From Convex Optimization to MDPs: A Review of First-Order, Second-Order and Quasi-Newton Methods for MDPs

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ABSTRACT

In this paper we present a review of the connections between classical algorithms for solving Markov Decision Processes (MDPs) and classical gradient-based algorithms in convex optimization. Some of these connections date as far back as the 1980s, but they have gained momentum in recent years and have led to faster algorithms for solving MDPs. In particular, two of the most popular methods for solving MDPs, Value Iteration and Policy Iteration, can be linked to first-order and second-order methods in convex optimization. In addition, recent results in quasi-Newton methods lead to novel algorithms for MDPs, such as Anderson acceleration. By explicitly classifying algorithms for MDPs as first-order, second-order, and quasi-Newton methods, we hope to provide a better understanding of these algorithms, and, further expanding this analogy, to help to develop novel algorithms for MDPs, based on recent advances in convex optimization.

Keywords: Markov Decision Process, convex optimization, first-order methods, value iteration, second-order methods, policy iteration, quasi-Newton methods, Anderson acceleration.

1 Introduction

Markov Decision Process (MDP) is a common framework modeling dynamic optimization problems, with applications ranging from reinforcement learning [Mnih et al., 2015] to healthcare [Goh et al., 2018, Grand-Clément et al., 2020] and wireless sensor networks [Alsheikh et al., 2015]. Most of the algorithms for computing an optimal control policy are variants of two algorithms: Value Iteration (VI) and Policy Iteration (PI). Over the last 40 years, a number of works have highlighted the strong connections between these algorithms and methods from convex optimization, even though computing an optimal policy is a non-convex problem. Most algorithms in convex optimization can naturally be classified as first-order, second-order and quasi-Newton methods, if the iterates rely on gradients and/or Hessian computations [Boyd and Vandenberghe, 2004]. The goal of this paper is to outline a unifying framework for the classification of algorithms for solving MDPs. In particular, we present a systematic review of the connections between Value Iteration and first-order methods, between Policy Iteration and second-order methods, and between variants of Value Iteration and quasi-Newton methods. We hope that this unifying view can help to develop novel fast algorithms for solving MDPs, by extending the latest advances in convex optimization to the MDP framework. On the other hand, solving MDPs through the lens of convex optimization motivates novel interesting questions and challenges in optimization, as the operators and objective functions do not satisfy classical structural properties (e.g., convexity and/or differentiability).

Outline. We introduce the MDP framework as well as the classical Value Iteration and Policy Iteration algorithms in Section 2. We highlight the recent connections between Value Iteration and first-order methods (Gradient Descent) in Section 3. The relations between Policy Iteration and second-order methods (Newton’s method) are presented in Section 4. We review Anderson Value Iteration, a quasi-Newton methods for MDPs, in Section 5. For the sake of completeness, in Appendix A, we present a detailed review of the results for Gradient Descent (along with acceleration and momentum), Newton’s method and quasi-Newton methods in convex optimization.
Notations. In this paper, \( n \) and \( A \) denote integers in \( \mathbb{N} \). The notation \( \Delta(A) \) refers to the simplex of size \( A \). We write \([n]\) for the set \( \{1, ..., n\} \).

2 Markov Decision Process (MDP)

2.1 Setting and notations

An MDP is defined by a tuple \((\mathcal{S}, \mathcal{A}, P, r, p_0, \lambda)\), where \( \mathcal{S} \) is the set of states and \( \mathcal{A} \) is the set of actions. In this review we focus on finite sets of states and actions, and we write \(|\mathcal{S}| = n < +\infty, |\mathcal{A}| = A < +\infty\). The kernel \( P \in \mathbb{R}^{n \times A \times n} \) models the transition rates from state-action pairs to the next states, and \( r \in \mathbb{R}^{n \times A} \) is the state-action reward. There is an initial distribution \( p_0 \in \mathbb{R}_+^n \) over the set of states, and a discount factor \( \lambda \in (0, 1) \).

A (stationary) policy \( \pi \in (\Delta(A))^n \) maps each state to a probability distribution over the set of actions \( \mathcal{A} \). For each policy \( \pi \), the value vector \( v^\pi \in \mathbb{R}^n \) is defined as

\[
v^\pi_s = \mathbb{E}^\pi_P \left[ \sum_{t=0}^{\infty} \lambda^t r_{s_0 a_t} \mid s_0 = s \right], \forall s \in \mathcal{S},
\]

where \((s_t, a_t)\) is the state-action pair visited at time \( t \). From the dynamic programming principle, \( v^\pi \) satisfies the following recursion:

\[
v^\pi_s = \sum_{a \in \mathcal{A}} \pi_{sa} \left( r_{sa} + \lambda \sum_{s' \in \mathcal{S}} P_{sas'} v^\pi_{s'} \right) = \sum_{a \in \mathcal{A}} \pi_{sa} \left( r_{sa} + \lambda P_{sa}^T v^\pi \right), \forall s \in \mathcal{S},
\]

where \( P_{sa} \in \Delta(n) \) is the probability distribution over the next state given a state-action pair \((s, a)\). We simply reformulate the previous equality as

\[
v^\pi = (I - \lambda P)^{-1} r^\pi,
\]

where \( P \in \mathbb{R}^{n \times n} \), \( r^\pi \in \mathbb{R}^n \) are the transition matrix and the one-step expected reward vector induced by the policy \( \pi \):

\[
P_{\pi,ss'} = \sum_{a \in \mathcal{A}} \pi_{sa} P_{sas'}, \forall (s, s') \in \mathcal{S} \times \mathcal{S}.
\]

The goal of the decision-maker is to compute a policy \( \pi^* \) that maximizes the infinite horizon expected discounted reward, defined as \( R(\pi) = p_0^\top v^\pi \). In this review we focus on Value Iteration (VI) and Policy Iteration (PI) algorithms to compute an optimal policy; we refer the reader to Puterman [2014] for a detailed discussion about other algorithms.

2.2 Value Iteration

Value Iteration was introduced by Bellman [Bellman, 1966]. Define the Bellman operator \( T : \mathbb{R}^n \to \mathbb{R}^n \), where for \( v \in \mathbb{R}^n \),

\[
T(v)_s = \max_{a \in \mathcal{A}} \left\{ r_{sa} + \lambda \cdot P_{sa}^T v \right\}, \forall s \in \mathcal{S}. \tag{2.1}
\]

The operator \( T \) is an order-preserving contraction of \((\mathbb{R}^n, \| \cdot \|_\infty)\), where for \( v, w \in \mathbb{R}^n \), we have \( v \leq w \Rightarrow T(v) \leq T(w) \), and \( \|T(v) - T(w)\|_\infty \leq \lambda \cdot \|v - w\|_\infty \). Crucially, note that \( T \) is not differentiable everywhere, because of the \( \max \) in its definition. The value iteration (VI) algorithm is defined as follows:

\[
v_0 \in \mathbb{R}^n, v_{t+1} = T(v_t), \forall t \geq 0. \tag{VI}
\]

The following theorem gives the convergence rate and stopping criterion for VI.

**Theorem 2.1** [Puterman [2014], Chapter 6.3].

1. The value vector \( v^* \) of the optimal policy \( \pi^* \) is the unique fixed point of the operator \( T \).

   2. Let \((v_t)_{t \geq 0}\) be generated by VI. For any \( t \geq 0 \), we have \( \|v_t - v^*\|_\infty \leq \lambda^t \|v_0 - v^*\|_\infty \).

   3. If \( \|v_t - v_{t+1}\|_\infty \leq \epsilon (1 - \lambda) (2\lambda)^{-1} \) then \( \|v^{\pi_t} - v^*\|_\infty \leq \epsilon \), where \( v^{\pi_t} \) is the value vector of \( \pi_t \), the policy attaining the maximum in each component of \( T(v_t) \).

Therefore, VI converges to \( v^* \) at a linear rate of \( \lambda \in (0, 1) \). However, in many applications, the future rewards are almost as important as the reward in the current period and \( \lambda \) is close to 1. In this case, the convergence of VI becomes very slow.
Value Computation. The problem of computing the value vector of a policy is also crucial, see for instance the Policy Evaluation step in the Policy Iteration algorithm, presented in the next section. The operator \( T_\pi : \mathbb{R}^n \to \mathbb{R}^n \) associated with a policy \( \pi \) is defined as

\[
T_\pi(v)_s = \sum_{a \in \mathcal{A}} \pi_{sa} (r_{sa} + \lambda P_{sa}^T v), \quad \forall \, s \in \mathcal{S}.
\] (2.2)

Note that \( T_\pi \) is an affine operator and a contraction for \( \| \cdot \|_\infty \). The unique fixed point of \( T_\pi \) is \( v^\pi \), the value vector of the policy \( \pi \). Therefore, the following algorithm is called Value Computation (VC):

\[
v_0 \in \mathbb{R}^n, v_{t+1} = T_\pi(v_t), \forall \, t \geq 0.
\] (VC)

For the same reason as Algorithm VI, the sequence of vectors \( (T_\pi(v_0))_{t \geq 0} \) generated by VC converges linearly to \( v^\pi \) with a rate of \( \lambda \), for any initial vector \( v_0 \) [Puterman, 2014].

2.3 Policy Iteration

Policy Iteration was developed by Howard [Howard, 1960] and Bellman [Bellman, 1966]. The algorithm runs as follow.

Algorithm 1 Policy Iteration (PI)

1: Initialize \( \pi_0 \) at random.
2: for \( t \geq 0 \) do
3:    (Policy Evaluation) Choose \( v_t = v^{\pi_t} \) the value vector of \( \pi_t \).
4:    (Policy Improvement) Choose \( \pi_{t+1} \) with \( \pi_{t+1}, s \in \Delta(A) \) solving \( \max \pi \sum_{a \in \mathcal{A}} \pi_a (r_{sa} + \lambda P_{sa}^T v_t), \forall \, s \in \mathcal{S} \).
5:    (Stopping Criterion) Stop when \( \pi_{t+1} = \pi_t \).
6:    Increment \( t \leftarrow t + 1 \).
7: end for

In the Policy Improvement step, Policy Iteration computes \( \pi_{t+1} \) as a greedy one-step update, given the value vector \( v^{\pi_t} \) related to \( \pi_t \). Prior to Ye [2011], the proofs of convergence of Policy Iteration [Howard, 1960, Bellman, 1966] relied on the fact that the policies \( \pi_1, ..., \pi_t \) monotonically improve the objective function. In particular, the Policy Improvement step guarantees that \( \pi_{t+1} \) is a strictly better policy than \( \pi_t \), unless \( \pi_{t+1} = \pi_t \). As there are only finitely many policies (in the case of finite numbers of states and actions sets), the Policy Iteration algorithm terminates. As a side note, Policy Iteration is equivalent to a block-wise update rule when running the simplex algorithm on the linear programming reformulation of MDPs.

In Theorem 4.2 in Ye [2011], the author proves that the number of iterations of Policy Iteration is bounded from above by

\[
n^2 A \frac{1}{1 - \lambda} \log \left( \frac{n^2}{1 - \lambda} \right).
\]

Note that this bound does not depend of the order of magnitude of the rewards \( r_{sa} \), so that the worst-case complexity of Policy Iteration is actually strongly polynomial (for fixed discount factor \( \lambda \)). It is remarkable that even though the proofs presented in Ye [2011] rely on simple tools from the theory of linear programming. These results have been further improved in Scherrer [2016] for MDPs and extended to two-player games in Hansen et al. [2013].

3 Value Iteration as a first-order method

From Section 2.2, finding an optimal policy is equivalent to solving \( v = T(v) \). It has been noted since as far back as Bertsekas [1995] that the operator \( v \mapsto (I - T)(v) \) can be treated as the gradient of an unknown function. We review here the results from Goyal and Grand-Clément [2019], where the authors build upon this analogy to define first-order methods for MDPs, extend Nesterov’s acceleration and Polyak’s momentum to MDPs and present novel lower bounds on the performances of value iteration algorithms. We also review novel connections between Mirror Descent [Nemirovski and Yudin, 1983], Primal-Dual Algorithm [Chambolle and Pock, 2016], and Value Iteration at the end of this section. A review of the classical results for first-order methods in convex optimization can be found in Appendix A.1.
3.1 First-order methods for MDPs

Considering that $v \mapsto (I - T)(v)$ as the gradient of an unknown function, Goyal and Grand-Clément [2019] define first-order methods for MDPs as follows.

**Definition 3.1.** An algorithm is a **first-order method** for MDPs if it produces a sequence of iterates $(v_t)_{t \geq 0}$ satisfying

$$v_0 = 0, v_{t+1} \in \text{span}\{v_0, \ldots, v_t, T(v_0), \ldots T(v_t)\}, \forall \ t \geq 0.$$ 

Clearly, VI is a first-order method for MDP. However, Policy Iteration is not, since it relies on the update $v_t = v^\pi_t$ (Policy Evaluation step).

Choice of parameters. Recall that for a differentiable, $\mu$-strongly convex, $L$-Lipschitz continuous function $f : \mathbb{R}^n \to \mathbb{R}$, the following inequalities hold: for all vectors $v, w \in \mathbb{R}^n$,

$$\mu \cdot \|v - w\| \leq \|\nabla f(v) - \nabla f(w)\|_2,$$

$$\|\nabla f(v) - \nabla f(w)\|_2 \leq L \cdot \|v - w\|.$$ 

In the case of the Bellman operator $T$, for any vectors $v, w \in \mathbb{R}^n$, the triangle inequality gives

$$(1 - \lambda) \cdot \|v - w\|_\infty \leq \|(I - T)(v) - (I - T)(w)\|_\infty,$$

$$\|(I - T)(v) - (I - T)(w)\|_\infty \leq (1 + \lambda) \cdot \|v - w\|_\infty.$$ 

In convex optimization, the constant $\mu$ and $L$ can be used to tune the step sizes of the algorithms. Because of the analogy between the last four equations, the choice of $\mu = 1 - \lambda$ and $L = 1 + \lambda$ will be used to tune the step sizes of the novel algorithms for MDPs, inspired from first-order methods in convex optimization. We also use the notation $\kappa = \mu / L$.

**Relaxed Value Iteration.** For $\alpha > 0$, Goyal and Grand-Clément [2019] consider the following **Relaxed Value Iteration** (RVI) algorithm:

$$v_0 \in \mathbb{R}^n, v_{t+1} = v_t - \alpha (v_t - T(v_t)), \forall \ t \geq 0. \tag{RVI}$$ 

Note the analogy with the Gradient Descent algorithm GD (presented in Appendix A.1), when $v \mapsto (I - T)(v)$ is the gradient of an unknown function. RVI is also considered in Kushner and Kleinman [1971], Porteus and Totten [1978] without an explicit connection to GD. Formal convergence guarantees are provided in Goyal and Grand-Clément [2019]; the main difficulty is the use of $\|\cdot\|_\infty$ instead of $\|\cdot\|_2$ in the properties that characterize the Bellman operator $T$. Recall that a sequence $(v_t)_{t \geq 0}$ converges linearly to $v^* \in \mathbb{R}^n$ at a rate of $\rho \in (0, 1)$ if $v_t = v^* + o(\rho^t)$.

**Proposition 3.2.** (Goyal and Grand-Clément [2019], Proposition 1). Consider $(v_t)_{t \geq 0}$ the iterates of RVI and $v^*$ the unique fixed point of the Bellman operator $T$. Recall that $\mu = 1 - \lambda$, $L = 1 + \lambda$ and $\kappa = \mu / L$.

1. The sequence $(v_t)_{t \geq 0}$ produced by RVI does converge at a linear rate to $v^*$ as long as $\alpha \in (0, 2/L)$.
2. For $\alpha = 1 = 2/(L + \mu)$, RVI converges linearly to $v^*$ at a rate of $\lambda = (1 - \kappa)/(1 + \kappa)$.

Note the analogy with the classical convergence results for GD (as presented in Proposition A.3 in Appendix A.1). In particular, VI is a first-order method for MDP which converges to $v^*$ at a rate of $(1 - \kappa)/(1 + \kappa)$, where $\kappa = \mu / L = (1 - \lambda)/(1 + \lambda)$. When $\lambda \to 1$, $\kappa \to 0$ and the convergence of VI becomes very slow. Also, the same results hold for VC, i.e., for a fixed policy $\pi$, VC converges to $v^\pi$ at a rate of $(1 - \kappa)/(1 + \kappa)$.

3.2 Nesterov’s acceleration and Polyak’s momentum for Value Iteration

**Nesterov Accelerated Value Iteration.** Still considering $v = T(v)$ as the gradient of some unknown function, it is possible to write an **Accelerated Value Iteration** (AVI), building upon Accelerated Gradient Descent AGD (see Appendix A.1 for the exact definition and convergence rates of AGD). In particular, Goyal and Grand-Clément [2019] consider the following algorithm.

$$v_0, v_1 \in \mathbb{R}^n, \begin{cases} h_t = v_t + \gamma \cdot (v_t - v_{t-1}), \\ v_{t+1} \leftarrow h_t - \alpha (h_t - T(h_t)), \end{cases}, \forall \ t \geq 1. \tag{AVI}$$ 

The choice of step sizes for AVI is the same as for AGD in convex optimization (e.g., Theorem A.4 in Appendix A.1) but here $\mu = (1 - \lambda), L = (1 + \lambda)$. This yields

$$\alpha = 1/(1 + \lambda), \gamma = \left(1 - \sqrt{1 - \lambda^2}\right)/\lambda. \tag{3.1}$$
Polyak’s Momentum Value Iteration. It is also possible to write a Momentum Value Iteration algorithm (MVI):

\[ v_0, v_1 \in \mathbb{R}^n, v_{t+1} = v_t - \alpha (v_t - T(v_t)) + \beta \cdot (v_t - v_{t-1}), \forall t \geq 1. \] (MVI)

The same choice as for Momentum Gradient Descent (MGD, see Theorem A.4 in Appendix A.1) gives the following step sizes:

\[ \alpha = 2/(1 + \sqrt{1 - \lambda^2}), \beta = (1 - \sqrt{1 - \lambda^2})/(1 + \sqrt{1 - \lambda^2}). \] (3.2)

Properties for non-affine operators. Unfortunately, algorithms AVI and MVI may diverge on some MDP instances. In all generality, the analysis of AVI and MVI is related to the computation of the joint spectral radius of a specific set of matrices, which is usually a hard problem [Tsitsiklis and Blondel, 1997]. Vieillard et al. [2020] obtain convergence of a momentum variant of Q-learning, under a strong assumption on the sequence of iterates (Assumption 1 in Vieillard et al. [2020], closely related to the joint spectral radius of the visited matrices), and study this assumption empirically. Despite this, the analysis of the case where \( T \) is an affine operator reveals some interesting connections on the convergence properties of AVI and MVI compared to AGD and MGD.

Convergence rate for affine operators. Let \( \pi \) be a policy and consider Algorithm AVI and Algorithm MVI where the Bellman operator \( T \) is replaced with the Bellman recursion operator \( T_\pi \), defined in (2.2). Goyal and Grand-Clément [2019] refer to these new algorithms as Accelerated Value Computation (AVC) and Momentum Value Computation (MVC), in reference to Algorithm VC. Under conditions on the spectral radius of \( P_\pi \), Goyal and Grand-Clément [2019] provide the following convergence rates.

**Theorem 3.3** (Goyal and Grand-Clément [2019], Theorem 1 and Theorem 2). Assume that \( \pi \) is a policy such that the spectrum of the transition matrix \( P_\pi \) is contained in \( \mathbb{R} \).

1. Let \( \alpha, \gamma \) be as in (3.1). Then Algorithm AVC converges to \( v^* \) at a linear rate of \( 1 - \sqrt{\kappa} \).

2. Let \( \alpha, \gamma \) be as in (3.2). Then Algorithm MVC converges to \( v^* \) at a linear rate of \((1 - \sqrt{\kappa})/(1 + \sqrt{\kappa})\).

Note the analogy between the convergence results for Accelerated Gradient Descent and Momentum Gradient Descent (as presented in Theorem A.4 in Appendix A.1) and Theorem 3.3 (for Accelerated Value Computation and Momentum Value Computation). In particular, when \( \lambda \to 1 \) then \( \kappa \to 0 \) and both AVC and MVC enjoy significantly better convergence rates than VC, which achieves a convergence rate of \((1 - \kappa)/(1 + \kappa)\). The proof essentially relies on reformulating the AVC and MVC updates as second-order recursions on the sequence of iterates and inspecting the eigenvalues of the matrix \( B_\pi \) defining the dynamics of the iterates. Under the assumption that the eigenvalues of \( P_\pi \) are real, the matrix \( B_\pi \) has maximum eigenvalue of at most \( 1 - \sqrt{\kappa} \) in the case of AVC, and at most \((1 - \sqrt{\kappa})/(1 + \sqrt{\kappa})\) in the case of MVC. Without this assumption, this spectral radius of the matrix \( B_\pi \) may be more than 1, so that \( (v_t)_{t \geq 0} \) may diverge (see Section 5.1.2 in Goyal and Grand-Clément [2019]). The proof in Goyal and Grand-Clément [2019] is close to the methods in Section 3 in Flammarion and Bach [2015], pertaining to the analysis of algorithms from convex optimization such as AGD, MGD, and linear averaging as second-order recursions.

The condition that the spectrum of \( P_\pi \) is contained in \( \mathbb{R} \) is satisfied, in particular, when \( P_\pi \) is the transition matrix of a reversible Markov chain. Akian et al. [2020] provide more general conditions on the spectrum of \( P_\pi \) for AVC to converge, and extend AVC to \( d \)-th order acceleration, where the inertial step involves the last \( d \) iterates and the convergence rate approaches \( 1 - \sqrt{\kappa} \).

### 3.3 Lower bound on any first-order method for MDP

While in convex optimization, AGD achieves the best worst-case convergence rate over the class of smooth, convex functions, this is not the case for AVI for MDPs. In particular, Goyal and Grand-Clément [2019] prove the following lower-bound on any first-order method for MDPs.

**Theorem 3.4** (Goyal and Grand-Clément [2019], Theorem 3). There exists an MDP instance \((\mathbb{S}, \lambda, P, r, R_0, \lambda)\) such that for any sequence of iterates \((v_t)_{t \geq 0}\) generated by a first-order method for MDPs, the following lower bound holds for any step \( t \in \{1, \ldots, n - 1\} \):

\[ \|v_t - v^*\|_\infty \geq \Omega (\lambda^t). \]

The proof of Theorem 3.4 is constructive and relies on a hard MDP instance (which is neither reversible nor ergodic). Note the analogy with the lower-bound on the convergence rates of any first-order methods in convex optimization (e.g.,
as presented in Theorem A.5 in Appendix A.1), but recall that it is VI that converges at a linear rate of \( \lambda \). Therefore, for MDPs, it is Value Iteration that attains the lower bound on the worst-case complexity of first-order methods. This is in stark contrast with the analogous result in convex optimization (as presented in Theorem A.5), where AGD attains the worst-case convergence rate. It may be possible to obtain sharper lower-bounds for the convergence rate of first-order methods for MDPs by restraining the MDP instance to be reversible or ergodic.

### 3.4 Connections with mirror descent and other first-order methods

We review here some other algorithms from first-order convex optimization that have been recently used to design novel Value Iteration algorithms.

**Mirror Descent.** The authors in Geist et al. [2019] show that adding a regularization term in the Bellman update (2.1) yields an algorithm resembling Mirror Descent (MD, [Nemirovski and Yudin, 1983]). In particular, let \( c_{s,v} = (r_{sa} + \lambda P_{sa}^T v) \in \mathbb{R}^n \) for \( v \in \mathbb{R}^n \). Then the Bellman operator \( T \) as in (2.1) can rewritten as

\[
T(v)_s = \max_{\pi \in \Delta(A)} \langle \pi, c_{s,v} \rangle.
\]

Let \( D : \Delta(A) \times \Delta(A) \to \mathbb{R} \) be a given Bregman divergence (for instance, \( D(x, y) = \|x - y\|_2^2 \) or the Kullback-Leibler divergence). Geist et al. [2019] define an algorithm resembling Mirror Descent and Value Iteration MD-VI as follows:

\[
v_t \in \mathbb{R}^n, \pi_0 \text{ random } \begin{cases} \pi_{t+1,s} = \arg \max_{\pi \in \Delta(A)} \langle \pi, c_{s,v_t} \rangle + D(\pi, \pi_{t,s}), \forall s \in S, \forall t \geq 0. \\ v_{t+1} = \langle \pi_{t+1,s}, c_{s,v_t} \rangle, \forall s \in S, \forall t \geq 0, \end{cases}
\]

It is also possible to replace the update on the value vector by

\[
v_{t+1} = \langle \pi_{t+1,s}, c_{v_t} \rangle + D(\pi, \pi_{t,s}), \forall s \in S, \forall t \geq 0,
\]

i.e., to include the regularization term in the update on \( v_{t+1} \). Geist et al. [2019] provide the convergence rate of MD-VI and a detailed discussion on the connection of MD-VI and other classical algorithms for reinforcement learning, e.g., Trust Region Policy Optimization (TRPO, see Section 5.1 in Geist et al. [2019]).

**Other approaches.** (Stochastic) Mirror Descent can also be used to solve the linear programming formulation of MDPs in min-max form [Jin and Sidford, 2020, Gong and Wang, 2020]. In Grand-Clément and Kroer [2020, 2021] the authors adapt the Primal-Dual Algorithm of Chambolle and Pock [2016] for solving robust MDPs and distributionally robust MDPs. While these papers provide new algorithms for solving MDPs, they do not highlight a novel connection between classical algorithms for MDPs and classical algorithms for convex optimization. The interested reader can find a concise presentation of connections between Frank-Wolfe algorithm [Frank et al., 1956], Mirror Descent, dual averaging [Nesterov, 2009] and algorithms for reinforcement learning in the review of Viellard et al. [2019]. We note that the algorithms for MDPs in Vieillard et al. [2019] are optimizing in the space of policies \( (\pi \in (\Delta(A))^n) \), thereby relying on methods from constrained convex optimization. In contrast, in this review we consider that Value Iteration and Policy Iteration are optimizing over the space of value vectors \( (v \in \mathbb{R}^n) \), thereby relying on methods from unconstrained convex optimization.

### 4 Policy Iteration as a second-order method

#### 4.1 Newton-Raphson with the Bellman operator

The relation between Policy Iteration and Newton-Raphson method dates as far back as Kalaba [1959] and Pollatschek and Avi-Itzhak [1969]. We adapt here the presentation of Puterman and Brumelle [1979]. A review of the classical results for second-order methods in convex optimization can be found in Appendix A.2.

**Jacobian of the gradient operator.** We introduce the following notations, which greatly simplify the exposition of the results. Let us write \( F : v \mapsto (I - T)(v) \). Note that in Section 3, we interpreted \( F \) as the gradient of an unknown function \( f : \mathbb{R}^n \to \mathbb{R} \). In order to develop second-order methods, one uses second-order information, i.e., the Hessian of the unknown function \( f : \mathbb{R}^n \to \mathbb{R} \), which is the Jacobian of the gradient operator \( F : \mathbb{R}^n \to \mathbb{R}^n \). However, the map \( F \) may not be differentiable at any vector \( v \). This is because the Bellman operator itself is not necessarily differentiable, as the maximum of some linear forms. Still, it is possible to give a closed-form expression for the Jacobian of \( F \), where it is differentiable. In particular, we have the following lemma.
Lemma 4.1. Recall that \( F(v) = (I - T)(v) \) and assume that \( F \) is differentiable at \( v \in \mathbb{R}^n \). Let \( J_v \) be the Jacobian of \( F \) at \( v \):

\[
J_v \in \mathbb{R}^{n \times n}, \ J_{v,ij} = \partial_j F_i(v), \ \forall (i,j) \in [n] \times [n].
\]

With this notation, we have \( J_v = I - \lambda P_{\pi(v)} \), where \( \pi(v) \) attains the \( \max \) in \( T(v) \). Additionally, \( J_v \) is invertible.

The fact that the matrix \( J_v \) is invertible directly follows from \( \lambda \in (0, 1) \) and \( P_{\pi(v)} \) being a stochastic matrix.

Analytic representation of Policy Iteration. Let us now analyze the update from \( v_t = v^{\pi_t} \) to \( v_{t+1} = v^{\pi_{t+1}} \) in Policy Iteration. In the policy improvement step, \( \pi_{t+1} \) is obtained as a row-wise solution of \( \max \{ r_x + \lambda P_x v^{\pi_t} \} \).

The vector \( v_{t+1} \) is then updated as the value vector \( v^{\pi_{t+1}} \) of \( \pi_{t+1} \). From the definition of the value vector, we have the recursion

\[
v^{\pi_{t+1}} = r^{\pi_{t+1}} + \lambda P_{\pi_{t+1}} v^{\pi_{t+1}},
\]

and we can invert this equation to obtain \( v^{\pi_{t+1}} = (I - \lambda P_{\pi_{t+1}})^{-1} r^{\pi_{t+1}} \). Note that with the notation of Lemma 4.1, we have \( v^{\pi_{t+1}} = J_v^{-1} r^{\pi_{t+1}} \).

Adding and subtracting \( v_t \) from the previous equality yields

\[
v^{\pi_{t+1}} = J_v^{-1} J_v v_t + J_v^{-1} r^{\pi_{t+1}} = v_t - J_v^{-1} (-r^{\pi_{t+1}} + J_v v_t).
\]

We can then replace \( J_v \), by its expression from Lemma 4.1 in the last part of the above equation to obtain

\[
v^{\pi_{t+1}} = v_t - J_v^{-1} (-r^{\pi_{t+1}} + (I - \lambda P_{\pi_{t+1}}) v_t) = v_t - J_v^{-1} (v_t - r^{\pi_{t+1}} - \lambda P_{\pi_{t+1}} v_t) = v_t - J_v^{-1} (v_t - T(v_t)), \tag{4.1}
\]

where (4.1) follows from the definition of \( \pi_{t+1} \) attaining the \( \max \) in \( T(v_t) \). Equation (4.1) brings down to the following theorem.

Theorem 4.2 (Puterman and Brumelle [1979], Theorem 2). Assume that \( F \) is differentiable at any vector \( v_t \) visited by Policy Iteration. With the notation of Lemma 4.1, the Policy Iteration algorithm can be written as

\[
v_{t+1} = v_t - J_v^{-1} F(v_t), \ \forall \ t \geq 1.
\]

This is exactly the update of the Newton-Raphson method in convex optimization for solving \( F(v) = 0 \) (as presented in Appendix A.2). Unfortunately, the map \( F \) may not be differentiable at any \( v \), because of the maximization term in the definition of the Bellman operator \( T \). Therefore, this reformulation does not immediately yields the convergence property and the convergence rate of Policy Iteration using the classical results from convex optimization (e.g., Theorem A.6 in Appendix A.2). Puterman and Brumelle [1979] provide some assumptions on the Lipschitzness of \( v \rightarrow J_v \) that lead to convergence of Policy Iteration, and this analysis was refined recently in Cvetković and Protasov [2020] for a variant of Policy Iteration. However, these assumptions are hard to verify on MDP instances; to the best of our knowledge, the most meaningful analysis of the convergence rate of Policy Iteration remains Ye [2011] and its refinements [Hansen et al., 2013, Scherrer, 2016]. Additional analysis of Policy Iteration based on value functions approximations (for Markov games) can be found in Pérolat et al. [2016].

4.2 Smoothing the Bellman Operator

Replacing the \( \max \) in the Bellman operator with a log-sum-exp reformulation, Kamanchi et al. [2019] obtain convergence of a log-sum-exp version of Policy Iteration directly from the convergence of the Newton-Raphson method. In particular, for \( \beta > 0 \), define \( T_\beta : \mathbb{R}^n \rightarrow \mathbb{R}^n \) such that

\[
T_\beta(v) = \frac{1}{\beta} \log \left( \sum_{s \in S} \exp \left( \beta \left( r_{sa} + \lambda P_{sa}^T v \right) \right) \right), \ \forall \ s \in S. \tag{4.2}
\]

The authors in Kamanchi et al. [2019] prove the following results. Note that the same type of results can be obtained by replacing the log-sum-exp transformation with a soft-max.
Theorem 4.3 (Kamanchi et al. [2019], Lemmas 2-3, Theorem 2). 
1. The operator $T_\beta$ is a $\lambda$-contraction for $\| \cdot \|_\infty$. In particular, $T_\beta$ has a unique fixed point $v^*_\beta \in \mathbb{R}^n$.

2. When $\beta \to +\infty$, we have $v^*_\beta \to v^*$ where $v^*$ is the fixed point of the Bellman operator $T$. In particular, 
\[ \|v^*_\beta - v^*\|_\infty \leq \beta^{-1} \cdot \lambda(1 - \lambda)^{-1} \log(A). \]

3. Let $(v_t)_{t \geq 0}$ be the sequence of iterates generated by the Newton-Raphson method $\text{NR}$ for solving the equation $v - T_\beta(v) = 0$. Then $(v_t)_{t \geq 0}$ converges to $v^*_\beta$ at a quadratic rate, for any initial point $v_0$.

5 A Quasi-Newton method for MDP: Anderson acceleration

Specific applications of Anderson accelerations to MDPs has been considered in Geist and Scherrer [2018] and Zhang et al. [2020]. We give here some motivations, the relation with Anderson acceleration in convex optimization and review the convergence results of Zhang et al. [2020]. A review of the classical results for quasi-Newton methods in convex optimization can be found in Appendix A.3.

Anderson acceleration for MDPs. We adapt here the presentation of Geist and Scherrer [2018]. Fix an integer $m \in \mathbb{N}$. In order to compute the next iterates $v_{t+1}$, Anderson VI computes weights $\alpha_{t,0}, \ldots, \alpha_{t,m}$ and updates $v_{t+1}$ as a linear combination of the last $(m+1)$-iterates $T(v_t), \ldots, T(v_{t-m})$:
\[ v_{t+1} = \sum_{i=0}^{m} \alpha_{t,i} T(v_{t-m+i}). \tag{5.1} \]

Crucially, the weights are updated to minimize the weighted residuals of the previous of the previous iterates, where the residual of a vector $v$ is $F(v) = v - T(v)$. In particular, the vector $\alpha_t$ is chosen to minimize the following least-square problem:
\[ \min_{\alpha \in \mathbb{R}^{m+1}} \| \sum_{i=0}^{m} \alpha_i (v_{t-m+i} - T(v_{t-m+i})) \|^2_2. \tag{5.2} \]

The program (5.2) is better understood if $T$ is a linear operator (or close to being linear). In particular, if $\sum_{i=0}^{m} \alpha_i T(v_{t-m+i}) \approx T(\sum_{i=0}^{m} \alpha_i v_{t-m+i})$, then
\[ \| \sum_{i=0}^{m} \alpha_i (v_{t-m+i} - T(v_{t-m+i})) \|^2_2 \approx \| T(\sum_{i=0}^{m} \alpha_i v_{t-m+i}) - \sum_{i=0}^{m} \alpha_i v_{t-m+i} \|^2_2. \]

Therefore, in this case the weights $\alpha$ are chosen so that the next vector $v(\alpha) = \sum_{i=0}^{m} \alpha_i v_{t-m+i}$ has a minimal $F(v(\alpha))$ in terms of $\ell_2$-norm, where $F(v) = (I - T)(v)$. The vector $v_{t+1}$ is then chosen as $T(v(\alpha))$.

Relation to Anderson Acceleration in Convex Optimization. We present here a reformulation of Anderson Value Iteration which highlights the connection to the Anderson updates in convex optimization (as presented in Appendix A.3). The connection between Anderson acceleration and quasi-Newton methods was first shown in Eyert [1996] and expanded in Fang and Saad [2009], Walker and Ni [2011], Zhang et al. [2020]. We follow the lines of Walker and Ni [2011] and Zhang et al. [2020] here. Note that there is a closed-form solution of (5.2) at period $t$ using variable eliminations. In particular, the optimization program (5.2) can be rewritten as
\[ \min_{\beta \in \mathbb{R}^m} \| F(v_t) - \Delta F_t \beta \|^2_2, \]
where $\Delta F_t \in \mathbb{R}^{n \times m}$ is defined as
\[
\Delta F_t = (\Delta F_{t-m+1}, \ldots, \Delta F_t) \in \mathbb{R}^{n \times m},
\]
\[
\Delta F_{t'} = F(v_{t'}) - F(v_{t'-1}) \in \mathbb{R}^n, \quad \forall \ t' \geq 0,
\]
and where $\beta \in \mathbb{R}^m$ relates to $\alpha \in \mathbb{R}^{m+1}$ as $\alpha_0 = 1 - \beta_m$ and $\alpha_t = \beta_t - \beta_{t-1}$ otherwise. If $\Delta F_t$ is non-singular, the solution $\beta_t$ to the least-square problem (5.2) is simply
\[ \beta_t = \left( \Delta F_t^\top \Delta F_t \right)^{-1} \Delta F_t^\top F(v_t). \]
Using the relation between $\alpha$ and $\beta$ and plugging this expression into the Anderson update (5.1) yields

$$G_t = I + (\Delta v_t - \Delta F_t) \left(\Delta F_t^\top \Delta F_t\right)^{-1} \Delta F_t^\top,$$

(AndVI-II)

$$v_{t+1} = v_t - G_tF(v_t),$$

with $\Delta v_t = (v_{t-m+1} - v_{t-m}, \ldots, v_t - v_{t-1}) \in \mathbb{R}^{n \times m}$. The update on the sequence $(G_t)_{t\geq 0}$ is the same as the Anderson update of type-II in convex optimization (see (And-II) in Appendix A.3). In particular, $G_t$ achieves the minimum of $G_t \mapsto \|G_t - I\|_F$, under the conditions that $G_t \Delta F = \Delta v_t$; we write $\| \cdot \|_F$ for the Frobenius norm on matrices. This last equation can be interpreted as an inverse multi-secant condition, since

$$G_t \Delta F_t = \Delta v_t \iff G_t \left(F(v_{t-m+1}) - F(v_{t-m+i-1})\right) = v_{t-m+i} - v_{t-m+i-1}, \forall i = 1, \ldots, m.$$

The authors in Zhang et al. [2020] also consider another version of Anderson Value Iteration, using the Anderson update of type-I (see (And-I) in Appendix A.3):

$$J_t = I + (\Delta F_t - \Delta v_t) \left(\Delta v_t^\top \Delta v_t\right)^{-1} \Delta v_t^\top,$$

(AndVI-I)

$$v_{t+1} = v_t - J_t^{-1}F(v_t).$$

The update AndVI-I approximates the Jacobian $J_{v_t}$ with $J_t$, whereas AndVI-II directly approximates $J_{v_t}^{-1}$ with $G_t$. In particular, $J_t$ achieves the minimum of $J_t \mapsto \|J_t - I\|_F$, under the condition that $J_t \Delta v_t = \Delta F_t$. It is possible to show that $G_t$ and $J_t$ are rank-one updates from $G_{t-1}$ and $J_{t-1}$ [Zhang et al., 2020]. The analysis of Anderson Value Iteration as in Algorithm AndVI-I and Algorithm AndVI-II is complex. In particular, the operator $T$ is not differentiable, so that it is not straightforward that AndVI-I or AndVI-II converge. In order to prove convergence of Algorithm AndVI-I, the authors in Zhang et al. [2020] present a stabilized version of the vanilla Anderson algorithm AndVI-I. This results in $\lim_{t \to +\infty} v_t = v^*$, but the convergence rate is not known (Theorem 3.1 in Zhang et al. [2020]), even though the algorithm enjoys good empirical performances, typically outperforming VI. Note that this is the first result on the convergence of Anderson Acceleration, without differentiability of the operator $T$ (and with fixed-memory, i.e., with $m$ fixed). There are many other quasi-Newton methods aside from Anderson acceleration, see Appendix A.3 for a short review of some classical methods. Note that in the practical implementation of Anderson acceleration, one typically chooses $1 \leq m \leq 5$. Therefore, algorithms based on information on only the last two iterates may still perform well. In particular, it could be possible to develop algorithms for MDPs, based on Broyden or BFGS updates.

The convergence of AndVI-II for non-differentiable operators (or a stabilized version of AndVI-II) also remains an open question.

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A short review of gradient-based methods in convex optimization

In this section we review the classical definitions, algorithms and convergence results from convex optimization (mostly referring to Boyd and Vandenberghe [2004], Nesterov [2013]). As the scope of gradient methods in convex optimization is extremely large, we only introduce here the notions that are relevant to our connections with algorithms for MDPs.
A.1 First-order methods

In this section we let $f : \mathbb{R}^n \to \mathbb{R}$ be a differentiable function. We start with the following definitions of first-order algorithms for convex optimization and linear convergence.

**Definition A.1.** A first-order algorithm produces a sequence $(x_t)_{t \geq 0}$ of points of $\mathbb{R}^n$ that only exploits first-order information at the previous iterates to compute the next iterate. In particular, a first-order algorithm satisfies the following condition:

$$x_{t+1} \in x_t + \text{span} \{ \nabla f(x_0), ..., \nabla f(x_t) \}, \forall \ t \geq 0.$$  

We also use the following definition of linear convergence in convex optimization.

**Definition A.2.** The convergence of $(x_t)_{t \geq 0}$ to $x_f$ a minimizer of $f$ is called linear of rate $\rho \in (0, 1)$ if

$$x_t = x_f + o \left( \rho^t \right).$$

**Gradient descent.** The gradient descent algorithm (Algorithm GD) with fixed step size $\alpha$ is described as:

$$x_0 \in \mathbb{R}^n, \ x_{t+1} = x_t - \alpha \nabla f(x_t), \forall \ t \geq 0. \quad \text{(GD)}$$

The intuition for GD comes from first-order Taylor expansions. In particular, if $f$ is differentiable, we can write $f(x + h) = f(x) + h^T \nabla f(x) + o(\|h\|_2)$. Now using the update rule for $x_{t+1}$ we obtain

$$f(x_{t+1}) = f(x_t - \alpha \nabla f(x_t)) = f(x_t) - \alpha \nabla f(x_t) + o(\|\nabla f(x_t)\|_2).$$

Therefore, for the right choice of step size $\alpha$ we can obtain $f(x_{t+1}) \leq f(x_t)$. In particular, the following proposition gives the rate of convergence of GD.

**Proposition A.3** (Nesterov [2013], Chapter 2.1.5)). Assume that $f$ is $\mu$-strongly convex and $L$-Lipschitz continuous. Let $x^*$ be the minimizer of $f$ and $\kappa = \mu / L$.

1. The sequence $(x_t)_{t \geq 0}$ produced by GD does converge at a linear rate to $x^*$ as long as $\alpha \in (0, 2/L)$.
2. For $\alpha = 2/(L + \mu)$, GD converges linearly to $x^*$ at a rate of $(1 - \kappa) / (1 + \kappa)$.
3. In both the previous cases, GD is a descent algorithm: $f(x_{t+1}) \leq f(x_t), \forall \ t \geq 1$.

Note that they are many other step sizes strategy, e.g., (backtracking) line search or choosing $\alpha_t = 1/t$ [Boyd and Vandenberghe, 2004].

**Acceleration and Momentum in convex optimization.** Two popular first-order algorithms building upon GD are Accelerated Gradient Descent (AGD) and Momentum Gradient Descent (MGD). In particular, Accelerated Gradient Descent [Nesterov, 1983, 2013] builds upon GD by adding an intermediate extrapolation step:

$$x_0, x_1 \in \mathbb{R}^n, \ \{ h_t = v_t + \gamma \cdot (x_t - x_{t-1}), \quad (x_{t+1} \leftarrow h_t - \alpha \nabla f(h_t), \quad \forall \ t \geq 1. \quad \text{(AGD)}$$

In contrast, Momentum Gradient Descent builds upon GD by adding a momentum term in the update:

$$x_0, x_1 \in \mathbb{R}^n, \ x_{t+1} = x_t - \alpha \nabla f(x_t) + \beta \cdot (x_t - x_{t-1}), \forall \ t \geq 1. \quad \text{(MGD)}$$

The following theorem presents the rates of convergence of AGD and MGD for fixed step sizes.

**Theorem A.4** (Ghadimi et al. [2015], Nesterov [2013]). Assume that $f$ is $\mu$-strongly convex and $L$-Lipschitz continuous. Let $x^*$ be the minimizer of $f$ and $\kappa = \mu / L$.

1. Let $\alpha = 1/L, \gamma = (\sqrt{L} - \sqrt{\mu}) / (\sqrt{L} + \sqrt{\mu})$. Then AGD converges to $x^*$ at a linear rate of $1 - \sqrt{\kappa}$.
2. Additionally, assume that $f$ is twice continuously differentiable. Let $\alpha = 4/(\sqrt{L} + \sqrt{\mu})^2, \beta = (\sqrt{L} - \sqrt{\mu})^2 / (\sqrt{L} + \sqrt{\mu})^2$. Then MGD converges to $x^*$ at a linear rate of $(1 - \sqrt{\kappa}) / (1 + \sqrt{\kappa})$.

When $\kappa$ is small, both AGD and MGD enjoy better convergence rates than $(1 - \kappa) / (1 + \kappa)$, the convergence rate of GD. Note that the convergence of MGD relies on stronger assumption than the convergence of AGD. In particular, MGD may diverge if $f$ is not twice continuously differentiable [Ghadimi et al., 2015]. Additionally, AGD and MGD are not descent algorithms: they do not necessarily produce estimates that result in a monotonically decreasing objective function. The objective value might increase for a few periods, before significantly decreasing afterward. This is known as oscillations, and we refer to O’donoghue and Candes [2015] for a detailed study of the oscillation effects of AGD.
Lower bounds on the convergence rates. Nesterov [2013] provides lower-bounds on the convergence rate of any first-order algorithm on the class of smooth, convex functions and on the class of smooth, strongly-convex functions. In particular, we recall the results for lower-bounds of first-order algorithms in smooth convex optimization. Nesterov [2013] considers \( n = +\infty \), i.e., considers the space \( \mathbb{R}^N \) of infinite sequences of scalars, and proves the following lower-bound for smooth, strongly-convex functions.

\[ \|x_t - x^*\|_2 \geq \left( \frac{1 - \sqrt{\kappa}}{1 + \sqrt{\kappa}} \right)^t \cdot \|x_0 - x^*\|_2. \]

The proof of Theorem A.5 relies on designing a hard instance. Additionally, Nesterov [2013] proves that the rate of convergence of AGD matches this lower-bound: AGD achieves the optimal rate of convergence over the class of smooth, convex functions, as well as over the class of smooth, strongly-convex functions. Note that MGD achieves a better worst-case convergence rate than AGD but requires that \( f \) is twice continuously differentiable. In particular, MGD may diverge if \( f \) is only continuously differentiable once [Ghadimi et al., 2015].

A.2 Second-order methods

We focus here on the Newton-Raphson method (NR). In particular, consider solving the equation \( F(x) = 0 \), for a function \( F: \mathbb{R}^n \to \mathbb{R}^n \). If the function \( F \) is differentiable, the Jacobian \( J_x \) of \( F \) is defined for any \( x \in \mathbb{R}^n \), where \( J_x \in \mathbb{R}^{n \times n} \), \( J_{x,ij} = \partial_j F_i(x) \), \( \forall \ (i, j) \in [n] \times [n] \). The NR algorithm is:

\[ x_0 \in \mathbb{R}^n, x_{t+1} = x_t - J_{x_t}^{-1} F(x_t), \forall t \geq 0. \]

(NR)

The intuition is as follows. If we linearize the condition \( F(x) = 0 \) close to \( x \), we have \( F(x + v) \approx F(x) + J_x v \), and \( F(x) + J_x v = 0 \iff v = -J_{x}^{-1} F(x) \). Therefore, for \( x \) close to \( x^* \), the point \( x - J_{x}^{-1} F(x) \) should be a good approximation of \( x^* \). Under some conditions, Algorithm NR converges to \( x^* \) a solution to \( F(x) = 0 \) and the convergence is quadratic.

Theorem A.6. [Boyd and Vandenberghe [2004] Section 9.5, Poczos and Tibshirani [CMU-1075]] Assume that:

1. \( J_x \succ \mu I, \forall x \in \mathbb{R}^n \).
2. \( \exists M \geq 0, \|J_x - J_y\|_2 \leq M \|x - y\|_2, \forall x, y \in \mathbb{R}^n \).

Then there is a unique solution \( x^* \) to \( F(x) = 0 \). Additionally, if \( x_0 \) is sufficiently close to \( x^* \): \( \|x_0 - x^*\|_2 \leq 2\mu/3M \), then there exists \( \rho > 0 \), such that Algorithm NR enjoys a quadratic convergence rate to \( x^* \):

\[ \|x_{t+1} - x^*\|_2 \leq \rho \|x_t - x^*\|_2^2. \]

In the case of quadratic convergence, the number of correct digits of \( x_t \) (compared to the optimal solution \( x^* \)) roughly doubles at every iteration of the Newton-Raphson method (see Equation (9.35) in Boyd and Vandenberghe [2004]). However, the convergence result of Theorem A.6 is very subtle. In particular, if \( x_0 \) is not close enough to \( x^* \), NR may diverge. Additionally, the quadratic convergence result may hold only for a fairly small region around \( x^* \). Without the assumptions presented in Theorem A.6, the algorithm may cycle or converge only in linear time. We refer the reader to Poczos and Tibshirani [CMU-1075] for some examples of problematic cases. We also refer the reader to Chapter 7, 10 and 12 of Ortega and Rheinboldt [2000] for a comprehensive analysis of Newton-Raphson method and its variations. Note that it is possible to start in a damped phase where we add a step size \( \alpha_t \), i.e., to start with \( x_{t+1} = x_t - \alpha_t J_{x_t}^{-1} F(x_t) \) for adequate choices of the step sizes \( \alpha_t \). During the damped phase, NR has linear convergence, before transitioning to quadratic convergence when the iterates are close enough to the optimal solution \( x^* \), see Boyd and Vandenberghe [2004], Section 9.5.1.

Relation to Newton’s method in convex optimization. We highlight here the relation between NR and Newton’s method. Let \( f \) be a \( \mu \)-strongly convex and twice-differentiable function. In this case, the Hessian \( \nabla^2 f(x) \) is defined at any point \( x \in \mathbb{R}^n \) and \( \nabla^2 f(x) \succ \mu I \). Newton’s method to find a minimizer of \( f \) follows from minimizing the second-order Taylor expansion of \( f \) at the current iterate and runs as follows:

\[ x_0 \in \mathbb{R}^n, x_{t+1} = x_t - \alpha_t \left( \nabla^2 f(x_t) \right)^{-1} \nabla f(x_t), \forall t \geq 0. \]  

(Newton)
The relation with NR comes from first-order optimality conditions. In order to minimize \( f \), we want to solve the system of equations \( \nabla f(x) = 0 \). We can then use NR on this equation in \( x \in \mathbb{R}^n \). The operator \( x \mapsto \nabla f(x) \) is differentiable (since \( f \) is twice-differentiable) and plays the role of the operator \( F \) in the Newton-Raphson algorithm. The first assumption in Theorem A.6 is equivalent to \( f \) being \( \mu \)-strongly convex and the second assumption in Theorem A.6 translates as \( \exists M \geq 0, \| \nabla^2 f(x) - \nabla^2 f(y) \|_2 \leq M \| x - y \|_2, \forall x, y \in \mathbb{R}^n \).

A.3 Quasi-Newton methods

Invented mostly in the 1960s and 1970s, quasi-Newton methods are still very popular to solve nonlinear optimization problems and nonlinear equations; their applications are very diverse and range from machine learning [Zhang et al., 2020] to quantum chemistry [Rohwedder and Schneider, 2011]. We describe here the Broyden updates for constructing successive approximations of the Jacobians, discuss their convergence rates and their relations to the BFGS update. We also detail Anderson acceleration, as it relates to Anderson Value Iteration for MDPs (see Section 5).

Let us consider again solving \( F(v) = 0 \) with \( F : \mathbb{R}^n \to \mathbb{R}^n \). Generally speaking, any method that replaces the exact computation of the (inverse of the) Jacobian matrices in the Newton-Raphson algorithm NR with an approximation is a quasi-Newton method. This can be useful when the derivative of \( F \) is hard to compute, or the dimension of the problem is very large. In particular, a quasi-Newton method constructs a sequence of iterates \( (x_t)_{t \geq 0} \) and a sequence of matrices \( (J_t)_{t \geq 0} \) such that \( J_t \) is an approximation of the Jacobian \( J_{x_t} \), for any \( t \geq 0 \) and

\[
x_0 \in \mathbb{R}^n, x_{t+1} = x_t - J_t^{-1}F(x_t), \forall t \geq 0.
\]

(A.1)

**Good and bad Broyden updates.** Recall that the motivation for the Newton-Raphson algorithm NR comes from linearization:

\[
F(x_t + v) \approx F(x_t) + J_{x_t}v.
\]

(A.2)

Therefore, when we compute the approximation \( J_t \) we want to maintain (A.2), which yields the so-called secant condition on \( J_t \):

\[
J_t \Delta x_t = \Delta F_t,
\]

(A.3)

for \( \Delta x_t = x_t - x_{t-1}, \Delta F_t = F(x_t) - F(x_{t-1}) \). Another common condition is that \( J_t \) and \( J_{t-1} \) coincides on the direction orthogonal to \( \Delta x_t \). This is the so-called no-change condition:

\[
J_t q = J_{t-1} q, \forall q \in \mathbb{R}^n \text{ such that } q^\top \Delta x_t = 0.
\]

(A.4)

There is a unique matrix satisfying (A.3)-(A.4) and this yields the type-I Broyden’s update:

\[
J_t = J_{t-1} + (\Delta F_t - J_{t-1} \Delta x_t) \frac{\Delta x_t^\top}{\Delta x_t \Delta x_t^\top}.
\]

(Broyden-I)

Note that this is a rank one update from \( J_{t-1} \). Additionally, Dennis and Moré [1977] show that the update (Broyden-I) minimizes \( J_t \mapsto \| J_t - J_t \|^2_F \), subject to the secant condition (A.3), where \( \| \cdot \|_F \) is the Frobenius norm. The update (Broyden-I) approximates the Jacobian and is sometimes called the good Broyden’s update, as it enjoys better empirical performances than the bad Broyden update (type-II), which directly approximates the inverse of the Jacobian by considering both the secant condition (A.3) and the no-change condition (A.4) in their inverse forms. In particular, the bad Broyden update finds a matrix \( G \) satisfying

\[
G_t \Delta F_t = \Delta x_t, G_t q = G_{t-1} q, \forall q \in \mathbb{R}^n \text{ such that } q^\top \Delta F_t = 0.
\]

(A.5)

This also has a closed-form solution:

\[
G_t = G_{t-1} + (\Delta x_t - G_t \Delta F_t) \frac{\Delta x_t^\top G_{t-1} \Delta x_t}{\Delta x_t \Delta x_t^\top}.
\]

(Broyden-II)

and \( G_t \) minimizes \( G_t \mapsto \| G_t - G_t \|^2_F \), subject to the secant condition in inverse form (A.5). Since \( G_t \) directly approximates the inverse of the Jacobian, the update on \( x_t \) is \( x_{t+1} = x_t - G_t F(x_t) \). Note that there are many other possible approximation schemes for the Jacobians (or their inverses). We refer the reader to Fang and Saad [2009], Section 2 and Section 3, for more details.
Superlinear Convergence of Broyden’s updates. Recall that the superlinear convergence of a sequence \((x_t)_{t \geq 0}\) to \(x^*\) is equivalent to \(\lim_{t \to \infty} \|x_{t+1} - x^*\|_2 / \|x_t - x^*\|_2 = 0\). The result for the local superlinear convergence of the quasi-Newton iterations with Broyden’s updates dates back to [Broyden et al. 1973]. Crucially, the next convergence results require differentiable operators.

Theorem A.7 ([Broyden et al. 1973], Theorems 3.2-3.4). Let \(x^* \in \mathbb{R}^n\) be such that \(F(x^*) = 0\). Assume that:

- \(F : \mathbb{R}^n \to \mathbb{R}^n\) is differentiable on \(\mathbb{R}^n\).
- \(J_{x^*}\) is non-singular.
- \(\exists M \in \mathbb{R}, \|J_x - J_{x^*}\|_2 \leq M\|x - x^*\|_2, \forall x \in \mathbb{R}^n\).

Then the following convergence results hold:

1. Let \((x_t)_{t \geq 0}\) following \(x_{t+1} = x_t - J_t^{-1} F(x_t)\) where \((J_t)_{t \geq 0}\) is updated following (Broyden-I).
   If \(x_0\) is close enough to \(x^*\), then \(\lim_{t \to \infty} x_t = x^*\) and the convergence is superlinear.

2. Let \((x_t)_{t \geq 0}\) following \(x_{t+1} = x_t - G_t F(x_t)\) where \((G_t)_{t \geq 0}\) is updated following (Broyden-II).
   If \(x_0\) is close enough to \(x^*\), then \(\lim_{t \to \infty} x_t = x^*\) and the convergence is superlinear.

Comparison to BFGS. Note that the updates (Broyden-I) and (Broyden-II) are rank-one updates. In contrast, the Broyden-Fletcher-Goldfarb-Shanno (BFGS) algorithm performs rank-two updates of the form

\[
J_t = J_{t-1} + \frac{\Delta F_i \Delta F_i^\top}{\Delta F_i^\top \Delta x_t} - \frac{J_{t-1} \Delta x_t (J_{t-1} \Delta x_t)^\top}{\Delta x_t^\top J_{t-1} \Delta x_t}.
\]  

(BFGS)

In particular, the update \(J_t\) from (BFGS) minimizes \(J_t \mapsto \|J_t - J_i\|_F\), subject to the secant condition (A.3) and \(J_t\) being symmetric. The second condition is because when \(F = v \mapsto \nabla f(v)\) for a function \(f\), then \(J_x = \nabla^2 f(x)\) is a symmetric matrix. The inverse of \(J_t\) can be computed efficiently given the inverse of \(J_{t-1}\) following the Sherman-Morrison formula.

Under the same assumption as Theorem A.7 and the assumption that \(J_{x^*}\) is symmetric definite positive, the quasi-Newton method (A.1) with updates (BFGS) converges locally and superlinearly to \(x^*\) (e.g., Theorem 5.5 and Corollary 5.6 in [Broyden et al. 1973]). Note that [Rodomanov and Nesterov 2021] provide explicit superlinear convergence rates (i.e., bounds on \(\|x_t - x^*\|_2\)) for a large class of quasi-Newton methods, including BFGS.

Anderson mixing. Let \(m \in \mathbb{N}\). In Anderson mixing [Anderson, 1965], the iterates \((x_t)_{t \geq 0}\) are still \(x_{t+1} = x_t - J_t^{-1} F(x_t)\), but the information about the last \(m + 1\) iterates \(x_t, \ldots, x_{t-m}\) and \(F(x_t), \ldots, F(x_{t-m})\) is used to update the approximation \(J_t\). In particular, the computation of \(J_t\) relies on the last \(m + 1\) secant conditions:

\[
J_t \Delta x_i = \Delta F_i, \forall i \in \{t, \ldots, t-m+1\}.
\]  

(A.6)

With the notation \(\Delta X_i = (\Delta x_{t-i+1}, \ldots, \Delta x_t) \in \mathbb{R}^{n \times m}, \Delta F_i = (\Delta F_{t-i+1}, \ldots, \Delta F_t) \in \mathbb{R}^{m \times n}\), Equation (A.6) can be concisely written as \(J_t \Delta X_i = \Delta F_i\). Minimizing \(J_t \mapsto \|J_t - I\|_F^2\) subject to (A.6) yields a closed-form solution as

\[
J_t = I + (\Delta F_t - \Delta X_i) \left(\Delta X_i^\top \Delta X_i\right)^{-1} \Delta X_i^\top.
\]  

(And-I)

The update (And-I) relies on the type-I Broyden update. The type-II Broyden update, which estimates \(G_t\) as an approximation of the inverse of the Jacobian, satisfies the multi-secant condition in inverse form: \(G_t \Delta F_t = \Delta X_i\). Subject to minimizing \(G_t \mapsto \|G_t - I\|_F^2\), the closed-form iteration for Anderson acceleration based on type-II Broyden update becomes

\[
G_t = I + (\Delta X_t - \Delta F_t) \left(\Delta X_t^\top \Delta F_t\right)^{-1} \Delta F_t^\top,
\]  

(And-II)

and the update on \(x_t\) simply becomes \(x_{t+1} = x_t - G_t F(x_t)\). Note that it can be proved that \(J_t\) and \(G_t\) are still rank one updates from the previous approximations \(J_{t-1}\) and \(G_{t-1}\) (Zhang et al. [2020], Proposition 3.1, and Rohwedder and Schneider [2011], Theorem 3.2).
Superlinear Convergence of Anderson Acceleration. The relations between Anderson acceleration and Broyden’s updates dates from Eyert [1996]. A proof of convergence of the full-memory version \((m = t\) at each iteration \(t\)) of (And-I) can be found in Gay and Schnabel [1978]. In particular, under the same assumption as Theorem A.7, and with \(m = t\) at each iteration \(t\), if \(x_0\) is close enough to \(x^*\) then Anderson acceleration converges superlinearly to \(x^*\). Local superlinear convergence, for a limited memory version \((m \in \mathbb{N}\) is fixed across iterations) of (And-II) is obtained in Rohwedder and Schneider [2011] (Theorem 5.2 and Theorem 5.4) under the same conditions. Under a contraction assumption, Toth and Kelley [2015] obtain local linear convergence of (And-II) (Theorem 2.3 and Theorem 2.4). Crucially, all these convergence results rely on the operator \(F\) being differentiable, which is not the case for MDPs (see the Bellman operator (2.1)). Zhang et al. [2020] obtain convergence results for (And-I) without the operator \(F\) being differentiable, and this result is discussed more at length in Section 5.