Domain decomposition methods with overlapping subdomains for time-dependent problems

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Abstract Domain decomposition (DD) methods for solving time-dependent problems can be classified by (i) the method of domain decomposition used, (ii) the choice of decomposition operators (exchange of boundary conditions), and (iii) the splitting scheme employed. To construct homogeneous numerical algorithms, overlapping subdomain methods are preferable. Domain decomposition is associated with the corresponding additive representation of the problem operator. To solve time-dependent problems with the DD splitting, different operator-splitting schemes are used. Various variants of decomposition operators differ by distinct types of data exchanges on interfaces. They ensure the convergence of the approximate solution in various spaces of grid functions.

1 Introduction

Domain decomposition methods are used for the numerical solution of boundary value problems for partial differential equations on parallel computers. In the theory of domain decomposition methods, modern investigations are most fully presented for stationary problems Quarteroni and Valli (1999); Toselli and Widlund (2005). Computational algorithms with and without overlap of subdomains are applied in synchronous (sequential) and asynchronous (parallel) methods.

Domain decomposition methods for unsteady problems are based on two basic approaches Samarski et al. (2002).
1. For the numerical solution of time-dependent problems, we use the standard implicit approximation in time. Domain decomposition methods are applied to solve the discrete problem at the new time level. The number of iterations in optimal iterative methods for domain decomposition does not depend on steps of discretization in time and space.

2. To solve unsteady problems, iteration-free domain decomposition algorithms are developed. We construct a special scheme of splitting into subdomains (regionally additive schemes).

A domain decomposition scheme is defined by a decomposition of the computational domain and by specifying a splitting of the problem operator. To construct decomposition operators, it is convenient to use the partition of unity for the computational domain.

In DD methods with overlap, we introduce a function associated with each subdomain, and this function takes value between zero and one. In the extreme case, the width of the overlap of subdomains is equal to the step of discretization in space. In this case, regionally additive schemes can be interpreted as non-overlapping domain decomposition schemes, where data exchange is achieved by setting proper boundary conditions for each subdomain.

Domain decomposition methods for unsteady problems include the following steps:

- Decomposition of a domain;
- Constructing operators of decomposition;
- Design of a splitting scheme;
- A study of convergence;
- Computational implementation.

These basic questions (without numerical implementation) are discussed in this paper using a boundary value problem for the second-order parabolic equation as an example.

2 Standard approximation

Assume that in a bounded domain $\Omega$, an unknown function $u(x, t)$ satisfies the following equation:

$$\frac{\partial u}{\partial t} - \sum_{\alpha=1}^{m} \frac{\partial}{\partial x_\alpha} \left( k(x) \frac{\partial u}{\partial x_\alpha} \right) = f(x, t), \quad x \in \Omega, \quad 0 < t \leq T, \quad (1)$$

where $k(x) \geq \kappa > 0$, $x \in \Omega$. Homogeneous Dirichlet boundary conditions are applied:

$$u(x, t) = 0, \quad x \in \partial \Omega, \quad 0 < t \leq T. \quad (2)$$
The initial condition seems like this:

$$u(x, 0) = u^0(x), \quad x \in \Omega. \quad (3)$$

Let $(\cdot, \cdot), \| \cdot \|$ be the scalar product and the norm in $L_2(\Omega)$, respectively:

$$(u, v) = \int_{\Omega} u(x)v(x)dx, \quad \|u\| = (u, u)^{1/2}.$$  

A symmetric, positive definite, bilinear form $d(u, v)$ such that

$$d(u, v) = d(v, u), \quad d(u, u) \geq \delta \|u\|^2, \quad \delta > 0,$$

is associated with a Hilbert space $H_d$ equipped with the following scalar product and norm:

$$(u, v)_d = d(u, v), \quad \|u\|_d = (d(u, u))^{1/2}.$$  

Suppose $t = t^n = n\tau$, $n = 0, 1, \ldots$, where $\tau > 0$ is a constant time step. A finite-dimensional space of finite elements is denoted by $\mathcal{V}^h$, and $u^n (u^n \in \mathcal{V}^h)$ stands for the approximate solution at the time level $t = t^n$. The original boundary value problem (1)–(3) is treated in the variational form:

$$\left( \frac{du}{dt}, v \right) + a(u, v) = (f, v), \quad \forall v \in H^1_0(\Omega), \quad 0 < t \leq T, \quad (4)$$

$$(u(0), v) = (u^0, v), \quad \forall v \in H^1_0(\Omega), \quad (5)$$

where

$$a(u, v) = \int_{\Omega} k(x) \text{grad} u \text{grad} v \, dx.$$  

We study the projection-difference scheme (schemes with weights) for (4), (5):

$$\left( y^{n+1} - y^n, \frac{1}{\tau} \right) + a(\sigma y^{n+1} + (1 - \sigma)y^n, v) = (f(\sigma t^{n+1} + (1 - \sigma)t^n), v), \quad (6)$$

$$(y^n, v) = (u^n, v), \quad \forall v \in \mathcal{V}^h, \quad n = 1, 2, \ldots, \quad (7)$$

where $\sigma$ is a number (weight). If $\sigma = 0$, then the scheme (6), (7) is the explicit (Euler forward-time) scheme; for $\sigma = 1$, we obtain the fully implicit (Euler backward-time) scheme; and $\sigma = 0.5$ yields the averaged (the so-called Crank–Nicolson) scheme. The condition

$$(v, v) + \left( \sigma - \frac{1}{2} \right) \tau a(v, v) \geq 0, \quad \forall v \in \mathcal{V}^h$$

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is necessary and sufficient for the stability of the scheme in the space $H_a$ Samarskii (2001); Samarskii et al. (2002).

3 Decomposition operators

To construct a domain decomposition scheme, we introduce the partition of unity for the computational domain $\Omega$ Laevsky (1987). Assume that the domain $\Omega$ consists of $p$ separate subdomains:

$$\Omega = \Omega_1 \cup \Omega_2 \cup \ldots \cup \Omega_p.$$  

Individual subdomains may be overlapped. With an individual subdomain $\Omega_\alpha$, $\alpha = 1, 2, \ldots, p$ we associate the function $\eta_\alpha(x)$, $\alpha = 1, 2, \ldots, p$ such that

$$\eta_\alpha(x) = \begin{cases} > 0, & x \in \Omega_\alpha, \\ 0, & x \notin \Omega_\alpha, \end{cases} \alpha = 1, 2, \ldots, p,$$

where

$$\sum_{\alpha=1}^{p} \eta_\alpha(x) = 1, \quad x \in \Omega.$$  

For problem (4), (5), we have

$$a(u, v) = \sum_{\alpha=1}^{p} a_\alpha(u, v), \quad (f, v) = \sum_{\alpha=1}^{p} (f_\alpha, v).$$

Here

$$(f_\alpha, v) = \int_\Omega \eta_\alpha(x)f(x, t)v \, dx$$

and (the standard decomposition)

$$a_\alpha(u, v) = \int_\Omega \eta_\alpha(x)k(x) \text{grad} u \text{grad} v \, dx, \quad \alpha = 1, 2, \ldots, p.$$  

Other variants of domain decompositon operators Vabishchevich (1989) are associated with the following forms:

$$a_\alpha(u, v) = \int_\Omega k(x) \text{grad} u \text{grad} (\eta_\alpha(x)v) \, dx,$$

$$a_\alpha(u, v) = \int_\Omega k(x) \text{grad} (\eta_\alpha(x)u) \text{grad} v \, dx, \quad \alpha = 1, 2, \ldots, p.$$  

Let us investigate the corresponding operator-splitting schemes. From problem (4), (5), we can go to the Cauchy problem for the evolutionary
equation of first order:
\[ B \frac{dy}{dt} + Ay = \varphi(t), \quad 0 < t \leq T, \]  
\[ y(0) = y^0. \]  
(8)

Here the mass matrix  
\[ B = B^* > 0, \]
and the stiffness matrix  
\[ A = A^* > 0. \]

For (8), (9), we have the following operator splitting:
\[ A = \sum_{\alpha=1}^{p} A_{\alpha}, \quad \varphi = \sum_{\alpha=1}^{p} \varphi_{\alpha} \]
with (the standard decomposition)
\[ A_{\alpha} = A_{\alpha}^* \geq 0, \quad \alpha = 1, 2, ..., p. \]

We arrive to the symmetrized equation:
\[ \frac{dw}{dt} + \tilde{A}w = \tilde{\varphi}(t), \quad 0 < t \leq T, \]
where  
\[ w = B^{1/2}v, \quad \tilde{A} = B^{-1/2}AB^{-1/2}, \quad \tilde{\varphi} = B^{-1/2}\varphi, \]
\[ \tilde{A} = \sum_{\alpha=1}^{p} \tilde{A}_{\alpha}, \quad \tilde{A}_{\alpha} = \tilde{A}_{\alpha}^* = B^{-1/2}A_{\alpha}B^{-1/2}, \quad \alpha = 1, 2, ..., p. \]

Now we can employ general results of the stability (correctness) theory for operator-difference schemes [Samarskii (2001); Samarskii et al. (2002)].

4 Splitting schemes

The investigation of domain decomposition schemes for time-dependent problems is based on consideration of the relevant splitting schemes [Vabishchevich (2014)]. Here we highlight the case of the two-component splitting \((p = 2)\). In this case, we can focus on the following methods:

- the Douglas-Rachford scheme;
- the Peaceman-Rachford scheme;
- Factorized schemes;
- Symmetric scheme of componentwise splitting.
In particular, the Douglas-Rachford scheme may be written as:

\[
\left( \frac{u^{n+1/2} - u^n}{\tau}, v \right) + a_1(u^{n+1/2}, v) + a_2(u^n, v) = (f^{n+1}, v),
\]

\[
\left( \frac{u^{n+1} - u^n}{\tau}, v \right) + a_1(u^{n+1/2}, v) + a_2(u^n, v) = (f^{n+1}, v), \quad \forall v \in V^h.
\]

The problem in the subdomain (explicit-implicit scheme) is formulated in the form:

\[
\left( \frac{u^{n+1/2}}{2} - u^n, v \right) + a_1(u^{n+1/2}, v) + a_2(u^n, v) = (\chi^n, v),
\]

\[
\left( u^{n+1}, v \right) + \tau a_2(u^{n+1}, v) = (\chi^{n+1/2}, v).
\]

In the more general case, we focus on factorized schemes with weights:

\[
\left( \frac{u^{n+1/2} - u^n}{\tau}, v \right) + a_1(\sigma u^{n+1/2} + (1 - \sigma)u^n, v) + a_2(u^n, v) = (f^{n+1/2}, v),
\]

\[
\left( \frac{u^{n+1} - u^n}{\tau}, v \right) + a_1(\sigma u^{n+1/2} + (1 - \sigma)u^n, v) + a_2(\sigma u^{n+1} + (1 - \sigma)u^n, v) = (f^{n+1/2}, v),
\]

\[
\forall v \in V^h, \quad n = 1, 2, ..., \quad (11)
\]

For \(\sigma = 1/2\), we obtain the Peaceman-Rachford scheme, whereas at \(\sigma = 1\) we have the Douglas-Rachford scheme.

The operator form of the factorized scheme seems like this:

\[
(B + \sigma \tau A_1)B^{-1}(B + \sigma \tau A_2)\frac{y^{n+1} - y^n}{\tau} + Ay^n = \varphi^n.
\]

The following estimate for stability takes place:

\[
\|(B + \sigma \tau A_2)y^{n+1}\|_{B^{-1}} \leq \|(B + \sigma \tau A_2)y^n\|_{B^{-1}} + \tau\|\varphi^n\|_{B^{-1}}.
\]

**Theorem 1.** For the error of the factorized regionally additive difference scheme (10), (11) with \(\sigma \geq 1/2\), the estimate

\[
\|(B + \sigma \tau A_2)z^{n+1}\|_{B^{-1}} \leq M \left( h^2 + \tau^2 + \left( \sigma - \frac{1}{2} \right) \tau + \sigma \tau \|\chi_2\|_{A} \right)
\]

holds.

For the minimal overlapping of subdomains, this estimate yields

\[
\|(B + \sigma \tau A_2)z^{n+1}\|_{B^{-1}} \leq M \left( h^2 + \tau^2 + \left( \sigma - \frac{1}{2} \right) \tau + \sigma \tau H^{-1/2}h^{-1/2} \right),
\]
where $H$ is the step of the coarse mesh. The scheme with $\sigma = 1/2$ does not increase the order of accuracy. In this case, the main error term is two times lower compared to $\sigma = 1$.

For multicomponent splitting, the basic classes of additive schemes [Marchuk 1990; Vabishchevich 2014] are the following:

- Schemes of componentwise splitting;
- Additively averaged schemes of summarized approximation;
- Regularized additive schemes;
- Vector additive schemes.

In particular, the classic serial version for the scheme of componentwise splitting yields:

$$
\left( \frac{u^{n+1}/p - u^{n+(\alpha-1)/p}}{\tau}, v \right) + a_\alpha(\sigma u^{n+1}/p + (1 - \sigma) u^{n+(\alpha-1)/p}, v) = (f^{n+1}_\alpha, v),
$$

$\alpha = 1, 2, ..., p.$

The right-hand side is written as:

$$
f^{n+1}_\alpha = \begin{cases} 0, & \alpha = 1, 2, ..., p - 1, \\ f^{n+1}, & \alpha = p. \end{cases}
$$

Unconditional stability is fulfilled for $\sigma \geq 0.5$.

Additively averaged schemes (parallel version) may be written in the form:

$$
\left( \frac{u^{n+1}_\alpha - u^n_\alpha}{p\tau}, v \right) + a_\alpha(\sigma u^{n+1}_\alpha + (1 - \sigma) u^n_\alpha, v) = (f^{n+1}_\alpha, v), \quad \alpha = 1, 2, ..., p,
$$

$$(u^{n+1}, v) = \frac{1}{p} \sum_{\alpha=1}^{p} (u^{n+1}_\alpha, v).$$

In constructing vector additive schemes [Vabishchevich 2014], instead of a single unknown $u(t)$, we consider $p$ unknowns $u_\alpha$, $\alpha = 1, 2, ..., p$, which are determined from the system:

$$
\left( \frac{du_\alpha}{dt}, v \right) + \sum_{\beta=1}^{p} a_\beta(u_\beta, v) = (f, v), \quad \alpha = 1, 2, ..., p, \quad 0 < t \leq T.
$$

For this system of equations, there are used the initial conditions

$$(u_\alpha(0), v) = (u^0, v), \quad \alpha = 1, 2, ..., p.$$
\[
\left( \frac{u_{n+1}^\alpha - u_n^\alpha}{\tau}, v \right) + \sum_{\beta=1}^{\alpha} a_\beta (u_{n+1}^\beta, v) + \sum_{\beta=\alpha+1}^{p} a_\beta (u_n^\beta, v) = (f^{n+1}, v), \\
(u_0^\alpha, v) = (u_0, v), \quad \alpha = 1, 2, ..., p.
\]

For the parallel version, we have
\[
\left( \frac{u_{n+1}^\alpha - u_n^\alpha}{\tau}, v \right) + a_\alpha (\sigma u_{n+1}^\alpha + (1 - \sigma)u_n^\alpha, v) + \sum_{\alpha \neq \beta=1}^{p} a_\beta (u_n^\beta, v) = (f^{n+1}, v), \\
\alpha = 1, 2, ..., p.
\]

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