ABELIAN STRICT APPROXIMATION IN AW*-ALGEBRAS
AND WEYL-VON NEUMANN TYPE THEOREMS

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Dedicated to Professor E. Effros on his 70th birthday

Abstract. In this paper, for a C*-algebra $A$ with $M = M(A)$ an AW*-algebra, or equivalently, for an essential, norm-closed, two-sided ideal $A$ of an AW*-algebra $M$, we investigate the strict approximability of the elements of $M$ from commutative C*-subalgebras of $A$. In the relevant case of the norm-closed linear span $A$ of all finite projections in a semi-finite AW*-algebra $M$ we shall give a complete description of the strict closure in $M$ of any maximal abelian self-adjoint subalgebra (masa) of $A$. We shall see that the situation is completely different for discrete respectively continuous $M$:

In the discrete case, for any masa $C$ of $A$, the strict closure of $C$ is equal to the relative commutant $C' \cap M$, while in the continuous case, under certain conditions concerning the center valued quasitrace of the finite reduced algebras of $M$ (satisfied by all von Neumann algebras), $C$ is already strictly closed. Thus in the continuous case no elements of $M$ which are not already belonging to $A$ can be strictly approximated from commutative C*-subalgebras of $A$.

In spite of this pathology of the strict topology in the case of the norm-closed linear span of all finite projections of a continuous semi-finite AW*-algebra, we shall prove that in general situations including also this case, any normal $y \in M$ is equal modulo $A$ to some $x \in M$ which belongs to an order theoretical closure of an appropriate commutative C*-subalgebra of $A$. In other words, if we replace the strict topology with order theoretical approximation, Weyl-von Neumann-Berg-Sikonia type theorems will hold in substantially greater generality.

Introduction

Let $A$ be a C*-algebra. The multiplier algebra of $A$ is the C*-subalgebra

$$\{x \in A^{**}; \ xa, \ ax \in A \ \text{for all} \ a \in A\}$$

of the second dual $A^{**}$ (see [Ped 2], Section 3.12 or [WO], Chapter 2). A natural locally convex vector space topology on $M(A)$, called the strict topology $\beta$, is defined by the seminorms

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It is complete and compatible with the duality between $M(A)$ and $A^*$. Hence the strict topology is weaker than the norm-topology on $M(A)$, but stronger than the restriction to $M(A)$ of the weak * topology of $A^{**}$.

We notice that for $A$ the $C^*$-algebra $K(H)$ of all compact linear operators on a complex Hilbert space $H$, $M(A)$ can be identified with the $C^*$-algebra $B(H)$ of all bounded linear operators on $H$ and on every bounded subset of $B(H)$ the strict topology coincides with the $s^*$-topology.

More generally, if $M$ is an $AW^*$-algebra (see [Kap 1] or [Be], §4 or [S-Z], §9) and $A$ is an essential, norm-closed, two-sided ideal of $M$, then, by a theorem of B. E. Johnson, $M$ can be identified with $M(A)$ (see [J] or [Ped 3]). Thus the pairs $(A, M(A))$, where $A$ is a $C^*$-algebra such that $M(A)$ is an $AW^*$-algebra, are exactly the pairs $(A, M)$, where $M$ is an $AW^*$-algebra and $A$ is an essential, norm-closed, two-sided ideal of $M$.

A relevant case of essential, norm-closed, two-sided ideal of an $AW^*$-algebra is the norm-closed linear subspace $A$ generated by all finite projections of a semi-finite $AW^*$-algebra $M$. Then there are central projections $p_1, p_2, p_3$ of $M$ with $p_1 + p_2 + p_3 = 1_M$ such that $Mp_1$ is finite, $Mp_2$ is properly infinite and discrete, while $Mp_3$ is properly infinite and continuous (see [Be], §15, Theorem 1). Since $Ap_1 = Mp_1$, the non-trivial cases are $Ap_2$ and $Ap_3$, with $M(Ap_3) = Mp_3$ properly infinite and discrete and $M(Ap_3) = Mp_3$ properly infinite and continuous.

In the previous paper [D-Z] we investigated the strict approximability of a normal element $x$ of $M(A)$ from a commutative $C^*$-subalgebra of $A$. More precisely, we say that $x$ belongs to the abelian strict closure of $A$ if there exists a commutative $C^*$-subalgebra $C_x$ of $A$ such that $x \in \overline{C_x}^\beta$. Abelian strict approximability is closely related to the classical Weyl-von Neumann-Berg-Sikonia (WNBS) Theorem, which claims that in the case of $A = K(H)$, $H$ a separable complex Hilbert space, every normal element of $M(A) = B(H)$ is of the form $a + x$ with $a \in A$ and $x$ in the abelian strict closure of $A$. For a general $\sigma$-unital $C^*$-algebra $A$, that is a $C^*$-algebra having a countable approximate unit, we proved a partial extension in [D-Z], Theorem 1, which implies that all elements $y \in M(A)$ are of the form $a + x_1 + x_2$, where $a \in A$ and $x_1 \in B_1, x_2 \in B_2$, where $B_1, B_2$ are separable $C^*$-subalgebras of $M(A)$ such that every normal element of $B_j, j = 1, 2$, belongs to the abelian strict closure of $A$. Moreover, if $y$ is self-adjoint then $x_1, x_2$ can be chosen self-adjoint, so in this situation $x_1, x_2$ themselves belong to the abelian strict closure of $A$.

We notice that if the multiplier algebra of a $\sigma$-unital $C^*$-algebra $A$ is of real rank zero (see [Br-Ped]), then, according to [M] and [Zh], the WNBS Theorem holds in the same formulation as in the classical case.

In this paper we discuss abelian strict approximability for a $C^*$-algebra $A$ which is the norm-closed linear subspace generated by all finite projections of some semi-finite $AW^*$-algebra $M$. Since the abelian strict closure of $A$ is the union of all $\overline{C_x}^\beta$ with $C$ a maximal abelian self-adjoint subalgebra (masa) of $A$, we are interested in describing $\overline{C_x}^\beta$ for any masa $C$ of $A$. We shall see that the situation is completely different for discrete respectively continuous $M(A) = M$.

In the discrete case $\overline{C_x}^\beta$ is equal to the relative commutant $C^\prime \cap M(A)$ (Theorem 1), while in the continuous case, under a certain condition on the centre valued quasitrace of the reduced $AW^*$-subalgebra of $M(A)$ by a finite projection of central
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support $1_{M(A)}$ (always satisfied if $M(A)$ is a von Neumann algebra), $C$ is already strictly closed (Theorem 3).

Consequently, if $M(A)$ is a properly infinite, continuous $AW^*$-algebra satisfying the above mentioned condition, then the unit of $M(A)$ does not belong to the abelian strict closure of $A$, that is it does not exist an approximate unit for $A$ contained in a commutative $*$-subalgebra of $A$. In particular, in this case $A$ is not $\sigma$-unital. We notice that it was already shown in [Ak-Ped], Proposition 4.5, that the norm-closed linear span of all finite projections of a type $\Pi_\infty$ factor is a non-$\sigma$-unital $C^*$-algebra. Nevertheless, also in this case WNBS type theorems can be proved. Indeed, if $A$ is the norm-closed linear subspace generated by all finite projections of some countably decomposable semi-finite $W^*$-factor $M$, then, according to [Z], Theorem 3.1, every normal $y \in M(A) = M$ is of the form $a + x$ with $a \in A$ and $x$ in the $s^*$-closure in $M$ of some masa $C$ of $A$. Since the $s^*$-closure of a commutative $*$-subalgebra of a $W^*$-algebra is equal to its monotone order closure (cf. [Kad 1] and [Ped 1]), it is natural to expect that for extensions of the WNBS Theorem to non-$\sigma$-unital $C^*$-algebras the strict closure should be replaced by an order theoretical closure. Along this line we prove several WNBS type theorems in a general setting which includes the case of the norm-closed linear span of all finite projections of a countably decomposable semi-finite $AW^*$-algebra.

More precisely, we prove that if $J$ is a norm-closed two-sided ideal of a (unital) Rickart $C^*$-algebra $M$, which has a countable “order theoretical approximate unit”, then any normal $y \in M$ is of the form $y = a + x$, where $a \in A$ is of arbitrarily small norm and $x$ belongs to the order theoretical closure of some masa of $J$ (Theorem 4 and the subsequent remark). Moreover, the above $x$ can be chosen as a particular infinite linear combination of a sequence of mutually orthogonal projections from $J$ (Theorems 5 and 6).

Since only little of the specific properties of Rickart $C^*$-algebras is used, we are left with the question, to which extent the above mentioned WNBS type theorems hold if $M$ is assumed to be only a $C^*$-algebra of real rank zero.

1 Abelian Strict Closure in Discrete $AW^*$-algebras

First we prove a general result concerning a masa $C$ of a $C^*$-algebra $A$, whose multiplier algebra is an $AW^*$-algebra, that is, according to the theorem of B. E. Johnson quoted in Introduction (see [J] or [Ped 3]), a masa $C$ of an essential, norm-closed, two-sided ideal $A$ of some $AW^*$-algebra. We notice that a part of this result holds for a masa of an essential, norm-closed, two-sided ideal of any Rickart $C^*$-algebra. We shall restrict us to unital Rickart $C^*$-algebras, because adjoining a unit to a non-unital Rickart $C^*$-algebra $M$, we obtain a unital Rickart $C^*$-algebra $\tilde{M}$ (see [Be], §5, Theorem 1 or [S-Z], 9.11.(1)) and it is easy to see that every essential, norm-closed, two-sided ideal of $M$ is an essential, norm-closed, two-sided ideal also of $\tilde{M}$.

Any essential two-sided ideal $J$ of a $C^*$-algebra $M$ induces a strict topology $\beta_J$ on $M$, defined by the seminorms

$$M \ni x \mapsto \|xa\| \text{ and } x \mapsto \|ax\|, \quad a \in J.$$  

With this definition, the usual strict topology on the multiplier algebra of a $C^*$-algebra $A$ is $\beta_A$.

For the basic facts concerning Rickart $C^*$-algebras and $AW^*$-algebras see [Be], §§ 3, 4 and 5, or [S-Z], §9.
Lemma 1. Let $M$ be a unital $C^*$-algebra, $J$ an essential, norm-closed, two-sided ideal of $M$, and $C$ a masa of $J$. By the strict topology on $M$ we shall understand $\beta_J$, which of course is the usual strict topology when $M$ is an $AW^*$-algebra and so can be identified with the multiplier algebra $M(J)$. Then

(i) every $x \geq 0$ in the strict closure of $C$ in $M$ belongs to the strict closure of $\{b \in C : 0 \leq b \leq x\}$ in $M$.

Let us next assume that $M$ is a Rickart $C^*$-algebra. Then

(ii) for every $0 \leq b \in C$ and every $\delta > 0$ there is a projection $f_\delta \in C$ such that

$$bf_\delta \geq \delta f_\delta, \quad b(1_M - f_\delta) \leq \delta(1_M - f_\delta),$$

so $C$ is the norm-closed linear span of its projections;

(iii) any projection $e$ in the strict closure of $C$ in $M$ belongs to the strict closure of $\{f \in C : f \leq e$ projection $\}$ in $M$;

(iv) any projection $e$ in the relative commutant $C' \cap M$ is the least upper bound of $\{f \in C : f \leq e$ projection $\}$ in the projection lattice of $M$, in particular $C' \cap M$ is a masa of $M$.

Finally, assuming $M$ to be an $AW^*$-algebra,

(v) the relative commutant $C' \cap M$ is the $AW^*$-subalgebra of $M$ generated by $C$, so $C' \cap M$ can be identified with $M(C)$;

(vi) the strict closure of $C$ in $M$ coincides with $C' \cap M$ if and only if $C$ contains a two-sided approximate unit for $J$, in which case the strict topology of $M(C) = C' \cap M$ is the restriction of the strict topology of $M(J) = M$.

Proof. The strict closure $\overline{C^\beta_J}$ of $C$ being an abelian $C^*$-subalgebra of $M(A)$, we have for every $b \in C$

$$(x - b)^*(x - b) \geq (x - \text{Re } b)^2 \geq (x - (\text{Re } b)_+)^2 \geq (x - b_0)^2,$$

where

$$b_0 = \frac{1}{2}(x + (\text{Re } b)_+ - |x - (\text{Re } b)_+|)$$

denotes the greatest lower bound of $x$ and $(\text{Re } b)_+$ in the Hermitian part of $\overline{C^\beta_J}$.

Since

$$0 \leq b_0 \leq (\text{Re } b)_+ \in C \subset J,$$

by [Ped 2], Prop. 1.4.5 we have $b_0 \in J$, so

$$b_0 \in C' \cap J = C.$$

Thus, for every $a \in J$ and $b \in C$ we have $\|(x - b)a\| \geq \|(x - b_0)a\|$ for some $0 \leq b_0 \leq x$ in $C$ and (i) follows.

For (ii) put

$$f_\delta = \text{ support of } (b - \delta 1_{A^\sim})_+ \text{ in } M.$$

Then $f_\delta$ commutes with every element of $C$ and

$$bf_\delta \geq \delta f_\delta, \quad b(1_M - f_\delta) \leq \delta(1_M - f_\delta).$$

In particular, $f_\delta \leq \frac{1}{\delta}b \in A$ and [Ped 2], Prop.1.4.5 yields $f_\delta \in J$. Consequently $f_\delta \in C' \cap A = C$.

For (iii) let $0 \neq a \in J$ and $\varepsilon > 0$ be arbitrary. According to (i) there exists $0 \leq b \leq \varepsilon$ in $C$ such that
\[ \| (e - b)a \| < \frac{\varepsilon}{2}, \]

Further, by (ii) there is a projection \( f \in C \) with
\[ bf \geq \frac{\varepsilon}{2\|a\|} f, \quad b(1_A^*-f) \leq \frac{\varepsilon}{2\|a\|} \cdot (1_A^*-f). \]

Then \( f \leq e \) and \( e - f \leq (e - bf)^2 \), so
\[
\| (e - f)a \| = \| a^*(e - f)a \|^{1/2} \leq \\
\leq \| a^*(e - bf)^2a \|^{1/2} = \| (e - bf)e \| \leq \\
\leq \| (e - b)e \| + \| b(1_A^* - f)e \| < \\
< \frac{\varepsilon}{2} + \frac{\varepsilon}{2\|a\|} \|a\| = \varepsilon
\]

For (iv) we have to show that if a projection \( g \in M \) majorizes all projections \( C \ni f \leq e \), then \( g \geq e \), that is \( e \) is equal to the greatest lower bound \( e \wedge g \) of \( e \) and \( g \) in the projection lattice of \( M \). Let us assume that
\[ e_o = e - e \wedge g \neq 0. \]

Since \( J \) is essential ideal in \( M \), there exists \( a \in J \) with \( ae_o \neq 0 \). Choosing some \( 0 < \delta < \|e_o a^* a e_o\| \) and putting
\[ e_1 = \text{support of } (e_o a^* a e_o - \delta 1_M)_{+} \text{ in } M, \]
we have
\[ 0 \neq e_1 \leq \frac{1}{\delta} e_o a^* a e_o \in J. \]

Clearly, \( e_1 \leq e_o \) and [Ped 2], Prop. 1.4.5 yields also \( e_1 \in J \). Furthermore, for every projection \( f \in C \) we get successively
\[ fe \in C' \cap J = C \text{ and } fe \leq e, \]
\[ fe \leq e \wedge g, \text{ hence } fe_o = (fe)e_o = 0, \]
\[ fe_1 = (fe_o)e_1 = 0. \]

Taking into account (ii), it follows that
\[ be_1 = 0 \text{ for all } b \in C, \]
in particular
\[ e_1 \in C' \cap J = C. \]

But then \( e_1 \leq e_o \leq e \) implies \( e_1 \leq e \wedge g \), which contradicts \( 0 \neq e_1 \leq e_o = e - e \wedge g \).

In particular, \( C' \cap M \) is commutative. For the proof we notice that, since \( C' \cap M \) is a Rickart \( C^* \)-subalgebra of \( M \) (see [Be], §5, Proposition 5 or [S-Z], 9.12.(1)), it is the norm-closed linear span of its projections (see e.g. [S-Z], 9.4) and therefore it is enough to show that any two projections \( e_1, e_2 \in C' \cap M \) commute. But the \(*\)-automorphism \( M \ni x \mapsto (2e_2 - 1_M)x(2e_2 - 1_M) \in M \) leaves fixed \( C \), hence also the least upper bound of any projection family in \( C \) in the projection lattice of \( M \). Therefore it leaves fixed \( e_1 \), that is \( e_1 e_2 = e_2 e_1 \).

Moreover, \( C' \cap M \) is a masa of \( M \). Indeed, if \( C_o \supset C' \cap M \) is a commutative subalgebra of \( M \), then \( C_o \supset C \) and thus we have also \( C_o \subset C_o' \cap M \subset C' \cap M \).
For (v) we first notice that $C' \cap M$ is an $AW^*$-subalgebra of $M$ containing $C$ (see [Be], §4, Proposition 8 or [S-Z], 9.24.(1)). Now let $N$ be any $AW^*$-subalgebra of $M$ containing $C$. By (iv) $N$ contains all projections from $C' \cap M$, hence $N \supset C' \cap M$. Consequently $C' \cap M$ is the $AW^*$-subalgebra of $M$ generated by $C$.

Further, $C$ is a two-sided ideal of $C' \cap M$:

$$b \in C \text{ and } y \in C' \cap M \implies by \in C' \cap J = C.$$ 

Moreover, it is essential, because a projection $e \in C' \cap M$ with $Ce = \{0\}$ belongs to the $AW^*$-subalgebra of $C' \cap M$ generated by $C$ only if $e = 0$. Hence we can identify $C' \cap M$ with $M(C)$ (see [J] or [Ped 3]).

Finally we prove (vi). If the strict closure of $C$ in $M$ is $C' \cap M \ni 1_M$, then there exists a net $(u_i)_i$ in $C$ with $u_i \to 1_M$ strictly in $M$, that is

$$\|a - u_i a\| \to 0 \text{ and } \|a - u_i a\| \to 0 \text{ for all } a \in J.$$ 

Conversely, let us assume that $C$ contains a two-sided approximate unit $(u_i)_i$ for $J$. Then the strict topology $\beta_C$ of $M(C) = C' \cap M$ agrees with the strict topology $\beta_J$ of $M(J) = M$ on every norm bounded subset of $C' \cap M$. Indeed, if $(y_\lambda)_\lambda$ is a norm bounded net in $C' \cap M$, convergent to $0$ with respect to $\beta_C$, and $0 \neq a \in J$, $\varepsilon > 0$ are arbitrary, then there exists $\iota_\alpha$ such that

$$\|y_\lambda\| \cdot \|a - u_{\iota_\alpha} a\| < \frac{\varepsilon}{2} \text{ for all } \lambda,$$

and then there exists some $\lambda_\alpha$ with

$$\|y_\lambda u_{\iota_\alpha}\| < \frac{\varepsilon}{2\|a\|} \text{ for every } \lambda \geq \lambda_\alpha.$$ 

It follows for every $\lambda \geq \lambda_\alpha$:

$$\|y_\lambda a\| \leq \|y_\lambda(a - u_{\iota_\alpha} a)\| + \|y_\lambda u_{\iota_\alpha} a\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2\|a\|} \|a\| = \varepsilon.$$ 

But $\beta_C$ is the finest locally convex vector space topology on $C' \cap M$ that agrees with $\beta_C$ on every norm bounded subset of $C' \cap M$ (see [T], Cor. 2.7). Thus the restriction of $\beta_J$ to $C' \cap M$, which is plainly finer than $\beta_C$, is actually equal to $\beta_C$. In particular, the $\beta_C$-density of $C$ in $M(C)$ implies the $\beta_J$-density of $C$ in $C' \cap M$.

It is well known that every commutative $AW^*$-algebra $Z$ is monotone complete (see e.g. [S-Z], 9.26, Proposition 1). If $M$ is an arbitrary $AW^*$-algebra, we call

$$\Phi : \{e \in M; e \text{ projection} \} \to Z^+$$

completely additive whenever, for every family $(e_i)_i$ of mutually orthogonal projections in $M$, we have

$$\Phi(\bigvee_i e_i) = \sum_i \Phi(e_i),$$

where the sum stands for the least upper bound in $Z^+$ of all finite sums of $\Phi(e_i)$’s.

Now we describe the strict closure of a masa of the norm-closed two-sided ideal generated by the finite projections of a discrete semi-finite $AW^*$-algebra:
Theorem 1 (on the abelian strict closure in discrete AW*-algebras). Let $M$ be a discrete AW*-algebra, $A$ the norm-closed linear span of all finite projections of $M$, and $C$ a masa of $A$. Then the strict closure of $C$ in $M(A) = M$ is equal to $C' \cap M$.

Proof. According to Lemma 1 (vi), we have to show that $C$ contains a two-sided approximate unit for $A$. Without loss of generality we may assume that $A \neq \{0\}$, hence $C \neq \{0\}$.

Let $(e_i)_{i \in I}$ be a maximal family of mutually orthogonal non-zero projections in $C$. Then

$$\bigvee_i e_i = 1_M.$$ 

Indeed, $e_o = 1_M - \bigvee_i e_i$ belongs to $C' \cap M$, so Lemma 4 (iv) yields $e_o = \bigvee \{f \in C; f \leq e_o \text{ projection}\}$. Thus $e_o \neq 0$ would imply the existence of some projection $0 \neq f \leq e_o$ in $C$, contradicting the maximality of $(e_i)_{i \in I}$.

Denoting by $Z$ the centre of $M$, we call central partition of $1_M$ any set of mutually orthogonal projections in $Z$ with least upper bound $1_M$. The projections

$$\bigvee_{p \in \mathcal{P}} (\sum_{i \in I_p} e_i) p, \quad \mathcal{P} \text{ central partition of } 1_M, \quad I_p \subset I \text{ finite for any } p \in \mathcal{P}$$

belong to $C' \cap M$ and are finite (see [Be], §15, Proposition 8), hence they belong to $C' \cap M = C$. We show that their family is an (increasing positive) approximate unit for $A$. For we have to prove that every finite projection $e$ in $M$ has the property

(P)

$$\Bigl\| (1_M - \bigvee_{p \in \mathcal{P}} (\sum_{i \in I_p} e_i) p) e \Bigr\| \leq \varepsilon.$$ 

But standard arguments show that every finite projection $e$ in $M$ is of the form

$$e = \bigvee_{n \geq 1} (e_{n,1} + \cdots + e_{n,n}) p_n,$$

where $p_n, n \geq 1$ are mutually orthogonal projections in $Z$ and, for every $n \geq 1, e_{n,1}, \ldots, e_{n,n}$ are mutually orthogonal abelian projections of central support $p_n$ (use [Be], §18, Exercises 3, 4 and Proposition 1), so it is enough to prove (P) for every abelian projection $e$ in $M$. Moreover, since every abelian projection is majorized by an abelian projection of central support $1_M$, without loss of generality we can restrict us to the case of an abelian projection $e$ of central support $1_M$.

For every $x \in M$ there exists a unique $\Phi_e(x) \in Z$ such that

$$exe = \Phi_e(x)e$$

(see [Be], §15, Proposition 6 and §5). Clearly, $\Phi_e : M \to Z$ is a conditional expectation and, according to [Kap 2], Lemma 7, it is completely additive on the projection lattice of $M$. Furthermore, $Z \ni z \mapsto ze \in Ze$ being *-isomorphism, we have

$$\|xe\|^2 = \|ex^*xe\| = \|\Phi_e(x^*xe)\| = \|\Phi_e(x^*x)\|, \quad x \in M.$$ 

Now, by the complete additivity of $\Phi_e$,

$$\sum_i \Phi_e(e_i) = \Phi_e(1_M) = 1_M.$$
Thus, according to [Kap 2], Lemma 5, for every $\varepsilon > 0$ there exist a central partition $\mathcal{P}$ of $1_M$ and finite sets $I_p \subset I, p \in \mathcal{P}$ such that
\[
\left\| \left( 1_M - \sum_{i \in I_p} \Phi(e_i) \right)p \right\| \leq \varepsilon^2 \quad \text{for all } p \in \mathcal{P}.
\]
But then we have for every $p \in \mathcal{P}$
\[
\left\| \left( 1_M - \sum_{i \in I_p} e_i \right)pe \right\|^2 = \left\| \Phi_e((1_M - \sum_{i \in I_p} e_i)p) \right\| = \left\| \left( 1_M - \sum_{i \in I_p} \Phi(e_i) \right)p \right\| \leq \varepsilon^2,
\]
so, taking into account [Kap 1], Lemma 2.5,
\[
\left\| \left( 1_M - \bigvee_{p \in \mathcal{P}} \left( \sum_{i \in I_p} e_i \right)p \right)e \right\| = \sup_{p \in \mathcal{P}} \left\| \left( 1_M - \sum_{i \in I_p} e_i \right)pe \right\| \leq \varepsilon.
\]

2 Abelian Strict Closure in Continuous $\text{AW}^*$-algebras

For the treatment of the case of continuous $M$ we need several lemmas on $\text{AW}^*$-algebras, which could be of interest for themselves. First we extend [Z], Lemma 2.2, concerning a Darboux property of normal functionals on von Neumann algebras without minimal projections, to the case of centre valued completely additive maps on the projection lattice of a continuous $\text{AW}^*$-algebra (similar results can be found in [Ars-Z] and, for tracial maps, in [Kad 2], Prop. 3.13, [Kaf], Prop. 27).

Lemma 2. Let $M$ be a continuous $\text{AW}^*$-algebra, $Z$ its centre, $C$ a masa of $M$, and
\[
\Phi : \{ e \in M; e \text{ projection } \} \to Z^+
\]
a completely additive map such that
\[
\Phi(ep) = \Phi(e)p, \quad e \in M \text{ and } p \in Z \text{ projections}.
\]
Then, for every projection $e \in C$,
\[
\{ z \in Z; 0 \leq z \leq \Phi(e) \} = \{ \Phi(f); e \geq f \in C \text{ projection} \}.
\]

Proof. a) First we prove that for every projection $0 \neq g \in C$ there exists a projection $0 \neq h \leq g$ in $C$ such that
\[
\Phi(h) \leq \frac{1}{2} \Phi(g).
\]
The case $\Phi(g) = 0$ being trivial, we can assume without loss of generality that $\Phi(g) \neq 0$.

Let $(g_i)_i$ be a maximal family of mutually orthogonal projections in $Cg$ such that $\Phi(g_i) = 0$ for every $i$. Put $g_1 = g - \bigvee_i g_i \in C$. Then
\[
\Phi(g_1) = \Phi(g) - \sum_i \Phi(g_i) = \Phi(g) \neq 0,
\]
so $g_1 \neq 0$. By the maximality of $(g_i)_i$, for no projection $0 \neq g' \leq g_1$ in $C$ can hold $\Phi(g') = 0$. 
Now there exists a projection \( g_2 \leq g_1 \) in \( C \) such that \( g_2 \notin Zg_1 \). For let us assume the contrary, that is that

\[ C = Zg_1 + C(1_M - g_1). \]

There exist projections \( h_1, h_2 \in M \) such that \( g_1 = h_1 + h_2 \) and \( h_1 \sim h_2 \) ([Be], §19, Th. 1) and then

\[ C \subset Zh_1 + Zh_2 + C(1_M - g_1) \]

and the maximal abelianness of \( C \) imply that

\[ C = Zh_1 + Zh_2 + C(1_M - g_1). \]

Thus

\[ h_1, h_2 \in Cg_1 = Zg_1. \]

But, denoting by \( z(g_1) \) the central support of \( g_1 \),

\[ Zz(g_1) \ni z \mapsto zg_1 \in Zg_1. \]

is a \( \ast \)-isomorphisms and it follows that \( h_1 \) and \( h_2 \) have orthogonal central supports, in contradiction to \( h_1 \sim h_2 \neq 0 \).

We claim that \( \Phi(g_2)\Phi(g_1 - g_2) \neq 0 \). Indeed, otherwise it would exist a projection \( p \in Z \) such that

\[ \Phi(g_2) = \Phi(g_2)p \text{ and } \Phi(g_1 - g_2)p = 0 \]

and it would follow successively

\[ \Phi(g_2(1_M - p)) = 0 \text{ and } \Phi((g_1 - g_2)p) = 0, \]

\[ (g_2(1_M - p)) = 0 \text{ and } (g_1 - g_2)p = 0, \]

\[ g_2 = g_2p = g_1p \in Zg_1. \]

Let \( q \in Z \) denote the support projection of \((\Phi(g_1) - 2\Phi(g_2))_+\). Then

\[ \Phi(g_1q) - 2\Phi(g_2q) = (\Phi(g_1) - 2\Phi(g_2))_+ \geq 0, \]

\[ \Phi(g_2q) \leq \frac{1}{2}\Phi(g_1q) \leq \frac{1}{2}\Phi(g_1) \leq \frac{1}{2}\Phi(g). \]

Similarly,

\[ \Phi((g_1 - g_2)(1_M - q)) \leq \frac{1}{2}\Phi(g). \]

But we can not have simultaneously

\[ \Phi(g_2q) = 0 \text{ and } \Phi((g_1 - g_2)(1_M - q)) = 0, \]

because this would imply

\[ \Phi(g_2)\Phi(g_1 - g_2) = \Phi(g_2q)\Phi(g_1 - g_2) + \Phi(g_2)\Phi((1_M - q)(g_1 - g_2)) = 0. \]

Therefore, putting \( h = g_2q \) if \( \Phi(g_2q) \neq 0 \) and \( h = (g_1 - g_2)(1_M - q) \) otherwise, \( h \) is a non-zero projection in \( C \), majorized by \( g \), such that \( \Phi(h) \leq \frac{1}{2}\Phi(g) \).

b) Now let \( e \in C \) be a projection and let \( x \in Z, 0 \leq z \leq \Phi(e) \) be arbitrary. Choose a maximal family \((f_i)\), of mutually orthogonal projections in \( Ce \) satisfying
\[ \sum_{i} \Phi(f_i) \leq z. \]

Then the projection \( f = \bigvee_{i} f_i \leq e \) belongs to \( C \) and
\[ \Phi(f) = \sum_{i} \Phi(f_i) \leq z. \]

We claim that actually \( \Phi(f) = z \).

For let us assume the contrary. Then there exist a projection \( 0 \neq p \in Z \) and \( \varepsilon > 0 \) such that
\[ (z - \Phi(f))p \geq \varepsilon p. \]

The projection \( g = (e - f)p \in C \) is not zero, because otherwise it would follow
\[ 0 = (\Phi(e) - \Phi(f))p \geq (z - \Phi(f))p \geq \varepsilon p, \]
contradicting \( p \neq 0, \varepsilon > 0 \). Choosing an integer \( n \geq 1 \) with \( 2^{-n}\|\Phi(e - f)\| \leq \varepsilon \), \( n \)-fold application of a) yields the existence of a projection \( 0 \neq h \leq g \) in \( C \) such that
\[ \Phi(h) \leq 2^{-n}\Phi((e - f)p) \leq \varepsilon p. \]

Since \( 0 \neq h \in Ce \) is orthogonal to every \( f_i \) and
\[ \Phi(h) + \sum_{i} \Phi(f_i) = \Phi(h) + \Phi(f) \leq \varepsilon p + \Phi(f) \leq z, \]
the maximality of \((f_i)_i\) is contradicted.
\[ \square \]

It is well known that if the projection family \((e_i)_i\) in a finite \( AW^* \)-algebra \( M \) is upward directed and, for some projection \( f \in M, e_i \prec f \) for all \( i \), then \( \bigvee_{i} e_i \prec f \) (see [Be], §33, Exercise 1). The above statement actually holds in any \( AW^* \)-algebra \( M \) under the only assumption of the finiteness of \( f \) (see Appendix, Cor. 1). Here we give a proof for this, assuming additionally that the projections \( e_i \) are the finite partial sums of a family of mutually orthogonal projections in \( M \):

**Lemma 3.** Let \( M \) be an \( AW^* \)-algebra, \( f \in M \) a finite projection, and \((e_i)_{i \in I}\) a family of mutually orthogonal projections in \( M \) such that
\[ \sum_{i \in F} e_i \prec f \text{ for every finite } F \subset I. \]

Then
\[ \bigvee_{i \in I} e_i \prec f. \]

**Proof** According to the theory of Murray-von Neumann equivalence for projections in \( AW^* \)-algebras, we can assume without loss of generality that either \( fMf \) is of type \( I_n \) for some natural number \( n \geq 1 \), or that it is continuous (see [Be], §15, Th.1, §18, Th. 2, §6, Cor. 2 of Prop. 4).

Let us first assume that \( fMf \) is of type \( I_n \). By the Zorn Lemma there exists a maximal set \( P \) of mutually orthogonal central projections in \( M \) such that
\[ \text{card } \{ i \in I; pe_i \neq 0 \} \leq n \text{ for every } p \in P. \]

We claim that \( \bigvee P = 1_M \). For let us assume that \( p_o = \bigvee P \neq 1_M \). Then we can find recursively \( n + 1 \) indices \( i_1, \ldots, i_{n+1} \in I \) such that
p_1 = (1_M - p_0)z(e_{i_1}) \ldots z(e_{i_{n+1}}) \neq 0,

where z(e_i) denotes the central support of e_i. By the assumption of the lemma there exist mutually orthogonal projections f_{i_1}, \ldots, f_{i_{n+1}} \leq f in M such that e_{i_j} \sim f_{i_j} for every 1 \leq j \leq n+1. For every 1 \leq j \leq n+1, the central support of p_1 f_{i_j} is p_1, so there exists an abelian projection g_j \leq p_1 f_{i_j} of central support p_1 (see [Be], §18, exercise 4). But then g_1, \ldots, g_{n+1} are mutually orthogonal, equivalent, non-zero projections in fMf (see [Be], §18, Prop.1), which contradicts [Be], §18, Prop. 4.

By the very orthogonal additivity of equivalence in AW*-algebras (see [Be], §11, Prop. 2) we conclude that

\[ \bigvee_{i \in I} e_i = \bigvee \left\{ \sum_{p e_i \neq 0} p e_{i_j}; p \in \mathcal{P} \right\} < \bigvee \{ p f; p \in \mathcal{P} \} = f. \]

Let us next assume that fMf is continuous and let \( x \mapsto x^z \) denote the centre valued dimension function of the finite AW*-algebra fMf (see [Be], Ch.6).

For every \( i \in I \) there exists a projection \( e_i' \leq f \) in M such that \( e_i \sim e_i' \). Since \( (e_i')^z \) does not depend on the choice of \( e_i' \), we can put

\[ e_i^z = (e_i')^z. \]

By the assumption of the lemma, for every finite \( F \subset I \) we can choose the projections \( e_i', i \in F \), mutually orthogonal and then

\[ \sum_{i \in F} e_i^z = \sum_{i \in F} (e_i')^z = \left( \sum_{i \in F} e_i' \right)^z \leq f. \]

It follows that all sums

\[ \sum_{i \in J} e_i^z \leq f, \quad J \subset I \]

exist in the monotone complete centre of fMf.

Now let us consider the set of all families of mutually orthogonal projections in fMf

\[ (f_i)_{i \in J} \text{ with } J \subset I, \]

for which \( f_i \sim e_i \) for every \( i \in J \). We can endowe this set with the partial order

\[ (f_i)_{i \in J} \leq (f'_i)_{i \in J'} \iff J \subset J' \text{ and } f_i = f'_i \text{ for all } i \in J. \]

By the Zorn lemma there exists a maximal element \( (f_i)_{i \in J} \) of the above partially ordered set. We claim that then \( J = I \). For let us assume the existence of some \( i_0 \in I \setminus J \). Since

\[ e_{i_0}^z + \left( \bigvee_{i \in J} f_i \right)^z = e_{i_0}^z + \sum_{i \in J} f_i^z \leq \sum_{i \in I} e_i^z \leq f, \]

that is

\[ e_{i_0}^z \leq \left( f - \bigvee_{i \in J} f_i \right)^z, \]

by [Be], §33, Th.3 (particular case of the above Lemma 5) there exists a projection \( f_{i_0} \leq f - \bigvee_{i \in J} f_i \) in M such that \( f_{i_0}^z = e_{i_0}^z = (e_{i_0}')^z \), hence \( f_{i_0} \sim e_{i_0}' \sim e_{i_0} \). But this contradicts the maximality of \( (f_i)_{i \in J} \).
By the general additivity of equivalence in $AW^*$-algebras (see [Be], §20, Th. 1) we can conclude also in this case that

$$\bigvee_{i \in I} e_i \sim \bigvee_{i \in I} f_i \leq f.$$ 

Let $M$ be a semi-finite $AW^*$-algebra, and $A$ the norm-closed linear span of all finite projections of $M$. We recall that then $M = M(A)$.

Let us call a masa $\tilde{C}$ of $M$ $M$-semi-finite if $\tilde{C} \cap A$ is an essential ideal of $\tilde{C}$ or, equivalently, if every non-zero projections in $\tilde{C}$ majorizes a non-zero projection in $\tilde{C} \cap A$ (cf. with [Kaf]. Def. 1). For $\tilde{C} \subset M$ are equivalent:

1) $\tilde{C}$ is an $M$-semi-finite masa of $M$;
2) $\tilde{C} = C' \cap M$ for some masa $C$ of $A$.

Indeed, 2) implies 1) by Lemma 1 (iv), while 1) $\Rightarrow$ 2) follows by noticing that, according to the $M$-semifiniteness of $\tilde{C}$, every projection in $\tilde{C}$ is the least upper bound of a family of mutually orthogonal projections from $C = \tilde{C} \cap A$, and so $C' \cap M = C' \cap M = \tilde{C}$, $C' \cap A = (C' \cap M) \cap A = \tilde{C} \cap A = C$.

The following result extends [Kad 2], Th. 3.18 and [Kaf], Cor. 31 in the case of an $M$-semi-finite masa:

**Theorem 2** (on labeling Murray-von Neumann equivalence classes). Let $M$ be a semi-finite $AW^*$-algebra, $A$ the norm-closed linear span of all finite projections of $M$, and $C$ a masa of $A$. Then

(i) for any projections $M \ni f \leq e \in C' \cap M$ there exists a projection $f \sim g \leq e$ in $C' \cap M$;
(ii) for any projections $M \ni f \leq e \in C' \cap M$ of equal central supports, $f$ finite and $e$ properly infinite, there is a set $\mathcal{P}$ of mutually orthogonal central projections in $M$ with $\bigvee \mathcal{P} = 1_M$ such that, for every $p \in \mathcal{P}$, $ep$ is the least upper bound in the projection lattice of $M$ of some family of mutually orthogonal projections from $C$, each one of which is equivalent in $M$ to $fp$.

**Proof.** (a) First we prove (i) in the case $e \in C$. Similarly as in the proof of Lemma 3, we can assume without loss of generality that either $eMe = eAe$ is of type $I_n$ for some natural number $n \geq 1$, or it is continuous.

If $eMe$ is of type $I_n$, by [Kad 2], Lemma 3.7 there exist mutually orthogonal projections $e_1, \ldots, e_n \in C$ with $\sum_{j=1}^n e_j = e$, such that each $e_j$ is abelian in $M$ and has the same central support in $M$ as $e$ (actually [Kad 2], Lemma 3.7 is proved only for von Neumann algebras, but an inspection of the proof shows that it works without any change also in the realm of the $AW^*$-algebras). On the other hand, using [Be], §18, Exercise 4 and Prop. 4, it is easy to see that there exist mutually orthogonal abelian projections $f_1, \ldots, f_n \in M$ with $\sum_{j=1}^n f_j = f$ and central supports $z(f) = z(f_1) \geq \cdots \geq z(f_n)$. By [Be], §18, Prop. 1 it follows that $f_j \sim e_j z(f_j)$ for all $1 \leq j \leq n$, so $f$ is equivalent to $C \ni \sum_{j=1}^n e_j z(f_j) \leq e$.

Now let us assume that $eMe$ is continuous and let $x \mapsto x^\sharp$ denote the centre valued dimension function of finite $AW^*$-algebra $eMe$. Then Lemma 2 yields the existence of a projection $C \ni g \leq e$ such that $g^\sharp = f^\sharp$, hence $g \sim f$. 

□
(b) Next we prove (i) in the case \( f \in A \).

By Lemma 1 (iv) there exists a family \( (e_i)_{i \in I} \) of mutually orthogonal projections in \( C \) such that

\[
e = \bigvee_{i \in I} e_i.
\]

Let \( \mathcal{P} \) be a maximal set of mutually orthogonal central projections in \( M \) such that, for every \( p \in \mathcal{P} \), there is a finite set \( F_p \subset I \) with

\[
fp < p \sum_{i \in F_p} e_i \in C.
\]

By the above part (a) of the proof, for every \( p \in \mathcal{P} \) there exists a projection \( g(p) \in C \) with \( fp \sim g(p) \leq p \sum_{i \in F_p} e_i \).

If \( \bigvee \mathcal{P} = 1_M \) then \( f = \bigvee \{ fp ; p \in \mathcal{P} \} \) is equivalent to \( C' \cap M \ni \bigvee \{ g(p) ; p \in \mathcal{P} \} \leq e \), so let us assume in the sequel that \( p_o = 1_M - \bigvee \mathcal{P} \neq 0 \).

By the maximality of \( \mathcal{P} \) and by the comparison theorem (see [Be], §14, Cor. 1 of Prop. 7) we have

\[
p_o \sum_{i \in F} e_i < f \quad \text{for every finite } F \subset I.
\]

According to Lemma 3 it follows that

\[
p_o e = \bigvee_{i \in I} p_o e_i < f,
\]

so by the Schröder-Bernstein theorem (see [Be], §12) we have

\[
fp_o \sim ep_o.
\]

Consequently \( f = fp_o + \bigvee \{ fp ; p \in \mathcal{P} \} \) is equivalent to

\[
C' \cap M \ni ep_o + \bigvee \{ g(p) ; p \in \mathcal{P} \} \leq e.
\]

(c) Now we prove (ii).

Let \( \mathcal{P} \) be a maximal set of mutually orthogonal central projections in \( M \) such that, for every \( p \in \mathcal{P} \), \( ep \) is the least upper bound in the projection lattice of \( M \) of some family of mutually orthogonal projections from \( C \), each one of which is equivalent in \( M \) to \( fp \). We claim that then \( \bigvee \mathcal{P} = 1_M \).

For let us assume that \( p_o = 1_M - \bigvee \mathcal{P} \neq 0 \). We notice that \( fp \neq 0 \) for any central projection \( 0 \neq p \leq p_o \) in \( M \) : indeed, otherwise \( p \) would be orthogonal to the common central support of \( f \) and \( e \), so \( ep = 0 \) would be equal to \( fp = 0 \in C \), in contradiction with the maximality of \( \mathcal{P} \).

Let \( (e_i)_{i \in I} \) be a maximal family of mutually orthogonal projections in \( C \) such that \( fp_o \sim e_i \leq ep_o \) for all \( i \in I \). By the comparison theorem there exists a central projection \( p_1 \leq p_o \) in \( M \) such that

\[
\left( ep_o - \bigvee_{i \in I} e_i \right) p_1 \prec fp_1,
\]

\[
\left( ep_o - \bigvee_{i \in I} e_i \right) (p_o - p_1) \succ f(p_o - p_1).
\]

Then \( p_1 \neq 0 \) : indeed, \( p_1 = 0 \) would imply

\[
A \ni fp_o \prec \bigvee_{i \in I} e_i \in C' \cap M
\]
and, by the above proved (b), it would exist a projection \( fp_0 \sim e' \leq e_0_\cap e_v \in (C' \cap M) \cap A = C \), contradicting the maximality of \((e_i)_{i \in I}\). Put
\[
e_0 = e_0 - \bigvee_{i \in I} e_i p_1 < fp_1.
\]
Then \( e_0 \) is finite and belongs to \( C' \cap M \), so it belongs to \( C' \cap A = C \). On the other hand, the proper infiniteness of \( e \) and \( e_0 \neq 0 \) imply that \( e_1 = e_0 + \bigvee_{i \in I} e_ip_1 \) is properly infinite. It follows that the set \( I \) is necessarily infinite, hence containing an infinite sequence \( \ell_1, \ell_2, \ldots \).

For every \( j \geq 1 \), \( e_0 \sim f p_1 \sim e_{i} p_1 \in C \) and the above proved a) yield the existence of some projection \( e_0 \sim e_{i}^{(1)} \leq e_{i} p_1 \) in \( C \). In particular, all projections \( e_{i}^{(1)} \) are equivalent, hence, the projections \( e_{i} p_1 \) being finite, the projections \( e_{i}^{(2)} = e_{i} p_1 - e_{i}^{(1)} \) are also all equivalent (see [Be], §17, Exercise 3). Consequently, the projections from \( C \)
\[
e_{i} = e_{o} + e_{i}^{(2)} \text{ and } e_{i}^{(2)} = e_{i}^{(1)} + e_{i}^{(2)}, \quad j \geq 2
\]
are all equivalent in \( M \) to \( e_{i}^{(1)} + e_{i}^{(2)} = e_{i} p_1 \sim f p_1 \). Clearly, they are mutually orthogonal and
\[
\bigvee_{j \geq 1} e_{i}^{(1)} = e_0 \vee \bigvee_{j \geq 1} e_{i}^{(2)} = e_0 \vee \bigvee_{j \geq 1} e_{i} p_1.
\]
Letting
\[
e_i' = e_i p_1 \text{ for } i \in I \setminus \{\ell_1, \ell_2, \ldots\},
\]
we conclude that all projections \( e_i' \), \( i \in I \), belong to \( C \) and are equivalent in \( M \) to \( f p_1 \). Moreover, they are mutually orthogonal and
\[
\bigvee_{i \in I} e_i' = \bigvee_{i \neq \ell_j} e_i' \vee \bigvee_{i \neq \ell_j} e_i = e_0 \vee \bigvee_{i \neq \ell_j} e_i p_1 \vee \bigvee_{i \neq \ell_j} e_i p_1 = e_0 \vee \bigvee_{i \neq \ell_j} e_i p_1 = e_0 \vee \bigvee_{i \in I} e_i p_1.
\]
But this contradicts the maximality of \( P \).

(d) Finally we prove (i) in full generality.

We can assume without loss of generality that either \( f \) is finite, or it is properly infinite. The case of finite \( f \) was already settled in (b), so it remains to consider only the case of properly infinite \( f \).

Choose some finite projection \( M \ni f_0 \leq f \) of the same central support as \( f \) (see [Be], §17, Exercise 19 iii)). According to the above proved (c), we can assume without loss of generality that there are families \((e_i)_{i \in I}\) and \((f_\kappa)_{\kappa \in K}\) of mutually orthogonal projections in \( M \) such that
\[
e_i \sim f_0 \sim f_\kappa \text{ for all } i \in I \text{ and } \kappa \in K
\]
\[
\bigvee_{i \in I} e_i = e, \quad \bigvee_{\kappa \in K} f_\kappa = f.
\]
If \( \text{card} \ K \leq \text{card} \ I \), that is if there exists an injective map \( K \ni \kappa \mapsto i(\kappa) \in I \), then the projection \( g = \bigvee_{\kappa \in K} e_i(\kappa) \leq e \) belongs to \( C' \cap M \) and is equivalent to \( \bigvee_{\kappa \in K} f_\kappa = f \). On the other hand, if \( I \leq \text{card} \ K \), then \( e = \bigvee_{i \in I} e_i \sim \bigvee_{\kappa \in K} f_\kappa = f \leq e \) and the Schröder-Bernstein theorem imply that \( e \sim f \).

Let us now prove the statement of [Kad 2], Th. 3.18 and [Kaf], Cor. 31 in the case of an \( M \)-semifinite masa of an arbitrary semifinite \( AW^* \)-algebra \( M \):
Corollary. Let $M$ be a semifinite $AW^*$-algebra, $A$ the norm-closed linear span of all finite projections of $M$, and $C$ a masa of $A$. If $e \in C' \cap M$ is a projection and $1 \leq n \leq \aleph_0$ is a cardinal number such that $e$ is the least upper bound of $n$ mutually orthogonal, equivalent projections from $M$, then there exist $n$ mutually orthogonal projections in $C' \cap M$, all equivalent in $M$, whose least upper bound is $e$.

Proof. It is enough to treat separately the case of finite respectively properly infinite $e$. If $e$ is finite, $n$ can be only a natural number. Let $f_1, \ldots, f_n$ be mutually orthogonal, equivalent projections in $M$ with $\sum_{j=1}^n f_j = e$. By (i) in the above theorem there exists a projection $f_1 \sim e_1 \leq e$ in $C$. Since $e$ is finite, it follows that $\sum_{j=2}^n f_j \sim e - e_1$, so we can apply again (i) in the above theorem to get a projection $f_2 \sim e_2 \leq e - e_1$ in $C$. By induction we obtain $n$ mutually orthogonal projections $e_1, \ldots, e_n \in C$ such that $f_j \sim e_j$ for all $j$ and $\sum_{j=1}^n e_j = e$.

Now let us assume that $e$ is properly infinite and consider a set $I$ of cardinality $n$. Choosing a finite projection $M \ni f \leq e$ of the same central support as $e$ (see [Be], §17, Exercise 19 iii)), (ii) in the above theorem entails the existence of a set $\mathcal{P}$ of mutually orthogonal central projections in $M$ with $\sqrt{\mathcal{P}} = 1_M$ such that, for every $p \in \mathcal{P}$, $ep$ is the least upper bound of some set $\mathcal{E}_p$ of mutually orthogonal projections from $C$, each one of which is equivalent in $M$ to $fp$. If $ep \neq 0$ then $\mathcal{E}_p$ must be infinite, so there exists a partition $(\mathcal{E}_{p,e})_{e \in I}$ of $\mathcal{E}_p$ in $n$ sets of equal cardinality. Then the projections $e_e = \bigvee_{ep \neq 0} \bigvee_{e \in I} \mathcal{E}_{p,e}$, $e \in I$, belong to $C' \cap M$, are mutually orthogonal and equivalent in $M$, and $\bigvee_{e \in I} e_e = e$.

Let $M$ be a finite $AW^*$-algebra with centre $Z$ and let $x \mapsto x^z$ denote its centre valued dimension function (see [Be], Ch. 6). It is known (see [Bl-Ha], II, 1) that $\Phi$ can be uniquely extended to a centre valued quasitrace on $M$, that is to a map $\Phi : M \to Z$ such that
- $\Phi$ is linear on commutative $*$-subalgebras of $M$,
- $\Phi(a + ib) = \Phi(a) + i\Phi(b)$ for all selfadjoint $a, b \in M$,
- $\Phi$ acts identically on $Z$,
- $0 \leq \Phi(x^*x) = \Phi(xx^*)$ for all $x \in M$,
and then
- $\Phi(a) \leq \Phi(b)$ whenever $a \leq b$ are selfadjoint elements of $M$,
- $\Phi$ is norm continuous, more precisely, $\|\Phi(a) - \Phi(b)\| \leq \|a - b\|$ for all selfadjoint $a, b \in M$.

We shall use the symbol $\natural$ to denote also the above $\Phi$.

According to classical results of F.J. Murray and J. von Neumann, the centre valued quasitrace of every finite $W^*$-algebra is additive, hence linear.

It is an open question, raised by I. Kaplansky, whether the centre valued quasitrace of every finite $AW^*$-algebra is additive. Recently U. Haagerup has proven that the answer to Kaplansky’s question is positive for any finite $AW^*$-algebra which is generated (as an $AW^*$-algebra) by an exact $C^*$-subalgebra (see [Haa], Th. 5.11, Prop. 3.12 and Lemma 3.7 (4)).

We notice that if $M$ is a finite $AW^*$-algebra and $n \geq 1$ is an integer, then the $*$-algebra $\text{Mat}_n(M)$ of all $n \times n$ matrices over $M$ is again a finite $AW^*$-algebra (see
[Be], §62). Denoting by $\sharp_n$ and $\sharp_n$ the respective centre valued quasitraces, it is easily seen that

$$n \cdot \begin{pmatrix} x & 0 & 0 \\ 0 & 0 & \ddots \\ 0 & \cdots & 0 \end{pmatrix}^{\sharp_n} = \begin{pmatrix} x^{\sharp_n} & 0 & 0 \\ 0 & x^{\sharp_n} & \ddots \\ 0 & \cdots & x^{\sharp_n} \end{pmatrix}, \quad x \in M.$$  

Moreover the additivity of $\sharp$ is equivalent with the validity of

$$2 \cdot \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}^{\sharp_2} = \begin{pmatrix} x_{11}^{\sharp_2} + x_{22}^{\sharp_2} & 0 \\ 0 & x_{11}^{\sharp_2} + x_{22}^{\sharp_2} \end{pmatrix}.$$  

Indeed, using the above equality, we get for all $0 \leq a, b \in M$

$$\begin{pmatrix} (a + b)^{\sharp} & 0 \\ 0 & (a + b)^{\sharp} \end{pmatrix} = 2 \cdot \begin{pmatrix} a + b & 0 \\ 0 & 0 \end{pmatrix}^{\sharp_2} = 2 \cdot \left[ \begin{pmatrix} a^{1/2} & b^{1/2} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a^{1/2} & 0 \\ b^{1/2} & 0 \end{pmatrix} \right]^{\sharp_2} = 2 \cdot \left[ \begin{pmatrix} a^{1/2} & 0 \\ b^{1/2} & 0 \end{pmatrix} \begin{pmatrix} a^{1/2} & b^{1/2} \\ 0 & 0 \end{pmatrix} \right]^{\sharp_2} = 2 \cdot \begin{pmatrix} a^{\sharp_2} + b^{\sharp_2} & 0 \\ 0 & a^{\sharp_2} + b^{\sharp_2} \end{pmatrix}.$$  

Conversely, assuming that $\sharp$ is additive, it is easy to verify that

$$\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}^{\sharp} \mapsto \frac{1}{2} \begin{pmatrix} x_{11}^{\sharp_2} + x_{22}^{\sharp_2} & 0 \\ 0 & x_{11}^{\sharp_2} + x_{22}^{\sharp_2} \end{pmatrix}$$

is a centre valued quasitrace on $\text{Mat}_2(M)$.

For a given $\delta > 0$, we say that the centre valued quasitrace $\sharp$ of a finite $AW^*$-algebra $M$ is $\delta$-subadditive (resp. $\delta$-superadditive) if the map $M_+ \ni a \mapsto (a^{\sharp})^{\delta}$ is subadditive (resp. superadditive). Clearly, $\delta$-subadditivity ($\delta$-superadditivity) of $\sharp$ implies its $\delta'$-subadditivity ($\delta'$-superadditivity) whenever $\delta' < \delta(\delta' > \delta)$. It was proven by U. Haagerup that $\sharp$ is always $\frac{1}{2}$-subadditive (see [Haa], Lemma 3.5 (1)) and it seems reasonable to conjecture that it is also always 2-superadditive (or, at least, $k$-superadditive for some $k \geq 1$).

We notice as a curiosity that, for any two projections $p, q$ in a finite $AW^*$-algebra $M$ with centre valued quasitrace $\sharp$,

$$(p + q)^{\sharp} = p^{\sharp} + q^{\sharp}.$$  

Indeed, since

$$\begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} = \begin{pmatrix} p & \pm q \\ 0 & \pm q \end{pmatrix} \begin{pmatrix} p & 0 \\ \pm q & q \end{pmatrix} = \begin{pmatrix} p & \pm pq \\ \pm qp & q \end{pmatrix} = \begin{pmatrix} p & 0 \\ \pm q & 0 \end{pmatrix} \begin{pmatrix} p & \pm q \\ 0 & 0 \end{pmatrix},$$

and $\begin{pmatrix} p & pq \\ q & -qp \end{pmatrix}, \begin{pmatrix} p & -pq \\ q & q \end{pmatrix}$ commute, we have
This can be deduced also from Haagerup’s result, taking to account that the $C^*$-algebra generated by two projections is of type $I$, hence nuclear, hence exact.

**Lemma 4.** Let $M$ be a finite $\AW^*$-algebra, whose centre valued quasitrace $\natural$ is $k$-superadditive for some $k \geq 1$. Let further $e_1, \ldots, e_n \in M$ be mutually equivalent projections with $\sum_{j=1}^n e_j = 1_M$. Then there exists a projection $e_1 \sim p \in M$ such that, for every projection $f \in \{e_1, \ldots, e_n\}' \cap M$,

$$f^2 \geq (1 - \|(1_M - f)p\|^2)n^{\frac{k}{k-1}}1_M.$$  

**Proof.** Let $v_1, \ldots v_n \in M$ be partial isometries such that

$$v_j^* v_j = e_1, \quad v_j v_j^* = e_j, \quad 1 \leq j \leq n.$$  

Since

$$\left(\frac{1}{\sqrt{n}} \sum_{j=1}^n v_j\right)^* \frac{1}{\sqrt{n}} \sum_{j=1}^n v_j = \frac{1}{n} \sum_{j_1, j_2=1}^n v_{j_1}^* v_{j_2} = \frac{1}{n} \sum_{j=1}^n v_j^* v_j = e_1,$$

$$p = \frac{1}{n} \sum_{j_1, j_2=1}^n v_{j_1} v_{j_2}^* = \frac{1}{\sqrt{n}} \sum_{j=1}^n v_j \left(\frac{1}{\sqrt{n}} \sum_{j=1}^n v_j\right)^*$$  
is a projection in $M$ equivalent to $e_1$.

Now let the projection

$$f \in \{e_1, \ldots, e_n\}' \cap M$$

be arbitrary and set $\delta = \|(1_M - f)p\|$. Since the case $\delta = 1$ is trivial, we can assume without loss of generality that $\delta < 1$. Then

$$\|p - pf p\| = \|(1_M - f)p\|^2 = \delta^2 < 1,$$

so $pf p \geq (1 - \delta^2)p$ is invertible in $pMp$. Thus the polar decomposition $f p = w \cdot |f p|$ exists in the $C^*$-algebra generated by $p$ and $f$ and we have

$$w^* w = p, \quad f p f = w(pf p)w^* \geq (1 - \delta^2)ww^*.$$  

Let us denote $\zeta = e^{\frac{i\pi}{n}}$. Then
\[ u = \sum_{j=1}^{n} \zeta^j e_j \in \{e_1, \ldots, e_n, f\}' \cap M \]
is unitary. Since
\[
u^m p u - m = \frac{1}{n} \sum_{j,j_1,j_2,j'=1}^{n} \zeta^{mj} e_{j_1} v_{j_2}^* \zeta^{-mj'} e_{j_1}' = \]
\[
= \frac{1}{n} \sum_{j_1,j_2=1}^{n} \zeta^{m(j_1-j_2)} v_{j_1} v_{j_2}^* = \]
and
\[
\sum_{m=1}^{n} \zeta^{mj} = 0 \text{ for every } 1 \leq j \leq n - 1,
\]
we have
\[
\sum_{m=1}^{n} u^m p u - m = \frac{1}{n} \sum_{j_1,j_2=1}^{n} \left( \sum_{m=1}^{n} \zeta^{m(j_1-j_2)} \right) v_{j_1} v_{j_2}^* = \]
\[
= \frac{1}{n} \sum_{j_1=1}^{n} n v_{j_1} v_{j_1}^* = 1_M \]
Therefore
\[
f = f \sum_{m=1}^{n} u^m p u - m f = \sum_{m=1}^{n} u^m (fpf) u - m \geq (1 - \delta^2) \sum_{m=1}^{n} u^m w w^* u - m
\]
and, using the superadditivity of \( \zeta \), we get
\[
f^{\sharp} \geq (1 - \delta^2) \left( \sum_{m=1}^{n} u^m w w^* u - m \right)^{\sharp} \geq \]
\[
\geq (1 - \delta^2) \left( \sum_{m=1}^{n} \left( (u^m w w^* u - m)^{\sharp} \right)^k \right)^{\frac{1}{k}} \]
\[
= (1 - \delta^2) \left( n (w^* w)^{\sharp} \right)^{\frac{1}{k}} = (1 - \delta^2) n^{\frac{1}{k}} p^{\sharp}.
\]
But \( p^{\sharp} = e_j^\sharp \) for all \( 1 \leq j \leq n \), so
\[
n p^{\sharp} = \sum_{j=1}^{n} e_j^\sharp = \left( \sum_{j=1}^{n} e_j \right)^{\sharp} = 1_M
\]
and we conclude that \( f^{\sharp} \geq (1 - \delta^2) n^{\frac{1}{k}} - 11_M. \)

Now we are ready to prove the following

**Theorem 3** (on the abelian strict closure in continuous semi-finite AW*-algebras).

Let \( M \) be a continuous semi-finite AW*-algebra such that, for some finite projection \( e_0 \in M \) of central support \( 1_M \) and some \( k \geq 1 \), the centre valued quasitrace of \( e_0 M e_0 \) is \( k \)-superadditive. Let further \( A \) denote the norm-closed linear span of all finite projections of \( M \), and \( C \) a masa of \( A \). Then the strict closure of \( C \) in \( M = M(A) \) is \( C \).
Proof. Let us assume that the strict closure $\overline{C}^\beta \subset C' \cap M$ of $C$ contains some $0 \leq x \notin C$.

(a) First we prove that then $\overline{C}^\beta$ contains some projection $e \notin C$.

For let $e_\delta$ denote, for every $\delta > 0$, the support of $(x - \delta 1_M)_+$ in the $AW^*$-subalgebra $C' \cap M$ of $M$. Then

$$xe_\delta \geq \delta e_\delta, \quad x(1_M - e_\delta) \leq \delta(1_M - e_\delta).$$

In particular, there exists $0 \leq y \in C' \cap M$ with $yx = e_\delta$. Moreover, $e_\delta \in \overline{C}^\beta$.

Indeed, by Lemma 1 (i) there is a net $(b_i)_i$ in $C$ with

$$0 \leq b_i \leq x$$

for all $i$ and $b_i \to x$ strictly.

Then $0 \leq yb_i \in C' \cap A = C$ for all $i$ and

$$\|(e_\delta - yb_i)a\| = \|y(x - b_i)a\| \leq \|y\| \cdot \|(x - b_i)a\| \to 0.$$

for every $a \in A$.

(b) Next we prove the existence of an infinite sequence of mutually orthogonal projections $0 \neq e_1, e_2, \ldots \in C$, all equivalent in $M$ to $e_oq_o$ for some projection $q_o$ in the centre $Z$ of $M$, such that $\bigvee_{n \geq 1} e_n \in \overline{C}^\beta$.

Let $e$ be a projection as in (a). Then $e$ is not finite, so there exists a projection $q \in Z$ such that $eq$ is properly infinite. But then, by the comparison theorem, there exists a projection $0 \neq q_o \in Z$ such that $e_oq_o \prec eq$. Since the central support of $e_o$ is $1_M$, we have $q_o \leq q$.

Now, according to (ii) in Theorem 2 (on labeling Murray-von Neumann equivalence classes), there exists a family $(e_i)_{i \in I}$ of mutually orthogonal projections in $C$, all equivalent in $M$ to $e_oq_o \neq 0$, such that $\bigvee_{i \in I} e_i = eq_o$. $I$ must be infinite, so it contains an infinite sequence $i_1, i_2, \ldots$. Put

$$e_n = e_{i_n}, \quad n \geq 1.$$

Then $\bigvee_{n \geq 1} e_n$ belongs to $\overline{C}^\beta$. Indeed, since $\bigvee_{n \geq 1} e_n \in C' \cap M$, if $(b_\kappa)_{\kappa}$ is a net in $C$ which converges strictly to $e$, then the net $(b_\kappa \bigvee_{n \geq 1} e_n)_{\kappa}$ is contained in $C$ and converges clearly to $e \bigvee_{n \geq 1} e_n = \bigvee_{n \geq 1} e_n$ in the strict topology of $M$.

(c) Finally we prove that the statement in (b) leads to a contradiction.

Let us denote by $\sharp$ the map $\bigcup_{n \geq 1} e_n M e_n \to Zq_o$ such that, for every $n \geq 1$, $e_n M e_n \ni x \mapsto x^\sharp e_n$ is the centre valued quasitrace of $e_n M e_n$. It is easy to see that $\sharp$ takes the same value in two projections from $\bigcup_{n \geq 1} e_n M e_n$ if and only if they are equivalent in $M$.

Let $n \geq 1$ be arbitrary and let $j_n = \left\lceil n^{\frac{k+1}{k}} \right\rceil \geq 1$ denote the integer part of $n^{\frac{k+1}{k}}$. According to the corollary of Theorem 2 (on labeling Murray-von Neumann equivalence classes), there exist projections

$$e_{n,1}, \ldots, e_{n,j_n} \in C,$$

such that

$$\sum_{j=1}^{j_n} e_{n,j} = e_n,$$

such that

$$e_{n,j}^k = \frac{1}{j_n} q_o \text{ for all } 1 \leq j \leq j_n.$$
Since $e_n \sim e_o q_o$, the centre valued quasitrace of $e_n M e_n$ is $k$-superadditive and Lemma 4 yields the existence of a projection $p_n \in e_n M e_n$ with $p_n^2 = \frac{1}{f_n} q_o$ such that, for every projection $g \in \{e_{n,1}, \ldots, e_{n,j_n}\}' \cap e_n M e_n$,
\[
g^2 \geq (1 - \| (e_n - g) p_n \|^2) \frac{1}{f_n} q_o \geq (1 - \| (e_n - g) p_n \|^2) \frac{1}{n} q_o.
\]

Now put $p = \bigvee_{n \geq 1} p_n$. Since $p_n^2 = \frac{1}{f_n} q_o$ and $\sum_{n \geq 1} \frac{1}{f_n} < +\infty$, using Lemma 2 it is easy to verify that $p$ is equivalent to a subprojection of the sum of finitely many $e_n$'s. In particular, $p$ is finite, that is $p \in A$. Therefore, $\bigvee_{n \geq 1} e_n$ being in $\overline{C^g}$, Lemma 1 (iii) yields the existence of a projection $\bigvee_{n \geq 1} e_n \geq f \in C$ with
\[
\left\| \left( \bigvee_{n \geq 1} e_n - f \right) p \right\| \leq \frac{1}{\sqrt{2}}.
\]
But then, for every $n \geq 1$, $fe_n$ is a projection in $C \cap e_n M e_n \subset \{e_{n,1}, \ldots, e_{n,j_n}\}' \cap e_n M e_n$ and the above yields
\[
(f e_n)^2 \geq (1 - \| (e_n - f e_n) p_n \|^2) \frac{1}{n} q_o \geq \frac{1}{2n} q_o.
\]
Since $\sum_{n \geq 1} \frac{1}{2n} = +\infty$, using again Lemma 2, it is easily seen that $f = \bigvee_{n \geq 1} (f e_n)$ is equivalent to $\bigvee_{n \geq 1} e_n$. In particular, $f$ is properly infinite, in contradiction with $f \in C \subset A$. □

3 Weyl-von Neumann-Berg-Sikonia type theorems

We recall that any Rickart $C^*$-algebra $M$ is $\sigma$-normal, what means that, for every increasing sequence $(e_k)_{k \geq 1}$ of projections in $M$, the least upper bound of $(e_k)_{k \geq 1}$ in the projection lattice of $M$ is actually its least upper bound in the ordered space $\overline{M}_h$ of all self-adjoint elements of $M$ (see [A-Go 2] or [Sa]). Therefore we shall speak in the sequel simply about the least upper bound of increasing sequences of projections in $M$.

Let us first prove a lemma about the sequential approximability of a projection in a Rickart $C^*$-algebra from a two-sided ideal:

Lemma 5. Let $M$ be a unital Rickart $C^*$-algebra, $J$ a two-sided ideal of $M$, and $f \in M$ a projection. Then the following statements are equivalent:

(a) there exists a sequence $(b_k)_{k \geq 1}$ of positive elements in $J$ such that $b_k \leq f$ for all $k \geq 1$ and every projection $e \in M$ with $b_k \leq e$, $k \geq 1$, satisfies $f \leq e$;

(b) there exists an increasing sequence $(f_k)_{k \geq 1}$ of projections in $J$, whose least upper bound in $M$ is $f$.

Proof. Let us assume that (a) holds and put
\[
f_{k,l} = \text{support of} \left( b_k - \frac{1}{l} 1_M \right)_+ \leq f, \quad k, l \geq 1,
\]
\[
f_n = \bigvee_{1 \leq k, l \leq n} f_{k,l} \text{ in the projection lattice of } M \leq f, \quad n \geq 1.
\]
Since \(b_k f_{k,l} \geq \frac{1}{l} f_{k,l} \), and so \(f_{k,l} \) can be factorized by \(b_k \leq f \), we have \(f \geq f_{k,l} \in \mathcal{J} \) for all \(k \) and \(l \). Further, using the validity of the Parallelogramm Law in all Rickart \(C^*\)-algebras (see [Be], §13, Th. 1), we obtain also \(f \geq f_n \in \mathcal{J} \), \(n \geq 1 \).

Now \((f_n)_{n \geq 1} \) is an increasing sequence, whose least upper bound in the projection lattice of \(M \) is \(f \). Indeed, if \(e \in M \) is a projection which majorizes every \(f_n \), hence every \(f_{k,l} \), then we have

\[
\| (1_M - e) b_k \| \leq \frac{1}{l}.
\]

Thus

\[
b_k = e b_k e \leq e \text{ for all } k \geq 1.
\]

and it follows that \(f \leq e \).

Conversely, (b) obviously implies (a) with \(b_k = f_k \).

\[\square\]

For unital Rickart \(C^*\)-algebras we have the following Weyl-von Neumann-Berg-Sikonia type result (cf. with [Z], Theorem 3.1 and [Ak-Ped], §4):

**Theorem 4.** Let \(M \) be a unital Rickart \(C^*\)-algebra, and \(\mathcal{J} \) a norm-closed two-sided ideal of \(M \), which contains a sequence of positive elements such that \(1_M \) is the only projection in \(M \) majorizing the sequence. Then, for any normal \(y \in M \) and every \(\varepsilon > 0 \), there exist a masa \(C \) of \(\mathcal{J} \) and an element \(x \) of the masa \(C' \cap M \) of \(M \), such that

1) \(C \) contains an increasing sequence of projections, whose least upper bound in \(M \) is \(1_M \),

2) \(y - x \in \mathcal{J} \) and \(\| y - x \| \leq \varepsilon \).

**Remark.** We notice that in Theorem 4 \(C' \cap M \) is the sequentially monotone closure of \(C \) in the following sense: every \(0 \leq a \in C' \cap M \) is the least upper bound in \(M_h \) of some increasing sequence of positive elements from \(\mathcal{J} \).

Indeed, if \((e_k)_{k \geq 1} \) is an increasing sequence of projections in \(C \), whose least upper bound in \(M \) is \(1_M \), then \((a^{1/2} e_k a^{1/2})_{k \geq 1} \) is an increasing sequence of positive elements from \(\mathcal{J} \), whose least upper bound in \(A_h \) is \(a^{1/2} 1_M a^{1/2} = a \) (see [S-Z], 9.14, the remark after Proposition 3).

\[\square\]

For the proof of Theorem 4 we need the next result on quasi-central approximate units, implicitly contained in [Z], Proposition 1.2:

**Lemma 6.** Let \(M \) be a unital Rickart \(C^*\)-algebra, \(\mathcal{J} \) an essential, norm-closed, two-sided ideal of \(M \), and \(B \subset M \) a commutative \(C^*\)-subalgebra. Then the upward directed set of all projections of \(\mathcal{J} \) contains a subnet \((e_i)_{i \in I} \) which, besides being automatically approximate unit for \(\mathcal{J} \), is quasi-central for \(B \), that is

\[
\lim_i \| e_i b - b e_i \| = 0 \text{ for all } b \in B.
\]

**Proof.** Passing to the Rickart \(C^*\)-subalgebra of \(M \) generated by \(B \) and \(1_M \) (see e.g. [S-Z], 9.11 (3)), we can assume without loss of generality that \(B \) is a Rickart \(C^*\)-subalgebra of \(M \) containing \(1_M \).
Let $P$ denote the set of all finite sets $P$ of projections from $B$ satisfying the equality $\sum_{p \in P} p = 1_M$ and set

$$I = \{ f \in A ; f \text{ projection } \} \times P.$$ 

We endow $I$ with a partial order by putting $(f_1, P_1) \leq (f_2, P_2)$ whenever $f_1 \leq f_2$ and the $C^*$-algebra $C^*(P_1)$ generated by $P_1$ is contained in $C^*(P_2)$ (that is the partition $P_2$ is a refinement of $P_1$). Clearly, in this way $I$ becomes an upward directed ordered set.

Let $\iota = (f, P) \in I$ be arbitrary. For every $p \in P$, the right support $r(fp)$ of $fp$ is equivalent in $M$ to the left support $l(fp) \leq f \in J$ (see [A] or [A-Go 1]), so it belongs to $J$. Thus

$$e_\iota = \sum_{p \in P} r(fp)$$

is a projection in $J$. Since every $r(fp) \leq p$ belongs to the commutant $P'$, also $e_\iota \in P'$. Furthermore,

$$f \leq e_\iota .$$

Indeed, for every $q \in P$,

$$fq = fq r(fq) = \sum_{p \in P} fq r(fp) = fqe_\iota ,$$

so

$$f = f \sum_{q \in P} q = \sum_{q \in P} fqe_\iota = fe_\iota \leq e_\iota .$$

It is easily seen that

$$\iota_1 \leq \iota_2 \Rightarrow e_{\iota_1} \leq e_{\iota_2} ,$$

so $(e_{\iota})_{\iota \in I}$ is a subnet of the upward directed set of all projections of $J$.

Now, the upward directed set of all projections $f$ of $J$ is an increasing approximate unit for $J$. Indeed, $\{ x \in J ; \lim_{f} \| x(1_M - f) \| = 0 \}$ is a norm-closed linear subspace of $J$ containing all projections from $J$, hence it is equal to $J$. Thus also the subnet $(e_{\iota})_{\iota \in I}$ is an approximate unit for $J$.

On the other hand, the norm-closed linear subspace $\{ b \in B ; \lim_{k} \| e_\iota b - be_\iota \| = 0 \}$ contains every projection from $B$: for any projection $p \in B$ and every $\iota = (f, P)$ with $p \in C^*(P)$ we have $e_\iota \in P' \cap A = C^*(P)' \cap J$, so $e_\iota p - pe_\iota = 0$. Consequently the above subspace of $B$ is actually equal to $B$.

**Proof of Theorem 4**. Put $y_1 = \frac{1}{2} (y + y^*)$, $y_2 = \frac{1}{2i} (y - y^*)$ and

$$p_j(\lambda) = \text{ support of } (y_j - \lambda 1_M) \text{ in } M , \quad \lambda \in \mathbb{R} .$$

Let further $\{ \lambda_1, \lambda_2, \ldots \}$ be the countable set of all rational numbers. Then

$$a = \sum_{k=1}^{\infty} 3^{-(2k-1)} (2p_1(\lambda_k) - 1_A^{\ast\ast}) + \sum_{k=1}^{\infty} 3^{-2k} (2p_2(\lambda_k) - 1_A^{\ast\ast}) + \frac{1}{2} 1_A^{\ast\ast} \in M ,$$

$$0 \leq a \leq 1_M$$
and it is easy to see that the $C^*$-subalgebra of $M$ generated by $a$ and $1_M$ contains all projections $p_j(\lambda), j = 1, 2, \lambda \in \mathbb{Q}$, hence also $y = y_1 + iy_2$. Therefore there exists a continuous function $f : [0, +\infty) \to \mathbb{C}$ such that $y = f(a)$. Furthermore, by a well known continuity property of the functional calculus (see e.g. [S-Z], 1.18 (5)), there exists some $\delta > 0$ such that

$$0 \leq b \in M, \|a - b\| \leq \delta \implies \|f(a) - f(b)\| \leq \varepsilon.$$ 

Now, by Lemma 5, there exists an increasing sequence $(f_k)_{k \geq 1}$ of projections in $\mathcal{J}$, whose least upper bound in $M$ is $1_M$. Using Lemma 6, we can then construct by induction a sequence $0 = e_o \leq e_1 \leq e_2 \leq \ldots$ of projections in $\mathcal{J}$ such that

$$f_k \leq e_k, \quad \|e_k - ae_k\| \leq 2^{-k-1}\delta.$$ 

Since the elements $e_k$ and $(e_k - e_{k-1})a(e_k - e_{k-1})$ of $\mathcal{J}$ are mutually commuting, there exists a masa $C$ of $\mathcal{J}$ containing all of them. Then $C$ contains the increasing projection sequence $(e_k)_{k \geq 1}$, whose least upper bound in $M$ is $1_M$.

Let us denote

$$b_o = a, \quad b_n = \sum_{k=1}^{n} (e_k - e_{k-1})a(e_k - e_{k-1}) + (1_M - e_n)a(1_M - e_n), \quad n \geq 1.$$ 

Then, for every $n \geq 1$,

$$b_{n-1} - b_n = (1_M - e_n)(1_M - e_n)(1_M - e_n) = (1_M - e_n) \cdot [e_n, e_n - ae_n] \cdot (1_M - e_n),$$

$$\|b_{n-1} - b_n\| \leq 2\|e_n - ae_n\| \leq 2^{-n}\delta.$$ 

It follows that $\sum_{n=1}^{\infty} \|b_{n-1} - b_n\| \leq \delta$, so the sequence $(b_n)_{n \geq 1}$ is norm convergent to some $b \in M(A)^+$ and

$$\|a - b\| = \lim_{n \to \infty} \|b_o - b_n\| \leq \delta.$$ 

Put $x = f(b)$.

We claim that $b \in C' \cap M$, hence also $x \in C' \cap M$. Since $C' \cap M$ is a masa of $M$ (see Lemma 1 (iv)), it is enough to prove that $b$ is commuting with all elements $a' \in C' \cap M \subset \{e_k, (e_k - e_{k-1})a(e_k - e_{k-1}); k \geq 1\}' \cap M$. For we notice that, for every $n \geq 1$,

$$b_n a' - a'b_n = (1_M - e_n)(aa' - a'a)(1_M - e_n),$$

hence

$$|b_n a' - a'b_n|^2 \leq (1_M - e_n)|aa' - a'a|^2(1_M - e_n) \leq \|aa' - a'a\|^2(1_M - e_n).$$

Therefore

$$|b_n a' - a'b_n|^2 \leq \|aa' - a'a\|^2(1_M - e_k), \quad n \geq k \geq 1$$

and, passing to limit for $n \to \infty$, we get for every $k \geq 1$

$$|ba' - a'b|^2 \leq \|aa' - a'a\|^2(1_M - e_k),$$
support of $|ba' - a'b|^2$ in $M$ is $\leq 1_M - e_k$.

Since the least upper bound of $(e_k)_{k \geq 1}$ in $M$ is $1_M$, it follows that $ba' - a'b = 0$.

Finally, according to the choice of $\delta$, $\|a - b\| \leq \delta$ implies that

$$\|y - x\| = \|f(a) - f(b)\| \leq \varepsilon.$$  

On the other hand,

$$a - b_n = \sum_{k=1}^{n} (b_{k-1} - b_k) = \sum_{k=1}^{n} (1_M - e_{k-1}) \cdot [e_k, e_ka - a\varepsilon_k] \cdot (1_M - e_{k-1}) \in J$$

implies by passing to the limit for $n \to \infty$ that $a - b \in J$. Using the Weierstrass Approximation Theorem, we infer that $y - x = f(a) - f(b) \in J$.

\[\Box\]

We shall prove that in Theorem 4 the element $x$ can be found under the form of an “infinite linear combination” of a sequence of mutually orthogonal projections from $J$. To this aim we need an appropriate understanding of the summation of series in Rickart $C^*$-algebras.

We recall that every commutative Rickart $C^*$-algebra $C$ is sequentially monotone complete (see e.g. [S-Z], 9.16, Proposition 1). Thus, if $(a_k)_{k \geq 1}$ is a sequence in $C^+$ such that the partial sums $\sum_{n=1}^{\infty} a_k, n \geq 1$, are bounded, then there exists the least upper bound in $C^+$

$$\sum_{k=1}^{\infty} a_k = \sup \left\{ \sum_{k=1}^{n} a_k; n \geq 1 \right\} \in C^+.$$

Let next $M$ be an arbitrary Rickart $C^*$-algebra, $(a_k)_{k \geq 1}$ a bounded sequence in $M^+$ such that the supports $s(a_k), k \geq 1$, are mutually orthogonal, and $(e_k)_{k \geq 1}$ a sequence of mutually orthogonal projections in $M$, for which $s(a_k) \leq e_k, k \geq 1$ (we can take, for example, $e_k = s(a_k)$). Then $\{a_k; k \geq 1\} \cup \{e_k; k \geq 1\}$ generates a commutative Rickart $C^*$-subalgebra $C$ of $M$, so there exists $a = \sum_{k=1}^{\infty} a_k \in C^+$. Moreover, $a$ is the least upper bound of the partial sums $\{\sum_{k=1}^{n} a_k; n \geq 1\}$ even in $M_h$. Indeed, by the $\sigma$-normality of the Rickart $C^*$-algebras, $\bigvee_{k=1}^{\infty} e_k$ is the least upper bound in $M_h$ of the sequence $(\bigvee_{k=1}^{n} e_k)_{n \geq 1}$ and it follows that

$$a = a^{1/2} \left( \bigvee_{k=1}^{\infty} e_k \right) a^{1/2}$$

is the least upper bound in $M_h$ of the increasing sequence $a^{1/2} \left( \bigvee_{k=1}^{n} e_k \right) a^{1/2} = \sum_{k=1}^{n} a_k, n \geq 1$

(see [S-Z], 9.14, the remark after Proposition 3). In particular, $a$ is the only element of $M_h$ satisfying the conditions

$$a e_k = a_k, k \geq 1, \quad s(a) \leq \bigvee_{k=1}^{\infty} e_k.$$

For sake of completeness we notice that, by the above characterization, if $(e_k)_{k \geq 1}$ is a sequence of mutually orthogonal projections in $M$, then $\sum_{k=1}^{\infty} e_k = \bigvee_{k=1}^{\infty} e_k$. 


Now let \( (x_k)_{k \geq 1} \) be a bounded sequence in \( M \) such that, denoting by \( I(x_k) \) the left support of \( x_k \) and by \( r(x_k) \) the right one, the projections \( I(x_k) \lor r(x_k), k \geq 1 \), are mutually orthogonal. Then we can define

\[
\sum_{k=1}^{\infty} x_k = \left( \sum_{k=1}^{\infty} (\text{Re } x_k)_+ - \sum_{k=1}^{\infty} (\text{Re } x_k)_- \right) + i \left( \sum_{k=1}^{\infty} (\text{Im } x_k)_+ - \sum_{k=1}^{\infty} (\text{Im } x_k)_- \right).
\]

It is easy to see that, if \( (e_k)_{k \geq 1} \) is any sequence of mutually orthogonal projections in \( M \) such that \( I(x_k) \lor r(x_k) \leq e_k, k \geq 1 \), then \( \sum_{k=1}^{\infty} x_k \) is the only element \( x \in M \), for which

\[
(*) \quad x e_k = e_k x = x_k, \quad k \geq 1, \quad I(x) \lor r(x) \leq \bigvee_{k=1}^{\infty} e_k.
\]

By the aboves, considering the direct product \( C^* \)-algebra

\[
\bigoplus_{k=1}^{\infty} e_k Me_k = \left\{ (y_k)_{k \geq 1} \in \prod_{k=1}^{\infty} e_k Me_k ; \sup_{k \geq 1} \|y_k\| < +\infty \right\},
\]

the mapping

\[
\bigoplus_{k=1}^{\infty} e_k Me_k \ni (y_k)_{k \geq 1} \longrightarrow \sum_{k=1}^{\infty} y_k \in M
\]

is well defined and it is an injective \(*\)-homomorphism. Consequently

\[
(**) \quad \left\| \sum_{k=1}^{\infty} x_k \right\| = \sup_{k \geq 1} \|x_k\|.
\]

Finally, let \( (e_k)_{k \geq 1} \) be a sequence of mutually orthogonal projections in \( M \), and \( (x_k)_{k \geq 1}, (y_k)_{k \geq 1} \in \bigoplus_{k=1}^{\infty} e_k Me_k \). Denoting by \( \overline{\text{lin}} \{ x_k - y_k ; k \geq 1 \} \) the norm-closed linear subspace of \( M \) generated by \( \{ x_k - y_k ; k \geq 1 \} \), we have

\[
(***) \quad \sum_{k=1}^{\infty} x_k - \sum_{k=1}^{\infty} y_k \in \overline{\text{lin}} \{ x_k - y_k ; k \geq 1 \} \quad \text{if} \quad \|x_k - y_k\| \longrightarrow 0.
\]

Indeed, according to (**), we have:

\[
\left\| \sum_{k=1}^{\infty} x_k - \sum_{k=1}^{n} y_k - \sum_{k=1}^{n} (x_k - y_k) \right\| = \sup_{k \geq n+1} \|x_k - y_k\| - \frac{n \to \infty}{0}.
\]

A slight modification of the proof of Theorem 4 yields the following Weyl-von Neumann-Berg-Sikonia type result, which is much closer to [Z], Theorem 3.1 than Theorem 4:

**Theorem 5.** Let \( M \) be a unital Rickart \( C^* \)-algebra, and \( \mathcal{J} \) a norm-closed two-sided ideal of \( M \), which contains a sequence of positive elements such that \( 1_M \) is the only projection in \( M \) majorizing the sequence. Then, for any normal \( y \in M \) and every \( \varepsilon > 0 \), there are

- a sequence \( (p_k)_{k \geq 1} \) of mutually orthogonal projections in \( \mathcal{J} \),
- a sequence \( (\lambda_k)_{k \geq 1} \) in the spectrum \( \sigma(y) \) of \( y \),

such that

1) the least upper bound of \( (p_n)_{n \geq 1} \) in \( M \) is \( 1_M \),
2) \( y - \sum_{k=1}^{\infty} \lambda_k p_k \in J \) and \( \| y - \sum_{k=1}^{\infty} \lambda_k p_k \| \leq \varepsilon \).

**Proof.** Repeating word for word the arguments from the first paragraph of the proof of Theorem 4, we get \( a \in M \) with \( 0 \leq a \leq 1_M \), a continuous function \( f : [0, +\infty) \to \mathbb{C} \) and \( \delta > 0 \), such that \( y = f(a) \) and

\[(\dagger) \quad 0 \leq b \in M, \|a - b\| \leq \delta \implies \|f(a) - f(b)\| \leq \varepsilon .\]

Subtracting from \( a \) an appropriate positive multiple of \( 1_M \) and modifying \( f \) correspondingly, if necessary, we can assume that \( 0 \in \sigma(a) \).

Choose a sequence \( \delta/3 = \delta_1 > \delta_2 > \ldots > 0 \) which converges to 0. According to the upper semicontinuity of the spectrum, there exist

\[ \eta_1 > \eta_2 > \ldots > 0 \]
\[ \wedge \quad \wedge \]
\[ \delta/3 = \delta_1 > \delta_2 > \ldots \]

such that the spectrum of every \( b \in M \) with \( \|a - b\| \leq \eta_k \) is contained in

\[ U_{\delta_k}(\sigma(a)) = \{ \mu \in \mathbb{C}; |\mu - \lambda(\mu)| < \delta_k \text{ for some } \lambda(\mu) \in \sigma(a) \}. \]

Arguing now again as in the proof of Theorem 4, we can construct a sequence \( 0 = e_0 \leq e_1 \leq e_2 \leq \ldots \) of projections in \( J \), whose least upper bound in \( M \) is \( 1_M \), such that

\[ \|e_k a - ae_k\| \leq 2^{-k-1} \eta_{k+1} \text{ for all } k \geq 1. \]

Setting then

\[ b_o = a, \]
\[ b_n = \sum_{k=1}^{n}(e_k - e_{k-1})a(e_k - e_{k-1}) + (1_M - e_n)a(1_M - e_n), \quad n \geq 1, \]

we have

\[ b_{n-1} - b_n = (1_M - e_{n-1}) \cdot [e_n, e_n a - ae_n] \cdot (1_M - e_{n-1}), \quad n \geq 1, \]

so \( \|b_{n-1} - b_n\| \leq 2^{-n}\eta_{n+1} \leq 2^{-n}\delta/3 \) and \( b_{n-1} - b_n \in J \). Therefore the sequence \( (b_n)_{n \geq 1} \) is norm convergent to some \( b_\infty \in M^+ \), for which \( \|a - b_\infty\| \leq \delta/3 \) and \( a - b_\infty \in J \).

We claim that

\[ b_\infty = \sum_{k=1}^{\infty}(e_k - e_{k-1})a(e_k - e_{k-1}). \]

Indeed, since

\[ b_n(e_k - e_{k-1}) = (e_k - e_{k-1})b_n = (e_k - e_{k-1})a(e_k - e_{k-1}), \quad n \geq k \geq 1, \]

by passing to the limit for \( n \to \infty \) we get

\[ b_\infty(e_k - e_{k-1}) = (e_k - e_{k-1})b_\infty = (e_k - e_{k-1})a(e_k - e_{k-1}), \quad k \geq 1. \]

Thus, taking into account that \( \bigvee_{k=1}^{\infty}(e_k - e_{k-1}) = \bigvee_{k=1}^{\infty} e_k = 1_M \), the description \((*)\) yields the desired equality.

We notice that, for every \( k \geq 1 \),

\[ (\infty) \quad \sigma((e_k - e_{k-1})a(e_k - e_{k-1})) \subset U_{\delta_k}(\sigma(a)). \]
Indeed, since the norm of
\[
a - \left( (e_k - e_{k-1})a(e_k - e_{k-1}) + (1_{A^*} - (e_k - e_{k-1}))a(1_{A^*} - (e_k - e_{k-1})) \right)
\]
was majorized by \(2 \left( \| e_k a - a e_k \| + \| e_{k-1} a - a e_{k-1} \| \right) \leq 2 \left( 2^{-k-2} \eta_{k+1} + 2^{-k-1} \eta_k \right) < \eta_k \),
by the choice of \( \eta_k \) we have
\[
\sigma \left( (e_k - e_{k-1})a(e_k - e_{k-1}) \right) \\
\subset \sigma \left( (e_k - e_{k-1})a(e_k - e_{k-1}) + (1_{A^*} - (e_k - e_{k-1}))a(1_{A^*} - (e_k - e_{k-1})) \right) \cup \{ 0 \}
\subset U_{\delta_k} (\sigma(a)) .
\]

For any \( k \geq 1 \), let \([r_1^{(k)}, r_2^{(k)}]\) denote the smallest compact interval in \( \mathbb{R} \) containing the spectrum \( \sigma \left( (e_k - e_{k-1})a(e_k - e_{k-1}) \right) \). Choose
\[
r_1^{(k)} = \mu_1^{(k)} < \ldots < \mu_j^{(k)} < \ldots < \mu_{j_k}^{(k)} = r_2^{(k)}
\]
in \( \sigma \left( (e_k - e_{k-1})a(e_k - e_{k-1}) \right) \) such that \( |\mu_j^{(k)} - \mu_{j-1}^{(k)}| \leq \eta_k \) for all \( 2 \leq j \leq j_k \). Then there exist mutually orthogonal projections \( (p_j^{(k)})_{1 \leq j \leq j_k} \) in \( \mathcal{J} \) such that
\[
\sum_{j=1}^{j_k} p_j^{(k)} = e_k - e_{k-1} \text{ and } \left\| (e_k - e_{k-1})a(e_k - e_{k-1}) - \sum_{j=1}^{j_k} \mu_j^{(k)} p_j^{(k)} \right\| \leq \eta_k .
\]

For example, we can set \( p_j^{(k)} = e_j^{(k)} - e_{j+1}^{(k)} , 1 \leq j \leq j_k \), where
\[
e_j^{(k)} = s \left( \left( (e_k - e_{k-1})a(e_k - e_{k-1}) - \mu_j^{(k)} (e_k - e_{k-1}) \right)_+ \right), \quad 1 \leq j \leq j_k
\]
and \( e_{j_k+1}^{(k)} = 0 \) (see e.g. [S-Z], 9.9, Proposition 1). Using \( \langle \infty \rangle \), we can find for every \( \mu_j^{(k)} \) some \( \lambda_j^{(k)} \in \sigma(a) \) with \( |\lambda_j^{(k)} - \mu_j^{(k)}| < \delta_k \) and then
\[
\left\| (e_k - e_{k-1})a(e_k - e_{k-1}) - \sum_{j=1}^{j_k} \lambda_j^{(k)} p_j^{(k)} \right\| \leq \eta_k + \delta_k < 2 \delta_k \leq 2 \delta/3 .
\]

Now \( \bigcup_{k=1}^{\infty} \{ p_j^{(k)} ; 1 \leq j \leq j_k \} \) consists of mutually orthogonal projections in \( M \), whose least upper bound in \( M \) is \( 1_M \), while \( \bigcup_{k=1}^{\infty} \{ \lambda_j^{(k)} ; 1 \leq j \leq j_k \} \subset \sigma(a) \). Set \( b = \sum_{k=1}^{j_k} \sum_{j=1}^{j_k} \lambda_j^{(k)} p_j^{(k)} \in M^+ \). Then \( \langle \infty \rangle \) yields
\[
\| b - b_\infty \| = \left\| \sum_{k=1}^{\infty} (e_k - e_{k-1})a(e_k - e_{k-1}) - \sum_{k=1}^{\infty} \sum_{j=1}^{j_k} \lambda_j^{(k)} p_j^{(k)} \right\| = \sup_{k \geq 1} \left\| (e_k - e_{k-1})a(e_k - e_{k-1}) - \sum_{j=1}^{j_k} \lambda_j^{(k)} p_j^{(k)} \right\| \leq 2 \delta/3 ,
\]
so \( \| a - b \| \leq \| a - b_\infty \| + \| b_\infty - b \| \leq \delta/3 + 2 \delta/3 = \delta \). On the other hand, since
\[
\left\| (e_k - e_{k-1})a(e_k - e_{k-1}) - \sum_{j=1}^{j_k} \lambda_j^{(k)} p_j^{(k)} \right\| < 2 \delta_k \longrightarrow 0 ,
\]
\( \in \mathcal{J} \)

\( \langle \infty \rangle \) implies that \( b_\infty - b \in \mathcal{J} \), hence \( a - b = (a - b_\infty) + (b_\infty - b) \in \mathcal{J} \).
Using the characterization (*), it is easy to deduce that

\[ f(b) = \sum_{k=1}^{\infty} \sum_{j=1}^{j_k} f(\lambda_j^{(k)}) p_j^{(k)}, \]

where, by the Spectral Mapping Theorem, \( \bigcup_{k=1}^{\infty} \{ f(\lambda_j^{(k)}) : 1 \leq j \leq j_k \} \) is contained in \( f(\sigma(a)) = \sigma(f(a)) = \sigma(y) \). On the other hand, \( \langle \sigma \rangle \) yields the norm estimation

\[ \|y - f(b)\| = \|f(a) - f(b)\| \leq \varepsilon. \]

Finally, using \( a - b \in \mathcal{J} \) and the Weierstrass Approximation Theorem, we infer also that \( y - f(b) = f(a) - f(b) \in \mathcal{J} \).

\[ \square \]

If in the above theorem we are not requiring the norm estimation in 2), then the coefficients \( \lambda_k \) can be chosen even in the essential spectrum of \( y \) modulo \( \mathcal{J} \):

**Theorem 6.** Let \( M \) be a unital Rickart \( C^* \)-algebra, and \( \mathcal{J} \) a norm-closed two-sided ideal of \( M \), which contains a sequence of positive elements such that \( 1_M \) is the only projection in \( M \) majorizing the sequence. For any normal \( y \in M \) there are

- a sequence \( (p_k)_{k \geq 1} \) of mutually orthogonal projections in \( \mathcal{J} \),
- a sequence \( (\lambda_k)_{k \geq 1} \) in the spectrum \( \sigma_{\mathcal{J}}(y) \) of the canonical image of \( y \) in the quotient \( C^* \)-algebra \( M/\mathcal{J} \)

such that

1) the least upper bound of \( (p_n)_{n \geq 1} \) in \( M \) is \( 1_M \),

2) \( y - \sum_{k=1}^{\infty} \lambda_k p_k \in \mathcal{J} \).

For the proof we need the next lifting result, which is essentially [Z], Proposition 2.1:

**Lemma 7.** Let \( M \) be a unital Rickart \( C^* \)-algebra, and \( \mathcal{J} \) a norm-closed two-sided ideal of \( M \). For any self-adjoint \( a \in M \) there exists a self-adjoint \( b \in M \) such that \( \sigma(b) = \sigma_{\mathcal{J}}(b) \) and \( a - b \in \mathcal{J} \).

**Proof.** A moment’s reflection shows that the proof of [Z], Proposition 2.1 works for \( M \) unital Rickart \( C^* \)-algebra instead of \( W^* \)-algebra.

\[ \square \]

**Proof of Theorem 6.** Repeating again the arguments from the first paragraph of the proof of Theorem 4, we get some \( a \in M \) with \( 0 \leq a \leq 1_M \) and a continuous function \( f : [0, +\infty) \to \mathbb{C} \) such that \( y = f(a) \). Now, according to Lemma 7, there exists a self-adjoint \( b \in M \) such that \( \sigma(b) = \sigma_{\mathcal{J}}(b) \) and \( a - b \in \mathcal{J} \). In particular, \( \sigma(b) = \sigma_{\mathcal{J}}(a) \subset [0, 1] \), and so \( 0 \leq b \leq 1_M \).

Let \( x \) denote the normal element \( f(b) \). Using the Weierstrass Approximation Theorem, we infer that \( y - x \in \mathcal{J} \), hence, by the Spectral Mapping Theorem, we have \( \sigma(x) = f(\sigma(b)) = f(\sigma_{\mathcal{J}}(a)) = \sigma_{\mathcal{J}}(y) \). Now Theorem 5 yields the existence of

- a sequence \( (p_k)_{k \geq 1} \) of mutually orthogonal projections in \( \mathcal{J} \),
- a sequence \( (\lambda_k)_{k \geq 1} \) in \( \sigma(x) = \sigma_{\mathcal{J}}(y) \),

such that the least upper bound of \( (p_n)_{n \geq 1} \) in \( M \) is \( 1_M \) and \( x - \sum_{k=1}^{\infty} \lambda_k p_k \in \mathcal{J} \). Then \( y - \sum_{k=1}^{\infty} \lambda_k p_k = (y - x) + (x - \sum_{k=1}^{\infty} \lambda_k p_k) \in \mathcal{J} \).

\[ \square \]
Let us say that a $C^*$-algebra $A$ is $\sigma$-subunital if there exists a sequence $(b_n)_{n \geq 1}$ in $A^+$, whose least upper bound in $M(A)_h$ is $1_{A^{**}}$. Clearly, if $A$ is $\sigma$-unital then it is $\sigma$-subunital. For commutative $A$ the two notions coincide. However, if $M$ is a countably decomposable type $\Pi_\infty$-factor and $A$ is the norm-closed linear span of all finite projections of $M$, then $A$ is not $\sigma$-unital (see [Ak-Ped], Prop. 4.5), but it is easily seen that it is $\sigma$-subunital.

We remark that the sequence $(b_n)_{n \geq 1}$ in the definition of the $\sigma$-subunitality can be considered a kind of “approximate unit with respect to the order structure”. Indeed, according to [S-Z], 9.14, the remark after Proposition 3, if the least upper bound of $(b_n)_{n \geq 1}$ in $M(A)_h$ is $1_{A^{**}}$ and $x \in M(A)$, then the least upper bound of the sequence $(x^*b_nx)_{n \geq 1}$ in $M(A)_h$ is $x^*x$.

By Theorems 5 and 6 we have:

**Corollary.** Let $A$ be a $\sigma$-subunital $C^*$-algebra, whose multiplier algebra $M(A)$ is a Rickart $C^*$-algebra. For any normal $y \in M(A)$ and any $\varepsilon > 0$ there exist
- a sequence $(p_k)_{k \geq 1}$ of mutually orthogonal projections in $A$,
- a sequence $(\lambda_k)_{k \geq 1}$ in the spectrum $\sigma(y)$ of $y$,

such that

1) the least upper bound of $(p_n)_{n \geq 1}$ in $M(A)_h$ is $1_{A^{**}}$,

2) $y - \sum_{k=1}^{\infty} \lambda_k p_k \in A$ and $\left\| y - \sum_{k=1}^{\infty} \lambda_k p_k \right\| \leq \varepsilon$.

Moreover, if we don’t require the second inequality in 2), then the sequence $(\lambda_k)_{k \geq 1}$ can be chosen even in the spectrum of the canonical image of $y$ in the corona algebra $C(A) = M(A)/A$.

□

In particular, the above corollary can be applied to $A = K(H)$, where $H$ is a separable complex Hilbert space, in which case the series $\sum_{k=1}^{\infty} \lambda_k p_k$ converges even with respect to the strict topology of $M(A) = B(H)$. This is the statement of the classical Weyl-von Neumann-Berg-Sikonia Theorem, but convergence with respect to the strict topology is used also in its subsequent extensions to $\sigma$-unital $C^*$-algebras with real rank zero multiplier algebra (see e.g. [M], [Br-Ped], [Zh], [H-Ro], [L1], [L2], [L3]).

On the other hand, in the early extension from [Z] of the Weyl-von Neumann-Berg-Sikonia Theorem to the norm-closed linear span $A$ of all finite projections of an arbitrary semifinite $W^*$-factor $M$, which for $M$ of type $\Pi_\infty$ turns out to be not $\sigma$-unital, the series $\sum_{k=1}^{\infty} \lambda_k p_k$ is proved to converge only with respect to the $s^*$-topology. The reason, why here a weaker topology than the strict topology should be used, is given by Theorem 3: if $M$ is a type $\Pi_\infty W^*$-factor and we assume that a sum $\sum_{k=1}^{\infty} \lambda_k p_k$ with $p_k \in A$ is strictly convergent, then, according to Theorem 3, we must have $\sum_{k=1}^{\infty} \lambda_k p_k \in A$. 

Appendix

We give here, for the convenience of the reader, a treatment of a set-theoretical result of T. Iwamura (see [Ma], Appendix II) and two applications to the theory of $AW^*$-algebras.

**Proposition.** Let $I, \leq$ be an upward directed partially ordered uncountable set. Then, there exist a well order $\preceq$ on $I$ and a family $(I_\iota)_{\iota \in I}$ of subsets of $I$ such that
- $I_\iota$ is upward directed for every $\iota \in I$,
- $\text{card } I_\iota < \text{card } I$, $\iota \in I$,
- $I_{\iota_1} \subset I_{\iota_2}$ whenever $\iota_1 \prec \iota_2$,
- $\bigcup_{\iota \in I} I_\iota = I$.

**Proof.** By Zermelo’s theorem there exists a well order $\preceq$ on $I$. We can choose it such that

$$(*) \quad \text{card } \{ \iota' \in I; \iota' \prec \iota \} < \text{card } I \text{ for every } \iota \in I.$$ 

Indeed, if there exists some $\iota \in I$ such that

$$\text{card } \{ \iota' \in I; \iota' \prec \iota \} = \text{card } I,$$

then there exists a smallest $\iota$ with respect to $\preceq$, having the above property. Choose for this $\iota$ a bijection

$$\Phi : I \to \{ \iota' \in I; \iota' \prec \iota \}$$

and replace $\preceq$ by the well order, according to which $\iota_1$ less or equal to $\iota_2$ means $\Phi(\iota_1) \preceq \Phi(\iota_2)$.

We notice that, $I$ being infinite, $(*)$ implies that $I$ does not contain a largest element with respect to $\preceq$.

Let us denote

$$J_\iota = \{ \iota' \in I; \iota' \prec \iota \}, \quad \iota \in I.$$ 

Then

$$\text{card } J_\iota < \text{card } I, \quad \iota \in I,$$

$$J_{\iota_1} \subset J_{\iota_2} \text{ whenever } \iota_1 \prec \iota_2,$$

$$\bigcup_{\iota \in I} J_\iota = I.$$ 

On the other hand, $I, \leq$ being upward directed, we can choose for each finite $F \subset I$ some $\iota(F) \in I$ such that

$$\iota \leq \iota(F) \text{ for all } \iota \in F.$$ 

Denote for every $J \subset I$

$$D_1(J) = J \cup \{ \iota(F); F \subset J \text{ finite } \}.$$ 

We notice that

$D_1(J)$ is finite for $J$ finite,

$$\text{card } D_1(J) = \text{card } J \text{ for } J \text{ infinite}$$

and

$$D_1(J_1) \subset D_1(J_2) \text{ whenever } J_1 \subset J_2.$$
Now we define by recursion
\[ D_{n+1}(J) = D_1(D_n(J)) \supset D_n(J), \quad n \geq 1 \text{ integer}, \]
\[ D_\omega(J) = \bigcup_{n \geq 1} D_n(J). \]
Then
\[ D_\omega(J) \text{ is countable for } J \text{ finite}, \]
\[ \operatorname{card} D_\omega(J) = \operatorname{card} J \text{ for } J \text{ infinite} \]
and
\[ D_\omega(J_1) \subset D_\omega(J_2) \text{ whenever } J_1 \subset J_2. \]
Moreover, \( D_\omega(J), \leq \) is upward directed for every \( J \subset I \).

Now, putting
\[ I_\iota = D_\omega(J_\iota), \quad \iota \in I, \]
it is easy to see that all conditions from the statement are satisfied. \( \square \)

The first corollary extends Lemma 3 (compare with [Be], §33, Exercise 1):

**Corollary 1.** Let \( M \) be an AW*-algebra, \( f \in M \) a finite projection, and \((e_\iota)_{\iota \in I}\) an upward directed family of projections in \( M \) such that
\[ e_\iota \prec f \text{ for all } \iota \in I. \]
Then
\[ \bigvee_{\iota \in I} e_\iota \prec f. \]

**Proof.** The case of countable \( I \) can be easily reduced to Lemma 6. Indeed, choosing a cofinal sequence \( \iota_1 \leq \iota_2 \leq \ldots \) in \( I \), we have
\[ \bigvee_{\iota \in I} e_\iota = \bigvee_{n \geq 1} e_{\iota_n} = e_{\iota_1} \vee \bigvee_{n \geq 1} (e_{\iota_n+1} - e_{\iota_n}) \]
and we can apply Lemma 3 to \( f \) and the family \( e_{\iota_1}, e_{\iota_2} - e_{\iota_1}, e_{\iota_3} - e_{\iota_2}, \ldots. \)

For the proof in the general case let \( f \in M \) be a finite projection and let us assume the existence of some upward directed family \((e_\iota)_{\iota \in I}\) of projections in \( M \) such that
\[ e_\iota \prec f \text{ for all } \iota \in I, \text{ but } \bigvee_{\iota \in I} e_\iota \neq f. \]
Choose among all such families one with \( I \) of the smallest cardinality. By the first part of the proof \( I \) is then uncountable.

Let the well order \( \preceq \) on \( I \) and the family \((I_\iota)_{\iota \in I}\) of subsets of \( I \) be as in the above proposition.

According to the minimality property of \( \operatorname{card} I \), we have
\[ p_\iota = \bigvee_{\iota' \in I_\iota} e_{\iota'} \prec f, \quad \iota \in I. \]
On the other hand,
\[ p_{\iota_1} \leq p_{\iota_2} \quad \text{whenever} \quad \iota_1 < \iota_2 , \]
\[
\bigvee_{\iota \in I} p_{\iota} = \bigvee_{\iota \in I} e_{\iota} .
\]
Consequently, denoting
\[
q_{\iota} = p_{\iota} - \bigvee_{\iota' \prec \iota} p_{\iota'} , \quad \iota \in I ,
\]
the projections \((q_{\iota})_{\iota \in I}\) are mutually orthogonal and
\[
\sum_{\iota \in F} q_{\iota} \prec f \quad \text{for any finite} \quad F \subset I .
\]
By Lemma 6 it follows that
\[
\bigvee_{\iota \in I} q_{\iota} \prec f .
\]
But
\[
\bigvee_{\iota \in I} q_{\iota} = \bigvee_{\iota \in I} p_{\iota} = \bigvee_{\iota \in I} e_{\iota} .
\]
Indeed, otherwise it would exist a smallest \(\iota \in I\) with respect to \(\preceq\) such that
\[(**)
\[
\bigvee_{\iota' \in I} q_{\iota'} .
\]
But then we would have
\[
\bigvee_{\iota'' \prec \iota} p_{\iota''} \leq \bigvee_{\iota' \in I} q_{\iota'} ,
\]
which contradicts \((**\).

For \(M\) an arbitrary \(AW^*\)-algebra and \(Z\) a commutative \(AW^*\)-algebra we call
\[
\Phi : \{ e \in M : e \text{ projection} \} \to Z^+ \]
normal if, for every upward directed family \((e_{\iota})_{\iota}\) of projections in \(M\), we have
\[
\Phi \left( \bigvee_{\iota} e_{\iota} \right) = \sup \Phi(e_{\iota}) ,
\]
where \(\sup\) denotes the least upper bound in \(Z^+\). Clearly,
\[\Phi \text{ normal } \Rightarrow \Phi \text{ completely additive},\]
but, using the above proposition similarly as in the proof of the Corollary 2, we get also the converse implication (which should be known, but for which we have no reference):

**Corollary 2.** Let \(M, Z\) be \(AW^*\)-algebras, \(Z\) commutative, and \(\Phi : \{ e \in M : e \text{ projection} \} \to Z^+\). Then
\[\Phi \text{ normal } \iff \Phi \text{ completely additive}.\]

In particular, the centre valued dimension function of a finite \(AW^*\)-algebra is normal (see [Be], §33, Exercise 4). Also, if \(M\) is a discrete \(AW^*\)-algebra and \(e \in M\) is an abelian projection of central support \(1_M\), then the map \(\Phi_e\) considered in the proof of Theorem 1 (on the abelian strict closure in discrete \(AW^*\)-algebras) is normal on the projection lattice of \(M\).
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