On broken zero modes of a string world sheet, and a correlation function of a 1/4 BPS Wilson loop and a 1/2 BPS local operator

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Abstract

We reconsider a gravity dual of a 1/4 BPS Wilson loop. In the case of an expectation value of the Wilson loop, it is known that broken zero modes of a string world sheet in the gravity side play important roles in the limit $\lambda \to \infty$ with keeping the combination $\lambda \cos^2 \theta_0$ finite. Here, $\lambda$ is the 't Hooft coupling constant and $\theta_0$ is a parameter of the Wilson loop. In this paper, we reconsider a gravity dual of a correlation function between the Wilson loop and a 1/2 BPS local operator with R charge $J$. We take account of contributions coming from the same configurations of the above-mentioned broken zero modes. We find an agreement with the gauge theory side in the limit $J \ll \sqrt{\lambda \cos^2 \theta_0}$.

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1 Introduction

In the context of the AdS/CFT correspondence [1][2][3], dual objects of Wilson loop operators are given in terms of string world sheets attached to the loops on the AdS boundary [4][5]. An expectation value of the Wilson loop in the gauge theory side, for example, is conjectured to be equal to a string disk partition function which is, in principle, given by a path integral of the string world sheet. In literature, the path integral is usually approximated by using a classical solution, and then a strong coupling limit of the gauge theory result is reproduced. A natural way to go beyond the classical string would be taking account of effects coming from small fluctuations around the classical solutions perturbatively. For developments in such a method, see the papers [6][7][8][9].
Besides such analyses, an interesting method has been proposed in [10] for the case of an expectation value of a 1/4 BPS Wilson loop operator. The operator depends on a parameter $\theta_0$, and for a special case with $\theta_0 = \pi/2$, there is a three-parameter family of string solutions corresponding to a single common Wilson loop. These three parameters correspond to zero modes of the string world sheet. An integration of the exact zero modes just gives a multiplicative factor. The author of the paper [10] further considered a limit in which $\theta_0$ is not exactly equal to $\pi/2$, but very close to it. In such a limit, the zero modes get a potential which is not exactly flat. Hence they are no longer exact zero modes. However, these modes are expected to be still much lighter than generic fluctuations in the limit $\lambda \to \infty$, $\theta_0 \to \pi/2$. Then the integration over such modes gives non-negligible contributions. In fact, it was found that by keeping the combination $\lambda' \equiv \lambda \cos^2 \theta_0$ finite, the integration over such broken zero modes reproduces the exact gauge theory result of the Wilson loop expectation value in the planar limit [10].

In this paper we apply the method of the broken zero modes to a gravity dual of a correlation function between the 1/4 BPS Wilson loop operator and a 1/2 BPS local operator with an R charge $J$. The same system has been studied in [11][12][13] based on the method developed in earlier works [14][15] by treating the string world sheet classically.\footnote{In [15], a correlation function between a 1/2 BPS Wilson loop and the 1/2 BPS local operator is studied by taking account of small fluctuations around a classical solution. It is found that such an analysis reproduces the gauge theory result in the limit $\lambda \to \infty$ with keeping $J^2/\sqrt{\lambda}$ finite. A relation between our result and the one in [15] is discussed in section 4.} We start with reconsidering the broken zero modes by giving an explicit form of them, and we check that the integration over the modes reproduces the result of [10]. Then we apply the method to the case of the correlation function. We find that our computation reproduces the gauge theory result approximately only in the limit $J \ll \sqrt{\lambda'}$. Deviations in other range is actually natural because our string configuration does not carry any angular momentum which corresponds to the R charge $J$ of the local operator. Hence only in the limit $J \ll \sqrt{\lambda'}$, the configuration is acceptable. We will see that the limit still allows us to go beyond the purely classical limit.

The paper is organized as follows. In section 2, we introduce the 1/4 BPS Wilson loop operator and review some known facts about the operator and its gravity dual. In section 3, we discuss an explicit form of the broken zero modes and show that their contributions reproduce the planar limit of the Wilson loop expectation value as discussed in [10]. In section 4, we apply the method of the broken zero modes to the correlation function of the 1/4 BPS Wilson loop and the 1/2 BPS local operator. Section 5 is devoted to a summary and discussions.
2 A review of 1/4 BPS Wilson loop operators

2.1 1/4 BPS Wilson loop operators and modified Bessel functions

A Wilson loop operator in the $\mathcal{N} = 4$ super Yang-Mills (SYM) theory is characterized by the shape of the loop $x_i(\tau) \, (i = 1 \ldots 4)$ in the Euclidean four-dimensional space and also another set of “coordinates” $\Theta_I(\tau) \, (I = 1 \ldots 6)$, which specifies how the operator depends on the 6 scalar fields $\Phi_I$ [4][5]:

$$ W(C) = \frac{1}{N} \text{tr} P \exp \oint \left( iA_i(\vec{x}(\tau)) \dot{x}_i(\tau) + \left| \dot{\vec{x}}(\tau) \right| \Theta_I(\tau) \Phi_I(\vec{x}(\tau)) \right) d\tau. \quad (2.1) $$

In this paper, we take the following choice [10]:

$$ \vec{x}(\tau) = ( a \cos \tau, a \sin \tau, 0, 0 ), \quad (2.2) $$
$$ \vec{\Theta}(\tau) = ( \sin \theta_0 \cos \tau, \sin \theta_0 \sin \tau, \cos \theta_0, 0, 0, 0 ). \quad (2.3) $$

The operator depends on two parameters $a$ and $\theta_0$. We assume that the range of $\theta_0$ is $0 \leq \theta_0 \leq \pi/2$. The operator preserves 1/4 of the supersymmetries in the SYM theory for a generic value of $\theta_0$.

The planar limit of the expectation value of the operator is studied in [10] based on the analysis of [16][17]. The result is given in terms of the modified Bessel function $I_1$ as follows:

$$ \langle W(C) \rangle = \frac{2}{\sqrt{\lambda'}} I_1(\sqrt{\lambda'}). \quad (2.4) $$

Here, as mentioned in the introduction, $\lambda'$ is related to the 't Hooft coupling constant $\lambda$ as $\lambda' = \lambda \cos^2 \theta_0$.

2.2 String solutions in the gravity side

In the gravity side, counterparts of (2.4) is formally given in terms of a string path integral as follows:

$$ \langle W(C) \rangle \leftrightarrow \int e^{-S}. \quad (2.5) $$

Here, the right hand side (the gravity side) is expressed only symbolically. $S$ is a total string world sheet action which includes a boundary term. We assume that the bulk part of it is the Polyakov type in the conformal gauge:

$$ S_{\text{bulk}} = \frac{\sqrt{\lambda}}{4\pi} \int d\tau d\sigma g_{MN} \partial_a \mathcal{X}^M(\tau, \sigma) \partial_a \mathcal{X}^N(\tau, \sigma). \quad (2.6) $$

Here $M$ and $N$ are ten-dimensional indices and $a$ is a world sheet index. $g_{MN}$ are the components of the metric for the AdS$_5 \times$S$^5$ background, while $\mathcal{X}^M(\tau, \sigma)$ are string coordinates which
are functions of world sheet coordinates \((\tau, \sigma)\). For simplicity, we use the unit in which the common radius of the AdS\(_5\) and the S\(_5\) is taken to be 1, then the ’t Hooft coupling constant \(\lambda\) and the Regge slope \(\alpha'\) are related by \(\sqrt{\lambda} = 1/\alpha'\). As for the boundary term, we assume the one proposed in [18] throughout this paper. In order to avoid unimportant complexities, detailed computations of the boundary terms are summarized in appendix A. The symbol \(\int\) in (2.5) expresses the path integral of the string world sheet, the precise definition of which is beyond the scope of the present paper. The boundary conditions for the AdS\(_5\) coordinates, other than the radial direction, are given by (2.2), while those for the S\(_5\) are given by (2.3).

In [10], the correspondence (2.5) for a generic value of \(\theta_0\) is studied by evaluating the path integral at solutions for equations of motion. Then the large \(\lambda'\) limit of the modified Bessel function is correctly reproduced. Let us briefly review the string solutions.

### 2.2.1 Classical solution: AdS\(_5\)-part

Let \(\vec{X} = (X_0, X_1, X_2, X_3, X_4, X_5)\) be coordinates of a flat R\(^{1,5}\) space with a line element

\[
d s^2 = d\vec{X} \cdot d\vec{X} = -dX_0^2 + dX_1^2 + dX_2^2 + dX_3^2 + dX_4^2 + dX_5^2. \quad (2.7)
\]

Here, the inner product “\(\cdot\)” for \(\vec{X}\) is defined with the signatures \((-\cdots, +\cdots, +\cdots)\). Then the Euclidean AdS\(_5\) can be expressed as a hypersurface defined by the following equation:

\[
\vec{X} \cdot \vec{X} = -X_0^2 + X_1^2 + X_2^2 + X_3^2 + X_4^2 + X_5^2 = -1. \quad (2.8)
\]

An interesting feature of the string solutions dual to the Wilson loop (2.1) with (2.2) and (2.3) is that the AdS\(_5\)-part of it is independent of the parameter \(\theta_0\). This means that it is the same configuration as the 1/2 BPS Wilson loop [14][18], which corresponds to the case with \(\sin \theta_0 = 0\). The explicit form of the solution in this coordinate system is given by

\[
X_0 = \coth \sigma, \quad X_1 = \cosech \sigma \cos \tau, \quad X_2 = \cosech \sigma \sin \tau, \quad X_3 = X_4 = X_5 = 0. \quad (2.9)
\]

Here, ranges of the world sheet coordinates \((\tau, \sigma)\) are \(0 \leq \tau < 2\pi\) and \(0 \leq \sigma \leq \infty\), respectively. The solution satisfies the following equations:

\[
\partial_\tau \vec{X} \cdot \partial_\tau \vec{X} = \partial_\sigma \vec{X} \cdot \partial_\sigma \vec{X} = \cosech^2 \sigma, \quad \partial_\tau \vec{X} \cdot \partial_\sigma \vec{X} = 0. \quad (2.10)
\]

Hence, the Virasoro constraints need to be “satisfied separately” by the AdS\(_5\)-part and the S\(_5\)-part.

In order to study the AdS/CFT correspondence, it is convenient to introduce the Poincaré coordinate \((z, \vec{x}) = (z, x_1, x_2, x_3, x_4)\) by the following coordinate transformation:

\[
X_0 = \frac{z^2 + (\vec{x})^2 + a^2}{2az}, \quad X_5 = \frac{z^2 + (\vec{x})^2 - a^2}{2az}, \quad X_i = \frac{x_i}{z} \quad (i = 1 \ldots 4), \quad (2.11)
\]
where \((\vec{x})^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2\). The line element in this coordinate is given by
\[
ds^2 = d\vec{X} \cdot d\vec{X} = \frac{dz^2 + (d\vec{x})^2}{z^2} ,
\]
and the solution (2.9) is expressed as follows:
\[
z = a \tanh \sigma , \quad x_1 = a \text{sech} \sigma \cos \tau , \quad x_2 = a \text{sech} \sigma \sin \tau , \quad x_3 = x_4 = 0.
\]
It is attached to the path (2.2) on the AdS\(_5\) boundary \(z = 0\), at \(\sigma = 0\).

### 2.2.2 Classical solutions: S\(^5\)-part

Let \(\vec{Y} = (Y_1, Y_2, Y_3, Y_4, Y_5, Y_6)\) be coordinates of a flat \(\mathbb{R}^6\) space with a line element
\[
ds^2 = d\vec{Y} \cdot d\vec{Y} = dY_1^2 + dY_2^2 + dY_3^2 + dY_4^2 + dY_5^2 + dY_6^2 .
\]
The inner product “\(\cdot\)” for \(\vec{Y}\) is defined with the all positive signatures. The \(S^5\) we consider is embedded in the space by the following equation:
\[
\vec{Y} \cdot \vec{Y} = Y_1^2 + Y_2^2 + Y_3^2 + Y_4^2 + Y_5^2 + Y_6^2 = 1.
\]
For a generic value of \(\theta_0\), there are two string solutions corresponding to the Wilson loop:
\[
Y_1 = \text{sech}(\sigma_0 \pm \sigma) \cos \tau , \quad Y_2 = \text{sech}(\sigma_0 \pm \sigma) \sin \tau , \quad Y_3 = \tanh(\sigma_0 \pm \sigma) , \quad Y_4 = Y_5 = Y_6 = 0.
\]
Here, the parameter \(\sigma_0\) is related to \(\theta_0\) by \(\tanh \sigma_0 = \cos \theta_0\), and the following boundary conditions are satisfied:
\[
\vec{Y} \big|_{\sigma=0} = (\sin \theta_0 \cos \tau , \sin \theta_0 \sin \tau , \cos \theta_0 , 0 , 0 , 0). \quad (2.17)
\]
As mentioned below (2.10), each configuration of (2.16) satisfies the following \(S^5\)-part of the Virasoro constraints:
\[
\partial_\tau \vec{Y} \cdot \partial_\sigma \vec{Y} = \partial_\sigma \vec{Y} \cdot \partial_\sigma \vec{Y} = \text{sech}^2(\sigma_0 \pm \sigma) , \quad \partial_\tau \vec{Y} \cdot \partial_\sigma \vec{Y} = 0. \quad (2.18)
\]

### 2.3 Zero modes and broken zero modes of the string world sheet

One of interesting developments in [10] about the 1/4 BPS Wilson loop is that the exact integral representation of the modified Bessel function is reproduced by taking account of broken zero modes. In the rest of this section, we focus on the \(S^5\)-part and discuss the idea of the broken zero modes. As for the AdS\(_5\)-part of the configuration, we assume (2.13).
2.3.1 Zero modes in the case $\theta_0 = \pi/2$

Let us consider the case with $\cos \theta_0 = 0$, i.e., $\theta_0 = \pi/2$. In this case, the boundary conditions for the $S^5$ coordinates are given by the following great circle on the $Y_1Y_2$-plane:

$$\bar{Y} \big|_{\sigma=0} = (\cos \tau, \sin \tau, 0, 0, 0). \quad (2.19)$$

There is a three-parameter family of string solutions satisfying the boundary condition. It is given by the following configuration

$$Y_1 = \text{sech} \sigma \cos \tau,$$
$$Y_2 = \text{sech} \sigma \sin \tau,$$
$$Y_3 = \tanh \sigma \cos \alpha,$$
$$Y_4 = \tanh \sigma \sin \alpha \cos \beta,$$
$$Y_5 = \tanh \sigma \sin \alpha \sin \beta \cos \gamma,$$
$$Y_6 = \tanh \sigma \sin \alpha \sin \beta \sin \gamma. \quad (2.20)$$

Here, $\alpha$, $\beta$ and $\gamma$ are the three constant parameters, the ranges of which are $0 \leq \alpha \leq \pi$, $0 \leq \beta \leq \pi$ and $0 \leq \gamma < 2\pi$. The solution (2.16), with $\sigma_0 = 0$, corresponds to the configuration (2.20) with $\alpha = 0, \pi$. These three parameters correspond to zero modes, i.e., the string action does not depend on them. In fact, the total action including the boundary term for these configurations turns out to be zero. Then, if we approximate the string path integral by integrating over only these zero modes, we obtain

$$\int \frac{d\Omega_3}{2\pi^2} e^{-S|_{0\text{-modes}}} = 1, \quad (S|_{0\text{-modes}} = 0). \quad (2.21)$$

Here, the integral measure is defined by $d\Omega_3 = d\alpha d\beta d\gamma \sin^2 \alpha \sin \beta$ and we have fixed the normalization so that the gauge theory result, $\langle W(C) \rangle = 1$, is reproduced. We use this unique normalization throughout this paper. This special case of the Wilson loop and its gravity dual are first studied in [19]. The existence of the three zero modes and also the fact that string total action becomes zero are found in that paper.

2.3.2 Broken zero modes for $\theta_0 \sim \pi/2$

For a generic value of $\theta_0$ there are only two solutions which are given by (2.16), and there are no zero modes around the solutions. This is because the non-vanishing third component $\cos \theta_0$ in (2.17) breaks the $S^3$ symmetry. Although the boundary condition has $S^2$ symmetry, rotations in $Y_4Y_5Y_6$-space do not generate any independent solutions since the world sheet (2.16) is sitting at the origin of that space. Evaluating the string path integral by the classical solution (2.16), the saddle point values of the modified Bessel function, $e^{\pm \sqrt{\lambda}}$, are reproduced [10].
Beyond such classical analyses, a remarkable success in the limit \( \theta_0 \sim \pi/2 \) is given in [10]. There, it is argued that for a small deviation from the \( S^3 \) symmetric case of \( \theta_0 = \pi/2 \), the symmetry is broken only slightly, and the modes corresponding to the zero modes in the symmetric case should still have significant contributions to the string path integral. By following [10], we call such modes as the “broken zero modes”. In fact, it is reported if we take the limit \( \lambda \to \infty \) and \( \cos \theta_0 \to 0 \) with keeping the combination \( \lambda' = \lambda \cos^2 \theta_0 \) finite, then the broken zero modes give rise to a finite potential \( S_{|\phi=0\text{-modes}} = -\cos \alpha \sqrt{\lambda'} \).‡ By taking account of only these broken zero modes, the string path integral is approximated as

\[
\int \frac{d\Omega_3}{2\pi^2} e^{-S_{|\phi=0\text{-modes}}} = \frac{2}{\pi} \int d\alpha \sin^2 \alpha e^{\cos \alpha \sqrt{\lambda'}} = \frac{2}{\sqrt{\lambda'}} I_1(\sqrt{\lambda'}). \tag{2.22}
\]

This agrees with the planar limit of the Wilson loop expectation value.

An important point here is that it is \textit{not} a finite \( \lambda \) result, but the large \( \lambda \) limit is taken. This implies that we do not need to care about corrections coming from other generic string fluctuations, since they are expected to be suppressed by inverse powers of \( \sqrt{\lambda} \). Note also that small corrections which are of higher orders with respect to \( \cos \theta_0 \) are suppressed since \( \cos \theta_0 = \sqrt{\lambda'}/\sqrt{\lambda} \) and \( \lambda' \) is now finite.

3 More about the broken zero modes

In this section, we discuss an explicit form of the \( S^5 \)-part of the broken zero modes and reproduce the result of [10]. We also give a more systematic argument by assuming a symmetric ansatz. As for the \( \text{AdS}_5 \)-part we assume the same configuration (2.13).

3.1 An explicit form of the broken zero modes

The idea is to construct a three-parameter family of string configurations satisfying the common boundary condition (2.17). In the case \( \theta_0 = \pi/2 \) the three parameters should be identical to those in (2.20). So, let us call them \( \alpha, \beta \) and \( \gamma \). We also expect that the configuration reduces to the solution (2.16) for specific values of the parameter, i.e., \( \alpha = 0, \pi \).

It is not difficult to check that the following configuration satisfies all of these requirements:

\[
\begin{align*}
Y_1 &= Y_1(\tau, \sigma, \alpha) = f(\sigma, \alpha) \cos \tau , \\
Y_2 &= Y_2(\tau, \sigma, \alpha) = f(\sigma, \alpha) \sin \tau , \\
Y_3 &= Y_3(\sigma, \alpha) = f(\sigma, \alpha)(\cosh \sigma_0 \sinh \sigma \cos \alpha + \sinh \sigma_0 \cosh \sigma), \\
Y_4 &= Y_4(\sigma, \alpha, \beta) = f(\sigma, \alpha) \sinh \sigma \sin \alpha \cos \beta , \\
Y_5 &= Y_5(\sigma, \alpha, \beta, \gamma) = f(\sigma, \alpha) \sinh \sigma \sin \alpha \sin \beta \cos \gamma , \\
Y_6 &= Y_6(\sigma, \alpha, \beta, \gamma) = f(\sigma, \alpha) \sinh \sigma \sin \alpha \sin \beta \sin \gamma,
\end{align*}
\tag{3.1}
\]

‡Note that our parameter \( \alpha \) is related to the one in the paper [10] by \( \alpha \leftrightarrow \pi - \alpha \).
where $f(\sigma, \alpha)$ is defined by

$$f(\sigma, \alpha) = \frac{1}{\cosh \sigma_0 \cosh \sigma + \sinh \sigma_0 \sinh \sigma \cos \alpha}. \quad (3.2)$$

The configuration also satisfies the following equations

$$\partial_\tau \vec{Y} \cdot \partial_\tau \vec{Y} = \partial_\sigma \vec{Y} \cdot \partial_\sigma \vec{Y} = f^2(\sigma, \alpha), \quad \partial_\tau \vec{Y} \cdot \partial_\sigma \vec{Y} = 0. \quad (3.3)$$

These equations mean that the Virasoro constraints are satisfied by the $S^5$-part separately from the AdS$_5$-part. Hence, the configuration (2.13) for the AdS$_5$-part can be used without any change.

### 3.2 $\alpha$, $\beta$, $\gamma$ as $S^5$ coordinates

Although it is not clear, (3.1) with (3.2) defines a coordinate system $(\tau, \sigma, \alpha, \beta, \gamma)$ of an $S^5$, which covers the entire $S^5$ if we take the ranges as follows:

$$0 \leq \tau < 2\pi, \quad 0 \leq \sigma \leq \infty, \quad 0 \leq \alpha \leq \pi, \quad 0 \leq \beta \leq \pi, \quad 0 \leq \gamma < 2\pi. \quad (3.4)$$

This property assures that different sets of the parameters $(\alpha, \beta, \gamma)$ give different string configurations, i.e., they are not related through any redefinition of world sheet coordinates $(\tau, \sigma)$.

Let us confirm the property briefly. First we introduce the radial coordinate $R$ in the $Y_4Y_5Y_6$-space:

$$R = \sqrt{Y_4^2 + Y_5^2 + Y_6^2}. \quad (3.5)$$

For (3.1), it is given by

$$R(\sigma, \alpha) = f(\sigma, \alpha) \sinh \sigma \sin \alpha, \quad (3.6)$$

and we find the following relation:

$$\tan \alpha Y_3(\sigma, \alpha) - \text{sech} \sigma_0 R(\sigma, \alpha) = \tanh \sigma_0 \tan \alpha, \quad (3.7)$$

where the function $Y_3(\sigma, \alpha)$ is defined in (3.1). For a given value of $\alpha$, $R(\sigma, \alpha)$ is a monotonically increasing function of $\sigma$. Hence, as $\sigma$ increases from 0 to $\infty$, the point $(Y_3, R) = (Y_3(\sigma, \alpha), R(\sigma, \alpha))$ moves monotonically along the line segment between the following two points:

$$\sigma = 0 : \begin{cases} Y_3 = \tanh \sigma_0, \\ R = 0, \end{cases} \quad \sigma = \infty : \begin{cases} Y_3 = \overline{Y}_3(\alpha) \equiv \frac{\cosh \sigma_0 \cos \alpha + \sinh \sigma_0}{\cosh \sigma_0 + \sinh \sigma_0 \cos \alpha}, \\ R = \overline{R}(\alpha) \equiv \frac{\sin \alpha}{\cosh \sigma_0 + \sinh \sigma_0 \cos \alpha}. \end{cases} \quad (3.8)$$

On the other hand, if we increase $\alpha$ from 0 to $\pi$, the end point $(Y_3, R) = (\overline{Y}_3(\alpha), \overline{R}(\alpha))$ of the line segment moves, again monotonically, on the semi-circle which is defined by $Y_3^2 + R^2 = 1$. 

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and \( R \geq 0 \). More precisely, \( Y_3(\alpha) \) is a monotonically decreasing function of \( \alpha \). This means that by changing \( \sigma \) and \( \alpha \), the point \((Y_3, R) = (Y_3(\sigma, \alpha), R(\sigma, \alpha))\) covers the half disk \( Y_3^2 + R^2 \leq 1, \ R \geq 0 \) exactly once. Next, if we choose some specific point on the half disk, then the corresponding slice of the \( S^5 \) is given by a direct product of an \( S^1 \) on the \( Y_1Y_2 \)-plane and an \( S^2 \) in the \( Y_4Y_5Y_6 \)-space, which are parametrized by the independent parameters \( \tau \) and \( (\beta, \gamma) \), respectively. Hence the entire \( S^5 \) is covered.

If we consider \((\tau, \sigma, \alpha, \beta, \gamma)\) as the \( S^5 \) coordinate system, the line element is given as follows:

\[
ds^2 = dY^2 \cdot dY = f^2(\sigma, \alpha) \left[ d\tau^2 + d\sigma^2 + \sinh^2 \sigma (d\alpha^2 + \sin^2 \alpha (d\beta^2 + \sin^2 \beta d\gamma^2)) \right].
\]

From this expression, we understand that if we consider \((\tau, \sigma)\) as the world sheet coordinates and \((\alpha, \beta, \gamma)\) as the constant parameters, then (3.3) is satisfied.

### 3.3 The broken zero modes and the modified Bessel function

By using (2.13) and (3.1), the total string action for the broken zero modes, \( S|_{\phi\text{-modes}} \), including the boundary term \( S_{\text{boundary}} \) is evaluated as follows:

\[
S|_{\phi\text{-modes}} = \frac{\sqrt{\lambda}}{4\pi} \int_0^{2\pi} d\tau \int_{\sigma_{\text{min}}}^{\infty} d\sigma \left( \partial_\tau \vec{X} \cdot \partial_\tau \vec{X} + \partial_\sigma \vec{X} \cdot \partial_\sigma \vec{X} + \partial_\tau \vec{Y} \cdot \partial_\tau \vec{Y} + \partial_\sigma \vec{Y} \cdot \partial_\sigma \vec{Y} \right) + S_{\text{boundary}}
\]

\[
= \sqrt{\lambda} \int_{\sigma_{\text{min}}}^{\infty} d\sigma \left( \cosech^2 \sigma + f^2(\sigma, \alpha) \right) + S_{\text{boundary}}
\]

\[
= \sqrt{\lambda} \left[ - \coth \sigma + \frac{1}{\cos \alpha} \left( \frac{\tanh \sigma \cos \alpha + \tanh \sigma_0}{1 + \tanh \sigma_0 \tanh \sigma \cos \alpha} \right) \right]_{\sigma_{\text{min}}}^{\infty} + S_{\text{boundary}}
\]

\[
= \sqrt{\lambda} \coth \sigma_{\text{min}} - \sqrt{\lambda} \tanh \sigma_0 \left[ \cos \alpha + \tanh \sigma_0 \frac{\sin^2 \alpha}{1 + \tanh \sigma_0 \cos \alpha} \right] + S_{\text{boundary}}. \tag{3.10}
\]

In the final expression, the cutoff parameter \( \sigma_{\text{min}} \) is set to be 0 except for the first term \( \sqrt{\lambda} \coth \sigma_{\text{min}} \) (and also for the boundary term). As explained in appendix A, this divergent term is cancelled by the boundary term \( S_{\text{boundary}} \). We take the limit \( \sigma_0 \to 0 \) and \( \lambda \to \infty \) with keeping \( \sqrt{\lambda} = \sqrt{\lambda} \tanh \sigma_0 = \sqrt{\lambda} \cos \theta_0 \) finite, and obtain

\[
S|_{\phi\text{-modes}} = - \cos \alpha \sqrt{\lambda} + \mathcal{O}(\tanh \sigma_0) = - \cos \alpha \sqrt{\lambda} + \mathcal{O}(1/\sqrt{\lambda}). \tag{3.11}
\]

This is the result given in [10] and the modified Bessel function is reproduced as (2.22).

### 3.4 Some general arguments for small \( \sigma_0 \)

In the previous subsection, we discussed the broken-zero-mode configurations in an ad hoc manner. If we assume small \( \sigma_0 \) from the beginning, it is also possible to restrict the configuration more systematically.

Let us start with the following four assumptions for the broken zero modes:
1. The configurations reduce to the exact zero modes (2.20) in the limit \( \sigma_0 \to 0 \).

2. The configurations preserve \( S^1 \) symmetry on the \( Y_1Y_2 \)-plane corresponding to \( \tau \).

3. \((\beta, \gamma)\) describe flat directions corresponding to the \( S^2 \) on the \( Y_4Y_5Y_6 \)-space.

4. The Virasoro constraints are satisfied separately by the \( S^5 \)-part and by the \( AdS^5 \)-part.

The first assumption is a necessary condition for the broken zero modes, while the others are just for simplification. A more generic argument is beyond the scope of this paper. The configurations which satisfy these assumptions are written as follows:

\[
\begin{align*}
Y_1 &= \left( \text{sech} \, \sigma + \tanh \sigma_0 y_1(\sigma, \alpha) + O(\tanh^2 \sigma_0) \right) \cos \tau, \\
Y_2 &= \left( \text{sech} \, \sigma + \tanh \sigma_0 y_1(\sigma, \alpha) + O(\tanh^2 \sigma_0) \right) \sin \tau, \\
Y_3 &= \tanh \sigma \cos \alpha + \tanh \sigma_0 y_2(\sigma, \alpha) + O(\tanh^2 \sigma_0), \\
Y_4 &= \left( \tanh \sigma \sin \alpha + \tanh \sigma_0 y_3(\sigma, \alpha) + O(\tanh^2 \sigma_0) \right) \cos \beta, \\
Y_5 &= \left( \tanh \sigma \sin \alpha + \tanh \sigma_0 y_3(\sigma, \alpha) + O(\tanh^2 \sigma_0) \right) \sin \beta \cos \gamma, \\
Y_6 &= \left( \tanh \sigma \sin \alpha + \tanh \sigma_0 y_3(\sigma, \alpha) + O(\tanh^2 \sigma_0) \right) \sin \beta \sin \gamma.
\end{align*}
\]

Here, \( y_1(\sigma, \alpha) \), \( y_2(\sigma, \alpha) \), \( y_3(\sigma, \alpha) \) are undetermined functions. The \( S^5 \) condition \( \vec{Y} \cdot \vec{Y} = 1 \) and the Virasoro constraint for the \( S^5 \)-part, \( \partial_\tau \vec{Y} \cdot \partial_\tau \vec{Y} = \partial_\sigma \vec{Y} \cdot \partial_\sigma \vec{Y} \), require

\[
\begin{align*}
0 &= \text{sech} \, \sigma y_1 + \tanh \sigma \cos \alpha y_2 + \tanh \sigma \sin \alpha y_3, \\
0 &= \frac{1}{\cosh \sigma} y_1 + \frac{\sinh \sigma}{\cosh^2 \sigma} \partial_\sigma y_1 - \frac{1}{\cosh^2 \sigma} \cos \alpha \partial_\sigma y_2 - \frac{1}{\cosh^2 \sigma} \sin \alpha \partial_\sigma y_3.
\end{align*}
\]

The boundary condition (2.17) requires

\[
y_1(0, \alpha) = 0, \quad y_2(0, \alpha) = 1, \quad y_3(0, \alpha) = 0.
\]

The solution for these conditions is given by

\[
y_1(\sigma, \alpha) = -\cos \alpha \frac{\sinh \sigma}{\cosh^2 \sigma}, \quad \cos \alpha y_2(\sigma, \alpha) + \sin \alpha y_3(\sigma, \alpha) = \frac{\cos \alpha}{\cosh^2 \sigma}.
\]

Although there remains one arbitrary function, which we can take \( y_2(\sigma, \alpha) \) or \( y_3(\sigma, \alpha) \) for example, it does not affect the computation of the string action.\footnote{Another constraint \( \partial_\tau \vec{Y} \cdot \partial_\sigma \vec{Y} = 0 \) is satisfied.} In fact, by using (3.12) and (3.16), the \( S^5 \)-part of the bulk action is computed as follows:

\[
S_{\text{bulk, } S^5|\phi \text{-modes}} = \frac{\sqrt{\lambda}}{4\pi} \int_0^{2\pi} d\tau \int_0^\infty d\sigma \left( \partial_\tau \vec{Y} \cdot \partial_\tau \vec{Y} + \partial_\sigma \vec{Y} \cdot \partial_\sigma \vec{Y} \right)
\]
\[
\sqrt{\lambda} \int_0^\infty d\sigma \left( \frac{1}{\cosh^2 \sigma} + 2 \tanh \sigma_0 \frac{y_1(\sigma, \alpha)}{\cosh \sigma} + O(\tanh^2 \sigma_0) \right) \quad (3.18)
\]

\[
\sqrt{\lambda} \left[ 1 - \cos \alpha \tanh \sigma_0 + O(\tanh^2 \sigma_0) \right]. \quad (3.19)
\]

It correctly reproduces the $S^5$-part of (3.10).

### 4 Correlation function between a Wilson loop and a local operator

Next let us consider the effect of the broken zero modes on the correlation function of the 1/4 BPS Wilson loop (2.1) and the following local operator:

\[
O_J = (2\pi)^J \sqrt{J\lambda} J \text{tr}(\Phi_3 + i\Phi_4)^J. \quad (4.1)
\]

The overall factor is taken so that the large $N$ limit of the two point function is normalized as

\[
\langle O_J^\dagger(\vec{x}_1) O_J(\vec{x}_2) \rangle = \frac{1}{|\vec{x}_1 - \vec{x}_2|^{2J}}. \quad (4.2)
\]

The large $N$ limit of the correlation function is well studied in the gauge theory side \cite{20,11,12}. It is given in terms of the modified Bessel functions as follows \cite{11}:

\[
\frac{\langle W(C) O_J(\vec{x}) \rangle}{\langle W(C) \rangle} = \frac{1}{2N} \frac{a^J}{\ell^{2J}} \sqrt{\lambda} I_J(\sqrt{\lambda}) I_1(\sqrt{\lambda}). \quad (4.3)
\]

Here we assume that the position of the local operator is $\vec{x} = (0, 0, 0, \ell)$, and also that the distance $\ell$ between the center of the loop $C$ and the position of the local operator is much larger than the radius $a$ of the loop, i.e., $a \ll \ell$. It was found in \cite{11} that the system with the 1/4 BPS Wilson loop and the 1/2 BPS local operator preserves 1/8 of the supersymmetries of the SYM theory. We review the supersymmetries in appendix C.

The gravity dual of the correlation function has been also well studied \cite{11,12,13,14,15}. The basic idea is to consider a propagation of a bulk mode between the local operator and the Wilson loop, both of which are assumed to be on the AdS boundary. In order to explain the idea, let us review the result of \cite{11} in subsection 4.1,\footnote{In \cite{11}, a two point function of the Wilson loops is considered in order to compute the correlation function. This is another standard approach \cite{14}. See \cite{23}, for example, for the same approach as the present paper.} where the string world sheet dual to the Wilson loop is treated as classical. Then in subsection 4.2, we take account of the contribution from the broken zero modes.
4.1 Review of an analysis without broken zero modes

In literature, the correlation function in the gravity side is computed based on the following correspondence:

\[
\langle W(C) O_J(\vec{x}, J) \rangle \leftrightarrow \delta \sum \frac{δ}{δ s^\alpha_0(\vec{x})} e^{-S[s^\alpha_0(\vec{x})]} \bigg|_{s^\alpha_0 = 0, \text{ classical}} \frac{1}{\sum e^{-S[s^\alpha_0(\vec{x})]} \bigg|_{s^\alpha_0 = 0, \text{ classical}}}, \tag{4.4}
\]

Here, the total string action \( S[s^\alpha_0(\vec{x})] \) includes the same boundary term \( S_{\text{boundary}} \) as before (see appendix A), and the bulk part of it is given by

\[
S[s^\alpha_0(\vec{x})]_{\text{bulk}} = \sqrt{\lambda} \int d\tau d\sigma G_{\mu\nu} \partial_\mu \chi^M \partial_\nu \chi^N. \tag{4.5}
\]

As explained shortly, the geometry \( G_{\mu\nu} \) depends on a source \( s^\alpha_0(\vec{x}) \) of the local operator \( O_J \). Both the denominator and the numerator of (4.4) are evaluated at \( s^\alpha_0 = 0 \) and also at classical world sheets as indicated by “classical”. The symbols \( \sum \) in (4.4) express summations over classical world sheet solutions. In the present case, we have two solutions which are given in (2.16).

The source \( s^\alpha_0(\vec{x}) \) is included in a small fluctuation \( h_{\mu\nu} \) of the geometry \( G_{\mu\nu} \) around the AdS\(_5 \times S^5 \) background \( g_{\mu\nu} \)

\[
G_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}. \tag{4.6}
\]

Equations of motion and constraints for such fluctuations are diagonalized in [22], and the metric fluctuation corresponding to the local operator \( O_J \) is given by

\[
h^{\text{AdS}}_{\mu\nu} = \left[ -\frac{6J}{5} g^{\text{AdS}}_{\mu\nu} + \frac{4}{J+1} D(\mu D\nu) \right] s^J(\vec{x}, z) \mathcal{Y}_J, \tag{4.7}
\]

\[
h^{S}_{\alpha\beta} = 2J g^{S}_{\alpha\beta} s^J(\vec{x}, z) \mathcal{Y}_J. \tag{4.8}
\]

Here, the ten-dimensional indices \( M, \ N \) are decomposed into two sets; \( \mu, \nu \) and \( \alpha, \beta \). The former set is the indices for the AdS\(_5\)-part and the latter is for the S\(_5\)-part. \( g^{\text{AdS}}_{\mu\nu} \) and \( g^{S}_{\alpha\beta} \) are the indicated components of the background metric \( g_{\mu\nu} \), while \( h^{\text{AdS}}_{\mu\nu} \) and \( h^{S}_{\alpha\beta} \) are their fluctuations. The round bracket in \( D(\mu D\nu) \) expresses the symmetric traceless part, and \( \mathcal{Y}_J \) is the spherical harmonics on the S\(_5\), which is defined by

\[
\mathcal{Y}_J = \frac{1}{2J+1} (Y_3 + i Y_4)^J. \tag{4.9}
\]

Here, \( Y_3 \) and \( Y_4 \) are the coordinates on the unit S\(_5\). The classical solution for the bulk field \( s^J(\vec{x}, z) \) is written in terms of a boundary function \( s^\alpha_0(\vec{x}) \) by using the Green’s function as

\[
s^J(\vec{x}, z) = \int d^4x' G(\vec{x}, z; \vec{x}') s^\alpha_0(\vec{x}'), \tag{4.10}
\]

\[
G(\vec{x}, z; \vec{x}') = c \left( \frac{z}{z^2 + (\vec{x} - \vec{x}')^2} \right)^J. \tag{4.11}
\]
The constant $c$ is given by
\[ c = \frac{2^{\frac{J}{2}} - 2 (J + 1)}{N \sqrt{J}}, \tag{4.12} \]
which is chosen so that the two point function of the local operators, computed in the gravity side, satisfies the same normalization as the gauge theory side (4.2).

In (4.4), the derivative with respect to the source $s_0^J(\vec{x})$ acts on $h_{MN}$ as
\[ \frac{\delta}{\delta s_0^J(\vec{x})} e^{-S[s_0^J(\vec{x})]} \bigg|_{s_0^J = 0} = e^{-S[0]} \left[ -\frac{\sqrt{\lambda}}{4\pi} \int d\tau d\sigma \left( \frac{\delta}{\delta s_0^J(\vec{x})} h_{MN} \right) \partial_a \mathcal{X}^M \partial_a \mathcal{X}^N \right]. \tag{4.13} \]

The AdS$_5$-part and the S$^5$-part of the derivatives are given as follows:
\[ \frac{\delta}{\delta s_0^J(\vec{x})} h_{\mu\nu}^{\text{AdS}} = \left[ -\frac{6J}{5} g_{\mu\nu}^{\text{AdS}} + \frac{4}{J + 1} D_\mu D_\nu \right] G(\vec{x}, z; \vec{x}^J) \mathcal{Y}_J \sim 2J \left[ -g_{\mu\nu}^{\text{AdS}} + \frac{2}{\ell^2} \delta_\mu^{\bar{\nu}} \right] c \frac{z^J}{\ell^2} \mathcal{Y}_J, \tag{4.14} \]
\[ \frac{\delta}{\delta s_0^J(\vec{x})} h_{\alpha\beta}^{\text{S}} = 2J g_{\alpha\beta}^{\text{S}} G(\vec{x}, z; \vec{x}^J) \mathcal{Y}_J \sim 2J g_{\alpha\beta}^{\text{S}} c \frac{z^J}{\ell^2} \mathcal{Y}_J. \tag{4.15} \]

Here, we have assumed that the distance $\ell$ between the center of the loop and the position of the local operator is much larger than the radius $a$ of the loop.

By using the solutions (2.13) and (2.16), the classical value of (4.13) is evaluated as
\[ e^{\pm \sqrt{\lambda}} \left[ \frac{\sqrt{\lambda}}{2} \frac{4J}{\ell^2 J} a^J \int_0^\infty d\sigma \left( \frac{1}{\cosh^2 \sigma} - \frac{1}{\cosh^2 (\sigma_0 \pm \sigma)} \right) \tanh^J \sigma \mathcal{Y}_J \right]. \tag{4.16} \]

Here, the spherical harmonics $\mathcal{Y}_J$ is now given by
\[ \mathcal{Y}_J = \frac{1}{2^{J/2}} \tanh^J (\sigma_0 \pm \sigma). \tag{4.17} \]

For simplicity, we consider only the small $\sigma_0$ limit. Then (4.16) is evaluated as follows:
\[ (\pm 1)^{J+1} e^{\pm \sqrt{\lambda}} \sqrt{\lambda} 4J c \frac{a^J}{\ell^2 J} \int_0^\infty d\sigma \frac{\tanh^{2J+1} \sigma}{\cosh^2 \sigma} = (\pm 1)^{J+1} \frac{1}{2N} \frac{a^J}{\ell^2 J} \sqrt{\lambda} e^{\pm \sqrt{\lambda}}. \tag{4.18} \]

This gives two contributions coming from the two string solutions. These are summed up in the numerator of (4.4). The denominator is the summation of $e^{\pm \sqrt{\lambda}}$. Then (4.18) correctly reproduces the prefactor of (4.3) in the large $\lambda'$ limit [11].

**As it is shown in appendix A, the derivative of the boundary term vanishes.**
4.2 Summing up broken zero modes

Let us now try to take account of quantum corrections coming from the broken zero modes. For this purpose, we further generalize the expression (4.4) so that the integration over these modes are included

\[
\langle W(C) \mathcal{O}_J(\vec{x}_J) \rangle \leftrightarrow \int \frac{d\Omega_3}{2\pi^2} \frac{\delta}{\delta s_0(\vec{x}_J)} e^{-S[\mathcal{O}_J(\vec{x}_J)]} \bigg|_{s_0=0, \phi\text{-modes}}.
\]

Here, the right hand side is not evaluated for the classical solution, but it is done so for the broken-zero-mode configurations (3.1) and (2.13). This fact is indicated in (4.19) as “\(\phi\text{-modes}\)”.

Since the expressions (4.14) and (4.15) for the functional derivatives are still valid, we just substitute the configuration (2.13) and (3.1) into them and obtain the following expression for the numerator of (4.19):

\[
\int \frac{d\Omega_3}{2\pi^2} e^{-S[0]}_{\phi\text{-modes}} \left[ -\frac{\sqrt{\lambda}}{4\pi} \int d\tau d\sigma \left( \frac{\delta}{\delta s_0(\vec{x}_J)} h_{MN} \right) \partial_a \mathcal{X}^M \partial^a \mathcal{X}^N \right] \bigg|_{\phi\text{-modes}}
\]

\[
= \int \frac{d\Omega_3}{2\pi^2} e^{\cos \alpha \sqrt{\lambda}} \left[ \sqrt{\lambda} \frac{4J}{2^J} \alpha^J \int d\sigma \left( \frac{1}{\cosh^2 \sigma} - f^2(\sigma, \alpha) \right) \tanh^J \sigma \mathcal{Y}_J \right].
\]

The spherical harmonics \(\mathcal{Y}_J\) is also evaluated by using (3.1) as

\[
\mathcal{Y}_J = \frac{f^J(\sigma, \alpha)}{2^{J/2}} \left( \cosh \sigma_0 \sinh \sigma \cos \alpha + \sinh \sigma_0 \cosh \sigma + i \sin \sigma \sin \alpha \cos \beta \right)^J
\]

\[
\sim \frac{1}{2^{J/2}} \tanh^J \sigma (\cos \alpha + i \sin \alpha \cos \beta)^J + \mathcal{O}(\tanh \sigma_0).
\]

Then the \(\sigma\)-integral reduces to the one in (4.18) and we are left with the following integral over the broken zero modes:

\[
\int \frac{d\Omega_3}{2\pi^2} e^{\cos \alpha \sqrt{\lambda}} \cos \alpha \left( \cosh \sigma \sin \sigma \cos \alpha + \sinh \sigma \cosh \sigma + i \sin \sigma \sin \alpha \cos \beta \right)^J.
\]

The integral with respect to \(\beta\) and \(\gamma\) results in the following \(\alpha\) integral:

\[
\int \frac{d\alpha}{2N \ell^{2J} \sqrt{JN}} \times \frac{1}{\pi J + 1} \int_0^{\pi} d\alpha e^{\cos \alpha \sqrt{\lambda}} \sin \alpha \cos \alpha \sin((J + 1)\alpha).
\]

By integrating with respect to \(\alpha\), we obtain

\[
(4.24) = \frac{1}{N \ell^{2J} \sqrt{JN}} \left( 1 - \frac{J + 2 I_{J+1}(\sqrt{\lambda})}{I_J(\sqrt{\lambda})} \right).
\]

\[\text{††}\text{The derivative of the boundary term again vanishes. See appendix A.}\]
Here, the following equation is used:

\[
\frac{1}{\pi(J+1)} \int_0^\pi da e^{\cos \alpha z} \sin \alpha \cos \alpha \sin((J+1)\alpha) = \frac{1}{z} I_J(z) - \frac{J+2}{z^2} I_{J+1}(z),
\]

which is proved in appendix B. Finally, dividing (4.25) by (2.22) we obtain the following result:

\[
(4.19) = \frac{1}{2N^2} C^J \frac{J_1(\sqrt{\lambda})}{I_1(\sqrt{\lambda})} \left( 1 - \frac{J+2}{\sqrt{\lambda}} \frac{I_J(\sqrt{\lambda})}{I_1(\sqrt{\lambda})} \right).
\]

\[(4.27)\]

is the main result in this paper. Since \( J \) is an integer satisfying \( J \geq 2 \), and also since the modified Bessel functions satisfy the relation \( I_J(\sqrt{\lambda}) \geq I_{J+1}(\sqrt{\lambda}) \) for these values of \( J \), the second term in the round bracket is subleading in the limit \( J/\sqrt{\lambda} \ll 1 \). This means that (4.27) reproduces the gauge theory result in that limit. In fact, deviations in other range is expected, because the string world sheet we consider does not carry any angular momentum and the conservation of it is broken. Only in the limit \( J \ll \sqrt{\lambda} \), the world sheet configuration is approximately acceptable. Hence it is natural that our computation in the gravity side reproduces the gauge theory result only in the limit. In order to go beyond such an approximation, we need to consider world sheet configurations by taking account of effects of the operator insertion. Such an analysis for the classical solution is given in [15][12][13]. It would be interesting future work to study such effects on the broken-zero-mode configurations.

We would like to emphasize that (4.27) does not give the exact form of the expected modified Bessel function. When we take the limit \( J/\sqrt{\lambda} \ll 1 \) in (4.27), it affects not only the second term in the round bracket but also the ratio of the modified Bessel function \( I_J(\sqrt{\lambda})/I_1(\sqrt{\lambda}) \) itself. Then the result is not equal to the gauge theory result (4.3), but the limit of it.

Fortunately, the limit \( J/\sqrt{\lambda} \ll 1 \) still allows the range beyond the purely classical limit. An example is the limit in which \( \sqrt{\lambda} \) is taken to be large while the combination \( J^2/\sqrt{\lambda} \) is kept finite. This limit is suggested from the following asymptotic expansion of the modified Bessel function [24][25]:

\[
I_J(z) \sim \frac{e^z}{\sqrt{2\pi z}} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2z)^n n!} \Gamma(J + n + \frac{1}{2}) + \frac{e^{-z(J+\frac{1}{2})\pi i}}{\sqrt{2\pi z}} \sum_{n=0}^{\infty} \frac{\Gamma(J + n + \frac{1}{2})}{(2z)^n n!} \Gamma(J - n + \frac{1}{2}).
\]

The ratio of the Gamma functions is given by

\[
\frac{\Gamma(J + n + \frac{1}{2})}{(2z)^n n! \Gamma(J - n + \frac{1}{2})} = \frac{1}{2^n n!} \frac{J^2 - \frac{1^2}{2^2} J^2 - \frac{3^2}{2^2} \cdots J^2 - \frac{(2n-1)^2}{2^2}}{z} \approx \frac{1}{n!} \left( \frac{J^2}{2z} \right)^n.
\]

\[(4.29)\]

Here, in the last expression, we took the large \( z \) limit with keeping the combination \( J^2/\sqrt{\lambda} \) finite. If we use (4.29) in the summation of (4.28), we obtain

\[
I_J(z) \sim \frac{e^z}{\sqrt{2\pi z}} \sum_{n=0}^{\infty} \frac{1}{n!} \left( -\frac{J^2}{2z} \right)^n + \frac{e^{-z(J+\frac{1}{2})\pi i}}{\sqrt{2\pi z}} \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{J^2}{2z} \right)^n = \frac{e^{z^2/J^2}}{\sqrt{2\pi z}} \frac{e^{-z(J+\frac{1}{2})\pi i + J^2/2z}}{\sqrt{2\pi z}}.
\]

\[(4.30)\]
Then (4.27) reduces to the following expression in the limit:

\[
(4.27) \sim \frac{1}{2N} \frac{a'}{\ell^2} \sqrt{J} \lambda e^{-\lambda' \frac{a'^2}{\ell^2}} \cdot
\]

(4.31)

Here, we have kept only the leading term in the asymptotic expansion of the modified Bessel functions.

The same limit of the modified Bessel function is considered previously in the case of the 1/2 BPS Wilson loop in [15]. In that paper, the string world sheet action is expanded with respect to small fluctuations to quadratic order. By integrating the fluctuations, the result corresponding to (4.31), with a replacement \( \lambda' \rightarrow \lambda \), is reproduced. It would be interesting to consider the small fluctuations in the case of the 1/4 BPS Wilson loop and study the relation between these two approaches.

5 Summary and discussions

In this paper we studied broken zero modes of string world sheet which exist in the case of the gravity dual of the 1/4 BPS Wilson loop operator. As proposed in [10], we consider the limit \( \cos \theta_0 \rightarrow 0, \ \lambda \rightarrow \infty \) with keeping \( \lambda' = \lambda \cos^2 \theta_0 \) finite. In this limit, the broken zero modes give significant contributions to the string path integral, while the effects of other generic fluctuations are expected to be negligible.

We started by giving an explicit form of the configuration of the broken zero modes which depends on three parameters \( \alpha, \beta \) and \( \gamma \). The configuration satisfies the correct boundary conditions and the Virasoro constraints. It also reduces to the known solutions at \( \alpha = 0, \pi \). In the case with \( \cos \theta_0 = 0 \), the three parameters are identical to the exact zero modes. Since we take the small \( \cos \theta_0 \) limit, the leading corrections for the string configuration with respect to \( \cos \theta_0 \) would be enough to compute the string path integral. However, having an explicit smooth configuration, which includes all order terms of the small parameter \( \cos \theta_0 \), makes our arguments more convincing. Another property of the configuration is that the five parameters \( (\tau, \sigma, \alpha, \beta, \gamma) \) form a smooth coordinate system of the \( S^5 \). This clearly shows that two configurations with different values of parameters \( (\alpha, \beta, \gamma) \) describe different configurations of the world sheet, i.e., they are not related by any redefinition of the world sheet coordinates \( (\tau, \sigma) \). We have checked that our configuration reproduces the modified Bessel function as found in [10].

Since our argument for the explicit configuration is completely ad hoc, we also tried to derive the form of the broken zero modes more systematically by imposing appropriate conditions on the world sheet. For simplicity, we assumed an ansatz that respects the \( S^1 \) symmetry corresponding to \( \tau \), and also the \( S^2 \) flat directions of \( \beta \) and \( \gamma \). This allows three arbitrary functions of \( \sigma \) and \( \alpha \). Although the conditions on the world sheet still leave one arbitrary function undetermined, we find that the result is not affected by the choice of it.
In section 4, we studied the gravity dual of the correlation function between the 1/4 BPS Wilson loop operator and the 1/2 BPS local operator by taking account of the broken zero modes. The resulting expression (4.27) agrees with the gauge theory result only in the limit $J \ll \sqrt{\lambda}$. Deviation at the outside of the range is attributed to the fact that the world sheet we consider does not carry any angular momentum. In other words, the string configuration is determined by neglecting the effects of the operator insertions. Hence the configuration is valid only in the limit $J \ll \sqrt{\lambda}$, in which the effects of the angular momentum is negligible. Fortunately, the limit $J \ll \sqrt{\lambda}$ still allows a check of the duality beyond the purely classical analysis. By considering the limit $J/\sqrt{\lambda} \ll 1$ with keeping the combination $J^2/\sqrt{\lambda}$ finite, we obtain the result (4.31) which includes non-trivial effects of the broken zero modes.

There are several points left for future works. One is how to study the case with finite $\sqrt{\lambda}$. For this purpose, we need to consider the string world sheet which carries the angular momentum. The string solutions used in [15][12][13] could be a good starting point for such studies. As mentioned at the end of the previous section, studying the relation between the method of the broken zero modes and the one in [15] would be also interesting future work. Another point which is not quite clear to the author is the treatment of the path integral measure and the derivative with respect to the source $s_0$. Since the $S^3$ measure for the broken zero modes comes from the target space geometry, it would depend on the source. In that case it seems to be not quite clear whether the derivative should be from the outside of the path integral and it hits the measure, or it is just inside the path integral. In the present paper, we took the latter choice as an assumption. We would like to address this issue in the future work.

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A Boundary terms

In the present case, the boundary term proposed in [18] can be written as follows:

$$S_{\text{boundary}} = - \int_0^{2\pi} d\tau \frac{\partial L}{\partial (\partial_z z)} \bigg|_{\sigma = \sigma_{\text{min}}} = -\frac{\sqrt{\lambda}}{2\pi} \int_0^{2\pi} d\tau (g_{z\mu}^{\text{AdS}} + h_{z\mu}^{\text{AdS}}) \partial_\sigma \mathcal{X}^\mu_z \bigg|_{\sigma = \sigma_{\text{min}}}. \quad (A.1)$$

Here, $g_{\mu\nu}^{\text{AdS}}$ is the AdS$_5$ metric and $h_{z\mu}^{\text{AdS}}$ is the fluctuation which depends on the source $s_0^J$ of the local operator. If we set $s_0^J = 0$, then the fluctuation $h_{z\mu}^{\text{AdS}}$ is zero, and by using (2.13),
the boundary term is evaluated as

$$S_{\text{boundary}}|_{s_0^J=0} = -\frac{\sqrt{\lambda}}{2\pi} \int_0^{2\pi} d\tau \frac{1}{z} \partial_\tau z \bigg|_{\sigma=\sigma_{\text{min}}} = -\frac{\sqrt{\lambda}}{\sinh \sigma_{\text{min}} \cosh \sigma_{\text{min}}}. \quad (A.2)$$

This term exactly cancels the first term of (3.10) in the limit $\sigma_{\text{min}} \to 0$.

Next let us check that the functional derivative with respect to $s_0^J(\vec{x},J)$ vanishes. In (4.13) and (4.20), there could be an additional term like

$$-\frac{\delta}{\delta s_0^J(\vec{x},J)} S_{\text{boundary}} = \frac{\sqrt{\lambda}}{2\pi} \int_0^{2\pi} d\tau \left( \frac{\delta}{\delta s_0^J(\vec{x},J)} h_{z\mu}^\dagger \right) \partial_\sigma \mathcal{X}^\mu z \bigg|_{\sigma=\sigma_{\text{min}}} \\
\sim \frac{\sqrt{\lambda}}{2\pi} \int_0^{2\pi} d\tau \left( 2J \frac{1}{z^2} c_{\ell J}^J \mathcal{X}^{\ell J} \right) \partial_\sigma z z \bigg|_{\sigma=\sigma_{\text{min}}} \\
= \frac{1}{2N} \sqrt{\lambda} \lambda (J+1) d^J (\cos \alpha + i \sin \alpha \cos \beta)^J \tanh^{2J-1} \sigma_{\text{min}} \frac{1}{\cosh^2 \sigma_{\text{min}}}. \quad (A.3)$$

From the final expression, we see that it vanishes in the limit $\sigma_{\text{min}} \to 0$, since $J$ is an integer greater than or equal to 2.

**B Properties of the modified Bessel functions**

Let us explain some properties of the modified Bessel function $I_J(z)$ which we use in the main text. In the following discussion, we assume that $J$ is a non-negative integer and $z \geq 0$.

The first point we explain is the $\alpha$-integral in (4.26), which may not be very clear. We start with the following integral formula for the modified Bessel function:

$$I_J(z) = \frac{1}{\pi} \int_0^\pi d\alpha \cos(J\alpha) e^{\cos \alpha z}, \quad (J = 0, 1, \ldots). \quad (B.1)$$

We can confirm this expression by checking that it satisfies the following recurrence formulas of the modified Bessel functions

$$I_J(z) - I_{J+2}(z) = \frac{2(J+1)}{z} I_{J+1}(z), \quad (B.2)$$

$$I_J(z) + I_{J+2}(z) = 2I_{J+1}(z),$$

and also it satisfies the initial condition

$$I_0(z) = \frac{1}{\pi} \int_0^\pi d\alpha e^{\cos \alpha z}. \quad (B.3)$$

Next by integrating by part and using (B.1), we find the following relation:

$$\frac{1}{\pi(J+1)} \int_0^\pi d\alpha e^{\cos \alpha z} \sin \alpha \sin((J+1)\alpha)$$
\[
= -\frac{1}{\pi (J+1) z} \int_0^\pi d\alpha \frac{\partial}{\partial \alpha} (e^{\cos \alpha z}) \sin((J+1)\alpha)
= \frac{1}{z} I_{J+1}(z).
\]

Finally we differentiate the first and the last expression of (B.4) with respect to \( z \) and obtain (4.26) as

\[
\frac{1}{\pi (J+1)} \int_0^\pi d\alpha e^{\cos \alpha z} \sin \alpha \cos \sin((J+1)\alpha) = -\frac{1}{z^2} I_{J+1}(z) + \frac{1}{z} I'_{J+1}(z)
= \frac{1}{z} I_J(z) - \frac{J+2}{z^2} I_{J+1}(z).
\]

Here, the above recurrence formulas (B.2) are again used when we go to the final expression.

The recurrence formulas can be also used to show the following inequality for \( z \geq 0 \):

\[
I_J(z) \geq I_{J+1}(z) \quad (J = 0, 1, \ldots).
\]

First, from (B.1) we have

\[
I_0(z) - I_1(z) = \frac{1}{\pi} \int_0^\pi d\alpha (1 - \cos \alpha) e^{\cos \alpha z} > 0.
\]

Next, from the second equation of (B.2), we find

\[
\frac{d}{dz} (I_{J+1}(z) - I_{J+2}(z)) = \frac{1}{2} \left[ (I_J(z) - I_{J+1}(z)) + (I_{J+2}(z) - I_{J+3}(z)) \right].
\]

From (B.7), (B.8) and the initial conditions\(^\dagger\) \( I_0(0) = 1 \) and \( I_J(0) = 0 \) \( (J = 1, 2, \ldots) \), we see that the relation (B.6) holds for \( z \geq 0 \).

\section{Supersymmetry}

Supersymmetries preserved by the system including a 1/4 BPS Wilson loop and a 1/2 BPS local operator is studied in [11]. Here, we review the analysis with taking account of the distance \( \ell \) between the local operator and the center of the Wilson loop operator.

The supersymmetries preserved by the Wilson loop operator (2.1), with (2.2) and (2.3), are given by the following conditions [10]:

\[
\left[ -i \sin \tau \gamma_1 + i \cos \tau \gamma_2 + \sin \theta_0 \cos \tau \gamma_5 + \sin \theta_0 \sin \tau \gamma_6 + \cos \theta_0 \gamma_7 \right] \left[ \epsilon_0 + a \left( \cos \tau \gamma_1 + \sin \tau \gamma_2 \right) \epsilon_1 \right] = 0.
\]

\(^\dagger\)For the range \( 0 < z < 2 \), we find \( I_J(z) - I_{J+1}(z) > I_J(z) - \frac{2(J+1)}{z} I_{J+1}(z) = I_{J+2}(z) > 0 \) \( (J = 0, 1, \ldots) \), which can be used as alternative initial conditions.
Here, we use the ten-dimensional notation. \( \gamma_M (M = 1, \ldots, 10) \) are the ten-dimensional gamma matrices which satisfy \( \{ \gamma_M, \gamma_N \} = 2 \delta_{MN} \). \( M = 1, 2, 3, 4 \) correspond to the four-dimensional space of the SYM theory, while \( M = 5, \ldots, 10 \) correspond to the reduced dimensions. Each spinor \( \epsilon_0 \) and \( \epsilon_1 \) has an opposite chirality, in the sense of ten-dimension, and generates the Poincaré supersymmetry and the conformal supersymmetry, respectively. These two spinors appear in the transformation of the bosonic field only in the following combination:

\[
\epsilon = \epsilon_0 + x^i \gamma_i \epsilon_1 .
\]

(C.2)

The independent conditions coming from (C.1) are summarized in [10]. For \( \theta_0 = 0 \), (C.1) relates \( \epsilon_0 \) and \( \epsilon_1 \) as follows:

\[
\epsilon_0 = i a \gamma_1 \gamma_2 \gamma_7 \epsilon_1 .
\]

(C.3)

On the other hand, for the other special case with \( \theta_0 = \pi/2 \), each spinor \( \epsilon_0 \) and \( \epsilon_1 \) satisfies the following common conditions independently:

\[
(1 - i \gamma_2 \gamma_5) \epsilon_0 = (1 + i \gamma_1 \gamma_6) \epsilon_0 = 0, \quad (1 - i \gamma_2 \gamma_5) \epsilon_1 = (1 + i \gamma_1 \gamma_6) \epsilon_1 = 0 .
\]

(C.4)

Finally, for a generic value of \( \theta_0 \), (C.1) reduces to the following conditions:

\[
\cos \theta_0 \epsilon_0 = a ( -i \gamma_1 + \sin \theta_0 \gamma_6) \gamma_7 \gamma_2 \epsilon_1 , \quad (1 - \gamma_1 \gamma_2 \gamma_5 \gamma_6) \epsilon_1 = 0 .
\]

(C.5)

In addition to these “Wilson loop conditions”, we need to impose the following conditions coming from the local operator \( \mathcal{O}_J \) located at \( \vec{x} = (0, 0, 0, \ell) \):

\[
\left[ 1 + i \gamma_7 \gamma_8 \right] [\epsilon_0 + \ell \gamma_4 \epsilon_1] = 0 .
\]

(C.6)

Let us summarize the combined conditions for each value of \( \theta_0 \).

- **\( \theta_0 = 0 \)**

  From (C.3) and (C.6), we find that the following condition is imposed on \( \epsilon_1 \)

\[
\left[ 1 + i \gamma_7 \gamma_8 \right] [i a \gamma_1 \gamma_2 \gamma_7 + \ell \gamma_4] \epsilon_1 = 0 .
\]

(C.7)

Since the matrix \( (i a \gamma_1 \gamma_2 \gamma_7 + \ell \gamma_4) \) is invertible, the condition (C.7) projects out half of the degrees of freedom (d.o.f.) of \( \epsilon_1 \), and then 8 of them survive. Since \( \epsilon_0 \) is determined by (C.3), the whole system including \( W(C) \) and \( \mathcal{O}_J \) preserves 8 supersymmetries.

- **\( \theta_0 = \pi/2 \)**

In this case, we may recombine \( \epsilon_0 \) and \( \epsilon_1 \) as

\[
\begin{cases}
\epsilon_+ = \epsilon_0 + \ell \gamma_4 \epsilon_1 , \\
\epsilon_- = \epsilon_1 - \ell \gamma_4 \epsilon_0 ,
\end{cases}
\quad \leftrightarrow \quad \begin{cases}
\epsilon_0 = \frac{1}{1 + \ell^2} (\epsilon_+ - \ell \gamma_4 \epsilon_-) , \\
\epsilon_1 = \frac{1}{1 + \ell^2} (\epsilon_- + \ell \gamma_4 \epsilon_+) ,
\end{cases}
\]

(C.8)
where the each spinor $\epsilon_{\pm}$ has 16 d.o.f. with an opposite chirality. Then the condition (C.4) can be equivalently imposed on $\epsilon_{\pm}$ as
\[
(1 - i\gamma_2\gamma_5)\epsilon_{\pm} = (1 + i\gamma_1\gamma_6)\epsilon_{\pm} = 0. \tag{C.9}
\]
These conditions allow only quarter of the each spinor $\epsilon_{\pm}$ and $4 + 4 = 8$ d.o.f. survive. Then the additional condition (C.6) reduces the d.o.f. of $\epsilon_+$ to the half of it, while $\epsilon_-$ is not affected. Then the whole system preserves $4 + \frac{1}{2} \times 4 = 6$ supersymmetries.

- $0 < \theta_0 < \pi/2$

From the first condition of (C.5) and (C.6), $\epsilon_1$ needs to satisfy the following condition:
\[
\left(1 + i\gamma_7\gamma_8\right)\Gamma\epsilon_1 = 0, \quad \left(\Gamma \equiv a\gamma_1\gamma_2\gamma_7 + a\sin\theta_0\gamma_2\gamma_6\gamma_7 + \ell\cos\theta_0\gamma_4\right). \tag{C.10}
\]
Since the matrix $\Gamma$ is invertible, (C.10) projects out half of the d.o.f. of $\epsilon_1$ and 8 of them survive. This is a generalization of the case with $\theta_0 = 0$. Now in the case of generic $\theta_0$, $\epsilon_1$ needs to satisfy the second condition of (C.5). Since $\Gamma$ and $\gamma_1\gamma_2\gamma_5\gamma_6$ commute, we can impose the following equivalent condition:
\[
(1 - \gamma_1\gamma_7\gamma_8)\Gamma\epsilon_1 = 0. \tag{C.11}
\]
Then, since $i\gamma_7\gamma_8$ and $\gamma_1\gamma_2\gamma_5\gamma_6$ commute, two conditions (C.10) and (C.11) allow $\frac{1}{2} \times \frac{1}{2} \times 16 = 4$ independent d.o.f. of the original spinor $\epsilon_1$. The other spinor $\epsilon_0$ is determined by the first equation of (C.5) and the whole system preserves $4 = \frac{1}{8} \times (16 + 16)$ supersymmetries.

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