Crossover from the parity-conserving pair contact process with diffusion to other universality classes

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The pair contact process with diffusion (PCPD) with modulo 2 conservation (PCPD2) \([2A \rightarrow 4A, \ 2A \rightarrow 0]\) is studied in one dimension, focused on the crossover to other well established universality classes: the directed Ising (DI) and the directed percolation (DP). First, we show that the PCPD2 shares the critical behaviors with the PCPD, both with and without directional bias. Second, the crossover from the PCPD2 to the DI is studied by including a parity-conserving single-particle process \((A \rightarrow 3A)\). We find the crossover exponent \(1/\phi_1 = 0.57(3)\), which is argued to be identical to that of the PCPD-to-DP crossover by adding \(A \rightarrow 2A\). This suggests that the PCPD universality class has a well defined fixed point distinct from the DP. Third, we study the crossover from a hybrid-type reaction-diffusion process belonging to the DP \([3A \rightarrow 5A, \ 2A \rightarrow 0]\) to the DI by adding \(A \rightarrow 3A\). We find \(1/\phi_2 = 0.73(4)\) for the DP-to-DI crossover. The inequality of \(\phi_1\) and \(\phi_2\) further supports the non-DP nature of the PCPD scaling. Finally, we introduce a symmetry-breaking field in the dual spin language to study the crossover from the PCPD2 to the DP. We find \(1/\phi_3 = 1.23(10)\), which is associated with a new independent route from the PCPD to the DP.

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I. INTRODUCTION

The pair contact process with diffusion (PCPD) is an interacting particle system with diffusion, pair annihilation \((2A \rightarrow 0)\), and creation of offspring by pairs \((2A \rightarrow 3A)\) \([1, 2, 3]\). It is well known that the PCPD scaling is definitely distinct from the directed percolation (DP) scaling in two dimensions and higher. However, for the one dimensional PCPD, there is still no consensus on its critical scaling with two different viewpoints such as the DP scaling after an extremely long crossover time \([4, 5, 6]\) and the new scaling distinct from the DP \([7, 8]\). Since the critical exponents of the PCPD are so similar to those of the DP in one dimension and the strong corrections to scaling are present, it would be difficult, if not impossible, to settle down the controversy via direct numerical simulations of the PCPD. On this account, the authors suggested two critical tests in the past. One is the driven PCPD (DPCPD) \([9, 10]\) and the other is the crossover \([7, 8]\). A recent brief review on these two approaches is available in Ref. \([2]\). In this paper, we continue to investigate the crossover behavior of the PCPD to other universality classes, in order to understand further the difference between the PCPD and the DP.

It is well established that the critical behavior of the PCPD is not affected by introducing modulo 2 (parity) conservation of the total particle number \([12, 14]\). That is, even if a pair branches two offspring \((2A \rightarrow 4A)\) rather than one, this model is believed to share the critical behavior with the PCPD and does not belong to the directed Ising (DI) \([15, 16, 17]\) or the parity-conserving (PC) class \([18, 19]\). In the context of the crossover, however, the parity-conserving PCPD provides a useful platform to study rich crossover behaviors compared to the non-conserving PCPD. Because of the special role of the parity conservation, we will refer to the parity-conserving PCPD as PCPD2 in this paper. The term PCPD will be preserved to refer to the universality class in what follows.

We introduce a single-particle branching process in the PCPD2 to study the crossover behavior from the PCPD to other universality classes. Depending on the number of offspring in the single-particle process, one can study the crossover either to the DP (by \(A \rightarrow 2A\)) as in Ref. \([8]\), or to the DI (see Sec. IV A). Since the PCPD is clearly independent from the DI class, we expect a non-trivial crossover scaling from the PCPD to the DI. However, this crossover scaling may not be independent from the PCPD-to-DP crossover scaling, if the PCPD universality class can be described by a well-defined fixed point and the parity conservation is irrelevant to the PCPD fixed point.

As discussed in our previous work \([20]\) and also in Sec. IV A, the crossover scaling depends only on the initial fixed point and its crossover operator, but not directly on the terminal fixed point. The initial PCPD fixed point is blind to the parity conservation \([12, 14]\). Therefore, in the PCPD point of view, the crossover operator associated with \(A \rightarrow 2A\) and \(A \rightarrow 3A\) may be undistinguishable, and thus have the same scaling dimension. Hence, we expect the same crossover exponent for the PCPD-to-DP and the PCPD-to-DI crossover scaling, which is numerically confirmed in Sec. IV A. The terminal fixed point (DP or DI), of course, depends crucially on the existence of the parity conservation in the reaction
dynamics.

More complex reaction-diffusion models with hybrid-type dynamics may be considered such that \( mA \to (m+k)A \) and \( nA \to (n-l)A \). These models are numerically shown \([12]\) and reasonably argued [see Sec. V in Ref. \([3]\)] to belong to the DP in one dimension when \( m > n \) and \( k, l > 0 \), in contrast to the cases of \( m = n \) (PCPD at \( m = n = 2 \) and TCPD at \( m = n = 3 \) \([12, 21]\)). The crossover from the diffusing \( \text{tp2} \ (3A \to 4A, 2A \to 0) \) to the DP by adding a single-particle reaction (\( A \to 0 \)) has been studied previously \([7]\) to find the typical “crossover” behavior between the identical (DP) universality classes as expected: a linear phase boundary and no diverging time scale. In this paper, we study the crossover from the diffusing \( \text{tp2} \ (3A \to 5A, 2A \to 0) \) with the parity conservation to the DI by adding \( A \to 3A \). A nontrivial crossover exponent for this DP-to-DI crossover is numerically estimated, which is different from the PCDP-to-DI crossover exponent. This finding again supports the distinct nature of the PCDP scaling from the DP.

When \( A \to 2A \) is added to \( \text{tp2} \), we again observe the trivial crossover between the identical (DP) classes. One may question how the \( \text{tp2} \) can distinguish the crossover operators associated with \( A \to 2A \) and \( A \to 3A \), in contrast to the PCDP. The answer is simply that the parity conservation at the level of single-particle reactions is relevant to the DP fixed point, but not to the PCDP fixed point. This again further supports the difference between the DP and the PCDP fixed point even in one dimension.

The parity conservation in one dimension makes it possible to map the PCDP2 to the nonequilibrium Ising spin dynamics with the \( Z_2 \) symmetry \([15, 17, 20]\). Although this mapping can be considered nominal due to the irrelevance of the parity conservation in the PCDP, one can still ask a question regarding the effect of the symmetry breaking field in the dual spin language. In terms of the original particle (kink) language, it implies an alternating directional bias in diffusion (or branching) of particles, which makes a pair of neighboring particles tightly bound. Then we can easily expect a simple DP-type dynamics of bound pairs (\( X \to 2X, X \to 0 \)). Thus, with the symmetry breaking field, we may find a new independent route from the PCDP to the DP, which is characterized by a new crossover exponent. (Similar independent routes are discussed for the DI-to-DP crossover in Ref. \([20]\)). This is also confirmed by numerical simulations.

This paper is organized as follows: Section \( \text{II} \) introduces the dynamic rules of the PCDP2 and shows that the PCDP2 shares the critical behavior with its cousin without conservation (PCPD). By introducing the relative bias between isolated particles and pairs, the PCDP2 exhibits the same critical scaling as in the DPCPD. In Sec. \( \text{III} \) we introduce the three crossover models by adding a single-particle reaction process to the PCDP2 and to \( \text{tp2} \), and by adding an alternating directional bias to the PCDP2, respectively. In Sec. \( \text{IV} \) the extensive numerical results are presented for these three models. Finally, the conclusion and the summary of the results follow in Sec. \( \text{V} \).

\section{PCPD2}

This section begins with the description of the PCDP2 dynamics adopted for simulations, and then reconfirms the PCDP behavior of the PCDP2. The dynamic rules for the PCDP2 are as follows:

\[
\begin{align*}
&A \emptyset \xrightarrow{D} \emptyset A, \quad \emptyset A \xrightarrow{D} A \emptyset, \\
&A A \emptyset \xrightarrow{Z_2} A A A, & \emptyset \emptyset A \xrightarrow{Z_2} A A A,
\end{align*}
\]

where \( A (\emptyset) \) stands for an occupied (vacant) site and the parameters on the top of arrows represent the transition rates. For convenience, we set \( \sigma_R + \sigma_L = 1 - p \) with \( 0 \leq p \leq 1 \). Except the biased branching model (see below), \( \sigma_R = \sigma_L \) is set in most cases. Each site can carry at most one particle (hard core exclusion).

The PCDP2 has been shown numerically to belong to the PCDP class \([12, 14]\). This may be understood by viewing the PCDP-type models as a coupled system of two particle species, where a particle pair corresponds to an \( X \) particle and a solitary particle to a \( Y \) particle \([11, 22, 23]\). Then, the PCDP dynamics is the combination of the DP-like dynamics of \( X \)'s and the binary annihilation diffusing dynamics of \( Y \)'s. The two species are coupled through transmutations. The only difference in the PCDP2 dynamics from the PCDP is the number of offspring in branching of \( X \) particles such as \( X \to 2X \) in the PCDP and \( X \to 3X \) in the PCDP2, which should be irrelevant in the DP dynamics of \( X \)'s with no parity conservation by \( X \to 0 \). Therefore, it is not surprising to find the PCDP2 in the PCDP class.

In this paper, we numerically study the PCDP2 with directional bias and check whether it shares the same critical behavior with the driven PCPD (DPCPD) \([12, 22]\). This may serve as another evidence supporting that the parity conservation in the PCDP2 is irrelevant. Being relevant the relative bias between isolated particles and pairs, the biased branching as well as the biased diffusion should entail the change of the universality class \([9]\). Since the data with the biased branching have not been reported in the literature, we consider here the biased branching for the PCDP2 by setting \( \sigma_L = 0 \) and \( \sigma_R = 1 - p \) in Eq. \( (1) \). The diffusion constant is set as \( D = \frac{1}{2} \).

Figure \( \text{I} \) summarizes the simulation results using the system size \( 2^{23} \). For each curve, 80 independent samples are collected. Up to the observation time, every sample has pairs, which minimally guarantees that the system does not feel the finite size effect. During simulations, we measured the particle density \( \rho(t) \), and the pair density \( \rho_p(t) \). Figure \( \text{I} \) shows how the density decays near criticality for the PCDP2 with the biased branching (\( \rho \sim t^{-\delta} \) with \( \delta = 1/2 \) in the long-time limit at criticality) and its inset shows the logarithmic corrections to the power law
scaling. The observed critical behavior is the same as that of the DPCPD \cite{9}, which reconfirms that the parity conservation is irrelevant in the PCPD.

Now we move to the unbiased PCPD2 with $\sigma_R = \sigma_L = (1-p)/2$ in Eq. $1$. In order to study the crossover behavior near the PCPD fixed point, it is necessary to measure the critical point of the PCPD2 as accurate as possible. To this end, we use the effective exponent of the density decay defined as

$$-\delta_{\text{eff}}(t) \equiv \frac{\ln(\rho(t)) - \ln(\rho(t/m))}{\ln m},$$  

with $m = 10$, which will drift to zero (−1/2) in the active (absorbing) phase with time and at criticality will saturate to the critical decay exponent $-\delta$ of the PCPD between 0 and −1/2. To locate the critical point, we make two assumptions. First, if $-\delta_{\text{eff}}(t)$ veers down substantially, we took this as the signal of the absorbing phase. On the other hand, if the effective exponent veers up and passes the DP value $\delta_{\text{DP}} \approx 0.1595$, we will conclude that the system is in the active phase. This must be true if $\delta$ for the PCPD is not smaller than $\delta_{\text{DP}}$ asymptotically, which has been confirmed in all of previous studies for the PCPD.

We simulated the PCPD2 at $p = 0.180 21$ and $p = 0.180 22$ with system size $5^2$ up to $t = 5 \times 10^8$. The diffusion constant is set as $D = \frac{1}{2}$. The numbers of independent samples are 120 and 30 for $p = 0.180 21$ and $0.180 22$, respectively. In these simulations, all samples survive up to the observation time in the sense that there is at least one pair. The resulting effective exponents are depicted in Fig. 2. Clearly, the PCPD2 with the annihilation probability $p = 0.180 22$ falls into the absorbing phase. At $p = 0.180 21$, the effective exponent drifts and passes well above the DP value. Hence the critical point should be located between these two values and we conclude $p_c = 0.180 215(5)$ for the PCPD2. Note that our estimate for $p_c$ is valid, regardless of the true asymptotic exponent value $\delta$ for the PCPD, unless $\delta$ is much smaller than $\delta_{\text{DP}}$.

III. CROSSOVER MODELS

In this section, we introduce three models, each of which describes the PCPD-to-DI, the DP-to-DI, and the PCPD-to-DP crossover, respectively.

A. PCPD-to-DI

First, introducing the single-particle branching dynamics which preserves the parity ($A \rightarrow 3A$) to the PCPD2, the crossover from the PCPD to the DI is studied. With this single-particle dynamics, it is clear that the branching processes by pairs ($2A \rightarrow 4A$) become irrelevant (a higher-order process) and thus the system must share the critical behavior with the branching annihilating random walks with two offspring (BAW2)\cite{23}, which belongs to the DI class in one dimension. We will refer to this model as the PCPD-to-DI crossover model, or simply PCPD-DI. To be specific, together with the dynamics in Eq. $1$, the single-particle branching process is included in a static...
The Monte Carlo simulation begins with a random selection of a particle among $N_t$ particles at time $t$. Let us denote the site on which the chosen particle resides by $n$. With probability $\frac{1}{2}$, we choose one of the directions (left or right) which will be denoted by $e \in \{1, -1\}$. If the $(n + e)$-th site is vacant, the particle at site $n$ hops to that site with probability $1 - w$. With probability $w$, two offspring are newly placed at two nearest neighbors of the site $n$, only if the $(n - e)$-th site is also vacant. On the other hand, if the $(n + e)$-th site is already occupied, both particles are removed from the system with probability $p$ or with probability $1 - p$, we choose two sites in a row, that is, either $(n + 2e, n + 3e)$ or $(n - e, n - 2e)$ with equal probability and branch two particles there once both sites are empty. If any of the conditions stated above is not satisfied, nothing happens. After the above attempt, time increases by $1/N_t$.

\[ \emptyset A \emptyset \overset{w}{\rightarrow} \text{AAA}. \]  

(3)

For a technical reason, the diffusion constant is now set as $D = (1 - w)/2$ with $0 \leq w \leq 1$.

Below is the algorithm how we simulate the PCPD-DI. The Monte Carlo simulation begins with a random selection of a particle among $N_t$ particles at time $t$. Let us denote the site on which the chosen particle resides by $n$. With probability $\frac{1}{2}$, we choose one of the directions (left or right) which will be denoted by $e \in \{1, -1\}$. If the $(n + e)$-th site is vacant, the particle at site $n$ hops to that site with probability $1 - w$. With probability $w$, two offspring are newly placed at two nearest neighbors of the site $n$, only if the $(n - e)$-th site is also vacant. On the other hand, if the $(n + e)$-th site is already occupied, both particles are removed from the system with probability $p$ or with probability $1 - p$, we choose two sites in a row, that is, either $(n + 2e, n + 3e)$ or $(n - e, n - 2e)$ with equal probability and branch two particles there once both sites are empty. If any of the conditions stated above is not satisfied, nothing happens. After the above attempt, time increases by $1/N_t$.

B. DP-to-DI

For comparison, we introduce a crossover model from the DP to the DI to check the difference of the PCPD and the DP in the crossover behavior to the DI via the same operation ($A \rightarrow 3A$). To this end, we need a DP-type model with the parity conservation. One example is the diffusing $\mathfrak{Tp}22$ with $3A \rightarrow 5A$ and $2A \rightarrow 0$. Including $A \rightarrow 3A$, the process $3A \rightarrow 5A$ becomes irrelevant as in the PCPD and the model belongs to the DI class. We refer to this model as the DP-DI. The algorithm is identical to the PCPD-DI except the triplet branching process: If the $(n + e)$-th site is occupied, then with probability $1 - p$ we look up the nearest-neighbor site of the pair in the right (site $n + (3 + e)/2$). If this site is occupied, two particles are generated at nearest neighbors of this triplet, in case both nearest neighbor sites are vacant.

C. PCPD-to-DP via a pair-binding route

Finally, we consider an alternating diffusion bias in the one-dimensional PCPD2. All particles diffuse with bias in one direction or the other, and neighboring particles have opposite directional bias. Neighboring particles may be separated by an arbitrary stretch of empty sites. If a particle hops to the right with rate $(1 + w)/2$ and to the left with $(1 - w)/2$, then the hopping rates for neighboring particles are just reversed. As the particles are created or annihilated pairwise locally in Eq. (1) and the hardcore repulsion prevents the crossing of particles, each particle can preserve the bias direction and the alternating diffusion bias in the system is still intact during the dynamics. Therefore, one may identify particles with different bias as different species of particles. For example, one may name a particle with diffusion bias to the right as an $A$ particle and a particle with diffusion bias to the left as a $B$ particle. Then the $AB$ ordering is preserved if one starts with a configuration with the $AB$ ordering under the dynamics of $AB \rightarrow \emptyset$, $AB\emptyset \rightarrow ABAB$, $BA \rightarrow \emptyset$, and $BA\emptyset \rightarrow BABA$ mimicking the PCPD2 dynamics. Without bias ($w = 0$), two species are indistinguishable and we recover the PCPD2.

At $w \neq 0$, this model belongs to the DP class. First, consider the extreme case ($w = 1$), where $A$ ($B$) always jumps to the right (left). After a transient time, all particles form $AB$ bound pairs which cannot split once bound. Afterwards, any diffusion is impossible and the dynamics becomes identical to the contact process (CP) of $X \rightarrow 0$ and $X \rightarrow 2X$ with $X \equiv AB$. For finite $w$, the biased diffusion will dominate in large scales and all particles form a little bit loose but still bound $AB$ pairs. Thus we expect the same critical behavior as in the case $w = 1$.

Since the CP belongs to the DP class, so does the model with $w \neq 0$. We emphasize that the fundamental difference between the PCPD and the DP lies in this pair binding property. The crossover from the PCPD to the DP via this pair-binding route can be studied near $w \simeq 0$.

We may map the PCPD2 onto the ferromagnetic Ising spin model by interpreting a particle as a domain wall of the Ising spins $15, 17, 20$. Then an $A$ and $B$ particle represent a $\uparrow \downarrow$ and $\downarrow \uparrow$ domain wall, respectively. The alternating bias in diffusion can be interpreted as a symmetry-breaking (SB) field favoring $\uparrow \downarrow$ for $w > 0$ ($w < 0$). So we may call the PCPD2 with finite $w$ as the symmetry-broken PCPD2 (SBPCPD). The crossover scaling property by the symmetry-breaking (or pair-binding) route may be different from the conventional one observed in Ref. [8], similar to various independent routes found in the DI-to-DP crossover [21]. It is interesting to note that the conventional crossover is also found by a single-particle reaction process ($A \rightarrow 2A$) breaking the parity conservation in the PCPD2.

We remark that the alternating bias (symmetry-breaking field) in the BAW2 also makes two neighboring particles form a bound pair, which triggers the crossover from the DI to the DP. Therefore the pair-binding process is the common underlying mechanism for the SB routes of the DI-to-DP and the PCPD-to-DP crossover. However, the corresponding crossover scaling exponents are different, which depend on the scaling property of the initial fixed points.

D. crossover scaling

The crossover can be described by the scaling function $\rho(t; w; \Delta) = t^{-\delta} F(\Delta w t, w^{\mu} t)$.

(4)
where $\Delta = p - p_c(0)$ with $p_c(0)$ being the critical point at $w = 0$ and $\mu_{\parallel} = \nu_{\parallel}/\phi$ with the crossover exponent $\phi$. The exponents $\delta$ and $\nu_{\parallel}$ are the density decay and the relaxation time exponent, respectively, at the initial fixed point ($w = 0$).

The crossover scaling function in Eq. (4) implies that the crossover time scale diverges as $w \to 0$ such that $\tau_{\text{cross}} \sim w^{-\delta}$ and the critical amplitude for the density decay at finite $w$’s should scale as $\mu_{\parallel}^w \sim w^\chi$ with $\chi = \mu_{\parallel}(\delta - \delta_{T})$ where $\delta_{T}$ is the density decay exponent at the terminal fixed point ($w \neq 0$). The crossover exponent $\phi$ can be most accurately measured from studying how the phase boundary $p_c(w)$ approaches the critical point $p_c(0)$ as $w \to 0$ such that $\Delta_c(w) \sim w^{1/\phi}$ with $\Delta_c(w) = p_c(w) - p_c(0)$.

### IV. NUMERICAL RESULTS

#### A. PCPD-DI

This section studies the PCPD-DI focusing on the crossover exponent $\phi$. The critical point for the PCPD-DI with nonzero $w$ can be accurately measured by using the known DI exponent value $\delta_{\text{DI}} \simeq 0.285$. The DI scalings for some $w$’s are shown in Fig. 2 and our findings about the critical points are summarized in Table II. The crossover time scale seems to diverge as expected in the $w \to 0$ limit, which is consistent with the scaling theory prediction of Eq. (4). However, the amplitude of the critical decay ($\lim_{t \to \infty} \rho_{\parallel}^{w\phi}$) does not seem to vary much with $w$. Similar observation was reported in Ref. [8] for the crossover from the PCPD to the DP; see Fig. 2 of Ref. [8]. Of course, this does not mean $\chi = 0$ or $\delta_{\text{PCPD}} = \delta_{\text{DI}}$. It just implies a rather big correction to scaling which results in the narrow crossover scaling regime near the

![FIG. 3: (Color online) Plots of $\rho(t)^{1/\phi}$ vs $t$ of the PCPD-DI for finite $w$’s at criticality in semi-logarithmic scales. The smaller $w$ is, the later the system enters the asymptotic scaling regime. Although $\delta_{\text{DI}} > \delta_{\text{PCPD}}$ is evident, the critical amplitudes do not seem to vary with $w$ up to $w = 10^{-5}$.](image)

![FIG. 4: (Color online) The phase boundary of the PCPD-DI near $w = 0$. The curves are well fitted by the function $p_c = p_c(0) + ax^{1/\phi}$ with $1/\phi = 0.57(3)$ and $a \approx 1.61$. Inset: Plot of $\Delta_c$ vs $w$ in the logarithmic scales. The slope of the straight line is 0.57.](image)

### TABLE I: The critical points of the PCPD-DI for various $w$’s. The numbers in parentheses indicate the error of the last digits.

| $w$    | $p_c(w)$  |
|--------|-----------|
| $0$    | 0.180 215(5) |
| $10^{-5}$ | 0.182 35(5)   |
| $5 \times 10^{-5}$ | 0.185 6(1)    |
| $10^{-4}$ | 0.188 35(5)   |
| $2 \times 10^{-4}$ | 0.192 2(1)    |
| $3 \times 10^{-4}$ | 0.195 4(1)    |
| $4 \times 10^{-4}$ | 0.198 05(5)   |
| $5 \times 10^{-4}$ | 0.200 5(1)    |

PCPD fixed point [8].

Nevertheless, the crossover exponent can be quite accurately estimated from the phase boundary. Using the data for the critical points in Table II we estimate the crossover exponents as $1/\phi = 0.57(3)$ which is summarized in Fig. 4. This value of the crossover exponent is very close to that reported in Ref. [8] which studies the crossover from the PCPD to the DP via adding single-particle reaction processes. We do not think this is a mere coincidence. Rather, we believe that they are identical.

To argue the equivalence, let us first reexamine the meaning of the crossover exponent. Consider two universality classes, say I (initial) and T (terminal), and an operator triggering the crossover from the I to the T fixed points. In case of the PCPD-DI, the $I$ fixed point resides at $(p = p_{c}(0), w = 0)$ and the $T$ fixed point at $(p = p_{c}(w^{*}), w = w^{*})$ with $w^{*} > 0$, usually distant from the $I$ fixed point. There are two relevant operators which make the $I$ fixed point unstable. One is the crossover operator responsible for the renormalization group (RG) flow in the direction of $w$ and eventually into the $T$ un-
versatility class. The other is the usual “thermal” operator responsible for the RG flow in the direction of the $p$ axis.

The crossover exponent describes how the RG flow behaves locally near the I fixed point and is given by the ratio of the scaling dimensions of these two relevant operators. These scaling dimensions should be decided entirely in terms of eigenvalues at the I fixed point \[28\]. Hence the T fixed point has no direct role in determining the crossover exponent. Therefore, whatever the T universality class is, the crossover exponent at the I fixed point may not change if the scaling dimensions of two relevant operators remain the same.

Now let us move on to our problem. We should emphasize that the PCPD is insensitive to the existence of the parity conservation. Therefore, the PCPD fixed point does not feel any intrinsic difference when we add one of the single-particle reaction processes of $A \rightarrow nA$ for any $n$. If the PCPD can be described by a well-defined fixed point, then the scaling dimension of the crossover operator associated with the $A \rightarrow nA$ process will not depend on $n$. Of course, the terminal (T) fixed point is sensitive to the parity conservation and thus belongs to the DP for odd $n$ or to the DI for even $n$. This distinction cannot be achieved by a perturbative RG method (like the crossover exponent study), but by studying the global RG flows.

Hence, we conclude that the crossover exponent should be the same for both the PCPD-to-DP and the PCPD-to-DI crossover via a single-particle reaction route. Our numerical confirmation of this equivalence, in turn, provides another evidence that the PCPD has a well-defined fixed point distinct from the DP.

### B. DP-DI

For comparison, we study the DP-DI model to study the crossover behavior from the DP to the DI mediated by the same operation ($A \rightarrow 3A$). We again confirmed that this hybrid-type multiple reaction model (the DP-DI at $w = 0$) does belong to the DP class (not shown here). As before, the critical points for finite $w$’s are determined, exploiting the DI critical scaling. The results are summarized in Table II and in Fig. 5.

If one stops the analysis at $w = 10^{-4}$, one might conclude that the phase boundary meets the vertical axis with a finite angle, and therefore the crossover exponent is 1. However, our more elaborated study with sufficiently small $w$’s shows that the phase boundary has a singularity with the crossover exponent $1/\phi = 0.73(4)$; see Fig. 5.

Now turn back to the scaling function in Eq. 4. If we fix $\Delta = 0$, the crossover scaling function takes the form

$$\rho(t) t^{\delta_{DP}} = c(w^{\mu_{||}} t)$$

where the numerical value of $\mu_{||} = 1^{DP}/\phi = 1.26(7)$ with $\delta_{DP} \simeq 0.16$ and $\nu^{P} \simeq 1.73$. In Fig. 6, we depicted the scaling collapse for the particle and pair densities for $10^{-6} \leq w \leq 10^{-5}$. All curves are collapsed into a single curve in the asymptotic regime, which means that the scaling function in Eq. 5 describes the behavior near $w = 0$ properly with the crossover exponent obtained before.

Hence we conclude that the crossover exponent for the PCPD to the DI is $1/\phi = 0.73(4)$ or $\phi = 1.37(8)$ which is different from that obtained from the PCPD-DI.

### C. SBPCPD

Finally, we study the SBPCPD where the alternating diffusion bias is included in the PCD2. We expect the crossover from the PCDP to the DP via a pair-binding route, which should be independent from that via a single-particle reaction route.

The strategy to find the critical points for finite $w$’s is the same as before, i.e., exploiting the power law decay of

| $w$       | $p_c$ of the DP-DI |
|-----------|-------------------|
| 0         | 0.029 376(1)      |
| $5 \times 10^{-6}$ | 0.029 530(5) |
| $10^{-5}$ | 0.029 620(5)      |
| $2 \times 10^{-5}$ | 0.029 775(5) |
| $3 \times 10^{-5}$ | 0.029 920(5) |
| $5 \times 10^{-5}$ | 0.030 185(5) |
| $10^{-4}$ | 0.030 85(5)       |
| $2 \times 10^{-4}$ | 0.032 1(1) |
| $4 \times 10^{-4}$ | 0.034 8(1) |
| $5 \times 10^{-4}$ | 0.036 20(5) |

FIG. 5: (Color online) The phase boundary of the DP-DI near $w = 0$. For $w \geq 10^{-4}$, the phase boundary looks almost a straight line, but the close-up of the region of $w \leq 5 \times 10^{-5}$ (panel located at the right bottom) shows a curvature which is fitted with the crossover exponent $1/\phi = 0.73(4)$. The panel at the left top is the plot of $\Delta_c$ vs $10^5w$ in the logarithmic scales. The slope of the straight line is 0.75.
the density with the DP critical exponent \( \delta_{\text{DP}} \) at criticality. The results are summarized in Table III. As can be seen from the table, the critical points for finite \( w \)'s are quite close to the critical point of the PCPD2. Hence, the accurate estimation of the critical points are necessary to estimate the crossover exponent with a reasonable precision. To this end, we simulated large systems (\( N = 2^{20} \)) for long times (up to \( t = 2 \times 10^7 \) for \( w = 10^{-2} \) and \( t = 5 \times 10^8 \) for \( w = 10^{-3} \)).

As before, we measure the crossover exponent from the singular behavior of the phase boundary. Rather than using the critical point of the PCPD2 to calculate \( \phi \), we fit the phase boundary using the trial function

\[
\rho_p(t) = \rho_c(0) + a \times t^{1/\phi} \\
\rho_p(t) = \rho_c(0) + a \times t^{1/\phi}
\]

with three fitting parameters \( \rho_c(0) \), \( a \), and \( 1/\phi \). The fitting using the whole data in Table III except \( \rho_c(0) \), gives \( \rho_c(0) \approx 0.180 214 7 \) and \( 1/\phi \approx 1.23 \). These results are summarized in Fig. 7. If we fix \( \rho_c(0) \) between 0.182 21 and 0.182 22 and use \( a \) and \( 1/\phi \) as two fitting parameters, the crossover exponent varies from \( 1/\phi = 1.13 \) to 1.33. So we estimate the crossover exponent as \( 1/\phi = 1.23(10) \).

For comparison, we consider the qP22 (4A \( \rightarrow \) 6A, 2A \( \rightarrow \) 0), belonging to the DP class. Adding the alternating diffusion bias \( w \) (symmetry-breaking field in the dual spin language), all particles form bound pairs \( (X = 2A) \), of which the dynamics is given by \( 2X \rightarrow 3X \) and \( X \rightarrow 0 \). Therefore, we expect the DP scaling even at nonzero \( w \). There is no true crossover between the same (DP) universality classes and so there are no diverging time scale \( (\chi = 0) \) as \( w \rightarrow 0 \). Figure 8 confirms our prediction.

| \( w \)     | \( \rho_c(w) \)   |
|-----------|------------------|
| 0         | 0.180 215 (5)    |
| 10\(^{-3}\) | 0.180 222 5 (25) |
| 2 \times 10\(^{-3}\) | 0.180 235 (5) |
| 3 \times 10\(^{-3}\) | 0.180 250 (5) |
| 4 \times 10\(^{-3}\) | 0.180 260 (5) |
| 5 \times 10\(^{-3}\) | 0.180 275 (5) |
| 8 \times 10\(^{-3}\) | 0.180 325 (5) |
| 10\(^{-2}\) | 0.180 360 (10) |

### FIG. 6: (Color online) Scaling collapse for the particle (\( \rho \)) and pair densities (\( \rho_p \)) in log-log scale with \( \Delta = 0 \); see Eq. 5. Both particle and pair densities are collapsed into a respective single curve in the asymptotic regime.

### FIG. 7: (Color online) \( \rho_c(w) - \rho_c(0) \) vs \( w \) plot of the SBPCPD in log-log scales. The symbols are results from simulation and the straight line is result of the fitting using the trial function

\[
\rho_p(t) = \rho_c(0) + a \times w^{1/\phi} \\
\rho_p(t) = \rho_c(0) + a \times w^{1/\phi}
\]

with three fitting parameters \( \rho_c(0) \), \( a \), and \( 1/\phi \). The fitting results are \( \rho_c(0) \approx 0.180 214 7 \) and \( 1/\phi \approx 1.23 \).

### FIG. 8: (Color online) Plots of \( \rho(t)^{1/\phi} \) vs \( t \) near criticality for the "crossover" model from the DP to the DP mediated by the symmetry breaking field for \( w = 0 \) (\( p = 0.031 405 \)), \( 5 \times 10^{-3} \) (\( p = 0.031 409 \)), and \( 10^{-2} \) (\( p = 0.031 419 \)). No clear distinction among curves is observed, which is the characteristics of the "crossover" model within the same universality class.
V. SUMMARY

To summarize, we introduced and studied two types of crossover models from the pair contact process with diffusion with the parity conservation (PCPD2). We study the crossover from the PCPD by adding the parity-conserving single-particle process to the directed Ising (DI) class and by adding the alternating bias (symmetry-breaking field in the spin language) to the directed percolation (DP) class. For comparison, we also explored the DP-to-DI crossover by studying the diffusing $tp^22$ with the parity-conserving single-particle process.

The crossover exponents were found numerically, which read $1/\phi_1 = 0.57(3)$, $1/\phi_2 = 0.73(4)$, and $1/\phi_3 = 1.23(10)$ respectively for the PCPD-to-DI, the DP-to-DI, and the PCPD-to-DP with the alternating bias. First, we argue that $\phi_1$ is identical to the PCPD-to-DP via a single-particle reaction route, which is numerically confirmed. The new crossover exponent $\phi_2$ found for the DP-to-DI crossover is clearly distinct from $\phi_1$, which supports the non-DP nature of the PCPD fixed point. Third, we found another independent crossover exponent $\phi_3$ for the PCPD-to-DP crossover, which is ascribed to the pair-binding route different from the conventional single-particle reaction route. Nontriviality of $\phi_3$ as well as $\phi_1$ provides another evidence to support that the PCPD is distinct from the DP.

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[1] M. Henkel and H. Hinrichsen, J. Phys. A: Math. Gen. 37, R117 (2004).
[2] S.-C. Park and H. Park, Eur. Phys. J. B 64, 415 (2008).
[3] M. J. Howard and U. C. Täuber, J. Phys. A 30, 7721 (1997).
[4] G.T. Barkema and E. Carlon, Phys. Rev. E 68 036113 (2003).
[5] H. Hinrichsen, Physica A 361, 457 (2006).
[6] F. Smallenburg and G.T. Barkema, Phys. Rev. E 78, 031129 (2008).
[7] S.-C. Park and H. Park, Phys. Rev. E 76, 051123 (2007).
[8] S.-C. Park and H. Park, Phys. Rev. E 73, 025105(R) (2006).
[9] S.-C. Park and H. Park, Phys. Rev. Lett. 94, 065701 (2005).
[10] S.-C. Park and H. Park, Phys. Rev. E 71, 016137 (2005).
[11] J. D. Noh and H. Park, Phys. Rev. E 69, 016122 (2004).
[12] J. Kockelkoren and H. Chaté, Phys. Rev. Lett. 90, 125701 (2003).
[13] H.-K. Janssen, F. van Wijland, O. Deloubrière, and U.C. Täuber, Phys. Rev. E 70, 056114 (2004).
[14] K. Park, H. Hinrichsen, and I.-M. Kim, Phys. Rev. E 63, 065103(R) (2001).
[15] M. H. Kim and H. Park, Phys. Rev. Lett. 73, 2579 (1994).
[16] H. Park and H. Park, Physica A 221, 97 (1995).
[17] W.M. Hwang, S. Kwon, H. Park, and H. Park, Phys. Rev. E 57, 6438 (1998).
[18] P. Grassberger, F. Krause, and T. von der Twer, J. Phys. A 17 L105 (1984).
[19] P. Grassberger, J. Phys. A 22, L1103 (1989).
[20] S.-C. Park and H. Park, Phys. Rev. E 78, 041128 (2008).
[21] K. Park, H. Hinrichsen, and I.-M. Kim, Phys. Rev. E 66, 025101(R) (2002).
[22] H. Hinrichsen, Physica A 291, 275 (2001).
[23] Lowest elementary excitations for reactions are pairs for the PCPD-type models. Similar analogy can be considered for general multiple reaction models.
[24] S.-C. Park, Eur. Phys. J. B 50, 327 (2006).
[25] H. Takayasus, A.Y. Tretyakov, Phys. Rev. Lett. 68, 3060 (1992).
[26] S. Kwon and H. Park, Phys. Rev. E 52, 5955 (1995).
[27] I.D. Lawrie and S. Sarbach, in Phase Transitions and Critical Phenomena, edited by C. Domb and J.L. Lebowitz (Academic Press, London, 1984), Vol. 9.
[28] J. Cardy, Scaling and Renormalization in Statistical Physics, (Cambridge University Press, 1996).