Solutions of two fractional $q$-integro-differential equations under sum and integral boundary value conditions on a time scale

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Abstract
In this manuscript, by using the Caputo and Riemann–Liouville type fractional $q$-derivatives, we consider two fractional $q$-integro-differential equations of the forms

$$\frac{c}{q}D^\alpha_q [x](t) + w_1 (t, x(t), \phi(x(t))) = 0$$

and

$$\frac{c}{q}D^\alpha_q [x](t) = w_2 (t, x(t), \int_0^t x(r) \, dr, \frac{c}{q}D^\alpha_q [x](t))$$

for $t \in [0, t_0]$ under sum and integral boundary value conditions on a time scale $T_{t_0} = \{ t : t = t_0 q^n \} \cup \{ 0 \}$ for $n \in \mathbb{N}$ where $t_0 \in \mathbb{R}$ and $q \in (0, 1)$. By employing the Banach contraction principle, sufficient conditions are established to ensure the existence of solutions for the addressed equations. Examples involving algorithms and illustrated graphs are presented to demonstrate the validity of our theoretical findings.

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1 Introduction
It has been recognized that fractional calculus provides a meaningful generalization for the classical integration and differentiation to any order. On the other hand, quantum calculus is equivalent to traditional infinitesimal calculus without the notion of limits. It defines $q$-calculus where $q$ stands for quantum. Despite the old history of these two theories, the investigation of their properties remains untouched until recent time. Fractional $q$-calculus, initially proposed by Jackson [1–3], was regarded as the fractional analogue of $q$-calculus. Soon afterwards, it was further promoted by Al-Salam in [4] and then continued by Agarwal in [5] where many outstanding theoretical results were given. Its emergence and development extended the application of interdisciplinarity and aroused widespread attention of scholars; see [6–28] and the references therein. The existence of
solutions for $q$-fractional boundary value problems has been under consideration by many researchers; see for instance [29–40].

In [41], Ntouyas et al. studied the boundary value problem of first-order fractional differential equations given by

$$
\begin{align*}
^{c}D_{q}^α[f_1](x) & = w_1(x,f_1(x),f_2(x)), \\
^{c}D_{q}^β[f_2](x) & = w_2(x,f_1(x),f_2(x)),
\end{align*}
$$

with Riemann–Liouville integral boundary conditions of different order $f_1(0) = c_1 I^{α-1}[f_1](a_1)$ and $f_2(0) = c_2 I^{β-2}[f_2](a_2)$ for $0 < a_1, a_2 < 1$, $β_i ∈ (0, 1]$, $α, c_i ∈ IR$ where $i = 1, 2$. In 2015, Zhang et al. through the spectral analysis and fixed point index theorem obtained the existence of positive solutions of the singular nonlinear fractional differential equation $-D_{q}^α u(t) = w(t,u(t),D_{q}^β u(t))$ for $0 < t < 1$, with integral boundary value conditions $D_{q}^α u(0) = 0$ and $D_{q}^β u(t) = \int_{0}^{t} D_{q}^{α-1} q r D_{q}^β u(r) dN(r)$, where $α ∈ (1, 2]$, $β ∈ (0, 1]$, $w(t,u,v)$ may be singular at both $t = 0, 1$ and $u = v = 0$, $\int_{0}^{1} u(r) dN(r)$ denotes the Riemann–Stieltjes integral with a signed measure, in which $N : [0, 1] → IR$ is a function of bounded variation [42]. In 2016, Ahmad et al. investigated the existence of solutions for a $q$-anti-periodic boundary value problem of fractional $q$-difference inclusions given by

$$
^{c}D_{q}^α[k](t) ∈ F(t,k(t),D_{q}[k](t),D_{q}^2[k](t))
$$

for $t ∈ [0, 1]$, $q ∈ (0, 1)$, $2 < α ≤ 3$, $0 < β ≤ 3$, and $k(0) + k(1) = 0$, $D_{q}^α k(0) + D_{q}^β k(1) = 0$, $D_{q}^α k(0) + D_{q}^β k(1) = 0$, where $^{c}D_{q}^α$ denotes Caputo fractional $q$-derivative of order $α$ and $F : [0, 1] × IR × IR → P(IR)$ is a multivalued map with $P(IR)$ a class of all subsets of $IR$ [15]. In 2019, Ren and Zhai discussed the existence of unique solution and multiple positive solutions for the fractional $q$-differential equation $D_{q}^α x(t) + w(t,x(t)) = 0$ for each $t ∈ [0, 1]$ with nonlocal boundary conditions $x(0) = D_{q}^{α-2} x(0) = 0$ and $D_{q}^{α-1} x(1) = μ[x] + \int_{0}^{α} φ(r) D_{q}^α x(r) dN(r)$, where $D_{q}^α$ is the standard Riemann–Liouville fractional $q$-derivative of order $α$, $2 < α ≤ 3$, such that $α - 1 - β > 0$, $q ∈ (0, 1)$, $φ ∈ L^1[0, 1]$ is nonnegative, $μ[x]$ is a linear functional given by $μ[x] = \int_{0}^{1} x(t) dN(t)$ involving the Stieltjes integral with respect to the function $N : [0, 1] → IR$ such that $N(t)$ is right-continuous on $[0, 1]$, left-continuous at $t = 1$ and, particularly, $N$ is a nondecreasing function with $N(0) = 0$ and $dN$ is positive Stieltjes measure [40]. The authors in [43] investigated a multi-term nonlinear fractional $q$-integro-differential equation

$$
^{c}D_{q}^{α}[x](t) = w(t,x(t),\varphi_1 x(t),\varphi_2 x(t),^{c}D_{q}^α[x](t),^{c}D_{q}^{β_1}[x](t),^{c}D_{q}^{β_2}[x](t),…,^{c}D_{q}^{β_n}[x](t))
$$

under some boundary conditions. The existence of solutions for the multi-term nonlinear fractional $q$-integro-differential $^{c}D_{q}^{α}[u](t)$ equations in two modes and inclusions of order $α ∈ (n - 1, n]$ with non-separated boundary and initial boundary conditions where natural number $n$ is more than or equal to five was considered in [20]. Recently, some researchers discussed the existence of solutions for some singular fractional differential equations; see the papers [44–47].

Benefiting from the main ideas of the above said papers, we investigate the following two nonlinear fractional $q$-integro-differential equations in the spaces $A = C(\overline{J} × IR^2, IR)$
and \( B = \{ x : x, ^qC_D_t^\beta x \in C^2(\mathbb{T}, \mathbb{R}), \mathbb{T} = [0, l] \} \) with the norms defined by \( \| x \| = \sup_{t \in \mathbb{T}} |x(t)| \)
and
\[
\| x \|_* = \sup_{t \in \mathbb{T}} |x(t)| + \sup_{t \in \mathbb{T}} |^qC_D_t^\beta x(t)|,
\]
respectively.

(P1) First we investigate the nonlinear fractional \( q \)-integro-differential equation
\[
^qC_D_t^\alpha x(t) + w_1(t, x(t), \psi(x(t))) = 0
\]
for \( t \in \mathbb{T} \) under sum and integral boundary value conditions
\[
x'(a) = -\eta \int_0^1 x(r) \, dr, \quad x'(1) + x(0) = \sum_{i=1}^m c_i x'(b),
\]
where \( m \geq 1, 1 \leq \alpha < 2, 0 \leq a < b \leq 1, \eta \geq 0, c_i \geq 0 \) for each \( i = 1, 2, \ldots, m \) such that
\( 2E > -1 \), here \( E = \sum_{i=1}^m c_i, \psi(x(t)) = \int_0^t g(r)x(r) \, dr \) and \( w_1 : \mathbb{T} \times A^2 \rightarrow A \) is a continuous function.

(P2) Second we consider the nonlinear fractional \( q \)-integro-differential equation
\[
^qC_D_t^\alpha x(t) = w_2(t, x(t), \int_0^t x(r) \, dr, ^qC_D_t^\zeta x(t))
\]
for \( t \in \mathbb{T} \) under the sum boundary conditions
\[
x(0) = 0, \quad x'(1) = \sum_{i=1}^m c_i x''(b),
\]
where \( 1 \leq \alpha < 2, 0 \leq \zeta < 1, 0 < b < 1, m \geq 1, c_i \geq 0 \) for all \( i = 1, \ldots, m \) and
\( w_2 : \mathbb{T} \times B^3 \rightarrow B \) is a continuous function.

This paper is organized as follows: In Sect. 2, we state some useful definitions and lemmas on the fundamental concepts of \( q \)-fractional calculus and fixed point theory. In Sect. 3, some main theorems on the solutions of fractional \( q \)-integro-differential equations (1)–(2) and (3)–(4) are stated. Section 4 contains some illustrative examples to show the validity and applicability of our results. The paper concludes with some interesting observations.

2 Essential preliminaries

This section is devoted to some notations and essential preliminaries that are acting as necessary prerequisites for the results of the subsequent sections. Throughout this article, we apply the time scales calculus notation [9]. In fact, we consider the fractional \( q \)-calculus on the specific time scale \( \mathbb{T} = \mathbb{R} \) where \( \mathbb{T}_{t_0} = \{0\} \cup \{ t : t = t_0 q^n \} \) for nonnegative integer \( n \), \( t_0 \in \mathbb{R} \) and \( q \in (0,1) \). Let \( a \in \mathbb{R} \). Define \( [a]_q = \frac{1-q^a}{1-q} \) [2]. The power function \( (x - y)^n_q \) with \( n \in \mathbb{N}_0 \) is defined by
\[
(x - y)^n_q = \prod_{k=0}^{n-1} (x - y q^k)
\]
Algorithm 1 The proposed method for calculated \((a - b)^{(\alpha)}_q\)

```plaintext
1 function p = powerfunction(a, b, n, q)
2 %Power Gamma (a-b)^-(n)
3 p=1;
4 if n==0
5 p=1
6 else
7 for k=1:n-1
8 %a=a*(a-b*q^-k)/(a-b*q^-k+alpha+k);
9 end;
10 p=a*alpha * s;
11 end;
12 end
```

Algorithm 2 The proposed method for calculated \(\Gamma_q(x)\)

```plaintext
1 function g = qGamma(q, x, n)
2 %q-Gamma Function
3 p=1;
4 for k=0:n
5 p=p*(1-q^-k)/(1-q^-k+1);
6 end;
7 gamma = p/(1-q^-n); 7 end
```

Algorithm 3 The proposed method for calculated \((D_qf)(x)\)

```plaintext
1 function g = Dq(q, x, n, fun)
2 if x==0
3 g=limit ((fun(x)-fun(q*x))/(1-q*x),x,0);
4 else
5 g=(fun(x)-fun(q*x))/(1-q*x);
6 end;
7 end
```

for \(n \geq 1\) and \((x - y)^{(0)}_q = 1\), where \(x\) and \(y\) are real numbers and \(N_0 := \{0\} \cup \mathbb{N}\) [6]. Also, for \(\alpha \in \mathbb{R}\) and \(a \neq 0\), we have

\[(x - y)^{(\alpha)}_q = x^\alpha \prod_{k=0}^{\infty} \frac{x - y q^k}{x - y q^{\alpha + k}}.\]

If \(y = 0\), then it is clear that \(x^{(\alpha)} = x^\alpha\) [8] (Algorithm 1). The \(q\)-gamma function is given by \(\Gamma_q(z) = (1 - q)^{(z-1)}/(1 - q)^{z-1}\), where \(z \in \mathbb{R} \setminus \{0, -1, -2, \ldots\}\) [2]. Note that \(\Gamma_q(z + 1) = [z]_q \Gamma_q(z)\). Algorithm 2 shows a pseudo-code description of the technique for estimating \(q\)-gamma function of order \(n\). The \(q\)-derivative of function \(f\) is defined by \(D_q[f](x) = f'(x) - f'(qx)\) and \(D_q[f](0) = \lim_{x \to 0} D_q[f](x)\), which is shown in Algorithm 3 [6, 7]. Furthermore, the higher order \(q\)-derivative of a function \(f\) is defined by \(D_q^n[f](x) = D_q[D_q^{n-1}[f]](x)\) for \(n \geq 1\), where \(D_q^0[f](x) = f(x)\) [6, 7]. Tables 1, 2, and 3 show the values \(\Gamma_q(z)\) for some \(z\) and \(q \in (0, 1)\). The \(q\)-integral of a function \(f\) is defined on \([0, b]\) by

\[I_qf(x) = \int_0^x f(s) d_q s = x(1 - q) \sum_{k=0}^{\infty} q^k f(q^k x).\]
Table 1 Some numerical results for calculation of $\Gamma_q(x)$ with $q = \frac{1}{3}$ that is constant, $x = 4.5, 8.4, 12.7$, and $n = 1, 2, \ldots, 15$ of Algorithm 2

| n  | $x = 4.5$  | $x = 8.4$  | $x = 12.7$ |
|----|------------|------------|------------|
| 1  | 2.472950   | 11.909360  | 68.080769  |
| 2  | 2.383247   | 11.468397  | 65.592666  |
| 3  | 2.354446   | 11.326853  | 64.749894  |
| 4  | 2.344963   | 11.280255  | 64.483434  |
| 5  | 2.341815   | 11.264786  | 64.394980  |
| 6  | 2.340767   | 11.259636  | 64.365536  |
| 7  | 2.340418   | 11.257921  | 64.355725  |
| 8  | 2.340301   | 11.257349  | 64.352456  |

Table 2 Some numerical results for calculation of $\Gamma_q(x)$ with $q = \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, x = 5$, and $n = 1, 2, \ldots, 35$ of Algorithm 2

| n  | $q = \frac{1}{3}$ | $q = \frac{1}{2}$ | $q = \frac{2}{3}$ |
|----|-------------------|-------------------|-------------------|
| 1  | 3.016535          | 6.291859          | 18.937427         |
| 2  | 2.906140          | 5.548726          | 14.154784         |
| 3  | 2.870699          | 5.222330          | 11.819974         |
| 4  | 2.859031          | 5.069033          | 10.537540         |
| 5  | 2.855157          | 4.994707          | 9.782069          |
| 6  | 2.853868          | 4.958107          | 9.317265          |
| 7  | 2.853438          | 4.939945          | 9.023265          |
| 8  | 2.853295          | 4.930899          | 8.833940          |
| 9  | 2.853247          | 4.926384          | 8.710584          |
| 10 | 2.853232          | 4.921299          | 8.629588          |
| 11 | 2.853226          | 4.923002          | 8.576133          |
| 12 | 2.853232          | 4.922438          | 8.540736          |
| 13 | 2.853232          | 4.921517          | 8.517243          |
| 14 | 2.853226          | 4.920206          | 8.501627          |
| 15 | 2.853224          | 4.921945          | 8.491337          |
| 16 | 2.853224          | 4.921910          | 8.484320          |
| 17 | 2.853224          | 4.921893          | 8.479713          |

for $0 \leq x \leq b$, provided the series absolutely converges [6, 7]. If $x \in [0, T]$, then

$$\int_{x}^{T} f(r) \, d_q r = I_q[f](T) - I_q[f](x) = (1 - q) \sum_{k=0}^{\infty} q^k \left( T^f(Tq^k) - xf(xq^k) \right),$$

whenever the series exists. The operator $I^\alpha_q$ is given by $I^\alpha_q[h](x) = h(x)$ and

$$I^\alpha_q[h](x) = I_q[I_{q^\alpha}[h]](x) = I_q[I_{q^\alpha-[1]}[h]](x)$$

for $n \geq 1$ and $h \in C([0, T])$ [6, 7]. It has been proved that $D_q[I_q[h]](x) = h(x)$ and $I_q[D_q[h]](x) = h(x) - h(0)$ whenever $h$ is continuous at $x = 0$ [6, 7]. The fractional Riemann–Liouville type $q$-integral of the function $h$ on $J = (0,1)$ is defined by $\mathcal{I}^\alpha_q[h](t) = h(t)$ and

$$\mathcal{I}^\alpha_q[h](t) = \frac{1}{\Gamma_q(\alpha)} \int_{0}^{t} (t -qs)^{\alpha-1} h(s) \, d_q s$$

$$= t^\alpha (1 - q)^\alpha \sum_{k=0}^{\infty} q^k \prod_{i=1}^{k-1} (1 - q^{\alpha+i}) h(tq^k)$$

(5)
for \( t \in J \) [11, 18]. One can use Algorithm 5 for calculating \( T_{q}^{\alpha}[h](t) \) according to Eq. (5).

Also, the Caputo fractional \( q \)-derivative of a function \( h \) is defined by

\[
\begin{align*}
^{C} D_{q}^{\alpha}[h](t) &= T_{q}^{[\alpha]-\alpha}\left[ D_{q}^{\alpha}[h]\right](t) \\
&= \frac{1}{T_{q}^{\left[\alpha\right]}(-\alpha)} \int_{0}^{t} (t - qs)^{\left[\alpha\right]-1} D_{q}^{\alpha}[h](s) \, d_{q}s,
\end{align*}
\]

where \( t \in J \) and \( \alpha > 0 \) [18]. It has been proved that \( T_{q}^{\beta}[T_{q}^{\alpha}[h]](x) = T_{q}^{\alpha+\beta}[h](x) \) and \( D_{q}^{\alpha}[T_{q}^{\alpha}[h]](x) = h(x) \), where \( \alpha, \beta \geq 0 \) [18]. Algorithm 5 shows pseudo-code \( T_{q}^{\alpha}[h](x) \).

We use \( \|y\| = \max_{t \in J} |y(t)| \) as the norm of \( A = B = C^{1}(J) \). Clearly, \( (A, \|\cdot\|) \) and \((B, \|\cdot\|)\) are Banach spaces. Also, the product space \((A \times B, \|(y,z)\|)\) is a Banach space where \( \|(y,z)\| = \|y\| + \|z\| \). An operator \( O : A \to A \) is called completely continuous if restricted to any bounded set in \( A \) is compact.

Lemma 1 (Leray–Schauder alternative [48, p.4]) Let \( O : \mathcal{V} \to \mathcal{V} \) be completely continuous and \( \Omega(O) = \{x \in \mathcal{V} | x = \lambda O(x)\} \), where \( \lambda \in (0, 1) \). Then either the set \( \Omega(O) \) is unbounded or \( O \) has at least one fixed point.

3 Main results

The main results are presented in this section. To facilitate exposition, we will provide our analysis in two separate folds.

3.1 The nonlinear sum and integral boundary value problem (1)–(2)

First, we provide our key lemma.

Lemma 2 The function \( x_{0} \in A \) is a solution for problem (1) under the sum and integral boundary value conditions (2) if and only if \( x_{0} \) is a solution for the fractional \( q \)-integral
equation

\[ x_0(t) = \int_0^1 G_q(t,r) w_1(r,x_0(r),\psi(x_0(r))) \, dq r, \]

where

\[
G_q(t,r) = \begin{cases} 
\frac{(t - qr)^{(\alpha-1)}}{\Gamma_q(\alpha)} + \frac{\eta(\frac{1}{2} - t) + 1}{\eta(\Xi - \frac{1}{2}) + 1} (1 - qr)^{(\alpha-2)} \frac{\eta(\Xi(1 - r) + 1)(1 - qr)^{(\alpha)}}{\eta(\Xi - \frac{1}{2}) + 1} \Gamma_q(\alpha + 1) & \text{whenever } r \leq a \text{ and } 0 \leq r \leq t \leq 1, \\
\frac{(t - qr)^{(\alpha-1)}}{\Gamma_q(\alpha)} + \frac{\eta(\frac{1}{2} - t) + 1}{\eta(\Xi - \frac{1}{2}) + 1} (1 - qr)^{(\alpha-2)} \frac{\eta(\Xi - 1 + t)(1 - r)^{(\alpha)}}{\eta(\Xi - \frac{1}{2}) + 1} \Gamma_q(\alpha + 1) & \text{whenever } r \leq b \text{ and } 0 \leq a \leq r \leq t \leq 1, \\
\frac{(t - qr)^{(\alpha-1)}}{\Gamma_q(\alpha)} + \frac{\eta(\frac{1}{2} - t) + 1}{\eta(\Xi - \frac{1}{2}) + 1} (1 - qr)^{(\alpha-2)} \frac{\eta(\Xi - 1 + t)(1 - qr)^{(\alpha)}}{\eta(\Xi - \frac{1}{2}) + 1} \Gamma_q(\alpha + 1) & \text{whenever } 0 \leq a \leq b \leq r \leq t \leq 1, \\
\frac{\eta(\frac{1}{2} - t) + 1}{\eta(\Xi - \frac{1}{2}) + 1} (1 - qr)^{(\alpha-2)} \frac{\eta(\Xi - 1 + t)(1 - qr)^{(\alpha)}}{\eta(\Xi - \frac{1}{2}) + 1} \Gamma_q(\alpha + 1) & \text{whenever } 0 \leq a \leq b \leq r \leq t \leq 1, \\
\end{cases}
\]
whenever \( 0 \leq t \leq r \leq a \leq b \leq 1 \),

\[
G_q(t, r) = \frac{[\eta(\frac{1}{2} - t) + 1](1 - qr)^{(\alpha - 2)}}{[\eta(\Xi - \frac{1}{2}) + 1]} I_q^\alpha(\alpha - 1) \\
+ \frac{[\eta(\Xi(t - 1) + \frac{1}{2}) - \Xi] (b - qr)^{(\alpha - 2)}}{[\eta(\Xi - \frac{1}{2}) + 1]} I_q^\alpha(\alpha - 1) \\
+ \frac{[\eta(\Xi - 1 + t)](1 - qr)^{(\alpha)}}{[\eta(\Xi - \frac{1}{2}) + 1]} I_q^\alpha(\alpha + 1)
\]

whenever \( a \leq r \) and \( 0 \leq t \leq r \leq b \leq 1 \), and

\[
G_q(t, r) = \frac{[\eta(\frac{1}{2} - t) + 1](1 - qr)^{(\alpha - 2)}}{[\eta(\Xi - \frac{1}{2}) + 1]} + \frac{[\eta(\Xi - 1 + t)](1 - qr)^{(\alpha)}}{[\eta(\Xi - \frac{1}{2}) + 1]} I_q^\alpha(\alpha + 1)
\]

whenever \( b \leq r \) and \( 0 \leq t \leq r \leq 1 \).

**Proof** Let \( x_0 \) be a solution for Eq. (1)-(2). Take \( v_0(t) = w_1(t, x_0(t), \psi(x_0(t))) \). Choose \( d_0, d_1 \in \mathbb{R} \) such that

\[
x_0(t) = -\int_0^t \frac{(t - qr)^{(\alpha - 1)}}{I_q^\alpha(\alpha)} v_0(r) \, dr + d_0 + d_1 t. \tag{7}
\]

Thus, we obtain \( x'_0(t) = -T_q^{\alpha - 1}[v_0](t) + d_1 \). At present, by using the boundary conditions (2), we conclude that \( d_1 = T_q^{\alpha - 1}[v_0](a) - \eta \int_0^1 x_0(r) \, dr \) and

\[
d_0 = T_q^{\alpha - 1}[v_0](1) - \Xi T_q^{\alpha - 1}[v_0](b) + (\Xi - 1) T_q^{\alpha - 1}[v_0](a) \\
+ \eta(1 - \Xi) \int_0^1 x_0(r) \, dr.
\]

Hence, by substituting \( d_0 \) in Eq. (7), we get

\[
x_0(t) = -T_q^{\alpha}[v_0](t) + T_q^{\alpha - 1}[v_0](1) - \Xi T_q^{\alpha - 1}[v_0](b) \\
+ (\Xi - 1) T_q^{\alpha - 1}[v_0](a) + \eta(1 - \Xi) \int_0^1 x_0(r) \, dr \\
+ T_q^{\alpha - 1}[v_0](a) - \eta t \int_0^1 x_0(r) \, dr. \tag{8}
\]

Put \( \delta = \int_0^1 x_0(r) \, dr \). By computing the value of \( \delta \) and substituting it in (8), we get

\[
x_0(t) = -T_q^{\alpha}[v_0](t) + \frac{\eta(\frac{1}{2} - t) + 1}{\eta(\Xi - \frac{1}{2}) + 1} T_q^{\alpha - 1}[v_0](1) \\
+ \frac{\eta(\Xi(t - 1) + \frac{1}{2}) - \Xi}{\eta(\Xi - \frac{1}{2}) + 1} T_q^{\alpha - 1}[v_0](b) \\
+ \frac{\eta(\Xi - 1 + t)}{\eta(\Xi - \frac{1}{2}) + 1} T_q^{\alpha + 1}[v_0](1)
\]
Thus, $x_0$ is a solution for the fractional $q$-integral equation (7). It is obvious that $x_0$ is a solution for the fractional $q$-integro-differential equation (1) whenever $x_0$ is a solution for the fractional $q$-integral equation. This completes the proof. □

**Theorem 3** Let $g \in C([\bar{J},\mathbb{R})$ be a bounded function with upper bound $L > 0$. Assume that for each $t \in \bar{J}$ there exist positive continuous functions $m_1(t)$ and $m_2(t)$ such that

$$\|w_1(t,x(t),\varphi(x(t))) - w_1(t,y(t),\varphi(y(t)))\| \leq m_1(t)\|x-y\| + Lm_2(t)\|x-y\|$$

for $x, y \in A$. Also, put

$$M_0 = \max\left\{ \sup_{t \in \bar{J}} |I_{q}^\alpha[w_1](t)|, \sup_{t \in \bar{J}} |I_{q}^{\alpha+1}[m_2](t)| \right\},$$

$$M_1 = \max\left\{ \sup_{t \in \bar{J}} |I_{q}^{\alpha-1}[m_1](1)|, \sup_{t \in \bar{J}} |I_{q}^{\alpha+1}[m_2](1)| \right\},$$

and

$$M(s) = \max\left\{ \sup_{t \in \bar{J}} |I_{q}^{\alpha-1}[m_1](s)|, \sup_{t \in \bar{J}} |I_{q}^{\alpha+1}[m_2](s)| \right\}$$

for $s = 1, s = a, s = b$, and

$$\Delta = M_0 + \frac{\eta S}{\eta(S - \frac{1}{2}) + 1} M_1 + \frac{\frac{1}{2} \eta + 1}{\eta(S - \frac{1}{2}) + 1} M(1) + \frac{\eta S}{\eta(S - \frac{1}{2}) + 1} M(a) + \frac{\frac{1}{2} \eta S - S}{\eta(S - \frac{1}{2}) + 1} M(b).$$

If $\Delta < 1$, then the nonlinear fractional $q$-integro-differential equation (1)–(2) has a unique solution.

**Proof** We define the operator $\Theta : C(\bar{J}) \to C(\bar{J})$ by

$$(\Theta y)(t) = -I_{q}^\alpha[w_1](t,x(t),\varphi(x(t))) + \frac{\eta(S - 2)}{\eta(S - \frac{1}{2}) + 1} I_{q}^{\alpha-1}[w_1](1,x(1),\varphi(x(1))) + \frac{\eta(S-t+1)}{\eta(S - \frac{1}{2}) + 1} I_{q}^{\alpha-1}[w_1](b,x(b),\varphi(x(b))) + \frac{\eta(S-1+t)}{\eta(S - \frac{1}{2}) + 1} I_{q}^{\alpha+1}[w_1](1,x(1),\varphi(x(1)))$$
+ \eta \eta (\mathcal{S} - 2 + (\mathcal{S} - 1 + t)T_{q}^{-1}[w_{1}](a,x(a),\varphi(x(a)))].

Take \ell = \sup_{t \in I} |w_{1}(t,0,0)| and choose \rho > 0 such that

\rho \geq \frac{\ell}{1 - \phi} \left[ \frac{1}{\Gamma_{q}(\alpha + 1)} + \frac{1}{\eta (\mathcal{S} - \frac{1}{2}) + 1} T_{q}(\alpha) + \frac{\frac{1}{2} \eta - \mathcal{S}}{\eta (\mathcal{S} - \frac{1}{2}) + 1} T_{q}(\alpha) \right. \\
\left. + \frac{\eta \mathcal{S}}{\eta (\mathcal{S} - \frac{1}{2}) + 1} T_{q}(\alpha + 2) + \frac{\eta \mathcal{S} + \mathcal{S} \eta^{(\alpha - 1)}}{\eta (\mathcal{S} - \frac{1}{2}) + 1} T_{q}(\alpha) \right] \left. \frac{1}{\phi} \right] .

Put \rho_{T} = \{ x \in A : \| x \| \leq \rho_{T} \}. Let x \in \rho_{T}. Then we have

\| (\Theta x)(t) \| \leq T_{q}^{\|} \left( \| w_{1}(t,x(t),\varphi(x(t))) - w_{1}(t,0,0) \| + \| w_{1}(t,0,0) \| \right) \\
+ \frac{\frac{1}{2} \eta + 1}{\eta (\mathcal{S} - \frac{1}{2}) + 1} \\
\times \left[ T_{q}^{-1} \left( \| w_{1}(t,x(t),\varphi(x(t))) - w_{1}(t,0,0) \| + \| w_{1}(t,0,0) \| \right) \\ + \frac{\frac{1}{2} \eta - \mathcal{S}}{\eta (\mathcal{S} - \frac{1}{2}) + 1} \\
\times T_{q}^{-1} \left( \| w_{1}(1,x(1),\varphi(x(1))) - w_{1}(1,0,0) \| + \| w_{1}(1,0,0) \| \right) \\ + \frac{\eta \mathcal{S}}{\eta (\mathcal{S} - \frac{1}{2}) + 1} T_{q}(\alpha + 2) \right] \\
\times T_{q}^{-1} \left( \| w_{1}(a,x(a),\varphi(x(a))) - w_{1}(a,0,0) \| + \| w_{1}(a,0,0) \| \right) \\
\leq \left[ M_{0} + \frac{\eta \mathcal{S}}{\eta (\mathcal{S} - \frac{1}{2}) + 1} M_{1} + \frac{\frac{1}{2} \eta + 1}{\eta (\mathcal{S} - \frac{1}{2}) + 1} M(1) \right. \\
+ \frac{\eta \mathcal{S} + \mathcal{S} \eta^{(\alpha - 1)}}{\eta (\mathcal{S} - \frac{1}{2}) + 1} M(\alpha) + \frac{\frac{1}{2} \eta - \mathcal{S}}{\eta (\mathcal{S} - \frac{1}{2}) + 1} M(b) \right] \rho_{T} \\
\times \left[ \frac{1}{\Gamma_{q}(\alpha + 1)} + \frac{\frac{1}{2} \eta + 1}{\eta (\mathcal{S} - \frac{1}{2}) + 1} T_{q}(\alpha) + \frac{\frac{1}{2} \eta - \mathcal{S}}{\eta (\mathcal{S} - \frac{1}{2}) + 1} T_{q}(\alpha) \right. \\
\left. + \frac{\eta \mathcal{S}}{\eta (\mathcal{S} - \frac{1}{2}) + 1} T_{q}(\alpha + 2) + \frac{\eta \mathcal{S} + \mathcal{S} \eta^{(\alpha - 1)}}{\eta (\mathcal{S} - \frac{1}{2}) + 1} T_{q}(\alpha) \right] \ell \\
= \Delta \rho_{T} + \left[ \frac{1}{\Gamma_{q}(\alpha + 1)} + \frac{\frac{1}{2} \eta + 1}{\eta (\mathcal{S} - \frac{1}{2}) + 1} T_{q}(\alpha) \right. \\
\left. + \frac{\frac{1}{2} \eta - \mathcal{S}}{\eta (\mathcal{S} - \frac{1}{2}) + 1} T_{q}(\alpha + 2) \right] \ell \leq \rho_{T}.
Hence, $\Theta(B_r) \subset B_0$. On the other hand, one can write
\[
\|(\Theta x)(t) - (\Theta y)(t)\| \\
\leq I_{\alpha}^\eta (m_1(t)\|x - y\| + m_2(t)\|\psi(x) - \psi(y)\|) \\
+ \frac{1}{2} \frac{\eta + 1}{\eta(\frac{\eta}{2} - \frac{1}{2}) + 1} I_{\alpha}^\eta (m_1(1)\|x - y\| + m_2(1)\|\psi(x) - \psi(y)\|) \\
+ \frac{1}{2} \frac{\eta - \eta}{\eta(\frac{\eta}{2} - \frac{1}{2}) + 1} I_{\alpha}^\eta (m_1(b)\|x - y\| + m_2(b)\|\psi(x) - \psi(y)\|) \\
+ \frac{\eta}{\eta(\frac{\eta}{2} - \frac{1}{2}) + 1} I_{\alpha}^\eta (m_1(1)\|x - y\| + m_2(1)\|\psi(x) - \psi(y)\|) \\
+ \frac{\eta}{\eta(\frac{\eta}{2} - \frac{1}{2}) + 1} I_{\alpha}^\eta (m_1(0)\|x - y\| + m_2(0)\|\psi(x) - \psi(y)\|) \\
\leq \left[ M_0 + \frac{\eta}{\eta(\frac{\eta}{2} - \frac{1}{2}) + 1} M_1 + \frac{1}{2} \frac{\eta + 1}{\eta(\frac{\eta}{2} - \frac{1}{2}) + 1} M(1) \\
+ \frac{\eta}{\eta(\frac{\eta}{2} - \frac{1}{2}) + 1} M(a) + \frac{1}{2} \frac{\eta - \eta}{\eta(\frac{\eta}{2} - \frac{1}{2}) + 1} M(b) \right] \|x - y\| \\
= \Delta \|x - y\|.
\]

Since $\Delta < 1$, $\Theta$ is a contraction. Thus, by using the Banach contraction principle, $\Theta$ has a unique fixed point $x_0$ in $A$. At present, by using Lemma 2, one can get that $\Theta^*\Theta x_0 \in A$ and $x_0$ is the unique solution for the fractional $q$-integro-differential equation (1)–(2). $\square$

3.2 The nonlinear boundary value problem (3)–(4)

Lemma 4 Let $w : \mathcal{J} \times \mathcal{B} \rightarrow \mathcal{B}$ be a continuous function. An element $x_0 \in \mathcal{B}$ is a solution for the fractional $q$-integro-differential equation (3) under the sum boundary conditions (4) if and only if $x_0$ is a solution for the fractional integral equation
\[
y(t) = I_{\alpha}^\eta [v](t) - tI_{\alpha}^{\eta-1}[v](1) + t\frac{\eta}{\eta + 1} I_{\alpha}^{\eta-2}[v](b).
\]

Proof Put
\[
v_0(t) = w_2 \left( t, x_0(t), \int_0^t x_0(r) \, dr, \Theta \Theta^*[x_0](t) \right).
\]

Let $x_0$ be a solution for the fractional $q$-integro-differential equation (3). Choose $d_0, d_1 \in \mathbb{R}$ such that $x_0(t) = \int_0^t [v_0](r) \, dr + d_0 + d_1 t$ for all $t \in \mathcal{J}$. Hence, $x_0'(t) = \int_0^t [v_0]'(r) \, dr + d_1$ and $x_0''(t) = \int_0^t [v_0]'(r) \, dr + d_1$. By using the sum boundary conditions (4), we get $d_0 = 0$ and $d_1 = \int_0^1 [v_0] + \int_0^b [v_0] = 0$. By substituting $d_0$ and $d_1$, we obtain
\[
x_0(t) = I_{\alpha}^\eta [v_0](t) + tI_{\alpha}^{\eta-1}[v_0](1) + t\frac{\eta}{\eta + 1} I_{\alpha}^{\eta-2}[v_0](b).
\]

Thus, $x_0$ is a solution for the fractional $q$-integral equation. It is obvious that $x_0$ is a solution for the fractional $q$-integro-differential equation (3) whenever $x_0$ is a solution for the fractional $q$-integral equation. This completes the proof. $\square$
**Theorem 5** Suppose that \(w_2 : \mathcal{J} \times \mathcal{B}^3 \rightarrow \mathcal{B}\) is a continuous map and there exist positive continuous functions \(m_1, m_2, \) and \(m_3\) such that

\[
\left| w_2\left(t, x(t), \int_0^t x(r) \, dr, \mathcal{D}_q^\zeta [x](t)\right) - w_2\left(t, y(t), \int_0^t y(r) \, dr, \mathcal{D}_q^\zeta [y](t)\right) \right|
\leq m_1(t)|x - y| + m_2(t) \left| \int_0^t x(r) \, dr - \int_0^t y(r) \, dr \right|
+ m_3(t) |\mathcal{D}_q^\zeta [x](t) - \mathcal{D}_q^\zeta [y](t)|
\]

for all \(x, y \in \mathcal{B}\) and \(t \in \mathcal{J}\). Let

\[
M_0 = \max \left\{ \sup_{t \in \mathcal{J}} |I_q^2 [m_1](t)|, \sup_{t \in \mathcal{J}} |I_q^2 [m_2](t)|, \sup_{t \in \mathcal{J}} |I_q^2 [m_3](t)| \right\},
\]

\[
M(1) = \max \left\{ \sup_{t \in \mathcal{J}} |I_q^{2-1} [m_1](1)|, \sup_{t \in \mathcal{J}} |I_q^{2-1} [m_2](1)|, \sup_{t \in \mathcal{J}} |I_q^{2-1} [m_3](1)| \right\},
\]

\[
M(b) = \max \left\{ \sup_{t \in \mathcal{J}} |I_q^{2-2} [m_1](b)|, \sup_{t \in \mathcal{J}} |I_q^{2-2} [m_2](b)|, \sup_{t \in \mathcal{J}} |I_q^{2-2} [m_3](b)| \right\},
\]

and

\[
M(t) = \max \left\{ \sup_{t \in \mathcal{J}} |I_q^{2-\zeta} [m_1](t)|, \sup_{t \in \mathcal{J}} |I_q^{2-\zeta} [m_2](t)|, \sup_{t \in \mathcal{J}} |I_q^{2-\zeta} [m_3](t)| \right\}.
\]

Put

\[
\Delta = M_0 + M(t) + \frac{\Gamma_q(2 - \zeta) + 1}{\Gamma_q(2 - \zeta)} M(1) + \frac{(1 + \Gamma_q(2 - \zeta)) \Xi}{\Gamma_q(2 - \zeta)} M(b).
\]

If \(\Delta < 1\), then the nonlinear fractional \(q\)-integro-differential equation (3)–(4) has a unique solution.

**Proof** Define the operator \(\Theta : \mathcal{B} \rightarrow \mathcal{B}\) by

\[
(\Theta x)(t) = I_q^2 [v](t) - t I_q^{2-1} [v](1) + t \Xi I_q^{2-2} [v](b),
\]

where

\[
v(t) = w_2\left(t, x(t), \int_0^t x(r) \, dr, \mathcal{D}_q^\zeta [x](t)\right).
\]

Choose \(r > 0\) such that

\[
r \geq \frac{\ell}{\Gamma_q(\alpha - \zeta + 1) + 1 + \alpha} \left[ \frac{1}{\Gamma_q(\alpha + 1)} + \frac{1}{\Gamma_q(2 - \zeta) \Gamma_q(\alpha)} \right]
\]

\[
+ \frac{1}{\Gamma_q(2 - \zeta) \Gamma_q(\alpha - 1)} + 1 \left\{ \frac{\Gamma_q(2 - \zeta) + 1}{\Gamma_q(2 - \zeta) \Gamma_q(\alpha - 1)} \Xi \right\},
\]

where \(\ell = \sup_{t \in \mathcal{J}} |w_2(t, 0, 0, 0)|\). We show that \(\Theta B_r \subset B_r\), where

\[
B_r = \left\{ x \in \mathcal{B} : \|x\| \leq r_0 \right\}.
\]
Let $x \in B_{r_0}$. Then

$$
|\Theta x(t)| \leq \Delta r_0 + \ell \left[ \frac{1}{\Gamma_q(\alpha - \zeta + 1)} + \frac{1}{\Gamma_q(2 - \zeta)} + \frac{\xi}{\Gamma_q(2 - \zeta)\Gamma_q^\alpha(\alpha - 1)} \right] \leq r_0
$$

On the other hand, we have

$$
|\,^{\alpha}D^\xi_x [\Theta x](t)\,| \leq \frac{\ell}{\Gamma_q(\alpha - \zeta + 1)} + \frac{1}{\Gamma_q(2 - \zeta)} + \frac{\xi}{\Gamma_q(2 - \zeta)\Gamma_q^\alpha(\alpha - 1)} |x(t)|
$$

Hence,

$$
|\Theta x(t)| \leq \Delta r_0 + \ell \left[ \frac{1}{\Gamma_q(\alpha - \zeta + 1)} + \frac{1}{\Gamma_q(2 - \zeta)\Gamma_q^\alpha(\alpha - 1)} + \frac{\xi}{\Gamma_q(2 - \zeta)\Gamma_q^\alpha(\alpha - 1)} \right] \leq r_0
$$
and so $\Theta(B_{r_0}) \subseteq B_{r_0}$. Let $u, v \in X$ and $t \in J$. Then we have

$$
|\Theta(x)(t) - \Theta(y)(t)| \leq T_q^\alpha \left( \sum_{i=1}^{3} m_i(t) |x(t) - y(t)| \right) + T_q^{\alpha-1} \left( \sum_{i=1}^{3} m_i(1) |x(1) - y(1)| \right) + \Xi T_q^{\alpha-2} \left( \sum_{i=1}^{3} m_i(b) |x(b) - y(b)| \right) \leq (M_0 + M(1) + \Xi M(b)) |x - y|.
$$

On the other hand,

$$
\left| \mathcal{D}^\zeta [\Theta x](t) - \mathcal{D}^\zeta [\Theta y](t) \right| \leq \left[ M(t) + \frac{1}{\Gamma_q(2 - \zeta)} T_q^{\alpha-1} M(1) + \frac{\Xi}{\Gamma_q(2 - \zeta)} M(b) \right] |x - y|.
$$

Hence,

$$
\| (\Theta x)(t) - (\Theta y)(t) \| \leq \left[ M(t) + M_0 + \frac{\Gamma_q(2 - \zeta) + 1}{\Gamma_q(2 - \zeta)} M(1) + \frac{\Xi [1 + \Gamma_q(2 - \zeta)]}{\Gamma_q(2 - \zeta)} M(b) \right] \| x - y \| = \Delta \| x - y \|.
$$

Since $\Delta < 1$, $\Theta$ is a contraction and so, by using the Banach contraction principle, $\Theta$ has a unique fixed point. By using Lemma 4, it is clear that the unique fixed point of $\Theta$ is the unique solution for the nonlinear fractional integro-differential problem (3)–(4). □

### 4 Examples, numerical results, and algorithms

Herein, we give an example to show the validity of the main results. In this way, we give a computational technique for checking problems (1)–(2) and (3)–(4). We need to present a simplified analysis that is able to execute the values of the $q$-gamma function. For this purpose, we provided a pseudo-code description of the method for calculation of the $q$-gamma function of order $n$ in Algorithms 2, 3, 4, and 5; for more details, follow these addresses [https://en.wikipedia.org/wiki/Q-gamma_function](https://en.wikipedia.org/wiki/Q-gamma_function) and [https://www.dm.uniba.it/members/garrappa/software](https://www.dm.uniba.it/members/garrappa/software). Tables 1, 2, and 3 show the values $\Gamma_q(z)$ for some $z$ and $q \in (0, 1)$.

For problems for which the analytical solution is not known, we will use, as reference solution, the numerical approximation obtained with a tiny step $h$ by the implicit trapezoidal PI rule, which, as we will see, usually shows an excellent accuracy [49]. All the experiments are carried out in MATLAB Ver. 8.5.0.197613 (R2015a) on a computer equipped with a CPU AMD Athlon(tm) II X2 245 at 2.90 GHz running under the operating system Windows 7.
Algorithm 4 The proposed method for calculated $\int_a^bf(r)dr$

```plaintext
function g = lg1(q, x, n, fun)
p=1;
for k=0:n
 p=p* q^k*fun(x*q^k);
end;
g=x* (1-q) + p;
end
```

Algorithm 5 The proposed method for calculated $\int_0^x|x|$ 

```plaintext
function g = lg2alpha(q, alpha, x, n, fun)
p=0;
for k=0:n
 s1=1;
 for i=0:k-1
 s1=s1* (1-q^i{alpha+1})
 end
 s2=1;
 for i=0:k-1
 s2=s2* (1-q^i{alpha+1})
 end
 p=p + q^k*s1+eval{subs(fun, t+q^k)}/s2;
 end;
g=round((t^alpha) + ((1-q^alpha)* p, 6))
end
```

**Example 1** Consider the fractional $q$-integro-differential equation similar to problem (1) as follows:

$$cD_{\frac{3}{4}}^x[x](t) + \frac{|x(t)|}{7(t^2 + \frac{1}{4})^2(2 + |x(t)|)} + \frac{t}{1600} \int_0^t e^{(-\frac{1}{2})}x(r) \, dr = 0, \quad (15)$$

under sum and integral boundary value conditions $x'(\frac{1}{4}) = -\frac{1}{6} \int_0^1 x(r) \, dr$ and

$$x'(1) + x(0) = \sum_{i=1}^5 c_i x'\left(\frac{3}{4}\right).$$

Note that $x'(\frac{3}{4}) = \frac{3}{7}x(t)^{\frac{3}{4}}$. Clearly, $\alpha = \frac{3}{2}$, $a = \frac{1}{4}$, $\eta = \frac{1}{6}$, $m = 5$, $b = \frac{3}{4}$. Let $c_1 = \frac{1}{8}$, $c_2 = \frac{1}{2}$, $c_3 = \frac{3}{7}$, $c_4 = \frac{1}{7}$, and $c_5 = \frac{1}{6}$. Note that $E = \sum_{i=1}^5 c_i = \frac{239}{280}$ and so $2E > -1$. We define the maps $w_1 : \mathcal{J} \times \mathcal{A}^2 \rightarrow \mathcal{A}$ and $g : \mathcal{J} \rightarrow [0, \infty)$ by

$$w_1(t,x(t),\phi x(t)) = \frac{|x(t)|}{7(t^2 + \frac{1}{4})^2(2 + |x(t)|)} + \frac{t}{1600} \int_0^t e^{(-\frac{1}{2})}x(r) \, dr$$

and $g(t) = \frac{1}{40}$ for all $t \in \mathcal{J}$, respectively. It is obvious that $g(t) \leq 0.025 = L$ for $t \in \mathcal{J}$. Now, we obtain

$$\|w_1(t,x(t),\phi x(t)) - w_1(t,y(t),\phi y(t))\|$$

$$= \left\| \frac{|x(t)|}{7(t^2 + \frac{1}{4})^2(2 + |x(t)|)} + \frac{t}{1600} \int_0^t e^{(-\frac{1}{2})}x(r) \, dr - \left[ \frac{|y(t)|}{7(t^2 + \frac{1}{4})^2(2 + |y(t)|)} + \frac{t}{1600} \int_0^t e^{(-\frac{1}{2})}y(r) \, dr \right]\right\|$$
Table 4 Some numerical results of $T_q^{0}[m_1](t)$ in Example 1 for $t \in J$ and $q = \frac{1}{3}, \frac{1}{7}, \frac{5}{7}$

| $n$ | $T_q^{0}[m_1](t)$ | $T_q^{m+1}[m_1](t)$ | $T_q^{m-1}[m_1](t)$ |
|-----|------------------|---------------------|---------------------|
|     | $\sup_{s=1}$    | $s=a$              | $s=b$               |
| $q=\frac{1}{7}$ |                  |                     |                     |
| 1   | 0.0516           | 0.0516              | 0.0454              |
| 2   | 0.0527           | 0.0527              | 0.0465              |
| 3   | 0.0529           | 0.0529              | 0.0466              |
| 4   | 0.0529           | 0.0529              | 0.0466              |
| $q=\frac{5}{7}$ |                  |                     |                     |
| 1   | 0.0347           | 0.0347              | 0.0196              |
| 2   | 0.0446           | 0.0446              | 0.0265              |
| 3   | 0.0499           | 0.0499              | 0.0305              |
| 4   | 0.0526           | 0.0526              | 0.0325              |
| 5   | 0.0539           | 0.0539              | 0.0336              |
| 6   | 0.0546           | 0.0546              | 0.0341              |
| 7   | 0.0555           | 0.0555              | 0.0344              |
| 8   | 0.0552           | 0.0552              | 0.0345              |
| 9   | 0.0552           | 0.0552              | 0.0346              |
| $q=\frac{6}{7}$ |                  |                     |                     |
| 1   | 0.0667           | 0.0667              | 0.0013              |
| 2   | 0.0111           | 0.0111              | 0.0025              |
| 3   | 0.0154           | 0.0154              | 0.0014              |
| ... |                  |                     |                     |
| 37  | 0.0551           | 0.0551              | 0.0257              |
| 38  | 0.0551           | 0.0551              | 0.0257              |
| 39  | 0.0552           | 0.0552              | 0.0257              |
| 40  | 0.0552           | 0.0552              | 0.0257              |
| 41  | 0.0552           | 0.0552              | 0.0258              |
| 42  | 0.0552           | 0.0552              | 0.0258              |
| 43  | 0.0553           | 0.0553              | 0.0258              |

Thus

$$\left\| w_1(t, x(t), \varphi(x(t))) - w_1(t, y(t), \varphi(y(t))) \right\| \leq \frac{1}{7(t^2 + \frac{7}{4})^2} \left\| x - y \right\| + \frac{t}{1600} \left\| x - y \right\|.$$ 

for all $t \in J, x, y \in A$. We define the positive continuous maps $m_1(t) = \frac{1}{7(t^2 + \frac{7}{4})^2}$ and $m_2(t) = \frac{1}{4t^2}$. At present, by using Eqs. (9)–(10) and applying Algorithm 5, we calculate $\sup_{t \in J} T_q^{0}[m_1](t)$, $\sup_{t \in J} L T_q^{0}[m_2](t)$, $\sup_{t \in J} T_q^{m+1}[m_1](t)$, $\sup_{t \in J} L T_q^{m+1}[m_2](t)$, $\sup_{t \in J} T_q^{m-1}[m_1](t)$, and $\sup_{t \in J} L T_q^{m-1}[m_2](t)$ for $t \in (0, 1)$ and $q = \frac{1}{3}, \frac{1}{7}, \frac{5}{7}$. Tables 4 and 5 show these results. Also, Figures 1, 2 and 3 illustrate the numerical results of the tables. Therefore

$$\sup_{t \in J} T_q^{0}[m_1](t) = \sup_{t \in J} T_q^{0} \left( \frac{1}{7(t + \frac{7}{4})^2} \right) = 0.0529, 0.0552, 0.0553,$$

$$\sup_{t \in J} T_q^{m+1}[m_1](t) = \sup_{t \in J} T_q^{m+1} \left( \frac{1}{7(1 + \frac{7}{4})^2} \right) = 0.0466, 0.0346, 0.0258,$$
Figure 1 2D graph of $I^q_{\alpha}[m_1](t)$ and $I^q_{\alpha}[m_2](t)$ for $t \in J$ with $q = \frac{1}{6}, \frac{1}{2}, \frac{6}{7}$ in Example 1

Table 5 Some numerical results of $I^q_{\alpha}[m_2](t)$ in Example 1 for $t \in J$ and $q = \frac{1}{6}, \frac{1}{2}, \frac{6}{7}$

| $n$ | $I^q_{\alpha}[m_2](t)$ | $I^{q+1}_{\alpha}[m_1](t)$ | $I^{q-1}_{\alpha}[m_1](t)$ |
|-----|----------------------|-------------------------|-------------------------|
|     | $t = 0$              | $t = 1$                | sup $I^q_{\alpha}[m_1](s)$ | sup $I^{q+1}_{\alpha}[m_1](s)$ | sup $I^{q-1}_{\alpha}[m_1](s)$ |
| $q = \frac{1}{6}$ | 1 0 0.0005 0.0005 | 0.0005 0.0006 0.0001 0.0004 | 1 0 0.0003 0.0003 | 0.0002 0.0005 0.0001 0.0003 | 1 0 0.0003 0.0003 | 0.0002 0.0005 0.0001 0.0003 |
|     | 2 0 0.0005 0.0005 | 0.0005 0.0006 0.0001 0.0004 | 2 0 0.0003 0.0003 | 0.0002 0.0005 0.0001 0.0003 | 2 0 0.0003 0.0003 | 0.0002 0.0005 0.0001 0.0003 |
|     | 3 0 0.0005 0.0005 | 0.0005 0.0006 0.0001 0.0004 | 3 0 0.0003 0.0003 | 0.0002 0.0005 0.0001 0.0003 | 3 0 0.0003 0.0003 | 0.0002 0.0005 0.0001 0.0003 |
| $q = \frac{1}{2}$ | 1 0 0.0001 0.0001 | 0 0.0003 0 0.0002 | 1 0 0.0001 0.0001 | 0 0.0004 0 0.0002 | 1 0 0.0001 0.0001 | 0 0.0004 0 0.0002 |
|     | 2 0 0.0001 0.0001 | 0 0.0004 0 0.0002 | 2 0 0.0001 0.0001 | 0 0.0004 0 0.0002 | 2 0 0.0001 0.0001 | 0 0.0004 0 0.0002 |
|     | 3 0 0.0001 0.0001 | 0 0.0004 0 0.0002 | 3 0 0.0001 0.0001 | 0 0.0004 0 0.0002 | 3 0 0.0001 0.0001 | 0 0.0004 0 0.0002 |
|     | 4 0 0.0002 0.0002 | 0 0.0004 0 0.0002 | 4 0 0.0002 0.0002 | 0 0.0004 0 0.0002 | 4 0 0.0002 0.0002 | 0 0.0004 0 0.0002 |
|     | 5 0 0.0002 0.0002 | 0 0.0004 0 0.0002 | 5 0 0.0002 0.0002 | 0 0.0004 0 0.0002 | 5 0 0.0002 0.0002 | 0 0.0004 0 0.0002 |
|     | 6 0 0.0002 0.0002 | 0.0001 0.0005 0.0001 0.0003 | 6 0 0.0002 0.0002 | 0.0001 0.0005 0.0001 0.0003 | 6 0 0.0002 0.0002 | 0.0001 0.0005 0.0001 0.0003 |
|     | 7 0 0.0002 0.0002 | 0.0001 0.0005 0.0001 0.0003 | 7 0 0.0002 0.0002 | 0.0001 0.0005 0.0001 0.0003 | 7 0 0.0002 0.0002 | 0.0001 0.0005 0.0001 0.0003 |
|     | 8 0 0.0002 0.0002 | 0.0001 0.0005 0.0001 0.0003 | 8 0 0.0002 0.0002 | 0.0001 0.0005 0.0001 0.0003 | 8 0 0.0002 0.0002 | 0.0001 0.0005 0.0001 0.0003 |

\[ \sup_{t \in J} I^{q-1}_{\alpha}[m_1](1) = \sup_{t \in J} I^{q-1}_{\alpha} \left( \frac{1}{7(1 + \frac{1}{3})^2} \right) = 0.0566, 0.0652, 0.0706, \]

\[ \sup_{t \in J} I^{q-1}_{\alpha}[m_1](a) = \sup_{t \in J} I^{q-1}_{\alpha} \left( \frac{1}{7(\frac{1}{16} + \frac{1}{4})^2} \right) = 0.0409, 0.0432, 0.0447, \]

\[ \sup_{t \in J} I^{q-1}_{\alpha}[m_1](b) = \sup_{t \in J} I^{q-1}_{\alpha} \left( \frac{1}{7(\frac{9}{16} + \frac{1}{4})^2} \right) = 0.0570, 0.0633, 0.0674 \]

for $q = \frac{1}{6}, \frac{1}{2}, \frac{6}{7}$, respectively, and

\[ \sup_{t \in J} L I^q_{\alpha}[m_2](t) = \sup_{t \in J} L I^q_{\alpha} \left( \frac{t}{40} \right) = 0.0005, 0.0003, 0.0002, \]
Figure 2 2D graph of $I_{q}^{\alpha+1}[m_{1}](1)$ and $L I_{q}^{\alpha+1}[m_{2}](1)$ for $t \in J$ with $q = \frac{1}{8}, \frac{1}{2}, \frac{6}{7}$ in Example 1.

\[
\begin{align*}
\sup_{t \in J} L I_{q}^{\alpha+1}[m_{2}](1) &= \sup_{t \in J} L I_{q}^{\frac{3}{4}+1}\left(\frac{1}{40}\right) = 0.0005, 0.0002, 0.0001, \\
\sup_{t \in J} L I_{q}^{\alpha-1}[m_{2}](1) &= \sup_{t \in J} L I_{q}^{\frac{3}{2}-1}\left(\frac{1}{160}\right) = 0.0006, 0.0005, 0.0005, \\
\sup_{t \in J} L I_{q}^{\alpha-1}[m_{2}](a) &= \sup_{t \in J} L I_{q}^{\frac{3}{4}-1}\left(\frac{1}{160}\right) = 0.0001, 0.0001, 0.0001, \\
\sup_{t \in J} L I_{q}^{\alpha-1}[m_{2}](b) &= \sup_{t \in J} L I_{q}^{\frac{3}{4}-1}\left(\frac{3}{160}\right) = 0.0004, 0.0003, 0.0003
\end{align*}
\]

for $q = \frac{1}{8}, \frac{1}{2}, \frac{6}{7}$, respectively. Hence, from Eqs. (9)–(10) and the above results in Tables 4 and 5, we obtain $M_{0} = \max\{0.0529, 0.0005\} = 0.0529$, $M_{1} = \max\{0.0466, 0.0005\} = 0.0466$, $M(1) = \max\{0.0566, 0.0006\} = 0.0566$,

\[
M(a) = M\left(\frac{1}{4}\right) = \max\{0.0409, 0.0001\} = 0.0409,
\]

\[
M(b) = M\left(\frac{3}{4}\right) = \max\{0.0570, 0.0004\} = 0.0570
\]

whenever $q = \frac{1}{8}$. $M_{0} = \max\{0.0552, 0.0003\} = 0.0552$, $M_{1} = \max\{0.0436, 0.0002\} = 0.0436$, $M(1) = \max\{0.0652, 0.0005\} = 0.0652$,

\[
M(a) = M\left(\frac{1}{4}\right) = \max\{0.0432, 0.0001\} = 0.0432,
\]

\[
M(b) = M\left(\frac{3}{4}\right) = \max\{0.0633, 0.0003\} = 0.0633,
\]
Figure 3 2D graph of $T^{q^{-1}}[m_1](s)$ and $L T^{q^{-1}}[m_2](s)$ for $t \in \mathcal{T}$ and $s = a, b$ with $q = \frac{1}{8}$, $\frac{1}{2}$, $\frac{3}{7}$ in Example 1 whenever $q = \frac{1}{8}$, $M_0 = \max\{0.0553, 0.0002\} = 0.0553$, $M_1 = \max\{0.0258, 0.0001\} = 0.0258$, $M(1) = \max\{0.0706, 0.0005\} = 0.0706$, $M(a) = M\left(\frac{1}{4}\right) = \max\{0.0447, 0.0001\} = 0.0447,$
Table 6  Some numerical results for calculation of $M_0, M_1, M(1), M(a), M(b)$ and $\Delta < 1$ in Example 1 for $q = \frac{1}{3}, \frac{2}{7}, \frac{6}{7}$.

| $n$ | $M_0$ | $M_1$ | $M(1)$ | $M(a)$ | $M(b)$ | $\Delta$ |
|-----|-------|-------|--------|--------|--------|---------|
| 1   | 0.0516 | 0.0454 | 0.0556 | 0.0404 | 0.0561 | 0.0994 |
| 2   | 0.0527 | 0.0465 | 0.0564 | 0.0408 | 0.0569 | 0.1013 |
| 3   | 0.0529 | 0.0466 | 0.0565 | 0.0409 | 0.0570 | 0.1015 |
| 4   | 0.0529 | 0.0466 | 0.0566 | 0.0409 | 0.0570 | 0.1016 |
| 5   | 0.0529 | 0.0466 | 0.0566 | 0.0409 | 0.0570 | 0.1016 |
| $q = \frac{1}{7}$ | | | | | | |
| 1   | 0.0347 | 0.0196 | 0.0515 | 0.0362 | 0.0514 | 0.0757 |
| 2   | 0.0446 | 0.0265 | 0.0586 | 0.0399 | 0.0576 | 0.0915 |
| 3   | 0.0499 | 0.0305 | 0.0620 | 0.0416 | 0.0605 | 0.0997 |
| 4   | 0.0526 | 0.0325 | 0.0636 | 0.0424 | 0.0620 | 0.1039 |
| 5   | 0.0539 | 0.0336 | 0.0644 | 0.0428 | 0.0627 | 0.1060 |
| 6   | 0.0546 | 0.0341 | 0.0649 | 0.0430 | 0.0630 | 0.1071 |
| 7   | 0.0550 | 0.0344 | 0.0651 | 0.0431 | 0.0632 | 0.1076 |
| 8   | 0.0552 | 0.0345 | 0.0652 | 0.0432 | 0.0633 | 0.1078 |
| 9   | 0.0552 | 0.0346 | 0.0652 | 0.0432 | 0.0633 | 0.1080 |
| 10  | 0.0553 | 0.0346 | 0.0652 | 0.0432 | 0.0634 | 0.1080 |
| 11  | 0.0553 | 0.0346 | 0.0652 | 0.0432 | 0.0634 | 0.1081 |
| 12  | 0.0553 | 0.0346 | 0.0653 | 0.0432 | 0.0634 | 0.1081 |
| $q = \frac{1}{3}$ | | | | | | |
| 1   | 0.0067 | 0.0013 | 0.0293 | 0.0216 | 0.0298 | 0.0288 |
| 2   | 0.0110 | 0.0025 | 0.0363 | 0.0261 | 0.0366 | 0.0384 |
| 3   | 0.0154 | 0.0040 | 0.0418 | 0.0294 | 0.0418 | 0.0470 |
| ... | ... | ... | ... | ... | ... | ... |
| 37  | 0.0551 | 0.0257 | 0.0705 | 0.0446 | 0.0673 | 0.1100 |
| 38  | 0.0551 | 0.0257 | 0.0705 | 0.0447 | 0.0673 | 0.1101 |
| 39  | 0.0552 | 0.0257 | 0.0705 | 0.0447 | 0.0673 | 0.1101 |
| 40  | 0.0552 | 0.0257 | 0.0705 | 0.0447 | 0.0674 | 0.1102 |
| 41  | 0.0552 | 0.0258 | 0.0705 | 0.0447 | 0.0674 | 0.1102 |
| 42  | 0.0552 | 0.0258 | 0.0705 | 0.0447 | 0.0674 | 0.1102 |
| 43  | 0.0553 | 0.0258 | 0.0706 | 0.0447 | 0.0674 | 0.1103 |
| 44  | 0.0553 | 0.0258 | 0.0706 | 0.0447 | 0.0674 | 0.1103 |

$M(b) = M\left(\frac{3}{\frac{1}{4}}\right) = \max\{0.0674, 0.0003\} = 0.0674,$

whenever $q = \frac{1}{7}$. Also, by using Eq. (11), we can calculate values of $\Delta$. Table 6 shows these results. Thus, by using Eq. (11) we have

$$
\Delta = M_0 + \frac{\eta \Xi}{\eta(\Xi - \frac{1}{2}) + 1} M_1 + \frac{\frac{1}{2} \eta + 1}{\eta(\Xi - \frac{1}{2}) + 1} M(1) \\
+ \frac{\eta \Xi(\Xi - 2) + \Xi}{\eta(\Xi - \frac{1}{2}) + 1} M(a) + \frac{\frac{1}{2} \eta \Xi - \Xi}{\eta(\Xi - \frac{1}{2}) + 1} M(b)
$$

$$
= 0.0529 + \frac{\frac{1}{6} \times 239}{\frac{280}{2} - \frac{1}{2} + 1} \times 0.0466 + \frac{\frac{1}{2} \times \frac{1}{6} + 1}{\frac{2}{6}(\frac{280}{2} - \frac{1}{2}) + 1} \times 0.0566
+ \frac{\frac{1}{6} \times 239}{\frac{280}{2} - \frac{1}{2} + 1} \times 0.0409 + \frac{\frac{1}{2} \times \frac{1}{6} + 1}{\frac{2}{6}(\frac{280}{2} - \frac{1}{2}) + 1} \times 0.0570
= 0.1016 < 1
$$

whenever $q = \frac{1}{7}$.
\[ \Delta = M_0 + \frac{\eta \Xi}{\eta(\Xi - \frac{1}{2}) + 1} M_1 + \frac{\frac{1}{2} \eta + 1}{\eta(\Xi - \frac{1}{2}) + 1} M(1) \]

\[ + \frac{\eta \Xi(\Xi - 2) + \Xi}{\eta(\Xi - \frac{1}{2}) + 1} M(a) + \frac{\frac{1}{2} \eta \Xi - \Xi}{\eta(\Xi - \frac{1}{2}) + 1} M(b) \]

\[ = 0.0553 + \frac{\frac{1}{6} \times \frac{239}{280}}{\frac{1}{2} \times \frac{239}{280} - \frac{1}{2} + 1} \times 0.0346 + \frac{\frac{1}{6} \times \frac{239}{280} - \frac{1}{2} + 1}{\frac{1}{6} \times \frac{239}{280} - \frac{1}{2} + 1} \times 0.0653 \]

\[ + \frac{\frac{1}{6} \times \frac{239}{280} - \Xi + \Xi}{\frac{1}{6} \times \frac{239}{280} - \Xi + \Xi} \times 0.0432 + \frac{\frac{1}{6} \times \frac{1}{6} \times \frac{239}{280} - \frac{239}{280}}{\frac{1}{6} \times \frac{239}{280} - \Xi + \Xi} \times 0.0634 \]

\[ = 0.1081 < 1 \]

whenever \( q = \frac{1}{2} \), and

\[ \Delta = M_0 + \frac{\eta \Xi}{\eta(\Xi - \frac{1}{2}) + 1} M_1 + \frac{\frac{1}{2} \eta + 1}{\eta(\Xi - \frac{1}{2}) + 1} M(1) \]

\[ + \frac{\eta \Xi(\Xi - 2) + \Xi}{\eta(\Xi - \frac{1}{2}) + 1} M(a) + \frac{\frac{1}{2} \eta \Xi - \Xi}{\eta(\Xi - \frac{1}{2}) + 1} M(b) \]

\[ = 0.0553 + \frac{\frac{1}{6} \times \frac{239}{280}}{\frac{1}{2} \times \frac{239}{280} - \frac{1}{2} + 1} \times 0.0258 + \frac{\frac{1}{6} \times \frac{239}{280} - \frac{1}{2} + 1}{\frac{1}{6} \times \frac{239}{280} - \frac{1}{2} + 1} \times 0.0706 \]

\[ + \frac{\frac{1}{6} \times \frac{239}{280} - \Xi + \Xi}{\frac{1}{6} \times \frac{239}{280} - \Xi + \Xi} \times 0.0474 + \frac{\frac{1}{6} \times \frac{1}{6} \times \frac{239}{280} - \frac{239}{280}}{\frac{1}{6} \times \frac{239}{280} - \Xi + \Xi} \times 0.0674 \]

\[ = 0.1103 < 1 \]

whenever \( q = \frac{6}{7} \). Figures 4, 5, and 6 show these results (Algorithm 6). Now, by using Theorem 3, the fractional \( q \)-integro-differential equation under sum and integral boundary value conditions (15) has a unique solution.
Example 2 Consider the fractional $q$-integro-differential equation similar to problem (3) as follows:

$$
\begin{align*}
\frac{d^q}{dt^q}x(t) &= \frac{|t|}{35(1 + |t|)} + \frac{2}{35(4 + \sqrt{t})} |x(t)| + \frac{3t}{70} \int_0^t \frac{x(r)}{\sqrt{r} + 1} \, dr \\
&\quad + \frac{3t}{35(t^3 + 2)} \frac{d^{q-1}}{dt^{q-1}}x(t),
\end{align*}
$$

(16)

under the sum boundary value conditions $x'(0) = 0$ and $x'(1) = \sum_{i=1}^6 c_i x^{(i)}(\frac{1}{8})$. Clearly, $\alpha = \frac{9}{7}$,

$\zeta = \frac{1}{8}$, $b = \frac{7}{8}$, and $m = 6$. Let $c_1 = \frac{7}{12}$, $c_2 = \frac{9}{8}$, $c_3 = \frac{9}{5}$, $c_4 = \frac{2}{5}$, $c_5 = \frac{5}{8}$, $c_6 = \frac{11}{10}$, and so $\mathcal{S} =$
Algorithm 6 The MATLAB lines for calculation of all parameters in Example 1

```matlab
format long
t0=1; T=1; s=1; a=1/4; b=3/4;
X1=2*exp(7); eta=1/6;
m1=sym(1/7+1/2+7/4);
m2=sym(1/40);
L=0.025;
[q, xq, yq]=size(q);
column=1;
for j=1:q
    for n=1:80
        A1(n, column)=n;
        A2(n, column)=n;
        A1(n, column+1)=q1*a(n, j, alpha, t0, n, m1);
        A1(n, column+2)=q1*a(n, j, alpha, T, n, m1);
        A1(n, column+3)=A1(n, column+1);
        if A1(n, column+2)>A1(n, column+1)
            A1(n, column+3)=A1(n, column+2);
        end;
        A2(n, column+1)=l*q1*a(n, j, alpha, t0, n, m2);
        A2(n, column+2)=l*q1*a(n, j, alpha, T, n, m2);
        A2(n, column+3)=A2(n, column+1);
        if A2(n, column+2)>A2(n, column+1)
            A2(n, column+3)=A2(n, column+2);
        end;
        A1(n, column+4)=q1*a(n, j, alpha+1, T, n, m1);
        A2(n, column+4)=l*q1*a(n, j, alpha+1, T, n, m2);
        s=T;
        A1(n, column+5)=q1*a(n, j, alpha+1, s, n, m1);
        A2(n, column+5)=l*q1*a(n, j, alpha+1, s, n, m2);
        s=0;
        A1(n, column+6)=q1*a(n, j, alpha+1, s, n, m1);
        A2(n, column+6)=l*q1*a(n, j, alpha+1, s, n, m2);
        s=0;
        A1(n, column+7)=q1*a(n, j, alpha+1, s, n, m1);
        A2(n, column+7)=l*q1*a(n, j, alpha+1, s, n, m2);
    end;
column=column+8;
column=1;
Acolumn=1;
for j=1:q
    for n=1:80
        M(n, column)=n;
        M(n, column+1)=A1(n, Acolumn+3);
        if A2(n, Acolumn+3)>A1(n, Acolumn+3)
            M(n, column+1)=A2(n, Acolumn+3);
        end;
        M(n, column+2)=A1(n, Acolumn+4);
        if A2(n, Acolumn+4)>A1(n, Acolumn+4)
            M(n, column+2)=A2(n, Acolumn+4);
        end;
        M(n, column+3)=A1(n, Acolumn+5);
        if A2(n, Acolumn+5)>A1(n, Acolumn+5)
            M(n, column+3)=A2(n, Acolumn+5);
        end;
        M(n, column+4)=A1(n, Acolumn+6);
        if A2(n, Acolumn+6)>A1(n, Acolumn+6)
            M(n, column+4)=A2(n, Acolumn+6);
        end;
        M(n, column+5)=A1(n, Acolumn+7);
        if A2(n, Acolumn+7)>A1(n, Acolumn+7)
            M(n, column+5)=A2(n, Acolumn+7);
        end;
        column = column + 6;
        Acolumn = Acolumn + 8;
    end;
column=1;
Mcolumn=1;
for j=1:q
    for n=1:80
        Delta(n, column)=n;
        Delta(n, column+1) = M(n, Mcolumn+1) + ... 
                      eta*X1*M(n, Mcolumn+2) / (eta*X1 -1/2 -1) + (eta/2 + ... 
                      1)*M(n, Mcolumn+3) / (eta*X1 -1/2 +1) + (eta + X1*X1 ...
Algorithm 6 (Continued)

\[
\sum_{i=1}^{5} c_i = \frac{792}{80} = 6.1083. \text{ We define the map } w_2 : \mathcal{J} \times \mathcal{B}^2 \rightarrow \mathcal{B} \text{ by }
\]

\[
w_2(t,x(t),y(t),z(t)) = \left| \frac{|t|}{35(1 + |t|)} + \frac{2}{35(4 + \sqrt{t})} x(t) \right|
\]

\[+ \frac{3t}{70} \int_0^t \frac{y(r)}{\sqrt{r + 1}} dr + \frac{3t}{35(t^3 + 2)} c \mathcal{D}^{1}_{\|} [z](t)\]

for all \( t \in \mathcal{J} \). Now, we get

\[
\left| w_2 \left( t, x(t), \int_0^t x(r) dr, c \mathcal{D}^{1}_{\|} [x](t) \right) - w_2 \left( t, y(t), \int_0^t y(r) dr, c \mathcal{D}^{1}_{\|} [y](t) \right) \right|
\]

\[
= \left| \frac{|t|}{35(1 + |t|)} + \frac{2}{35(4 + \sqrt{t})} x(t) + \frac{3t}{70} \int_0^t \frac{y(r)}{\sqrt{r + 1}} dr 
\]

\[+ \frac{3t}{35(t^3 + 2)} c \mathcal{D}^{1}_{\|} [x](t) - \left[ \frac{|t|}{35(1 + |t|)} + \frac{2}{35(4 + \sqrt{t})} y(t) 
\]

\[+ \frac{3t}{70} \int_0^t \frac{y(r)}{\sqrt{r + 1}} dr + \frac{3t}{35(t^3 + 2)} c \mathcal{D}^{1}_{\|} [y](t) \right] \right|
\]

\[
\leq \frac{2}{35(4 + \sqrt{t})} \| x - y \| + \frac{3t}{70} \| x - y \| + \frac{3t}{35(t^3 + 2)} \| x - y \|
\]

Thus

\[
\left| w_2 \left( t, x(t), \int_0^t x(r) dr, c \mathcal{D}^{1}_{\|} [x](t) \right) - w_2 \left( t, y(t), \int_0^t y(r) dr, c \mathcal{D}^{1}_{\|} [y](t) \right) \right|
\]

\[
\leq \left[ \frac{2}{35(4 + \sqrt{t})} + \frac{3t}{70} + \frac{3t}{35(t^3 + 2)} \right] \| x - y \|
\]

for all \( t \in \mathcal{J}, x, y \in \mathcal{B} \). We define the positive continuous maps

\[
m_1(t) = \frac{2}{35(4 + \sqrt{t})}, \quad m_2(t) = \frac{3t}{70}, \quad m_3(t) = \frac{3t}{35(t^3 + 2)}. \]

At present, by using Eqs. (12)–(13) and applying Algorithm 5, we calculate \( \sup_{t \in \mathcal{J}} T^\alpha_{\|} [m_i](t) \), \( \sup_{t \in \mathcal{J}} T^{\alpha-1}_{\|} [m_i](1) \), \( \sup_{t \in \mathcal{J}} T^{\alpha-2}_{\|} [m_i](b) \), \( \sup_{t \in \mathcal{J}} T^{\alpha-3}_{\|} [m_i](1) \) for \( i = 1, 2, 3 \). Tables 7, 8, and 9 show these results. Therefore

\[
\sup_{t \in \mathcal{J}} T^\alpha_{\|} [m_1](t) = \sup_{t \in \mathcal{J}} T^\alpha_{\|} \left( \frac{2}{35(4 + \sqrt{t})} \right) = 0.0106, 0.0089, 0.0077,
\]

\[
\sup_{t \in \mathcal{J}} T^{\alpha-1}_{\|} [m_1](1) = \sup_{t \in \mathcal{J}} T^{\alpha-1}_{\|} \left( \frac{2}{175} \right) = 0.0118, 0.0125, 0.0128,
\]

for all \( t \in \mathcal{J} \). Therefore
Table 7 Some numerical results of $I_{q}^{m_{1}}(t)$, $I_{q}^{m_{1}−1}(1)$, $I_{q}^{m_{1}−2}(b)$, and $I_{q}^{m_{1}−\varepsilon}(t)$ in Example 2 for $t \in J$ and $q = \frac{1}{8}, \frac{1}{2}, \frac{6}{7}$

| $n$ | $\sup_{t \in J} I_{q}^{m_{1}}(t)$ | $\sup_{t \in J} I_{q}^{m_{1}−1}(1)$ | $\sup_{t \in J} I_{q}^{m_{1}−2}(b)$ | $\sup_{t \in J} I_{q}^{m_{1}−\varepsilon}(t)$ |
|-----|---------------------------------|---------------------------------|---------------------------------|---------------------------------|
| 1   | $0.00104$                       | $0.00116$                       | $0.00116$                       | $0.00106$                       |
| 2   | $0.00106$                       | $0.00118$                       | $0.00115$                       | $0.00108$                       |
| 3   | $0.00106$                       | $0.00118$                       | $0.00115$                       | $0.00108$                       |
| 4   | $0.00106$                       | $0.00118$                       | $0.00115$                       | $0.00108$                       |
| $q = \frac{1}{2}$ |                               |                                |                                |                                |
| 1   | $0.00058$                       | $0.00095$                       | $0.00119$                       | $0.0062$                       |
| 2   | $0.00073$                       | $0.011$                         | $0.00113$                       | $0.0077$                       |
| 3   | $0.00081$                       | $0.0118$                       | $0.0011$                       | $0.0086$                       |
| 4   | $0.00085$                       | $0.0121$                       | $0.00109$                       | $0.009$                        |
| 5   | $0.00087$                       | $0.0123$                       | $0.00108$                       | $0.0092$                       |
| 6   | $0.00088$                       | $0.0124$                       | $0.00108$                       | $0.0093$                       |
| 7   | $0.00089$                       | $0.0125$                       | $0.00108$                       | $0.0094$                       |
| 8   | $0.00089$                       | $0.0125$                       | $0.00107$                       | $0.0094$                       |
| 9   | $0.00089$                       | $0.0125$                       | $0.00107$                       | $0.0094$                       |
| 10  | $0.00089$                       | $0.0125$                       | $0.00107$                       | $0.0094$                       |
| 11  | $0.00089$                       | $0.0125$                       | $0.00107$                       | $0.0095$                       |
| $q = \frac{6}{7}$ |                               |                                |                                |                                |
| 1   | $0.00099$                       | $0.0041$                       | $0.0147$                       | $0.001$                        |
| 2   | $0.00144$                       | $0.0055$                       | $0.0134$                       | $0.0017$                       |
| 3   | $0.0002$                        | $0.0066$                       | $0.0126$                       | $0.0024$                       |
| ... |                                |                                |                                |                                |
| 11  | $0.00074$                       | $0.0126$                       | $0.0105$                       | $0.0081$                       |
| 12  | $0.00075$                       | $0.0126$                       | $0.0104$                       | $0.0082$                       |
| 13  | $0.00075$                       | $0.0127$                       | $0.0104$                       | $0.0082$                       |
| 14  | $0.00076$                       | $0.0127$                       | $0.0104$                       | $0.0083$                       |
| 15  | $0.00076$                       | $0.0127$                       | $0.0104$                       | $0.0083$                       |
| 16  | $0.00076$                       | $0.0127$                       | $0.0104$                       | $0.0083$                       |
| 17  | $0.00077$                       | $0.0128$                       | $0.0104$                       | $0.0084$                       |
| 18  | $0.00077$                       | $0.0128$                       | $0.0104$                       | $0.0084$                       |
| 19  | $0.00077$                       | $0.0128$                       | $0.0104$                       | $0.0084$                       |
| 20  | $0.00077$                       | $0.0128$                       | $0.0104$                       | $0.0084$                       |
| 21  | $0.00077$                       | $0.0128$                       | $0.0104$                       | $0.0085$                       |
| 22  | $0.00078$                       | $0.0129$                       | $0.0104$                       | $0.0085$                       |

$$\begin{align*}
\sup_{t \in J} I_{q}^{m_{1}−2}(b) &= \sup_{t \in J} I_{q}^{m_{1}−2} \left( \frac{2}{35(4 + \sqrt{7})} \right) = 0.0115, 0.0107, 0.0104, \\
\sup_{t \in J} I_{q}^{m_{1}−\varepsilon}(t) &= \sup_{t \in J} I_{q}^{m_{1}−\frac{\varepsilon}{2}} \left( \frac{2}{35(4 + \sqrt{7})} \right) = 0.0108, 0.0095, 0.0085
\end{align*}$$

for $q = \frac{1}{8}, \frac{1}{2}, \frac{6}{7}$, respectively,

$$\begin{align*}
\sup_{t \in J} I_{q}^{m_{2}}(t) &= \sup_{t \in J} I_{q}^{m_{2}} \left( \frac{3t}{70} \right) = 0.0343, 0.0184, 0.0109, \\
\sup_{t \in J} I_{q}^{m_{1}−1}(1) &= \sup_{t \in J} I_{q}^{m_{1}−1} \left( \frac{3}{70} \right) = 0.0391, 0.0315, 0.0269, \\
\sup_{t \in J} I_{q}^{m_{2}−2}(b) &= \sup_{t \in J} I_{q}^{m_{2}−2} \left( \frac{1}{30} \right) = 0.0357, 0.0367, 0.0374, \\
\sup_{t \in J} I_{q}^{m_{2}−\varepsilon}(t) &= \sup_{t \in J} I_{q}^{m_{2}−\frac{\varepsilon}{2}} \left( \frac{3t}{70} \right) = 0.0349, 0.0198, 0.0123
\end{align*}$$
Table 8 Some numerical results of $I^q \alpha_m(t)$, $I^{q-1} \alpha_m(1)$, $I^{q-2} \alpha_m(b)$, and $I^{q-\zeta} \alpha_m(t)$ in Example 2 for $t \in J$ and $q = \frac{1}{8}, \frac{1}{2}, 6$, respectively.

| $n$ | $\sup_{t \in J} I^q \alpha_m(t)$ | $\sup_{t \in J} I^{q-1} \alpha_m(1)$ | $\sup_{t \in J} I^{q-2} \alpha_m(b)$ | $\sup_{t \in J} I^{q-\zeta} \alpha_m(t)$ |
|-----|---------------------------------|---------------------------------|---------------------------------|---------------------------------|
| $q = \frac{1}{8}$ | 0.0343 | 0.0391 | 0.0357 | 0.0349 |
| 1 | 0.0343 | 0.0391 | 0.0357 | 0.0349 |
| 2 | 0.0343 | 0.0391 | 0.0357 | 0.0349 |
| 3 | 0.0343 | 0.0391 | 0.0357 | 0.0349 |
| $q = \frac{1}{2}$ | 0.0167 | 0.0299 | 0.0373 | 0.018 |
| 1 | 0.0179 | 0.0311 | 0.0368 | 0.0193 |
| 2 | 0.0183 | 0.0314 | 0.0368 | 0.0197 |
| 3 | 0.0184 | 0.0315 | 0.0367 | 0.0198 |
| $q = 6$ | 0.0029 | 0.0144 | 0.0434 | 0.0036 |
| 1 | 0.0045 | 0.018 | 0.0407 | 0.0054 |
| 2 | 0.0058 | 0.0206 | 0.0394 | 0.0069 |
| $q = \frac{6}{7}$ | 0.0107 | 0.0267 | 0.0375 | 0.0122 |
| 1 | 0.0108 | 0.0268 | 0.0374 | 0.0122 |
| 2 | 0.0108 | 0.0268 | 0.0374 | 0.0123 |
| 3 | 0.0109 | 0.0269 | 0.0374 | 0.0123 |
| 4 | 0.0109 | 0.0269 | 0.0374 | 0.0124 |
| 5 | 0.0109 | 0.0269 | 0.0374 | 0.0124 |

For $q = \frac{1}{8}, \frac{1}{2}, 6$, respectively, and

$$\sup_{t \in J} I^q \alpha_m(t) = \sup_{t \in J} I^q \alpha_m(1) = \sup_{t \in J} I^q \alpha_m(b) = \sup_{t \in J} I^q \alpha_m(t)$$

we obtain $M_0 = 0.0343$, $M(1) = 0.0391$, $M(b) = 0.0357$, $M(t) = 0.0349$ whenever $q = \frac{1}{8}$, $M_0 = 0.0184$, $M(1) = 0.0315$, $M(b) = 0.0367$, $M(t) = 0.0198$ whenever $q = \frac{1}{2}$, $M_0 = 0.0110$, $M(1) = 0.0270$, $M(b) = 0.0374$, $M(t) = 0.0125$ whenever $q = 6$. Also, by using Eq. (14), we can calculate values of $\Delta$. Table 10 shows these results. Thus, by using Eq. (14), we have

$$\Delta = M_0 + M(t) + \frac{\Gamma_q(2 - \zeta)}{\Gamma_q(2 - \zeta)} + \frac{\Gamma_q(2 - \frac{1}{q}) + 1}{\Gamma_q(2 - \frac{1}{q})} \times 0.0391$$

$$= 0.0343 + 0.0349 + \frac{\Gamma_q(2 - \frac{1}{8}) + 1}{\Gamma_q(2 - \frac{1}{8})} \times 0.0391$$

$$+ \frac{(1 + \Gamma_q(2 - \frac{1}{8})) \times 6.1083}{\Gamma_q(2 - \frac{1}{8})} \times 0.0357 = 0.5859 < 1$$
Table 9 Some numerical results of $I_{q}^{α}[m_3](t)$, $I_{q}^{α−1}[m_3](1)$, $I_{q}^{α−2}[m_3](b)$, and $I_{q}^{α−ζ}[m_3](t)$ in Example 2 for $t \in J$ and $q = \frac{1}{8}, \frac{1}{2}, \frac{6}{7}$

| $n$ | $\text{sup } I_{q}^{α}[m_3](t)$ | $\text{sup } I_{q}^{α−1}[m_3](1)$ | $\text{sup } I_{q}^{α−2}[m_3](b)$ | $\text{sup } I_{q}^{α−ζ}[m_3](t)$ |
|-----|-------------------------------|-------------------------------|-------------------------------|-------------------------------|
| 1   | 0.0231                        | 0.0262                        | 0.0288                        | 0.0234                        |
| 2   | 0.0231                        | 0.0262                        | 0.0288                        | 0.0234                        |
| q = $\frac{1}{8}$ |                               |                               |                               |                               |
| 1   | 0.0123                        | 0.0213                        | 0.0297                        | 0.0133                        |
| 2   | 0.0136                        | 0.0226                        | 0.0293                        | 0.0146                        |
| 3   | 0.0139                        | 0.0229                        | 0.0292                        | 0.0149                        |
| 4   | 0.0140                        | 0.0223                        | 0.0292                        | 0.0149                        |
| 5   | 0.0140                        | 0.0223                        | 0.0292                        | 0.0150                        |
| 6   | 0.0140                        | 0.0223                        | 0.0292                        | 0.0150                        |
| q = $\frac{1}{2}$ |                               |                               |                               |                               |
| 1   | 0.0021                        | 0.0101                        | 0.0346                        | 0.0026                        |
| 2   | 0.0034                        | 0.0131                        | 0.0322                        | 0.0041                        |
| 3   | 0.0046                        | 0.0154                        | 0.0313                        | 0.0054                        |
| 4   | 0.0057                        | 0.0173                        | 0.0303                        | 0.0066                        |
| q = $\frac{6}{7}$ |                               |                               |                               |                               |
| 1   | 0.0093                        | 0.0213                        | 0.0290                        | 0.0105                        |
| 2   | 0.0094                        | 0.0213                        | 0.0290                        | 0.0106                        |
| 3   | 0.0094                        | 0.0214                        | 0.0290                        | 0.0106                        |
| 4   | 0.0095                        | 0.0214                        | 0.0290                        | 0.0107                        |
| 5   | 0.0095                        | 0.0215                        | 0.0290                        | 0.0107                        |
| 6   | 0.0095                        | 0.0215                        | 0.0290                        | 0.0108                        |
| 7   | 0.0095                        | 0.0215                        | 0.0290                        | 0.0108                        |
| 8   | 0.0095                        | 0.0215                        | 0.0290                        | 0.0108                        |
| 9   | 0.0096                        | 0.0215                        | 0.0290                        | 0.0108                        |
| 10  | 0.0096                        | 0.0215                        | 0.0290                        | 0.0108                        |

Figure 7 2D graphs of $M_0, M(1), M(b), M(t)$ on $t \in J$ for $q = \frac{1}{2}$ in Example 2

whenever $q = \frac{1}{8}$,

\[
\Delta = M_0 + M(t) + \frac{\Gamma_q(2 - \zeta) + 1}{\Gamma_q(2 - \zeta)} M(1) \\
+ \frac{(1 + \Gamma_q(2 - \zeta)) \times 6.1083}{\Gamma_q(2 - \zeta)} M(b)
\]
Table 10 Some numerical results for calculation of $M_0$, $M(1)$, $M(b)$, and $\Delta < 1$ in Example 2 for $q = \frac{1}{5}, \frac{2}{7}, \frac{6}{7}$

| $n$ | $M_0$ | $M(1)$ | $M(b)$ | $M(t)$ | $\Delta$ |
|-----|-------|--------|--------|--------|---------|
| $q = \frac{1}{5}$ | | | | | |
| 1 | 0.0343 | 0.0391 | 0.0357 | 0.0349 | 0.5855 |
| 2 | 0.0343 | 0.0391 | 0.0357 | 0.0349 | 0.5859 |
| 3 | 0.0343 | 0.0391 | 0.0357 | 0.0349 | 0.5859 |
| 4 | 0.0343 | 0.0391 | 0.0357 | 0.0349 | 0.5859 |
| $q = \frac{2}{7}$ | | | | | |
| 1 | 0.0167 | 0.0299 | 0.0373 | 0.018 | 0.5275 |
| 2 | 0.0179 | 0.0311 | 0.0368 | 0.0193 | 0.5426 |
| 3 | 0.0183 | 0.0314 | 0.0368 | 0.0197 | 0.5504 |
| 4 | 0.0184 | 0.0315 | 0.0367 | 0.0198 | 0.5542 |
| 5 | 0.0184 | 0.0315 | 0.0367 | 0.0198 | 0.5561 |
| 6 | 0.0184 | 0.0315 | 0.0367 | 0.0198 | 0.5571 |
| 7 | 0.0184 | 0.0315 | 0.0367 | 0.0198 | 0.5576 |
| 8 | 0.0184 | 0.0315 | 0.0367 | 0.0198 | 0.5578 |
| 9 | 0.0184 | 0.0315 | 0.0367 | 0.0198 | 0.5579 |
| 10 | 0.0184 | 0.0315 | 0.0367 | 0.0198 | 0.5580 |
| 11 | 0.0184 | 0.0315 | 0.0367 | 0.0198 | 0.5580 |
| 12 | 0.0184 | 0.0315 | 0.0367 | 0.0198 | 0.5580 |
| $q = \frac{6}{7}$ | | | | | |
| 1 | 0.0029 | 0.0144 | 0.0373 | 0.0036 | 0.4067 |
| 2 | 0.0045 | 0.018 | 0.0407 | 0.0054 | 0.4166 |
| 3 | 0.0058 | 0.0206 | 0.0394 | 0.0069 | 0.4317 |
| ... | ... | ... | ... | ... | ... |
| 13 | 0.0107 | 0.0267 | 0.0375 | 0.0122 | 0.5218 |
| 14 | 0.0108 | 0.0268 | 0.0372 | 0.0122 | 0.5253 |
| 15 | 0.0108 | 0.0268 | 0.0374 | 0.0123 | 0.5283 |
| 16 | 0.0109 | 0.0269 | 0.0374 | 0.0123 | 0.5308 |
| 17 | 0.0109 | 0.0269 | 0.0374 | 0.0124 | 0.533 |
| 18 | 0.0109 | 0.0269 | 0.0374 | 0.0124 | 0.5348 |
| 19 | 0.0109 | 0.0269 | 0.0374 | 0.0124 | 0.5364 |
| 20 | 0.011 | 0.0269 | 0.0374 | 0.0124 | 0.5378 |
| 21 | 0.011 | 0.0269 | 0.0374 | 0.0124 | 0.5389 |
| 22 | 0.011 | 0.027 | 0.0374 | 0.0125 | 0.5399 |
| 23 | 0.011 | 0.027 | 0.0374 | 0.0125 | 0.5408 |
| 24 | 0.011 | 0.027 | 0.0374 | 0.0125 | 0.5415 |
| ... | ... | ... | ... | ... | ... |
| 53 | 0.011 | 0.027 | 0.0374 | 0.0125 | 0.5457 |
| 54 | 0.011 | 0.027 | 0.0374 | 0.0125 | 0.5458 |
| 55 | 0.011 | 0.027 | 0.0374 | 0.0125 | 0.5458 |

\[
= 0.0184 + 0.0198 + \frac{\Gamma_q(2 - \frac{1}{5}) + 1}{\Gamma_q(2 - \frac{1}{5})} \times 0.0315 \\
+ \frac{(1 + \Gamma_q(2 - \frac{1}{5})) \times 6.1083}{\Gamma_q(2 - \frac{1}{5})} \times 0.0367 = 0.5580 < 1
\]

whenever $q = \frac{1}{5}$, and

\[
\Delta = M_0 + M(t) + \frac{\Gamma_q(2 - \zeta) + 1}{\Gamma_q(2 - \zeta)}M(1) + \frac{(1 + \Gamma_q(2 - \zeta)) \times M(b)}{\Gamma_q(2 - \zeta)}
\]

\[
= 0.0110 + 0.0125 + \frac{\Gamma_q(2 - \frac{1}{5}) + 1}{\Gamma_q(2 - \frac{1}{5})} \times 0.0270
\]
whenever $q = \frac{6}{7}$. Figures 7, 8, and 9 show these results (Algorithm 7). Now, by using Theorem 5, the fractional $q$-integro-differential equation under sum boundary value conditions (16) has a unique solution.

5 Conclusion
The $q$-integro-differential boundary equations and their applications represent a matter of high interest in the area of fractional $q$-calculus and its applications in various areas.
Algorithm 7 The MATLAB lines for calculation of all parameters in Example 2

```matlab
function [A1, A2, A3] = calculate_parameters(q, alpha, beta, t, n, m)

% MATLAB code for calculating parameters

% Initialize parameters

% Define parameters

% Calculate parameters

% Output parameters

end
```
Algorithm 7 (Continued)

| Line | Description |
|------|-------------|
| 77   | `if A3(n,Acolumn+3)>M1` |
| 78   | `M1=A3(n,Acolumn+3);` |
| 79   | `end;` |
| 80   | `M(n,column+1)=M1;` |
| 81   | `M2=A1(n,Acolumn+4);` |
| 82   | `if A2(n,Acolumn+6)>M2` |
| 83   | `M2=A2(n,Acolumn+4);` |
| 84   | `end;` |
| 85   | `if A3(n,Acolumn+6)>M2` |
| 86   | `M2=A3(n,Acolumn+4);` |
| 87   | `end;` |
| 88   | `M(n,column+2)=M2;` |
| 89   | `M3=A1(n,Acolumn+5);` |
| 90   | `if A2(n,Acolumn+5)>M3` |
| 91   | `M3=A2(n,Acolumn+5);` |
| 92   | `end;` |
| 93   | `if A3(n,Acolumn+5)>M3` |
| 94   | `M3=A3(n,Acolumn+5);` |
| 95   | `end;` |
| 96   | `M(n,column+3)=M3;` |
| 97   | `M4=A1(n,Acolumn+8);` |
| 98   | `if A2(n,Acolumn+8)>M4` |
| 99   | `M4=A2(n,Acolumn+8);` |
| 100  | `end;` |
| 101  | `if A3(n,Acolumn+8)>M4` |
| 102  | `M4=A3(n,Acolumn+8);` |
| 103  | `end;` |
| 104  | `M(n,column+4)=M4;` |
| 105  | `column=column+5;` |
| 106  | `Acolumn=Acolumn+3;` |
| 107  | `end;` |
| 108  | `column=1;` |
| 109  | `Mcolumn=1;` |
| 110  | `for j=1:yq` |
| 111  | `for n=1:80` |
| 112  | `Delta(n,column)=n;` |
| 113  | `G=qGamma(q(1), 2-zeta, n);` |
| 114  | `Delta(n,column+1)=M(n, Mcolumn+1) + M(n, Mcolumn+4) + ...` |
| 115  | `(G+1) * M(n, Mcolumn+2) / (1+G) * Xi * M(n, Mcolumn+3) / G;` |
| 116  | `end;` |
| 117  | `column=column+2;` |
| 118  | `Mcolumn=Mcolumn+5;` |
| 119  | `end;` |

of science and technology. q-integro-differential boundary value problems occur in the mathematical modeling of a variety of physical operations. The end of this article is to investigate a complicated case by utilizing an appropriate basic theory. In this manner, we prove the existence of a solution for two new q-integro-differential equations under sum and integral boundary conditions (1)–(2) and (3)–(4) on a time scale and show the perfect numerical effects for the problem which confirmed our results.

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Authors’ contributions
The authors declare that the study was realized in collaboration with equal responsibility. All authors read and approved the final manuscript.

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