Dynamics of Generalized Hydrodynamics: Hyperbolic and Pseudohyperbolic Burgers Equations

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Abstract

The equations of continuum hydrodynamics can be derived from the Boltzmann equation, which describes rarefied gas dynamics at the kinetic level, by means of the Chapman-Enskog expansion. This expansion assumes a small Knudsen number, and as a consequence, the hydrodynamics equations are able to successfully describe sound propagation when the frequency of a sound wave is much higher than the collision frequency of the particles. When both frequencies become comparable, these equations give a poor account of the experimental measurements. A series of generalized hydrodynamic equations has been introduced in the literature along the years in order to improve the continuous description of small scale properties of fluid flow, as ultrasound propagation. We will describe herein some of the simplified models that has been proposed so far.
One of the main challenges of nonequilibrium statistical mechanics is to obtain accurate macroscopic descriptions of microscopic processes. An important example is the derivation of the hydrodynamics, Euler and Navier-Stokes, equations from the Boltzmann equation \([1, 2]\), which describes rarefied gas dynamics at the kinetic level. This program can be carried out by means of the Chapman-Enskog expansion \([3, 4]\), assuming a small Knudsen number, which is the ratio among the mean free path of the gas molecules and the macroscopic characteristic length. While the Navier-Stokes equations represent a remarkable success of the theoretical study of fluid mechanics, it is well known that the spectral properties of their solution do not agree with experimental data for short wavelengths \([5]\). Consequently, the Navier–Stokes equations are able to successfully describe sound propagation when the frequency of a sound wave is much higher than the collision frequency of the particles. When both frequencies become comparable, these equations give a poor account of the experimental measurements. In order to solve this problem, one is tempted to derive generalized hydrodynamic equations continuing the Chapman-Enskog expansion to higher orders \([6]\), to obtain the so called Burnett and supra–Burnett equations, but they have never achieved any notable success \([7]\).

Of course, studying and analyzing the Euler, Navier–Stokes, Burnett and supra–Burnett equations is rather complicated as they are composed of a large number of terms. This fact justified the introduction of several simplified models for hydrodynamics. One of them is the inviscid Burgers equation

\[
\partial_t u + u \partial_x u = 0,
\]

which describes the evolution of the pointwise fluid velocity \(u = u(x, t)\). This equation expresses the free flight of the fluid particles, and experience non–existence of the solution, in the form of finite time shock and rarefaction waves \([8]\). Actual particles do interact, and this interaction can be introduced phenomenologically into the equation for fluid motion by means of a viscosity term, to obtain the viscous Burgers equation

\[
\partial_t u + u \partial_x u = \nu \partial_x^2 u,
\]

where \(\nu\) is the fluid viscosity. This equation can be explicitly solved by means of the Hopf–Cole transformation \([8]\), and the exact formula reveals that the solution is regular for all times. However, contrarily to what happens in the inviscid case, in which perturbations propagate linearly in time, the viscous case supports infinitely fast propagation of distur-
bances. The hyperbolic modification of the Burgers equation

\[ \mu \partial_t^2 u + \partial_t u + u \partial_x u = \nu \partial_x^2 u, \tag{3} \]

where \( \mu \) is the fluid inertia, can be introduced phenomenologically in order to take into account the memory effects coming from the finite size of the temporal lapse among gas molecules collisions. One would in principle expect that such a wave form of the hyperbolic Burgers equation is able to support finite propagation of disturbances, a fact that can also be proven rigourously [9].

Beyond phenomenology, it is possible to find more fundamental justifications of the wave form of the hyperbolic Burgers equation. One possibility was suggested by Rosenau [10], who found that the telegraphers equation (which is actually the linearization about \( u = 0 \) of the hyperbolic Burgers equation)

\[ \mu \partial_t^2 u + \partial_t u = \nu \partial_x^2 u, \tag{4} \]

reproduces the spectrum of its microscopic counterpart, the persistent random walk, almost exactly. Using this fact, he claimed that the Chapman-Enskog expansion should be substituted by a different expansion keeping space and time on equal footing. This procedure would preserve the hyperbolic nature of the resulting equations, and thus the nice spectral properties of the solution, at least in the linear regime [10]. An expansion of this type was carried out by Khonkin [11], who found new equations for the momentum and energy fluxes which, in contrast to the classical Navier-Stokes and Fourier laws, depend on the first time derivative of these fluxes. This dependence implies in turn the appearance of a term proportional to the second order time derivative of the velocity, among others, in the corresponding modified Navier-Stokes equations, which become now hyperbolic. However, it was already argued by Rosenau that hyperbolicity in union with nonlinear hydrodynamical evolution might result in the non-existence of the solution. A similar idea was proposed later [12], where hyperbolicity was introduced to take into account memory effects in the hydrodynamic description of the flow, and to get rid of the infinite speed of signal propagation. In order to understand the interplay between hyperbolicity and nonlinear convection, the hyperbolic Burgers equation was studied in [12] by means of linear and numerical analyses. One of the conclusions of this work is that this equation has blowing up solutions under certain circumstances. Note also that, apart from its use as a toy model for generalized
hydrodynamics, the hyperbolic Burgers equation has been employed as mathematical model of traffic flow [13, 14].

The hyperbolic Burgers equation has been rigorously analyzed a number of times. In [15], the global existence in time of the solution was proved for small enough initial conditions, together with the convergence of this solution to the corresponding one of the viscous Burgers equation in the limit \( \mu \to 0 \). The non-existence of a global in time solution of a certain class of nonlinear wave equations was proven in [16]; the hyperbolic Burgers equation can be shown to belong to this class. The temporal asymptotic behavior of the solution to the hyperbolic Burgers equation was proved in [17] to be the same as the one of the viscous Burgers equation, in those cases in which this solution is global in time. The shock wave dynamics of initial discontinuous profiles was studied in [14, 18] by means of singular surface theory and numerical approximations. In [9] we proved the finite time blow-up of the solution to the hyperbolic Burgers equation provided its initial conditions where large enough. We showed that for regular, large enough and compactly supported initial conditions the solution \( u \) to the hyperbolic Burgers equation obeys

\[
\lim_{t \to t_b} \|u\|_{L^\infty(R)} = \infty,
\]

for some finite \( t_b \). We have proven additionally some other properties of the solution to the hyperbolic Burgers equation. It cannot develop any other form of non-existence of the solution, apart from blow-up, provided the initial conditions are regular enough. This prohibits in particular the finite time formation of shock or acceleration waves out of regular initial data. In more general terms we can say that we have proven that the solution is regular and compactly supported as long as it is bounded. See [9] for details of the proofs.

One sees that after introducing inertia in the viscous Burgers equation to get the hyperbolic Burgers equation one recovers both the desirable finite speed of propagation of disturbances and the undesirable finite time non-existence of the solution. To prevent the singularities a higher order viscosity term was introduced in [12], and so the pseudohyperbolic Burgers equation

\[
\mu \partial_t^2 u + \partial_t u + u \partial_x u = \nu \partial_x^2 u + \lambda \partial_x^2 \partial_t u,
\]

was found. Note that herein we have assumed a constant hyperviscosity \( \lambda > 0 \), contrary to the \( x \)-dependent one in [12], which sign was not defined either (in this last case the equation becomes linearly ill-posed). Contrary to the conjecture raised in [12], our preliminary
numerical analyses\cite{9} have shown that the solution to the pseudohyperbolic Burgers equations, which propagates perturbations infinitely fast as one could na"{i}vely expect, also blows up in finite time provided the initial conditions are large enough. In this case, however, blow-up requires much larger initial conditions than in the case of the hyperbolic Burgers equation. In this sense we can say that hyperviscosity implies a partial regularization of the solution, but it still unable to unconditionally prevent blow-up.

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