The Dirichlet problem for semi-linear equations

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Abstract

We study the Dirichlet problem for the semi–linear partial differential equations in the simple connected domains $D$ in $\mathbb{C}$ the linear part of which is written in a divergent (anisotropic !) form. Thanking to a factorization theorem established by us earlier in [25], the problem is reduced to the Dirichlet problem for the corresponding quasilinear Poisson equation in the unit disk $D$. On the basis of the potential theory, that makes possible to prove the existence of the weak solutions of the class $C \cap W^{1,2}_{\text{loc}}$ for the given semi–linear equations in arbitrary domains $D$ with the so–called quasihyperbolic boundary condition, generally speaking, without the standard $(A)$–condition and the known outer cone condition.

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1 Introduction

Given a domain $D$ in $\mathbb{C}$, denote by $M^{2 \times 2}_{\mathbb{K}}(D)$ the class of all $2 \times 2$ symmetric matrix function $A(z) = \{a_{jk}(z)\}$ with measurable entries and $\det A(z) = 1$, satisfying the uniform ellipticity condition

$$\frac{1}{K} |\xi|^2 \leq \langle A(z) \xi, \xi \rangle \leq K |\xi|^2 \quad \text{a.e. in } D$$

for every $\xi \in \mathbb{C}$ where $1 \leq K < \infty$. Further we study the semilinear equations

$$\text{div} \left[ A(z) \nabla u(z) \right] = f(u(z)), \quad z \in D$$

with continuous functions $f : \mathbb{R} \to \mathbb{R}$ either bounded or $f(t)/t \to 0$ as $t \to \infty$ which describe many physical phenomena in anisotropic inhomogeneous media.
The equations (1.2) are closely relevant to the so-called Beltrami equations. Let \( \mu : D \to \mathbb{C} \) be a measurable function with \( |\mu(z)| < 1 \) a.e. The equation

\[ \omega \bar{z} = \mu(z) \cdot \omega_z \]  

where \( \omega \bar{z} = (\omega_x + i\omega_y)/2, \omega_z = (\omega_x - i\omega_y)/2, z = x + iy, \omega_x \) and \( \omega_y \) are partial derivatives of the function \( \omega \) in \( x \) and \( y \), respectively, is said to be a Beltrami equation. The equation (1.3) is said to be nondegenerate if \( ||\mu||_\infty < 1 \). The homeomorphic solutions of nondegenerate Beltrami’s equations (1.3) with all the first generalized derivatives by Sobolev are called quasiconformal mappings, see e.g. [1] and [37].

We say that a quasiconformal mapping \( \omega \) satisfying (1.3) is agreed with \( A \in M^{2 \times 2}_K(D) \) if

\[ |\mu(z)| \leq \frac{K-1}{K+1} \text{ a.e. in } D. \]  

(1.5)

Vice versa, given a measurable function \( \mu : D \to \mathbb{C} \), satisfying (1.5), one can invert the algebraic system (1.4) to obtain the matrix function \( A \in M^{2 \times 2}_K(D) \):

\[ A(z) = \left( \begin{array}{cc} \frac{1-|\mu|^2}{1+|\mu|^2} & \frac{-2\mu \bar{\mu}}{1+|\mu|^2} \\ \frac{-2\mu \bar{\mu}}{1-|\mu|^2} & \frac{1+|\mu|^2}{1-|\mu|^2} \end{array} \right). \]  

(1.6)

By the existence theorem for (1.3), see e.g. Theorem V.B.3 in [1] and Theorem V.1.3 in [37], any \( A \in M^{2 \times 2}_K(D) \) generates a quasiconformal mapping \( \omega : D \to \mathbb{D} \).

We also would like to pay attention to a strong interaction between linear and non-linear elliptic systems in the plane and quasiconformal mappings. The most general first order linear homogeneous elliptic system with real coefficients can be written in the form \( f \bar{z} + \mu(z)f_z + \nu(z)f \bar{z} = 0 \), with measurable coefficients \( \mu \) and \( \nu \) such that \( |\mu| + |\nu| \leq (K-1)/(K+1) < 1 \). This equation is a particular case of a non-linear first order system \( f \bar{z} = H(z, f_z) \) where \( H : G \times \mathbb{C} \to \mathbb{C} \) is Lipschitz in the second variable,

\[ |H(z, w_1) - H(z, w_2)| \leq \frac{K-1}{K+1} |w_1 - w_2|, \quad H(z, 0) \equiv 0. \]
The principal feature of the above equation is that the difference of two solutions need not solve the same equation but each solution can be represented as a composition of a quasiconformal homeomorphism and an analytic function. Thus quasiconformal mappings become the central tool for the study of these non-linear systems. A rather comprehensive treatment of the present state of the theory is given in the excellent book of Astala, Iwaniec and Martin [2]. This book contains also an exhaustive bibliography on the topic. In particular, the following fundamental Harmonic Factorization Theorem for the uniformly elliptic divergence equations

$$\text{div } A(z, \nabla u) = 0, \ z \in \Omega,$$  \hspace{1cm} (1.7)

holds, see [2], Theorem 16.2.1: Every solution \( u \in W^{1,2}_{\text{loc}}(\Omega) \) of the equation (1.7) can be expressed as the composition \( u(z) = h(f(z)) \) of a quasiconformal homeomorphism \( f : \Omega \rightarrow G \) and a suitable harmonic function \( h \) on \( G \).

The main goal of this paper is to point out another application of quasiconformal mappings to the study of some semi-linear partial differential equations, linear part of which contains the elliptic operator in the divergence form \( \text{div } [A(z)\nabla u(z)] \).

A fundamental role in the study of the posed problem will play Theorem 4.1 in [25], that can be considered as a suitable counterpart to the mentioned above Factorization theorem: a function \( u : D \rightarrow \mathbb{R} \) is a weak solution of (1.2) in the class \( C \cap W^{1,2}_{\text{loc}}(D) \) if and only if \( u = U \circ \omega \) where \( \omega : D \rightarrow \mathbb{D} \) is a quasiconformal mapping agreed with \( A \) and \( U \) is a weak solution in the class \( C \cap W^{1,2}_{\text{loc}}(D) \) of the quasilinear Poisson equation

$$\triangle U(w) = J(w) \cdot f(U(w)) \ , \quad w \in \mathbb{D} ,$$ \hspace{1cm} (1.8)

\( J \) denotes the Jacobian of the inverse quasiconformal mapping \( \omega^{-1} : \mathbb{D} \rightarrow D \).

Note that the mapping \( \omega^* := \omega^{-1} \) is extended to a quasiconformal mapping of \( \mathbb{C} \) onto itself if \( \partial D \) is the so–called quasicircle, see e.g. Theorem II.8.3 in [37]. By one of the main Bojarski results, see [10], the generalized derivatives
of quasiconformal mappings in the plane are locally integrable with some power $q > 2$. Note also that its Jacobian $J(w) = |\omega_w^*|^2 - |\omega_{\bar{w}}^*|^2$, see e.g. I.4(9) in [1]. Consequently, in this case $J \in L^p(D)$ for some $p > 1$.

In this connection, recall that the image of the unit disk $D$ under a quasiconformal mapping of $\mathbb{C}$ onto itself is called a **quasidisk** and its boundary is called a **quasicircle** or a **quasiconformal curve**. Recall also that a **Jordan’s curve** is a continuous one-to-one image of the unit circle in $\mathbb{C}$. As known, such a smooth ($C^1$) or Lipschitz curve is a quasiconformal curve and, at the same time, quasiconformal curves can be even locally non-rectifiable as it follows from the well-known Van Koch snowflake example, see e.g. the point II.8.10 in [37]. The recent book [21] contains a comprehensive discussion and numerous characterizations of quasidisks, see also [1], [20] and [37].

By Theorem 4.7 in [3], cf. also Theorem 1 and Corollary in [9], the Jacobian of a quasiconformal homeomorphism $\omega^* : D \to D$ is in $L^p(D)$, $p > 1$, iff $D$ satisfies the **quasihyperbolic boundary condition**, i.e.

$$k_D(z, z_0) \leq a \cdot \ln \frac{d(z_0, \partial D)}{d(z, \partial D)} + b \quad \forall z \in D$$  \hspace{1cm} (1.9)

for some constants $a$ and $b$ and a fixed point $z_0 \in D$ where $k_D(z, z_0)$ is the **quasihyperbolic distance** between the points $z$ and $z_0$ in the domain $D$,

$$k_D(z, z_0) := \inf_\gamma \int_\gamma \frac{ds}{d(\zeta, \partial D)}.$$  \hspace{1cm} (1.10)

Here $d(\zeta, \partial D)$ denotes the Euclidean distance from a point $\zeta \in D$ to the boundary of $D$ and the infimum is taken over all rectifiable curves $\gamma$ joining the points $z$ and $z_0$ in $D$.

The notion of domains with the quasihyperbolic boundary condition was introduced in [23] but, before it, was first applied in [9]. Note that such domains can be not satisfying the (A)-condition as well as the outer cone condition and not Jordan at all, see Sections 4 and 5. Note also that, generally speaking not rectifiable, quasidisks and, in particular, smooth and Lipschitz domains satisfy the quasihyperbolic boundary condition.
In Section 2 we give the necessary backgrounds for the Poisson equation \( \Delta u(z) = g(z) \) due to the theory of the Newtonian potential and the theory of singular integrals in \( \mathbb{C} \). First of all, correspondingly to the key fact of the potential theory, Proposition 1, the Newtonian potential
\[
N_g(z) := \frac{1}{2\pi} \int_\mathbb{C} \ln |z - w| g(w) \, dm(w)
\] (1.11)
of arbitrary integrable densities \( g \) of charge with compact support satisfies the Poisson equation in a distributional sense, see Corollary 1. Moreover, \( N_g \) is continuous for \( g \in L^p(\mathbb{C}) \) and, furthermore, the Newtonian operator \( N : L^p(\mathbb{C}) \to C(\mathbb{C}) \) is completely continuous for \( p > 1 \), Theorem 1. An example in Proposition 2 shows that \( N_g \) for \( g \in L^1(\mathbb{C}) \) can be not continuous and even not in \( L^\infty_{\text{loc}}(\mathbb{C}) \). Theorem 2 says on additional properties of regularity of \( N_g \) depending on a degree of integrability of \( g \). Finally, resulting Corollary 2 states the existence, representation and regularity of solutions to the Dirichlet problem for the Poisson equation with continuous boundary data.

Section 3 contains one of the main results Theorem 3 on the existence of regular solutions of the Dirichlet problem to the quasilinear Poisson equation
\[
\Delta u(z) = h(z) \cdot f(u(z))
\] (1.12)
in the unit disk \( \mathbb{D} \) for arbitrary continuous boundary data. In general, we assume that the function \( h : \mathbb{D} \to \mathbb{R} \) is in the class \( L^p(\mathbb{D}) \), \( p > 1 \), and the continuous function \( f : \mathbb{R} \to \mathbb{R} \) is either bounded or with non-decreasing \( |f| \) of \( |t| \) and
\[
\lim_{t \to \infty} \frac{f(t)}{t} = 0 ,
\] (1.13)
without any assumptions on the sign and zeros of the right hand side in (1.12). The degree of regularity of solutions depends first of all on a degree of integrability of the multiplier \( h \). The proof of Theorem 3 is realized by the Leray–Schauder approach. This result is extended to arbitrary smooth \( (C^1) \) domains in Corollary 3. The proof of the latter is obtained from Theorem 3 through the fundamental results of Carathéodory–Osgood–Taylor and Warschawski on the boundary behavior of conformal mappings between Jordan’s domains in \( \mathbb{C} \).
Section 4 includes the main result of the present paper Theorem 4 on the existence of regular weak solutions of the Dirichlet problem for semi-linear equations (1.2) in Jordan domains $D$ in $\mathbb{C}$ satisfying the quasihyperbolic boundary condition with arbitrary continuous boundary data $\varphi : \partial D \to \mathbb{R}$. Theorem 4 states the existence of a weak solution $u : D \to \mathbb{R}$ of the equation (1.2) in the class $C \cap W^{1,2}_{\text{loc}}(\Omega)$ which is locally Hölder continuous in $D$ and continuous in $\overline{D}$ with $u|_{\partial D} = \varphi$. If in addition $\varphi$ is Hölder continuous, then $u$ is Hölder continuous in $\overline{D}$. Moreover, $u = U \circ \omega$ where $U$ is a weak solution of the quasilinear Poisson equation (1.8) and $\omega : D \to \mathbb{D}$ is a quasiconformal mapping agreed with $A$. In Lemma 2 we show that there exist Jordan domains $D$ in $\mathbb{C}$ with the quasihyperbolic boundary condition but without the standard $(A)$–condition and, consequently, without the known outer cone condition.

Section 5 contains Theorem 5 which is the extension of Theorem 4 to arbitrary bounded simple connected (not Jordan!) domains $D$ in $\mathbb{C}$ with the quasihyperbolic boundary condition formulated in terms of the prime ends by Caratheodory. Finally, in Section 6 we give applications of the obtained results to various kind of absorption, reaction-diffusion problems, equations of a heated plasma and the combustion in anisotropic and inhomogeneous media, Theorem 6 and 7, see also Corollaries 6–8.

### 2 Potentials and the Poisson equation

Let $\mathbb{D}$ be the unit disk in the complex plane $\mathbb{C}$. For $z$ and $w \in \mathbb{D}$ with $z \neq w$, set

$$G(z, w) := \ln \left| \frac{1 - z \overline{w}}{z - w} \right| \quad \text{and} \quad P(z, e^{it}) := \frac{1 - |z|^2}{|1 - z e^{-it}|^2}$$

(2.1)

be the Green function and Poisson kernel in $\mathbb{D}$. If $\varphi \in C(\partial \mathbb{D})$ and $g \in C(\overline{\mathbb{D}})$, then a solution to the Poisson equation

$$\Delta f(z) = g(z)$$

(2.2)

satisfying the boundary condition $f|_{\partial \mathbb{D}} = \varphi$ is given by the formula

$$f(z) = P_\varphi(z) - G_g(z)$$

(2.3)
where
\[ P_\varphi(z) = \frac{1}{2\pi} \int_0^{2\pi} P(z, e^{it}) \varphi(e^{-it}) \, dt, \quad G_g(z) = \int_D G(z, w) g(w) \, dm(w), \quad (2.4) \]

see e.g. [28], p. 118-120. Here \( m(w) \) denotes the Lebesgue measure in \( \mathbb{C} \).

In this section, we give the representation of solutions of the Poisson equation in the form of the Newtonian (normalized antilogarithmic) potential that is more convenient for our research and, on this basis, we prove the existence and representation theorem for solutions of the Dirichlet problem to the Poisson equation under the corresponding conditions of integrability of sources \( g \).

Correspondingly to 3.1.1 in [44], given a finite Borel measure \( \nu \) on \( \mathbb{C} \) with compact support, its potential is the function \( p_\nu : \mathbb{C} \to (-\infty, \infty) \) defined by
\[ p_\nu(z) = \int_{\mathbb{C}} \ln |z - w| d\nu(w). \quad (2.5) \]

**Remark 1.** Note that the function \( p_\nu \) is subharmonic by Theorem 3.1.2 and, consequently, it is locally integrable on \( \mathbb{C} \) by Theorem 2.5.1 in [44]. Moreover, \( p_\nu \) is harmonic outside the support of \( \nu \).

This definition can be extended to finite charges \( \nu \) with compact support (named also signed measures), i.e., to real valued sigma-additive functions on Borel sets in \( \mathbb{C} \), because of \( \nu = \nu^+ - \nu^- \) where \( \nu^+ \) and \( \nu^- \) are Borel measures by the well-known Jordan decomposition, see e.g. Theorem 0.1 in [35].

The key fact is the following statement, see e.g. Theorem 3.7.4 in [44].

**Proposition 1.** Let \( \nu \) be a finite charge with compact support in \( \mathbb{C} \). Then
\[ \Delta p_\nu = 2\pi \cdot \nu \quad (2.6) \]
in the distributional sense, i.e.,
\[ \int_{\mathbb{C}} p_\nu(z) \Delta \psi(z) \, dm(z) = 2\pi \int_{\mathbb{C}} \psi(z) \, d\nu(z) \quad \forall \ \psi \in C_0^\infty(\mathbb{C}). \quad (2.7) \]
Here as usual $C_0^\infty(\mathbb{C})$ denotes the class of all infinitely differentiable functions $\psi : \mathbb{C} \to \mathbb{R}$ with compact support in $\mathbb{C}$, $\triangle = \partial_2^{x^2} + \partial_2^{y^2}$ is the Laplace operator and $d m(z)$ corresponds to the Lebesgue measure in $\mathbb{C}$.

**Corollary 1.** In particular, if for every Borel set $B$ in $\mathbb{C}$

$$\nu(B) := \int_B g(z) \, dm(z) \quad (2.8)$$

where $g : \mathbb{C} \to \mathbb{R}$ is an integrable function with compact support, then

$$\triangle N_g = g \quad , \quad (2.9)$$

where

$$N_g(z) := \frac{1}{2\pi} \int_{\mathbb{C}} \ln |z-w| \, g(w) \, dm(w) \quad , \quad (2.10)$$

in the distributional sense, i.e.,

$$\int_{\mathbb{C}} N_g(z) \triangle \psi(z) \, dm(z) = \int_{\mathbb{C}} \psi(z) g(z) \, dm(z) \quad \forall \, \psi \in C_0^\infty(\mathbb{C}) \quad . \quad (2.11)$$

Here the function $g$ is called a density of charge $\nu$ and the function $N_g$ is said to be the Newtonian potential of $g$.

The next statement on continuity in the mean of functions $\psi : \mathbb{C} \to \mathbb{R}$ in $L^q(\mathbb{C})$, $q \in [1, \infty)$, with respect to shifts is useful for the study of the Newtonian potential, see e.g. Theorem 1.4.3 in [46], cf. also Theorem III(11.2) in [45]. Here we give its direct proof arguing by contradiction.

**Lemma 1.** Let $\psi \in L^q(\mathbb{C})$, $q \in [1, \infty)$, have a compact support. Then

$$\lim_{\Delta z \to 0} \int_{\mathbb{C}} |\psi(z + \Delta z) - \psi(z)|^q \, dm(z) = 0 \quad . \quad (2.12)$$

The shift of a set $E \subset \mathbb{C}$ by a complex vector $\Delta z \in \mathbb{C}$ is the set

$$E + \Delta z := \{ \xi \in \mathbb{C} : \xi = z + \Delta z , z \in E \} \quad .$$
Proof. Let us assume that there is a sequence \( \Delta z_n \in \mathbb{C}, n = 1, 2, \ldots \), such that \( \Delta z_n \to 0 \) as \( n \to \infty \) and, for some \( \delta > 0 \) and \( \psi_n(z) := \psi(z + \Delta z_n) \),

\[
I_n := \left[ \int_{\mathbb{C}} |\psi_n(z) - \psi(z)|^q \, dm(z) \right]^{\frac{1}{q}} \geq \delta \quad \forall \ n = 1, 2, \ldots . \tag{2.13}
\]

Denote by \( K \) the closed disk in \( \mathbb{C} \) centered at 0 with the minimal radius \( R \) that contains the support of \( \psi \). By the Luzin theorem, see e.g. Theorem 2.3.5 in [19], for every prescribed \( \varepsilon > 0 \), there is a compact set \( C \subset K \) such that \( g|_C \) is continuous and \( m(K \setminus C) < \varepsilon \). With no loss of generality, we may assume that \( C \subset K_\ast \) where \( K_\ast \) is a closed disk in \( \mathbb{C} \) centered at 0 with a radius \( r \in (0, R) \) and, moreover, that \( C_n \subset K \), where \( C_n := C - \Delta z_n \), for all \( n = 1, 2, \ldots \). Note that \( m(C_n) = m(C) \) and then \( m(K \setminus C_n) < \varepsilon \) and, consequently, \( m(K \setminus C_\ast_n) < 2\varepsilon \), where \( C_\ast_n := C \cap C_n \), because \( K \setminus C_\ast_n = (K \setminus C_n) \cup (K \setminus C) \).

Next, setting \( K_n = K - \Delta z_n \), we see that \( K \cup K_n = C_\ast \cup (K \setminus C_\ast_n) \cup (K_n \setminus C_\ast) \) and that \( K_n \setminus C_\ast_n + \Delta z_n = K \setminus C_\ast_n \). Hence by the triangle inequality for the norm in \( L^p \) the following estimate holds

\[
I_n \leq 4 \cdot \left[ \int_{K \setminus C_\ast_n} |\psi(z)|^q \, dm(z) \right]^{\frac{1}{q}} + \left[ \int_{C_\ast_n} |\psi_n(z) - \psi(z)|^q \, dm(z) \right]^{\frac{1}{q}} \quad \forall \ n = 1, 2, \ldots .
\]

By construction the both terms from the right hand side can be made to be arbitrarily small, the first one for small enough \( \varepsilon \) because of absolute continuity of indefinite integrals and the second one for all large enough \( n \) after the choice of the set \( C \). Thus, the assumption (2.13) is disproved. \( \square \)

**Theorem 1.** Let \( g : \mathbb{C} \to \mathbb{R} \) be in \( L^p(\mathbb{C}) \), \( p > 1 \), with compact support. Then \( N_g \) is continuous. A collection \( \{N_g\} \) is equicontinuous on compacta if the collection \( \{g\} \) is bounded by the norm in \( L^p(\mathbb{C}) \) with supports in a fixed disk \( K \). Moreover, under these conditions, on each compact set in \( \mathbb{C} \)

\[
\|N_g\|_C \leq M \cdot \|g\|_p . \tag{2.14}
\]
The corresponding statement on the continuity of integrals of potential type in $\mathbb{R}^n$, $n \geq 3$, can be found in [46], Theorem 1.6.1.

**Proof.** By the Hölder inequality with $\frac{1}{q} + \frac{1}{p} = 1$ we have that

$$|N_g(z) - N_g(\zeta)| \leq \frac{\|g\|_p}{2\pi} \left[ \int_{K} |\ln|z - w| - \ln|\zeta - w| |^q \, dm(w) \right] =$$

$$= \frac{\|g\|_p}{2\pi} \left[ \int_{C} |\psi_\zeta(\xi + \Delta z) - \psi_\zeta(\xi) |^q \, dm(\xi) \right]^\frac{1}{q},$$

where $\xi = \zeta - w$, $\Delta z = z - \zeta$, $\psi_\zeta(\xi) := \chi_{K+\zeta}(\xi) \ln|\xi|$. Thus, the first conclusion follows by Lemma 1 because $\ln|\xi| \in L^q_{\text{loc}}(C)$ for all $q \in [1, \infty)$.

The second conclusion follows by the continuity of the integral from the right hand side in the above estimate with respect to the parameter $\zeta \in \mathbb{C}$. Indeed,

$$\|\psi_\zeta - \psi_\zeta, \|_q = \left\{ \int_{\Delta} \ln|\xi| |^q \, dm(\xi) \right\}^\frac{1}{q},$$

where $\Delta$ denotes the symmetric difference of the disks $K + \zeta$ and $K + \zeta*$. Thus, the statement follows from the absolute continuity of the indefinite integral.

The third conclusion similarly follows through the direct estimate

$$|N_g(\zeta)| \leq \frac{\|g\|_p}{2\pi} \left[ \int_{K} |\ln|\zeta - w| |^q \, dm(w) \right] = \frac{\|g\|_p}{2\pi} \left[ \int_{C} \psi_\zeta(\xi) |^q \, dm(\xi) \right]^\frac{1}{q} \square$$

**Proposition 2.** There exist functions $g \in L^1(C)$ with compact support whose potentials $N_g$ are not continuous, furthermore, $N_g \notin L^\infty_{\text{loc}}$.

**Proof.** Indeed, let us consider the function

$$g(z) = \omega(|z|), \quad z \in \overline{D}, \quad g(z) \equiv 0, \quad z \in \mathbb{C} \setminus \overline{D},$$

where

$$\omega(t) = \frac{1}{t^2} (1 - \ln t)^\alpha, \quad t \in (0, 1], \quad \alpha \in (1, 2), \quad \omega(0) = \infty.$$
Setting $\Omega(t) = t \cdot \omega(t)$, we see that, firstly,

$$\int_D |g(w)| \, d m(w) = 2\pi \int_0^1 \Omega(t) \, d t = 2\pi \int_0^1 \frac{d \ln t}{(1 - \ln t)^\alpha} = \frac{2\pi}{\alpha - 1}$$

and, secondly,

$I := N_g(0) = \int_0^1 \Omega(t) \ln t \, d t = \left[ \ln t \int_0^t \Omega(\tau) \, d \tau \right]_0^1 - \left( t \int_0^t \Omega(\tau) \, d \tau \right) \, d t =

= \frac{1}{\alpha - 1} \cdot \left( \left[ \frac{\ln t}{(1 - \ln t)^{\alpha - 1}} \right]_0^1 + \frac{1}{t(1 - \ln t)^{\alpha - 1}} \right) =

= \frac{1}{\alpha - 1} \cdot \left[ (1 - \ln t)^{1 - \alpha} - \frac{3 - \alpha}{2 - \alpha} \cdot (1 - \ln t)^{2 - \alpha} \right]_0^1 = -\infty \, . \quad \Box$

The following theorem on the Newtonian potentials is important to obtain solutions of the Dirichlet problem to the Poisson equation of a higher regularity.

**Theorem 2.** Let $g : \mathbb{C} \to \mathbb{R}$ have compact support. If $g \in L^1(\mathbb{C})$, then $N_g \in L^r_{\text{loc}}$ for all $r \in [1, \infty)$, $N_g \in W^{1,q}_{\text{loc}}$ for all $q \in [1, 2)$, moreover, $N_g \in W^{2,1}_{\text{loc}}$, $\nabla^2 N_g = g$ a.e.

If $g \in L^p(\mathbb{C})$, $p > 1$, then $N_g \in W^{2,p}_{\text{loc}}$, $\nabla N_g = g$ a.e. and, moreover, $N_g \in W^{1,q}_{\text{loc}}$ for $q > 2$, consequently, $N_g$ is locally Hölder continuous. If $g \in L^p(\mathbb{C})$, $p > 2$, then $N_g \in C^{1,\alpha}_{\text{loc}}$ where $\alpha = (p - 2)/p$.

In this connection, recall the definition of the formal complex derivatives:

$$\frac{\partial}{\partial z} := \frac{1}{2} \left\{ \frac{\partial}{\partial x} - i \cdot \frac{\partial}{\partial y} \right\} \, , \quad \frac{\partial}{\partial \overline{z}} := \frac{1}{2} \left\{ \frac{\partial}{\partial x} + i \cdot \frac{\partial}{\partial y} \right\} \, , \quad z = x + iy \, .$$

The elementary algebraic calculations show their relation to the Laplacian

$$\Delta := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4 \cdot \frac{\partial^2}{\partial z \partial \overline{z}} = 4 \cdot \frac{\partial^2}{\partial \overline{z} \partial z}$$

**Proof.** Note that $N_g$ is the convolution $\psi * g$, where $\psi(\zeta) = \ln |\zeta|$, and hence $N_g \in L^r_{\text{loc}}$ for all $r \in [1, \infty)$, see e.g. Corollary 4.5.2 in [29]. Moreover, as
well-known $\frac{\partial \psi g}{\partial z} = \frac{\partial \psi}{\partial z} * g$ and $\frac{\partial \psi g}{\partial \z}$, see e.g. (4.2.5) in [29], and in addition by elementary calculations

$$\frac{\partial}{\partial z} \ln |z - w| = \frac{1}{2} \frac{1}{z - w}, \quad \frac{\partial}{\partial \z} \ln |z - w| = \frac{1}{2} \frac{1}{\z - \w}.$$  

Consequently,

$$\frac{\partial N_g(z)}{\partial z} = \frac{1}{4} Tg(z), \quad \frac{\partial N_g(z)}{\partial \z} = \frac{1}{4} Tg(z),$$

where $Tg$ and $Tg$ are the well-known integral operators

$$Tg(z) := \frac{1}{\pi} \int_\mathbb{C} g(w) \frac{d m(w)}{z - w}, \quad Tg(z) := \frac{1}{\pi} \int_\mathbb{C} g(w) \frac{d m(w)}{\z - \w}.$$  

Thus, all the rest conclusions for $g \in L^1(\mathbb{C})$ follow by Theorems 1.13–1.14 in [17]. If $g \in L^p(\mathbb{C})$, $p > 1$, then $N_g \in W^{1,q}_{\text{loc}}$, $q > 2$, by Theorem 1.27, (6.27) in [17], consequently, $N_g$ is locally Hölder continuous, see e.g. Theorem 8.22 in [23], and $N_g \in W^{2,p}_{\text{loc}}$ by Theorems 1.36–1.37 in [17]. If $g \in L^p(\mathbb{C})$, $p > 2$, then $N_g \in C^{1,\alpha}_{\text{loc}}$ with $\alpha = \frac{p - 2}{p}$ by Theorem 1.19 in [17]. □

Note that the corresponding Newtonian potentials $N_g$ in $\mathbb{R}^n$, $n \geq 3$, also belong to $W^{2,p}_{\text{loc}}$ if $g \in L^p(\mathbb{C})$ for $p > 1$ with compact support, see e.g. [24], Theorem 9.9.

By Theorem 2 and the known Poisson formula, see e.g. I.D.2 in [32], we come to the following consequence on the existence, regularity and representation of solutions for the Dirichlet problem to the Poisson equation in the unit disk $\mathbb{D}$ where we assume the charge density $g$ to be extended by zero outside $\mathbb{D}$.

**Corollary 2.** Let $\varphi : \partial \mathbb{D} \to \mathbb{R}$ be a continuous function and $g : \mathbb{D} \to \mathbb{R}$ belong to the class $L^p(\mathbb{D})$, $p > 1$. Then the function $U := N_g - P_{N_g^*} + P_\varphi$, $N_g^* := N_g|_{\partial \mathbb{D}}$, is continuous in $\overline{\mathbb{D}}$ with $U|_{\partial \mathbb{D}} = \varphi$, belongs to the class $W^{2,p}_{\text{loc}}(\mathbb{D})$ and $\Delta U = g$ a.e. in $\mathbb{D}$. Moreover, $U \in W^{1,q}_{\text{loc}}(\mathbb{D})$ for some $q > 2$ and $U$ is locally Hölder continuous. If in addition $\varphi$ is Hölder continuous, then $U$ is Hölder continuous in $\overline{\mathbb{D}}$. If $g \in L^p(\mathbb{D})$, $p > 2$, then $U \in C^{1,\alpha}_{\text{loc}}(\mathbb{D})$ where $\alpha = (p - 2)/p$.  

Furthermore, we have

$$U = \lim_{r \to 0} U_r,$$ where $U_r$ is the classical solution in the disk $B_r(z_0)$.  

Moreover, $U = \lim_{r \to 0} U_r$ is the classical solution in the disk $B_r(z_0)$.
Remark 2. The Hölder continuity of $U$ for Hölder continuous $\varphi$ follows from the corresponding result for the integral of the Cauchy type over the unit circle, see e.g. Theorem 1.10 in [47], because of the Poisson kernel $P(z,e^{it}) = \text{Re} \frac{e^{it}+e^{-it}}{e^{it}-e^{-it}}$. Note also by the way that a generalized solution of the Dirichlet problem to the Poisson equation in the class $C(\overline{D}) \cap W^{1,2}_{loc}(D)$ is unique at all, see e.g. Theorem 8.30 in [24]. One can show that the integral operators in Theorem 2 and Corollary 2 are completely continuous (it is clear from the corresponding theorems in [47] mentioned under the proof of Theorem 2), cf. e.g. [30] and [31]. However, for our goals it is sufficient that the operator $N_g : L^p(D) \rightarrow C(D)$ is completely continuous by Theorem 1 for $p > 1$, see the proof of Theorem 3 further.

3 The case of the quasilinear Poisson equations

The case is reduced to the Poisson equation by the Leray–Schauder approach.

Theorem 3. Let $\varphi : \partial D \rightarrow \mathbb{R}$ be a continuous function, $h : D \rightarrow \mathbb{R}$ be a function in the class $L^p(D)$, $p > 1$, and let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function with the nondecreasing function $|f|$ of $|t|$ such that

$$\lim_{t \rightarrow +\infty} \frac{f(t)}{t} = 0. \quad (3.1)$$

Then there is a continuous function $U : \overline{D} \rightarrow \mathbb{R}$ with $U|_{\partial D} = \varphi$, $U|_{D} \in W^{2,p}_{loc}(D)$ and

$$\triangle U(z) = h(z) \cdot f(U(z)) \quad \text{for a.e. } z \in D. \quad (3.2)$$

Moreover, $U \in W^{1,q}_{loc}(D)$ for some $q > 2$ and $U$ is locally Hölder continuous in $D$. If in addition $\varphi$ is Hölder continuous, then $U$ is Hölder continuous in $\overline{D}$. Furthermore, if $p > 2$, then $U \in C^{1,\alpha}_{loc}(\mathbb{D})$ where $\alpha = (p-2)/p$. In particular, $U \in C^{1,\alpha}_{loc}(\mathbb{D})$ for all $\alpha \in (0,1)$ if $h \in L^\infty(\mathbb{D})$.

Proof. If $\|h\|_p = 0$ or $\|f\|_C = 0$, then the Poisson integral $P_\varphi$ gives the desired solution of the Dirichlet problem for equation (3.2), see e.g. I.D.2 in [32]. Hence we may assume further that $\|h\|_p \neq 0$ and $\|f\|_C \neq 0$. 
By Theorem 1 and the maximum principle for harmonic functions, we obtain the family of operators $F(g; \tau) : L^p(\mathbb{D}) \to L^p(\mathbb{D})$, $\tau \in [0, 1]$

$$F(g; \tau) := \tau h \cdot f(N_g - \mathcal{P}N_g + \mathcal{P}_\varphi), \quad N_g^* := N_g|_{\partial \mathbb{D}}, \quad \forall \ \tau \in [0, 1]$$ (3.3)

which satisfies all groups of hypothesis H1-H3 of Theorem 1 in [38].

H1). First of all, $F(g; \tau) \in L^p(\mathbb{D})$ for all $\tau \in [0, 1]$ and $g \in L^p(\mathbb{D})$ because by Theorem 1 $f(N_g - \mathcal{P}N_g + \mathcal{P}_\varphi)$ is a continuous function and, moreover, by (2.14)

$$\|F(g; \tau)\|_p \leq \|h\|_p |f(2M\|g\|_p + \|\varphi\|_C)| < \infty \quad \forall \ \tau \in [0, 1].$$

Thus, by Theorem 1 in combination with the Arzela–Ascoli theorem, see e.g. Theorem IV.6.7 in [13], the operators $F(g; \tau)$ are completely continuous for each $\tau \in [0, 1]$ and even uniformly continuous with respect to the parameter $\tau \in [0, 1]$.

H2). The index of the operator $F(g; 0)$ is obviously equal to 1.

H3). By inequality (2.14) and the maximum principle for harmonic functions, we have the estimate for solutions $g \in L^p$ of the equations $g = F(g; \tau)$:

$$\|g\|_p \leq \|h\|_p |f(2M\|g\|_p + \|\varphi\|_C)| \leq \|h\|_p |f(3M\|g\|_p)|$$

whenever $\|g\|_p \geq \|\varphi\|_C/M$, i.e. then it should be

$$\frac{|f(3M\|g\|_p)|}{3M\|g\|_p} \geq \frac{1}{3M\|h\|_p}$$ (3.4)

and hence $\|g\|_p$ should be bounded in view of condition (3.1).

Thus, by Theorem 1 in [38] there is a function $g \in L^p(\mathbb{D})$ such that $g = F(g; 1)$ and, consequently, by Corollaries 2 the function $U := N_g - \mathcal{P}N_g + \mathcal{P}_\varphi$ gives the desired solution of the Dirichlet problem for the quasilinear Poisson equation (3.2). \(\square\)

**Remark 3.** As it is clear from the proof, condition (3.1) can be replaced by the weaker one

$$\limsup_{t \to +\infty} \frac{|f(t)|}{t} < \frac{1}{3M\|h\|_p}$$ (3.5)

where $M$ is the constant from estimate (2.14). Moreover, Theorem 3 is valid if $f$ is an arbitrary continuous bounded function.
Corollary 3. Let $D$ be a smooth Jordan’s domain in $\mathbb{C}$, $\Phi : \partial D \to \mathbb{R}$ be a continuous function, $H : D \to \mathbb{R}$ be a function in the class $L^p(D)$, $p > 1$, and let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function which is either bounded or satisfying \( (3.1) \) with nondecreasing $|f| \text{ of } |t|$. Then there is a continuous function $u : \overline{D} \to \mathbb{R}$ with $u|_{\partial D} = \Phi$, $u \in W^{2,p}_{\text{loc}}(D)$, $\quad \triangle u(\zeta) = H(\zeta) \cdot f(u(\zeta)) \quad \text{for a.e. } \zeta \in D. \quad (3.6)$

Moreover, $u \in W^{1,q}_{\text{loc}}(D)$ for some $q > 2$ and $u$ is locally Hölder continuous in $D$. If in addition $\Phi$ is Hölder continuous, then $u$ is Hölder continuous in $\overline{D}$. Furthermore, if $p > 2$, then $u \in C^{1,\alpha}_{\text{loc}}(D)$ where $\alpha = (p - 2)/p$.

In particular, $u \in C^{1,\alpha}_{\text{loc}}(D)$ for all $\alpha \in (0, 1)$ if $h \in L^\infty(D)$. If in addition $\Phi$ is Hölder continuous on $\partial D$ with some order $\beta \in (0, 1)$, then $u$ is Hölder continuous in $D$ with the same order.

Proof. Let $\omega$ be a conformal mapping of $D$ onto $\mathbb{D}$. By the Caratheodory-Osgood-Taylor theorem, $\omega$ is extended to a homeomorphism $\tilde{\omega}$ of $\overline{D}$ onto $\overline{\mathbb{D}}$, see [14] and [42], see also [5] and Theorem 3.3.2 in the monograph [16]. Then, setting $\varphi = \Phi \circ \tilde{\omega}^{-1}|_{\partial \mathbb{D}}$, we see that the function $\varphi : \partial \mathbb{D} \to \mathbb{R}$ is continuous. Let $h = J \cdot H \circ \Omega$ where $\Omega$ is the inverse mapping $\omega^{-1} : \mathbb{D} \to D$ and $J$ is its Jacobian $J = |\Omega'|^2$. By the known Warschawski result, see Theorem 2 in [49], its derivative $\Omega'$ is extended by continuity onto $\overline{\mathbb{D}}$. Consequently, $J$ is bounded and the function $h$ is of the same class in $\mathbb{D}$ as $H$ in $D$. Let $U$ be a solution of the Dirichlet problem from Proposition 1 for the equation \( (3.2) \) with the given $\varphi$ and $h$. Note that $\omega' = 1/\Omega' \circ \omega$ is also extended by continuity onto $\overline{D}$ because $\Omega' \neq 0$ on $\partial \mathbb{D}$ by Theorem 1 in [49]. Thus, $u = U \circ \omega$ is the desired solution of the Dirichlet problem for the equation \( (3.6) \). $\square$

4 Case of inhomogeneous and anisotropic media

By the mentioned above factorization theorem from [25], the study of semi-linear equations \( (1.2) \) in Jordan domains $D$ is reduced, by means of a suitable
Theorem 4. Let $D$ be a Jordan’s domain in $\mathbb{C}$ satisfying the quasihyperbolic boundary condition, $A \in M^2_{K^2}(D)$, $\varphi : \partial D \to \mathbb{R}$ be a continuous function, $f : \mathbb{R} \to \mathbb{R}$ be a continuous function which is either bounded or with nondecreasing $|f|$ of $|t|$ such that
\[
\lim_{t \to +\infty} \frac{f(t)}{t} = 0 .
\]
Then there is a weak solution $u : D \to \mathbb{R}$ of the equation (1.2) which is locally Hölder continuous in $D$ and continuous in $\overline{D}$ with $u|_{\partial D} = \varphi$. If in addition $\varphi$ is Hölder continuous, then $u$ is Hölder continuous in $D$.

Corollary 4. In particular, under hypotheses of Theorem 4 on $D$, $\varphi$ and $f$, there is a weak solution $U$ of the quasilinear Poisson equation
\[
\triangle U(z) = f(U(z)) \quad \text{for a.e. } z \in D .
\]
which is locally Hölder continuous in $D$ and continuous in $\overline{D}$ with $U|_{\partial D} = \varphi$. If in addition $\varphi$ is Hölder continuous, then $U$ is Hölder continuous in $\overline{D}$.

In the case (4.3), $A$ is the unit matrix and $\eta$ can be taken in the class $C^\infty_0(D)$.

Proof. By Theorem 4.1 in [25], if $u$ is a week solution of (1.2), then $u = U \circ \omega$ where $\omega$ is a quasiconformal mapping of $D$ onto the unit disk $\mathbb{D}$ agreed with $A$ and $U$ is a week solution of the equation (3.2) with $h = J$, where $J$ stands for the Jacobian of $\omega^{-1}$. It is also easy to see that if $U$ is a week solution of (3.2) with $h = J$, then $u = U \circ \omega$ is a week solution of (1.2). It allows us to reduce the Dirichlet problem for equation (1.2) with continuous boundary function $\varphi$ in the simply connected Jordan domain $D$ to the Dirichlet problem for the equation...
in the unit disk $D$ with the continuous boundary function $\psi = \varphi \circ \omega^{-1}$.

Indeed, $\omega$ is extended to a homeomorphism of $\overline{D}$ onto $\overline{D}$, see e.g. Theorem I.8.2 in [37]. Thus, the function $\psi$ is well defined and really is continuous on the unit circle.

It is well-known that the quasiconformal mapping $\omega$ is locally Hölder continuous in $D$, see Theorem 3.5 in [11]. Taking into account the fact that $D$ is a Jordan’s domain in $\mathbb{C}$ satisfying the quasihyperbolic boundary condition, we can show that both mappings $\omega$ and $\omega^{-1}$ are Hölder continuous in $\overline{D}$ and $\overline{D}$, correspondingly. Indeed, $\omega = H \circ \Omega$ where $\Omega$ is a conformal (Riemann) mapping of $D$ onto $\mathbb{D}$ and $H$ is a quasiconformal mapping of $\mathbb{D}$ onto itself. The mappings $\Omega$ and $\Omega^{-1}$ are Hölder continuous in $\overline{D}$ and in $\overline{D}$, correspondingly, by Theorem 1 and its corollary in [9]. Next, by the reflection principle $H$ can be extended to a quasiconformal mapping of $\mathbb{C}$ onto itself, see e.g. I.8.4 in [37], and, consequently, $H$ and $H^{-1}$ are also Hölder continuous in $\overline{D}$, see again Theorem 3.5 in [11]. The Hölder continuity of $\omega$ and $\omega^{-1}$ in closed domains follows immediately.

Now it is easy to see that if $\varphi$ is Hölder continuous, then $\psi$ is also so, and all the conclusions of Theorem 1 follow from Proposition 1. \(\Box\)

Recall that a domain $D$ in $\mathbb{R}^n$, $n \geq 2$, is called satisfying the \textbf{(A)-condition} if

\[
\text{mes } D \cap B(\zeta, \rho) \leq \Theta_0 \cdot \text{mes } B(\zeta, \rho) \quad \forall \, \zeta \in \partial D \, , \, \rho \leq \rho_0 \quad (4.4)
\]

for some $\Theta_0$ and $\rho_0 \in (0,1)$, see 1.1.3 in [36]. Recall also that a domain $D$ in $\mathbb{R}^n$, $n \geq 2$, is said to be satisfying the \textbf{outer cone condition} if there is a cone that makes possible to be touched by its top to every boundary point of $D$ from the completion of $D$ after its suitable rotations and shifts. It is clear that the outer cone condition implies (A)-condition.

\textbf{Remark 4.} Note that quasidisks $D$ satisfy (A)-condition. Indeed, the quasidisks are the so-called QED-domains by Gehring–Martio, see Theorem 2.22 in [22], and the latter satisfy the condition

\[
\text{mes } D \cap B(\zeta, \rho) \geq \Theta_* \cdot \text{mes } B(\zeta, \rho) \quad \forall \, \zeta \in \partial D \, , \, \rho \leq \text{dia}D \quad (4.5)
\]
for some $\Theta_s \in (0,1)$, see Lemma 2.13 in [22], and quasidisks (as domains with quasihyperbolic boundary) have boundaries of the Lebesgue measure zero, see e.g. Theorem 2.4 in [3]. Thus, it remains to note that, by definition, the completions of quasidisks $D$ in the the extended complex plane $\mathbb{C} := \mathbb{C} \cup \{\infty\}$ are also quasidisks up to the inversion with respect to a circle in $D$.

Probably the first example of a simply connected plane domain $D$ with the quasihyperbolic boundary condition which is not a quasidisk was constructed in [9], Theorem 2. However, this domain had $(A)$–condition. Probably one of the simplest example of a domain $D$ with the quasihyperbolic boundary condition and without $(A)$–condition is the union of 3 open disks with the radius 1 centered at the points 0 and 1 $\pm$ $i$. It is clear that the domain has zero interior angle in its boundary point 1 and by Remark 4 it is not quasidisk.

5 The Dirichlet problem in terms of prime ends

The simplest example of a domain $D$ with the quasihyperbolic boundary condition and simultaneously without $(A)$–condition is disc with a cut along its radius.

Before to formulate the corresponding results for non-Jordan domains, let us recall the necessary definitions of the relevant notions and notations. Namely, we follow Caratheodory [15] under the definition of the prime ends of domains in $\mathbb{C}$, see also Chapter 9 in [16]. First of all, recall that a continuous mapping $\sigma : I \to \mathbb{C}$, $I = (0,1)$, is called a {Jordan arc} in $\mathbb{C}$ if $\sigma(t_1) \neq \sigma(t_2)$ for $t_1 \neq t_2$. We also use the notations $\sigma$, $\overline{\sigma}$ and $\partial \sigma$ for $\sigma(I)$, $\overline{\sigma(I)}$ and $\sigma(I) \setminus \sigma(I)$, correspondingly.

A {cross–cut} of a simply connected domain $D \subset \mathbb{C}$ is a Jordan arc $\sigma$ in the domain $D$ with both ends on $\partial D$ splitting $D$.

A sequence $\sigma_1, \ldots, \sigma_m, \ldots$ of cross-cuts of $D$ is called a {chain} in $D$ if:

(i) $\overline{\sigma_i} \cap \overline{\sigma_j} = \emptyset$ for every $i \neq j$, $i, j = 1, 2, \ldots$
(ii) \( \sigma_m \) splits \( D \) into 2 domains one of which contains \( \sigma_{m+1} \) and another one \( \sigma_{m-1} \) for every \( m > 1 \);

(iii) \( \delta(\sigma_m) \to 0 \) as \( m \to \infty \) where \( \delta(\sigma_m) \) is the diameter of \( \sigma_m \) with respect to the Euclidean metric in \( \mathbb{C} \).

Correspondingly to the definition, a chain of cross-cuts \( \sigma_m \) generates a sequence of domains \( d_m \subset D \) such that \( d_1 \supset d_2 \supset \ldots \supset d_m \supset \ldots \) and \( D \cap \partial d_m = \sigma_m \). Chains of cross-cuts \( \{\sigma_m\} \) and \( \{\sigma'_k\} \) are called equivalent if, for every \( m = 1, 2, \ldots \), the domain \( d_m \) contains all domains \( d'_k \) except a finite number and, for every \( k = 1, 2, \ldots \), the domain \( d'_k \) contains all domains \( d_m \) except a finite number, too. A prime end \( P \) of the domain \( D \) is an equivalence class of chains of cross-cuts of \( D \). Later on, \( E_D \) denote the collection of all prime ends of a domain \( D \) and \( \overline{D}_P = D \cup E_D \) is its completion by its prime ends.

Next, we say that a sequence of points \( p_l \in D \) is convergent to a prime end \( P \) of \( D \) if, for a chain of cross–cuts \( \{\sigma_m\} \) in \( P \), for every \( m = 1, 2, \ldots \), the domain \( d_m \) contains all points \( p_l \) except their finite collection. Further, we say that a sequence of prime ends \( P_l \) converge to a prime end \( P \) if, for a chain of cross–cuts \( \{\sigma_m\} \) in \( P \), for every \( m = 1, 2, \ldots \), the domain \( d_m \) contains chains of cross–cuts \( \{\sigma'_k\} \) in all prime ends \( P_l \) except their finite collection.

A basis of neighborhoods of a prime end \( P \) of \( D \) can be defined in the following way. Let \( d \) be an arbitrary domain from a chain in \( P \). Denote by \( d^* \) the union of \( d \) and all prime ends of \( D \) having some chains in \( d \). Just all such \( d^* \) form a basis of open neighborhoods of the prime end \( P \). The corresponding topology on \( E_D \) and, respectively, on \( \overline{D}_P \) is called the topology of prime ends. The continuity of functions on \( E_D \) and \( \overline{D}_P \) will be understood with respect to this topology or, the same, with respect to the above convergence.

**Theorem 5.** Let \( D \) be a bounded simply connected domain in \( \mathbb{C} \) satisfying the quasihyperbolic boundary condition, \( A \in M_{K}^{2\times 2}(D), \varphi : E_D \to \mathbb{R} \) be a continuous function, \( f : \mathbb{R} \to \mathbb{R} \) be either a continuous bounded function or a
continuous function with the nondecreasing function $|t|$ such that

$$\lim_{t \to +\infty} \frac{f(t)}{t} = 0.$$  (5.1)

Then there is a weak solution $u : D \to \mathbb{R}$ of the equation (1.2) which is locally Hölder continuous in $D$ and continuous in $\overline{D}_P$ with $u|_{E_D} = \varphi$.

**Corollary 5.** In particular, under hypotheses of Theorem 5 on $D$, $\varphi$ and $f$, there is a weak solution $U$ of the quasilinear Poisson equation (4.3) which is locally Hölder continuous in $D$ and continuous in $\overline{D}_P$ with $U|_{E_D} = \varphi$.

**Proof.** Again by Theorem 4.1 in [25], if $u$ is a week solution of (1.2), then $u = U \circ \omega$ where $\omega$ is a quasiconformal map of $D$ onto the unit disk $\mathbb{D}$ agreed with $A$ and $U$ is a week solution of the equation (3.2) with $h = J$, the Jacobian of $\omega^{-1}$. Similarly, if $U$ is a week solution of (3.2) with $h = J$, then $u = U \circ \omega$ is a week solution of (1.2).

Hence the Dirichlet problem for (1.2) in the domain $D$ will be reduced to the so for (3.2) in $\mathbb{D}$ with the corresponding boundary function $\psi = \varphi \circ \omega^{-1}$. The existence and continuity of the boundary function $\psi$ in the case of an arbitrary bounded simply connected domain $D$ is a fundamental result of the theory of the boundary behavior of conformal and quasiconformal mappings. Namely,

$$\omega^{-1} = H \circ \Omega$$

where $\Omega$ stands for a quasiconformal automorphism of the unit disk $\mathbb{D}$ and $H$ is a conformal mapping of $\mathbb{D}$ onto $\Omega$. It is known that $\Omega$ can be extended to a homeomorphism of $\overline{\mathbb{D}}$ onto itself, see e.g. Theorem I.8.2 in [37]. Moreover, by the well-known Caratheodory theorem on the boundary correspondence under conformal mappings, see e.g. Theorems 9.4 and 9.6 in [16], the mapping $H$ is extended to a homeomorphism of $\overline{\mathbb{D}}$ onto $\overline{D}_P$. Thus, the function $\psi$ is well defined and really continuous on the unit circle.

Moreover, $\omega$ is locally Hölder continuous in $\mathbb{D}$, see e.g. Theorem 3.5 in [11]. Thus, by Theorem 4.1 in [25], Theorem 5 follows from Theorem 3. □
6 Some applied corollaries

The interest in this subject is well known both from a purely theoretical point of view, due to its deep relations to linear and nonlinear harmonic analysis, and because of numerous applications of equations of this type in various areas of physics, differential geometry, logistic problems etc., see e.g. [13], [17], [24], [27], [33], [34], [39], [41], [48] and the references therein.

In particular, the mathematical modelling of some reaction-diffusion problems leads to the study of the corresponding Dirichlet problem for the equation (1.12) with specified right hand side. Following [4], a nonlinear system can be obtained for the density $u$ and the temperature $T$ of the reactant. Upon eliminating $T$ the system can be reduced to a scalar problem for the concentration

$$\Delta u = \lambda \cdot f(u)$$

(6.1)

where $\lambda$ stands for a positive constant.

In the excellent book by M. Marcus and L. Veron [39] the reader can find a comprehensive analysis of the Dirichlet problem for the semi-linear equation

$$\Delta u(z) = f(z, u(z))$$

(6.2)

in smooth ($C^2$) domains $D$ in $\mathbb{R}^n$, $n \geq 3$, with boundary data in $L^1$. Here $t \to f(\cdot, t)$ is a continuous mapping from $\mathbb{R}$ to a weighted Lebesgue’s space $L^1(D, \rho)$, and $z \to f(z, \cdot)$ is a non-decreasing function for every $z \in D$, $f(z, 0) \equiv 0$, such that

$$\lim_{t \to \infty} \frac{f(z, t)}{t} = \infty$$

(6.3)

uniformly with respect to the parameter $z$ in compact subsets of $D$.

It turns out that the density of the reactant $u$ may be zero in a closed interior region $D_0$ called a dead core. If, for instance in the equation (6.1), $f(u) = u^q$, $q > 0$, a particularization of the results in Chapter 1 of [17] shows that a dead core may exist if and only if $0 < q < 1$ and $\lambda$ is large enough. See also the corresponding examples of dead cores in [25]. We have by Theorem 4 the following:
Theorem 6. Let \(D\) be a Jordan’s domain in \(\mathbb{C}\) satisfying the quasihyperbolic boundary condition, \(A \in M_2^{2 \times 2}(D), \varphi : \partial D \to \mathbb{R}\) be a continuous function. Then there is a weak solution \(u : D \to \mathbb{R}\) of the semilinear equation

\[
\text{div} \left[ A(z) \nabla u(z) \right] = u^q(z), \quad 0 < q < 1,
\]

which is locally Hölder continuous in \(D\) and continuous in \(\overline{D}\) with \(u|_{\partial D} = \varphi\). If in addition \(\varphi\) is Hölder continuous, then \(u\) is Hölder continuous in \(D\).

Recall that under a weak solution to the equation (6.4) we understand a function \(u \in C \cap W^{1,2}_{1,\text{loc}}(\Omega)\) such that, for all \(\eta \in C \cap W^{1,2}_{0}(D),\)

\[
\int_{D} (A(z)\nabla u(z), \nabla \eta(z)) \, dm(z) + \int_{D} u^q(z) \eta(z) \, dm(z) = 0.
\]

(6.5)

We have also the following significant consequence of Corollary 3.

Corollary 6. Let \(D\) be a smooth Jordan’s domain in \(\mathbb{C}\) and \(\varphi : \partial D \to \mathbb{R}\) be a continuous function. Then there is a weak solution \(U\) of the quasilinear Poisson equation

\[
\Delta U(z) = U^q(z), \quad 0 < q < 1,
\]

which is continuous in \(\overline{D}\) with \(U|_{\partial D} = \varphi\) and \(U \in C^{1,\alpha}_{\text{loc}}(D)\) for all \(\alpha \in (0,1)\). If in addition \(\varphi\) is Hölder continuous with some order \(\beta \in (0,1)\), then \(U\) is also Hölder continuous in \(\overline{D}\) with the same order.

Recall also that certain mathematical models of a heated plasma lead to nonlinear equations of the type (6.1). Indeed, it is known that some of them have the form \(\Delta \psi(u) = f(u)\) with \(\psi'(0) = +\infty\) and \(\psi'(u) > 0\) if \(u \neq 0\) as, for instance, \(\psi(u) = |u|^q - 1u\) under \(0 < q < 1\), see e.g. [7], [8] and [17], p. 4. With the replacement of the function \(U = \psi(u) = |u|^q \cdot \text{sign} u\), we have that \(u = |U|^Q \cdot \text{sign} U, Q = 1/q\), and, with the choice \(f(u) = |u|^{q^2} \cdot \text{sign} u\), we come to the equation \(\Delta U = |U|^q \cdot \text{sign} U = \psi(U)\).

Of course, the similar results can be formulated for this case, for instance:
Corollary 7. Let $D$ be a smooth Jordan’s domain in $\mathbb{C}$ and $\varphi : \partial D \to \mathbb{R}$ be a continuous function. Then there is a weak solution $U$ of the quasilinear Poisson equation

$$\nabla U(z) = |U(z)|^{q-1}U(z), \quad 0 < q < 1,$$

which is continuous in $\overline{D}$ with $U|_{\partial D} = \varphi$ and $U \in C^{1,\alpha}_{\text{loc}}(D)$ for all $\alpha \in (0,1)$. If in addition $\varphi$ is Hölder continuous with some order $\beta \in (0,1)$, then $U$ is also Hölder continuous in $\overline{D}$ with the same order.

In the combustion theory, see e.g. [6], [33] and the references therein, the following model equation

$$\frac{\partial u(z,t)}{\partial t} = \frac{1}{\delta} \cdot \Delta u + e^u, \quad t \geq 0, \quad z \in D,$$

occupies a special place. Here $u \geq 0$ is the temperature of the medium and $\delta$ is a certain positive parameter.

We restrict ourselves by stationary solutions of the equation and its generalizations in anisotropic and inhomogeneous media although our approach makes it possible to consider the parabolic case, see [25]. Namely, by Theorem 4 we have the following statement:

Theorem 7. Let $D$ be a Jordan’s domain in $\mathbb{C}$ satisfying the quasihyperbolic boundary condition, $A \in M_{2\times 2}^K(D)$, $\varphi : \partial D \to \mathbb{R}$ be a continuous function. Then there is a weak solution $U : D \to \mathbb{R}$ of the semilinear equation

$$\text{div} \left[ A(z) \nabla U(z) \right] = \delta \cdot e^{-U(z)} , \quad \delta > 0,$$

which is locally Hölder continuous in $D$ and continuous in $\overline{D}$ with $u|_{\partial D} = \varphi$. If in addition $\varphi$ is Hölder continuous, then $u$ is Hölder continuous in $\overline{D}$.

Finally, we obtain also the following consequence of Corollary 3.

Corollary 8. Let $D$ be a smooth Jordan’s domain in $\mathbb{C}$ and $\varphi : \partial D \to \mathbb{R}$ be a continuous function. Then there is a weak solution $U$ of the quasilinear Poisson equation

$$\nabla U(z) = \delta \cdot e^{-U(z)} , \quad \delta > 0,$$

(6.10)
which is continuous in $\overline{D}$ with $U|_{\partial D} = \varphi$ and $U \in C^{1,\alpha}_{\text{loc}}(D)$ for all $\alpha \in (0,1)$. If in addition $\varphi$ is Hölder continuous with some order $\beta \in (0,1)$, then $U$ is also Hölder continuous in $\overline{D}$ with the same order.

Thus, the results on regular solutions for the quasilinear Poisson equations (3.2) and the comprehensively developed theory of quasiconformal mappings in the plane, see e.g. the monographs [1], [12], [26], [37] and [40], are good basic tools for the further study of equations (1.2). The latter opens up a new approach to the study of a number of semi-linear equations of mathematical physics in anisotropic and inhomogeneous media.

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