Chiral Algebras and the Höhn–Stolz Conjecture

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Abstract

Recent studies of two-dimensional sigma models with \((0, 2)\) supersymmetry have revealed that the chiral algebras of these models can vanish due to instanton effects. In this paper we discuss the relation of this phenomenon to the Kähler case of the Höhn–Stolz conjecture, which asserts that the Witten genus vanishes for closed string manifolds admitting a Riemannian metric of positive Ricci curvature.

1 Introduction

In his classic paper [W], Witten unraveled a beautiful connection between the elliptic genus discovered by Ochanine [O] and a certain two-dimensional quantum field theory, known as the supersymmetric sigma model. Extending his idea to the case with no left-moving fermions, he was naturally led to discover a new genus. The Witten genus, as it is now called, associates to every closed string manifold \(M\) of dimension \(4k\) an integral modular form \(W(M)\) of weight \(2k\).

About a decade after the discovery of the Witten genus, Höhn and Stolz independently made the following

Conjecture (Höhn, Stolz [S]). Let \(M\) be a closed string manifold. If \(M\) admits a Riemannian metric of positive Ricci curvature, then the Witten genus \(W(M)\) vanishes.

The aim of this paper is to explain how recent developments in the study of two-dimensional supersymmetric sigma models provide a new approach to the Kähler case of the Höhn–Stolz conjecture. This approach is based on a remarkable phenomenon that certain sigma models with \((0, 2)\) supersymmetry exhibit, which was first predicted by Witten [W3] and subsequently confirmed by Tan and the author [TY]—specifically, the phenomenon that the chiral algebras associated to these models vanish due to instanton effects.
The emergence of a chiral algebra (or vertex algebra) is a characteristic property of two-dimensional quantum field theories with (0, 2) supersymmetry. Although this fact had been realized [V] as early as a few years after the appearance of Witten’s paper [W], it had not been paid much attention by physicists until relatively recently. During the last several years, however, there have been significant advances in our understanding of the chiral algebras of (0, 2) supersymmetric sigma models, thanks largely to the pioneering works of Kapustin [K] and Witten [W3]. What inspired these advances is the theory of chiral differential operators developed earlier in mathematics. The author hopes that the results presented here from physics give some inspirations to mathematicians.

We begin our discussion in the next section by reviewing how the Witten genus arises in two-dimensional supersymmetric sigma models. Then, in Section 3, we describe how to construct a chiral algebra using (0, 2) supersymmetry in the case when the target space is Kähler. In Section 4, we outline the argument for the vanishing phenomenon of the chiral algebra and explain its relation to the Höhn–Stolz conjecture. In the last section, we discuss what this phenomenon implies for the geometry of loop spaces.

2 Supersymmetry and the Witten genus

Let $M$ be a closed spin manifold of dimension $d = 4k$. We will consider the supersymmetric sigma model with no left-moving fermions that has target space $M$. This theory has a hermitian operator $Q_+$ satisfying the (0, 1) supersymmetry algebra

\[ \{Q_+, Q_+\} = H - P, \quad \{Q_+, (-1)^{F_R}\} = 0, \]

where $H$ and $P$ are the generators of translations respectively in time and space directions, and $(-1)^{F_R}$ is the operator that counts the number of right-moving fermions modulo two. With a suitable choice of the complex structure on the two-dimensional spacetime manifold, the operator $H - P$ generates antiholomorphic translations, $H - P \propto \partial_{\bar{z}}$. States annihilated by $Q_+$ are called supersymmetric. Supersymmetric states are those states that have $H - P = 0$.

On the torus $\mathbb{C}/2\pi(\mathbb{Z} + \tau\mathbb{Z})$ with $\tau$ in the upper half-plane, the partition function of the theory is computed by

\[ Z(M; q) = \text{Tr} \left( (-1)^{F_R} q^{(H-P)/2} \bar{q}^{(H-P)/2} \right). \]

Here $q = e^{2\pi i \tau}$ and the trace is taken in the Hilbert space of states. Since the supercharge $Q_+$ pairs bosonic ($F_R$ even) and fermionic ($F_R$ odd) states
not in its kernel, $Z(M; q)$ receives contributions only from supersymmetric states. The operator $P$ takes values in $-d/24 + \mathbb{Z}$; the shift by $-d/24$ comes from the regularization of an infinite sum. Thus we can write $Z(M; q)$ in a $q$-series as

$$Z(M; q) = q^{-d/24} \sum_{n=0}^{\infty} q^n \text{Tr}_{\mathcal{H}_n} (-1)^{F_R}. \quad (3)$$

The coefficients are the indices of $Q_+$ in the spaces $\mathcal{H}_n$ of supersymmetric states with $P = -d/24 + n$. The Witten genus of $M$ is then given by

$$W(M) = \eta(q)^d Z(M; q), \quad (4)$$

with $\eta(q) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n)$ the Dedekind eta function. Since the Witten genus and the partition function differ only by a factor, we will mainly talk about the latter which is a physically more natural object to consider.

The geometric meaning of the partition function becomes clear if we canonically quantize the theory. Quantization identifies states with spinors on the free loop space $L M$ and $(-1)^{F_R}$ with the chirality operator. The supercharge $Q_+$, on the other hand, is identified with the Dirac operator $\mathcal{D}_{LM}$ on $LM$ coupled to the Killing vector field $V$ generating rotations of loops:

$$Q_+ = \mathcal{D}_{LM} + t \Gamma \cdot V. \quad (5)$$

Here $\Gamma$ denotes Clifford multiplication and $t \in \mathbb{R}^+$ is a parameter of the theory.\footnote{There are two ways to think about this parameter. The volume of the target space vs. the radius of the loop.} Taking the limit $t \to 0$ in the expression\footnote{W2}, we learn that the partition function is the $S^1$-equivariant index of the Dirac operator on the loop space with respect to the natural circle action\footnote{W2}.

Taking the limit $t \to \infty$, we obtain a formula that expresses $Z(M; q)$ in terms of characteristic classes of $M$. Since $H - P$ contains the potential $t^2 |V|^2$, supersymmetric states localize in this limit to the zeros of $V$. The zeros are the constant loops forming a copy of $M$ inside $LM$. Such localized states can be thought of as spinors on $M$, taking values in various vector bundles; an approximate supersymmetric state with $P = -d/24 + n$ is a spinor with values in the bundle given by the coefficient of $q^n$ in the series

$$\bigotimes_{m=1}^{\infty} S_q^m(TM) = 1 + qTM + q^2 (TM + S^2(TM \otimes TM)) + \cdots, \quad (6)$$

where the symmetric power operation $S$ of a vector bundle is defined by $S_q(E) = 1 + qS(E) + q^2S^2(E) + \cdots$. On these localized states, $Q_+$ acts as
the Dirac operator $\mathcal{D}_M$ on $M$. Thus we obtain

$$Z(M; q) = q^{-d/24} \text{index} \left( \mathcal{D}_M \otimes \bigotimes_{m=1}^{\infty} S_q^{m}(TM \otimes \mathbb{C}) \right)$$

$$= q^{-d/24} \left\langle \hat{A}(M) \text{ch} \left( \bigotimes_{m=1}^{\infty} S_q^{m}(TM \otimes \mathbb{C}) \right), [M] \right\rangle. \quad (7)$$

Mathematically, this formula defines the Witten genus via the relation (4).

Although the above definition of the Witten genus $W(M)$ works for any closed spin manifold $M$, the underlying quantum field theory does not. For the theory to be well defined, $M$ must be string (that is, spin and has $p_1(M)/2 = 0$). In addition, the theory suffers from the usual problem of infinities of quantum field theory. To ensure that these infinities can be removed by renormalization (at least to one-loop level), one must impose that the Ricci curvature of $M$ be positive or else vanish. Hence, one expects that these conditions are necessary for the partition function $Z(M; q)$ to be interpreted as the $S^1$-equivariant index of the Dirac operator on $\mathcal{L}M$. The modularity of $W(M)$ for $M$ a string manifold follows from the diffeomorphism invariance of the theory. Actually, the weaker condition that $p_1(M) = 0$ rationally is sufficient for this since the Witten genus can be computed in the perturbative approximation, by which one loses the torsion information of anomalies.

3 Chiral Algebras

Now suppose that $M$ is Kähler. In this case, the Dirac operator on $M$ splits into two pieces as $\mathcal{D}_M = \mathcal{D}_M^+ + \mathcal{D}_M^- [\mathcal{L}M]$. Each of these pieces squares to zero, $\mathcal{D}_M^2 = \mathcal{D}_M^- = 0$, and is the adjoint of the other, $\mathcal{D}_M = \mathcal{D}_M^\dagger$. Furthermore, the spinor bundle has a $\mathbb{Z}$-grading under which $\mathcal{D}_M$ has degree one and $\mathcal{D}_M^\dagger$ degree minus one. The $r$th $\mathcal{D}_M$-cohomology group of the spinor bundle is isomorphic to the space of harmonic spinors of degree $r$.

Similarly, in the Kähler case the supercharge splits as $Q_+ = Q + Q^*$, with $Q$ satisfying the $(0, 2)$ supersymmetry algebra

$$Q^2 = 0, \quad \{Q, Q^*\} = \frac{1}{2} (H - P). \quad (8)$$

There is also a grading given by $R$-charge, which coincides modulo two with $(-1)^F_R$ and under which $Q$ has charge one and $Q^*$ charge minus one. This is a $\mathbb{Z}$-grading at the level of perturbation theory, but reduced to a $\mathbb{Z}/2n\mathbb{Z}$-grading by instanton effects for $c_1(M) \neq 0$, where $2n$ is the greatest divisor
of $c_1(M)$. (Recall that $M$ is spin, thus $c_1(M) \equiv 0 \mod 2$.) The $r$th $Q$-cohomology group computed in the Hilbert space of states is isomorphic to the space of supersymmetric states of R-charge $r$.

What makes a quantum field theory with $(0, 2)$ supersymmetry interesting for us is that one can construct another $\mathbb{Z}/2n\mathbb{Z}$-graded $Q$-cohomology, namely the $Q$-cohomology in the space of local operators. The action of $Q$ on a local operator $\mathcal{O}$ is given by supercommutator, $\{Q, \mathcal{O}\} = Q\mathcal{O} \mp \mathcal{O}Q$ depending on whether $\mathcal{O}$ is bosonic or fermionic. The $Q$-cohomology of states and local operators are isomorphic when the theory is conformally invariant via the state-operator correspondence. This is not the case in general.

An important property of the $Q$-cohomology of local operators is that its elements vary holomorphically on the spacetime. By this we mean the following. Let $\mathcal{O}$ be a $Q$-closed local operator, $\{Q, \mathcal{O}\} = 0$, and $[\mathcal{O}]$ its $Q$-cohomology class. Recalling that $H - P \propto \partial \bar{z}$, we have

$$\partial \bar{z} \mathcal{O} \propto \{H - P, \mathcal{O}\} = \{(Q, Q^*), \mathcal{O}\} = \{Q, \{Q^*, \mathcal{O}\}\}. \quad (9)$$

Thus $\partial \bar{z} \mathcal{O}$ is $Q$-exact, and $\partial \bar{z}$ annihilates $Q$-cohomology classes.

Another important property is that the $Q$-cohomology of local operators has natural operator product expansions (OPE) inherited from those of the underlying theory:

$$[\mathcal{O}_i(z)] \cdot [\mathcal{O}_j(z')] = [\mathcal{O}_i(z) \cdot \mathcal{O}_j(z')] \sim \sum_k c_{ij}^k (z - z') [\mathcal{O}_k(z')]. \quad (10)$$

The coefficient functions $c_{ij}^k (z - z')$ are holomorphic for $z \neq z'$, but can have poles at $z = z'$.

The holomorphic $Q$-cohomology of local operators, equipped with this OPE structure, is the chiral algebra of the $(0, 2)$ supersymmetric theory. The chiral algebra forms a holomorphic sector of the theory, in which correlation functions are holomorphic in the insertion points as long as no two points coincide. For the topological A-model (considered as a $(0, 2)$ supersymmetric theory), these correlation functions generalize Gromov–Witten invariants.

The chiral algebra of a quantum field theory with $(0, 2)$ supersymmetry has the same structure as the chiral algebras (vertex algebras) of conformal field theories do, except that the grading by conformal weight is missing. In fact, the chiral algebra is in general not conformally invariant, and certainly not for our sigma model if $c_1(M) \neq 0$. Though our model is classically conformally invariant, quantum mechanically the conformal invariance is broken if the Ricci curvature of $M$ is nonzero. Correspondingly, though the energy-momentum tensor $T_{zz}$ lies in the $Q$-cohomology classically, there exists a
Q-closed local operator $\theta$ associated to $c_1(M)$, which, under perturbative quantum corrections, satisfies the relation

$$\{Q, T_{zz}\} = \partial_z \theta.$$  \hspace{1cm} (11)

If $c_1(M) = 0$, then $\theta$ is $Q$-exact and one can find quantum corrections for $T_{zz}$ to make it $Q$-closed again. However, this is not possible if $c_1(M) \neq 0$. Hence, the chiral algebra lacks an energy-momentum tensor in this case.

In view of this, one may be surprised by the fact that even though the chiral algebra has in general no conformal invariance, in a slightly different variant of the theory—obtained by the so-called “twisting” procedure—there is a grading by conformal weight at the perturbative level. It was shown by Witten [W3] that with this additional grading, the perturbative chiral algebra of the twisted model is isomorphic to the cohomology of a sheaf $\mathcal{D}_{ch}^M$ of chiral differential operators, introduced by Malikov et al. [MSV]. In particular, when $c_1(M) = 0$, the perturbative chiral algebra is conformally invariant and, by the state-operator correspondence, its character computes the partition function:

$$\text{ch} H^*(M; \mathcal{D}_{ch}^M) = Z(M; q).$$  \hspace{1cm} (12)

By the same token, Kapustin [K] showed that the perturbative chiral algebra of the A-model is isomorphic to the cohomology of a sheaf of chiral de Rham complex. The case in which the theory is coupled to a general “gauge bundle” $E$ was considered by Tan [T] and found related to the construction of Gorbounov et al. [GMS]. The A-model corresponds to $E = TM$, the holomorphic tangent bundle of $M$.

Beyond perturbation theory, the gradings by R-charge and conformal weight can be violated by instantons, which are holomorphic maps from the spacetime manifold to $M$. Thus, these perturbative relations between the chiral algebras and chiral differential operators may be altered drastically.

4 The Vanishing “Theorem”

In [W3], Witten made a remarkable prediction. He claimed that the chiral algebras of sigma models of $(0, 2)$ supersymmetry, though perturbatively infinite-dimensional, can nevertheless vanish nonperturbatively in the presence of instantons. More specifically, he predicted that this occurs for

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On the torus (or any Riemann surface with trivial canonical bundle), the twisted and untwisted models are equivalent and have the same partition function.
\( M = \mathbb{C}P^1 \) via an instanton-induced relation
\[ \{ Q, \theta \} \sim 1. \tag{13} \]

In the case of \( M = \mathbb{C}P^1 \), the chiral algebra is \( \mathbb{Z}/2\mathbb{Z} \)-graded and the operator \( \{ Q, \theta \} \), which has R-charge two perturbatively, has chance to be equal to the identity operator 1 nonperturbatively. This relation shows that 1 = 0 in the \( Q \)-cohomology and therefore the chiral algebra is identically zero.

Witten’s prediction was confirmed by an explicit computation in [TY], where this result was also extended to all complete flag manifolds. In fact, one has a more general vanishing “theorem” which has been “proved” in [Y] at the physicists’ level of rigor:

**Theorem** (Yagi [Y]). Let \( M \) be a closed Kähler string manifold with \( c_1(M) > 0 \). If \( M \) contains a rational curve with trivial normal bundle, then the chiral algebra of a (0, 2) supersymmetric sigma model with no left-moving fermions vanishes for target space \( M \).

The normal bundle \( N_{C/M} \) of a rational curve \( C \subset M \) is a holomorphic vector bundle defined by the exact sequence
\[ 0 \rightarrow T_C \rightarrow T_M|_C \rightarrow N_{C/M} \rightarrow 0. \tag{14} \]

The condition on the normal bundle says that
\[ N_{C/M} \cong \mathcal{O}^{\oplus d/2-1}. \tag{15} \]

This ensures that the chiral algebra is \( \mathbb{Z}/2\mathbb{Z} \)-graded by R-charge.

We can only give the outline of the “proof” here. Suppose that we have an instanton-induced relation \( \{ Q, \theta \} = \mathcal{O} \). The local operator \( \mathcal{O} \) is \( Q \)-closed, but need not be \( Q \)-exact in perturbation theory, since we are assuming that this relation arises only after instantons are taken into account. Let us look at the R-charge zero, weight zero part of \( \mathcal{O} \). A local operator of R-charge zero and weight zero is a function on \( M \). Since \( \mathcal{O} \) varies holomorphically on the compact complex manifold \( M \), this function must be constant modulo \( Q \)-exact terms. Applying the computation from the case of \( M = \mathbb{C}P^1 \), one sees that this constant is nonzero on the rational curve \( C \), hence on the whole manifold \( M \). The positive weight part of \( \mathcal{O} \), on the other hand, can be shown to be all \( Q \)-exact. Thus, we again have the relation \( \{ Q, \theta \} \sim 1 \), the equation 1 = 0 holds in the \( Q \)-cohomology, and the chiral algebra vanishes in the presence of instantons.

The intuitive picture behind the vanishing “theorem” is as follows. When the normal bundle of \( C \) is trivial, \( C \) can be translated in every direction in
and we obtain a family of rational curves. This family must sweep out the whole manifold \( M \), and the contributions to \( \{ Q, \theta \} \) from its members add up to a nonzero constant. For example, in the case where \( M \) is a flag manifold the transitive group action generates such a family.

We are now ready to explain the relation of the vanishing phenomenon to the Höhn–Stolz conjecture. It is based on the following observation: the \( Q \)-cohomology of states vanishes if the chiral algebra does.

The argument is very simple. Notice that the \( Q \)-cohomology of states is naturally a module over the chiral algebra, with the action of \( [ \mathcal{O} ] \) on \( [ | \Psi \rangle ] \) defined by \( [ \mathcal{O} ] : [ | \Psi \rangle ] = [ \mathcal{O} | \Psi \rangle ] \). If the chiral algebra vanishes, then \( [1] = 0 \) and for any \( Q \)-closed state \( | \Psi \rangle \) we have

\[
| \Psi \rangle = [1] \cdot [ | \Psi \rangle ] = 0.
\]

Therefore, any \( Q \)-closed state is \( Q \)-exact; in other words, the \( Q \)-cohomology of states vanishes.

Now, recall that the \( Q \)-cohomology of states is isomorphic to the space of supersymmetric states. Therefore, when this vanishes the partition function, which counts the number of bosonic supersymmetric states minus that of fermionic ones at each level of \( P \), vanishes as well.

It would be interesting to see to what extent our vanishing “theorem” generalizes to other Kähler manifolds.

## 5 Loop-Space Lichnerowicz

A natural geometric framework for understanding the two-dimensional supersymmetric sigma model with target space \( M \) is the loop space \( \mathcal{L}M \). Hence, the vanishing of the chiral algebra should have implications for the geometry of this space. One such implication is that \( \mathcal{L}M \) has no harmonic spinors. This is in accordance with the original argument of Stolz, who interpreted the conjecture as arising from a loop-space analog of the Lichnerowicz theorem for finite-dimensional spin manifolds with positive scalar curvature.

We have already explained that the partition function of the sigma model with target space \( M \) is the \( S^1 \)-equivariant index of the supercharge \( Q_+ \), and this is equal to the \( S^1 \)-equivariant index of the Dirac operator \( \mathcal{D}_{\mathcal{L}M} \) on \( \mathcal{L}M \). When \( M \) is Kähler, one can actually say more: the space of supersymmetric states is isomorphic to the space of harmonic spinors on \( \mathcal{L}M \). The reason is that \( Q \) is related to the degree one part \( D_{\mathcal{L}M} \) of \( \mathcal{D}_{\mathcal{L}M} \) by the conjugation

\[
Q = e^{A_0} D_{\mathcal{L}M} e^{-A_0},
\]
where $A_0$ is the symplectic action functional with Hamiltonian set to zero. Thus, the two operators have an isomorphic cohomology.

The $Q$-cohomology of states vanishes if the chiral algebra does. Therefore, the vanishing of the chiral algebra implies that there exist no harmonic spinors on $\mathcal{LM}$.

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