Electric Vehicle Charging Station Network Equilibrium Models and Pricing Schemes
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Abstract—In this paper, we develop and apply a framework to characterize network equilibrium traffic and charge patterns in an electric transportation network, mainly focused on the effect of heterogeneities in charging prices across different charging stations. Furthermore, we design optimal charging station prices that can jointly manage societal congestion and electricity costs. Our framework explores the effect of cost-minimizing users preferring different charging stations based on their charge amount and the charging stations’ energy prices, and how this endogenous energy vs. route choice connection defines the threshold-based structure of flows at Wardrop equilibria. We discuss the uniqueness properties of the equilibria and showcase how first-best marginal cost-based pricing can enforce the socially optimal travel and charge behavior in the network, but only under certain charging station congestion models.

I. INTRODUCTION
Recharging a plug-in battery Electric Vehicle (EV) takes significantly more time than refueling an Internal Combustion Engine Vehicle (ICEV). Hence, congestion at charging stations is emerging as a major issue, especially in urban areas with large EV populations. As the travel range of EVs keeps increasing, “charge rage” due to long wait times at charging stations is gradually replacing range anxiety in EV users [1]. To address this issue, there is a clear need for new traffic assignment methods that consider the effect of pricing strategies (or other forms of congestion control schemes) employed at charging stations on their traffic pattern. Pricing strategies can also help manage the effect of EV charging load on the power grid. It is clear that value of time, heterogeneous charging prices, and mobility patterns/needs play central roles in how users choose to charge at different charging stations, as users consider a trade-off between energy costs and time spent en route or waiting to find a free Electric Vehicle Supply Equipment (EVSE) at a charging station.

In this paper, we aim to capture how individual users’ charging decisions affect aggregate congestion and electricity costs by studying the so-called Traffic and Charge Assignment Problem (TCAP). We will study the special threshold-based structure of network flows at equilibrium due to the effect of charging prices on users’ path decisions. Our user-equilibrium model of traffic assignment allows us to characterize the uniqueness properties of the traffic and charge equilibrium model under certain conditions. Our Socially-Optimal (SO) traffic and charge assignment model allows us to design mobility-aware charging fees that would jointly reduce wait times at charging stations and control the impact of the vehicles on the electricity distribution network. The charging fees that we propose are a form of congestion pricing, which has been studied extensively in multi-commodity flow networks [2], [3] and more specifically for optimal toll design in transportation networks [4].

Prior Art: A number of papers have considered the EV charging station control problem as admission control problem in a queuing network, with queue stability and uniform station utilization as the main goal, see, e.g., [5], [6], [7], [8], [9]. Variations in electricity prices across stations are not considered. In [10], the authors consider two service types at a charging station, mathematically modeled as two queuing systems. Another relevant line of work is focused on developing routing and charging models for individual vehicles, considering energy constraints. For example, in [11], the authors study the routing problem of electric vehicles to increase the driving range based on traffic conditions while considering electric vehicles’ battery storage. In [12], the authors study the problem of online charging and routing for individual EVs who aim to minimize their travel cost. A series of past works study the problem of optimally placing charging infrastructure in an electric transportation network. For example, in [13], the authors optimize the location of charging lanes to minimize the social cost. Additionally, in [14], [15], the authors consider the problem of deploying charging stations in transportation networks in order to minimize the travel cost of EV users. Finally, several authors have considered the effect of electric vehicles on equilibrium traffic patterns. For example, in [16], [17] and [18], the authors study a mixed flow of battery electric vehicles and gasoline vehicles and their effects on network equilibrium. However, charging station pricing mechanisms and or the effects of charging station demands on network equilibrium are not considered. In [19], the authors study the path-constrained traffic assignment problem considering the driving range of electric vehicles. They model the driving range of users based on their battery capacity, which is random for each user. In [20], the authors study the equilibrium models for a network of electric vehicles considering recharging time and driving range. In their model, they assume all the charging stations have the same unit price of electricity for all users, i.e., the effect of charging prices on users’ decisions is not considered.

Synopsis: The remaining sections are organized as follows. In Section II, we present the basic elements of our electric transportation and charging network model. In Section III,
we present the cost model for individual users who aim to minimize their travel and charging costs. In Section IV, we present a threshold based energy cost-versus-time equilibrium model for electric transportation and charging networks. Section V discusses the SO assignment problem. Finally, Section VI contains our numerical experiment.

II. SYSTEM MODEL

We consider a multi-commodity network characterized by EV users that are traveling between different origin and destination pairs and would like to charge their vehicles en-route at public charging stations. We model the transportation network as a connected digraph \( G = (V, A) \), where \( V \) is the set of nodes, \( A \) is the set of road arcs, and a subset of nodes \( J \subseteq V \) contain charging facilities that EV users may enter. Assuming that the population of EV users is large, each individual EV user carries an infinitesimal amount of traffic and energy demand. Furthermore, we assume the population is homogeneous in their value of time. We will denote their time value of money as \( \alpha \).

A. Individual EV User Characteristics

Each EV user in the network is associated with an origin-destination (OD) pair \((o,d) \in \mathcal{O}\) and a charge requirement of \( \epsilon \) units of energy.

1) Origin-destination (OD) pair: The charging needs of each EV user originates at a source node \( o \in V \) and must be fulfilled en route to a destination \( d \in V \). Note that the origin of the charging trip might not necessarily coincide with the node from which the user starts their travel. It is simply the location as which the user decides to charge while traveling.

2) Energy request \( \epsilon \): In modeling the energy needs of the EVs, we make two important assumptions. First, we assume each EV user has an inelastic energy demand, i.e., the energy demand is not a function of charging station prices. Second, we assume that this charging demand is the same irrespective of the charging station \( j \in J \) that the user chooses for charging. This means that an EV user’s energy need does not change significantly in a single trip. We note that the first assumption is not critical, as elasticity of energy demand as a function of charging station prices can be included in our model through the use of elastic demand functions, which we refrain from due to brevity. However, the second assumption is critical to the tractability of our model and allows us to characterize the structural properties of equilibrium for charging demand. While we acknowledge that our simplified model does not fully capture the mobility constraints of individual EVs, we believe that, at the macro scale, this approach is sufficiently realistic for the purposes of price design for congestion management in public charging station networks, which should only be concerned with aggregate statistics. For example, in urban areas, travel ranges for BEVs now average above 100 miles, while the average single-trip distance is merely 5.95 miles.

B. Charging Network Operator (CNO)

The Charging Network Operator (CNO) is the entity that is in charge of designing the prices that manage congestion and electricity usage at the charging stations. While the characteristics of each individual EV user are private and unknown to the CNO, we assume that the CNO has some degree of knowledge of the mobility patterns and energy requests of EV users based on statistics collected on the transportation network. Specifically, we assume that the CNO has access to the following information:

- Mobility Patterns of EV users: We model the trip rate of EV users between OD pair \((o,d)\) according to a Poisson process with mean rate \( \lambda_{od} \), which is known to the CNO, and is considered inelastic.
- Statistics of Charge Requests: In this study, battery charge demand of each EV is treated as a continuous random variable that varies across the population of EVs that travel between the same OD pair. We model the energy request of each EV user as an i.i.d. random variable \( \mathcal{E} \) distributed according to a general distribution \( g_E(\epsilon) \). Denote the minimum and maximum possible energy requests for an individual EV as \( \epsilon_{\min} \) and \( \epsilon_{\max} \). Without loss of generality, and purely for brevity of notation, we assume that \( g_E(\epsilon) \) does not vary between different OD pairs, and that it is bounded below away from zero in its compact support \([\epsilon_{\min}, \epsilon_{\max}]\).

C. Charging Stations

Public charging stations are located at a limited number of nodes \( j \in J \) across the network. Without loss of generality, we assume that all stations can provide similar charging rates of \( 1/\gamma \) units of energy per unit time. Each station can be equipped with behind-the-meter solar generation that can be used to serve EV charging demand. Any excess charging demand is served from the grid, and is billed at the appropriate locational rate to the CNO. Due to the limited number of EV supply equipment (EVSE) available at each station, which limits the number of EVs that can be simultaneously plugged in at each station, the limited charging rate of each EVSE, as well as potential activities that the drivers may undertake while their EV is parked and charging, congestion can occur at charging stations. For now, we assume congestion to be a function of average number of charging requests arriving at the station (denoted by \( \lambda_j \)), leading to a expected cost \( T_j(\lambda_j) \) associated with finding a free EVSE at the station \( j \). Information about expected search times is known to individual users. We assume that \( T_j(\lambda_j) \) is a strictly increasing and continuously differentiable function. In Section V-D, we will discuss the implications of using other forms of queueing based models for charging stations.

\footnote{Mathematically, the effect of different charging rates on the user equilibrium can be captured by adding the term \( \alpha^{-1} \gamma_j \) to each station’s price of electricity. Hence, the additional time spent waiting for charge acts as a penalty for charging stations with lower charging rates, and more so for vehicles with larger charging demand.}
Note that the availability of expected wait times but not dynamic wait time information is key in this work. Given this information, individuals choose the minimum expected cost path that would allow them to travel to their destination and charge en route, which we discuss next.

III. THE INDIVIDUAL USER DECISION PROBLEM

For an EV user with OD pair \((o, d)\), an acyclic path \(p \in \mathcal{P}\) on the transportation network \(G\) is feasible if it connects nodes \(o\) and \(d\) via a number of consecutive road arcs and enters a single charging station \(j \in J\) located at a node on path \(p\). We define \(P_{od} \subset \mathcal{P}\) as the set of feasible paths for OD pair \((o, d)\), and denote \(|P_{od}| = K_{od}\). Note that if all charging stations are not available to all users traveling between an OD pair, we need to consider them under separate classes.

A. User Costs

Each EV user’s goal is to choose the path \(p \in P_{od}\) that minimizes their expected trip costs, which consists of travel time costs, wait costs at charging stations to find a free EVSE and for the vehicle to be charged, as well as monetary costs due to charging fees collected by the CNO. We discuss these cost components next.

1) Latency Costs on Road Arcs: To preserve simplicity of exposition, we assume that each road arc \(a \in A\) has a constant latency \(t_a > 0\) that is independent of the mean flow rate \(x_a\) of users on that road arc. This is a reasonable assumption provided that the flow of EVs is small relative to the flow of other vehicles on the network, which is the case in transportation systems at the time of this writing [22]. However, we should note that, should the need arise, the results in this paper can be readily extended to handle separable cost functions for road arcs.

2) Latency Costs at Charging Stations: The expected waiting times for finding an empty charging spot at charging stations, \(T_j(\lambda_j)\), is considered as known for each user. Hence, for each user, the average sojourn time of an EV user with energy request \(\epsilon\) at any charging station \(j \in J\) is the sum of the charging time \(\gamma \epsilon\) and the average waiting time:

\[
\text{average sojourn time} = \gamma \epsilon + T_j(\lambda_j).
\]

Given a congestion \(\lambda\) at charging stations, the mean latency of any path \(p_i \in P\) is the sum of the travel times along road arcs and the average sojourn times at charging stations:

\[
l_{pi}(\lambda) = \sum_{a \in A} \delta_{ata}t_a + \sum_{j \in J} \delta_{j}(\gamma \epsilon + T_j(\lambda_j)).
\]

3) Cost of Charging: For each usage at charging station \(j \in J\), we assume that EV users will pay a one time plug-in fee \(\tau_j\), along with an electricity price \(\upsilon_j\) for each unit of charge that is purchased. The payment for an EV user with demand \(\epsilon\) at charging station \(j\) is then:

\[
\text{payment at charging station } j = \upsilon_j \epsilon + \tau_j.
\]

We provide more specifics on the objectives of the CNO and how the CNO would select \(\tau_j\) and \(\upsilon_j\) in subsequent sections.

B. The Shortest Path Problem

Our key behavioral assumption is that each individual EV user with OD pair \((o, d)\) and energy demand \(\epsilon\) would solve for the path \(p_i \in P_{od}\) with the smallest expected total cost:

\[
C_{od}^p(\epsilon) = \left[ \sum_{a \in A} \delta_{ata}t_a + \sum_{j \in J} \delta_{j}(\gamma \epsilon + T_j(\lambda_j)) \right] + \alpha \sum_{j \in J} \delta_{j}(\tau_j + \upsilon_j \epsilon).
\]

This path choice model is the underlying framework for the development of a traffic assignment model that explicitly considers users with heterogeneous energy needs and how their decisions affects the overall congestion in a Wardropian equilibrium [23]. The important observation is that even for users within the same OD pair that choose the same path and enter the same charging station, travel costs could be different, since each individual EV has a unique energy need.

IV. NETWORK EQUILIBRIUM MODEL

Recall our previous assumption that energy requests are iid distributed for any OD pair according to a distribution function \(q_\epsilon(\epsilon)\). Note from [4] that for any EV user traveling between OD pair \((o, d)\), the total cost of a path \(p_i \in P_{od}\) varies depending on the energy request \(\epsilon\). Informally put, we will see that the dependence of the individual’s costs on their energy requirement \(\epsilon\) determines the structure of the equilibrium. A user with higher energy needs would travel further for cheaper charging prices, whereas someone with low charge requirements would rather charge at a nearby station with potentially higher prices. Accordingly, the path flows are not invariant within the population of users traveling between the same OD pair, and the distribution of charging amounts at each station does not coincide with the network level charge distribution \(q_\epsilon(\epsilon)\). Instead, we will see that, for each OD pair, the span of potential energy values \([\epsilon_{\text{min}}, \epsilon_{\text{max}}]\) can be partitioned into intervals such that, within each interval, the OD pair flow is assigned to a unique path. The reader should note that this structure naturally has similarities to that of the Wardrop equilibrium of non-electric heterogeneous travellers with a continuous value of time distribution [24, 25].

A. The Set of Feasible Flows

Let \(f_{od}^i \in [0, \infty)\) be the average flow rate of EV users of OD pair \((o, d)\) traveling on the \(i\)-th path \(p_i \in P_{od}\), such that \(q_{od} = \sum_{i=1}^{K_{od}} f_{od}^i\). We define \(f = \{f_{od}^i\}^{K_{od}}\) as the column vector of flows for all combinations of EV user types and paths. The conservation of flow constraints apply to the arrival rate of users at charging stations as well as arc flow rates. Accordingly, the mean arrival rate of EV users at any charging station \(j \in J\) satisfies:

\[
\lambda_j = \sum_{(o,d) \in \mathcal{O}} \sum_{i=1}^{K_{od}} \delta_{ji} f_{od}^i.
\]
where \( \delta_{ji} = 1 \) if path \( p_i \) enters charging station \( j \) and is zero otherwise. Similarly, we define the mean flow rate of EV users on any road arc \( a \in A \) as:

\[
x_a = \sum_{(o,d) \in O} \sum_{i=1}^{K_{od}} \delta_{ai} f_{od}^i,
\]

where \( \delta_{ai} = 1 \) if arc \( a \) is on path \( p_i \) and is zero otherwise.

**B. A Threshold-Based Wardropian Equilibrium Structure**

In this section, we characterize the structure of the Wardrop equilibria. Following the classical definition of Wardrop’s first principle (or the extension proposed by Holden [26]), we know that, at the state of equilibrium, for each OD pair, no trip-maker can decrease their experienced trip cost with respect to their own energy requirement \( \epsilon \) by unilaterally changing paths.

For notational convenience, let us denote the electricity price at the station on path \( p_i \in P_{od} \) as \( \theta^i_{od} \), i.e.,

\[
\theta^i_{od} = \sum_{j \in J} \delta_{ji} v_j.
\]

Furthermore, we assume that the paths in \( P_{od} \) are indexed such that their electricity prices \( \theta^i_{od} \) are in a decreasing order, i.e., for all paths \( p_i \) and \( p_{i+1} \) in \( P_{od} \), we have \( \theta^i_{od} \geq \theta^{i+1}_{od} \).

**Lemma IV.1.** If a path flow pattern \( f \) satisfies Wardrop’s first principle, for each alternative paths \( p_i \) and \( p_{i+1} \) connecting one OD pair \( (o,d) \) such that \( i \leq k \), the energy requirement \( \epsilon \) of any user (if any at all) traveling on path \( p_i \) is less than or equal to the energy requirement \( \epsilon' \) of any user (if any at all) traveling on path \( p_{i+1} \).

**Proof.** For any user traveling on path \( i \), we know that at equilibrium, we must have \( \theta^i_{od} = \theta^i_{od} + l_{pi} + \alpha \sum_{j \in J} \delta_{ji} v_j \leq \theta^i_{od} + l_{pi} + \alpha \sum_{j \in J} \delta_{ji} v_j \). On the other hand, for any user traveling on path \( k \), \( \theta^i_{od} \leq \theta^i_{od} + l_{pi} + \alpha \sum_{j \in J} \delta_{ji} v_j \). Accordingly, as we know that \( \theta^i_{od} \geq \theta^i_{od} \), we must have \( \epsilon - \epsilon' \leq 0 \).

Given this observation, the next theorem characterizes the structure of flows at equilibrium as a function of energy thresholds for each OD pair (see Fig. 1), which will be useful in finding the equilibrium as the solution of a nonlinear program.

**Theorem IV.2.** The equilibrium traffic pattern \( f_{od}^i = f_{od}^i \) for each OD pair \( (o,d) \) is characterized by a vector of energy thresholds \( \pi_{od} = (\pi^0_{od}, \pi^1_{od}, \ldots, \pi^{K_{od} - 1}_{od}, \pi^K_{od}) \), where \( \pi^0_{od} = \epsilon_{min} \) and \( \pi^K_{od} = \epsilon_{max} \). All customers with \( \pi^i_{od} \leq \epsilon \leq \pi^i+1_{od} \) will choose path \( p_i \). Moreover, for any unused paths \( p_i \) and \( p_{i+1} \),

\[
\pi^i_{od} = \frac{\psi^i+1_{od} - \psi^i_{od}}{\alpha(\theta^i_{od} - \theta^{i+1}_{od})},
\]

where for path \( p_i \) in pair \( (o,d) \),

\[
C_{od}^i(\epsilon) = \psi^i_{od} + \alpha \theta^i_{od} \epsilon,
\]

\[
\psi^i_{od} = \sum_{a \in A} \delta_{ai} t_a + \sum_{j \in J} \delta_{ji}(\gamma \epsilon + T_j(\lambda_j)) + \alpha \sum_{j \in J} \sum_{j \in J} \delta_{ji} \tau_j
\]

and the vector \( \pi_{od} \) characterizes the flow on each path \( p_i \) in \( (o,d) \) by:

\[
f^i_{od} = q_{od} \left( \int_{\pi^i_{od} - 1}^{\pi^i_{od}} g_{\epsilon}(x) dx \right).
\]

For any unused path \( i + 1 \), \( \pi^i_{od} = \pi^{i+1}_{od} \).

**Proof.** First, we prove that if the specific threshold-based traffic pattern specified by (8) is established, we have equilibrium. We prove this by contradiction. Suppose this flow pattern that is not an equilibrium. This means that there will exist a user with energy demand \( \epsilon' \) assigned to path \( p_{i+1} \) (\( \pi^i_{od} \leq \epsilon' \)) who has incentive to change his or her path to \( p_k \) (without loss of generality we assume \( k < i + 1 \)). We use induction on \( k \) to show contradiction. For the user to prefer path \( k = i \), we must have \( C_{od}^i(\epsilon') < C_{od}^{i+1}(\epsilon') \), i.e.,

\[
\psi^{i+1}_{od} + \alpha \theta^i_{od} \epsilon' < \psi^{i+1}_{od} + \alpha \theta^{i+1}_{od} \epsilon' \Rightarrow \epsilon' < \frac{\psi^{i+1}_{od} - \psi^{i}_{od}}{\alpha(\theta^i_{od} - \theta^{i+1}_{od})} = \pi^i_{od}
\]

which is contradictory to \( \pi^i_{od} \leq \epsilon' \). Now, let us assume that the claim holds for all paths \( p_{k+1}, p_{k+2}, \ldots, p_i \), which mean that we must have

\[
\frac{\psi^{i+1}_{od} - \psi^{i+1}_{od}}{\alpha(\theta^{k+1}_{od} - \theta^{i+1}_{od})} < \frac{\psi^{i+1}_{od} - \psi^i_{od}}{\alpha(\theta^i_{od} - \theta^{i+1}_{od})}
\]

\[
\alpha \theta^i_{od} (\psi^{i+1}_{od} - \psi^{k+1}_{od}) < \left( \alpha \theta^{i+1}_{od} (\psi^{i+1}_{od} - \psi^{i}_{od}) - \alpha \theta^{i+1}_{od} (\psi^{i+1}_{od} - \psi^{i}_{od}) \right).
\]

This means that for used path \( p_k \), we should have:

\[
\pi^i_{od} \leq \epsilon' < \frac{\psi^{i+1}_{od} - \psi^i_{od}}{\alpha(\theta^i_{od} - \theta^{i+1}_{od})}.
\]
However, given that \( \pi_{od}^k \leq \pi_{od}^i \) by definition, we have:

\[
\frac{\psi_{od}^{k+1} - \psi_{od}^k}{\alpha(\theta_{od}^k - \theta_{od}^{k+1})} < \frac{\psi_{od}^{i+1} - \psi_{od}^i}{\alpha(\theta_{od}^i - \theta_{od}^{i+1})} \implies \\
\psi_{od}^{k+1} - \psi_{od}^k < \psi_{od}^{i+1} - \psi_{od}^i \implies \\
\alpha \theta_{od}^k (\psi_{od}^{k+1} - \psi_{od}^k) - \alpha \theta_{od}^i (\psi_{od}^{i+1} - \psi_{od}^i) - \\
\alpha \theta_{od}^{k+1} (\psi_{od}^{k+1} - \psi_{od}^k) + \alpha \theta_{od}^{i+1} (\psi_{od}^{i+1} - \psi_{od}^i) < \\
\alpha (\theta_{od}^k - \theta_{od}^{k+1}) (\psi_{od}^{k+1} - \psi_{od}^k). \tag{16}
\]

If we use (14), we can write:

\[
\frac{\psi_{od}^{i+1} - \psi_{od}^i}{\alpha(\theta_{od}^i - \theta_{od}^{i+1})} < \frac{\psi_{od}^{i+1} - \psi_{od}^i}{\alpha(\theta_{od}^i - \theta_{od}^{i+1})} = \pi_{od}^i \tag{17}
\]

which is contradictory to (15). Hence no user will have incentive to change his or her path if the specific threshold-based traffic pattern by \([8]\) is established.

To prove the reverse, we can see that it easily follows from Lemma IV.1 that only threshold based flows can exist at equilibrium. Hence, we only need to prove that no other threshold based flow except for that described by (8) can be an equilibrium. Therefore, users with energy demand \( \epsilon \leq \beta_{od}^i \) will choose path \( p_i \), and no user has any incentive to change their path to path \( p_{i+1} \). Hence, \( \psi_{od}^i + \alpha \beta_{od}^i \leq \psi_{od}^{i+1} + \alpha \beta_{od}^{i+1} \) + \( \epsilon \) that leads to \( \epsilon \leq \pi_{od}^i \) that means \( \beta_{od}^i =\pi_{od}^i \) which is contradictory to existence of another set of thresholds that can be equilibrium pattern. \( \Box \)

**Corollary IV.2.1.** At equilibrium, for any utilized paths \( p_i \) and \( p_{i+1} \), we must have:

\[
C_{od}^{p_i}(\pi_{od}^i) = C_{od}^{p_{i+1}}(\pi_{od}^i). \tag{18}
\]

Next we look at how we can determine these thresholds and discuss the uniqueness properties of the equilibrium.

**C. The UE Optimization Problem**

In this section, we develop a nonlinear minimization problem whose solution(s) satisfy the stated user equilibrium conditions. Our approach is inspired by that of \([24]\) in characterizing UE flows for with a continuous value of conditions. Our approach is inspired by that of \([24]\) in characterizing UE flows for with a continuous value of conditions. Our approach is inspired by that of \([24]\) in characterizing UE flows for with a continuous value of conditions. Our approach is inspired by that of \([24]\) in characterizing UE flows for with a continuous value of conditions. Our approach is inspired by that of \([24]\) in characterizing UE flows for with a continuous value of conditions. Our approach is inspired by that of \([24]\) in characterizing UE flows for with a continuous value of conditions.

We first state a lemma that slightly reformulates the equilibrium conditions.

**Theorem IV.3.** The vector flow \( f \) is a user equilibrium traffic pattern in the charging station network if and only if it solves the following side-constrained traffic assignment problem:

\[
\begin{align*}
\text{minimize} & \quad \sum_{a \in A} x_a t_a + \alpha \sum_{j \in J} \lambda_j T_j(x) + \\
& \quad \sum_{i,j \in J} \int_0^{\lambda_j} T_j(x) dx + \\
& \quad \sum_{i,j \in J} \sum_{k \in K} \alpha \left( Q_{od}^k - Q_{od}^{i-1} \right) \left( \theta_{od}^{i-1} - \theta_{od}^k \right)
\end{align*}
\]

subject to

\[
\begin{align*}
& \forall a \in A: \quad x_a = \sum_{(o,d) \in O} \delta_{od} f_{od}^a, \tag{19a} \\
& \forall j \in J: \quad \lambda_j = \sum_{(o,d) \in O} \delta_{od} f_{od}^j, \tag{19b} \\
& \forall \left( o, d \right): \quad q_{od} = \sum_{i=1}^{K_{od}} f_{od}^i, \tag{19c} \\
& \forall \left( o, d \right) \forall i: \quad Q_{od}^i = \sum_{k=1}^{K_{od}} f_{od}^k, \tag{19d} \\
& \forall \left( o, d \right) \forall i: \quad f_{od}^i \geq 0. \tag{19e}
\end{align*}
\]

**Proof.** We first state a lemma that slightly reformulates the equilibrium conditions.

**Lemma IV.4.** The flow \( f \) is a user equilibrium traffic pattern if and only if, it satisfies the following conditions:

1. \( \forall \left( o, d \right), j : f_{od}^j \geq 0, \)
2. for every \( OD \) pair, for every utilized path \( i,l \):

\[
\begin{align*}
\psi_{od}^i + \sum_{k=1}^{K_{od}-1} G^{-1}(\frac{Q_{od}^k}{q_{od}}) \alpha \left( \theta_{od}^k - \theta_{od}^{k+1} \right) \\
= \psi_{od}^i + \sum_{k=1}^{K_{od}-1} G^{-1}(\frac{Q_{od}^k}{q_{od}}) \alpha \left( \theta_{od}^k - \theta_{od}^{k+1} \right) \\
\end{align*}
\]

**Proof.** Proof of the first condition is trivial. For the second condition, we know from (8) that

\[
\frac{\psi_{od}^{i+1} - \psi_{od}^i}{\alpha(\theta_{od}^i - \theta_{od}^{i+1})} = G^{-1}(\frac{Q_{od}^i}{q_{od}}). \tag{21}
\]

Therefore, for every utilized path \( i \) in pair (\( o,d \)):

\[
\psi_{od}^i + \sum_{k=1}^{K_{od}-1} G^{-1}(\frac{Q_{od}^k}{q_{od}}) \alpha \left( \theta_{od}^k - \theta_{od}^{k+1} \right) = \psi_{od}^{K_{od}}, \tag{22}
\]
which proves the second condition.

Let us now write the Lagrangian of (19) as follows:

\[ L_{eq}(f_{od}^i, Q_{od}^i, q_{od}, f_{od}^i, z_{od}) = \sum_{a \in A} x_a t_a + \alpha \sum_{j \in J} \lambda_j T_j + \sum_{j \in J} \int_0^{\lambda_j} T_j(x)dx + \]

\[ \alpha \sum_{(o,d) \in O} q_{od} \left( \sum_{i=1}^{K_{od}} \theta_{od}^i \left[ E\left( \frac{Q_{od}^i}{q_{od}} \right) - E\left( \frac{Q_{od}^i}{q_{od}} \right) \right] \right) + \]

\[ \sum_{(o,d) \in O} \sum_{i=1}^{K_{od}} \left( Q_{od}^i - \sum_{k \leq i} f_{od}^k \right) \]

(23)

Partial derivative of (23) with respect to \( f_{od}^i \) and \( Q_{od}^i \) are as follows:

\[ \frac{\partial L_{eq}}{\partial f_{od}^i} = \psi_{od}^i - \sum_{k=1}^{K_{od}-1} v_{od}^k - z_{od} \]

(24)

\[ \frac{\partial L_{eq}}{\partial Q_{od}^i} = \alpha \left( \theta_{od}^i - \theta_{od}^{i+1} \right) G^{-1}(Q_{od}^i) + v_{od}^i \]

(25)

In equilibrium, for every two utilized path \( i, l \) we must have:

\[ \frac{\partial L_{eq}}{\partial Q_{od}^i} = \frac{\partial L_{eq}}{\partial Q_{od}^l} = 0, \]

(26)

\[ \frac{\partial L_{eq}}{\partial f_{od}^i} = \frac{\partial L_{eq}}{\partial f_{od}^l} = 0, \]

(27)

which are equivalent to the conditions put forth in Lemma [IV.4], proving that the solution of (19) is the user equilibrium traffic pattern of network.

Theorem [IV.5] considers conditions for the uniqueness of the equilibrium flow and the thresholds.

**Theorem IV.5.** Under strictly increasing searching time functions of the form \( T_j(\lambda_j) \) at all charging stations, the equilibrium path flows as well as the thresholds, \( G^{-1}(Q^i/q) \), are unique.

The proof is provided in the Appendix.

**D. A note on modeling charging stations as queues**

At first glance, it could seem that the ideal choice for modeling the wait time \( T_j(\lambda_j) \) at charging stations is as the average time spent waiting in a queueing system. For example, one could have a wait time of the following form:

\[ T_j(\lambda_j) = \frac{\lambda_j / \mu}{1 - (\lambda_j / \mu) \cdot c \mu}, \]

(28)

where \( \mu \) is the expected service time of each user in the system, and \( c \) is the capacity of the charging station. The key challenge with queueing-type models arises in how we define \( \mu \), the expected service time of each user here. In order for our previous model to hold, i.e., for \( T_j(\lambda_j) \) to be simply a uni-variate, strictly-increasing and convex function of \( \lambda_j \), we have to assume that the expected service time of the station, \( \mu \) is a constant. This means that the expected amount of energy required by the users at each station, which is an endogenous parameter in our model and is a function of users’ charging decisions, should not have a direct effect on their average service times in the queueing system. This is an appropriate model when users park and charge their vehicle while performing an activity such as shopping or dining. However, if this is not the case, and charging is the sole purpose of the users for coming to the station, we would need to make \( \mu \) a function of the quantiles \( Q_{od}^i \) in order to appropriately model the mapping between the pdf of service times for that specific station and its average wait time. We discuss the implications of adopting a model that can capture this connection in this section.

Let us assume that \( 1/\mu \) corresponds to the average charge amount (given the constant rate of charge), and \( c \) denotes the station’s capacity, e.g., number of chargers times their rate of charge. The quantity \( \rho = \lambda / \mu \) is the so called utilization ratio of a charging station, and is essentially equivalent to the expected charging demand of the station. Hence, as a first step, we will derive an analytical mapping for the expected electricity consumption of charging stations (a surrogate for \( \rho \) here), as a function of the variables \( Q_{od}^i \).

**Lemma IV.6.** The expected electricity consumption in charging station \( j \) is

\[ \left( \sum_{(o,d) \in O} q_{od} \sum_{i=1}^{K_{od}} \delta_{ij} \left[ E\left( \frac{Q_{od}^i}{q_{od}} \right) - E\left( \frac{Q_{od}^{i-1}}{q_{od}} \right) \right] \right). \]

Proof. Note that since the demand \( q_{od} \) for each type of EV user is given by a Poisson process, it follows from Theorem [IV.2] that the mean arrival rate at any charging station \( j \in J \) also forms a Poisson process with rate

\[ \sum_{(o,d) \in O} q_{od} \sum_{i=1}^{K_{od}} \delta_{ij} \mathbb{P}(G^{-1}(Q_{od}^{i-1} / q_{od}) \leq x \leq G^{-1}(Q_{od}^i / q_{od})) . \]

Hence, we can write that the average electricity consumption in each charging station \( j \) by using the linearity of expectation and using Wald’s equation as follows:

\[ \sum_{(o,d) \in O} q_{od} \sum_{i=1}^{K_{od}} \delta_{ij} \mathbb{E}(G^{-1}(Q_{od}^{i-1} / q_{od})), \]

(29)

Then we use the substitution \( l = G(x) \) or \( x = G^{-1}(l) \), we will have:

\[ \sum_{(o,d) \in O} q_{od} \sum_{i=1}^{K_{od}} \delta_{ij} \int_{G^{-1}(Q_{od}^{i-1} / q_{od})} G^{-1}(l)G^{-1}(l)g(G^{-1}(l))dl, \]

(31)

and using the derivative of inverse function rule, this yields:

\[ \sum_{(o,d) \in O} q_{od} \sum_{i=1}^{K_{od}} \delta_{ij} \int_{Q_{od}^{i-1} / q_{od}} G^{-1}(l)dl. \]

(32)
Therefore, the expected total electricity consumption in each charging station \( j \), denoted by \( U_j(Q) \), is given by:

\[
U_j(Q) = \sum_{(o,d)\in O} \sum_{d=1}^{K_{od}} q_{od} \left[ E\left(\frac{Q^i_{od}}{q_{od}}\right) - E\left(\frac{Q^i_{od} - 1}{q_{od}}\right) \right], \tag{33}
\]

Equipped with this, we can now create a wait time resembling the form presented in (28) by modifying the wait times associated with finding a free EVSE at a charging station in [4] with a function of the following form:

\[
\text{expected wait time} = \frac{1}{c} T_j \left( \frac{1}{c} U_j(Q) \right) \epsilon, \tag{34}
\]

where \( T_j(x) = x/(1-x) \) corresponds to the term \( \frac{\lambda_j/c}{1-(\lambda_j/cp)} \) in (28), and \( \epsilon \) replaces \( 1/\mu \) for each user.

While we remove the details for brevity, we can show that the structure of the Wardrop equilibrium of the network given this new wait time formula is still threshold based, and can be characterized by replacing the term \( J_{0,\lambda} T_j(x)dx \) in (19) with \( \int_0^{1/2} \frac{U_j(Q)}{Q} T_j(x)dx \). This is straightforward to show given the following lemma.

**Lemma IV.7.** The derivative of \( U_j(Q) \) with respect to the upper and lower quantiles \( Q^i_{od} \) and \( Q^i_{od} \) of path \( i \) is respectively equal to \( G^{-1}\left(Q^i_{od}\right) \) and \( G^{-1}\left(Q^i_{od}\right) \), the upper and lower energy thresholds of path \( i \).

However, while this all seems positive, the function \( U_j(Q) \) is generally not convex in \( Q \), leading to the following result.

**Proposition IV.8.** The traffic flow pattern in a threshold-based EV equilibrium may not be unique in path flows when adopting wait times of the form given in (34).

Accordingly, to preserve the uniqueness properties of threshold-based traffic equilibria, we have decided to adopt waiting times of the form \( T_j(\lambda_j) \). We next study the socially optimal flow of each path, and we will derive the plug-in fees to guide the equilibrium solution towards a socially optimal traffic and energy footprint.

### V. Socially Optimal Pattern

In the social optimal (SO) view, the CNO has the objective to minimize the total latency and electricity cost occurred by EV users over all the network. Therefore, the socially optimal flow is the solution of following optimization:

\[
\text{minimize} \quad \sum_{i\in A} x_{at} t_a + \sum_{j\in J} \lambda_j T_j(\lambda_j) + \alpha \sum_{j\in J} D_j\left(U_j(Q)\right), \tag{35}
\]

subject to (19b) - (19g), 

where \( D(\cdot) \) is a strictly convex and continuously differentiable electricity cost function for each charging station, and is a function of the expected total energy consumption \( U_j(Q) \) of that charging station, given by (33).

The CNO’s goal is to calculate the plug-in fees and electricity prices for each charging station that drive the equilibrium flow pattern to be equal to the SO pattern.

**Proposition V.1.** If the condition set forth in Theorem IV.3 is satisfied (i.e., if the threshold-based equilibrium is unique), there exists an anonymous charging station pricing scheme that can enforce the system optimum flow.

**Proof.** The proof is straightforward. By comparing the KKT conditions of (35) and (19) for used path \( p_i \), we must have:

\[
\frac{\partial L_{eq}}{\partial q_{od}} = \frac{\partial L_{so}}{\partial q_{od}} \quad \Rightarrow \quad \sum_{j\in J} \delta_{j} T_j(\lambda_j) = \sum_{j\in J} \delta_{j} T_j'(\lambda_j) \tag{37}
\]

\[
\frac{\partial L_{eq}}{\partial Q_{od}} = \frac{\partial L_{so}}{\partial Q_{od}} \quad \Rightarrow \quad \alpha \sum_{j\in J} \frac{\partial U_j(Q)}{\partial Q_{od}} = \alpha \sum_{j\in J} \frac{\partial D_j\left(U_j(Q)\right)}{\partial U_j(Q)} \tag{38}
\]

Accordingly, to derive charging station prices that induce the system-optimal flow pattern as a UE, the CNO can simply set the plug-in fee \( (\tau_j) \) and the cost of electricity \( (v_j) \) at charging station \( j \) as follows:

\[
\tau_j = \lambda_j T_j'(\lambda_j) \tag{39}
\]

\[
v_j = \frac{\partial D_j\left(U_j(Q)\right)}{\partial U_j(Q)}. \tag{40}
\]

Next we show our result with the numerical simulation.

### VI. Numerical Experiment

In this section, we study the Wardrop equilibrium and social welfare of an electric transportation and charging network, and showcase the effect of charging station plug-in fees and prices in (39) and (40). For the transportation network, we use the road map of the SF bay area from [27], considering Tesla supercharger, Nema 14-50, and roadster charging stations as shown in Fig. 2.

![Fig. 2: SF Bay area map with Tesla supercharger, Tesla Nema 14-50, and Tesla roadster charging station.](image-url)
We design a transportation network based on Fig. 2 where we have three OD pairs, \((o_1, d_1), (o_2, d_2), (o_3, d_3)\), and 6 charging stations, \(X_1, \ldots, X_6\), and all road arcs are considered bidirectional (as we do not include road congestion costs in our model). For each OD pair, we consider 4 different paths that include a charging station stop. The network is shown in Fig. 3.

Fig. 3: Transportation network model of Fig. 2

We assume each road arc has a constant latency \(t_a = 1\). Moreover, we choose the charging rate \(\gamma = 1\), and time value of money \(\alpha = 5\). We consider 24 time slots with varying travel rates, and we solve separate static TACPs for each time slot. Three separate days are simulated, and the average number of travelers between each OD pair for each day is shown in Fig. 4.

Fig. 4: Average number of travelers in each OD pair for each day

The searching time function \(T_j(\lambda_j)\) to capture congestion due to limited EVSEs available at each charging station was chosen as:

\[
T_j = \begin{cases} 
\frac{\lambda_j}{10^{\lambda_j}}, & \lambda_j \leq X_j, \\
0, & \text{o.w},
\end{cases}
\]

with \(X_j\) shown in Table II.

| Station index | 1 | 2 | 3 | 4 | 5 | 6 |
|---------------|---|---|---|---|---|---|
| \(X_j\)       | 12| 10| 15| 10| 8 | 12|
| \(\nu_j\)     | 10| 5 | 7 | 6 | 8 | 9 |

TABLE II: Charging station capacities \(X_j\) and electricity prices \(\nu_j\) (in \$/kWh)

Moreover, we assume that energy demand of users takes value from a uniform distribution from 0 to 80 kWh (\(U[0, 80]\)). Therefore, the CDF will be \(G(x) = \frac{x}{80}\), and \(E(x) = \frac{x^2}{160}\). Given these parameters, we study the network traffic pattern and energy consumption under two scenarios:

1) Wardrop equilibrium: Here we study the traffic and charge pattern of the network under no form of marginal pricing. In this case, we assume that the plug-in fees are equal to zero, and the charging price at each station is equal to the grid price, as shown in Table II, which we consider as exogenous parameters. The result of Theorem IV.3 allows us to calculate the equilibrium traffic flow, and we can calculate the average energy consumption of customers in each charging station based as shown in Lemma IV.6.

2) Socially Optimal flow and electricity consumption: The result of optimization (35)-(36) gives the socially optimal flow, and again we can calculate the average energy consumption of customers in each charging station as shown in Lemma IV.6. As we would like to capture the effect of customers’ aggregate charging decisions on their electricity costs, and we have taken grid prices as exogenous parameters, we assume some charging stations are equipped with on-site (behind-the-meter) solar PV generation. Hence, we consider the energy cost function \(D_j(.)\) for each charging station \(j\) to be of the following form:

\[
D_j(x) = \zeta_j(x - s_j(t)),
\]

where \(\zeta_j(.)\) specifies the monetary cost of any electricity consumption above the on-site solar generation amount available at each charging station (which we assume is available for free). We use solar data from [29] for September 2018.

Fig. 5: Average energy consumption in both Wardrop equilibrium and Socially optimal scenarios over three days.

To see the effect of our transportation-aware marginal pricing of electricity, let us consider the average energy consumption at station 4, under both Wardrop equilibrium and Socially optimal scenarios. Figure 5 compares the average energy consumption under these scenarios against the solar energy generation \(s_4(t)\). In Wardrop equilibrium, users have no information about the generated solar energy in each charging station. Hence, their charging decisions do not

\footnote{Note this is simply due to our set up, where the total charging load is small. At larger scales, one could consider looking at joint pricing in transportation and power systems, which we have studied in a previous work [28].}
not take the availability of free solar energy into account. However, as shown in Fig. 5, the flow of users under the SO scenario is such that the demand at the charging station is better aligned with the solar energy production.

Let us also look at EVSE search costs at charging stations. We calculate \( \sum_{j \in J} \lambda_j T_j(\lambda_j) \) as the search cost at charging stations. For Wardrop equilibrium, the search cost peak is equal to \( 2.3e+4 \), and its average is \( 6.07e+3 \). For the socially optimal scenario, the search cost at the peak is \( 1.8e+4 \), and in average is \( 5.76e+3 \). Additionally, for the latency cost on roads we calculate \( \sum_{a \in A} x_a a \). For Wardrop equilibrium the latency cost at the peak is \( 991.27 \), and its average is \( 492.56 \). For the socially optimal scenario the latency cost at the peak is \( 781.2 \), and in average \( 484.41 \). Therefore, in the social optimal scenario, users will encounter less waiting/drive time at both charging stations and roads.

VII. CONCLUSIONS AND FUTURE WORK

We defined the so called Traffic and Charge Assignment Problem (TCAP) for EVs traveling a multi-commodity transportation network and requiring charge at a fast charging station en route. We studied the special threshold-based structure of equilibrium flows due to variations in electricity prices across different stations, and we discussed conditions under which the Wardrop equilibrium is unique. This allows us to design congestion and electricity prices for charging stations that can drive the overall traffic and energy footprint of the network towards that of the SO pattern. A further benefit we would like to point out is that traffic assignment plays an important role in the transportation planning process, and the same would apply to charging station capacity planning when considering mobility and energy costs jointly. Building capacity under an already congested distribution system feeder or in a neighborhood with no EV traffic would not be wise. Our TCAP model can also help with these decisions. On other other hand, we also observed that is not very difficult to have multiple equilibria in electricity price-aware EV routing once you generalize your model to consider more realistic effects. Moreover, the threshold-based routing models we observed also lead to non-convex programs for determining the socially optimal flow pattern. Coupled with the potential complications that could arise due to non-homogeneous value of time from users, the limited range of EVs, or elasticity of charging demand, intractability also becomes a concern. As such, one could argue that allowing geographical energy price discrimination in charging stations might not necessarily be implementable under model-based EV demand management. Staying in a energy price agnostic regime where charging station prices simply vary solely in terms of the plug-in cost could lead to more well-behaved congestion pricing models (clearly at the cost of efficiency). Alternatively, one could consider model-free learning-based approaches or differentiated pricing schemes where optimal pricing and routing schemes can be designed for the setting where users cannot directly choose which station they can charge their vehicle at. Instead, they specify their preferences for charging (which will lead to different charging prices), after which an operator directly assigns them to a station nearby.

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We will take the first and the second order derivative of (44) with respect to $f_i^j$. First order partial derivatives of (44) with respect to $Q_i$ ($i = 1, \ldots, K - 1$) are as follows:

$$\frac{\partial a_{od}(f)}{\partial Q_i} = (\theta_i - \frac{1}{q})G^{-1}(\frac{Q_i}{q}),$$

(45)

We can calculate the derivative of (44) with respect to $f_i^j$ as follows:

$$\frac{\partial a_{od}(f)}{\partial f_i} = \sum_{k=1}^{K-1} \frac{\partial a_{od}(f)}{\partial Q_k^j} \cdot \frac{\partial Q_k}{\partial f_i} = \sum_{k=1}^{K-1} \left( (\theta_k - \frac{1}{q})G^{-1}(\frac{Q_k}{q}) \right) \Gamma_{ki},$$

(47)

where $\frac{\partial Q_k}{\partial f_i} = \Gamma_{ki}$, with $\Gamma_{ki}$ given by:

$$\Gamma_{ki} = \begin{cases} 1 & i \leq k \\ 0 & \text{otherwise} \end{cases}$$

Then, we calculate the second order derivative of (44):

$$\frac{\partial^2}{\partial f_i \partial f_j} a_{od}(f) = \sum_{k=1}^{K-1} \left( [\theta_k - \frac{1}{q}y_i(f)] \Gamma_{ki} \Gamma_{kl} \right),$$

(48)

where $y_i(f)$ is:

$$y_i(f) = \frac{\partial G^{-1}(Q_i^j/q)}{\partial Q_i^j}.$$  

(49)

For proving the convexity, we will show the Hessian of (44) is non-negative with respect to $f_i$, i.e., we consider the non-negativity of:

$$\sum_{i,l} \frac{\partial^2}{\partial f_i \partial f_l} a_{od}(f) h_i h_l = \sum_{i,l} \sum_{k=1}^{K-1} \left[ (\theta_k - \theta_k^{+1}) \left( \frac{y_i(f)}{q} \right) \Gamma_{ki} \Gamma_{kl} h_i h_l \right] = \sum_{k=1}^{K-1} \left[ (\theta_k - \theta_k^{+1}) \left( \frac{y_i(f)}{q} \right) \sum_{i,l} \Gamma_{ki} \Gamma_{kl} h_i h_l \right] = \sum_{k=1}^{K-1} \left[ (\theta_k - \theta_k^{+1}) \left( \frac{y_i(f)}{q} \right) \left( \sum_{i,l} \Gamma_{ki} \Gamma_{kl} h_i h_l \right)^2 \right].$$

(50)

The coefficients $(\theta_k - \theta_k^{+1}) (y_i(f)/q)$ are non-negative, since $y_i(f) = \frac{\partial G^{-1}(Q_i^j/q)}{\partial Q_i^j}$ is the derivative of the inverse of an increasing function (the cumulative density function) and prices are in decreasing order ($\theta^k \geq \theta^{+1}$), therefore, those coefficients are non-decreasing. The non-negativeness of Hessian of (44) with respect to $f_i$ proves the convexity of (44). Summing over all O-D pairs will prove the convexity of (42). Therefore, because the waiting time function, $T_j(\lambda_j)$, is strictly increasing, the optimization problem (19) is strictly convex, and hence has a unique solution. Thus, the equilibrium pattern as well as energy thresholds are unique.