JORDAN-SCHWINGER-TYPE REALIZATIONS OF
THREE-DIMENSIONAL POLYNOMIAL ALGEBRAS

V. SUNIL KUMAR and B. A. BAMBAH,∗
School of Physics, University of Hyderabad
Hyderabad - 500046, India
∗bbsp@uohyd.ernet.in

R. JAGANNATHAN
The Institute of Mathematical Sciences
C.I.T. Campus, Tharamani, Chennai - 600113, India
jagan@imsc.ernet.in

A three-dimensional polynomial algebra of order $m$ is defined by the commutation relations
\[ [P_0, P_\pm] = \pm P_\pm, \quad [P_+, P_-] = \phi^{(m)}(P_0) \]
where $\phi^{(m)}(P_0)$ is an $m$-th order polynomial in $P_0$ with the coefficients being constants or central elements of the algebra. It is shown that two given mutually commuting polynomial algebras of orders $l$ and $m$ can be combined to give two distinct $(l + m + 1)$-th order polynomial algebras. This procedure follows from a generalization of the well known Jordan-Schwinger method of construction of $su(2)$ and $su(1, 1)$ algebras from two mutually commuting boson algebras.

Keywords: Polynomial algebras; Higgs algebra; cubic algebras; quadratic algebras; Jordan-Schwinger realization.

PACS Nos.: 02.20.-a, 02.20.Sv

1 Introduction

A three-dimensional polynomial algebra of order $m$ is defined by the commutation relations
\[ [P_0, P_\pm] = \pm P_\pm, \]
\[ [P_+, P_-] = \phi^{(m)}(P_0) = \sum_{j=0}^{m} a_j P_0^j, \quad a_m \neq 0, \quad (1.1) \]
where the coefficients \( \{ a_j \} \) are constants or central elements of the algebra. It should be noted that in this case the Jacobi identity does not impose any restriction on the values of the coefficients \( \{ a_j \} \). In general,

\[
\mathcal{C} = P_+ P_+ + g(P_0 - 1) = P_- P_+ + g(P_0),
\]

is a Casimir operator of the algebra (1.1) where \( g(P_0) \) is defined by the relation

\[
g(P_0) - g(P_0 - 1) = \phi^{(m)}(P_0). \tag{1.3}
\]

Note that \( g(P_0) \) is an \((m+1)\)-th order polynomial in \( P_0 \) and can be determined uniquely, up to an additive constant, from the relation (1.3). In the following we shall take the \( g \)-function to be defined uniquely without the constant term.

It may be noted that the canonical boson algebra corresponds to the case \( m = 0 \) when \( \phi^{(0)} \) is just a constant. The \( su(2) \) and \( su(1, 1) \) algebras belong to \( m = 1 \) corresponding to a monomial \( \phi^{(1)} \). If \( m = 2 \) we have a quadratic algebra and if \( m = 3 \) we have a cubic algebra. A well known cubic algebra is the Higgs algebra (1.4) where \( h \) can be positive or negative. Such polynomial algebras, and their supersymmetric versions including anticommutation relations, represent the nonlinear symmetry or dynamical algebras in several physical problems in quantum mechanics, statistical physics, field theory, Yang-Mills-type gauge theories, integrable systems, quantum optics, etc. (e.g., see Refs.[1-27]). Hence a general mathematical study of such algebras is of interest. Here we shall be concerned only with polynomially deformed three dimensional Lie algebras and present a Jordan-Schwinger-like method of combining lower order polynomial algebras to get higher order polynomial algebras generalizing the earlier works on the Higgs algebra\(^1\) and quadratic algebras\(^{25,26}\).

2 \hspace{1em} \textbf{su}(2) and \textbf{su}(1, 1)

Let us briefly recall the construction of \( su(2) \) and \( su(1, 1) \) algebras starting with two boson algebras. Let \((a_+, a_-)\) and \((b_+, b_-)\) be two mutually commuting boson creation-annihilation operator pairs. Let \( N_a = a_+ a_- \) and \( N_b = b_+ b_- \) be the corresponding number operators. As is well known, \((J_0, J_+, J_-)\)
defined by

\[ J_0 = \frac{1}{2}(N_a - N_b), \quad J_+ = a_+b_-, \quad J_- = a_-b_+, \]

satisfy the \( su(2) \) algebra,

\[ [J_0, J_\pm] = \pm J_\pm, \quad [J_+, J_-] = 2J_0. \]

In this Jordan-Schwinger realization of \( su(2) \), \( N_a + N_b \) is seen to be a central element: if

\[ \mathcal{L}_J = \frac{1}{2}(N_a + N_b), \]

then,

\[ [\mathcal{L}_J, J_0, \pm] = 0. \]

The \( g \)-function in this case is \( g(J_0) = J_0(J_0 + 1) \) and hence the Casimir operator is

\[ \mathcal{C}_J = J_+J_- + J_0(J_0 - 1) = J_-J_+ + J_0(J_0 + 1). \]

In an analogous way, \((K_0, K_+, K_-)\) defined by

\[ K_0 = \frac{1}{2}(N_a + N_b), \quad K_+ = a_+b_+, \quad K_- = a_-b_-, \]

satisfy the algebra

\[ [K_0, K_\pm] = \pm K_\pm, \quad [K_+, K_-] = -(2K_0 + 1). \]

Calling \( K_0 + \frac{1}{2} \) as \( K_0 \) this algebra becomes the standard \( su(1, 1) \) algebra

\[ [K_0, K_\pm] = \pm K_\pm, \quad [K_+, K_-] = -2K_0. \]

Now,

\[ \mathcal{L}_K = \frac{1}{2}(N_a - N_b) \]

is a central element of the algebra:

\[ [\mathcal{L}_K, K_0, \pm] = 0. \]

The \( g \)-function is \( g(K_0) = -K_0(K_0 + 1) \) and hence the Casimir operator is

\[ \mathcal{C}_K = K_+K_- - K_0(K_0 - 1) = K_-K_+ - K_0(K_0 + 1). \]
3 Jordan-Schwinger-like construction of polynomial algebras

Let us now generalize the above construction of $su(2)$ and $su(1,1)$ algebras leading to polynomial algebras. This work follows from an observation in Ref. [10] on the construction of the Higgs cubic algebra (1.4) starting with mutually commuting $su(2)$ and $su(1,1)$ algebras (when $h > 0$) or two mutually commuting $su(1,1)$ algebras (when $h < 0$) and also a generalization of our earlier work\textsuperscript{25,26} in which we have constructed four classes of quadratic algebras combining a boson algebra with $su(2)$ and $su(1,1)$. Let $(L_0, L_\pm)$ and $(M_0, M_\pm)$ be the generating sets of two mutually commuting polynomial algebras of order $l$ and $m$, respectively. Then, using these two algebras as building blocks, we can construct two distinct polynomial algebras of order $l+m+1$ analogous to $su(1,1)$ and $su(2)$. To this end we proceed as follows.

Let

$$J_0 = \frac{1}{2}(L_0 - M_0), \quad J_+ = \mu L_+ M_-, \quad J_- = \mu L_- M_+,$$

(3.1)

in analogy with $su(2)$. Then, it is easily seen that these generate an algebra with the commutation relations :

$$[J_0, J_\pm] = \pm J_\pm,$$

$$[J_+, J_-] = \mu^2 \{[C_M - g_M (\mathcal{L}_J - J_0 - 1)] \phi^{(l)}(\mathcal{L}_J + J_0) - [C_M - g_M (\mathcal{L}_J + J_0 - 1)] \phi^{(m)}(\mathcal{L}_J - J_0)\},$$

(3.2)

where

$$\mathcal{L}_J = \frac{1}{2}(L_0 + M_0)$$

(3.3)

is a central element of the algebra and $C_L, C_M, g_L, \text{ and } g_M$ are the Casimir operators and the $g$-functions of the $L$-algebra and the $M$-algebra, respectively. Let us call this algebra (3.2) as $\mathcal{J}$. It is straightforward to see that $\mathcal{J}$ is a polynomial algebra of order $l+m+1$.

Now, in analogy with $su(1,1)$, let

$$K_0 = \frac{1}{2}(L_0 + M_0), \quad K_+ = \mu L_+ M_+, \quad K_- = \mu L_- M_-.$$

(3.4)

The corresponding commutation relations are :

$$[K_0, K_\pm] = \pm K_\pm,$$
\[ [\mathcal{K}_+, \mathcal{K}_-] = \mu^2 \{ [\mathcal{C}_L - g_L(\mathcal{K}_0 + \mathcal{L}_\mathcal{K} - 1)] \phi^{(m)}(\mathcal{K}_0 - \mathcal{L}_\mathcal{K}) \\
+ [\mathcal{C}_M - g_M(\mathcal{K}_0 - \mathcal{L}_\mathcal{K})] \phi^{(l)}(\mathcal{K}_0 + \mathcal{L}_\mathcal{K}) \}, \quad (3.5) \]

where

\[ \mathcal{L}_\mathcal{K} = \frac{1}{2}(L_0 - M_0) \quad (3.6) \]

is a central element of the algebra and \( \mathcal{C}_L, \mathcal{C}_M, g_L \) and \( g_M \) are the Casimir operators and the \( g \)-functions of the \( L \)-algebra and the \( M \)-algebra, respectively. Let us call the algebra (3.5) as \( \mathcal{K} \). It is clear that the polynomial algebra \( \mathcal{K} \), distinct from \( \mathcal{J} \), is also of order \( l + m + 1 \).

### 4 Examples

Let us first consider an example from our earlier work\(^\text{25,26}\). With \((J_0, J_\pm)\) as the generators of the \( su(2) \) algebra and \((a_+, a_-, N)\) as the boson operators it can be easily verified that

\[ Q_0 = \frac{1}{2}(J_0 - N), \quad Q_+ = \mu J_+ a_-, \quad Q_- = \mu J_+ a_+ \quad (4.1) \]

satisfy a quadratic algebra

\[ [Q_0, Q_\pm] = \pm Q_\pm, \quad [Q_+, Q_-] = -\mu^2 \{ 3Q_0^2 + (2\mathcal{L}_Q - 1)Q_0 - [\mathcal{C}_J + \mathcal{L}_Q(\mathcal{L}_Q + 1)] \}, \quad (4.2) \]

where

\[ \mathcal{L}_Q = \frac{1}{2}(J_0 + N) \quad (4.3) \]

is a central element of the algebra and \( \mathcal{C}_J \) is the \( su(2) \) Casimir operator. This is a \( J \)-type quadratic algebra resulting from the fusion of the \( su(2) \) algebra and a boson algebra. Fusion of a boson algebra \((l = 0)\) with \( su(2) \) and \( su(1,1) \) algebras \((m = 1)\) in this way leads to four classes of quadratic algebras which have been studied in detail by us earlier\(^\text{25,26}\).

Let us now consider a few other examples. First, let us start with a quadratic algebra \((l = 2)\) and combine it with a boson algebra \((m = 0)\) to get a cubic algebra \((l + m + 1 = 3)\). Any quadratic algebra is of the generic form

\[ [Q_0, Q_\pm] = \pm Q_\pm, \quad [Q_+, Q_-] = aQ_0^2 + bQ_0 + c, \quad (4.4) \]
where \((a, b, c)\) commute with \((Q_0, Q_\pm)\). The corresponding \(g\)-function is
\[
g(Q_0) = \frac{a}{3}Q_0^3 + \frac{1}{2}(a + b)Q_0^2 + \frac{1}{6}(a + 3b + 6c)Q_0,
\]
and the Casimir operator is
\[
C_Q = Q_+Q_0 + \frac{a}{3}Q_0^3 - \frac{1}{2}(a - b)Q_0^2 + \frac{1}{6}(a - 3b + 6c)Q_0 - \frac{1}{3}(a - 3c - 1)
= Q_-Q_0 + \frac{a}{3}Q_0^3 + \frac{1}{2}(a + b)Q_0^2 + \frac{1}{6}(a + 3b + 6c)Q_0.
\]

Following the procedure prescribed above we define
\[
C_0 = \frac{1}{2}(Q_0 - N), \quad C_+ = \mu Q_+ a_-, \quad C_- = \mu Q_- a_+.
\]
and
\[
\mathcal{L}_C = \frac{1}{2}(Q_0 + N).
\]
Then, we get the cubic algebra
\[
\begin{align*}
[C_0, C_\pm] &= \pm C_\pm, \\
[C_+, C_-] &= -\mu^2 \left\{ \frac{4a}{3}C_0^3 + \frac{1}{2}(4a\mathcal{L}_C - a + 3b)C_0^2 \\
&\quad - \left[ (a - b)\mathcal{L}_C - \frac{1}{6}(a - 3b + 12c) \right] C_0 \\
&\quad - \left[ C_0 + a\mathcal{L}_C + \frac{1}{2}(a + b)\mathcal{L}_C^2 - \frac{1}{6}(a - 3b)\mathcal{L}_C \\
&\quad - \frac{1}{3}(a - 3c - 1) \right] \right\},
\end{align*}
\]
where \(\mathcal{L}_C\) is a central element of the algebra.

As the next example, let us combine mutually commuting \(\text{su}(2)\) and \(\text{su}(1,1)\) algebras to get the Higgs cubic algebra \([1,2]\) with \(h > 0\), following Ref.[10]. Let \((J_0, J_\pm)\) and \((K_0, K_\pm)\) be, respectively, the generators of mutually commuting \(\text{su}(2)\) and \(\text{su}(1,1)\) algebras. Then,
\[
H_0 = \frac{1}{2}(J_0 - K_0), \quad H_+ = \mu J_+ K_-, \quad H_- = \mu J_- K_+.
\]

6
generate the cubic algebra

\[
[H_0, H_\pm] = \pm H_\pm, \\
[H_+, H_-] = \mu^2 \left\{ 4H_0^3 + \left[ 2(C_J - C_K) + 4L_H^2 \right] H_0 + 2(C_J + C_K)\mathcal{L}_H \right\},
\]

(4.11)

where

\[
\mathcal{L}_H = \frac{1}{2}(J_0 + K_0)
\]

(4.12)
is a central element of the algebra and \( C_J \) and \( C_K \) are the Casimir operators of the \( J \) and \( K \) algebras, respectively. The Higgs algebra (1.4) with \( h > 0 \) can be now identified with this cubic algebra by taking \( \mu^2 = h \) and \( C_J + C_K = 0 \), and suitably choosing the value of \( \mathcal{L}_H \).

As the final example, let us combine two mutually commuting \( su(1,1) \) algebras to get the Higgs cubic algebra (1.4) with \( h < 0 \), again following Ref.[10]. Let \((L_0, L_\pm)\) and \((M_0, M_\pm)\) be the generators of two mutually commuting \( su(1,1) \) algebras. Then,

\[
H_0 = \frac{1}{2}(L_0 - M_0), \quad H_+ = \mu L_+ M_-, \quad H_- = \mu L_- M_+,
\]

(4.13)
generate the cubic algebra

\[
[H_0, H_\pm] = \pm H_\pm, \\
[H_+, H_-] = -\mu^2 \left\{ 4H_0^3 + \left[ 2(C_L + C_M) - 4L_H^2 \right] H_0 - 2(C_L - C_M)\mathcal{L}_H \right\},
\]

(4.14)

where

\[
\mathcal{L}_H = \frac{1}{2}(L_0 + M_0)
\]

(4.15)
is a central element of the algebra \( C_L \) and \( C_M \) are the Casimir operators of the \( L \) and \( M \) algebras respectively. The Higgs algebra (1.4) with \( h < 0 \) can be now identified with this cubic algebra by taking \( \mu^2 = |h| \) and \( C_L = C_M \), and suitably choosing the value of \( \mathcal{L}_H \).

The last two examples show that the observation in Ref.[10] that the Higgs algebra can be obtained by combining mutually commuting \( su(2) \) and \( su(1,1) \) algebras, or two mutually commuting \( su(1,1) \) algebras, in the Jordan-Schwinger way, is a special case of a generalized Jordan-Schwinger method of constructing polynomial algebras. Now, it should be noted that
two distinct cubic algebras are obtained whenever a boson algebra is combined with a quadratic algebra or an $su(2)$ or $su(1,1)$ algebra is combined with another $su(2)$ or $su(1,1)$ algebra. Thus it is clear that there are several classes of cubic algebras of which the Higgs algebra is a special case. It should be interesting to study these algebras in detail.

5 Conclusion

Generalizing the method of construction of the Higgs algebra found in Ref.[10] and the method of construction of quadratic algebras described in Refs.[25,26] we have shown how two mutually commuting polynomial algebras of order $l$ and $m$ can be combined in the Jordan-Schwinger way to get two distinct polynomial algebras of order $l + m + 1$. The simplest example of this construction is the Jordan-Schwinger realization of $su(2)$ and $su(1,1)$, linear algebras corresponding to order $m = 1$, starting with two commuting boson algebras which are algebras of order $m = 0$. By combining a boson algebra with $su(2)$ or $su(1,1)$ we get four classes of quadratic algebras. Combining a boson algebra and these quadratic algebras or combining an $su(2)$ or $su(1,1)$ algebra with another $su(2)$ or $su(1,1)$ algebra one can get several classes of cubic algebras of which the Higgs algebra is a special case. Higher order algebras can be generated similarly by combining lower order algebras. It should be noted that the above construction leads to polynomial algebras in which the coefficients of the polynomials are central elements which are defined in terms of the Casimir operators of the original algebras with which one starts or a combination of their generators $L_0$ and $M_0$. This construction also helps find some irreducible representations of the constructed three dimensional polynomial algebras starting with the irreducible representations of the underlying $L$ and $M$ algebras. For example, in the case of each of the four classes of quadratic algebras we have obtained in Refs.[25,26] some irreducible representations have been found starting with the irreducible representations of the $su(2)$, $su(1,1)$ and boson algebras. Then, an interesting problem, which would help understand the classification and representation theory of three dimensional polynomial algebras, is: Given a three dimensional polynomial algebra with certain numerical coefficients is it possible to identify it with a particular type of three dimensional polynomial algebra generated by the fusion of two lower order algebras and corresponding to certain numerical values of the central elements?
References

[1] P. W. Higgs, *J. Phys. A: Math. Gen.* **12** 309 (1979).

[2] M. Lakshmanan and K. Eswaran, *J. Phys. A: Math. Gen.* **8** 1658 (1975).

[3] E. K. Sklyanin, *Funct. Anal. Appl.* **16** 263 (1982).

[4] T. Curtwright and C. Zachos, *Phys. Lett. B* **243** 237 (1990).

[5] A. P. Polychronakos, *Mod. Phys. Lett. A* **5** 2325 (1990).

[6] M. Roček, *Phys. Lett. B* **255** 554 (1991).

[7] O. F. Gal’bert, Ya. I. Granovskii and A. S. Zhedanov, *Phys. Lett. A* **153** 177 (1991).

[8] Ya. I. Granovskii, A. S. Zhedanov and I. M. Lutzenko, *J. Phys. A: Math. Gen.* **24** 3887 (1991).

[9] K. Schoutens, A. Sevrin and P. van Nieuwenhuizen, *Phys. Lett. B* **255** 549 (1991).

[10] A. S. Zhedanov, *Mod. Phys. Lett. A* **7** 507 (1992).

[11] V. P. Karassiov, *J. Sov. Laser Res.* **13** 188 (1992).

[12] D. Bonatsos, C. Daskaloyannis and K. Kokkotas K, *Phys. Rev. A* **48** 3407 (1993).

[13] C. Quesne, *Phys. Lett. A* **193** 245 (1994).

[14] P. Létourneau and L. Vinet, *Ann. Phys.* **243** 144 (1995).

[15] J. Van der Jeugt and R. Jagannathan, *J. Math. Phys.* **36** 4507 (1995).

[16] B. Abdesselam, J. Beckers, A. Chakrabarti and N. Debergh, *J. Phys. A: Math. Gen.* **29** 3075 (1996).

[17] J. de Boer, F. Harmsze and T. Tijn, *Phys. Rep.* **272** 139 (1996).

[18] V. I. Man’ko, G. Marmo, E. C. G. Sudarshan and F. Zaccaria, *Phys. Scr.* **55** 528 (1997).
[19] D. J. Fernández and V. Hussin, *J. Phys. A: Math. Gen.* **32** 3603 (1999).

[20] B. Roy and P. Roy, *Quantum Semiclass. Opt.* **1** 341 (1999).

[21] C. Quesne, *Phys. Lett. A* **272** 313 (2000). (*Erratum: 275* 313 (2000)).

[22] V. Sunil Kumar, B. A. Bambah, R. Jagannathan, P. K. Panigrahi and V. Srinivasan, *Quantum Semiclass. Opt.* **1** 126 (2000).

[23] S. M. Klishevich and M. S. Plyushchay, *Nucl. Phys. B* **616** 403 (2001).

[24] J. Delgado, E. C. Yustas, L. L. Sánchez-Soto and A. B. Klimov, arXiv:quant-ph/0102026

[25] V. Sunil Kumar, B. A. Bambah and R. Jagannathan, *J. Phys. A: Math. Gen.* **34** 8583 (2001).

[26] V. Sunil Kumar, *Aspects of polynomial algebras and their physical applications*, Ph.D. Thesis (submitted to University of Hyderabad, Hyderabad, India, 2002), arXiv:math-ph/0203047

[27] L. Jonke and S. Meljanac, arXiv:hep-th/0203245