UPPER BOUNDS FOR NUMERICAL RADIUS INEQUALITIES INVOLVING OFF-DIAGONAL OPERATOR MATRICES

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Abstract. In this paper, we establish some upper bounds for numerical radius inequalities including of $2 \times 2$ operator matrices and their off-diagonal parts. Among other inequalities, it is shown that if $T = \begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}$, then

$$\omega(T) \leq 2^{r - 2} \| f^{2r}(|X|) + g^{2r}(|Y^*|) \|^\frac{1}{2} \| f^{2r}(|Y|) + g^{2r}(|X^*|) \|^\frac{1}{2}$$

and

$$\omega(T) \leq 2^{r - 2} \| f^{2r}(|X|) + f^{2r}(|Y^*|) \|^\frac{1}{2} \| g^{2r}(|Y|) + g^{2r}(|X^*|) \|^\frac{1}{2},$$

where $X, Y$ are bounded linear operators on a Hilbert space $\mathcal{H}$, $r \geq 1$ and $f, g$ are nonnegative continuous functions on $[0, \infty)$ satisfying the relation $f(t)g(t) = t$ ($t \in [0, \infty)$). Moreover, we present some inequalities involving the generalized Euclidean operator radius of operators $T_1, \ldots, T_n$.

1. Introduction

Let $\mathcal{B}(\mathcal{H})$ denote the $C^*$-algebra of all bounded linear operators on a Hilbert space $\mathcal{H}$. In the case when $\dim \mathcal{H} = n$, we identify $\mathcal{B}(\mathcal{H})$ with the matrix algebra $\mathbb{M}_n$ of all $n \times n$ matrices with entries in the complex field. An operator $A \in \mathcal{B}(\mathcal{H})$ is said to be contraction, if $A^*A \leq I$. The numerical radius of $T \in \mathcal{B}(\mathcal{H})$ is defined by

$$\omega(T) := \sup\{|\langle Tx, x \rangle| : x \in \mathcal{H}, \| x \| = 1\}.$$

It is well known that $\omega(\cdot)$ defines a norm on $\mathcal{B}(\mathcal{H})$, which is equivalent to the usual operator norm. In fact, $\frac{1}{2}\| \cdot \| \leq \omega(\cdot) \leq \| \cdot \|$; see [9]. An important inequality for $\omega(A)$ is the power inequality stating that $\omega(A^n) \leq \omega(A)^n$ ($n = 1, 2, \cdots$). For further information about the properties of numerical radius inequalities we refer the reader to [1, 5, 13] and references therein. Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces, and consider the direct sum $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$. With respect to this decomposition, every operator $T \in \mathcal{B}(\mathcal{H})$ has a $2 \times 2$ operator matrix representation $T = [T_{ij}]$ with entries $T_{ij} \in \mathcal{B}(\mathcal{H}_j, \mathcal{H}_i)$, the

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\end{footnotesize}
space of all bounded linear operators from $\mathcal{H}_j$ to $\mathcal{H}_i$ ($1 \leq i, j \leq 2$). Operator matrices provide a usual tool for studying Hilbert space operators, which have been extensively studied in the literatures. Let $A \in \mathbb{B}(\mathcal{H}_1, \mathcal{H}_1)$, $B \in \mathbb{B}(\mathcal{H}_2, \mathcal{H}_1)$, $C \in \mathbb{B}(\mathcal{H}_1, \mathcal{H}_2)$ and $D \in \mathbb{B}(\mathcal{H}_2, \mathcal{H}_2)$. The operator \[
abla \begin{bmatrix} A & 0 \\ 0 & B \\ C & 0 \end{bmatrix} \] is called the diagonal part of \[
abla \begin{bmatrix} A & B \\ C & D \end{bmatrix} \] and \[
abla \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} \] is the off-diagonal part.

The classical Young inequality says that if $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, then $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$ for positive real numbers $a, b$. In [3], the authors showed that a refinement of the scalar Young inequality as follows \[
abla \left( a \frac{1}{p} b \frac{1}{q} \right)^m + \frac{r_0^m}{(a \frac{m}{p} - b \frac{m}{q})} \leq \left( \frac{a}{p} + \frac{b}{q} \right)^m \] where $r_0 = \min \{ \frac{1}{p}, \frac{1}{q} \}$ and $m = 1, 2, \ldots$. In particular, if $p = q = 2$, then
\[
(\frac{a^2}{2} + b^2) + \frac{(a - b)^2}{2} \leq 2^{-m}(a + b)^m. \tag{1.1}
\]

It has been shown in [8], that if $T \in \mathbb{B}(\mathcal{H})$, then
\[
\omega(T) \leq \frac{1}{2} \| |T| + |T^*| \|, \tag{1.2}
\]
where $|T| = (T^*T)^{1/2}$ is the absolute value of $T$. Recently [2], the authors extended this inequality for off-diagonal operator matrices of the form $T = \begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix} \in \mathbb{B}(\mathcal{H}_1 \oplus \mathcal{H}_2)$ as follows
\[
\omega(T) \leq \frac{1}{2} \| |X| + |Y^*| \|^{1/2} \| |X^*| + |Y| \|^{1/2}. \tag{1.3}
\]
Let $T_1, T_2, \ldots, T_n \in \mathbb{B}(\mathcal{H})$. The functional $\omega_p$ of operators $T_1, \ldots, T_n$ for $p \geq 1$ is defined in [11] as follows
\[
\omega_p(T_1, \ldots, T_n) := \sup_{\|x\|=1} \left( \sum_{i=1}^n \| \langle T_ix, x \rangle \|_p \right)^{\frac{1}{p}}.
\]
If $p = 2$, then we have the Euclidean operator radius of $T_1, \ldots, T_n$ which was defined in [10]. In [13], the authors showed that an upper bound for the functional $\omega_p$
\[
\omega_p^p(T_1, \ldots, T_n) \leq \frac{1}{2} \left\| \sum_{i=1}^n \left( f^{2p}(|T_i|) + g^{2p}(|T^*_i|) \right) \right\| - \inf_{\|x\|=1} \zeta(x),
\]
where $T_i \in \mathbb{B}(\mathcal{H})$ ($i = 1, 2, \ldots, n$), $f, g$ are nonnegative continuous functions on $[0, \infty)$ such that $f(t)g(t) = t$ ($t \in [0, \infty)$), $p \geq 1$ and
\[
\zeta(x) = \frac{1}{2} \sum_{i=1}^n \left( \langle f^{2p}(|T_i|)x, x \rangle^{1/2} - \langle g^{2p}(|T^*_i|)x, x \rangle^{1/2} \right)^2.
\]
In this paper, we show some inequalities involving powers of the numerical radius for off-diagonal parts of $2 \times 2$ operator matrices. In particular, we extend inequalities (1.2) and (1.3) for nonnegative continuous functions $f, g$ on $[0, \infty)$ such that $f(t)g(t) = t (t \in [0, \infty))$. Moreover, we present some inequalities including the generalized Euclidean operator radius $\omega_p$.

2. Main Results

To prove our first result, we need the following lemmas.

**Lemma 2.1.** [6, 14] Let $X \in \mathcal{B}(\mathcal{H})$. Then

(a) $\omega(X) = \max_{\theta \in \mathbb{R}} \| \text{Re}(e^{i\theta}X) \| = \max_{\theta \in \mathbb{R}} \| \text{Im}(e^{i\theta}X) \| .$

(b) $\omega \left( \begin{bmatrix} 0 & X \\ X & 0 \end{bmatrix} \right) = \omega(X).$

The next lemma follows from the spectral theorem for positive operators and Jensen inequality; see [7].

**Lemma 2.2.** Let $T \in \mathcal{B}(\mathcal{H})$, $T \geq 0$ and $x, y \in \mathcal{H}$ be any vectors. If $f, g$ are nonnegative continuous functions on $[0, \infty)$ which are satisfying the relation $f(t)g(t) = t (t \in [0, \infty))$, then

\[ |\langle Tx, y \rangle|^2 \leq \langle f^2(|T||x), x \rangle \langle g^2(|T^r|)y, y \rangle. \]

**Proof.** Let $r \geq 1$ and $x \in \mathcal{H}$ such that $\|x\| \leq 1$. Fix $u = \frac{x}{\|x\|}$. Using the McCarty inequality we have $\langle Tu, u \rangle^r \leq \langle T^r u, u \rangle$, whence

\[ \langle Tx, x \rangle^r \leq \|x\|^{2r-2} \langle T^r x, x \rangle \]

\[ \leq \langle T^r x, x \rangle \quad \text{(since $\|x\| \leq 1$ and $2r - 2 \geq 0$)}. \]

Hence, we get the first inequality. The proof of the second inequality is similar. $\square$

**Lemma 2.3.** [7, Theorem 1] Let $T \in \mathcal{B}(\mathcal{H})$ and $x, y \in \mathcal{H}$ be any vectors. If $f, g$ are nonnegative continuous functions on $[0, \infty)$ which are satisfying the relation $f(t)g(t) = t (t \in [0, \infty))$, then

\[ |\langle Tx, y \rangle|^2 \leq \langle f^2(|T||x), x \rangle \langle g^2(|T^r|)y, y \rangle. \]
Now, we are in position to demonstrate the main results of this section by using some ideas from [2, 13].

**Theorem 2.4.** Let \( T = \begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix} \in B(\mathcal{H}_1 \oplus \mathcal{H}_2) \), \( r \geq 1 \) and \( f, g \) be nonnegative continuous functions on \([0, \infty)\) satisfying the relation \( f(t)g(t) = t \) (\( t \in [0, \infty) \)). Then

\[
\omega^r(T) \leq 2^{r-2} \| f^{2r}(|X|) + g^{2r}(|Y^*|) \|^{\frac{1}{2}} \| f^{2r}(|Y|) + g^{2r}(|X^*|) \|^{\frac{1}{2}}
\]

and

\[
\omega^r(T) \leq 2^{r-2} \| f^{2r}(|X|) + f^{2r}(|Y^*|) \|^{\frac{1}{2}} \| g^{2r}(|Y|) + g^{2r}(|X^*|) \|^{\frac{1}{2}}.
\]

**Proof.** Let \( x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathcal{H}_1 \oplus \mathcal{H}_2 \) be a unit vector (i.e., \( \| x_1 \|^2 + \| x_2 \|^2 = 1 \)). Then

\[
| \langle Tx, x \rangle |^r \\
= | \langle X x_2, x_1 \rangle + \langle Y x_1, x_2 \rangle |^r \\
\leq (| \langle X x_2, x_1 \rangle | + | \langle Y x_1, x_2 \rangle |)^r \quad \text{(by the triangular inequality)} \\
\leq 2^r \left( | \langle X x_2, x_1 \rangle | + | \langle Y x_1, x_2 \rangle | \right)^r \quad \text{(by the convexity \( f(t) = t^r \))} \\
\leq 2^r \left( \left( \| f^{2r}(|X|) x_2, x_1 \|^{\frac{1}{2}} \| g^{2r}(|X^*|) x_1, x_1 \|^{\frac{1}{2}} \right)^r \\
+ \left( \| f^{2r}(|Y|) x_1, x_1 \|^{\frac{1}{2}} \| g^{2r}(|Y^*|) x_2, x_2 \|^{\frac{1}{2}} \right)^r \right) \quad \text{(by Lemma 2.3)} \\
\leq 2^r \left( \| f^{2r}(|X|) x_2, x_2 \|^{\frac{1}{2}} \| g^{2r}(|X^*|) x_1, x_1 \|^{\frac{1}{2}} + \| f^{2r}(|Y|) x_1, x_1 \|^{\frac{1}{2}} \| g^{2r}(|Y^*|) x_2, x_2 \|^{\frac{1}{2}} \right) \\
\quad \text{(by Lemma 2.2(a))} \\
\leq 2^r \left( \| f^{2r}(|X|) x_2, x_2 \| + \| g^{2r}(|X^*|) x_2, x_2 \| \right)^{\frac{1}{2}} \\
\times \left( \| f^{2r}(|Y|) x_1, x_1 \| + \| g^{2r}(|X^*|) x_1, x_1 \| \right)^{\frac{1}{2}} \quad \text{(by the Cauchy-Schwarz inequality)} \\
= 2^r \left( \| f^{2r}(|X|) + g^{2r}(|Y^*|) x_2, x_2 \|^{\frac{1}{2}} \| f^{2r}(|Y|) + g^{2r}(|X^*|) x_1, x_1 \|^{\frac{1}{2}} \\
\leq 2^r \left( \| f^{2r}(|X|) + g^{2r}(|Y^*|) \|^{\frac{1}{2}} \| f^{2r}(|Y|) + g^{2r}(|X^*|) \|^{\frac{1}{2}} \| x_1 \| \| x_2 \| \\
\leq 2^r \left( \| f^{2r}(|X|) + g^{2r}(|Y^*|) \|^{\frac{1}{2}} \| f^{2r}(|Y|) + g^{2r}(|X^*|) \|^{\frac{1}{2}} \right)^{\frac{1}{2}} \left( \| x_1 \|^2 + \| x_2 \|^2 \right) \quad \text{(by the arithmetic-geometric mean inequality)} \\
= 2^r \left( \frac{4}{2} \right) \left( \| f^{2r}(|X|) + g^{2r}(|Y^*|) \|^{\frac{1}{2}} \| f^{2r}(|Y|) + g^{2r}(|X^*|) \|^{\frac{1}{2}} \right)^{\frac{1}{2}} \right).
Hence, we get the first inequality. Now, applying this fact

\[
| \langle Tx, x \rangle |^r \\
= | \langle Xx, x \rangle + \langle Yx, x \rangle |^r \\
\leq (| \langle Xx, x \rangle | + | \langle Yx, x \rangle |)^r \
\text{(by the triangular inequality)} \\
\leq \frac{2^r}{2} (| \langle Xx, x \rangle |^r + | \langle Yx, x \rangle |^r) \
\text{(by the convexity } f(t) = t^r \text{)} \\
\leq \frac{2^r}{2} \left( \left( f^2(|X|)_{x_2, x_2} \right)^{\frac{1}{2}} \left( g^2(|X^*|)_{x_1, x_1} \right)^{\frac{1}{2}} \right)^r \\
\quad + \left( g^2(|Y|)_{x_1, x_1} \right)^{\frac{1}{2}} \left( f^2(|Y^*|)_{x_2, x_2} \right)^{\frac{1}{2}} \right)^r \
\text{(by Lemma 2.3) } (2.1)
\]

and a similar argument to the proof of the first inequality we have the second inequality and this completes the proof of the theorem. □

Theorem 2.4 includes a special case as follows.

**Corollary 2.5.** Let \( T = \begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix} \in \mathbb{B}(\mathcal{H}_1 \oplus \mathcal{H}_2) \), \( 0 \leq p \leq 1 \) and \( r \geq 1 \). Then

\[
\omega_r(T) \leq 2^{r-2} \left \| |X|^{2r} + |Y^*|^{2r(1-p)} \right \|^\frac{1}{2} \left \| |Y|^{2r} + |X^*|^{2r(1-p)} \right \|^\frac{1}{2}
\]

and

\[
\omega_r(T) \leq 2^{r-2} \left \| |X|^{2r} + |Y^*|^{2r} \right \|^\frac{1}{2} \left \| |Y|^{2r(1-p)} + |X^*|^{2r(1-p)} \right \|^\frac{1}{2}.
\]

**Proof.** The result follows immediately from Theorem 2.4 for \( f(t) = t^p \) and \( g(t) = t^{1-p} \) \((0 \leq p \leq 1)\). □

**Remark 2.6.** Taking \( f(t) = g(t) = t^\frac{1}{2} \) \((t \in [0, \infty))\) and \( r = 1 \) in Theorem 2.4, we get (see [2, Theorem 4])

\[
\omega(T) \leq \frac{1}{2} \left \| |X| + |Y^*| \right \|^\frac{1}{2} \left \| |Y| + |X^*| \right \|^\frac{1}{2},
\]

where \( T = \begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix} \in \mathbb{B}(\mathcal{H}_1 \oplus \mathcal{H}_2) \).

If we put \( Y = X \) in Theorem 2.4, then by using Lemma 2.1(b) we get an extension of Inequality (1.2).
Corollary 2.7. Let $X \in \mathcal{B}(\mathcal{H})$, $r \geq 1$ and $f, g$ be nonnegative continuous functions on $[0, \infty)$ satisfying the relation $f(t)g(t) = t$ ($t \in [0, \infty)$). Then

$$\omega^r(X) \leq 2^{r-2} \left\| f^{2r}(|X|) + g^{2r}(|X^*|) \right\|$$

and

$$\omega^r(X) \leq 2^{r-2} \left\| f^{2r}(|X|) + f^{2r}(|X^*|) \right\|^\frac{1}{2} \left\| g^{2r}(|X|) + g^{2r}(|X^*|) \right\|^\frac{1}{2}.$$

Corollary 2.8. Let $X, Y \in \mathcal{B}(\mathcal{H})$ and $0 \leq p \leq 1$. Then

$$\omega^p(XY) \leq 2^{r-2} \left\| |X|^{2rp} + |Y^*|^{2r(1-p)} \right\|^\frac{1}{2} \left\| |Y|^{2rp} + |X^*|^{2r(1-p)} \right\|^\frac{1}{2}$$

and

$$\omega^p(XY) \leq 2^{r-2} \left\| |X|^{2rp} + |Y^*|^{2rp} \right\|^\frac{1}{2} \left\| |Y|^{2r(1-p)} + |X^*|^{2r(1-p)} \right\|^\frac{1}{2}$$

for $r \geq 1$.

Proof. It follows from the power inequality $\omega^\frac{1}{2} (T^2) \leq \omega^r (T)$ that

$$\omega^\frac{1}{2} (T^2) = \omega^\frac{1}{2} \left( \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix} \right) = \max \left\{ \omega^\frac{1}{2} (XY), \omega^\frac{1}{2} (YX) \right\}.$$

The required result follows from Corollary 2.5. □

Corollary 2.9. Let $X, Y \in \mathcal{B}(\mathcal{H})$ and $r \geq 1$. Then

$$\|X \pm Y^*\|^r \leq 2^{2r-2} \left( \|X\|^r + \|Y^*\|^r \right)^\frac{1}{2} \left( \|Y\|^r + \|X^*\|^r \right)^\frac{1}{2}.$$

In particular, if $X$ and $Y$ are normal operators, then

$$\|X \pm Y\|^r \leq 2^{2r-2} \left( \|X\|^r + \|Y\|^r \right). \quad (2.2)$$

Proof. Applying Lemma 2.1(a) and Corollary 2.5 (for $p = \frac{1}{2}$), we have

$$\|X + Y^*\|^r = \|T + T^*\|^r \leq 2^r \max_{\theta \in \mathbb{R}} \|\text{Re} (e^{i\theta}T)\|^r = 2^r \omega^r (T) \leq 2^{2r-2} \left( \|X\|^r + \|Y^*\|^r \right)^\frac{1}{2} \left( \|Y\|^r + \|X^*\|^r \right)^\frac{1}{2}$$
where $T = \begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}$. Similarly,

$$
\|X - Y^*\|_r = \|T - T^*\|_r \\
\leq 2^r \max_{\theta \in \mathbb{R}} \|\text{Im} (e^{i\theta} T)\|_r \\
= 2^r \omega^r (T) \\
\leq 2^{2r-2} \|X\|_r + |Y^*|^{\frac{r}{2}} \|Y\|_r + |X^*|^{\frac{r}{2}}
$$

Hence we get the desired result. For the particular case, observe that $|Y^*| = |Y|$ and $|X^*| = |X|$.

□

**Remark 2.10.** It should be mentioned here that inequality (2.2), which has been given earlier, is a generalized form of the well-known inequality (see [4]): if $A$ and $B$ are normal operators, then

$$
\|X + Y\| \leq \|X\| + |Y| . 
$$

(2.3)

The normality of $X$ and $Y$ are necessary that means Inequality (2.3) is not true for arbitrary operators $X$ and $Y$; see [12]

In the next theorem, we show another upper bound for numerical radius involving off-diagonal operator matrices.

**Theorem 2.11.** Let $T = \begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix} \in \mathcal{B}(\mathcal{H}_1 \oplus \mathcal{H}_2)$, $r \geq 1$ and $f$, $g$ be nonnegative continuous functions on $[0, \infty)$ satisfying the relation $f(t)g(t) = t$ ($t \in [0, \infty)$). Then

$$
\omega^{2r}(T) \leq 4^{r-2} \left( \frac{\| (f^{2r}(|X|) + g^{2r}(|Y^*|))^p \|}{p^2} + \frac{\| (f^{2r}(|Y|) + g^{2r}(|X^*|))^q \|}{q^2} \right)
$$

and

$$
\omega^{2r}(T) \leq 4^{r-2} \left( \frac{\| (f^{2r}(|X|) + f^{2r}(|Y^*|))^p \|}{p^2} + \frac{\| (g^{2r}(|Y|) + g^{2r}(|X^*|))^q \|}{q^2} \right),
$$

where $\frac{1}{p} + \frac{1}{q} = 1$ and $p \geq 1$. 
Proof. If $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathcal{H}_1 \oplus \mathcal{H}_2$ is a unit vector, then by a similar argument to the proof of Theorem 2.4 we have

$$\left| \langle T\mathbf{x}, \mathbf{x} \rangle \right|^r = \left| \langle Xx_2, x_1 \rangle + \langle Yx_1, x_2 \rangle \right|^r$$

$$\leq \left( \left| \langle Xx_2, x_1 \rangle \right| + \left| \langle Yx_1, x_2 \rangle \right| \right)^r \quad \text{(by the triangular inequality)}$$

$$\leq \frac{2^r}{2} \left( \left| \langle Xx_2, x_1 \rangle \right|^r + \left| \langle Yx_1, x_2 \rangle \right|^r \right) \quad \text{(by the convexity } f(t) = t^r)$$

$$\leq \frac{2^r}{2} \left( \left( \langle f^2(|X|)x_2, x_1 \rangle \right)^\frac{1}{r} \left( \langle g^2(|X^*|)x_1 \rangle \right)^\frac{1}{r} \right)^r \quad \text{(by Lemma 2.3)}$$

$$\leq \frac{2^r}{2} \left( \left( \langle f^2(|X|)x_2, x_1 \rangle \right)^\frac{1}{r} \left( \langle g^2(|X^*|)x_2, x_1 \rangle \right)^\frac{1}{r} \right)^r \quad \text{(by Lemma 2.2(a))}$$

$$\leq \frac{2^r}{2} \left( \left( \langle f^2(|X|) + g^2(|X^*|) \rangle x_2, x_1 \right)^\frac{1}{r} \left( \langle f^2(|Y|) + g^2(|Y^*|) \rangle x_1 \right)^\frac{1}{r} \right) \quad \text{(by the Cauchy-Schwarz inequality)}$$

$$\leq \frac{2^r}{2} \left( \left( \langle f^2(|X|) + g^2(|X^*|) \rangle x_2, x_1 \right)^\frac{1}{r} \left( \langle f^2(|Y|) + g^2(|Y^*|) \rangle x_1 \right)^\frac{1}{r} \right) \quad \text{(by the Young inequality)}$$

$$\leq \frac{2^r}{2} \left( \left( \langle f^2(|X|) + g^2(|Y^*|) \rangle x_2, x_1 \right)^\frac{1}{r} \left( \langle f^2(|Y|) + g^2(|X^*|) \rangle x_1 \right)^\frac{1}{r} \right) \quad \text{(by Lemma 2.2(a))}$$

$$\leq \frac{2^r}{2} \left( \left( \langle f^2(|X|) + g^2(|Y^*|) \rangle x_2, x_1 \right)^\frac{1}{r} \left( \langle f^2(|Y|) + g^2(|X^*|) \rangle x_1 \right)^\frac{1}{r} \right) \quad \text{(by Lemma 2.2(a))}$$

Let $\alpha = \left\| \frac{(f^2(|X|) + g^2(|Y^*|))^\frac{1}{r}}{p} \right\|^{\frac{1}{r}} x_2 \right\| + \left\| \frac{(f^2(|Y|) + g^2(|X^*|))^\frac{1}{r}}{q} \right\|^{\frac{1}{r}} x_1 \right\|$. It follows from

$$\max_{\|x_1\|^2 + \|x_2\|^2 = 1} \left( \alpha \|x_1\| + \beta \|x_2\| \right) = \max_{\theta \in [0, 2\pi]} (\alpha \sin \theta + \beta \cos \theta) = \sqrt{\alpha^2 + \beta^2}$$
and Inequality (2.4) that we deduce

\[ |\langle Tx, x \rangle|^r \leq \frac{2^r}{2} \left( \left\| \frac{f^{2r}(|X|) + g^{2r}(|Y^*|)}{p^2} \right\|^\frac{2}{r} + \left\| \frac{f^{2r}(|Y|) + g^{2r}(|X^*|)}{q^2} \right\|^\frac{2}{q} \right). \]

Taking the supremum over all unit vectors \( x \in \mathcal{H}_1 \oplus \mathcal{H}_2 \) we get the first inequality. Now, according to inequality (2.1) and the same argument in the proof of the first inequality, we obtain the second inequality. \( \square \)

**Remark 2.12.** If \( T = \begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix} \in \mathbb{B}(\mathcal{H}_1 \oplus \mathcal{H}_2) \) and \( \frac{1}{p} + \frac{1}{q} = 1 \), then by using Theorem 2.4 and the Young inequality we obtain the inequalities

\[ \omega^r(T) \leq 2^{r-2} \left( \left\| \frac{f^{2r}(|X|) + g^{2r}(|Y^*|)}{p} \right\|^\frac{2}{r} + \left\| \frac{f^{2r}(|Y|) + g^{2r}(|X^*|)}{q} \right\|^\frac{2}{q} \right) \]

and

\[ \omega^r(T) \leq 2^{r-2} \left( \left\| \frac{f^{2r}(|X|) + f^{2r}(|Y^*|)}{p} \right\|^\frac{2}{r} + \left\| \frac{g^{2r}(|Y|) + g^{2r}(|X^*|)}{q} \right\|^\frac{2}{r} \right), \]

where \( r \geq 1 \) and \( f, g \) are nonnegative continuous functions on \([0, \infty)\) satisfying the relation \( f(t)g(t) = t \ (t \in [0, \infty)) \). Now, Theorem 2.11 shows some other upper bounds for \( \omega(T) \).

In the special case of Theorem 2.11 for \( Y = X \) and \( p = q = 2 \), we have the next result.

**Corollary 2.13.** Let \( X \in \mathbb{B}(\mathcal{H}), \ r \geq 1 \) and \( f, g \) be nonnegative continuous functions on \([0, \infty)\) satisfying the relation \( f(t)g(t) = t \ (t \in [0, \infty)) \). Then

\[ \omega^{2r}(X) \leq 2^{2r-3} \left\| \left( f^{2r}(|X|) + g^{2r}(|X^*|) \right)^2 \right\| \]

and

\[ \omega^{2r}(T) \leq 2^{2r-4} \left( \left\| \left( f^{2r}(|X|) + f^{2r}(|X^*|) \right)^2 \right\| + \left\| \left( g^{2r}(|X|) + g^{2r}(|X^*|) \right)^2 \right\| \right). \]

Applying Inequality (1.1) we obtain the following theorem.
Theorem 2.14. Let $T = \begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix} \in \mathcal{B}(\mathcal{H}_1 \oplus \mathcal{H}_2)$ and $f$, $g$ be nonnegative continuous functions on $[0, \infty)$ satisfying the relation $f(t)g(t) = t$ ($t \in [0, \infty)$). Then for $r \geq 1$

$$\omega^r(T) \leq 2^{r-2} \left( \|f^{2r}(|X|) + g^{2r}(|Y^*|)\| + \|f^{2r}(|Y|) + g^{2r}(|X^*|)\| \right) - 2^{r-2} \inf_{\|x_1, x_2\| = 1} \zeta(x_1, x_2),$$

where

$$\zeta(x_1, x_2) = \left( (f^{2r}(|X|) + g^{2r}(|Y^*|)) x_2, x_2 \right) \frac{1}{2} - \left( (f^{2r}(|Y|) + g^{2r}(|X^*|)) x_1, x_1 \right) \frac{1}{2}.$$

Proof. Let $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathcal{H}_1 \oplus \mathcal{H}_2$ be a unit vector. Then

$$\|\langle Tx, x \rangle\|^r$$

$$= \|\langle Xx_2, x_1 \rangle + \langle Yx_1, x_2 \rangle\|^r$$

$$\leq \left( \|\langle Xx_2, x_1 \rangle\| + \|\langle Yx_1, x_2 \rangle\| \right)^r \quad \text{(by the triangular inequality)}$$

$$\leq \frac{2^r}{2} (\|\langle Xx_2, x_1 \rangle\|^r + \|\langle Yx_1, x_2 \rangle\|^r) \quad \text{(by the convexity $f(t) = t^r$)}$$

$$\leq \frac{2^r}{2} \left( \left( f^{2r}(|X|) x_2, x_2 \right) \frac{1}{2} \|g^{2r}(|X^*|) x_1, x_1\| \frac{1}{2} \left( f^{2r}(|Y|) x_1, x_1 \right) \frac{1}{2} \|g^{2r}(|Y^*|) x_2, x_2\| \frac{1}{2} \right)$$

(by Lemma 2.3)

$$\leq \frac{2^r}{2} \left( (f^{2r}(|X|) + g^{2r}(|Y^*|)) x_2, x_2 \right) \frac{1}{2} \left( f^{2r}(|Y|) x_1, x_1 \right) \frac{1}{2} \|g^{2r}(|X^*|) x_1, x_1\| \frac{1}{2}$$

$$\leq \frac{2^r}{4} \left( (f^{2r}(|X|) + g^{2r}(|Y^*|)) x_2, x_2 \right) \frac{1}{2} \left( f^{2r}(|Y|) + g^{2r}(|X^*|) \right) x_1, x_1 \right)$$

$$- \frac{2^r}{4} \left( (f^{2r}(|X|) + g^{2r}(|Y^*|)) x_2, x_2 \right) \frac{1}{2} \left( f^{2r}(|Y|) + g^{2r}(|X^*|) \right) x_1, x_1 \right) \frac{1}{2}$$

(by Inequality (1.1))

$$\leq \frac{2^r}{4} \left( \|f^{2r}(|X|) + g^{2r}(|Y^*|)\| + \|f^{2r}(|Y|) + g^{2r}(|X^*|)\| \right)$$

$$- \frac{2^r}{4} \left( \left( f^{2r}(|X|) + g^{2r}(|Y^*|) \right) x_2, x_2 \right) \frac{1}{2} \left( f^{2r}(|Y|) + g^{2r}(|X^*|) \right) x_1, x_1 \right) \frac{1}{2} \right)^2.$$

Taking the supremum over all unit vectors $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathcal{H}_1 \oplus \mathcal{H}_2$ we get the desired inequality. \qed
If we put \( Y = X \) in Theorem 2.14, then we get next result.

**Corollary 2.15.** Let \( X \in \mathcal{B}(\mathcal{H}) \) and \( f, g \) be nonnegative continuous functions on \([0, \infty)\) satisfying the relation \( f(t)g(t) = t \ (t \in [0, \infty)) \). Then for \( r \geq 1 \)
\[
\omega^r(X) \leq 2^{r-1}\| f^{2r}(|X|) + g^{2r}(|X^*|) \| - 2^{r-2} \inf_{\| (x_1, x_2) \| = 1} \zeta(x_1, x_2),
\]
where
\[
\zeta(x_1, x_2) = \left( \left( (f^{2r}(|X|) + g^{2r}(|X^*|)) x_2, x_2 \right)^\frac{1}{2} - \left( (f^{2r}(|X|) + g^{2r}(|X^*|)) x_1, x_1 \right)^\frac{1}{2} \right)^2.
\]

**Remark 2.16.** If \( x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathcal{H}_1 \oplus \mathcal{H}_2 \) is a unit vector, then by using the inequality
\[
|\langle Tx, x \rangle| \leq |\langle X x_2, x_1 \rangle + \langle Y x_1, x_2 \rangle| \leq (|\langle X x_2, x_1 \rangle| + |\langle Y x_1, x_2 \rangle|)^r \leq \frac{2^r}{2} (|\langle X x_2, x_1 \rangle|^r + |\langle Y x_1, x_2 \rangle|^r)
\]
and the same argument in the proof if Theorem 2.14 we get the following inequality
\[
\omega^r(T) \leq \frac{2^r}{4} (\| f^{2r}(|X|) + f^{2r}(|Y^*|) \| + \| g^{2r}(|Y|) + g^{2r}(|X^*|) \|) - \frac{2^r}{4} \inf_{\| (x_1, x_2) \| = 1} \zeta(x_1, x_2),
\]
where \( T = \begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix} \in \mathcal{B}(\mathcal{H}_1 \oplus \mathcal{H}_2) \), \( f, g \) are nonnegative continuous functions on \([0, \infty)\) satisfying the relation \( f(t)g(t) = t \ (t \in [0, \infty)) \), \( r \geq 1 \) and
\[
\zeta(x_1, x_2) = \left( \left( (f^{2r}(|X|) + f^{2r}(|Y^*|)) x_2, x_2 \right)^\frac{1}{2} - \left( (g^{2r}(|Y|) + g^{2r}(|X^*|)) x_1, x_1 \right)^\frac{1}{2} \right)^2.
\]

3. Some upper bounds for \( \omega_p \)

In this section, we obtain some upper bounds for \( \omega_p \). We first show the following theorem.

**Theorem 3.1.** Let \( \tilde{S}_i = \begin{bmatrix} A_i & 0 \\ 0 & B_i \end{bmatrix}, \tilde{T}_i = \begin{bmatrix} 0 & X_i \\ Y_i & 0 \end{bmatrix} \) and \( \tilde{U}_i = \begin{bmatrix} C_i & 0 \\ 0 & D_i \end{bmatrix} \) be operators matrices in \( \mathcal{B}(\mathcal{H}_1 \oplus \mathcal{H}_2) \) \( (1 \leq i \leq n) \) such that \( A_i, B_i, C_i \) and \( D_i \) are contractions.
Then
\[
\omega^p_p(\tilde{S}_1^* \tilde{T}_1 \tilde{U}_1, \ldots, \tilde{S}_n^* \tilde{T}_n \tilde{U}_n)
\leq 2^{p-2} \sum_{i=1}^n \left\| D_i^* f^{2p}(|X_i|) D_i + B_i^* g^{2p}(|Y_i^*|) B_i \right\|^\frac{p}{2} \left\| C_i^* f^{2p}(|Y_i|) C_i + A_i^* g^{2p}(|X_i^*|) A_i \right\|^\frac{p}{2},
\]
and
\[
\omega^p_p(\tilde{S}_1^* \tilde{T}_1 \tilde{U}_1, \ldots, \tilde{S}_n^* \tilde{T}_n \tilde{U}_n)
\leq 2^{p-2} \sum_{i=1}^n \left\| D_i^* f^{2p}(|X_i|) D_i + B_i^* f^{2p}(|Y_i^*|) B_i \right\|^\frac{p}{2} \left\| C_i^* g^{2p}(|Y_i|) C_i + A_i^* g^{2p}(|X_i^*|) A_i \right\|^\frac{p}{2},
\]
where \( p \geq 1 \).

**Proof.** For any unit vector \( \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathcal{H}_1 \oplus \mathcal{H}_2 \) we have

\[
\sum_{i=1}^n |(T_i \mathbf{x}, \mathbf{x})|^p
= \sum_{i=1}^n |(A_i^* X_i D_i x_2, x_1) + (B_i^* Y_i C_i x_1, x_2)|^p
\leq \sum_{i=1}^n \left( (|A_i^* X_i D_i x_2, x_1| + |B_i^* Y_i C_i x_1, x_2|) \right)^p \quad \text{(by the triangular inequality)}
\leq \frac{2^p}{2} \sum_{i=1}^n |A_i^* X_i D_i x_2, x_1|^p + |B_i^* Y_i C_i x_1, x_2|^p \quad \text{(by the convexity } f(t) = t^p \text{)}
= \frac{2^p}{2} \sum_{i=1}^n |X_i D_i x_2, A_i x_1|^p + |Y_i C_i x_1, B_i x_2|^p
\leq \frac{2^p}{2} \sum_{i=1}^n \left( f^2(|X_i|) D_i x_2, D_i x_2 \right)^{\frac{p}{2}} \left( g^2(|Y_i^*|) A_i x_1, A_i x_1 \right)^{\frac{p}{2}}
+ \left( f^2(|Y_i|) C_i x_1, C_i x_1 \right)^{\frac{p}{2}} \left( g^2(|X_i^*|) B_i x_2, B_i x_2 \right)^{\frac{p}{2}} \quad \text{(by Lemma 2.3)}
\leq \frac{2^p}{2} \sum_{i=1}^n \left( f^{2p}(|X_i|) D_i x_2, D_i x_2 \right)^{\frac{p}{2}} \left( g^{2p}(|Y_i^*|) A_i x_1, A_i x_1 \right)^{\frac{p}{2}}
+ \left( f^{2p}(|Y_i|) C_i x_1, C_i x_1 \right)^{\frac{p}{2}} \left( g^{2p}(|X_i^*|) B_i x_2, B_i x_2 \right)^{\frac{p}{2}} \quad \text{(by Lemma 2.2(a))}
= \frac{2^p}{2} \sum_{i=1}^n \left( D_i^* f^{2p}(|X_i|) D_i x_2, x_2 \right)^{\frac{p}{2}} \left( A_i^* g^{2p}(|X_i^*|) A_i x_1, x_1 \right)^{\frac{p}{2}}
+ \left( C_i^* f^{2p}(|Y_i|) C_i x_1, x_1 \right)^{\frac{p}{2}} \left( B_i^* g^{2p}(|Y_i^*|) B_i x_2, x_2 \right)^{\frac{p}{2}}
\leq \frac{2^p}{2} \sum_{i=1}^n \left( D_i^* f^{2p}(|X_i|) D_i x_2, x_2 \right)^{\frac{p}{2}} \left( A_i^* g^{2p}(|X_i^*|) A_i x_1, x_1 \right)^{\frac{p}{2}}
\times \left( C_i^* f^{2p}(|Y_i|) C_i x_1, x_1 \right)^{\frac{p}{2}} \left( B_i^* g^{2p}(|Y_i^*|) B_i x_2, x_2 \right)^{\frac{p}{2}} \quad \text{(by the Cauchy-Schwarz inequality)}
\]
\[
\frac{2p}{2} \sum_{i=1}^{n} \left( \langle (D_i f^{2p}(|X_i|)D_i + B_i^* g^{2p}(|Y_i^*|)B_i) x_2, x_2 \rangle \right)^{\frac{1}{2}} \\
\times \left( \langle (C_i f^{2p}(|Y_i|)C_i + A_i^* g^{2p}(|X_i^*|)A_i) x_1, x_1 \rangle \right)^{\frac{1}{2}} \\
\leq \frac{2p}{2} \sum_{i=1}^{n} \|D_i f^{2p}(|X_i|)D_i + B_i^* g^{2p}(|Y_i^*|)B_i\|^{\frac{1}{2}} \|C_i f^{2p}(|Y_i|)C_i + A_i^* g^{2p}(|X_i^*|)A_i\|^{\frac{1}{2}} \|x_1\| \|x_2\| \\
= \frac{2p}{2} \sum_{i=1}^{n} \|D_i f^{2p}(|X_i|)D_i + B_i^* g^{2p}(|Y_i^*|)B_i\|^{\frac{1}{2}} \\
\times \|C_i f^{2p}(|Y_i|)C_i + A_i^* g^{2p}(|X_i^*|)A_i\|^{\frac{1}{2}} \left( \frac{\|x_1\|^2 + \|x_2\|^2}{2} \right) \\
= \frac{2p}{4} \sum_{i=1}^{n} \|D_i f^{2p}(|X_i|)D_i + B_i^* g^{2p}(|Y_i^*|)B_i\|^{\frac{1}{2}} \|C_i f^{2p}(|Y_i|)C_i + A_i^* g^{2p}(|X_i^*|)A_i\|^{\frac{1}{2}}.
\]

Taking the supremum over all unit vectors \( x \in \mathcal{H}_1 \oplus \mathcal{H}_2 \) we obtain the first inequality. Using the inequality

\[
\sum_{i=1}^{n} | \langle T_i x, x \rangle |^p \\
= \sum_{i=1}^{n} | \langle A_i^* X_i D_i x_2, x_1 \rangle + \langle B_i^* Y_i C_i x_1, x_2 \rangle |^p \\
\leq \sum_{i=1}^{n} ( | \langle A_i^* X_i D_i x_2, x_1 \rangle | + | \langle B_i^* Y_i C_i x_1, x_2 \rangle | )^p \quad \text{(by the triangular inequality)} \\
\leq \frac{2p}{2} \sum_{i=1}^{n} | \langle A_i^* X_i D_i x_2, x_1 \rangle |^p + | \langle B_i^* Y_i C_i x_1, x_2 \rangle |^p \quad \text{(by the convexity } f(t) = t^p) \\
= \frac{2p}{2} \sum_{i=1}^{n} | \langle X_i D_i x_2, A_i x_1 \rangle |^p + | \langle Y_i C_i x_1, B_i x_2 \rangle |^p \\
\leq \frac{2p}{2} \sum_{i=1}^{n} \langle f^2(|X_i|) D_i x_2, D_i x_2 \rangle^{\frac{r}{2}} \langle g^2(|X_i^*|) A_i x_1, A_i x_1 \rangle^{\frac{t}{2}} \\
+ \langle g^2(|Y_i|) C_i x_1, C_i x_1 \rangle^{\frac{r}{2}} \langle f^2(|Y_i^*|) B_i x_2, B_i x_2 \rangle^{\frac{t}{2}} \\
\quad \text{(by Lemma 2.3)}
\]

and a similar fashion in the proof of the first inequality we reach the second inequality.

\(\square\)

In the special case of Theorem 3.1 for \( A_i = B_i = C_i = D_i = I \ (1 \leq i \leq n) \) we have the next result.
Corollary 3.2. Let $T_i = \begin{bmatrix} 0 & X_i \\ Y_i & 0 \end{bmatrix} \in \mathcal{B} (\mathcal{H}_1 \oplus \mathcal{H}_2) \ (1 \leq j \leq n)$. Then

$$\omega^p(T_1, T_2, \ldots, T_n) \leq 2^{p-2} \sum_{i=1}^{n} \left\| f^{2p}(|X_i|) + g^{2p}(|Y_i|^*) \right\|^{\frac{1}{2}} \left\| f^{2p}(|Y_i|) + g^{2p}(|X_i|^*) \right\|^{\frac{1}{2}}$$

and

$$\omega^p(T_1, T_2, \ldots, T_n) \leq 2^{p-2} \sum_{i=1}^{n} \left\| f^{2p}(|X_i|) + f^{2p}(|Y_i|^*) \right\|^{\frac{1}{2}} \left\| g^{2p}(|Y_i|) + g^{2p}(|X_i|^*) \right\|^{\frac{1}{2}}$$

for $p \geq 1$.

If we put $f(t) = g(t) = t^\frac{1}{2} \ (t \in [0, \infty))$, then we get the next result.

Corollary 3.3. Let $T_i = \begin{bmatrix} 0 & X_i \\ Y_i & 0 \end{bmatrix} \in \mathcal{B} (\mathcal{H}_1 \oplus \mathcal{H}_2) \ (1 \leq j \leq n)$. Then

$$\omega^p(T_1, T_2, \ldots, T_n) \leq 2^{p-2} \sum_{i=1}^{n} \left\| |X_i|^p + |Y_i|^p \right\|^{\frac{1}{2}} \left\| |Y_i|^p + |X_i|^p \right\|^{\frac{1}{2}}$$

for $p \geq 1$.

Theorem 3.4. Let $T_i = \begin{bmatrix} A_i & B_i \\ C_i & D_i \end{bmatrix} \in \mathcal{B} (\mathcal{H}_1 \oplus \mathcal{H}_2) \ (1 \leq i \leq n)$ and $p \geq 1$. Then

$$\omega^p(T_1, \ldots, T_n) \leq 2^{-p} \left( \omega (A_i) + \omega (D_i) + \sqrt{(\omega (A_i) - \omega (D_i))^2 + (\| B_i \| + \| C_i \|)^2} \right)^p.$$ 

In particular,

$$\omega \left( \begin{bmatrix} A & B \\ C & D \end{bmatrix} \right) \leq \frac{1}{2} \left( \omega (A) + \omega (D) + \sqrt{(\omega (A) - \omega (D))^2 + (\| B \| + \| C \|)^2} \right).$$

Proof. Let $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ be a unit vector in $\mathcal{H}_1 \oplus \mathcal{H}_2$. Then
\[ |\langle T_i x, x \rangle| = \left| \left\langle \begin{bmatrix} A_i & B_i \\ C_i & D_i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\rangle \right| \]
\[ = \left| \left\langle \begin{bmatrix} A_i x_1 + B_i x_2 \\ C_i x_1 + D_i x_2 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\rangle \right| \]
\[ = |\langle A_i x_1, x_1 \rangle + \langle B_i x_2, x_1 \rangle + \langle C_i x_1, x_2 \rangle + \langle D_i x_2, x_2 \rangle| \]
\[ \leq |\langle A_i x_1, x_1 \rangle| + |\langle B_i x_2, x_1 \rangle| + |\langle C_i x_1, x_2 \rangle| + |\langle D_i x_2, x_2 \rangle| \]

Thus,

\[ \omega_p^p(T_1, \ldots, T_n) \]
\[ = \sup_{||x||=1} \sum_{i=1}^n |\langle T_i x, x \rangle|^p \]
\[ \leq \sup_{||x_1||^2+||x_2||^2=1} \sum_{i=1}^n (|\langle A_i x_1, x_1 \rangle| + |\langle B_i x_2, x_1 \rangle| + |\langle C_i x_1, x_2 \rangle| + |\langle D_i x_2, x_2 \rangle|)^p \]
\[ \leq \sum_{i=1}^n \left( \sup_{||x_1||^2+||y||^2=1} (|\langle A_i x_1, x_1 \rangle| + |\langle B_i x_2, x_1 \rangle| + |\langle C_i x_1, x_2 \rangle| + |\langle D_i x_2, x_2 \rangle|)^p \right) \]
\[ \leq \sum_{i=1}^n \left( \sup_{||x_1||^2+||x_2||^2=1} \left( \omega (A_i) \||x_1||^2 + \omega (D_i) \||x_2||^2 + (\||B_i|| + \||C_i|| \||x_1|| \||x_2||) \right)^p \right) \]
\[ = \sum_{i=1}^n \left( \sup_{\theta \in [0,2\pi]} \left( \omega (A_i) \cos^2 \theta + \omega (D_i) \sin^2 \theta + (\||B_i|| + \||C_i|| \cos \theta \sin \theta) \right)^p \right) \]
\[ = 2^{-p} \sum_{i=1}^n \left( \omega (A_i) + \omega (D_i) + \sqrt{(\omega (A_i) - \omega (D_i))^2 + (\||B_i|| + \||C_i||)^2} \right)^p. \]

This completes the proof. \( \square \)

For \( A_i = D_i \) and \( B_i = C_i \) \((1 \leq i \leq n)\) we get the following result.

**Corollary 3.5.** Let \( T_i = \left[ \begin{array}{cc} \pm A_i & \pm B_i \\ \pm B_i & \pm A_i \end{array} \right] \) be an operator matrix with \( A_i, B_i \in \mathbb{B}(\mathcal{H}) \) \((1 \leq i \leq n)\). Then for all \( p \geq 1 \),

\[ \omega_p^p(T_1, \ldots, T_n) \leq \sum_{i=1}^n (\omega (A_i) + \||B_i||)^p. \]
In particular, if $A, B \in \mathbb{B}(\mathcal{H})$, then
\[
\omega \left( \begin{bmatrix} \pm A & \pm B \\ \pm B & \pm A \end{bmatrix} \right) \leq \omega(A) + \|B\|.
\]

If we take $B_i = C_i = 0 \ (1 \leq i \leq n)$ in Theorem 3.4, then we get the following inequality.

**Corollary 3.6.** Let $T_i = \begin{bmatrix} A_i & 0 \\ 0 & D_i \end{bmatrix} \in \mathbb{B}(\mathcal{H}_1 \oplus \mathcal{H}_2) \ (1 \leq i \leq n)$. Then for all $p \geq 1$,
\[
\omega^p(T_1, \ldots, T_n) \leq \sum_{i=1}^{n} \max(\omega^p(A_i), \omega^p(D_i)).
\]

For $C_i = D_i = 0 \ (1 \leq i \leq n)$ we obtain a result that generalize and refine the inequality $\omega \left( \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} \right) \leq \omega(A) + \frac{\|B\|}{2}$.

**Corollary 3.7.** Let $T_i = \begin{bmatrix} A_i & B_i \\ 0 & 0 \end{bmatrix} \in \mathbb{B}(\mathcal{H}_1 \oplus \mathcal{H}_2) \ (1 \leq i \leq n)$ and $p \geq 1$. Then
\[
\omega^p(T_1, \ldots, T_n) \leq 2^{-p} \sum_{i=1}^{n} \left( \omega(A_i) + \sqrt{\omega^2(A_i) + \|B_i\|^2} \right)^p.
\]

In particular,
\[
\omega \left( \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} \right) \leq \frac{1}{2} \left( \omega(A) + \sqrt{\omega^2(A) + \|B\|^2} \right).
\]

If we put $A_i = D_i = 0 \ (1 \leq i \leq n)$, then we deduce

**Corollary 3.8.** Let $T_i = \begin{bmatrix} 0 & B_i \\ C_i & 0 \end{bmatrix} \in \mathbb{B}(\mathcal{H}_1 \oplus \mathcal{H}_2) \ (1 \leq i \leq n)$ and $p \geq 1$. Then
\[
\omega^p(T_1, \ldots, T_n) \leq 2^{-p} \sum_{i=1}^{n} \left( \|B_i\| + \|C_i\| \right)^p.
\]

In particular, if $B \in \mathbb{B}(\mathcal{H}_2, \mathcal{H}_1)$ and $C \in \mathbb{B}(\mathcal{H}_1, \mathcal{H}_2)$, then
\[
\omega \left( \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} \right) \leq \frac{1}{2} \left( \|B\| + \|C\| \right).
\]

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