MINIMAL TWIST OF ALMOST COMMUTATIVE GEOMETRIES

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In memoriam John Madore

Abstract. We classify the twists of almost commutative spectral triples that keep the Hilbert space and the Dirac operator untouched. The involved twisting operator is shown to be the product of the grading of a manifold by a finite dimensional operator, which is not necessarily a grading of the internal space. Necessary and sufficient constraints on this operator are listed.

1. Introduction

The Higgs field might be a probe of the internal structure of spacetime. This idea - pioneered in [17] - has been fully implemented in the framework of noncommutative geometry [10]. It yields a description of the Standard Model of fundamental interactions (including massive neutrinos [6]) as a pure gravity theory on an almost commutative geometry [11], that is on a space which is the product of a usual continuous manifold $\mathcal{M}$ by some internal matricial structure.

This product is mathematically well defined in terms of spectral triple. The latter consists in an involutive algebra $\mathcal{A}$ acting faithfully on some Hilbert space $\mathcal{H}$, together with a selfadjoint operator $D$ with compact resolvent, such that the commutator $[D,a]$ is bounded for any $a$ in $\mathcal{A}$. With additional axioms, spectral triples furnish a purely algebraic characterization of riemannian (spin) manifolds [12], as well as their generalisation to the noncommutative setting [11].

A twisted spectral triple is defined similarly, except that the commutator is no longer required to be bounded. Instead, one asks for an automorphism $\rho$ of $\mathcal{A}$ such that the twisted commutator

$$[D,a]_\rho := Da - \rho(a)D$$

is bounded for any $a$ in $\mathcal{A}$. Such twists have been introduced in [13] with some mathematical motivations. Later, they show to be useful for physical applications as well, for they offer a way to build models beyond the Standard Model [15, 16]. In particular, by twisting the spectral triple of the Standard Model in a minimal way, that is keeping the Hilbert space and Dirac operator untouched (only the algebra is modified), one produces extra bosonic fields without altering the fermionic content of the theory.

There exists a general procedure to obtain such minimal twist, recalled in section 2, which uses a grading of the spectral triple [19]. Recall that the later is a selfadjoint operator $\Gamma$ on $\mathcal{H}$, squaring to the identity $I$ and anticommuting with $D$, such that $[\Gamma,a] = 0$ for any $a \in \mathcal{A}$. The aim of this note is to understand which of these properties are necessary: is the twist doable using a twisting operator that is not a grading?

We first list in section 3 some basic properties expected from the twisting operator. Then we proceed at the light of three conditions that must be satisfied by a real twisted spectral triple: the boundedness of the twisted commutator (1) in section 4, the order zero condition in section 5 and the twisted first-order condition in section 6. The resulting constraints are listed in propositions 5.1 and 6.1. The grading operator is not the only solution.

For almost-commutative geometries, assuming the twisting operator is the product of an operator $\mathcal{T}$ acting on the spinor space of $\mathcal{M}$ with an operator $T_F$ acting on the internal space, these constraints are shown to be equivalent to their reduction to the internal space (corollaries 5.1.1 and 6.1.1). Although the boundedness of the commutator forces $\mathcal{T}$ to be the grading of the manifold, $T_F$ is not necessarily a grading of the internal space.
2. Minimal twist

A minimal twist of a (real, graded) spectral triple \((\mathcal{A}, \mathcal{H}, D)\) is [19, Def. 3.2] a (real, graded) twisted spectral triple \((\mathcal{A} \otimes \mathcal{B}, \mathcal{H}, D)_{\rho}\) where \(\mathcal{B}\) is a involutive algebra with unit \(1_{\mathcal{B}}\), \(\rho\) an automorphism of \(\mathcal{A} \otimes \mathcal{B}\) and the representation \(\pi\) of the latter on \(\mathcal{H}\) is such that

\[
\pi(a \otimes 1_{\mathcal{B}}) = \pi_0(a) \quad \forall a \in \mathcal{A},
\]

where \(\pi_0\) is the representation of \(\mathcal{A}\) on \(\mathcal{H}\) from the initial spectral triple.

If this initial spectral triple is graded, then there always exists a minimal twist with\(^1\) \(\mathcal{B} = \mathbb{C}^2\) and \(\rho\) the automorphism of \(\mathcal{A} \otimes \mathbb{C}^2 \simeq \mathcal{A} \oplus \mathcal{A}\) given by the flip

\[
\rho((a, a')) := (a', a) \quad \forall a, a' \in \mathcal{A} \otimes \mathbb{C}^2.
\]

The construction of this minimal twist-by-grading,

\[(\mathcal{A} \otimes \mathbb{C}^2, \mathcal{H}, D)_{\rho},\]

starts with the following observation: by definition the grading \(\Gamma\) commutes with the algebra, so the projections \(\frac{1 + \Gamma}{2}\) on its eigenspaces \(\mathcal{H}_\pm\) define two independent involutive representations

\[
\pi_\pm(a) := \left(\frac{1 \pm \Gamma}{2} \pi_0(a)\right)_{\mathcal{H}_\pm}
\]

of \(\mathcal{A}\) on \(\mathcal{H}_\pm\). Their direct sum

\[
\pi(a, a') := \frac{1 + \Gamma}{2} \pi_0(a) + \frac{1 - \Gamma}{2} \pi_0(a') \quad \forall a, a' \in \mathcal{A}
\]

is a representation of \(\mathcal{A} \otimes \mathbb{C}^2\) that satisfies the properties of a twisted spectral triple [19, Prop.3.8] as well as condition (2).

This twist-by-grading is the only possible minimal twist for the spectral triple naturally associated to an (even dimensional) closed riemannian spin manifold \(\mathcal{M}\) [19, Prop.4.2], namely

\[
C^\infty(\mathcal{M}), \quad L^2(\mathcal{M}, S), \quad \mathcal{D} = -i \sum_{\mu=1}^{\dim \mathcal{M}} \gamma^\mu \nabla_\mu
\]

where the unital algebra \(C^\infty(\mathcal{M})\) of smooth functions on \(\mathcal{M}\) acts by multiplication on the Hilbert space \(L^2(\mathcal{M}, S)\) of square integrable spinors,

\[
(\pi_M(f)\psi)(x) := f(x)\psi(x) \quad \forall \psi \in L^2(\mathcal{M}, S), x \in \mathcal{M},
\]

and \(\mathcal{D}\) is the Dirac operator associated with the spin structure. This spectral triple is graded with grading the product \(\gamma_M\) of the gamma matrices.

The unicity of this twist no longer holds true for an almost commutative geometry

\[
\mathcal{A} = C^\infty(\mathcal{M}) \otimes \mathcal{A}_F, \quad \mathcal{H} = L^2(\mathcal{M}, S) \otimes \mathcal{H}_F, \quad D = \mathcal{D} \otimes 1_F + \gamma_M \otimes D_F,
\]

that is the product of (7) with a finite dimensional graded spectral triple \((\mathcal{A}_F, \mathcal{H}_F, D_F)\) with grading \(\Gamma_F\) (in the equation above \(1_F\) is the identity operator on \(\mathcal{H}_F\)). The representation

\[
\pi_0 = \pi_M \otimes \pi_F
\]

of \(\mathcal{A}\) on \(\mathcal{H}\) is the product of \(\pi_M\) with the representation \(\pi_F\) of \(\mathcal{A}_F\) on \(\mathcal{H}_F\) given by the finite dimensional spectral triple. If \(\pi_F\) is irreducible, then any minimal twist is necessarily by \(\mathcal{B} = \mathbb{C}^2\) but the representation \(\pi\) is not necessarily the one given in (6), as explained below. If \(\pi_F\) is not irreducible, there exists minimal twists with \(\mathcal{B}\) different from \(\mathbb{C}^2\) [19, Corr.4.5].

**Remark 2.1.** The dimension of \(\mathcal{M}\) has to be even so that the spectral triple (7) admits a grading \(\gamma_M\). The odd dimensional case should be investigated elsewhere.

\(^1\)To fix notation we assume that \(\mathcal{A}\) is a complex algebra, but the results also hold for real algebras.
3. Twisting Operator

The point of this note is to investigate which properties of the grading $\Gamma$ are necessary to
build a minimally twisted partner to a usual spectral triple. The commutativity with the initial
representation of $\mathcal{A}$ is important to get two independent representations $\pi_{\pm}$, but to what extend
are the commutation properties of $\Gamma$ with $D$ and (in case of a real spectral triple) with the
real structure $J$ relevant?

Given a spectral triple $(\mathcal{A}, \mathcal{H}, D)$, we thus consider an operator in $\mathcal{B}(\mathcal{H})$, which shares all the
properties of a grading but the commutation properties with $D$ and $J$. Namely $T$ is selfadjoint,
$T^2=I$, the degeneracy of both its eigenvalues $\pm 1$ are non-zero and $T$ commutes with the
representation $\pi_0$ of $\mathcal{A}$ on $\mathcal{H}$. The latter is thus the direct sum $\pi_+ \oplus \pi_-$ of the two involutive
representations of $\mathcal{A}$ on the eigenspaces $\mathcal{H}_\pm$ of $T$ given by
\begin{equation}
\pi_{\pm}(a) = \left( \frac{1 \pm T}{2} \pi_0(a) \right)_\mathcal{H}_\pm.
\end{equation}
As in (6), the operator $T$ allows to define a representation of $\mathcal{A} \otimes \mathbb{C}^2$ on $\mathcal{H}$
\begin{equation}
\pi(a,a') := \pi_+(a) \oplus \pi_-(a') = \frac{1 + T}{2} \pi_0(a) \oplus \frac{1 - T}{2} \pi_0(a').
\end{equation}
To avoid domain issues, we assume that $T \mathcal{H} \subset \text{Dom } D$. We call $T$ a twisting operator.

For an almost commutative geometry (9), we further assume that
\begin{equation}
T = T \otimes T_F
\end{equation}
where $T \in L^2(\mathcal{M},S)$ and $T_F \in \mathcal{B}(\mathcal{H}_F)$. A bounded operator on $\mathcal{H}$ is not necessarily of this
form, but could be (the closure of) a sum of such operators. However, we restrict to operators
(13), for they already pave the way to interesting physical applications beyond the Standard
Model. The selfadjointness of $T$ is to guarantee that the representations $\pi_{\pm}$ are involutive. This
does not imply that $T$ and $T_F$ are selfadjoint. However one may always restrict to this case.

**Lemma 3.1.** Let $T = T \otimes T_F$ be a selfadjoint operator on $L^2(\mathcal{M},S) \otimes \mathcal{H}_F$ that squares to $I$. 
Then there exist two selfadjoint operators $\tilde{T}$ on $L^2(\mathcal{M},S)$ and $\tilde{T}_F$ on $\mathcal{H}_F$, squaring to the
identity, such that $T = \tilde{T} \otimes \tilde{T}_F$.

**Proof.** The matrix $T \otimes T_F$ being non-zero (otherwise $T$ does not square to $I$) is positive, thus it
admits at least one real eigenvalue $\lambda > 0$, with associated eigenvectors $\psi \in \mathcal{H}_F$, and all the
other non-zero eigenvalues are also strictly positive. For any $\varphi \in L^2(\mathcal{M},S)$, one has
\begin{equation}
T^\dagger T(\varphi \otimes \psi) = T^\dagger T \varphi \otimes T^\dagger_T F \psi = \lambda T^\dagger T \varphi \otimes \psi.
\end{equation}
On the other side, by hypothesis $T^\dagger T = I$, that is $T^\dagger T(\varphi \otimes \psi) = \varphi \otimes \psi$. Therefore
\begin{equation}
(\lambda T^\dagger T - \mathbb{1}_M) \varphi \otimes \psi = 0 \quad \forall \varphi \in L^2(\mathcal{M},S)
\end{equation}
meaning that $T^\dagger T$ coincides with the operator of multiplication of spinors by $\lambda^{-1}$. Repeating
the analysis for another non-zero eigenvalue $\lambda'$ shows that $T^\dagger T$ coincides with the multiplication
by $\lambda'^{-1}$, so $\lambda' = \lambda$. This allows to define
\begin{equation}
\tilde{T} = \lambda^{-\frac{1}{2}} T, \quad \tilde{T}_F = \lambda^\frac{1}{2} T_F,
\end{equation}
such that $T = \tilde{T} \otimes \tilde{T}_F$ with
\begin{equation}
T^\dagger T = \lambda^{-\frac{1}{2}} T^\dagger \lambda^{-\frac{1}{2}} T = \lambda^{-1} T^\dagger T = \mathbb{1}_M.
\end{equation}
From $T^\dagger T = T T^\dagger = I$ then follows that $\tilde{T}^\dagger_F \tilde{T}_F = \tilde{T}_F \tilde{T}^\dagger_F = \mathbb{1}_F$, that is $\tilde{T}_F$ is unitary, and so is $T_F$.
To show that $\tilde{T}$ and $T_F$ are selfadjoint, let us apply $T = T^\dagger$ on $\varphi \otimes \Psi$ where $\Psi$ is an eigenvector
of $T_F$, with eigenvalue $\tau \in \mathbb{C}$, $|\tau| = 1$. Using $\tilde{T}_F^\dagger \Psi = \tau^{-1} \Psi$, one obtains
\begin{equation}
\tilde{T} \varphi \otimes \tau \Psi = \tilde{T}^\dagger \varphi \otimes \tau^{-1} \Psi \quad \forall \varphi \in L^2(\mathcal{M},S),
\end{equation}
meaning that $\tau^{-1} \tilde{T}^\dagger = \tau \tilde{T}$. Redefining $\tau \tilde{T} \rightarrow \tilde{T}$ (that is $\tau^{-1} \tilde{T}^\dagger \rightarrow \tilde{T}^\dagger$), the previous equation shows that $\tilde{T}$ is selfadjoint. The selfadjointness of $\tilde{T}_F$ then follows from the one of $T$. 

\hfill \Box
4. Boundedness of the commutator

We investigate the conditions imposed, on the twisting operator (13) of an almost commutative geometry, by the boundedness of the twisted commutator

\[ [\hat{\theta} \otimes \mathbb{I}_F + \gamma^5 \otimes D_F, \pi(a, a')]_\rho, \]

for \( \pi \) the representation (12) and \( \rho \) the flip (3). For simplicity, we restrict to the case \( \pi_F \) is irreducible, that is \( \mathcal{B} = \mathbb{C}^2 \).

**Lemma 4.1.** For an almost commutative geometry, the twisted commutator \([\hat{\theta} \otimes \mathbb{I}_F, \pi(a, a')]_\rho\) is bounded for any \((a, a') \in \mathcal{A} \otimes \mathbb{C}^2\) if and only if \(T\) anticommutes with \(\hat{\theta}\).

**Proof.** Using

\[ (\hat{\theta} \otimes \mathbb{I}_F)(I \pm T) = (I \mp T)(\hat{\theta} \otimes \mathbb{I}_F) \pm \{\hat{\theta} \otimes \mathbb{I}_F, T\}, \]

one obtains, omitting the symbol \(\pi_0\),

\[ (\hat{\theta} \otimes \mathbb{I}_F, \pi(a, a'))_\rho = (\hat{\theta} \otimes \mathbb{I}_F)\pi(a, a') - \pi(a', a)(\hat{\theta} \otimes \mathbb{I}_F), \]

\[ = (\hat{\theta} \otimes \mathbb{I}_F)\left(\frac{I + T}{2}a + \frac{I - T}{2}a'\right) - \left(\frac{I + T}{2}a + \frac{I - T}{2}a\right)(\hat{\theta} \otimes \mathbb{I}_F), \]

\[ = \frac{I - T}{2}[\hat{\theta} \otimes \mathbb{I}_F, a] + \frac{1}{2} \{\hat{\theta} \otimes \mathbb{I}_F, T\} \frac{I + T}{2}(a - a'). \]

For \(a = f \otimes m\), then \([\hat{\theta} \otimes \mathbb{I}_F, a] = [\hat{\theta}, f] \otimes m\) is bounded, being \((\mathcal{A}, \mathcal{H}, D)\) a spectral triple. The same is true for an arbitrary \(a \in \mathcal{A}\), and also for \([\hat{\theta} \otimes \mathbb{I}_F, a']\). So the first two terms in (23) are bounded.

If \(T\) anticommutes with \(\hat{\theta} \otimes \mathbb{I}_F\), the last term in (23) is zero, so that (21) is bounded.

Conversely, assume (21) is bounded for any \((a, a') \in \mathcal{A} \otimes \mathbb{C}^2\). This means that the last term in (23) is bounded. For \(a - a' = 1 \otimes m\) with 1 the constant function \(f(x) = 1\) on \(\mathcal{M}\), then this last term is (up to a factor \(\frac{1}{2}\))

\[ \{\hat{\theta} \otimes \mathbb{I}_F, T\}(a - a') = \{\hat{\theta}, T\} \otimes T F m. \]

This is bounded if and only if \(\{\hat{\theta}, T\}\) is bounded. For \(\psi\) on \(\mathcal{H}_+\) (+1 eigenspace of \(T\)), one has

\[ \{\hat{\theta}, T\} \psi = \hat{\theta} \psi + T \hat{\theta} \psi = (I + T)\hat{\theta} \psi \]

meaning \(\{\hat{\theta}, T\}\) coincides with \((I + T)\hat{\theta}\) which is an unbounded operator, unless it is zero. So the restriction of \(\{\hat{\theta}, T\}\) to \(\mathcal{H}_-\) is zero. A similar argument holds for the restriction to \(\mathcal{H}_-\).

Hence the result. \(\square\)

The finite part of an almost commutative geometry only involves bounded operator. Therefore the boundedness of the twisted commutator (19) only depends on the property of \(T\).

**Proposition 4.2.** The twisted commutator (19) is bounded if, and only if, \(T = \pm \gamma_\mathcal{M}\).

**Proof.** The twisted commutator \([\gamma^5 \otimes D_F, \pi(a, a')]_\rho\) is bounded, whether or not \(T\) anticommutes with \(\gamma^5 \otimes D_F\). So (19) is bounded iff \([\hat{\theta} \otimes \mathbb{I}_2, \pi(a, a')]_\rho\) is bounded, that is by Prop. 4.1 iff \(T\) anticommutes with \(\hat{\theta}\). Explicitly, with \(\nabla_\mu = \partial_\mu + \omega_\mu\) where \(\omega_\mu\) is the spin connection, this means

\[ \{-i \gamma_\mu \partial_\mu, T\} + \{-i \gamma_\mu \omega_\mu, T\} = 0. \]

The second term is bounded, being \(\omega_\mu\) bounded. By the Leibniz rule satisfied by \(\hat{\theta}\), one has

\[ \{-i \gamma_\mu \partial_\mu, T\} = -i \gamma_\mu \partial_\mu - i T \gamma_\mu \partial_\mu = -i \gamma_\mu (\partial_\mu T) - i \gamma_\mu T \partial_\mu - i \gamma_\mu \partial_\mu T, \]

\[ = -i \gamma_\mu (\partial_\mu T) - i \{\gamma_\mu, T\} \partial_\mu. \]

The first term is bounded, the second one unbounded. For (26) to hold, both the bounded part and the unbounded parts must be zero. The latter condition is equivalent to \(\{\gamma_\mu, T\} = 0\), so \(T = \lambda \gamma_\mathcal{M}\), for the only operator that anti-commutes with all the \(\gamma\)’s matrices are the multiple of \(\gamma_\mathcal{M}\). By lemma 3.1 \(T\) is selfadjoint - which forces \(\lambda\) to be real - and \(T^2 = \mathbb{I}_\mathcal{M}\), which reduces the choice to \(\lambda = \pm 1\). \(\square\)
5. Order-Zero Condition

Not all the axioms of noncommutative geometry have been adapted to the twisted context. However, those most relevant for physics (i.e. regarding gauge transformations) do make sense for a twisted spectral triple \([19, 20]\). Especially, a real structure for a twisted spectral triple is defined as in the non-twisted case, that is an antilinear operator \(J\) such that
\[
J^2 = \epsilon I,\quad JD = \epsilon' DJ,\quad JT = \epsilon'' TJ
\]
for some \(\epsilon, \epsilon', \epsilon'' \in \{-1, 1\}\) (those three signs defines the \(KO\)-dimension of the triple), which implements a representation \(\pi^0\) of the opposite algebra \(A^\circ\),
\[
\pi^0_\beta(a^\circ) = J\pi_0(a^\ast)J^{-1}
\]
asking to commutes with the one of \(A\), \(J\) defined as in the non-twisted case, that is an antilinear operator
\[
\pi \circ \circ (a^\ast) = J\pi_0(a^\ast)J^{-1}
\]
for some \(\epsilon, \epsilon' \in \{1, -1\}\) (omitting the symbol of representation and denoting with an hat the adjoint action of \(J\)).

The first term is zero by the order zero condition. The second term is zero for any \(\alpha, \beta \in A\).

This is the order zero condition (the first-order condition, discussed in the next section).

The twist-by-grading of a real twisted spectral triple \((A, H, D)\) automatically satisfies the order zero condition, with the same real structure. Namely, if (31) holds for \((A, H, D)\), then for \(\pi\) the representation (6) of \(A \otimes \mathbb{C}^2\) defined by the grading one has
\[
[\pi(a, a'), J\pi(b^\ast, b'^\ast)]J^{-1} = 0 \quad \forall (a, a'), (b, b') \in A \otimes \mathbb{C}^2.
\]

We work out below the conditions such that the same holds true for the representation \(\pi\) (12) induced by the twisting operator.

**Proposition 5.1.** Let \((A, H, D)\) be a real spectral triple with real structure \(J\). If \(A\) is a unital algebra, then the order zero condition (32) for the representation \(\pi\) in (12) holds true if and only if
\[
[T, JTJ^{-1}] = 0 \quad \text{and} \quad [\pi_0(a), JTJ^{-1}] = 0 \quad \forall a \in A.
\]

**Proof.** By easy manipulations, one has
\[
[\pi(a, a'), J\pi(b, b')]J^{-1} = \frac{1}{2} \left( [\pi_0(a) + \frac{1}{2} (1 + T) \pi_0(a')]J^{-1} - [\pi_0(b) + \frac{1}{2} (1 - T) \pi_0(b')]J^{-1} \right),
\]

where we write
\[
\alpha := a + a', \quad \alpha' := a - a', \quad \beta := b + b', \quad \beta' := b - b'.
\]

The order zero condition is equivalent to (34) being zero for any \(\alpha, \alpha', \beta, \beta' \in A\), that is (omitting the symbol of representation and denoting with an hat the adjoint action of \(J\), e.g. \(\hat{T} = JTJ^{-1}\))
\[
[\alpha, \hat{\beta}] = 0, \quad [\alpha, JT\beta J^{-1}] = 0,
\]
\[
[T\alpha, \hat{\beta}] = 0, \quad [T\alpha, JT\beta J^{-1}] = 0 \quad \forall \alpha, \beta \in A.
\]

The first condition (36) is the order zero condition for \((A, H, D)\), so it is always true by hypothesis. The second condition (37) writes
\[
0 = [\alpha, \hat{T} \hat{\beta}] = \hat{T} [\alpha, \hat{\beta}] + [\alpha, \hat{T}] \hat{\beta}.
\]
The first term is zero by the order zero condition. The second term is zero for any \(\alpha, \beta\) if and only if (consider the case \(\beta = \text{the unit of } A\))
\[
[\alpha, \hat{T}] \quad \forall \alpha \in A.
\]

For the same reasons, the first equation (37) written as
\[
0 = T[\alpha, \hat{\beta}] + [T, \hat{\beta}]\alpha
\]
is equivalent to \([T, \hat{\beta}] = 0\) that is, multiplying by \(J^{-1}\) on the left and \(J\) on the right,
\begin{equation}
0 = J^{-1}TJ\beta - \beta J^{-1}TJ = [J^{-1}TJ, \beta].
\end{equation}
Remembering that \(J^{-1} = \epsilon J\), this is equivalent to (39). Remains the second condition (37), which is equivalent to
\begin{equation}
\hat{T}[T\alpha, \hat{\beta}] + [T\alpha, \hat{T}]\hat{\beta} = 0,
\end{equation}
It implies (consider \(\alpha = \beta = 1\_A\))
\begin{equation}
[T, \hat{T}] = 0.
\end{equation}
Therefore the order zero condition implies (39) and (43). To show that these two conditions are sufficient, the only points that remains to show is that they imply the second condition (37), that is (42). The first term of this equation is zero as soon as (39) holds (as shown studying the first term of (37)). The second term is obviously zero as soon as both \(T\) and \(\alpha\) commute with \(\hat{T}\).

For an almost commutative geometry the conditions (33) are equivalent to their restrictions to the finite dimensional spectral triple. To show this, let us absorb the sign ambiguity of proposition 4.2 redefining \(T_F \rightarrow \pm T_F\). The twisting operator (13) of an almost commutative geometry is thus
\begin{equation}
T = \gamma_M \otimes T_F.
\end{equation}
Since \(\gamma_M\) commutes with the representation \(\pi_M\) (8), requiring \(T\) to commute with the representation \(\pi_0\) (10) implies
\begin{equation}
[T_F, \pi_F(m)] = 0 \ \ \forall m \in A_F.
\end{equation}

**Corollary 5.1.1.** The minimal twist of a real, almost commutative, geometry by \(T\) as in (44) satisfies the order zero condition if and only if
\begin{equation}
[T_F, J_F T_F J_F^{-1}] = 0 \ \ \text{and} \ \ [T_F, J_F \pi_F(m) J_F^{-1}] = 0 \ \ \forall m \in A_F.
\end{equation}

**Proof.** Since \(J\gamma_M = \epsilon'' \gamma_M J\), then
\begin{equation}
JTJ^{-1} = J\gamma_M J^{-1} \otimes J_F T_F J_F^{-1} = \epsilon'' \gamma_M \otimes J_F T_F J_F^{-1}.
\end{equation}
The conditions (33) then become
\begin{equation}
\epsilon'' \otimes [T_F, J_F T_F J_F^{-1}] = 0, \ \ \epsilon'' \gamma_M f \otimes [m, J_F T_F J_F^{-1}] = 0 \ \ \forall f \otimes m \in \mathcal{A}.
\end{equation}
This is equivalent to (46) using that \([m, J_F T_F J_F^{-1}] = 0\) is equivalent to \([T_F, J_F m J_F^{-1}] = 0\).

Since \(T_F\) commutes with the representation of \(A_F\) by (45), the two conditions (46) are automatically satisfied if \(T_F\) commutes or anticommutes with \(J_F\), as required for a grading operator. However, by the order zero condition of the initial triple, these conditions together with (45) are also satisfied if \(T_F\) is the representation of any element in the center of \(A_F\), not necessarily a grading. Furthermore these are not the only possibilities: think for instance of a spectral triple obtained by taking a subalgebra \(A_F'\) of a spectral triple \((A_F, H_F, D_F)\). Then any elements in the center of \(A_F\) can be taken as \(T_F\), even if it is not in \(A_F'\).

6. Twisted first order condition

The operator \(T_F\) is further constrained if one takes into account the twisted version of the first order condition:
\begin{equation}
[[D, \pi((a, a'))]_\rho, J\pi_0((b^*, b'^*))J^{-1}]_{\rho^*} = 0 \ \ \forall (a, a'), (b, b') \in A \otimes \mathbb{C}^2
\end{equation}
where \(\rho^*\) is the automorphism of the opposite algebra \((A \otimes \mathbb{C}^2)^o \simeq A^o \oplus A^o\) induced by \(\rho\):
\begin{equation}
\rho^o((a^o, a'^o)) := (\rho^{-1}((a, a'))^o = (a', a)^o = (a'^o, a^o).
\end{equation}
Proposition 6.1. Consider the minimal twist by $T$ of a real spectral triple, such that the order zero condition holds. Then the twisted first-order condition holds if and only if

\[ \{ D, T \}, JTJ^{-1} = 0 \quad \text{and} \quad \{ [D, T], J \pi_0(a) J^{-1} \} = 0 \quad \forall a \in \mathcal{A}. \]

Proof. For the representation (12) and omitting the symbol of representation $\pi_0$, one has

\[ [D, \pi(a, a')]_{\rho} = \frac{1}{2} D \left( (\mathbb{I} + T)a + (\mathbb{I} - T)a' \right) - \frac{1}{2} \left( (\mathbb{I} + T)a' + (\mathbb{I} - T)a \right) D = \frac{1}{2} [D, a+a'] + \frac{1}{2} \{ D, T(a-a') \}, \]

while, denoting $\hat{b} := JbJ^{-1}$ the conjugation by $J$,

\[ J\pi(b, b')J^{-1} = \frac{1}{2} J \left( (\mathbb{I} + T)b + (\mathbb{I} - T)b' \right) J^{-1} = \frac{1}{2} (\hat{b} + \hat{b}') + JT(b-b')J^{-1}. \]

So the twisted commutator (49) (with $b, b'$ instead of $b^*, b'^*\ast$) is the sum of a term of order 0 in $T$,

\[ \frac{1}{4} \left[ [D, a+a'], \hat{b} + \hat{b}' \right] \]

a term of order 1,

\[ \frac{1}{4} \left[ \left\{ D, T(a-a') \right\}, \hat{b} + \hat{b}' \right] \]

and a term of order 2,

\[ \frac{1}{4} \left\{ \left\{ D, T(a-a') \right\}, JT(b-b')J^{-1} \right\}. \]

The term of order 0 is always zero by the first order condition of the initial triple. The first component of the term of order 1 must vanish independently: indeed, for $a = a'$, the second term in (54) as well as the term of order 2 are zero, so the twisted first-order condition reduces to

\[ \left\{ D, \alpha \right\}, \hat{T}\hat{\beta}' = 0 \quad \forall \alpha = a + a', \beta' = b - b'. \]

By the twisted first-order condition of the initial triple, one has

\[ \left\{ [D, \alpha], \hat{T}\hat{\beta}' \right\} = [D, \alpha] \hat{T}\hat{\beta}' + \hat{T}\hat{\beta}' [D, \alpha] = [D, \alpha] \hat{T}\hat{\beta}' + \hat{T}[D, \alpha] \hat{\beta}', \]

\[ = \left\{ D, \hat{T} \right\} \alpha \hat{\beta}' - \alpha \left\{ D, \hat{T} \right\} \hat{\beta}', \]

where the second equation follows developing the commutators, and using that $\hat{T}$ commutes with $\alpha$ by proposition 5.1. In particular, for $\beta = 1_{\mathcal{A}}$ one gets that (56) - hence the twisted first-order condition - implies

\[ \left\{ \left\{ D, \hat{T} \right\}, \alpha \right\} = 0 \quad \forall \alpha \in \mathcal{A}. \]

As well, the term of order 2 must vanish independently: for $a = -a'$, $b' = -1_{\mathcal{A}}$, $b = -2b'$ then the term of order 1 vanishes, so the twisted first-order condition reduces to

\[ \left\{ \left\{ D, T\alpha' \right\}, \hat{T} \right\} = 0 \quad \forall \alpha' = a - a' \in \mathcal{A}. \]

In particular, for $\alpha' = 1_{\mathcal{A}}$, one gets that the twisted first-order condition implies

\[ \left\{ \left\{ D, T \right\}, \hat{T} \right\} = 0. \]

Therefore (59) and (61) are necessary to get the twisted first-order condition. Let us show they are sufficient conditions. If (59) holds, then (58) vanishes for any $\alpha, \hat{\beta}'$, meaning the first component of the term of order 1 vanishes for any $a, a', b, b'$. The same is true for the second component since, with $\beta = \hat{b} + \hat{b}'$, the later writes

\[ \left\{ [D, \alpha'] T, \hat{\beta} \right\} = [D, \alpha'] T \hat{\beta} + \hat{\beta} \left\{ \left\{ D, T \right\}, \hat{\beta} \right\} \]

where we use

\[ \left\{ D, \alpha' \right\} = [D, \alpha'] T + \alpha' \left\{ D, T \right\}. \]
obtained by direct computation, with $T$ commuting with $\alpha'$ by definition of twisting operator. The first commutator in (62) vanishes because $\hat{\beta}$ commutes with both $[D, \alpha']$ (by the first order condition of the initial triple) and with $T$ (by the second equation (33) rewritten as $[T, \hat{\alpha}] = 0$ for all $a \in \mathcal{A}$). The second commutator in (62) vanishes as well since $\hat{\beta}$ commutes with both $\alpha'$ (by the order zero condition of the initial triple) and with $\{D, T\}$ by (59) rewritten as
\[
\{\{D, T\}, \hat{\alpha}\} = 0 \quad \forall a \in \mathcal{A}.
\]
Finally, by (63) the term of order 2 writes
\[
\left\{ \left\{ D, T \alpha' \right\}, \hat{T} \beta' \right\} = \left\{ [D, \alpha']T, \hat{T} \beta' \right\} + \left\{ \alpha' \{D, T\}, \hat{T} \beta' \right\}.
\]
The second anti-commutator vanishes, for $\hat{T} \beta'$ anticommutes with $\{D, T\}$ (by (61) and (64)) but commutes with $\alpha$ (by (33) and the order zero condition). The first anti-commutator vanishes as well, for $\hat{T} \beta'$ commutes with $T$ (by the first equation (33) and the order zero condition) while it anticommutes with $[D, \alpha']T$. The latter assertion follows from the observation that $\hat{T} \beta'$ anticommutes with $[D, \alpha']$, since $\beta'$ commutes with it (by the first order condition of the initial triple) while $T$ anti-commutes with $[D, \alpha']$, as can be seen from (64) noticing that
\[
\left\{ \{D, T\}, \hat{\alpha}\right\} = DT\hat{\alpha} + TD\hat{\alpha} - \hat{\alpha}DT - \hat{\alpha}TD,
\]
\[
= [D, \hat{\alpha}]T + T[D, \hat{\alpha}] = \{[D, \hat{\alpha}], T\}.
\]
So (61) and (59) (or equivalently (64)) are equivalent with the twisted first-order condition. Hence the result. \[\square\]

For an almost commutative geometry, these conditions are equivalent to their restriction of the finite dimensional space.

**Corollary 6.1.1.** Consider the minimal twist of a real, almost commutative, geometry by $T = \gamma_M \otimes T_F$ such that the order zero condition holds. Then the twisted first-order condition holds if and only if
\[
\{\{D_F, T_F\}, J_F T_F J^{-1}_F\} = 0 \quad \text{and} \quad \{\{D_F, T_F\}, J_F \pi_F(m) J^{-1}_F\} = 0 \quad \forall m \in \mathcal{A}_F.
\]

**Proof.** For $D$ as in (9) and $T = \gamma_M \otimes T_F$, one has
\[
\{D, T\} = \{\gamma_M \otimes D_F, \gamma_M \otimes T_F\} = I \otimes \{D_F, T_F\},
\]
so the second equation (51) is equivalent to $\{\{D_F, T_F\}, J_F T_F J^{-1}_F\} = 0$. The first equation (51) reads
\[
0 = \{I \otimes \{D_F, T_F\}, J \gamma_M J^{-1} \otimes J_F T_F J^{-1}_F\} = J \gamma_M J^{-1} \otimes \{\{D_F, T_F\}, J_F T_F J^{-1}_F\}
\]
which is equivalent to the first equation (68). \[\square\]

These conditions are automatically satisfied if $T_F$ anticommutes with $D_F$, that is if the twisting operator $T$ is a grading. But this may not be the only possibility.

**7. Conclusion**

In recent time several modifications of the framework of noncommutative geometry have been proposed to get extra scalar fields, beyond the Standard Model: removing the first order condition \cite{8, 7}, twisting the real structure \cite{4, 21, 14, 17}, introducing non-associativity \cite{3}, working in the framework of geometric background \cite{2, 1} (see \cite{9} for a recent review). The relation of some of these procedures with the minimal twist presented here have been investigated in \cite{22} regarding the removal of the first order condition \cite{22}, and in \cite{5} regarding the twisting of the real structure (see also \cite{18}).

The main result of this note is that the twist-by-grading is not the only possibility for minimally twisting the Standard Model. It is true that the minimal twist of an almost commutative geometry by a twisting operator of the form $T \otimes T_F$ is a twisted spectral triple if and only if
\( T = \gamma_M \). However, \( T_F \) does not need to be a grading of the finite dimensional space. If one requires the order zero condition, then
\[
[T_F, J_F T_F J_F^{-1}] = 0 \quad \text{and} \quad [T_F, J_F \pi_F(m) J_F^{-1}] = 0 \quad \forall m \in \mathcal{A}_F,
\]
and if one requires the twisted first-order condition, then
\[
\{\{D_F, T_F\}, J_F T_F J_F^{-1}\} = 0 \quad \text{and} \quad \{\{D_F, T_F\}, J_F \pi_F(m) J_F^{-1}\} = 0 \quad \forall m \in \mathcal{A}_F.
\]
These conditions hold in more generality, for the minimal twist of an arbitrary (real) spectral triple, as shown in Propositions 5.1 and 6.1. Classifying all the solutions of these constraints for the spectral triple of the Standard Model will be the object of a future work.

References

[1] F. Besnard. Extensions of the noncommutative standard model and the weak order one condition. arXiv 2011.02708, 2021.
[2] F. Besnard. A U(1)-BL extension of the Standard Model from noncommutative geometry. J. Math. Phys., 62:012301, 2021.
[3] L. Boyle and S. Farnsworth. The standard model, the Pati-Salam model, and Jordan geometry. New J. Phys., 22(arXiv: 1910.11888):073023, 2020.
[4] T. Brzeziński, N. Ciccoli, L. Dabrowski, and A. Sitarz. Twisted reality condition for Dirac operators. Math. Phys. Anal. Geo., 19(3:16), 2016.
[5] T. Brzeziński, L. Dabrowski, and A. Sitarz. On twisted reality conditions. Lett. Math. Phys., 109(3):643–659, 04 2019.
[6] A. H. Chamseddine, A. Connes, and M. Marcoli. Gravity and the standard model with neutrino mixing. Adv. Theor. Math. Phys., 11:991–1089, 2007.
[7] A. H. Chamseddine, A. Connes, and W. van Suijlekom. Beyond the spectral standard model: emergence of Pati-Salam unification. JHEP, 11:132, 2013.
[8] A. H. Chamseddine, A. Connes, and W. van Suijlekom. Inner fluctuations in noncommutative geometry without first order condition. J. Geom. Phy., 73:222–234, 2013.
[9] A. H. Chamseddine and W. D. van Suijlekom. A survey of spectral models of gravity coupled to matter. In: Chamseddine A., Consani C., Higson N., Khalkhali M., Moscovici H., Yu G. (eds) Advances in Noncommutative Geometry. Springer, pages 1–51, 2019.
[10] A. Connes. Noncommutative Geometry. Academic Press, 1994.
[11] A. Connes. Gravity coupled with matter and the foundations of noncommutative geometry. Commun. Math. Phys., 182:155–176, 1996.
[12] A. Connes. On the spectral characterization of manifolds. J. Noncom. Geom., 7(1):1–82, 2013.
[13] A. Connes and H. Moscovici. Type III and spectral triples. Traces in number theory, geometry and quantum fields, Aspects Math. Friedt. Vieweg, Wiesbaden, E38:57–71, 2008.
[14] L. Dabrowski, F. D’Andrea, and A. M. Magee. Twisted reality and the second-order condition. Math. Phys. Anal. Geo., 24(13), 12 2021.
[15] A. Devastato, F. Lizzi, and P. Martinetti. Grand Symmetry, Spectral Action and the Higgs mass. JHEP, 01:042, 2014.
[16] A. Devastato and P. Martinetti. Twisted spectral triple for the standard model and spontaneous breaking of the grand symmetry. Math. Phys. Anal. Geo., 20(2):43, 2017.
[17] M. Dubois-Violette, J. Madore, and R. Kern. Classical bosons in a noncommutative geometry. Class. Quantum Grav., 6:1709, 1989.
[18] M. Goffeng, B. Mesland, and A. Rennie. Untwisting twisted spectral triples. International Journal of Mathematics, 30(14), 03 2019.
[19] G. Landi and P. Martinetti. On twisting real spectral triples by algebra automorphisms. Lett. Math. Phys., 106:1499–1530, 2016.
[20] G. Landi and P. Martinetti. Gauge transformations for twisted spectral triples. Lett. Math. Phys., 108:2589–2626, 2018.
[21] A. M. Magee and L. Dabrowski. Gauge transformations of spectral triples with twisted real structures. J. Math. Phys., 62:083502, 2020.
[22] P. Martinetti and J. Zanchettin. Twisted spectral triples without the first order condition. arXiv 2103.15643, 2021.