Inverse scattering and solitons in $A_{n-1}$ affine Toda field theories II

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Abstract

New single soliton solutions to the affine Toda field theories are constructed, exhibiting previously unobserved topological charges. This goes some of the way in filling the weights of the fundamental representations, but nevertheless holes in the representations remain. We use the group doublecross product form of the inverse scattering method, and restrict ourselves to the rank one solutions.
1 Introduction

This paper should be considered as the sequel to our earlier paper [1], where some single soliton solutions to the $A_{n-1}$ affine Toda field theories were computed by studying the space-time evolution of the residues of simple poles in the underlying loop group, which is a variation of the inverse scattering method. In the original paper this was carried out with the aim of finding new single soliton solutions with new topological charges which would fill some of the missing weights in the fundamental representations, and thereby go some way towards confirming the affine quantum group symmetry of the model. Filling all the weights is needed to justify the S-matrices calculated in [2]. The reader can refer to McGhee [3] for the calculation of the previously known topological charges. The attempt in the original paper failed. Although extra solutions were found, it was discovered that they simply represented modes of oscillation around the standard solutions, and were not fundamentally new, and certainly did not yield any new topological charge sectors. It was also seen how these solutions could be obtained by restricting multi-soliton solutions. In this paper, we shall exhibit some new solutions which lead to new topological charges, and which have the correct weights which lie in the appropriate fundamental representations. It will also be easy enough to see that the new solitons have the correct masses, corresponding to the multiplet which they belong to, and that they scatter classically in the correct way, i.e. they have the same time delays [4] when they scatter with other solitons. However not all the missing charges are found by this construction, in particular no new charges are found for $A_3$.

We shall also show that the new solutions can be obtained by restricting more complicated previously known multi-soliton solutions, so that the new solutions could have been found without using inverse scattering. However, it has become clear that inverse scattering has the advantage of giving a unified approach to the possible ‘variations on the theme of a single soliton’. While these solutions can be derived from multi-solitons, it is not at all obvious to see how to restrict the momenta of the multi-solitons to effectively give a single soliton.

We use a loop group, defined in [1], which consists of matrix valued meromorphic functions on $\mathbb{C} \setminus \{0\}$ satisfying a certain symmetry condition. The affine Toda equation is solved by reversing the order of factorisation between an analytic element and a meromorphic element regular at 0 and $\infty$. However in the earlier paper we used a formulation of the meromorphic function which was not the most general.

In this paper, we modify the procedure used in [1], and for the rank one case manage to determine the space-time evolution of the kernel of the residue of the meromorphic function independently to that of the image, where previously a special projection for the residue was chosen. The projection was originally chosen so that the inverse loops had a particularly attractive form, which was necessary because we could not determine the space-time evolution of the kernel of the residue, only the image. In fact with this projection, the kernel was fixed completely by the image, and this allowed us to side-step the issue about the kernel, without meeting the problem head on.

2 Preliminaries

We recall the setup and notation used in [1]. The affine Toda field equations for the affine algebra $\hat{g}$ follow from the zero-curvature condition

$$[\partial_+ + A_+, \partial_- + A_-] = 0,$$

where $A_\pm$ are given, in terms of the spectral parameter $\lambda$, by

$$A_\pm = \pm \frac{1}{2} \partial_\pm (u.H) \pm \lambda^{\pm 1} \mu e^{\mp \frac{1}{2} \theta u.H} E_{\pm 1} e^{\mp \frac{1}{2} \theta u.H},$$

(2.1)
where $x_\pm = t \pm x$, and $H$ is the Cartan-subalgebra of $g$. In a basis where $H$ is diagonal and for $g = A_{n-1}$, in the vector representation the $n \times n$ matrices $E_{\pm 1}$ are defined to be

$$E_{+1} = E_{-1}^\dagger = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

We are interested in solutions $\Phi(\lambda)$ to the linear system:

$$\partial_\pm \Phi = \Phi A_\pm. \quad (2.3)$$

The simple vacuum solutions are given by $u$ constant with $e^{\beta u. H/2}$ commuting with $E_{+1}$ and $E_{-1}$. In this case we define

$$A_+ = J(\lambda) = \mu \lambda E_{+1}, \quad A_- = K(\lambda) = -\mu \lambda^{-1} E_{-1}. \quad (2.4)$$

The equations

$$\partial_+ \Phi_0 = \Phi_0 J \quad \text{and} \quad \partial_- \Phi_0 = \Phi_0 K,$$

have the simple exponential solution

$$\Phi_0 = C e^{Jx_+ + Kx_-},$$

and if we subtract off this solution from $\Phi$ by setting

$$\phi = \Phi_0^{-1} \Phi,$$

we see that $\phi$ obeys the equations

$$\partial_+ \phi = \phi A_+ - J \phi \quad \text{and} \quad \partial_- \phi = \phi A_- - K \phi. \quad (2.5)$$

To construct soliton solutions, we suppose that $\phi$ is a meromorphic function of $\lambda$ which is regular at $0$ and $\infty$. The classical ‘vacuum’ element $a(x, t)$ is defined by $a(x, t) = e^{-J(\lambda)x_+ - K(\lambda)x_-}$, and is an analytic function of $\lambda \in \mathbb{C} \setminus \{0\}$. In [1], the group factorisation result

$$a(x, t).\phi(\lambda, 0, 0) = \phi(\lambda, x, t).b, \quad (2.6)$$

where $b(x, t)$ is another analytic function of $\lambda$, is established.

We also define the diagonal matrix

$$U = \begin{pmatrix} 1 & \omega^{-1} & \omega^{-2} & \cdots & \omega^{-(n-1)} \\ \omega^{-1} & \omega & \omega^2 & \cdots & \omega^{n-1} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \omega^{-(n-1)} & \omega^{-(n-2)} & \cdots & \omega & 1 \end{pmatrix}, \quad (2.7)$$

where we recall that $\omega = e^{2\pi i/n}$. With these definitions for $U$ and $E_{\pm 1}$, it is easy to check the relations

$$UE_{\pm 1} U^\dagger = \omega_{\pm 1} E_{\pm 1}. \quad (2.8)$$

It follows that we can consider solutions $\phi(\lambda)$ that satisfy the condition

$$U \phi(\lambda) U^\dagger = \phi(\omega \lambda). \quad (2.9)$$
3 The form of the meromorphic loops

We begin by considering a meromorphic \( \phi \) which satisfies the symmetry condition (2.9). If \( \phi \) has a pole at \( \alpha \), then by (2.9) it also has poles at \( \{ \omega \alpha, \omega^2 \alpha, \ldots, \omega^{n-1} \alpha \} \). The most general form for \( \phi \) satisfying (2.9) with a single set of poles is

\[
\phi(\lambda) = \left( \frac{\lambda \xi}{\lambda - \alpha} + \frac{\lambda U \xi U^\dagger}{\lambda - \omega \alpha} + \cdots + \frac{\lambda U^{n-1} \xi U^{n-1}}{\lambda - \omega^{n-1} \alpha} + k \right) e^{-\beta u.H/2},
\]

where

\[
k = \begin{pmatrix}
k_1^n & 0 & \cdots & 0 \\
0 & k_2^n & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
k_1^n & 0 & \cdots & k_n^n
\end{pmatrix}.
\]

The factor of \( e^{-\beta u.H/2} \) is introduced for convenience, as we shall see in a moment. By summing geometric series we write, using \( z = \frac{\lambda}{\alpha} \),

\[
\frac{\lambda \xi}{\lambda - \alpha} + \frac{\lambda U \xi U^\dagger}{\lambda - \omega \alpha} + \cdots + \frac{\lambda U^{n-1} \xi U^{n-1}}{\lambda - \omega^{n-1} \alpha} = \frac{nz - 1}{z^n - 1} \begin{pmatrix}
z^{n-1} \zeta_{11} & z \zeta_{12} & \cdots & z^{n-2} \zeta_{1n} \\
z^{n-2} \zeta_{12} & z^{n-1} \zeta_{22} & \cdots & z^{n-3} \zeta_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
z \zeta_{n-1} & z^2 \zeta_{n-1} & \cdots & z^n \zeta_{nn}
\end{pmatrix}. \tag{3.3}
\]

Now we must discuss how to fix the diagonal matrix \( k \). As in equation (6.1) of \([1]\) we impose the condition

\[
\phi(\infty) = 1.e^{-\beta u.H/2}, \tag{3.4}
\]

or equivalently

\[
k_i^n = 1 - n a_{ii}, \quad i = 1, \ldots, n. \tag{3.5}
\]

In this paper we will only discuss the case where the matrix residue \( \xi \) is of rank one. The most general rank one matrix can be written

\[
\xi = \begin{pmatrix}
v_1 & 0 & 0 & \cdots & 0 \\
v_2 & 0 & 0 & \cdots & 0 \\
\vdots & 0 & 0 & \cdots & 0 \\
v_n & 0 & 0 & \cdots & 0
\end{pmatrix} \begin{pmatrix}
w_1 & w_2 & \cdots & w_n \\
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0
\end{pmatrix}. \tag{3.6}
\]

Now we can calculate

\[
\det(\phi e^{\beta u.H/2}) = \frac{\lambda^n - (k_1 k_2 \cdots k_n \alpha)^n}{\lambda^n - \alpha^n}. \tag{3.7}
\]

Now we examine \( \phi(\lambda)^{-1} \), which is required in subsequent sections. From the determinant (3.7), \( \phi(\lambda)^{-1} \) only has one set of poles at \( \lambda = \gamma, \omega \gamma, \ldots, \omega^{n-1} \gamma \), for \( \gamma = \tau \alpha \), where for convenience we set \( \tau = k_1 k_2 \cdots k_n \). The condition \( U \phi(\lambda) U^\dagger = \phi(\omega \lambda) \) implies \( U \phi(\lambda)^{-1} U^\dagger = \phi(\omega \lambda)^{-1} \), so \( \phi(\lambda)^{-1} \) can be written in the same form as \( \phi(\lambda) \), i.e. equation (3.1), but with different matrices instead of \( \xi \) and \( k \). It is possible to show that

\[
\phi(\lambda)^{-1} = e^{\beta u.H/2} \left( \frac{\lambda \xi}{\lambda - \gamma} + \frac{\lambda U \xi U^\dagger}{\lambda - \omega \gamma} + \cdots + \frac{\lambda U^{n-1} \xi U^{n-1}}{\lambda - \omega^{n-1} \gamma} + k^{-1} \right), \tag{3.8}
\]
where $\xi$ is given explicitly by

$$
\xi = \left( \frac{\tau^n}{k_1^n} \right) \begin{pmatrix}
u_1 & 0 & \cdots & 0 \\
v_2(\tau^{-1}k_1^n) & 0 & \cdots & 0 \\
v_3(\tau^{-2}k_1^n k_2^n) & 0 & \vdots & 0 \\
v_4(\tau^{-3}k_1^n k_2^n k_3^n) & 0 & \vdots & 0 \\
\vdots & 0 & \vdots & 0 \\
v_n(\tau^{-(n-1)}k_1^n k_2^n \cdots k_{n-1}^n) & 0 & \cdots & 0 \\
\end{pmatrix}
$$

$$
\times \begin{pmatrix}
w_1 & w_2(\tau k_2^{-n}) & w_3(\tau^2 k_2^{-n} k_3^{-n}) & w_4(\tau^3 k_2^{-n} k_3^{-n} k_4^{-n}) & \cdots & w_n(\tau^{n-1} k_2^{-n} k_3^{-n} \cdots k_n^{-n}) \\
0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0 \\
\end{pmatrix}.
$$

(3.9)

4 Reconstruction of the meromorphic loop

We shall take the dot product of vectors to be $u.v = u_1v_1 + \ldots + u_nv_n$ in terms of components. This omits the more usual complex conjugate, but we shall find this form convenient for constructing matrices.

It is our task in this section to reconstruct the meromorphic function $\phi$ from the data $V = \text{Im} \left( \text{res} \phi(\lambda) \big|_{\lambda=\alpha} \right)$ and $W = \text{Ker} \left( \text{res} \phi(\lambda)^{-1} \big|_{\lambda=\gamma} \right)$, provided the extra complex number $\tau$ is given. The subspace $W$ is $n-1$ dimensional, and we shall describe it by the vector $s$ which is perpendicular to $W$, i.e. $s.m=0$ for all $m \in W$. The vector $s$ is only determined up to scalar multiple, but we can read off the ratios of its components from equation (3.9) as

$$
\frac{w_2 \tau}{w_1 k_2^n} = \frac{s_2}{s_1}, \quad \frac{w_3 \tau^2}{w_1 k_2^n k_3^n} = \frac{s_3}{s_1}, \quad \frac{w_4 \tau^3}{w_1 k_2^n k_3^n k_4^n} = \frac{s_4}{s_1}, \\
\vdots & = \vdots \\
\frac{w_n \tau^{n-1}}{w_1 k_2^n k_3^n \cdots k_n^n} = \frac{s_n}{s_1}.
$$

(4.1)

From the condition (3.3) we could write each $k_i$ in terms of $v$ and $w$. However we have no direct knowledge of the vector $w$, we just know $v$ and $s$ up to scalar multiples. We combine (3.3) with (1.1), and write $p_i = nv_is_i$, to get the following results:

$$
k_2^n = 1 - nw_2v_2
$$

$$
= 1 - \frac{p_2}{\tau} \frac{w_2}{s_1} \frac{k_2^n}{s_1},
$$

(4.2)

Thus

$$
k_2^n = \frac{1}{1 + \frac{p_2}{\tau} \frac{w_2}{s_1}},
$$
Now
\[ k^n_3 = 1 - \frac{p^n w_1}{s_1} k^n_3 k^n_3, \]
so
\[ k^n_3 = \frac{\alpha \cdot n}{w_1} + \frac{p^n}{\tau} + \frac{p^n}{\tau^2} + \cdots + \frac{p_{n-2} \cdot 2}{\tau^{n-2}} + \frac{p_{n-1}}{\tau^{n-1}}, \]
and continuing in this way, we see that
\[ k^n_i = \frac{\alpha \cdot n}{w_1} + \frac{p^n}{\tau} + \frac{p^n}{\tau^2} + \cdots + \frac{p_{n-2} \cdot 2}{\tau^{n-2}} + \frac{p_{n-1}}{\tau^{n-1}}, \]
for \( i = 2, \ldots n \). The condition (3.7) also gives
\[ k^n_1 = 1 - p_1 \frac{w_1}{s_1} \] (4.4)
If we use the fact that
\[ k^n_1 k^n_2 \cdots k^n_n = \tau^n, \]
then
\[ \frac{(\frac{\alpha \cdot n}{w_1} - p_1)}{\frac{\alpha \cdot n}{w_1} + \frac{p^n}{\tau} + \frac{p^n}{\tau^2} + \cdots + \frac{p_{n-2} \cdot 2}{\tau^{n-2}} + \frac{p_{n-1}}{\tau^{n-1}}} = \tau^n, \]
resulting in
\[ \frac{s_1}{w_1} = \frac{p_1 + p_n \tau + p_{n-1} \tau^2 + p_{n-2} \tau^3 + \cdots + p_2 \tau^{n-1}}{1 - \tau^n}. \]
Substitution back into (4.4) gives
\[ k^n_1 = \frac{\tau(p_1 + p_n \tau + \cdots + p_2 \tau^{n-2} + p_1 \tau^{n-1})}{p_1 + p_n \tau + p_{n-1} \tau^2 + \cdots + p_2 \tau^{n-1}}, \]
and for general \( i \), substituting into (4.3) and with the understanding of taking \( j \) mod \( n \) in \( p_j \):
\[ k^n_i = \frac{\tau(p_1 + p_{n-1} \tau^2 + \cdots + p_1 + p_{n-1} \tau) + \cdots + p_{i-1} \tau^2 + \cdots + p_{n-1} \tau^{n-1}}{p_1 + p_{n-1} \tau + p_{n-2} \tau^2 + \cdots + p_{i-1} \tau^2 + \cdots + p_{n-1} \tau^{n-1}}. \]
We have thus reconstructed the \( k_i \) from the vectors \( v \) and \( s \), together with \( \tau \).

5 The rank one solution

We begin by specifying the initial meromorphic loop \( \phi_0 \) with poles at \( \{\alpha, \omega \alpha, \ldots, \omega^{n-1} \alpha\} \). The factorisation (2.0) solves the space-time evolution of \( \phi \), given some initial \( \phi_0 \). Thus \( a \cdot \phi_0 = \phi \cdot \mathbf{b} \), where \( a \) and \( b \) are analytic on \( \mathbb{C} \setminus \{0\} \), and we see that the pole position \( \alpha \) of \( \phi \) is independent of space-time. The inverse \( \phi_0^{-1} \) has poles at \( \{\gamma, \omega \gamma, \ldots, \omega^{n-1} \gamma\} \). From the factorisation \( \phi_0^{-1} \cdot a^{-1} = b^{-1} \cdot \phi^{-1} \) we see that \( \gamma \) for \( \phi^{-1} \) also does not depend on space-time, and from this we deduce that \( \tau \) must be a constant.

As \( \phi_0 \) is specified, we know
\[ V_0 = \text{Im} \left( \text{res} \phi_0(\lambda) \right)_{\lambda = \alpha} \quad \text{and} \quad W_0 = \text{Ker} \left( \text{res} \phi_0(\lambda)^{-1} \right)_{\lambda = \gamma}. \]
For the factorisation \( a \cdot \phi_0 = \phi \cdot b \), we take the residues at \( \lambda = \alpha \) of both sides of the equation. Since \( b(\alpha) \) is invertible, by comparing images it follows that
\[ V = \text{Im} \left( \text{res} \phi(\lambda) \right)_{\lambda = \alpha} = a(\alpha)V_0. \]
If we invert the factorisation to get $\phi^{-1} \cdot \phi^{-1} = b^{-1} \cdot \phi^{-1}$, taking the residues of both sides at $\lambda = \gamma$, and applying a vector $m$ to both sides, we get

$$\left( \text{res } \phi(\lambda)^{-1} \bigg|_{\lambda=\gamma} \right) a(\gamma)^{-1} m = b(\gamma)^{-1} \left( \text{res } \phi(\lambda)^{-1} \bigg|_{\lambda=\gamma} \right) m.$$  

From this it follows that

$$a(\gamma)^{-1} m \in \text{Ker } \left( \text{res } \phi(\lambda)^{-1} \bigg|_{\lambda=\gamma} \right) \quad \text{if and only if} \quad m \in \text{Ker } \left( \text{res } \phi(\lambda)^{-1} \bigg|_{\lambda=\gamma} \right),$$

thus

$$W = \text{Ker } \left( \text{res } \phi(\lambda)^{-1} \bigg|_{\lambda=\gamma} \right) = a(\gamma)W_0.$$  

We will take the one dimensional subspace $V_0$ to be spanned by the vector $v_0$, and then $V$ will be spanned by $v = a(\alpha)v_0$. The $n - 1$ dimensional subspace $W_0$ will be specified by its perpendicular vector $s_0$. Then the perpendicular vector $s$ describing $W$ is defined by $s = (a(\gamma)^T)^{-1}s_0$.

As $\lambda = 0$ is a regular point of $\phi(\lambda)$, from the linear system (2.5) we have

$$\phi(0)^{-1}E_{-1}\phi(0) = e^{-\beta u \cdot H/2} E_{-1}e^{\beta u \cdot H/2}. \quad (5.1)$$

Since $\phi(0)$ is diagonal we must have

$$\frac{1}{\tau} \begin{pmatrix} k_1^n & 0 & \cdots & 0 \\ 0 & k_2^n & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & k_n^n \end{pmatrix} = e^{\beta u \cdot H}. $$

Thus to determine $e^{\beta u \cdot H}$ we have to find the $k_i$ from our known quantities $v$, $s$ and $\tau$, which is what we did in the last section. Writing the fundamental weights $\{\lambda_i : i = 1, \ldots, n - 1\}$ in terms of the weights of the basic representation (the eigenvalues of $H$), we get the solution:

$$e^{-\beta \lambda_i \cdot u} = \frac{p_1 + p_n \tau + p_{n-1} \tau^2 + \cdots + p_2 \tau^{n-1}}{p_n + p_{n-1} \tau + \cdots + p_2 \tau^{n-2} + p_1 \tau^{n-1}} \quad i = 1, \ldots, n - 1, \quad (5.2)$$

6 Discussion of the rank one result

We recall first the standard soliton solutions to these theories and the rules for composing these solutions together. For

$$e^{-\beta \lambda_i \cdot u} = \frac{\tau_i}{\tau_0}, \quad i = 1, \ldots, n - 1,$$  

recall that the standard one-soliton solution of species $j$ is

$$\tau_i = 1 + Q \omega^j W_j(\theta), \quad i = 0, \ldots, n - 1, \quad (6.2)$$

where $W_j(\theta) = e^{m_j(\theta)}W_j(\theta)$. The constant $m_j$ is proportional to the mass of the soliton of species $j$, which is $m_j = 2 \sin \frac{\pi j}{n}$. This solution can be added to some solution $\tau_i^{\text{old}}$, already in the soliton sector of the theory, by the rule

$$\tau_i^{\text{new}} = \tau_i^{\text{old}} + Q \omega^j W_j(\theta) + \text{cross-terms},$$

6
where the cross-terms are formed by the rule:
For any term \(AW_{j_1}(\theta_1)W_{j_2}(\theta_2) \cdots W_{j_r}(\theta_r)\) in \(\tau^\text{old}_i\), we pick up a cross-term

\[QAW_{j_1}(\theta_1)X^{j_2}(\theta - \theta_2)X^{j_3}(\theta - \theta_3) \cdots X^{j_r}(\theta - \theta_r)W_j(\theta)W_{j_1}(\theta_1)W_{j_2}(\theta_2) \cdots W_{j_r}(\theta_r).\]

The interaction coefficient \(X^{jr}(\theta)\) has the definition

\[
X^{jr}(\theta) = \left(e^\theta - e^{\frac{\pi i(j-r)}{m}}\right)\left(e^\theta - e^{\frac{\pi i(r-1)}{m}}\right). \tag{6.3}
\]

Thus with this rule, we can immediately write down the standard two-soliton solution, starting from the standard one-soliton solution above:

\[\tau_1 = 1 + Q_1\omega^jW_j(\theta_1) + Q_2\omega^{jr}W_r(\theta_2) + Q_3\omega^{(j+r)}X^{jr}(\theta_1 - \theta_2)W_j(\theta_1)W_r(\theta_2). \tag{6.4}\]

We now consider the solution derived by the rank one inverse scattering method, equation (6.2), and discuss how it is composed of \(W\)'s, and how it can be recovered by considering multi-soliton solutions.

Let \(e_i : i = 0, \ldots, n - 1\) be the simultaneous eigenvectors of \(E_{+1}\) and \(E_{-1}\) (recall that \([E_{+1}, E_{-1}] = 0\)

\[
e_i = \begin{pmatrix} 1 \\ \omega^i \\ \omega^{2i} \\ \vdots \\ \omega^{(n-1)i} \end{pmatrix}, \quad \text{where} \quad E_{\pm 1}e_i = \omega^{\pm i}e_i : \quad i = 0, \ldots, n - 1.
\]

Then

\[
a(\alpha)e_j = e^{-\alpha\mu E_{+1}x_+ + \alpha^{-1}\mu E_{-1}x_- - e_j} = e^{-\mu\alpha\omega^jx_+ + \mu\alpha^{-1}\omega^{-j}x_- - e_j}.
\]

We set

\[W_j = e^{-\mu\alpha(\omega^{-j}x_+ + \mu\alpha^{-1}(\omega^{-j}x_- - e_j)}, \tag{6.5}\]

fixing the phase of \(\alpha\) by setting \(\alpha = i\omega^{-j/2}e^{-\theta}\), then

\[W_j(\theta) = e^{m_j(e^{-\theta}x_+ - e^\theta x_-)}, \tag{6.2}\]

as before. We also introduce the set \(B_j\), as in [3], defined as the set of all integers \(k\) such that

\[m_j > m_k \cos\left(\frac{(j-k)\pi}{n}\right).\]

Then we can define, for \(k \in B_j\),

\[W_k(\theta + \frac{(j-k)\pi i}{n}) = e^{m_k(e^{-\theta}(\omega^{-j}x_+ + \mu\alpha^{-1}(\omega^{-j}x_- - e_j))}.\]

Given the initial image vector \(v_0 = e_0 + \sum_{k \in B_j} Q_k e_k + Q_j e_j\), the image \(v\) evolves as

\[v = e_0 + \sum_{k \in B_j} Q_k W_k(\theta + \frac{(j-k)\pi i}{n})e_k + Q_j W_j(\theta)e_j. \tag{6.5}\]

This defines so-called ‘right moving’ modes for the standard soliton of species \(j\). As shown in [3], these can be formed by composing solitons together, and repeatedly setting \(X^{kj}(\theta_k - \theta_j) = 0\) for \(k \in B_j\), by choosing values of \(\theta_k\) which lie at a zero, i.e. the second zero in \(\theta_j\).
We can also define ‘left movers’, by considering the evolution of the vector \( s \), orthogonal to the kernel. Pick the initial \( s_0 \), as \( s_0 = e_0 + \sum_{k \in B_r} \hat{Q}_k e_k + \hat{Q}_r e_r \), then recalling that \( s = (a(\tau)}{T} s_0 \),

\[
s = e_0 + \sum_{k \in B_r} \hat{Q}_k W_k (\hat{\theta} - \frac{(r - k)\pi i}{n}) e_k + \hat{Q}_r W_r (\hat{\theta}) e_r, \tag{6.6}
\]

where

\[
e^{\hat{\theta}} = \omega^{\frac{i + r}{2}} \tau^{-1} e^\theta. \tag{6.7}
\]

These left movers were not considered in [3], but arise because there are always two zeros to the interaction coefficient \( X^{kr}(\theta) \), at \( \theta = \nu \) and at \( \theta = -\nu \), and we take the alternative case to that considered for the image.

By expanding out the new solution [5, 2], it is possible to see that it is the result of combining a standard soliton of species \( j \) with right moving modes of oscillation with a standard soliton of species \( r \) with left-moving modes. This expansion will be done below. It is only necessary to pick a value of \( \tau \) or \( \hat{\theta} \) to ensure that the mass of the resultant solution is what we are looking for. This is done by again picking a zero of the interaction coefficient \( X^{jr}(\theta - \hat{\theta}) \). Important cross terms arise when left-movers interact with right-movers, and it is these which lead to the new topological charge sectors.

The new solutions [5, 2] are given by

\[
\tau_i = p_i + p_{i-1} \tau + p_{i-2} \tau^2 + \cdots + p_{i+1} \tau^n, \quad i = 0, \ldots, n - 1,
\]

with the understanding that \( p_0 = p_n \), and the index \( j \) of \( p_j \) is taken modulo \( n \). We first take the simplest situation with no extra modes, we write \( W_j \) for \( W_j(\theta) \), and \( \tilde{W}_r \) for \( W_r(\hat{\theta}) \):

\[
v_i = 1 + Q_j \omega^j(i-1) W_j,
\]

\[
s_i = 1 + \tilde{Q}_r \omega^r(i-1) \tilde{W}_r.
\]

Now \( p_i = n v_i s_i \), so

\[
\tau_i = (1 + Q_j \omega^j(i-1) W_j)(1 + \tilde{Q}_r \omega^r(i-1) \tilde{W}_r) + \tau(1 + Q_j \omega^j(i-2) W_j)(1 + \tilde{Q}_r \omega^r(i-2) \tilde{W}_r) + \cdots + \tau^n(1 + Q_j \omega^j W_j)(1 + \tilde{Q}_r \omega^r W_r)
\]

\[
= (1 + \tau + \tau^2 + \cdots + \tau^n) + Q_j \omega^j W_j(\omega^{-j} + \omega^{-2j} \tau + \cdots + \tau^n) + \tilde{Q}_r \omega^r \tilde{W}_r(\omega^{-r} + \omega^{-2r} \tau + \cdots + \tau^n) + Q_j \tilde{Q}_r \omega^{(j+r)} \tilde{W}_r W_j (\omega^{-(r+j)} + \omega^{-2(r+j)} \tau + \cdots + \tau^n). \tag{6.8}
\]

We divide \( \tau_i \) by \( (1 + \tau + \tau^2 + \cdots + \tau^n) \), and let

\[
\hat{Q}_r' = \hat{Q}_r \frac{(\omega^{-r} + \omega^{-2r} \tau + \cdots + \tau^n)}{(1 + \tau + \cdots + \tau^n)}.
\]

\[
Q_j' = Q_j \frac{(\omega^{-j} + \omega^{-2j} \tau + \cdots + \tau^n)}{(1 + \tau + \cdots + \tau^n)}.
\]

Then

\[
\tau_i = 1 + Q_j' \omega^j W_j + \hat{Q}_r' \omega^r \tilde{W}_r + X^{jr}(\theta - \hat{\theta}) Q_j' \hat{Q}_r' \omega^{(j+r)} W_j \tilde{W}_r,
\]

where we have made the identification

\[
X^{jr}(\theta - \hat{\theta}) = \frac{(\omega^{-(r+j)} + \omega^{-2(r+j)} \tau + \cdots + \tau^n)(1 + \tau + \cdots + \tau^n)}{(\omega^{-j} + \omega^{-2j} \tau + \cdots + \tau^n)(\omega^{-r} + \omega^{-2r} \tau + \cdots + \tau^n)}.
\]

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Using the formula for the sum of a geometric progression four times on this expression shows that it is equal to the formula (6.3), after using (5.7). Thus we have derived the two-soliton solution (6.4).

This shows how the interaction coefficient \( X^{ij}(\theta) \) appears from a \( W_j \) term in \( v \) combining with the \( W_r \) term in \( s \). From this it is easy to see how the correct interaction coefficients are picked up when other modes in \( v_i \) (or \( s_i \)) combine with modes from \( s_j \) (or \( v_j \)) respectively.

It is also amusing to note that the standard breather solutions, for example mentioned in [1], considered as a bound state of a soliton and anti-soliton, (and an analytic continuation in the relative rapidity to give a total real energy and momentum), are contained as a simple case of the solutions (5.2). These are recovered if no extra modes are taken, \( j = r \), and \( \tau \) has some special values. The breathers are of course single extended objects, so we would hope to find them in this way.

7 Some explicit new single solitons

We have to be careful to choose modes so that any resultant exponential behaviour in \( x \) does not exceed the behaviour of the final term in the tau function, which is the term we have selected to provide the mass of the soliton we are interested in, see [1] for a discussion of this. To illustrate how we obtain new topological charges we shall focus on the largest fundamental representation (the middle node of the Dynkin diagram) of the \( A_n \), \( n \) odd, theories. This representation always has two previously known topological charges. For the combined modes not to exceed the mass term, we must take \( n \geq 5 \), so the new rank one solutions fail to solve the \( A_3 \) theories where the middle representation is of dimension 6, so that there are four missing charges.

So we consider \( A_5 \) in detail, this being the next simplest case, the first case where we get something new. The middle fundamental representation \( V_5 \), corresponding to the soliton of species 3, has dimension 20. There are therefore 18 missing charges.

Consider the ‘right movers’ around a standard soliton of species 3,

\[
\tau_j^{(1)} = 1 + \omega^j Q_1 W_1(\theta - \frac{2\pi i}{6}) + \omega^{5j} Q_5 W_5(\theta + \frac{2\pi i}{6}) + \omega^{3j} Q_3 W_3(\theta),
\]

and the left-movers around a standard soliton of species 3,

\[
\tau_j^{(2)} = 1 + \omega^j \tilde{Q}_1 W_1(\theta + \frac{2\pi i}{6}) + \omega^{5j} \tilde{Q}_5 W_5(\theta - \frac{2\pi i}{6}) + \omega^{3j} \tilde{Q}_3 W_3(\theta).
\]

We have seen that the rank one solutions (5.2) are given by combining these solitons with left-moving and right-moving modes together. We have set the relative rapidity of these two solitons to zero so that the resultant mass is the same as a species 3 soliton, the final term in each tau function remains, and there are no higher terms in \( x \). Thus the combined soliton, which we interpret as a single soliton, is

\[
\tau_j = 1 + (\omega^j Q_1 + \omega^{5j} \tilde{Q}_5) W_1(\theta - \frac{2\pi i}{6}) + (\omega^{5j} Q_5 + \omega^j \tilde{Q}_1) W_1(\theta + \frac{2\pi i}{6}) + X^{11}(-\frac{4\pi i}{6}) \omega^j \tilde{Q}_1 \tilde{Q}_1 W_1(\theta - \frac{2\pi i}{6}) W_1(\theta + \frac{2\pi i}{6}) + X^{55}(-\frac{4\pi i}{6}) \omega^{5j} \tilde{Q}_5 \tilde{Q}_5 W_5(\theta + \frac{2\pi i}{6}) W_5(\theta - \frac{2\pi i}{6}) + X^{51}(0) Q_5 \tilde{Q}_1 W_5(\theta - \frac{2\pi i}{6}) W_1(\theta + \frac{2\pi i}{6}) + X^{15}(0) Q_1 \tilde{Q}_5 W_1(\theta - \frac{2\pi i}{6}) W_5(\theta - \frac{2\pi i}{6}) + \omega^{3j}(Q_3 + \tilde{Q}_3) W_3(\theta).
\]

We know that

\[
X^{11}(-\frac{4\pi i}{6}) = X^{55}(-\frac{4\pi i}{6}) = \frac{3}{2} \quad \text{and} \quad X^{15}(0) = X^{51}(0) = \frac{3}{4}.
\]
From the formula \( m_j = \sin \left( \frac{n_j \pi}{3} \right) \), we calculate

\[
W_1(\theta) = W_2(\theta) = e^{\frac{1}{2}(e^{-g_{x_+}-e^{g_{x_-}}})} \quad \text{and} \quad W_3(\theta) = e^{(e^{-g_{x_+}-e^{g_{x_-}}})},
\]
so \( W_3(0) = e^{2x} \). Also

\[
W_1(-\frac{2\pi i}{6}) = e^{\frac{1}{2}(e^{-\frac{5}{3}x_+}-e^{-\frac{5}{3}x_-})} = e^{\frac{1}{2}x + i \frac{\pi}{3} t}, \quad (7.3)
\]
and

\[
W_1\left( \frac{2\pi i}{6} \right) = e^{\frac{1}{2}x - i \frac{\pi}{3} t}.
\]

For \( \theta = 0 \), when we are working in the rest-frame of the soliton, equation (7.1) becomes

\[
\tau_j = 1 + (\omega^j Q_1 + \omega^{5j} \tilde{Q}_5)e^{\frac{1}{2}x + i \frac{\pi}{3} t} + (\omega^{5j} Q_5 + \omega^j \tilde{Q}_1)e^{\frac{1}{2}x - i \frac{\pi}{3} t} + \frac{3}{2} \left\{ \omega^{3j} Q_1 \tilde{Q}_1 + \omega^{4j} Q_5 \tilde{Q}_5 + \frac{1}{2} \left( Q_1 \tilde{Q}_5 e^{i \sqrt{3} t} + Q_5 \tilde{Q}_1 e^{-i \sqrt{3} t} \right) \right\} e^x + \omega^{3j} (Q'_5)e^{2x}. \quad (7.4)
\]

Notice that as promised the intermediate term in curly brackets has an exponential behaviour \( e^x \) which does not beat the last term \( e^{2x} \) at large \( x \), thus this solution indeed has a mass equal to that of the solitons of species 3. If the construction were repeated for the middle soliton of the \( A_3 \) theories, then this intermediate term would have an exponential behaviour exactly equal to the final term. Given the time dependence in the intermediate term, this is not allowed.

**Structure of the singularities**

We graphically examine the behaviour of \( \tau_j, j = 0, \ldots, 5 \), as we vary the complex parameters \( Q_1, \tilde{Q}_1, Q_5, \tilde{Q}_5, Q'_5 \). We seek values where the solutions \( \tau_j \) are free from zeros, so that the field \( u \) (recall equation (6.1)) is free from singularities, for all \( x \in \mathbb{R} \) and \( t \in \mathbb{R} \).

Consider the equation \( \tau_j = 0 \). To simplify this equation we partially split the \( x \) and \( t \) dependence by rearranging it as

\[
e^{-\frac{t}{2}} + \frac{3}{2} \left\{ \omega^{3j} Q_1 \tilde{Q}_1 + \omega^{4j} Q_5 \tilde{Q}_5 + \frac{1}{2} \left( Q_1 \tilde{Q}_5 e^{i \sqrt{3} t} + Q_5 \tilde{Q}_1 e^{-i \sqrt{3} t} \right) \right\} e^{\frac{x}{2}} + \omega^{3j} (Q'_5)e^{\frac{2x}{3}} = 0.
\]

We plot the left and right hand sides of this equation, for all \( x \in \mathbb{R} \), and for all \( t \in \mathbb{R} \). The right-hand side will always be an ellipse, in fact we will mostly consider the degenerate case when this ellipse is a line or a point. The \( t \) dependence of the left hand-side complicates matters, and we will have to pick certain values of \( t \), i.e. ‘extremal’ values, and plot curves for all \( x \in \mathbb{R} \), these curves come from and leave infinity. From the graph of the left-hand side it will be a simple matter to read off the winding numbers around the origin of the tau functions and thus compute the topological charges. We also remark that if we set \( Q_1, \tilde{Q}_1, Q_5, \tilde{Q}_5 \) to zero, we recover the known solution, equation (6.2), for a soliton of species 3. We aim to tune these parameters to enter a new topological charge sector, so we must increase \( Q_1, \tilde{Q}_1, Q_5, \tilde{Q}_5 \) from zero through a singular region into a region free from singularities. The winding number will then automatically be different from that previously known.

It makes sense to do this tuning in a systematic manner, so we pick a particular tau function which we want to tune into a new charge sector, \( \tau_0 \) say. We have to tune the curve given by the left-hand side through and past the ellipse centred at the origin, given by the right-hand side. This could be difficult to do since the size of the perturbation given by the term in curly brackets depends on the size of the ellipse: the ellipse can increase in size as we increase the perturbation. Therefore we choose values so that the ellipse for this tau function collapses to a point at the origin, i.e \( Q_1 = -\tilde{Q}_5, Q_5 = -\tilde{Q}_1 \), thus making it much easier to tune the curve through the ellipse. We then do further tuning, within this parameter space, to make sure that the other tau functions, \( \tau_j : j = 1, \ldots, 5 \), are free from zeroes. We will also expect that the other tau functions will not be tuned into new topological charge sectors, but
this has to be seen explicitly in the graphs. We can then repeat the whole construction by focusing in turn on \( \tau_1, \tau_2, \) etc, instead of \( \tau_0. \)

It turns out that the tuning required is rather delicate, so we shall present the result as a fait accompli, exhibiting precise values of \( Q_1, Q_2, Q_5, Q_5, \) and \( Q'_3 \) which yield a new sector. We will not attempt to prove statements about completeness, that is whether all possible charge sectors have been found.

These precise values are (noting that \( Q_1 = -\tilde{Q}_5, \ Q_5 = -\tilde{Q}_1 \)

\[
Q_1 = \tilde{Q}_1 = \frac{3}{2} e^{-i\pi/4} \quad \text{and} \quad Q'_3 = e^{i\pi/4}.
\]

We discuss \( \tau_0, \) which has been selected to give a new topological charge, first. We check that \( \tau_0 \) is free from zeroes. For \( j = 0 \) equation (7.3) becomes

\[
e^{-x/2} \left( 1 + \frac{3}{2} e^{x} (2 - \cos(\sqrt{3}t)) Q'_1 + Q'_1 e^{2x} \right) = 0 ,
\]

and if we substitute \( p = 2 - \cos(\sqrt{3}t) \) we get

\[
e^{-x/2} \left( 1 + \frac{3}{2} e^{x} p e^{-41i\pi/42} e^{x} + e^{i\pi/6} e^{2x} \right) = 0 .
\] (7.6)

The imaginary part of (7.6) is

\[
e^{-x/2} \left( - \frac{3}{2} e^{x} p \sin \left( \frac{41\pi}{42} \right) e^{x} + \frac{1}{2} e^{2x} \right) = 0 ,
\]

which can be solved for \( e^x \) to give

\[ e^x = 2p \left( \frac{3}{2} \right)^3 \sin \left( \frac{41\pi}{42} \right) . \]

The real part of (7.6) is

\[
e^{-x/2} \left( 1 + p \frac{3}{2} e^{x} \cos \left( \frac{41\pi}{42} \right) e^{x} + \frac{\sqrt{3}}{2} e^{2x} \right),
\]

and substituting the previous expression for \( e^x \) in this gives

\[
e^{-x/2} \left( 1 + p^2 \frac{3}{2} e^{x} (2 - \cos(\sqrt{3}t)) Q'_1 + Q'_1 e^{2x} \right) = 0 ,
\]

which is the co-ordinate on the real axis where the curve passes through that axis. In the permitted region for \( p, 1 \leq p \leq 3, \) this value is always strictly negative and we conclude that the curve \( \tau_0, \) parametrised by \( x, \) never passes through the origin at any time \( t. \)

The following two graphs (1a) and (1b) show the left-hand side of equation (7.3) with \( j = 0 \) at the two possible extremal values in time \( t. \) As discussed beforehand, the right-hand side of (7.3) has collapsed to a point at the origin. If time is switched on, the curve oscillates between these two. The thin lines shows the same curve, for the same value of \( Q'_3, \) but with all intermediate terms zero, i.e. \( Q_1 = \widetilde{Q}_1 = Q_5 = \widetilde{Q}_5 = 0. \) These show that a new topological charge sector has been found for \( \tau_0. \)

The next simplest case is \( \tau_3. \) The conditions \( Q_1 = -\widetilde{Q}_5, Q_5 = -\widetilde{Q}_1 \) also mean that the right-hand side of (7.3) is a point. Graphs (2a) and (2b) show the two extremal curves. For \( j = 3 \) equation (7.3) becomes

\[
e^{-x/2} \left( 1 + \frac{3}{2} e^{x} (2 - \cos(\sqrt{3}t)) Q''_1 - Q''_1 e^{2x} \right) = 0 ,
\]

and the imaginary part of this is

\[
- \left( \frac{3}{2} \right)^3 p \sin \left( \frac{41\pi}{42} \right) e^{x} - \frac{1}{2} e^{2x} = 0 .
\]
For real $x$, this has no solutions and the curve $\tau_3$ remains in the lower half plane and does not pass through the origin at any time $t$. This shows that there is no new sector.

The cases $\tau_1$ and $\tau_2$ are more delicate. For $\tau_2$, graphs (3a) and (3b) show the ellipse given by the right-hand side of (7.4) (in fact this has degenerated to a line), and the two curves given by the extremal values of $t$. In both cases these curves hit, or at least seem to touch, the ellipse. Also note that in (3b) the curve lies in the same position as the standard case $Q_1 = \tilde{Q}_1 = Q_5 = \tilde{Q}_5 = 0$. This can also be seen in some of the other graphs. We shall give two arguments for demonstrating that this situation is free from zeroes. The first is slightly more intuitive, but not completely rigorous: Firstly, we expand (3a), where the curve appears to touch the ellipse, but we plot the point on the ellipse given by the precise value of $t$, $\sqrt{2}t = \pi$, corresponding to that curve. The expanded graph (3c) shows that there is no touching, and the point is below and to the left of the curve, in the way of the motion of the curve when we switch on $t$. This point is at the tip of the ellipse. Graph (3d) is a similarly expanded graph, at the other extremal value of $t$, that is an expansion of (3b). The point on the ellipse at that value of $t$, $\sqrt{2}t = \frac{\pi}{2}$, is positioned to the left and below the curve, at the origin in fact. Now the curve oscillates between these extremal cases, but the point on the ellipse oscillates at precisely half this frequency. There are two periods in the motion of the curve during a single period of the point on the ellipse. Since this point has moved half-way down its length during the time the curve has moved between the extremal curves, and the extremal curve in (3b) intersects the ellipse less than half-way down its length, we can intuitively infer that the point always stays ahead of the curve throughout the motion during this half-period, with the separation along the line increasing. During the other half-period when the curve moves back to its starting position, the point on the ellipse moves further down away from the curve. In the next half-period it moves back to look like graph (3d), but the point is now travelling up the ellipse, and the curve is travelling towards its starting position. This is just the reverse motion of the initial half-period, and is free from a singularity. This is not a completely rigorous proof, because the motion of the intersection of the curve with the line is not quite sinusoidal, the tolerances may be such that we still get an intersection in the first half-period. To disprove this we observe that equation (7.4) for $\tau_2$, omitting the final term $\omega^3(Q_3')e^{2x}$, can be written as (where we must remember that $\omega = e^{\frac{3\pi}{4}}$)

$$1 + i\sqrt{3}(Q_1 e^{i\frac{3\pi}{4}t} + \tilde{Q}_1 e^{-i\frac{3\pi}{4}t})e^x - 3(\frac{\tilde{Q}_1 e^{i\frac{3\pi}{4}t} + \tilde{Q}_1 e^{-i\frac{3\pi}{4}t}}{2})e^x,$$

which rather surprisingly is a perfect square,

$$\left(1 + i\sqrt{3}e^{i\frac{3\pi}{4}t} + \tilde{Q}_1 e^{-i\frac{3\pi}{4}t}e^x\right)^2.$$

Inserting back the final term of (7.4) means that we must check that the equation

$$1 + i\sqrt{3}(Q_1 e^{i\frac{3\pi}{4}t} + \tilde{Q}_1 e^{-i\frac{3\pi}{4}t})e^x = \pm i\sqrt{Q_3}e^x,$$

has no solutions, or rearranging slightly,

$$e^{-x} + i\sqrt{Q_3}e^x = -i\sqrt{3}(Q_1 e^{i\frac{3\pi}{4}t} + \tilde{Q}_1 e^{-i\frac{3\pi}{4}t}). \quad (7.7)$$

Now $x$ and $t$ have completely decoupled into left and right hand sides of this equation, and we can therefore plot graphs of the left and right hand sides, for both of the possible signs. These are shown in (4a) and (4b). There are no intersections and therefore $\tau_2$ is free from zeroes, and the winding number is unchanged.

We treat $\tau_1$ in a similar fashion, graphs (5a) and (5b) show the ellipse in comparison to the curve. It is also possible to show by completing the square, in a similar way to $\tau_2$, that $\tau_1$ is free from zeroes.

A brief calculation will show that $\tau_4(x, t) = \tau_2(x, t + 2\pi/\sqrt{3})$ and $\tau_5(x, t) = \tau_1(x, t + 2\pi/\sqrt{3})$. Using the results for $\tau_1$ and $\tau_2$, it is clear that $\tau_4$ and $\tau_5$ are never zero for any value of $x$ and $t$, and that the winding numbers are unchanged.
We now discuss the actual values of the topological charges which have been obtained. We note that for the given value of $Q'_3$, and $Q_1 = \tilde{Q}_1 = Q_5 = \tilde{Q}_5 = 0$, we have the topological charge $\Delta$, where $\alpha_i$ are the simple roots:

$$\Delta = -\frac{1}{2} \alpha_1 - \frac{1}{2} \alpha_3 - \frac{1}{2} \alpha_5.$$  

We have seen how at the special values of $Q$, $\tau_0$ has been tuned into a new sector, but the sectors of $\tau_j : j = 1, \ldots, 5$, are unchanged in comparison with the thin lines, the case where $Q_1 = \tilde{Q}_1 = Q_5 = \tilde{Q}_5 = 0$. A change in winding number by $n \in \mathbb{Z}$ of $\tau_j$ has the effect of changing the charge by $n \alpha_j$, with the natural understanding that $\alpha_0 = - \sum_{i=1}^{5} \alpha_i$.

This gives rise to the topological charge:

$$\Delta = \frac{1}{2} \alpha_1 + \alpha_2 + \frac{1}{2} \alpha_3 + \alpha_4 + \frac{1}{2} \alpha_5.$$  

The analysis can be repeated by focussing on the other $\tau_j$, $j = 1, \ldots, 5$, with

$$Q_1 = -\omega^{4j} \tilde{Q}_5, \quad Q_5 = -\omega^{-4j} \tilde{Q}_1.$$
If \( j \) is even we can pick \( Q_1 \) in terms of the old \( Q_1 \) as \( \omega^{-j}Q_1 \), and \( \bar{Q}_1 \) as \( \omega^{-j}\bar{Q}_1 \). It is easy to see that the only difference from the previous case is a permutation of the tau functions, and therefore the other \( \tau \)'s must be free from zeroes and the winding numbers are unchanged. On the other hand, if \( j \) is odd, we must pick a different \( Q_1 \) and \( \bar{Q}_1 \) unrelated to before. It is possible to do this, and the analysis for showing that \( \tau_j \)'s winding number is changed, and the others are free from zeroes and their winding numbers unchanged, is essentially identical to the \( j \) even case.

Treating \( \tau_1 \) gives a new topological charge. Hence we find the charge
\[
\Delta = \frac{1}{2} \alpha_1 - \frac{1}{2} \alpha_3 - \frac{1}{2} \alpha_5.
\]

Treating \( \tau_2 \) gives
\[
\Delta = -\frac{1}{2} \alpha_1 - \alpha_2 - \frac{1}{2} \alpha_3 - \frac{1}{2} \alpha_5.
\]

Treating \( \tau_3 \) gives:
\[
\Delta = -\frac{1}{2} \alpha_1 + \frac{1}{2} \alpha_3 - \frac{1}{2} \alpha_5.
\]
Figure (3a): Eqn. (7.5) for $\tau_2$ at extremal value of $t$

Figure (3b): Eqn. (7.5) for $\tau_2$ at other extremal value of $t$

Treating $\tau_4$ gives:

$$\Delta = -\frac{1}{2} \alpha_1 - \frac{1}{2} \alpha_3 - \alpha_4 - \frac{1}{2} \alpha_5.$$ 

Treating $\tau_5$ gives:

$$\Delta = -\frac{1}{2} \alpha_1 - \frac{1}{2} \alpha_3 + \frac{1}{2} \alpha_5.$$ 

This gives 6 new charges which are weights of the fundamental representation $V_3$. We can get 6 more by negating all these. These arise because we can pick the topological charge of the standard solution, when $Q_1 = 0$, to be the other of the two possibilities, when $Q_3'$ is moved through the real axis. It is then possible to find suitable $Q_1$, $Q_1$, $Q_5$, $\bar{Q}_5$, so that this statement is true.

Therefore there are now $12 + 2 = 14$ topological charges of the fundamental representation $V_3$ filled by classical soliton solutions. These do not include the highest weight of $V_3$. As $V_3$ is 20 dimensional, there are 6 as yet unfilled ones. It is curious that 6 is precisely the number of $\tau$ functions in this model, which suggests that the charges can be found by similar methods, since the tuning method described above shows that new topological charges can be found in sets of 6.
We also remark that the first attempt at analysing the solution (7.4) was to consider $Q_5 = \bar{Q}_5 = 0$, where the time dependence in the intermediate term in curly brackets of (7.5) does not appear. In order to tune the curve for $\tau_0$, say, past the ellipse given by the right-hand side, it is necessary to choose quite large values for the parameters $Q_1$, $\bar{Q}_1$ (in comparison with the values actually chosen), and to choose the phase of $Q_1\bar{Q}_1$ within a narrow range of acceptable values. These choices mean that at least one of the other tau functions obviously hits zero, with no possibility of fine adjustment of the $Q$'s. Therefore the case considered here is indeed the simplest.

We conclude by observing that the time delay of this new soliton when it interacts with a standard soliton of a different, or the same, species, given by (6.2), is the same as that of standard solitons interacting with each other. This is because the $W$ with the highest exponential behaviour in $x$, for each of the two solitons, determines the time delay, see 4. The $X$ factor which arises when these two $W$'s combine in the multi-soliton solution (see the rules at the beginning of section 5) is essentially the exponential of the time delay. This $X$ factor is the same regardless of whether there are intermediate terms in the tau functions: these are negligible asymptotically, and do not contribute when the time delay is calculated. This argument is equally valid for the new solitons interacting with
each other. Therefore the new solitons are exactly what are required to fill weights in the fundamental representations.

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Figure (5a): Eqn. (7.5) for $\tau_1$ at extremal value of $t$

Figure (5b): Eqn. (7.5) for $\tau_1$ at other extremal value of $t$

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