We investigate consequences of the effective colour-dielectric formulation of lattice gauge theory using the light-cone Hamiltonian formalism with a transverse lattice [1]. As a quantitative test of this approach, we have performed extensive analytic and numerical calculations for 2 + 1-dimensional pure gauge theory in the large $N$ limit. We study the structure of coupling constant space for our effective potential by comparing with results available from conventional Euclidean lattice Monte Carlo simulations of this system. In particular, we calculate and measure the scaling behaviour of the entire low-lying glueball spectrum, glueball wavefunctions, string tension, asymptotic density of states, and deconfining temperature.

The recent Euclidean Lattice Monte Carlo (ELMC) simulations of Teper [2] have shown that pure non-Abelian gauge theory behaves much the same way in three as in four dimensions: a discrete set of massive boundstates are generated by a linearly confining string-like force. Moreover, Teper has performed calculations for $N = 2, 3,$ and $4,$ allowing an extrapolation to large $N$. The large-$N$ limit is convenient, though not essential, for our Light-Cone Transverse Lattice (LCTL) formulation. Our formulation offers the rare possibility of describing the parton, constituent, and string behaviour of hadrons in one framework. The relationship between these pictures, each very different but equally successful, remains one of the outstanding enigmas of QCD.

We characterise a dielectric formulation as one in which gluon fields, or rather the $SU(N)$ group elements they generate, are replaced by collective variables
which represent an average over the fluctuations on short distance scales, represented by complex $N \times N$ matrices $M$. These dielectric variables carry colour and form an effective gauge field theory with classical action minimised at zero field, meaning that colour flux is expelled from the vacuum at the classical level. The price one pays for starting with a simple vacuum structure is that the effective action will be largely unknown and must be investigated per se.

Starting with the Wilson lattice action [3], we take the continuum limit in the $x^0$ and $x^2$ directions, leaving the transverse direction $x^1$ discrete. Replacing the link variables $U \in SU(N)$ with $N \times N$ complex matrices $M$, one derives a transverse lattice action whose form was first suggested in Ref. [4]

$$A = \int dx^0 dx^2 \sum_{x_1} \left( \text{Tr} \left\{ D_\alpha M_{x_1} (D^\alpha M_{x_1})^\dagger \right\} - \frac{a}{4g^2} \text{Tr} \left\{ F_{\alpha \beta} F^{\alpha \beta} \right\} - V_{x_1} [M] \right)$$

(1)

where $\alpha, \beta \in \{0, 2\}$ and

$$D_\alpha M_{x_1} = (\partial_\alpha + i A_\alpha(x^1)) M_{x_1} - i M_{x_1} A_\alpha(x^1 + a).$$

(2)

$M_{x_1}$ lies on the link between $x^1$ and $x^1 + a$ while $A_\alpha(x^1)$ is associated with the site $x^1$. $V_{x_1} [M]$ is a purely transverse gauge invariant effective potential. Next we introduce light-cone co-ordinates $x^\pm = x^\mp = (x^0 \pm x^2)/\sqrt{2}$ and quantise by treating $x^+$ as canonical time. The theory has a conserved current

$$J^\alpha_{x_1} = i \left[ M_{x_1} \overset{\leftrightarrow}{D^\alpha} M_{x_1}^\dagger + M_{x_1 - a}^\dagger \overset{\leftrightarrow}{D^\alpha} M_{x_1 - a} \right]$$

(3)

at each transverse lattice site $x^1$. If we pick the light-cone gauge $A_- = 0$ the non-propagating field $A_+$ satisfies a simple constraint equation at each transverse site $(\partial_\ldots)^2 A_+(x^1) = g^2 J^+_x / a$. Solving this constraint leaves an action in terms of the dynamical fields $M_{x_1}$:

$$A = \int dx^+ dx^- \sum_{x_1} \text{Tr} \left\{ \partial_\mu M_{x_1} \partial^\mu M_{x_1}^\dagger + \frac{g^2}{2a} J^{+}_{x_1} \frac{1}{(\partial_-)^2} J^{+}_{x_1} \right\} - V_{x_1} [M].$$

(4)

At large $N$, Eguchi-Kawai reduction [3] introduces considerable simplification. For $P^1 = 0$ the theory is isomorphic to one compactified on a one-link transverse lattice, id est we can simply drop the argument $x^1$ or $l$ from $M$ in all of the previous expressions. Effectively one is now dealing with a $1 + 1$-dimensional gauge theory coupled to a complex scalar field in the adjoint representation (with self-interactions).

For the transverse effective potential $V[M]$, we will include all Wilson loops and products of Wilson loops up to fourth order in link fields $M$:

$$V[M] = \mu^2 \text{Tr} \left\{ M M^\dagger \right\} + \frac{\lambda_1}{aN} \text{Tr} \left\{ M M^\dagger M M^\dagger \right\}$$

$$+ \frac{\lambda_2}{aN} \text{Tr} \left\{ M M^\dagger M^\dagger M \right\} + \frac{\lambda_3}{aN^2} \text{Tr} \left\{ M^\dagger M \right\} \text{Tr} \left\{ M^\dagger M \right\}$$

(5)
Note that the last term above, which might appear suppressed at large $N$, is in fact non-zero only for 2 particle Fock states.

Let us introduce creation/annihilation operators

$$M_{x^1}(x^-) = \frac{1}{\sqrt{4\pi}} \int_0^{\infty} \frac{dk}{k} \left( a^-_{-1}(k, x^1) e^{-ikx^-} + (a^+_{+1}(k, x^1))^\dagger e^{ikx^-} \right).$$

In the associated Fock space, we include only states annihilated by the charge integral $\int dx^- J^+$. This gives a Hilbert space formed from all possible closed Wilson loops of link modes $a_{\pm}$ on the transverse lattice. Thus, a typical $p$-link loop will be something like

$$\text{Tr} \left\{ a^+_{+1}(k_1) a^+_{-1}(k_2) a^-_{-1}(k_3) \cdots a^+_{+1}(k_p) \right\} |0\rangle$$

where the number of $+1$'s equals the number of $-1$'s, $\sum_{m=1}^p k_m = P^+$, and $k_m \geq 0$. At large $N$ we need only study the dynamics of single connected Wilson loops in the Hilbert space since the loop-loop coupling constant is of order $1/N$. These loops may be thought of as ‘bare’ glueballs, and the problem is to find the linear combinations that are on mass shell. Neglecting $k_m = 0$, which is consistent with expanding about the $M = 0$ solution of the dielectric regime, the Fock vacuum is an eigenstate of the full light-cone Hamiltonian $P^- |0\rangle = P^+ |0\rangle = 0$.

The theory possesses several discrete symmetries. Charge conjugation induces the symmetry $C : a^+_{+1, i,j} \leftrightarrow a^+_{-1, i,j}$. There are two orthogonal reflection symmetries $P_1$ and $P_2$ either of which may be used as ‘parity’. If $P_1 : x^1 \rightarrow -x^1$, we have $P_1 : a^+_{+1, i,j} \leftrightarrow a^+_{-1, i,j}$. If rotational symmetry has been restored in the theory, states of spin $J \neq 0$ should form degenerate $P_1$ doublets $|+J\rangle \pm |-J\rangle$. We use “spectroscopic notation” $|J|^P_{C}$ to classify states.

For $\lambda_1$ and $\lambda_2$ small there is very little mixing between Fock states of different number of link modes $p$. In this case a mass eigenstate $|\Psi\rangle$ has predominantly a fixed $p$, the mass increasing with $p$. For a given $p$, the energy also tends to increase with the number of nodes in the wavefunction $f$ due to the $J (\partial_-)^2 J$ term \[], which is in fact a positive contribution. Thus one expects the lowest two eigenstates to be approximately

$$\int_0^{P^+} dk f_{+1,-1}(k, P^+ - k) \text{Tr} \left\{ a^+_{+1}(k) a^+_{-1}(P^+ - k) \right\} |0\rangle$$

with the lowest state having a symmetric wavefunction $f_{+1,-1}(k, P^+ - k)$, corresponding to $0^{++}$, and first excited state having $f_{+1,-1}$ antisymmetric with one node, corresponding to $0^{--}$. The next highest states are either a 4-link state with positive symmetric wavefunctions $f_{+1,+1,-1,-1}$ and $f_{+1,-1,+1,-1}$ or a symmetric 2-link state with $f_{+1,-1}$ having two nodes. In the glueball spectrum we identify the latter states as $0^{++}_+$ and $2^{++}$, respectively, although actual eigenstates are a mixture of these.
In order to fix the coupling constants in the effective potential, we perform a least \( \chi^2 \) fit to Teper’s ELMC large \( N \) extrapolated spectrum. As we shall later show, the mass in units of the coupling \( m^2 = \mu^2 a/(g^2 N) \) is a measure of the lattice spacing while the other terms in our effective potential \( \lambda_1, \lambda_2, \) and \( \lambda_3 \) must be found from the fitting procedure. We will also determine \( g^2 N/a \) based on a fit, which we check self-consistently with measurements of the string tension.

In our numerical solutions we restrict the number of link fields \( p \) in our basis states and discretise momenta by demanding antiperiodicity of the fields in \( x^- \to x^- + L \). In order to minimise errors associated with these truncations, we take the spectra for various \( K \) and \( p \) truncations and extrapolate to the continuum, \( K, p \to \infty \). Since we cannot measure \( |J| \) directly, we only classified states according to \( P_1 \) and \( C \) during the fitting process. We estimate our one-sigma errors from finite \( K \) and \( p \)-truncation to be roughly 0.05\( M^2 \). As we fit the various couplings as a function of \( m^2 \), we find a narrow strip in parameter space where we obtain good agreement with the ELMC spectrum; see Fig. 1. As we shall see, moving along this strip corresponds to changing the lattice spacing \( a \). The strip, where \( \chi^2 \) has a local minimum, disappears when \( m^2 \) is sufficiently large, indicating that for large enough lattice spacing our truncation of the effective potential is no longer a good approximation.

The numerical bound from absence of tachyons is shown in Fig. 1 as a zero-mass surface. As the transverse lattice spacing vanishes the mass gap should vanish in lattice units. The fixed point for this, which we believe lies somewhere at negative \( m^2 \), should lie on the zero-mass surface, but is inaccessible to us.
in the dielectric regime $m^2 > 0$. Nevertheless the scaling trajectory should gradually approach the zero-mass surface if it is to eventually encounter the fixed point.

In Fig. 2, we have plotted a typical spectrum along the scaling trajectory together with the ELMC results. For graphing purposes, we assigned $|J|$ to our spectrum based on a best fit to Teper’s results. Although the overall fit with the conventional lattice results is quite good, we see two deficiencies of our spectrum that cannot be attributed to $K$ or $p$-truncation errors. First, we see that the lowest $0^{-+}$ state is too low in energy. Second, we see that the lowest parity doublet $2^{++}$ is not quite degenerate. We believe that these discrepancies must be due to our truncation of the effective potential. Finally, we have made no prediction for the lowest $1^{++}$ state since it lies too high in the $|J|^{|J|}$ spectrum.

To measure the string tension in the $x^1$ direction (before Eguchi-Kawai reduction) consider a lattice with $n$ transverse links and periodic boundary conditions. Constructing a basis of Polyakov loops, “winding modes,” that wind once around this lattice, one may extract from the lowest eigenvalue $M^2$ vs $n$ the lattice string tension $a\sigma = \Delta M_n / \Delta n$. String theory arguments indicate that oscillations of the winding mode transverse to itself yield a form $M^2/\sigma = \sigma a^2 n^2 - \pi/3$, the constant correction being due to the Casimir energy [6]. Fig. 3 shows a typical $M^2$ vs $n$ plot for winding modes where we see a good fit to a quadratic. The constant term -3.5, however, does not agree well.
Fig. 3. — (a) Lowest $M^2$ eigenvalue vs $n$ for winding modes. Here, $K = 10.5$ or 11, $p \leq n + 4$, and the couplings are taken from Fig. 2. Also shown is a numerical fit to $1.92n^2 - 3.5$. The dashed line is from an analytic estimate. (b) The lattice spacing in units of the string tension along the scaling trajectory. Here we plot $a^2\sigma$ vs $m^2$.

with the expected value of $-\pi/3$.

As a consistency check, we use $\sigma a^2$ from the string tension measurement together with $g^2N/(a\sigma)$ from the spectrum to form the ratio $\sigma g^2$ (as measured by us) to $\sigma g^2$ (as measured by ELMC). The result is equal to one $\pm 5\%$ all along the scaling trajectory. If we then assume that our $\sigma$ is equal to the ELMC value of $\sigma$, we can determine the lattice spacing $a$ in units of the string tension. This demonstrates that the mass $m^2 = \mu^2 a/(g^2N)$ determines the lattice spacing and that the continuum limit occurs at $m^2 < 0$; see Fig. 3.

Another quantity of interest is the deconfinement temperature $T_c$ which may be obtained from the Hagedorn behaviour [7] of the asymptotic density of mass eigenstates $\rho(M) \sim M^{-\alpha} \exp(M/T_c)$. The canonical partition function diverges for $T > T_c$. If $\alpha > (D + 1)/2$ it is a phase transition, beyond which the canonical and microcanonical ensembles are inequivalent. If $\alpha < (D + 1)/2$ the ensembles are equivalent and $T_c$ represents a limiting temperature — the free energy diverges at $T_c$. It is essential to demonstrate that the spectrum is sufficiently converged in $K$, thus we only fit to the states between 0.5 < \log(t) < 5.5 for the $K = 7$ data in Fig. 4. A numerical fit gives,

$$\log(t) = 3.99 + \frac{M}{0.78\sqrt{\sigma}} - 5.86 \log \left( \frac{M}{\sqrt{\sigma}} \right).$$

Since the density of states is $\rho(M) = dt/dM$, we find $T_c = 0.78\sqrt{\sigma}$ with an estimated error of at least 10%. Due to the large error, this result is compatible with the Euclidean lattice result [2, 8]. The fact that the power correction
Fig. 4. — \( \ln t \) vs the mass of the \( t \)-th eigenvalue \( M_t \) in units of the string tension. Here we have applied a cutoff in \( K \) only. Also shown is a least squares fit to the \( K = 7 \) data in the range \( 0.5 < \log(t) < 5.5 \) and couplings from Fig. 2.

\( \alpha \approx 4.86 \) in the density \( \rho(M) \) is much larger than \( (D + 1)/2 = 2 \) means we can safely say that the thermodynamic free energy remains finite at \( T_c \), marking a true phase transition (presumably deconfinement) rather than a limiting temperature. In addition, we have an analytic estimate suggesting that \( T_c \) lies in the range \( 0.81 \sqrt{\sigma} < T_c < 0.98 \sqrt{\sigma} \), also in agreement with the ELMC result.

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