Effects of first-order chemical reaction and melting heat on hybrid nanoliquid flow over a nonlinear stretched curved surface with shape factors

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Abstract
The aim of the current study is to disclose the results of shape factor analysis of the flow of a hybrid nanofluid over a curved sheet. The flow is caused by a stretchable curved sheet. Mathematical modeling and analyses have been performed in the presence of curvature, melting heat and heterogeneous-homogeneous reactions. Autocatalysis and the coefficients of the reactant are dealt with in a similar manner. The physical properties of the fluid, including the fluid velocity, the heat and mass transfer properties, the skin friction and the Nusselt number have been acquired and analyzed under the influences of the dimensionless curvature, melting and heterogeneous–homogeneous reaction variables. Boundary layer approximations are used in the mathematical formulation. Suitable transformations have been used to transform differential equations into nonlinear ordinary differential equations. The resulting nonlinear system of equations has been analyzed via the matlab bvp4c solver. Comparisons of nanoliquids with the hybrid nanoliquid are presented through graphs and tables. The results of this analysis show that the skin friction and the heat transfer rate in the hybrid nanofluid are seen more prominently than those of the nanofluid for larger values of the curvature parameter $k$.

Keywords
Hybrid nanofluid, nanofluid flow, viscous fluid, nonlinear curved sheet, melting heat transfer, homogeneous-heterogeneous reactions

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Introduction
Boundary layer flow of viscous fluids has seen extensive use in industry because of the widespread use of stretching moveable surfaces with peculiar velocities. This type of flow is promising for implementation in a variety of industrial fields, including fabrication of prominences in polymer plates using a die or extraction of plastic sheets. The melted plastic is extracted from a slit during the sheet structuring process, consequently leaving the sheet stretchable and allowing the desired thickness to be achieved. The mechanical features of refined products depend on the cooling rate and the extending features of the products during these procedures. Boundary layer flow via a moving stretchable sheet was explored for the first time by Sakiadis¹,². After this initial work,¹,² researchers documented stretchable plates in boundary layer flow in various directions. Crane³ introduced three-dimensional analysis for

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incompressible laminar flow. However, closed form results for the Crane problem have been very scarce in fluid mechanics to date. Boundary layer flow research has broadened into numerous areas to explore other physical features of the flow, including suction/blowing and heat transmission. In addition, many associated problems have been developed, with each incorporating different aspects to provide a precise solution. Accurate analytical results for the Crane study \(^2\) were investigated by McLeod and Rajagopal. \(^4\) Stretched flow via injection/suction processes was researched by Gupta et al. \(^5\) Wang \(^6\) extended the axisymmetric flow into a three-dimensional stretching sheet. The stretching flow \(^7\)-\(^11\) was studied extensively for both Newtonian and non-Newtonian fluids. The above research on linear and nonlinear extendable sheets focused on flat sheets only and few attempts have been made to analyze flow over stretchable curved sheets. The pioneering investigation of linearly stretchable curved sheets has been documented in the literature. \(^1\) Subsequent studies of the use of liquid flow via curved extendable plates in industry have incorporated flow in the curved jaws of machines. \(^1\)\(^3\)-\(^1\)\(^9\)

Nanofluids, which are composed of nanoparticles submerged in base liquids, have seen several real applications in numerous fields, including cancer treatment, temperature reduction in electronic tools, the nuclear field, drug delivery and many other areas. Because of their significance in the aforementioned areas, nanofluids have become a challenging research topic in recent years. The characteristics of final industrial products are largely dependent on heat transport aspects and thermal conduction, and low-thermal-conductivity fluids are mostly encountered in manufacturing industries. Nanoparticles are being developed to overcome this inadequacy in fluids for industrial applications. The pioneering analysis of these engineered liquids was introduced by Choi and Eastman. \(^2\) Because of the unique features of nanofluids, they are being used increasingly widely in medical and engineering sciences in heat exchangers, targeted drug delivery, mechanical cooling, extraction of geothermal forces and many other fields.

Hybrid nanofluids (HBNFs) have been applied extensively in the cooling of machines and motors, bio-technologies and numerous other areas. An HBNF is a mixture of two differently-sized nanoparticles and a base liquid. An HBNF was analyzed experimentally for the first time in 2007 by Jana et al. \(^2\)\(^1\) A comprehensive review of the characteristics of HBNFs was later provided by Sarkar et al. \(^2\) They concluded that HBNFs enhance both pressure and heat transfer. Subsequent investigations \(^2\)\(^3\)-\(^2\) and further studies cited therein can be referred to for further analyses of HBNFs.

The solidification and melting features of materials have remained a prominent research topic because of their promising links to innovative technologies and industries. Research scientists tend to place importance on technologies that provide more durable, effective and low-cost energy storage. In the heat and power industries, lost heat recovery and solar energy applications are interlinked with these materials. Chemical storage and latent heat and sensible heat energies are used frequently for energy storage based on these techniques. Latent heat is more appropriate economically and there is an effective energy storage trend based on altering the phase of the storage materials. The thermal energy is preserved in the material through its latent heat by melting and is regained by later freezing the same material. The melting of ice chunks in steam was studied by Roberts. \(^2\)\(^8\) Further literature reviews on aspects of melting and freezing were cited in several articles. \(^2\)\(^9\)-\(^3\)\(^4\)

Heterogeneous–homogeneous reactions are implicated in a variety of chemical reaction phenomena. These phenomena are engaged together in both heterogeneous and homogeneous reactions. Various reactions proceed gradually or do not occur instantaneously in the presence of a catalyst. Within fluids and on catalyst surfaces, different reaction rates occur for the different reactant species. These reactions \(^3\)\(^5\)-\(^3\)\(^6\) are commonly used in processes that include food processing, fog formation and dispersion, polymer processing, combustion and biochemical processing. Merkin \(^3\)\(^7\) proposed the pioneering concept of isothermal homogeneous–heterogeneous reactions. Autocatalyst diffusivity and reactants were also investigated in boundary layer flows by Chaudhary and Merkin. \(^3\)\(^8\) The impact of diffusion species in nanofluids when varying the sheet thickness was reported by Hayat et al. \(^3\)\(^9\) A computational study of these reactions for heat transport applications was performed by Waqas. \(^4\)\(^0\)

Molybdenum disulphide (MoS\(_2\)) has been used as a catalyst, a hydrogenation nano-catalyst and a lubricant. MoS\(_2\) has unique features that include chemical inertness, a photo-corrosion property and anisotropy with a layered structure. Graphene has properties that include high-speed electrons. Polymer electrical conduction could therefore be enhanced by using graphene within the polymer matrix. The thermal conductivity of the graphene oxide (GO) shape factor could be accelerated by several orders of magnitude easily. Graphene has been found to have one of the highest in-plane thermal conductivities among the materials considered in this field.

The shape factors of the nanoparticles play an important role in advancement of heat transfer. However, very little consideration has been given to the heat transfer through exact nanoparticle shapes. The current analysis technique describes the flow of a
water-based HBNF (MoS$_2$ + GO/water) on a curved nonlinear stretching surface, including the characteristics of heterogeneous–homogeneous reactions and melting heat. To the best of our knowledge, nobody has previously tried to investigate HBNF flow over a curved nonlinear stretchable surface with melting heat effects and homogeneous–heterogeneous reactions with shape factors. The associated problems have been modeled first and then analyzed using the MATLAB bvp4c solver to provide numerical results. This approach is beneficial when compared with other methods, i.e. the Runge–Kutta integration method, the shooting technique, and the Keller box method, because of its rapid convergence.

**Thermal characteristics of base liquid with shape factors**

The composition of the HBNF comprises MoS$_2$(f$_1$) (molybdenum disulphide) and GO(f$_2$) (graphene oxide) in a base liquid. According to the literature,$^{46}$ the mathematical form of the volume fraction for an HBNF is defined as

$$\phi_{\text{hbf}} = \frac{V_{\text{MoS}_2} + V_{\text{GO}}}{V_f} = \phi_1 + \phi_2,$$

where $V_{\text{MoS}_2}$, $V_{\text{GO}}$ and $V_f$ are the volume of MoS$_2$, the volume of GO and the total volume of the fluid, respectively, and $\phi_1$ and $\phi_2$ are the volume fractions of MoS$_2$ and GO, respectively. The thermal characteristics of the base liquid and the nanoparticles are presented in Table 1. Table 2 compares the thermophysical characteristics of the MoS$_2$–water nanoliquid and the MoS$_2$ + GO hybrid nanoliquid.

**Formulation**

Consider a two-dimensional incompressible HBNF flow over a curved sheet with radius $R$. The curved surface is stretchable in a nonlinear manner. The curved surface reduces to a flat sheet for larger values of $R$ (i.e. $R \gg \infty$). The curvature of the curved sheet depends on the distance $R$ from the center. The water-based hybrid nanoliquid is a mixture of molybdenum disulphide (MoS$_2$ ($\phi_1$)) and graphene oxide (GO ($\phi_2$)) nanoparticles. The nonlinear velocity (i.e. $u_s(s) = c_s s^p$) of the stretchable curve sheet is considered in the $sr$-directions, as illustrated in Figure 1. In addition, the effects of melting features and heterogeneous–homogeneous reactions are included in the formulation. It is assumed that $T_m$ (melting temperature) > $T_\infty$ (ambient temperature) and that $T_0$ (solid surface

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**Table 1.** Characteristics of base liquid and nanoparticles.$^{47}$

| Characteristics | Base liquid | Nanoscale particles |
|-----------------|------------|---------------------|
| Pr              | 6.2        | 1800                |
| $\rho$ (kg/m$^3$) | 997        | 5060                |
| $\beta_1$ (K$^{-1}$) | 21        | 6.30×10$^7$         |
| $k_1$ (W/mK)     | 0.613      | 2.84×10$^{-4}$      |
| $c_p$ (J/kgK)    | 4179       | 904.4               |

$Pr$: Prandtl number; $\rho$: density; $\beta_1$: thermal expansion; $k_1$: thermal conductivity; $c_p$: heat capacity.

**Table 2.** Thermophysical properties of hybrid nanoliquid (MoS$_2$ + GO)($\phi_2$) and nanoliquid (MoS$_2$)($\phi_1$).$^{37}$

| Properties         | Hybrid nanoliquid | Nanoliquid |
|--------------------|-------------------|------------|
| Density            | $\rho_{\text{hbf}} = [(1-\phi_2)(1-\phi_1)\rho_r + \phi_1 \rho_1 + \phi_2 \rho_2]$ | $\rho_{\text{hbf}} = [(1-\phi_2)(1-\phi_1)\rho_r + \phi_1 \rho_1]$ |
| Viscosity          | $\mu_{\text{hbf}} = \mu_r (1-\phi_1)^{1-\phi_2}$ | $\mu_{\text{hbf}} = \mu_r (1-\phi_1)^{1-\phi_2}$ |
| Thermal expansion  | $(\rho\beta)_{\text{hbf}} = \phi_2 (\rho \beta_2)_{\text{MoS}_2} + (1-\phi_2)(1-\phi_1)(\rho \beta_1)_{\text{GO}} + \phi_1 \rho \beta_1_{\text{GO}}$ | $(\rho\beta)_{\text{hbf}} = \phi_2 (\rho \beta_2)_{\text{MoS}_2} + (1-\phi_2)(1-\phi_1)(\rho \beta_1)_{\text{GO}} + \phi_1 \rho \beta_1_{\text{GO}}$ |
| Thermal conductivity | $k_{\text{hbf}} = k_1 + (m-1)k_{\text{MoS}_2} - (m-1)\phi_1(k_1 - k_{\text{MoS}_2}) + \phi_2(k_1 - k_{\text{GO}})$ | $k_{\text{hbf}} = k_1 + (m-1)k_1 - (m-1)\phi_1(k_1 - k_{\text{MoS}_2}) + \phi_2(k_1 - k_{\text{GO}})$ |
| Heat capacity      | $(\rho \varepsilon_\phi)_{\text{hbf}} = \phi_2 (\rho \varepsilon_\phi)_{\text{MoS}_2} + (1-\phi_2)(1-\phi_1)(\rho \varepsilon_\phi)_{\text{GO}} + \phi_1 (\rho \varepsilon_\phi)_{\text{GO}}$ | $(\rho \varepsilon_\phi)_{\text{hbf}} = \phi_2 (\rho \varepsilon_\phi)_{\text{MoS}_2} + (1-\phi_2)(1-\phi_1)(\rho \varepsilon_\phi)_{\text{GO}} + \phi_1 (\rho \varepsilon_\phi)_{\text{GO}}$ |

$Pr$: Prandtl number; $\rho$: density; $\beta_1$: thermal expansion; $k_1$: thermal conductivity; $c_p$: heat capacity; $\mu_1$: dynamic viscosity; $\kappa$: thermal conductivity; $\rho$: density; $\kappa$: thermal conductivity; $\varepsilon_\phi$: heat capacity.
temperature) < \( T_m \). The melting heat features are defined mathematically as follows:

\[
k_{\text{hnf}} \frac{\partial T}{\partial r} = \rho_{\text{hnf}}(\lambda_1 + C_s(T_m - T_0))u(s, 0),
\]

where \( k_{\text{hnf}} \) is the thermal conduction of the HBNF, \( \rho_{\text{hnf}} \) is the density of the HBNF, and \( \lambda_1 \) and \( C_s \) are the liquid latent heat and the solid specific heat, respectively.

The isothermal homogeneous reaction is defined as:

\[
A_1 + 2B_1 \rightarrow 3B_1, \quad \text{rate} \quad k_{\text{d}}a_b^2,
\]

and for cubic auto-catalysis, the heterogeneous reaction on the catalyst surface is defined as:

\[
A_1 \rightarrow B_1, \quad \text{rate} \quad k_e a,
\]

where \( k_d \) and \( k_e \) are the rate constants and \( a \) and \( b \) are the concentrations of species \( A_1 \) and \( B_1 \), respectively.

**Momentum analysis**

The momentum balance equation reveals how the diffusion of the nanoliquid molecules in the boundary layer is caused by the nonlinear stretchable curved sheet. The resultant problems after application of approximation theory are given by:

\[
R \frac{\partial w}{\partial s} + \frac{\partial}{\partial r}((r + R)u) = 0,
\]

\[
\frac{1}{r + R} w^2 = \frac{1}{\rho_{\text{hnf}}} \frac{\partial p}{\partial r},
\]

\[
u_{\text{hnf}} \frac{\partial^2 w}{\partial r^2} + \frac{1}{r + R} \frac{\partial w}{\partial r} - \frac{1}{(r + R)^2} w
\]

\[
= -\frac{1}{\rho_{\text{hnf}}} \frac{R}{r + R} \frac{\partial p}{\partial s},
\]

with the following conditions:

\[
w = w_u(s) = c_1 s^n, \quad u = 0 \quad \text{at} \quad r = 0
\]

\[
w \rightarrow 0, \quad \frac{\partial w}{\partial r} \rightarrow 0 \quad \text{as} \quad r \rightarrow \infty,
\]

where \( w \) and \( u \) are the velocity components, and \( \rho_{\text{hnf}}, p, c_1, 0 \) and \( n \) represent the HBNF density, the pressure, the stretching constant and the power law index (where \( n = 1 \) indicates linear stretching, \( n \neq 1 \) represents...
nonlinear stretching and \( n = 0 \) corresponds to a constant stretching surface), respectively.

**Energy analysis**

Here, we examine how the temperature depends on the melting heat transfer. Therefore, we have the balance of energy equation as follows:

\[
k_{h nf} \left( \frac{\partial^2 T}{\partial r^2} + \frac{1}{r + R} \frac{\partial T}{\partial r} \right) = (\rho c_p)_{h nf} \left( \frac{\partial T}{\partial r} + \frac{R}{r + R} \frac{\partial T}{\partial s} \right),
\]

(9)

with the conditions that

\[
T = T_m \text{ at } r = 0, \quad T \to T_\infty \text{ as } r \to \infty,
\]

(10)

where \((c_p)_{h nf}, k_{h nf}\) and \(T\) represent the specific heat, the thermal conductivity, and the temperature, respectively.

**Mass transfer analysis**

The mass balance equation shows how the concentration of an HBNF varies through homogeneous-heterogeneous reactions. The resulting problems after application of approximation theory are:

\[
D_{A_1} \left( \frac{\partial^2 a}{\partial r^2} + \frac{1}{r + R} \frac{\partial a}{\partial r} \right) - k_{dA} b^2 = \frac{\partial a}{\partial r} + \frac{R}{r + R} \frac{\partial a}{\partial s},
\]

(11)

\[
D_{B_1} \left( \frac{\partial^2 b}{\partial r^2} + \frac{1}{r + R} \frac{\partial b}{\partial r} \right) + k_{dB} b^2 = \frac{\partial b}{\partial r} + \frac{R}{r + R} \frac{\partial b}{\partial s},
\]

(12)

with the conditions that

\[
D_{A_1} \frac{\partial a}{\partial r} = k_{dA}, \quad D_{B_1} \frac{\partial b}{\partial r} = -k_{dA} \text{ at } r = 0, \quad a \to a^*, \quad b \to 0 \text{ as } r \to \infty,
\]

(13)

where \(D_{A_1}\) and \(D_{B_1}\) represent the diffusion coefficients for species \(a, b\), respectively.

By letting

\[
w = c_1 s^\eta f'(\xi),
\]

\[
u = -\frac{R}{r + R} \sqrt{c_1 \nu_0 s^\eta - \frac{1}{2} n + 1} f(\xi) + \frac{n - 1}{2} \xi f'(\xi),
\]

\[
\xi = \sqrt{\frac{c_1 s^\eta}{\nu_0 r}}, \quad p = c_0^2 \rho s^2 \psi(\xi), \quad k = \sqrt{\frac{c_1 s^\eta}{\nu_0}} R,
\]

\[
\theta(\xi) = \frac{T - T_m}{T_\infty - T_m},
\]

\[
a = a^\infty \Phi(\xi), \quad b = a^\infty h(\xi),
\]

(14)

equation (5) is then balanced trivially and then equations (6 – 14) lead to the following form:

\[
\frac{\partial P}{\partial \xi} = \frac{f'^2}{\xi + k},
\]

(15)

\[
2n k + \frac{(n + 1) \xi k \partial P}{2(\xi + k)} \frac{\partial P}{\partial \xi} = g_1 \left( f'' + \frac{f'''}{\xi + k} - \frac{f'}{\xi + k}^2 \right)
\]

\[
- \left( g_2 \frac{2n k + (n + 1) \xi \xi}{2(\xi + k)} \right) f'' + \frac{g_2}{2(\xi + k)} f'''
n + \frac{g_2}{2(\xi + k)} f''',
\]

(16)

\[
\frac{1}{Pr} \left( g_3 \theta'' + \frac{\theta'}{\xi + k} \right) + \frac{k}{\xi + k} \frac{1}{2} \left( n + 1 \right) f \theta' = 0,
\]

(17)

\[
\frac{1}{Sc} \left( \Phi'' + \frac{\Phi'}{\xi + k} \right) + \frac{k}{\xi + k} \frac{1}{2} \left( n + 1 \right) f \Phi' - K_1 \Phi h^2 = 0,
\]

(18)

\[
\frac{\delta}{Sc} \left( h'' + \frac{h'}{\xi + k} \right) + \frac{k}{\xi + k} \frac{1}{2} \left( n + 1 \right) f h' + K_1 \Phi h^2 = 0,
\]

(19)

with

\[
\begin{cases}
g_3 \beta \theta' + g_2 \left( \frac{k}{\xi + k} \right) Pr \left( \frac{n + 1}{2} f + \frac{n - 1}{2} \xi f' \right) = 0, \\
f = 0, \quad f' = 1, \quad \theta = 0, \quad \Phi = K_2 \Phi, \quad \delta h' = -K_2 \Phi \text{ at } \xi = 0, \\
f' \to 0, \quad f'' \to 0, \quad \theta \to 1, \quad \Phi \to 1, \quad h \to 0 \text{ as } \xi \to \infty.
\end{cases}
\]

(20)

In the above,
\[
g_1 = \left\{ \frac{1}{(1 - \phi_1)^{2.5}(1 - \phi_2)^{2.5}} \right\},
\]
\[
g_2 = \left\{ (1 - \phi_2) \left( (1 - \phi_1) + \phi_1 \frac{\rho_2}{\rho_f} \right) + \phi_2 \frac{\rho_1}{\rho_f} \right\},
\]
\[
g_3 = \left\{ \frac{\frac{k_{x_1} + (m - 1)k_f}{k_{x_1} - k_f}(m - 1)\phi_1(k_f - k_{x_1})}{(k_{x_1} - k_f)(1 - m)} + \phi_2 k_f - k_{x_1} \right\},
\]
\[
g_4 = (1 - \phi_2)(1 - \phi_1) + \frac{\rho_{p_f} \phi_1}{\rho_{p_f} \phi_1} + \phi_2 \frac{\rho_{p_f} \phi_1}{\rho_{p_f} \phi_1}.
\]

The Schmidt number, the Prandtl number, the dimensionless curvature variable, the melting variable, the ratio of the diffusion coefficients, the homogeneous reaction variable and the heterogeneous reaction variable are denoted by \( Sc \), \( Pr \), \( k \), \( \beta \), \( \delta \), \( K_1 \) and \( K_2 \), respectively. The dimensionless variables that appeared in the equations above can be defined as follows:

\[
k = R \sqrt{\frac{c_1 s^{\alpha-1}}{v_f}}, \quad \beta = \frac{c_p(T_c - T_m)}{\lambda_1 + c_s(T_m - T_0)},
\]
\[
Pr = \frac{\mu_c c_p}{k_f}, \quad \delta = \frac{D_{Bi}}{D_{A_1}}, \quad K_1 = \frac{a_1^{2.5} k_f}{c_1 s^{\alpha-1}},
\]
\[
Sc = \frac{v_f}{D_{A_1}}, \quad K_2 = \frac{k_e}{D_{A_1} \sqrt{c_1 s^{\alpha-1}}}.
\]

By simplifying equations (15) and (16), we obtain:

\[
f'''' + \frac{1}{g_1} \left[ 2f'''' + \frac{f''''}{(\xi + k)^2} + \frac{f'''}{\xi + k} \right]
- \left( \frac{3n}{2} - g_2 + \frac{\xi}{\xi + k} \right) \frac{k}{\xi + k} f''
- \frac{g_2 n + 1}{2} \xi + k \xi f'' + \frac{g_2 n + 1}{2} k \xi f'''
- \frac{g_2 n + 1}{2} k \xi f'' + \frac{g_2 n + 1}{2} (\xi + k)^2 f'''
- \frac{g_2(n + 1)}{2} k \xi f''
= 0,
\]

where the pressure from equation (16) is given as:

\[
P = \frac{f'''''}{2nk} \left[ g_1 f'''' + \frac{f'''''}{\xi + k} \right] - \frac{f''''}{\xi + k}
- \left( \frac{g_2 n + 1}{2} \xi + k \right) \frac{f''''}{\xi + k}
+ \frac{g_2 (n + 1)k}{2(\xi + k)^2} f''''
\]

For equal diffusion coefficients, where \( D_{A_1} = D_{Bi} \) and thus \( \delta = 1 \),

\[
\Phi(\xi) + h(\xi) = 1.
\]

Equations (18) and (19) then yield

\[
\frac{1}{Sc} \left( \Phi'' + \frac{\Phi}{\xi + k} \right) + \frac{k}{\xi + k} \left( \frac{n + 1}{2} \right) f\Phi'
- K_1 \Phi(1 - \Phi)^2 = 0,
\]

with

\[
\Phi'(0) = K_2 \Phi(0), \quad \Phi(\infty) \to 1.
\]

**Physical quantities**

The skin friction coefficient \( C_f \) and the local Nusselt number \( Nu_t \) can be given in the following forms:

\[
C_f = \frac{2\tau_w}{\rho w_u^2}, \quad Nu_t = \frac{q_w}{k_f(T_m - T_c)}.
\]

where \( \tau_w \) and \( q_w \) represent the shear stress and the heat flux, respectively, i.e.:

\[
\tau_w = \mu c_s \frac{\partial w}{\partial r} \bigg|_{r = 0}, \quad q_w = -K_{huf}(\frac{\partial T}{\partial r}) \bigg|_{r = 0},
\]

and the dimensionless quantities included above are defined as

\[
\frac{C_f \sqrt{Re_s}}{2} = \frac{1}{(1 - \phi_1)^{2.5}(1 - \phi_2)^{2.5}} (f''(0) - \frac{1}{k} f'(0)),
\]

\[
\frac{Nu_t}{\sqrt{Re_s}} = -K_{huf} \frac{\theta'(0)}{k_f}
\]

where \( Re_s = \frac{u_{ref}}{v_f} \) denotes the local Reynolds number.

**Results and discussion**

This section incorporates a graphical description of the dimensionless variables for the fluid velocity, the temperature, the nanofluid concentration, the skin friction and the Nusselt number. The solid lines in the figures represent the values for the nanofluid ((MoS\(_2\)/water)) and the dashed lines represent the values for the HBNF (MoS\(_2\) + GO/water). The values of the dimensionless
momentum boundary layer thickness in the HBNF
enhancement of the fluid velocity. The velocity and the
produced on the surface for the fluid and results in
larger radius of curvature. This reduces the drag force
because a larger curvature parameter corresponds to a
momentum boundary layer thickness. This occurs
correspond to enhancements of the velocity and the

Table 3. Effects of variations in \( \phi_1 \) (volume fraction) on skin friction coefficient (\( C_f \)) and Nusselt number (\( N_u \)) for nanoliquid and HBNF when \( \phi_2 = 0.00001, k = 5, \beta = 0.1, n = 2, K_1 = 0.5 \) and \( K_2 = Sc = 1.2 \).

| Quantity | Brick shape (\( m = 3.7 \)) | Cylinder shape (\( m = 4.9 \)) | Platelet shape (\( m = 5.7 \)) | Blade shape (\( m = 8.6 \)) |
|----------|-------------------------------|-------------------------------|-------------------------------|-------------------------------|
| \( \phi_1 \) | Nanofluid Hybrid nanofluid | Nanofluid Hybrid nanofluid | Nanofluid Hybrid nanofluid | Nanofluid Hybrid nanofluid |
| \( C_f \) | 0.01 | 1.3801 1.3774 | 1.3801 1.3773 | 1.3800 1.3772 | 1.3799 1.3770 |
| | 0.012 | 1.3883 1.3855 | 1.3882 1.3854 | 1.3882 1.3853 | 1.3880 1.3851 |
| | 0.014 | 1.3965 1.3937 | 1.3964 1.3936 | 1.3963 1.3935 | 1.3962 1.3932 |
| | 0.012 | 0.6916 0.8683 | 0.6949 0.8683 | 0.6971 0.8717 | 0.7052 0.8837 |
| | 0.014 | 0.6947 0.8684 | 0.6994 0.8754 | 0.7025 0.8801 | 0.7136 0.8903 |
| \( N_u \) | 0.01 | 1.3801 1.3774 | 1.3801 1.3773 | 1.3800 1.3772 | 1.3799 1.3770 |
| | 0.012 | 1.3883 1.3855 | 1.3882 1.3854 | 1.3882 1.3853 | 1.3880 1.3851 |
| | 0.014 | 1.3965 1.3937 | 1.3964 1.3936 | 1.3963 1.3935 | 1.3962 1.3932 |
| | 0.012 | 0.6916 0.8683 | 0.6949 0.8683 | 0.6971 0.8717 | 0.7052 0.8837 |
| | 0.014 | 0.6947 0.8684 | 0.6994 0.8754 | 0.7025 0.8801 | 0.7136 0.8903 |

Table 4. Effects of variations in \( k \) on \( C_f \) and \( N_u \) for nanoliquid and HBNF when \( \phi_1 = \phi_2 = 0.01, \beta = 0.1, n = 2, K_1 = 0.5 \) and \( K_2 = Sc = 1.2 \).

| Quantity | Brick shape (\( m = 3.7 \)) | Cylinder shape (\( m = 4.9 \)) | Platelet shape (\( m = 5.7 \)) | Blade shape (\( m = 8.6 \)) |
|----------|-------------------------------|-------------------------------|-------------------------------|-------------------------------|
| \( k \) | Nanofluid Hybrid nanofluid | Nanofluid Hybrid nanofluid | Nanofluid Hybrid nanofluid | Nanofluid Hybrid nanofluid |
| \( C_f \) | 5 | 1.6565 1.7071 | 1.6564 1.7071 | 1.6564 1.7070 | 1.6564 1.7070 |
| | 10 | 1.5449 1.5925 | 1.5449 1.5924 | 1.5449 1.5924 | 1.5449 1.5923 |
| | 15 | 1.5100 1.5565 | 1.5100 1.5565 | 1.5100 1.5565 | 1.5100 1.5564 |
| \( N_u \) | 5 | 0.6916 0.8756 | 0.6949 0.8860 | 0.6971 0.8930 | 0.7052 0.9188 |
| | 10 | 0.4633 0.6231 | 0.4660 0.6325 | 0.4678 0.6387 | 0.4743 0.6619 |
| | 15 | 0.3796 0.5349 | 0.3822 0.5440 | 0.3839 0.5501 | 0.3902 0.5725 |

Variables (i.e. \( k = \beta = 0.1, \phi_1 = \phi_2 = 0.01, n = 2, K_1 = 0.4, K_2 = Sc = 1.2 \)) remain constant, except for the variables given in the figures. Tables 1-2 list the thermophysical and structural properties of both the nanofluid and the HBNF. Tables 3 - 4 compare the skin friction and the Nusselt number values for the (MoS\(_2\))/water nanofluid and the (MoS\(_2\) + GO/water) HBNF with different shape factors. Table 5 presents a comparison of the results of the current study with those given by Sajid et al.\(^{12}\) for \( k \).

### Velocity field

Figure 2 shows the influence of the curvature parameter \( k \) on the velocity profile (\( f'(\xi) \)). Higher values of \( k \) correspond to enhancements of the velocity and the momentum boundary layer thickness. This occurs because a larger curvature parameter corresponds to a larger radius of curvature. This reduces the drag force produced on the surface for the fluid and results in enhancement of the fluid velocity. The velocity and the momentum boundary layer thickness in the HBNF (MoS\(_2\) + GO/water) case are notably higher than the corresponding properties of the nanofluid (MoS\(_2\)/water). Figure 3 displays the effect of \( n \) (the power law index) on the velocities of the nanofluid (MoS\(_2\)/water) and the HBNF(MoS\(_2\) + GO/water). The velocity and the momentum boundary layer thickness for both fluids decrease in tandem with an increasing power law index \( n \). Physically, this indicates that the concavity of the sheet increases for larger values of \( n \). Therefore, a greater drag force is produced for the fluid movement and the velocity thus decreases. It is

Table 5. Comparison of skin friction coefficient (\( -\frac{1}{2}C_f(Re)^{0.5} \)) with the results of Sajid et al.\(^{12}\) in the limiting case for variations in \( k \) when \( \phi_1 = \phi_2 = K_1 = K_2 = \xi = 0, Pr = Sc = 1.0 \) and \( n = 1 \).

| \( k \) | Skin friction coefficient results from Sajid et al.\(^{12}\) | Calculated skin friction coefficient \( -\frac{1}{2}C_f(Re)^{0.5} \) this work |
|-------|-------------------------------------------------------------|-------------------------------------------------------------|
| 5.0   | 0.75763                                                     | 0.75421                                                     |
| 10    | 0.87349                                                     | 0.87200                                                     |
| 20    | 0.93561                                                     | 0.93322                                                     |
| 30    | 0.95686                                                     | 0.95432                                                     |
| 40    | 0.96759                                                     | 0.96150                                                     |
| 50    | 0.97405                                                     | 0.97322                                                     |
| 100   | 0.98704                                                     | 0.98624                                                     |
| 200   | 0.99356                                                     | 0.99098                                                     |
| 1000  | 0.99880                                                     | 0.99716                                                     |
however noted that the velocity for the \( \text{MoS}_2 + \text{GO}/\text{water} \) based nanofluid is higher when compared with that of the \( \text{MoS}_2/\text{water} \) based nanofluid. The effects of the shape factors (for shapes including bricks, cylinders, platelets and blades) represented by \( m \) on the velocity profile \( f'(\xi) \) for the \( \text{MoS}_2 + \text{GO}/\text{water} \) HBNF and the \( \text{MoS}_2/\text{water} \) nanofluid are presented in Figure 4. The velocities for the different shape factors are observed to show slight differences. The shape factors are observed to have a more prominent effect in the case of the hybrid nanoliquid than for the normal nanoliquid.

Temperature field

The effects of the curvature variable \( k \) on the temperature profiles of the \( \text{MoS}_2 + \text{GO}/\text{water} \) HBNF and the \( \text{MoS}_2/\text{water} \) nanofluid are shown in Figure 5. The temperature declines at larger \( k \) values for both fluids. Larger values of \( k \) correspond to a reduction of the resistive force for the fluid particles and consequently to a reduction in the heat produced as a result of friction. Therefore, the temperature of the fluid will then decrease. The temperature is observed to be higher for the \( \text{MoS}_2 + \text{GO}/\text{water} \) HBNF than for the nanofluid \( \text{MoS}_2/\text{water} \). Higher values of \( n \) correspond to increments in the temperature profile for both the \( \text{MoS}_2 + \text{GO}/\text{water} \) HBNF and the \( \text{MoS}_2/\text{water} \) nanofluid (see Figure 6). In fact, as the value of \( n \) increases, a resistive force develops in the fluid that causes increased internal friction between the fluid particles, which causes the temperature of the fluid to increase. Figure 7 illustrates the impact of the melting variable on the dimensionless temperature \( \theta(\xi) \). The temperatures of both the nanofluid and the HBNF decreased as \( \beta \) increased. A larger \( \beta \) value results in more heat shifting from the hot liquid to the melting surface as a result of convective flow. Furthermore, the temperature of the \( \text{MoS}_2/\text{water} \) nanofluid is higher when compared with that of the \( \text{MoS}_2 + \text{GO}/\text{water} \) HBNF. Figure 8
illustrates the influence of the shape factor $m$ on the temperature profile. The temperatures of both fluids decrease at larger values of $m$. This occurs because addition of nanoparticles to the fluid enhances the thermal conductivity of fluid and more heat is then emitted from the fluid; consequently, the fluid temperature then declines. The temperature decline in the case of the MoS$_2$/water nanofluid is seen to be more prominent than that in the MoS$_2$ + GO/water HBNF case. Furthermore, higher temperatures are noted for bricks with $m = 3.7$ (red lines), cylinders with $m = 4.9$ (green lines), platelets with $m = 5.7$ (magenta lines) and blades with $m = 8.6$ (black lines).

**Concentration field**

Higher values of the homogeneous and heterogeneous reaction variables cause the fluid concentration profiles to decline (see Figures 9-10). Higher values of the homogeneous reaction variable $K_1$ correspond to homogeneity of the fluid concentration within the boundary layer thickness. In contrast, a low value of $K_1$ results in homogeneity occurring near the surface of the curved sheet. The reaction rate at the plate surface is enhanced by reducing the heterogeneous reaction.
parameter $K_2$. The concentration is noted to be similar for both the nanofluid and hybrid fluid types. A larger Schmidt number tends to enhance the profile concentration (see Figure 11). The solutal layer thickness decreases when $Sc$ increases because of the inverse relationship between $Sc$ and the mass diffusivity. The concentrations are observed to be the same for both fluids. Figures 12-13 present the streamlines for the nanofluid and the HBNF, respectively. Figure 14 shows the isotherms for the HBNF. The normal temperature distribution through the fluid is also shown.

Table 3 shows that the magnitude of the skin friction is higher in the case of the hybrid nanoliquid when compared with that of the nanoliquid with variations in $\phi_1$. Furthermore, the skin friction has been noted to be higher for brick-shaped nanoparticles ($m = 3.7$) and lower for blade-shaped nanoparticles ($m = 8.6$). The heat transfer rate in the HBNF case is observed to have a higher magnitude than that of the nanoliquid for higher values of $\phi_1$. The heat transfer rate was observed to be lower for brick-shaped nanoparticles with $m = 3.7$ and higher for blade-shaped nanoparticles ($8.6$). Table 4 shows that the drag coefficient in the hybrid nanoliquid is higher than that in the normal nanoliquid. The drag coefficient for the blade-shaped nanoparticles ($m = 8.6$) is shown to be lower when compared with those for the other shape factors. The heat transfer rate for the HBNF is higher than that for the nanofluid (see Table 4). Table 4 shows that the heat transfer rate reaches its maximum value following addition of blade-shaped nanoparticles, while the addition of
brick-shaped nanoparticles causes it to reach a minimum value when compared with use of the other nanosized particles. The hydrodynamics boundary layer for the hybrid nanoliquid is observed to be higher than that for the normal nanoliquid (see Figure 4). The temperature in the HBNF case is also observed to be higher than that of the normal nanoliquid. Higher and lower temperatures are noted for brick-shaped and blade-shaped nanoparticles, respectively, when compared with the other shape factors (see Figure 8).

**Findings**

This analysis has disclosed the results for the flow of a molybdenum sulphide ($\text{MoS}_2$) and graphene oxide ($\text{GO}$) hybrid nanofluid toward a curled nonlinear curved stretching sheet. The heat and mass balance equations for the hybrid nanofluid ($\text{MoS}_2 + \text{GO}/\text{water}$) have been studied with respect to the effects of heterogeneous-homogeneous reactions and melting heat. The main points raised by the analysis are presented below.

- Larger values of the curvature $k$ and the shape factor enhance the flow velocity and the opposite trend is shown for increasing power index $n$. Additionally, the velocity for the hybrid nanofluid ($\text{MoS}_2 + \text{GO}/\text{water}$) is higher than that of the nanofluid ($\text{MoS}_2$).
- The fluid temperature declines at higher values of $k$, $\beta$ and $m$ and it also decreases with decreasing $n$ for both the nanofluid and the hybrid nanofluid. The thermal layer thickness and the temperature are observed to increase more for the nanoliquid than for the hybrid nanoliquid.
- Heterogeneous and homogeneous reaction variables show the same behavior for both fluids but the opposite was noted for the Schmidt number.
- The hybrid nanofluid ($\text{MoS}_2 + \text{GO}/\text{water}$) shows higher heat transfer and a higher drag coefficient versus $k$ when compared with the corresponding values for the nanofluid ($\text{MoS}_2$). The drag coefficient and the heat transfer at the surface are enhanced by increasing the volume fraction $\phi_1$ for both the hybrid nanofluid ($\text{MoS}_2 + \text{GO}/\text{water}$) and the nanofluid ($\text{MoS}_2$). The magnitude of the heat transfer in the hybrid nanofluid case is higher than that of the normal nanofluid for the same volume fraction while the reverse behavior is observed with respect to the skin friction.

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