A Relativistic Extension of Cowling’s Theorem.

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Abstract. Cowling’s theorem within the framework of non relativistic dynamo theory states that an axisymmetric magnetic field can not supported by dynamo action due to an axisymmetric conducting fluid flow. In this work we shall examine whether effects of spacetime curvature may modify the conclusion of this theorem. We study an axially symmetric magnetic field $B$ generated by an axisymmetric conducting fluid on a non singular static, spherically symmetric and asymptotically flat with one end spacetime. For such background geometry at first we derive a set of equations describing the evolution of the poloidal and toroidal components of the magnetic field generated by an axisymmetric conducting fluid flow. The background curvature influence the evolution of the poloidal and toroidal component via the Ricci curvature and non local contributions manifesting themselves as the gradient in the magnitude of the timelike and axial Killing vector fields. Despite the presence of those contributions, we shall show that an axisymmetric magnetic field can not maintained by dynamo action as long as:

1) The axially symmetric fluid flow is divergence free.
2) The background geometry correspond to a static spherical star of constant density with compactness ratio $\varepsilon = \frac{2GM}{c^2R}$, in the range $\varepsilon \in [0, \frac{2}{9}]$.

For an arbitrary static spherical stellar model the conclusion remains intact provided specific integrals involving the eq of state are negative definite.

1. Introduction.
Astrophysical and cosmological observations spanning a period of about a century lead to the conclusion that we may live in a magnetized universe. Every cosmic object that has been placed under observational scrutiny exhibits some form of magnetic activity. Cosmic structures of planetary, stellar, galactic, cluster and super-cluster scale possess their own magnetic fields. The properties of the observed fields vary substantially from one cosmic object to another. A dipole like structure approximately of 1 G for our own earth and planets, to gigantic ($10^8 - 10^{12}$)G fields of pulsars and $(10^{14} - 10^{15})$G field of magnetars and anomalous $X-ray$ pulsars. The magnetic fields of galactic structures are in general very weak, they are in the $\mu$G range, but they exhibit coherence and fluctuating component. For instance the field of the Milky way is few $\mu$G coherent on length scales over $Kpc$ but it is accompanied by a secondary fluctuating component over length scales $\simeq 100 pc$, the characteristic length scale of inter stellar turbulence. For an update of the observed properties and detection of cosmic magnetic fields, consult [1, 2, 3]. The existence of those fields raises some delicate questions: What is their origin? Are they subject
to dissipation? Are they manufactured via processes taking place within the cosmic bodies or do they originate in processes taking place in the early universe? Does there exist a single unified principle that underlies their existence? Are they telling us anything about the large scale structure of the universe? Those issues are open and constitute subjects of scientific debate. However there is a general consensus amongst the scientific community that inductive effects due to the motion of highly conducting fluids and plasmas play an important role in cosmic magnetism. The idea that conducting fluids may be responsible for large scale magnetism goes back to Larmor. Motivated by Hale’s discovery, of the solar magnetic field, Larmor in 1918 proposed that the motion of a conducting fluid in the presence of a magnetic field can act as a mechanism for the generation and maintenance of a large scale magnetism. Even though at first Larmor proposal offered a natural framework capable of explaining the solar field, Cowling in a landmark paper [4, 5] presented arguments showing that an axisymmetric magnetic field can not supported by dynamo action due to an axisymmetric conducting fluid flow. His conclusion verified by a number of independent investigations and within the framework of non relativistic dynamo theory, the statement that an axisymmetric magnetic field can not supported by a dynamo action due to a conducting fluid is referred as Cowling’s theorem. For a recent review regarding the status of this theorem consult [6].

The theorem originally has been formulated and proven within the framework of a non relativistic setting and as it has been emphasized in [6] it is centered in the long time behavior of solutions of a system of parabolic eqs. On the other hand, for a large class of astrophysical and cosmological settings, the observed fields find themselves in regions where the curvature of the spacetime is not any longer weak, case of pulsars or early universe and possibly the topology of the underlying spacetime may be non trivial case of multiply connected universes, wormholes etc. If the observed fields are maintained and generated by the motion of conducting fluids it is natural to examine the impact of the spacetime curvature and non trivial topology upon such generation process. However such program is meet with difficulties. In an arbitrary spacetime electromagnetic processes are described via the covariant form of Maxwell’s eqs which involve as primary variables the components of the Maxwell tensor $F_{\mu\nu}$ and the four current $J^\mu$. Moreover and in contrast to Minkowski spacetime where the Poincare group singles out the family of inertial observers, in an arbitrary spacetime preferred families of observers may not exist and thus the concept of the electric or magnetic fields become observer dependent. One has to resort to a covariant formulation of dynamo theory. Such formulation should involve only covariant objects such as the components of the Maxwell tensor and invariant properties of the fluid flow and is discussed in details in [7]. In this work we shall adopt the formalism of [7] for the particular case where the background correspond to a static spherically symmetric and asymptotically flat with one end spacetime. Such spacetime admits a well defined family of preferred observers the Killing observers and this property allows us to formulate the dynamo eqs in close analogy to the non relativistic setting. We derive a system of eqs describing an axially symmetric magnetic field generated by an axially symmetric fluid flow. Via those eqs in section II, we address the issue whether an axisymmetric fluid flow taking place in a spherical region that may be thought as the interior of a compact star, can support an axisymmetric magnetic field. Even though our model is restrictive, nevertheless it possess the advantage that it makes clear the influence of the background curvature upon the dynamo process. We shall show in sections II and III, that the background curvature influence the evolution of an axisymmetric magnetic field via the Ricci curvature as well as other non local terms involving the gradients of the timelike and-or axial Killing field of the background geometry. Despite the presence of those terms we shall show in section III that Cowling’s theorem, at least for the model considered, remains intact. In the next section we shall formulate the dynamo eqs and shall specify our model in more details.
2. Relativistic Dynamo Equations.

We recall that for an arbitrary spacetime \((M, g)\) the covariant form of Maxwell’s equations in CGS-ESU system are:

\[
\nabla_{\mu} F^{\mu \nu} = -\frac{4\pi}{c} J^{\nu}, \quad \nabla_{[\mu} F_{\nu \gamma]} = 0.
\]

where \((F^{\mu \nu}, J^{\mu})\) stand respectively for the coordinate components of the Maxwell tensor and four current. We shall assume that this \((M, g)\) admits a smooth timelike congruence of future directed world lines and denote by \(u\) the future directed tangent field normalized according to \(g(u, u) = -1\). It follows that \(h_{\mu \nu} = \delta_{\mu \nu} + u^{\mu} u_{\nu}\), acts as a projection tensor, i.e., \(h_{\mu \nu} h^{\nu \gamma} = h_{\mu \gamma}\), while the acceleration \(a_{\nu}\), rotation \(\omega_{\mu \nu}\), shear \(\sigma_{\mu \nu}\) and expansion \(\Theta\) of the congruence are defined from the covariant decomposition \(\nabla_{\mu} u_{\nu} = \omega_{\nu \mu} + \sigma_{\nu \mu} + \frac{1}{3} \Theta h_{\nu \mu} - a_{\nu} u_{\mu}\), \(\omega_{\nu \mu} = -\omega_{\mu \nu}\), \(\sigma_{\nu \mu} = \sigma_{\mu \nu}\), \(\sigma^{\mu} = 0\). Relative to this congruence the Maxwell tensor \(F_{\mu \nu}\) can be decomposed according to:

\[
F_{\mu \nu} = u_{\mu} E_{\nu} - u_{\nu} E_{\mu} + \varepsilon_{\mu \sigma \tau} u^{\sigma} B^{\tau}, \quad E_{\mu} = F_{\mu \nu} u^{\nu}, \quad B_{\mu} = -\frac{1}{2} \varepsilon_{\mu \nu}^{\sigma \tau} F_{\sigma \tau} u^{\nu}.
\]

where \((E_{\mu}, B_{\nu})\) are the coordinate components of the electric and magnetic field seen by the \(u\)-observes. The 3+1 congruence decomposition of Maxwell eqs can be obtained by combining (1,2) and after algebra yield:

\[
\nabla_{\mu} E^{\mu} = -\varepsilon_{\nu \sigma \tau} \omega^{\mu \nu} u^{\sigma} B^{\tau} + a_{\mu} E^{\mu} + 4\pi \rho, \quad \nabla_{\mu} B^{\mu} = \varepsilon_{\nu \sigma \tau} \omega^{\mu \nu} u^{\sigma} E^{\tau} + a_{\mu} B^{\mu},
\]

\[
u \mu E^{\nu} \left( \omega_{\nu} + \sigma_{\nu} - \frac{2}{3} h_{\nu}^{\mu} \Theta + u_{\nu} a_{\mu} \right) E^{\mu} + \varepsilon^{\nu \mu \sigma \tau} \left( u_{\mu} a_{\sigma} B_{\tau} + u_{\mu} \nabla_{\sigma} B_{\tau} + \omega_{\mu \sigma} B_{\tau} \right),
\]

\[
u \mu B^{\nu} \left( \sigma_{\nu} - \frac{2}{3} h_{\nu}^{\mu} \Theta + u_{\nu} a_{\mu} \right) B^{\mu} - \varepsilon^{\nu \mu \sigma \tau} \left( u_{\nu} a_{\sigma} E_{\tau} + u_{\nu} \nabla_{\sigma} E_{\tau} + \omega_{\nu \sigma} E_{\tau} \right)
\]

\[
u \mu \left( u_{\nu} \omega_{\nu \mu} B_{\tau} + u_{\nu} \omega_{\nu \mu} E_{\tau} \right) = 0.
\]

while the conservation equation \(\nabla_{\mu} J^{\mu} = 0\) reads \(u^{\mu} \nabla_{\mu} (\rho c) + \Theta \rho + \nabla_{\mu} J_{\mu}^{\nu} = 0\) where \(\rho c = -J^{\mu} u_{\mu}\), \(J^{i}(u) = h_{\nu}^{i} J^{\nu}\) represent the charge density and electric current seen by the \(u\)-observer.

We shall specialize eqs (3-5) for a non singular, static spherically symmetric spacetime \((M, g)\) representing the interior of a perfect fluid spherical star, that joins smoothly to a part of the Schwarzschild manifold representing the exterior of the star. For such spacetime there exist a preferred chart so that \(g\) takes the form:

\[
g = -V^{2} dt^{2} + \frac{dr^{2}}{1 - \frac{2m(r)}{r}} + r^{2} (d\theta^{2} + \sin^{2}\theta d\phi^{2}).
\]

where \(t \in (-\infty, \infty), r \in [0, \infty)\). The timelike congruence of world lines with \(u = u^{\mu} \frac{\partial}{\partial x^{\mu}} = \frac{1}{V} \frac{\partial}{\partial t}\) shall be referred as the Killing congruence and it is easily seem from (6) that this congruence satisfy \(\omega_{\mu \nu} = \sigma_{\mu \nu} = \Theta = 0\), \(a_{\mu} = V^{-1} \nabla_{\mu} V\). Moreover from (6) it follows that the intrinsic metric \(\gamma\) on a \(t=\text{cte}\) hypersurface has the form:

\[
\gamma = h_{r}^{2} dr^{2} + h_{\theta}^{2} d\theta^{2} + h_{\phi}^{2} d\phi^{2}, \quad h_{r} = \left(1 - \frac{2m(r)}{r}\right)^{-1/2}, \quad h_{\theta} = r, \quad h_{\phi} = r \sin \theta.
\]
Let
\[
\begin{align*}
e_0 &= \frac{1}{V} \frac{\partial}{\partial t}, \\
e_r &= \frac{1}{r} \frac{\partial}{\partial r}, \\
e_\theta &= \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta}, \\
e_\varphi &= \frac{1}{r \sin \theta} \frac{\partial}{\partial \varphi}.
\end{align*}
\]
be the orthonormal basis carried by a Killing observer at \((t, r, \theta, \varphi)\). Since \(h^\mu_\nu E^\nu = h^\mu_\nu B^\nu = 0\), the electric field \(E\) and magnetic field \(B\) measured by the Killing observers can be written as:
\[
E(t, r, \theta, \varphi) = E^t e_t + E^\theta e_\theta + E^\varphi e_\varphi, \quad B(t, r, \theta, \varphi) = B^t e_t + B^\theta e_\theta + B^\varphi e_\varphi.
\]
where above and hence forth indices with a caret will be always be considered as frame indices. Eliminating the coordinate component in favor of frame components from eqs (3-5) it follows where above primed quantities refer to quantities measured in the proper frame, \(\gamma\) defined by (7).

We shall assume here after that the flow relative to Killing observers satisfy
\[
\nabla \cdot E = 4\pi \rho, \quad \nabla \cdot B = 0,
\]
(10)
\[
\nabla \times (VB) = \frac{4\pi}{c} VJ + \frac{1}{c} \frac{\partial E}{\partial t}, \quad \nabla \times (VE) = -\frac{1}{c} \frac{\partial B}{\partial t},
\]
(11)
where \(\nabla\cdot, \nabla\times\) stand for the divergence and curl operator formed using the components of the spatial metric \(\gamma\) defined by (7).

We shall adopt eqs (10-11) for the particular case where \((E,B)\) are generated by the flow of a conducting fluid. The flow takes in the interior of the star of areal radius \(R\), and shall denote by \(V\) its four velocity. The fluid is threaded by a magnetic field \(B\) that extends into the exterior free space. We shall ignore any influence of the induced field upon the fluid flow, and moreover shall assume that the fluid flow and generated fields have negligible energy and stresses so that to be considered as a test fields on the geometry described by (6). The conducting properties of the fluid are described by an electrical conductivity \(\sigma\) taken as real and constant. Relative to the orthonormal basis \(\{e_0, e_r, e_\theta, e_\varphi\}\) defined by (8), the velocity field \(V\) may be expressed in the form:
\[
V = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} e_0 + \frac{v^i}{c\sqrt{1 - \frac{v^2}{c^2}}} e_i = \frac{1}{V} \frac{\partial}{\partial x^0} + \frac{v^i}{c} \frac{\partial}{\partial x^i} + O\left(\frac{v^2}{c^2}\right), \quad i = r, \theta, \varphi.
\]
(12)
where \(v^i = \nabla^i(r, \theta, \varphi)\), stand for components of the three velocity relative to the Killing observers. We shall assume here after that the flow relative to Killing observers satisfy \(\nabla \cdot V \ll 1\) and this assumption allows us to pass to the second equality in (12). One of the important ingredient regarding the electromagnetic field generated by a fluid flow concerns the structure of Ohm’s law. Within the non relativistic setting this law reads \(J' = \sigma E'\), where \(J'\) and \(E'\) stand for the current density and electric field relative to the rest frame of the flow. However in the presence of curvature the conduction electrons will be affected by inertial forces. Elsewhere [8] we have study the form of Ohm’s law in an arbitrary background geometry. If \(n\) stands for the electron density of conducting electrons as measured in the proper reference frame of the flow, then the conduction current \(J' = -en\nu\) satisfies:
\[
J' = \sigma E' + \frac{e\tau}{m} \frac{J' \times B'}{c} - ne\tau c^2 a(1 + a \cdot x) - ne\tau c^2 R
\]
\[
+ 2e^2 \tau (J' \times \omega) - 2e^2 ne\tau ((x \times \omega) \times \omega),
\]
(13)
where in above primed quantities refer to quantities measured in the proper frame, \(a, \omega\) are the four acceleration and rotation vector of the proper frame, while \(x^i\) are the spatial components relative to the proper frame. Moreover \(\sigma = \frac{e^2 n \nu}{m}\) is the electrical conductivity of the fluid with \(\tau\) the conduction time scale while \(R = R^i_{\delta k\delta} x^k e^j\) with the components of the Riemann evaluated.
at the world line representing the origin of the proper frame. As it stands this electric current \( J' \) exhibits a complex structure. If however we define by \( l_f \) and \( l_g \) the characteristic length scales where flow and gravity change significantly and \( l \) by the mean free path of conducting electrons, the contribution to \( J \) from the Riemann and other terms are of the order \((\frac{1}{l_f}), (\frac{1}{l_g})\) respectively.

The term involving the coriolis force is of the order \( \frac{\tau}{\omega} \) while the second term involves \( \tau \omega_B \), where \( \omega_B \equiv \frac{c B}{mc} \) is the gyrofrequency of conduction electrons. We shall assume that the conduction time scale \( \tau \) is the dominant time scale and moreover the mean free path \( l \) is much less than other length scales. Under those condition the dominant contribution to the conduction current in the rest frame of the flow is described by \( J' = \sigma E' \). Combining this assumption with \( \frac{|V|^2}{c} \ll 1 \) it follows that \( J \) relative to the Killing observer has the form:

\[
J = \sigma \left( E + \frac{v \times B}{c} \right), \tag{14}
\]

where \( v = v^i e_i \) is the three velocity of the flow relative to Killing observer and \((E,B)\) are the electric and magnetic fields measured by Killing observers. Combining (14) with 'Amperes’s law' we arrive at:

\[
\nabla \times (VB) = \frac{4\pi}{c} \sigma \left( E + \frac{v \times B}{c} \right) + \frac{1}{c} \frac{\partial E}{\partial t}, \tag{15}
\]

and upon neglecting the displacement current, and taking a curl in this eqs combined with to Faraday’s law we find that \( B \) satisfies:

\[
\frac{\partial B}{\partial t} = \nabla \times (v \times VB) - \nabla \times \eta \left( \nabla \times VB \right), \quad \nabla \cdot B = 0, \quad |x| < R. \tag{16}
\]

where here after \( \eta = \frac{c^2}{4\pi \sigma} \) denotes the magnetic diffucivity of the fluid. In the exterior region and in the absence of any external current distribution, \( B \) obeys:

\[
\nabla \times (VB) = 0, \quad \nabla \cdot B = 0, \quad |x| > R. \tag{17}
\]

As long \( \eta \neq 0 \), \( B \) will be continuous across the star’s surface and would decay to zero at spacelike infinity at least as a dipole field, i.e, \( \lim_{|x| \to \infty} B = \mathcal{O} \left( \frac{1}{|x|^3} \right) \). The electric field \( E \) and any induced volume or surface charged density \( \rho \) can be computed once \( B \) has been determined. Therefore the system (16) completely determines the generated electromagnetic field provided some initial field distribution \( B(x,t=0) = B_0(x) \) has been specified.

We shall now analyze the system (16,17) under the assumptions that \( v \) and \( B \) are both axisymmetric with respect to an axis which without loss of generality shall identify with the axis defined by the rotational Killing vector field \( \xi_x = \frac{\partial}{\partial x} \). In this work we shall assume that all fields describing the fluid flow as well as initial data for the field \( B \) to be sufficiently smooth so that all subsequent operations are well defined. Of course we are aware that this assumption is over-simplified and effects of finite differentiability ought to be incorporated. It is our hope that a more detail and complete discussion of the theorem will be discussed in the near future.

With this comments in mind, relative to the coordinate gauge of (7) we introduce the poloidal \( B^P \) and toroidal \( B^T \) part of \( B \) via \( B = B^T + B^P \) with \( B^T = B^T e_r + B^T e_\phi \) and \( B^P = B^P e_r \), with a similar splitting for the velocity field \( v \). This splitting is well defined and as long as the background geometry is spherically symmetric then \( \nabla \times (VB^T), \nabla \times (VB^P) \) are respectively poloidal and toroidal fields. Based on properties of toroidal and poloidal fields (for an overview and further properties of such fields see [9]), it follows from the induction eq that in the star’s interior \( B^T \) and \( B^P \) satisfy:

\[
\frac{\partial B^T}{\partial t} = \nabla \times (v^T \times VB^T + v^T \times VB^P) - \nabla \times (\eta \nabla \times VB^T), \quad \nabla \cdot B^T = 0, \quad |x| < R. \tag{18}
\]
\[ \frac{\partial \mathbf{B}^P}{\partial t} = \nabla \times (\mathbf{v}^P \times \mathbf{V} \mathbf{B}^P) - \nabla \times (\eta \nabla \times \mathbf{V} \mathbf{B}^P), \quad \nabla \cdot \mathbf{B}^P = 0, \quad |\mathbf{x}| < R. \]  

(19)

while in the exterior region \( \mathbf{B}^T \) and \( \mathbf{B}^P \) obey:

\[ \nabla \times (\mathbf{V} \mathbf{B}^P) = \nabla \cdot (\mathbf{B}^P) = 0, \quad \mathbf{B}^T = 0, \quad |\mathbf{x}| > R. \]  

(20)

Across the star’s surface the field \( \mathbf{B}^T \) and \( \mathbf{B}^P \) are continuous and \( \mathbf{B}^P \) obeys:

\[ |\mathbf{B}^P||_R = 0, \quad \lim_{|\mathbf{x}| \to \infty} |\mathbf{B}^P| = \mathcal{O}(\frac{1}{|\mathbf{x}|^3}). \]  

(21)

where in above and here after the symbol \([...]|_R = 0\) denotes the jump across the surface \( R \).

The representation \( \mathbf{B}^P = \nabla \times (A(t, r, \theta) \mathbf{e}_\varphi) \), for some smooth scalar \( A(t, r, \theta) \), eliminates the constrain \( \nabla \cdot \mathbf{B}^P = 0 \), and (19) implies:

\[ \frac{\partial \mathbf{A}}{\partial t} \mathbf{e}_\varphi = (\mathbf{V} \mathbf{v}^P - \eta \nabla \mathbf{V}) \times (\nabla \times (A \mathbf{e}_\varphi)) - \eta \mathbf{V} \nabla \times (\nabla \times (A \mathbf{e}_\varphi)) + \nabla g, \]  

(22)

where \( g \) is an arbitrary function. Due to axial symmetry the toroidal contribution of this \( g \) vanishes and it follows from the this eq via elementary manipulations that the Stokes function \( X = h_\varphi A \) satisfies:

\[ \frac{1}{V} \frac{\partial}{\partial t} \left( \frac{X}{h_\varphi} \right) = -\frac{1}{h_\varphi} \frac{\partial}{\partial t} (\mathbf{v}^P \cdot \nabla) X + \eta \left[ \frac{1}{h_\varphi} \nabla^2 X - \frac{2 \nabla X \cdot \nabla \log h_\varphi}{h_\varphi} \right], \quad |\mathbf{x}| < R. \]  

(23)

where \( \nabla^2 \) stands for the Laplacian operator of the spatial metric and in above he have defined \( \mathbf{v}^P = \mathbf{v}^P - \eta \nabla \mathbf{V} \). The exterior poloidal field \( \mathbf{B}^P \) can be represented in the form \( \mathbf{B}^P = \nabla \times (A \mathbf{e}_\varphi) = \nabla \times \left( \frac{X \mathbf{e}_\varphi}{h_\varphi} \right) \), and eqs (20) yield:

\[ \frac{1}{h_\varphi} \nabla^2 X - \frac{2}{h_\varphi} \nabla X \cdot \nabla \log h_\varphi = 0, \quad |\mathbf{x}| > R. \]  

(24)

where we have used the same symbol \( X \) to represent the interior and exterior Stokes function. The condition (21) require \( X \) to satisfy \( |X||_R = 0, \lim_{|\mathbf{x}| \to \infty} X(t, r, \theta) = \mathcal{O}(\frac{1}{|\mathbf{x}|^3}) \). Similar reduction holds true for the eq determining the toroidal field \( \mathbf{B}^T \). Axial symmetry implies that \( \mathbf{B}^T = B(t, r, \theta) \mathbf{e}_\varphi \) and this representation satisfies \( \nabla \cdot \mathbf{B}^T = 0 \). Substituting \( \mathbf{B}^T = B(t, r, \theta) \mathbf{e}_\varphi \) and \( \mathbf{v}^T = \mathbf{v}^\varphi(t, r, \theta) \mathbf{e}_\varphi \) in (19) and by appealing to vector identities it can be shown that \( B(t, r, \theta) \) satisfies (Details of such calculations are discussed in [7]):

\[ \frac{\partial \mathbf{B}}{\partial t} = -h_\varphi \mathbf{v}^P \cdot \nabla \left( \frac{\mathbf{V} \mathbf{B}^T}{h_\varphi} \right) + (\nabla \cdot \mathbf{v}^P) \mathbf{V} \mathbf{B}^T + \eta \left[ \frac{1}{h_\varphi} \nabla \cdot h_\varphi^2 \nabla \left( \frac{\mathbf{V} \mathbf{B}^T}{h_\varphi} \right) + 2 \mathbf{BV} \nabla^2 \log h_\varphi \right] \]

\[ + \left[ \mathbf{v}^\varphi (\mathbf{B}^P \cdot \nabla \mathbf{V}) + h_\varphi \left( \mathbf{V} \mathbf{B}^P \cdot \nabla \left( \frac{\mathbf{v}^\varphi}{h_\varphi} \right) \right) \right], \quad |\mathbf{x}| < R. \]  

(25)

and of course \( B \equiv 0 \) for \( |\mathbf{x}| \geq R \).

For the rest of this section we shall derive some integral identities that are consequences of (23,24,25). At first we multiply (23) by \( X \) and after some manipulations we obtain:

\[ \frac{1}{2V} \frac{\partial X^2}{\partial t} = \frac{1}{2} \nabla \cdot \mathbf{v}^P - K X^2 - \eta \nabla X \cdot \nabla X + \nabla \cdot \mathbf{A}, \]  

(26)
where $K$ and $A$ are defined by:

$$K \equiv -\frac{1}{2} \frac{\nabla^2 V}{V} + \frac{1}{2} \frac{\nabla V \cdot \nabla V}{V^2} + \nabla^2 (\log h \phi).$$

(27)

$$A = \eta X \nabla X - \eta X^2 \nabla \log h \phi + \frac{X^2}{2} \tilde{v}.$$  

(28)

Using $d\Omega \equiv \sqrt{\gamma} dr d\theta d\varphi$ as the proper volume element on any $t=$const hypersurface we integrate (26) over the star’s interior and with the help of the divergence theorem, (26) yields:

$$\frac{1}{2} \frac{d}{dt} \int \frac{X^2}{V} d\Omega = \int [\frac{1}{2} \nabla \cdot \mathbf{v}^P + \eta K] X^2 d\Omega - \eta \int \nabla X \cdot \nabla X d\Omega + \oint A \cdot ds,$$

(29)

where the surface integral is evaluated over the star’s surface with $ds$ parallel to the outwardly pointing unit normal $\mathbf{n}$ of the star’s surface. Identical manipulations of (24) yield after an integration in the star’s exterior region:

$$\int K X^2 d\Omega - \int \nabla X \cdot \nabla X d\Omega + \oint A \cdot ds = 0.$$

(30)

where $K$ structurally identical to (27) with the sole exception that all term are evaluated in the star’s exterior region and the surface integral in (30) is computed over the star’s surface with an outward pointing normal $-\mathbf{n}$ and a two-sphere at infinity. Presently the vector field $A$ reads:

$$A = X \nabla X - X^2 \nabla \log h \phi + \frac{X^2}{2} \nabla V.$$  

(31)

Since by assumption the exterior Schwarzschild geometry joins smoothly to interior geometry, across the star’s surface and the velocity field satisfies $\mathbf{v}^P \cdot \mathbf{n} |_R = 0$, the interior and exterior field $A$ defined by (28–31) are continuous across the star’s surface. Moreover due to the asymptotic behavior of the Stoke’s function $X$ and the asymptotically flat nature of the geometry, the surface integral in (30) evaluated at the asymptotic region integrates to zero. Those considerations allows us to combine (30) and (29) into a single relation of the form:

$$\frac{1}{2} \frac{d}{dt} \int \frac{X^2}{V} d\Omega = \int [\frac{1}{2} \nabla \cdot \mathbf{v}^P + \eta K] X^2 d\Omega - \eta \int \nabla X \cdot \nabla X d\Omega.$$

(32)

where the integral in the left hand side is evaluated on the star’s interior, while the integrals in the right hand side are computed over the entire $t=$constant spacelike hypersurface.

Suppose that on some initial hypersurface $t = 0$ smooth axisymmetric distributions of $\mathbf{B}^P(x, t = 0)$ and $\mathbf{B}^T(x, t = 0)$ has been specified. To the future of $t = 0$ the advection terms in (23,25) would amplify the initial distribution until the diffusion term that would start operating. This occurs at a characteristic time greater or equal to Ohm’s timescale $T_0$ taken approximately as $T_0 = \frac{4\sigma a L}{c}$ (for subletties around the definition of this time scale in the presence of curvature and decay of an initial $\mathbf{B}$ field distribution on a curved spacetime see [9, 10]). It is natural to assume that the effects of advection would be balanced by ohmic dissipation in a way that a steady state is established. It is this steady state solution that we are interest to probe. For such steady state solution, since the background geometry is independent upon the hypersurface $t=$constant, the left hand side of (32) vanishes. If moreover the velocity field is assumed to obey $\nabla \cdot \mathbf{v}^P = 0$ and $K \leq 0$, then (32) implies that any steady state solution of (23,24) would have necessary $X=0$ and thus $\mathbf{B}^P = 0$. 


We shall show that the vanishing of $B^P$ has as a consequence the decay of the toroidal field. At first setting $B^P = 0$ in (25) we obtain:

$$\frac{\partial}{\partial t} \frac{B}{h^2} = -v^P \cdot \nabla \left( \frac{V B}{h} \right) + \eta \left[ \frac{1}{h^2} \nabla \cdot h^2 \nabla \left( \frac{V B}{h} \right) + \frac{2 V B}{h} \nabla^2 \log h \right], \quad |x| < R. \quad (33)$$

and upon multiplying both sides by $\frac{V B}{h}$ we obtain:

$$\frac{1}{2} \frac{\partial}{\partial t} \left( \frac{V B^2}{h^2} \right) = -\frac{1}{2} \left( \frac{V B}{h} \right)^2 \nabla \cdot v^P - \eta \nabla \left( \frac{V B}{h} \right) \cdot \nabla \left( \frac{V B}{h} \right) + \eta \left( \frac{V B}{h} \right)^2 \nabla^2 \log h \quad (34)$$

where the vector field $A_1$ is defined by

$$A_1 = \eta \left( \frac{V B}{h} \right) \nabla \left( \frac{V B}{h} \right) + \left( \frac{V B}{h} \right)^2 \nabla \log h - \frac{1}{2} \left( \frac{V B}{h} \right) v^P. \quad (35)$$

Remembering that $B^T = 0$ in the star’s exterior, an integration of (34) over the star’s interior yields:

$$\frac{1}{2} \frac{d}{dt} \int \frac{V B^2}{h^2} d\Omega = \int \left[ -\frac{1}{2} \left( \frac{V B}{h} \right)^2 \nabla \cdot v^P - \eta \nabla \left( \frac{V B}{h} \right) \cdot \nabla \left( \frac{V B}{h} \right) + \eta \left( \frac{V B}{h} \right)^2 \nabla^2 \log h \right] d\Omega + \oint A_1 \cdot ds. \quad (36)$$

where the surface integral is taken along the star’s surface. Since however $B|_R = 0$ and $v^P \cdot n|_R = 0$ it follows from (35) that $A|_R = 0$ and thus the surface integral vanishes. If as for the case of the poloidal assume that $\nabla \cdot v^P = 0$ it follows that the right hand side of (36) is negative definite provided $\nabla^2 (\log h)$ ≤ 0 in the star’s interior region. Assuming for the moment that this is the case it follows from (36) that any steady state solution has $B = 0$ in the interior of the star. In summary, as long as $K \leq 0$ throughout the spacetime and $\nabla^2 \log h \leq 0$ in the star’s interior and $\nabla \cdot v^P = 0$, then any steady state solution of (16,17) is the trivial solution.

3. The influence of the spacetime curvature.

The results of the previous section show that Ohmic dissipation would dominate the effects of the advection and drives any non singular solution of (23,24,25) into the trivial solution $B^T = B^P = 0$, provided $\nabla^2 \log h$ and $K$ are negative definite. In this section we shall discuss the nature of those contribution. At first it is worth mentioning that both terms are geometric invariants. In fact nothing that $V^2 = -g(\xi, t, \xi, t)$ and $h^2 = g(\xi, \xi)$ where $\xi, t$ stand for the hypersurface orthogonal and Killing fields admitted by $g$, then (6,7) imply (a detail derivation of the following formulae are discussed in [7]):

$$\nabla^2 (\log h) = \frac{D^\alpha D_\alpha h}{h} - \frac{D^\alpha h \cdot D_\alpha h}{h^2} - \frac{R_{\alpha \beta \xi(\xi) \xi(\xi)}}{h^2} - \frac{\nabla^\alpha V \nabla_\alpha h}{h V}, \quad \alpha, \beta = 0, 1, 2, 3,$$

$$K = \frac{1}{2} \left[ -\frac{R_{\alpha \beta \xi(\xi) \xi(\xi)}}{V^2} - \frac{2 R_{\alpha \beta \xi(\xi) \xi(\xi)}}{h V} + \frac{\nabla^\alpha V \nabla_\alpha V}{V^2} - \frac{2 \nabla^\alpha V \nabla_\alpha h}{h V} \right] \quad (37)$$

Those representations, show that for an arbitrary static and spherically symmetric metric $g$ there is no fundamental reason that the right hand side of (37) should be negative definite. In
such event we can not any longer conclude from the integral identities (25,36) that a steady state solution would be trivial. However equally well we can not any longer conclude that an axisymmetric field B can be maintained by dynamo action. Our method of establishing Cowling’s theorem simoly breaks down. An investigation of Cowling’s theorem for such background would require a more sophisticated treatment than the one presental in this work. Fortunately for the cases where g is a solution of Einstein eqs with source a perfect fluid distribution i.e $T_{\mu\nu} = (\rho + P)u_\mu u_\nu + Pg_{\mu\nu}$, the structure of the invariants are amenable to analytical treatment. In such event it follows from (7) that:

$$\nabla^2 \log h_\varphi = - \left[ \frac{m(r)}{r^3} + 4\pi \rho \right].$$

and as long as the energy density is positive then $\nabla^2 \log h_\varphi$ is negative definite. On the other hand K takes the form:

$$K = \frac{1}{2} \left[ k \frac{3\rho + P}{2} - \frac{D^3 V D_\varphi V}{V^2} + \frac{2D^3 V D_\varphi h_\varphi}{h_\varphi V} \right],$$

and by appealing to one of the eqs of relativistic stellar structure [11], i.e

$$\frac{1}{V} \frac{dV}{dr} = g_{rr} \left[ \frac{k P}{2} + \frac{m(r)}{r^3} \right],$$

it follows that (39):

$$K = -\frac{1}{2} \left[ \frac{3(\rho + P)}{2} - 2\frac{m(r)}{r^3} - g_{rr} \left( \frac{k P}{2} + \frac{m(r)}{r^3} \right)^2 r^2 \right],$$

In the exterior region K reduces to:

$$K = \frac{g_{rr} m^2(R)}{2} - \frac{m(R)}{r^3}.$$

where $m(R) = \int_0^R \rho(r')r'^2 dr'$ in the total ADM mass of the star. In the form (41,42) and for any static spherical model the right hand side of (41) can be computed explicitly. In particular

$$K = -\frac{1}{2(1 - \frac{2m(R)}{r})} \frac{m(R)}{r^3} \frac{4r - 9m(R)}{r},$$

and this K is negative definite provided the compactness ratio $\varepsilon = \frac{2Gm(R)}{c^2 R}$ of the background star satisfies $\varepsilon \leq \frac{8}{9}$. At the value $\frac{8}{9}$, in the borderline separating static spherical model, $K(R) = 0$, while for the unrealistic case $\varepsilon > \frac{8}{9}$ and $K(R) > 0$. Still the overall sign of K is difficult to access. It is however determined for the case of incompressible stellar model i.e the case of a constant density star. For such background, the pressure $P(r)$ can be written in the form [11]:

$$P(r) = \frac{\rho (1 - \varepsilon)^{1/2} - (1 - \varepsilon x^2)^{1/2}}{(1 - \varepsilon x^2)^{1/2} - 3(1 - \varepsilon)^{1/2}}, \quad x = \frac{r}{R},$$

while $1 - \frac{2m(r)}{r} = 1 - \varepsilon x^2$, with $\varepsilon = \frac{2Gm(R)}{c^2 R}$. Substituting the above form an P(r) in (41) we obtain:

$$6(3\alpha - \beta)^2 \frac{K}{k\rho} = -[37 - 36\varepsilon - 15(1 - \varepsilon)^{1/2}(1 - \varepsilon x^2)^{1/2} - 2\varepsilon x^2],$$

where $\alpha = (1 - \varepsilon)^{1/2}$, $\beta = (1 - \varepsilon x^2)^{1/2}$. It follows that the expression in the parenthesis is always positive definite for all $\varepsilon \in [0, \frac{8}{9}]$ and thus K is negative definite in the star’s interior.
4. Conclusion.
In this work we have shown that the effects of the background curvature, within the model considered, do not alter the conclusion of the non relativistic form of Cowling’s theorem. There however two comments regarding this conclusion. The first one concerns the status of the theorem for the case where the background spherical star is not any longer of constant density. In such event whenever $K \leq 0$ in the interior of the star $K$ the theorem remains valid. Even though we believe that $K \leq 0$ whenever the energy momentum tensor satisfies the weak and strong energy condition, we have not been able to show this property analytically. The second comment concerns the compressibility assumption imposed upon the fluid flow. This assumption appears as a necessary condition for our proof to go through. A proof establishing the theorem without invoking this assumption is welcomed. Unfortunately it is very difficult to get a handle on the effects of compressibility. The basic identities (35,39) eqs (18,19) are modified by the term $\nabla \cdot \mathbf{V} P$ and the overall sign of this term is very difficult to determine. However judging from the results valid in the non relativistic regime, we expect the theorem to hold true for a general fluid flow. In any case a proof establishing the theorem without invoking the incompressibility assumption is welcomed. Finally a comment regarding the nature of the theorem for the case of stationary axisymmetric background spacetime. Electrodynamics on such a background becomes a more complex problem. The so called gravitomagnetic effect modifies the structure of eqs (10-14) and as a consequence the proof presented in this work breaks down. Whether the theorem remain valid for a stationary axisymmetric spacetime for the moment is unknown. The issue is under active investigation and we hope to come back to that issue in the near future.

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