A PTAS for $k$-hop MST on the Euclidean plane: Improving Dependency on $k$

Jittat Fakcharoenphol∗ Nonthaphat Wongwattanakij†

June 22, 2021

Abstract

For any $\epsilon > 0$, Laue and Matijević [CCCG’07, IPL’08] give a PTAS for finding a $(1 + \epsilon)$-approximate solution to the $k$-hop MST problem in the Euclidean plane that runs in time $(n/\epsilon)^{O(k/\epsilon)}$.

In this paper, we present an algorithm that runs in time $(n/\epsilon)^{O(\log k \cdot (1/\epsilon)^2 \cdot \log^2 (1/\epsilon))}$. This gives an improvement on the dependency on $k$ on the exponent, while having a worse dependency on $\epsilon$.

As in Laue and Matijević, we follow the framework introduced by Arora for Euclidean TSP. Our key ingredients include exponential distance scaling and compression of dynamic programming state tables.

1 Introduction

Given a set $S$ of $n$ points in 2-dimensional Euclidean space, an integer $k$, and a root node $r \in S$, we would like to find a spanning tree with minimum cost rooted at $r$ such that any path from $r$ to any point contains at most $k$ edges. We refer to a spanning trees satisfying this condition as a $k$-hop spanning tree.

This problem has applications in network design [7, 8, 9], distributed system design [15], and wireless networks [11].

For any $\epsilon > 0$, Laue and Matijević [14] present a polynomial-time approximation scheme (PTAS) for this problem on the plane that runs in time $n^{O(k/\epsilon)}$ for finding a $(1 + \epsilon)$-approximate solution. They follow the general framework of random dissection by Arora [3, 4] for finding good approximate solutions for instances in Euclidean metric. The dynamic programming structure of [14] (reviewed in Section 3.1) follows the approach from the PTAS for the $k$-median problem by Arora, Raghavan, and Rao [5].

In this work, we give a novel trade-off between the hop bound $k$ and the approximation requirement $\epsilon$. Namely, we reduce the dependency on $k$ on the exponent of the running time from $k$ down to $\log k$, while suffering a factor of $(1/\epsilon) \log^2 (1/\epsilon)$ increment. This might not be of an important concern in practice, but we would like to note that in many other problems the hop bound can be seen as “hard” constraints; therefore, (doubly) exponentially improvements on the dependency might indicate possibilities for further improvements.

When the points are on a metric and $k = 2$, Alfandari and Paschos [1] show that the problem is MAX-SNP-hard; thus it is unlikely to have a PTAS.

When points are randomly distributed in the $d$-dimensional Euclidean space, Clementi et al. [10] prove a lowerbound on the cost of $k$-hop MST and show that a divide-and-conquer heuristic finds a solution matching the lowerbound.

For general metrics, Althaus, Funke, Har-Peled, Könemann, Ramos, and Skutella [2] present an $O(\log n)$-approximation algorithm based on low distortion tree embeddings. Kantor and Peleg [12] present a constant factor approximation algorithm that runs in time $1.52 \cdot 9^{k-2}$.

The problem can be generalized to the $k$-hop Steiner tree problem by allowing the set of points required to be connected to $r$ to be $X \subseteq S$. Böhm et al. [6] gives an $n^{O(k)}$ exact algorithm for this problem when the points are from the metrics induced by graphs of bounded tree width.

∗Email: jittat@gmail.com, Department of Computer Engineering, Kasetsart University, Bangkok, Thailand. Supported by the Thailand Research Fund, Grant RSA-6180074.

†Email: nonthaphat.wo@ku.th, Department of Computer Engineering, Kasetsart University, Bangkok, Thailand. Supported by Graduate Student Scholarship, Faculty of Engineering, Kasetsart University.
We review the PTAS framework in Section 2 and review the dynamic programming approach of [14] in Section 3.1. Section 3.2 discusses how we change the dynamic programming table to reduce the dependency on $k$. We give full descriptions of the dynamic programming in Section 3.4 and its analysis in Section 4.

2 Preliminaries

We begin with the description of the bounding box and quadtree dissection based on techniques of Arora [3] which is also used by Laue and Matijević [14].

2.1 Bounding box

Let the bounding box be the smallest axis-aligned square inside which the set of points lie. As in previous work (e.g., [14, 3]), we can scale and translate all the points into the bounding box of side length $L = O(n)$. We can also move all the points into the closest grid point, while increasing the cost of the $k$-hop MST by at most $\epsilon$ fraction of the optimal cost. To see this, note that the cost of the optimal solution is at least $\Omega(n/\epsilon)$, the increased cost for each edge in the grid-aligned solution is at most 4, incurring the cost at most $4n$ for the entire solution, which is only an $\epsilon$ fraction of $\Omega(n/\epsilon)$.

2.2 Quadtree dissection

A dissection of the bounding box is a recursive partition of a square into four equal and smaller squares by vertical and horizontal lines. We recurse until the square has unit length or it contains at most one point. We call each square obtained from the procedure a box in the dissection. This recursive partition forms a quadtree, whose nodes are boxes, rooted at the node representing the bounding box. We assign levels to the boxes in dissection as the level of their associated nodes in the quadtree. The root node has level 0. There are at most $O(L^2)$ nodes and its depth is $O(\log L) = O(\log(n/\epsilon))$. We say that vertical and horizontal lines are of level $i$ if we recursively partition a square at level $i$ to level $i + 1$ by these lines.

For each box, we introduce the set of pre-specified points on its side called portals. It is hard to find a solution when each solution can cross a box at any position. Therefore, we place $m$ equally spaced portals on each side of a box, with a total of $4m$ portals per box, to enforce a solution to cross each box only at these portals. If every edge of a solution of $k$-hop MST crosses the sides of each box in the dissection only at its portals, we say that the solution is portal respecting.

To deal with an increased cost of the optimal portal-respecting $k$-hop MST solution, we use randomized shift described as follows. Let $a, b$ be a positive integer, the $(a, b)$-shift dissection is defined by shifting all the lines with $x$ and $y$-coordinate by $a$ and $b$ respectively, and then modulo with $L$. In other words, a vertical line with $x$-coordinate $X$ will move to $x$-coordinate $(X + a) \mod L$ and a horizontal line with $y$-coordinate $Y$ will move to $y$-coordinate $(Y + b) \mod L$. The following is a key lemma from Arora [3].

**Lemma 1** Let $m = O(\log L/\epsilon) = O(\log(n/\epsilon)/\epsilon)$ and choose two positive integers $a$ and $b$ at random such that $0 \leq a, b < L$, with probability at least $1/2$, there is an optimal portal-respecting solution with respect to the $(a, b)$-shift dissection of cost at most $(1 + \epsilon)$ times the optimal $k$-hop MST solution.

We, later on, focus only on finding good portal-respecting solutions.

3 Dynamic Programming

We apply the dynamic programming approach to find the portal respecting solution that has approximation ratio $(1 + \epsilon)$ as in Laue and Matijević [14]. We first review the approach used by [14], which is introduced by Arora, Raghavan, and Rao [5] in Section 3.1. We give an overview of our improvements in Section 3.2 and provide the description of the algorithm in Sections 3.3 and 3.4.
Figure 1: Examples of inside and outside assignments: outside\(_i\)(p) is the distance from portal \(p\) to the closest point \(y\) of hop-level \(i\) outside the box, and inside\(_i\)(q) is the distance from portal \(q\) to the closest point \(w\) of hop-level \(i\) inside the box. Note that outside\(_i\)(p) provides information for connecting \(x\) inside the box, and inside\(_i\)(q) provides information for connecting \(z\) outside the box (in the higher level of the recursion).

### 3.1 Review: the Tables of Laue and Matijević

Later on, to distinguish from the levels of boxes, we refer to the hop distance of a point from the root in the \(k\)-MST solution as its hop-level. Consider a box \(B\) with side length \(l\), we can find the optimal portal respecting \(k\)-MST solution inside \(B\) if we have information on points outside of \(B\). Suppose we want to assign a point \(x \in B\) to hop-level \(i\) in the tree. To do so optimally, we need to know the closest point \(y\) with hop-level \(i - 1\). This point might be outside \(B\); therefore, while we work on box \(B\), we need to specify this as a requirement for this particular solution.

The approach introduced by [5] is to represent these requirements approximately with two assignments inside and outside on the portals. We would later “guess” the values of these assignments (by enumerating all possible approximate values) when working on box \(B\).

To illustrate the idea, we would first start by describing the assignments outside and inside in an “ideal” setting, where we keep all distances exact. For portal \(p\) and hop-level \(i\), outside\(_i\)(\(p\)) is the distance from \(p\) to the closest point outside the box with hop-level \(i\) and inside\(_i\)(\(p\)) is the distance from \(p\) to the closest point inside the box with hop level \(i\). One can view the inside and outside assignments as a specification for a particular subproblem in box \(B\) which includes a provided “outside” condition in the outside assignment and a requirement from inside assignment that has to be satisfied (by providing points with appropriate hop-levels and portal-distances “inside” the box). See Figure 1 for an illustration on inside and outside assignments and Figure 2 for interactions between these assignments between levels of recursion.

Since the actual values for inside and outside are reals, it is not possible to enumerate their values during the dynamic programming table evaluation. We would instead deal with approximate values by rounding these values up to be multiples of \(l/m\). Note that since forcing the solution to be portal-respecting already introduces an additional error of \(l/m\) for each tree-edge going through a portal, the analysis in Lemma 4 can be modified slightly to account for an additional \(l/m\) additive error for each tree edge in the solution.

We would keep in assignments outside and inside approximate distances. Consider box \(B\) whose side length is \(l\). For the inside\(_i\) assignments, we know that the distance between any two points inside box \(B\) is at most \(2l\), and since we can tolerate \(l/m\) errors, we only keep inside\(_i\)(\(p\)) \(\in \{0, l/m, 2l/m, \ldots, 2l, \infty\}\), for portal \(p\), where \(\infty\) represents the fact that no nodes of hop-level \(i\) are inside \(B\). Similarly, for the outside assignments, we know that any two points in the bounding box can be at most \(2L\) apart; thus, we can have outside\(_i\)(\(p\)) \(\in \{0, l/m, 2l/m, \ldots, 2L, \infty\}\), for portal \(p\).

For each subproblem for box \(B\), we want to find the minimum cost solution satisfying the inside and outside assignments.

The table entry

\[
\text{Table}(B, \text{inside}_0, \ldots, \text{inside}_{k-1}, \text{outside}_0, \ldots, \text{outside}_{k-1})
\]

keeps the minimum cost solution for box \(B\) that respects the specified inside and outside assignments.

The key observation from [5, 4] is that the distance assignment between adjacent portals can differ by at most \(2l/m\); this implies that the number of possible assignments of inside\(_i\) is \(2m \cdot 3^{4m}\) per box.
Figure 2: An example showing 2 levels of the recursive quadtree dissection. Point \( r \) is the root of the \( k \)-MST, points \( a, b, c \) are at hop-level 1, and points \( d, e \) are at hop-level 2. Consider two boxes \( B_1 \) and \( B_2 \) at level 1. Inside \( B_1 \), the cost for point \( b \), which connects to \( r \) through portal \( p_3 \), can be computed using \( \text{outside}_0(p_3) \). Inside \( B_2 \), the cost for point \( d \) of hop-level 2, which connects to \( b \) through portal \( p_6 \) can be computed using \( \text{outside}_1(p_6) \). Note that box \( B_1 \), which provides the connection for \( d \), must guarantee the “inside” distance through portal \( p_5 \) (which is at the same location as portal \( p_6 \) of \( B_2 \)). Observe that, in this case, \( \text{inside}_1(p_5) = \text{outside}_1(p_6) \).

With the same observation, the number of possible assignments of \( \text{outside}_i \) is \( 2^{Lm/l} \cdot 3^{4^m} \) per box. In total we have \( 4L^2 \cdot 3^{8mk} \) possible assignments we have to “guess” during the dynamic programming evaluation per box. Plugging in values for \( L \) and \( m \) yields that we have \( n^{O(k/\epsilon)} \) assignments for each box. The running time of [14] depends essentially on this number of assignments, because to compute all table entries for box \( B \) with a particular \( \text{inside-outside} \) assignment, one have to go over all possible assignments of 4 child boxes of \( B \) and check their compatibility (in \( O(m^4) \) time); if the number of assignments per box is at most \( T \), the running time is bounded by \( T \cdot T^4 \cdot m^4 = n^{O(k/\epsilon)} \).

### 3.2 Reducing the table size

We describe two basic ideas for reducing the table size. Let \( \delta = \delta_\epsilon \) be a constant depending on \( \epsilon \), to be defined later. Our goal is to ensure that the multiplicative error incurred in each subproblem is at most \( 1 + c \cdot \delta \), for some constant \( c \). Section 4.1 analyzes the value for \( \delta \) and bounds the total error.

#### 3.2.1 Scaling the distances

Consider box \( B \) whose side length is \( l \). Consider a portal \( p \) of \( B \).

Instead of using a fixed linear scale \( l/m \), we use exponentially increasing distance scale. This is a standard technique; see, e.g., Kollipoulos and Rao [13]. For the \( \text{inside} \) assignments, we choose distances from the set \{0, \( l/m \), \((1 + \delta)l/m \), \((1 + \delta)^2l/m \), \((1 + \delta)^3l/m \), ...\}. This reduces the number of states for each portal from \( O(m) \) to \( O(\log_{1+\delta} m) \). We also use the same trick for \( \text{outside} \), i.e., we allows the distances from the set \{0, \( l/m \), \((1 + \delta)l/m \), \((1 + \delta)^2l/m \), ... L\}. The number of possible distance values is \( O(\log_{1+\delta} L/m) \).

Using exponentially increasing distance scales clearly reduces the number of states, however, disparity between the actual distance when merging subproblems incurs additional multiplicative error to our solution. We would bound this error in Subsection 4.1.

We also slightly change how the distance assignments \( \text{inside} \) and \( \text{outside} \) are represented in the box. For each portal, instead of keeping the distance for each hop-level \( i \), we keep for each distance scale the minimum hop-level \( i \) of nodes within that distance. Formally, let \( \gamma_0 = 0, \gamma_1 = (1 + \delta)l/m \), and, in general, \( \gamma_j = (1 + \delta)^j l/m \), be distance level thresholds. When consider distance level \( j \), for \( j \geq 0 \), for portal \( p \), we look for the minimum hop-level of a node with distance at most \( \gamma_j \). Therefore, for
each distance level, there are $k^{O(m)}$ possible value assignments. Considering all distance levels, there are $k^{O(m \log_{1+\delta} L)}$ value assignments, which are too many. We need another idea to reduce this number.

### 3.2.2 Granularity for hop changes

We can further reduce the number of possible values for adjacent portals. Since adjacent portals in $B$ are $l/m$ apart, if we can tolerate larger error, we can let portals share the same closest point. More specifically, since we can tolerate a factor of $1 + \delta$ multiplicative error, if the actual distance is at least $(l/m)(1/\delta)$, additional error of $l/m$ is acceptable.

Consider distance level $j'$ where

$$(1 + \delta)^{j'} (l/m) \geq \frac{2}{\delta} (l/m).$$

In that level, we can let two adjacent portals $p$ and $p'$ share the closest point. This reduces the number of variables at this level by half. Suppose that at distance level 1 there are $k^{O(m)} \leq k^m$ value assignments for some constant $c$, in this level $j'$, there will be $k^{m/2}$ value assignments. More over at level $j''$ where $(1 + \delta)^{j''} \geq 4/\delta$, we can further combine portals and there will be only $k^{m/4}$ value assignments. Let $\beta$ be the smallest integer such that

$$(1 + \delta)^{\beta} \geq \frac{2}{\delta},$$

i.e.,

$$\beta \geq \frac{\log(2/\delta)}{\log(1 + \delta)}.$$  

Thus, the number of possible value assignments decreases at every $\beta$ distance levels. This implies that the number of assignments is

$$(k^m)^\beta, (k^{m/2})^\beta, (k^{m/4})^\beta, \ldots,$$

which is $k^{O(m\beta)} = (\varepsilon/n)^{O(\beta \log k/\varepsilon)}$, since $m = O(\log L/\varepsilon)$. We would work out the value of $\beta$ later in Section 4.2.

### 3.3 The Table

From the discussion in Section 3.2, we formally describe our compressed table. Consider box $B$ with side length $l$ whose portals are $p_1, p_2, \ldots, p_{4m}$. Recall that $\gamma_j = (1 + \delta)^{j-1} l/m$ is the $j$-th distance level threshold for $B$. Let $\alpha_i = 1 + \log_{1+\delta} m$ and $\alpha_o = 1 + \log_{1+\delta} L$ be the number of distance levels for inside and outside assignments. For a set of portals $p_s, \ldots, p_t$ of $B$, let

- $ilevel_j(s,t)$ for $j = 0, 1, \ldots, \alpha_i$ be the minimum hop-level $i$ satisfying $inside_i(p_d) \leq \gamma_j$ for some $s \leq d \leq t$,
- $olevel_j(s,t)$ for $j = 0, 1, \ldots, \alpha_o$ be the minimum hop-level $i$ satisfying $outside_i(p_d) \leq \gamma_j$ for some $s \leq d \leq t$.

Also recall $\beta = \lceil \log_{1+\delta}(2/\delta) \rceil$. Instead of assigning values $inside$ and $outside$ we keep the following assignments. For levels $j = 0, 1, \ldots, \beta - 1$ and $1 \leq d \leq 4m$, assignments $ilevel_j(d, d)$ and $olevel_j(d, d)$. For levels $j = \beta, \beta + 1, \ldots, 2\beta - 1$ and $1 \leq d \leq 2m$, assignments $ilevel_j(2d - 1, 2d)$ and $olevel_j(2d - 1, 2d)$. In general, for $f = 1, 2, \ldots$, for levels $j = f \cdot \Delta, f \cdot \beta + 1, \ldots, (f + 1) \cdot \beta - 1$, and $1 \leq d \leq 4m/2^f$, assignments

$$ilevel_j(2^f(d - 1) + 1, 2^f d),$$

and

$$olevel_j(2^f(d - 1) + 1, 2^f d).$$

We refer to this set of assignments as a compressed representation. As noted in Subsection 3.2.2 the number of assignment variables in a box for the first $\beta$ levels is $O(m\beta) = O(m \log_{1+\delta}(2/\delta))$. The number decreases exponentially for every $\beta$ levels; thus there are at most $O(m \log_{1+\delta}(2/\delta))$ variables, implying the total number of $k^{O(m \log_{1+\delta}(2/\delta))}$ assignments per box.
From distance assignments \textit{inside}_i and \textit{outside}_i described in [14], one can obtain the compressed representation in polynomial time. Furthermore, from a compressed representation, the original assignments \textit{inside} and \textit{outside} can also be recovered in polynomial time with, possibly, missing values. This issue can be dealt with by filling them up with \( \infty \). With this polynomial-time transformation, to compute the dynamic programming table, we can use the algorithm of Laue and Matijevi\v{c} [14] with only slight modification.

### 3.4 Computing The Table

As mentioned in the previous section, we can use the dynamic programming algorithm of Laue and Matijevi\v{c} [14] to solve the problem. Here we describe a slight modification of their algorithm (to handle missing values) in detail for completeness. Note that the time bound for their algorithm essentially depends on the total number of states, not the merging procedure.

#### 3.4.1 Base case

Consider box \( B \) which is the leaf of the recursive quadtree dissection. Let \( \text{dist}(u,v) \) be the Euclidean distance between points \( u \) and \( v \). As in [14], we consider two base cases.

**Case 1:** Box \( B \) contains root \( r \). In this case, the box may contain other points, but they all lie at the same position as \( r \).

- For any non-root node inside box \( B \), we create an edge between them and root \( r \) (with cost 0). We set a table entry of box \( B \) with cost 0 if for each portal \( p_i \),
  1. \( \text{outside}_i(p) = \infty \), for \( 0 \leq i \leq k - 1 \),
  2. \( \text{inside}_i(p) = \infty \), for \( 1 \leq i \leq k - 1 \), and
  3. \( \text{dist}(r,p) \leq \text{inside}_0(p) \).

- For all other entries of box \( B \), we set their costs to be \( \infty \).

**Case 2:** All points inside box \( B \) are at the same grid point \( u \) and box \( B \) does not contain root \( r \).

Consider each possible entry for \( B \). We first ensure that the \textit{inside} assignments can be satisfied at hop-level \( i \), i.e., that \( \text{dist}(u,p) \leq \text{inside}_i(p) \) for each portal \( p \) and \( \text{inside}_v(p) = \infty \) for all \( i \neq i' \). If this condition is not satisfied, we set the cost of this entry to \( \infty \).

We have to connect a point at \( u \) to some point outside \( B \) at hop-level \( i - 1 \). As mentioned in Laue and Matijevi\v{c} [14], there are two possible cases: (1) either each point in \( B \) is connected with its own tree edge through a portal \( p \) (where each point pays a separate connection cost) or (2) only one point \( q \) is connected with its own tree edge through a portal (paying the cost) while the others connect through \( q \) (paying 0 cost) with one additional hop-level. Note that the latter case is always cheaper, but it might violate the hop constraint; therefore only applicable when the hop-level \( i < k \).

More precisely, when hop-level \( i < k \), we set the cost for this entry to be

\[
\min_{p'} \text{outside}_{i-1}(p') + \text{dist}(u,p'),
\]

where \( p' \) ranges over all portals of \( B \). Otherwise, when \( i = k \), we set the cost to be

\[
|B| \cdot \left( \min_{p'} \text{outside}_{i-1}(p') + \text{dist}(u,p') \right),
\]

where \( |B| \) is the number of points in box \( B \) and \( p' \) ranges over all portals of \( B \).

#### 3.4.2 Merging process

Let \( B \) be the box in dessection with children \( B_1, B_2, B_3 \), and \( B_4 \). For each assignment of \textit{inside} and \textit{outside} of \( B \), we consider all assignments \textit{inside}^{(j)} and \textit{outside}^{(j)} of its children \( B_j \), for \( 1 \leq j \leq 4 \), to compute \( B \)'s entry of the table. We have to check (1) if some \( B_j \) provides the required \textit{inside}, and (2) if the children satisfy their own \textit{outside}^{(j)} requirements with some \textit{inside}^{(j')} or \( B \) appropriately propagates the \textit{outside} requirements. If all conditions are satisfied, we set the cost of \( B \)'s entry to be the minimum of the sum of the children costs. Otherwise, we set the cost to be \( \infty \).
4 Analysis

4.1 Error bounds

During the merging process, for each level, we can incur additional multiplicative factor of $(1 + \delta)$. Since there are $O(\log(n/\epsilon)) \leq c \log(n/\epsilon)$ recursive levels, if we let $\delta$ to be such that $(1 + \delta)^{c \log(n/\epsilon)} \leq (1 + \epsilon)$, we can ensure that the error is not too large. However, this implies that $\delta \leq \epsilon/\log(n/\epsilon)$, which is too small, and we have to take an additional $\log n$ on the exponent, resulting in a quasi-polynomial time algorithm.

We shall give a better analysis on the error so that we only need $\delta = O(\epsilon/\log(1/\epsilon))$. Consider an edge $(u, v)$ in an optimal solution whose length is $\ell$. From the randomized dissection, we know that the expected portal respecting length of this edge is at most $(1 + \epsilon)\ell = \ell + \epsilon\ell$. We analyze additional errors due to exponential distance level scaling.

Let box $B$ be the box in the dissection that contain both $u$ and $v$. Clearly, all relevant boxes are those in $B$ that contain $u$ or $v$. First note that boxes containing $u$ or $v$ whose side lengths are at most $\epsilon \ell$ contribute to at most $O(\epsilon \ell)$ additive errors; thus we only consider boxes containing $u$ and $v$ whose side lengths are at least $\epsilon \ell$.

We first deal with relevant boxes containing $u$ in $B$ whose side lengths are at most $4 \cdot \ell$. The number of recursive levels is $\log\left(\frac{4}{\epsilon}\right) = O(\log(1/\epsilon))$. Each level the multiplicative error can be at most $(1 + \delta)$. For relevant boxes in $B$ whose sides are larger than $4\ell$, we claim that $(u, v)$ can pass through at most 2 of them, incurring at most another $(1 + \delta)^2$ factor. To see this, suppose that $(u, v)$ passes through one vertical side of box $B'$ at level $j$ whose side length is larger than $4\ell$. Since $(u, v)$’s length is $\ell$, it cannot reach another vertical line at level $j - 1$. The same argument applies to horizontal sides.

Thus, to ensure that the accumulative multiplicative error over all these levels is at most $(1 + \epsilon)$, we need $\delta$ to satisfy
\[
(1 + \delta)^{\log(1/\epsilon)+2} \leq 1 + \epsilon.
\]
Note that since $1 + x \leq e^x$, the following inequality suffices
\[
(1 + \delta)^{\log(1/\epsilon)+2} \leq e^{\delta \log(1/\epsilon)+2} \leq 1 + \epsilon;
\]
thus, by taking log, we need to ensure that $\delta = O\left(\frac{\log(1+\epsilon)}{\log(1/\epsilon)}\right) = O\left(\frac{\epsilon}{\log(1/\epsilon)}\right)$, because $\epsilon/2 \leq \log(1 + \epsilon)$, for $\epsilon \leq 1$.

4.2 Running time

Again note that the running time mainly depends on the number of assignments per box. From Section 3.2.2, we show that the number of assignments for each box is
\[
k^{O(m\beta)} = (n/\epsilon)^{O(\beta \log k/\epsilon)}.
\]
We are left to bound $\beta$, the number of distance levels needed before we can combine portals. Recall that we need
\[
\beta \geq \log(2/\delta)/\log(1 + \delta).
\]
Since $\delta \leq 1$, we know that $\delta/2 \leq \log(1 + \delta)$. Combining this with the bound from the previous section, we can let
\[
\beta = O\left(\frac{\log(1/\epsilon/\epsilon)}{\epsilon/\log(1/\epsilon)}\right) = O\left(\frac{\log(1/\epsilon) \cdot \log(1/\epsilon)/\epsilon}{\epsilon}\right) = O\left(\frac{\log^2(1/\epsilon)}{\epsilon}\right)
\]
since $\log(1/\epsilon/\epsilon) = \log(1/\epsilon) + \log(1/\epsilon) = O(\log(1/\epsilon))$.

Let $T$ be the maximum number of assignments per box. Clearly since the merging process takes at most $O(T^2)$-time per box, we only need to analyze the number of assignments. From the discussion in Section 3.3, the number of assignments $T$ is
\[
k^{O(\beta m)} = k^{O(\beta \log(n/\epsilon)/\epsilon)} = (n/\epsilon)^{O(\beta \log k/\epsilon)} = (n/\epsilon)^{O((\log^2(1/\epsilon)/\epsilon) \cdot (\log k/\epsilon))} = (n/\epsilon)^{O((\log k \cdot (\log(1/\epsilon)/\epsilon)^2))}
\]
7
Thus, we have the running time bound as claimed.

References

[1] Laurent Alfandari and Vangelis Th. Paschos. Approximating minimum spanning tree of depth 2. *International Transactions in Operational Research*, 6(6):607–622, 1999.

[2] Ernst Althaus, Stefan Funke, Sariel Har-Peled, Jochen Könemann, Edgar A. Ramos, and Martin Skutella. Approximating k-hop minimum-spanning trees. *Operations Research Letters*, 33(2):115 – 120, 2005.

[3] S. Arora. Polynomial time approximation schemes for euclidean tsp and other geometric problems. In *Proceedings of 37th Conference on Foundations of Computer Science*, pages 2–11, 1996.

[4] S. Arora. Nearly linear time approximation schemes for euclidean tsp and other geometric problems. In *Proceedings 38th Annual Symposium on Foundations of Computer Science*, pages 554–563, 1997.

[5] Sanjeev Arora, Prabhakar Raghavan, and Satish Rao. Approximation schemes for euclidean k-median and related problems. In *Proceedings of the Thirtieth Annual ACM Symposium on Theory of Computing*, STOC ’98, page 106–113, New York, NY, USA, 1998. Association for Computing Machinery.

[6] Martin Böhm, Ruben Hoeksma, Nicole Megow, Lukas Nölke, and Bertrand Simon. Computing a Minimum-Cost k-Hop Steiner Tree in Tree-Like Metrics. In Javier Esparza and Daniel Král, editors, *45th International Symposium on Mathematical Foundations of Computer Science (MFCS 2020)*, volume 170 of *Leibniz International Proceedings in Informatics (LIPICS)*, pages 18:1–18:15, Dagstuhl, Germany, 2020. Schloss Dagstuhl–Leibniz-Zentrum für Informatik.

[7] Paz Carmi, Lilach Chaitman-Yerushalmi, and Bat-Chen Ozeri. Minimizing the sum of distances to a server in a constraint network. *Computational Geometry*, 80:1 – 12, 2019.

[8] Paz Carmi, Lilach Chaitman-Yerushalmi, and Ohad Trabelsi. On the bounded-hop range assignment problem. In Frank Dehne, Jörg-Rüdiger Sack, and Ulrike Stege, editors, *Algorithms and Data Structures*, pages 140–151, Cham, 2015. Springer International Publishing.

[9] Paz Carmi, Lilach Chaitman-Yerushalmi, and Ohad Trabelsi. Bounded-hop communication networks. *Algorithmica*, 80(11):3050–3077, November 2018.

[10] Andrea E. F. Clementi, Miriam Di Ianni, Angelo Monti, Massimo Lauria, Gianluca Rossi, and Riccardo Silvestri. Divide and conquer is almost optimal for the bounded-hop mst problem on random euclidean instances. In Andrzej Pelc and Michel Raynal, editors, *Structural Information and Communication Complexity*, pages 89–98, Berlin, Heidelberg, 2005. Springer Berlin Heidelberg.

[11] M. Haenggi. Twelve reasons not to route over many short hops. In *IEEE 60th Vehicular Technology Conference, 2004. VTC2004-Fall. 2004*, volume 5, pages 3130–3134 Vol. 5, 2004.

[12] Erez Kantor and David Peleg. Approximate hierarchical facility location and applications to the shallow steiner tree and range assignment problems. volume 3998, pages 211–222, 05 2006.

[13] Stavros G. Kolliopoulos and Satish Rao. A nearly linear-time approximation scheme for the euclidean k-median problem. *SIAM J. Comput.*, 37(3):757–782, June 2007.

[14] Sören Laue and Domagoj Matijević. Approximating k-hop minimum spanning trees in euclidean metrics. *Information Processing Letters*, 107(3):96 – 101, 2008.

[15] Kerry Raymond. A tree-based algorithm for distributed mutual exclusion. *ACM Trans. Comput. Syst.*, 7(1):61–77, January 1989.