Electric charge in hyperbolic motion: the special conformal transformation solution

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Abstract
As a simple application of special conformal transformations, we derive the electromagnetic field produced by an electric charge in hyperbolic motion. Unlike other purely algebraic derivations, here we develop a more intuitive geometrical description, based on Minkowski diagrams.

Keywords: conformal symmetry, Maxwell’s equations, hyperbolic motion, special conformal transformation

(Some figures may appear in colour only in the online journal)

1. Introduction

Just a few years after the birth of the special relativity theory, Cunningham [1] and Bateman [2, 3] discovered that Maxwell’s equations are also invariant under conformal transformations. The conformal group is the largest group that conserves null line elements, and consists of space-time translations, proper and improper homogeneous Lorentz transformations, dilatation (or scale) transformations, and special conformal (or acceleration) transformations. Special conformal transformations are described by the formula

\[ x'^\mu = \frac{x^\mu + b^\alpha x_\alpha^2}{1 + 2b \cdot x + b^2 x^2}, \]

where \( b \cdot x = b^\alpha x_\alpha, \) \( b^2 = b^\alpha b_\alpha, \) and \( x^2 = x^\alpha x_\alpha. \) A special conformal transformation can be decomposed into an inversion in the origin, followed by a translation by a constant vector \( b^\mu, \) and then a second inversion in the new origin. The inverse transformation of (1) has the translation in the opposite direction:

\[ x^\mu = \frac{x'^\mu - b^\alpha x'^\alpha}{1 - 2b \cdot x' + b^2 x'^2}. \]
The denominator in (1) or (2) may vanish. Transformation (1) has singularity points on a lightcone with vertex at \( v^\mu = \frac{m^2}{E^2} \). Transformation (2) has singularity points on a lightcone with vertex at \( v^\mu = \frac{m^2}{E^2} \).

The original work of Cunningham and Bateman dealt with inversion operations. After it was discovered that special conformal transformations can transform a straight line (a particle at rest) into the two branches of a hyperbola (two particles in hyperbolic motion), this result was used to derive the total electromagnetic field produced by two such particles (the so-called Born solution) [4]. It is quite surprising how one particle can transform into two particles, in the presence of singularity points on its worldline. The Born solution itself is strange, because it adds together a retarded field (produced by the particle on one branch of the hyperbola) with an advanced field (produced by the particle on the other branch of the hyperbola). These conceptual difficulties have, most likely, prevented the effective use of special conformal transformations in undergraduate and graduate classical electrodynamics textbooks. Due to the fundamental role of symmetry in theoretical physics, and the importance of hyperbolic motion in special relativity, we believe that a more accessible treatment of special conformal transformations is very much needed.

The conceptual difficulties mentioned above can be circumvented if, instead of transforming one straight line into the two branches of a hyperbola, we transform just one branch of the hyperbola into one straight line segment. In other words, we use the inverse of the special conformal transformation that was previously used by other authors. This method allows us to derive the electromagnetic field produced by just one electric charge in hyperbolic motion, not the Born solution, as a simple application of special conformal transformations.

In section 2 we look at an electric charge in hyperbolic motion, describing its worldline and the electric field produced at a point that is simultaneous with the center of the hyperbola. In section 3 we perform a particular special conformal transformation, and as a result the hyperbolic worldline of the source particle turns into a straight line segment parallel to the time axis. In section 4 we look at an electric charge at rest, describing its worldline and the electric field produced. In section 5 we review the rules for the transformation of the electromagnetic field strength tensor under special conformal transformations. We use the special conformal transformation from section 2, and we derive the electric field of a charge at rest from the electric field of a charge in hyperbolic motion. Then we use the inverse of the special conformal transformation from section 2, and we derive the electric field of a charge in hyperbolic motion from the electric field of a charge at rest. For pedagogical reasons, we demonstrate both the direct and the inverse transformations. In the Conclusions section we point out the need to stay away from the singularity points of the transformations.

2. Electric charge in hyperbolic motion

The electromagnetic field produced by an electric charge in hyperbolic motion was calculated a long time ago by Sommerfeld [5], based on the retarded electromagnetic four-potential of Liénard and Wiechert. As noticed by Bondi and Gold [6], the solution based on retarded potentials is valid only when the speed of the electric charge is less than the speed of light in vacuum. Sommerfeld’s geometrical derivation takes place in Minkowski space, where a point has coordinates \( (x, y, z, i\epsilon t) \). The introduction of an imaginary time coordinate has allowed Minkowski to describe a pure Lorentz transformation (a boost) as a rotation of the coordinate axes through an imaginary angle [7]. Here we obtain the same electromagnetic field, this time using special conformal transformations. To facilitate the comparison of results, and to give
our intuition the benefit of a geometrical description, we also use Minkowski diagrams in our derivation. The same Minkowski metric applies to both Minkowski space and flat conformal space. The notation, the Minkowski diagrams, and the calculations in this paper are related to calculations from ‘Electric charge in hyperbolic motion: the early history’ [8], and a reading of this article is suggested before proceeding further.

Consider a source particle with electric charge $e$ in hyperbolic motion, with a trajectory in the $(x,ict)$ plane. The center $O$ of the hyperbola coincides with the origin of the reference frame. The source particle is on the hyperbola branch with positive $x$ coordinates, as shown in figure 1. A point $Q$ on the worldline of the particle has coordinates $X_Q = (a \cos(\psi), 0, 0, a \sin(\psi))$, where $\psi$ is an imaginary angle measured from the $Ox$ axis to the $OQ$ ray, the positive direction being towards the future. We also define $\theta = - \psi$, where $\theta$ is an imaginary angle measured from the $Ox$ axis to the $OQ$ ray, the positive direction being towards the past. We use $\psi$ to refer to any point on the hyperbola, and we use $\theta$ to refer to the point $Q$ on the hyperbola that is connected to the field point $P$ by a light signal.

Consider a test particle at point $P$, with coordinates $X_P = (\rho, y, z, 0)$. We work in a reference frame in which the test particle at $P$ and the center $O$ of the hyperbola are simultaneous. If necessary, a Lorentz boost can always bring us to such a reference frame. The projection of point $P$ (the only point above the page) onto the $(x,ict)$ plane (the plane of the paper) is point $S$, with coordinates $X_S = (\rho, 0, 0, 0)$.

Figure 1. The electric field at point $P$, produced by the source charge at $Q$, has the direction of ray $MP$. This Minkowski diagram assumes that $OR < OS$, where $OR = a$ and $OS = \rho$. Point $A$ is the vertex of the lightcone of singularity points, and $OA = a$. 
The position four-vector from \( Q \) to \( P \) is \( X = X_P - X_Q \). We want point \( P \) to be on the future lightcone of point \( Q \). Due to this retardation condition, angle \( \psi \) is negative and angle \( \theta \) is positive. Because points \( Q \) and \( P \) are connected by a light signal, we know that \( X \cdot X = 0 \). As a result
\[
\cos(\theta) = \frac{\rho^2 + a^2 + y^2 + z^2}{2a\rho}.
\] (3)

The electric field produced by the source charge from \( Q \) at point \( P \) is
\[
E_x = \frac{e\rho - a\cos(\theta)}{ar^3},
\] (4)
\[
E_y = \frac{e\rho y}{ar^3},
\] (5)
\[
E_z = \frac{e\rho z}{ar^3},
\] (6)
where
\[
r = -i\rho \sin(\theta),
\] (7)
as explained in ‘Electric charge in hyperbolic motion: the early history’ [8]. Assuming that \( e > 0 \), the magnitude of the electric field is
\[
|E| = \sqrt{E_x^2 + E_y^2 + E_z^2} = \frac{e}{r^2}.
\] (8)

The magnetic field, in the reference frame considered, is zero.

In order to establish the connection between the electromagnetic field of a charge in hyperbolic motion and the electromagnetic field of a charge at rest, we start by eliminating any reference to \( \theta \) in the above formulas. Substitution of (3) into (4) gives
\[
E_x = \frac{e(\rho^2 - a^2 - y^2 - z^2)}{2ar^3},
\] (9)
and the electric field takes the form
\[
\vec{E} = \frac{e}{r^3} \left( \frac{\rho^2 - a^2 - y^2 - z^2}{2a}, \frac{\rho y}{a}, \frac{\rho z}{a} \right).
\] (10)

The square of the radial distance \( r \) takes the form
\[
r^2 = -\rho^2 \sin^2(\theta) = \rho^2(\cos^2(\theta) - 1) = \rho^2\left( \frac{\rho^2 + a^2 + y^2 + z^2}{2a\rho} \right)^2 - \rho^2
\]
\[
= \frac{\rho^4 + a^4 + y^4 + z^4 - 2\rho^2a^2 + 2\rho^2y^2 + 2\rho^2z^2 + 2a^2y^2 + 2a^2z^2 + 2y^2z^2}{4a^2}.
\] (11)

3. The transformation of the coordinates

We perform the special conformal transformation that has a four-vector \( b = \left( \frac{1}{a}, 0, 0, 0 \right) \). As a result we have \( b^2 = \frac{1}{a^2} \).
A point \( Q \) on the worldline of the source particle has coordinates

\[
X_Q = (a \cos(\psi), 0, 0, a \sin(\psi)),
\]

where \( \psi \) is an imaginary angle that goes from \(-i\infty\) to \(i\infty\). In equation (1) we substitute \( x = X_Q \), and we calculate \( x^2 = a^2, b \cdot x = \cos(\psi) \), and \( 1 + 2b \cdot x + b^2 x^2 = 2(1 + \cos(\psi)) \). After the special conformal transformation, point \( Q' \) has coordinates

\[
X_{Q'} = \left( \frac{a}{2}, 0, 0, -\frac{a \sin(\psi)}{2(1 + \cos(\psi))} \right).
\]

We notice that the spatial coordinate is constant, and that the temporal coordinate \( \text{ict}' \) goes from an initial value given by

\[
\lim_{\psi \to -i\infty} \frac{a \sin(\psi)}{2(1 + \cos(\psi))} = \frac{ia}{2} \lim_{\omega \to -i\infty} \frac{\sin(\omega)}{1 + \cosh(\omega)} = \frac{ia}{2},
\]

to a final value given by

\[
\lim_{\psi \to i\infty} \frac{a \sin(\psi)}{2(1 + \cos(\psi))} = \frac{ia}{2} \lim_{\omega \to i\infty} \frac{\sin(\omega)}{1 + \cosh(\omega)} = \frac{ia}{2},
\]

where \( \psi = i\omega \) and \( \omega \) is a real number.

In particular, when \( \psi = 0 \), point \( R \) has coordinates

\[
X_R = (a, 0, 0, 0),
\]

and, after the special conformal transformation, point \( R' \) has coordinates

\[
X_{R'} = \left( \frac{a}{2}, 0, 0, 0 \right).
\]

Point \( P \) has coordinates

\[
X_P = (\rho, y, z, 0).
\]

In equation (1) we substitute \( x = X_P \), and we calculate \( x^2 = \rho^2 + y^2 + z^2, b \cdot x = \frac{\rho}{\sigma}, \) and \( 1 + 2b \cdot x + b^2 x^2 = \left(1 + \frac{\rho}{\sigma}\right)^2 + \frac{y^2 + z^2}{\sigma^2} \equiv \sigma \). We notice that \( \sigma \) is a positive real number. After the special conformal transformation, point \( P' \) has coordinates

\[
X_{P'} = \left( \frac{a \rho(a + \rho) + a(y^2 + z^2)}{(a + \rho)^2 + y^2 + z^2}, \frac{a^2 y}{(a + \rho)^2 + y^2 + z^2}, \frac{a^2 z}{(a + \rho)^2 + y^2 + z^2}, 0 \right).
\]

In particular, when \( y = 0 \) and \( x = 0 \), point \( S \) has coordinates

\[
X_S = (\rho, 0, 0, 0),
\]

and, after the special conformal transformation, point \( S' \) has coordinates

\[
X_{S'} = \left( \frac{a \rho}{a + \rho}, 0, 0, 0 \right),
\]

which also means that point \( S' \) is no longer the projection of point \( P' \) (the only point above the page) onto the \((x', \text{ict}')\) plane (the plane of the paper).
The special conformal transformation preserves the ordering of points $S'$ and $R'$ on the $Ox'$ axis. If $\rho > a$ then $\frac{\rho a}{a + \rho} > \frac{a}{2}$, and if $\rho < a$ then $\frac{\rho a}{a + \rho} < \frac{a}{2}$.

4. Electric charge at rest

Consider a source particle with electric charge $e$ at rest, with a worldline given by $X_{Q'} = \left(\frac{a}{t'}, 0, 0, ict'\right)$, as shown in figure 2. The calculation of the electric field at $P'$ is not affected by our assumption that $-ia/2 < ict' < ia/2$. When the time $t'$ is zero the source particle has coordinates $X_{Q'}$ given by (17), and the test particle has coordinates $X_{P'}$ given by (19).

Since

$$\frac{a \rho (a + \rho) + a(y^2 + z^2)}{(a + \rho)^2 + y^2 + z^2} - \frac{a}{2} = \frac{a \rho^2 - a^2 + y^2 + z^2}{2(a + \rho)^2 + y^2 + z^2},$$

the position four-vector $X_{R'P'} = X_{P'} - X_{R'}$, from point $R'$ to point $P'$, is

$$X_{R'P'} = \left(\frac{a \rho^2 - a^2 + y^2 + z^2}{2(a + \rho)^2 + y^2 + z^2}, \frac{a^2 y}{(a + \rho)^2 + y^2 + z^2}, \frac{a^2 z}{(a + \rho)^2 + y^2 + z^2}, 0\right).$$

The square of the distance from point $R'$ to point $P'$ is
The Coulomb electric field produced at point \( P' \) by the source charge at rest at point \( R' \) is

\[
E'_x = \frac{e}{r'^3} \frac{a^2 \rho^2 - a^2 + y^2 + z^2}{2(a + \rho)^2 + y^2 + z^2},
\]

\[
E'_y = \frac{e}{r'^3} \frac{a^2 y}{(a + \rho)^2 + y^2 + z^2},
\]

\[
E'_z = \frac{e}{r'^3} \frac{a^2 z}{(a + \rho)^2 + y^2 + z^2},
\]

which we can also write in the equivalent form

\[
\vec{E}' = \frac{e}{r'^3} \begin{pmatrix} \rho^2 - a^2 + y^2 + z^2 & y & z \\ 2a & \sigma & \sigma \end{pmatrix}.
\]

Assuming that \( e > 0 \), the magnitude of the electric field is

\[
|\vec{E}'| = \sqrt{E'_x^2 + E'_y^2 + E'_z^2} = \frac{e}{r'^2}.
\]

5. The transformation of the electric field

In Minkowski space, when the magnetic field is zero, as it is in the two reference frames considered, the electromagnetic field strength tensor takes the form

\[
F = ||F_{\alpha\beta}|| = \begin{pmatrix} 0 & 0 & 0 & -iE_x \\ 0 & 0 & 0 & -iE_y \\ 0 & 0 & 0 & -iE_z \\ iE_x & iE_y & iE_z & 0 \end{pmatrix}.
\]

We introduce the matrices

\[
\mathbb{M} = ||M_{\alpha\beta}|| = \begin{vmatrix} \frac{\partial x^\alpha}{\partial x^\beta} \end{vmatrix},
\]

\[
\mathbb{W} = ||W_{\alpha\beta}|| = \begin{vmatrix} \frac{\partial x^\alpha}{\partial x^\beta} \end{vmatrix},
\]

where \( \alpha \) is a row index, and \( \beta \) is a column index.

As discussed by Nicholas Wheeler [9], under the special conformal transformation (1) the covariant \( F_{\alpha\beta} \) transforms like a tensor.
$F_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial \chi^\mu} \frac{\partial x^\beta}{\partial \chi^\nu} F_{\alpha\beta}(x), \quad (33)$

and the contravariant $F^{\alpha\beta}$ transforms like a tensor density of weight 1:

$F^{\mu\nu}(x') = W \frac{\partial x^\mu}{\partial \chi^\alpha} \frac{\partial x^\nu}{\partial \chi^\beta} F^{\alpha\beta}(x), \quad (34)$

where $W$ is the Jacobian determinant of the transformation $x \mapsto x'$:

$W = |\mathcal{W}| = \frac{1}{(1 - 2b \cdot x + b^2x^2)^{\frac{3}{2}}} = (1 + 2b \cdot x + b^2x^2)^{\frac{3}{2}} = \sigma^{\frac{3}{2}}. \quad (35)$

A similar set of equations is obtained for the inverse special conformal transformation (2).

Equation (33) becomes

$F'_{\mu\nu}(x) = \frac{\partial x^{\alpha'}}{\partial \chi^\mu} \frac{\partial x^{\beta'}}{\partial \chi^\nu} F'_{\alpha'\beta'}(x'), \quad (36)$

and equation (34) becomes

$F'^{\mu\nu}(x) = M \frac{\partial x^{\mu'}}{\partial \chi^{\alpha'}} \frac{\partial x^{\nu'}}{\partial \chi^{\beta'}} F'^{\alpha'\beta'}(x'), \quad (37)$

where $M$ is the Jacobian determinant of the transformation $x' \mapsto x$:

$M = |\mathcal{M}| = \frac{1}{(1 + 2b \cdot x + b^2x^2)^{\frac{3}{2}}} = \frac{1}{\sigma^{\frac{3}{2}}} \quad (38)$

In flat conformal space the Lorentz invariant $F_{\alpha\beta}$ transforms like a scalar density of weight 1:

$F'_{\mu\nu}(x') F'^{\mu\nu}(x') = W F_{\alpha\beta}(x) F^{\alpha\beta}(x). \quad (39)$

Since in the two reference frames considered the magnetic field is zero, $F_{\alpha\beta} F^{\alpha\beta} = -2E^2 = -2e^2/r^4$, $F'_{\mu\nu} F'^{\mu\nu} = -2E'^2 = -2e'/r'^4$, and equation (39) provides an alternative derivation of the equation (24), $r' = r/\sigma$.

From equation (1) we calculate [10]

$M'_{\mu} = \frac{\partial x'^\mu}{\partial x^\alpha} = \frac{\delta_{\mu}^{\alpha} + 2b^{\mu} x_{\alpha}}{1 + 2b \cdot x + b^2x^2} = \frac{(x^\mu + b^\mu x^2)(2h_{\mu} + 2x_{\mu}b^2)}{(1 + 2b \cdot x + b^2x^2)^2}, \quad (40)$

and from equation (2) we calculate

$W'_{\mu} = \frac{\partial x'^\mu}{\partial x^\alpha} = \frac{\delta_{\mu}^{\alpha} - 2b^{\mu} x_{\alpha}}{1 - 2b \cdot x' + b^2x'^2} = \frac{(x^\mu - b^\mu x^2)(-2h_{\mu} + 2x_{\mu}b^2)}{(1 - 2b \cdot x' + b^2x'^2)^2}. \quad (41)$

The equations (33), (34), (36), (37) can be used to calculate one electric field from the other one, and this is most easily done by writing these equations in matrix form:

$F' = \mathcal{W}^T F \mathcal{W}, \quad (42)$

$F' = W M \mathcal{F} M^T \mathcal{F}^T, \quad (43)$

$F = M^T F' \mathcal{M}, \quad (44)$

$F = M \mathcal{W} F' \mathcal{W}^T \quad (45)$

Because the matrices $\mathcal{M}$ and $\mathcal{W}$ are inverses of each other, it is obvious that equations (42) and (44) are equivalent, and also equations (43) and (45).
Direct substitution of \( b = \left( \frac{1}{a}, 0, 0, 0 \right) \) and of \( x = X_P = (\rho, y, z, 0) \) into equation (40) gives

\[
\mathcal{M} = \begin{pmatrix}
A & B & C & 0 \\
D & E & F & 0 \\
G & H & I & 0 \\
0 & 0 & 0 & J
\end{pmatrix},
\]

where

\[
A = \frac{\partial x^1}{\partial x^1} = \left( 1 + 2\frac{\rho}{a} \right) \frac{1}{\sigma} \rho^2 + y^2 + z^2 \left( \frac{2}{a} + \frac{2\rho}{a^2} \right) \frac{1}{\sigma^2}
\]
\[
= \frac{1}{\sigma} - \frac{2(y^2 + z^2)}{a^2\sigma^2} = \frac{a^2 + 2a\rho + \rho^2 - y^2 - z^2}{a^2\sigma^2},
\]

(47)

\[
B = \frac{\partial x^1}{\partial x^2} = \frac{2y}{a} \frac{1}{\sigma} \rho^2 + y^2 + z^2 \left( \frac{2y}{a} + \frac{1}{a^2\sigma^2} \right) = \frac{2y(a + \rho)}{a^2\sigma^2},
\]

(48)

\[
C = \frac{\partial x^1}{\partial x^3} = \frac{2z}{a} \frac{1}{\sigma} \rho^2 + y^2 + z^2 \left( \frac{2z}{a} + \frac{1}{a^2\sigma^2} \right) = \frac{2z(a + \rho)}{a^2\sigma^2},
\]

(49)

\[
D = \frac{\partial x^2}{\partial x^1} = -y \left( \frac{2}{a} + \frac{2\rho}{a^2} \right) \frac{1}{\sigma^2} = \frac{2y(a + \rho)}{a^2\sigma^2},
\]

(50)

\[
E = \frac{\partial x^2}{\partial x^3} = \frac{1}{\sigma} - \frac{2y^2}{a^2\sigma^2} = \frac{a^2 + 2a\rho + \rho^2 - y^2 + z^2}{a^2\sigma^2},
\]

(51)

\[
F = \frac{\partial x^2}{\partial x^4} = -\frac{2yz}{a^2\sigma^2},
\]

(52)

\[
G = \frac{\partial x^3}{\partial x^1} = -z \left( \frac{2}{a} + \frac{2\rho}{a^2} \right) \frac{1}{\sigma^2} = \frac{2z(a + \rho)}{a^2\sigma^2},
\]

(53)

\[
H = \frac{\partial x^3}{\partial x^2} = -\frac{2yz}{a^2\sigma^2},
\]

(54)

\[
I = \frac{\partial x^3}{\partial x^4} = \frac{1}{\sigma} - \frac{2z^2}{a^2\sigma^2} = \frac{a^2 + 2a\rho + \rho^2 + y^2 - z^2}{a^2\sigma^2},
\]

(55)

\[
J = \frac{\partial x^4}{\partial x^1} = \frac{1}{\sigma}.
\]

(56)

Direct substitution of matrix (46) into equation (43) gives the electric field \( \vec{E}' \) as a function of the electric field \( \vec{E} \):

\[
E_{x'} = W(\xi_{x} + B\xi_{y} + C\xi_{z})J,
\]

(57)

\[
E_{y'} = W(\xi_{x} + E\xi_{y} + F\xi_{z})J,
\]

(58)

\[
E_{z'} = W(G\xi_{x} + H\xi_{y} + I\xi_{z})J.
\]

(59)

The substitution of the coefficients (47)–(56), of the Jacobian determinant (35), and of the electric field (10) into equations (57)–(59) produces the electric field (28). During this
calculation we use the relations \( r = \sigma r' \) and \( a^2 \sigma = a^2 + 2a\rho + \rho^2 + y^2 + z^2 \). For the calculation of \( E'_r \) we also need the factorization:

\[
\rho^4 - a^4 + y^4 + z^4 + 2a\rho^3 - 2a^3\rho - 2y^2z^2 + 2y^2z^2 + 2a\rho y^2 + 2a\rho z^2
\]

\[
= (\rho^2 - a^2 + y^2 + z^2)(a^2 + 2a\rho + \rho^2 + y^2 + z^2).
\]  

(60)

The electric field of a charge at rest is in this way obtained:

\[
E'_r = \sigma \left( \frac{(a^2 + 2a\rho + \rho^2 - y^2 - z^2)}{a^2\sigma^2} e (\rho^2 - a^2 - y^2 - z^2) \right)
\]

\[
+ \frac{2 \cdot 2y(a + \rho)}{a^2\sigma^2} e \frac{\rho y}{r^3 a} + \frac{2 \cdot 2z(a + \rho)}{a^2\sigma^2} e \frac{\rho z}{r^3 a} \frac{1}{\sigma}
\]

\[
= \frac{e\sigma^3}{r^3} (\rho^4 - a^4 + y^4 + z^4 + 2a\rho^3 - 2a^3\rho - 2y^2z^2 + 2y^2z^2 + 2a\rho y^2 + 2a\rho z^2)
\]

\[
= \frac{e}{2\sigma} (\rho^2 - a^2 + y^2 + z^2).
\]  

(61)

\[
E'_r = \sigma \left( -\frac{2y(a + \rho)}{a^2\sigma^2} e \frac{\rho y}{r^3 a} + \frac{2 \cdot 2z(a + \rho)}{a^2\sigma^2} e \frac{\rho z}{r^3 a} \frac{1}{\sigma} \right)
\]

\[
= \frac{e\sigma^3}{r^3} (a^3y + 2a^2\rho y + a\rho^2y + ay^3 + ayz^2)
\]

\[
= \frac{e}{a^2\sigma^2} y.
\]  

(62)

\[
E'_r = \sigma \left( -\frac{2z(a + \rho)}{a^2\sigma^2} e \frac{\rho y}{r^3 a} + \frac{2 \cdot 2y(a + \rho)}{a^2\sigma^2} e \frac{\rho z}{r^3 a} \frac{1}{\sigma} \right)
\]

\[
= \frac{e\sigma^3}{r^3} (a^3z + 2a^2\rho z + a\rho^2z + ayz^2 + az^3)
\]

\[
= \frac{e}{a^2\sigma^2} z.
\]  

(63)

Direct substitution of matrix (46) into equation (44) gives the electric field \( \mathbf{\tilde{E}} \) as a function of the electric field \( \mathbf{E} \):

\[
E_r = (AE'_r + DE'_\theta + GE'_\phi)J,
\]

\[
E_\theta = (BE'_r + EE'_\theta + HE'_\phi)J,
\]

\[
E_\phi = (CE'_r + FE'_\theta + IE'_\phi)J.
\]

The substitution of the coefficients (47)–(56) and of the electric field (28) into equations (64)–(66) produces the electric field (10). For the calculation of \( E_r \) we also need the factorization:

\[
\rho^4 - a^4 + y^4 + z^4 + 2a\rho^3 - 2a^3\rho - 2y^2z^2 + 2y^2z^2 + 2a\rho y^2 + 2a\rho z^2
\]

\[
= (\rho^2 - a^2 + y^2 + z^2)(a^2 + 2a\rho + \rho^2 + y^2 + z^2).
\]  

(67)
The electric field of a charge in hyperbolic motion is in this way obtained:

\[
E_x = \left( \frac{(a^2 + 2a\rho + \rho^2 - y^2 - z^2)}{\sigma^2 r^3} \right) e \cdot \left( \frac{\rho^2 - a^2 + y^2 + z^2}{\sigma^2 r^3} \right) \\
+ \frac{2a \cdot (-2y)(a + \rho)}{2a \cdot a^2 \sigma^2 r^3} e \cdot \left( \frac{y}{r^3 \sigma} \right) + \frac{2a \cdot (-2z)(a + \rho)}{2a \cdot a^2 \sigma^2 r^3} e \cdot \left( \frac{z}{r^3 \sigma} \right) \left( \frac{1}{\sigma} \right) \sigma \\
= \frac{e}{\sigma^3 r^3} \left( \frac{(a^2 + 2a\rho + \rho^2 - y^2 - z^2)}{2a \sigma} \right) \\
= \frac{e}{\sigma^3 r^3} \left( \frac{\rho^2 - a^2 - y^2 - z^2}{2a \sigma} \right). \tag{68}
\]

\[
E_y = \left( \frac{\xi}{a^2 \sigma^2} \frac{\xi}{a^2 \sigma^2} \right) e \cdot \left( \frac{\rho^2 - a^2 + y^2 + z^2}{r^3 \sigma} \right) \\
+ \frac{a \cdot (-2yz)}{a \cdot a^2 \sigma^2 r^3} e \cdot \left( \frac{y}{r^3 \sigma} \right) + \frac{a \cdot (a^2 + 2a\rho + \rho^2 + y^2 - z^2)}{a \cdot a^2 \sigma^2 r^3} e \cdot \left( \frac{z}{r^3 \sigma} \right) \left( \frac{1}{\sigma} \right) \sigma \\
= \frac{e}{\sigma^3 r^3} \left( \frac{\rho^2 y + 2a\rho^2 y + \rho^2 y + \rho y^3 + \rho y^2 z - \rho y^2 z + \rho y^2 z + \rho y^2 z}{a \sigma} \right) \\
= \frac{e \cdot \rho y}{r^3 a}. \tag{69}
\]

\[
E_z = \left( \frac{\xi}{a^2 \sigma^2} \frac{\xi}{a^2 \sigma^2} \right) e \cdot \left( \frac{\rho^2 - a^2 + y^2 + z^2}{r^3 \sigma} \right) \\
+ \frac{a \cdot (-2yz)}{a \cdot a^2 \sigma^2 r^3} e \cdot \left( \frac{y}{r^3 \sigma} \right) + \frac{a \cdot (a^2 + 2a\rho + \rho^2 + y^2 - z^2)}{a \cdot a^2 \sigma^2 r^3} e \cdot \left( \frac{z}{r^3 \sigma} \right) \left( \frac{1}{\sigma} \right) \sigma \\
= \frac{e}{\sigma^3 r^3} \left( \frac{\rho^2 z + 2a\rho^2 z + \rho^2 z + \rho y^2 z + \rho y^2 z + \rho y^2 z}{a \sigma} \right) \\
= \frac{e \cdot \rho z}{r^3 a}. \tag{70}
\]

As a side note, in the situation when the restrictions \( b = \left( \frac{1}{\sigma}, 0, 0, 0 \right) \) and of \( x = X_P = (\rho, y, z, 0) \) do not apply, substitution of (40) into (34), or of (41) into (33), produces the most general formula relating the electromagnetic fields before and after the special conformal transformation. This formula was also obtained by Barut and Haugen [11] with the help of a six-dimensional linear representation of the conformal group.

6. Conclusions

The conformal symmetry of Maxwell’s equations remains a topic still avoided by most electrodynamics textbooks. Some conceptual difficulties seem to bear the blame for this situation.

When we derive the electromagnetic field of an accelerated electric charge, starting with one electric charge at rest (worldline \( A' C' D' Q' R' E' B' A' \) in figure 2) and applying the special conformal transformation (2), we notice the apparition of two particles (worldlines \( D Q R E \) and \( B A C \) in figure 1). Fulton and Rohrlich [12], giving credit to Gürsey for valuable comments, have analyzed this transformation in detail, providing Minkowski diagrams similar to ours.

We also notice the temporal inversion of points \( B \) and \( C \), which indicates possible violations of causality. Rosen [13] has analyzed this outcome in detail, showing that the reversal of temporal ordering happens only when there is a singularity point on the trajectory between the two spacepoints considered. Because of the singularity point at \( A \), worldline \( B A C \)
is split in two. \(AB\) transforms into \(A'B'\), and \(AC\) transforms into \(A''C'\), with points \(A'\) and \(A''\) at infinity.

The Coulomb electric field of an electric charge at rest fills the whole of spacetime, and it was assumed that the electric field after the special conformal transformation has the same property. Since the electric field produced by just one electric charge in hyperbolic motion does not fill the whole of spacetime (even if we combine the retarded field with the advanced field), the apparition of the second particle seemed necessary. Codirla and Osborn [4] justify in this way the derivation of the Born solution: ‘The fields obtained by conformal transformation are nonzero everywhere for all time and are, of course, solutions of Maxwell’s equations. They are related to, but not identical with, the standard retarded, or advanced, solutions, since these are zero on half of space-time.’

We do not embrace the assumption that the electromagnetic field, after the special conformal transformation, should fill all of the spacetime, instead we limit our investigation to regions away from the lightcone of singularity points. This restriction was made clear by Fulton, Rohrlich, and Witten [14]: ‘The fact that the transformation is singular is in no way disturbing. One must merely obey the injunction to stay away from the singularities in discussing any physical process. Using this transformation, we discuss physical processes only for regions in space-time for which the transformation is nonsingular.’

In our approach the electric charge at rest is not represented by the infinite worldline \(A''C'DQREBA\), but by the finite worldsegment \(DQRE\). In this way we avoid the singularity points, with all their related conceptual difficulties. Although the infinite worldline \(A''C'DQRE'B'A'\) in flat conformal space may look like the infinite worldline of a particle at rest in Minkowski space, in truth they are not the same. This is because the special conformal transformation (1), acting on a particle with constant rest mass \(m\) in hyperbolic motion in Minkowski space, will change the rest mass \(m'\) of the stationary particle along its worldline in flat conformal space, according to the formula [10]:

\[
m' = m(1 + 2b \cdot x + b^2 x^2).
\] (71)

In particular, \(m' = 0\) at the endpoints of the worldsegment \(DQRE\), where the source particle reaches the lightcone of singularity points. This important fact can be easily overlooked because the electric charge of a particle, unlike its rest mass, is invariant under conformal transformations.

We also know that the selection of the retarded electromagnetic potential, at the expense of the advanced one, reduces the symmetry group of classical relativistic physics to the inhomogeneous Lorentz group plus dilatations [15]. The conformal symmetry of Maxwell’s equations could play an important role in theories of time-symmetric electrodynamic interaction. In particular, the physical systems described in figures 1 and 2 exhibit temporal symmetry. In these two cases the same electric fields are obtained from retarded potentials, from advanced potentials, or from a linear combination of both. As noticed by Boulware [16], for this time-symmetric configuration, ‘[…] under time reversal, retarded fields are transformed into advanced fields, hence the retarded field is equal to the advanced field.’ This property holds true even after a Lorentz transformation, revealing the time-symmetric nature of the electromagnetic field produced by an electric charge in hyperbolic motion.

In conclusion, we hope that our intuitive, geometrical derivation presented here will become a pedagogical example of how to use the conformal symmetry of Maxwell’s equations in order to derive the electromagnetic field produced by an electric charge in hyperbolic motion.
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