Multiparametric geometry of numbers and its application to splitting transference theorems

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Abstract
In this paper we consider a multiparametric version of Wolfgang Schmidt and Leonard Summerer’s parametric geometry of numbers. We apply this approach in two settings: the first one concerns weighted Diophantine approximation, the second one concerns Diophantine exponents of lattices. In both settings we use multiparametric approach to define intermediate exponents. Then we split the weighted version of Dyson’s transference theorem and an analogue of Khintchine’s transference theorem for Diophantine exponents of lattices into chains of inequalities between the intermediate exponents we define based on the intuition provided by the parametric approach.

Keywords Parametric geometry of numbers · Diophantine approximation with weights · Lattice exponents · Transference principle

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1 Introduction
Parametric geometry of numbers was introduced several years ago by Schmidt and Summerer in [1,2]. It allowed to look at Diophantine problems from a different angle and gave a strong impulse to the development of Diophantine approximation. In this paper we consider a slightly more general setting, which we prefer to call multiparametric.

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1.1 Multiparametric geometry of numbers

Let $\Lambda$ be a full rank lattice in $\mathbb{R}^d$ of covolume 1. Let $| \cdot |$ denote the supremum norm. Set

$\mathcal{B} = \left\{ z \in \mathbb{R}^d \mid |z| \leq 1 \right\}, \quad \mathcal{T} = \left\{ \tau = (\tau_1, \ldots, \tau_d) \in \mathbb{R}^d \mid \tau_1 + \cdots + \tau_d = 0 \right\},$

and for each $\tau \in \mathcal{T}$ set

$\mathcal{B}_\tau = \text{diag}(e^{\tau_1}, \ldots, e^{\tau_d})\mathcal{B}.$

Let $\lambda_k(\mathcal{B}_\tau) = \lambda_k(\mathcal{B}_\tau, \Lambda), k = 1, \ldots, d,$ denote the $k$th successive minimum, i.e. the infimum of positive $\lambda$ such that $\lambda \mathcal{B}_\tau$ contains at least $k$ linearly independent vectors of $\Lambda.$ Finally, for each $k = 1, \ldots, d,$ let us set

$L_k(\tau) = L_k(\Lambda, \tau) = \log \left( \lambda_k(\mathcal{B}_\tau, \Lambda) \right), \quad S_k(\tau) = S_k(\Lambda, \tau) = \sum_{1 \leq j \leq k} L_j(\Lambda, \tau).$

Many problems in Diophantine approximation can be interpreted as questions concerning the asymptotic behaviour of $L_k(\tau)$ and $S_k(\tau).$ Different problems require different subsets of $\mathcal{T}$ along which $\tau$ is supposed to tend to infinity. In this paper we show that Diophantine approximation with weights requires considering one-dimensional and two-dimensional subspaces of $\mathcal{T},$ whereas the whole $\mathcal{T}$ equipped with appropriate exhaustion leads us to Diophantine exponents of lattices. In both those settings there exist transference theorems. A particular aim of this paper is to apply parametric geometry of numbers to split those transference theorems into chains of inequalities between intermediate exponents. The first result of this kind belongs to Laurent [3] (see also [4]), who developed some ideas of Schmidt [5] and obtained a splitting of Khintchine’s transference inequalities. It appears that the language of parametric geometry of numbers is very well fit for the purpose of splitting transference theorems. The main tools are provided by Proposition 2.4 (see Sect. 2), which we believe to be of interest in itself.

In the case considered by Schmidt and Summerer there is a very strong result by Roy [6], which allows considering instead of $L_k(\tau), S_k(\tau)$ certain functions obeying rather simple formal laws. It is a challenging problem to obtain an analogue of Roy’s theorem for the multiparametric setting. However, for the purposes of the current paper, statements like Proposition 2.4 describing local behaviour of $L_k(\tau)$ and $S_k(\tau)$ are already enough.
1.2 Diophantine approximation with weights

Given a matrix

$$
\Theta = \begin{pmatrix}
\theta_{11} & \cdots & \theta_{1m} \\
\vdots & \ddots & \vdots \\
\theta_{n1} & \cdots & \theta_{nm}
\end{pmatrix} \in \mathbb{R}^{n \times m}, \ n + m = d,
$$

and a real $\gamma$, it is questioned in the most classical ‘non-weighted’ setting whether the system of inequalities

$$\begin{cases}
|x|^m \leq t \\
|\Theta x - y|^n \leq t^{-\gamma}
\end{cases} \quad (1)
$$

admits nonzero solutions in $(x, y) \in \mathbb{Z}^m \oplus \mathbb{Z}^n$ for large values of $t$. Here, as before, $|\cdot|$ denotes the supremum norm.

In multiplicative Diophantine approximation the supremum norm is replaced with the geometric mean. For instance, the famous Littlewood conjecture, one of the most challenging problems in multiplicative Diophantine approximation, asserts that for every $\theta_1, \theta_2 \in \mathbb{R}$ and every $\varepsilon > 0$ there are arbitrarily large $t$ such that the system of inequalities

$$\begin{cases}
|x| \leq t \\
|\theta_1 x - y_1| \cdot |\theta_2 x - y_2| \leq e^{-1} t
\end{cases} \quad (2)
$$

admits nonzero solutions in $(x, y_1, y_2) \in \mathbb{Z}^3$.

Diophantine approximation with weights is in a sense an intermediate step between those two settings. Given weights $\sigma = (\sigma_1, \ldots, \sigma_m) \in \mathbb{R}^m$, $\rho = (\rho_1, \ldots, \rho_n) \in \mathbb{R}^n$,

$$\sigma_1 \geq \cdots \geq \sigma_m > 0, \ \rho_1 \geq \cdots \geq \rho_n > 0, \ \sum_{j=1}^m \sigma_j = \sum_{i=1}^n \rho_i = 1,$$

the supremum norm is replaced with the weighted norms $|\cdot|_\sigma$ and $|\cdot|_\rho$,

$$|x|_\sigma = \max_{1 \leq j \leq m} |x_j|^{1/\sigma_j} \quad \text{for } x = (x_1, \ldots, x_m),$$

$$|y|_\rho = \max_{1 \leq i \leq n} |y_i|^{1/\rho_i} \quad \text{for } y = (y_1, \ldots, y_n).$$

Respectively, instead of (1), the system

$$\begin{cases}
|x|_\sigma \leq t \\
|\Theta x - y|_\rho \leq t^{-\gamma}
\end{cases} \quad (2)
$$

is considered. Clearly, when all the $\sigma_j$ are equal to $1/m$ and all the $\rho_i$ are equal to $1/n$, (2) turns into (1).
Definition 1.1 The weighted Diophantine exponent \( \omega_{\sigma, \rho}(\Theta) \) is defined as the supremum of real \( \gamma \) such that the system (2) admits nonzero solutions in \((x, y) \in \mathbb{Z}^{m+n}\) for some arbitrarily large \( t \).

It is well known that, as a rule, there is a relation between problems concerning \( \Theta \) and problems concerning the transposed matrix \( \Theta^\top \). This relation is provided by the so called transference principle discovered by Khintchine in [7]. Khintchine’s result concerns row and column matrices, and it was generalised to the case of arbitrary matrices by Dyson [9]. Recently, in paper [8], the following transference result was obtained. It generalises Dyson’s transference theorem to the weighted setting.

**Theorem 1.1** Set \( \omega = \omega_{\sigma, \rho}(\Theta) \) and \( \omega^\top = \omega_{\rho, \sigma}(\Theta^\top) \). Then

\[
\omega \geq \frac{(\sigma_m^{-1} - 1) + \rho_n^{-1} \omega^\top}{\sigma_m^{-1} + (\rho_n^{-1} - 1) \omega^\top}.
\]

In Sect. 3 we interpret (3) in terms of parametric geometry of numbers and split it into a chain of inequalities for intermediate exponents we define therein.

1.3 Diophantine exponents of lattices

For each \( z = (z_1, \ldots, z_d) \in \mathbb{R}^d \) let us set

\[
\Pi(z) = \prod_{1 \leq i \leq d} |z_i|^{1/d}.
\]

Let \( \Lambda \) be a full rank lattice in \( \mathbb{R}^d \) of covolume 1. Its **Diophantine exponent** is defined as

\[
\omega^x(\Lambda) = \sup \left\{ \gamma \in \mathbb{R} \right| \Pi(z) \leq |z|^{-\gamma} \text{ for infinitely many } z \in \Lambda \right\},
\]

where \(|\cdot|\) is again the supremum norm. It follows from Minkowski’s convex body theorem that \( \omega^x(\Lambda) \) is nonnegative for every \( \Lambda \).

Consider the dual lattice

\[
\Lambda^* = \left\{ w \in \mathbb{R}^d \right| \langle w, z \rangle \in \mathbb{Z} \text{ for each } z \in \Lambda \right\},
\]

where \( \langle \cdot, \cdot \rangle \) is the inner product.

In this setting the phenomenon of transference can also be observed. The following result was obtained in [10].

**Theorem 1.2** If one of the exponents \( \omega^x(\Lambda) \) or \( \omega^x(\Lambda^*) \) is zero, then so is the other. If they are nonzero, then

\[
1 + \omega^x(\Lambda)^{-1} \leq (d - 1)^2(1 + \omega^x(\Lambda^*)^{-1}).
\]

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In Sect. 4 we interpret (4) in terms of parametric geometry of numbers and split it into a chain of inequalities for intermediate exponents we define therein.

The rest of the paper is organised as follows. In Sect. 2 we study local properties of the functions \( L_k \) and \( S_k \). In Sects. 3 and 4 we apply the results of Sect. 2 to the theory of Diophantine approximation with weights and to the theory of Diophantine exponents of lattices, respectively. In both cases we define intermediate exponents of two types (Sects. 3.1–3.3 and 4.1–4.2) based on the intuition provided by the parametric approach and by the Schmidt–Summerer exponents that naturally arise within the parametric geometry of numbers. In Sects. 3.4 and 4.3 we perform the announced splitting of Theorems 1.1 and 1.2 respectively. Section 3.5 complements the case of weighted Diophantine approximation with a diagram demonstrating the geometry of the two-dimensional subspace of \( T \) corresponding to the given weights from the point of view of the transference phenomenon. Finally, in Sects. 3.6 and 4.4, we prove two series of transference relations for intermediate exponents of the first type. As a corollary of those relations, we obtain in Sect. 3.7 a transference theorem for the inhomogeneous weighted Diophantine approximation proved recently by Ghosh et al. [11].

2 Local properties of \( L_k \) and \( S_k \)

In this section we use Minkowski’s second theorem and Mahler’s theorem on successive minima of compound bodies to derive some important properties of \( L_k(\tau) \) and \( S_k(\tau) \).

For each \( \tau \in T \) set
\[
|\tau|_+ = \max_{1 \leq i \leq d} \tau_i, \quad |\tau|_- = |-\tau|_+ = -\min_{1 \leq i \leq d} \tau_i.
\]

Clearly,
\[
|\tau| = \max(|\tau|_-, |\tau|_+),
\]
\[
|\tau|_+/ (d - 1) \leq |\tau|_- \leq (d - 1) |\tau|_+ .
\]

Proposition 2.1 The functions \( L_k(\tau) \) enjoy the following properties:

(i) \( L_1(\tau) \leq \cdots \leq L_d(\tau) \);
(ii) \( 0 \leq -L_1(\tau) \leq |\tau|_+ + O(1) \);
(iii) \( L_d(\tau) \leq |\tau|_- + O(1) \);
(iv) each \( L_k(\tau) \) is continuous and piecewise linear.

Moreover, the implied constants in both \( O(1) \) depend only on \( \Lambda \).

Proof Statement (i) follows immediately from the definition of successive minima. The inequality \( L_1(\tau) \leq 0 \) is a corollary of Minkowski’s convex body theorem. The rest of (ii) and (iii) is provided by

\[
e^{-|\tau|_+} B_\tau \subset B \implies \lambda_1(e^{-|\tau|_+} B_\tau) \geq \lambda_1(B) \implies \lambda_1(B_\tau) \geq e^{-|\tau|_+} \lambda_1(B)
\]
and
\[ e^{\|\tau\|} \cdot B \supset B \implies \lambda_d (e^{\|\tau\|} \cdot B) \leq \lambda_d (B) \implies \lambda_d (B) \leq e^{\|\tau\|} \cdot \lambda_d (B). \]

Here, as before, we denote by \( B \) the cube that is the unit ball for the supremum norm.

Let us prove (iv). The fact that \( L_k(\tau) \) is continuous is easily verified. Let us show that \( L_k(\tau) \) is piecewise linear. For each nonzero \( z \in \mathbb{R}^d \) let us denote by \( \lambda_z (B) \) the infimum of all positive \( \lambda \) such that \( \lambda B \) contains \( z \), and set \( L_z(\tau) = \log (\lambda_z (B)) \). If \( z = (z_1, \ldots, z_d) \), then
\[ \lambda_z (B) = \max_{1 \leq i \leq d} (|z_i| e^{-\tau_i}), \quad L_z(\tau) = \max_{1 \leq i \leq d, z_i \neq 0} \left( \log |z_i| - \tau_i \right), \]
i.e. \( L_z(\tau) \) is continuous and piecewise linear. Furthermore, given an arbitrary \( \tau \in T \) and an arbitrary bounded neighbourhood \( U(\tau) \subseteq T \), there are finitely many points \( z \in \Lambda \) such that for every \( \tau' \in U(\tau) \) we have \( L_k(\tau') = L_z(\tau') \). Therefore, since \( L_k(\tau) \) is continuous, it is also piecewise linear.

**Proposition 2.2** The functions \( S_k(\tau) \) enjoy the following properties:

1. \(-\log d! \leq S_d(\tau) \leq 0;\)
2. \( k + 1 \leq S_k(\tau) \leq S_{k+1}(\tau) \leq \frac{d - k - 1}{d - k} S_k(\tau); \)
3. \((d - 1) S_1(\tau) \leq S_{d-1}(\tau) \leq S_1(\tau)/(d - 1); \)
4. \( S_{d-1}(\tau) = -L_d(\tau) + O(1). \)

**Proof** Statement (i) follows from Minkowski’s second theorem, which states that
\[ \frac{1}{d!} \leq \prod_{1 \leq k \leq d} \lambda_k (B) \leq 1. \]

Furthermore, statement (i) of Proposition 2.1 and statement (i) of the current Proposition imply
\[ S_k(\tau) \leq k L_{k+1}(\tau) \]
and
\[ S_k(\tau) + (d - k) L_{k+1}(\tau) \leq S_d(\tau) \leq 0. \]

Hence
\[ \frac{1}{k} S_k(\tau) \leq L_{k+1}(\tau) \leq \frac{-1}{d - k} S_k(\tau), \]
and (ii) follows.

Applying statement (ii) recursively, we get statement (iii).

As for statement (iv), it is an immediate corollary of statement (i). \( \square \)
We remind that $\Lambda^*$ denotes the dual lattice.

**Proposition 2.3** For every $\tau \in T$ we have

\[(i) \quad -\log d \leq L_k(\Lambda, \tau) + L_{d+1-k}(\Lambda^*, -\tau) \leq \log d!, \quad k = 1, \ldots, d; \]
\[(ii) \quad -k \log d \leq S_k(\Lambda, \tau) - S_{d-k}(\Lambda^*, -\tau) \leq (k + 1) \log d!, \quad k = 1, \ldots, d - 1. \]

**Proof** In his paper [12] Mahler proved that for a parallelepiped $B_\tau$ and its polar cross-polytope $B_\tau^\circ$ we have

\[1 \leq \lambda_k(B_\tau, \Lambda) \lambda_{d+1-k}(B_\tau^\circ, \Lambda^*) \leq d!. \]  

\[\text{(6)}\]

Since

\[B_\tau^\circ = (D_\tau B)^\circ = D_\tau^{-1} B^\circ = D_{-\tau} B^\circ, \]

we have

\[d^{-1} B_{-\tau} \subset B_\tau^\circ \subset B_{-\tau}. \]

Therefore, (6) implies

\[\frac{1}{d} \leq \lambda_k(B_\tau, \Lambda) \lambda_{d+1-k}(B_{-\tau}, \Lambda^*) \leq d!. \]

Taking the logarithm of all terms, we get (i).

Furthermore, statement (i) implies

\[-k \log d \leq S_k(\Lambda, \tau) + (S_d(\Lambda^*, -\tau) - S_{d-k}(\Lambda^*, -\tau)) \leq k \log d!, \]

\[\text{(7)}\]

whereas by statement (i) of Proposition 2.2

\[0 \leq -S_d(\Lambda^*, -\tau) \leq \log d!. \]

\[\text{(8)}\]

Summing up (7) and (8), we get (ii).

**Proposition 2.4** For every $\tau \in T$ we have

\[(i) \quad L_k(\Lambda, \tau) = -L_{d+1-k}(\Lambda^*, -\tau) + O(1), \quad k = 1, \ldots, d; \]
\[(ii) \quad S_k(\Lambda, \tau) = S_{d-k}(\Lambda^*, -\tau) + O(1), \quad k = 1, \ldots, d - 1; \]
\[(iii) \quad S_1(\Lambda, \tau) \leq \cdots \leq \frac{S_k(\Lambda, \tau)}{k} \leq \cdots \leq \frac{S_{d-1}(\Lambda, \tau)}{d - 1} \leq \frac{S_d(\Lambda, \tau)}{d} = O(1); \]
\[(iv) \quad \frac{S_1(\Lambda, \tau)}{d - 1} \geq \cdots \geq \frac{S_k(\Lambda, \tau)}{d - k} \geq \cdots \geq S_{d-1}(\Lambda, \tau). \]

Here we assume that the implied constants depend only on $d$.

**Proof** Statements (i), (ii) follow from Proposition 2.3. Statements (iii), (iv) follow from statements (i), (ii) of Proposition 2.2.
Corollary 2.1  For every $\tau \in T$ we have

$$S_1(\Lambda, \tau) \leq \frac{S_1(\Lambda^*, -\tau)}{d - 1} + O(1),$$

where the implied constant depends only on $d$.

As we shall see in the next two sections, Corollary 2.1 is the core of both Theorems 1.1 and 1.2. Evidently, statements (ii) and (iii) of Proposition 2.4 split Corollary 2.1 into a chain of inequalities between the corresponding values of $S_k$.

3 Weighted exponents and multiparametric geometry of numbers

3.1 Intermediate exponents

As in Sect. 1.2, let us fix an $n \times m$ real matrix $\Theta$ and weights $\sigma = (\sigma_1, \ldots, \sigma_m) \in \mathbb{R}^m$, $\rho = (\rho_1, \ldots, \rho_n) \in \mathbb{R}^n$ such that

$$\sigma_1 \geq \cdots \geq \sigma_m > 0, \quad \rho_1 \geq \cdots \geq \rho_n > 0, \quad \sum_{j=1}^m \sigma_j = \sum_{i=1}^n \rho_i = 1.$$

Let us supplement $\omega_{\sigma, \rho}(\Theta)$ with the following family of intermediate exponents.

**Definition 3.1**  Let $k$ be an integer, $1 \leq k \leq d$. We define the $k$th weighted Diophantine exponent $\omega_{\sigma, \rho}^{(k)}(\Theta)$ as the supremum of real $\gamma$ such that the system (2) admits $k$ linearly independent solutions in $(x, y) \in \mathbb{Z}^{m+n}$ for some arbitrarily large $t$.

Clearly, $\omega_{\sigma, \rho}(\Theta) = \omega_{\sigma, \rho}^{(1)}(\Theta)$.

**Definition 3.2**  Let $k$ be an integer, $1 \leq k \leq d$. We define the $k$th weighted uniform Diophantine exponent $\hat{\omega}_{\sigma, \rho}^{(k)}(\Theta)$ as the supremum of real $\gamma$ such that the system (2) admits $k$ linearly independent solutions in $(x, y) \in \mathbb{Z}^{m+n}$ for every $t$ large enough.

For every $k$ we have

$$\omega_{\sigma, \rho}^{(k)}(\Theta) \geq \hat{\omega}_{\sigma, \rho}^{(k)}(\Theta) \geq 0,$$

since every parallelepiped determined by (2) with $\gamma \leq 0$ and $t \geq 1$ contains a basis of $\mathbb{Z}^{m+n}$. Let us interpret the exponents just defined in terms of multiparametric geometry of numbers.

The choice of a lattice is rather standard for problems concerning systems of linear forms, and it does not depend on the weights. We set

$$\Lambda = \Lambda(\Theta) = \left( \begin{array}{cc} I_m \\ -\Theta I_n \end{array} \right) \mathbb{Z}^d, \quad d = m + n.$$
As for the subset of $\mathcal{T}$ along which $\tau$ is supposed to tend to infinity, one could say that it should be a one-dimensional subspace, and this would essentially be true. At least, it is literally true when the weights are trivial (i.e. when all the $\sigma_j$ are equal to $1/m$, and all the $\rho_i$ are equal to $1/n$; a detailed description of this case is given in [13]). However, if the weights are nontrivial, there appears a whole family of one-dimensional subspaces, all of which should be taken into account. Let us set

$$e_1 = e_1(\sigma, \rho) = (1 - d\sigma_1, \ldots, 1 - d\sigma_m, 1, \ldots, 1),$$

$$e_2 = e_2(\sigma, \rho) = (1, \ldots, 1, 1 - d\rho_n, \ldots, 1 - d\rho_1).$$

(9)

It is clear that in the trivially weighted setting $e_1$ and $e_2$ are proportional, whereas in the case of non-trivial weights they span a two-dimensional subspace of $\mathcal{T}$. Let us also set for each $\gamma \in \mathbb{R}$

$$\mu = \mu(\sigma, \rho, \gamma) = -e_1 + \gamma e_2.$$ 

(10)

**Definition 3.3** Given $\Lambda = \Lambda(\Theta)$, $\mu = \mu(\sigma, \rho, \gamma)$, and $k \in \{1, \ldots, d\}$, the quantities

$$\varphi_k(\Lambda, \mu) = \lim \inf_{s \to +\infty} \frac{L_k(\Lambda, s\mu)}{s}, \quad \bar{\varphi}_k(\Lambda, \mu) = \lim \sup_{s \to +\infty} \frac{L_k(\Lambda, s\mu)}{s}$$

are called the Schmidt–Summerer lower and upper exponents of the first type, and the quantities

$$\Phi_k(\Lambda, \mu) = \lim \inf_{s \to +\infty} \frac{S_k(\Lambda, s\mu)}{s}, \quad \bar{\Phi}_k(\Lambda, \mu) = \lim \sup_{s \to +\infty} \frac{S_k(\Lambda, s\mu)}{s}$$

are called the Schmidt–Summerer lower and upper exponents of the second type.

**Remark 3.1** It is easily verified that, if $\mu = (\mu_1, \ldots, \mu_d) = \mu(\sigma, \rho, \gamma), \gamma \neq 0$, then $\sigma, \rho$, and $\gamma$ are uniquely restored from $\mu$ by the relations

$$\gamma = \frac{1}{m} \left( \sum_{j=1}^{m} \mu_j - n \right), \quad \sigma_j = \frac{1}{d} (\mu_j + 1 - \gamma), \quad \rho_i = \frac{1}{d\gamma} (\gamma - 1 - \mu_{d+1-i}),$$

$$j = 1, \ldots, m, \quad i = 1, \ldots, n.$$

Given arbitrary $s, \gamma \in \mathbb{R}$ and $\mu = (\mu_1, \ldots, \mu_d) = \mu(\sigma, \rho, \gamma)$, we shall consider the parallelepipeds

$$\mathcal{P}(s, \gamma) = \left\{ z = (z_1, \ldots, z_d) \in \mathbb{R}^d \mid |z_j| \leq e^{s\sigma_j}, \quad |z_{d+1-i}| \leq e^{-s\rho_i}, \quad j = 1, \ldots, m \right\},$$

$$\mathcal{Q}(s, \gamma) = \left\{ z = (z_1, \ldots, z_d) \in \mathbb{R}^d \mid |z_k| \leq e^{s\mu_k}, \quad k = 1, \ldots, d \right\}.$$

(11)
Proposition 3.1  The exponent \( \omega^{(k)}_{\sigma, \rho}(\Theta) \) (resp. \( \hat{\omega}^{(k)}_{\sigma, \rho}(\Theta) \)) equals the supremum of \( \gamma \in \mathbb{R} \) such that \( P(s, \gamma) \) contains \( k \) linearly independent points of \( \Lambda \) for some arbitrarily large \( s \) (resp. for every \( s \) large enough).

The exponent \( \varphi_k(\Lambda, \mu) \) (resp. \( \overline{\varphi}_k(\Lambda, \mu) \)) equals the infimum of \( \chi \in \mathbb{R} \) such that \( e^{s\chi}Q(s, \gamma) \) contains \( k \) linearly independent points of \( \Lambda \) for some arbitrarily large \( s \) (resp. for every \( s \) large enough).

Proof The statements immediately follow from Definitions 3.1, 3.2, 3.3.

Proposition 3.2  Given \( \Lambda = \Lambda(\Theta) \), \( \mu = \mu(\gamma) = \mu(\sigma, \rho, \gamma) \), and \( k \in \{1, \ldots, d\} \), we have

\[
\omega^{(k)}_{\sigma, \rho}(\Theta) = \gamma \iff \varphi_k(\Lambda, \mu(\gamma)) = 1 - \gamma, \\
\hat{\omega}^{(k)}_{\sigma, \rho}(\Theta) = \gamma \iff \overline{\varphi}_k(\Lambda, \mu(\gamma)) = 1 - \gamma. \tag{13}
\]

Besides that, \( \varphi_k(\Lambda, \mu(\gamma)) + \gamma \) and \( \overline{\varphi}_k(\Lambda, \mu(\gamma)) + \gamma \) are increasing as functions of \( \gamma \).

Proof By (10)

\[
\mu_j = d\sigma_j + \gamma - 1, \quad j = 1, \ldots, m, \\
\mu_{d+1-i} = -d\rho_i + \gamma - 1, \quad i = 1, \ldots, n. \tag{14}
\]

For every \( s, \gamma \in \mathbb{R} \) we have by (14)

\[
P(ds, \gamma) = e^{s(1-\gamma)}Q(s, \gamma). \tag{15}
\]

Hence, in view of Proposition 3.1, (13) follows.

It also follows from (15) that for each \( \gamma_1, \gamma_2, \chi \in \mathbb{R}, 0 \leq \gamma_1 \leq \gamma_2 \), we have

\[
e^{s(\chi-\gamma_2)}Q(s, \gamma_2) \subset e^{s(\chi-\gamma_1)}Q(s, \gamma_1), \tag{16}
\]

as \( P(ds, \gamma_2) \subset P(ds, \gamma_1) \). Hence the monotonicty of \( \varphi_k(\Lambda, \mu(\gamma)) + \gamma \) and \( \overline{\varphi}_k(\Lambda, \mu(\gamma)) + \gamma \) follows.

Corollary 3.1  Within the hypothesis of Proposition 3.2, we have

\[
\omega^{(k)}_{\sigma, \rho}(\Theta) \geq \gamma \iff \varphi_k(\Lambda, \mu(\gamma)) \leq 1 - \gamma, \\
\hat{\omega}^{(k)}_{\sigma, \rho}(\Theta) \geq \gamma \iff \overline{\varphi}_k(\Lambda, \mu(\gamma)) \leq 1 - \gamma.
\]

3.2 Intermediate exponents of the second type

The exponents \( \omega^{(k)}_{\sigma, \rho}(\Theta) \) and \( \hat{\omega}^{(k)}_{\sigma, \rho}(\Theta) \) are analogues in the classical setting of Schmidt–Summerer exponents of the first type. The point of view at all those exponents provided by Proposition 3.1 proposes a natural analogue of Schmidt–Summerer exponents of
the second type. Before giving the corresponding definition, let us introduce some notation.

Given \( k \in \{1, \ldots, d\} \), let us consider the \( k \)th exterior power \( \bigwedge^k(\mathbb{R}^d) \). It is a \((\binom{d}{k})\)-dimensional space. We shall write its elements as \( Z = (Z_{i_1}, \ldots, Z_{i_k}) \), assuming that \( i_1 < \cdots < i_k \) and that \((i_1, \ldots, i_k)\) ranges through the set of all \( k \)-element subsets of \((1, \ldots, d)\).

Let us also consider the lattice \( \bigwedge^k(\Lambda) \) spanned (over \( \mathbb{Z} \)) by the elements of the form \( z_1 \wedge \cdots \wedge z_k, \quad z_1, \ldots, z_k \in \Lambda \).

Given a parallelepiped

\[
P = \left\{ z = (z_1, \ldots, z_d) \in \mathbb{R}^d \mid |z_i| \leq c_i, \quad i = 1, \ldots, d \right\},
\]

let us denote

\[
P^{[k]} = \left\{ Z = (Z_{i_1}, \ldots, Z_{i_k}) \in \bigwedge^k(\mathbb{R}^d) \mid |Z_{i_1}, \ldots, Z_{i_k}| \leq \prod_{j=1}^k c_{i_j} \right\}.
\]

(17)

Let also \( P(s, \gamma), Q(s, \gamma) \) be defined by 11, 12, and let \( \Lambda = \Lambda(\Theta) \).

**Definition 3.4** Let \( k \) be an integer, \( 1 \leq k \leq d \). We define the \( k \)th weighted Diophantine exponent \( \Omega^{(k)}_{\sigma, \rho}(\Theta) \) of second type as the supremum of real \( \gamma \) such that \( P^{[k]}(s, \gamma) \) contains a nonzero element of \( \bigwedge^k(\Lambda) \) for some arbitrarily large \( s \).

Clearly, \( \Omega^{(1)}_{\sigma, \rho}(\Theta) = \omega^{(1)}_{\sigma, \rho}(\Theta) = \omega_{\sigma, \rho}(\Theta) \).

**Definition 3.5** Let \( k \) be an integer, \( 1 \leq k \leq d \). We define the \( k \)th weighted uniform Diophantine exponent \( \hat{\Omega}^{(k)}_{\sigma, \rho}(\Theta) \) of second type as the supremum of real \( \gamma \) such that \( P^{[k]}(s, \gamma) \) contains a nonzero element of \( \bigwedge^k(\Lambda) \) for every \( s \) large enough.

Since \( \det(\bigwedge^k(\Lambda)) = 1 \) and \( \text{vol}(P^{[k]}(s, 1)) = 2^{\binom{d}{k}} \), it follows by Minkowski’s convex body theorem that

\[
\Omega^{(k)}_{\sigma, \rho}(\Theta) \geq \hat{\Omega}^{(k)}_{\sigma, \rho}(\Theta) \geq 1.
\]

**Proposition 3.3** Given \( \Lambda = \Lambda(\Theta), \mu = \mu(\gamma) = \mu(\sigma, \rho, \gamma), \) and \( k \in \{1, \ldots, d\} \), we have

\[
\Omega^{(k)}_{\sigma, \rho}(\Theta) = \gamma \iff \frac{\Phi_k(\Lambda, \mu(\gamma))}{k} = 1 - \gamma.
\]

\[
\hat{\Omega}^{(k)}_{\sigma, \rho}(\Theta) = \gamma \iff \frac{\overline{\Phi}_k(\Lambda, \mu(\gamma))}{k} = 1 - \gamma.
\]

(18)

Besides that, \( \frac{\Phi_k(\Lambda, \mu(\gamma))}{k} + \gamma \) and \( \frac{\overline{\Phi}_k(\Lambda, \mu(\gamma))}{k} + \gamma \) are increasing as functions of \( \gamma \).
Proof It follows from Mahler’s theory of compound bodies that
\[
\lambda_1(\mathcal{Q}^{[k]}(s, \gamma), \bigwedge^k(\Lambda)) \asymp \prod_{i=1}^{k} \lambda_i(\mathcal{Q}(s, \gamma), \Lambda)
\]
with the implied constant depending only on \(d\).

Thus, the exponent \(\Phi_k(\Lambda, \mu)\) (resp. \(\hat{\Phi}_k(\Lambda, \mu)\)) equals the infimum of \(\chi \in \mathbb{R}\) such that \(e^{x \chi} \mathcal{Q}^{[k]}(s, \gamma)\) contains a nonzero element of \(\bigwedge^k(\Lambda)\) for some arbitrarily large \(s\) (resp. for every \(s\) large enough).

The rest of the argument is pretty much the same as in the proof of Proposition 3.2. Instead of (15) we have
\[
P^{[k]}(d, s, \gamma) = e^{x(1-\gamma)} \mathcal{Q}^{[k]}(s, \gamma),
\]
and instead of (16) we have
\[
e^{x(\chi-\gamma_2)} \mathcal{Q}^{[k]}(s, \gamma_2) \subseteq e^{x(\chi-\gamma_1)} \mathcal{Q}^{[k]}(s, \gamma_1),
\]
for every \(\gamma_1, \gamma_2, \chi \in \mathbb{R}, 0 \leq \gamma_1 \leq \gamma_2\).

Hence (18) and monotonicity follow. \(\square\)

Corollary 3.2 Within the hypothesis of Proposition 3.3, we have
\[
\Omega_{\sigma, \rho}(\Theta) \geq \gamma \iff \frac{\Phi_k(\Lambda, \mu(\gamma))}{k} \leq 1 - \gamma,
\]
\[
\hat{\Omega}_{\sigma, \rho}(\Theta) \geq \gamma \iff \frac{\hat{\Phi}_k(\Lambda, \mu(\gamma))}{k} \leq 1 - \gamma.
\]

3.3 Dual problem

Along with the system (2), i.e. the system
\[
\begin{cases}
|x|_\sigma \leq t \\
|\Theta x - y|_\rho \leq t^{-\gamma}
\end{cases}
\]
let us consider the “dual” system
\[
\begin{cases}
|y|_\rho \leq t \\
|\Theta^\top y - x|_\sigma \leq t^{-\delta}
\end{cases}
\] (19)

Given \(k \in \{1, \ldots, d\}\), the exponent \(\omega_{\rho, \sigma}(\Theta^\top)\) (resp. \(\hat{\omega}_{\rho, \sigma}(\Theta^\top)\)) is defined by Definition 3.1 (resp. 3.2) as the supremum of real \(\delta\) such that the system (19) admits \(k\) linearly independent solutions in \((x, y) \in \mathbb{Z}^{m+n}\) for some arbitrarily large \(t\) (resp. for every \(t\) large enough).
The previous section already provides an interpretation of the exponents \( \omega_{\rho,\sigma}^{(k)}(\Theta^\top) \), \( \hat{\omega}_{\rho,\sigma}^{(k)}(\Theta^\top) \). It suffices to swap the triple \((\Theta, \sigma, \rho)\) for \((\Theta^\top, \rho, \sigma)\). However, in the context of the transference phenomenon, it is very useful to give another interpretation involving the dual lattice for \( \Lambda(\Theta) \) and preserving the subspace generated by \( e_1 \) and \( e_2 \). So, let us consider the dual lattice

\[
\Lambda^* = \left( I_m \Theta_1^\top I_n \right) \mathbb{Z}^d
\]

and set for each \( \delta \in \mathbb{R} \)

\[
\mu^* = \mu^*(\sigma, \rho, \delta) = \delta e_1 - e_2,
\]

where \( e_1 \) and \( e_2 \) are defined by (9).

**Proposition 3.4** Given \( \Lambda^* = \left( I_m \Theta_1^\top I_n \right) \mathbb{Z}^d \), \( \mu^* = \mu^*(\delta) = \mu^*(\sigma, \rho, \delta) \), and \( k \in \{1, \ldots, d\} \), we have

\[
\omega_{\rho,\sigma}^{(k)}(\Theta^\top) = \delta \iff \varphi_k(\Lambda^*, \mu^*(\delta)) = 1 - \delta,
\]

\[
\hat{\omega}_{\rho,\sigma}^{(k)}(\Theta^\top) = \delta \iff \overline{\varphi}_k(\Lambda^*, \mu^*(\delta)) = 1 - \delta.
\]

(21)

Besides that, \( \varphi_k(\Lambda^*, \mu^*(\delta)) + \delta \) and \( \overline{\varphi}_k(\Lambda^*, \mu^*(\delta)) + \delta \) are increasing as functions of \( \delta \).

**Proof** The proof repeats, mutatis mutandis, that of Proposition 3.2. We should replace \( \mu(\gamma) \) with \( \mu^*(\delta) \),

\[
\mu_j^* = -d \sigma_j \delta + \delta - 1, \quad j = 1, \ldots, m,
\]

\[
\mu_{d+1-i}^* = d \rho_i + \delta - 1, \quad i = 1, \ldots, n,
\]

and consider, instead of \( P(s, \gamma) \), \( Q(s, \gamma) \), the parallelepipeds

\[
P^*(s, \delta) = \left\{ z = (z_1, \ldots, z_d) \in \mathbb{R}^d \left| \begin{array}{c}
|z_j| \leq e^{-s\sigma_j \delta}, \\
|z_{d+1-i}| \leq e^{s\rho_i},
\end{array} \right. \right\},
\]

\[
Q^*(s, \delta) = \left\{ z = (z_1, \ldots, z_d) \in \mathbb{R}^d \left| z_k \leq e^{s\mu_k^*}, \quad k = 1, \ldots, d \right. \right\}.
\]

Then, same as (15) and (16), we get

\[
P^*(ds, \delta) = e^{s(1-\delta)} Q^*(s, \delta)
\]

and

\[
e^{s(\chi-\delta_2)} Q^*(s, \delta_2) \subset e^{s(\chi-\delta_1)} Q^*(s, \delta_1),
\]

for each \( \delta_1, \delta_2, \chi \in \mathbb{R}, \, 0 \leq \delta_1 \leq \delta_2 \). Hence (21) and monotonicity follow. \( \square \)
Proposition 3.5 Given $\Lambda^* = \Lambda^*(\Theta)$, $\mu^* = \mu^*(\delta) = \mu^*(\sigma, \rho, \delta)$, and $k \in \{1, \ldots, d\}$, we have

$$\Omega_{\rho, \sigma}^{(k)}(\Theta^T) = \delta \iff \frac{\Phi_k(\Lambda^*, \mu^*(\delta))}{k} = 1 - \delta,$$

$$\hat{\Omega}_{\rho, \sigma}^{(k)}(\Theta^T) = \delta \iff \frac{\overline{\Phi}_k(\Lambda^*, \mu^*(\delta))}{k} = 1 - \delta. \quad (22)$$

Besides that, $\frac{\Phi_k(\Lambda^*, \mu^*(\delta))}{k} + \delta$ and $\frac{\overline{\Phi}_k(\Lambda^*, \mu^*(\delta))}{k} + \delta$ are increasing as functions of $\delta$.

The proof of Proposition 3.5 is obtained by changing the proof of Proposition 3.3 in the very same way we obtained the proof of Proposition 3.4 from that of Proposition 3.2.

3.4 Application of general theory and splitting Theorem 1.1

Let us apply Proposition 2.4 and Corollary 2.1 for $\Lambda = \Lambda(\Theta)$, $\tau = s\mu$. Dividing every relation thus obtained by $s$ and sending $s$ to $+\infty$, we get the following statements on the Schmidt–Summerer exponents.

Proposition 3.6 Given an arbitrary $\gamma \in \mathbb{R}$, let $\Lambda = \Lambda(\Theta)$, $\mu = \mu(\sigma, \rho, \gamma)$. Then

(i) $\varphi_k(\Lambda, \mu) = -\overline{\varphi}_{d+1-k}(\Lambda^*, -\mu)$,

$\varphi_k(\Lambda, \mu) = -\varphi_{d+1-k}(\Lambda^*, -\mu)$; $k = 1, \ldots, d$;

(ii) $\Phi_k(\Lambda, \mu) = \Phi_{d-k}(\Lambda^*, -\mu)$,

$\overline{\Phi}_k(\Lambda, \mu) = \overline{\Phi}_{d-k}(\Lambda^*, -\mu)$, $k = 1, \ldots, d - 1$;

(iii) $\Phi_1(\Lambda, \mu) \leq \cdots \leq \frac{\Phi_k(\Lambda, \mu)}{k} \leq \cdots \leq \frac{\Phi_{d-1}(\Lambda, \mu)}{d-1} \leq \frac{\Phi_d(\Lambda, \mu)}{d} = 0$,

$\overline{\Phi}_1(\Lambda, \mu) \leq \cdots \leq \frac{\overline{\Phi}_k(\Lambda, \mu)}{k} \leq \cdots \leq \frac{\overline{\Phi}_{d-1}(\Lambda, \mu)}{d-1} \leq \frac{\overline{\Phi}_d(\Lambda, \mu)}{d} = 0$;

(iv) $\frac{\Phi_1(\Lambda, \mu)}{d-1} \geq \cdots \geq \frac{\Phi_k(\Lambda, \mu)}{d-k} \geq \cdots \geq \frac{\Phi_{d-1}(\Lambda, \mu)}{d-1}$,

$\frac{\overline{\Phi}_1(\Lambda, \mu)}{d-1} \geq \cdots \geq \frac{\overline{\Phi}_k(\Lambda, \mu)}{d-k} \geq \cdots \geq \frac{\overline{\Phi}_{d-1}(\Lambda, \mu)}{d-1}$.

Corollary 3.3 $\Phi_1(\Lambda, \mu) \leq \frac{\Phi_1(\Lambda^*, -\mu)}{d-1}$.

We claim that Corollary 3.3 implies Theorem 1.1. To show this, let us set for each $\delta \geq 0$

$$\gamma_\delta = \begin{cases} 
\frac{(\sigma_m^{-1} - 1) + \rho_n^{-1}\delta}{\sigma_m^{-1} + (\rho_n^{-1} - 1)\delta}, & \text{if } \delta \geq 1, \\
\frac{(\sigma_1^{-1} - 1) + \rho_1^{-1}\delta}{\sigma_1^{-1} + (\rho_1^{-1} - 1)\delta}, & \text{if } \delta \leq 1.
\end{cases} \quad (23)$$
It is easy to see that the function $\delta \mapsto \gamma_\delta$ monotonously maps $[0, +\infty]$ onto the segment $[1 - \sigma_1, (1 - \rho_n)^{-1}]$, and that $\delta = 1$ if and only if $\gamma_\delta = 1$ (see Fig. 1). We naturally assume that $(1 - \rho_n)^{-1} = +\infty$ in case $\rho_n = 1$ (which holds if and only if $n = 1$).

The inverse of (23) is

$$
\delta = \begin{cases} 
(1 - \sigma_m^{-1}) + \sigma_m^{-1} \gamma_\delta, & \text{if } \gamma_\delta \geq 1, \\
\rho_n^{-1} + (1 - \rho_n^{-1}) \gamma_\delta, & \text{if } \gamma_\delta \leq 1.
\end{cases}
$$

Lemma 3.1 For each $k \in \{1, \ldots, d\}$ and every $\delta \geq 0$ we have

$$
\varphi_k(\Lambda^*, \mu^*(\delta)) \leq 1 - \delta \implies \varphi_k(\Lambda^*, -\mu(\gamma_\delta)) \leq (d - 1)(1 - \gamma_\delta),
$$

$$
\overline{\varphi}_k(\Lambda^*, \mu^*(\delta)) \leq 1 - \delta \implies \overline{\varphi}_k(\Lambda^*, -\mu(\gamma_\delta)) \leq (d - 1)(1 - \gamma_\delta).
$$

Proof For each $s \geq 0$ and $\delta \geq 0$ let us set

$$
s_\delta = \begin{cases} 
s(\rho_n^{-1} + (1 - \rho_n^{-1}) \gamma_\delta), & \text{if } \delta \geq 1, \\
s(\rho_1^{-1} + (1 - \rho_1^{-1}) \gamma_\delta), & \text{if } \delta \leq 1.
\end{cases}
$$

Then

$$
s_\delta \delta = \begin{cases} 
s((1 - \sigma_m^{-1}) + \sigma_m^{-1} \gamma_\delta), & \text{if } \delta \geq 1, \\
s((1 - \sigma_1^{-1}) + \sigma_1^{-1} \gamma_\delta), & \text{if } \delta \leq 1,
\end{cases}
$$

and we have for each $j \in \{1, \ldots, m\}$, $i \in \{1, \ldots, n\}$

$$
s_\delta \delta \geq s((1 - \sigma_j^{-1}) + \sigma_j^{-1} \gamma_\delta),
$$

$$
s_\delta \delta \leq s(\rho_i^{-1} + (1 - \rho_i^{-1}) \gamma_\delta).
$$

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Upon some minor calculations, we conclude from (25) that
\[ e^{s(1-\delta)} Q^* (s, \delta) \subset e^{s(d-1)(1-\gamma\delta)} Q(-s, \gamma\delta), \] (26)
where \( Q \) and \( Q^* \) are as in the proofs of Propositions 3.2 and 3.4. Hence the desired statement follows immediately.

\[ \square \]

**Theorem 3.1** For every \( \delta \geq 1 \) we have
\[ \varphi_1 (\Lambda^*, \mu^*(\delta)) \leq 1 - \delta \iff \varphi_1 (\Lambda, \mu(\gamma\delta)) \leq 1 - \gamma\delta. \]

**Proof** It suffices to apply Lemma 3.1 with \( k = 1 \), Corollary 3.3, and the fact that the exponents \( \varphi_1 \) and \( \varphi_1^* \) are the same.

\[ \square \]

Theorem 3.1, in view of Corollary 3.1, is a reformulation of Theorem 1.1. Note that the condition \( \delta \geq 1 \) reflects the fact that, by Minkowski’s convex body theorem, we always have \( \omega_{\sigma, \rho}(\Theta) \geq 1 \), in contrast to \( \omega_{(k)}(\Theta) \) with \( k \geq 2 \), which can attain values in the interval \([0, 1)\).

The key ingredient in the proof of Theorem 3.1 is Corollary 3.3. Lemma 3.1 can be considered as a rather important but technical statement: its essence is shown by the inclusion (26) performing a proper rescaling. In its term, Corollary 3.3 gets split by statements (ii) and (iii) of Proposition 3.6 into a sequence of inequalities between the Schmidt–Summerer exponents \( \varphi_1^k(\Lambda, \mu) \). Applying Proposition 3.3, Corollary 3.2, statement (iii) of Proposition 3.6, and, of course, Lemma 3.1, we get the following splitting of Theorem 1.1.

**Theorem 3.2** Set \( \Omega_1 = \Omega_{(k)}^{(k)}(\sigma, \rho)(\Theta) \) and \( \Omega_1^\top = \Omega_{(k)}^{(k)}(\rho, \sigma)(\Theta^\top) \) for each \( k = 1, \ldots, d \). Then
\[ \Omega_1 \geq \cdots \geq \Omega_k \geq \cdots \geq \Omega_{d-1} \geq \left( \frac{\sigma_1^{-1} - 1}{\sigma_1^{-1}} + \rho_1^{-1} \Omega_1^\top \right) \frac{\sigma_m^{-1} - 1}{\sigma_m^{-1}} \frac{\rho_n^{-1} - 1}{\rho_n^{-1}} \Omega_1^\top.(3.5) \]

### 3.5 Transference diagram

Application of Lemma 3.1 and Corollary 3.3 leads us from \( \mu^*(\delta) \) to \( \mu(\gamma\delta) \) via \( -\mu(\gamma\delta) \). Those three points belong to the subspace spanned by \( e_1, e_2 \). Lemma 3.2 and Fig. 2 below demonstrate how they are related.

**Lemma 3.2** Suppose the weights are nontrivial (i.e. either \( \sigma_1 \neq \sigma_m, \rho_1 \neq \rho_n \)).

If \( \delta = 1 \), then \( -\mu(\gamma\delta) = \mu^*(\delta) \).

If \( n > 1 \) (equivalently, \( \rho_1 < 1, \rho_n < 1 \)) and \( \delta \neq 1 \), then the line through the points \( \mu^*(\delta) \) and \( -\mu(\gamma\delta) \) passes through a point \( \nu \), that depends only on the sign of \( \delta - 1 \). This point can be expressed explicitly as
\[ \nu = \begin{cases} \frac{1}{1 - \rho_1^{-1}}(\sigma_1^{-1} e_1 + \rho_1^{-1} e_2), & \text{if } \delta > 1, \\ \frac{1}{1 - \rho_1^{-1}}(\sigma_1^{-1} e_1 + \rho_1^{-1} e_2), & \text{if } \delta < 1. \end{cases} \]
If \( n = 1 \) (equivalently, \( \rho_1 = \rho_n = 1 \)) and \( \delta \neq 1 \), then the line through the points \( \mu^*(\delta) \) and \( -\mu(\gamma \delta) \) is parallel to either \( \sigma^{-1}_m \mathbf{e}_1 + \mathbf{e}_2 \), or \( \sigma^{-1}_1 \mathbf{e}_1 + \mathbf{e}_2 \), according to whether \( \delta > 1 \), or \( \delta < 1 \).

The proof is elementary and we leave it to the reader. The case \( n > 1, \delta > 1 \) is illustrated by Fig. 2.

### 3.6 Transference equalities for intermediate weighted Diophantine exponents

One might ask whether there is a chain of inequalities between the Schmidt–Summerer exponents of the first type that splits Theorem 1.1 in a way similar to that described above. Such a result would split Theorem 1.1 into a chain of inequalities between the exponents \( \omega_{\sigma, \rho}(\Theta) \). We do not know the answer, though we doubt that such a chain of inequalities exists.

However, there is a relation, and a very nice one, between the exponents \( \omega^{(k)}_{\sigma, \rho}(\Theta) \) and the exponents \( \omega^{(k)}_{\rho, \sigma}(\Theta^\top) \). It generalises the corresponding relation that holds in the case of trivial weights (see [13, Corollary 8.5]). It is given by Theorem 3.3 below.

**Lemma 3.3** For each \( k \in \{1, \ldots, d\} \) and every \( \gamma > 0 \) we have

\[
\varphi_k(\Lambda^*, -\mu(\gamma)) = \gamma \cdot \varphi_k(\Lambda^*, \mu^*(\gamma^{-1})),
\]

\[
\overline{\varphi}_k(\Lambda^*, -\mu(\gamma)) = \gamma \cdot \overline{\varphi}_k(\Lambda^*, \mu^*(\gamma^{-1})).
\]

**Proof** By (10) and (20) we have \( -\mu(\gamma) = e_1 - \gamma e_2 = \gamma \mu^*(\gamma^{-1}) \). Therefore,

\[
\frac{L_k(\Lambda^*, -s \mu(\gamma))}{s} = \frac{L_k(\Lambda^*, s \gamma \mu^*(\gamma^{-1}))}{s} = \gamma \cdot \frac{L_k(\Lambda^*, s' \mu^*(\gamma^{-1}))}{s'},
\]

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where \( s' = s \gamma \). Hence the desired relations follow. \( \square \)

**Theorem 3.3** For every \( k \in \{1, \ldots, d\} \) we have

\[
\omega_{\sigma, \rho}^{(k)}(\Theta) \cdot \omega_{\rho, \sigma}^{(d+1-k)}(\Theta^\top) = 1, \quad \hat{\omega}_{\sigma, \rho}^{(k)}(\Theta) \cdot \omega_{\rho, \sigma}^{(d+1-k)}(\Theta^\top) = 1. \tag{27}
\]

Here it is assumed that if any of the factors is zero, then the other one is equal to \( +\infty \), and vice versa.

**Proof** Proving the first equality in (27) will suffice, as each of the two equalities can be turned into the other by simple swapping the tuple \((\Theta, \sigma, \rho, k)\) for \((\Theta^\top, \rho, \sigma, d+1-k)\).

Suppose \( \gamma > 0 \). Applying successively Proposition 3.2, statement (i) of Proposition 3.6, Lemma 3.3, and Proposition 3.4, we get

\[
\omega_{\sigma, \rho}^{(k)}(\Theta) = \gamma \iff \varphi_k(\Lambda, \mu(\gamma)) = 1 - \gamma \iff \varphi_{d+1-k}(\Lambda^*, -\mu(\gamma)) = 1 - 1 \iff \omega_{\rho, \sigma}^{(d+1-k)}(\Theta^\top) = 1 - 1.
\]

Taking into account Corollary 3.1, we get in a similar way that

\[
\omega_{\sigma, \rho}^{(k)}(\Theta) < \gamma \iff \hat{\omega}_{\rho, \sigma}^{(d+1-k)}(\Theta^\top) > 1 - 1, \quad \omega_{\sigma, \rho}^{(k)}(\Theta) > \gamma \iff \hat{\omega}_{\rho, \sigma}^{(d+1-k)}(\Theta^\top) < 1 - 1.
\]

Thus, (27) holds for all possible values of the factors, including zero and \( +\infty \). \( \square \)

### 3.7 Concerning inhomogeneous approximation

There is an important class of Diophantine problems that concerns the inhomogeneous setting. Given \( \eta = (\eta_1, \ldots, \eta_n) \in \mathbb{R}^n \), consider the system

\[
\begin{aligned}
| |x|_{\sigma} & \leq t \\
|\Theta x - y - \eta|_{\rho} & \leq t^{-\gamma}
\end{aligned} \tag{28}
\]

instead of (2). The inhomogeneous weighted Diophantine exponent \( \omega_{\sigma, \rho}(\Theta, \eta) \) (resp. the inhomogeneous uniform weighted Diophantine exponent \( \hat{\omega}_{\sigma, \rho}(\Theta, \eta) \)) is defined as the supremum of real \( \gamma \) such that the system (28) admits nonzero solutions in \((x, y) \in \mathbb{Z}^{m+n}\) for some arbitrarily large \( t \) (resp. for every \( t \) large enough).

In [11], Ghosh et al. proved the following inequalities, the “non-weighted” version of which belongs to Laurent and Bugeaud [14]:

\[
\omega_{\sigma, \rho}(\Theta, \eta) \geq \frac{1}{\hat{\omega}_{\rho, \sigma}(\Theta^\top)}, \quad \hat{\omega}_{\sigma, \rho}(\Theta, \eta) \geq \frac{1}{\omega_{\rho, \sigma}(\Theta^\top)}, \tag{29}
\]

where \( \hat{\omega}_{\rho, \sigma}(\Theta^\top) \) stands for \( \omega_{\rho, \sigma}^{(1)}(\Theta^\top) \).
In view of Theorem 3.3 (29) is equivalent to

\[ \omega_{\mathbf{\sigma}, \rho}(\Theta, \eta) \geq \omega^{(d)}_{\mathbf{\sigma}, \rho}(\Theta), \quad \hat{\omega}_{\mathbf{\sigma}, \rho}(\Theta, \eta) \geq \hat{\omega}^{(d)}_{\mathbf{\sigma}, \rho}(\Theta). \quad (30) \]

In such a form the inequalities are very easy to prove with the help of the classical argument, that goes back to Khintchine and Jarník. Indeed, if \( \omega^{(d)}_{\mathbf{\sigma}, \rho}(\Theta, \eta) > \gamma \) (resp. if \( \hat{\omega}^{(d)}_{\mathbf{\sigma}, \rho}(\Theta, \eta) > \gamma \)), then there is a constant \( c > 0 \) such that the parallelepiped \( c\mathcal{P}(s, \gamma) \) (in the notation of the proof of Proposition 3.2) contains a fundamental parallelepiped of \( \Lambda \) for some arbitrarily large \( s \) (resp. for every \( s \) large enough). Therefore, given arbitrary \( \xi \in \mathbb{R}^d \), the shifted parallelepiped \( \mathcal{P}(s, \gamma) + \xi \) contains a point of \( \Lambda \) for some arbitrarily large \( s \) (resp. for every \( s \) large enough). Hence \( \omega_{\mathbf{\sigma}, \rho}(\Theta, \eta) \geq \gamma \) (resp. \( \hat{\omega}_{\mathbf{\sigma}, \rho}(\Theta, \eta) \geq \gamma \)), and (30) follows.

Thus, (29) is a consequence of Theorem 3.3.

4 Lattice exponents and multiparametric geometry on numbers

4.1 Intermediate exponents

Let \( \Lambda \) be an arbitrary full rank lattice in \( \mathbb{R}^d \) of covolume 1. Let also \( \Pi(\cdot) \) be as defined in Sect. 1.3.

Same as in the case of Diophantine approximation with weights, let us supplement \( \omega^\times(\Lambda) \) with two families of intermediate exponents. For every \( \mathbf{v} = (v_1, \ldots, v_d) \in \mathbb{R}^d \) let us denote

\[ \mathcal{P}(\mathbf{v}) = \left\{ \mathbf{z} = (z_1, \ldots, z_d) \in \mathbb{R}^d \mid |z_i| \leq |v_i|, \ i = 1, \ldots, d \right\}. \]

Definition 4.1 Let \( k \) be an integer, \( 1 \leq k \leq d \). We define the \( k \)th Diophantine exponent \( \omega^\times_k(\Lambda) \) as the supremum of real \( \gamma \) such that \( \mathcal{P}(\mathbf{v}) \) contains \( k \) linearly independent points of \( \Lambda \) for some \( \mathbf{v} \in \mathbb{R}^d \) satisfying \( \Pi(\mathbf{v}) = |\mathbf{v}|^{-\gamma} \) with \( |\mathbf{v}| \) however large.

Clearly, \( \omega^\times(\Lambda) = \omega^\times_1(\Lambda) \).

Definition 4.2 Let \( k \) be an integer, \( 1 \leq k \leq d \). We define the \( k \)th uniform Diophantine exponent \( \hat{\omega}^\times_k(\Lambda) \) as the supremum of real \( \gamma \) such that \( \mathcal{P}(\mathbf{v}) \) contains \( k \) linearly independent points of \( \Lambda \) for every \( \mathbf{v} \in \mathbb{R}^d \) satisfying \( \Pi(\mathbf{v}) = |\mathbf{v}|^{-\gamma} \) with \( |\mathbf{v}| \) large enough.

Let us interpret the exponents just defined in terms of multiparametric geometry of numbers.

Every norm in \( \mathbb{R}^d \) induces a norm in \( \mathcal{T} \) (we remind that \( \mathcal{T} \) is the space of parameters defined in Sect. 1.1). Particularly, the supremum norm \( |\cdot| \). As for the functionals induced by \( |\cdot|_+ \) and \( |\cdot|_- \) (defined at the beginning of Sect. 2), they are not norms for \( d \geq 3 \), the corresponding “unit balls” are simplices and are not symmetric w.r.t. the origin. However, \( |\cdot|_+ \) cannot be neglected, as it is the image of the supremum norm under the logarithmic mapping: if \( \mathbf{z} = (z_1, \ldots, z_d), z_i > 0, i = 1, \ldots, d, \) and
\[ z_{\text{log}} = (\log z_1, \ldots, \log z_d), \text{ then} \]

\[ \log |z| = |z_{\text{log}}|_+. \]

The functional \(| \cdot |_+\) will be most important for us in this section. However, some of the statements we prove are valid for an arbitrary functional one can choose to measure \(\tau\), it only needs to generate an exhaustion of \(T\).

Let \(f\) be an arbitrary non-negative function on \(T\) such that the sets

\[ S(\lambda) = \{ \tau \in T \mid f(\tau) \leq \lambda \}, \lambda > 0, \]

form a monotone exhaustion of \(T\), i.e. every \(S(\lambda)\) is compact, \(T = \bigcup_{\lambda > 0} S(\lambda)\), and \(S(\lambda')\) is a subset of the (relative) interior of \(S(\lambda'')\), if \(\lambda' < \lambda''\). Particularly, for such \(f\), we have \(f(\tau) \to +\infty\) as \(|\tau| \to +\infty\).

**Definition 4.3** Given \(\Lambda, f\), and \(k \in \{1, \ldots, d\}\), the quantities

\[ \varphi_k(\Lambda, f) = \liminf_{|\tau| \to \infty} \frac{L_k(\Lambda, \tau)}{f(\tau)}, \quad \overline{\varphi}_k(\Lambda, f) = \limsup_{|\tau| \to \infty} \frac{L_k(\Lambda, \tau)}{f(\tau)} \]

are called the **Schmidt–Summerer lower and upper exponents of the first type**, and the quantities

\[ \Phi_k(\Lambda, f) = \liminf_{|\tau| \to \infty} \frac{S_k(\Lambda, \tau)}{f(\tau)}, \quad \overline{\Phi}_k(\Lambda, f) = \limsup_{|\tau| \to \infty} \frac{S_k(\Lambda, \tau)}{f(\tau)} \]

are called the **Schmidt–Summerer lower and upper exponents of the second type**.

**Proposition 4.1** Let \(f(\tau) = |\tau|_+\). Then, for each \(k = 1, \ldots, d\), we have

\[ \omega_k^x(\Lambda) \geq \hat{\omega}_k^y(\Lambda) \geq -1 + \frac{1}{d}, \quad (31) \]

\[ -1 \leq \varphi_k(\Lambda, f) \leq \overline{\varphi}_k(\Lambda, f) \leq d - 1, \quad (32) \]

and

\[ (1 + \omega_k^x(\Lambda))(1 + \varphi_k(\Lambda, f)) = 1, \]

\[ (1 + \hat{\omega}_k^y(\Lambda))(1 + \overline{\varphi}_k(\Lambda, f)) = 1, \quad (33) \]

assuming that \(\omega_k^x(\Lambda) = +\infty\) whenever \(\varphi_k(\Lambda, f) = -1\), and \(\hat{\omega}_k^y(\Lambda) = +\infty\) whenever \(\overline{\varphi}_k(\Lambda, f) = -1\).

**Proof** Let us first prove (32). It follows from statements (ii), (iii) of Proposition 2.1, and relation (5) that

\[ L_1(\tau)/|\tau|_+ \geq -1 + O(1/|\tau|_+), \]
\[ L_d(\tau)/|\tau|_+ \leq |\tau|_-/|\tau|_+ + O(1/|\tau|_+) \leq d - 1 + O(1/|\tau|_+), \]

as \(|\tau| \to \infty\), with the implied constants in all the \(O(\cdot)\) depending only on \(\Lambda\). Hence (32) follows immediately.

Let us prove (33) and (31). For each \(\gamma \in \mathbb{R}\) let us set

\[ \mathcal{H}_\gamma = \left\{ v \in \mathbb{R}^d_0 \mid \prod (v) = |v|^{-\gamma} \right\} \]

By Definitions 4.1, 4.2 the exponent \(\omega_k^\times(\Lambda)\) (resp. \(\hat{\omega}_k^\times(\Lambda)\)) equals the supremum of \(\gamma \in \mathbb{R}\) such that \(\mathcal{P}(v)\) contains \(k\) linearly independent points of \(\Lambda\) for some \(v \in \mathcal{H}_\gamma\) with \(|v|\) however large (resp. for every \(v \in \mathcal{H}_\gamma\) with \(|v|\) large enough). By Definition 4.3 the exponent \(\varphi_k(\Lambda, f)\) (resp. \(\hat{\varphi}_k(\Lambda, f)\)) equals the infimum of \(\chi \in \mathbb{R}\) such that \(e^{f(\tau)}B_\tau\) contains \(k\) linearly independent points of \(\Lambda\) for some \(\tau \in \mathcal{T}\) with \(f(\tau)\) however large (resp. for every \(\tau \in \mathcal{T}\) with \(f(\tau)\) large enough).

For every \(\gamma > -1\) let us define \(\chi(\gamma)\) by the relation

\[ (1 + \chi(\gamma))(1 + \gamma) = 1. \]

Let us also consider the bijection between \(\mathcal{H}_\gamma\) and \(\mathcal{T}\) determined by

\[ v = (v_1, \ldots, v_d) \mapsto \tau(v) = \left( \log \left( v_1/\prod(v) \right), \ldots, \log \left( v_d/\prod(v) \right) \right). \]

Then for every \(v \in \mathcal{H}_\gamma\) with \(\gamma > -1\)

\[ f(\tau(v))\chi(\gamma) = \log(\prod(v)). \quad (34) \]

Indeed, if \(\gamma = 0\), then both sides of (34) are equal to zero, whereas if \(\gamma \neq 0\), we have

\[ f(\tau(v)) = |\tau(v)|_+ = \log |v| - \log(\prod(v)) = -(1 + \gamma^{-1}) \log(\prod(v)) = \chi(\gamma)^{-1} \log(\prod(v)). \]

It follows from (34) that for every \(\gamma > -1\) and every \(v \in \mathcal{H}_\gamma\)

\[ \mathcal{P}(v) = \prod(v)B_{\tau(v)} = e^{f(\tau(v))\chi(\gamma)}B_{\tau(v)}. \]

Hence, taking into account the reformulation of Definitions 4.1, 4.2, 4.3 given above, we obtain that for every \(\gamma > -1\)

\[ \omega_k^\times(\Lambda) = \gamma \iff \varphi_k(\Lambda, f) = \chi(\gamma), \]

\[ \hat{\omega}_k^\times(\Lambda) = \gamma \iff \hat{\varphi}_k(\Lambda, f) = \chi(\gamma). \quad (35) \]

Since (32) holds, (35) implies both (33) and (31), as the correspondence \(\gamma \mapsto \chi(\gamma)\) induces a bijection between \([-1 + 1/d, +\infty]\) and \([-1, d - 1]\). \(\square\)
4.2 Intermediate exponents of the second type

In accordance with the notation introduced in Sect. 3.2, let us set for each \( v = (v_1, \ldots, v_d) \in \mathbb{R}^d \)

\[
P^{[k]}(v) = \left\{ Z = (Z_{i_1, \ldots, i_k}) \in \bigwedge^k (\mathbb{R}^d) \mid |Z_{i_1, \ldots, i_k}| \leq \prod_{j=1}^k |v_{i_j}| \right\}.
\]

Let us also consider the following functionals generalizing both the supremum norm and \( \Pi(\cdot) \). Given \( v = (v_1, \ldots, v_d) \in \mathbb{R}^d \), set

\[
|v|^{[k]} = \max_{1 \leq i_1 < \cdots < i_k \leq d} \prod_{1 \leq j \leq k} |v_{i_j}|^{1/k}.
\]

Clearly, \( |v|^{[1]} = |v| \) and \( |v|^{[d]} = \Pi(v) \).

**Definition 4.4** Let \( k \) be an integer, \( 1 \leq k \leq d \). We define the \( k \)th Diophantine exponent \( \Omega^\times_k(\Lambda) \) of the second type as the supremum of real \( \gamma \) such that \( P^{[k]}(v) \) contains a nonzero element of \( \bigwedge^k (\Lambda) \) for some \( v \in \mathbb{R}^d \) satisfying \( \Pi(v) = (|v|^{[k]})^{-\gamma} \) with \( |v| \) however large.

**Definition 4.5** Let \( k \) be an integer, \( 1 \leq k \leq d \). We define the \( k \)th uniform Diophantine exponent \( \hat{\Omega}^\times_k(\Lambda) \) of the second type as the supremum of real \( \gamma \) such that \( P^{[k]}(v) \) contains a nonzero element of \( \bigwedge^k (\Lambda) \) for every \( v \in \mathbb{R}^d \) satisfying \( \Pi(v) = (|v|^{[k]})^{-\gamma} \) with \( |v| \) large enough.

Clearly, \( \Omega^\times_1(\Lambda) = \omega^\times_1(\Lambda) = \omega^\times(\Lambda) \).

Furthermore, since \( \det \left( \bigwedge^k (\Lambda) \right) = 1 \) and \( \text{vol} \left( P^{[k]}(v) \right) = 2^k \), provided \( \Pi(v) = 1 \), it follows by Minkowski’s convex body theorem that

\[
\Omega^\times_k(\Lambda) \geq \hat{\Omega}^\times_k(\Lambda) \geq 0.
\]

Moreover, for \( k = d \) the exponents degenerate: we have

\[
\Omega^\times_d(\Lambda) = \hat{\Omega}^\times_d(\Lambda) = 0.
\]

In order to prove an analogue of Proposition 4.1 for the exponents of the second type, let us consider the image of \( |\cdot|^{[k]} \) under the logarithmic mapping. Let us set

\[
|\tau|^{[k]}_+ = \max_{1 \leq i_1 < \cdots < i_k \leq d} \frac{\tau_{i_1} + \cdots + \tau_{i_k}}{k}.
\]

Clearly, \( |\tau|^{[1]}_+ = |\tau|_+ \) and \( |\tau|^{[d]}_+ = 0 \) for every \( \tau \in T \). It is also easily verified that each \( |\cdot|^{[k]}_+ \) with \( k \in \{1, \ldots, d-1\} \) generates an exhaustion of \( T \), whereas \( |\cdot|^{[d]}_+ \) does not.
Proposition 4.2 Let $k \in \{1, \ldots, d-1\}$ and let $f(\tau) = |\tau|^k$. Then, along with (36), we have
\[-k \leq \Phi_k(\Lambda, f) \leq \Phi^*_k(\Lambda, f) \leq 0\]
and
\[
\left(1 + \Omega^\times_k(\Lambda)\right) \left(1 + \frac{\Phi_k(\Lambda, f)}{k}\right) = 1, \\
\left(1 + \hat{\Omega}^\times_k(\Lambda)\right) \left(1 + \frac{\Phi_k(\Lambda, f)}{k}\right) = 1,
\]
assuming that $\Omega^\times_k(\Lambda) = +\infty$ whenever $\Phi_k(\Lambda, f) = -k$, and $\hat{\Omega}^\times_k(\Lambda) = +\infty$ whenever $\Phi_k(\Lambda, f) = -k$.

Proof It follows from Mahler’s theory of compound bodies that
\[\lambda_1\left(B_{\tau}[k], \Lambda^k(\Lambda)\right) \asymp \prod_{i=1}^k \lambda_i\left(B_\tau, \Lambda\right)\]
with the implied constant depending only on $d$.

Thus, the exponent $\Phi_k(\Lambda, f)$ (resp. $\Phi^*_k(\Lambda, f)$) equals the infimum of $\chi \in \mathbb{R}$ such that $e^{f(\tau)\chi}B_{\tau}[k]$ contains a nonzero element of $\Lambda^k(\Lambda)$ for some arbitrarily large $|\tau|$ (resp. for every $|\tau|$ large enough).

Following the ideas of the proof of Proposition 4.1, let us set for each $\gamma \geq 0$
\[\mathcal{H}^k_\gamma = \left\{v \in \mathbb{R}^d \mid \Pi(v) = \left(|v|^k\right)^{-\gamma}\right\}\]
Since $1 \leq k \leq d-1$, it is easily verified that for each $v \in \mathbb{R}^d$ with nonzero $\Pi(v)$ there is a unique positive $\lambda$ such that $\Pi(\lambda v) = \left(|\lambda v|^k\right)^{-\gamma}$. By Definitions 4.4, 4.5 the exponent $\Omega^\times_k(\Lambda)$ (resp. $\hat{\Omega}^\times_k(\Lambda)$) equals the supremum of $\gamma \in \mathbb{R}$ such that $\mathcal{P}^k[v]$ contains a nonzero element of $\Lambda^k(\Lambda)$ for some $v \in \mathcal{H}^k_\gamma$ with $|v|$ however large (resp. for every $v \in \mathcal{H}^k_\gamma$ with $|v|$ large enough).

For every $\gamma \geq 0$ let us define $\chi(\gamma)$ by the relation
\[a(1 + \chi(\gamma))(1 + \gamma) = 1.\]
Let us also consider the same bijection between $\mathcal{H}^k_\gamma$ and $T$, as in the proof of Proposition 4.1, i.e.
\[v = (v_1, \ldots, v_d) \mapsto \tau(v) = \left(\log\left(v_1 / \Pi(v)\right), \ldots, \log\left(v_d / \Pi(v)\right)\right).\]
Then, for each $\gamma > 0$,
\[f(\tau(v)) = |\tau(v)|^k_+ = \log|v|^k - \log(\Pi(v)).\]
\[ -(1 + \gamma^{-1}) \log(\Pi(v)) = \chi(\gamma)^{-1} \log(\Pi(v)). \]

Hence, same as (34), we have for each \( \gamma \geq 0 \) and every \( v \in \mathcal{H}^{[k]}_\gamma \)
\[ f(\tau(v))\chi(\gamma) = \log(\Pi(v)), \]
i.e.
\[ \mathcal{P}^{[k]}(v) = \Pi(v)^k \mathcal{B}^{[k]}_{\tau(v)} = e^{k f(\tau(v))\chi(\gamma)} \mathcal{B}^{[k]}_{\tau(v)}. \]

Therefore, for every \( \gamma \geq 0 \)
\[ \Omega_k^X(\Lambda) = \gamma \iff \Phi_k(\Lambda, f) = k \chi(\gamma), \]
\[ \hat{\Omega}_k^X(\Lambda) = \gamma \iff \overline{\Phi}_k(\Lambda, f) = k \chi(\gamma). \]

Taking into account (36) and the fact that the correspondence \( \gamma \mapsto \chi(\gamma) \) induces a bijection between \([0, +\infty]\) and \([-1, 0]\), we get all the desired relations. \( \square \)

### 4.3 Application of general theory and splitting Theorem 1.2

Let us set
\[ f^*(\tau) = f(-\tau). \]

Clearly, \( |\tau| \to \infty \) if and only if \( f(\tau) \to \infty \). Hence, dividing all the relations provided by Proposition 2.4 and Corollary 2.1 by \( f(\tau) \) and sending \( \tau \) to infinity, we get the following statements on the Schmidt–Summerer exponents, analogous to Proposition 3.6 and Corollary 3.3.

**Proposition 4.3** Given \( \Lambda \) and \( f \), we have

(i) \( \varphi_k(\Lambda, f) = -\varphi_{d+1-k}(\Lambda^*, f^*), \)
\( \varphi_k(\Lambda, f) = -\varphi_{d+1-k}(\Lambda^*, f^*); \ k = 1, \ldots, d; \)

(ii) \( \Phi_k(\Lambda, f) = \Phi_{d-k}(\Lambda^*, f^*), \)
\( \overline{\Phi}_k(\Lambda, f) = \overline{\Phi}_{d-k}(\Lambda^*, f^*), \ k = 1, \ldots, d-1; \)

(iii) \( \Phi_1(\Lambda, f) \leq \cdots \leq \frac{\Phi_k(\Lambda, f)}{k} \leq \cdots \leq \frac{\Phi_{d-1}(\Lambda, f)}{d-1} \leq \frac{\Phi_d(\Lambda, f)}{d} = 0, \)
\( \overline{\Phi}_1(\Lambda, f) \leq \cdots \leq \frac{\overline{\Phi}_k(\Lambda, f)}{k} \leq \cdots \leq \frac{\overline{\Phi}_{d-1}(\Lambda, f)}{d-1} \leq \frac{\overline{\Phi}_d(\Lambda, f)}{d} = 0; \)

(iv) \( \frac{\Phi_1(\Lambda, f)}{d-1} \geq \cdots \geq \frac{\Phi_k(\Lambda, f)}{d-k} \geq \cdots \geq \frac{\Phi_{d-1}(\Lambda, f)}{d-1}, \)
\( \frac{\overline{\Phi}_1(\Lambda, f)}{d-1} \geq \cdots \geq \frac{\overline{\Phi}_k(\Lambda, f)}{d-k} \geq \cdots \geq \frac{\overline{\Phi}_{d-1}(\Lambda, f)}{d-1}. \)

**Corollary 4.1** \( \Phi_1(\Lambda, f) \leq \frac{\Phi_1(\Lambda^*, f^*)}{d-1}. \)
As in the case of Diophantine approximation with weights, we claim that Corollary 4.1 implies Theorem 1.2. To show this, we need the following simple observation.

**Lemma 4.1** Let $f(\tau) = |\tau|_+$. Then, for each $k \in \{1, \ldots, d\}$, the signs of $\varphi_k(\Lambda^*, f)$ and $\varphi_k(\Lambda^*, f^*)$ coincide, as well as do the signs of $\varphi_k(\Lambda^*, f)$ and $\varphi_k(\Lambda^*, f^*)$. Furthermore, we have

$$\frac{|\varphi_k(\Lambda^*, f)|}{d-1} \leq (d-1)|\varphi_k(\Lambda^*, f)|, \quad \frac{|\varphi_k(\Lambda^*, f)|}{d-1} \leq (d-1)|\varphi_k(\Lambda^*, f)|. \quad (37)$$

**Proof** The sign of $\varphi_k(\Lambda^*, f)$ (resp. $\varphi_k(\Lambda^*, f^*)$) does not depend on the choice of $f$, it depends only on whether $L_k(\tau) \geq 0$ for some $\tau$ with $|\tau|$ however large (resp. for every $\tau$ with $|\tau|$ large enough).

Furthermore, it follows from (5) that

$$\frac{f(\tau)}{d-1} \leq f^*(\tau) \leq (d-1)f(\tau). \quad (38)$$

This immediately implies (37). \qed

**Theorem 4.1** Let $f(\tau) = |\tau|_+$. Then

$$\varphi_1(\Lambda, f) \leq \frac{\varphi_1(\Lambda^*, f)}{(d-1)^2}. \quad \Phi_1$$

**Proof** The exponents $\varphi_1$ and $\Phi_1$ are the same. By statement (iii) of Proposition 4.3 they are nonpositive. It remains to apply Lemma 4.1 with $k = 1$ and Corollary 4.1. \qed

Similar to the case of Diophantine approximation with weights, Theorem 4.1 is a reformulation of Theorem 1.2, due to Proposition 4.1 and the observation that for nonzero $x, y \in \mathbb{R}$ the relation $(1+x)(1+y) = 1$ is equivalent to $x^{-1} + y^{-1} + 1 = 0$. The key ingredient in the proof of Theorem 4.1 is Corollary 4.1. In its turn, Corollary 4.1 gets split by statements (ii) and (iii) of Proposition 4.3 into a sequence of inequalities between the Schmidt–Summerer exponents $\Phi_k(\Lambda, f)$. We should note, however, that such a splitting involves only $f = |\cdot|_+^k$ and thus cannot be immediately interpreted as a result concerning lattice exponents of the second type. In order to perform such a result, let us make some observations concerning the functionals $|\cdot|_+^k$.

**Lemma 4.2** For each $\tau \in T$ and each $k = 1, \ldots, d-1$ we have

$$|\tau|_+^{[k]} = \frac{d-k}{k} \cdot |\tau|_+^{[d-k]}. \quad [k]$$

In other words, if $f = |\cdot|_+^k$, then $f^* = \frac{d-k}{k} |\cdot|_+^{[d-k]}$. \qed
Proof Suppose $\tau = (\tau_1, \ldots, \tau_d)$, $\tau_1 \geq \cdots \geq \tau_d$. Then, since $\tau_1 + \cdots + \tau_d = 0$,

$$|\tau|_+^{[k]} = \frac{1}{k} \sum_{j=1}^{k} \tau_j = \frac{d-k}{k} \cdot \frac{1}{d-k} \sum_{j=1}^{d} (-\tau_j) = \frac{d-k}{k} \cdot |\tau|_+^{[d-k]}.$$

□

Corollary 4.2 Let $f_k(\tau) = |\tau|_+^{[k]}$, $k = 1, \ldots, d - 1$. Then, for every such $k$, we have

$$\frac{\Phi_k(\Lambda, f_k)}{k} = \frac{\Phi_{d-k}(\Lambda^n, f_{d-k})}{d-k}.$$

Proof It suffices to apply statement (ii) of Proposition 4.3 and Lemma 4.2 to establish such a result. □

Lemma 4.3 For each $\tau \in \mathcal{T}$ and each $k = 1, \ldots, d - 1$ we have

$$\frac{k(d-k-1)}{(d-k)(k+1)} |\tau|_+^{[k]} \leq |\tau|_+^{[k+1]} \leq |\tau|_+.$$

Proof Suppose $\tau = (\tau_1, \ldots, \tau_d)$, $\tau_1 \geq \cdots \geq \tau_d$. Then, since $\tau_1 + \cdots + \tau_d = 0$,

$$\frac{1}{k} \sum_{j=1}^{k} \tau_j \geq \tau_{k+1} \geq \frac{1}{d-k} \sum_{j=k+1}^{d} \tau_j = \frac{-1}{d-k} \sum_{j=1}^{k} \tau_j.$$

Hence

$$\frac{d-k-1}{d-k} \sum_{j=1}^{k} \tau_j \leq \sum_{j=1}^{k+1} \tau_j \leq \frac{k+1}{k} \sum_{j=1}^{k} \tau_j,$$

and the desired inequality follows. □

Corollary 4.3 Let $f_k(\tau) = |\tau|_+^{[k]}$, $k = 1, \ldots, d - 1$. Then, for every $k \in \{1, \ldots, d - 2\}$, we have

$$\frac{d-k}{k} \cdot \frac{\Phi_k(\Lambda, f_k)}{k} \leq \frac{d-k-1}{k+1} \cdot \frac{\Phi_{k+1}(\Lambda, f_{k+1})}{k+1}.$$

Proof It suffices to apply statement (iii) of Proposition 4.3, Lemma 4.3, and the fact that the Schmidt–Summerer exponents of the second type, as well as the functions $S_k$, are nonpositive (see statement (i) of Proposition 2.2). □

Applying Corollaries 4.2, 4.3, Proposition 4.2, and the fact that for nonzero $x, y \in \mathbb{R}$ the relation $(1+x)(1+y) = 1$ is equivalent to $x^{-1} + y^{-1} + 1 = 0$, we get the following splitting of Theorem 1.2.
Theorem 4.2 If any of the exponents $\Omega_1^x(\Lambda), \ldots, \Omega_{d-1}^x(\Lambda), \Omega_1^x(\Lambda^*), \ldots, \Omega_{d-1}^x(\Lambda^*)$ is zero, then so are all the others. If they are nonzero, then

$$\frac{1 + \Omega_1^x(\Lambda)^{-1}}{d - 1} \leq \cdots \leq \frac{k}{d - k}(1 + \Omega_k^x(\Lambda)^{-1}) \leq \cdots \leq (d - 1)(1 + \Omega_{d-1}^x(\Lambda)^{-1})$$

and $\Omega_{d-1}^x(\Lambda) = \Omega_1^x(\Lambda^*)$.

4.4 Transference inequalities for intermediate Diophantine exponents of lattices

We conclude with another transference result, one concerning the exponents $\varphi_k^x(\Lambda, f)$, $\overline{\varphi}_k(\Lambda, f)$, and thus, the exponents $\omega_k^x(\Lambda), \hat{\omega}_k^x(\Lambda)$. Combining statement (i) of Proposition 4.3 with Lemma 4.1 immediately produces the following statement.

Theorem 4.3 Let $f(\tau) = |\tau|_+$. Then, for each $k \in \{1, \ldots, d\}$, the exponents $\varphi_k^x(\Lambda, f), \overline{\varphi}_{d+1-k}(\Lambda^*, f)$ have different signs, or are simultaneously zero. Besides that,

$$|\varphi_k(\Lambda, f)| \leq (d - 1)|\overline{\varphi}_{d+1-k}(\Lambda^*, f)|,$$

$$|\overline{\varphi}_k(\Lambda, f)| \leq (d - 1)|\varphi_{d+1-k}(\Lambda^*, f)|.$$ 

Applying again Proposition 4.1 and the fact that for nonzero $x, y \in \mathbb{R}$ the relation $(1 + x)(1 + y) = 1$ is equivalent to $x^{-1} + y^{-1} + 1 = 0$, we get the following reformulation of Theorem 4.3.

Theorem 4.4 Let $f(\tau) = |\tau|_+, k \in \{1, \ldots, d\}$.

The exponents $\omega_k^x(\Lambda), \hat{\omega}_{d+1-k}(\Lambda^*)$ have different signs, or are simultaneously zero.

If they are nonzero, let $A$ be the positive one of them, and let $B$ be the negative one. Then

$$\frac{A^{-1} + 1}{(d - 1)} \leq -(B^{-1} + 1) \leq (d - 1)(A^{-1} + 1). \quad (39)$$

Remark 4.1 Theorem 4.4 is an analogue of Theorem 3.3. However, one can easily notice that (39) is an inequality, whereas (27) is an equality. The reason for this difference is (38), implied by the choice of $f$. The latter in its turn is implied by the choice of the supremum norm to define $\omega^x(\Lambda)$. If $f$ were symmetric, then $f = f^*$ would hold instead of (38), and the factor $(d - 1)$ would vanish on both sides of (39), turning (39) into

$$A^{-1} + B^{-1} = -2.$$ 

We must admit though, that no such choice of $f$ seems as natural from the point of view of the definition of lattice exponents, as $f = |\cdot|_+$. 

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