Asymptotic expansions for ratios of products of gamma functions

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Abstract

An asymptotic expansion for a ratio of products of gamma functions is derived.

2000 Mathematics Subject Classification: Primary 33B15; Secondary 33C20

Keywords and phrases: Gamma function, generalized hypergeometric functions

1 Introduction

An asymptotic expansion for a ratio of products of gamma functions has recently been found [2], which, with

\[ s_1 = b_1 - a_1 - a_2, \]  

may be written

\[ \frac{\Gamma(a_1 + n)\Gamma(a_2 + n)}{\Gamma(b_1 + n)\Gamma(-s_1 + n)} = 1 + \sum_{m=1}^{M} \frac{(s_1 + a_1)_m(s_1 + a_2)_m}{(1)_m(1 + s_1 - n)_m} + O(n^{-M-1}) \]  

(2)
as $n \to \infty$. Here use is made of the Pochhammer symbol

$$(x)_n = x(x+1) \cdots (x+n-1) = \Gamma(x+n)/\Gamma(x).$$

The special case when $b_1 = 1$ of this formula (2) had been stated earlier by Dingle [3], and there were proofs by Paris [8] and Olver [6,7].

The proof of (2) is based on the formula for the analytic continuation near unit argument of the Gaussian hypergeometric function. For the more general hypergeometric functions

$$p+1 F_p \left( \begin{array}{c} a_1, a_2, \ldots, a_{p+1} \\ b_1, \ldots, b_p \end{array} | z \right) = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \cdots (a_{p+1})_n}{(b_1)_n \cdots (b_p)_n (1)_n} z^n, \quad (|z| < 1),$$

the analytic continuation near $z = 1$ is known too, and this raises the question as to whether a sufficiently simple asymptotic expansion can be derived in a similar way for a ratio of products of more gamma function factors. This is indeed the case, and it is the purpose of this work to present such an expansion.

## 2 Derivation of the asymptotic expansion

The analytic continuation of the hypergeometric function near unit argument may be written

$$\frac{\Gamma(a_1)\Gamma(a_2) \cdots \Gamma(a_{p+1})}{\Gamma(b_1) \cdots \Gamma(b_p)} p+1 F_p \left( \begin{array}{c} a_1, a_2, \ldots, a_{p+1} \\ b_1, \ldots, b_p \end{array} | z \right) = \sum_{m=0}^{\infty} g_m(0) (1-z)^m + (1-z)^{s_p} \sum_{m=0}^{\infty} g_m(s_p) (1-z)^m,$$

where

$$s_p = b_1 + \cdots + b_p - a_1 - a_2 - \cdots - a_{p+1}$$

and the coefficients $g_m$ are known. While the $g_m(0)$ are not needed for the present purpose, the $g_m(s_p)$ are

$$g_m(s_p) = (-1)^m \frac{(a_1 + s_p)_m (a_2 + s_p)_m \Gamma(-s_p - m)}{(1)_m} \times \sum_{k=0}^{m} \frac{(-m)_k}{(a_1 + s_p)_k (a_2 + s_p)_k} A_k^{(p)},$$
where the coefficients $A_{k}^{(p)}$ will be shown below.

The left-hand side $L$ of (4) is

$$L = \sum_{n=0}^{\infty} \frac{\Gamma(a_1 + n)\Gamma(a_2 + n)\cdots \Gamma(a_{p+1} + n)}{\Gamma(b_1 + n)\cdots \Gamma(b_p + n)\Gamma(1 + n)} z^n. \quad (7)$$

The asymptotic behaviour, as $n \to \infty$, of the coefficients of this power series is governed \[4\][5][10] by the terms $R$ on the right-hand side which, at $z = 1$, are singular,

$$R = \sum_{m=0}^{\infty} g_m(s_p)(1 - z)^{s_p + m}. \quad (8)$$

Expanded by means of the binomial theorem in its hypergeometric-series-form, this is

$$R = \sum_{m=0}^{\infty} g_m(s_p) \sum_{n=0}^{\infty} \frac{(-s_p - m)n}{\Gamma(1 + n)} z^n. \quad (9)$$

Interchanging the order of summation (and making use of the reflection formula of the gamma function) we may get

$$R = \sum_{n=0}^{\infty} \frac{\Gamma(-s_p + n)}{\Gamma(1 + n)} \sum_{m=0}^{\infty} (-1)^m g_m(s_p) \frac{1}{\Gamma(-s_p - m)(1 + s_p - n)m} z^n. \quad (10)$$

Comparison of the coefficients of the two power series for $R$ and $L$, which asymptotically, as $n \to \infty$, should agree, then leads to

$$\frac{\Gamma(a_1 + n)\Gamma(a_2 + n)\cdots \Gamma(a_{p+1} + n)}{\Gamma(b_1 + n)\cdots \Gamma(b_p + n)\Gamma(-s_p + n)} \sim \sum_{m=0}^{\infty} (-1)^m g_m(s_p) \frac{1}{\Gamma(-s_p - m)(1 + s_p - n)m}. \quad (11)$$

Inserting $g_m$ from (6) and keeping the first $M + 1$ terms of the asymptotic series, we get

**Theorem 1**

$$\frac{\Gamma(a_1 + n)\Gamma(a_2 + n)\cdots \Gamma(a_{p+1} + n)}{\Gamma(b_1 + n)\cdots \Gamma(b_p + n)\Gamma(-s_p + n)} = 1 \quad (12)$$

$$+ \sum_{m=1}^{M} \frac{(a_1 + s_p)_m(a_2 + s_p)_m}{(1)_m(1 + s_p - n)_m} \sum_{k=0}^{m} \frac{(-m)_k}{(a_1 + s_p)_k(a_2 + s_p)_k} A_{k}^{(p)} + O(n^{-M-1})$$

as $n \to \infty$, where $s_p = b_1 + \cdots + b_p - a_1 - a_2 - \cdots - a_{p+1}$. 3
The simple formula (2) above corresponding to $p = 1$ can be recovered from this theorem if we define $A_0^{(1)} = 1$, $A_k^{(1)} = 0$ for $k > 0$, so that then the sum over $k$ is equal to 1 and disappears. The coefficients for larger $p$ can be found in [11], but a few of them are here displayed again for convenience:

$$A_k^{(2)} = \frac{(b_2 - a_3)_k (b_1 - a_3)_k}{k!}, \quad (13)$$

$$A_k^{(3)} = \sum_{k_2=0}^{k} \frac{(b_3 + b_2 - a_4 - a_5 + k_2)_{k-k_2}(b_1 - a_3)_{k-k_2}(b_3 - a_4)_{k_2}(b_2 - a_4)_{k_2}}{(k - k_2)!k_2!}, \quad (14)$$

$$A_k^{(4)} = \sum_{k_2=0}^{k} \frac{(b_4 + b_3 + b_2 - a_5 - a_4 - a_5 + k_2)_{k-k_2}(b_1 - a_3)_{k-k_2}}{(k - k_2)!} \times \sum_{k_3=0}^{k_2} \frac{(b_4 + b_3 - a_5 - a_4 + k_3)_{k_2-k_3}(b_2 - a_4)_{k_2-k_3}}{(k_2 - k_3)!} \times \frac{(b_4 - a_5)_{k_3}(b_3 - a_5)_{k_3}}{k_3!}, \quad (15)$$

For $p = 3, 4, \ldots$, several other representations are possible [11], such like

$$A_k^{(3)} = \frac{(b_3 + b_2 - a_4 - a_3)_k (b_1 - a_3)_k}{k!} \times 3F_2 \left( \begin{array}{c} b_3 - a_4, b_2 - a_4, -k \\ b_3 + b_2 - a_4 - a_3, 1 + a_3 - b_1 - k \end{array} \bigg| 1 \right) \quad (16)$$

or

$$A_k^{(3)} = \frac{(b_1 + b_3 - a_3 - a_4)_k (b_2 + b_3 - a_3 - a_4)_k}{k!} \times 3F_2 \left( \begin{array}{c} b_3 - a_3, b_3 - a_4, -k \\ b_1 + b_3 - a_3 - a_4, b_2 + b_3 - a_3 - a_4 \end{array} \bigg| 1 \right). \quad (17)$$

For $p = 2$, the formula (12) may be written simply as

$$\frac{\Gamma(a_1 + n)\Gamma(a_2 + n)\Gamma(a_3 + n)}{\Gamma(b_1 + n)\Gamma(b_2 + n)\Gamma(-s_2 + n)} = 1 \quad (18)$$

$$+ \sum_{m=1}^{M} \frac{(a_1 + s_2)_m(a_2 + s_2)_m}{(1)_m(1 + s_2 - n)_m} 3F_2 \left( \begin{array}{c} b_2 - a_3, b_1 - a_3, -m \\ a_1 + s_2, a_2 + s_2 \end{array} \bigg| 1 \right) + O(n^{-M-1}),$$

where $s_2 = b_1 + b_2 - a_1 - a_2 - a_3$. 

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3 Additional comments

The derivation of the theorem is based on the continuation formula (1) which holds, as it stands, only if \( s_p \) is not equal to an integer. Nevertheless, the theorem is valid without such a restriction. This can be verified if the derivation is repeated starting from any of the continuation formulas for the exceptional cases [1]. Instead of or in addition to the binomial theorem, the expansion

\[
(1 - z)^m \ln(1 - z) = \sum_{n=1}^{\infty} c_n z^n,
\]

is then needed for integer \( m \geq 0 \), where

\[
c_n = -\frac{1}{n} (-1)^m \frac{\Gamma(1 + m) \Gamma(n - m)}{\Gamma(n)}
\]

for \( n > m \), while the coefficients are not needed here for \( n \leq m \).

The theorem has been proved here for any sufficiently large positive integer \( n \) only. On the basis of the discussion in [2], it can be suspected that the theorem may be theoretically valid (although less useful) in the larger domain of the complex \( n \)-half-plane \( \Re(s_p + a_1 + a_2 - 1 + n) \geq 0 \).

Expansions for ratios of even more general products of gamma functions are treated in a recent monograph by Paris and Kaminski [9].

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