Localization for constrained martingale problems and optimal conditions for uniqueness of reflecting diffusions in 2-dimensional piecewise smooth domains

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We prove existence and uniqueness for semimartingale reflecting diffusions in 2-dimensional piecewise smooth domains with varying, oblique directions of reflection on each “side”, under geometric, easily verifiable conditions. Our conditions are optimal in the sense that, in the case of a polygonal domain, they reduce to the conditions of Dai and Williams (1996), which are necessary for existence of Reflecting Brownian Motion.

Our argument exploits the equivalence between stochastic differential equations with reflection and constrained martingale problems. We prove that uniqueness holds for the constrained martingale problem if and only if it holds locally. This result is valid more generally for all constrained martingale problems that satisfy a certain condition, including some where the boundary operator is non-local.

**Key words:** reflecting diffusion; oblique reflection; nonsmooth domain; cusp; constrained martingale problem

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1 Introduction

Reflecting diffusions in nonsmooth domains have been studied since the early 1980s. Despite this long history, there is no general existence and uniqueness result in the literature for curved, piecewise smooth domains or cones, not even for the semimartingale case and not even in di-
mension 2. This notwithstanding the fact that there are significant applications, for instance in stochastic networks (see e.g. Kang et al. (2009) or Kang and Williams (2012)).

Exhaustive results exist only for normal reflection (Tanaka (1979), Saisho (1987), Bass and Hsu (1991), Bass (1996), DeBlassie and Toby (1993), etc.), for Brownian motion in a wedge with constant direction of reflection on each side (Varadhan and Williams (1985), Williams (1985)), for Brownian motion in a smooth cone with radially constant direction of reflection (Kwon and Williams (1991)) and for semimartingale reflecting Brownian motion in a convex polyhedral domain with constant direction of reflection on each face (Taylor and Williams (1993) and Dai and Williams (1996)). In the case of a simple polyhedral domain, the assumptions of Dai and Williams (1996) are necessary for existence of a semimartingale Brownian motion.

For a piecewise smooth domain with varying oblique direction of reflection on each “face”, the best available result is Dupuis and Ishii (1993). Unfortunately, the Dupuis and Ishii (1993) result is proved under a condition that is not easy to verify and leaves out many very natural examples. (See e.g. Remark 3.6.) In fact, the Dupuis and Ishii (1993) condition does not reduce to the assumptions of Dai and Williams (1996) in the case of a polyhedral domain.

More recently, existence and uniqueness of a semimartingale reflecting diffusion has been proved by Costantini and Kurtz (2018) in a 2-dimensional cusp with varying, oblique directions of reflection on each “side” and by Costantini and Kurtz (2021) in a $d$-dimensional domain with one singular point that near the singular point can be approximated by a smooth cone, with varying, oblique direction of reflection on the smooth part of the boundary. In the cusp case, even starting at the cusp, with probability one, the process never hits it again. In contrast, in the case when the domain can be approximated by a cone, the process can hit the singular point infinitely many times. Therefore the study of this case requires a new ergodic theorem for inhomogeneous subprobability transition kernels. The conditions under which the above results are proved are geometric in nature and easily verifiable. A quite general existence result for piecewise smooth domains in $\mathbb{R}^d$, even with cusp like points, has been obtained in Costantini and Kurtz (2019), leaving the question of uniqueness.

In dimension two, piecewise smooth domains look locally like smooth domains or like domains with one singular point. Consequently, by a localization argument, one should be able to exploit the results of Costantini and Kurtz (2018) and Costantini and Kurtz (2021) to give conditions for uniqueness of semimartingale reflecting diffusions. In this paper we carry out this program. The conditions we find are geometric and easy to verify and of course allow for cusps and for points where the boundary is smooth but the direction of reflection has a discontinuity. The same conditions allow to apply the results of Costantini and Kurtz (2019) to obtain existence as well. They are optimal in the sense that for a polygonal domain they reduce to the conditions of Dai and Williams (1996).

The existence proof in Costantini and Kurtz (2019) makes use of the equivalence between solutions of a stochastic differential equation with reflection (SDER) and natural solutions of the corresponding constrained martingale problem (CMP), introduced in Kurtz (1990) and Kurtz (1991) and further studied in Costantini and Kurtz (2015). Here we exploit this equivalence to localize the uniqueness problem for the SDER.

In Section 2, we introduce CMPs stopped at the exit from an open set and show that uniqueness holds for the natural solution of a CMP in a given domain if and only if it holds for the natural solution of the CMP stopped at the exit from each open set belonging to an open covering of the domain. This result holds for general CMPs and is of independent interest.
Then, in Section 3, we combine the above localization result with the uniqueness results in Costantini and Kurtz (2018) and Costantini and Kurtz (2021) to obtain global uniqueness for the natural solution of the CMP corresponding to an SDER in a piecewise smooth domain with varying, oblique direction of reflection on each “side”. As mentioned above, existence follows by Costantini and Kurtz (2019). By the equivalence between natural solutions of the CMP and solutions of the SDER, existence and uniqueness transfer to the SDER.

We will use the following notation: For a metric space $E$, $B(E)$ will denote the $\sigma$-algebra of Borel sets and $\mathcal{P}(E)$ will denote the set of probability measures on $(E, B(E))$; For a stochastic process $Z$, $\mathcal{F}_t^Z := \sigma(Z(s), s \leq t)$ and $\mathcal{F}_t^{Z_u} := \cap_{s \leq t} \mathcal{F}_s^Z$; Finally the superscript $T$ denotes the transpose of a matrix and $B_r(0)$ denotes a ball in $\mathbb{R}^d$ of radius $r$ and center the origin.

## 2 Localization for constrained martingale problems

Let $E$ be a compact metric space, $E_0$ an open subset of $E$, and let $A \subset C(E) \times C(E)$ with $(1, 0) \in A$. Let $U$ also be a compact metric space, let $\Xi$ be a closed subset of $(E - E_0) \times U$ and assume that, for every $x \in E - E_0$, there is some $u \in U$ such that $(x, u) \in \Xi$. Let $B \subset C(E) \times C(\Xi)$ with $(1, 0) \in B$, $\mathcal{D} := \mathcal{D}(A) \cap \mathcal{D}(B)$ and assume $\mathcal{D}$ is dense in $C(E)$. The intuition is that $A$ is the generator for a process in $\Xi$ and that $B$ determines controls that constrain the process to remain in $E_0$ or, more precisely, in $\overline{E_0}$.

Let $\mathcal{L}_U$ be the space of Borel measures $\mu$ on $[0, \infty) \times U$ such that $\mu([0, t] \times U) < \infty$ for all $t > 0$. $\mathcal{L}_U$ is topologized so that $\mu_n \in \mathcal{L}_U \to \mu \in \mathcal{L}_U$ if and only if

$$\int_{[0, \infty) \times U} f(s, u) \mu_n(ds \times du) \to \int_{[0, \infty) \times U} f(s, u) \mu(ds \times du)$$

for all continuous $f$ with compact support in $[0, \infty) \times U$. It is possible to define a metric on $\mathcal{L}_U$ that induces the above topology and makes $\mathcal{L}_U$ into a complete, separable metric space. Also let $\mathcal{L}_\Xi$ be defined analogously. For any $\mathcal{L}_U$-valued ($\mathcal{L}_\Xi$-valued) random variable $L$, for each $t \geq 0$, $L([0, t] \times \cdot)$ is a random measure on $U (\Xi)$. We will occasionally use the notation $L(t) := L([0, t] \times \cdot)$.

For a nondecreasing path $l_0 \in D_{[0, \infty]}[0, \infty)$ with $l_0(0) = 0$, we define

$$(l_0)^{-1}(t) := \inf\{s \geq 0 : l_0(s) > t\},$$

(2.1)

where we adopt the usual convention that the infimum of the empty set is $\infty$. Of course, if $l_0$ is strictly increasing $(l_0)^{-1}$ is just the inverse of $l_0$. In addition, for every path $y \in D_{[0, \infty]}[0, \infty)$ such that $\lim_{t \to \infty} y(t)$ exists, we will use the notation $y(\infty) := \lim_{t \to \infty} y(t)$.

The controlled martingale problem for $(A, E_0, B, \Xi)$, the constrained martingale problem for $(A, E_0, B, \Xi)$ and natural solutions of the constrained martingale problem for $(A, E_0, B, \Xi)$ have been introduced and studied in Kurtz (1990), Kurtz (1991), Costantini and Kurtz (2015) and Costantini and Kurtz (2019): we refer to these papers for definitions and properties, although the definitions for constrained and controlled martingale problems can be recovered from the definitions below by taking $U = E$ or $U = E_0$.

In this section, given an open subset $U$ of $E$, we introduce the notions of stopped controlled martingale problem for $(A, E_0, B, \Xi; U)$, of stopped constrained martingale problem for
Remark 2.2

Let \( (A, E_0, B, \Xi; U) \) and of natural solution of the stopped constrained martingale problem for \( (A, E_0, B, \Xi; U) \) and study their relations with the corresponding unstopped objects. Our main goals are Theorems 2.12 and 2.14 which correspond to Theorems 4.6.1 and 4.6.3 of Ethier and Kurtz (1986) for martingale problems. A natural solution of the stopped constrained martingale problem for \( (A, E_0, B, \Xi; U) \) is obtained by time-changing a solution of the stopped controlled martingale problem for \( (A, E_0, B, \Xi; U) \) (see below for precise definitions): Roughly speaking, in order to transfer the results of Section 4.6 of Ethier and Kurtz (1986) to constrained martingale problems, what we need to show is that, under suitable conditions, the ”stopping” and the ”time-changing” can be exchanged.

Note that the set \( E \) here corresponds to \( E \cup F_1 \) in Costantini and Kurtz (2019) and that for Lemma 2.3 below we do not need Condition 3.5 c) of Costantini and Kurtz (2019).

**Definition 2.1**

Let \( Y^U \) be a process in \( D_E[0, \infty) \), \( \lambda^U_0 \) be a nonnegative, nondecreasing process such that

\[
\lambda^U_0(t) = \int_{[0,t]} 1_{E_0}(Y^U(s))d\lambda^U_0(s) \quad \text{a.s.,} 
\]

and \( \Lambda^U \) be a \( \mathcal{L}_U \)-valued random variable such that

\[
\lambda^U_1(t) := \Lambda^U([0,t] \times \mathcal{U}) = \int_{[0,t] \times \mathcal{U}} 1_\Xi(Y^U(s), u)\Lambda^U(ds \times du). 
\]

Define

\[
\theta^U := \inf\{t \geq 0 : Y^U(t) \notin U \text{ or } Y^U(t-) \notin U\}. 
\]

\((Y^U, \lambda^U_0, \Lambda^U)\) is a solution of the stopped, controlled martingale problem for \((A, E_0, B, \Xi; U)\) if

\[
(Y^U, \lambda^U_0, \Lambda^U)(t) = (Y^U, \lambda^U_0, \Lambda^U)(t \wedge \theta^U), \quad \forall t \geq 0 \quad \text{a.s.,} 
\]

\[
\lambda^U_0(t) + \lambda^U_1(t) = t \wedge \theta^U, \quad \forall t \geq 0 \quad \text{a.s.,} 
\]

and

\[
f(Y^U(t)) - f(Y^U(0)) = \int^t_0 Af(Y^U(s))d\lambda^U_0(s) - \int^t_0 Bf(Y^U(s), u)\Lambda^U(ds \times du) 
\]

is a \( \{\mathcal{F}_s^Y, \lambda^U_0, \Lambda^U\} \)-martingale for all \( f \in \mathcal{D} \). Since \((2.5)\) is right continuous, it is also a \( \{\mathcal{F}_s^Y, \lambda^U_0, \Lambda^U\} \)-martingale.

**Remark 2.2**

Let \((Y, \lambda_0, \Lambda_1)\) be a solution of the controlled martingale problem for \((A, E_0, B, \Xi)\). Then, setting

\[
\theta := \inf\{t \geq 0 : Y(t) \notin U \text{ or } Y(t-) \notin U\}, 
\]

\((Y, \lambda_0, \Lambda_1)(\cdot \wedge \theta)\) is a solution of the stopped controlled martingale problem for \((A, E_0, B, \Xi; U)\).

**Lemma 2.3**

Suppose that for every \( \nu \in \mathcal{P}(E) \) there exists a solution of the controlled martingale problem for \((A, E_0, B, \Xi)\) with initial distribution \( \nu \).

Then, for every solution of the stopped controlled martingale problem for \((A, E_0, B, \Xi; U)\), \((Y^U, \lambda^U_0, \Lambda^U_1)\), there exists a solution \((Y, \lambda_0, \Lambda_1)\) of the controlled martingale problem for \((A, E_0, B, \Xi)\) such that, with \( \theta \) defined by \((2.6)\), \((Y^U, \lambda^U_0, \Lambda^U_1, \theta^U)\) has the same distribution as \((Y(\cdot \wedge \theta), \lambda_0(\cdot \wedge \theta), \Lambda_1(\cdot \wedge \theta), \theta)\).
Proof. The proof is a suitable modification of the proof of Lemma 4.5.16 of Ethier and Kurtz (1986): see Appendix A. \[\square\]

Definition 2.4 A process $X^U$ in $D_{E_0}(0, \infty)$ is a solution of the stopped constrained martingale problem for $(A, E_0, B, \Xi; U)$ if there exists a $\mathcal{L}_\Xi$-valued random variable $\Lambda^U$ such that, setting

$$\tau^U := \inf\{t \geq 0 : X^U(t) \notin U \text{ or } X^U(t-) \notin U\},$$

$(X^U, \Lambda^U)$ satisfies

$$(X^U, \Lambda^U)(t) = (X^U, \Lambda^U)(t \wedge \tau^U) \text{ a.s.}$$

and

$$f(X^U(t)) - f(X^U(0)) - \int_0^{t \wedge \tau^U} Af(X^U(s))ds - \int_{[0,t] \times \Xi} Bf(x,u)\Lambda^U(ds \times dx \times du) \quad (2.8)$$

is a $\{\mathcal{F}^X_{t \wedge \tau^U}, \Lambda^U\}$-local martingale for all $f \in D$. Since $(2.8)$ is right continuous, it is also a $\{\mathcal{F}^X_t, \Lambda^U\}$-local martingale.

Definition 2.5 A solution $X^U$ of the stopped constrained martingale problem for $(A, E_0, B, \Xi; U)$ is natural, if there exists a solution $(Y^U, \lambda^U_0, \Lambda^U_1)$ of the stopped controlled martingale, with the property that the event $\{\theta^U = \infty, \lim_{s \to \infty} \lambda^U_0(s) < \infty\}$ has zero probability, such that

$$X^U(t) = Y^U((\lambda^U_0)^{-1}(t))$$

and

$$\Lambda^U([0,t] \times C) := \int_{[0,(\lambda^U_0)^{-1}(t)] \times C} 1_C(Y^U(s), u)\Lambda^U_1(ds \times du), \quad C \in \mathcal{B}(\Xi), \text{ a.s..} \quad (2.9)$$

(Note that, a.s., if $\lim_{s \to \infty} \lambda^U_0(s) = t_0 < \infty$, then $\theta^U < \infty$ and $(\lambda^U_0)^{-1}(t) = \infty$ for all $t \geq t_0$, so that, for $t \geq t_0$, $Y^U((\lambda^U_0)^{-1}(t)) = Y^U(\infty) = Y^U(\theta^U)$).

Definition 2.6 Uniqueness holds for natural solutions of the stopped constrained martingale problem for $(A, E_0, B, \Xi; U)$ if any two solutions with the same initial distributions have the same distribution on $D_{E_0}(0, \infty)$.

In the sequel we assume the following condition on the controlled martingale problem for $(A, E_0, B, \Xi)$ and the open set $U$.

Condition 2.7

(i) For each $\nu \in \mathcal{P}(E)$ there exists a solution $(Y, \lambda_0, \Lambda_1)$ of the controlled martingale problem for $(A, E_0, B, \Xi)$ with initial distribution $\nu$.

(ii) For each solution $(Y, \lambda_0, \Lambda_1)$ of the controlled martingale problem for $(A, E_0, B, \Xi)$,

$$\lim_{t \to \infty} \lambda_0(t) = \infty \quad \text{a.s..}$$

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(iii) For each solution \((Y, \lambda_0, \Lambda_1)\) of the controlled martingale problem for \((A, E_0, B, \Xi)\), there exists a sequence of \(\\{Y^\gamma_{\lambda_0, \Lambda_1}\}\) - stopping times \(\{\gamma_n\}\) such that \(\gamma_n \to \infty\) a.s. and \(\mathbb{E}[\lambda_0^{-1}(\gamma_n)] < \infty\) for each \(n\).

(iv) For each solution \((Y, \lambda_0, \Lambda_1)\) of the controlled martingale problem for \((A, E_0, B, \Xi)\), \(X(t) := Y(\lambda_0^{-1}(t))\), \(\tau\) defined as

\[
\tau := \inf\{t \geq 0 : X(t) \notin U \text{ or } X(t^-) \notin U\} \quad \text{a.s.} \tag{2.10}
\]

and \(\theta\) defined by (2.6),

\[
\lambda_0^{-1}(\tau) = \theta \quad \text{a.s.}
\]

Remark 2.8 For each solution \((Y^U, \lambda_0^U, \Lambda_1^U)\) of the stopped controlled martingale problem for \((A, E_0, B, \Xi; U)\), \(X^U(t) := Y^U(\lambda_0^U(t))\), \(\tau^U\) defined by (2.7), and \(\theta^U\) defined by (2.4), Condition 2.7 (iv) and Lemma 2.3 imply that,

\[
\tau^U = \lambda_0^U(\theta^U) \quad \text{a.s.}
\]

In fact, it always holds

\[
\tau^U \geq \lambda_0^U(\theta^U) \quad \text{a.s.}
\]

On the other hand, by Lemma 2.3, we can suppose, without loss of generality, that

\[
(Y^U, \lambda_0^U, \Lambda_1^U, \theta^U, X^U) = (Y(\cdot \wedge \theta), \lambda_0(\cdot \wedge \theta), \Lambda_1(\cdot \wedge \theta), \theta, Y((\lambda_0(\cdot \wedge \theta))^{-1}(\cdot \wedge \theta)).
\]

Then, by Condition 2.7 (iv),

\[
X^U(\tau) = X^U(\lambda_0(\theta)) = Y(\theta) = Y(\lambda_0^{-1}(\tau)) = X(\tau), \quad \text{a.s.}
\]

and

\[
X^U(\tau^-) = \lim_{s \to \tau^-} Y((\lambda_0(\cdot \wedge \theta))^{-1}(s \wedge \theta) = \lim_{s \to \tau^-} Y((\lambda_0)^{-1}(s \wedge \lambda_0^{-1}(\tau)))
\]

\[
= \lim_{s \to \tau^-} Y(\lambda_0^{-1}(s \wedge \tau)) = X(\tau^-) \quad \text{a.s.}
\]

Therefore

\[
\tau^U \leq \tau = \lambda_0(\theta) = \lambda_0^U(\theta^U) \quad \text{a.s.}
\]

Proposition 2.9 Suppose Condition 2.7 (i) is verified. If each solution of the controlled martingale problem for \((A, E_0, B, \Xi)\) satisfies \(\lambda_0(t) > 0\) for all \(t > 0\) a.s., then \(\lambda_0\) is strictly increasing a.s. for each solution, and Condition 2.7 is verified for every open set \(U\).

Proof. (ii) and the fact that \(\lambda_0\) is strictly increasing follow from Lemmas 3.3 and 3.4 of Costantini and Kurtz (2019). In turn, the fact that \(\lambda_0\) is strictly increasing immediately implies (iv). As in the proof of Corollary 3.9 of Costantini and Kurtz (2019), (iii) is verified by

\[
\gamma_n := \lambda_0(n).
\]
Remark 2.10  The controlled martingale problems corresponding to reflecting diffusions will usually satisfy the assumptions of Proposition 2.9 (e.g. see Lemma 6.8 of Costantini and Kurtz (2014)). However there are significant examples of controlled martingale problems for which Condition 2.7 is verified although the assumptions of Proposition 2.9 are not. For instance, this is typically the case for the non-local boundary conditions of Section 7.1 of Costantini and Kurtz (2019), if the interior operator \( A \) is a diffusion operator.

Lemma 2.11  Under Condition 2.7, for every natural solution \( X^U \) of the stopped constrained martingale problem for \((A, E_0, B, \Xi; U)\), there exists a natural solution \( X \) of the constrained martingale problem for \((A, E_0, B, \Xi)\) such that, with \( \tau \) defined by (2.10), \( X(\cdot \land \tau) \) has the same distribution as \( X^U(\cdot) \).

Proof. Let \( X^U(\cdot) = Y^U((\lambda_0^U)^{-1} (\cdot)) \) for some solution \((Y^U, \lambda_0^U, \Lambda_1^U)\) of the stopped controlled martingale problem for \((A, E_0, B, \Xi; U)\), and let \((Y, \lambda_0, \Lambda_1)\) be the solution of the controlled martingale problem for \((A, E_0, B, \Xi)\) constructed in Lemma 2.3. Let \( \theta \) be defined by (2.6). By Condition 2.7 (i), (ii) and (iii) and Theorem 3.6 of Costantini and Kurtz (2019),

\[
X(\cdot) := Y(\lambda_0^{-1}(\cdot))
\]

is a natural solution of the constrained martingale problem for \((A, E_0, B, \Xi)\). Then, by Condition 2.7 (iv),

\[
X(t \land \tau) = Y(\lambda_0^{-1}(t) \land \theta) = Y(\lambda_0(\cdot \land \theta)^{-1}(t) \land \theta), \quad t \geq 0,
\]

and the assertion follows from the fact that the distribution of \( Y(\lambda_0(\cdot \land \theta)^{-1}(\cdot) \land \theta) \) is the distribution of \( Y^U((\lambda_0^U)^{-1}(\cdot)) \), i.e. of \( X^U(\cdot) \).

\[\square\]

Theorem 2.12  Under Condition 2.7, if uniqueness holds for natural solutions of the constrained martingale problem for \((A, E_0, B, \Xi)\), then it holds for natural solutions of the stopped constrained martingale problem for \((A, E_0, B, \Xi; U)\).

Proof. The assertion follows immediately from Lemma 2.11

\[\square\]

Lemma 2.13  Under Condition 2.7, for every solution \((Y^U, \lambda_0^U, \Lambda_1^U)\) of the stopped controlled martingale problem for \((A, E_0, B, \Xi; U)\), \( X^U(\cdot) := Y^U((\lambda_0^U)^{-1}(\cdot)) \) is a natural solution of the stopped constrained martingale problem for \((A, E_0, B, \Xi; U)\) with \( \Lambda^U \) defined by (2.3).

Proof. By Remark 2.8 \((X^U, \Lambda U)(\cdot) = (X^U, \Lambda U)(\cdot \land \tau^U)\) a.s.. By Lemma 2.3 we can suppose, without loss of generality, that

\[
(Y^U, \lambda_0^U, \Lambda_1^U, \theta^U, X^U) = (Y(\cdot \land \theta), \lambda_0(\cdot \land \theta), \Lambda_1(\cdot \land \theta), \theta, Y((\lambda_0(\cdot \land \theta))^{-1}(\cdot) \land \theta)).
\]

Then, by Condition 2.7 (ii), the event

\[
\{\theta^U = \infty, \lim_{s \to \infty} \lambda_0^U(s) < \infty\} = \{\theta = \infty, \lim_{s \to \infty} \lambda_0(s) < \infty\}
\]
has zero probability. Finally, let \( \{\gamma_n\} \) be the sequence of random variables of Condition 2.7 (iii) and define

\[
\gamma_n^U := \begin{cases} n & \text{if } \theta \leq n, \\ \gamma_n & \text{if } \theta > n. \end{cases}
\]

Then \( \gamma_n^U \to \infty \) a.s., for each \( n \gamma_n^U \) is a \( \{\mathbb{X}^U, \lambda_0^U, \Lambda_1^U\} \)-stopping time for each \( n \) and

\[
(\lambda_0^U)^{-1}(\gamma_n^U) \wedge \theta^U = \lambda_0^{-1}(\gamma_n^U) \wedge \theta \leq n + \lambda_0^{-1}(\gamma_n) \quad \text{a.s.}
\]

Therefore

\[
\mathbb{E} \left[ \int_{[0,t \wedge \gamma_n^U] \times \mathbb{X}} \mathbb{X} f(x,u) \Lambda^U(ds \times dx \times du) \right] \\
\leq \| \mathbb{X} f \| \mathbb{E} \left[ (\lambda_0^U)^{-1}(t \wedge \gamma_n^U) \wedge \theta^U) \right] \\
\leq \| \mathbb{X} f \| \mathbb{E} \left[ (\lambda_0^U)^{-1}(t \wedge \gamma_n^U) \wedge \theta^U) \right] < \infty,
\]

so that (2.8) is a local martingale. \( \square \)

**Theorem 2.14** Suppose there exist open subsets \( U_k \subset E, k = 1, 2, \ldots \), with \( E = \bigcup_{k=1}^\infty U_k \), such that for each \( k \) uniqueness holds for natural solutions of the stopped, constrained martingale problem for \( (A, E_0, B, \Xi; U_k) \). Then, under Condition 2.7 uniqueness holds for natural solutions of the constrained martingale problem for \( (A, E_0, B, \Xi) \).

**Proof.** First of all note that the arguments of the proof of point (a) of Theorem 4.2.2 of Ethier and Kurtz (1986) apply to constrained martingale problems as well, so that it is sufficient to prove that any two natural solutions of of the constrained martingale problem for \( (A, E_0, B, \Xi) \) with the same initial distribution have the same one-dimensional distributions. The proof of Theorem 4.6.2 of Ethier and Kurtz (1986) essentially carries over. The only thing we have to check is that, with \( V_i \) and \( P_i \) as in Theorem 4.6.2 of Ethier and Kurtz (1986), (in particular, for each \( i, V_i = U_k \) for some \( k \) \( P_i \) is the distribution of a natural solution of the stopped constrained martingale problem for \( (A, E_0, B, \Xi; V_i) \). To see this, let \( X \) be a natural solution of the constrained martingale problem for \( (A, E_0, B, \Xi) \), and suppose \( X(\cdot) = Y(\lambda_0^{-1}(\cdot)) \) for some solution \( (Y, \lambda_0, \Lambda_1) \) of the controlled martingale problem for \( (A, E_0, B, \Xi) \). (Note that \( Y \) denotes a different object in the proof of Theorem 4.6.2.) Let

\[
\theta_0 := 0, \quad \theta_i := \inf \{ t \geq \theta_{i-1} : Y(t) \notin V_i \text{ or } Y(t-) \notin V_i \}, \quad i \geq 1,
\]

\[
\rho_i := \inf \{ t \geq 0 : Y(\theta_{i-1} + t) \notin V_i \text{ or } Y((\theta_{i-1} + t)--) \notin V_i \}, \quad \text{on } \{ \theta_{i-1} < \infty \},
\]

and, for \( i \) such that \( \mathbb{P}(\lambda_0(\theta_{i-1}) < \infty) = \mathbb{P}(\theta_{i-1} < \infty) > 0 \)

\[
Q_i(D) := \mathbb{E} \left[ e^{-\beta \lambda_0(\theta_{i-1})} 1_{\{\lambda_0(\theta_{i-1}) < \infty\}} 1_D \left( Y(\theta_{i-1} + \cdot \wedge \rho_i), \lambda_0(\theta_{i-1} + \cdot \wedge \rho_i) - \lambda_0(\theta_{i-1}), \Lambda_1^{\theta_{i-1} \wedge \rho_i}(\cdot) \right) \right] \\
\leq \mathbb{E} \left[ e^{-\beta \lambda_0(\theta_{i-1})} 1_{\{\lambda_0(\theta_{i-1}) < \infty\}} \right],
\]

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where \( \beta \) is a positive number, \( D \in \mathcal{B}(D_E[0, \infty) \times C_{[0,\infty)[0, \infty) \times \mathcal{L}_U) \) and \( \Lambda^\theta_i \) is the measure on \([0, \infty) \times U\) defined by \( \Lambda^\theta_i([0, t] \times C) := \Lambda_1([\theta_i, \theta_i + t \wedge \rho_i] \times C) \). Then, by Lemma 2.11 of Costantini and Kurtz (2019) and Remark 2.2, the coordinate process on \( DE[0, \infty) \times C[0, \infty)[0, \infty) \times L_U \), \( \lambda_{\theta_i} \), is the measure on \([0, \infty) \times U\) defined by \( \Lambda_{\theta_i}([\theta_i, t] \times C) := \Lambda_1([\theta_i, \theta_i + t \wedge \rho_i] \times C) \). Then, by Lemma 2.13, under \( Q_i \), \( \eta((l_0)^{-1}(\cdot)) \) is a natural solution of the stopped constrained martingale problem for \((A, E_0, B, \Xi; V_i)\).

On the other hand, with \( \tau_i \) defined by (2.10), and

\[
q_i := \inf\{t \geq 0 : X(\tau_i + t) \notin V_i \text{ or } X((\tau_i + t)^{-}) \notin V_i\},
\]

(note that \( q_i \) is denoted as \( \eta_i \) in Ethier and Kurtz (1986)) we have

\[
\lambda_0^{-1}(\tau_i + t \wedge q_i) = \theta_i + (\lambda_0(\theta_i + \cdot \wedge \rho_i) - \lambda_0(\theta_i))^{-1}(t) \wedge \rho_i,
\]

so that the distribution \( P_i \) of Theorem 4.6.2 of Ethier and Kurtz (1986),

\[
P_i(C) := \frac{\mathbb{E}[e^{-\beta \tau_i} \mathbf{1}_{\{\tau_i < \infty\}} X((\tau_i + \cdot \wedge q_i))]}{\mathbb{E}[e^{-\beta \tau_i} \mathbf{1}_{\{\tau_i < \infty\}}]},
\]

can be written as

\[
P_i(C) = \frac{\mathbb{E}[e^{-\beta \lambda_0(\theta_i)} \mathbf{1}_{\{\lambda_0(\theta_i) < \infty\}} Y(\theta_i + (\lambda_0(\theta_i + \cdot \wedge \rho_i) - \lambda_0(\theta_i))^{-1}(\cdot) \wedge \rho_i))]}{\mathbb{E}[e^{-\beta \lambda_0(\theta_i)} \mathbf{1}_{\{\lambda_0(\theta_i) < \infty\}}]}
\]

\[
= \mathbb{E}^{Q_i}(\mathbf{1}_C(\eta((l_0)^{-1}(\cdot))))
\]

that is \( P_i \) is the distribution of \( \eta((l_0)^{-1}(\cdot)) \) under \( Q_i \).

3 Existence and uniqueness of reflecting diffusions in a 2-dimensional, piecewise smooth domain

We consider a domain \( D \) satisfying the following condition.

**Condition 3.1**

(i) \( D \) is a bounded domain that admits the representation

\[
D = \bigcap_{i=1}^{m} D^i,
\]
where, for \( i = 1, \ldots, m \), \( D^i \) is a bounded domain defined as
\[
D^i := \{ x : \psi^i(x) > 0 \}, \quad \psi^i \in C^1(\mathbb{R}^2), \quad \inf_{x : \psi^i(x) = 0} |\nabla \psi^i(x)| > 0,
\]
and
\[
\overline{D} = \bigcap_{i=1}^m D^i.
\]
The representation is minimal in the sense that, for \( j = 1, \ldots, m \),
\[
D \subset \bigcap_{i \neq j} D^i,
\]
where \( \subset \) denotes strict inclusion.

For \( x \in \partial D^i \), we denote by \( n^i(x) \) the unit, inward normal to \( D^i \) at \( x \), i.e. \( n^i(x) := \frac{\nabla \psi^i(x)}{|\nabla \psi^i(x)|} \).

(ii) For \( x^0 \in \bigcup_{i=1}^m \partial D^i \) and
\[
I(x^0) := \{ i : x^0 \in \partial D^i \},
\]
the set \( \{ x \in \bigcup_{i=1}^m \partial D^i : |I(x^0)| > 1 \} \) is finite. We call a point \( x^0 \in \partial D \) such that \( |I(x^0)| > 1 \) a corner and assume \( |I(x^0)| = 2 \) at every corner.

(iii) Let \( x^0 \) be a corner and \( I(x^0) = \{ i, j \} \). If \( n^i(x^0) \neq -n^j(x^0) \) (then we say that \( x^0 \) is a cone point),
\[
\limsup_{x \in \partial D^i \setminus \{ x^0 \}, x \to x^0} \frac{|n^i(x) - n^i(x^0)|}{|x - x^0|} < \infty, \quad \limsup_{x \in \partial D^j \setminus \{ x^0 \}, x \to x^0} \frac{|n^i(x^0) \cdot (x - x^0)|}{|x - x^0|^2} < \infty,
\]
for \( l = i, j \). If \( n^j(x^0) = -n^i(x^0) \) (then we say that \( x^0 \) is a cusp point), \( D \cap B_r(x^0) \) is connected for all \( r > 0 \) small enough, and
\[
\lim_{x \in \partial D^i \cap \partial D \setminus \{ x^0 \}, z \in \partial D \cap \partial D \setminus \{ x^0 \}, |z-x|, |z-x^0|, |z-x| \to x^0} \frac{(x - x^0) \cdot n^i(x^0)}{(z - x) \cdot n^i(x^0)} = L,
\]
for some finite \( L \).

Remark 3.2 A piecewise \( C^1 \) domain \( D \) admits infinitely many representations as in Condition 3.1(i) and it may be that some representations verify all assumptions in Condition 3.1 and others do not. In all our results we only need that there exists a representation that verifies Condition 3.1. It may be convenient to use different representations with different properties (see the proof of Theorem 3.10).

Define the inward normal cone at \( x^0 \in \partial D \) as
\[
N(x^0) := \left\{ n : \liminf_{x \in \partial D \setminus \{ x^0 \}, x \to x^0} \frac{(x - x^0)}{|x - x^0|} \cdot n \geq 0 \right\}.
\]
For $I(x^0) = \{i, j\}$, if $x^0$ is a cone point, clearly $N(x^0)$ is the closed, convex cone generated by $n^i(x^0)$ and $n^j(x^0)$. If $x^0$ is a cusp point, by the assumption that $D \cap \partial B_r(0)$ is connected for all $r > 0$ small enough, there exists one and only one unit vector $\tau(x^0)$ such that

$$
\tau(x^0) \cdot n^i(x^0) = 0 \quad \text{and} \quad \lim_{x \to x^0, x \notin D} \frac{\tau(x^0) \cdot (x - x^0)}{|x - x^0|} = 1. \quad (3.3)
$$

Then

$$
N(x^0) = \{u \in \mathbb{R}^2 : u \cdot \tau(x^0) \geq 0\}. \quad (3.4)
$$

**Remark 3.3** Let $x^0$ be a corner, $I(x^0) = \{i, j\}$, and suppose $\psi^i, \psi^j \in C^2(\mathbb{R}^2)$. Then, if $x^0$ is a cone point, Condition 3.1 (iii) is always verified; if $x^0$ is a cusp point Condition 3.1 (iii) is verified if

$$
\tau(x^0) \cdot \left( \frac{D^2\psi^j(x^0)}{\left| \nabla \psi^j(x^0) \right|} + \frac{D^2\psi^i(x^0)}{\left| \nabla \psi^i(x^0) \right|} \right) \tau(x^0) \neq 0.
$$

The set of possible directions of reflection on the boundary of $D$ is defined by vector fields $g^i$, $i = 1, \ldots, m$ satisfying the following condition.

**Definition 3.4** For $i = 1, \ldots, m$, let $g^i : \mathbb{R}^2 \to \mathbb{R}^2$ be a vector field of unit length on $\partial D^i$. For $x^0 \in \partial D$, define

$$
G(x^0) := \left\{ \sum_{i \in I(x^0)} \eta_i g^i(x^0), \eta_i \geq 0 \right\}. \quad (3.5)
$$

**Condition 3.5**

(i) For $i = 1, \ldots, m$, $g^i$ is a Lipschitz continuous vector field such that

$$
\inf_{x \in \partial D^i} g^i(x) \cdot n^i(x) > 0.
$$

(ii) For every $x^0 \in \partial D$, there exists a unit vector $e(x^0) \in N(x^0)$ such that

$$
e(x^0) \cdot g > 0, \quad \forall g \in G(x^0) - \{0\}.
$$

**Remark 3.6** As mentioned in the Introduction, the best result available in the literature for a piecewise smooth domain with varying directions of reflection on each “face” is Dupuis and Ishii (1993). A very simple example that shows how the Dupuis and Ishii (1993) assumptions may not be satisfied is the following. Let $D^1$ be the unit ball centered at $(1, 0)$, and let $D$ be its intersection with the upper half plane. Of course $D$ can be represented as $D := D^1 \cap D^2$, where $D^2$ is a bounded $C^1$ domain. Let $n^i, i = 1, 2$, denote the unit, inward normal to $D^i$, and

$$
g^i(x) \equiv \begin{bmatrix} \cos(\vartheta) & \sin(\vartheta) \\ -\sin(\vartheta) & \cos(\vartheta) \end{bmatrix} n^i(x), \quad \vartheta \text{ a constant angle}, \ \frac{\pi}{4} \leq \vartheta < \frac{\pi}{2}.
$$
Then, at $x^0 = (0, 0)$ and at $x^0 = (2, 0)$, it can be proved by contradiction that there is no convex compact set that satisfies (3.7) of Dupuis and Ishii (1993). Conditions 3.1 and 3.5 are instead satisfied.

In fact, in the case of a convex polygon, Conditions 3.1 and 3.5 are instead satisfied. In this sense our conditions are optimal.

**Remark 3.7** Conditions 3.1 and 3.5 allow for boundary points $x^0$ at which the boundary is actually smooth, but the direction of reflection has a discontinuity, i.e.

$$n^i(x^0) = n^j(x^0), \quad g^i(x^0) \neq g^j(x^0), \quad i, j \in I(x^0).$$

Finally, the drift $b$ and the dispersion coefficient $\sigma$ satisfy the following condition.

**Condition 3.8**

1. $b : \mathbb{R}^2 \to \mathbb{R}^2$ and $\sigma : \mathbb{R}^2 \to \mathbb{R}^{2 \times 2}$ are Lipschitz continuous.

2. For every corner $x^0$, $\sigma(x^0)$ is non-singular.

In much of the literature, a semimartingale reflecting diffusion is defined as a solution of a stochastic differential equation with reflection. We recall the definition below, for the convenience of the reader.

**Definition 3.9** Let $D$ be a bounded domain and, for $x \in \partial D$, let $G(x)$ be a closed, convex cone such that $\{ (x, u) \in \partial D \times \partial B_1(0) : u \in G(x) \}$ is closed. Let $b : \mathbb{R}^2 \to \mathbb{R}^2$ and $\sigma : \mathbb{R}^2 \to \mathbb{R}^{2 \times 2}$ be bounded, measurable functions, and $\nu \in \mathcal{P}(\overline{D})$. A stochastic process $X$ is a solution of the stochastic differential equation with reflection in $\overline{D}$ with coefficients $b$ and $\sigma$, cone of directions of reflection $G$, and initial distribution $\nu$, if $X(0)$ has distribution $\nu$, there exist a standard Brownian motion $W$, a continuous, non-decreasing process $\lambda$, and a process $\gamma$ with measurable paths, all defined on the same probability space as $X$, such that $W(t + \cdot) - W(t)$ is independent of $\mathcal{F}_t^X,W,\lambda,\gamma$, for all $t \geq 0$, and the equation

$$X(t) = X(0) + \int_0^t b(X(s))ds + \int_0^t \sigma(X(s))dW(s) + \int_0^t \gamma(s)d\lambda(s), \quad t \geq 0,$$

$$\gamma(t) \in G(X(t)), \quad |\gamma(t)| = 1, \quad d\lambda - a.e., \quad t \geq 0,$$

$$X(t) \in \overline{D}, \quad \lambda(t) = \int_0^t 1_{\partial D}(X(s))d\lambda(s), \quad t \geq 0,$$

is satisfied a.s.

Given an initial distribution $\nu \in \mathcal{P}(\overline{D})$, weak uniqueness or uniqueness in distribution holds if all solutions of (3.6) with $P\{X(0) \in \cdot\} = \nu$ have the same distribution on $C_D[0, \infty)$.

A stochastic process $\tilde{X}$ is a weak solution of (3.6) if there is a solution $X$ of (3.6) such that $\tilde{X}$ and $X$ have the same distribution.
Let \( E := \overline{D}, \) \( A \) denote the operator
\[
D(A) := C^2(\overline{D}), \quad Af(x) := b(x) \cdot \nabla f(x) + \frac{1}{2} \text{tr}((\sigma \sigma^T)(x)D^2 f(x)),
\] (3.7)
and define
\[
U := \partial B_1(0), \quad \Xi := \{(x,u) \in \partial D \times U : u \in G(x)\}, \quad (3.8)
\]
\[
B : C^2(\overline{D}) \to C(\Xi), \quad Bf(x,u) := \nabla f(x) \cdot u.
\]

As shown in Theorem 6.12 of Costantini and Kurtz (2019), under quite general assumptions, every solution of (3.6) is a natural solution of the constrained martingale problem for \((A,D,B,\Xi)\) and every natural solution of the constrained martingale problem for \((A,D,B,\Xi)\) is a weak solution of (3.6).

In this section, first we prove existence of a solution to (3.6) by showing that, under Conditions 3.1, 3.5 and 3.8, the assumptions of Section 6 of Costantini and Kurtz (2019) are verified. Next we check that the assumptions of Costantini and Kurtz (2018) and Costantini and Kurtz (2021) are verified locally: By the localization results of Section 2 this yields global uniqueness.

**Theorem 3.10** Under Conditions 3.1, 3.5 and 3.8 for every initial distribution \( \nu \in P(\overline{D}) \), there exists a strong Markov solution of (3.6) with initial distribution \( \nu \).

**Proof.** We are going to show that \( D \) admits a representation such that the assumptions of Theorem 6.13 of Costantini and Kurtz (2019) are verified. (See Remark 3.2.) Note that the assumption of Costantini and Kurtz (2019) that \( D, D^1, \ldots, D^m \) are simply connected is redundant: it is enough to assume that the domains are connected, as we are doing here.

At every cone point \( x^0 \), (6.3) and Condition 6.2 of Costantini and Kurtz (2019) are clearly verified. In particular, Condition 6.2 (c) is verified for every \( I \subseteq I(x^0) \), hence for every \( I \in I(x^0) \).

Let the corner \( x^0 \) be a cusp point. We suppose, without loss of generality, that \( x^0 = 0, I(0) = \{1,2\} \), and we write \( n^1 \) for \( n^1(0) \) and \( n^2 \) for \( n^2(0) \). Let \( \tau = \tau(0) \) be the vector defined in (3.3). Let \( \Delta \) be a bounded domain with \( C^1 \) boundary, such that, for some \( r_0 > 0 \),
\[
\Delta \cap B_{r_0}(0) = \{ x \in B_{r_0}(0) : x \cdot \tau > 0 \}.
\]

We can suppose, without loss of generality, that \( \Delta \supset \overline{D} \setminus \{0\} \), so that
\[
D = \bigcap_{i=1}^m D^i \cap \Delta, \quad D = \bigcap_{i=1}^m D^i \cap \overline{\Xi}.
\]

With the addition of the extra domain \( \Delta \), the normal cone \( N(0) \) satisfies (6.3) of Costantini and Kurtz (2019). By defining the direction of reflection on \( \partial \Delta, \gamma \), to be the inward normal direction, we have \( \gamma(0) = \tau \), so that the cone of directions of reflection at 0, \( G(0) \), does not change.

---

1This is a rephrasing of Theorem 6.12 of Costantini and Kurtz (2019) with the terminology that we are adopting here, which differs from the terminology of Costantini and Kurtz (2014) in the use of the term "weak solution".
Now let $r_0 > 0$ be small enough that $\overline{B_{r_0}(0)}$ contains no other corners than 0. Without loss of generality we can take $(\tau, n^1)$ as the basis of the coordinate system. In order to see that Condition 6.2 (c) of [Costantini and Kurtz 2019] is verified, observe that $D$ can be represented as

$$D = \tilde{D}^1 \cap \tilde{D}^2 \cap \bigcap_{i \geq 3} D^i \cap \Delta,$$

where $\bigcap_{i \geq 3} D^i = \mathbb{R}^2$ if $m = 2$,

$$\Delta = \{x : \tilde{\psi}^0(x) > 0\}, \quad \tilde{\psi}^0 \in C^1(\mathbb{R}^2),$$

$$\tilde{D}^i = \{x : \tilde{\psi}^i(x) > 0\}, \quad i = 1, 2,$$

$$\tilde{\psi}^i(x_1, x_2) := \psi^i(|x_1|, x_2)[1 - \chi(\frac{2}{r_0}(|x| - \frac{r_0}{2}))] + \psi^i(x_1, x_2)\chi(\frac{2}{r_0}(|x| - \frac{r_0}{2})),$$  

$i = 1, 2,$

$\psi^i$ is the function defining $D^i$ and $\chi$ is a smooth, nondecreasing function such that $\chi(t) = 0$ for $t \leq 0$, $\chi(t) = 1$ for $t \geq 1$. Then $\tilde{\mathcal{I}}(0)$, defined by (6.9) of [Costantini and Kurtz 2019] for $\Delta$, $\tilde{D}^1$, $\tilde{D}^2$, is

$$\tilde{\mathcal{I}}(0) = \{\emptyset, \{0\}, \{1\}, \{2\}, \{0, 1\}, \{0, 2\}, \{1, 2\}\},$$

and Condition 6.2 (c) of [Costantini and Kurtz 2019] is satisfied at 0.

By iterating the above construction for each cusp point of $\partial D$, we obtain a representation of $D$ that satisfies the assumptions of Theorem 6.13 of [Costantini and Kurtz 2019].

**Lemma 3.11** Let $x^0 \in \partial D$ be a corner, $r_0$ be small enough that $\partial D \cap \overline{B_{r_0}(x^0)}$ contains no other corners and $U := \overline{D} \cap \overline{B_{r_0}(x^0)}$.

Then, under Conditions 3.7, 3.9 and 3.8, uniqueness holds for natural solutions of the stopped constrained martingale problem for $(A, D, B, \Xi; U)$.

**Proof.** We suppose, without loss of generality, that $x^0 = 0$, $I(0) = \{1, 2\}$, and we write $n^1$, $g^1$, $n^2$, $g^2$ for $n^1(0)$, $g^1(0)$, etc..

Let $\tilde{U} \subseteq D$ be a bounded domain with boundary of class $C^1$ at every point except 0, such that $\overline{\tilde{U} \cap B_{r_0}(0)} = \overline{D \cap B_{r_0}(0)}$ and denote by $\tilde{n}(x)$ the unit, inward normal to $\tilde{U}$ at $x \in \partial \tilde{U} - \{0\}$. Let $\tilde{G}(x) := \{\eta \tilde{g}(x) : \eta \geq 0\}$ for $x \in \partial \tilde{U} - \{0\}$, where $\tilde{g} : \mathbb{R}^2 - \{0\} \rightarrow \mathbb{R}^2$ is some locally Lipschitz continuous vector field, of unit length on $\partial \tilde{U} - \{0\}$, such that $\tilde{g}(x) \cdot \tilde{n}(x) > 0$ for $x \in \partial \tilde{U} - \{0\}$ and $\tilde{G}(x) = G(x)$ for $x \in (\partial \tilde{U}) \cap \overline{B_{r_0}(0)} - \{0\}$. Set $\tilde{G}(0) := G(0)$ and

$$\tilde{\Xi} := \{(x, u) \in \partial \tilde{U} \times U : u \in \tilde{G}(x)\}.$$

Suppose that 0 is a cone point and let

$$\tilde{\mathcal{K}} := \{u \in \mathbb{R}^2 : u \cdot n^1 > 0, u \cdot n^2 > 0\},$$

if $n^1 \neq n^2$, and

$$\tilde{\mathcal{K}} := \{u \in \mathbb{R}^2 : u \cdot n^1 > 0\},$$

if $n^1 = n^2$. Then, for $x^0 = 0$, we have

$$\tilde{\mathcal{I}}(0) = \{\emptyset, \{0\}, \{1\}, \{2\}, \{0, 1\}, \{0, 2\}, \{1, 2\}\},$$
if \( n^1 = n^2 \). Then it can be checked by elementary computations that Condition 3.1 implies that \( \tilde{U} \) and \( \tilde{K} \) satisfy Conditions 3.1 (i) and (ii) of Costantini and Kurtz (2021). As far as Condition 3.3 of Costantini and Kurtz (2021) is concerned, (i), (ii) and (iv) follow immediately from Condition \( 3.5 \) (i) and (ii). As for (iii), if \( n^1 = n^2 \), Condition 3.3 (ii) implies that \( G(0) - \{0\} \subseteq \tilde{K} \). If \( n^1 \neq n^2 \), Condition 3.3 implies that the matrix

\[
\begin{bmatrix}
n^1 \cdot g^1 & n^2 \cdot g^1 \\
n^1 \cdot g^2 & n^2 \cdot g^2
\end{bmatrix}
\]

is a completely-S matrix. Then its transpose is also completely-S (Lemma 3 of Reiman and Williams (1988)), which is equivalent to (iii) of Condition 3.3 of Costantini and Kurtz (2021). Therefore, by Theorem 3.25 of Costantini and Kurtz (2021), uniqueness holds for natural solutions of the constrained martingale problem for \((\tilde{A}, \tilde{U}, B, \tilde{\Xi})\). Moreover, it is shown in the proof of Theorem 3.23 of Costantini and Kurtz (2021) that, for each \( \nu \in \mathcal{P}(\tilde{U}) \), there exists a solution of the controlled martingale problem for \((\tilde{A}, \tilde{U}, B, \tilde{\Xi})\) with initial distribution \( \nu \). Together with Lemma 3.16 of Costantini and Kurtz (2021), this ensures that the assumptions of Proposition 2.9 and hence Condition 2.7 are verified.

Now suppose that 0 is a cusp point. Let \( \tau = \tau(0) \) be the vector in (3.3) and take \((\tau, n^1)\) as the basis of the coordinate system. By the implicit function theorem there exist \( r_1 > 0, r_2 > 0, r_1^2 + r_2^2 \leq r_0^2 \), and continuously differentiable functions \( \tilde{\psi}^1 \) and \( \tilde{\psi}^2 \) defined on \([-r_1, r_1]\), with values in \([-r_2, r_2]\), such that \( \tilde{\psi}^1(0) = \tilde{\psi}^2(0) = 0 \) and, for \((x_1, x_2) \in [-r_1, r_1] \times [-r_2, r_2]\),

\[
\begin{align*}
\psi^1(x_1, x_2) > 0 & \iff x_2 > \tilde{\psi}^1(x_1), \\
\psi^1(x_1, x_2) = 0 & \iff x_2 = \tilde{\psi}^1(x_1), \\
\psi^2(x_1, x_2) > 0 & \iff x_2 < \tilde{\psi}^2(x_1), \\
\psi^2(x_1, x_2) = 0 & \iff x_2 = \tilde{\psi}^2(x_1).
\end{align*}
\]

Then \( \tilde{\psi}^1 \) and \( \tilde{\psi}^2 \) satisfy Condition 2.1 of Costantini and Kurtz (2018). In addition, taking into account (3.4), Condition 3.5 ensures that \( \tilde{\varrho} \) satisfies Condition 2.3 of Costantini and Kurtz (2018). Therefore Theorems 3.1 and 4.7 of Costantini and Kurtz (2018), together with Theorem 6.12 of Costantini and Kurtz (2019), give uniqueness for natural solutions of the constrained martingale problem for \((\tilde{A}, \tilde{U}, B, \tilde{\Xi})\). Moreover, in the proof of Theorem 4.1 of Costantini and Kurtz (2018), a solution of the controlled martingale problem for \((\tilde{A}, \tilde{U}, B, \tilde{\Xi})\) with initial distribution the Dirac measure at 0 is constructed and it is shown that, for that solution, \( \lambda_0 \) (denoted as \( K_0 \) there) is strictly increasing. Exactly the same arguments allow to construct a solution of the controlled martingale problem for \((\tilde{A}, \tilde{U}, B, \tilde{\Xi})\) with an arbitrary initial distribution \( \nu \in \mathcal{P}(\tilde{U}) \) and to show that \( \lambda_0 \) is strictly increasing for each solution of the controlled martingale problem for \((\tilde{A}, \tilde{U}, B, \tilde{\Xi})\). Hence, by Proposition 2.9, Condition 2.7 is satisfied.

Then Theorem 2.12 yields that uniqueness holds for natural solutions of the stopped constrained martingale problem for \((\tilde{A}, \tilde{U}, B, \tilde{\Xi}; U)\). A solution \( X^U \) of the stopped constrained martingale problem for \((A, D, B, \Xi; U)\) is not necessarily a solution of the stopped constrained martingale problem for \((\tilde{A}, \tilde{U}, B, \tilde{\Xi}; U)\) because its initial distribution might charge \( \tilde{D} \cap (\tilde{U})^c \). However if \( X^U \) and \( \tilde{X}^U \) are two solutions of the stopped constrained martingale problem for \((A, D, B, \Xi; U)\) with the same initial distribution,

\[
Z^U(t) := \begin{cases}
X^U(t), & t \geq 0, \text{ if } X^U(0) \in U, \\
z^0, & t \geq 0, \text{ if } X^U(0) \notin U,
\end{cases}
\]

\[
\tilde{Z}^U(t) := \begin{cases}
\tilde{X}^U(t), & t \geq 0, \text{ if } \tilde{X}^U(0) \in U, \\
z^0, & t \geq 0, \text{ if } \tilde{X}^U(0) \notin U,
\end{cases}
\]
where \( z^0 \) is some fixed point in \( \overline{U} - U \), are two solutions of the stopped constrained martingale problem for \((A, \overline{U}, B, \Xi; U)\) with the same initial distribution. Therefore \( Z^U \) and \( \tilde{Z}^U \) have the same distribution and so do \( X^U \) and \( \tilde{X}^U \).

**Theorem 3.12** Under Conditions 3.7, 3.5 and 3.8 for every initial distribution \( \nu \in \mathcal{P}(\overline{D}) \), uniqueness in distribution holds for solutions of (3.6) with initial distribution \( \nu \). The solution is a strong Markov process.

**Proof.** As we have seen in the proof of Theorem 3.10 \( D, G, b \) and \( \sigma \) satisfy the assumptions of Section 6 of Costantini and Kurtz (2019), therefore Theorems 6.7 and Lemma 6.8 of Costantini and Kurtz (2019) ensure that the assumptions of Proposition 2.9 and hence Condition 2.7 are satisfied.

Let \( x^1, x^2, \ldots, x^M \) be the corners of \( \overline{D} \), \( r_0 > 0 \) be such that \( x^h \not\in B_{r_0}(x^k) \) for \( h \neq k \). Let

\[
U^k := \overline{D} \cap B_{r_0}(x^k), \quad k = 1, \ldots, M, \quad U^{M+1} := \overline{D} \cap \left( \bigcup_{k=1}^{M} B_{r_0/2}(x^k) \right)^c.
\]

By Lemma 3.11, uniqueness holds for natural solutions of the stopped constrained martingale problems for \((A, \overline{D}, B, \Xi; U^k)\), for \( k = 1, \ldots, M \). As for the stopped constrained martingale problem for \((A, \overline{D}, B, \Xi; U^{M+1})\), one can consider a domain \( \overline{U}^{M+1} \subseteq D \) with \( C^1 \) boundary, such that \( \overline{U}^{M+1} \cap \left( \bigcup_{k=1}^{M} B_{r_0/2}(x^k) \right)^c = \overline{D} \cap \left( \bigcup_{k=1}^{M} B_{r_0/2}(x^k) \right)^c \) and a Lipschitz continuous direction of reflection \( \eta^{M+1} \) on \( \partial \overline{U}^{M+1} \) such that \( \tilde{G}^{M+1}(x) := \{ \eta \tilde{G}^{M+1}(x), \eta \geq 0 \} = G(x) \) for \( x \in \partial \overline{U}^{M+1} \cap \left( \bigcup_{k=1}^{M} B_{r_0/2}(x^k) \right)^c \), and argue as in Lemma 3.11 but using Corollary 5.2 (Case 2) of Dupuis and Ishii (1993) and Theorem 6.12 of Costantini and Kurtz (2019), to obtain that uniqueness holds for natural solutions of the stopped constrained martingale problems for \((A, \overline{D}, B, \Xi; U^{M+1})\).

Then the assertion follows by Theorem 2.14 and by Theorem 6.12 of Costantini and Kurtz (2019).

**A Proof of Lemma 2.3**

Let \( P^U \) denote the distribution of \((Y^U, \lambda^U, \Lambda^U)\), \( \nu \) denote the distribution of \( Y^U(\theta^U) \) and \( P \) denote the distribution of a solution of the controlled martingale problem for \((A, E_0, B, \Xi)\) with initial distribution \( \nu \). Let \( Q \) be the probability measure on \( D_E[0, \infty) \times C_{[0, \infty)}[0, \infty) \times \mathcal{L}_U \times [0, \infty) \times D_E[0, \infty) \times C_{[0, \infty)}[0, \infty) \times \mathcal{L}_U \) defined by

\[
Q(D_1 \times D_2) := \int_E \mathbb{E}^{P^U}[\mathbf{1}_{D_1}(\eta^1, l^1_0, L^1_1, \theta) | \eta^1(\theta) = y] \mathbb{E}^P[\mathbf{1}_{D_2}(\eta^2, l^2_0, L^2_1) | \eta^2(0) = y] \nu(dy) \tag{1.9}
\]

where \( (\eta^1, l^1_0, L^1_1, \theta, \eta^2, l^2_0, L^2_1) \) is the coordinate random variable in \( D_E[0, \infty) \times C_{[0, \infty)}[0, \infty) \times \mathcal{L}_U \times [0, \infty) \times D_E[0, \infty) \times C_{[0, \infty)}[0, \infty) \times \mathcal{L}_U \). \( D_1 \) is a Borel subset of \( D_E[0, \infty) \times C_{[0, \infty)}[0, \infty) \times \mathcal{L}_U \times [0, \infty) \times D_E[0, \infty) \times C_{[0, \infty)}[0, \infty) \times \mathcal{L}_U \) and \( D_2 \) is a Borel subset of \( D_E[0, \infty) \times C_{[0, \infty)}[0, \infty) \times \mathcal{L}_U \).
Define, for \( t \geq 0, C \in \mathcal{B}(\mathcal{U}) \),

\[
Y(t) := \begin{cases} 
\eta^1(t), & t < \theta \\
\eta^2(t - \theta), & t \geq \theta,
\end{cases}
\]

\[
\lambda_0(t) := \begin{cases} 
\ell_0^1(t), & t < \theta \\
\ell_0^2(t - \theta) + \ell_0^1(\theta), & t \geq \theta,
\end{cases}
\]

\[
\Lambda_1([0, t] \times C) := \begin{cases} 
L_1^1([0, t] \times C) & t < \theta, \\
L_2^1([0, t - \theta] \times C) + L_1^1([0, \theta] \times C), & t \geq \theta,
\end{cases}
\]

\[
\theta := \theta
\]

Then the distribution of \((Y, \lambda_0, \Lambda_1)(\cdot \wedge \theta)\) under \( Q \) is \( P^U \). In particular \( \theta \) as defined above agrees \( Q \)-a.s. with \( \theta \) as defined in (2.6).

Let us show that \((Y, \lambda_0, \Lambda_1)\) is a solution of the controlled martingale problem for \((A, E_0, B, \Xi)\).

To this end we need to show that, for arbitrary \( 0 = t_0 < t_1 < \ldots t_n < t_{n+1} \), denoting

\[
R := f(Y(t_{n+1})) - f(Y(t_n)) - \int_{t_n}^{t_{n+1}} Af(Y(s))d\lambda_0(s) - \int_{[t_n, t_{n+1}] \times \mathcal{U}} Bf(Y(s), u)\Lambda_1(ds \times du),
\]

it holds, for arbitrary continuous functions \( h_k \) and \( H_k \) and \( C_k \in \mathcal{B}(\mathcal{U}) \),

\[
\mathbb{E}\left[ R \prod_{k=1}^{n} h_k(Y(t_k)) H_k(\lambda_0(t_k) - \lambda_0(t_{k-1}), \Lambda_1((t_{k-1}, t_k] \times C_k)) \right] = 0.
\]

Observing that

\[
\lambda_0(t_k) - \lambda_0(t_{k-1}) = \lambda_0(t_k \vee \theta) - \lambda_0(t_{k-1} \vee \theta) + \lambda_0(t_k \wedge \theta) - \lambda_0(t_{k-1} \wedge \theta),
\]

\[
\Lambda_1((t_{k-1}, t_k] \times C_k) = \Lambda_1((t_{k-1} \vee \theta, t_k \vee \theta] \times C_k) + \Lambda_1((t_{k-1} \wedge \theta, t_k \wedge \theta] \times C_k)
\]

we see that we can replace \( H_k(\lambda_0(t_k) - \lambda_0(t_{k-1}), \Lambda_1((t_{k-1}, t_k] \times C_k)) \) by the product

\[
H_k^\vee(\lambda_0(t_k \vee \theta) - \lambda_0(t_{k-1} \vee \theta), \Lambda_1((t_{k-1} \vee \theta, t_k \vee \theta] \times C_k))
\times H_k^\wedge(\lambda_0(t_k \wedge \theta) - \lambda_0(t_{k-1} \wedge \theta), \Lambda_1((t_{k-1} \wedge \theta, t_k \wedge \theta] \times C_k)),
\]

where \( H_k^\vee \) and \( H_k^\wedge \) are arbitrary continuous functions such that \( H_k^\vee(0, 0) = H_k^\wedge(0, 0) = 1 \). Analogously we can split \( R \) as

\[
R = R^\vee + R^\wedge,
\]

\[
R^\vee := f(Y(t_{n+1} \vee \theta)) - f(Y(t_n \vee \theta))
- \int_{t_n \vee \theta}^{t_{n+1} \vee \theta} Af(Y(s))d\lambda_0(s)
- \int_{(t_n \vee \theta, t_{n+1} \vee \theta] \times \mathcal{U}} Bf(Y(s), u)\Lambda_1(ds \times du),
\]

\[
R^\wedge := f(Y(t_{n+1} \wedge \theta)) - f(Y(t_n \wedge \theta))
- \int_{t_n \wedge \theta}^{t_{n+1} \wedge \theta} Af(Y(s))d\lambda_0(s)
- \int_{(t_n \wedge \theta, t_{n+1} \wedge \theta] \times \mathcal{U}} Bf(Y(s), u)\Lambda_1(ds \times du),
\]
so that we reduce to proving that

\[
\mathbb{E}^Q \left[ R^\lor \prod_{k=1}^n h_k(Y(t_k)) H_k^\lor(\lambda_0(t_k \lor \theta) - \lambda_0(t_{k-1} \lor \theta), \Lambda_1((t_{k-1} \lor \theta, t_k \lor \theta] \times C_k)) \right] = 0, \tag{1.10}
\]

and that

\[
\mathbb{E}^Q \left[ R^\land \prod_{k=1}^n h_k(Y(t_k)) H_k^\land(\lambda_0(t_k \land \theta) - \lambda_0(t_{k-1} \land \theta), \Lambda_1((t_{k-1} \land \theta, t_k \land \theta] \times C_k)) \right] = 0. \tag{1.11}
\]

Noting that

\[
R^\land = R^\land 1_{\theta > t_n},
\]

and that

\[
1_{\theta > t_n} \prod_{k=1}^n H_k^\lor(\lambda_0(t_k \lor \theta) - \lambda_0(t_{k-1} \lor \theta), \Lambda_1((t_{k-1} \lor \theta, t_k \lor \theta] \times C_k)) = 1_{\theta > t_n},
\]

we see, by computations analogous to those of Lemma 4.5.16 of \textit{Ethier and Kurtz} (1986), that the left hand side of (1.11) equals zero.

In order to see that (1.10) is verified, define

\[
\vartheta_m := \left\lfloor \frac{m\theta}{m} \right\rfloor,
\]

\[
R_m^\lor := f(\eta^2(t_{n+1} \lor \vartheta_m - \vartheta_m)) - f(\eta^2(t_n \lor \vartheta_m - \vartheta_m)) - \int_{t_n \lor \vartheta_m - \vartheta_m}^{t_{n+1} \lor \vartheta_m - \vartheta_m} Af(\eta^2(s))dl^2(s)
\]

\[
- \int_{(t_n \lor \vartheta_m - \vartheta_m, t_{n+1} \lor \vartheta_m - \vartheta_m]} Bf(\eta^2(s), u)L^2_1(ds \times du),
\]

and consider

\[
\prod_{t_k < \vartheta_m} h_k(\eta^2(t_k)) H_k^\lor(l_k^1(t_k \lor \vartheta_m) - l_{k-1}^1(t_k \lor \vartheta_m), L_1^1((t_{k-1} \lor \vartheta_m, t_k \lor \vartheta_m] \times C_k))
\]

\[
\prod_{t_k \geq \vartheta_m} h_k(\eta^2(t_k - \vartheta_m)) \tag{1.12}
\]

\[
H_k^\lor(l_k^2(t_k \lor \vartheta_m - \vartheta_m) - l_{k-1}^2(t_k \lor \vartheta_m - \vartheta_m),
L_1^2((t_{k-1} \lor \vartheta_m - \vartheta_m, t_k \lor \vartheta_m - \vartheta_m] \times C_k))
\]

\[
H_k^\lor(l_k^2(t_k \land \vartheta_m - \vartheta_m) - l_{k-1}^2(t_k \land \vartheta_m - \vartheta_m),
L_1^2((t_{k-1} \land \vartheta_m - \vartheta_m, t_k \land \vartheta_m - \vartheta_m] \times C_k)),
\]

Noting that

\[
\prod_{t_k < \vartheta_m} H_k^\lor(l_k^1(t_k \lor \vartheta_m) - l_{k-1}^1(t_k \lor \vartheta_m), L_1^1((t_{k-1} \lor \vartheta_m, t_k \lor \vartheta_m] \times C_k)) = 1,
\]

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\[
\prod_{t_{k-1} \geq \vartheta_m} H_k^\vartheta(\ell_0^k(t_k \wedge \vartheta_m - \vartheta_m)) - \ell_0^k(t_{k-1} \wedge \vartheta_m - \vartheta_m), L^2_1((t_{k-1} \wedge \vartheta_m - \vartheta_m, t_k \wedge \vartheta_m - \vartheta_m] \times C_k)) = 1,
\]

and that
\[
R_m^\vartheta = R_m^\vartheta 1_{\vartheta_m < t_{m+1}},
\]
we find, by computations analogous to those of Lemma 4.5.16 of Ethier and Kurtz (1986), that the expectation of \( R \) under \( Q \) equals zero. Since \( R \) converges pointwise and boundedly to \( R \), \( \mathbb{E} \left[ R(t) \right] = 0 \) for all \( t \). Thus, \( R \) is the required solution.

\[\Box\]

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