Black Holes in
Three-Dimensional Topological Gravity

by

S. Carlip
Department of Physics
University of California
Davis, California 95616 USA
[e-mail: carlip@dirac.ucdavis.edu]

J. Gegenberg
Department of Mathematics and Statistics
University of New Brunswick
Fredericton, New Brunswick Canada E3B 5A3
[e-mail: lenin@math.unb.ca]

R. B. Mann
Department of Applied Mathematics and Theoretical Physics
Silver Street
Cambridge University
Cambridge U.K. CB2 9EW
and
Department of Physics
University of Waterloo
Waterloo, Ontario Canada N2L 3G1
[e-mail: rbm20@damtp.cambridge.ac.uk]
Abstract

We investigate the black hole solution to (2+1)-dimensional gravity coupled to topological matter, with a vanishing cosmological constant. We calculate the total energy, angular momentum and entropy of the black hole in this model and compare with results obtained in Einstein gravity. We find that the theory with topological matter reverses the identification of energy and angular momentum with the parameters in the metric, compared with general relativity, and that the entropy is determined by the circumference of the inner rather than the outer horizon. We speculate that this results from the contribution of the topological matter fields to the conserved currents. We also briefly discuss two new possible (2+1)-dimensional black holes.
1 The (2+1)-Dimensional Black Hole

It is a peculiar feature of general relativity in 2+1 dimensions that any solution of the Einstein field equations

\[ G_{\mu\nu} = 8\pi GT_{\mu\nu} - \Lambda g_{\mu\nu}, \]  

(1.1)

with vanishing stress-energy tensor has constant curvature. Despite this limitation, Bañados et al. [1] have made the interesting observation that when \( \Lambda = -1/\ell^2 < 0 \), the field equations have a black hole solution, characterized by the metric

\[ ds^2 = -N^2 dt^2 + N^{-2} dr^2 + r^2 (N^\phi dt + d\phi)^2, \]  

(1.2)

\[ -\infty < t < \infty, \quad 0 < r < \infty, \quad 0 \leq \phi \leq 2\pi, \]

with lapse and angular shift functions

\[ N^2(r) = -M + \frac{r^2}{\ell^2} + \frac{J^2}{4r^2}, \quad N^\phi(r) = -\frac{J}{2r}. \]  

(1.3)

As a space of constant curvature, this geometry can be obtained directly from anti-de Sitter space by means of appropriate identifications, as discussed in Ref. [2]. When \( M > 0 \) and \( |J| \leq M\ell \), the solution has an outer event horizon at \( r = r_+ \), where

\[ r_+^2 = \frac{M\ell^2}{2} \left\{ 1 + \left[ 1 - \left( \frac{J}{M\ell} \right)^2 \right]^{1/2} \right\}, \]  

(1.4)

and an inner horizon at \( r_- = J\ell/2r_+ \).

The parameters \( J \) and \( M \) have been shown to be the quasilocal angular momentum and mass of the black hole [3]; alternatively they can be expressed in terms of Casimir invariants in a gauge-theoretic formulation of (2 + 1)-dimensional gravity [4]. The parameter \( M \) can also be expressed in terms of the initial energy density of a disk of collapsing dust in AdS space [5].

For later reference, it will be useful to display the first order formulation of Einstein gravity and the first order form of the black hole solution. We suppose that \( M^3 \) is a smooth orientable 3-manifold whose cotangent bundle has SO(2,1) as its structure group. The fibers of \( T^*M^3 \) are three-dimensional
vector spaces which come equipped with a “natural” metric $\eta_{ab}$ and volume element $\epsilon_{abc}$. A smooth frame field, or triad, on $M^3$ is a set of three independent one-form fields $E^a$, and a spin connection $\omega_a$ on $M^3$ is an SO(2,1) connection. In terms of these fields, the Einstein action takes the simple form

$$ I = \int L = -\frac{1}{2} \int_{M^3} \left( E^a \wedge R_a[\omega] - \frac{\Lambda}{3} \epsilon_{abc} E^a \wedge E^b \wedge E^c \right), \quad (1.5) $$

where the curvature $R_a[\omega]$ is

$$ R_a[\omega] = d\omega_a + \frac{1}{4} \epsilon_{abc} \omega_b \wedge \omega_c. \quad (1.6) $$

We use the convention $\epsilon^{012} = +1$, and our units are such that $8\pi G = 1$. Variation of $I$ with respect to $\omega_a$ yields the condition

$$ D_\omega E^a = dE^a + \frac{1}{2} \epsilon^{abc} \omega_b \wedge E_c = 0, \quad (1.7) $$

which is the usual relationship between $E^a$ and the connection (that is, the condition that $\omega^a$ be torsion-free). Variation with respect to the triad yields

$$ R_a = \Lambda \epsilon_{abc} E^b \wedge E^c, \quad (1.8) $$

which is equivalent to Eq. (1.1) upon insertion of Eq. (1.7).

The black hole (1.2) can be now described by the spacetime triad

$$ E^0 = \sqrt{\nu^2(r) - 1} \left( \frac{r^+}{\ell} dt - r^- d\phi \right), $$

$$ E^1 = \ell \sqrt{\nu^2(r) - 1} d\phi, $$

$$ E^2 = \nu(r) \left( r^+ dt - \frac{r^-}{\ell} d\phi \right), \quad (1.9) $$

and the compatible spin connection

$$ \omega^0 = -2\sqrt{\nu^2(r) - 1} \left( \frac{r^+}{\ell} d\phi - \frac{r^-}{\ell^2} dt \right), $$

$$ \omega^1 = 0, $$

$$ \omega^2 = -2\nu(r) \left( \frac{r^+}{\ell^2} dt - \frac{r^-}{\ell} d\phi \right), \quad (1.10) $$

where $\nu^2(r) := \frac{r^2 - r^2}{r^2 - r^2}$. 

3
2 BCEA Gravity

There is another three-dimensional theory of gravity that admits the asymptotically AdS black hole as a solution. This theory, developed by two of the present authors [7], minimally couples topological matter to Einstein gravity in 2+1 dimensions, in effect replacing the cosmological constant by a simple set of matter fields. We shall call this model “BCEA theory.”

We start with a triad \( E^a \) and a spin connection \( A^a \) as in the last section, but without imposing the torsion-free condition (1.7). Now let \( B^a \) and \( C^a \) be two additional one-form “matter” fields. The action for BCEA theory is

\[
I = \int L = -\frac{1}{2} \int_{M^3} (E^a \wedge R_a[A] + B^a \wedge D(A)C_a),
\]

(2.1)

where \( D(A) \) is the covariant derivative with respect to the connection \( A^a \) and the curvature \( R_a[A] \) is given by (1.6).

The stationary points of \( I \) are determined by the field equations

\[
\begin{align*}
R_a[A] &= 0, \\
D(A)B^a &= 0, \\
D(A)C_a &= 0, \\
D(A)E^a + \frac{1}{2} \varepsilon^{abc} B_b \wedge C_c &= 0.
\end{align*}
\]

(2.2)

Because of the term in \( B_b \wedge C_c \) in the last equation of motion, the triad \( E^a \) is not, in general, compatible with the spin connection \( A_a \). Nevertheless, the equations of motion above determine a Lorentzian geometry on \( TM^3 \): if we define a one-form field \( Q_a \) by the requirement

\[
\varepsilon^{abc} (Q_b \wedge E_c - B_b \wedge C_c) = 0,
\]

(2.3)

then the equation of motion for the \( E^a \) can be written as

\[
dE^a + \frac{1}{2} \varepsilon^{abc} \omega_b \wedge E_c = 0, \quad \omega_a := A_a + Q_a.
\]

(2.4)

Eq. (2.4) may be recognized as the condition that the frame field \( E^a \) be compatible with the (nonflat) spin connection \( \omega_a \). We may thus interpret BCEA theory as a model of (2+1)-dimensional gravity with a triad \( E^a \) and
a connection $\omega_a$ coupled to matter fields $B^a$ and $C^a$. Alternatively, the model may be viewed as a “teleparallel” theory of gravity, with a triad $E^a$ and a flat—but not torsion-free—connection $A^a$, again coupled to matter fields $B^a$ and $C^a$. In either case, the geometry is determined by the metric $g_{\mu\nu} = \eta_{ab} E^a_\mu E^b_\nu$.

The action functional $I$ is invariant under a twelve-parameter group whose infinitesimal generators are

\[
\begin{align*}
\delta B^a &= D_{(A)} \rho^a + \frac{1}{2} \epsilon^{abc} B_b \tau_c, \\
\delta C^a &= D_{(A)} \lambda^a + \frac{1}{2} \epsilon^{abc} C_b \tau_c, \\
\delta E^a &= D_{(A)} \xi^a + \frac{1}{2} \epsilon^{abc} (E_b \tau_c + B_b \lambda_c + C_b \rho_c), \\
\delta A^a &= D_{(A)} \tau^a.
\end{align*}
\] (2.5)

This group may be recognized as $I(\text{ISO}(2,1))$, where the notation $I G$ denotes the semi-direct product of the Lie group $G$ with its own Lie algebra $L_G$. Like the action for ordinary Einstein gravity in three dimensions \[3\], the BCEA action can be obtained from a Chern-Simons functional, now for the gauge group $I(\text{ISO}(2,1))$.

\section{The Black Hole Solution in BCEA Theory}

We shall now demonstrate that the asymptotically anti-de Sitter black hole of section 1 is also a solution of BCEA theory. The computation is simplest in the gauge $A^a = 0$, for which the field equations for $B$ and $C$ reduce to the condition that the one-forms $B^a$ and $C^a$ be closed. It is then easy to show that

\[
\begin{align*}
B^0 &= -r_+ d\phi + \frac{r_+}{\ell} dt, \\
B^1 &= -\ell d(\nu + \sqrt{\nu^2 - 1}), \\
B^2 &= \frac{r_+}{\ell} dt - r_- d\phi, \\
\end{align*}
\] (3.1)

and

\[
C^0 = -\frac{1}{\ell} B^0,
\]
\[ C^1 = d(\nu - \sqrt{\nu^2 - 1}), \]
\[ C^2 = \frac{1}{\ell} B^2, \]

(3.2)

together with the black hole triad of equation (1.9) and the connection \( A^a = 0 \), satisfy the BCEA equations of motion.

As discussed above, the triad \( E^a \) determines the spacetime geometry, so the solution given by Eqs. (1.9), (3.1), and (3.2) still has an interpretation as a black hole. To obtain more information, we can investigate the conserved quantities, or Noether currents, associated with the symmetries of the solution. For the conventional black hole, for example, the charge and mass are the conserved charges associated with spatial translations and rotations at infinity, and the entropy is the charge associated with an appropriate Killing vector at the horizon. It is instructive to understand the analogous statements for the BCEA solution.

4 Noether Currents

We shall follow the methods of Wald, which we briefly summarize for the case of a three-dimensional spacetime (see [8] for details).

A. Noether Charges a la Wald

Let \( L[\phi] \) be a Lagrangian three-form (in three-dimensional spacetime), where \( \phi \) represents an arbitrary set of fields. Under a variation \( \delta \phi \),

\[ \delta L = \Xi \delta \phi + d\Theta[\phi, \delta \phi], \]

(4.1)

where the field equations are \( \Xi = 0 \) and \( \Theta \) is a boundary term, constructed locally from \( \phi \) and \( \delta \phi \), that determines the symplectic structure of the theory. For the action to be invariant under a symmetry transformation

\[ \phi \rightarrow \phi + \delta_g \phi, \quad g \in G, \]

(4.2)

it is clearly necessary that

\[ \delta_g L = d\alpha[\phi, \delta_g \phi], \]

(4.3)
for some two-form \( \alpha \). Combining Eq. (4.2) and Eq. (4.3), we see that \( dj[g] = 0 \) when the equations of motion are satisfied, where

\[
j[g] = \Theta[\phi, \delta_g \phi] - \alpha[\phi, \delta_g \phi]. \tag{4.4}
\]

The two-form \( j[g] \) is the Hodge dual of the usual Noether current associated with the symmetry generated by \( g \); its integral over a Cauchy surface \( C \) gives a conserved charge \( q[g] \). Note that when \( j \) is exact, that is, \( j = dq \), then the integral that gives \( q[g] \) reduces to an integral at \( \partial C \). This is the case when the symmetry group is the group of diffeomorphisms, and explains why the energy in general relativity can be written as an integral at spatial infinity.

We now specialize to the case of diffeomorphism invariance. Let \( \delta_\zeta \phi = L_\zeta \phi \) be a diffeomorphism generated by a vector field \( \zeta \). It is then easy to show that

\[
j[\zeta] = \Theta[\phi, L_\zeta \phi] - \zeta \cdot L. \tag{4.5}
\]

(\( L \) is the Lie derivative, and the centered dot denotes contraction of a vector with the first index of a form.) The Noether charges now have simple physical interpretations: if \( t^\mu \) generates an asymptotic time translation and \( \varphi^\mu \) generates an asymptotic rotation, then Wald has shown [8] that the canonical energy and angular momentum are

\[
E = \int_\infty (Q[t] - t \cdot G) \\
O = -\int_\infty Q[\varphi]. \tag{4.6}
\]

Here the integrals are taken along a circle of constant time and infinite radius, and \( G \) is defined by the condition

\[
\delta_0 \int_\infty t \cdot G = \int_\infty t \cdot \Theta[\phi, \delta_0 \phi], \tag{4.7}
\]

for variations \( \delta_0 \) lying within the space of solutions of the equations of motion.

**B. Noether Charges for Einstein Gravity**

For the first-order Einstein action of Eq. (1.5), it is easy to see that

\[
\Theta = \frac{1}{2} E^a \wedge \delta \omega_a. \tag{4.8}
\]
and a simple computation gives a current (4.3) of
\[ j[\zeta] = dQ[\zeta], \quad Q[\zeta] = -\frac{1}{2} E^a \zeta \cdot \omega_a. \] (4.9)

The canonical energy (4.6) for the black hole solution of Eqs. (1.9)–(1.10) is then easy to calculate: we obtain
\[ \mathcal{E} = \pi M, \] (4.10)
while the canonical angular momentum \( \mathcal{J} \) is
\[ \mathcal{J} = \pi J. \] (4.11)

C. Noether Charges for the BCEA System

For the BCEA system, there are two sets of symmetries that may give rise to interesting Noether charges, the diffeomorphisms and the I(ISO(2,1)) transformations (2.5). These symmetries are not independent. On shell,
\[ \mathcal{L}_\zeta A^a = D(\zeta \cdot A^a), \]
\[ \mathcal{L}_\zeta B^a = D(\zeta \cdot B^a) + \frac{1}{2} \epsilon^{abc} B_b(\zeta \cdot A^c), \]
\[ \mathcal{L}_\zeta C^a = D(\zeta \cdot C^a) + \frac{1}{2} \epsilon^{abc} C_b(\zeta \cdot A^c), \]
\[ \mathcal{L}_\zeta E^a = D(\zeta \cdot E^a) + \frac{1}{2} \epsilon^{abc} [E_b\zeta(\cdot A^c) + C_b(\zeta \cdot B_c) + B_b(\zeta \cdot C_c)], \] (4.12)
and the transformations (4.12) are precisely the gauge transformations (2.3) with parameters
\[ \tau^a = \zeta \cdot A^a, \quad \rho^a = \zeta \cdot B^a, \quad \lambda^a = \zeta \cdot C^a, \quad \hat{\zeta}^a = \zeta \cdot E^a. \] (4.13)
The diffeomorphisms are thus equivalent to I(ISO(2,1)) gauge transformations on shell, as one expects in a topological theory.

From the Lagrangian (2.1), it is evident that
\[ \Theta = \frac{1}{2} (E^a \wedge \delta A_a + B^a \wedge \delta C_a). \] (4.14)

For the transformations (2.3), it then follows that
\[ j[g] = dQ[g], \quad Q[g] = -\frac{1}{2} (E^a \tau_a + B^a \lambda_a), \] (4.15)
on shell. The conserved charge is thus
\[ q[g] = \int_C \Theta[g] = -\frac{1}{2} \int_{\infty} \left( E^a \tau_a + B^a \lambda_a \right). \tag{4.16} \]

In particular, consider the BCEA black hole. For the integral (4.16) to exist, we must restrict ourselves to parameters with the asymptotic behavior
\[ \tau_a \sim \frac{1}{r} \hat{\tau}_a, \quad \lambda_a \sim \hat{\lambda}_a, \tag{4.17} \]
where \( \hat{\tau}_a \) and \( \hat{\lambda}_a \) are constants (or possibly functions of \( \phi \)). Then
\[ q[\hat{\tau}_a, \hat{\lambda}_a] = -\pi(\hat{\tau}_0 r_- + \hat{\lambda}_2 r_+) \left( \frac{\ell \hat{\tau}_0}{r_- + r_+} \right)^{1/2} + \pi(\hat{\lambda}_0 r_+ + \hat{\lambda}_2 r_-). \tag{4.18} \]
The charges are thus determined by the constants \( r_\pm \), as one might expect.

As a special case, we may consider transformations parametrized as in (4.13), that is, \( I(ISO(2,1)) \) transformations that correspond to diffeomorphisms. We find conserved charges
\[ q[\hat{\zeta}, \hat{\zeta}_t] = -\pi(\hat{\zeta}_0 r_- + \hat{\zeta}_2 r_+) \left( \frac{\ell \hat{\zeta}_0}{r_- + r_+} \right) + \frac{\pi \ell J}{\ell} \hat{\zeta}_t. \tag{4.19} \]

To determine the canonical energy, we must add in the term \( t \cdot G \) of Eq. (4.6), but it is not hard to check that this gives no further contribution. The charges associated with asymptotic time translations and rotations are the mass and angular momentum, as expected, but in the wrong order—the mass appears as the charge for \( \zeta_\phi \), while the angular momentum is the charge for \( \zeta_t \)!

A possible interpretation of this result is to attribute the mass and charge of Eq. (4.19) to a combination of the black hole and the “matter” fields \( B \) and \( C \). That is, we can write
\[ M_{\text{total}} = M_{bh} + M_{B-C}, \]
\[ J_{\text{total}} = J_{bh} + J_{B-C}, \tag{4.20} \]
with
\[ M_{B-C} = \frac{J_{B-C}}{\ell} = \frac{J}{\ell} - M. \tag{4.21} \]
It is perhaps unsurprising to find that the $B$ and $C$ fields carry angular momentum—they are, after all, sources of torsion in the field equations (2.2). We do not, however, have a good explanation for the extremal condition $M_{B-C} = -J_{B-C}/\ell$.

D. Thermodynamics

In the first item of Ref. [8], Wald shows that black hole entropy can also be derived as a Noether charge. Unfortunately, that derivation breaks down in the first-order formulation. However, we can still use the first law of thermodynamics to determine an entropy for the BCEA system.

To do so, we choose a constant $\Omega_H$ such that the Killing vector

$$\chi^\mu = t^\mu + \Omega_H \varphi^\mu,$$

(4.22)

vanishes at the horizon. The surface gravity $\kappa$ is then

$$\kappa^2 = -\frac{1}{2} \nabla^\mu \chi^\nu \nabla_\mu \chi_\nu,$$

(4.23)

and the first law of black hole dynamics takes the form

$$\frac{\kappa}{2\pi} \delta S = \delta E - \Omega_H \delta J.$$

(4.24)

For the $(2+1)$-dimensional black hole (1.2), in particular, it is easily checked that

$$\Omega_H = -N^\phi (r_+), \quad \kappa = \frac{r_+^2 - r_-^2}{r_+^2 \ell^2}.$$

(4.25)

The relationship to standard thermodynamics may then be established by means of the Euclidean path integral: it is shown in Refs. [1, 3] that the black hole temperature obtained from the path integral is $\beta = 2\pi/\kappa$, so (4.24) is just the first law of thermodynamics.

For the black hole in Einstein gravity, (4.10)–(4.11) gives

$$\delta E - \Omega_H \delta J = \frac{2\pi}{\ell^2} \left\{ r_+ \delta r_+ + r_- \delta r_- - \frac{r_-}{r_+} (r_+ \delta r_- + r_- \delta r_+) \right\} = 2\pi \kappa \delta r_+.$$

(4.26)

The entropy (1.24) is thus

$$S = 4\pi^2 r_+,$$

(4.27)

\[1\] In particular, the ansatz used to define Wald’s $\tilde{Q}$ fails.
agreeing with the conventional expression in our choice of units. For the BCEA black hole, on the other hand, (4.19) gives

\[
\delta \mathcal{E} - \Omega_H \delta \mathcal{J} = \frac{2\pi}{\ell^2} \left\{ r_+ \delta r_- + r_- \delta r_+ - \frac{r_-}{r_+} (r_+ \delta r_+ + r_- \delta r_-) \right\} = 2\pi \kappa \delta r_-.
\]

Hence

\[
S = 4\pi^2 r_-. \tag{4.29}
\]

Once again, the parameters of the conventional black hole have been interchanged: this time, the roles of \( r_+ \) and \( r_- \) are reversed.

5 Conclusion

We have seen that the (2+1)-dimensional black hole does not require a negative cosmological constant. Rather, the cosmological constant can be replaced by a suitable distribution of “matter.” In itself, this is perhaps not a very surprising observation, although the division of mass and angular momentum between the metric and the \( B \) and \( C \) fields is rather unexpected. But it is certainly interesting that even such a simple system of topological matter can give rise to a black hole.

One may also examine the BCEA model in second order form, using (2.3) and (2.4) to express the connection \( A^a \) in terms of the fields \( E^a, B^a, \) and \( C^a \) in the action. In this case, it turns out that the \( B \) and \( C \) fields contribute to the Noether current at spatial infinity, but not at the event horizon. The result is that, as in the first order form, the locally constructed expression for the entropy does not satisfy the first law of thermodynamics. This is probably generic in models where matter fields are coupled to the spin connection, i.e., to first derivatives of the metric.

The BCEA system is, in fact, quite powerful, and our results suggest an interesting direction to search for additional black hole solutions. As noted in section 1, the standard (2+1)-dimensional black hole can be obtained from anti-de Sitter space by a set of identifications. It is natural to ask whether this procedure can lead to black hole solutions if we begin with a different homogeneous geometry.
Homogeneous Riemannian metrics have been classified by Thurston [10], and their Wick rotated metrics provide a useful starting point. Our preliminary results are that “black-hole-like” solutions can be obtained from at least three of these geometries, $H^3$ (which gives the usual black hole), $H^2 \times E^1$, and SOL. In particular, if we let
\[ f(r) := \frac{r^2}{\ell^2} - M, \quad (5.1) \]
the metrics
\[ ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + d\phi^2, \quad (5.2) \]
(obtained from $H^2 \times E^1$ by identifications) and
\[ ds^2 = -\frac{f}{M\ell^2}dt^2 + \frac{1}{\ell^2f}dr^2 + \frac{M}{f}e^{-2t/\ell}d\phi^2, \quad (5.3) \]
(obtained from SOL) satisfy the BCEA equations of motion for suitable choices of $B$ and $C$. We do not yet understand the detailed behavior of these solutions, but we believe them to be of some interest.

ACKNOWLEDGEMENTS
One of the authors (SC) was supported in part by National Science Foundation grant PHY-93-57203 and Department of Energy grant DE-FG03-91ER40674. Two of the authors (JG and RBM) would like to acknowledge the partial support of the Natural Sciences and Engineering Research Council of Canada.

References

[1] M. Bañados, C. Teitelboim and J. Zanelli, Phys. Rev. Lett. 69, 1849 (1992).

[2] M. Bañados, M. Henneaux, C. Teitelboim, and J. Zanelli, Phys. Rev. D48, 1506 (1992).

[3] J. D. Brown, J. Creighton and R. B. Mann, Phys. Rev. D, accepted for publication, gr-qc/9405007, Un. of Waterloo preprint WATPHYS-TH-94-02.
[4] D. Cangemi, M. Leblanc and R. B. Mann, Phys. Rev. D48, 3606 (1993).

[5] R. B. Mann and S. F. Ross, Phys. Rev. D47, 3319 (1993).

[6] E. Witten, Nuc. Phys. B311, 46 (1988).

[7] S. Carlip and J. Gegenberg, Phys. Rev. D44, 424 (1991).

[8] R. M. Wald, Phys. Rev. D48, 3427 (1993); J. Lee and R. M. Wald, J. Math. Phys. 31, 725 (1990); V. Iyer and R. M. Wald, Phys. Rev. D50, 846 (1994).

[9] S. Carlip and C. Teitelboim, “Aspects of Black Hole Quantum Mechanics and Thermodynamics in 2+1 Dimensions”, gr-qc/9405070, IAS preprint IASSNS-HEP-94/34 and U.C. Davis preprint UCD-94–13 (1994).

[10] W. Thurston, The Geometry and Topology of Three Manifolds, Princeton Univ. lecture notes, 1978; P. Scott, Bull. London Math. Soc. 15, 401 (1983).