Remote information concentration using a bound entangled state

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Remote information concentration, the reverse process of quantum telecloning, is presented. In this scheme, quantum information originally from a single qubit, but now distributed into three spatially separated qubits, is remotely concentrated back to a single qubit via an initially shared entangled state without performing any global operations. This entangled state is an unlockable bound entangled state and we analyze its properties.

Quantum entanglement has generated a great deal of interest recently as the most important resource in quantum information processing. The protocols of super dense coding \cite{1}, quantum teleportation \cite{2} and telecloning \cite{3} cannot be performed without some form of entanglement between the parties involved: classical correlations alone can never achieve the quantum efficiency arising from entanglement. Given that entanglement is a resource, it is important to be able to quantify it in order to deduce how effectively we can process information. There have been a number of suggestions for quantifying entanglement \cite{3}, but the most fruitful method comes from a procedure known as entanglement distillation \cite{4}. In this procedure, two distant users, Alice and Bob, share a certain number of entangled pairs all in the same state $\rho$. They then are allowed to perform local operations and communicate classically with each other (LOCC). The question is how many maximally entangled pairs they can obtain in this way. The limit of distillation in the infinite number of initial copies of $\rho$ is known as the entanglement of distillation \cite{4}. A natural question to ask is: which states $\rho$ can be distilled to maximally entangled states? Separable states $\rho = \sum_i \rho_A^i \otimes \rho_B^i$ are clearly non-distillable. Surprisingly, however, a recent important discovery by the Horodecki family showed that there are also some entangled states which cannot be distilled \cite{5}. These states have appropriately been called bound entangled. They are peculiar as entanglement has to be invested in creating them by LOCC, but this invested entanglement cannot then be recovered by LOCC. Bound entanglement has been studied extensively in the last two years \cite{6}, nevertheless no information processing protocol has been found where bound entangled states perform better than just classically correlated states. Therefore it has seemed that they are useless for quantum information processing and that we always need to use some form of “free” (unbound) entanglement to achieve greater-than-classical efficiency. However, as we show in this letter, this is not the case.

We present an important protocol where bound entanglement can be utilized effectively and performs better than any classically correlated states. This protocol is remote information concentration, the inverse of telecloning \cite{3}. Quantum telecloning, as its name suggests, combines teleportation and cloning in such a way that a sender teleports an unknown qubit state $|\phi\rangle = \alpha|0\rangle + \beta|1\rangle$ to a number of spatially separated receivers simultaneously. These teleported qubits cannot, of course, be exact replicas of the original qubit due to the linear laws of quantum evolution (“no-cloning theorem”) \cite{8}. However it has been shown that fidelities as high as allowed by the non-exact cloning (known as optimal cloning \cite{9}) can be achieved. The optimal cloning state for $|\phi\rangle$ is represented by a three qubit state

$$|\psi_c\rangle = \frac{\sqrt{1}}{\sqrt{2}} \left\{ |00\rangle + \frac{1}{\sqrt{2}} |11\rangle + |01\rangle + |10\rangle \right\}$$

where the first qubit is an ancilla and the last two qubits are two optimal clones. Now the question we ask is: once a state has been telecloned to spatially separated parties, can it then be recreated using only LOCC? The answer is yes and surprisingly involves a recently constructed unlockable bound entangled state \cite{10}.

The four particle unlockable bound entangled state presented by Smolin \cite{10} is

$$\rho_{ab} = \frac{1}{4} \sum_{i=0}^3 |\Phi^i\rangle \langle \Phi^i| \otimes |\Phi^i\rangle \langle \Phi^i|$$

where $|\Phi^i\rangle$ represents the four Bell states, $|\Phi^0\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$, $|\Phi^1\rangle = (|00\rangle - |11\rangle)/\sqrt{2}$, $|\Phi^2\rangle = (|01\rangle + |10\rangle)/\sqrt{2}$ and $|\Phi^3\rangle = (|01\rangle - |10\rangle)/\sqrt{2}$. This state is not distillable if we do not allow joint quantum operations (i.e. if all four parties only operate locally), and is therefore, a bound entangle state. However, if we allow a two qubit joint operation, i.e. Bell joint measurement on any two qubits, we can obtain a maximally
entangled state for the other two qubits via LOCC. Thus this state is unlockable. The unlocking mechanism is based on a joint operation for two out of four qubits.

Before we explain remote information concentration, we briefly summarize the forward process, telecloning. We focus on the 1 to 2 telecloning and its reverse in this letter. Generalizations to more qubits are possible and will be investigated elsewhere. The telecloning scheme allows direct distribution of optimal clones from a single original qubit state $|\phi\rangle$ to spatially separated parties using LOCC. In the telecloning scheme, we use an initially shared entangled channel (telecloning state)

$$|\xi_{tc}\rangle = \frac{1}{\sqrt{3}} (|00\rangle |00\rangle + |11\rangle |11\rangle)$$

(3)

$$+ \frac{1}{2} (|01\rangle + |01\rangle)(|01\rangle + |01\rangle).$$

(4)

where the first qubit is an input port of the distributor, the second qubit is an output port for the ancilla, and the third and forth qubits are output ports for the optimal clones. The telecloning protocol is similar to teleportation; the distributor performs a Bell joint measurement between the unknown state and the input port qubit, and then the receivers, who hold output port qubits, perform a single qubit operations depending on the distributor’s measurement result.

Now we present our remote information concentration scheme. From the distributed optimal cloning qubits shared by the spatially separated parties (Alice who holds the ancilla qubit and Bob and Charlie who each hold a clone qubit), the original single qubit state is recreated at the location of an receiver, David in our scheme: $|\psi_i\rangle_{ABC} \rightarrow |\phi_i\rangle_D$. We employ the unlockable bound entangled state (Eq. (3)) as an entangled channel for this scheme. The four qubits of the unlockable bound entangled state are initially distributed to Alice, Bob and Charlie (input port qubits) and David (output port qubit). The three senders, Alice, Bob and Charlie, individually perform Bell joint measurements between their qubits of the optimal cloning state and their input port qubits. We stress that no global operation is allowed between qubits belonging to different parties. One of the four possible outcomes $\{\Phi^i\}$ is obtained by the measurement of a party. All three senders classically communicate their measurement results with David. ($2 \times 3 = 6$ bits of classical information is communicated in total.) Each Bell measurement result $\{\Phi^i\}$ is associated with the corresponding Pauli operators $\{\sigma_i\}$, where $\sigma_9 \equiv 1$, $\sigma_1 \equiv x$, $\sigma_2 \equiv z$, and $\sigma_3 \equiv \sigma_2 \cdot \sigma_x$. David performs a Pauli operation $\sigma_j$, which is the product (up to a global phase factor) of the three Pauli operators associated with the three Bell measurements, on his output port qubit. The output port of David is now in the original state $|\phi\rangle$.

A schematic picture of this protocol is shown in Fig. 1. Since we do not allow joint operations on spatially separated qubits, the information channel in our scheme is indeed bound entangled. It is surprising that a bound entangled state can actually be useful for “transmitting” quantum information. In the following, we analyze this feature from two points of view: remote quantum operation and entanglement structure.

Remote quantum operation is performance of (global) unitary operations on remote qubits: A unitary operation $U$ is implemented by an initially shared entangled channel, Bell measurement, classical communication and (simple) single qubit operations, instead of directly running a quantum circuit. This is a generalization of quantum teleportation. Telecloning and quantum information distribution via entanglement are examples of remote quantum operation for $1 \rightarrow N$ quantum optimal cloning with $d$-level particles, which requires one input port and $2N - 1$ output ports. More general cases requiring more than one input port have been studied by Gottesman and Chuang in the context of “quantum computation using teleportation” and “quantum software”. The initially shared entanglement in the remote quantum operation scheme functions as quantum software. According to their result, unitary operations which belong to the Clifford group can be implemented remotely, if we restrict the single qubit operations to be the Pauli operations. The compounding qubits of the shared entanglement need not be in the same location. In this case, the share entanglement functions as a transmission channel as well as quantum software. We consider this most restricted case of all-separated qubits.

To implement a unitary operation $U$ on a state of three input qubits ($|\psi\rangle$), the entangled channel consists of three input port qubits and three output port qubits. For unitary operations that can be decomposed into CNOT (controlled NOT) and Hadamard gates, which are members of the Clifford group, the entangled channel state is given by

$$|\xi\rangle = \sum_{k=0}^{2^3} |\tilde{k}\rangle \otimes U |\tilde{k}\rangle,$$

(5)

where $\tilde{k}$ is a 3-bit binary number, for example, $\tilde{0} = 000$. 

![FIG. 1. Schematic picture showing the concentration of information from Alice, Bob and Charlie at the remote receiver, David, using an unlockable bound entangled state.](image-url)
\( \tilde{1} = 001, \ldots, \tilde{7} = 111 \). The first three qubits are the input ports and the last three qubits are the output ports. All the qubits of this channel are spatially separated from each other. We assume that Alice, Bob, Charlie, David, Elizabeth, Fred each hold one qubit of the channel (in this order). Alice, Bob and Charlie individually perform Bell joint measurements on their input qubits (in the state \( |\psi_{ABC}\rangle \), the qubits to be processed) and the input port qubits. David, Elizabeth and Fred perform an appropriate Pauli operation depending on the measurement results of Alice, Bob and Charlie. The mapping between measurement results and Pauli operations is initially agreed. The final state of David, Elizabeth and Fred is \( U |\psi\rangle_{DEF} \).

Now we return to reverse optimal cloning. We define a reverse cloning unitary operator \( U_r \)

\[
U_r |\psi_c\rangle = |\phi\rangle \otimes \sqrt{\frac{2}{3}} \left( |00\rangle + \frac{|01\rangle + |10\rangle}{2} \right),
\]

where the last two qubits are ancillas that are disentangled from the first qubit which holds the concentrated single qubit information. Note that \( U_r \) does not initialize the ancilla qubits after the operation into the conventional ancilla state \( |00\rangle \). \( U_r \) can be decomposed into just CNOT gates as shown in Fig.2. Thus the reverse cloning operation is in the Clifford group and can be performed by remote quantum operation. \( U_r \) is explicitly given in the computational basis by

\[
U_r = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}.
\]

Inserting \( U_r \) in Eq.(5), we obtain the channel for the remote reverse cloning:

\[
|\xi_{rc}\rangle = \frac{1}{2\sqrt{2}} \left\{ (|000\rangle + |111\rangle) |00\rangle + (|010\rangle + |101\rangle) |01\rangle + (|001\rangle + |110\rangle) |10\rangle + (|011\rangle + |100\rangle) |11\rangle \right\}.
\]

In this expression, Alice, Bob, Charlie and David hold the first, second, third and fourth qubits, respectively. The last two qubits are ancillas and may be separated from the other qubits (their location is irrelevant). From Eq.(8), we obtain the unlockable bound entangled state \( \rho_{ab} \), if we trace out the ancilla variables. Since the operations performed on ancillas do not effect the output port qubit, we can trace out the ancilla variables from the beginning. Then this remote quantum operation is equivalent to remote information concentration. This is why the unlockable bound entangled state actually functions as a channel for remote information concentration.
state), which is initially “closed” for transmission. The unbound entanglement of the input state provides quantum correlation among the qubits of the bound entangled state. The quantum correlation “opens” the channel for transmitting concentrated single qubit information from distributed in three qubits of the input state. Entanglement of the input optimal cloning state and the unboundable bound entangled channel state function in a complementary fashion. This result explains the importance of the ancilla qubit in the optimal cloning state, since the ancilla qubit is necessary for holding entanglement.

Another interesting observation is that the unclockable bound entangled state is also valid for remotely concentrating information from the spatially separate 3-qubit error correction state: \( |\psi_e\rangle_{ABC} = \alpha |000\rangle + \beta |111\rangle \rightarrow |\phi\rangle_D \). The procedure is similar to the case of optimal cloning. The only difference is a modification to the mapping to Pauli operations. David performs an additional \( \sigma_2 \) if the measurement results from Bob or Charlie, but not both, belong to the set \( \{|\Phi^0\rangle, |\Phi^1\rangle\} \). In this case, we may again consider that the (unbound) entanglement of the input state \( |\psi_e\rangle \) “opens” the bound entangled channel for transmitting concentrated single qubit information from the input state. If we consider the quantum state \( |\phi\rangle \) as a quantum key \([16]\), remote information concentration together with information distribution \([11]\) may allow more secure distribution of the quantum key to David via spatially-separated, branched repeaters Alice, Bob and Charlie.

Finally, we show that no classically correlated state can achieve the same task (c.f. \([4]\)). In optimal cloning scheme, due to the linearity of quantum transformations, mixed states can be cloned as well as pure. The same of course holds for teleporting. We consider the case when the qubit to be telecloned is maximally entangled with another qubit of George. After teleporting the qubit state into the qubits of Alice, Bob and Charlie, we perform the reverse process and remotely concentrate information at the location of David. Consequently, the qubits of David and George become entangled. If a shared state with only classical correlation could perform this remote information concentration, the procedure would create entanglement between David and George. This, however, is not possible: entanglement cannot be increased by LOCC. Therefore no classically correlated state can perform remote information concentration.

In this letter we have presented remote information concentration, the reverse process of quantum teleporting. It was shown that, surprisingly, the state needed for this operation is a bound entangled state. We have analyzed the remote information concentration scheme from two points of view, considering remote quantum operations and analyzing the entanglement structure of the bound state and the input state. We have shown that the unclockable bound entangled state is a reduced density matrix for the entanglement channel of remote reverse cloning, if we trace out the ancilla qubits of the output state. From our entanglement structure analysis, we have found that the functions of the entanglement of the optimal cloning state and the unclockable bound entangled state are complementary. We have also shown that the unclockable bound entangled state can be used for remotely concentrating information from a distributed 3-qubit error correction state, which may be useful for secure transmission of a quantum key. Furthermore, we showed that no purely classically correlated state can achieve this task. We hope that our work would stimulate more research into the nature of entanglement and its general usefulness in quantum information processing.

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