CODIMENSION 2 CYCLES ON PRODUCTS OF PROJECTIVE HOMOGENEOUS SURFACES

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Abstract. In the present paper, we provide general bounds for the torsion in the codimension 2 Chow groups of the products of projective homogeneous surfaces. In particular, we determine the torsion for the product of four Pfister quadric surfaces and the maximal torsion for the product of three Severi-Brauer surfaces. We also find an upper bound for the torsion of the product of three quadric surfaces with the same discriminant.

Contents

1. Introduction 1
2. Two filtrations on the Grothendieck ring 3
3. Grothendieck group of the product of Severi-Brauer varieties 5
4. Product of Conics 7
4.1. Four Conics 13
4.2. Galois cohomology and torsion groups 16
5. Product of Severi-Brauer surfaces 13
5.1. Three Severi-Brauer surfaces 20
6. Product of Quadric surfaces 24
6.1. Three quadric surfaces with the same discriminant. 28
References 28

1. Introduction

Let $X$ be a projective homogenous variety under the action of a semisimple group $G$ over an algebraically closed field $F$. The Chow group $CH(X)$ of algebraic cycles modulo the rational equivalence relation is well-understood as well as its ring structure. Namely, the Chow group of $X$ is a free abelian group with the basis of Schubert cycles. For an arbitrary base field $F$, this is no longer true: the Chow group $CH(X)$ can have torsion. Indeed, by a transfer argument the problem of determining the Chow group of $X$ over an arbitrary field $F$ reduces to computing its torsion subgroup.

For codimension $d \leq 1$, the Chow group $CH^d(X)$ is torsion-free. A nontrivial torsion first appears in codimension 2 cycles on $X$ and the exact structure of the torsion subgroup is known in many cases. For a projective quadric $X$, the torsion subgroup of $CH^2(X)$ is either 0 or $\mathbb{Z}/2\mathbb{Z}$ [6]. For a Severi-Brauer variety $X$, it is shown that the torsion subgroup of $CH^2(X)$ is either 0 or a cyclic group if the corresponding simple algebra satisfies certain
For a simple simply connected group $G$ splits over $F(X)$, it is known that the torsion subgroup of $\text{CH}^2(X)$ is a cyclic group generated by the Rost invariant \cite{2}. However, the only partial results are known for their products. In this case, the structure of the torsion subgroup is more complicated.

As a first step in determining torsion in codimension 2 cycles on the product of flag varieties, we consider in this paper the product of two dimensional flag varieties of the same type. For an integer $n \geq 1$, this can be divided into two classes: the set of all products of $n$ Severi-Brauer surfaces, denoted by $SB_n$, and the set of all products of $n$ quadric surfaces, denoted by $Q_n$. The latter has a special subclass $PQ_n$ consisting of all product of $n$ Pfister quadric surfaces. Here, we view the product of two conics as the product of two Pfister quadric surfaces since they have the same torsion subgroup in codimension 2 cycles by \cite[Corollary 2.5]{3}. To measure the size of the torsion subgroup of codimension 2 cycles, we introduce the following notation:

$$M(A) = \max_{X \in A} |\text{CH}^2(X)_{\text{tors}}|,$$

where $A$ is any of $SB_n$, $Q_n$, and $PQ_n$. Hence, the torsion subgroup of any element of $A$ is an elementary abelian group whose order is a divisor of $M(A)$. We denote by $C_n$ the set of all products of $n$ conics, thus we have $M(PQ_n) = M(C_n)$.

It is well-known that $M(SB_1) = M(Q_1) = M(C_2) = 1$. In \cite{10} Peyre proved that $M(C_3) = 2$, thus $M(PQ_3) = 2$. In \cite{11}, \cite{3} and \cite{5} Izhboldin and Karpenko proved that $M(SB_2) = 3$ and $M(Q_2) = 2$. As it is showed in \cite[Theorem 2.1]{11} and \cite[Theorem 4.1]{10}, the torsion subgroup of codimension 2 cycles is closely related to a relative Galois cohomology group in degree 3 and the above results were used to describe the cohomology groups. Moreover, Izhboldin and Karpenko’s result was the key part of their results on isotropy of quadratic forms.

The main goal of this paper is to extend their results to arbitrary $n$. First, we determine the maximal torsion $M(C_n) (= M(PQ_n))$ in Corollary 4.4, which gives a general lower bound of $M(Q_n)$. In particular, we determine the torsion in the gamma filtration of the product of four conics in Theorem 4.6 as well as its torsion in Chow group. Secondly, we find a general lower bound of $M(SB_n)$ in Proposition 5.2. Especially, we show that the bound is sharp for the product of three Severi-Brauer surfaces in Theorem 5.4. The results $M(PQ_4) = 2^5$ and $M(SB_3) = 3^8$ give the first examples such that the torsion subgroup $\text{CH}^2(X)_{\text{tors}}$ is not cyclic in their classes. In the last part, we provide an elementary proof of Izhboldin and Karpenko’s result for the product of two quadric surfaces and find an upper bound for the torsion of the product of three quadric surfaces with the same discriminant in Theorem 6.1, Proposition 6.3 and Theorem 6.5

As Karpenko showed in \cite{7} and \cite{8}, the topological filtration and the gamma filtration on the Grothendieck ring can be used to find the torsion in Chow groups of codimension 2 of projective homogeneous varieties. Moreover, the torsion in the filtrations can be computed by studying the divisibility of certain polynomials produced by the characteristic classes on the Grothendieck ring. We use this general approach to find the torsion in codimension 2 cycles on the product of projective homogeneous surfaces together with some additional combinatorial arguments.

This paper is organized as follows. In Sections 2 and 3 we recall basics of the topological filtration and the gamma filtration on the Grothendieck ring as well as their torsion subgroups of the product of Severi-Brauer varieties. In Section 4 we determine the bound of
the torsion in codimension 2 cycles on the product of \( n \) conics or the product of \( n \) Pfister quadric surfaces. In particular, we determine the torsion of the product of four conics or the product of four Pfister quadric surfaces in terms of the indexes of the corresponding algebras. In the last part of this section, we also present its application to a Galois cohomology group in degree 3. In Section 5 we find a general lower bound of the torsion in codimension 2 cycles on the product of \( n \) Severi-Brauer surfaces. Using this bound, we determine the maximal torsion subgroup of the Chow group of codimension 2 cycles on the product of three Severi-Brauer surfaces. In the last section, we recover a result of Izhboldin and Karpenko and extend it to the product of three quadric surfaces with the same discriminant.

In the present paper, \( A_{\text{tors}} \) denotes the torsion subgroup of an abelian group \( A \) and \( I_n = \{1, \ldots, n\} \) for any integer \( n \geq 1 \). We denote by \( \min \{ \cdot \} \) and \( \max \{ \cdot \} \) the minimum integer of a set and the maximum of a set, respectively.

2. Two filtrations on the Grothendieck ring

In this section, we briefly recall definitions and properties of the topological filtration and the gamma filtration on the Grothendieck ring \( K \) of a smooth projective variety (see [1] and [5] for details). We also provide a useful fact concerning the torsion part of these two filtrations on a smooth projective homogeneous variety.

Let \( X \) be a smooth projective variety and let \( K^\bullet (X) \) be the Grothendieck ring of \( X \). The topological filtration

\[
K(X) = T^0(X) \supset T^1(X) \supset \ldots
\]

is given by the ideal \( T^d(X) \) generated by the class \( [\mathcal{O}_Y] \) of the structure sheaf of a closed subvariety \( Y \) of codimension at least \( d \). We write \( T^d/d+1(X) \) for the quotient \( T^d(X)/T^{d+1}(X) \).

Let \( \Gamma^0(X) = K(X) \) and let \( \Gamma^1(X) \) be the kernel of the rank map \( K(X) \to \mathbb{Z} \). The gamma filtration

\[
K(X) = \Gamma^0(X) \supset \Gamma^1(X) \supset \ldots
\]

is given by the ideals \( \Gamma^d(X) \) generated by the product \( \gamma_{d_1}(x_1) \cdots \gamma_{d_i}(x_i) \) with \( x_j \in \Gamma^1(X) \) and \( d_1 + \cdots + d_i \geq d \), where \( \gamma_d \) is the gamma operation on \( K(X) \). For instance, we have \( \gamma_0(x) = 1 \) and \( \gamma_1 = \text{id} \), where \( x \in K(X) \). Indeed, the gamma operation defines the Chern class \( c_j(x) := \gamma_j(x - \text{rank}(x)) \) with values in \( K \).

For any \( d \geq 0 \), the gamma filtration \( \Gamma^d(X) \) is contained in the topological filtration \( T^d(X) \). For small degree \( d = 1, 2 \), two filtrations coincide. Moreover, the second quotient of the topological filtration can be identified with the codimension 2 cycles so that we have:

\[
(1) \quad \Gamma^{2/3}(X) \to T^{2/3}(X) = \text{CH}^2(X).
\]

Now we assume that \( X \) is a smooth projective homogeneous variety over a field \( F \). Let \( E \) be a splitting field of \( X \). Then, by [11] Proposition 3.4 we have

\[
(2) \quad T^d(X) = \Gamma^d(X) = \Gamma^d(X_E) \cap K(X)
\]

for \( d = 1, 2 \).

Let \( \mathcal{F} \) be either the gamma-filtration \( \Gamma \) or the topological filtration \( T \) on \( K(X) \). Applying the Snake lemma to the commutative diagram involving the exact sequences \( 0 \to
\[ \mathcal{F}^{d+1}(X) \to \mathcal{F}^{d}(X) \to \mathcal{F}^{d/d+1}(X) \to 0 \] and the one over a splitting field \( E \), we have the following useful formula \([7\) Proposition 2]:

\[ |\bigoplus \mathcal{F}^{d/d+1}(X)_{\text{tors}}| \cdot |K(X_E)/K(X)| = \prod_{d=1}^{\dim(X)} |\mathcal{F}^{d/d+1}(X_E)/\text{Im}(\text{res}^{d/d+1})|, \]

where \( \text{res}^{d/d+1} : \mathcal{F}^{d/d+1}(X) \to \mathcal{F}^{d/d+1}(X_E) \) is the restriction map.

3. Grothendieck group of the product of Severi-Brauer varieties

We now recall the Grothendieck group of a product of Severi-Brauer varieties. In addition, we state some basic facts about codimension 2 cycles of a product of Severi-Brauer varieties.

Let \( A_i \) be a central simple \( F \)-algebra of degree \( d_i \) for \( 1 \leq i \leq n \). Consider the restriction map

\[ K(\prod_{i=1}^{n} \text{SB}(A_i)) \to K(\prod_{i=1}^{n} \mathbb{P}^{d_i-1}_E), \]

where the corresponding Severi-Brauer variety \( \text{SB}(A_i) \) over a splitting field \( E \) is identified with the projective space \( \mathbb{P}^{d_i-1}_E \). The latter ring in (4) is isomorphic to the quotient ring \( \mathbb{Z}[x_1, \ldots, x_n]/((x_1-1)^{d_1}, \ldots, (x_n-1)^{d_n}) \), where \( x_i \) is the pullback of the class of the tautological line bundle on \( \mathbb{P}^{d_i-1}_E \). Then by \([12\) §8 Theorem 4.1\) (see also \([10\) Proposition 3.1\]) the image of the map (4) coincides with the sublattice with basis

\[ \{ \text{ind}(A_1^{\otimes i_1} \otimes \cdots \otimes A_n^{\otimes i_n}) \cdot x_1^{i_1} \cdots x_n^{i_n} \mid 0 \leq i_j \leq d_j - 1, 1 \leq j \leq n \}. \]

Let \( X \) be the product of Severi-Brauer varieties \( \text{SB}(A_i) \) as above. For codimension 2 cycles, one can simplify the computation of torsion subgroup by using \([4\) Proposition 4.7\): if \( (A'_1, \ldots, A'_{n'}) = (A_1, \ldots, A_n) \) in the Brauer group \( \text{Br}(F) \), then

\[ \text{CH}^2(X)_{\text{tors}} \simeq \text{CH}^2(X')_{\text{tors}}, \]

where \( X' = \prod_{i=1}^{n'} \text{SB}(A'_i) \).

Now we restrict our attention to \( p \)-primary algebras. Let \( A_1, \ldots, A_n \) be central simple algebras of \( p \)-power degree with given indices \( \text{ind}(A_1^{\otimes i_1} \otimes \cdots \otimes A_n^{\otimes i_n}) \) for all nonnegative integers \( i_1, \ldots, i_n \). Let \( X \) be the product of \( \text{SB}(A_1), \ldots, \text{SB}(A_n) \). Then, by \([8\) Corollary 2.15\] the map (11) induces a surjection on torsion subgroups \( \Gamma^{2/3}(X)_{\text{tors}} \to \text{CH}^2(X)_{\text{tors}} \). Moreover, by \([4\) Theorem 4.5\] and \([9\) Proposition III.1\] there is a product \( X \) of Severi-Brauer varieties \( \text{SB}(A_i) \) of algebras \( A_i \) with \( \text{ind}(A_1^{\otimes i_1} \otimes \cdots \otimes A_n^{\otimes i_n}) = \text{ind}(A_1^{\otimes i_1} \otimes \cdots \otimes A_n^{\otimes i_n}) \) such that the above surjective map becomes bijective map

\[ \Gamma^{2/3}(X)_{\text{tors}} \simeq \text{CH}^2(X)_{\text{tors}}. \]

This variety \( \tilde{X} \) will be called a generic variety corresponding to \( X \).
4. Product of Conics

In the present section, we provide a general bound for the torsion in codimension 2 cycles of the product of $n$ conics in Corollary 1.2. In case of the product of four conics, we determine its torsion in terms of the indexes of the corresponding algebras in Theorem 4.3. In the last subsection, we present its application to a Galois cohomology group.

The following example was proved in a different way [10 Corollary 3.9].

**Example 4.1.** Consider the product of two conics $X = SB(Q_1) \times SB(Q_2)$, where $Q_1$ and $Q_2$ are quaternions. Let $a = \text{ind}(Q_1)$, $b = \text{ind}(Q_2)$, and $\text{ind}(Q_1 \otimes Q_2) = c$. If $ab = 1$, then the Chow group of $X$ is torsion free, thus we may assume that $ab \geq 1$. By (3), we have a basis $\{1, ax_1, bx_2, cx_1x_2\}$ of $K(X)$, where $x_1$ and $x_2$ are the pullbacks of the classes of the line bundles on the projective line over a splitting field $E$. Therefore, we have $|K(X)/K(X)| = abc$. By substitution $y_n = x_n - 1$ for $n = 1, 2$, we have a different basis $\{1, ay_1, by_2, cy_1y_2 + y_1 + y_2\}$ of $K(X)$. Let $\alpha_n = [T^{n/n+1}(X_E) / \text{Im}(\text{res}^{n/n+1})]$. If $c \geq 2$, then by (2) we have $ay_1, by_2 \in T^1(X)$ and $cy_1y_2 \in T^2(X)$. Therefore, $\bigoplus T^{n/n+1}(X_{\text{tors}})$ is trivial. Otherwise, by the diagonal embedding $SB(Q_1) \hookrightarrow X$ we have $y_1 + y_2 \in \text{Im}(\text{res}^{1/2})$, thus $\alpha_1 \leq \min\{a, b\}$. As $2y_1y_2 \in T^2(X)$, we get $\alpha_2 \leq 2$. Hence, $\bigoplus T^{n/n+1}(X_{\text{tors}})$ is trivial as well. In any case, $\text{CH}^2(X)$ is trivial. In particular, $\mathcal{M}(C_2) = \mathcal{M}(PQ_2) = 0$.

We determine the Chow group of codimension 2 of the product of three conics.

**Proposition 4.2.** (cf. [10] Proposition 6.1, Proposition 6.3) Let $Q_1, Q_2, Q_3$ be quaternions and $X = SB(Q_1) \times SB(Q_2) \times SB(Q_3)$. Then, we have $\mathcal{M}(C_3) = \mathcal{M}(PQ_3) = 0$. Moreover, $\text{CH}^2(X)_{\text{tors}}$ is trivial except the cases where the division algebras $Q_i$ satisfying $\text{ind}(Q_i \otimes Q_j) = \text{ind}(Q_1 \otimes Q_2 \otimes Q_3) = 2$ or 4 for all $1 \leq i \neq j \leq 3$ and $\text{CH}^2(X)_{\text{tors}} = \mathbb{Z}/2\mathbb{Z}$ in these cases, where $X$ is the corresponding generic variety.

**Proof.** Let $d = \text{ind}(Q_1 \otimes Q_2 \otimes Q_3)$ and $e_{ij} = \text{ind}(Q_i \otimes Q_j)$ for $1 \leq i \neq j \leq 3$. If one of $\text{ind}(Q_i)$, $e_{ij}$, and $d$ is 1, then by (6) and Example 4.1 $\text{CH}^2(X)_{\text{tors}}$ is trivial. Therefore, by (6) we have the following basis

$$\{1, 2y_i, e_{ij}y_iy_j, dy_1y_2y_3 + y_1y_2 + y_1y_3 + y_2y_3\}$$

of $K(X)$, where $y_i = x_i - 1$, $x_i$ is the pullback of the tautological line bundle on the projective line over a splitting field $E$, and $e_{ij}, d \geq 2$.

If either $d = 8$ or $d = 4$ and $e_{ij} = 2$ for some $i, j$, then the set $K(X) \cap T^3(X_E)$ has only one element $dy_1y_2y_3$ of the basis. As $e_{ij}y_iy_j \in T^2(X)$ and $2y_k \in T^1(X)$ for $k \neq i, j$, we have $dy_1y_2y_3 = 2e_{ij}y_iy_jy_k = 2y_k(e_{ij}y_iy_j) \in T^3(X)$, which implies that $T^{2/3}(X)_{\text{tors}} = \text{CH}^2(X)_{\text{tors}}$ is trivial.

Now we assume that $d = e_{ij} = 2$ or 4 for all $1 \leq i \neq j \leq 3$. Then, we have $|K(X)/K(X)| = 2^3e_{ij}^2d$. Let $\alpha_n = [\Gamma^{n/n+1}(X_E)/\text{Im}(\text{res}^{n/n+1})]$. Then, we obtain $\alpha_1 \leq 2^3$, $\alpha_2 \leq e_{ij}^3$, and $\alpha_3 \leq 2d$ as $2dy_1y_2y_3 = 2y_1(dy_3y_2) \in \Gamma^3(X)$. Hence, $|\oplus \Gamma^{n/n+1}(X)_{\text{tors}}| \leq 2$. As $c_2(dx_1x_2x_3) = \binom{d}{2}(6y_1y_2y_3 + 2y_1y_2 + 2y_1y_3 + 2y_2y_3) \in \Gamma^2(X)$, we have $dy_1y_2y_3 \in \Gamma^2(X)$. Observe that $\Gamma^3(X)$ is generated by $\Gamma^1(X) \cdot \Gamma^2(X)$ and any element of $\Gamma^1(X) \cdot \Gamma^2(X)$ is divisible by $2d$. Therefore, the class of $dy_1y_2y_3$ gives a torsion of order 2 in $\Gamma^{2/3}(X)$. Hence, we have $\Gamma^{2/3}(X)_{\text{tors}} = \mathbb{Z}/2\mathbb{Z}$ and $\text{CH}^2(X)_{\text{tors}} = \mathbb{Z}/2\mathbb{Z}$ by (7).
We determine the maximal torsion in Chow group of codimension 2 of the product of \(n\) conics or the product of \(n\) Pfister quadric surfaces.

**Proposition 4.3.** Let \(n \geq 2\) and \(1 \leq i \leq n\) be integers, \(Q_i\) quaternion algebras satisfying \(\text{ind}(\otimes_{k=1}^{m} Q_{i_k}) = 2\) for any \(1 \leq m \leq n\) and all distinct \(i_k\), and \(X = \prod_{i=1}^{n} \text{SB}(Q_i)\). Then, the torsion subgroup \(\text{CH}^2(X)_{\text{tors}}\) of the corresponding generic variety \(X\) is

\[
(\mathbb{Z}/2\mathbb{Z})^{\oplus N}, \quad \text{where } N = 2^n - \left(\binom{n}{2}\right) + n + 1.
\]

In particular, \(\mathcal{M}(C_n) = \mathcal{M}(PQ_n) \geq 2^N\).

**Proof.** Let \(1 \leq i \leq n\) and let \(Q_i\) be a quaternion such that \(\text{ind}(\otimes_{k=1}^{m} Q_{i_k}) = 2\) for any \(1 \leq m \leq n\) and all distinct \(i_k\), and \(X = \prod_{i=1}^{n} \text{SB}(Q_i)\). Then, by [5] we have a basis \(\{1, 2x_{i_1} \cdots x_{i_m}\}\) of \(K(X)\), where \(x_i\) is the pullback of the class of the tautological line bundle on the projective line over a splitting field \(E\). Let \(y_i = x_i - 1\). Consider another basis \(\{1, 2y_{i_1} \cdots y_{i_m}\}\) of \(K(X)\).

Let \(j \geq 1\). As any element of \(\Gamma^{2j+1}(X)\) is divisible by \(2j+1\), we have

\[
2^{j}y_{i_1} \cdots y_{i_k} \in \Gamma^{2j}(X)\setminus \Gamma^{2j+1}(X)
\]

for any \(2j+1 \leq k \leq n\). Moreover, if \(2j+1 \leq k \leq 2j+2\), then we obtain

\[
2^{j+1}y_{i_1} \cdots y_{i_k} \in \Gamma^{2j+2}(X).
\]

Then, it follows from (8) and (9) that for any \(j \geq 1\) the class of \(2^{j}y_{i_1} \cdots y_{i_k}\) generates a subgroup of \(\Gamma^{2j/2j+1}(X)_{\text{tors}}\) of order 2. By the divisibility of an element of \(\Gamma^{2j+1}(X)\), any two subgroups generated by different classes of the elements \(2^{j}y_{i_1} \cdots y_{i_k}\) and \(2^{j}y_{i_1} \cdots y_{i_{k'}}\) have trivial intersection. Hence, we have

\[
(\mathbb{Z}/2\mathbb{Z})^{\oplus N_j} \subseteq \Gamma^{2j/2j+1}(X)_{\text{tors}},
\]

where \(N_j = \sum_{k=2j+1}^{n} \binom{n}{k}\).

Let \(\beta_i = \lfloor \Gamma^{2j+1}(X_E)/\text{Im(res}^{i/2j+1})/\lfloor K^i(X_E)/K^i(X)\rfloor\rfloor\), where \(K^i(X_E)\) (resp. \(K^i(X)\)) is the codimension \(i\) part of \(K(X_E)\) (resp. \(K(X)\)). Then, \(K^i(X_E)\) (resp. \(K^i(X)\)) is the base of \(K(X)\) that \(\beta_1, \beta_2 \leq 1\) and

\[
\beta_i \leq 2^{[i+1/2]^2 - 1}(\binom{n}{i}) = 2^{i+1/2}\binom{n}{i}/2^{\binom{n}{i}}
\]

for any \(3 \leq i \leq n\). Hence, by (3) we have

\[
\begin{align*}
\sum_{i=3}^{n} \left| \Gamma^{i/2j+1}(X)_{\text{tors}} \right| \leq 2\sum_{i=3}^{n} \left[ 2^{[i+1/2]^2 - 1}(\binom{n}{i}) \right],
\end{align*}
\]

Therefore, it follows from (10) and (11) that for any \(j \geq 1\)

\[
\Gamma^{2j/2j+1}(X)_{\text{tors}} = (\mathbb{Z}/2\mathbb{Z})^{N_j},
\]

As \(N_1 = N\), the result follows from (7).

**Corollary 4.4.** For any \(n \geq 2\), we have \(\mathcal{M}(C_n) = \mathcal{M}(PQ_n) = 2^N\).
Proof. Let \( X = \prod_{i=1}^{n} \text{SB}(Q_i) \in C_n \) such that \( \text{ind}(Q_i) = 2 \) for all \( i \). For \( 1 \leq m \leq n \), set \( e_{i_1 \ldots i_m} = \max\{\text{ind}(Q_{i_1}^{j_1} \otimes \cdots \otimes Q_{i_m}^{j_m})\} \), where the maximum ranges over \( 0 \leq j_1, \ldots, j_m \leq 1 \). If \( m \geq 3 \), then \( e_{i_1 \ldots i_m} y_{i_1} \cdots y_{i_{m-1}} \in T^2(X) \). As \( 2y_{i_m} \in T^1(X) \), we have

\[
2e_{i_1 \ldots i_m} y_{i_1} \cdots y_{i_m} \in T^3(X). 
\]

Since the subgroup \( T^3(X_E) \cap K(X) \) is generated by \( e_{i_1 \ldots i_m} y_{i_1} \cdots y_{i_m} \) for \( m \geq 3 \), it follows from (1), (2), and (12) that \( \text{tors} \leq 2^N \). If \( \text{ind}(Q_i) = 1 \) for some \( i \), then by (6) the computation of the upper bound for the torsion subgroup reduces to the case of product of less than \( n \) conics. Hence, we have \( \mathcal{M}(C_n) \leq 2^N \), thus by Proposition 4.3 we obtain the result.

4.1. Four Conics. Let \( Q_i \) be a quaternion algebra over a field \( F \) for \( 1 \leq i \leq 4 \). Consider the product \( X \) of the corresponding conics \( \text{SB}(Q_i) \). If \( \text{ind}(Q_i) = 1 \) for some \( i \), then by (6) the problem to find torsion in \( \text{CH}^2(X) \) is reduced to the case of product of three conics (Proposition 4.2). Hence, we may assume that \( \text{ind}(Q_i) = 2 \) for all \( i \). Let \( g_{ij} = \text{ind}(Q_i \otimes Q_j) \), \( h_i = \text{ind}(Q_j \otimes Q_k \otimes Q_l) \), and \( d = \text{ind}(Q_1 \otimes Q_2 \otimes Q_3 \otimes Q_4) \) for all integers \( i, j, k, l \) such that \( \{i, j, k, l\} = I_4 \). For the same reason, we may assume that \( g_{ij}, h_i, d \geq 2 \). By (6), we have the following basis of \( K(X) \)

\[
\{1, 2x_i, g_{ij}x_ix_j, h_ix_jx_kx_l, dx_1x_2x_3x_4\},
\]

where \( x_i, x_j, x_k, x_l \) are the pullbacks of the classes of the tautological line bundles on the projective lines. By substitution \( y_i = x_i - 1 \) for all \( 1 \leq i \leq 4 \), it follows from (13) that we have another basis \( K(X) \)

\[
\{1, 2y_i, g_{ij}y_iy_j, h_i(y_jy_ky_l + y_jy_ky_l + y_jy_ky_l) + \sum y_iy_jy_k + \sum y_iy_j\}.
\]

Now we will set up some notations, which will be used in this subsection. For any integer \( m \in \{2, 4, 8\} \), we set

\[
H_m = \{1 \leq i \leq 4 \mid h_i = m\}.
\]

Let \( J \) be the set of all indices \( \{12, 13, 14, 23, 24, 34\} \) of \( g_{ij} \). We consider the decompositions of \( J \):

\[
J = J_1 \cup J_2 \cup J_3 = K_i \cup L_i,
\]

where \( J_1 = \{12, 34\}, J_2 = \{13, 24\}, J_3 = \{14, 23\}, K_i = \{jk, jl, kl\} \), and \( L_i = J \setminus K_i \) for \( j < k < l \). We set

\[
G = \{ij \in J \mid g_{ij} = 2\}.
\]

We will use the following lemma to find bounds for the torsion in \( \text{CH}^2(X) \).

**Lemma 4.5.** Let \( 1 \leq m \leq 3 \) and \( 1 \leq p < q < r \leq 4 \) be integers and let \( i, j, k, l \) be integers such that \( \{i, j, k, l\} = I_4 \). Then, in codimension 2, 3, and 4 respectively, we have

\[
\text{Im}(\text{res}^{2/3}) \geq \begin{cases} 2 \sum y_py_q & \text{if } d = 2, \
2(y_iy_j + y_iy_k + y_jy_k) & \text{if } h_l = 2, 
\end{cases}
\]
If follows from the case of 4.2. the proof of Lemma 4.5 (2).

d complete the proof of Lemma 4.5 (3).

in Lemma 4.5 (2)(3) follow from (14) and (2).

which completes the proof of Lemma 4.5 (1). Multiplying (15) by respectively, we obtain the results in Lemma 4.5 (2). Multiplying 4 have the result in Lemma 4.5 (3).

Proof. If d = 2, then we have

\[
c_2(2x_1x_2x_3x_4) = 2(y_1y_2y_3y_4 + \sum y_p y_q y_r + \sum y_p y_q) \in \Gamma^2(X),
\]

thus, we obtain \( c_1(2x_i)c_1(2x_j)c_2(2x_1x_2x_3x_4) = 8y_1y_2y_3y_4 \in \Gamma^4(X) \), which imply the results in Lemma 4.5 (1), (3), respectively. Since \( 8 \sum y_p y_q y_r \in \Gamma^3(X) \), the result in Lemma 4.5 (2) follows from

\[
c_1(2x_i)c_2(2x_1x_2x_3x_4) = 12y_1y_2y_3y_4 + 4y_i(y_j y_k + y_j y_l + y_k y_l) \in \Gamma^3(X) \)

and

\[
c_1(2x_1x_2x_3x_4)c_2(2x_1x_2x_3x_4) = 72y_1y_2y_3y_4 + 12 \sum y_p y_q y_r \in \Gamma^3(X).
\]

If \( h_l = 2 \), then it follows from \( 15 \) and \( 14 \) that

\[
2(y_iy_j y_k + y_i y_j + y_i y_k + y_j y_k) \in \Gamma^2(X),
\]

which completes the proof of Lemma 4.5 (1). Multiplying \( 15 \) by \( c_1(2x_k) \) and \( c_1(2x_l) \), respectively, we obtain the results in Lemma 4.5 (2). Multiplying \( 4y_iy_j y_k \) by \( c_1(2x_i) \), we have the result in Lemma 4.5 (3).

If \( |G \cap J_m| = 2 \) for some \( m \), then we have \( g_{ij} = g_{kl} = 2 \) for some \( i, j, k, l \), thus the results in Lemma 4.5 (2)(3) follow from \( 14 \) and \( 2 \). If \( g_{ij} = 2 \), then it follows from \( 13 \) and \( 2 \) that

\[
2y_iy_j c_1(2y_k), 2y_iy_j c_1(2y_l) \in \Gamma^3(X).
\]

Multiplying the first element in \( 16 \) by \( c_1(2x_i) \), we obtain the result in Lemma 4.5 (3). If \( d = 4 \), then we have \( c_4(4x_1 x_2 x_3 x_4) = 24y_1y_2y_3y_4 \in \Gamma^4(X) \). As \( 16y_1y_2y_3y_4 \in \Gamma^4(X) \), we complete the proof of Lemma 4.5 (3).

If \( |G| \geq 4 \), then we have \( |G \cap J_m| = 2 \) for some \( m \), hence the result in Lemma 4.5 (2) follows from the case of \( |G \cap J_m| = 2 \). If \( g_{ij} = g_{ik} = 2 \), then it follows from \( 14 \) and \( 2 \) that

\[
2y_iy_j c_1(2x_k), 2y_iy_j c_1(2x_l), 2y_iy_k c_1(2x_l) \in \Gamma^3(X).
\]

If \( |G \cap K_i| = 3 \), then we get \( g_{jk} = g_{jl} = g_{kl} = 2 \), thus, by the same argument we complete the proof of Lemma 4.5 (2). □

We determine the codimension 2 cycles of the product of four conics as in Proposition 4.2.
Theorem 4.6. The torsion in Chow group of codimension 2 of the product $X$ of four conics is trivial if it satisfies one of the conditions \( \text{(17)}, \text{(22)}, \text{(23)}, \text{(25)}, \text{(31)}, \text{(32)}, \text{(33)}, d = 16 \). Otherwise, the torsion subgroup admits all elementary abelian group whose order is a divisor of $2^5$.

Proof. Let $m$ be an integer in $I_3$, $Q_i$ a quaternion division algebra over $F$ and $X$ the product of the corresponding conics $SB(Q_i)$ for $1 \leq i \leq 4$. Set

$$
\beta_i = |\Gamma^{i/i+1}(X_E)/\text{Im(res}^{i/i+1})|/|K^i(X_E)/K^i(X)|,$$

where $E$ is a splitting field of $X$ and $K^i(X_E)$ (resp. $K^i(X)$) is the codimension $i$ part of $K(X_E)$ (resp. $K(X)$). We find upper bounds of $\beta_i$ using case by case analysis. Note that we have $\beta_1 \leq 1$.

We begin with some observations on $d$: if $d = 2^4$, then we have $h_i = d/2$ for all $i$, which implies that $g_{ij} = d/4$ for all $1 \leq i \neq j \leq 4$. Therefore, we obtain $\beta_i \leq 1$ for all $i$, thus, $|\oplus \Gamma^{i/i+1}(X)_{\text{tors}}| \leq 1$, i.e., $\text{CH}^2(X)_{\text{tors}}$ is trivial. Hence, we may assume that $2 \leq d \leq 2^3$. Note also that we have

$$
d \leq 4 \text{ if } |H_2| \geq 1, \text{ and } |H_8| = 0 \text{ if } d = 2,$$

thus we only consider the cases where $d = 4, 8$ (resp. $d = 2, 4$) in the first 3 cases (resp. the last 4 cases) of the following. In case where $|H_2| = 4$, we have $d \in \{2, 4, 8\}$.

Case: $|H_8| \geq 3$. In this case, we have $g_{ij} = 4$ for all $i, j$. It follows from Lemma 4.5 (1) that $\beta_2 \leq (4^5 \cdot 2^{-|H_2|})/4^6$. By Lemma 4.5 (2), we obtain

$$
\beta_3 \leq \begin{cases} 
(8^2 \cdot 4^2)/8^3 \cdot 2 & \text{if } |H_2| = 1, \\
2 |H_4| & \text{otherwise}.
\end{cases}
$$

It follows by Lemma 4.5 (3) that $\beta_4 \leq 2$. Hence, by (32) the order of the group $\oplus \Gamma^{i/i+1}(X)_{\text{tors}}$ is nontrivial except the case where

\( |H_2| = 1, |H_8| = 3 \).

If $|H_8| = 4$, $d = 8$, then by the divisibility of elements in $\Gamma^3(X)$ the class of $8y_1 y_2 y_3 y_4$ gives a torsion element of $\Gamma^{2/3}(X)_{\text{tors}}$ of order 2. As $\beta_1 \beta_2 \beta_3 \beta_4 \leq 2$, we have

\( \Gamma^{2/3}(X)_{\text{tors}} = \mathbb{Z}/2\mathbb{Z} \).

Similarly, if $|H_8| = 3, h_i = 4$ and $d = 4$, then we have

\( \Gamma^{2/3}(X)_{\text{tors}} = (\mathbb{Z}/2\mathbb{Z})^{\oplus 2} \)

generated by the classes of $4y_j y_k y_l$ and $4(y_i y_j y_k + y_i y_j y_l + y_i y_k y_l + y_1 y_2 y_3 y_4)$.

Case: $|H_8| = 2$. A simple calculation of index implies that $0 \leq |G| \leq 1$. Hence, it follows from Lemma 4.5 (1) that

$$
\beta_2 \leq (4^6 - |G| - |H_2| \cdot 2 |G| + |H_2|)/4^6 - |G| \cdot 2 |G|.
$$

If $|G| = |H_2| = 0$, then we have $\beta_3 \leq 8^4/(8^2 \cdot 4^2)$. Otherwise, by Lemma 4.5 (2) we obtain

$$
\beta_3 \leq (3 - |G| - |H_2| \cdot 4^1 + |G| + |H_2|)/8^2 \cdot 4^2 - |H_2| \cdot 2 |H_2|.
$$
In summary, we have

$$\beta_3 \leq \begin{cases} 
1 & \text{if } |G| = 1, |H_2| = 0 \text{ or } |G| = |H_2| = 1, \\
2 & \text{if } |G| = 0, |H_2| = 1 \text{ or } |H_2| = 2, \\
2^2 & \text{otherwise.}
\end{cases}$$

It follows from Lemma 4.5 (3) that

$$\beta_4 \leq \begin{cases} 
1 & \text{if } |G| \geq 1, |H_2| = 0, d = 8, \\
2 & \text{otherwise.}
\end{cases}$$

Therefore, it follows by the same argument as in the previous case that $|\oplus \Gamma^{i/i+1}(X)_{\text{tors}}|$ is nontrivial except the case where

$$|H_2| = 2 \text{ or } |G| = |H_2| = 1 \text{ or } |G| = 1, |H_2| = 0, d = 8.$$ 

**Case:** $|H_8| = 1$. By this assumption, we have $0 \leq |G| \leq 3$. It follows by the same argument as in the previous case that we have the same upper bounds in (20) and (21) for $\beta_2$ and $\beta_4$, respectively. Let $\Gamma = G \cap (\cap_{i \in H_n K_i})$ for $n = 2, 4$. If $|G| = |G_2| = |H_2| = 1$ (resp. $|G| = |G_4| = 1, |H_2| = 2$), then we have $h_i = 2, h_l = 8, h_j = h_k = 4$ (resp. $h_j = 2, h_k = 4$), and $g_{il} = 2$ (resp. $g_{kl} = 2$) for $\{i, j, k, l\} = I_4$, which implies that $\beta_3 \leq 4^4/(8 \cdot 4^2 \cdot 2)$ (resp. $\beta_3 \leq 8 \cdot 4^3/(8 \cdot 4^2 \cdot 2)$) by Lemma 4.5 (2). If $|H_2| = 0, |G| \geq 2$ (resp. $|H_2| = 0, |G| = 0$), then by Lemma 4.5 (2) we obtain $\beta_3 \leq 8 \cdot 4^3/(8 \cdot 4^2)$ (resp. $\beta_3 \leq 8^4/(8 \cdot 4^3)$). Otherwise, by Lemma 4.5 (2) we have

$$\beta_3 \leq \min\{1, 8^{3-|G|+|H_2|}\} \cdot \min\{4^4, 4^{1+|G|+|H_2|}\}/(8 \cdot 4^{3-|H_2|} \cdot 2|H_2|).$$

Therefore, we conclude that

$$\beta_3 \leq \begin{cases} 
1 & \text{if } 0 \leq |H_2| \leq 1, |G| \geq 2 \text{ or } |H_2| = |G_2| = |G| = 1, \\
2^2 & \text{if } 1 \leq |H_2| \leq 2, |G| = 0 \text{ or } |H_2| = 2, |G| = |G_4| = 1, \\
2^3 & \text{if } |H_2| = |G| = 0.
\end{cases}$$

By the same argument, the order of the torsion $|\oplus \Gamma^{i/i+1}(X)_{\text{tors}}|$ is nontrivial except the following cases:

$$|G| \geq 2, |H_2| = 0, d = 8 \text{ or } |G| \geq |H_2||H_4| = 2 \text{ or } |H_2||H_4| = 2, |G| = |G_2| = 1 \text{ or } |H_2| = 3.$$ 

If $d = 8$, $|G| = 6$, and $h_i = 8$, then by the same argument as above we obtain

$$\Gamma^{2/3}(X)_{\text{tors}} = (\mathbb{Z}/2\mathbb{Z})^{\otimes 3} \text{ and } \Gamma^{3/4}(X)_{\text{tors}} = \mathbb{Z}/2\mathbb{Z}$$
generated by $4y_iy_ky_l$, $4y_iy_jy_l$, $4y_iy_jy_k$, and $8y_iy_2y_3y_4$, respectively.

**Case:** $|H_4| = 4$. By Lemma 4.5 (1), we have

$$\beta_2 \leq \begin{cases} 
(4^{5-|G|} \cdot 2^{G+1})/4^{6-|G|} \cdot 2^{G} & \text{if } d = 2, |G| \neq 6, \\
(4^{6-|G|} \cdot 2^{G}/4^{6-|G|} \cdot 2^{G}) & \text{otherwise.}
\end{cases}$$
It follows from Lemma 4.5 (2) that

\[ 4^4 \cdot \beta_3 \leq \begin{cases} 
4^4 & \text{otherwise,} \\
8 \cdot 4^3 & \text{if } d \in \{4, 8\}, 2 \leq |G| = |G \cap L_i| \leq 3 \text{ for some } i, \\
8^2 \cdot 4^2 & \text{if } d \in \{4, 8\}, |G| = 1, \\
8^4 & \text{if } d \in \{4, 8\}, |G| = 0. 
\end{cases} \]

By Lemma 4.5 (3) and (14), we obtain

\[ \beta_4 \leq \begin{cases} 
1 & \text{if } d = 8, |G| \neq 0 \text{ or } d = 4, |G \cap J_m| = 2 \text{ for some } m, \\
2 & \text{otherwise,} \\
2^2 & \text{if } d = 2, |G| \neq 6, |G \cap J_m| \neq 2 \forall m. 
\end{cases} \]

Applying the same argument, we have that the order of \( \oplus \Gamma^{i+1}(X)_{\text{tors}} \) is nontrivial except the following cases: for some \( m \) and \( i \)

\[ (25) \quad d = 2, |G| \neq 6, |G \cap J_m| = 2 \text{ or } d = 4, |G \cap J_m| = 2 \text{ or } d = 8, |G \cap K_i| = 3. \]

If \( |G| = 0 \) and \( d = 4 \), then by (14) and (2) one has \( h_i y_1 y_2 y_3 y_4 \in \Gamma^2(X) \setminus \Gamma^3(X) \) for all \( \{i, j, k, l\} = I_4 \) since any element of \( \Gamma^3(X) \) is divisible by \( 2^3 \). Therefore, by (14) and Lemma 4.5 (3) the classes of \( h_i y_1 y_2 y_3 y_4 \) give torsion elements of \( \Gamma^2(X) \) of order 2. Moreover, the subgroups generated by these classes have trivial intersection by the divisibility of an element of \( \Gamma^3(X) \). Hence, it follows from Corollary 4.4 that

\[ (26) \quad \Gamma^2(X)_{\text{tors}} = (\mathbb{Z}/2\mathbb{Z})^\oplus 5, \quad \text{thus } \text{CH}^2(\tilde{X})_{\text{tors}} = (\mathbb{Z}/2\mathbb{Z})^\oplus 5, \]

where \( \tilde{X} \) is the corresponding generic variety. If \( |G| = 0 \) and \( d = 8 \), then by the same argument as above the classes of \( h_i y_1 y_2 y_4 \) generate different subgroups of \( \Gamma^2(X)_{\text{tors}} \) of order 2 and the class of \( 8 y_1 y_2 y_3 y_4 = 2 y_1 (4 y_2 y_3 y_4) \) generates a subgroup of \( \Gamma^3(4)(X)_{\text{tors}} \) of order 2. Therefore, it follows Corollary 4.4 that

\[ (27) \quad \Gamma^2(4)(X)_{\text{tors}} = (\mathbb{Z}/2\mathbb{Z})^\oplus 4 \quad \text{and} \quad \Gamma^3(4)(X)_{\text{tors}} = \mathbb{Z}/2\mathbb{Z}. \]

Case: \( |H_2| = 4 \). By (14) and Lemma 4.5 (1), one has

\[ (28) \quad \beta_2 \leq \begin{cases} 
\frac{d}{2^5} & \text{otherwise,} \\
\frac{d}{2^4} & \text{if } |G| = d = 4 \text{ or } |G| = 3, d = 2, \\
\frac{d}{2^3} & \text{if } |G| = 4, d = 2 \text{ or } |G| = 5, d = 4, \\
\frac{d}{2^2} & \text{if } |G| = 6, d = 4 \text{ or } |G| = 5, d = 2, \\
\frac{d}{2} & \text{if } |G| = 6, d = 2. 
\end{cases} \]

It follows from (14) that \( \beta_3 \leq 2^4 = 8^4/4^4 \). By Lemma 4.3 (3), we have

\[ (29) \quad \beta_4 \leq \begin{cases} 
\frac{2^2}{d} & \text{if } |G| \geq 2, |G \cap J_m| = 2 \text{ for some } m, \\
\frac{2^4}{d} & \text{otherwise.} 
\end{cases} \]

Applying the same argument together with the upper bounds \( \beta_i \), we have \( \Gamma^2(X)_{\text{tors}} \neq 0 \) for any case. In particular, if \( |G| = 6 \) and \( d = 2 \), then by Proposition 3.3 we have

\[ (30) \quad \Gamma^2(X)_{\text{tors}} = (\mathbb{Z}/2\mathbb{Z})^\oplus 5, \quad \text{thus } \text{CH}^2(\tilde{X})_{\text{tors}} = (\mathbb{Z}/2\mathbb{Z})^\oplus 5. \]
Case: \(|H_2| = 3, |H_4| = 1\). Let \(H_2' = \{i \in H_2 \mid |K_i \cap G| = 3\}\). Then, by the same argument as in the previous case we have the same upper bound (28) for \(\beta_2\) if we add one case \(|G| = 3, d = 4, |H_2'| = 1\) to the second line of (28). It follows by Lemma 4.5 (2) that \(\beta_3 \leq 2^{3}\). By applying the same argument as in the previous case we have the same upper bound (29) for \(\beta_4\). It follows by the same argument as in the previous case that \(|\oplus \Gamma^{i+1}(X)_{\text{tors}}|\) is nontrivial except the case:

\[(31) \quad |G| = 2, |G \cap J_m| = 2 \quad \text{or} \quad |G| = 3, |G \cap J_m| = 2, d = 4 \quad \text{for some } m.\]

Case: \(|H_2| = 2, |H_4| = 2\). By Lemma 4.5 (1), we have

\[
\beta_2 \leq \begin{cases} 
\frac{d}{2^4} & \text{otherwise,} \\
\frac{d}{2^3} & \text{if } 3 \leq |G| \leq 4, |H_2'| = 1 \text{ or } |G| = 4, |H_2'| = 0, d = 2 \text{ or } |G| = 5, |H_2'| \leq 1, d = 4, \\
\frac{d}{2^2} & \text{if } |G| = 5, |H_2'| = 2 \text{ or } |G| = 5, |H_2'| \leq 1, d = 2 \text{ or } |G| = 6, d = 4, \\
\frac{d}{2} & \text{if } |G| = 6, d = 2. 
\end{cases}
\]

It follows from Lemma 4.5 (2) that

\[
\beta_3 \leq \begin{cases} 
2^2 & \text{otherwise,} \\
2^3 & \text{if } |G| = 0, d = 4 \text{ or } |G| = |G_2| = 1, d = 4. 
\end{cases}
\]

Applying the same argument as in the previous case, one has the same upper bound (29) for \(\beta_4\). Therefore, by the same argument, the order of the torsion \(|\oplus \Gamma^{i+1}(X)_{\text{tors}}|\) is nontrivial except the following cases: for some \(m\) and \(i\)

\[(32) \quad |G| = |G \cap J_m| = 2 \quad \text{or} \quad |G| = |G \cap K_i| = 3 \quad \text{or} \quad |G| = 4, |H_2'| = 0, d = 4.\]

Case: \(|H_2| = 1, |H_4| = 3\). By Lemma 4.5 (1), we obtain

\[
\beta_2 \leq \begin{cases} 
\frac{d}{2^3} & \text{otherwise,} \\
\frac{d}{2^2} & \text{if } 3 \leq |G| \leq 5, |H_2'| = 1 \text{ or } |G| = 5, |H_2'| = 0, d = 2 \text{ or } |G| = 6, d = 4, \\
\frac{d}{2} & \text{if } |G| = 6, d = 2. 
\end{cases}
\]

In codimension 3, it follows from Lemma 4.5 (2) that

\[
\beta_3 \leq \begin{cases} 
2 & \text{otherwise,} \\
2^2 & \text{if } |G| = |G_2| = 1, d = 4, \\
2^3 & \text{if } |G| = 0, d = 4. 
\end{cases}
\]

In codimension 4, it follows by Lemma 4.5 (2) that

\[
\beta_4 \leq \begin{cases} 
1 & \text{if } 2 \leq |G| \leq 3, |G \cap J_m| = 2 \text{ for some } m, d = 4 \text{ or } |G| \geq 4, d = 4, \\
2 & \text{otherwise,} \\
2^2 & \text{if } 0 \leq |G| \leq 3, |G \cap J_m| \neq 2 \forall m, d = 2. 
\end{cases}
\]

Hence, by the same argument as above the order of \(|\oplus \Gamma^{i+1}(X)_{\text{tors}}|\) is nontrivial except the following cases:

\[(33) \quad 2 \leq |G| \leq 3, |G \cap J_m| = 2 \quad \text{for some } m \text{ or } |G| = 4, |H_2'| = 0 \quad \text{or} \quad |G| = 5, |H_2'| = 0, d = 4.\]

Finally, the second statement of the theorem follows from (18), (19), (24), (26), and (27). \(\square\)
\textbf{Remark 4.7.} Indeed, one can easily show that the upper bounds $\beta_1\beta_2\beta_3\beta_4$ for each case of the proof of Theorem 4.6 are sharp.

4.2. \textbf{Galois cohomology and torsion groups.} As mentioned in Section 1, the torsion subgroup of the Chow group of codimension 2 cycles can be used to measure how far is a relative Galois cohomology from being a decomposable subgroup (generated by the class of $A_i$ below) [10, Theorem 4.1]. Namely, for an $F$-variety $X = \prod_i SB(A_i) \in C_n$ or $SB_n$ we have

\begin{equation}
(34) \quad CH^2(X)_{\text{tors}} \simeq H^3(F(X)/F, Q/\mathbb{Z}(2))/ \oplus_i H^1(F, Q/\mathbb{Z}(1)) \cup [A_i],
\end{equation}

where $H^3(F(X)/F, Q/\mathbb{Z}(2))$ denotes the kernel of $H^3(F, Q/\mathbb{Z}(2)) \to H^3(F(X), Q/\mathbb{Z})$ of Galois cohomology groups with coefficient in $Q/\mathbb{Z}(2)$ and $[A_i]$ denotes the class in the Brauer group $Br(F) = H^2(F, Q/\mathbb{Z}(1))$. Therefore, our main results (Corollary 4.4, Proposition 5.2, and Theorem 5.4) tell us how large indecomposable subgroups we can have.

Moreover, by [10, Remark 4.1] there is a canonical injection from a Galois cohomology group with the finite coefficient $\mu_n^{\otimes 2}$ to the torsion subgroup:

\begin{equation}
(35) \quad H^3(F(X)/F, \mu_n^{\otimes 2})/ \oplus_i H^1(F, \mu_n) \cup [A_i] \hookrightarrow CH^2(X)_{\text{tors}}.
\end{equation}

Therefore, if the torsion subgroup $CH^2(X)_{\text{tors}}$ is trivial, then one can write the relative Galois cohomology group in terms of decomposable subgroups with the finite coefficient $\mu_n$. For instance, if $X \in C_4$ satisfying one of the conditions (17), (22), (23), (25), (31), (32), (33), $d = 16$ in Theorem 4.6 we obtain $|CH^2(X)_{\text{tors}}| = 1$, thus by (35) we have

$$
H^3(F(X)/F, \mathbb{Z}/2\mathbb{Z}) = \oplus_{i=1}^4 H^1(F, \mathbb{Z}/2\mathbb{Z}) \cup [Q_i].
$$

5. \textbf{PRODUCT OF SEVERI-BRAUER SURFACES}

In this section, we find a general lower bound of the torsion in Chow group of codimension 2 of the product of $n$ Severi-Brauer surfaces in Proposition 5.2 and prove that the lower bound is sharp for $n = 3$ in Theorem 5.4.

To prove Proposition 5.2 we shall need the following lemma.

\textbf{Lemma 5.1.} \textit{Let $p$ be an odd prime and let $n \geq 2$ and $m_i$ be integers such that $1 \leq m_i \leq p-1$ for all $1 \leq i \leq n$. Let $\Phi$ be the polynomial $\prod_{i=1}^n (s_i + 1)^{m_i} - 1$ in $\mathbb{Z}[s_1, \ldots, s_n]$. Then, the alternating sum $\sum_{j_1=\ldots=j_n=1}^{p-1} \phi j_1+\cdots+j_n C_{j_1\cdots j_n}$ in the quotient $\mathbb{Z}[s_1, \ldots, s_n]/(s_1^p, \ldots, s_n^p)$ is divisible by $p^2$, where $C_{j_1\cdots j_n}$ is the coefficient of the monomial $s_1^{j_1}\cdots s_n^{j_n}$ in $\Phi$.}

\textit{Proof.} Let $t_n = (s_n + 1)^{\text{gcd}}$. For any $1 \leq j_1, \ldots, j_{n-1} \leq p - 1$, we write $\sum_{k=1}^p d_{j_1\cdots j_{n-1}k} k^k$ for the coefficient of $s_1^{j_1}\cdots s_n^{j_n}$ in $\Phi$. Let $e_{j_1\cdots j_{n-1}}$ be the coefficient of $s_1^{j_1}\cdots s_n^{j_n}$ in $\Psi_n := (s_1 + 1)^{\text{gcd}} \cdots (s_{n-1} + 1)^{\text{gcd}}$. By expanding each factor $(s_i + 1)^{m_i} = (s_i + 1)^{m_i}$ of $\Psi_n$, we have

\begin{equation}
(36) \quad e_{j_1\cdots j_{n-1}} = \left( \begin{array}{c} p \\ j_1 \end{array} \right) m_1 + \alpha_1 \right) \cdots \left( \begin{array}{c} p \\ j_{n-1} \end{array} \right) m_{n-1} + \alpha_{n-1},
\end{equation}

where $p^2 \mid \alpha_i$ for all $1 \leq i \leq n - 1$.}
We prove by induction on \(n\). Assume \(n = 2\). First, observe that \(C_{j_0} = 0\) in the quotient \(\mathbb{Z}[s_1, s_2]/(s^p_1, s^p_2)\). Hence, we have
\[
\sum_{j_2=1}^{p-1} (-1)^{j_2} C_{j_1j_2} = \sum_{j_2=0}^{p-1} (-1)^{j_2} C_{j_1j_2} = \sum_{j_2=0}^{p} d_{j_1k} \sum_{i=0}^{p-1} (-1)^i \left( \frac{m_{2k}}{i} \right) = \sum_{j_2=0}^{p} d_{j_1k} \left( \frac{m_{2k} - 1}{p - 1} \right),
\]
which implies that
\[
\sum_{j_1=1}^{p-1} (-1)^{j_1j_2} C_{j_1j_2} = \sum_{j_1=1}^{p-1} \sum_{j_2=0}^{p} d_{j_1k} \left( \frac{m_{2k} - 1}{p - 1} \right) (-1)^{j_1}. \tag{37}
\]
For each \(1 \leq k \leq p - 1\), we have
\[
p \mid \left( \frac{m_{2k} - 1}{p - 1} \right). \tag{38}
\]
As
\[
\Phi = \sum_{i=0}^{p} \left( \frac{p}{i} \right) (s_1 + 1)^{m_i} t^i_2 (-1)^{p-i},
\]
for any \(1 \leq k \leq p\) we obtain
\[
p \mid d_{j_1k}. \tag{39}
\]
From (38) and (39), it suffices to show that
\[
p^2 \mid \sum_{j_1=1}^{p-1} d_{j_1p} \left( \frac{m_{2p} - 1}{p - 1} \right) (-1)^{j_1}. \tag{40}
\]
Since \(d_{j_1p} = c_{j_1}\) and \(\sum_{j_1=1}^{p-1} \left( \frac{p}{j_1} \right) (-1)^{j_1} = 0\), the divisibility in (40) follows from (36).

Now we assume that the result holds for \(n - 1\). By the induction hypothesis, it is enough to show that \(\sum_{j_1=\ldots=j_{n-1}=1, j_n=0}^{p-1} (-1)^{j_1+\ldots+j_n} C_{j_1\ldots j_n}\) is divisible by \(p^2\). Applying the same argument as in the case \(n = 2\), we have
\[
\sum_{j_1=\ldots=j_{n-1}=1, j_n=0}^{p-1} (-1)^{j_1+\ldots+j_n} C_{j_1\ldots j_n} = \sum_{j_1=\ldots=j_{n-1}=1}^{p-1} \sum_{k=1}^{p} d_{j_1\ldots j_{n-1}k} \left( \frac{m_{nk} - 1}{p - 1} \right) (-1)^{j_1+\ldots+j_{n-1}}. \tag{41}
\]
We have \(p \mid \left( \frac{m_{nk} - 1}{p - 1} \right)\) for each \(1 \leq k \leq p - 1\). By the same argument with \(t_n\), we also have \(p \mid d_{j_1\ldots j_{n-1}k}\) for each \(1 \leq k \leq p\). For \(k = p\), we have \(p^2 \mid d_{j_1\ldots j_{n-1}p}\) by (36). Therefore, the result immediately follows.

We will provide lower bounds of the torsion in Chow group of codimension 2 of the product of \(n\) Severi-Brauer surfaces, which generalize the case of \(n = 2\) in [4, Proposition 6.3].

**Proposition 5.2.** Let \(n \geq 2\) and \(1 \leq i \leq n\) be integers, \(A_i\) a central simple algebra satisfying \(\text{ind}(A_1 ^{\otimes j_1} \otimes \cdots \otimes A_n^{\otimes j_n}) = 3\) for any integers \(0 \leq j_1, \ldots, j_n \leq 2\), not all equal to 0,
and \( X = \prod_{i=1}^{n} SB(A_i) \). Then, the torsion subgroup \( CH^2(X)_{\text{tors}} \) of the corresponding generic variety \( X \) contains
\[
\left( \mathbb{Z}/3\mathbb{Z} \right)^{\oplus N}, \text{ where } N = 2^n + 4\binom{n}{3} - (n+1).
\]

In particular, \( MN(SB_n) \geq 3^N \).

Proof. Let \( 1 \leq i \leq n \) and let \( A_i \) be a central simple algebra such that \( \text{ind}(\otimes_{k=1}^{n} A_k) = 3 \) for any integers \( 0 \leq j_1, \ldots, j_n \leq 2 \), and \( X = \prod_{i=1}^{n} SB(A_i) \). Then, by \( 1 \), we have a basis \( \{1, 3x_i^1 \cdots x_i^m \} \) of \( K(X) \), where \( x_i \) is the pullback of the class of the tautological line bundle on the projective plane and \( 1 \leq m \leq n \).

Let \( y_i = x_i - 1 \). Consider another sequence \( \beta \) of \( (44) \) such that \( y_i \) has trivial intersection with any subgroup generated by any element of \( (42) \). Moreover, by the same argument \( \beta \) is divisible by \( 9 \), which is impossible. Therefore, any element of \( (44) \) gives a torsion of \( \Gamma \) is in \( \Gamma \).

Let \( y = 3 \) be either \( 1 \) or a product of \( 3 \). Then we have \( y^3 = 0 \). Therefore, any element of \( (44) \) gives a torsion of \( \Gamma \) of order \( 3 \).

We show that any two subgroups generated by any two elements of \( (44) \) have trivial intersection. Let \( b_{pq} \) and \( b_{tu} \) be two different elements of \( (44) \). Then, by applying Lemma \( 5.1 \) with \( y_p = y_q = -1, y_{tl} = 0 \) for all \( l \neq p, q \leq n \) we obtain \( 3 \) is divisible by \( 9 \), which is a contradiction. Hence, the subgroups generated by \( b_{pq} \) and \( b_{tu} \) have trivial intersection. Moreover, by the same argument the subgroup generated by \( b_{pq} \) has trivial intersection with any subgroup generated by any element of \( (44) \).

Let \( z \) be either \( 1 \) or a product of \( x_1, \ldots, x_n \) which does not contain any of \( x_p, x_q \), and \( x_r \). Consider the sequence \( \beta'_{pqr} \) consisting of the coefficient of \( b_{pqr}/3 \) in
\[
(42) \quad c_3(3x_p^2x_qx_r), c_3(3x_p^2x_qx_r), c_3(3x_p^2x_qx_r), c_3(3x_p^2x_qx_r), c_3(3x_p^2x_qx_r), c_3(3x_p^2x_qx_r),
\]
\[
c_3(3x_p^2x_qx_r), c_3(3x_p^2x_qx_r), c_3(3x_p^2x_qx_r), \text{ respectively.}
\]

Then, by a direct calculation, we have \( \beta'_{pqr} = (66, 30, 30, 132, 132, 60, 264, 15) \). Hence, each element of \( \beta'_{pqr} - \beta'_{qpr}, \beta'_{pqr} - \beta'_{rqp}, \text{ and } \beta'_{qpr} - \beta'_{rqp} \) is divisible by \( 9 \), i.e.,
\[
(43) \quad 9 | \beta'_{pqr} - \beta'_{qpr}, \beta'_{pqr} - \beta'_{rqp}, \beta'_{qpr} - \beta'_{rqp}.
\]

Consider another sequence \( \beta_{pqr} \) consisting of the coefficient of \( b_{pqr}/3 \) in \( (42) \). Then, we have \( \beta_{pqr} = (12, 12, 12, 24, 24, 24, 48, 6) \). Therefore, we have
\[
(44) \quad 9 | \beta_{pqr} - \beta'_{pqr}, \beta_{pqr} - \beta'_{qpr}, \beta_{pqr} - \beta'_{rqp}.
\]
Let \( b_{pqr}' \) and \( b_{tuv}' \) be two different elements of \( B_2 \). If \( b_{pqr}' \pm b_{tuv}' \in \Gamma^2(X) \), then by applying Lemma 5.1 with \( y_p = y_q = y_r = -1, y_l = 0 \) for all \( 1 \leq l \neq p, q, r \leq n \) we obtain 3 is divisible by 9, which is impossible. Therefore, the subgroups generated by \( b_{pqr}' \) and \( b_{tuv}' \) have trivial intersection. Assume that \( b_{pqr}' \pm b_{tuv}' \in \Gamma^3(X) \) or \( b_{pqr}' \pm d_{s_1 \cdots s_n} \in \Gamma^3(X) \). Then, this contradicts the divisibility in (43). Hence, the subgroup generated by \( b_{pqr}' \) has trivial intersection with any subgroup generated by any element of \( B_2 \) and \( D_s \).

Let \( b_{pqr} \) and \( b_{tuv} \) be two different elements of \( B_2 \). Suppose that \( b_{pqr} \pm b_{tuv} \in \Gamma^3(X) \). Then, by applying Lemma 5.1 with \( y_i = -1 \) and \( y_l = 0 \) for all \( 1 \leq l \neq i, \ldots, s \leq n \) we obtain \(-3\) is divisible by 9, which is a contradiction. Therefore, the subgroups generated by \( b_{pqr} \) and \( b_{tuv} \) have trivial intersection. If \( b_{pqr} \pm d_{s_1 \cdots s_n} \in \Gamma^3(X) \), then we obtain a contradiction by the divisibility in (44). Hence, the subgroup generated by \( b_{pqr} \) has trivial intersection with any subgroup generated by any element of \( D_s \).

Let \( 3 \leq s, s' \leq n \) and let \( d_{s_1 \cdots s_n} \) and \( d_{s'_1 \cdots s'_n} \) be two different elements of \( D_s \) and \( D_{s'} \), respectively. If \( d_{s_1 \cdots s_n} \pm d_{s'_1 \cdots s'_n} \in \Gamma^3(X) \), then by applying Lemma 5.1 with \( y_i = \cdots = y_i = -1 \) and \( y_l = 0 \) for all \( 1 \leq l \neq i_1, \ldots, s \leq n \) we have 3 is divisible by 9, which is a contradiction. It follows that the torsion subgroup \( \Gamma^{2/3}(X)_{\text{tors}} \) contains \((\mathbb{Z}/3\mathbb{Z})^\oplus N\), so does \( \text{CH}^2(X)_{\text{tors}} \).

\[
5.1. \text{Three Severi-Brauer surfaces.} \text{ We consider the product of three Severi-Brauer surfaces. Let } A_1, A_2, A_3 \text{ be a central simple algebras of degree 3 over } F \text{ and let } X \text{ be the product of the corresponding Severi-Brauer surfaces } SB(A_1), SB(A_2), SB(A_3). \text{ If one of } A_1, A_2, A_3 \text{ is split, then by (6) the problem to compute torsion in Chow group of codimension 2 is reduced to the case of product of two Severi-Brauer varieties, which was done in [4, Theorem 5.1]. Therefore, we may assume that } \text{ind}(A_m) = 3 \text{ for all } 1 \leq m \leq 3. \text{ Let } e_i = \text{ind}(A_j \otimes A_k), f_i = \text{ind}(A_j^{\otimes 2} \otimes A_k), d = \text{ind}(A_1 \otimes A_2 \otimes A_3), \text{ and } g_i = \text{ind}(A_i^{\otimes 2} \otimes A_j \otimes A_k) \text{ for all } i, j, k \text{ such that } \{i, j, k\} = I_3. \text{ For the same reason, we may assume that } e_i, f_i, d, g_i \geq 3. \text{ By (5), we have the following basis of } K(X) \]

\[
\{1, 3x_m, 3x_m^2, e_ix_jx_k, f_{ij}x_k, dx_1x_2x_3, e_ix_j^2x_k, g_{ij}x_ix_jx_k, g_{ij}x_ix_j^2x_k, dx_1^2x_2x_3^2\},
\]

where \( x_m \) is the pullback of the class of the tautological line bundle on the projective plane. We will need the following lemma to find upper bounds of the torsion.

**Lemma 5.3.** Let \( i, j, k \) be integers such that \( \{i, j, k\} = I_3 \) and let \( y_m = x_m - 1 \) for all \( 1 \leq m \leq 3 \). Then, we have

1. \( 3y_m^2 \in \Gamma^2(X) \) in any case and \( 3y_jy_k \in \Gamma^2(X), 3y_j^2y_k^2 \in \Gamma^{3/4}(X) \) if \( e_i = 3 \),
2. \( 3y_jy_k - 3y_jy_k^2 \in \Gamma^{3/4}(X) \) if \( f_i = 3 \),
3. \( 3\sum_{m,l=1}^3 y_my_l \in \text{Im}(\text{res}^{2/3}), 6y_1y_2y_3 + 3\sum_{m,l=1}^3 y_m^2y_l \in \Gamma^{3/4}(X) \) and \( 9y_1^2y_2^2y_3 \in \Gamma^6(X) \) if \( d = 3 \),
4. \( 3\sum_{m,l=1}^3 y_my_l + 3(y_1y_2^2 + y_1^2y_2) + 12y_1y_2y_3 \in \Gamma^{3/4}(X), 3(y_jy_k - y_ky_j - y_ky_j) \in \text{Im}(\text{res}^{2/3}) \) if \( g_i = 3 \),
5. \( 3y_j^2y_k^2 - 3y_jy_2y_3 \in \Gamma^3(X) \) if \( f_m = g_m = e_m = 3 \) for all \( m \),
6. \( 3y_j^2y_k^2 - 6y_1y_2y_3 \in \Gamma^3(X) \) if \( f_m = g_m = e_m = d = 3 \) for all \( m \).
Proof. (1) By (15), we have \( \gamma_2(3x_m - 3) = 3y_m^2 \in \Gamma^2(X) \). Since \( 3y_jy_k \in K(X) \), this element is in \( \Gamma^2(X) \). The rest of them follow from direct computation of \( c_3(3x_jx_k) \).

(2) By (15) and Lemma 5.3 (1), the elements \( 9y_j^2y_k = 3y_j^2(3y_k) \) and \( 9y_jy_k^2 = 3y_j(3y_k^2) \) are in \( \Gamma^3(X) \), the result follows from the calculation of \( c_3(3x_j^2x_k) \).

(3) By (15), we obtain \( 3(y_1y_2y_3 + \sum_{m=1}^3 y_m y) \in \Gamma^2(X) \). Thus, the first inclusion immediately follows. As \( 27(y_1y_2y_3)^2 \in \Gamma^6(X) \), the rest of them follow from the computations of \( c_3(3x_1x_2x_3) \) and \( c_6(6x_1x_2x_3) \).

(4) By the calculation of \( c_3(3x_j^2x_kx_l) \), the first inclusion follows. By (15) we get \( 3x_j^2x_kx_l \in \Gamma^2(X) \). Hence, the second inclusion follows by expanding the element \( 3(y_i + 1)^2(y_j + 1)(y_k + 1) \).

(5) By direct calculation, we have

\[
150(y_1^2y_2^2y_3 - y_1^2y_2y_3^2) = -6(c_3(3x_1x_2^2x_3) + c_3(3x_1x_2^3)) + 6(c_3(3x_1x_2^2x_3) + c_3(3x_1x_2^3)) + 3(c_3(3x_1x_2^2x_3) + c_3(3x_2^2x_3)) - 3(c_3(3x_1x_2^2x_3) + c_3(3x_2^2x_3)) - 9c_3(3x_1^2x_2) + 9c_3(3x_1^2x_3) + 42c_3(3x_1x_2) - 42c_3(3x_1x_3).
\]

Since \( 3y_1^2(3y_2^2y_3 - 3y_1^2y_2^2y_3) \in \Gamma^4(X) \), the result follows.

(6) It follows by a direct computation that

\[
150y_1^2y_2^2y_3^2 - 300y_1^2y_3^2y_3 = 4(c_3(3x_1x_2^2x_3) + c_3(3x_1x_2^3)) + 6(c_3(3x_1x_2^2x_3) + c_3(3x_1x_2^3)) + 2(c_3(3x_1x_2^2x_3) + c_3(3x_1x_2^3)) - 2(c_3(3x_1x_2^2x_3) + c_3(3x_1x_2^3)) + 8(c_3(3x_1^2x_2x_3) + c_3(3x_1^2x_3)) - 8(c_3(3x_1^2x_2x_3) + c_3(3x_1^2x_3)) + 3c_3(3x_1^2x_2^3) - 16c_3(3x_1x_2x_3) - 36c_3(3x_1x_2).
\]

As \( 9y_1^2y_2^2y_3 \in \Gamma^4(X) \) by Lemma 5.3 (3) and \( 3y_1^2(3y_2^2y_3) \in \Gamma^4(X) \), the result follows. \( \Box \)

Applying Proposition 5.2 we prove the main result of this section.

Theorem 5.4. The maximal torsion in Chow group of codimension 2 of the product of three Severi-Brauer surfaces is \((\mathbb{Z}/3\mathbb{Z})^8\). In other words, \( \mathcal{M}(SB_3) = 3^8 \).

Proof. Let \( A_1, A_2, A_3 \) be division algebras of degree 3 over \( F \) and let \( X \) be the product of the corresponding Severi-Brauer surfaces \( SB(A_1), SB(A_2), SB(A_3) \). Set

\[ \beta_n = |\Gamma^{n/n+1}(X_E)/\text{Im}(\text{res}^{n/n+1})|/|K^n(X_E)/K^n(X)|, \]

where \( E \) is a splitting field of \( X, 1 \leq n \leq 6 \), and \( K^n(X_E) \) (resp. \( K^n(X) \)) is the codimension \( n \) part of \( K(X_E) \) (resp. \( K(X) \)).

We shall find upper bounds of \( \beta_n \) for \( 1 \leq n \leq 6 \). First of all, by (15) we have \( \beta_1 \leq 1 \). For the rest of them, we will find upper bounds using case by case analysis. Let \( i, j, k \) be integers such that \( \{i, j, k\} = I_3, f = f_1f_2f_3, g = g_1g_2g_3, e = e_1e_2e_3, G = \{1 \leq m \leq 3 \mid y_m = 3\} \) and \( H = \{1 \leq m \leq 3 \mid f_m = 9\} \). Observe that if one of \( \{e_1, e_2, e_3\} = 3 \) (say, \( e_i = 3 \)), then \( d, g_m \leq 9 \) for \( 1 \leq m \leq 3 \) and \( 9y_1y_2y_3 = 3y_jy_k(3y_i) \in \Gamma^3(X) \) by Lemma 5.3 (1).
Case: $e_1 = e_2 = e_3 = 3$. By Lemma 5.3 (1), we have $\beta_2 \leq 1$. If $d = 3$, then by Lemma 5.3 (1) and (3) we get $3y_1y_2y_3 \in \text{Im}(\text{res}^{3/4})$. Hence, by Lemma 5.3 (1), (2), and (4) we have

$$f^2 \cdot \beta_3 \leq \begin{cases} 3^6 & \text{if } |H| = 0, \\ 3^3 \min \{f_i, g_j\} & \text{if } |H| = 1, f_k = 9, \\ 3^3 \max \{3^{4-|G|}, 3^2\} & \text{if } |H| = 2, f_i = 3, \\ 3^3 \max \{3^6-|G|, 3^4\} & \text{if } |H| = 3. \end{cases}$$

It follows from 5.3 (1) that $9y_1y_2y_3^2, 9y_1^2y_2^2 \in \text{Im}(\text{res}^{4/5})$, thus we obtain $\beta_4 \leq 3^{12}/3^3g$. Again, by Lemma 5.3 (1), we have $3y_1y_2^2y_3 + 3j_jy_2^2 \in \text{Im}(\text{res}^{4/5})$. Therefore, we have $\beta_5 \leq 3^6/g$. It follows from Lemma 5.3 (3) that we have $\beta_6 \leq 3^5/3^2$. Finally, by 5.3 we conclude that

$$|\oplus \Gamma^{n/n+1}(X)_{\text{tors}}| \leq \begin{cases} 3^{16}/g^2 & \text{if } |H| = 0, \\ 3^{13} \min \{g_i, g_j\}/g^2 & \text{if } |H| = 1, f_k = 9, \\ 3^{10} \max \{3^{4-|G|}, 3^2\}/g^2 & \text{if } |H| = 2, \\ 3^7 \max \{3^6-|G|, 3^4\}/g^2 & \text{if } |H| = 3. \end{cases}$$

The maximum upper bound of (46) is $3^{10}$ when $g_m = f_m = 3$ for all $1 \leq m \leq 3$ and $d \in \{3, 9\}$. If $g_m = f_m = d = 3$, then by Lemma 5.3 (5) and (6) we have $a := 3y_1^2y_2^2y_3^2 - 3y_1^2y_2^2y_3^2 - 3y_1^2y_2^2y_3^2 \in \Gamma^3(X)\backslash \Gamma^4(X)$ as any element of $\Gamma^4(X)$ is divisible by 9. Hence, the classes of $a$ and $b$ give torsion elements of $\Gamma^{3/4}(X)$ of order 3 since $3a = 3y_1^2(3y_2^2y_3^2) - 3y_1^2(3y_2^2y_3^2), 3b = 9y_1^2y_2^2y_3^2 - 3y_1^2(3y_2^2y_3^2) \in \Gamma^4(X)$. Moreover, we have $a - b, a + b \notin \Gamma^3$ as any element of $\Gamma^4(X)$ is divisible by 9, thus the subgroups generated by $a$ and $b$ have trivial intersection. By Proposition 5.2 we have

$$\Gamma^{2/3}(X)_{\text{tors}} = (\mathbb{Z}/3\mathbb{Z})^{\oplus 8} \text{ and } \Gamma^{3/4}(X)_{\text{tors}} = (\mathbb{Z}/3\mathbb{Z})^{\oplus 2}.$$ 

In this case, by (47) we have

$$\text{CH}^2(\bar{X})_{\text{tors}} = (\mathbb{Z}/3\mathbb{Z})^{\oplus 8},$$

where $\bar{X}$ is the corresponding generic variety. If $g_m = f_m = 3$ and $d = 9$, then by the same argument the class $a$ is a torsion of $\Gamma^{3/4}(X)$. Moreover, we have either $9y_1^2y_2^2y_3^2$ is a torsion of order 3 in $\Gamma^{3/4}(X) \oplus \Gamma^{5/6}(X)$ or $\beta_6 \leq 1$. Therefore, we obtain

$$|\oplus \Gamma^{n/n+1}(X)_{\text{tors}}| \leq 3^8.$$ 

Case: $e_i = e_j = 3$ and $e_k = 9$. By Lemma 5.3 (1), (3) and (4) we have

$$\beta_2 \leq (3^5 \min \{d, g_1, g_2, g_3\})/3^3e.$$ 

If $d = 9$, then by Lemma 5.3 (1), (2), and (4) we have

$$f^2 \cdot \beta_3 \leq \begin{cases} 3^2 \min \{9, g_k\} & \text{if } |H| = 0, \\ 3^2 f_i f_k g_j g_k & \text{if } |H| = 1, f_k = 9, \\ 3^2 f_i f_k g_j g_k & \text{if } |H| = 1, f_k = 9, \\ 3^{1+1} g & \text{if } |H| \geq 2, \end{cases}$$
where \(s\) and \(t\) are integers such that \(\{s, t\} = \{i, j\}\). Similarly, if \(d = 3\), then it follows by Lemma 5.3 (1), (2), (3), and (4) that

\[
f^2 \cdot \beta_3 \leq \begin{cases} 
3 f \min\{g, g_m\} & \text{if } |H| = 0, \\
3 f_i f_k g_s \min\{g, g_k\} & \text{if } |H| = 1, f_s = 9, \\
3 f_i f_j \max\{3^{i-|G|}, 3^2\} & \text{if } |H| = 1, f_k = 9, \\
3^{|H|+1} g & \text{if } |H| \geq 2, 
\end{cases}
\]

where the minimum of the first inequality ranges over \(1 \leq m \leq 3\). By Lemma 5.3 (1), we have \(9 y_i y_j y_k^2, 9 y_i^2 y_j^2 \in \text{Im}(\text{res}^{4/5})\). Hence, \(\beta_4 \leq 3^{12}/(3^4 g)\).

By Lemma 5.3 (1), we have \(3 y_i^2 (3 y_j^2 y_k + 3 y_j y_k^2) \in \text{Im}(\text{res}^{5/6})\). If \(f_i = 3\) (resp. \(f_j = 3\)), then by Lemma 5.3 (2) we obtain \(3 y_i^2 (3 y_j^2 y_k - 3 y_j y_k^2) \in \text{Im}(\text{res}^{5/6})\) (resp. \(3 y_j^2 (3 y_i^2 y_k - 3 y_i y_k^2) \in \text{Im}(\text{res}^{5/6})\)). Moreover, if \(d = 3\), then by Lemma 5.3 (3) \(3 y_k^2 (-y_j y_j y_k^2 + 3\sum_{m=1}^3 y_m y_i y_j) = 9 y_i^2 y_j y_k^2 + 9 y_i y_j^2 y_k^3 \in \text{Im}(\text{res}^{5/6})\). Therefore, we conclude that \(\beta_5 \leq 3^6/g\) (resp. \(3^7/g\)) if one of \(f_i, f_j, d \) is 3 (resp. otherwise). By Lemma 5.3 (3), we have \(\beta_6 \leq 3\). In conclusion, we have

\[|\oplus \Gamma^{n/n+1}(X)_{\text{tors}}| \leq 3^6 \text{ for all cases.}\]

**Case:** \(e_i = 3\) and \(e_j = e_k = 9\). By Lemma 5.3 (1), (3) and (4) we obtain

\[\beta_2 \leq (3^4 \min\{g, g_k\} \min\{d, g_i\})/3^3 e.\]

In codimension 3, by Lemma 5.3 (1), (2), (3), and (4) we have

\[
f^2 \cdot \beta_3 \leq \begin{cases} 
3 f g j g k & \text{if } |H| = 0, d = 9, \\
3 f g_i \min\{g, g_k\} & \text{if } |H| = 0, d = 3, \\
3 f_j f_k g_j g_k & \text{if } |H| = 1, f_i = 9, d = 9, \\
3 f_j f_k \max\{3^{i-|G|}, 9\} & \text{if } |H| = 1, f_i = 9, d = 3, \\
3^2 g & \text{if } |H| = 1, f_i \neq 9, \\
3^{|H|+2} g & \text{if } |H| \geq 2, 
\end{cases}
\]

By Lemma 5.3 (1), we get \(3 y_i^2 (3 y_j y_k), 3 y_i (3 y_j^2 y_k + 3 y_j y_k^2) \in \text{Im}(\text{res}^{4/5})\). Hence, it follows from Lemma 5.3 (2) that \(\beta_4 \leq 3^6/g\) (resp. \(3^7/g\)) if \(|H| = 3\) (resp. otherwise).

By Lemma 5.3 (1), we obtain \(3 y_i^2 (3 y_j^2 y_k + 3 y_j y_k^2) \in \text{Im}(\text{res}^{5/6})\). Therefore, by Lemma 5.3 (2) and (3) we have

\[g \cdot \beta_5 \leq \begin{cases} 
g & \text{if } |H| \leq 1 \text{ or } d = 3, \\
g^2 \cdot 27 & \text{if } |H| = 2, d \neq 3, \\
9 \cdot 27^2 & \text{if } |H| = 3, d \neq 3. 
\end{cases}\]

Finally, by Lemma 5.3 (3) \(\beta_6 \leq 3\). Therefore, for all cases we have

\[|\oplus \Gamma^{n/n+1}(X)_{\text{tors}}| \leq 3^6.\]

**Case:** \(e_1 = e_2 = e_3 = 9\). By Lemma 5.3 (1), (3) and (4) we have

\[\beta_2 \leq 3^3 \max\{1/3^3, 1/3^{|G|+3/d}\}/3^3.\]
In codimension 3, by Lemma 5.3 (2), (3) and (4) we obtain \( \beta_3 \leq (3^3 f g)/d^2 \). It follows from Lemma 5.3 (4) that
\[
geg \cdot \beta_4 \leq \begin{cases} 
9^5 \cdot 27 & \text{if } |H| \leq 1, \\
9^4 \cdot 27^2 & \text{if } |H| = 2, \\
9^3 \cdot 27^3 & \text{if } |H| = 3.
\end{cases}
\]

In codimension 5, by Lemma 5.3 (2) and (3) we obtain
\[
g \cdot \beta_5 \leq \begin{cases} 
9^3 & \text{if } d = 3, \\
9^2 \cdot 27 & \text{if } d \neq 3, |H| \leq 1, \\
9 \cdot 27^2 & \text{if } d \neq 3, |H| = 2, \\
27^3 & \text{if } d \neq 3, |H| = 3.
\end{cases}
\]

Finally, by Lemma 5.3 (3) we have \( \beta_6 \leq 3^3 / \max\{9, d\} \). Therefore, for all cases we obtain
\[
\bigoplus \Gamma^{n/n+1}(X)_{\text{tors}} \leq 3^7.
\]

In conclusion, the result follows from (47), (48), (49), (50), and (51). \( \square \)

6. Product of Quadric surfaces

In this section, we obtain upper bounds for the torsion in Chow group of codimension 2 of the product of two quadric surfaces in Theorem 6.1 and the product of three quadric surfaces with the same discriminant in Theorem 6.5. In the case of the product of two quadric surfaces, we also provide a sharp lower bound in the gamma filtration in Proposition 6.3.

Let \( F \) be a field of characteristic different from 2 and let \( q = \langle c, -a, -b, ab \rangle \) be a non-degenerate quadratic form over \( F \) of rank 4 for \( c, a, b \in F^\times \). If the discriminant is trivial, then the quadric surface corresponding to the form \( q \) is birational to \( \mathbb{P}^1 \times \text{SB}(Q) \), where \( Q \) is the quaternion \( F \)-algebra determined by \( a \) and \( b \). Otherwise, the quadric is isomorphic to \( R_{L/F}(\text{SB}(Q)) \), where \( R_{L/F} \) is the Weil restriction over a quadratic field \( L = F(\sqrt{c}) \). We shall write \( \text{disc} Q \) for the discriminant \( c \).

Consider two quadric surfaces with the corresponding quaternions \( Q_1, Q_2 \) and quadratic extensions \( L_1, L_2 \) as above. We set
\[
(52) \quad X = \begin{cases} 
\text{SB}(Q_1) \times \text{SB}(Q_2) & \text{if } \text{disc} Q_1 = 1, \\
\text{SB}(Q_1) \times R_{L_2/F}(\text{SB}(Q_2)) & \text{if } \text{disc} Q_1 = 1 \neq \text{disc} Q_2, \\
R_{L_1/F}(\text{SB}(Q_1)) \times R_{L_2/F}(\text{SB}(Q_2)) & \text{if } \text{disc} Q_1 \neq 1.
\end{cases}
\]

Then, by [3, Corollary 2.5] the torsion in codimension 2 cycles of \( X \) of the first and second cases of (52) is isomorphic to that of the product of two quadric surfaces. Therefore, it suffices to consider \( X \) for the torsion in codimension 2 cycles of the product of two quadric surfaces. We call \( X \) the variety associated to the product of two quadric surfaces.

Consider the last case of (52). If \( \text{ind}(Q_1)_{L_1} = \text{ind}(Q_1)_{L_2} = 1 \), then the associated variety \( X \) has torsion-free Chow groups. Thus, we may assume that \( \text{ind}(Q_1)_{L_1} = 2 \). We choose a splitting field \( E \) of \( X \) as follows. If \( \text{ind}(Q_2)_{L_2} = 1 \), then we take a maximal subfield \( \neq L_2 \) of \( Q_1 \) for \( E \). Otherwise, we take for \( E \) a common maximal subfield \( \neq L_1, L_2 \) of \( Q_1 \) and
Q_2 if \( \text{ind}(Q_1 \otimes Q_2) \leq 2 \) or the tensor product of maximal subfields \( E_1(\neq L_2) \) of \( Q_1 \) and \( E_2(\neq L_1) \) of \( Q_2 \) if \( \text{ind}(Q_1 \otimes Q_2) = 4 \). Hence, \( d := [E : F] = 4 \) if \( \text{ind}(Q_1 \otimes Q_2) = 4 \) and \( d = 2 \) otherwise. For the second case [52], we choose a splitting field \( E \) in the same way.

The theorem below was proven in [3]. Here, we give an elementary proof which does not use any cohomological method and \( K \) theory of quadrics. Moreover, we find upper bound of the total torsion in the topological filtration of the product of two quadric surfaces with nontrivial discriminants.

**Theorem 6.1.** [3 Theorems 5.1, 5.7, 5.8, 5.9] Let \( X \) be the variety associated to the product of two quadric surfaces. Then, the torsion subgroup \( CH^2(X)_{\text{tors}} \) is either trivial or \( \mathbb{Z}/2\mathbb{Z} \), i.e., \( \mathcal{M}(Q_2) \leq 2 \).

**Proof.** Let \( Q_i \) be a quaternion algebra for \( i = 1, 2 \), \( X \) the variety associated to the product of two quadric surfaces of \( Q_i \), and \( E \) the splitting field of \( X \). If \( \text{disc} Q_i = 1 \) for all \( i \), then the variety \( X \) is torsion free. From now on we only consider the other cases. To apply [3] we shall find upper bounds of \( \alpha_n := |T^{n/n+1}(X_E)/\text{Im}(\text{res}^{n/n+1})| \) for each of the following 3 cases.

*Case: \( L_1L_2 := L_1 \otimes L_2 \) is a biquadratic field extension. Let \( L = L_1L_2 \). Then, we have \( X_E = R_{EL_1/E}(\mathbb{P}^1) \times R_{EL_2/E}(\mathbb{P}^1) \) and \( X_{EL} = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \), which have torsion-free Chow groups. For each \( 1 \leq k \leq 4 \), let \( x_k \) be the pullback of the class of tautological line bundle on the projective space in \( K(X_{EL}) \). We set \( x_{i_1} \cdots x_{i_k} = x_{i_1} \cdots x_{i_k} \) for \( 1 \leq i_1 < \ldots < i_k \leq 4 \). By the action of the Galois group of \( EL/E \), we have the following bases of \( K(X_{EL}) \) and \( K(X) \), respectively:

\[
(53) \quad \{1, x_{2i-1} + x_{2i}, x_{12}, x_{34}, (x_1 + x_2)(x_3 + x_4), x_{12}(x_3 + x_4), x_{34}(x_1 + x_2), x_{1234}\}
\]

\[
(54) \quad \{1, e_i(x_{2i-1} + x_{2i}), x_{12}, x_{34}, f(x_1 + x_2)(x_3 + x_4), e_2x_{12}(x_3 + x_4), e_1x_{34}(x_1 + x_2), x_{1234}\}
\]
where \( e_i = \text{ind}(Q_i) \) and \( f = \text{ind}(Q_1 \otimes Q_2) \). Then, we have \( |K(X_{EL})/K(X)| = e_1^2 e_2^2 f \). If \( e_1 = e_2 = 1 \), then \( f = 1 \). Hence, we may assume that \( e_1 e_2 \geq 2 \).

Let \( y_k = x_k - 1 \). Set \( y_{i_1} \cdots y_{i_k} = y_{i_1} \cdots y_{i_k} \) for \( 1 \leq i_1 < \ldots < i_k \leq 4 \). We will use other bases for \( K(X_{EL}) \) the basis \( (53) \) by replacing \( x_k \) by \( y_k \) and for \( K(X) \)

\[
(55) \quad \{1, e_i(y_{2i-1} + y_{2i}), z_{12}, z_{34}, f(y_1 + y_2)(y_3 + y_4), e_2z_{12}(y_3 + y_4), e_1z_{34}(y_1 + y_2), z_{12}z_{34}\}
\]
where \( z_{12} = y_1 + y_2 + y_1 y_2 \) and \( z_{34} = y_3 + y_4 + y_3 y_4 \).

It follows from \( z_{12}z_{34} \in K(X) \) that \( y_1 + y_2, y_3 + y_4 \in \text{Im}(\text{res}^{1/2}) \), thus we have \( \alpha_1 = 1 \).

By the same argument, we get \( (y_1 + y_2)(y_3 + y_4) \in \text{Im}(\text{res}^{2/3}) \). In addition, it follows from the basis \( (53) \) that \( e_1 y_{12}, e_2 y_{34} \in T^2(X) \). Hence, \( \alpha_2 \leq e_1 e_2 \).

If \( \text{ind}(Q_1 \otimes Q_2) = 1 \), then it follows from closed embeddings \( \text{SB}(Q_1) \times R_{L_1/E}(\text{SB}(Q_1)) \hookrightarrow X \) and \( R_{L_1/E}(\text{SB}(Q_1)) \times \text{SB}(Q_1) \hookrightarrow X \) that we have \( y_{34}(y_1 + y_2), y_1 y_2(y_3 + y_4) \in \text{Im}(\text{res}^{3/4}) \), respectively. Otherwise, it follows from \( e_{1}y_{12}.(y_3 + y_4) \) and \( e_{2}y_{34}.(y_1 + y_2) \) that \( e_{1}y_{12}.(y_3 + y_4), e_{2}y_{34}.(y_1 + y_2) \in \text{Im}(\text{res}^{3/4}) \). Therefore, \( \alpha_3 \leq 1 \) if \( \text{ind}(Q_1 \otimes Q_2) = 1 \) and \( \alpha_3 \leq e_1 e_2 \) otherwise.

By a transfer argument, we have \( \alpha_4 \leq d \). Hence, we obtain

\[
|\circledast T^{n/n+1}(X)_{\text{tors}}| \leq \begin{cases} 1 & \text{if } \text{ind}(Q_1 \otimes Q_2) = 1, \\ d/f & \text{otherwise}. \end{cases}
\]
If \( f = 1 \), then \( \text{ind}(Q_1 \otimes Q_2) \leq 2 \), thus \( d = 2 \). Therefore, \( |T^{2/3}(X)_{\text{tors}}| \leq 2 \). Moreover, the order of the group \( \oplus T^{n/n+1}(X)_{\text{tors}} \) is trivial if \( f = 4 \) or \( f = d = 2 \) or \( \text{ind}(Q_1 \otimes Q_2) = 1 \).

**Case:** \( L_1 = L_2 \). Let \( L = L_1 = L_2 \). Then, \( X_E = R_{E/E}(\mathbb{P}^1) \times R_{E/E}(\mathbb{P}^1) \). Applying the same argument with the previous case, we have the basis \( (53) \) (resp. \( (54) \)) replacing \( (x_1 + x_2)(x_3 + x_4) \) (resp. \( f(x_1 + x_2)(x_3 + x_4) \)) with two elements \( x_{13} + x_{24} \) and \( x_{14} + x_{23} \) (resp. \( f(x_{13} + x_{24}) \) and \( f(x_{14} + x_{23}) \)) for \( K(X_E) \) (resp. \( K(X) \)). As \( e_1 = 2 \), we have \( |K(X_E)/K(X)| = 4e_2^2f^2 \).

Similarly, we use other bases for \( K(X_E) \) the basis by replacing \( x_k \) with \( y_k \) and for \( K(X) \) the basis replacing \( f(y_1 + y_2)(y_3 + y_4) \) with two elements \( f(y_{13} + y_{24} + \sum_{k=1}^4 y_k) \) and \( f(y_{14} + y_{23} + \sum_{k=1}^4 y_k) \).

By the same argument used in the previous case, we have \( \alpha_1 = 1 \). In codimension 2, we have \( 2y_{12}, e_2y_{23}, (y_1 + y_2)(y_3 + y_4) \in \text{Im}(\text{res}^{2/3}) \). If \( f = 1 \), then \( e_2 = d = 2 \), and \( X = R_{L/F}(\text{SB}(Q_1)) \times R_{L/F}(\text{SB}(Q_2)) \), thus by the diagonal embedding \( R_{L/F}(\text{SB}(Q_1)) \hookrightarrow X \) the sum of all elements of codimension 2 in the basis is contained in the group \( \text{Im}(\text{res}^{2/3}) \). Moreover, if \( f \neq 1 \), then \( f(y_{13} + y_{24}) \), \( f(y_{14} + y_{23}) \in T^2(X) \). Therefore, we obtain \( \alpha_2 \leq 2e_2f \).

If \( f = 1 \), then we have \( z_{12}(y_{12} + y_{34} + y_{13} + y_{24} + y_{14} + y_{23}) = \text{Im}(\text{res}^{3/4}) \). Otherwise, we obtain \( 2y_{12}(y_3 + y_4), e_2y_{23}(y_1 + y_2) \in \text{Im}(\text{res}^{3/4}) \). Hence, we have \( \alpha_3 \leq 2 \) if \( f = 1 \) and \( \alpha_3 \leq 2e_2 \) otherwise. Finally, we get \( \alpha_4 \leq d \), thus

\[
|\oplus T^{n/n+1}(X)_{\text{tors}}| \leq \begin{cases} 1 & \text{if } f = 1, \\ d/f & \text{otherwise.} \end{cases}
\]

Hence, the group \( \oplus T^{n/n+1}(X)_{\text{tors}} \) is trivial except the case where \( f = 2 \) and \( d = 4 \). In the latter case, we can further reduce the upper bound of \( \alpha_4 \) to 2 if \( 2y_{1234} \in T^4(X) \). Hence, the group \( \oplus T^{n/n+1}(X)_{\text{tors}} \) is trivial in this case. If \( 2y_{1234} \notin T^4(X) \), then the class \( 2y_{1234} = 2y_{12} \cdot z_{34} - z_{12} \cdot 2(y_{13} + y_{24}) \in T^3(X) \) gives a torsion element of order 2 in \( T^{3/4}(X)_{\text{tors}} \). Therefore, the group \( T^{3/4}(X)_{\text{tors}} \) is trivial in all cases.

**Case:** one of disc \( Q_i \) is trivial. Let \( \text{disc} Q_1 = 1 \). Then, we have \( X_E = \mathbb{P}^1 \times R_{L_E/E}(\mathbb{P}^1) \). We have the following bases of \( K(X_E) \) and \( K(X) \), respectively:

\[
\{1, x_1, x_2 + x_3, x_{23}, x_1(x_2 + x_3), x_{123}\} \text{ and } \{1, e_1x_1, e_2(x_2 + x_3), x_{23}, f(x_1(x_2 + x_3), e_1x_{123}\},
\]

where \( e_1 = \text{ind}(Q_1)_{L_2}, e_2 = \text{ind}(Q_2)_{L_2}, f = \text{ind}(Q_1 \otimes Q_2)_{L_2} \). It follows that we obtain \( |K(X_E)/K(X)| = e_1^2e_2f \). We will use other bases for \( K(X_E) \) the above basis by replacing \( x_k \) by \( y_k \) and for \( K(X) \)

\[
\{1, e_1y_1, e_2(y_2 + y_3), z_{23}, f(y_1 + 1)(y_2 + y_3), e_1y_1z_{23}\},
\]

where \( z_{23} = y_2 + y_3 + y_{23} \).

Obviously, we have \( \alpha_1 \leq e_1 \). In codimension 2, we have \( e_2y_{23}, e_1y_1(y_2 + y_3) \in T^2(X) \). If \( f = 1 \), then \( (y_{12} + y_{13} + y_{24} + z_{23} - 2(y_2 + y_3) = y_{12} + y_{13} + y_{23} \in T^2(X) \). Hence, we get \( \alpha_2 \leq \min\{e_1, e_2\} \) if \( f = 1 \) and \( \alpha_3 \leq e_1e_2 \) otherwise. Finally, we have \( \alpha_4 \leq d \), thus the same upper bound \( (53) \) is obtained for the order of the group \( \oplus T^{n/n+1}(X)_{\text{tors}} \). Hence, the group \( \oplus T^{n/n+1}(X)_{\text{tors}} \) is trivial if \( f = 1 \) or \( f = d = 2 \) or \( f = 4 \). Assume that \( f = 2 \) and \( d = 4 \). Then it follows from \( (57) \) that \( 2y_{123} \in T^2(X) \). If \( 2y_{123} \in T^3(X) \), then we have \( \alpha_3 \leq 2 \). Therefore, the group \( \oplus T^{n/n+1}(X)_{\text{tors}} \) is trivial in this case. Otherwise, the class of
$2y_{123}$ gives a torsion element of order 2 in $T^{2/3}(X)$. In any case, we have $|T^{2/3}(X)_{\text{tors}}| \leq 2$. In conclusion, the result follows from (1).

**Remark 6.2.** Observe that the proof of Theorem 6.1 indicates when the torsion in Chow group of codimension 2 of $X$ is trivial. Moreover, the proof still works if we replace the topological filtration by the gamma filtration.

Now we provide a nontrivial torsion subgroup in the gamma filtration:

**Proposition 6.3.** With the above notations, we have $\Gamma^{2/3}(X)_{\text{tors}} = \mathbb{Z}/2\mathbb{Z}$ if $L$ is a biquadratic extension, $f = 2$, and $d = 4$.

**Proof.** It follows from the basis (55) that $2y_{1234} \in K(X)$. Hence, by (2) we have $2y_{1234} \in \Gamma^2(X)$. As $d = 4$, $4y_{1234} \in \Gamma^3(X)$. We show that this element is not contained in $\Gamma^3(X)$ by computing the Chern classes of the elements in the basis (54).

Consider the basis (54) of $\Gamma(X)$ with $f = e_i = 2$, where $i = 1, 2$. It follows from Whitney formula that we have $c_1(2(x_{2i-1} + x_{2i})) = 2(y_{2i-1} + y_{2i}), c_1(x_{2i-1} + x_{2i}) = y_{2i-1} + y_{2i}$, and $c_2(x_{2i-1} + x_{2i}) = y_{2i-1}y_{2i}$. Therefore, we have

$$c_2(2(x_{2i-1} + x_{2i})) = 4y_{2i-1}y_{2i}, c_1(x_{2i-1}x_{2i})^2 = 2y_{2i-1}y_{2i}, \text{ and } c_1(x_{2i-1}x_{2i})^3 = 0.$$  

Similarly, we obtain $c_j(2(x_{2i-1} + x_{2i})) = 0$ for $3 \leq j \leq 4$.

Let $z = (x_1 + x_2)(x_3 + x_4)$, $z' = (y_1 + y_2)(y_3 + y_4)$, $u = y_{123} + y_{124}$, and $v = y_{134} + y_{234}$. Then, by a direct computation, we have

$$c_j(z) = \begin{cases} 
2(\sum_{k=1}^4 y_k) + z' & \text{for } j = 1, \\
2y_{1234} + 3z' + 4(u + v + y_{12} + y_{34}) & \text{for } j = 2, \\
4(u + v + 3y_{1234}) & \text{for } j = 3, \\
2y_{1234} & \text{for } j = 4.
\end{cases}$$

Since $c_2(2z) = c_1(z)^2 + 2c_2(z)$, it follows from (59) that

$$c_2(2z) = 8y_{1234} + 14z' + 16(u + v + y_{12} + y_{34}).$$

As $c_3(2z) = 2(c_1(z)c_2(z) + c_3(z))$ and $c_4(2z) = 2(c_1(z)c_3(z) + c_4(z)) + c_2(z)^2$, it follows from (59) that

$$c_j(2z) \equiv 0 \mod 4 \text{ for } j = 3, 4.$$

Let $w = x_{12}(x_3 + x_4)$. Then, we obtain $c_1(w) = 2z_{12} + y_3 + y_4 + z' + u$ and $c_2(w) = 4y_{1234} + 3u + 2(y_{12} + v) + y_{13} + y_{14} + y_{23} + y_{24} + y_{34}$. Therefore, we have

$$c_j(2w) = \begin{cases} 
2(8y_{1234} + 9u + 4v + 6y_{12} + 3y_{13} + 3y_{23} + 3y_{14} + 3y_{24} + 2y_{34}) & \text{for } j = 2, \\
4(10y_{1234} + 3u + 2v) & \text{for } j = 3, \\
8y_{1234} & \text{for } j = 4.
\end{cases}$$

Let $w' = x_{34}(x_1 + x_2)$. Then, we have the Chern classes (62) for $2w'$ by replacing 1, 2, 3, 4 with 3, 4, 1, 2, respectively.

It follows from (58) that $c_1(x_{12})^2c_1(x_{34}) = c_1(x_{12})^2c_1(x_{1234}) = 2(u + y_{1234}), c_1(x_{34})^2c_1(x_{12}) = c_1(x_{34})^2c_1(x_{1234}) = 2(v + y_{1234})$. Since $c_1(x)$ is divisible by 2 for any element $x \in \{2z, 2w, 2w', 2(x_{2i-1} + x_{2i})\}$, one can see easily that the subgroup generated by the products of three of the first Chern classes of any element of the basis (54) is generated by
$2(u + y_{1234})$ and $2(v + y_{1234})$ modulo 4. Similarly, using \([58], [60], [62]\) one sees that the subgroup generated by the products of the first and second Chern classes of any element of the basis \([54]\) is also generated by $2(u + y_{1234})$ and $2(v + y_{1234})$ modulo 4. It follows from \([61]\) and \([62]\) that the third and fourth Chern classes of $2z, 2w, 2w'$ is divisible by 4. Therefore, the subgroup $\Gamma^3(X)$ is generated by $2(u + y_{1234})$ and $2(v + y_{1234})$ modulo 4. Hence, $2y_{1234}$ is not contained in $\Gamma^3(X)$ and this element gives a torsion of $\Gamma^2/\Gamma^3(X)$ of order 2. The result immediately follows from Remark \([62]\). \(\square\)

**Remark 6.4.** If $\bar{X}$ is a corresponding generic variety to $X$ in Proposition \([63]\) then we obtain $\text{CH}^2(\bar{X}) = \mathbb{Z}/2\mathbb{Z}$, which recovers a theorem of Izhboldin and Karpenko \([53]\) Theorem 14.1]. Indeed, it is possible to find such a variety by showing that the gamma filtration for the variety $R_{L_1/F}(\text{SB}(Q'_1)) \times R_{L_2/F}(\text{SB}(Q'_2)) \times R_{L_3/F}(\text{SB}(2, Q'_1 \otimes Q'_2))$ is torsion free, where $L = L_1L_2$ is a biquadratic extension and $\text{SB}(2, Q'_1 \otimes Q'_2)$ is the generalized Severi-Brauer variety of rank 2 left ideals in the biquaternion algebra $(Q'_1 \otimes Q'_2)_L$.

### 6.1. Three quadric surfaces with the same discriminant.

In this subsection we consider the product of three quadric surfaces with the same discriminant. Let $Q_1, Q_2, Q_3$ be three quaternion $F$-algebras and let $L$ be the quadratic extension over $F$ corresponding to three quadric surfaces with the same discriminant as above.

We set

\[
(63) \quad X = \begin{cases} 
\text{SB}(Q_1) \times \text{SB}(Q_2) \times \text{SB}(Q_3) & \text{if } \text{disc} \, Q_i = 1, \\
R_{L/F}(\text{SB}(Q_1)) \times R_{L/F}(\text{SB}(Q_2)) \times R_{L/F}(\text{SB}(Q_3)) & \text{otherwise}
\end{cases}
\]

and call it the variety associated to the product of three quadric surfaces with the same discriminant. Then, by the same argument as in the case of two quadric surfaces, the torsion in codimension 2 cycles of $X$ is isomorphic to that of the product of three quadric surfaces with the same discriminant.

Let $h = \text{ind}(Q_1 \otimes Q_2 \otimes Q_3)$, $J = \{\{1, 2\}, \{3, 4\}, \{5, 6\}\}$, and $f_{pq} = \text{ind}(Q_{\text{max}\{r,s\}/2} \otimes Q_{\text{max}\{t,u\}/2})$ for any $\{\{p, q\}, \{r, s\}, \{t, u\}\} = J$. Set

$F_m = \{\{p, q\} \in J \mid f_{pq} = m\}$ for $m = 1, 2, 4$.

Consider the second case of \([63]\). If $\text{ind}(Q_i)_L = 1$ for all $1 \leq i \leq 3$, then the variety $X$ has torsion-free Chow groups, thus we may assume that $\text{ind}(Q_1)_L \neq 1$. We choose a splitting field $E$ of $X$ as follows. If $\text{ind}(Q_2)_L = \text{ind}(Q_3)_L = 1$, then we take a maximal subfield for $E$. If $\text{ind}(Q_2)_L = 2$ and $\text{ind}(Q_3)_L = 1$, then we take for $E$ a common maximal subfield of $Q_1$ and $Q_2$ if $\text{ind}(Q_1 \otimes Q_2) \leq 2$ or the tensor product of maximal subfields of $Q_1$ and $Q_2$ if $\text{ind}(Q_1 \otimes Q_2) = 4$.

Now we may assume that $\text{ind}(Q_i)_L = 2$ for all $1 \leq i \leq 3$. If $h = 8$, then we take for $E$ the tensor product of maximal subfields of $Q_i$. If $h = 1$, then $|F_2| = 3$, thus we take the tensor product of a common maximal subfield of $Q_1$ and $Q_2$ and a maximal subfield of $Q_3$ for $E$. If $|F_1| \geq 2$, then we have $(Q_1)_L \simeq (Q_2)_L \simeq (Q_3)_L$, thus we take for $E$ a maximal subfield of $Q_1$. If $h \in \{2, 4\}$, $|F_1| \leq 1$, and $|F_4| \geq 1$, then there exist $Q_i$ and $Q_j$ such that $\text{ind}(Q_i \otimes Q_j) = 4$ for some $1 \leq i \neq j \leq 3$, thus we take for $E$ the tensor product of maximal subfields of $Q_i$ and $Q_j$ which also splits the remaining quaternion algebra. If $h \in \{2, 4\}$, $|F_1| = 1$, and $|F_4| = 0$, then there exist $Q_i$ and $Q_j$ such that $(Q_i)_L \simeq (Q_j)_L$ for
some $1 \leq i \neq j \leq 3$, thus we take for $E$ the tensor product of a maximal subfield of $Q_i$ and a maximal subfield of the remaining quaternion algebra. Hence, we have

$$d := [E : F] = \begin{cases} 
2 & \text{if } |F_1| \geq 2, \\
4 & \text{if } h = 1 \text{ or } h \in \{2, 4\}, |F_1| \leq 1, \\
8 & \text{if } h = 8.
\end{cases}$$

Theorem 6.5. The torsion subgroup in the codimension 2 Chow group of the product of three quadric surfaces with the same discriminant is contained in $(\mathbb{Z}/2\mathbb{Z})^\oplus 7$.

Proof. Let $Q_i$ be a quaternion $F$-algebra for $1 \leq i \leq 3$ such that the corresponding quadrics have the same discriminant. Let $X$ be the associated variety to the product of three quadric surfaces of $Q_i$ and $E$ be the splitting field of $X$ as above. If the discriminant is trivial, the result follows from Proposition 4.2. Hence, we may assume that the discriminant is non-trivial, thus we have $X_E = R_{EL/E}(\mathbb{P}^1) \times R_{EL/E}(\mathbb{P}^1) \times R_{EL/E}(\mathbb{P}^1)$ and $X_{EL} = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

For each $1 \leq k \leq 6$, let $x_k$ be the pullback of the class of tautological bundle on the projective line in $K(X_E)$. Set $x_{i_1} \cdots x_{i_k}$ for $1 \leq i_1 < \ldots < i_k \leq 6$. It follows from the action of the Galois group $\mathbb{Z}/2\mathbb{Z}$ that we have the bases of $K(X_E)$ and $K(X)$, respectively:

$$\{1, x_p + x_q, x_{pq}, x_{pr} + x_{qs}, x_{rs}(x_p + x_q), x_{prt} + x_{qsu}, x_{pqrs}, x_{pqrst}(x_t + x_u), x_{pqrs}(x_t + x_u), x_{pqrs}(x_t + x_u)\}$$

$$\{1, e_{pq}(x_p + x_q), x_{pq}, f_{tu}(x_{pr} + x_{qs}), e_{pq}x_{rs}(x_p + x_q), g(x_{prt} + x_{qsu}), x_{pqrs}, f_{pq}x_{pq}(x_{rt} + x_{su}), e_{tu}x_{pqrs}(x_t + x_u), x_{pqrs}(x_t + x_u)\},$$

where $e_{pq} = \text{ind}(Q_{\text{max}} \{p, q\}/2)$, $g = \text{ind}(Q_1 \otimes Q_2 \otimes Q_3)_L$, and $p, q, r, s, t, u$ range over $\{p, q\}, \{r, s\}, \{t, u\} = J$. Then, we have $|K(X_E)/K(X)| = (e_{12}e_{34}e_{56}f_{12}f_{34}f_{56}g)^4$.

Let $y_k = x_k - 1$ and $y_{i_1} \cdots y_{i_k}$ for $1 \leq i_1 < \ldots < i_k \leq 6$. To simplify the computation, we shall use other bases for $K(X_E)$ that we have the bases replacing $x_k$ with $y_k$ and $K(X)$

$$\{1, e_{pq}(y_p + y_q), z_{pq}f_{tu}(z_{pr} + z_{qs}), e_{pq}z_{rs}(y_p + y_q), g(z_{prt} + z_{qsu}), z_{pq}z_{rs}, f_{pq}z_{pq}(z_{rt} + z_{su}), z_{pq}z_{rs}(z_{tu})\},$$

where $z_{pq} = y_py_q + y_p + y_q$, $z_{pr} = y_pr + y_p + y_r$, and $z_{prt} = y_pr + z_{pr} + y_p + y_r + y_t$.

Let $\alpha_n = |T^{n/n+1}(X_E)/\text{Im}(\text{res}^{n+1})|$. We will find upper bounds of $\alpha_n$ for $1 \leq n \leq 6$. Observe that any basis element of $K(X_E)$ multiplied by $d$ is contained in the image of the restriction map. Since $z_{pq} \in K(X)$, we get $y_p + y_q \in \text{Im}(\text{res}^{1/2})$. Hence, we have $\alpha_1 = 1$. We divide the proof into three cases.

Case 1: $f_{pq} \neq 1$ for all $\{p, q\} \in J$, i.e., $|F_1| = 0$. It follows from the basis of $K(X)$ that $e_{pq}y_{pq} \in T^2(X)$ and $(y_p + y_q)(y_r + y_s) \in \text{Im}(\text{res}^{2/3})$ for any $\{p, q\} \neq \{r, s\} \in J$. Thus, we obtain $f_{tu}(y_{pr} + y_{qs}) \in T^2(X)$ for any $\{t, u\} \in J$. Therefore, we have

$$\alpha_2 \leq e_{12}e_{34}e_{56}f_{12}f_{34}f_{56}. $$

Since $z_{pq} \in T^1(X)$, we have $e_{rs}y_{rs} \cdot (y_p + y_q) \in \text{Im}(\text{res}^{3/4})$. Moreover, as $z_{12}z_{34}z_{56} \in T^3(X)$ we obtain $\sum y_{prt} + y_{qsu} \in \text{Im}(\text{res}^{3/4})$, where the sum ranges over all such elements in the
basis. As \( f_{tu} \neq 1 \), the element \( f_{tu}(y_{pr} + y_{qs}) \cdot (y_t + y_u) \) is contained in the group \( \text{Im}(\text{res}^{3/4}) \).

Hence, we have

\[
\alpha_3 \leq (e_{12}e_{34}e_{56})^2 f_{12}f_{34}f_{56}d / \max\{f_{12}, f_{34}, f_{56}\}. 
\]

As \( e_{pq}y_{pq} \cdot f_{pq}(y_{rt} + y_{su}) \in T^2(X) \), we obtain \( e_{pq}e_{rqs}y_{pq} \cdot e_{pq}y_{pq} \cdot f_{pq}(y_{rt} + y_{su}) \in T^4(X) \). As \( e_{pq}y_{pq} \cdot z_{rs} \cdot z_{tu} \in \text{Im}(\text{res}^{4/5}) \), we have \( \alpha_4 \leq \prod_{p \in J} e_{pq} \min\{d, e_{12}e_{34}e_{56}/e_{pq}\} \min\{d, f_{pq}e_{pq}\} \).

It follows from \( e_{pq}y_{pq} \in T^2(X) \) that we have \( \alpha_5 \leq \prod_{p \in J} \min\{d, e_{12}e_{34}e_{56}/e_{pq}\} \). Obviously, \( \alpha_6 \leq d \). Hence, we obtain

\[
| \oplus T^{n/n+1}(X)_{\text{tors}} | \leq \frac{d^2}{g^4 \max\{f_{12}, f_{34}, f_{56}\} \prod_{p \in J} \min\{d, e_{12}e_{34}e_{56}/e_{pq}\} \min\{d, f_{pq}e_{pq}\}}. 
\]

Let \( B \) be the right-hand side of the inequality of (66). We first consider the case where \( G := g = h \). Then, one can compute \( B \) for each subcase of \( g = h \). Consider a subcase where \( G = 4 \) and \( |F_2| = 1 \). Then, \( f_{pq} = 2 \) for some \( pq \in J \). If \( 2y_{rstu}y_{pq} \in \text{Im}(\text{res}^{5/6}) \), thus we can reduce the upper bound \( B(= 2^2) \) to 1. Hence, we have \( | \oplus T^{n/n+1}(X)_{\text{tors}} | \leq 1 \). Otherwise, by Lemma 6.6 (1) below we obtain either \( | \oplus T^{n/n+1}(X)_{\text{tors}} | \leq 2^4, T^{3/4}(X)_{\text{tors}} = T^{1/5}(X)_{\text{tors}} = \mathbb{Z}/2\mathbb{Z} \) or \( | \oplus T^{n/n+1}(X)_{\text{tors}} | \leq 2, T^{3/4}(X)_{\text{tors}} = \mathbb{Z}/2\mathbb{Z} \). Therefore, \( T^{2/3}(X)_{\text{tors}} = 0 \) in any case.

Let \( E_2 = \{ \{p, q\} \in J \mid e_{pq} = 2 \} \). Then, we may assume that \( |E_2| \geq 1 \). If \( G = 1 \), then we have \( |F_2| = |E_2| = 3 \) and \( d = 4 \). Therefore, by Lemma 6.6 (2) we can reduce the upper bound \( 2^6 \) (resp. \( 2^{10} \)) obtained by (65) (resp. (65)) to \( 2^5 \) (resp. \( 2^8 \)).

Applying the same argument, together with Lemma 6.6 to each subcase of \( g = h \), we obtain

\[
| T^{2/3}(X)_{\text{tors}} | \leq \begin{cases} 1 & \text{if } G = 4, \ 0 \leq |F_2| \leq 2 \text{ or } G = |E_2| = d = 2, |F_2| = 3, \\ 2^2 & \text{if } G = 8 \text{ or } G = 4 \text{ (resp. 2)}, |F_2| = 3, |E_2| = 3, \\ 2^4 & \text{if } G = |E_2| = 2, |F_2| = 3, d = 4 \text{ or } G = 2, 0 \leq |F_2| \leq 2, \\ 2^6 & \text{if } G = 1. \end{cases} 
\]

Now we consider the case where \( H := g = h/2 \). By the same argument, we have the same upper bound (67) for \( | T^{2/3}(X)_{\text{tors}} | \) if \( G = H = 1, 2 \). Similarly, if \( H = 4 \), then we obtain

\[
| T^{2/3}(X)_{\text{tors}} | \leq \begin{cases} 1 & \text{if } H = 4, |F_2| = 2, |E_2| = 2, \\ 2^2 & \text{if } H = 4, 2 \leq |F_2| \leq 3, |E_2| = 3, \\ 2^5 \text{ (resp. } 2^4 \text{)} & \text{if } H = 4, |F_2| = 0 \text{ (resp. } |F_2| = 1). \end{cases} 
\]

Case: \( |F_1| = 1 \). Let \( f_{1u} = 1, e = e_{pq} = e_{rs}, \) and \( e' = e_{1u} \). For any number \( t \), we write \( m(t) \) for \( \min\{d, t\} \). Then, Lemma 6.6 (3) and the above argument implies that

\[
| \oplus T^{n/n+1}(X)_{\text{tors}} | \leq (e^2 e' f_{pq} f_{rs}) (e^2 e' m(f_{pq}) m(f_{rs}) d) (edm(e^2 m(ee')^2 m(f_{pq} e) m(f_{rs} e)) / (e^2 e' f_{pq} f_{rs} g)^4, 
\]

where each term inside the parentheses of the numerator is the upper bound of \( \alpha_n \) for \( 2 \leq n \leq 6 \).
By the same argument used above, Lemma 6.6 (1) and (2) yield

\[
T^{2/3}(X)_{\text{tors}} \leq \begin{cases} 
2 & \text{if } G \text{ (or } H) = 2, |F_2| = 2, |E_2| = 2, \\
2^2 & \text{if } G \text{ (or } H) = 2, |F_2| = 2, |E_2| = 1 \text{ or } G \text{ (or } H) = 1, \\
2^7 & \text{if } G \text{ (or } H) = 2, 0 \leq |F_2| \leq 2, |E_2| = 3.
\end{cases}
\]

Case: \(|F_1| \geq 2\). Then, we have \(|F_1| = |E_2| = 3, d = 2, \) and \(g \in \{1, 2\}\). It follows from Lemma 6.6 (3) that \(y_{pr} + y_{qs} - (y_{pq} + y_{rs}) \in T^2(X)\) for all \(\{p, q\} \neq \{r, s\} \in J\). Since we have \(2y_{pq} \in T^2(X)\) and \((y_p + y_q)(y_r + y_s) \in \text{Im}(\text{res}^{2/3})\) for all \(\{p, q\} \neq \{r, s\} \in J\), we obtain \(\alpha_2 \leq 2^3\).

As \(d = 2\) and \(\sum y_{pr}t + y_{qs}u \in \text{Im}(\text{res}^{3/4})\), where the sum ranges over all such elements in the basis, it follows from Lemma 6.6 (3) that \(\alpha_3 \leq 2^6\). Similarly, by Lemma 6.6 (3) we have \(\alpha_4 \leq 2^9\). As \(d = 2\), we obtain \(\alpha_5 \leq 2^4\) and \(\alpha_6 \leq 2\). In conclusion, we have

\[
\text{res}^{n/n+1}(X)_{\text{tors}} \leq 2^7/g^4
\]

for \(1 \leq g \leq 2\).

The result follows from (67), (68), (69), and (70).

\[\square\]

**Lemma 6.6.** With the above notation, the followings hold:

1. If \(f_{pq} = e_{rs}\) (resp. \(f_{pq} = e_{tu}\)), then \(e_{rsy_{rstu}}\) (resp. \(e_{etu_{rstu}}\) \(\in T^3(X)\)),

2. If \(f_{pq} = e_{rs}y_{rstu}\) (resp. \(f_{pq} = e_{tu_{rstu}}\) \(\in T^4(X)\)),

3. Moreover, if in addition \(e_{rsy_{rstu}}\) (resp. \(e_{etu_{rstu}}\)) \(\notin T^4(X)\), then we have a subgroup \((e_{rsy_{rstu}}\) (resp. \(e_{etu_{rstu}}\)) \(\subseteq T^{3/4}(X)_{\text{tors}}\) of order \(e_{tu}\) (resp. \(e_{rs}\)). If in addition \(e_{rsy_{rstu}}\) (resp. \(e_{etu_{rstu}}\)) \(\notin T^5(X)\) and \(e_{etu_{123456}}\) \(\in T^5(X)\), then we obtain a subgroup \((e_{rsy_{rstu}}\cdot (y_p + y_q)(y_{rstu}(y_p + y_q))) \subseteq T^{4/5}(X)_{\text{tors}}\) of order \(e_{tu}\) (resp. \(e_{rs}\)). If in addition \(e_{pq}e_{123456}\) (resp. \(e_{pq}e_{tu_{123456}}\)) \(\notin T^6(X)\), then we have a subgroup \((e_{pq}e_{rsy_{123456}}\cdot e_{pq}e_{tu_{123456}}\)) \(\subseteq T^{5/6}(X)_{\text{tors}}\) of order \(e_{tu}\) (resp. \(e_{rs}\)).

**Proof.** (1) For simplicity, we give the proof for the case of \(f_{pq} = e_{rs}\). In this case, we have

\[
e_{rsy_{rstu}} \cdot (y_{tu} + y_{tu} + y_{tu}) - (y_{rs} + y_{rs} + y_{tu}) \cdot f_{pq}(y_{rst} + y_{rs}) = e_{rsy_{rstu}} \in T^3(X).
\]

It follows from the previous result that \(e_{rsy_{rstu}} \cdot z_{pq} \in T^4(X)\) and \(e_{pq}y_{pq} \cdot e_{rsy_{rstu}} \in T^5(X)\).

Since we have \(e_{etu_{tu}} \cdot e_{rsy_{rs}} \in T^4(X)\), \(e_{rsy_{rs}} \cdot e_{tu_{tu}} \cdot z_{pq} - e_{tu_{tu}}e_{123456} \in T^5(X)\) and \(e_{pq}y_{pq} \cdot e_{rsy_{rs}} \cdot e_{tu_{tu}} \in T^6(X)\), the second statement immediately follows.
(2) If \( g = 1 \), then \( z_{pr} + z_{qs} = (z_{pq} + z_{rs} + z_{tu}) \in T^2(X) \), which implies the first result. As \( y_r + y_s, y_p + y_q \in \text{Im}(\text{res}^{1/2}) \), the remaining results follow by multiplication the first result by these elements.

(3) Since \( f_{tu} = 1 \), we have \( y_{pr} + y_{qs} = (y_{pq} + y_{rs}) = (z_{pq} + z_{rs}) \in T^2(X) \). As \( y_p + y_q \in \text{Im}(\text{res}^{1/2}) \), the second and third results follow from \( (y_p + y_q)(y_{pr} + y_{qs} - y_{pq} - y_{rs}) \in \text{Im}(\text{res}^{3/4}) \) and \( (y_r + y_s)(y_p + y_q - y_{rs}) \in \text{Im}(\text{res}^{4/5}) \), respectively.

The assumption implies that \( (Q^{(p,q)}_{\max}/2)_{L} \simeq (Q^{(r,s)}_{\max}/2)_{L} =: Q_{L} \). Hence, the last one follows from the closed embedding \( R_{L/F}(SB(Q)) \times R_{L/F}(SB(Q^{(t,u)}_{\max}/2)) \hookrightarrow R_{L/F}(SB(Q)) \times R_{L/F}(SB(Q^{(t,u)}_{\max}/2)) \). □

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