New results for the Hankel two-wavelet multipliers

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ABSTRACT
The aim of this paper is to improve and generalize some results stated in Ghobber (A variant of the Hankel multiplier, Banach J Math Anal. 2018;12(1):144–166). We also give new results about the boundedness and compactness of Hankel two-wavelet multipliers on the space $L^p_+(\mathbb{R})$ for $1 \leq p \leq \infty$.

1. Introduction
It is well known that when we consider radial functions, the Fourier transform becomes the Hankel transform of order $d/2 − 1$. Indeed, taking the Hankel transform of order $\alpha$, with $\alpha \geq −1/2$, as

$$H_\alpha (g)(\lambda) = \int_0^\infty g(r) j_\alpha (\lambda r) \, dr,$$

where $j_\alpha$ is the modified Bessel function, it is verified that $\mathcal{F}(f)(\xi) = H_{d/2 − 1} \psi(\|\xi\|)$, where $\mathcal{F}(f)$ is the classical Fourier transform of $f$ on $\mathbb{R}^d$.

Very recently, many authors have been interested to the behaviour of the Hankel transform with respect to several problems already studied for the Fourier transform, like generalized potentials [1,2], multipliers [3,4], uncertainty inequalities [5–7], wavelets [8], time-frequency concentration and localization [3,9], variation diminishing [10], and so on.

One of the aims of the Fourier transform is the study of the theory of wavelet multipliers. This theory was initiated by He and Wong [11], developed in the paper [12] by Du and Wong, and detailed in the book [13] by Wong.

Recently, Ghobber [3], with the aid of the harmonic analysis associated with the Bessel operator, has defined and studied the Hankel wavelet multipliers. In the same paper [3], Ghobber has given a trace formula for the Hankel wavelet multiplier as a bounded linear operator in the trace class from $L^2(\mathbb{R}_+, \, dv_\alpha)$ into $L^2(\mathbb{R}_+, \, dv_\alpha)$ in terms of the symbol and the two admissible wavelets.

The aim of this paper is to improve, generalize the results given in [3], and to give other new results on the $L^p$ boundedness and compactness of Hankel two-wavelet multipliers.

The remainder of this paper will be arranged as follows.

In Section 2, we state some basic notions and results from harmonic analysis associated with the Bessel operator that will be needed throughout this paper.

The aim of Section 3 is to survey and revisit some results for the Hankel two-wavelet multipliers. More precisely, we give an explicit formula for the function that occurs in the lower bound for the trace norm as that of class Hankel two-wavelet multipliers. A result concerning the trace of products of Hankel two-wavelet multipliers is proved. In this section, we also introduce the generalized Landau–Pollak–Slepian operator which allows us to give an example of a Hankel two-wavelet multiplier. Our result generalize and improve Ghober’s result concerning the Landau–Pollak–Slepian operator from the Hankel one-wavelet multiplier.

Section 4 is devoted to study the $L^p$ boundedness and compactness of these two-wavelet multipliers when suitable conditions on the symbols and the two admissible wavelets are satisfied.

2. Preliminaries
This section deals with some basic notions and results in harmonic analysis associated with the Bessel operator that will be needed in the sequel. For more details, we refer the interested reader to [14] for instance.
Throughout this paper, we fix $\alpha \geq -\frac{1}{2}$. We denote by:

- $\mathbb{R}_+ = [0, \infty)$.
- $S_\alpha(\mathbb{R})$ the Schwartz space of even rapidly decreasing functions on $\mathbb{R}$.
- $D_\alpha(\mathbb{R})$ the space of even $C^\infty$-functions on $\mathbb{R}$ which are of compact support.
- $C^p_\alpha(\mathbb{R})$ the space of even functions of class $C^p$ on $\mathbb{R}$.
- For $p \in [1, \infty]$, $p'$ denotes as in all that follows, the conjugate exponent of $p$.
- $L^p_\alpha(\mathbb{R}_+)$, $1 \leq p \leq \infty$, the space of measurable functions $f$ on $\mathbb{R}_+$ such that

$$
\|f\|_{L^p_\alpha(\mathbb{R}_+)} := \left( \int_{\mathbb{R}_+} |f(x)|^p \, dv_\alpha(x) \right)^{1/p} < \infty,
$$

if $1 \leq p < \infty$,

$$
\|f\|_{L^\infty_\alpha(\mathbb{R}_+)} := \text{ess sup}_{x \in \mathbb{R}_+} |f(x)| < \infty.
$$

For $p = 2$, we provide the space $L^2_\alpha(\mathbb{R}_+)$ with the inner product

$$
(f, g)_{L^2_\alpha(\mathbb{R}_+)} := \int_{\mathbb{R}_+} f(x)\overline{g(x)} \, dv_\alpha(x).
$$

Let $\Delta_\alpha$ be the Bessel operator defined by

$$
\Delta_\alpha = \frac{d^2}{dx^2} + \frac{2\alpha + 1}{x} \frac{d}{dx}.
$$

Then, the following problem

$$
\Delta_\alpha u(x) = -\lambda^2 u(x), \quad \lambda \in \mathbb{C},
$$

$$
u(0) = 1,
$$

$$
u'(0) = 0
$$

admits a unique solution $j_\alpha(\lambda \cdot)$, where $j_\alpha$ is the function defined by

$$
j_\alpha(x) = \begin{cases}
2^{\alpha} \Gamma(\alpha + 1) \frac{j_\alpha(x)}{x^\alpha}, & \text{if } x \neq 0, \\
1, & \text{if } x = 0
\end{cases}
$$

and $j_\alpha$ is the Bessel function of the first kind and index $\alpha$ (see [15, 16]). The function $j_\alpha$ has the following Poisson representation formula:

$$
\forall x \in \mathbb{R}_+,
$$

$$
j_\alpha(x) = \begin{cases}
\frac{\Gamma(\alpha + 1)}{\sqrt{\pi} \Gamma(\alpha + \frac{1}{2})} \int_{-\infty}^{\infty} \frac{1}{(1 + t^2)^{\frac{\alpha}{2}} e^{-itx}} \, dt, & \text{if } \alpha > -\frac{1}{2}; \\
\cos x, & \text{if } \alpha = -\frac{1}{2}
\end{cases}
$$

and for all $n \in \mathbb{N}$, we have

$$
|j_\alpha^{(n)}(x)| \leq 1.
$$

The Hankel translation operator $\tau_\alpha$ is defined on $L^p_\alpha(\mathbb{R}_+)$ for all $x \in \mathbb{R}_+$ by

$$
\tau_\alpha(f)(y) = \frac{\Gamma(\alpha + 1)}{\sqrt{\pi} \Gamma(\alpha + \frac{1}{2})} \int_{\mathbb{R}_+} f(y(\sin \theta)^{2\alpha} \cos \theta) \, d\theta,
$$

if $\alpha > -\frac{1}{2}$,

$$
\frac{1}{2} \int_0^\infty f(x+y) + f(x-y) \, dy,
$$

if $\alpha = -\frac{1}{2}$.

(2.4)

It is well known that the Hankel translation operator satisfies the following product formula

$$
\forall (x, y) \in \mathbb{R}_+ \times \mathbb{R}_+,
$$

$$
\tau_\alpha(j_\alpha(\lambda \cdot))(y) = j_\alpha(\lambda x) j_\alpha(\lambda y).
$$

(2.5)

For every $f \in L^p_\alpha(\mathbb{R}_+), p \in [1, \infty)$, the function $\tau_\alpha(f)$ belongs to $L^p_\alpha(\mathbb{R}_+)$ and we have

$$
\|	au_\alpha(f)\|_{L^p_\alpha(\mathbb{R}_+)} \leq \|f\|_{L^p_\alpha(\mathbb{R}_+)}. 
$$

(2.6)

The Hankel convolution product of $f$ and $g$ in $L^1_\alpha(\mathbb{R}_+)$ is defined by

$$
\forall x \in \mathbb{R}_+, \quad f \ast_\alpha g(x) = \int_0^\infty \tau_\alpha(f)(s) g(s) \, ds.
$$

(2.7)

The Young inequality for the Hankel convolution product $\ast_\alpha$ reads as follows: if $p, q, r \in [1, \infty]$ are such that $1/p + 1/q = 1 + 1/r$, then for all $f \in L^p_\alpha(\mathbb{R}_+)$ and $g \in L^q_\alpha(\mathbb{R}_+)$, the function $f \ast_\alpha g$ belongs to the space $L^r_\alpha(\mathbb{R}_+)$, with the following inequality

$$
\|f \ast_\alpha g\|_{L^r_\alpha(\mathbb{R}_+)} \leq \|f\|_{L^p_\alpha(\mathbb{R}_+)} \|g\|_{L^q_\alpha(\mathbb{R}_+)}. 
$$

(2.8)

The Hankel transform $\mathcal{H}_\alpha$ is defined on $L^1_\alpha(\mathbb{R}_+)$ by

$$
\forall \lambda \in \mathbb{R}_+, \quad \mathcal{H}_\alpha(f)(\lambda) = \int_0^\infty f(x) j_\alpha(\lambda x) \, dv_\alpha(x).
$$

(2.9)

The following properties hold.

(i) For $f \in L^1_\alpha(\mathbb{R}_+)$ we have

$$
\|\mathcal{H}_\alpha(f)\|_{L^2_\alpha(\mathbb{R}_+)} \leq \|f\|_{L^1_\alpha(\mathbb{R}_+)}. 
$$

(2.10)

(ii) (Inversion formula) Let $f \in L^1_\alpha(\mathbb{R}_+)$ be such that $\mathcal{H}_\alpha(f) \in L^1_\alpha(\mathbb{R}_+)$, then we have

$$
f(x) = \int_0^\infty \mathcal{H}_\alpha(f)(\lambda) j_\alpha(\lambda x) \, dv_\alpha(\lambda), \quad a.e. x \in \mathbb{R}_+.
$$

(2.11)

(iii) (Paley-Wiener theorem) The Hankel transform $\mathcal{H}_\alpha$ is a topological isomorphism from $S_\alpha(\mathbb{R})$ onto itself.

(iv) (Plancherel’s formula) The Hankel transform $\mathcal{H}_\alpha$ can be extended to an isometric isomorphism from $L^2_\alpha(\mathbb{R}_+)$ onto itself and we have

$$
\|\mathcal{H}_\alpha(f)\|_{L^2_\alpha(\mathbb{R}_+)} = \|f\|_{L^2_\alpha(\mathbb{R}_+)}. 
$$

(2.12)
(v) (Parseval’s formula) For all \( f, g \in L^2_\sigma(\mathbb{R}_+) \), we have
\[
\int_0^\infty f(x)\overline{g(x)} \, dv_\sigma(x) = \int_0^\infty \mathcal{H}_\sigma(f)(\lambda)\overline{\mathcal{H}_\sigma(g)(\lambda)} \, dv_\sigma(\lambda). \tag{2.13}
\]

3. Hankel two-wavelet multipliers

3.1. Schatten-von Neumann classes

Notation. We denote by

- \( \mathcal{P}(\mathbb{N}) \), \( 1 \leq p < \infty \), the set of all infinite sequences of real (or complex) numbers \( x := (x_j)_{j \in \mathbb{N}} \), such that
  \[
  \|x\|_p := \left( \sum_{j=1}^{\infty} |x_j|^p \right)^{1/p} < \infty, \quad \text{if } 1 \leq p < \infty,
  \]
  \[
  \|x\|_\infty := \sup_{j \in \mathbb{N}} |x_j| < \infty.
  \]

For \( p = 2 \), we provide this space \( \ell^2(\mathbb{N}) \) with the scalar product
\[
\langle x, y \rangle_2 := \sum_{j=1}^{\infty} x_jy_j.
\]

- \( B(L^p_\sigma(\mathbb{R}_+)) \), \( 1 \leq p < \infty \), the space of bounded operators from \( L^p_\sigma(\mathbb{R}_+) \) into itself.

Definition 3.1:
(i) The singular values \( s_n(A) \) of a compact operator \( A \) in \( B(L^2_\sigma(\mathbb{R}_+)) \) are the eigenvalues of the positive self-adjoint operator \( |A| = \sqrt{A^*A} \).
(ii) For \( 1 \leq p < \infty \), the Schatten class \( S_p \) is the space of all compact operators whose singular values lie in \( \mathcal{P}(\mathbb{N}) \). The space \( S_p \) is equipped with the norm
\[
\|A\|_{S_p} := \left( \sum_{n=1}^{\infty} (s_n(A))^p \right)^{1/p}. \tag{3.1}
\]

Remark 3.1: We mention that the space \( S_2 \) is the space of Hilbert-Schmidt operators, and \( S_1 \) is the space of trace class operators.

Definition 3.2: The trace of an operator \( A \) in \( S_1 \) is defined by
\[
\text{tr}(A) = \sum_{n=1}^{\infty} \langle Av_n, v_n \rangle_{L^2_\sigma(\mathbb{R}_+)}. \tag{3.2}
\]

where \( (v_n)_n \) is any orthonormal basis of \( L^2_\sigma(\mathbb{R}_+) \).

Remark 3.2: If \( A \) is positive, then
\[
\text{tr}(A) = \|A\|_{S_1}. \tag{3.3}
\]

Moreover, a compact operator \( A \) on the Hilbert space \( L^2_\sigma(\mathbb{R}_+) \) is Hilbert-Schmidt, if the positive operator \( A^*A \) belongs to the space of trace class \( S_1 \). With this, we have
\[
\|A\|_{S_2} := \|A\|_{S_2} = \|A^*A\|_{S_1} = \text{tr}(A^*A) = \sum_{n=1}^{\infty} \|Av_n\|_{L^2_\sigma(\mathbb{R}_+)}^2. \tag{3.4}
\]

for any orthonormal basis \( (v_n)_n \) of \( L^2_\sigma(\mathbb{R}_+) \).

Definition 3.3: We also define the space \( S_\infty := B(L^2_\sigma(\mathbb{R}_+)) \), equipped with the norm
\[
\|A\|_{S_\infty} := \sup_{v \in L^2_\sigma(\mathbb{R}_+)} \|Av\|_{L^2_\sigma(\mathbb{R}_+)}. \tag{3.5}
\]

Remark 3.3: It is obvious that \( S_p \subset S_q \), for any \( 1 \leq p \leq q \leq \infty \).

3.2. Revisiting Hankel two-wavelet multipliers

The aim of this subsection is to survey and revisit some results for the Hankel two-wavelet multipliers.

Definition 3.4: Let \( u, v, \sigma \) be measurable functions on \( \mathbb{R}_+ \), we define the Hankel two-wavelet multiplier operator noted by \( \mathcal{P}_{u,v}(\sigma) \), on \( L^p_\sigma(\mathbb{R}_+) \), \( 1 \leq p \leq \infty \), by
\[
\mathcal{P}_{u,v}(\sigma)(f)(y) = \int_0^\infty \sigma(\xi) \mathcal{H}_\sigma(uf)(\xi)j_\alpha(y\xi) \overline{v(y)} \, dv_\sigma(\xi), \quad y \in \mathbb{R}_+. \tag{3.6}
\]

In accordance with the different choices of the symbols \( \sigma \) and the different continuities required, we need to impose different conditions on \( u \) and \( v \). We then obtain an operator on \( L^p_\sigma(\mathbb{R}_+) \).

It is often more convenient to interpret the definition of \( \mathcal{P}_{u,v}(\sigma) \) in a weak sense, that is, for \( f \in L^p_\sigma(\mathbb{R}_+) \), \( p \in [1, \infty] \), and \( g \in L^p_\sigma(\mathbb{R}_+) \), we have
\[
(\mathcal{P}_{u,v}(\sigma)(f), g)_{L^2_\sigma(\mathbb{R}_+)} = \int_0^\infty \sigma(\xi) \mathcal{H}_\sigma(uf)(\xi)\overline{\mathcal{H}_\sigma(vg)(\xi)} \, dv_\sigma(\xi). \tag{3.7}
\]

In the sequel of this section, \( u \) and \( v \) denote two any functions in \( L^p_\sigma(\mathbb{R}_+) \cap L^\infty_\sigma(\mathbb{R}_+) \) satisfying the following condition
\[
\|u\|_{L^p_\sigma(\mathbb{R}_+)} = \|v\|_{L^p_\sigma(\mathbb{R}_+)} = 1.
\]

Below, we recall fourth results that have been proved by Gobber [3].

Proposition 3.1 [(3)]: Let \( \sigma \) be in \( L^p_\sigma(\mathbb{R}_+) \), \( 1 \leq p \leq \infty \). Then there exists a unique bounded linear operator \( \mathcal{P}_{u,v}(\sigma) : L^p_\sigma(\mathbb{R}_+) \rightarrow L^p_\sigma(\mathbb{R}_+) \), such that
\[
\|\mathcal{P}_{u,v}(\sigma)\|_{S_\infty} \leq \|u\|_{L^p_\sigma(\mathbb{R}_+)} \|v\|_{L^p_\sigma(\mathbb{R}_+)}(p^{-1}/p) \|\sigma\|_{L^p_\sigma(\mathbb{R}_+)}.
\]
Proposition 3.2 ([3]): Let $\sigma$ be a symbol in $L_p^p(\mathbb{R}^+)$, $1 \leq p < \infty$. Then the Hankel two-wavelet multiplier $P_{u,v}(\sigma)$ is compact.

Proposition 3.3 ([3]): Let $\sigma$ be in $L_p^p(\mathbb{R}^+)$, Then $P_{u,v}(\sigma) : L_2^2(\mathbb{R}^+) \rightarrow L_2^2(\mathbb{R}^+)$ is in $S_1$, with

$$\|P_{u,v}(\sigma)v\|_{L_2^2(\mathbb{R}^+)} \leq \|\sigma\|_{L_2^2(\mathbb{R}^+)}$$

and the following trace formula

$$\text{tr}(P_{u,v}(\sigma)) = \int_0^\infty \sigma(\xi)\langle \tilde{u}_\sigma(\xi), \tilde{v}_\sigma(\xi)\rangle_{L_2^2(\mathbb{R}^+)} d\nu(\xi).$$

(3.10)

Proposition 3.4 ([3]): Let $\sigma$ be in $L_p^p(\mathbb{R}^+)$, $1 \leq p \leq \infty$. Then, the Hankel two-wavelet multiplier $P_{u,v}(\sigma) : L_2^2(\mathbb{R}^+) \rightarrow L_2^2(\mathbb{R}^+)$ is in $S_p$ and we have

$$\|P_{u,v}(\sigma)v\|_{L_2^2(\mathbb{R}^+)} \leq \left\{\|u\|_{L_2^2(\mathbb{R}^+)}\|v\|_{L_2^2(\mathbb{R}^+)}\right\}^{(p-1)/p}\|\sigma\|_{L_2^2(\mathbb{R}^+)}.$$

(3.11)

In the remainder of this subsection we establish some new results for the Hankel two-wavelet multipliers.

Proposition 3.5: Let $p \in [1, \infty)$. The adjoint of linear operator

$$P_{u,v}(\sigma) : L_p^p(\mathbb{R}^+) \rightarrow L_p^p(\mathbb{R}^+)$$

is $P_{v,u}(\tilde{\sigma}) : L_p^p(\mathbb{R}^+) \rightarrow L_p^p(\mathbb{R}^+)$. 

Proof: For any $f \in L_p^p(\mathbb{R}^+)$ and $g \in L_p^p(\mathbb{R}^+)$, (3.7) immediately implies that

$$\langle P_{u,v}(\sigma)(f), g \rangle_{L_2^2(\mathbb{R}^+)} = \int_0^\infty \sigma(\xi)\langle \tilde{u}_\sigma(\xi), \tilde{g}(\xi)\rangle_{L_2^2(\mathbb{R}^+)} d\nu(\xi).$$

(3.12)

Thus we get

$$P_{u,v}^*(\sigma) = P_{v,u}(\tilde{\sigma}).$$

■

Theorem 3.1: Let $\sigma$ be in $L_p^p(\mathbb{R}^+)$. Then we have

$$\frac{2}{\|u\|_{L_p^2(\mathbb{R}^+)}^2 + \|v\|_{L_p^2(\mathbb{R}^+)}^2} \|\tilde{\sigma}\|_{L_2^2(\mathbb{R}^+)} \leq \|P_{u,v}(\sigma)\|_{S_1},$$

where $\tilde{\sigma}$ is given by

$$\tilde{\sigma}(\xi) = \langle P_{u,v}(\sigma)\rangle\tilde{\sigma}(\xi), \quad \xi \in \mathbb{R}^+.$$  

(3.14)

Proof: First, it is easy to see that $\tilde{\sigma}$ belongs to $L_2^2(\mathbb{R}^+)$. Since $\sigma$ is in $L_2^2(\mathbb{R}^+)$, by Proposition 3.4, $P_{u,v}(\sigma)$ is in

S_2. Using [13, Theorem 2.2], there exists an orthonormal basis $\{\psi_j, j = 1, 2, \ldots\}$ for the orthogonal complement of the kernel of the operator $P_{u,v}(\sigma)$, consisting of eigenvectors of $|P_{u,v}(\sigma)|$ and $\{\psi_j, j = 1, 2, \ldots\}$ an orthonormal set in $L_2^2(\mathbb{R}^+)$, such that

$$P_{u,v}(\sigma)(\hat{f}) = \sum_{j=1}^\infty \hat{s}_j(\hat{f}, \psi_j)\hat{\psi}_j$$

(3.15)

where $s_j, j = 1, 2, \ldots$ refer to the positive singular values of $P_{u,v}(\sigma)$ corresponding to $\psi_j$. Then we get

$$\|P_{u,v}(\sigma)v\|_{L_2^2(\mathbb{R}^+)} = \sum_{j=1}^\infty s_j \|\hat{f}(\hat{\psi}_j)v(\psi_j)\|_{L_2^2(\mathbb{R}^+)}.$$
is a function in $L^2(\mathbb{R}_+)\ such\ that\ \|u\|_{L^2(\mathbb{R}_+)} = 1$. Then, the Hankel two-wavelet multipliers $P_{u,v}(\sigma_1)$, $P_{u,v}(\sigma_2)$ are positive trace class operators and

$$
\| (P_{u,v}(\sigma_1)P_{u,v}(\sigma_2))^n \|_{S_1} \leq (\text{tr}(P_{u,v}(\sigma_1)))^{n}(\text{tr}(P_{u,v}(\sigma_2)))^{n} = \|P_{u,v}(\sigma_1)\|_{S_1} \|P_{u,v}(\sigma_2)\|_{S_1}^{n}
$$

for any natural number $n$.

**Proof:** By Theorem 1 in the paper [17], we know that if $A$ and $B$ are in the trace class $S_1$ and are positive operators, then

$$
\forall n \in \mathbb{N}, \quad \text{tr}(AB)^n \leq (\text{tr}(A))^n(\text{tr}(B))^n.
$$

So, if we take $A = P_{u,v}(\sigma_1)$, $B = P_{u,v}(\sigma_2)$ and we invoke the previous remark, we obtain the desired result, so completing the proof. 

### 3.3. The generalized Landau–Pollak–Slepian operator

The purpose of this subsection is to give an example of a Hankel two-wavelet multiplier which improve and generalize Ghober’s result concerning the Landau–Pollak–Slepian operator from the Hankel one-wavelet multiplier.

Let $R, R_1$ and $R_2$ be positive numbers. We define the linear operators

$$
Q_R : L^2_a(\mathbb{R}_+) \rightarrow L^2_a(\mathbb{R}_+), \quad P_{R_1} : L^2_a(\mathbb{R}_+) \rightarrow L^2_a(\mathbb{R}_+),
$$

$$
P_{R_2} : L^2_a(\mathbb{R}_+) \rightarrow L^2_a(\mathbb{R}_+)
$$

by

$$
Q_Rf = \chi_{[0,R]}f, \quad P_{R_1}f = \mathcal{H}_a(\chi_{[0,R_1]}\mathcal{H}_a(f)), \quad P_{R_2}f = \mathcal{H}_a(\chi_{[0,R_2]}\mathcal{H}_a(f)),
$$

where $\chi_{[0,s]}$ stands for the characteristic function of the interval $[0,s]$.

Adapting the proof of Proposition 20.1 in the book [13], we prove the following.

**Proposition 3.6:** The linear operators

$$
Q_R, P_{R_1}, P_{R_2} : L^2_a(\mathbb{R}_+) \rightarrow L^2_a(\mathbb{R}_+)
$$

are self-adjoint projections.

The bounded linear operator $P_{R_2}Q_R P_{R_1} : L^2_a(\mathbb{R}_+) \rightarrow L^2_a(\mathbb{R}_+)$ is called the generalized Landau–Pollak–Slepian operator. We can show that the generalized Landau–Pollak–Slepian operator is in fact a Hankel two-wavelet multiplier.

**Theorem 3.2:** Let $u$ and $v$ be the functions on $\mathbb{R}_+$ defined by

$$
u = \frac{1}{\sqrt{v_0((0,R_2))}} X((0,R_2)), \quad v = \frac{1}{\sqrt{v_0((0,R_1))}} X((0,R_1)),
$$

where

$$
\forall s > 0, \quad v_0((0,s)) := \int_0^s d v_0(x) = \frac{x^{\alpha + 2}}{(\alpha + 2)!}.
$$

Then the generalized Landau–Pollak–Slepian operator

$$
P_{R_2}Q_R P_{R_1} : L^2_a(\mathbb{R}_+) \rightarrow L^2_a(\mathbb{R}_+)
$$

is unitary equivalent to a scalar multiple of the Hankel two-wavelet multiplier

$$
P_{u,v}(\chi_{[0,R_1]} : L^2_a(\mathbb{R}_+) \rightarrow L^2_a(\mathbb{R}_+).$$

In fact

$$
P_{R_2}Q_R P_{R_1} = C_a(R_1,R_2) \mathcal{H}_a(P_{u,v}(\chi_{[0,R_1]})) \mathcal{H}_a,
$$

where

$$
C_a(R_1,R_2) := \sqrt{v_0((0,R_1))v_0((0,R_2))} = \frac{(R_1R_2)^{\alpha + 1}}{2^{\alpha + 1}(\alpha + 2)!}. \quad (3.18)
$$

**Proof:** It is easy to see that $u$ and $v$ belong to $L^2_a(\mathbb{R}_+) \cap L^2\infty(\mathbb{R}_+)$ and

$$
\|u\|_{L^2(\mathbb{R}_+)} = \|v\|_{L^2(\mathbb{R}_+)} = 1.
$$

By simple calculations we find

$$
(P_{u,v}(\chi_{[0,R_1]})(f), g)_{L^2(\mathbb{R}_+)}
$$

$$
= \frac{1}{C_a(R_1,R_2)} \int_0^\infty x_{[0,R_1]}(\xi) P_{R_1}(\mathcal{H}_a(f))(\xi) \mathcal{H}_a(g)(\xi) \ dv_0(\xi)
$$

$$
= \frac{1}{C_a(R_1,R_2)} \int_0^\infty \mathcal{H}_a(f)(\xi) P_{R_1}(\mathcal{H}_a(g))(\xi) \ dv_0(\xi)
$$

$$
= \frac{1}{C_a(R_1,R_2)} (Q_{R_1}P_{R_1}(\mathcal{H}_a(f)), P_{R_1}(\mathcal{H}_a(g)))_{L^2(\mathbb{R}_+)}
$$

$$
= \frac{1}{C_a(R_1,R_2)} (P_{R_1}Q_{R_1}(\mathcal{H}_a(f)), \mathcal{H}_a(g))_{L^2(\mathbb{R}_+)}
$$

$$
= \frac{1}{C_a(R_1,R_2)} (\mathcal{H}_aP_{R_1}Q_{R_1}(\mathcal{H}_a(f)), g)_{L^2(\mathbb{R}_+)}
$$

for all $f, g$ in $S_a(\mathbb{R})$ and hence the proof is complete.

The next result gives a formula for the trace of the generalized Landau–Pollak–Slepian operator.

**Corollary 3.2:** We have

$$
\text{tr}(P_{R_2}Q_R P_{R_1}) = \int_0^R \int_0^{\min(R_1,R_2)} |f_\alpha(y\xi)|^2 \ dv_1(y) \ dv_0(\xi).
$$

**Proof:** The result is an immediate consequence of Theorem 3.2 and Proposition 3.3.
4. $L^p$ boundedness and compactness of $\mathcal{P}_{u,v}(\sigma)$

4.1. Boundedness for symbols in $L^1_\alpha(R_+)$

Let $\sigma \in L^1_\alpha(R_+)$, $v \in L^1_\alpha(R_+)$ and $u \in L^p_\alpha(R_+)$, with $1 \leq p \leq \infty$.

We wish to establish that $\mathcal{P}_{u,v}(\sigma)$ is a bounded operator on $L^p_\alpha(R_+)$. Let us start with the following propositions.

Proposition 4.1: Let $\sigma$ be in $L^1_\alpha(R_+)$, $u \in L^\infty_\alpha(R_+)$ and $v \in L^1_\alpha(R_+)$, then the Hankel two-wavelet multiplier $\mathcal{P}_{u,v}(\sigma) : L^1_\alpha(R_+) \rightarrow L^1_\alpha(R_+)$ is a bounded linear operator and we have

$$
\|\mathcal{P}_{u,v}(\sigma)\|_{B(L^1_\alpha(R_+))} \leq \|u\|_{L^\infty_\alpha(R_+)} \|v\|_{L^1_\alpha(R_+)} \|\sigma\|_{L^1_\alpha(R_+)}.
$$

(4.1)

Proof: For every function $f$ in $L^1_\alpha(R_+)$, we have from the relations (3.6), (2.10) and (2.3)

$$
\begin{align*}
\|\mathcal{P}_{u,v}(\sigma)(f)\|_{L^1_\alpha(R_+)} & \leq \int_0^\infty \int_0^\infty |\sigma(\xi)| \|H_\alpha(u(f))(\xi)\|_{L^1_\alpha([0,\infty))} d\nu_\alpha(\xi) d\nu_u(\xi) \\
& \leq \|f\|_{L^2_\alpha(R_+)} \|u\|_{L^\infty_\alpha(R_+)} \|v\|_{L^1_\alpha(R_+)} \|\sigma\|_{L^1_\alpha(R_+)}.
\end{align*}
$$

Thus

$$
\|\mathcal{P}_{u,v}(\sigma)\|_{B(L^1_\alpha(R_+))} \leq \|u\|_{L^\infty_\alpha(R_+)} \|v\|_{L^1_\alpha(R_+)} \|\sigma\|_{L^1_\alpha(R_+)}.
$$

(4.2)

Proposition 4.2: Let $\sigma$ be in $L^1_\alpha(R_+)$, $u \in L^1_\alpha(R_+)$ and $v \in L^\infty_\alpha(R_+)$, then the Hankel two-wavelet multiplier $\mathcal{P}_{u,v}(\sigma) : L^\infty_\alpha(R_+) \rightarrow L^\infty_\alpha(R_+)$ is a bounded linear operator and we have

$$
\|\mathcal{P}_{u,v}(\sigma)\|_{B(L^\infty_\alpha(R_+))} \leq \|u\|_{L^1_\alpha(R_+)} \|v\|_{L^\infty_\alpha(R_+)} \|\sigma\|_{L^1_\alpha(R_+)}.
$$

(4.4)

Remark 4.1: Proposition 4.2 is also a corollary of Proposition 4.1, since the adjoint of

$$
\mathcal{P}_{v,u}(\tilde{\sigma}) : L^1_\alpha(R_+) \rightarrow L^1_\alpha(R_+)
$$

is $\mathcal{P}_{u,v}(\sigma) : L^\infty_\alpha(R_+) \rightarrow L^\infty_\alpha(R_+)$.

Using an interpolation of Propositions 4.1 and 4.2, we get the following result.

Theorem 4.1: Let $u$ and $v$ be functions in $L^1_\alpha(R_+) \cap L^\infty_\alpha(R_+)$. Then for all $\sigma$ in $L^1_\alpha(R_+) \cap L^\infty_\alpha(R_+)$, there exists a unique bounded linear operator

$$
\mathcal{P}_{u,v}(\sigma) : L^1_\alpha(R_+) \rightarrow L^p_\alpha(R_+), \quad 1 \leq p \leq \infty
$$

such that

$$
\begin{align*}
\|\mathcal{P}_{u,v}(\sigma)\|_{B(L^p_\alpha(R_+))} & \leq \|u\|_{L^1_\alpha(R_+)}^{1/p'} \|v\|_{L^1_\alpha(R_+)} \|\sigma\|_{L^1_\alpha(R_+)}^{1/p} \\
& \leq \|v\|_{L^p_\alpha(R_+)} \|\sigma\|_{L^1_\alpha(R_+)}.
\end{align*}
$$

(4.3)

With a Schur technique, we can obtain an $L^p_\alpha$ boundedness result as in the previous theorem, but the estimate for the norm $\|\mathcal{P}_{u,v}(\sigma)\|_{B(L^p_\alpha(R_+))}$ is cruder.

Theorem 4.2: Let $\sigma$ be in $L^1_\alpha(R_+)$ and $u,v$ be in $L^1_\alpha(R_+) \cap L^\infty_\alpha(R_+)$. Then there exists a unique bounded linear operator $\mathcal{P}_{u,v}(\sigma) : L^p_\alpha(R_+) \rightarrow L^p_\alpha(R_+)$, $1 \leq p \leq \infty$ such that

$$
\|\mathcal{P}_{u,v}(\sigma)\|_{B(L^p_\alpha(R_+))} \leq \max(\|u\|_{L^1_\alpha(R_+)} \|v\|_{L^\infty_\alpha(R_+)}),
$$

$$
\|u\|_{L^p_\alpha(R_+)} \|v\|_{L^\infty_\alpha(R_+)} \|\sigma\|_{L^1_\alpha(R_+)}.
$$

(4.4)

Proof: Let $N$ be the function defined on $R_+ \times R_+$ by

$$
N(y,z) = \int_0^\infty \sigma(\xi) u(\xi y) \overline{v}(\xi z) d\nu_u(\xi) d\nu_v(\xi).
$$

(4.5)

We have

$$
\mathcal{P}_{u,v}(\sigma)(f)(y) = \int_0^\infty N(y,z) f(z) d\nu_v(z).
$$

By simple calculations, it is not hard to see that

$$
\int_0^\infty |N(y,z)| d\nu_u(y) \leq \|u\|_{L^\infty_\alpha(R_+)} \|v\|_{L^1_\alpha(R_+)} \|\sigma\|_{L^1_\alpha(R_+)},
$$

$$
\quad \quad y \in R_+.
$$

and

$$
\int_0^\infty |N(y,z)| d\nu_v(z) \leq \|u\|_{L^1_\alpha(R_+)} \|v\|_{L^\infty_\alpha(R_+)} \|\sigma\|_{L^1_\alpha(R_+)},
$$

$$
\quad \quad y \in R_+.
$$

Thus by Schur Lemma (cf. [18]), we can conclude that

$$
\mathcal{P}_{u,v}(\sigma) : L^p_\alpha(R_+) \rightarrow L^p_\alpha(R_+).$$
is a bounded linear operator for any \(1 \leq p \leq \infty\), and we have
\[
\|P_{u,v}(\sigma)\|_{B(L^p_v(\mathbb{R}^+))} \leq \max(\|u\|_{L^1_u(\mathbb{R}^+)} \|v\|_{L^\infty_v(\mathbb{R}^+)},
\|u\|_{L^\infty_u(\mathbb{R}^+)} \|v\|_{L^1_v(\mathbb{R}^+)} \|\sigma\|_{L^1_u(\mathbb{R}^+)}).
\]

**Remark 4.2:** The previous theorem tells us that the unique bounded linear operator on \(L^p_v(\mathbb{R}^+)\), \(1 \leq p \leq \infty\), obtained by interpolation in Theorem 4.1 is in fact the integral operator on \(L^p_v(\mathbb{R}^+)\) with kernel \(N\) given by (4.5).

We can give another version of the \(L^p\)-boundedness. Firstly we generalize and improve Proposition 4.2.

**Proposition 4.3:** Let \(\sigma\) be in \(L^1_v(\mathbb{R}^+)\), \(v \in L^p_v(\mathbb{R}^+)\) and \(u \in L^r_u(\mathbb{R}^+)\), for \(1 < p \leq \infty\). Then the Hankel two-wavelet multiplier
\[
P_{u,v}(\sigma) : L^p_v(\mathbb{R}^+) \rightarrow L^p_v(\mathbb{R}^+)
\]
is a bounded linear operator, and we have
\[
\|P_{u,v}(\sigma)\|_{B(L^p_v(\mathbb{R}^+))} \leq \|u\|_{L^r_u(\mathbb{R}^+)} \|v\|_{L^p_v(\mathbb{R}^+)} \|\sigma\|_{L^1_u(\mathbb{R}^+)}. 
\]
(4.6)

**Proof:** For any \(f \in L^p_v(\mathbb{R}^+)\), consider the linear functional
\[
I_f : L^r_u(\mathbb{R}^+) \rightarrow \mathbb{C}
\]
\[
g \mapsto \langle g, P_{u,v}(\sigma)(f) \rangle_{L^2_v(\mathbb{R}^+)},
\]
From the relation (3.7)
\[
|\langle P_{u,v}(\sigma)(f), g \rangle_{L^2_v(\mathbb{R}^+)}| 
\]
\[
\leq \int_{0}^{\infty} |\sigma(\xi)| \|H_v(uf)(\xi)\| \|H_v(vg)(\xi)\| \, dv_\xi(\xi)
\]
\[
\leq \|\sigma\|_{L^1_u(\mathbb{R}^+)} \|H_v(uf)\|_{L^2_v(\mathbb{R}^+)} \|H_v(vg)\|_{L^2_v(\mathbb{R}^+)}. 
\]
Using the relation (2.9), (2.3) and Hölder’s inequality, we get
\[
|\langle P_{u,v}(\sigma)(f), g \rangle_{L^2_v(\mathbb{R}^+)}| 
\]
\[
\leq \|\sigma\|_{L^1_u(\mathbb{R}^+)} \|u\|_{L^r_u(\mathbb{R}^+)} \|v\|_{L^p_v(\mathbb{R}^+)} \|f\|_{L^p_v(\mathbb{R}^+)} \|g\|_{L^p_v(\mathbb{R}^+)}. 
\]
Thus, the operator \(I_f\) is a continuous linear functional on \(L^p_v(\mathbb{R}^+)\), and the operator norm
\[
\|I_f\|_{B(L^p_v(\mathbb{R}^+))} \leq \|u\|_{L^r_u(\mathbb{R}^+)} \|v\|_{L^p_v(\mathbb{R}^+)} \|f\|_{L^p_v(\mathbb{R}^+)} \|\sigma\|_{L^1_u(\mathbb{R}^+)}. 
\]
As \(I_f(g) = \langle g, P_{u,v}(\sigma)(f) \rangle_{L^2_v(\mathbb{R}^+)\rangle}\), by the Riesz representation theorem, we have
\[
\|P_{u,v}(\sigma)(f)\|_{L^p_v(\mathbb{R}^+)} = \|I_f\|_{B(L^p_v(\mathbb{R}^+))}
\]
\[
\leq \|u\|_{L^r_u(\mathbb{R}^+)} \|v\|_{L^p_v(\mathbb{R}^+)} \|f\|_{L^p_v(\mathbb{R}^+)} \|\sigma\|_{L^1_u(\mathbb{R}^+)}. 
\]
which concludes the proof.
and
\[ \frac{\theta}{1} + \frac{1 - \theta}{2} = \frac{1}{r}. \]

By the definition of \( I \), we have
\[ \| P_{u,v}(\sigma) \|_{B(L^p(R_+))} \leq C_1 \| \sigma \|_{L^p(R_+)}. \]

As the adjoint of \( P_{u,v}(\sigma) \) is \( P_{v,u}(\sigma) \), so \( P_{u,v}(\sigma) \) is a bounded linear map on \( L^p_0(R_+) \) with its operator norm
\[ \| P_{u,v}(\sigma) \|_{B(L^p_0(R_+))} = \| P_{v,u}(\sigma) \|_{B(L^p_0(R_+))} \leq C_2 \| \sigma \|_{L^p_0(R_+)}, \]

where
\[ C_2 = (\| u \|_{L^1(R_+)} \| v \|_{L^2(R_+)} \| v \|_{L^2(R_+)}^{(1-\theta)/2}. \]

Using an interpolation of (4.13) and (4.14), we have that, for any \( p \in [r,r'] \),
\[ \| P_{u,v}(\sigma) \|_{B(L^p_0(R_+))} \leq C_1^\frac{1}{t} \| \sigma \|_{L^p_0(R_+)}, \]
with
\[ \frac{t}{r} + \frac{1 - t}{r'} = \frac{1}{p}. \]

\section{Compactness of \( P_{u,v}(\sigma) \)}

\textbf{Proposition 4.4:} Under the same hypothesis as in Theorem 4.1, the Hankel two-wavelet multiplier \( P_{u,v}(\sigma) : L^1_0(R_+) \rightarrow L^1_0(R_+) \) is compact.

\textbf{Proof:} Let \( (f_n)_{n \in \mathbb{N}} \in L^1_0(R_+) \) such that \( f_n \rightharpoonup 0 \) weakly in \( L^1_0(R_+) \) as \( n \rightarrow \infty \). It is enough to prove that
\[ \lim_{n \rightarrow \infty} \| P_{u,v}(\sigma)(f_n) \|_{L^1_0(R_+)} = 0. \]

We have
\[ \| P_{u,v}(\sigma)(f_n) \|_{L^1_0(R_+)} \leq \int_0^\infty \int_0^\infty |\sigma(\xi)| |f_n(\xi)| |u(\xi)| d\nu_u(\xi) d\nu_v(\xi). \]

Now using the fact that \( f_n \rightharpoonup 0 \) weakly in \( L^1_0(R_+) \), we deduce that
\[ \forall \xi, y \in R_+, \]
\[ \lim_{n \rightarrow \infty} |\sigma(\xi)| |f_n(\xi)| |u(\xi)| \|v(y)\| = 0. \]

(4.16)

On the other hand, since \( f_n \rightharpoonup 0 \) weakly in \( L^1_0(R_+) \) as \( n \rightarrow \infty \), then there exists a positive constant \( C \) such that
\[ \| f_n \|_{L^1_0(R_+)} \leq C. \]

Hence by simple computations we get
\[ \forall \xi, y \in R_+, \]
\[ |\sigma(\xi)| |f_n(s(\xi))| \|u(\xi)\| \|v(y)\| \leq C |\sigma(\xi)| \|u\|_{L^\infty(R_+)} \|v\|, \]

(4.17)

Moreover, from Fubini’s theorem and relation (2.3), we have
\[ \int_0^\infty \int_0^\infty |\sigma(\xi)| |f_n(\xi)| |u(\xi)| \|v(y)\| d\nu_u(\xi) d\nu_v(\xi) \]
\[ \leq C \|u\|_{L^\infty(R_+)} \int_0^\infty \|v(y)\| d\nu_v(\xi) \]
\[ \leq C \|u\|_{L^\infty(R_+)} \|v\| L^1_0(R_+) \|\sigma\|_{L^1_0(R_+)} < \infty. \]

(4.18)

Thus from the relations (4.15)–(4.18) and the Lebesgue dominated convergence theorem we deduce that
\[ \lim_{n \rightarrow \infty} \| P_{u,v}(\sigma)(f_n) \|_{L^1_0(R_+)} = 0 \]
and the proof is complete.

In the following, we give two results for compactness of the Hankel two-wavelet multipliers.

\textbf{Theorem 4.5:} Under the hypothesis of Theorem 4.1, the bounded linear operator
\[ P_{u,v}(\sigma) : L^p_0(R_+) \rightarrow L^p_0(R_+) \]

is compact for \( 1 \leq p \leq \infty \).

\textbf{Proof:} From the previous proposition, we only need to show that the conclusion holds for \( p = \infty \). In fact, the operator \( P_{u,v}(\sigma) : L^\infty_0(R_+) \rightarrow L^\infty_0(R_+) \) is the adjoint of the operator \( P_{v,u}(\sigma) : L^\infty_0(R_+) \rightarrow L^\infty_0(R_+) \), which is compact by the previous proposition. Thus by the duality property, \( P_{u,v}(\sigma) : L^\infty_0(R_+) \rightarrow L^\infty_0(R_+) \) is compact. Finally, by an interpolation of the compactness on \( L^1_0(R_+) \) and on \( L^\infty_0(R_+) \) such as that given in the book [20, p.202–203] by Bennett and Sharply, we conclude the proof.

The following result is an analogue of Theorem 4.4 for compact operators.

\textbf{Theorem 4.6:} Under the hypotheses of Theorem 4.4, the bounded linear operator
\[ P_{u,v}(\sigma) : L^p_0(R_+) \rightarrow L^p_0(R_+) \]

is compact for all \( p \in [r,r'] \).

\textbf{Proof:} The result is an immediate consequence of an interpolation of Proposition 3.4 and Proposition 4.4. See again the book [20, p.202–203] for the interpolation used therein.

\textbf{Remark 4.3:} Our hope that this work motivates the researchers to study the two-wavelet multipliers for the Bessel-Struve transform on \( R^d \), [21].
5. Conclusions

In the present paper, we have successfully studied many spectral theorems associated with the Hankel two-wavelet multipliers. The obtained results have a novelty and contribution to the literature, and they improve and generalize the results of Ghobber [3].

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