Pandora’s box lid: geometry near the apparent horizon

Daniel R. Terno
Department of Physics & Astronomy, Macquarie University, Sydney NSW 2109, Australia and Institute for Quantum Science and Engineering, Department of Physics, Southern University of Science and Technology, Shenzhen 518055, Guangdong, China

Formation of a trapped region with a singularity-free apparent horizon in finite time of a distant observer has important physical consequences. In spherical symmetry it implies that only two classes of solutions of the Einstein equations are possible. In both cases the null energy condition (NEC) is violated and an expanding trapped region leads to a firewall. The weighted time average of the energy density for an observer crossing this firewall is negative and exceeds the maximal NEC violation that quantum fields can produce. For a contracting trapped region both metrics approach the ingoing Vaidya metric with decreasing mass. None of the solutions leads to a static limit. Only one class of solutions allows for a test particle to cross the apparent horizon, and produces a black hole at the end of a thin shell collapse. These results significantly constrain the regular black hole models.

I. INTRODUCTION

Black holes are probably the most celebrated prediction of classical general relativity (GR) [1, 2]. Quantum effects modify the Einstein equations [3–5] and allow matter to violate the classical energy conditions [1, 2, 6]. Both results affect the final stages of the gravitational collapse, making plausible three distinct types of the ultra-compact objects (UCOs) as its end. [7, 8].

The first possibility is that the event horizon and the singularity are formed, even if their onset and properties are modified by quantum effects. The final result of the evolution may be a black hole remnant [9]. Another scenario envisages formation of horizonless UCOs. The third option is a black hole that has an apparent horizon, but no event horizon or singularity.

Current observations of astrophysical black holes — UCOs with dynamically-determined mass and spin [10–12] — only weakly constrain these scenarios. Given the apparent tension between quantum mechanics and general relativity, issues of logical consistency of models and the information loss problem [13, 14], it is important to understand what each scenario entails.

There are different opinions on what makes a UCO a black hole [15]. However, the strongest degree of consensus is that it should have a trapped spacetime region, whose boundary is the apparent horizon [1, 2, 10]. A trapped region is a domain where both ingoing and outgoing future-directed null geodesics emanating from a spacelike two-dimensional surface with spherical topology have negative expansion [1, 2, 16]. The apparent horizon is the outer boundary of the trapped region and the defining feature of a physical black hole (PBH). [17, 18]. To be physically relevant the apparent horizon should form in a finite time of a distant observer. Here we investigate the consequences of having a PBH to actually form.

The simplest setting to investigate is a spherically-symmetric collapse, where the apparent horizon is unambiguously defined in all foliations that respect this symmetry [19].

Building on the results of Refs. [18, 20, 21] we describe the two possible classes of the near-horizon geometries, discuss their properties, and consider the implications for singularity-free black hole models.

II. GEOMETRY NEAR THE APPARENT HORIZON: TWO CLASSES OF SOLUTIONS

We assume validity of semiclassical gravity. That means we use classical notions (horizons, trajectories, etc.), and describe dynamics via the Einstein equations \( G_{\mu\nu} = 8\pi T_{\mu\nu} \), where the standard left-hand side is equated to the expectation value \( T_{\mu\nu} = \langle \hat{T}_{\mu\nu} \rangle \) of the renormalized energy-momentum tensor (EMT). The latter represents both the collapsing matter and the created excitations of the quantum fields, but no specific properties of the state \( \omega \) are assumed.

Boundaries of the trapped region are nonsingular in classical GR [1, 2], a requirement that is typically assumed to extend to the semiclassical regime. We implement this property by requiring that the scalars \( T := T_{\mu\nu}^{\mu\nu} \), and \( \Sigma := T_{\mu\nu}T^{\mu\nu} \) are finite. The Einstein equations imply that \( 64\pi^2 \Sigma = R_{\mu\nu}R^{\mu\nu} \) and \( 8\pi T = -R \), where \( R_{\mu\nu} \) and \( R \) are the Ricci tensor and the Ricci scalar, respectively. Finite values of these scalars are a necessary regularity condition, and additional tests may be required.

A general spherically-symmetric metric in Schwarzschild coordinates is given by

\[
ds^2 = -e^{2h(t,r)}f(t,r)dt^2 + f(t,r)^{-1}dr^2 + r^2d\Omega^2,
\]

where \( r \) is the areal radius. The Misner-Sharp mass [10, 16, 22] \( C(t,r) \) is invariantly defined via

\[
1 - C/r := \partial_\mu r \partial^\mu r,
\]

and thus the function \( f(t,r) = 1 - C(t,r)/r \) is invariant under general coordinate transformations. The apparent horizon is located at the Schwarzschild radius \( r_h \) that is the largest root of \( f(t,r) = 0 \) [16, 19]. The function \( h(t,r) \) plays the role of an integrating factor in the coordinate transformations and may contain information about the potential hairs of the stationary PBHs [2, 23].

It is convenient to introduce

\[
\tau_t := e^{-2h}T_{tt}, \quad \tau^r := T^{rr}, \quad \tau^r_t := e^{-h}T_{rt},
\]
and represent the Misner-Sharp mass as
\[ C = r_g(t) + W(t, r - r_g), \] (4)

where the definition of the apparent horizon implies
\[ W(t, 0) = 0, \quad W(t, x) < x. \] (5)

In this notation the three of the Einstein equations become
\[ G_{tt} = \frac{\partial_t W}{r^2} = 8\pi \tau_t, \quad (6) \]
\[ G_{rr} = \frac{\partial_r C}{r^2} = 8\pi e^k \tau_r, \quad (7) \]
\[ G^{rr} = \frac{\partial_r h}{r} = 4\pi (\tau_t + \tau_r), \quad (8) \]

Following Ref. [18] requirements of the horizon regularity and existence of solutions of the Einstein equations allow the full classification of the EMT and the metric in the near-horizon region. Solutions are labelled by a single power \( k \) that describes convergence or divergence of the three functions \( \tau_t, \tau_r \) and \( \tau_t^r \) when \( r \rightarrow r_g \).

Regularity of the apparent horizon is expressed as a condition on the potentially divergent parts of the curvature scalars. For \( k < 1 \) the necessary regularity conditions are
\[ T = (\tau^r - \tau_t)/f \rightarrow g_1(t)f^{\kappa_1}, \quad (9) \]
\[ \Upsilon = ((\tau^r)^2 + (\tau_t)^2 - 2(\tau_t^r)^2)/f^2 \rightarrow g_2(t)f^{\kappa_2}, \quad (10) \]
for some \( g_{1,2}(t) \) and \( \kappa_{1,2} \geq 0 \). We can exclude \( T^\theta_\theta = T^\phi_\phi \) from the consideration because the Einstein equations imply \( T^\theta_\theta/f^k \rightarrow 0 \) (Appendix A1). However, additional regularity checks are required.

Only the classes \( k = 0 \) (divergent energy density and pressure) and \( k = 1 \) (finite non-zero values of energy density and pressure at the apparent horizon) are self-consistent. They are described below. Leading terms of the solutions for \( k < 1 \) are given in Appendix A2.

Part of the subsequent analysis for the case of contracting apparent horizon, \( r_g' < 0 \), employs the description of PBHs using the advanced null coordinate \( \nu \),
\[ dt = e^{-h}(e^{h+} dv - f^{-1} dr). \] (11)

A general spherically-symmetric metric in \((v, r)\) coordinates is
\[ ds^2 = -e^{2h+} \left( 1 - \frac{C_+}{r} \right) dv^2 + 2e^{h+} dvdr + r^2 d\Omega. \] (12)

For \( r_g' > 0 \) a similar analysis proceeds using the retarded null coordinate. Imposing the finiteness conditions on the Ricci scalar \( \bar{R} \) (Appendix A3) at the apparent horizon \( r_+(v) = C_+(v, v) = r_g(t) \), we obtain that as \( r \rightarrow r_g \equiv r_+ \),
\[ C_+(v, r) = r_+(v) + w_+(v)(r - r_+) + w_{1+}^2 (r - r_+)^2 + \ldots, \] (13)
\[ h_+(v, r) = \chi_+(v)(r - r_+) + \ldots, \] (14)

for some functions \( w_+, w_{2+} \) and \( \chi_+ \), where the condition \( w_+ \leq 1 \) is due to the definition of the Schwarzschild radius.

Components of the EMT are related by
\[ \theta_v := e^{-2h+} \Theta_{vv} = \tau_t, \quad (15) \]
\[ \theta_{vr} := e^{-h+} \Theta_{vr} = (\tau_t^r - \tau_t)/f, \quad (16) \]
\[ \theta_r := \Theta_{rr} = (r^r + \tau_t - 2\tau_t^r)/f^2, \quad (17) \]

where \( \Theta_{\mu\nu} \) label the components in \((v, r)\) coordinates. The limits \( \theta_{\mu\nu}^r = \lim_{r \rightarrow r_g} \theta_{\mu\nu} \) are
\[ \theta_+^v = (1 - w_+)^{-1} r'^_+ - \frac{w_+}{8\pi r_+^2}, \quad \theta_+^r = -\frac{w_+}{8\pi r_+^2}, \quad \theta_+^+ = \frac{\chi_+}{4\pi r_+}. \] (18)

### A. Divergent density and pressure

Solutions of the first class satisfy
\[ \tau_t = -\Upsilon^2(t)f^k, \quad \tau_r = -\Upsilon^2(t)f^k, \quad \tau_t^r = \pm \Upsilon^2 f^k, \] (19)

where \( \Upsilon^2(t) \) is some function of time, \( k < 1 \), and the higher-order terms are omitted. The upper (lower) signs of \( \tau_t^r \) correspond to growth (evaporation) of the PBH.

The null energy condition (NEC) that requires \( T_{\mu\nu}l^\mu l^\nu \geq 0 \) for all null vectors \( l^\nu \). It is violated for all values of \( k < 1 \). Static non-vacuum solutions with \( \tau_t^r \rightarrow 0 \) are impossible for \( k < 1 \), as the regularity condition Eq. (10) cannot be satisfied unless all three components are zero. Violations of the NEC are bounded by quantum energy inequalities (QEIs) [24]. Moreover, for a growing PBH, \( r_g' > 0 \), in the reference frame of an infalling massive test particle the energy density (as well as the pressure and the flux), diverge (Appendix B1). Such a transient firewall leads to a violation of the quantum energy inequality [25], that is shown by repeating the analysis of Ref [21]. Henceforth we consider only the solutions with \( r_g' < 0 \).

A comparison of Eqs. (15) and (18) with Eq. (19) shows that only the case \( k = 0 \) is allowed, with \( \Upsilon^2 = -\theta_+^+ \). In this case the leading terms in the metric functions in \((t, r)\) coordinates are given as power series in terms of \( x := r - r_g \)
\[ C = r_g - w\sqrt{x} + \frac{1}{3} x + \ldots, \quad h = -\frac{1}{2} \ln \frac{x}{\xi} + \frac{4}{3w^2}\sqrt{x} + \ldots \] (20)

where \( w^2 := 16\pi \Upsilon^2 r_g^3 \) and the higher-order terms depend on the specific properties of the EMT [20]. The function \( \xi(t) \) is determined by the choice of the time frame. Eq. (7) results in the consistency condition
\[ r_g'/\sqrt{\xi} = \pm 4\sqrt{\pi} \Upsilon \sqrt{r_g} = \pm w/r_g. \] (21)

For a static observer the energy density \( \rho = -T^t_t \) the pressure \( p = T^r_r \) and the flux diverge at the apparent horizon. On the other hand, in the reference frame of the infalling observer on
an arbitrary radial trajectory \((T_A(\tau), R_A(\tau), 0, 0)\) these quantities are

\[ \rho_A = p_A = \phi_A = -\frac{\gamma^2}{4R^2}, \]  

(22)
at the horizon crossing. Additional properties of this metric are discussed in Refs. [18, 21].

Further relations between the EMT components near the apparent horizon are obtained as follows. A point on the apparent horizon has the coordinates \((v, r_g)\) and \((t, r_g)\) in the two coordinate systems. Moving from \(r_+(v)\) along the line of constant \(v\) by \(\delta r\) leads to the point \((t + \delta t, r_g + \delta r)\). Eq. (11) implies \(\delta t = \delta r/r'_g\). Hence the EMT components are related at the first order in \(\delta r\) by

\[ \partial_v \theta_v^+ = -2\Upsilon \Upsilon'/r'_g + \alpha, \]  

(23)
\[ \partial_v \theta_v^+ + \frac{1}{r_+} \theta_v^{\alpha \beta} = -2\Upsilon \Upsilon'/r'_g + \beta, \]  

(24)
\[ \partial_v \theta_v^+ + 2 \frac{1}{r_+} \theta_v^{\alpha \beta} = -2\Upsilon \Upsilon'/r'_g + \gamma, \]  

(25)
where \(\partial_v \theta_v^+ := \partial_v \theta_v\big|_{r=r_g}\), \(\alpha(t) := \partial_{\tau^x} | r=r_g\), \(\beta := \partial_{\tau^x} \tau^x | r=r_g\), \(\gamma := \partial_{\tau^x} \tau^x | r=r_g\). As a result, the subleading terms satisfy

\[ \alpha + \gamma = 2\beta. \]  

(26)

This metric approaches the pure ingoing Vaidya metric with decreasing mass, which is the usual near-horizon approximation when the backreaction from Hawking radiation is taken into account [26, 27]. The triple limit \(\tau, \tau^x, r^x \rightarrow -\Upsilon^2\) is observed in the \textit{ab initio} calculations of the renormalized energy-momentum tensor on the Schwarzschild background [28].

### B. Finite density and pressure

For \(k \geq 1\) Eqs. (9) and (10) do not impose any constraints, and different components of the energy-momentum tensor can converge to zero at different rates. However, only the case \(k = 1\), where at the leading order in \(f\)

\[ \tau = E(t)f, \quad \tau^x = P(t)f, \quad \tau^x = \Phi(t)f, \]  

(27)
allows for a solution with \(r'_g \neq 0\) (Appendix B 3). These solutions exhibit a finite pressure and a finite density at the apparent horizon, \(\rho(t, r_g) = E\) and \(p(t, r_g) = P\), respectively.

Then the Misner-Sharp mass is given by

\[ C = r_g(t) + 8\pi E r_g^2 x + \ldots, \quad 8\pi E r_g^2 < 1. \]  

(28)
The strict inequality follows from Eq. (32), as \(8\pi r_g^2 E = 1\) is incompatible with \(r_g \neq 1\). Consistency of Eqs. (7) and (8) results in

\[ \frac{4\pi(E + P)r_g^2}{1 - 8\pi E r_g^2} = -1, \]  

(29)
ensures the necessary logarithmic divergence of \(h\),

\[ h = -\ln x/\xi(t) + \omega(t)x + \ldots, \]  

(30)
for some \(\xi(t) > 0\) and \(\omega(t)\), that results in a regular expression

\[ r'_g = 8\pi \Phi r_g. \]  

(31)

Requiring the Ricci scalar to be finite (Appendix B 3) at \(r_g\) imposes the constraint

\[ (1 - 8\pi E r_g^2)\xi = \pm r'_g r_g, \]  

(32)
where the upper (lower) signs correspond to the expansion (contraction) of the apparent horizon.

As a result, the energy density at the apparent horizon defines two other parameters,

\[ P = \frac{-1 + 4\pi E r_g^2}{4\pi r_g^2}, \quad \Phi = \pm \frac{1 - 8\pi E r_g^2}{8\pi r_g^2}, \]  

(33)
where the upper (lower) sign corresponds to growth (evaporation).

The \((t, r)\) block of the EMT is given in Appendix B 2. The NEC is violated in both cases. For example, for \(r'_g < 0\) and the outward pointing null vector \(k^\mu\) as \(r \rightarrow r_g\)

\[ T_{\mu\nu}k^\mu k^\nu \approx -\frac{1}{16\pi r_g^2}. \]  

(34)

For \(\Phi > 0\) (an expanding trapped region, \(r'_g > 0\)) in the reference frame of an infalling observer the energy density diverges. This transient firewall leads to the violation of the QEI, similarly to the \(k = 0\) case (Appendix B 2).

The metric of Eq. (12) describes the \(k = 1\), \(\Phi < 0\) case only if \(w_w \equiv 1\). Compatibility with Eqs. (16) and (17) results in the relations

\[ 8\pi r_g^2 (E - \Phi) = 1, \quad E + P - 2\Phi = 0, \]  

(35)
that are automatically satisfied due to Eq. (33).

We no show that it is impossible to fall into a \(k = 1\) black hole. Consider for simplicity a massless test particle. It is convenient to parameterize the radial ingoing null geodesic \((T_A, R_A)\) by its radial coordinate, \(\lambda = -R_A\). Then

\[ \frac{dT}{d\lambda} = \frac{e^{-h(T_A, R_A)}}{f(T_A, R_A)} = \frac{r_g(1 - \omega_x(R_A - r_g))}{(1 - 8\pi E r_g^2)\xi} + \ldots \]  

(36)
Similarly, the rate of change of the coordinate time with respect to the proper time of an infalling massive test particle is also finite.

To possibility of the horizon crossing is conveniently monitored by the gap function [29, 30],

\[ X(\lambda) := R_A - r_g(T_A(\lambda)), \]  

(37)
whose rate of change \(X_\lambda = 1 - r_g(T_A/\lambda)\) is indicative of the final fate of the test particle. Expanding it in powers of \(X\) we find that

\[ X_\lambda = -\omega X + \mathcal{O}(X^2). \]  

(38)
If \(\omega < 0\) then once certain minimal coordinate distance is reached, the gap has to increase. If \(\omega > 0\) then the gap will close exponentially slow,

\[ X \approx X_0 \exp(-\int_{\lambda_0}^{\lambda} |\omega|d\lambda), \]  

(39)
and thus crossing of the apparent horizon \((X = 0)\) of an evaporation \(k = 0\) PBH never happens. The same conclusion is obtained by considering a massive test particle and the proper time parametrization.

These results cast doubt on the possibility that \(k = 1\) black holes can actually form. A thin dust shell, with a flat metric inside and a curved metric outside, provides the simplest tractable model of the collapse. The classical Schwarzschild exterior leads to the well-known result of a finite proper time of the collapse and an infinite collapse time \(t\) according to the clock of a distant observer. By using the Vaidya metrics to emulate the effects of evaporation, one obtains results that depend on their choice [20].

By assuming the outgoing Vaidya metric with decreasing mass (which satisfies the NEC and thus cannot lead to the formation of a PBH in finite coordinate time \(t\)), the apparent horizon is never formed, but the shell either becomes superluminal [31] or develops a surface pressure at the coordinate distance \(x \sim w^2\) from the Schwarzschild radius [20, 30]. On the other hand, the ingoing Vaidya metric of Eq. (12) leads to the horizon formation in finite time according to both clocks [20]. However, if the exterior is modelled by Eq. (1) with \(k = 1\) metric functions (Eqs. (28) and (30)), Eq. (39) indicates that the shell’s collapse will never be complete, even if the exterior metric violates the NEC.

Consider now a static solution. Assume first that \(\Phi < 0\) as in the dynamical case. The minimal static condition \(r_0^* = 0\) requires

\[
\frac{4\pi(E + P)}{1 - 8\pi E r_g^2} = -\lambda, \tag{40}
\]

\(\lambda < 1\) to hold. The Ricci scalar is finite only if either the density takes the extreme allowed value \(E = (8\pi r_g^2)^{-1}\), or \(\lambda = \frac{1}{2}\) (Appendix B3). Using Eq. (35) (that still holds up to the end of the dynamical phase), we obtain

\[
\Phi = 0, \quad E = -P = 1/(8\pi r_g^2), \tag{41}
\]

in both cases. The NEC is not violated so the solution cannot be realized in finite time \(t\).

A genuine static solution with all metric function being independent of time is possible only if \(\tau^r = 0\). If \(h \neq 0\) there is no general requirement \(\rho = -p\), but Eqs. (23)–(25) imply \(E = -P\).

III. REGULAR BLACK HOLES

Whether the motivation is to construct a geodetically complete spacetime, to resolve the information loss paradox, or to illustrate the effects of quantum gravity, models of regular black holes (RBHs) envisage a trapped region with a singularity-free core (see e.g., [32–37] and the reviews [9, 38, 39]). Considerations of a geometric nature [40], as well as constraints from the effective field theory of quantum gravity [41, 42] restrict these models. Here we explore the impact the results of Sec. II make on them.

Many of the proposed static models [32, 34, 36] assume finite density and pressure at the horizon and thus belong to the class \(k = 1\). However, without a breakdown of semiclassical physics such solutions cannot be realized. Leaving aside the doubts about viability of the dynamical \(k = 1\) solutions, the static situation \((E = -P, \Phi = 0)\) still cannot arise at finite \(t\), as in this case the NEC is satisfied and the apparent horizon is hidden from Bob by the event horizon. Appearance of the feature that the model is intended to prevent indicates its breakdown.

Even the asymptotic case cannot be realized without some radical departures from the semiclassical physics. The zero flux limit can be produced only if the scenario of Eq. (41) is realized. However, in the limit \(8\pi r_g^2 E \to 1\) the formerly regular terms in the curvature scalars diverge (Appendix B3).

The leading behaviour of the function \(h\) of the \(k = 0\) solutions matches the regular static scenario with \(k = 1, \lambda = \frac{1}{2}\). Nevertheless, the latter is not a suitably defined limit of the former. First, some mechanism should freeze the apparent horizon and thus push \(T\) in \(\tau = -T^2 + \alpha x + \ldots\), etc., to zero. This is exactly opposite of the expected semiclassical behavior [2, 14, 27]. Moreover, to avoid a discontinuous change in the Misner-Sharp mass the linear terms in Eqs. (20) and (28) should match. However

\[
E = \frac{1}{24\pi r_g^2} \neq \frac{1}{8\pi r_g^2} = -P, \tag{42}
\]

where the first value of \(E\) is obtained by matching with the linear part of Eq. (20) and the second value results from Eq. (41).

A dynamical model of Refs. [34, 35] uses \((v, r)\) coordinates and the minimal modification of the Vaidya metric by setting

![FIG. 1. Schematic depiction of the evolution of a PBH from the point of view of a distant observer. The dark blue line represents the apparent horizon and the double dark red line represent the inner horizon. The trapped region is cross-hatched. The NEC-violating region (blue spread, dashed boundary) appears prior to the formation of the first marginally trapped surface \((F, r_F)\) and covers part of the trapped region. Its outer boundary is not constrained by our considerations. The thin black line traces the surface of the collapsing body is traced up to the NEC-violating region.](image)
with
\[ C_+(v) = \frac{2m(v)r^3}{r^3 + 2m(v)b^2}, \quad h_+ = 0, \tag{43} \]
for some \( b > 0 \) and \( m(v) \). When \( m \gg b \) the approximate locations of the apparent horizon and the inner horizon are given by
\[ r_g \approx 2m - \frac{b^2}{2m}, \quad r_{in} \approx \frac{5b}{4} - \frac{3b^2}{32m}, \tag{44} \]
respectively, and the non-zero components of the energy-momentum tensor at the apparent horizon are
\[ \Theta_{vv} \approx \frac{m'(v)}{16\pi m^2(v)}, \quad \Theta_{vr} \approx -\frac{3b^2}{128m^4}. \tag{45} \]
This model belongs to \( k = 0 \) class and is consistent with existence of the apparent horizon that was formed at a finite time of a distant observer.

However, it is unsuitable for describing formation and growth of a PBH. Leaving aside the issue of a transient firewall that accompanies this process, the NEC is not violated in this model for \( m'(v) > 0 \), and thus the apparent horizon, if exists, is hidden behind the even horizon that was purportedly eliminated. In fact, no model that uses \((u, r)\) coordinates and has a regular function \( h_+(u, r) \) can describe growth of a PBS, as in this case \( \tau_t \to +\Upsilon^2 \)
\[ \frac{\partial_r h_+}{r} = 4\pi \Theta_{rr} \to \frac{16\pi}{f^2} \Upsilon^2. \tag{46} \]
Since the energy density and pressure are negative in the vicinity of the apparent horizon and positive in the vicinity of the inner horizon [17, 21] there should be density and pressure jumps at the intersection of the two horizons, making problematic the blanket requirement of continuity of density and pressure. If we accept that violations of the QEI is a sufficient reason to discount the growth of trapped region, the horizon structure of a regular black hole is schematically shown on Fig. 1.

In this case the model with the metric functions (43) cannot describe the first stages of the evolution of the trapped region, even if \( C_{\text{new}} < 0 \). For a PBH of Fig. 1 both the apparent horizon and the inner horizon develop from a single trapped surface that appears at some \( t_F \) and meet again at \( t_E \), possibly forming a remnant. The Misner-Sharp mass of Eq. (43) allows a latter possibility (at \( m(v) = 3\sqrt{3}b/4 \), but not the former one, as Eq. (44) indicates.

IV. DISCUSSION

We have seen that in the vicinity of the apparent horizon a singular nature of Schwarzschild coordinates serves a useful purpose. Scaling of the suitably selected functions of the EMT components with the powers of \( f = (1 - C(t, r)/r) \) allows to classify solutions of the Einstein equations. Only two types of solutions with \( k = 0, 1 \) are possible. Both violate the NEC and result in a firewall at the expanding apparent horizon. Only \( k = 0 \) solutions allow to a collapsing thin shell to form a black hole or for a test particle to cross the apparent horizon. These failures cast doubts on the physical relevance of the \( k = 1 \) solutions.

Analysis of the inner regions of RBHs leads to the arguments indicating the need for physics beyond standard model to support such objects[37, 43]. Our analysis of the near-horizon regions indicates that \( k = 0 \) models of evaporating RBHs are no more exotic than any UCO with an apparent horizon. On the other hand, complete regularity (finite values of density and pressure for both static and infalling observers) of \( k = 0 \) may be impossible to realize without significant modification of the semiclassical gravity.

ACKNOWLEDGMENTS

Useful discussions with Valentina Baccetti, Robert Mann and Sebastian Murr are gratefully acknowledged.

Appendix A: Solutions with \( k < 1 \)

1. Behaviour of \( T_{\theta}^{\theta} \)

The regularity conditions
\[ T = -\frac{\tau_t}{f} + \frac{\tau_r}{f} + 2T_{\theta}^{\theta}, \tag{A1} \]
\[ \Xi = \left(\frac{\tau_t}{f}\right)^2 + \left(\frac{\tau_r}{f}\right)^2 - 2\left(\frac{\tau_t}{f}\right)^2 + 2(T_{\theta}^{\theta})^2 \tag{A2} \]
force the leading term in \( T_{\theta}^{\theta} \) to diverge (or slowly converge) at \( r_g \) in the same way as the other three terms. Set
\[ \Xi_1 := \lim_{r \to r_g} \frac{\tau_t}{f}, \quad \Xi_2 := \lim_{r \to r_g} \frac{\tau_r}{f}, \quad \Xi_3 := \lim_{r \to r_g} T_{\theta}^{\theta}, \quad \Xi_4 := \lim_{r \to r_g} \frac{\tau_r}{f}, \tag{A3} \]
(and focus on the leading terms only). The two conditions become
\[ -\Xi_1 + 2\Xi_3 + 2\Xi_2 = 0, \quad \Xi_1^2 + 2\Xi_3^2 + 2\Xi_2^2 - 2\Xi_3 = 0, \tag{A5} \]
Taking \( \Xi_1 \) and \( \Xi_2 \) as the independent variables we find
\[ \Xi_3 = \frac{1}{2}(\Xi_1 - \Xi_2), \quad \Xi_4 = \pm \frac{1}{2}\sqrt{3\Xi_1^2 + 3\Xi_2^2 - 2\Xi_3\Xi_1}. \tag{A6} \]
The Einstein equation (6) does not change; the leading terms of the Misner-Sharp mass are
\[ C = r_g - w_1x^j(2-k), \tag{A7} \]
where \( w_1 = (8(2-k)r_{\text{g}}^{3-k}\Xi_1)^{1/(2-k)} \). The limiting form of Eq. (8) now becomes
\[ \frac{\partial_r h}{r} = 4\pi(\Xi_1 + \Xi_2)x^j w_1^{k-2} \frac{w_1^{\frac{k-2}{2}}}{x} = -\frac{\Xi_1 + \Xi_2}{2(2-k)\Xi_1 x}. \tag{A8} \]
The leading term of the solution are
\[ h = -\frac{\Xi_1 + \Xi_2}{2(2-k)\Xi_1} \ln \frac{x}{\xi}. \] (A9)

Hence consistency of Eq. (7) imposes
\[ -\frac{\Xi_1 + \Xi_2}{2(2-k)\Xi_1} = -\frac{1}{2-k}, \] (A10)
resulting in \( \Xi_1 = \Xi_2 \) and
\[ \Xi_3 = 0, \quad \Xi_4 = \pm \Xi_1. \] (A11)

2. Leading terms of the solutions, \( k < 1 \)

For regular \( T^\theta_\theta \equiv T^\phi_\phi \) Eq. (19) applies and as \( r \to r_g \) the Einstein equations with divergent terms become
\[
\partial_r C = -8\pi r_g^2 \gamma^2 f^{-k-1}, \] (A12)
\[
\partial_t C = \pm 8\pi r_g^2 \gamma^2 \gamma^{-k} f^k, \] (A13)
\[
\partial_r h = 8\pi r_g^2 \gamma^2 f^{k-2}. \] (A14)

The leading terms of the metric functions are then
\[ C = r_g(t) - w_1 x^{1/(2-k)}, \quad w_1 \gamma^{-k} = 8(2-k)\pi r_g^3 \gamma^2, \] (A15)
and
\[ h = -\frac{1}{2-k} \ln \frac{x}{\xi}, \] (A16)
and Eq. (A13) results in the constraint
\[ \pm r_g' = \frac{w_1 \gamma^{1/(2-k)}}{r_g}, \] (A17)

3. Regular solutions in \( (v, r) \) coordinates.

Existence of the apparent horizon constrains the Misner-Sharp function to have the form
\[ C_+ = r_+(v) + w_1^+ (v) x^{1-\alpha_1} + w_2^+ (v) x^{1-\alpha_1+\alpha_2+\ldots}, \] (A18)
where \( x = r - r_+ \), and \( \alpha_1 < 1, \alpha_2 > 0 \), while we keep the function \( h \) unconstrained,
\[ h_+ = h_+(v) \ln x/\xi(v) + h_1^+ (v) x^{\beta_1} + h_2^+ (v) x^{\beta_1+\beta_2} + \ldots. \] (A19)

Explicit evaluation of the Ricci scalar \( R \) for the metric (12) with the above functions results in a number of the divergent terms, that as \( x \to 0 \) behave as its various powers. The curvature scalar is finite if the coefficients of all such divergent powers cancel. However, it is possible only if \( h_+ = 0 \), as well as all the coefficients of all fractional powers that are less than two.

Appendix B: Some properties of the solutions

1. Firewall at the apparent horizon, \( k < 1 \)

For a radially infalling massive particle the four-velocity components are related by
\[ T_A = \frac{\sqrt{F + R_A^2}}{e^H} \approx \frac{|\dot{R}|}{e^H} + \frac{e^H}{2R_A}, \] (B1)
where \( H = h(T_A, R_A) \) and \( F = f(T_A, R_A) \). For \( k < 1 \) means that the four-velocity of an infalling observer at the leading order is given by
\[ u_A^\mu = |\dot{R}| \left( \frac{r_g}{w_1 \xi^{1/(2-k)}}, -1, 0, 0 \right), \] (B2)
while the leading terms in the \((t, r)\) block of the EMT are
\[ T_{ab} = \frac{T^{(2-u^2)}_{w^2}}{x r_g^{k-2}} \left( \frac{w_1^2 \xi^{2/(2-k)}}{r_g^2 \pm w_1 \xi^{1/(2-k)} / r_g} \right), \] (B3)
where the upper (lower) sign corresponds to \( r_g' < 0 \) (\( r_g' > 0 \)), respectively. In the former case the divergent terms in the energy density in the frame of an infalling observer cancel out. However, for the expanding apparent horizon its negative and divergent,
\[ \rho_A = T_{\mu\nu} u_A^\mu u_A^\nu \approx -\frac{4\dot{R}^2 \gamma^2 r_g^{2-k}}{w_1^{2-k} X}, \] (B4)
where \( X = R_A - r_g \).

2. Firewall at the apparent horizon, \( k = 1 \).

The leading terms of the metric functions are
\[ C = r_g + 8\pi E r_g^2 x, \quad h = -\ln x/\xi. \] (B5)
The four-velocity of an infalling observer at the leading order is then given by
\[ u_A^\mu = \left| \dot{R} \right| \left( \frac{r_g}{\xi (1 - 8\pi E r_g^2)}, -1, 0, 0 \right), \] (B6)
and the leading terms in the \((t, r)\) block of the EMT are
\[ T_{ab} = -\frac{1}{r_g x} \left( 8\pi E (1 - 8\pi E r_g^2) \xi^2 / (2 - 8\pi E r_g^2), \right. \] (B7)
\[ \left. r_g^2 \quad (2 - 8\pi E r_g^2) / (1 - 8\pi E r_g^2) \right). \]
In the case of expansion the function \( \xi \) satisfies
\[ (1 - 8\pi E r_g^2) \xi = +r_g r_g', \] (B8)
and the resulting energy density diverges as
\[ \rho_A \approx -\frac{4\dot{R}^2}{r_g X}. \] (B9)
For spacetimes of small curvature explicit expressions that bound time-averaged energy density for a geodesic observer were derived in Ref. [25]. For any Hadamard state $\omega$ and a sampling function $f(\tau)$ of compact support, negativity of the expectation value of the energy density $\rho_A$ as seen by a geodesic observer on a trajectory $\gamma(\tau)$ is bounded by

$$\int_{\gamma} f^2(\tau) \rho d\tau \geq -B(R, f, \gamma), \quad (B10)$$

where $B > 0$ is a bounded function that depends on the trajectory, the Ricci scalar and the sampling function [25].

Consider a growing apparent horizon, $r_g' > 0$. For a macroscopic black hole the curvature at the apparent horizon is low and its radius does not appreciably change while the observer (a massive test particle) moves in its vicinity. Then $X \approx R$, and for a given geodesic trajectory we can choose $f \approx 1$ at the horizon crossing and $f \to 0$ within the NEC-violating domain. As the trajectory passes through $X_0 + r_g \to r_g$ the l.h.s of Eq. (B10) behaves as

$$\int_{\gamma} f^2(\tau) \rho d\tau \approx - \int_{\gamma} \frac{4R^2 d\tau}{r_g^2 X} \approx \int_{\gamma} \frac{4\hat{R} dX}{r_g X} \propto \log X_0 \to -\infty, \quad (B11)$$

where we used $\hat{R} \sim \text{const}$. The r.h.s of Eq. (B10) remains finite, and thus the QEI is violated.

### 3. The Ricci Scalar

For a dynamical solution in the case $k = 1$ with the metric functions given by Eqs. (28) and (30) expansion of the Ricci scalar near the apparent horizon gives

$$R = -\frac{(1 - 8\pi Er_g^2)^2 \xi^2 - r_g^2 x^2}{r_g (1 - 8\pi Er_g^2) k^2 x} + \mathcal{O}(x^3). \quad (B12)$$

It is finite if and only if Eq. (32) is satisfied.

The Ricci scalar diverges if the evaporating black hole freezes ($r_g \to 0$), as the regular (as a function of $x$) part of $R$ contains a clearly divergent term

$$R_0 = \frac{1}{r_g^2 x^2}. \quad (B13)$$

If the metric function $h$ has a proportionality coefficient that is different from one,

$$h = -\lambda \ln x/\xi + \ldots, \quad (B14)$$

the Ricci scalar may have a number of divergent terms. However, it will contain a particular term

$$R_{-1} = \frac{\lambda(2\lambda - 1)(1 - 8\pi Er_g^2)}{r_g x}, \quad (B15)$$

that will be finite only if $\lambda = 0, \frac{1}{2}$ or $E = (8\pi r_g)^{-1}$. Since to have an evolving apparent horizon with the flux $\tau_g^f = \Phi f^{k_\phi}$, $k_\phi > 1$ it is necessary to have $\lambda = k_\phi$, we see that this is impossible.

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[1] S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Space-Time* (Cambridge University Press, 1973).
[2] V. P. Frolov and I. D. Novikov, *Black Holes: Basic Concepts and New Developments*, (Kluwer, Dordrecht, 1998).
[3] N. D. Birrel and P. C. W. Davies, *Quantum Fields in Curved Space* (Cambridge University Press, Cambridge, 1986).
[4] J. F. Donoghue, arXiv:1209.3511 (2012); N. E. J. Bjerrum-Bohr, J. F. Donoghue, B. R. Holstein, L. Planté, and P. Vanhove, Phys. Rev. Lett. **114**, 061301 (2015).
[5] C. Rovelli and F. Vidotto, *Covariant Loop Quantum Gravity* (Cambridge University Press, Cambridge, 2014)
[6] P. Martín-Moruno and M. Visser, *Classical and Semi-classical Energy Conditions, in Wormholes, Warp Drives and Energy Conditions*, edited by F. N. S. Lobo (Springer, New York, 2017), p. 193.
[7] V. Cardoso and P. Paní, Nature Astr. **1**, 586 (2017); V. Cardoso and P. Paní, Living Rev. Relat. **22**, 4 (2019).
[8] L. Barack, V. Cardoso, S. Nissanka, and T. P. Sotiriou (eds.), *Black holes, gravitational waves and fundamental physics: a roadmap*, Class. Quant. Grav. **36**, 143001 (2019).
[9] P. Chen, Y. C. Ong, and D.-h. Yeom, Phys. Reports **603**, 1 (2015).
[10] C. Bambi, *Black Holes: a Laboratory for Testing Strong Gravity* (Springer, Singapore, 2017); C. Bambi, Ann. Phys. (Berlin) **530**, 1700430 (2018).
[11] LIGO Scientific Collaboration and Virgo Collaboration, Phys. Rev. X **9**, 031040 (2019).
[12] Event Horizon Telescope Collaboration, Astrophys. J. Lett. **875**, L1 (2019).
[13] R. B. Mann, *Black Holes: Thermodynamics, Information, and Firewalls* (Springer, New York, 2015); V. Baccetti, V. Hussain and D. R. Terno, Entropy **19**, 17 (2017).
[14] D. Harlow, Rev. Mod Phys. **88**, 015002 (2016); W. G. Unruh and R. M. Wald, Rep. Prog. Phys. **80**, 092002 (2017); D. Marolf Rep. Prog. Phys. **80**, 092001 (2017).
[15] E. Curiel, Nature Astron. **3**, 27 (2019).
[16] V. Faraoni, *Cosmological and Black Hole Apparent Horizons*, (Springer, Heidelberg, 2015).
[17] V. P. Frolov, arXiv:1411.6981 (2014).
[18] V. Baccetti, R. B. Mann, S. Murk, and D. R. Terno, Phys. Rev. D **99**, 124014 (2019).
[19] V. Faraoni, G. F. R. Ellis, J. T. Firouzjaee, A. Helou, and I. Musco, Phys. Rev. D **95**, 024008 (2017).
[20] V. Baccetti, S. Murk, and D. R. Terno, Phys. Rev. D **100**, 064054 (2019).
[21] D. R. Terno, Phys. Rev. D **100**, 124025 (2019).
[22] C. W. Misner and D. H. Sharp, Phys. Rev. **136**, B571 (1964).
[23] P. T. Chruściel, J. L. Costa, and M. Heusler, Living Rev. Rel.
15, 7 (2012).

[24] C. J. Fewster, *Quantum Energy Inequalities*, in *Wormholes, Warp Drives and Energy Conditions*, edited by F. N. S. Lobo (Springer, New York, 2017), p. 215.

[25] E.-A. Kontou and K. D. Olum, Phys. Rev. D 91, 104005 (2015).

[26] J. M. Bardeen, Phys. Rev. Lett. 46, 382 (1981).

[27] R. Brout, S. Massar, R. Parentani, and P. Spindel, Phys. Rep. 260, 329 (1995).

[28] A. Levi and A. Ori, Phys. Rev. Lett. 117, 231101 (2016).

[29] H. Kawai, Y. Matsuo, Y. Yokokura, Int. J. Mod. Phys. A 28, 1350050 (2013).

[30] I. Nagle, R. B. Mann and D. R. Terno, Nucl. Phys. B 936, 18 (2018).

[31] P. Chen, W. G. Unruh, C-H. Wu, and D.-h. Yeom, Phys. Rev. D 97, 064045 (2018).

[32] J. M. Bardeen, in *Proceedings of International Conference GR5*, (Tbilisi, USSR, 1968), p. 174 (cited in [9]).

[33] V. P. Frolov and G. A. Vilkovisky, Phys. Lett. 106B, 307 (1981).

[34] S. Hayward, Phys. Rev. Lett. 96, 031103 (2006).

[35] V. P. Frolov, JHEP 05(2014), 049 (2014).

[36] R. Carballo-Rubio, F. Di Filippo, S. Liberati, C. Pacillo, and M. Visser, JHEP 07(2018) 023 (2018).

[37] R. Brustein and A. J. M. Medved, Phys. Rev. D 99, 064019 (2019).

[38] S. D. Mathur, Fortsch. Phys. 53, 793 (2005).

[39] S. Ansoldi, arXiv:0802.0330 [gr-qc] (2008).

[40] R. Carballo-Rubio, F. Di Filippo, S. Liberati, and M. Visser, arXiv:/1908.03261 (2019); R. Carballo-Rubio, F. Di Filippo, S. Liberati, and M. Visser, arXiv:/1911.11200 (2019).

[41] V. P. Frolov, M. Markov, and V. F. Mukhanov, Phys. Lett. B 216, 272 (1989).

[42] T. De Lorenzo, C. Pacilio, C. Rovelli, and S. Speziale, Gen. Relat. Grav. 47, 41 (2015).

[43] R. Brustein and A. J. M. Medved, Fortsch. Phys. 67, 1900058 (2019).