LOCAL RIGIDITY FOR ACTIONS OF KAZHDAN GROUPS ON NON COMMUTATIVE $L_p$-SPACES

BACHIR BEKKA

Abstract. Let $\Gamma$ be a discrete group and $\mathcal{N}$ a finite factor, and assume that both have Kazhdan’s Property (T). For $p \in [1, +\infty)$, $p \neq 2$, let $\pi : \Gamma \to O^+(L_p(\mathcal{N}))$ be a homomorphism to the group $O^+(L_p(\mathcal{N}))$ of linear bijective complete isometries of the $L_p$-space of $\mathcal{N}$. There are two actions $\pi^l$ and $\pi^r$ of $\Gamma$ by automorphisms of $\mathcal{N}$ associated to $\pi$ and given by $\pi^l(g)x = (\pi(g)1)^*\pi(g)(x)$ and $\pi^r(g)x = \pi(g)(x)(\pi(g)1)^*$ for $g \in \Gamma$ and $x \in \mathcal{N}$. Assume that $\pi^l$ and $\pi^r$ are ergodic. We prove that $\pi$ is locally rigid, that is, the orbit of $\pi$ under $O^+(L_p(\mathcal{N}))$ is open in $\text{Hom}(\Gamma, O^+(L_p(\mathcal{N})))$. As a corollary, we obtain that, if moreover $\Gamma$ is an ICC group, then the embedding $g \mapsto \text{Ad}(\lambda(g))$ is locally rigid in $O^+(L_p(\mathcal{N}(\Gamma)))$, where $\mathcal{N}(\Gamma)$ is the von Neumann algebra generated by the left regular representation $\lambda$ of $\Gamma$.

1. Introduction

Let $\Gamma$ be a discrete group and $G$ a topological group. A group homomorphism $\pi_0 : \Gamma \to G$ is locally rigid if every sufficiently small deformation of $\pi_0$ is trivial, in the sense that it is given by conjugation by elements from $G$. More precisely, let $\text{Hom}(\Gamma, G)$ be the set of all homomorphisms $\pi : \Gamma \to G$ endowed with the topology of pointwise convergence on $\Gamma$. The group $G$ acts on $\text{Hom}(\Gamma, G)$ by conjugation:

$$\text{Ad}(g)\pi(\gamma) = g\pi(\gamma)g^{-1} \quad \text{for all} \quad g \in G, \gamma \in \Gamma.$$ 

We say that $\pi_0$ is locally rigid if its $G$-orbit in $\text{Hom}(\Gamma, G)$ is open.

Local rigidity was proved for cocompact lattices in semisimple real Lie groups by Calabi, Vesentini, Selberg, and Weil, culminating in the following result: let $G$ be a semisimple Lie group without compact factors and assume that $G$ is not locally isomorphic to $SL_2(\mathbb{R})$; then,
for every irreducible cocompact lattice $\Gamma$, the canonical embedding of $\Gamma$ in $G$ is locally rigid (see [Ragh72, Chapter VII]).

Groups with Kazhdan’s Property (T) are defined by a rigidity property of their unitary group representations and play an important role in a large variety of subjects (for an account on Kazhdan’s groups, see the monography [BeHV08]). It is natural to study local rigidity for homomorphisms from such groups to various topological groups $G$. As an example, it was shown in [Rapi99, Theorem 1] that, if $\Gamma$ is a (discrete) Kazhdan group, then every unitary representation $\Gamma \to U(n)$ is locally rigid in $GL_n(C)$. In recent years, there has been an increasing interest in local rigidity for homomorphisms with “infinite dimensional” groups as targets. For instance, a striking result in [FiMa05] shows that every action of a Kazhdan group by isometries on a compact Riemannian manifold is locally rigid in its group of diffeomorphisms. For an overview on local rigidity for actions of groups on various manifolds, see [Spat04] and [Fish07].

In this paper, we study local rigidity for homomorphisms of discrete Kazhdan groups into the group of linear isometries of non-commutative $L_p$-spaces, that is, the $L_p$-spaces associated to a von Neumann algebra. Recently, Property (T) has been studied in the framework of group actions by isometries on Banach spaces and more specifically on $L_p(X, \mu)$ for a measure space $(X, \mu)$; see [BFGM07]. Some of the results from [BFGM07] were extended in [Oliv12] to non-commutative $L_p$-spaces.

Recall that a von Neumann algebra $\mathcal{N}$ is said to be finite if there exists a faithful normal finite trace $\tau$ on $\mathcal{N}$. Let $1 \leq p < \infty$; the non-commutative $L_p$-space $L_p(\mathcal{N})$ is the completion of $\mathcal{N}$ with respect to the norm defined by $\|x\|_p = (\tau(|x|^p))^{1/p}$ for $x \in \mathcal{N}$. For a survey on these spaces, see [PiXu03].

The von Neumann algebra $\mathcal{N}$ is a factor if the centre of $\mathcal{N}$ is reduced to the scalar operators. When $\mathcal{N}$ is a finite factor, then either $\mathcal{N}$ is finite dimensional, in which case $\mathcal{N}$ is isomorphic to a matrix algebra $M_n(C)$ equipped with the usual (normalized) trace, or $\mathcal{N}$ is a so-called type $II_1$ factor.

An important class of examples of type $II_1$ factors is given by ICC groups. Recall that the group $\Gamma$ is ICC if its conjugacy classes, except $\{e\}$, are infinite. In this case, the von Neumann algebra $\mathcal{N}(\Gamma)$ of $\Gamma$ is a (finite) factor. Recall that $\mathcal{N}(\Gamma)$ is the von Neumann algebra generated by the left regular representation $\lambda$ of $\Gamma$ on $\ell_2(\Gamma)$; thus, $\mathcal{N}(\Gamma)$ is the closure for the strong operator topology of the linear span of $\{\lambda(g) : g \in \Gamma\}$ in the algebra $\mathcal{B}(\ell_2(\Gamma))$. 
A notion of Kazhdan's property (T) for von Neumann algebras was defined in [CoJo85] (see Section 2.3 below) and it was shown there that, for an ICC group $\Gamma$, the factor $\mathcal{N}(\Gamma)$ has Property (T) if and only if the group $\Gamma$ has Kazhdan's property (T).

Let $1 \leq p < \infty$. The orthogonal group $O(L_p(\mathcal{N}))$ of $L_p(\mathcal{N})$, that is, the group of bijective linear isometries of $L_p(\mathcal{N})$, is a topological group when endowed with the strong operator topology (see Section 2.1 below). Observe that, since $L_p(\mathcal{N})$ contains $\mathcal{N}$ as dense subspace, every automorphism of $\mathcal{N}$ extends to a unique isometry of $L_p(\mathcal{N})$. In this way, we identify the group of automorphisms of $\mathcal{N}$ with a subgroup of $O(L_p(\mathcal{N}))$. The group $O(L_p(\mathcal{N}))$ for $p \neq 2$ was described in [Yead82] (see Theorem 2.1 below). It follows from this description that $O(L_p(\mathcal{N}))$ contains a subgroup $O^+(L_p(\mathcal{N}))$ of at most index 2 such that, for every $U$ in $O^+(L_p(\mathcal{N}))$, the mappings

$$U^l : x \mapsto U(1)^*U(x) \quad \text{and} \quad U^r : x \mapsto U(x)U(1)^*$$

are automorphisms of $\mathcal{N}$. It turns out (see Section 2.1) that $O^+(L_p(\mathcal{N}))$ is the group of complete isometries of $L_p(\mathcal{N})$, that is, the isometries $U$ of $L_p(\mathcal{N})$ such that $\text{id} \otimes U$ is an isometry of $L_p(M_n(\mathbb{C}) \otimes \mathcal{N})$ for every $n \in \mathbb{N}$, where $M_n(\mathbb{C}) \otimes \mathcal{N} \cong M_n(\mathcal{N})$ is the finite factor consisting of the $\mathcal{N}$-valued $n \times n$-matrices. If

$$\pi : \Gamma \to O^+(L_p(\mathcal{N}))$$

is a group homomorphism, we obtain in this way two actions of $\Gamma$ by automorphisms on $\mathcal{N}$, given by homomorphisms

$$\pi^l : \Gamma \to \text{Aut}(\mathcal{N}) \quad \text{and} \quad \pi^r : \Gamma \to \text{Aut}(\mathcal{N});$$

we call $\pi^l$ and $\pi^r$ the actions of $\Gamma$ by automorphisms associated to $\pi$ (see Section 2.2). Recall that an action $\theta : \Gamma \to \text{Aut}(\mathcal{N})$ of $\Gamma$ by automorphisms on a von Neumann algebra $\mathcal{N}$ is ergodic if the fixed point algebra

$$\mathcal{N}^\theta = \{x \in \mathcal{N} : \theta_g(x) = x \quad \text{for all} \quad g \in \Gamma\}$$

consists only of the scalar multiples of 1.

Here is our main result.

**Theorem 1.1.** Let $\Gamma$ be a discrete group and $\mathcal{N}$ a finite factor, and assume that both have Property (T). For $p \in [1, +\infty)$, $p \neq 2$, let $\pi : \Gamma \to O^+(L_p(\mathcal{N}))$ be a homomorphism from $\Gamma$ to the group of complete isometries of $L_p(\mathcal{N})$. Assume that the associated actions $\pi^l$ and $\pi^r$ of $\Gamma$ by automorphisms on $\mathcal{N}$ are both ergodic. Then $\pi$ is locally rigid in $O^+(L_p(\mathcal{N}))$. 
Thus, there exists a neighbourhood \( \mathcal{V} \) of \( \pi \) such that every \( \rho \) in \( \mathcal{V} \) is conjugate to \( \pi \) by some \( U \) in \( \mathbf{O}^+(L_p(\mathcal{N})) \). In fact, we will determine explicitly, in terms of Kazhdan pairs for \( \Gamma \) and \( \mathcal{N} \), such a neighbourhood \( \mathcal{V} \) for which \( U \) can be chosen to be close the identity (see Remark 3.1 below).

As we will see in Section 4, the various assumptions made in the statement of Theorem 1.1 are necessary in one form or another.

Since \( \mathcal{N}(\Gamma) \) is a factor when \( \Gamma \) is an ICC group, the action \( g \mapsto \text{Ad}(\lambda(g)) \) of \( \Gamma \) by automorphisms of \( \mathcal{N}(\Gamma) \) is ergodic; so, the following corollary is an immediate consequence of the previous theorem.

**Corollary 1.2.** Let \( \Gamma \) be an ICC group with Kazhdan’s Property (T). The embedding \( g \mapsto \text{Ad}(\lambda(g)) \) of \( \Gamma \) in \( \mathbf{O}^+(L_p(\mathcal{N}(\Gamma))) \) is locally rigid, for \( p \in [1, +\infty), \ p \neq 2 \).

**Remark 1.3.** There is another natural action of \( \Gamma \) by isometries on \( L_p(\mathcal{N}(\Gamma)) \): it is given by the embedding

\[
\pi_0 : \Gamma \rightarrow \mathbf{O}^+(L_p(\mathcal{N}(\Gamma))), \ g \mapsto \lambda(g),
\]

where \( \lambda(g) \) denotes the extension from \( \mathcal{N}(\Gamma) \) to \( L_p(\mathcal{N}(\Gamma)) \) of multiplication from the left by the unitary \( \lambda(g) \). As will be seen below (Example 4.2), \( \pi_0 \) may fail to be locally rigid when \( \Gamma \) is an ICC group with Property (T). It can be shown that, if we view \( \pi_0 \) as a homomorphism in the unitary group \( \mathbf{U}(\mathcal{N}(\Gamma)) \) of \( \mathcal{N}(\Gamma) \), then \( \pi_0 \) is locally rigid.

Apart from Yeadon’s description of the group of isometries of \( L_p(\mathcal{N}) \) for \( p \neq 2 \), the proof of Theorem 1.1 depends on the following three ingredients: the first one (see Proposition 2.3 below) is that \( \mathbf{O}^+(L_p(\mathcal{N})) \) is isomorphic, as topological group, to an appropriate subgroup of the group of isometries of the Hilbert space \( L_2(\mathcal{N}) \); the second ingredient is the fact (see [CoJo85]) that the group of outer automorphisms of a factor with Property (T) is discrete; the third ingredient is an extension of a result from [Kawa91] showing that certain 1-cohomology classes associated to actions of a Kazhdan group by automorphisms of a finite factor are open (see Proposition 2.10). For this, we use in a crucial way a rigidity property of projective unitary representations of Kazhdan groups from [NiPS07].

This paper is organized as follows. In Section 2 we collect the ingredients necessary for the proof of our main result. Section 3 is devoted to the proof of Theorem 1.1. In Section 4 we present counter-examples in relation with the various assumptions made in the statement of Theorem 1.1.
Acknowledgments. We thank Mikael de la Salle and Eric Ricard for helpful comments.

2. Preliminaries

Let $\mathcal{N}$ be a finite factor, fixed throughout this section.

2.1. The group of isometries of $L_p(\mathcal{N})$. The following result is a corollary of Yeadon’s description of the linear (not necessarily surjective) isometries of the non-commutative $L_p$ space of a semi-finite von Neumann algebra for $p \neq 2$ (see [Yead82, Theorem 2]). For an extension of Yeadon’s result to arbitrary (not necessarily semi-finite) von Neumann algebras, see [Sher05].

**Theorem 2.1.** ([Yead82]) Let $1 \leq p < \infty$ and $p \neq 2$. A linear mapping $U : L_p(\mathcal{N}) \to L_p(\mathcal{N})$ is a surjective isometry if and only if there exists a unique pair $(u, \theta)$ consisting of a unitary $u \in \mathcal{N}$ and an automorphism or an anti-automorphism $\theta$ of $\mathcal{N}$ such that

$$U(x) = u\theta(x) \quad \text{for all } x \in \mathcal{N}. $$

As this result is not explicitly stated in [Yead82], we indicate how it follows from there. Since $U$ is surjective, by [Yead82, Theorem 2], there exist a normal Jordan isomorphism $J : \mathcal{N} \to \mathcal{N}$, a unitary $u \in \mathcal{N}$, and a positive self-adjoint operator $B$ affiliated with $\mathcal{N}$ such that $U(x) = uBJ(x)$ for all $x \in \mathcal{N}$. Since $\mathcal{N}$ is factor, $J$ is either an automorphism or an anti-automorphism (see [BrRo79, Proposition 3.2.2]) and $B = 1$. The group $O(L_p(\mathcal{N}))$ of linear bijective isometries of $L_p(\mathcal{N})$ is a topological group when equipped with the strong operator topology; this is the topology for which a fundamental system family of neighbourhoods of $U$ in $O(L_p(\mathcal{N}))$ is given by subsets of the form

$$\{ V \in O(L_p(\mathcal{N})) : \|V(x_i) - U(x_i)\|_p \leq \varepsilon \}$$

for $x_1, \ldots, x_n \in L_p(\mathcal{N})$ and $\varepsilon > 0$.

The set of isometries $U$ given by $U(x) = u\theta(x)$ for a unitary $u \in \mathcal{N}$ and an automorphism $\theta$ of $\mathcal{N}$ is a subgroup of index at most 2 in $O(L_p(\mathcal{N}))$ and will be denoted by $O^+(L_p(\mathcal{N}))$.

It follows from Yeadon’s result and from [JiRS05, Theorem 2] that the subgroup $O^+(L_p(\mathcal{N}))$ can be intrinsically characterized inside $O(L_p(\mathcal{N}))$ as the subgroup of the complete isometries (or as the subgroup of 2-isometries) of $L_p(\mathcal{N})$ in the sense of operator spaces, that is, the isometries $U$ of $L_p(\mathcal{N})$ such that $id \otimes U$ is an isometry of $L_p(M_n(\mathbb{C}) \otimes \mathcal{N})$ for every $n \in \mathbb{N}$ (or such that $id \otimes U$ is an isometry of $L_p(M_2(\mathbb{C}) \otimes \mathcal{N})$).
We identify $\text{Aut}(\mathcal{N})$ and the unitary group $\text{U}(\mathcal{N})$ of $\mathcal{N}$ with subgroups of $\text{O}(L_p(\mathcal{N}))$, endowed with the topology induced by that of $\text{O}(L_p(\mathcal{N}))$; observe that, as is easy to show, this topology on $\text{U}(\mathcal{N})$ coincides with the topology induced by its embedding in $L_p(\mathcal{N})$ given by $u \mapsto u1$.

**Corollary 2.2.** For $1 \leq p < \infty$ and $p \neq 2$, the group $\text{O}^+(L_p(\mathcal{N}))$ is isomorphic as topological group to the topological semi-direct product $\text{U}(\mathcal{N}) \rtimes \text{Aut}(\mathcal{N})$, given by the natural action of $\text{Aut}(\mathcal{N})$ on $\text{U}(\mathcal{N})$.

**Proof** The fact that $\text{O}^+(L_p(\mathcal{N}))$ is isomorphic as abstract group to $\text{U}(\mathcal{N}) \rtimes \text{Aut}(\mathcal{N})$ is a consequence of Yeadon’s theorem. Moreover, evaluation at $1 \in \mathcal{N}$ shows that the projection $\text{O}^+(L_p(\mathcal{N})) \to \text{U}(\mathcal{N}) \rtimes \text{Aut}(\mathcal{N})$ is continuous; it follows that $\text{O}^+(L_p(\mathcal{N}))$ and $\text{U}(\mathcal{N}) \rtimes \text{Aut}(\mathcal{N})$ are isomorphic as topological groups.

We will henceforth identify $\text{O}^+(L_p(\mathcal{N}))$ with $\text{U}(\mathcal{N}) \rtimes \text{Aut}(\mathcal{N})$; for $U \in \text{O}^+(L_p(\mathcal{N}))$, we will often write $U = (u, \theta)$ for $u \in \text{U}(\mathcal{N})$ and $\theta \in \text{Aut}(\mathcal{N})$ and refer to the pair $(u, \theta)$ as the Yeadon decomposition of $U$.

We observe that every $U = (u, \theta)$ in $\text{O}^+(L_p(\mathcal{N}))$ defines, for every $1 \leq q < \infty$, a linear bijective isometry of $L_q(\mathcal{N})$ by the same formula: $U(x) = u\theta(x)$ for all $x$ in the dense subspace $\mathcal{N}$ of $L_q(\mathcal{N})$. One obtains in this way a mapping

$$\Phi_{p,q} : \text{O}^+(L_p(\mathcal{N})) \to \text{O}^+(L_q(\mathcal{N})).$$

For $q \neq 2$, this mapping is of course surjective; this is not the case for $q = 2$ (if $\mathcal{N}$ is infinite dimensional) and we define the $\mathcal{N}$-unitary group $\text{O}^+_N(L_2(\mathcal{N}))$ of the Hilbert space $L_2(\mathcal{N})$ to be the image of $\Phi_{p,2}$; thus $\text{O}^+_N(L_2(\mathcal{N}))$ is the group of unitary operators $U$ of $L_2(\mathcal{N})$ which have a Yeadan decomposition $U = (u, \theta)$ for $u \in \text{U}(\mathcal{N})$ and $\theta \in \text{Aut}(\mathcal{N})$.

The next proposition will allow us to transfer representations in $\text{O}^+(L_p(\mathcal{N}))$ to representations in the $\mathcal{N}$-unitary group of $L_2(\mathcal{N})$. Its proof uses in a crucial way properties of the Mazur map, which is the (non linear) mapping $M_{p,q} : L_p(\mathcal{N}) \to L_q(\mathcal{N})$ defined by

$$M_{p,q}(x) = u|x|^{\frac{p}{q}}$$

for $x \in L_p(\mathcal{N})$ with polar decomposition $x = u|x|$. 
Proposition 2.3. Let $1 \leq p, q < \infty$ and $p, q \neq 2$. The mappings
\[ \Phi_{2,p} : O_X^+(L_2(N)) \to O^+(L_p(N)), \Phi_{p,2} : O_X^+(L_p(N)) \to O_X^+(L_2(N)), \]
and \[ \Phi_{p,q} : O^+(L_p(N)) \to O^+(L_q(N)) \]
are continuous. In particular, the groups $O^+(L_p(N))$ for $p \neq 2$ and $O_X^+(L_2(N))$ are mutually isomorphic as topological groups.

**Proof** For $U \in O^+(L_p(N))$ with Yeadon decomposition $(a, \theta)$ and $x \in N$ with polar decomposition $x = u|x|$, we have
\[ U(x) = a \theta(u) \theta(|x|) = a \theta(u) \theta(|x|^{\frac{q}{p}})^{\frac{p}{q}} = M_{p,q}(a \theta(M_{q,p}(x))) = M_{p,q} \circ U \circ M_{q,p}(x), \]
so that
\[ \Phi_{p,q}(U) = M_{p,q} \circ U \circ M_{q,p} \quad \text{for all } U \in O^+(L_p(N)). \]
It is known that (even for a general von Neumann algebra $N$) the restriction $M_{p,q} : B_1(L_p(N)) \to B_1(L_q(N))$ of $M_{p,q}$ to the unit ball $B_1(L_p(N))$ is uniformly continuous (see [Rayn02, Lemma 3.2]; a more precise result is proved in [Rica14]: $M_{p,q}$ is min\{p/q, 1\}–Hölder continuous on $B_1(L_p(N))$). Since
\[ \|\Phi_{p,q}(U)(x) - \Phi_{p,q}(V)(x)\|_q = \|M_{p,q}(U(M_{q,p}(x)) - M_{p,q}(V(M_{q,p}(x)))\|, \]
for $U, V \in O^+(L_p(N))$ and $x \in L_q(N)$, the proposition follows. \[ \square \]

2.2. Group representations by linear isometries on $L_p(N)$. Let $\Gamma$ be a discrete group and $p \in [1, +\infty)$, $p \neq 2$, fixed throughout this section.

Given a group homomorphism $\theta : \Gamma \to Aut(N), g \mapsto \theta_g$, we denote by $Z^1(\Gamma, \theta)$ the set of all corresponding 1-cocycles, that is the set of mappings $u : \Gamma \to U(N)$ such that
\[ u_{gh} = u_g \theta_g(u_h) \quad \text{for all } g, h \in \Gamma. \]
Two 1-cocycles $u$ and $v$ are cohomologous, if there exists $w \in U(N)$ such that
\[ v_g = w u_g \theta_g(w^*) \quad \text{for all } g \in \Gamma. \]
The set of 1-coboundaries $B^1(\Gamma, \theta)$ is the set of 1-cocycles which are cohomologous to the trivial cocycle $g \mapsto 1$.

For a mapping $\pi : \Gamma \to O^+(L_p(N))$ or $\pi : \Gamma \to O_X^+(L_2(N))$, we have corresponding mappings $u : \Gamma \to U(N)$ and $\theta : \Gamma \to Aut(N)$ given by the Yeadon decomposition $\pi(g) = (u_g, \theta_g)$ for every $g \in \Gamma$. We will refer to $\pi = (u, \theta)$ as the Yeadon decomposition of $\pi$. 
The proof of following proposition is straightforward.

**Proposition 2.4.** Let \( \pi : \Gamma \to O^+(L_p(\mathcal{N})) \) or \( \pi : \Gamma \to O^+_{N}(L_2(\mathcal{N})) \) be a mapping with Yeadon decomposition \( \pi = (u, \theta) \). The following conditions are equivalent.

(i) \( \pi \) is a group homomorphism;
(ii) \( \theta : \Gamma \to Aut(\mathcal{N}) \) is a group homomorphism and \( u : \Gamma \to U(\mathcal{N}) \) is a 1-cocycle with respect to \( \theta \).

Given \( \pi \in Hom(\Gamma, O^+(L_p(\mathcal{N}))) \), with Yeadon decomposition \( (u, \theta) \), there are two associated actions \( \pi^l \) and \( \pi^r \) in \( Hom(\Gamma, Aut(\mathcal{N})) \) defined by

\[
\pi^l(g) = \theta_g \quad \text{and} \quad \pi^r(g) = Ad(u_g) \theta_g \quad \text{for all} \quad g \in \Gamma.
\]

(For \( u \in U(\mathcal{N}) \), \( Ad(u) \) denotes the automorphism of \( \mathcal{N} \) given by \( Ad(u)x = u x u^* \) for \( x \in \mathcal{N} \).)

Recall that \( G = O^+(L_p(\mathcal{N})) \) or \( G = O^+_{N}(L_2(\mathcal{N})) \) acts on the set of all mappings \( \pi : \Gamma \to G \) by conjugation

\[
Ad(U)\pi(g) = U\pi(g)U^{-1} \quad \text{for all} \quad U \in G, g \in \Gamma.
\]

**Proposition 2.5.** Let \( \pi \) and \( \rho \) be homomorphisms from \( \Gamma \) to \( G = O^+(L_p(\mathcal{N})) \) or \( G = O^+_{N}(L_2(\mathcal{N})) \), with Yeadon decompositions \( \pi = (u, \theta) \) and \( \rho = (v, \alpha) \). The following conditions are equivalent:

(i) \( \rho \) belongs to the \( G \)-orbit of \( \pi \);
(ii) there exists \( \varphi \in Aut(\mathcal{N}) \) such that \( \alpha = Ad(\varphi)(\theta) \) and such that \( v \) is cohomologous to \( Ad(\varphi)(u) : g \mapsto \varphi(u_g) \) in \( Z^1(\Gamma, Ad(\varphi)(\theta)) \).

**Proof** For an element \( U = (w, \varphi) \) in \( G \), one computes that the Yeadon decomposition \( (v, \alpha) \) of \( Ad(U)\pi \) is given by

\[
\alpha_g = Ad(\varphi)(\theta_g) = \varphi \theta_g \varphi^{-1}, \quad v_g = w \varphi(u_g)(Ad(\varphi)(\theta_g))(w^*)
\]

for every \( g \in \Gamma \) and the claim follows.

### 2.3. Groups and factors with Kazhdan’s Property (T)

We recall (see [BeHV08]) that a (discrete) group \( \Gamma \) has Kazhdan’s Property (T) if there exist a finite subset \( S \) of \( \Gamma \) and \( \varepsilon > 0 \) with the following property: if a unitary representation \( \pi : \Gamma \to U(\mathcal{H}) \) of \( \Gamma \) in a Hilbert space \( \mathcal{H} \) has a \( (S, \varepsilon) \)-invariant unit vector, that is, a unit vector \( v \in \mathcal{H} \) with

\[
\|\pi(s)v - v\| \leq \varepsilon \quad \text{for all} \quad s \in S,
\]

then there exists a non-zero \( \Gamma \)-invariant vector in \( \mathcal{H} \). The pair \( (S, \varepsilon) \) is called a Kazhdan pair for \( \Gamma \). Moreover, if this is the case, then for every \( \delta > 0 \) and every \( (S, \delta\varepsilon) \)-invariant unit vector \( v \), there exists a
$\Gamma$-invariant vector $w \in \mathcal{H}$ with $\|v - w\| \leq \delta$ (see Proposition 1.1.9 in \cite{BeHV08}).

We shall need the extension from \cite{NiPS07} of the previous result to projective unitary representations; recall that a projective unitary representation of $\Gamma$ in a Hilbert space $\mathcal{H}$ is a mapping $\pi$ from $\Gamma$ to the unitary group $U(\mathcal{H})$ of $\mathcal{H}$ such that, for every $g, h \in G$, there exists a scalar $\mu_{g, h} \in \mathbb{S}^1 = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ with

$$\pi(g)\pi(h) = \mu_{g, h}\pi(gh)$$

for all $g, h \in \Gamma$.

A projective unitary representation $\pi$ determines a homomorphism

$$\tilde{\pi} : \Gamma \to \text{PU}(\mathcal{H}) = U(\mathcal{H})/\mathbb{S}^1$$

to the projective unitary group $\text{PU}(\mathcal{H})$ of $U(\mathcal{H})$, where $\mathbb{S}^1$ is identified with the subgroup of scalar multiples of the identity operator $\text{Id}_\mathcal{H}$; conversely, every lift $\pi : \Gamma \to U(\mathcal{H})$ of a homomorphism $\tilde{\pi} : \Gamma \to \text{PU}(\mathcal{H})$ is a projective unitary representation of $\Gamma$. The mapping $\mu : \Gamma \times \Gamma \to \mathbb{S}^1, (g, h) \mapsto \mu_{g, h}$ is a 2-cocycle, that is, it satisfies the identity

$$\mu_{g, k}\mu_{g, hk} = \mu_{gh, k}\mu_{g, h}$$

for all $g, h, k \in \Gamma$.

If $\mu : \Gamma \times \Gamma \to \mathbb{S}^1$ is a 2-coboundary, that is, if there exists a mapping $\lambda : \Gamma \to \mathbb{S}^1$ such that

$$\mu_{g, h} = \lambda_g\lambda_h\lambda_{gh}$$

for all $g, h \in \Gamma$,

then $\pi$ gives rise to a genuine representation $\pi : \Gamma \to U(\mathcal{H})$, defined by

$$\pi(g) = \lambda_g\pi(g)$$

for all $g \in \Gamma$ and inducing the same homomorphism $\Gamma \to \text{PU}(\mathcal{H})$ as $\pi$.

Given a projective unitary representation $\pi : \Gamma \to U(\mathcal{H})$ and a subset $S$ of $\Gamma$ and $\varepsilon > 0$, we will say that a unit vector $v \in \mathcal{H}$ is projectively $(S, \varepsilon)$-invariant if, for every $s \in S$, there exists $\alpha_s \in \mathbb{C}$ such that $\|\pi(s)v - \alpha_s v\| \leq \varepsilon$.

The following result is proved in \cite{NiPS07} in the more general situation of a pair of groups with the relative Property (T). When $\Gamma$ has Property (T), with a Kazhdan pair $(S, \varepsilon)$, one checks easily that the proof of Lemma 1.1 of \cite{NiPS07} yields exactly the following result.

**Theorem 2.6.** (\cite{NiPS07}) Let $\Gamma$ be a Kazhdan group, with a Kazhdan pair $(S, \varepsilon)$. Fix $\delta$ with $0 < \delta < 1$. Let $\pi : \Gamma \to U(\mathcal{H})$ be a projective unitary representation of $\pi$, with corresponding 2-cocycle $\mu : \Gamma \times \Gamma \to \mathbb{S}^1$, and let $v \in \mathcal{H}$ be unit vector which is projectively $(S, \varepsilon \delta^2/56)$-invariant. Then there exists a mapping $\lambda : \Gamma \to \mathbb{S}^1$ with $\mu_{g, h} = \lambda_g\lambda_h\lambda_{gh}$ for all $g, h \in \Gamma$ and a vector $v_0 \in \mathcal{H}$ such that

$$\|v - v_0\| \leq \delta \quad \text{and} \quad \pi(g)v_0 = \lambda_gv_0$$
for all \( g \in \Gamma \). In particular, \( \mu \) is a coboundary.

We now recall Property (T) for von Neumann algebras as defined in [CoJo85].

Let \( \mathcal{N} \) be finite factor. A **Hilbert bimodule** over \( \mathcal{N} \) is a Hilbert space \( \mathcal{H} \) carrying two commuting normal representations, one of \( \mathcal{N} \) and one of the opposite algebra \( \mathcal{N}_0 \); we will write \( v \mapsto xvy \) for all \( v \in \mathcal{H}, \ x, y \in \mathcal{N} \).

The factor \( \mathcal{N} \) is said to have Property (T) if there exist a finite subset \( F \) of \( \mathcal{N} \) and \( \varepsilon' > 0 \) such that the following property holds: if a Hilbert bimodule \( \mathcal{H} \) for \( \mathcal{N} \) contains a unit vector \( v \) which is \((F, \varepsilon')\)-central, that is, which is such that

\[
\|xv - vx\| \leq \varepsilon' \quad \text{for all } x \in F,
\]

then \( \mathcal{H} \) has a non-zero central vector, that is, a non-zero vector \( w \in \mathcal{H} \) such that \( xw = wx \) for all \( x \in \mathcal{N} \). Moreover, one can choose \((F, \varepsilon')\)-central unit vector \( v \), there exists a central vector \( w \in \mathcal{H} \) with \( \|v - w\| \leq \delta \) (see Proposition 1 in [CoJo85]). We call such a pair \((F, \varepsilon')\) a **Kazhdan pair** for \( \mathcal{N} \).

It was shown in [CoJo85] (see Proposition 12.1.19 in [BrOz08]) that the subgroup \( \text{Inn}(\mathcal{N}) \) of inner automorphisms of \( \mathcal{N} \) (that is, the subgroup of automorphisms of the form \( \text{Ad}(u) \) for \( u \in \mathcal{U}(\mathcal{N}) \)) is open in \( \text{Aut}(\mathcal{N}) \). Here, \( \text{Aut}(\mathcal{N}) \) is endowed with the topology of pointwise \( L_2 \)-convergence (we could also take the induced topology from \( \mathcal{O}^+(L_p(\mathcal{N})) \) as above for any \( 1 \leq p < \infty \)). We will need a quantitative estimate, in terms of a Kazhdan pair \((F, \varepsilon')\), for the distance to 1 of the unitary operators defining the appropriate inner automorphisms.

**Proposition 2.7.** Let \( \mathcal{N} \) be a finite factor with Property (T). Let \((F, \varepsilon')\) be a Kazhdan pair for \( \mathcal{N} \). Let \( 0 < \delta < 1 \) and let \( \mathcal{V}_\delta \) be the neighbourhood of the trivial automorphism \( \text{id}_\mathcal{N} \) given by

\[
\mathcal{V}_\delta = \{ \theta \in \text{Aut}(\mathcal{N}) : \|\theta(x) - x\|_2 \leq \varepsilon'\delta/2 \text{ for all } x \in F \}.
\]

Then \( \mathcal{V}_\delta \) is contained in \( \text{Inn}(\mathcal{N}) \). More precisely, for every \( \theta \) in \( \mathcal{V}_\delta \), there exists \( u \) in \( \mathcal{U}(\mathcal{N}) \) with \( \theta = \text{Ad}(u) \) and \( \|u - 1\|_2 \leq \delta \).

**Proof** We follow the standard proof that \( \text{Inn}(\mathcal{N}) \) is open in \( \text{Aut}(\mathcal{N}) \) as given, for instance, in the proof of Proposition 12.1.19 in [BrOz08].

Let \( \theta \in \mathcal{V}_\delta \). We define a bimodule structure on \( L_2(\mathcal{N}) \) over \( \mathcal{N} \) by

\[
v \mapsto \theta(x) vy \quad \text{for all } v \in L_2(\mathcal{N}), \ x, y \in \mathcal{N}.
\]

Then \( 1 \in L_2(\mathcal{N}) \) is a unit vector which is \((F, \varepsilon'\delta/2)\)-central. Hence, there exists a central vector \( w \in L_2(\mathcal{N}) \) with \( \|w - 1\|_2 \leq \delta/2 \). Let
Let \( w = u|w| \) be the polar decomposition of \( w \), viewed as a densely defined operator on \( L_2(\mathcal{N}) \) affiliated to \( \mathcal{N} \). Then, \( |w| \) is in the center of \( \mathcal{N} \) and hence \( |w| = \lambda 1 \) for some \( \lambda > 0 \). It follows that \( u \) is a unitary element in \( \mathcal{N} \) such that \( \theta = \text{Ad}(u) \). As \( \|w\|_2 = \lambda \), we have \( |1-\lambda| \leq \|w-1\|_2 \leq \delta/2 \) and therefore

\[
\|u - 1\|_2 \leq \|u - w\|_2 + \|w - 1\|_2 = |1-\lambda| + \|w - 1\|_2 \leq \delta. \]

2.4. Projective 1-cocycles for actions of Kazhdan groups. In the sequel, we will need to deal with mappings \( \Gamma \to U(\mathcal{N}) \) which are 1-cocycles for an action of \( \Gamma \) on \( \mathcal{N} \) modulo scalars in the following sense (cocycles of this type appear in Section 1.3 of [Pop06], where they are called weak 1-cocycles).

**Definition 2.8.** Let \( \Gamma \) be a group, \( \mathcal{N} \) a von Neumann algebra, and \( \theta : \Gamma \to \text{Aut}(\mathcal{N}) \) a homomorphism. A *projective 1-cocycle* for \( \theta \) is a mapping \( u : \Gamma \to U(\mathcal{N}) \) such that, for every \( g, h \in \Gamma \), there exists a scalar \( \mu_{g,h} \in \mathbb{S}^1 \) with

\[
u_g \theta_g(u_h) = \mu_{g,h} u_{gh}.
\]

Two projective 1-cocycles \( u \) and \( v \) are cohomologous if there exist \( w \) in \( U(\mathcal{N}) \) and a mapping \( \lambda : \Gamma \to \mathbb{S}^1 \) such that

\[
u_g = \lambda_g w u_g \theta_g(w^*) \quad \text{for all } g \in \Gamma.
\]

A *projective coboundary* is a projective 1-cocycle which is cohomologous to the trivial cocycle \( g \mapsto 1 \).

The following lemma, which can be checked by a straightforward computation, shows that projective cocycles appear naturally.

**Lemma 2.9.** Let \( \Gamma \) be a group, \( \mathcal{N} \) a factor, and \( \theta : \Gamma \to \text{Aut}(\mathcal{N}) \) a homomorphism. For a mapping \( u : \Gamma \to U(\mathcal{N}) \), the following properties are equivalent.

(i) \( u \) is a projective 1-cocycle for \( \theta \);
(ii) the mapping \( g \mapsto \text{Ad}(u_g)\theta_g \) is a homomorphism from \( \Gamma \) to \( \text{Aut}(\mathcal{N}) \).

We denote by \( Z^1_{\text{proj}}(\Gamma, \theta) \) and by \( B^1_{\text{proj}}(\Gamma, \theta) \) the set of projective 1-cocycles and coboundaries for \( \theta \). We equip \( Z^1_{\text{proj}}(\Gamma, \theta) \) with the topology of pointwise \( L_2 \)-convergence: a sequence \( (u^{(n)})_n \) in \( Z^1_{\text{proj}}(\Gamma, \theta) \) converges to \( u \in Z^1_{\text{proj}}(\Gamma, \theta) \) if \( \lim_n \|u^{(n)}_g - u_g\|_2 = 0 \) for every \( g \in \Gamma \).

Assume now that \( \Gamma \) has Property (T). We will need to know that cohomology classes in \( Z^1_{\text{proj}}(\Gamma, \theta) \) are open. This is not true in general even for classes in \( Z^1(\Gamma, \theta) \) and even when \( \theta \) is ergodic (see Examples 4 and 8 in [Kawa91]). However, the following result was shown in
Moreover, as in the proof of Proposition 2.7, we have \( \| K_{\text{Wau91}, \text{Theorem 7}} \). Let \( u \in Z^1(\Gamma, \theta) \) be such that the action of \( \Gamma \) on \( \mathcal{N} \) given by \( g \mapsto \text{Ad}(u_g)\theta_g \) is ergodic; then the equivalence class of \( u \) is open in \( Z^1(\Gamma, \theta) \). Following the same proof and making crucial use of Theorem 2.6, we now show that a quantitative version of this result is true for projective 1-cocycles.

**Proposition 2.10.** Let \( \Gamma \) be a group with Kazhdan’s Property (T), with a Kazhdan pair \( (S, \varepsilon) \). Let \( \mathcal{N} \) be a finite factor and \( \theta : \Gamma \to \text{Aut}(\mathcal{N}) \) a homomorphism. Let \( u : \Gamma \to U(\mathcal{N}) \) be a projective 1-cocycle for \( \theta \). Assume that the action of \( \Gamma \) on \( \mathcal{N} \) given by \( g \mapsto \text{Ad}(u_g)\theta_g \) is ergodic. Fix \( 0 < \delta < 1 \) and let \( \mathcal{U}_\delta \) be the neighbourhood of \( u \) in \( Z^1_{\text{proj}}(\Gamma, \theta) \) defined by

\[
\mathcal{U}_\delta = \{ v \in Z^1_{\text{proj}}(\Gamma, \theta) : \| v_s - u_s \|_2 \leq \varepsilon \delta^2/112 \text{ for all } s \in S \}. 
\]

Then every \( v \in \mathcal{U}_\delta \) is cohomologous to \( u \). More precisely, for every \( v \in \mathcal{U}_\delta \), there exists \( w \in U(\mathcal{N}) \) with \( \| w - 1 \|_2 \leq \delta \) and a mapping \( \lambda : \Gamma \to S^1 \) such that \( v_g = \lambda_g w u_g \theta_g (w^*) \) for all \( g \in \Gamma \).

**Proof.** We adapt the proof from [Kawa91] Theorem 7, making it quantitative at the appropriate places. Let \( v \in \mathcal{U}_\delta \). For every \( g \in \Gamma \), let \( \pi(g) \) be the unitary operator on \( L_2(\mathcal{N}) \) given by

\[
\pi(g)x = u_g \theta_g (x) v_g^* \quad \text{for all } x \in \mathcal{N}.
\]

Since \( u \) and \( v \) are projective 1-cocycles for \( \theta \), the mapping \( \pi : g \mapsto \pi(g) \) is a projective unitary representation of \( \Gamma \), as is easily checked. Let \( w : \Gamma \times \Gamma \to S^1 \) be the corresponding 2-cocycle. Observe that \( 1 \in L_2(\mathcal{N}) \) is a unit vector which is \((S, \varepsilon \delta^2/112)-\)invariant. Hence, it follows from Theorem 2.6 that there exists a mapping \( \lambda : \Gamma \to S^1 \) with

\[
\mu_{g,h} = \lambda_g \lambda_h \bar{\lambda}_{gh} \quad \text{for all } g, h \in \Gamma
\]

and a vector \( b \in L_2(\mathcal{N}) \) such that \( \| b - 1 \| \leq \delta/2 \) and

\[
\pi(g)b = \lambda_g b \quad \text{for all } g \in \Gamma.
\]

Thus, \( b \neq 0 \) and \( u_g \theta_g (b)v_g^* = \lambda_g b \) for every \( g \in \Gamma \). We view \( b \) as a densely defined operator on \( L_2(\mathcal{N}) \) affiliated to \( \mathcal{N} \). Taking adjoints, we see that the positive operator \( bb^* \in L^1(\mathcal{N}) \) is fixed by the extension to \( L^1(\mathcal{N}) \) of \( \text{Ad}(u_g)\theta_g \) for every \( g \in \Gamma \). Since \( g \mapsto \text{Ad}(u_g)\theta_g \) is ergodic, it follows that \( bb^* = \beta 1 \) for some \( \beta > 0 \). Then \( w := b^*/\sqrt{\beta} \) is a unitary element in \( \mathcal{N} \) such that

\[
v_g = \bar{\lambda}_g w u_g \theta_g (w^*) \quad \text{for all } g \in \Gamma.
\]

Moreover, as in the proof of Proposition 2.7, we have \( \| w - 1 \|_2 \leq \delta \). \( \blacksquare \)
One can improve upon the constant defining $V_\delta$ in the previous proposition, when one deals with genuine 1-cocycles instead of projective ones; indeed, in this case, the projective unitary representation appearing in the proof is a true unitary representation and one checks that the following statement holds.

**Proposition 2.11.** Let $\Gamma$, $(S, \varepsilon)$, $\mathcal{N}$ and $\theta$ be as in Proposition 2.10.

Let $u : \Gamma \to \mathbf{U}(\mathcal{N})$ be a 1-cocycle for $\theta$ such that $g \mapsto \text{Ad}(u_g)\theta_g$ is ergodic. For $0 < \delta < 1$, set

$$U_\delta = \{ v \in Z^1(\Gamma, \theta) : \|v_s - u_s\|_2 \leq \varepsilon \delta/2 \text{ for all } s \in S \}.$$ 

Then, for every $v \in U_\delta$, there exists $w \in \mathbf{U}(\mathcal{N})$ with $\|w - 1\|_2 \leq \delta$ such that

$$v_g = w u_g \theta_g(w^*) \quad \text{for all } g \in \Gamma. \quad \blacksquare$$

### 3. Proof of Theorem 1.1

Let $\pi : \Gamma \to \mathbf{O}^+(L_p(\mathcal{N}))$ be a group homomorphism for $p \neq 2$. Then $\Phi_{p,2} \circ \pi$ is a group homomorphism from $\Gamma$ to the $\mathcal{N}$-unitary group $\mathbf{O}_\mathcal{N}^+(L_2(\mathcal{N}))$ as defined in Section 2.1, where $\Phi_{p,2} : \mathbf{O}^+(L_p(\mathcal{N})) \to \mathbf{O}_\mathcal{N}^+(L_2(\mathcal{N}))$ is the identity mapping. By Proposition 2.3, $\mathbf{O}^+(L_p(\mathcal{N}))$ and $\mathbf{O}_\mathcal{N}^+(L_2(\mathcal{N}))$ are topologically isomorphic groups. Hence, to prove that $\pi : \Gamma \to \mathbf{O}^+(L_p(\mathcal{N}))$ is locally rigid amounts to prove that $\Phi_{p,2} \circ \pi : \Gamma \to \mathbf{O}_\mathcal{N}^+(L_2(\mathcal{N}))$ is locally rigid. So, we can replace $\pi$ by $\Phi_{p,2} \circ \pi$.

Recall (see Section 2.2) that every $\rho \in \text{Hom}(\Gamma, \mathbf{O}_\mathcal{N}^+(L_2(\mathcal{N})))$ has a Yeadon decomposition $\rho = (v, \alpha)$ with $\alpha \in \text{Hom}(\Gamma, \text{Aut}(\mathcal{N}))$ and $v \in Z^1(\Gamma, \alpha)$. Moreover, by Corollary 2.2 and Proposition 2.3, $\mathbf{O}_\mathcal{N}^+(L_2(\mathcal{N}))$ is topologically isomorphic to the topological semi-direct product $\mathbf{U}(\mathcal{N}) \rtimes \text{Aut}(\mathcal{N})$.

Let $\pi = (u, \theta)$ be the Yeadon decomposition of the homomorphism $\pi : \Gamma \to \mathbf{O}_\mathcal{N}^+(L_2(\mathcal{N}))$.

Recall that the associated homomorphisms $\pi^l, \pi^r \in \text{Hom}(\Gamma, \text{Aut}(\mathcal{N}))$ are given by

$$\pi^l(g) = \theta_g \quad \text{and} \quad \pi^r(g) = \text{Ad}(u_g)\theta_g$$

for every $g \in \Gamma$.

Assume now that $\Gamma$ and $\mathcal{N}$ have both Property (T) and that $\pi^l$ and $\pi^r$ are ergodic. Let $(S, \varepsilon)$ be a Kazhdan pair for $\Gamma$ and $(F, \varepsilon')$ a Kazhdan pair for $\mathcal{N}$. We can assume that $S$ is a generating set for $\Gamma$, since $\Gamma$ is finitely generated (see Theorem 1.3.1 in [BeHV08]).
Fix $0 < \delta < 1$ and define $\mathcal{V}_\delta$ to be the neighbourhood of $\pi$ in $\text{Hom}(\Gamma, \mathcal{O}_N^+(L_2(N)))$ consisting of all $\rho = (v, \alpha) \in \text{Hom}(\Gamma, \mathcal{O}_N^+(L_2(N)))$ such that

1. $\|v_s - u_s\|_2 \leq \delta \varepsilon / 4$ for all $s \in S$ and
2. $\|\alpha_s(x) - \theta_s(x)\|_2 \leq \frac{\delta^2 \varepsilon^3 \varepsilon'}{2^7 \cdot 112}$ for all $s \in S$ and all $x \in F$.

Let $\rho \in \mathcal{V}_\delta$ with Yeadon decomposition $\rho = (v, \alpha)$.

**Claim.** There exist $a, b \in U(N)$ with $\|a - 1\|_2 \leq \delta$ and $\|b - 1\|_2 \leq \delta$

such that

$$\alpha_g = \text{Ad}(a)\theta_g\text{Ad}(a^*)$$ and $v_g = b u_g(\text{Ad}(a)\theta_g\text{Ad}(a^*)) (b^*)$

for all $g \in \Gamma$. In particular, once proved, this claim will show that $\rho$ is in the $\Gamma$-orbit of $\pi$ (see Proposition 2.5). The proof will be carried out in four steps.

- **First step:** We claim that there exists a projective 1-cocycle $w$ in $Z^1_{\text{proj}}(\Gamma, \theta)$ with the following properties:

  $$\alpha_g = \text{Ad}(w_g)\theta_g$$ for all $g \in \Gamma$ and

  $$\|w_s - 1\|_2 \leq \frac{\delta^2 \varepsilon^3}{2^6 \cdot 112}$$ for all $s \in S$.

Indeed, by (2) above, for every $s \in S$ and $x \in F$, we have

$$\|\theta_s^{-1}(\alpha_s(x)) - x\|_2 = \|\alpha_s(x) - \theta_s(x)\|_2 \leq \frac{\varepsilon' \delta^2 \varepsilon^3}{2^7 \cdot 112}.$$ 

Hence, it follows from Proposition 2.7 that, for every $s \in S$, there exists $w_s$ in $U(N)$ with

$$\|w_s - 1\|_2 \leq \frac{\delta^2 \varepsilon^2}{2^6 \cdot 112}$$

and such that $\alpha_s = \text{Ad}(w_s)\theta_s$.

Now, $\alpha$ and $\theta$ are group homomorphisms from $\Gamma$ to $\text{Aut}(N)$ and $\text{Inn}(N)$ is a normal subgroup in $\text{Aut}(N)$. Moreover, we have just shown that the homomorphisms $p \circ \alpha$ and $p \circ \theta$ from $\Gamma$ to the quotient group $\text{Aut}(N)/\text{Inn}(N)$ agree on the generating set $S$, where

$$p : \text{Aut}(N) \to \text{Aut}(N)/\text{Inn}(N)$$

is the canonical projection. It follows that $p \circ \alpha = p \circ \theta$ on $\Gamma$. Hence, we can extend $S \mapsto U(N), s \mapsto w_s$ to a mapping $w : \Gamma \mapsto U(N)$ such that

$$\alpha_g = \text{Ad}(w_g)\theta_g$$ for all $g \in \Gamma$. 

By Lemma 2.9, \(w\) is a projective 1-cocycle for \(\theta\).

• **Second step:** We claim that there exist \(a \in U(N)\) and a mapping \(\lambda: \Gamma \to S^1\) such that such that
  \[ w_g = \lambda_g a \theta_g(a^*) \quad \text{for all } g \in \Gamma \]
  (that is, \(w\) is in \(B_{\text{proj}}^1(\Gamma, \theta)\)) and such that
  \[ \|a - 1\|_2 \leq \frac{\delta \varepsilon}{2^5}. \]
  Indeed, the action of \(\Gamma\) given by \(\pi^l = \theta\) is ergodic and
  \[ \|w_s - 1\|_2 \leq \frac{\delta^2 \varepsilon^2}{2^6 \cdot 112} = \left(\frac{\delta \varepsilon}{2^5}\right)^2 \frac{\varepsilon}{112} \quad \text{for all } s \in S. \]
  for every \(s \in S\). Hence, the claim follows from Proposition 2.10 applied to the trivial cocycle \(u: g \mapsto 1\).

• **Third step:** Let \(a \in U(N)\) be as in the second step. We claim that
  \[ \alpha_g = \theta^{\text{Ad}(a)} \quad \text{that is,} \quad \alpha_g = \text{Ad}(w_g) \theta_g \]
  Indeed, this follows from the fact that \(\alpha_g = \text{Ad}(w_g) \theta_g\) and \(w_g = \lambda_g a \theta_g(a^*)\) for every \(g \in \Gamma\).

  Let \(u' = \text{Ad}(a)u\) be the cocycle in \(Z^1(\Gamma, \alpha) = Z^1(\Gamma, \theta^{\text{Ad}(a)})\) defined by
  \[ u'_g = au_g a^* \quad \text{for all } g \in \Gamma. \]

• **Fourth step:** We claim that there exists \(b \in U(N)\) with \(\|b - 1\|_2 \leq \delta\)
  such
  \[ v_g = bu'_g \theta^{\text{Ad}(a)}(b^*) \quad \text{for all } g \in \Gamma. \]
  Indeed, by (1) above and the choice of \(a\), we have
  \[ \|v_s - u'_s\|_2 = \|v_s - au_s a^*\|_2 \leq \|v_s - u_s\|_2 + \|au_s - u_s a\|_2 \]
  \[ \leq \|v_s - u_s\|_2 + \|(a - 1)u_s\|_2 + \|u_s(a - 1)\|_2 \]
  \[ \leq \|v_s - u_s\|_2 + 2\|a - 1\|_2 \]
  \[ \leq \frac{\delta \varepsilon}{4} + 2 \frac{\delta \varepsilon}{8} = \frac{\delta \varepsilon}{2}, \]
  for every \(s \in S\). Moreover, the action of \(\Gamma\) given by \(\pi^r(g) = \text{Ad}(u_g) \theta_g\)
  for \(g \in \Gamma\) is ergodic. Hence, the action given by
  \[ g \mapsto \text{Ad}(u'_g) \theta_g = \text{Ad}(a) \pi^r(g) \text{Ad}(a^*) \]
  is also ergodic. The claim follows now from Proposition 2.11.

This concludes the proof of the theorem. ■
Remark 3.1. For a fixed $0 < \delta < 1$, the set $\mathcal{V}_\delta$ given above is a neighbourhood of $\pi$ in $O^+_{\mathcal{N}}(L_2(\mathcal{N}))$ such that every $\rho \in \mathcal{V}_\delta$ is conjugate to $\pi$ by some $U = (b, \text{Ad}(a))$ in $O^+_{\mathcal{N}}(L_2(\mathcal{N}))$ for which $a$ and $b$ are $\delta$-close to the identity in the $L_2$-norm.

Using estimates for the Mazur map $M_{p,q}$, we can determine a neighbourhood $\mathcal{V}_{\delta,p}$ of $\pi$ in $O^+(L_p(\mathcal{N}))$ for $p \neq 2$, with the same properties. Indeed, by [Rica14], there exist constants $C = C_p$ (of order $p$) and $D$ (independent of $p$) such that

$$
\|M_{p,2}(x) - M_{p,2}(y)\|_2 \leq C\|x - y\|_p,
$$
$$
\|M_{2,y}(x') - M_{2,y}(y')\|_p \leq D\|x' - y'\|_2^\beta,
$$

where $\alpha = \min\{p/2, 1\}$ and $\beta = \min\{2/p, 1\}$, for $x, y$ in the unit ball in $L_p(\mathcal{N})$ and $x', y'$ in the unit ball in $L_2(\mathcal{N})$.

We can clearly assume that, for every $x$ in the Kazhdan set $F$ for $\mathcal{N}$, we have $\|x\|_2 = 1$ and hence $\|M_{2,y}(\theta(x))\|_p = 1$ for every $\theta \in \text{Aut}(\mathcal{N})$, since $\|M_{2,y}(\theta(x))\|_p = \|\theta(M_{2,y}(x))\|_p = \|x\|_2^{2/p}$.

Set $\delta_p = (\frac{\delta}{\pi})^{1/\beta}$ and

$$
\varepsilon_p = \left(\frac{\delta_p\varepsilon}{4C}\right)^{1/\alpha}, \quad \varepsilon'_p = \left(\frac{\delta_p^2\varepsilon^3\varepsilon'}{C \cdot 2^7 \cdot 112}\right)^{1/\alpha}.
$$

Let $\mathcal{V}_{\delta_p}$ to be the neighbourhood of $\pi = (u, \theta)$ in $\text{Hom}(\Gamma, O^+(L_p(\mathcal{N})))$ consisting of all $\rho = (v, \alpha)$ in $\text{Hom}(\Gamma, O^+(L_p(\mathcal{N})))$ such that

$$
\|v_s - u_s\| \leq \varepsilon_p \quad \text{for all } s \in S \text{ and }
$$
$$
\|\alpha_s(M_{2,y}(x)) - \theta_s(M_{2,y}(x))\|_p \leq \varepsilon'_p \quad \text{for all } s \in S \text{ and all } x \in F.
$$

Then every $\rho \in \mathcal{V}_{\delta_p}$ is conjugate to $\pi$ by some $U = (b, \text{Ad}(a))$ in $O^+(L_p(\mathcal{N}))$ for which $a$ and $b$ are $\delta$-close to the identity in the $L_p$-norm.

4. On the assumptions in the statement of Theorem 1.1

We present counterexamples in relation with the various assumptions made in Theorem 1.1.

Example 4.1. If the finite factor $\mathcal{N}$ does not have Property (T), the conclusion of Theorem 1.1 may not be true. Indeed, let $\mathcal{R}$ be the hyperfinite type $II_1$-factor. M. Choda constructed in [Chod85] a continuous family $(\theta_t)_{t \in [0,1]}$ of actions of the group $\Gamma = SL_n(\mathbb{Z})$ for $n \geq 2$ (recall that $SL_n(\mathbb{Z})$ has Property (T) for $n \geq 3$) by automorphisms on $\mathcal{R}$, which are ergodic and mutually non conjugate in $\text{Aut}(\mathcal{R})$ for irrational $t$. It follows from Proposition 2.5 that, for any $1 \leq p < \infty$, $p \neq 2$ and
any irrational \( t \), the homomorphisms \( \pi_t : \Gamma \to \text{O}^+(L_p(\mathcal{R})) \) defined by these actions are mutually non conjugate in \( \text{O}^+(L_p(\mathcal{R})) \) and hence are not locally rigid. In fact, a more general result in [Pop06, Corollary 0.2] implies that any Kazhdan group admits a continuous family of actions by automorphisms on \( \mathcal{R} \) which are mutually non conjugate.

**Example 4.2.** The conclusion of Theorem [11] might also fail, if any one of the associated actions \( \pi^t \) and \( \pi^r \) by automorphisms of \( \mathcal{N} \) is not ergodic. A counter-example may be obtained by a slight modification of Example 8 in [Kawa91] as follows.

Let \( H \) be an ICC group with Kazhdan’s property (for instance \( H = SL_3(\mathbb{Z}) \)). Set \( \Gamma = H \times H \) and \( \mathcal{N} = \mathcal{N}(\Gamma) \). Then \( \Gamma \) is an ICC group with Kazhdan’s property and \( \mathcal{N} \) can be identified with the tensor product \( \mathcal{N}(H) \otimes \mathcal{N}(H) \) of von Neumann algebras, with trace \( \tau = \tau_H \otimes \tau_H \), where \( \tau_H \) is the canonical trace on \( \mathcal{N}(H) \). For \( 1 \leq p < \infty, p \neq 2 \), let \( \pi_0 \) denote the embedding \( \Gamma \to \text{O}^+(L_p(\mathcal{N})) \) given by \( g \mapsto \lambda(g) \); observe that the associated action \( \pi^t_0 \) is the trivial action, while the action \( \pi^r_0 \), which is given by \( g \mapsto \text{Ad}(\lambda(g)) \), is ergodic.

Since \( \mathcal{N}(H) \) is a factor of type \( II_1 \), for every \( t \in [0, 1] \), there exists a projection \( p_t \) in \( \mathcal{N}(H) \) with \( \tau_H(p_t) = t \). For \( g = (h_1, h_2) \in \Gamma \) and \( t \in [0, 1] \), let \( u^{(t)}_g \in \mathcal{N}(\Gamma) \) be defined by

\[
u^{(t)}_g = \lambda(h_1) \otimes p_t + 1 \otimes (1 - p_t).
\]

Then \( u^{(t)}_g \) is unitary and

\[
u^{(t)} : \Gamma \to \text{U}(\mathcal{N}), g \mapsto u^{(t)}_g
\]

is a group homomorphism. For \( g \in \Gamma \), let \( \pi_t(g) \) denote the isometry of \( L_p(\mathcal{N}) \) with Yeadon decomposition \( (\lambda(u^{(t)}_g), \text{Ad}(u^{(t)}_g)) \), that is,

\[
\pi_t(g)x = \lambda(g)u^{(t)}_{g^{-1}}\text{Ad}(u^{(t)}_g)(x) = \lambda(g)xu^{(t)}_{g^{-1}} \quad \text{for all} \quad x \in L_p(\mathcal{N}).
\]

Then \( \pi_t : \Gamma \to \text{O}^+(L_p(\mathcal{N})) \) is a group homomorphism.

We claim that \( \pi_0 \) is not locally rigid. For this, it suffices to show (see Proposition [23]) that \( \pi_0 \) is not locally rigid when viewed as automorphism with values in the \( \mathcal{N} \)-unitary group \( \text{O}^+_{\mathcal{N}}(L_2(\mathcal{N})) \).

For \( g = (h_1, h_2) \in \Gamma \) and \( t \in [0, 1] \), we have

\[
\|u^{(t)}_g - 1\|_2^2 = 2(1 - \tau(u^{(t)}_g)) = 2t(1 - \tau_H(\lambda(h_1))) \leq 2t
\]

and hence \( \lim_{t \to 0} \|u^{(t)}_g - 1\|_2 = 0 \). It follows that \( \lim_{t \to 0} \pi_t(g) = \pi_0(g) \) in \( \text{O}^+_{\mathcal{N}}(L_2(\mathcal{N})) \) for every \( g \in \Gamma \).
Assume, by contradiction, that \( \pi_0 \) is locally rigid. Then, by Proposition 2.5, for \( t > 0 \) sufficiently small, \( \text{Ad}(u_g) \) is conjugate to the trivial automorphism \( x \mapsto x \) in \( \text{Aut}(\mathcal{N}) \) and hence \( \text{Ad}(u_g) \) is the trivial automorphism for every \( g \in \Gamma \). This is a contradiction, as \( u_g \) is not a scalar multiple of the identity for \( g = (h_1, h_2) \) with \( h_1 \neq e \).

Similarly, one can show that the embedding \( \rho_0 : \Gamma \to \text{O}^+(L_p(\mathcal{N})) \), given by

\[
\rho_0(g) : x \mapsto x \lambda(g) = \lambda(g) \text{Ad}(\lambda(g^{-1}))(x),
\]

is not locally rigid; here, it is the associated action \( \rho_0^* \) which is ergodic, while \( \rho_0^* \) is the trivial action.

**Example 4.3.** The conclusion of Theorem 1.1 does not hold in general for actions by isometries on the classical (commutative) \( L_p \)-spaces. We give a counter-example for actions on the sequence space \( \ell_p \) (more involved counter-examples can be found for actions on the space \( L_p(0,1) \)).

Let \( \Gamma \) be an arbitrary group which is not a torsion group; thus \( \Gamma \) contains a subgroup \( A \) isomorphic to \( \mathbb{Z} \).

There exists a family \( (\chi_t)_{t \in [0,1]} \) of unitary characters \( \chi_t \) of \( A \) with \( \chi_t \neq 1 \) for \( t \neq 0 \), \( \chi_0 = 1 \) and such that

\[
\lim_{t \to 0} \chi_t(a) = 1 \quad \text{for all} \quad a \in A.
\]

Choose a set of representatives \( X \) for the left cosets of \( \Gamma \) modulo \( A \) with \( e \in X \); so, \( \Gamma = XA \). Let \( c : \Gamma \times X \to A \) be the cocycle defined by

\[
gx \in X c(g, x) \quad \text{for all} \quad g \in \Gamma, \ x \in X.
\]

We transfer the natural \( \Gamma \)-action on \( \Gamma/A \) to an action \( (g, x) \mapsto g(x) \) of \( \Gamma \) on \( X \cong \Gamma/A \), by setting

\[
g(x) = gx c(g, x)^{-1} \quad \text{for all} \quad g \in \Gamma, \ x \in X.
\]

For every \( t \in \mathbb{R} \) and every \( p \in [1, +\infty[ \), the operator \( \pi_t(g) \) on \( \ell_p(X) \), defined by

\[
\pi_t(g)f(x) = \chi_t(c(g^{-1}, x))f(g^{-1}(x)) \quad \text{for all} \quad f \in \ell_p(X), \ x \in X,
\]

is an isometry and \( \pi_t : \Gamma \to \text{O}(\ell_p(X)) \) is a homomorphism. (For \( p = 2 \), \( \pi_t \) is the unitary representation of \( \Gamma \) induced by the character \( \chi_t \) of \( A \). Observe also that \( \pi_0 \) is the quasi-regular representation of \( \Gamma \) in \( \ell_p(X) \cong \ell_p(\Gamma/A) \).)

We have \( \lim_{t \to 0} \pi_t(g) = \pi_0(g) \) in \( \text{O}(\ell_p(X)) \), for every \( g \in \Gamma \). Indeed, by linearity and density, it suffices to check that

\[
\lim_{t \to 0} \| \pi_t(g)\delta_x - \pi_0(g)\delta_x \|_p = 0,
\]

where \( \delta_x \) is the Dirac measure at \( x \).
where $\delta_x$ is the Dirac function at $x \in X$. This is the case, since
\[
\|\pi_t(g)\delta_x - \pi_0(g)\delta_x\|_p = |\chi_t(c(g^{-1}, g(x)) - 1|.
\]

Let $t \neq 0$ and $p \neq 2$. We claim that $\pi_t$ does not belong to the $O(\ell_p(X))$-orbit of $\pi_0$. Indeed, assume, by contradiction, that there exists $U = U_t$ in $O(\ell_p(X))$ such that
\[
\pi_t(g) = U\pi_0(g)U^{-1}
\]
for all $g \in \Gamma$.

By Banach characterization of the isometries of $\ell_p(X)$ from \cite[Bana32, Chap. XI]{}, there exists a function $\alpha : X \to S^1$ and a bijective mapping $\phi : X \to X$ such that
\[
(Uf)(x) = \alpha(x)f(\phi(x))
\]
for all $f \in \ell_p(X), x \in X$.

One computes that $U\pi_t(g^{-1})U^{-1} = \pi_0(g^{-1})$ amounts to the equation
\[
\alpha(x)\alpha(\varphi^{-1}g\varphi(x))^{-1}\chi_t(c(g, \varphi(x)))f(\varphi^{-1}g\varphi(x)) = f(g(x)),
\]
for all $f \in \ell_p(X)$ and $x \in X$. It follows from this that $\varphi^{-1}g\varphi = g$ on $X$ and, consequently,
\[
\alpha(x)\alpha(g(x))^{-1}\chi_t(c(g, \varphi(x))) = 1
\]
for all $g \in \Gamma, x \in X$. (\ast)

Let $x = \varphi^{-1}(e)$ and $g \in A$. Then, $c(g, e) = g$ and $g(e) = e$. Hence, we have
\[
g(x) = g(\varphi^{-1}(e)) = \varphi^{-1}(g(e)) = \varphi^{-1}(e) = x
\]
If follows from (\ast) that $\chi_t(g) = 1$ for all $g \in A$ and this is a contradiction.

REFERENCES

\cite{Bana32} S. Banach. Théorie des opérations linéaires, Hafner Publishing Company, 1932.
\cite{BFGM07} U. Bader, A. Furman, T. Gelander, and N. Monod. Property (T) and rigidity for actions on Banach spaces. Acta Math. 198, 57–105, 2007.
\cite{BeHV08} B. Bekka, P. de la Harpe and A. Valette. Kazhdan’s Property (T), Cambridge Univ. Press, 2008.
\cite{BrRo79} O. Bratteli and D. Robinson. Operator Algebras and Quantum Statistical Mechanics I, Springer-Verlag 1979.
\cite{BrOz08} N. Brown and N. Ozawa. C*-algebras and finite-dimensional approximations, American Mathematical Society 2008.
\cite{Chod85} M. Choda. A continuum of non-conjugate property T actions of $SL(n, \mathbb{Z})$ on the hyperfinite $II_1$-factor. Math. Japon. 30, 133–150, 1985.
\cite{CoJo85} A. Connes and V. Jones. Property (T) for von Neumann algebras. Bull. London Math. Soc. 17, 51-62 (1985).
\cite{Fish07} D. Fisher. Local rigidity of group actions: past, present, future. In: Dynamics, Ergodic Theory and Geometry, p.45–98, Cambridge Univ. Press, 2007.
D. Fisher and G. Margulis. Almost isometric actions, property (T), and local rigidity. *Invent. Math.* **162**, 19–80, 2005.

M. Junge, Z.J. Ruan, and D. Sherman. A classification for 2-isometries of noncommutative \( L_p \)-spaces. *Israel J. Math.* **150**, 285–314, 2005.

Y. Kawahigashi. Cohomology of actions of discrete groups on factors of type \( II_1 \). *Pac. J. Math.* **149**, 303–317, 1991.

R. Nicoara, S. Popa and R. Sasyk. On \( II_1 \)-factors arising from 2-cocycles of \( w \)-rigid groups. *J. Funct. Anal.* **242**, 230–246, 2007.

B. Olivier. Kazhdan’s property (\( T \)) with respect to non-commutative \( L_p \)-spaces. *Proc. Amer. Math. Soc.* **140**, 4259–4269, 2012.

G. Pisier and Q. Xu. Non-commutative \( L_p \)-spaces. Handbook of the geometry of Banach spaces, Vol. 2, pp. 1459–1517, North-Holland 2003.

S. Popa. Some rigidity results for non-commutative Bernoulli shifts. *J. Funct. Anal.* **230**, 273–328, 2006.

M.S. Raghunathan. *Discrete subgroups of Lie groups*, Springer-Verlag, 1972.

A. Rapinchuk. On the finite dimensional unitary representations of Kazhdan groups. *Proc. Amer. Math. Soc.* **127**, 1557–1562, 1999.

D. Sherman. Noncommutative \( L_p \) structure encodes exactly Jordan structure. *J. Funct. Anal.* **221**, 150–166, 2005.

Y. Raynaud. On ultrapowers of non commutative \( L_p \) spaces. *J. Operator Theory* **48**, 41–68, 2002.

E. Ricard. Holder estimates for the non commutative Mazur maps. Preprint 2014, [arXiv:1407.8343v3](http://arxiv.org/abs/1407.8343v3).

R.J. Spatzier. An invitation to rigidity theory. In: Modern dynamical systems and applications, p.211–231, Cambridge Univ. Press, 2004.

F. J. Yeadon. Isometries of noncommutative \( L^p \)-spaces. *Math. Proc. Cambridge Philos. Soc.* **90**, 41–50, 1981.

**Bachir Bekka**, IRMAR, UMR-CNRS 6625 Université de Rennes 1, Campus Beaulieu, F-35042 Rennes Cedex, France

*E-mail address:* bachir.bekka@univ-rennes1.fr