Quasilinear viscous approximations to scalar conservation laws

Ramesh Mondal*, S. Sivaji Ganesh† and S. Baskar‡

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Abstract

We prove the convergence of quasilinear parabolic viscous approximations to the entropy solution (in the sense of Bardos-Leroux-Nedelec) of a scalar conservation law, considered on a bounded domain in $\mathbb{R}^d$.

1 Introduction

Let $\Omega$ be a bounded domain in $\mathbb{R}^d$ with smooth boundary $\partial \Omega$. For $T > 0$, denote $\Omega_T := \Omega \times (0, T)$. The aim of this article is to prove that the sequence of solutions (called quasilinear viscous approximations) of the generalized viscosity problems

\begin{align*}
  u_t + \nabla \cdot f(u) &= \varepsilon \nabla \cdot (B(u) \nabla u) &\text{in } \Omega_T, \\
  u(x, t) &= 0 &\text{on } \partial \Omega \times (0, T), \\
  u(x, 0) &= u_0(x) &\text{on } \Omega,
\end{align*}

indexed by $\varepsilon > 0$, converges to the entropy solution (in the sense of [2]) of the initial-boundary value problem (IBVP) for the scalar conservation laws given by

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*ramesh@math.iitb.ac.in
†siva@math.iitb.ac.in
‡baskar@math.iitb.ac.in
\[ u_t + \nabla \cdot f(u) = 0 \quad \text{in } \Omega_T, \quad (1.2a) \]
\[ u(x, t) = 0 \quad \text{on } \partial \Omega \times (0, T), \quad (1.2b) \]
\[ u(x, 0) = u_0(x) \quad x \in \Omega. \quad (1.2c) \]
as \( \varepsilon \to 0 \). In this context, we have three results (Theorem 1.1, Theorem 1.2 and Theorem 1.3) depending on the regularity of the data. First we establish the convergence under more regular hypothesis on the data (see Hypothesis A), then use it to prove the result for data with lesser regularity (see Hypothesis B) and finally, we prove the result for data with lesser regularity and initial data without having compact essential support in \( \Omega \) (see Hypotheses C). We now state the hypotheses, and then the main results of this article.

**Hypothesis A**

1. Let \( f \in (C^4(\mathbb{R}))^d, f' \in (L^\infty(\mathbb{R}))^d \), and denote
   \[
   \|f'\|_{(L^\infty(\mathbb{R}))^d} := \max_{1 \leq j \leq d} \sup_{y \in \mathbb{R}} |f'_j(y)|.
   \]

2. Let \( B \in C^3(\mathbb{R}) \cap L^\infty(\mathbb{R}) \), and there exists an \( r > 0 \) such that \( B \geq r \).

3. Let \( 0 < \beta < 1 \), and \( u_0 \in C^{4+\beta}_c(\Omega) \), where
   \[
   C^{4+\beta}_c(\Omega) := \{ u_0 \in C^{4+\beta}(\Omega) ; \text{supp}(u_0) \text{ is compact in } \Omega \},
   \]
   and we denote \( I = [-\|u_0\|_\infty, \|u_0\|_\infty] \).

For definitions of Hölder spaces used in this work, we refer the reader to [13].

**Hypothesis B**

1. Let \( f \in (C^1(\mathbb{R}))^d, f' \in (L^\infty(\mathbb{R}))^d \), and denote
   \[
   \|f'\|_{(L^\infty(\mathbb{R}))^d} := \max_{1 \leq j \leq d} \sup_{y \in \mathbb{R}} |f'_j(y)|.
   \]

2. Let \( B \in C^3(\mathbb{R}) \cap L^\infty(\mathbb{R}) \), and there exists an \( r > 0 \) such that \( B \geq r \).

3. Let \( u_0 \) belong to the space \( W^{1,\infty}_c(\Omega) \) consisting of those elements of \( W^{1,\infty}(\Omega) \) whose essential support is a compact subset of \( \Omega \), and we denote \( I = [-\|u_0\|_\infty, \|u_0\|_\infty] \).
Hypothesis C

1. Let $f \in (C^1(\mathbb{R}))^d$, $f' \in (L^\infty(\mathbb{R}))^d$, and denote
\[
\|f'\|_{(L^\infty(\mathbb{R}))^d} := \max_{1 \leq j \leq d} \sup_{y \in \mathbb{R}} |f'_j(y)|.
\]

2. Let $B \in C^3(\mathbb{R}) \cap L^\infty(\mathbb{R})$, and there exists an $r > 0$ such that $B \geq r$.

3. Let $u_0 \in H^1_0(\Omega) \cap C(\overline{\Omega})$. Let $u_{0\varepsilon}$ be in $D(\Omega)$ such that for all $\varepsilon > 0$, there exist constants $A, C > 0$ such that $\|u_{0\varepsilon}\|_{L^\infty(\Omega)} \leq A$, $\|\Delta u_{0\varepsilon}\|_{L^1(\Omega)} \leq C$ and $u_{0\varepsilon} \to u_0$ in $H^1_0(\Omega)$ as $\varepsilon \to 0$. Denote $I := [-A, A]$.

**Theorem 1.1** Let $f, u_0, B$ satisfy Hypothesis A. Then the sequence of solutions to (1.1) converges a.e. to the unique entropy solution of (1.2) as $\varepsilon \to 0$.

In the literature, convergence of viscous approximations where $B \equiv 1$ are obtained under the assumption that the initial condition $u_0$ is a function of bounded variation. We are able to generalize this result to the case of a non-constant $B$ only when $u_0$ belongs to the space $W^{1,\infty}_c(\Omega)$, and is the content of the next result.

**Theorem 1.2** Let $f, u_0, B$ satisfy Hypothesis B. Then the sequence of solutions to (6.143) converges a.e. to the unique entropy solution of (1.2) as $\varepsilon \to 0$.

In Theorem 1.2, we dealt with initial data with compact essential support in $\Omega$. But in the following result, we are able to capture the unique entropy solution of IBVP for conservation laws (1.2) as the a.e. limit of quasilinear viscous approximations with initial data in $H^1_0(\Omega) \cap C(\overline{\Omega})$.

**Theorem 1.3** Let $f, u_0, B$ satisfy Hypothesis C. Then the sequence of solutions to (7.178) converges a.e. to the unique entropy solution of (1.2) as $\varepsilon \to 0$.

Problems of the form (1.1) are of interest in many physical situations. For instance, they appear in the viscous shallow water problem [5], and also in the equations of gas dynamics for viscous heat conducting fluid in eulerian coordinates [18, p.256]. Apart from the physical point of view, the study of viscous IBVP along with the convergence results play an important role in the numerical analysis of hyperbolic conservation laws. For instance, Kurganov and Liu [12] proposed a finite volume method for solving general
multi-dimentional system of conservation laws, in which they introduce an adaptive way of adding viscosity in the shock regions in order to capture numerically stable entropy solution of hyperbolic conservation laws. As a result, the coefficient of the added numerical viscosity appears as a function of the solution and hence the convergence of the scheme to the entropy solution is similar to the problem addressed in this article.

When \( \Omega = \mathbb{R} \) and \( B \equiv 1 \), Oleinik [14] established the convergence of viscous approximations to the entropy solution of (1.2). This result is generalized to \( \Omega = \mathbb{R}^d \) by Kruzhkov [11]. For a special class of \( 2 \times 2 \) systems of hyperbolic conservation laws, DiPerna [6] showed the convergence of viscous approximations to a weak solution of the corresponding conservation laws. Bianchini and Bressan [3] established the BV estimates of viscous approximations (with the viscosity coefficient being the identity matrix) and showed that the a.e. limit of the viscous approximations is the unique entropy solution of the corresponding system of conservation laws.

For hyperbolic problems posed on a bounded domain, it is incorrect to impose boundary conditions as information from initial conditions propagates throughout the domain, and the information reaching the points on the boundary may not coincide with the prescribed boundary conditions. In other words IBVP for hyperbolic problems are ill-posed in general [2]. Thus we need to give meaning to the way boundary conditions are realized. The entropy condition of Bardos-Leroux-Nedelec (BLN entropy condition) introduced in [2] gives a way to interpret the boundary conditions. Since the BLN entropy condition is introduced for solutions which are of bounded variation as these functions have a well-defined trace on the boundary, it can not be extended to a problem where the solutions are not of bounded variation. For entropy conditions we refer the reader to Otto [15], Carrillo [4], Vallet [19] in the \( L^\infty \)-setting, and to Porretta and Vovelle [16], Ammar et al. [1] in \( L^1 \)-setting. Dubois and Le Floch [7] derived a boundary entropy condition satisfied by the limit of viscous approximations for a system of conservation laws in the domain \( \{(x, t) : x > 0, t > 0\} \), assuming that the sequence of quasilinear viscous approximations \( \{u^\varepsilon\} \) is bounded in \( W^{1,1}_{loc}(\mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^d) \) and converges in \( L^1_{loc}(\mathbb{R}^+ \times \mathbb{R}^+) \) to a limit function \( u \).

Theorem 1.1 and Theorem 1.2 are proved by showing that the sequence \( \{u^\varepsilon\} \) has an a.e. convergent subsequence, and its limit is an entropy solution of (1.2). Since entropy solution is unique [2], we conclude that the entire sequence of viscous approximations \( \{u^\varepsilon\} \) converges a.e. to the unique entropy solution. Existence of an a.e. convergent subsequence of \( \{u^\varepsilon\} \) is proved by establishing a uniform BV-estimate on the sequence \( u^\varepsilon \). When \( B \equiv 1 \), Bardos et al. [2] established such a BV-estimate by employing clever
multipliers. Their technique is used to obtain estimates on time derivative of \( u^\varepsilon \), and the same technique does not yield \( L^1(\Omega_T) \) estimates on the gradient of \( u^\varepsilon \). However, using a different multiplier, we establish a bound on gradients of \( u^\varepsilon \).

The plan of the paper is as follows. In section 2, we show the well-posedness of solution to the IBVP (1.1). In section 3, we prove the higher regularity of solutions of (1.1) using an existence-cum-regularity result in Hölder spaces. In section 4, we prove the BV estimates on solutions to (1.1). In section 5, we prove Theorem 1.1 in section 6, we prove Theorem 1.2 and finally in section 7, we prove Theorem 1.3.

2 Well-posedness of generalized viscosity problem

Even though we assume higher smoothness on initial condition \( u_0 \) and viscosity coefficient \( B \) in Theorem 1.1 and Theorem 1.2 for the purposes of this section it is enough to assume that \( u_0 \in L^2(\Omega) \), and \( B \) is continuous on \( \mathbb{R} \) and bounded.

2.1 Existence

We show the existence of a weak solution \( u^\varepsilon \) to the IBVP (1.1) in the standard function space \( W(0,T) \) defined by

\[
W(0,T) := \left\{ u \in L^2(0,T; H^1_0(\Omega)) : u_t \in L^2(0,T; H^{-1}(\Omega)) \right\},
\]

in the following sense: For a.e. \( 0 \leq t \leq T \), and for every \( v \in H^1_0(\Omega) \) the following equalities hold:

\[
\langle u_t, v \rangle + \varepsilon \sum_{j=1}^{d} \int_{\Omega} B(u) \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_j} \, dx + \sum_{j=1}^{d} \int_{\Omega} f_j'(u) \frac{\partial u}{\partial x_j} v \, dx = 0, \tag{2.3a}
\]

\[
u(0) = u_0(x), \tag{2.3b}
\]

where \( \langle \cdot, \cdot \rangle \) denotes the duality pairing between \( H^{-1}(\Omega) \) and \( H^1_0(\Omega) \). Note that the condition (2.3b) is meaningful since the space \( W(0,T) \) can be identified with a subset of \( C([0,T]; L^2(\Omega)) \). Since \( \varepsilon \) remains fixed throughout this section, we write \( u \) instead of \( u^\varepsilon \).

Existence of a solution to the problem (1.1) is obtained as a limit of a sequence of approximations \( u_m \) given by Galerkin method. In Galerkin method, we take a sequence of functions \( (w_k) \) which is an orthogonal basis for the space \( H^1_0(\Omega) \), and an orthonormal basis of \( L^2(\Omega) \). For each \( m \in \mathbb{N} \), let \( V_m \) denote the subspace of \( L^2(\Omega) \) spanned by the first
functions, namely, \( w_1, w_2, \cdots, w_m \) of the sequence \((w_k)\). The Galerkin approximation \( u_m \in V_m \) is required to satisfy the equation (2.3a) with \( u = u_m \) for every \( v \in V_m \) along with the initial condition

\[
(u_m(0), w_k) = (u_0, w_k), \quad k = 1, 2, \cdots, m.
\]  

(2.4)

Thus \( u_m \) satisfies for \( k = 1, 2, \cdots, m \) and for a.e. \( t \in [0, T] \)

\[
\langle u'_m, w_k \rangle + \epsilon \sum_{j=1}^{d} \int_{\Omega} B(u_m) \frac{\partial u_m}{\partial x_j} \frac{\partial w_k}{\partial x_j} dx + \sum_{j=1}^{d} \int_{\Omega} f'_j(u_m) \frac{\partial u_m}{\partial x_j} w_k dx = 0, \quad k = 1, 2, \ldots, m
\]

(2.5)

and the initial condition (2.4). Since \( u_m : [0, T] \to H^1_0(\Omega) \), we may write

\[
u_m(t) = \sum_{k=1}^{m} c^k_m(t) \ w_k.
\]

(2.6)

Substituting the expression (2.6) in (2.5) and (2.4) yields the following initial value problem for the system of ordinary differential equations satisfied by the coefficients \( c^k_m \) in (2.6) given by

\[
\begin{align*}
c^k_m' + \epsilon \sum_{j=1}^{d} \left( B(u_m) \frac{\partial u_m}{\partial x_j}, \frac{\partial w_k}{\partial x_j} \right) + \sum_{j=1}^{d} \left( f'_j(u_m) \frac{\partial u_m}{\partial x_j}, w_k \right) &= 0, \quad k = 1, 2, \ldots, m, \\
c^k_m(0) &= (u_0, w_k),
\end{align*}
\]

(2.7a)

(2.7b)

using the fact that the duality product \( \langle u'_m, w_k \rangle \) reduces to the \( L^2(\Omega) \) inner product of \( u'_m, w_k \in L^2(\Omega) \).

Setting \( c_m(t) = (c^1_m(t), c^2_m(t), \cdots, c^m_m(t))^T \), and \( g = (g_1, g_2, \cdots, g_m)^T \), the system of equations (2.7) may be written as \( c'_m = g(c_m) \), where

\[
g_i(c_m) = -\epsilon \sum_{j=1}^{d} \left( B \left( \sum_{k=1}^{m} c^k_m w_k \right) \sum_{k=1}^{m} c^k_m \frac{\partial w_k}{\partial x_j} \frac{\partial w_i}{\partial x_j} \right) + \sum_{j=1}^{d} \left( f'_j \left( \sum_{k=1}^{m} c^k_m(t) \ w_k \right) \sum_{k=1}^{m} c^k_m(t) \frac{\partial w_k}{\partial x_j} w_i \right),
\]

(2.8)

for \( i = 1, 2, \cdots, m \). It is a simple matter to check that the following inequality holds, in view of the expressions (2.8):

\[
\|g(c)\|_2 \leq P \|c\|_2 \quad \text{for all } c \in \mathbb{R}^m,
\]

(2.9)
where \( \| \cdot \|_2 \) denotes the Euclidean norm in \( \mathbb{R}^m \), and \( P \) is given by

\[
P = \sqrt{\sum_{i=1}^{d} M_{g_i}^2},
\]

where

\[
M_{g_i} = \left\{ \varepsilon \| B \|_{L^\infty(\mathbb{R})} \sum_{j=1}^{d} \left( \sum_{k=1}^{m} \left( \| \frac{\partial w_k}{\partial x_j} \|_{L^2(\Omega)}^2 \| \frac{\partial w_i}{\partial x_j} \|_{L^2(\Omega)}^2 \right)^\frac{1}{2} 
+ M \sum_{j=1}^{d} \left( \sum_{k=1}^{m} \left( \| \frac{\partial w_k}{\partial x_j} \|_{L^2(\Omega)}^2 \| w_{ij} \|_{L^2(\Omega)}^2 \right)^\frac{1}{2} \right) \right\},
\]

\[
M = \max_{1 \leq j \leq d} \left( \sup \left\{ |f_j'(y)| \mid y \in \mathbb{R} \right\} \right). \tag{2.10}
\]

Since \( g \) is continuous, it follows from a result in the theory of ordinary differential equations \[20\] p.73 that the initial value problem \((2.7)\) has a global solution. Thus each of the Galerkin approximations \( u_m \) are defined on the interval \([0, T]\).

The following result gives an a priori bound on the Galerkin approximations \( u_m \) and their derivatives \( u'_m \), which helps in extracting a subsequence of the sequence \((u_m)\) that serves our purpose (of establishing the existence of solutions to IBVP for generalized viscosity problem). Since its proof is on the lines of \[8\] p.354, we omit the same. The only difference in its proof when compared to \[8\] p.354 is the presence of terms involving the nonlinearity \( f_j'(\cdot), B(\cdot) \) which is estimated by its \( L^\infty(\mathbb{R}) \) norm.

**Theorem 2.1 (Energy Estimate)** There exists a constant \( C > 0 \) such that for every \( m \in \mathbb{N} \),

\[
\max_{0 \leq t \leq T} \| u_m \|_{L^2(\Omega)} + \| u'_m \|_{L^2(0,T;H^1_0(\Omega))} + \| u'_m \|_{L^2(0,T;H^{-1}(\Omega))} \leq C \| u_0 \|_{L^2(\Omega)} \tag{2.11}
\]

The next result asserts the existence of convergent subsequence of Galerkin approximations.

**Theorem 2.2** There exists a \( u \in L^2(0,T;L^2(\Omega)) \) and a subsequence of the sequence of Galerkin approximations \((u_m)\) that converges to \( u \) in \( L^2(0,T;L^2(\Omega)) \) as \( n \to \infty \). Consequently, a further subsequence of it converges pointwise for a.e. \( t \in [0,T] \) and a.e. \( x \in \Omega \).
We prove Theorem 2.2 by closely following the arguments from [13, p.469]. However for the sake of clarity, we provide all the details.

Lemma 2.1 [13, p.469] For each \( k \in \mathbb{N} \) and \( m \geq k \), let \( c^k_m \) be the coefficients defined by (2.6). Then

(i). there exists a subsequence \((m^{(1)}, m^{(2)}, \cdots, m^{(n)}, \cdots)\) of indices \( m \), and a sequence \((c^k)^{(i)}\) of functions in \( C[0,T]\) such that for each fixed \( k \in \mathbb{N} \) the sequence \((c^k_{m^{(n)}})_{m^{(n)} \geq k}\) converges to \( c^k \) in \( C[0,T] \).

(ii). for each fixed \( t \in [0,T] \), the sequence \((c^k)^{(i)}\) obtained in (i) belongs to \( \ell^2 \).

Proof:

Recall that for each \( k \in \mathbb{N} \) and every \( m \geq k \), the function \( c^k_m : [0,T] \to \mathbb{R} \) is given by

\[
  c^k_m(t) = (u_m(\cdot,t), w_k).
\]  

(2.12)

Uniform boundedness of the sequence \((c^k_m)_{m \geq k}\) is a direct consequence of the energy estimate (see Theorem 2.1). By a simple calculation involving the weak formulation (2.5) and using the Cauchy-Schwarz and Hölder inequalities, we arrive at

\[
  \left| c^k_m(t + h) - c^k_m(t) \right| \leq \|u_m\|_{L^2(0,T;H^1_0(\Omega))} \left( \epsilon \sum_{j=1}^d \left\| \frac{\partial w_k}{\partial x_j} \right\|_{L^2(\Omega)} + \sum_{j=1}^d \|f'_{j}\|_{\infty} \right) \sqrt{h}.
\]

Using the energy estimate and the last inequality, equicontinuity of the sequence \((c^k_m)_{m \geq k}\) follows. Applying Ascoli-Arzela theorem with \( k = 1 \), we conclude the existence of a subsequence \((m^{(1)}_1, m^{(2)}_1, \cdots, m^{(n)}_1, \cdots)\) of indices \( m \), and a \( c^1 \in C[0,T] \) such that the sequence \((c^1_{m^{(n)}_1})\) converges to \( c^1 \) uniformly on \([0,T]\). Applying Ascoli-Arzela theorem with \( k = 2 \) for the sequence \((c^2_{m^{(n)}_1})\), we get a further subsequence of the subsequence \((m^{(n)}_1)\) denoted by \( m^{(n)}_2 \), and a \( c^2 \in C[0,T] \) such that the sequence \((c^2_{m^{(n)}_2})\) converges to \( c^2 \) uniformly on \([0,T]\). Continuing in this fashion, we get a sequence \((c^k)\) of functions in \( C[0,T] \). By a diagonal argument we obtain the subsequence \((m^{(n)}_m)\) for which the assertion (i) of the lemma holds. For reasons of notational convenience we still index this subsequence using \( m \in \mathbb{N} \).

Let us turn to the proof of (ii) now. For each \( t \in [0,T] \) we have

\[
  \left| c^k(t) \right| = \lim_{m \to \infty} \left| (u_m(t), w_k) \right|,
\]
and hence we get
\[
\sum_{k=1}^{\infty} |c_k(t)|^2 = \sum_{k=1}^{\infty} \left( \lim_{m \to \infty} \left| (u_m(t), w_k) \right| \right)^2 = \lim_{p \to \infty} \left( \lim_{m \to \infty} \left( \sum_{k=1}^{p} \left| (u_m(t), w_k) \right|^2 \right) \right).
\]

Using Bessel’s inequality, the last equation yields
\[
\sum_{k=1}^{\infty} |c_k(t)|^2 \leq \limsup_{m \to \infty} \| u_m \|_{L^2(\Omega)}^2.
\]

Using the energy estimate again, we get \((c^k) \in \ell^2\). This completes the proof of the lemma.

In the proof of Theorem 2.2 the following lemma from [13] will be used.

**Lemma 2.2** [13, p.72] Let \(\{\psi_k\}_{k=1}^{\infty}\) be an orthonormal basis for \(L^2(\Omega)\). Then for any \(\nu > 0\), there exists a number \(N_\nu\) such that for all \(v \in H^1(\Omega)\) the inequality
\[
\|v\|_{L^2(\Omega)} \leq \left( \sum_{k=1}^{N_\nu} (v, \psi_k)^2 \right)^{\frac{1}{2}} + \nu \|v\|_{H^1(\Omega)}.
\]
(2.13)

holds.

**Proof of Theorem 2.2**: Let the subsequence of indices asserted by Lemma 2.1 still be denoted by \(m\) by dropping the superscript. For proving Theorem 2.2, it is enough to show that the sequence \(u_m\) is a Cauchy sequence in \(L^2(0, T; L^2(\Omega))\). Let \(\nu > 0\) be given. Let \(m_1, m_2\) be any two natural numbers satisfying \(m_1 \geq m_2\). Applying Lemma 2.2 with the given \(\nu > 0\), we get a \(N_\nu\) such that the inequality (2.13) holds for \(v = u_{m_1} - u_{m_2}\). Thus we have
\[
\|u_{m_1} - u_{m_2}\|_{L^2(\Omega)} \leq \left( \sum_{k=1}^{N_\nu} (u_{m_1} - u_{m_2}, w_k)^2 \right)^{\frac{1}{2}} + \nu \|u_{m_1} - u_{m_2}\|_{H^1(\Omega)}.
\}
(2.14)

Squaring the above inequality and using the inequality \(2ab \leq a^2 + b^2\) which is valid for any \(a, b \in \mathbb{R}\), we get
\[
\|u_{m_1} - u_{m_2}\|_{L^2(\Omega)}^2 \leq (1 + \nu) \sum_{k=1}^{N_\nu} (u_{m_1} - u_{m_2}, w_k)^2 + (\nu^2 + \nu) \|u_{m_1} - u_{m_2}\|_{H^1(\Omega)}^2.
\]
(2.15)
On the other hand, assertion (ii) of Lemma 2.1 says that the function

\[ q(x,t) := \sum_{k=1}^{\infty} c^k(t) w_k \]

is such that \( q(.,t) \in L^2(\Omega) \). Note that \( (u_m - q, w_j) = (u_m, w_j) - e^j(t) \). In view of (2.12), we have \( (u_m - q, w_j) \to 0 \) uniformly in \([0,T]\). Hence \( (u_m - q, w_j) \) is a Cauchy sequence in \( C[0,T] \). Thus, for the given \( \nu > 0 \), and the integer \( N_\nu > 0 \), there exists an \( N \in \mathbb{N} \) such that for \( m_1 \geq m_2 \geq N \) we get

\[ \sup_{t \in [0,T]} \left\{ \sum_{k=1}^{N_\nu} \left| (u_{m_1} - u_{m_2}, w_k) \right|^2 \right\} < \nu. \tag{2.16} \]

Integrating w.r.t. \( t \) variable on both sides of the inequality (2.15), and using (2.16), we obtain

\[ \int_0^T \| u_{m_1} - u_{m_2} \|^2_{L^2(\Omega)} dt \leq (1 + \nu) T \nu + (\nu^2 + \nu) \| u_{m_1} - u_{m_2} \|^2_{L^2(0,T;H^1_0(\Omega))} \tag{2.17} \]

for all \( m_1 \geq m_2 \geq N \). In view of the inequality

\[ \| u_{m_1} - u_{m_2} \|^2_{L^2(0,T;H^1_0(\Omega))} \leq 2 \left( \| u_{m_1} \|^2_{L^2(0,T;H^1_0(\Omega))} + \| u_{m_2} \|^2_{L^2(0,T;H^1_0(\Omega))} \right), \]

using energy estimate, the inequality (2.17) becomes

\[ \| u_{m_1} - u_{m_2} \|^2_{L^2(0,T;H^1_0(\Omega))} \leq (1 + \nu) T \nu + 4C^2(\nu^2 + \nu) \| u_0 \|^2_{L^2(\Omega)}. \]

This shows that the sequence \( (u_m) \) is Cauchy in \( L^2(0,T;L^2(\Omega)) \). This completes the proof of the theorem.

We now prove the existence of a weak solution to the problem (1.1).

**Theorem 2.3 (Existence of a weak solution)** There exists a weak solution of the generalized viscosity problem (1.1).

**Proof:**

Let \( u \) be as asserted by Theorem 2.2. We need to show that the function \( u \) satisfies (2.3a) and the initial condition (2.3b). Note that

\[ S = \left\{ \sum_{k=1}^{N} d^k(t) w_k : d^k \in C^1([0,T]), \ N \in \mathbb{N} \right\} \]
is a dense subspace of $L^2(0,T;H^1_0(\Omega))$. Let $v \in S$ be chosen such that $N \leq m$. Then by definition, we have the set of $N$ $C^1$ functions $\{d^k\}$. Multiplying the equation (2.5) by $d^k$, summing over $k = 1, 2, \ldots, N$, and integrating over the interval $[0,T]$, we get

$$
\int_0^T \langle u'_m, v \rangle \, dt + \epsilon \sum_{j=1}^d \int_0^T \left( \frac{\partial u_m}{\partial x_j}, B(u_m) \frac{\partial v}{\partial x_j} \right) \, dt + \sum_{j=1}^d \int_0^T \left( \frac{\partial u_m}{\partial x_j}, f'_j(u_m)v \right) \, dt = 0. 
$$

(2.18)

As a consequence of the energy estimate, Theorem 2.2, and using Banach-Alaoglu theorem, there exists a subsequence $\{u'_m\}_{l=1}^\infty$ of $\{u'_m\}_{m=1}^\infty$ (as usual, we drop the index $l$) such that $u'_m \rightharpoonup u'$ in $L^2(0,T;H^{-1}(\Omega))$. Therefore, we have

$$
\int_0^T \langle u'_m, v \rangle \, dt \rightarrow \int_0^T \langle u', v \rangle \, dt, \quad \forall v \in L^2(0,T;H^1_0(\Omega)).
$$

(2.19)

For $j = 1, 2, \ldots, d$, since $f'_j$ are $C^1$ functions and $B$ is a continuous function, by Theorem 2.2, for a.e. $(x,t) \in \Omega_T$ we have the following convergences as $m \rightarrow \infty$:

$$
f'_j(u_m) \rightarrow f'_j(u), \quad B(u_m) \frac{\partial v}{\partial x_j} \rightarrow B(u) \frac{\partial v}{\partial x_j}, \quad \text{in } L^2(0,T;L^2(\Omega)).
$$

(2.20)

By energy estimate (2.11), we have

$$
\frac{\partial u_m}{\partial x_j} \rightharpoonup \frac{\partial u}{\partial x_j} \quad \text{in } L^2(0,T;L^2(\Omega)).
$$

(2.21)

Using the information (2.19)-(2.21) in the equation (2.18), we get for each $v \in S$ with $N \leq m$,

$$
\int_0^T \langle u', v \rangle \, dt + \epsilon \sum_{j=1}^d \int_0^T \left( \frac{\partial u}{\partial x_j}, B(u) \frac{\partial v}{\partial x_j} \right) \, dt + \sum_{j=1}^d \int_0^T \left( f'_j(u) \frac{\partial v}{\partial x_j} \right) \, dt = 0.
$$

Since $S$ is dense in $L^2(0,T;H^1_0(\Omega))$, the above equation holds for all $v \in L^2(0,T;H^1_0(\Omega))$ and can be written as

$$
\int_0^T \langle u', v \rangle \, dt + \epsilon \sum_{j=1}^d \int_0^T \left( B(u) \frac{\partial u}{\partial x_j}, \frac{\partial v}{\partial x_j} \right) \, dt + \sum_{j=1}^d \int_0^T \left( f'_j(u) \frac{\partial u}{\partial x_j}, v \right) \, dt = 0. \quad (2.22)
$$

By choosing the test functions $v(x,t) = \varphi(t)w(x)$ where $\varphi \in \mathcal{D}(0,T)$, $w \in H^1_0(\Omega)$, we conclude that $u$ satisfies (2.3a).
We now turn to the proof of (2.3b). We know that $u \in L^2(0,T;H^1_0(\Omega))$ and $u' \in L^2(0,T;H^{-1}(\Omega))$. This implies that $u \in C([0,T];L^2(\Omega))$ and therefore $u(0) \in L^2(\Omega)$.

Choosing $v = c(t)w_k$ where $v \in C^1[0,T]$ is such that $c(0) = 1,c(T) = 1$ in the equation (2.22), integration by parts yields

$$
\int_0^T -(u,v') \ dt + \epsilon \sum_{j=1}^d \int_0^T \left( B(u) \frac{\partial u}{\partial x_j}, \frac{\partial v}{\partial x_j} \right) \ dt + \sum_{j=1}^d \int_0^T \left( f'_j(u) \frac{\partial u}{\partial x_j}, v \right) \ dt = (u(0),v(0)).
$$

(2.23)

Again from (2.18), get

$$
\int_0^T -(u_m,v') \ dt + \epsilon \sum_{j=1}^d \int_0^T \left( B(u_m) \frac{\partial u_m}{\partial x_j}, \frac{\partial v}{\partial x_j} \right) \ dt \\
+ \sum_{j=1}^d \int_0^T \left( f'_j(u_m) \frac{\partial u_m}{\partial x_j}, v \right) \ dt = (u_m(0),v(0)).
$$

(2.24)

Now passing to the limit in (2.24) as $m \to \infty$, we get

$$
\int_0^T -(u,v') \ dt + \epsilon \sum_{j=1}^d \int_0^T \left( B(u) \frac{\partial u}{\partial x_j}, \frac{\partial v}{\partial x_j} \right) \ dt + \sum_{j=1}^d \int_0^T \left( f'_j(u) \frac{\partial u}{\partial x_j}, v \right) \ dt = (u(0),v(0)).
$$

(2.25)

Since $(w_k)$ is an orthonormal for $L^2(\Omega)$, the equations (2.23) and (2.25) yield $u(0) = u_0$.

This completes the proof of the theorem.

2.2 Uniqueness

Any solution of generalized viscosity problem (1.1) satisfies the following maximum principle [9, p.60]:

**Theorem 2.4 (Maximum principle)** Let $f : \mathbb{R} \to \mathbb{R}^d$ be a $C^1$ function and $u_0 \in L^\infty(\Omega)$. Then any solution $u$ of generalized viscosity problem (1.1) in $W(0,T)$ satisfies the bound

$$
||u^\varepsilon(\cdot,t)||_{L^\infty(\Omega)} \leq ||u_0||_{L^\infty(\Omega)} \ a.e. \ t \in (0,T).
$$

(2.26)

We adapt the proof of a result in [13, p.150] concerning linear equations to our situation, and obtain the following uniqueness theorem.

**Theorem 2.5 (Uniqueness theorem)**[13, p.150] Let $f \in (C^1(\mathbb{R}))^d, B \in C^1(\mathbb{R}) \cap L^\infty(\mathbb{R}), u_0 \in L^\infty(\Omega)$. Then the generalized viscosity problem (1.1) has at most one solution in $W(0,T)$. 12
Let \( u_1 \) and \( u_2 \) be solutions of \((1.1)\) in \( W(0, T) \) and denote \( w := u_1 - u_2 \). In view of \((2.22)\), the function \( w \) satisfies the following equation for all \( \eta \in L^2(0, T; H^1_0(\Omega)) \):

\[
\int_0^T \int_\Omega \nu^{-1}(\alpha) \langle w_t, \eta \rangle \mu^1_\nu(\alpha) \, dt + \varepsilon \sum_{j=1}^d \int_0^T \int_\Omega \left( B(u_1) \frac{\partial u_1}{\partial x_j} - B(u_2) \frac{\partial u_2}{\partial x_j} \right) \frac{\partial \eta}{\partial x_j} \, dx \, dt \\
- \sum_{j=1}^d \int_0^T \int_\Omega (f_j(u_1) - f_j(u_2)) \frac{\partial \eta}{\partial x_j} \, dx \, dt = 0. \tag{2.27}
\]

Using fundamental theorem of calculus, we may write

\[
B(u_1) \frac{\partial u_1}{\partial x_j} - B(u_2) \frac{\partial u_2}{\partial x_j} = w \int_0^1 B'(\tau u_1 + (1 - \tau) u_2) \left[ \tau (u_1)_{x_j} + (1 - \tau) (u_2)_{x_j} \right] d\tau \\
+ \frac{\partial w}{\partial x_j} \int_0^1 B(\tau u_1 + (1 - \tau) u_2) \, d\tau,
\]

\[
f_j(u_1) - f_j(u_2) = w(x, t) \int_0^1 f_j'(\tau u_1 + (1 - \tau) u_2) \, d\tau, \quad j = 1, 2, \ldots, d.
\]

Thus the function \( w \) satisfies, in view of the equation \((2.27)\), for all \( \eta \in H^1(\Omega \times (0, T)) \) such that \( \eta(x, T) = 0 \) on \( \Omega \) and \( \eta = 0 \) on \( \partial \Omega \times (0, T) \),

\[
\int_0^T \int_\Omega \nu^{-1}(\alpha) \langle w_t, \eta \rangle \mu^1_\nu(\alpha) \, dt + \varepsilon \sum_{j=1}^d \int_0^T \int_\Omega \left( \tilde{a}(x, t) \frac{\partial w}{\partial x_j} + \tilde{b}_j(x, t) w \right) \frac{\partial \eta}{\partial x_j} \, dx \, dt \\
- \sum_{j=1}^d \int_0^T \int_\Omega \tilde{c}_j(x, t) w \frac{\partial \eta}{\partial x_j} \, dx \, dt = 0. \tag{2.28}
\]

where

\[
\tilde{a}(x, t) := \int_0^1 B(\tau u_1 + (1 - \tau) u_2) \, d\tau,
\]

\[
\tilde{b}_j(x, t) := \int_0^1 B'(\tau u_1 + (1 - \tau) u_2) \left[ \tau (u_1)_{x_j} + (1 - \tau) (u_2)_{x_j} \right] \, d\tau, \tag{2.29a}
\]

\[
\tilde{c}_j(x, t) := \int_0^1 f_j'(\tau u_1 + (1 - \tau) u_2) \, d\tau. \tag{2.29b}
\]

Thus \( w \) is a weak solution of

\[
\mathcal{L} w \equiv w_t - \varepsilon \sum_{j=1}^d \frac{\partial}{\partial x_j} \left( \tilde{a}(x, t) w_{x_j} \right) - \varepsilon \sum_{j=1}^d \frac{\partial}{\partial x_j} \left( \tilde{b}_j(x, t) w \right) + \sum_{j=1}^d \frac{\partial}{\partial x_j} \left( \tilde{c}_j(x, t) w \right) = 0 \quad \text{in} \ \Omega_T
\]

satisfying \( w(x, t) = 0 \) for \( (x, t) \in \partial \Omega \times (0, T) \), and \( w(x, 0) = 0 \) for \( x \in \Omega \). Let us introduce a sequence of auxiliary IBVPs corresponding to operators \( \mathcal{L}_m \), which are obtained by
regularizing the coefficients of the formal adjoint to $\mathcal{L}$, defined by the IBVP

$$\mathcal{L}_m(\eta) = h(x,t) \quad \text{in } \Omega_T,$$

$$\eta(x,t) = 0 \quad \text{on } \partial \Omega \times [0,T],$$

$$\eta(x,T) = 0 \quad \text{for } x \in \Omega,$$

where

$$\mathcal{L}_m(\eta) := -\eta_t - \varepsilon \sum_{j=1}^{d} \frac{\partial}{\partial x_j} (a^m(x,t)\eta_{x_j}) + \varepsilon \sum_{j=1}^{d} b^m_j(x,t)\eta_{x_j} - \sum_{j=1}^{d} c^m_j(x,t)\eta_{x_j},$$

where the coefficient functions $a^m, b^m_j, c^m_j$ are obtained by first extending the functions $\tilde{a}, \tilde{b}_j, \tilde{c}_j$ by zero outside $\Omega \times (0,T)$ and then regularizing the resultant functions using a sequence $\rho^m_k$, and then restricting the regularized functions to $\Omega_T$. Here we use standard sequence of mollifiers, but defined for $(x,t) \in \mathbb{R}^{d+1}$.

Study of the auxiliary problems (2.30) is based on the following existence-cum-higher regularity theorem for linear parabolic problems:

**Theorem 2.6** [13, p.320] Let $l > 0$ be a real number such that $l \notin \mathbb{N}$. Let the coefficients of the differential operator appearing in the IBVP

$$\frac{\partial u}{\partial t} - \sum_{i,j=1}^{d} a_{ij}(x,t) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{d} a_i(x,t) \frac{\partial u}{\partial x_i} + a(x,t)u = h(x,t) \quad \text{in } \Omega_T$$

$$u = \Phi \quad \text{on } \partial \Omega \times [0,T]$$

$$u(x,0) = u_0(x) \quad \text{on } \Omega \times \{t = 0\}$$

belong to the class $C^{l+2/l}(\overline{\Omega_T})$ and the boundary $\partial \Omega$ belongs to the class $C^{l+2}$. Then for any $h \in C^{l+2/l}(\overline{\Omega_T}), u_0 \in C^{l+2}(\overline{\Omega}), \Phi \in C^{l+2,1+2/l}(\overline{\Omega \times [0,T]})$ satisfying the compatibility condition of order $[l/2] + 1$ i.e., for $k = 0, 1, \ldots, [l/2] + 1$ and $x \in \partial \Omega$,

$$\left| \frac{\partial^k u}{\partial t^k}(x,t) \right|_{t=0} = \left| \frac{\partial^k \Phi}{\partial t^k}(x,t) \right|_{t=0}$$

holds. Then the problem (2.32) has a unique solution in $C^{l+2,1+2/l}(\overline{\Omega_T})$. Further $u$ satisfies

$$|u|_{l+2,\Omega_T} \leq C \left( |h|_{l,\Omega_T} + |u_0|_{l+2,\Omega} + |\Phi|_{l+2,\Omega_T} \right).$$

We have the following result concerning the auxiliary problems (2.30):

**Theorem 2.7** Let $m \in \mathbb{N}$, $0 < l < 1$. Let $h$ be in $C^{l,1}(\overline{\Omega_T})$ such that $h(.,T) = 0$. Then the IBVP (2.30) has a unique classical solution $\eta^m$. Further
1. for all $m \in \mathbb{N}$ and $(x, t) \in \overline{\Omega_T}$, we have
\[
\left| \eta^m(x, t) \right| \leq e^T \max_{(x, t) \in \Omega_T} \left| h(x, t) \right|.
\] (2.34)

2. there exists a $C > 0$ such that for all $m \in \mathbb{N}$, we have
\[
\| \nabla \eta^m \|_{L^2(\Omega_T)} \leq C.
\] (2.35)

**Proof of Theorem 2.7** Introducing the following change of variables
\[
\xi(x, t) = x, \quad \tau(x, t) = T - t,
\] (2.36)
and setting
\[
\eta(x, t) := U(\xi(x, t), \tau(x, t)),
\] (2.37)
the backward parabolic equation satisfied by $\eta$ in the IBVP (2.30) is transformed into a forward parabolic equation that $U$ satisfies and the IBVP satisfied by $U$ is given by
\[
U_r(\xi, \tau) - \varepsilon \sum_{j=1}^d \alpha^m(\xi, \tau)U_{\xi_j}(\xi, \tau) - \sum_{j=1}^d \beta^m_j(\xi, \tau)U_{\xi_j}(\xi, \tau) = \overline{h}(\xi, \tau) \text{ in } \Omega_T, \quad \text{(2.38a)}
\]
\[
U(\xi, \tau) = 0 \text{ on } \partial \Omega \times [0, T], \quad \text{(2.38b)}
\]
\[
U(\xi, 0) = 0 \text{ on } \Omega, \quad \text{(2.38c)}
\]
where
\[
\alpha^m(\xi, \tau) = a^m(\xi, T - \tau),
\]
\[
\beta^m_j(\xi, \tau) = \varepsilon \frac{\partial a^m(\xi, T - \tau)}{\partial \xi_j} - \varepsilon b^m_j(\xi, T - \tau) + c^m_j(\xi, T - \tau),
\]
\[
\overline{h}(\xi, \tau) = h(\xi, T - \tau).
\] (2.39)

On applying Theorem 2.6 to the IBVP (2.38), in view of (2.37), we get the existence of a unique classical solution $\eta^m$ to the IBVP (2.30).

The estimate (2.34) follows by applying Theorem 2.1 of [13, p.13] to the IBVP (2.38), and using equation (2.37).

We now turn to the proof of the estimate (2.35). Note that
\[
\int_0^T \int_\Omega h \eta^m \ dx \ dt = \int_0^T \int_\Omega \mathcal{L}_m(\eta^m) \eta^m \ dx \ dt
\]
\[
= - \int_0^T \int_{\mu^{-1}(0)} (\eta^m_\mu, \eta^m_\mu)_{\mu^{-1}(0)} \ dx \ dt + \varepsilon \sum_{j=1}^d \int_0^T \int_{\Omega} a^m(x, t) (\eta^m_{\xi_j})^2 \ dx \ dt
\]
\[
+ \varepsilon \sum_{j=1}^d \int_0^T \int_{\Omega} b^m_j(x, t) \eta^m_{\xi_j} \eta^m - \sum_{j=1}^d \int_0^T \int_{\Omega} c^m_j(x, t) \eta^m_{\xi_j} \eta^m \ dx \ dt.
\] (2.40)
Using $a^m \geq r$, $\eta^m(x, T) = 0$ and $\frac{d}{dt} \langle \eta^m, \eta^m \rangle = \frac{1}{2} \frac{d}{dt} \| \eta^m \|_{L^2(\Omega)}^2$ in equation (2.40), we arrive at

$$
\varepsilon \sum_{j=1}^d \int_0^T \int_\Omega \left( \eta_{x j}^m \right)^2 \, dx \, dt + \frac{1}{2} \| \eta^m(0) \|_{L^2(\Omega)}^2 \leq \| h \|_{L^\infty(\Omega_T)} \| \eta^m \|_{L^\infty(\Omega_T)} \text{Vol}(\Omega_T) + \varepsilon \| \eta^m \|_{L^\infty(\Omega_T)} \sum_{j=1}^d \int_0^T \int_\Omega \left| b_j^m(x, t) \right| \left| \eta_{x j}^m \right| \, dx \, dt \\
+ \| \eta^m \|_{L^\infty(\Omega_T)} \sum_{j=1}^d \int_0^T \int_\Omega \left| c_j^m(x, t) \right| \left| \eta_{x j}^m \right| \, dx \, dt.
$$

Let $\varepsilon < 1$. Using Hölder inequality, the inequality (2.41) yields

$$
\varepsilon \sum_{j=1}^d \| \eta_{x j}^m \|_{L^2(\Omega_T)}^2 \leq \| h \|_{L^\infty(\Omega_T)} \| \eta^m \|_{L^\infty(\Omega_T)} \text{Vol}(\Omega_T) + \sum_{j=1}^d \gamma_j \| \eta_{x j}^m \|_{L^2(\Omega_T)},
$$

where

$$
\gamma_j := \| \eta^m \|_{L^\infty(\Omega_T)} \| b_j^m \|_{L^2(\Omega_T)} + \| \eta^m \|_{L^\infty(\Omega_T)} \| c_j^m \|_{L^2(\Omega_T)}.
$$

Since $b_j^m \to \hat{b}_j$ and $c_j^m \to \hat{c}_j$ in $L^2(\Omega_T)$, there exist $R_j > 0$ such that for $m \in \mathbb{N}$

$$
\| b_j^m \|_{L^2(\Omega_T)} \leq R_j, \quad \| c_j^m \|_{L^2(\Omega_T)} \leq R_j.
$$

Using (2.34) and inequalities (2.43) in (2.42), we obtain

$$
\varepsilon \sum_{j=1}^d \| \eta_{x j}^m \|_{L^2(\Omega_T)}^2 \leq \| h \|_{L^\infty(\Omega_T)}^2 \text{Vol}(\Omega_T) + 2e^T \| h \|_{L^\infty(\Omega_T)} \sum_{j=1}^d R_j \| \eta_{x j}^m \|_{L^2(\Omega_T)}. \quad (2.44)
$$

From (2.44), we get for any $\alpha > 0$

$$
\varepsilon \sum_{j=1}^d \| \eta_{x j}^m \|_{L^2(\Omega_T)}^2 \leq \| h \|_{L^\infty(\Omega_T)}^2 \text{Vol}(\Omega_T) + \sum_{j=1}^d \alpha \| \eta_{x j}^m \|_{L^2(\Omega_T)}^2 + \frac{e^{2T} \| h \|_{L^\infty(\Omega_T)}^2}{\alpha} \sum_{j=1}^d R_j^2.
$$

(2.45)

By choosing an $\alpha$ satisfying $0 < \alpha < \frac{e^{2T}}{2}$, the inequality (2.45) yields

$$
\varepsilon \sum_{j=1}^d \| \eta_{x j}^m \|_{L^2(\Omega_T)}^2 \leq \alpha T \| h \|_{L^\infty(\Omega_T)}^2 \text{Vol}(\Omega_T) + \frac{e^{2T} \| h \|_{L^\infty(\Omega_T)}^2}{\alpha} \sum_{j=1}^d R_j^2.
$$

(2.46)
This proves (2.35), and completes the proof of Theorem 2.7.

**Proof of Theorem 2.5:**

Recall that $w := u - v$, where $u$ and $v$ are solutions to (1.1) in $W(0, T)$. We want to show that $w = 0$ a.e. $(x, t) \in \Omega_T$. For that, it is enough to show that for all $h \in C^{l, \frac{1}{2}}(\Omega_T)$ with $h(., T) = 0$ the following equality holds:

$$\int_{\Omega_T} w h \, dx \, dt = 0.$$  \hspace{1cm} (2.47)

For a given $h \in C^{l, \frac{1}{2}}(\Omega_T)$ with $h(., T) = 0$, let $\eta^m$ denote solution to the auxiliary IBVP (2.30) corresponding to the operator $L_m$. Note that

$$\int_{\Omega_T} w h \, dx \, dt = \int_{\Omega_T} w L_m(\eta^m) \, dx \, dt$$

$$= -\int_0^T \int_{\Omega_T} \langle \eta^m_t, w \rangle_{H^l(\Omega)} \, dt + \int_{\Omega_T} a^m(x, t) \eta^m_{x_j} w x_j \, dx \, dt$$

$$+ \varepsilon \int_{\Omega_T} b^m_j(x, t) \eta^m_{x_j} w \, dx \, dt - \int_{\Omega_T} c^m_j(x, t) \eta^m_{x_j} w \, dx \, dt.$$  \hspace{1cm} (2.48)

Applying integration by parts formula in $W(0, T)$ [17, p.427] and using (2.28) in (2.48), we get

$$\int_{\Omega_T} w h \, dx \, dt = \varepsilon \sum_{j=1}^d \int_{\Omega_T} \left[ a^m(x, t) - \hat{a}(x, t) \right] \eta^m_{x_j} w x_j \, dx \, dt$$

$$+ \varepsilon \sum_{j=1}^d \int_{\Omega_T} \left[ b^m_j(x, t) - \hat{b}_j(x, t) \right] \eta^m_{x_j} w \, dx \, dt$$

$$- \sum_{j=1}^d \int_{\Omega_T} \left[ c^m_j(x, t) - \hat{c}_j(x, t) \right] \eta^m_{x_j} w \, dx \, dt.$$  \hspace{1cm} (2.49)

Since $a^m \rightarrow \hat{a}$ in $L^2(\Omega_T)$, by passing to a subsequence, we may also assume that $a^m \rightarrow \hat{a}$ a.e. $\Omega_T$. We will continue to index the subsequence by $m$ itself. By a similar argument, we obtain pointwise convergences for the other sequences $b^m_j, c^m_j$. Thus in each of the three integral terms on RHS of (2.49), we have a sequence that goes to zero pointwise. It follows easily from the bounds given by Theorem 2.7 that these integrands also satisfy hypothesis of dominated convergence theorem. Thus we conclude that the RHS of (2.49) converges to zero as $m \rightarrow \infty$, and thereby completing the proof of the theorem.
2.3 Continuous dependence

Note that we have shown the IBVP (1.1) admits a unique solution in the space $W(0,T)$. This solution also turns out to be a classical solution i.e., it belongs to the space $C^{2+\beta,2+\beta}(\Omega_T)$. The following theorem follows by applying a result from [13, p.452].

**Theorem 2.8** Let $f, B, u_0$ satisfy Hypothesis A. Then there exists a unique solution $u^\varepsilon$ of (1.1) in the space $C^{2+\beta,2+\beta}(\Omega_T)$. Further, for each $i = 1, 2, \cdots, d$ the second order partial derivatives $u^\varepsilon_{x_i t}$ belong to $L^2(\Omega_T)$.

Let $u_0, v_0$ satisfy Hypothesis A. Whenever $u, v$ are solutions to the IBVP (1.1) belonging to $C^{2+\beta,2+\beta}(\Omega_T)$ satisfying the initial conditions $u_0, v_0$ respectively, the function $w := u - v$ satisfies a linear parabolic equation with variable coefficients that depend on $u$ and $v$. Applying the maximum principle for linear parabolic equations [13, p.13] yields the following continuous dependence result.

**Theorem 2.9 (Continuous Dependence)** Let $f, B, u_0, v_0$ satisfy Hypothesis A. Let $u, v$ be classical solutions to the IBVP (1.1) belonging to $C^{2+\beta,2+\beta}(\Omega_T)$ satisfying the initial conditions $u_0, v_0$ respectively. Then there exists a constant $C$ depending only on $T$ such that

$$
\|u - v\|_{L^\infty(\Omega_T)} \leq C\|u_0 - v_0\|_{L^\infty(\Omega)}.
$$

Thus we conclude the well-posedness of the IBVP (1.1).

3 Higher regularity result for viscous approximations

In this section we establish the following result which shows that the classical solutions to the IBVP (1.1) possess higher regularity when the initial conditions are more regular.

**Theorem 3.1 (higher regularity)** Let $f, B, u_0$ satisfy Hypothesis A. Then the solutions of the IBVP (1.1) belong to the space $C^{4+\beta,4+\beta}(\Omega_T)$. Further, $u^\varepsilon_{tt} \in C(\Omega_T)$.

Due to the presence of a quasilinear higher order term in the equation (1.1a), the standard methods of inductively proving higher regularity results by differentiating the equation as many times as needed fail. This is because the form of the equation obtained after differentiating the equation (1.1a) each differentiation is different. Since
we know that \( u^\varepsilon \) are classical solutions to (1.1), we apply the higher regularity result Theorem 2.6 for linear parabolic equations repeatedly, after deriving a linear parabolic equation satisfied by the solution to (1.1).

**Proof of Theorem 3.1:**

Since \( u^\varepsilon \) is a classical solution to the IBVP (1.1), we have

\[
\begin{align*}
u_i^\varepsilon - \sum_{j=1}^{d} \varepsilon B(u^\varepsilon) \frac{\partial^2 u^\varepsilon}{\partial x_j^2} + \sum_{i=1}^{d} \left( f'_i(u^\varepsilon) - \varepsilon B'(u^\varepsilon) \frac{\partial u^\varepsilon}{\partial x_i} \right) \frac{\partial u^\varepsilon}{\partial x_i} &= 0 \quad \text{in } \Omega_T, \\
u^\varepsilon(x,t) &= 0 \quad \text{on } \partial \Omega \times (0,T), \\
u^\varepsilon(x,0) &= u_0(x) \quad x \in \Omega.
\end{align*}
\]

(3.50a) (3.50b) (3.50c)

Thus we may view \( u^\varepsilon \) as a solution of the following IBVP for a linear parabolic equation

\[
\begin{align*}
v_i^\varepsilon - \sum_{j=1}^{d} \varepsilon B(u^\varepsilon) \frac{\partial^2 v^\varepsilon}{\partial x_j^2} + \sum_{i=1}^{d} \left( f'_i(u^\varepsilon) - \varepsilon B'(u^\varepsilon) \frac{\partial v^\varepsilon}{\partial x_i} \right) \frac{\partial v^\varepsilon}{\partial x_i} &= 0 \quad \text{in } \Omega_T, \\
v^\varepsilon(x,t) &= 0 \quad \text{on } \partial \Omega \times (0,T), \\
v^\varepsilon(x,0) &= u_0(x) \quad x \in \Omega.
\end{align*}
\]

(3.51a) (3.51b) (3.51c)

Thus we are in a position to utilize Theorem 2.6 after duly checking the relevant hypotheses on the coefficient functions

\[
B(u^\varepsilon), \ f'_i(u^\varepsilon), \ B'(u^\varepsilon) u^\varepsilon_{x_i} \ (i = 1, 2, \ldots, d),
\]

and initial-boundary data of the problem (3.51). Hypotheses on coefficient functions are checked by first writing out the relevant Hölder difference quotient, and estimating the same using mean value theorem, and bounding it from above by using the boundedness of the domain \( \Omega \), and of the partial derivatives of \( u^\varepsilon \), regularity of \( B, f \). Hence we give only a sample of these proofs. Strictly speaking we cannot apply mean value theorem since the domain \( \Omega \) may not be convex. However it turns out that Hölder continuous functions can always be extended to bigger convex domains with the same Hölder regularity, and preserving Hölder norms, see for example [13, p.297]. Thus we may extend \( u^\varepsilon \in C^{2+\beta, \frac{2+\beta}{2}}(\Omega_T) \) (by Theorem 2.8) to a bigger convex domain, and we still use \( u^\varepsilon \) to denote the extended function. Thus we are in a position to apply mean value theorem using the extended functions.

**Step 1:** We show that the coefficient functions have the property

\[
B(u^\varepsilon), \ f'_i(u^\varepsilon), \ B'(u^\varepsilon) u^\varepsilon_{x_i} \in C^{1+\beta, \frac{1+\beta}{2}}(\Omega_T).
\]
Note that for $k \in \{1, 2, \cdots, d\}$,
\[
\left[ \frac{\partial B(u^\varepsilon)}{\partial x_k} \right]_{x, \beta, \Omega_T} = \sup_{(x,t),(x',t) \in \Omega_T, x \neq x'} \frac{\left| B'(u^\varepsilon(x,t))u_{x_k}^\varepsilon(x,t) - B'(u^\varepsilon(x',t))u_{x_k}^\varepsilon(x',t) \right|}{|x-x'|^\beta},
\]
\[
\leq \sup_{(x,t),(x',t) \in \Omega_T, x \neq x'} \frac{\left| (B'(u^\varepsilon(x,t))u_{x_k}^\varepsilon(x,t) - B'(u^\varepsilon(x',t))u_{x_k}^\varepsilon(x',t)) \right|}{|x-x'|^\beta} + \sup_{(x,t),(x',t) \in \Omega_T, x \neq x'} \frac{\left| (B'(u^\varepsilon(x',t))u_{x_k}^\varepsilon(x,t) - B'(u^\varepsilon(x',t))u_{x_k}^\varepsilon(x',t)) \right|}{|x-x'|^\beta}.
\]
(3.52)

Applying mean value theorem, the inequality (3.52) yields
\[
\left[ \frac{\partial B(u^\varepsilon)}{\partial x_k} \right]_{x, \beta, \Omega_T} \leq \sup_{(x,t) \in \Omega_T} \left| \frac{\partial u^\varepsilon}{\partial x_k} \right| \sup_{(x,t),(x',t) \in \Omega_T, x \neq x'} \left| \nabla_x \left( B'(u^\varepsilon) \right) (\xi(x,t),(x',t)) \right| |x-x'|^{1-\beta}
\]
\[
+ \|B'\|_{L^\infty(I)} \left[ \frac{\partial u^\varepsilon}{\partial x_k} \right]_{x, \beta, \Omega_T}.
\]
(3.53)

Applying similar arguments, we get
\[
\left[ \frac{\partial B(u^\varepsilon)}{\partial x_k} \right]_{t, \frac{\beta}{2}, \Omega_T} \leq \sup_{(x,t) \in \Omega_T} \left| \frac{\partial u^\varepsilon}{\partial x_k} \right| \sup_{(x,t),(x',t) \in \Omega_T, t \neq t'} \left| \frac{\partial}{\partial t} \left( B'(u^\varepsilon) \right) (\xi(x,t),(x',t)) \right| |t-t'|^{1-\beta}
\]
\[
+ \|B'\|_{L^\infty(I)} \left[ \frac{\partial u^\varepsilon}{\partial x_k} \right]_{t, \frac{\beta}{2}, \Omega_T}.
\]
(3.54)

Note that the quantities
\[
\left[ \frac{\partial u^\varepsilon}{\partial x_k} \right]_{x, \beta, \Omega_T}, \left[ \frac{\partial u^\varepsilon}{\partial x_k} \right]_{t, \frac{\beta}{2}, \Omega_T}
\]
(3.55)
are finite due to the fact that $u^\varepsilon \in C^{2+\beta,\frac{4+\beta}{2}}(\Omega_T)$, and also since it was extended with the same regularity to a bigger convex domain. In view of this observation, and the fact that $\text{diam}(\Omega) < \infty$, $B \in C^3(\mathbb{R})$, therefore RHS of (3.53) and (3.54) are finite.

Let us now show that $f_i'(u^\varepsilon) \in C^{1+\beta,\frac{1+\beta}{2}}(\Omega_T)$. Note that
\[
\left[ \frac{\partial f_i'(u^\varepsilon)}{\partial x_k} \right]_{x, \beta, \Omega_T} = \sup_{(x,t),(x',t) \in \Omega_T, x \neq x'} \frac{\left| f_i'(u^\varepsilon(x,t))u_{x_k}^\varepsilon(x,t) - f_i'(u^\varepsilon(x',t))u_{x_k}^\varepsilon(x',t) \right|}{|x-x'|^\beta},
\]
\[
\leq \sup_{(x,t),(x',t) \in \Omega_T, x \neq x'} \frac{\left| (f_i'(u^\varepsilon(x,t)) - f_i'(u^\varepsilon(x',t)))u_{x_k}^\varepsilon(x,t) \right|}{|x-x'|^\beta} + \sup_{(x,t),(x',t) \in \Omega_T, x \neq x'} \frac{\left| f_i'(u^\varepsilon(x',t)) (u_{x_k}^\varepsilon(x,t) - u_{x_k}^\varepsilon(x',t)) \right|}{|x-x'|^\beta}.
\]
(3.56)
\]
Using mean value theorem in (3.56), we obtain
\[
\left[ \frac{\partial f_i'(u^\varepsilon)}{\partial x_k} \right]_{(t, \frac{1}{2}, \Omega_T)} \leq \sup_{(x,t),(x',t') \in \Omega_T, x \neq x'} \left| \nabla_x \left( f_i''(u^\varepsilon) \right) \right| \left( \xi(x,t),(x',t) \right) \left| u^\varepsilon_{x_k} \right| |x - x'|^{1-\beta} \\
+ \| f_i'' \|_{L^\infty(t)} \sup_{(x,t),(x',t') \in \Omega_T, x \neq x'} \left| \nabla_x \left( u^\varepsilon_{x_k} \right) \right| |x - x'|^{1-\beta}.
\]

(3.57)

Since \( u^\varepsilon \in C^{2,1}(\Omega_T) \), \( f \in C^3(\mathbb{R}) \) and \( \text{diam}(\Omega) < \infty \), therefore the RHS of (3.57) is finite.

We will now show that \( \left[ \frac{\partial f_i'(u^\varepsilon)}{\partial x_k} \right]_{(t, \frac{1}{2}, \Omega_T)} < \infty \). Note that
\[
\left[ \frac{\partial f_i'(u^\varepsilon)}{\partial x_k} \right]_{(t, \frac{1}{2}, \Omega_T)} = \sup_{(x,t),(x',t') \in \Omega_T, t \neq t'} \left| f_i''(u^\varepsilon(x,t)) u^\varepsilon_{x_k}(x,t) - f_i''(u^\varepsilon(x',t')) u^\varepsilon_{x_k}(x,t) \right| |t - t'|^{\frac{\beta}{2}} \\
\leq \sup_{(x,t),(x',t') \in \Omega_T, t \neq t'} \left| f_i'' \left( u^\varepsilon(x,t) \right) - f_i'' \left( u^\varepsilon(x',t') \right) \right| \left| u^\varepsilon_{x_k} \right| |t - t'|^{\frac{\beta}{2}} \\
+ \| f_i'' \|_{L^\infty(t)} \sup_{(x,t),(x',t') \in \Omega_T, t \neq t'} \left| u^\varepsilon_{x_k}(x,t) - u^\varepsilon_{x_k}(x,t') \right| |t - t'|^{\frac{\beta}{2}}.
\]

Applying mean value theorem, we get
\[
\left[ \frac{\partial f_i'(u^\varepsilon)}{\partial x_k} \right]_{(t, \frac{1}{2}, \Omega_T)} \leq T^{-\frac{\beta}{2}} \sup_{(x,t),(x',t') \in \Omega_T, t \neq t'} \left| \frac{\partial}{\partial t} \left( f_i''(u^\varepsilon) \right) \right| \left( \xi(x,t),(x',t') \right) \sup_{(x,t) \in \Omega_T} \left| \nabla u^\varepsilon \right| \\
+ \| f_i'' \|_{L^\infty(t)} \left[ u^\varepsilon_{x_k} \right]_{(t, \frac{1}{2}, \Omega_T)}.
\]

(3.58)

Since \( u^\varepsilon \in C^{2,1}(\Omega_T) \), the RHS of (3.58) is finite.

Checking of \( B'(u^\varepsilon) u^\varepsilon_{x_1} \in C^{1+\beta,\frac{4+\beta}{2}}(\Omega_T) \) follows on similar lines, and we shall omit the computations.

Since \( u_0 \) has compact support in \( \Omega \), it satisfies the compatibility conditions (2.33), an application of Theorem 2.6 asserts the existence of a unique solution \( u^\varepsilon \) to the IBVP (3.51) in \( C^{3+\beta,\frac{3+\beta}{2}}(\Omega_T) \). Since \( C^{3+\beta,\frac{3+\beta}{2}}(\Omega_T) \) is a subset of \( C^{2+\beta,\frac{4+\beta}{2}}(\Omega_T) \), it follows that \( u^\varepsilon = v^\varepsilon \). Thus \( u^\varepsilon \in C^{3+\beta,\frac{3+\beta}{2}}(\Omega_T) \).

**Step 2:** By following the procedure outlined and implemented in Step 1, it can be shown that the coefficient functions belong to the space \( C^{2+\beta,\frac{4+\beta}{2}}(\Omega_T) \), and by using the information which is coming from Steps 1, namely, \( u^\varepsilon \in C^{3+\beta,\frac{3+\beta}{2}}(\Omega_T) \). Since \( u_0 \) has compact support in \( \Omega \), it satisfies the compatibility conditions (2.33), an application of Theorem 2.6 asserts the existence of a unique solution \( z^\varepsilon \) to the IBVP (3.51) in
4 BV estimates

In this section we prove the following result concerning BV estimates on the sequence of viscous approximations \((u^\varepsilon)\). For definition of BV functions, we refer the reader to [9].

**Theorem 4.1 (BV estimate)** Let \(f, B, u_0\) satisfy Hypothesis A. Let \((u^\varepsilon)\) be the sequence of solutions to the IBVP (1.1). Then there exists a \(C > 0\) such that for all \(\varepsilon > 0\), the following estimate holds:

\[
\|\partial u^\varepsilon / \partial t\|_{L^1(\Omega_T)} + \|\nabla u^\varepsilon\|_{L^1(\Omega_T)}^d \leq C. \tag{4.61}
\]

Further there exists a subsequence \((u^{\varepsilon_k})\) of \((u^\varepsilon)\), and a function \(u \in L^1(\Omega_T)\) such that

\[
\begin{align*}
    u^{\varepsilon_k} \to u & \text{ in } L^1(\Omega_T), \tag{4.62} \\
    u^{\varepsilon_k} \to & u \text{ a.e. } (x, t) \in \Omega_T \tag{4.63}
\end{align*}
\]

as \(k \to \infty\).

Bardos et. al. [2] established uniform \(L^1\)-estimates on first order derivatives of \(u^\varepsilon\), when \(B(u) \equiv 1\), using suitable multipliers. When \(B\) is a non-constant function, we are unable
to estimate the first order spatial derivatives of $u^\epsilon$ using their multipliers, and we use a different multiplier for this purpose.

Let $sg_n$ ($n \in \mathbb{N}$) be the sequence of functions which converges pointwise to the signum function $sg$ which are defined for $s \in \mathbb{R}$ by

$$sg_n(s) = \begin{cases} 
1 & \text{if } s > \frac{1}{n}, \\
ns & \text{if } |s| \leq \frac{1}{n}, \\
-1 & \text{if } s < -\frac{1}{n}, 
\end{cases} \quad sg(s) = \begin{cases} 
1 & \text{if } s > 0, \\
0 & \text{if } s = 0, \\
-1 & \text{if } s < 0. 
\end{cases}$$

The following result will be used in proving the required BV estimate.

Lemma 4.1 [2, p.1020] Let $v \in C^1(\Omega)$. Then

$$\lim_{n \to \infty} \int_{\{x \in \Omega : |v(x)| < \frac{1}{n}\}} |\nabla v| \, dx = 0.$$  

Proof of Theorem 4.1: We prove Theorem 4.1 in three steps. The estimate (4.61) will be established in the first two steps, and in the third step we deduce (4.62).

Step 1: We show the existence of constant $C_1 > 0$ such that for every $\epsilon > 0$,

$$\left\| \frac{\partial u^\epsilon}{\partial t} \right\|_{L^1(\Omega_T)} \leq C_1. \quad (4.64)$$

Differentiating the equation (1.1a) with respect to $t$, multiplying by $sg_n \left( \frac{\partial u^\epsilon}{\partial t} \right)$ and integrating over $\Omega$, we get

$$\int_{\Omega} u^\epsilon_{tt} sg_n(u^\epsilon_t) \, dx + \sum_{j=1}^{d} \int_{\Omega} \left[ \frac{\partial}{\partial x_j} \left( f_j'(u^\epsilon) \frac{\partial u^\epsilon}{\partial t} \right) \right] sg_n \left( \frac{\partial u^\epsilon}{\partial t} \right) \, dx$$

$$= \epsilon \sum_{j=1}^{d} \int_{\Omega} sg_n \left( \frac{\partial u^\epsilon}{\partial t} \right) \frac{\partial}{\partial x_j} \left( B'(u^\epsilon) \frac{\partial u^\epsilon}{\partial t} \frac{\partial u^\epsilon}{\partial x_j} + B(u^\epsilon) \frac{\partial u^\epsilon}{\partial x_j} \left( \frac{\partial u^\epsilon}{\partial t} \right) \right) \, dx. \quad (4.65)$$

Using integration by parts in (4.65) and using $sg_n \left( \frac{\partial u^\epsilon}{\partial t} \right) = 0$ on $\partial \Omega \times (0, T)$, we have

$$\int_{\Omega} u^\epsilon_{tt} sg_n(u^\epsilon_t) \, dx = \sum_{j=1}^{d} \int_{\Omega} f_j'(u^\epsilon) \frac{\partial u^\epsilon}{\partial t} sg_n' \left( \frac{\partial u^\epsilon}{\partial t} \right) \frac{\partial}{\partial x_j} \left( \frac{\partial u^\epsilon}{\partial t} \right) \, dx$$

$$- \epsilon \sum_{j=1}^{d} \int_{\Omega} B'(u^\epsilon) \frac{\partial u^\epsilon}{\partial t} \frac{\partial u^\epsilon}{\partial x_j} sg_n' \left( \frac{\partial u^\epsilon}{\partial t} \right) \frac{\partial}{\partial x_j} \left( \frac{\partial u^\epsilon}{\partial t} \right) \, dx$$

$$- \epsilon \sum_{j=1}^{d} \int_{\Omega} B(u^\epsilon) \left( \frac{\partial}{\partial x_j} \left( \frac{\partial u^\epsilon}{\partial t} \right) \right)^2 sg_n' \left( \frac{\partial u^\epsilon}{\partial t} \right) \, dx. \quad (4.66)$$
We now prove that first two terms on the RHS of (4.66) tend to zero as $\varepsilon \to 0$. That is,

$$\lim_{n \to \infty} \sum_{j=1}^{d} \int_{\Omega} f'_j(u^\varepsilon) \frac{\partial u^\varepsilon}{\partial t} s g'_n \left( \frac{\partial u^\varepsilon}{\partial t} \right) \frac{\partial}{\partial x_j} \left( \frac{\partial u^\varepsilon}{\partial t} \right) dx = 0, \quad (4.67)$$

$$\lim_{n \to \infty} \sum_{j=1}^{d} \int_{\Omega} B'(u^\varepsilon) \frac{\partial u^\varepsilon}{\partial t} \frac{\partial u^\varepsilon}{\partial x_j} s g'_n \left( \frac{\partial u^\varepsilon}{\partial t} \right) \frac{\partial}{\partial x_j} \left( \frac{\partial u^\varepsilon}{\partial t} \right) dx = 0. \quad (4.68)$$

**Proof of (4.67):** Since $\left| \frac{\partial u^\varepsilon}{\partial t} \right| s g'_n \left( \frac{\partial u^\varepsilon}{\partial t} \right) < 1$, note that

$$\left| \sum_{j=1}^{d} \int_{\{x \in \Omega : |\frac{\partial u^\varepsilon}{\partial t}| < \frac{1}{n}\}} \frac{\partial u^\varepsilon}{\partial t} s g'_n \left( \frac{\partial u^\varepsilon}{\partial t} \right) (f'_1(u^\varepsilon), \ldots, f'_d(u^\varepsilon)) \cdot \nabla \left( \frac{\partial u^\varepsilon}{\partial t} \right) dx \right| \leq \sqrt{d} \max_{1 \leq j \leq d} \left( \sup_{y \in I} |f'_j(y)| \right) \int_{\{x \in \Omega : |\frac{\partial u^\varepsilon}{\partial t}| < \frac{1}{n}\}} \left| \nabla \left( \frac{\partial u^\varepsilon}{\partial t} \right) \right| dx. \quad (4.69)$$

Applying Lemma 4.1 with $v = \frac{\partial u^\varepsilon}{\partial t}$, the inequality (4.69) gives (4.67).

**Proof of (4.68):** Observe that

$$\left| \varepsilon \sum_{j=1}^{d} \int_{\Omega} B'(u^\varepsilon) \frac{\partial u^\varepsilon}{\partial t} \frac{\partial u^\varepsilon}{\partial x_j} s g'_n \left( \frac{\partial u^\varepsilon}{\partial t} \right) \frac{\partial}{\partial x_j} \left( \frac{\partial u^\varepsilon}{\partial t} \right) dx \right| \leq \varepsilon \sqrt{d} \|B'\|_{L^\infty(I)} \max_{1 \leq j \leq d} \left( \left\| \frac{\partial u^\varepsilon}{\partial x_j} \right\|_{L^\infty(\Omega_T)} \right) \int_{\{x \in \Omega : |\frac{\partial u^\varepsilon}{\partial t}| < \frac{1}{n}\}} \left| \nabla \left( \frac{\partial u^\varepsilon}{\partial t} \right) \right| dx. \quad (4.70)$$

Applying Lemma 4.1 with $v = \frac{\partial u^\varepsilon}{\partial t}$, the inequality (4.70) yields (4.68).

Since the third term on RHS of (4.71) is non-positive for every $\varepsilon > 0$, on taking limit supremum on both sides of (4.66) yields

$$\limsup_{n \to \infty} \int_{\Omega} u^\varepsilon_t s g_n(u^\varepsilon_t) dx \leq 0 \quad (4.71)$$

in view of (4.67) and (4.68). Note that the limit supremum in (4.71) is actually a limit, and as a consequence we get

$$\int_{\Omega} \frac{\partial}{\partial t} |u^\varepsilon_t| dx \leq 0 \quad (4.72)$$

Integrating w.r.t. $t$ on both sides of (4.72), and applying Fubini’s theorem yields

$$\int_{0}^{t} \int_{\Omega} \frac{\partial}{\partial t} |u^\varepsilon_t| d\tau dx \leq 0 \quad (4.73)$$
Thus we get
\[
\int_{\Omega} (|u^\varepsilon(x,t)| - |u^\varepsilon_i(x,0)|) \, dx \leq 0 \tag{4.74}
\]

From the equation (1.1a), we get
\[
\frac{\partial u^\varepsilon}{\partial t}(x,0) = \varepsilon \sum_{j=1}^{d} \left[ B(u_0) \frac{\partial^2 u^\varepsilon}{\partial x_j^2}(x,0) + B'(u_0) \left( \frac{\partial u^\varepsilon}{\partial x_j} \right)^2(x,0) \right] - \sum_{j=1}^{d} f_j(u_0) \frac{\partial u^\varepsilon}{\partial x_j}(x,0) \tag{4.75}
\]

Since \( u_0 \in C^{4+\beta} (\Omega) \), taking
\[
\frac{C_1}{T d \text{Vol}(\Omega)} = \|B\|_{L^\infty(\Omega)} \max_{1 \leq j \leq d} \left( \sup_{x \in \Omega} \left| \frac{\partial^2 u_0}{\partial x_j^2} \right| \right) + \|B' \|_{L^\infty(\Omega)} \max_{1 \leq j \leq d} \left( \sup_{x \in \Omega} \left| \frac{\partial u_0}{\partial x_j} \right|^2 \right) + \|f' \|_{(L^\infty(\Omega))^d} \|\nabla u_0\|_{(L^\infty(\Omega))^d}, \tag{4.76}
\]
we get (4.64).

**Step 2:** In Step 2, we show that
\[
\|\nabla u^\varepsilon\|_{(L^1(\Omega_T))^d} = \|\nabla u_0\|_{(L^1(\Omega))^d}. \tag{4.77}
\]

Generalized viscosity problem (1.1) can be written in the following non-divergence form
\[
u^\varepsilon_i + \nabla \cdot f(u^\varepsilon) = \varepsilon \sum_{j=1}^{d} \left( B'(u^\varepsilon) \left( \frac{\partial u^\varepsilon}{\partial x_j} \right)^2 + B(u^\varepsilon) \frac{\partial^2 u^\varepsilon}{\partial x_j^2} \right) \quad \text{in } \Omega_T, \tag{4.78a}
\]
\[
u^\varepsilon(x,t) = 0 \quad \text{on } \partial \Omega \times (0,T), \tag{4.78b}
\]
\[
u^\varepsilon(x,0) = u_0(x) \quad x \in \Omega, \tag{4.78c}
\]

For \( t \in (0,T) \) and \( i \in \{1,2,\ldots,d\} \), differentiating the equation (4.78a) with respect to \( x_i \), multiplying by \( sgn \left( \frac{\partial u^\varepsilon}{\partial x_i} \right) \) and using \( \frac{\partial u^\varepsilon}{\partial x_i} \in H^1(\Omega \times (0,T)) \), we get
\[
\frac{\partial}{\partial t} \left( \frac{\partial u^\varepsilon}{\partial x_i} \right) sgn \left( \frac{\partial u^\varepsilon}{\partial x_i} \right) + \sum_{j=1}^{d} sgn \left( \frac{\partial u^\varepsilon}{\partial x_j} \right) \frac{\partial^2 u^\varepsilon}{\partial x_i \partial x_j} f_j(u^\varepsilon) \tag{4.79}
\]

Summing (4.79) over \( i = 1,2,\ldots,d \) and integrating over \( \Omega \), we obtain
\[
\sum_{i=1}^{d} \int_{\Omega} \frac{\partial}{\partial t} \left( \frac{\partial u^\varepsilon}{\partial x_i} \right) sgn \left( \frac{\partial u^\varepsilon}{\partial x_i} \right) \, dx + \sum_{i,j=1}^{d} \int_{\Omega} sgn \left( \frac{\partial u^\varepsilon}{\partial x_j} \right) \frac{\partial}{\partial x_i} \left( f_j(u^\varepsilon) \frac{\partial u^\varepsilon}{\partial x_j} \right) \, dx \tag{4.80}
\]
We pass to limit as $n \to \infty$ in each of the terms in (4.80) below.

(i) Passage to limit on RHS of (4.80) as $n \to \infty$:

Using integration by parts on RHS of (4.80), we get

$$
\varepsilon \sum_{i,j=1}^{d} \int_{\Omega} s_{n} \left( \frac{\partial u^{\varepsilon}}{\partial x_{i}} \right) \frac{\partial}{\partial x_{i}} \left( B'(u^{\varepsilon}) \left( \frac{\partial u^{\varepsilon}}{\partial x_{j}} \right)^{2} + B(u^{\varepsilon}) \frac{\partial^{2} u^{\varepsilon}}{\partial x_{j}^{2}} \right) \, dx
$$

$$
= -\varepsilon \sum_{i,j=1}^{d} \int_{\Omega} s'_{n} \left( \frac{\partial u^{\varepsilon}}{\partial x_{i}} \right) \frac{\partial^{2} u^{\varepsilon}}{\partial x_{i}^{2}} \left( B'(u^{\varepsilon}) \left( \frac{\partial u^{\varepsilon}}{\partial x_{j}} \right)^{2} + B(u^{\varepsilon}) \frac{\partial^{2} u^{\varepsilon}}{\partial x_{j}^{2}} \right) \, dx
$$

$$
+ \varepsilon \sum_{i,j=1}^{d} \int_{\partial \Omega} s_{n} \left( \frac{\partial u^{\varepsilon}}{\partial x_{i}} \right) \left( B(0) \frac{\partial^{2} u^{\varepsilon}}{\partial x_{j}^{2}} + B'(0) \left( \frac{\partial u^{\varepsilon}}{\partial x_{j}} \right)^{2} \right) \sigma_{i} \, d\sigma. \quad (4.81)
$$

Firstly, we want to show that

$$
\lim_{n \to \infty} \sum_{i,j=1}^{d} \int_{\Omega} s'_{n} \left( \frac{\partial u^{\varepsilon}}{\partial x_{i}} \right) \frac{\partial^{2} u^{\varepsilon}}{\partial x_{i}^{2}} \left( B'(u^{\varepsilon}) \left( \frac{\partial u^{\varepsilon}}{\partial x_{j}} \right)^{2} + B(u^{\varepsilon}) \frac{\partial^{2} u^{\varepsilon}}{\partial x_{j}^{2}} \right) \, dx = 0. \quad (4.82)
$$

For $i \in \{1, 2, \cdots, d\}$, denote

$$
A_{i}^{\varepsilon} := \left\{ x \in \Omega : \frac{\partial u^{\varepsilon}}{\partial x_{i}} = 0 \right\}.
$$

Since $\frac{\partial u^{\varepsilon}}{\partial x_{i}} \in C^{1}(\Omega)$, Stampacchia’s theorem (see [10]) gives $\nabla \left( \frac{\partial u^{\varepsilon}}{\partial x_{i}} \right) = 0$ a.e. $x \in A_{i}^{\varepsilon}$. In particular, $\frac{\partial^{2} u^{\varepsilon}}{\partial x_{i}^{2}} = 0$ a.e. $x \in A_{i}^{\varepsilon}$. For $i \in \{1, 2, \cdots, d\}$, if $\Omega \setminus A_{i}^{\varepsilon} = \emptyset$, then (4.82) follows trivially. Assuming that $\Omega \setminus A_{i}^{\varepsilon} \neq \emptyset$, for each $i, j \in \{1, 2, \cdots, d\}$ and for each $x \in \Omega \setminus A_{i}^{\varepsilon}$, we have

$$
sg'_{n} \left( \frac{\partial u^{\varepsilon}}{\partial x_{i}} \right) \frac{\partial^{2} u^{\varepsilon}}{\partial x_{i}^{2}} \left( B'(u^{\varepsilon}) \left( \frac{\partial u^{\varepsilon}}{\partial x_{j}} \right)^{2} + B(u^{\varepsilon}) \frac{\partial^{2} u^{\varepsilon}}{\partial x_{j}^{2}} \right) \to 0 \text{ as } n \to \infty. \quad (4.83)
$$

Note that

$$
\left| - \sum_{i,j=1}^{d} \int_{\Omega \setminus A_{i}^{\varepsilon}} sg'_{n} \left( \frac{\partial u^{\varepsilon}}{\partial x_{i}} \right) \frac{\partial^{2} u^{\varepsilon}}{\partial x_{i}^{2}} \left( B'(u^{\varepsilon}) \left( \frac{\partial u^{\varepsilon}}{\partial x_{j}} \right)^{2} + B(u^{\varepsilon}) \frac{\partial^{2} u^{\varepsilon}}{\partial x_{j}^{2}} \right) \, dx \right|
$$

$$
\leq \sum_{i,j=1}^{d} \int_{\Omega \setminus A_{i}^{\varepsilon}} sg'_{n} \left( \frac{\partial u^{\varepsilon}}{\partial x_{i}} \right) \frac{\partial^{2} u^{\varepsilon}}{\partial x_{i}^{2}} \left| B'(u^{\varepsilon}) \left( \frac{\partial u^{\varepsilon}}{\partial x_{j}} \right)^{2} + B(u^{\varepsilon}) \frac{\partial^{2} u^{\varepsilon}}{\partial x_{j}^{2}} \right| \, dx. \quad (4.84)
$$

Let $t \in (0, T)$ be fixed. For $i \in \{1, 2, \cdots, d\}$, let $x_{i0}^{\varepsilon} \in A_{i}^{\varepsilon}$. Since $\Omega$ is bounded, there exists $R' > 0$ such that $\Omega \subset B(x_{i0}^{\varepsilon}, R')$, where $B(x_{i0}, R')$ denotes the open ball with center at $x_{i0}$ and having radius $R'$. Consequently, $\Omega \subset B(x_{i0}^{\varepsilon}, 2R')$. 26
For each $n \in \mathbb{N}$ and $i \in \{1, 2, \cdots, d\}$, denote
\[
C_{n,i}^e := \left\{ x \in \Omega \setminus A_i^e : 0 < \left| \frac{\partial u^e}{\partial x_i}(x,t) \right| \leq \frac{1}{n} \right\}.
\] (4.85)

Note that for each $n \in \mathbb{N}$, we have $C_{(n+1),i}^e \subseteq C_{n,i}^e$. Let $n_0 \in \mathbb{N}$ be such that for all $n \geq n_0$, the following inclusion holds:
\[
C_{n,i}^e \subseteq B(x_{i0}^e, \frac{n}{2}).
\]

Define a function $\rho \in C^\infty(\mathbb{R}^d)$ by
\[
\rho(x) := \begin{cases} \rho \exp \left( -\frac{1}{1-|x-x_{i0}^e|^2} \right), & \text{if } x \in B(x_{i0}^e, 1) \\ 0, & \text{if } x \notin B(x_{i0}^e, 1), \end{cases}
\] (4.86)

where the constant $k_\varepsilon$ is chosen so that
\[
\int_{\mathbb{R}^d} \rho(x) \, dx = 1.
\] (4.87)

Denote the sequence of mollifiers $\rho_{n,i} : \mathbb{R}^d \to \mathbb{R}$ by
\[
\rho_{n,i}(x) := \begin{cases} k_\varepsilon n^d \exp \left( -\frac{n^2}{n^2 - |x-x_{i0}^e|^2} \right), & \text{if } x \in B(x_{i0}^e, n) \\ 0, & \text{if } x \notin B(x_{i0}^e, n) \end{cases}
\] (4.88)

Since for each $n \geq n_0$,
\[
s g_n^\varepsilon \left( \frac{\partial u^e}{\partial x_i} \right) := \begin{cases} n & \text{if } 0 \leq \left| \frac{\partial u^e}{\partial x_i} \right| \leq \frac{1}{n}, \\ 0 & \text{if } \left| \frac{\partial u^e}{\partial x_i} \right| > \frac{1}{n}, \end{cases}
\] (4.89)

the RHS of (4.84) can be rewritten as
\[
\sum_{i,j=1}^{d} \int_{\Omega \setminus A_i^e(x)} s g_n^\varepsilon \left( \frac{\partial u^e}{\partial x_i} \right) \left| \frac{\partial^2 u^e}{\partial x_i^2} \left( B'(u^e) \left( \frac{\partial u^e}{\partial x_j} \right)^2 + B(u^e) \frac{\partial^2 u^e}{\partial x_j^2} \right) \right| \, dx
\]
\[
= \sum_{i,j=1}^{d} \int_{B(x_{i0}^e, \frac{n}{2})} \chi_{\Omega \setminus A_i^e(x)} s g_n^\varepsilon \left( \frac{\partial u^e}{\partial x_i} \right) \left| \frac{\partial^2 u^e}{\partial x_i^2} \left( B'(u^e) \left( \frac{\partial u^e}{\partial x_j} \right)^2 + B(u^e) \frac{\partial^2 u^e}{\partial x_j^2} \right) \right| \, dx
\]
\[
= \sum_{i,j=1}^{d} \int_{B(x_{i0}^e, \frac{n}{2})} \chi_{\Omega \setminus A_i^e(x)} \frac{s g_n^\varepsilon \left( \frac{\partial u^e}{\partial x_i} \right)}{\rho_{n,i}(x)} \rho_{n,i}(x) \left| \frac{\partial^2 u^e}{\partial x_i^2} \left( B'(u^e) \left( \frac{\partial u^e}{\partial x_j} \right)^2 + B(u^e) \frac{\partial^2 u^e}{\partial x_j^2} \right) \right| \, dx.
\] (4.90)
For all \( n \geq n_0 \) and for \( x \in B(x_{i0}^\varepsilon, \frac{n}{2}) \), we have

\[
sg'_n \left( \frac{\partial u^\varepsilon}{\partial x_i} \right) \rho_{n,i}(x) := \begin{cases} 
\frac{1}{k_{\varepsilon} n^d} e^{\frac{n^2}{|x - x_{i0}^\varepsilon|^2}} & \text{if } x \in B(x_{i0}^\varepsilon, \frac{n}{2}) \cap C_{n,i}^\varepsilon, \\
0 & \text{if } x \notin B(x_{i0}^\varepsilon, \frac{n}{2}) \cap C_{n,i}^\varepsilon.
\end{cases}
\]

(4.91)

For \( x \in B(x_{i0}^\varepsilon, \frac{n}{2}) \), we have

\[
\left| \frac{sg'_n \left( \frac{\partial u^\varepsilon}{\partial x_i} \right)}{\rho_{n,i}(x)} \right| \leq \frac{1}{k_{\varepsilon} n^{d-1}} e^{\frac{4}{3}}.
\]

Since \( n \in \mathbb{N} \), the integrand on the last line of (4.90) is dominated by

\[
\frac{1}{k_{\varepsilon}} e^{\frac{4}{3}} \rho_{n,i} \left| \frac{\partial^2 u^\varepsilon}{\partial x_i^2} \right| \left( B'(u^\varepsilon) \left( \frac{\partial u^\varepsilon}{\partial x_j} \right)^2 + B(u^\varepsilon) \left( \frac{\partial^2 u^\varepsilon}{\partial x_j^2} \right) \right),
\]

which is integrable as \( u^\varepsilon \in C^{4+\beta,\frac{4+\beta}{2}}(\Omega_T) \) and

\[
\int_{\mathbb{R}} \rho_{n,i}(x) \, dx = 1.
\]

Applying dominated convergence theorem on RHS of (4.90), we conclude (4.82).

Next, we want to show that

\[
\lim_{n \to \infty} \sum_{i,j=1}^{d} \int_{\partial \Omega} sg_n \left( \frac{\partial u^\varepsilon}{\partial x_i} \right) \left( B(0) \frac{\partial^2 u^\varepsilon}{\partial x_j^2} + B'(0) \left( \frac{\partial u^\varepsilon}{\partial x_j} \right)^2 \right) \sigma_i \, d\sigma
\]

\[
= \sum_{i,j=1}^{d} \int_{\partial \Omega} sg \left( \frac{\partial u^\varepsilon}{\partial x_i} \right) \left( B(0) \frac{\partial^2 u^\varepsilon}{\partial x_j^2} + B'(0) \left( \frac{\partial u^\varepsilon}{\partial x_j} \right)^2 \right) \sigma_i \, d\sigma.
\]

(4.92)

Note that (4.92) follows by applying dominated convergence theorem, on noting that the sequence of integrands converges to the required limit pointwise, and is pointwise bounded by

\[
B(0) \left| \frac{\partial^2 u^\varepsilon}{\partial x_j^2} \right| + |B'(0)| \left( \frac{\partial u^\varepsilon}{\partial x_j} \right)^2,
\]

which is an integrable function as \( u^\varepsilon \in C^{4+\beta,\frac{4+\beta}{2}}(\Omega_T) \) and \( \text{surface measure}(\partial \Omega) < \infty \).
Using (4.82), (4.92) in (4.81), we get

\[ \varepsilon \lim_{n \to \infty} \sum_{i,j=1}^{d} \int_{\Omega} \sigma_{n} \left( \frac{\partial u_{\varepsilon}}{\partial x_{i}} \right) \frac{\partial}{\partial x_{i}} \left( B'(u_{\varepsilon}) \left( \frac{\partial u_{\varepsilon}}{\partial x_{j}} \right)^{2} + B(u_{\varepsilon}) \frac{\partial^{2} u_{\varepsilon}}{\partial x_{j}^{2}} \right) dx \]

\[ = \varepsilon \sum_{i,j=1}^{d} \int_{\partial \Omega} \sigma \left( \frac{\partial u_{\varepsilon}}{\partial x_{i}} \right) \left( B(0) \frac{\partial^{2} u_{\varepsilon}}{\partial x_{j}^{2}} + B'(0) \left( \frac{\partial u_{\varepsilon}}{\partial x_{j}} \right)^{2} \right) \sigma_{i} \; d\sigma. \quad (4.93) \]

From the equation (1.1a), we have

\[ \varepsilon \sum_{j=1}^{d} \left( B'(u_{\varepsilon}) \left( \frac{\partial u_{\varepsilon}}{\partial x_{j}} \right)^{2} + B(u_{\varepsilon}) \frac{\partial^{2} u_{\varepsilon}}{\partial x_{j}^{2}} \right) = u_{\varepsilon} + \sum_{j=1}^{d} f_{j}(u_{\varepsilon}) u_{\varepsilon} \text{ in } \Omega_{T}. \quad (4.94) \]

Using (4.94) on RHS of (4.93), we get

\[ \varepsilon \lim_{n \to \infty} \sum_{i,j=1}^{d} \int_{\Omega} \sigma_{n} \left( \frac{\partial u_{\varepsilon}}{\partial x_{i}} \right) \frac{\partial}{\partial x_{i}} \left( B'(u_{\varepsilon}) \left( \frac{\partial u_{\varepsilon}}{\partial x_{j}} \right)^{2} + B(u_{\varepsilon}) \frac{\partial^{2} u_{\varepsilon}}{\partial x_{j}^{2}} \right) dx \]

\[ = \sum_{i,j=1}^{d} \int_{\partial \Omega} \sigma \left( \frac{\partial u_{\varepsilon}}{\partial x_{i}} \right) f_{j}(0) \frac{\partial u_{\varepsilon}}{\partial x_{j}} \sigma_{i} \; d\sigma. \quad (4.95) \]

(ii) Passage to limit in the second term on LHS of (4.80) as \( n \to \infty \):

Using integration by parts, we have

\[ \sum_{i,j=1}^{d} \int_{\Omega} \sigma_{n} \left( \frac{\partial u_{\varepsilon}}{\partial x_{i}} \right) \frac{\partial}{\partial x_{i}} \left( f_{j}(u_{\varepsilon}) \frac{\partial u_{\varepsilon}}{\partial x_{j}} \right) dx \]

\[ = - \sum_{i,j=1}^{d} \int_{\Omega} \sigma_{n} \left( \frac{\partial u_{\varepsilon}}{\partial x_{i}} \right) \frac{\partial^{2} u_{\varepsilon}}{\partial x_{i}^{2}} f_{j}(u_{\varepsilon}) \frac{\partial u_{\varepsilon}}{\partial x_{j}} dx + \sum_{i,j=1}^{d} \int_{\partial \Omega} f_{j}(0) \frac{\partial u_{\varepsilon}}{\partial x_{j}} \sigma_{n} \left( \frac{\partial u_{\varepsilon}}{\partial x_{i}} \right) \sigma_{i} \; d\sigma \quad (4.96) \]

Using the arguments given to prove assertions (4.82) and (4.92), we obtain

\[ \lim_{n \to \infty} \sum_{i,j=1}^{d} \int_{\Omega} \sigma_{n} \left( \frac{\partial u_{\varepsilon}}{\partial x_{i}} \right) \frac{\partial^{2} u_{\varepsilon}}{\partial x_{i}^{2}} f_{j}(u_{\varepsilon}) \frac{\partial u_{\varepsilon}}{\partial x_{j}} dx = 0, \quad (4.97) \]

\[ \lim_{n \to \infty} \sum_{i,j=1}^{d} \int_{\partial \Omega} f_{j}(0) \frac{\partial u_{\varepsilon}}{\partial x_{j}} \sigma_{i} d\sigma = \sum_{i,j=1}^{d} \int_{\partial \Omega} f_{j}(0) \frac{\partial u_{\varepsilon}}{\partial x_{j}} \sigma_{i} d\sigma = \sum_{i,j=1}^{d} \int_{\partial \Omega} f_{j}(0) \frac{\partial u_{\varepsilon}}{\partial x_{j}} \sigma_{i} d\sigma, \quad (4.98) \]

On using equations (4.95), (4.97), and (4.98), the equation (4.80) yields

\[ \sum_{i=1}^{d} \int_{\Omega} \frac{\partial}{\partial t} \left( \frac{\partial u_{\varepsilon}}{\partial x_{i}} \right) \sigma \left( \frac{\partial u_{\varepsilon}}{\partial x_{i}} \right) dx = \frac{\partial}{\partial t} \left( \sum_{i=1}^{d} \int_{\Omega} \left| \frac{\partial u_{\varepsilon}}{\partial x_{i}} \right| dx \right) = 0. \quad (4.99) \]
The last equation implies that

$$\| \nabla u^\varepsilon \|_{(L^1(\Omega_T))^d} = \| \nabla u_0 \|_{(L^1(\Omega))^d},$$

which is nothing but (4.77). Equations (4.64) and (4.77) together give (4.61).

**Step 3:** Denote the total variation of $u^\varepsilon$ by $TV_{\Omega_T}(u^\varepsilon)$. Since for each $\varepsilon > 0$, $u^\varepsilon \in H^1(\Omega_T)$, the total variation of $u^\varepsilon$ is given by (4.61), i.e.,

$$TV_{\Omega_T}(u^\varepsilon) = \left( \int_{\Omega_T} \left( \partial u^\varepsilon / \partial t \right)^2 + \| \nabla u^\varepsilon \|^2_{(L^1(\Omega_T))^d} \right) dt.$$  \hspace{1cm} (4.100)

Equation (4.61) shows that $\{ TV_{\Omega_T}(u^\varepsilon) \}_{\varepsilon > 0}$ is bounded. Since $BV(\Omega_T) \cap L^1(\Omega_T)$ is compactly imbedded in $L^1(\Omega_T)$ \cite{9}, therefore there exists a subsequence $(u^{\varepsilon_k})$ and a function $u \in L^1(\Omega_T)$ such that $u^{\varepsilon_k} \to u$ in $L^1(\Omega_T)$ as $k \to \infty$. We still denote the subsequence by $(u^{\varepsilon})$ and since $u^{\varepsilon} \to u$ in $L^1(\Omega_T)$ as $\varepsilon \to 0$, there exists a further subsequence $(u^{\varepsilon_k})$ such that we have (4.62).

**Theorem 4.2** Let $f$, $B$, $u_0$ satisfy Hypothesis A. Let $u^\varepsilon$ be the unique solution to generalized viscosity problem (1.1). Then

$$\sum_{j=1}^d \left( \varepsilon \| \partial u^\varepsilon / \partial x_j \|_{L^2(\Omega_T)} \right)^2 \leq \frac{1}{2r} \| u_0 \|^2_{L^2(\Omega)}$$ \hspace{1cm} (4.101)

**Proof:**

Substituting $v = u^\varepsilon$ in the equation (2.3a) of generalized viscosity problem (1.1) and using chain rule and then integration by parts formula, for a.e. $t \in (0, T)$, we obtain

$$\langle u^\varepsilon_t, u^\varepsilon \rangle + \varepsilon \sum_{j=1}^d \int_\Omega B(u^\varepsilon) \frac{\partial u^\varepsilon}{\partial x_j} \frac{\partial u^\varepsilon}{\partial x_j} dx - \sum_{j=1}^d \int_\Omega f_j(u^\varepsilon) \frac{\partial u^\varepsilon}{\partial x_j} dx = 0.$$ \hspace{1cm} (4.102)

Integrating (4.102) with respect to $t$ over the interval $(0, T)$ and using the formula

$$\langle u^\varepsilon_t, u^\varepsilon \rangle = \frac{1}{2} \frac{d}{dt} \| u^\varepsilon(t) \|^2_{L^2(\Omega)},$$

we get

$$\frac{1}{2} \| u(T) \|^2_{L^2(\Omega)} + \varepsilon r \sum_{j=1}^d \int_0^T \int_\Omega \left( \frac{\partial u^\varepsilon}{\partial x_j} \right)^2 dx dt = \frac{1}{2} \| u_0 \|^2_{L^2(\Omega)} + \sum_{j=1}^d \int_0^T \int_\Omega f_j(u^\varepsilon) \frac{\partial u^\varepsilon}{\partial x_j} dx dt.$$ \hspace{1cm} (4.103)
From equation \((4.103)\), we obtain
\[
\varepsilon r \sum_{j=1}^{d} \int_{0}^{T} \int_{\Omega} \left( \frac{\partial u^{\varepsilon}}{\partial x_{j}} \right)^{2} \, dx \, dt \leq \frac{1}{2} \| u_{0} \|^{2}_{L^{2}(\Omega)} + \sum_{j=1}^{d} \int_{0}^{T} \int_{\Omega} f_{j}(u^{\varepsilon}) \frac{\partial u^{\varepsilon}}{\partial x_{j}} \, dx \, dt. \tag{4.104}
\]
For \(j = 1, 2 \cdots, d\), denote
\[
g_{j}(s) := \int_{0}^{s} f_{j}(\lambda) \, d\lambda.
\]
Since for \(j = 1, 2 \cdots, d\), each \(f_{j} : \mathbb{R} \to \mathbb{R}\) is continuous, in view of fundamental theorem of calculus, we have
\[
g'_{j}(u^{\varepsilon}) = f_{j}(u^{\varepsilon}), \tag{4.105}
\]
for a.e. \((x, t) \in \Omega_{T}\). Substituting \(g'_{j}(u^{\varepsilon}) = f_{j}(u^{\varepsilon})\) and using chain rule in inequality \((4.104)\), we obtain
\[
\varepsilon r \sum_{j=1}^{d} \int_{0}^{T} \int_{\Omega} \left( \frac{\partial u^{\varepsilon}}{\partial x_{j}} \right)^{2} \, dx \, dt \leq \frac{1}{2} \| u_{0} \|^{2}_{L^{2}(\Omega)} + \sum_{j=1}^{d} \int_{0}^{T} \int_{\Omega} \frac{\partial g_{j}(u^{\varepsilon})}{\partial x_{j}} \, dx \, dt. \tag{4.106}
\]
Using integration by parts formula and \(u^{\varepsilon} = 0\) for a.e. \((x, t) \in \partial \Omega \times (0, T)\) in inequality \((4.106)\), we get
\[
\varepsilon r \sum_{j=1}^{d} \int_{0}^{T} \int_{\Omega} \left( \frac{\partial u^{\varepsilon}}{\partial x_{j}} \right)^{2} \, dx \, dt \leq \frac{1}{2} \| u_{0} \|^{2}_{L^{2}(\Omega)} + \sum_{j=1}^{d} \int_{0}^{T} \int_{\partial \Omega} g_{j}(0) \nu_{j} \, d\sigma \, dt. \tag{4.107}
\]
Since \(g_{j}(0) = 0\), the inequality \((4.107)\) reduces to \((4.101)\).

## 5 Proof of Theorem 1.1

In this section, we want to show that the a.e. limit of the subsequence of solutions to generalized viscosity problem \((1.1)\) obtained in Theorem 4.1 is an entropy solution to scalar conservation laws \((1.2)\). We do not prove the uniqueness of the entropy solution as it was already established in [2]. The notion of entropy employed here was introduced by Bardos-Leroux-Nedelec [2] for IBVPs for conservation laws, which is defined below.

**Definition 5.1** Let \(u \in BV(\Omega \times (0, T))\) and \(\gamma(u)\) be the trace of \(u\) on \(\partial \Omega\). The function \(u\) is said to be an entropy solution of IBVP \((1.2)\) if for all \(k \in \mathbb{R}\) and for all nonnegative \(\phi \in C^{2}(\overline{\Omega} \times (0, T))\) with compact support in \(\overline{\Omega} \times (0, T)\), the inequality
\[
\int_{0}^{T} \int_{\Omega} \left\{ |u - k| \frac{\partial \phi}{\partial t} + s g(u - k) \, (f(u) - f(k)) \cdot \nabla \phi \right\} \, dx \, dt + \int_{0}^{T} \int_{\partial \Omega} s g(k) \, (f(\gamma(u)) - f(k)) \cdot \sigma \, \phi \, d\sigma \, dt \geq 0 \tag{5.108}
\]
holds, and \( u \) satisfies the initial condition \((1.2c)\) almost everywhere in \( \Omega \).

We once again drop the subscript \( k \) in the subsequence \( u^k \). Since uniqueness of an entropy solution is already established by Bardos et al. in [2], we conclude that the full sequence \((u^\varepsilon)\) converges to the unique entropy solution to \((1.2)\).

**Proof of Theorem 1.1:**

We prove Theorem 1.1 in two steps. In Step 1, we show that \( u \) is a weak solution to IBVP \((1.2)\) and in Step 2, we show that weak solution \( u \) is an entropy solution to IBVP \((1.2)\).

**Step 1:** Let \( \phi \in \mathcal{D}(\Omega \times [0,T)) \). Multiplying the equation \((1.1a)\) by \( \phi \), integrating over \( \Omega_T \), and using integration by parts formula, we get

\[
\int_0^T \int_\Omega u^\varepsilon \frac{\partial \phi}{\partial t} \, dx \, dt + \varepsilon \sum_{j=1}^d \int_0^T \int_\Omega B(u^\varepsilon) \frac{\partial u^\varepsilon}{\partial x_j} \frac{\partial \phi}{\partial x_j} \, dx \, dt + \sum_{j=1}^d \int_0^T \int_\Omega f^j(u^\varepsilon) \frac{\partial \phi}{\partial x_j} \, dx \, dt = \int_\Omega u^\varepsilon(x,0) \phi(x,0) \, dx \, dt.
\]

(5.109)

We pass to the limit as \( \varepsilon \to 0 \) in (5.109) in four steps.

**Step 1A:** (Passage to limit in the first term on LHS of (5.109))

Since for a.e. \((x,t) \in \Omega_T\),

\[
u^\varepsilon \frac{\partial \phi}{\partial t} \to u \frac{\partial \phi}{\partial t} \quad \text{as} \quad \varepsilon \to 0,
\]

and the quantity \( u^\varepsilon \frac{\partial \phi}{\partial t} \) is uniformly bounded by \( \|u_0\|_{L^\infty(\Omega)} \|\frac{\partial \phi}{\partial t}\| \), bounded convergence theorem yields

\[
\lim_{\varepsilon \to 0} \int_0^T \int_\Omega u^\varepsilon \frac{\partial \phi}{\partial t} \, dx \, dt = \int_0^T \int_\Omega u \frac{\partial \phi}{\partial t} \, dx \, dt.
\]

(5.110)

**Step 1B:** (Passage to limit in the second term on LHS of (5.109))

Note that for each \( j \in \{1,2,\cdots,d\}, \)

\[
\left| \varepsilon \int_0^T \int_\Omega B(u^\varepsilon) \frac{\partial u^\varepsilon}{\partial x_j} \frac{\partial \phi}{\partial x_j} \, dx \, dt \right| \leq \|B\|_\infty \varepsilon \int_0^T \int_\Omega \left| \frac{\partial u^\varepsilon}{\partial x_j} \right| \left| \frac{\partial \phi}{\partial x_j} \right| \, dx \, dt \leq \|B\|_\infty \sqrt{\varepsilon} \left( \sqrt{\varepsilon} \left\| \frac{\partial u^\varepsilon}{\partial x_j} \right\|_{L^2(\Omega_T)} \right) \left\| \frac{\partial \phi}{\partial x_j} \right\|_{L^2(\Omega_T)}.
\]

(5.111)

In view of (4.101), the sequence of numbers \( \left\{ \sqrt{\varepsilon} \left\| \frac{\partial u^\varepsilon}{\partial x_j} \right\|_{L^2(\Omega_T)} \right\}_{\varepsilon \geq 0} \) is bounded, and using
sandwich theorem the inequality (5.111) gives
\[
\lim_{\varepsilon \to 0} \varepsilon \sum_{j=1}^{d} \int_{0}^{T} \int_{\Omega} B(u^\varepsilon_j) \frac{\partial u^\varepsilon}{\partial x_j} \frac{\partial \phi}{\partial x_j} \, dx \, dt = 0. \tag{5.112}
\]

**Step 1C:** (Passage to limit in the third term on LHS of (5.109))

In this step, we show that
\[
\lim_{\varepsilon \to 0} \sum_{j=1}^{d} \int_{0}^{T} \int_{\Omega} f_j(u^\varepsilon) \frac{\partial \phi}{\partial x_j} \, dx \, dt = \int_{0}^{T} \int_{\Omega} f_j(u) \frac{\partial \phi}{\partial x_j} \, dx \, dt. \tag{5.113}
\]

For each \( j \in \{1, 2, \cdots, d\} \) and a.e. \((x, t) \in \Omega_T\), we have
\[
f_j(u^\varepsilon) \frac{\partial \phi}{\partial x_j} \to f_j(u) \frac{\partial \phi}{\partial x_j} \text{ as } \varepsilon \to 0.
\]

For each \( j \in \{1, 2, \cdots, d\} \), using continuity of \( f_j : \mathbb{R} \to \mathbb{R} \) and maximum principle of sequence of solutions to generalized viscosity problem (1.1), we observe that the integrand on LHS of (5.113) is pointwise bounded by
\[
\max_{1 \leq j \leq d} \{ |f_j(x)| \mid x \in I \} \left| \frac{\partial \phi}{\partial x_j} \right|
\]
which is integrable as \( \text{Vol}(\Omega_T) < \infty \).

Applying dominated convergence theorem on LHS of (5.113), we have (5.113).

**Step 1D:** (Passage to limit in the RHS of (5.109))

In this step, we prove that
\[
\lim_{\varepsilon \to 0} \int_{0}^{T} \int_{\Omega} u^\varepsilon(x, 0) \phi(x, 0) \, dx \, dt = \int_{0}^{T} \int_{\Omega} u(x, 0) \phi(x, 0) \, dx \, dt. \tag{5.114}
\]

For a.e. \( x \in \Omega \), we have
\[
u^\varepsilon(x, 0) \phi(x, 0) \to u^\varepsilon(x, 0) \phi(x, 0) \text{ as } \varepsilon \to 0.
\]

The integrand on LHS of (5.114) is a.e. bounded by \( \|u_0\|_{L^\infty(\Omega)} \left| \phi(x, 0) \right| \) which is integrable as \( \text{Vol}(\Omega_T) < \infty \).

Applying dominated convergence theorem on LHS of (5.114), we have (5.114).

Using equations (5.110), (5.112), (5.113), (5.114) in (5.109), we have
\[
\int_{0}^{T} \int_{\Omega} u \frac{\partial \phi}{\partial t} \, dx \, dt + \sum_{j=1}^{d} \int_{0}^{T} \int_{\Omega} f_j(u) \frac{\partial \phi}{\partial x_j} \, dx \, dt = \int_{0}^{T} \int_{\Omega} u(x, 0) \phi(x, 0) \, dx \, dt. \tag{5.115}
\]
In view of a result [2, p.1026] that \( u(x,t) = u_0(x) \text{ a.e. } x \in \Omega \), from (5.115), we have
\[
\int_0^T \int_\Omega u \frac{\partial \phi}{\partial t} \, dx \, dt + \sum_{j=1}^d \int_0^T \int_\Omega f_j(u) \frac{\partial \phi}{\partial x_j} \, dx \, dt = \int_\Omega u_0(x) \phi(x,0) \, dx \, dt.
\]
Therefore \( u \) is a weak solution of IBVP for conservation law (1.2).

**Step 2:** We want to show that weak solution \( u \) of IBVP (1.2) belongs to \( BV(\Omega_T) \), and satisfies inequality (5.108). Since \( u \) is the \( L^1(\Omega_T) \) limit of a sequence of BV functions, it follows from [2, p.1021] that \( u \in BV(\Omega_T) \). Let \( k \in \mathbb{R} \) and \( \phi \in C^2(\Omega \times (0,T)) \) such that \( \phi \geq 0 \) and has compact support in \( \Omega \times (0,T) \). Multiplying the first equation of IBVP (1.1) by \( sgn(u^\varepsilon - k) \phi \) and integrating over \( \Omega \times (0,T) \), we get
\[
\int_0^T \int_\Omega \frac{\partial u^\varepsilon}{\partial t} sgn(u^\varepsilon - k) \phi \, dx \, dt + \sum_{j=1}^d \int_0^T \int_\Omega \frac{\partial}{\partial x_j} (f_j(u^\varepsilon)) \cdot sgn(u^\varepsilon - k) \phi \, dx \, dt
\]
\[
= \varepsilon \sum_{j=1}^d \int_0^T \int_\Omega \frac{\partial}{\partial x_j} \left( B(u^\varepsilon) \frac{\partial u^\varepsilon}{\partial x_j} \right) sgn(u^\varepsilon - k) \phi \, dx \, dt. \quad (5.116)
\]
Using integration by parts in (5.116), we arrive at
\[
\int_0^T \int_\Omega \left\{ \int_k^{u^\varepsilon} sgn(y - k) \, dy \right\} \frac{\partial \phi}{\partial t} \, dx \, dt + \int_0^T \int_\Omega (f(u^\varepsilon) - f(k)) \cdot \nabla \phi \cdot sgn(u^\varepsilon - k) \, dx \, dt
\]
\[
+ \int_0^T \int_\Omega (f(u^\varepsilon) - f(k)) \cdot \nabla u^\varepsilon \cdot sgn'(u^\varepsilon - k) \phi \, dx \, dt = \varepsilon \int_0^T \int_\Omega B(u^\varepsilon) \cdot \nabla u^\varepsilon \cdot \nabla \phi \cdot sgn(u^\varepsilon - k) \, dx \, dt
\]
\[
+ \varepsilon \int_0^T \int_\Omega sgn' \cdot sgn(u^\varepsilon - k) \phi \, dx \, dt + \varepsilon \int_0^T \int_{\partial \Omega} B(0) \nabla u^\varepsilon \cdot \sigma \, sgn(k) \phi \, d\sigma \, dt
\]
\[
- \int_0^T \int_{\partial \Omega} (f(0) - f(k)) \cdot \sigma \, sgn(k) \phi \, d\sigma \, dt. \quad (5.117)
\]
We want to prove the entropy inequality (5.108) by passing to the limit in (5.117) in two steps. In Step 2A, we pass to the limit in (5.117) as \( n \to \infty \) and then in Step 2B, we pass to the limit as \( \varepsilon \to 0 \) in the resultant from Step 2A.

**Step 2A:**

(i) (Passage to the limit in the first term on LHS of (5.117) as \( n \to \infty \)):
Note that, for a.e. \((x,t) \in \Omega_T\), we have
\[
\left( \int_k^{u^\varepsilon} sgn(y - k) \, dy \right) \frac{\partial \phi}{\partial t} \to \left( \int_k^{u^\varepsilon} sgn(y - k) \, dy \right) \frac{\partial \phi}{\partial t} \text{ as } n \to \infty.
\]
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Observe that
\[
\left| \left( \int_k^{u^\varepsilon} s g_n(y - k) \, dy \right) \frac{\partial \phi}{\partial t} \right| \leq |u^\varepsilon - k| \left| \frac{\partial \phi}{\partial t} \right|. \tag{5.118}
\]

Since \( Vol(\Omega_T) < \infty \), the RHS of (5.118) is integrable. An application of dominated convergence theorem gives
\[
\lim_{n \to \infty} \int_0^T \int_\Omega \left( \int_k^{u^\varepsilon} s g_n(y - k) \, dy \right) \frac{\partial \phi}{\partial t} \, dx \, dt = \int_0^T \int_\Omega \left( \int_k^{u^\varepsilon} s g(y - k) \, dy \right) \frac{\partial \phi}{\partial t} \, dx \, dt.
\tag{5.119}
\]

(ii) (Passage to the limit in the second term on LHS of (5.117) as \( n \to \infty \)): For \( a.e. \) \((x, t) \in \Omega_T \), we have
\[
(f(u^\varepsilon) - f(k)) \cdot \nabla \phi \cdot s g_n(u^\varepsilon - k) \to (f(u^\varepsilon) - f(k)) \cdot \nabla \phi \cdot s g(u^\varepsilon - k) \text{ as } n \to \infty.
\]
Note that
\[
|(f(u^\varepsilon) - f(k)) \cdot \nabla \phi \cdot s g_n(u^\varepsilon - k)| \leq d \left[ \max_{1 \leq j \leq d} \left( \sup_{y \in I} |f_j(y)| \right) + \max_{1 \leq j \leq d} \left( |f_j(k)| \right) \right] \max_{1 \leq j \leq d} \left( \sup_{(x, t) \in \Omega_T} \left| \frac{\partial \phi}{\partial x_j} \right| \right)
\tag{5.120}
\]

Since \( Vol(\Omega_T) < \infty \), we have the RHS of (5.120) is integrable. An application of dominated convergence theorem gives
\[
\lim_{n \to \infty} \int_0^T (f(u^\varepsilon) - f(k)) \cdot \nabla \phi \cdot s g_n(u^\varepsilon - k) \, dx \, dt = \int_0^T (f(u^\varepsilon) - f(k)) \cdot \nabla \phi \cdot s g(u^\varepsilon - k) \, dx \, dt.
\tag{5.121}
\]

(iii) (Passage to the limit in the third term on LHS of (5.117) as \( n \to \infty \)): For \( j = 1, 2, \ldots, d \), applying mean value theorem to \( f_j : \mathbb{R} \to \mathbb{R} \), we get
\[
f_j(u^\varepsilon) - f_j(k) = f_j'(\xi^j_{u^\varepsilon,k})(u^\varepsilon - k).
\tag{5.122}
\]
Denote
\[
f'(\xi_{u^\varepsilon,k}) := \left( f_1'(\xi^1_{u^\varepsilon,k}), f_2'(\xi^2_{u^\varepsilon,k}), \ldots, f_d'(\xi^d_{u^\varepsilon,k}) \right).
\]
Applying (5.122) and \(|(u^\varepsilon - k)s g_n'(u^\varepsilon - k)| \leq 1\), we have
\[
\left| \int_0^T \int_\Omega (f(u^\varepsilon) - f(k)) \cdot \nabla u^\varepsilon \cdot s g_n'(u^\varepsilon - k) \phi \, dx \, dt \right| \\
\leq \int_0^T \int_{\{x \in \Omega : |u^\varepsilon - k| < \frac{1}{n} \}} \left| f'(\xi_{u^\varepsilon,k}) \right| \left| \nabla (u^\varepsilon - k) \right| |\phi| \, dx \, dt, \\
\leq \|\phi\|_{L^\infty(\Omega_T)} \sqrt{d} \max_{1 \leq j \leq d} \left( \sup_{y \in I} |f'_j(y)| \right) \int_0^T \int_{\{x \in \Omega : |u^\varepsilon - k| < \frac{1}{n} \}} \left| \nabla (u^\varepsilon - k) \right| \, dx \, dt.
\tag{5.123}
\]
Denote
\[ g_n(t) := \int_{\{x \in \Omega : |u^\varepsilon - k| < \frac{1}{n}\}} |\nabla (u^\varepsilon - k)| \, dx. \]

Applying Lemma 4.1 with \( v = (u^\varepsilon - k) \), we get
\[ \lim_{n \to \infty} g_n(t) = 0. \quad (5.124) \]

For all \( t \in (0, T) \) and \( n \in \mathbb{N} \), we have \( g_n(t) \leq \int_{\Omega} |\nabla (u^\varepsilon - k)| \, dx \). Since \( u^\varepsilon \in C^{4+\beta, \frac{4+\beta}{2}}(\Omega_T) \),
\[ \int_{0}^{T} \int_{\Omega} |\nabla (u^\varepsilon - k)| \, dx \, dt < \infty. \]

An application of dominated convergence theorem yields
\[ \lim_{n \to \infty} \left( \int_{0}^{T} \int_{\Omega} (f(u^\varepsilon) - f(k)) \cdot \nabla u^\varepsilon s g_n(u^\varepsilon - k) \phi \, dx \, dt \right) = \int_{0}^{T} \lim_{n \to \infty} g_n(t) \, dt = 0 \quad (5.125) \]

(iv) (Passage to the limit in the first term on RHS of (5.117) as \( n \to \infty \): For a.e. \((x,t) \in \Omega_T\),
\[ B(u^\varepsilon) (\nabla u^\varepsilon \cdot \nabla \phi) s g_n(u^\varepsilon - k) \to B(u^\varepsilon) (\nabla u^\varepsilon \cdot \nabla \phi) s g(u^\varepsilon - k) \text{ as } n \to \infty. \]

Note that
\[ |B(u^\varepsilon) (\nabla u^\varepsilon \cdot \nabla \phi) s g_n(u^\varepsilon - k)| \leq \|B\|_{L^\infty(I)} \|\nabla u^\varepsilon\|_{L^\infty(\Omega_T)^d} \|\nabla \phi\|_{L^\infty(\Omega_T)^d}. \quad (5.126) \]

Since \( \text{Vol}(\Omega_T) < \infty \), the RHS of (5.126) is integrable and applying dominated convergence theorem, we conclude
\[ \lim_{n \to \infty} \int_{0}^{T} \int_{\Omega} B(u^\varepsilon) (\nabla u^\varepsilon \cdot \nabla \phi) s g_n(u^\varepsilon - k) \, dx \, dt = \int_{0}^{T} \int_{\Omega} B(u^\varepsilon) (\nabla u^\varepsilon \cdot \nabla \phi) s g(u^\varepsilon - k) \, dx \, dt. \quad (5.127) \]

(v) (Passage to the limit in the third term on RHS of (5.117) as \( n \to \infty \): For a.e. \((x,t) \in \partial \Omega \times (0, T)\), we have
\[ B(0) \frac{\partial u^\varepsilon}{\partial \sigma} s g_n(k) \phi \to B(0) \frac{\partial u^\varepsilon}{\partial \sigma} s g(k) \phi \text{ as } n \to \infty. \]

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Since $|s_{g_n}(k)| \leq 1$, we have
\begin{equation}
\left| B(0) \frac{\partial u^\varepsilon}{\partial \sigma} \, s_{g_n}(k) \phi \right| \leq B(0) \max_{1 \leq j \leq d} \left( \sup_{(x,t) \in \Omega_T} \left| \frac{\partial u^\varepsilon}{\partial x_j} \right| \right) |\phi|, \tag{5.128}
\end{equation}
the RHS of (5.128) is integrable as surface measure($\partial \Omega \times (0,T)) < \infty$. An application of dominated convergence theorem gives
\begin{equation}
\lim_{n \to \infty} \int_0^T \int_{\partial \Omega} B(0) \frac{\partial u^\varepsilon}{\partial \sigma} \, s_{g_n}(k) \phi \, dx \, dt = \int_0^T \int_{\partial \Omega} B(0) \frac{\partial u^\varepsilon}{\partial \sigma} \, s_{g}(k) \phi \, dx \, dt. \tag{5.129}
\end{equation}

(vi) (Passage to the limit in the fourth term on RHS of (5.117) as $n \to \infty$): The proof of
\begin{equation}
\lim_{n \to \infty} \int_0^T \int_{\partial \Omega} s_{g_n}(k) \left( f(0) - f(k) \right) \cdot \sigma \, d\sigma \, dt = \int_0^T \int_{\partial \Omega} s_{g}(k) \left( f(0) - f(k) \right) \cdot \sigma \, d\sigma \, dt. \tag{5.130}
\end{equation}
is similar to the proof of (5.129).

Using (5.119),(5.121),(5.125),(5.127),(5.129),(5.130) and
\begin{equation}
\varepsilon \int_0^T \int_{\Omega} B(u^\varepsilon) \, |\nabla u^\varepsilon|^2 \, s_{g_n}(u^\varepsilon - k) \, \phi \, dx \, dt \geq 0,
\end{equation}
in (5.117), we get
\begin{align*}
\int_0^T \int_{\Omega} \left\{ \int_k^{u^\varepsilon} s_{g}(y - k) \, dy \right\} \frac{\partial \phi}{\partial t} \, dx \, dt + \int_0^T \int_{\Omega} \left( f(u^\varepsilon) - f(k) \right) \cdot \nabla \phi \, s_{g}(u^\varepsilon - k) \, dx \, dt \\
\geq \varepsilon \int_0^T \int_{\Omega} B(u^\varepsilon) \left( \nabla u^\varepsilon \cdot \nabla \phi \right) \, s_{g}(u^\varepsilon - k) \, dx \, dt + \varepsilon \int_0^T \int_{\partial \Omega} B(0) \nabla u^\varepsilon \cdot \sigma \, s_{g}(k) \phi \, d\sigma \, dt \\
- \int_0^T \int_{\partial \Omega} \left( f(0) - f(k) \right) \cdot \sigma \, s_{g}(k) \phi \, d\sigma \, dt.
\end{align*}
\tag{5.131}

Step 2B:

(i) (Passage to the limit in the first term on LHS of (5.131) as $\varepsilon \to 0$): Denote
\begin{equation}
g(z) := \int_0^z s_{g}(y - k) \, dy.
\end{equation}
Since \( g \) is an absolutely continuous function, for \( a.e. \ (x, t) \in \Omega_T, \) we have
\[
\lim_{\varepsilon \to 0} g(u^\varepsilon) = g(u).
\]

Therefore, as \( \varepsilon \to 0, \) for \( a.e. \ (x, t) \in \Omega_T, \) we obtain
\[
g(u^\varepsilon) \frac{\partial \phi}{\partial t} \to g(u) \frac{\partial \phi}{\partial t}
\]

Observe that
\[
\left| \left( \int_k^{u^\varepsilon} s g(y - k) \, dy \right) \frac{\partial \phi}{\partial t} \right| \leq |u^\varepsilon - k| \left| \frac{\partial \phi}{\partial t} \right| \leq \left( \|u_0\|_{L^\infty(\Omega)} + |k| \right) \left| \frac{\partial \phi}{\partial t} \right|.
\]  
(5.132)

Since \( \text{Vol}(\Omega_T) < \infty, \) the RHS of (5.132) is integrable. Applying dominated convergence theorem, we get
\[
\lim_{\varepsilon \to 0} \int_0^T \int_\Omega \left\{ \int_k^{u^\varepsilon} s g(y - k) \, dy \right\} \frac{\partial \phi}{\partial t} \, dx \, dt = \int_0^T \int_\Omega |u - k| \frac{\partial \phi}{\partial t} \, dx \, dt
\]  
(5.133)

(ii) (Passage to the limit in the second term on LHS of (5.131) as \( \varepsilon \to 0: \) We know that for \( a.e. \ (x, t) \in \Omega_T, \)
\[s g(u^\varepsilon - k) \to s g(u - k) \text{ as } \varepsilon \to 0.
\]

Therefore for \( a.e. \ (x, t) \in \Omega_T, \) we have
\[
((f(u^\varepsilon) - f(k)) \cdot \nabla \phi) \ s g(u^\varepsilon - k) \to ((f(u) - f(k)) \cdot \nabla \phi) \ s g(u - k) \text{ as } \varepsilon \to 0.
\]

Observe that
\[
|((f(u^\varepsilon) - f(k)) \cdot \nabla \phi) \ s g(u^\varepsilon - k)| \leq d \left[ \max_{1 \leq j \leq d} \left( \sup_{y \in I} |f_j(y)| \right) + \max_{1 \leq j \leq d} |f_j(k)| \right] \times \left[ \sup_{(x, t) \in \Omega_T} \left| \frac{\partial \phi}{\partial x_j} \right| \right].
\]  
(5.134)

Since \( \text{Vol}(\Omega_T) < \infty, \) therefore the RHS of (5.134) is integrable. An application of dominated convergence theorem yields
\[
\lim_{\varepsilon \to 0} \int_0^T \int_\Omega (f(u^\varepsilon) - f(k)) \cdot \nabla \phi \ s g(u^\varepsilon - k) \, dx \, dt = \int_0^T \int_\Omega (f(u) - f(k)) \cdot \nabla \phi \ s g(u - k) \, dx \, dt.
\]  
(5.135)
(iii) (Passage to the limit in the first term on RHS of (5.131) as $\varepsilon \to 0$:) Using
\[ \| \nabla u^\varepsilon \|_{(L^1(\Omega_T))^d} \leq \| \nabla u_0 \|_{(L^1(\Omega_T))^d}, \]
we observe that
\[ \varepsilon \int_0^T \int_\Omega B(u^\varepsilon) \nabla u^\varepsilon \cdot \nabla \phi \; sg(u^\varepsilon - k) \; dx \; dt \leq \varepsilon \| B \|_{L^\infty(I)} \max_{1 \leq j \leq d} \left( \sup_{(x,t) \in \overline{\Omega_T}} \left| \frac{\partial \phi}{\partial x_j} \right| \right) \| \nabla u_0 \|_{(L^1(\Omega_T))^d}. \]
(5.136)

Passing to the limit as $\varepsilon \to 0$ in (5.136), we conclude
\[ \lim_{\varepsilon \to 0} \varepsilon \int_0^T \int_\Omega B(u^\varepsilon) \nabla u^\varepsilon \cdot \nabla \phi \; sg(u^\varepsilon - k) \; dx \; dt = 0. \] (5.137)

(iv) (Passage to the limit in the second term on RHS of (5.131) as $\varepsilon \to 0$:) We want to compute
\[ \lim_{\varepsilon \to 0} \left\{ \varepsilon \int_0^T \int_{\partial \Omega} B(0) \phi \frac{\partial u^\varepsilon}{\partial \sigma} \; d\sigma \; dt \right\}. \]
For $\delta > 0$, let $\rho_\delta \in C^2(\overline{\Omega})$ be functions introduced by Kruzhkov (see [11, 2]) having the properties
\[
\begin{align*}
\rho_\delta & = 1 \text{ on } \partial \Omega \\
\rho_\delta & = 0 \text{ on } \{ x \in \Omega : \text{dist}(x, \partial \Omega) \geq \delta \} \\
0 & \leq \rho_\delta \leq 1 \text{ on } \Omega \\
|\nabla \rho_\delta| & \leq \frac{C}{\delta},
\end{align*}
\]
where $C$ is independent of $\delta$. Observe that
\[ \varepsilon \int_0^T \int_{\partial \Omega} B(0) \phi \frac{\partial u^\varepsilon}{\partial \sigma} \; d\sigma \; dt = \varepsilon \int_0^T \int_\Omega \nabla (B(u^\varepsilon) \nabla u^\varepsilon) \; \phi \; \rho_\delta \; dx \; dt \]
\[ + \varepsilon \int_0^T \int_\Omega B(u^\varepsilon) \nabla u^\varepsilon \cdot \nabla (\phi \; \rho_\delta) \; dx \; dt. \] (5.138)
Using the equation (1.1a) and integration by parts in (5.138), we get
\[
\begin{align*}
\varepsilon \int_0^T \int_{\partial \Omega} B(0) \phi \frac{\partial u^\varepsilon}{\partial \sigma} \; d\sigma \; dt & = - \int_0^T \int_\Omega \left\{ u^\varepsilon \frac{\partial \phi}{\partial t} + f(u^\varepsilon) \cdot \nabla \phi \right\} \rho_\delta \; dx \; dt - \int_0^T \int_\Omega \phi \; f(u^\varepsilon) \cdot \nabla \rho_\delta \; dx \; dt \\
& \quad + \varepsilon \int_0^T \int_\Omega \left[ B(u^\varepsilon) \{ (\nabla u^\varepsilon \cdot \nabla \phi) \rho_\delta + (\nabla u^\varepsilon \cdot \nabla \rho_\delta) \phi \} \right] \; dx \; dt \\
& \quad + \int_0^T \int_{\partial \Omega} \phi \; f(0) \cdot \sigma \; d\sigma \; dt.
\end{align*}
\]
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Using property $|\nabla \rho_\delta| \leq \frac{C}{\delta}$ and $\|\nabla u^\varepsilon\|_{(L^1(\Omega_T))^d} \leq \|\nabla u_0\|_{(L^1(\Omega_T))^d}$, we observe that

$$|\varepsilon B(u^\varepsilon) \{(\nabla u^\varepsilon \cdot \nabla \phi) \rho_\delta + (\nabla u^\varepsilon \cdot \nabla \rho_\delta) \phi\}|$$

$$\leq \varepsilon \left( \max_{1 \leq j \leq d} \left( \sup_{(x,t) \in \Omega_T} \left| \frac{\partial \phi}{\partial x_j} \right| \right) + \frac{C}{\delta} \|\phi\|_{L^\infty(\Omega_T)} \right) \|B\|_{L^\infty(I)} \|\nabla u^\varepsilon\|_{(L^1(\Omega_T))^d}$$

$$\leq \varepsilon \left( \max_{1 \leq j \leq d} \left( \sup_{(x,t) \in \Omega_T} \left| \frac{\partial \phi}{\partial x_j} \right| \right) + \frac{C}{\delta} \|\phi\|_{L^\infty(\Omega_T)} \right) \|B\|_{L^\infty(I)} \|\nabla u_0\|_{(L^1(\Omega_T))^d}. \quad (5.139)$$

Passing to the limit as $\varepsilon \to 0$ in (5.139), we get

$$\lim_{\varepsilon \to 0} \varepsilon \int_0^T \int_\Omega B(u^\varepsilon) \{(\nabla u^\varepsilon \cdot \nabla \phi) \rho_\delta + (\nabla u^\varepsilon \cdot \nabla \rho_\delta) \phi\} \, dx \, dt = 0. \quad (5.140)$$

Applying integration by parts in the second term on RHS of (5.139), we have

$$\lim_{\varepsilon \to 0} \varepsilon \int_0^T \int_{\partial \Omega} B(0) \phi \frac{\partial u^\varepsilon}{\partial \sigma} \, d\sigma \, dt = - \int_0^T \int_\Omega \left\{ u \frac{\partial \phi}{\partial t} + f(u) \cdot \nabla \phi \right\} \rho_\delta \, dx \, dt$$

$$- \int_0^T \int_\Omega \nabla \cdot (\phi f(u)) \rho_\delta \, dx \, dt$$

$$+ \int_0^T \int_{\partial \Omega} \phi \left( f(0) - f(\gamma(u)) \right) \cdot \sigma \, d\sigma \, dt. \quad (5.141)$$

Passing to the limit as $\delta \to 0$ on RHS of (5.141), we get

$$\lim_{\varepsilon \to 0} \varepsilon \int_0^T \int_{\partial \Omega} B(0) \phi \frac{\partial u^\varepsilon}{\partial \sigma} \, d\sigma \, dt = \int_0^T \int_{\partial \Omega} \phi \left( f(0) - f(\gamma(u)) \right) \cdot \sigma \, d\sigma \, dt. \quad (5.142)$$

Using (5.133), (5.135), (5.137), (5.142) in (5.131), we get the required entropy inequality (5.108).

### 6 Proof of Theorem 1.2

Denote by $f_\varepsilon$ and $u_0\varepsilon$, the regularizations of the flux function $f = (f_1, f_2, \cdots, f_d)$ and initial condition $u_0$ of IBVP (1.1), using the standard sequence of mollifiers $\tilde{\rho}_\varepsilon$ defined on $\mathbb{R}$, and $\rho_\varepsilon$ defined on $\mathbb{R}^d$, respectively. They are given by $f_\varepsilon := (f_1\varepsilon, f_2\varepsilon, \cdots, f_d\varepsilon)$ where

$$f_j\varepsilon := f_j * \tilde{\rho}_\varepsilon \ (j = 1, 2, \cdots, d), \text{ and } u_{0\varepsilon} := u_0 * \rho_\varepsilon.$$
Consider the IBVP for regularized generalized viscosity problem
\[
\begin{align*}
\varepsilon \partial_t u^\varepsilon + \nabla \cdot f^\varepsilon(u^\varepsilon) &= \nabla \cdot (B(u^\varepsilon) \nabla u^\varepsilon) \quad \text{in } \Omega_T, \quad (6.143a) \\
u^\varepsilon(x,t) &= 0 \quad \text{on } \partial \Omega \times (0,T), \quad (6.143b) \\
u^\varepsilon(x,0) &= u_0^\varepsilon(x) \quad x \in \Omega, \quad (6.143c)
\end{align*}
\]

**Lemma 6.1** Let \( f, B, u_0 \) satisfy Hypothesis B. Then there exists a unique solution of (6.143) in \( C^{1+\beta, \frac{1+\beta}{2}}(\Omega_T) \), for every \( 0 < \beta < 1 \) and the following estimates hold:
\[
\begin{align*}
\| u^\varepsilon \|_{L^\infty(\Omega)} &\leq \| u_0 \|_{L^\infty(\Omega)}, \\
TV_{\Omega_T}(u^\varepsilon) &\leq TTV_{\Omega}(u_0).
\end{align*}
\]

Also, there exists a constant \( C > 0 \) such that
\[
\begin{align*}
\left\| \frac{\partial u^\varepsilon}{\partial t} \right\|_{L^1(\Omega_T)} &\leq CT\| B \|_{L^\infty(I)} Vol(\Omega) TV_{\Omega}(u_0) + T Vol(\Omega) \sum_{j=1}^d \| B' \|_{L^\infty(I)} \left\| \frac{\partial u_0}{\partial x_j} \right\|_{L^\infty(\Omega)}^2 \\
&\quad + T\| f' \|_{L^\infty(I)} \sum_{j=1}^d \left\| \frac{\partial u_0}{\partial x_j} \right\|_{L^\infty(\Omega)}.
\end{align*}
\] (6.146)

Furthermore, there exists a subsequence \( \{u_{\varepsilon_k}\}_{k=1}^\infty \) of \( u^\varepsilon \) and a function \( u \) in \( L^1(\Omega_T) \) such that \( u_{\varepsilon_k} \to u \) a.e. in \( \Omega_T \), and also in \( L^1(\Omega_T) \) as \( k \to \infty \).

The following result follows from [9, p.67] which is useful in proving Theorem 6.1 and we omit its proof.

**Lemma 6.2** Let \( u_0 \in W^{1,\infty}_c(\Omega) \). Then \( u_{0\varepsilon} \) satisfies the following bounds
\[
\begin{align*}
\| u_{0\varepsilon} \|_{L^\infty(\Omega)} &\leq \| u_0 \|_{L^\infty(\Omega)}, \\
\| \nabla u_{0\varepsilon} \|_{(L^1(\Omega))^d} &\leq TV_{\Omega}(u_0)
\end{align*}
\] (6.147, 6.148)

There exists a constant \( C > 0 \) such that for all \( \varepsilon > 0 \), \( u_{0\varepsilon} \) satisfies
\[
\left\| \Delta u_{0\varepsilon} \right\|_{L^1(\Omega)} \leq \frac{C}{\varepsilon} TV_{\Omega}(u_0). 
\] (6.149)

**Proof of Lemma 6.1** Note that \( f^\varepsilon \in (C^\infty(\mathbb{R}))^d \) and \( \| f^\varepsilon \|_{(L^\infty(\Gamma))^d} \leq \| f' \|_{(L^\infty(\Gamma))^d} < \infty \). Since \( u_0 \in W^{1,\infty}_c(\Omega) \), the function \( u_{0\varepsilon} \) belongs to the space \( C^{\infty}(\overline{\Omega}) \) and also has compact
support in \( \Omega \) for sufficiently small \( \varepsilon \). As a consequence the initial-boundary data of the regularized generalized viscosity problem (6.143) satisfies compatibility conditions of orders 0, 1, 2 which are required to apply Theorem 3.1. Applying Theorem 3.1, we get the existence of a unique solution \( u^\varepsilon \) in \( C^{4+\beta,4+\beta/2}(\Omega_T) \) for regularized generalized viscosity problem (6.143), and \( u^\varepsilon_{tt} \in C(\Omega_T) \).

By maximum principle (Theorem 2.4), we conclude that \( u^\varepsilon \) satisfies

\[
\|u^\varepsilon\|_{L^\infty(\Omega)} \leq \|u_{0\varepsilon}\|_{L^\infty(\Omega)} \text{ a.e. } t \in (0,T).
\]

Combining (6.147) with (6.150), we get (6.144).

Using equations (4.72) and (4.75) (from Step 1 in the proof of Theorem 4.1) with \( f = f^\varepsilon \) and \( u_0 = u_{0\varepsilon} \), we get

\[
\int_\Omega \left| \frac{\partial u^\varepsilon}{\partial t}(x,t) \right| \, dx \leq \varepsilon \int_\Omega B(u_{0\varepsilon}) \left| \Delta u_{0\varepsilon} \right| \, dx + \int_\Omega \left| B'(u_{0\varepsilon}) \right| \left( \frac{\partial u_{0\varepsilon}}{\partial x_j} \right)^2 \, dx + \sum_{j=1}^d \int_\Omega \left| f^\varepsilon_j(u_{0\varepsilon}) \right| \left| \frac{\partial u_{0\varepsilon}}{\partial x_j} \right| \, dx.
\]

Since \( u_0 \in W^{1,\infty}(\Omega) \), we have

\[
\|u_{0\varepsilon}\|_{L^\infty(\Omega)} \leq \|u_0\|_{L^\infty(\Omega)}, \quad (6.152a)
\]

\[
\left\| \frac{\partial u_{0\varepsilon}}{\partial x_j} \right\|_{L^\infty(\Omega)} = \left\| \frac{\partial u_0}{\partial x_j} * \rho_\varepsilon \right\|_{L^\infty(\Omega)} \leq \left\| \frac{\partial u_0}{\partial x_j} \right\|_{L^\infty(\Omega)}. \quad (6.152b)
\]

In view of (7.205) and (6.152), we obtain

\[
\int_\Omega \left| \frac{\partial u^\varepsilon}{\partial t}(x,t) \right| \, dx \leq \varepsilon \|B\|_{L^\infty(I)} \int_\Omega \left| \Delta u_{0\varepsilon} \right| \, dx + \varepsilon \Vol(\Omega) \sum_{j=1}^d \left\| B'(u_{0\varepsilon}) \left( \frac{\partial u_{0\varepsilon}}{\partial x_j} \right)^2 \right\|_{L^\infty(\Omega)} + \sum_{j=1}^d \left\| f^\varepsilon_j(u_{0\varepsilon}) \right\|_{L^\infty(I)} \left\| \frac{\partial u_{0\varepsilon}}{\partial x_j} \right\|_{L^\infty(\Omega)}.
\]

We may assume that \( \varepsilon < 1 \). Using (6.149) in (6.153) gives

\[
\int_\Omega \left| \frac{\partial u^\varepsilon}{\partial t}(x,t) \right| \, dx \leq C \|B\|_{L^\infty(I)} \Vol(\Omega) \TV(\Omega) + \Vol(\Omega) \sum_{j=1}^d \left\| B'(u_{0\varepsilon}) \left( \frac{\partial u_{0\varepsilon}}{\partial x_j} \right)^2 \right\|_{L^\infty(\Omega)} + \sum_{j=1}^d \left\| f^\varepsilon(u_{0\varepsilon}) \right\|_{L^\infty(I)} \left\| \frac{\partial u_{0\varepsilon}}{\partial x_j} \right\|_{L^\infty(\Omega)}.
\]

\[
(6.154)
\]
Integrating on both sides of the last inequality w.r.t. $t$ on the interval $[0, T]$ yields (6.146).

Applying the conclusion of Step 2 in the proof of Theorem 4.1, namely (4.100), with $f = f_ε$ and $u_0 = u_0ε$ yields

$$TV_Ω(u^ε) \leq TV_Ω(u_0ε).$$

Applying Lemma 6.2, we get

$$TV_Ω(u^ε) \leq TV_Ω(u_0) \quad (6.155)$$

Once again integrating w.r.t. $t$ over the interval $[0, T]$ on both sides of the inequality (6.155) yields (6.145).

Using the compact embedding of $BV(Ω_T) \cap L^1(Ω_T)$ in $L^1(Ω_T)$, we conclude that there exists a subsequence $\{u^{εk}\}_{k=1}^∞$ of $u^ε$ and a function $u$ in $L^1(Ω_T)$ such that we have $u^{εk} → u$ in $L^1(Ω_T)$ as well as pointwise a.e. $Ω_T$, as $k → ∞$. This completes the proof of Lemma 6.1.

We still denote the subsequence $\{u^{εk}\}_{k=1}^∞$ by $\{u^ε\}$.

We now show that the a.e. limit of a sequence of solutions to the regularized generalized viscosity problem is a weak solution for IBVP (1.2).

**Proof of Theorem 1.2**: We will show that the function $u$ whose existence is asserted by Lemma 6.1 is indeed an entropy solution to (1.2). For notational convenience, we still denote the subsequence $\{u^{εk}\}_{k=1}^∞$ (as asserted by Lemma 6.1) by $\{u^ε\}$. The proof of Theorem 1.2 is divided into two steps. In the first step, we show that $u$ is a weak solution for the IBVP (1.2). In the second step, we show that $u$ satisfies an entropy inequality for the IBVP (1.2) in the sense of (5.108). Note that $u \in BV(Ω_T)$ as it the $L^1(Ω_T)$ limit of a sequence of BV functions [2, p.1021]. Thus it then follows that $u$ is an entropy solution.

**Step 1**: Let $φ \in D(Ω \times [0, T))$. Multiplying the first equation of (6.143) by $φ,$
integrating over $\Omega_T$ and using integration by parts, we get
\[
\int_0^T \int_{\Omega} u^\varepsilon \frac{\partial \phi}{\partial t} \, dx \, dt + \varepsilon \sum_{j=1}^{d} \int_0^T \int_{\Omega} B(u^\varepsilon) \frac{\partial u^\varepsilon}{\partial x_j} \frac{\partial \phi}{\partial x_j} \, dx \, dt + \sum_{j=1}^{d} \int_0^T \int_{\Omega} f_{j\varepsilon}(u^\varepsilon) \frac{\partial \phi}{\partial x_j} \, dx \, dt
\]
\[
= \int_{\Omega} u^\varepsilon(x,0) \phi(x,0) \, dx \, dt.
\] (6.156)

We would like to pass to the limit as $\varepsilon \to 0$ in the equation (6.156), and obtain
\[
\int_0^T \int_{\Omega} u \frac{\partial \phi}{\partial t} \, dx \, dt + \sum_{j=1}^{d} \int_0^T \int_{\Omega} f_j(u) \frac{\partial \phi}{\partial x_j} \, dx \, dt = \int_{\Omega} u_0(x) \phi(x,0) \, dx \, dt,
\] (6.157)

thereby we conclude that $u$ is a weak solution for the IBVP (1.2). Note that we have
\[
\lim_{\varepsilon \to 0} \int_0^T \int_{\Omega} u^\varepsilon \frac{\partial \phi}{\partial t} \, dx \, dt = \int_0^T \int_{\Omega} u \frac{\partial \phi}{\partial t} \, dx \, dt,
\] (6.158)
\[
\lim_{\varepsilon \to 0} \sum_{j=1}^{d} \int_0^T \int_{\Omega} B(u^\varepsilon) \frac{\partial u^\varepsilon}{\partial x_j} \frac{\partial \phi}{\partial x_j} \, dx \, dt = 0,
\] (6.159)
\[
\lim_{\varepsilon \to 0} \int_0^T \int_{\Omega} u^\varepsilon(x,0) \phi(x,0) \, dx \, dt = \int_0^T \int_{\Omega} u_0(x) \phi(x,0) \, dx \, dt,
\] (6.160)

whose proofs follow on similar lines as those of (5.110), (5.112) and (5.114) respectively.

For $j = 1, 2, \ldots, d$, we want to show that
\[
\lim_{\varepsilon \to 0} \int_0^T \int_{\Omega} f_{j\varepsilon}(u^\varepsilon) \frac{\partial \phi}{\partial x_j} \, dx = \int_0^T \int_{\Omega} f_j(u) \frac{\partial \phi}{\partial x_j} \, dx.
\] (6.161)

Note that
\[
\left| \int_0^T \int_{\Omega} \left( f_{j\varepsilon}(u^\varepsilon) - f_j(u) \right) \frac{\partial \phi}{\partial x_j} \, dx \right| \leq \int_0^T \int_{\Omega} \left| f_{j\varepsilon}(u^\varepsilon) - f_j(u) \right| \left| \frac{\partial \phi}{\partial x_j} \right| \, dx \, dt,
\]
\[
\leq \int_0^T \int_{\Omega} \left| f_{j\varepsilon}(u^\varepsilon) - f_j(u^\varepsilon) \right| \left| \frac{\partial \phi}{\partial x_j} \right| \, dx \, dt
\]
\[
+ \int_0^T \int_{\Omega} \left| f_j(u^\varepsilon) - f_j(u) \right| \left| \frac{\partial \phi}{\partial x_j} \right| \, dx \, dt.
\] (6.162)

For $j = 1, 2, \ldots, d$, $f_j$ is continuous, therefore $f_{j\varepsilon} \to f_j$ uniformly on compact sets as $\varepsilon \to 0$. Observe that
\[
\int_0^T \int_{\Omega} \left| f_{j\varepsilon}(u^\varepsilon) - f_j(u^\varepsilon) \right| \left| \frac{\partial \phi}{\partial x_j} \right| \, dx \leq \left\| f_{j\varepsilon} - f_j \right\|_{L^\infty(I)} \int_0^T \int_{\Omega} \left| \frac{\partial \phi}{\partial x_j} \right| \, dx \, dt.
\] (6.163)
Using sandwich theorem in (6.163), and dominated convergence theorem in the second term on RHS of the inequality (6.162), we get (6.161). This completes the proof of (6.157).

Step 2: We want to show that $u$ satisfies entropy inequality (5.108). Let $k \in \mathbb{R}$ and $\phi \in C^2(\overline{\Omega} \times (0, T))$ such that $\phi \geq 0$ and has compact support in $\overline{\Omega} \times (0, T)$. Multiplying the equation (6.143a) by $s g_n(u^\varepsilon - k) \phi$ and integrating over $\Omega \times (0, T)$, we get

$$
\int_0^T \int_\Omega \frac{\partial u^\varepsilon}{\partial t} s g_n(u^\varepsilon - k) \phi \, dx \, dt + \sum_{j=1}^d \int_0^T \int_\Omega \frac{\partial}{\partial x_j} (f_j(u^\varepsilon)) s g_n(u^\varepsilon - k) \phi \, dx \, dt
$$

$$
= \varepsilon \sum_{j=1}^d \int_0^T \int_\Omega \frac{\partial}{\partial x_j} \left( B(u^\varepsilon) \frac{\partial u^\varepsilon}{\partial x_j} \right) s g_n(u^\varepsilon - k) \phi \, dx \, dt. \tag{6.164}
$$

Integrating by parts in (6.164) yields

$$
\int_0^T \int_\Omega \left\{ \int_k^y \frac{\partial \phi}{\partial t} \, dy \right\} s g_n(y - k) \, dx \, dt + \int_0^T \int_\Omega (f(u^\varepsilon) - f(k)) \cdot \nabla \phi \ s g_n(u^\varepsilon - k) \, dx \, dt
$$

$$
+ \int_0^T \int_\Omega (f(u^\varepsilon) - f(k)) \cdot \nabla u^\varepsilon \ s g_n^\varepsilon(u^\varepsilon - k) \phi \, dx \, dt = \varepsilon \int_0^T \int_\Omega B(u^\varepsilon) (\nabla u^\varepsilon \cdot \nabla \phi) \ s g_n(u^\varepsilon - k) \, dx \, dt
$$

$$
+ \varepsilon \int_0^T \int_\Omega B(u^\varepsilon) |\nabla u^\varepsilon|^2 \ s g_n^\varepsilon(u^\varepsilon - k) \phi \, dx \, dt + \varepsilon \int_0^T \int_{\partial \Omega} B(0) \nabla u^\varepsilon \cdot \sigma \ s g_n(k) \phi \, d\sigma \, dt
$$

$$
- \int_0^T \int_{\partial \Omega} (f(u^\varepsilon(0)) - f(u^\varepsilon(k))) \cdot \sigma \ s g_n(k) \phi \, d\sigma \, dt. \tag{6.165}
$$

We prove the entropy inequality (5.108) by passing to the limit in (6.165), first as $n \to \infty$ (in Step 2A), and then pass to the limit as $\varepsilon \to 0$ in the equation resulting from Step 2A (in Step 2B). Here we follow the arguments of [2].

Step 2A: Using the conclusion of Step 1 of the proof of Theorem 1.1 with $f = f^\varepsilon$, we pass to the limit in (6.165) as $n \to \infty$ and obtain the following inequality

$$
\int_0^T \int_\Omega \left\{ \int_k^y \frac{\partial \phi}{\partial t} \, dy \right\} s g_n(y - k) \, dx \, dt + \int_0^T \int_\Omega (f(u^\varepsilon) - f(k)) \cdot \nabla \phi \ s g(u^\varepsilon - k) \, dx \, dt
$$

$$
\geq \varepsilon \int_0^T \int_\Omega B(u^\varepsilon) (\nabla u^\varepsilon \cdot \nabla \phi) \ s g(u^\varepsilon - k) \, dx \, dt + \varepsilon \int_0^T \int_{\partial \Omega} B(0) \nabla u^\varepsilon \cdot \sigma \ s g'(k) \phi \, d\sigma \, dt
$$

$$
- \int_0^T \int_{\partial \Omega} (f(u^\varepsilon(0)) - f(u^\varepsilon(k))) \cdot \sigma \ s g(k) \phi \, d\sigma \, dt. \tag{6.166}
$$
Step 2B: Following the proofs of (5.133), (5.137), we get

\[
\lim_{\varepsilon \to 0} \int_0^T \int_\Omega \left\{ \int_k^{u_\varepsilon} s g(y - k) \, dy \right\} \frac{\partial \phi}{\partial t} \, dx \, dt = \int_0^T \int_\Omega |u - k| \frac{\partial \phi}{\partial t} \, dx \, dt, \tag{6.167}
\]

\[
\lim_{\varepsilon \to 0} \varepsilon \int_0^T \int_\Omega B(u_\varepsilon) \nabla u_\varepsilon \cdot \nabla \phi \, s g(u_\varepsilon - k) \, dx \, dt = 0. \tag{6.168}
\]

We pass to limit as \( \varepsilon \to 0 \) in the second term on LHS of (6.166), and prove

\[
\lim_{\varepsilon \to 0} \int_0^T \int_\Omega \left( f_\varepsilon(u_\varepsilon) - f_\varepsilon(k) \right) \cdot \nabla \phi \, s g(u_\varepsilon - k) \, dx \, dt = \int_0^T \int_\Omega \left( f(u) - f(k) \right) \cdot \nabla \phi \, s g(u - k) \, dx \, dt, \tag{6.169}
\]

by an application of dominated convergence theorem. Firstly, we show that integrands on LHS converge to the integrand on the RHS of the equation (6.169). Observe that

\[
|f_{j\varepsilon}(u_\varepsilon) - f_j(u)| \leq |f_{j\varepsilon}(u_\varepsilon) - f_j(u_\varepsilon)| + |f_j(u_\varepsilon) - f_j(u)|, \leq \|f_{j\varepsilon} - f_j\|_{L^\infty(I)} + |f_j(u_\varepsilon) - f_j(u)|. \tag{6.170}
\]

For each \( j \in \{1, 2, \cdots, d\} \), \( f_j \in C(\mathbb{R}) \). Therefore \( f_{j\varepsilon} \to f_j \) uniformly on compact sets of \( \mathbb{R} \). Since for a.e. \((x, t) \in \Omega_T \), \( u_\varepsilon(x, t) \in I \), we have

\[
\|f_{j\varepsilon} - f_j\|_{L^\infty(I)} \to 0 \text{ as } \varepsilon \to 0. \tag{6.171}
\]

For each \( j \in \{1, 2, \cdots, d\} \), since for a.e. \((x, t) \in \Omega_T \), \( u_\varepsilon \to u \) as \( \varepsilon \to 0 \), we have

\[
f_j(u_\varepsilon) \to f_j(u) \text{ as } \varepsilon \to 0. \tag{6.172}
\]

Using equations (6.171) and (6.172) in (6.170), we get

\[
f_{j\varepsilon}(u_\varepsilon) \to f_j(u) \text{ a.e. } (x, t) \in \Omega_T, \text{ as } \varepsilon \to 0. \tag{6.173}
\]

As \( \varepsilon \to 0 \), it is easy to observe that

\[
sg(u_\varepsilon - k) \to sg(u - k) \text{ a.e. in } \Omega_T, \tag{6.174}
\]

\[
f_{j\varepsilon}(k) \to f_j(k). \tag{6.175}
\]

Using the information from (6.173), (6.174), and (6.175), we conclude that for a.e. \((x, t) \in \Omega_T \),

\[
\lim_{\varepsilon \to 0} ((f_\varepsilon(u_\varepsilon) - f_\varepsilon(k)) \cdot \nabla \phi \, s g(u_\varepsilon - k)) = (f(u) - f(k)) \cdot \nabla \phi \, s g(u - k). \]
Since \(|f_jε(u^ε)| \leq \|f_j\|_{L^∞(I)}\) and \(|f_jε(k)| \leq |f_j(k)|\), therefore the integrand on LHS of the equation \((6.169)\) is bounded by
\[
\sum_{j=1}^{d} (\|f_j\|_{L^∞(I)} + |f_j(k)|) \left| \frac{∂φ}{∂x_j} \right|
\]
which is integrable over \(Ω_T\) as \(φ\) has compact support in \(\overline{Ω} \times (0,T)\). Applying dominated convergence theorem, we get \((6.169)\).

Following the proof of \((5.142)\), we get
\[
\lim_{ε→0} \left\{ ε \int_0^T \int_{\partialΩ} B(0)φ \frac{∂u^ε}{∂σ} dσ dt \right\} = \int_0^T \int_{\partialΩ} φ (f(0) - f(γ(u))) \cdot σ dσ dt. \quad (6.176)
\]
An application of dominated convergence theorem gives
\[
\lim_{ε→0} \int_0^T \int_{\partialΩ} (f^ε(0) - f^ε(k)) \cdot σ s_g(k) φ dx dt = \int_0^T \int_{\partialΩ} (f(0) - f(k)) \cdot σ s_g(k) φ dx dt.
\quad (6.177)
\]
Using the information from \((6.167),(6.168),(6.169),(6.176),(6.177)\) in \((6.166)\), we get the required entropy inequality \((5.108)\). We do not prove the uniqueness of the entropy solution as it was already established in \([2]\).

7 Proof of Theorem 1.3:

Let \(f_ε\) be as defined in the regularized generalized viscosity problem \((6.143)\) and \((u_{0ε})\) be as given in Hypotheses C. We introduce the IBVP for viscosity problem
\[
\begin{align*}
    u_ε^t + ∇ \cdot f_ε(u^ε) &= ε ∇ \cdot (B(u^ε) ∇ u^ε) & \text{in } Ω_T, \quad (7.178a) \\
    u^ε(x, t) &= 0 & \text{on } ∂Ω \times (0, T), \quad (7.178b) \\
    u^ε(x, 0) &= u_{0ε}(x) & \text{in } Ω, \quad (7.178c)
\end{align*}
\]
The next result deals with the existence of a uniformly bounded sequence \((u_{0ε})\) in \(D(Ω)\) with all the properties as mentioned in Hypothesis C.

**Lemma 7.1** Let \(u_0 \in H^1_0(Ω) \cap C(\overline{Ω})\). Then there exists a sequence \((u_{0ε})\) in \(D(Ω)\) such that the following properties hold.
1. $u_{0\varepsilon} \to u_0$ in $H^1(\Omega)$ as $\varepsilon \to 0$. \hfill (7.179)

2. For all $\varepsilon > 0$, there exists a constant $A > 0$ such that
\[ \|u_{0\varepsilon}\|_{L^\infty(\Omega)} \leq A. \] \hfill (7.180)

3. For all $\varepsilon > 0$ small enough, there exists a constant $C > 0$ such that
\[ \|\Delta u_{0\varepsilon}\|_{L^1(\Omega)} \leq C\varepsilon. \] \hfill (7.181)

Denote
\[ Q := \{ y \in \mathbb{R}^d : |y_i| < 1, i = 1, 2, \ldots, d \}, \quad Q^+ := \{ y \in Q : y_d > 0 \}, \quad \Gamma := \{ y \in Q : y_d = 0 \}. \]

The following result is used to prove Lemma 7.1.

**Proposition 7.1** [21, p.31] Let $u \in H^1(Q^+) \cap C(\overline{Q^+})$ and $u = 0$ near $\partial Q^+ \setminus \Gamma$. If $u = 0$ on the bottom $\Gamma$, then $u \in H^1_0(Q^+)$. 

**Proof of Lemma 7.1:**

**Step 1:** The proof of (7.179).

Since $\partial \Omega$ is smooth, for a given point $x_0 \in \partial \Omega$, there exists a neighbourhood $U_0$ of $x_0$ and a smooth invertible mapping $\Psi_0 : Q \to U_0$ such that
\[ \Psi_0(Q^+) = U_0 \cap \Omega, \quad \Psi_0(\Gamma) = U_0 \cap \partial \Omega, \]

where $Q^+$, $\Gamma$ as defined above. Let $\eta_0 \in \mathcal{D}(U_0)$. We consider $\eta_0 u_0$ instead of $u_0$ in the neighbourhood $U_0$ of $x_0$. Define $v : Q^+ \to \mathbb{R}$ by
\[ v(y) := (\eta_0 u_0)(\Psi(y)). \]

Since $\eta_0 \in \mathcal{D}(U_0)$, therefore $v \equiv 0$ in a neighbourhood of the upper boundary and the lateral boundary of $Q^+$. Since $u_0 = 0$ on $\partial \Omega$, for $y \in \Gamma$, we have $v(y) = 0$. Therefore an application of Proposition 7.1 we have $v \in H^1_0(Q^+)$. Let $\tilde{v}$ be the extension of $v$ to the whole $Q$ by setting $\tilde{v} = 0$ in $Q \setminus Q^+$. Let $\tilde{\rho}_\varepsilon : \mathbb{R} \to \mathbb{R}$ be the standard sequence of mollifiers. Define $J_\varepsilon \tilde{v}(y) : Q^+ \to \mathbb{R}$ by
\[ J_\varepsilon \tilde{v}(y) := \int_Q \tilde{\rho}_\varepsilon(y_1 - z_1) \tilde{\rho}_\varepsilon(y_2 - z_2) \cdots \tilde{\rho}_\varepsilon(y_{d-1} - z_{d-1}) \tilde{\rho}_\varepsilon(y_d - z_d - 2\varepsilon) \tilde{v}(z) \, dz. \] \hfill (7.182)
Observe that $J_\varepsilon \bar{v} \to v$ in $H^1_0(Q^+)$. For clarity, we only show that $(J_\varepsilon \bar{v})$ has compact support. Since $\bar{v}$ is zero in a neighbourhood of the upper boundary and the lateral boundary of $Q^+$, we only show that $J_\varepsilon \bar{v}$ is zero in $\{y \in Q^+; \ 0 < y_d < \varepsilon\}$. We know that $\rho_\varepsilon(y_d - z_d - 2\varepsilon) = 0$ whenever $|y_d - z_d - 2\varepsilon| \geq \varepsilon$. Let us compute

$$|y_d - z_d - 2\varepsilon| = |z_d + \varepsilon + (\varepsilon - y_d)|,$$

$$\geq z_d + \varepsilon > \varepsilon. \tag{7.183}$$

Therefore $(J_\varepsilon \bar{v})$ has compact support in $Q^+$. As a result, the function $(J_\varepsilon \bar{v}(\Psi_0^{-1}(x)))$ belongs to $C^1_\varepsilon(\Omega \cap U_{x_0})$ and

$$J_\varepsilon \bar{v}(\Psi_0^{-1}(x)) \to \eta u_0 \text{ in } H^1(\Omega).$$

Since $\partial \Omega$ is compact, there exists $x_1, x_2, \cdots, x_N \in \partial \Omega$ and $U_1, U_2, \cdots, U_N$ such that $\partial \Omega \subset \bigcup_{i=1}^N U_i$. Choose $U_{N+1} \subset \subset \Omega$ such that

$$\overline{\Omega} \subset \bigcup_{i=1}^{N+1} U_i.$$

For $i = 1, 2, \cdots, N, N + 1$, let $\eta_i$ be a partition of unity associated to $U_i$. For $i = 1, 2, \cdots, N$, let $(u_{0i})$ be the sequences corresponding to $U_i$, $\eta_i$ obtained as above manner, i.e., $u_{0i} = J_\varepsilon \bar{v}_i(\Psi_i^{-1}(x))$. Let $\rho_\varepsilon : \mathbb{R}^d \to \mathbb{R}$ be sequence of mollifiers. Then $u_{0N+1} := (\eta_{N+1} u_0) * \rho_\varepsilon \to \eta_{N+1} u_0$ uniformly on $\overline{U}_{N+1}$ as $\varepsilon \to 0$ and $(\eta_{N+1} u_0) * \rho_\varepsilon \to \eta_{N+1} u_0$ in $H^1(\Omega)$.

Denote

$$u_{0\varepsilon}(x) := \sum_{i=1}^{N+1} u_{0i}(x).$$

It is clear that $u_{0\varepsilon} \to u_0$ in $H^1(\Omega)$ and we obtain (7.179).

**Step 2:** The proof of (7.180).

Applying change of variable $\frac{-z}{\varepsilon} = p$ in (7.182), we get

$$J_\varepsilon^{-1} \bar{v}(y) := \int_{y_1-\varepsilon}^{y_1+\varepsilon} \cdots \int_{y_d-\varepsilon}^{y_d+\varepsilon} \frac{1}{\varepsilon^d} \tilde{\rho}(\frac{y_1 - z_1}{\varepsilon}) \cdots \tilde{\rho}(\frac{y_d - z_d - 2\varepsilon}{\varepsilon}) \bar{v}(z) \, dz,$$

$$= \int_{-1}^{1} \cdots \int_{-1}^{1} \int_{-1}^{1} \tilde{\rho}(p_1) \tilde{\rho}(p_2) \cdots \tilde{\rho}(p_{d-1}) \tilde{\rho}(p_d - 2) \bar{v}(x - \varepsilon p) (-1)^d \, dp \quad \tag{7.184}$$

Taking modulus on both sides of (7.184), for $i \in \{1, 2, \cdots, N\}$, we get

$$\|J_\varepsilon^{-1} \bar{v}_i(\Psi_i^{-1})\|_{L^\infty(\Omega)} \leq 2^d \left(\|\rho\|_{L^\infty(\mathbb{R})}\right)^d \|\eta_i u_0\|_{L^\infty(\Omega)}. \tag{7.185}$$

In view of (7.185), for $i = 1, 2, \cdots, N$, there exist constants $C_i > 0$ such that

$$\|u_{0i}\|_{L^\infty(\Omega)} \leq C_i \tag{7.186}$$
Again, since $u_0 \in C(\Omega)$, then $\|\eta_0 u_0 \ast \rho \|_{L^\infty(\Omega)} \leq \|\eta_0 u_0\|_{L^\infty(\Omega)}$ on $U_{N+1}$.

Taking $A = \max \left\{ C_1, C_2, \cdots, C_d, \|\eta_0 u_0\|_{L^\infty(\Omega)} \right\}$, we get (7.180).

**Step 3:** The proof (7.181).

For $i \in \{1, 2, \cdots, N\}$, denote

$$\Psi_i^{-1}(x) := (\Psi_{i1}^{-1}(x), \Psi_{i2}^{-1}(x), \cdots, \Psi_{id}^{-1}(x)).$$

We recall $\tilde{v}_i$ as given in Step 1, i.e., for $i \in \{1, 2, \cdots, N\}$, $\tilde{v}_i$ are given by $(\eta_0 u_0) (\Psi_i(y))$ on $Q^+$ and zero in $Q \setminus Q^+$. Then $u_{\tilde{v}_i}(x) = J_\varepsilon \tilde{v}_i (\Psi_i^{-1}(x))$. Denote $h_\varepsilon : \mathbb{R} \to \mathbb{R}$ by

$$h_\varepsilon(p) := \tilde{\rho}(p_1)\tilde{\rho}(p_2) \cdots \tilde{\rho}(p_{d-1})\tilde{\rho}(p_d - 2\varepsilon).$$

Denote $Q_\varepsilon^+ := \{y \in Q^+; \text{dist}(y, \partial Q^+) > \varepsilon\}$. Then for $y \in Q_\varepsilon^+$, observe that

$$J_\varepsilon \tilde{v}_i(y) := (h_\varepsilon \ast \tilde{v}_i)(y). \quad (7.187)$$

Since $\text{supp}(J_\varepsilon \tilde{v}_i)$ is contained in $Q_\varepsilon^+$, therefore we consider $J_\varepsilon \tilde{v}_i(y)$ only on $Q_\varepsilon^+$. In $Q_\varepsilon^+$, we have

$$\frac{\partial}{\partial x_l} J_\varepsilon \tilde{v}_i(y) = \frac{\partial}{\partial x_l} \left( (h_\varepsilon \ast \tilde{v}_i) (\Psi_i^{-1}(x)) \right),$$

$$= \sum_{j=1}^{d} \left( h_\varepsilon \ast \frac{\partial \tilde{v}_i}{\partial y_j} \right) (\Psi_i^{-1}(x)) \frac{\partial \Psi_{ij}^{-1}(x)}{\partial x_l}. \quad (7.188)$$

Differentiating (7.188) with respect to $x_l$, we get

$$\frac{\partial^2}{\partial x_l^2} J_\varepsilon \tilde{v}_i(\Psi_i^{-1}(x)) = \sum_{j=1}^{d} \frac{\partial}{\partial x_l} \left[ \left( h_\varepsilon \ast \frac{\partial \tilde{v}_i}{\partial y_j} (\Psi_i^{-1}(x)) \right) \frac{\partial \Psi_{ij}^{-1}(x)}{\partial x_l} \right],$$

$$= \sum_{j,k=1}^{d} \frac{\partial}{\partial y_k} \left( h_\varepsilon \ast \frac{\partial \tilde{v}_i}{\partial y_j} \right) (\Psi_i^{-1}(x)) \frac{\partial \Psi_{ik}^{-1}(x)}{\partial x_l} \frac{\partial \Psi_{ij}^{-1}(x)}{\partial x_l},$$

$$+ \sum_{j=1}^{d} \left( h_\varepsilon \ast \frac{\partial \tilde{v}_i}{\partial y_j} \right) (\Psi_i^{-1}(x)) \frac{\partial^2}{\partial x_l^2} (\Psi_i^{-1}(x)), \quad (7.189)$$
Summing over $l = 1, 2, \ldots, d$, we get
\[
\Delta_x J_\varepsilon \tilde{v}_i(\Psi_i^{-1}(x)) = \sum_{j,k,l=1}^d \left( \frac{\partial h_\varepsilon}{\partial y_k} \ast \frac{\partial v_i}{\partial y_j} \right) (\Psi_i^{-1}(x)) \frac{\partial \Psi_i^{-1}(x)}{\partial x_l} \frac{\partial \Psi_i^{-1}(x)}{\partial x_i} \\
+ \sum_{j,l=1}^d \left( h_\varepsilon \ast \frac{\partial v_i}{\partial y_j} \right) (y) \frac{\partial^2}{\partial x_l^2} \left( \Psi_i^{-1}(x) \right).
\]
(7.190)

For $y \in Q^+$, we know
\[
\left( h_\varepsilon \ast \frac{\partial v_i}{\partial y_j} \right) (y) = \int_Q h_\varepsilon(y - z) \frac{\partial v_i}{\partial y_j}(z) \, dz.
\]
(7.191)

Using the change of variable $\frac{y - z}{\varepsilon} = r$ in (7.191), we get
\[
\left( h_\varepsilon \ast \frac{\partial v_i}{\partial y_j} \right) (y) = \int_{Q^+} h_\varepsilon(y - z) \frac{\partial v_i}{\partial y_j}(z) \, dz.
\]
(7.192)

Taking modulus on both sides of (7.192), we have
\[
\left| \left( h_\varepsilon \ast \frac{\partial v_i}{\partial y_j} \right) (y) \right| \leq \|\tilde{\rho}\|_{L^\infty(\mathbb{R})} \int_{Q^+} \left| \frac{\partial v_i}{\partial y_j}(y - r) \right| \, dr,
\]
\[
\int_{Q^+} \left| \left( h_\varepsilon \ast \frac{\partial v_i}{\partial y_j} \right) (y) \right| \leq \|\tilde{\rho}\|_{L^\infty(\mathbb{R})} \int_{Q^+} \left| \frac{\partial v_i}{\partial y_j}(y - \varepsilon r) \right| \, dy \, dr,
\]
(7.193)

Since $\text{Vol}(Q) < \infty$, $\tilde{v}_i \in H^1(Q)$, therefore the RHS of (7.193) is bounded by $2^d \|\tilde{\rho}\|_{L^\infty(\mathbb{R})} \int_{Q} \left| \frac{\partial v_i}{\partial y_j} \right| \, dy$, which is independent of $\varepsilon$. Since $\left| \left( h_\varepsilon \ast \frac{\partial v_i}{\partial y_j} \right) (y) \right|$ is bounded on $Q^+$, therefore $\left| \left( h_\varepsilon \ast \frac{\partial v_i}{\partial y_j} \right) (\Psi_i^{-1}(x)) \right|$ is bounded on $U_i \cap \Omega$ by the same constant.

Next for $y \in Q^+$, we compute
\[
\left( \frac{\partial h_\varepsilon}{\partial y_k} \ast \frac{\partial v_i}{\partial y_j} \right) (y) = \int_{Q^+} \frac{\partial h_\varepsilon}{\partial y_k}(y - k) \frac{\partial v_i}{\partial y_j}(z) \, dz,
\]
\[
= \int_Q \tilde{\rho}_\varepsilon(y_1 - z_1) \tilde{\rho}_\varepsilon(y_2 - z_2) \cdots \frac{\partial}{\partial y_k} \left( \tilde{\rho}_\varepsilon(y_k - z_k) \right) \cdots \tilde{\rho}_\varepsilon(y_d - z_d - 2\varepsilon) \frac{\partial v_i}{\partial y_j}(z) \, dz.
\]
(7.194)

Using change of variable $\frac{y - z}{\varepsilon} = r$ and taking modulus on both sides of (7.194), we get
\[
\left| \left( \frac{\partial h_\varepsilon}{\partial y_k} \ast \frac{\partial v_i}{\partial y_j} \right) (y) \right| \leq \frac{1}{\varepsilon} \|\tilde{\rho}\|_{L^\infty(\mathbb{R})} \|\tilde{\rho}'\|_{L^\infty(\mathbb{R})} \int_{Q^+} \left| \frac{\partial v_i}{\partial y_j}(y - \varepsilon r) \right| \, dr.
\]
(7.195)
Integrating over \( Q^+ \), we arrive at
\[
\int_{Q^+} \left| \frac{\partial \psi(x)}{\partial x_j} \right|^d \, dy \leq \frac{2^d}{\varepsilon} \rho_j^d \| L^\infty(\mathbb{R}) \| \| \rho \| L^\infty(\mathbb{R}) \int_Q \left\| \frac{\partial \psi}{\partial y_j} \right\| \, dy. \tag{7.196}
\]
Therefore we have
\[
\int_{\Omega} \left| \nabla \psi_i(x) \right|^d \, dx \leq \sum_{j,k,l=1}^d \| \nabla \psi_{ik} \|_{L^\infty(\Omega \cap U_i)} \| \nabla \psi_{ij} \|_{L^\infty(\Omega \cap U_i)} \int_{Q^+} \left\| \frac{\partial \psi_i}{\partial y_j} \right\| \, dy,
\]
\[
\leq \sum_{j,k,l=1}^d \left( 2^d \| \rho_j \|_{L^\infty(\mathbb{R})} \| \nabla \psi_{ik} \|_{L^\infty(\Omega \cap U_i)} \| \nabla \psi_{ij} \|_{L^\infty(\Omega \cap U_i)} \right) \int_{Q^+} \left\| \frac{\partial \psi_i}{\partial y_j} \right\| \, dy. \tag{7.197}
\]
For \( i \in \{1, 2, \ldots, N\} \), denote
\[
C_i := \sum_{j,k,l=1}^d \left( 2^d \| \rho_j \|_{L^\infty(\mathbb{R})} \| \nabla \psi_{ik} \|_{L^\infty(\Omega \cap U_i)} \| \nabla \psi_{ij} \|_{L^\infty(\Omega \cap U_i)} \right) \int_{Q^+} \left\| \frac{\partial \psi_i}{\partial y_j} \right\| \, dy.
\]
Since \( \eta_{N+1} \in \mathcal{D}(U_{N+1}) \), \( \eta_{N+1} u_0 \in H_0^1(\Omega) \). Let \( 0 < \varepsilon < \frac{\text{dist}(\text{supp}(\eta_{N+1} u_0), \partial \Omega)}{2} \) be small enough. For \( l \in \{1, 2, \ldots, d\} \), on \( \Omega_\varepsilon \), we compute
\[
\Delta (\eta_{N+1} u_0 * \rho_\varepsilon)(x) := \sum_{l=1}^d \left( \frac{\partial}{\partial x_l} (\eta_{N+1} u_0) \ast \frac{\partial \rho_\varepsilon}{\partial x_l} \right)(x),
\]
\[
= \sum_{l=1}^d \int_{\Omega} \frac{\partial}{\partial x_l} \rho_\varepsilon(x - z) \frac{\partial}{\partial x_l} (\eta_{N+1} u_0)(z) \, dz. \quad (7.198)
\]
Using change of variable \( \frac{x - u}{\varepsilon} = r \) in (7.198), we get
\[
\Delta (\eta_{N+1} u_0 * \rho_\varepsilon)(x) = \frac{1}{\varepsilon} \sum_{l=1}^d \int_{\Omega} \rho'(r) \frac{\partial}{\partial x_l} (\eta_{N+1} u_0)(x - \varepsilon r) (-1)^d \, dr. \quad (7.199)
\]
For \( i \in \{1, 2, \ldots, d\} \), since \( \text{supp}(\frac{\partial}{\partial x_i} (\eta_{N+1} u_0)) \subset \text{supp}(\eta_{N+1} u_0) \) and \( \text{supp}(\frac{\partial}{\partial x_i} (\rho_\varepsilon)) \subset \text{supp}(\rho_\varepsilon) \), \( \Delta (\eta_{N+1} u_0 * \rho_\varepsilon) = 0 \) on \( (\Omega_\varepsilon)^c \). From equation (7.199), we get
\[
\int_{\Omega} |\Delta (\eta_{N+1} u_0 * \rho_\varepsilon)| \, dx \leq \frac{1}{\varepsilon} \rho' \|L_\infty(\mathbb{R}^d)\sum_{l=1}^d \int_{\Omega} \left| \frac{\partial}{\partial x_l} (\eta_{N+1} u_0)(x - \varepsilon r) \right| \, dr
\]
\[
\leq \frac{1}{\varepsilon} \rho' \|L_\infty(\mathbb{R}^d)\sum_{l=1}^d \int_{\Omega} \left| \frac{\partial}{\partial x_l} (\eta_{N+1} u_0)(r) \right| \, dr. \quad (7.200)
\]
Denote
\[
C_{N+1} := \|\rho'\|L_\infty(\mathbb{R}^d)\sum_{l=1}^d \int_{\Omega} \left| \frac{\partial}{\partial x_l} (\eta_{N+1} u_0)(r) \right| \, dr.
\]
Taking \( C = \sum_{p=1}^{N+1} C_p \), we get (7.181). This completes the proof of Lemma 7.1.

In the next result, we show the existence of a unique solution of (7.178) in \( C^{4+\beta, \frac{4+\beta}{2}}(\Omega_T) \) and an a.e. convergent subsequence of the quasilinear viscous approximations \( (u^\varepsilon) \).

**Lemma 7.2** Let \( f, B, u_0 \) satisfy Hypothesis C. Then there exists a unique solution of (7.178) in \( C^{4+\beta, \frac{4+\beta}{2}}(\Omega_T) \), for every \( 0 < \beta < 1 \) and the following estimates hold:
\[
\|u^\varepsilon\|_{L_\infty(\Omega)} \leq A, \quad (7.201)
\]
\[
TV_{\Omega_T}(u^\varepsilon) \leq T \left( (\text{Vol}(\Omega))^{\frac{1}{2}} \sum_{j=1}^d \sqrt{C_j} \right). \quad (7.202)
\]
Also, there exists a constant \( C > 0 \) such that
\[
\left\| \frac{\partial u^\varepsilon}{\partial t} \right\|_{L_1(\Omega_T)} \leq T \left( C \|B\|_{L_\infty(I)} + \|B'\|_{L_\infty(I)} \sum_{j=1}^d C_j + (\text{Vol}(\Omega))^{\frac{1}{2}} \|f'\|_{L_\infty(I)} \sum_{j=1}^d \sqrt{C_j} \right). \quad (7.203)
\]
Furthermore, there exists a subsequence \( \{u^\varepsilon_k\}_{k=1}^\infty \) of \( u^\varepsilon \) and a function \( u \) in \( L^1(\Omega_T) \) such that \( u^\varepsilon_k \to u \) a.e. in \( \Omega_T \), and also in \( L^1(\Omega_T) \) as \( k \to \infty \).

**Proof:**

Note that \( f_\varepsilon \in (C^\infty(\mathbb{R}))^d \) and \( \|f'_\varepsilon\|_{(L^\infty(\Omega))^d} \leq \|f'\|_{(L^\infty(\Omega))^d} < \infty \). Since \( u_0\varepsilon \in D(\Omega) \), the initial-boundary data of the viscosity problem (7.178) satisfies compatibility conditions of orders 0, 1, 2 which are required to apply Theorem 3.1. Applying Theorem 3.1, we get the existence of a unique solution \( u^\varepsilon \) in \( C^{4+\beta,4+\beta/2}(\overline{\Omega_T}) \) for the viscosity problem (7.178), and \( u^\varepsilon_{tt} \in C(\overline{\Omega_T}) \).

By maximum principle (Theorem 2.4), we conclude that \( u^\varepsilon \) satisfies

\[
\|u^\varepsilon\|_{L^\infty(\Omega)} \leq \|u_0\varepsilon\|_{L^\infty(\Omega)} \text{ a.e. } t \in (0,T).
\] (7.204)

Combining (7.180) with (7.204), we get (7.201).

Using equations (4.72) and (4.75) (from Step 1 in the proof of Theorem 4.1) with \( f = f_\varepsilon \) and \( u_0 = u_0\varepsilon \), we arrive at

\[
\int_\Omega \left| \frac{\partial u^\varepsilon}{\partial t}(x,t) \right| \, dx \leq \varepsilon \int_\Omega B(u_0\varepsilon) \, |\Delta u_0\varepsilon| \, dx + \int_\Omega \left| B'(u_0\varepsilon) \right| \left( \frac{\partial u_0\varepsilon}{\partial x_j} \right)^2 \, dx + \sum_{j=1}^d \int_\Omega |f'_{\varepsilon}\varepsilon(u_0\varepsilon)| \left| \frac{\partial u_0\varepsilon}{\partial x_j} \right| \, dx.
\] (7.205)

Since \( u_0\varepsilon \to u \) in \( H^1_0(\Omega) \), for \( i \in \{1,2,\ldots,d\} \) and for all \( \varepsilon > 0 \), there exist \( C_i > 0 \) such that

\[
\left\| \frac{\partial u_0\varepsilon}{\partial x_i} \right\|_{L^2(\Omega)}^2 \leq C_i.
\] (7.206)

Using Hölder inequality and (7.206), we have

\[
\left\| \frac{\partial u_0\varepsilon}{\partial x_i} \right\|_{L^1(\Omega)} \leq (\text{Vol}(\Omega))^{\frac{1}{2}} \left\| \frac{\partial u_0\varepsilon}{\partial x_i} \right\|_{L^2(\Omega)} \leq (\text{Vol}(\Omega))^{\frac{1}{2}} \sqrt{C_i}.
\] (7.207)

In view of (7.206), (7.207) and (7.181), we obtain

\[
\int_\Omega \left| \frac{\partial u^\varepsilon}{\partial t}(x,t) \right| \, dx \leq C \|B\|_{L^\infty(I)} + \varepsilon \|B'\|_{L^\infty(I)} \sum_{j=1}^d C_i + (\text{Vol}(\Omega))^{\frac{1}{2}} \|f'_{\varepsilon}\|_{L^\infty(I)} \sum_{j=1}^d \sqrt{C_i}.
\] (7.208)
We may assume that \( \varepsilon < 1 \). Using \( \|f'_\varepsilon\|_{L^\infty(I)} \leq \|f'\|_{L^\infty(I)} \) in (7.208), we get
\[
\int_{\Omega} \left| \frac{\partial u^\varepsilon}{\partial t}(x,t) \right| \, dx \leq \left( C \|B\|_{L^\infty(I)} + \|B'_\varepsilon\|_{L^\infty(I)} \sum_{j=1}^{d} C_i + (\text{Vol}(\Omega))^{\frac{\varepsilon}{2}} \|f'\|_{L^\infty(I)} \sum_{j=1}^{d} \sqrt{C_i} \right).
\]
(7.209) Integrating on both sides of the last inequality w.r.t. \( t \) on the interval \([0, T]\) yields (7.203). Using the conclusion of Step 2 in the proof of Theorem 4.1 namely (4.100), with \( f = f_\varepsilon \) and \( u_0 = u_{0\varepsilon} \) yields
\[
TV_\Omega(u^\varepsilon) \leq TV_\Omega(u_{0\varepsilon}).
\]
(7.210) Applying (7.207) in (7.210), we get
\[
TV_\Omega(u^\varepsilon) \leq (\text{Vol}(\Omega))^{\frac{\varepsilon}{2}} \sum_{j=1}^{d} \sqrt{C_i}.
\]
(7.211) Once again integrating w.r.t. \( t \) over the interval \([0, T]\) on both sides of the inequality (7.211) yields (7.201). We now use the compact embedding of \( BV(\Omega_T) \cap L^1(\Omega_T) \) in \( L^1(\Omega_T) \) to conclude that there exists a subsequence \((u^{\varepsilon_k})\) of \((u^\varepsilon)\) and a function \( u \) in \( L^1(\Omega_T) \) such that \( u^{\varepsilon_k} \to u \) as \( k \to \infty \) in \( L^1(\Omega_T) \) as well as a.e. \((x,t) \in \Omega_T\). We still denote the subsequence \((u^{\varepsilon_k})\) by \((u^\varepsilon)\).

**Proof of Theorem 1.3**: We now show that the function \( u \) as asserted in Lemma 7.1 is the unique entropy solution to (1.2). We prove entropy solution in two steps. In Step 1, we show that \( u \) is a weak solution to (1.2) and in Step 2, we show that weak solution \( u \) is the unique entropy solution to (1.2).

**Step 1**: Let \( \phi \in \mathcal{D}(\Omega \times [0, \infty)) \). Multiplying (7.178a) by \( \phi \) and using integration by parts, we arrive at
\[
\int_{0}^{T} \int_{\Omega} u^\varepsilon \frac{\partial \phi}{\partial t} \, dx \, dt + \varepsilon \sum_{j=1}^{d} \int_{0}^{T} \int_{\Omega} B(u^\varepsilon) \frac{\partial u^\varepsilon}{\partial x_j} \frac{\partial \phi}{\partial x_j} \, dx \, dt + \sum_{j=1}^{d} \int_{0}^{T} \int_{\Omega} f_{j\varepsilon}(u^\varepsilon) \frac{\partial \phi}{\partial x_j} \, dx \, dt
= \int_{\Omega} u_{0\varepsilon} \phi(x,0) \, dx \, dt.
\]
(7.212)
Observe that
\[ \left| \int_{\Omega} (u_{0\varepsilon}(x) - u_0(x)) \phi(x,0) \, dx \right| \leq \| \phi \|_{L^\infty(\Omega_T)} \| u_{0\varepsilon} - u_0 \|_{L^1(\Omega)}. \] (7.213)

Since \( u_{0\varepsilon} \to u_0 \) in \( H^1_0(\Omega) \), therefore \( u_{0\varepsilon} \to u_0 \) in \( L^1(\Omega) \) as \( \varepsilon \to 0 \). Using
\[ \int_{\Omega} u_{0\varepsilon} \phi(x,0) \, dx \to \int_{\Omega} u_0 \phi(x,0) \, dx \quad \text{as} \quad \varepsilon \to 0 \]

and passing to the limit in (7.212) as \( \varepsilon \to 0 \) by following Step 1 in the proof of Theorem 1.2, we conclude that \( u \) is a weak solution of (1.2), i.e., for all \( \phi \in \mathcal{D}(\Omega \times [0,T)) \), the limit \( u \) satisfies
\[ \int_0^T \int_{\Omega} u \frac{\partial \phi}{\partial t} \, dx \, dt + \sum_{j=1}^d \int_0^T \int_{\Omega} f_j(u) \frac{\partial \phi}{\partial x_j} \, dx \, dt = \int_{\Omega} u_0(x) \phi(x,0) \, dx \, dt, \] (7.214)

**Step 2:** Following Step 2 in the proof of Theorem 1.2, we conclude that the limit \( u \) satisfies the entropy inequality (5.108). We do not prove the uniqueness of the entropy solution as it was already established in [2].

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### References

[1] K. Ammar, P. Wittbold and J. Carrillo, *Scalar conservation laws with general boundary condition and continuous flux function*, J. Differential Equations, 228(1), pp. 111-139, 2006.

[2] C. Bardos, A.Y. Leroux, and J.-C. Nédélec, *First order quasilinear equations with boundary conditions*, Comm. Partial Differential Equations, 4(9), pp. 1017-1034, 1979.

[3] S. Bianchini and A. Bressan, *Vanishing viscosity solutions of nonlinear hyperbolic systems* Ann. of Math. (2), 161(1), pp. 223-342, 2005.
[4] J. Carrillo, *Entropy solutions for nonlinear degenerate problems*, Arch. Ration. Mech. Anal., 147(4), pp. 269-361, 1999.

[5] G.-Q. Chen and M. Perepelitsa, *Shallow water equations: viscous solutions and inviscid limit*, Z. Angew. Math. Phys., 63(6), pp. 1067-1084, 2012.

[6] R.J. DiPerna, *Convergence of approximate solutions to conservation laws*, Arch. Rational Mech. Anal., 82(1), pp. 27-70, 1983.

[7] F. Dubois and P. LeFloch, *Boundary conditions for nonlinear hyperbolic systems of conservation laws*, J. Differential Equations, 71(1), pp. 93-122, 1988.

[8] L.C. Evans, *Partial differential equations*, Graduate Studies in Mathematics, Vol.19, American Mathematical Society, 1998.

[9] E. Godlewski and P.-A. Raviart, *Hyperbolic systems of conservation laws*, Mathématiques and Applications, Ellipses, 1991.

[10] S. Kesavan, *Topics in functional analysis and applications*, Wiley, 1989.

[11] S.N. Kružkov, *First order quasilinear equations with several space variables*, Math. USSR Sbornik, 10(2), pp. 217-243, 1970.

[12] A. Kurganov and Y. Liu, *New adaptive artificial viscosity method for hyperbolic systems of conservation laws*, J. Comput. Phys., 231(24), pp. 8114-8132, 2012.

[13] O.A. Ladyženskaja, V. A. Solonnikov and N.N Ural’ceva, *Linear and Quasi-linear Equations of Parabolic Type*, American Mathematical Society, 1988.

[14] O.A. Oleĭnik, *Discontinuous solutions of non-linear differential equations*, Amer. Math. Soc. Transl. (2), 26, pp. 95-172, 1963.

[15] F. Otto, *Initial-boundary value problem for a scalar conservation law*, C. R. Acad. Sci. Paris Sér. I Math., 322(8), pp. 729-734, 1996.

[16] A. Porretta and J. Vovelle, *L¹ solutions to first order hyperbolic equations in bounded domains*, Comm. Partial Differential Equations, 28, no.1-2, pp. 381-408, 2003.

[17] S. Salsa, *Partial differential equations in action: from modelling to theory*, Springer-Verlag Italia, 2008.

[18] J. Smoller, *Shock waves and reaction-diffusion equations*, Springer, 1994.
[19] G. Vallet, *Dirichlet problem for a nonlinear conservation law*, Rev. Mat. Complut., 13(1), pp. 231-250, 2000.

[20] I.I. Vrabie, *Differential equations: An introduction to basic concepts, results and applications*, World Scientific, 2004.

[21] W. Zhuoqun, Y. Jingxue and W. Chunpeng, *Elliptic and parabolic equations*, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2006.