A *-autonomous Category of Banach Spaces – Corrected

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Abstract

We describe a $\mathbb{C}$-linear additive *-autonomous category of Banach spaces. Please note that a correction has been appended to the original version 1 which is maintained here for reference. Also, a proposed example of a *-autonomous category of topological $\mathbb{C}$-linear spaces has been added to version 2.

1 Introduction

First, we describe an elementary *-autonomous monoidal category:

$$\text{Ban}(\mathbb{N}) \subset \text{Ban}$$

where $\text{Ban}$ is the usual symmetric monoidal closed category of complex Banach spaces and linear contractions, and $\text{Ban}(\mathbb{N})$ is the replete full subcategory of $\text{Ban}$ determined by the Hilbert spaces $\ell_2(X)$ for $X \in \{0, 1, 2, \ldots, \mathbb{N}\}$. We also consider the additive aspects of the corresponding $\mathbb{C}$-linearisations:

$$\mathbb{C}\text{Ban}(\mathbb{N}) \subset \mathbb{C}\text{Ban}$$

The resulting $\mathbb{C}$-linear and additive *-autonomous category $\mathbb{C}\text{Ban}(\mathbb{N})$ is analogous to the “completion” of any (symmetric) compact closed category to a *-autonomous monoidal category by adding both an object-at-zero and an object-at-infinity. In this case, the object-at-zero is $\ell_2(0)$, while the object-at-infinity is $\ell_2(\mathbb{N})$. Another example is the multiplicative group of positive real numbers with both 0 and $\infty$ adjoined, which similarly forms a *-autonomous category (with $r \otimes \infty = \infty$ for $r \neq 0$ and $0 \otimes \infty = 0$).

This article probably contains nothing that is essentially new.

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2 The \(*\)-autonomous Category \(\text{CBan}(\mathbb{N})\)

First we note that a norm-decreasing linear map:

\[
\bigoplus^n C \to B \quad \text{(n finite)}
\]

corresponds to a set \(\{b_1, b_2, \ldots, b_n\}\) of \(n\) points in \(B\) such that:

\[
||b_1 + b_2 + \cdots + b_n||^2 \leq n
\]

Then we deduce that the canonical map:

\[
\bigoplus^n C \to [B, \bigoplus^n B]
\]

is norm-decreasing and so gives a map:

\[
\left(\bigoplus^n C\right) \otimes B \to \bigoplus^n B
\]

for all Banach spaces \(B\).

In particular, taking \(B = \bigoplus^m C\), we obtain:

\[
\mathbb{C}^n \otimes \mathbb{C}^m \to \mathbb{C}^{n \times m}
\]

if we write \(\mathbb{C}^n = \bigoplus^n \mathbb{C}\), etc.

We also obtain a natural transformation \(\alpha_B:\)

\[
\text{Ban}(\mathbb{C}^n, [\mathbb{C}^m, B]) \xrightarrow{\alpha_B} \text{Ban}(\mathbb{C}^{n \times m}, B)
\]

\[
\text{Bil}(\mathbb{C}^n \times \mathbb{C}^m, B)
\]

which is thus a monomorphism. So, by the Yoneda lemma, and the natural isomorphism:

\[
\text{Ban}(\mathbb{C}^n \otimes \mathbb{C}^m, B) \cong \text{Ban}(\mathbb{C}^n, [\mathbb{C}^m, B])
\]

we have an epimorphism:

\[
\mathbb{C}^{n \times m} \to \mathbb{C}^n \otimes \mathbb{C}^m
\]

in \(\text{Ban}\). But \(\mathbb{C}^n \otimes \mathbb{C}^m\) is a topological completion of the algebraic tensor product of \(\mathbb{C}^n\) and \(\mathbb{C}^m\). Hence we have:

**Proposition 1.** \(\mathbb{C}^n \otimes \mathbb{C}^m \to \odot \mathbb{C}^{n \times m}\) in \(\text{Ban}\)

**Proposition 2.** We also have \(\text{colim}_{n \in X} \bigoplus^n \mathbb{C} \cong \ell_2(X) = \bigoplus X \mathbb{C}\) in \(\text{Ban}\) for all sets \(X\).
The proof is straightforward.

**Corollary 1.** We have $\ell_2(X) \otimes \ell_2(Y) \cong \ell_2(X \times Y)$ in $\text{Ban}$ for all $X, Y \in \{0, 1, 2, \ldots, N\}$

The proof follows from Propositions 1 and 2, and the preservation of colimits by $\otimes$ in $\text{Ban}$.

Each object in $\text{Ban}(\mathbb{N})$ is reflexive as a Banach space, so we get:

**Proposition 3.** The monoidal replete full subcategory

$$\text{Ban}(\mathbb{N}) \subset \text{Ban}$$

is a $*$-autonomous monoidal category.

Proof:

$$\text{Ban}(\mathbb{N})(A \otimes B, C) = \text{Ban}(A \otimes B, C)$$

$$\cong \text{Ban}(B \otimes A, [[C, C], C])$$

$$\cong \text{Ban}(B, [A, [[C, C], C]])$$

$$\cong \text{Ban}(B, [[C, C], [A, C]])$$

$$\cong \text{Ban}(B \otimes [C, C], [A, C])$$

$$= \text{Ban}(\mathbb{N})(B \otimes C^*, A^*)$$

where $A^*$ denotes $[A, C]$, etc., so there is a natural isomorphism:

$$\text{Ban}(\mathbb{N})(A \otimes B, C) \cong \text{Ban}(\mathbb{N})(A, (B \otimes C^*)^*)$$

Now consider the $\mathbb{C}$-linearisations

$$\mathbb{C}\text{Ban}(\mathbb{N}) \subset \mathbb{C}\text{Ban}$$

where $\mathbb{C}\text{Ban}(\mathbb{N})$ is automatically $*$-autonomous from Proposition 3. First, note that the (finite) direct sum $A \oplus B$ of two Banach spaces $A$ and $B$ (which is the $\mathbb{C}$-vector space product $A \times B$ with the norm $\|a, b\| = \sqrt{||a||^2 + ||b||^2}$ ) becomes a biproduct in $\mathbb{C}\text{Ban}$. This is fairly immediate from the fact that we have bijective contraction maps

$$A + B \rightarrow A \oplus B \quad A \oplus B \rightarrow A \times B$$

in $\text{Ban}$, where the coproduct $A + B$ in $\text{Ban}$ is the $\mathbb{C}$-vector space product $A \times B$ with the norm

$$\|a, b\| = ||a|| + ||b||$$

while the product $A \times B$ in $\text{Ban}$ is the $\mathbb{C}$-vector space product $A \times B$ with the norm

$$\|(a, b)\| = \max (||a||, ||b||)$$
Then the canonical diagram:

\[
\begin{align*}
\text{cBan}(A, B \oplus C) & \xrightarrow{i} \text{cBan}(A, B) \oplus \text{cBan}(A, C) \\
\text{cBan}(A, B \times C) & \cong \text{cBan}(A, B) \otimes \text{cBan}(A, C)
\end{align*}
\]

commutes in \text{Vect}_C, so \(i\) is in injection, and is also a retraction, hence, it is an isomorphism. Similarly, the diagram

\[
\begin{align*}
\text{cBan}(A \oplus B, C) & \xrightarrow{j} \text{cBan}(A, C) \oplus \text{cBan}(B, C) \\
\text{cBan}(A + B, C) & \cong \text{cBan}(A, C) \otimes \text{cBan}(B, C)
\end{align*}
\]

commutes in \text{Vect}_C, so then \(j\) also is an isomorphism. Thus:

**Proposition 4.** The \(\mathbb{C}\)-linear category \(\text{cBan}(\mathbb{N})\) is additive and \(*\)-autonomous.

Finally, we note that the braid groupoid \(\mathbb{B}\) generates a convolution category

\[
[\mathbb{B}, \text{cBan}(\mathbb{N})]_{f.s.}
\]

consisting of the functors

\[
F : \mathbb{B} \longrightarrow \text{cBan}(\mathbb{N})
\]

of finite support, and the natural transformations between them. Here the (lax) tensor product is given by:

\[
F \ast G(l) = \bigoplus_{m,n} \mathbb{B}(m+n, l) \cdot (F(m) \otimes G(n))
\]

where \(\mathbb{B}(m, n) \cdot B\) is defined to be the countable direct sum

\[
\bigoplus_{\mathbb{B}(m, n)} B
\]

in \(\text{Ban}\) for each Banach space \(B\). This yields a lax monoidal functor category (with lax unit \(\mathbb{B}(0, -) \cdot \mathbb{C}\)). The functor category

\[
[\mathbb{B}, \text{cBan}(\mathbb{N})]_{f.s.}
\]

also has on it the pointwise tensor product

\[
F \otimes G(l) = F(l) \otimes G(l)
\]

which is \(\mathbb{C}\)-linear, additive, and \(*\)-autonomous.
Correction

The canonical contraction map

$$l_2(X) \otimes l_2(Y) \rightarrow l_2(X \times Y)$$

in Proposition 1 and Corollary 1 is not in general an isomorphism in $\text{Ban}$ (as pointed out by Dr. Yemon Choi) which negates the overall purpose of the article.

One “alternative”, that seems not too significant or useful, is to consider the compact-closed category $\text{fdhilb}$ of finite-dimensional Hilbert spaces, and adjoin an abstract terminal object called “$l_2(\mathbb{N})$”. One then obtains a symmetric monoidal *-autonomous category extending the structure of $\text{fdhilb}$ on defining also:

$$l_2(X) \otimes l_2(\mathbb{N}) = l_2(\mathbb{N})$$ for $X \neq 0$

$$l_2(0) \otimes l_2(\mathbb{N}) = l_2(0)$$

with $l_2(0)^* = l_2(\mathbb{N})$ and $l_2(\mathbb{N})^* = l_2(0)$ instead of $l_2(0)^* = l_2(0)$.

3 A *-autonomous category of $\mathbb{C}$-linear spaces

We shall somewhat overstate the main result we need:

**Proposition 5.** If $\Pi B_x$ is a topological product of complex Banach spaces and $V$ is a $\mathbb{C}$-subspace (subspace topology) of $\Pi B_x$, then any continuous $\mathbb{C}$-linear map from $V$ to $\mathbb{C}$ extends to $\Pi B_x$.

The proof follows from that of Kaplan\[4\] Theorem 1, together with the Hahn-Banach Theorem. We need the special case $B_x = \mathbb{C}$ for all $x \in X$.

Let $\text{Vect}(\mathbb{C})$ denote the monoidal closed category of topological $\mathbb{C}$-linear spaces and continuous $\mathbb{C}$-linear maps, with the pointwise internal-hom $(-, -)$ and tensor product. Let $\mathcal{P}(\mathbb{C})$ denote the full reflective subcategory of $\text{Vect}(\mathbb{C})$ determined by the $\mathbb{C}$-subspaces of powers of $\mathbb{C}$. Then $\mathcal{P}(\mathbb{C})$ is complete and closed under exponentiation in $\text{Vect}(\mathbb{C})$, hence is monoidal closed by \[2\], and in fact has all small colimits since $\text{Vect}(\mathbb{C})$ does.

**Lemma 1.** The canonical map $V \rightarrow (((V, \mathbb{C}), \mathbb{C})$ is a surjection for all $V \in \text{Vect}(\mathbb{C})$.

**Proof.** Consider the following diagram, which is natural in $V$:

$$\begin{array}{cccc}
\text{hom}(\mathbb{C}, V) \cdot (\mathbb{C}, \mathbb{C}) & \stackrel{h_V}{\longrightarrow} & V \\
\downarrow^{f_V} & & \downarrow^{\text{can}} \\
(\mathbb{C}^{\text{hom}(\mathbb{C}, V)}, \mathbb{C}) & \stackrel{g_V}{\longrightarrow} & ((V, \mathbb{C}), \mathbb{C})
\end{array}$$
which is easily seen to commute by applying the Yoneda lemma to \( V \in \text{Vect}(\mathbb{C}) \), where “\(_{\cdot}\)” denotes copower in \( \text{Vect}(\mathbb{C}) \). Then \( g_V \) is a surjection by Proposition 5, and \( f_V \) is a surjection because any continuous \( \mathbb{C} \)-linear map \( \text{C}^{\text{hom}(\mathbb{C}, V)} \rightarrow \mathbb{C} \) factors (uniquely) through projection onto some finite power \( \mathbb{C}^n \) where \( n \subset \text{hom}(\mathbb{C}, V) \). So \( V \rightarrow ((V, \mathbb{C}), \mathbb{C}) \) is a surjection. 

Now the objects of \( \mathcal{P}(\mathbb{C}) \) are precisely the subspaces \( V \leq (W, \mathbb{C}) \) for some \( W \in \text{Vect}(\mathbb{C}) \), so, since

\[
\begin{array}{ccc}
V & \xrightarrow{\text{can}} & ((V, \mathbb{C}), \mathbb{C}) \\
\downarrow & & \downarrow \\
(W, \mathbb{C})
\end{array}
\]

commutes for such a \( V \), we get \( V \cong ((V, \mathbb{C}), \mathbb{C}) \) by lemma 1. Hence, from the proof of Proposition 3, we get:

**Proposition 6.** \( \mathcal{P}(\mathbb{C}) \) is a \( \mathbb{C} \)-linear \(^*\)-autonomous category.

**References**

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