STRUCTURE OF GRADIENT CONFORMAL SOLITONS AND ITS APPLICATIONS

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Abstract. The gradient $k$-Yamabe soliton is an interesting generalization of the gradient Yamabe soliton. We consider the most general notion of the gradient Yamabe soliton, that is, the gradient conformal soliton which includes the gradient $k$-Yamabe soliton. In this paper, we completely reveal the structure of complete gradient conformal solitons under some assumption, and give some applications to gradient $k$-Yamabe solitons. These results improve many results obtained so far. Furthermore, we give an affirmative partial answer to the Yamabe soliton version of Perelman’s conjecture.

1. Introduction

An $n$-dimensional Riemannian manifold $(M^n, g)$ is called a gradient Yamabe soliton if there exists a smooth function $F$ on $M$ and a constant $\rho \in \mathbb{R}$, such that

$$\nabla\nabla F = (R - \rho)g,$$

where, $R$ is the scalar curvature on $M$. If $F$ is constant, $M$ is called trivial.

One of the most interesting problem of the Yamabe soliton is the Yamabe soliton version of Perelman’s conjecture, that is, “Any complete (steady) gradient Yamabe soliton with positive sectional curvature under some natural assumption is rotationally symmetric”. The problem was first considered by P. Daskalopoulos and N. Sesum [7]. They showed that any locally conformally flat complete gradient Yamabe soliton with positive sectional curvature is rotationally symmetric. Later, G. Catino, C. Mantegazza and L. Mazzieri [5] and H.-D. Cao, X. Sun and Y. Zhang [4] also considered the same problem. In particular, Cao, Sun and Zhang showed that any locally conformally flat complete
gradient Yamabe soliton with positive scalar curvature is rotationally symmetric.

To understand the gradient Yamabe soliton, many generalizations of it have been introduced.

(1) Almost gradient Yamabe solitons [2]:
For smooth functions $F$ and $\rho$ on $M$,
\[(R - \rho)g = \nabla\nabla F.\]

(2) Gradient $k$-Yamabe solitons [5]:
For a smooth function $F$ on $M$ and $\rho \in \mathbb{R}$,
\[2(n - 1)(\sigma_k - \rho)g = \nabla\nabla F,\]
where, $\sigma_k$ denotes the $\sigma_k$-curvature of $g$, that is,
\[\sigma_k = \sigma_k(g^{-1}A) = \sum_{i_1 < \cdots < i_k} \mu_{i_1} \cdots \mu_{i_k} \text{ (for } 1 \leq k \leq n),\]
where, $A = \frac{1}{n-2} (\text{Ric} - \frac{1}{2(n-1)}Rg)$ is the Schouten tensor and $\mu_1, \cdots, \mu_n$ are the eigenvalues of the symmetric endomorphism $g^{-1}A$. Here, Ric is the Ricci tensor of $M$.

(3) $h$-almost gradient Yamabe solitons [17]:
For smooth functions $F$, $\rho$ and $h$ ($h > 0$ or $h < 0$) on $M$,
\[(R - \rho)g = h\nabla\nabla F.\]

To consider all these solitons, we consider the gradient conformal soliton defined by G. Catino, C. Mantegazza and L. Mazzieri [5]:

**Definition 1.1** ([5]). Let $(M, g)$ be an $n$-dimensional Riemannian manifold. For smooth functions $F$ and $\varphi$ on $M$, $(M, g, F, \varphi)$ is called a gradient conformal soliton if it satisfies
\[(1.1) \quad \varphi g = \nabla\nabla F.\]
If $F$ is constant, $M$ is called trivial. The function $F$ is called the potential function.

**Remark 1.2.** By the definition, all results in this paper can be applied to gradient Yamabe solitons, gradient $k$-Yamabe solitons, almost gradient Yamabe solitons and $h$-almost gradient Yamabe solitons.

We also remark that gradient conformal solitons were studied by Cheeger-Colding ([6], see also [10]).

In this paper, we completely reveal the structure of gradient conformal solitons under some assumption.
Theorem 1.3. Any nontrivial complete gradient conformal soliton \((M^n, g, F, \varphi)\) with \(\varphi, F \geq 0\) and
\[
\int_{M \setminus B(x_0, R_0)} \frac{F}{d(x_0, x)^2} < +\infty,
\]
for some \(x_0 \in M\) and \(R_0 > 0\) is
\[
([0, \infty) \times \mathbb{S}^{n-1}, ds^2 + a^2 \bar{g}_S, as + b, 0),
\]
where, \(\bar{g}_S\) is the round metric on \(\mathbb{S}^{n-1}\), and \(a > 0\) and \(b \geq 0\).

As a corollary, we also completely reveal the structure of gradient \(k\)-Yamabe solitons and all other generalizations of gradient Yamabe solitons with the similar assumption.

Corollary 1.4. Any nontrivial complete gradient \(k\)-Yamabe soliton \((M^n, g, F)\) with \(\sigma_k \geq \rho\), \(F \geq 0\) and
\[
\int_{M \setminus B(x_0, R_0)} \frac{F}{d(x_0, x)^2} < +\infty,
\]
for some \(x_0 \in M\) and \(R_0 > 0\) is
\[
([0, \infty) \times \mathbb{S}^{n-1}, ds^2 + a^2 \bar{g}_S, as + b, 0),
\]
where, \(a > 0\) and \(b \geq 0\).

Here we remark that Corollary 1.4 improves Theorem 4 in [1]. Furthermore, by Theorem 1.3, we give an affirmative partial answer to the Yamabe soliton version of Perelman’s conjecture, that is, any complete (steady) gradient Yamabe soliton with positive sectional curvature (under some natural assumption) is rotationally symmetric.

Corollary 1.5. Any nontrivial complete steady gradient Yamabe soliton \((M^n, g, F)\) with positive scalar curvature, \(F \geq 0\) and
\[
\int_{M \setminus B(x_0, R_0)} \frac{F}{d(x_0, x)^2} < +\infty,
\]
for some \(x_0 \in M\) and \(R_0 > 0\) is rotationally symmetric. In particular, it is
\[
([0, \infty) \times \mathbb{S}^{n-1}, ds^2 + a^2 \bar{g}_S, as + b, 0),
\]
where, \(a > 0\) and \(b \geq 0\).

The remaining sections are organized as follows. Section 2 is devoted to the proof of Theorem 1.3. In Section 3, we give some classifications of complete gradient conformal solitons. In Section 4, we also give some triviality results of complete gradient conformal solitons.
2. Proof of Theorem 1.3

To show Theorem 1.3, we use the following result shown by the author [13].

**Theorem 2.1** ([13]). A nontrivial complete gradient conformal soliton \((M^n, g, F, \varphi)\) is either

1. compact and rotationally symmetric, or
2. the warped product

\[
(\mathbb{R}, ds^2) \times |\nabla F| (N^{n-1}, \bar{g}),
\]

where, the scalar curvature \(\bar{R}\) of \(N\) satisfies

\[|\nabla F|^2 R = \bar{R} - (n - 1)(n - 2)\varphi^2 - 2(n - 1)g(\nabla F, \nabla \varphi),\]

or

3. rotationally symmetric and equal to the warped product

\[(\mathbb{R}, ds^2) \times |\nabla F| (S^{n-1}, \bar{g}_S),\]

where, \(\bar{g}_S\) is the round metric on \(S^{n-1}\).

Furthermore, the potential function \(F\) depends only on \(s\).

Therefore, to consider the Yamabe soliton version of Perelman’s conjecture, we only have to consider (2) of Theorem 2.1.

**Proof of Theorem 1.3.** We first take a cut off function \(\eta\) on \(M\) satisfying that

\[
\begin{cases}
0 \leq \eta(x) \leq 1 & (x \in M), \\
\eta(x) = 1 & (x \in B(x_0, R)), \\
\eta(x) = 0 & (x \notin B(x_0, 2R)), \\
|\nabla \eta| \leq \frac{C}{R} & (x \in M), \text{ for some constant } C \text{ independent of } R, \\
\Delta \eta \leq \frac{C}{R^2} & (x \in M), \text{ for some constant } C \text{ independent of } R,
\end{cases}
\]
where, \( B(x_0, R) \) and \( B(x_0, 2R) \) are the balls centered at a fixed point \( x_0 \in M \) with radius \( R \) and \( 2R \), respectively. By \( \Delta F = n \varphi \), one has

\[
0 \leq \int_{B(x_0,2R)} \eta \varphi = \frac{1}{n} \int_{B(x_0,2R)} \eta \Delta F = \frac{1}{n} \int_{B(x_0,2R) \setminus B(x_0,R)} \Delta \eta F \leq \frac{C}{nR^2} \int_{B(x_0,2R) \setminus B(x_0,R)} F.
\]

Take \( R \rightarrow +\infty \). By the assumption, we have that the right hand side of (2.2) goes to 0. Therefore, we have

\[
\int_{M} \varphi = 0.
\]

Since \( \varphi \geq 0 \), one has \( \varphi = 0 \), hence

\[
\nabla \nabla F = 0.
\]

By Theorem 2.1, we have 3 types of gradient conformal solitons.

Case 1. \( M \) is compact and rotationally symmetric.

By (2.3), \( \Delta F = 0 \). Since \( M \) is compact, by the standard Maximum principle, we have that \( F \) is constant.

Case 2. \( M \) is the warped product

\[
(\mathbb{R}, ds^2) \times |\nabla F| \left( N^{n-1}, \bar{g} \right).
\]

By (2.3), we have

\[
\nabla |\nabla F|^2 = 2 \nabla_j \nabla_i F \nabla_j F = 0.
\]

Hence, \( \nabla F \) is a constant vector field. Since \( F \) depends only on \( s \in \mathbb{R} \), one can get that

\[
\nabla F = F'(s) \partial_s,
\]

where, \( F'(s) \) is a constant, say \( a \). If \( a \neq 0 \), one has \( F(s) = as + b \geq 0 \) on \( \mathbb{R} \), which cannot happen. Hence, \( F \) is constant. Therefore, \( M \) is trivial.

Case 3. \( M \) is rotationally symmetric and equal to the warped product

\[
([0, \infty), ds^2) \times |\nabla F| (\mathbb{S}^{n-1}, \bar{g}_S).
\]

By the same argument as in Case 2, we have \( F(s) = as + b \geq 0 \) on \([0, \infty)\). Therefore, one has \( a > 0 \) and \( b \geq 0 \).
3. Other classifications of gradient conformal solitons

In this section, we also give classification results, part of which are improvements of [12] and [1]. In particular, we completely reveal the structure of gradient $k$-Yamabe solitons and gradient Yamabe solitons under the assumption as in Theorem 3 of [12].

Proposition 3.1. Let $(M^n, g, F, \varphi)$ be a nontrivial complete gradient conformal soliton. Assume that $M$ satisfies the following (A), (B) or (C).

(A) The function $\varphi$ satisfies that $|\varphi| \in L^1(M)$, $\int_M \text{Ric}(\nabla F, \nabla F) \leq 0$, and $|\nabla F|$ has at most linear growth on $M$.

(B) The function $\varphi$ is nonnegative, and $|\nabla F|$ has at most linear growth on $M$. Let $u$ be a non-constant solution of

$$\begin{cases} 
\Delta u + h(u) = 0, \\
\int_M h(u)\langle \nabla u, \nabla F \rangle \geq 0,
\end{cases}$$

for $h \in C^1(\mathbb{R})$, and the function $|\nabla u|$ satisfies

$$\int_{B(x_0, R)} |\nabla u|^2 = o(\log R), \text{ as } R \to +\infty.$$ 

(C) The manifold $M$ is parabolic and nontrivial with $\text{Ric}(\nabla F, \nabla F) \leq 0$ and $|\nabla F| \in L^\infty(M)$.

Then, $M$ is either

(1) $$(\mathbb{R} \times N^{n-1}, ds^2 + a^2\bar{g}, as + b, 0),$$

where, $a, b \in \mathbb{R}$, or

(2) $$([0, \infty) \times S^{n-1}, ds^2 + a^2\bar{g}_S, as + b, 0),$$

where, $\bar{g}_S$ is the round metric on $S^{n-1}$, and $a, b \in \mathbb{R}$.

We first show some formulas which will be used later. For any gradient conformal soliton, we have

$$\Delta \nabla_i F = \nabla_i \Delta F + R_{ij} \nabla_j F,$$

$$\Delta \nabla_i F = \nabla_k \nabla_k \nabla_i F = \nabla_k (\varphi g_{ki}) = \nabla_i \varphi,$$

and

$$\nabla_i \Delta F = \nabla_i (n \varphi) = n \nabla_i \varphi.$$

Hence, we have

$$\begin{align*}
(n - 1)\nabla_i \varphi + R_{ij} \nabla_j F &= 0,
\end{align*}$$

(3.1)
where, $R_{ij}$ is the Ricci tensor of $M$. Therefore, one has
\begin{equation}
\langle \nabla \varphi, \nabla F \rangle = -\frac{1}{n-1} \text{Ric}(\nabla F, \nabla F).
\end{equation}
By applying $\nabla_t$ to the both sides of (3.1), we obtain
\begin{equation}
(n-1)\nabla_t \nabla_i \varphi + \nabla_t R_{ij} \cdot \nabla_j F + R_{ij} \nabla_t \nabla_j F = 0.
\end{equation}
Taking the trace, we obtain
\begin{equation}
(n-1)\Delta \varphi + \frac{1}{2} g(\nabla R, \nabla F) + R \varphi = 0.
\end{equation}
We observe the following proposition.

**Proposition 3.2.** Any compact gradient conformal soliton with
\[
\int_M \text{Ric}(\nabla F, \nabla F) \leq 0
\]
is trivial.

**Proof.** By $\Delta F = n \varphi$ and (3.2), we have
\[
\int_M \varphi^2 = \frac{1}{n} \int_M \varphi \Delta F
= -\frac{1}{n} \int_M \langle \nabla \varphi, \nabla F \rangle
= -\frac{1}{n(n-1)} \int_M \text{Ric}(\nabla F, \nabla F) \leq 0.
\]
Thus, one has $\varphi = 0$ and $\nabla \nabla F = 0$. Hence, we have $\Delta F = 0$. By the standard maximum principle, we have that $M$ is trivial. \hfill \Box

By using the above arguments, we show Proposition 3.1.

**Proof of Proposition 3.1**

(A) If $M$ is compact, by Proposition 3.2, $M$ is trivial.

Therefore, we assume that $M$ is noncompact. By $\Delta F = n \varphi$ and (3.2), we have
\[
\int_{B(x_0, R)} \varphi^2 = \frac{1}{n} \int_{B(x_0, R)} \varphi \Delta F
= -\frac{1}{n} \left\{ \int_{B(x_0, R)} \langle \nabla \varphi, \nabla F \rangle + \int_{\partial B(x_0, R)} \varphi \langle \nu, \nabla F \rangle \right\}
= -\frac{1}{n} \left\{ \int_{B(x_0, R)} -\frac{1}{n-1} \text{Ric}(\nabla F, \nabla F) + \int_{\partial B(x_0, R)} \varphi \langle \nu, \nabla F \rangle \right\}
\leq \frac{1}{n(n-1)} \int_{B(x_0, R)} \text{Ric}(\nabla F, \nabla F) + CR \int_{\partial B(x_0, R)} |\varphi|,
\]
where, \( \nu \) is the outward unit normal to the boundary \( \partial B(x_0, R) \). Since \( |\varphi| \in L^1(M) \), by taking \( R \to +\infty \), one has

\[
CR \int_{\partial B(x_0, R)} |\varphi| \to 0.
\]

Therefore, by the assumption, we have

\[
\int_M \varphi^2 = 0,
\]

hence, \( \varphi = 0 \).

By Theorem 2.1, we have 3 types of gradient conformal solitons.

**Case 1.** \( M \) is compact. This case cannot happen.

**Case 2.** \( M \) is the warped product

\[
(\mathbb{R}, ds^2) \times_{|\nabla F|} (N^{n-1}, \bar{g}).
\]

Since \( \nabla \nabla \nabla F = 0 \), we have

\[
\nabla |\nabla F|^2 = 2 \nabla_j \nabla_i F \nabla_i F = 0.
\]

Hence, \( \nabla F \) is a constant vector field. Therefore, we have \( F(s) = as + b \).

**Case 3.** \( M \) is rotationally symmetric and equal to the warped product

\[
([0, \infty), ds^2) \times_{|\nabla F|} (S^{n-1}, \bar{g}_S).
\]

By the same argument as in Case 2, we have that \( M \) is

\[
([0, \infty) \times S^{n-1}, ds^2 + a^2 \bar{g}_S, as + b, 0).
\]

(B) We take a cut off function \( \eta \) on \( M \) satisfying that

\[
\eta(r) = \begin{cases} 
1 & r \leq R, \\
2 - \frac{\log r}{\log R} & r \in [R, R^2], \\
0 & r \geq R^2.
\end{cases}
\]

By the soliton equation, we have

\[
\int_M \varphi |\nabla u|^2 \eta^2 = \int_{B(x_0, R^2)} \nabla_i \nabla_j F \nabla_i u \nabla_j \eta^2
\]

\[
= -\int_{B(x_0, R^2)} \nabla_i F \nabla_j \nabla_i u \nabla_j \eta^2 + \nabla_i F \Delta u \nabla_j \eta^2
\]

\[
- \int_{B(x_0, R^2)} \nabla_i F \nabla_i u \nabla_j \eta^2.
\]
Substituting
\[-\int_{B(x_0,R^2)} \nabla_i F \nabla_j \nabla_i u \nabla_j u \eta^2 = \frac{1}{2} \int_{B(x_0,R^2)} n \varphi |\nabla u|^2 \eta^2 + |\nabla u|^2 \langle \nabla F, \nabla \eta \rangle,\]
into (3.6), we have
\[\int_M \varphi |\nabla u|^2 \eta^2 = \frac{1}{2} \int_{B(x_0,R^2)} n \varphi |\nabla u|^2 \eta^2 + 2 \eta |\nabla u|^2 \langle \nabla F, \nabla \eta \rangle + \int_{B(x_0,R^2)} \langle \nabla F, \nabla u \rangle \eta^2 - 2 \eta \langle \nabla F, \nabla u \rangle \langle \nabla u, \nabla \eta \rangle.\]
Therefore, we have
\[\frac{n-2}{2} \int_M \varphi |\nabla u|^2 \eta^2 = - \int_{B(x_0,R^2)} \eta |\nabla u|^2 \langle \nabla F, \nabla \eta \rangle - \int_{B(x_0,R^2)} \langle \nabla F, \nabla u \rangle \eta^2 + 2 \int_{B(x_0,R^2)} \eta |\nabla u|^2 |\nabla F||\nabla \eta| - \int_{B(x_0,R^2)} \langle \nabla F, \nabla u \rangle \eta^2 \leq \frac{C}{\log R} \int_{B(x_0,R^2) \setminus B(x_0,R)} |\nabla u|^2 - \int_{B(x_0,R^2)} \langle \nabla F, \nabla u \rangle \eta^2,\]
where, the last inequality follows from $|\nabla F| \leq Cr$ near infinity and the definition of the cut off function $\eta$. Take $R \nearrow +\infty$. From this and the assumption, we have
\[\frac{n-2}{2} \int_M \varphi |\nabla u|^2 \eta^2 \leq - \int_M \langle \nabla F, \nabla u \rangle \eta^2 \leq 0.\]
Since $u$ is a non-constant solution, we have $\varphi = 0$. Therefore, we have $\nabla \nabla F = 0$.

By Theorem 2.1, we have 3 types of gradient conformal solitons.

Case 1. $M$ is compact and rotationally symmetric.

Since $M$ is compact, by the standard Maximum principle, we have that $F$ is constant. Therefore, $M$ is trivial.

Cases 2 and 3 are considered by the same argument as in (A).
(C) Since (3.2) and $\Delta F = n\varphi$, by a direct computation,

$$
\frac{1}{2} \Delta |\nabla F|^2 = |\nabla \nabla F|^2 + \text{Ric}(\nabla F, \nabla F) + \langle \nabla F, \nabla \Delta F \rangle \\
= |\nabla \nabla F|^2 - \frac{1}{n-1} \text{Ric}(\nabla F, \nabla F) \\
\geq 0,
$$

where, the last inequality follows from the assumption. Since $M$ is parabolic and $\sup |\nabla F| < +\infty$, we have that $|\nabla F|$ is constant. Therefore, we have $\nabla \nabla F = 0$. By the same argument as in (B), we complete the proof. \hfill \Box

4. SOME TRIVIALITY RESULTS OF GRADIENT CONFORMAL SOLITONS

In this section, we give triviality results of gradient conformal solitons under some assumption. We first recall parabolicity of $M$: A Riemannian manifold $M$ is parabolic, if every subharmonic function on $M$ which is bounded from above is constant (see [9]).

**Proposition 4.1.** There exists no nontrivial complete gradient conformal soliton $(M^n, g, F, \varphi)$ with one of the following is satisfied.

(A) The potential function satisfies that $F \geq K > 0$ for some $K \in \mathbb{R}$, $\varphi \leq 0$, and one of the following is satisfied (i) $M$ is parabolic, (ii) $|\nabla F| \in L^1(M)$, (iii) $F^{-1} \in L^p(M)$ for some $p > 1$, or (iv) $M^n$ has linear volume growth.

(B) The Ricci curvature satisfies that $\text{Ric}(\nabla F, \nabla F) \leq 0$, and $|\nabla F| \in L^p(M)$ for $p > 1$.

**Remark 4.2.** Proposition 4.1 indicates that Theorems 2 and 6 of [1] are trivial. In fact, by Proposition 4.1, a complete gradient $k$-Yamabe soliton with the assumption of Theorem 2 of [1] is trivial. Therefore, it satisfies that $\sigma_k = \rho$. By Proposition 4.1, there exist no such a soliton which satisfies the "if" part of Theorem 6 of [1].

**Proof of Proposition 4.1.**

(A) If $M$ is compact, by $\Delta (-F) = -n\varphi \geq 0$, the standard Maximum principle shows that $F$ is constant. Therefore, we assume that $M$ is noncompact.

(i) Since $-F \leq -K$, and $\varphi \leq 0$, we have

$$
\Delta (-F) = -n\varphi \geq 0.
$$
Since $M$ is parabolic, $-F$ is constant. Therefore, $M$ is trivial.

(ii) By a direct computation,

\begin{equation}
\text{div}\nabla(-F) = \Delta(-F) = -n\varphi \geq 0.
\end{equation}

By Theorem 2.1, we have 3 types of gradient conformal solitons.

Case 1. $M$ is compact and rotationally symmetric.

This case cannot happen.

To consider Cases 2 and 3, let us recall the following:

**Lemma 4.3** (\cite{3}). Let $X$ be a smooth vector field on a complete non-compact Riemannian manifold, such that, $\text{div}X$ does not change the sign on $M$. If $|X| \in L^1(M)$, then $\text{div}_M X = 0$.

By Lemma 4.3, we have $\Delta F = 0$. Hence, we have $\varphi = 0$, and $\nabla\nabla F = 0$.

Case 2. $M$ is the warped product

\[ (\mathbb{R}, ds^2) \times_{|\nabla F|} (N^{n-1}, \bar{g}). \]

Since $\nabla\nabla F = 0$, we have

\[ \nabla|\nabla F|^2 = 2\nabla_j \nabla_i F \nabla_i F = 0. \]

Hence, $\nabla F$ is a constant vector field. Set $|\nabla F| = a$. If $a \neq 0$,

\[ a \text{Vol} (\mathbb{R} \times N^{n-1}) = +\infty. \]

From this and the assumption, we have $a = 0$. Therefore, $F$ is constant, and $M$ is trivial.

Case 3. $M$ is rotationally symmetric and equal to the warped product

\[ ([0, \infty), ds^2) \times_{|\nabla F|} (\mathbb{S}^{n-1}, \bar{g}s). \]

By the same argument as in Case 2, we have that $F$ is constant.

(iii) Since $\Delta(-F) \geq 0$, one has

\[ \Delta F^{-1} = 2F^{-3}|\nabla F|^2 - \Delta F^{-2} \geq 0. \]

By the Yau’s Maximum principle, one has that $F^{-1}$ is constant.
(iv) By Lemma 6.3 in [15] and the assumption, we have

\[
\int_{B(x_0,R)} |\nabla F^{-1}|^2 \leq \frac{C}{R^2} \int_{B(x_0,2R)} F^{-2} \leq \frac{C}{R^2 K^2} \text{Vol}(B(x_0,2R)) \leq \frac{\bar{C}}{RK^2}.
\]

Take \( R \nearrow +\infty \). The righthand side of (4.2) goes to 0. Therefore, we have that \( F \) is constant.

(B) By \( \Delta F = n \varphi \) and (3.2), we have

\[
\frac{1}{2} \Delta |\nabla F|^2 = |\nabla \nabla F|^2 + \text{Ric}(\nabla F, \nabla F) + \langle \nabla F, \nabla \Delta F \rangle
\]

\[
= |\nabla \nabla F|^2 - \frac{1}{n-1} \text{Ric}(\nabla F, \nabla F).
\]

From this and \( \text{Ric}(\nabla F, \nabla F) \leq 0 \), we have

\[
\Delta |\nabla F|^2 \geq 0.
\]

If \( M \) is compact, by the standard Maximum principle, we have that \( |\nabla F| \) is constant.

Assume that \( M \) is noncompact. By the Yau’s Maximum principle, \( |\nabla F| \) is constant.

By (4.3),

\[
|\nabla \nabla F|^2 - \frac{1}{n-1} \text{Ric}(\nabla F, \nabla F) = 0.
\]

Therefore, we have \( \nabla \nabla F = 0 \).

By Theorem 2.1, we have 3 types of gradient conformal solitons.

Case 1. \( M \) is compact and rotationally symmetric.

Since \( \Delta F = 0 \) and \( M \) is compact, by the standard Maximum principle, we have that \( F \) is constant.

Cases 2 and 3 are considered by the same argument as in (A)-(ii).

□

Conflict of interest

There is no conflict of interest in the manuscript.
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