Sharp inequalities for dyadic $A_1$ weights

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Abstract. We show how the Bellman function method can be used to obtain sharp inequalities for the maximal operator of a dyadic $A_1$ weight on $\mathbb{R}^n$. Using this approach, we determine the optimal constants in the corresponding weak-type estimates. Furthermore, we provide an alternative, simpler proof of the related maximal $L^p$-inequalities, originally shown by Melas.

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1. Introduction. A locally integrable nonnegative function $w$ on $\mathbb{R}^n$ is called a dyadic $A_1$ weight if it satisfies the condition

$$\frac{1}{|Q|} \int_Q w(x) dx \leq C \text{ess inf}_{x \in Q} w(x)$$

(1.1)

for any dyadic cube $Q$ in $\mathbb{R}^n$. This is equivalent to saying that

$$M_d w(x) \leq C w(x) \quad \text{for almost all } x \in \mathbb{R}^n,$$

(1.2)

where $M_d$ is the dyadic maximal operator, given by

$$M_d w(x) = \sup \left\{ \frac{1}{|Q|} \int_Q w(t) dt : x \in Q, Q \subset \mathbb{R}^n \text{ a dyadic cube} \right\}.$$

The smallest $C$ for which (1.1) (equivalently (1.2)) holds is called the dyadic $A_1$ constant of $w$ and is denoted by $[w]_1$. A classical result of Coifman and
Fefferman [2] states that any $A_1$ weight satisfies the reverse Hölder inequality
\[
\left( \frac{1}{|Q|} \int_Q w(x)^p \, dx \right)^{1/p} \leq \frac{c}{|Q|} \int_Q w(x) \, dx
\]  \hfill (1.3)
for certain $p > 1$ and $c \geq 1$ which depend only on the dimension $n$ and the value of $[w]_1$. The exact information on the range of possible $p$’s was studied by Melas [3] (see also [1] for related results in the non-dyadic case). Here is the precise statement.

**Theorem 1.1.** Let $w$ be a dyadic weight on $\mathbb{R}^n$. Then for every $p$ such that

\[
1 \leq p < p_0(n, [w]_1) := \frac{\log(2^n)}{\log \left( 2^n - \frac{2^n - 1}{[w]_1} \right)}
\]

and for every dyadic cube $Q$, we have

\[
\frac{1}{|Q|} \int_Q (M_d w(x))^p \, dx \leq \frac{2^n - 1}{\left( 2^n - \frac{2^n - 1}{[w]_1} \right)^p - 2^n} \left( \frac{1}{|Q|} \int_Q w(x) \, dx \right)^p.
\]  \hfill (1.4)

Both the range of $p$ and the corresponding constant in (1.4) are best possible.

This result implies that the range of admissible exponents $p$ in the reverse Hölder inequality (1.3) is at least $[1, p_0(n, [w]_1))$. To prove that both intervals are actually equal, Melas [3] constructed, for any $\lambda > 1$, a dyadic weight on $[0, 1]^n$ such that $[w]_1 = \lambda$ and $\int_{[0,1]^n} w(x)^{p_0(n,\lambda)} \, dx = \infty$.

The purpose of this paper is to study the corresponding weak-type estimates. We will prove the following result.

**Theorem 1.2.** Let $w$ be a dyadic weight on $\mathbb{R}^n$, and let $1 \leq p \leq p_0(n, [w]_1)$. Then for every dyadic cube $Q$, we have

\[
\frac{1}{|Q|} |\{ x \in Q : M_d w(x) > 1 \}| \leq \left( \frac{1}{|Q|} \int_Q w(x) \, dx \right)^p.
\]  \hfill (1.5)

Both the range of $p$ and the constant 1 are already best possible in the estimate

\[
\frac{1}{|Q|} |\{ x \in Q : w(x) > 1 \}| \leq \left( \frac{1}{|Q|} \int_Q w(x) \, dx \right)^p.
\]

A few words about the proof. Using a standard dilation argument, it is enough to establish (1.5) for $Q = [0, 1]^n$. In fact, we will prove the estimate in a wider context of probability spaces equipped with a tree-like structure similar to the dyadic one. Next, while Melas’ proof of Theorem 1.1 is combinatorial and rests on a clever linearization of the dyadic maximal operator, our approach will be entirely different and will exploit the properties of a certain special function. In the literature, this type of argument is called the Bellman function method and has been applied recently in various settings: see e.g. [4–8], and references therein.
The paper is organized as follows. Section 2 contains some preliminary definitions. The description of the Bellman method can be found in Sect. 3, and it is applied in two final parts of the paper: in Sect. 4 we present the study of the weak type estimate, while in Sect. 5 we provide an alternative proof of Melas’ result.

2. Measure spaces with a tree-like structure. Assume that \((X, \mathcal{F}, \mu)\) is a given non-atomic probability space. We assume that it is equipped with an additional tree structure.

**Definition 2.1.** Let \(\alpha \in (0, 1]\) be a fixed number. A sequence \(T = (T_n)_{n\geq 0}\) of partitions of \(X\) is said to be \(\alpha\)-splitting if the following conditions hold.

(i) We have \(T_0 = \{X\}\) and \(T_n \subset \mathcal{F}\) for all \(n\).

(ii) For any \(n \geq 0\) and any \(E \in T_n\), there are pairwise disjoint sets \(E_1, E_2, \ldots, E_m \in T_{n+1}\) whose union is \(E\) and such that \(\frac{|E_i|}{|E|} \geq \alpha\) for all \(i\).

Let us stress that the number \(m\) in (ii) may be different for different \(E\).

**Example.** Assume that \(X = (0, 1]^n\) is the unit cube of \(\mathbb{R}^n\) with Borel subsets and Lebesgue’s measure. Let \(T_k\) be the collection of all dyadic cubes of volume \(2^{-kn}\) contained in \(X\) (i.e., products of intervals of the form \((a2^{-k}, (a+1)2^{-k}]\), where \(a \in \{0, 1, 2, \ldots, 2^k - 1\}\)). Then \(T = (T_n)_{n\geq 0}\) is \(2^{-n}\)-splitting.

In what follows, we will restrict ourselves to \(\alpha \leq 1/2\) since for \(\alpha > 1/2\) there is only one \(\alpha\)-splitting tree: \(T = (\{X\}, \{X\}, \{X\}, \ldots)\). Let us define the maximal operator and \(A_1\) class corresponding to the structure \(T\).

**Definition 2.2.** Given a probability space \((X, \mathcal{F}, \mu)\) with a sequence \(T\) as above, we define the corresponding maximal operator \(M_T\) as

\[
M_T f(x) = \sup \left\{ \frac{1}{\mu(I)} \int_I |f| \mu : x \in I \in T \right\}
\]

for any \(f \in L^1(X, \mathcal{F}, \mu)\). We will also use the notation \(M^n_T\) for the truncated maximal operator, associated with \(T^n = (T_0, T_1, \ldots, T_{n-1}, T_n, T_n, T_n, \ldots)\).

**Definition 2.3.** A nonnegative integrable function \(w\) is an \(A_1\) weight with respect to \(T\) if there is a finite constant \(C\) such that

\[
\frac{1}{|E|} \int_E w(x) dx \leq C \text{essinf}_{x \in E} w(x)
\]

for any \(E \in T\). This is equivalent to saying that

\[
M_T w(x) \leq C w(x)
\]

for almost all \(x \in X\). The smallest \(C\) for which the above holds is called the \(A_1\) constant of \(w\) and will be denoted by \([w]_1\).
3. On the method of proof. Now we will describe the technique which will be used to establish the inequalities announced in Section 1. Throughout this section, \( c > 1, \alpha \in (0, 1/2] \) are fixed constants. Distinguish the following subset of \( \mathbb{R}^3_+ \):

\[
D = D_c = \{ (x, y, z) \in \mathbb{R}^3_+ : y \leq x \leq cy, z \leq cx \}.
\]

Let \( \Phi, \Psi : \mathbb{R}_+ \to \mathbb{R} \) be two given functions, and assume that we want to show that

\[
\int_X \Phi(M^n_T w(x)) \, dx \leq \left( \int_X w(x) \, dx \right) \Psi(n), \quad n = 0, 1, 2, \ldots,
\]

for any \( A_1 \) weight \( w \) with respect to an \( \alpha \)-splitting tree \( T \), such that \([w]_1 \leq c\). The key idea in the study of this problem is to construct a special function \( B = B_{c, \alpha, \Phi, \Psi} : D \to \mathbb{R} \), which satisfies the following conditions.

1° We have \( B(x, y, x \lor z) = B(x, y, z) \) for any \( (x, y, z) \in D \).
2° We have \( B(x, y, z) \geq \Phi(z) \) for any \( (x, y, z) \in D \).
3° We have \( B(x, y, x) \leq \Psi(x) \) for all \( x, y \) such that \( (x, y, x) \in D \).
4° For any \( (x, y, z) \in D \) there exists \( A = A(x, y, z) \in \mathbb{R} \) such that whenever \( (x', y', z) \in D \) satisfies \( x' \leq \frac{c-1+\alpha}{c\alpha} x \) and \( y' \geq y \), then

\[
B(x', y', z) \leq B(x, y, z) + A(x, y, z)(x' - x). \quad (3.2)
\]

A few remarks concerning these conditions are in order. The condition 1° is a technical assumption which enables the proper handling of the maximal operator. The conditions 2° and 3° are appropriate majorizations. The most complicated (and most mysterious) condition is the last one. To shed some light on it, observe that it yields the following concavity-type property of \( B \).

**Lemma 3.1.** Let \( (x, y, z) \) be a fixed point belonging to \( D \), and let \( n \geq 2 \) be an arbitrary integer. Let \( \alpha_1, \alpha_2, \ldots, \alpha_n \) be positive numbers which sum up to 1, such that \( \alpha_i \geq \alpha \) for each \( i \). Assume further that \( (x_1, y_1, z), (x_2, y_2, z), \ldots, (x_n, y_n, z) \in D \) satisfy

\[
x = \sum_{i=1}^{n} \alpha_i x_i \quad \text{and} \quad y = \min\{y_1, y_2, \ldots, y_n\}.
\]

Then

\[
B(x, y, z) \geq \sum_{i=1}^{n} \alpha_i B(x_i, y_i, z). \quad (3.3)
\]

**Proof.** Apply 4° to \( x' = x_i, y' = y_i, i = 1, 2, \ldots, n \), multiply both sides by \( \alpha_i \), and finally sum the obtained inequalities. Then, as the result, we get (3.3). Thus, all we need is to verify whether the requirements for \( x', y' \) appearing in 4° are fulfilled. The inequality \( y_i \geq y \) is assumed in the statement of the lemma. Furthermore, by the definition of \( D \),
\[ x = \sum_{i=1}^{n} \alpha_i x_i + \sum_{j \neq i} \alpha_j y_j \geq \alpha_i x_i + (1 - \alpha_i) y \geq \alpha x_i + (1 - \alpha) y \geq \alpha x_i + (1 - \alpha) x/c, \]

which yields the desired bound \( x_i \leq \frac{c-1+\alpha}{\alpha} x. \) \( \square \)

We turn to the main result of this section.

**Theorem 3.2.** Suppose that \((X, \mathcal{F}, \mu)\) is a probability space equipped with an \(\alpha\)-splitting tree \(T\). If there is a function \(B = B_{c, \alpha, \Phi, \Psi}\) satisfying \(1^\circ - 4^\circ\), then (3.1) holds for any \(A_1\) weight \(w\) with respect to \(T\) such that \([w]_1 \leq c.\)

**Proof.** Let \(w\) be as in the statement. Define two sequences \((w_n)_{n \geq 0}, (v_n)_{n \geq 0}\) of measurable functions on \(X\) as follows. Given an integer \(n\), an element \(E\) of \(T_n\), and a point \(x \in E\), set

\[ w_n(x) = \frac{1}{\mu(E)} \int_E w(t) d\mu(t) \quad \text{and} \quad v_n(x) = \text{essinf}_{t \in E} w(t). \]

The following interplay between these objects will be important to us. Let \(n, E\) be as above, and let \(E_1, E_2, \ldots, E_m\) be the elements of \(T_{n+1}\) whose union is \(E\). Then we easily check that

\[ \frac{1}{\mu(E)} \int_E w_n(t) d\mu(t) = \sum_{i=1}^{m} \frac{\mu(E_i)}{\mu(E)} \cdot \frac{1}{\mu(E_i)} \int_{E_i} w_{n+1}(t) d\mu(t) \]

and

\[ v_n|_E = \min\{v_{n+1}|_{E_1}, v_{n+1}|_{E_2}, \ldots, v_{n+1}|_{E_m}\}. \]

Furthermore, the inequality \([w]_1 \leq c\) implies that the triple \((w_n, v_n, M_T^n w)\) takes values in \(D\). These conditions, combined with Lemma 3.1, yield the inequality

\[ \int_{E} B(w_n(t), v_n(t), M_T^n w(t)) d\mu(t) \geq \int_{E} B(w_{n+1}(t), v_{n+1}(t), M_T^{n+1} w(t)) d\mu(t). \]

Indeed, we have \(M_T^{n+1} w = M_T^n w \vee w_{n+1}\), so by \(1^\circ\),

\[ B(w_{n+1}(t), v_{n+1}(t), M_T^{n+1} w(t)) = B(w_{n+1}(t), v_{n+1}(t), M_T^n w(t)), \quad t \in E. \]

It remains to use (3.3) with \(x = w_n|_E, y = v_n|_E, z = M_T^n w|_E, x_i = w_{n+1}|_{E_i}, y_i = v_{n+1}|_{E_i}, \) and \(\alpha_i = \mu(E_i)/\mu(E)\). Summing over all \(E \in T_n\), we get

\[ \int_X B(w_n(t), v_n(t), M_T^n w(t)) d\mu(t) \geq \int_X B(w_{n+1}(t), v_{n+1}(t), M_T^{n+1} w(t)) d\mu(t) \]

and therefore, by induction,

\[ \int_X B(w_0(t), v_0(t), M_T^0 w(t)) d\mu(t) \geq \int_X B(w_n(t), v_n(t), M_T^n w(t)) d\mu(t). \]
However, the left-hand side equals

$$B \left( \int_X w(t) dt, \inf_{t \in X} w(t), \int_X w(t) dt \right),$$

and hence the application of $2^\circ$ and $3^\circ$ completes the proof of (3.1).

\[\square\]

4. A sharp weak-type estimate. The principal result of this section is the following.

**Theorem 4.1.** Suppose that $(X, \mathcal{F}, \mu)$ is a probability space equipped with an $\alpha$-splitting tree $T$. Then for any $A_1$ weight $w$ with respect to $T$ and any $p$ satisfying

$$1 \leq p \leq p_0(\alpha, [w]_1) := \frac{\log \alpha}{\log \left( \frac{[w]_1 + \alpha^{-1}}{[w]_1 \alpha} \right)},$$

we have

$$\mu(\{x \in X : M_T w(x) > 1\}) \leq \left( \int_X w(x) dx \right)^p. \quad (4.1)$$

The range of $p$ and the constant 1 are already the best possible in

$$\mu(\{x \in X : w(x) > 1\}) \leq \left( \int_X w(x) dx \right)^p. \quad (4.2)$$

Before we proceed, let us establish the following technical fact.

**Lemma 4.2.** For any $c \geq 1$ and $\alpha \in (0, 1/2]$, we have $(p_0(\alpha, c) - 1)c \leq p_0(\alpha, c)$.

*Proof.* The claim is equivalent to $1/p_0(\alpha, c) \geq 1 - 1/c$, or

$$\log \left( \frac{c\alpha}{c + \alpha - 1} \right) \leq \log \alpha^{1-1/c}.$$

Substituting $x = 1/c \in [0, 1]$ and working a little bit turns this bound into

$$\alpha^x \leq 1 + (\alpha - 1)x,$$

which is evident: the left-hand side is convex as a function of $x$, and both sides are equal when $x \in \{0, 1\}$. \[\square\]

4.1. **Proof of (4.1).** We may assume that $[w]_1 > 1$, since otherwise $w$ is constant and the assertion holds true. If the average $\int_X w$ is at least 1, then the inequality is trivial. So, suppose that $\int_X w < 1$; then it suffices to prove the weak-type estimate for $p = p_0(\alpha, [w]_1)$. In view of Theorem 3.2, all we need is to construct an appropriate special function corresponding to $c = [w]_1 > 1$, $\alpha \in (0, 1/2]$, $\Phi(z) = \chi_{\{z \geq 1\}}$ and $\Psi(x) = x^p$. Indeed, this will yield (3.1), and
letting $n$ go to $\infty$ will complete the proof. Introduce $B = B_{c,\alpha,\Phi,\Psi} : D \to \mathbb{R}_+$ by

$$B(x, y, z) = \begin{cases} 
   c^p y^{p-1}(x - y)/(c - 1) & \text{if } y \leq c^{-1} \text{ and } x \lor z < 1, \\
   (x - y)/(1 - y) & \text{if } y > c^{-1} \text{ and } x \lor z < 1, \\
   1 & \text{if } x \lor z \geq 1.
\end{cases}$$

We will exploit the following auxiliary property of $B$.

**Lemma 4.3.** For fixed $x, z > 0$, the function $B(x, \cdot, z) : y \mapsto B(x, y, z)$ is nonincreasing on $[x/c, x]$.

**Proof.** We may assume that $x, z < 1$, since otherwise the claim is obvious. Note that for $y \geq x/c$ we have

$$\frac{\partial}{\partial y} [y^{p-1}(x - y)] = y^{p-2}((p - 1)x - py) \leq y^{p-1}((p - 1)c - p) \leq 0$$

in light of Lemma 4.2. Furthermore, for any $y < 1$,

$$\frac{\partial}{\partial y} \left[ \frac{x - y}{1 - y} \right] = \frac{x - 1}{(1 - y)^2} < 0.$$

Since $B$ is continuous, this gives the desired monotonicity. \qed

Now we turn to the verification that $B$ satisfies the conditions $1^\circ - 4^\circ$. The first two of them are obvious, so let us look at $3^\circ$. By Lemma 4.3, it suffices to prove the majorization for $y = x/c$. But then the estimate is clear: both sides are equal when $x < 1$, and for $x \geq 1$ the inequality takes the form $1 \leq x^p$.

Finally, we will check $4^\circ$ with

$$A(x, y, z) = \begin{cases} 
   c^p y^{p-1}(c - 1) & \text{if } y \leq c^{-1} \text{ and } x \lor z < 1, \\
   1/(1 - y) & \text{if } y > c^{-1} \text{ and } x \lor z < 1, \\
   0 & \text{if } x \lor z \geq 1.
\end{cases}$$

We may and do assume that $x \lor z < 1$ since otherwise the right-hand side of (3.2) is equal to $1$ and there is nothing to prove. By the preceding lemma, it suffices to show (3.2) under the assumption that

$$y' = (x'/c) \lor y. \tag{4.3}$$

Suppose first that $y > c^{-1}$; then (3.2) becomes

$$B(x', y', z) \leq \frac{x' - y}{1 - y}.$$  

If $x' \geq 1$, then this bound is clear; if $x' < 1$, then $y' = y$ (see (4.3)) and thus both sides are equal. Finally, assuming that $y \leq c^{-1}$, we see that (3.2) reads

$$B(x', y', z) \leq \frac{c^p}{c - 1} y^{p-1}(x' - y).$$

If $x' \leq cy$, then $y' = y$ (see (4.3)) and hence both sides are equal. If $x' > cy$, then (4.3) implies that $y' = x'/c$ and the inequality becomes

$$(x')^p \leq \frac{c^p}{c - 1} y^{p-1}(x' - y),$$
or, after the substitution \( t = x'/y \),

\[
    t^p \leq \frac{c^p}{c-1} (t - 1). \tag{4.4}
\]

We have \( t > c \) by the assumption we have just made above. On the other hand, exploiting the requirements appearing in \( 4^p \), we get

\[
    t = \frac{x'}{y} \leq \frac{cx'}{x} \leq \frac{c - 1 + \alpha}{\alpha}.
\]

It suffices to note that the left-hand side of (4.4) is a convex function of \( t \) and both sides are equal for the extremal values of \( t \): \( t = c \) and \( t = (c - 1 + \alpha)/\alpha \) (the equality for the latter value of \( t \) is just the definition of \( p_0(\alpha, c) \)).

### 4.2. Sharpness

It is obvious that the constant 1 cannot be improved in (4.2): consider a constant weight \( w \equiv \lambda > 1 \), and let \( \lambda \downarrow 1 \). To show that the weak-type estimate cannot hold with exponents larger than \( p_0(\alpha, c) \), we will construct an appropriate example; a related object can be found in [3].

Suppose that \((X, \mathcal{F}, \mu)\) is a probability space equipped with an \( \alpha \)-splitting tree \( T \), such that there is a monotone sequence \( X = E_0 \supset E_1 \supset E_2 \supset \ldots \), with \( E_n \in \mathcal{T}_n \) and \( \mu(E_n) = \alpha^n \). For \( x \in X \), put \( N(x) = \sup\{ n \geq 0 : x \in E_n \} \); this is well-defined since \( E_0 = X \). Moreover, \( N(x) < \infty \) almost everywhere, because the sets \( E_i \) shrink to a set of measure zero. Define a weight \( w \) by

\[
    w(x) = \left( \frac{c - 1 + \alpha}{c \alpha} \right)^{N(x)}, \quad x \in X.
\]

In other words, we have \( w(x) = [(c - 1 + \alpha)/(c \alpha)]^n \), where \( n \) is the unique number such that \( x \in E_n \setminus E_{n+1} \). Then \( w \) is in the \( A_1 \) class and \( [w]_1 = c \). To see this, pick \( x \in X \) and let \( n \) be the unique integer such that \( x \in E_n \setminus E_{n+1} \). The only elements of \( T \) which contain \( x \) are \( E_0, E_1, \ldots, E_n \), so

\[
    M_T w(x) = \max_{0 \leq k \leq n} \left\{ \frac{1}{\mu(E_k)} \int_{E_k} w(t) \mu(t) \right\}.
\]

However, by the definition of \( w \), we easily compute that

\[
    \frac{1}{\mu(E_k)} \int_{E_k} w(t) \mu(t) = \frac{1}{\mu(E_k)} \sum_{\ell \geq k} \mu(E_{\ell \setminus E_{\ell+1}}) \left( \frac{c - 1 + \alpha}{c \alpha} \right)^k
\]

and hence

\[
    M_T w(x) = c \left( \frac{c - 1 + \alpha}{c \alpha} \right)^n = cw(x).
\]

Putting \( k = 0 \) in the above calculation gives \( \int_X w = c \). Fix \( q \geq 1, \lambda > 1 \), and consider the weight \( \tilde{w} = \lambda [c \alpha/(c - 1 + \alpha)]^n \). It satisfies \( [\tilde{w}]_1 = c \) and

\[
    \mu(\{ x \in X : \tilde{w}(x) > 1 \}) \geq \mu(E_n) = \frac{1}{c \lambda} \left[ \left( \frac{c - 1 + \alpha}{c \alpha} \right)^q \alpha \right]^n (\int_X w(x) dx)^q.
\]
Now, if we put \( q = p_0(\alpha, c) \), then the expression in the square brackets is equal to 1. Therefore, if \( q \) is larger than \( p_0(\alpha, c) \), then the constant on the right-hand side explodes as \( n \to \infty \). This shows that the threshold \( p_0(\alpha, c) \) in the weak-type estimate cannot be improved.

5. Melas’ theorem revisited. Now we use the method developed in Section 3 to obtain the following version of Theorem 1.1. For a fixed \( c \geq 1 \), \( \alpha \in (0, 1/2] \), and \( 1 \leq p < p_0(\alpha, c) \), let

\[
C = C_{c,\alpha,p} = \frac{1 - \alpha}{1 - \alpha c^{-\alpha}} \left( \frac{c - 1}{c^\alpha} \right)^p
\]

(when \( \alpha = 2^{-n} \) and \( c = [w]_1 \), this is exactly the constant appearing in (1.4)).

**Theorem 5.1.** Suppose that \((X, \mathcal{F}, \mu)\) is a probability space equipped with an \( \alpha \)-splitting tree \( T \). Then for any \( A_1 \) weight \( w \) with respect to \( T \), we have

\[
\int_X (\mathcal{M}_T w(x))^p \, dx \leq C_{[w]_1,\alpha,p} \left( \int_X w(x) \, dx \right)^p.
\]

(5.1)

Both the range of \( p \) and the constant \( C_{c,\alpha,p} \) are best possible.

**Proof.** We only show (5.1), for the construction of the extremal examples, the reader is referred to [3]. We may assume \( c = [w]_1 > 1 \). Define the functions \( \Phi(z) = z^p, \Psi(x) = Cx^p \), and consider \( B = B_{c,\alpha,\Phi,\Psi} : D \to \mathbb{R} \), given by

\[
B(x, y, z) = (c - 1)^{-1} (x \lor z)^{p-1} [(C - 1)cx + (c - C)(x \lor z)].
\]

It is easy to show that this function enjoys the conditions 1°, 2°, and 3°; we leave the details to the reader. Finally, we will prove 4° with

\[
A(x, y, z) = (c - 1)^{-1} (x \lor z)^{p-1} (C - 1)c.
\]

The estimate (3.2) can be rewritten in the form

\[
(x' \lor z)^{p-1} [(C - 1)cx' + (c - C)(x' \lor z)] \\
\leq (x \lor z)^{p-1} [(C - 1)cx + (c - C)(x \lor z)].
\]

If \( x' \leq (x \lor z) \), then both sides are equal; if \( x' > (x \lor z) \), then the bound becomes

\[
C(c - 1)(x')^p \leq (x \lor z)^{p-1} ((C - 1)cx' + (c - C)(x \lor z)),
\]

or, after the substitution \( t = x'/(x \lor z) \),

\[
C(c - 1)t^p \leq (C - 1)ct + c - C.
\]

(5.2)

However, we have \( t > 1 \) and \( t \leq (c - 1 + \alpha)/(c\alpha) \) (see the assumptions appearing in 4°). It suffices to note that the left-hand side of (5.2) is a convex function, and that both sides are equal for \( t \in \{1, (c-1+\alpha)/(c\alpha)\} \). Thus, (3.1) gives the claim for truncated maximal operator, and letting \( n \to \infty \) completes the proof, by the use of Lebesgue’s monotone convergence theorem. \( \square \)
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