Teleportation with partially entangled states

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The relations of antilinear maps, bipartite states and quantum channels is summarized. Antilinear maps are applied to describe bipartite states and entanglement. Teleportation is treated in this general formalism with an emphasis on conditional schemes applying partially entangled pure states. It is shown that in such schemes the entangled state shared by the parties, and those measured by the sender should “match” each other.

1 Introduction

A fundamental part of quantum mechanics is the description of the evolution of quantum states. In the lack of measurements, the evolution of the quantum state of closed systems can be described by unitary operators. Measurements are described by projections onto the eigenstates of the measured quantity. In more general approach to the question of quantum state evolution, the system in argument is considered as a part of a larger subsystem. This approach gives rise to concepts of general quantum channels and generalized (POVM) measurements.

In the context of quantum information and communication, quantum entanglement is a central issue. General quantum kinematics of multipartite systems harbors lots of secrets still. Quantum teleportation\(^[1]\) is probably the most frequently quoted application of entanglement, and its experimental feasibility\(^[2, 3, 4]\) has further increased its relevance.

The methods developed for representations of quantum channels can be used successfully in the description of entanglement and teleportation. The results in this paper are motivated by these kind of methods. Section 2 summarizes some facts concerning the relations between quantum states of bipartite systems, channels, antilinear and antiunitary maps. Some of these have already found application in quantum information theory, but those use a fixed antilinear map, which is related to a specific maximally entangled state. We also present another possibility, namely we consider different antilinear maps, which is an alternative description of all pure bipartite states. This approach provides us with a very convenient description of quantum teleportation, including all schemes applying a pure (but not necessarily maximally entangled) resource.

Popescu\(^[5]\) pointed out that teleportation is also possible using mixed states but with a fidelity less than 1. In a recent paper of Banaszek\(^[6]\) a protocol using a partially entangled state was optimized for average fidelity. Horodecki et al.\(^[7]\) presented a formula for the fidelity of such imperfect teleportation schemes.

In conditional schemes on the other hand, fidelity can be one but at the cost that the process sometimes fails, so the probability of successfulness (also called efficiency) is less than one. This is the case in conclusive teleportation\(^[8, 9]\), where partially entangled or even mixed states and
generalized measurements (POVM) are considered. Using nonunitary transformation at Bob’s side also makes the process probabilistic [10].

The main part of our paper is based on the formalism summarized in Section 2. Our description is completely independent of the dimensions of the Hilbert-spaces involved, and we do not even need to fix a basis. In Section 3 we give a general condition for conditional teleportation in terms of the applicable entangled states and joint measurements. In Section 4 we show that partially entangled states are capable of conditional teleportation with fidelity one, but only if the outcome of the measurement performed by Alice and the state shared by the parties “match” each other. Section 5 summarizes our results.

2 States, channels and antilinear maps

Consider a bipartite system with subsystems A and B. The subsystems are described by the Hilbert-spaces \( \mathcal{H}_A \) and \( \mathcal{H}_B \), thus the pure states of the system are in \( \mathcal{H}_A \otimes \mathcal{H}_B \). Let \( \dim \mathcal{H}_A = \dim \mathcal{H}_B = N \). Let \( \{ |i \rangle_A \} \) and \( \{ |i \rangle_B \} \) \( (i = 0, \ldots, N - 1) \) denote the computational bases on \( \mathcal{H}_A \) and \( \mathcal{H}_B \), and let

\[
|\Psi^+\rangle_{AB} = \frac{1}{\sqrt{N}} \sum_{i=0}^{N-1} |i \rangle_A |i \rangle_B \quad (1)
\]

be a maximally entangled state of the system. All other maximally entangled states of the system can be obtained from \( |\Psi^+\rangle \) by local unitary transformations. The set of density matrices of system A will be denoted by \( \mathcal{S}_A \). A quantum channel \( \mathcal{A} \) is a completely positive, trace-preserving, and hermiticity preserving \( \mathcal{S}_A \to \mathcal{S}_A \) map.

Keeping \( |\Psi^+\rangle \) in mind, we may introduce the relative state representation [11] of states and operators on \( \mathcal{H}_A \). Any pure state \( |\Psi\rangle_A \in \mathcal{H}_A \) can be described by an (unnormalized) index state \( |\Psi^*\rangle_B \in \mathcal{H}_B \) so that

\[
|\Psi\rangle_A = B \langle \Psi^* | \Psi^+ \rangle_{AB} \quad (2)
\]

The state in argument is obtained as a partial inner product of its index state and the maximally entangled state \( |\Psi^+\rangle \). The mapping creating the index state from the original state,

\[
L_{|\Psi^+\rangle} : \mathcal{H}_A \to \mathcal{H}_B, \quad L_{|\Psi^+\rangle} |\Psi\rangle_A = |\Psi^*\rangle_B \quad (3)
\]

is antilinear, and in fact \( \sqrt{N} L_{|\Psi^+\rangle} \) is antiunitary. Indeed, expanding an arbitrary \( |\Psi\rangle_A \) on the computational basis,

\[
L_{|\Psi^+\rangle} |\Psi\rangle_A = L_{|\Psi^+\rangle} \sum_i C_i |i\rangle_A = \frac{1}{\sqrt{N}} \sum_i C_i^* |i\rangle_B \quad (4)
\]

from which the above properties follow.

The introduction of \( L \) via \( |\Psi^+\rangle \) is also useful in describing channels \( \mathcal{A} \). Let us have the compound system in the state \( |\Psi^+\rangle_{AB} \), and send subsystem A through the channel \( \mathcal{A} \) while doing nothing with subsystem B. The effect of the channel on any pure state \( |\Psi\rangle_A \) of system A is then obtained by the partial inner product with the corresponding index state:

\[
\mathcal{A} (|\Psi\rangle_A \langle \Psi|) = N B \langle \Psi^* | (\mathcal{A} \otimes I_B) (|\Psi^+\rangle_{AB} \langle \Psi^+|) |\Psi^*\rangle_B , \quad (5)
\]

where \( |\Psi^*\rangle_B = L_{|\Psi^+\rangle} |\Psi\rangle_A \), and \( I_B \) stands for the identity operator. This is the so called relative state representation of channels, which is widely used to describe them. But even more can be
stated \[7\]. An affine isomorphism between the set of all $S_A$ channels on $S_A$, and the set of bipartite states $\varrho_{AB} \in S_{H_A \otimes H_B}$ with maximally mixed partial trace, i.e. with the property

$$\text{tr}_A \varrho_{AB} = \frac{1}{N} I_B,$$

(6)
can be found similarly to Eq. \[5\]. The bipartite state corresponding to a channel can be obtained from $|\Psi^+\rangle$ by applying the channel on system A and doing nothing with system B:

$$\varrho_{AB} = (S_A \otimes I_B) \left( |\Psi^+\rangle_{AB} \langle AB | \Psi^+\rangle \right).$$

(7)

As $|\Psi^+\rangle_{AB}$ has a maximally mixed partial trace, and this property cannot be changed by local operations, \[6\] will also hold for the $\varrho_{AB}$ obtained in Eq. \[7\]. The isomorphism has been found by the Horodecki et al. \[7\], who have discussed it in detail, and have used it for the description of teleportation channels.

So far we have considered the antiunitary map $\sqrt{N} L_{i \langle \Phi} \rangle$, arising from the maximally entangled state $|\Psi^+\rangle$. This is a useful tool in the description of channels and states. Let us follow the reverse way now. We may use the set of antilinear $H_A \rightarrow H_B$ maps in order to describe pure states in $H_A \otimes H_B$. As relative state representation is also based on $L$, changing this map can give rise to different relative state representations.

Consider a bipartite pure state $|\Phi\rangle_{AB} \in H_A \otimes H_B$. We may write it on the computational basis as $|\Phi\rangle_{AB} = \sum_{ij} C_{ij} |i\rangle_A \otimes |j\rangle_B$. We define the $H_A \rightarrow H_B$ antilinear operator $L_{i \langle \Phi}$ such that $L_{i \langle \Phi} |i\rangle_A = \sum_j C_{ij} |j\rangle_B$. Thus we can write

$$|\Phi\rangle_{AB} = \sum_i |i\rangle_A \otimes (L_{i \langle \Phi} |i\rangle_A).$$

(8)

For any bipartite pure state $|\Phi\rangle_{AB} \in H_A \otimes H_B$, there uniquely exists an antilinear operator $L_{i \langle \Phi}$ defined this way.

Because of the antilinear property $L_{i \langle \Phi}$, \[8\] is independent of the actual computational basis chosen on $H_A$. Let $C_{AB}$ denote the set of bounded antilinear operators $L : H_A \rightarrow H_B$ which have finite norm (that is, $\text{tr}(L^\dagger L) < \infty$, where the adjoint of $L$ is defined by the relation $\langle f | L e \rangle = \langle L^\dagger f | e \rangle$):

$$C_{AB} = \{ L : H_A \rightarrow H_B \text{ bound antilinear} \, | \, \text{tr}(L^\dagger L) < \infty \}. $$

(9)

$C_{AB}$ forms a Hilbert space (the scalar product is $\langle L, L' \rangle = \text{tr}(L^\dagger L)$, which is conjugate linear in the first argument). It is shown in Ref. \[12\] that \[8\] establishes a unitary isomorphism between $C_{AB}$ and $H_A \otimes H_B$ in a natural way. Every pure bipartite state $|\Phi\rangle_{AB} \in C_{AB}$ can uniquely be described by an antilinear operator $L_{i \langle \Phi} \in C_{AB}$ such that $\text{tr}(L_{i \langle \Phi}^\dagger L_{j \langle \Phi}) = 1$. Conversely, every such $L$ describes a pure bipartite state.

Now let us characterize maximally entangled states and possible relative state representations. By maximally entangled state we mean a pure bipartite state with maximally mixed partial trace \[6\]. The partial traces of bipartite states over systems $A$ and $B$ are $LL^\dagger$ and $L^\dagger L$ respectively. Thus the state \[8\] is maximally entangled if and only if $LL^\dagger = N^{-1} I_B$ and $L^\dagger L = N^{-1} I_A$. This is equivalent to that $\sqrt{N} L$ is antiunitary. On the other hand, the operator $L_{i \langle \Phi}$ gives rise to a relative state representation if and only if $\{ L_{i \langle \Phi} | i \rangle \}_{i=0,\ldots,N-1}$ forms an orthogonal basis on $H_B$, which means, that $\sqrt{N} L_{i \langle \Phi}$ is antiunitary. Relative state representations can be defined via maximally entangled states.
3 Probabilistic teleportation with partially entangled states

In this section we apply the antilinear description of bipartite states introduced in Section 2 for quantum teleportation. Suppose that system A prepared in the unknown state $|\Phi\rangle_A$ is to be teleported, and systems B and C shared by the parties (Alice and Bob) are in a partially entangled state $|\sigma\rangle_{BC}$. We will call this shared state in what follows. The shared state is described by the antilinear map $L_{|\sigma\rangle}$. Systems A and B are located at Alice who performs a joint projective measurement on them. Suppose that its outcome corresponds to the projection onto the state $|\sigma_q\rangle_{AB}$. In the followings, we will regard only this outcome, thus our teleportation scheme will be probabilistic, conditional one.

To have common computational bases in the description of the shared state and the state the measurement project onto, we expand $|\sigma_q\rangle$ in the following way:

$$|\sigma_q\rangle = \sum_i (L_q|i\rangle \otimes |i\rangle),$$

where $L_q \in C_{BA}$. One can characterize the states corresponding to nondegenerate measurement outcomes by bounded antilinear operators $L_q: \mathcal{H}_B \rightarrow \mathcal{H}_A$ such that $\text{tr}(L_q^* L_q) = 1$. Note that $L_q$ is unique disregarding a unit complex phase factor.

The projection resolution of every joint observable of $A$ and $B$, which has nondegenerate eigenvalues, is (up to phase factors) uniquely described by an orthonormal basis $L_q$ in $C_{BA}$. Those measurements whose nondegenerate outcomes are represented by projections onto mutually orthogonal maximally entangled states, are called measurements of Bell type \[13\]. Every Bell measurement can be described by an orthonormal basis $L_q$ in $C_{BA}$ such that $(\dim \mathcal{H}_A)^{1/2} L_q$ is antiunitary for every $q$.

Now we will calculate the teleportation channel. By this we mean a function $f_q: \mathcal{H}_A \rightarrow \mathcal{H}_C$ that relates the input state, and the state of system C after the measurement. Note that, although the reversing unitary transformation is usually also included in the definition of teleportation channel, our terminology is more convenient here, as we investigate linearity and reversibility. At the beginning, the three systems are in the state $|\Phi\rangle_A \otimes |\sigma\rangle_{BC}$. The probability of the outcome $q$ under consideration is given by

$$p_q(|\Phi\rangle_A) = \|(\langle \sigma_q |_{AB} AB (|\sigma_q\rangle \otimes I_C) |\Phi\rangle_A \otimes |\sigma\rangle_{BC})\|_2^2 = \|\sum_i (\langle \Phi |_{\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}_B} L_q^* L_q |i\rangle B)\|^2.$$  

$$= \sum_i L_q(i \otimes |i\rangle_{\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}_B})^2 = \|LL_q^* |\Phi\rangle_A\|^2.$$  

On condition that the measurement yields the outcome $q$, the state of system C can be written as

$$\frac{1}{\sqrt{p_q(|\Phi\rangle_A)}} \sum_i (AB |\sigma_q\rangle |i\rangle_B \otimes |\Phi\rangle_A) L |e_i\rangle_B = \frac{1}{\sqrt{p_q(|\Phi\rangle_A)}} LL_q^* |\Phi\rangle_A.$$  

The teleportation channel for the outcome $q$ is

$$f_q: \mathcal{H}_A \rightarrow \mathcal{H}_C, \quad f_q(|\Phi\rangle_A) = \frac{LL_q^* |\Phi\rangle_A}{\|LL_q^* |\Phi\rangle_A\|}.$$  

If the input state is given by the density operator $\rho_{in}$ then the probability of the outcome $q$ is

$$p_q(\rho_{in}) = \text{tr}_A(L_q L^\dagger LL_q^\dagger \rho_{in})$$.
and the output state is
\[ \rho_{\text{out}} = \frac{LL_q^\dagger \rho_{\text{in}} LL_q L_i}{\text{tr}_A(LL_q L_i^\dagger LL_q \rho_{\text{in}})}. \] (15)

We have defined a special quantum operation based on the teleportation scheme of Ref. [1]. One can obtain from (14) that this operation is a generalized (POVM) measurement of the input state and the positive operator representing it is \( LL_q L_i^\dagger LL_q \).

The channel \( f_q \) has to be reversible, so that we can obtain a teleported state identical to the original input state. We call the channel \( f_q \) reversible, if it is injective, that is, for different input state \( |\Phi\rangle_A \) \( (||\Phi\rangle_A|| = 1) \) the corresponding output state \( f_q(|\Phi\rangle_A) \) is different. We remark, that the reversibility of teleportation channels has also been investigated in Ref. [1]. We adopt a more general definition here. Reversibility means that every input state can be recovered (theoretically) from the output state. One can easily verify that this condition is equivalent to that the linear operator \( LL_q^\dagger : \mathcal{H}_A \rightarrow \mathcal{H}_C \) is injective.

It may be the case, however, that the channel \( f_q \) is not linear. This way, the input state can be recovered from the output only using some sophisticated nonlinear transformations, which may not be realistic. Therefore, it is a natural requirement for the channel to be linear.

We show that if the teleportation channel is reversible, then its linearity is equivalent to that the probability (11) of the outcome \( q \) is independent of the input state \( |\Phi\rangle_A \). Suppose that \( |\Phi\rangle_1 \) and \( |\Phi\rangle_2 \) are linearly independent, and let \( (\alpha_1|\Phi\rangle_1 + \alpha_2|\Phi\rangle_2) \) be such that \( ||\alpha_1|\Phi\rangle_1 + \alpha_2|\Phi\rangle_2|| = 1 \). From the linearity condition \( f_q(\alpha_1|\Phi\rangle_1 + \alpha_2|\Phi\rangle_2) = \alpha_1 f_q(|\Phi\rangle_1) + \alpha_2 f_q(|\Phi\rangle_2) \), one can obtain:
\[
\alpha_1 \left( \frac{1}{||LL_q^\dagger(|\alpha_1|\Phi\rangle_1 + \alpha_2|\Phi\rangle_2)||} - \frac{1}{||LL_q^\dagger|\Phi\rangle_1||} \right) LL_q^\dagger |\Phi\rangle_1 \\
+ \alpha_2 \left( \frac{1}{||LL_q^\dagger(|\alpha_1|\Phi\rangle_1 + \alpha_2|\Phi\rangle_2)||} - \frac{1}{||LL_q^\dagger|\Phi\rangle_2||} \right) LL_q^\dagger |\Phi\rangle_2 = 0. \] (16)

Since \( f_q \) is injective, \( LL_q^\dagger |\Phi\rangle_1 \) and \( LL_q^\dagger |\Phi\rangle_2 \) are also linearly independent. Then (16) implies that their coefficients are zero, that is, the probability (11) of the outcome \( q \) is independent of the input state \( |\Phi\rangle_A \). Reversely, if (11) is independent of \( |\Phi\rangle_A \), then \( LL_q^\dagger \) is injective and \( f_q \) is linear. We can conclude that the condition that “the probability of the measurement outcome \( q \) does not depend on the input state” (that is, Alice learns nothing about the input state due to the measurement) is equivalent to that the teleportation channel is linear. Moreover, it can be proven in a way not detailed here that the linearity of the channel is equivalent to its unitarity—therefore its unitary reversibility.

4 Entanglement matching

In this section we answer the question what measurements provide fidelity 1 teleportation for an arbitrarily given (even partially) entangled shared state. Suppose that the shared state \( |\sigma\rangle \) is described by an invertible antilinear operator \( L \). If Bob applies a unitary transformation \( U_q : \mathcal{H}_C \rightarrow \mathcal{H}_C \) which may depend on the result \( q \) of Alice’s measurement, then the final state of system \( C \) reads \( |\text{out}\rangle_C = \frac{1}{\sqrt{p_q}} U_q LL_q^\dagger |\Phi\rangle_A \). Let \( i_{AC} \) be a unitary isomorphism between \( \mathcal{H}_A \) and \( \mathcal{H}_C \) so that we can compare the states of systems \( A \) and \( C \). The teleportation condition is
\[
\frac{1}{\sqrt{p_q}} U_q LL_q^\dagger = i_{AC} \] (17)
which also guarantees that \( p_q(|\Phi\rangle_A) \) is independent of \(|\Phi\rangle_A\). From this we conclude that a measurement with an outcome described by the antilinear operator

\[
L_q = \sqrt{p_q} i_{AC} U_q L^{-1\dagger}
\]

supports fidelity 1 conditional teleportation. The appropriate recovering unitary transformation applied by Bob is to be \( U_q \).

Although \( p_q \) in (11) depends on \( L_q \), this can be resolved by the fact that \( L_q \) has a norm \( \text{tr}_B(L_q L_q^\dagger) = 1 \). Then we obtain that the probability is

\[
p_q = \left[ \text{tr}_B \left( (L_q^\dagger L_q)^{-1} \right) \right]^{-1}.
\]  

For an arbitrary entangled shared pair described by invertible \( L \), the set of measurement outcomes providing fidelity 1 conditional teleportation is given by the set

\[
\mathcal{M}_L = \left\{ L_q = \frac{i_{AC} U L^{-1\dagger}}{\sqrt{\text{tr}_B(L^{-1} L^{-1\dagger})}} \mid U \text{ is unitary} \right\}.
\]

Thus not every possible measurement outcome allows teleportation, only those described by \( \mathcal{M}_L \). The measurement and the shared state should be “matched” to each other. This can be regarded as a generalization of “entanglement matching” introduced in Ref. [10].

It is worth to note that (19) is the same for every outcome \( q \) that matches the shared state in the above sense. The probability of a successful outcome depends only on the shared state. Another important result is that the set \( \mathcal{M}_L \) of matching outcomes is spanned by local unitary transformations: if one finds a measurement outcome which enables probabilistic teleportation, then every matching outcome can be obtained from it by a local unitary transformation on system \( A \).

We give an example for entanglement matching. Suppose that the antilinear operator \( L \) describing the shared state \(|\sigma\rangle_{BC}\) is given by the following matrix:

\[
L = \begin{pmatrix}
\alpha_1 \\
\vdots \\
\alpha_n
\end{pmatrix}, \quad
|\sigma\rangle_{AB} = \sum_i \alpha_i |i\rangle_B |i\rangle_C,
\]

where all \( \alpha_i \) are nonzero (consider a Schmidt decomposition for example). Taking that the unitary transformation \( U_q \) is identity, we obtain from (18) that a matching measurement outcome is given by

\[
L_1 = \left( \sum_i \frac{1}{|\alpha_i|^2} \right)^{-1/2} \begin{pmatrix}
1/\alpha_1^* \\
\vdots \\
1/\alpha_n^*
\end{pmatrix},
\]

\[
|\sigma_1\rangle_{AB} = \left( \sum_i \frac{1}{|\alpha_i|^2} \right)^{-1/2} \sum_i \frac{1}{\alpha_i} |i\rangle_A |i\rangle_B.
\]

Thus if Alice measures an observable with an eigenstate equal to \(|\sigma_1\rangle_{AB}\) then that measurement outcome implements a conditional teleportation.
5 Conclusion

We have summarized the relations between quantum channels, bipartite states and antilinear operators, focusing the description of bipartite pure states with the latter. Applying this description, we have characterized all possible conditional teleportation schemes. We have found that the independence of the probability of a measurement outcome on the input state is a necessary and sufficient condition of the linearity of the transformation to be applied by the receiver. We have generalized the concept of “entanglement matching”, which means that in schemes under consideration the entangled state shared by the parties, and those measured by the sender should “match” each other.

The results presented here show that this formalism is applicable of describing entanglement and quantum teleportation in a quite general way. This method may be also applicable for the treatment of entanglement between systems described by Hilbert-spaces of different dimensionality. It can also have consequences regarding the description of teleportation and related phenomena in the framework of quantum operations.

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