ON LOG CANONICAL DIVISORS THAT ARE LOG QUASI-NUMERICALLY POSITIVE

SHIGETAKA FUKUDA

Abstract. Let \((X, \Delta)\) be a four-dimensional log variety that is projective over the field of complex numbers. Assume that \((X, \Delta)\) is not Kawamata log terminal (klt) but divisorial log terminal (dlt). First we introduce the notion of “log quasi-numerically positive”, by relaxing that of “numerically positive”. Next we prove that, if the log canonical divisor \(K_X + \Delta\) is log quasi-numerically positive on \((X, \Delta)\) then it is semi-ample.

Contents

1. Introduction 1
2. The subadjunction theory of Kawamata-Shokurov 2
3. Reduction of the non-klt but dlt case, to the klt case in lower dimensions 3
4. Proof of the main theorem 4
References 4

1. Introduction

Throughout the paper every variety is projective over the field of complex numbers. We follow the notation and terminology of the proceedings [7] of “the second Utah seminar”.

Definition 1.1. A \(\mathbb{Q}\)-Cartier \(\mathbb{Q}\)-divisor \(L\) on a projective variety \(X\) is numerically positive (nup, for short) if \((L, C) > 0\) for every curve \(C\) on \(X\). A nef \(\mathbb{Q}\)-divisor \(L\) on \(X\) is quasi-numerically positive (quasi-nup, for short) if there exists a union \(V\) of at most countably many Zariski-closed subsets \(\subseteq X\) such that \((L, C) > 0\) for every curve \(C\) not contained in \(V\). A quasi-nup \(\mathbb{Q}\)-divisor \(L\) on \(X\) is log quasi-numerically positive (log quasi-nup, for short) on a divisorial log terminal (dlt) variety \((X, \Delta)\) if \(L|_B\) is quasi-nup for every non-Kawamata log terminal (non-klt) center \(B\) (in other words, for every \(B \in C_{\text{non-klt}}(X, \Delta)\), under the notation of Section 2).

Of course, the nupness (resp. the log quasi-nupness) implies the log quasi-nupness (resp. the quasi-nupness). In the case where \((X, \Delta)\) is Kawamata log terminal (klt), the quasi-nupness is equivalent to the log quasi-nupness.

Recently F. Ambro ([1]) reduced the famous log abundance conjecture, which claims that the nef log canonical divisors should be semi-ample, for klt varieties.

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to the log minimal model conjecture and the problem of semi-ampleness of the quasi-nup log canonical divisors:

**Problem 1.2.** Assume that $(S, D)$ is klt and $K_S + D$ is quasi-nup. Is $K_S + D$ semi-ample?

With regard to Problem 1.2 we consider the following

**Problem 1.3.** Assume that $(X, \Delta)$ is not klt but dlt and $K_X + \Delta$ is log quasi-nup on $(X, \Delta)$. Is $K_X + \Delta$ semi-ample?

We note that the log abundance conjecture (including Problems 1.2 and 1.3) for dlt varieties is known to be true in dimension $\leq 3$ (\cite{3, 6}). The subadjunction theory of Kawamata-Shokurov (in Section 2) and a uniruledness theorem of Mori-Miyaoka type (due to Matsuki \cite{9}) enable us to reduce Problem 1.3 (where $|\Delta| \neq 0$) in dimension $n$ to Problem 1.2 (where $|D| = 0$) in dimension $\leq n - 1$ (see Proposition 3.4) and obtain the following main theorem.

**Theorem 1.4.** Assume that $(X, \Delta)$ is not klt but dlt, dim $X = 4$ and $K_X + \Delta$ is log quasi-nup on $(X, \Delta)$. Then $K_X + \Delta$ is semi-ample.

In the case where $X$ is smooth and $\Delta$ is reduced and with only simple normal crossings, the theorem was proved in \cite{4}.

2. **The subadjunction theory of Kawamata-Shokurov**

We recall the subadjunction theory of Kawamata (cf. \cite{3} Lemma 5-1-9) and Shokurov (\cite{10} Subsection 3.2.3), clarify the notion of minimal non-klt centers and fix the relevant notation.

A log variety $(X, \Delta)$ consists of a normal variety $X$ and an effective $\mathbb{Q}$-divisor $\Delta$ on $X$ such that $|\Delta|$ is reduced. The log variety $(X, \Delta)$ is said to be divisorial log terminal (dlt, for short) if $K_X + \Delta$ is $\mathbb{Q}$-Cartier and there exists a projective log resolution $f : Y \to X$ such that $K_Y + f_*^{-1}\Delta = f^*(K_X + \Delta) + E$ with the properties that $|E| \geq 0$, Exc($f$) is a divisor and Supp($f_*^{-1}\Delta) \cup$ Exc($f$) is a simply normal crossing divisor. (Moreover if $|\Delta| = 0$, the log variety $(X, \Delta)$ is said to be Kawamata log terminal (klt, for short).)

We set $D_i := f_*^{-1}\Delta_i$, where $|\Delta| = \sum_{i=1}^l \Delta_i$ and $\Delta_i$ is a prime divisor. Define $\text{Strata}_f(X, \Delta) := \{\Gamma; k \geq 1, 1 \leq i_1 < i_2 < \ldots < i_k \leq l, \Gamma$ is an irreducible component of $D_{i_1} \cap D_{i_2} \cap \ldots \cap D_{i_k} \neq \emptyset\}$. The set of non-klt centers $C_{\text{non-klt}}(X, \Delta) := \{f(\Gamma); \Gamma \in \text{Strata}_f(X, \Delta)\}$ is known not to depend on the choice of $f$. Note that Exc($f) \nsubseteq \Gamma$ for every $\Gamma \in \text{Strata}_f(X, \Delta)$, because Exc($f$) is a divisor and Supp($\sum_{i=1}^l D_i) \cup$ Exc($f$) is a simply normal crossing divisor. Thus if for every $B \in C_{\text{non-klt}}(X, \Delta)$, the morphism $f$ is isomorphic over some suitable neighborhood of the generic point of $B$. Therefore $f(\Gamma_1) \subset f(\Gamma_2)$ if and only if $\Gamma_1 \subset \Gamma_2$, for $\Gamma_1, \Gamma_2 \in \text{Strata}_f(X, \Delta)$. Consequently the set of minimal non-klt centers $M C_{\text{non-klt}}(X, \Delta) := \{B; B$ is a minimal element (with respect to inclusion) of $C_{\text{non-klt}}(X, \Delta)\}$ coincides with the set $\{f(\Gamma); \Gamma$ is a minimal element of $\text{Strata}_f(X, \Delta)\}$.

Now we focus on the subvariety $\Delta_i$ of $X$. From Kollár-Mori (\cite{8} Corollary 5.52), $\Delta_i$ is normal and from the subadjunction theorem (Kawamata-Matsuda-Matsuki \cite{5} Lemma 5-1-9)), $\text{Diff}_\Delta(\Delta_i) \geq 0$, where $(K_X + \Delta)|_{\Delta_i} = K_{\Delta_i} + \text{Diff}_\Delta(\Delta_i - \Delta_i)$. Put $f_i := f|_{\Delta_i}$ and $\text{Strata}_f(X, \Delta)|_{\Delta_i} := \{\Gamma \in \text{Strata}_f(X, \Delta); \Gamma \subseteq \Delta_i\}$. Note that
thus $K_{D_j} + f_j^{-1}(\Delta - \Delta_i)|_{D_j} = f_j^*(K_{\Delta_j} + \text{Diff}_{\Delta_j}(\Delta - \Delta_i)) + E|_{D_j}$ and that $(f_j^{-1}(\Delta - \Delta_i)|_{D_j} - E|_{D_j}) = \sum\{D_j|_{D_j} : j \neq i; D_j \cap D_i \neq \emptyset\} + F$ (where $F$ is some divisor such that Supp $F$ does not contain any irreducible component of $D_j|_{D_j} \neq 0$ and that $-|F|$ is effective and $f_j$-exceptional because $\text{Diff}_{\Delta_j}(\Delta - \Delta_i) \geq 0$). Here $\text{Exc}(f_j) \not\subseteq \Gamma$ for any $\Gamma \in \text{Strata}_{\Delta}(X, \Delta)|_{D_j}$, since $\text{Exc}(f_j) \subseteq \text{Exc}(f)|D_i$. Hence, by considering a suitable embedded resolution of $\text{Exc}(f_j) \subseteq D_i$, we obtain that $(\Delta_i, \text{Diff}_{\Delta_i}(\Delta - \Delta_i))$ is dlt (Shokurov [10, Subsection 3.2.3] and Fujino [2, Proof of Theorem 0.1]).

Then $C_{\text{non-klt}}(\Delta_i, \text{Diff}_{\Delta_i}(\Delta - \Delta_i)) = \{f_i(\Gamma) ; \Gamma \in \text{Strata}_{\Delta}(X, \Delta)|_{D_i}\}$ and $C_{\text{non-klt}}(X, \Delta) = \bigcup_{j=1}^n C_{\text{non-klt}}(\Delta_j, \text{Diff}_{\Delta_j}(\Delta - \Delta_j) \cup \{\Delta_j\})$. Note that $f(\Gamma_1) \subset f(\Gamma_2) \subset \Delta_i$ if and only if $\Gamma_1 \subset \Gamma_2 \subset D_i$ for $\Gamma_1, \Gamma_2 \in \text{Strata}_{\Delta}(X, \Delta)$. Therefore $C_{\text{non-klt}}(\Delta_i, \text{Diff}_{\Delta_i}(\Delta - \Delta_i)) = C_{\text{non-klt}}(X, \Delta) \cap C_{\text{non-klt}}(\Delta_i, \text{Diff}_{\Delta_i}(\Delta - \Delta_i))$.

We define the maximal dimension of minimal non-klt centers by $l(X, \Delta) := \max\{\dim B : B \in C_{\text{non-klt}}(X, \Delta)\}$ in the case where $(X, \Delta)$ is not klt but dlt. Of course, $l(X, \Delta) \leq \dim X - 1$ in this case.

3. REDUCTION OF THE NON-KLT BUT DLT CASE, TO THE KLT CASE IN LOWER DIMENSIONS

We reduce Problem 1 to Problem 2 in lower dimensions.

**Proposition 3.1.** Let $(X, \Delta)$ be a log variety that is not klt but dlt and whose log canonical divisor $K_X + \Delta$ is log quasi-np on $(X, \Delta)$. Assume that Problem 2 has an affirmative answer in dimension $\leq l(X, \Delta)$ (the maximal dimension of minimal non-klt centers). Then $K_X + \Delta$ is semi-ample.

**Proof.** We shall prove the proposition by induction on $n := \dim X$, heavily relying on the notation introduced in Section 2.

Note that $(K_X + \Delta)|_{\Delta_i} = K_{\Delta_i} + \text{Diff}_{\Delta_i}(\Delta - \Delta_i)$ is log quasi-np on the dlt variety $(\Delta_i, \text{Diff}_{\Delta_i}(\Delta - \Delta_i))$ from the fact that $(C_{\text{non-klt}}(\Delta_i, \text{Diff}_{\Delta_i}(\Delta - \Delta_i)) \cup \{\Delta_i\}) \subseteq C_{\text{non-klt}}(X, \Delta)$. Because $MC_{\text{non-klt}}(\Delta_i, \text{Diff}_{\Delta_i}(\Delta - \Delta_i)) = C_{\text{non-klt}}(X, \Delta)$, the inequality $l(\Delta_i, \text{Diff}_{\Delta_i}(\Delta - \Delta_i)) \leq l(X, \Delta)$ holds in the case where $(\Delta_i, \text{Diff}_{\Delta_i}(\Delta - \Delta_i))$ is not klt. Therefore we know that $(K_X + \Delta)|_{\Delta_i}$ is semi-ample from the induction hypothesis in this case. The $\mathbb{Q}$-divisor $(K_X + \Delta)|_{\Delta_i}$ is semi-ample also in the case where $(\Delta_i, \text{Diff}_{\Delta_i}(\Delta - \Delta_i))$ is klt, because the value of $l(X, \Delta)$ becomes $n - 1$ and hence the assumption of the theorem applies. Thus $((K_X + \Delta)|_{B})^{\dim B} > 0$ for every $B \in C_{\text{non-klt}}(X, \Delta) = \bigcup_{j=1}^n C_{\text{non-klt}}(\Delta_j, \text{Diff}_{\Delta_j}(\Delta - \Delta_j) \cup \{\Delta_j\})$.

Next we show that $(K_X + \Delta)^n > 0$. By assuming that $(K_X + \Delta)^n = 0$, we shall imply the contradiction. Note that $-K_Y f^*(K_X + \Delta)^{n-1} = (f_*^{-1}(\Delta - E) f^*(K_X + \Delta))^n = (f_*^{-1}(\Delta - E) f^*(K_X + \Delta))^n \geq (f_*^{-1}(\Delta - E) f^*(K_X + \Delta))^n = ((K_X + \Delta)|_{\Delta_i})^{\dim \Delta_i} > 0$ because $E$ is $f$-exceptional and $\Delta_i \in C_{\text{non-klt}}(X, \Delta)$. Thus from Matsuki (the uniruledness theorem of Mori-Miyaoka type [3]), $Y$ is covered by $f^*(K_X + \Delta)$-trivial curves. Therefore also $X$ is covered by $(K_X + \Delta)$-trivial curves. This is a contradiction, because $K_X + \Delta$ is quasi-np! So we have that $(K_X + \Delta)^n > 0$.

Consequently $K_X + \Delta$ becomes nef and log big on $(X, \Delta)$ (i.e. $K_X + \Delta$ is nef, $(K_X + \Delta)^n > 0$ and also $((K_X + \Delta)|_{B})^{\dim B} > 0$ for every $B \in C_{\text{non-klt}}(X, \Delta)$) and thus $K_X + \Delta$ is semi-ample, by virtue of the base point free theorem of Reid-type (Fujino [2]).
4. Proof of the main theorem

Proof of Theorem 1.4. Note that $l(X, \Delta) \leq \dim X - 1 = 3$. Thus the assumption of Proposition 3.1 is satisfied, from the log abundance theorem (3, 6) for klt varieties in dimensions 2 and 3. □

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Faculty of Education, Gifu Shotoku Gakuen University, Yanaizu-cho, Gifu 501-6194, Japan
E-mail address: fukuda@ha.shotoku.ac.jp