Spin-Entropy

Davi Geiger and Zvi M. Kedem

Courant Institute of Mathematical Sciences
New York University, New York, New York 10012

Abstract

In classical physics, entropy quantifies the randomness of large systems, where the complete specification of the state, though possible in theory, is impossible in practice. In quantum physics, despite its inherently probabilistic nature, the concept of entropy has been elusive. The von Neumann entropy, currently adopted in quantum information and computing, models only the randomness associated with unknown specifications of a state and is zero for pure quantum states, and thus cannot quantify the inherent randomness of its observables. Our goal is to provide such quantification.

This paper focuses on the quantification of the randomness associated with observed spin values of a pure quantum, given an axis $\hat{z}$. To this end, we define a spin-entropy whose minimum is $\ln 2\pi$, reflecting the uncertainty principle for the spin observables. We also extend the concept to quantum mixed states.

The spin-entropy attains local minima for entangled Bell states, and local maxima for disentangled states. The spin-entropy may be useful for analyzing physical phenomena and developing robust quantum computational processes.
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INTRODUCTION

The quantification of randomness associated with a quantum state is a subject that bears analogies with the classical state. In classical physics, entropy was introduced to handle large ensembles of particles, where the complete specification of the state is not practical. However, it has been an elusive concept in quantum physics. Von Neumann entropy [19] quantifies the lack of knowledge an observer has about the quantum state, i.e., it quantifies the randomness in specifying the degrees of freedom (DOFs) of a quantum state. However, it does not quantify the intrinsic randomness of the observables associated with a quantum state whose DOFs are fully known. Thus, for all single particles and for all entangled particles von Neumann entropy is zero, even though only probabilities are knowable about the set of observables. The Stern–Gerlach experiment [9] illustrates the quantum scenario for spin, where spin $\frac{1}{2}$ particles with a $z$ up direction state are prepared and, despite having zero von Neuman entropy, randomness on the spin values along any direction perpendicular to $z$ is observed. Clearly, this uncertainty is due to the intrinsic randomness of the observables given the prepared quantum state, the fully specified $z$ up state.

Our focus is spin systems. We propose a definition of spin-entropy in order to quantify the randomness of the observables once the DOFs of a quantum spin state have been specified. We adopt the geometric quantization method of the spin to create a spin phase space, in which the randomness of the state can be quantified. We show that this entropy has the minimum value of $\ln 2\pi$, due to the uncertainty principle for spin. We extend the definition to mixed states and to quantum field theory where the number of particles is also an observable.

In this new quantification of randomness, we explore spin entanglement and disentanglement of two particles. Entanglement states tend to disentangle in nature, posing a challenge to build quantum computing processes, and thus maintaining entanglement is a topic of much interest in quantum information and quantum com-
puting. A better understanding of temporal evolution of such states will influence
the choice of quantum processes for constructing robust quantum algorithms and
computers. For the case of two particles, the more entangled the states are the lower
is the entropy. For three fermions of spin $\frac{1}{2}$ we suggest that maximum entangled
states should be defined by the entropy value. The lower the entropy of an entangled
state, the greater the entanglement.

**Previous Work**

Wehrl entropy [20] was introduced to approximate a classical entropy from a
quantum state. Lieb studied coherent spin states to evaluate Wehrl entropy for spin
states [13, 14]. As in the case of spatial coordinates, spin coherent states constitute
an overcomplete set of states. For spin $\frac{1}{2}$, all spin states up to a global phase are
coherent states. For spin 1 a large set of states are coherent states, including two
of the eigenstates of the spin operator $S_z$. Wehrl entropy is minimized for all such
coherent states [14, 17].

When examining entangled and disentangled states, Wehrl entropy is minimized
for disentangled states. However, reflecting on the randomness of the observables,
and due to the entanglement, the entropy of entangled states should be lower than
that of disentangled ones. We argue that such general overcomplete basis and the
arbitrariness of the choice of coherent states to define the probability distribution,
prevents Wehrl entropy from accurately quantifying the randomness associated with
the spin state observables. We argue that a proper probability distribution satisfying
all Kolmogorov axioms of a probability is required to quantify the randomness of
the observables.

For the case of two particles, the behaviors of von Neumann entropy and Wehrl
entropy of entanglement have been studied by tracing out the states of one particle
to obtain a mixture of states for the other particle. A salient difference between
our proposed entropy and von Neumann entropy and Wehrl entropy is exhibited
during the temporal evolution of an entangled state to a disentangled one. While
our proposed entropy increases the more the state disentangle, the von Neumann
entropy and Wehrl entropy decreases the more the states disentangle.

SPIN PHASE SPACE

We first briefly review geometric aspects of spin, including the geometric quan-
tization method that leads to the spin phase space.

The spin matrix associated with a particle can be specified, e.g., [6], as
\[
\vec{S} = S_x \hat{x} + S_y \hat{y} + S_z \hat{z} \quad \text{and} \quad S^2 = S_x^2 + S_y^2 + S_z^2,
\]
\[\{S_a, S_b\} = i\hbar S_c, \quad \text{where } a, b, c \text{ is a cyclic permutation of } x, y, z, \text{ and}\]
\[\{S^2, S_a\} = 0, \quad \text{for } a = x, y, z.\]

The spin value of a particle is a Casimir invariant but it is not possible to know
the values of the spin projections in all directions in three dimensions. Knowing
the value of the spin along the $z$ direction implies that the $x$ or the $y$ direction spin
values are unknowable. This uncertainty reflects the close relation between spin
matrices, their unitary transformations, and the rotation group SO(3). For spin $1/2$
particles, any spin state is reachable from any other spin state via a $2 \times 2$ unitary
transformation, which is a local isomorphism (and a global homomorphism) to the
SO(3) group. For spin 1, the matrices are unitarily similar to SO(3), and they can
be transformed into generators of SO(3) via unitary transformations.

Two observations, which we describe next, lead us to adopting the Geometric
Quantization (GQ) method. One is the relevance of the SO(3) group to modeling
spin states which leads to the quantizing of the sphere itself. The other is that at
any given time, a spin observable is the spin value along one chosen direction, say $z$. 
direction, and thus the uncertainty is between the $z$ direction and not-the-$z$ direction.

**The Geometric Quantization Approach to Spin Phase Space**

The phase space of a spin is derived from quantizing the sphere as it is developed by the GQ method, e.g., see [4, 16, 22], and we summarize it now. The sphere is the surface of the ball with a radius of the spin magnitude $s\hbar$.

On the sphere, the $z$ values are specified by the polar representation $s\hbar \cos \theta$, while the values on the intersection with planes perpendicular to the $z$ axis are specified by the azimuth angle $\phi$. Treating the sphere as a phase space, one assigns a rescaled symplectic 2-form in spherical polar coordinates $d\omega = dp \wedge dq = s\hbar \sin \theta \, d\theta \wedge d\phi$, and so the Lagrangian is $\mathcal{L} = p\dot{q} = s\hbar \cos \theta \, \dot{\phi}$, and the action is $S_L = \int \mathcal{L} \, dt = s\hbar \int \cos \theta \, d\phi$. From this Lagrangian, the GQ method derives the uncertainty commutation relation

$$[\phi, s\hbar \cos \theta] = i\hbar,$$

where the conjugate pair of eigenvalues $(\phi, s\hbar \cos \theta) \in [0, 2\pi] \times \{-s\hbar, \ldots, s\hbar\}$ forms a spin phase space with a finite Hilbert space volume $2(2s + 1)\pi\hbar$.

Note that the rotation operator $e^{-i\frac{S_z}{\hbar} \phi}$ of an angle $\phi$ around the $z$ axis describes the polarization angle in the $x$-$y$ plane. Thus we will refer to the angle $\phi$ as the polarization angle. In order to create physical quantities with the polarization operator $\phi$, we must constrain $\phi$ to a periodic function with period $2\pi$, i.e., to values that are a function of $e^{i\phi}$.

At the northern pole ($\cos \theta = 1$) and the southern pole ($\cos \theta = -1$) the angles $\phi$ are not defined. Thus, in the basis $|\phi\rangle$ that diagonalizes $\phi$ we have two operators, namely $s\hbar \cos \theta = s\hbar - i\hbar \frac{\partial}{\partial \phi}$ for the northern hemisphere and $s\hbar \cos \theta = -s\hbar - i\hbar \frac{\partial}{\partial \phi}$ for the southern hemisphere.
In the basis $|\phi\rangle$, the eigenstates of the operator $S_z = s\hbar \cos \theta$ are

$$|\xi_{s,m}\rangle = \int |\phi\rangle \langle \phi|\xi_{s,m}\rangle \ d\phi = \int \psi_{s,m}(\phi) |\phi\rangle \ d\phi,$$

where $m = -s, \ldots, s$, and

$$\psi_{s,m}(\phi) = \begin{cases} \frac{1}{\sqrt{2\pi}} e^{i(s+m)\phi}, & m \geq 0 \quad \text{(northern hemisphere)}; \\ \frac{1}{\sqrt{2\pi}} e^{i(-s+m)\phi}, & m < 0 \quad \text{(southern hemisphere).} \end{cases}$$ (1)

The two solutions in (1) are periodic in $\phi$ and differ by a phase (gauge) transformation of $e^{-i 2s\phi}$.

Consider a particle state $|\xi_s\rangle$ of spin magnitude $s\hbar$. This state in the basis of the eigenvectors of $S_z = s\hbar \cos \theta$ and $S^2$ is

$$|\xi_s\rangle = \sum_{m=-s}^{s} \alpha_{s,m} |\xi_{s,m}\rangle,$$

with $1 = \sum_{m=-s}^{s} |\alpha_{s,m}|^2$. In the basis of the conjugate variable $\phi$, the state is

$$|\xi_s\rangle = \int_0^{2\pi} |\phi\rangle \langle \phi|\xi_s\rangle \ d\phi = \int_0^{2\pi} |\phi\rangle \sum_{m=-s}^{s} \alpha_{s,m} \langle \phi|\xi_{s,m}\rangle \ d\phi$$

$$= \int_0^{2\pi} \sum_{m=-s}^{s} \alpha_{s,m} \psi_{s,m}(\phi) |\phi\rangle \ d\phi = \int_0^{2\pi} \lambda_s(\phi) |\phi\rangle \ d\phi,$$ (2)

where

$$\lambda_s(\phi) = \langle \phi|\xi_s\rangle = \sum_{m=-s}^{s} \alpha_{s,m} \psi_{s,m}(\phi).$$ (3)

Thus, for a state $|\xi_s\rangle$ with density matrix $\rho_s = |\xi_s\rangle \langle \xi_s|$ the probabilities of the phase space are the product of the probabilities $\{\rho_{s,m} = \langle \xi_{s,m}| \rho_s |\xi_{s,m}\rangle = |\alpha_{s,m}|^2\}$ with the probability densities $\{\rho_s(\phi) = \langle \phi| \rho_s |\phi\rangle = |\lambda_s(\phi)|^2\}$.  

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The Spin-Entropy in Spin Phase Space

Our goal is to quantify the randomness given a spin state. In order to capture the randomness of the observables without double counting them, we specify the spin conjugate operators that creates the phase space. The phase space captures all the randomness of the observables of a spin state.

We define the spin-entropy of a pure quantum state \( \ket{\xi_s} \) with spin \( s \) in spin phase space to be

\[
S = S^z + S^{z^+} = - \sum_{m=-s}^{s} \rho_{s,m} \ln \rho_{s,m} - \int \rho_s(\phi) \ln \rho_s(\phi) \, d\phi.
\]

The first term is the Shannon entropy capturing the randomness of the spin value along the \( z \) axis. The second term is differential entropy capturing the randomness of the spin value in the plane perpendicular to the \( z \) axis, i.e., the entropy of the polarization angle \( \phi \). We define intrinsic-spin-information, represented by \( \Gamma \), as the inverse of the entropy, i.e.,

\[
\Gamma = \frac{-1}{\sum_{m=-s}^{s} |\alpha_{s,m}|^2 \ln |\alpha_{s,m}|^2 + \int |\lambda_s(\phi)|^2 \ln |\lambda_s(\phi)|^2 \, d\phi}.
\]

\( \Gamma \) is non-negative and quantifies how much it is known about the observables of a state, where infinity indicates full knowledge of all observable values simultaneously.

We conjecture that the spin-entropy (4) depends only on the variables that define the spin-entropy’s component along the \( z \) axis. This is further elaborated and utilized in Conjecture 1. We also conjecture that the ordering (ranking) of the states according to the spin-entropy can be obtained solely from the spin-entropy component along the \( z \) axis. This conjecture implies that states with maximum and minimum spin-entropy values are the ones with maximum and minimum spin-
entropy component along the $z$ axis.

**Illustrative Example**

To illustrate, consider the case of a state with $\alpha_{s,m} = \sqrt{2\pi/2s+1} \psi_{s,m}^*(\phi_0)$. Then, we get the uniform distribution $|\alpha_{s,m}|^2 = 1/2s+1$, i.e., thus a state with the lowest intrinsic-spin-information along the $z$ axis. Then

$$
\lambda_s(\phi) = \frac{1}{\sqrt{2\pi \cdot 2s+1}} \sum_{m=-s}^s e^{i((2\theta(m)-1)s+m)(\phi-\phi_0)}
= \frac{1}{\sqrt{2\pi \cdot (2s+1)}} \left(1 - \text{mod}(2s, 2) + 2 \sum_{j=1}^{2s} \cos(j(\phi - \phi_0))\right),
$$

where $j_{\text{min}} = s + 1 - \frac{1}{2} \text{mod}(2s, 2)$, and mod is the modulus function. Note that the distribution $\lambda_s(\phi)$ becomes more concentrated around $\phi_0$ as $s$ increases. The intrinsic-spin-information about the spin on the plane $z^\perp$ increases as $s$ increases. However, the total intrinsic-spin-information reduces because of the increase in spin-entropy along $z$. For a fermion of spin $\frac{1}{2}$ the spin-entropy becomes

$$
S = \ln 2 + \ln \pi - \frac{1}{\pi} \int_0^{2\pi} \cos^2(\phi - \phi_0) \ln \cos^2(\phi - \phi_0) \, d\phi
= \ln 2 + \ln \pi + \ln \frac{4}{e^\Gamma} \approx 1.531 + \ln 2,
$$

and $\Gamma \approx 0.450$. For spin 1, $S \approx 1.270 + \ln 3$ with $\Gamma \approx 0.422$.

**Minimum Spin-Entropy**

The third law of thermodynamics establishes zero as the minimum of thermodynamics entropy. The use of differential entropy for the term $S_{s,\phi}^{z^\perp}$ may suggest that the proposed entropy could be negative, but we will establish a positive minimum
for it.

In quantum mechanics, ignoring the spin and focusing on the spatial DOFs, the entropic uncertainty principle \([1, 3, 11]\), establishes that

$$S_x + S_p \geq 3 \ln(e \hbar \pi),$$

with equality for the normal distribution. A similar bound can now be derived for the spin-entropy.

**Lemma 1** (Hausdorff-Young Inequality Holds). The Hausdorff-Young inequality holds for the spin states phase space, i.e., for \(p \in (1, 2]\) and for every \(q\) satisfying

\[
1 = \frac{1}{p} + \frac{1}{q},
\]

\[
\left( \sum_{m=-s}^{s} |\alpha_{s,m}|^q \right)^{1/q} \leq \left( (2\pi)^{\frac{p-2}{2}} \int_0^{2\pi} |\lambda_s(\phi)|^p \, d\phi \right)^{1/p}.
\]

**Proof.** Rewrite (3) as

$$\lambda_s(\phi) = \sum_{m=-\infty}^{\infty} \alpha_{s,m} \psi_{s,m}(\phi).$$

(5)

where \(\alpha_{s,m} = 0\) for \(|m| > s\). Note that the sum on \(m\) is either on all even or all odd multiples of \(\frac{1}{2}\), depending on whether \(2s\) is even or odd. Then, we rewrite (5) as

$$\lambda_s(\phi) = \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} \alpha_{s,m} e^{i[(2\theta(m)-1) s + m] \phi} = \frac{1}{\sqrt{2\pi}} \sum_{j=-\infty}^{\infty} \beta_{s,j} e^{ij \phi},$$

(6)

where \(j \in \mathbb{Z}, \beta_{s,j} = \alpha_{s,m}\) for \(j = (2\theta(m) - 1) s + m\) and \(\beta_{s,j} = 0\) otherwise. Clearly, for any value of \(r\)

\[
\sum_{j=-\infty}^{\infty} |\beta_{s,j}|^r = \sum_{m=-\infty}^{\infty} |\alpha_{s,m}|^r.
\]

Then (6) describes a Fourier series expansion of the periodic complex-valued func-
tion $\lambda_s(\phi)$. The Fourier inverse is

$$\frac{1}{\sqrt{2\pi}} \int_0^{2\pi} \lambda_s(\phi) e^{-i\phi} \, d\phi = \beta_{s,j} \quad \text{or} \quad \int_0^1 \sqrt{2\pi} \lambda_s(2\pi \mu) e^{-2\pi i \mu} \, d\mu = \beta_{s,j}.$$  

Thus, the Hausdorff-Young inequality

$$\left( \sum_{j=-\infty}^{\infty} |\beta_{s,j}|^q \right)^{1/q} \leq \left( \int_0^1 \left| \sqrt{2\pi} \lambda_s(2\pi \mu) \right|^p \, d\mu \right)^{1/p} \quad (7)$$

holds. We can then replace back the variables $\alpha_{s,m}$’s and $\phi = 2\pi \mu$ into (7), completing the proof.

**Theorem 1.** The spin-entropy satisfies the inequality

$$S \geq \ln 2\pi,$$  

with equality attained for the eigenstates of the spin operator $S_z = \hbar \cos \theta$.

**Proof.** This proof is an adaptation to the specific spin phase space of the work of [1, 3, 11].

First, by Lemma [4] we can apply Beckner’s theorem [1]

$$\left( \sum_{j=-\infty}^{\infty} |\beta_{s,j}|^q \right)^{1/q} \geq \frac{\left( \sum_{m=-s}^{s} |\alpha_{s,m}|^q \right)^{1/q}}{(2\pi)^{p/2} \int_0^{2\pi} |\lambda_s(\phi)|^p \, d\phi} \quad (7')$$

where $1 < p \leq 2$, $\frac{1}{p} + \frac{1}{q} = 1$, and $n$ is the dimensionality of the space in which the functions are defined. For the sphere, $n = 2$. Applying the logarithm to both sides of the inequality and noting that all the quantities are non-negative, we get

$$\ln \left( \sum_{m=-s}^{s} |\alpha_{s,m}|^q \right)^{1/q} \geq \ln \left( \sum_{m=-s}^{s} |\alpha_{s,m}|^q \right)^{1/q} \quad (9)$$

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Substituting $p = 2\alpha$ and $q = 2\beta$, we get the constraint $\alpha/1-\alpha = -\beta/1-\beta \in (1, \infty)$ from $1 = 1/p + 1/q$. Multiplying the terms on both sides of (9) by either one of these equal and positive expressions $\alpha/1-\alpha$ or $-\beta/1-\beta$, and rearranging terms, we get

$$\frac{1}{1 - \alpha} \ln \left( \int_0^{2\pi} |A_x(\phi)|^{2\alpha} d\phi \right) + \frac{1}{1 - \beta} \ln \left( \sum_{m=-s}^{s} |\alpha_{s,m}|^{2\beta} \right) \geq \frac{1}{1 - \alpha} \ln \left( \frac{1}{2\alpha} \right) + \frac{1}{1 - \beta} \ln \left( \frac{1}{2\beta} \right) + \ln 2\pi,$$

where the two first terms are the Rényi’s differential entropy and the Rényi’s entropy, forming an entropic inequality. Thus, by taking the limits $\alpha \to 1^-$ and $\beta \to 1^+$, we obtain (8).

The minimum entropy is reached for the eigenstates of $S_z$, $|\xi_{s,z}\rangle = |\xi_{s,m_0}\rangle$, for any eigenvalue $\hbar m_0$ where $-s \leq m_0 \leq s$. Then $\alpha_{s,m} = \delta_{m,m_0}$, and so $\lambda_s(\phi) = \psi_{s,m_0}$. The probabilities are then $|\alpha_{s,m}|^2 = \delta_{m,m_0}$, and so the Shannon entropy is zero. The probability density is then $|\lambda_s(\phi)|^2 = 1/2\pi$, with differential entropy $\ln 2\pi$. □

Thus, for the intrinsic-spin-information $0 \leq \Gamma \leq 1/\ln 2\pi$.

Two observations:

1. The spin-entropic principle inequality for a bounded variable $\phi$ and an integer variable $m$ resulting from the Fourier series transformation reaches its minimum for the eigenstates of the $S_z, S^2$ operators, with corresponding $S_z$ eigenvalues $m$’s.

2. Preparing a system to align a spin state with a particular direction, an eigenstate $|\xi_{s,m=s}\rangle$ of a $z$ axis, as it is done in many experiments with spin up along $z$, provides the knowledge of the spin along that direction. Thus, the intrinsic-spin-information is gained by such a preparation of the lowest spin-entropy state.
Extending the Spin-entropy to Mixed States

Mixed states extend the Hilbert space of specified quantum states, or pure states, to quantum states that are not fully specified and instead are described by a classical probabilistic combination of pure states. Let a spin mixed state be defined via the density matrix

$$\rho^M_s = \sum_{i=1}^{N} \gamma_i |\xi^i_s\rangle \langle \xi^i_s|,$$

where $|\xi^i_s\rangle = \sum_{m=-s}^{s} \alpha^{i}_{s,m} |\xi_{s,m}\rangle$, $i = 1, \ldots, N$ are pure states. The randomness present in a pure state is concentrated in the observables of the spin phase space. However, for mixed states additional randomness exists, that of specifying the state itself and is captured by the probabilities $\gamma_i$. Then, the distributions after a choice of $z$ axis are

$$\rho^\gamma_{s,m,i} = \gamma_i \langle \xi_{s,m} | \xi^i_s \rangle \langle \xi^i_s | \xi_{s,m} \rangle = \gamma_i |\alpha^i_{s,m}|^2,$$

$$\rho^\gamma_{s,m} = \langle \xi_{s,m} | \rho^M_s | \xi_{s,m} \rangle = \sum_{i=1}^{N} \rho^\gamma_{i,s,m} = \sum_{i=1}^{N} \gamma_i |\alpha^i_{s,m}|^2,$$

with the normalizations $1 = \sum_{i=1}^{N} \sum_{m=-s}^{s} \rho^\gamma_{i,s,m}$ and $1 = \sum_{m=-s}^{s} \rho^M_{s,m}$. Thus, extending (4) to include mixed states, the spin-entropy is then

$$S^M_s = - \sum_{i=1}^{N} \sum_{m=-s}^{s} \rho^\gamma_{s,m,i} \ln \rho^\gamma_{s,m,i} - \sum_{i=1}^{N} \int \rho^\gamma_{s,i}(\phi) \ln \rho^\gamma_{s,i}(\phi) \, d\phi,$$

$$= S^{vN} + \sum_{i=1}^{N} \gamma_i S^i_s, \quad (10)$$

where von Neumann entropy is given by $S^{vN} = - \sum_{i=1}^{N} \gamma_i \ln \gamma_i$ and $S^i_s = - \sum_{m=-s}^{s} |\alpha^i_{s,m}|^2 \ln |\alpha^i_{s,m}|^2 + \int \lambda^i_s(\phi) \ln \lambda^i_s(\phi) \, d\phi$ is the entropy of each pure state in the mixture. This entropy is always larger than von Neumann entropy, since we
also consider the randomness of the observables and for every \( i \), \( S^i_s \geq 0 \), and so \( \sum_{i=1}^{N} \gamma_i S^i_s \geq 0 \).

Note that if one focuses only on the randomness of the observables, the distributions in spin phase space are

\[
\rho^M_{s,m} = \langle \xi_{s,m} | \rho^M | \xi_{s,m} \rangle = \sum_{i=1}^{N} \gamma_i |a^i_{s,m}|^2,
\]

\[
\rho^M_s(\phi) = \langle \phi | \rho^M | \phi \rangle = \sum_{i=1}^{N} \gamma_i \left( \sum_{m'=-s}^{s} \alpha^i_{s,m'} \psi_{s,m'}(\phi) \right)^* \sum_{m'=-s}^{s} \alpha^i_{s,m'} \psi_{s,m'}(\phi).
\]

**Pure System with Multiple Particles and Quantum Field Theory**

Consider the spin of a pure system with \( N \) particles of the same species, each with spin \( s \). The maximum spin \( s_{\text{max}} \) of the system occurs when all the \( N \) particles are aligned yielding \( s_{\text{max}} = Ns \). For a state with spin value \( s_{\text{max}} \), the possible \( z \) values are \( m = -s_{\text{max}}, \ldots, s_{\text{max}} \). However, for the system with \( N \) particles all different values \( s' \in [s_{\text{min}}, s_{\text{max}}] \) can occur, where \( s_{\text{min}} = s \mod(N, 2) \) for fermions and \( s_{\text{min}} = 0 \) for bosons. For each \( s' \), all possible \( z \) components must be considered, so \( m_{s'} = -s', \ldots, s' \). The eigenstate basis of the \( S_z, S^2 \) operators associated with the system is then

\[
\left\{ |\xi_{s',m}^{N,s} \rangle \right\} = \bigcup_{s'=s_{\text{min}}}^{s_{\text{max}}} \bigcup_{m'=-s'}^{s'} \left| \xi_{s',m'} \right\rangle.
\]

For example, two fermions with \( s = 1/2 \) will produce four \( z \) states, namely three states with \( s' = 1 \) and \( m' = -1, 0, 1 \), and the fourth state with \( s' = 0 \) and \( m' = 0 \). Thus, the eigenstate basis of \( S_z, S^2 \) operators for two fermions is \( \left\{ |\xi_{1,1} \rangle, |\xi_{1,0} \rangle, |\xi_{1,-1} \rangle, |\xi_{0,0} \rangle \right\} \).

In order to extend the spin-entropy to \( N \) particles of the same species, we first describe
the density matrix of a state to be

$$\rho^{N,s} = |\xi^{N,s}\rangle \langle \xi^{N,s}| = \left( \sum_{s' = s_{\min}}^{s_{\max}} \sum_{m = -s'}^s \alpha_{s',m} |\xi_{s',m}\rangle \right) \left( \sum_{s'' = s_{\min}}^{s_{\max}} \sum_{m' = -s''}^s \alpha_{s'',m'} |\xi_{s'',m'}\rangle \right)^*.$$  

Thus, projecting (11) onto the $\hat{z}$-basis and onto the $|\phi\rangle$ basis we derive the density functions in spin phase space for $s' \in [s_{\min}, s_{\max}]$ as

$$\rho^{N,s}_{m} = |\langle \xi_{s',m} | \rho^{N,s} | \xi_{s',m}\rangle|^2, \quad \text{for} \quad m \in [-s', s'], \quad \text{and}$$

$$\rho^{N,s}(\phi) = \langle \phi | \rho^{N,s} | \phi\rangle = |\lambda_s(\phi)|^2 = \sum_{m = -s'}^{s'} \sum_{m' = -s'}^{s'} \alpha_{s',m} \alpha_{s',m'}^* \psi_{s',m}(\phi) \psi_{s',m'}^*(\phi).$$

Then the spin-entropy (4) of the system is

$$S_{N,s} = S_{N,s}^z + S_{N,s}^{\pm}$$

$$= - \sum_{s' = s_{\min}}^{s_{\max}} \sum_{m = -s'}^s \rho^{N,s}_{m} \ln \rho^{N,s}_{m} - \sum_{s' = s_{\min}}^{s_{\max}} \int \rho^{N,s}(\phi) \ln \rho^{N,s}(\phi) \ d\phi$$

$$= - \sum_{s' = s_{\min}}^{s_{\max}} \left( \sum_{m = -s'}^{s'} |\alpha_{s',m}|^2 \ln |\alpha_{s',m}|^2 - \int |\lambda_s(\phi)|^2 \ln |\lambda_s(\phi)|^2 \ d\phi \right). \ (12)$$

where the normalization is $1 = \sum_{s' = s_{\min}}^{s_{\max}} \sum_{m = -s'}^s \rho^{N,s}_{m}$ and $1 = \sum_{s' = s_{\min}}^{s_{\max}} \int |\lambda_s(\phi)|^2 \ d\phi$, and the phase space to describe a system of $N$ particles of spin $s$ consists of all the spheres with radii $\hbar s$, for $s \in [s_{\min}, s_{\min} + 1, \ldots, s_{\max} - 1, s_{\max}].$

In quantum field theory (QFT), superpositions of states with any number of
particles of spin species \( s \) are also considered. One can write a state as

\[
|\xi_{\text{QFT}}\rangle = \sum_{N=1}^{\infty} \sum_{s_{\text{min}} = s_{0}}^{s_{\text{max}}} \sum_{m=-s'}^{s'} \alpha_{s',m}^{N} |\xi_{s',m}\rangle ,
\]

where \( \sum_{N=1}^{\infty} |\gamma_{N}|^2 = 1 \) and both, \( s_{\text{min}} \) and \( s_{\text{max}} \), depend on \( N \) and \( s \). Due to the orthogonality of the states with different number of particles, it is straightforward to extend (12) to obtain

\[
S_{\text{QFT},s} = \sum_{N=1}^{\infty} |\gamma_{N}|^2 \left( S_{N,s}^z + S_{N,s}^{z+} \right) - \sum_{N=1}^{\infty} |\gamma_{N}|^2 \ln |\gamma_{N}|^2 .
\]

Thus, in QFT the role of the magnitude square of the complex valued coefficient \( \gamma_{N} \) resembles the mixed states coefficients to the entropy (10).

**The \( z \) axis**

The formulation of a spin operator and corresponding eigenstates requires a choice of a \( z \) axis. This is evident from the spin phase space where the quantization of the spin is along the \( z \) axis, and perpendicular to it a continuous polarization variable \( \phi \in [0, 2\pi) \) is assigned. A choice of \( z \) axis must be made to construct the quantum state and assign to it a unique entropy. Approaches such as [14] average their proposed entropy over all possible 3D rotations to eliminate the \( z \) axis bias, ending up with a spin-entropy that bundles a large set of quantum states into one value. Instead, we investigate a physical property that causes the break of the isotropy of the 3D space for spin states. For the Stern–Gerlach (SG) experiment, the direction of the applied magnetic field does define the \( z \) axis, splitting spins according to a positive \( z^+ \) or negative \( z^- \) direction eigenstates.

For a system of particles with total spin \( S \) its spin Hamiltonian can be described by

\[
H = \frac{\alpha h}{2m} (g B_e + (g - 1) B_t) \cdot S
\]

where \( \alpha \) is the scalar product, \( g \) is the gyromagnetic
ratio, \( q \) the charge of the system, \( m \) its inertial mass, \( B_e \) is the external magnetic field applied to the spin system, and \( B_1 = \frac{q E v}{\sqrt{1 - \frac{v^2}{c^2}}} \) is the magnetic field in the rest frame of the system when the charged system is moving in an electric field \( E \) with velocity \( v \). Thus, the preferred direction that breaks the isotropy of the 3D space and defines the \( z \) axis is \( g B_e + (g - 1) B_1 \).

Consider the example of photon emission by an excited hydrogen atom in state \( 2p_z (n = 2, l = 1, m = 1) \) transitioning to state \( 1s (n = 1, l = 0, m = 0) \). The \( z \) axis defines the quantum state associated with the numbers \( m = 1, 0, -1 \), i.e., the excited state already broke the isotropy and consequently the angular momentum conservation leads to the emission of a photon on a plane perpendicular to this \( z \) axis.

**SPIN-ENTROPY FOR ONE PARTICLE**

We first analyze the spin-entropy for a fermion with spin value \( \frac{1}{2} \), then we analyze a massive boson with spin 1, and we conclude this section by analyzing the photon entropy.

**Spin \( \frac{1}{2} \)**

A spin state of a particle with spin \( \frac{1}{2} \), represented by a set of two orthonormal eigenstates \( |+\rangle = |\xi_{1/2,1/2}\rangle = (1, 0)^T \) and \( |-\rangle = |\xi_{1/2,-1/2}\rangle = (0, 1)^T \) of the operators \( (S^2, S_z) \) with associated eigenvalues \( (s = 1/2, m = \pm 1/2) \), is

\[
|\xi_{1/2,z}\rangle = e^{i\varphi} \left( e^{iv \cos \theta_\alpha} |+\rangle + \sin \theta_\alpha |-\rangle \right) = e^{i\varphi} \left( e^{iv \cos \theta_\alpha, \sin \theta_\alpha} \right)^T
\]

with \( \theta_\alpha \in [0, \frac{\pi}{2}] \) and \( \varphi, v \in [0, \pi) \).
Figure 1. A plot of spin-entropy (14) vs \( \theta_\alpha \) \( \in [0, \frac{\pi}{2}] \), for \( s = \frac{1}{2} \).

**Proposition 1** (spin-entropy for \( s = \frac{1}{2} \)). The spin-entropy of a spin state with \( s = \frac{1}{2} \), described by (13), is

\[
S_{z/2}(\theta_\alpha) = -\cos^2 \theta_\alpha \ln \cos^2 \theta_\alpha - \sin^2 \theta_\alpha \ln \sin^2 \theta_\alpha + \ln 2\pi \\
- \frac{1}{2\pi} \int_0^{2\pi} (1 + \sin 2\theta_\alpha \cos 2\phi) \ln (1 + \sin 2\theta_\alpha \cos 2\phi) \, d\phi ,
\]

(14)

**Proof.** A state \( |\xi_{1/2}\rangle \) assigns the probability distribution along the \( z \) axis eigenstates \( P_z = \left( \langle +|\xi_{1/2}\rangle \langle +|\xi_{1/2}\rangle \right)^T = \left( \cos^2 \theta_\alpha, \sin^2 \theta_\alpha \right)^T \). Thus, from (2) and (5) we get \( |\xi_{1/2,\phi}\rangle = \int e^{i\phi} \frac{1}{\sqrt{2\pi}} \left( e^{i\theta_\alpha} \cos \theta_\alpha + e^{-i\theta_\alpha} \sin \theta_\alpha \right) |\phi\rangle \, d\phi \), and so \( \rho_{\theta_\alpha,\nu}(\phi) = \frac{1}{2\pi} (1 + \sin 2\theta_\alpha \cos(2\phi + \nu)) \). \( \square \)

For a visualization of the entropy, see Figure[1]

Note the Wehrl’s entropy for any state \( |\xi_{1/2}\rangle \) is a constant since all spin \( 1/2 \) states are the same coherent state up to a phase \[13, 14\], yielding the same entropy.
Spin 1

For massive particles with spin \( s = 1 \), the spin matrices are

\[
S_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad S_y = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad S_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad S^2 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix},
\]

yielding a basis representation formed with the eigenvectors of \( S_z \) and \( S^2 \): \(|\uparrow\rangle_z = |\xi_{1,1}\rangle = (1, \ 0, \ 0)^T \), \(|\rightarrow\rangle_z = |\xi_{1,0}\rangle = (0, \ 1, \ 0)^T \), \(|\downarrow\rangle_z = |\xi_{1,-1}\rangle = (0, \ 0, \ 1)^T \)\). A general state of spin \( s = 1 \) in the basis aligned with the \( z \) axis is

\[
|\xi_z\rangle = e^{i\phi_x} (\sin \theta_\alpha \cos \theta_\beta \ e^{i\phi_x} \ |\uparrow\rangle_z + \cos \theta_\alpha \ |\rightarrow\rangle_z + \sin \theta_\alpha \sin \theta_\beta \ e^{i\phi_z} \ |\downarrow\rangle_z ), \quad (15)
\]

with \( \theta_\alpha, \theta_\beta \in [0, \frac{\pi}{2}] \), and \( \phi_x, \phi_z, \phi_y \in [0, 2\pi) \). Following (2), we write the state in the \(|\phi\rangle\) basis as

\[
|\xi_z\rangle = \frac{e^{i\phi_y}}{\sqrt{2\pi}} \int (\sin \theta_\alpha \cos \theta_\beta \ e^{i(\phi_x+2\phi)} + \cos \theta_\alpha \ e^{i\phi} + \sin \theta_\alpha \sin \theta_\beta \ e^{i(\phi_z-2\phi)} ) \ |\phi\rangle \ d\phi.
\]

\( (16) \)

Proposition 2. The spin-entropy of a particle with spin \( s = 1 \) in a state given by (15) is

\[
S_1 = -\int_0^{2\pi} \rho_1(\phi, \varphi_x, \varphi_z, \theta_\alpha, \theta_\beta) \ln \rho_1(\phi, \varphi_x, \varphi_z, \theta_\alpha, \theta_\beta) \ d\phi \\
+ S_c(\cos^2 \theta_\alpha) + \sin^2 \theta_\alpha \ S_c(\cos^2 \theta_\beta), \quad (17)
\]

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where

\[
S_c(\cos^2 \theta) = -\cos^2 \theta \ln \left( \cos^2 \theta \right) - (1 - \cos^2 \theta) \ln \left( 1 - \cos^2 \theta \right)
\]

\[
\rho_1(\phi) = \frac{1}{2\pi} \left| \sin \theta_\alpha \cos \theta_\beta e^{i(\varphi_x + 2\phi)} + \cos \theta_\alpha e^{i\phi} + \sin \theta_\alpha \sin \theta_\beta e^{i(\varphi_z - 2\phi)} \right|^2
\]

\[
= \frac{1}{2\pi} \left[ 1 + \sin^2 \theta_\alpha \sin(2\theta_\beta) \cos(4\phi + \varphi_x - \varphi_z)
+ \sin(2\theta_\alpha)(\cos \theta_\beta \cos(\phi + \varphi_x) + \sin \theta_\beta \cos(3\phi - \varphi_z)) \right].
\]

Proof. Computing the three probabilities associated with state \(|\xi_z\rangle\) and deriving its entropy term \(S_z^\perp\) yields

\[
S_z^\perp = -\cos^2 \theta_\alpha \ln \cos^2 \theta_\alpha - \sin^2 \theta_\alpha \left[ \ln \sin^2 \theta_\alpha - S_c(\cos^2 \theta_\beta) \right].
\]

The entropy term \(S_z^\perp\) is derived from the probability density associated with the state \(|\xi_{z^\perp}\rangle\) from (16), and so

\[
\rho_1(\phi) = \frac{1}{2\pi} \left| \sin \theta_\alpha \cos \theta_\beta e^{i(\varphi_x + 2\phi)} + \cos \theta_\alpha e^{i\phi} + \sin \theta_\alpha \sin \theta_\beta e^{i(\varphi_z - 2\phi)} \right|^2.
\]

\[
\square
\]

Observed in simulations and in (14), we conjecture that

**Conjecture 1.** The spin-entropy depends only on the variables that define the spin-entropy component along the z axis. In particular, the spin-entropy (17) of a state given by (15) can be simplified to

\[
S_1(\theta_\alpha, \theta_\beta) = -\int_0^{2\pi} \rho(\phi, \theta_\alpha, \theta_\beta) \ln \rho(\phi, \theta_\alpha, \theta_\beta) \, d\phi
+ S_c(\cos^2 \theta_\alpha) + \sin^2 \theta_\alpha S_c(\cos^2 \theta_\beta),
\]

(18)
Figure 2. Spin-entropy for $s = 1$ (18) vs $(\theta_\alpha, \theta_\beta)$. Note that for $\theta_\alpha = 0$ and for $\theta_\alpha = \frac{\pi}{2}$, $\theta_\beta = 0, \frac{\pi}{2}$, describing the three eigenstates along the $z$ axis, the spin-entropy reaches its minimum value $1.838$, approximating $\ln 2\pi$.

where

\[
\rho(\phi, \theta_\alpha, \theta_\beta) = \frac{1}{2\pi} | \sin \theta_\alpha \cos \theta_\beta e^{i2\phi} + \cos \theta_\alpha e^{i\phi} + \sin \theta_\alpha \sin \theta_\beta e^{-i2\phi}|^2
\]

By Theorem 1, the spin-entropy is minimized at $\theta_\alpha = 0$, at $(\theta_\alpha = \pi/2, \theta_\beta = 0)$, and at $(\theta_\alpha = \pi/2, \theta_\beta = \pi/2)$, the three eigenstates along the $z$ axis. Thus, preparing a spin state orientation by aligning it with an axis, reduces the entropy. The entropy given by (18) is visualized in Figure 2.

In order to compare (18) to Wehrl’s spin-entropy, we consider the spin 1 coherent states

\[
|s, \alpha\rangle = \left( \cos \frac{\theta}{2} \right)^2 |1, -1\rangle + \sqrt{2} \cos \frac{\theta}{2} \sin \frac{\theta}{2} e^{i\phi} |1, 0\rangle + \left( \sin \frac{\theta}{2} \right)^2 e^{i2\phi} |1, 1\rangle . \quad (19)
\]

The cases where $\theta$ is 0 or $\pi$ show that the states $|1, -1\rangle$ and $|1, 1\rangle$ are coherent states, and thus minimize the Wehrl spin-entropy. However, state $|1, 0\rangle$ cannot be a coherent state and therefore has higher Wehrl’s spin-entropy. For our spin-entropy, all eigenstates of $S_z$ minimize the spin-entropy. Moreover, we note that coherent
states (19) correspond to the general spin 1 state (15) via the mapping
\[ \phi = \varphi_z = -\varphi_x, \quad \sin \theta = \sqrt{\frac{2 \sin 2\theta_B}{1 + \sin 2\theta_B}} = \sqrt{2} \cos \theta_x \]

There is a one-to-one map between \( \theta \) and \( 2\theta_B \), both in the range \([0, \pi]\). Thus, changes in the angle \( \theta_x \) from the constraint \( \cos \theta_x = \sqrt{\sin^2 \theta_B / (1 + \sin 2\theta_B)} \) will increase Wehrl’s entropy above its minimum.

### Photon Entropy

Photon is a massless spin 1 particle. Its group representation is ISO(2) or E(2), with the gauge transformation accounting for the transverse direction. Then, this results in a particle with helicity \( \pm 1 \) and a two dimensional polarization field in a plane propagating along, or against, the helicity direction. Formally one creates two circular polarized states, a right-hand side, \( |h_+\rangle \), and a left-hand side, \( |h_-\rangle \), which are the quantum states of the polarization field. Given an \( x, y, z \) coordinate system, we write \( |h_+\rangle = \frac{1}{\sqrt{2}} (1, i)^T \) and \( |h_-\rangle = \frac{1}{\sqrt{2}} (1, -i)^T \) in the basis \( |x\rangle = (1, 0)^T, \ |y\rangle = (0, 1)^T \) in the plane perpendicular to \( z \). The photon-spin operator along \( z \) is then
\[ S_z = \hbar \left( |h_+\rangle \langle h_+| - |h_-\rangle \langle h_-| \right) = \hbar \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \]

with the two eigenstates, \( |h_+\rangle, |h_-\rangle \) and eigenvalues \( \pm 1 \), which are the helicity values. The general state of the photon is then \( |\psi\rangle = \alpha_+ |h_+\rangle + \alpha_- |h_-\rangle \) where \( 1 = |\alpha_+|^2 + |\alpha_-|^2 \). The DOFs of the photon spin are captured by these two coefficients, which produce the probabilities \( |\alpha_+|^2, |\alpha_-|^2 \) for the photon to be in each of the helicities eigenvectors.
ENTANGLEMENT

We study entanglement of two fermions and three fermions (triplets) with $s = 1/2$ for each fermion.

Two Fermions

Consider a system with two fermions with spin $s = 1/2$, say $A$ and $B$. We focus on their entanglement. Consider the spin basis, the product of individual spin eigenstates along the $z$ axis,

$$|++angle = |+angle^A |+angle^B, \quad |+-angle = |+angle^A |-angle^B,$$

$$|->angle = |-angle^A |+angle^B, \quad |--angle = |-angle^A |-angle^B .$$

**Theorem 2** (Two spin $s = 1/2$ entanglement). Consider a system with two identical particles, each with $s = 1/2$, in a subspace of spin $0$ along $z$. The Ising-like spin-spin interaction is described by a Hamiltonian $H = \begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix}$. A spinless environment state $|e\rangle$ with energy $\hbar \gamma$ produces an orthogonal basis of the system $|+-\rangle \otimes |e\rangle$ and $|--\rangle \otimes |e\rangle$. Assume that due to the presence of the environment the two system states interact via a Hamiltonian $H^1 = \hbar \begin{pmatrix} 0 & -i \omega \\ i \omega & 0 \end{pmatrix}$ with eigenvalues $\pm \omega$. Let the initial state be entangled $|\Psi^+, e\rangle = \frac{1}{\sqrt{2}}(|+-\rangle + |--\rangle) \otimes |e\rangle$. Then, the spin-entropy of this state $|\Psi^+(t), e(t)\rangle = U(t) |\Psi^+, e\rangle = e^{-i\frac{(E+\hbar \gamma) t}{\hbar} H^1} |\Psi^+, e\rangle$ is

$$S(t) = -\sin^2(\omega t) \ln \sin^2(\omega t) - \cos^2(\omega t) \ln \cos^2(\omega t)$$

$$- \int_0^{2\pi} \frac{1 + \sin(2\omega t) \cos \phi}{2\pi} \ln \frac{1 + \sin(2\omega t) \cos \phi}{2\pi} d\phi,$$

which is invariant under $E$ and $\gamma$, and reaches its maximum at time $T = \frac{\pi}{4 \omega}$, when
the state is disentangled with probability 1.

Proof. The Hamiltonian can be diagonalized as

\[ H + \hbar \gamma 1 + H^I = \frac{\hbar}{2} \begin{pmatrix} 1 - i & \frac{E}{\hbar} + \gamma + \omega & 0 \\ i - 1 & 0 & \frac{E}{\hbar} + \gamma - \omega \\ i - 1 & \frac{E}{\hbar} + \gamma + \omega & 1 - i \end{pmatrix}. \]

Thus, the unitary evolution operator \( U(t) \) is

\[ U(t) = e^{-i(\frac{E}{\hbar} + \gamma)t} \begin{pmatrix} \cos \omega t & \sin \omega t \\ -\sin \omega t & \cos \omega t \end{pmatrix}. \]

To evaluate the spin-entropy, we examine the spin phase space for the two fermions case. Spin matrices associated with the two fermions can be written in terms of Pauli matrices \( \sigma_x, \sigma_y, \sigma_z \) of the individual fermions as

\[ S_{x,y,z} = \frac{1}{2} \sigma_{x,y,z} \otimes I + I \otimes \frac{1}{2} \sigma_{x,y,z} \quad \text{and} \quad S^2 = S_x^2 + S_y^2 + S_z^2, \]

written in the basis of products of single particle eigenstates along \( z \), namely \( |++\rangle, |+-\rangle, |-+\rangle, |--\rangle \). Some of these vectors however are not eigenstates of \( S^2 \). A common eigenbasis to \( S^2 \) and \( S_z \), written in terms of the product of single particle eigentstates, is given by \{ \( |++\rangle, \frac{1}{\sqrt{2}}(|+-\rangle + |--\rangle), \frac{1}{\sqrt{2}}(|+-\rangle - |--\rangle), |--\rangle \} \).

In particular, we are exploring the subspace with spin 0 along the \( z \) axis (\( m = 0 \)), i.e., the subspace generated by the two Bell states

\[ |\Psi^+\rangle = |\xi_{2,1,0}\rangle = \frac{1}{\sqrt{2}}(|++\rangle + |--\rangle), \quad |\Psi^-\rangle = |\xi_{2,0,0}\rangle = \frac{1}{\sqrt{2}}(|+-\rangle - |--\rangle), \]

which are entangled states and eigenstates of \( S_z \) with \( m = 0 \). The basis transformation from \( |+\rangle\langle -|, |+\rangle\langle -| \) to \( |\Psi^+\rangle, |\Psi^-\rangle \) is \( B = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \) and so, in the basis
$|\Psi^+, e\rangle, |\Psi^-, e\rangle$ the unitary evolution operator is

$$U_z(t) = BU(t)B^{-1} = e^{-i(\frac{\xi}{\pi} + \gamma)t} \begin{pmatrix} \cos \omega t & -\sin \omega t \\ \sin \omega t & \cos \omega t \end{pmatrix},$$

where the subscript in $U_z$ indicates that this representation is associated with eigenstates of the spin operators $S_z, S^2$. Thus, $|\Psi^+(t), e(t)\rangle = U_z(t)|\Psi^+, e\rangle = e^{-i(\frac{\xi}{\pi} + \gamma)t} \begin{pmatrix} \cos \omega, & \sin \omega \end{pmatrix}^T$ with probabilities $P(t) = \begin{pmatrix} \cos^2 \omega t, & \sin^2 \omega t \end{pmatrix}^T$. We now examine the entropy term of the state $|\Psi^+(t), e(t)\rangle$ in the conjugate basis $|\phi, e\rangle$.

From (2)

$$|\Psi^+(z, t), e(t)\rangle = \frac{e^{-i(\frac{\xi}{\pi} + \gamma)t}}{\sqrt{2\pi}} \int \left( e^{i\phi \cos \omega t + \sin \omega t} \right) |\phi, e\rangle \, d\phi,$$

implying that $\rho_{z^+}(\phi, t) = \frac{1}{2\pi}(1 + \sin(2\omega t) \cos \phi)$ and the entropy (12) can be written as

$$S_{2,1/2}(t) = S_{2,1/2}^z(t) + S_{2,1/2}^{z^-}(t) = -\cos^2 \omega t \ln \cos^2 \omega t - \sin^2 \omega t \ln \sin^2 \omega t$$

$$- \int \frac{1}{2\pi} (1 + \sin(2\omega t) \cos \phi) \ln \frac{1}{2\pi} (1 + \sin(2\omega t) \cos \phi) \, d\phi,$$

and readily evaluated. There is an oscillation of period $\pi/2\omega$. The entropy is maximized at $T = \pi/4\omega$, where $|\Psi^+(T), e(T)\rangle = e^{-i(\frac{\xi}{\pi} + \gamma)\frac{\pi}{2\omega}} \begin{pmatrix} 1, & 1 \end{pmatrix}^T = e^{-i(\frac{\xi}{\pi} + \gamma)\frac{\pi}{2\omega}} |+\!, -e\rangle$, when with probability 1, the spin state is in the disentangled state $|+\!, -e\rangle$.

For a visualization of this entropy $S_{2,1/2}(t)$ and of its two components, $S_{2,1/2}^z(t)$ and $S_{2,1/2}^{z^-}(t)$, see Figure 3a.

We point out that the environment was only modeled as one state, i.e., we did not consider the pioneer work [12, 23] of decoherence. It would be interesting in future research to study how the proposed spin-entropy for mixed states can impact...
Figure 3. a. Spin-entropy \((20)\) and its components \(S^z(t), S^z_{\perp}(t)\). b. Spin-entropy vs von Neumann entropy. The magnitudes are significantly different, and their maxima and minima occur in opposition, at values of \(\omega t\) that are multiples of \(\pi/4\).

the environment on quantum states.

**Comparison with Von Neuman Entropy and Werhl Entropy**

Work exists exploring the information content of entangled physical systems considering von Neumann entropy, e.g., \([2, 5, 18, 21]\). In that approach the required choice of basis functions is the product of one-particle eigenstates \(|+\rangle \langle -|, |+\rangle \langle -\rangle\), so that one can trace out over one-particle spin eigenstates. Thus, in this basis, the initial state is represented as \(|\Psi^+, e\rangle = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})\) and its evolution in this basis is described by

\[
|\Psi^+(t), e(t)\rangle = U(t)|\Psi^+, e\rangle = e^{-i(\frac{\pi}{4} + \Gamma)t} \begin{pmatrix} \sin (\omega t + \frac{\pi}{4}) \\ \cos (\omega t + \frac{\pi}{4}) \end{pmatrix}.
\]

After tracing out any of the two particles states, the probabilities of the resulting mixed state are \(P_+(t) = \sin^2 (\omega t + \frac{\pi}{4})\) and \(P_-(t) = \cos^2 (\omega t + \frac{\pi}{4})\) with von Neumann entropy

\[
S^{vN}(t) = -\cos^2 \left(\omega t + \frac{\pi}{4}\right) \log \cos^2 \left(\omega t + \frac{\pi}{4}\right) - \sin^2 \left(\omega t + \frac{\pi}{4}\right) \log \sin^2 \left(\omega t + \frac{\pi}{4}\right),
\]

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which is also plotted in Figure 3b for a visual comparison with the spin-entropy of the original pure state.

Wehrl’s entropy for such mixed state was evaluated by [15] to conclude that, like von Neumann entropy, disentangled states minimize Wehrl entropy.

Disentanglement occurs at time steps \( t = \frac{\pi}{4\omega} + n\frac{\pi}{2\omega}; \ n = 0, 1, \ldots \). The spin-entropy is maximized at these times, while the tracing out procedure leads to the minimum of the von Neumann entropy and the Wehrl’s entropy. The 0 value for the von Neumann entropy is to be expected since this mixed state becomes a pure state at disentanglement. In contrast, the Bell states (the eigenstates of \( S_z, S^2 \) for \( m = 0 \)) minimize the spin-entropy and maximize the von Neumann entropy. These min/max entropy differences are, technically, due to the spin basis. For the spin-entropy, the spin basis needs to be the eigenfunctions of the two-particles spin operators \( S_z, S^2 \), while for the von Neumann entropy, one must trace out the eigenstates of the individual particle eigenstates. The large entropy magnitude difference everywhere is due to the spin-entropy’s contribution from the \( \phi \) variable, the conjugate variable to \( z = \cos \theta \).

**Triplets**

Given a triplet set of spin 1/2 particles, the total-spin matrices \( S_{x,y,z} \) in the basis of products of single particle eigenstates are

\[
S_{x,y,z} = \sigma_{x,y,z} \otimes I \otimes I + I \otimes \sigma_{x,y,z} \otimes I + I \otimes I \otimes \sigma_{x,y,z}
\]
where the Pauli matrices are written in the single particle $z$ axis eigenvectors basis. 

A common basis for both, $S_z$ and $S^2$ is given by the vectors \[8\]

\[
\begin{align*}
|\xi_{1,2}^{1,2}\rangle &= |+++angle, \\
|\xi_{1,2}^{1,1}\rangle &= \frac{1}{\sqrt{3}} (|+-+-|+--+|+--||-++), \\
|\xi_{1,2}^{1,-1}\rangle &= \frac{1}{\sqrt{3}} (|--+|++-|--+|-++), \\
|\xi_{1,2}^{1,-2}\rangle &= |--\rangle, \\
|\xi_{1,2}^{1,1}(\theta+)\rangle &= \frac{\cos \theta_+}{\sqrt{2}} |+\rangle \otimes (|+-\rangle - |--\rangle) + \frac{\sin \theta_+}{\sqrt{2}} (|+-\rangle - |--\rangle) \otimes |+\rangle, \\
|\xi_{1,2}^{1,1}(\theta-)\rangle &= \frac{\cos \theta_-}{\sqrt{2}} |--\rangle \otimes (|+-\rangle - |--\rangle) + \frac{\sin \theta_-}{\sqrt{2}} (|+-\rangle - |--\rangle) \otimes |--\rangle (21)
\end{align*}
\]

where the parameters $\theta_+, \theta_- \in [0, 2\pi)$ characterize the degeneracy of the subspace of spin magnitude $s = 1/2$, a four dimensional subspace with only two eigenvalues.

**Maximally Entangled States**

According to our spin-entropy, the minimum $\ln 2\pi$ is associated with the eigenstates \[21\], i.e., entangled states $|\xi_{1,2}^{1,2}\rangle, |\xi_{1,2}^{1,1}\rangle$ and entangled subspaces $|\xi_{1,2}^{1,1}(\theta+)\rangle, |\xi_{1,2}^{1,1}(\theta-)\rangle$. We propose that the minimum entropy be the criterion to define *maximally entangled states* among all the entangled states. Thus entangled state $W [7]$, $|\psi_W\rangle = \frac{1}{\sqrt{3}} (|+-\rangle + |--\rangle + |--\rangle)$, is a *maximally entangled state* since it is the eigenstate $|\xi_{1,2}^{1,2}\rangle$ and thus has lowest entropy.

In contrast, consider the entangled state GHZ [10], $|\psi_{GHZ}\rangle = \frac{1}{\sqrt{2}} (|++\rangle + |--\rangle)$. Clearly, it is not an eigenstate of either $S^2$ or $S_z$. The entropy will be larger than for all entangled eigenstates of $S^2, S_z$ and thus it is not a *maximally entangled state*.

Ranking entangled states according to the spin-entropy provides an information content evaluation that may be helpful when devising quantum physical processes.
The lower the entropy, the further away from decoherence.

CONCLUSIONS

The concept of spin-entropy in a spin phase space is proposed. The spin phase space of a particle is defined via the already existing Geometric Quantization method that quantize a sphere surface. The operators associated with the spherical polar representation do not commute, yielding the uncertainty principle for the spin values in phase space. The states in spin phase space are the simultaneous projections of a spin state onto the $z$ axis eigenstates and onto the polarization angle states, which generates the plane perpendicular to the $z$ axis. The proposed spin-entropy captures the randomness present in the spin state for a specified $z$ axis. The $z$ axis for a system of fermions is defined as the direction of the magnetic field present in the system, and it can vary over time. In the case of a photon, the $z$ axis is the direction of the propagation, where the helicity is defined. The formulation is general for a system of many spin particles and extends to quantum fields. We studied spin-entropy for single particles with spin $1/2$, spin 1, photons, and for two and three entangled fermions of spin $1/2$.

We have examined entangled states and their time evolution. Bell’s entangled states that are eigenstates of the spin $S_z, S^2$ operators with total zero spin value along the $z$ axis have lower spin-entropy than that of the product of one particle states (disentangled states). In contrast, the von Neumann entropy and Wehrl entropy are maximized at Bell’s entangled states and minimized at disentangled states. We studied the dynamics of entangled states with an Ising-like Hamiltonian model of the interaction between spins, and a model of the environment as one state and its impact on a Hamiltonian mediating the interaction between Bell’s entangled states. In a simulation, the time evolution of such Bell entangled states the entropy increases up to its maximum value, and disentanglement then occurs.
We then analyzed some quantum states of three fermions of spin $\frac{1}{2}$ and suggested that maximum entangled states should be defined by the entropy value. The lower the entropy of an entangled state, the larger the entanglement.

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