Geometrical Formulation of Adiabatic Pumping as a Heat Engine

Yuki Hino and Hisao Hayakawa

Yukawa Institute for Theoretical Physics, Kyoto University, Kitashirakawa-ouwake cho, Sakyo-ku, Kyoto 606-8502, Japan

(Dated: March 13, 2020)

We investigate a heat engine under an adiabatic (Thouless) pumping process. In this process, the extracted work and lower bound on dissipated availability are characterized by a vector potential and a Riemannian metric, respectively. We derive a trade-off relation between the power and effective efficiency. We also explicitly calculate the trade-off relation as well as the power and efficiency for a spin-boson model coupled to two reservoirs.

I. INTRODUCTION

Adiabatic pumping is a process where an average current is generated even in the absence of an average bias under slow and periodic modulation of multiple parameters of the system. Thouless first proposed the theory of adiabatic pumping for an isolated quantum system [1, 2]. He showed that electrons can be transported by applying a time-periodic potential to one-dimensional isolated quantum systems under a periodic boundary condition. He also clarified that the charge transport in this system is essentially induced by a Berry-phase-like quantity in the space of the modulation parameters [1–4]. This phenomenon has been observed experimentally in various processes such as charge transport [5–12] and a spin pumping process [13]. Later, Brouwer extended the Thouless pumping in the isolated system to that in an open quantum system [14]. It has been recognised that the essence of Thouless pumping can be described by a classical master equation in which the Berry-Sinitsyn-Nemenman (BSN) phase is the generator of the pumping current [15, 16]. There are various papers on geometrical pumping processes in terms of scattering theory [17–23], classical master equations [24–32] and quantum master equations [33–38]. The extended fluctuation theorem for adiabatic pumping processes has also been studied [39, 40].

Geometrical concepts were also introduced in the context of finite-time thermodynamics [41]. The key concept was the thermodynamic length, which is originally introduced for macroscopic systems [42–46]. Later, this approach has been applied to a classical nanoscale system [47], a closed quantum system [48] and an open quantum system [49].

Recently, Brandner and Saito have established the geometrical formulation for microscopic heat engines in the adiabatic regime [50]. In this approach, the properties of the working system are described by a vector potential and a Riemannian metric in the space of control parameters. If one defines a driving protocol, an effective flux and a length are assigned to the protocol. Then, they provide the extracted work and lower bound on dissipated availability. On the other hand, Giri and Goswami proposed a quantum heat engine which includes the effect of the BSN phase by controlling temperatures of reservoirs [51].

Nevertheless, we cannot apply the previous approaches to a heat engine undergoing an adiabatic pumping process with equal average temperatures in both reservoirs, because (i) Ref. [50] only considers systems coupled to a single reservoir closed to an equilibrium steady state and (ii) Ref. [51] only considers the situation where the temperature of one of the reservoirs is always higher than that of the other one. Note that in this case, the outcome is dominated by the dynamical phase. Namely, it is difficult to observe the geometrical contribution in such a system. In this paper, therefore, we extend the geometrical formulation of Refs. [50] and [51] to a system which interacts with two reservoirs with equal average temperature under adiabatic modulation of thermodynamic quantities of the reservoirs and the target system. In the adiabatic regime, we obtain a geometrical representation of the extracted work and a lower bound of dissipated availability. We also derive a trade-off relation between the power and effective efficiency.

The organization of this paper is follows. In Sec. II, we explain the setup and geometrical formulation for describing the heat engine under an adiabatic pumping process. In Sec. III we apply our method to a two-level spin-boson system coupled to two reservoirs to validate our formulation. Finally, we discuss and summarize our results in Sec. IV. In Appendix A, we present detailed calculations to support the description in the main text.
Figure 1. A schematic of the total system which consists of the target system and the left and right reservoirs. We control the temperatures of the reservoirs $T^L$, $T^R$ and the parameter $\lambda$ of the system Hamiltonian $H(\lambda)$ by the external device. $P$ is the instantaneous power and $J^L$ and $J^R$ are the heat currents from the system to the left and right reservoirs, respectively.

II. GENERAL FRAMEWORK

A. Setup

In this paper, we consider a small system $S$ coupled to two reservoirs $L$ and $R$ (Fig. 1). Let us consider the situation that the reservoir $\alpha = L$ or $R$ is characterised by only the temperature $T^\alpha$. We also assume that the system $S$ is characterised by discrete states $i = 1, \ldots, n$ as in quantum cases. The Hamiltonian $H(\lambda)$ of the system $S$ is given by $H(\lambda) := \langle E_i(\lambda) \delta_{ij} \rangle$, where $E_i(\lambda)$ is the energy of the state $i$, $\lambda$ is a control parameter in the system and $\delta_{ij}$ is the Kronecker’s delta, respectively. In this paper, we control the set of parameters $\Lambda := (\Lambda^\mu) := (\lambda, T^L, T^R)$ for $\mu = W, L, R$ with period $\tau_p$. We assume that the dynamics of $S$ follows the master equation

$$\frac{d}{d\theta} |p(\theta)\rangle = e^{-1} K(\Lambda_\theta) |p(\theta)\rangle. \quad (1)$$

Here we have introduced the dimensionless time $\theta := (t - t_0)/\tau_p$ and the subscript on $\Lambda_\theta$ denotes the $\theta$-dependence of the modulation parameters $\Lambda$. We have also introduced the dimensionless control speed $\epsilon := \tau_{\text{mix}}/\tau_p$, where $t_0$ is the time after which the system has reached a periodic state and $\tau_{\text{mix}}$ is the characteristic time scale of the coupling between the system and the reservoirs. We have also introduced the vector $|p(\theta)\rangle := (p_1(\theta), \ldots, p_n(\theta))^T$, where $p_i(\theta)$ is the probability of state $i$ at $\theta$, $K(\Lambda_\theta) = \sum_{\alpha=L,R} K^{\alpha}(\Lambda_\theta) := \sum_{\alpha=L,R} \Lambda_\theta^{\alpha}(k_{ij}^\alpha(\Lambda_\theta))$ is the transition matrix and its $(ij)$-component $k_{ij}^\alpha(\Lambda_\theta)$ is the transition rate of $j \to i$ due to interaction with the reservoir $\alpha$ at $\theta$. We assume that the $\theta$-dependence of $K(\Lambda_\theta)$ only appears through the control parameters $\Lambda_\theta$.

B. Thermodynamic quantities

In this subsection, we introduce the thermodynamic quantities which characterize the heat engine. Let us introduce the dissipated availability $A$ [11]:

$$A := \int_0^1 d\theta \dot{T}_\theta \dot{\sigma}_{\text{ex}}(\theta), \quad (2)$$

where $1/\dot{T}_\theta := (1/T^L_\theta + 1/T^R_\theta)/2$ is the effective temperature. $\dot{\sigma}_{\text{ex}}(\theta)$ is the excess entropy production rate defined as

$$\dot{\sigma}_{\text{ex}}(\theta) := \dot{\sigma}_S(\theta) + \sum_{\alpha=L,R} J^{\alpha}(\theta)/T^\alpha_\theta, \quad (3)$$

where $\sigma_S(\theta)$ is the entropy of $S$ given as $\sigma_S(\theta) := -\sum_{i=1}^n p_i(\theta) \ln p_i(\theta)$, $J^{\alpha}(\theta)$ is the excess heat current from $S$ to reservoir $\alpha$ given as $J^{\alpha}(\theta) := J^{\alpha}(\theta) - \bar{J}^{\alpha}(\theta)$, where $J^{\alpha}(\theta) := -1|H(\lambda_\theta)K^{\alpha}(\Lambda_\theta)p(\theta)|$ is the total heat current from $S$ to reservoir $\alpha$ and $\bar{J}^{\alpha}(\theta) := -1|H(\lambda_\theta)K^{\alpha}(\Lambda_\theta)p^{\alpha}(\theta)|$ is the instantaneous steady state heat current from $S$ to reservoir $\alpha$. The excess entropy production rate $\dot{\sigma}_{\text{ex}}(\theta)$ satisfies the generalized Clausius inequality, i.e. $\dot{\sigma}_{\text{ex}}(\theta) \geq 0$ when the degree of nonequilibrium, i.e. the temperature difference between the two reservoirs is small [23–25]. The equality is achieved in the quasi-static limit ($\epsilon \to 0$). Therefore $A \geq 0$, where the equality is again achieved in the quasi-static limit.

The dissipated availability $A$ can be decomposed into three parts as $A = U - W - \Psi$. Here $U$ is the effective thermal energy given as

$$U := -\int_0^1 d\theta \dot{\sigma}_S(\theta)$$

$$= -\int_0^1 d\theta [\dot{\Lambda}_\theta^W f^L(\theta) + \dot{\Lambda}_\theta^R f^R(\theta)], \quad (4)$$

where

$$f^L(\theta) := -2 \left( \frac{T^R_\theta}{T^L_\theta + T^R_\theta} \right)^2 \sum_i p_i(\theta) \ln p_i(\theta), \quad (5)$$

$$f^R(\theta) := -2 \left( \frac{T^L_\theta}{T^L_\theta + T^R_\theta} \right)^2 \sum_i p_i(\theta) \ln p_i(\theta). \quad (6)$$

The work $W$ extracted from $S$ is given as

$$W := \int_0^1 d\theta \dot{\Lambda}_\theta^W f^W(\theta), \quad (7)$$

where

$$f^W(\theta) := -\langle 1|\partial_{\lambda} H(\lambda_\theta) |p(\theta)\rangle. \quad (8)$$
The correction term $\Psi$ arising from the presence of two reservoirs is given as

$$\Psi := \int_0^1 d\theta \Delta_\theta (J^L_{ex} - J^R_{ex})$$

$$= \int_0^1 d\theta \Delta^R_\theta \psi^\mu(\theta),$$

where $\Delta_\theta := (T^L_\theta - T^R_\theta)/(T^L_\theta + T^R_\theta)$. Here, $\psi^\mu(\theta)$ is given as

$$\psi^\mu(\theta) := \langle |\hat{\psi}^\mu(\Lambda_\theta)| p(\theta) \rangle$$

(9)

with

$$\hat{\psi}^\mu(\Lambda) := -\Delta \langle |H(\lambda)| K^L(\Lambda) - K^R(\Lambda) \rangle_k \partial_\mu$$

(10)

where $K^+(\Lambda_\theta)$ is the pseudo-inverse of $K(\Lambda_\theta)$, which is given in Eq. (11). Then the dissipated availability $A$ can be simplified as

$$A = -\int_0^1 d\theta (f^\mu(\theta) + \psi^\mu(\theta)) \Delta^R_\theta \geq 0.$$  

(12)

Let us introduce the ratio

$$\eta := \frac{W}{U - \Psi} = \frac{W}{W + A}.$$  

(13)

By the non-negativity of $A$, this quantity $\eta \leq 1$ so that $\eta$ can be considered as a measure of the efficiency [50]. Let us call $\eta$ the effective efficiency for later discussion. In the quasi-static limit, the equality in Eq. (12) is achieved, so $\eta = 1$. However in this limit the average power

$$P := \epsilon W$$

(14)

becomes zero. Finite-speed driving increases the power at the cost of the effective efficiency. We will discuss this trade-off relation in the next subsection.

C. Efficiency and Power in the Adiabatic Regime

In this section, we consider the thermodynamic properties of the pumping process in the adiabatic regime where the modulation of the external parameters is much slower than the relaxation rate of the system. In this regime, the solution of Eq. (1) can be expanded in terms of $\epsilon$ as

$$|p(\theta) \rangle \simeq |p^{ss}(\Lambda_\theta) \rangle + \epsilon |p^\mu(\Lambda_\theta) \rangle |\Lambda^R_\theta \rangle + O(\epsilon^2),$$

(15)

where $|p^{ss}(\Lambda_\theta) \rangle$ is the instantaneous steady state at $\theta$ which satisfies $K(\Lambda_\theta)|p^{ss}(\Lambda_\theta) \rangle = 0$. The first order correction $|p^\mu(\Lambda_\theta) \rangle$ is expressed as a function of $|p^{ss}(\Lambda_\theta) \rangle$ as in Eq. (10). By using Eq. (19), $f^\mu(\theta)$ and $\psi^\mu(\theta)$ can be written as

$$f^\mu(\theta) \simeq f^\mu_{ss}(\Lambda_\theta) + \epsilon f^\mu(\Lambda_\theta) \Lambda^R_\theta + O(\epsilon^2),$$

$$\psi^\mu(\theta) \simeq \psi^\mu_{ss}(\Lambda_\theta) + \epsilon \psi^\mu(\Lambda_\theta) \Lambda^R_\theta + O(\epsilon^2).$$

(16)

(17)

Here $f^\mu_{ss}(\Lambda)$ and $\psi^\mu_{ss}(\Lambda)$ are steady forces defined as

$$f^\mu_{ss}(\Lambda) := \langle |f^\mu(\Lambda) | p^{ss}(\Lambda) \rangle,$$

$$\psi^\mu_{ss}(\Lambda) := \langle |\psi^\mu(\Lambda) | p^{ss}(\Lambda) \rangle,$$

(18)

(19)

where $f^\mu(\Lambda)$ and $\psi^\mu(\Lambda)$ are force operators defined as

$$f^\mu(\Lambda) := -\partial_\lambda H(\lambda),$$

$$\psi^\mu(\Lambda) := \langle |\hat{\psi}^\mu(\Lambda) | p^{ss}(\Lambda) \rangle,$$

(20)

(21)

(22)

and $f^\mu_{11}(\Lambda)$ and $\psi^\mu_{11}(\Lambda)$ are adiabatic response coefficients defined as

$$f^\mu_{11}(\Lambda) := \langle |\hat{f}^\mu(\Lambda) | p^\mu(\Lambda) \rangle,$$

$$\psi^\mu_{11}(\Lambda) := \langle |\hat{\psi}^\mu(\Lambda) | p^\mu(\Lambda) \rangle,$$

(23)

(24)

By using Eqs. (16) and (17). The dissipated availability $A$ can be written as

$$A \simeq \epsilon \int_0^1 d\theta g^{\mu\mu}(\Lambda_\theta) \Lambda^R_\theta \Lambda^R_\theta + O(\epsilon^2),$$

(25)

where

$$g^{\mu\mu}(\Lambda) := -\frac{1}{2} \left[ f^\mu_{11}(\Lambda) + f^\mu_{11}(\Lambda) + \psi^\mu(\Lambda) + \epsilon \psi^\mu(\Lambda) \right].$$

(26)

By using the Cauchy-Schwartz inequality, it can be shown that $A$ is bounded as

$$A \simeq \epsilon \mathcal{L}^2 + O(\epsilon^2),$$

(27)

where

$$\mathcal{L} := \int_{\gamma} \sqrt{g^{\mu\mu}(\Lambda)} d\Lambda^\mu d\Lambda^\nu$$

(28)

is the thermodynamic length along the path $\gamma$ of the modulation in the parameter space [44, 50].

The average power $P$ per cycle is given as

$$P \simeq \epsilon W + O(\epsilon^2)$$

(29)

where

$$W := \int_0^1 d\theta \Lambda^W \hat{f}^W(\Lambda_\theta)$$

$$= -\int_{\gamma} \langle \Lambda^\mu(\Lambda) d\Lambda^\mu \rangle$$

(30)
is the adiabatic work along $\gamma$ and
\[ A^\gamma(\Lambda) := \Lambda^W \delta \mu f_0^W(\Lambda) \] (31)
is the thermodynamic vector potential.

From Eq. (27), the efficiency (18) is bounded as
\[ \eta \simeq 1 - \frac{\lambda}{\mathcal{W}} + O(\epsilon^2) \]
\[ \leq 1 - \epsilon \frac{\mathcal{L}^2}{\mathcal{W}} + O(\epsilon^2). \] (32)
Therefore we obtain the geometrical trade-off relation between the average power and the effective efficiency:
\[ P \leq \left( \frac{\mathcal{W}}{\mathcal{L}} \right)^2 (1 - \eta) + O(\epsilon^2). \] (33)

III. APPLICATION TO THE TWO-LEVEL MODEL

![Two-level system diagram](image)

Figure 2. A schematic of a two-level spin-boson model.

In this section, we apply the general framework to the two-state spin-boson model to explicitly calculate the efficiency and the power. We assume that the Hamiltonian $H(\lambda)$ and the transition matrix $K(\Lambda)$ are given as
\[ H(\lambda) := \begin{pmatrix} 0 & 0 \\ 0 & E \lambda \end{pmatrix}, \] (34)
\[ K(\Lambda) := \sum_{\alpha=L,R} \begin{pmatrix} -n^\alpha(\Lambda) & n^\alpha(\Lambda) + 1 \\ n^\alpha(\Lambda) & -n^\alpha(\Lambda) - 1 \end{pmatrix}, \] (35)
where $E$ is the periodic average of the energy difference between the two states and $\lambda$ is a dimensionless control parameter. Here, $n^\alpha(\Lambda)$ is the Bose distribution function of reservoir $\alpha$ given as $n^\alpha(\Lambda) := (\exp[\alpha \lambda E] - 1)^{-1}$. We assume that the left and right reservoirs have the same chemical potential, which is absorbed into the energy $E$ in the system.

We consider the control protocol of $\Lambda = (\lambda, T^L, T^R)$ given as
\[ \lambda_\theta = 1 + \lambda_A \cos[2\pi \theta], \] (36)
\[ T^L_\theta / E = \hat{T}^L_C + \hat{T}^L_A \sin[2\pi \theta], \] (37)
\[ T^R_\theta / E = \hat{T}^R_C + \hat{T}^R_A \sin[2\pi (\theta - \delta)], \] (38)
where $\delta$ is the phase difference which leads to the temperature difference between the left and right reservoirs at each $\theta$. For simplicity, we set $\lambda_A = \hat{T}^L_A = \hat{T}^R_A = 0.1$, $\hat{T}^L_C = \hat{T}^R_C = 1$. To find out where the framework in Sec. III can be used, we calculate the excess entropy production rate $\dot{\Sigma}_{\text{ex}}$ given in Eq. (3). Figure 3 shows that $\dot{\Sigma}_{\text{ex}} \geq 0$ when $0 \leq \delta \leq 0.10$. For later discussion, we only focus on this region.

Next, we investigate the effect of the correction term $\Psi$. Here, $g^{\mu\nu}(\Lambda_\theta)$ can be decomposed into
\[ g^{\mu\nu}(\Lambda_\theta) = \tilde{g}^{\mu\nu}(\Lambda_\theta) + h^{\mu\nu}(\Lambda_\theta), \] (39)
where
\[ \tilde{g}^{\mu\nu}(\Lambda_\theta) := -\frac{1}{2} (f_1^{\mu\nu}(\Lambda_\theta) + f_1^{\nu\mu}(\Lambda_\theta)), \] (40)
\[ h^{\mu\nu}(\Lambda_\theta) := -\frac{1}{2} (\psi_1^{\mu\nu}(\Lambda_\theta) + \psi_1^{\nu\mu}(\Lambda_\theta)). \] (41)
Figure 4 indicates that $g^{\mu\nu}(\Lambda_\theta) \gg h^{\mu\nu}(\Lambda_\theta)$, although the value of $g^{\mu\nu}(\Lambda_\theta)\Lambda_\theta^{\mu\nu}\Lambda_\theta^{\nu\mu}$ is strongly influenced by $\delta$. Indeed, $g^{\mu\nu}(\Lambda_\theta)\Lambda_\theta^{\mu\nu}\Lambda_\theta^{\nu\mu}$ decreases as $\delta$ increases (Fig. 4). As a result, the thermodynamic length $\mathcal{L}$ decreases as $\delta$ increases (Fig. 5). Similarly as $\delta$ increases, the adiabatic work $W$ decreases (Fig. 6), thus leading to the decrease of the average power $P$ (Fig. 6). On the other hand, the effective efficiency $\eta$ increases because the ratio $\mathcal{L}^2/\mathcal{W}$ decreases as $\delta$ increases (Fig. 7).

IV. CONCLUSION

In this paper, we successfully extended the geometrical thermodynamics formulated in Ref. [50] to a system coupled to two slowly modulated reservoirs, i.e. the adiabatic (Thouless) pumping system. In the adiabatic regime, the extracted work can be written as the line integral of the thermodynamic vector potential $\mathcal{W}$ along the path of the manipulation in the parameter space. On the other hand, the lower bound of the dissipated availability can be written as the thermodynamic length $\mathcal{L}^2/\mathcal{W}$ along the path. These results are valid only when the phase difference $\delta$ which corresponds to the temperature difference between two reservoirs is not large. By using these results, we obtained the geometrical trade-off relation (38) between the power and effective efficiency. We applied these results to a two-level spin-boson system to obtain the explicit values of the power and effective efficiency. In contrast to Ref. [51], we have analyzed a pumping system with two reservoirs of the same average temperature. Thanks to this setup, the geometrical contribution plays the dominant role in the thermodynamics of the heat engine.

Our future tasks are as follows: (i) Because our analysis is restricted to the weak nonequilibrium case where the phase difference $\delta$ is small, we will have to try to...
extend our analysis to systems far from equilibrium. In such a regime, the excess entropy production rate $\dot{\sigma}_{ex}$ in the form of Eq. (3) is not positive and we have to consider higher order terms to recover the positivity of $\dot{\sigma}_{ex}$ [52–54]. (ii) Because the present method is restricted to the adiabatic case, we will have to try to extend the analysis to the non-adiabatic regime. In Ref. [55], we obtained the non-adiabatic solution of a classical master equation and geometrical representation of the non-adiabatic current in two level system. We expect to apply these methods to investigate the non-adiabatic effect in the heat engine. (iii) Because we only focus on a classical system, we will have to try to extend our analysis to quantum systems in which quantum coherence plays an important role. Ref. [56] showed that quantum coherence reduces the performance of slowly driven heat engines. On the other hand, it was shown that coherence can enhance the performance of heat engines in Ref. [56]. Therefore, we will have to analyze full quantum systems to clarify whether the coherence can improve the efficiency in the heat engine undergoing an adiabatic pumping process.

ACKNOWLEDGEMENTS

The authors thank Hiroyasu Tajima and Ken Funo for fruitful discussions. The authors also thank Ville Paasonen for his critical reading of this manuscript. This work is partially supported by a Grant-in-Aid of MEXT for Scientific Research (Grant No. 16H04025).

Appendix A: Slow driving perturbation theory

In this appendix, we explain the outline of the perturbation theory of the master equation with slowly modulated parameters. First, we expand the solution of Eq. (1) in terms of $\epsilon$ as

$$|p(\theta)\rangle = \sum_{n=0}^{\infty} \epsilon^n |p_n(\theta)\rangle.$$  (A1)
Since the normalization condition \( \langle p(\theta) \rangle = 1 \) holds for any \( \epsilon \), \(|p_n(\theta)\rangle\) satisfies
\[
\langle 1 | p_0(\theta) \rangle = 1, \quad \langle 1 | p_n(\theta) \rangle = 0 \ (n \geq 1). \tag{A2} \]

Substituting these into Eq. \( (1) \), we obtain
\[
K(\Lambda_\theta)|p_0(\theta)\rangle = 0, \tag{A4}
K(\Lambda_\theta)|p_n(\theta)\rangle = \frac{d}{d\theta}|p_{n-1}(\theta)\rangle \ (n \geq 1). \tag{A5}
\]
Equation \( (A3) \) means that \(|p_0(\theta)\rangle\) is the instantaneous steady state of \( K(\Lambda_\theta) \):
\[
|p_0(\theta)\rangle = |p^{ss}(\Lambda_\theta)\rangle. \tag{A6}
\]
To calculate \(|p_n(\theta)\rangle \ (n \geq 1)\), we introduce the pseudo-inverse \( K^+(\Lambda) \) of \( K(\Lambda) \), which satisfies following conditions
\[
K^+(\Lambda)K(\Lambda) = 1 - |p^{ss}(\Lambda)\rangle\langle 1 |, \tag{A7}
K(\Lambda)K^+(\Lambda) = 1 - |p^{ss}(\Lambda)\rangle\langle 1 |, \tag{A8}
K^+(\Lambda)|p^{ss}(\Lambda)\rangle = 0, \tag{A9}
\langle 1 |K^+(\Lambda) = 0. \tag{A10}
\]
In particular, if \( K(\Lambda) \) is diagonalizable, \( K(\Lambda) = \sum_m \phi_m(\Lambda)|r_m(\Lambda)\rangle\langle l_m(\Lambda) | \), \( K^+(\Lambda) \) can be written as
\[
K^+(\Lambda) = \sum_{m \neq 0} \phi_m(\Lambda)^{-1}|r_m(\Lambda)\rangle\langle l_m(\Lambda) |, \tag{A11}
\]
where \( \phi_m(\Lambda) \) is the eigenvalue and \(|r_m(\Lambda)\rangle, \langle l_m(\Lambda) | \) are the corresponding right and left eigenvectors of \( K(\Lambda) \).
Here we note that \( \phi_0(\Lambda) = 0 \), then \(|r_0(\Lambda)\rangle = |p^{ss}(\Lambda)\rangle\) and \(|l_0(\Lambda)\rangle = |1\rangle \). Here we assume that these eigenstates do not degenerate.

By using \( K^+(\Lambda) \), Eq. \( (A5) \) can be written as
\[
|p_n(\theta)\rangle = K^+(\Lambda_\theta) \frac{d}{d\theta}|p_{n-1}(\theta)\rangle = \left( K^+(\Lambda_\theta) \frac{d}{d\theta} \right)^n |p^{ss}(\Lambda_\theta)\rangle. \tag{A12}
\]
The explicit expression for $|p_1(\theta)\rangle$ is given by

$$|p_1(\theta)\rangle = K^+(A_\theta) \frac{d}{d\theta} |p^s(A_\theta)\rangle = |p_1^s(A_\theta)\rangle \Lambda^\mu_\theta,$$

(A13)

where

$$|p_1^s(A)\rangle := K^+(A) \partial_\mu |p^s(A)\rangle.$$

(A14)

Ignoring terms of $O(\varepsilon^2)$ and higher in Eq. (A1), we obtain Eq. [15] of the main text.

[1] D. J. Thouless, Phys. Rev. B 27, 6083 (1983).
[2] Q. Niu and D. J. Thouless, J. Phys. A 17, 2453 (1984).
[3] M. V. Berry, Proc. R. Soc. London Ser. A 392, 45 (1984).
[4] D. Xiao, M.-C. Chang, and Q. Niu, Rev. Mod. Phys. 82, 1950 (2010).
[5] L. P. Kouwenhoven, A. T. Johnson, N. C. van der Vaart, C. J. P. M. Harmans, and C. T. Foxon, Phys. Rev. Lett. 67, 1626 (1991).
[6] H. Pothier, P. Lafarge, C. Urbina, D. Esteve, and M. H. Devoret, Europhys. Lett. 17, 249 (1992).
[7] M. Switkes, C. M. Marcus, K. Campman, and A. C. Gossard, Science 285, 1905 (1999).
[8] A. Fuhrer, C. Fasth, and L. Samuelson, Appl. Phys. Lett. 91, 052109 (2007).
[9] B. Kaestner, V. Kashcheyevs, G. Hein, K. Pierz, U. Siegner, and H. W. Schumacher, Appl. Phys. Lett. 92, 192106 (2008).
[10] S. J. Chorley, J. Frake, G. C. Smith, G. A. C. Jones, and M. R. Buitema, Appl. Phys. Lett. 100, 143104 (2012).
[11] S. Nakajima, T. Tomita, S. Taie, T. Ichinose, H. Ozawa, L. Wang, M. Troyer and Y. Takahashi, Nature Physics 12, 296 (2016).
[12] M. Lohse, C. Schweitzer, O. Zilberberg, M. Aidelsburger and I. Bloch, Nature Physics 12, 350 (2016).
[13] S. K. Watson, R. M. Potok, C. M. Marcus, and V. Umansky, Phys. Rev. Lett. 91, 258301 (2003).
[14] P. W. Brouwer, Phys. Rev. B 58, R10135 (1998).
[15] N. A. Sinitsyn and I. Nemenman, Europhys. Lett. 77, 58001 (2007).
[16] N. A. Sinitsyn and I. Nemenman, Phys. Rev. Lett. 99, 220408 (2007).
[17] J. E. Avron, A. Elgart, G. M. Graf, and L. Sadun, Phys. Rev. B 62, R10618 (2000).
[18] M. Moskalets and M. Büttiker, Phys. Rev. B 64, 201305(R) (2001).
[19] J. N. H. J. Cremers and P. W. Brouwer, Phys. Rev. B 65, 115333 (2002).
[20] A. Andreev and A. Kamenev, Phys. Rev. Lett. 85, 1294 (2000).
[21] Y. Makhlin and A. D. Mirlin, Phys. Rev. Lett. 87, 276803 (2001).
[22] I. L. Aleiner and A. V. Andreev, Phys. Rev. Lett. 81, 1286 (1998).
[23] E. R. Mucciolo, C. Chamon, and C. M. Marcus, Phys. Rev. Lett. 89, 146802 (2002).
[24] J. M. R. Parrondo, Phys. Rev. E 57, 7297 (1998).
[25] O. Usmani, E. Lutz, and M. Büttiker, Phys. Rev. E 66, 021111 (2002).
[26] R. D. Astumian, Phys. Rev. Lett. 91, 118102 (2003).
[27] R. D. Astumian, Proc. Natl. Acad. Sci. USA 104, 19715 (2007).
[28] S. Rahav, J. Horowitz, and C. Jarzynski, Phys. Rev. Lett. 101, 140602 (2008).
[29] V. Y. Chernyak, J. R. Klein, and N. A. Sinitsyn, J. Chem. Phys. 136, 154107 (2012).
[30] V. Y. Chernyak, J. R. Klein, and N. A. Sinitsyn, J. Chem. Phys. 136, 154108 (2012).
[31] J. Ren, P. Hänggi, and B. Li, Phys. Rev. Lett. 104, 170601 (2010).
[32] T. Sagawa and H. Hayakawa, Phys. Rev. E 84, 051110 (2011).
[33] F. Renzoni and T. Brandes, Phys. Rev. B 64, 245301 (2001).
[34] T. Brandes and T. Vorrath, Phys. Rev. B 66, 075341 (2002).
[35] E. Kota, R. Aguado, and G. Platero, Phys. Rev. Lett. 94, 107202 (2005).
[36] J. Splettsstoesser, M. Governale, J. König, and R. Fazio, Phys. Rev. B 74, 085305 (2006).
[37] T. Yuge, T. Sagawa, A. Sugita, and H. Hayakawa, Phys. Rev. B 86, 235308 (2012).
[38] T. Yuge, T. Sagawa, A. Sugita, and H. Hayakawa, J. Stat. Phys. 153, 412 (2013).
[39] K. L. Watanabe and H. Hayakawa, Phys. Rev. E 96, 022118 (2017).
[40] Y. Hino and H. Hayakawa, arXiv:1908:10597 (2019).
[41] Bjarne Andresen, Angew. Chem. Int. Ed. 50, 2690 (2011).
[42] F. Weinhold, J. Chem. Phys. 63, 2479 (1975).
[43] G. Ruppeiner, Phys. Rev. A 20, 1608 (1979).
[44] P. Salamon and R. S. Berry, Phys. Rev. Lett. 51, 1127 (1983).
[45] F. Schlägl, Z. Phys. B 59, 449 (1985).
[46] G. Ruppeiner, Rev. Mod. Phys. 67, 605 (1995).
[47] G. E. Crooks, Phys. Rev. Lett. 99, 190602 (2007).
[48] S. Deffner and E. Lutz, Phys. Rev. E 87, 022143 (2013).
[49] M. Scandi and M. Perarnau-Llobet, Quantum 3, 19 (2019).
[50] K. Brandner and K. Saito, Phys. Rev. Lett. 124, 040602 (2020).
[51] S. K. Giri and H. P. Goswami, Phys. Rev. E 96, 052129 (2017), ibid 99, 022104 (2019).
[52] T. S. Komatsu, N. Nakagawa, S.-I. Sasa, H. Tasaki, Phys. Rev. Lett. 100, 230602 (2008).
[53] T. S. Komatsu, N. Nakagawa,S.-I. Sasa, H. Tasaki, J. Stat. Phys. 134, 401(2009).
[54] T. S. Komatsu, N. Nakagawa,S.-I. Sasa, H. Tasaki, J. Stat. Phys. 159, 1237 (2015).
[55] K. Fujii, H. Hayakawa, Y. Hino and K. Takahashi, arXiv:1909.02202 (2019).
[56] A. A. Svidzinsky, K. E. Dorfman, M. O. Scully, Coherent Opt. Phenom. 1, 7 (2012).