A graph-based equilibrium problem for the limiting distribution of non-intersecting Brownian motions at low temperature

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Abstract

We consider \( n \) non-intersecting Brownian motion paths with \( p \) prescribed starting positions at time \( t = 0 \) and \( q \) prescribed ending positions at time \( t = 1 \). The positions of the paths at any intermediate time are a determinantal point process, which in the case \( p = 1 \) is equivalent to the eigenvalue distribution of a random matrix from the Gaussian unitary ensemble with external source. For general \( p \) and \( q \), we show that if a temperature parameter is sufficiently small, then the distribution of the Brownian paths is characterized in the large \( n \) limit by a vector equilibrium problem with an interaction matrix that is based on a bipartite planar graph. Our proof is based on a steepest descent analysis of an associated \((p + q) \times (p + q)\) matrix valued Riemann-Hilbert problem whose solution is built out of multiple orthogonal polynomials. A new feature of the steepest descent analysis is a systematic opening of a large number of global lenses.

Keywords: non-intersecting Brownian motions, Karlin-McGregor theorem, vector potential theory, graph theory, multiple orthogonal polynomials, Riemann-Hilbert problem, Deift-Zhou steepest descent analysis.

1 Introduction

This paper deals with non-intersecting one-dimensional Brownian motions with prescribed starting and ending positions. This model has already been discussed in various regimes. For the case of one starting point and one ending point it is known that the positions of the paths at any intermediate time have the same distribution (up to trivial scaling) as the eigenvalues of a Gaussian Unitary Ensemble from random matrix theory [18]. Moreover, as the number of paths tends to infinity and after appropriate scaling, the paths fill out an ellipse in the \( tx \)-plane, see Figure 1.

In the case of one starting point and two or more ending points the positions of the paths have the same distribution as the eigenvalues of a Gaussian Unitary Ensemble with external source. This model is described by multiple Hermite polynomials. As the number of paths tends to infinity, the paths fill out a more complicated region whose boundary has cusp points. The limiting distributions can be computed in terms of an algebraic curve known as Pastur’s equation \[3, 5, 6, 24, 28, 29\]. See Figure 2 for an illustration of the case of two ending points.

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Figure 1: Non-intersecting Brownian motions with one starting and one ending point. As the number of paths tends to infinity, the paths fill out an ellipse in the $tx$-plane.

Figure 2: Non-intersecting Brownian motions with one starting and two ending positions.

The case of one ending point and two or more starting points is equivalent due to the time reversal symmetry in the model.

Much less is known for the general case of $p \geq 2$ starting points and $q \geq 2$ ending points. For example, it is not known whether there exists an underlying random matrix model for this case. What is known is that the model is described by multiple Hermite polynomials of mixed type [9] which have a characterization in terms of a $(p + q) \times (p + q)$ matrix valued Riemann Hilbert problem. Calculations for the limiting distributions of paths in the large $n$ limit were done for very specific cases with $p = q = 2$ in [10, 16] based on the spectral curve (analogue of the Pastur’s equation) that could be computed in these special cases, see also the related works [1, 25].

It is the goal of this paper to study the case of general $p \geq 2$ and $q \geq 2$ with methods from potential theory, more precisely with vector equilibrium problems with external fields.

Equilibrium problems with external fields were developed in approximation theory in the context of orthogonal polynomials, Padé approximation, and polynomial approximation with varying weights, see [30, 31]. They are also a powerful tool in the study of unitary random matrix ensembles [11, 12]. Vector equilibrium problems were studied in the context of Hermite-Padé approximation and the associated multiple orthogonal polynomials [2, 27].
Figure 3: Non-intersecting Brownian motions with two starting and two ending positions. The starting and ending positions are sufficiently far apart so that around time $t = 1/2$, there are three groups of paths in the large $n$ limit.

For $p = q = 2$ we are considering a situation such as the one shown in Figure 3, where a certain fraction of the paths starts in each of the two starting points, and ends at each of the two ending points. As the number of paths increases we see the following situation. For small time the paths are in two separate groups that emanate from the two starting positions. At a certain time one of the groups splits into two, leading to a situation of three separate groups of paths. Then at a later time two of the groups come together and we end up with two groups that end at the two ending points.

Our results will deal with the situation at times where there are three groups of paths, or for general $p$ and $q$, where there are the maximal number (namely $p + q - 1$) of groups of paths.

There is an alternative possible scenario in which the two groups of paths first merge into one group and later split again into two groups of paths. This will happen if the starting and ending positions are sufficiently close to each other. The first scenario happens if the starting and ending positions are relatively far away from each other. Below we will actually distinguish the two scenarios in terms of a temperature parameter $T$ so that for small $T$ we have the situation with the three groups of paths.

2 Statement of results

2.1 Assumptions

Let $p \geq 2$ and $q \geq 2$. We fix $p$ starting points $a_1, \ldots, a_p$ which we assume to be ordered as

$$a_1 > a_2 > \cdots > a_p,$$

and $q$ ending points $b_1, \ldots, b_q$ with

$$b_1 > b_2 > \cdots > b_q.$$

For a given (large) $n$ we consider $n$ non-intersecting Brownian motion paths and we assume that $n_k$ of the paths start at $a_k$ and that $m_l$ of the paths end at $b_l$ for $k = 1, \ldots, p$ and $l = 1, \ldots, q$. Thus

$$\sum_{k=1}^{p} n_k = \sum_{l=1}^{q} m_l = n.$$
Since the paths are non-intersecting, the numbers \( n_k \) and \( m_l \) also determine for each \( k = 1, \ldots, p \) and \( l = 1, \ldots, q \), the number \( n_{k,l} \) of paths that start at \( a_k \) and end at \( b_l \). We call the fractions

\[
    t_{k,l}^{(n)} = \frac{n_{k,l}}{n}
\]  

(2.3)

the finite \( n \) transition numbers. Note that

\[
    t_{k,l}^{(n)} \geq 0, \quad \sum_{k=1}^{p} \sum_{l=1}^{q} t_{k,l}^{(n)} = 1. 
\]  

(2.4)

As \( n \to \infty \), we assume that the finite \( n \) transition numbers have limits

\[
    t_{k,l} = \lim_{n \to \infty} t_{k,l}^{(n)}
\]  

(2.5)

which are the limiting transition numbers. It is convenient to arrange the (finite \( n \) and limiting) transition numbers into \( p \times q \) matrices

\[
    \left( t_{k,l}^{(n)} \right)_{k=1, \ldots, p; l=1, \ldots, q}, \quad \left( t_{k,l} \right)_{k=1, \ldots, p; l=1, \ldots, q}. 
\]

To avoid degenerate cases, we assume that each row and column of the matrix \( (t_{k,l})_{k=1, \ldots, p; l=1, \ldots, q} \) has at least one non-zero entry.

The assumption that the paths are non-intersecting puts a number of constraints on the numbers \( n_{k,l} \) and on the limiting transition numbers \( t_{k,l} \). Indeed, not all \( a_k \) can be connected to all \( b_l \) and certain transition numbers must be zero. The constraints on the transition numbers are easy to visualize in terms of a weighted bipartite graph

\[ G = (V, E, t), \]  

(2.6)

with vertices

\[ V = \{a_1, \ldots, a_p\} \uplus \{b_1, \ldots, b_q\}, \quad \text{(disjoint union)}, \]

edges

\[ E = \{(a_k, b_l) \in V \times V \mid t_{k,l} > 0\}, \]

and a weight function

\[ t : E \to (0, 1] : (a_k, b_l) \mapsto t_{k,l}. \]  

(2.7)

**Example 1.** The graph \( G = (V, E, t) \) associated with Figure 3 is shown in Figure 4. The graph has four vertices and three edges, each of them with weight \( 1/3 \).

\[
\begin{align*}
    a_1 & \quad \quad \frac{1}{3} \quad \quad b_1 \\
    a_2 & \quad \quad \frac{1}{3} \quad \quad b_2 \\
\end{align*}
\]

Figure 4: The graph associated with Figure 3.
Example 2. For a more complicated example we consider a situation with $p = 2$ starting points and $q = 4$ ending points as in Figure 5.

The matrix of transition numbers is

$$
(t_{k,l})_{k=1,2, l=1,\ldots, 4} = \begin{pmatrix}
\frac{4}{30} & \frac{4}{30} & 0 & 0 \\
0 & \frac{4}{30} & \frac{7}{30} & \frac{11}{30}
\end{pmatrix}
$$

(2.8)

and the graph $G$ associated with (2.8) is shown in Figure 6.

Figure 6: The graph associated with the transition numbers (2.8).

The constraints on the transition numbers are contained in the following obvious result that we state without proof.

**Proposition 2.1.** The graph $G$ has the following properties:

(a) $G$ has at most $p+q-1$ edges. For each $i = 1, \ldots, p+q-1$, there is at most one non-vanishing transition number $t_{k,l}$ with $k + l - 1 = i$.

(b) $G$ is a connected graph if and only if the number of edges is equal to $p+q-1$.

(c) $G$ has no cycles (and so $G$ is a tree if $G$ is connected).

In [10, 16] the special case of transition numbers

$$
(t_{k,l})_{k=1,2, l=1,2} = \begin{pmatrix}
\frac{1}{2} & 0 \\
0 & \frac{1}{2}
\end{pmatrix}
$$

In [10, 16] the special case of transition numbers

$$
(t_{k,l})_{k=1,2, l=1,2} = \begin{pmatrix}
\frac{1}{2} & 0 \\
0 & \frac{1}{2}
\end{pmatrix}
$$
was considered. This example leads to a non-connected graph.

In the case of a connected graph the structure of the matrix of transition numbers \((t_{k,l})\) is easy to describe.

**Proposition 2.2.** Suppose that the graph \(G\) is connected. Then the non-zero entries of the matrix \((t_{k,l})_{k,l}\) are situated on a right-down path starting at the top left entry \((1,1)\) and ending at the bottom right entry \((p,q)\). The steps in the path are either by one unit to the right (a right step) or one unit down (a down step).

In Examples 1 and 2 we have

\[
\begin{pmatrix}
\times & 0 \\
\times & \times
\end{pmatrix}, \quad \begin{pmatrix}
\times & \times & 0 & 0 \\
0 & \times & \times & \times
\end{pmatrix},
\]

respectively.

In this paper we consider only connected graphs, and we will make the following assumption.

**Assumption 2.3.** We assume that the graph \(G\) is connected. That is, we assume that \(G\) has \(p+q-1\) edges, and for every \(i = 1, \ldots, p+q-1\) there is exactly one non-vanishing transition number \(t_{k,l}\) with \(k+l-1 = i\), and we define

\[
k(i) = k, \quad l(i) = l, \quad \text{if} \quad t_{k,l} > 0, \quad \text{and} \quad k + l - 1 = i.
\]

It follows from the assumption and from \((2.5)\) that also for large enough \(n\), the finite \(n\) transition numbers have the same non-zero pattern. Thus \(t^{(n)}_{k,l} > 0\) if and only if \(t_{k,l} > 0\).

In what follows we follow the convention that \(i\) labels the edges \(E\) of the graph, and so we identify \(i\) with the edge \((a_{k(i)}, b_{l(i)})\) of the graph.

### 2.2 Non-intersecting Brownian motions

We consider Brownian motions having transition probability density

\[
P(t, x, y) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-y)^2}
\]

whose overall variance

\[
\sigma^2 = \frac{T}{n}
\]

is proportional to \(1/n\) where \(n\) is the number of Brownian paths. We interpret the proportionality constant \(T > 0\) as a temperature variable.

In this paper we consider small temperature \(T\). We show that for small \(T\) the paths at time \(t\) have a limiting mean distribution that is characterized by a vector equilibrium problem. In the first theorem we state the existence of a limiting mean distribution.

**Theorem 2.4.** Consider \(n\) independent Brownian motions with transition probability \((2.9)\) – \((2.10)\) conditioned so that

- the paths are non-intersecting in the time interval \((0,1)\),
- \(n_{k,l}\) of the paths start at \(a_k\) and end at \(b_l\), for each \(k = 1, \ldots, p\) and \(l = 1, \ldots, q\).
Assume that as \( n \to \infty \), the finite \( n \) transition numbers \( n_{k,l}/n \) converge to \( t_{k,l} \), and that the corresponding graph \( G \) is connected.

Let \( t \in (0,1) \). Then there exists a \( T^* = T^*(t) > 0 \) so that for all \( T \in (0,T^*) \) the limiting mean distribution of the positions of the paths at time \( t \) exists, and is supported on the union of \( p + q - 1 \) disjoint intervals \( \bigcup_{i=1}^{p+q-1} [\alpha_i, \beta_i] \) with a density \( \rho_i \) on the \( i \)th interval.

The vector of measures \( (\mu_1, \ldots, \mu_{p+q-1}) \) where \( d\mu_i(x) = \rho_i(x)\, dx \) for \( i = 1, \ldots, p + q - 1 \) is the minimizer of a vector equilibrium problem that will be described in the next subsection.

The proof of Theorem 2.4 will be based on a Deift-Zhou steepest descent analysis of the Riemann-Hilbert problem in Section 2.5. The details of the steepest descent analysis will be described in Sections 5–7, and the proof of Theorem 2.4 will then be given in Section 8.

Remark 2.5. The special case where \( p = 1, q = 2 \) and \( m_1 = m_2 = n/2 \) was studied in \([3, 5, 6]\). In \([5]\) the energy functional (2.15) is given for this special case, but it was not used in the further analysis. Instead the results in \([5]\) were stated and proved in terms of an algebraic curve (Pastur’s equation).

Remark 2.6. As already mentioned, for the case \( p = 1 \) and \( q \geq 2 \) the non-intersecting Brownian motions are distributed like the eigenvalues of the Gaussian unitary ensemble with external source.

There are more general random matrix ensembles with external source, which however do not have an equivalent interpretation in terms of non-intersecting paths. The case of a quartic potential was studied in \([25]\). An analogue of Theorem 2.4 is valid in this case as well, provided that a suitable analogue of the temperature \( T \) is sufficiently small.

Note that Theorem 2.4 gives only a result for \( T \) sufficiently small, but it does not specify how small \( T \) should be. We expect that the theorem remains valid for all temperatures \( T \) that are such that at time \( t \) we have the maximal number of groups of paths. At a critical temperature \( T_{crit}(t) \) we expect that two (or maybe more) neighboring intervals \( [\alpha_i, \beta_i] \) and \( [\alpha_{i+1}, \beta_{i+1}] \) merge and a Pearcey phase transition occurs. However we were unable to prove this.

### 2.3 Equilibrium problem

The logarithmic energy of a measure \( \mu \) on \( \mathbb{R} \) is defined as usual by

\[
I(\mu) = \iint \log \frac{1}{|x-y|} \, d\mu(x) \, d\mu(y).
\]  (2.11)

The mutual energy of two measures \( \mu, \nu \) is defined by

\[
I(\mu, \nu) = \iint \log \frac{1}{|x-y|} \, d\mu(x) \, d\nu(y).
\]  (2.12)

We write for \( i = 1, \ldots, p + q - 1 \),

\[
x_i(t) = (1-t)a_{k(i)} + tb_{l(i)} , \quad 0 < t < 1
\]  (2.13)

and given \( t \in (0,1) \) we define the quadratic functions

\[
V_i(x) = \frac{1}{2t(1-t)} (x - x_i(t))^2 , \quad x \in \mathbb{R}
\]  (2.14)

for \( i = 1, \ldots, p+q-1 \). These functions will play the role of external fields in the vector equilibrium problem that is relevant for our problem.
Definition 2.7. Fix $t \in (0, 1)$ and let $T > 0$. Consider the energy functional

$$E(\mu_1, \ldots, \mu_{p+q-1}) = \sum_{i,j=1}^{p+q-1} a_{i,j} I(\mu_i, \mu_j) + \frac{1}{T} \sum_{i=1}^{p+q-1} \int V_i(x) \, d\mu_i(x), \quad (2.15)$$

where $V_i$ is defined in (2.14), and the interaction matrix $A = (a_{i,j})$ has entries

$$a_{i,j} = \begin{cases} 
1 & \text{if } i = j, \\
\frac{1}{2} & \text{if } i \neq j \text{ and } k(i) = k(j) \text{ or } l(i) = l(j), \\
0 & \text{otherwise}.
\end{cases} \quad (2.16)$$

Then the vector equilibrium problem consists in minimizing the energy functional (2.15) over all vectors of positive measures $(\mu_1, \ldots, \mu_{p+q-1})$ supported on the real line for which

$$\int d\mu_i = t_{k(i), l(i)}, \quad \text{for } i = 1, \ldots, p + q - 1. \quad (2.17)$$

One may understand the energy minimization problem in Definition 2.7 as follows. To each of the edges of the graph $G = (V, E, t)$ we associate a measure $\mu_i$, $i = 1, \ldots, p + q - 1$, of total mass equal to the weight $t_{k(i), l(i)}$ of that edge. This measure represents a distribution of charged particles on the real line that repel each other due to the diagonal term $a_{i,i} I(\mu_i, \mu_i)$ in (2.15). For the particles of different measures $\mu_i$, $\mu_j$, $i \neq j$, there are two possibilities. The first case is when the $(i, j)$ entry of (2.16) equals $1/2$. This happens if the edges corresponding to $i$ and $j$ are adjacent in the graph $G$. Then there is repulsion between the measures $\mu_i$ and $\mu_j$ but with a strength that is only half as strong as the repulsion for each individual measure. The second case is when the $(i, j)$ entry of (2.16) equals zero. In that case there is no direct interaction between the measures $\mu_i$ and $\mu_j$.

The last term of (2.15) is a sum of external field terms due to the action of the external field $\frac{1}{T} V_i(x)$ on the measure $\mu_i$. The energy minimizer $(\mu_1, \ldots, \mu_{p+q-1})$ in Definition 2.7 then corresponds to the equilibrium distribution of charged particles under the energy functional (2.15).

Proposition 2.8. The interaction matrix $A$ is positive definite.

Proof. It is easy to check that the interaction matrix $A$ is equal to

$$A = \frac{1}{2} B^T B \quad (2.18)$$

where $B$ is the incidence matrix of the graph $G$. That is, we choose a numbering $k = 1, \ldots, p + q$ of the vertices, and then we have

$$B = (b_{k,i})_{k=1,\ldots,p+q, i=1,\ldots,p+q-1}
$$

where $b_{k,i} = 1$ if vertex $k$ is incident to edge $i$, and 0 otherwise.

From (2.18) we get that $A$ is positive semi-definite, and for any column vector $x$ of length $p + q - 1$ we have

$$x^T A x = \frac{1}{2} \|Bx\|^2 \geq 0. \quad (2.19)$$
Now assume that $Bx = 0$. Consider a leaf of $G$, i.e., a vertex which is incident to exactly one edge. Then $B$ has exactly one zero in the row corresponding to this vertex, and from $Bx = 0$ it follows that the component of $x$ corresponding to the edge that is incident to the leaf vanishes. Since $G$ is a tree (see Proposition 2.1(c)) we can then gradually undress $G$ by peeling off leaves one by one and we conclude in this way that all components of $x$ are equal to 0. Thus $x = 0$ if $Bx = 0$, which implies in view of (2.19) that $A$ is positive definite.

Corollary 2.9. The vector equilibrium problem of Definition 2.7 has a unique solution $(\mu_1, \ldots, \mu_{p+q-1})$ and each measure $\mu_i$ is compactly supported.

Proof. The interaction matrix is positive definite by Proposition 2.8. The external fields $V_i$ in the energy functional (2.15)–(2.16) have enough increase at $\pm \infty$ so that standard arguments of potential theory as in [11, 27, 30] can be used to establish the existence and uniqueness of the minimizer as well as the fact that each measure $\mu_i$ is supported on a compact set.

Our next theorem describes the structure of the solution of the vector equilibrium problem for small $T$.

Theorem 2.10. Fix $t \in (0, 1)$, and let $(t_{k,l})$ be a matrix of transition numbers. Then there exists $T^* > 0$ (the same $T^*$ that makes Theorem 2.4 work) so that for every $T \in (0, T^*)$ the following holds.

(a) Each $\mu_i$ is supported on an interval

$$\text{supp}(\mu_i) = [\alpha_i, \beta_i], \quad i = 1, \ldots, p + q - 1.$$  

The intervals $[\alpha_i, \beta_i]$ are pairwise disjoint and satisfy

$$\beta_{i+1} < \alpha_i, \quad i = 1, \ldots, p + q - 2.$$  

(b) The measure $\mu_i$ has a density $\rho_i$ with respect to Lebesgue measure which is real analytic and positive in the open interval $(\alpha_i, \beta_i)$ and vanishes like a square root at the endpoints of $[\alpha_i, \beta_i]$, i.e., there exist non-zero constants $\rho_i^{(1)}$ and $\rho_i^{(2)}$ such that

$$\rho_i(x) = \rho_i^{(1)} \sqrt{x - \alpha_i} + O((x - \alpha_i)^{3/2}) \quad \text{as } x \downarrow \alpha_i,$$  

$$\rho_i(x) = \rho_i^{(2)} \sqrt{\beta_i - x} + O((\beta_i - x)^{3/2}) \quad \text{as } x \uparrow \beta_i.$$  

The proof of Theorem 2.10 will be given in Section 3.1.

2.4 Special cases

In Subsections 2.4.1–2.4.3 our main Theorem 2.4 will be illustrated for some special cases.

2.4.1 The case $p = 1$: Angelesco-type interaction

For the case $p = 1$ of one starting point, and an arbitrary number $q$ of ending points, the graph $G$ has a single vertex $a_1$ on the left which is connected to each of the vertices $b_1, \ldots, b_q$ on the right. For example, if $q = 3$ the graph has the form shown in Figure 7.

The energy functional (2.15)–(2.16) is then equal to

$$E(\mu_1, \ldots, \mu_{p+q-1}) := \sum_{i=1}^{p+q-1} I(\mu_i) + \frac{1}{2} \sum_{i \neq j} I(\mu_i, \mu_j) + \frac{1}{T} \sum_{i=1}^{p+q-1} \int V_i(x) \, d\mu_i(x).$$  

(2.23)
Figure 7: A graph $G$ with $p = 1$ starting points and $q = 3$ ending points.

The functional (2.23) is exactly the one familiar from the theory of Angelesco systems [21]. All off-diagonal entries in the interaction matrix $A$ in (2.16) are equal to $1/2$. For the example in Figure 7 the interaction matrix is

$$A = \begin{pmatrix}
1 & 1/2 & 1/2 \\
1/2 & 1 & 1/2 \\
1/2 & 1/2 & 1
\end{pmatrix}. \quad (2.24)$$

2.4.2 The ‘zigzag’ case: nearest neighbor interaction

Next we consider the case where $p = q$ and the corresponding lattice path follows a zigzag line. The graph $G$ is then just a chain of vertices: see Figure 8.

Figure 8: For $p = q = 3$, Figure 8(a) shows a graph $G$ which has zigzag form. Figure 8(b) shows the same graph written as a chain.

The energy functional (2.15) now takes the form

$$E(\mu_1, \ldots, \mu_{p+q-1}) := \sum_{i=1}^{p+q-1} I(\mu_i) + \sum_{i=1}^{p+q-2} I(\mu_i, \mu_{i+1}) + \frac{1}{T} \sum_{i=1}^{p+q-1} \int V_i(x) \, d\mu_i(x). \quad (2.25)$$

We see that the interaction matrix of (2.25) is tridiagonal with diagonal entries equal to 1, and entries on the first sub- and superdiagonal equal to $1/2$. For the example in Figure 8 the
interaction matrix equals

\[
A = \begin{pmatrix}
1 & 1/2 & 0 & 0 & 0 \\
1/2 & 1 & 1/2 & 0 & 0 \\
0 & 1/2 & 1 & 1/2 & 0 \\
0 & 0 & 1/2 & 1 & 1/2 \\
0 & 0 & 0 & 1/2 & 1
\end{pmatrix}.
\] (2.26)

Note that the tridiagonal structure of the interaction matrix means that there is only nearest neighbor interaction. The neighboring measures repel each other, since all signs in the interaction matrix are positive.

In a Nikishin system the interaction matrix is also tridiagonal, but the off-diagonal entries are \(-1/2\) instead of \(1/2\), see [21].

### 2.4.3 The general case: block nearest neighbor interaction with Angelesco-type blocks

Finally we consider the graph \(G\) in Figure 6. In this case the interaction matrix equals

\[
A = \begin{pmatrix}
1 & 1/2 & 0 & 0 & 0 \\
1/2 & 1 & 1/2 & 0 & 0 \\
0 & 1/2 & 1 & 1/2 & 1/2 \\
0 & 0 & 1/2 & 1 & 1/2 \\
0 & 0 & 1/2 & 1/2 & 1
\end{pmatrix}.
\] (2.27)

Note that this interaction matrix is a mixture of the nearest neighbor and Angelesco interaction matrices. More precisely one could say that \((2.27)\) has ‘block’ nearest neighbor interaction, where each of the blocks in turn has an Angelesco-type interaction, and with subsequent blocks intersecting in exactly one entry. For the matrix in \((2.27)\) these building blocks are

\[
\begin{pmatrix}
1 & 1/2 \\
1/2 & 1
\end{pmatrix}, \quad \begin{pmatrix}
1 & 1/2 \\
1/2 & 1
\end{pmatrix}, \quad \begin{pmatrix}
1 & 1/2 & 1/2 \\
1/2 & 1 & 1/2 \\
1/2 & 1/2 & 1
\end{pmatrix}.
\]

### 2.4.4 Comparison with [21]

A graph-based vector equilibrium problem was also considered in the work of Gonchar, Rakhmanov, and Sorokin [21]. The rule for building the interaction matrix from a graph is similar to the one in this paper. The vector equilibrium problem is also labeled by the edges of a graph, which in [21], however, is a directed rooted tree. The off-diagonal entries of the interaction matrix are non-zero if the corresponding two edges have a common vertex. The entry is \(1/2\) if the two edges have a common initial vertex, and \(-1/2\) if the final vertex of one edge agrees with the initial vertex of the other.

In our case, we only have \(1/2\) since the latter situation cannot happen.

### 2.5 About the proof of Theorem 2.4

The proof of Theorem 2.4 uses the connection of non-intersecting Brownian motions with prescribed starting and ending points with a determinantal point process and an associated Riemann-Hilbert problem. We recall this connection.
2.5.1 Determinantal point process

The positions at time \( t \in (0, 1) \) of \( n \) non-intersecting Brownian motions, starting at distinct \( a_j \), \( j = 1, \ldots, n \), and ending at distinct positions \( b_j \), \( j = 1, \ldots, n \), have the joint probability density function

\[
\frac{1}{Z} \det (P(t, a_i, x_j))_{i,j=1}^n \cdot \det (P(1 - t, x_i, b_j))_{i,j=1}^n,
\]

with the transition probability density \( P \) defined in (2.9) and with \( Z \) a normalization constant.

This is a consequence of a theorem of Karlin and McGregor [23]. (For applications of the discrete version of the Karlin-McGregor theorem see e.g. [22].) In the confluent limit where \( n_k \) of the starting positions come together at \( a_k \), \( k = 1, \ldots, p \), and \( m_l \) of the ending positions come together at \( b_l \) for \( l = 1, \ldots, q \), the joint p.d.f. for the positions of the paths at time \( t \) can be written as

\[
P(x_1, \ldots, x_n) = \frac{1}{Z} \det (f_i(x_j))_{i,j=1}^n \cdot \det (g_i(x_j))_{i,j=1}^n,
\]

(2.28)

for certain functions \( f_i, g_i \), that are built out of the \( p + q \) functions

\[
w_{1,k}(x) = e^{-\frac{1}{2\sigma^2}(x-a_k)^2}, \quad k = 1, \ldots, p,
\]

(2.29)

\[
w_{2,l}(x) = e^{-\frac{1}{2(1-t)\sigma^2}(x-b_l)^2}, \quad l = 1, \ldots, q,
\]

(2.30)

see e.g. [9].

The p.d.f. (2.28) defines a determinantal point process (in fact a biorthogonal ensemble, see [7]) with a correlation kernel \( K(x, y) \) that is such that

\[
P(x_1, \ldots, x_n) = \frac{1}{n!} \det (K(x_i, x_j))_{i,j=1}^n
\]

(2.31)

and for each \( m = 1, \ldots, n \),

\[
\int \cdots \int P(x_1, \ldots, x_n) \, dx_{m+1} \cdots dx_n = \frac{(n-m)!}{n!} \det (K(x_i, x_j))_{i,j=1}^m.
\]

(2.32)

In particular for \( m = 1 \), we have that

\[
\frac{1}{n} K(x, x)
\]

is the mean density of paths.

2.5.2 Riemann-Hilbert problem

The kernel \( K \) can be described in terms of the following Riemann-Hilbert problem (RH problem) introduced in [9].

**RH problem 2.11.** The RH problem consists in finding a matrix-valued function \( Y : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}^{(p+q) \times (p+q)} \) such that

1. \( Y \) is analytic in \( \mathbb{C} \setminus \mathbb{R} \);
2. For \( x \in \mathbb{R} \), the limiting values

\[
Y_+(x) = \lim_{z \to x, \text{Im} \, z > 0} Y(z), \quad Y_-(x) = \lim_{z \to x, \text{Im} \, z < 0} Y(z)
\]
exist and satisfy
\[ Y_+(x) = Y_-(x) \begin{pmatrix} I_p & W(x) \\ 0 & I_q \end{pmatrix}, \]
(2.33)
where \( I_k \) denotes the identity matrix of size \( k \), and where \( W(x) \) denotes the rank-one matrix (outer product of two vectors)
\[ W(x) = \begin{pmatrix} w_{1,1}(x) \\ \vdots \\ w_{1,p}(x) \end{pmatrix} \begin{pmatrix} w_{2,1}(x) & \cdots & w_{2,q}(x) \end{pmatrix} \]
(2.34)
with \( w_{1,k}(x), k = 1, \ldots, p \), and \( w_{2,l}(x), l = 1, \ldots, q \) given by (2.29) and (2.30).

(3) As \( z \to \infty \), we have that
\[ Y(z) = (I_{p+q} + O(1/z)) \text{diag}(z^{n_1}, \ldots, z^{n_p}, z^{-m_1}, \ldots, z^{-m_q}). \]
(2.35)

The RH problem has a unique solution that can be described in terms of multiple Hermite polynomials. This is shown in [9], generalizing results in [19, 32]. According to [9] the correlation kernel is expressed in terms of the solution to the RH problem as
\[ K(x, y) = \frac{1}{2\pi i(x - y)} \begin{pmatrix} 0 & \cdots & 0 & w_{2,1}(y) & \cdots & w_{2,q}(y) \end{pmatrix} Y_+^{-1}(y) Y_+(x) \begin{pmatrix} w_{1,1}(x) \\ \vdots \\ w_{1,p}(x) \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \]
(2.36)

It is also worth noticing that if \( Y_{1,1}(z) \) denotes the top leftmost \( p \) by \( p \) block of the RH matrix \( Y(z) \), then the determinant of \( Y_{1,1}(z) \) equals the average characteristic polynomial
\[ \det Y_{1,1}(z) = \mathbb{E} \left[ \prod_{j=1}^{n} (z - x_j) \right], \]
where the expectation \( \mathbb{E} \) is taken according to the joint probability density (2.28). This was shown in [4] for the case \( p = 1 \) and in [15] for the case of general \( p \) and \( q \).

2.5.3 Asymptotic analysis
We analyze the RH problem (2.11) in the limit described in Theorem 2.4. That is, we take \( \sigma^2 = T/n \) and we let \( n_k \to \infty \), \( m_l \to \infty \), so that
\[ \frac{n_{k,l}}{n} \to \frac{t_{k,l}}{n} \quad \text{as} \quad n \to \infty. \]

If, for each \( n \), we denote the correlation kernel (2.36) by \( K_n \), then the limiting density of paths is
\[ \rho(x) = \lim_{n \to \infty} \frac{1}{n} K_n(x, x), \quad x \in \mathbb{R}. \]
(2.37)

The proof of Theorem 2.4 will be based on a steepest descent analysis of the RH problem (2.11). As a byproduct of this analysis we can also show that the local scaling limits of the limiting
distribution of the Brownian motions are those familiar from random matrix theory, i.e., they are described in terms of the sine kernel in the bulk and the Airy kernel at the edge. We will not discuss this any further and refer to the papers [5, 6, 10, 28] for a more detailed analysis in a similar context.

2.6 Outline of the rest of the paper

The rest of this paper is organized as follows. In Section 3 we establish general properties of the equilibrium problem, in particular leading to the proof of Theorem 2.10. In Section 4 we define the ξ-functions, λ-functions and the associated Riemann surface. These functions are used in Section 5 to normalize the RH problem at infinity. In Section 6 we apply Gaussian elimination to the RH problem by opening global lenses, thereby making the RH problem locally of size $2 \times 2$. This construction is fully systematic and may have interest in its own right. The remaining steps of the RH steepest descent analysis are described in Section 7. Finally, in Section 8 we prove the main Theorem 2.4.

3 Properties of the equilibrium measures

In this section we study the vector equilibrium problem of Definition 2.7. In particular we prove Theorem 2.10. In particular we prove Theorem 2.10.

3.1 Proof of Theorem 2.10

The main difficulty is in the proof of part (a) of Theorem 2.10. For this we use the following lemma. Recall that $t \in (0, 1)$ is fixed.

**Lemma 3.1.** For every $\varepsilon > 0$ and $\tau > 0$, there exists $T_{\varepsilon} > 0$ so that for every $T \in (0, T_{\varepsilon})$ the following holds.

If $(\mu_1, \ldots, \mu_{p+q-1})$ is the minimizer for the energy functional (2.15) under the normalization (2.17) with

$$\min_{i} t_{k(i), l(i)} \geq \tau > 0,$$

(3.1)

then the support of $\mu_i$ is contained in $[x_i(t) - \varepsilon, x_i(t) + \varepsilon]$ for every $i = 1, \ldots, p + q - 1$.

**Proof.** We are going to prove the following upper and lower bounds for the quantity

$$E^* := E(\mu_1, \ldots, \mu_{p+q-1})$$

(which depends on the chosen normalization (2.17)).

(a) There exist constants $C_1 > 0$ and $C_{2, \delta}$ such that for every $\delta > 0$ and $T > 0$ we have

$$E^* \leq C_{2, \delta} + C_1 \frac{\delta^2}{T}. \quad (3.2)$$

The constant $C_1$ is independent of $\delta$ and $T$, and $C_{2, \delta}$ is independent of $T$.

(b) There exist positive constants $C_3$, and $C_4$, such that for every $T \in (0, 1]$ and every $i = 1, \ldots, p + q - 1$, we have

$$E^* \geq -C_3 + \frac{C_4}{T} (x - x_i(t))^2, \quad \text{for } x \in \text{supp}(\mu_i). \quad (3.3)$$

In addition, all constants $C_1$, $C_{2, \delta}$, $C_3$ and $C_4$ can be taken independently of the normalization (2.17), and only $C_4$ depends on $\tau$.\]
Proof of (a): We may assume that $\delta > 0$ is sufficiently small so that
\[
\min_i (x_i(t) - x_{i+1}(t)) \geq 2\delta.
\]
Then the intervals $[x_i(t) - \delta, x_i(t) + \delta]$ are mutually disjoint.

We fix transition numbers $t_{k(i),l(i)} \geq 0$ so that $\sum_i t_{k(i),l(i)} = 1$. On each of the intervals $[x_i(t) - \delta, x_i(t) + \delta]$ we choose a measure $\lambda_i$ of finite energy with total mass 1 (for example, an appropriately rescaled and centered semi-circle law will do) and we put $\nu_i = t_{k(i),l(i)} \lambda_i$. We choose the measures $\nu_i$ independent of $T$ and so
\[
\sum_{i,j=1}^{p+q-1} a_{i,j} I(\nu_i, \nu_j) \tag{3.4}
\]
does not depend on $T$. The sum (3.4) depends continuously on the numbers $t_{k(i),l(i)}$ and so
\[
C_{2,\delta} = \max \sum_{i,j=1}^{p+q-1} a_{i,j} I(\nu_i, \nu_j) \tag{3.5}
\]
exists, where the maximum is taken over all choices $t_{k(i),l(i)} \geq 0$ with $\sum_i t_{k(i),l(i)} = 1$.

Since $V_i(x) \leq \frac{\delta^2}{2t(1-t)}$ on $[x_i(t) - \delta, x_i(t) + \delta]$, we also have
\[
\int V_i(x) d\nu_i(x) \leq \frac{\delta^2}{2t(1-t)}
\]
and
\[
\sum_{i=1}^{p+q-1} \int V_i(x) d\nu_i(x) \leq C_1 \delta^2 \tag{3.6}
\]
with a constant $C_1$ that is independent of $T$ and $\delta$, and also of the $t_{k(i),l(i)}$. Then part (a) follows from (3.5), (3.6), and the fact that
\[
E^* \leq E(\nu_1, \ldots, \nu_{p+q-1}) = \sum_{i,j=1}^{p+q-1} a_{i,j} I(\nu_i, \nu_j) + \frac{1}{T} \sum_{i=1}^{p+q-1} \int V_i(x) d\nu_i(x).
\]

Proof of (b): For each $i$ we take a measure $\nu_i$ of total mass $\|\nu_i\| = \|\mu_i\|$ and satisfying for some constant $K \geq 1$,
\[
\nu_i \leq K \mu_i, \quad \text{for } i = 1, \ldots, p + q - 1. \tag{3.7}
\]
Then $(1 - \lambda)\mu_i + \lambda \nu_i$ is a positive measure for every $\lambda \in [-1/K, 1]$, and so
\[
E^* \leq E(((1 - \lambda)\mu_1 + \lambda \nu_1, \ldots, (1 - \lambda)\mu_{p+q-1} + \lambda \nu_{p+q-1}) \tag{3.8}
\]
for every $\lambda \in [-1/K, 1]$, with equality for $\lambda = 0$. Thus the $\lambda$-derivative of the right-hand side of (3.8) vanishes for $\lambda = 0$, which leads to
\[
E^* = \sum_{i,j=1}^{p+q-1} a_{i,j} I(\mu_i, \nu_j) + \frac{1}{2T} \sum_{i=1}^{p+q-1} \int V_i(x) \left(d\mu_i(x) + d\nu_i(x)\right). \tag{3.9}
\]
From the elementary inequality $|x - y| \leq \sqrt{x^2 + 1} \sqrt{y^2 + 1}$ it follows that

\[
I(\mu_i, \nu_j) = \int \int \log \frac{1}{|x - y|} d\mu_i(x) d\nu_j(y) \\
\geq -\frac{1}{2} \int \int \log(x^2 + 1) d\mu_i(x) d\nu_j(y) - \frac{1}{2} \int \int \log(y^2 + 1) d\mu_i(x) d\nu_j(y) \\
\geq -\frac{1}{2} \int \log(x^2 + 1) (d\mu_i(x) + d\nu_j(x))
\]

where for the last inequality we used the facts that $\log(x^2 + 1) \geq 0$ and $\|\nu_j\| \leq 1$, $\|\mu_i\| \leq 1$. Since all $a_{i,j} \geq 0$ it then follows from (3.9) that

\[
E^* \geq -\frac{1}{2} \sum_{i,j=1}^{p+q-1} a_{i,j} \int \log(x^2 + 1) (d\mu_i(x) + d\nu_j(x)) + \frac{1}{2T} \sum_{i=1}^{p+q-1} \int V_i(x) (d\mu_i(x) + d\nu_i(x))
\]

\[
= \frac{1}{2T} \sum_{i=1}^{p+q-1} \int \left( V_i(x) - T \sum_{j=1}^{p+q-1} a_{i,j} \log(x^2 + 1) \right) (d\mu_i(x) + d\nu_i(x)). \tag{3.10}
\]

For $i = 1, \ldots, p + q - 1$ and $T \leq 1$, the functions $V_i(x) - T \sum_{j=1}^{p+q-1} a_{i,j} \log(x^2 + 1)$ are bounded from below and assume their minimum value in a fixed compact interval (independent of $i$ and $T \leq 1$). This is clear from the explicit form (2.14) of the functions $V_i$, and from the fact that $\sum_{j=1}^{p+q-1} a_{i,j} \leq \frac{p+q}{2}$ for each $i$. Thus

\[
\left. V_i(x) - T \sum_{j=1}^{p+q-1} a_{i,j} \log(x^2 + 1) \right\} \geq \frac{1}{2} V_i(x) - C_0 T \geq -C_0 T, \quad x \in \mathbb{R},
\]

for some constant $C_0 > 0$ independent of $T \leq 1$ and $i$.

Using this in (3.10) we find that for each $i = 1, \ldots, p + q - 1$,

\[
E^* \geq -C_3 + \frac{1}{4T} \int V_i(x) d\nu_i(x), \tag{3.11}
\]

where $C_3 > 0$ is a constant independent of $i$ and $T \leq 1$. The inequality (3.11) (with the same constant $C_3$) holds for all measures $\nu_i$ with $\|\nu_i\| = t_{k(i),l(i)}$ and satisfying (3.7) for some $K$. For every $x$ in the support of $\mu_i$, we can approximate the point mass $\|\mu_i\| \delta_x$ by such $\nu_i$. It thus follows from (3.11) that

\[
E^* \geq -C_3 + \frac{1}{4T} V_i(x) t_{k(i),l(i)}, \quad \text{ for } x \in \text{supp}(\mu_i).
\]

If $t_{k(i),l(i)} \geq \tau$ as in (3.1), then we obtain (3.3) with

\[
C_4 = \frac{1}{4} \cdot \frac{1}{2t(1-t)} \cdot \tau.
\]

**Conclusion of the proof:** Combining (3.2) and (3.3) we find that there exist positive constants $C_5 = C_1/C_4$ and $C_{6,\delta} = (C_{2,\delta} + C_3)/C_4$ so that

\[
(x - x_i(t))^2 \leq C_5 \delta^2 + C_{6,\delta} T, \quad \text{ for } x \in \text{supp}(\mu_i),
\]
for every \(i = 1, \ldots, p + q - 1, \ T \leq 1\) and \(\delta > 0\).

Let \(\varepsilon > 0\) be given. Choose first \(\delta > 0\) so that \(C_5 \delta^2 \leq \frac{1}{2} \varepsilon^2\) and then choose \(T_\varepsilon \in (0, 1]\) so that \(C_6, \delta T_\varepsilon \leq \frac{1}{2} \varepsilon^2\). Then for every \(i = 1, \ldots, p + q - 1\) and \(T \leq T_\varepsilon\),

\[
(x - x_i(t))^2 \leq \varepsilon^2, \quad \text{for } x \in \text{supp}(\mu_i),
\]

and Lemma 3.1 follows. \(\square\)

Having Lemma 3.1 the proof of Theorem 2.10 is rather straightforward.

Proof. (Proof of Theorem 2.10)

(a) Let \(\varepsilon > 0\) be such that the intervals \([x_i(t) - \varepsilon, x_i(t) + \varepsilon]\), \(i = 1, \ldots, p + q - 1\), are disjoint. From Lemma 3.1 we know that there exists \(T_\varepsilon > 0\) so that for \(T < T_\varepsilon\) the support of \(\mu_i\) is contained in \([x_i(t) - \varepsilon, x_i(t) + \varepsilon]\).

Take \(i = 1, \ldots, p + q - 1\), and fix the other measures \(\mu_j, j \neq i\). From (2.12), (2.15) we see that the measure \(\mu_i\) is the equilibrium measure in an effective external field

\[\sum_{j \neq i} a_{i,j} \int \log \frac{1}{|x-y|} d\mu_j(y) + \frac{1}{T} V_i(x), \tag{3.12}\]

and it is also the minimizer if we restrict to measures with total mass \(t_{k(i),l(i)}\) that are supported on \([x_i(t) - \varepsilon, x_i(t) + \varepsilon]\). On this interval the external field (3.12) is strictly convex (we use that the measures \(\mu_j\) with \(j \neq i\) are supported outside \([x_i(t) - \varepsilon, x_i(t) + \varepsilon]\)). The convexity implies that the support of \(\mu_i\) is an interval, see e.g. [30] Theorem IV 1.10.

(b) The real analyticity of the density of \(\mu_i\) in the interior of the support follows from [12], since the effective external field (3.12) is real analytic on the support of \(\mu_i\).

The convexity of (3.12) implies that the density of \(\mu_i\) does not vanish in the interior of the support, and has square root behavior at the endpoints, see e.g. [8] Lemma 3.5]. \(\square\)

### 3.2 Varying \(n\)

In the situation of Theorem 2.4 we have for each finite \(n\), the number \(n_{k,l}\) of paths going from \(a_k\) to \(b_l\). The finite \(n\) transition numbers are

\[t_{k,l}^{(n)} = \frac{n_{k,l}}{n}\]

and in the limit we have

\[\lim_{n \to \infty} t_{k,l}^{(n)} = t_{k,l} \quad \text{for } k = 1, \ldots, p, \ l = 1, \ldots, q. \tag{3.13}\]

The equilibrium problem depends on the transition numbers by means of the normalizations (2.17). For a finite \(n\), we use the equilibrium problem for a vector of measures \((\mu_1, \ldots, \mu_{p+q-1})\) that have total masses

\[
\int d\mu_i = t_{k(i),l(i)}^{(n)}, \quad \text{for } i = 1, \ldots, p + q - 1 \tag{3.14}
\]

instead of (2.17). Then the minimizer \((\mu_1^{(n)}, \ldots, \mu_{p+q-1}^{(n)})\) will also depend on \(n\).

Because of Theorem 2.10 and (3.13), there is a \(T_0 > 0\), so that for every \(T < T_0\), each \(\mu_i^{(n)}\) is supported on an interval \([\alpha_i^{(n)}, \beta_i^{(n)}]\) (depending on \(n\)) so that parts (a) and (b) of Theorem 2.10 hold. As \(n \to \infty\), we have that \(\mu_i^{(n)} \to \mu_i\) and

\[
\alpha_i^{(n)} \to \alpha_i, \quad \beta_i^{(n)} \to \beta_i.
\]
for every $i$.

In the steepest descent analysis that follows we will fix a large enough $n$. We will work with the $n$-dependent measures $\mu_i^{(n)}$ and intervals $[\alpha_i^{(n)}, \beta_i^{(n)}]$, but for ease of notation we will usually not write the superscript $(n)$. We trust that this will not lead to confusion. However, we will write $t_{k,l}^{(n)}$. A property that will be used a number of times is that

$$ n_{k,l}^{(n)} = n_{k,l} \quad \text{is an integer.} \quad (3.15) $$

4 Riemann surface, $\xi$-functions, $\lambda$-functions

4.1 Variational conditions

As said before, we take a large $n$ and consider the vector equilibrium problem with normalization (3.14) and we assume that $T$ is sufficiently small so that Theorem 2.10 applies. So the measure $\mu_i$ is supported on the interval $[\alpha_i, \beta_i]$.

The variational conditions associated with the vector equilibrium problem (2.15) are as follows. For each $i = 1, \ldots, p + q - 1$, there is a constant $L_i \in \mathbb{R}$ so that

$$ 2 \sum_j a_{i,j} \int \log \frac{1}{|x-y|} \, d\mu_j(y) + \frac{1}{T} V_i(x) \left\{ \begin{array}{ll} = L_i, & x \in [\alpha_i, \beta_i], \\ \geq L_i, & x \in \mathbb{R} \setminus [\alpha_i, \beta_i]. \end{array} \right. \quad (4.1) $$

We use $F_i$ to denote the Cauchy transform of the measure $\mu_i$,

$$ F_i(z) := \int_{\alpha_i}^{\beta_i} \frac{1}{z-x} \, d\mu_i(x), \quad (4.2) $$

for $i = 1, \ldots, p + q - 1$. The function $F_i(z)$ is analytic on $\mathbb{C} \setminus [\alpha_i, \beta_i]$. By taking the derivative of (4.1) and using the Sokhotski-Plemelj formula it follows that

$$ -F_{i,+}(x) - F_{i,-}(x) - 2 \sum_{j \neq i} a_{i,j} F_j(x) + \frac{1}{T} V_i'(x) = 0, \quad x \in [\alpha_i, \beta_i]. \quad (4.3) $$

Lemma 4.1. The variational inequality (4.1) is strict for $x \in [\beta_{i+1}, \alpha_i) \cup (\beta_i, \alpha_{i-1}]$, where we put $\alpha_0 = +\infty$ and $\beta_{p+q} = -\infty$.

Proof. On both gaps $(\beta_{i+1}, \alpha_i)$ and $(\beta_i, \alpha_{i-1})$, the left-hand side of (4.1) is a real analytic function of $x$ whose first derivative is

$$ -2 \sum_j a_{i,j} F_j(x) + \frac{1}{T} V_i'(x) \quad (4.4) $$

and whose second derivative is

$$ 2 \sum_j a_{i,j} \int_{\alpha_i}^{\beta_i} \frac{1}{(x-y)^2} \, d\mu_j(y) + \frac{1}{T} V_i''(x). \quad (4.5) $$

Each term in (4.5) is positive and so the left-hand side of (4.1) is strictly convex on both $(\beta_{i+1}, \alpha_i)$ and $(\beta_i, \alpha_{i-1})$, which proves the lemma. \hfill $\square$
4.2 Riemann surface

We construct a Riemann surface $\mathcal{R}$ as follows, compare with [17]. The Riemann surface has $p + q$ sheets which we denote by $\mathcal{R}_j$, $j = 1, \ldots, p + q$. Each sheet is associated with a vertex of the graph, that is, with either a starting point $a_k$ or an ending point $b_l$. We choose the numbering so that $\mathcal{R}_k$ is associated with $a_k$ for $k = 1, \ldots, p$, and $\mathcal{R}_{p+l}$ is associated with $b_l$ for $l = 1, \ldots, q$.

Recall that we use $i$ as a label for the edges of the graph, and we write $k = k(i)$ and $l = l(i)$ if $i$ labels the edge $(a_k, b_l)$. Then the $p + q$ sheets are defined as

$$\mathcal{R}_k = \overline{\mathbb{C}} \setminus \bigcup_{i: k(i) = k} [\alpha_i, \beta_i], \quad k = 1, \ldots, p,$$

$$\mathcal{R}_{p+l} = \overline{\mathbb{C}} \setminus \bigcup_{i: l(i) = l} [\alpha_i, \beta_i], \quad l = 1, \ldots, q.$$  

The sheets are connected as follows. For each $i = 1, \ldots, p + q - 1$ we have that $\mathcal{R}_{k(i)}$ is connected to sheet $\mathcal{R}_{p+l(i)}$ along the interval $[\alpha_i, \beta_i]$ in the usual crosswise manner. See Figure 9 for a picture of the Riemann surface for the example in Figure 4. Note that the cuts $[\alpha_i, \beta_i]$ are always between a sheet $\mathcal{R}_k$ which is among the first $p$ sheets and a sheet $\mathcal{R}_{p+l}$ which is among the last $q$ sheets. The cuts strictly move to the left if we go from one sheet to the next among the first $p$ sheets, and similarly, among the last $q$ sheets.

The Riemann surface depends on $n$, since the endpoints $\alpha_i^{(n)}$ and $\beta_i^{(n)}$ depend on $n$. The $\xi$- and $\lambda$-functions that we define from the Riemann surface will also depend on $n$. We will not indicate the $n$-dependence, as already mentioned before.

The Riemann surface $\mathcal{R}$ has $p + q$ sheets and $2(p + q - 1)$ simple branch points. Therefore by Hurwitz’s formula (see e.g. [26]) its genus $g$ satisfies

$$2g - 2 = -2(p + q) + 2(p + q - 1) = -2$$

so that $g = 0$. The fact that the genus is zero will be helpful in the construction of the global parametrix in Section 7.2.
4.3 \(\xi\)-functions

We define the \(\xi\)-functions as follows. Recall that \(F_i\) is given by (4.2).

\[
\xi_k(z) = - \sum_{i: k(i) = k} F_i(z) + \frac{1}{T} (z - a_k), \quad k = 1, \ldots, p, \tag{4.8}
\]

\[
\xi_{p+i}(z) = \sum_{i: l(i) = l} F_i(z) - \frac{1}{T(1 - t)} (z - b_l), \quad l = 1, \ldots, q. \tag{4.9}
\]

We consider \(\xi_j\) as an analytic function on the \(j\)th sheet \(R_j\) of the Riemann surface with a pole at infinity. Moreover, these functions define a global meromorphic function on \(R\) as the following result shows.

**Theorem 4.2.** Consider the function \(\xi_j(z)\) on the \(j\)th sheet \(R_j\), \(j = 1, \ldots, p + q\). Then these functions are compatible along the cuts \([\alpha_i, \beta_i]\) of the Riemann surface in the sense that

\[
\left\{ \begin{array}{l}
\xi_{k(i),+}(x) = \xi_{p+1(i),-}(x), \\
\xi_{k(i),-}(x) = \xi_{p+1(i),+}(x),
\end{array} \right. \quad x \in [\alpha_i, \beta_i], \tag{4.10}
\]

for every \(i = 1, \ldots, p + q - 1\). Hence the \(\xi_j\)-functions can be extended to a global meromorphic function \(\xi\) defined on the Riemann surface \(R\).

**Proof.** Fix \(i \in \{1, \ldots, p + q - 1\}\). On the interval \([\alpha_i, \beta_i]\) we have by definition (4.3)–(4.10) that

\[
\xi_{k(i),+}(x) - \xi_{p+1(i),-}(x) = -\sum_{j: k(j) = k(i)} F_{j,+}(x) - \sum_{j: l(j) = l(i)} F_{j,-}(x) + \frac{1}{T(1 - t)} (x - (1 - t)a_{k(i)} - tb_{l(i)})
\]

\[
= -F_{i,+}(x) - F_{i,-}(x) - \left( \sum_{j \neq i: k(j) = k(i) \text{ or } l(j) = l(i)} F_j(x) \right) - \frac{1}{T} V'_i(x), \tag{4.11}
\]

where we used the definition (2.13)–(2.14) of \(V_i\).

If \(j \neq i\) is such that \(k(j) = k(i)\) or \(l(j) = l(i)\) then \(a_{i,j} = 1/2\). For all other \(j \neq i\) we have \(a_{i,j} = 0\). Therefore

\[
\left( \sum_{j \neq i: k(j) = k(i) \text{ or } l(j) = l(i)} F_j(x) \right) = 2 \sum_{j \neq i} a_{i,j} F_j(x).
\]

Using this in (4.11) and recalling the variational equality (4.3), we see that that \(\xi_{k(i),+}(x) = \xi_{p+1(i),-}(x)\) for \(x \in [\alpha_i, \beta_i]\).

The other equality \(\xi_{k(i),-}(x) = \xi_{p+1(i),+}(x)\) follows in exactly the same way.

In the following two lemmas we collect some more properties of the \(\xi\)-functions that will be needed in what follows.

**Lemma 4.3.** The \(\xi\)-functions have the following behavior as \(z \to \infty\)

\[
\xi_k(z) = \frac{1}{T} (z - a_k) - \frac{n_k}{nz} + O \left( \frac{1}{z^2} \right), \quad k = 1, \ldots, p, \tag{4.12}
\]

\[
\xi_{p+i}(z) = -\frac{1}{T(1 - t)} (z - b_l) + \frac{m_l}{nz} + O \left( \frac{1}{z^2} \right), \quad l = 1, \ldots, q. \tag{4.13}
\]

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Proof. From (4.2) and (3.14) it follows that

\[
F_i(z) = \frac{1}{z} + O\left(\frac{1}{z^2}\right) = \frac{i^{(n)}_{k(i),l(i)}}{z} + O\left(\frac{1}{z^2}\right)
\]

(4.14)
as \(z \to \infty\). Recall that we work with \(n\)-dependent transition numbers. Since

\[
\sum_{i: k(i) = k} i^{(n)}_{k(i),l(i)} = \frac{n_k}{n}, \quad \sum_{i: l(i) = l} i^{(n)}_{k(i),l(i)} = \frac{m_l}{n}
\]

the asymptotic behaviors (4.12) and (4.13) follow immediately from the definitions (4.8)–(4.9).

Lemma 4.4. For any \(i = 1, \ldots, p + q - 1\) we have

\[
\int_{C_i} \xi_{k(i)}(z) \, dz = -2\pi i \frac{n_k}{n},
\]

(4.15)

\[
\int_{C_i} \xi_{p+l(i)}(z) \, dz = 2\pi i \frac{n_k}{n},
\]

(4.16)

\[
\int_{C_i} \xi_j(z) \, dz = 0, \quad j \in \{k(i), p+l(i)\},
\]

(4.17)

where \(C_i\) denotes a counterclockwise contour surrounding the interval \([\alpha_i, \beta_i]\) and not enclosing any point of the other intervals \([\alpha_j, \beta_j]\), \(j \neq i\).

Proof. From (4.2), (4.14) and the definition of \(C_i\) it follows that

\[
\int_{C_i} F_i(z) \, dz = 2\pi i i^{(n)}_{k(i),l(i)}
\]

and

\[
\int_{C_i} F_j(z) \, dz = 0 \quad \text{if } j \neq i.
\]

The lemma then follows from the definitions (4.8)–(4.9).

Remark 4.5. Our definition of the \(\xi\)-functions differs slightly from the one used in [5]. If \(\tilde{\xi}_j\) denote the \(\xi\)-functions in [5] then we have

\[
\xi_j(z) = \tilde{\xi}_j(z) - \frac{z}{T(1-t)}, \quad j = 1, \ldots, p + q.
\]

In the present form the formulae are more symmetric.

4.4 \(\lambda\)-functions

We define the \(\lambda\)-functions as

\[
\lambda_k(z) = c_k + \int_{\beta_k}^z \xi_k(s) \, ds, \quad k = 1, \ldots, p,
\]

(4.18)

\[
\lambda_{p+l}(z) = c_{p+l} + \int_{\beta_{l}}^z \xi_{p+l}(s) \, ds, \quad l = 1, \ldots, q,
\]

(4.19)
where
\[ i_k := \min \{ i \mid k(i) = k \}, \quad \text{and} \quad \hat{i}_l := \min \{ i \mid l(i) = l \}, \]
and where the path of integration in the integrals (4.18) and (4.19) lies in \( \mathbb{C} \setminus (-\infty, \beta_{i_k}) \) and \( \mathbb{C} \setminus (-\infty, \beta_{i_l}) \), respectively. The functions \( \lambda_k(z) \) and \( \lambda_{p+l}(z) \) are defined with a branch cut along the intervals \( (-\infty, \beta_{i_k}] \) and \( (-\infty, \beta_{i_l}] \), respectively.

We choose the constants \( c \) in the following way.

**Lemma 4.6.** We can (and do) choose the constants \( c \) in (4.18)–(4.19) in such a way that

\[
\Re(\lambda_k,+, (\beta_i)) = \Re(\lambda_{p+l},+, (\beta_i)), \tag{4.20}
\]

for every \( i = 1, \ldots, p + q - 1 \).

**Proof.** We use the fact that the graph \( G = (V, E, t) \) is a tree. We can iteratively ‘undress’ this tree as follows. We start with \( G_1 = G \). Next we choose a leaf vertex \( v \) and set

\[
G_2 = G_1 \setminus v,
\]
i.e., \( G_2 \) is the tree obtained by removing the leaf and its corresponding edge from the tree \( G_1 \). We iteratively repeat this procedure and obtain in this way a chain of nested trees

\[
G = G_1 \supset G_2 \supset \cdots \supset G_{|V|}, \tag{4.21}
\]

where each \( G_i \) is obtained from \( G_{i-1} \) by removing one leaf. Obviously the last non-empty tree \( G_{|V|} \) in this chain consists of a single vertex \( v_j \).

We freely choose the corresponding constant \( c_j \). Next we use induction on \( k = |V| - 1, \ldots, 2, 1 \), to fill in all the remaining constants \( c_j \) so that each time (4.20) is satisfied. This is possible since \( G_{i-1} \setminus G_i \) consists of a single vertex \( v_j \), which is a leaf of \( G_{i-1} \), and hence we have exactly one condition (4.20) to fix the integration constant \( c_j \) of this leaf. Thus we see that the conditions (4.20) can indeed be imposed on the constants \( c_j \) in (4.18)–(4.19).

Properties of the \( \lambda \)-functions that we will need are stated in the following lemmas. The first is a reformulation of the variational conditions (4.11).

**Lemma 4.7.** We have for \( i = 1, \ldots, p + q - 1 \),

\[
\Re(\lambda_k,+, (x) - \lambda_{p+l},+, (x)) \begin{cases} = 0, & x \in [\alpha_i, \beta_i], \\ \geq 0, & x \in \mathbb{R} \setminus [\alpha_i, \beta_i]. \end{cases} \tag{4.22}
\]

Strict inequality in (4.22) holds for \( x \in [\beta_{i+1}, \alpha_i] \cup (\beta_i, \alpha_i - 1] \) (where \( \alpha_0 = +\infty \) and \( \beta_{p+q} = -\infty \)).

**Proof.** Observe that by (4.3)–(4.9) and (4.18)–(4.19) we have that

\[
\lambda_k(z) = \sum_{i : k(i) = k} \int \log \frac{1}{z - x} \, d\mu_i(x) + \frac{1}{2T} (z - a_k)^2 + \tilde{c}_k, \tag{4.23}
\]

\[
\lambda_{p+l}(z) = -\sum_{i : l(i) = l} \int \log \frac{1}{z - x} \, d\mu_i(x) - \frac{1}{2T(1-t)} (z - b_l)^2 + \tilde{c}_{p+l}, \tag{4.24}
\]

for certain real constants \( \tilde{c}_j, j = 1, \ldots, p + q \). Therefore, for \( i = 1, \ldots, p + q - 1 \),

\[
\Re(\lambda_k(z) - \lambda_{p+l}(z)) = 2 \int \log \frac{1}{|z - x|} \, d\mu_i(x) + \sum_{j \neq i : k(j) = k(i) \text{ or } l(j) = l(i)} \int \log \frac{1}{|z - x|} \, d\mu_j(x) + \frac{1}{T} \Re V_i(z) + \frac{1}{2T} (a_{k(i)} - b_{l(i)})^2 + \tilde{c}_{k(i)} - \tilde{c}_{p+l}, \tag{4.25}
\]

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where we used the definition \((2.14)\) of \(V_i\). Then by the variational conditions \((4.1)\) we have

\[
\Re \left( \lambda_{k(i),+}(x) - \lambda_{p+l(i),+}(x) \right) \geq L_i + \frac{1}{2T} (a_{k(i)} - b_{l(i)})^2 + \hat{c}_{k(i)} - \hat{c}_{p+l(i)}
\]

for \(x \in \mathbb{R}\) with equality for \(x \in [\alpha_{i}, \beta_{i}]\). The constant in the right-hand side of \((4.26)\) is equal to zero because the constants \(c_j\) are chosen so that \((4.20)\) holds.

The strict inequality for \(x \in [\beta_{i+1}, \alpha_{i}] \cup (\beta_{i}, \alpha_{i-1}]\) is a consequence of Lemma 4.1.

**Lemma 4.8.** As \(z \to \infty\) we have that

\[
\lambda_k(z) = \frac{1}{2Tt}(z - a_k)^2 - \frac{n_k}{n} \log z + \hat{c}_k + O \left( \frac{1}{z} \right), \quad k = 1, \ldots, p, \quad (4.27)
\]

\[
\lambda_{p+l}(z) = -\frac{1}{2T(1-t)}(z - b_l)^2 + \frac{m_l}{n} \log z + \hat{c}_{p+l} + O \left( \frac{1}{z} \right), \quad l = 1, \ldots, q. \quad (4.28)
\]

**Proof.** This follows from the definitions \((4.18)\) and \((4.19)\) and the asymptotic behavior \((4.12)\) and \((4.13)\) of the \(\xi\)-functions.

The next lemma will be a consequence of Lemma 4.4.

**Lemma 4.9.** For \(k = 1, \ldots, p\), we have that

\[
\exp(n(\lambda_{k,+}(x) - \lambda_{k,-}(x))) = 1, \quad \text{for} \quad x \in \mathbb{R} \setminus \bigcup_{i: k(i) = k} [\alpha_i, \beta_i]. \quad (4.29)
\]

For \(l = 1, \ldots, q\), we have that

\[
\exp(n(\lambda_{p+l,+}(x) - \lambda_{p+l,-}(x))) = 1, \quad \text{for} \quad x \in \mathbb{R} \setminus \bigcup_{i: \ell(i) = l} [\alpha_i, \beta_i]. \quad (4.30)
\]

**Proof.** Fix \(k = 1, \ldots, p\) and let \(x \in \mathbb{R} \setminus \bigcup_{i: k(i) = k} [\alpha_i, \beta_i]\). By definition of \(\lambda_k\) and by contour deformation we have that

\[
\lambda_{k,+}(x) - \lambda_{k,-}(x) = \int_C \xi_k(z) \, dz
\]

where \(C\) is some closed contour surrounding some of the intervals \([\alpha_i, \beta_i]\). From Lemma 4.3 it follows that each of the enclosed intervals gives a contribution to the integral of the form \(\pm 2\pi i t_{k,i}^{(n)}\). Since each \(t_{k,i}^{(n)}\) is a rational number with denominator \(n\), indeed \(t_{k,i}^{(n)} = n_{k,i}/n\), we conclude that \(n(\lambda_{k,+}(x) - \lambda_{k,-}(x))\) is a multiple of \(2\pi i\), and so we obtain \((4.29)\).

The proof of \((4.30)\) is similar.

**Lemma 4.9** will be important for the steepest descent analysis in Section 5.

5 First transformation of the RH problem: Normalization at infinity

In the following sections we describe the steepest descent analysis \(Y \mapsto X \mapsto T \mapsto S \mapsto R\) of the RH problem \((2.11)\). Throughout the steepest descent analysis the following simple lemma will be repeatedly used.
Lemma 5.1. Assume that the matrix function $Y(z)$ satisfies the jump condition $Y_+(x) = Y_-(x)J(x)$ for $x \in \mathbb{R}$. Let $A(z)$ and $B(z)$ be matrix functions with $A(z)$ entire and $B(z)$ analytic in $\mathbb{C} \setminus \mathbb{R}$. Then

$$X(z) := A(z)Y(z)B(z)$$

satisfies the jump condition

$$X_+(x) = X_-(x) \left( B^{-1}_-(x)J(x)B_+(x) \right).$$

The point will be to choose appropriate transformation matrices $A(z)$ and $B(z)$ in order to bring the RH problem 2.11 to a simple form.

Let $Y$ be the solution to the original RH problem 2.11. The first transformation $Y \mapsto X$ serves to normalize the RH problem at infinity. To this end we define $X = X(z)$ as

$$X(z) = L^{-n}Y(z)D(z)^n, \quad (5.1)$$

where we define the diagonal matrices

$$D(z) = \text{diag} \left( \exp \left( \lambda_k(z) - \frac{1}{2T}(z - a_k)^2 \right) \right)_{k=1}^p,$$

$$\left( \exp \left( \lambda_{p+l}(z) + \frac{1}{2T(1-t)}(z - b_l)^2 \right) \right)_{l=1}^q, \quad (5.2)$$

and

$$L = \text{diag} (\exp(\hat{c}_1), \ldots, \exp(\hat{c}_p+q)), \quad (5.3)$$

where the constants $\hat{c}_j$ are as in (4.27)–(4.28). From Lemma 5.1 it follows by straightforward calculations that $X = X(z)$ satisfies the following RH problem.

**RH problem 5.2.**

1. $X$ is analytic in $\mathbb{C} \setminus \mathbb{R}$;
2. For $x \in \mathbb{R}$, we have that

$$X_+(x) = X_-(x) \begin{pmatrix} J_{1,1}(x) & J_{1,2}(x) \\ 0 & J_{2,2}(x) \end{pmatrix}, \quad (5.4)$$

where the blocks $J_{1,1}$, $J_{1,2}$ and $J_{2,2}$ (of sizes $p \times p$, $p \times q$ and $q \times q$, respectively) have the following form.

a) $J_{1,2}$ is a full $p \times q$ matrix with entries

$$J_{1,2}(x)_{k,l} = \exp \left( n(\lambda_{p+l,+(x)} - \lambda_{k,-}(x)) \right), \quad x \in \mathbb{R}. \quad (5.5)$$

b) Outside of the intervals $[\alpha_i, \beta_i]$, $J_{1,1}$ and $J_{2,2}$ are identity matrices

$$J_{1,1}(x) = I_p, \quad J_{2,2}(x) = I_q, \quad x \in \mathbb{R} \setminus \bigcup_{i=1}^{p+q-1} [\alpha_i, \beta_i]. \quad (5.6)$$
(c) On the interval $[\alpha_i, \beta_i]$, $J_{1,1}$ and $J_{2,2}$ are diagonal matrices with ones on the diagonal, except for

\[
(J_{1,1}(x))_{k(i),k(i)} = \exp \left(n(\lambda_{k(i),+}(x) - \lambda_{k(i),-}(x))\right), \quad x \in (\alpha_i, \beta_i),
\]  

and

\[
(J_{2,2}(x))_{l(i),l(i)} = \exp \left(n(\lambda_{p+l(i),+}(x) - \lambda_{p+l(i),-}(x))\right), \quad x \in (\alpha_i, \beta_i).
\]  

(3) As $z \to \infty$, we have that

\[
X(z) = I_{p+q} + O(1/z).
\]  

Here we used Lemma 4.9 to see that the diagonal entries of (5.6), as well as most of the diagonal entries of (5.7)–(5.8) are equal to 1. On the other hand, we used the asymptotic conditions (4.27)–(4.28) and (5.1)–(5.3) to see that the RH problem for $X$ is normalized at infinity in the sense of (5.9).

Let us illustrate the jump matrices for the example with $p = q = 2$ as in Figures 4 and 9. In that case the jump conditions are written as

\[
X_+ = X_- \begin{pmatrix}
\exp(n(\lambda_1+ - \lambda_1-)) & 0 & \exp(n(\lambda_3+ - \lambda_1-)) & \exp(n(\lambda_4 - \lambda_1-)) \\
0 & 1 & \exp(n(\lambda_3+ - \lambda_2)) & \exp(n(\lambda_4 - \lambda_2)) \\
0 & 0 & \exp(n(\lambda_3+ - \lambda_3-)) & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]  

on the interval $[\alpha_1, \beta_1]$,

\[
X_+ = X_- \begin{pmatrix}
1 & 0 & \exp(n(\lambda_3+ - \lambda_1)) & \exp(n(\lambda_4 - \lambda_1)) \\
0 & \exp(n(\lambda_2+ - \lambda_2-)) & \exp(n(\lambda_3+ - \lambda_2-)) & \exp(n(\lambda_4 - \lambda_2-)) \\
0 & 0 & \exp(n(\lambda_3+ - \lambda_3-)) & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]  

on the interval $[\alpha_2, \beta_2]$,

\[
X_+ = X_- \begin{pmatrix}
1 & 0 & \exp(n(\lambda_3 - \lambda_1)) & \exp(n(\lambda_4+ - \lambda_1)) \\
0 & \exp(n(\lambda_2+ - \lambda_2)) & \exp(n(\lambda_3 - \lambda_2)) & \exp(n(\lambda_4+ - \lambda_2)) \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & \exp(n(\lambda_4+ - \lambda_4-))
\end{pmatrix}
\]  

on the interval $[\alpha_3, \beta_3]$, and

\[
X_+ = X_- \begin{pmatrix}
1 & 0 & \exp(n(\lambda_3 - \lambda_1)) & \exp(n(\lambda_4 - \lambda_1)) \\
0 & 1 & \exp(n(\lambda_3 - \lambda_2)) & \exp(n(\lambda_4 - \lambda_2)) \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]  

on $\mathbb{R} \setminus \bigcup_{i=1}^{3} [\alpha_i, \beta_i]$.

### 6 Second transformation of the RH problem: Gaussian elimination and opening of global lenses

On the interval $[\alpha, \beta]$ we have that $\lambda_{p+l(i),+} - \lambda_{k(i),-}$ is a purely imaginary constant. So the entry

\[
(J_{1,2}(x))_{k(i),l(i)}, \quad x \in [\alpha, \beta]
\]  

(6.1)
is constant with modulus one. It would be an ideal situation if, except for (6.1), all entries of the matrix $J_{1,2}(x)$ for $x \in \mathbb{R}$, are exponentially decaying as $n \to \infty$. That would happen if for every $k = 1, \ldots, p$, and $l = 1, \ldots, q$, we have

(a) if $t_{k,l}^{(n)} = 0$ then
$$\Re \lambda_{p+l} (x) < \Re \lambda_k (x), \quad x \in \mathbb{R},$$

(b) if $t_{k,l}^{(n)} > 0$ then
$$\Re \lambda_{p+l} (x) < \Re \lambda_k (x), \quad x \in \mathbb{R} \setminus [\alpha_i, \beta_i],$$

where $i = k + l - 1$.

However, this will not happen in general, and that is why we need a second transformation $X \mapsto T$ in the steepest descent analysis of the RH problem in which a number of unwanted entries of the jump matrices are eliminated. In particular those entries of $J_{1,2}(x)$ that could potentially be exponentially increasing as $n \to \infty$.

To this end we will open global lenses [3] and apply Gaussian elimination inside each of the lenses. This construction will be systematic and may have interest in its own right. The proof that appropriate lenses exist will be a consequence of the maximum principle for subharmonic functions.

The opening of global lenses can be conveniently described in terms of the right-down path in Proposition 2.2. We start at the top left entry $(1, 1)$ of the matrix of transition numbers $(t_{k,l}^{(n)})$ and walk along the path until we arrive at the bottom right entry $(p, q)$. During this walk, we will open global lenses in an appropriate way. The precise action to perform depends on whether we are taking a vertical (down) or a horizontal (right) step along the path.

### 6.1 Construction of global lenses: Vertical step

First we construct the global lenses for a vertical step along the lattice path. Thus assume that $i \in \{1, \ldots, p + q - 2\}$ is such that

$$\begin{align*}
(k(i), l(i)) &= (k, l), \\
(k(i + 1), l(i + 1)) &= (k + 1, l),
\end{align*}$$

for certain $k = 1, \ldots, p - 1$ and $l = 1, \ldots, q$.

From (4.27) we obtain the asymptotic behavior

$$\lambda_{k+1}(z) - \lambda_k(z) = \frac{1}{Tt} (a_k - a_{k+1}) z + O(\log |z|),$$

as $z \to \infty$.

We also note that $\Re \lambda_k$ and $\Re \lambda_{k+1}$ are well-defined and continuous on $\mathbb{C}$. Indeed, we have by (4.23) that

$$\begin{align*}
\Re (\lambda_k(z)) &= \sum_{j; k(j) = k} \int \log \frac{1}{|z - x|} d\mu_j(x) + \frac{1}{2Tt} \Re (z - a_k)^2 + \tilde{c}_k, \\
\Re (\lambda_{k+1}(z)) &= \sum_{j; k(j) = k+1} \int \log \frac{1}{|z - x|} d\mu_j(x) + \frac{1}{2Tt} \Re (z - a_{k+1})^2 + \tilde{c}_{k+1},
\end{align*}$$

for certain constants $\tilde{c}_k$ and $\tilde{c}_{k+1}$.

The representations (6.5) and (6.6) also show the following.
Lemma 6.1. The function \( z \mapsto \Re (\lambda_{k+1}(z) - \lambda_k(z)) \) is superharmonic on \( \mathbb{C} \setminus \bigcup_{j: k(j) = k} [\alpha_j, \beta_j] \), and subharmonic on \( \mathbb{C} \setminus \bigcup_{j: k(j) = k+1} [\alpha_j, \beta_j] \).

Proof. It is a standard fact from potential theory that any function of the form \( z \mapsto \int \log \frac{1}{|z - x|} \, d\mu(x) \), with \( \mu \) a positive measure with compact support, is superharmonic on \( \mathbb{C} \) and harmonic on \( \mathbb{C} \setminus \mathrm{supp}(\mu) \), see e.g. [30, Chapter 0]. Thus by (6.5) and (6.6) we have that \( z \mapsto \Re \lambda_{k+1}(z) \) is superharmonic on \( \mathbb{C} \) and harmonic on \( \mathbb{C} \setminus \bigcup_{j: k(j) = k} [\alpha_j, \beta_j] \), while \( z \mapsto -\Re \lambda_k(z) \) is subharmonic on \( \mathbb{C} \) and harmonic on \( \mathbb{C} \setminus \bigcup_{j: k(j) = k+1} [\alpha_j, \beta_j] \).

Since the two sets \( \bigcup_{j: k(j) = k} [\alpha_j, \beta_j] \) and \( \bigcup_{j: k(j) = k+1} [\alpha_j, \beta_j] \) are disjoint, the lemma follows. \( \Box \)

We next define the open sets \( \Omega_+, \Omega_- \subset \mathbb{C} \) as follows

\[
\Omega_+ := \{ z \in \mathbb{C} \mid \Re (\lambda_{k+1}(z) - \lambda_k(z)) > 0 \} \quad (6.7)
\]

\[
\Omega_- := \{ z \in \mathbb{C} \mid \Re (\lambda_{k+1}(z) - \lambda_k(z)) < 0 \} \quad (6.8)
\]

We also denote

\[
\Omega_{+, \infty} := \{ z \in \Omega_+ \mid \exists \text{ a connected path in } \Omega_+ \text{ from } z \text{ to } \infty \} \quad (6.9)
\]

\[
\Omega_{-, \infty} := \{ z \in \Omega_- \mid \exists \text{ a connected path in } \Omega_- \text{ from } z \text{ to } \infty \} \quad (6.10)
\]

In other words, \( \Omega_{+, \infty} \) is the union of the unbounded connected components of \( \Omega_+ \), and similarly for \( \Omega_{-, \infty} \).

The open sets \( \Omega_{+, \infty}, \Omega_{-, \infty} \) satisfy the following properties.

Lemma 6.2. (a) For each \( \varepsilon > 0 \) there exists \( R > 0 \) so that

\[
\{ z \in \mathbb{C} \mid |z| > R, -\pi/2 + \varepsilon < \arg z < \pi/2 - \varepsilon \} \subset \Omega_{+, \infty}, \quad (6.11)
\]

and

\[
\{ z \in \mathbb{C} \mid |z| > R, \pi/2 + \varepsilon < \arg z < 3\pi/2 - \varepsilon \} \subset \Omega_{-, \infty}. \quad (6.12)
\]

In particular, \( \Omega_{+, \infty} \) lies to the right of \( \Omega_{-, \infty} \).

(b) Both \( \Omega_{+, \infty} \) and \( \Omega_{-, \infty} \) are connected.

Proof. Part (a) follows from (6.1) and the fact that \( a_k > a_{k+1} \). Part (b) follows from part (a) in a similar way as in [16, Proof of Lemma 2.4], to which we refer for further details. \( \Box \)

Lemma 6.3. We have

\[
\alpha_i \in \Omega_{+, \infty} \quad \text{and} \quad \beta_{i+1} \in \Omega_{-, \infty}, \quad (6.13)
\]

where we recall \( i \) is related to \( k \) as in (6.2) and (6.3).

Proof. By applying the variational conditions (4.22) twice, first with the index \( i \) and then with \( i + 1 \), we obtain

\[
\Re (\lambda_{k+1}(x) - \lambda_k(x)) = \Re (\lambda_{k+1}(x) - \lambda_{p+1}(x)) \geq 0, \quad x \in [\alpha_i, \beta_i]. \quad (6.14)
\]

and the inequality (6.14) is strict for \( x = \alpha_i \) because of the statement about the strict inequality in Lemma 4.7. Hence \( \alpha_i \in \Omega_+ \), and in a similar way we obtain \( \beta_{i+1} \in \Omega_- \).
To show that $\alpha_i$ belongs to the unbounded component of $\Omega_+$ we argue as in [16] Proof of Lemma 2.4. What we use is that $\bigcup_{j: k(j) = k} [\alpha_j, \beta_j]$ lies to the right of $\bigcup_{j: k(j) = k+1} [\alpha_j, \beta_j]$, and that $\alpha_i$ is the left-most point of $\bigcup_{j: k(j) = k} [\alpha_j, \beta_j]$, and that $\beta_{i+1}$ is the right-most point of $\bigcup_{j: k(j) = k+1} [\alpha_j, \beta_j]$.

The proof is by contradiction. Suppose that $\alpha_i$ does not belong to the unbounded component of $\Omega_+$. Then the set

$$\Omega_{+\alpha_i} := \{ z \in \Omega_+ \mid \exists \text{ a connected path in } \Omega_+ \text{ from } z \text{ to } \alpha_i \}$$

(6.15)
is bounded, it is symmetric with respect to the real line, and it contains $\alpha_i$. Also $\text{Re}(\lambda_{k+1} - \lambda_k)$ is zero on the boundary of $\Omega_{+\alpha_i}$ and strictly positive inside of $\Omega_{+\alpha_i}$ by construction. Since subharmonic functions satisfy a maximum principle, it follows that $\text{Re}(\lambda_{k+1} - \lambda_k)$ is not subharmonic on all of $\Omega_{+\alpha_i}$. Then by Lemma 6.1 we conclude that $\Omega_{+\alpha_i}$ has a nonempty intersection with $\bigcup_{j: k(j) = k+1} [\alpha_j, \beta_j]$. Any point of intersection lies strictly to the left of $\beta_{i+1}$, since $\beta_{i+1} \in \Omega_-$ and $\beta_{i+1}$ is the right-most point of $\bigcup_{j: k(j) = k+1} [\alpha_j, \beta_j]$. Then, because of symmetry in the real axis, it follows that $\Omega_{+\alpha_i}$ surrounds the point $\beta_{i+1}$. The set

$$\Omega_{-\beta_{i+1}} := \{ z \in \Omega_- \mid \exists \text{ a connected path in } \Omega_- \text{ from } z \text{ to } \beta_{i+1} \}$$

(6.16)
is then bounded, and it does not intersect with $\bigcup_{j: k(j) = k} [\alpha_j, \beta_j]$. By Lemma 6.1 we then have that $\text{Re}(\lambda_{k+1} - \lambda_k)$ is superharmonic on $\Omega_{-\beta_{i+1}}$. However, $\text{Re}(\lambda_{k+1} - \lambda_k)$ is zero on the boundary, and strictly negative inside of $\Omega_{-\beta_{i+1}}$, which gives a contradiction with the minimum principle for superharmonic functions.

Thus $\alpha_i$ belongs to the unbounded component of $\Omega_+$, and similarly $\beta_{i+1}$ belongs to the unbounded component of $\Omega_-$. \[\square\]

Let $X_0 > 0$ be such that

$$\text{Re}(\lambda_{k'}(x) - \lambda_{l'}(x)) > 0,$$

for all $x \in (-\infty, -X_0) \cup (X_0, \infty)$ and all $k' = 1, \ldots, p$, $l' = 1, \ldots, q$. The existence of such a constant $X_0$ follows from (4.27) and (4.28).

The next result is an immediate consequence of Lemma 6.3.

**Theorem 6.4.** There exist two simple closed contours

$$\Gamma_{+,i} \subset \Omega_{+,\infty} \quad \text{and} \quad \Gamma_{-,i} \subset \Omega_{-,\infty}$$

such that

(a) $\Gamma_{+,i}$ surrounds the interval $[\alpha_i, \beta_1]$.

(b) $\Gamma_{-,i}$ surrounds the interval $[\alpha_{p+q-1}, \beta_{i+1}]$.

(c) both $\Gamma_{-,i}$ and $\Gamma_{+,i}$ intersect the interval $[\beta_{i+1}, \alpha_i]$ in exactly one point, which we denote by $x_i$, $y_i$, respectively. We have $x_i < y_i$.

(d) both $\Gamma_{-,i}$ and $\Gamma_{+,i}$ have one extra intersection point with the real axis, which lies inside the interval $(-\infty, -X_0) \cup (X_0, \infty)$, respectively.

As in (3), the contours $\Gamma_{+,i}$ and $\Gamma_{-,i}$ will be called global lenses.

**Remark 6.5.** Instead of taking closed contours, one might also take $\Gamma_{+,i}$ and $\Gamma_{-,i}$ to be unbounded, tending to infinity in the right and left half of the complex plane, respectively, and both intersecting the real line in exactly one point in the line segment $(\beta_{i+1}, \alpha_i)$. This is the construction that was used in [14].

An illustration of Theorem 6.3 is shown in Figure 10. When convenient we also define $x_0 = X_0$, $y_0 = \infty$, $x_{p+q-1} = -\infty$, and $y_{p+q-1} = -X_0$. 28
6.2 Construction of global lenses: Horizontal step

Next we construct the global lenses for a horizontal step along the lattice path. Thus assume that \( i \in \{1, \ldots, p + q - 2\} \) is such that

\[
(k(i), l(i)) = (k, l),
\]
\[
(k(i + 1), l(i + 1)) = (k, l + 1),
\]

for certain \( k = 1, \ldots, p \) and \( l = 1, \ldots, q - 1 \).

The analysis of Section 6.1 can be adapted to the present case. Let us discuss the main points. We now define the open sets \( \Omega^+ \) and \( \Omega^- \) as follows

\[
\Omega^+ := \{ z \in \mathbb{C} | \text{Re} (\lambda_{p+l}(z) - \lambda_{p+l+1}(z)) > 0 \} \]
\[
\Omega^- := \{ z \in \mathbb{C} | \text{Re} (\lambda_{p+l}(z) - \lambda_{p+l+1}(z)) < 0 \}.
\]

The unbounded regions \( \Omega^+ \cup, \Omega^- \cup \subset \mathbb{C} \) are again defined as in (6.9)–(6.10). Then Lemma 6.2 remains valid.

Using the above definitions, Lemma 6.3 and Theorem 6.4 both remain valid as well. Thus we can construct the global lenses \( \Gamma^+ \) and \( \Gamma^- \) in exactly the same way as before.

6.3 Gaussian elimination step

Now we will show how to apply Gaussian elimination inside the global lenses. First, it might be worthwhile to recall the basic idea of Gaussian elimination in our present context, see e.g. [20]. Let

\[
J = uv^T = \begin{pmatrix} u_1 \\ \vdots \\ u_p \end{pmatrix} \begin{pmatrix} v_1 & \ldots & v_q \end{pmatrix}
\]

be a rank-one matrix of size \( p \) by \( q \). Assume that one multiplies \( J \) on the left with the matrix

\[
I_p - \frac{u_{k+1}}{u_k} e_{k+1} e_k^T.
\]
Here and below we use \( e_k \) to denote the column vector with all entries equal to zero, except for the \( k \)th entry which equals 1. The length of \( e_k \) will be clear from the context. Note that the outer product \( e_{k+1} e_k^T \) in (6.22) is the matrix with all zero entries except for the \((k+1, k)\) entry which equals one.

The multiplication with (6.22) on the left is equivalent to applying an elementary row operation to the rows of \( J \), where to row \( k + 1 \) one adds \(-\frac{u_{k+1}}{u_k}\) times row \( k \). This row operation is such that the entries of row \( k + 1 \) of \( J \) are eliminated.

Similarly, assume that one multiplies \( J \) on the right with a transformation matrix of the form (6.22)–(6.23) is to define a new RH matrix (5.4), cf. (5.5). The mechanism to multiply the jump matrix on the left or on the right with a transformation matrix \( B(z) \) and subsequently apply Lemma 5.1.

Now we are ready to describe the Gaussian elimination in detail. This will be the next transformation \( X \mapsto T \) in the steepest descent analysis of the RH problem.

**Algorithm 6.6. (The transformation \( X \mapsto T \))**

1. **(Initialization.)** We initialize \( T(z) := X(z) \).

2. **(Forward sweep.)** For each \( i = 1, \ldots, p + q - 2 \) we open the global lens \( \Gamma_{-,i} \) in Theorem 6.4 and we update, in case of a vertical step (6.22)–(6.23),

\[
T(z) = \begin{cases} 
T(z) \left( I_{p+q} + \exp(n(\lambda_{k+1}(z) - \lambda_k(z))) e_k e_{k+1}^T \right), & \text{inside the lens } \Gamma_{-,i} \\
T(z), & \text{elsewhere,}
\end{cases}
\]

and in case of a horizontal step (6.17)–(6.18),

\[
T(z) = \begin{cases} 
T(z) \left( I_{p+q} - \exp(n(\lambda_{p+i}(z) - \lambda_{p+i+1}(z))) e_{p+i} e_{p+i+1}^T \right), & \text{inside the lens } \Gamma_{-,i} \\
T(z), & \text{elsewhere.}
\end{cases}
\]

3. **(Backward sweep.)** For each \( i = p + q - 2, \ldots, 2, 1 \) we open the global lens \( \Gamma_{+,i} \) in Theorem 6.4 and we update, in case of a vertical step (6.2)–(6.3),

\[
T(z) = \begin{cases} 
T(z) \left( I_{p+q} + \exp(n(\lambda_k(z) - \lambda_{k+1}(z))) e_{k+1} e_k^T \right), & \text{inside the lens } \Gamma_{+,i} \\
T(z), & \text{elsewhere,}
\end{cases}
\]

and in case of a horizontal step (6.17)–(6.18),

\[
T(z) = \begin{cases} 
T(z) \left( I_{p+q} - \exp(n(\lambda_{p+i+1}(z) - \lambda_{p+i}(z))) e_{p+i} e_{p+i+1}^T \right), & \text{inside } \Gamma_{+,i} \\
T(z), & \text{elsewhere.}
\end{cases}
\]

Incidently, we note that the forward and backward sweeps in the above algorithm commute. But one is not allowed to change the order in which the index \( i \) varies inside the sweeps.
It is easy to see that Algorithm 6.6 does not change the jump matrix in (5.4) except for its top right submatrix \( J_{1,2}(x) \). To see what happens with the latter, we have to distinguish between different regions of the complex plane.

First assume that \( x \) belongs to one of the intervals \( (y_j, x_{j-1}) \supset [\alpha_j, \beta_j] \), \( j = 1, \ldots, p + q - 1 \). (Recall that \( y_p + q - 1 = -X_0 \) and \( x_0 = X_0 \).) From Theorem 6.3, we see that \( x \) lies inside the global lens \( \Gamma_{-i} \) precisely when \( i = 1, \ldots, j - 1 \). Hence the ‘relevant’ indices in the forward sweep in Algorithm 6.6 are \( i = 1, \ldots, j - 1 \). During the corresponding operations, the entries in rows \( 1, 2, \ldots, k(j) - 1 \) and columns \( 1, 2, \ldots, l(j) - 1 \) of the matrix \( J_{1,2}(x) \) are cancelled by Gaussian elimination.

On the other hand, Theorem 6.3 shows that \( x \) lies inside the global lens \( \Gamma_{+i} \) precisely when \( i = p + q - 2, p + q - 3, \ldots, j \). Hence the relevant indices in the backward sweep in Algorithm 6.6 are \( i = p + q - 2, p + q - 3, \ldots, j \). During the corresponding operations, the entries in rows \( p, p - 1, \ldots, k(j) + 1 \) and columns \( q, q - 1, \ldots, l(j) + 1 \) of \( J_{1,2}(x) \) are cancelled by Gaussian elimination.

It follows that at the end of the two sweeps in Algorithm 6.6 all the entries of the rank-one matrix \( J_{1,2}(x) \) are eliminated, except for the \( (k(j), l(j)) \) entry which equals

\[
\exp(n(\lambda_{p+l(j),+}(x) - \lambda_{k(j),-}(x))).
\]

Recall that in the above description, we assumed that \( x \in (y_j, x_{j-1}) \supset [\alpha_j, \beta_j] \). Next, let us assume that \( x \) belongs to one of the gaps \( (x_j, y_j) \) for certain \( j \). We can then repeat the above arguments and find that at the end of Algorithm 6.6 all the entries of \( J_{1,2}(x) \) are eliminated except for two of them. In case of a vertical step (6.2)–(6.3) these are the \( (k(j), l(j)) \) and \( (k(j) + 1, l(j)) \) entries, which are given by

\[
\exp(n(\lambda_{p+l(j)}(x) - \lambda_{k(j)}(x))), \quad \exp(n(\lambda_{p+l(j)}(x) - \lambda_{k(j)+1}(x))),
\]

respectively. But by the variational inequality in (4.22), which is strict according to Lemma 4.1, we see that both entries are exponentially small for \( n \to \infty \). A similar argument applies in case of a horizontal step (6.17)–(6.18).

Finally we should note that by the operations in Algorithm 6.6 the RH matrix \( T(z) \) also has a jump on each of the contours \( \Gamma_{+i} \) and \( \Gamma_{-i} \). For example, in case of a vertical step (6.2)–(6.3) the jump matrix on the contour \( \Gamma_{+i} \) takes the form

\[
I_{p+q} \pm \exp(n(\lambda_k(z) - \lambda_{k+1}(z))) e_{k+1} e_k^T,
\]

see Algorithm 6.6. But by our assumption that \( \Gamma_{+i} \subset \Omega_+ \) we see that this jump matrix is uniformly exponentially close to the identity matrix when \( n \to \infty \). A similar argument holds for the jumps along the contours \( \Gamma_{-i} \subset \Omega_- \).

Summarizing, we established that \( T \) satisfies the following RH problem.

**RH problem 6.7.**

1. \( T \) is analytic on \( \mathbb{C} \setminus (\mathbb{R} \cup \bigcup_{i=1}^{p+q-2}(\Gamma_{+i} \cup \Gamma_{-i})) \).

2. For \( x \in \mathbb{R} \cup \bigcup_{i=1}^{p+q-2}(\Gamma_{+i} \cup \Gamma_{-i}) \) we have that

\[
T_+(x) = T_-(x) J_T(x)
\]

where \( J_T(x) \) satisfies the following

\[
\text{(6.24)}.
\]
Figure 11: The figure shows the contours in the RH problem for the matrix $T$ for the example of Figures 4 and 9. We have three intervals $[\alpha_i, \beta_i], i = 1, 2, 3$, and four global lenses $\Gamma_{\pm,i}, i = 1, 2$.

(a) For $x \in (y_i, x_{i-1}) \supset [\alpha_i, \beta_i], i = 1, \ldots, p+q-1$, we have that $J_T(x)$ equals the identity matrix, except for the $2 \times 2$ block lying on the intersection of rows and columns $k(i)$ and $p+l(i)$, which is given by

$$
\begin{pmatrix}
\exp(n(\lambda_{k(i)+1}, (x) - \lambda_{k(i)-1}, (x))) & \exp(n(\lambda_{p+l(i)+1}, (x) - \lambda_{k(i)-1}, (x))) \\
0 & \exp(n(\lambda_{p+l(i)+1}, (x) - \lambda_{p+l(i)-1}, (x)))
\end{pmatrix}
$$

(6.25)

(b) For $x \in (-\infty, y_{p+q-1}) \cup \bigcup_i (y_i, x_i) \cup (x_0, \infty)$ we have that $J_T(x)$ is exponentially close to the identity matrix as $n \to \infty$, both uniformly as well as in $L^2$ sense.

(c) For $x \in \bigcup_i (\Gamma_{+,i} \cup \Gamma_{-,i})$, the jump matrix $J_T$ is also exponentially close to the identity matrix as $n \to \infty$ in uniform sense (and therefore also in $L^2$ since the contours $\Gamma_{\pm,i}$ are compact).

(3) As $z \to \infty$, we have that

$$
T(z) = I_{p+q} + O(1/z).
$$

(6.26)

Let us illustrate this RH problem for the example in Figures 4 and 9. Then we have three intervals $[\alpha_i, \beta_i], i = 1, 2, 3$, and four global lenses between them: see Figure 11.

The jump conditions (6.24)–(6.25) can now be written as

$$
T_+ = T_-egin{pmatrix}
\exp(n(\lambda_{1+} - \lambda_{1-})) & 0 & \exp(n(\lambda_{3+} - \lambda_{1-})) & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & \exp(n(\lambda_{3+} - \lambda_{3-})) & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
$$

on the interval $(y_1, x_0) \supset [\alpha_1, \beta_1]$,

$$
T_+ = T_-egin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \exp(n(\lambda_{1+} - \lambda_{2-})) & \exp(n(\lambda_{1+} - \lambda_{2-})) & 0 \\
0 & 0 & \exp(n(\lambda_{3+} - \lambda_{2-})) & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
$$
on the interval \((y_2, x_1) \supset [\alpha_2, \beta_2]\), and

\[
T_+ = T_- \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \exp(n(\lambda_{2,+} - \lambda_{2,-})) & 0 & \exp(n(\lambda_{4,+} - \lambda_{4,-})) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \exp(n(\lambda_{4,+} - \lambda_{4,-})) \end{pmatrix}
\]

on the interval \((y_3, x_2) \supset [\alpha_3, \beta_3]\). The jump matrices on the remaining contours \((-\infty, y_3)\), \((x_2, y_2)\), \((x_1, y_1)\), \((x_0, \infty)\), \(\Gamma_{+,1}\), \(\Gamma_{-,1}\), \(\Gamma_{+,2}\) and \(\Gamma_{-,2}\) in Figure 11 are all exponentially close to the identity matrix as \(n \to \infty\).

7 Final transformations of the RH problem

The jump matrices \(J_T\) in the RH problem for \(T\) are nontrivial only in the \(2 \times 2\) block given by (6.25) on the interval \([\alpha_i, \beta_i]\).

7.1 Third transformation: Opening of the local lenses

In the transformation \(T \to S\) of the RH problem we transform the oscillatory entries of the jump matrix (6.24)–(6.25) along each interval \([\alpha_i, \beta_i]\) into exponentially decaying ones. To this end we open a local lens around the interval \([\alpha_i, \beta_i]\) = \([\alpha_i, \beta_i]\). Since the RH problem is locally of size 2 by 2 this can be done in the standard way [11] [13].

For each \(i = 1, \ldots, p + q - 1\) we open a lens around the interval \([\alpha_i, \beta_i]\), \(i = 1, \ldots, p + q - 1\) as in Figure 12 so that

\[
\text{Re}(\lambda_{k(i)} - \lambda_{p+1(i)}) < 0
\]

on the lips of the lens. It follows from an argument based on the Cauchy-Riemann equations that this is indeed possible, cf. [11]. The lenses are small (so called local lenses), and in particular they do not intersect with the global lenses \(\Gamma_{\pm,i}\). We use \(L_{+,i}\) and \(L_{-,i}\) to denote the upper and lower lip of the lens, respectively.
We define the matrix function $S$ as follows

$$
S(z) = \begin{cases} 
T(z) \left( I_{p+q} - \exp(n(\lambda_{k(i)}(z) - \lambda_{p+l(i)}(z))) e_{p+l(i)} e_{k(i)}^T \right), & \text{in the upper part of the lens around } [\alpha_i, \beta_i], \\
T(z) \left( I_{p+q} + \exp(n(\lambda_{k(i)}(z) - \lambda_{p+l(i)}(z))) e_{p+l(i)} e_{k(i)}^T \right), & \text{in the lower part of the lens around } [\alpha_i, \beta_i], \\
T(z), & \text{outside of all the lenses.}
\end{cases}
$$

(7.1)

Then $S$ satisfies the following RH problem.

**RH problem 7.1.**

1. $S$ is analytic in $\mathbb{C} \setminus (\mathbb{R} \cup \bigcup_i (\Gamma_{+,i} \cup \Gamma_{-,i}) \cup \bigcup_i (L_{+,i} \cup L_{-,i}))$;

2. For $x \in \mathbb{R} \cup \bigcup_i (\Gamma_{+,i} \cup \Gamma_{-,i}) \cup \bigcup_i (L_{+,i} \cup L_{-,i})$ we have that

$$
S_+(x) = S_-(x) J_S(x)
$$

(7.2)

where $J_S(x)$ satisfies the following

(a) For $x \in [y_i, x_{i-1}] \cup L_{+,i} \cup L_{-,i}$ we have that $J_S(x)$ is the identity matrix except for the $2 \times 2$ block on the intersection of rows and columns $k(i)$ and $p + l(i)$, which is given by

$$
\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \text{for } x \in [\alpha_i, \beta_i],
$$

(7.3)

and by

$$
\begin{pmatrix} \exp(n(\lambda_{k(i)} - \lambda_{p+l(i)})) & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{for } x \in L_{\pm,i},
$$

(7.4)

and by

$$
\begin{pmatrix} \exp(n(\lambda_{k(i),+}(x) - \lambda_{k(i),-}(x))) & \exp(n(\lambda_{p+l(i),+}(x) - \lambda_{p+l(i),-}(x))) \\ 0 & \exp(n(\lambda_{p+l(i),+}(x) - \lambda_{p+l(i),-}(x))) \end{pmatrix},
$$

(7.5)

for $x \in [y_i, \alpha_i] \cup [\beta_i, x_{i-1}]$.

(b) On the other parts of the contour we have $J_S(x) = J_T(x)$ and $J_S(x)$ is exponentially close to the identity matrix as $n \to \infty$, both uniformly and in the $L^2$ sense.

3. As $z \to \infty$, we have that

$$
S(z) = I_{p+q} + O(1/z).
$$

(7.6)

From standard arguments based on the Cauchy–Riemann conditions, it follows that the local lenses $L_i$ can be chosen so that the jumps on $L_i$ in (7.4) are uniformly exponentially close to the identity matrix, away for a neighborhood of the endpoints $\alpha_i$, $\beta_i$.

### 7.2 Global parametrix

In this subsection we build a global parametrix $P^{(\infty)}(z)$, which will be a good approximation to the RH problem away from the endpoints $\alpha_i$, $\beta_i$, $i = 1, \ldots, p + q - 1$. The construction will be quite similar to the one in [10].

We will construct the matrix function $P^{(\infty)}(z)$ such that it satisfies the following RH problem, obtained from RH problem (7.1) by ignoring all exponentially small entries of the jump matrices.
RH problem 7.2.

(1) \( P^{(\infty)}(z) \) is analytic in \( \mathbb{C} \setminus \bigcup_{i=1}^{p+q-1}[\alpha_i, \beta_i] \);

(2) For \( x \in \bigcup_i(\alpha_i, \beta_i) \), we have that
\[
P^{(\infty)}_+(x) = P^{(\infty)}_-(x) J_{P^{(\infty)}}(x) \tag{7.7}
\]
where the jump matrix \( J_{P^{(\infty)}}(x) \) equals
\[
J_{P^{(\infty)}}(x) = \begin{pmatrix}
I_{k(i) - 1} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & I_{p+i(i) - 1 - k(i)} & 0 & 0 \\
0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & I_{q-i(i)}
\end{pmatrix}, \tag{7.8}
\]
for \( x \in (\alpha_i, \beta_i) \).

(3) As \( z \to \infty \), we have that
\[
P^{(\infty)}(z) = I_{p+q} + O(1/z). \tag{7.9}
\]

To solve this RH problem, we will use the fact that the Riemann surface \( \mathcal{R} \) in Section 4.2 has genus zero. From general algebraic geometry [26], this implies the existence of a rational parametrization
\[
\xi = \xi(v), \quad z = z(v) \tag{7.10}
\]
where \( v \) runs through the extended complex plane \( \mathbb{C} \) (Riemann sphere).

The \( v \)-plane is then partitioned into \( p + q \) disjoint open sets \( \Omega_j \), \( j = 1, \ldots, p + q \), where \( \Omega_j \) is defined as the inverse image under (7.9) of the \( j \)th sheet \( \mathcal{R}_j \) of the Riemann surface. Correspondingly we have \( p + q \) inverse functions \( v_j(z) \) of (7.9) such that
\[
v_j: \mathcal{R}_j \rightarrow \Omega_j, \quad j = 1, \ldots, p + q, \tag{7.11}
\]
is a bijection. We use \( v_j(\infty) \) to denote the image under this map of the point at infinity of the \( j \)th sheet \( \mathcal{R}_j \), \( j = 1, \ldots, p + q \). Hence \( v_j(\infty) \in \Omega_j \).

For \( i = 1, \ldots, p + q - 1 \), the common boundary of \( \Omega_{k(i)} \) and \( \Omega_{p+i(i)} \) in the \( v \)-plane, is an analytic curve \( C_i \) with a natural partition
\[
C_i = C_{+i} \cup C_{-i}, \tag{7.12}
\]
where \( C_{+i} \) is the image of the upper side of the cut \([\alpha_i, \beta_i]\) under the mapping \( v_k \), and \( C_{-i} \) is the image of the lower side. The two parts \( C_{\pm i} \) meet at two points \( \gamma_i^{(1)} \) and \( \gamma_i^{(2)} \), which are the images of the endpoints \( \alpha_i \) and \( \beta_i \) of the cut \([\alpha_i, \beta_i]\), respectively.

Define the polynomial
\[
g(v) = \prod_{i=1}^{p+q-1} \left( v - \gamma_i^{(1)} \right) \left( v - \gamma_i^{(2)} \right), \tag{7.13}
\]
and its square root
\[
\sqrt{g(v)}
\]

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which is defined as an analytic function in the \( v \)-plane, with a cut along the disjoint union of arcs \( \bigcup_j \mathcal{C}_{+,i} \). We assume \( \sqrt{g(v)} \sim v^{p+q-1} \) as \( v \to \infty \).

We then construct a global parametrix \( P^{(\infty)}(z) \) as in \cite{10}. We define for \( z \in \mathbb{C} \setminus \bigcup_i [\alpha_i, \beta_i] \),

\[
P^{(\infty)}(z) = \left( f_i(v_j(z)) \right)_{i,j=1}^{p+q}, \quad f_i(v) = \frac{l_i(v)}{\sqrt{g(v)}}, \tag{7.13}
\]

where \( l_i \) is the Lagrange interpolation polynomial for the points \( v_1(\infty), \ldots, v_{p+q}(\infty) \). That is, \( l_i \) is a polynomial of degree \( p + q - 1 \) so that

\[
f_i(v_j(\infty)) = \delta_{ij}, \quad j = 1, \ldots, p + q.
\]

The fact that \( P^{(\infty)}(z) \) in (7.13) satisfies conditions (1) and (3) in RH problem 7.2 is immediate. For the jump condition (2) we need to show that

\[
\begin{align*}
&f_i(v_{k(i),+}(x)) = -f_i(v_{p+i(i),-}(x)), \\
&f_i(v_{p+i(i),+}(x)) = f_i(v_{k(i),-}(x)),
\end{align*}
\quad x \in [\alpha_i, \beta_i]. \tag{7.14}
\]

These relations reduce to

\[
\begin{align*}
&f_{i,+}(v) = -f_{i,-}(v), \quad \text{for } v \in \mathcal{C}_{+,i}, \tag{7.15} \\
&f_{i,+}(v) = f_{i,-}(v), \quad \text{for } v \in \mathcal{C}_{-,i} \tag{7.16}
\end{align*}
\]

and these jumps follow from (7.13), since we have chosen the square root in \( \sqrt{g(v)} \) with a cut along the union of arcs \( \bigcup_j \mathcal{C}_{+,i} \).

### 7.3 Local parametrices around the endpoints: Airy parametrices

In a small disk around the endpoints \( \alpha_i \) and \( \beta_i \) of the interval \( [\alpha_i, \beta_i] \) we construct a local parametrix \( P^{(\text{Airy})}(z) \) involving Airy functions. Since the RH problem is locally of size \( 2 \times 2 \) and the equilibrium measures all vanish as a square root, this can be done in the standard way \cite{11}. We omit the details.

### 7.4 Final transformation of the RH problem

Using the global parametrix \( P^{(\infty)} \) of Section 7.2 and the local parametrices \( P^{(\text{Airy})} \) of Section 7.3 we define the final transformation \( S \mapsto R \) of the RH problem by

\[
R(z) = \begin{cases} 
S(z) (P^{(\text{Airy})}^{-1}(z)), & \text{in the disks around } \alpha_i, \beta_i, \quad i = 1, \ldots, p + q - 1, \\\nS(z) (P^{(\infty)}^{-1}(z)), & \text{elsewhere}.
\end{cases} \tag{7.17}
\]

From the construction of the parametrices it then follows that \( R \) satisfies the following RH problem.

**RH problem 7.3.**

1. \( R(z) \) is analytic in \( \mathbb{C} \setminus \Sigma_R \) where \( \Sigma_R \) is the contour shown in Figure 13.
2. \( R \) has jumps \( R_+ = R_- J_R \) on \( \Sigma_R \), where \( J_R(z) = I_{p+q} + O(1/n) \), on the boundaries of the disks,

\[
J_R(z) = I_{p+q} + O(e^{-c|\alpha_n|z+1}), \quad \text{on the other parts of } \Sigma_R,
\]

for some constant \( c > 0 \).
(3) \( R(z) = I_{p+q} + O(1/z) \) as \( z \to \infty \).

As \( n \to \infty \), the jump matrix \( J_R \) tends to the identity matrix both in \( L^\infty(\Sigma_R) \) and in \( L^2(\Sigma_R) \). Then as in [11, 13, 14] we may conclude that

\[
R(z) = I_{p+q} + O\left(\frac{1}{n(|z| + 1)}\right)
\]

as \( n \to \infty \), uniformly for \( z \) in the complex plane. This completes the RH steepest descent analysis.

8 Proof of Theorem 2.4

Now we are ready to prove the main Theorem 2.4 by unfolding the transformations of the RH steepest descent analysis. Compare with the proofs in the earlier papers [5, 6, 10].

For a finite \( n \) we define the function \( \rho^{(n)} \) as

\[
\rho^{(n)}(x) = \frac{1}{\pi} \text{Im} \xi^{(n)}_{k(i),+}(x), \quad x \in [\alpha_i^{(n)}, \beta_i^{(n)}], \quad i = 1, \ldots, p + q - 1,
\]

\[
\rho^{(n)}(x) = 0, \quad x \in \mathbb{R} \setminus \bigcup_{i=1}^{p+q-1} [\alpha_i^{(n)}, \beta_i^{(n)}].
\]

Here we write \( \xi^{(n)}_{k(i),+}(x) \) and \( \alpha_i^{(n)}, \beta_i^{(n)} \) to emphasize the \( n \)-dependence.

We recall that

\[
\text{Im} \xi^{(n)}_{k(i),+} = -\text{Im} \xi^{(n)}_{k(i),-} = -\text{Im} \xi^{(n)}_{p+l(i),+} = \text{Im} \xi^{(n)}_{p+l(i),-}
\]
on the interval \( [\alpha_i^{(n)}, \beta_i^{(n)}] \), so one has in fact several equivalent ways of expressing (8.1).

By (4.8), (8.1) and the Stieltjes-Perron inversion formula one has that

\[
\rho^{(n)}(x) = \frac{d\mu^{(n)}_{k(i)}(x)}{dx}, \quad x \in [\alpha_i^{(n)}, \beta_i^{(n)}].
\]

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As $n \to \infty$, we have that $\alpha_i^{(n)} \to \alpha_i, \beta_i^{(n)} \to \beta_i$ and

$$\lim_{n \to \infty} \rho^{(n)}(x) = \rho_i(x), \quad x \in (\alpha_i, \beta_i),$$

(8.3)

where

$$\rho_i(x) = \frac{d\mu_i(x)}{dx}, \quad x \in [\alpha_i, \beta_i]$$

is the density of the $i$th component $\mu_i$ of the minimizer $(\mu_1, \ldots, \mu_{p+q-1})$ of the vector equilibrium problem with transition numbers $(t_{k,l})$.

Now we show that the $\rho_i$ give indeed the limiting distribution of the non-intersecting Brownian motions. To this end we will use (2.37). We start with the expression for the correlation kernel $K_n(x, y)$.

$$K_n(x, y) = \frac{1}{2\pi i(x - y)} \begin{pmatrix} w_{1,1}(y) & \cdots & 0 & w_{2,1}(y) & \cdots & w_{2,q}(y) \end{pmatrix} Y_+^{-1}(y)Y_+(x) \begin{pmatrix} w_{1,1}(x) \\ \vdots \\ w_{1,p}(x) \\ \vdots \\ 0 \end{pmatrix}.$$  

From the first transformation $Y \to X$ in (5.1)–(5.3) we get (we do not explicitly write the $n$-dependence in the $\lambda$-functions)

$$K_n(x, y) = \frac{1}{2\pi i(x - y)} \begin{pmatrix} 0 & \cdots & 0 & e^{n\lambda_{p+1},+(y)} & \cdots & e^{n\lambda_{p+q},+(y)} \end{pmatrix} X_+^{-1}(y)X_+(x) \begin{pmatrix} e^{-n\lambda_{1,+,}(x)} \\ \vdots \\ e^{-n\lambda_{p,+,}(x)} \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$  

From the second transformation $X \to T$ we obtain for $x, y \in (\alpha_i^{(n)}, \beta_i^{(n)})$,

$$K_n(x, y) = \frac{1}{2\pi i(x - y)} \left( e^{n\lambda_{p+1},+(y)}e^{T_{p+1}(i)} \right) T_+^{-1}(y)T_+(x) \left( e^{-n\lambda_{1,+,}(x)} + e^{n\lambda_{p,+,}(x)} \right).$$

From the third transformation $T \to S$ in (7.1) we get

$$K_n(x, y) = \frac{1}{2\pi i(x - y)} \left( -e^{n\lambda_{k,(i),+}(y)}e^{T_{k,(i)}} + e^{n\lambda_{p+1},+(y)}e^{T_{p+1}(i)} \right) \times S_+^{-1}(y)S_+(x) \left( e^{-n\lambda_{k,(i),+}(x)} + e^{-n\lambda_{p+1},+(x)} \right),$$

(8.4)

for $x, y \in (\alpha_i^{(n)}, \beta_i^{(n)})$. Defining the function $h_n$ by

$$h_n(x) := -\operatorname{Re}(\lambda_{k,(i),+}(x)) = -\operatorname{Re}(\lambda_{p+1,(i),+}(x)), \quad x \in [\alpha_i^{(n)}, \beta_i^{(n)}],$$

(8.5)

we see that (5.4) can be rewritten as

$$K_n(x, y) = \frac{e^{n(h_n(x) - h_n(y))}}{2\pi i(x - y)} \left( -e^{n\operatorname{Im}(\lambda_{k,(i),+}(y))}e^{T_{k,(i)}} + e^{-n\operatorname{Im}(\lambda_{k,(i),+}(y))}e^{T_{p+1}(i)} \right) \times S_+^{-1}(y)S_+(x) \left( e^{-n\operatorname{Im}(\lambda_{k,(i),+}(x))} + e^{n\operatorname{Im}(\lambda_{k,(i),+}(x))} \right),$$

(8.6)
for \( x, y \in (\alpha_i, \beta_i) \).

Now from (7.18) it follows by standard arguments (e.g. [5, Section 9]) that

\[
S^{-1}(y)S(x) = I + O(x - y), \quad \text{as } y \to x
\]

uniformly in \( n \). Hence (8.6) takes the form

\[
K_n(x, y) = e^{n(h_n(x) - h_n(y))} \left( \frac{\sin(n \Im (\lambda_{k(i),+}(x) - \lambda_{k(i),+}(y)))}{\pi(x - y)} + O(1) \right), \quad (8.7)
\]

for \( x, y \in (\alpha_i, \beta_i) \), where the \( O(1) \) term holds uniformly in \( n \). Then by letting \( y \to x \) and using l’Hôpital’s rule we find

\[
K_n(x, x) = \frac{n}{\pi} \Im (\zeta^{(n)}_{k(i),+}(x)) + O(1),
\]

for \( x \in (\alpha_i, \beta_i) \), or equivalently

\[
K_n(x, x) = n \rho^n(x) + O(1),
\]

by virtue of (8.3). It follows from (8.3) that

\[
\lim_{n \to \infty} \frac{1}{n} K_n(x, x) = \rho_i(x), \quad x \in (\alpha_i, \beta_i). \quad (8.8)
\]

In a similar way one can prove that

\[
\lim_{n \to \infty} \frac{1}{n} K_n(x, x) = 0, \quad x \in \mathbb{R} \setminus \bigcup_i [\alpha_i, \beta_i]. \quad (8.9)
\]

This completes the proof of Theorem 2.4.

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