1}|^2\), \(X_k^{(2)} = |b_{n-1}u_{k,n-1}|^2\), or \(X_k^{(1)} = |b_{n-1}u_{k,n-1}|^2, X_k^{(2)} = |b_{1}u_{k1}|^2\).

Finally, recall that the system of vectors \(u_1, \ldots, u_{n-1}\) is orthonormal, so the matrix \(U = [u_{k,j}]_{k,j=1}^{n-1}\) is unitary. Therefore,
\[\sum_{k=1}^{n-1} |u_{k1}|^2 = 1.\]
Since \(X_k = |b_nu_{k1}|^2\), we have
\[\sum_{k=1}^{n-1} X_k^{(1,2)} = \sum_{k=1}^{n-1} |b_nu_{k1}|^2 = |b_n|^2 \sum_{k=1}^{n-1} |u_{k1}|^2 = |b_n|^2.\] (3.27)

Thus, (3.20) and (3.21) imply
\[|b_n| = \left(\sum_{k=1}^{n-1} (-1)^{n-k}(\chi_n(\mu_k) + 2 \Re \beta) \pm \sqrt{|\chi_n(\mu_k) + 2 \Re \beta|^2 - 4\beta^2} \right)^{1/2},\] (3.28)
and
\[|u_{k1}|^2 = \frac{(-1)^{n-k}(\chi_n(\mu_k) + 2 \Re \beta) \pm \sqrt{|\chi_n(\mu_k) + 2 \Re \beta|^2 - 4\beta^2}}{2|b_n|^2|\chi_{n-1}(\mu_k)|} > 0,\] (3.29)
for \(k = 1, \ldots, n-1.\)
By the same reasoning as above and by Remark 3.8, we obtain that
\[|b_{n-1}| = \left(\sum_{k=1}^{n-1} (-1)^{n-k}(\chi_n(\mu_k) + 2 \Re \beta) \pm \sqrt{|\chi_n(\mu_k) + 2 \Re \beta|^2 - 4\beta^2} \right)^{1/2},\] (3.30)
and
\[|u_{k,n-1}|^2 = \frac{(-1)^{n-k}(\chi_n(\mu_k) + 2 \Re \beta) \pm \sqrt{|\chi_n(\mu_k) + 2 \Re \beta|^2 - 4\beta^2}}{2|b_{n-1}|^2|\chi_{n-1}(\mu_k)|} > 0,\] (3.31)
for \(k = 1, \ldots, n-1.\)
If $\beta \in \mathbb{R}\setminus\{0\}$, then the formulae (3.28) - (3.29) can be represented in the following form:

$$|b_n| = \left(\sum_{k=1}^{n-1} \frac{(-1)^{n-k}(\chi_n(\mu_k) + 2\beta) \pm \sqrt{\chi_n(\mu_k)(\chi_n(\mu_k) + 4\beta)}}{2|\chi_n-1(\mu_k)|}\right)^{\frac{1}{2}},$$

and

$$|u_{k1}|^2 = \frac{(-1)^{n-k}(\chi_n(\mu_k) + 2\beta) \pm \sqrt{\chi_n(\mu_k)(\chi_n(\mu_k) + 4\beta)}}{2|b_n|^2|\chi_n-1(\mu_k)|} > 0,$$

for $k = 1, \ldots, n-1$.

$$|b_{n-1}| = \left(\sum_{k=1}^{n-1} \frac{(-1)^{n-k}(\chi_n(\mu_k) + 2\beta) \pm \sqrt{\chi_n(\mu_k)(\chi_n(\mu_k) + 4\beta)}}{2|\chi_n-1(\mu_k)|}\right)^{\frac{1}{2}},$$

and

$$|u_{k,n-1}|^2 = \frac{(-1)^{n-k}(\chi_n(\mu_k) + 2\beta) \pm \sqrt{\chi_n(\mu_k)(\chi_n(\mu_k) + 4\beta)}}{2|b_{n-1}|^2|\chi_n-1(\mu_k)|} > 0,$$

for $k = 1, \ldots, n-1$.

### 4 Inverse problems for matrices in $\mathcal{J}_n$

In this section, we show that the necessary conditions (3.16) – (3.17) on the spectra of the matrices in the class $\mathcal{J}_n$ are also sufficient. This means that given $2n$ numbers $\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_{n-1}, \beta$ satisfying (3.10) or (3.17), we can reconstruct a matrix $J_n \in \mathcal{J}_n$ whose eigenvalues are $\lambda_1, \ldots, \lambda_n$, the eigenvalues of its leading principal submatrix $J_{n-1}$ are $\mu_1, \ldots, \mu_{n-1}$, and $\beta = b_1 \cdots b_n$.

Thus, in this section, we consider the subclass $\mathcal{J}_n$ of the class $\mathcal{J}_n$ consisting of matrices of the form

$$\mathcal{J}_n = \begin{pmatrix} \hat{c}_1 & \hat{b}_1 & 0 & \ldots & 0 & \hat{b}_n \\ \hat{b}_1 & \hat{c}_2 & \hat{b}_2 & \ldots & 0 & 0 \\ 0 & \hat{b}_2 & \hat{c}_3 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & \hat{c}_{n-1} & \hat{b}_{n-1} \\ \hat{b}_n & 0 & 0 & \ldots & \hat{b}_{n-1} & \hat{a}_n \end{pmatrix},$$

with real $\hat{c}_k, \hat{b}_k$, $k = 1, \ldots, n-1$ and complex (real or nonreal) $\hat{a}_n$ and $\hat{b}_n$. This subclass has the following important property.

**Lemma 4.1.** For any matrix in the class $\mathcal{J}_n$, there exists a matrix in the class $\mathcal{J}_n$ with the same spectral data, that is, with the same spectra of the matrix itself and of its leading $(n-1) \times (n-1)$ submatrix, and the same number $\beta$ (see (2.7)).

**Proof.** Indeed, by (3.10) the spectrum of a matrix $J_n$ in the class $\mathcal{J}_n$ depends on $a_n, \mu_k$, and $\alpha_k$, $k = 1, \ldots, n-1$, where $\mu_k$ are the eigenvalues of the leading principal submatrix $J_{n-1}$. At the same time, by (3.10), the numbers $\alpha_k$ depend on $\mu_k, \beta$, and $|b_{1u_{k1}}|^2$, where $\beta$ is defined in (2.7).

Thus, given a matrix $J_n$ in the class $\mathcal{J}_n$, we must find a matrix $\mathcal{J}_n$ in the class $\mathcal{J}_n$ with the same $a_n, \beta$, the spectrum of the submatrix $J_{n-1}$, and the same numbers $|b_{1u_{k1}}|^2$.

So we consider a matrix

$$J_n = \begin{pmatrix} c_1 & b_1 & 0 & \ldots & 0 & \hat{c}_n \\ b_1 & c_2 & b_2 & \ldots & 0 & 0 \\ 0 & b_2 & c_3 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & \hat{c}_{n-1} & b_{n-1} \\ b_n & 0 & 0 & \ldots & \hat{b}_{n-1} & \hat{a}_n \end{pmatrix},$$
in the class $\mathcal{J}_n$ and construct the matrix

$$
\tilde{\mathcal{J}}_n = \begin{pmatrix}
\hat{c}_1 & \hat{b}_1 & 0 & \cdots & 0 & \hat{a}_n \\
\hat{b}_1 & \hat{c}_2 & \hat{b}_2 & \cdots & 0 & \hat{b}_n \\
0 & \hat{b}_2 & \hat{c}_3 & \cdots & 0 & \hat{b}_{n-1} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & \hat{c}_{n-1} & \hat{b}_{n-1} \\
\hat{b}_n & 0 & 0 & \cdots & \hat{b}_{n-1} & \hat{a}_n
\end{pmatrix},
$$

such that

$$
\hat{c}_j = c_j, \quad j = 1, \ldots, n - 1,
$$

$$
\tilde{b}_k = |b_k|, \quad k = 1, \ldots, n - 1,
$$

$$
\tilde{a}_n = a_n,
$$

and

$$
\tilde{b}_n = \frac{b_1 b_2 \cdots b_n}{|b_1 b_2 \cdots b_{n-1}|} = \frac{\beta}{b_1 b_2 \cdots b_{n-1}}.
$$

(4.3)

Obviously, $\tilde{\mathcal{J}}_n \in \tilde{\mathcal{J}}_n$.

The formulae (4.2) show that the spectra of the submatrices $J_{n-1}$ and $\tilde{\mathcal{J}}_{n-1}$ coincide (see [14]):

$$
\sigma(\tilde{\mathcal{J}}_n) = \sigma(J_n) = \{\mu_1, \ldots, \mu_{n-1}\}.
$$

Moreover, from (4.3) we have

$$
\beta = b_1 b_2 \cdots b_n = \hat{b}_1 \hat{b}_2 \cdots \hat{b}_n = \tilde{\beta}.
$$

Therefore, to establish that $\sigma(\tilde{\mathcal{J}}_n) = \sigma(J_n)$ it suffices to prove that

$$
(\tilde{b}_1 \tilde{u}_{k1})^2 = |b_1 u_{k1}|^2, \quad k = 1, \ldots, n - 1.
$$

(4.4)

The formula (2.10) and the equalities (4.2) imply that

$$
\sum_{k=1}^{n-1} |b_1 u_{k1}|^2 \lambda - \mu_k = |b_1|^2 e_1^T (\lambda I_{n-1} - J_{n-1})^{-1} e_1 =
$$

$$
= |b_1|^2 \text{det}(\lambda I_{n-2} - J_{n-2}) \text{det}(\lambda I_{n-1} - J_{n-1}) =
$$

$$
= \tilde{b}_1^2 e_1^T (\lambda I_{n-1} - J_{n-1})^{-1} e_1 = \sum_{k=1}^{n-1} \frac{(\tilde{b}_1 \tilde{u}_{k1})^2}{\lambda - \mu_k},
$$

so (4.4) holds, as required.

Now we establish that the conditions (3.16)–(3.17) on the spectral data are also sufficient to reconstruct finitely many matrices in the class $\tilde{\mathcal{J}}_n$ with this spectral data. We consider 4 cases.

**Theorem 4.2.** Let $\beta$ be a nonreal number, and let $\{\mu_k\}_{k=1}^{n-1}$ be a set of real distinct numbers. Suppose that $\{\lambda_j\}_{j=1}^n$ is a set of complex numbers such that $\text{Im} \lambda_j > 0$, $j = 1, \ldots, n - 1$.

Then there exists a matrix $\tilde{\mathcal{J}}_n$ in the class $\tilde{\mathcal{J}}_n$ with $\text{Im} \tilde{a}_n > 0$ and $\beta = \tilde{b}_1 \cdots \tilde{b}_n$ such that $\sigma(\tilde{\mathcal{J}}_n) = \{\lambda_1, \ldots, \lambda_n\}$ and $\sigma(\tilde{\mathcal{J}}_{n-1}) = \{\mu_1, \ldots, \mu_{n-1}\}$ if and only if

$$
(-1)^{n-k} \chi_n(\mu_k) > 0 \quad \text{and} \quad |\chi_n(\mu_k) + 2 \text{Re} \beta| \geq 2|\beta|, \quad k = 1, \ldots, n - 1,
$$

(4.5)

where $\chi_n(\lambda) = \prod_{j=1}^n (\lambda - \lambda_j)$.

Moreover, if there are $m$ $(0 \leq m \leq n - 1)$ equalities in (3.16), then there exist exactly $2^{n-m-1}$ matrices in the class $\tilde{\mathcal{J}}_n$ with this spectral data.
Proof. The necessity of the conditions (4.5) is provided by Theorem 3.4.

Suppose that we have a nonreal number $\beta$, distinct real numbers $\{\mu_k\}_{k=1}^{n-1}$, and the numbers $\{\lambda\}_{j=1}^{n}$ in the open upper half-plane, and suppose that they satisfy (4.5).

We also assume that there are exactly $m$, $0 \leq m \leq n-1$, equalities in (4.5). Then one has exactly $2^{n-m-2}$ ways to construct the number $|\hat{b}_n|$ and the corresponding values $|u_{k1}|$, $k = 1, \ldots, n-1$, by the formulas (3.28) and (3.29). The conditions (4.5) guarantee the positivity of the obtained numbers $|\hat{b}_n|$ and $|u_{k1}|$, $k = 1, \ldots, n-1$. Indeed, (4.5) imply positivity of the numbers $(-1)^n k_1\chi(\mu_k + 2 \text{Re} \beta)$, $k = 1, \ldots, n-1$, as we established in the proof of Theorem 3.4. At the same time, positivity of these numbers and the conditions (4.5) obviously imply positivity of the numbers $X^{(1,2)}$ defined in (5.3). Thus, the conditions (4.5) guaranty the existence of positive numbers $|\hat{b}_n|$ and $|u_{k1}|^2$, $k = 1, \ldots, n-1$.

With the values $|u_{k1}|^2$, $k = 1, \ldots, n-1$, we construct the rational function

$$
\sum_{k=1}^{n-1} |u_{k1}|^2 \lambda - \mu_k := \psi(\lambda) / \chi_{n-1}(\lambda),
$$

(4.6)

where the polynomial $\psi(\lambda)$ of degree $n-2$ is uniquely determined by the numbers $|u_{k1}|^2$ and $\mu_k$, $k = 1, \ldots, n-1$. Moreover, since $|u_{k1}|^2 > 0$ by construction, the function $\psi(\lambda)/\chi_{n-1}(\lambda)$ maps the upper half-plane to the lower half-plane of the complex plane, so the zeroes of $\psi$ are real and simple and interlace the zeroes of $\chi_{n-1}$.

If we know the function (4.6), we can always reconstruct a unique Jacobi matrix of the form

$$
\left(\begin{array}{cccc}
\hat{c}_1 & \hat{b}_1 & 0 & \ldots \\
\hat{b}_1 & \hat{c}_2 & \hat{b}_2 & \ldots \\
0 & \hat{b}_2 & \hat{c}_3 & \ldots \\
\vdots & \vdots & \vdots & \ddots \\
0 & 0 & 0 & \ldots & \hat{c}_{n-2} & \hat{b}_{n-2} \\
0 & 0 & 0 & \ldots & \hat{b}_{n-2} & \hat{c}_{n-1}
\end{array}\right) = J_{n-1},
$$

(4.7)

where $\hat{b}_1, \ldots, \hat{b}_{n-1} > 0$, $\hat{c}_1, \ldots, \hat{c}_{n-1} \in \mathbb{R}$. There exist a few algorithms to make such a reconstruction [20] [19] [16] [15]. Thus, given the numbers $\mu_k$ and $|u_{k1}|^2$ the rational function (4.6) uniquely determines the matrix $J_{n-1}$ whose eigenvalues are $\mu_k$, while $\psi(\lambda) = \chi_{n-2}(\lambda)$. Furthermore, we put

$$
\hat{a}_n := \left(\sum_{j=1}^{n} \lambda_j - \sum_{k=1}^{n-1} \mu_k\right)
$$

and

$$
\hat{b}_n := |\hat{b}_n| e^{i \arg \beta}
$$

and

$$
\hat{b}_{n-1} := \frac{\beta}{\hat{a}_n} \cdot \hat{b}_{n-2} \hat{b}_{n-2}
$$

So we constructed a matrix of the form (4.1) such that $\sigma(J_{n-1}) = \{\mu_1, \ldots, \mu_{n-1}\}$ and $\hat{b}_1 \cdots \hat{b}_n = \beta$. To finish the proof it suffices to establish that $\sigma(J_n) = \{\lambda_1, \ldots, \lambda_n\}$. To do this we consider the following formula obtained from (3.19) – (3.24)

$$
\frac{\det(\lambda I_n - J_n)}{\det(\lambda I_{n-1} - J_{n-1})} = \lambda + \hat{a}_n - \sum_{k=1}^{n-1} \hat{b}_nu_{k1} + \hat{b}_{n-1}u_{k,n-1}|^2 - \frac{\lambda - \mu_k}{\lambda - \mu_k},
$$

(4.8)

where $u_{k1}$ and $u_{k,n-1}$ are the first and the last entries of the orthonormal eigenvector $u_k$ of the submatrix $J_{n-1}$ corresponding to the eigenvalue $\mu_k$, $k = 1, \ldots, n-1$. These entries are related as in (2.6), and $u_{k1}$ satisfies (3.29) by construction. Then the numbers $\hat{b}_n$ and $|u_{k,n-1}|^2$, $k = 1, \ldots, n-1$, must satisfy the equalities (3.30) – (3.31). Thus $|\hat{b}_nu_{k1}|^2$ and $|\hat{b}_{n-1}u_{k,n-1}|^2$ are the solutions of the equation (3.19) so we have

$$
|\hat{b}_nu_{k1}|^2 + |\hat{b}_{n-1}u_{k,n-1}|^2 = -\frac{\chi_n(\mu_k) + 2 \text{Re} \beta}{\chi_{n-1}^{(1,2)}(\mu_k)}.
$$
This formula together with \((2.6)\) gives us
\[
|b_n u_{k1} + b_{n-1} u_{k,n-1}|^2 = |b_n u_{k1}|^2 + |b_{n-1} u_{k,n-1}|^2 + 2 \text{Re}(b_n u_{k1} \overline{b_{n-1} u_{k,n-1}}) =
\]
\[-\frac{\chi_n(\mu_k) + 2 \text{Re} \beta}{\chi_n^{-1}(\mu_k)} + \frac{2 \text{Re} \beta}{\chi_n^{-1}(\mu_k)} = \frac{-\chi_n(\mu_k)}{\chi_n^{-1}(\mu_k)} = \prod_{j=1}^{n} (\mu_k - \lambda_j)
\]
so \(\det(\lambda I_n - \hat{J}_n) = \prod_{j=1}^{n} (\lambda - \lambda_j)\), as required. \(\square\)

**Theorem 4.3.** Let \(\beta\) be a nonreal number, and let \(\{\mu_k\}_{k=1}^{n-1}\) and \(\{\lambda_j\}_{j=1}^{n}\) be sets of distinct real numbers with no common elements.

Then there exists a matrix \(\hat{J}_n\) in the class \(\hat{J}_n\) with \(\hat{a}_n \in \mathbb{R}\) and \(\beta = \hat{b}_1 \cdots \hat{b}_n\) such that \(\sigma(\hat{J}_n) = \{\lambda_1, \ldots, \lambda_n\}\) and \(\sigma(\hat{J}_{n-1}) = \{\mu_1, \ldots, \mu_{n-1}\}\) if and only if the conditions \((4.5)\) hold.

Moreover, if there are \(m\) \((0 \leq m \leq n-1)\) equalities in \((4.5)\), then there exist exactly \(2^{n-m-1}\) matrices in the class \(\hat{J}_n\) with these spectral data.

For real nonzero \(\beta\) we have the following results.

**Theorem 4.4.** Let \(\beta\) be a real nonzero number, and let \(\{\mu_k\}_{k=1}^{n-1}\) be a set of real distinct numbers. Suppose that \(\{\lambda_j\}_{j=1}^{n}\) is a set of complex numbers in the closed upper half-plane of the complex plane, \(\text{Im} \lambda_j \geq 0\), \(j = 1, \ldots, n-1\).

Then there exists a matrix \(\hat{J}_n\) in the class \(\hat{J}_n\) with \(\text{Im} \hat{a}_n > 0\) and \(\beta = \hat{b}_1 \cdots \hat{b}_n\) such that \(\sigma(\hat{J}_n) = \{\lambda_1, \ldots, \lambda_n\}\) and \(\sigma(\hat{J}_{n-1}) = \{\mu_1, \ldots, \mu_{n-1}\}\) if and only if
\[
(-1)^{n-k} \chi_n(\mu_k) \geq 0, \quad k = 1, \ldots, n-1,
\]
and
\[
|\chi_n(\mu_k)| \geq 4(-1)^{n-k-1} \beta, \quad k = 1, \ldots, n-1,
\]
(4.9)
where \(\chi_n(\lambda) = \prod_{j=1}^{n} (\lambda - \lambda_j)\).

Moreover, if there are exactly \(m_1\) equalities in \((4.9)\) and exactly \(m_2\) equalities in \((4.10)\) \((0 \leq m_1 + m_2 \leq n-1)\), then there exist exactly \(2^{n-m_1-m_2-1}\) matrices in the class \(\hat{J}_n\) with these spectral data.

**Theorem 4.5.** Let \(\beta\) be a real number, and let \(\{\mu_k\}_{k=1}^{n-1}\) and \(\{\lambda_j\}_{j=1}^{n}\) be sets of distinct real numbers that may have common elements.

Then there exists a matrix \(\hat{J}_n\) in the class \(\hat{J}_n\) with \(\hat{a}_n \in \mathbb{R}\) and \(\beta = \hat{b}_1 \cdots \hat{b}_n\) such that \(\sigma(\hat{J}_n) = \{\lambda_1, \ldots, \lambda_n\}\) and \(\sigma(\hat{J}_{n-1}) = \{\mu_1, \ldots, \mu_{n-1}\}\) if and only if the numbers \(\{\mu_k\}_{k=1}^{n-1}\) and \(\{\lambda_j\}_{j=1}^{n}\) satisfy the inequalities \((4.9)\) and \((4.10)\).

Moreover, if there are exactly \(m_1\) equalities in \((4.9)\) and exactly \(m_2\) equalities in \((4.10)\) \((0 \leq m_1 + m_2 \leq n-1)\), then there exist exactly \(2^{n-m_1-m_2-1}\) matrices in the class \(\hat{J}_n\) with these spectral data.

Theorems 4.3, 4.5 can be proved in the same way as we proved Theorem 4.2. Note that the numbers \(m_1\) and \(m_2\) do not exceed a half of \(n\) (approximately). The exact upper bounds on \(m_1\) and \(m_2\) depend on the parity of \(n\) and on the sign of \(\beta\), and can be obtained from the inequalities \((4.9)\) and \((4.10)\).

Theorem 4.5 was established in [26, Theorems 6–7]. As well as in [26, Corollary 8] we note that in Theorems 4.2, 4.3 the constructed matrix \(\hat{J}_n\) is unique if and only if \(m = n-1\) or \(m_1 + m_2 = n-1\). This is possible for specific distributions of the \(\lambda_j\) and the \(\mu_k\).

### 5 Conclusions

In this paper, we showed that the technique developed by Y.-H. Xu and E.-X. Jiang [26] for periodic Jacobi matrices can be extended to a class of complex band matrices. We also gave some new explanations of the technique using the theory of the location of roots of polynomials. Our results extend the results of Arlinskii and Tsekhonovskii [3] (see also [4]) who studied a one-dimensional imaginary perturbation of symmetric real Jacobi matrices.

We believe that the combination of the methods of the works [26] and [4] can be helpful to study direct and inverse spectral problems of some other classes of band matrices.
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