Expanding alternating André permutations of the second kind: left-to-right minima, right-to-left minima and min-max paths.

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Abstract

The aim of this work was to investigate a new class of permutations enumerated by Euler numbers. This class is obtained expanding a constrained subset of André permutations of the second kind. The constraint is given by the \(d\)-alternating property, i.e. alternating starting with a descent, while expansion is provided by an application of the restriction operator which has been already used in the literature to define - for example - simsun permutations.

We focus on left-to-right minima and right-to-left minima statistics. Both of them are strongly related to the tree-structure of the considered permutations. The connection is given in terms of min-max paths defined on binary increasing trees.

As a corollary to some of our results, we indicate the presence of a link between our permutations and those called cycle-up-down, recently introduced by Deutsch and Elizalde.

1 Introduction

The aim of this work was to study a new class of permutations counted by Euler numbers. This class is derived from André permutations of the second kind.

André permutations of the second kind (André permutations) have been introduced in \([8]\) and extensively studied in the literature especially because of their relations with other combinatorial structures \([9, 11, 15]\). For instance the \(cd\)-index of the Boolean algebra may be computed by summing the \(cd\)-variation monomials of André permutations \([15]\).

The \(n\)-th Euler number \(e_n\) counts André permutations of size \(n\). The first terms are \(e_0 = 1, e_1 = 1, e_2 = 1, e_3 = 2, e_4 = 5, e_5 = 16, \ldots\) Euler numbers give the enumeration of several other combinatorial structures. In particular they also count rooted binary unordered \(increasing\) trees. Indeed in \([8]\) the authors describe a bijection - denoted here by \(\phi\) - which maps André permutations to this class of trees and viceversa.

Here we construct a new class of permutations associated with the sequence \((e_n)_{n}\). We do this by expanding a constrained subset of André permutations. The constraint is given by considering only those permutations which are \(d\)-alternating. This is equivalent to say that they are \(\phi\)-associated with \(strictly-binary\) increasing trees. The expansion is provided by an application of the restriction operator. This last tool has been already used in the literature to define simsun permutations (see \([4]\) and related references).

What we consider in this manuscript is indeed the class of restrictions of \(d\)-alternating André permutations.

In Section \(3.1\) we provide the enumeration with respect to the size which is given by Euler numbers.

In Section \(3.2\) we find the bivariate generating function which counts the restrictions according to two parameters, i.e. the size and the number of left-to-right minima. To achieve this result we make use of a nice correspondence between the set of left-to-right minima and a particular path, called the \(min-path\), of the associated increasing tree. As a
result, fixing the number of left-to-right minima, we provide a nice combinatorial formula which completely describes the desired enumeration in terms of Euler numbers.

In Section 3.3 we study the number of right-to-left minima. We give a functional equation for the associated biivariate generating function. We show how the number of restrictions of size \( n + 1 \) with 2 right-to-left minima is related to the total number of left-to-right minima in restrictions of size \( n \). Finally, we study restrictions with a generic - but fixed - number of right-to-left minima, providing an asymptotic bound for the probability of a random restriction of given size to have a certain number of right-to-left minima.

Interestingly, as a corollary to some of our results, we observe a connection with the class of cycle-up-down permutations [5] and this will need to be further investigated.

2 Preliminaries

The set of permutations of size \( n \) is denoted by \( S_n \). If \( \pi = (\pi_1 \pi_2 \ldots \pi_n) \in S_n \) the set of its left-to-right minima is denoted by \( \text{lrm}(\pi) \) and its elements are those entries \( \pi_i \) such that if \( j < i \), then \( \pi_i < \pi_j \). We denote by \( \text{rlm}(\pi) \) the set of right-to-left minima and we recall to the reader that \( \pi_i \in \text{rlm}(\pi) \) if \( j > i \) implies \( \pi_i < \pi_j \).

A binary increasing tree is a rooted, un-ordered tree with nodes of outdegree 0, 1 or 2. Nodes of outdegree 0 are also called the leaves of the tree. Moreover, for such a tree, we require that each of the \( n \) nodes is bijectively labelled by a number in \( \{1, 2, \ldots, n\} \) in a way that going from the root to any leaf we always find an increasing sequence of numbers. If \( x \) and \( y \) are two nodes, we write \( x < y \) when the label of \( x \) is less than the label of \( y \). The set of binary increasing trees is denoted by \( B \) while we use the symbol \( B_n \) to denote the subset of \( B \) made of those trees with \( n \) nodes.

Observe that, according to [8], each tree in \( B \) can be drawn in the plane in a unique way respecting the following two conditions \( c_1 \) and \( c_2 \):

\( c_1 \) if a node has only one child, then this child is drawn on the right of its direct ancestor;

\( c_2 \) if a node \( x \) has two children \( y \) and \( z \) with \( y < z \), then \( y \) is drawn on the left of \( x \) while \( z \) on the right.

In Fig. 1 we show those trees belonging to \( B_4 \) respecting the previous two conditions.

The set of André permutations (of the second kind) \( \mathcal{A} \) can be defined in several equivalent ways, see for example Section 2 of [11]. Since \( \mathcal{A}_n \) is a subset of \( S_n \) equinumerous to \( B_n \), we choose to characterize the mentioned permutations according to the following injective map \( \phi : B_n \rightarrow S_n \) (see [5]):

1) given \( t \in B_n \), draw \( t \) according to \( c_1 \) and \( c_2 \);

2) each leaf collapses into its direct ancestor whose label is then modified receiving on the left the label of the left child (if any) and on the right the label of its right child.

We obtain in this way a new tree whose nodes are labelled with sequences of numbers;

3) starting from the obtained tree go to step 2).

The algorithm \( \phi \) ends when the tree \( t \) is reduced to a single node whose label is then a permutation \( \phi(t) \) of size \( n \).

The set \( \mathcal{A}_n \) can be seen as \( \mathcal{A}_n = \{ \phi(t) \in S_n : t \in B_n \} \). Looking at Fig. 1 the corresponding permutations in \( \mathcal{A}_4 \) are (from left to right) (2314), (1234), (2134), (1324) and (2413).
Andre permutations, as binary increasing trees, are enumerated, with respect to the size, by the so called Euler numbers \((e_n)_{n \geq 0}\) whose exponential generating function satisfies

\[
\int E(z)^2dz = 2E(z) - z - 2
\]

and therefore is equal to

\[
E(z) = \sec(z) + \tan(z).
\]

The first terms of the sequence are: 1, 1, 1, 2, 5, 16, 61, 272, 1385, ... and they correspond to entry A000111 in [11].

Given a permutation \(\pi \in S_n\), a restriction of \(\pi\) is a permutation obtained from \(\pi\) by considering only the entries 1, 2, ..., \(k\), with \(1 \leq k \leq n\). If \(\sigma\) is a restriction of \(\pi\), then we write \(\sigma \triangleleft \pi\). Restrictions have been used in the literature for example to define simsun permutations. Here we use restrictions as an expansion operator. Indeed, if \(X\) is a class of permutations, we can define \(R^X = \{\sigma \triangleleft \pi : \pi \in X\}\). Observe that \(X \subseteq R^X\) and in particular, for Andre permutations, we have a strict inclusion given that - for example - \((21) < (213) \in A\) and \((21) \notin A\). It follows that, in general, the quantity \(|R^A \cap S_n|\) can be greater than \(e_n\).

It is then of interest to ask for a constraint on \(A\) such that, expanding the resulting subset \(\hat{A}\), gives back for \((R^A \cap S_n)\) the original enumerative sequence \((e_n)\). That is what we do with the next definitions.

Given a permutation \(\pi = (\pi_1 \pi_2 \ldots \pi_n)\), a descent (resp. a ascent) in \(\pi\) is an entry \(\pi_i\) with \(1 \leq i < n\) such that \(\pi_i > \pi_{i+1}\) (resp. \(\pi_i < \pi_{i+1}\)). A permutation is \(d\)-alternating when it starts with a descent and it has neither two consecutive descents nor two consecutive ascents. The subset of \(A_n\) made of those permutations which are \(d\)-alternating is denoted by \(\hat{A}_n\) and \(\hat{A} = \bigcup \hat{A}_n\).

If we define a tree in \(B\) which does not have nodes of outdegree 1 to be strictly-binary and we let \(\hat{B}_n\) be the corresponding subset of \(B_n\), we can observe the following correspondence

**Proposition 1** Let \(\pi \in A_n\), then \(\pi \in \phi(\hat{B}_n)\) if and only if \(\pi \in \hat{A}_n\).

**Proof.** It is a well known fact that Andre permutations do not have two consecutive descents [11]. It is then sufficient to show that, given \(\pi \in A_n\), \(\pi \in \phi(\hat{B}_n)\) is equivalent to say that \(\pi\) starts with a descent and that it does not contain two consecutive ascents.

(\(\Rightarrow\)) To start observe that if \(\pi\) starts with a ascent, then it cannot be \(\phi\)-associated with a strictly-binary tree. Furthermore, if \(\pi\) has two consecutive ascents, say \(\pi = (\ldots \pi_i \pi_{i+1} \pi_{i+2} \ldots)\) with \(\pi_i < \pi_{i+1} < \pi_{i+2}\), let \(t \in B_n\) such that \(\phi(t) = \pi\). Looking at \(\pi_i, \pi_{i+1}\) and \(\pi_{i+2}\) as nodes of \(t\) we must have that \(\pi_{i+1}\) belongs to the right subtree of \(\pi_i\) and, similarly, \(\pi_{i+2}\) is a node in the right subtree of \(\pi_{i+1}\). Furthermore note that \(\pi_{i+1}\) cannot have a left subtree, otherwise \(\pi_{i+1}\) would not be next to \(\pi_i\) in \(\phi(t)\). It follows that \(\pi_{i+1}\) is a node of outdegree 1 in \(t\), which means \(t \notin \hat{B}_n\).

(\(\Leftarrow\)) Let us now take \(\pi = \phi(t)\) with \(t\) having a node \(x\) of outdegree 1. Let \(w\) be the first node we encounter in the path from \(x\) to the root of \(t\) such that \(x\) is in the right subtree of \(w\); such a \(w\) must exist otherwise \(\pi\) would start with a rise. Let \(y\) be the (right) child of \(x\). If \(t'\) is the (possibly empty) left subtree of \(y\), then, applying \(\phi\), we obtain \(\pi = \phi(t) = (\ldots w x \phi(t') y \ldots)\) where, either in \(w x \phi(t')\) or in \(w x y\), we find two consecutive ascents. It follows that \(\pi \notin \hat{A}\). \(\square\)

The previous proposition allows us to switch from \(d\)-alternating Andre permutations to strictly-binary increasing trees and vice versa.

Let us finally define the object of our work: we consider

\[
R^\hat{A}_n = \{\sigma \in S_n : \exists \pi \in \hat{A} \text{ with } \sigma \triangleleft \pi\}
\]

and

\[
R^\hat{A} = \bigcup_i R^\hat{A}_i.
\]

According to the definition the reader can check that \(R^\hat{A}_n\) is made of the following 16 permutations
Observe that in general $\mathcal{R}_n^\mathcal{A} \not\subseteq \mathcal{A}_n$. Indeed we have that $(326514) \prec (3265714) \in \mathcal{A}$ but $(326514) \notin \mathcal{A}_n$.

In Section 5.2 we will focus on left-to-right minima statistics using the symbol $\mathcal{R}_n^\mathcal{A,m}$ to denote the subset of $\mathcal{R}_n^\mathcal{A}$ made of those permutations $\pi$ with $|\text{rhm}(\pi)| = m$, while, in Section 5.3 we will study right-to-left minima using $\mathcal{R}_n^\mathcal{A,m}$ to refer to the subset $|\text{rlm}(\pi)| = m$.

### 3 Enumeration of $\mathcal{R}_n^\mathcal{A}$ and $\mathcal{R}_n^\mathcal{A,m}$

In this section we provide the enumeration of $\mathcal{R}_n^\mathcal{A}$ and $\mathcal{R}_n^\mathcal{A,m}$ using the existing correspondence between permutations and binary increasing trees.

#### 3.1 The cardinality of $\mathcal{R}_n^\mathcal{A}$

The two criteria $c_1$) and $c_2$) given in Section 2 which were used to draw each binary increasing tree in a unique and well defined way, are not the only possible ones. Indeed, one can consider the criterion $c_2$) plus the dual version of $c_1$). To be more precise, if in a tree a node has only one child, then this child is drawn on the left of its direct ancestor and not on the right as it was in $c_1$).

Trees drawn according to $c_2$) and the new condition described above are said to be left oriented. Given such a tree, we can still apply the procedure $\phi$ of Section 2 - without considering step 1) - to define a permutation which is not, in general, an André one.

In what follows we show that a permutation belongs to $\mathcal{R}_n^\mathcal{A}$ if and only if it is the permutation coming through $\phi$ from a left oriented binary increasing tree of size $n$. As a corollary we also have that $|\mathcal{R}_n^\mathcal{A}| = |\mathcal{B}_n| = e_n$.

**Proposition 2** Given a permutation $\pi \in \mathcal{S}_n$, we have that $\pi$ is $\phi$-associated with a left oriented binary increasing tree of size $n$ if and only if $\pi \in \mathcal{R}_n^\mathcal{A}$.

**Proof.** ($\Rightarrow$) Take the left oriented tree corresponding to $\pi$ and, to every node with outdegree 1, add a child with label a number greater than $n$. Let $t$ be the resulting tree. Then $t \notin \mathcal{B}$ and $\pi \prec \phi(t) \notin \mathcal{A}$.

($\Leftarrow$) Suppose $\pi \prec \sigma \in \mathcal{A}_m$, with $n \leq m$. Let $t \in \mathcal{B}_m$ be the tree associated with $\sigma$. Remove from $t$ all the nodes labelled by a number $k > n$. The obtained tree is left oriented and $\pi$ comes from $t$ applying steps 2) and 3) of $\phi$. $\square$

Thus we have

**Corollary 1** The cardinality of $\mathcal{R}_n^\mathcal{A}$ is given by the $n$-th Euler number $e_n$.

#### 3.2 Left-to-right minima and the min-path

In the previous section we have shown that permutations in $\mathcal{R}_n^\mathcal{A}$ correspond to those permutations coming from left oriented trees when we apply steps 2) and 3) of $\phi$. Now we want to refine the bijection described in the proof of Proposition 2 to consider also the set of left-to-right minima of such permutations. To do this we need the following definition.

Given a binary increasing tree $t$, consider the following path: we start from the root of $t$ and at each step we move to the child with the least label. In Fig. 2 we show a tree $t$ where such path, which will be called the min-path of $t$, is highlighted.

The next proposition states the relation between left-to-right minima of a permutation in $\mathcal{R}_n^\mathcal{A}$ and the min-path of the corresponding tree in $\mathcal{B}$.
Proposition 3 If \( \pi \in R_{n}^{\tilde{A}} \), then \( lrm(\pi) \) consists of the labels belonging to the min-path of \( \phi^{-1}(\pi) \).

Proof. First, observe that when a tree is left oriented then its min-path corresponds to the path going from the root to the leftmost leaf.

If we consider \( t = \phi^{-1}(\pi) \), then \( t \) is a left oriented binary increasing tree.

If a node \( x \) belongs to the min-path of \( t \), then in \( \pi \) the entries which are on the left of \( x \) correspond to the nodes belonging to the left subtree of \( x \). Since \( t \) is increasing we have that \( x \in lrm(\pi) \).

If a node \( x \) does not belong to the min-path of \( t \), then, going from \( x \) to the root of \( t \), we find a node of outdegree 2 such that \( x \) belongs to its right subtree. Let \( z \) be the left child of this node and \( y \) the right one. Then \( z < y \preceq x \) which implies \( z < x \). Moreover, applying the procedure \( \phi \), the entry \( z \) will be placed on the left of \( x \). From this we have that \( x \notin lrm(\pi) \). □

As an example consider the binary increasing tree presented in Fig. 2. The associated permutation is \( \pi = (11 7 6 10 9 5 8 2 1 13 4 14 3 12) \) and then the set \( lrm(\pi) = \{11, 7, 6, 5, 2, 1\} \) has the same entries of the min-path.

Based on Propositions 2 and 3 we can obtain the enumeration of \( R_{n,m}^{\tilde{A}} \) by counting the number of binary increasing trees with respect to the size and to the length of the min-path.

3.2.1 A multivariate generating function for \( B \)

We now introduce a recursive construction for binary increasing trees. In particular we construct each tree belonging to \( B_{n+1} \) by adding a new node to a tree in \( B_{n} \). This construction, denoted by \( \Theta \), is then translated into a functional equation. Solving the equation yields a multivariate exponential generating function counting binary increasing trees with respect to size and to the length of the min-path.

Given a tree \( t \in B_{n} \), \( \Theta \) simply adds the node labelled \( 'n + 1' \) as a child of a node of \( t \) having outdegree less than two. More precisely, if \( o(t) \) (resp. \( p(t) \)) is the number of nodes with outdegree 0 (resp. 1) in \( t \), then \( \Theta \) applied to \( t \) produces \( o(t) + p(t) \) elements of \( B_{n+1} \) each time adding the new node labelled \( n + 1 \) as a child of the nodes counted in \( o(t) + p(t) \). In Fig. 3 the first levels of the generating tree resulting from \( \Theta \) are shown.
Note that, if \( q(t) \) is the number of nodes with outdegree two in a tree \( t \in B_n \), then \( o(t) = q(t) + 1 \) and \( o(t) + p(t) + q(t) = n \). From these relations it follows that \( p(t) = n - 2o(t) + 1 \). The construction \( \Theta \) can be translated into the succession rule (1) (see [1] [2])

\[
(o, l, n) \rightarrow (o, l + 1, n + 1) (o + 1, l, n + 1)^{n-2o+1}.
\]

Note that each tree is represented in (1) by the values of its parameters \( o, l \) and \( n \), where \( l \) represents the length of the min-path (i.e. \( l = \text{lrm}(\phi(t)) \)) while \( n \) corresponds to the size. In particular, given a tree \( t \) with parameters \( o = o(t) \) and \( n = n(t) \), the application of \( \Theta \) on \( t \) produces \( o \) new trees having size \( n + 1 \) and the same value for \( o \) and \( n - 2o + 1 \) new trees having both the size and the number of nodes of outdegree zero augmented by one.

The starting point of the construction is the unique tree of size one which is associated with the label \((1, 1, 1)\).

Now consider the exponential generating function

\[
F(x, y, z) = \sum_{t \in B} x^{o(t)} y^{l(t)} z^{n(t)} \frac{1}{n(t)!},
\]

rule (1) can be then translated as follows

\[
F(x, y, z) = xyz + \sum x^{o'} y^l z^n \frac{(o-1)x^{o'}y^{l+1}z^{n+1}}{n+1!} + \sum x^{o'} y^{l+1} z^n \frac{x^{o'}y^{l+1}z^{n+1}}{n+1!} + \sum x^{o'} y^{l} z^{n+1} \frac{(n-2o+1)(x^{o'}y^{l}z^{n+1})}{n+1!},
\]

We obtain that

\[
F(x, y, z)(1 - xz) - xyz = x(1 - 2x) \sum x^{o-1} y^{l} z^{n+1} \frac{1}{n+1!} + (y - 1) \sum x^{o} y^{l} z^{n+1} \frac{1}{n+1!} + xz F(x, y, z).
\]

and differentiating both sides with respect to the variable \( z \) we have

\[
(1 - x - y) F(x, y, z) - xy = x(1 - 2x) \frac{\partial F}{\partial x}(x, y, z) + (x - 1) \frac{\partial F}{\partial z}(x, y, z).
\]

The boundary condition to the previous first order partial differential equation can be given as

\[
F(x, y, 0) = 0.
\]

One can find the desired solution applying the method of characteristics (see [12]) or using some computer algebra tools (like Mathematica or Maple). In the latter case, after
the substitution $x = 1$ into the solution $F(x, y, z)$ and performing some manipulations, one finds that, when $n \geq 2$ and $2 \leq l \leq n$,

$$|R_{n,l}^A| = [y^{l-1} \left( \frac{\partial^{n-1} \tilde{F}}{\partial z^{n-1}} \right)_{z=0}],$$

with

$$\tilde{F}(y, z) = \left( \frac{1}{1 - \sin(z)} \right)^y.$$

In other words, we have the following result

**Proposition 4** The (shifted) exponential generating function counting the permutations in $R^A$ with respect to the size and number of left-to-right minima is given by

$$\tilde{F}(y, z) = \left( \frac{1}{1 - \sin(z)} \right)^y = \sum_{t \in B} y^{l(t)-1} z^{n(t)-1} / (n(t) - 1)!.$$

Considering $\tilde{F}(1, z)$ provides the (shifted) exponential generating function for Euler numbers.

The first terms of $|R_{n,l}^A|$ are thus given by the following table.

| n/l | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 |
|-----|----|----|----|----|----|----|----|----|----|
| 2   | 1  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  |
| 3   | 1  | 1  | 0  | 0  | 0  | 0  | 0  | 0  | 0  |
| 4   | 1  | 3  | 1  | 0  | 0  | 0  | 0  | 0  | 0  |
| 5   | 2  | 7  | 6  | 1  | 0  | 0  | 0  | 0  | 0  |
| 6   | 5  | 20 | 25 | 10 | 1  | 0  | 0  | 0  | 0  |
| 7   | 16 | 70 | 105| 65 | 15 | 1  | 0  | 0  | 0  |
| 8   | 61 | 287| 490| 385| 140| 21 | 1  | 0  | 0  |
| 9   | 272| 1356|2548|2345|1120|266|28 | 1  | 0  |
| 10  | 1385|7248|14608|15204|8715|2772|462|36 | 1  |

Note that Euler numbers are the entries of the first column. Furthermore observe that, looking at the table column by column, one has

$$\left( \frac{\partial^l \tilde{F}}{\partial y^l} \right)_{y=0} = [-\ln (1 - \sin(z))]^l$$

and then

$$\frac{1}{l!} [-\ln (1 - \sin(z))]^l = \sum_{t \in B, (l(t)=l+1)} z^{n(t)-1} / (n(t) - 1)!. $$

Given that $\int E(z)dz = -\ln (1 - \sin(z))$, as a corollary we have

**Proposition 5** For every fixed $l \geq 1$

$$\frac{1}{l!} \left( \sum_{n \geq 1} \frac{e_{n-1} z^n}{n!} \right)^l = \sum_{n \geq l} \frac{|R_{n+1,l+1}^A|}{n!} z^n,$$

where $e_0 = 1, e_1 = 1, e_2 = 1, e_3 = 2, e_4 = 5, e_5 = 16, e_6 = 61, ...$ are Euler numbers.

We conclude this section recalling that in Chapter 7 of [13] the author studies a family of polynomials corresponding to the rows of the previous table. He also shows a criterion according to which each row defines a partition of the set of up-down permutations of a given size. Furthermore, in [5] the authors prove that the rows of the previous table also provide the enumeration of the so called cycle-up-down permutations with respect to the size and to the number of cycles. It is then natural to ask for a bijection between the permutations in $R_{n+1}^A$ and the cycle-up-down ones of size $n$ enlightening the correspondence between left-to-right minima and cycles.
3.3 Right-to-left minima and the max-path

In the previous section we have enumerated the permutations in \( R^A \) with respect to the size and to the number of left-to-right minima. We have also seen how the last parameter corresponds to the length of a particular path in the associated class of trees.

It is interesting to observe that a similar correspondence holds also between the number of right-to-left minima and the length of another particular path defined for trees in \( B \). If \( t \in B \), the max-path of \( t \) is made of those nodes visited according to the following procedure. Start from the root of \( t \) and, at each step (until possible), if the current node has outdegree two move to the child having the biggest label. It is important to remark that the max-path does not represent a dual version of the previously defined min-path. Indeed, according to the procedure for the max-case, we visit a new node only if the current one has outdegree two.

Looking at the tree presented in Fig. 2 its max-path is composed by the nodes in \( \{1, 3, 12\} \) which also corresponds to the set of right-to-left minima \( rlm(\pi) \), where \( \pi = (1176109582113414312) \).

More in general we can state the following

**Proposition 6** If \( \pi \in R^A \), then \( rlm(\pi) \) consists of the labels belonging to the max-path of \( \phi^{-1}(\pi) \).

**Proof.** Observe that, when a tree \( t = \phi^{-1}(\pi) \) is drawn according to its left orientation, the max-path of \( t \) is made of those nodes we can visit starting from the root and performing only right steps. Then, if \( \pi_1 \) belongs to the max-path and \( j > i \), \( \pi_j \) must be a descendant of \( \pi_i \) in \( t \) and then \( \pi_i < \pi_j \). Vice versa, if \( \pi_1 \) does not belong to the max-path of \( t \), going from the root of \( t \) to \( \pi_i \) we have to perform at least one left step. Corresponding to this left step we find an entry of \( \pi \) which has a smaller value than \( \pi_i \) and it is placed on its right. Then \( \pi_i \) cannot be a right-to-left minima of \( \pi \). \( \square \)

As a first step, one can proceed in the enumeration of right-to-left minima using the same generating-tree approach of the previous section. In this case, given a tree \( t \), we denote by \( r(t) \) the cardinality of its max-path, by \( o(t) \) (resp. \( n(t) \)) the number of leaves (resp. the size) while \( d(t) \) is defined as \( d(t) = n(t) - 2o(t) + 1 \) (i.e. the number of nodes of outdegree 1). Furthermore we say that a tree is in the class \( A \) if the last node of its max-path has outdegree 1, while we say that it is in the class \( B \) otherwise. The recursive construction \( \Theta \) defined in Section 3.2.1 gives, in this case, the following set of rules:

\[
\begin{align*}
(o, r, n)_{B, d \geq 0} & \rightarrow (o, r, n + 1)_{B, d + 1}^{d-1} (o, r, n + 1)_{A, d + 1} (o + 1, r, n + 1)_{B, d - 1}^d \\
(o, r, n)_{A, d > 0} & \rightarrow (o, r, n + 1)_{A, d + 1} (o + 1, r + 1, n + 1)_{B, d - 1} (o + 1, r, n + 1)_{A, d - 1}^{d-1}
\end{align*}
\]

where the construction starts with the tree made of one node whose label is then \((1,1,1)_{B,0}\). Note that we can increase the value of the parameter \( r \) only starting from a tree in the class \( A \).

It seems natural at this point to consider

\[
G = G(x, w, v, z) = \sum_{t \in B} x^{o(t)}w^{r(t)}v^{d(t)}z^{n(t)}
\]

as the sum

\[
G = \sum_{k \geq 0} (G_{A,k}(x, w, v, z) + G_{B,k}(x, w, v, z))
\]

where \( G_{A,k} \) (resp. \( G_{B,k} \)) is the ordinary generating function counting those trees in the class \( A \) (resp. \( B \)) having \( d = k \).

The functional equations look then as follows
\[ G_{A,0} = 0, \]
\[ G_{A,k} = -\frac{xz}{v} G_{A,k+1} + xz \frac{\partial G_{A,k+1}}{\partial v} + xz\frac{\partial G_{A,k-1}}{\partial x} + vzG_{B,k-1}, \]
\[ G_{B,0} = xwz + \frac{xwz}{v} G_{A,1} + xz \frac{\partial G_{A,1}}{\partial v}, \]
\[ G_{B,k} = \frac{xwz}{v} G_{A,k+1} + xz \frac{\partial G_{B,k+1}}{\partial v} + xz\frac{\partial G_{B,k-1}}{\partial x} - vzG_{B,k-1} \quad (\text{with } k > 0). \]

Then, considering that \( G_{B,0} \) does not depend on \( v \), we can write
\[ G_A = \sum_{k \geq 0} G_{A,k} \]
\[ = -\frac{xz}{v} G_A + xz \frac{\partial G_A}{\partial v} + xz\frac{\partial G_A}{\partial x} + vzG_B \quad \text{and} \]
\[ G_B = \sum_{k \geq 0} G_{B,k} \]
\[ = xwz + \frac{xwz}{v} G_A + xz \frac{\partial G_B}{\partial v} + xz\frac{\partial G_B}{\partial x} - vzG_B. \]

We are not able to solve the system of equations for \( G_A \) and \( G_B \) but we can still use it, as a recursive algorithm, to find the number of trees belonging to \( B_n \) having the cardinality of the max-path equal to \( r \). Based on Proposition 6 this value is the number of permutations in \( \tilde{R}_n^A \) having a fixed number \( r \) of right-to-left minima.

**Recursion:** Let us define \( G_A^{(n)} \) as the polynomial
\[ G_A^{(n)} = \sum_{t \in B_n \cap A} x^{o(t)} w^{r(t)} v^{d(t)} z^n, \]
and analogously consider
\[ G_B^{(n)} = \sum_{t \in B_n \cap B} x^{o(t)} w^{r(t)} v^{d(t)} z^n. \]

We are interested in the polynomial \( G^{(n)} = G_A^{(n)} + G_B^{(n)}. \)

The starting point of the procedure is given by \( G_A^{(1)} = 0 \) and \( G_B^{(1)} = xwz. \) By induction, given \( G_A^{(n)} \) and \( G_B^{(n)} \), we can compute
\[ G_A^{(n+1)} = -\frac{xz}{v} G_A^{(n)} + xz \frac{\partial G_A^{(n)}}{\partial v} + xz\frac{\partial G_A^{(n)}}{\partial x} + vzG_B^{(n)} \quad \text{and} \]
\[ G_B^{(n+1)} = \frac{xwz}{v} G_A^{(n)} + xz \frac{\partial G_B^{(n)}}{\partial v} + xz\frac{\partial G_B^{(n)}}{\partial x} - vzG_B^{(n)}. \]

Using the previously defined procedure we have computed the entries of the following table showing, for all \((n, r) \in \{1, \ldots, 10\}^2\), the number of permutations in \( \tilde{R}_n^A \) having \( r \) right-to-left minima.

| n/r | 1   | 2   | 3   | 4   | 5   | 6   | 7   | 8   | 9   | 10  |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 0   | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   |
| 1   | 1   | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   |
| 2   | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   |
| 3   | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   |
| 4   | 2   | 3   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   |
| 5   | 5   | 10  | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0   |
| 6   | 16  | 38  | 7   | 0   | 0   | 0   | 0   | 0   | 0   | 0   |
| 7   | 61  | 165 | 45  | 1   | 0   | 0   | 0   | 0   | 0   | 0   |
| 8   | 272 | 812 | 288 | 13  | 0   | 0   | 0   | 0   | 0   | 0   |
| 9   | 1385| 4478| 1936| 136 | 1   | 0   | 0   | 0   | 0   | 0   |
| 10  | 7936| 27408| 13836| 1320| 21  | 0   | 0   | 0   | 0   | 0   |
In the first column we find (shifted) Euler numbers. It is also interesting to observe that the entries in the second column

\[ 1, 3, 10, 38, 165, 812, 4478, 27408, 184529, 1356256, 10809786, 92892928, \ldots \]

belong to sequence A186367 of [14]. This sequence counts the number of cycles in all cycle-up-down permutations of size \( n \) (see also [5]) and, furthermore, it is strongly related to the total number of left-to-right minima in the permutations of \( R^A \) having fixed size. Indeed, we will prove that the exponential generating function associated with the non-zero entries of the column \( r = 2 \) of the table above is given by

\[
\left( \frac{\partial \tilde{F}}{\partial y} \right)_{y=1} = -\frac{\ln(1 - \sin(z))}{1 - \sin(z)},
\]

where \( \tilde{F} \) is the same of Proposition [4].

In order to prove the correspondence, we observe that each tree in \( B \) having max-path of length exactly two can be decomposed as shown in Fig. 4.

In particular, note that tree \( t_1 \) must contain at least one node (labelled with 2) while tree \( t_2 \) could be empty. The class of trees consisting of a (possibly empty) tree appended to a node - denoted by \( k \) in Fig. 4 - is counted by the exponential generating function

\[
f_{t_2}(z) = \sum_{m>0} \frac{\tilde{e}_m z^m}{m!} = \int E(z) = -\ln(1 - \sin(z)) \text{ (with } \tilde{e}_m = e_{m-1})
\]

while

\[
f_{t_1}(z) = \sum_{n>0} \frac{e_n z^n}{n!} = E(z) - 1
\]

counts those trees having at least one node.Appending \( t_1 \) of size \( n \) and \( t_2 \) of size \( m - 1 \) as shown in Fig. 4 we can build exactly

\[
\binom{n + m - 1}{m} = \frac{(n + m - 1)!}{m!(n-1)!}
\]

different trees. Indeed we have to merge the total order given by \( k \) and the nodes of \( t_2 \) into the order of \( t_1 \) placing \( k \) after 2. It follows that, in the previous table, the entries \( n \geq 1 \) of the column \( r = 2 \) correspond to the coefficients of the following exponential generating function

\[
g_2(z) = \sum_{n>0} \sum_{m>0} \frac{e_n \tilde{e}_m z^{n+m+1}}{(n+m+1)! (m!)(n-1)!} = \sum_{n>0} \sum_{m>0} \frac{e_n \tilde{e}_m z^{n+m+1}}{(n+m)(n+m+1)(m!)(n-1)!}.
\]

Finally observe that
Given the above calculations, we obtain the following result

**Proposition 7** The following equality holds for all $n \geq 2$:

\[
|\mathcal{R}_{n+1,2}^A| = \sum_{l \geq 2} (l - 1) \cdot \{ \pi \in \mathcal{R}_n^A : |\text{lr}(\pi)| = l \} = (n - 1)! \cdot \left[ z^{n - 1} \right] \left( -\ln(1 - \sin(z)) \right)
\]

from which we have the next corollary

**Corollary 2** For $n \geq 2$:

\[
|\mathcal{R}_{n+1,2}^A| + |\mathcal{R}_n^A| = \sum_{l \geq 2} l \cdot \{ \pi \in \mathcal{R}_n^A : |\text{lr}(\pi)| = l \}
\]

Observe that, starting from (3) and dividing by $|\mathcal{R}_n^A|$, one obtains that the expected number of left-to-right minima in a random permutation of $\mathcal{R}_n^A$ is given by $1 + \frac{|\mathcal{R}_{n+1,2}^A|}{|\mathcal{R}_n^A|}$.

### 3.3.1 Fixing the number of right-to-left minima

Given what we have shown above, it is interesting to investigate more in details what happens when we fix the number of right-to-left minima in $\mathcal{R}_n^A$. Let

\[
g_r(z) = \sum_n \frac{|\mathcal{R}_n^A|}{n!} \cdot z^n.
\]

Looking at the associated tree-decomposition, the situation looks as shown in Fig. 5. A tree with max-path of length $r$ is obtained by appending to the same root a tree of max-path $r - 1$ and a generic tree of size greater than zero. We must put the root of the last tree above the root of the first one. Furthermore, recall that we can distinguish among two possible trees of max-path size $r$ depending on the out-degree of the last node belonging to the max-path. If this has out-degree one, then we say that the tree belongs to the class $A_r$ otherwise it is in $B_r$. We denote by $g_r^A(z)$ and $g_r^B(z)$ the associated generating function and obviously $g_r(z) = g_r^A(z) + g_r^B(z)$.

If $r > 0$, one has

\[
g_r^A = \sum_{t_1 \in B} \sum_{t_2 \in A_r} \frac{z^{n_1+n_2+1}}{(n_1+n_2+1)!} \cdot \binom{n_1+n_2-1}{n_1-1}
\]

and similarly for $g_r^B$. 

Figure 5: Tree-decomposition for a fixed number of right-to-left minima. The max-path is made of the nodes $1, k_1, k_2, k_3, k_4$. 

\[
g_2'' = f_2 \cdot f_1' = \ln \left( \frac{1}{1 - \sin(z)} \right) \cdot \frac{1}{1 - \sin(z)} = \left( \frac{\partial \tilde{F}}{\partial y} \right)_{y=1}.
\]
It follows that we can write two independent recursions for \( g^A_r(z) \) and \( g^B_r(z) \), namely

\[
(g^A_r)' = g^A_{r-1} \cdot (E(z) - 1)', \quad \text{with } g^A_0 = 1
\]

and

\[
(g^B_r)' = g^B_{r-1} \cdot (E(z) - 1)', \quad \text{with } g^B_1 = z.
\]

Applying the first one to compute \( g^A_1 \) gives

\[
g^A_1 = \int E(z) - 1 \, dz = - \ln(1 - \sin(z)) - z
\]

and then we obtain a recursion for \((g^A_r)' = (g^A_r + g^B_r)'\) when \( r > 1 \). Indeed we have the following result which shows the link between two contiguous generating functions in the family \((g_r)_r\).

**Proposition 8** For \( r > 1 \), the family of generating functions \((g_r)_r\) satisfies

\[
g_r(z) = \int \int g_{r-1}(z) \cdot E'(z) \, dz
\]

being \( E'(z) = \left( \frac{1}{1 - \sin(z)} \right) \) and \( g_1(z) = \int E(z) \, dz = - \ln(1 - \sin(z)) \).

Observe that as a corollary, if we define \( g = g(w, z) = \sum_{r \geq 1} w^r g_r(z) \), one has the following equation

\[
g = wg_1 + \sum_{r \geq 2} w^r g_r = wg_1 + \sum_{r \geq 2} w^r \left( \int \int E' g_{r-1} \right) = wg_1 + \left( \int \int w E' \sum_{r \geq 2} w^{r-1} g_{r-1} \right)
\]

\[
= wg_1 + w \left( \int \int E' g \right) = w \left( \int E \right) + w \left( \int \int E' g \right)
\]

which gives

\[
\frac{\partial^2 g}{\partial z^2} = wE' + wE' g.
\]

Considering now \( \tilde{g} = g + 1 \) we have - as expected - the following result

**Corollary 3** The bivariate generating function \( \tilde{g}(w, z) = 1 + \sum_{r \geq 1} w^r g_r(z) \) satisfies the differential equation

\[
\frac{\partial^2 \tilde{g}}{\partial z^2} = wE'(z) \cdot \tilde{g},
\]

where \( E'(z) = \left( \frac{1}{1 - \sin(z)} \right) \).

Unfortunately equation (5) does not give an explicit solution for \( \tilde{g} \). Still, as we will see later, it can be used to explore the structure of the solution in a neighbourhood of the singularity \( z = \pi/2 \).

Let us now focus on the exact computation of \( g_r \). To do that one can apply the result of Proposition 8 together with the fact that \( E = E(z) \) satisfies \( \int E^2 = 2E - z - 2 \). Here we compute explicitly the generating function \( g_r \) for the first values of \( r \), say \( r = 1, 2, 3 \).

Indeed, if we define

\[
\int^{(1)} H = \int \int \ldots \int H(z) \, dz,
\]

for \( r = 1, 2 \) we have

\[
g_1 = \int E
\]

\[
g_2 = \left( \int^{(2)} \left( \int E \right) E' \right) = \left( \int \left( \int E \right) E \right) - \int^{(2)} E^2
\]

\[
= \frac{1}{2} \left( \int E \right)^2 - \int (2E - z - 2)
\]

\[
= \frac{1}{2} \left( \int E \right)^2 - 2 \left( \int E \right) + \frac{z^2}{2} + 2z,
\]
while, for $r = 3$, we obtain:

\[
g_3 = \left( \int (\frac{E}{2})^2 E' \right) - 2 \left( \int (\frac{E}{2}) E' \right) + \left( \int (\frac{z^2}{2} + 2z) E' \right)
\]

\[
= \left( \int (\frac{E}{2})^2 \right) - \left( \int (\frac{E}{2}) E' \right) - 2 \left[ \left( \int (\frac{E}{2}) \right) - \int (E^2) \right]
\]

\[
+ \left( \frac{z^2}{2} + 2z \right) \left( \int E \right) + (-2z - 4) \left( \int (\frac{E}{2}) \right) + 3 \left( \int (E^3) \right)
\]

\[
= \frac{1}{6} \left( \int E \right)^3 - \left[ \left( \int (2E - z - 2) \left( \int E \right) \right) - \left( \int (2E - z - 2)E \right) \right]
\]

\[
- 2 \left[ \frac{1}{2} \left( \int E \right)^2 - \left( \int (2E - z - 2) \right) \right] + \left( \frac{z^2}{2} + 2z \right) \left( \int E \right)
\]

\[
+ (-2z - 4) \left( \int (\frac{E}{2}) \right) + 3 \left( \int (E^3) \right)
\]

\[
= \frac{1}{6} \left( \int E \right)^3 - 2 \left( \int E \right)^2 + \left( 8 + 2z + \frac{z^2}{2} \right) \left( \int E \right) - 2z^2 - 8z
\]

\[
+ (-2z - 4) \left( \int (\frac{E}{2}) \right) + 4 \left( \int (E^3) \right).
\]

For values of $r$ greater than 3 the exact computation of $g_r$ becomes more difficult. In this case we can still use the results of Proposition 8 to obtain asymptotic estimates of the coefficients $[z^n]g_r(z)$. Using standard methods of analytic combinatorics (see [7]) it is sufficient to know an approximation of the function $g_r$ near its dominant singularity to describe the behaviour of $([z^n]g_r(z))$. Using this approach, we provide, for a given value of $r$, an estimate of the probability for a random permutation $\pi \in \tilde{R}_n$ to have $|\pi^{lm}| = r$. To illustrate the idea we start, as a first step, considering just rough estimates.

Near the dominant singularity $z = \pi/2$ we have

\[
E'(z) = \frac{1}{1 - \sin(z)} = \frac{2}{(\pi/2 - z)^2} + O(1)
\]

and, for every $A > 0$,

\[
g_1 = \int E(z)dz = \ln \left( \frac{1}{1 - \sin(z)} \right) = -2 \ln \left( \frac{\pi}{2} - z \right) + O(1) = O \left( \left( \frac{\pi}{2} - z \right)^{-A} \right).
\]

Then, as a first approximation, one has

\[
(g_1 \cdot E')(z) = O \left( \left( \frac{\pi}{2} - z \right)^{-2-A} \right),
\]

which gives by Proposition [8] and Th. VI.9 of [7]

\[
g_2(z) = O \left( \left( \frac{\pi}{2} - z \right)^{-A} \right).
\]

We remark that, by the mentioned theorem, we can obtain a singular approximation of $g_2$ by integrating, according to classical rules, the singular expansion of $(g_1 \cdot E')$.

Iterating the procedure one has that, independently on $r$, for every $A > 0$

\[
g_r(z) = O \left( \left( \frac{\pi}{2} - z \right)^{-A} \right).
\]
Applying Th. VI.3 of [7] to (5) gives that, when \( n \) is large, for every \( A > 0 \)

\[
|\mathcal{R}_{n,r}^A| = [z^n]g_r(z) = O\left(\left(\frac{2}{\pi}\right)^n \cdot n^{A-1}\right).
\]

(9)

A more precise analysis can be now carried out, using exactly the same procedure, to provide a refinement of (9) showing the dependance of \(|\mathcal{R}_{n,r}^A|\) on the parameter \( r \). One can indeed iteratively consider more terms in the estimates for \( g_r \) and \((g_r \cdot E')\) and then use a more general version of Th. VI.9 of [7]. Indeed, according to Th. 7 of [6] and to the related references, in order to obtain a singular approximation of \( g_{r+1} \) we are allowed to classically integrate the singular expansions of \((g_r \cdot E')\) even when this is of the form \((\pi/2-z)^{-\alpha} \cdot (\ln(\pi/2-z))^\beta\), where \( \alpha \neq 1 \) and \( \beta \) is a positive integer. Furthermore, this applies also to error terms with the same structure, i.e., the corresponding \( O \)-transfer holds true. This property is in our case very useful because, for example, one can exactly compute integrals of the form

\[
\int \left[\ln\left(\frac{\pi}{2} - z\right)\right]^r \frac{\left(\frac{\pi}{2} - z\right)^{r+1}}{r+1} + \cdots,
\]

where the remaining term is \( O\left(-\left[\ln\left(\frac{\pi}{2} - z\right)\right]^r\right) \) for \( z \) near \( \pi/2 \).

Based on this one finds that for \( z \to \pi/2 \)

\[
g_r(z) = \frac{2^r}{r!} \cdot \left[-\ln\left(\frac{\pi}{2} - z\right)\right]^r + O\left(\left[-\ln\left(\frac{\pi}{2} - z\right)\right]^{r-1}\right)
\]

(10)

which gives, by Th. VI.4 of [7], that for \( n \to \infty \)

\[
[z^n]g_r(z) = \frac{2^r}{r!} \cdot [z^n]\left[-\ln\left(\frac{\pi}{2} - z\right)\right]^r + [z^n]\left[O\left(\left[-\ln\left(\frac{\pi}{2} - z\right)\right]^{r-1}\right)\right].
\]

By Th. VI.2 of [7] (see special cases) one has that there is a constant \( c_{1,r} > 0 \) such that

\[
[z^n]\left[-\ln\left(\frac{\pi}{2} - z\right)\right]^r \sim c_{1,r} \left(\frac{2}{\pi}\right)^n \cdot n^{-1} \cdot [\ln(n)]^{r-1} \quad (n \to \infty)
\]

while, using Th. VI.3 of [7], one finds that there is \( c_{2,r} > 0 \) such that

\[
[z^n]\left[O\left(\left[-\ln\left(\frac{\pi}{2} - z\right)\right]^{r-1}\right)\right] \leq c_{2,r} \left(\frac{2}{\pi}\right)^n \cdot n^{-1} \cdot [\ln(n)]^{r-1} \quad (n \to \infty).
\]

All together this gives

\[
\frac{2^r}{r!} \cdot c_{1,r} \leq \frac{[z^n]g_r(z)}{\left(\frac{2}{\pi}\right)^n \cdot n^{-1} \cdot [\ln(n)]^{r-1}} \leq c_{1,r} + c_{2,r},
\]

(11)

which says that the ratio

\[
|\mathcal{R}_{n,r}^A| \quad \frac{n! \cdot \left(\frac{2}{\pi}\right)^n \cdot n^{-1} \cdot [\ln(n)]^{r-1}}
\]

is bounded from both sides.

In conclusion we remark that, even if the recursion of Proposition 8 requires a double integration, we can still efficiently use it to compute the singular expansions at \( z = \pi/2 \) of the functions \( g_r \). Indeed, near the mentioned singularity, we are allowed to use standard integration and the integrals involved are explicitly solvable.

Comparing the value of \(|\mathcal{R}_{n,r}^A|/n!\) as described in (11) with the probability for a random permutation of size \( n \) to be in \( \mathcal{R}_{n,r}^A \), i.e. \( \frac{1}{n!} \cdot \left(\frac{2}{\pi}\right)^n \), we obtain the following result

**Proposition 9** For \( n \) large, the probability for a random permutation in \( \mathcal{R}_{n,r}^A \) to have \( r \) right-to-left minima satisfies

\[
P\left(|r\ln(\pi)| = r \mid x \in \mathcal{R}_{n,r}^A\right) = \frac{c_r}{n^{-1} \cdot [\ln(n)]^{r-1}} \leq C_r,
\]

where \( c_r \) and \( C_r \) are two positive constants.
Figure 6: Plot of $P \left( |\text{rlm}(\pi)| = 2 |\pi \in \mathbb{R}^4 \right) / (n^{-1}\ln(n))$.

In Fig. 6 we show for $n$ large the behaviour of $|\mathbb{R}^4_n|/|\mathbb{R}^4_n|$ divided by $n^{-1}\ln(n)$.

**Structural properties of $\tilde{g}$ near the singularity.** To conclude our asymptotic analysis we go back to equation (5) to describe a structural property of the solution $\tilde{g}$.

Indeed, treating $w$ as a constant, we can apply Th. VII.9 of [7] finding that near the regular singular point $z = \pi/2$ the desired solution $\tilde{g}$ can be expressed as

$$\tilde{g} = a_w \cdot \left( \frac{\pi}{2} - z \right)^{1-\sqrt{1+8w^2}} A_w \left( z - \frac{\pi}{2} \right) + b_w \cdot \left( \frac{\pi}{2} - z \right)^{1-\sqrt{1+8w^2}} B_w \left( z - \frac{\pi}{2} \right),$$

where $w$ could appear in $a_w, A_w(z), b_w, B_w(z)$ and the functions $A_w(z), B_w(z)$ are analytic at $z = 0$.

It is interesting to note that, taking $a_w = 0$ and $b_w = B_w = 1$, one obtains

$$\tilde{g}_\alpha = \left( \frac{\pi}{2} - z \right)^{1-\sqrt{1+8w^2}}$$

whose expansion at $w = 0$ looks as

$$\tilde{g}_\alpha = 1 - 2w \ln(\pi/2 - z) + w^2 \left( 4 \ln(\pi/2 - z) + 2 \left( \ln(\pi/2 - z) \right)^2 \right) + \cdots.$$

Based on (7), this reflects well enough the asymptotic behaviour of the expressions for $g_1, g_2$ and $g_3$ which have been previously computed. More in general, we can consider $(w^r[\tilde{g}_\alpha](z)$ as an approximation of the desired $g_r(z)$ near the singularity $z = \pi/2$. This can be justified recalling what is shown in (10) and observing that $\tilde{g}_\alpha$ satisfies

$$\frac{\partial^2 \tilde{g}_\alpha}{\partial z^2} = w \cdot \frac{2}{(\pi/2 - z)^2} \cdot \tilde{g}_\alpha$$

which is obtained by substituting in (59), i.e. the defining equation for $\tilde{g}$, the term $E'(z)$ by $2/(\pi/2 - z)^2$, the latter being the main part of the singular approximation (6). Therefore, as for $g_r(z)$, we have that in a neighbourhood of $z = \pi/2$

$$(w^r[\tilde{g}_\alpha](z) = \frac{2r}{r!} \left[ - \ln \left( \frac{\pi}{2} - z \right) \right]^r + O \left( \left[ - \ln \left( \frac{\pi}{2} - z \right) \right]^{r-1} \right).$$

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References

[1] C. Banderier, M. Bousquet-Melou, A. Denise, P. Flajolet, D. Gardy, and D. Gouyou-Beauchamps. *Generating functions for generating trees*. In: Proceedings of 11-th formal power series and algebraic combinatorics, 40-52 (1999).

[2] E. Barcucci, A. Del Lungo, E. Pergola, R. Pinzani, *ECO: a methodology for the enumeration of combinatorial objects*, Journal of Difference Equations and Applications, Vol. 5 (1999) 435-490.

[3] F. Bergeron, P. Flajolet, B. Salvy, *Varieties of increasing trees*. In J.-C. Raoult, editor, CAAP’92, volume 581 of Lecture Notes in Computer Science, pages 24-48, 1992. Proceedings of the 17th Colloquium on Trees in Algebra and Programming, Rennes, France, February (1992).

[4] C-O. Chow, W. C. Shiu, *Counting Simsun permutations by Descents*, Ann. Comb. 15 (2011) 625-635.

[5] E. Deutsch, S. Elizalde, *Cycle-up-down permutations*, Australas. J. Combin. 50 (2011) 187-199

[6] J. A. Fill, P. Flajolet, N. Kapur, *Singularity Analysis, Hadamard Products and Tree Recurrences*, J. Comput. Appl. Math. 174 (2005) 271-313.

[7] P. Flajolet, R. Sedgewick, *Analytic Combinatorics*, Cambridge University press, (2009).

[8] D. Foata, M. P. Schützenberger, *Nombres d’Euler et permutations alternantes*, In: Bose, R.C. (Ed.) A survey of combinatorial theory, pp. 173-187, North-Holland, Amsterdam (1973) 173-187.

[9] D. Foata and V. Strehl, *Euler numbers and variations of permutations*. In: Colloquio Internazionale sulle Teorie Combinatorie, 1973, Tome I (Atti Dei Convegni Lincei 17, 119-131), Accademia Nazionale dei Lincei, Rome (1976).

[10] R. L. Graham, D. Knuth, O. Patashnik, *Concrete Mathematics*, Addison-Wesley (1989).

[11] G. Hetyei, *On the cd-Variation Polynomials of André and Simsun Permutations*, Discrete Comput. Geom., 16 (1996) 259-275.

[12] F. B. Hildebrand, *Methods of Applied Mathematics*, Dover, (1992).

[13] W. P. Johnson, *Some polynomials associated with up-down permutations*, Discrete Mathematics, 210 (2000) 117-136.

[14] N. J. A. Sloane, *The On-Line Encyclopedia of Integer Sequences*, available at: http://oeis.org/

[15] R. P. Stanley, *Flag f-vectors and the cd-index*, Math. Z. 216 (1994) 483-499.