Algorithmic randomness and Ramsey properties of countable homogeneous structures.

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Abstract

We study, in the context of algorithmic randomness, the closed amenable subgroups of the symmetric group \( S_\infty \) of a countable set. In this paper we address this problem by investigating a link between the symmetries associated with Ramsey Fraïssé order classes and algorithmic randomness.

Keywords: Martin-Löf randomness, topological dynamics, amenable groups, Fraïssé limits, Ramsey theory.

1 Introduction

The focus of this work is, as for example in [4, 5], on the problem of understanding the symmetries that transform a recursively presented universal structure, which in this paper will be a Fraïssé limit of finite first order structures, to a copy of such a structure which is Martin-Löf random relative to a canonical \( S_\infty \)-invariant measure on the class of all universal structures of the given type. Here \( S_\infty \) is the symmetric group of a countable set, with the pointwise convergence topology. This investigation leads to a link between the symmetries associated with the so-called discernable flows in structural Ramsey theory and algorithmic randomness.

Glasner and Weiss [6] showed that there exists a unique measure on the set of linear orderings of the natural numbers (seen as a subset of Cantor space) that is invariant under the canonical action of the symmetric group of the natural numbers. The author [5] showed that this measure is computable and studied the associated Martin-Löf (ML) random points, which, due to the uniqueness of the Glasner-Weiss

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measure, may be regarded as random linear orders. The author showed that any
ML-random linear order has the order type of the rationals. Moreover, it was shown
that recent work by Kechris and Sokic [12] implies that no random linear order can
be the extension of the universal poset (the Fraïssé limit of finite posets) to a linear
order. In [5] a study was made of so-called “randomizers”. These are permutations
of the natural numbers that transform a computable (Cantor) rational linear order
into a random one. It was also proven in [5] that any such randomizer cannot be
an automorphism of the universal poset.

The aim of this paper is to generalise these results to a broader class of Fraïssé
limits $F_0$ of Ramsey classes, the automorphism groups $\text{Aut}(F_0)$ not being amenable.
Again, as in [5], this paper relies heavily on the groundbreaking paper [10] by Kechris,
Pestov and Todorcevic. The arguments in this paper require some understanding
of the subtle interplay between structural Ramsey theory and topological dynamics
as is beautifully explicated in the paper [10]. This paper has been written in such
a way that it should be accessible to a non-specialist in Ramsey theory.

2 Preliminaries on amenable groups

Let $G$ be a topological group and $X$ a compact Hausdorff space. A dynamical
system $(X,G)$ (or a $G$-flow on $X$) is given by a jointly continuous action of $G$ on $X$. If $(Y,G)$ is a second dynamical system, then a $G$-morphism $\pi : (X,G) \rightarrow (Y,G)$ is
a continuous mapping $\pi : X \rightarrow Y$ which intertwines the $G$-actions, i.e.,the diagram

$$
\begin{array}{ccc}
G \times X & \xrightarrow{\alpha} & X \\
\downarrow{\pi} \quad \downarrow{\pi} & & \quad \downarrow{\pi} \\
G \times Y & \xrightarrow{\beta} & Y \\
\end{array}
$$

commutes with $\alpha, \beta$ being the group actions.

An isomorphism is a bijective homomorphism. A subflow of $(X,G)$ is a $G$-flow
on a compact subspace $Y$ of $X$ with the action of $G$ on $X$ restricted to the action
on $Y$. A $G$-flow is minimal if it has no proper subflows. Every dynamical system
has a minimal subflow (Zorn).

The following fact, first proven by Ellis (1949) [2], is central to the theory of
dynamical systems:

**Theorem 1** Let $G$ be a Hausdorff topological group. There exists, up to
$G$-isomorphism, a unique minimal dynamical system, denoted by $(M(G),G)$, such
that for every minimal dynamical system $(X,G)$ there exists a $G$-epimorphism

$$
\pi : (M,G) \rightarrow (X,G),
$$

and any two such universal systems are isomorphic.

The flow $(M(G),G)$ is called the universal minimal flow of $G$.

We next introduce the notion of amenable groups.
Definition 1 A topological group $G$ is amenable if, whenever $X$ is a non-empty compact Hausdorff space and $\pi$ is a continuous action of $G$ on $X$, then there is a $G$–invariant Borel probability measure on $X$.

This means that, for every $G$-flow on a compact space $X$, there is a measure $\nu$ on the Borel algebra of $X$, such that, $\nu(X) = 1$ and, for every $g \in G$ and Borel subset $U$ of $X$,

$$\nu(gU) = \nu(U).$$

3 Fraïssé limits and their recursive representations

In the sequel, $L$ will stand for the signature of a relational structure. Moreover, $L$ will always be finite and the arities of the relational symbols will all be $\geq 1$. The definitions that follow were introduced by Fraïssé in 1954.

The age of an $L$-structure $X$, written $\operatorname{Age}(X)$, is the class of all finite $L$-structures (defined on finite ordinals) which can be embedded as $L$-structures into $X$. The structure $X$ is homogeneous (some authors say ultrahomogeneous) if, given any isomorphism $f : A \rightarrow B$ between finite substructures of $X$, there is an automorphism $g$ of $X$ whose restriction to $A$ is $f$. A class $K$ of finite $L$-structures has the amalgamation property if, for structures $A, B_1, B_2$ in $K$ and embeddings $f_i : A \rightarrow B_i$ ($i = 1, 2$) there is a structure $C$ in $K$ and there are embeddings $g_i : B_i \rightarrow C$ ($i = 1, 2$), such that the following diagram commutes:

Suppose $K$ is a countable class of finite $L$-structures, the domains of which are finite ordinals such that

1. if $A$ is a finite $L$-structure defined on some finite ordinal, if $B \in K$ and if there is an embedding of $A$ into $B$, then $A \in K$;

2. the class $K$ has the amalgamation property.

Then, Fraïssé showed that there is a countable homogeneous structure $X$ such that $\operatorname{Age}(X) = K$. Moreover, $X$ is unique up to isomorphism. The (essentially) unique $X$ is called the Fraïssé limit $K$ of $K$. Note that, conversely, the age $K$ of a countable homogeneous structure has properties (1) and (2). We shall frequently call a countable structure which is isomorphic to a Fraïssé limit a universal structure.

A recursive representation of a countably infinite $L$-structure $X$ is a bijection $\phi : X \rightarrow \mathbb{N}$ such that, for each $R \in L$, if the arity of $R$ is $n$, then the relation $R^\phi$ defined on $\mathbb{N}^n$ by

$$R^\phi (x_1, x_2, \ldots, x_n) \iff R (\phi^{-1}(x_1), \ldots, \phi^{-1}(x_n)),$$
is recursive. If we identify the underlying set of $X$ with $\mathbb{N}$ via $\phi$ and each $R$ with $R^\phi$, we call the resulting structure a recursive $\mathcal{L}$-structure on $\mathbb{N}$ and we say it has a recursive representation on $\mathbb{N}$.

If $X$ is countable and homogeneous and if $\text{Age}(X)$ has an enumeration $A_0, A_1, A_2, \ldots$, possibly with repetition, with the property that there is a recursive procedure that yields, for each $i \in \mathbb{N}$, and $R \in \mathcal{L}$, the underlying set $A(i)$ of $A_i$ together with the interpretation of $R$ in $A(i)$, then we call $(A_i : i \in \mathbb{N})$ a recursive enumeration of $\text{Age}(X)$. It follows from the construction of Fraïssé limits from their ages, that one can construct a recursive representation of $X$ from a recursive enumeration of its age. (Conversely, it is trivial to derive a recursive enumeration of $\text{Age}(X)$ from a recursive representation of $X$.) It is therefore not difficult to find recursive representations for Fraïssé limits of classes $K$ from recursive enumerations of their ages.

**Theorem 2** Suppose $\mathbb{C}$ and $\mathbb{D}$ are countable recursively represented $\mathcal{L}$-structures on $\mathbb{N}$ with the same age. Suppose that they are both homogeneous. Then there is a recursive isomorphism from $\mathbb{C}$ to $\mathbb{D}$.

Proof. As mentioned in [5], the model-theoretic back-and-forth argument as discussed, for example, on pp 161-162 of Hodges [7] is constructive relative to the recursive representations of the homogeneous structures $\mathbb{C}$ and $\mathbb{D}$.

4 Structural Ramsey theory in a model theoretic context

In this section we summarise the results from [10] which underly the formalisation and proof of the main theorem of this paper. Unless otherwise stated all the proofs of the statements made here can be found in [10].

Let $K$ be the age of some countable $\mathcal{L}$-structure. For $A, \pi \in K$ we denote by $A_\pi$ the set of all the (model-theoretic) structure-preserving embeddings of $\pi$ in $A$. For a natural number $r \geq 1$ and for $\pi, A, B \in K$ we introduce the predicate $B \rightsquigarrow (A)_r^\pi$ (Erdős-notation) to mean:

$$B \rightsquigarrow (A)_r^\pi \iff (\forall B^\pi \chi \quad \exists A B ^\alpha \quad A ^\alpha \odot B ^\pi \quad \chi ^\alpha ) .$$

Here $\alpha : A^\pi \rightarrow B^\pi$ is the mapping that takes an embedding $\pi \xrightarrow{\alpha} A$ to the induced embedding $\pi \xrightarrow{\alpha^\pi} B$.

In other words, $B \rightsquigarrow (A)_r^\pi$ iff: for every $r$-colouring $\chi$ of the set $B^\pi$ consisting of the embeddings of $\pi$ in $B$ (copies of $\pi$ in $B$), there is an embedding $\alpha$ of $A$ into $B$ such that $\chi^\alpha$ is a constant. This means that $\chi$ assigns a constant value on all the embeddings of $\pi$ into the image $A' \subset B$ of $A$ under $\alpha$.

We shall call an age $K$ a Ramsey age if, for all $\pi, A \in K$ with $A^\pi \neq \emptyset$, and all natural numbers $r \geq 1$, there is some $B \in K$ such that $B \rightsquigarrow (A)_r^\pi$. 
Assume \( \mathcal{L} \) is a countable signature containing a distinguished binary relation symbol \( < \). An order structure \( A \) for the signature \( \mathcal{L} \) with the distinguished symbol \( < \), is a structure \( A \) for which the interpretation \( <^A \) of the symbol \( < \) in \( A \) is a total ordering.

An order class \( \mathbf{K} \) for \( \mathcal{L} \) is one for which all \( A \in \mathbf{K} \) are order structures (relative to the distinguished \( < \)).

Let \( \mathcal{L}_0 \) be the signature obtained by removing the distinguished symbol \( < \) from \( \mathcal{L} \). For any \( \mathcal{L} \)-structure \( A \), denote by \( A_0 \) the \( \mathcal{L}_0 \)-structure which is the reduct of \( A \) to \( \mathcal{L}_0 \). This means that \( A_0 \) is the structure \( A \) where the distinguished order \( < \) interpreted as a total order \( <^A \) in \( A \) is being ignored.

Let \( \mathbf{K} \) be a Fraïssé order class. Denote by \( \mathbf{K}_0 \) the class of all reducts \( A_0 \) for some \( A \in \mathbf{K} \). Write \( F \) for the Fraïssé limit of \( \mathbf{K} \). We now discuss when \( \mathbf{K}_0 \) is also a Fraïssé class with limit \( F_0 \), the latter being the reduct of \( F \) to the signature \( \mathcal{L}_0 \).

Then Kechris et al (p 135) showed that \( \mathbf{K}_0 \) is a Fraïssé class with limit \( F_0 \), (which is the reduct of \( F \) to the signature \( \mathcal{L}_0 \)) iff the Fraïssé order class \( \mathbf{K} \) is reasonable.

Note that, in this case, the underlying sets of \( F \) and \( F_0 \) are the same. Moreover, we can write

\[
F = (F_0, <_0),
\]

for some linear ordering \( <_0 \) on the underlying set of \( F_0 \).

We consider the continuous action of the automorphism group \( \text{Aut}(F_0) \) on the (topological) space of all linear orderings on the set \( F_0 \), which is the underlying set of the structure \( F_0 \). Write \( X_\mathbf{K} \subset \{0, 1\}^{F_0 \times F_0} \) for the orbit topological closure of the action of \( \text{Aut}(F_0) \) on the linear ordering \( <_0 \), i.e.,

\[
X_\mathbf{K} = \overline{\text{Aut}(F_0). <_0}.
\]

This set is clearly a closed, hence compact, subset of the Baire space \( \{0, 1\}^{F_0 \times F_0} \). Moreover, it is clearly also an \( \text{Aut}(F_0) \)-invariant subset of \( \{0, 1\}^{F_0 \times F_0} \) under the natural action of \( \text{Aut}(F_0) \) on the latter space. We have thus obtained an \( \text{Aut}(F_0) \)-flow on \( X_\mathbf{K} \).

This flow can be defined for any reasonable (in the technical sense as explained above) Fraïssé order class \( \mathbf{K} \). I will call it the discerning flow associated with the reasonable Fraïssé order class \( \mathbf{K} \).

**Remark.** The author uses the terminology discerning in acknowledgement of Ramsay's pioneering work in developing his theorem in the context what now would be considered as a study of indiscernibles in model theory.

If, in addition to being a Fraïssé order class, the class \( \mathbf{K} \) is Ramsey, then every minimal subflow of the discernable flow is isomorphic to the universal minimal flow of \( \text{Aut}(F_0) \).
The Fraïssé order class $K$ is said to have the ordering property if for every $A_0 \in K_0$, there is a $B_0 \in K_0$ such for any linear ordering $<$ on $A_0$ and every linear ordering $<_1$ on $B_0$, where both $<$, $<_1$ are restrictions of $<_0$, there is an embedding of $(A_0, <)$ into $(B_0, <_1)$.

The discerning flow associated with the Fraïssé order class $K$ is itself minimal iff $K$ has the ordering property.

We also extract the following remark from [10].

**Proposition 1** If $K$ is a Fraïssé order class which is Ramsey and has the ordering property, then a total order $\xi$ belongs to the discerning flow $X_K$ iff for any $A$ in the age of $F_0$ it is the case that $<^A$ is the restriction of $\xi$ to $A$.

We shall make substantial use of this remark in the sequel.

**Example.** It is known (see, for example [3] that the class $P$ (finite posets, linear extensions) is Ramsey and has the ordering property. As was noted in [10], this has the implication that the discerning Aut($P_0$)-flow is thus a universal minimal flow. It acts on the space $X_P$ consisting of the linear extensions of the universal poset $P_0$. Using these facts, Kechris and Sokić (2011) [12] recently showed that the automorphism group of $P_0$ is not amenable. These results do imply that the set of linear extensions of the Fraïssé limit of finite posets are all, in a definite sense, nonrandom, at least from the point of view of algorithmic randomness as was shown in [5]. This result will be placed in a broader context in what follows.

## 5 Martin-Löf random countable orders

Let $S_\infty$ be the group of permutations of a countable set, which, without loss of generality, we may take to be $\mathbb{N}$. We place on $S_\infty$ the pointwise convergence topology. Let $(\mathbb{N} \times \mathbb{N})^\neq$ denote the set of ordered pairs $(i, j)$ of natural numbers with $i \neq j$. Write $\mathcal{M}$ for the set of total orders on $\mathbb{N}$. We identify $\mathcal{M}$ with a subset of $\{0, 1\}^{(\mathbb{N} \times \mathbb{N})^\neq}$ by identifying a total order $<$ on $\mathbb{N}$ with the function $\xi : (\mathbb{N} \times \mathbb{N})^\neq \to \{0, 1\}$ given by

$$
\xi(x, y) = 1 \iff x < y, \quad x, y \in \mathbb{N}.
$$

The total order associated with $\xi$ will be denoted by $<_\xi$. We topologise $\mathcal{M}$ via the natural injection

$$
\mathcal{M} \to \{0, 1\}^{(\mathbb{N} \times \mathbb{N})^\neq},
$$

where the (Baire) space $\{0, 1\}^{(\mathbb{N} \times \mathbb{N})^\neq}$ has the product topology. As such $\mathcal{M}$ is a closed hence compact subspace of $\{0, 1\}^{(\mathbb{N} \times \mathbb{N})^\neq}$.

The group $S_\infty$ acts continuously on $\mathcal{M}$ if, for $\xi \in \mathcal{M}$ and $\sigma \in S_\infty$, we define the total order $\sigma \xi$ by:

$$
x <_{\sigma \xi} y \iff \sigma^{-1} x <_{\xi} \sigma^{-1} y, \quad x, y \in \mathbb{N}.
$$

Since $S_\infty$ is an amenable group, there is an $S_\infty$-invariant measure on $\mathcal{M}$. In fact, Glasner and Weiss (2002) [6] showed that there is exactly one such measure (i.e., the flow on $\mathcal{M}$ is uniquely ergodic). Their proof is based on an ergodic argument. Let us denote this measure by $\mu$. I shall refer to this measure as in [3] as the Glasner-Weiss measure.
We write $\mathcal{M}_f$ for the set of finite total orders on some subset of $\mathbb{N}$. For $\ell \in \mathcal{M}_f$, denote by $Z_\ell$ the set of $\xi \in \mathcal{M}$, such that $\xi$ is an extension of $\ell$. These sets are the cylinder subsets of $\mathcal{M}$. Write $Z_0$ for the class of events of the form $Z_\ell$ for some $\ell \in \mathcal{M}_f$ and $Z$ for the algebra generated by $Z_0$. Note that the $\sigma$-algebra generated by $Z$ is exactly the Borel algebra on $\mathcal{M}$.

For $Z \in Z_0$ we write $Z^0$ for the complement of $Z$ and $Z^1$ for $Z$. Let $(T_i)_{i \in \mathbb{N}}$ be any enumeration of the algebra $Z$ generated by $(Z_\ell)_{\ell \in \mathcal{M}_f}$ in such a way that one can effectively retrieve from a given $i \in \mathbb{N}$, the corresponding $T_i$ as a finite union of sets $T$ of the form
\[ T = Z_{\ell_1}^{\delta_1} \cap \ldots \cap Z_{\ell_k}^{\delta_k}, \]
where each $\ell_i$ is in $\mathcal{M}_f$ and $\delta_i \in \{0, 1\}$ for $i = 1, \ldots, k$. We call any such enumeration a recursive representation of $Z$.

The Glasner-Weiss measure $\mu$ is computable in the following sense:

**Theorem 3** Denote by $\mu$ the Glasner-Weiss measure on the Borel-algebra of $\mathcal{M}$. Let $(T_i : i < \omega)$ be a recursive representation of the algebra $Z$. There is an effective procedure that yields, for $i, k \in \mathbb{N}$, a binary rational $\beta_k$ such that
\[ |\mu(T_i) - \beta_k| < 2^{-k}. \]

A proof of this result appears in [5].

**Definition 2** A set $A \subset \mathcal{M}$ is of constructive measure 0, if, for some recursive representation of $(T_i : i \in \mathbb{N})$ of $Z$, there is a total recursive $\phi : \mathbb{N}^2 \to \mathbb{N}$ such that
\[ A \subset \bigcap_n \bigcup_m T_{\phi(n,m)} \]
and $\mu\left(\bigcup_m T_{\phi(n,m)}\right)$ converges effectively to 0 as $n \to \infty$.

**Definition 3** A total order $\xi$ is said to be $\mu$-Martin-Löf random if $\xi$ is in the complement of every subset $B$ of $\mathcal{M}$ of constructive measure 0.

### 6 The main theorem

Write $ML_\mu \subset \mathcal{M}$ for the set of $\mu$-Martin-Löf random total orders.

**Theorem 4** Let $K$ be a recursive Fraïssé order class which is Ramsey and has the ordering property. Write
\[ \mathbb{F} = (\mathbb{F}_0, <) \]
for its Fráïssé limit and $X_K$ for the associated discerning flow. Fix some recursive representation of $\mathbb{F}$. Note that
\[ X_K \subset \mathcal{M}. \]
If some element of $X_K$ is $\mu$-Martin-Löf random, then the automorphism group $\text{Aut}(\mathbb{F}_0)$ is amenable. Equivalently, if $\text{Aut}(\mathbb{F}_0)$ is not amenable, then
\[ ML_\mu \cap X_K = \emptyset. \]
Consequently, if for some $\xi \in X_K$ and some automorphism $\pi$ of $\mathbb{F}_0$ it is the case that the linear order $\pi \xi$ is $\mu$-Martin-Löf random, then $\text{Aut}(\mathbb{F}_0)$ is amenable.
Proof: Note that a topological group $G$ is amenable iff its universal minimal flow $M(G)$ has a $G$-invariant probability measure. Indeed, let $\nu$ be an invariant measure on $M(G)$. Consider any $G$-flow on some compact Hausdorff space $X$. By Zorn's lemma there is a minimal subflow $Y$ and a $G$-embedding $i$ of $Y$ into $X$. Therefore, there are $G$-morphisms

$$M(G) \xrightarrow{\pi} Y \xrightarrow{i} X.$$ 

Let $\rho$ be the pushout measure of $\nu$ under $i\pi$. In other words, for every Borel subset $A$ of $X$, we set

$$\rho(A) = \nu(i^{-1}\pi^{-1}A).$$

Then $\rho$ is an invariant measure on $X$. The converse is trivial, since $M(G)$ is a compact $G$-flow.

We introduce a number of (standard) recursion-theoretic concepts and terminology: A sequence $(A_n)$ of sets in $\mathbb{Z}$ is said to be enumerable if for each $n$, the set $A_n$ is of the form $T_{\phi(n)}$ for some total recursive function $\phi : \omega \to \omega$ and some effective enumeration $(T_i)$ of $\mathbb{Z}$. (Note that the sequence $(A^c_n)$, where $A^c_n$ is the complement of $A_n$, is also an $\mathbb{Z}$-enumerable sequence.) In this case, we call the union $\bigcup_n A_n$ an $\Sigma^0_1$ set. A set is a $\Pi^0_1$ set if it is the complement of a $\Sigma^0_1$ set. It is of the form $\bigcap_n A_n$, for some $\mathbb{Z}$-semirecursive sequence $(A_n)$.

We shall also need the following observation. (In the language of algorithmic randomness, it states the well-known fact that the notion of Martin-Löf randomness is stronger than that of Kurtz randomness. A proof of this observation, in the present context, can be found in [5]. For more on Kurtz randomness, the reader is referred to the book [1] by Downey and Hirschfeldt.)

**Lemma 1** If $A$ is a $\Sigma^0_1$ subset of $\mathcal{M}$ and if $\mu(A) = 1$, then $ML_\mu$ is contained in $A$. In particular, if $B$ is a $\Pi^0_1$ subset of $\mathcal{M}$ that contains some element of $ML_\mu$, then $\mu(B) > 0$.

It follows from Proposition [4] that a total order $\xi$ belongs to $X_K$ iff for any $A$ in the age of $F_0$ it is the case that $<^A$ is the restriction of $\xi$ to $A$. Therefore, since $K$ is a recursive order class, the relation

$$\xi \in X_K$$

is $\Pi^0_1$ definable over $\mathcal{M}$. It follows from Lemma [1] that, if

$$ML_\mu \cap X_K \neq \emptyset,$$

then $\mu(X_K) > 0$. This means that $\mu$ is a nonzero $\text{Aut}(F_0)$-invariant measure on the flow $X_K$.

Since $K$ is a Ramsey order class, $X_K$ is the universal minimal flow associated with the group $\text{Aut}(F_0)$. By universality we can conclude that any $\text{Aut}(F_0)$-flow on a compact Hausdorff space will admit a nonzero $\text{Aut}(F_0)$-invariant measure. In particular, $\text{Aut}(F_0)$ is an amenable topological group.

The second part now follows from the observation that if $\xi \in X_K$ and $\pi$ is an automorphism of $F_0$ then $\pi\xi$ will also belong to $X_K$. 

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[1] Downey, R. G., and Hirschfeldt, D. R. (2010). *Algorithmic Randomness and Complexity*. Springer.
[2] Kurtz, S. (1972). *An introduction to recursive analysis*. Lecture Notes in Mathematics, 251. Springer.
[3] Martin-Löf, P. (1966). *The definition of random sequences*. Information and Control, 9, 602-619.
[4] Proposition: [4].
[5] Lemma: [1].
Corollary 1  Fix a recursive representation of the universal poset $\mathbb{P}_0$ on the natural numbers $\mathbb{N}$. Let $\mathcal{M}(\mathbb{P}_0)$ be the class of linear extensions of $\mathbb{P}_0$. Write $ML_\mu$ for the set of total orders on $\mathbb{N}$ that are Martin-Löf random relative to the Glasner-Weiss probability measure $\mu$. Then

$$ML_\mu \cap \mathcal{M}(\mathbb{P}_0) = \emptyset.$$ 

Proof: The result is a direct consequence of the fact that $\text{Aut}(\mathbb{P}_0)$ is not an amenable group. [12].

7  Open problems

The following theorem is in [5].

Theorem 5  Write $Q$ for the set of total orders on $\mathbb{N}$ which are isomorphic to the Cantor rational order $\eta$. Then

$$ML_\mu \subset Q.$$ 

In particular,

$$\mu(Q) = 1.$$ 

This observation has the following consequence.

Theorem 6  For a total order $\eta$, set

$$S_\mu(\eta) := \{ \sigma \in S_\infty : \sigma \eta \in ML_\mu \}.$$ 

Then $S_\mu(\eta) \neq \emptyset$ iff $\eta$ is a rational Cantor order.

Proof. By Theorem 5 if $\eta$ were not rational, the corresponding set $S_\mu(\eta)$ must be the empty set. If $\eta$ is rational, then the class $Q$ is exactly the orbit of $\eta$ under the action of $S_\infty$. Since both $Q$ and $ML_\mu$ have $\mu$-measure one, it follows that

$$\mu(Q \cap ML_\mu) = 1,$$ 

and, therefore, that $S_\mu(\eta) \neq \emptyset$.

Following [5], note that, if $\pi \in S_\infty$, then

$$S_\mu(\eta)\pi^{-1} = S_\mu(\pi \eta).$$  \hspace{1cm} (2)$$

Indeed, for $\alpha \in S_\mu(\eta)$, we have $\alpha \pi^{-1}(\pi \eta) = \alpha \eta \in ML_\mu$ and hence $\alpha \pi^{-1} \in S_\mu(\pi \eta)$. Conversely, if $\tau \in S_\mu(\pi \eta)$, then $\tau \pi \eta \in ML_\mu$, i.e., $\tau \pi \in S_\mu(\eta)$, and, so, $\tau \in S_\mu(\eta)\pi^{-1}$.

If $\eta_1, \eta_2 \in Q$, there is some $\pi \in S_\infty$ such that $\eta_2 = \pi \eta_1$. Moreover, if $\eta_1, \eta_2$ were both recursive, the permutation $\pi$ could also be chosen to be recursive. (See Theorem 2). Write $S_r$ for the class of recursive permutations of $\mathbb{N}$. We let $S_r$ act
on the right on the class $\Sigma$ of all sets of the form $S_\mu(\tau)$ with $\tau$ a recursive rational order on $\mathbb{N}$. The action is given by

$$\Sigma \times S_r \longrightarrow \Sigma,$$

$$(S_\mu(\tau), \pi) \mapsto S_\mu(\tau)\pi^{-1}, \; \pi \in S_r, \; \tau \in \mathcal{Q}_r,$$

where $\mathcal{Q}_r$ denotes the class of all recursive rational orders on $\mathbb{N}$. It follows from the preceding arguments that this $S_r$-action will have a single orbit, i.e., the action is transitive. Set

$$\mathcal{S} = \bigcup_{\tau \in \mathcal{Q}_r} S_\mu(\tau).$$

If we choose any fixed $\eta \in \mathcal{Q}_r$, we also have

$$\mathcal{S} = \bigcup_{\pi \in S_r} S_\mu(\eta)\pi^{-1}.$$ We shall call the permutations in $\mathcal{S}$ Martin-Löf randomizers. These are the permutations that transform some recursive rational order to one which is $\mu$-Martin-Löf random.

These arguments show that an understanding of $\mathcal{S}$ can be be attained from any single $S_\mu(\tau)$ for a single recursive rational order $\tau$ modulo the recursive permutations in $S_\infty$.

Let $\mathbf{K}$ be a recursive Fraïssé order class which is Ramsey and has the ordering property. Write again

$$\mathcal{F} = (\mathbb{F}_0, <)$$

for its Fraïssé limit and $X_K$ for the associated discerning flow. The arguments of this paper show that, if $\tau \in \mathcal{Q}_r \cap X_K$, then the presence of elements in $\text{Aut}(\mathbb{F}_0)$ which are Martin-Löf randomizers of $\tau$ is related to the amenability of the group $\text{Aut}(\mathbb{F}_0)$. Indeed, for some $\pi \in \text{Aut}(\mathbb{F}_0)$ to be a Martin-Löf randomizer of any $\tau$ as above is a generic property, in the sense that this very fact forces the group $\text{Aut}(\mathbb{F}_0)$ to be amenable! The problem still remains to identify the class of Martin-Löf randomizers.

Let $\mathbf{L}$ be the Fraïssé order class consisting of all pairs $(L, <)$ where $L$ is a lattice with underlying set a finite ordinal and with $<$ being a total order on the underlying set of $L$ which is a linear extension of the partial order on $L$. As far as the author knows, it is unknown whether $\mathbf{L}$ is Ramsey and whether it has the ordering property. The author has discussed this problem with specialists in Ramsey theory and it would appear that this problem is wide open. Writing $\mathcal{L} = (\mathbb{L}_0, <)$ for the Fraïssé limit of $\mathbf{L}$, it is also an interesting open problem to relate $X_L$ to $ML_\mu$ and thus perhaps gaining an understanding of the amenability or not of $\text{Aut}(\mathbb{L}_0)$. Note that if $\mathbf{L}$ were Ramsey with the ordering property, then $\text{Aut}(\mathbb{L})$ would be an extremely amenable group. This would mean that its universal minimal flow is a singleton.
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