THE RIEMANN-ROCH THEOREM FOR GRAPHS AND THE RANK
IN COMPLETE GRAPHS

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ABSTRACT. The paper by M. Baker and S. Norine in 2007 introduced a new parameter on configurations of graphs and gave a new result in the theory of graphs which has an algebraic geometry flavour. This result was called Riemann-Roch formula for graphs since it defines a combinatorial version of divisors and their ranks in terms of configuration on graphs. The so called chip firing game on graphs and the sandpile model in physics play a central role in this theory. In this paper we give a presentation of the theorem of Baker and Norine in purely combinatorial terms, which is more accessible and shorter than the original one. An algorithm for the determination of the rank of configurations is also given for the complete graph $K_n$. This algorithm has linear arithmetic complexity. The analysis of number of iterations in a less optimized version of this algorithm leads to an apparently new parameter which we call the prerank. This parameter and the classical area parameter provide an alternative description to some well known $q,t$-Catalan numbers. Restricted to a natural subset of configurations, the two natural statistics degree and rank in Riemann-Roch formula lead to a distribution which is described by a generating function which, up to a change of variables, is a symmetric fraction involving two copies of Carlitz $q$-analogue of the Catalan numbers.

We consider the following solitary game on an undirected connected (non oriented) graph $G = (X,E)$ with no loops: at the beginning integer values $u_i$ are attributed to the $n$ vertices $x_1, x_2, \ldots x_n$ of the graph, these values can be positive or negative. At each step a toppling can be performed by the player on a vertex $x_i$, it consists in subtracting $d_i$ (the number of neighbors of $x_i$) to the amount $u_i$ and adding 1 to all the amounts $u_j$ of the neighbors $x_j$ of $x_i$. In this operation the amount of vertex $x_i$ may become negative. The aim of the player is to find a sequence of toppling operations which will end with a configuration where all the $u_i$ are non negative. Since the sum of the $u_i$ is invariant by toppling, a necessary condition to succeed is that in the initial configuration this sum should be non negative. We will see that this condition is not sufficient.

This game has much to do with the chip firing game (see [4], [3]) and the sandpile model (see [1], [8], [9]), for which recurrent configurations were defined and proved to be canonical representatives of the classes of configurations equivalent by a sequence of topplings, (for a more algebraic treatment see also [20]).

The game was introduced and studied in detail by Baker and Norine ([2]) who introduced a new parameter on graph configurations: the rank. One characteristic of the rank $\rho(u)$ of a configuration $u$ is that it is non negative if and only if one can get from $u$ a positive configuration by performing a sequence of topplings. For this parameter they

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obtain a simple formula expressing a symmetry similar to the Riemann-Roch formula for surfaces (a classical reference to this formula is the book by [13]).

Our aim here is to give a simple presentation of this Riemann-Roch like theorem and study the values of this parameter when \( G \) is the complete graph on \( n \) vertices. For these graphs, it was noticed (see Proposition 2.8. in [6]) that the recurrent configurations correspond to the parking functions which play a central role in combinatorics. We obtain a simple algorithm to compute the rank in that case, of linear complexity, while there is no known polynomial time algorithm to compute that rank for arbitrary graphs (see [18]). This algorithm suggests the introduction of a new parameter on parking functions and Dyck paths which we call prerank for which we study some properties in relation with other known parameters on Dyck paths. The degree is the other statistic on configuration involved by Baker and Norine theorem. The restriction to sorted parking configurations select exactly one representent of each possible run of the algorithm up to the symmetries of the complete graph with a pointed vertex. We prove that the generating function of sorted parking configurations according to their degree, their rank and the size of complete graph, is, up to a change of variable, a symmetric fraction in two copies of Carlitz q-analogue of Catalan’s numbers.

1. Configurations on a graph

Let \( G = (X, E) \) be a multi-graph with \( n \) vertices, where \( X = \{x_1, x_2, \ldots, x_n\} \) is the vertex set and \( E \) is a symmetric matrix such that \( e_{i,j} \) is the number of edges with endpoints \( x_i, x_j \), hence \( e_{i,j} = e_{j,i} \). In all this paper \( n \) denotes the number of vertices of the graph \( G \) and \( m \) the number of its edges. We suppose that \( G \) is connected and has no loops, so that \( e_{i,i} = 0 \) for all \( i \).

We will consider configurations on a graph, these are elements of the discrete lattice \( \mathbb{Z}^n \). Each configuration \( u \) may be considered as assigning (positive or negative) tokens to the vertices. When there is no possibility of confusion the symbol \( x_i \) will also denote the configuration in which the value 1 is assigned to vertex \( x_i \) and the value 0 is assigned to all other vertices.

The degree of the configuration \( u \) is the sum of the \( u_i \)'s it is denoted by \( \text{deg}(u) \).

1.1. The Laplacian configurations. These configurations correspond to the rows of the Laplacian matrix of a graph, a classical tool in Algebraic Graph Theory.

The Laplacian configuration \( \Delta^{(i)} \) is given by: \( \Delta^{(i)} = d_i x_i - \sum_{j=1}^{n} e_{i,j} x_j \), where \( d_i = \sum_{j=1}^{n} e_{i,j} \) is the degree of the vertex \( x_i \). This \( n \) configurations which degrees are 0 play a central role throughout this paper.

We denote by \( L_G \) the subgroup of \( \mathbb{Z}^n \) generated by the \( \Delta^{(i)} \), and two configurations \( u \) and \( v \) will be said toppling equivalent if \( u - v \in L_G \), which will also be written as \( u \sim_{L_G} v \).

In the sandpile model, the transition from configuration \( u \) to the configuration \( u - \Delta^{(i)} \) is allowed only if \( u_i \geq d_i \) and is called a toppling, it is called a firing in the theory of chip firing games.

Notice that \( \sum_{i=1}^{n} \Delta^{(i)} = 0 \) and that for a connected graph this is the unique relation satisfied by the \( \Delta^{(i)} \), moreover the principal minors of the Laplacian matrix are all equal to the number of spanning trees of the graph.
1.2. **Recurrent configurations.** We use here the notation usually considered in the sandpile model, so that we will call *sandpile configuration* a configuration \( u \) such that \( u_i \geq 0 \) for all \( i < n \). This corresponds to the fact that in the sandpile model the vertex \( x_n \) is considered as a sink collecting tokens, so that the number of tokens of the sink is not considered there.

**Definition 1.** In the sandpile model a toppling on vertex \( x_i \), where \( i \neq n \) may occur in a sandpile configuration only if \( u_i \geq d_i \). A sandpile configuration is stable if no toppling can occur, that is \( u_i < d_i \) for all \( i < n \).

Notice that when a toppling occurs in the sandpile model, the configuration \( u - \Delta^{(i)} \) is also a sandpile configuration.

The toppling operation for a sandpile configuration will be denoted by \( u \rightarrow v \). We also write:

\[
u \ast v
\]

if \( u \) and \( v \) are sandpile configurations and if \( v \) is obtained from \( u \) by a sequence of toppling operations. Notice that \( u \ast v \) implies \( u \sim_{L_G} v \).

Sequences of topplings may be performed in any order until a stable configuration is attained as the following proposition states, the proof of which may be found in [10] or in [17] pages 42 and 70.

**Proposition 1.** For any sandpile configuration \( u \) there exists a unique stable configuration such \( u' \) that \( u \ast u' \).

A configuration is *recurrent* in an evolving system if it could be observed after a long period of the evolution of the system. In the case of the sandpile model, the system is considered to evolve by adding a token in any cell at random and then applying topplings until a stable configuration is reached. This translates into the following notion which is central:

**Definition 2.** A configuration \( u \) is recurrent if it is stable and there exists a sandpile configuration \( v \neq 0 \) such that \( u + v \ast u \).

The following important result, giving canonical representatives in the classes of the relation \( \sim_{L_G} \) is obtained in [7, 3, 6] by different ways.

**Theorem 1.** For any configuration \( u \) there exists a unique recurrent configuration \( v \) such that \( u \sim_{L_G} v \).

In order to characterize the recurrent configurations D. Dhar used the configuration \( \Delta^{(n)} \) and proposed the following algorithm.

**Theorem 2. Burning Algorithm.** The stable configuration \( u \) is recurrent if and only if

\[
u - \Delta^{(n)} \ast u
\]

Moreover in this sequence of topplings each vertex topples exactly once.

This algorithm can be translated into a characterization, giving:
Corollary 1. A stable configuration \( u \) is recurrent if and only if for any subset \( Y \) of \( X \setminus \{x_n\} \) there is at least an \( x_k \) in \( Y \) such that its degree in the subgraph spanned by \( Y \) is greater than or equal to \( u_k \), more precisely if the following condition is satisfied:

\[
    u_k \geq \sum_{x_i \in Y} e_{i,k}
\]  

(1.1)

Proof Let \( u \) be a recurrent configuration, and \( Y \) be a subset of \( X \), then by Dhar's Burning Algorithm, starting from the configuration \( u - \Delta^{(n)} \) there is a sequence of topplings of the vertices in which any vertex topples. We may suppose that the vertices are numbered in the order in which they topple, \( x_1 \) just after \( x_n \), then \( x_2 \) and so on until \( x_{n-1} \) then for allowing a toppling at vertex \( x_i \) each \( u_i \) has to satisfies the condition:

\[
    d_i \leq u_i + \sum_{j=1}^{i-1} e_{i,j}
\]

Now for any subset \( Y \) of \( X \), let \( k \) be the smallest integer such that \( x_k \in Y \), then since there is no \( x_i \in Y \) with \( i \) less than \( k \) we have:

\[
    d_k \geq \sum_{j=1}^{k-1} e_{j,k} + \sum_{x_i \in Y} e_{i,k}
\]

Putting \( i = k \) in the first inequality and the two inequalities together gives the result.

Conversely if \( u \) is a stable configuration satisfying condition 1.1, we build a toppling sequence starting with vertex \( x_n \), then taking as \( x_1 \) the vertex in \( Y = \{x_1, x_2, \ldots, x_{n-1}\} \) satisfying \( u_1 \geq \sum_{i=2}^{n-1} e_{1,i} \), this vertex can topple after \( x_n \) has since in that case \( u_1 + e_{1,n} \geq \sum_{i=2}^{n} e_{1,i} = d_1 \). Then at each step, a vertex \( x_j \) such that \( u_j \geq \sum_{i=j+1}^{n-1} e_{i,j} \) exists taking \( Y = X \setminus \{x_n, x_1, x_2, \ldots x_{j-1}\} \), this vertex can topple at this stage. We have thus built a sequence of topplings proving that \( u \) is recurrent. \( \square \)

1.3. Parking configurations. We consider a kind of dual notion to that of recurrent configuration, such configurations are often called parking configurations since in the case of complete graphs, these are exactly the parking functions, a central object in combinatorics.

Definition 3. A sandpile configuration \( u \) on a graph \( G \) is a parking configuration if for any subset \( Y \) of \( \{x_1, x_2, \ldots, x_{n-1}\} \) there is a vertex \( x_k \) in \( Y \) such that \( u_k \) is less than the number of edges which incident to \( x_k \) and a vertex out of \( Y \). More precisely if the exists \( x_k \in Y \) such that \( u_k < \sum_{x_i \not\in Y} e_{i,k} \).

In other words a sandpile configuration \( u \) is a parking configuration if and only if there is no toppling of all the vertices in a subset \( Y \) of \( \{x_1, x_2, \ldots x_{n-1}\} \) leaving all the \( u_i \geq 0 \). For this reason these configurations are also called superstable (see [20]).

Proposition 2. Let \( u \) be a stable configuration and let \( \delta \) be the configuration such that \( \delta_i = d_i - 1 \). Define \( \beta(u) = \delta - u \). Then \( u \) is recurrent if and only if \( \beta(u) \) is a parking configuration.
Proof. It suffices to compare Corollary 1 and Definition 3 and to notice that:

\[ d_k = \sum_{x_j \notin Y} e_{k,j} + \sum_{x_j \in Y} e_{k,j} \]

□

Corollary 2. For any configuration \( u \) there exists a unique parking configuration \( v \) such that \( u \sim_{LG} v \).

Proof. For any configuration \( u \) let \( v \) be the recurrent configuration such that \( v \sim_{LG} \delta - u \) then \( \beta(v) \) is a parking configuration such that \( u \sim_{LG} \beta(v) \). □

In this paper we will often consider the parking configuration in a class as a representative of this class. A parking configuration \( u \) in a graph with \( n \) vertices will be written as \( u = (u, s) \), where \( u \) is sequence of length \( n \) and \( s \) is an integer such that:

\[ u = (u_1, u_2, \ldots, u_{n-1}) \quad s = u_n \quad (1.2) \]

and an integer \( s \) representing the number of tokens in \( x_n \).

Parking configurations and acyclic orientations. An orientation of \( G \) is a directed graph obtained from \( G \) by orienting each edge, so that one end vertex is called the head and the other vertex is called the tail. A directed path in such a graph consists of a sequence of edges such that the head of an edge is equal to the tail of the subsequent one.

The orientation is acyclic if there is no directed circuit, i.e. a directed path starting and ending at the same vertex. We associate to any parking configuration \( u \) an acyclic orientation by:

Proposition 3. For any parking configuration \( u \) on \( G = (X, E) \) there exists at least one acyclic orientation \( \overrightarrow{G} \) such that for any vertex \( x_i, i \neq n \), \( u_i \) is strictly less than its indegree \( d_i^- \) (i.e. the number of edges with head \( x_i \)).

Proof. We orient the edges using an algorithm that terminates after \( n \) steps. Consider \( Y = \{x_1, x_2, \ldots, x_{n-1}\} \). From the definition of parking configurations, there is one vertex \( x_i \) such that \( u_i < e_{i,n} \) then orient all these \( e_{i,n} \) edges from \( x_n \) to \( x_i \), and remove \( x_i \) from \( Y \). Repeat the following operation until \( Y \) is empty:

- Find \( x_k \) in \( Y \) such that \( u_k < \sum_{x_j \notin Y} e_{k,j} \); orient all the edges joining any vertex \( j \) outside \( Y \) to \( x_k \) from \( x_j \) to \( x_k \) and remove \( x_k \) from \( Y \).

□

2. Configurations on the complete graph

2.1. Configuration classes in \( K_n \). In the complete graph \( K_n \) each of the \( n \) vertices has all the \( n - 1 \) other vertices as neighbors.

For a configuration \( u \) in the complete graph \( K_n \) the determination of the parking configuration equivalent to it is facilitated by the following lemma and its corollaries:
Lemma 1. A configuration $u$ of $K_n$ is toppling equivalent to 0 if and only if the two following conditions are satisfied:

$$\deg(u) = 0 \quad \text{and} \quad \forall i, j \leq n \; u_i = u_j \pmod{n}$$

Proof. The necessary condition follows from the fact that these relations are not modified by any toppling and are satisfied by the parking configuration equivalent to 0 which is equal to $(0, 0, \ldots, 0, 0)$.

The sufficient condition is obtained by induction on the sum of the $|u_i|$. More precisely, for a given configuration $u$ satisfying the conditions above, let $i$ be such that $|u_i|$ is maximal, replacing $u$ by $-u$ if necessary, allows to assume $u_i > 0$. Consider $v = u - \Delta^{(i)}$, this configuration satisfies the conditions of the Lemma; then proving that $\sum_i |v_i| < \sum_i |u_i|$ will allow to use the inductive hypothesis and obtain $v \sim_{LG} 0$ and hence $u = v + \Delta^{(i)} \sim_{LG} 0$.

For this aim we consider two cases:

- If $u_i \geq n$ then $|v_i| = |u_i| - n + 1$ moreover since $\deg(u) = 0$ there exists at least one $k$ such that $u_k < 0$ and $|v_k| = |u_k + 1| = |u_k| - 1$ giving:

$$\sum_j |v_j| \leq \sum_j |u_j| - 2$$

- If $u_j < n$ then there are $k_1$ of the $j$ such that $u_j = u_i > 0$ and $n - k_1$ such that $u_j = u_i - n < 0$. For a $j$ such that $j \neq i$ and $u_j = u_i$ we have $v_j = u_j + 1$ and for the others $j \neq i$ we have $|v_j| = |u_j| - 1$. Using the maximality of $|u_i|$ we have also $|v_i| = n - 1 - |u_i|$. This gives similarly $\sum_j |v_j| \leq \sum_j |u_j| - 2$. Which ends the proof.

Notice that this characterization may also be considered as given by the toppling invariants defined in [11, equation(3.11)].

Using the fact that $u \sim_{LG} v$ if and only if $u - v \sim_{LG} 0$ we obtain:

Corollary 3. Two configurations $u$ and $v$ are toppling equivalent in $K_n$ if and only if the following holds

$$\deg(u) = \deg(v) \quad \text{and for any} \; 1 \leq i, j \leq n, \; u_i - u_j = v_i - v_j \pmod{n}.$$ 

Another consequence is the following:

Corollary 4. Any class of the toppling equivalence on $K_n$ contains exactly $n$ sandpile configurations such that $0 \leq u_i < n$ for all $i < n$, all the stable configurations of the class are among them.

Proof. For each class of the toppling equivalence there is a parking configuration $u$ which clearly satisfies the condition. Now for any $k = 1, 2, \ldots, n - 1$ the configuration $u^{(k)}$ given by

$$u_i^{(k)} = u_i + k \pmod{n}$$

for $i < n$ and $u_n^{(k)} = \deg(u) - \sum_{i=1}^{n-1} u_i^{(k)}$, is a configuration equivalent to $u$ and all these configurations are distinct.
Conversely a configuration \( v \) satisfying \( v_i \leq n - 1 \) for all \( i < n \) equivalent to \( u \) is equal to one of the \( u^{(k)} \), where \( k = u_1 - u_1^{(k)} \pmod{n} \)

As a direct consequence of Corollary 3 we have an efficient algorithm computing from any configuration of \( K_n \) an equivalent configuration with relatively small values of the \( u_i \) for \( i < n \):

**Algorithm.** Given a configuration \( u \) one can find a configuration \( v \) toppling equivalent to \( u \) and such that \( 0 \leq v_i < n \) for any \( 1 \leq i \leq n - 1 \) by setting:

\[
v_i = u_i - u_1 \pmod{n} \text{ for } i < n, \quad \text{and } \quad v_n = \deg(u) - \sum_{i=1}^{n-1} v_i.
\]

Notice that this algorithm performs \( O(n) \) arithmetic operations on integers.

### 2.2. Parking configurations.

Since in the complete graph \( K_n \) the \( n \) vertices have all the \( n - 1 \) other vertices as neighbors, a configuration \( u \) is a parking configuration if for any subset \( Y \) of \( \{x_1, x_2 \cdots, x_{n-1}\} \) containing \( p \) vertices there is at least one vertex \( x_k \in Y \) such that \( u_k < n - p \). The sequences \( u_1, u_2, \ldots, u_{n-1} \) satisfying this condition are well known in combinatorics and are called parking functions, they are characterized by the simpler following condition:

**Proposition 4.** A sequence \( (u_1, u_2, \ldots, u_{n-1}) \) corresponds to the first \( n - 1 \) values of a parking configuration of \( K_n \), if and only if after reordering it as a weakly increasing sequence \( (v_1, v_2, \ldots, v_n) \) one has \( v_i < i \), for all \( i = 1, \ldots, n - 1 \).

From a configuration \( v \) such that for \( u_i < n \) for each \( i < n \), one builds the parking configuration equivalent to \( u \) by using the following notion of exceedence:

**Definition 4.** Let \( v \) be a configuration such that \( 0 \leq v_i < n \) for \( 1 \leq i < n \), for each integer \( j \) such that \( 1 \leq j < n \) we define the exceedence \( e_j \) by:

\[
e_j = |\{i|1 \leq i < n, v_i < j\}| - j
\]

Notice that \( v \) is a parking configuration if and only if none of the \( e_j \) is a negative integer, moreover we have:

**Proposition 5.** Let \( v \) be as above and let \( k \) be the minimal index such that

\[
e_k = \min_{1 \leq j < n} e_j
\]

If \( e_k \geq 0 \) then \( v \) is a parking configuration, else the configuration \( v' \) defined below is the parking configuration equivalent to \( v \)

\[
v'_i = \begin{cases} v_i - k \text{ if } i < n \text{ and } v_i \geq k \\ v_i + n - k \text{ if } i < n \text{ and } v_i < k \\ \deg(v) - \sum_{i=1}^{n-1} v'_i \text{ if } i = n \end{cases}
\]
The fact that \( v \sim_{L_G} v' \) follows directly from Corollary 3 since we have that \( u_i - v_i \) is equal either to \( k \) or to \( k - n \). It is also possible to prove directly that \( v' \) is a parking configuration, we prefer however to introduce the terminology of words and to show that this result is a consequence of the well known Cyclic Lemma (see \([12]\)). This is the aim of the next section.

2.3. Dyck words. We consider words on the alphabet with two letters \( \{a, b\} \). We denote \( |f|_x \) (where \( x \in \{a, b\} \)) the number of occurrences of the letter \( x \) in the word \( f \). The word \( g \) is a prefix of \( f \) if \( f = gh \).

For any sequence \((u_1, u_2, \ldots, u_{n-1})\) such that \( u_i \leq u_{i+1} \) for \( 1 \leq i < n - 1 \), satisfying \( u_1 \geq 0 \) and \( u_{n-1} < n \) we associate bijectively the word \( \phi(u) \) containing \( n - 1 \) occurrences of \( a \) and \( n \) occurrences of \( b \) such that the \( i \)-th occurrence of \( a \) in \( \phi(u) \) is preceded by exactly \( u_i \) occurrences of \( b \), or equivalently the prefix of \( \phi(u) \) ending with the \( i \)-th occurrence of the letter \( a \) contains exactly \( u_i \) occurrences of the letter \( b \). For a configuration \((u_1, u_2, \ldots, u_{n-1})\) such that \( 0 \leq u_i < n \) for all \( i \) we define \( \phi(u) \) as equal to \( \phi(v) \), where \( v \) is the sequence obtained from \( u \) by reordering it in (weekly) increasing order. Notice that \( \phi(u) \) has length \( 2n - 1 \) and ends with an occurrence of \( b \). Recall that a Dyck word \( f \) is a word having an equal number of occurrences of letters \( a \) and \( b \) and such that any prefix of it has no more occurrences of the letter \( b \) than that of the letter \( a \).

**Lemma 2.** Let \( u \) be a configuration of \( K_n \) such that \( u_i < n \) for all \( i < n \) and let \( f = \phi(u) \), then the exceedences \( e_j \) of \( u \) may be read on the word \( f \) in the following way, let \( f^{(j)} \) be the prefix of \( f \) ending with the \( j \)-th occurrence of \( b \) then:

\[
e_j = |f^{(j)}|_a - |f^{(j)}|_b
\]

Moreover \( f \) is a Dyck word followed by an occurrence of \( b \) if and only if \( u \) is a parking configuration.

**Proof** The number occurrences of \( a \) in \( f^{(j)} \), for \( j = 1, 2, \ldots, n \) is clearly equal to the number of \( i \) such that \( u_i < j \), hence the definition of the exceedence gives:

\[
e_j = |\{i|1 \leq i < n, u_i < j\}| - j = |f^{(j)}|_a - j = |f^{(j)}|_a - |f^{(j)}|_b
\]

Moreover, \( f \) is a Dyck word followed by an occurrence of \( b \) if and only if \(|f^{(j)}|_a - |f^{(j)}|_b| \geq 0 \) for all \( j < n \), which is equivalent to \( e_j \geq 0 \) and hence the fact that \( u \) is a parking configuration.

\( \square \)

The classical Cyclic Lemma may be stated as follows:

**Lemma 3.** Any word \( f \) with \( n \) occurrences of \( b \) and \( n - 1 \) occurrences of \( a \) admits a unique factorization

\[
f = gh
\]
such that \( hg \) is a Dyck word, moreover \( gb \) is the shortest prefix of \( f \) such that \(|gb|_a - |gb|_b| \) is minimal.

Using this Lemma we are now able to return to the proof of Proposition 5.

**Proof** (of Proposition 5)
Let \( v \) be a configuration satisfying the conditions stated in Definition 4 and set \( f = \phi(u) \). Consider the decomposition \( f = gbh \) given by the Cyclic Lemma and let \( k = |g|_b \). Then examine the values of \(|h'|_a - |h'|_b\) for the prefixes \( h' \) of \( h \) ending with an occurrence of \( b \): these are equal to \(|gbh'|_a - |gbh'|_b - (|gb|_a - |gb|_b)| \), that is \(|gbh'|_a - |gbh'|_b - n + k| \). This shows that in \( hgb \) these values for prefixes of \( h \) correspond to the exceedences \( e_i \) of \( v' \) for \( i < k \). A similar argument shows that the values for the prefixes \( hg' \) correspond to the exceedences of \( e_i \) for \( i \geq k \). So that \( \phi(v') = hgb \); since \( hg \) is a Dyck word these values are non negative and so are the exceedences, showing that \( v' \) is a parking configuration.

\[ \square \]

### 3. Effective configurations

We come back in this section with general graphs \( G \) not necessarily equal to a complete graph, define the notion of effective configuration and recall the main results of [2], the proofs we give in this section are more or less a reformulation in our terms of the proofs given in [2]. The game described in the introduction can be translated in determining if a configuration is effective with the following definition of effectiveness:

**Definition 5.** A configuration \( u \) is positive if \( u_i \geq 0 \) for all \( i \). A configuration \( u \) is effective if there exists a positive configuration \( v \) toppling equivalent to \( u \) that is such that \( u - v \in L_G \).

Since two equivalent configurations by \( \sim_{L_G} \) have the same degree, it is clear that a configuration with negative degree is not effective. However we will prove that configurations with positive degree are not necessarily effective as these two examples show:
3.1. Configuration associated to an acyclic orientation of $G$. As already seen in Section 1.3 an orientation of $G$ is a directed graph obtained from $G$ by orienting each edge, that is distinguishing for each edge with end points $x_i$ and $x_j$ which one is the head and the other being the tail. The orientation is acyclic if there is no directed circuit. Let $\overrightarrow{G}$ be an acyclic orientation of $G$, we define the configuration $u_{\overrightarrow{G}}$ by:

$$(u_{\overrightarrow{G}})_{i} = d_i - 1$$

Where $d_i$ is the number of edges which have head $x_i$. The configuration represented in the right of Figure 1 is equal to $u_{\overrightarrow{G}}$ for the orientation of $G$ represented in Figure 2.

Proposition 6. The configuration associated to an acyclic orientation of $G$ is non effective.

Proof We will show that for any linear combination $v = \sum_{i=1}^{n} a_i \Delta^{(i)}$ the sum $w$ of $v$ and $u_{\overrightarrow{G}}$ is not a positive configuration. Let $\varepsilon_{i,j}$ denote the number of edges with head $x_j$ and tail $x_i$. Then $\varepsilon_{i,j} = \varepsilon_{i,j} + \varepsilon_{j,i}$ (but notice that since the orientation is acyclic one of the two values in the sum above is equal to 0).
For any vertex \( x_i \) of \( G \) we have \( d_i^- = \sum_{j=1}^{n} \varepsilon_{j,i} \) so that:

\[
    w_i = -1 + \sum_{j=1}^{n} \varepsilon_{j,i} + a_i d_i - \sum_{j=1}^{n} a_j \varepsilon_{i,j}
\]

Using \( d_i = \sum_{j=1}^{n} \varepsilon_{j,i} \) and decomposing each \( \varepsilon_{i,j} \) into \( \varepsilon_{i,j} + \varepsilon_{j,i} \) gives:

\[
    w_i = -1 + \sum_{j=1}^{n} \varepsilon_{j,i} + a_i \sum_{j=1}^{n} (\varepsilon_{i,j} + \varepsilon_{j,i}) - \sum_{j=1}^{n} a_j (\varepsilon_{i,j} + \varepsilon_{j,i})
\]

(3.1)

Giving:

\[
    w_i = -1 + \sum_{j=1}^{n} (1 + a_i - a_j) \varepsilon_{j,i} + \sum_{j=1}^{n} (a_i - a_j) \varepsilon_{i,j}
\]

(3.2)

If there is a unique minimal value, say \( a_k \) among the \( a_i \), that is \( a_k < a_i \) for all \( i \neq k \) then since the \( a_i \) are integers \( 1 + a_k - a_j \leq 0 \) and \( w_k < 0 \).

If there are many \( a_i \)'s attaining the minimal value take \( k \) be such that \( a_k \) be among them and \( \varepsilon_{j,k} = 0 \) for all the other minima \( j \), the existence of such a \( k \) follows from the acyclicity of \( \overrightarrow{G} \). Then for this \( k \) we have \( w_k < 0 \).

\( \square \)

3.2. Characterisation of effective configurations. The following Theorem is the central result in [2].

**Theorem 3.** A configuration \( u \) is effective if and only if the parking configuration \( v = parking(u) \) equivalent to \( u \) is such that \( v_n \geq 0 \). Moreover for any configuration \( u \) one and only one of the following assertions is satisfied:

1. \( u \) is effective
2. There exists an acyclic orientation \( \overrightarrow{G} \) such that \( u \overrightarrow{G} - u \) is effective.

**Proof** Let \( u \) be a configuration and let \( v = parking(u) \) be the parking configuration in its class. If \( v_n \geq 0 \) then \( u \) is effective since it is equivalent to \( v \) which is positive. If \( v_n < 0 \) then the acyclic orientation \( \overrightarrow{G} \) of \( G \) given by Propostion 3 is such that the in degree \( d_i^- \) of each vertex \( x_i \) except \( x_n \) in \( \overrightarrow{G} \) satisfies \( v_i < d_i^- \), since \( (u \overrightarrow{G})_n \geq -1 \) we have \( v \leq u \overrightarrow{G} \) proving that \( v \) is not effective, so is \( u \) since \( u \sim_{L_G} v \).

Let \( u \) be non effective, consider the parking configuration \( v \) equivalent to \( u \) and the acyclic orientation given by Proposition 3 let \( w = u \overrightarrow{G} - v \), then for \( i \neq n \) we have

\[
    w_i = d_i^- - 1 - u_i \geq 0
\]

And since \( v_n < 0 \):

\[
    w_n = -1 + v_n \geq 0
\]

Hence since \( u \) and \( v \) are in the same class, so are \( u \overrightarrow{G} - u \) and \( u \overrightarrow{G} - v \) showing that \( u \overrightarrow{G} - u \) is effective.

Notice that \( u \) and \( u \overrightarrow{G} - u \) cannot be both effective since there sum \( u \overrightarrow{G} \) would be too, contradicting Proposition 6.
Corollary 5. Any configuration $u$ with degree greater than $m - n$ is effective.

Proof If $u$ such that $\text{deg}(u) > m - n$ is not effective, by the above theorem there exists an acyclic orientation $\overrightarrow{G}$ of $G$ such that $u_{\overrightarrow{G}} - u$ is. But the degree of this configuration is negative, giving a contradiction. □

Proposition 7. Let $T_G(x,y)$ be the Tutte polynomial of the graph $G$, and let $t_i$ be the integer coefficients given by:

$$T(1,y) = \sum_{i=0}^{m-n+1} t_i x^i$$

Then the number of non equivalent effective configurations of degree $d$ is given by:

$$\sum_{k=m-n+1-d}^{m-n+1} t_k$$

Proof In [19] the level of a recurrent configuration $u$ was defined as

$$\text{level}(u) = \sum_{i=0}^{n-1} u_i - m + d_n$$

where $d_n$ is the degree of the vertex $x_n$.

It was proved that this level varies from 0 to $m - n + 1$ and that the number of recurrent configurations of level $p$ and such that $x_n = q$ does not depend on $q$ and is equal to the coefficient $t_p$ of $y^p$ in the evaluation of the Tutte polynomial $T_G(x,y)$ of $G$ for $x = 1$. A bijective proof of this result was given in [5].

Using the bijection $\beta$ defined in Proposition 2 we have that the number of parking configurations $v$ such that $\sum_{i=1}^{n-1} v_i = j$ and a given value for $v_n$ is equal to the number of recurrent configurations $u$ such that:

$$\sum_{i=1}^{n-1} u_i = \sum_{i=1}^{n-1} (d_i - 1 - v_i) = 2m - d_n - (n - 1) - j$$

and $u_n = d_n - 1 - v_n$, which is the number of recurrent configurations of level $k = m - n + 1 - j$ and a given value of $u_n$. This number is equal to $t_k$.

In order that the configuration $v$ of degree $d$ to be effective we must have $v_n \geq 0$ so that $k$ must be greater or equal to 0 and not greater than $m - n + 1$, thus ending the proof. □

4. The rank of configurations

From now on it will be convenient to denote positive configurations using letters $f, g \cdots$ and configurations with no particular assumptions on them by letters $u, v, w$. 
4.1. Definition of the rank.

**Definition 6.** The rank \( \rho(u) \) of a configuration is the integer defined by:

- If \( u \) is non-effective it is equal to \(-1\)
- If \( u \) is effective, it is the largest integer \( r \) such that for any positive configuration \( f \) of degree \( r \) the configuration \( u - f \) is effective.

Denoting \( P \) the set of positive configurations and \( E \) the set of effective configurations this definition can given by the following compact formula which is valid in both cases:

\[
\rho(u) + 1 = \min_{f \in P, u - f \in E} \deg(f)
\]

In other words, let \( u \) be a configuration of rank \( \rho(u) \) and \( f \) be a positive configuration such that \( \deg(f) \leq \rho(u) \) then \( u - f \) is effective; moreover there exists a positive configuration \( g \) of degree \( \rho(u) + 1 \) such that \( u - g \) is not effective.

**Definition 7.** A positive configuration \( f \) is a proof for the rank \( \rho(u) \) of an effective configuration \( u \) if \( u - f \) is non-effective and \( u - h \) is effective for any positive configuration \( h \) such that \( \deg(h) < \deg(f) \).

Notice that if \( f \) is a proof for \( \rho(u) \) then \( \rho(u) = \deg(f) - 1 = \deg(f) + \rho(u - f) \).

**Proposition 8.** A configuration \( u \) of degree greater than \( 2m - 2n \) has rank

\[
r = \deg(u) - m + n - 1
\]

**Proof** We first show that for any positive configuration \( f \) such that \( \deg(f) = r \), the configuration \( u - f \) is effective. This follows from \( \deg(u - f) = \deg(u) - r = m - n + 1 \) by Corollary 5.

We now build a positive configuration \( f \) of degree \( r + 1 \) such that \( u - f \) is not effective. Consider any acyclic orientation \( G \) of \( G \) and let \( v = u - u_G \) then \( v \) is effective since its degree is equal to \( \deg(u) - m + n \) hence greater than \( m - n \). Let \( f \) be the positive configuration such that \( v \sim u_G f \), then \( u - f \) is such that

\[
\begin{align*}
    u_G & \sim u - v \sim u_G u - f
\end{align*}
\]

so that \( u - f \) is not effective by Proposition 6.

4.2. Riemann-Roch like theorem for graphs. We give here a proof of the following theorem first proved in [2] which is shorter and simpler than the original one.

**Theorem 4.** Let \( \kappa \) be the configuration such that \( \kappa_i = d_i - 2 \) for all \( 1 \leq i \leq n \), so that \( \deg(\kappa) = 2(m - n) \). Any configuration \( u \) satisfies:

\[
\rho(u) - \rho(\kappa - u) = \deg(u) + n - m
\]
Proof The main ingredient for the proof is to use Theorem 3 and remark that for any acyclic orientation $\vec{G}$, the orientation $\bar{G}$ of $G$ obtained from $\vec{G}$ by reversing the orientations of all the edges is such that: $u_{\vec{G}} + u_{\bar{G}} = \kappa$.

Let $u$ be any configuration we first give an upper bound for $\rho(\kappa - u)$, we define $f$ to be a proof for the rank of $u$ if $u$ is effective, and to be equal to 0 if $u$ is not effective. So that $\rho(u) = \deg(f) - 1$ in both cases.

Since $u - f$ is not effective, we have by Theorem 3 that there exists an acyclic orientation $\vec{G}$ of $G$ such that $u_{\vec{G}} - (u - f)$ is effective, hence equivalent to a positive configuration $g$. This may be written as:

$$u_{\vec{G}} - (u - f) \sim_{L_G} g \quad (4.1)$$

Now consider the orientation $\bar{G}$ of $G$ obtained from $\vec{G}$ by reversing the orientations of all the arrows, clearly $u_{\vec{G}} + u_{\bar{G}} = \kappa$. Hence adding $u_{\bar{G}}$ to both sides of (4.1) we have:

$$\kappa - (u - f) \sim_{L_G} g + u_{\bar{G}} \quad (4.2)$$

which may be written as:

$$(\kappa - u) - g \sim_{L_G} u_{\bar{G}} - f$$

Giving that $\kappa - u - g$ is non effective since the reverse of an acyclic orientation is also acyclic. Hence by the definition of the rank we have

$$\rho(\kappa - u) < \deg(g) \quad (4.3)$$

The degree of $g$ is obtained form (4.1) giving:

$$\deg(g) = \deg(u_{\vec{G}}) - \deg(u) + \deg(f) = m - n - \deg(u) + \rho(u) + 1$$

and:

$$\rho(\kappa - u) < m - n - \deg(u) + \rho(u) + 1 \quad (4.4)$$

Now to obtain a lower bound for $\rho(\kappa - u)$ we exchange the roles of $u$ and $\kappa - u$ giving:

$$\rho(u) < m - n - \deg(\kappa - u) + \rho(u) + 1 \quad (4.5)$$

Since $\deg(\kappa - u) = 2(m - n) - \deg(u)$, inequality (4.5) may be written as:

$$\rho(u) + m - n - \deg(u) - 1 < \rho(\kappa - u) \quad (4.6)$$

Comparing inequalities (4.4) and (4.6), and noticing that the rank is an integer gives

$$\rho(u) + m - n - \deg(u) = \rho(\kappa - u)$$

hence proving the Theorem. □

section On the rank of configurations in the complete graph

We are interested here in an algorithm allowing to compute the rank of configurations on the complete graph $K_n$. 


4.3. **Some useful remarks on the rank.** We begin by some simple facts satisfied by the rank on any graph $G$.

**Lemma 4.** Let $u$ be a configuration on a graph $G$ and $g$ be a positive configuration then:

$$
\rho(u) \leq \rho(u + g) \leq \rho(u) + \deg(g)
$$

**Proof** It is clear from the definition of the rank that increasing the values of the components of a configuration cannot decrease the value of the rank, this proves the first part of the inequality. For the second part, let $f$ be a proof for the rank of $u$, then $u + g - (g + f) = u - f$ is non effective, so that the rank of $u + g$ is strictly less than $\deg(f + g)$ but $\deg(f) = \rho(u) + 1$, giving:

$$
\rho(u + g) < \rho(u) + 1 + \deg(g)
$$

which is the expected result.

**Corollary 6.** Let $u$ be a configuration on a graph $G$ and $u'$ be the configuration obtained by adding 1 to one of the $u_i$'s; so that for one $j$, $u'_j = u_j + 1$ and for all $i \neq j$, $u'_i = u_i$. Then:

$$
\rho(u) \leq \rho(u') \leq \rho(u) + 1
$$

**Proof** It suffices to apply Lemma 4 to the positive configuration $g$ such that $g_i = 1$ and $g_j = 0$ for $j \neq i$.

**Lemma 5.** Let $u$ be a configuration on a graph $G$ and $g$ be a positive configuration such that :

$$
\rho(u - g) = \rho(u) - \deg(g)
$$

then for each positive configuration $g'$ such that for all $j$, $g'_j \leq g_j$ we have:

$$
\rho(u - g') = \rho(u) - \deg(g')
$$

**Proof** Since for all $j$, $g'_j \leq g_j$ we can write $g = g' + g''$, where $g''$ is a positive configuration. By Lemma 4 we have:

$$
\rho(u) \leq \rho(u - g') + \deg(g')
$$

Applying again the same Lemma and since $u - g' = u - g + g''$ we get:

$$
\rho(u - g') \leq \rho(u - g) + \deg(g'')
$$

But it is assumed that $\rho(u - g) = \rho(u) - \deg(g)$ giving:

$$
\rho(u - g') \leq \rho(u) - \deg(g) + \deg(g'') = \rho(u) - \deg(g')
$$

Hence proving the Lemma.
4.4. Main fact on the rank in $K_n$.

**Proposition 9.** Let $u$ be a positive configuration on the complete graph such that for some $i, u_i = 0$ and let $\varepsilon^{(i)}$ be the configuration such that $\varepsilon^{(i)}_i = 1$ and $\varepsilon^{(i)}_j = 0$ for $j \neq i$ then:

$$\rho(u) = \rho(u - \varepsilon^{(i)}) + 1$$

**Proof** Let $f$ be a proof for $\rho(u)$, since $u - f$ is not effective there exists at least one $j$ such that $u_j < f_j$. Consider the two configurations:

$$v = u - f_j \varepsilon^{(i)}$$

these two configurations have the same components but in different order since $v_j = w_i = u_j - f_j$ and $v_i = w_j = 0$, hence by the symmetry of $K_n$ they have the same rank. Giving:

$$\rho(v) = \rho(w)$$

Since $f$ is a proof for $\rho(u)$ we have $\rho(u) = \deg(f) - 1 = \deg(f) + \rho(u - f)$. Hence applying Lemma 5 with $g = f$ and $g' = f_j \varepsilon^{(i)}$ we obtain:

$$\rho(u - g') = \rho(u) - \deg(g') = \rho(u) - f_j$$

Hence since $v = u - g'$, we have also:

$$\rho(w) = \rho(v) = \rho(u) - f_j$$

Since $f_j - u_j \geq 1$, we can apply again Lemma 5, this time with $g = u_j \varepsilon^{(j)} + (f_j - u_j) \varepsilon^{(i)}$ and $g' = \varepsilon^{(i)}$, giving:

$$\rho(u - \varepsilon^{(i)}) = \rho(u) - 1$$

This result does not hold for any graph, the subtraction of 1 on the $i$-th coordinate of configuration $u$ with $u_i = 0$ may leave the rank invariant as shows the following example.

**Remark 1.** The configuration $u = (0,1,0,1,0,1)$ on the wheel graph $W_5$ given in the left of the Figure 4.4 below has rank 0, as has the configuration $u' = u - (0,0,1,0,0,0)$.

**Proof** Indeed the configuration $u$ is effective, we first show that it has rank 0. Indeed, notice that $v = u - (0,0,0,0,1,0)$ is not effective since the acyclic orientation given on the right part of the Figure 4.4 gives a configuration $w$ such that $w_i \geq v_i$ for all $i$. On the other hand the configuration $u' = (0,1,-1,1,0,1)$ is toppling equivalent to $(1,2,0,2,1,-4)$ and to $(0,0,2,0,0,0)$ hence it is effective and has also rank equal to 0. □

**Proposition 10.** Let $u$ be a configuration on the complete graph such that there exists a permutation $\alpha \in \mathcal{S}_n$ satisfying $u_{\alpha(i)} = i - 1$ for $i = 1, 2, \ldots, n - 1$ then:

$$\rho(u) = \begin{cases} -1 & \text{if } u_{\alpha(n)} \geq 0 \\ u_{\alpha(n)} & \text{otherwise} \end{cases}$$
Proof Since the automorphism group of $K_n$ is $S_n$ we may suppose that the configuration $u$ is equal to $(0, 1, 2, \ldots, n - 2, a)$, where $a = u_{\alpha(n)}$. The configuration is a parking configuration so that by Proposition 3 it is not effective if $a < 0$, giving the first part of the formula.

For the second part we have for the degree of $u$:

$$\deg(u) = a + \frac{(n - 1)(n - 2)}{2}$$

Moreover the number of edges of the complete graph is $m = \frac{n(n - 1)}{2} = \deg(u) - a - (n - 1)$

So that we can apply Proposition 8 when $\deg(u) > 2m - 2n$, that is when $a > m - n - 1$. When this condition is satisfied Proposition 8 gives:

$$\rho(u) = \deg(u) - m + n - 1 = a$$

It is easy to check that when $a = 0$ the configuration $u$ is effective and subtracting 1 to $a$ gives a non effective configuration. Hence the rank of the configuration $u$ when $a = 0$ is 0. Since by Corollary 6 while adding 1 to $a$ from $a = 0$ to $a = m - n$ the rank increases at most by 1 at each step ending the proof.

\[\square\]

4.5. Algorithm. The two Propositions above give a recursive algorithm in order to compute the rank of a configuration. It consists in determining first the parking configuration $v$ equivalent to $u$; if $v_n$ is negative then the rank is -1. If $v_n \geq 0$, we use the fact that for a parking configuration there is necessarily an $i < n$ such that $v_i = 0$. So that the determination of the rank of the configuration $w = v - \varepsilon^{(i)}$ and adding 1 to the value obtained gives the rank of $u$. This algorithm terminates since one obtains recursively after at most $\deg(v)$ steps a non effective configuration. Notice that the rank is exactly the number of recursive calls of the algorithm minus 1. However we may improve this algorithm by making it to stop when the configuration attained at some step satisfies the conditions stated in Proposition 10.
The main difficulty of the algorithm consists in obtaining at each step the parking configuration toppling equivalent to a given one, but this can be simplified supposing that the configurations are sorted at each step. That is reordering the \( u_i \) such that \( u_{i-1} \leq u_i \) for all \( 1 < i < n \).

Let us consider as an example the computation of the rank of the configuration \( u = (3, 1, 3, 4, -1) \) of \( K_5 \).

The configuration \( (0, 3, 0, 1, 6) \) is the parking configuration toppling equivalent to \( u \). It is preferable to write the coordinates in increasing order obtaining a new configuration with the same rank: \( (0, 0, 1, 3, 6) \). A first step is to subtract \( \varepsilon^{(1)} \) giving \( (-1, 0, 1, 3, 6) \) and by the above Proposition, since \( v = (0, 1, 2, 4, 2) \), we get \( j = 4 \) \( w = (1, 2, 3, 0, 3) \) and \( \rho(u) = \rho(w) + 1 \). Reordering gives \( w' = (0, 1, 2, 3, 3) \).

At the second step we get the configuration \( (-1, 1, 2, 3, 3) \) and the parking configuration \( (0, 1, 2, 3, 2) \) after toppling and reordering. The third step gives \( (-1, 1, 2, 3, 2) \) and the parking configuration \( (3, 0, 1, 2, 1) \). Two other steps are necessary to obtain the non-effective configuration \( (1, 2, 0, 3, -1) \) of rank \(-1\) giving \( \rho(u) = 4 \).

Notice that an application of Proposition 11 would have given the result after the first step, since this Proposition 11 gives \( \rho(0, 1, 2, 3, k) = k \) for \( k \geq -1 \).

A first glance at this algorithm shows that the complexity of the determination of the rank of the configuration \( u \) in \( K_n \) is in \( O(nD) \), where \( D \) is the degree of \( u \). But this could be lowered to \( O(n) \) using some observations on Dyck words.

4.6. Working on Dyck words. We use here the notation of Section 2.3 to which we add some items. We denote by \( \varepsilon \) the empty word and the height of a word \( f \) denoted by \( \delta(f) \) is the value of \( |f|_a - |f|_b \). The first return to origin decomposition of a Dyck word \( f \) is given by \( f = agbh \) where \( g \) and \( h \) are Dyck words. A Dyck word is primitive if \( h = \varepsilon \) in this decomposition. The factorization of a Dyck word into primitive factors is given by \( f = ag_1bag_2\ldots ag_kb \) where the \( g_i \)'s are Dyck words.

Let \( u = (u_1, u_2, \ldots, u_{n-1}, u_n) \) be a parking configuration of \( K_n \) and let \( i \) be an index such that \( i < n, u_i = 0 \), the configuration \( u' = u - \varepsilon^{(i)} \) has a negative value \((-1)\) in vertex \( x_i \), the equivalent configuration \( u' = u - \varepsilon^{(i)} - \Delta^{(n)} \) is such that \( u'_1 = 0 \), \( u'_n = u_n - (n - 1) \) and \( u'_j = u_j + 1 \) for all other values of \( j \). Since \( u \) is a parking configuration, \( u' \) satisfies \( u'_j \leq n \) for all \( j < n \) so that we can apply Proposition 5 in order to determine the parking configuration \( v' \) equivalent to it. This gives:

**Proposition 11.** If \( \phi(u) = agbh \), where \( g, h \) are Dyck words then:

\[
\phi(v') = habgb \quad \text{and} \quad v'_n = u_n - |agb|_a
\]  

**Proof** Since \( u'_j = u_j + 1 \) for all is easy to check that : \( \phi(u') = abgh \). The computation of the parking function equivalent to \( u' \) translates into the computation of the conjugate of \( \phi(u') \) which is a Dyck word followed by an occurrence of \( b \); from the above decomposition of \( \phi(u') \) one gets that this conjugate is \( habgb \). A computation of the degree of \( u' \) and \( v' \) which must be equal shows that \( v'_n = u'_n + (n - 1 - |agb|_a) \) gives the value of \( v'_n \). \( \Box \)

Let \( \phi'(u) \) denote for a parking configuration \( u \) the Dyck word obtained by deleting from \( \phi(u) \) the letter \( b \) at the end of it. This result above shows that instead of considering
the configuration $u$, it is preferable to work with $(f = \phi'(u), s = u_n)$ a pair consisting of a Dyck word $f$ and integer $s$; and to define two function $\theta$ and $\psi$ on these pairs, where $\theta(f, s)$ is a Dyck word and $\psi(f, s)$ an integer given using the first return to the origin decomposition and:

$$
\theta(agr, s) = habg \quad \psi(agr, s) = s - |ab|_a.
$$

(4.8)

So that the rank $r$ of $u$ is obtained by iterating $\theta$ and $\psi$ until reaching a negative number for the second component of the pair, this may be written:

$$
r + 1 = \min_{k \geq 0} \{ \psi^k(\phi'(u), u_n) < 0 \}
$$

the algorithm on configurations could then be translated in terms of Dyck words as:

- $f := \phi'(u); s := u_n; r := -1$
- while $s \geq 0$ do
  - $f := \theta(f, s); s := \psi(f, s); r := r + 1$
- od
- return $r$

Iterating this Proposition a few times gives:

**Lemma 6.** Let $u$ be a parking configuration and $f$ be the Dyck word associated to it. Let $f = af_1bf_2\cdots af_kb$ be the factorization of $f$ into its primitive factors. Then

$$
\theta^k(f, s) = abf_1abf_2\cdots abf_k \quad \psi^k+1(f, s) = s - (n - 1)
$$

Moreover a parking configuration $v$ such that $\phi(v) = f_1abf_2\cdots abf_kab$ and $v_n = s - (n - 1)$ when $v_n \geq 0$ satisfies: $\rho(v) = \rho(u) - k$.

**Remark 2.** Notice that the heights $h_1, h_2, \ldots, h_{n-1}$ of all the prefixes ending by $b$ in the $f$ and the heights $h'_1, h'_2, \ldots, h'_{n-1}$ are such that $h'_i = h_i$ if $h_i = 0$ and $h'_i = h_i - 1$ if $h_i > 0$.

**Theorem 5.** Let $u$ be a parking configuration on $K_n$, let $f = \phi'(u)$ and $h_1, h_2, \ldots, h_{n-1}$ be the sequence of heights in $f$. Then the rank of $u$ is given by:

$$
1 + \rho(u) = \sum_{i=1}^{n-1} \max(0, q - h_i + \chi(i \leq r))
$$

(4.9)

Where $q$ and $r$ are the quotient and remainder of the division of $u_n + 1$ by $n - 1$, and $\chi(i \leq r)$ is equal to 1 if $i \leq r$ and to 0 otherwise.

**Proof** When $u_n < 0$ all the terms in the sum in the right hand side of Equation (4.9) are equal to 0 so that the formula gives $\rho(u) = -1$ as expected.

For $u_n \geq 0$, we proceed by induction on $q$ the quotient of $u_n + 1$ by $n - 1$ and we consider the decomposition of $\phi'(u)$ into factors:

$$
\phi'(u) = f = af_1b\cdots af_kb
$$
• If \( q = 0 \) then \( 0 \leq u_n < n - 1 \) hence \( \rho(u) \) is determined by the position of \( u_n \) in the intervals given by the lengths of the prefixes \( f' \) of \( f \) satisfying \( \delta(f') = 0 \) these are \( af_1 ba f_2 b \cdots af_i b \) of \( f \) for \( i = 1, 2, \ldots k \). More precisely let \( n_i \) be given for \( 1 \leq i \leq k \) by
\[
n_i = |af_1 ba f_2 b \cdots af_i b|_a
\]
and let us denote \( n_0 = 0 \). Then we have \( \psi^i(f, u_n) = u_n - n_i \) for \( i = 1, 2, \ldots, k+1 \) so that \( \rho(u) \) is given by the first \( i \) for which \( \psi^i(f, u_n) \) is negative which may be translated by:
\[
\rho(u) = i \iff n_i \leq u_n < n_{i+1}
\]
since \((g, s) = \theta(f, u_n)\) is such that \( s < 0 \).

On the other hand the \( n_i \) can be characterized by \( h_{n+1} = 0 \) giving:
\[
1 + \rho(u) = |\{j | j \leq r, h_j = 0\}|
\]
But this is exactly what equation 4.9 gives.

• If \( q > 0 \) let us consider \( g = \theta^k(f, u_n) \) and notice that \( \psi^{k+1}(f, u_n) = u_n - (n - 1) \) and let \( v \) be such that \( \phi'(v) = f, v_n = u_n - (n - 1) \). Let then the quotient of \( v_n \) by \( n \) is \( q' = q - 1 \) and the remainder is also \( r \). Let \( h'_1, h'_2, \ldots, h'_{n-1} \) be the sequence of heights of \( g \) ending. Applying the inductive hypothesis we have
\[
1 + \rho(v) = \sum_{i=1}^{n-1} \text{Max}(0, q - 1 - h'_i + \chi(i \leq r))
\]
By Lemma 6, we have \( \rho(u) = \rho(v) + k \) and \( g = f_1 ab f_2 b \cdots ab f_k ab \).

It is easy to check since the heights \( h'_i \) in \( g \) are such that \( h'_i = h_i - 1 \) if \( h_i \neq 0 \) and \( h'_i = 0 \) when \( h_i = 0 \) we have \( \text{Max}(0, q - 1 - h'_i + \chi(i \leq r)) = \text{Max}(0, q - h_i \chi(i \leq r)) \) when \( h_i \neq 0 \) and \( \text{Max}(0, q - 1 - h'_i + \chi(i \leq r)) = \text{Max}(0, q - h_i \chi(i \leq r)) - 1\) when \( h_i = 0 \) from which we get
\[
1 + \rho(v) = \sum_{i=1}^{n-1} \text{Max}(0, q - h_i + \chi(i \leq r)) - k
\]
The proof ends by reminding that \( \rho(u) = \rho(v) + k \).

An exemple of calculation:
\[
u = (0, 0, 0, 0, 1, 1, 1, 4, 7, 7, 9, 26)
\]
For this configuration we have: \( n = 11, u_n = 26, q = 2, r = 7 \).

| \( i \) | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| \( u_i \) | 0 | 0 | 0 | 1 | 1 | 1 | 4 | 7 | 7 | 9 |
| \( h_i \) | 0 | 1 | 2 | 2 | 3 | 4 | 2 | 0 | 1 | 0 |
| \( q + \chi(i \leq r) \) | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 2 | 2 | 2 |
| \( q + \chi(i \leq r) - h_i \) | 3 | 2 | 1 | 1 | 0 | -1 | 1 | 2 | 1 | 2 |

Adding the positive values of \( q + \chi(i \leq r) \) gives \( 1 + \rho(u) = 13 \) so that \( \rho(u) = 12 \) This calculation is illustrated in Figure 4.
5. A new parameter for Dyck Paths

5.1. Prerank and coheights. The algorithm used to determine the rank of a configuration of the complete graph suggests the introduction of a parameter on Dyck paths (or Dyck words) which we call prerank. In this Section we show how this parameter behaves with other known parameters.

We denote \( \theta \) the mapping associating to the Dyck path \( f \) with first return to the origin decomposition \( f = aghb \) the Dyck path \( habg \). And we consider the Dyck word \((ab)^n\) consisting of the concatenation of \( n \) two-letters words \( ab \).

**Definition 8.** The prerank \( \rho(f) \) of the Dyck path \( f \), of length \( 2n \), is the smallest integer \( k \) such that \( k \) operations \( \theta \) are needed to reach the word \((ab)^n\), or in other words such that: \( \theta^k(f) = (ab)^n \).

Notice that \( \rho((ab)^n) = area((ab)^n) = 0 \), also one can prove that \( \rho(a^n b^n) = area(a^n b^n) = \frac{n(n-1)}{2} \), this suggests that pre-rank and area could be equal, but this is not always the case since for example \( \rho(abaabb) = 2 \) while \( area(abaabb) = 1 \).

The prerank may be calculated using the notion of coheight, which may be defined as follows:

Let the heights of the prefixes followed by an occurrence of \( a \) of a Dyck word \( f \) of length \( 2n \) be: \( h_1, h_2, \ldots, h_n \), let \( m \) be the largest integer such that \( h_m \) is maximal among the \( h_i \), then the co-heights of the prefixes of \( f \) are given by the formula

\[
T_i = \begin{cases} 
    h_m - h_i & \text{if } i \leq m \\
    h_m - h_i - 1 & \text{otherwise}
\end{cases}
\]  

(5.1)

The following lemma allows to determine the prerank

![Figure 4. Calculation of the rank on a Dyck path](image)
Lemma 7. The prerank $\rho(f)$ of the Dyck word $f$ is given by the sum of the coheights of the prefixes of $f$ followed by an occurrence of the letter $a$.

Proof We proceed by induction on the value of $\rho(f)$. If $\rho(f) = 0$, then $f = (ab)^n$ and $h_i = 0$ for all $1 \leq i \leq n$ hence $h_m = 0$ and $m = n$, then all the coheights are equal to 0 in accordance with the Lemma.

Let $f$ be such that $\rho(f) \neq 0$ and let $agbh$ be the first return to the origin decomposition of $f$, we have $\rho(f) = 1 + \rho(habg)$. We proceed to the comparison of the coheights of $f$ and $habg$ considering two cases according to the position $m$ of the largest prefix of $f$ with maximal height. Denote $k = |agb|_a$.

- If $k < m$ then the sequence of heights of $f$ may be written

$$h_1 = 0, h_2, \ldots, h_k = 0, \ldots h_m, \ldots, h_n$$

and that of $habg$:

$$h_{k+1} \ldots h_m, \ldots, h_n, 0, 0, h_2', \ldots h_k'$$

where $h_i' = h_i - 1$ for $i = 2, \ldots k$.

These sequences are such that the values of the heights which decrease by one from the first to the second make a move from before $h_m$ to after $h_m$; hence their corresponding coheights do not vary. Those who do not decrease give also equal coheights. The only difference is that $h_1 = 0$ moving from before $h_m$ to after it and stays equal to 0, so that the corresponding coheight decreases by one. Hence the sum of the sequence of coheights of $habg$ is one less than the sum of the sequence of those of $f$. Since $\rho(f) = 1 + \rho(habg)$, the inductive hypothesis proves the assertion in the Lemma.

- If $k \geq m$. Then the sequences of heights are this time

$$h_1 = 0, h_2, \ldots, h_m, \ldots, h_k = 0, \ldots, h_n$$

for $f$ and

$$h_{k+1} \ldots h_n, 0, 0, h_2', \ldots h_m', \ldots h_k'$$

for $habg$, where $h_i' = h_i - 1$ for $i = 2, \ldots, k$. The maximal height decreases by 1 for $habg$, so that the coheights in $f$ and the corresponding ones in $habg$ are equal except as above for the first coheight, for which the corresponding one decreases by 1. A similar argument to above ends the proof.

\[\square\]

5.2. The mapping $\Phi$. For any Dyck word $f$ we consider its sequence of heights $h_1, h_2, \ldots h_n$ and the largest integer $m$ such that $h_m$ is maximal among the $h_i$‘s. We consider also the decomposition of $f$ as $f = f'abf''$ such that $|f'|_a = m$, and denote

$$\Phi(f) = a\tilde{f}^b\tilde{f}''$$

Where we use the notation: if $g = x_1x_2 \ldots x_p$ then $\tilde{g} = x_p \ldots x_2x_1$. In the sequel we will say sequence of heights of a Dyck word $f$ as a shortening for: sequence of heights of the prefixes of $f$ followed by an occurrence of the letter $a$.

We recall the definition of the parameter $\text{dinv}$ introduced by M. Haiman [14] which is obtained from the sequence of heights $h_1, h_2, \ldots, h_n$ of $f$. 


Definition 9. For a Dyck word $f$ the parameter $\text{dinv}(f)$ is equal to the number of elements of the set $\text{DINV}(f)$ of pairs $(i,j)$ such that $i < j$ and $h_i = h_j$ or $h_j = h_i - 1$.

The main result of this subsection is:

Proposition 12. The mapping $\Phi$ is an involution and for any Dyck word $f$:

\[
\rho(f) = \text{area}(\Phi(f)) \quad \text{and} \quad \text{dinv}(f) = \text{dinv}(\Phi(f))
\]

Before proving this Proposition we need to compare the sequence of heights in a word $f$ and that in the word $\tilde{f}$

Lemma 8. Let $f$ be any word on $\{a,b\}^*$ containing $n$ occurrences of the letter $a$ and let $h_1, h_2, \ldots, h_n$ be its sequence of heights, and let $p = |f|_a - |f|_b$ the sequence $h'_1, h'_2, \ldots, h'_n$ of heights of $\tilde{f}$ is given by

\[
h'_{n-i+1} = p - 1 - h_i
\]

Proof (of Lemma) Let $f'a f''$ be such that $f'$ contains $i - 1$ occurrences of the letter $a$, then $h_i = \delta(f')$ and since $\tilde{f} = f'' a f'$, $h'_{n-i+1} = \delta(f'')$.

Computing $\delta(f)$ we have

\[
\delta(f) = \delta(f') + 1 + \delta(f'') = p
\]

\[\square\]

Proof (of Proposition 12) Let $h_1, \ldots, h_m, \ldots, h_n$ be the sequence of heights of $f$, and $m$ be the largest integer such that $h_m$ is maximal among the $h_i$'s. Let $h'_i$ be the sequence of heights in $\Phi(f)$. Since $h'_m = h_m - h_1 = h_m$ and $h'_j \leq h_m$ the largest $j$ such that $h'_j$ is maximal among the $h'_i$ is equal to $m$. Hence $\Phi(\Phi(f)) = f$.

Denote $f = f' a f''$ where $|f'| = |f''| = m$ then $\delta(f') = h_m + 1$ and $\delta(f'') = -h_m$. Then by Lemma 8 the sequence of heights $h'_1, h'_2, \ldots, h'_m, \ldots, h'_n$ of $\Phi(f)$ is given by:

\[
h'_i = \begin{cases} h_m - h_{m-i+1} & \text{if } i \leq m \\ h_m + (-h_m - (h_{n-i+1} - h_m - 1)) & \text{otherwise} \end{cases}
\]

(5.2)

Since the second value is $h_m - 1 - h_{n-i+m+1}$, this proves that the sequence of heights in $\Phi(f)$ is a rearrangement of the sequence of the coheights of $f$ proving the first part of the Proposition.

For the second part of the Proposition we consider an element $(i,j)$ in $\text{dINV}(f)$ and denote $a = h_i, b = h_j$; in $\Phi(f)$, these values $a, b$ become $a', b'$ and are such that $h_i' = a'$ and $h_j' = b'$. We consider three cases depending on the values of $i$:

1. If $i, j \leq m$ then
   \[
a' = h_m - a, b' = h_m - b, i' = m - i + 1, j' = m - j + 1
   \]
   so that $j' < i'$ and $(j', i') \in \text{DINV}(\Phi(f))$.
2. If $i > m$ then
   \[
a' = h_m - 1 - a, b' = h_m - 1 - b, i' = m + n - i + 1, j' = m + n - j + 1
   \]
   so that $j' < i'$ and $(j', i') \in \text{DINV}(\Phi(f))$.
3. If $i \leq m$ and $j > m$ then $i' \leq m, j' > m, a' = h_m - a, b' = h_m - 1 - b$. So that $b' = a' - 1$ if $a = b$ and $b' = a'$ if $a = b + 1$. Hence $(i', j') \in \text{DINV}(\Phi(f))$. 

5.3. Link with Haglund’s function $\zeta$. In [16] page 50] J. Haglund introduces a mapping $\zeta$ on Dyck words and shows that this mapping keeps invariant the value of $dinv$. We will give a relationship between $\zeta$ and $\Phi$.

Before let us recall how $\zeta(f)$ is build from the sequence of heights $\eta = (h_1, h_2, \ldots, h_n)$ of $\eta$. Let $k$ be the maximal values of the $h_i$, then the subsequences $\eta^{(0)}, \eta^{(1)}, \ldots, \eta^{(k)}$ where $\eta^{(i)}$ contains all the occurrences of $i$ and $i - 1$ in $\eta$ in the same order as they appear in $\eta$. Hence $\eta^{(0)}$ contains only occurrences of 0, and $\eta^{(k+1)}$ only occurrences of $k$. Then $\zeta(f)$ is obtained by concatenating the words $\overline{\eta}^{(i)}$ obtained from the $\eta^{(i)}$'s replacing occurrences of $i$ by $a$ and those of $i - 1$ by $b$.

Example for $f = aabaabbabbaabaababb$ we have $\eta = (0, 1, 1, 2, 1, 0, 1, 1, 2, 1)$, so that:

$$\eta^{(0)} = (0, 0), \eta^{(1)} = (0, 1, 1, 1, 0, 1, 1, 1), \eta^{(2)} = (1, 1, 2, 1, 1, 2, 1), \eta^{(3)} = (2, 2)$$

Giving

$$\overline{\eta}^{(0)} = aa, \overline{\eta}^{(1)} = baaabaaa, \overline{\eta}^{(2)} = bbabbbab, \overline{\eta}^{(3)} = bb$$

and $\zeta(f) = aabaabaababababbb$.

Consider the mapping $R$ on words consisting in writing them from the right to left and replacing $a$ by $b$ and $b$ by $a$ such that $R(f_1 f_2 \cdots f_{n-1} f_n) = \overline{f_n f_{n-1} \cdots f_2 f_1}$ where $\overline{a} = b, \overline{b} = a$.

**Proposition 13.** The mappings $\Phi$ and $\zeta$ satisfy $R(\zeta(f)) = \zeta(\Phi(f))$.

In the example above we have $\Phi(f) = aabaabbabbaabaabbbab$ which has for sequence of heights$(0, 1, 1, 2, 1, 0, 1, 1, 2, 0)$ giving $\zeta(\Phi(f)) = aabaabaabababbbabbbab$ which is equal to $R(\zeta(f))$.

**Proof** Consider the sequence $\eta = (h_1, h_2, \ldots, h_n)$ of heights of $f$, let $k$ be the maximal value of the $h_i$ and $m$ the largest integer such that $h_m = k$. It is convenient for the proof to divided the $\eta^{(i)}$'s leading to the definition of $\zeta(f)$ in two parts $\overline{\eta}^{(i)}$ and $\overline{\eta}^{(i)}$ corresponding respectively to $(h_1, \ldots, h_m)$ and to $(h_{m+1}, \ldots, h_n)$, this gives since $\overline{\eta}^{(k+1)}$ is the empty word:

$$R(\zeta(f)) = R(\overline{\eta})^{(k+1)} R(\overline{\eta})^{(k)} R(\overline{\eta})^{(k)} \cdots R(\overline{\eta})^{(0)} R(\overline{\eta})^{(0)}$$

Consider now the sequence $\mu = (h'_1, h'_2, \ldots, h'_m, \ldots, h'_n)$ of heights of $\Phi(f)$, we divide also the sequences $\mu^{(i)}$ into to two sequences $\mu^{(i)}$ and $\mu^{(i)}$ giving:

$$\zeta(\Phi(f)) = \overline{\mu}^{(0)} \overline{\mu}^{(1)} \overline{\mu}^{(1)} \cdots \overline{\mu}^{(k)} \overline{\mu}^{(k)} \overline{\mu}^{(k+1)}$$

seen in the Proof of Proposition [12] that

$$h'_i = \begin{cases} h_m - h_{m-i+1} & \text{if } i \leq m \\ h_m - 1 - h_{m+n-i} & \text{otherwise} \end{cases}$$

(5.3)

But this exactly means that : $\overline{\mu}^{(i)} = R(\overline{\eta})^{(k+1-i)}$ and $\overline{\mu}^{(i)} = R(\overline{\eta})^{(k-i)}$, hence proving this Proposition. □
section: Enumerative study of the \((\text{degree, rank})\) bistatistic on sorted parking configurations

In this section, we are interested in the distribution of the degrees and ranks on sorted configurations on \(K_n\). Hence we consider the following generating function \(u\) ranges among all sorted parking configurations on \(K_n\),

\[
K_n(d, r) = \sum_u d^{\deg(u)} r^{\rho(u)}.
\]

In this sum, \(u\) runs over all parking configurations of \(K_n\) such that \(u_1 \leq u_2 \leq \ldots \leq u_{n-1}\). Notice that this generating function is a formal sum which is a Laurent series, since \(\rho(u)\) may be equal to \(-1\) and the \(\deg(u)\) can be any integer in \(\mathbb{Z}\).

We will provide two successive descriptions of \(K_n(d, r)\) after a change of variables leading to another series \(L_n(x, y)\) which is a more usual power sum and symmetric in \(x\) and \(y\). In the first description, we explain the occurrence of a factor \(\frac{1-xy}{(1-x)(1-y)}\) in \(L_n(x, y)\). In the second description, we show that the second factor of \(L_n(x, y)\) is a simple fraction combining two copies of the Carlitz \(q\)-analogue of Catalan numbers counting the area of Dyck paths.

### 5.4. Dyck paths dividing a strip or a skew cylinder into two regions.

Consider the Dyck path \(w\) representing the configuration \((u_1, u_2, \ldots, u_{n-1})\) as the one in Figure 4 embedded in an infinite strip of height \(n-1\) where the coordinates of each cell are elements of \(\mathbb{Z} \times \mathbb{Z}/(n-1)\mathbb{Z}\) as shown in Figure 5. By convention, the starting point of the Dyck path is the bottom right corner \((a)\) of the cell of coordinates \((0, 0)\), hence its ending point is the top right corner \((b)\) of the cell \((1, n-2)\). It is also convenient to add one more East step to the Dyck path, ending in the point represented as \(c\) in Figure 4.

When tracing the Dyck path \(w\) in this strip, the cells fall in two regions, either in the region at the left of the Dyck path or in the one at the right. The cells in the regions may be characterized by a comparison with the heights \(h_1, h_2, \ldots, h_{n-1}\) in the Dyck word as follows: the cell with coordinates \((i, j)\) is in the region on the left if and only if \(i \leq -h_{j+1}\), and is on the one on the right otherwise.

A geometric excursion. A geometric illustration of our computations may be now given. Considering the strip as defining a skew cylinder \(\text{Cyl}_n\) where the top side of each cell \((i, n-2)\) is glued to the bottom side of the cell \((i-2, 0)\). One may remark that this gluing implies that the corners corresponding to the points denoted \(a\) and \(c\) in our Figure become equal. This skew cylinder is not completely disconnected by the embedded Dyck path \(w\): the cell \((2, n-2)\), at right in the strip is connected via its top side to the cell \((0, 0)\) at right in the strip. Adding this side as the last step of the embedded path, now described by the word \(wb\), leads to an embedded path, called the extended Dyck path, disconnecting the skew cylinder. The two components are the same in the strip as in the skew cylinder. We denote by \(\text{Cyl}[w]\) this skew cylinder, cut by the embedded path \(wb\), where the semi-length of \(w\) defines implicitly \(n\).

We draw a curve traversing the whole set of cells going from cell of coordinates \((i, j)\) to that of coordinates \((i, j+1)\) if \(j < n-1\) and from \((i, n-1)\) to \((i-1, 0)\). This allows to label the cells in the order in which they are traversed by the curve, starting
Figure 5. The coordinates in the strip of height 10

from 0 for the cell (0, 0) then 1 for the cell (0, 1) and so on. Notice that we can extend
the numbering to the cells (i, j) with i > 0 using the same convention so that the cell
(1, n − 2) has label −1, the cell (1, n − 3) has label −2 and so on. More generally the
label of the cell (i, j) is given by:

\[ \text{label}(i, j) = j - (n - 1)i \]

Figure 6 illustrates this description with a skew cylinder (or a strip) where cells are
labeled in the order of visit by the curve. The skew cylinder is obtained by gluing the
shaded grey cell on the top row with the bottom row such that cell with the same label
are identified. The result is that the vertex (0, 0) is also the top right corner of the cell
labeled by −n. The extended Dyck path is drawn in blue and corresponds to the Dyck
word \( w = a a b a a b b b a b b a b b a b \).

This labeling may also be explained by its reverse map \( S \) from \( \mathbb{Z} \) to \( \mathbb{Z} \times (n - 1)\mathbb{Z} \)
defined by

\[ S(k) = (-q, r) \]

where \( q \) and \( r \) are the quotient and the remainder of division of \( k \) by \( n - 1 \).

Let us denote \( lw(u_n) \) the number of cells in the left region with label less or equal
to \( u_n \), and by \( rw(u_n) \) the number of cells in the right region with label greater than
\( u_n \). Later we will gives relations between these parameters and the rank and degree of
sorted configurations.

We define the power series \( L_n(x, y) \) as the generating function according to left and
right weight of cells \( (i, j) \) of all cut strip or cut skew cylinder, cut by any Dyck word \( w \)
Figure 6. Image of the trajectory of previous example [see Figure 3, section 5.4] in the skew cylinder of semi-length $n - 1$:

$$L_n(x, y) := \sum_w \sum_{(i,j) \in \text{Cyl}[w]} x^{rw(i,j)} y^{lw(i,j)}.$$  

The following Lemma is a direct consequence of Theorem 5:

**Lemma 9.** For any sorted parking configuration $u = u_1, u_2, \ldots, u_{n-1}, u_n$, consider the Dyck path defined by $(u_1, u_2, \ldots, u_{n-1})$, then the cell labelled $u_n$, the rank and degree of $u$ are linked by the following relations:

$$lw(u_n) = \rho(u) + 1 \quad (5.4)$$

and

$$rw(u_n) = \left(\frac{n - 1}{2}\right) + \rho(u) - \deg(u). \quad (5.5)$$

**Proof** For the first part we use Theorem 5 giving:

$$1 + \rho(u) = \sum_{j=1}^{n-1} \max(0, q + \chi(i \leq r) - h_j)$$

where $u_n + 1 = (n - 1)q + r$.

Notice that a cell $(i, j)$ with label less or equal to $u_n$ satisfies $j - i(n - 1) \leq u_n$ giving:

$$j - i(n - 1) < (n - 1)q + r$$

Hence for each $j = 0, 1, \ldots, r - 1$ the number of $i$ such that the cell $(i, j)$ is in the left region and has label less or equal to $u_n$ is equal to the number of indices $i$
such that \( i \geq -q \) and \( i \leq -h_{j+1} \) but this number is \( \text{Max}(0, 1 + q - h_{j+1}) \). For each \( j = r, r + 1, \ldots, n - 2 \) the number of these indexes is \( \text{Max}(0, q - h_{j+1}) \) proving relation \([5.4]\).

For the second part the number of cells with non negative labels in the right region is equal to the area of the Dyck path which is equal to \( \sum_{i=1}^{n-1} h_i \), hence the number \( rw(u_n) \) of such cells with label greater than \( u_n \) is \( \sum_{i=1}^{n-1} h_i - lw(u_n) \) which expresses that the \( u_n + 1 \) non negative numbers less or equal to \( u_n \) are either on the left region or on the right one, and from \( \sum_{i=1}^{n-1} h_i = \binom{n-1}{2} - \sum_{i=1}^{n-1} a_i. \)

\[ \square \]

5.5. Factorization of the series. The first description of \( K_n(x, y) \) requires some additional definition on words. For any word \( f \) on the alphabet \( \{a, b\} \) we define its weight \( W(f) \) which is a monomial in \( x \) and \( y \) as follows. Consider the heights \( h_1, h_2, \ldots, h_p \) of the prefixes of \( f \) followed by the occurrence of a letter \( a \) and let \( \alpha = \sum_{h_i \geq 0} (h_i + 1), \) \( \beta = \sum_{h_i < 0} (-h_i - 1) \) then \( W(f) = x^\alpha y^\beta. \) The main result of this section is the following:

**Proposition 14.** Let \( A_n \) be the set of words having \( n \) occurrences of the letter \( b \) and \( n - 1 \) occurrences of the letter \( a \). For any \( n \geq 2 \), we have

\[
K_n(d, r) = \frac{d^{\binom{n-1}{2} - 1}}{r} L_n\left(\frac{1}{d}, rd\right)
\]

where

\[
L_n(x, y) = \frac{(1 - xy)}{(1 - x)(1 - y)} \left( \sum_{bfb \in A_n} W(bfb) - \sum_{afa \in A_n} W(afa) \right)
\]

5.5.1. **Comments on this proposition.** We start by some comments on the structure of \( L_n(x, y) \) that will appear along the proof.

First we remark that the expression of \( L_n(x, y) \) has the factor

\[
H(x, y) := \frac{(1 - xy)}{(1 - x)(1 - y)} = 1 + \sum_{i \geq 1} (x^i + y^i)
\]

that may be obviously interpreted as a generating function of cells of an infinite hook shape whose corner is cell \((0, 0)\) and a cell \((i, j)\) is weighted by \(x^i y^j\). This factor \(H(x, y)\) comes from a mix of two kind of geometric sums of common ratios \(x\), respectively \(y\).

We also remark that, ignoring the common factor \(H(x, y)\), we obtain a finite sum on factors and since \(W(f)\) is a monomial the second factor is a polynomial in \(x\) and \(y\). This polynomial which a sum on factors related to cyclic conjugates of \(wb\). This polynomial has not positive coefficients when the second sum do not vanish, but we will prove that \(H(x, y)P[w](x, y)\) is a series in \(x\) and \(y\) with positive coefficients.

The last remark is that one may not guess immediately the symmetry

\[
L_n(x, y) = L_n(y, x).
\]
Our combinatorial interpretation will show that this symmetry is obvious in our setting and also that it is inherited from the natural involution in Baker and Norine’s theorem. Let $w$ be a non-empty Dyck word of semi-length $n - 1$ and $u'$ the related sorted parking function. We map one-to-one the configurations of the trajectory $((u', u_n))_{u_n \in \mathbb{Z}}$ to the cell of the skew cylinder of circumference $n - 1$ cut by $w$ by assigning to it the cell with label $u_n$.

The inversion of relations between the pairs $(lw(S(s + 1)), rw(S(s + 1)))$ and $(deg((u', s)), \rho((u', s)))$ leads to

$$
deg((u', s)) = \left(\frac{n-1}{2}\right) + lw(u_n) - 1 - rw(u_n)
$$

whose translation in terms of series is the first relation of Proposition 14 between $K_n(d, r)$ and an evaluation of $L_n(x, y)$.

5.5.2. Mixing geometric sums delimited by $aa$ and $bb$ factors of the cut $wb$. Let $w$ be a Dyck word. We may consider the sequence of cells in order of their labels $w$ (spiral) curve that visit each cell of the cut skew cylinder $\text{Cyl}[w]$. In this sequence are found cells in the left and the right region. A cell with label $k < 0$, is in the right region. If the cell has label $k$ satisfying $k > 1 + n(n - 3)$, then it is at the left of the triangle $T(n)$ which contains the $w$ cut path hence it is in the left region. We thus know that there is an odd number $2m + 1$ ($m \geq 0$) of indices $k_1 \leq k_2 \leq \ldots k_{2m+1}$ such that the cells with labels $k_i$ and $k_i + 1$ are in different regions. In other words the finite sequence $k_i$ for $i = 1 \ldots 2m + 1$ collects the labels of the cells just before a crossing of the path $w$. On the example in Figure 12, $m = 5$ and $k_1, k_2, \ldots k_{2m+1} = -1, 0, 6, 7, 8, 11, 16, 23, 25, 34, 35$. We will call the integer $m$, the crossing multiplicity and the sequence $k_1, \ldots k_{2m+1}$ the crossing indices of the Dyck word $w$.

Lemma 10. Let $w$ be a Dyck word, we consider the sequence $k \in \mathbb{Z}$ in the cut skew cylinder $\text{Cyl}[w]$, $m$ its crossing multiplicity and $k_1, \ldots k_{2m+1}$ its crossing indices. For any $k \in \mathbb{Z}$ consider the cell with label $k$ and denote $M(k)$ the monomial $x^{rw(k)}y^{lw(k)}$. We have

$$
\sum_{k \in \mathbb{Z}} M(k) = H(x, y) \left( \sum_{i=0}^{m} M(k_{2i+1}) - \sum_{i=1}^{m} M(k_{2i}) \right).
$$

Proof Observe first the two following local facts:

- If $k$ is the label of a cell in the left region, then $lw(k) = lw(k - 1) + 1$ and $rw(k) = rw(k - 1)$, so that:

  $$
x^{rw(k)}y^{lw(k)} = y \times x^{rw(k-1)}y^{lw(k-1)}
$$

- If $k$ is the label of a cell in the right region, then $lw(k) = lw(k - 1)$ and $rw(k) = rw(k - 1) - 1$ so that

  $$
x^{rw(k)}y^{lw(k)} = \frac{1}{x} \times x^{rw(k-1)}y^{lw(k-1)}.
$$
Let \( a < b \), assume that for any \( k \) such that \( a + 1 \leq k \leq b \), the cell with label \( k \) lies in the left region, then a repeated use of the first observation shows that:

\[
\sum_{k=a+1}^{b} M(k) = \sum_{k=1}^{b-a} y^k M(a) = \frac{y - y^{b-a+1} M(a)}{1 - y} = \frac{y}{1 - y} (M(a) - M(b))
\]

where in the first equality we use that each cell with label \( k \) lies in the left region for \( k = a + 1, \ldots, b \) and the third equality uses \( M(b) = y^{b-a} M(a) \).

In the particular case where \( b = +\infty \) the term \( M(b) \) vanishes, so that:

\[
\sum_{k=a+1}^{\infty} M(k) = \frac{y M(a)}{1 - y}
\]

On the other hand, assume that for \( k = a + 1, \ldots, b \) each cell with label \( k \) is in the right region, similarly we have:

\[
\sum_{k=a+1}^{b} M(k) = \sum_{k=0}^{b-a-1} x^k M(b) = \frac{1 - x^{b-a}}{1 - x} M(b) = \frac{1}{1 - x} (M(b) - M(a))
\]

In the particular case where \( a = -\infty \), the terms \( M(a) \) vanishes so that:

\[
\sum_{k=-\infty}^{b} M(k) = \frac{M(b)}{1 - x}
\]

We split the sequence of cells with labels \( k \in \mathbb{Z} \) into maximal factors delimited by its sequence of crossing indices where all cells lie alternatively in the right region and the left region, beginning by the right one and ending with the left one.

\[
\sum_{k \in \mathbb{Z}} M(k) = \sum_{k=-\infty}^{k_1} M(k) + \sum_{i=1}^{2m} \left( \sum_{k=k_{i+1}}^{k_i} M(k) \right) + \sum_{k=k_{2m+1}+1}^{+\infty} M(k)
\]

For \( k \leq k_1 \) the cells of label \( k \) are in the right region, and for \( k > k_{2m+1} \) they are in the left region so that:

\[
\sum_{k \leq k_1} M(k) = \frac{M(k_1)}{1 - x} \quad \text{and} \quad \sum_{k > k_{2m+1}} M(k) = \frac{y M(k_{2m+1})}{1 - y}
\]

For \( k_{2i-1} < k \leq k_{2i} \) the cells of label \( k \) lie in the left region so that:

\[
\sum_{k = k_{2i-1}+1}^{k_{2i}} M(k) = \frac{y}{1 - y} (M(k_{2i-1}) - M(k_{2i}))
\]
For \( k_{2i} < k \leq k_{2i+1} \) the cells of label \( k \) lie in the right region so that:

\[
\sum_{k=k_{2i+1}}^{k_{2i+1}} M(k) = \frac{1}{1-x} (M(k_{2i+1}) - M(k_{2i}))
\]

Observe that in the sum \( \sum_{k \in \mathbb{Z}} M(k) \) the monomials \( M(k_{2i}) \) appear with a minus sign and the monomials \( M(k_{2i+1}) \) with a plus sign, moreover each one appears twice with factors \( \frac{y}{1-y} \) and \( \frac{1}{1-x} \) adding these two factors gives \( H(x, y) \) hence:

\[
\sum_{k \in \mathbb{Z}} M(k) = \left( \sum_{i=0}^{m} H(x, y)M(k_{2i+1}) \right) + \left( \sum_{i=1}^{m} (-H(x, y))M(k_{2i}) \right)
\]

\[\square\]

5.5.3. Evaluation of the monomials for the crossing indices of the Dyck path. In this section we translate the evaluation of \( M(k_i) \) by a formula involving the Dyck paths.

We first observe that a crossing point in the Dyck path described by the word \( w \) corresponds to a factor \( aa \) or \( bb \) in \( w \). Indeed since the Dyck path consists of North steps and East steps, the curve defined by a trajectory consisting in visiting the cells in order of their labels can only cross the path when there are two consecutive steps in the same direction. Hence the crossing cells \( k_{2i} \) correspond to factors \( aa \) in \( w \) while the crossing cells \( k_{2i+1} \) correspond to factors \( bb \). More precisely a cell with label \( k_{2i} \) is immediately on the left of the a North step of the Dyck path followed by another North step, while a cell with label \( k_{2i+1} \) is just below of an East step of the Dyck path followed by another East step. In order to take also into account the case \( k_1 = -1 \) it is convenient to add an occurrence of \( b \) at the end of \( w \), that is to consider the word \( wb \) instead of \( w \).

In the example considered above the correspondence between factors and crossing cells is given by the following table:

\[
\begin{array}{cccccccccccccccc}
\text{a} & \text{a} & \text{a} & \text{b} & \text{a} & \text{a} & \text{a} & \text{b} & \text{b} & \text{b} & \text{b} & \text{b} & \text{a} & \text{b} & \text{b} & \text{b} \\
0 & 11 & 23 & 24 & 35 & 25 & 16 & 6 & 7 & 8 & -1
\end{array}
\]

In the sequel we will consider that the label \( k_i \) of a crossing cell is also the label of the corresponding occurrence of a letter in the Dyck word. We will also define the conjugate \( \tau(wb, k) \) of Dyck word \( w \) followed by a letter \( b \), where \( k \) is the label of the occurrence of \( b \) appearing in \( w = w_1bw_2 \) as the the word \( bw_2w_1 \).

Let us examine more precisely what we do in order to compute \( rw(k) \) and \( lw(k) \) for a cell with label \( k \) corresponding to an occurrence of the factor \( bb \). This computation may be decomposed as the sum of \( n-1 \) values \( rw_j(k) \) and \( lw_j(w) \) each one corresponding to one row \( j \) of the strip. To determine \( rw_j(k) \) we count the number of cells with coordinates \((i, j)\) which lie in the right region and have a label greater than \( k \), similarly \( lw_j(k) \) is the number of cells with coordinates \((i, j)\) which lie in the left region and have a label less than or equal to \( k \). Since labels of cells in the same row decrease from left to right, for each \( j \) at least one of two values \( rw_j(k) \) and \( lw_j(k) \) is equal to 0.

In the following Lemma we use the definition of \( W(f) \) for a word \( f \) given before Proposition 14.
Lemma 11. Let \( w \) be a Dyck word and \( k \) be a crossing index of \( w \) in the cut skew cylinder \( \text{Cyl}[w] \). Then we have

\[
M(k_i) = W(\tau(w, k_i))
\]

Where \( \tau(w, k_i) \) is the conjugate of the word \( wb \) given as above by the position of the letter with label \( k_i \).

Proof Let \((i_1, j_1)\) be the coordinates of the cell with label \( k \) which we suppose to correspond to a factor \( bb \). Similar arguments may be used to prove the case where \( k \) is the label of a cell corresponding to the factor \( aa \). We have \( rw(k) = \sum_{j=0}^{n-2} rw_j(k) \) and \( lw(k) = \sum_{j=0}^{n-2} lw_j(k) \), the values \( rw_j(w) \) are determined by the value \((i_2, j)\) of the coordinates of the cell situated immediately on the left of the north step of the Dyck word situated at row \( j \). More precisely we have:

- If \( j > i_1 \) then
  \[
  rw_j(k) = \text{Max}(0, i_1 - i_2) \quad lw_j(k) = \text{Max}(0, i_2 - i_1)
  \]
- If \( j \leq i_1 \) then
  \[
  rw_j(k) = \text{Max}(0, i_1 - i_2 - 1) \quad lw_j(k) = \text{Max}(0, i_2 - i_1 + 1)
  \]

But the values \( i_1 - i_2 \) and \( i_1 - i_2 - 1 \) are exactly those of heights of \( \tau(w, k_i) \) increased by 1.

The proof of Proposition 14 follows directly from Lemmas 9, 10, and 11.

5.6. An involution explaining \( L_n(x, y) = L_n(y, x) \). For any \( n \geq 2 \), the symmetry

\[
L_n(x, y) = L_n(x, y)
\]

is deduced from the following involution \( \Psi \) on the cells in the skew cylinders \( \langle \text{Cyl}[w] \rangle_w \) where \( w \) is a Dyck word of semi-length \( n - 1 \).

The cell \( u_n \) in the skew cylinder \( \text{Cyl}[w] \) is mapped to the cell \( \psi_2(u_n, w) \) in the skew cylinder \( \text{Cyl}[\psi_1(w)] \) as follows:

\[
\Psi(w, u_n) = (\psi_1(w), \psi_2(u_n, w)) = (\Phi(w), k_{2m+1} - u_n)
\]

where \( \Phi(w) \) is the involution exchanging area and prerank in Proposition 12. \( m \) is the crossing multiplicity of \( w \) and \( k_{2m+1} \) the last crossing index.

Proposition 15. Let \( rw(u_n) \) and \( lw(u_n) \) be the right and left weight of the cell \( u_n \) in the skew cylinder \( \text{Cyl}[w] \). Let \( rw(\psi_2(u_n, w)) \) and \( lw(\psi_2(u_n, w)) \) be the right and left weight of the cell \( \psi_2(u_n, w) \) in skew cylinder \( \text{Cyl}[\psi_1(w)] \). We have

\[
\begin{align*}
  rw(u_n) &= lw(\psi_2(u_n, w)) \\
  lw(u_n) &= rw(\psi_2(u_n, w))
\end{align*}
\]

Proof This involution corresponds to the reversing of the ordering of the spiral curve along the cells of the skew cylinder. This reversing corresponds in Figure 6 to a rotation of rows to place at the top the row with the cell \( k_{2m+1} \) of the last crossing and then an exchange of left and right and also top and bottom. The shift \(+k_{2m+1}\) in the definition of \( \psi_2(u_n, w) \) implies that the extended embedded path \( \Phi(w)b \) still starts in bottom right corner of cell labeled by 0. In this transformation the cells in the right region with label greater than \( u_n \) become exactly the cells in the left region with label
less or equal to \( \psi(u_n, 2) \) showing first equality of the Proposition. The proof of the second equality is similar.

This symmetry is also proved less explicitly by the obvious symmetry in \( x \) and \( y \) in the expression of \( L_n(x, y) \) in Theorem [3]. The careful reader may observe that we do not enter into details claiming that this involution is also related to the involution \( u \rightarrow \kappa - u \) in Baker and Norine’s theorem.

5.7. A generating function for \( L_n(x, y) \). Proposition [14] is a finite and combinatorial description of \( K_n(r, d) \) via an evaluation of \( L_n(x, y) \) for \( n \geq 2 \). In order to make deeper connection with other chapters of combinatorics, we consider the generating function

\[
\mathbb{L}(x, y; z) = \sum_{n \geq 1} L_n(x, y) z^{n-1}
\]

and give a simple formula for it involving two copies of the Carlitz \( q \) analog series for Catalan numbers.

Before entering into the computation of this function, we need to give a value for \( L_1 \).

On the complete graph \( K_1 \). On \( K_1 \), a configuration is limited to the value \( u_1 \) assigned to the unique vertex. Its degree is \( u_1 \) and its rank is \(-1\) if \( u_1 < 0 \) and equal to \( u_1 \) otherwise so that:

\[
K_1(d, r) = \frac{1}{r} \frac{1}{1 - \frac{1}{d}} + \frac{1}{1 - rd}
\]

Applying the formula

\[
K_n(d, r) = \frac{d^{(n-1)} - 1}{r} L_n\left(\frac{1}{d}, rd\right)
\]

gives

\[
L_1(x, y) = \frac{x}{1 - x} + \frac{1}{1 - y} = H(x, y)
\]

5.7.1. A formula for the generating function of \( L_n(x, y) \). The generating series of the Tutte polynomials of the complete graphs, were proved to have compact expressions (see [15, chapter 5], [21, equation (17)]).

Notice that these polynomials enumerate the spanning trees (in bijection with recurrent configurations) of \( K_n \) using two parameters external and internal activity.

For the series \( \mathbb{L}(x, y; z) \) can also be expressed by a compact formula as a “simple” rational function of \( z \) and two copies of Carlitz \( q \)-analogue:

\[
C(q, z) = \sum_{n \geq 0} \sum_{w \in D_n} q^{\text{area}(w)} z^n
\]

where \( D_n \) denotes the set of Dyck words of semi-length \( n \).

**Theorem 6.** We have

\[
\mathbb{L}(x, y; z) = \frac{(1 - xy) C(x, xz) + C(y, yz) - C(x, xz)C(y, yz)}{(1 - x)(1 - y) - C(x, xz)zC(y, yz)}.
\]
Proof. For \( n = 1 \), we use the previously computed generating function \( L_1(x, y) = H(x, y) \).

For \( n \geq 2 \), we use the description of \( L_n(x, y) \) in Proposition 14. To estimate \( \sum_{n \geq 2} L_n(x, y) z^{n-1} \) we have to compute the generating function of words on the alphabets \( \{a, b\} \) starting and ending by different occurrences of the same letter and containing one more occurrence of the letter \( b \) than that of the letter \( a \). Since there is a negative sign to the weight if the first letter is \( a \) and a positive sign if the first letter is \( b \) we discuss separately this two cases. We denote by \( X \) the language consisting of all Dyck words, including the empty word denoted by \( \epsilon \). We denote \( \alpha \) the morphism on words on alphabet \( \{a, b\} \) exchanging the letters \( a \) and \( b \), so that \( \alpha(a) = b \) and \( \alpha(b) = a \) and denote \( Y = \alpha(X) \).

Let \( f = bf''b \) be a word containing \( n \) occurrences of the letter \( b \), \( n - 1 \) occurrences of the letter \( a \) (and such that the first and the last letters are occurrences of \( b \)).

This word may be interpreted as the sequence of steps of a walk on the line \( \mathbb{Z} \), a letter \( a \) corresponding to an increment, and a letter \( b \) to a decrement. We assume that this walk starts at 1 and hence ends at 0 and decompose it at each step from 1 to 0 or 0 to 1. In terms of language, this decomposition leads to the following non-ambiguous description of these words:

\[ bYa(XbYa)^*Xb. \]

The weight \( W[f](x, y) \) is distributed on the occurrences of the letter \( a \) as in its definition. We add a factor \( z \) to each occurrence of the letter \( a \) in order to take into account the size of words. Each factor \( X \) corresponds to a “Dyck” walk starting and ending at 1, so compared to the usual definition, the height of a step is incremented by one. Hence the generating function of this factor is \( C(x, xz) \). Similarly, each factor \( Y \) has generating function \( C(y, yz) \). Each \( a \) step from 0 to 1 is weighted by \( z \) so the previous language decomposition leads to the generating function

\[
\frac{C(x, xz)zC(y, yz)}{1 - C(x, xz)zC(y, yz)}.
\]

We repeat a similar proof for words whose extremal letters are two different occurrences of the letter \( a \). The interpretation in terms of walk on the line and its decomposition according to the same steps leads to the formula

\[
(X\{\epsilon\})b(YaXb)^*(Y\{\epsilon\})
\]

where at some steps we had to exclude the empty word to guarantee the extremal occurrences of letter \( a \). This leads to the generating function for this case

\[
\frac{(C(x, xz) - 1)(C(y, yz) - 1)}{1 - C(x, xz)zC(y, yz)}.
\]

Hence, the difference of the two generating functions gives

\[
\sum_{n \geq 2} L_n(x, y) z^{n-1} = \frac{(1 - xy)}{(1 - x)(1 - y)} \frac{C(x, xz)zC(y, yz) - (C(x, xz) - 1)(C(y, yz) - 1)}{1 - C(x, xz)zC(y, yz)}.
\]

This formula added to the case \( n = 1 \) ends the proof of our Theorem. □
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THE RIEMANN-ROCH THEOREM FOR GRAPHS AND THE RANK IN COMPLETE GRAPHS

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ABSTRACT. The paper by M. Baker and S. Norine in 2007 introduced a new parameter on configurations of graphs and gave a new result in the theory of graphs which has an algebraic geometry flavor. This result was called Riemann-Roch formula for graphs since it defines a combinatorial version of divisors and their ranks in terms of configuration on graphs. The so called chip firing game on graphs and the sandpile model in physics play a central role in this theory.

In this paper we give a presentation of the theorem of Baker and Norine in purely combinatorial terms, which is more accessible and shorter than the original one.

An algorithm for the determination of the rank of configurations is also given for the complete graph $K_n$. This algorithm has linear arithmetic complexity. The analysis of number of iterations in a less optimized version of this algorithm leads to an apparently new parameter which we call the prerank. This parameter and the parameter $\text{dinv}$ provide an alternative description to some well known $q,t$-Catalan numbers. Restricted to a natural subset of configurations, the two natural statistics degree and rank lead to a distribution which is described by a generating function which, up to a change of variables and a rescaling, is a symmetric fraction involving two copies of Carlitz $q$-analogue of the Catalan numbers.

We consider the following solitary game on an undirected (non oriented) connected graph $G = (X, E)$ without loops: at the beginning integer values $f_i$ are attributed to the $n$ vertices $x_1, x_2, \ldots, x_n$ of the graph, these values can be positive or negative and define a configuration $f$. At each step a toppling can be performed by the player on a vertex $x_i$, it consists in subtracting $d_i$ (the number of edges incident to $x_i$) to the amount $f_i$ and for each neighbor $x_j$ of $x_i$ increase $f_j$ by the number of edges between these two vertices. In this operation the amount of vertex $x_i$ may become, or stay negative. The aim of the player is to find a sequence of toppling operations which will end with a configuration where all the $f_i$ are non negative. Since the sum of the $f_i$ is invariant by toppling, a necessary condition to succeed is that in the initial configuration this sum should be non negative. We will see that this condition is not sufficient.

This game has much to do with the chip firing game (see [6], [5]) and the sandpile model (see [1], [10], [11]), for which recurrent configurations where defined and proved to be canonical representatives of the classes of configurations equivalent by a sequence of topplings (for a more algebraic treatment see also [24]).

The game was introduced and studied in detail by Baker and Norine in [2] who also introduced a new parameter on graph configurations : the rank. One characteristic of the rank $\rho(f)$ of a configuration $f$ is that it is non negative if and only if one can

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get from \( f \) a configuration non-negative on every vertices by performing a sequence of topplings. For this parameter they obtain a simple formula expressing a symmetry similar to the Riemann-Roch formula for surfaces and curves (a classical reference to this formula is the book by H. Farkas and I. Kra [15]).

Our aim here is to give a simple presentation of this Riemann-Roch like theorem for all graphs and study the values of this parameter in the particular case where \( G \) is a complete graph on \( n \) vertices. For these complete graphs, it was noticed (see Proposition 2.8. in [8]) that the recurrent configurations correspond to the parking functions which play a central role in combinatorics. We obtain a simple algorithm to compute the rank in that case, of linear arithmetic complexity, while there is no known polynomial time algorithm to compute that rank for arbitrary graphs (see [3] [22]). This algorithm suggests the introduction of a new parameter on parking functions and Dyck paths which we call prerank. For this parameter we study some properties in relation with other known parameters on Dyck paths. The degree is the other statistic on configuration involved by Baker and Norine theorem. The restriction to sorted parking configurations select exactly one representative of each possible run of the algorithm up to the symmetries of the complete graph with a pointed vertex. We prove that the generating function of sorted parking configurations according to their degree, their rank and the size of complete graph, is, up to a change of variable and a rescaling, a symmetric fraction in two copies of Carlitz q-analogue of Catalan’s numbers.

1. Configurations on a graph

Let \( G = (X, E) \) be a multi-graph with \( n \) vertices, where \( X = \{x_1, x_2, \ldots, x_n\} \) is the vertex set and \( E \) is a symmetric matrix such that \( e_{i,j} \) is the number of edges with endpoints \( x_i, x_j \), hence \( e_{i,j} = e_{j,i} \). In all this paper \( n \) denotes the number of vertices of the graph \( G \) and \( m \) the number of its edges. We suppose that \( G \) is connected and has no loops, so that \( e_{i,i} = 0 \) for all \( i \).

We will consider configurations on a graph, these are elements of the discrete lattice \( \mathbb{Z}^n \). Each configuration \( f \) may be considered as assigning (positive or negative) tokens to the vertices. The symbol \( \varepsilon^{(i)} \) will denote the configuration in which the value 1 is assigned to vertex \( x_i \) and the value 0 is assigned to all other vertices.

The degree of the configuration \( f \) is the sum of the \( f_i \)'s it is denoted by \( \text{deg}(f) \).

1.1. The Laplacian configurations. These configurations correspond to the rows of the Laplacian matrix of a graph, a classical tool in Algebraic Graph Theory.

The Laplacian configuration \( \Delta^{(i)} \) is given by: \( \Delta^{(i)} = d_i \varepsilon^{(i)} - \sum_{j=1}^{n} e_{i,j} \varepsilon^{(j)} \), where \( d_i = \sum_{j=1}^{n} e_{i,j} \) is the degree of the vertex \( x_i \). This \( n \) configurations which degrees are equal to 0 play a central role throughout this paper.

We denote by \( L_G \) the subgroup of \( \mathbb{Z}^n \) generated by the \( \Delta^{(i)} \), and two configurations \( f \) and \( g \) will be said toppling equivalent if \( f - g \in L_G \), which will also be written as \( f \sim_{L_G} g \).

In the sandpile model, the transition from configuration \( f \) to the configuration \( f - \Delta^{(i)} \) is allowed only if \( f_i \geq d_i \) and is called a toppling, it is called a firing in the theory of chip firing games, here we omit this condition and perform topplings even if \( f_i < d_i \).
Notice that $\sum_{i=1}^{n} \Delta^{(i)} = 0$ and that for a connected graph this is the unique relation (up to multiplication by a constant) satisfied by the $\Delta^{(i)}$, moreover the principal minors of the Laplacian matrix are all equal to the number of spanning trees of the graph.

1.2. **Recurrent configurations.** We use here the notation usually considered in the sandpile model, so that we will call sandpile configuration a configuration $f$ such that $f_i \geq 0$ for all $i < n$. This corresponds to the fact that in the sandpile model the vertex $x_n$ is considered as a sink collecting tokens, so that the number of tokens of the sink is not considered in this context.

**Definition 1.** In the sandpile model, a toppling on vertex $x_i$, where $i \neq n$, may occur in a sandpile configuration only if if $f_i \geq d_i$. A sandpile configuration $f$ is stable if no toppling can occur, that is $f_i < d_i$ for all $i < n$.

Notice that when a toppling occurs in the sandpile model, the configuration $f - \Delta^{(i)}$ is also a sandpile configuration.

The toppling operation for a sandpile configuration will be denoted by $f \rightarrow g$. We also write:

$$f \circlearrowright g$$

if $f$ and $g$ are sandpile configurations and if $g$ is obtained from $f$ by a sequence of toppling operations meeting only sandpile configurations. Notice that $f \circlearrowright g$ implies $f \sim_{LG} g$.

Sequences of topplings may be performed in any order until a stable configuration is attained as the following proposition states, the proof of which may be found in [12] or in [20] pages 42, and 70.

**Proposition 1.** For any sandpile configuration $f$ there exists a unique stable configuration $f'$ such that $f \rightarrow f'$.

A configuration is recurrent in an evolving system if it could be observed after a long period of the evolution of the system. In the case of the sandpile model, the system is considered to evolve by adding a token in any cell at random and then applying topplings until a stable configuration is reached. This translates into the following notion which is central:

**Definition 2.** A configuration $f$ is recurrent if it is stable and there exists a sandpile configuration $g \neq 0$ such that $f + g \rightarrow f$.

The following important result, giving canonical representatives in the classes of the relation $\sim_{LG}$ is obtained in [9, 5, 8] by different ways.

**Theorem 1.** For any configuration $f$ there exists a unique recurrent configuration $g$ such that $f \sim_{LG} g$.

In order to characterize the recurrent configurations D. Dhar used the configuration $\Delta^{(n)}$ and proposed the following algorithm.
Theorem 2. Burning Algorithm. The stable configuration $u$ is recurrent if and only if
\[ f - \Delta^{(n)} \xrightarrow{*} f \]
Moreover in this sequence of topplings each vertex different from $x_n$ topples exactly once.

This algorithm can be translated into another characterization, giving:

Corollary 1. A stable configuration $f$ is recurrent if and only if for any non-empty subset $Y$ of $X \setminus \{x_n\}$ there is at least an $x_k$ in $Y$ such that its degree in the subgraph spanned by $Y$ is greater than or equal to $f_k$, more precisely if the following condition is satisfied:
\[ f_k \geq \sum_{x_i \in Y} e_{i,k} \]

**Proof:** Let $f$ be a recurrent configuration, and $Y$ be a subset of $X$, then by Dhar’s Burning Algorithm, starting from the configuration $f - \Delta^{(n)}$ there is a sequence of topplings of the vertices in which any vertex topples. We may suppose that the vertices are numbered in the order in which they topple, $x_1$ just after $x_n$, then $x_2$ and so on until $x_{n-1}$ then for allowing a toppling at vertex $x_i$ each $f_i$ has to satisfies the condition:
\[ d_i \leq f_i + \sum_{j=1}^{i-1} e_{i,j} \]

Now for any subset $Y$ of $X$, let $k$ be the smallest integer such that $x_k \in Y$, then since there is no $x_i \in Y$ with $i$ less than $k$ we have:
\[ d_k \geq \sum_{j=1}^{k-1} e_{j,k} + \sum_{x_i \in Y} e_{i,k} \]
Putting $i = k$ in the first inequality and the two inequalities together gives the result.

Conversely if $f$ is a stable configuration satisfying condition (1.1) we build a toppling sequence starting with vertex $x_n$, then taking as $x_1$ the vertex in $Y = \{x_1, x_2, \ldots, x_{n-1}\}$ satisfying $f_1 \geq \sum_{i=2}^{n-1} e_{1,i}$, this vertex can topple after $x_n$ since in that case $f_1 + e_{1,n} \geq \sum_{i=2}^{n} e_{1,i} = d_1$. Then at each step, a vertex $x_j$ such that $f_j \geq \sum_{i=j+1}^{n-1} e_{i,j}$ exists taking $Y = X \setminus \{x_n, x_1, x_2, \ldots, x_{j-1}\}$, this vertex can topple at this stage. We have thus built a sequence of topplings proving that $f$ is recurrent.

\[ \square \]

1.3. Parking configurations. We consider a kind of dual notion to that of recurrent configuration, such configurations are often called parking configurations since in the case of complete graphs, these are exactly the parking functions, a central object in combinatorics.

**Definition 3.** A sandpile configuration $f$ on a graph $G$ is a parking configuration if for any subset $Y$ of $X \setminus \{x_n\}$ there is a vertex $x_k$ in $Y$ such that $f_k$ is less than the number of edges which are incident to $x_k$ and a vertex out of $Y$. More precisely if the exists $x_k \in Y$ such that $f_k < \sum_{x_i \notin Y} e_{i,k}$. 
In other words a sandpile configuration $f$ is a parking configuration if and only if there is no toppling of all the vertices in a subset $Y$ of $\{x_1, x_2, \ldots, x_{n-1}\}$ leaving all the $f_i \geq 0$. For this reason these configurations are also called superstable (as for instance in [24]).

**Proposition 2.** Let $f$ be a stable configuration and let $\delta$ be the configuration such that $\delta_i = d_i - 1$. Define $\beta(f) = \delta - f$. Then $f$ is recurrent if and only if $\beta(f)$ is a parking configuration.

**Proof:** It suffices to compare Corollary 1 and Definition 3 and to notice that:

$$d_k = \sum_{x_j \not\in Y} e_{k,j} + \sum_{x_j \in Y} e_{k,j}$$

\(\square\)

**Corollary 2.** For any configuration $f$ there exists a unique parking configuration $g$ such that $f \sim_{LG} g$.

**Proof:** For any configuration $f$ let $g$ be the recurrent configuration such that $g \sim_{LG} \delta - f$ then $\beta(g)$ is a parking configuration such that $f \sim_{LG} \beta(g)$. \(\square\)

In this paper we will often consider the parking configuration in a class as a representative of this class. A parking configuration $f$ in a graph with $n$ vertices will be represented by the subsequence consisting of this first $n-1$ terms and an integer $s$ such that:

$$(f_1, f_2, \ldots, f_{n-1}) \quad s = f_n$$

hence $s$ represents the number of tokens on the distinguished often called ”sink” vertex $x_n$.

**Parking configurations and acyclic orientations.** An orientation of $G$ is a directed graph obtained from $G$ by orienting each edge, so that one end vertex becomes the head and the other one the tail. A directed path in such a graph consists of a sequence of edges such that the head of an edge is equal to the tail of the subsequent one.

The orientation is acyclic if there is no directed circuit, i.e. a directed path starting and ending at the same vertex. We associate to any parking configuration $f$ an acyclic orientation by:

**Proposition 3.** For any parking configuration $f$ on $G = (X, E)$ there exists at least one acyclic orientation $\overrightarrow{G}$ such that for any vertex $x_i$, $i \neq n$, $f_i$ is strictly less than its indegree $d_i^-$ (i.e. the number of edges with head $x_i$).

**Proof:** We orient the edges using an algorithm that terminates after $n$ steps. Consider $Y = \{x_1, x_2, \ldots, x_{n-1}\}$. From the definition of parking configurations, there is at least one vertex $x_i$ such that $f_i < e_{i,n}$ then orient all these $e_{i,n}$ edges from $x_n$ (the tail) to $x_i$ (the head), and remove $x_i$ from $Y$. Repeat the following operation until $Y$ is empty:
• Find $x_k$ in $Y$ such that $f_k < \sum_{x_j \not\in Y} e_{k,j}$; orient all the edges joining any vertex $j$ outside $Y$ to $x_k$ from $x_j$ to $x_k$ and remove $x_k$ from $Y$.

In the preceding proof one may recognize a scheduling of topplings related to the Dhar criterion applied to the recurrent configuration $\beta(f)$. Notice that more precise results involving maximal parking configurations are given in [4].

2. Configurations on the complete graph

2.1. Configuration classes in $K_n$. In the complete graph $K_n$ each of the $n$ vertices has all the $n - 1$ other vertices as neighbors via a simple edge.

For a configuration $f$ in the complete graph $K_n$ the determination of the parking configuration equivalent to it is facilitated by the following lemma and its corollaries:

**Lemma 1.** A configuration $f$ of $K_n$ is toppling equivalent to 0 if and only if the two following conditions are satisfied:

$$\deg(f) = 0 \quad \text{and} \quad \forall i, j \leq n \quad f_i = f_j \pmod{n}$$

**Proof:** The necessary condition follows from the fact that these relations are not modified by any toppling and are satisfied by the parking configuration equivalent to 0 which is $(0, \ldots, 0)$ and by all the $\Delta^{(i)}$.

The sufficient condition is obtained by induction on the sum of the $|f_i|$. More precisely, for a given configuration $f$ satisfying the conditions above, let $i$ be such that $|f_i|$ is maximal. Replacing $f$ by $-f$ if necessary, allows to assume $f_i > 0$.

We consider two cases:

- If $f_i > n$ then, since $\deg(f) = 0$, there exists $j$ such that $f_j < 0$. Let us consider $g = f - \Delta^{(i)} + \Delta^{(j)}$. We have $g_k = f_k$ for $k \not\in \{i, h\}$ and $g_i = f_i - n, g_j = f_j + n$ hence this configuration satisfies also the conditions of the Lemma; and $\sum_i |g_i| < \sum_i |f_i|$ which allows us to use the inductive hypothesis and obtain $g \sim_{L_G} 0$ and hence $f = g + \Delta^{(i)} - \Delta^{(j)} \sim_{L_G} 0$.

- If $0 < f_i \leq n$, the equations above imply that the only possible values for $f_j$ are either $f_i$ or $f_i - n$. Let $k \geq 1$ denote the number of $j$ such that $f_j = f_i$. Since $\deg(f) = 0$ we have:

$$kf_i + (n - k)(f_i - n) = 0$$

hence $k = n - f_i$. Let $I = \{j | f_j = f_i\}$ then it is then not difficult to check that:

$$f = \sum_{j \in I} \Delta^{(j)}.$$ 

Notice that this characterization may also be considered as given by the toppling invariants defined in [13, equation(3.11)].

Using the fact that $f \sim_{L_G} g$ if and only if $f - g \sim_{L_G} 0$ we obtain:
Corollary 3. Two configurations $f$ and $g$ are toppling equivalent in $K_n$ if and only if the following holds

$$\text{deg}(f) = \text{deg}(g) \quad \text{and for any } 1 \leq i, j \leq n, \; f_i - f_j = g_i - g_j \; (\text{mod } n).$$

Another consequence is the following:

Corollary 4. Any class of the toppling equivalence on $K_n$ contains exactly $n$ sandpile configurations such that $0 \leq f_i < n$ for all $i < n$, all the stable configurations of the class are among them.

Proof: For each class of the toppling equivalence there is a parking configuration $f$ which clearly satisfies the condition. Now for any $k = 1, 2, \ldots, n - 1$ the configuration $f^{(k)}$ given by

$$f_i^{(k)} = f_i + k \; (\text{mod } n)$$

for $i < n$ and $f_n^{(k)} = \text{deg}(f) - \sum_{i=1}^{n-1} f_i^{(k)}$, is a configuration equivalent to $f$ and all these configurations are distinct.

Conversely a configuration $g$ satisfying $g_i \leq n - 1$ for all $i < n$ equivalent to $f$ is equal to one of the $f^{(k)}$, where $k = f_1 - g_1 \; (\text{mod } n)$.

As a direct consequence of Corollary 3 we have an efficient algorithm computing from any configuration of $K_n$ an equivalent configuration with relatively small values of the $f_i$ for $i < n$:

**Algorithm.** Given a configuration $f$ of $K_n$ one can find a configuration $g$ toppling equivalent to $f$ and such that $0 \leq g_i < n$ for any $1 \leq i \leq n - 1$ by setting:

$$g_i = f_i - f_1 \; (\text{mod } n) \; \text{for } i < n, \quad \text{and} \quad g_n = \text{deg}(f) - \sum_{i=1}^{n-1} g_i.$$

Notice that this algorithm performs $O(n)$ arithmetic operations on integers.

### 2.2. Parking configurations

Since in the complete graph $K_n$ the $n$ vertices have all the $n - 1$ other vertices as neighbors, a configuration $f$ is a parking configuration if for any subset $Y$ of $\{x_1, x_2, \ldots, x_{n-1}\}$ containing $p$ vertices there is at least one vertex $x_k \in Y$ such that $f_k < n - p$. The sequences $f_1, f_2, \ldots, f_{n-1}$ satisfying this condition are well known in combinatorics and are called parking functions, they are characterized by the simpler following condition:

**Proposition 4.** A sequence $(f_1, f_2, \ldots, f_{n-1})$ corresponds to the first $n - 1$ values of a parking configuration of $K_n$, if and only if after reordering it as a weakly increasing sequence $(g_1, g_2, \ldots, g_{n-1})$ one has $g_i < i$, for all $i = 1, \ldots, n - 1$. 
2.3. Dyck words. We consider words on the alphabet with two letters \{a, b\}. For a word \(w\), we denote \(|w|_x\) (where \(x \in \{a, b\}\)) the number of occurrences of the letter \(x\) in the word \(w\). Hence the length \(|w|\) of \(w\) is equal to \(|w|_a + |w|_b\), we use also the mapping \(\delta\) associating to any word \(w\) the integer \(\delta(w) = |w|_a - |w|_b\). The word \(u\) is a prefix of \(w\) if \(w = uv\), this prefix is strict if \(u \neq w\). Two words \(w, w'\) are conjugate, if they may be written like \(w = uv, w' = vu\). For any positive integer \(n\), we denote \(A_n\) the set of words containing \(n - 1\) occurrences of the letter \(a\) and \(n\) occurrences of \(b\). Notice that a word \(w\) is in \(A_n\) if and only if \(|w| = 2n - 1\) and \(\delta(w) = -1\). The set \(D_n\) of Dyck words followed by an occurrence of \(b\) will have a central role in the study of configurations on \(K_n\), it is defined as follows,

**Definition 4.** The subset \(D_n\) of \(A_n\) is the set of words \(w\) such that \(\delta(u) \geq 0\) for any strict prefix \(u\) of \(w\).

A Dyck word is a word \(w\) such that \(|w|_a = |w|_b\) and \(\delta(u) \geq 0\) for any prefix \(u\) of \(w\), hence \(w\) is a Dyck word if and only if \(wb \in D_n\) for \(n = |w|_a + 1\).

The classical Cyclic Lemma (see [14]) may be stated as follows:

**Lemma 2.** Any word \(w\) of \(A_n\) admits a unique factorization

\[ w = uv \]

such that \(vu\) is in \(D_n\). Moreover \(u\) is the shortest prefix of \(w\) such that \(\delta(u)\) is minimal among all prefixes of \(w\).

We will often consider configurations \(f\) satisfying the following condition:

\[ 0 \leq f_1 \leq f_2 \leq \ldots \leq f_{n-1} \leq n \quad (2.1) \]

**Definition 5.** To any configuration satisfying condition \((2.1)\) we associate the word \(\phi_1(f)\) of \(A_n\) such that the prefix of \(\phi_1(f)\) ending with the \(i\)-th occurrence of the letter \(a\) contains exactly \(f_i\) occurrences of the letter \(b\).

**Proposition 5.** Let \(f, g\) be two configurations of \(K_n\) such that their first \(n - 1\) values satisfy condition \((2.1)\). If \(\deg(f) = \deg(g)\), \(\phi_1(f) = uv, \phi_1(g) = vu\) then:

\[ g \sim_{LG} f' = (f_{p+1}, \ldots, f_{n-1}, f_1, \ldots, f_p, f_n) \]

where \(p = |u|_a\).

**Proof:** Let \(q\) be the number of occurrences of \(b\) in \(u\), and \(p' = n - 1 - p\). Since \(\phi_1(f) = uv\) and \(\phi_1(g) = vu\) we have:

\[
g_i = \begin{cases} f_{p+i} - q & \text{if } 0 < i < n - p \\ f_{i-p'} + n - q & \text{if } n - p \leq i < n \end{cases}
\]

This shows that \(g_i - f'_i = -q\) if \(i < n - p\) and \(g_i - f'_i = n - q\), if \(i \geq n - p\) but all these are equal \(\mod n\), and the result follows from Corollary \([3]\) since \(\deg(f') = \deg(f) = \deg(g)\).

**Remark 1.** Let \(f, g, u, v, p, q\) be as in the Proposition above then:

\[ g_n = f_n + n(q - p) - q.\]
Proof: The first \( n - 1 \) values of \( f \) and \( g \) are such that \( n - p - 1 \) values pf \( f \) are decreased by \( q \) and \( p \) others are increased by \( n - q \) giving: \( \sum_{i=1}^{n-1} g_i = \sum_{i=1}^{n-1} f_i - q(n - p - 1) + (n - q)p \), hence computing \( \deg(g) \) we have:

\[
\deg(g) = g_n + \sum_{i=1}^{n-1} g_i = g_n + \sum_{i=1}^{n-1} f_i + n(p - q) + q
\]

The result follows from \( \deg(g) = \deg(f) = f_n + \sum_{i=1}^{n-1} f_i \). \( \square \)

We introduce two mappings \( \phi \) and \( \psi \), on configurations of \( K_n \) the first associates to any configuration a word in \( D_n \), and the second one associates to it an integer, these two values will give many informations on the configuration.

**Definition 6.** Let \( f \) be any configuration on \( K_n \), \( g \) the parking configuration such that \( f \sim_{L_G} g \) and \( g' \) the configuration obtained from \( g \) by reordering the \( n - 1 \) first terms in weakly increasing order, then the mapping \( \phi \) and \( \psi \) are defined by:

\[
\phi(f) = \phi_1(g'), \quad \psi(f) = g_n.
\]

Notice that from the definition of parking configurations, for any \( i \) the \( i \)-th occurrence of \( a \) in \( \phi(f) \) is preceded by less than \( i \) occurrences of \( b \), showing that \( \phi(f) \) is a word in \( D_n \). Since there is a unique parking configuration in any class of \( \sim_{L_G} \) we have that \( f \sim_{L_G} g \) implies \( \phi(f) = \phi(g) \) and \( \psi(f) = \psi(g) \).

**Lemma 3.** Let \( f, g \) be two configurations such that the first \( n - 1 \) entries of \( g \) are obtained by permuting those of \( f \), then \( \phi(f) = \phi(g) \)

**Proof:** Let \( \alpha \) be the permutation such that \( f_i = g_{\alpha(i)} \) for \( 1 \leq i \leq n - 1 \) and let \( f - \sum_{i=1}^{n-1} a_i \Delta(i) \) be the parking configuration equivalent to \( f \), then \( g - \sum_{i=1}^{n-1} a_{\alpha(i)} \Delta(\alpha(i)) \) is the parking configuration equivalent to \( g \), showing that these two parking configurations differ only by a permutation of their first \( n - 1 \) entries. Hence their sorted versions are equal ending the proof. \( \square \)

**Proposition 6.** For any configuration \( f \) satisfying condition (2.1), the word \( \phi(f) \) is the unique conjugate of \( \phi_1(f) \) which is an element of \( D_n \).

**Proof:** Let \( \phi_1(f) = uv \) be such that \( vu \) is an element of \( D_n \), and let \( g \) be the configuration such that \( \deg(g) = \deg(f) \) and \( \phi_1(g) = vu \). Clearly \( g \) is a parking configuration and is by Proposition 3 such that a permutation of it is \( \sim_{L_G} \) equivalent to \( f \). By the preceding Lemma we have that \( \phi(f) = \phi(g) = \phi_1(g) \). \( \square \)

A linear time algorithm to compute the parking configuration equivalent to a given configuration. The preceding proof gives also an algorithm finding the recurrent configuration of any configuration, this algorithm performs \( O(n) \) arithmetic operations on integers. Starting from any configuration \( f \), find the equivalent configuration \( f' \) which first \( n - 1 \) values are non negative and not greater than \( n \), as described at the end of section 2.1.
Then sort the first \( n - 1 \) first entries of \( f' \) (this can be done in linear time, since the sorted values are between 0 and \( n - 1 \)) to obtain \( f'' \). Let \( w = \phi_1(f'') \) and determine the decomposition \( w = uw \) by identifying in linear time the shortest prefix \( u \) of \( w \) such that \( \delta(u) \) is minimal. Determine \( q = |u|_b \) then deduce the parking configuration \( g \) by:

\[
g_i = \begin{cases} f'_i - q & \text{if } f'_i \geq q \\ f'_i + n - q & \text{if } f'_i < q \end{cases}
\]

We extract from the end of this algorithm and the proof of Proposition 6 a corollary on the sorted parking configuration toppling equivalent to a configuration \( f \).

We will say that two configurations \( f \) and \( g \) of \( K_n \) are toppling equivalent after permutation, if there exists a permutation \( \sigma \in S_{n-1} \) such that \( f' = \sigma(f) = (f_{\sigma(1)}, \ldots, f_{\sigma(n-1)}, f_n) \) is such that \( f' \sim_{L_G} g \).

**Corollary 5.** Let \( w \) be any word of \( A_n \), and let \( w = uv \) be the factorisation of \( w \) given by the cyclic lemma (Lemma 2). Then two configurations \( f \) and \( g \) satisfying equation (2.1) and the equalities:

\[
\phi_1(f) = uv, \quad \phi_1(g) = vu, \quad \deg(f) = \deg(g)
\]

are such that \( g \) is a parking configuration which is toppling equivalent after permutation to the configuration \( f \).

3. **Effective configurations**

We come back in this section with general graphs \( G \) not necessarily equal to a complete graph, define the notion of \( L_G \)-effective configuration and recall the main results of [2], the proofs we give in this section are more or less a reformulation in our terms of the proofs of them given in [2]. The game described in the introduction can be translated in determining if a configuration is \( L_G \)-effective with the following definition of effectiveness:

**Definition 7.** A configuration \( f \) is effective if \( f_i \geq 0 \) for all \( i \). A configuration \( f \) is \( L_G \)-effective if there exists an effective configuration \( g \) toppling equivalent to \( f \) (recall that this means \( f - g \in L_G \)).

Since two equivalent configurations by \( \sim_{L_G} \) have the same degree, it is clear that a configuration with negative degree is not \( L_G \)-effective. However we will prove that configurations with positive degree are not necessarily \( L_G \)-effective as these two examples show:
3.1. **Configuration associated to an acyclic orientation of** $G$. As already seen in Section 1.3, an orientation of $G$ is a directed graph obtained from $G$ by orienting each edge, that is distinguishing for each edge with end points $x_i$ and $x_j$ which one is the head and the other being the tail. The orientation is acyclic if there is no directed circuit. Let $\overrightarrow{G}$ be an acyclic orientation of $G$, we define the configuration $f_{\overrightarrow{G}}$ by:

$$(f_{\overrightarrow{G}})_i = d_i^\rightarrow - 1$$

Where $d_i^\rightarrow$ is the number of edges which have head $x_i$. The configuration represented in the right of Figure 1 is equal to $f_{\overrightarrow{G}}$ for the orientation of $G$ represented in Figure 2.

**Proposition 7.** The configuration associated to an acyclic orientation of $G$ is not $L_G$-effective.

**Proof:** We will show that for any linear combination $g = \sum_{i=1}^{n} a_i \Delta^{(i)}$ the sum $h$ of $g$ and $f_{\overrightarrow{G}}$ is not an effective configuration. Let $\varepsilon_{i,j}$ denote the number of edges with head $x_j$ and tail $x_i$. Then $\varepsilon_{i,j} = \varepsilon_{i,j} + \varepsilon_{j,i}$ (but notice that since the orientation is acyclic, at least one of the two values in the sum above is equal to 0).
For any vertex $x_i$ of $G$ we have $d_i^- = \sum_{j=1}^{n} \varepsilon_{j,i}$ so that:

$$\eta_i = -1 + \sum_{j=1}^{n} \varepsilon_{j,i} + a_i d_i - \sum_{j=1}^{n} a_j \varepsilon_{i,j}$$

Using $d_i = \sum_{j=1}^{n} e_{j,i}$ and decomposing each $\varepsilon_{i,j}$ into $\varepsilon_{i,j} + \varepsilon_{j,i}$ gives:

$$\eta_i = -1 + \sum_{j=1}^{n} \varepsilon_{j,i} + a_i \sum_{j=1}^{n} (\varepsilon_{i,j} + \varepsilon_{j,i}) - \sum_{j=1}^{n} a_j (\varepsilon_{i,j} + \varepsilon_{j,i})$$

(3.1)

Giving:

$$\eta_i = -1 + \sum_{j=1}^{n} (1 + a_i - a_j) \varepsilon_{j,i} + \sum_{j=1}^{n} (a_i - a_j) \varepsilon_{i,j}$$

(3.2)

If there is a unique minimal value, say $a_k$ among the $a_i$, that is $a_k < a_i$ for all $i \neq k$ then since the $a_i$ are integers $1 + a_k - a_j \leq 0$ and $\eta_k < 0$.

If there are many $a_i$’s attaining the minimal value take $k$ be such that $a_k$ be among them and $\varepsilon_{j,k} = 0$ for all the other minima $j$, the existence of such a $k$ follows from the acyclicity of $\overrightarrow{G}$. Then for this $k$ we have $\eta_k < 0$.

\[ \square \]

3.2. Characterisation of $L_G$-effective configurations. The following Theorem is the central result in [2].

**Theorem 3.** A configuration $f$ is $L_G$-effective if and only if the parking configuration $g$ equivalent to $f$ is such that $g_n \geq 0$. Moreover for any configuration $f$ one and only one of the following assertions is satisfied:

1. $f$ is $L_G$-effective
2. There exists an acyclic orientation $\overrightarrow{G}$ such that $f_{\overrightarrow{G}} - f$ is $L_G$-effective.

**Proof:** Let $f$ be a configuration and let $g$ be the parking configuration in its class. If $g_n \geq 0$ then $f$ is $L_G$-effective since it is equivalent to $g$ which is effective. If $g_n < 0$ then the acyclic orientation $\overrightarrow{G}$ of $G$ given by Propostion [3] is such that the indegree $d_i^-$ of each vertex $x_i$ except $x_n$ in $\overrightarrow{G}$ satisfies $g_i < d_i^-$, since $(f_{\overrightarrow{G}})_n \geq -1$ we have $g \leq f_{\overrightarrow{G}}$ proving that $g$ is not $L_G$-effective, so is $f$ since $f \sim_{L_G} g$.

Let $f$ be non $L_G$-effective, consider the parking configuration $g$ equivalent to $f$ and the acyclic orientation given by Proposition [3] let $h = f_{\overrightarrow{G}} - g$. Then for $i \neq n$ we have

$$\eta_i = d_i^- - 1 - f_i \geq 0$$

and since $g_n < 0$:

$$\eta_n = -1 + g_n \geq 0.$$

Hence since $f$ and $g$ are in the same class, so are $f_{\overrightarrow{G}} - u$ and $f_{\overrightarrow{G}} - v$ showing that $f_{\overrightarrow{G}} - u$ is $L_G$-effective.

Notice that $f$ and $f_{\overrightarrow{G}} - f$ cannot be both $L_G$-effective since their sum $f_{\overrightarrow{G}}$ would be too, contradicting Proposition [7].
Corollary 6. Any configuration $f$ with degree greater than $m - n$ is $L_G$-effective.

Proof: If $f$ such that $\text{deg}(f) > m - n$ is not $L_G$-effective, by the above theorem there exists an acyclic orientation $\overrightarrow{G}$ of $G$ such that $f_{\overrightarrow{G}} - f$ is. But the degree of this configuration is negative, giving a contradiction. \hfill \Box

Proposition 8. Let $T_G(x, y)$ be the Tutte polynomial of the graph $G$, and let $t_i$ be the integer coefficients given by:

$$T_G(1, y) = \sum_{i=0}^{m-n+1} t_i y^i$$

Then the number of non equivalent $L_G$-effective configurations of degree $d$ is given by:

$$\sum_{k=m-n+1-d}^{m-n+1} t_k$$

Proof:

In [23] the level of a recurrent configuration $f$ was defined as

$$\text{level}(f) = \sum_{i=0}^{n-1} f_i - m + d_n$$

where $d_n$ is the degree of the vertex $x_n$.

It was proved that this level varies from 0 to $m - n + 1$ and that the number of recurrent configurations of level $p$ and such that $x_n = q$ does not depend on $q$ and is equal to the coefficient $t_p$ of $y^p$ in the evaluation of the Tutte polynomial $T_G(x, y)$ of $G$ for $x = 1$. A bijective proof of this result was given in [7].

Using the bijection $\beta$ defined in Proposition 2 we have that the number of parking configurations $g$ such that $\sum_{i=1}^{n-1} g_i = j$ and a given value for $g_n$ is equal to the number of recurrent configurations $f$ such that:

$$\sum_{i=1}^{n-1} f_i = \sum_{i=1}^{n-1} (d_i - 1 - g_i) = 2m - d_n - (n - 1) - j$$

and $f_n = d_n - 1 - g_n$, which is the number of recurrent configurations of level $k = m - n + 1 - j$ and a given value of $f_n$. This number is equal to $t_k$.

In order that the configuration $g$ of degree $d$ to be $L_G$-effective we must have $g_n \geq 0$ so that $k$ must be greater or equal to 0 and not greater than $m - n + 1$, thus ending the proof. \hfill \Box

The generating function for non-equivalent $L_G$-effective configurations according to the degree counted by the variable $y$ is $\frac{y^{m-n+1}}{1-y}T_G(1, y^{-1})$. 
4. The rank of configurations

From now on it will be convenient to denote effective configurations using greek letters \( \lambda, \mu \) and configurations with no particular assumptions on them by letters \( f, g, h \).

4.1. Definition of the rank.

**Definition 8.** The rank \( \rho(f) \) of a configuration \( f \) is the integer equal to:

- \(-1\), if \( f \) is not \( L_G \)-effective,
- or, if \( f \) is \( L_G \)-effective, the largest integer \( r \) such that for any effective configuration \( \lambda \) of degree \( r \) the configuration \( f - \lambda \) is \( L_G \)-effective.

Denoting \( \mathbb{P} \) the set of effective configurations and \( \mathbb{E} \) the set of \( L_G \)-effective configurations this definition can be given by the following compact formula which is valid in both cases:

\[
\rho(f) + 1 = \min_{\lambda \in \mathbb{P}, f - \lambda \notin \mathbb{E}} \deg(\lambda)
\]

In other words let \( f \) be a configuration of rank \( \rho(f) \) and \( \lambda \) be an effective configuration such that \( \deg(\lambda) \leq \rho(f) \) then \( f - \lambda \) is \( L_G \)-effective; moreover there exists an effective configuration \( \mu \) of degree \( \rho(f) + 1 \) such that \( f - \mu \) is not \( L_G \)-effective.

An immediate consequence of this definition is that if \( \deg(f) < 0 \) or if \( f = f \rightarrow G \) for an acyclic orientation \( \vec{G} \) then the rank of \( f \) is \(-1\). Moreover if two configurations \( f \) and \( g \) are such that \( f_i \leq g_i \) for all \( i \) then \( \rho(f) \leq \rho(g) \).

**Definition 9.** An effective configuration \( \mu \) is a proof for the rank \( \rho(f) \) of an \( L_G \)-effective configuration \( f \) if \( f - \mu \) is not \( L_G \)-effective and \( f - \lambda \) is \( L_G \)-effective for any effective configuration \( \lambda \) such that \( \deg(\lambda) < \deg(\mu) \).

Notice that if \( \lambda \) is a proof for \( \rho(f) \) then \( \rho(f) = \deg(\lambda) - 1 = \deg(\lambda) + \rho(f - \lambda) \).

**Proposition 9.** A configuration \( f \) of degree greater than \( 2m - 2n \) has rank

\[
r = \deg(f) - m + n - 1
\]

**Proof:** We first show that for any effective configuration \( \lambda \) such that \( \deg(\lambda) = r \), the configuration \( f - \lambda \) is \( L_G \)-effective. This follows from \( \deg(f - \lambda) = \deg(f) - r = m - n + 1 \) by Corollary \( \text{[6]} \).

We now build a effective configuration \( \lambda \) of degree \( r + 1 \) such that \( f - \lambda \) is not \( L_G \)-effective. Consider any acyclic orientation \( \vec{G} \) of \( G \) and let \( g = f - f \rightarrow \vec{G} \) then \( g \) is \( L_G \)-effective since its degree is equal to \( \deg(f) - m + n \) hence greater than \( m - n \). Let \( \lambda \) be the effective configuration such that \( g \sim_{L_G} \lambda \), then \( f - \lambda \) is such that

\[
f \rightarrow \vec{G} \sim_{L_G} f - g \sim_{L_G} f - \lambda
\]

so that \( f - \lambda \) is not \( L_G \)-effective by Proposition \( \text{[7]} \). And the Proposition results from:

\[
\deg(\lambda) = \deg(g) = \deg(f) - \deg(f \rightarrow \vec{G}) = \deg(f) - m + n = r + 1
\]

\( \square \)
4.2. Riemann-Roch like theorem for graphs. We give here a proof of the following theorem first proved in [2] which we estimate shorter and simpler than the original one.

**Theorem 4.** Let $\kappa$ be the configuration such that $\kappa_i = d_i - 2$ for all $1 \leq i \leq n$, so that $\text{deg}(\kappa) = 2(m - n)$. Any configuration $f$ satisfies:

$$
\rho(f) - \rho(\kappa - f) = \text{deg}(f) + n - m
$$

**Proof:** The main ingredient for the proof is to use Theorem 3 and remark that for any acyclic orientation $\rightarrow \rightarrow G$ the orientation $\leftarrow \leftarrow G$ of $G$ obtained from $\rightarrow \rightarrow G$ by reversing the orientations of all the edges is such that: $f_{\rightarrow \rightarrow G} + f_{\leftarrow \leftarrow G} = \kappa$.

Let $f$ be any configuration we first give an upper bound for $\rho(\kappa - f)$, we define $\lambda$ to be a proof for the rank of $f$ if $f$ is $L_G$-effective, and to be equal to 0 if $f$ is not $L_G$-effective. So that $\rho(f) = \text{deg}(\lambda) - 1$ in both cases.

Since $f - \lambda$ is not $L_G$-effective, we have by Theorem 3 that there exists an acyclic orientation $\rightarrow \rightarrow G$ of $G$ such that $f_{\rightarrow \rightarrow G} - (f - \lambda)$ is $L_G$-effective, hence equivalent to an effective configuration $\mu$. This may be written as:

$$
f_{\rightarrow \rightarrow G} - (f - \lambda) \sim_{L_G} \mu \tag{4.1}
$$

Now consider the orientation $\leftarrow \leftarrow G$ of $G$ obtained from $\rightarrow \rightarrow G$ by reversing the orientations of all the arrows, clearly $f_{\rightarrow \rightarrow G} + f_{\leftarrow \leftarrow G} = \kappa$. Hence adding $f_{\leftarrow \leftarrow G}$ to both sides of (4.1) we have:

$$
\kappa - (f - \lambda) \sim_{L_G} \mu + f_{\leftarrow \leftarrow G} \tag{4.2}
$$

which may be written as:

$$
(\kappa - f) - \mu \sim_{L_G} f_{\leftarrow \leftarrow G} - \lambda
$$

Giving that $\kappa - f - \mu$ is not $L_G$-effective since the reverse of an acyclic orientation is also acyclic. Hence by the definition of the rank we have

$$
\rho(\kappa - f) < \text{deg}(\mu) \tag{4.3}
$$

The degree of $\mu$ is obtained from (4.1) giving:

$$
\text{deg}(\mu) = \text{deg}(f_{\rightarrow \rightarrow G}) - \text{deg}(f) + \text{deg}(\lambda) = m - n - \text{deg}(f) + \rho(f) + 1
$$

and:

$$
\rho(\kappa - f) < m - n - \text{deg}(f) + \rho(f) + 1 \tag{4.4}
$$

Now to obtain a lower bound for $\rho(\kappa - f)$ we exchange the roles of $f$ and $\kappa - f$ giving:

$$
\rho(f) < m - n - \text{deg}(\kappa - u) + \rho(\kappa - u) + 1 \tag{4.5}
$$

Since $\text{deg}(\kappa - u) = 2(m - n) - \text{deg}(f)$, inequality (4.5) may be written as:

$$
\rho(f) + m - n - \text{deg}(f) - 1 < \rho(\kappa - f) \tag{4.6}
$$

Comparing inequalities (4.4) and (4.6), and noticing that the rank is an integer gives

$$
\rho(f) + m - n - \text{deg}(f) = \rho(\kappa - f)
$$

hence proving the Theorem. □
5. On the rank of configurations in the complete graph

We are interested here in an algorithm allowing to compute the rank of configurations on the complete graph $K_n$.

5.1. Some useful remarks on the rank. We begin by some simple facts satisfied by the rank on any graph $G$.

**Lemma 4.** Let $f$ be a configuration on a graph $G$ and $\mu$ be an effective configuration then:

$$\rho(f) \leq \rho(f + \mu) \leq \rho(f) + \deg(\mu)$$

**Proof:** It is clear from the definition of the rank that increasing the values of the components of a configuration cannot decrease the value of the rank, this proves the first part of the inequality. For the second part, let $\lambda$ be a proof for the rank of $f$, then $f + \mu - (\mu + \lambda) = f - \lambda$ is not $L_G$-effective, so that the rank of $f + \mu$ is strictly less than $\deg(\lambda + \mu)$ but $\deg(\lambda) = \rho(f) + 1$, giving:

$$\rho(f + \mu) < \rho(f) + 1 + \deg(\mu)$$

which is the expected result. $\square$

**Corollary 7.** Let $f$ be a configuration on a graph $G$ and $f'$ be the configuration obtained by adding 1 to one of the components of $f$; so that for one $j$, $f'_j = f_j + 1$ and for all $i \neq j, f'_i = f_i$. Then:

$$\rho(f) \leq \rho(f') \leq \rho(f) + 1$$

**Proof:** It suffices to apply Lemma 4 to the effective configuration $\mu$ such that $\mu_j = 1$ and $\mu_i = 0$ for $j \neq i$. $\square$

**Lemma 5.** Let $f$ be a configuration on a graph $G$ and $\mu$ be an effective configuration such that :

$$\rho(f - \mu) = \rho(f) - \deg(\mu)$$

then for each effective configuration $\mu'$ such that for all $j, \mu'_j \leq \mu_j$ we have:

$$\rho(f - \mu') = \rho(f) - \deg(\mu')$$

**Proof:** Since for all $j, \mu'_j \leq \mu_j$ we can write $\mu = \mu' + \mu''$, where $\mu''$ is an effective configuration. By Lemma 4 we have:

$$\rho(f) \leq \rho(f - \mu') + \deg(\mu') \quad (5.1)$$

Applying again the same Lemma and since $f - \mu' = f - \mu + \mu''$ we get:

$$\rho(f - \mu') \leq \rho(f - \mu) + \deg(\mu'')$$

But it is assumed that $\rho(f - \mu) = \rho(f) - \deg(\mu)$ giving:

$$\rho(f - \mu') \leq \rho(f) - \deg(\mu) + \deg(\mu'') = \rho(f) - \deg(\mu') \quad (5.2)$$

Comparing equations (5.1) and (5.2) ends the proof of the Lemma. $\square$
5.2. Main fact on the rank in $K_n$.

**Proposition 10.** Let $f$ be a parking and $L_G$-effective configuration on the complete graph such that for some $i$, $f_i = 0$ and let $\varepsilon^{(i)}$ be the configuration such that $\varepsilon_i^{(i)} = 1$ and $\varepsilon_j^{(i)} = 0$ for $j \neq i$ then:

$$\rho(f) = \rho(f - \varepsilon^{(i)}) + 1$$

**Proof:** Let $\lambda$ be a proof for $\rho(f)$, since $f - \lambda$ is not $L_G$-effective there exists at least one $j$ such that $f_j < \lambda_j$. Consider the two configurations:

$$g = f - \lambda_j \varepsilon^{(j)}$$
$$h = f - f_j \varepsilon^{(j)} - (\lambda_j - f_j) \varepsilon^{(i)}$$

these two configurations have the same components but in different order (since $g_j = \eta_j = f_j - \lambda_j$ and $g_i = \eta_i = 0$), hence by the symmetry of $K_n$ they have the same rank. Giving:

$$\rho(g) = \rho(h)$$

Since $\lambda$ is a proof for $\rho(f)$ we have $\rho(f) = \deg(\lambda) - 1 = \deg(\lambda) + \rho(f - \lambda)$. Hence applying Lemma 5 with $\mu = \lambda$ and $\mu' = \lambda_j \varepsilon^{(j)}$ we obtain:

$$\rho(f - \mu') = \rho(f) - \deg(\mu') = \rho(f) - \lambda_j$$

Hence since $g = f - \mu'$, we have also:

$$\rho(h) = \rho(g) = \rho(f - \lambda_j)$$

Since $\lambda_j - f_j \geq 1$, we can apply again Lemma 5, this time with $\mu = f_j \varepsilon^{(j)} + (\lambda_j - f_j) \varepsilon^{(i)}$ and $\mu' = \varepsilon^{(i)}$, giving:

$$\rho(f - \varepsilon^{(i)}) = \rho(f) - 1$$

This result does not hold for any graph, the subtraction of 1 on the $i$-th coordinate of configuration $f$ with $f_i = 0$ may leave the rank invariant as shows the following example.

**Remark 2.** The configuration $f = (0,1,0,1,0,1)$ on the wheel graph $W_5$ given in the left of the Figure has rank 0, as has the configuration $f' = f - (0,0,1,0,0,0)$.

**Proof:** Indeed the configuration $f$ is $L_G$-effective, we first show that it has rank 0. Indeed, notice that $g = f - (0,0,0,1,0,0,1)$ is not $L_G$-effective since the acyclic orientation given on the right part of the Figure gives a configuration $\eta$ such that $\eta_i \geq g_i$ for all $i$. On the other hand the configuration $f' = (0,1,-1,1,0,1)$ is toppling equivalent to $(1,2,0,2,1,-4)$ via a toppling of $x_6$ and then to $(0,0,2,0,0,0)$ via topplings of $x_1,x_2,x_4$ and $x_5$, hence it is $L_G$-effective and has also rank equal to 0.

**Proposition 11.** Let $f$ be a configuration on the complete graph such that there exists a permutation $\alpha \in S_n$ satisfying $f_{\alpha(i)} = i - 1$ for $i = 1,2,\ldots,n-1$ then:

$$\rho(f) = \begin{cases} -1 & \text{if } f_{\alpha(n)} < 0 \\ f_{\alpha(n)} & \text{otherwise} \end{cases}$$
**Figure 3.** Two configurations on the Wheel Graph with ranks 0 and -1

**Proof:** Since the automorphism group of $K_n$ is $S_n$ we may suppose that the configuration $f$ is equal to $(0, 1, 2, \ldots, n-2, a)$, where $a = f_\alpha(n)$. The configuration is a parking configuration so that by Proposition 3 it is not $L_G$-effective if $a < 0$, giving the first part of the formula.

For the second part we have for the degree of $f$:

$$\text{deg}(f) = a + \frac{(n-1)(n-2)}{2}$$

Moreover the number of edges of the complete graph is $m = \frac{n(n-1)}{2}$ giving $\text{deg}(f) = a + m - (n-1)$ So that we can apply Proposition 9 when $\text{deg}(f) > 2m - 2n$, that is when $a > m - n - 1$. When this condition is satisfied Proposition 9 gives:

$$\rho(f) = \text{deg}(f) - m + n - 1 = a$$

It is easy to check that when $a = 0$ the configuration $f$ is $L_G$-effective and subtracting 1 to $a$ gives a non $L_G$-effective configuration. Hence the rank of the configuration $f$ when $a = 0$ is 0. By Corollary 7 while adding 1 to $a$ from $a = 0$ to $a = m - n$ the rank increases at most by 1 at each step, and it has to go from 0 to $a$, hence it increases exactly by 1 at each step. This ends the proof.

□

5.3. **Algorithm.** The two Propositions above give a recursive greedy algorithm in order to compute the rank of a configuration $f$ in $K_n$. It consists in determining first the parking configuration $g$ equivalent to $f$; if $g_n$ is negative then the rank is -1. If $g_n \geq 0$, use the fact that for a parking configuration there is necessarily an $i < n$ such that $g_i = 0$, otherwise the configuration is not superstable; then the rank of the configuration $g$ is equal to the rank of $h = g - \varepsilon(i)$ increased by 1. This algorithm terminates since one obtains recursively after at most $\text{deg}(g)$ steps a non $L_G$-effective configuration. Notice that the rank is exactly the number of recursive calls of the algorithm minus 1. A first
improvement of this algorithm consists in making it to stop when the configuration attained at some step satisfies the conditions stated in Proposition 11.

The main difficulty of the algorithm consists in obtaining at each step the parking configuration toppling equivalent to a given one, but this can be simplified supposing that the configurations are sorted at each step. That is reordering the \( f_i \) such that \( f_{i-1} \leq f_i \) for all \( 1 < i < n \), this reordering do not modify the rank.

Let us consider as an example the computation of the rank of the configuration \( f = (3, 1, 3, 4, -1) \) of \( K_5 \).

The configuration \((0, 3, 0, 1, 6)\) is the parking configuration toppling equivalent to \( f \). It is preferable to write the coordinates in increasing order obtaining a new configuration with the same rank: \((0, 0, 1, 3, 6)\). A first step is to subtract \( \varepsilon(1) \) giving \((-1, 0, 1, 3, 6)\). Then to reach the toppling equivalent parking configuration, we topple once the sink to obtain \((0, 1, 2, 4, 2)\) which is not a parking one due to the fourth vertex and after its toppling we obtain the expected parking configuration which is \( h = (1, 2, 3, 0, 3) \). From Proposition 10 we have \( \rho(f) = \rho(h) + 1 \). Reordering gives \( h' = (0, 1, 2, 3, 3) \) and concludes this first step.

At the second step we get the configuration \((-1, 1, 2, 3, 3)\) and the parking configuration \((0, 1, 2, 3, 2)\) after toppling and reordering. The third step gives \((-1, 1, 2, 3, 2)\) and the parking configuration \((3, 0, 1, 2, 1)\). Two other steps are necessary to obtain the non \( L_G \)-effective configuration \((1, 2, 0, 3, -1)\) of rank \(-1\) giving \( \rho(f) = 4 \).

Notice that an application of Proposition 11 would have given the result after the first step, since this Proposition 11 gives \( \rho(0, 1, 2, 3, k) = k \) for \( k \geq -1 \).

A first glance at this algorithm shows that the complexity of the determination of the rank of the configuration \( f \) in \( K_n \) is in \( O(nD) \), where \( D \) is the degree of \( f \). But this could be lowered to \( O(n) \) using some observations on Dyck words.

### 5.4. Working on Dyck words

We use here the notation of Section 2.3 to which we add some items. We denote by \( \varepsilon \) the empty word and the height of a word \( w \) is given by \( \delta(w) \), the value of \( |w|_a - |w|_b \). Recall that a \( w \) is a Dyck word if \( wb \in D_n \) where \( n = |w|_a + 1 \). The first return decomposition of a Dyck word \( w \) is given by \( w = aubv \) where \( u \) and \( v \) are Dyck words. The factorization of a word \( w \) of \( D_n \) into primitive factors is given by \( w = aw_1baw_2b \cdots aw_kbbb \) where the \( w_i \)’s are Dyck words.

Let \( f = (f_1, f_2, \ldots, f_{n-1}, f_n) \) be a parking configuration of \( K_n \) such that \( f_1 \leq f_2 \cdots \leq f_{n-1} \), then \( f_1 = 0 \). The configuration \( f - \varepsilon(1) \) has a negative value \((-1)\) in vertex \( x_1 \), the equivalent configuration \( f' = f - \varepsilon(1) - \Delta(a) \) is such that \( f'_1 = 0 \), \( f'_n = f_n - (n - 1) \) and \( f'_j = f_j + 1 \) for all other values of \( j \). Since \( f \) is a parking configuration, \( f' \) satisfies equation \((2.1)\) so that we can apply Proposition 9 in order to determine the parking configuration \( g \) equivalent to it. This gives:

**Proposition 12.** For a parking configuration \( f \) of \( K_n \) such that \( f_1 \leq f_2 \cdots \leq f_{n-1} \) and \( \phi(f) = aubvb \), where \( u, v \) are Dyck words, the configuration \( f - \varepsilon(1) \) is such that:

\[
\phi(f - \varepsilon(1)) = vabub \quad \text{and} \quad \psi(f - \varepsilon(1)) = f_n - |aub|_a
\]

**Proof:** The existence of the decomposition \( \phi(f) = aubvb \) is a consequence of the algorithm computing parking configurations via Dyck words. Let \( f' \) be as above, since
$f'_j = f_j + 1$ for all $j \notin \{i, n\}$ and $f'_1 = 0$, it is easy to check that $\phi_1(f') = abuv$, the conjugate of this word which is in $D_n$ is $vabub$. Applying Proposition 6 we have

$$\phi(f') = vabub$$

Remark 1 gives

$$\psi(f') = f'_n + n(q - p) - q$$

where $p = |abub|_a$ and $q = |abub|_b$, giving $q - p = 1$, hence: $\psi(f') = f'_n + n - |u|_b - 2$ and the result follows from $f'_n = f_n - n + 1$.

□

This result above shows that instead of considering the parking configuration $f$, it is preferable to work with $(w = \phi(f), s = f_n)$ a pair consisting of word $w$ in $D_n$ and an integer $s$. Define two functions $\theta_1$ and $\theta_2$, where $\theta_1(w)$ is a word in $D_n$ and $\theta_2(w, s)$ is an integer, they are both given using the first return to the origin decomposition $aubvb$ of $w$:

$$\theta_1(aubvb) = vabub \quad \theta_2(aubvb, s) = s - |aub|_a. \quad (5.4)$$

Hence, one loop iteration starting from the pair $(w, s)$ leads to the pair

$$\Lambda(w, s) = (\theta_1(w), \theta_2(w, s)). \quad (5.5)$$

So that the rank $r$ of $f$ is obtained by iterating $\Lambda$ until reaching a negative number for the second component of the pair, this may be written:

$$r + 1 = \min_{k \geq 0} \{ \theta^k_2(\phi(f), f_n) < 0 \}$$

where we use the slight abuse of notation $\theta^k_2(f, s)$ to denote the second component of $\Lambda^k(w, s)$. The algorithm on parking configurations could then be translated in terms of Dyck words as:

- $w := \phi(f); s := f_n; r := -1$
- **while** $s \geq 0$ **do**
  - $s := \theta_2(w, s); w := \theta_1(w); r := r + 1$
- **od**
- return $r$

Iterating this Proposition a few times gives:

**Lemma 6.** Let $f$ be a parking configuration and $w \in D_n$ be equal to $\phi(f)$. Let $w = aw_1baw_2b \cdots aw_kb bb$ be the factorization of $w$ into its primitive factors. Then

$$\theta^k(w) = abw_1abw_2 \cdots abw_kbb \quad \theta^k_2(w, s) = s - (n - 1)$$

Moreover any parking configuration $g$ such that $\phi(g) = w_1abw_2 \cdots abw_kab$ and $g_n = s - (n - 1) \geq 0$ satisfies: $\rho(g) = \rho(f) - k$.

In the sequel we will use extensively the following sequence associated to a word $w$ containing $m$ occurrences of the letter $a$
Definition 10. The sequence of heights $\eta_1, \eta_2, \ldots, \eta_m$ of $w$ is such that

$$ \eta_i = |w(i)|_a - |w(i)|_b $$

where $w^{(i)}$ is the prefix of $w$ followed by the $i$-th occurrence of the letter $a$.

Let $w \in D_n$, $w = aw_1b \cdots aw_kb$ be the decomposition of $w$ into primitive factors and $\eta_1, \eta_2, \ldots, \eta_{n-1}$ be its sequence of heights. Let $\eta'_1, \eta'_2, \ldots, \eta'_{n-1}$ be that of $\theta_1^k(w)$, then:

$$ \eta'_i = \eta_i \text{ if } \eta_i = 0 \text{ and } \eta'_i = \eta_i - 1 \text{ if } \eta_i > 0. $$

Theorem 5. Let $f$ be a parking configuration on $K_n$, let $w = \phi(f)$ and $\eta_1, \eta_2, \ldots, \eta_{n-1}$ be the sequence of heights in $w$. Then the rank of $f$ is given by:

$$ 1 + \rho(f) = \sum_{i=1}^{n-1} \text{Max}(0, q - \eta_i + \chi(i \leq r)) \tag{5.6} $$

Where $q$ and $r$ are the quotient and remainder of the division of $f_n + 1$ by $n - 1$, and $\chi(i \leq r)$ is equal to 1 if $i \leq r$ and to 0 otherwise.

Proof: When $f_n < 0$ all the terms in the sum in the right hand side of Equation (5.6) are equal to 0 so that the formula gives $\rho(f) = -1$ as expected.

For $f_n \geq 0$, we proceed by induction on $q$ the quotient of $f_n + 1$ by $n - 1$ and we consider the decomposition of $\phi(f)$ into factors:

$$ \phi(f) = w = aw_1b \cdots aw_kb $$

- If $q = 0$ then $0 \leq f_n < n - 1$ hence $\rho(f)$ is determined by the position of $f_n$ in the intervals given by the lengths of the prefixes $fw'$ of $w$ satisfying $\delta(w') = 0$ these are $aw_1baw_2b \cdots aw_ib$ for $i = 1, 2, \ldots, k$. More precisely let $n_i$ be given for $1 \leq i \leq k$ by

$$ n_i = |aw_1baw_2b \cdots aw_ib|_a $$

and let us denote $n_0 = 0$. Then we have $\theta_2^k(w, f_n) = f_n - n_i$ for $i = 1, 2, \ldots, k+1$ so that $\rho(f)$ is given by the first $i$ for which $\theta_2^k(w, f_n)$ is negative which may be translated by:

$$ \rho(f) = i \iff n_i \leq f_n < n_{i+1}. $$

On the other hand the $n_i$ can be characterized by $\eta_{n_i} = 0$ giving:

$$ 1 + \rho(f) = |\{j| j \leq r, \eta_j = 0\}| $$

But this is exactly what equation (5.6) gives.

- If $q > 0$ let us consider $g = \theta_1^k(w)$ and notice that $\theta_2^{k+1}(w, f_n) = f_n - (n - 1)$ and let $g$ be such that $\phi(g) = w, g_n = f_n - (n - 1)$. Let then the quotient of $g_n$ by $n$ is $q' = q - 1$ and the remainder is also $r$. Let $\eta'_1, \eta'_2, \ldots, \eta'_{n-1}$ be the sequence of heights of $v$. Applying the inductive hypothesis we have

$$ 1 + \rho(g) = \sum_{i=1}^{n-1} \text{Max}(0, q - 1 - \eta'_i + \chi(i \leq r)) $$

By Lemma 6, we have $\rho(f) = \rho(g) + k$ and $v = w_1abw_2b \cdots abw_kab$. 


It is easy to check since the heights $\eta'_i$ in $g$ are such that $\eta'_i = \eta_i - 1$ if $\eta_i \neq 0$ and $\eta'_i = 0$ when $\eta_i = 0$ we have $\text{Max}(0, q - 1 - \eta'_i + \chi(i \leq r)) = \text{Max}(0, q - \eta_i \chi(i \leq r))$ when $\eta_i \neq 0$ and $\text{Max}(0, q - 1 - \eta'_i + \chi(i \leq r)) = \text{Max}(0, q - \eta_i \chi(i \leq r) - 1)$ when $\eta_i = 0$ from which we get

$$1 + \rho(g) = \sum_{i=1}^{n-1} \text{Max}(0, q - \eta_i + \chi(i \leq r)) - k$$

The proof ends by reminding that $\rho(f) = \rho(g) + k$.

\[\square\]

An exemple of calculation:

$$f = (0, 0, 0, 1, 1, 1, 4, 7, 7, 9)$$

For this configuration we have: $n = 11, f_n = 26, q = 2, r = 7$.

\[
\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|}
\hline
i & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
\hline
f_i & 0 & 0 & 0 & 1 & 1 & 1 & 4 & 7 & 7 & 9 \\
\eta_i & 0 & 1 & 2 & 2 & 3 & 4 & 2 & 0 & 1 & 0 \\
q + \chi(i \leq r) & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 2 & 2 & 2 \\
q + \chi(i \leq r) - \eta_i & 3 & 2 & 1 & 1 & 0 & -1 & 1 & 2 & 1 & 2 \\
\hline
\end{array}
\]

Adding the positive values of $q + \chi(i \leq r)$ gives $1 + \rho(f) = 13$ so that $\rho(f) = 12$.

This formula will be rephrased in terms of the representation of a Dyck word by a path in $\mathbb{Z}^2$ in a subsequent section and in terms of skew cylinders in the appendix.

6. A new parameter for Dyck words

6.1. Prerank and coheights. The algorithm used to determine the rank of a configuration of the complete graph suggests the introduction of a parameter on Dyck words (or Dyck words) which we call \textit{prerank}. In this Section we show how this parameter behaves with other known parameters, it may be skipped by the reader only interested by the rank on complete graphs.

We denote $\theta$ the mapping associating to the Dyck word $w$ with first return to the origin decomposition $w = aubv$ the Dyck path $vabu$, notice that $\theta_1(wb) = \theta(w)b$. And we consider the Dyck word $(ab)^n$ consisting of the concatenation of $n$ two-letters words $ab$.

\textbf{Definition 11.} The prerank $\rho(w)$ of the Dyck path $w$, of length $2p$, is the integer $k$ such that $k$ operations $\theta$ are needed to reach the word $(ab)^p$, or equivalently the smallest $k$ such that: $\theta^k(w) = (ab)^p$.

Recall that the \textit{area} of a Dyck word is equal to the sum of the elements of its sequence of heights. So that we have $\text{area}(aubv) = \text{area}(u) + \text{area}(v) + |u|_a$. Notice that $\rho((ab)^n) = \text{area}((ab)^n) = 0$, one can also prove that $\rho(a^n b^n) = \text{area}(a^n b^n) = \frac{n(n-1)}{2}$, this suggests that pre-rank and area could be equal, but it is not always the case since for example $\rho(abaabb) = 2$ while $\text{area}(abaabb) = 1$. 
The prerank may be calculated using the notion of coheight, which may be defined as follows:

Let the heights of the prefixes followed by an occurrence of $a$ of a Dyck word $w$ of length $2p$ be: $\eta_1, \eta_2, \ldots \eta_p$, let $m$ be the largest integer such that $\eta_m$ is maximal among the $\eta_i$, then the sequence of co-heights of $w$ is given by the formula

$$\bar{\eta}_i = \begin{cases} 
\eta_m - \eta_i & \text{if } i \leq m \\
\eta_m - \eta_i - 1 & \text{otherwise}
\end{cases} \quad (6.1)$$

The following lemma allows to determine the prerank

**Lemma 7.** The prerank $\rho(w)$ of the Dyck word $w$ is given by the sum of the elements of its sequence of coheights.

**Proof:** We proceed by induction on the value of $\rho(w)$. If $\rho(w) = 0$, then $w = (ab)^p$ and $\eta_i = 0$ for all $1 \leq i \leq n$ hence $\eta_m = 0$ and $m = n$, then all the coheights are equal to 0 in accordance with the Lemma.

Let $w$ be such that $\rho(w) \neq 0$ and let $aubv$ be the first return to the origin decomposition of $w$, we have $\rho(w) = 1 + \rho(vabu)$. We proceed to the comparison of the coheights of $w$ and those of $vabu$ considering two cases according to the position $m$ of the largest prefix of $w$ with maximal height. Denote $k = |aub|_a$.

- If $k < m$ then the sequence of heights of $w$ may be written
  $$\eta_1 (= 0), \eta_2, \ldots, \eta_k (= 0), \ldots, \eta_m, \ldots, \eta_n$$
  and that of $vabu$:
  $$\eta_{k+1}, \ldots, \eta_m, \ldots, \eta_n, 0, 0, \eta'_2, \ldots, \eta'_k$$
  where $\eta'_i = \eta_i - 1$ for $i = 2, \ldots k$.
  The values of the heights which decrease by one from the first sequence to the second one, are moved from before $\eta_m$ to after $\eta_m$; hence their corresponding coheights do not vary. Those who do not decrease give also equal coheights. The only difference is that $\eta_1 = 0$ moving from before $\eta_m$ to after it and remains equal to 0, so that the corresponding coheight decreases by one. Hence the sum of the sequence of coheights of $vabu$ is one less than the sum of the sequence of those of $f$. Since $\rho(w) = 1 + \rho(vabu)$, the inductive hypothesis proves the assertion in the Lemma.
- If $k \geq m$. Then the sequences of heights may be written:
  $$\eta_1 (= 0), \eta_2, \ldots, \eta_m, \ldots, \eta_k (= 0), \ldots, \eta_n$$
  for $f$ and
  $$\eta_{k+1} \ldots \eta_n, 0, 0, \eta'_2, \ldots, \eta'_m \ldots \eta'_k$$
  for $vabu$, where $\eta'_i = \eta_i - 1$ for $i = 2, \ldots k$. The maximal height decreases by $1$ for $vabu$, so that the coheights in $w$ and the corresponding ones in $vabu$ are equal except as above for the first coheight, for which the value decreases by $1$. A similar argument as above ends the proof.

$\blacksquare$
6.2. The mapping \( \Phi \). For a word \( w \) of \( D_n \) we consider its sequence of heights \( \eta_1, \eta_2, \ldots, \eta_{n-1} \) and the largest integer \( \eta_m \) such that \( \eta_m \) is maximal among the \( \eta_i \)’s and define \( \Phi(w) \) as follows:

**Definition 12.** Consider the decomposition of \( w \in D_n \) as \( w = uv \) such that \( |u|_a = m \), where \( m \) is defined above, then

\[
\Phi(w) = \tilde{u}\tilde{v}
\]

We use here the notation: if \( w = x_1 x_2 \ldots x_p \) where \( x_i \) are letters, then \( \tilde{w} = x_p \ldots x_2 x_1 \).

It is not difficult to prove that the word \( \Phi(w) \) is in \( D_n \), by showing that if \( w' \) is a strict prefix of \( w \) then \( \delta(w') \geq 0 \).

**Remark 3.** For any word \( w \) of \( D_n \), the word \( \Phi(w) \) is the conjugate of \( \tilde{w} \) which is a Dyck word followed by an occurrence of \( b \). Consequently \( \Phi(\Phi(w)) = w \).

**Proof:** With the notation in Definition 12 we have

\[
\tilde{w} = \tilde{v}\tilde{u}.
\]

A conjugate of this word is \( \tilde{u}\tilde{v} = \Phi(w) \), and this word is in \( D_n \). The cyclic lemma insures the unicity of such a word among the conjugates of \( \tilde{w} \), thus ending the proof.

We recall the definition of the parameter \( \text{dinv} \) of \( w \) introduced by M. Haiman [16] which is obtained from the sequence of heights \( \eta_1, \eta_2, \ldots, \eta_k \) of a Dyck word \( w \) of length \( 2k \). We use here the same definition for \( \text{DINV}(w) \) when \( w \in D_n \), since it makes no difference for the sequence of heights to consider a Dyck word \( w \) or the word \( wb \) which is in \( D_n \). Also for \( w \) in \( D_n \) we define \( \text{area}(w) = \text{area}(w') = \sum_{i=1}^{n-1} \eta_i \) and \( \rho(w) = \rho(w') \), where \( w' \) is given by: \( w = wb \).

**Definition 13.** For a word \( w \in D_n \) the parameter \( \text{dinv}(w) \) is equal to the number of elements of the set \( \text{DINV}(w) \) of pairs \((i, j)\) such that \( i < j \) and \( \eta_i = \eta_j \) or \( \eta_j = \eta_i - 1 \).

The main result of this subsection is:

**Proposition 13.** The mapping \( \Phi \) is an involution, and for any \( w \in D_n \) we have:

\[
\rho(w) = \text{area}(\Phi(w)) \quad \text{dinv}(w) = \text{dinv}(\Phi(w))
\]

Before proving this Proposition we need to compare the sequence of heights in a word \( w \) and that of the word \( \tilde{w} \)

**Lemma 8.** Let \( w \) be any word on \( \{a, b\}^* \) containing \( n \) occurrences of the letter \( a \), \( \eta_1, \eta_2, \ldots, \eta_n \) be its sequence of heights, and let \( p = |w|_a - |w|_b \). The sequence \( \eta'_1, \eta'_2, \ldots, \eta'_n \) of heights of \( \tilde{w} \) is given by:

\[
\eta'_{n-i+1} = p - 1 - \eta_i.
\]

**Proof:** (of Lemma) Let \( w = uav \) be such that \( u \) contains \( i-1 \) occurrences of the letter \( a \), then \( \eta_i = \delta(u) \) and since \( \tilde{w} = \tilde{v}a\tilde{u}, |\tilde{v}a|_a = n-i+1, \eta'_{n-i+1} = \delta(\tilde{v}) = \delta(v) \).

Computing \( \delta(w) \) we have

\[
\delta(w) = \delta(u) + 1 + \delta(v) = p
\]
Since the second value is \( \eta \) by Lemma 8, the sequence of heights \( \eta \) and denote \( \Phi(\cdot) \) the Proposition.\( \eta \) is maximal among the \( \eta \)’s. Let \( \eta' \) be the sequence of heights in \( \Phi(w) \). Since \( \eta'_m = \eta_m - \eta_1 = \eta_m \) and \( \eta'_j \leq \eta_m - \) the largest \( j \) such that \( \eta'_j \) is maximal among the \( \eta' \) is equal to \( m \).

Denote \( w = w'abw'' \) where \(|w'| = m \) then \( \delta(w') = \eta_m + 1 \) and \( \delta(w'') = -\eta_m \). Then by Lemma 8 the sequence of heights \( \eta'_1, \eta'_2, \ldots, \eta'_m, \ldots, \eta'_n \) of \( \Phi(w) \) is given by:

\[
\eta'_i = \begin{cases} 
\eta_m - \eta_{m-i+1} & \text{if } i \leq m \\
\eta_m + (-\eta_m - (\eta_{n-i+1+m} - \eta_m - 1)) & \text{otherwise.}
\end{cases} \tag{6.2}
\]

Since the second value is \( \eta_m - 1 - \eta_{m-i+m+1} \), this proves that the sequence of heights in \( \Phi(w) \) is a rearrangement of the sequence of the coheights of \( w \) proving the first part of the Proposition.

For the second part of the Proposition we consider an element \((i, j)\) in \( DINV(w) \) and denote \( a = \eta_i, b = \eta_j \) in \( \Phi(w) \), these values \( a, b \) become \( a', b' \) and are such that \( \eta_i = a' \) and \( \eta_j = b' \). We consider three cases depending on the values of \( i \):

1. If \( i, j \leq m \) then
   \[
a' = \eta_m - a, b' = \eta_m - b, i' = m - i + 1, j' = m - j + 1
   \]
   so that \( j' < i' \) and \((j', i') \in DINV(\Phi(w))\).

2. If \( i > m \) then
   \[
a' = \eta_m - 1 - a, b' = \eta_m - 1 - b, i' = m + n - i + 1, j' = m + n - j + 1
   \]
   so that \( j' < i' \) and \((j', i') \in DINV(\Phi(w))\).

3. If \( i \leq m \) and \( j > m \) then \( i' \leq m, j' > m \) \( a' = \eta_m - a, b' = \eta_m - b \). So that
   \[
b' = a' - 1 \text{ if } a = b \text{ and } b' = a' \text{ if } a = b + 1.
   \]
   Hence \((i', j') \in DINV(\Phi(w))\).

\( \Box \)

### 6.3. Link with Haglund’s function \( \zeta \).\(^{19}\)

In [19] page 50 J. Haglund introduces a mapping \( \zeta \) on Dyck words and shows that this mapping keeps invariant the value of \( \text{dinv} \). We will give a relationship between \( \zeta \) and \( \Phi \).

Before lets us recall how \( \zeta(w) \) is build from the sequence of heights \( \eta = \eta_1, \eta_2, \ldots, \eta_m \) of \( w \). Let \( k \) be the maximal values of the \( \eta_i \) then the subsequences \( \eta^{(0)}, \eta^{(1)}, \ldots, \eta^{(k)}, \eta^{(k+1)} \) where \( \eta^{(i)} \) contains all the occurrences of \( i \) and \( i - 1 \) in \( \eta \) in the same order as they appear in \( \eta \). Hence \( \eta^{(0)} \) contains only occurrences of \( 0 \), and \( \eta^{(k+1)} \) only occurrences of \( k \). Then \( \zeta(w) \) is obtained by concatenating the words \( \overline{\eta}^{(i)} \) obtained from the \( \eta^{(i)} \)’s replacing occurrences of \( i \) by \( a \) and those of \( i - 1 \) by \( b \).

Example for \( w = aababbaababaaabbab \) we have \( \eta = (0, 1, 1, 2, 1, 0, 1, 1, 2, 1) \), so that:

\[
\eta^{(0)} = (0, 0), \eta^{(1)} = (0, 1, 1, 1, 0, 1, 1, 1), \eta^{(2)} = (1, 1, 2, 1, 1, 1, 2, 1), \eta^{(3)} = (2, 2)
\]

Giving

\[
\overline{\eta}^{(0)} = aa, \overline{\eta}^{(1)} = baaabaaa, \overline{\eta}^{(2)} = bbabbbab, \overline{\eta}^{(3)} = bb
\]
and $\zeta(w) = aabaaabaababbabbbab$.  

Consider the mapping $R$ on words consisting in writing them from the right to left and replacing $a$ by $b$ and $b$ by $a$ such that $R(w_1 w_2 \cdots w_{n-1} w_n) = \overline{\overline{w}_n \overline{w}_{n-1} \cdots \overline{w}_2 \overline{w}_1}$ where $\overline{a} = b, \overline{b} = a$.

**Proposition 14.** The mappings $\Phi$ and $\zeta$ satisfy $R(\zeta(w)) = \zeta(\Phi(w))$.

In the example above we have $\Phi(w) = aabaaabbaababbab$ which has for sequence of heights(0, 1, 1, 2, 1, 0, 1, 1, 2, 0) giving $\zeta(\Phi(w)) = aaabaaabaabbbabbbabb$ which is equal to $R(\zeta(w))$.

**Proof:** Consider the sequence $\eta = \eta_1, \eta_2, \ldots, \eta_n$ of heights of $w$, let $k$ be the maximal value of the $\eta_i$ and $m$ the largest integer such that $\eta_m = k$. It is convenient for the proof to divide the $\overline{\eta}^{(i)}$'s leading to the definition of $\zeta(w)$ in two parts $\overline{\eta}^{(i)}$ and $\overline{\eta}^\ast(i)$ corresponding respectively to $(\eta_1, \ldots, \eta_m)$ and to $(\eta_{m+1}, \ldots, \eta_n)$, this gives since $\overline{\eta}^\ast(k+1)$ is the empty word:

$$R(\zeta(w)) = R(\overline{\eta}^{(k+1)})(\overline{\eta}^\ast(k)R(\overline{\eta}^{(k)}) \cdots R(\overline{\eta}^\ast(0))R(\overline{\eta}^{(0)})$$

Consider now the sequence $\mu = \mu_1, \mu_2, \ldots, \mu_m, \ldots, \mu_n$ of heights of $\Phi(w)$, we divide also the sequences $\mu^{(i)}$ into to two sequences $\mu^{(i)}$ and $\mu^\ast(i)$ giving:

$$\zeta(\Phi(w)) = \overline{\mu}^\ast(0)\overline{\mu}^{(0)}\overline{\mu}^{(1)}\overline{\mu}^{\ast(1)} \cdots \overline{\mu}^{(k)}\overline{\mu}^{\ast(k)}\overline{\mu}^{(k+1)}$$

We have seen in the Proof of Proposition 13 that

$$h'_i = \begin{cases} 
\eta_m - \eta_{n-i+1} & \text{if } i \leq m \\
\eta_m - 1 - \eta_{m+n-i} & \text{otherwise}
\end{cases} \quad (6.3)$$

But this exactly means that: $\overline{\mu}^{(i)} = R(\overline{\eta}^{(k+1-i)})$ and $\overline{\mu}^\ast(i) = R(\overline{\eta}^\ast(k-i))$, hence proving this Proposition. \qed
In this section we will represent a sorted parking configuration $f$ of $K_n$ by a Dyck path in a strip of the plane $\mathbb{Z}^2$ consisting of $n-1$ infinite rows of cells. The word $\phi(f)$ will be drawn by a path in which north steps correspond to occurrences of the letter $a$ and east steps correspond to occurrences of the letter $b$. This will allow to determine the rank and the degree of $f$ by counting some cells determined by the value $f_n$.

We consider here a strip of $n-1$ rows in the plane consisting of $n-1$ infinite rows of cells and we label these cells of this strip with the integers in $\mathbb{Z}$, as follows: we start with the label 0 given to the cell at the north-west of the origin, then we continue using labels from 1 to $n-2$ along the main diagonal (dashed in our Figure) until the line $y = n-1$ is reached; we repeat this labeling giving the label $k + n - 1$ to the cell lying on the west of the cell with label $k$ and continue in the north-east direction. We also label the cells in the diagonals on the left of the main with negative integers one giving the label $k - (n-1)$ to the cell on the east of one labelled $k$ so that the labels increase by 1 when allowing a diagonal in the north east direction. Notice that the cell at the north of a cell labeled $k$ has label $n + k$. Notice that this labeling process may be considered as an illustration of the euclidian division by $n-1$, since the cell in row $i$ are those whose labels are equal to $i - 1 \ mod \ (n-1)$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{labels.png}
\caption{Labels of the cells in the strip}
\end{figure}

For the representation of the sorted parking configuration $f$ we draw the Dyck path corresponding to $\phi(f)$ starting at the origin and where a north step corresponds to each occurrence of the letter $a$ and an east step corresponds to each occurrence of the letter $b$. Consider each row $i$ ($1 \leq i < n$), composed with the cells lying between the lines $y = i - 1$ and $y = i$, the value of $f_i$ is equal to the number of cells lying between the
\(y\)-axis and the north step of the path in this row. In addition, \(\eta_i\) is the number of cells in row \(i\) between the North step of the Dyck path crossing this row and the main diagonal. Consider the euclidian quotient \(q\) of \(f_n + 1\) by \(n - 1\) such that \(f_n + 1 = q(n - 1) + r\) then the value \(\max(0, q + \chi(i \leq r) - \eta_i)\) given in Theorem 5 is the number of these cells situated on the left of the North step in row \(i\) which labels are less or equal to \(f_n\).

This fact is illustrated in Figure 4.

![Figure 4](image)

**Figure 4.** Computation of the rank on a Dyck path

Our configuration \(f\) is represented by the Dyck word \(aaabaaabbbabbaabbab\) and the value \(f_n = 26\), where \(n = 11\) for our example. On the right of the Figure one can read the configuration \(f\) from bottom to top, each value \(f_i\) corresponds to the number of cells between the \(y\)-axis and the north step in the row \(i\). On the left are displayed the values of \(q + \chi(i \leq r) - \eta_i\) determined by the cells at the left of the path lying which coordinates \((i, j)\) satisfy \(i \geq j - q\) for \(j \leq r\) and \(i \geq j - q\) for \(j > r\). The number of these cells colored in green is equal to \(\rho(f) + 1\).

**Remark 4.** Theorem 5 may be interpreted by stating that the rank \(\rho(f)\) is one less than the number of cells lying on the left of the Dyck path associated to \(f\) and having a label not greater than \(f_n\).

The Dyck path separates the strip of height \(n - 1\) into two regions, one on the left of and the other on its right. Consider the following sets of integers associated to a Dyck path \(w\).

**Definition 14.** For a path \(w\) in \(D_n\) we define \(\text{lastright}(w)\) as the highest label of a cell lying in the right region. For any integer \(s\), \(\text{left}(w, s)\) is the number of cells lying in the left region whose labels are not greater than \(s\), and \(\text{right}(w, s)\) is the number of cells in the right region whose labels are strictly greater than \(s\).
Lemma 9. Let \( w \in D_n \) be such that \( w = uv \) where \( u \) is the longest prefix of \( w \) with maximal value of \( \delta(u) \). Then the value of \( \text{lastright}(w) \) is equal to \( n(q - p) - q - 1 \) where \( p = |u|_a \) and \( q = |u|_b \).

Proof: Notice that the labels of the cells increase by \( n \) while going from one cell to that at the top of it and by \( n - 1 \) when going from one cell to that on that at the left of it. Hence the cell situated at the right of the unitary path joining \((x, y)\) to \((x, y + 1)\) has label equal to \( ny - (x + 1)(n - 1) \). In each row the cell with highest label lying in the right region is stated immediately at the right of the north step of the path in the row. This label is higher in row \( i \) than in row \( j \) if the heights \( \eta_i \) and \( \eta_j \) satisfy \( \eta_i > \eta_j \) or if \( \eta_i = \eta_j \) and \( i > j \), this shows that \( \text{lastright}(w) \) is the label of the cell in the row corresponding to the longest prefix of \( w \) with maximal height. \( \Box \)

Proposition 15. Let \( f \) be any sorted parking configuration of \( K_n \), and let \( w = \phi(f) \) then:

\[
\text{left}(w, f_n) = \rho(u) + 1 \tag{7.1}
\]

and

\[
\text{right}(w, f_n) = \left( \frac{n - 1}{2} \right) + \rho(f) - \deg(f). \tag{7.2}
\]

Proof:

The first equation follows immediately from Remark 4. We discuss two cases for the computation of parameter right.

If \( f_n \geq 0 \) the number of elements of the set of the right cells with non negative labels is equal to the area of the Dyck word \( w \) which is equal to

\[
\sum_{i=1}^{n-1} \eta_i = \left( \frac{n - 1}{2} \right) - \sum_{i=1}^{n-1} f_i = \left( \frac{n - 1}{2} \right) - \deg(f) + f_n
\]

Among these cells \((f_n + 1 - \text{left}(w, f_n))\) have labels not greater than \( f_n \). This gives

\[
\text{right}(w, f_n) = \left( \frac{n - 1}{2} \right) - \deg(f) + f_n - (f_n + 1 - \text{left}(w, f_n)) = \left( \frac{n - 1}{2} \right) - \deg(f) + \rho(f)
\]

If \( f_n < 0 \), the set cells in the right region with label greater than \( f_n \) are those the cells between the Dyck path and the main diagonal, and in addition all the cells with negative labels \(-1, -2, \ldots, f_n + 1\) hence:

\[
\text{right}(w, f_n) = \left( \left( \frac{n - 1}{2} \right) - (\deg(f) - f_n) \right) - (f_n + 1)
\]

as expected since in this case \( \rho(f) = -1 \).

The calculation of \( \rho(f) \) and \( \deg(f) \) is illustrated below with the same configuration \( f \) but with \( f = 13 \).
Baker and Norine’s theorem applied to the configuration $f$ involves the computation of the rank for the configurations $f$ and $\kappa - f$. The map $f \rightarrow \kappa - f = \Psi(f)$ is an involution.

The sorted parking configuration toppling and permuting equivalent to any configuration $f \in K_n$ is described by $(\phi(f), \psi(f))$ in Definition 4.

See the appendix for an alternative approach. For the readers who skipped the preceding section we recall that for a word $w$ in $D_n$ we denote $\Phi(w)$ the conjugate of $\tilde{w}$ which is an element of $D_n$. Hence $\Phi(w) = \tilde{u} \tilde{v}$, such that $w = uv$ and $u$ is the longest prefix of $w$ with maximal value by $\delta$.

**Proposition 16.** Let $f$ be any configuration on $K_n$ and $w = \phi(f)$, then $\phi(\kappa - f) = \Phi(w)$. Moreover $\psi(\kappa - f) = \text{last} \text{right}(w) - 1 - \psi(f)$

**Proof:**

Let $f'$ be the parking configuration toppling equivalent to $f$, so that it satisfies $w = \phi(f')$, let us denote $g$ the configuration $\kappa - f'$ then we have $g \sim_{L_G} \kappa - f$ and using $\kappa = (n - 3, n - 3, \ldots, n - 3, n - 3) \sim_{L_G} (n - 2, n - 2, \ldots, n - 2, -2)$:

$$g = (n - 2 - f'_1, n - 2 - f'_2, \ldots n - 2 - f'_{n-1}, -2 - f'_n).$$

Notice that, since if $i < n$ we have $0 \leq f'_i < n - 1$, the $g_i$ satisfy $0 \leq g_i < n$ for $1 \leq i < n$. Moreover $f'_i \leq f'_j$ if and only if $g_i \geq g_j$. This last fact implies that the configurations $f''$ and $g'$ obtained by reordering the first $n - 1$ values of $f'$ and $g$ respectively satisfy $g'_i = n - 2 - f''_{n-i}$. A consequence is that the words $w = \phi(f) = \phi_1(f'')$ and $v = \phi_1(g')$
are such that: \[ w = ubb \quad \text{and} \quad v = \bar{ubb} \]

Hence \( v \) is a conjugate of \( \bar{w} = bb\bar{u} \).

By Remark 3 we have that \( \Phi(w) \) is the conjugate of \( v \) which is in \( D_n \), and it is also equal to \( \phi(g') \) by Proposition 6, since \( v = \phi_1(g') \). Now since \( g' \) is obtained by permuting the first \( n - 1 \) values of \( g \) and \( g \sim L_\kappa \), \( \kappa - f \) we have:

\[ \Phi(w) = \phi(g') = \phi(\kappa - f). \]

We now compute \( \psi(\kappa - f) \). We consider two configurations satisfying condition (2.1) and use Remark 1. These two configurations are \( g' \) and the parking configuration \( g'' \) toppling equivalent to \( g' \), indeed in each one the first \( n - 1 \) values are in weakly increasing order and they satisfy condition (2.1). Moreover their degrees are equal since they are toppling equivalent. Notice that \( g''_n = g_n = -2 - \psi(f) \) and \( g''_n = \psi(\kappa - f) \). In order to apply Remark 1 we have to compare \( \phi_1(g') = v = ubb \) and \( \phi_1(g'') = \Phi(w) \) which are two conjugates word, the second one being an element of \( D_n \). Let \( v_1 \) be the smallest prefix of \( v \) which has minimal value by \( \delta \) then:

\[ v = v_1v_2 \quad \text{and} \quad \Phi(w) = v_2v_1 \]

Now denote \( p = |v_1|_a \) and \( q = |v_1|_b \), and by Remark 1 we get:

\[ \psi(\kappa - f) = g''_n = g'_n + n(q - p) - q = -\psi(f) + n(q - p) - (q + 2) \]

Using Lemma 9 we have

\[ \lambda_{right}(w) = n(q - p) - (q + 1) \]

which ends the proof.

\[ \square \]

**Corollary 8.** The involution associating \((w, s)\) to \((w', s')\), where \( w' = \Phi(w) \) and \( s' = \lambda_{right}(w) - s - 1 \) is such that

\[ \lambda_{left}(w, s) = \lambda_{right}(w', s') \quad \text{and} \quad \lambda_{left}(w', s') = \lambda_{right}(w, s) \]

**Proof:** We give here a short proof using Riemann-Roch theorem for graphs (i.e. Theorem 4), a more direct proof is proposed in the appendix. Let \( f \) be a configuration such that \( \phi(f) = w \) and \( \psi(f) = s \), this configuration may be supposed to be a parking configuration such that its first \( n - 1 \) values are sorted, hence we have \( f_n = s \) and \( w' = \phi(\kappa - f) \), \( s' = \psi(\kappa - f) \). Theorem 4 gives:

\[ \rho(f) = \rho(\kappa - f) + \deg(f) - \left( \frac{n}{2} \right) + n \]

(7.3)

Giving:

\[ \rho(\kappa - f) = \rho(f) - \deg(f) + \left( \frac{n - 1}{2} \right) - 1 \]

(7.4)

By the first part of Proposition 16 we have that \( \rho(\kappa - f) = \lambda_{left}(w', s') - 1 \) and that the right hand side of equation (7.4) is equal to \( \lambda_{right}(w, s) - 1 \) this gives \( \lambda_{left}(w', s') = \lambda_{right}(
right\((w, s)\). Using the fact that the mapping exchanging \((w, s)\) and \((w', s')\) is an involution ends the proof. 

8. Enumerative study of the \((degree, rank)\) bistatistic on sorted parking configurations

In this section, we are interested in the distribution of the degrees and ranks on sorted configurations on \(K_n\). For this purpose we consider the following generating function

\[
K_n(d, r) = \sum_f d^{\deg(u)} r^{\rho(f)}.
\]

In this sum, \(f\) runs over all parking configurations of \(K_n\) such that \(f_1 \leq f_2 \leq \ldots \leq f_{n-1}\). Notice that this generating function is a formal sum which is a Laurent series, since \(\rho(f)\) may be equal to \(-1\) and \(\deg(u)\) may be any integer in \(\mathbb{Z}\).

Lorenzini [21] consider a similar generating function summed over parking configurations, not necessarily sorted, and call it the two variable zeta function of the underlying graph.

In preceding section, we remark two linear combinations of the rank and degree statistics are more natural with respect to the involution \(f \rightarrow \kappa - f\) appearing in Riemann-Roch theorem for graphs. This leads to a change of the variables \(r, d\) of \(K_n(r, d)\) into variables \(x, y\) of a new generating function

\[
L_n(x, y) = \sum_{w \in D_n} \sum_{s = -\infty}^{+\infty} x^{left(w, s)} y^{right(w, s)}
\]

which is a more tractable power series in \(x\) and \(y\). In addition, the involution \(\Psi\) in preceding section shows that this new series satisfies the symmetric relation

\[
L_n(x, y) = L_n(y, x).
\]

Our main result is that via this ordering of summations \(L_n(x, y)\) may be described by a sum of geometric sums of two alternating ratio, which share as a common factor the rational fraction \(H(x, y) = \frac{1-xy}{(1-x)(1-y)}\).

We conclude this enumerative study by the summation on all these binomial words that is related, up to a rescaling due to change of variables, to \(\sum_{n \geq 1} K_n(d, r) z^{n-1}\) and appears to be a rational function in \(x, y, z\) and two copies of a Carlitz \(q\)-analogue of Catalan numbers, each counting area below Dyck paths.

8.1. Change of variable from \(K_n(d, r)\) to \(L_n(x, y)\). The relation between the bistatistics \((deg, \rho)\) and \((lw, rw)\) are reversible. Hence the generating function \(K_n(d, r)\) is equivalent up to a change of variables to the generating function \(L_n(x, y)\) defined above.

Corollary 9. (of Proposition [13]) For any \(n \geq 2\), we have

\[
K_n(d, r) = \frac{d^{\binom{n-1}{2} - 1}}{r} L_n\left(\frac{1}{d}, rd\right).
\]

(8.1)
8.2. Sums on all the \( D_n \). Our first finite description of \( L_n(x, y) \) requires some additional definition on words. For any word \( f \) on the alphabet \( \{a, b\} \) we define its weight \( W(f) \) which is a monomial in \( x \) and \( y \) as follows. Consider the heights \( \eta_1, \eta_2, \ldots, \eta_p \) of the prefixes of \( f \) followed by the occurrence of a letter \( a \) and let \( \alpha = \sum_{\eta_i \geq 0} (\eta_i + 1) \), \( \beta = \sum_{\eta_i < 0} (-\eta_i - 1) \) then \( W(f) = x^\alpha y^\beta \).

**Proposition 17.** Let \( A_n \) be the set of words having \( n \) occurrences of the letter \( b \) and \( n - 1 \) occurrences of the letter \( a \). For any \( n \geq 2 \), we have

\[
L_n(x, y) = H(x, y) \left( \sum_{bfb \in A_n} W(bfb) - \sum_{afa \in A_n} W(afa) \right)
\]

where

\[
H(x, y) = \frac{(1 - xy)}{(1 - x)(1 - y)} = 1 + \sum_{i \geq 1} (x^i + y^i).
\]

8.2.1. **Mixing geometric sums delimited by \( aa \) and \( bb \) factors of the word \( wb \).** The first step in the proof of this proposition sums the bistatistic \((lw, rw)\) on all cells of a fixed cut skew cylinder \( \text{Cyl}[w] \), here defined by a Dyck word \( w \) of size \( n - 1 \). A cell with label \( k < 0 \), is in the right region. If the cell has label \( k \) satisfying \( k > 1 + n(n - 3) \), then it is at the left of the triangle \( T(n) \) of corners \((0, 0), (n - 1, 0) \) and \((n - 1, n - 1) \) which contains the \( w \) cut path hence it is in the left region. We thus know that there is an odd number \( 2m + 1 \) \((m \geq 0)\) of indices \( k_1 \leq k_2 \leq \ldots \leq k_{2m+1} \) such that the cells with labels \( k_i \) and \( k_i + 1 \) are in different regions. In other words the finite sequence \( k_i \) for \( i = 1 \ldots 2m + 1 \) collects the labels of the cells just before a crossing of the path \( w \). On the example in Figure 8, \( m = 5 \) and \( k_1, k_2, \ldots, k_{2m+1} = -1, 0, 6, 7, 8, 11, 16, 23, 25, 34, 35 \). We will call the integer \( m \), the crossing multiplicity and the sequence \( k_1, \ldots, k_{2m+1} \) the crossing indices of the Dyck word \( w \).

**Lemma 10.** Let \( w \) be a Dyck word, we consider the sequence \( k \in \mathbb{Z} \) in the cut skew cylinder \( \text{Cyl}[w] \), \( m \) its crossing multiplicity and \( k_1, \ldots, k_{2m+1} \) its crossing indices. For any \( k \in \mathbb{Z} \) consider the cell with label \( k \) and denote \( M(k) \) the monomial \( x^{rw(k)} y^{lw(k)} \) We have

\[
\sum_{k \in \mathbb{Z}} M(k) = H(x, y) \left( \sum_{i=0}^m M(k_{2i+1}) - \sum_{i=1}^m M(k_{2i}) \right).
\]

**Proof:** Observe first the two following local facts:

- If \( k \) is the label of a cell in the left region, then \( lw(k) = lw(k - 1) + 1 \) and \( rw(k) = rw(k - 1) \), so that:
  \[
x^{rw(k)} y^{lw(k)} = y \times x^{rw(k-1)} y^{lw(k-1)}
  \]

- If \( k \) is the label of a cell in the right region, then \( lw(k) = lw(k - 1) \) and \( rw(k) = rw(k - 1) - 1 \) so that
  \[
x^{rw(k)} y^{lw(k)} = \frac{1}{x} \times x^{rw(k-1)} y^{lw(k-1)}.
  \]
Let \( a < b \), assume that for any \( k \) such that \( a + 1 \leq k \leq b \), the cell with label \( k \) lies in the left region, then a repeated use of the first observation shows that:

\[
\sum_{k=a+1}^{b} M(k) = \sum_{k=1}^{b-a} y^k M(a) = \frac{y - y^{b-a+1}}{1 - y} M(a) = \frac{y}{1 - y} (M(a) - M(b))
\]

where in the first equality we use that each cell with label \( k \) lies the left region for \( k = a + 1, \ldots, b \) and the third equality uses \( M(b) = y^{b-a} M(a) \).

In the particular case where \( b = +\infty \) the term \( M(b) \) vanishes, so that:

\[
\sum_{k=a+1}^{\infty} M(k) = \frac{y M(a)}{1 - y}
\]

On the other hand, assume that for \( k = a + 1, \ldots, b \) each cell with label \( k \) is in the right region, similarly we have:

\[
\sum_{k=a+1}^{b} M(k) = \sum_{k=1}^{b-a} \left( \frac{1}{x} \right)^k M(a) = \sum_{k=0}^{b-a-1} x^k M(b) = \frac{1 - x^{b-a}}{1 - x} M(b) = \frac{1}{1 - x} (M(b) - M(a))
\]

In the particular case where \( a = -\infty \), the terms \( M(a) \) vanishes so that:

\[
\sum_{k=-\infty}^{b} M(k) = \frac{M(b)}{1 - x}
\]

We split the sequence of cells with labels \( k \in \mathbb{Z} \) into maximal factors delimited by its sequence of crossing indices where all cells lie alternatively in the right region and the left region, beginning by the right one and ending with the left one.

\[
\sum_{k \in \mathbb{Z}} M(k) = \sum_{k=-\infty}^{k_1} M(k) + \sum_{i=1}^{2m} \left( \sum_{k=k_i+1}^{k_{i+1}} M(k) \right) + \sum_{k=k_{2m+1}+1}^{+\infty} M(k)
\]

For \( k \leq k_1 \) the cells of label \( k \) are in the right region, and for \( k > k_{2m+1} \) they are in the left region so that:

\[
\sum_{k \leq k_1} M(k) = \frac{M(k_1)}{1 - x} \quad \text{and} \quad \sum_{k > k_{2m+1}} M(k) = \frac{y M(k_{2m+1})}{1 - y}
\]

For \( k_{2i-1} < k \leq k_{2i} \) the cells of label \( k \) lie in the left region so that:

\[
\sum_{k=k_{2i-1}+1}^{k_{2i}} M(k) = \frac{y}{1 - y} (M(k_{2i-1}) - M(k_{2i}))
\]
For $k_{2i} < k \leq k_{2i+1}$ the cells of label $k$ lie in the right region so that:

$$\sum_{k=k_{2i}+1}^{k_{2i+1}} M(k) = \frac{1}{1-x} (M(k_{2i+1}) - M(k_{2i}))$$

Observe that in the sum $\sum_{k \in \mathbb{Z}} M(k)$ the monomials $M(k_{2i})$ appear with a minus sign and the monomials $M(k_{2i+1})$ with a plus sign, moreover each one appears twice with factors $\frac{1}{1-y}$ and $\frac{1}{1-x}$ adding these two factors gives $H(x,y)$ hence:

$$\sum_{k \in \mathbb{Z}} M(k) = \left( \sum_{i=0}^{m} H(x,y)M(k_{2i+1}) \right) + \left( \sum_{i=1}^{m} (-H(x,y))M(k_{2i}) \right)$$

8.2.2. Evaluation of the monomials for the crossing indices of the Dyck path. In this section we translate the evaluation of $M(k_i)$ by a formula involving the Dyck paths.

We first observe that a crossing point in the Dyck path described by the word $w$ corresponds to a factor $aa$ or $bb$ in $w$. Indeed since the Dyck path consists of North steps and East steps, the curve defined by a trajectory consisting in visiting the cells in order of their labels can only cross the path when there are two consecutive steps in the same direction. Hence the crossing cells $k_{2i}$ correspond to factors $aa$ in $w$ while the crossing cells $k_{2i+1}$ correspond to factors $bb$. More precisely a cell with label $k_{2i}$ is immediately on the left of the a North step of the Dyck path followed by another North step, while a cell with label $k_{2i+1}$ is just below of an East step of the Dyck path followed by another East step. In order to take also into account the case $k_1 = -1$ it is convenient to add an occurrence of $b$ at the end of $w$, that is to consider the word $wb$ instead of $w$.

In the example considered above the correspondence between factors and crossing cells is given by the following table:

|     | a | a | a | b | a | a | b | b | b | a | b | b | a | b | b | b |
|-----|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0   | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 11  | 23| 24| 35| 25| 16| 6 | 7 | 8 | 1 |

In the sequel we will consider that the label $k_i$ of a crossing cell is also the label of the corresponding occurrence of a letter in the Dyck word. We will also define the conjugate $\tau(wb,k)$ of Dyck word $w$ followed by a letter $b$, where $k$ is the label of the occurrence of $b$ appearing in $w = w_1bw_2$ as the the word $bw_2w_1$.

Let us examine more precisely what we do in order to compute $rw(k)$ and $lw(k)$ for a cell with label $k$ corresponding to an occurrence of the factor $bb$. This computation may be decomposed as the sum of $n-1$ values $rw_j(k)$ and $lw_j(w)$ each one corresponding to one row $j$ of the strip. To determine $rw_j(k)$ we count the number of cells with coordinates $(i,j)$ which lie in the right region and have a label greater than $k$, similarly $lw_j(k)$ is the number of cells with coordinates $(i,j)$ which lie in the left region and have a label less than or equal to $k$. Since labels of cells in the same row decrease from left to right, for each $j$ at least one of two values $rw_j(k)$ and $lw_j(k)$ is equal to 0.

In the following Lemma we use the definition of $W(f)$ for a word $f$ given before Proposition 17.
Lemma 11. Let \( w \) be a Dyck word and \( k \) be a crossing index of \( w \) in the cut skew cylinder \( \text{Cyl}[w] \). Then we have

\[
M(k_i) = W(\tau(w, k_i))
\]

Where \( \tau(w, k_i) \) is the conjugate of the word \( w_b \) given as above by the position of the letter with label \( k_i \).

Proof: Let \((i_1, j_1)\) be the coordinates of the cell with label \( k \) which we suppose to correspond to a factor \( bb \). Similar arguments may be used to prove the case where \( k \) is the label of a cell corresponding to the factor \( aa \). We have \( rw(k) = \sum_{j=0}^{n-2} rw_j(k) \) and \( lw(k) = \sum_{j=0}^{n-2} lw_j(k) \), the values \( rw_j(w) \) are determined by the value \((i_2, j)\) of the coordinates of the cell situated immediately on the left of the North step of the Dyck word situated at row \( j \). More precisely we have:

- If \( j > i_1 \) then
  \[
  rw_j(k) = \max(0, i_1 - i_2) \quad \text{and} \quad lw_j(k) = \max(0, i_2 - i_1)
  \]
- If \( j \leq i_1 \) then
  \[
  rw_j(k) = \max(0, i_1 - i_2 - 1) \quad \text{and} \quad lw_j(k) = \max(0, i_2 - i_1 + 1)
  \]

But the values \( i_1 - i_2 \) and \( i_1 - i_2 - 1 \) are exactly those of heights of \( \tau(w, k_i) \) increased by 1. \( \square \)

The proof of Proposition 17 follows directly from Lemmas 10 and 11.

8.3. Sum on all cut skew cylinders of any circumferences. Proposition 17 is a finite and combinatorial description of \( K_n(r, d) \) via an evaluation of \( L_n(x, y) \) for \( n \geq 2 \). In order to make deeper connection with other chapters of combinatorics, we consider the generating function

\[
L(x, y; z) = \sum_{n \geq 1} L_n(x, y) z^{n-1}
\]

and give a simple formula for it involving two copies of the Carlitz \( q \) analog series for Catalan numbers.

Before entering into the computation of this function, we need to give a value for \( L_1 \).

**On the complete graph \( K_1 \).** On \( K_1 \), a configuration is limited to the value \( f_1 \) assigned to the unique vertex. Its degree is \( f_1 \) and its rank is \(-1\) if \( f_1 < 0 \) and equal to \( f_1 \) otherwise so that:

\[
K_1(d, r) = \frac{1}{r} \frac{\frac{1}{d} + \frac{1}{1-rd}}{1 - \frac{1}{d} + \frac{1}{1-rd}}
\]

Applying the formula

\[
K_n(d, r) = \frac{d^{(n-1)-1}}{r} L_n(\frac{1}{d}, rd)
\]

gives

\[
L_1(x, y) = \frac{x}{1-x} + \frac{1}{1-y} = H(x, y)
\]
A formula for the generating function of \( L_n(x, y) \). The generating series of the Tutte polynomials of the complete graphs, were proved to have compact expressions (see [17, chapter 5], [25, equation (17)]).

Notice that these polynomials enumerate the spanning trees (in bijection with recurrent configurations) of \( K_n \) using two parameters external and internal activity.

For the series \( L(x, y; z) \) can also be expressed by a compact formula as a “simple” rational function of \( z \) and two copies of Carlitz \( q \)-analogue:

\[
C(q, z) = \sum_{n \geq 0} \sum_{w \in D_n} q^{\text{area}(w)} z^n
\]

where \( D_n \) denotes the set of Dyck words of semi-length \( n \).

**Theorem 6.** We have

\[
L(x, y; z) = \frac{(1 - xy)}{(1 - x)(1 - y)} \frac{C(x, xz) + C(y, yz) - C(x, xz)C(y, yz)}{1 - C(x, xz)zC(y, yz)}
\]

**Proof:**

- For \( n = 1 \), we use the previously computed generating function \( L_1(x, y) = H(x, y) \).
- For \( n \geq 2 \), we use the description of \( L_n(x, y) \) in Proposition 17. To estimate \( \sum_{n \geq 2} L_n(x, y)z^{n-1} \) we have to compute the generating function of words on the alphabets \( \{a, b\} \) starting and ending by different occurrences of the same letter and containing one more occurrence of the letter \( b \) than that of the letter \( a \). Since there is a negative sign to the weight if the first letter is \( a \) and a positive sign if the first letter is \( b \) we discuss separately this two cases. We denote by \( X \) the language consisting of all Dyck words, including the empty word denoted by \( \epsilon \). We denote \( \alpha \) the morphism on words on alphabet \( \{a, b\} \) exchanging the letters \( a \) and \( b \), so that \( \alpha(a) = b \) and \( \alpha(b) = a \) and denote \( Y = \alpha(X) \).

Let \( f = b\alpha^n a \) be a word containing \( n \) occurrences of the letter \( b \), \( n - 1 \) occurrences of the letter \( a \) (and such that the first and the last letters are occurrences of \( b \)).

This word may be interpreted as the sequence of steps of a walk on the line \( \mathbb{Z} \), a letter \( a \) corresponding to an increment, and a letter \( b \) to a decrement. We assume that this walk starts at 1 and hence ends at 0 and decompose it at each step from 1 to 0 or 0 to 1. In terms of language, this decomposition leads to the following non-ambiguous description of these words:

\[
b\alpha(\alpha X \alpha)a^*(X)b.
\]

The weight \( W[f](x, y) \) is distributed on the occurrences of the letter \( a \) as in its definition. We add a factor \( z \) to each occurrence of the letter \( a \) in order to take into account the size of words. Each factor \( X \) corresponds to a “Dyck” walk starting and ending at 1, so compared to the usual definition, the height of a step is incremented by one. Hence the generating function of this factor is \( C(x, xz) \). Similarly, each factor \( Y \) has generating function \( C(y, yz) \). Each \( a \) step from 0 to 1 is weighted by \( z \) so the previous language decomposition leads to the generating function

\[
\frac{C(x, xz)zC(y, yz)}{1 - C(x, xz)zC(y, yz)}
\]

We repeat a similar proof for words whose extremal letters are two different occurrences of the letter \( a \). The interpretation in terms of walk on the line and its
decomposition according to the same steps leads to the formula

\[(X\{\epsilon\})b(YaXb)^*(Y\{\epsilon\})\]

where at some steps we had to exclude the empty word to guarantee the extremal occurrences of letter \(a\). This leads to the generating function for this case

\[\frac{(C(x, xz) - 1)(C(y, yz) - 1)}{1 - C(x, xz)zC(y, yz)}.\]

Hence, the difference of the two generating functions gives

\[\sum_{n \geq 2} L_n(x, y)z^{n-1} = \frac{(1 - xy) C(x, xz)zC(y, yz) - (C(x, xz) - 1)(C(y, yz) - 1)}{(1 - x)(1 - y) 1 - C(x, xz)zC(y, yz)}.\]

This formula added to the case \(n = 1\) ends the proof of our Theorem. \(\square\)
APPENDIX A. AN ALTERNATIVE USING POINTED CUT SKEW CYLINDERS

In the main part of this paper the proofs of the main results concerning the rank in \( K_n \) are given using a classical object, namely the set of Dyck words. We wish to introduce in this appendix another combinatorial object which gives a wider geometric insight and guided some of our intuitions on significant parts of this work. In addition, this setting seems also adapted to the similar study on the bipartite complete graphs \( K_{m,n} \). This case is out of the scope of this paper but we have work in progress about it. Hence, in this appendix we present quickly these objects, skipping potentially tedious details by pointing to the related detailed proofs in the paper. A consequence of this lake of detailed proof is that propositions are there called statements, even if we estimate that the reader can figure out the details from our preceeding proofs to avoid such a drop in status.

A.1. Some operators on compact sorted configurations. We deal only with the complete graph \( K_n \) for some fixed \( n \geq 2 \). Let \( f \in \mathbb{Z}^n \) and \( g \in \mathbb{Z}^n \) be generic configurations on \( K_n \), we recall that we fix the vertex \( n \) to be the sink used to define the parking configurations. The toppling equivalence relation \( f \sim_{L_G} g \) induces classes.

We represent the class containing the configuration \( f \) by the existing and unique parking configuration denoted \( \text{park}(f) \), which is toppling equivalent to \( f \). We also consider the \((S_{n-1})\)-permuting equivalence, \( f \sim_{S_{n-1}} g \) if \( f_n = g_n \) and there exists a permutation \( \sigma \in S_{n-1} \) on the elements \( \{1, \ldots, n-1\} \) such that \( f_i = g_{\sigma(i)} \) also denoted \( f = \sigma.g \). We represent the class containing the configuration \( f \) by the existing and unique sorted configuration denoted \( \text{sort}(f) = \sigma.f = g \) where \( \sigma \) is a sorting permutation in \( S_{n-1} \) leading to \( g \) whose first \( n-1 \) entries \( g_1 \leq g_2 \leq \ldots \leq g_{n-1} \) are weakly increasing and \( f_n = g_n \). Hence any configuration \( f \) can be represented by the sorted configuration \( \text{sort}(f) \) and some permutation \( \sigma \in S_{n-1} \), where \( \sigma \) is not necessarily unique. These two distinct toppling and permuting equivalences leads to the toppling and permuting equivalence: \( f \sim_{L_G,S_{n-1}} g \) if there exists \( \sigma \in S_{n-1} \) such that \( f \sim_{L_G} \sigma.g \), notice that in that case the ranks and degrees of the configurations \( f \) and \( g \) are equal. Since sorting the \( n-1 \) first entries, preserves the property of being a parking configuration, the toppling and permuting class of a configuration \( f \) is represented by the existing and unique sorted and parking configuration \( \text{sort} \circ \text{park}(f) \) which is toppling and permuting equivalent to \( f \).

Most of the computations for the rank on \( K_n \) may manipulate only sorted configurations, via a sort after each elementary step.

**Definition 15.** A configuration \( f \) is called compact if any two of the first \( n-1 \) entries differ by at most \( n \), that is: \( \max_{1 \leq i,j \leq n-1} |f_i - f_j| \leq n. \)

If sorted parking configurations are motre precise to state our results, it appears that the weaker notion of sorted compact configuration is sufficient to describe configurations appearing in most of our computations in our proofs for \( K_n \).

First, for any configuration \( f \), we denote by \( \text{compact}(f) \), the result \( g \) of the algorithm in Section 2.1 defined for \( i < n \) by \( g_i = f_i - f_1 \text{mod} n \) and \( g_n = \deg(f) - \sum_{i=1}^{n-1} g_i \). This map is the initial projection of the configuration \( f \) into a compact configuration \( \text{compact}(f) \). Then the computations manipulate exclusively compact configurations.
Definition 16. We define the operator $T$ on sorted configurations by
\[ T(f) = \text{sort}(f - \Delta^{(n-1)}) \]
Hence $T(f)$ is obtained from $f$ by toppling the vertex $n - 1$ of maximal value among the $\{f_i\}_{i=1,\ldots,n-1}$ and then sorting the resulting configuration.

A key property is that this operator becomes invertible when restricted to compact sorted configurations. Indeed, explicit computation shows that for a compact sorted configuration $f$ we have:
\[ T((f_1, \ldots, f_{n-1}, f_n)) = (f_{n-1} - (n-1), f_1 + 1, f_2 + 1, \ldots, f_{n-2} + 1, f_n + 1) \]
Since the compact assumption in a sorted configuration $g$ is equivalent to $g_{n-1} \leq g_1 + n$.
Hence the permutation sorting into a parking configuration this case is the cyclic permutation $\tau \in S_{n-1}$ defined by $\tau(i) = i + 1$ for $i < n - 1$ and $\tau(n - 1) = 1$. In addition, when restricted to sorted compact configurations, the inverse of the operator is explicitly given by
\[ T^{-1}((f_1, \ldots, f_{n-1}, f_n)) = (f_2 - 1, \ldots, f_{n-2} - 1, f_{n-1} - 1, f_1 + (n-1), f_n - 1) \]
Moreover for a compact sorted configuration $f$, the $(n-1)$-th power of $T^{-1}$ corresponds to the toppling of the sink:
\[ T^{n-1}(f) = f + \Delta^{(n)} \]
Indeed one can show that

Statement 1. For any sorted compact configuration $f$ there exists some power $i \in \mathbb{Z}$ such that
\[ \text{sort} \circ \text{park}(f) = T^i(f) \]
A key point in this proof is that in a stable sorted compact configuration $f$, a subset $Y$ of maximal cardinality which is a counter example to the parking assumption is described as $\{n-1, n-2, \ldots, n-k\}$, hence the toppling of all the vertices of this set are described by $T^k$. One can then conclude via the following naive algorithm computing $\text{sort} \circ \text{park}(f)$ almost according to the element of its definition for a compact sorted configuration $f$:
- While at least one of the $n - 1$ first entries is negative, topple the sink (this is equivalent to do the inverse of $T^{n-1}$).
- While there is a subset $Y$ giving a counter example to parking assumption, pick the subset $Y$ of maximal cardinality and topple its vertices. ($T^{|Y|}$ at each loop iteration).

The unicity of a sorted parking configuration in a toppling and permuting equivalent class allows to reformulate this result in:

Statement 2. The set of sorted compact configurations toppling and permuting equivalent to the sorted compact configuration $f$ is exactly $\{T^i(f)\}_{i \in \mathbb{Z}}$ where all these powers of $T$ are distinct.
All these powers are distinct since $T$ increments exactly by 1 the value on the sink $n$.

A.2. Pointed cut skew cylinders to represent previous operators. The doubly pointed cut skew cylinders defined below appear to be the natural combinatorial objects which describe $\{T^i(\text{sort} \circ \text{park} \circ \text{compact}(f))\}_{i \in \mathbb{Z}}$, they also describe the graphical
description of our algorithm computing the rank on $K_n$ and later the analysis of this algorithm.

In the following we also call vertex a point of the plane $\mathbb{Z}^2$.

**Definition 17.** The skew cylinder $C_n$ of circumference $2n-1$ is obtained from the usual two dimensional grid $\mathbb{Z}^2$ by identifying two vertices of coordinates $(i, j)$ and $(i', j')$ if and only if $(i', j') = (i, j) + k(n, n-1)$ for some $k \in \mathbb{Z}$ also denoted $(i, j) \sim_{(n, n-1)} (i', j')$.

The skew cylinder may be described as illustrated in Figure 7 by a strip made up of $n-1$ rows, each row containing an infinite number of cells. In this Figure we have added two virtual copies of rows above and below the strip, these express the neighborhood in the skew cylinder for each cell in the top and bottom row of the strip. Notice that traversing the cylinder from one cell to the one at the north-east of it, defines a unique diagonal which visits all the cells of the strip.

This kind of skew cylinder is far from new since it already appeared for example in 1941 in a work of Kramers and Wannier [18].

The unique north-east diagonal allows to label the vertices (i.e points in $\mathbb{Z}^2$) as follows:

**Definition 18.** (of labeling and spiral traversal of vertices in a skew cylinder) Label 0 the origin of the skew cylinder and then from the vertex labeled $i$, the vertices of label $i+1$, and $i-1$ are obtained by a diagonal north-east step $(+1, +1)$ for the first and a south-west step $(-1, -1)$ for the second. Hence, the visit of vertices in increasing order according to the label in $\mathbb{Z}$ describes what we call a spiral traversal of the vertices.

Notice that this spiral traversal corresponds to a visit of the single diagonal of the skew cylinder in north-east direction.

We will show how it is related both to the powers $T^i(\text{sort}\circ\text{park}(f))$ and to the order of labeling in our algorithm computing the rank on $K_n$.

This cylinder $C_n$ is said of circumference $2n-1$ since disconnecting it by a general cut, which is a self-avoiding loop starting from the origin and made up of north or east steps, requires $2n-1$ elementary steps: $n-1$ north steps and $n$ east steps. If one maps a north step to the letter $a$ and an east step to the letter $b$, it obtains an obvious bijection between general cuts of $C_n$ and the words of $A_n$. We will impose that the cut is described by a word $w$ of $D_n \subseteq A_n$, these cuts are in bijection with the set of all $n-1$ first entries of the sorted parking configurations on $K_n$. For $w \in D_n$, we denote $C_n[w]$ the skew cylinder disconnected by the cut described by the word/path $w$ starting from the origin.

**Definition 19.** A doubly pointed cut skew cylinder $C_n[w](s, x)$ is defined by a triplet $(w, s, x) \in D_n \times \mathbb{Z} \times \mathbb{Z}$ where $s$ and $x$ describe a pointer on the vertices with labels $s$ and respectively $x$ in the cut skew cylinder $C_n[w]$.

The two pointers may be equal ($s = x$) but if $s \neq x$ then $C_n[w](s, x)$ and $C_n[w](x, s)$ are distinct.

The following explicit mapping cyltoconf is a bijection between the set of doubly pointed cut skew cylinders and the infinite set of compact sorted configurations on $K_n$.

**Definition 20.** For any doubly pointed cut skew cylinder $C[w](s, x)$, the configuration

$$f = \text{cyltoconf}(C[w](s, x))$$
Figure 7. In black, the skew cylinder $C_4$ as a strip. The label of a cell is the label of its bottom right corner in the spiral traversal. The wrapping conditions are described by the grey rows which are translated copies of the top and bottom rows. Notice that starting from the vertex labeled by 0 on the bottom (black) row one reaches its copy on the top (grey) row after any combination of 4 north steps and 5 east steps and such a path disconnects the cylinder in two parts.

is defined by $f_n = s$ and for $i < n$,

$$f_i = \frac{1}{n-1} (x_i - d_i)$$

where $x_i = x + n(i - 1)$ and $d_i$ is the label of the vertex which is the start of a north step of the cut $w$ and on the same row as the vertex of label $x_i$.

This definition has a graphical interpretation by drawing a segment of $n-1$ north steps starting from the vertex labeled $x$, the values of the $f_i$’s are equal to the relative position of a north steps of the segment and the north step of the cut $w$ in the same row.

Figure 8.
Figure 8 contains three examples of cyltoconf for the cut $w = abaababbabb \in D_5$ drawn in blue. The dashed blue line helps to check that $w \in D_5$ showing the factorization $w = w'b$ where $w'$ is a Dyck word. The label in a cell is the label of its bottom right corner.

- cyltoconf$(w, s, x_1) = (2, 3, 3, 4, 6, s)$ (defined by cells pointed by black and red circles in Cyl$[w]$)
- cyltoconf$(w, s, x_1') = (2, 4, 6, 7, 7, s)$ (defined by cells pointed by black and green circles in Cyl$[w]$)
- cyltoconf$(w, s, x_1'') = (−8, −7, −7, −6, −4, s)$ (defined by cells pointed by black and orange circles in Cyl$[w]$).

By convention, the cell containing $c_1$ is labeled by 0, then $s = −13$, $x_1 = 10$, $x_1' = 18$ and $x_1'' = −40$.

The motivation for introducing the map cyltoconf comes from the following description of the operators on sorted compact configurations.

**Statement 3.** For any sorted parking configuration $f$, we have

$$\text{sort} \circ \text{park}(f) = \text{cyltoconf}((C_n[\phi(f)], \psi(f), 0)).$$

Moreover, for any doubly pointed cut skew cylinder $(C_n[w], x, s)$ we have

$$T(\text{cyltoconf}((C_n[w], s, x))) = \text{cyltoconf}((C_n[w], s + 1, x - 1)),$$

$$\text{cyltoconf}((C_n[w], s, x)) - \Delta^{(n)} = \text{cyltoconf}((C_n[w], s - (n - 1), x + (n - 1))),$$

$$\text{cyltoconf}((C_n[w], s, x)) + \epsilon^{(n)} = \text{cyltoconf}((C_n[w], s + 1, x)),$$

$$\text{sort}(−\text{cyltoconf}((C_n[w], s, x))) = \text{cyltoconf}((C_n[\Phi(w)], −s, \text{lastright}(w) − x − k)).$$

Where $k = n(n − 3) + 1$.

The end of this section is devoted to give some elements of proof for this statement. Adding a final east step to the (blue) Dyck path in Figure 8 leads to recognize the cut of a skew cylinder starting at the origin which is the bottom right corner of cell labeled by 0.

This example should convince that for any sorted parking configuration $f$ we have

$$f = \text{park}(f) = \text{cyltoconf}((C_n[\phi(f)], \psi(f), 0))$$

where the north steps used as $(x_i)_{i=1...n-1}$ are the north steps on the $y$-axis. In addition, inspection shows that for any doubly pointed cut skew cylinder we have

$$T(\text{cyltoconf}((C_n[w], s, x))) = \text{cyltoconf}((C_n[w], s + 1, x - 1)).$$

These two first remarks are combined to show that cyltoconf is a bijection from the doubly pointed cut skew cylinders of circumference $2n - 1$ to the sorted compact configurations described as

$$\{T^i(f) | f \text{ sorted parking configuration on } K_n, i \in \mathbb{Z}\}.$$

Iterating the description of (the inverse of) $T$ we obtain

$$\text{cyltoconf}((C_n[w], s, x)) - \Delta^{(n)} = \text{cyltoconf}((C_n[w], s - (n - 1), x + (n - 1))),$$

and obviously the increment by one of the value on the sink is described by

$$\text{cyltoconf}((C_n[w], s, x)) + \epsilon^{(n)} = \text{cyltoconf}((C_n[w], s + 1, x)).$$

A less obvious relation describing the operator $f \mapsto \text{sort}(−f)$ is a key to understand globally many properties of involutions of this paper and explain the origin and name of the parameter lastright$(w)$:
sort(−cyltoconf((C_n[w], s, x))) = cyltoconf((C_n[Φ(w)], −s, lastright(w) − x − k)).

The proof of this relation is the subject of the three following paragraphs. We define two symmetries on skew cylinders. Then we show a link by graphic superimposition between C_n[w] and C_n[Φ(w)] described by the preceding symmetries. Finally we refine this superimposition to the definitions of f and sort(−f) by cyltoconf to obtain the expected relation.

First we define two symmetries on skew cylinders involved in a link between C_n[w] and C_n[Φ(w)].

Definition 21. (shift and flip on skew cylinders)

The unit diagonal shift on Z^2 is defined by shift(((i, j)) = (i + 1, j + 1). This symmetry is compatible with the equivalence ∼_{(n,n−1)} defining C_n hence induces a shift with similar notations on skew cylinder C_n.

The flip symmetry flip(((i, j)) = (−i, −j) on the two dimensional grid Z^2 is also compatible with the relation ∼_{(n,n−1)} hence induces a flip with similar notations on skew cylinder C_n.

The shift on skew cylinder maps each vertex to the next according to the spiral traversal. The compatibility of the flip with ∼_{(n,n−1)} comes from a multiplication by −1 of the relation assuming (i, j) ∼_{(n,n−1)} (i′, j′) which leads to a one proving (−i, −j) ∼_{(n,n−1)} (−i′, −j′). We notice that the image F(C_n) by the flip symmetry of the skew cylinder C_n of circumference 2n − 1 is C_n except that the spiral traversal of vertices is reversed since the vertex labeled by x becoming labeled −x.

Figure 9. Superimposition of cyltoconf((C_8[aaabaabbbababbb], s, 35)) and shift^{33} • flip(cyltoconf((C_8[aaabaabbbababbb], −s, −58))))

A key element of proof is the superimposition of the graphic descriptions of most notions in C_n[w] and shift^{lastright(w)+2(n−1)} • flip(C_n[Φ(w)]). Figure 9 provides an example of such a superimposition. There, each vertex x has two labels, the (green) label [x]_w
in $C_n[w]$ drawn, as previously, in the bottom right corner of the top left cell incident to the vertex and the (red) label $[x]_{\Phi(w)}$ in shift$^{\text{lastright}(w)+2(n-1)} \circ \text{flip}(C_n(\Phi(w)))$ drawn in the top left corner of the bottom right cell incident to the vertex. To prove such superimposition, one remarks that the image of the cut $w \in D_n$ by flip is the general cut $\tilde{w} \in A_n$. The shift is then iterated so that the conjugate $\Phi(w)$ of $\tilde{w}$ in $D_n$ starts at the origin. It means that the vertex of label $[0]_{\Phi(w)}$ is also the vertex reached after reading the prefix $u$ in the factorisation $w = uv$ such that $u$ is the longest prefix with maximal value by $\delta$. Hence the vertex labeled by $[0]_{\Phi(w)}$ is also labeled in $C_n[w]$ by

$$[n|u|_a - (n - 1)|u|_b]_w = [\text{lastright}(w) + 2n - 1]_w,$$

where may be defined by $\text{lastright}(w) = n|u|_a - (n - 1)(|u|_b + 1)$. On the example in Figure 3, $n = 8$, $|u|_a = 4$, $|u|_b = 1$, $\text{lastright}(w) = 8 \times 4 - 7 \times (1 + 1) = 18$, $\text{lastright}(w) + 2n - 1 = 18 + 2 \times 8 - 1 = 33$ and the vertex labeled by $[0]_{\Phi(w)}$ is also labeled by $[33]_w$. This remark helps to determine the number of iterations of shift for the superimposition: the flip maps the vertex $[\text{lastright}(w) + 2n - 1]_w$ in $C_n[w]$ to the vertex $[-(\text{lastright}(w) + 2n - 1)]_w$ in $C_n[w]$, then $(\text{lastright}(w) + 2n - 1)$ shift maps this vertex to $[0]_{\Phi(w)}$ in $C_n[\Phi(w)]$.

This superimposition is then refined to $f = \text{cyltoconf}((C_n[w], s, x))$ and $\text{sort}(-f) = \text{cyltoconf}((C_n[\Phi(w)], -s, \text{lastright}(w) - x - k))$. Indeed, in the definition of $f$ by cyltoconf the segment of $n - 1$ north steps starting from vertex labeled $[x]_w$ and ending in vertex $[x + n(n - 1)]_w$ in $C_n[w]$ is superimposed, up to reverse, to the image of the similar segment used in the definition of some configuration $g$ in $C_n[\Phi(w)]$. It appears that $g = \text{sort}(-f)$.

Indeed, first the permutation used in this case of sort is $\omega \in S_{n-1}$ defined by $\omega(i) = n - i$ for $i < n$. This matches the reverse of segment induced by the flip symmetry. Then each row we observe that $g_i = -f_{n-i}$ for some $i < n$ since both definitions use the same pair of north steps from the segment and the cut but there relative position change of sign due to the flip symmetry. The case of the value on the sink does not corresponds to a superimposition but the map $s \rightarrow -s$ obviously satisfies the constraint $g_n = -f_n$. We obtain the expected relation by observing that in the superimposition described using shift$^{\text{lastright}(w)+2n-1} \circ \text{flip}$, the vertex labeled by $[x + n(n - 1)]_w$ in $C_n[w]$ is labeled by

$$[(\text{lastright}(w) + 2n - 1) - x - n(n - 1)]_{\Phi(w)} = [\text{lastright}(w) - k]_{\Phi(w)}$$

in $C_n[\Phi(w)]$. On the example in Figure 3, we have $n = 8$, $x = 35$, the (green) segment of $n - 1$ north steps defining cyltoconf($C_8[\text{aabaaabbbabbb}]$, $s, 35$) starts in $[35]_w$ and ends in $[91]_w = [35 + 7 \times 8]_w$. The (red) segment of $n - 1$ south steps defining cyltoconf($C_8[\text{aabaaabbbabbb}]$, $-s, -58$) starts in $[33 - 91]_{\Phi(w)} = [-58]_{\Phi(w)}$.

A.3. Revisiting some properties of discussed involutions. The preceeding superimposition gives an alternative setting to describe the involution $\Phi$. Since the shift and flip maps act on vertices, they can act on steps seen as a pair of vertices. Using the identification between general cuts and paths of $A_n$, we may describe $\Phi$ as follows:

**Statement 4.** Let $w \in D_n$, we have

$$\Phi(w) = \text{shift}^{\text{lastright}(w)+2n-1} \circ \text{flip}(w).$$
Definition 22. Let \( w \in D_n \). We consider on each row \( i \) of \( C_n[w] \), from bottom to top, the contact \( c_i \) which is the label in \( C_n[w] \) of the vertex where start the north step crossing this row \( i \). Then

\[
\text{cdinv}(w) = |\{(i, j) | 1 \leq i < j \leq n - 1 \text{ and } |c_i - c_j| \leq n - 1\}|
\]

On the example in Figure 9, we have

\[(c_i)_{i=1\ldots n-1} = 0, 8, 9, 17, 25, 19, 13\]

and

\[
\text{cdinv}(w) = |\{\{8, 9\}, \{8, 13\}, \{9, 13\}, \{13, 17\}, \{13, 19\}, \{17, 19\}, \{19, 25\}\}| = 7.
\]

This definition is motivated by the following statement:

Statement 6. Let \( w = w'b \in D_n \), then

\[
\text{dinv}(w') = \text{cdinv}(w)
\]

and

\[
\text{cdinv}(\Phi(w)) = \text{dinv}(w).
\]

The preceding example for the definition of \( \text{cdinv} \) helps to convince the reader that this one case definition of pairs for \( \text{cdinv}(w) \) coincide with the two case definition of \( \text{dinv}(w') \). More precisely, the vertices of label \([c_i]_w\) and \([c_i + n - 1]_w\) are on the same row, and for any pair \( \{c_i, c_j\} \) counted for \( \text{dinv} \), with \( c_i < c_j \), the two cases in the definition of \( \text{dinv} \) corresponds to the discussion if the row of the vertex 0 belongs to one of the row of \([c_i + k]_w\) for some \( k = 0, 1, \ldots c_j - c_i \). Hence \( \text{cdinv}(w) = \text{dinv}(w') \).
The contacts are defined as the starts of vertical steps, and up to orientation, these steps are preserved in the superimposition of $C_n[w]$ and $\text{shift}_{\text{lastright}(w)+2n-1}\circ\text{flip}(C_n[\Phi(w)])$. In this setting, the contacts of $\Phi(w)$ are the opposite vertices to the vertical of the contacts of $w$. More precisely, starting from the contact of label $[c_i]_w$, the opposite vertex to the related north step is of label $[c_i+n]_w$ in $C_n[w]$, equivalently of label $[\text{lastright}(w)+2n-1-(c_i+n)]_{\Phi(w)}$ in $C_n[\Phi(w)]$. Hence the images of the contacts $([c_i]_w)_{i=1,...,n-1}$ are the contacts $\{c'_i\}_{i=1,...,n-1} = \{[\text{lastright}(w)+n-1-c_i]_{\Phi(w)}\}_{i=1,...,n-1}$.

We notice that $c'_i - c'_j = -(c_i - c_j)$ so the map $\{c_i, c_j\} \rightarrow \{c'_i, c'_j\}$ defines a bijection between the pairs counted for $\text{cdinv}(w)$ and those counted for $\text{cdinv}(\Phi(w))$, showing that $\text{cdinv}(w) = \text{cdinv}(\Phi(w))$.

On the example in Figure 9, we have the contact of $\Phi(w)$ which are

$$(c'_i)_{i=1,...,n-1} = 0, 8, 16, 17, 25, 12, 6$$

and

$$\text{cdinv}(\Phi(w)) = |\{(0, 6), (6, 8), (6, 12), (8, 12), (12, 16), (12, 17), (16, 17)\}| = 7.$$ 

For example the pair $\{0, 6\}$ in the computation of $\text{cdinv}(\Phi(w))$ corresponds to the pair $\{19, 25\}$ in the computation of $\text{cdinv}(w)$.

**ψ via the operators related to cyltoconf.** The operators on compact sorted configurations that we describe via the map cyltoconf allows a derivation via this setting of a preceding explicit description of the involution $\Psi(f) = \text{sort} \circ \text{park}(\kappa - f)$ on sorted parking configurations. Indeed, we decompose $\Psi$ as follows

$$\Psi(f) = T_{\text{lastright}(w)-1-(n-3)} \circ (g \rightarrow g - \Delta^{(n)})^{n-3} \circ (g \rightarrow g + \epsilon^{(n)})^{n(n-3)} \circ (g \rightarrow \text{sort}(-g))(f)$$

where $w$ is defined in the doubly pointed cut skew cylinder $(C_n[w], s, 0)$ related by cyltoconf to the sorted recurrent configuration $f$. Each operator in this decomposition is described in preceding subsection via cyltoconf, hence applying these descriptions we also obtain

**Statement 7.** For any $w \in D_n$ and $s \in \mathbb{Z}$, we have

$$\Psi(\text{cyltoconf}(C_n[w], s, 0)) = \text{cyltoconf}(C_n[\Phi(w)], \text{lastright}(w) - 1 - s, 0)$$

**ψ and a refined superimposition.** This involution $\Psi$ can also be described globally by a refinement of the superimposition and in particular an appropriate description of $rw(f)$ and $lw(f)$ from its doubly pointed cut skew cylinder $(C_n[w], s, 0)$.

We associate to any doubly pointed cut skew cylinder $(C_n[w], s, 0)$ the **partial spiral traversal** which visits the vertices labeled from $-\infty$ to $s + n$. In Figure 10 $n = 8$ and for $s = 10$, the green arrows suggests the partial traversal from $-\infty$ to $s + n = 18$, this last visited vertex labeled by $[18]_w$ being underlined by a black dot. A cell is **visited** by a partial traversal when at least three of its corners are visited otherwise it is called unvisited. In Figure 10 for $s + n = 18$, the cells visited are those crossed by the green arrows. Since the (blue) cut $w$ disconnect the skew cylinder into two connected components of cells adjacent by sides, we also consider the cells in the terminal (left) component of the complete spiral traversal and the complementary initial (right) component. Hence a cell is either **terminal** or **initial** according to the component
it belongs. In this setting, the number \( vister(C_n[w], s, 0) \) of visited and terminal cells corresponds to the cells used in Figure 6 to compute the rank hence

\[ vister(C_n[w], s, 0) = \rho(f) + 1. \]

On Figure 10, these cells are marked by a red disk. Similarly the number \( unvini(C_n[w], s, 0) \) of unvisited and initial cells are related to the rank and degree of \( f \) by

\[ unvini(C_n[w], s, 0) = \left( \frac{n-1}{2} \right) + \rho(f) - \deg(f). \]

On Figure 10, these cells are marked by a green disk.

**Statement 8.** Let \( f = cyltoconf(C_n[w], s, 0) \) be a sorted recurrent configuration, we have

\[ (lw(f), rw(f)) = (vister(C_n[w], s, 0), unvini(C_n[w], s, 0)). \]

The refinement of the superimposition of \( C_n[w] \) and shift \( lastright(w)+2n-1 \circ flip(C_n[\Phi(w)]) \) consider for the partial traversal of \( C_n[w] \) ending in the vertex \( [s+n]_w \), the almost complementary partial traversal of \( C_n[\Phi(w)] \) also ending in the same vertex. This vertex was underlined by the black dot in Figure 10 and the complementary partial traversal is described by the red arrows. We notice that each cell is visited by exactly one of these two partial traversals. More precisely, we notice that this superimposition exchanges visited and unvisited cells and also terminal and initial cells. Hence the related involution exchanges \( vister \) and \( unvini \) statistics.

The last vertex \( [s+n]_w \) of the partial traversal related to \( (C_n[w], s, 0) \) is labeled by \( [lastright(w)+2n-1-(s+n)]_{\Phi(w)} = [s'+n]_{\Phi(w)} \) in \( C_n[\Phi(w)] \) where \( s' \) is the undetermined value for the configuration defined in \( C_n[\Phi(w)] \). Hence \( s' = lastright(w) - 1 - s \) and we notice that this involution exchanging \( lw(f) \) and \( rw(f) \) is indeed \( \Psi \).

On the example of Figure 10 for \( s = 10 \), the last visited vertex by the partial traversal in \( C_n[w] \) is \( s + n = [18]_w \). The label of this vertex in \( C_n[\Phi(w)] \) is \( [15]_{\Phi(w)} = [s' + 8]_{\Phi(w)} \) so the superimposed configuration is defined on its sink by \( s' = 7 \) which corresponds to \( lastright(w) - 1 - s = 18 - 1 - 10 = 7 \).
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