On the Interpretation Of the Local Dark Matter

H. L. Helfer

Department of Physics and Astronomy; Laboratory for Laser Energetics
University of Rochester, Rochester NY 14627
lary@astro.pas.rochester.edu

ABSTRACT

The cause of the extended rotation curves of galaxies is investigated. It is shown that conventional sources and most exotic sources for the needed gravitational fields are implausible. We suggest spatial fluctuations in a scalar field, similar to the inflation field, are responsible for the gravitational fields. These fluctuations play the role of ‘dark’ halos around galaxies. They take $\sim 10^5$ yrs to develop and could not have been important in the early days of the universe. When galaxies are clustered, a $\Lambda$ term appears naturally in this theory. The universe’s present energy density associated with these scalar field variations is $\Omega \sim 1/2 - 2/3$. A possible scenario is suggested in which the cosmic scale factor $R(t)$ would have experienced a recent acceleration. A discussion of further observations and theoretical work needed to resolve some ambiguities in the theory is given.

Subject headings: galactic rotation, dark matter, inflation, dark halo, quintessence

1. Introduction

This paper speculates that the local dark matter found in halos of galaxies may actually be fluctuations in a scalar field. The general distribution of dark matter has been summarized by Trimble (1985); Ostriker (1993). By ‘local dark matter’ (LDM) we refer to the unexplained source of the extended rotation curves, $V_{rot}(r) \text{ vs. } r$, of the Milky Way galaxy (Fich & Tremaine 1991) and other galaxies. This definition means that the LDM may not be the sole component of the dark matter considered in cosmology. But we deal with this restricted class because the Sun is on the ‘flat’ portion of the Milky Way’s rotation curve and we can calculate the LDM’s density in our vicinity; from this one can derive severe constraints upon the halo’s possible composition.

The rotation curves of dwarf, elliptical, and spiral galaxies have been reviewed by Casertano & van Gorkom (1991), de Zeeuw & Franx (1991), and Sofue & Rubin (2001). While a correlation is seen between the shapes of the rotation curves and the surface brightness and morphology of galaxies, modeling them requires a mixture of baryonic matter and LDM for which the effective density falls off as $r^{-2}$ at large $r$.

So, rotation curves are generally observed to be composite with the LDM contribution dominating the rotation curves of normal spiral galaxies outside their central core regions; see Rubin (1980, 1982, 1985). They may be schematized by a linear rise of the rotation velocity $V$ to a value $V_h$ at a galactocentric distance $r = R_h$, followed by a constant rotation velocity ($\pm \sim 10 \text{ km s}^{-1}$) for $R_h < r < R_0$; here $R_0$ is set only by observational limits of finding radiant baryonic material. About 74% of field spirals show these simple rotation curves (Sofue & Rubin 2001).

We will ignore those galaxies in which large baryonic contributions to the total mass make it difficult to extract the LDM’s effect on the rotation curve. These include: galaxies with peculiar shaped rotation curves in $r \leq R_h$ (Sofue et al 1998); compact bright galaxies which may have declining velocity curves for $r > \sim 2 - 3R_h$ (Casertano & van Gorkom 1991); and low surface brightness galaxies for which $V_h$ is small < 100 km s$^{-1}$, and there is clear evidence of large
amounts of baryonic material at $r \geq 20$ kpc, (Pickering et al. 1997; Quillen & Pickering 1997; McGaugh, Rubin, & Blok 2001); we regard these as nascent galaxies.

Observations of the rotation curve of the Milky Way galaxy (Clemens 1985; Merrifield 1992) show it is composite, with this typical underlying LDM distribution. For our galaxy, adopting a solar galactocentric distance of $R_0 \approx 7.8$ kpc (Fich & Tremaine 1991), one has $R_h \approx 5$ kpc and $V_R \approx 200 - 210$ km s$^{-1}$. The limiting value $R_l$ for which $V = V_R$ is uncertain; it is generally assumed to be at least 25 kpc and is more likely to be at least 50–60 kpc and may well be $>\sim 200$ kpc (Fich & Tremaine 1991; Kuijken & Lynden-Bell 1992). In some spirals, one actually observes $R_l > 60$ kpc. If the LDM is the same as the dark halo material used in galaxy-galaxy gravitational lens models, then $R_l > 200$ kpc is appropriate (Guzik & Seljak 2002). We shall adopt $R_l > 10 R_h$ in calculations.

For other spiral galaxies, generally $R_h$ has about the same value. However while $V_R$ often is $\approx 200$ km s$^{-1}$, values in the range $\sim 100$ to 250 km s$^{-1}$ do occur. There also is a smaller class of galaxies of interest in which the rise is more gradual, $R_h \approx 10$ kpc, with the rise from $r = 0$ to $r = R_h$ better represented by a convex curve, see examples in Persic & Salucci (1995); Generally these have not been observed much beyond $R_h$.

The Milky Way schematized velocity curve, assuming spherical symmetry, gives for $r \leq R_h$, $\langle \rho \rangle = 6.4 \times 10^{-24}$ g cm$^{-3} = 0.086$ $M_\odot$ pc$^{-3} = 4.0$ hydrogen atoms cm$^{-3}$. For $r \geq R_h$ one has $\rho_{LDM} = \frac{1}{2} \rho \left( R_h/r \right)^2$. At the Sun’s galactocentric distance, $R_0$, one has $\rho_0 = 0.87 \times 10^{-24}$ g cm$^{-3} = 0.012$ $M_\odot$ pc$^{-3} = 0.54$ hydrogen atoms cm$^{-3}$. Masses of interest are: $M(R_h) = 4.5 \times 10^{10} M_\odot$; $M(r \geq R_h) = (r/R_0) M(R_h)$; and $M(R_0) = 7.0 \times 10^{10} M_\odot = M_G$. For $r < R_h$ the calculated mean density may not be too useful for representing the LDM because only a contribution for the central spheroid bulge has been subtracted and an uncertain stellar disk contribution has been ignored. In this region, however, the density of the LDM must fall appreciably below the $r^{-2}$ law (for otherwise the horizontal portion of the velocity curve would continue inward.) The value of $R_h$ depends somewhat upon this uncertain baryonic disk contribution to the rotation curve for $r \leq R_0$ (Kuijken & Gilmore 1989a;b;c). However, for $r > R_0$, the calculated masses and densities are dominated by the LDM contributions (e.g. Merrifield (1992)) and using these calculated densities and masses for representing the LDM cannot overestimate the true values by more than a factor $\sim 2$ if the rotation velocity is to have the observed value and lack of $r$–dependence. Similarly the value of $R_h$ may be somewhat underestimated.

In our vicinity where $\rho \propto r^{-2}$, the LDM column height, is $\int_0^\infty n \, dz = \pi R_0 n_0/2 = 2.0 \times 10^{22}$ hydrogen atoms cm$^{-2} = 145 M_\odot$ pc$^{-2}$; up to 1 kpc the projected surface density is $11.8 M_\odot$ pc$^{-2}$. If the LDM were hot gas, the emission measure would be $\int_0^\infty n^2 \, dz = \pi R_0 n_0^2/4 = 1.8 \times 10^3$ cm$^{-6}$ pc.

In the next section, we show that no population of observed astronomical material can play the role of the LDM. We also show that proposed exotic sources, massive neutrinos, primordial mini-black holes, black fluids, etc. are highly unlikely to be the LDM.

In the third section, a scalar field source term is examined which produces gravitational accelerations similar to that observed. The fourth section contains proposals for observational tests and a discussion of some uncertainties in the theory that still need to be resolved.

2. Unacceptable LDM Sources

2.1. No Interstellar Constituents

The observed LDM mass density for $r = R_0$ is very much higher than the observed energy densities of magnetic field, cosmic rays, and radiant energy; these cannot be primary constituents of the LDM. The LDM cannot have a significant component of ordinary interstellar gas, for the column densities needed are much higher than those observed; see pg 525 of Allen’s Astrophysical Quantities (Cox 1999), hereafter referred to as AQ. A very high temperature medium is excluded by the low emission measures observed in the x-ray region (Marshall & Clark 1984; McCammon & Sanders 1990). One cannot make the LDM out of, say $5 - 10 M_G$ of He gas, without providing an explanation as to where the missing $\sim 15 - 30 M_G$ of H, expected by observed abundances, disappeared.

Nor can the LDM be explained by large dust particles. By weight, dust is comprised of ‘metals’ with
universal abundance (by weight) \( Z \approx 0.02 \). For every 1 \( M_\odot \) of LDM attributed to dust one must account for a missing 50 \( M_\odot \) of H & He. Consequently all particulate matter up to the size of Uranus and Neptune can be excluded, as well as very small compact molecular clouds (Alves, Lada, & Lada 2001).

2.2. No Stellar Constituents

The mass distribution in the solar vicinity is given by Table 19.9 of \( AQ \). The only low luminosity stars that could contribute significantly to the LDM are some low mass main sequence stars and white dwarfs/neutron stars. Looking at edge-on galaxies, the surface brightness for any class of stars with a constant \( L/M \) should fall off as \( 1/r \); this is not observed implying only very low luminosity stars could be a main LDM constituent. Trimble (\( AQ \), p.530) gives as an lower limit for our galaxy, \( M/L \sim 18 \) to \( r = 35 \) kpc; this corresponds to stars with \( M \sim 0.1 M_\odot \) (\( AQ \), p.487). But the integrated stellar mass function up to \( M = 0.1 - 0.2 M_\odot \) is only 22% – 35% of the total main sequence stellar mass function (\( AQ \), pg 488). Therefore, if the LDM of \( \sim 4 M_\odot \) seen up to \( r = 35 \) kpc is due to such low mass stars, one must account for a missing 12 – 20 \( M_\odot \) of main sequence stars associated with them by current ideas of star formation. Similarly, white dwarfs and neutron stars are excluded by the same type of argument. In their formation at least a comparable mass of material is expelled into the interstellar medium and it is not observed.

Now, these arguments cannot exclude the possibility that a very old population of stars was formed (or captured) with an initial mass function favoring objects of \( 1 - 100 M_{\text{Jupiter}} \) or that populations of small black holes were somehow produced \textit{ab initio}. Normally, such proposals are not detailed enough to discuss either the lack of expected associated matter or the type of physics needed to constrain the mass range. It is also possible that massive mesons or some other exotic particles are present. These possibilities can be restricted by the more general physical arguments, given in the next subsections.

2.3. ‘Dark Fluid’ Hydrodynamics

Suppose the energy-momentum tensor of the LDM can be represented by a simple fluid with scalar pressure, \( P = P(\rho) \). First, assume hydrostatic equilibrium. Then, for the outer region \( r > R_h \), using \( \rho \propto r^{-2} \), one can solve for an equation of state. One finds \( P = \frac{1}{2} V^2 \rho + P_0 \) where \( P_0 \propto \rho V^2/2c^2 \equiv \text{constant} \), so that for \( r > R_h \) we have an isothermal sphere solution. For finite \( \rho \) at \( r = 0 \), the isothermal Bonnor-Ebert solutions, Ebert (1955); Bonnor (1956), discussed by Alves, Lada, & Lada (2001), are appropriate. They have too small a range in which \( \rho \propto r^{-2} \). Also, for our thermal velocity, \( \sim V_h \), they are Jeans unstable for \( R_L > \sim 20 \) kpc and hence not acceptable. So, hydrostatic equilibrium with a simple equation of state, is unlikely. But, this argument would not necessarily exclude a gas in a state of partial ionization.

Non-hydrostatic equilibrium models can be rejected. Postulating centrifugally supported ellipsoidal LDM matter runs into another serious problem. Suppose \( V^2/g = (1 - \xi)V_h^2/r \) for sufficiently large \( r > R_h \); then the effective thermal velocity is reduced by a factor \( \sqrt{\xi} \). However, one would then have to explain why the total LDM angular momentum increases \( \propto r^2 \).

Turbulent support by means of a term \( \nabla \cdot \mathbf{v}_t \) is unlikely. Mass conservation requires \( 4 \pi R^2 \rho \langle v_r \rangle \ll 1 M_\odot /10^{10} \) yrs; this gives as an upper limit \( \langle v_r \rangle \sim 1 \) km s\(^{-1} \). It would be difficult to maintain this low average radial velocity over the entire circumference at \( r = R_0 \) unless \( \langle v_r \rangle \) is small, say \( \sim 5 \) km s\(^{-1} \). Using \( \langle v_r^2 \rangle^{1/2} \) as representative turbulent velocity, one would require a characteristic turbulent length scale \( < \sim 5 \) pc to balance the gravitational acceleration \( V^2_{0}/R_0 \). Again, a mechanism for providing this small scale would be required.

Since \( \rho \propto r^{-2} \) is observed, mass conservation would limit outward (or inward) streaming of massive particles to \( \langle v_r \rangle \sim 1 \) km/s\(^{-1} \). This is an unlikely restriction on any hypothesized relativistic particles.

2.4. Galaxy-Galaxy Collisions

Galaxy-galaxy collisions involving halos pose difficulties for models of the LDM involving particles not interacting with ordinary matter nor ca-
pable of radiating. If the particles undergo collisions among themselves, then the halos should relax \textit{adiabatically} after a galaxy-galaxy collision, not isothermally (as is observed). If particle-particle collisions are unimportant (as \textit{e.g.} mini-black holes), then the particles’ average velocity must be of order \( V_h \) or less for them to remain bound to a galaxy. Then if two galaxies with halos collide, one knows that the particles’ trajectories would be modified by the non-spherically symmetric gravitational potential present during the long galaxy collision. After the galaxies separate, there would be no restoring force to re-establish sphericity for the halos, and the time evolution of the particles’ distribution function would be governed by Liouville’s equation.

This is a good argument which is not yet conclusive because we do not have observational studies of halo asymmetries at large \( r \). In principle it is possible to study the aftermaths of halo collisions. About 5\% of the galaxies are in rich clusters of galaxies (see Bahcall \textit{AQ}, p.613) and in the core regions, \( \sim 1.5h_{-1} \) Mpc radius, the average galaxy halo with \( R_h > 50 \) kpc experiences at least one halo-halo collision in \( 10^{10} \) yrs with a short collision time < \( 10^9 \) yrs, the collision frequency being \( \propto R_h^2 \). Similarly, In galaxy groups and poor clusters, which contain \( \sim 50\% \) of the galaxies, the average halo with radius \( R_h > 250 \) kpc experiences collisions in \( 10^{10} \) yrs. In some groupings, of course such as the Milky Way-M31 Local System even smaller halos should show severe distortion if composed of particles. Field galaxies should generally keep undistorted their primitive halo structure.

Assuming halos show approximate spherical symmetry out to \( \sim 50 \) kpc, we need conclude that any exotic LDM candidate must be able to radiate efficiently during galaxy-galaxy collisions, either at presently undetectable wavelengths (\( \nu < 10 \) MHz) or in very low energy particle emission (\textit{e.g.} pairs of neutrinos) to avoid adiabatic relaxation in collisions, and that, if considered as a fluid, the equation of state cannot be reduced to a simple \( P = \rho \) relation if gravitational instability is to be avoided.

3. The Scalar Field

As an alternate to specifying radiative properties and thermodynamics of unknown particles, we suggest that a class of fluctuations in a scalar field \( \phi \) form gravitational potential wells into which baryonic matter may flow, forming luminous galaxies in their center regions. These potential wells have gravitational mass and are the dark halos. There is no reason to assume all halos have luminous galaxies associated with them, or that all luminous galaxies are associated with these dark halo potential wells.

The halo potential wells will be specified by the energy momentum tensor \( T_{ab} \) determined by \( \delta(\sqrt{-g}\mathcal{L}) = -\sqrt{-g}(T_{ab}/2)\delta g^{ab} \) once a Lagrangian \( \mathcal{L}(\phi) \) is chosen. The associated field equation is given by \( \delta \mathcal{L}/\delta \phi = 0 \). We use as a Lagrangian \( \mathcal{L} = -\frac{1}{2} \alpha^2 L_\phi^2 + \Lambda \) where \( L_{ij} = \phi_i \phi_j + m_i m_j [\phi^2 (1 - \frac{\alpha^2}{2} \phi^2) - \lambda] \). Here \( m_i \) is an assigned timelike vector, \( m^a m_a = m^2 \), and the coupling constant is the dimensionless \( \alpha^2 \), not the usual \( \kappa = 8\pi G c^{-2} \) used for fluids. If one substitutes \( \Psi = \alpha \phi \), and \( \Lambda = \lambda = 0 \), then \( \mathcal{L} \) is the Ginzberg & Landau (1950) Lagrangian used in studying phase transitions and the Higgs field and introduced into astronomy for the inflation field by Guth (1981). The parameters \( \Lambda \) & \( \lambda \) do not affect the behavior of \( \phi \); they are introduced to allow us to shape the extent of each individual dark matter halo. The energy-momentum tensor then is \( T_{ij} = \alpha^2 (L_{ij} - \frac{1}{2} g_{ij} L_\phi^2) + g_{ij} \Lambda \). The form of the wave equation in flat space is:

\[
\partial^2_{tt} \phi - \nabla^2 \phi - m^2 \phi (1 - \phi^2) = 0.
\] (1)

Since \( T_{ii} \propto \phi^2 \), the requirement that a field theory has an effective mass density falling off as \( r^{-2} \) at large \( r \) and finite at the origin, basically forces the use of form of the wave equation in flat space to be that shown, restricting permissible forms of \( \mathcal{L} \). The wave operator gives the \( r^{-2} \) behavior for the effective density at large distances; the sign of the 'mass' term \( m^a m_a \) is opposite that used in the Klein-Gordon equation because we need solutions finite at the origin; and a non-linear term must be introduced in the potential term to limit growth. We use a scalar field because it is simple and it is conceptually economical to see if a descendant of the inflation field can play a contemporary role.

3.1. Approximate Halo Solutions

We summarize the discussion of the flat space solutions given in the Appendix. We assume that
the field always has a background of infinitesimal fluctuations, similar to those seen in the CMB, and consider only solutions which could have grown from these fluctuations \( \phi(r, t = 0) \sim 0 \) and remain finite. The nonlinear cubic term has two significant functions: (1) it limits the growth of time-dependent solutions; and (2), it may provide a strong interaction between two sets of fluctuations. Aside from these, the term has only a modest effect on the analytic form of the two classes of solutions found.

The first class of solutions are the large wavenumber modes, \( k^2 > m^2 \). They represent traveling waves, \( t\)-waves, for short, whose frequencies are amplitude dependent. \[ \text{[Wave packets of the } t\text{-waves are problematical; they would act as tachyons since the group velocity } d\omega/dk = k/\omega \text{ is greater than that of light.]} \] These \( t\)-waves are stable, non-localizable, and remain infinitesimal for \( t > 0 \); for this reason they will be mostly ignored in this paper.

The second class of solutions, composed of modes with \( k^2 < m^2 \), do not ‘travel’ and can grow to finite amplitude. With appropriate boundary conditions, they exhibit anharmonic periodic motion, first growing exponentially fast from a small initial value until \( \phi^2 = \langle \phi^2 \rangle \leq 1 \); then deceleration takes place and the solutions reach their maximum amplitudes and then decrease once more to their initial amplitudes. The rise time from the fluctuation level seen in the cosmic microwave background, to the maximum value is \( \sim 1 - 5 \times 10^9 \) yrs for \( a^2 \sim \frac{1}{4} - 1 \), and \( m^{-1} \sim 3 \) kpc \( \sim 10^4 \) yrs (see below).

They could not have arisen early in the history of the universe. They are standing waves (or \( s\)-waves.)

We suggest these represent galactic halos. Because their time derivatives make relatively small contributions to the energy-momentum tensor, (see section A.3) time variations for the \( s\)-waves can usually be neglected and the steady state solution, \( k^2 \approx m^2 \) is our model for a typical galactic halo.

The approximate steady state solution \(^1\) is, using \( m_r = m_g = m_\phi = 0 \) and for \( ct \) in formulation. We regard \( a^2 \) as small, and ignore the small \( (\partial_r \phi)^2 \) terms and terms of order \( \sim V^2_B/c^4 \). The time dependent \( s\)-wave halo solutions are of the approximate form:

\[
\phi = [1 + h^2 \sin \Theta(h^2, r)]^\frac{1}{2} \phi_s \quad \text{where } 0 \leq h^2(r) < 1 \quad \text{and } \Theta
\]

\[ \phi_s \approx a \frac{\sin m_r r}{mr} \approx a \quad \text{for } \hat{m}r < \pi/2; \]  

\[ \phi_s \approx a \frac{\sin(m(r - a^2 \pi/4))}{mr} \approx \frac{\sin(mr)}{mr} \quad \text{for } \hat{m}r > \pi. \]

Here, \( a \) is an arbitrary amplitude \( a^2 \leq 1 \).

One may use the steady state solution as representative for evaluating the energy momentum tensor for \( s\)-waves, using the approximations of equations (2) & (3). For the radically symmetric interior Schwarzschild metric, \( dr^2 = B(r)dt^2 - A(r)dr^2 - r^2d\Omega^2 \), standard relations (see, e.g., Weinberg (1972)) give:

\[
rA^{-1} = r - \int [\kappa \rho_0 + \Lambda] r^2 \, dr
\]

\[
\ln AB = \int [2\kappa \rho_0] r \, dr + \text{constant};
\]

\[
f \equiv -\Gamma_{tt} = -(2A)^{-1} \partial_t B. \tag{4}
\]

where

\[
\kappa \rho_0 = \frac{1}{2} \alpha^2 [(m^2 \phi^2 + (\partial_r \phi)^2) - \lambda m^2], \tag{5}
\]

and \( \mathcal{C}^2 f \) is the radial gravitational acceleration a non-relativistic particle experiences because of this \( s\)-wave solution.

For the case \( \Lambda = \lambda = 0 \),

\[
r(1 - A^{-1}) = \frac{1}{2} \alpha^2 \mathcal{C}^2 r [1 - \sin^2 \frac{mr}{(mr)}] \tag{6}
\]

\[
f = -\alpha^2 \mathcal{C}^2 (2r)^{-1} [1 - \sin 2mr/(2mr)] \equiv -(\mathcal{C}^2/M_{\text{halo}}/r^2). \tag{7}
\]

For spherically symmetric metrics, there are two different definitions of the mass if \( AB \neq \text{constant} \); the first is the volume integral of the energy density (found in the solution for \( A \)) and the second is the effective mass determining the differential acceleration of a test body, \( f \) (found in the solution for \( \partial_t B \)). We give mass formulae according to the second definition. Outside a cluster, where \( AB = 1 \), the mass definitions are equivalent.

The parameters of the theory are easily determined. The rotation velocity, \(- (V(r)/c)^2 / r \) is determined by \( f \). Since \( R_{\text{h}} \approx 5 \) kpcs for the

\[ \text{is an elliptic integral (See the Appendix).} \]

\(^1\)We use \( m_r = m_g = m_\phi = 0 \) and for \( ct \) in formulation. We regard \( a^2 \) as small, and ignore the small \( (\partial_r \phi)^2 \) terms and terms of order \( \sim V^2_B/c^4 \). The time dependent \( s\)-wave halo solutions are of the approximate form:
Milky Way, one determines $m^{-1} = 3.2$ pc by setting $2mR_h = \pi$. Similarly, one finds $a_{\text{int}}^2 \sim 2(\ln(\lambda_0) + 1)^{-1}$ rather than on $m$ there.

3.1.1. Isolated Physical Halos

The infinite halo solution shown above is for one halo in an otherwise empty universe. It needs modification because exceedingly small values of $\phi$ should not significantly contribute to the energy momentum tensor. There is no obvious physical interpretation for such contributions when they produce densities much less than mean baryonic densities in the neighboring intergalactic medium, or much less than the critical cosmological density $\rho_c$. In studying halo structures, we may suppress these low background density contributions by introducing a reference energy level $\lambda = \lambda_0 > 0$, or a related limiting value $r = R_t$, by imposing other physics. One introduces a ‘cut-off’ for the halo $T_{sb}$ by choosing $\lambda$ to require that at $r = R_t \equiv (a/m)\lambda_0^{-1/2}$ one has $\kappa\rho_0 = 0$ and $d\ln(AB)/d\ln r = 0$. Then, we may join an exterior Schwarzschild solution for $r > R_t$ to the interior halo solution, in effect regarding $R_t$ as the end of the physical halo. Suppose $mr \gg 1$. The halo mass interior to $R_t$ is

$$ (G/c^2)M(R_t) = (1/6)a^2a^2R_t, \quad (8) $$

The gravitational acceleration for the interior region $r \leq R_t$ is then given by

$$ f = a^2a^2(2r)^{-1}[1-\sin 2mr/(2mr)]+r\alpha_0a^2m^2/3. \quad (9) $$

So that $f \to -(G/c^2)M(R_t)/R_t^2$ as $r \to R_t$; for $r \geq R_t$, one has $f = -a^2a^2R_t/6r^2$. This represents an isolated physical halo appropriate for field galaxies. Since the quantity $f$ is an observable, one should see that the rotational velocity does decrease in the outer parts of halos, $V(r) = V_0[1 - (r/R_t)^2]^{1/2}$ for $r \leq R_t$; again observations could determine $R_t$. [Note that we are explicitly ignoring $1/r^2$ contributions to $f$ from any more distant ‘point’ sources.]

For this case, when a halo is limited by the density of surrounding baryonic matter, a very crude estimate for $R_t$ can be made. Suppose we consider a halo galaxy surrounded by baryonic dwarf galaxies, each of mass $10^6 - 10^7 M_\odot$ with a mean spacing $s = 120 - 250$ kpc. [We used $\Omega_{\text{dwarf}} = \Omega_{\text{baryon}}/3 \approx 0.01$ Bristow & Phillips (1994), a cosmological critical density $\rho_c = 2.0h^2 \times 10^{-29}$, and $h^2 = 0.5$ in making these estimates.] For the Milky Way-like galaxies the halo density becomes comparable to the dwarf galaxies’ mean baryonic density when $R_t \sim s$. Since dwarf galaxies tend to cluster around large galaxies, this seems to be a reasonable order-of-magnitude guess for $R_t$ when intergalactic baryonic matter is present. This gives $M(s) = 5 \times 10^7 M_\odot$, values which are $1/3$ of that given by equation (7) (when $\lambda = 0$). Otherwise, an upper limit $R_t \sim 1.6 h^{-1}$ Mpc for Milky Way-like field galaxies results when the halo density becomes comparable to $\rho_c$. The uncertainty in specifying $\lambda$ (or $R_t$) does not affect the value of $f$ in the inner parts of halos.

3.1.2. Clustered Physical Halos

These formulae hold only when other halos are not close-by, i.e. $R_t \ll R_0$ where $R_0$ is half the mean spacing between halos (galaxies); we refer to this situation as case I solutions. For field galaxies $R_0 \sim 2.2$ Mps. In crowded regions such as clusters, one has another situation (case II solutions) where $R_t \geq R_0$; the halo boundary must be re-examined because the $\phi$–field may make significant contributions to the energy density there. We find it necessary to introduce a $\Lambda \neq 0$ term in order to define reference background levels appreciably greater than zero.

In the next section we note that a solution for $\phi$ can be broken up into many effectively independent parts if their centers are far from one another. Assuming this, one can consider that $\phi$ can be represented by cluster of similar dark halos with mean spacing between halo centers $r = 2R_0 \gg m^{-1}$. Represent each halo solution by a cell with a central ‘bump’ on top of a ‘plateau’, the plateau of one cell joining smoothly onto the plateaux of the adjacent cells. In the weak gravi-
tional approximation we can treat each cell separately. Consider two adjacent halos with central amplitudes $a_1, a_2$. Because we are limited by our spherical symmetry assumption, look only at the forces and densities along the lines connecting their centers. Then, at the join point between cells, $r = r_1$, the individual halo densities $\kappa_{\text{halo}} \equiv (a^2/2)[a_1^2 r_1^{-2} - \lambda m^2] + \Lambda$ match, providing $a_1^2 r_1^{-2} = a_2^2 r_2^{-2}$, where $r_1 + r_2 \equiv 2 \hat R_0$ (and $mr_1, mr_2 \gg 1$). We can then regard this location as defining the edges of the halo bump’s interior Schwarzschild solution, by forcing $d \ln AB/dr = 0$ there with the choice $\lambda m^2 = a_2^2 r_2^{-2}$; an exterior solution holds for $r > r_1$. The underlying background density (our ‘plateau’) will be represented by the $\Lambda$ term; because of the local spherical symmetry assumption $\Lambda$ is constant in a cell. If now we choose as the uniform background density $\rho_{\text{halo}} = a^2 \lambda m^2/2$ we can restore a halo’s interior density $\rho_{\text{halo}}$ to be that of the $\phi$–field, its physical value. One has:

$$f = -\alpha^2[a_2^2/2r - \lambda m^2/3] + \Lambda r/3 \to 0 \quad (10)$$

as $r \to r_1$. For $r_1 = r_2 = \hat R_0$ one finds each halo bump has a mass $(G/\alpha^2)M_0 = (1/6)a^2 \alpha^2 \hat R_0$, superimposed on the mean background density level $\rho_{\Lambda} = \frac{1}{3}\langle \rho_{\text{halo}} \rangle \equiv (3/8 \pi) M(\hat R_0)/\hat R_0^3$.

In each cell $\Lambda r/3$ acts like a differential tidal force; the $\Lambda$–term causes the entire cluster to experience an expansion force. Our choice for $\Lambda$ requires that the total acceleration on a test mass vanish at cell interfaces.

For example, suppose we have 125 galaxies like the Milky Way, with $\langle a^2 \rangle \alpha^2 = 8.8 \times 10^{-7}$, in a cluster with radius $R_{cl} = 1.5 \, h^{-1} \, \text{Mpc}$. [We ignore the possibility of dwarf baryonic galaxies in the cluster limiting the value of $R_{cl}$] Then $\hat R_0 = 300 h^{-1} \, \text{kpc}$. The typical galaxy halo mass is $\sim 13 h^{-1} \, M_G$ and the total mass of the cluster dark matter is $\sim 1.7 \times 10^{14} h^{-1} \, M_\odot$, of which one-third is due to the background $\Lambda$–term. This is in the range of rich cluster mass estimates given by N. Balcac (in AQ). The agreement is surprisingly good since we have ignored halo-halo collisions, resulting from the galaxies’ appreciable velocities, and ignored density structure within the cluster. For $N$ galaxies in a cluster, the total cluster mass is $\propto N^{2/3}$ so that this estimate adequately represents poor clusters as well if we use e.g. $N = 8$.

Finally, a similar argument must limit the range of applicability of the exterior Schwarzschild solutions in case I models; the total force on a test mass must vanish when $r = \hat R_0$, the mean spacing between galaxies. We must introduce a $\Lambda$ term. Then, for $\hat R_0 \geq r \geq R_1$, the force is $f = -a^2 \alpha^2 R_1 (6r^2)^{-1} - \Lambda r/3$ where $\Lambda = \frac{1}{2}a^2 \alpha^2 R_1 \hat R_0^3$.

In effect the distinction between case I or Case II is whether or not the halo is limited by background baryon density or $\phi$–field density (or by force-balance).

In actual cases, such as treating the Local Group of galaxies, the convenience of assuming spherical symmetry should be replaced by the weak field approximations for the metric tensor, requiring continuity of the metric and its first derivatives across interfaces. In this case, the equations of the interfaces between halos will be more complicated and $\Lambda$ and $\lambda$ will be functions of position. A particular halo, such as that of the Milky Way may combine elements of both cases I & II. The rotation curve for the Milky Way should be different in the directions toward and away from Andromeda at large $r$ because the rotation curve (determined by $f$) is $\Lambda$–dependent.

The requirement $\Lambda > 0$ is not dictated by the physics of the $\phi$–Lagrangian but is required to meet boundry conditions we imposed by first subtracting off surrounding material. As used, it is a representation of Mach’s principle since it explicitly appears as a ‘background’ mean density induced by the proximity of other halos.

### 3.2. Close Interactions

Because the theory is nonlinear, wave interactions between close galactic halos are much more complicated than in simple linear potential theory. But, some simple features are easy to see. Suppose one considers a case in which one wants to represent $\phi$ as the sum of two components, $\phi = \eta(r, t) + \xi(r, t)$. For example, one may represent the established LDM field of one galaxy and the other the influence of that of another passing galaxy. The time of interaction between colliding galaxies is $\sim 10^9 \, \text{yr}$, much longer than the characteristic scalar field vibration time $\sim 10^4 - 10^5 \, \text{yrs}$. The coupling between the fields induced by the cubic terms may be studied by writing equation (1) in two parts:
\[ [\partial_{tt} - \nabla^2] \eta = m^2 \eta (1 - 3\xi^2) - m^2 \eta^3, \quad (11) \]
\[ [\partial_{tt} - \nabla^2] \xi = m^2 \xi (1 - 3\eta^2) - m^2 \xi^3. \quad (12) \]

If $3\xi^2, 3\eta^2 \ll 1$, the two components act as independent s-wave solutions, or independent halos, not affecting each other except through gravitational interaction (see the Appendix). If the inequalities do not hold, then the mixed cubic terms cause $\xi$ and $\eta$ to act like Mathieu functions, permitting resonant interactions to develop during the long collision time. We shall use a crude model. Suppose we can time average some terms, $\xi^2 \eta \to \langle \xi^2 \rangle \eta$, $\eta^2 \xi \to \langle \eta^2 \rangle \xi$, regarding these averages to be very slowly varying functions of position. Then, e.g. a quasi-static solution for $\xi$ is given by using $m_{eff}^2 = m^2 (1 - 3\langle \eta^2 \rangle)$. Consequently, if $\langle \eta^2 \rangle > 1/3$, a steady state solution for $\xi$ is not allowed; the original dark halo of this galaxy will be modified and may not recover after the collision. One needs a close collision, centers passing within $\sim 10$ kpc, for this to occur. In the past this may have been more frequent; if so, one expects that an equilibrium population of dark halos will have $\langle \xi^2 \rangle < 1/3$. Since $m_{eff}$ may be considerably smaller than $m$ when $\langle \eta^2 \rangle \to 1/3$ it is possible that many of second class of galaxies, those with very long (convex) rise distances, $R_h$, to be the result of recent halo interactions. [Another way of getting long rise distances is to require $a \to 1$, since $m$ should be replaced by $m$ for small $r$; see equation (2).]

In such close halo-halo collisions, the structure of the halos are also changed by the gravitational interactions between the halos themselves and with any ordinary matter trapped in the halos' gravitational wells. Section A.5 gives estimates of the gravitational forces involved. It is possible that some dark halos have been stripped of much of their entrapped luminous material.

We have avoided discussing t-waves, regarding them as infinitesimal. But they may not be, in regions where halos collide or in earlier epochs of the universe. Suppose $\xi$, represent a collection of 'strong' t-wave fluctuations; then, in their presence, the stability of s-wave halos will be affected.

4. Discussion

In Section 2 we eliminated all more conventional potential sources for the LDM. This paper explores the idea that dark halo galaxies are fluctuations in a scalar field. The s-wave halo solutions studied are successful in being necessarily spherical in their inner regions and in producing the form of the observed gravitational acceleration. In its simplest form, the only parameter that can vary from galaxy to galaxy is the square of an amplitude, $a^2 < 1$, which controls the depth of the gravitational well of the LDM. Also, there are parameters $m^2$ & $\alpha^2$ which have been determined from the Milky Way’s rotation curve; we expect them to have these values for all galaxy halos at the present epoch $z \approx 0$ if our Lagrangian is complete.

Finally, there is a parameter $R_l$ which influences the gravitational attraction in the very outer regions of a halo. Its value may depend upon the density of neighboring galaxies. We have guessed that $R_l > 300$ kpc for an average halo, but a much smaller value might be needed in a few galaxies (Casertano & van Gorkom 1991).

4.0.1 Tests and Utility Of the Theory

The most important observational tests revolve around the question: What physics determines the values of $\alpha^2$ and $m^2$ (and of $R_l$)? If this is the inflation field then these parameters must have varied over large time scales and be functions of $z$.

If $m^2$ is constant at the present cosmic epoch, then angular measurements of $R_h$ in spiral galaxies would provide a new cosmic distance scale. Attempts to establish such a distance scale would provide a test for the assumption $m^2$ is constant. One might be able to place limits on the variation of $\alpha^2$ with time. If, e.g. $\alpha^2 \propto (1+z)^{3/2}$ then also $V_h^2 \propto (1+z)^{3/2}$ and the rotation curves of galaxies at $z \approx 2$ would show larger amplitudes than those nearby.

Because we may have omitted possibly important coupling terms to matter and radiation, it is possible that both $\alpha^2$ and $m^2$ may have much different values in the immediate neighborhood of stars, compact clusters, or galactic centers. In principle, this possibility could be restricted by setting constraints upon the rotation of the lines of apsides in double star systems and in the Sag.
A*-S2 system (Schödel et al 2002). Analyzing the velocity fields of colliding galaxies and compact galaxy clusters provides both a test and a utility of the simple theory. The non-linear terms should cause halo interaction forces stronger than the ones predicted by Newtonian theory and halos should feel a stronger than expected gravitational attraction to ordinary matter.

Since baryonic matter can flow into the gravitational wells of LDM fluctuations and contribute additional source terms to the gravitational fields, the rotation curves of galaxies will be composite. Use of this scalar theory to subtract out the LDM contribution should allow determinations of \( L/M \) for stars in the inner regions of galaxies. One test is therefore whether these determinations make sense. Suppose there are equal opportunities for baryonic matter to have settled in most of the large LDM wells. Chose galaxies with similar colors for their central regions, so that \( \langle L/M \rangle \) is the same for these galaxies' central region stars. Then the central luminosity within \( R_b \approx 5 \) kpc in field galaxies would be a function of \( a^2 \), measured by \( V_b^2 \). This would produce a Tully-Fisher relation but restricted to the very inner parts of galaxies, \( r \leq R_b \). For low luminosity galaxies, the halo gravitational potential may actually be determinable in the center regions. Then, by Liouville's theorem, one would expect to observe star densities \( \propto \exp[\beta \int f \, dr] \).

4.0.2. The Equivalent Cosmological Fluid

In order to get a useful interpretation of the energy momentum tensor for \( \phi \)-field fluctuations regarded as individual dark matter halos, it was necessary to introduce reference energy and momentum levels by subtracting from the basic field \( T_{ab} \) a fluid energy-momentum tensor, \( \hat{T}(\lambda, \Lambda) \), where
\[
\kappa \rho = \frac{1}{2} \alpha^2 m^3 \lambda - \Lambda, \quad \kappa \rho = \frac{1}{2} \alpha^2 m^3 \lambda + \Lambda, \\
U^a = m^2 / m, \quad \text{and} \quad \hat{T}(\lambda, \Lambda)_{ab} = \partial_U T_{ab} = 0.
\]
In crowded fields of galaxies, \( \Lambda \) represents an explicit underlying background field density term used in determining the metric coefficient \( A \). It seems this feature is unavoidable in any theory in which the source terms are derived from a Lagrangian formulation. Fluctuations in a field must be referred to a reference level. Because the reference energy density level in a Lagrangian normally is not included but must be specified in Einstein's equation, one has a choice of either initially augmenting the Lagrangian or of introducing \( \hat{T}(\lambda, \Lambda) \) as an additional required source term. In performing the large-scale averages of source terms required in cosmology, the intimate connection between the \( \phi \) field \( T_{ab} \) and \( \hat{T}(\lambda, \Lambda) \) can be lost and two separate 'independent' averages may appear. One wonders if in the remote past when fluctuations in the Lagrangians for the weak and strong forces were more important contributors to the total energy density whether such \( \lambda, \Lambda \) terms contributed significantly to the early cosmological \( \Lambda \) term.

A fluid representation of the energy momentum tensor for a particular spherical s-wave solution when \( \lambda = \Lambda = 0 \) is not rewarding because \( T_{xx}, T_{yy}, T_{zz} \) are not equal and vary from place to place; it is only by averaging over the surface of an entire sphere \( r = \text{constant} \) that a pressure can be defined: \( 3 \rho \equiv \langle T_{xx} \rangle + \langle T_{yy} \rangle + \langle T_{zz} \rangle = \langle T_{tt} \rangle = \hat{\rho} \). The same relation holds for averaged t-waves. However, for a cosmological fluid, one consisting of a great many individual s-wave regions in a unit volume, one may not adopt the same equation of state used for representing light, \( \langle p \rangle = \langle \rho \rangle / 3 \), because in this case, in general \( \lambda, \Lambda \neq 0 \). I am skeptical that spatial averaging can be dismissed as a trivial problem, for the link between a source field and (some of) the cosmological constant \( \Lambda \)-term can easily be lost. The conservation laws \( (\hat{T}^{ab} - T^{ab})_{,b} = 0 \) necessarily involve \( \lambda, \Lambda \), modifying the definition of the effective density and pressure. Therefore, the cosmic time evolution of the s-wave field density fluctuations need not be \( \propto R(t)^{-4} \), where \( R \) is the cosmic scale factor, because the distribution of the halos themselves must be considered.

Suppose we assume we can replace galaxies by halos, each of the same mass, \( M = (c^2/6G)(a^2)R_e \) (see equation (9)), where \( R \) is the effective maximum size of the representative s-wave halo. Suppose they were equidistant from one another with a number density \( \psi \). Assume that in the recent past, \( \alpha \) was constant and halos were neither destroyed or created. If \( R \) was an assigned multiple of \( m^{-1} \), the mass of each halo decouples from the general cosmic expansion and the LDM contribution to the cosmic mass density would scale as \( M \psi \propto R(t)^{-3} \). For case II solutions when the halo mass is \( \propto R_0 = \psi^{-1/3}/2 \), the LDM halo contribution and the \( \Lambda \) term would scale as \( R(t)^{-2} \).
The present minimum value for $\psi$ is probably a mass (or luminosity) weighted Schechter function (see AQ, p.581), $\psi = 0.014 h^4$ halos Mpc$^{-3}$. Then, for case II, using $R_0 = \psi^{-1/3}/2$, the values of $\alpha^2 \psi^2$ found for the Milky Way galaxy and, $M(R_0) \psi + \rho_\Lambda \equiv \Omega_{LDM} \rho_c$, one finds $\Omega_{LDM} \sim 0.45$, independent of $h$, with $\rho_\Lambda \sim 0.15 \rho_c$. Since there can be halos not associated with luminous galaxies, it is reasonable to consider $\psi$ for halos is $\sim 1.5$ larger than that used; then the values of $\Omega$ would then be $4/3$ larger. These are conservative estimates, for suppose we just guess that each galaxy has a halo mass ten times its luminous baryon mass, then $\Omega_{LDM} \sim 10 \Omega_{baryon} \sim 0.3$. The possibility that $\Omega_{LDM} > \Omega_{baryon} + \Omega_{other}$ is appreciable and $\rho_\Lambda$ from clustering LDM cannot be ignored. If so, the current cosmic expansion rate is then determined by the LDM with $\mathcal{R}(t) \propto t$.

This estimate permits us to point out an interesting scenario. Suppose we take at present $\Omega_{other} \sim 1/3$ ($\equiv (1/2)\Omega_{LDM}$), scaling in time as ordinary matter, $\propto \mathcal{R}(t)^{-3}$. Then going back in time to when $\mathcal{R} = 1/3$ one sees the reverse would have been true, $\Omega_{other} \sim 2 \Omega_{LDM}$ and then the scale factor would have had a different time dependence, $\mathcal{R}(t) \propto t^{2/3}$. This would mean that in the time between $z = 2$ and the present, the universe would have been observed to experience an accelerated growth rate. Such an acceleration may have been observed. [For a discussion of the observations of Perlmutter, Riess and their very many collaborators see Perlmutter (2003).] The argument can be inverted: If these observations hold up, it would be reasonable to conclude $\Omega_{other} \propto \mathcal{R}(t)^{-3}$.

5. Summary of Theoretical Uncertainties

What we have found is that the $\phi$-field variations, as presented, can explain features of observed galaxy rotation curves. Coupling to other matter fields is not presently needed since a nonlinear theory can produce its own source term; however, future observations could easily require inclusion of such terms.

There are some computational investigations needed to verify that the time-varying s-wave solutions will settle down quickly enough to be represented by the steady state solutions. Also, the fluctuation spectrum in physics is represented by the action of a stochastic force on the RHS of equation (1). In order to specify the cosmology of the $\phi$-field, the time evolution of this force is needed. At the very least it has to be shown that the evolution of $\phi$ variations from the observed spectrum of fluctuations seen in the cosmic microwave background can produce the presently observed galaxy clustering. In investigating this problem, one would effectively determine the time dependence of the $\phi$-field back to the time of photon decoupling.

Because of the uncertainties in knowing how $\alpha^2$ & $m^2$ varied with time, and what role should be ascribed the t-waves in the very distant past, we simply do not know whether the scalar field proposed here is compatible with any of the many scalar theory suggestions for ‘dark matter’ or ‘dark energy’ which have already been proposed. Conventionally one sets for the spatially averaged scalar field for the dark matter a fluid representation, $p = (\partial_t \phi)^2/2 - V(\phi) - \langle (\nabla \phi)^2 \rangle /6$ and $\rho = (\partial_t \phi)^2/2 + V(\phi) - \langle (\nabla \phi)^2 \rangle /2$, with the spatial derivative terms normally ignored. [See Kolb & Turner (1990) and comprehensive reviews of dark matter theory by Peebles & Ratra (2003) & Bernardeau (2003).] We require that the averaged spatial derivatives be included. For us, the cosmological average for $p$ is the same, (substituting $(\partial_t \phi)^2$ for $(\nabla \phi)^2$ and using $V(\phi) = -\frac{1}{2}m^2 \phi^2(1 - \phi^2/2)$), but $\rho$ is different $\rho = (\partial_t \phi)^2/2 - V(\phi) + \langle (\partial_t \phi)^2 \rangle /2$, even when the necessary terms $\lambda$ & $\Lambda$ are ignored. The two forms are not compatible, even for t-waves. [The differences arise from the fact we used in the Lagrangian a factor $m^2 m_c$ rather than $m^2$; one gets a different energy-momentum tensor from the conventional one when the variation $\delta L/\delta g_{ab}$ is performed. We prefer our form of the Lagrangian because we do not get basic changes in the algebraic form of the Lagrangian if we consider simple transformations in $g_{ab}$ like $g_{ab} \rightarrow -g_{ab}$.]
A. APPENDIX

A.1. Representations of the s-Wave Solutions

We consider the flat space field equation (1) with \( A = B = 1 \), requiring solutions be initially infinitesimal. There are two classes of solutions. Consider initially local mode excitations, \( \phi \propto \exp(\imath(\omega t + k \cdot r)) \); then the dispersion relation for the modes become:

\[
\omega^2 = k^2 - m^2(1 - \langle \phi^* \phi \rangle). \tag{A1}
\]

For \( \omega^2 > 0 \) we have t-wave solutions which remain infinitesimal. The second class, s-wave solutions, correspond to \( \omega^2 < 0 \); they are non-traveling waves and may grow to finite amplitude, their smallest values being determined by boundary conditions. The Fourier transforms \( \tilde{\phi}(k, t) \) for the s-wave solutions must have the feature that \( \tilde{\phi}(k, t) \sim 0 \) for \( k^2 > m^2 \) (since these Fourier modes always remain infinitesimal.) Consider the rest frame of a point treated as the origin. This restriction implies that \( \phi \) must fall off rapidly for \( r > \pi / 2m \) and is effectively large only in the small central region.

For simplicity we consider \( \phi \) to be a real field. All s-wave solutions finite at the origin, can be written as \( \hat{a}(t) + \phi(r, t) \). Then

\[
\nabla^2 \phi = -m^2 \phi(1 - \hat{f}^2), \tag{A2}
\]

where \( \hat{f}(r, t)^2 \leq 1 \) is a slowly varying bounded function [This follows from taking the Fourier transform of \( \nabla^2 \phi \) and applying the mean value theorem.] All s-wave solutions must be of this form including the steady state solutions, \( \hat{f}^2 = \phi^2 \). For solutions \( \phi \to 0 \) as \( r \to \infty \), we require \( \hat{a} = 0 \). The time dependence is then given by:

\[
\partial_t^2 \phi = m^2 \phi(\hat{f}^2 - \phi^2). \tag{A3}
\]

With suitable boundary conditions, some solutions are stationary and some oscillate\(^2\) with \( \hat{f}^2 \) representing a ‘mean’ value of \( \phi^2 \).

We need consider only those solutions for which, at large \( t \), the maximum values of \( \phi^2 \) decrease as \( r \) increases; then \( \hat{f}^2 \) must also decrease with \( r \) because it is bounded by the maxima and minima of \( \phi^2 \). When the oscillations are of low amplitude, one finds \( f \approx \phi_s \), one of the steady state solutions discussed below.

More generally, we shall assume that in time both \( \langle \hat{f}^2 \rangle \) and \( \phi^2 \) approach one of the steady state values \( \phi_s^2 \). Since the frequency \( \sim m\sqrt{\hat{f}^2} \) also decreases with \( r \), the time variation of the solutions discussed below, is effectively confined to the central regions and the steady-state solution adequately represents the outer regions.

A.1.1. Radial Symmetry

Only spherically symmetric solutions can dominate in an inner radial region, since the non-spherical modes cannot satisfy the restriction \( k_{\text{transverse}}^2 = \ell(\ell + 1)/r^2 < m^2 \) until \( r \) is large; e.g. for \( m \sim 3 \) kpc, the \( \ell = 2 \) mode cannot possibly be significant until \( r > \sim 7 \) kpc. If many other isolated halo regions of s-wave solutions exist they will cause nonradial interactions in each other's outer regions. Considering the case of all regions being of the same minimal size; then \( \ell = 6 \) is appropriate for regions in contact and each would need a minimum core radius \( > 18 \) kpc for such interactions to be represented by s-waves; if this restriction is not true t-waves will be generated by the interaction.

A.2. The Steady-State Halo Solution

Equation (A2), with \( \hat{f}^2 = \phi^2 \), may be replaced by an integral equation using a Greens’ function and regarding the nonlinear term as a source. In the general case, the formal steady state solution, can be

\(^2\)The boundary condition at \( t = 0 \) is that \( (\partial_t \phi)^2 \) is small, satisfying equation (A7) with \( 0 < a_s^2 < \langle f^2 \rangle \). If this condition is not met, the solutions will exponentially decay.
represented by a convergent series expansion in the amplitude \( a < 1 \) at \( r = 0 \). One has \( \phi_n = a(\phi_0 + a^2\phi_1 + a^4\phi_2 + \ldots) \), where \( \phi_0 = \sin mr/mr \) (plus angular terms) for an isolated halo. The \( \phi_n \) can be related to one another by e.g.

\[
\phi_1(r) = \frac{1}{4\pi} m^2 \int \frac{\cos m|\mathbf{r} - \mathbf{r}'|}{|\mathbf{r} - \mathbf{r}'|} \phi_0^3(\mathbf{r}') d^3r. \tag{A4}
\]

Alternatively, a static (or time averaged) spherically symmetric real solution can be obtained by successive approximations. Put \( \phi_n = (f_n/r) \sin g_n \); then

\[
g_n = \int [m^2(1 - \phi_{n-1}^2) + \partial_t^2 f_{n-1}]^{1/2} dr, \tag{A5}
\]

where \( f_n^2 = (a/m)\partial_r g_n \) is required as a constraint. To lowest order we use \( \phi_0 = (a/mr) \sin mr \) and \( \partial_r^2 f = 0 \). Successive approximations amount to an expansion in the amplitude \( a \) at \( r = 0 \). Since it is probable that \( a^2 < 1/3 \), the next approximation should be sufficient:

\[
\phi \approx \phi_1 = (a/mr)[(1 - a^2)(1 - \phi_0^2)]^{-1/4} \sin \int m[1 - 5\phi_0^4/6]^{1/2} dr \simeq (a/mr\sqrt{(1 - a^2)}) \sin m[1 - \phi_0^2]^{1/2} dr. \tag{A6}
\]

### A.3. Time dependent Halo Solutions

There is a class of time dependent halo solutions which are related to the steady state solution, permitting us to estimate the time it takes the s-waves to develop. In equation (A2), for some time interval, replace \( \hat{f}^2 \) by a time averaged value, \( \langle \hat{f}^2 \rangle \), to simplify the time dependence; then equation (A2) can be integrated:

\[
(\partial_t \phi)^2 = \frac{1}{2} m^2 \langle \hat{f}^2 \rangle (\phi^2 - a_0^2)(a_1^2 - \phi^2), \tag{A7}
\]

where \( a_0^2(r) \) is the value of \( \phi^2(r) \) at the lower turning point and \( a_1^2(r) \equiv 2\langle \hat{f}^2 \rangle - a_0^2(r) \) is its value at the upper turning point. Using the notation \( a_0^2 \equiv \langle \hat{f}^2 \rangle(1 - h^2(r)) \) for defining \( h^2(r) \), the solution may be written as:

\[
\phi^2 = \langle \hat{f}^2 \rangle H \quad \text{where} \quad H \equiv [1 + h^2(r)\sin \theta], \tag{A8}
\]

with the time dependence given by the standard elliptic integral

\[
[2m^2 \langle \hat{f}^2 \rangle]^{1/2} (t - t_0(r)) = \int_{-\pi/2}^\theta H^{-1/2} d\theta. \tag{A9}
\]

For small values of \( h^2 \) this is basically Kepler’s equation relating the mean and eccentric anomalies. For the steady state solution, \( h^2 = 0 \).

A solution still requires a value of \( \langle \hat{f}^2 \rangle \). One may regard equations (2) & (7) as defining \( \langle \hat{f}^2 \rangle \) in terms of \( \phi^2 \), by successive substitutions. If this value is substituted into equation (A3), one can solve to get the actual form of \( \phi(r,t) \). An adequate approximation, for \( h^2 \simeq 1 \), is to ignore the spacial variation of \( H \) and replace \( \phi \) by \( \phi_s\sqrt{H} \), where \( \phi_s \) now is the (approximate) steady state equation given in the form of equation (A6):

\[
\phi \simeq \sqrt{H} \cdot (a/m) \sin \int m(1 - a^2 \phi_0^2)^{1/2} dr. \tag{A10}
\]

\(^3\)The formal procedure is to use the specification of \( \phi(r,t = 0) \) in equation (A2) to solve for \( \hat{f}(r,0) \); using this and the specification of \( \partial_t \phi(r,t = 0) \) in equation (A7), one calculates \( a_0^2(r) \). The integration, equation (A9), then gives the time evolution in the vicinity of \( t = 0 \). Because the time varying amplitude so obtained is \( r \)-dependent, \( \phi(r,\delta t) \) will have a different \( r \)-dependence than \( \phi(r,0) \); this causes \( \hat{f}(r,\delta t) \) to change, requiring the process to be iterated for the next time step. While \( \hat{f} \) is bounded, it too will oscillate with time. Equation (A3) forces mode-mixing and some generation of t-waves which travel away from the halo. This produces some damping of the time dependent s-wave solution. The use of \( \langle \hat{f}^2(r) \rangle \) permits a reasonable estimate of the rise time and of the value of \( \langle \phi(r)^2 \rangle \).
For evaluating components of the energy momentum tensor we note time derivatives of s-wave solutions normally can be ignored since \( \dot{\phi}^2 \sim m^2(\dot{f}^2)\phi^2 < m^2\phi^2 \sim (\partial_r\phi)^2 \).

A.4. The Rise Time

We now consider the rise time of these s-waves. From the rotation curves, we estimate \( m^{-1} \sim 3\) kpc so that the time scale is measured in units of \( m^{-1} \sim 10^4\) yr. For the case \( a_0^2 \ll \dot{f}^2 \), the rise time, \( t_1 \), from \( a_0 \) to \( a_1 \) is then approximately,

\[
m\dot{f}t_1 \approx \ln[2\sqrt{2\dot{f}/a_0}].
\]

The initial variations in \( a_0^2 \propto T_{11} \) are assumed to be \( \sim 1 \times 10^{-4} \), equal to the flux density variations seen in the cosmic microwave background. Our solutions are limited to \( r < R_\odot \) where \( \phi(R_\odot)^2 \sim 10^{-4} \), for the solutions cannot be carried out to values less than the fluctuation levels from which they arose. This determines \( R_\odot \approx 100m^{-1}\sqrt{a^2} \sim 170\) kpc for \( a^2 = \frac{1}{3} \) and rise times \( \sim 1 - 5 \times 10^5\) yr for the central region \( mr < 10 \). Therefore the characteristic times of oscillation and of growth to full amplitude are short compared to present galactic internal dynamic time scales, but comparable to the age of the universe at the time of photon decoupling.

A.5. Curvature and Restricted Two-Body Problems

For a Schwarzschild metric, the wave equation is:

\[
0 = \partial^2_\theta\phi - (B/Ar^2)\partial_r(r^2\partial_r\phi) - B\nabla^2_\phi - m^2\phi(1 - \phi^2) + \frac{1}{2} \left[ \partial_\phi \partial_t \ln(A/B) - \partial_\phi \partial_r(B/A) \right],
\]

where

\[
\nabla^2_\phi \equiv (r^2\sin^2\theta)^{-1}[\sin\theta\partial_\theta(\sin\theta\partial_\phi) + \partial^2_\phi].
\]

For the weak gravitational fields considered in this paper, the coefficients \( A, B \sim 1 \) plus terms of order \( (V_h/c)^2 \) and are static. The term \( \partial_\phi \partial_t \partial_r(B/A) \), for a time-dependent halo, produces damping with a characteristic inverse time \( \sim (V_h/c)^2m \) and can be neglected. The term \( \partial_\phi \partial_r(B/A) \) is of order \( \sim (V_h/c)^2r^{-1}\partial_r\phi \) and produces a very slight distortion of s-wave solutions centered at the origin. So, neglecting terms of order \( (V_h/c)^2 \) the flat space wave equation is sufficient to evaluate contributions to the energy momentum tensor, in equations (4)-(6).

However, the gradient term, \( \propto \partial_r(B/A) \), does determine the motion of the centroid of a distant small ‘test’ halo, one which makes a negligible contribution to the energy-momentum tensor. For suppose we change the independent variables in the field equation \( (t, x) \rightarrow (t, w) \), the local coordinates centered on the test halo, \( x = u(t) + w \), and as before, set \( A, B \approx 1 \). In terms of the new independent variables, the wave equation becomes the normal one for flat space, \( \partial^2_{tt} - \nabla^2\phi - m^2\phi(1 - \phi^2) + \ldots = 0 \), providing one sets

\[
\ddot{u} = -\frac{1}{2} \left. \frac{d}{dr} \frac{(B/A)}{r} \right|_{r = u} \hat{F},
\]

Here \( u = |u| \), two very small terms, of order \( \dot{u}^2m^2\phi \) and \( \dot{u}m\omega\phi \) have been neglected, and we have used a Taylor’s expansion of \( \frac{d}{dr} \frac{(B/A)}{r} \) assuming \( u \gg w \). This is an equation of motion for the centroid of the distant test halo in the field of a central halo or star. [The next order term in the expansion gives the tidal force on the test halo.]

If the central object generating the metric is an ordinary baryonic mass or black hole, one has \( AB = 1 \) and \( A^{-1} = -2GM/r \); consequently the test halo experiences twice the gravitational attraction that a test mass would feel. If the central object is a s-wave halo, and \( w \gg m \) then \( A \approx \) constant, since \( M \propto r \); the distant test halo experiences the same acceleration \( f \) a test mass experiences (see equation 76)). Since halo-halo interactions are wave phenomena; the halos do not really make separate contributions to the energy-momentum tensor. In close massive halo-halo interactions, the energy-momentum tensor will show interference phenomena and the wave equation, because of the non-linear terms, will also contain interference
terms. Consequently interactions between close massive halos will not be as simple as those involving ‘test’ halos.
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