Application of Exp-function method to the generalized Burgers-Huxley equation

Changbum Chun
School of Liberal Arts, Korea University of Technology and Education, Cheonan City, Chungnam 330-708, Republic of Korea
Email: cbchun@kut.ac.kr, Fax: +82-41-560-1378

Abstract. In this paper, the Exp-function method is applied to obtain generalized solitary solutions of the generalized Burgers-Huxley equation. It is shown that the Exp-function method, with the help of symbolic computation, provides a powerful mathematical tool for solving nonlinear equations arising in mathematical physics.

1. Introduction
The investigation of exact traveling wave solutions of nonlinear evolution equations plays an important role in the study of nonlinear physical phenomena in various areas such as physics, chemistry, biology, fluid dynamics, plasma, optical fibers and others of engineering. Many effective methods for obtaining exact solutions of the nonlinear equations have been proposed and applied such as tanh-function method [1], F-expansion method [2], variational iteration method [3-4], homotopy perturbation method [5-6] and others, see [7] for a complete review on the field.

Recently, He and Wu [8] proposed a straightforward and concise method, called Exp-function method, which reasonably play a unifying role among the above-mentioned methods to obtain generalized solitary solutions and periodic solutions of nonlinear evolution equations. This method has been successfully applied to many nonlinear evolution equations (see [8-10] and the references therein). In this paper we extend the Exp-function method to finding new and generalized solitary solutions to generalized Burgers-Huxley equation [11] which is encountered in the description of many non-linear wave phenomena

\[ u_t + \alpha u^\delta u_x - u_{xx} = \beta u (1-u^\delta) (u^\delta - \gamma), \]  

where \( \alpha, \beta \geq 0 \) are real constants, \( \delta \) is a positive integer and \( \gamma \in (0, 1) \).

2. Exp-function Method
Using the transformation

\[ \eta = k x + \omega t \]  

Eq. (1) becomes an ordinary differential equation, which reads

\[ \omega u' + \alpha k u^\delta u' - k^2 u^\delta - \beta u (1-u^\delta) (u^\delta - \gamma) = 0, \]
where prime denotes the differential with respect to $\eta$.

According to the Exp-function method \[8\], we assume that the solution of Eq. (3) can be expressed in the form:

$$u(\eta) = \sum_{n=-c}^{d} a_n \exp(n\eta) - \sum_{m=-p}^{q} b_m \exp(m\eta),$$

where $c, d, p$ and $q$ are positive integers which are unknown to be further determined, $a_n$ and $b_m$ are unknown constants.

Eq. (4) can be rewritten in an alternative form as follows:

$$u(\eta) = \frac{a_c \exp(c\eta) + \cdots + a_d \exp(-d\eta)}{b_p \exp(p\eta) + \cdots + b_q \exp(-q\eta)}.$$ (5)

In order to determine values of $c$ and $p$, we balance the linear term of highest order in Eq. (3) with the highest order nonlinear term. By simple calculation, we have

$$u'' = \frac{c_1 \exp[(c + 2p + 2\delta p)\eta] + \cdots}{c_2 \exp[(3 + 2\delta) p\eta] + \cdots},$$

and

$$u^{2\delta+1} = \frac{c_3 \exp[(c + 2p + 2\delta c)\eta] + \cdots}{c_4 \exp[(3 + 2\delta) p\eta] + \cdots},$$

where $c_i$ are determined coefficients only for simplicity. Balancing highest order of Exp-function in Eqs. (6) and (7), we have

$$c + 2p + 2\delta p = c + 2p + 2\delta c,$$

so that,

$$p = c.$$ (9)

Similarly to determine values of $d$ and $q$, we balance the linear term of lowest order in Eq. (3)

$$u'' = \frac{\cdots + d_1 \exp[-(d + 2q + 2\delta q)\eta]}{\cdots + d_2 \exp[-(3 + 2\delta) q\eta]},$$

and

$$u^{2\delta+1} = \frac{\cdots + d_3 \exp[-(d + 2q + 2\delta d)\eta]}{\cdots + d_4 \exp[(3 + 2\delta) q\eta]},$$

where $d_i$ are determined coefficients only for simplicity. Balancing lowest order of Exp-function in Eqs. (10) and (11), we have

$$d + 2q + 2\delta q = d + 2q + 2\delta d,$$

so that,
In the sequel, we consider the case \( \alpha = \beta = \delta = 1 \) only for the sake of illustration and find nontrivial solutions of Eq. (1). By nontrivial solutions we mean that they are solutions of Eq. (1) except the solutions \( u = 0, u = 1 \) and \( u = \gamma \). The other cases can be treated exactly the same way to find the corresponding solutions.

**Case 1:** \( p = c = 1 \) and \( d = q = 1 \)

We can freely choose the values of \( c \) and \( d \), but we will illustrate the final solutions do not strongly depend upon the choice of values of \( c \) and \( d \). For simplicity, we set \( p = c = 1 \) and \( d = q = 1 \), the trial solution, Eq. (5) becomes

\[
u(\eta) = \frac{a_0 \exp(\eta) + a_{-1} \exp(-\eta)}{\exp(\eta) + b_0 + b_{-1} \exp(-\eta)}.
\]

Substituting Eq. (14) into (3) and using the Maple 8, equating to zero the coefficients of all powers of \( \exp(n \eta) \) yields a set of algebraic equations for \( a_1, a_0, a_{-1}, b_0, b_{-1}, k \) and \( \omega \). Solving the system of algebraic equations with the help of Maple gives the following sets of solutions:

\[
\begin{align*}
\{a_1 &= b_0 = a_0 = 0, a_{-1} = a_{-1}, b_{-1} = a_{-1}, k = -1/4, \omega = (4\gamma - 1)/8\}, \\
\{a_1 &= b_0 = a_0 = 0, a_{-1} = a_{-1}, b_{-1} = a_{-1}, k = 1/2, \omega = (\gamma - 1)/2\}, \\
\{a_1 &= b_0 = a_0 = 0, a_{-1} = \gamma b_{-1}, b_{-1} = b_{-1}, k = -\gamma/4, \omega = (\gamma^2 + 4\gamma)/8\}, \\
\{a_1 &= b_0 = a_0 = 0, a_{-1} = \gamma b_{-1}, b_{-1} = a_{-1}, k = \gamma/2, \omega = (\gamma^2 + \gamma)/2\}, \\
\{a_1 &= a_{-1} = b_{-1} = 0, a_0 = b_0, b_0 = b_0, k = -\gamma/2, \omega = (\gamma^2 + 4\gamma)/4\}, \\
\{a_1 &= a_{-1} = b_{-1} = 0, a_0 = b_0, b_0 = b_0, k = \gamma, \omega = -\gamma^2 + \gamma\}, \\
\{a_1 &= a_{-1} = b_{-1} = 0, a_0 = b_0, b_0 = b_0, k = -1/2, \omega = (4\gamma - 1)/4\}, \\
\{a_1 &= 0, a_{-1} = b_{-1} = \gamma b_{-1}, b_0 = \frac{\gamma^2 b_{-1} + a_0^2}{a_0 \gamma}, a_0 = a_0, b_{-1} = b_{-1}, k = -\gamma/2, \omega = (\gamma^2 + 4\gamma)/4\}, \\
\{a_1 &= 0, a_{-1} = b_{-1} = \gamma b_{-1}, b_0 = \frac{\gamma^2 b_{-1} + a_0^2}{a_0 \gamma}, a_0 = a_0, b_{-1} = a_{-1}, k = 1, \omega = -1 + \gamma\}, \\
\{a_1 &= 0, a_{-1} = \gamma b_{-1}, b_0 = \frac{\gamma^2 b_{-1} + a_0^2}{a_0 \gamma}, a_0 = a_0, b_{-1} = b_{-1}, k = \gamma, \omega = -\gamma^2 + \gamma\}, \\
\{a_0 &= a_{-1} = b_{-1} = 0, a_1 = 1, b_0 = b_0, k = -1, \omega = 1 - \gamma\},
\end{align*}
\]
\{ a_0 = a_{-1} = b_{-1} = 0, a_1 = 1, b_0 = b_0, k = 1/2, \ \omega = (1-4\gamma)/4 \}, \quad (15i)

\{ a_0 = a_{-1} = b_{-1} = 0, a_1 = \gamma, b_0 = b_0, k = -\gamma, \ \omega = \gamma^2 - \gamma \}, \quad (15m)

\{ a_0 = a_{-1} = b_{-1} = 0, a_1 = \gamma, b_0 = b_0, k = \gamma/2, \ \omega = (\gamma^2 - 4\gamma)/4 \}, \quad (15n)

\{ a_0 = a_{-1} = b_0 = 0, a_1 = 1, b_{-1} = b_{-1}, k = -1/2, \ \omega = (1-\gamma)/2 \}, \quad (15o)

\{ a_0 = a_{-1} = b_0 = 0, a_1 = 1, b_{-1} = b_{-1}, k = 1/4, \ \omega = (1-4\gamma)/8 \}, \quad (15p)

\{ a_0 = a_{-1} = b_0 = 0, a_1 = \gamma, b_{-1} = b_{-1}, k = -\gamma/2, \ \omega = (\gamma^2 - \gamma)/2 \}, \quad (15q)

\{ a_0 = a_{-1} = b_0 = 0, a_1 = \gamma, b_{-1} = b_{-1}, k = \gamma/4, \ \omega = (\gamma^2 - 4\gamma)/8 \}, \quad (15r)

\{ a_{-1} = 0, a_1 = 1, b_0 = (b_{-1} + a_0^2)/a_0, a_0 = a_0, b_{-1} = b_{-1}, \ k = -1, \ \omega = 1-\gamma \}, \quad (15s)

\{ a_{-1} = 0, b_0 = \frac{\gamma^2 b_{-1} + a_0^2}{a_0 \gamma}, a_1 = \gamma, a_0 = a_0, b_{-1} = b_{-1}, k = -\gamma, \ \omega = \gamma^2 - \gamma \}, \quad (15t)

\{ a_{-1} = 0, b_0 = (b_{-1} + a_0^2)/a_0, a_1 = 1, a_0 = a_0, b_{-1} = b_{-1}, k = 1/2, \ \omega = (1-4\gamma)/4 \}, \quad (15u)

\{ a_{-1} = 0, b_0 = \frac{\gamma^2 b_{-1} + a_0^2}{a_0 \gamma}, a_1 = \gamma, a_0 = a_0, b_{-1} = b_{-1}, k = \gamma/2, \ \omega = (\gamma^2 - 4\gamma)/4 \}, \quad (15v)

\{ a_0 = b_0 = 0, a_1 = 1, a_{-1} = \gamma b_{-1}, b_{-1} = b_{-1}, k = (\gamma - 1)/2, \ \omega = (1-\gamma^2)/2 \}, \quad (15w)

\{ a_0 = b_0 = 0, a_1 = 1, a_{-1} = \gamma b_{-1}, b_{-1} = b_{-1}, k = (1-\gamma)/4, \ \omega = (1-\gamma^2)/8 \}, \quad (15x)

\{ a_1 = 1, a_{-1} = \frac{\gamma(a_0 b_0 (\gamma + 1) - a_0^2 - b_0^2 \gamma)}{(\gamma - 1)^2}, b_{-1} = \frac{1}{\gamma} a_{-1}, a_0 = a_0, b_0 = b_0, k = -1+\gamma, \ \omega = 1-\gamma^2 \}, \quad (15y)

\{ a_1 = 1, a_{-1} = \frac{\gamma(a_0 b_0 (\gamma + 1) - a_0^2 - b_0^2 \gamma)}{(\gamma - 1)^2}, b_{-1} = \frac{1}{\gamma} a_{-1}, a_0 = a_0, b_0 = b_0, k = \frac{1}{2} - \gamma, \ \omega = \frac{1}{4} - \gamma^2 \}, \quad (15z)

\{ a_0 = b_0 = 0, a_{-1} = b_{-1}, a_1 = \gamma, b_{-1} = b_{-1}, k = (\gamma - 1)/4, \ \omega = (\gamma^2 - 1)/8 \}, \quad (16a)

\{ a_0 = b_0 = 0, a_{-1} = b_{-1}, a_1 = \gamma, b_{-1} = b_{-1}, k = (1-\gamma)/2, \ \omega = (\gamma^2 - 1)/2 \}, \quad (16b)
Although each set of solutions listed above yields the solution of Eq. (1), here, we discuss two sets only. Substituting Eq. (15y) into (14) yields the following solution of Eq. (1):

\[
\eta = (\gamma - 1)t + (1 - \gamma^2)t
\]

where \( \eta = (\gamma - 1)x + (1 - \gamma^2)t \) and \( a_0, b_0 \) are free parameters.

Substituting Eq. (15r) into (14) yields the solitary solution of Eq. (1):

\[
u = \frac{\gamma \exp(\eta + a_0)}{\exp(\eta) + b_1 \exp(-\eta)}, \quad \eta = \frac{\gamma}{4} x + \left(\frac{\gamma^2}{8} - \frac{\gamma}{2}\right) t,
\]

where \( b_1 \) is a free real number. In particular, in case \( b_1 = 1 \), Eq. (18) reduces to the solution obtained in [10]

\[
u = \frac{\gamma \exp(\eta)}{\exp(\eta) + \exp(-\eta)} = \frac{\gamma}{2} (1 + \tanh \eta), \quad \eta = \frac{\gamma}{4} x + \left(\frac{\gamma^2}{8} - \frac{\gamma}{2}\right) t.
\]

**Case 2:** \( p = c = 2 \) and \( d = q = 2 \)

If we set \( p = c = 2 \) and \( d = q = 2 \), then the trial function, Eq. (5) becomes

\[
u(\eta) = \frac{a_2 \exp(2\eta) + a_1 \exp(\eta) + a_0 + a_{-1} \exp(-\eta) + a_{-2} \exp(-2\eta)}{b_2 \exp(2\eta) + b_1 \exp(\eta) + b_0 + b_{-1} \exp(-\eta) + b_{-2} \exp(-2\eta)}
\]

There are some free parameters in Eq. (20), we set \( b_2 = 1, b_1 = b_{-1} = 0 \) for simplicity, the trial function, Eq. (20) is simplified as follows

\[
u(\eta) = \frac{a_2 \exp(2\eta) + a_1 \exp(\eta) + a_0 + a_{-1} \exp(-\eta) + a_{-2} \exp(-2\eta)}{\exp(2\eta) + b_0 + b_{-2} \exp(-2\eta)}
\]

By the same manipulation as illustrated above, we obtained forty three sets of algebraic solutions which in turn give nontrivial solutions of Eq. (1). Here, we present the following sets of algebraic solutions only:

\[
\{a_2 = a_1 = a_0 = a_{-1} = b_0 = 0, a_{-2} = \gamma b_{-2}, b_{-2} = b_{-2}, k = \gamma / 4, \omega = (\gamma - \gamma^2) / 4, \}
\]

\[
\{a_2 = 1, a_1 = a_{-1} = 0, a_{-2} = \gamma [a_0 b_0 (\gamma + 1) - a_0^2 - b_0^2 \gamma], b_{-2} = \frac{1}{\gamma} a_{-2}, a_0 = a_0, b_0 = b_0, \}
\]

\[
\{k = (\gamma - 1) / 2, \omega = (1 - \gamma^2) / 2 \}
\]
\begin{align}
\{a_2 = \gamma, a_{-2} = 0, a_i = \frac{-\gamma^2 b_i}{a_{-2}}, a_0 = \frac{-a_{-2}}{\gamma b_{-2}}, b_i = \frac{-a_{-2} a_i + \gamma^2 b_i b_{-2}}{\gamma^2 a_{-2} b_{-2}}, a_{-1} = a_{-2}, b_{-2} = b_{-1}, k = \frac{\gamma}{2}, \omega = \frac{\gamma^2}{4} - \gamma \}. 
\end{align} 

Substituting (22) into (21) gives the following solution

\begin{align}
\eta = \frac{\gamma}{4} \chi + \left( \frac{\gamma^2}{4} - \frac{\gamma^2}{4} \right) t.
\end{align}

If we set \( b_{-2} = 1 \) in Eq. (25), then we can recover the solution presented in [12]:

\begin{align}
\eta = \frac{\gamma}{2} \chi + \frac{\gamma^2}{2} \tanh \left[ \frac{\gamma}{2} \chi + \left( \frac{\gamma^2}{2} - \frac{\gamma^2}{2} \right) t \right].
\end{align}

It should be noted that substituting (23) into (21) we can recover the solution (17).

If we consider Eq. (24), then we obtain the new generalized solitary solution of Eq. (1)

\begin{align}
\eta = \frac{\gamma^2 a_{-1}^2 b_{-2}^2 \gamma \exp(2\eta) + a_{-2} \exp(-\eta)}{\gamma^2 a_{-1}^2 b_{-2}^2 \exp(2\eta) + b_{-2} \exp(-2\eta)} - \frac{\gamma^4 a_{-1}^4 \exp(\eta) - \gamma a_{-1}^4}{\gamma^2 a_{-1}^2 b_{-2}^2 \exp(2\eta) + b_{-2} \exp(-2\eta)} - a_{-1} - \frac{\gamma^4 b_{-2}^4}{\gamma^2 a_{-1}^2 b_{-2}^2 \exp(2\eta) + b_{-2} \exp(-2\eta)}.
\end{align}

where \( \eta = \frac{\gamma}{2} \chi + \left( \frac{\gamma^2}{4} - \gamma \right) t \), and \( a_{-1} \) and \( b_{-2} \) are free parameters.

**Case 3:** \( p = c = 2 \) and \( d = q = 1 \)

We consider the case \( p = c = 2 \) and \( d = q = 1 \), then the trial function, Eq. (5) can be expressed as

\begin{align}
u(\eta) = \frac{a_2 \exp(2\eta) + a_1 \exp(\eta) + a_0 + a_{-1} \exp(-\eta)}{\exp(2\eta) + b_1 \exp(\eta) + b_0 + b_{-1} \exp(-\eta)}.
\end{align}

By the same manipulation as illustrated above, we obtained fifty-four sets of algebraic solutions which in turn yield the nontrivial solutions of Eq. (1). Here, we present the following sets only.

\begin{align}
\{a_2 = a_1 = a_0 = b_1 = b_0 = 0, a_{-1} = a_{-2}, k = -1/6, \omega = (4\gamma - 1)/12 \},
\end{align}

\begin{align}
\{a_2 = a_1 = a_0 = b_1 = b_0 = 0, a_{-1} = \gamma b_{-1}, b_{-1} = b_{-2}, k = \gamma / 3, \omega = (\gamma^2 + \gamma) / 3 \},
\end{align}

\begin{align}
b_0 = (a_1 b_1 + a_2 b_2 + a_0 \gamma - b_1 \gamma - a_0 \gamma - a_3^2 b_1) / \gamma (\gamma - 1), b_1 = b_{-1}, k = (\gamma - 1) / 2, \omega = (\gamma^2 - 1) / 4 \}.
\end{align}

Here, we only discuss the set (31), in this case we obtain the new solitary solution of Eq. (1)

\begin{align}
u(\eta) = \frac{\gamma (\gamma - 1) [\gamma \exp(2\eta) + a_1 \exp(\eta) + a_0 + a_{-1} \exp(-\eta)]}{\gamma (\gamma - 1) [\exp(2\eta) + b_1 \exp(\eta) + a_{-1} \exp(-\eta)] + a_{-1} \exp(\eta) - b_1 \gamma - a_0 \gamma - a_{-1}^2 + a b_1}.
\end{align}

where \( \eta = \frac{\gamma}{2} \chi + \frac{\gamma^2}{4} t \), \( a_{-1} \) is defined by Eq. (31), and \( a_0, a_1, b_1 \) are free parameters.

In particular, if we set \( a_1 = \gamma b_1 \) in Eq. (32), then we obtain the solution
\[ u(\eta) = \frac{\gamma[\gamma \exp(2\eta) + \gamma b_1 \exp(\eta) + a_0]}{\gamma[\exp(2\eta) + b_1 \exp(\eta)] + a_0} \]  

(33)

3. Conclusion
The Exp-function method with the symbolic computation package Maple has been successfully applied to obtain as many new generalized solitary solutions to generalized Burgers-Huxley equation. The results reveal that the Exp-function method is straightforward, concise and a promising tool for solving a wide class of nonlinear evolution equations arising in mathematical physics. Application of the Exp-function method to difference-differential equations was given in Refs.[13,14].

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