Unified equation of state for neutron stars and supernova cores using the nuclear energy-density functional theory

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Abstract. We present a unified approach to the equation of state (EoS) of dense matter at any temperature, based on the nuclear energy-density functional (EDF) theory. Both homogeneous and inhomogeneous phases can be treated consistently. In particular, we have constructed three different EoSs of cold catalyzed matter for a wide range of densities from $\sim 10^5$ g cm$^{-3}$ to $\sim 10^{15}$ g cm$^{-3}$. For this purpose, we have employed generalized Skyrme functionals fitted to essentially all experimental nuclear mass data and constrained to reproduce properties of homogeneous nuclear matter as obtained from many-body calculations. We have applied these unified EoSs to compute the structure of cold isolated neutron stars (NSs).

1. Introduction
Type II Supernovae (SNe) represent the end point of stellar evolution for stars with a mass greater than about $8 M_\odot$. When the core of massive stars exceeds the Chandrasekhar mass, it undergoes a gravitational collapse followed by a bounce leading to the formation of a shock wave which expels the outer layers of the star and triggers the explosion. The core shrinks and its density increases up to a few times the nuclear matter saturation density, thus producing a dense compact remnant: a NS or a black hole. Because of the wide range of conditions encountered in the collapsing core (densities from $10$ to $10^{15}$ g cm$^{-3}$, temperatures from 0 to 100 MeV, and electron fraction $Y_e$ from 0 - pure neutron matter - to 0.6), a unified description of the matter is a very challenging task. Indeed, the interiors of SN cores and NSs are expected to exhibit very different phases of matter, from ordinary nuclei to homogeneous nuclear matter. The EDF theory allows for a consistent and computationally tractable treatment of these various phases, both at zero and at finite temperatures. At very high densities, matter might contain other particles like hyperons or even deconfined quarks (see e.g. Ref. [1]). The EDF theory can be extended to take into account these additional particles, but here we will only consider nucleons and leptons. We have constructed a set of three different unified EoSs describing all regions of the NS, from the surface to the core, as described in the next section.
2. Construction of the energy-density functional

The EDFs that we consider here are based on generalized Skyrme effective nucleon-nucleon interactions [2, 3]:

\[
v_{ij} = t_0(1 + x_0 P_\sigma) \delta (r_{ij}) + \frac{1}{2} t_1(1 + x_1 P_\sigma) \frac{1}{h^2} \left[ p_{ij}^2 \delta (r_{ij}) + \delta (r_{ij}) p_{ij}^2 \right]
+ t_2(1 + x_2 P_\sigma) \frac{1}{h^2} p_{ij} \delta (r_{ij}) \cdot p_{ij} + \frac{1}{6} t_3(1 + x_3 P_\sigma) n(r)^\alpha \delta (r_{ij})
+ \frac{1}{2} t_4(1 + x_4 P_\sigma) \frac{1}{h^2} \left[ p_{ij}^2 n(r)^\beta \delta (r_{ij}) + \delta (r_{ij}) n(r)^\beta p_{ij}^2 \right]
+ t_5(1 + x_5 P_\sigma) \frac{1}{h^2} p_{ij} n(r)^7 \delta (r_{ij}) \cdot p_{ij} + \frac{1}{h^2} W_0(\sigma_i + \sigma_j) \cdot p_{ij} \times \delta (r_{ij}) \cdot p_{ij},
\]

supplemented with a microscopic contact pairing interaction [3, 4]. In Eq. (1), \( r_{ij} = r_i - r_j \), \( r = (r_i + r_j)/2 \), \( p_{ij} = -i h (\nabla_i - \nabla_j) \), is the relative momentum, \( P_\sigma \) is the two-body spin-exchange operator, and \( n(r) = n_n(r) + n_p(r) \) is the total local density, \( n_n(r) \) and \( n_p(r) \) being the neutron and proton densities, respectively. The \( t_4 \) and \( t_5 \) terms are the density-dependent generalizations of the \( t_1 \) and \( t_2 \) terms, respectively. These EDFs were fitted to the 2149 measured masses of nuclei with \( N \) and \( Z \geq 8 \) given in the 2003 AME [5], with an rms deviation of 0.58 MeV. These EDFs are therefore well-suited for describing the properties of neutron-rich nuclei found in the outer crust of a NS. The masses were obtained by adding to the Hartree-Fock-Bogoliubov (HFB) energy a phenomenological Wigner term and correction term for the spurious collective energy. The EDFs were simultaneously constrained to reproduce various properties of homogeneous nuclear matter as obtained from many-body calculations using realistic two- and three-nucleon interactions. Three different functionals BSk19, BSk20 and BSk21 have been constructed, constrained to reproduce three different neutron matter EoSs at \( T = 0 \) thus reflecting the current lack of knowledge of the high-density behaviour of nuclear matter (see Fig. 1 in Ref. [3]). In particular, BSk19 was adjusted to the softest EoS of neutron matter known to us, the EoS of Friedman and Pandharipande obtained from the realistic Urana \( \nu_{14} \) nucleon-nucleon force with the three-body force TNI [6], BSk20 was fitted to the EoS of Akmal et al. labeled “A18 + \delta v + UIX” in Ref. [7], whereas BSk21 was constrained to reproduce the very stiff “V18” EoS from Li and Schulze [8]. Furthermore, i) spurious spin and spin-isospin instabilities in nuclear matter that generally plague existing Skyrme functionals have been eliminated for all densities and temperatures prevailing in NS and SN cores [9], ii) a qualitatively realistic distribution of the potential energy among the four spin-isospin channels in nuclear matter has been obtained, iii) the isovector effective mass is found to be smaller than the isoscalar effective mass, in agreement with both experiments and many-body calculations. With these features, our EDFs can be reliably applied to all regions of the NS: the outer crust, the inner crust, and the liquid core.

In Fig. 1, we have plotted for our three functionals the free energy per baryon and the pressure of hot neutron matter, as a function of the baryon number density. For comparison, the many-body calculations at \( T = 0 \) from Friedman and Pandharipande [6], Akmal et al. [7], Li and Schulze [8] discussed previously, together with microscopic calculations at finite temperature based on the Self-Consistent Green Function (SCGF) method with CD Bonn potential [10] are also shown. As can be seen in Fig. 1, our functionals are in close agreement with these calculations, at least for densities up to nuclear saturation density, even though they were only fitted to realistic EoSs of neutron matter at zero temperature. We therefore conclude that our functionals are also suitable for describing the hot dense matter found in SN cores and proto-neutron stars.
Figure 1. Free energy per baryon (upper panels) and pressure (lower panels) of neutron matter as a function of the number baryon density, at temperatures $T = 0, 10, 20 \text{ MeV}$. The microscopic calculations at $T = 0$ are taken from Friedman and Pandharipande [6] (labeled as “FP”), Akmal et al. [7] (labeled as “APR”), and Li and Schulze [8] (labeled as “LS2”), while those based on Self-Consistent Green Function with CD Bonn potential (labeled as “SCGF”) are taken from Ref. [10].

3. Unified equation of state of neutron stars

3.1. The microscopic model

We recognize in the NS three distinct regions: i) the outer crust where nuclei are arranged in a body-centered lattice and coexist with a gas of relativistic electrons, ii) the inner crust where nuclei coexist with both electrons and unbound neutrons and iii) the liquid core which consists of a uniform mixture of nucleons and leptons. All regions were treated consistently using the same EDF at $T = 0$ under the assumption of $\beta$ equilibrium. The EoS of the outer crust was calculated in the framework of the BPS model [11] using the latest experimental atomic masses complemented with theoretical masses obtained coherently from our HFB mass model (see Ref. [12] for details). With increasing density, the equilibrium nuclei become more and more neutron-rich until neutrons start to drip out. This transition, which defines the boundary between the outer crust and the inner crust, occurs when the neutron chemical potential exceeds the neutron rest mass. For the inner crust, we employed the temperature-dependent extended Thomas-Fermi method (up to the 4th order) with proton shell effects added via the Strutinsky integral theorem [13]. This method, originally developed for calculating the EoS of SN cores, is a computationally fast approximation to the full Hartree-Fock equations. Neutron shell corrections, which are known to be much smaller than proton shell corrections [14], are neglected. In order to further reduce the computational work, we assumed that nuclear
clusters are spherical and we made use of the Wigner-Seitz approximation. In addition, the local nucleon number densities were parametrized as \((q = n \text{ or } p \text{ for neutrons and protons respectively})\) \(n_q(r) = n_{Bq} + n_{\Lambda q} f_q(r)\), in which \(n_{Bq}\) is a constant background term, while \(f_q(r)\) has the form of a modified Fermi distribution [15] given by

\[
f_q(r) = \frac{1}{1 + \exp \left\{ \left( \frac{C_q - R}{r - R} \right)^2 - 1 \right\} \exp \left( \frac{r - C_q}{a_q} \right)},
\]

\(R\) being the radius of the Wigner-Seitz cell. The parameters \(C_q\) determine the size of the clusters whereas \(a_q\) take into account the diffuseness of the clusters surface. In this “damped” form of the usual simple Fermi profile all density derivatives vanish at the surface of the cell, thereby ensuring a smooth matching of the nucleon distributions between adjacent cells, and also satisfying the necessary conditions for the validity of the underlying semi-classical expressions. The EoS was obtained by minimizing, at a fixed average baryon density \(\bar{n}\), the free energy per nucleon \(f\). This procedure is equivalent to the minimization of the Gibbs free energy per nucleon \(g\) but the former is computationally much more convenient. At zero temperature, \(f\) coincides with the internal energy per nucleon \(e\) which can be expressed as (see Refs. [13, 15] for details)

\[
e = e_{\text{nuc}} + e_e + e_C - Y_e Q_{n,\beta},
\]

where \(e_e\) is the electron kinetic energy per nucleon, \(e_C\) the Coulomb energy per nucleon and \(Y_e Q_{n,\beta}\) takes into account the neutron-proton mass difference (\(Q_{n,\beta} = 0.728\) MeV is the neutron \(\beta\)-decay energy and \(Y_e\) the electron fraction). The nuclear energy \(e_{\text{nuc}}\) is given by

\[
e_{\text{nuc}} = \frac{4\pi}{A} \int_{\text{cell}} r^2 E_{\text{ETF}}(r) dr + e_{\text{sh}}^p,
\]

where \(A\) is the total number of nucleons in the Wigner-Seitz cell, \(E_{\text{ETF}}(r)\) is the ETF approximation to the energy density \(E_{\text{Sky}}(r)\) given by the generalized Skyrme functional. Proton shell effects were added perturbatively using the Strutinsky integral method and are contained in \(e_{\text{sh}}^p\). The minimization of \(e\) with respect to the eight parameters \((Z, N,\) plus the three parameters of the Wigner-Seitz cell for each nucleon type) was performed using the CERN routine MINUIT.

The EoS of the core was calculated under the assumption of a uniform plasma of neutrons, protons, electrons and muons in \(\beta\) equilibrium.

### 3.2. Results

In Fig. 2, we show the EoSs corresponding to the three different functionals BSk19, BSk20 and BSk21 for all regions of a NS. For practical purpose, we have fitted our EoSs using the analytical representation proposed in Eq. (14) of Ref. [16]. The new coefficients of the fit are calculated using a least-squares method. The fits reproduce the tabulated EoSs with an error below 4% except near the crust-core boundary where the error may reach about 10%.

We have applied our unified EoSs to compute the mass-radius relationship of non-rotating NSs. The NS structure is determined using the LORENE library [17, 18]. Figure 3 shows the gravitational mass as a function of the circumferential radius of static NSs for our three EoSs. In particular, the maximum NS masses for our three different functionals BSk19, BSk20 and BSk21 are 1.86 \(M_\odot\), 2.15 \(M_\odot\), 2.28 \(M_\odot\) respectively. The shaded area corresponds to the largest (up-to-date) mass (1.97 ± 0.04 \(M_\odot\)) of the recently observed millisecond pulsar J1614–2230 [19]. This observation rules out the softest EoS BSk19.
Figure 2. Upper panel: pressure $P$ versus mass-energy density $\rho$ of cold catalyzed matter. Rarefied tabular data are indicated by symbols, while the solid lines show the analytical representation for non-rotational configurations discussed in the text. Lower panel: relative difference between the data and the fit.

Figure 3. Gravitational mass versus circumferential radius for non-rotating neutron stars. The configurations are calculated using three different equations of state, namely BSk19, BSk20 and BSk21.
4. Conclusion
We have presented three different unified EoSs of NS based on the EDF theory. These functionals are based on generalized Skyrme forces which were fitted to essentially all experimental nuclear mass data and, in addition, were constrained to reproduce several properties of homogeneous nuclear matter as obtained from microscopic calculations using realistic nucleon-nucleon potentials. These EoSs have also been applied to compute the static structure of NSs. We have found that the EoSs BSk20 and BSk21 are compatible with the recent observation of a high-mass NS. Although we have presented the EoSs for the limiting case of cold catalyzed matter as relevant for cold non-accreting NS, our method can be easily extended to the hot dense matter of SN cores.

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References
[1] Haensel P., Potekhin A. Y., and Yakovlev D. G. 2007, “Neutron Stars I. Equation of State and Structure”, Astrophysics and Space Science Library, 326
[2] Chamel N., Goriely S., and Pearson J. M. 2009, Phys. Rev. C, 80, 065804
[3] Goriely S., Chamel N., and Pearson J. M. 2010, Phys. Rev. C 82, 035804
[4] Chamel N. 2010, Phys. Rev. C 82, 014313
[5] Audi G., Wapstra A. H., and Thibault C. 2003, Nucl. Phys. A 729, 337
[6] Friedman B., and Pandharipande V. R. 1981, Nucl. Phys. A 361, 502
[7] Akmal A., Pandharipande V. R., and Ravenhall D. G. 1998, Phys. Rev. C 58, 1804
[8] Li Z. H., and Schulze H. J. 2008, Phys. Rev. C 78, 028801
[9] Chamel N., and Goriely S. 2010, Phys. Rev. C 82, 045804
[10] Rios A., Polls A., and Vilaña I. 2009, Phys. Rev. C 79, 025802
[11] Baym G., Pethick C., and Sutherland P. 1971, Astrophys. J., 170, 299
[12] Pearson J. M., Goriely S., and Chamel N. 2011, Phys. Rev. C 83, 065810
[13] Onsi M., Dutta A. K., Chatri H., Goriely S., Chamel N., and Pearson J. M. 2008, Phys. Rev. C 77, 065805
[14] Chamel N., Naimi S., Khan E., and Margueron J. 2007, Phys. Rev. C 75, 055806
[15] Pearson J. M., Chamel N., Goriely S. and Ducoin C., submitted to Phys. Rev. C
[16] Haensel P., and Potekhin A. Y. 2004, Astron. Astrophys. 428, 191
[17] http://www.laorene.obspm.fr
[18] Gourgoulhon E. 2010, Preprint arXiv:1003.5015
[19] Demorest P. B., Pennucci T., Ransom S. M., Roberts M. S. E., and Hessels J. W. T. 2010, Nature 467, 1081