A mixed finite element method with piecewise linear elements for the biharmonic equation on surfaces

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The biharmonic equation with Dirichlet and Neumann boundary conditions discretized using the mixed finite element method and piecewise linear (with the possible exception of boundary triangles) finite elements on triangular elements has been well-studied for domains in $\mathbb{R}^2$. Here we study the analogous problem on polyhedral surfaces. In particular, we provide a convergence proof of discrete solutions to the corresponding smooth solution of the biharmonic equation. We obtain convergence rates that are identical to the ones known for the planar setting. Our proof focuses on three different problems: solving the biharmonic equation on the surface, solving the biharmonic equation in a discrete space in the metric of the surface, and solving the biharmonic equation in a discrete space in the metric of the polyhedral approximation of the surface. We employ inverse discrete Laplacians to bound the error between the solutions of the two discrete problems, and generalize a flat strategy to bound the remaining error between the discrete solutions and the exact solution on the curved surface.

Keywords: biharmonic equation; polyhedral surfaces; mixed finite elements; discrete geometry.

1. Introduction

We consider the biharmonic equation on smooth surfaces embedded in three-dimensional Euclidean space: given a function $f$ on a smooth surface $\Gamma$ with smooth boundary $\partial \Gamma$, find a function $u$ such that

$$\Delta^2 u = f,$$

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where $\Delta_\Gamma$ is the Laplace–Beltrami operator on the smooth surface $\Gamma$. This Laplacian arises from the Riemannian metric $g$ on $\Gamma$, where $g$ is inherited from ambient three-dimensional Euclidean space. If boundaries are present, boundary conditions must be taken into account (where we assume that $\Gamma$ has a smooth boundary). We consider Dirichlet and Neumann boundary conditions,

$$u = \frac{\partial u}{\partial n} = 0 \quad \text{at the boundary},$$

where $\frac{\partial u}{\partial n}$ denotes the co-normal derivative of $u$ at the boundary – the scalar product of the function’s gradient and the boundary normal. For flat surfaces, this problem is sometimes referred to as the clamped thin plate problem. If no boundaries are present, $f$ and $u$ must have zero mean, i.e.,

$$\int_{\Gamma} f \, dx = \int_{\Gamma} u \, dx = 0.$$

In this paper, we use a mixed approach, which corresponds to solving a linear system of two equations: in $u_1$ (which corresponds to the solution $u$), and in $u_2$ (which corresponds to the Laplacian $\Delta_\Gamma u$ of the solution). Given $f \in L^2(\Gamma)$, the (smooth) mixed formulation takes the following form: Find $u_1 \in H^1_0(\Gamma)$, $u_2 \in H^1(\Gamma)$ such that

$$(u_2, \xi)_1 = (f, \xi)_0 \quad \forall \xi \in H^1_0(\Gamma) \quad \text{and} \quad (u_1, \eta)_1 = (u_2, \eta)_0 \quad \forall \eta \in H^1(\Gamma).$$

Here the Sobolev spaces $L^2(\Gamma)$ and $H^1_0(\Gamma)$ are equipped with the inner products

$$(u, v)_0 = \int_{\Gamma} uv \, dx \quad \text{and} \quad (u, v)_1 = \int_{\Gamma} g(\nabla_F u, \nabla_F v) \, dx,$$

respectively, where $g$ denotes the Riemannian metric on $\Gamma$, and $\nabla_F$ denotes the gradient on $\Gamma$. The mixed method can be formulated for any $u_1 \in H^1_0(\Gamma)$ and $u_2 \in H^1(\Gamma)$, and has a unique solution such that $u_1 \in H^1_0(\Gamma) \cap H^1(\Gamma)$ and $u_2 \in H^2(\Gamma)$, as we assumed smooth $\partial \Gamma$ (Gazzola et al., 2010).

In order to solve (1.3) numerically, we use a corresponding mixed finite element method on a polyhedral surface $\Gamma_h$ that is nearby the smooth surface $\Gamma$ in the sense of conditions (C1-C4) from Section 3.1. In particular, we consider $\Gamma_h$ to be a mesh with piecewise flat triangles with straight edges (with the exception of boundary triangles, which can have curved edges along the boundary – see Figure 1), together with a bijection $\Psi : \Gamma_h \to \Gamma$ that is defined via the closest point projection. As finite elements we use the space $\hat{S}_h$ of piecewise linear elements on $\Gamma_h$ (with the exception of boundary triangles, which can have modified elements). Then the discrete mixed formulation on $\Gamma_h$ is: Find $\hat{u}_1^h \in \hat{S}_{h,0}$, $\hat{u}_2^h \in \hat{S}_h$ such that

$$\int_{\Gamma_h} \nabla_{\Gamma_h} \hat{u}_1^h \cdot \nabla_{\Gamma_h} \xi \, dx = \int_{\Gamma_h} \hat{f} \xi \, dx \quad \forall \xi \in \hat{S}_{h,0} \quad \text{and} \quad \int_{\Gamma_h} \nabla_{\Gamma_h} \hat{u}_1^h \cdot \nabla_{\Gamma_h} \eta \, dx = \int_{\Gamma_h} \hat{u}_2^h \eta \, dx \quad \forall \eta \in \hat{S}_h, \quad (1.5)$$

where $\hat{f} = f \circ \Psi$ is the evaluation of the right hand side $f$ from (1.3) on $\Gamma_h$, and $\hat{S}_{h,0} := \hat{S}_h \cap H^1_0(\Gamma_h)$.

In our approach, every mesh $\Gamma_h$ is required to have uniformly shape-regular triangles. Moreover, we consider sequences of triangle meshes that converge to a given smooth limit surface such that both positions and normals converge at a certain rate (to be specified later). Letting $\hat{u}_1^h = \hat{u}_1^h \circ \Psi^{-1}$ and $\hat{u}_2^h = \hat{u}_2^h \circ \Psi^{-1}$ denote the liftings of the discrete solutions from $\Gamma_h$ to $\Gamma$, we show:

- $L^2$-convergence of $\hat{u}_1^h$ to $u_1$ of order $h$ (Theorem 5.3);
• $H^1$-convergence of $u_h^1$ to $u_1$ of order $h^2$ (Corollary 5.1);
• $L^2$-convergence of $u_h^2$ to $u_2$ of order $\sqrt{h}$ (Theorem 5.2);

where $h$ is the maximum edge length of the approximating triangles of the mesh. If no boundaries are present, the problem becomes simpler and we observe that better convergence rates can be obtained.

Before continuing with an overview of our proof, we provide a few historical remarks on mixed finite elements and finite elements for curved surfaces for context.

**Mixed finite elements for the biharmonic equation in $\mathbb{R}^2$.** Ciarlet & Raviart (1973) introduce the mixed finite element method for the biharmonic problem. Their work informs the functional analysis framework that we use here. They solve the same system of equations that we end up solving (in the flat case), but only show convergence for higher-order ($\geq 2$) Lagrangian finite elements. Their approach is later expanded by Monk (1987) to deal with boundary smoothness problems caused by triangulating (in the flat case).

Scholz (1978) proves that the mixed finite element method with linear, first-order Lagrange elements can be used to solve the biharmonic problem, and he gives an error bound of $h \log^2 h$ in the $L^2$ norm of the solution. The result by Scholz is central to understanding the convergence of the linear finite element method for the biharmonic equation, and forms the basis of our proof. The result is remarkable, since it shows convergence of the method, even though the standard convergence conditions for mixed finite elements (the inf-sup conditions (Boffi et al., 2013)) are not fulfilled. Scholz’s error estimate is not optimal, as it relies on an $L^\infty$ estimate of the Ritz projection error by Nitsche (1978). An application of a later, better bound for the same interpolation error (Rannacher & Scott, 1982) gives convergence of order $h$.

Oukit & Pierre (1996) provide an analysis of the biharmonic equation with Dirichlet and Neumann boundary conditions that combines a hybrid approach (which they call Hermann–Miyoshi) and the mixed approach (which they call Ciarlet–Raviart). Their analysis holds for first and second order elements. The result by Scholz (1978) is recovered in the limit $p \to \infty$, where $p$ is the degree of the $L^p$ space used in their estimate (5.39). An alternative approach to solving the biharmonic equation using mixed finite elements is the decomposition into four linear equations, such as done by Behrens & Guzmán (2011) (which leads to superconvergence of the solution) and Li & Zhang (2017).

**Finite elements on curved surfaces.** Dziuk (1988) generalizes the standard result for solving the Poisson equation with linear finite elements from $\mathbb{R}^2$ to smooth surfaces by working with inscribed meshes, i.e., requiring that vertices of the approximating mesh be contained in the limit surface. His approach to analyzing discretizations of curved surfaces has since been used in advanced numerical methods for curved surfaces (Dziuk & Elliott, 2007; Demlow & Dziuk, 2007; Olshanskii et al., 2009; Du et al., 2011; Dziuk & Elliott, 2013b). An overview of methods to discretize the Laplace–Beltrami operator on curved surfaces can be found, for example, in the works of Dziuk & Elliott (2013a); Bonito et al. (2020).

Wardetzky (2006) and Hildebrandt et al. (2006) generalize Dziuk’s result to non-inscribed meshes. We also work with this generalized setting here, as non-inscribed meshes are prevalent in various applications, e.g., in geometry processing. More specifically, the setting that we consider here, namely discretizations of the biharmonic equation (and its related Helmholtz problem) using a mixed formulation with linear Lagrange elements have been popular in practice (Desbrun et al., 1999; Sorkine et al.,
2004; Bergou et al., 2006; Garg et al., 2007; Tosun, 2008; Jacobson et al., 2010, 2011, 2012). In this article, we provide a justification for the use of the linear mixed finite element method for such applications.

Meanwhile, other methods for solving the biharmonic equation on surfaces exist in the literature. Larsson & Larson (2017) use the discontinuous Galerkin approach to achieve a method for surfaces without boundary where the $L^2$ error is of order $h$. Cockburn & Demlow (2016) use a different kind of discontinuous Galerkin method as well as non-conforming mixed finite elements (Raviart–Thomas, Brezzi–Douglas–Marini, Brezzi–Douglas–Fortin–Marini). Elliott & Herbert (2020) analyze a generalized framework for the solution of a variety of fourth-order problems formulated as saddle point problems in a Dziuk-like setting. Fourth-order problems and their discrete formulations also arise in fluid dynamics, see, e.g., (Reusken, 2018).

**Proof Strategy.** The above mentioned *liftings* $u_1^h$ and $u_2^h$ of discrete Lagrange functions $\hat{u}_1^h$ and $\hat{u}_2^h$ from $\mathcal{I}_h$ to $\Gamma$ via the closest point projection enable us to compare discrete and smooth solutions. This is necessitated by the geometric difference between the triangle mesh and the smooth limit surface (since otherwise there would be no way to compare the two solutions). Our proof requires certain approximation properties in order for the closest point projection to yield a bijection. These requirements are necessitated by the geometric difference between the triangle mesh and the smooth limit surface (since otherwise there would be no way to compare the two solutions). Our proof requires certain approximation properties in order for the closest point projection to yield a bijection. These requirements are fulfilled, e.g., for incribed meshes as considered by Dziuk (1988), i.e., the case where mesh vertices reside on the smooth surface and the triangles are uniformly shape regular. Our setting is more general than the case of inscribed meshes. We detail our setting in Section 3.

We denote by $S_h$ and $S_{h,0}$ the finite element spaces $\hat{S}_h$ and $\hat{S}_{h,0}$ lifted to the smooth surface $\Gamma$ (using the closest point projection $\Psi$), and we denote by $(\cdot, \cdot)_{0,h}$ and $(\cdot, \cdot)_{1,h}$ the $L^2$ and $H^1_0$ inner products lifted from $\mathcal{I}_h$ to $\Gamma$. Using these lifted inner products, (1.5) becomes

$$
(u_2^h, \xi)_{1,h} = (f, \xi)_{0,h} \quad \forall \xi \in S_{h,0} \quad \text{and} \quad (u_2^h, \eta)_{1,h} = (\hat{u}_2^h, \eta)_{0,h} \quad \forall \eta \in S_h.
$$

However, lifting alone does not suffice for directly proving error estimates for $(u_1^h - \hat{u}_1^h)$ and $(u_2^h - \hat{u}_2^h)$. Indeed, a problem arises from the difference between the metric on the polygonal surface (which is piecewise flat) lifted to $\Gamma$ and the smooth metric $g$. This difference implies that the Hilbert spaces $(S_{h,0}, (\cdot, \cdot)_{1,h})$ and $(S_h, (\cdot, \cdot)_{1,h} + (\cdot, \cdot)_{0,h})$ for the discrete solutions are not subspaces of the Hilbert spaces $(H^1_0(\Gamma), (\cdot, \cdot)_1)$ and $(H^1(\Gamma), (\cdot, \cdot)_1 + (\cdot, \cdot)_0)$ for the smooth solutions since, although the sets are respectively subsets, the metrics differ.

Therefore, we introduce the following *auxiliary* discrete mixed problem: find $\hat{u}_1^h \in S_{h,0}$, $\hat{u}_2^h \in S_h$ such that

$$
(\hat{u}_2^h, \xi)_{1} = (f, \xi)_{0} \quad \forall \xi \in S_{h,0} \quad \text{and} \quad (\hat{u}_2^h, \eta)_{1} = (\hat{u}_2^h, \eta)_{0} \quad \forall \eta \in S_h,
$$

where the inner products are those arising from the smooth metric $g$. Using this approach, the spaces $(S_h, (\cdot, \cdot)_0)$ and $(S_{h,0}, (\cdot, \cdot)_1)$ are indeed subspaces of the Hilbert spaces $(L^2(\Gamma), (\cdot, \cdot)_{0})$ and $(H^1_0(\Gamma), (\cdot, \cdot)_1)$, respectively. Notice that this problem is only an auxiliary problem for our proof; its operators are never computed in practice.

Considering the auxiliary mixed problem is central to our approach, since it allows us to adapt the proof of Scholz (1978), which treats the case of convergence for the mixed biharmonic problem using linear elements for the case of flat domains in $\mathbb{R}^2$. In the planar case, one has $u_1^h = \hat{u}_1^h$ and $u_2^h = \hat{u}_2^h$ by construction. Scholz (1978) splits up the proof for the planar case into showing that $u_2^h$ converges to $u_2$ and that $u_2^h$ converges to $u_1$. 


In the curved case, a similar argument only works to show that $\tilde{u}^2_h$ converges to $u_2$, and that $\tilde{d}_h^2$ converges to $u_1$. Indeed, in order to bound the error between $u_2$ and $u_2^h = \tilde{d}_h^2$ in the flat setting, Scholz invokes an $L^m$ estimate for the Ritz projection. His result relies on a suboptimal bound by Nitsche (1978). An application of a later, better bound for the same interpolation error (Rannacher & Scott, 1982) yields the above-mentioned convergence rates in the flat setting. In order to adapt this analysis to the curved setting, we rely on an $L^m$ estimate for the Ritz projection provided by Demlow (2009). While Demlow (2009) works with inscribed meshes, his result can be adapted to our more general setup, resulting in a bound of the error between $\tilde{u}_1^2$ and $u_2$ when combined with Scholz’s approach.

In our setup, we must consider an additional step in order to bound the error between $u_2^h$ and $\tilde{d}_h^2$, which finally leads to a bound on the error between $u_1^h$ and $u_1$. To this end, we adapt the classical formulation for the mixed biharmonic problem introduced by Ciarlet & Raviart (1973), which requires the definition of the following function spaces:

$$V := \{(v_1, v_2) \in H^1_0 \times L^2 \mid (v_1, \mu)_1 = (v_2, \mu) \forall \mu \in H^1\},$$

$$\tilde{V}^h := \{(v_1, v_2) \in S_h \times S_{h, L^2} \mid (v_1, \mu)_1 = (v_2, \mu) \forall \mu \in S_h\},$$

$$V^h := \{(v_1, v_2) \in S_{h, L^2} \times S_{h, L^2} \mid (v_1, \mu)_{1, h} = (v_2, \mu)_{0, h} \forall \mu \in S_h\},$$

where the space $S_{h, L^2}$ is the space $S_h$, but with the $L^2$ norm instead of its usual $H^1$ norm. Using these spaces, we formulate the mixed biharmonic problem employing the Riesz map that results from an inner product that is different from the spaces’ product metric. We detail this construction in Section 4. Notice that the space $\tilde{V}^h$ is absent in the classical formulation of Ciarlet and Raviart, as it corresponds to our auxiliary mixed problem.

It requires beyond a mere generalization of Scholz’s proof to control the error between $u_2^h$ and $\tilde{d}_h^2$. We do so by bounding the geometric error between the function spaces $V^h$ and $\tilde{V}^h$. While the need for controlling geometric errors is also present when studying solutions to the Poisson equation on surfaces (Dziuk, 1988; Wardetzky, 2006), these results cannot be directly applied to our setting. We bound the difference between elements of the spaces $V^h$ and $\tilde{V}^h$ by using inverse discrete Laplacians for constructing a map between these function spaces. Using inverse discrete Laplacians introduces an error of order $h^{-1}$, which we manage to contain using a geometric error bound of order $h^{\frac{1}{2}}$, adapted from the work of Wardetzky (2006). This yields an error of order $h^{\frac{1}{2}}$, which exactly corresponds to the error estimate for $u_2$ given in Scholz (1978) for the planar case.

**OVERVIEW.** We detail the mixed biharmonic problem of the smooth setting in Section 2. We then continue with the description of discrete function spaces and differential operators on polyhedral surfaces in Section 3, where we consider the case of non-inscribed meshes and their relation to smooth surfaces that they discretize. In Section 4, we continue with the description of the approach of Ciarlet & Raviart (1973) adapted to our setting. In Section 5 we provide our convergence proof for the mixed biharmonic problem on curved surfaces.

**2. The Biharmonic Equation on Smooth Surfaces**

Let $\Gamma$ be a compact smooth surface with smooth boundary or no boundary, embedded into $\mathbb{R}^3$. We denote by $L^p$ the usual $L^p$-spaces on surfaces, and we let $W^{k, p}$ denote the Sobolev space with $k$ weak derivatives in $L^p$. We let $H^k := W^{k, 2}$, and we denote by $H^1_0 \subseteq H^1$ the subspace of functions with zero trace along the boundary (for surfaces with nonempty boundary), or those functions that have zero mean.
(for surfaces without boundary). Whenever the domain is omitted, these spaces are implied to be defined over a smooth surface \( \Gamma \).

We denote the metric tensor on \( \Gamma \) by \( g(\cdot, \cdot) \), i.e., the restriction of the inner product on \( \mathbb{R}^3 \) to the tangent spaces of \( \Gamma \). The metric induces the \( L^2 \) inner product \((\cdot, \cdot)_0 \) and the \( H^1_0 \) inner product \((\cdot, \cdot)_1 \),

\[
(u, v)_0 = \int_{\Gamma} u v \, d\sigma \\
(u, v)_1 = \int_{\Gamma} g(\nabla u, \nabla v) \, d\sigma
\]

(2.1)

The norm on \( H^1 \) is induced by the inner product \((\cdot, \cdot)_0 + (\cdot, \cdot)_1 \).

**Definition 2.1** For \( f \in L^2 \), the biharmonic equation is defined as follows: find \( u \in H^1_0 \cap H^4 \), such that

\[
\Delta^2 u = f,
\]

(2.2)

where \( \Delta^2 \) is the positive semidefinite Laplace–Beltrami operator on \( \Gamma \). Additionally,

- if \( \Gamma \) has a boundary, then zero Dirichlet and Neumann boundary conditions apply, \( u = \frac{\partial u}{\partial n} = 0 \);
- if \( \Gamma \) is closed, \( f \) must have zero mean, i.e., \( \int_{\Gamma} f \, d\sigma = 0 \).

The biharmonic equation has a corresponding weak formulation. For \( f \in L^2 \), find \( u \in H^2_0 \) such that

\[
\int_{\Gamma} \Delta^2 u \Delta^2 v \, d\sigma = \int_{\Gamma} f v \, d\sigma \quad \forall v \in H^2_0
\]

(2.3)

where \( H^2_0 \) is the subspace of \( H^2 \) with zero Dirichlet and Neumann boundary conditions. If \( \Gamma \) is a closed surface, one additionally requires that \( \int_{\Gamma} f \, d\sigma = 0 \), and one looks for \( u \) such that \( \int_{\Gamma} \Delta^2 u \, d\sigma = 0 \).

We assume that there is a unique solution such that \( u \in H^4 \), \( \|u\|_{H^4} \leq C\|f\|_{L^2} \). For closed surfaces this follows from the fact that the biharmonic equation decouples into two Poisson equations (given that \( f \) integrates to zero), and for planar domains, it follows from Gazzola et al. (2010, Section 2.5.2). Additionally, we assume the standard existence and regularity estimates for the Poisson equation: for \( g \in L^p \), \( 1 < p < \infty \) there is a unique \( w \in W^{2,p} \) with Dirichlet boundary conditions such that, weakly, \( \Delta^2 w = g \) and \( \|w\|_{W^{2,p}} \leq C \|g\|_{L^p} \). See, for example, the work of Grisvard (2011, Section 2) for planar domains or Dziuk & Elliott (2013a) for smooth surfaces.

With \( u_1 := u \), and using the intermediate variable \( u_2 := \Delta^2 u_1 \), (2.3) can be rewritten in its mixed form (Monk, 1987, (1.4))

\[
(u_2, \xi)_1 = (f, \xi)_0 \quad \forall \xi \in H^1_0, \\
(u_1, \eta)_1 = (u_2, \eta)_0 \quad \forall \eta \in H^1
\]

(2.4)

We refer to this system of equations as the smooth mixed formulation of the biharmonic equation with Dirichlet and Neumann boundary conditions (this system was also mentioned in (1.7)). It can be formulated for any \( u_1 \in H^1_0 \) and \( u_2 \in H^1 \). By Ciarlet (2002, Theorem 7.1.1), (2.4) has a unique solution such that \( u_1 \in H^1_0 \cap H^2 \), so by our assumptions that \( \Gamma \) is a smooth surface with smooth boundary the mixed problem has a unique solution such that \( u_1 \in H^1_0 \cap H^4 \) and \( u_2 \in H^2 \).
3. Discretization

3.1 Discretizing the surface

In the discrete setting, we work with a triangulated surface, i.e., a connected topological manifold of dimension two, piecewise consisting of flat triangles. Boundary edges of triangles along the surface boundary are allowed to be curved as long as the curve remains in the plane of the triangle. In the planar case, where \( \Gamma \subseteq \mathbb{R}^2 \) is a flat surface embedded in the plane only (and not, as in our general case, embedded in \( \mathbb{R}^3 \)), triangle meshes are only needed to discretize the function space \( H^1_0 \) in which the solution lives. In the case of a surface \( \Gamma \subseteq \mathbb{R}^3 \), however, the mesh is also used to discretize the geometry itself. To deal with the error introduced by the discretization, we employ the setting of Wardetzky (2006), which we explain in this section.

**Definition 3.1** (Reach) Let \( X \) be a topologically closed subset of \( \mathbb{R}^3 \). The medial axis of \( X \) is the set of those points in \( \mathbb{R}^3 \) that do not have a unique closest point in \( X \). The *reach* of \( X \) is the distance of \( X \) to its medial axis, and we say that an object lies in the reach of \( X \) if the object is closer to \( X \) than the medial axis of \( X \).

Let \( \Gamma_h \) be a triangle mesh, where all triangles are flat, and interior triangles have straight edges while boundary triangles are allowed to have curved edges along the boundary as long as these curved edges remain within the triangle plane. Then we can define the following map:

**Definition 3.2** Let \( \Gamma_h \) lie within the reach of \( \Gamma \). The closest point projection is the map \( \Psi : \Gamma_h \to \Gamma \) defined via

\[
\Psi(q) = \arg\min_{p \in \Gamma} \|q - p\|_{\mathbb{R}^3}.
\]

If \( \Psi \) is bijective, we define the inverse closest point projection as \( \Phi = \Psi^{-1} \). Notice that in this case, \( \Phi \) maps any \( p \in \Gamma \) to the closest intersection of the line through \( p \) parallel to the normal of \( \Gamma \) at \( p \) with \( \Gamma_h \), see Figure 1.
Throughout, we require certain conditions of our mesh. These conditions are automatically satisfied for any shape-regular triangle mesh that is inscribed into a smooth surface $\Gamma$ (Wardetzky, 2006, Section 3.5). Indeed, the following conditions are fulfilled by the setting considered in the work of Dziuk (1988) and others (with minor modifications at the boundary).

(C1) The triangles of $\Gamma_h$ are uniformly shape regular, i.e., there exist constants $\kappa, K > 0$ such that every triangle contains a circle of radius $\kappa h$ and is contained in a circle of radius $Kh$.

(C2) The polyhedral surface $\Gamma_h$ is a normal graph over the smooth surface $\Gamma$, i.e., $\Gamma_h$ lies within the reach of $\Gamma$ and the closest point projection $\Psi$ is a bijective function. In particular, the boundary of $\Gamma_h$ is bijectively mapped to the boundary of $\Gamma$, where triangle edges along the boundary of $\Gamma_h$ are allowed to be curved, but must remain in the plane of their respective triangle, see Figure 1.

(C3) The distance of every point under the closest point projection is bounded by $Ch^\gamma$ for some $\gamma \geq \frac{3}{2}$.

(C4) Assuming (C2), the triangle normals of $\Gamma_h$ approximate the normals of $\Gamma$ in the sense that, at every point $p \in \Gamma$ that maps to an interior point $\Phi(p)$ of a triangle on $\Gamma_h$, the angle between the surface normal of $\Gamma$ at $p$ and the triangle normal of $\Gamma_h$ at $\Phi(p)$ is bounded by $Ch^\varepsilon$ for some $\varepsilon \geq 1$.

**Definition 3.3** We define the approximation parameter of the mesh $\Gamma_h$ as $\sigma := \min(\gamma, 2\varepsilon) \geq \frac{3}{2}$. Our bounds will depend on this combination of the mesh’s pointwise and normal approximation quality.

Condition (C2) might seem difficult to satisfy for meshes that have nonempty boundary, since we require that the boundary of $\Gamma$ maps exactly to the boundary of $\Gamma_h$ under $\Phi = \Psi^{-1}$. However, this condition is similar to the condition of Scholz (1978) in the flat case: Consider a straight-edged triangle mesh within the reach of $\Gamma$. Let boundary vertices be inscribed into the boundary of $\Gamma$ such that every triangle has at most two vertices on the boundary of the mesh. For every boundary triangle $T$, replace the straight boundary edge by a curved edge in the plane of $T$ such that the closest point projection $\Psi$ becomes surjective. This yields a piecewise flat surface $\Gamma_h$ for which condition (C2) is satisfied. This makes condition (C2) very similar to the condition of Scholz (1978) in the flat case, which requires triangles with curved edges that exactly match the boundary of the smooth domain.

**Remark 3.1** Similar conditions to (C1-C4) are being used by Wardetzky (2006) to show convergence of the finite element discretization of the Poisson equation. Condition (C2), however, is not present in the work of Wardetzky (2006) as it pertains to bijectivity at the boundary. Because of that, Wardetzky’s result on the convergence of the finite element method for the Poisson equation, Theorem 3.3.3, only holds for solutions that are supported sufficiently far away from the boundary. With Condition (C2), and with the finite element spaces that will be defined in Definition 3.6, the estimates of Wardetzky (2006) for the finite element solutions of the Poisson equation extend from the case of solutions that are compactly supported away from the boundary to the general case of solutions $u \in H^1_0 \cap H^2$. This mirrors similar work on polygonal meshes where boundary edges are allowed to be curved, as long as they bijectively map to the surface boundary (Scholz, 1978; Demlow & Dziuk, 2007; Demlow, 2009).

**Remark 3.2** The numerical method described in this work uses triangles whose edges are allowed to be curved at the boundary to fulfill the bijectivity constraint from (C2). It might be possible to relax condition (C2) to only require triangles with straight edges and vertices that are inscribed into the boundary.

\[\text{We adopt the convention that, wherever a constant } C \text{ occurs, the words "there is a constant } C > 0, \text{ dependent only on the surface } \Gamma \text{ and mesh regularity parameters" are implied.}\]
boundary. However, such a relaxation could lead to lower convergence rates. An example of such a construction in the flat case can be found, for example, in the work Monk (1987). Other alternatives for weakening the bijectivity constraint from (C2) are given by the approach of Elliott & Ranner (2012) who employ a piecewise polynomial boundary, or by the approach of Burman et al. (2018) who weakly enforce a nonhomogeneous boundary condition on straight-edged boundary triangles.

Using Conditions (C1-C4), we can relate the metric and the function spaces of the mesh \( \Gamma_h \) to the metric and the function spaces of \( \Gamma \). Let \( g \) denote the metric tensor on the smooth surface \( \Gamma \). Notice that the polyhedral surface \( \Gamma_h \) can be regarded as a so-called Riemannian manifold, see Troyanov (1986). Indeed, the surface \( \Gamma_h \) carries a Riemannian metric \( g_{\Gamma_h} \) that is flat almost everywhere except at the mesh vertices, which are singularities for the metric. Notice in particular that the metric \( g_{\Gamma_h} \) is smooth across triangle edges since any pair of adjacent triangles can be isometrically mapped to the flat plane; as a consequence, the metric \( g_{\Gamma_h} \) does not “see” triangle edges (they are intrinsically flat).

We can then use the inverse closest point projection \( \Phi \) in order to pull back the cone manifold’s metric \( g_{\Gamma_h} \) from \( \Gamma_h \) to \( \Gamma \). This results in a metric \( g_h \) on \( \Gamma \), defined everywhere except at the preimage under \( \Phi \) of edges of \( \Gamma_h \). More precisely, we define

\[
g_h(\mathbf{X}, \mathbf{Y}) := g_{\Gamma_h}(d\Phi(\mathbf{X}), d\Phi(\mathbf{Y})) = g_{\mathbb{R}^3}(d\Phi(\mathbf{X}), d\Phi(\mathbf{Y})) \quad \text{a.e.,}
\]

where \( \mathbf{X} \) and \( \mathbf{Y} \) are arbitrary smooth tangential vector fields on \( \Gamma \), \( d\Phi \) is the Jacobian of \( \Phi \), and where \( g_{\mathbb{R}^3} \) denotes the standard Euclidean metric of ambient three space. Since \( \Gamma_h \) depends on the chosen triangle mesh, and \( \Phi \) depends on the choice of \( \Gamma_h \), all three expressions depend on \( h \).

DEFINITION 3.4 Define the unique matrix field \( A \) on \( \Gamma \) that relates the pulled back metric \( g_h \) to the smooth metric \( g \) almost everywhere. This matrix field is defined by requiring that

\[
g_h(\mathbf{X}, \mathbf{Y}) = g(A\mathbf{X}, \mathbf{Y}) \quad \text{a.e.} \tag{3.1}
\]

holds for all smooth vector fields \( \mathbf{X} \) and \( \mathbf{Y} \) on \( \Gamma \).

Consider the \( L^2 \) and \( H^1 \) inner products on the polyhedral surface pulled back to the smooth surface \( \Gamma \) via \( \Phi \). Using the matrix field \( A \), these inner products can be conveniently be expressed as

\[
(u, v)_{0,h} := \int_\Gamma uv|\det A|^\frac{1}{2} \, dx \quad \text{for } u, v \in L^2(\Gamma),
\]

\[
(u, v)_{1,h} := \int_\Gamma g(A^{-1}\nabla_{\Gamma} u, \nabla_{\Gamma} v)|\det A|^\frac{1}{2} \, dx \quad \text{for } u, v \in H^1(\Gamma),
\]

respectively.

We adopt the convention that for every norm, the same norm subscripted with \( h \) implies that the norm is taken with respect to the metric \( g_h \) lifted from \( \Gamma_h \) to \( \Gamma \). For example, \( \| \cdot \|_{L^2_h} \) is the \( L^2 \) norm in the metric \( g_h \).

REMARK 3.3 The discretization described in Section 3.1 follows the approach of Wardetzky (2006), but parallels to some extent the theory of Dziuk (1988) (who considers inscribed meshes and \( \sigma = 2 \)). The metric distortion tensor \( A \) corresponds to a combination of Dziuk’s operators \( P, P_h, I - dH \).

The significance of using these inner products together with the lifting defined by \( \Phi \) lies in the fact that this allows us to work on the smooth surface \( \Gamma \), even when considering operations on the polyhedral surface \( \Gamma_h \), thus simplifying the comparison between solutions to differential equations. We cannot
compute finite element operators on \( \Gamma \) without numerical integration, but we can compute them on the piecewise triangular \( \Gamma_h^2 \). Consequently, from now on we will exclusively work on the smooth surface \( \Gamma \). In Section 4, this will allow us to introduce a discrete mixed finite element problem (with solutions \( u_{h1}, u_{h2} \)) on the surface \( \Gamma \) that is equivalent to the discrete problem (with solutions \( \tilde{u}_{h1}, \tilde{u}_{h2} \)). It is this new discrete mixed problem on \( \Gamma \) with the modified inner products from (3.2) that allows us to compute error bounds that also hold for \( \tilde{u}_{h1}, \tilde{u}_{h2} \).

In order to bound certain geometric errors for our finite element spaces later on, we make use of explicit bounds on the entries of the matrix field \( A \) that describes the pulled back metric \( g_h \) in terms of the smooth metric \( g \). From now on, the statement “for small enough \( h \)” is implied everywhere.

**Lemma 3.1** It holds that

\[
\|A - \text{Id}\|_{L^\infty} \leq Ch^\sigma, \\
\left\| \left| \det A \right|^{\frac{1}{2}} - 1 \right\|_{L^\infty} \leq Ch^\sigma, \\
\left\| \left| \det A \right|^{\frac{1}{2}} A^{-1} - \text{Id} \right\|_{L^\infty} \leq Ch^\sigma, \\
\left\| \left| \det A \right|^{-\frac{1}{2}} A - \text{Id} \right\|_{L^\infty} \leq Ch^\sigma,
\]

where the \( L^\infty \) norm is the essential supremum over the operator norms of the respective matrix fields. The scalar \( \sigma > 0 \) depends on approximation properties of the mesh and is defined in Definition 3.3.

**Proof.** By Wardetzky (2006, Theorem 3.2.1), for any point on \( \Gamma \) where \( A \) is defined and for any orthonormal tangent frame there exists a matrix decomposition \( A = PQP \) such that \( P, Q \) can be diagonalized (possibly in different bases) as

\[
P = \begin{pmatrix}
1 - \phi \kappa_1 & 0 \\
0 & 1 - \phi \kappa_2
\end{pmatrix}, \\
Q = \begin{pmatrix}
\frac{1}{\langle N,N_h \rangle} & 0 \\
0 & 1
\end{pmatrix},
\]

where \( \phi \) is the pointwise distance between \( \Gamma \) and \( \Gamma_h \) under the map \( \Phi \), \( N \) and \( N_h \) are the surface normals of \( \Gamma \) and \( \Gamma_h \) respectively, and \( \kappa_1, \kappa_2 \) are the principal curvatures of the surface. Therefore,

\[
\left| \det A \right|^{\frac{1}{2}} = \frac{|1 - \phi \kappa_1| |1 - \phi \kappa_2|}{|N \cdot N_h|} \approx \frac{|1 - \phi \kappa_1 - \phi \kappa_2|}{|N \cdot N_h|} \approx \frac{|1 - \phi \kappa_1 - \phi \kappa_2|}{|1 - \angle(N,N_h)^2/2|},
\]

where \( \angle(N,N_h) \) denotes the unsigned angle between the two vectors \( N, N_h \) in \( \mathbb{R}^3 \) and we have dropped higher order terms in \( \phi \) and \( \angle(N,N_h) \). A simple Taylor expansion then gives

\[
\left| \det A \right|^{\frac{1}{2}} - 1 \approx -\phi \kappa_1 - \phi \kappa_2 + \frac{1}{2} \angle(N,N_h)^2,
\]

which proves the estimate for \( \left\| \left| \det A \right|^{\frac{1}{2}} - 1 \right\|_{L^\infty} \), given that \( |\phi| \leq C h^\gamma (C3), |\angle(N,N_h)| \leq C h^\epsilon (C4), \) and \( \sigma = \min(\gamma, 2\epsilon) \) (Definition 3.3). A similar argument works for the other three expressions.

\(^2\) with some exceptions at the boundary
For inscribed meshes and $\sigma = 2$, Lemma 3.1, parallels the inequality on $A_h$ found in Dziuk (1988, Section 5).

**Lemma 3.2** Let $L^2_h(\Gamma)$ denote the $L^2$ space on $\Gamma$ where integration happens with the volume element of the lifted metric $g_h$. Let $H^1(\Gamma), H^1_0(\Gamma), W^{1,\infty}(\Gamma)$ denote the $H^1, H^1_0, W^{1,\infty}$ spaces on $\Gamma$ where the gradient is taken with respect to the lifted metric $g_h$, and integration happens with the volume element of the lifted metric $g_h$.

The following equalities hold as equalities of sets:

$$L^2(\Gamma) = L^2_h(\Gamma),$$
$$H^1(\Gamma) = H^1_h(\Gamma),$$
$$H^1_0(\Gamma) = H^1_0(\Gamma),$$
$$W^{1,\infty}(\Gamma) = W^{1,\infty}_h(\Gamma),$$

where $\Phi$ denotes the action of $\Phi$ mapping between sets.

The norms of the respective spaces are all equivalent independently of the choice of $h$ (for $h$ small enough).

**Proof.** Since $\Phi$ is bijective, every function on $\Gamma$ can be uniquely identified with a function on $\Gamma_h$, using $\Phi$ to lift functions.

Under the map $\Phi$, the inner products $(\cdot, \cdot)_{0,h}$ and $(\cdot, \cdot)_{1,h}$ that generate the norms of the spaces $L^2_h(\Gamma), H^1_h(\Gamma), \text{and } H^1_0(\Gamma)$ are exactly the inner products that generate the norms of the spaces $L^2(\Gamma), H^1(\Gamma), \text{and } H^1_0(\Gamma)$, respectively. Then (3.2) and Lemma 3.1 imply the respective norm equivalences with $L^2(\Gamma), H^1(\Gamma), \text{and } H^1_0(\Gamma)$. The pointwise estimates in the proof of Lemma 3.1 imply the equivalence of $W^{1,\infty}(\Gamma)$ and $W^{1,\infty}_h(\Gamma)$, as well as $W^{1,\infty}_h(\Gamma)$ under the map $\Phi$. □

**Remark 3.4** The equivalence of norms independently of $h$ also implies a Poincaré inequality for the pulled back cone metric norms, where the constant is independent of $h$,

$$\|u\|_{L^2_h} \leq C\|u\|_{H^1_0}, \quad \forall u \in H^1_0,$$
$$\|u - u_{F,h}\|_{L^2_h} \leq C\|u\|_{H^1_0} \quad \forall u \in H^1,$$

where $u_{F,h}$ is the average of $u$ with respect to integration in the pulled back metric $g_h$. These Poincaré inequalities can be derived from the smooth inequalities on $\Gamma$ using Lemma 3.2.

In order to quantify the differences between the inner products with respect to the smooth metric $g$ and the pulled back metric $g_h$, we introduce the following bilinear forms:

**Definition 3.5** The difference bilinear forms are defined as follows:

$$c(u,v) := (u,v)_0 - (u,v)_{0,h},$$
$$d(u,v) := (u,v)_1 - (u,v)_{1,h}. \quad (3.3)$$
LEMMA 3.3 The difference bilinear forms satisfy
\[ |c(u,v)| \leq Ch^2 \|u\|_{L^2} \|v\|_{L^2}, \quad \forall u,v \in L^2, \]
\[ |d(u,v)| \leq Ch^2 \|u\|_{H^1_0} \|v\|_{H^1_0}, \quad \forall u,v \in H^1. \]

Proof. This is a consequence of Lemma 3.1 and the Cauchy–Schwarz inequality. \(\square\)

3.2 Discretizing the function spaces

Having discretized the geometry of \(\Gamma\), we now turn to the approximation of function spaces.

DEFINITION 3.6 Let \(\hat{S}_h\) be the space of continuous functions on the mesh \(\Gamma_h\) that are linear within each triangle. On boundary triangles (which might have a curved edge), an isoparametric modification is applied (Zl´amal, 1973; Scott, 1973), which projects a curved edge of a triangle onto a straight edge while keeping the vertices fixed and minimizing distortion.\(^3\) On the resulting triangle, functions are required to be linear. \(\hat{S}_{h,0} := \hat{S}_h \cap H^1_0(\Gamma_h)\) is the space of finite element functions that evaluate to zero at the boundary.

Let \(S_h\) be the lift of the function space \(\hat{S}_h\) under the inverse of the closest point projection \(\Phi\). The space \(S_h\) has domain \(\Gamma\), and is a subset of \(H^1\) (as a set of functions). Analogously, we define the discrete space \(S_{h,0} := S_h \cap H^1_0\).

REMARK 3.5 Because the inner products \((\cdot, \cdot)_{0,h}, (\cdot, \cdot)_{1,h}\) defined in (3.2) arise from the mesh’s metric lifted to the surface \(\Gamma\), the Hilbert spaces \((S_{h,0}, (\cdot, \cdot)_{1,h}), (S_h, (\cdot, \cdot)_{1,h} + (\cdot, \cdot)_{0,h})\) are isometric to the corresponding Hilbert spaces generated by \(\hat{S}_{h,0}, \hat{S}_h\) (using the metric \(g_{\hat{\Gamma}}\)) on \(\hat{\Gamma}_h\).

Using inner products (2.1) derived from the smooth surface’s metric \(g\) leads to different Hilbert spaces \((S_{h,0}, (\cdot, \cdot)_1), (S_h, (\cdot, \cdot)_1 + (\cdot, \cdot)_0)\) that are not isometric to \((S_{h,0}, (\cdot, \cdot)_{1,h}), (S_h, (\cdot, \cdot)_{1,h} + (\cdot, \cdot)_{0,h})\). This difference will lead us to formulate two different mixed finite element problems, (4.2) and (4.1), each using different metrics.

REMARK 3.6 The number of degrees of freedom of \(S_h\) is the number of mesh vertices. The number of degrees of freedom of \(S_{h,0}\) is the number of mesh vertices minus the number of boundary vertices.

REMARK 3.7 The behavior of the discrete functions spaces on the boundary triangles is the same as in the work of Scholz (1978), who states: “Let \(S_h = S_h(\Gamma)\) be the space of continuous functions which are linear in each triangle of \(\Gamma_h\) with the usual modification for the curved elements (see Ciarlet & Raviart (1972); Zl´amal (1973)).” (Scholz, 1978, p. 2).

The discrete spaces come with various interpolation operators.

DEFINITION 3.7 Let \(I_h : H^2 \rightarrow S_h\) denote the per-vertex interpolation operator,\(^4\) i.e., \((I_h u)(p) = u(p)\) for all nodes.

---

\(^3\)This treatment of the boundary does not appear in the work of Wardetzky (2006). The main results of Wardetzky, however, remain true if an optimal isoparametric element is chosen (Bernardi, 1989), with minor modifications to the proofs, as long as Conditions (C1-C4) hold, see Remark 3.1.

\(^4\)The Sobolev embedding theorem implies that \(H^2(\Gamma) \subseteq C^0(\Gamma)\), which justifies pointwise interpolation.
Moreover, $R_h, R_h^{(b)} : H^1 \to S_h$ and $R_{h,0}, R_{h,0}^{(b)} : H^1_{0} \to S_{h,0}$ are the Ritz projection operators,

\[ (u - R_h u, \eta)_1 = (u - R_h^{(b)} u, \eta)_1, \quad \forall \eta \in S_h, \ u \in H^1 \]

\[ (u - R_h u, 1)_0 = (u - R_h^{(b)} u, 1)_0, \quad \forall \eta \in S_h, \ u \in H^1 \]

\[ (u - R_{h,0} u, \xi)_1 = (u - R_{h,0}^{(b)} u, \xi)_1, \quad \forall \xi \in S_{h,0}, \ u \in H^1_{0} \]

Ritz projections are a widely used tool in finite element analysis (Rannacher & Scott, 1982; Du et al., 2011; Dziuk & Elliott, 2013a). They are an important ingredient to show convergence of the discrete solution on the surface to the exact solution on the surface.

Standard arguments together with the equivalence of norms from Lemma 3.2 yield:

**LEMMA 3.4** The Ritz projection is $H^1$-stable, i.e.,

\[ \|R_{h,0} u\|_{H^1_{0}} + \|R_{h,0}^{(b)} u\|_{H^1_{0}} \leq C \|u\|_{H^1_{0}}, \quad \forall u \in H^1_{0} \]

\[ \|R_h u\|_{H^1} + \|R_h^{(b)} u\|_{H^1} \leq C \|u\|_{H^1}, \quad \forall u \in H^1 \]

Moreover, the interpolation operators satisfy certain interpolation inequalities. While these results are classical for flat domains, they require more work in the curved regime due to the presence of second derivatives in standard interpolation estimates.

**LEMMA 3.5** For $u \in H^1_{0} \cap H^2$ one has

\[ \|u - I_h u\|_{H^1_{0}} \leq C h \|u\|_{H^2} \]

\[ \|u - R_{h,0} u\|_{H^1_{0}} \leq C h \|u\|_{H^2} \]

\[ \|u - R_{h,0}^{(b)} u\|_{H^1_{0}} \leq C h \|u\|_{H^2} \]

Analogous results hold for $H^1$-functions that are nonzero at the boundary using the appropriate interpolation operators.

Let $T = \Psi(\hat{T}) \subset \Gamma$ denote a curved triangle that is the image of a flat triangle $\hat{T} \subset \Gamma_h$ under the closest point projection. Suppose that $u \in H^1_{0}$ is continuous and lies in $H^2(T)$ (resp. $W^{2,\infty}(T)$) for each such curved triangle $T$. Then one has

\[ \|u - I_h u\|_{\hat{H}^1_{0}} \leq C h \|u\|_{\hat{H}^2} \]

\[ \|u - I_h u\|_{\hat{H}^1_{0}} \leq C h \|u\|_{\hat{W}^{2,\infty}} \]

where the tilde above the norm indicates a per-triangle norm, summed over all triangles $T$ in the triangulation: \( \|u\|_{\hat{H}^2}^2 = \sum_T \|u\|_{H^2(T)}^2, \ \|u\|_{\hat{W}^{2,\infty}} = \max_T \|u\|_{W^{2,\infty}(T)} \).

**Proof.** The estimate in the first line of (3.4) follows from the first line of (3.5). The estimates in the second and third lines of (3.4) follow from the first line of (3.4) and the $H^1$-stability from Lemma 3.4. It thus remains to show (3.5).
4. Mixed Finite Elements

With the discrete geometry and discrete function spaces in place, we can now turn towards discretizing the problem (2.3) in its mixed form (2.4).

Using the two inner products (2.1) and (3.2) on \( \Gamma \) leads to two discrete mixed problems. In practice, one solves the discrete problem (1.5) on the mesh, which, using the inner products (3.2), is equivalent to

\[
\begin{align*}
(u_1^h, \xi)_{1,h} &= (f, \xi)_{0,h}, \\
(u_1^h, \eta)_{1,h} &= (u_2^h, \eta)_{0,h} \quad \forall \eta \in S_h,
\end{align*}
\]

(4.1)

where \( u_1^h \in S_{h,0}, \ u_2^h \in S_h, \) and \( f \in L^2 \). Having formulated an equivalent problem to (1.5) on the surface \( \Gamma \) itself, we can now compare \( u_1^h \) to \( u_1 \) and \( u_2^h \) to \( u_2 \), and attempt to derive an error estimate.

We additionally make use of a similar discrete problem with respect to the inner products (2.1), i.e.,

\[
\begin{align*}
(\tilde{u}_2^h, \xi)_{1} &= (f, \xi)_0, \\
(\tilde{u}_1^h, \eta)_{1} &= (\tilde{u}_2^h, \eta)_0 \quad \forall \eta \in S_h,
\end{align*}
\]

(4.2)
where \( \tilde{u}_h \in S_{h,0}, \tilde{u}_h^2 \in S_h, \) and \( f \in L^2 \). This problem is only an auxiliary problem for our proof. Its operators are never computed in practice. If the surface has no boundary, the solutions of (4.2) and (4.1) additionally have to fulfill the zero mean property from Definition 2.1.

In the planar case of Scholz (1978), where \( \Gamma \subseteq \mathbb{R}^2 \) is a planar domain, the two discrete problems (4.2) and (4.1) coincide.

Existence and uniqueness for (4.2) and (4.1) follow from an argument by Ciarlet (2002, Section 7), which we repeat here for convenience.

**Definition 4.1** We define the following three Hilbert spaces:

\[
V := \{(v_1, v_2) \in H^1_0 \times L^2 \mid (v_1, \mu)_1 = (v_2, \mu)_0 \forall \mu \in H^1\},
\]

\[
\tilde{V}^h := \{(v_1, v_2) \in S_{h,0} \times S_{h}\}_{L^2} \mid (v_1, \mu)_1 = (v_2, \mu)_0 \forall \mu \in S_h\},
\]

\[
V^h := \{(v_1, v_2) \in S_{h,0} \times S_{h}\}_{L^2} \mid (v_1, \mu)_{1,h} = (v_2, \mu)_{0,h} \forall \mu \in S_h\},
\]

where the space \( S_{h,0} \) is the space \( S_h \), but with the \( L^2 \) norm instead of its usual \( H^1 \) norm.

As an immediate consequence of Poincaré’s inequality we obtain that the resulting product norms on these spaces are equivalent to (simpler) norms that we heavily rely on going forward:

**Lemma 4.1** The product norms on \( V, \tilde{V}^h, V^h \) are equivalent to the norms induced by the inner products

\[
\begin{align*}
((v_1, v_2), (w_1, w_2)) &\mapsto (v_2, w_2)_0 \quad \text{on } V, \tilde{V}^h, \\
((v_1, v_2), (w_1, w_2)) &\mapsto (v_2, w_2)_{0,h} \quad \text{on } V^h.
\end{align*}
\]

**Proof.** The symmetric bilinear forms defined by (4.4) are indeed positive definite since \( v_2 = 0 \) implies \( v_1 = 0 \) for all elements \( (v_1, v_2) \) in \( V, \tilde{V}^h, V^h \). Poincaré’s inequality implies that

\[
\|v_1\|_{H^1_0}^2 = (v_1, v_1)_1 = (v_2, v_1)_0 \leq \|v_2\|_{L^2} \|v_1\|_{L^2} \leq C \|v_2\|_{L^2} \|v_1\|_{H^1_0}.
\]

Therefore, \( \|v_1\|_{H^1_0} \leq C \|v_2\|_{L^2} \), which proves the lemma for \( V \). By Remark 3.4, an identical derivation holds for \( \tilde{V}^h, V^h \). \( \square \)

**Definition 4.2** On the linear spaces \( V, \tilde{V}^h, V^h \) we can define the functionals

\[
J((v_1, v_2)) := \frac{1}{2} (v_2, v_2)_0 - F((v_1, v_2)) \quad \text{on } V \text{ and } \tilde{V}^h,
\]

\[
J_h((v_1, v_2)) := \frac{1}{2} (v_2, v_2)_{0,h} - F((v_1, v_2)) \quad \text{on } V^h,
\]

for a dual function \( F \in V', (\tilde{V}^h)' \), or \( (V^h)' \), respectively. The explicit dual functions used to solve the mixed finite element problems are introduced in Lemma 4.2.

As a direct consequence of Lemma 4.1 and the Riesz representation theorem, the functionals from (4.5) have unique minimizers:

**Lemma 4.2** (Existence and uniqueness for the mixed biharmonic problem) The functionals \( J \) on \( V, J \) on \( \tilde{V}^h \), and \( J_h \) on \( V^h \) have unique minimizers \( u := (u_1, u_2), \tilde{u}_h := (\tilde{u}_h^1, \tilde{u}_h^2) \), and \( u^h := (u^h_1, u^h_2) \), respectively.
Denoting by \( R : V \rightarrow V' \), \( \tilde{R}^h : \tilde{V}^h \rightarrow (\tilde{V})'^h \), and \( R^h : V^h \rightarrow (V^h)' \) the respective Riesz maps, and using the inner products defined in Lemma 4.1, these minimizers solve the mixed biharmonic equation and can be written as

\[
\mathcal{R} u = F, \quad \tilde{R}^h \tilde{u}^h = \tilde{F}^h, \quad R^h u^h = F^h, \tag{4.6}
\]

for \( F(v_1, v_2) := (v_1, f)_0 \) on \( V \), \( \tilde{F}^h(v_1, v_2) := (v_1, f)_0 \) on \( \tilde{V}^h \), and \( F^h(v_1, v_2) := (v_1, f)_{0,h} \) on \( V^h \).

**Remark 4.1** Lemma 4.2 ensures existence and uniqueness of the mixed formulation of the biharmonic equation. In particular, if the domain is a smooth surface with smooth boundary, then the respective minimizer on \( V \) solves the weak biharmonic equation (2.3). Notice, however, that the respective minimizer on \( V \) does not necessarily solve the weak biharmonic equation (2.3) if the domain does not satisfy appropriate regularity conditions, e.g., if the domain has reentrant corners (Stylianou, 2010, Section 4.3). This does not pose a problem for the smooth solution \( u_1 \), since we work with smooth surfaces for which standard regularity estimates hold. However, similar issues impact the discrete solutions, since Lemma 4.2 can only be used to show that the \( H^1_0 \) norms of \( \tilde{u}^h_1, u^h_1 \) are bounded independently of \( h \), and likewise, that the \( L^2 \) norms of \( \tilde{u}^h_2, u^h_2 \) are bounded independently of \( h \). We cannot infer boundedness of the \( H^1 \) norms of \( \tilde{u}^h_1, u^h_1 \) independently of \( h \) – these norms can (and in certain cases will) blow up as \( h \) decreases. We address this issue in the next section.

We end this section with a table visualizing all of our three settings, the functions spaces used for each of them, and the solutions that each of them contains (Table 1). It is important to note here that all functions and function spaces are defined exclusively on the smooth surface \( \Gamma \), making comparisons between functions from different settings possible.

### 5. Convergence of the Numerical Method

It is somewhat surprising that the derivatives of \( \tilde{u}^h_2, u^h_2 \) appear in the linear systems that we are solving, but the \( L^2 \)-norms of these derivatives cannot be bounded independently of \( h \). This indeed complicates the task of bounding errors between solutions of (2.4), (4.2), and (4.1). Scholz (1978) elegantly solves this issue by utilizing the Ritz projection in order to cancel contributions of derivatives of \( \tilde{u}^h_2 \). In the curved case, an argument similar to Scholz’s’ only serves to show that \( \tilde{u}^h_2 \) converges to \( u_2 \), and \( \tilde{u}^h_1 \) converges to \( u_1 \). In particular, for the case of curved geometries, one also must account for the approximation of the curved surface by a piecewise flat surface, i.e., to show that \( u^h_2 \) converges to \( \tilde{u}^h_2 \). This is precisely why the curved case is more intricate than the flat one.

#### 5.1 Convergence of the discrete problem on the mesh to the discrete problem on the surface

In this section and Section 5.2 we treat the case of surfaces with boundary; the case of surfaces without boundary (treated later) is significantly simpler.

So far, our treatment for the mixed biharmonic problem has considered the smooth setting alongside the two discrete settings. The next step, however, only works in the two discrete settings. We define a discrete Laplace operator that maps into the \( L^2 \)-like space \( S_{h, L^2} \) from \( S_{h, 0} \).
LEMMA 5.1 (Discrete Laplacians) There exist bounded linear and injective operators $\tilde{L}^h, L^h : S_{h,0} \to S_{h,L^2}$ such that

\[
\begin{align*}
(v_1, \tilde{L}^h v_1) &\in \tilde{V}^h & \forall v_1 \in S_{h,0}, \\
(v_1, L^h v_1) &\in V^h & \forall v_1 \in S_{h,0}.
\end{align*}
\]

(5.1)

Moreover, every element in $\tilde{V}^h$ can be written as the pair $(v_1, \tilde{L}^h v_1)$, and every element in $V^h$ can be written as the pair $(v_1, L^h v_1)$.

Proof. We prove the lemma for $\tilde{L}^h$; the proof for $L^h$ is similar. For all $(v_1, v_2) \in \tilde{V}^h$ we have that

\[(v_1, \mu)_1 = (v_2, \mu)_0 &\quad \forall \mu \in S_h.
\]

Written as a discrete linear equation, the right-hand side involves the mass matrix $M$ for piecewise linear (except, potentially, on boundary triangles) Lagrange finite elements. This matrix is invertible. We can thus define $\tilde{L}^h := M^{-1} \Sigma$, where $\Sigma$ denotes the discrete Laplacian stiffness matrix with columns in $S_{h,0}$ and rows in $S_h$. The resulting operator is well-defined and linear. Injectivity follows from the solvability...
of the Poisson equation. Indeed, \((v_1, \eta) = 0 \forall \eta \in S_{0,0}\) has a unique solution \(v_1 = 0 \in S_{0,0}\). This implies that the discrete Laplacian stiffness matrix \(\Sigma\) is injective, and hence \(\bar{L}^h\) is injective.

It remains to show that every element in \(\bar{V}^h\) can be written as a pair \((v_1, \bar{L}^h v_1)\). In order to see that, let \((v_1, v_2) \in \bar{V}^h\). Then, the definition of \(\bar{V}^h\) implies that

\[
0 = (0, \mu) = (v_2 - \bar{L}^h v_1, \mu) \quad \forall \mu \in S_h.
\]

Therefore, \(\bar{L}^h v_1 = v_2\).

**Remark 5.1** The linear operators \(\bar{L}^h, L^h\) are bounded, as they are discrete operators. This bound, however, is not independent of \(h\).

The next result is central for relating solutions from the two discrete spaces.

**Lemma 5.2 (Inverse estimate)** We have that

\[
\left\| \bar{L}^h v_1 - L^h v_1 \right\|_{L^2} \leq C h^{\sigma - 1} \| v_1 \|_{H_0^1} \quad \forall v_1 \in S_{0,0},
\]

where the constant \(C\) is independent of \(h\), and where \(\sigma\) was defined in Definition 3.3.

**Proof.** For \(\mu \in S_{h, L^2}\) and \(c, d\) as defined in Definition 3.5, it holds that

\[
\left( \bar{L}^h v_1 - L^h v_1, \mu \right)_0 = \left( \bar{L}^h v_1, \mu \right)_0 - \left( L^h v_1, \mu \right)_{0,h} - c(L^h v_1, \mu) = (v_1, \mu)_1 - (v_1, \mu)_{1,h} - c(L^h v_1, \mu)
\]

\[
= d(v_1, \mu) - c(L^h v_1, \mu) \leq C h^{\sigma} \| v_1 \|_{H_0^1} \| \mu \|_{H_0^1} + C h^{\sigma} \left\| L^h v_1 \right\|_{L^2} \| \mu \|_{L^2}.
\]

Using the standard inverse estimate, we have that \(\| L^h v_1 \|_{H_0^1} \leq C h^{-1} \| L^h v_1 \|_{L^2}\) (Braess, 2007, II 6.8).

Therefore,

\[
\left\| L^h v_1 \right\|_{L^2}^2 \leq C \left\| L^h v_1 \right\|_{L^2}^2 = C \left( v_1, L^h v_1 \right)_{1,h} \leq C h^{-1} \| v_1 \|_{H_0^1} \| L^h v_1 \|_{L^2}^2,
\]

which proves the lemma after applying another inverse estimate to \(\mu\).

**Lemma 5.3** Let \(W^h : \bar{V}^h \to V^h\) be the linear map such that

\[
W^h \left( (v_1, \bar{L}^h v_1) \right) = (v_1, L^h v_1).
\]

\(W^h\) is well-defined, bounded independently of \(h\), invertible, and the inverse is also bounded independently of \(h\). Additionally,

\[
\left\| (W^h)^* W^h - \text{Id} \right\| \leq C h^{\sigma - 1},
\]

where \((W^h)^*\) is the linear adjoint of \(W^h\).
We now show boundedness of $W^h$; a similar argument works to show boundedness of the inverse. Let $(v_1, \tilde{L}^h v_1) \in \tilde{V}^h$. Then

$$\left\| \left( W^h \left( v_1, \tilde{L}^h v_1 \right) \right) \right\|_{L^2} \leq C \left\| L^h v_1 \right\|_{L^2} + C \left\| L^h v_1 - \tilde{L}^h v_1 \right\|_{L^2} \leq C \left\| L^h v_1 \right\|_{L^2} + C \left\| L^h v_1 - \tilde{L}^h v_1 \right\|_{L^2},$$

where the last inequality follows from Lemma 5.2. Using that $\sigma > 1$ and the equivalence of norms from Lemma 4.1, $W^h$ is bounded independently of $h$.

We have

$$\left\| (W^h)^* W^h - \text{Id} \right\| = \sup_{(v_1, L^h v_1) \in \tilde{V}^h} \frac{1}{\left\| L^h v_1 \right\|_{L^2}^2} \left( \left( L^h v_1, L^h v_1 \right)_{0,h} - \left( \tilde{L}^h v_1, \tilde{L}^h v_1 \right)_0 \right),$$

where, again, we have used the equivalence of norms from Lemma 4.1. Now,

$$\left( L^h v_1, L^h v_1 \right)_{0,h} - \left( \tilde{L}^h v_1, \tilde{L}^h v_1 \right)_0 = \left( L^h v_1, L^h v_1 - \tilde{L}^h v_1 \right)_{0,h} + \left( L^h v_1, \tilde{L}^h v_1 \right)_{0,h} - \left( \tilde{L}^h v_1, \tilde{L}^h v_1 \right)_0$$

$$= \left( L^h v_1, L^h v_1 - \tilde{L}^h v_1 \right)_{0,h} + \left( v_1, \tilde{L}^h v_1 \right)_{1,h} - \left( v_1, \tilde{L}^h v_1 \right)_1 \right)$$

$$\leq \left\| L^h v_1 \right\|_{L^2} \left\| L^h v_1 - \tilde{L}^h v_1 \right\|_{L^2} + d(v_1, \tilde{L}^h v_1),$$

where $d$ is the bilinear difference form from Definition 3.5, and where we used the definition of $\tilde{L}^h, L^h$ from Lemma 5.1. By the equivalence of norms from Lemma 3.2, Lemma 5.2, and (5.3),

$$\left\| L^h v_1 \right\|_{L^2} \left\| L^h v_1 - \tilde{L}^h v_1 \right\|_{L^2} \leq C \left\| L^h v_1 \right\|_{L^2} \left\| L^h v_1 - \tilde{L}^h v_1 \right\|_{L^2} \leq C \left\| L^h v_1 \right\|_{L^2} \left\| v_1 \right\|_{H^1_H}.$$  (5.5)

By Lemma 3.3 as well as the standard inverse estimate for discrete Lagrangian functions that states $\|w\|_{H^1} \leq C^{-1} \|w\|_{L^2}$ for all $w \in S_h$ (Braess, 2007, II 6.8),

$$\left| d(v_1, \tilde{L}^h v_1) \right| \leq C \left\| v_1 \right\|_{H^1_H} \left\| \tilde{L}^h v_1 \right\|_{H^1_H} \leq C \left\| v_1 \right\|_{H^1_H} \left\| \tilde{L}^h v_1 \right\|_{L^2}.$$  (5.6)

Substituting (5.5) and (5.6) into (5.4) and applying Lemma 4.1 then gives

$$\left| \left( L^h v_1, L^h v_1 \right)_{0,h} - \left( \tilde{L}^h v_1, \tilde{L}^h v_1 \right)_0 \right| \leq C \left\| \tilde{L}^h v_1 \right\|_{L^2}^2,$$

which proves the lemma. $\square$
We denote by $(W^h)' : (V^h)' \to (\tilde{V}^h)'$ the dual operator to $W^h$. $(W^h)'$ is bounded independently of $h$, it is invertible, and its inverse is bounded independently of $h$. This is true by the same argument as the one for $W^h$. If $f : V^h \to \mathbb{R}$,

$$
\left( (W^h)'(f) \left( (v_1, \tilde{L}^h v_1) \right) \right) = (f \circ W^h) \left( (v_1, \tilde{L}^h v_1) \right) = f \left( (v_1, L^h v_1) \right),
$$

$$
\left\| (W^h)'(f) \left( (v_1, \tilde{L}^h v_1) \right) \right\| \leq \|f\|_{(V^h)'(V^h)} \left\| \tilde{L}^h v_1 \right\|_{L^2} \text{ by Lemma 4.1}
$$

$$
\leq \|f\|_{(V^h)'(V^h)} \left\| \tilde{L}^h v_1 \right\|_{L^2} \text{ by Lemma 5.2}
$$

$$
\leq \|f\|_{(V^h)'(V^h)} \left\| \tilde{L}^h v_1 \right\|_{\tilde{V}^h} \text{ by Lemma 4.1}.
$$

The operators $W^h$, $(W^h)'$, and $(W^h)^*$ provide the tool for relating the two discrete problems.

We mention a lemma that is true for maps between Hilbert spaces in general, independent of our particular setting:

**Lemma 5.4** Let $\Lambda : X \to Y$ be a bijective bounded linear map between Hilbert spaces, and let $R_X : X \to X'$, $R_Y : Y \to Y'$ be the respective Riesz maps. Let $\alpha \in X'$ and $\beta \in Y'$ be given. Let $\tilde{x} \in X$ be such that

$$
R_X(\tilde{x}) = \alpha \text{ and let } \tilde{y} \in Y \text{ be such that } R_Y(\tilde{y}) = \beta.
$$

Then we have that

$$
\|\Lambda \tilde{x} - \tilde{y}\| \leq \|\text{Id} - \Lambda^* \Lambda\| \|\alpha\| \|\Lambda^{-1}\| + \|\beta - (\Lambda')^{-1} \alpha\|.
$$

**Proof.** Let $y \in Y$ arbitrary such that $\|y\| = 1$, and let $\tilde{x} := \Lambda^{-1} y$. Then

$$
\langle \Lambda \tilde{x} - \tilde{y}, y \rangle = \langle \Lambda \tilde{x}, y \rangle - \langle \tilde{y}, y \rangle = \langle \Lambda \tilde{x}, \Lambda x \rangle - \langle \tilde{x}, x \rangle + \alpha(x) - \beta(y)
$$

$$
\leq \|\text{Id} - \Lambda^* \Lambda\| \|\tilde{x}\| \|\Lambda^{-1} y\| + \|\beta - (\Lambda')^{-1} \alpha\| \|y\|,
$$

which proves the lemma, since $\|\tilde{x}\| = \|\alpha\|$, and $y$ was arbitrary with norm 1. \qed

We can now bound the error between the solutions of our two discrete problems from Lemma 4.2.

**Lemma 5.5** Consider the following two linear problems for $\tilde{F}^h \in (\tilde{V}^h)'$, $F^h \in (V^h)'$,

$$
\tilde{R}^h u^h = \tilde{F}^h \quad \text{and} \quad R^h u^h = F^h.
$$

Then we have that

$$
\|W^h \tilde{u}^h - u^h\|_{V^h} \leq C h^{\sigma-1} \left\| \tilde{F}^h \right\|_{(\tilde{V}^h)'} + C \left\| F^h - ((W^h)')^{-1} \tilde{F}^h \right\|_{(V^h)'}.
$$

**Proof.** Combining Lemma 5.3 and Lemma 5.4 with $X = \tilde{V}^h, Y = V^h$, the statement immediately follows. \qed

The main result of this section relates the two discrete systems (4.2) and (4.1) that we use in our mixed finite element method.

**Theorem 5.1** Let $\tilde{u}^h_1, \tilde{u}^h_2$ solve problem (4.2), and let $u^h_1, u^h_2$ solve problem (4.1). Then

$$
\left\| u^h_2 - \tilde{u}^h_2 \right\|_{L^2} \leq C h^{\sigma-1} \left\| f \right\|_{L^2}.
$$
Proof. Let \( \bar{u}^h = (\bar{u}_1^h, \bar{u}_2^h) \), \( u^h = (u_1^h, u_2^h) \), \( \bar{F}^h((v_1, v_2)) = (f, v_1)_0 \), \( F^h((v_1, v_2)) = (f, v_1)_0 \). Then

\[
\|u_2^h - \bar{u}_2^h\|_{L^2} \leq C \|L^h u_1^h - L^h \bar{u}_1^h\|_{L^2} + \|L^h \bar{u}_1^h - L^h \bar{u}_1^h\|_{L^2},
\]

(by (5.2))

\[
\leq C \|u^h - W^h \bar{u}^h\|_{V_h} + \|L^h \bar{u}_1^h - L^h \bar{u}_1^h\|_{L^2} \quad \text{(by Lemma 5.2)}
\]

\[
\leq C \|u^h - W^h \bar{u}^h\|_{V_h} + Ch^\sigma-1 \|u_1^h\|_{H_0^\sigma} \quad \text{(by Lemma 4.1)}
\]

\[
\leq C \|F^h - ((W^h)^{-1}F^h)\|_{L^2(V_h')} + Ch^\sigma-1 \|F^h\|_{L^2(V_h')}. \quad \text{(by Lemma 5.5)}
\]

It remains to deal with the right-hand sides of the last inequality. By the equivalence of norms,

\[
\|F^h\|_{L^2(V_h')} \leq C \|f\|_{L^2}. \quad \text{For } (v_1, L^h v_1) \in V^h \text{ we have}
\]

\[
\left( F^h - ((W^h)^{-1}F^h) \left( (v_1, L^h v_1) \right) \right) = (f, v_1)_0 - (f, v_1)_0 = c(f, v_1), \text{ and hence,}
\]

\[
\left( F^h - ((W^h)^{-1}F^h) \left( (v_1, L^h v_1) \right) \right) \leq Ch^\sigma \|f\|_{L^2} \|v_1\|_{L^2} \quad \text{(by Lemma 4.1)}
\]

\[
\leq Ch^\sigma \|f\|_{L^2} \|L^h v_1\|_{L^2},
\]

\[
\left| F^h - ((W^h)^{-1}F^h) \right|_{L^2(V_h')} \leq Ch^\sigma \|f\|_{L^2}.
\]

This proves the result. \(\square\)

5.2 Convergence of the discrete problem on the surface to the exact solution

Having successfully bounded the error between the discrete problem on the mesh (with solution \( (u_1^h, u_2^h) \)) and the discrete problem on the surface (with solution \( (\bar{u}_1^h, \bar{u}_2^h) \)), we move on to bounding the error between \( (\bar{u}_1^h, \bar{u}_2^h) \) and the exact solution, \( (u_1, u_2) \). Our proof follows the roadmap laid out by Scholz (1978). However, we require considerable adjustments to extend this approach to curved surfaces.

We start with an extension of Scholz’s Lemma to curved surfaces, using a theorem by Demlow (2009) in place of Scholz’s use of a result by Nitsche (1978). Demlow works in the setting of inscribed meshes of Dziuk (1988) and adapts a result by Schatz (1998) from the flat setting. Demlow’s analysis can be adapted to our setting fulfilling conditions (C1-C4) by using Lemma 3.2, Lemma 3.3, and Lemma 3.5. Alternatively, one could also consider generalizing the \( L^\infty \) estimate of Rannacher & Scott (1982) (who offer an improved Nitsche-type bound) to our setting. For details, we refer to the work of Stein (2020, Section 5.2.3).

**Lemma 5.6** (Scholz’s Lemma) Let \( u \in H^1_0 \cap W^{2,\infty} \). Let \( \eta \in \mathcal{S}_h \). Then

\[
\left| (u - R_{h,0} u, \eta) \right|_1 \leq C \sqrt{h} \|u\|_{W^{2,\infty}} \|\eta\|_{L^2}.
\]

**Proof.** Let \( \xi \in \mathcal{S}_{h,0} \) interpolate \( \eta \) on all interior vertices of the mesh. Let \( \varphi := \eta - \xi \).

By the definition of Ritz projection, we have that

\[
(u - R_{h,0} u, \eta) = (u - R_{h,0} u, \varphi) \quad \text{on } \mathcal{S}_{h,0}.
\]
As \( \varphi \) is only supported on the boundary triangles \( T_\partial \), the last equation can be simplified to

\[
\left| \left( u - R_{h,0} u, \eta \right) \right| = \sum_{T \in T_\partial} \left| \nabla T (u - R_{h,0} u) \cdot \nabla \varphi \right| dx \leq C h^2 \left\| u - R_{h,0} u \right\|_{W^{1,\infty}} \sum_{T \in T_\partial} \| \varphi \|_{W^{1,\infty}(T)} ,
\]

where we used the fact that the area of a triangle is bounded by \( Ch^2 \), where the \( C \) depends on the triangle regularity constants.

By the standard inverse estimate we can conclude that \( \| \varphi \|_{W^{1,\infty}(T)} \leq C h^{-1} \| \varphi \|_{L^\infty(T)} \). By definition, \( \| \varphi \|_{L^\infty(T)} \leq C \| \eta \|_{L^\infty(T)} \). Moreover, using a per-triangle calculation, we obtain that \( \| \eta \|_{L^\infty(T)} \leq C h^{-1} \| \eta \|_{L^2(T)} \). Thus we conclude

\[
\left| \left( u - R_{h,0} u, \eta \right) \right| \leq C \left\| u - R_{h,0} u \right\|_{W^{1,\infty}} \sum_{T \in T_\partial} \| \eta \|_{L^2(T)} \leq C h^2 \left\| u - R_{h,0} u \right\|_{W^{1,\infty}} \| \eta \|_{L^2} ,
\]

where we used the fact that the number of triangles in \( T_\partial \) is \( \sim h^{-1} \).

The estimate by Demlow (2009, Theorem 3.2) (which carries over to our setting with non-inscribed triangle meshes with minor modifications) states that

\[
\| R_{h,0} v \|_{W^{1,\infty}} \leq C \| v \|_{W^{1,\infty}} \quad \text{for all } v \in W^{1,\infty} .
\]

Together with Lemma 3.5, this leads to

\[
\left\| u - R_{h,0} u \right\|_{W^{1,\infty}} \leq \left\| u - I_h u \right\|_{W^{1,\infty}} + \left\| I_h u - R_{h,0} u \right\|_{W^{1,\infty}} = \left\| u - I_h u \right\|_{W^{1,\infty}} + \left\| R_{h,0} (I_h u - u) \right\|_{W^{1,\infty}} 
\leq \left\| u - I_h u \right\|_{W^{1,\infty}} + C \left\| I_h u - u \right\|_{W^{1,\infty}} \leq C h \left\| u \right\|_{W^{2,\infty}} ,
\]

which proves the lemma. \( \square \)

Using this lemma we can now estimate the error in \( u_2 \). This mirrors the first part of Theorem 1 by Scholz (1978), but we achieve a bound of order \( \sqrt{h} \) instead of Scholz’s \( \sqrt{h} \log h \) due to the improved Lemma 5.6.

**Theorem 5.2** Let \( u_1, u_2 \) solve the smooth mixed biharmonic problem (2.4), and let \( u_1^h, u_2^h \) solve the discrete mixed biharmonic problem (4.1) on the mesh. Then one has

\[
\| u_2 - u_2^h \|_{L^2} \leq C \sqrt{h} \| f \|_{L^2} .
\]

**Proof.** Using Lemma 3.5, we obtain that

\[
\| u_2^h - u_2 \|_{L^2} \leq \| u_2^h - R_h u_2 \|_{L^2} + \| R_h u_2 - u_2 \|_{L^2} \leq \| R_h u_2 \|_{L^2} + C h \| u_2 \|_{H^2} . \tag{5.8}
\]

Using Theorem 5.1 (together with the fact that \( \sigma \geq \frac{3}{2} \)) and (5.8), we have

\[
\| u_2 - u_2^h \|_{L^2} \leq \| u_2^h - u_2 \|_{L^2} + \| u_2^h - u_2^h \|_{L^2} \leq \| u_2^h - R_h u_2 \|_{L^2} + C h \| u_2 \|_{H^2} + C \sqrt{h} \| f \|_{L^2} . \tag{5.9}
\]

Using (5.9) and the regularity estimate for the smooth problem, we thus obtain

\[
\| u_2 - u_2^h \|_{L^2} \leq \| u_2^h - R_h u_2 \|_{L^2} + C \sqrt{h} \| f \|_{L^2} . \tag{5.10}
\]
It remains to bound \( \| \tilde{u}_2^h - R_h u_2 \|_{L^2} \). To this end, we first note that
\[
\left( \tilde{u}_1^h - R_{h,0} u_1, \tilde{u}_2^h - R_h u_2 \right)_1 = \left( \tilde{u}_1^h - R_{h,0} u_1, \tilde{u}_2^h - u_2 \right)_1 = 0 ,
\]
using the definition of the Ritz projection for the first equality and using the smooth and discrete formulations of the mixed biharmonic problems for the second equality.

Thus we can compute
\[
\| \tilde{u}_2^h - R_h u_2 \|_{L^2}^2 = \left( \tilde{u}_2^h - R_h u_2, \tilde{u}_2^h - u_2 \right)_0 - \left( \tilde{u}_1^h - R_{h,0} u_1, \tilde{u}_2^h - R_h u_2 \right)_1 \\
= \left( u_2 - R_h u_2, \tilde{u}_2^h - u_2 \right)_0 + \left( R_{h,0} u_1 - u_1, \tilde{u}_2^h - R_h u_2 \right)_1 \\
\leq \| u_2 - R_h u_2 \|_{L^2} \| \tilde{u}_2^h - R_h u_2 \|_{L^2} + \left( R_{h,0} u_1 - u_1, \tilde{u}_2^h - R_h u_2 \right)_1 ,
\]
where we again used the (smooth and discrete) formulations of the mixed biharmonic problems. The first of the two summands can be estimated using the estimates for the Ritz projection from (3.4). The second summand is covered by Lemma 5.6 and the fact that \( \| u_1 \|_{W^{2,\infty}} \leq C \| u \|_{H^4} \leq C \| f \|_{L^2} \). Division by \( \| \tilde{u}_2^h - R_h u_2 \|_{L^2} \) then gives
\[
\| \tilde{u}_2^h - R_h u_2 \|_{L^2} \leq C \sqrt{h} \| f \|_{L^2} .
\]
Together with (5.10) this proves the theorem. \( \square \)

It remains to compute the error in \( u_1 \). The next theorem follows the second part of Theorem 1 by Scholz (1978), but requires additional work due to the curved geometries. Because of Lemma 5.6, we achieve convergence of order \( h \) here.

**Theorem 5.3** We have that
\[
\| u_1 - u_1^h \|_{L^2} \leq C h \| f \|_{L^2} .
\]

**Proof.** Since \( u_1 - u_1^h \in H^1_0 \), by assumption the biharmonic equation \( \Delta^2 w = u_1 - u_1^h \) with zero Dirichlet and Neumann boundary conditions has a unique solution \( w \in H_0^1 \cap H^4 \). As before, we use the geometrical convention that the Laplacian be positive semidefinite.

We use the mixed biharmonic PDEs, Ritz projection, and integration by parts repeatedly to obtain
\[
\| u_1 - u_1^h \|_{L^2}^2 = \left( u_1 - u_1^h, \Delta^2 w \right)_0 = \left( u_1 - u_1^h, \Delta w \right)_1 \\
= \left( u_1 - u_1^h, \Delta w - R_h \Delta w \right)_0 + \left( u_2 - u_2^h, R_h \Delta w \right)_0 + d(u_1^h, R_h \Delta w) + c(u_2^h, R_h \Delta w) \\
= \left( u_1 - u_1^h, \Delta w - R_h \Delta w \right)_1 + \left( u_2 - u_2^h, R_h \Delta w - \Delta w \right)_0 + \left( u_2 - u_2^h, w - R_{h,0} w \right)_1 \\
- d(u_2^h, R_{h,0} w) + c(f, R_{h,0} w) - d(u_1^h, R_h \Delta w) + c(u_2^h, R_h \Delta w) .
\]
Using (3.4), the first term of the last expression can be bounded by
\[
\left| \left( u_1 - u_1^h, \Delta g w - R_h \Delta g w \right)_1 \right| = \left| \left( u_1 - R_{h,0} u_1, \Delta g w - R_h \Delta g w \right)_1 \right| \leq C h^2 \| f \|_{L^2} \| w \|_{H^4} .
\]
Using (3.4) and Theorem 5.2, the bound for the second term is
\[ \left| \left( u_2 - u_2^h, \Delta \Gamma w - R_h \Delta \Gamma w \right)_0 \right| \leq \left| \left( R_h u_2 - u_2^h, \Delta \Gamma w - R_h \Delta \Gamma w \right)_0 \right| + \left( u_2 - R_h u_2, \Delta \Gamma w - R_h \Delta \Gamma w \right)_0 \]
\[ \leq Ch^\frac{3}{2} \| f \|_{L^2} \| w \|_{H^4} . \]

We can bound the third term as follows,
\[ \left| \left( u_2 - u_2^h, w - R_h w \right)_1 \right| \leq \left| \left( u_2 - R_h u_2, w - R_h w \right)_1 \right| + \left| \left( u_2^h - R_h u_2, w - R_h w \right)_1 \right| \]
\[ \leq Ch^2 \| u_2 \|_{H^2} \| w \|_{H^2} + \left( u_2^h - R_h u_2, w - R_h w \right)_1 \] (by (3.4))
\[ \leq Ch^2 \| u_2 \|_{H^2} \| w \|_{H^2} + Ch \| f \|_{L^2} \| w \|_{W^2} \] (by Lemma 5.6 and Theorem 5.2)
\[ \leq Ch \| f \|_{L^2} \| w \|_{H^4} . \]

Finally, three of the remaining terms can be bounded as
\[ d(u_1^h, R_h \Delta \Gamma w) + c(u_1^h, R_h \Delta \Gamma w) + c(f, R_h w) \leq Ch^\sigma \| f \|_{L^2} \| w \|_{H^4} , \]
where we used Lemma 3.3 and (3.4).

In order to bound the last remaining term, observe that
\[ \left| d(u_1^h, R_h w) \right| \leq Ch^\sigma \left| u_1^h \right|_{H^1} \| R_h w \|_{H^1} \leq Ch^\sigma \left| u_1^h \right|_{H^1} \| w \|_{H^1} , \]
\[ \left| u_1^h \right|_{H^1} \leq \| R_h u_2 \|_{H^1} + \left| u_2 - R_h u_2 \right|_{H^1} \leq C \| u_2 \|_{H^1} + Ch^{-1} \left| u_2 - R_h u_2 \right|_{L^2} \]
\[ \leq C \| f \|_{L^2} + Ch^{-\frac{1}{2}} \| f \|_{L^2} \leq Ch^{-\frac{1}{2}} \| f \|_{L^2} , \]
\[ \left| d(u_1^h, R_h w) \right| \leq Ch^{\sigma - \frac{1}{2}} \| f \|_{L^2} \| w \|_{H^1} . \]

where we used Lemma 3.3, Lemma 3.4, and Theorem 5.2.

Using that \( \Delta \Gamma w = u_1 - u_1^h \), we obtain \( \| w \|_{H^4} \leq C \| u_1 - u_1^h \|_{L^2} \). Together with the assumption that \( \sigma \geq \frac{3}{2} \), these estimates show that
\[ \left| u_1 - u_1^h \right|_{L^2} \leq Ch \| f \|_{L^2} \left| u_1 - u_1^h \right|_{L^2} , \]
which proves the theorem. \qed

A simple corollary provides a convergence rate of \( h^{\frac{3}{2}} \) for the gradient of \( u_1 \).

**Corollary 5.1** We have that
\[ \left| u_1 - u_1^h \right|_{H^1_0} \leq Ch^\frac{3}{2} \| f \|_{L^2} \]

**Proof.** Using the mixed biharmonic problem, it follows that
\[ \left( u_1^h - u_1, u_1^h - u_1 \right)_1 = \left( u_1^h - u_1, u_1^h - u_1 \right)_0 + d(u_1^h, u_1^h - u_1) - c(u_1^h, u_1^h - u_1) , \]
\[ \left| u_1 - u_1^h \right|_{H^1_0} \leq Ch^\frac{3}{2} \| f \|_{L^2} + Ch^\frac{3}{2} \| f \|_{L^2} + Ch^{\sigma + 1} \| f \|_{L^2} , \]
where we applied the estimates from Lemma 3.3, the fact that the solution of the discrete problem is bounded, and going through the Ritz approximation as an intermediate. Since we assumed that \( \sigma \geq \frac{3}{2} \), this proves the corollary.

5.3 The no-boundary case

Here we provide the proof for the case of empty boundary, which is much simpler than the case of a nonempty boundary. This case is also effectively handled in the work of Elliott et al. (2019).

A similar analysis (with minor modifications) to the one presented in this section also yields corresponding convergence results for the case of a surface with boundaries and boundary conditions \( \Delta \Gamma u = u = 0 \) on \( \partial \Gamma \).

If there is no boundary, the mixed formulation decouples, in the sense that \( u_1, u_2 \in H^1_0 \) solve the decoupled Poisson equations

\[
(u_2, \xi)_1 = (f, \xi) \quad \forall \xi \in H^1_0 ,
\]

\[
(u_1, \eta)_1 = (u_2, \eta) \quad \forall \eta \in H^1_0 .
\]

Notice that unlike the case of nonempty boundaries, we here have \( u_2, \eta \in H^1_0 \) instead of \( u_2, \eta \in H^1 \), where \( H^1_0 \) is the subspace of functions in \( H^1 \) that integrate to zero; see (2.3). The same pertains to the corresponding discrete formulations. In this case, we obtain a better convergence rate:

**Theorem 5.4** It holds that

\[
\| u^h_2 - u_2 \|_{L^2} + h \| u^h_2 - u_2 \|_{H^1_0} \leq Ch^2 \| f \|_{L^2} ,
\]

\[
\| u^h_1 - u_1 \|_{L^2} + h \| u^h_1 - u_1 \|_{H^1_0} \leq Ch^2 \| f \|_{L^2} .
\]

**Proof.** By Wardetzky (2006, Theorem 3.3.3) it holds that

\[
\| u^h_2 - u_2 \|_{L^2} + h \| u^h_2 - u_2 \|_{H^1_0} \leq Ch^2 \| f \|_{L^2} .
\]

To bound the error in \( u^h_1 \) we turn to the solution \( v_1 \in S_{h,0} \) of the discrete Poisson problem

\[
\begin{pmatrix} v^h_1, \eta \end{pmatrix}_{1,h} = (u_2, \eta)_0, \quad \forall \eta \in S_{h,0} .
\]

As \( v^h_1 \) is the solution to a discrete Poisson problem, we obtain

\[
\| v^h_1 - u_1 \|_{L^2} + h \| v^h_1 - u_1 \|_{H^1_0} \leq Ch^2 \| u_2 \|_{L^2} \leq Ch^2 \| f \|_{L^2} .
\]

As for the error between \( u^h_1 \) and \( v^h_1 \), we know that

\[
\| u^h_1 - v^h_1 \|_{H^1_0} \leq C \left( \| u^h_2 - u_2, u^h_1 - v^h_1 \|_{0,h} \right) \leq C \| u^h_2 - u_2 \|_{L^2} \| u^h_1 - v^h_1 \|_{H^1_0} ,
\]

\[
\| u^h_1 - v^h_1 \|_{H^1_0} \leq C \| u^h_1 - u_2 \|_{L^2} \leq Ch^2 \| f \|_{L^2} .
\]
Combining (5.12) and (5.13), and using the Poincaré inequality, we obtain that

\[ \| u_h^1 - u_1 \|_{L^2} + h \| u_h^1 - u_1 \|_{H^1_0} \leq C h^2 \| f \|_{L^2}, \]

which proves the theorem.

\[ \square \]

6. Algorithm & Experiments

The numerical method whose convergence we have shown in Section 5 is implemented by solving the following linear system for \( u_1 \in \mathbb{R}^{\dim \mathring{S}_{h,0}}, \) and \( u_2 \in \mathbb{R}^{\dim \mathring{S}_h}:

\[
\begin{pmatrix}
L & M \\
L^T & -M
\end{pmatrix}
\begin{pmatrix}
u_1 \\
u_2
\end{pmatrix}
= \begin{pmatrix} f \\
0
\end{pmatrix}.
\]

(6.1)

Here we use the piecewise linear Lagrange basis functions \((\phi_i)_i\) as a basis for the finite element space \( \mathring{S}_{h,0} \), and the piecewise linear Lagrange basis functions \((\rho_i)_i\) is the basis for the finite element space \( \mathring{S}_h \). The linear system (6.1) corresponds to the discrete problem on the discrete mesh (1.5), and is equivalent to the problem (4.1) (via the map \( \Phi \)), for which Section 5 proves convergence estimates (lifting (1.5) from \( \Gamma_h \) to \( \Gamma \) results in the equivalent (4.1)). The solutions \( u_h^1, u_h^2 \) (which are equivalent to \( u_1, u_2 \)) are recovered via the sums

\[ u_h^1 = \sum_i (u_1)_i \phi_i \quad \text{and} \quad u_h^2 = \sum_i (u_2)_i \rho_i. \]

(6.2)

In (6.1), \( L \) denotes the sparse Laplacian stiffness matrix,

\[ L = \left( \int_{\Gamma_h} \nabla \phi_i \cdot \nabla \rho_j \, dx \right)_{i,j} \in \mathbb{R}^{\dim \mathring{S}_{h,0} \times \dim \mathring{S}_h}, \]

(6.3)

\( M \) denotes the mass matrix,

\[ M = \left( \int_{\Gamma_h} \rho_i \rho_j \, dx \right)_{i,j} \in \mathbb{R}^{\dim \mathring{S}_h \times \dim \mathring{S}_h}, \]

(6.4)

and \( f \) is the right-hand side,

\[ f = \left( \int_{\Gamma_h} \phi_i (f \circ \Psi) \, dx \right)_i \in \mathbb{R}^{\dim \mathring{S}_{h,0}}, \]

(6.5)

where \( \Psi \) is defined in Definition 3.2.

(6.3)-(6.4) are the standard stiffness and mass matrices for the Poisson equation on triangle meshes, as they appear, e.g., in the work of Dziuk (1988).

We performed a variety of numerical experiments with our method in MATLAB R2020a, using the gptoolbox (Jacobson et al., 2021) and triangle (Shewchuk, 2005) libraries. For all our experiments, errors are computed by measuring the difference between the method’s solution and the exact solution, projected onto the finite element space. The exact solutions and the right-hand sides are sampled from the exact functions pointwise at vertices in the parameter space, and these samples are then used as degrees of freedom of the discrete finite element spaces. Implementation details are provided in the supplemental material, along with the MATLAB code used to generate the images.
Figure 2. Solving the biharmonic equation on a hemisphere (log-log plot).

6.1 Spherical cap

Figure 2 shows our method used to solve the biharmonic equation on a hemisphere. We observe convergence to the exact solution, \( (\pi/2 - \theta)^2 \theta^5 \), where \( \theta \) is the colatitude of the spherical coordinate system, as predicted by the theory. We observe no convergence in the \( H^1 \) norm of \( u_2 \), where the theory makes no guarantees. Convergence to the exact solution is observed with rates at least as good as predicted by our theory. It can be seen in the solution plot for \( u_2 \) that the function is oscillating strongly near the boundary – which is due to the fact that \( u_2^h \) converges only in the \( L^2 \) norm, but not in the \( H^1 \) norm. This corresponds to the theoretical intuition from Remark 4.1 that the Ciarlet theory offers no way to directly estimate the norm of the derivative of \( u_2^h \): we can only control the \( L^2 \) norm.

As discussed in Remark 3.2, it might be possible to relax \( (C2) \) to allow for triangles with straight edges and vertices inscribed into the boundary instead of requiring that the closest point projection is a global bijection. Figure 3 shows an example of this conjecture in action for a spherical cap (not a hemisphere). In this example, the colatitude runs from 0 to \( \pi/4 \), and \( (C2) \) is not exactly fulfilled (as edges of boundary triangles are straight). Regardless, we observe convergence to the exact solution, \( (\pi/4 - \theta)^2 \theta^5 \), where \( \theta \) is the colatitude of the spherical coordinate system. We still empirically observe convergence, even if \( (C2) \) is relaxed in the above manner.

6.2 Schwarz’s Lantern

In Figure 4 the importance of the triangle regularity conditions are demonstrated. We solve the biharmonic equation with exact solution \( (\cos \varphi)(\sin \pi z)z(1 - z) \), where \( \varphi \) is the angular coordinate and \( z \) is

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5In Figures 2-4, the curved boundaries of the smooth surface become straight when projected onto the triangles, thus boundary triangles with straight edges fulfill \( (C2) \). This is no longer true of the boundary of the spherical cap in Figure 3.
Fig. 3. Solving the biharmonic equation on a spherical cap with colatitude from 0 to $\pi/4$ (log-log plot).

Fig. 4. Solving the biharmonic equation on a Schwarz lantern perturbed on the order of $(\text{mean edge length})^2$ (log-log plot).
the $z$-coordinate of the cylindrical coordinate system on a Schwarz lantern perturbed on the order of $(\text{mean edge length})^2$. The standard Schwarz lantern fulfills conditions (C1-C4) if it fulfills the triangle regularity condition. Triangle regularity, in turn, is satisfied if $m \sim n$, where $m$ is the number of vertices along the equator, and $n$ is the number of vertices along the axis of rotational symmetry. In this case convergence is observed (in fact, we even observe a rate of $h^2$, which is better than predicted). If, on refinement, $m$ increases much more quickly than $n$, such as when $m \sim n^2$, the mixed finite element method ceases to converge. This is a standard result, and not at all surprising, since, in this case, not even the surface area of the discrete mesh converges to the surface area of the respective smooth cylinder under refinement (see, for example, the book of Morvan (2008, Section 3.1.3)).

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