A Generalized Curvature of a Generalized Envelope

Tahir H. Ismail
ibrahim.hamad@su.edu.krd
College of computers Sciences and Mathematics
University of Mosul
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Ibrahim O. Hamad
ibrahim.hamad@su.edu.krd
College of Sciences
Salahaddin University
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ABSTRACT

In this paper we study one of the applications of a generalized curvature [3] on the generalized envelope of a family of lines given in [7], [8], using some concepts of nonstandard analysis given by Robinson, A. [5] and axiomatized by Nelson, E.

Keywords: infinitesimals, monad, envelope, generalized curvature

1- Introduction:

The following definitions and notations are needed throughout this paper.

Every concept concerning sets or elements defined in classical mathematics is called standard [4].

Any set or formula which does not involve new predicates “standard, infinitesimals, limited, unlimited…etc” is called internal, otherwise it is called external [2], [4].

A real number \( x \) is called unlimited if and only if \(|x| > r\) for all positive standard real numbers, otherwise it is called limited [2].

A real number \( x \) is called infinitesimal if and only if \(|x| < r\) for all positive standard real numbers \( r \) [2].
Two real numbers $x$ and $y$ are said to be infinitely close if and only if $x - y$ is infinitesimal and denoted by $x \cong y$ [2], [6].

If $x$ is a limited number in $\mathbb{R}$, then it is infinitely close to a unique standard real number, this unique number is called the standard part of $x$ or shadow of $x$ denoted by $st(x)$ or $^0x$ [2], [4].

If $x$ is a real limited number, then the set of all numbers, which are infinitely close to $x$, is called the monad of $x$ and denoted by $(x)$ [2], [4].

A curve $\nu$ is called envelope of a family of curves $\{\gamma_\alpha\}$ depending on a parameter $\alpha$, if at each of its points, it is tangent to at least one curve of the family $\{\gamma_\alpha\}$, and if each of its segments is tangent to an infinite set of these curves [1].

The projective homogenous plane over $\mathbb{R}$, denoted by $\mathbb{P}^2_\mathbb{R}$ is the set:

$$\mathbb{P}^2_\mathbb{R} = \mathbb{R}^2 \cup \{\text{one point at } \infty \text{ for each equivalence classes of parallel lines }\},$$

we denoted it by (PHP) [1].

The projective homogeneous coordinates of a point $p(x, y) \in \mathbb{R}^2$ are $(x\alpha, y\alpha, \alpha)$, where $\alpha$ is any nonzero number, we denote it by (PHC).

In this sense the projective homogeneous coordinates of any point is not unique. [1]

By a parameterized differentiable curve, we mean a differentiable map $\gamma : I \to \mathbb{R}^3$ of an open interval $I = (a, b)$ of the real line $\mathbb{R}$ into $\mathbb{R}^3$ such that: $\gamma(t) = (x(t), y(t), z(t)) = x(t)e_1 + y(t)e_2 + z(t)e_3$, and $x$, $y$, and $z$ are differentiable at $t$; it is also called spherical curve [2].

**Definition 1.1** [7]

Let $A = \gamma(t)$ be a standard point on the curve $\gamma$, then the following cases occur for the point $A$ with the existence of the order of derivatives of $\gamma$:

1- If $\gamma' \neq 0$, $\gamma'' \neq 0$ and $\gamma' \cdot \gamma'' \neq 0$ then the point is called biregular point.

2- If $\gamma' \neq 0$ then the point is called regular point.

3- If $\gamma' \neq 0$ and $\gamma' \cdot \gamma'' \neq 0$ then the point is called only regular point, and we say that the point is only regular point of order $p$-I if $\gamma' \neq 0$ and $\gamma' = \gamma'' = \cdots = \gamma^{(p-1)} = 0$, but $\gamma' \cdot \gamma^{(p)} \neq 0$. In this case we say that $p$ is the order of the first vector derivative not collinear with $\gamma'$.

4- If $\gamma' = 0$ then the point is called singular point. In general if $\gamma' = \gamma'' = \cdots = \gamma^{(p-1)} = 0$ but $\gamma^{(p)} \neq 0$, then the point is called singular point of order $p$. 

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Theorem 1.2 [7]
Let $\gamma$ be a standard curve of order $C^n$ and $A$ be a standard singular point of order $p-I$ on $\gamma$; and let $B$ and $C$ be two points infinitely close to the point $A$, then the generalized curvature of $\gamma$ at the point denoted by $K_\varphi$ and given by

$$K_\varphi = (p \varphi)^{\frac{2}{2}} \frac{|x^{(p)}y^{(q)} - x^{(q)}y^{(p)}|}{q!(p)!, q!(q)!}$$

where $q$ is the order of the first vector derivative of $\gamma$ not collinear with $\gamma'$.

Theorem 1.3 [7]
If $p_k(t) = r_k(t) = q_k(t) = 0$ for $1 \leq k \leq n$ (n standard) and $p_n(t)$, $r_n(t)$, $q_n(t)$ are not all zeros, then the PHC points of $\gamma(t)$ are of the form $(p_n(t), r_n(t), q_n(t))$ which does not depend on $e$. Thus, we get the generalized nonclassical form of the envelope curve $\gamma(t)$ as follows:

$$(x(t), y(t)) = \begin{pmatrix} X(t) \\ Y(t) \\ Z(t) \end{pmatrix} = \begin{pmatrix} u^{(n)}(t)v(t) - w^{(n)}(t)v(t) \\ w^{(n)}(t)u(t) - u^{(n)}(t)v(t) \\ u^{(n)}(t)v(t) - v^{(n)}(t)u(t) \end{pmatrix}$$

2- A Generalized Curvature of the Envelope of a Family of Lines

Throughout this section, we give a curvature formula for the envelope of a family of lines $L_t : u(t)x + v(t)y + w(t)z = 0$ represented by the components $u$, $v$, and $w$.

It is clear that every two infinitely closed points (points in the same monad) on the envelope curve of a family of lines determine two infinitely close lines in that monad.

That is, $\forall A(t_0), B(t_0) \in \gamma(t_0)$, where $B(t_0 + \alpha) \in m(A(t_0))$ there exists a line $L_{t_0 + \alpha} \in \{t_1\}$ such that $L_{t_0 + \alpha} > L_{t_0}$ in $m(A(t_0))$, where $m(A(t_0))$ denotes the monad of the point $A$, where $\alpha$ is an infinitesimal number.

For finding curvature formula of the envelope of a family of lines, we follow the following algorithm.

1. Find the envelope curve using Theorem 1.3 according to the case under consideration.
2. Find the singularity and collinearity order of the envelope curve.
3. Consider three infinitely closed points $A(t_0), B(t_0 + \alpha)$ and $C(t_0 + \beta)$ on the envelope curve $g(t)$ such that
Apply the generalized curvature formula given in Theorem 1.2 at the points $A(t_o)$, $B(t_o+\alpha)$ and $C(t_o+\beta)$. Where $\alpha$ and $\beta$ are infinitesimal numbers.

The following theorems will give a new formula of the generalized curvature of the envelope of a family of lines.

**Theorem 2.1**

Let $A=\gamma(t_o)$ be a regular point of the envelope curve $\gamma$ of the family $L_t : u(t)X + v(t)Y + w(t)Z = 0$ in PHC, then the generalized curvature $K_G$ of the envelope curve at a point $A$ is given by

\[
K_G = \frac{\left| r' (q''q' - r''q') + (p''q'q'r' - p'q'r'q') \right|}{2 \left| p'q'r' + q'r'' + r'q'' \right|^2}, \quad \ldots (2.1.1)
\]

where $p(t)$, $q(t)$ and $r(t)$ are as given in Theorem 1.3 for $n=1$.

**Proof:**

Let $A=\gamma(t_o)$ be a standard point on the envelope of the curve $\gamma$, and $B=\gamma(t_o+\alpha)$, $C=\gamma(t_o+\beta)$ be two points infinitely close to $A$. Let $L_t$, $L_{t+a}$ and $L_{t+b}$ be three lines of the family $\{L_t\}$ having $A$, $B$ and $C$ as contact points with the envelope curve, respectively. Then;

\[
L_t : u(t)X + v(t)Y + w(t)Z = 0,
\]

\[
L_{t+a} : u(t+a)X + v(t+a)Y + w(t+a)Z = 0,
\]

\[
L_{t+b} : u(t+b)X + v(t+b)Y + w(t+b)Z = 0.
\]

Since, the point $A$ is regular, then Theorem 1.3 for $n=1$ is satisfied, and therefore $\gamma(t) = (p1(t), r1(t), q1(t))$.

Using the spherical case of the generalized curvature given in Theorem 1.2 for a curve $\gamma = (x(t), y(t), z(t))$, we get

\[
K_G = \frac{\left| (y'z'' - y''z')^2 + (x'z'' - x''z')^2 + (x'y'' - x''y')^2 \right|}{2 \left| x'^2 + y'^2 + z'^2 \right|^2}, \quad \ldots (2.1.2)
\]

Now replacing each of $x$, $y$ and $z$ by $p1(t)$, $r1(t)$ and $q1(t)$, respectively, we get the required result. ■
Let $A = \gamma(t_0)$ be a singular point of the envelope curve $\gamma$ of order $n-1$, and let $m$ be the order of the first nonzero derivative which is not collinear with $\gamma^{(n)}(t)$, that is, $\gamma'(t) = \gamma''(t) = \cdots = \gamma^{(n-m)}(t) \neq 0$, and $\gamma^{(n-m)}(t) = \cdots = \gamma^{(n-1)}(t) = 0$. Then, the generalized curvature $K_G$ of the envelope curve $\gamma$ at the points of the monad of $A$ is given by

$$
\left(\begin{array}{c}
\frac{n!}{m!} \left[ \left( \frac{a}{q} \right)^{(n)} \left( \frac{b}{q} \right)^{(m)} - \frac{a}{q} \frac{b}{q} \right)^{(m)} \right] \\
\frac{m}{2n} \left[ \left( \frac{a}{q} \right)^{(m)} + \left( \frac{b}{q} \right)^{(m)} \right]^{m+n}
\end{array} \right) \cdots (2.2.1)
$$

Moreover, the Cartesian coordinate of the generalized curvature $K_G$ of the envelope curve $\gamma$ at the points of the monad of $A$ is given by

$$
\left(\begin{array}{c}
\frac{n!}{m!} \left[ \left( \frac{a}{q} \right)^{(n)} \left( \frac{b}{q} \right)^{(m)} - \frac{a}{q} \frac{b}{q} \right)^{(m)} \right] \\
\frac{m}{2n} \left[ \left( \frac{a}{q} \right)^{(m)} + \left( \frac{b}{q} \right)^{(m)} \right]^{m+n}
\end{array} \right) \cdots (2.2.2)
$$

where $n$ and $m$ are positive integer numbers.

**Proof:**

First, applying the spherical case of the generalized curvature given in **Theorem 1.2** at $x = p_1(t)$, $y = r_1(t)$ and $z = q_1(t)$, we get the generalized curvature formula (2.2.1). Since the point $(p_1(t), r_1(t), q_1(t))$ in $PHC$ is equivalent to the point $(p_1(t)/q_1(t), r_1(t)/q_1(t), I)$, so again, applying the spherical case of generalized curvature, we get

$$
K_G = \left(\begin{array}{c}
\frac{n!}{m!} \left[ \left( \frac{a}{q} \right)^{(n)} \left( \frac{b}{q} \right)^{(m)} - \frac{a}{q} \frac{b}{q} \right)^{(m)} \right] \\
\frac{m}{2n} \left[ \left( \frac{a}{q} \right)^{(m)} + \left( \frac{b}{q} \right)^{(m)} \right]^{m+n}
\end{array} \right) \cdots (2.2.3)
$$

Thus, putting $x = p_1(t)/q_1(t), y = r_1(t)/q_1(t)$ and $z = I$, in (2.2.3), we obtain the formula (2.2.2).

**Corollary 2.3**

Let $A = \gamma(t_0)$ be a singular point of the envelope curve $\gamma$ satisfying the hypothesis of **Theorem 2.2**.

Moreover, let the coefficient vector $(u(t), v(t), w(t))$ of the envelope curve has a singularity of order $n-1$, then the generalized curvature $K_G$ of the envelope curve $\gamma$ at points in the monad of $A$ is given by
\((n!)^m \left[ r_n(t) q_n(t) - r_n'(t) q_n'(t) \right]^2 + \left[ p_n(t) q_n(t) - p_n'(t) q_n'(t) \right]^2 + \left[ p_n(t) r_n(t) - p_n'(t) r_n'(t) \right]^2 \right]^\frac{1}{2} \ldots (2.3.1)\)

\(m! \left[ p_n''(t) + q_n''(t) + r_n''(t) \right]^{\frac{1}{2}}\)

and the cartesian coordinate curvature \(K_G(t)\) of the envelope curve \(\gamma\) at \(A\) is given by

\[(n!)^m \left[ \begin{array}{c} p_n(t) \\ q_n(t) \\ r_n(t) \end{array} \right] - \left[ \begin{array}{c} p_n'(t) \\ q_n'(t) \\ r_n'(t) \end{array} \right] \ldots (2.3.2)\]

**Proof:**

By Theorem 2.2 we have

\[K_G(t) = \left( \begin{array}{c} n! \end{array} \right)^m \left[ \begin{array}{c} r_n(t) q_n(t) - r_n'(t) q_n'(t) \\ p_n(t) q_n(t) - p_n'(t) q_n'(t) \\ p_n(t) r_n(t) - p_n'(t) r_n'(t) \end{array} \right]^2 \cdot \left[ p_n''(t) + q_n''(t) + r_n''(t) \right]^{\frac{1}{2}}\]

Since the coefficient vector \((u(t), v(t), w(t))\) of the envelope curve has a singularity of order \(n-1\), so we get

\[u(t) = v(t) = w(t) = \cdots = u^{(n-1)}(t) = v^{(n-1)}(t) = w^{(n-1)}(t) = 0,\]

and \((u^{(n)}(t), v^{(n)}(t), w^{(n)}(t)) \neq 0)\)

Therefore,

\[
\begin{align*}
p^{(n)}(t) &= v^{(n)}(t)w(t) - w^{(n)}(t)v(t) = p_n(t) \\
r^{(n)}(t) &= w^{(n)}(t)u(t) - u^{(n)}(t)w(t) = r_n(t) \\
q^{(n)}(t) &= u^{(n)}(t)v(t) - v^{(n)}(t)u(t) = q_n(t)
\end{align*}
\]

Hence, the result of the first part is proved.

To prove the second part put \(x = p_n(t)/q_n(t)\), \(y = r_n(t)/q_n(t)\) and \(z = 1\) and then apply the spherical curvature formula (2.2.3) to obtain the formula (2.3.2). \(\blacksquare\)

**Corollary 2.4**

Let \(A = \gamma(t_0)\) be a singular point of the envelope curve \(\gamma\) satisfying the hypothesis of Theorem 2.2. Let \(\gamma(t) = (p(t), r(t), q(t))\) be such that \(q(t)\) has a nonzero constant value, then the generalized curvature \(K_G\) of the envelope curve \(\gamma\) at points of the monad of \(A\) is given by

\[(n!)^m \left[ \begin{array}{c} p^{(m)}(t) r^{(m)}(t) - p^{(m)}(t) r^{(m)}(t) \\ p^{(m)}(t) r^{(m)}(t) - p^{(m)}(t) r^{(m)}(t) \\ p^{(m)}(t) r^{(m)}(t) - p^{(m)}(t) r^{(m)}(t) \end{array} \right]^2 \cdot q^{(m)}(t) \ldots (2.4.1)\]

\[m! \left[ p''(t) + r''(t) \right]^{\frac{1}{2}}\]
Proof:
Without loss of generality we use the cartesian coordinate form \((2.2.2)\) of Theorem 2.2 to obtain

\[
K_G(t) = \frac{(n!)^m}{m!} \left[ \left( \frac{p(t)}{q(t)} \right)^{(n)} - \left( \frac{p(t)}{q(t)} \right)^{(n'-m)} \left( \frac{r(t)}{q(t)} \right)^{(m)} \right] \quad \ldots (2.4.2)
\]

Since the value of \(q(t)\) is constant, we get

\[
K_G(t) = \frac{(n!)^m}{m!} \left[ \frac{1}{q} \left\{ \left( \frac{p(t)}{q(t)} \right)^{(n)} - \left( \frac{p(t)}{q(t)} \right)^{(n'-m)} \left( \frac{r(t)}{q(t)} \right)^{(m)} \right\} \right] \frac{m-n}{2n}.
\]

Remark 2.5
If \(q(t)=0\) then, by using either equation \((2.2.1)\) or the equation \((2.3.1)\), we can find a spherical generalized curvature \(K_G\), but it does not represent a real curvature of the envelope curve. We shall call such value of curvature Ideal Curvature of a curve \(\gamma\) at points of the monad of \(A=\gamma(t_0)\).

Example 2.6
Consider the family of lines \(2x - 3ty + t^3 = 0\)

By applying the algorithm given at the beginning of this section, we get

\[
\begin{align*}
u &= 2 \\ v &= 3t \\ w &= 2t^3
\end{align*}
\]

Now we determine the singularity and collinearity

\[
\begin{align*}
\gamma(0) &= (0,0) \\ \gamma'(0) &= (0,0) \\ \gamma''(0) &= (0,2) \\ \gamma'''(0) &= (12,0)
\end{align*}
\]

Thus \(\gamma\) has a first singularity order (that is \(n=2\)) and the order of collinearity is equal to 3. The envelope curve \(\gamma(t)\) is given by

\[
(X \in (t), Y \in (t), Z \in (t)) = (v'(t)w(t)-w'(t)v(t), w'(t)u(t)-u'(t)w(t), u'(t)v(t)-v'(t)u(t)) = (6t^3, 6t^2, 12)
\]
Since the value of $q(t)$ is constant, so using Corollary 2.4, we get,
\[
K_6 = \frac{\left(2!\right)^3}{3!} \left| \frac{p^{(3)}(t) - p^{(3)}(t)}{q^{(2)}(t)} \right|^2 = \frac{1}{\sqrt{6}} \cdot \sqrt{12} = \sqrt{2}
\]
\[
= \frac{1}{6} \cdot 2 = \frac{1}{3}
\]
Note that if we use the cartesian coordinate, we find that $\gamma(t)$ is equal to
\[
(x(t), y(t)) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} v(t)w(t) - w'(t)v(t) \\ u'(t)v(t) - v'(t)u(t) \end{pmatrix}
\]
Here $\gamma$ also has a first singularity order (that is $n=2$) and the order of collinearity is equal to 3. Thus by using the usual two dimensional forms of the generalized curvature, we get, (see Figure 2.3)
\[
K_6 = \frac{\left(2!\right)^3}{3!} \left| \frac{x^{(3)}(t) - y^{(3)}(t)}{\left(x^{(3)}(t)^2 + y^{(3)}(t)^2\right)^{3/2}} \right|^2 = \frac{1}{3} \cdot \left| 3t - 0 - 1 \right| = \sqrt{2}
\]
\[
\text{Figure 2.3}
\]

**Remark:** The graph of the equation of the above example is plotted with specific software **Omnigraph V3.1b-2005.**
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