Hilbert schemes of a surface and Euler characteristics

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September 22, 1998

Abstract

We use basic algebraic topology and Ellingsrud-Stromme results on the Betti numbers of punctual Hilbert schemes of surfaces to compute a generating function for the Euler characteristic numbers of the Douady spaces of “n-points” associated with a complex surface. The projective case was first proved by L. Göttsche.

0 Introduction

This note is dedicated to the determination of the Euler characteristic numbers $e(X^{[n]})$ of the Douady spaces of “n-points” $X^{[n]}$ associated with a complex surface $X$; see Theorem 2.1.

For $X$ algebraic (the Douady space is replaced by the Hilbert scheme) see [3], [10] and [4]. In the projective case L. Göttsche [9] used P. Deligne’s solution to the Weil Conjectures and Ellingsrud-Stromme’s determination of the Betti numbers of punctual Hilbert schemes [7]. Later, Göttsche-Soergel [10] proved the quasi-projective case using the methods of perverse sheaves. J. Cheah [4] obtained similar results using the theory of Mixed Hodge structures and [7].

The proof presented here, valid without the algebraicity restriction on $X$, uses [7] and elementary algebraic topology.

Acknowledgments. I wish to thank the Max-Planck-Institut für Mathematik in Bonn for its warm hospitality and stimulating environment. I would like to thank Pavlos Tzermias for useful conversations.

1 Notation and preliminaries

The topological Euler characteristic of a topological space $Y$, with respect to singular cohomology with compact supports, is denoted by $e(Y)$. If a statement involves this quantity $e(Y)$, it is understood that part of the statement consists of the assertion that $e(Y)$ is defined.

The term complex surface denotes, here, a smooth, connected and complex analytic surface with separable topology.

Let $X$ be a complex surface and $n \in \mathbb{N}^*$. Denote by $X^{[n]}$ the Douady Space of zero-dimensional analytic subspaces of $X$ of length $n$; in the algebraic case this notion is replaced by the one of the Hilbert scheme; note that the “local differential geometries” of the Hilbert scheme and of the Douady spaces run parallel so that one can translate most of the proofs in [3] in the analytic context. The spaces $X^{[n]}$ are connected, complex manifolds of dimension $2n$. The corresponding Barlet space (cf. [7]) of zero-dimensional cycles in $X$ of total multiplicity $n$ can be identified with the symmetric product $X^{(n)} = X^n / \Sigma_n$, where $\Sigma_n$ denotes the symmetric group over $n$ elements and it acts by permuting the

*Partially supported by N.S.F. Grant DMS 9701779 and by an A.M.S. Centennial Fellowship.
factors of $X^n$. There is a natural proper holomorphic map $\pi : X^{[n]} \to X^{(n)}$ which is a resolution of the singularities of $X^{(n)}$ (cf. [4]); we call this morphism the Douady-Barlet morphism.

There is a natural stratification for this morphism given by the partitions of $n$. Let $P(n)$ denote the set of partitions of the natural number $n$:

$$P(n) := \left\{ \nu = (\nu_1, \ldots, \nu_k) \mid k \in \mathbb{N}^*, \nu_j \in \mathbb{N}^*, \nu_1 \geq \nu_2 \geq \ldots \geq \nu_k > 0, \sum_{j=1}^{k} \nu_j = n \right\}.$$ 

The cardinality of $P(n)$ is denoted by $p(n)$. Every partition $\nu = (\nu_1, \ldots, \nu_k)$ can also be represented by a sequence of $n$ nonnegative integers:

$$\nu = (\nu_1, \ldots, \nu_k) \iff \{\alpha_1, \ldots, \alpha_n\},$$

where $\alpha_i$ is the number of times that the integer $i$ appears in the sequence $(\nu_1, \ldots, \nu_k)$. Clearly $n = \sum_{i=1}^{n} \nu_i$. The number $k$ associated with $\nu$ is called the length of $\nu$ and it is usually denoted by $\lambda(\nu)$. The set $P(n)$ is in natural one to one correspondence with a set of mutually disjoint subsets of $X^{(n)}$ which are locally closed for the Zariski topology on analytic spaces: associate with every $\nu \in P(n)$ the space

$$S^\nu_X = S^\nu_{(\nu_1, \ldots, \nu_k)}X := \left\{ \sum_{j=1}^{k} \nu_j x_j \in X^{(n)} \mid x_{j_1} \neq x_{j_2} \text{ if } j_1 \neq j_2 \right\}.$$ 

The space $X^{(n)}$ is the disjoint union of these locally closed subsets:

$$X^{(n)} = \coprod_{\nu \in P(n)} S^\nu_X. \quad (1)$$

Define $\tilde{S}^\nu_X := \pi^{-1}(S^\nu_X)$. The space $X^{[n]}$ is the disjoint union of these locally closed subsets:

$$X^{[n]} = \coprod_{\nu \in P(n)} \tilde{S}^\nu_X. \quad (2)$$

Given $m \in \mathbb{N}^*$, the analytic isomorphism class of the fiber over a point $x \in X = S^{m}_{(m)}X$ of the Douady-Barlet morphism $\pi : X^{[m]} \to X^{(m)}$ are independent of the surface $X$ and of the point $x \in X$; since they are isomorphic to the fiber in question when $X$ is the affine plane $\mathbb{A}^2 = \mathbb{C}^2$ and the point $x$ is the origin, we denote them by $\mathbb{A}^2_{[0]}$.

**Theorem 1.1** (Cf. [3]) $e\left(\mathbb{A}^2_{[0]}\right) = p(m)$.

Results of Fogarty’s [5] (see also [3]), adapted in this analytic context, ensure that the restriction of $\pi, \pi : \tilde{S}^\nu_X \to S^\nu_X$, is a locally trivial fiber bundle in the analytic Zariski topology with fiber $F_\nu$,

$$F_\nu \simeq \prod_{j=1}^{k} \mathbb{A}^{2(c_j)}_{[0]}.$$ \quad (3)

We shall deal only with complex analytic spaces $Z$ for which the singular cohomology with compact supports satisfies $\dim_{\mathbb{Q}} H^*_c(W, \mathbb{Q}) < \infty$; in particular the Euler numbers $e(Z)$ are defined. Alexander-Spanier cohomology, singular cohomology and sheaf cohomology with constant coefficients (all with compact supports) are naturally isomorphic to each other on the spaces that we will consider.

Alexander-Spanier’s Theory has good properties with respect to closed subsets. We shall need two of these properties. In what follows we assume that $W, Y, Z$ and $Z_i$ are analytic spaces with finite
dimensional rational cohomology with compact supports. Let $Z = \bigsqcup_{i=1}^n Z_i$ where $\{Z_i\}$ is a finite collection of mutually disjoint locally closed subsets of $Z$. Then

$$e(Z) = \sum_{i=1}^n e(Z_i).$$

(4)

Let $f : W \to Z$ be a continuous map which is locally trivial, with fiber $Y$, with respect to the stratification $\Sigma_i$, i.e. $f^{-1}(Z_i) \simeq \Sigma_i \times Y$ for every $i = 1, \ldots, n$. Then

$$e(W) = e(Y) e(Z).$$

(5)

It should be now clear what we should be aiming at: reduce the computation of $e(X^{[n]})$ to the computation of $e\left(\overline{S_n^\nu X}\right)$ via (4). The numbers $e\left(\overline{S_n^\nu X}\right)$ will be computed using (5) and (3). The resulting sum will turn out to be a multinomial expansion.

2 Statement of the result and the numbers $e\left(\overline{S_n^\nu X}\right)$

The following is the main result of this note.

**Theorem 2.1** Let $X$ be a smooth complex surface such that $\dim Q H^*_c(X, Q) < \infty$ and with topological Euler characteristic $e(X)$. Let $X^{[n]}$ be the associated Douady spaces of “$n$-points.” The following is a generating function for $e\left(X^{[n]}\right)$:

$$\sum_{n=0}^\infty e\left(X^{[n]}\right) q^n = \prod_{k=1}^\infty \left(\frac{1}{1-q^k}\right)^{e(X)}. $$

(6)

**Remark 2.2** The assumptions of the theorem are satisfied for any algebraic surface.

The proof will be preceded by some elementary calculations.

**The structure of $S_{\nu}^n X$.** Let $\nu = (1, \ldots, 1) \in P(n)$. The stratum $S_{(1, \ldots, 1)}^n X$ is the unique Zariski open subset in $X^{(n)}$ belonging to the stratification (4). Let $D^n$ be the big diagonal in the cartesian product $X^n$. The stratum $S_{(1, \ldots, 1)}^n X$ is the quotient of $X^n \setminus D^n$ under the natural free action of the symmetric group over $n$ elements $\Sigma_n$. It follows that $S_{(1, \ldots, 1)}^n X$ is smooth of complex dimension $2n$. Each stratum $S_{\nu}^n X$, $\nu \in P(n)$, is built from these basic strata $S_{(1, \ldots, 1)}^m X$ for $m \leq n$. Here is how. Let $\nu = (\nu_1, \ldots, \nu_k) \leftrightarrow \{\alpha_1, \ldots, \alpha_n\}$ be the same partition in the two different pieces of notation. There is a natural isomorphism:

$$S_{\nu}^n X \simeq \left( \prod_{i=1, \alpha_i \neq 0}^n \overline{S_{(1, \ldots, 1)}^{\alpha_i} X} \right) \setminus \Delta,$$

(7)

where $\Delta$ is the closed set

$$\Delta := \left\{ (C_1, \ldots, C_l) \in \prod_{i=1, \alpha_i \neq 0}^n S_{(1, \ldots, 1)}^{\alpha_i} X \mid \text{Supp}(C_h) \cap \text{Supp}(C_j) \neq \emptyset \text{ for some } h \neq j \right\}.$$

It follows that each stratum $S_{\nu}^n X$ is smooth, connected and of complex dimension $2k = 2\lambda(\nu)$.

**The numbers $e(S_{\nu}^n X)$.** Recalling the convention $0! = 1$, we have the following elementary
Lemma 2.3 Let $X$ be a smooth complex analytic surface with $\dim_{\mathbb{Q}} H^*(X, \mathbb{Q}) < \infty$, $e = e(X)$, $n \in \mathbb{N}^*$ and $P(n) \ni \nu = (\nu_1, \ldots, \nu_k) \leftrightarrow \{\alpha_1, \ldots, \alpha_n\}$. Then

$$e(S^n \nu X) = \frac{1}{\alpha_1! \cdots \alpha_n!} e(e - 1) \ldots (e - (\lambda(\nu) - 1)).$$

(8)

Proof. We first compute $e(S^m(1, \ldots, 1) X)$ for every $m$. Since $S^m(1, \ldots, 1) X = (X^m \setminus D^m)/\Sigma_m$, and the action is free, we have that, by the multiplicativity of the number $e$ under finite covering maps, $m! e(S^m(1, \ldots, 1) X) = e(X^m \setminus D^m)$.

We now prove that $e(X^m \setminus D^m) = e(e - 1) \ldots (e - (m - 1))$ by induction on $m$. The case $m = 1$ is trivial. Assume that the above formula is true for $m - 1$ and let us prove it for $m$. We have a commutative diagram

$$
\begin{array}{ccc}
X^m \setminus D^m & \ni & (X^{m-1} \setminus D^{m-1}) \times X \\
\downarrow \text{proj}_{1, \ldots, m-1} & \simeq & \text{proj}_1 \\
X^{m-1} \setminus D^{m-1}
\end{array}
$$

where the horizontal map is the natural open embedding, the vertical map is the projection to the first $(m - 1)$ coordinates and the “south-west” map is the projection onto the first factor. The map $\text{proj}_1$ admits $(m - 1)$ natural sections $(x_1, \ldots, x_{m-1}) \mapsto (x_1, \ldots, x_{m-1}, x_l)$, where $l = 1, \ldots, m - 1$. The images of these sections are pairwise disjoint. Their union is a closed subset of $(X^{m-1} \setminus D^{m-1}) \times X$ isomorphic to $(m - 1)$ disjoint copies of $X^{m-1} \setminus D^{m-1}$; the complement of this set is the open subset which we identify with $X^m \setminus D^m$ via the open immersion in the diagram above. The additive and multiplicative properties of the number $e$ and the inductive hypothesis give that

$$e(X^m \setminus D^m) = [e(e - 1) \ldots (e - (m - 1 - 1))e] - [e(e - 1) \ldots (e - (m - 1 - 1))(m - 1)]$$

(9)

$$= e(e - 1) \ldots (e - (m - 1)).$$

(10)

This proves that

$$m! e(S^m(1, \ldots, 1) X) = e(X^m \setminus D^m) = e(e - 1) \ldots (e - (m - 1)).$$

(11)

As to the case of $S^n_X$ we note that, because of the isomorphism given in (7), this space is the quotient of $X^{-1+\ldots+\nu_1\ldots+\nu_n=\lambda(\nu)} \setminus D^{\lambda(\nu)}$ under the free action of the group $\Sigma_{\alpha_1} \times \ldots \times \Sigma_{\alpha_n}$ induced by the individual actions of the groups $\Sigma_{\alpha_i}$ on the factors $X^{\alpha_i}$. The formula follows from the multiplicativity of Euler characteristics under finite covering maps. □

Remark 2.4 It is an amusing exercise to derive Macdonald formula for the Euler characteristics of symmetric products using Lemma 2.3 and the multinomial expansion; i.e. prove that

$$\sum_{n=0}^{\infty} e\left(\begin{array}{c}
X^{(\alpha)}
\end{array}\right) q^n = \left(\frac{1}{1 - q}\right)^{e(X)}.$$

(12)

The numbers $\nu(S^n_X)$. By virtue of Lemma 2.3, Theorem 1.1 and (7), the basic additive and multiplicative properties of cohomology with compact support give the following

Lemma 2.5 Let $X$ be a complex surface with finite Betti numbers. Then

$$e(S^n_X) = e\left(\prod_{j=1}^{\lambda(\nu)} E_{j,0}^{\nu_j}\right) e(S^n_X) = p(\nu) \ldots p(n) \frac{1}{\alpha_1! \ldots \alpha_n!} e(e - 1) \ldots (e - (\lambda(\nu) - 1))$$

$$= p(\nu) \left(\sum_{\alpha_1 \ldots \alpha_n} \frac{1}{\alpha_1! \ldots \alpha_n!} e(e - 1) \ldots (e - (\lambda(\nu) - 1))\right).$$

(13)
3 Proof of Theorem 2.1

Proof of Theorem 2.1. By virtue of (3), (4) and Lemma 2.3 we have

\[ e(X^{[n]}) = \sum_{\nu \in P(n)} e(S_{\nu}X) \]

\[ = \sum_{k=1}^{n} \sum_{\nu \in P(n)} \frac{p(1)^{\alpha_1} \ldots p(n)^{\alpha_n}}{\alpha_1! \ldots \alpha_n!} e(e - 1) \ldots (e - (k - 1)). \]

The last summand is the expansion of the r.h.s. of equation (3). Let us check this for \( X \) homotopically equivalent to \( X \). Is \( X \) for the differentiable structure of \( X \)?

Remark 3.1 The generating function of Theorem 2.1 exhibits modular behavior (cf. [3]).

Remark 3.2 If \( X \) is not algebraic, then the approach in [4] cannot be used to prove Theorem 2.1. The paper [10] restricts itself to the algebraic context. However, that restriction is unnecessary. This is remarked in [5], whose main purpose is to prove the necessary decomposition theorem in the form of an explicit quasi-isomorphism of complexes.

Remark 3.3 For \( X \) algebraic the numbers \( e(X^{[n]}) \) and the orbifold Euler characteristics \( e(X^n, \Sigma_n) \) coincide. Similarly, we see that because of Theorem 2.1 this “coincidence” occurs for any complex surface, i.e. not necessarily algebraic. This is explained by the fact that the groups \( K(X^{[n]} \otimes \mathbb{Q}) \) and the \( \Sigma_n \)-Equivariant K-Theory of \( X^n \), \( K_{\Sigma_n}(X^n) \otimes \mathbb{Q} \), are naturally isomorphic. This fact is due to B. Bezrukavnikov and V. Ginzburg [2] and, independently, to L. Migliorini and myself [5].

Question 3.4 The numbers \( e(X^{[n]}) \) are invariant in the class of complex analytic smooth surfaces homotopically equivalent to \( X \). Is \( X^{[n]} \) a topological invariant of \( X \)? To be precise, let \( X \) and \( Y \) be complex surfaces homeomorphic to each other; is \( X^{[n]} \) homeomorphic to \( Y^{[n]} \)? Is \( X^{[n]} \) an invariant for the differentiable structure of \( X \)?
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