Thermodynamic Properties of Kehagias-Sfetsos Black Hole &
KS/CFT Correspondence

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Abstract

We speculate on various thermodynamic features of the inner horizon ($H^-$) and outer horizons ($H^+$) of Kehagias-Sfetsos (KS) black hole (BH) in the background of Hořava Lifshitz gravity. We compute particularly the area product, area sum, area minus and area division of the BH horizons. We find that they all are not showing universal behavior whereas the product is a universal quantity [Pradhan P., Phys. Lett. B, 747 (2015) 64]. Based on these relations, we derive the area bound of all horizons. From the area bound we derive the entropy bound and irreducible mass bound for all the horizons ($H^\pm$). We also observe that the First law of BH thermodynamics and Smarr-Gibbs-Duhem relations do not hold for this BH. The underlying reason behind this failure due to the scale invariance of the coupling constant. Moreover, we compute the Cosmic-Censorship-Inequality for this BH which gives the lower bound for the total mass of the spacetime and it is supported by cosmic censorship conjecture. Finally, we discuss the KS/CFT (Conformal Field Theory) correspondence via a thermodynamic procedure.

1 Introduction

Recently the general relativity community and the string theory community have become quite interested in examining the thermodynamic features of $H^-$ and $H^+$ [1, 2, 3, 4, 5, 6, 8, 9]. Of particular interest are relations that are independent of mass, so called ADM (Arnowitt-Deser-Misner) mass, and then these relations are said to be “universal” in BH physics. They are novel in the sense that they involve the thermodynamic quantities defined at multi-horizons, i.e. the Cauchy (inner) horizon and event (outer) horizons of the spherically symmetric charged, axisymmetric charged and axisymmetric non-charged BH. For example, let us consider first spherically symmetric charged, axisymmetric charged and axisymmetric non-charged BH. For example, let us consider first spherically symmetric charged BH, i.e., Reissner Nordstrøm (RN) BH, the mass-independent relation for both the horizons ($H^\pm$) becomes

\[ A_+ A_- = (4\pi Q^2)^2 \quad \text{or} \quad S_+ S_- = (\pi Q^2)^2. \quad (1) \]

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For spinning non-charged BH, i.e. for Kerr BH, the mass independent relations are

\[ A_+ A_- = (8\pi J)^2 \text{ or } S_+ S_- = (2\pi J)^2 . \tag{2} \]

Finally, for charged spinning BH, these relations should read

\[ A_+ A_- = (8\pi J)^2 + (4\pi Q^2)^2 \]

or

\[ S_+ S_- = (2\pi J)^2 + (\pi Q^2)^2 . \tag{3} \]

Remarkably, all these thermodynamic relations are independent of the mass parameter therefore it should be treated as a universal quantity.

For the BPS (Bogomol'ni-Prasad-Sommerfield) class of BHs, the area product formula \[H^\pm\] should be written as

\[ A_+ A_- = (8\pi)^2 \left( \sqrt{N_L} + \sqrt{N_R} \right) \left( \sqrt{N_L} - \sqrt{N_R} \right) = N, \ N \in \mathbb{N}, N_L \in \mathbb{N}, N_R \in \mathbb{N} . \tag{4} \]

where the integers \(N_L\) and \(N_R\) should be defined as excitation numbers of the left- and right- moving sectors of a weakly-coupled 2D conformal field theory (CFT). Resultantly, the entropy product formula of \(H^\pm\) becomes

\[ S_+ S_- = (2\pi)^2 \left( \sqrt{N_L} + \sqrt{N_R} \right) \left( \sqrt{N_L} - \sqrt{N_R} \right) = N, \ N \in \mathbb{N}, N_L \in \mathbb{N}, N_R \in \mathbb{N} . \tag{5} \]

This implies that the product of the entropy of \(H^\pm\) is an integer quantity \[10\].

The product formulae that we would like to derive in this work, either area or entropy product of inner horizon and outer horizons could be used to determine whether the corresponding Bekenstein-Hawking entropy may be written as a Cardy formula, therefore providing some evidences for a CFT description of the corresponding microstates \[3, 4, 11\]. This boosts the study of the properties of the inner horizon thermodynamics in contrast with the outer horizon thermodynamics.

In our previous study \[7\], we derived the surface area product, BH entropy product, surface temperature product, Komar energy product and specific-heat product for this BH. Besides the area or entropy product it should be known what happens in case of area sum, area minus and area division. For this reason we extend our study by computing area sum, entropy sum, temperature sum and specific-heat sum of all the horizons. We expect that the quantization area product formula that we have found from our
previous investigation and from present study provides a strong indication that there exists an universal near-horizon structure for more general class of BHs. This indicates the possibility that the microscopic degrees of freedom may admit a dual field theoretic explanation that generalizes the 2D CFT duals.

Thus in this Letter, we wish to examine various thermodynamic features (besides the area or entropy product) of Kehagias-Sfetsos BH [12] in Hořava Lifshitz gravity [13,14,15]. We have considered both the inner horizon and outer horizons to further understanding the microscopic nature of BH entropy both interior and exterior. Moreover using these relations, we derive the area bound of all horizons. From area bound we derive entropy bound and irreducible mass bound for both the horizons.

One aspect that has not been studied previously is so called the Cosmic-Censorship-Inequality or the Cosmic Censorship Bound [17]. It should require the cosmic-censorship hypothesis [16] (See [18, 19, 20, 21, 22]) and which is an important inequality in general relativity which relates the total mass of the spacetime in terms of the $H^+$ area, and for Schwarzschild BH it should be minimum i.e.

$$M \geq \sqrt{\frac{A_+}{16\pi}}. \tag{6}$$

This brilliant idea was first given by Penrose in 1973 [16].

This inequality has an important implication in BH physics that it indicates the lower bound on the energy for a time-symmetric initial Cauchy data set which satisfies the Einstein equations, and which has also satisfied the dominant energy condition and which has no naked singularities.

The structure of the paper is as follows. In the second section, we shall describe various thermodynamic features of Kehagias-Sfetsos BH in Hořava Lifshitz gravity, we also calculate the different thermodynamic bound in different subsections. In the third section, we discuss the KS/CFT correspondence using thermodynamic procedure. Finally, we conclude our discussions in the last section.

2 Thermodynamic Properties of KS BH in Hořava Lifshitz gravity

At Lifshitz point, Hořava [13,14,15] has given a beautiful theory for general relativity which is renormalizable and UV complete. It can be reduced to Einstein’s general relativity which is so called Riemannian Penrose Inequality. It was first examined and proved by Huisken et al. [24]. This inequality has an important application in gravitational collapse and using Cauchy data it could be solved the Einstein equations. Finally, it has another interesting application to solve the Yamabe problem [18]. It should be noted that Riemannian Penrose Inequality satisfied the Riemannian positive mass theorem [23].
relativity at large scales for the value of dynamical coupling constant $\lambda = 1$. We have not mentioned here in detail the ADM formalism because it has been already mentioned in [7]. Since we are interested in this work to study the thermodynamic properties of Kehagias-Sfetsos (KS) BH [12] in Hořava Lifshitz (HL) gravity, thus the metric of KS BH [25, 26, 12, 27, 28, 29] is given by

$$ ds^2 = -F(r)dt^2 + \frac{dr^2}{F(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2). $$

(7)

where,

$$ F(r) = 1 - \sqrt{4\mathcal{M}\omega r + \omega^2 r^4} + \omega^2 r^2, $$

(8)

and $\mathcal{M}$ is an integration constant derived from equations of motion of KS action. This constant is treated as “mass” parameter in HL gravity. For $r \gg (\mathcal{M}/\omega)^{\frac{1}{3}}$, one obtains the result of a Schwarzschild BH.

The BH horizons occur at $F(r) = 0$:

$$ r_\pm = \mathcal{M} \pm \sqrt{\mathcal{M}^2 - \frac{1}{2\omega}}. $$

(9)

where $r_+$ is the event horizon and $r_-$ is the Cauchy horizon respectively. As long as

$$ \mathcal{M}^2 - \frac{1}{2\omega} \geq 0, $$

(10)

then the KS metric describes a BH, otherwise it has a naked singularity. When $\mathcal{M}^2 - \frac{1}{2\omega} = 0$, we find the extremal KS BH.

The product and sum of horizon radii become

$$ r_+ r_- = \frac{1}{2\omega} \quad \text{and} \quad r_+ + r_- = 2\mathcal{M}. $$

(11)

The area [7] of this BH is given by

$$ A_\pm = 4\pi \left(2\mathcal{M}r_\pm - \frac{1}{2\omega}\right). $$

(12)

Their product [7] and sum yield

$$ A_+ A_- = \frac{4\pi^2}{\omega^2} \quad \text{and} \quad A_+ + A_- = 4\pi \left(4\mathcal{M}^2 - \frac{1}{\omega}\right). $$

(13)

It is remarkable that the area product of KS BH is independent of mass but the area sum is not independent of the mass parameter.
For completeness, we further compute the area minus and area division:

\[ A_+ - A_- = \pm 16\pi M \sqrt{M^2 - \frac{1}{2\omega}}. \]  

(14)

and

\[ \frac{A_+}{A_-} = \frac{r_+^2}{r_-^2}. \]  

(15)

Again, the sum of area inverse is found to be

\[ \frac{1}{A_+} + \frac{1}{A_-} = \frac{\omega^2}{\pi} \left( 4M^2 - 1 \right). \]  

(16)

and the minus of area inverse is computed to be

\[ \frac{1}{A_+} - \frac{1}{A_-} = \mp \frac{4\omega^2 M}{\pi} \sqrt{M^2 - \frac{1}{2\omega}}. \]  

(17)

It indicates that they are all mass dependent relations.

Likewise, the entropy product and entropy sum of \( H^\pm \) become

\[ S_- S_+ = \frac{\pi^2}{4\omega^2} \text{ and } S_- + S_+ = \pi \left( 4M^2 - \frac{1}{\omega} \right). \]  

(18)

For record, we also compute the entropy minus of \( H^\pm \) as

\[ S_+ - S_- = \pm 4\pi M \sqrt{M^2 - \frac{1}{2\omega}}. \]  

(19)

and the entropy division of \( H^\pm \) as

\[ \frac{S_+}{S_-} = \frac{r_+^2}{r_-^2}. \]  

(20)

Again, the sum of entropy inverse is found to be

\[ \frac{1}{S_+} + \frac{1}{S_-} = \frac{4\omega^2}{\pi} \left( 4M^2 - \frac{1}{\omega} \right). \]  

(21)

and the minus of entropy inverse is

\[ \frac{1}{S_+} - \frac{1}{S_-} = \mp \frac{16\omega^2 M}{\pi} \sqrt{M^2 - \frac{1}{2\omega}}. \]  

(22)
The Hawking [32] temperature on $\mathcal{H}^\pm$ reads

$$T_\pm = \frac{\omega(r_\pm - M)}{2\pi(1 + \omega r_\pm^2)}. \quad (23)$$

Their product [7] and sum yield

$$T_+ T_- = \frac{\omega (1 - 2M^2 \omega)}{2\pi^2(1 + 16M^2 \omega)}$$

and

$$T_+ + T_- = \frac{4\omega M (1 - 2M^2 \omega)}{\pi(1 + 16M^2 \omega)}. \quad (24)$$

It may be noted that surface temperature product and sum both depend on mass, thus they are not universal in nature. It is shown that for KS BH

$$T_+ S_+ + T_- S_- = \frac{8\omega M \sqrt{M^2 - \frac{1}{4\omega}}}{1 + 16\omega M^2}. \quad (25)$$

In general, this relation is for RN BH or Kerr BH [10]

$$T_+ S_+ + T_- S_- = 0. \quad (26)$$

It is in fact a mass independent (universal) relation and implies that $T_+ S_+ = -T_- S_-$ should be taken as a criterion whether there is a 2D CFT dual for the BHs in the Einstein gravity and other diffeomorphism gravity theories [11, 30]. This universal relation also indicates that the left and right central charges are equal i.e., $c_L = c_R = 12J$ which is holographically dual to 2D CFT [39]. But for KS BH, it follows from Eq. (25) that it is mass dependent and it does not vanishes as in Eq. (26) that means the central charges of the left moving sectors and right moving sectors are not equal. This is an interesting observation for KS BH in HL gravity wherea Einstein gravity does not possesses such type of features. It is also interesting to mentioned that except the area (or entropy) product and irreducible mass product all the thermodynamic relations of KS BH are mass dependent.

### 2.1 Smarr Formula for HL BH on $\mathcal{H}^\pm$

Smarr [31] had first derived the mass parameter can be expressed as in terms of area, angular momentum and charge for Kerr-Newman BH. On the otherhand, Hawking [32] has been speculated that the BH area always increases. Therefore the BH area is indeed
a constant quantity over the $\mathcal{H}^\pm$. Analogously, the area of both the horizons for KS BH in HL gravity is given by

$$A_\pm = 4\pi \left[ 2M^2 - \frac{1}{2\omega} \pm 2M\sqrt{M^2 - \frac{1}{2\omega}} \right]$$

(27)

Alternatively, the mass parameter could be expressed as, in terms of horizons ($\mathcal{H}^\pm$),

$$M^2 = \frac{A_\pm}{16\pi} + \frac{\pi}{4\omega^2 A_\pm} + \frac{1}{4\omega}.$$  

(28)

Form the above relation we can easily derived the *Cosmic-Censorship-Inequality* for KS BH

$$\mathcal{M} \geq \sqrt{\frac{A_\pm}{16\pi} + \frac{\pi}{4\omega^2 A_\pm} + \frac{1}{4\omega}}.$$  

(29)

Actually, Penrose derived it for $\mathcal{H}^+$ only. We here suggest this inequality is valid for $\mathcal{H}^-$ also.

After differentiation, we get the mass differential as

$$dM = T_\pm dA_\pm + \Phi_\pm^\omega d\omega$$

(30)

where,

$$T_\pm = \text{Effective surface tension for horizons}$$

$$= \frac{1}{\mathcal{M}} \left( \frac{1}{32\pi} - \frac{\pi}{8\omega^2 A_\pm^2} \right) = \frac{\partial \mathcal{M}}{\partial A_\pm}. $$

(31)

$$\Phi_\pm^\omega = \text{Effective potential for horizons due to } \omega$$

$$= -\frac{1}{\mathcal{M}} \left( \frac{\pi}{4\omega^3 A_\pm} + \frac{1}{8\omega^2} \right)$$

$$= -\frac{1}{4\omega^2 r_\pm} = \frac{\partial \mathcal{M}}{\partial \omega}.$$  

(32)

(33)

It is well known that for spherically symmetric RN BH, the Smarr-Gibbs-Duhem relation is satisfied by the following condition:

$$\frac{M}{2} - T_\pm S_\pm - \frac{\Phi_\pm}{2} Q = 0.$$  

(34)

where the symbols are used as usual for RN BH. But for KS BH this relation is

$$\frac{M}{2} - T_\pm S_\pm - \frac{\Phi_\pm^\omega}{2\omega} = \frac{2 + 5\omega r_\pm^2}{8\omega r_\pm(1 + \omega r_\pm^2)} \neq 0.$$  

(35)
It indicates that the Smarr-Gibbs-Duhem relation do not satisfied for KS BH in HL
gravity. Followed by the first law of BH thermodynamics which is also satisfied for this
BH. The reason should be due to the scale invariance of the coupling constant $\omega$. This
observation is essential here because we have not seen such a type of discussion in the
literature regarding the KS BH in HL gravity.

It should be emphasized that when we add the AdS term to this BH then the both
first law of thermodynamics and Smarr-Gibbs-Duhem relations have satisfied which has
been explicitly examined in [38]. Where the author derived the generalized Smarr relation
in AdS space which has include a pressure-volume term and the thermodynamic mass,
ADM mass, Brown-York mass and Holland-Ishibashi-Marolf mass could also be defined.
But it is interesting to note that with out pressure-volume term the first law and Smarr
relation do not satisfied at all. This is one of the key results of our work.

2.2 Area Bound of KS BH for $\mathcal{H}^\pm$

Using the above thermodynamic relations, we are now able to derive the entropy bound
of both the horizons. Using the inequality equation (10) one can obtain $M^2 \geq \frac{1}{2\omega}$. Since
$r_+ \geq r_-$, one can get $A_+ \geq A_- \geq 0$. Then the area product gives

$$A_+ \geq \sqrt{A_+ A_-} = \frac{2\pi}{\omega} \geq A_-.$$  \hspace{1cm} (36)

and the area sum gives

$$4\pi \left(4M^2 - \frac{1}{\omega}\right) = A_+ + A_- \geq$$

$$A_+ \geq \frac{A_+ + A_-}{2} = 2\pi \left(4M^2 - \frac{1}{\omega}\right).$$  \hspace{1cm} (37)

Thus the area bound for $\mathcal{H}^+$ satisfies

$$2\pi \left(4M^2 - \frac{1}{\omega}\right) \leq A_+ \leq 4\pi \left(4M^2 - \frac{1}{\omega}\right).$$  \hspace{1cm} (38)

and the area bound for $\mathcal{H}^-$ satisfies

$$0 \leq A_- \leq \frac{2\pi}{\omega}.$$  \hspace{1cm} (39)

2.3 Entropy Bound for $\mathcal{H}^\pm$

Analogously, as $r_+ \geq r_-$, one can get $S_+ \geq S_- \geq 0$. Then the entropy product gives

$$S_+ \geq \sqrt{S_+ S_-} = \frac{\pi}{2\omega} \geq S_-.$$  \hspace{1cm} (40)
and the entropy sum gives
\[ \pi \left( 4M^2 - \frac{1}{\omega} \right) = S_+ + S_+ \geq S_+ \geq \]
\[ \frac{S_+ + S_-}{2} = \pi \left( 2M^2 - \frac{1}{2\omega} \right). \]  
(41)
Thus the entropy bound for $\mathcal{H}^+$ satisfies
\[ \pi \left( 2M^2 - \frac{1}{2\omega} \right) \leq S_+ \leq \pi \left( 4M^2 - \frac{1}{\omega} \right). \]  
(42)
and the entropy bound for $\mathcal{H}^-$
\[ 0 \leq S_- \leq \frac{\pi}{2\omega}. \]  
(43)

2.4 Irreducible mass bound for $\mathcal{H}^\pm$

Christodoulou [33] had given a relation between surface area of the $\mathcal{H}^+$ and irreducible mass, which can be written as
\[ \mathcal{M}_{\text{irr},+}^2 = \frac{A_+}{16\pi} = \frac{S_+}{4\pi}. \]  
(44)
It is now well known that this relation is valid for CH too. That means
\[ \mathcal{M}_{\text{irr},-}^2 = \frac{A_-}{16\pi} = \frac{S_-}{4\pi}. \]  
(45)
Now the the product and sum of the irreducible mass for both the horizons are
\[ \mathcal{M}_{\text{irr},+}\mathcal{M}_{\text{irr},-} = \frac{1}{8\omega} \]
and
\[ \mathcal{M}_{\text{irr},+}^2 + \mathcal{M}_{\text{irr},-}^2 = \mathcal{M}^2 - \frac{1}{4\omega}. \]  
(46)
From the area bound, we get the irreducible mass bound for KS BH
\[ \frac{\sqrt{4M^2 - \frac{1}{\omega}}}{2\sqrt{2}} \leq \mathcal{M}_{\text{irr},+} \leq \frac{\sqrt{4M^2 - \frac{1}{\omega}}}{2}. \]  
(47)
and
\[ 0 \leq \mathcal{M}_{\text{irr},-} \leq \sqrt{\frac{1}{8\omega}}. \]  
(48)
Eq. [47] is nothing but the Penrose inequality, which is the first geometric inequality for BHs [21].
2.5 Temperature Bound for $\mathcal{H}^\pm$

In BH thermodynamics, temperature is an important parameter. So there must exist temperature bound relation on the horizons. As is when $r_+ \geq r_-$, one must obtain $T_+ \geq T_- \geq 0$. Then the temperature product gives

$$T_+ \geq \sqrt{T_+ T_-} = \sqrt{\frac{\omega (1 - 2M^2\omega)}{2\pi^2 (1 + 16M^2\omega)}} \geq T_-.$$  \hspace{1cm} (49)

and the temperature sum gives

$$\frac{4\omega M (1 - 2M^2\omega)}{\pi (1 + 16M^2\omega)} = T_+ + T_- \geq T_+ \geq \frac{2\omega M (1 - 2M^2\omega)}{\pi (1 + 16M^2\omega)}. \hspace{1cm} (50)$$

Thus, the temperature bound for $\mathcal{H}^+$

$$\frac{2\omega M (1 - 2M^2\omega)}{\pi (1 + 16M^2\omega)} \leq T_+ \leq \frac{4\omega M (1 - 2M^2\omega)}{\pi (1 + 16M^2\omega)}. \hspace{1cm} (51)$$

and the temperature bound for $\mathcal{H}^-$

$$0 \leq T_- \leq \sqrt{\frac{\omega (1 - 2M^2\omega)}{2\pi^2 (1 + 16M^2\omega)}}. \hspace{1cm} (52)$$

2.6 Bound on heat capacity $C_\pm$ for $\mathcal{H}^\pm$

In BH thermodynamics, the specific heat can be defined as

$$C_\pm = \frac{\partial M}{\partial T_\pm}. \hspace{1cm} (53)$$

which is an important parameter to determine the thermodynamic properties in BH physics. In our previous work\cite{7}, we derived in detail the expression for specific heat for both the horizons. It is given by

$$C_\pm = \frac{2\pi (2\omega r_\pm^2 - 1)(1 + \omega r_\pm^2)^2}{\omega (1 + 5\omega r_\pm^2 - 2\omega^2 r_\pm^4)}. \hspace{1cm} (54)$$

Their product \cite{7} and sum on $\mathcal{H}^\pm$ yields

$$C_+ C_- = \frac{\pi^2 (1 - 2M^2\omega)(1 + 16M^2\omega)^2}{2\omega^2 (2 + 13\omega M^2 - 16\omega^2 M^4)}. \hspace{1cm} (55)$$
and
\[ C_+ + C_- = \]
\[ \frac{\pi}{\omega^2} \frac{(128\omega^4\mathcal{M}^6 + 8\omega^3\mathcal{M}^4 - 42\omega^2\mathcal{M}^2 + 4\omega\mathcal{M}^2 + 2\omega - 1)}{(2 + 13\omega\mathcal{M}^2 - 16\omega^2\mathcal{M}^4)}. \] (56)

Using \( \mathcal{M}^2 \geq \frac{1}{\omega} \) with the product of heat capacity and the sum of heat capacity, we get the bound on heat capacity for both the horizons. For \( \mathcal{H}^+ \)
\[ C_+ \leq \]
\[ \frac{\pi}{\omega^2} \frac{(128\omega^4\mathcal{M}^6 + 8\omega^3\mathcal{M}^4 - 42\omega^2\mathcal{M}^2 + 4\omega\mathcal{M}^2 + 2\omega - 1)}{(2 + 13\omega\mathcal{M}^2 - 16\omega^2\mathcal{M}^4)}. \] (57)

and for \( \mathcal{H}^- \)
\[ 0 \leq C_- \leq \]
\[ \sqrt{\frac{\pi}{\omega^2} \frac{(128\omega^4\mathcal{M}^6 + 8\omega^3\mathcal{M}^4 - 42\omega^2\mathcal{M}^2 + 4\omega\mathcal{M}^2 + 2\omega - 1)}{(2 + 13\omega\mathcal{M}^2 - 16\omega^2\mathcal{M}^4)}}. \] (58)

It should be mentioned that all the above thermodynamic formulae might be suggested the possibility of an explanation for the microscopic nature of such BHs in terms of a field theory in more than two dimensions.

3 KS/CFT Correspondence

In this section, we would like to prove that the central charges \( c_R \) and \( c_L \) of the right and left moving sectors of the dual CFT in KS/CFT correspondence are not same. To do this we should calculate the thermodynamic parameters in left moving sectors and right moving sectors by using the definitions of \( \beta_{R,L} = \beta_+ \pm \beta_- \), \( \beta_\pm = \frac{1}{T_\pm} \), \( \Phi_{\omega R,L} = \frac{\beta_+ \Phi_+ \pm \beta_- \Phi_-}{2\beta_{R,L}} \)
and \( S_{R,L} = \frac{(S_+ \mp S_-)}{2} [34, 35, 36] \). Now we could easily derive the temperature and entropy for left moving sectors and right moving sectors as
\[ T_L = \frac{1}{8\pi \mathcal{M}}, \quad T_R = \frac{\omega}{\pi} \sqrt{\mathcal{M}^2 - \frac{1}{2\omega}} \]
\[ S_L = \frac{\pi}{2} \left( 4\mathcal{M}^2 - \frac{1}{\omega} \right), \quad S_R = 2\pi \mathcal{M} \sqrt{\mathcal{M}^2 - \frac{1}{2\omega}} \]
\[ \Phi_{\omega R} = \frac{1}{64\mathcal{M} \omega^2}, \quad \Phi_{\omega R} = -\frac{3\mathcal{M}}{4\omega (1 + 4\omega \mathcal{M}^2)}. \] (59)
The first law of thermodynamics could be rewritten as in terms of right and left moving sectors of dual CFT

\[
\frac{dM}{2} = T_R dS_R + \Phi_R^\omega d\omega . \tag{60}
\]

\[
= T_L dS_L + \Phi_L^\omega d\omega . \tag{61}
\]

Using Eq. (60) & Eq. (61), one could determine the first law of thermodynamics for left moving sectors and right moving sectors of dual CFT

\[
d\omega = \frac{T_L}{\Phi_R^\omega - \Phi_L^\omega} dS_L - \frac{T_R}{\Phi_R^\omega - \Phi_L^\omega} dS_R . \tag{62}
\]

Using above Eq. (62), one can find the dimensionless temperature of the left and right moving sectors of the dual CFT correspondence

\[
T_L^\omega = \frac{T_L}{\Phi_R^\omega - \Phi_L^\omega}, \quad T_R^\omega = \frac{T_R}{\Phi_R^\omega - \Phi_L^\omega} . \tag{63}
\]

For KS BH, the values are

\[
T_L^\omega = - \frac{8\omega^2 (1 + 4\omega M^2)}{\pi (1 + 52\omega M^2)}. \tag{64}
\]

&

\[
T_R^\omega = - \frac{64\omega^3 M \sqrt{M^2 - \frac{1}{2\omega}}}{\pi (1 + 52\omega M^2)}. \tag{65}
\]

Now we compute the central charges \[11, 37\] in left and right moving sectors of the KS/CFT correspondence using the Cardy formula

\[
S_L^{\omega} = \frac{\pi^2}{3} c_L T_L^\omega, \quad S_R^{\omega} = \frac{\pi^2}{3} c_R T_R^\omega . \tag{66}
\]

Therefore the central charges of dual CFT are

\[
c_L^{\omega} = \frac{3 (1 - 4\omega M^2)(1 + 52\omega M^2)}{16\omega^3 (1 + 4\omega M^2)}. \tag{67}
\]

&

\[
c_R^{\omega} = - \frac{3 (1 + 52\omega M^2)}{32\omega^3}. \tag{68}
\]

From the above calculation we prove that

\[
c_L \neq c_R . \tag{69}
\]
Now we could see what happens in the extreme limit?

\[
T_L = \frac{\sqrt{2} \omega}{8\pi}, \quad T_R = 0
\]
\[
S_L = \frac{\pi}{2\omega}, \quad S_R = 0
\]
\[
\Phi_L = \frac{\sqrt{2} \omega}{64\sqrt{2} \omega}, \quad \Phi_R = -\frac{1}{4\sqrt{2} \omega}. \tag{70}
\]

Analogously the central charges are

\[
c_L = -\frac{27}{16\omega^3}. \tag{71}
\]

&

\[
c_R = -\frac{81}{32\omega^3}. \tag{72}
\]

Thus, the ratio of \(c_L\) and \(c_R\) is given by

\[
\frac{c_L}{c_R} = \frac{2}{3}. \tag{73}
\]

As we have said earlier in Eq. 26 and in Eq. 69 the central charges are not equal thus we could not find macroscopic Bekenstein-Hawking entropy of extreme KS BH. This is an interesting result of KS BH in HL gravity.

4 Discussion:

In order to understand the BH entropy (both outer as well as inner) at the microscopic level, we studied thermodynamic properties of KS BH in HL gravity. We computed various thermodynamic formula for this BH. We speculated that area sum, area minus and area division are mass dependent quantities, whereas the product is a mass independent quantity.

Based on these relations, we computed area bound, entropy bound, irreducible mass bound, temperature bound and specific-heat bound. The upper area bound of outer horizon is actually the Penrose-like inequality in BH mechanics. Due to the scale invariance of the coupling constant parameter \(\omega\), we showed that the First law of BH thermodynamics and Smarr-Gibbs-Duhem relations do not satisfied for this BH. Finally, we derived the Cosmic-Censorship-Inequality for this BH which has an important implications in Cosmic-Censorship-Conjecture.

We proved that the central charges of KS BH in HL gravity are not equal and do not produce the macroscopic Bekenstein-Hawking entropy of the extreme KS BH which is a drawback of HL gravity.
In conclusion, these thermodynamic product formulae suggests further evidence for the crucial role of both inner horizon and outer horizon for understanding the microscopic nature of BH entropy (both interior and exterior) which is the prime aim in quantum gravity.

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