QUASI-ISOMETRIC RIGIDITY OF SUBGROUPS AND FILTERED ENDS

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Abstract. Let \( G \) and \( H \) be quasi-isometric finitely generated groups and let \( P \leq G \); is there a subgroup \( Q \) (or a collection of subgroups) of \( H \) whose left cosets coarsely reflect the geometry of the left cosets of \( P \) in \( G \)? We explore sufficient conditions for a positive answer.

The article considers pairs of the form \((G, P)\) where \( G \) is a finitely generated group and \( P \) a finite collection of subgroups, there is a notion of quasi-isometry of pairs, and quasi-isometrically characteristic collection of subgroups. A subgroup is qi-characteristic if it belongs to a qi-characteristic collection. Distinct classes of qi-characteristic collections of subgroups have been studied in the literature on quasi-isometric rigidity, we list in the article some of them and provide other examples.

The first part of the article proves: if \( G \) and \( H \) are finitely generated quasi-isometric groups and \( P \) is a qi-characteristic collection of subgroups of \( G \), then there is a collection of subgroups \( Q \) of \( H \) such that \((G, P)\) and \((H, Q)\) are quasi-isometric pairs.

The second part of the article studies the number of filtered ends \( \tilde{e}(G, P) \) of a pair of groups, a notion introduced by Bowditch, and provides an application of our main result: if \( G \) and \( H \) are quasi-isometric groups and \( P \leq G \) is qi-characteristic, then there is \( Q \leq H \) such that \( \tilde{e}(G, P) = \tilde{e}(H, Q) \).

1. Introduction

Let \( G \) be a finitely generated group with a chosen word metric \( \text{dist}_G \), and denote the Hausdorff distance between subsets of \( G \) by \( \text{hdist}_G \). If \( \mathcal{P} \) is a finite collection of subgroups, then \( G/\mathcal{P} \) denote the set of left cosets \( g\mathcal{P} \) with \( g \in G \) and \( \mathcal{P} \in \mathcal{P} \). In this article, we consider pairs of the form \((G, \mathcal{P})\).

Consider two pairs \((G, \mathcal{P})\) and \((H, \mathcal{Q})\). An \((L, C, M)\)-quasi-isometry \( q : G \rightarrow H \) is an \((L, C, M)\)-quasi-isometry of pairs \( q : (G, \mathcal{P}) \rightarrow (H, \mathcal{Q}) \) if the relation

\[ \{(A, B) \in G/\mathcal{P} \times H/\mathcal{Q} : \text{hdist}_H(q(A), B) < M\} \]

satisfies that the projections into \( G/\mathcal{P} \) and \( H/\mathcal{Q} \) are surjective.

A collection of subgroups \( \mathcal{P} \) of \( G \) is quasi-isometrically characteristic (or shorter qi-characteristic) in \( G \) if \( \mathcal{P} \) is finite, each \( P \in \mathcal{P} \) has finite index in its commensurator, and every \((L, C, M)\)-quasi-isometry \( q : G \rightarrow G \) is an \((L, C, M)\)-quasi-isometry of pairs \( q : (G, \mathcal{P}) \rightarrow (G, \mathcal{P}) \) where \( M = M(G, \mathcal{P}, L, C) \), see Definition 2.3 and Theorem 2.9.

In the study of quasi-isometric rigidity, qi-characteristic collections appear in the literature. For example:

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• Kapovich and Leeb [KL97, Theorem 1.1 and Theorem 4.10] proved that the geometric decomposition of a Haken manifold is preserved by quasi-isometries. The geometric components induce a qi-characteristic collection of the fundamental group of the manifold.

• Behrstock, Drutu and Mosher [BDM09, Theorems 4.1 and 4.8] proved that relative hyperbolicity with respect to non-relatively hyperbolic groups is a quasi-isometry invariant. Their results imply that the collection of maximal parabolic subgroups is qi-characteristic, see Theorem 3.1.

• Lafont, Frigerio, and Sisto [FLS15, Lemma 2.19 and Proposition 8.35] proved that in an irreducible higher graph manifolds the collection of walls is invariant under quasi-isometry, and the corresponding subgroups form a qi-characteristic collection.

The main result of the first part of this article can be interpreted as an abstraction of a common technique used in the study of quasi-isometric rigidity, and in particular, in the works cited above.

**Theorem 1.1 (Corollary 4.2).** Let $G$ be a finitely generated group, let $\mathcal{P}$ be a finite qi-characteristic collection of subgroups of $G$. If $H$ is a finitely generated group and $q : G \to H$ is a quasi-isometry, then there is a qi-characteristic collection of subgroups $\mathcal{Q}$ of $H$ such that $q : (G, \mathcal{P}) \to (H, \mathcal{Q})$ is a quasi-isometry of pairs.

For additional examples of qi-characteristic collections see Example 3.4. The proof of Theorem 1.1 is the content of Section 4, and can be described as follows. The left multiplication action of $H$ on itself by isometries is pushed via the quasi-isometry $q$ and its quasi-inverse to a quasi-action of $H$ on $G$ with uniform constants. The qi-characteristic hypothesis implies that there is a finite collection of left cosets $\mathcal{F}$ in $G/\mathcal{P}$ whose $H$-translates reach all elements of $G/\mathcal{P}$ up to uniform Hausdorff distance. For each $gP \in \mathcal{F}$, the collection of elements of $H$ that fixed $gP$ up to finite Hausdorff distance turns out to be a subgroup. In this way the subgroups in $\mathcal{Q}$ arise as $H$-stabilizers of the elements in $\mathcal{F}$. Then an argument shows that $q : (G, \mathcal{P}) \to (H, \mathcal{Q})$ is a quasi-isometry of pairs. Since $\mathcal{P}$ is qi-characteristic in $G$, then it follows that $\mathcal{Q}$ is qi-characteristic in $H$ as well.

We traced back the idea of the argument proving Theorem 1.1 to the work of Schwartz [Sch95] related to quasi-isometric rigidity of finite volume hyperbolic manifolds. Our argument follows patterns in the works of Kapovich and Leeb [KL97] on quasi-isometric rigidity of Haken manifolds, Drutu and Sapir [DS05, §5.2] on quasi-isometric rigidity of relative hyperbolicity, and Mosher, Sageev and White [MSW11] on quasi-isometric rigidity of fundamental groups of finite graphs of groups. Recent work by A. Margolis [Mar19] contains an analogous statement to Theorem 1.1 in the case that $G$ is a Poincare duality group and $\mathcal{P}$ consists of a single subgroup that is almost normal, a condition, that is in a sense opposite to qi-characteristic.

As a sample application of Theorem 1.1, we prove a quasi-isometric rigidity result for the number filtered ends over qi-subcharacteristic subgroups which are defined in the next paragraph. Let $G$ be a finitely generated group and let $P$ be a subgroup. The number of filtered ends $\tilde{e}(G, P)$ of the pair $(G, P)$ was introduced by Bowditch [Bow02], under the name of coends, in his study of JSJ splittings of one-ended groups. The number of filtered ends coincides with the algebraic number of ends of the pair $(G, P)$ introduced by Kropholler and Roller [KR89], see [Bow02] for the equivalence. The number of filtered ends does not coincide with the number of relative ends $e(G, P)$ introduced by Houghton [Hon74], but there are several...
relations including the inequality $e(G, P) \leq \tilde{e}(G, P)$ and equality in the case that $P$ is normal and finitely generated, for an account see Geoghegan’s book [Geo08 Chapter 14].

A subgroup $P$ of a finitely generated group $G$ is \textit{qi-characteristic} if $P$ belongs to a qi-characteristic collection of subgroups of $G$, see Theorem 2.9.

**Corollary 1.2.** Let $G$ and $H$ be finitely generated quasi-isometric groups. If $P \leq G$ is a qi-subcharacteristic subgroup, then there is a qi-subcharacteristic subgroup $Q \leq H$ such that $\tilde{e}(G, P) = \tilde{e}(H, Q)$.

This corollary is a consequence of Theorem 1.1 together with the main result of Section 5 that is stated below. We develop the notion of filtered ends $E(X, C)$ for pairs where $X$ is a metric space and $C$ is a subspace, parallel to the treatment by Geoghegan [Geo08]; the main difference is that we use arbitrary metric spaces instead of CW-complexes. This alternative approach allows to study this invariant in the framework of coarse geometry. For a pair $(G, P)$ where $G$ is a finitely generated group with a word metric and $P$ is a subgroup, $\tilde{e}(G, P)$ is the cardinality of $E(G, P)$. In Section 5.4 we explain the equivalence with Bowditch notion of coends [Bow02].

**Theorem 1.3** (Theorem 5.2). Let $X$ and $Y$ be metric spaces, and $C \subseteq X$ and $D \subseteq Y$. If $f: X \to Y$ is a quasi-isometry such that $h\text{dist}(f(C), D)$ is finite, then $E(f): E(X, C) \to E(Y, D)$ is a bijection.

**Proof of Corollary 1.2.** Let $q: G \to H$ be a quasi-isometry. Since $P$ is a qi-characteristic subgroup, there is a qi-collection $\mathcal{P}$ of $G$ that contains $P$. By Theorem 1.1 there is a qi-characteristic collection $Q$ of $H$ such that $q: (G, \mathcal{P}) \to (H, Q)$ is a quasi-isometry of pairs. After composing $q$ with a left translation of $H$ if necessary, there is a subgroup $Q$ in $Q$ such that $h\text{dist}_H(q(P), Q) < \infty$. Then Theorem 1.3 implies that $\tilde{e}(G, P) = \tilde{e}(H, Q)$. □

**Organization.** The rest of the article is organized in four parts. Section 2 discusses the definitions of quasi-isometry of pairs, qi-characteristic collections of subspaces in the context of metric spaces, and a characterization of qi-characteristic collections of subgroups, Theorem 2.9. Section 3 discusses some examples of qi-characteristic collections arising from the theory of relatively hyperbolic groups, Theorem 3.1 and Corollary 3.3. That section also cites other examples and non-examples of qi-characteristic collections from the literature. Section 4 contains the proof of the main result of the article, Theorem 4.1, from which Theorem 1.1 is an immediate corollary. The last part of the article, Section 5, develops the notion of filtered ends, from a metric perspective, in order to prove Theorem 5.2 which contains the statement of Theorem 1.3.

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2. Qi-characteristic collections

This part discusses the definitions of quasi-isometry of pairs, and qi-characteristic collections of subspaces in the context of metric spaces. Section 2.1 contains the main result of this part, Theorem 2.9, which is a characterization of qi-characteristic collections of subgroups. Section 2.2 contains some results on quasi-isometries of pairs for future reference.

Definition 2.1. Let $X$ and $Y$ be metric spaces. A map $q: X \rightarrow Y$ is an $(L,C)$-quasi-isometry if:

1. $q$ is $(L,C)$-coarse Lipschitz:
   \[ \text{dist}(q(x), q(y)) \leq L \text{dist}(x,y) + C \]
2. There is an $(L,A)$-coarse Lipschitz map $\bar{q}: Y \rightarrow X$, called a quasi-inverse of $q$, such that:
   \[ \text{dist}(\bar{q} \circ q(x), x) \leq C, \quad \text{dist}(q \circ \bar{q}(y), y) \leq C \]
   for all $x \in X$ and $y \in Y$.

A map $q: X \rightarrow Y$ is an $(L,C)$-quasi-isometric map if the restriction $q: X \rightarrow q(X)$ is an $(L,C)$-quasi-isometry.

The following definitions borrow ideas of the work of Kapovich and Lee on the quasi-isometry invariance of the geometric decomposition of Haken manifolds [KL97 §5.1]. These ideas have been used in a similar fashion in other works, for example [DS05, BDM09, MSW11, FLS15].

Definition 2.2. Let $X$ and $Y$ be metric spaces, let $A$ and $B$ be collections of subspaces of $X$ and $Y$ respectively. A quasi-isometry $q: X \rightarrow Y$ is a quasi-isometry of pairs $q: (X,A) \rightarrow (Y,B)$ if there is $M > 0$:

1. For any $A \in A$, the set \{ $B \in B$: $\text{hdist}_Y(q(A), B) < M$ \} is non-empty.
2. For any $B \in B$, the set \{ $A \in A$: $\text{hdist}_Y(q(A), B) < M$ \} is non-empty.

In this case, if $q: X \rightarrow Y$ is a $(L,C)$-quasi-isometry, then $q: (X,A) \rightarrow (Y,B)$ is called a $(L,C,M)$-quasi-isometry. If there is a quasi-isometry of pairs $(X,A) \rightarrow (Y,B)$ we say that $(X,A)$ and $(Y,B)$ are quasi-isometric pairs.

Quasi-isometries of pairs have recently attracted the attention of other researchers in group theory, see for example [BH20, HH19, GT21, Genar] (in the last reference the notion appears implicitly, see Example 3.4).

Definition 2.3. Let $X$ be a metric space with metric $\text{dist}$. A collection of subspaces $A$ is called quasi-isometrically characteristic, or for short qi-characteristic, if the following properties hold:

1. For any $L \geq 1$ and $C \geq 0$ there is $M = M(L,C) > 0$ such that any $(L,C)$-quasi-isometry $q: X \rightarrow X$ is an $(L,C,M)$-quasi-isometry of pairs $q: (X,A) \rightarrow (X,A)$.
2. Every bounded subset $B \subset X$ intersects only finitely many non-coarsely equivalent elements of $A$; where $A, A' \in A$ are coarsely equivalent if their Hausdorff distance is finite.
For any $A \in \mathcal{A}$ the set $\{ A' \in \mathcal{A} : \text{hdist}(A, A') < \infty \}$ is bounded as a subspace of $(\mathcal{A}, \text{hdist})$.

**Remark 2.4.** The first condition in Definition 2.3 could be interpreted as non-positively curved property of $(X, A)$. That the constant $M$ only depends on $L$ and $C$ is reminiscent of the property that in $\delta$-hyperbolic spaces the images of any pair of $(L, C)$-quasi-geodesics $\mathbb{R} \to X$ are either at Hausdorff distance bounded by a constant $D = D(\delta, L, C)$, or they are at infinite Hausdorff distance. On the other hand, there are simple examples of pairs $(X, A)$ where every $(L, C)$-quasi-geodesic $q : X \to X$ is a $(L, C, M_q)$-quasi-isometry of pairs $(X, A)$, the set of constants $M_q$ is unbounded, and the second and third conditions of Definition 2.3 hold for $A$; see Example 2.6(1).

The verification of the following proposition is left to the reader.

**Proposition 2.5.** Suppose that $q : (X, A) \to (Y, B)$ is a quasi-isometry of pairs. Then $A$ is qi-characteristic if and only if $B$ is qi-characteristic.

**Example 2.6.** The following examples illustrate that conditions (1)-(3) in Definition 2.3 are independent of each other.

1. Consider the pair $(X, A)$ where $X$ is the $n$-dimensional Euclidean space and $A$ consists of a single compact subset of $X$. Note that any quasi-isometry $q : X \to X$ is a quasi-isometry of the pair $q : (X, A) \to (X, A)$. On the other hand, the collection $A$ is not qi-characteristic in $X$, the first condition fails by considering translations, while the conditions (2) and (3) hold.

2. Let $X = \mathbb{H}^n$ be the $n$-dimensional hyperbolic space and let $A$ be the set of all geodesic lines in $X$. Then condition (2) does not hold while conditions (1) and (3) hold. Condition (1) holds because $\mathbb{H}^n$ is a hyperbolic space in the sense of Gromov.

3. Let $X$ be the real line and let $A = \{ \{n\} : n \in \mathbb{Z} \}$. Then conditions (1) and (2) hold, but (3) does not.

### 2.1. Qi-characteristic collections of subgroups.

**Definition 2.7.** Let $G$ be a finitely generated group, and let $\mathcal{P}$ be a collection of subgroups of $G$. The collection $\mathcal{P}$ is qi-characteristic if $G/\mathcal{P}$ is a qi-characteristic collection of subspaces of $G$.

A subgroup of $G$ is a qi-subcharacteristic subgroup if it belongs to a qi-characteristic collection of subgroups of $G$.

**Remark 2.8** (Simplified Notation). Let $G$ be a finitely generated group, and let $\mathcal{P}$ be a collection of subgroups. For the rest of this section, by a quasi-isometry of pairs $(G, \mathcal{P}) \to (G, \mathcal{P})$ we mean a quasi-isometry of pairs $(G, G/\mathcal{P}) \to (G, G/\mathcal{P})$ in the sense of Definition 2.2.

Recall that the commensurator of a subgroup $P$ of a group $G$ is the subgroup of $G$ defined as

$$\text{Comm}_G(P) = \{ g \in G : P \cap gPg^{-1} \text{ is a finite index subgroup of } P \text{ and } gPg^{-1} \}.$$

**Theorem 2.9.** Let $G$ be a finitely generated group. A collection of subgroups $\mathcal{P}$ is qi-characteristic if and only if

1. For any $L \geq 1$ and $C \geq 0$ there is $M = M(L, C) > 0$ such that any $(L, C)$-quasi-isometry $q : G \to G$ is an $(L, C, M)$-quasi-isometry of pairs $q : (G, \mathcal{P}) \to (G, \mathcal{P})$. 


Proof. To show that the first statement implies the second, suppose every \( P \in \mathcal{P} \) has finite index in its commensurator. The statement of the theorem is a consequence of the following lemmas.

**Lemma 2.10.** [MSW11, Lemma 2.2] Let \( G \) be a finitely generated group and let \( B \) and \( C \) subgroups. Then for any \( k > 0 \) there is \( M > 0 \) such that

\[
B \cap N_k(C) \subseteq N_M(B \cap C)
\]

where \( N_k(C) \) and \( N_M(B \cap C) \) denote the closed neighborhoods of \( C \) and \( B \cap C \) in \( G \) with respect to \( \text{dist}_G \), respectively.

**Lemma 2.11.** Let \( G \) be a finitely generated group. For any subgroup \( P \) of \( G \) and \( g \in G \), \( \text{hdist}(P, gP) < \infty \) if and only if \( g \in \text{Comm}_G(P) \).

Proof. The following two equivalences are immediate:

1. \( \text{hdist}(P, gP) < \infty \) if and only if \( \text{hdist}(P, gPg^{-1}) < \infty \).
2. \( P \) is contained in finite neighborhood of \( P \cap gPg^{-1} \) if and only if \( g \in \text{Comm}_G(P) \).

Suppose that \( \text{hdist}(P, gPg^{-1}) < \infty \). Lemma 2.10 implies that \( P \) is contained in finite neighborhood of \( P \cap gPg^{-1} \). Therefore \( g \in \text{Comm}_G(P) \). Conversely, suppose \( g \in \text{Comm}_G(P) \). Then \( P \) is contained in finite neighborhood of \( P \cap gPg^{-1} \). Analogously, since \( g^{-1} \in \text{Comm}_G(P) \), using a left translation, it follows that \( gPg^{-1} \) is contained in finite neighborhood of \( P \cap gPg^{-1} \). The two last statements imply that \( \text{hdist}(P, gPg^{-1}) \) is finite.

**Lemma 2.12.** Let \( \mathcal{P} \) be a finite collection of subgroups of a finitely generated group \( G \). The following statements are equivalent:

1. For any \( P \in \mathcal{P} \), \( P \) has finite index in its commensurator.
2. For any \( P \in \mathcal{P} \), the set \( \{ Q \in G/P : \text{hdist}(P, Q) < \infty \} \) is finite.
3. For any \( P \in \mathcal{P} \), the set \( \{ Q \in G/P : \text{hdist}(P, Q) < \infty \} \) is bounded as a subspace of \( (G/P, \text{hdist}) \).

Proof. To show that the first statement implies the second, suppose every \( P \in \mathcal{P} \) has finite index in its commensurator. By Lemma 2.11 for any \( P \in \mathcal{P} \) the set \( \{ gP \in G/P : \text{hdist}(P, gP) < \infty \} \) is finite. Since the collection \( \mathcal{P} \) is finite, for any \( P \in \mathcal{P} \), the set \( \{ Q \in G/P : \text{hdist}(P, Q) < \infty \} \) is a finite union finite sets, hence it is finite. It is immediate that the second statement implies the third one. That the third statement implies the first one is a consequence of Lemma 2.11.

Proof of Theorem 2.9. Observe that if \( \mathcal{P} \) is a qi-characteristic collection of subgroups of a finitely generated group, then \( \mathcal{P} \) is finite. Then the theorem is a direct consequence of Lemma 2.12.

### 2.2. Other remarks on quasi-isometric pairs

In this part we record a couple of propositions on quasi-isometries of pairs over commensurable groups, and an additional proposition at the end, for future reference. The results of this subsection are not used in the rest of the article.

**Proposition 2.13.** Suppose that \( H \) is a finite index subgroup of a finitely generated group \( G \). Let \( \mathcal{P} \) be a finite collection of subgroups of \( G \). Then there is a finite collection of subgroups \( \mathcal{Q} \) of \( H \) such that the inclusion \( H \hookrightarrow G \) induces a quasi-isometry of pairs \( q: (H, \mathcal{Q}) \) to \( (G, \mathcal{P}) \).
Here \( Q = \{ Q_i : i \in I \} \) where \( Q_i = g_iP_i g_i^{-1} \cap H \) and \( \{ g_iP_i : i \in I \} \) is a collection of representatives of the orbits of the \( H \)-action on \( G/P \) by multiplication on the left.

**Proof.** The finite collection \( P \) is a collection of orbit representatives of the \( G \)-action on \( G/P \) by left multiplication. Since \( H \) is finite index in \( G \), the induced \( H \)-action on \( G/P \) has finitely many orbits. Let \( g_1P_1, \ldots, g_kP_k \) be representatives of the \( H \)-orbits of \( G/P \), let \( Q_i = g_iP_i g_i^{-1} \cap H \), and define

\[
Q = \{ Q_i : 1 \leq i \leq k \}.
\]

Since \( H \) is finite index in \( G \), for any \( g \in G \) and \( P \in P \), the index of \( gPg^{-1} \cap H \) as a subgroup of \( gPg^{-1} \) is finite, in particular, \( \text{hdist}_G(g^{-1}Pg \cap H, g^{-1}Pg) \) is finite. Let

\[
D_1 = \max \{ \text{hdist}_G(g_iP_i g_i^{-1} \cap H, g_iP_i g_i^{-1}) : 1 \leq i \leq k \}
\]

and

\[
D_2 = \max \{ \text{dist}_G(e, g_i) : 1 \leq i \leq k \}.
\]

Observe that for any \( h \in H \) and \( 1 \leq i \leq k \),

\[
\text{hdist}_G(hQ_i, hg_iP_i) = \text{hdist}_G(g_iP_i g_i^{-1} \cap H, g_iP_i)
\]

\[
\leq \text{hdist}_G(g_iP_i g_i^{-1} \cap H, g_iP_i g_i^{-1}) + \text{hdist}_G(g_iP_i g_i^{-1}, g_iP_i)
\]

\[
\leq D_1 + D_2.
\]

To conclude we argue that the inclusion \( i : H \hookrightarrow G \) is indeed a quasi-isometry of pairs \((H, H/Q) \to (G, G/P)\). Let \( D > D_1 + D_2 \). Suppose that \( P \in G/P \). Since \( P = hg_iP_i \) for some \( h \in H \) and \( g_iP_i \), let \( R = hQ_i \) and observe that inequality (1) implies that \( \text{hdist}_G(R, P) < D \) and \( R \in H/Q \). Analogously, suppose that \( Q \in H/Q \). Since \( Q = hQ_i \) for some \( h \in H \) and \( Q_i \), let \( S = hQ_i \) and note that \( \text{hdist}_G(Q, S) < D \) by inequality (1), and \( S \in G/P \). \( \square \)

**Example 2.14.** Let \( G \) be a finitely generated infinite group with finitely many conjugacy classes of finite subgroups, and let \( P \) be a collection of representatives of conjugacy classes of finite subgroups. Suppose that \( H \) is a torsion-free finite index subgroup of \( G \). If \( Q \) is the set containing only the trivial subgroup of \( H \), then inclusion \( H \hookrightarrow G \) is a quasi-isometry of pairs \((H, Q) \hookrightarrow (G, P)\).

**Proposition 2.15.** Suppose that \( H \) is a finite index subgroup of a finitely generated group \( G \). Let \( Q \) be a finite collection of subgroups of \( H \). Suppose that for each \( Q_1 \in Q \) and \( g \in G \), there is \( Q_2 \in Q \) and \( h \in H \) such that \( gQ_1 g^{-1} = hQ_2 h^{-1} \). Then the inclusion \( H \hookrightarrow G \) is a quasi-isometry of pairs \((H, Q) \hookrightarrow (G, Q)\).

**Proof.** Consider the left \( H \)-action on \( G/Q \) by multiplication on the left. Note that this \( H \)-action preserves the Hausdorff distance \( \text{hdist}_G \) between subsets of \( G \). Since \( \text{hdist} \) is a metric on \( G/Q \), the \( H \)-action on \( G/Q \) is by isometries. By definition, \( H/Q \) is a subset of \( G/Q \).

To prove that the inclusion \( H \hookrightarrow G \) is a quasi-isometry of pairs \((H, Q) \to (G, Q)\), it is enough to verify that

\[
\max \{ \text{hdist}_G(gQ, H/Q) : gQ \in G/Q \} < \infty,
\]

where

\[
\text{hdist}_G(gQ, H/Q) := \min \{ \text{hdist}_G(gQ, hQ') : hQ' \in H/Q \}.
\]
Since $H$ is finite index subgroup of $G$, and $Q$ is a finite collection, it follows that the $H$-action on $G/Q$ has finitely many orbits. Let $\mathcal{R}$ be a collection of orbit representatives of the $H$-action on $G/Q$.

Let $gQ \in \mathcal{R}$. By hypothesis, there is $h \in H$ and $Q' \in Q$ such that $gQg^{-1} = hQ'h^{-1}$. Therefore
\[
hdist_G(gQ, hQ') \leq \hdist_G(gQ, gQg^{-1}) + \hdist_G(hQ'h^{-1}, hQ')
\]
and hence
\[
\text{dist}_G(e, g) + \text{dist}_G(e, h) < \infty,
\]

Since $\mathcal{R}$ is a finite set,
\[
D = \max\{\text{hdist}_G(gQ, H/Q) : gQ \in \mathcal{R}\} < \infty
\]
is a well defined integer.

Since $H$ is a subgroup of $G$, the subset $H/Q$ of $G/Q$ is $H$-invariant. Therefore
\[
\text{hdist}_G(gQ, H/Q) = \text{hdist}_G(hgQ, H/Q)
\]
for every $gQ \in \mathcal{R}$ and $h \in H$.

Since $\mathcal{R}$ is a collection of representatives of orbits of $G/Q$, it follows
\[
\text{hdist}_G(gQ, H/Q) \leq D
\]
for every $gQ \in G/Q$.

**Proposition 2.16.** Let $P$ and $Q$ be finite collections of subgroups of the finitely generated groups $G$ and $H$ respectively. Suppose that $q : (G, \mathcal{P}) \to (H, \mathcal{Q})$ is a quasi-isometry of pairs. Let $P \in \mathcal{P}$, $Q \in \mathcal{Q}$ and suppose $\text{hdist}(q(P), hQ) < \infty$ for some $h \in H$. If $P$ is finitely generated then $Q$ is finitely generated and quasi-isometric to $P$.

**Proof.** By post-composing $q$ with the left multiplication by $h^{-1}$, we can assume that $\text{hdist}(q(P), Q) < \infty$. Hence there is a quasi-isometry $g : (P, \text{dist}_G) \to (Q, \text{dist}_H)$. Let $S$ be a finite generating set of $P$, and let $\sigma = \max\{\text{dist}_G(e, s) : s \in S\}$. Then it follows that $P$ is a $\sigma$-coarsely connected space with respect to $\text{dist}_G$, that means for every pair of points $a, b$ in $P$ there is a sequence $a = x_0, x_1, \ldots, x_\ell = b$ such that $\text{dist}_G(x_i, x_{i+1}) \leq \sigma$. Then the quasi-isometry $g$ implies that there is $\sigma' > 0$ such that $Q$ is $\sigma'$-coarsely connected with respect to $\text{dist}_H$. Let $\Delta$ be the graph with vertex set $Q$ and an edge between any pair of points at distance at most $\sigma'$ with respect to $\text{dist}_H$. Then $\Delta$ is a connected graph, and since $(H, \text{dist}_H)$ is locally finite, $\Delta$ is also a locally finite graph. Note that $Q$ acts freely and cocompactly on $\Delta$ and therefore, by the Schwartz-Milnor lemma, $Q$ is finitely generated.

Note that a finitely generated subgroup of a finitely generated group is coarsely embedded. Let $S$ and $T$ be finite generating sets of $P$ and $Q$ respectively. Then, we have that $\text{ld}_P : (P, \text{dist}_S) \to (P, \text{dist}_G)$ is a coarse equivalence, and analogously for $\text{ld}_Q$. Therefore $\text{ld}_P \circ g \circ \text{ld}_Q$ is a coarse equivalence between $P$ and $Q$. It is well known observation by Gromov that every coarse-equivalence between finitely generated groups is a quasi-isometry.

A finitely generated subgroup $P$ of a finitely generated group $G$ is undistorted if, by considering the corresponding words metrics induced by finite generating sets, the inclusion $P \hookrightarrow G$ is a quasi-isometric map.
**Question 2.17.** Let $P$ be a finitely generated qi-subcharacteristic subgroup of a finitely generated group $G$. Is $P$ an undistorted subgroup of $G$?

3. QI-characteristic collections from Relative hyperbolicity, and other examples.

This part discuss examples of qi-characteristic collections arising from relative hyperbolicity, and cite other examples from the literature. Following the convention in [BDM09], if a group contains no collection of proper subgroups with respect to which is relatively hyperbolic, then we say that the group is *not relatively hyperbolic* (NRH). The following theorem is a consequence of a corollary of work by Behrstock, Druţu, and Mosher [BDM09, Theorem 4.1] and Theorem 2.3.

**Theorem 3.1.** Let $G$ be a finitely generated group hyperbolic relative to a finite collection $\mathcal{P}$ of NRH finitely generated subgroups. Then $\mathcal{P}$ is a qi-characteristic collection of $G$.

**Proof.** Relative hyperbolicity implies $\mathcal{P}$ is an almost malnormal collection [Osi06], in particular, $P = \text{Comm}_{Q}(P)$ for any $P \in \mathcal{P}$. The collection $G/P$ is coarsely discrete, that is, if $g_1P_1, g_2P_2 \in G/P$ are at finite Hausdorff distance then $g_1P_1 = g_2P_2$.

Observe that, if $g_1P_1, g_2P_2 \in G/P$, and $g_1P_1 \subseteq \mathcal{N}_{k}(g_2P_2)$ for some $k$, then $g_1P_1 = g_2P_2$. Indeed, under the hypothesis, Lemma 2.10 implies that $g_1P_1g_1^{-1}$ is a subset of a finite neighborhood of $g_1P_1g_1^{-1} \cap g_2P_2g_2^{-1}$; since $g_1P_1g_1^{-1}$ is an NRH, in particular it is infinite, and then malnormality implies that $g_1P_1g_1^{-1} = g_2P_2g_2^{-1}$; therefore $P_1 = P_2$ and $g_2^{-1}g_1P_1 = P_1$.

The proof of [BDM09] Theorem 4.8 shows that for every $L \geq 1$ and $C \geq 0$, there is $M = M(L, C, G, \mathcal{P}) > 0$ such that for any $A \in G/\mathcal{P}$ and any $(L, C)$-quasi-isometric embedding $q: A \to G$ there is $B \in G/\mathcal{P}$ such that $q(A) \subseteq \mathcal{N}_{M}(B)$. Hence, for any $(L, C)$-quasi-isometry $q: G \to G$ and any $A \in G/\mathcal{P}$ there is $B \in G/\mathcal{P}$ such that $q(A) \subseteq \mathcal{N}_{M}(B)$.

To conclude the proof we show that any $(L, C)$-quasi-isometry $q: G \to G$ is an $(L, C, LM + 2C)$-quasi-isometry of pairs $q: (G, \mathcal{P}) \to (G, \mathcal{P})$. Let $A \in G/\mathcal{P}$. By the statement of the previous paragraph, there is $B \in G/\mathcal{P}$ such that

\[ q(A) \subseteq \mathcal{N}_{M}(B). \]

Let $\tilde{q}: G \to G$ be a quasi-inverse of $q$ as defined in Definition 2.11. It follows that $\tilde{q}(B) \subseteq \mathcal{N}_{M}(A')$ where $A' \in G/\mathcal{P}$, by the previous paragraph. On the other hand, (2) implies that $A \subseteq \mathcal{N}_{LM + 2C}(\tilde{q}(B))$, and therefore $A \subseteq \mathcal{N}_{LM + M + 2C}(A')$. Now, the statement of the second paragraph implies that $A = A'$. It follows that $\tilde{q}(B) \subseteq \mathcal{N}_{M}(A)$, and by applying $q$ both sides we obtain that

\[ B \subseteq \mathcal{N}_{LM + 2C}(q(A)) \]

This last equation and (2) imply that $\text{hdist}(q(A), B) \leq LM + 2C$. □

**Proposition 3.2.** Let $G$ be a finitely generated group and let $\mathcal{P} \cup \{H\}$ be a qi-characteristic collection of finitely generated undistorted subgroups. If for any $P \in \mathcal{P}$ there is no quasi-isometric embedding $P \to H$, then $\mathcal{P}$ is a qi-characteristic collection of $G$. 

Proof. Let \( L \geq 1 \) and \( C \geq 0 \). Then there is \( M = M(L, C) \) such that any \((L, C)\)-quasi-isometry \( q: G \to G \) is a \((L, C, M)\)-quasi-isometry of pairs \( q: (G, \mathcal{P} \cup \{H\}) \to (G, \mathcal{P} \cup \{H\}) \). The assumption on \( H \) implies \( q(P) \) can not be at finite Hausdorff distance from a left coset of \( H \). Therefore \( q: (G, \mathcal{P}) \to (G, \mathcal{P}) \) is an \((L, C, M)\)-quasi-isometry of pairs.

Corollary 3.3. Let \( G \) and \( \mathcal{P} = \{P_0, \ldots, P_n\} \) be as in Theorem 3.1. Suppose that there is no quasi-isometric embedding \( P_i \to P_0 \) for \( 1 \leq i \leq n \). Then \( \{P_1, \ldots, P_n\} \) is a qi-characteristic collection.

Proof. Since each \( P \in \mathcal{P} \) is a finitely generated NRH group, Theorem 3.1 implies that \( \mathcal{P} \) is a qi-characteristic collection of \( G \). The subgroups \( \mathcal{P} \) are undistorted in \( G \), see [DS05]. By Proposition 3.2 it follows that \( \{P_1, \ldots, P_n\} \) is a qi-characteristic collection of subgroup of \( G \). \( \square \)

Example 3.4. Recall that a subgroup is qi-subcharacteristic if it belongs to a qi-characteristic collection.

1. A finite subgroup of an finitely generated infinite group is not qi-subcharacteristic.
2. The NRH hypothesis of Theorem 3.1 is necessary, for instance, if \( F \) is a free group of finite rank then a maximal cyclic subgroup is not qi-subcharacteristic. There is a quasi-isometry of \( F \) that maps an infinite geodesic preserved by a non-trivial element of \( F \) to a geodesic that is preserved by no element of \( F \).
3. Let \( A \) and \( B \) be finitely generated NRH groups endowed with word metrics with a common finite subgroup \( C \). By Corollary 3.3 if there is no quasi-isometric embedding \( A \to B \), then \( \{A\} \) is a qi-characteristic collection of subgroups of \( A \ast_C B \).
4. In contrast to the previous example, let \( G = \mathbb{Z}^2 \ast \mathbb{Z}^2 \) and let \( H \) be the left hand side factor. While \( H \) is a qi-characteristic subgroup by Theorem 3.1 the collection \( \{H\} \) is not qi-characteristic. The second and third conditions of the Theorem 2.9 hold, but the first does not. Specifically, a quasi-isometry that flips the two factors sends \( H \) to a space that is at infinite Hausdorff distance of any of its left cosets.
5. Let \( n \geq 2 \) and consider the Baumslag-Solitar group \( BS(1, n) = \langle a, t | tat^{-1} = a^n \rangle \). The distorted cyclic subgroup \( \langle a \rangle \) is not qi-characteristic since it has infinite index in its commensurator.

The subgroup \( \langle t \rangle \) does not form a qi-characteristic collection. We sketch the argument using a construction that appears in the work of Farb and Mosher [FM98] on quasi-isometric rigidity of solvable Baumslag-Solitar groups. They use a particular metric on the Cayley complex \( X_n \) of \( BS(1, n) \) together with the projection \( \pi: X_n \to T_n \) to the Bass-Serre tree. Let us recall a few properties: the inverse image \( \pi^{-1}(L) \) of any coherently oriented proper line \( L \) of \( T_n \) is an isometrically embedded hyperbolic plane \( H \); all hyperbolic planes of \( X_n \) arise in this way and can be simultaneously identified with the upper half plane model of \( \mathbb{H}^2 \) so that inverse image \( \pi^{-1}(x) \) for \( x \in L \) correspond to an horocycle based at \( \infty \in \partial \mathbb{H}^2 \). In this way, the parabolic isometry \( q: \mathbb{H}^2 \to \mathbb{H}^2 \) given by \( z \mapsto z + 1 \) preserves horocycles based at \( \infty \), and hence it induces an isometry \( q: X_n \to X_n \) such that \( \pi \circ q = \pi \). The isometry \( q \) preserves each hyperbolic plane of \( X_n \), and each of these planes corresponds to a unique left coset of \( \langle t \rangle \) which can be identified with
a particular vertical geodesic. Since any two hyperbolic planes of $X_n$ are at infinite Hausdorff distance, and any two distinct geodesics of $\mathbb{H}^2$ are at infinite Hausdorff distance, it follows that $q(\langle t \rangle)$ is at infinite Hausdorff distance of every left coset of $\langle t \rangle$.

(6) Consider an amalgamated product $G = \mathbb{Z}^3 *_{\mathbb{Z}} \mathbb{Z}^3$, where $\mathbb{Z}$ corresponds to a maximal infinite cyclic subgroup in both factors, and let $P$ be the collection consisting of the two $\mathbb{Z}^3$ factors. The work of Papasoglu [Pap05, Theorem 7.1] implies that every $(L, C, M_q)$-quasi-isometry of $q: G \to G$ is $(L, C, M_q)$-quasi-isometry of pairs $q: (G, P) \to (G, P)$ for some constant $M_q$. To show that $P$ is qi-characteristic we need to show that $M_q$ can be chosen so that it depends only of $L$ and $C$, and not on $q$. We do not know whether the constant $M_q$ can be chosen so that it only depends on $L$ and $C$. Provided that is true, $P$ would be another example of a qi-characteristic collection.

(7) If $F$ is a finite group and $H$ is a finitely presented one-ended group, then the collection consisting of only the subgroup $H$ of the wreath product $G = F \wr H$ forms a qi-characteristic collection. This is a result of Genevois and Tessera [GT21, Theorem 1.88 and Proof of Theorem 7.3].

(8) Certain graph products of finite groups contain qi-characteristic collections, see [Genar, Fact 3.14].

4. Quasi-isometry Invariance of QI-Characteristic collections

In this section we prove the following theorem which is the main result of the article.

**Theorem 4.1.** Let $X$ be a metric space and let $A$ be a qi-characteristic collection of subspaces. If $G$ is a finitely generated group quasi-isometric to $X$, then there is a finite collection of subgroups $P$ such that $(G, G/P)$ and $(X, A)$ are quasi-isometric pairs.

The following corollary is an immediate consequence.

**Corollary 4.2.** Let $G$ be a finitely generated group, let $P$ be a qi-characteristic collection of subgroups of $G$. If $H$ is a finitely generated group quasi-isometric to $G$, then there is a qi-characteristic collection of subgroups $Q$ of $H$ such that $(H, Q)$ and $(G, P)$ have the same quasi-isometry type.

**Proof of Theorem 4.1.** Let $q: G \to X$ and $\bar{q}: X \to G$ be $(L_0, C_0)$-quasi-isometries such that $q \circ \bar{q}$ and $\bar{q} \circ q$ are at distance less than $C_0$ from the identity maps on $X$ and $G$ respectively. Let $x_0 = q(e)$ where $e$ is the identity element in $G$ and assume, without loss of generality, that $\bar{q}(x_0) = e$. For $g \in G$, let $g: G \to G$ denote the isometry given by $x \mapsto gx$; let $q_g: X \to X$ denote the composition $q_g = q \circ g \circ \bar{q}$. Then the following statements can be easily verified:

- For $g \in G$, $q_g: X \to X$ is an $(L, C)$-quasi-isometry where $L = L_0^2$ and $C = L_0C_0 + C_0 + 1 > C_0$.
- $(G$ quasi-acts on $X$) For $g_1, g_2 \in G$, the map $q_{g_1} \circ q_{g_2}$ is at distance at most $C$ from the map $q_{g_1} \circ q_{g_2}$; and the map $q_{g_1} \circ q_{g_1^{-1}}$ is at distance at most $2C$ from the identity.
- $(G$ acts $C_0$-transitively on $X$) For every $x, y \in X$ there is $g \in G$ such that $dist_G(x, q_g(y)) \leq C_0$. 


For $A \in \mathcal{A}$, define
\[\text{St}(A) = \{ g \in G : \text{hdist}(q_g(A), A) < \infty \}.\]

**Step 1.** For any $A \in \mathcal{A}$, $\text{St}(A)$ is a subgroup of $G$.

**Proof.** Let $g_1, g_2 \in \text{St}(A)$, then
\[
\text{hdist}(q_{g_1^{-1} g_2}(A), A) \leq L \text{hdist}(q_{g_1} \circ q_{g_1^{-1} g_2}(A), q_{g_1}(A)) + C
\leq L \text{hdist}(q_{g_2}(A), q_{g_1}(A)) + LC + C
\leq L \text{hdist}(q_{g_1}(A), A) + L \text{hdist}(q_{g_2}(A), A) + LC + C < \infty.
\]
Hence $g_1^{-1} g_2 \in \text{St}(A)$.

**Step 2.** There is a constant $M_1 > 0$ and a finite subset $\mathcal{F}$ of $\mathcal{A}$ such that:

1. For any $A \in \mathcal{A}$ and $g \in G$ there is $A' \in \mathcal{A}$ such that $\text{hdist}(q_g(A), A') < M_1$.
2. For any $A \in \mathcal{A}$ and $a \in A$ there is $g \in G$ and $B \in \mathcal{F}$ such that $\text{dist}(q_g(a), x_0) \leq C_0$ and $\text{hdist}(q_g(A), B) \leq M_1$ and $\text{hdist}(q_{g^{-1}}(B), A) \leq M_1$.
3. For any $A, A' \in \mathcal{A}$, $\text{hdist}(A, A') \leq M_1$ or $\text{hdist}(A, A') = \infty$.

**Proof.** Let $M = M(L, C)$ be provided by Definition 2.3 for the collection $\mathcal{A}$. This constant satisfies the first item. Since $\mathcal{A}$ is qi-characteristic, the collection
\[\mathcal{D} = \{ A \in \mathcal{A} : \text{there is } a \in A \text{ such that } \text{dist}(x_0, a) < M + C \}\]
contains only finitely many non-coarse equivalent elements. In particular, there is a finite subset $\mathcal{F} = \{ B_1, \ldots, B_m \}$ of $\mathcal{A}$ such that any element of $\mathcal{D}$ is at finite Hausdorff distance from an element of $\mathcal{F}$. Since $\mathcal{A}$ is qi-characteristic, there is a constant $K > 0$ such that for any $A \in \mathcal{A}$ and $B \in \mathcal{F}$, if $\text{hdist}(A, B) < \infty$ then $\text{hdist}(A, B) < K$.

Let $A \in \mathcal{A}$ and $a \in A$. Since $G$ quasi-acts $C_0$-transitively on $X$, there is $g \in G$ such that $\text{dist}(q_g(a), x_0) \leq C_0 < C$. Since $\mathcal{A}$ is qi-characteristic, there is $B' \in \mathcal{A}$ such that $\text{hdist}(q_g(A), B') < M$. Observe that $B' \in \mathcal{D}$, since $\text{dist}(q_g(a), x_0) \leq C$ and $q_g(a) \in q_g(A)$ imply that $\text{dist}(x_0, B') \leq \text{dist}(x_0, q_g(a)) + \text{dist}(q_g(a), B') \leq C + M$. Hence there is $B \in \mathcal{F}$ such that $\text{hdist}(B, B') < K$, and therefore $\text{hdist}(q_g(A), B) \leq C + M + K$. Since $\text{hdist}(A, q_{g^{-1}}(B)) \leq L \text{hdist}(q_g(A), B) + 2C$, the first and second statement hold with the constant $M_0 = L(C + M + K) + 2C$.

For the third item, let $A, A' \in \mathcal{A}$ such that $\text{hdist}(A, A') < \infty$. Then, by previous paragraph, there is $g \in G$ and $B \in \mathcal{F}$ such that $\text{hdist}(q_g(A), B) \leq M_0$. Since $\mathcal{A}$ is qi-characteristic, there is $B' \in \mathcal{A}$ such that $\text{hdist}(q_g(A'), B') \leq M_0$. It follows that $\text{hdist}(B, B')$ is finite and hence bounded by $K$. Therefore
\[
\text{hdist}(A, A') \leq L \text{hdist}(q_g(A), q_g(A')) + C
\leq L(2M_0 + K) + C.
\]

The proof concludes by defining $M_1$ as the constant on the right of the previous inequality.

**Step 3.** There is a constant $M_2$ with the following property. For every $A \in \mathcal{A}$
\[\text{St}(A) = \{ g \in G : \text{hdist}(q_g(A), A) \leq M_2 \}.\]
Proof. Suppose \( g \in St(A) \). Since \( A \) is \( \alpha \)-characteristic, there is \( A' \in A \) such that \( \text{dist}(\gamma_q(A), A') \leq M_1 \). Since \( \text{dist}(A, q_g(A)) < \infty \), we have that \( \text{dist}(A, A') < \infty \) and hence

\[
\text{dist}(A, q_g(A)) \leq \text{dist}(A, A') + \text{dist}(A', q_g(A)) \leq 2M_1.
\]

As a consequence,

\[
St(A) = \{ g \in G : \text{dist}(\gamma_q(A), A) \leq 2M_1 \}
\]

and to conclude let \( M_2 = 2M_1 \) is defined. \( \square \)

From here on let \( M \) be \( \max\{M_1, M_2, LM_1 + 3C\} + 1 \).

Step 4. If \( A, A' \in A \), \( g \in G \) and \( \text{dist}(\gamma_q(A), A') \) is finite, then

\[
gSt(A)g^{-1} = St(A')
\]

Proof. Let \( h \in St(A') \). Note that

\[
\text{dist}(\gamma_{g^{-1}h}(A), A) \leq 1
\]

\[
\leq L \text{dist}(\gamma_h \circ q_g(A), q_g(A)) + 5LC
\]

\[
\leq L \text{dist}(\gamma_h \circ q_g(A), q_g(A')) + L \text{dist}(A', q_g(A)) + 5LC
\]

\[
\leq L^2 \text{dist}(q_g(A), A') + L \text{dist}(q_h(A), A') + L \text{dist}(A', q_g(A)) + 6LC < \infty.
\]

Hence \( g^{-1}h \in St(A) \), and we conclude that \( gSt(A)g^{-1} \supseteq St(A') \). The other inclusion is proved analogously. \( \square \)

The following step is a version of [KL97, Lemma 5.2].

Step 5. There is \( D_0 > 0 \) such that for any \( A \in A \) and \( a \in A \), then

\[
\text{dist}(A, St(A)a) \leq D_0
\]

where \( St(A)a = \{ q_g(a) | g \in St(A) \} \).

Proof. Recall \( F = \{ B_1, \ldots, B_m \} \). Let

\[
I = \{ (i, j) : \text{there is } g \in G \text{ such that } \text{dist}(q_g(B_i), B_j) \leq M \}
\]

and note that

\[
I = \{ (i, j) : \text{there is } g \in G \text{ such that } \text{dist}(q_g(B_i), B_j) < M \}
\]

by Step 2. For each \( (i, j) \in I \) choose \( g_{i,j} \in G \) such that \( \text{dist}(q_{g_{i,j}}(B_i), B_j) \leq M \). Let

\[
T = \max\{ \text{dist}(q_{g_{i,j}}(x_0), x_0) : (i, j) \in I \} < \infty.
\]

The constant \( D_0 = D_0(L, C, M, T) \) is defined at the end of the proof and it is larger than \( M \). Let \( A \in A \) and \( a \in A \). Observe \( St(A)a \) is contained in the \( M \)-neighborhood of \( A \). We show below that for any \( b \in A \) there is \( \gamma \in St(A) \) such that \( \text{dist}(q_\gamma(a), b) \leq D_0 \), which implies \( \text{dist}(A, St(A)a) \leq D_0 \).

Let \( b \in A \). Since \( G \) acts \( C_0 \)-transitively on \( X \), there are \( \alpha, \beta \in G \) such that

\[
\text{dist}(q_\alpha(a), x_0) \leq C_0 \quad \text{and } \text{dist}(q_\beta(b), x_0) \leq C_0.
\]
By Step 2 there are $B_i$ and $B_j$ in $\mathcal{F}$ such that $\text{hdist}(q_\alpha(A), B_i) \leq M$ and $\text{hdist}(q_\beta(A), B_j) \leq M$, therefore $(i,j) \in I$. To simplify notation, let $g$ denote the corresponding element $g_{i,j}$. Let $\gamma = \beta^{-1}g\alpha$ and note that $\gamma \in St(A)$ since

$$
\text{hdist}(q_\gamma(A), A) \leq L \text{hdist}(q_\gamma \circ q_\alpha(A), q_\beta(A)) + 4LC
$$

$$
\leq L \text{hdist}(q_\gamma \circ q_\alpha(A), q_\beta(B_1)) + L \text{hdist}(q_\beta(B_1), B_j) + L \text{hdist}(B_j, q_\beta(A)) + 4LC
$$

$$
\leq (L^2M + LC) + LM + LM + 4LC.
$$

Moreover,

$$
\text{dist}(q_\alpha(a), b) \leq L \text{dist}(q_\gamma \circ q_\alpha(a), q_\beta(b)) + 4LC
$$

$$
\leq L \text{dist}(q_\gamma \circ q_\alpha(a), q_\beta(x_0)) + L \text{dist}(q_\beta(x_0), x_0) + L \text{dist}(x_0, q_\beta(b)) + 4LC
$$

$$
\leq (L^2C_0 + LC) + LT + LC_0 + 4LC = D'.
$$

Let $D_0 = \max\{D', M\}$. $\square$

**Step 6.** Let $St(A) = \{St_M(A) : A \in \mathcal{A}\}$. Then $G$ acts (from the left) on $St(A)$ by conjugation with finitely many orbits. Moreover $\{St(A) : A \in \mathcal{F}\}$ contains a representative of each $G$-orbit.

**Proof.** First, we verify that the action is well defined. Let $g \in G$ and $A \in \mathcal{A}$. Since $\mathcal{A}$ is $\alpha$-characteristic there is $A'$ such that $\text{hdist}(q_\beta(A), A') < M$. Then, by Step 4 we have that $g\text{St}(A)g^{-1} = \text{St}(A')$.

To verify that the action has finitely many orbits, let $A \in \mathcal{A}$. Then by Step 2 there is $g \in G$ and $B \in \mathcal{F}$ such that $\text{hdist}(q_\beta(A), B) \leq M$. Hence, by Step 4 $g\text{St}(A)g^{-1} = \text{St}(B)$. Therefore $\{\text{St}(B) : B \in \mathcal{F}\}$ contains a collection of representatives of the $G$-orbits of $\text{St}(A)$; since $\mathcal{F}$ is finite the claim follows. $\square$

**Step 7.** Let $g \in G$, $U \subset G$ and $a, b \in X$. The following statements hold.

- $\text{dist}(q(g), q_\alpha(x_0)) \leq L\text{C}_0 + C_0$,
- $\text{hdist}(q(U), q_\alpha(x_0)) \leq L\text{C}_0 + C_0$, and
- $\text{hdist}(Ua, Ub) \leq L\text{dist}(a, b) + C$.

Here $Ux_0$ denotes the set $\{q_\gamma(x_0) : g \in U\}$, and $Ua$ and $Ub$ are defined analogously.

**Proof.** For the first inequality, recall that $\bar{q}(x_0) = e$, and note that

$$
\text{dist}(q_\gamma(x_0), q(g)) \leq L_0 \text{dist}(g \circ \bar{q}(x_0), \bar{q} \circ q(g)) + C_0 \leq L\text{C}_0 + C_0.
$$

The second statement follows from the first one. The third inequality is a consequence of $q_\beta$ being an $(L, C)$-quasi-isometry for every $g \in G$. $\square$

Define

$$
K_1 = 3(LM + LC + M + 10C + D_0), \quad K_2 = 3(LK_1 + 10C) > K_1
$$

and

$$
D = 5K_2
$$

Suppose

$$
\mathcal{F} = \{B_1, \ldots, B_m\}
$$

and let $P_i = \text{St}(B_i)$ and define

$$
\mathcal{P} = \{P_1, \ldots, P_m\}.
$$

**Step 8.** For any $g \in G$ and $P \in \mathcal{P}$, the set $\{A \in \mathcal{A} : \text{hdist}(q(gP), A) < \infty\}$ is bounded in $(\mathcal{A}, \text{hdist})$. Moreover, there is $A \in \mathcal{A}$ such that $\text{hdist}(q(gP), A) \leq D$. 

Proof. The first statement is a direct consequence of the third item of the definition of qi-characteristic. It is left to prove the existence of $A \in \mathcal{A}$ such that $\text{hdist}(q(gP), A) \leq D$. By definition of $\mathcal{P}$, there is $B \in \mathcal{F}$ such that $P = St(B)$. Since $\mathcal{A}$ is qi-characteristic, there is $A \in \mathcal{A}$ such that

$$hdist(q_g(B), A) < M < K_2.$$  

(3)

By the triangle inequality,

$$hdist(q(gP), A) \leq hdist(q(gP), q_g(q(P))) + hdist(q_g(q(P)), q_g(B)) + hdist(q_g(B), A) \leq hdist(q(gP), q_g(q(P))) + hdist(q_g(q(P)), q_g(B)) + K_2$$

(4)

To conclude the proof, we show below that the two terms on the last line of inequality (4) are bounded by $K_2$, and hence $hdist(q(gP), A) < D$. For the first term,

$$hdist(q_g(q(P)), q_g(B)) \leq L hdist(q(P), B) + C$$

(5)

since $q_g$ is an $(L_0, C_0)$-quasi-isometry. To argue that $hdist(q_g(q(P)), q_g(B)) < K_2$ is enough to show that $hdist(q(P), B) < K_1$. Since $B \in \mathcal{F}$, there is $b \in B$ such that $\text{dist}(x_0, b) < M + C$. The triangle inequality implies

$$hdist(q(P), B) \leq hdist(q(P), P x_0) + hdist(P x_0, Pb) + hdist(Pb, B).$$

By Step 9

$$hdist(q(P), P x_0) \leq LC_0 + C_0,$$

and

$$hdist(P x_0, Pb) \leq L \text{dist}(x_0, b) + C \leq L(M + C) + C.$$

By Step 9

$$hdist(Pb, B) \leq D_0.$$  

(10)

Then inequalities (8), (9), and (10) imply that the expression on the right of (7) is bounded by $K_1$ which completes the proof.

Step 9. For any $A \in \mathcal{A}$, the set \{ $gP \in G/\mathcal{P}$: $hdist(q(gP), A) < \infty$ \} is finite, and there is $gP \in G/\mathcal{P}$ such that $hdist(q(gP), A) \leq D$.

Proof. Let $A \in \mathcal{A}$. Let $P \in \mathcal{P}$ and suppose $hdist(q(P), A) < \infty$ and $hdist(q(gP), A) < \infty$. Then

$$hdist(q_g(A), A) \leq hdist(q_g(A), q(gP)) + hdist(q(gP), A) < \infty$$

and hence $g \in St(A)$. Step 8 implies that $hdist(gP, P) \leq LM + C$. Since $G$ is locally finite, there are only finitely many left cosets $gP$ such that $hdist(q(gP), A) < \infty$. Since $\mathcal{P}$ is finite, it follows that the set \{ $gP \in G/\mathcal{P}$: $hdist(q(gP), A) < \infty$ \} is a finite union of finite sets and hence finite.

Now we prove that there is $gP \in G/\mathcal{P}$ such that $hdist(q(gP), A) \leq D$. By Step 2, there is $g \in G$ and $B \in \mathcal{F}$ such that $hdist(q_g(B), A) \leq M$. By Step 4, $St(A) = gSt(B)g^{-1}$. Let $P \in \mathcal{P}$ such that $P = St(B)$. Since $B \in \mathcal{F}$ there is $b \in B$
such that \( \text{dist}(x_0, b) \leq M + C \). Let \( a \in A \) such that \( \text{dist}(q_g(b), a) \leq M \). It follows that

\[
\begin{align*}
\text{hdist}(gPb, gPg^{-1}a) &\leq \text{hdist}(gPb, gPg^{-1}a) + \text{hdist}(gPg^{-1}a, gPg^{-1}a) \\
&\leq L \text{dist}(b, q_g^{-1}(a)) + C + C \\
&\leq L(M + C + 2C) + 2C \leq K_2.
\end{align*}
\]

By Step 7 and Step 5,

\[
\text{hdist}(q(gP), A) \leq \text{hdist}(q(gP), gPx_0) + \text{hdist}(gPx_0, gPb) + \text{hdist}(gPb, gPg^{-1}a) + \text{hdist}(gPg^{-1}a, A) \\
\leq (LC_0 + C_0) + (L \text{dist}(x_0, b) + C) + K_2 + D_0 \\
\leq K_2 + K_2 + K_2 + D_0 < D.
\]

\[\square\]

To conclude the proof of the theorem, observe that \( q : G \to X \) is a quasi-isometry of pairs \( q : (G, G/P) \to (X, A) \) as a consequence of Steps 8 and 9.

\[\square\]

5. Filtered Ends of Pairs

In this section, the following result is proved.

**Definition 5.1.** Let \( \text{QPMet} \) be the category whose objects are pairs \((X, C)\) where \( X \) is a metric space and \( C \) is a non-empty subspace; morphisms \((X, C) \to (Y, D)\) are quasi-isometric maps \( f : X \to Y \) such that \( D \subseteq f(C)^{+r} \) for some \( r \geq 0 \), where \( f(C)^{+r} \) is the \( r \)-neighborhood of \( f(C) \); and the composition law is the standard composition of functions.

**Theorem 5.2.** There is a covariant functor

\[ E : \text{QPMet} \to \text{Sets} \]

with the following properties:

1. If \( f : (X, C) \to (X, C) \) is a morphism such that \( f \) is at distance at most \( r \) from the identity function on \( X \), then \( E(f) \) is the identity morphism of \( E(X, C) \).
2. If \( f : X \to Y \) is a quasi-isometry, \( C \subseteq X \) and \( D \subseteq Y \) and \( \text{hdist}(f(C), D) \) is finite, then \( E(f) : E(X, C) \to E(Y, D) \) is a bijection.
3. If \( X \) is a proper geodesic metric space and \( C \subseteq X \) is compact, then there is a natural bijection \( E(X, C) \to \text{Ends}(X) \), where \( \text{Ends}(X) \) is defined as in \[ BH99 \].
4. If \( X \) is the Cayley graph with the combinatorial path metric of a finitely generated group \( G \) and \( P \leq G \), then the cardinality of \( E(X, P) \) coincides with the number of coends \( \tilde{e}(G, P) \) as defined in Bowditch \[ Bow02 \].

**Definition 5.3.** Given a pair \((X, C)\) in \( \text{QPMet} \), the set \( E(X, C) \) is called the set of filtered ends of \((X, C)\).

**Remark 5.4.** In Geoghegan’s book \[ Geo08 \ Section 14.3\] there is an alternative approach to filtered ends that also coincides with Bowditch’s approach \[ Bow02 \]. Geoghegan remarks the existence of a functor analogous to the one in Theorem 5.2 specifically, from the category whose objects are well-filtered CW-complexes of locally finite type and morphisms are filtered maps. This alternative framework is
suitable to study filtered ends from an algebraic topological perspective. Geoghegan’s functor also satisfies properties (3) and (4) as a natural consequence of the definition, while properties (1) and (2) are not addressed as the book does not approach coarse geometry aspects of filtered ends. Our approach is suitable to study filtered ends from a coarse geometrical point of view.

5.1. Coarsely connected components. Let \((X, \text{dist})\) be a metric space. A subset \(A\) of \(X\) is \(\sigma\)-coarsely connected if for any pair of points \(a, b \in A\) there is a finite sequence \(a = x_0, x_1, \ldots, x_n = b\) of elements of \(A\) such that \(\text{dist}(x_i, x_{i+1}) \leq \sigma\) for each index \(i\). We call such a sequence a \(\sigma\)-quasi-path from \(a\) to \(b\) in \(A\).

For \(\sigma \geq 0\), let \(C(X, \sigma)\) denote the collection of \(\sigma\)-coarsely connected components of \(X\). Observe that if \(\sigma' \geq \sigma\) then there is a function

\[
\rho: C(X, \sigma) \to C(X, \sigma')
\]

that satisfies that \(A \subseteq \rho(A)\) for any \(A \in C(X, \sigma)\). Let \(C_\infty(X, \sigma)\) be the collection of unbounded components in \(C(X, \sigma)\). Observe that if \(\sigma = 0\) then \(C_\infty(X, \sigma)\) is the empty set.

**Remark 5.5.** For \(\sigma \leq \sigma'\), the function

\[
\rho: C_\infty(X, \sigma) \to C_\infty(X, \sigma')
\]

is not necessarily surjective nor injective. As an example, fix an integer \(m > 1\) and let \(A = \{n \in \mathbb{Z} : n = 0\ or\ n \geq m\}\) consider \(X = \mathbb{R} \times A\) with the metric \(d = \text{dist}(\alpha \times \ell, \beta \times k)\) defined as follows: if \(\alpha = \beta\) then \(d = |\ell - k|\); if \(\alpha \neq \beta\) then \(d = |\alpha - \beta| + |\ell| + |k|\). In this case, \(C_\infty(X, \sigma)\) has cardinality 1 for \(0 < \sigma < 1\), infinite cardinality if \(1 \leq \sigma < m\), and cardinality 1 again for \(\sigma \geq m\). In particular \(\rho\) is not surjective if \(\sigma = 1/2\) and \(\sigma' = 1\); and it is not injective if \(\sigma = 1\) and \(\sigma' = m\).

Note that the failure of surjectivity arises in the case that there are elements of \(C_\infty(X, \sigma')\) that are partitioned into bounded \(\sigma\)-connected components.

**Proposition 5.6.** If \(f: X \to Y\) is a \((\lambda, \epsilon)\)-coarse Lipschitz map and \(\sigma \geq 0\), then there is an induced function \(f_*: C(X, \sigma) \to C(Y, \lambda \sigma + \epsilon)\) where \(f_*(A)\) is the unique element of \(C(Y, \lambda \sigma + \epsilon)\) containing \(f(A)\). In particular, there is an induced function

\[
f_*: C_\infty(X, \sigma) \to C_\infty(Y, \lambda \sigma + \epsilon)
\]

**Proof.** Let \(A \in C(X, \sigma)\). Then \(f(A)\) is \(\lambda \sigma + \epsilon\) coarsely connected and hence there is a unique \(B \in C(Y, \lambda \sigma + \epsilon)\) such that \(f(A) \subseteq B\). Observe that if \(A \in C(X, \sigma)\) is unbounded, then \(f(A)\) is unbounded and the second statement follows. \(\Box\)

5.2. Definition of the set of filtered ends \(E(X, C)\). Let \((I, \preceq)\) denote the directed set where \(I = (0, \infty) \times [0, \infty)\) and for \(\alpha, \beta \in I\),

\[
\alpha \preceq \beta \iff \text{if and only if } \alpha_1 \leq \beta_1 \text{ and } \alpha_2 \geq \beta_2.
\]

We use the following notation, for \(\alpha = (\sigma, \mu) \in I\) let

\[
I_\alpha = \{(x, y) \in I : \sigma \leq x \text{ and } \mu \leq y\},
\]

and

\[
J_\alpha = \{(\sigma, y) \in I : \mu \leq y\}.
\]

For a positive real number \(\sigma\), let

\[
I_\sigma = \{(x, y) \in I : \sigma \leq x \text{ and } 0 \leq y\},
\]
and
\[ J_\sigma = \{(\sigma, y) \in I : 0 \leq y \}. \]

**Remark 5.7.** Observe that \( J_\sigma \) is a coinitial subset of \( I_\alpha \). In particular, \( J_\sigma \) is a coinitial subset of \( I_\tau \).

Let \((X, \text{dist})\) be a metric space, and let \( C \) be a subset of \( X \). For \( \alpha = (\sigma, \mu) \in I \), let \( C_\alpha \) denote the set of unbounded \( \sigma \)-coarsely connected components of \( X - C^{+\mu} \),
\[ C_\alpha = C_{(\sigma, \mu)} = C_\infty(X - C^{+\mu}, \sigma), \]
where \( C^{+\mu} \) denote the \( \mu \)-neighborhood of \( C \).

**Remark 5.8.** Let \( A \) and \( B \) sets, let \( f : A \to B \) be a function. Suppose \( A \) and \( B \) are collections of subsets of \( A \) and \( B \) respectively. If any pair of distinct elements of \( B \) are disjoint, then there is at most one function \( g : A \to B \) with the property that \( f(C) \subseteq g(C) \) for any \( C \in A \).

**Proposition 5.9.** Let \( \alpha, \beta \in I \) and suppose that \( \alpha \preceq \beta \). Then there is a unique function
\[ \rho_{\alpha, \beta} : C_\alpha \to C_\beta \]
that satisfies that \( A \subseteq \rho_{\alpha, \beta}(A) \) for any \( A \in C_\alpha \). In particular, if \( \alpha \preceq \beta \preceq \gamma \) then
\[ \rho_{\beta, \gamma} \circ \rho_{\alpha, \beta} = \rho_{\alpha, \gamma}. \]

**Proof.** The uniqueness of the function is clear since \( C_\beta \) is a collection of disjoint subsets of \( X \), see Remark 5.8. To define the function we consider cases:

1. If \( \alpha_1 = \beta_1 \), then \( \rho_{\alpha_1, \beta_1} \) is a particular case of (11).
2. If \( \alpha_1 = \beta_1 \), then \( \rho_{\alpha_1, \beta_1} \) is the function induced by the inclusion \( X - C^{+\alpha_2} \to X - C^{+\beta_2} \). By Proposition 5.6, we have that \( A \subseteq \rho_{\alpha_1, \beta_1}(A) \) for any \( A \in C_\alpha \).
3. Suppose \( \alpha_1 < \beta_1 \) and \( \alpha_2 > \beta_2 \). Let \( \gamma = (\alpha_1, \beta_2) \) and \( \delta = (\beta_1, \alpha_2) \). Then
\[ \alpha \preceq \gamma \preceq \beta \text{ and } \alpha \preceq \delta \preceq \beta. \]
Observe that \( \rho_{\gamma, \delta} \circ \rho_{\alpha, \gamma} \) and \( \rho_{\beta, \gamma} \circ \rho_{\alpha, \beta} \) are functions from \( C_\alpha \) to \( C_\beta \) with the required property. By uniqueness, both functions are equal and they define \( \rho_{\alpha, \beta} \).

**Definition 5.10** (Filtered Ends at scale \( \sigma \)). Let \( X \) be a metric space, \( C \) be a subset of \( X \), and \( \sigma \geq 0 \). The direct system \( \{C_i, \rho_{i,j} : i, j \in J_\sigma \} \) is called the \( \sigma \)-ends-system for \( (X, C, \sigma) \), and the inverse limit is denoted as \( (E(X, C, \sigma), \psi_*) \); in symbols
\[ E(X, C, \sigma) = \varprojlim_{\mu} C_\infty(X - C^{+\mu}, \sigma) = \varprojlim_{\mu} C_{(\sigma, \mu)} = \lim_{i \in J_\sigma} C_i. \]

For \( \alpha \in I \), let
\[ E(X, C, J_\alpha) = \lim_{i \in J_\alpha} C_i \]
the inverse limit of the direct system \( \{C_i, \rho_{i,j} : i, j \in J_\alpha \} \).

**Remark 5.11.** Let \( \sigma > 0 \) and \( r > 0 \). Then there is a canonical bijection
\[ \text{Id} : E(X, C, \sigma) \to E(X, C^{+r}, \sigma). \]

Indeed, for any \( i, j \in J_\sigma \), if \( i \preceq j \) then \( J_i \) is a coinitial subset of \( J_j \) and hence there is a natural bijection \( \tau_{i,j} : E(X, C, J_i) \to E(X, C, J_j) \). Moreover, if \( i, j, k \in J_\sigma \) and \( i \preceq j \preceq k \), then \( \tau_{j,k} \circ \tau_{i,j} = \tau_{i,k} \). Hence all the sets \( E(X, C, J_i) \) for \( i \in J_\sigma \) can be naturally identified.
Example 5.12. The following examples illustrate that for reasonably well-behaved metric spaces, the number of filtered ends stabilizes for large scales. See Proposition 5.15 for a general result for geodesic spaces.

1. Let $X$ be the Euclidean plane and let $C$ be an infinite line. Then $E(X, C, \sigma)$ is a set of cardinality two if and only if $\sigma > 0$. Note that $E(X, C, 0)$ is empty.

2. Let $X$ be the set of pairs of integers with the Manhattan distance, that is, the distance between $(x_1, y_1)$ and $(x_2, y_2)$ is $|x_1 - x_2| + |y_1 - y_2|$. Let $C$ be the points of the form $(n, 0)$ with $n \in \mathbb{Z}$. Then $E(X, C, \sigma)$ has cardinality two if $\sigma \geq 1$, and $E(X, C, \sigma)$ is empty otherwise.

3. Let $X$ be the subspace of the Euclidean plane consisting of two vertical lines at distance two, and two horizontal lines at distance one. Let $C$ consists of a single point. Then $E(X, C, \sigma)$ has cardinality 8 if $\sigma < 1$, 6 if $1 \leq \sigma < 2$, and 4 if $\sigma \geq 2$.

Corollary 5.13. For any pair of non-negative real numbers $\sigma \leq \sigma'$, the inclusion of $I_{\sigma'}$ into $I_{\sigma}$ induces a canonical function $\varphi_{\sigma, \sigma'} : E(X, C, \sigma) \rightarrow E(X, C, \sigma')$.

In particular, if $\sigma \leq \sigma' \leq \sigma''$ then

$$\varphi_{\sigma, \sigma''} = \varphi_{\sigma', \sigma''} \circ \varphi_{\sigma, \sigma'}.$$ 

Proof. Note that there is an isomorphism of directed systems $J_{\sigma} \rightarrow J_{\sigma'}$ given by $i = (\sigma, y) \mapsto j = (\sigma', y)$. In particular, if $i, j \in J_{\sigma}$ and $i \preceq j$, then there is a commutative diagram

$$C_i \xrightarrow{\rho_{i, j}} C_j$$

$$\Downarrow \rho_{i, i*} \Downarrow \rho_{j, j*}$$

$$C_{i*} \xrightarrow{\rho_{i*, j*}} C_{j*}$$

which, via the universal property of inverse limits, induces the morphism $\varphi_{\sigma, \sigma'}$. The first statement follows directly from Proposition 5.9. The uniqueness in the universal properties implies that $\varphi_{\sigma, \sigma''} = \varphi_{\sigma', \sigma''} \circ \varphi_{\sigma, \sigma'}$ if $\sigma \leq \sigma' \leq \sigma''$. \qed

Definition 5.14. The space of filtered ends $E(X, C)$ of the pair $(X, C)$ is defined as the direct limit

$$E(X, C) = \lim_{\sigma} E(X, C, \sigma)$$

of the system $(E(X, C, \sigma), \varphi_{\sigma, \sigma'} : \sigma, \sigma' \in [0, \infty))$. This system is called the direct ends-system for $(X, C)$. The number of filtered ends $e(X, C)$ of the pair $(X, C)$ is defined as the cardinality of $E(X, C)$.

5.3. Filtered ends in geodesic spaces. In this part, we prove two results. First, that for geodesic metric spaces the number of filtered ends does not depend on the scale, Proposition 5.15. We expect this result to hold for a broader class of metric spaces but a more general statement would not be discussed in this note.

Proposition 5.15. Let $X$ be a geodesic metric space, and let $C$ be a subset of $X$. For any $0 < \sigma \leq \sigma'$, the function $\varphi_{\sigma, \sigma'} : E(X, C, \sigma) \rightarrow E(X, C, \sigma')$ is a bijection. In particular, the induced function $E(X, C, \sigma) \rightarrow E(X, C)$ is a bijection.
The second result is that, for geodesic proper metrics spaces, if $C$ is compact then $E(X, C)$ is the standard set of ends of $X$. An analogous result in the development of filtered by Geoghegan can be found in [Geo08 Propositions 13.4.7 and 14.3.1].

**Proposition 5.16.** Let $X$ be a proper geodesic metric space and let $C$ be a non-empty compact subset of $X$, then there is a natural bijection $E(X, C) \to \text{Ends}(X)$ as defined in [BH99].

The proofs of Propositions 5.15 and 5.16 rely on the following definition and lemma.

**Definition 5.17** (The set of $C$-proper infinite $\sigma$-rays $\mathcal{P}(X, C, \sigma)$). An infinite $\sigma$-path $p$ in $X$ is an infinite sequence $x_0, x_1, \ldots$ of points in $X$ such that $\text{dist}(x_i, x_{i+1}) \leq \sigma$ for all $i$. For such a path $p$, if $x_0 \in C$ we say that $p$ starts at $C$; and we say that $p$ is $C$-proper $\sigma$-ray if $\{n: x_n \in C^{+\mu}\}$ is finite for every $\mu \geq 0$. The set of all $C$-proper infinite $\sigma$-paths starting at $C$ is denoted as $\mathcal{P}(X, C, \sigma)$.

**Lemma 5.18.** Let $\sigma \geq 0$. Then there is a surjective function $\psi_\sigma: \mathcal{P}(X, C, \sigma) \to E(X, C, \sigma)$ with the following properties:

1. If $\sigma' \geq \sigma$, then the following diagram commutes
   
   $\begin{array}{ccc}
   \mathcal{P}(X, C, \sigma) & \xrightarrow{\psi_\sigma} & \mathcal{P}(X, C, \sigma') \\
   \downarrow & & \downarrow \\
   E(X, C, \sigma) & \xrightarrow{\varphi_{\sigma, \sigma'}} & E(X, C, \sigma')
   \end{array}$

2. Suppose $p, q \in \mathcal{P}(X, C, \sigma)$ and for every $\mu \geq 0$, there are points $x$ of $p$ and $y$ of $q$ that are connected by a $\sigma$-path in $X - C^{+\mu}$. Then $\psi_\sigma(p) = \psi_\sigma(q)$.

**Proof.** First we define the function $\psi_\sigma: \mathcal{P}(X, C, \sigma) \to E(X, C, \sigma)$. Let $p$ be an element of $\mathcal{P}(X, C, \sigma)$, notice that for any $\mu \geq 0$, all but finitely many elements of $p$ belong to a unique element $A_\mu$ of $\mathcal{C}_\infty(X - C^{+\mu}, \sigma)$. Moreover, if $\mu' \geq \mu$, then $A_{\mu'} \subseteq A_\mu$. It follows that the collection $(A_\mu)$, and in particular $p$, determines a unique element of $E(X, C, \sigma)$ that we define as $\psi_\sigma(p)$. The two statements of the lemma now follow directly from the definition of $\psi_\sigma$ for $\sigma > 0$. \qed

**Proof of Proposition 5.15** If $X$ is bounded, then there is nothing to prove. Assume that $X$ is unbounded, and observe that $E(X, C, \sigma) \neq \emptyset$ for every $\sigma > 0$.

Let us prove that $\varphi_{\sigma, \sigma'}$ is injective. Recall $E(X, C, \sigma) = \lim_{t \in J_\sigma} C_t$ where $C_t = C_\infty(X - C^{+\mu}, \sigma)$ if $i = (\sigma, \mu)$. Let $A = (A_t)$ and $B = (B_t)$ be elements of $E(X, C, \sigma)$. Suppose that $\varphi_{\sigma, \sigma'}(A) = \varphi_{\sigma, \sigma'}(B)$. Let $k \in J_\sigma$, with $k = (\sigma, \mu)$. Let $t = (\sigma, \mu + \sigma')$ and observe that $A_t \subseteq A_k$ and analogously $B_t \subseteq B_k$. Now, let $m = (\sigma', \mu + \sigma')$ and observe that $A_t$ and $B_t$ are subsets of $A_m = B_m$; this last equality follows from the assumption $\varphi_{\sigma, \sigma'}(A) = \varphi_{\sigma, \sigma'}(B)$. Let $x \in A_t$ and $y \in B_t$. Then, since $x, y \in A_m$, there is a $\sigma'$-path $x = x_0, x_1, \ldots, x_n = y$ in $X - C^{+\mu + \sigma'}$. By choosing a geodesic between any two consecutive points in the $\sigma'$-path one sees that, there is a polygonal continuous path from $x$ to $y$ in $X - C^{+\mu}$. It follows that there is a $\sigma$-path from $x$ to $y$ in $X - C^{+\mu}$, and therefore $A_k$ and $B_k$ are the same $\sigma$-coarsely connected component of $X - C^{+\mu}$. Since $k \in J_\sigma$ was arbitrary, it follows that $A = B$.

Now let us prove that $\varphi_{\sigma, \sigma'}$ is surjective. By Lemma 5.18 it is enough to show that for every $p \in \mathcal{P}(X, C, \sigma')$ there is a $q \in \mathcal{P}(X, C, \sigma)$ such that $\psi_{\sigma'}(p) = \psi_\sigma(q)$. 

Let $m$ be such that $\sigma' \leq m \sigma$. Let $p$ be the sequence $x_0, x_1, \ldots$ and suppose it is an element of $\mathcal{P}(X, C, \sigma')$. Since $X$ is geodesic, there is a finite $\sigma$-path $q_i$ from $x_i$ to $x_{i+1}$ of length at most $m$ for all $i$. Then the $\sigma$-path $q$ formed by $q_0, q_1, \ldots$ is an element of $\mathcal{P}(X, C, \sigma)$. It is a consequence of the second part of Lemma 5.15 that $\psi_{\sigma'}(p) = \psi_{\sigma}(q)$. 

Proof of Proposition 5.16. Let $x_0 \in C$. Any $C$-proper coarse ray $(x_i)_{i \in \mathbb{N}}$ induces a proper ray $r : [0, \infty) \to X$ such that $r(0) = x_0$ and $r(i) = x_i$ for each $i \in \mathbb{Z}_+$. By [BH99, I.8 Lemma 8.29], any two geodesic rays represent the same element of $\text{Ends}(X)$ if for every $\mu > 0$ they can be connected by a $\sigma$-path in $X - C^{\epsilon, \mu}$. Then Lemma 5.18 implies that there is a natural bijection $E(X, C, 1) \to \text{Ends}(X)$. The result follows applying Proposition 5.15.

5.4. Equivalence of filtered ends of pairs $\hat{e}(G, P)$ with Bowditch’s coends.

Definition 5.19. Let $G$ be a finitely generated group and let $P$ be a subgroup. Let $X$ be a Cayley graph of $G$ with respect to a finite generating set and with the edge path metric.

1. The number of filtered ends $\hat{e}(G, P)$ is defined as the cardinality of $E(X, P)$.

2. (Bowditch’s coends) The number of coends $\hat{e}(G, P)$ is defined as follows. Let $\mathcal{J}_0(P)$ be the collection of non-empty connected $P$-invariant subgraphs of $X$ with compact quotient under $P$. Given $A \in \mathcal{J}_0(P)$, let $\mathcal{C}_\infty(A)$ be the set of components of $X - A$ that are not contained in any other element of $\mathcal{J}_0(P)$. If $A \subseteq B$ and both are elements of $\mathcal{J}_0(P)$, then there is a surjective map $\mathcal{C}_\infty(B) \to \mathcal{C}_\infty(A)$. The cardinality of the inverse limit of the system $(\mathcal{C}_\infty(A) : A \in \mathcal{J}_0(P))$ is the number of coends of the pair $(G, P)$.

Remark 5.20. Observe that $P^{\epsilon, \mu}$, the $\mu$-neighborhood of $P$, is an element of $\mathcal{J}_0(P)$. Moreover, $\mathcal{C}_\infty(P^{\epsilon, \mu})$ coincides with $C_\infty(X - P^{\epsilon, \mu}, \sigma)$ for any $\sigma > 0$ (see Proposition 5.15). Since $(C_\infty(X - P^{\epsilon, \mu}, \sigma) : \mu > 0)$ is a cofinal system of $(\mathcal{C}_\infty(A) : A \in \mathcal{J}_0(P))$, we conclude that the number of filtered ends and the number of coends of $(G, P)$ coincide.

5.5. Induced functions. Let $f : X \to Y$ be a $(\lambda, \epsilon)$-quasi-isometric map. Let $C \subseteq X$ and $D \subseteq Y$ and suppose $D \subseteq f(C)^{r\tau}$ for some $r \geq 0$.

For $a \geq 0$, let

$$a^f = \lambda a + \epsilon.$$ (12)

Let $\alpha = (0, \lambda(\epsilon + r))$ and $\beta = (\epsilon, 0)$, consider the order preserving bijective function

$$I_\alpha \to I_\beta, \quad (\sigma, \mu) \mapsto \left(\lambda \sigma + \epsilon, \frac{1}{\lambda} \mu - \epsilon - r\right).$$

For $i \in I_\alpha$, let $i^f$ denote its image by this function.

Let $C_i$ and $D_i$ denote the objects of the $I_\alpha$-ends-system and $I_\beta$-ends-system of $(X, C)$ and $(Y, D)$ respectively.

Lemma 5.21. For each $i = (\sigma, \mu) \in I_\alpha$, there is a unique function $f_i : C_i \to D_i$, that satisfies $f(A) \subseteq f_i(A)$ for any $A \in C_i$. 

Proof. Since \( f \) is a \((\lambda, \epsilon)\)-quasi-isometric map, if \( \text{dist}(x, C) \geq \mu \) then \( \text{dist}(f(x), f(C)) \geq \mu/\lambda - \epsilon \geq r \) since \( \mu \geq \lambda (\epsilon + r) \). Since \( D \subseteq f(C)^{+r} \), it follows that \( \text{dist}(f(x), D) \geq \mu/\lambda - \epsilon - r \geq 0 \). Hence the restriction \( f: (X - C^{+\mu}) \to (Y - f(C)^{+\mu}(\lambda/\epsilon - r)) \) is a well-defined \((\lambda, \epsilon)\)-coarse Lipschitz map and Proposition 5.6 can be invoked to obtain \( f_i \). Since \( D_{i,j} \) is a collection of disjoint subsets of \( Y \), the function \( f_i \) is unique by Remark 5.8. \( \square \)

**Lemma 5.22.** For any \( i, j \in I_{\alpha} \), if \( i \leq j \) then there is a commutative diagram

\[
\begin{array}{ccc}
C_i & \xrightarrow{f_i} & D_{i,j} \\
\downarrow & & \downarrow \\
C_j & \xrightarrow{f_j} & D_{j,i}
\end{array}
\]

where the vertical arrows are transition functions of the \((\alpha\text{-ends-system}) \) and \((\alpha^f\text{-ends-system}) \) of \((X, C) \) and \((Y, D) \) respectively.

**Proof.** This is indeed a commutative diagram since \( D_{j,i} \) is a collection of pairwise disjoint sets, and therefore there is only function \( \xi: C_i \to D_{j,i} \) with the property that \( f(A) \subseteq \xi(A) \) for any \( A \in C_i \), see Remark 5.8. \( \square \)

**Definition 5.23** (Induced functions). Let \( f: X \to Y \) be a \((\lambda, \epsilon)\)-quasi-isometric map. Let \( C \subseteq X \) and \( D \subseteq Y \) and suppose \( D \subseteq f(C)^{+r} \) for some \( r \geq 0 \). Let \( 0 < a \leq b \).

The function \( f_{a,b}: E(X, C, a) \to E(Y, D, b) \) is defined as the composition

\[
f_{a,b} = \varphi_{a,b} \circ f_{a,a^f},
\]

where \( \varphi_{a,b}: E(Y, D, a^f) \to E(Y, D, b) \) is a transition function provided by Corollary 5.12. The function \( f_{a,a^f} \) is defined as follows. The commutative squares (13) for \( i \) and \( j \) in \( J_{(a,\lambda(\epsilon+r))} \) induce a function

\[
f_{a,a^f}: E(X, C, a) \to E(Y, D, a^f)
\]

by taking inverse limits of the \( D_{i,j} \)'s and then the inverse limits of the \( C_i \)'s. Indeed, the inverse limits induce a function

\[
f_{a,a^f}: E(X, C, J_{(a,\lambda(\epsilon+r))}) \to E(Y, D, J_{(a^f, 0)}),
\]

and by Remark 5.11

\[
E(X, C, a) = E(X, C, J_{(a,\lambda(\epsilon+r))}) \quad \text{and} \quad E(Y, D, J_{(a^f, 0)}) = E(Y, D, a^f).
\]

**Proposition 5.24.** Let \( f: X \to Y \) be a \((\lambda, \epsilon)\)-quasi-isometric map. Let \( C \subseteq X \) and \( D \subseteq Y \) and suppose \( D \subseteq f(C)^{+r} \) for some \( r \geq 0 \).

If \( a \leq c, a^f \leq b, c^f \leq d \) and \( b \leq d \), then there is a commutative diagram

\[
\begin{array}{ccc}
E(X, C, a) & \xrightarrow{f_{a,b}} & E(Y, D, b) \\
\downarrow & & \downarrow \\
E(X, C, c) & \xrightarrow{f_{c,d}} & E(Y, D, d)
\end{array}
\]

where the vertical arrows are transition functions of the \((\alpha\text{-ends-system}) \) and \((\alpha^f\text{-ends-system}) \) of \((X, C) \) and \((Y, D) \) respectively.
Proposition 5.25. For the identity map \( \mathrm{id} : (X, C) \to (X, C) \) and \( 0 < a \leq b \), the function \( \mathrm{id}_{a,b} : E(X, C, a) \to E(X, C, b) \) is \( \varphi_{a,b} \).

Proof. It is enough to verify that for \( a \leq b \), there is a commutative diagram

\[
\begin{array}{ccc}
E(X, C, a) & \xrightarrow{f_{a,a'}} & E(Y, D, a') \\
\downarrow & & \downarrow \\
E(X, C, b) & \xrightarrow{f_{b,b'}} & E(Y, D, b')
\end{array}
\]

where the vertical arrows are transition functions of the direct ends-system of \((X, C)\) and \((Y, D)\) respectively. This follows Definition 5.23 and the commutative squares \( \square \). \( \blacksquare \)

Proposition 5.26. Let \( f : X \to Y \) be a function at distance at most \( r \) from the identity. If \( 0 < a \) and \( a' \leq b \), then \( f_{a,b} : E(X, C, a) \to E(X, C, b) \) is \( \varphi_{a,b} \).

Proof. It is enough to prove that \( f_{a,a'} : E(X, C, a) \to E(X, C, a') \) is \( \varphi_{a,a'} \). Consider \( f : X \to X \) as a \((1,2r)\)-quasi-isometric map from the identity and let \( \sigma > 0 \). Consider the induced functions \( f_i : C_i \to C_i \) for \( i \in J_{a'} \); here \( i \in J_{(0,3r)} \) and if \( i = (\sigma, \mu) \) then \( i' = (\sigma + 2r, \mu - 3r) \).

By definition, the functions \( f_{a,a'} \) and \( \varphi_{a,a'} \) are obtained as the inverse limits of the functions \( f_i : C_i \to C_i \) and \( \rho_{i',i} \) respectively. To complete the proof we show that \( f_i = \rho_{i',i} \). By definition of \( f_i \), \( f(A) \subseteq f_i(A) \) for any \( A \subseteq C_i \). The assumption on \( f \) implies that \( \mathrm{hdist}(A, f(A)) \leq r \). Since \( f_i(A) \) is an unbounded \((\sigma + 2r)\)-coarsely connected component of \( X - C^{\mu - 3r} \) and \( \mathrm{hdist}(A, f(A)) \leq r \), it follows that \( A \subseteq f_i(A) \). By Proposition 4.5, \( f_i = \rho_{i',i} \). \( \blacksquare \)

Proposition 5.27. Let \( f : X \to Y \), \( g : Y \to Z \) and \( h = g \circ f \) be a quasi-isometric maps with some chosen constants. Let \( C \subseteq X \), \( D \subseteq Y \) and \( E \subseteq Z \) and suppose \( D \subseteq f(C)^{+r} \), \( E \subseteq g(D)^{+r'} \) and \( E \subseteq h(C)^{+r''} \) for some \( r, r', r'' \geq 0 \).

Let \( a > 0 \). Let \( b, c \) and \( d \) denote \( a^i, a^0f \), and \((a^i)^g \) respectively. Then, by choosing the quasi-isometric constants for \( h \) appropriately, one can assume that \( c \leq d \), and there is a commutative diagram,

\[
\begin{array}{ccc}
E(Y, D, b) & \xrightarrow{g_{b,d}} & E(Z, E, d) \\
\downarrow & & \uparrow \\
E(X, C, a) & \xrightarrow{(g_{a,\varepsilon})_{a,c}} & E(Z, E, c)
\end{array}
\]

where the vertical arrow into \( E(Z, E, d) \) is a transition function of the direct ends-system for \((Z, E)\).
Proof. If $i, j \in I_\alpha$ and $i \leq j$ then $i^{gf} \leq (ij)^f$ and there is a commutative diagram

$$
\begin{array}{ccc}
C_i & \xrightarrow{f_i} & D_{ij} \\
\downarrow (gf)_i & & \downarrow (gf)_{ij} \\
E_{i^{gf}} & \xrightarrow{(gf)_{ij}} & E_{(ij)^f} \\
\downarrow (gf)_{ij} & & \downarrow (gf)_{ij} \\
C_j & \xrightarrow{f_j} & D_{j} \\
\end{array}
$$

where all vertical arrows and all arrows in the front face are transition functions, and all other arrows are induced by $f$, $g$ and $g \circ f$ accordingly. Indeed, observe that the back face and side faces are particular cases of Lemma 5.22, the front face is a square diagram of transition functions in the ends-system for $(Z, E)$, so all these faces are commutative diagrams. That the top and bottom faces are commutative diagrams follows from the same argument as in Lemma 5.22. The proposition follows from the commutative cubes taking the corresponding inverse limits.

5.6. Filtered ends as a functor. Consider the category $\text{QPMet}$ whose objects are pairs $(X, C)$ where $X$ is a metric space and $C$ is a subspace; morphisms $(X, C) \to (Y, D)$ are quasi-isometric maps $f: X \to Y$ such that $D \subseteq f(C)^{+r}$ for some $r \geq 0$; and the composition law is the standard composition of functions.

Proposition 5.28. There is a covariant functor

$$
E: \text{QPMet} \to \text{Sets}
$$

that maps a pair $(X, C)$ to $E(X, C)$.

Proof. Let $f: (X, C) \to (Y, D)$ be a morphism of $\text{QPMet}$. Proposition 5.24 allows us to define the morphism $E(f): E(X, C) \to E(Y, D)$ as follows. For any $a > 0$ and $b > a^f$ there is a function $f_{a,b}: E(X, C, a) \to E(Y, D, b)$, and then taking direct limits we obtain $E(f)$. That $E(\text{id}_{E(X, C)}) = \text{id}_{E(X, C)}$ is a consequence of Proposition 5.25. For morphisms $f: (X, C) \to (Y, D)$ and $g: (Y, D) \to (Z, E)$, that $E(g \circ f) = E(g) \circ E(f)$ is a consequence of Proposition 5.27.

The following proposition follows directly from Proposition 5.26.

Proposition 5.29. Let $f: X \to X$ be a function at finite distance from the identity. Then $E(f)$ is the identity map of $E(X, C)$.

5.7. Quasi-isometry invariance of Filtered Ends.

Proposition 5.30. Let $X$ and $Y$ be metric spaces, and $C \subseteq X$ and $D \subseteq Y$. If $f: X \to Y$ is a quasi-isometry such that $\text{hdist}(f(C), D)$ is finite, then $E(f): E(X, C) \to E(Y, D)$ is a bijection.

Proof. Consider $g: Y \to X$ a quasi-isometry such that $g \circ f$ and $f \circ g$ are at finite distance from the corresponding identity functions. Note that $g \circ f$ and $f \circ g$ induced morphisms $(X, C) \to (X, C)$ and $(Y, D) \to (Y, D)$ of $\text{QPMet}$ respectively. By Proposition 5.28 both $E(g) \circ E(f)$ and $E(f) \circ E(g)$ are the identity morphisms of $E(X, C)$ and $E(Y, D)$ respectively and in particular they are bijections. Therefore $E(f)$ is a bijection.
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