Statistical Query Algorithms and Low-Degree Tests Are Almost Equivalent

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Abstract

Researchers currently use a number of approaches to predict and substantiate information-computation gaps in high-dimensional statistical estimation problems. A prominent approach is to characterize the limits of restricted models of computation, which on the one hand yields strong computational lower bounds for powerful classes of algorithms and on the other hand helps guide the development of efficient algorithms. In this paper, we study two of the most popular restricted computational models, the statistical query framework and low-degree polynomials, in the context of high-dimensional hypothesis testing. Our main result is that under mild conditions on the testing problem, the two classes of algorithms are essentially equivalent in power. As corollaries, we obtain new statistical query lower bounds for sparse PCA, tensor PCA and several variants of the planted clique problem.

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1 Introduction

Information-computation tradeoffs are ubiquitous in high dimensional statistics. As the amount and quality of the data increase, inference and estimation tasks often require fewer computational resources, creating an information-computation gap between the signal-to-noise ratios at which the problem is information-theoretically solvable and at which computationally efficient algorithms are known. This phenomenon is widespread, appearing in estimation of a sparse vector from linear observations, low-rank matrix estimation, sparse principal component analysis, subgraph recovery, random constraint satisfaction, dictionary learning, tensor completion, covariance estimation, phase retrieval, graph matching, and well beyond (c.f., [Don06, CRT06, FB96, CT07, LDP07, RFP10, JNS13, CMP10, RCLV13, JOH, CSV13, ACV14, ACBL12, Mon15, Fei02, JL09, BR13b, RBE10, SWW12, FHT08]). Tradeoffs between computational resources and statistical accuracy are also widely observed empirically in machine learning: both increasing model size and using more iterations of gradient descent to fit models to training data often improve generalization [JT18, SHN+18, NKB+19, KMH+20]. However, we lack a comprehensive theory that explains or predicts information-computation gaps.

In classical complexity theory, computational (in)tractability is explained by organizing problems into equivalence classes via efficient reductions. While this approach has strong merits, it is challenging to carry out in statistical settings (as discussed at length in [BB20]). Despite recent advances (e.g. [BR13a, MW15, HWX15, BBH18, ZX18, BB19, BBH19, LZ20, BB20]), it’s too early to tell whether a complete theory of information-computation gaps based on reductions is possible.

Currently, the predominant form of rigorous evidence for information-computation gaps is lower bounds against restricted models of computation. Here, the goal is to characterize the signal-to-noise ratio needed by specific algorithms for estimation tasks, sometimes taking this as a proxy for the signal-to-noise ratio required by polynomial time algorithms more generally. So far, such lower bounds have typically been proved separately for each statistical estimation problem, for each distribution over data, and for each model of computation. For instance, consider the planted clique problem, where the goal is to find a clique of size \( k \) placed at random in random graph on \( n \) vertices. The problem is solvable by exhaustive search for \( k \gg \log n \), but all known polynomial-time algorithms require \( k = \Omega(\sqrt{n}) \); the planted clique conjecture postulates that the problem is computationally hard if \( k = o(\sqrt{n}) \). The foundational work [Jer92] showed lower bounds for Markov-Chain Monte-Carlo methods. [FK03] prove lower bounds against Lovász–Schrijver semidefinite programs, and lower bounds against stronger Sum-of-Squares semidefinite programs were developed later in [BHK+19, DM15, MPW15, HKP+18]. [FGR+17] rule out algorithms for a similar problem in the statistical query model, while [ABDR+18, Ros08, Ros14] study proof and circuit complexity. Most of these lower bounds rule out algorithms for any \( k = o(\sqrt{n}) \).

Taken together, these works constitute some evidence for the planted clique conjecture. However, the proliferation of lower bounds suggests a need for unifying principles, especially because this story is repeated for numerous statistical estimation problems: lower bounds against a variety of restricted computational models are proven independently, all usually pointing to the same signal-to-noise ratios tolerated by efficient algorithms. This appears to be a miracle: why, for so many distinct problems, should so many restricted computational models point to the same signal-to-noise thresholds for efficient algorithms? (E.g., \( k \geq \Omega(\sqrt{n}) \) for planted clique.) We ask:

Are some or all of these restricted models equivalent in power? Do lower bounds in some models imply lower bounds in others?

If a single class of algorithms were to turn out to be at least as powerful as any of the other popular computational models for an interesting class of statistics problems, then numerous lower
bounds could be replaced with a single bound. One might hope to achieve this objective by giving reductions between computational models, establishing a hierarchy among them and quelling the proliferation of lower bounds.

In this paper, we make a small step towards this goal. Under mild conditions, we establish the equivalence of two popular frameworks for lower bounds on restricted models of computation for high-dimensional hypothesis testing: statistical dimension and low-degree polynomials. Statistical dimension is closely related to statistical query (SQ) algorithms, and our results also show that algorithms based on low-degree polynomials are at least as powerful as SQ algorithms.

1.1 Hypothesis Testing and Models of Computation

Hypothesis Testing. We consider simple-versus-simple hypothesis testing problems in which we have one null distribution \(D_\emptyset\) over \(\mathbb{R}^n\), and a family of alternative distributions \(S = \{D_u\}_{u \in S}\) over the same space, with a prior distribution \(\mu\) on \(S\).

Under the null hypothesis \(H_0\) we are given samples \(x_1, \ldots, x_m \in \mathbb{R}^n\) generated independently according to \(D_\emptyset\), whereas under the alternative hypothesis \(H_1\) the samples are instead generated according to \(D_u\) for \(u \sim \mu\) (we often write \(u \sim S\)). The objective is to determine which hypothesis is correct. One example is the sparse principal component analysis problem (sparse PCA), where \(D_\emptyset = \mathcal{N}(0, I_n)\), \(S = \{D_u\}\) where for each \(u \in \mathbb{R}^n\) with \(||u|| = 1\) and \(\rho n\) nonzero entries, \(D_u = \mathcal{N}(0, I_n + 0.1 u u^\top)\), and \(\mu\) taken uniform over \(S\)—here, the testing problem amounts to detecting the presence of the sparse rank-one spike.

Testing problems are of great interest in their own right; moreover, to give a lower bound for an estimation problem, it is often sufficient to show that a related hypothesis testing problem is hard (see, e.g., [BB20] – estimation and testing are related similarly to search and decision in worst-case complexity).

Since we study a model of computation (low degree polynomials) which most naturally outputs real rather than Boolean values, we will use the following notion of a successful test between \(H_0, H_1\).

Definition 1.1 (\(\beta\)-distinguisher). We call a function \(p : \mathbb{R}^{n \times m} \to \mathbb{R}\) of \(m\) vectors \(x = x_1, \ldots, x_m \in \mathbb{R}^n\) an \(m\)-sample \(\beta\)-distinguisher for a testing problem \(D_\emptyset\) vs. \(S\) if \(|E_{x \sim D_\emptyset} p(x) - E_{u \sim S} E_{x \sim D_u} p(x)| \geq \beta \cdot \sqrt{\text{Var}_{x \sim D_\emptyset} p(x)}\). If \(\beta > 1\), we call \(p\) a good distinguisher.

A hypothesis test with small probability of error automatically furnishes a good distinguisher. The converse is not necessarily true; though one might naturally try to apply thresholding to a distinguisher to obtain a hypothesis test, a good distinguisher may have large variance under the alternative hypothesis \(H_1\), so there is only a one-sided error guarantee. Thus, from the perspective of lower bounds, ruling out the existence of a \(\beta\)-distinguisher in a restricted computational model is at least as strong as ruling out the existence of a small-error hypothesis test (in that model).

Low Degree Polynomials. Given \(m\) samples \(x = x_1, \ldots, x_m \in \mathbb{R}^n\), our first model of computation is allowed to output the value of any fixed polynomial \(p(x)\) of bounded degree, usually constant or logarithmic in \(m, n\). Note that this model allows polynomials in all \(m\) samples jointly, not just empirical averages over \(m\) samples of the form \(\frac{1}{m} \sum_{i=1}^m p(x_i)\).

An extraordinary variety of high-dimensional hypothesis testing algorithms boil down to evaluating low-degree polynomials: for example, most spectral algorithms, the method of moments, algorithms based on small-subgraph statistics, and message passing algorithms (see [KWB19, Hop18]).

1 As we discuss below, this problem is unlike planted clique in that the number of samples rather than the signal per sample governs information-theoretic and computational complexity.

2 Here, \(\beta > 1\) is chosen to guarantee bounded one-sided error under Chebyshev’s inequality.
And, although faster implementations are often possible, any degree-$k$ polynomial can be evaluated in time $(nm)^{O(k)}$ by evaluating all monomials.

A recent line of work characterizes the limitations of such algorithms by ruling out the existence of low-degree distinguishers: such lower bounds are now known in the computationally-hard regimes of planted clique [BHK+19], stochastic block model [HS17, BBKW19], sparse principal component analysis [DKWB19], tensor principal component analysis [KWB19], and more. Remarkably, excluding problems with unusual algebraic structure [HW20], the (non)existence of a low-degree distinguisher closely tracks the (non)existence of any known poly-time hypothesis test.

**Statistical Queries and Statistical Dimension.** Our second model of computation is the statistical query (SQ) model $\text{VSTAT}(m)$. $\text{VSTAT}(m)$ algorithms access a distribution $D$ over $\mathbb{R}^n$ via queries $\phi : \mathbb{R}^n \rightarrow [0,1]$ to an oracle. For each query $\phi$, the oracle returns $E_{x \sim D} \phi(x) + \zeta$, for an adversarially chosen $\zeta \in \mathbb{R}$ with $|\zeta| \leq \max(\frac{1}{m}, \sqrt{E[\phi(1-E[\phi])]}). This approximates $E_D \phi$ with the same accuracy as an $m$-sample empirical estimate under the guarantees of Bernstein’s inequality.

The SQ model was first proposed as a framework for designing noise-tolerant algorithms [FGR17], and is a popular restricted model of computation for studying information-computation tradeoffs (see e.g. [FGR+17, FPV18, DKS17], as well as numerous supervised learning problems). An algorithm which makes $q$ queries to $\text{VSTAT}(m)$ is a proxy for an algorithm running in time $q$ on $m$ samples, albeit an imperfect one, since (1) the queries $\phi$ need not be polynomial-time computable, and (2) each query $\phi$ is permitted to be a function of only a single sample (whereas a general polynomial time algorithm may be allowed to, for instance, compare pairs of samples).

We will treat the SQ model via statistical dimension, a complexity measure on hypothesis testing problems which implies lower bounds against SQ algorithms. Most existing SQ lower bounds are proved by analyzing one of a few possible notions of statistical dimension. We use a mild strengthening of the statistical dimension introduced by [FGR+17].

**Definition 1.2 (Statistical Dimension).** Let $D_\emptyset$ vs. $S$ be a testing problem with prior $\mu$. For $D_u \in S$, define the relative density $\overline{D}_u(x) = \frac{D_u(x)}{D_\emptyset(x)}$, and the inner product $\langle f, g \rangle = E_{x \sim D_\emptyset} f(x)g(x)$. The statistical dimension $\text{SDA}(S, \mu, m)$ measures tails of $\langle \overline{D}_u, \overline{D}_v \rangle - 1$ with $u, v$ drawn independently from $\mu$.

$$\text{SDA}(S, \mu, m) = \max \left\{ q \in \mathbb{N} : E_{u, v \sim \mu} \left| \langle \overline{D}_u, \overline{D}_v \rangle - 1 \right| | A \rangle \leq \frac{1}{m} \text{ for all events } A \text{ s.t. } \Pr_{u, v \sim \mu} (A) \geq \frac{1}{n} \right\}.$$  

Often we will write $\text{SDA}(m)$ or $\text{SDA}(S, m)$ when $S$ and/or $\mu$ are clear from context.

We offer some intuition about the definition, which may be opaque at first. The quantity $\langle \overline{D}_u, \overline{D}_v \rangle - 1$ is equivalent to $E_{x \sim D_u} \frac{\Pr_{D_u[x]}[x]}{\Pr_{D_\emptyset}[x]} - 1$; that is, the centered average of the likelihood ratio of $D_u$ to $D_\emptyset$ over samples from $D_u$. When this quantity is at least $\delta$, $D_u$ and $D_v$ may have common events that allow one to distinguish them both from $D_\emptyset$ with probability $\delta'$. The statistical dimension quantifies the measure of pairs of distributions (according to $\mu$) with no such common events.

In [FGR+17], it is shown that the statistical dimension is a lower bound on the query complexity of hypothesis testing with a VSTAT oracle.

**Theorem 1.3 (Theorem 2.7 of [FGR+17]).** Let $D_\emptyset$ be a null distribution and $S$ be a set of alternate distributions over $\mathbb{R}^n$. Then any (randomized) statistical query algorithm which solves the hypothesis testing problem of $D_\emptyset$ vs. $S$ with probability at least $(1 - \delta)$ requires at least $(1 - \delta)\text{SDA}(S, m)$ queries to $\text{VSTAT}(m/3)$ (corresponding to $m/3$ samples).

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1. We remark on technical differences between our setup and that of [FGR+17] in Appendices A.1 and A.2.
2. We extend their result to our notion of SDA via a near-identical argument in Appendix A.2.
1.2 Our Results

Our main result is a surprisingly tight equivalence, under mild conditions, between statistical dimension and the minimum degree of any good distinguisher.

Summarizing the discussion of running times and sample complexities above, we might hope to equate $m$-sample distinguishers of degree $k$ (which can be evaluated in time $(nm)^{O(k)}$) with $2^{O(k)}$-query VSTAT$(m)$ algorithms. To understand the conditions under which this is possible, we first observe that planted clique already furnishes a counterexample—a case where a single-query SQ algorithm exists but there is no corresponding low-degree distinguisher. Concretely, to detect a $k$-clique planted in a graph $G$ from $G(n, 1/2)$, for any $k \gg \log n$ it suffices to make the single query $\phi(G) = 1$ if $G$ contains a $k$-clique) to VSTAT$(4)$. By contrast, it is known that no degree $o(\log^2 n)$ polynomial successfully distinguishes for any $k < n^{1/2-\varepsilon}$ [BHK$^+$19].

The issue here is that there is a high-degree function of a single sample which solves planted clique—that function can be used as a statistical query. As a condition for equivalence between statistical dimension and low-degree distinguishers, therefore, we must insist that such high-degree one-sample distinguishers do not exist. Our main theorem applies under the following niceness condition, which asks for just slightly more: no high degree function of a very small number of samples is a nontrivial distinguisher.

While niceness rules out problems like planted clique (which is what we want), we will see that it allows “many-sample” problems such as sparse PCA—precisely the type of problems for which the SQ model can capture interesting information-computation gaps. After our main theorem statement (Remark 1.9) we describe a principled approach to transform one-shot problems like planted clique into many-sample problems, so that they can also be studied with our techniques.

Definition 1.4 (($\delta, k$)-nice). Fix a null distribution $D_\emptyset$ on $\mathbb{R}^N$. Call a function $p : \mathbb{R}^{N \times k} \rightarrow \mathbb{R}$ of $k$ vectors $x_1, \ldots, x_k \in \mathbb{R}^N$ $k$-purely high degree if it is orthogonal to all functions $f(x_1, \ldots, x_k)$ which have degree at most $k$ in one of $x_1, \ldots, x_k$—that is, $E_{x_1, \ldots, x_k \sim D_\emptyset} p(x_1, \ldots, x_k)f(x_1, \ldots, x_k) = 0$ for all such $f$. The testing problem $D_\emptyset, \{D_u\}_{u \in S}$ is $(\delta, k)$-nice if no $k$-purely high-degree function of $k$ samples is a $\delta$-distinguisher.

We emphasize that $(\delta, k)$-niceness concerns hardness of a testing problem when given very few samples—we typically think of $k = O(1)$ or $k = \text{polylog } N$. We will show that almost any reasonable multi-sample testing problem which is not too easy to solve with $k$ samples becomes nice after the addition of a small amount of noise. The following is stated for a coordinate-wise resampling noise process—it follows from standard arguments about noise operators and high-degree functions. In Section 5 we give versions allowing a broad class of noise processes (additive Gaussian noise, random restriction, etc.).

Fact 1.5 (See Theorem 5.2). Let $S = \{D_u\}, D_\emptyset$ be a testing problem on $\mathbb{R}^N$ and suppose that $D_\emptyset = D_\emptyset^{\otimes N}$ is a product distribution. Let $k \in \mathbb{N}$ and suppose that $S, D_\emptyset$ does not have a $k$-sample $C$-distinguisher. Let $S' = \{D'_u\}$, where to sample $x' \sim D'_u$ we first sample $x \sim D_u$ and then each coordinate $x_i$ is independently replaced with a fresh sample from $D$ with probability $p \in [0, 1]$. Then $D_\emptyset$ versus $S'$ is $(C(1 - p)^k, k)$-nice.

Many natural high-dimensional hypothesis testing problems are robust to noise (including the main examples we have mentioned so far), and remain qualitatively unchanged by the addition of some form of noise captured by our theorems. The typical effect is a small decrease in the signal-to-noise ratio in each sample. In typical applications, $C = O(1)$, and when working with $m$ samples we will want roughly $(m^{-k/2}, k)$-niceness, which we can achieve by taking $k \approx \log m$ and $\rho$ a small constant, so that $S$ and $S'$ are very similar. In this case, our main theorem will
lead to \((\log m)^2\)-degree distinguishers, whereas brute-force algorithms would correspond to degree \(Nm \gg (\log m)^2\) – with more refined definitions later on, in many cases (e.g. Planted Clique) we can avoid the logarithmic loss and replace \((\log m)^2\) with \(\log m\).

**Main Theorem.** We turn to our main theorem. On first reading we suggest the interpretation that \(m' = m\) and \(k\) is constant or logarithmic in \(m\).

**Theorem 1.6** (Main Theorem, see Theorem 3.1 and Theorem 4.1). Let \(D_{\emptyset}\) vs. \(S\) be an \((m^{-k/2}/4, k)\)-nice testing problem on \(\mathbb{R}^N\) for some even \(k > 0\).

1. If there is some \(0 \leq m' \leq m\) such that \(\text{SDA}(S, m') \leq \left(\frac{2m}{m'}\right)^{k/2}\) (in particular, if there is an \(\text{SQ}\) algorithm making \(o(2^{k/2})\) queries to \(\text{VSTAT}(m/3)\)), then there is a good \(4mk\)-sample distinguisher \(p\) which has degree \(d \leq k^2\) and

2. if there is a degree \(k\) function \(p\) which is a good \(m\)-sample distinguisher, then there exists \(m' \leq m\) such that \(\text{SDA}(S, m') \leq \left(\frac{2m}{m'}\right)^{O(k)}\) (e.g. \(\text{SDA}(S, m) \leq 2^{O(k)}\)).

Using Fact 1.5, we already see that Theorem 1.6 applies to any noisy testing problem. Even without adding noise, our next theorem shows that the guarantees of Theorem 1.6 apply to some problems with additional structure – for instance, if \(D_{\emptyset}\) and the \(D_u\)'s are all product distributions. (This is the case even though such problems may not be nice; we are still able to apply a variant of the proof of Theorem 1.6.) This leads to slightly tighter results, especially for problems where the difference between degree \(\log m\) and \(\text{poly}(\log m)\) distinguishers is important.

**Theorem 1.7** (Gaussian or Independent Coordinates, see Theorems 6.1 & 6.3). Let \(S = \{D_u\}, D_{\emptyset}\) be a testing problem on \(\mathbb{R}^N\) with one of the following structures:

- \(D_{\emptyset} = \mathcal{N}(0, I_N)\) is the standard Gaussian distribution and each \(D_u = \mathcal{N}(u, I_N)\) for some vector \(u \in \mathbb{R}^N\)
- \(D_{\emptyset}\) and all \(D_u\) are product measures on \(\{\pm 1\}^N\)

Let \(m, k \in \mathbb{N}\) with \(k \ll m\) and suppose that \(S, D_{\emptyset}\) has no \(k\)-sample \(2^k\)-distinguisher. Then the conclusion of Theorem 1.6 holds for \(S\) (with the upper bound on \(d\) in part 1 replaced by \(d \leq O(k)\)).

Even with the additional requirements, Theorem 1.7 captures numerous interesting problems – spiked matrix and tensor models, variants of random constraint satisfaction and linear equations, community detection, and beyond.

**Remark 1.8** (Simulation Arguments Are Lossy). A natural approach to prove a theorem like Theorem 1.6 would be to naively simulate \(\text{SQ}\) algorithms by low-degree distinguishers and vice versa. However, direct simulation arguments that we are aware of (for instance, taking each monomial in a low-degree distinguisher to be an \(\text{SQ}\) query) at best relate \(\text{SDA}(S, m)\) to low-degree distinguishers on \(\text{poly}(m)\) samples (or vice versa). By contrast, Theorem 1.6 translates between \(\text{SDA}(S, m)\) and low-degree distinguishers on approximately \(m\) samples – this is crucial for most applications, where information-computation gaps occur on the scale of \(m\) versus \(\text{poly}(m)\) samples.

We remark as well that the statistical dimension is a lower bound on the \(\text{SQ}\) complexity, but does not always offer a tight characterization. There are problems for which polynomial-query \(\text{VSTAT}\) \(\text{SQ}\) algorithms require polynomially more samples than suggested by the statistical dimension, for example, in random constraint satisfaction problems [FPV18]. Hence, sometimes a low-degree distinguishers may exist for \(m\) samples even if no polynomial-query \(\text{VSTAT}(m)\) algorithms exist.

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\(^5\)As mentioned above, Theorems 3.1 and 4.1 are stated in terms of a more refined notion of degree (defined in Section 2) which allows us in many cases to improve the bound to \(d \leq O(k)\), which is the best we can hope for.
and as a consequence simulation arguments will not tightly characterize the existence of low-degree distinguishers.

Our proof of Theorem 1.6 directly relates statistical dimension to the minimum degree of a distinguisher, without a simulation argument. We also give (Appendix B) a different proof of a slightly weaker version of part 1 of Theorem 1.6,\(^6\) which is based on a simulation-style argument (though it has a non-constructive component) of an algorithm making calls to VSTAT via a low-degree distinguisher without poly(m) losses.

**Remark 1.9 (One-Shot Versus Multi-Sample Problems).** Theorem 1.6 only applies to nice testing problems. In particular, niceness rules out many “one-shot” problems which are information-theoretically easy to solve with a single sample, such as the usual formulation of planted clique, where the SQ model does not make sense – the model originates in PAC learning, where having many independent samples is fundamental. By contrast, low-degree tests can still be formulated for one-shot problems.

To give evidence of hardness for a one-shot problem in the SQ framework, one must first formulate a multi-sample version. For instance, the SQ lower bounds of [FGR⁺17] for planted clique treat a “bipartite” version where each sample is the adjacency list of a node in a bipartite graph. These multi-sample formulations are often ad hoc, which is problematic, as the choice of multi-sample version can significantly affect the resulting statistical query complexity.

Based on Theorem 1.6, we propose a canonical approach to translate one-shot problems into nice many-sample problems: decrease the per-sample signal-to-noise ratio (e.g., clique size versus graph density in planted clique) until the resulting problem is information-theoretically unsolvable given \(O(1)\) independent samples, while simultaneously increasing the number of samples appropriately. For example, in a Gaussian model, one sample from \(\mathcal{N}(u, I)\) is equivalent to \(m\) samples from \(\mathcal{N}(\sqrt{m}u, I)\). In numerous cases – additively Gaussian models and planted clique, for example – this yields problems which are polynomial-time equivalent to the underlying one-shot problem (see Section 7). For an illustration, see the Tensor PCA problem discussed in and above Corollary 1.11.

### 1.2.1 Overview of Techniques

**Proof Sketch of Theorem 1.6.** We outline the proof of case (1) of our main theorem; case (2) follows a similar argument in reverse. We argue contrapositively, starting with the hypothesis that there is no good degree \(k^2\) \(m\)-sample distinguisher. For this sketch, we ignore the case \(m' < m\) and consider the goal of proving a lower bound on the statistical dimension \(\text{SDA}(S, m)\). Unpacking the definition of \(\text{SDA}\), this amounts to the tail bound \(\mathbb{E}_{u,v\sim S}[|\langle \overline{D}_u, \overline{D}_v \rangle - 1| | A] \lesssim 1/m\) for any event \(A\) of probability roughly \(2^{-k}\). This tail bound will be implied by an upper bound on the \(k\)-th moment – our goal will be to show \(\mathbb{E}_{u,v\sim S}(\langle \overline{D}_u, \overline{D}_v \rangle - 1)^k \lesssim m^{-k}\).

Simple manipulations (which rely on the independence of the samples) show that the maximum value of \(\alpha\) such that there is a \(k\)-sample \(\alpha\)-distinguisher is given by the related quantity \(\alpha = \sqrt{\mathbb{E}_{u,v\sim S}(\langle \overline{D}_u, \overline{D}_v \rangle)^k - 1}\). To see why, recall that a \(k\)-sample \(\beta\)-distinguisher is a function of \(k\) samples, \(p(x_1, \ldots, x_k)\) that satisfies \(\beta \cdot (\text{Var}_{D_u^{\otimes k}} p)^{1/2} \leq |\mathbb{E}_{u\sim S} \mathbb{E}_{D_u^{\otimes k}} p - \mathbb{E}_{D_u^{\otimes k}} p| = |\langle p, \mathbb{E}_u D_u^{\otimes k} - 1 \rangle_{D_u^{\otimes k}}|^7\). By rescaling we may without loss of generality consider \(p\) with \(\text{Var}_{D_u^{\otimes k}} p =\)

\(^6\)The quantitative bounds we obtain are identical to Theorem 1.6; the theorem is weaker because the existence of a VSTAT algorithm is a stronger assumption than an upper bound on the statistical dimension.

\(^7\)Here we have used the notation that for a distribution \(D\), \((f,g)_D = \mathbb{E}_{x\sim D} f(x)g(x)\) and \(D^{\otimes k}\) is the joint distribution of \(k\) random samples from \(D\), and for a function \(f(x), f^{\otimes k}(x_1, \ldots, x_k) = \prod_{i=1}^k f(x_i)\).
⟨p, p⟩_{D_2^k} = 1. So now by Cauchy-Schwarz and by the independence of the samples,

\[ \beta = \left| \langle p, E_{u \sim S} \overline{D}_u^{\otimes k} - 1 \rangle_{D_2^k} \right| \leq \sqrt{E_{u,v \sim S} \left| \langle \overline{D}_u^{\otimes k} - 1, \overline{D}_v^{\otimes k} - 1 \rangle_{D_2^k} \right|} = \sqrt{E_{u,v \sim S} \langle \overline{D}_u, \overline{D}_v \rangle_D^{k} - 1}, \]

where in the final step we have used that \( \overline{D}_u^{\otimes k} \) is a density and so \( \langle \overline{D}_u^{\otimes k}, 1 \rangle = 1 \), as well as the independence of the samples. By choosing the \( p \) for which the Cauchy-Schwarz is tight, we have our conclusion.

Thus, pretending for the sake of this overview that the \( k \)-th moment \( E_{u,v}((\overline{D}_u, \overline{D}_v) - 1)^k \approx E_{u,v}(\overline{D}_u, \overline{D}_v)^k - 1 \), to show that \( E_{u,v \sim S}((\overline{D}_u, \overline{D}_v) - 1)^k \lesssim m^{-k} \), it suffices for us to rule out \( k \)-sample \( m^{-k/2} \)-distinguishers. Since by assumption \( D_2 \) versus \( S \) is \( (m^{-k/2}, k) \) nice, such a distinguisher could not be \( k \)-purely high degree. Via a careful application of Hölder’s inequality (Lemma 3.4), we are able to show that it suffices to consider only functions of purely high degree or purely low degree. The main challenge is now to rule out a low-degree \( k \)-sample \( m^{-k/2} \) distinguisher – that is, we need to show that every function \( p(x_1, \ldots, x_k) \) with degree at most \( k \) in each sample \( x_i \) has

\[ \left| E_{u \sim S} E_{D_2^{\otimes k}} p - E_{D_2^{\otimes k}} p \right| \lesssim m^{-k/2} \sqrt{\text{Var}_{D_2^{\otimes k}} p}. \tag{1} \]

Since we are analyzing \( k \)-sample distinguishers, it is not \textit{a priori} clear how such a \( 1/ \text{poly}(m) \) bound on the distinguishing power can appear, especially given that \( m \gg k \). Our key insight is that this strong quantitative bound follows from the assumption that there is no good degree-\( k^2 \) \( m \)-sample distinguisher:

**Lemma 1.10** (Key Lemma, Informal – see Claim 3.3, Lemma 3.5). \textit{If there is no good \( m \)-sample degree-\( k^2 \) distinguisher for the testing problem \( D_2 \) versus \( S \), then no function \( p(x_1, \ldots, x_k) \) with degree at most \( k \) in each sample is an \( m^{-k/2} \)-distinguisher.}

Once the (very careful) setup is in place, this lemma follows from elementary Fourier analysis, exploiting independence of samples. Nonetheless, we find it striking that a relatively mild assumption on the distinguishing power of low degree polynomials of \( m \) samples can be boosted into a strong quantitative bound on the distinguishing power of low degree polynomials of \( k \ll m \) samples. This lemma leads to (1), finishing the proof.

**Niceness of Noise-Robust Problems.** To show that noise-robust testing problems satisfy the niceness criterion (Fact 1.5 and its generalizations in Section 5), we again use Fourier Analysis; for some types of noise our arguments are entirely standard, exploiting the attenuation of high-degree functions under i.i.d. noise. We also allow for noise processes which make sense for problems with combinatorial structure which would be adversely affected by i.i.d. coordinate-wise noise (e.g. hypergraph planted clique) – showing that these also lead to nice testing problems uses similar ideas but requires more care.

**Avoiding Niceness for Product and Gaussian Distributions.** Finally, we overview the proof of Theorem 1.7. We need to avoid the use of the niceness assumption that we described in the overview above of the proof of Theorem 1.6. That is, we need a different way to rule out high-degree \( k \)-sample \( m^{-k/2} \)-distinguishers. Roughly speaking, we show that under either the product or Gaussian assumptions, a high-degree \( k \)-sample \( \alpha \)-distinguisher cannot exist unless a low-degree one does – then we follow the argument above to rule out low-degree \( k \)-sample \( m^{-k/2} \) distinguishers. This argument turns on the fact that, for Gaussian and product distributions, high-degree moments are simple functions of low-degree moments. (See Lemmas 6.2 and 6.4 for the details.)
1.2.2 Applications: New Information-Computation Lower Bounds “For Free”

We use our equivalence theorems to obtain new information-computation lower bounds for a number of testing problems. We obtain new lower bounds against SQ algorithms for tensor PCA (Corollary 8.4), (Hypergraph) Planted Clique and Planted Dense Subgraph (8.14), and sparse PCA (8.22), and we obtain new lower bounds against low-degree distinguishers for Gaussian mixture models (8.29) and Gaussian Graphical Models (8.32). Our bounds are obtained essentially “for free” by starting with known SDA or degree lower bounds, then applying Theorem 1.6 and its derivatives. (One exception is the Gaussian Graphical Models bound, for which we prove an SQ lower bound from scratch. Interestingly, for this problem, it seems easier to prove SDA lower bounds than degree lower bounds.)

In the case of planted clique, in addition to capturing the “bipartite” model of [FGR+17], we also prove lower bounds for a new multi-sample version, in which we receive $m$ independent copies of the adjacency matrix of $G(n, p^{1/m})$ or $G(n, p^{1/m})$ with the same planted $k$-clique. We show in Lemma 7.3 that our version is information-theoretically and computationally equivalent to the standard version of planted clique (albeit with slightly higher-than-usual edge density $p > 1/2$), a property not shared by the bipartite model. This is an example of our approach to transforming one-sample problems into many-sample ones by weakening the per-sample signal-to-noise ratio.

For the sake of illustration, we state our result for Tensor PCA here, and defer formal statements of our lower bounds for the other problems to Section 8. Tensor PCA is a well-studied higher-order generalization of the principal components analysis problem (see e.g. [RM14, HSS15, LML+17, WEAM19, AGJ+20]). It is typically stated as a “one-shot” problem: distinguish a 3-tensor $G$ with i.i.d. entries from $\mathcal{N}(0, 1)$ from a planted tensor of the form $G + \lambda u \otimes^3$, where $G$ is as before, $\lambda > 0$, and $u$ is a unit vector. In Lemma 7.2 we show that this problem is in fact equivalent (both statistically and computationally) to the following $m$-sample problem: distinguish between i.i.d. $G_1, \ldots, G_m$ and $G_1 + \frac{1}{\sqrt{m}} u \otimes^3, \ldots, G_m + \frac{1}{\sqrt{m}} u \otimes^3$.

By combining known bounds against low-degree distinguishers [HKP+17, KWB19] with Theorem 1.6, we obtain a new SQ lower bound against the multi-sample version of Tensor PCA:

**Corollary 1.11** (SQ lower bound for Tensor PCA (special case of Corollary 8.4)). Let $D_{\emptyset} = \mathcal{N}(0, I_n)$ and for unit $u \in \mathbb{R}^n$ let $D_u = \mathcal{N}(u \otimes^3, I_n^{3/2})$. Let $\mathcal{S}$ be the uniform distribution on \{\{Du\}_{u \in \{\pm 1/\sqrt{n}\}^n}\}. Any SQ algorithm solving the testing problem $\mathcal{S}$ versus $D_{\emptyset}$ requires at least $n^{\omega(1)}$ queries to VSTAT($n^{3/2}/(\log n)^{O(1)}$).

Up to logarithmic factors, this SQ lower bound matches the best known polynomial-time algorithms, which require at least $m \geq \Omega(n^{3/2})$ samples (or, for the one-shot problem, $\lambda \geq \Omega(n^{3/4})$) [HSS15]. We discuss the information-computation tradeoff in greater detail in Section 8.1. We note that similar bounds for tensor PCA were obtained concurrently and independently in [DH20].

1.3 Prior Work

Researchers have long been aware of the information-computation gap phenomenon, with early work showing such gaps in artificially constructed learning problems [DGR00, Ser99, SSST12] and more recent work focusing on algorithms that trade off between statistical and computational efficiency [SSS08, BKR+11, SSST12, CJ13, CX16]. Our goal here is to establish an equivalence between large classes of algorithms for a wide range of problems in high-dimensional statistics – low-degree distinguishers and SQ algorithms. Several prior works have a similar theme: in related contexts, [HKP+17] shows that Sum-of-Squares semidefinite programs are no more powerful than a restricted
class of spectral algorithms\textsuperscript{8} for hypothesis testing, and [FGV17] shows that a restricted class of convex programs is captured by SQ algorithms.

Several related lines of work establish algorithm-independent or structural properties of high dimensional statistics problems which imply hardness results against restricted models of computation – statistical dimension being one example. Other examples come from statistical physics, where overlap gaps and, more generally, solution-space geometry are related to performance of algorithms such as Markov-Chain Monte Carlo and message passing, with early work focusing primarily on random constraint satisfaction [JMS04, ACO08, IKKM12], and more recent work studying other optimization and hypothesis testing problems [GS14, GZ19, GJW20, AGJ\textsuperscript{20}, AWZ20, GJS19].

More broadly, information-computation tradeoffs have been studied in many restricted computational models: e.g. message-passing algorithms (see [MM09, ZK16] for overviews; we highlight recent work [WEAM19] focusing on running time versus information tradeoffs), Markov-Chain Monte Carlo (e.g. [Jer92, AGJ\textsuperscript{20}]), and Sum-of-Squares semidefinite programs (see e.g. [Gri01, RRS17, KMOW17] or [RSS18] for a survey). In our view, charting the formal connections among all these lenses on information-computation tradeoffs – the statistical physics approach, SQ models, low-degree tests, message-passing algorithms, Markov-Chain Monte Carlo methods, Sum-of-Squares, etc. – is an excellent direction for future investigation.

Statistical Query Model. The SQ model was proposed by Kearns as a framework for designing noise-tolerant algorithms for PAC learning [Kea98]. Blum et al. shortly thereafter introduced statistical query dimension [BFJ\textsuperscript{+}94] as a framework for proving lower bounds on SQ algorithms for supervised learning. The SQ framework has since been generalized to hypothesis testing and estimation [FGR\textsuperscript{+}17, FPV18].

An advantage of SQ lower bounds is their implications for other algorithms: since many algorithms can be implemented with SQ oracle access, SQ lower bounds immediately imply lower bounds against a number of other algorithms, including some convex programs, gradient descent, and more (see e.g. [FGV17]).

SQ lower bounds abound in the study of high-dimensional learning – recent examples are in robust statistics [DKS17, DKS19], polytopes [KS07], neural nets [GGJ\textsuperscript{+}20], and more. In this work, we derive new SDA lower bounds for sparse PCA and for tensor PCA – SQ lower bounds for tensor PCA also appear in the concurrent work of [DH20], who also obtain bounds for estimation.

Statistical dimension may not be a complete characterization of the query complexity in the VSTAT model, in that there are problems for which the statistical dimension is \( q \) but we do not know any \( q \)-query VSTAT algorithms. A complete characterization is given in [Fel12]. In light of this, our results equate the power of low-degree distinguishers with a computational model that is at least as powerful as VSTAT. There are a number of other statistical query models for hypothesis testing problems defined in the literature, for example the MVSTAT oracle of [FPV18]. An interesting open problem is whether a more direct equivalence (via simulation argument) can be achieved in an alternative SQ model.

Low-Degree Tests. Using low-degree polynomials to prove computational lower bounds is a classical idea in theoretical computer science; see e.g. [Bei93] on the polynomial method in circuit complexity. Their recent study as a restricted model of computation for high-dimensional estimation and hypothesis testing problems emerged implicitly in the literature on Sum-of-Squares lower bounds [BHK\textsuperscript{+}19], then more explicitly in [HS17, HKP\textsuperscript{+}17]. See [KWB19] for a survey.

\textsuperscript{8}This class of spectral algorithms, to our knowledge, is not captured by low-degree distinguishers.
Recent works prove lower bounds against low-degree tests for the Sherrington-Kirkpatrick spin glass model [BKW19], tensor PCA [HKP+17], sparse PCA [HKP+17], planted dense subgraphs [SW20], and more. The lower bound approach has also inspired algorithms, for instance for (mixed-membership) community detection [HS17], graph matching in correlated Erdős-Rényi graphs [BHK+19], and sparse PCA [DKWB19].

**Organization.** Section 2 contains preliminaries; the proofs of parts 1 and 2 of Theorem 1.6 follow in Sections 3 and 4. In Section 5 we obtain corollaries for noise robust problems (generalizations of Fact 1.5) and in Section 6 we derive even stronger corollaries for product measures (Theorem 1.7). Section 7 contains a discussion of the cloning methodology for transforming a one-shot problem to an appropriate multi-sample problem for the SQ framework. Section 8 applies our main results to obtain new lower bounds for a number of testing problems.

Appendices A.1 and A.2 give some further details on statistical dimension. Appendix B gives an argument showing how VSTAT algorithms can be simulated directly by low-degree distinguishers. Some calculations are postponed to Appendices C and D.

## 2 Preliminaries

We study hypothesis testing problems \( D_\Theta \) vs. \( S = \{D_u\}_{u \in S} \) with a prior \( \mu \) over \( S \). We frequently write \( u \sim S \) or \( u \sim S \) to indicate that \( D_u \) is sampled from \( S \) according to the marginal \( \mu \). We use \( \overline{D}_u \) to refer to the likelihood ratio or relative density \( \frac{D_u}{\overline{D}_u} \), where the background measure \( D_\Theta \) will be clear from context. We always assume that the likelihood ratio is finite and that \( E_{x \sim D_\Theta} (D_u(x)/D_\Theta(x))^2 < \infty \), for every \( D_u \). This holds if \( D_\Theta, D_u \) have finite support and the support of \( D_u \) is contained in that of \( D_\Theta \); it can also be enforced for continuous distributions by mild truncation of tails.

For \( \mathbb{R} \)-valued functions \( f, g \), let the inner product \( \langle f, g \rangle_{D_\Theta} = E_{x \sim D_\Theta} f(x)g(x) \) and the corresponding norm \( \|f\|_{D_\Theta} = \langle f, f \rangle_{D_\Theta}^{1/2} \). We drop the subscript \( D_\Theta \) when \( D_\Theta \) is clear from context. Note that always, \( \langle \overline{D}_u, 1 \rangle = 1 \). For a distribution \( D \) and an integer \( k \), let \( D_{\Theta}^k \) denote the joint distribution of \( k \) independent samples from \( D \). We will often use \( \langle f^k, g^k \rangle_{D_{\Theta}^k} = \langle f, g \rangle_{D_{\Theta}^k} \), which is a consequence of independence.

For \( D_\Theta \) over \( \mathbb{R}^n \), \( d \) a non-negative integer, and any function \( f : \mathbb{R}^n \to \mathbb{R} \), we let \( f(x)_{\leq d} \) denote the orthogonal (w.r.t. \( D_\Theta \)) projection of \( f \) to the span of functions of degree at most \( d \) in \( x \). We similarly define \( f_{<d}, f_{\geq d}, f_{>d} \), and \( f_{\leq d} \).

**Ruling Out Distinguishers in Subspaces via Small Norms.** We will repeatedly use the folklore fact that the optimal \( m \)-sample low-degree test for a problem \( S, D_\Theta \) has a canonical form: it is the projection of the \( m \)-sample likelihood ratio \( E_{u \sim S} \overline{D}_u^{\otimes m} \) to the span of functions of low degree. In fact, a more general statement is true (which we have essentially proved in Section 1.2.1):

**Fact 2.1.** Let \( D_\Theta \) vs. \( S \) be a testing problem on \( \mathbb{R}^n \). Let \( C \) be a linear subspace of functions \( p : (\mathbb{R}^n)^{\otimes m} \to \mathbb{R} \), and let \( \Pi_C \) be the orthogonal projection to the subspace \( C \). Then

\[
\arg \max_{p \in C} \left| \frac{E_{u \sim S} E_{D_u^{\otimes m}} p - E_{D_\Theta^{\otimes m}} p}{\Pi_C \left( E_{u \sim S} \overline{D}_u^{\otimes m} - 1 \right)} \right| \Pi_C \left( E_{u \sim S} \overline{D}_u^{\otimes m} - 1 \right)_{D_\Theta^{\otimes m}}.
\]
Letting \( p = \frac{\Pi_C( E_{u \sim S}(D_u^\otimes m) - 1)}{\| \Pi_C( E_{u \sim S}(D_u^\otimes m) - 1) \|} \) be the optimizer of the above program, observe also that

\[
\mathbb{E}_{u \sim S} \mathbb{E}_{D_u^\otimes m} p - \mathbb{E}_{D_u^\otimes m} p = \left\| \Pi_C \left( \mathbb{E}_{u \sim S}(D_u^\otimes m) - 1 \right) \right\|_{D_u^\otimes m}.
\]

Consequently,

**Fact 2.2.** If \( \left\| \Pi_C( E_{u \sim S}(D_u^\otimes m) - 1) \right\| \leq \varepsilon \), then \( D_\otimes \) vs. \( S \) has no \( m \)-sample \( \varepsilon \)-distinguisher in \( C \).

**Samplewise Degree.** Rather than directly ruling out distinguishers of low degree, it will be convenient for us to introduce a notion of degree which agrees with the product structure (across samples) of \( D_u^\otimes m \).

**Definition 2.3** (Samplewise degree). For integers \( m, n \geq 1 \), we say that a function \( f : (\mathbb{R}^n)^\otimes m \to \mathbb{R} \) has samplewise degree \((d,k)\) if \( f(x_1, \ldots, x_m) \) can be written as a linear combination of functions which have degree at most \( d \) in each \( x_i \), and nonzero degree in at most \( k \) of the \( x_i \)'s.

Note that a function of samplewise degree \((d,k)\) has degree at most \( d \cdot k \), and a function of degree \( d \) has samplewise degree at most \((d,d)\).

In order to rule out low-degree distinguishers, we will rule out low-samplewise degree distinguishers using Fact 2.2. We denote the orthogonal projection of \( f : (\mathbb{R}^n)^\otimes m \to \mathbb{R} \) to the span of samplewise degree \((d,k)\) functions by \( f \downarrow_{d,k} \). We define the following quantity:

**Definition 2.4** (Low degree likelihood ratio). For a hypothesis testing problem \( D_\otimes \) vs. \( S = \{D_v\} \), the \( m \)-sample \((d,k)\)-low degree likelihood ratio function is the projection of the \( m \)-sample likelihood ratio \( E_{u \sim S}(D_u^\otimes m) \) to the span of non-constant functions of sample-wise degree at most \((d,k)\):

\[
\left( \mathbb{E}_{u \sim S}(D_u^\otimes m) - 1 \right) \downarrow_{d,k} = \mathbb{E}_{u \sim S}(D_u^\otimes m) \downarrow_{d,k} - 1.
\]

We refer to this function as the \((d,k)\)-LDLR\(_m\). Abusing terminology, we also use \((d,k)\)-LDLR\(_m\) to refer to the norm of the low degree likelihood ratio, \( \| E_{u \sim S}(D_u^\otimes m) \downarrow_{d,k} - 1 \| \).

### 3 Bounds on Degree Imply Bounds on Statistical Dimension

In this section, we prove part 1 of Theorem 1.6, showing that an upper bound on the low-degree likelihood ratio’s norm (LDLR) implies lower bounds on the statistical dimension.

**Theorem 3.1** (LDLR to SDA Lower Bounds). Let \( d, k \in \mathbb{N} \) with \( k \) even and \( S = \{D_v\}_{v \in S} \) be a collection of probability distributions with prior \( \mu \) over \( S \). Suppose that \( S \) satisfies:

1. The \( k \)-sample high-degree part of the likelihood ratio is bounded by \( \| E_{u \sim S}(D_u^\otimes d) \otimes k \| \leq \delta \).
2. For some \( m \in \mathbb{N} \), the \((d,k)\)-LDLR\(_m\) is bounded by \( \| E_{u \sim S}(D_u^\otimes m) \downarrow_{d,k} - 1 \| \leq \varepsilon \).

Then for any \( q \geq 1 \), it follows that

\[
\text{SDA} \left( S, \frac{m}{q^2/k (2/3k + 2/km)} \right) \geq q.
\]
Notice that for a \((m^{-k/2}/4, k)\)-nice testing problem, Condition 1 of Theorem 3.1 holds with \(d = k\) and \(\delta = m^{-k/2}/4\) (by definition). So for \((m^{-k/2}/4, k)\)-nice problems with no good 4\(mk\)-sample degree \(k^2\) distinguisher (and therefore no good samplewise degree \((k, k)\) distinguisher), setting \(q = (2m/m!)^{k/2}\) in Theorem 3.1 implies that SDA\((S, \Theta(m'/k)) \supseteq (2m/m!)^{k/2}\), which establishes the contrapositive of part 1 of Theorem 1.6. In subsequent sections, we will demonstrate that the niceness condition holds for many natural hypothesis testing problems (or in some cases, holds if the \((d, k)\)-LDLR\(_m\) is small). Combining these conditions with Theorem 3.1 will yield Theorems 5.2, 6.1 and 6.3.

**Proof of Theorem 3.1.** For overview see Section 1.2. Let \(X\) be the random variable \(X = |(\overline{D}_u, \overline{D}_v) - 1|\) for \(u, v \sim S\) sampled independently according to the prior \(\mu\). By definition, SDA\((S, \frac{1}{q}) \supseteq q\) if \(E[X | A] \leq t\) for all events \(A\) over the choice of \(u, v\) of probability at least \(\frac{1}{q}\). So our goal is to show that \(E[X | A] \leq q^{2/k}(\frac{k}{m} + \frac{2}{k} + \delta^{2/k})\). We relate \(E[X | A]\) to moments of \(X\) via Hölder’s inequality:

**Fact 3.2.** If \(x\) is a real-valued random variable and \(A\) is any event then \(E[|x| | A] \leq \left(\frac{E[|x|^k]}{P(A)}\right)^{1/k}\).

We prove the fact below for completeness. Since we have assumed that \(k\) is even,

\[
E[X^k] = \mathbb{E}_{u,v \sim S}((\overline{D}_u, \overline{D}_v)_{DS} - 1)^k = \mathbb{E}_{u,v \sim S}((\overline{D}_u - 1, \overline{D}_v - 1)_{DS})^k = \left\|\mathbb{E}_{u \sim S}(\overline{D}_u - 1)^{\otimes k}\right\|_{DS}^2,
\]

where we have first used that \((\overline{D}_u, 1) = 1\) for all \(u \in S\), and then the independence of the samples. Applying Fact 3.2,

\[
\max_{A \text{ s.t. } \Pr_{u,v \sim S}[A] \geq \frac{1}{q}} \mathbb{E}_{u,v \sim S}[|(\overline{D}_u, \overline{D}_v) - 1| | A] \leq \left(q \cdot \left\|\mathbb{E}_{u \sim S}(\overline{D}_u - 1)^{\otimes k}\right\|\right)^{2/k}.
\]

(2)

Now, applying Hölder’s inequality (see Lemma 3.4 below), we can split the degree \(\leq d\) and degree \(> d\) parts of \(\overline{D}_u - 1\) in our bound on the right-hand side,

\[
\left\|\mathbb{E}_{u \sim S}(\overline{D}_u - 1)^{\otimes k}\right\|^{2/k} \leq \left\|\mathbb{E}_{u \sim S}(\overline{D}_u^{\leq d} - 1)^{\otimes k}\right\|^{2/k} + \left\|\mathbb{E}_{u \sim S}(\overline{D}_u^{> d} - 1)^{\otimes k}\right\|^{2/k}.
\]

(3)

The second right-hand-side term is bounded by \(\delta^{2/k}\) from Condition 1. So, it remains to bound the first term. This is our crucial “boosting” step. We employ the following structural claim, which uses the independence of the samples to relate the correlation of the \((d, k)\) projections of \(m\)-sample likelihood ratios to the correlation of the \((d, k)\) projections of \(k\)-sample likelihood ratios, with \(k \ll m\):

**Claim 3.3.** Let \(D_u, D_v\) be distributions with relative densities \(\overline{D}_u, \overline{D}_v\). Then their \((d, k)\)-projections are related as follows:

\[
\langle (\overline{D}_u^{\leq d})_{DS}, (\overline{D}_v^{\leq d})_{DS} \rangle - 1 = \sum_{t=1}^{k} \binom{m}{t} \cdot (\langle \overline{D}_u^{\leq d}, \overline{D}_v^{\leq d} \rangle - 1)^t.
\]

We give the (simple) proof of this claim below. Now, by linearity of expectation, the squared \((d, k)\)-LDLR\(_m\) is equal to

\[
\left\|\mathbb{E}_{u \sim S}(\overline{D}_u^{\leq d})^{\otimes k} - 1\right\|^2 = \mathbb{E}_{u,v \sim S} \langle (\overline{D}_u^{\leq d})_{DS}, (\overline{D}_v^{\leq d})_{DS} \rangle - 1 = \mathbb{E}_{u,v \sim S} \sum_{t=1}^{k} \binom{m}{t} \left(\langle \overline{D}_u^{\leq d}, \overline{D}_v^{\leq d} \rangle - 1\right)^t,
\]

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where in the final equality we applied Claim 3.3. So Condition 2 \( (\| \mathbf{E}_u (\mathbf{D}_u^\otimes m) \|_{d,k} \leq \varepsilon) \) combined with the above implies that

\[
\varepsilon^2 \geq \left\| \mathbf{E}_{u \sim S} (\mathbf{D}_u^\otimes m)_{d,k} - 1 \right\|^2 - \left\| \mathbf{E}_{u \sim S} (\mathbf{D}_u^\otimes m)_{d,k-1} - 1 \right\|^2 = \binom{m}{k} \cdot \mathbf{E}_{u \sim S} \left( (\mathbf{D}_u^\otimes d, \mathbf{D}_v^\otimes d) - 1 \right)^k \geq 0.
\]

Dividing through by \( \binom{m}{k} \) we have \( \mathbf{E}_{u,v}(\mathbf{D}_u^\otimes d, \mathbf{D}_v^\otimes d) - 1)^k = \| \mathbf{E}_u (\mathbf{D}_u^\otimes d - 1)^\otimes k \| \leq \varepsilon^2 \left( \frac{1}{\varepsilon} \right)^{k} \leq \varepsilon^2 \left( \frac{1}{m} \right)^k. \)

Combining this with Equations (2) and (3) finishes the proof.

We now prove the outstanding claims, in order of mathematical interest.

**Proof of Claim 3.3.** We write \( \mathbf{D}_u = 1 + (\mathbf{D}_u^\otimes d - 1) + \mathbf{D}_u^\otimes d \). Expanding the tensor power,

\[
(\mathbf{D}_u^\otimes m)_{d,k} = \sum_{A \subseteq [m], B \subseteq [m] \setminus A} \left( 1 \otimes (\mathbf{D}_u^\otimes d - 1) \otimes (\mathbf{D}_u^\otimes d)^\otimes [m] \setminus (A \cup B) \right)_{d,k}.
\]

Now, \( \mathbf{D}_u^\otimes d \) is orthogonal to all functions of degree at most \( d \). So the projection

\[
\left( 1 \otimes (\mathbf{D}_u^\otimes d - 1) \otimes (\mathbf{D}_u^\otimes d)^\otimes [m] \setminus (A \cup B) \right)_{d,k} = 0
\]

unless \( A \cup B = [m] \), and hence

\[
(\mathbf{D}_u^\otimes m)_{d,k} = \sum_{A \subseteq [m]} \left( 1 \otimes (\mathbf{D}_u^\otimes d - 1) \otimes [m] \setminus A \right)_{d,k}.
\]

Furthermore, if \( |[m] \setminus A| > k \), then \( 1 \otimes (\mathbf{D}_u^\otimes d - 1) \otimes [m] \setminus A \) is orthogonal to every function depending on at most \( k \) samples. So again applying the projection to degree-(\( d,k \)),

\[
(\mathbf{D}_u^\otimes m)_{d,k} = \sum_{B \subseteq [m], |B| \leq k} 1 \otimes [m] \setminus B \otimes (\mathbf{D}_u^\otimes d - 1) \otimes B.
\]

Observe also that if \( B, B' \subseteq [m] \) and \( B \neq B' \), then

\[
\left( 1 \otimes [m] \setminus B \otimes (\mathbf{D}_u^\otimes d - 1) \otimes B, 1 \otimes [m] \setminus B' \otimes (\mathbf{D}_u^\otimes d - 1) \otimes B' \right) = 0.
\]

So we have

\[
\langle (\mathbf{D}_u^\otimes m)_{d,k}, (\mathbf{D}_v^\otimes m)_{d,k} \rangle - 1 = \sum_{B \subseteq [m], B \neq \emptyset} \langle (\mathbf{D}_u^\otimes d - 1, \mathbf{D}_v^\otimes d - 1) \rangle_{|B|},
\]

which, by the independence of samples, proves the claim.

**Lemma 3.4.** Let \( D_\otimes \) be a null distribution and \( S = \{ D_u \}_{u \in S} \) be a set of alternate distributions with \( D_u \)'s density relative to \( D_\otimes \) density given by \( \mathbf{D}_u \) for each \( u \in S \). Let \( k, d \geq 1 \) be integers with \( k \) even. Then the centered \( k \)-sample likelihood ratio may be bounded in terms of the \( k \)-sample-homogeneous low-degree part and the \( k \)-sample-homogeneous high degree part:

\[
\left\| \mathbf{E}_{u \sim S} (\mathbf{D}_u - 1)^\otimes k \right\|^{2/k} \leq \left\| \mathbf{E}_{u \sim S} (\mathbf{D}_u^\otimes d - 1)^\otimes k \right\|^{2/k} + \left\| \mathbf{E}_{u \sim S} (\mathbf{D}_u^\otimes d)^\otimes k \right\|^{2/k}.
\]
Proof. By the triangle inequality, Hölder’s inequality and the fact that \( k \) is even, we have that

\[
\mathbb{E}_{u,v} \left[ (\langle D_u, D_v \rangle - 1)^k \right] = \mathbb{E}_{u,v} \left[ (\langle D_u^\leq d, D_v^\leq d \rangle - 1 + \langle D_u^> d, D_v^> d \rangle)^k \right] \\
\leq \mathbb{E}_{u,v} \left[ \left( \langle D_u^\leq d, D_v^\leq d \rangle - 1 \right) + \left| \langle D_u^> d, D_v^> d \rangle \right| \right]^k \\
\leq \sum_{\ell=0}^k \binom{k}{\ell} \mathbb{E}_{u,v} \left[ \left( \langle D_u^\leq d, D_v^\leq d \rangle - 1 \right)^\ell \mathbb{E}_{u,v} \left[ \left( \langle D_u^> d, D_v^> d \rangle \right)^{k-\ell} \right] \right] \\
= \left( \mathbb{E}_{u,v} \left[ \langle D_u^\leq d, D_v^\leq d \rangle - 1 \right] \right)^{1/k} + \left( \mathbb{E}_{u,v} \left[ \langle D_u^> d, D_v^> d \rangle \right] \right)^{1/k},
\]

and the conclusion now follows because \( \langle D_u, 1 \rangle = 1 \) for all \( u \in S \), which implies \( \mathbb{E}_{u,v}(\langle D_u, D_v \rangle - 1)^k = \mathbb{E}_{u,v}(1 - \mathbb{E}[\langle D_u, D_v \rangle - 1]^{\leq k}) \) and \( \mathbb{E}_{u,v}(\langle D_u^\leq d, D_v^\leq d \rangle - 1)^k = \mathbb{E}_{u,v}(1 - \mathbb{E}[\langle D_u^> d, D_v^> d \rangle]^{\leq k}) \). \( \square \)

Proof of Fact 3.2. Observe that

\[
\mathbb{E}[\|x\| A] = \frac{\mathbb{E}[\|x\| \cdot 1[A]]}{\text{Pr}[A]} \leq \frac{\mathbb{E}[\|x\|^{1/k} \mathbb{E}[1[A]]^{1-1/k}]}{\text{Pr}[A]} = \left( \frac{\mathbb{E}[\|x\|^{1/k}]}{\text{Pr}[A]} \right)^{1/k},
\]

where we have applied Hölder’s inequality. \( \square \)

We encapsulate the conclusion of the boosting argument above in the following standalone lemma, which will be useful later:

**Lemma 3.5** (Samplewise-LDLR boosting). If the \((d, k)\)-LDLR\(_m\) for the hypothesis testing problem of \( D_\emptyset \) vs \( \{D_v\}_{v \in S} \) is bounded, then the moments of the low-degree single-sample LR are also bounded, by

\[
\| \mathbb{E}_{u \sim S} (D_u^{\leq d} - 1)^{\leq k} \| \leq \left( \frac{1}{m} \right)^k \| \mathbb{E}_{u \sim S} (D_u^{\leq m})^{\leq d, k} - 1 \|.
\]

The proof is identical to the end of the proof of Theorem 3.1.

## 4 Bounds on Statistical Dimension Imply Bounds on Degree

In this section, we show that lower bounds on the statistical dimension imply that the low-degree likelihood ratio norm is small (hence ruling out good low-degree distinguishers). We will prove the following theorem:

**Theorem 4.1.** Let \( S \) be a hypothesis testing problem on \( \mathbb{R}^N \) with respect to null hypothesis \( D_\emptyset \). Let \( m, k \in \mathbb{N} \) with \( k \) even. Suppose that for all \( 0 \leq m' \leq m \), SDA(\( S, m' \)) \( \geq 100^k \cdot (m/m')^k \). (In particular, SDA(\( S, m \)) \( \geq 100^k \).) Then for all \( d \), \( \| \mathbb{E}_{u \sim S} (D_u^{\leq m})^{\leq d, k} - 1 \|^{1/2} \leq 1 \).

The key lemma to prove Theorem 4.1 is the following, which translates the bound SDA(\( S, m' \)) \( \geq 100^k \cdot (m/m')^k \) to a bound on the moments of \( \langle D_u, D_v \rangle - 1 \).

**Lemma 4.2.** In the setting of Theorem 4.1, for any \( t \leq k/8 \), \( \mathbb{E}_{u,v \sim S}(\langle D_u, D_v \rangle - 1)^t \leq 4 \cdot (1/100m)^t \).

Now we prove Theorem 4.1.
Proof of Theorem 4.1. We use Claim 3.3 and Lemma 4.2 to obtain
\[
\mathbb{E}_{u,v \sim S} \left< \left( \mathcal{D}_u \right)^m \right|_{k/8}, \left( \mathcal{D}_v \right)^m \right|_{k/8} \leq \sum_{t=1}^{k/8} \binom{m}{t} \mathbb{E}_{u,v \sim S} \left< \left( \mathcal{D}_u \mathcal{D}_v \right)^t - 1 \right> \leq \sum_{t=1}^{k/8} \binom{m}{t} 4 \cdot \left( \frac{1}{100m} \right)^t.
\]
Using \( \binom{m}{t} \leq (me/t)^t \), we find that this is at most \( 4 \sum_{t=1}^{k/8} \left( \frac{e}{100t} \right)^t \leq 4(e/100 - 1) \leq 1 \). But for all \( d \in \mathbb{N} \) we have
\[
\| \mathbb{E}_{u \sim S} \left( \mathcal{D}_u \right)^m \leq d, k/8 - 1 \|^2 \leq \mathbb{E}_{u,v \sim S} \left< \left( \mathcal{D}_u \right)^m \leq d, k/8, \left( \mathcal{D}_v \right)^m \leq k/8 \right>
\]
which completes the proof. \( \square \)

We turn to the proof of Lemma 4.2. We need the following basic fact to relate the moments and tails of \( \langle D_u, D_v \rangle - 1 \). (The proof is straightforward calculus; see e.g. Appendix A.2 of [HL19].)

**Fact 4.3.** Let \( X \) be an \( \mathbb{R} \)-valued random variable. For every \( p > q > 0 \), \( E \left| X \right|^q \leq \left( 2 \sup_A \Pr[A] \cdot (EX \left| A \right|)^p \right)^{q/p} \cdot \frac{p}{p-q} \). (The supremum is taken over all events \( A \).)

Proof of Lemma 4.2. Let \( X = \langle D_u, D_v \rangle - 1 \) be the \( \mathbb{R} \)-valued random variable given by two random draws \( u, v \sim S \). Our assumption \( \Pr[D_S, m'] \geq 100^k \cdot (m/m')^k \) for all \( m' \leq m \) implies that for every event \( A \) of probability \( \alpha \geq 100^{-2k} \cdot (m'/m)^{2k} \) we have \( \mathbb{E}[X \left| A \right] \leq 1/m' \). Rearranging, for all events \( A \) of probability \( \alpha \), we have \( \mathbb{E}[X \left| A \right] \leq \frac{1}{100m^{2/k}} \). So for any \( t \leq k/2 \),
\[
\sup_A \Pr(A) \cdot (EX \left| A \right|)^t \leq \sup_{\alpha \geq 0} \alpha^{1-2t/k} \cdot \left( \frac{1}{100m} \right)^t \leq \left( \frac{1}{100m} \right)^t.
\]
So applying Fact 4.3 for any \( t \leq k/8 \),
\[
EX^t \leq 4 \cdot (1/100m)^t.
\]

\( \square \)

5 Specialization to Noise-Robust Problems

In this section, we observe that Theorem 3.1 immediately applies to noise-robust problems, as noise-robustness implies a bound on the high-degree part of the LR.

5.1 Noise Operators

We define a class of Markov operators which generalize the Gaussian and discrete noise operators. Recall that a Markov operator \( T \) is a linear operator such that if \( f \) is a probability density, then so is \( Tf \).

**Definition 5.1** ((\( d, \epsilon \))-Markov operator). Let \( D_\varnothing \) be a probability measure on \( \mathbb{R}^N \) (or a discrete distribution on \( \Omega^N \) for some finite set \( \Omega \)), inducing an inner product on functions \( f, g : \mathbb{R}^N \rightarrow \mathbb{R} \) (or \( f, g : \Omega^N \rightarrow \mathbb{R} \)) by \( \langle f, g \rangle = \mathbb{E}_{x \sim D_\varnothing} f(x)g(x) \). Let \( \ell_2 = \{ f : \mathbb{R}^N \rightarrow \mathbb{R} \text{ s.t. } \mathbb{E}_{x \sim D_\varnothing} f(x)^2 \leq \infty \} \). Let \( d \in \mathbb{N} \), and let \( \ell_2^d \) be the orthogonal complement of \( \text{span}\{ f \in \ell_2 : f \text{ has degree } (d - 1) \} \) with respect to \( \langle \cdot, \cdot \rangle \).

Any hypothesis testing problem \((D_\varnothing, S)\) and Markov operator \( T : \ell_2 \rightarrow \ell_2 \) induce another hypothesis testing problem \((D_\varnothing, TS)\) by applying \( T \) to each of the distributions \( D_\varnothing \in S \). We call a Markov operator \( T \) a \((d, \epsilon)\)-operator if
\[
\ell_2^d \subseteq \text{span}\{ f \in \ell_2 : f \text{ is an eigenfunction of } T \text{ with eigenvalue } \lambda \text{ such that } |\lambda| \leq \epsilon \}.
\]
Our main examples are the Ornstein-Uhlenbeck operator $U_\rho$ (a.k.a. the Gaussian noise operator) and the discrete noise operator $T_\rho$, both of which are $(d, \rho^d)$ operators. In both cases, the testing problems $(D_\emptyset, T\mathcal{S})$ will be noisy versions of original problems $(D_\emptyset, \mathcal{S})$. However, we will use a different family of noise operators to treat certain statistical problems where there is planted structure which is not robust to independent entrywise noise, such as planted clique.

5.2 Results for Noise-Robust Problems

**Theorem 5.2.** Let $d, k \in \mathbb{N}$ with $k$ even and $\mathcal{S} = \{D_v\}_{v \in \mathcal{S}}$ be a collection of probability distributions, let $\overline{D}_u$ be the relative density of $D_u$ with respect to $D_\emptyset$. Let $T$ be a $(d + 1, \rho^{d+1})$ Markov operator. Suppose that the $k$-sample likelihood ratio is bounded by $\|E_u \overline{D}_u^{\otimes k}\|^2 \leq C^k$, and the noised $(d, k)$-LDLR$_m$ is bounded by $\|E_u (T \overline{D}_u \otimes m)^{\leq d, k} - 1\| \leq \varepsilon$. Then it follows that for any $q \geq 1$,

$$\text{SDA} \left( \mathcal{S}, \frac{1}{q^{2/k}} \left( \frac{k^2}{k^{2/k}} + \rho^{2(d+1)m}C^m \right) \right) \geq q.$$

**Proof.** Since $T$ is a $(d + 1, \rho^{d+1})$ Markov Operator by assumption, the $k$-sample high-degree part of the LR is bounded by

$$\left\| E_u (T \overline{D}_u^{\geq d})^{\otimes k} \right\|^2 \leq \rho^{2(d+1)k} \cdot \left\| E_u (D_u^{\otimes d})^{\otimes k} \right\|^2 \leq \rho^{2(d+1)k} \cdot \left\| E_u (D_u^{\otimes k}) \right\|^2 \leq \rho^{2(d+1)k} \cdot C^k.$$

Applying Theorem 3.1 now completes the proof of this theorem. \hfill \Box

5.3 Robustness to Random Restrictions

Some problems of interest are not noise-robust under nontrivial $(\rho, d)$-operators. For example, consider the (bipartite) planted clique problem—the clique structure is not preserved if the coordinates are resampled independently.\(^9\) To accommodate such problems, we generalize Theorem 5.2 to a different class of noise operators: random restrictions. A random restriction fixes a random subset of coordinates, then applies noise to the remaining coordinates across all of the samples.

**Definition 5.3** (Random Restriction). Let $T$ be a Markov operator on $\mathbb{R}^N$. Given a subset $R \subset [N]$, let $T^R$ be the Markov operator on $\mathbb{R}^N$ that applies $T$ to all entries except those in $R$. Given a set of probability distributions $\mathcal{S}$ and a prior $\mu$ over $\mathcal{S}$, the $(T,s)$-random restriction of $\mathcal{S}$ is the set of distributions

$$\mathcal{S}' = \left\{ T^R D \mid D \in \mathcal{S}, R \subset [N] \right\}$$

equipped with the prior $\mu'$ where a sample $T^R D \sim \mu'$ is generated sampling $D \sim \mu$ and sampling $R$ by including every coordinate in $[N]$ independently with probability $\frac{1}{s}$. Denote the distribution on subsets as $\mathcal{R}_N(s)$.

We will often abuse notation and let $T^R$ stand in for $(T^{\otimes n})^R$ when $T$ is a noise operator on $\mathbb{R}$.

For simplicity we restrict our attention to distributions $D_v$ over the boolean hypercube $\{\pm 1\}^n$, and to null distributions $D_\emptyset$ which are product measures for which all biases are the same, $D_\emptyset = D_0^{\otimes N}.$\(^10\) We now have the following lemma:

\(^9\)In the bipartite version, we further require that the resampling procedure be dependent across samples.

\(^10\)We expect that a near-identical proof will extend to the case when $D_\emptyset$ is a product measure with arbitrary coordinate biases.
Lemma 5.4. Let $D_\emptyset$ be a product measure over $\{\pm 1\}^N$. Let $d, k \in \mathbb{N}$, let $T$ be a $(1, \rho)$-operator over $\{\pm 1\}$ (with respect to the measure induced by $D_\emptyset$ on a single coordinate). Then for $S = \{D_v\}_{v \in S}$ a family of distributions over $\{\pm 1\}^N$ with prior $\mu$, we have that the $(T, s)$-random restriction $S', \mu'$ of $S$ has degree $(> d, = k)$ bounded by

$$\left\| E_{R \sim R_N(s)} E_{u \sim \mu} \left( T R D_u \right) \right\|^2 \leq \max \left\{ 4^{d+1} \rho^{2(d+1)k}, \left( \frac{2s}{n} \right)^{2(d+1)} \right\} \cdot \left\| E_{u \sim \mu} (D_u) \right\|^k.$$

Proof. We will abuse notation and let $T R$ simultaneously denote the noise operator on $(\mathbb{R}^N)^k$ that applies $T R$ independently to each copy of $\mathbb{R}^N$. Let $D = E_{u \sim \mu} (D_u)^k$ and let $\hat{D}(\alpha_1, \alpha_2, \ldots, \alpha_k)$ denote the Fourier character of $D$ at the subsets $\alpha_1, \alpha_2, \ldots, \alpha_k \subseteq [N]$. By the definition of $T R$, we have that

$$\hat{T R D}(\alpha_1, \alpha_2, \ldots, \alpha_k) = \rho^{\sum_{i=1}^k |\alpha_i \cap R|} \cdot \hat{D}(\alpha_1, \alpha_2, \ldots, \alpha_k)$$

for any $\alpha_1, \alpha_2, \ldots, \alpha_k \subseteq [N]$. Let $T'$ denote the operator $E_{R \sim R_N(s)} T R$ and observe that

$$\hat{T R D}(\alpha_1, \alpha_2, \ldots, \alpha_k) = E_{R \sim R_N(s)} \left[ \hat{D}(\alpha_1, \alpha_2, \ldots, \alpha_k) \right].$$

Now by Hölder’s inequality, we have that

$$E_{R \sim R_N(s)} \left[ \rho^{\sum_{i=1}^k |\alpha_i \cap R|} \right] \leq \prod_{i=1}^k E_{R \sim R_N(s)} \left[ \rho^{|\alpha_i \cap R|} \right]^{1/k} = \prod_{i=1}^k E_{R \sim R_N(s)} \left[ \prod_{j \in \alpha_i} \rho^{k-1(j \not\in R)} \right]^{1/k} = \left( \frac{s}{N} + (1 - \frac{s}{N}) \rho \right)^{\sum_{i=1}^k |\alpha_i|/k},$$

where the final equality follows from the fact that the events $1(j \not\in R)$ are independent and occur with probability $1 - \frac{s}{N}$ under $R \sim R_N(s)$. Now by Parseval’s inequality, we have that

$$\left\| E_{R \sim R_N(s)} E_{u \sim \mu} \left( T R D_u \right)^k \right\|^2 = \sum_{|\alpha_1|, |\alpha_2|, \ldots, |\alpha_k| > d} \hat{T R D}(\alpha_1, \alpha_2, \ldots, \alpha_k)^2 \leq \sum_{|\alpha_1|, |\alpha_2|, \ldots, |\alpha_k| > d} \left( \frac{s}{N} + (1 - \frac{s}{N}) \rho \right)^{2\sum_{i=1}^k |\alpha_i|/k} \cdot \hat{D}(\alpha_1, \alpha_2, \ldots, \alpha_k)^2 \leq \left( \frac{s}{N} + (1 - \frac{s}{N}) \rho \right)^{2(d+1)} \sum_{|\alpha_1|, |\alpha_2|, \ldots, |\alpha_k| > d} \hat{D}(\alpha_1, \alpha_2, \ldots, \alpha_k)^2 \leq \left( \frac{s}{N} + \rho \right)^{2(d+1)} \cdot \left\| E_{u \sim \mu} (D_u)^k \right\|^2.$$

The lemma then follows from the fact that $s/N + \rho^k \leq \max\{2\rho^k, 2s/N\}$. \hfill \square

Applying Theorem 3.1 yields the following Corollary:

Corollary 5.5. Let $D_\emptyset$ be a product measure over $\{\pm 1\}^N$. Let $d, k \in \mathbb{N}$ with $k$ even, let $T$ be a $(1, \rho)$-operator over $\{\pm 1\}$ (with respect to the measure induced by $D_\emptyset$ on a single coordinate). Let $S = \{D_v\}_{v \in S}$ a family of distributions over $\{\pm 1\}^N$ with prior $\mu$ over $S$, and let $D_u$ be the relative density of $D_u$ with respect to $D_\emptyset$. Suppose that the $k$-sample likelihood ratio is bounded by
\[ \| \mathbf{E}_u \mathbf{D}_u^\otimes k \|^2 \leq C^k, \text{ and suppose that the } (T, s)\text{-randomly restricted alternate hypothesis class } S, \mu' \text{ has } (d, k)\text{-LDLR}_m \text{ bounded,} \]

\[ \left\| \mathbf{E}_{R \sim \mathcal{R}_N(s)} \mathbf{E} (T^R \mathbf{D}_u^\otimes m) \right\|_{d, k} - 1 \leq \varepsilon, \]

Then it follows that for any \( q \geq 1, \)

\[ \text{SDA} \left( S', \mu', \frac{m}{q^2/k} \right) \left\{ k^{2/k} + \max \left\{ 4^{(d+1)/k} p^{2(d+1)}, \left( \frac{2s}{n} \right)^{2(d+1)/k} Cm \right\} \right\}^{-1} \geq q. \]

**Remark 5.6** (Comparison to Theorem 5.2). As long as \( k = \Omega(d) \), \( 4^{(d+1)/k} = O(1) \) and thus this theorem can be viewed as a natural extension of Theorem 5.2, recovering (essentially) the same result when \( s = 0. \)

In Section 8.2, we show that Corollary 5.5 implies an equivalence between distinguishers and statistical queries for a number of models such as planted clique, in which the planted structure is not robust to independent noise.

### 5.3.1 Random Subtensor Restrictions

In the above, we treated random restrictions in which coordinates in \([N]\) are fixed independently. In tensor- and matrix-problems, where \( \{\pm 1\}^N \) is identified with \( \{\pm 1^N\}^\otimes p \) for an integer \( p \), the natural notion of random restriction restricts to a random principal minor \( \{\pm 1\}^N \otimes p \). Below, we will generalize Corollary 5.5 to this type of random restriction.

Let \( \mathcal{R}_n(s) \) be as in the section above, and for \( R \in \mathcal{R}_n(s) \) let \( R^\otimes p \) denote the set of all coordinates in \( \{\pm 1\}^N \otimes p \) where all \( p \) modes lie in \( R \).

**Lemma 5.7.** Let \( p, s, n, k, d \in \mathbb{N} \) and \( \rho \in (0, 1) \) with \( 2s \leq n, 2^{p/k} \rho \leq 1 \). Let \( D_\otimes \) be a product measure over \( \{\pm 1\}^N \) where \( N = n^p \), and let \( T \) be a \((1, \rho)\)-operator over \( \{\pm 1\}\) (with respect to the measure induced by \( D_\otimes \) on a single coordinate). Then for \( S = \{D_u\}_{u \in S} \) a family of distributions over \( \{\pm 1\}^N \otimes p \) with prior \( \mu \), we have that the \((T, s)\)-random restriction \( S', \mu' \) of \( S \) has degree \((> d, = k)\) bounded by

\[ \left\| \mathbf{E}_{R \sim \mathcal{R}_n(s)} \mathbf{E} (T^R \mathbf{D}_u^\otimes p) \right\|_{d, k}^2 \leq \max \left\{ 4^{d+1} \rho^{(d+1)/k}, \left( \frac{2s}{n} \right)^{2(d+1)/k} \right\} \cdot \left\| \mathbf{E}_{u \sim S'} \mathbf{D}_u^\otimes k \right\|_{d, k}^2. \]

**Proof.** As in Lemma 5.4, let \( \mathbf{D} = \mathbf{E}_{u \sim \mu} (\mathbf{D}_u)^\otimes k \) with Fourier coefficients \( \widehat{D}(\alpha_1, \alpha_2, \ldots, \alpha_k) \) for any sequence of subsets \( \alpha_1, \alpha_2, \ldots, \alpha_k \subseteq [n]^p \). Similarly, let \( T' = \mathbf{E}_{R \sim \mathcal{R}_n(s)} T^R \mathbf{D}_u^\otimes p \). Applying Hölder’s inequality just as in the proof of Lemma 5.4, we have that

\[ \widehat{T'} \widehat{D}(\alpha_1, \alpha_2, \ldots, \alpha_k) = \mathbf{E}_{R \sim \mathcal{R}_n(s)} \left[ \sum_{i=1}^k |\alpha_i \cap (R^\otimes p)^-| \right] \widehat{D}(\alpha_1, \alpha_2, \ldots, \alpha_k) \]

\[ \leq \left( \prod_{i=1}^k \mathbf{E}_{R \sim \mathcal{R}_n(s)} \left[ \prod_{(i_1, i_2, \ldots, i_p) \in \alpha_i} \rho^{k \cdot 1(3s \in [p], i_a \notin R)} \right]^{1/k} \right) \cdot \widehat{D}(\alpha_1, \alpha_2, \ldots, \alpha_k) \quad (5) \]

\[ \text{We also remark that the } (2s/N)^{2(d+1)} \text{ factor in Lemma 5.4 cannot in general be improved. In particular, when } \rho = 0, \text{ the diagonal Fourier coefficients of the form } \widehat{T'} \widehat{D}(\alpha, \alpha, \ldots, \alpha) \text{ are exactly equal to } (s/N)^{|\alpha|} \cdot \widehat{D}(\alpha, \alpha, \ldots, \alpha). \]

However, other Fourier coefficients are scaled down more heavily under \( T' \) and it is possible to improve the bound in Lemma 5.4 under further assumptions about the Fourier coefficients of \( \mathbf{D} \).
We now will prove the following claim which will complete the proof of the lemma.

**Claim 5.8.** For any $\alpha \subseteq [n]^p$, so long as $2^{p/k} \rho \leq 1$ and $2s \leq n$,

$$
E_{R \sim \mathcal{R}_n(s)} \left[ \prod_{(i_1, i_2, \ldots, i_p) \in \alpha} \rho^{k \cdot 1(3a \in [p], i_a \notin R)} \right] \leq \max \left\{ 2^{\frac{2}{p} |\alpha|} \frac{k}{\rho^{p/2}}, \left( \frac{2s}{n} \right)^{\left( \frac{1}{2} |\alpha| \right)/p} \right\}. \tag{6}
$$

*Proof.* Let $V(\alpha) = \{i \in [n] \mid \exists (i_1, \ldots, i_p) \in \alpha, a \in [p] \text{ s.t. } i = i_a \}$ be the set of indices of $[n]$ that appear in $\alpha$. For each $i \in V(\alpha)$, let $d_i \geq 1$ be the total number of times $i$ appears as an index in $\alpha$. Since $|\rho| \leq 1$ and $\mathbf{1}(3a \in [p], i_a \notin R) \leq \frac{1}{p} \sum_{a \in [p]} \mathbf{1}(i_a \notin R)$, we have that

$$
E_{R \sim \mathcal{R}_n(s)} \left[ \prod_{(i_1, \ldots, i_p) \in \alpha} \rho^{k \cdot 1(3a \in [p], i_a \notin R)} \right] \leq E_{R \sim \mathcal{R}_n(s)} \left[ \prod_{i \in V(\alpha)} \rho^{k \cdot d_i \cdot 1(\bar{i} \notin R)} \right]
$$

$$
= \prod_{i \in V(\alpha)} E \left[ \rho^{k \cdot d_i \cdot 1(\bar{i} \notin R)} \right]
$$

$$
= \prod_{i \in V(\alpha)} \left( \frac{s}{n} + \left( 1 - \frac{s}{n} \right) \rho^{k \cdot d_i} \right)
$$

$$
\leq 2^{|V(\alpha)|} \cdot \max_{U \subseteq V(\alpha)} \left( \frac{s}{n} \right)^{|V(\alpha) \setminus U|} \cdot \frac{k}{\rho^{p/2}} \sum_{i \in U} d_i,
$$

$$
\leq \max_{U \subseteq V(\alpha)} \left( \frac{2s}{n} \right)^{|V(\alpha) \setminus U|} \cdot (2^{p/k} \rho)^{\frac{k}{p}} \sum_{i \in U} d_i,
$$

where to obtain the third line we have used the independence of the events $1(\bar{i} \notin R)$, in the penultimate line we have bounded the product expansion by its maximum term, and in the final line we have used that $d_i \geq 1$ for all $i \in U$. If $\sum_{i \in U} d_i \geq \frac{1}{2} |\alpha|$, then since $2s \leq n$ and $2^{p/k} \rho \leq 1$ we have $(\frac{s}{n})^{V(\alpha) \setminus U} \left( 2^{p/k} \rho \right)^{\frac{k}{p}} \sum_{i \in U} d_i \leq (2^{p/k} \rho)^{\frac{k}{p}} \frac{1}{2} |\alpha|$, and we have our conclusion. Otherwise suppose $\sum_{i \in U} d_i < \frac{1}{2} |\alpha|$ and consider the set tuples $\alpha'$ which do not contain elements from $U$. We have that $|\alpha'| \geq \frac{1}{2} |\alpha|$, because the elements of $U$ participate in at most $\sum_{i \in U} d_i$ tuples. Further, $|\alpha'| \leq (|V(\alpha) \setminus U|)^p$, since this is the number of distinct tuples of at most $p$ elements that can be formed from the elements of $V(\alpha) \setminus U$. Thus $|V(\alpha) \setminus U| \geq (\frac{1}{2} |\alpha|)^{1/p}$, and the bound now follows because $(\frac{s}{n})^{V(\alpha) \setminus U} \left( 2^{p/k} \rho \right)^{\frac{k}{p}} \sum_{i \in U} d_i \leq (\frac{2s}{n})^{(\frac{1}{2} |\alpha|)^{1/p}}$. \hfill \Box

Combining Equations (5) and (6) with a similar application of Parseval’s inequality as in Equation (4) from Lemma 5.4 now completes the proof of the lemma. \hfill \Box

Combining this lemma with Theorem 3.1 now yields that LDLR bounds for problems that can be realized as random submatrix or subtensor restrictions imply SQ lower bounds, as in Corollary 5.5 in the previous section. We remark that the bounds in Lemma 5.7 are nearly tight.\footnote{When $\rho = 0$, the diagonal Fourier coefficients corresponding to submatrices are given by $\hat{T}^i \hat{D}(R^{\otimes p}, \ldots, R^{\otimes p}) = (s/n)^{|R|} \hat{D}(R^{\otimes p}, \ldots, R^{\otimes p})$. This implies that the $(\frac{s}{n})^{(d + 1)/p}$ factor in the exponent of $(2s/n)^2(\frac{2}{d+1})^{1/p}$ in Lemma 5.7 is necessary.}
Remark 5.9. A final setting of interest (e.g. for multi-sample planted clique) is when $N = \binom{n}{p}$ and the indices of samples are identified with subsets in $\binom{[n]}{p}$. The natural notion of a random restriction is then to subsets of the form $\left( \frac{R}{p} \right) \in \binom{[n]}{p}$ where $R \sim R_n(s)$. Lemma 5.7 can be seen to handle this case as well: repeating the argument identically, but considering only tuples $(i_1, \ldots, i_p)$ with $i_1 < \cdots < i_p$, yields the following theorem.

**Theorem 5.10.** Let $p, s, n, k, d \in \mathbb{N}$ and $p \in (0, 1)$ with $2s \leq n$, $2^p/k \rho \leq 1$. Let $D_\varnothing$ be a product measure over $\{\pm\}^N$ where $N = \binom{n}{p}$, and let $T$ be a $(1, \rho)$-operator over $\{\pm\}$ (with respect to the measure induced by $D_\varnothing$ on a single coordinate). Then for $S = \{D_v\}_{v \in S}$ a family of distributions over $\{\pm\}^\binom{[n]}{p}$ with prior $\mu$, we have that the $(T, s)$-random restriction $S', \mu'$ of $S$ has degree $(> d, = k)$ bounded by

$$\left\| \mathbb{E}_{R \sim R_n(s)} \mathbb{E}_{u \sim \mu} \left( T_{(n)} \mathcal{D}_u^{(d)} \right)^{\otimes k} \right\|^2 \leq \max \left\{ 4^{d+1} \rho^{(d+1)/k/p}, \left( \frac{2s}{n} \right)^{2(d+1)/p} \right\} \cdot \left\| \mathbb{E}_{u \sim S} \mathcal{D}_u^{(d)} \right\|^2.$$

6 Specialization to Distributions with Independent Coordinates

In this section, we prove Theorems 6.1 and 6.3. In each case, we bound the high-degree part of the LR in terms of the LDLR and then apply Theorem 3.1 to deduce the result.

6.1 Identity-Covariance Gaussians

**Theorem 6.1.** Let $k$ be an even integer. For the null distribution $D_\varnothing = \mathcal{N}(0, I_n)$ and alternate distributions $S = \{D_v\}_{v \in S}$ with $D_v = \mathcal{N}(v, I_n)$, let $\mathcal{D}_u$ be the relative density of $D_u$ with respect to $D_\varnothing$. Suppose that the $2k$-sample likelihood ratio is bounded by $\| \mathbb{E}_u \mathcal{D}_u^{\otimes 2k} \|^2 \leq C^k$, and the $(1, 4k)$-LDLR$_m$ is bounded by $\| \mathbb{E}_u (\mathcal{D}_u)^{\leq 1, 4k} - 1 \| \leq \varepsilon$. Then for any $q \geq 1$,

$$\text{SDA} \left( S, \frac{m}{q^2/k} \mathcal{D}_u^{1/k} \left( \frac{1}{\varepsilon^{1/k} \left( 4^2k(1+C) \right)^{1/m}} \right) \right) \geq q.$$ 

We first will prove a lemma bounding the high-degree part of the LR in terms of its low-degree part.

**Lemma 6.2.** Let $S = \{D_u\}_{u \in S}$ be a set of identity-covariance Gaussian distributions, where $D_u = \mathcal{N}(u, I_n)$ and $D_\varnothing = \mathcal{N}(0, I_n)$. For each $u \in S$, let $\mathcal{D}_u$ be the relative density of $D_u$ with respect to $D_\varnothing$. For any integers $d, k \geq 1$ with $k$ even,

$$\left\| \mathbb{E}_u (\mathcal{D}_u^{(d)})^{\otimes k} \right\|^{2/k} \leq \frac{1}{(d+1)!} \mathbb{E}_{u,v} \left[ (\mathcal{D}_u^{\leq 1} \mathcal{D}_v^{\leq 1} - 1)^{2k(d+1)} \right]^{1/2k} \left( 1 + \| \mathbb{E}_u (\mathcal{D}_u^{\otimes 2k}) \|^2 \right)^{1/2k}.$$

**Proof.** We will exploit some properties of identity-covariance Gaussians. Let $\exp^{>d}(x) = \sum_{l=d+1}^{\infty} \frac{x^l}{l!}$ be truncation error of the degree-$d$ Taylor approximation of $\exp(x)$ about 0. In this setting, for each $u, v \in S$, it is shown in [KWB19] (Theorem 2.6) that

$$\langle \mathcal{D}_u^{\geq d}, \mathcal{D}_v^{\geq d} \rangle_{D_\varnothing} = \exp^{>d}(\langle u, v \rangle). \quad (7)$$

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By Taylor’s theorem, we have that \( \exp^{>d}(x) \) is bounded by
\[
|\exp^{>d}(x)| \leq \left| \frac{x^d}{(d+1)!} \cdot \exp(x) \right|,
\]
For some function \( \xi(x) \) with \( \text{sign}(\xi(x)) = \text{sign}(x) \) and \( |\xi(x)| \leq |x| \). Thus, using that \( k \) is even,
\[
\left\| \mathbb{E}_{u,v} (\mathcal{D}^{>d}_{u})^{k} \right\|^2 = \mathbb{E}_{u,v} \left[ \left( \mathcal{D}^{>d}_{u}, \mathcal{D}^{>d}_{v} \right)^k \right] \\
= \mathbb{E}_{u,v} \left[ \exp^{>d}(\langle u, v \rangle) \right]^k \\
\leq \mathbb{E}_{u,v} \left[ \frac{|\langle u, v \rangle|^{d+1}}{(d+1)!} \exp(|\xi(x)|) \right]^k \\
\leq \left( \frac{1}{(d+1)!} \right)^k \sqrt{\mathbb{E}_{u,v} \left[ \langle u, v \rangle^{2dk+2k} \right] \mathbb{E}_{u,v} \left[ \exp(\xi(x))^{2k} \right]} \\
\leq \left( \frac{1}{(d+1)!} \right)^k \sqrt{\mathbb{E}_{u,v} \left[ \langle u, v \rangle^{2dk+2k} \right] \mathbb{E}_{u,v} \left[ 1 + \exp(\xi(x))^{2k} \right]} \\
= \left( \frac{1}{(d+1)!} \right)^k \sqrt{\mathbb{E}_{u,v} \left[ \langle \mathcal{D}^{-1}_{u}, \mathcal{D}^{-1}_{v} \rangle - 1 \right]^{2dk+2k} \left( 1 + \mathbb{E}[\langle \mathcal{D}^{<d}_{u}, \mathcal{D}^{<d}_{v} \rangle^{2k}] \right)}.
\]
The fourth line follows from Cauchy-Schwarz, and the fifth line uses that \( \text{sign}(\xi(x)) = \text{sign}(x) \) and therefore \( 1 + \exp(x) \geq \max(1, \exp(x)) \geq |\exp(\xi(x))| \). The final line then follows from (7). Substituting this back in for the above, we have our desired conclusion.

**Proof of Theorem 6.1.** We will show that a more general result holds given \( \| \mathbb{E}_{u}(\mathcal{D}^{<m}_u)^{d,2k(d+1)} - 1 \| \leq \varepsilon \), and then set \( d = 1 \). By Lemma 3.5, we have that
\[
\left\| \mathbb{E}_{u}(\mathcal{D}^{<1}_u - 1)^{\otimes 2k(d+1)} \right\|^2 \leq \left\| \mathbb{E}_{u}(\mathcal{D}^{<d}_u - 1)^{\otimes 2k(d+1)} \right\|^2 \leq \frac{\varepsilon^2}{m}.
\]
Therefore Lemma 6.2 implies that
\[
\left\| \mathbb{E}_{u}(\mathcal{D}^{>d}_u)^{\otimes k} \right\|^{2/k} \leq \frac{1}{(d+1)!} \cdot \frac{\varepsilon^{1/k}}{(2k(d+1))^{1/2k}} \left( 1 + C^k \right)^{1/2k} \\
\leq \frac{1 + C}{(d+1)!} \cdot \frac{\varepsilon^{1/k}(2k(d+1))^{d+1}}{m^{d+1}} \\
\leq (1 + C) \cdot \frac{\varepsilon^{1/k}(2ke)^{d+1}}{m^{d+1}}
\]
using Stirling’s approximation to the factorials and the fact that \( \binom{d}{b} \geq (a/b)^b \). Since \((d,k)\)-LDLR \( m \leq (d,2k(d+1))\)-LDLR \( m \), we also have that \( \| \mathbb{E}_{u}(\mathcal{D}^{<m}_u)^{d,k} - 1 \| \leq \varepsilon \). Now applying Theorem 3.1 to the \((d,k)\)-LDLR \( m \) and then setting \( d = 1 \) completes the proof of the theorem.

### 6.2 Product Measures Over the Boolean Hypercube

**Theorem 6.3.** Let \( k \) be an even integer. Let \( S = \{D_u\}_{u \in S} \) be a set of product distributions over the \( n \)-dimensional hypercube. Let \( D_\emptyset \) be any product measure over \( \{\pm1\}^n \) with no fixed coordinates,
and let \( \overline{D_u} \) be the relative density of \( D_u \). Suppose that the \( 2k \)-sample likelihood ratio is bounded by \( \| E_u \overline{D_u}^{\otimes 2k} \|^2 \leq C^k \), and the \((1, 4k)\)-LDLR \( m \) is bounded by \( \| E_u \overline{D_u}^{\otimes m} \|_{1, 4k} \leq \varepsilon \). Then for any \( q \geq 1 \),

\[
\text{SDA} \left( S, \frac{m}{q^{2/k} \varepsilon^{1/k} k} \left( \frac{1}{\varepsilon^{1/k} k + 16kC^{1/2}} \right) \right) \geq q.
\]

We again will prove a lemma bounding the high-degree part of the LR in terms of its low-degree part.

**Lemma 6.4.** Let \( S = \{ D_u \}_{u \in S} \) be a set of product distributions over the \( n \)-dimensional hypercube. Let \( D \) be any product measure over \( \{\pm 1\}^n \) with no fixed coordinates, and let \( \overline{D_u} \) be the relative density of \( D_u \). For any integers \( d, k \geq 1 \) with \( k \) even,

\[
\left\| E_u (\overline{D_u}^{\otimes d})^{\otimes 2k} \right\|^2 \leq E_{u,v \sim S} \left[ \left( \| \overline{D_u} \|_{1, 1} \| \overline{D_v} \|_{1, 1} \right)^{2k(d+1)} \right]^{1/2} \left\| E_u \overline{D_u}^{\otimes 2k} \right\|.
\]

**Proof.** As in Lemma 6.2, \( \| E_u (\overline{D_u}^{\otimes d})^{\otimes 2k} \| = E_{u,v \sim S} \left( \| \overline{D_u} \|_{1, 1} \| \overline{D_v} \|_{1, 1} \right)^{2k(d+1)} \). We let \( \chi_i(x) \) be the unique function such that \( E_{x \sim D} \chi_i(x) = 0 \), \( E_{x \sim D} \chi_i(x)^2 = 1 \), and \( \chi_i(x) \geq 0 \) when \( x_i = 1 \). For convenience, we associate each \( u \in S \) with a vector \( u \in \mathbb{R}^n \) as follows: if \( D_u \) is the (unique) product measure \( P_u \) over \( \{\pm 1\}^n \) with \( E_{x \sim D_u} [\chi_i(x)] = u_i \). Let \( e_k : \mathbb{R}^n \to \mathbb{R} \) be the \( k \)th elementary symmetric polynomial:

\[
e_k(x) = \sum_{S \subseteq [n], |S| = k} \prod_{i \in S} x_i.
\]

For any \( t \in [n] \), using standard Fourier analysis over the Boolean hypercube one can see that

\[
\langle \overline{D_u}^{\otimes t}, \overline{D_v}^{\otimes t} \rangle = \sum_{S \subseteq [n], |S| = t} E_{D_u} \left[ \prod_{i \in S} \chi_i(x) \right] E_{D_v} \left[ \prod_{i \in S} \chi_i(x) \right] = \sum_{S \subseteq [n], |S| = t} \prod_{i \in S} u_i v_i = e_t(u \circ v),
\]

where \( u \circ v \in \mathbb{R}^n \) is the Hadamard (or “entrywise”) product of \( u \) and \( v \). So we may re-express

\[
\langle \overline{D_u}^{\otimes d}, \overline{D_v}^{\otimes d} \rangle = \sum_{t = d+1}^{n} e_t(u \circ v). \tag{8}
\]

We will exploit the following claims regarding polynomials in \( u \circ v \) and the elementary symmetric polynomials:

**Claim 6.5.** Let \( A \) be any multiset of elements from \( [n] \), and for a vector \( x \in \mathbb{R}^n \) denote by \( x^A = \prod_{i \in A} x_i \). Then, for any set \( S \subseteq \mathbb{R}^n \),

\[
E_{u,v \sim S} (u \circ v)^A = E_{u,v \sim S} \prod_{i \in A} u_i v_i = \left( E_{u \sim S} u^A \right)^2 \geq 0.
\]

The proof of Claim 6.5 is evident from the expression above. One consequence is the following:

**Claim 6.6.** Let \( p : \mathbb{R}^{n+1} \to \mathbb{R} \) be any polynomial which is a sum of monomials with non-negative coefficients, let \( S \subseteq \mathbb{R}^n \) and for each \( u \in S \) let there be a \( \lambda_u \in \mathbb{R} \). Then for any integers \( a, b \geq 1 \),

\[
E_{u,v} [e_{a+b}(u \circ v) \cdot p(u \circ v)] \leq E_{u,v} [e_a(u \circ v) \cdot e_b(u \circ v) \cdot p(u \circ v)].
\]
Proof. For any \( x \in \mathbb{R}^n \), we can expand the product

\[
e_a(x)e_b(x) = \sum_{A \subseteq [n]} x^A \sum_{B \subseteq [n]} x^B = \sum_{i=0}^{\min\{a, b\}} \sum_{I \subseteq [n]} x^{2I} \sum_{S,T \subseteq [n], |S|=a-|I|=i} x^{S \cup T},
\]

where we have arranged the second sum according to the intersection size \( i \) that a monomial from \( e_a \) and a monomial from \( e_b \) may have. Extracting the \( i = 0 \) summand, we have that

\[
\sum_{S,T \subseteq [n], |S|=a, |T|=b} x^{S \cup T} = \binom{a+b}{a} e_{a+b}(x),
\]

since each set \( S \cup T \) is counted in this sum \( \binom{a+b}{a} \) times. Write \( p(x') = \sum_{C} \hat{p}_C \cdot (x')^C \) where the sum is over monomials. Therefore we have that

\[
e_a(x) \cdot e_b(x) \cdot p(x') = \binom{a+b}{a} e_{a+b}(x) \cdot p(x') + q(x) p(x'),
\]

where \( q(x) \) (the summation over over \( i > 0 \)) is a sum of monomials with non-negative coefficients. The claim now follows from taking expectations on both sides and applying Claim 6.5.

Given these facts and (8), we can deduce the following upper bound:

\[
\mathbb{E}_{u,v} \left[ \langle \mathcal{D}^d_u, \mathcal{D}^d_v \rangle^k \right] = \mathbb{E}_{u,v} \left[ \left( \sum_{t=d+1}^n e_t(u \circ v) \right)^k \right]
\]

\[
= \mathbb{E}_{u,v} \left[ \sum_{t=d+1}^n e_t(u \circ v) \cdot \left( \sum_{t=d+1}^n e_t(u \circ v) \right)^{k-1} \right]
\]

\[
\leq \mathbb{E}_{u,v} \left[ \sum_{t=d+1}^n e_{d+1}(u \circ v) \cdot e_{t-(d+1)}(u \circ v) \cdot \left( \sum_{t=d+1}^n e_t(u \circ v) \right)^{k-1} \right]
\]

\[
= \mathbb{E}_{u,v} \left[ e_{d+1}(u \circ v) \cdot \left( \sum_{s=0}^{n-d-1} e_s(u \circ v) \right) \left( \sum_{t=d+1}^n e_t(u \circ v) \right)^{k-1} \right],
\]

Where to obtain the inequality we have applied Claim 6.6 with \( p = \left( \sum_{t=d+1}^n e_t(u \circ v) \right)^{k-1}, a = d+1, \) and \( b = t - d - 1 \). Repeating this for the \( k-1 \) remaining powers, we have

\[
\leq \mathbb{E}_{u,v} \left[ e_{d+1}(u \circ v) \cdot \left( \sum_{s=0}^{n-d-1} e_s(u \circ v) \right)^k \right]
\]

\[
\leq \mathbb{E}_{u,v} \left[ e_{d+1}(u \circ v) \cdot \left( \sum_{s=0}^n e_s(u \circ v) \right)^k \right]
\]

\[
= \mathbb{E}_{u,v} \left[ e_{d+1}(u \circ v) \cdot \langle \mathcal{T}_u, \mathcal{T}_v \rangle^k \right],
\]

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where in the second-to-last line we have used Claim 6.5 to add the terms for \( s = n - d, \ldots, n \) as they contribute positively to the expectation. Applying Cauchy-Schwarz to the conclusion of the above display,

\[
\mathbb{E}_{u,v} \left[ \mathcal{D}_{u}^{>d}, \mathcal{D}_{v}^{>d} \right]^{k} \leq \sqrt{\mathbb{E}_{u,v} \left[ (D_{u}, D_{v})^{2k} \right]} \leq \sqrt{\mathbb{E}_{u,v} \left[ \mathcal{D}_{u}^{<d+1}, \mathcal{D}_{v}^{<d+1} \right]^{2k(d+1)}} \leq \mathbb{E}_{u,v} \left[ \mathcal{D}_{u}^{<d+1}, \mathcal{D}_{v}^{<d+1} \right]^{2k(d+1)} \equiv \mathbb{E}_{u,v} \left[ \mathcal{D}_{u}^{<d} \right]^{2k(d+1)},
\]

where we have used that \( \mathbb{E}_{u,v} \left( (\mathcal{D}_{u}^{<d+1}, \mathcal{D}_{v}^{<d+1}) - 1 \right)^{2k(d+1)} \geq \mathbb{E}_{u,v} (e_{d+1}(u \circ v))^{2k}, \) again by applying Claim 6.5 in a similar manner to the proof of Claim 6.6. This completes the proof. \( \square \)

**Proof of Theorem 6.3.** As in the proof of Theorem 6.3, we will show that a more general result holds given \( \| \mathbb{E}_{u}(\mathcal{D}_{u}^{m})^{d,2k(d+1)} - 1 \| \leq \varepsilon, \) and then set \( d = 1. \) By Lemma 3.5, we have that

\[
\left\| \mathbb{E}_{u}(\mathcal{D}_{u}^{\leq d} - 1) \otimes 2k(d+1) \right\|^{2} \leq \left\| \mathbb{E}_{u}(\mathcal{D}_{u}^{\leq d} - 1) \otimes 2k(d+1) \right\|^{2} \leq \frac{\varepsilon^{2}}{(2k(d+1))^{2k/d+1}}.
\]

The same application of Lemma 3.5 as in the proof of Theorem 6.3 and Lemma 6.2 imply that

\[
\left\| \mathbb{E}_{u}(\mathcal{D}_{u}^{>d})^{k} \right\|^{2/k} \leq \frac{C^{1/2} \varepsilon^{1/k}}{(2k(d+1))^{2k/d+1}} \leq \frac{C^{1/2} \varepsilon^{1/k} (2k(d+1))^{d+1}}{m^{d+1}}
\]

using the fact that \( \binom{a}{b} \geq (a/b)^{b}. \) As in the proof of Theorem 6.3, we have that \( \| \mathbb{E}_{u}(\mathcal{D}_{u}^{m})^{d,1} - 1 \| \leq \varepsilon. \) Applying Theorem 3.1 to the \((d, k)\)-LDLR\(_{m}\) and then setting \( d = 1 \) completes the proof of the theorem. \( \square \)

### 7 Diluting the Power of Statistical Queries via Cloning: Leveling the Playing Field

As discussed in Remark 1.9, many average-case problems of interest such as planted clique and tensor PCA do not have a natural notion of samples. In contrast, the SQ framework requires problem formulations involving multiple samples. In this section we describe how to convert certain single sample problems into multiple-sample problems, and then address the question of how to choose the number of samples so that the SQ complexity of the resulting problem captures the computational complexity of the original problem (as predicted by e.g. low-degree tests).

**Multi-sample formulations of single-sample problems.** The idea is to apply an SQ bound to a “diluted” or “cloned” version of the single-sample problem, wherein each “dilute” sample carries little information compared to a single sample. When multiple cloned samples can be combined into one original sample in polynomial time, a lower bound against the cloned problem implies a lower bound against the original problem (within the framework of polynomial time algorithms).

We first state a general and somewhat obvious sufficient condition for the existence of an average-case reduction from a multi-sample problem to a single-sample problem. A computational lower bound for the multi-sample problem is then transferred to the single-sample problem via the reduction.

**Fact 7.1.** Let \( D_{\Theta} \) and \( S = \{D_{u}\}_{u \in S} \) be distributions on \( \mathbb{R}^{N} \) and let \( \mu \) be a prior over \( S. \) Let \( \{P_{\theta}\}_{\theta \in \Omega} \) be an exponential family of distributions on \( \mathbb{R}^{N} \) with sufficient statistic \( T \) that can be computed in time polynomial in the size of its input. Suppose that for each distribution \( D \in \{D_{\Theta}\} \cup S, \) there is
a $\theta = \theta(D)$ such that if $Y_1, \ldots, Y_m \overset{i.i.d.}{\sim} P_\theta$ then $T(Y_1, \ldots, Y_m) \sim D$. Then if there is no polynomial time algorithm testing between $H_0 : (Y_1, \ldots, Y_m) \sim P_{\theta(D_0)}^{\otimes m}$ versus $H_1 : (Y_1, \ldots, Y_m) \sim P_{\theta(D_1)}^{\otimes m}$ where $u \sim \mu$, with Type I$+\text{II}$ error $1 - \varepsilon$, then the same is true for the original testing problem.

If one can efficiently generate $m$ samples $Y_1, \ldots, Y_m$ as described in the fact just above given the single sample $X$, then the mapping is invertible, which implies that no signal is lost and the single and multi-sample versions of the problem are computationally and statistically equivalent. Note that by the definition of sufficient statistic it is possible to generate samples with given sufficient statistic, but it is not always possible to do so efficiently (assuming the widely believed computational complexity conjecture $\text{RP} \neq \text{NP}$) [BGS14, Mon14].

We now describe two examples where simple randomized algorithms show that it is possible to generate samples efficiently given a sufficient statistic. In the first, the data consists of unit variance Gaussians, for which the mean is the sufficient statistic.

**Lemma 7.2** (Gaussian Cloning). There is a randomized algorithm taking as input a real number $x$ and outputting $m$ independent random variables $Y_1, \ldots, Y_m$ such that for any $\mu \in \mathbb{R}$ if $x \sim \mathcal{N}(\mu, 1)$, then $Y_i \sim \mathcal{N}(\mu/\sqrt{m}, 1)$.

We will give the proof in Appendix C. In the second example, we show that the planted clique problem has an equivalent multi-sample version. Given a subset $U \subseteq [n]$, let $\mathcal{G}(n, U, \gamma)$ denote the distribution of $G(n, \gamma)$ conditioned on the vertices in $U$ forming a clique (again see Appendix C for a proof). This reduction is a mild variant of Bernoulli Cloning in [BBH18], which corresponds to the regime where $m = O(1)$.

**Lemma 7.3** (Planted Clique Cloning). There is an algorithm that when given $m$ independent samples from $\mathcal{G}(n, U, \gamma)$ for any $U \subseteq [n]$, efficiently produces a single instance distributed according to $\mathcal{G}(n, U, \gamma^m)$. Conversely, there is an efficient algorithm taking a graph as input and producing $m$ random graphs, such that given an instance of planted clique $\mathcal{G}(n, U, \gamma)$ with unknown clique position $U$, produces $m$ independent samples from $\mathcal{G}(n, U, \gamma^{1/m})$.

The same equivalence holds in the hypergraph formulation of planted clique. The Gaussian cloning algorithm runs in $\text{poly}(m)$ time given access to an oracle for sampling standard normal random variables. When applied entry-wise, this cloning procedure can be used to show average-case equivalences between single and multi-sample variants of problems with Gaussian noise such as tensor PCA and the spiked Wigner model. Furthermore, increasing the number of samples from 1 to $m$ dilutes the level of signal in the problem exactly by a factor of $1/\sqrt{m}$. The planted clique cloning algorithm runs in $\text{poly}(m, n)$ randomized time. This again shows a precise tradeoff between the level of signal and number of samples $m$ – as the ambient edge density varies as $\gamma$ to $\gamma^{1/m}$ with the number of samples $m$.

**Choosing the number of samples.** The number of queries used by statistical query algorithms is a proxy for runtime. However, the statistical query framework allows queries that cannot be computed in polynomial time, and for this reason can lead to predictions that do not correspond to polynomial time algorithms. For example, a naive application of the statistical query framework in [FGR$^+$17] to the planted clique problem treats an instance as a single sample from the planted clique distribution has a single-query $\text{VSTAT}(\frac{1}{2})$ algorithm, using the $\{0, 1\}$ query: does the graph $G$ have a clique of size at least $k$?

For this reason, prior SQ lower bounds for planted clique [FGR$^+$17] consider instead the planted biclique problem in a bipartite graph, and furthermore, assumed that i.i.d. data is generated by
observing a random column from the adjacency matrix. While this is an interesting problem to study, it is not known to be equivalent to planted clique, the original problem of interest. More troubling is that this approach of generating samples fails badly for hypergraph planted clique. If one views a sample as a random slice of the adjacency tensor, then statistical query algorithms can perform an exhaustive search over what amounts to an instance of planted clique and this succeeds if at least one sample contains a planted clique, which occurs with positive probability once one has \( n/k \) samples.

The methodology described earlier in this section of converting a single-sample problem to many-sample problem is applicable to a broad class of problems and thus gives a unified way of addressing a variety of problems within the SQ framework. If we are free to study multi-sample versions of problems, it remains to specify the correct number of samples in order to obtain meaningful predictions within the SQ framework. As noted in the introduction, a prescription is suggested by Theorem 1.6: we should dilute the signal so that each the problem is information-theoretically unsolvable from \( O(1) \) samples. Concretely, we convert to a hypothesis testing problem with \( m \) samples, \( D_\emptyset^\otimes m \) vs. \( D_u^\otimes m \) where \( \| E_u D_u \| = O(1) \).

8 Example Applications

8.1 Tensor PCA

Problems 8.1 (Tensor Principal Components Analysis (PCA)). For \( n, r \) positive integers, \( \lambda \in \mathbb{R} \), and \( S = \{ \pm \frac{1}{\sqrt{n}} \}^n \), the \( n \)-dimensional \( r \)-tensor PCA with signal strength \( \lambda \) problem is the following many-vs-one hypothesis testing problem:

- Null: a tensor in \( (\mathbb{R}^n)^{\otimes r} \) with independent standard Gaussian entries, \( D_\emptyset = \mathcal{N}(0, I_{nr}) \).
- Alternate: uniform mixture of \( D_u = \mathcal{N}(\lambda \cdot u^{\otimes r}, I_{nr}) \) over \( u \in S \).

Variations on the tensor PCA problem are possible; for example one may insist that the tensors be symmetric, or that \( S \) be a different subset of \( S^{n-1} \).

Claim 8.2. For any integers \( k, n, r \geq 2 \) satisfying \( k \lambda^2 < \frac{n}{2} \), the \( k \)-sample likelihood ratio for the \( n \)-dimensional \( r \)-tensor PCA problem with signal strength \( \lambda \) is bounded by

\[
\left\| E_{u \sim S} D_u^{\otimes k} \right\|^2 \leq \sqrt{\frac{2\pi}{1 - \frac{2k\lambda^2}{n}}}.
\]

We prove this claim in Appendix D.1.

Claim 8.3. For any integers \( n, r, k, m \) and real number \( \lambda \) which satisfy \( 2em \lambda^2 k^{(r-2)/2} \leq n^{r/2} \), the \( (1,k) \)-LDLR \( m \) for the \( m \)-sample, dimension-\( n \) tensor PCA problem with signal strength \( \lambda \) is bounded by

\[
\left\| E_{u \sim (D_u^{\otimes m})^{\otimes k}} \right\|^2 \leq \frac{2^{e^{r+1}} m \lambda^2 k^{(r-2)/2}}{n^{r/2}}.
\]

The proof is a straightforward calculation which appears in [HKP+17, KWB19]—these works consider the single-sample version, but it is not difficult to see that their bounds imply ours. For completeness we give a full proof in Appendix D.1. Together these claims are sufficient to deduce the following Corollary of Theorem 6.1.
Corollary 8.4. For integers $k,n,m,r$ and real numbers $\lambda, \delta$ with $\delta \in (0,1)$ satisfying
\[
|\lambda| \leq \min \left( \sqrt{\frac{n}{(4k)^{(r-2)/r}}} \, \frac{1}{2em}, \sqrt{(1-\delta) \frac{n}{4k}} \right), \text{ and } 4e^2k \left( 1 + \frac{2\pi}{\delta} \right)^{1/k} \leq \frac{m}{2}.
\]
then for the $n$-dimensional $r$-tensor PCA problem with signal strength $\lambda$, for all $q \geq 1$, $SDA(\frac{m}{sq^{1/k}k}) \geq q$.

Proof. By Claims 8.2 and 8.3 and our assumptions, we have that
\[
\left\| E_u(D_u)^{\otimes 2k} \right\|^2 \leq \sqrt{\frac{2\pi}{1-\frac{4k\lambda^2}{n}}} \leq \sqrt{\frac{2\pi}{\delta}}, \quad \left\| E_u(D_u)^{\otimes em} \right\|^2 \leq \frac{2em\lambda^2(4k)^{(r-2)/2}}{nr/2} \leq 1.
\]
We instantiate Theorem 6.1 with $C = \left( \frac{2\pi}{\delta} \right)^{1/k}$ and $\varepsilon = 1$, and using our assumption on $\delta$ we have our conclusion. \hfill \Box

Comparison with prior work and predictions. In the literature, it is most common to consider the single-sample version of tensor PCA; for translations’ sake, notice that $m$ samples from $\mathcal{N}(\lambda u^{\otimes r}, I_{nr})$ are equivalent to a single sample from $\mathcal{N}(\sqrt{m}\lambda u^{\otimes r}, I_{nr})$, since the sum of the samples is a sufficient statistic. So we compare the $m$-sample problem to the single-sample hypothesis testing problem with signal strength $\sqrt{m}\lambda$. Similarly, we compare the VSTAT($M$) to the single-sample hypothesis testing problem with signal strength $\sqrt{M}\lambda$.

Applying this transformation, the best $n^k$-time algorithms for the $n$-dimensional $r$-tensor PCA problem requires signal strength $\sqrt{m}\lambda \geq \tilde{\Omega} \left( \sqrt{k} \left( \frac{n}{k} \right)^{r/4} \right)$ [BGL17, RRS17, WEAM19]. To see that this is consistent with the obtained VSTAT($M$) bound with $M = \frac{m}{sekq^{1/k}}$, note that by Theorem A.5 our bound implies that any $q = 2^k$-query algorithm requires the “adjusted signal strength” to satisfy either $\lambda^2k = \Omega(\sqrt{n})$ (which we will discuss below) or
\[
\sqrt{M|\lambda|} \geq \left( \frac{n}{(4k)^{(r-2)/r}} \right)^{r/4} \sqrt{\frac{1}{16ekq^{2/k}}} = \Omega \left( \frac{1}{2} \left( \frac{n}{k} \right)^{r/4} \right).
\]
In the $k \gg \log n$ regime, this is equivalent to the performance of the best-known algorithms up to a factor of $\tilde{O}(\sqrt{k})$.

We remark as well that the condition $\lambda^2k < O(\sqrt{n})$ is necessary to rule out statistical query algorithms which use brute force on individual samples. If $\lambda^2 > 100n$, then there is a single-query SQ algorithm for the many-vs-one hypothesis testing problem: for a given sample $T \in (\mathbb{R}^n)^{\otimes r}$, simply query whether there exists some vector $x \in \{\pm \frac{1}{\sqrt{n}}\}^n$ which achieves $|\langle x^{\otimes r}, T \rangle| \geq \frac{1}{2} \lambda$. When $|\lambda| \geq 10\sqrt{n}$, it is easy to see that for $T \sim D_{\theta}$ this query will return false with high probability; this follows from the fact that $\langle x^{\otimes r}, T \rangle \sim \mathcal{N}(0, I_{nr})$. On the other hand, for any $T \sim D_u$, this query will return true with high probability for similar reasons.

8.2 Planted Clique and Planted Dense Subgraph

In this section, we consider several formulations of planted clique (PC) and planted dense subgraph (PDS). We begin by using our results to reproduce SQ lower bounds for “bipartite” formulations previously considered in the SQ literature [FGR+17], and then give new SQ lower bounds for non-bipartite multi-sample formulations.

\[13\] No effort has been made to optimize the constants, which may be improved using, e.g., chaining arguments.
8.2.1 Bipartite Models

The classical planted clique problem is a single-sample problem, which makes it incompatible with the SQ framework. In an effort to address the complexity of the PC problem, the authors of [FGR+17] give an SQ lower bounds for the following related problem: “bipartite planted clique” where each column of the resulting adjacency matrix is treated as an i.i.d. sample from a mixture distribution.

**Problem 8.5** (Bipartite Planted Dense Subgraph/Planted Clique). Given \( K, N \in \mathbb{N} \) and \( 0 < q < p \leq 1 \), bipartite planted dense subgraph with edge densities \( p \) and \( q \) is the following simple-vs-simple hypothesis testing problem:

- Null: independent Bernoulli random variables \( D_\emptyset = \text{Ber}(q)^{\otimes N} \).
- Alternate: the mixture of \( D_u = \frac{K}{N} \cdot D_u' + (1 - \frac{K}{N}) \cdot \text{Ber}(q)^{\otimes N} \) over random subsets \( u \subseteq [N] \), sampled by including each element of \([N]\) in \( u \) independently with probability \( K/N \). Here, \( D_u' \) is the distribution of \( x \in \{0,1\}^N \) with independent entries and \( \Pr[x_i = 1] = p \) if \( i \in u \) and \( \Pr[x_i = 1] = q \) otherwise.

The bipartite planted clique problem is the bipartite PDS problem with \( p = 1 \).

**LDLR and \( k \)-sample LR bounds.** The following claims carry out standard computations to identify the relevant quantities needed to apply our main theorems. These calculations are deferred to Appendix D.2. Let \( \mu \) denote the distribution over \( u \) described in the alternate hypothesis above.

**Claim 8.6.** For any \( K, N, k, d, m \in \mathbb{N} \), define \( \gamma = \frac{(p-q)^2}{q(1-q)} \). Then the \((d,k)\)-LDLR\(_m\) for bipartite PDS is bounded \( \| E_{u \sim \mu} (D_u^{\otimes m})^{d,k} - 1 \| = O_N(1) \) if

\[
\frac{K^2}{N} \cdot \max \left\{ \frac{m}{N}, (1 + \gamma)^k \right\} \leq 1 - \Omega_N(1).
\]

**Claim 8.7.** For any \( K, N, k \in \mathbb{N} \), the \( k \)-sample LR is bounded by \( \| E_{u \sim \mu} D_u^{\otimes k} \| = O_N(1) \) if

\[
\frac{K^2}{N} \cdot \max \left\{ \frac{k}{N}, (1 + \gamma)^k \right\} \leq 1 - \Omega_N(1)
\]

where \( \gamma = \frac{(p-q)^2}{q(1-q)} \).

**Implications of our results.** Given these computations, we now can deduce the following implication of Corollary 5.5.

**Corollary 8.8.** Suppose that \( K = \Theta(N^{1/2-\delta}) \) for some small constant \( \delta > 0 \) and \( 0 < q < p \leq 1 \) are constants. Then for bipartite PC and PDS with \( N \) vertices, edge densities \( 0 < q < p \leq 1 \) and planted dense subgraph size \( K \), it holds that SDA\((N) = N^{\omega(1)}\).

**Proof.** Let \( T \) be the noise operator that resamples independently from \( \text{Ber}(q) \), so \( T \) is a \((1,0)\)-operator. Note that bipartite PDS with \( K = \Theta(N^{1/2-\delta}) \) can be realized as a random restriction with noise operator \( T \) of bipartite PDS with \( K = \Theta(N^{1/2-\delta/2}) \), restriction probability \( s/N = N^{-\delta/2} \) and noise parameter \( \rho = 0 \). Suppose that \( d, k = \Theta((\log N)^{c_1}) \) where \( c_1 \in (0,1) \) and \( d/k \sim c_2 \) where \( c_2 \) is a sufficiently large constant. If again \( m = \Theta(N^{1+\delta}) \), then the parameters for both the restricted and unrestricted bipartite PDS instances satisfy condition (1) in Claims 8.6 and 8.7. Now consider applying Corollary 5.5 with dimension lower bound \( q' \sim 2^{k(\log N)^{c_3}} \) for some constant \( c_3 \in (1-c_1,1) \). If \( c_2 \) is sufficiently large, then \((2s/N)^{2(d+1)/k} m = o(1)\) and we have that SDA\((N) \geq q' = N^{\omega(1)}\). \(\square\)
Remark 8.9. Our generic noise-robustness result (Theorem 5.2) also recovers this lower bound in the case of bipartite PDS when $p < 1$. We choose $T$ to be the $(1, \rho)$-noise operator that resamples entries independently from $\text{Ber}(q)$ with probability $1 - \rho = \frac{p - q}{q}$. Then the distributions $D_u$ can be realized by applying $T$ entrywise to an instance of bipartite PC with edge density $q$. Note that the parameters $d \sim c_1 \log N$ for a sufficiently large constant $c_1$, $k \sim c_2 \log N$ for a sufficiently small constant $c_2$, $K = \Theta(N^{1/2 - \delta})$ and $m = \Theta(N^{1+\delta})$ satisfy condition (1) in Claims 8.6 and 8.7 for both the bipartite PDS instance in question and the bipartite PC instance before applying $T$. Now apply Theorem 5.2 with dimension lower bound $q' \sim 2^{k(\log N)^{c_3}}$ for some constant $c_3 \in (0, 1)$. If $c_1$ is sufficiently large, then $\rho^{2(d+1)m} = o(1)$ and it again follows that $\text{SDA}(N) \geq q' = N^{\omega(1)}$. We also remark that, unlike in our previous applications of our main results where we set $q' = 2^k$, we must take $q' = 2^{\omega(k)}$ in this application of our noise-robustness theorem to show superpolynomial SQ lower bounds.

Comparison to prior work and predictions. Corollary 8.8 recovers the $K = \Theta(N^{1/2 - \delta})$ barrier from [FGR+17] at which the SDA for bipartite PC/PDS with constant edge densities ceases to be poly$(N)$. Despite being the consequence of a much more general theorem on random restrictions, our results for bipartite PC/PDS also nearly recover the precise SDA lower bounds from [FGR+17]. In [FGR+17], for planted clique with edge density $1/2$, it is shown that $\text{SDA}(\frac{N^2}{2^{d+1}K^2}) \geq N^{2\delta^3}$ for all $\ell \leq K$. Fine-tuning our parameter choices in Corollary 8.8 yields that $\text{SDA}(\frac{N^2}{2^{d+1}K^2}) \geq N^{\Omega(\ell)}$ for any constant $\epsilon > 0$, which matches the bound from [FGR+17] up to arbitrarily small polynomial factors in the sample complexity.

8.2.2 Multi-Sample Hypergraph Planted Clique

We now consider a variant of planted clique where the observations consist of multiple samples from the planted clique distribution. As discussed in Section 7, there is a natural tradeoff between the number of samples $m$ and edge density $q$ for which this variant has an average-case equivalence with ordinary PC. In this section, we will treat a generalization of this variant to $s$-uniform hypergraphs (including the case $s = 2$ corresponding to simple graphs).

Let $G_s(N, q)$ denote the Erdős-Rényi distribution over $s$-uniform hypergraphs, where each $s$-subset of $[N]$ is included as a hyperedge independently with probability $q$. Given a subset $u \subseteq [N]$, let $G_s(N, u, q)$ denote the hypergraph where hyperedges among the vertices within $u$ are always included and all other hyperedges are included independently with probability $q$. Throughout this section, we will treat $s$ as a fixed positive integer constant.

Problem 8.10 (Multi-Sample Hypergraph PC). Given $s, K, N \in \mathbb{N}$ with $N \gg K \gg s \geq 2$ and $q \in (0, 1)$, the multi-sample $s$-uniform hypergraph planted clique problem with edge density $q$ is the following hypothesis testing problem:

- Null: the Erdős-Rényi hypergraph $D_\emptyset = G_s(N, q)$.
- Alternate: uniform mixture of $D_u = G_s(N, u, q)$ over $K$-subsets $u \subseteq [N]$.

The complexity of multi-sample hypergraph PC as $m$ and $q$ vary. To the best of our knowledge, multi-sample hypergraph PC has not been considered in this generality before. However, because of the average-case equivalence from Section 7, its complexity can be extrapolated exactly from that of ordinary hypergraph planted clique, i.e. when $m = 1$. For $m = 1$, its complexity conjecturally behaves as follows (as a function of $q$):
1. If $q$ is near constant with $N^{-o(1)} \leq q \leq 1 - N^{-o(1)}$, then the threshold at which polynomial-time algorithms begin to solve the distinguishing problem is $K^2 = N^{1-o(1)}$, which is consistent with the threshold in the classical setting of $q = \frac{1}{2}$.

2. If $q$ is polynomially small with $q = \Theta(N^{-\alpha})$ for some $\alpha > 0$, then the clique number of $\mathcal{G}_s(N, q)$ is constant and the problem begins to be easy when $K = \Theta(1)$.

3. If $q$ is very close to 1 with $q = 1 - \Theta(N^{-\alpha})$ for some $\alpha \in (0, 1)$, then polynomial-time algorithms begin to solve the distinguishing problem at the shifted threshold $K^2 = \Theta(N^{1+\alpha/s})$.

The best known algorithm in the last regime simply counts the total number of edges. In the graph case when $s = 2$, it was shown in [BBHI18] that the PC conjecture with $q = 1/2$ implies a lower bound up to the barrier $K^2 = \tilde{\Theta}(N^{1+\alpha/2})$ when $q = 1 - \Theta(N^{-\alpha})$. We remark that, in this regime, recovering the vertices in the planted clique is conjectured to be a harder problem that only becomes easy at larger values of $K$. Our focus in this section will be on the transition in the first parameter regime, when $N^{-o(1)} \leq q \leq 1 - N^{-o(1)}$.

As discussed in Section 7, there is a natural average-case equivalence between the single and multi-sample problems. Specifically, hypergraph PC with $m$ samples and edge density $q$ is equivalent to hypergraph PC with $m = 1$ sample and edge density $q^m$. Thus the parameter regime of interest corresponds to the $q$ with $\frac{1}{mN^o(1)} \leq 1 - q \ll \frac{\log N}{m}$. We remark that at $1 - q = \Theta(\frac{\log N}{m})$, the distinguishing problem undergoes a (conjecturally sharp) transition algorithmically easy. Specifically, taking the bit-wise AND of the edge indicators across the different samples corresponds to a single-sample instance of hypergraph PC with edge density $q^m = N^{-\Theta(1)}$, which can be solved in polynomial time whenever $K$ is a sufficiently large constant.

As also discussed in Section 7, another concern when choosing $m$ is the existence of inefficient algorithms that can be implement with a small number of VSTAT($m$). Let $h(G) \in \{0, 1\}$ be the indicator that $G$ has a clique of size $K$. While $h$ is NP-hard to compute, the single query of $h$ to a VSTAT($\Theta(1)$) oracle will solve the distinguishing problem unless $1 - q$ is sufficiently small. The expected number of cliques of size $K$ in $\mathcal{G}_s(N, q)$ is

$$\left(\frac{N}{K}\right) q^\binom{K}{s} \leq \exp \left( K \log N - \frac{1 - q}{q} \cdot \binom{K}{s} \right) = o(1)$$

as long as $1 - q \geq C K^{1-s} \log N$ for a sufficiently large constant $C$. Thus unless $1 - q = O(K^{1-s} \log N)$, Markov’s inequality implies that $\mathcal{G}_s(N, q)$ has no clique of size $K$ with probability $1 - o(1)$ and the SQ query of $h$ solves the distinguishing problem where no polynomial time algorithms are known to succeed. Thus to make the performance of SQ and polynomial-time algorithms comparable, it seems necessary to restrict to $q$ with $1 - q = O(K^{1-s} \log N)$. As will be shown in Claim 8.13, this threshold is also roughly when the $k$-sample LR begins to have a constant-sized norm. To summarize this discussion, the natural choices of $m$ and $q$ are:

- sufficiently large $q$ with $q = 1 - O(K^{1-s} \log N)$; and
- $m$ such that $q$ lies in the range $\frac{1}{mN^o(1)} \leq 1 - q \ll \frac{\log N}{m}$.

Note that this requires we take $m = \tilde{\Omega}(K^{s-1})$ samples.

**Remark 8.11.** A different natural alternative formulation of hypergraph PC views the adjacency lists of individual vertices as independent samples, as in bipartite PC. However, since each adjacency list is itself an $(s - 1)$-uniform hypergraph, in this model a single-query SQ algorithm succeeds whenever $s > 2$: ask if the adjacency list contains a clique of size at least $K$. For this reason, the bipartite model is not appropriate for the SQ framework.
Choice of prior $\mu$. We now discuss why the choice of prior $\mu$ over the clique vertex set $u$ differs in the definitions of multi-sample hypergraph PC and bipartite PDS. The prior $\mu$ in which each vertex is included in the clique independently with probability $K/N$ was used in defining bipartite PDS because it is more convenient to work with when computing the LDLR, $k$-sample LR and applying our main results.

However, a subtle technical issue arises in multi-sample PC that precludes using this prior. The underlying problem is that $D_\emptyset$ and the mixture of $D_u$ induced by this prior do not necessarily converge in $\chi^2$ divergence even when they converge in total variation. This is because $\chi^2$ divergence is large if certain tail events have very mismatched probabilities while total variation is not. Specifically, the probability the mixture of $D_u$ contains a clique of size $t \gg K$ is at least $\Pr[\text{Bin}(N, K/N) \geq t]$, which is much larger than the probability that $D_\emptyset$ contains a clique of size $t$. This issue causes the average correlations defining SDA and the key quantity $\|E_{u \sim \mu} D_u^{\otimes k}\|$ to be very different between the two priors. Specifically, carrying out a similar computation as in Claim 8.13 for the prior where each vertex is included with probability $K/N$ yields that $\|E_{u \sim \mu} D_u^{\otimes k}\|$ is only $O_N(1)$ for much smaller values of $\gamma$.

The important properties of the prior $\mu$ used in this section, where $u$ is a random $K$-subset of $[N]$, are that: (1) $u$ is symmetric; (2) the size of $u$ concentrates around $K$; and (3) the distribution of $|u|$ has very small upper tails. In particular, replacing $\mu$ with any prior that chooses a clique size from the interval $[CK, K]$ for some constant $C > 0$ and then chooses a random clique of this size would not affect the bounds in either Claim 8.12 or Claim 8.13.

**LDLR and $k$-sample LR bounds.** The following claims bound the LDLR and $k$-sample LR in multi-sample hypergraph PC in order to verify the conditions needed to apply our main results. Their proofs are standard computations and deferred to Appendix D.2. Let $\mu$ denote the uniform distribution over $K$-subsets $u \subseteq [N]$.

**Claim 8.12.** For any $s, K, N, k, d, m \in \mathbb{N}$, the $(d, k)$-LDLR$_m$ for multi-sample hypergraph PC satisfies that $\|E_{u \sim \mu}(D_u^{\otimes m}) \leq_{d, k} 1\| = O_N(1)$ if the following conditions are satisfied:

$$\gamma \cdot \max\{m, (ksd)^s\} = O_N(1) \quad \text{and} \quad \frac{2^{sk}e^2K^2}{N} = 1 - \Omega_N(1)$$

where $\gamma = \frac{1-q}{q}$.

**Claim 8.13.** For any $K, N, k \in \mathbb{N}$, the $k$-sample LR is bounded by $\|E_{u \sim \mu} D_u^{\otimes k}\| = O_N(1)$ if the following condition are satisfied:

$$K^2 \leq 3N \quad \text{and} \quad \gamma \leq \frac{1}{2k} \cdot K^{1-s} \log \left( \frac{N}{K^2} \right)$$

where $\gamma = \frac{1-q}{q}$.

**Implications of our results and comparison to conjectured complexity barriers.** We now can deduce the implications of our main theorems.

**Corollary 8.14.** Suppose that $s$ is a fixed constant, $K = \Theta(N^{1/2-\delta})$ for some small constant $\delta > 0$ and $q \in (0, 1)$ satisfies $q \geq 1 - c_1K^{-1-s}$ for a sufficiently small constant $c_1 > 0$. Then for multi-sample hypergraph PC with $N$ vertices, clique size $K$ and edge density $q$, it holds that SDA $\left( \Theta \left( \frac{1}{(1-q)^s} \right) \right) \geq N^{\Omega(\log t)}$ for any $t \geq (\log N)^{1+\Omega(1)}$. 

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Proof. In multi-sample hypergraph PC, each $D_u$ is a product measure on the hypercube and Theorem 6.3 applies. Consider setting the parameters $d = 1$, $k = c_2 \log N$ for a sufficiently small constant $c_2 > 0$, $K = \Theta(N^{1/2-\delta})$ for a constant $\delta > 0$ and the number of samples $m$ to be $m = c_3/(1-q)$ for some constant $c_3 > 0$. Note that $m$ is polynomially large in $N$. It now can be verified that, if $c_2$ is sufficiently small, then these parameters satisfy the conditions in Claim 8.12 and, if $c_1$ is sufficiently small, they also satisfy the condition in Claim 8.13. Now consider applying Theorem 6.3 with SDA lower bound $q' = N^{2/3} \log t / \log \log N$. It can be verified that this implies SDA($\Theta(m/t)$) $\geq q'$, proving the corollary.

Setting $t = (\log N)^{1+\delta'}$ for some small $\delta' > 0$ recovers the predicted $K = \Theta(N^{1/2-\delta})$ computational barrier in the SQ model for multi-sample hypergraph PC in the regime $mN^{o(1)} \leq 1 - q \leq O\left(\frac{1}{m}\right)$ of interest. It is worth noting that the loss of the $t = (\log N)^{1+\Omega(1)}$ factor in $m$ on applying Theorem 6.3 means that we cannot arrive at $m$ and $q$ satisfying that $1-q = \Theta(1/m)$ exactly. Under the average-case equivalence from Section 7, this corresponds to single-sample hypergraph PC with exactly constant edge densities. However, this constraint does not affect the tightness of Corollary 8.14, as the resulting lower bound still corresponds to a single-sample instance of hypergraph PC with a nearly constant edge density in the range $N^{-o(1)} \leq q \leq 1 - N^{-o(1)}$ and thus $K^2 = N^{1+o(1)}$ is still the conjectured computational barrier.

Remark 8.15. Our partial noise robustness results imply SQ lower bounds in multi-sample hypergraph PC, with a slightly different choice of the prior $\mu$. Let $\mu'$ be the prior formed by choosing a clique size according to $\text{Bin}(K, N^{-\delta})$ and then choosing a vertex set of this size uniformly at random from $[N]$ to be the planted clique, where $\delta > 0$ is a small constant. As in the discussion above, since $\text{Bin}(K, N^{-\delta})$ has zero probability mass above $K$, Claims 8.12 and 8.13 can be adapted to accommodate this different prior. Furthermore, this prior concentrates will around $K N^{-\delta} = \Theta(N^{1/2-2\delta})$ if $K = \Theta(N^{1/2-\delta})$.

If $T$ is the $(1,0)$ noise operator that resamples independently from $\text{Ber}(q)$, then $m$-sample hypergraph PC with the prior $\mu'$ can be realized as a subtensor random restriction of the type in Theorem 5.10 of $m$-sample hypergraph PC with the prior $\mu$. In particular, it can be realized with the noise operator $T$, restriction probability $N^{-\delta}$ and correlation parameter $\rho = 0$. Now consider setting the parameters $d = c_2^{-1} \log N$, $k = c_2 \log N$ for a sufficiently small constant $c_2 > 0$, $K = \Theta(N^{1/2-\delta})$ for a constant $\delta > 0$ and the edge density $q$ and number of samples $m$ to again be $m = c_3/(1-q)$. If $c_1$ and $c_2$ are sufficiently small, then the conditions in Claims 8.12 and 8.13 are met. Adapting the arguments in these claims to accommodate $\mu'$ yields that the relevant LDLR and $k$-sample LR are both $O_N(1)$. Now consider applying Theorem 5.10 together with Theorem 3.1, similarly to as in Corollary 5.5, again with the SDA lower bound $q' = N^{2/3} \log t / \log \log N$. If $c_2$ is sufficiently small, then $(N^{-\delta})^{2k-1} \sqrt{d+1}/m = o(1)$ and we recover the same lower bound as in Corollary 8.14 for the prior $\mu'$.

8.3 Spiked Wishart PCA

The spiked Wishart model is a well-studied model for understanding sparse PCA. We consider the following, standard version the problem. As with the other problems considered here, many variations of this problem exist in the literature, see e.g. [PWB+18] for a more detailed discussion.

Problem 8.16 (Sparse PCA with Wishart Noise). For a positive integer $n$, $\rho \in [0,1]$, and $\lambda \in [0,\infty)$, the sparse PCA with Wishart noise problem is the following many-vs-one hypothesis testing problem:
• Null: $m$ i.i.d. samples from the standard normal Gaussian, i.e. $D_0 = \mathcal{N}(0, I_n)$.

• Alternate: $m$ i.i.d. samples from a Gaussian with randomly spiked covariance. Specifically, sample a vector $s$ via the following process. First draw $s' \in \{-1,0,1\}^n$ so that each entry of $s'$ is independent and distributed as

$$s'_i = \begin{cases} 
0 & \text{with probability } 1 - \rho; \\
-1 & \text{with probability } \rho/2; \\
+1 & \text{with probability } \rho/2.
\end{cases}$$

Then, if $\|s'\|^2 > 2\rho n$, let $s = 0$, otherwise let $s = \frac{1}{\sqrt{\rho n}}s'$. Finally, draw $m$ samples from $D_s = \mathcal{N}(0, I_n + \lambda ss^T)$. Denote the distribution over $s$ by $S_\rho$.

The choice of constant 2 in this model is arbitrary and can be replaced by any constant larger than 1. By a Chernoff bound, for $\rho = \omega(1/n)$, $s \neq 0$ with high probability. Note that this problem is naturally stated as a multi-sample problem.

Unfortunately, while the null hypothesis for this problem is the standard normal Gaussian, it does not cleanly fit into the framework of Theorem 6.1, as the alternate hypotheses are not additive shifts of $\mathcal{N}(0, I_n)$. However, the $(d, k)$–LDLR$_m$ for this problem still has a nice form, which allows us apply our main theorem.

Recall the Hermite basis for $D_\mathbb{R}^\otimes t$ is the set of polynomials over $(\mathbb{R}^n)^t$ given by $\{H_\alpha\}$, where $H_\alpha$ is parametrized by multi-indices $\alpha = (\alpha_1, \ldots, \alpha_t) \in (\mathbb{N}^n)^t$. For any multi-index $\alpha \in \mathbb{N}^n$, and any $x \in \mathbb{R}^n$, let $x^\alpha = \prod_{i=1}^n x_i^{\alpha_i}$. Then, we have the following bound from [BKW19]:

**Lemma 8.17** (Lemma 5.8 in [BKW19]). Let $(\alpha_1, \ldots, \alpha_t) \in (\mathbb{N}^n)^t$. Then, we have:

$$\left( \mathbb{E}_{u \sim S_\rho} (\bar{D}_u, H_\alpha) \right)^2 \geq \begin{cases} 0 & \text{if } |\alpha_i| \text{ are even}; \\
|\alpha_i| \cdot \prod_{i=1}^t \frac{|\alpha_i|!}{\alpha_i!} & \text{otherwise.}
\end{cases}$$

As a result, we have the following:

**Lemma 8.18.** Let $t, d \in \mathbb{N}$. Suppose that $np^2 \leq 1$, and that $dt\lambda \leq \rho n$. Then, we have:

$$\left\| \mathbb{E}_{u \sim S_\rho} (\bar{D}_u \leq d) \right\| ^2 \leq 2 \left( \frac{d^2 k \lambda}{\rho n} \right)^{2t}.$$

We prove Lemma 8.18 in Appendix D.3. Together with Claim 3.3, this immediately implies:

**Corollary 8.19.** Let $t, d$ be as in Lemma 8.18. Let $m$ be so that $m \leq \frac{\rho^2 n^2}{4d^2 k^2}$. Then

$$\left\| \mathbb{E}_{u \sim S_\rho} (\bar{D}_d \leq m) \right\| ^2 \leq O(1).$$

We now seek to bound the norm of the high degree part of the correlation. To do so, we rely on the following lemma:

**Lemma 8.20** ([BKW19]). Let $\phi(x) = (1 - 4x)^{-1/2}$, and let $\phi^{\leq d}(x) = \sum_{t=0}^d \left( \frac{2^t}{t!} \right) x^t$ and $\phi^{> d}(x) = \sum_{t=d+1}^\infty \left( \frac{2^t}{t!} \right) x^t$ denote the low degree approximation and the approximation error of the degree $d$ Taylor approximation to $\phi(x)$ at zero, respectively. Then

$$\left\| \mathbb{E}_{u \sim S_\rho} (\bar{D}_u) \right\| ^2 = \mathbb{E}_{u, v \sim S_\rho} \left[ \phi^{> [d/2]} \left( \frac{\lambda^2 (u, v)^2}{4} \right) \right].$$
As a result, we obtain the following bound:

**Lemma 8.21.** Assume that $2nk(d + 1)n^2 \leq 1$. For $\lambda < 1/2$ and $d$ even, we have:

$$\left\| E_{u \sim \mathcal{S}_D} \left( D_u^{\otimes k} \right)^2 \right\| \leq \left( \frac{\lambda^2}{4m} \right)^{(d+1)}.$$

The proof closely resembles the proof of Lemma 6.2, and we defer it to Appendix D.3. Combining Corollary 8.19 and Lemma 8.21 with Theorem 3.1, we obtain:

**Corollary 8.22.** Let $d, k \in \mathbb{N}$. Let $\lambda \leq 1/4$, let $\rho$ be so that $2nk(d + 1)n^2 \leq 1$, let $m$ be so that $m \leq \frac{(\rho m)^2}{d^{4k^2}x^2}$. Then $\text{SDA}(\mathcal{S}, \tilde{O}(m/k)) \geq 2^k$.

**Comparison to prior work and predictions.** The Wishart model for spiked PCA has two, well-studied regimes, the sparse PCA model, where the sparsity, governed by $\rho$, is sublinear in $n$, typically $n\rho^2 \leq 1$, and the dense regime, when $\rho = \Theta(1)$. In the dense regime, the celebrated BBP transition [BAP+05] gives an exact prediction of when detection is computationally possible, and the computational limits in terms of the low degree likelihood ratio are known to exactly match these predictions [PWB+18, DKS17, BKW19]. In particular, it is predicted that when $\rho$ is a fixed universal constant, recovery is possible if and only if $m \geq n/\lambda^2$. While it is possible to plug in the machinery here with the LDLR bounds attained in [BKW19], it appears to be an inherent limitation of the SDA framework for proving SQ lower bounds that it cannot predict exact (i.e. including constants) thresholds. Thus, while we can attain SQ lower bounds matching the BBP transition up to constants, we cannot prove SQ lower bounds up to the transition.

For this reason, the calculations in the previous section primarily focus on the sparse regime. The problem is well-studied in this setting, and the best known sample complexity for this problem is $m = \Omega \left( \frac{\rho m^2 \log n}{\lambda^2} \right)$ [DBG08, BR13b]. In contrast, information theoretically $m = \Omega \left( \frac{\rho m \log n}{\lambda^2} \right)$ samples suffice. There is a slew of evidence [BR13a, HKP+17, BB19] that suggests that this is the best possible. Note that the SQ lower bounds and LDLR lower bounds we obtain witness this gap, up to logarithmic factors. To the best of our knowledge, prior to our work there were no LDLR lower bounds for sparse PCA in the $\rho \leq 1/\sqrt{n}$ regime, and existing SQ lower bounds required $\lambda = o(1)$ and $\rho = n^{-7/8}$ [WGL15].

### 8.4 Testing Gaussian Mixture Models

In this section, we prove LDLR bounds for robustly testing Gaussian Mixtures. We use the SDA bounds of [DKS17] in an almost black-box fashion (we must modify their proofs a little bit to account for the different notions of statistical dimension considered).

**Problem 8.23** (Testing Gaussian Mixture Models). For $n, s$ positive integers and $\varepsilon \in (0, 1)$, the $(1 - \varepsilon)$-separated Gaussian $s$-mixture model testing problem is the following hypothesis testing problem:

- **Null:** $\mathcal{N}(0, I_n)$
- **Alternate:** uniform over $\mathcal{S} = \{ D_U \}_{U \in \mathcal{S}}$ for some $\mathcal{S} \subset \times_s \mathbb{R}^{n-1}$, where each $D_U$ for $U = u_1, \ldots, u_s$ is a mixture of $\mathcal{N}(u_1, I), \ldots, \mathcal{N}(u_s, I)$ satisfying the conditions $d_{\text{TV}}(D_{u,v}, D_{\emptyset}) \geq 0.25$ and $d_{\text{TV}}(\mathcal{N}(u_1, I), \mathcal{N}(u_j, I)) \geq 1 - \varepsilon$ for all $i \neq j \in [s]$.

In [DKS17], the authors show lower bounds on the SDA for this problem—however, because the lower bounds are for product-SDA, we must make some mild modifications to their proofs. We use the following building blocks:
Lemma 8.24 (Lemma 3.4 of [DKS17]). Suppose $A$ is a distribution over $\mathbb{R}$ which matches $m$ moments of $\mathcal{N}(0, 1)$. For each $u \in S^{n-1}$, define the distribution with probability density function $D_u(x) = A(\langle x, u \rangle) \cdot \gamma_{\perp u}(x)$, where $\gamma_{\perp u}$ is the projection of $D_\emptyset = \mathcal{N}(0, I_n)$ orthogonal to $u$. Letting $\overline{D}_u$ be the relative density of $D_u$ with respect to $D_\emptyset$, we have that for any $u, v \in S^{n-1}$, 
\[
|\langle \overline{D}_u, \overline{D}_v \rangle - 1| \leq |\langle u, v \rangle|^{m+1} \cdot \|\overline{A}\|^2,
\]
for $\overline{A}$ the relative density of $A$ with respect to $\mathcal{N}(0, 1)$.

Lemma 8.25 (Lemma 3.7 of [DKS17]). For any $c \in (0, \frac{1}{2})$, there is a set $S$ of $2^{\Omega(nc)}$ unit vectors in $\mathbb{R}^n$ so that for each $u, v \in S$ with $u \neq v$, $|\langle u, v \rangle| \leq O(n^{c-1/2})$.

Now, we use the following propositions of [DKS17], which selects a distribution $A$ for the GMM testing problem:

Proposition 8.26 (Proposition 4.2 of [DKS17]). For any $\varepsilon \in (0, 1)$, $c \in (0, \frac{1}{2})$, and integer $s \geq 1$ there exists a distribution $A$ on $\mathbb{R}$ that is a mixture of $s$ Gaussians $A_1, \ldots, A_s$ with $d_{TV}(A_i, A_j) \geq 1 - \varepsilon$ for all $i \neq j \in [s]$. Further, $\|\overline{A}\|^2 \leq \exp(O(s)) \log \frac{1}{\varepsilon}$ and $A$ agrees with $\mathcal{N}(0, 1)$ on $2s - 1$ moments, and if we construct $\{D_u\}_{u \in S}$ as described in Lemmas 8.24 and 8.25, then each $D_u$ is a mixture of $s$ Gaussians and further for all $u, v \in S$, $d_{TV}(D_u, D_v) \geq \frac{1}{2}$.

Putting these together, we have the following instance of the GMM testing problem:

Problem 8.27 ($(1 - \varepsilon)$-separated GGM testing instance from [DKS17]). For $n, \ell$ positive integers and any $\varepsilon \in (0, 1)$, let $A$ be the mixture of $\ell$ Gaussians described in Proposition 8.26 and let $S$ be the subset of $S^{n-1}$ described in Lemma 8.25 with $c = 0.26$. Consider the following instance of the $(1 - \varepsilon)$-separated Gaussian $\ell$-mixture model testing problem:

- Null: $D_\emptyset = \mathcal{N}(0, I_n)$
- Alternate: Uniform over the set of distributions $S = \{D_u\}_{u \in S'}$, where $D_u(x) = A(\langle x, u \rangle) \cdot \gamma_{\perp u}(x)$ and $S'$ is the subset of $u \in S$ with $d_{TV}(D_u, D_\emptyset) \geq \frac{1}{4}$ (note $|S'| \geq \frac{1}{4}|S|$).

We note that Problem 8.27 is a valid instance of the $(1 - \varepsilon)$-separated Gaussian $\ell$-mixture testing problem: since from Proposition 8.26 $A$ is a one-dimensional mixture of $\ell$ Gaussians with pairwise total variation distance $\geq 1 - \varepsilon$, each $D_u$ is also a mixture of $\ell$ Gaussians with pairwise total variation distance $\geq 1 - \varepsilon$. Proposition 8.26 also guarantees that for each $u \neq v$, $d_{TV}(D_u, D_v) \geq \frac{1}{2}$. By the triangle inequality, we have that $d_{TV}(D_u, D) + d_{TV}(D_v, D) \geq d_{TV}(D_u, D_v) \geq \frac{1}{2}$, which implies that for at least half of $u \in S$, $d_{TV}(D_u, D_v) \geq 1/4$, and this half is exactly $S'$.

Putting these lemmas together, we have the following easy corollary:

Corollary 8.28. Let $\ell, n$ be integers with $n$ sufficiently large and $n^{\ell+1} \leq 2^{n^{1/4}}$. Let $S = \{D_u\}_{u \in S'}$ be as described in Problem 8.27. Then there exists a constant $c$ so that for all integers $n$ sufficiently large, for any $q \geq 1$,
\[
\text{SDA} \left( S, \left( \frac{(n/c)^{(\ell+1)/5}}{\log \frac{1}{\varepsilon} \left( 1 + \frac{q^2}{2n^2} \right)} \right) \right) \geq q.
\]

Proof. We have that $\text{Pr}_{u, v \sim S}[u = v] = \frac{1}{|S'|}$. Since Problem 8.27 uses the construction from Lemma 8.25 with $c = 0.26$, for $n$ sufficiently large $|S'| \geq 2^{n^{0.55}}$ and $|\langle u, v \rangle| \leq n^{-1/5}$ for all $u \neq v \in S'$. Since Lemma 8.24 furnishes a bound on the correlation for $u \neq v$, for any event $\mathcal{E}$,
\[
\mathbb{E}_{u, v \sim \mu} \left[ |\langle \overline{D}_u, \overline{D}_v \rangle - 1| \mid \mathcal{E} \right] \leq \min \left( 1, \frac{1}{|S'| \text{Pr}[\mathcal{E}]} \right) \cdot \|\overline{A}\|^2 + \max \left( 0, 1 - \frac{1}{|S'| \text{Pr}[\mathcal{E}]} \right) \cdot \frac{1}{n^{(\ell+1)/5}} \|\overline{A}\|^2,
\]

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and substituting our bound on $|S'|$, using that $|A|^2 \leq \log \frac{1}{\ell} C^\ell$ for some constant $C$, and using the assumption that $n^{(\ell+1)/5} 2^{n^{0.255}} \leq 2^{n^{1/4}}$, we have our conclusion. □

Applying Theorem 4.1, we deduce the following bound:

**Corollary 8.29.** There exists a real number $c \geq 0$ so that for any $\varepsilon \in (0, 1)$ and integer $\ell$, there exists $n$ sufficiently large that for any even integer $k \ll n^{1/8}$ and any $m \leq \frac{(n/\varepsilon)^{(\ell+1)/5}}{2 \log \frac{1}{\varepsilon}}$, the $(1-\varepsilon)$-separated Gaussian $\ell$-mixture model testing problem $S = \{D_u\}_{u \in S}$ vs. $D_\emptyset$ described in Problem 8.27 has $(\infty, k)$-LDLR$_m$ bounded by

$$\left\| E_{u \sim S} (D_u^{2m}) \right\|_{\ell, k} - 1 \leq 1.$$ 

**Proof.** Let $m = \frac{(n/\varepsilon)^{(\ell+1)/5}}{2 \log \frac{1}{\varepsilon}}$. We notice that $|\langle D_u, D_v \rangle - 1| \leq \exp(O(\varepsilon)) \log \frac{1}{\varepsilon} \leq m^{1/10}$ always, since $\varepsilon, \ell$ are fixed constants. Hence we meet the condition of Theorem 4.1 that $\| E_{u \sim S} (D_u^{2m}) \|_{\ell, k} \leq m^{k/10}$.

Applying Corollary 8.28 with $q = \sqrt{2^{n^{3/4}} \frac{m}{m'}}$, we have that for all $1 \leq m' \leq m$,

$$\text{SDA}(S, m') \geq \sqrt{2^{n^{3/4}} \frac{m}{m'}} \geq \left( \frac{100m}{m'} \right)^k$$

for any $k \leq n^{249}$. This concludes the argument. □

**Comparison with prior work and predictions** The lower bound Corollary 8.29 is consistent with the SQ lower bounds of [DKS17], suggesting efficient algorithms for learning a mixture of $\ell$ Gaussians in $n$ dimensions, each separated in total-variation distance, requires $d^{O(\ell)}$ samples. Information-theoretically, only $\text{poly}(n, \ell)$ samples are required in this setting, although the information-theoretic sample complexity becomes exponential in $\ell$ if the Gaussians are not required to have total variation distance close to 1 [MV10]. An algorithm using time and samples $d^{\text{poly}(k)}$ is known [MV10].

### 8.5 Gaussian Graphical Models

In this section, we prove an SDA lower bound for a hypothesis testing problem over Gaussian Graphical Models, and then show that this implies a LDLR lower bound for the same problem. We will not succeed in establishing evidence for information computation gaps—the point of this example is to illustrate the utility of Theorem 4.1, for a setting where LDLR lower bounds are highly intractable while SDA bounds are approachable.

In Gaussian Graphical models, we observe samples $x_1, \ldots, x_m \sim \mathcal{N}(\mu, \Theta^{-1})$, where $\Theta$ is a sparse positive semidefinite matrix—since it is sparse, it is thought of as a graph. The goal is to get algorithms for estimating $\Theta$ which do not depend on its condition number, and which take advantage of the graph sparsity. The relevant parameters are the maximum degree $d$ and the non-degeneracy parameter $\kappa := \min_{i,j \in [n]} |\Theta_{ij}| / \sqrt{\Theta_{ii} \Theta_{jj}}$.

**Problem 8.30** (Gaussian Graphical Models: planted $d$-regular subgraph). For $n > s > d$ positive integers and $\kappa \in \mathbb{R}$ with $\kappa \sqrt{d} < \frac{1}{6}$, the $\kappa$-nondegenerate $d$-sparse $s$-planted $n$-dimensional planted regular subgraph Gaussian Graphical Model ($\kappa, d, s, n)$-prsGGM) problem is the following many-vs-one hypothesis testing problem:
Null: $D_{\varnothing} = \mathcal{N}(0, I_n)$.

Alternate: uniform mixture of $D_u = \mathcal{N}(0, (I_n + \kappa \Delta_u)^{-1})$, over $u \sim S$, where each $u$ is sampled by choosing $s$ of $n$ indices uniformly at random, and then planting a randomly signed random $d$-regular graph on those indices (conditioned on the graph having all eigenvalues bounded in magnitude by $2\sqrt{d}$), then taking $\Delta_u$ to be the adjacency matrix of that graph.

We will prove the following Lemma, from which we obtain an LDLR lower bound as a corollary of Theorem 4.1:

**Lemma 8.31.** For any integer $d$ sufficiently large, any $s \gg d$ sufficiently large, any $n \gg s$ sufficiently large, and $\kappa \in (0, \frac{1}{6\sqrt{d}})$ such that the following holds: If $S$ vs. $D_{\varnothing}$ is an instance of the $(\kappa, d, s, n)$-prsGGM problem, then for any even integer $k$ and $q \geq 1$,

$$\text{SDA} \left( S, \left( \frac{n}{q^2 s^2} \right)^{1/k} \frac{1}{\exp\left(\frac{1}{2} s \kappa^2\right) - 1} \right) \geq q,$$

and further,

$$E_{u,v} \langle D_u, D_v \rangle^k \leq \left( 1 + \left( \frac{2^2}{n} \right)^{1/k} \left( \exp\left(\frac{1}{2} s \kappa^2\right) - 1 \right) \right)^k.$$

We give the proof of this Lemma in Appendix D.4. Combining Lemma 8.31 with Theorem 4.1 gives us the following corollary:

**Corollary 8.32.** For any integer $d$ sufficiently large, any $s \gg d$ sufficiently large, any $n \gg s$ sufficiently large, and $\kappa \in (0, \frac{1}{6\sqrt{d}})$ such that the following holds: If $S$ vs. $D_{\varnothing}$ is an instance of the $(\kappa, d, s, n)$-prsGGM problem, then for any even integers $k, t$ and $m \leq \frac{1}{k} \left( \frac{n}{s^2} \right)^{1/k} \frac{1}{\exp\left(\frac{1}{2} s \kappa^2\right) - 1}$ with $s \kappa^2 \leq \frac{k}{16} \log m$, the $m$-sample $(t, \Omega(k))$-LDLR$_m$ is bounded:

$$\left\| E_{u \sim S} (D_u^{\otimes m})^{t,k/2} - 1 \right\| \leq 1.$$

**Comparison with prior work and predictions.** For an arbitrary Gaussian Graphical Model with maximum degree $d$, $\kappa$-nondegeneracy, and dimension $n$, information-theoretically, $m \geq \frac{\log n}{\kappa^2}$ samples are required [WWR10], and the fastest known algorithms for $m = \Theta(\frac{\kappa^2}{\log n})$ run in time $n^{O(d)}$ [KKMM19], though faster algorithms are known for more structured cases [KKMM19, RWR+11]. Given the current state of the literature, it is not clear whether it is possible to achieve the information-theoretic limit with $n^{O(d)}$ time algorithms.

Our bounds are not strong enough to give evidence for an information-computation gap: for signal-to-noise ratios corresponding to $m = \Theta(\frac{\log n}{\log \kappa})$ samples, by choosing $s = \log n$ and $\kappa$ small enough we can rule out SQ algorithms with fewer than $\sqrt{n/(d \log^4 n)}$ queries, or degree-$O(\frac{\log n}{\log d})$ polynomial distinguishers (these bounds degrade as $d$ increases, instead of the other way around). We do not expect that this bound is tight, and our bound from Lemma 8.31 might easily be improved with a more careful analysis. But, because the matrices that we use are well-conditioned, and because there are algorithms for well-conditioned matrices that require fewer samples, it is unlikely that the hypothesis testing problem we consider will give evidence for this information-computation tradeoff, even if analyzed optimally.
However, this example does illustrate that it is possible to obtain a bound depending on the sparsity and non-degeneracy; in this, it highlights the usefulness of Theorem 4.1. In the GGM problem, any set of alternate hypotheses $S$ by definition involves Gaussian distributions whose inverse covariance matrices are easy to describe, but the covariance matrices themselves are not; this would make calculating the LDLR directly extremely arduous, even for our toy example of alternate distributions. However, calculating some bound on the SDA is relatively tractable, and Theorem 4.1 lets us draw conclusions for the LDLR.

### 8.6 Sparse Parity with Noise

Theorem 5.2 shows that if for the hypothesis testing problem $T_\rho S$ vs $D_\emptyset$, the $(s-1,k)$-LDLR$_m$ is bounded by $\varepsilon$, and $\|E_u(T_u)\|^2 \leq O(1)$, and $\rho^{2s} = O(\frac{1}{m})$, then at least $2^k$ queries to VSTAT$(O(m/k))$ are necessary. The following example illustrates that this dependence on $\rho$ is tight.

**Problem 8.33.** The following is the $2^k$-subset of s-sparse parities problem:

- **Null:** $D_\emptyset$ is uniform over $\{\pm 1\}^n$.
- **Alternate:** For $S$ an arbitrary subset of $\binom{\binom{n}{s}}{s}$ with $|S| = 2^k$, define $S = \{D_u\}_{u \in D}$, where for each $u \in S$ we take $D_u$ uniform over $x \sim \{\pm 1\}^n$ conditioned on $x^u = 1$.

**Claim 8.34.** For any $\rho \in [-1,1]$ and $T_\rho$ the standard Boolean noise operator, and any integer $m$, the many-vs-one $2^k$-subset of s-sparse parities problem $D_\emptyset$ vs $S = \{D_u\}$ has

$$\|E_u\sim_S(T_\rho D_u^m)\|_{s-1,\infty} - 1 = 0.$$

**Proof.** This is because each $D_u$ has no Fourier mass on degrees 1 through $s-1$.

**Claim 8.35.** For the many-vs-one $2^k$-subset of s-sparse parities problem,

$$\|E_u\sim_S(D_u^k)\| \leq 2.$$

**Proof.** For each $u \neq v$, $(D_u, D_v) = 1$, and $(D_u, D_v) = 2$. We then use the fact that $|S| \leq 2^k$ to calculate,

$$\|E_u(D_u^k)\| = E_{u,v\sim S}(D_u, D_v) = \frac{1}{|S|} \cdot 2^k + (1 - \frac{1}{|S|}) \cdot 1 \leq 2.$$

Together, the above claims demonstrate that we meet the conditions of Theorem 5.2. However, there is also a $2^k$-query VSTAT$(\rho^{-2s})$ algorithm:

**Claim 8.36.** There is a $2^k$ query VSTAT$(\rho^{-2s})$ algorithm for the $\rho$-noisy $2^k$-subset of s-sparse parities problem, $T_\rho S$ vs. $D_\emptyset$.

**Proof.** The algorithm is as follows: for each $u \in S$, take the query $\phi_u(x) = \frac{1}{2}(1 + x^u)$. Under null, $E_{D_\emptyset} \phi_u = \frac{1}{2}$. Under $T_\rho D_u$, $E_{T_\rho D_u} \phi_u = \frac{1}{2}(1 + \rho^s)$. Thus, a VSTAT$(\rho^{-2s})$ algorithm can distinguish these cases.

Hence, the requirement in Theorem 5.2 that $\rho^{2s} = O(\frac{1}{m})$ is tight.
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A SDA, Product-SDA, and Simple-vs-Simple Hypothesis Testing

We make several remarks here on technical differences between our hypothesis testing and statistical dimension setup and those of [FGR+17]. First, our version of statistical dimension bounds $\mathbb{E} \left[ \left| \langle D_u, D_v \rangle - 1 \right| \mid A \right]$ for all events $A$ in the joint distribution of $u, v \sim \mu$, while [FGR+17] considers only $A$ of the form $A = B \otimes B$ for some event $B$ in $\mu$.\footnote{For this reason, we use $\Pr(A) \geq 1/q^2$ in our definition, rather than the more natural $\Pr(A) \geq 1/q$, to maintain consistency with [FGR+17].} Our version corresponds to a stronger computational model, in the sense that a lower bound on $\text{SDA}(S, m)$ implies a lower bound on the statistical dimension of [FGR+17]. While we are not aware of any natural high-dimensional testing problems where these notions diverge, we give an artificial example where they differ in Appendix A.1. Second, the problems considered in [FGR+17] are many vs. one (simple vs. composite) hypothesis testing problems, but in Appendix A.2 we show that statistical dimension implies lower bounds on SQ algorithms in our simple vs. simple hypothesis testing setting as well.\footnote{The difference between these two settings is the presence of the prior $\mu$.} Notationally, we write $\text{SDA}(S, m)$ where [FGR+17] writes $\text{SDA}(S, D_{\otimes, \frac{1}{m}})$.

A.1 Counterexample to Equivalence of Two Notions of Statistical Dimension

In this appendix we construct a testing problem which shows that the definition of statistical dimension we use in this paper can differ from the statistical dimension of [FGR+17]. For reference, we repeat both definitions here.

Let $D_u$ vs. $S$ be a testing problem with prior $\mu$. For $D_u, D_v \in S$, we write as usual the relative density $\overline{D}_u(x) = \frac{D_u(x)}{\Pr_x[D_u(x)]}$ (and $\overline{D}_v$ for $v$), and the inner product $\langle \overline{D}_u, \overline{D}_v \rangle = \mathbb{E}_{x \sim D_x} \overline{D}_u(x) \overline{D}_v(x)$. We have used the following notion of statistical dimension:

**Definition A.1** (SDA).

$$\text{SDA}(S, m) = \max \left\{ q \in \mathbb{N} : \mathbb{E}_{u, v \sim \mu} \left[ \left| \langle \overline{D}_u, \overline{D}_v \rangle - 1 \right| \mid A \right] \leq \frac{1}{m} \text{ for all events } A \text{ s.t. } \Pr_{u, v \sim \mu} (A) \geq \frac{1}{q} \right\}. $$

The work [FGR+17] employs the a different, weaker notion, which we term product-SDA or $\text{SDA}_\otimes$ to distinguish it from the above:

**Definition A.2** (Product SDA).

$$\text{SDA}_\otimes(S, m) = \max \left\{ q \in \mathbb{N} : \mathbb{E}_{u, v \sim \mu} \left[ \left| \langle \overline{D}_u, \overline{D}_v \rangle - 1 \right| \mid A_u, A_v \right] \leq \frac{1}{m} \text{ for all events } A_u \text{ s.t. } \Pr_{u \sim \mu} (A) \geq \frac{1}{q} \right\}. $$

In the definition of product-SDA, the event $A_u \land A_v$ is a product of events occurring for a single samples $u, v \sim \mu$, rather than an event over the joint distribution of two samples $u, v \sim \mu$. In the
definition of SDA, we use $1/q^2$ so that the event $A$ has probability equal to the probability of the event \( \{ u \in A_u, v \in A_v \} \), where $u \in A_u$ has probability $1/q$ according to $\mu$.

Since the value of the product-SDA is the value of an optimization problem over a larger set than our notion of SDA, it is clear that $\text{SDA}_\times(m) \geq \text{SDA}(m)$. We will sketch a proof of the following claim, which demonstrates an example for which this inequality is far from equality.

**Claim A.3.** For every $n \in \mathbb{N}$ there is a number $t(n)$ and a family $\mathcal{S} = \{ D_i \}_{i \in [n]}$ of distributions over $[n]$ such that for the hypothesis testing problem $\mathcal{S}, D_\emptyset$ for $D_\emptyset$ the uniform distribution over $[n]$, $\text{SDA}(\mathcal{S}, t(n)) \leq O(1)$ while $\text{SDA}_\times(\mathcal{S}, t(n)) \geq n^{\Omega(1)}$.

We turn to our construction. Regarding notation in what follows: for vectors in $\mathbb{R}^n$, which we typically denote by lower-case letters, $\langle v, w \rangle$ is the usual Euclidean inner product $\langle v, w \rangle = \sum_{i \in n} v_i w_i$. For functions $F : [n] \to \mathbb{R}$, which we denote by upper-case letters, $\langle F, G \rangle$ is given by $E_{i \sim [n]} F(i) G(i)$ (this is merely a difference in normalization). We will use the following claim.

**Claim A.4.** Let $v_1, \ldots, v_n \in \mathbb{R}^n$. Let $v_{\max} = \max_i \| v_i \|_\infty$ be the largest-magnitude entry in any $v_i$, and let $\alpha = \max_i | \langle v_i, 1 \rangle |/\sqrt{n}$, where $1$ denotes the all-1’s vector. Then there exists a family of distributions $D_1, \ldots, D_n$ on $[n]$ such that, if $\overline{D}_i$ is the density of $D_i$ relative to the uniform distribution on $[n]$, then $\langle \overline{D}_i, \overline{D}_j \rangle - 1 = \frac{1}{4v_{\max}^2} (\langle v_i, v_j \rangle \pm \alpha^2)$.

**Proof.** Let $w_i = v_i - \langle v_i, 1 \rangle \cdot 1/n$. By construction, $\langle w_i, 1 \rangle = 0$. Let $\overline{D}_i : [n] \to \mathbb{R}$ be the function $\overline{D}_i(k) = \frac{1}{2v_{\max}} (w_{ik} + 2v_{\max})$. Then by construction $E_{i \sim [n]} \overline{D}_i(k) = 1$ and $\overline{D}_i(j) \geq 0$ for all $i, j$, so $\overline{D}_i$ is a density relative to the uniform distribution on $[n]$. Furthermore,

$$E_{k \sim [n]} \overline{D}_i(k) \overline{D}_j(k) - 1 = \frac{1}{n} \frac{1}{4v_{\max}^2} \langle w_i, w_j \rangle = \frac{1}{n} \frac{1}{4v_{\max}^2} (\langle v_i, v_j \rangle - \langle v_i, 1 \rangle \langle v_j, 1 \rangle /n) = \frac{1}{n} \frac{1}{4v_{\max}^2} (\langle v_i, v_j \rangle \pm \alpha^2)$$

as desired.

Now we will construct a random testing problem and sketch its analysis. Let $G$ be an $n \times n$ symmetric matrix with i.i.d. entries from $N(0, 1)$. Let $M = G + 3\sqrt{n}I$. With probability at least $0.99$ the following all hold (by standard concentration of measure):

- $M \succeq 0$, since the least eigenvalue of $G$ is at most $2\sqrt{n}$ in magnitude, with high probability.
- If $v_1, \ldots, v_n \in \mathbb{R}^n$ are such that $\langle v_i, v_j \rangle = M_{ij}$, then $| \langle v_i, 1 \rangle | /\sqrt{n} \leq O(\sqrt{\log n}/n^{1/4})$ for all $i$, by rotation-invariance of $M$.
- $\max_i \| v_i \|_\infty \leq O(\sqrt{\log n}/n^{1/4})$, again by rotation invariance.

Let $\beta = \max_i \| v_i \|_\infty$. By Claim A.4, there is a family of distributions $D_1, \ldots, D_n$ on $[n]$ such that for all $i, j$,

$$| \langle \overline{D}_i, \overline{D}_j \rangle - 1 | = \frac{1}{n} \frac{1}{4\beta^2} (\langle v_i, v_j \rangle \pm O(n^{1/4})).$$

Now, for all constant $q$, we can find a subset of $n^2/q^2$ entries of $M_{ij}$ such that $M_{ij} = \langle v_i, v_j \rangle \geq \Omega(\sqrt{\log q})$. So there is some constant $C$ such that for all constant $q$,

$$\text{SDA} \left( \{ D_i \}, \frac{Cn\beta^2}{\sqrt{\log q}} \right) \leq q^2.$$
On the other hand, we consider product-SDA – we aim to show that product-SDA(\{D_t\}, C_n q^2) \gg q^2. Take any subset S \subseteq [n] of size s. Then
\[
\frac{1}{n4\beta^2} \mathbf{E}_{i,j \sim S} | \langle v_i, v_j \rangle | + O(\log n/\sqrt{n}) \leq \frac{1}{4n\beta^2} \left[ (1 + o(1)) \mathbf{E}_{g \sim N(0,1)} |g| + \frac{1}{s} \cdot O(\sqrt{n}) + O(\log n/\sqrt{n}) \right].
\]
We can take s as small as \(n^{1-o(1)}\) and still have \(\mathbf{E}_{i,j \sim S} | \langle D_i, D_j \rangle - 1 | \ll \sqrt{n} q\), so SDA(\{D_t\}, C_n q^2) \geq n^{o(1)}.

A.2 Statistical Dimension as a Lower Bound for Hypothesis Testing

Here, we extend the argument of [FGR+17] which relates the product-statistical dimension to the SQ complexity of many-to-one hypothesis testing to simple hypothesis tests and our more powerful notion of statistical dimension.

**Theorem A.5.** Let \(S = \{D_u\}\) vs. \(D_\varnothing\) be a hypothesis testing problem with prior \(\mu\) on \(S\). Let \(q, k \in \mathbb{N}\) with \(k\) even. If SDA(\(\frac{2}{t}\)) \(\geq q\), then no \(q\)-query VSTAT(\(\frac{1}{t}\)) algorithm solves the hypothesis testing problem \(S\) vs. \(D_\varnothing\).

**Proof.** We prove the contrapositive. Let the distributions be supported on \(X\). Suppose there is a \(q\)-query VSTAT(\(1/t\)) algorithm for the testing problem. Then there must be some \(h: X \to [0,1]\) which distinguishes between \(D_\varnothing\) and \(D_u \sim S\) with probability at least \(\frac{1}{t}\) over the choice of \(D_u\) given oracle access to VSTAT(\(1/t\)). Without loss of generality with \(\mathbf{E}_{D_\varnothing} h < \frac{1}{2}\), as this affects \(p\) by a factor of at most 2. Let \(a := \mathbf{E}_{D_\varnothing} h\), and let \(a_u = \mathbf{E}_{D_u} h\).

Whenever \(h\) succeeds in distinguishing \(D_u\) from \(D_\varnothing\), by definition of VSTAT(\(1/t\)) we have that for every \(u\) for which \(h\) is successful,
\[
\min \left( \sqrt{ta(1-a)}, \sqrt{ta_u(1-a_u)} \right) \leq |\langle D_u - 1, h \rangle|.
\]
By Lemma 3.5 of [FGR+17] (a simple calculation), using the fact that \(a \leq \frac{1}{2}\), this further implies that
\[
\sqrt{\frac{ta}{3}} \leq |\langle D_u - 1, h \rangle|.
\]
Now for any even \(k \in \mathbb{N}\) we have that
\[
\mathbf{Pr}_{u \sim \mu}[h \text{ succeeds on } D_u] \cdot \sqrt{\frac{ta}{3}} \leq \mathbf{E}_{u \sim \mu} |\langle D_u - 1, h \rangle| \cdot 1[h \text{ succeeds on } D_u]
\]
\[
= \mathbf{E}_{u \sim \mu} \langle D_u - 1 \rangle \cdot \text{sign}(\langle D_u - 1, h \rangle) \cdot 1[h \text{ succeeds on } D_u], h
\]
\[
\leq \|h\| \cdot \mathbf{E}_{u,v \sim \mu} |\langle D_u - 1, D_v - 1 \rangle| \cdot 1[h \text{ succeeds on } D_u, D_v]
\]
\[
= \sqrt{a} \cdot \mathbf{E}_{u,v \sim \mu} |\langle D_u, D_v \rangle - 1| \cdot 1[h \text{ succeeds on } D_u, D_v],
\]
where in the penultimate line we have chosen the worst-case signs, and in the final line we have used that \(\|h\| = \sqrt{a}\). Now, we square the above expression and divide by \(\mathbf{Pr}_{u \sim \mu}[h \text{ succeeds on } D_u]^2\):
\[
\frac{t}{3} \leq \mathbf{E}_{u,v \sim \mu} \left[ |\langle D_u, D_v \rangle - 1 | | h \text{ succeeds on } D_u, D_v \right],
\]
where we have used that \(u, v \sim \mu\) independently. Furthermore, again by the independence of \(u, v \sim \mu\), \(\mathbf{Pr}_{u,v \sim \mu}[h \text{ succeeds on } D_u, D_v] \geq \frac{1}{q^2}\). So by definition of SDA, if VSTAT(\(1/t\)) succeeds then SDA(\(3/t\)) \(\leq q\). \(\square\)
B VSTAT Algorithms Imply Low-Degree Distinguishers

In this section, we will give a direct argument that the existence of a VSTAT algorithm implies the existence of a good low-degree algorithm. We will prove the following theorem, which recovers a nearly identical parameter dependence to Theorem 3.1 and successfully transfers lower bounds against low-degree algorithms to statistical query algorithms. However, since SDA is not a characterization for VSTAT, and \( q \)-query VSTAT(\( m \)) algorithms may fail even when SDA(\( m \)) \( < q \), Theorem 3.1 is stronger.

**Theorem B.1** (VSTAT Algorithms to LDLR). Let \( d, k, m, q \in \mathbb{N} \) with \( k \) even, and \( \tau, \eta \in (0, 1) \). Let \( D_\emptyset \) be a null distribution over \( \mathbb{R}^n \), and let \( S = \{ D_u \}_{u \in S} \) be a collection of alternative probability distributions, with \( \bar{D}_u \) the relative density of \( D_u \) with respect to \( D_\emptyset \). Suppose that the \( k \)-sample high-degree part of the likelihood ratio of \( S \) is bounded by \( \| E_{u \sim S} (\bar{D}_u^d)^\otimes k \| \leq \delta \).

If there is a (randomized) \( q \)-query VSTAT(\( 1/\tau \)) algorithm which solves the many-vs-one hypothesis testing problem of \( D_\emptyset \) vs. \( S = \{ D_u \}_{u \in S} \) with probability at least \( 1 - \eta \), then it must follow that

\[
\tau \leq \frac{4q^{2/k}}{m(1 - \eta)^{2/k}} \left( k \cdot \left\| E_{u \sim S} (\bar{D}_u^d)^\otimes_k - 1 \right\|^{2/k} + \delta^{2/k} m \right).
\]

The proof of this theorem will consist of two lemmas. The first uses a VSTAT algorithm to construct a good polynomial test of sample-wise degree \((\infty, k)\).

**Lemma B.2.** Let \( m, q \) be non-negative integers, let \( k \) be a non-negative even integer, and let \( \tau > 0 \) and \( \eta \in [0, 1] \). If there is a (randomized) \( q \)-query VSTAT(\( 1/\tau \)) algorithm which solves the many-vs-one hypothesis testing problem of \( D_\emptyset \) vs. \( S = \{ D_u \}_{u \in S} \) with probability at least \( 1 - \eta \), then there is a polynomial \( f : (\mathbb{R}^n)^\otimes m \rightarrow \mathbb{R} \) of sample-wise degree \((\infty, k)\) such that

\[
\mathbb{E}_{u \sim S} \mathbb{E}_{D_\emptyset^m} f \geq (1 - \eta) \sqrt{\left( \frac{m}{k} \right) \cdot \left( \frac{\tau}{2} \right)^k}, \quad \mathbb{E}_{D_\emptyset^m} f = 0, \quad \text{and} \quad \sqrt{\mathbb{E}_{D_\emptyset^m} f^2} \leq q.
\]

Furthermore, \( f = \mathbb{E}_{g \sim \Psi} \sum_{i_1, \ldots, i_k \in [m]} \prod_{t=1}^k g(x_{i_t}) \), for \( \Psi \) a distribution over functions \( g : \mathbb{R}^n \rightarrow \mathbb{R} \) with \( \mathbb{E}_{D_\emptyset} g = 0 \).

**Proof.** Let \( \Psi = \psi_1, \ldots, \psi_q : \mathbb{R}^n \rightarrow [0, 1] \) be any sequence of \( q \) statistical queries, and without loss of generality assume that \( 0 < \mathbb{E}_{D_\emptyset} \psi_t \leq \frac{1}{2} \) for all \( t \in [q] \). Call \( p_t = \mathbb{E}_{D_\emptyset} \psi_t \), and define \( \overline{\psi}_t(x) := \frac{1}{\sqrt{p_t}} (\psi_t(x) - p_t) \), the re-centered and re-normalized version of \( \psi_t \) so that \( \mathbb{E}_{D_\emptyset} \overline{\psi}_t(x) = 0 \), and \( \mathbb{E}_{D_\emptyset} \overline{\psi}_t(x)^2 \leq 1 \). Define \( f_\Psi : (\mathbb{R}^n)^\otimes m \rightarrow \mathbb{R} \) by

\[
f_\Psi(x_1, \ldots, x_m) = \sum_{t=1}^q \left( \sqrt{\frac{1}{(m)^k}} \sum_{i_1, \ldots, i_k \in [m]} \prod_{t=1}^k \overline{\psi}_t(x_{i_t}) \right).
\]

Since the second summation is over products over \( \overline{\psi}_t \) applied to independent samples,

\[
\mathbb{E}_{D_\emptyset^m} [f_\Psi] = \sum_{t=1}^q \left( \sqrt{\frac{1}{(m)^k}} \sum_{i_1, \ldots, i_k \in [m]} \prod_{t=1}^k \mathbb{E}_{D_\emptyset} \psi_t \right) = 0.
\]
Similarly, for any $\Psi, \Psi'$ we have
\[
\mathbb{E}_{D_\omega}^u f_{\psi} f_{\psi'} = \sum_{s, t \in [q]} \mathbb{E}_{D_\omega}^u \left[ \left( \sqrt{\frac{1}{k}} \sum_{i_1 < i_2 < \cdots < i_k} \prod_{\ell=1}^k \bar{\psi}_t(x_{i_\ell}) \right) \left( \sqrt{\frac{1}{k}} \sum_{i_1 < i_2 < \cdots < i_k} \prod_{\ell=1}^k \bar{\psi}_t(x_{i_\ell}) \right) \right] \\
\leq q^2 \cdot \max_{\psi \in \bar{\Psi} \cup \bar{\Psi}'} \mathbb{E}_{D_\omega} \left[ \left( \sqrt{\frac{1}{k}} \sum_{i_1 < i_2 < \cdots < i_k} \prod_{\ell=1}^k \bar{\psi}(x_{i_\ell}) \right)^2 \right] \leq q^2,
\]
where the final inequality follows because for $i_1 < \cdots < i_k$ and $j_1 < \cdots < j_k$,
\[
\mathbb{E}_{D_\omega} \left[ \prod_{\ell=1}^k \bar{\psi}(x_{i_\ell}) \prod_{\ell=1}^k \bar{\psi}(x_{j_\ell}) \right] = 1[(i_1, \ldots, i_k) = (j_1, \ldots, j_k)] \cdot (\mathbb{E}_{D_\omega} \bar{\psi}^2)^k,
\]
And because $\mathbb{E}_{D_\omega} \bar{\psi}^2 \leq 1$. Therefore, for any distribution $Q$ over $\Psi$,
\[
\mathbb{E}_{D_\omega}^u \left[ \mathbb{E}_{\Psi \sim Q}^u f_{\psi} \right] \leq 0, \quad \text{and} \quad \mathbb{E}_{D_\omega} \left[ \left( \mathbb{E}_{\Psi \sim Q}^u f_{\psi} \right)^2 \right] \leq q^2.
\]

Now, supposing that $Q$ is a distribution over $\Psi$ so that with probability at least $1 - \eta$ over $u \sim S$, the queries in $\Psi$ give a $\text{VSTAT}(1/\tau)$ algorithm for distinguishing $D_u, D_\omega$; that is, with probability at least $1 - \eta$ over $u \sim S, \Psi \sim Q$, we have the event
\[
\mathcal{E} := \left\{ \max_{t \in [q]} \mathbb{E}_{D_u} \bar{\psi}_t - \mathbb{E}_{D_\omega} \bar{\psi}_t \geq \max \left( \tau, \sqrt{\tau p_t(1 - p_t)} \right) \right\} \implies \left\{ \max_{t \in [q]} \mathbb{E}_{D_u} \bar{\psi}_t \geq \sqrt{\frac{\tau}{2}} \right\},
\]
where we have used the definition of $\bar{\psi}_t$ and the fact that $(1 - p_t) > \frac{1}{2}$ by assumption. This implies
\[
\mathbb{E}_{u \sim Q} \mathbb{E}_{\Psi \sim Q}^u f_{\psi} = \mathbb{E}_{u \sim Q} \mathbb{E}_{\Psi \sim Q}^u \left[ \sum_{t=1}^q \mathbb{E}_{D_u} \left[ \left( \sqrt{\frac{1}{k}} \sum_{i_1 < i_2 < \cdots < i_k} \prod_{\ell=1}^k \bar{\psi}_t(x_{i_\ell}) \right) \right] \right] \\
= \mathbb{E}_{u \sim Q} \left[ \sum_{t=1}^q \sqrt{\binom{m}{k}} \left( \mathbb{E}_{D_u} \bar{\psi}_t \right)^k \right] \quad (\text{independence of the } x_\ell \text{'s}) \\
\geq (1 - \eta) \mathbb{E}_{u \sim Q} \left[ \sum_{t=1}^q \sqrt{\binom{m}{k}} \left( \mathbb{E}_{D_u} \bar{\psi}_t \right)^k \mid \mathcal{E} \right] \\
\geq (1 - \eta) \sqrt{\binom{m}{k}} \cdot \left( \sqrt{\frac{\tau}{2}} \right)^k,
\]
where in the third line we use the law of conditional expectation and the fact that $k$ is even to drop the expectation in the event $\mathcal{E}$, and in the final line we use the implication of $\mathcal{E}$ and the fact that $k$ is even. Letting $f := \mathbb{E}_{\Psi \sim Q} f_{\psi}$, our conclusion now follows by linearity of expectation. \hfill \Box

We now will show that if the $k$-sample high-degree part of the likelihood ratio of $S$ is bounded, then a good polynomial test of sample-wise degree $(\infty, k)$ also implies one of samplewise degree
\((d,k)\). We remark that the resulting test is not necessarily the degree \((d,k)\)-projection \(f^{\leq d,k}\) of the degree \((\infty,k)\) test \(f\). We instead bound the distance between \(f\) and \(f^{\leq d,k}\) directly by \((d,k)\)-LDLR \(m\). This amounts to showing that if \(f\) and \(f^{\leq d,k}\) are far, then there must be a different good polynomial test of sample-wise degree \((d,k)\). This argument is carried out below.

**Lemma B.3.** Let \(D_\emptyset\) vs. \(S\) be a hypothesis testing problem over \(\mathbb{R}^n\), and suppose that the \(k\)-sample high-degree part of the likelihood ratio of \(S\) is bounded, \(\left\| E_{u \sim S} (D_u^{\otimes m})^{\leq d,k} \right\| \leq \delta\). Let \(\Psi\) be a distribution over functions from \(\mathbb{R}^n \rightarrow \mathbb{R}\). If \(f : (\mathbb{R}^n)^{\otimes m} \rightarrow \mathbb{R}\) is a sample-wise degree-\((\infty,k)\) polynomial of the form

\[
f(x_1, \ldots, x_m) = E_{g \sim \Psi} \sum_{i_1, \ldots, i_k \in [m]} \prod_{i=1}^k g(x_{i_\ell}),
\]

and \(E_{D_\emptyset} g = 0\) for all \(g \sim \Psi\), then we have that

\[
\left( \left\| E_{u \sim S} (D_u^{\otimes m})^{\leq d,k} - 1 \right\|^{2/k} + \delta^{2/k} \cdot \left( \begin{array}{c} m \\ k \end{array} \right)^{1/k} \right)^{k/2} \geq \frac{1}{2} \cdot \frac{E_u \left\| E_{D_\emptyset} f \right\|_2^2}{\left\| E_{D_\emptyset} f \right\|_2^2}.
\]

**Proof.** Since the samples \(x_1, \ldots, x_m \sim D_u^{\otimes m}\) are independent and identically distributed, the moments of \(f\) under the \(m\)-sample distribution \(D_u^{\otimes m}\) are within a multiplicative factor of the moments of one of the summands under the \(k\)-sample distribution \(D_u^{\otimes k}\),

\[
E_{u \sim D_u^{\otimes m}} f = E_{g \sim \Psi} \sum_{i_1, \ldots, i_k \in [m]} \prod_{i=1}^k g(x_{i_\ell}) = \left( \begin{array}{c} m \\ k \end{array} \right) \cdot E_{g \sim \Psi} \left( \prod_{i=1}^k g(x_{i_\ell}) \right).
\]

For any \(g \sim \Psi\), let \(g^{\leq d}\) be its sample-wise degree \((d,\infty)\) projection, and let \(g^{\otimes k}(x_1, \ldots, x_k) = \prod_{i=1}^k g(x_i)\). We have that

\[
E_{g \sim \Psi} \left( \prod_{i=1}^k g(x_{i_\ell}) \right) = \left\langle E_u (D_u^{\otimes d,k}), E_{g \sim \Psi} g^{\otimes k} \right\rangle
\]

\[
= E_u E_{D_u^{\otimes k}} g^{\otimes k} - \left\langle E_u (D_u^{\otimes k} - (D_u^{\leq d,k})^{\otimes k}), E_{g \sim \Psi} g^{\otimes k} \right\rangle
\]

\[
\geq E_u E_{D_u^{\otimes k}} g^{\otimes k} - \left\| E_{g \sim \Psi} g^{\otimes k} \right\| \cdot \left\| E_u (D_u^{\otimes k} - (D_u^{\leq d,k})^{\otimes k}) \right\|
\]

by Cauchy-Schwarz. Now note that \(E_u (D_u^{\leq d,k})^{\otimes k}\) is the orthogonal projection of \(E_u D_u^{\otimes k}\) onto the set of degree-\((d,k)\) polynomials. This set contains all constant polynomials and the projection of \(E_u (D_u^{\leq d,k})^{\otimes k}\) onto the set of constant polynomials is 1. Combining this with Lemmas 3.4, we have

\[
\left\| E_u (D_u^{\otimes k} - (D_u^{\leq d,k})^{\otimes k}) \right\|^2 \leq \left\| E_u D_u^{\otimes k} - 1 \right\|^2 = E_{u,v} \left( (D_u, D_v) - 1 \right)^k
\]

\[
\leq \left( \frac{1}{\left( \begin{array}{c} m \\ k \end{array} \right)^{1/k}} \cdot \left\| E_{u \sim S} (D_u^{\otimes m})^{\leq d,k} - 1 \right\|^{2/k} + \delta^{2/k} \right)^k
\]

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where the last line is from Lemma 3.5. Returning to (9), by linearity of projection to sample-wise degree \((d, k)\) and since \(f\) is already sample-wise degree-\((\infty, k)\), we have that

\[
\mathbb{E}_{u} \mathbb{E}_{D_u^\otimes m} f_{\leq d,k} = \mathbb{E}_{u} \mathbb{E}_{D_u^\otimes m} f - \binom{m}{k} \cdot \left\| \mathbb{E}_{g \sim \Psi} g^{\otimes k} \right\| \left( \frac{1}{\binom{m}{k}}^{1/2} \cdot \left\| \mathbb{E}_{u \sim S} (D_u^\otimes m)^{\leq d,k} - 1 \right\|^{2/k} + \delta^{2/k} \right)^{k/2},
\]

where we used the independence of the samples to equate \(\binom{m}{k}\) \(\mathbb{E}_{g \sim \Psi} \mathbb{E}_{D_u^\otimes k} g^{\otimes k}\) and \(\mathbb{E}_{u} \mathbb{E}_{D_u^\otimes m} f\).

By independence of samples, the terms \(\prod_{\ell=1}^{k} g(x_i)\) and \(\prod_{\ell=1}^{k} h(x_i)\) are uncorrelated when \(x \sim D_u^\otimes\), unless \(i_1, \ldots, i_k = j_1, \ldots, j_k\). Using the fact that for every \(g \sim \Psi\), \(\mathbb{E}_{D_u} g = 0\), and the independence of the samples, this implies that

\[
\mathbb{E}_{D_u^\otimes m} f^2 = \mathbb{E}_{g \sim \Psi} \sum_{i_1, \ldots, i_k \in [m]} \mathbb{E}_{D_u^\otimes m} \left[ \prod_{\ell=1}^{k} g(x_i) h(x_i) \right]
\]

\[
= \mathbb{E}_{g \sim \Psi} \left( \begin{pmatrix} m \\ k \end{pmatrix} \cdot \mathbb{E}_{D_u^\otimes m} \left[ \prod_{\ell=1}^{k} g(x_i) h(x_i) \right] \right) = \left( \begin{pmatrix} m \\ k \end{pmatrix} \cdot \left\| \mathbb{E}_{g \sim \Psi} g^{\otimes k} \right\|^2. \]

Therefore we have that

\[
\left\| \mathbb{E}_{u \sim S} (D_u^\otimes m)^{\leq d,k} - 1 \right\| \geq \frac{\mathbb{E}_{u} \mathbb{E}_{D_u^\otimes m} f_{\leq d,k}}{\mathbb{E}_{D_u^\otimes m} f^{\leq d,k}} \frac{\mathbb{E}_{u} \mathbb{E}_{D_u^\otimes m} f}{\mathbb{E}_{D_u^\otimes m} f^2}
\]

\[
\geq \frac{\mathbb{E}_{u} \mathbb{E}_{D_u^\otimes m} f}{\mathbb{E}_{D_u^\otimes m} f^2} \cdot \left( \left\| \mathbb{E}_{u \sim S} (D_u^\otimes m)^{\leq d,k} - 1 \right\|^{2/k} + \delta^{2/k} \cdot \left( \begin{pmatrix} m \\ k \end{pmatrix}^{1/k} \right)^{k/2} \right). \tag{12}
\]

The first inequality follows from the fact that the left-hand side gives the optimal signal to noises ratio among all sample-wise degree-\((d, k)\) polynomials for the distinguishing problem of \(D_u^\otimes\) versus \(\mathbb{E}_{u} D_u^\otimes m\) (see Section 2). The second inequality follows since \(f_{\leq d,k}\) is a projection of \(f\) onto a convex set, and the final inequality follows by combining (10) and (11). Finally, note that

\[

\left\| \mathbb{E}_{u \sim S} (D_u^\otimes m)^{\leq d,k} - 1 \right\| \leq \left( \left\| \mathbb{E}_{u \sim S} (D_u^\otimes m)^{\leq d,k} - 1 \right\|^{2/k} + \delta^{2/k} \cdot \left( \begin{pmatrix} m \\ k \end{pmatrix}^{1/k} \right)^{k/2} \right),
\]

Applying this after rearranging (12) now completes the proof of the lemma.

Theorem B.1 now follows immediately on applying these two lemmas.

**Proof of Theorem B.1.** Let \(f\) be as in Lemma B.2. Combining Lemmas B.2 and B.3 now yields that

\[
q^{-1}(1 - \eta) \sqrt{\left( \begin{pmatrix} m \\ k \end{pmatrix} \cdot \frac{\tau}{2} \right)^k} \leq \frac{\mathbb{E}_{u} \mathbb{E}_{D_u^\otimes m} f}{\mathbb{E}_{D_u^\otimes m} f^2}
\]

\[
\leq 2 \left( \left\| \mathbb{E}_{u \sim S} (D_u^\otimes m)^{\leq d,k} - 1 \right\|^{2/k} + \delta^{2/k} \cdot \left( \begin{pmatrix} m \\ k \end{pmatrix}^{1/k} \right)^{k/2} \right). \tag{12}
\]

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Rearranging and upper bounding $2^{1+2/k} \leq 4$ yields that
\[
\tau \leq \frac{4q^{2/k}}{(1 - \eta)^{2/k}} \left( \frac{1}{\binom{m}{k}} \cdot \mathbb{E}_{u \sim S} \left( \mathcal{T}_{u}^{\otimes m} \right)^{\otimes d,k} - 1 \right)^{2/k} + \delta^{2/k}.
\]
The fact that $(m/k)^k \leq \binom{m}{k}$ now completes the proof of the theorem. \qed

C Proofs of Cloning Facts

Lemma (Restatement of Lemma 7.2). There is a randomized algorithm taking as input a real number $x$ and outputting $m$ independent random variables $Y_1, \ldots, Y_m$ such that for any $\mu \in \mathbb{R}$ if $x \sim \mathcal{N}(\mu, 1)$, then $Y_i \sim \mathcal{N}(\mu/\sqrt{m}, 1)$.

Proof. Let $U \in \mathbb{R}^{n \times m}$ be a matrix with all entries in the first column equal to $1/\sqrt{m}$ and with remaining columns chosen so that $U$ is orthogonal, i.e., $U^\top U = I_m$. Generate independent variables $Z_2, \ldots, Z_m \sim \mathcal{N}(0, 1)$ and let $Z = (X, Z_2, \ldots, Z_m)^\top$. Now put $Y = UZ$. Note that $Z \overset{d}{=} \mu \cdot e_1 + W$, where $W \sim \mathcal{N}(0, I_m)$ and $e_1$ is the first standard basis vector, and the result follows since $UW \overset{d}{=} W$. \qed

Lemma (Restatement of Lemma 7.3). There is an algorithm that when given $m$ independent samples from $\mathcal{G}(n, U, \gamma)$ for any $U \subseteq [n]$, efficiently produces a single instance distributed according to $\mathcal{G}(n, U, \gamma^m)$. Conversely, there is an efficient algorithm taking a graph as input and producing $m$ random graphs, such that given an instance of planted clique $\mathcal{G}(n, U, \gamma)$ with unknown clique position $U$, produces $m$ independent samples from $\mathcal{G}(n, U, \gamma^{1/m})$.

Proof. The first direction is immediate: given $Y_1, \ldots, Y_m \sim \mathcal{G}(n, U, \gamma)$, form the graph $X$ by letting $X_e = \prod_{i \in [m]} Y_{i,e}$. For the other direction, we will show how to produce $m$ independent Bernoulli variables with appropriate bias from a single Bernoulli. The claim for planted clique will then follow immediately by applying the procedure to the edge indicators of the input graph.

Suppose that $p \in \{ \gamma, 1 \}$ for some $\gamma \in [0, 1]$. We describe how to map a single $x \sim \text{Bern}(p)$ to $(y_1, \ldots, y_m) \sim \text{Bern}(p^{1/m})^{\otimes m}$ without knowing which is the true value of $p$. Given input $x = 1$, output $y_1 = \ldots = y_m = 1$. Now suppose $x = 0$. Let $y = v$ for each $v \in \{0,1\}^m \setminus \{1\}$ with probability $(\gamma^{|v_1|/m} (1 - \gamma^{1/m})^{m - |v_1|} )/(1 - \gamma)$, where $|v_1| = \sum v_i$ is the number of ones in $v$. Note that this probability mass function is exchangeable and thus can be sampled in $\text{poly}(m)$ time as follows. First sample the support size $|y_1| \in \{0, 1, \ldots, m - 1\}$, which has distribution explicitly given by $\Pr(|y_1| = x) = \binom{m}{x} \gamma^x/m (1 - \gamma^{1/m})^{m-x}/(1 - \gamma)$ since the distribution of $y$ is exchangeable. Then produce $y$ by sampling a random binary string in $\{0,1\}^m$ with support size exactly $|y_1|$, uniformly at random.

To check that the output distribution of $(y_1, \ldots, y_m)$ is indeed $\text{Bern}(p^{1/m})^{\otimes m}$ for $p \in \{ \gamma, 1 \}$, first observe that if $p = 1$ then $x = 1$ deterministically and so too are $y_1, \ldots, y_m$. If $p = \gamma$, then
\[
\Pr(y = v) = \gamma \cdot 1_{v = 1} + (1 - \gamma) \cdot 1_{v \neq 1} \cdot \frac{\gamma^{|v_1|/m} (1 - \gamma^{1/m})^{m - |v_1|}}{1 - \gamma} = \left( \gamma^{1/m} \right)^{|v_1|} (1 - \gamma^{1/m})^{m - |v_1|},
\]
which is precisely the probability mass function of $\text{Bern}(\gamma^{1/m})^{\otimes m}$. \qed

D Omitted Calculations from Applications

In this section, we include the calculations omitted from Section 8.
D.1 Tensor PCA

Claim (Restatement of Claim 8.2). For any integers $k, n$, and $r \geq 2$ satisfying $k\lambda^2 < \frac{n}{2}$, the $k$-sample likelihood ratio for the $n$-dimensional $r$-tensor PCA problem with signal strength $\lambda$ is bounded by

$$\left\| E_{u \sim S} T_u^{\otimes k} \right\|^2 \leq \sqrt{\frac{2\pi}{1 - \frac{2k\lambda^2}{n}}}.$$  

Proof. To obtain the first conclusion, we expand

$$\left\| E_{u \sim S} T_u^{\otimes k} \right\|^2 = E_{u,v} (D_u, T_v)^k = E \exp(k\lambda \langle u, v \rangle^r),$$

where for the final equality we have used a simple calculation analogous to that in the proof of Proposition 2.5 of [KWB19]. Since $\langle u, v \rangle$ for $u, v$ sampled uniformly independently from $S$ is distributed as the mean of $n$ Rademacher random variables, we have that $\Pr[|\langle u, v \rangle| > C/\sqrt{n}] \leq 2\exp(-C^2/2)$, and $|\langle u, v \rangle| \leq 1$. So we have

$$E \exp(k\lambda \langle u, v \rangle^r) \leq E \exp(k\lambda^2 |\langle u, v \rangle|^r) \leq 2 \int_0^\sqrt{n} \exp \left( k\lambda^2 \left( \frac{C}{\sqrt{n}} \right)^r - \frac{C^2}{2} \right) dC \leq 2 \int_0^\sqrt{n} \exp \left( -\frac{1}{2} \left( 1 - \frac{2k\lambda^2}{n} \right) C^2 \right) dC \leq \sqrt{\frac{2\pi}{1 - \frac{2k\lambda^2}{n}}}.$$

where to obtain the second line we have substituted $C = \sqrt{n}$ for $r - 2$ copies of $C$, and to obtain the final conclusion we have used that $2k\lambda^2 < n$ and the expression for the Gaussian probability density function. \hfill \Box

Claim (Restatement of Claim 8.3). For any integers $n, r, k, m$ and real number $\lambda$ which satisfy $2em\lambda^2 k(r-2)/2 \leq n^{r/2}$, the $(1, k)$-LDLR$_m$ for the $m$-sample, dimension-$n$ tensor PCA problem with signal strength $\lambda$ is bounded by

$$\left\| E_{u \sim D_u^{\otimes m}} T_u^{\otimes 1, k} \right\|^2 \leq \frac{2e^{r+1}m\lambda^2 k^{(r-2)/2}}{n^{r/2}}.$$

Proof. For a given $D_u = \mathcal{N}(\lambda u^{\otimes r}, I_{n^r})$, from $D_u^{\otimes m}$ we have $m$ samples samples be $\{T_i\}_{i=1}^m$ with each $T_i = \lambda u^{\otimes r} + G_i$, where $G_i \sim \mathcal{N}(0, I_{n^r})$ are independent across samples. We will use the Fourier basis for $(D_u^{\otimes m})^{<1, k} = 1$, which is given by

$$\chi_S : S \in \bigcup_{t=1}^k \left( \begin{bmatrix} [n]^r \end{bmatrix} \right)^{\otimes t} \times \left( \begin{bmatrix} [m] \end{bmatrix} \right)^t,$$

that is, for each $S = \{(A_t, j_t)\}_{t=1}^T$, which specifies a collection $(A_1, \ldots, A_T)$ of $t$ indices in $(\mathbb{R}^n)^{\otimes r}$ and $t$ sample indices $(j_1, \ldots, j_t)$ in $[m]$, we take $\chi_S(T_1, \ldots, T_m) = \prod_{t=1}^T (m)^{j_t}$. For any such $S$ with $|S| = t$, we may compute

$$E_{u \sim D_u} E_{T_1, \ldots, T_m \sim D_u} [\chi_S(T_1, \ldots, T_m)] = E_{u \sim D_u} \prod_{t=1}^T \left( \lambda u^{A_t} + G_t^{(A_t)} \right) = \left( \frac{\lambda}{\sqrt{n^r}} \right)^{|S|} \cdot 1[|S| \text{ is even}],$$

where by “$S$ is even” we mean that the multiset $\cup_{t=1}^T A_t$ contains every $i \in [n]$ with even multiplicity. This is because the indices $j_1, \ldots, j_t \in [m]$ are all distinct, so any term in the expansion of the
product with nonzero degree in the \(G^{(t)}_{A_t}\) variables has expectation 0, and for any multiset of indices \(B \subset [n]^r\), \(E_u u^B = 0\) if any index appears in \(B\) with odd multiplicity, and \(E_u u^B = n^{-|B|/2}\) otherwise.

The even \(S\) of size \(t\) for a fixed set of samples \(j_1, \ldots, j_t \in \binom{[m]}{t}\) are in bijection with \(t\)-edge hypergraph with hyperedges from \([n]^r\) in which every vertex has even degree. Since there can be at most \(rt/2\) vertices in such a hypergraph, and once the vertex set is fixed there are at most \(\frac{r!}{2^{rt/2}(r/2)!}\) ways of choosing an even hypergraph on them according to the configuration model (assign every vertex 2 half-edges, assign every hyperedge \(r\) half-edges, and then count the number of distinct matchings),

\[
|\{S \mid |S| = t, S\text{ even}\}| \leq \binom{m}{t} \cdot n^{rt/2} \cdot \left(\frac{rt!}{2^{rt/2}(r/2)!}\right) \leq \left(\frac{em}{t}\right)^t \cdot n^{rt/2} \cdot (t)^{rt/2},
\]

where we have applied Stirling’s approximation and used that \(r \geq 2\). Thus, we can bound the LDLR,

\[
\left\|E_u(D_u^{(m)})^{\leq 1, k} \right\|^2 = \sum_{t=1}^{k} \left\|\{S \mid |S| = t, S\text{ even}\}\right\| \cdot E_{u \sim D_u^{(m)}}[\chi_S]^2
\leq \sum_{t=1}^{k} \left(\frac{emt^{(r-2)/2}n^{rt/2}}{n^{rt/2}}\right)^t \cdot \left(\frac{\lambda}{n^{rt/2}}\right)^{2t}
\leq \sum_{t=1}^{k} \left(\frac{em\lambda^2t^{(r-2)/2}}{n^{rt/2}}\right)^t \leq \sum_{t=1}^{k} \left(\frac{em\lambda^2k^{(r-2)/2}}{n^{rt/2}}\right)^t \leq 2\frac{em\lambda^2k^{(r-2)/2}}{n^{rt/2}},
\]

where in the final line we have used that \(2em\lambda^2k^{(r-2)/r} \leq n^{rt/2}\) and the fact that the sum is geometric. \(\square\)

### D.2 Planted Clique

**Claim** (Restatement of Claim 8.6). For any \(K, N, k, d, m \in \mathbb{N}\), define \(\gamma = \frac{(p-q)^2}{q(1-q)}\). Then the \((d,k)\)-LDLR\(_m\) for bipartite PDS is bounded \(\|E_{u \sim \mu}(D_u^{(m)})^{\leq d, k} - 1\| = O_N(1)\) if

\[
\frac{K^2}{N} \cdot \max \left\{ \frac{m}{N}, (1 + \gamma)^k \right\} \leq 1 - \Omega_N(1).
\]

**Proof.** We will compute the Fourier coefficients of \(\overline{D} = E_{u \sim \mu} D_u^{(m)}\) as a function on \(\{0, 1\}^{m \times N}\). For each \(m\)-tuple of subsets \(\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)\) where \(\alpha_i \subseteq [N]\), define the Fourier character

\[
\chi_\alpha(x) = \prod_{i=1}^{m} \prod_{j \in \alpha_i} \frac{x_{ij} - q}{\sqrt{q(1-q)}}
\]

for each \(x \in \{0, 1\}^{m \times N}\). Note that the \(\chi_\alpha\) form an orthogonal basis with respect to \(D_u^{(m)}\). For each \(\alpha\), let \(L(\alpha) = \alpha_1 \cup \alpha_2 \cup \cdots \cup \alpha_m\) and \(R(\alpha) = \{ i \in m : \alpha_i \neq \emptyset \}\). A direct computation yields that the Fourier coefficients of \(\overline{D}\) are given by

\[
\widehat{D}(\alpha) = E_{u \sim \mu} E_{x \sim D_u^{(m)}} \chi_\alpha(x) = \left(\frac{K}{N}\right)^{|L(\alpha)| + |R(\alpha)|} \gamma^{1/2 \sum_{i=1}^{m} |\alpha_i|}
\]

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By Parseval’s identity, we now have that
\[
\left\| \mathbf{E}_{u \sim \mu} (\mathcal{D}_u^m)^{\otimes d,k} - 1 \right\|^2 = \sum_{t=1}^{k} \binom{m}{t} \sum_{1 \leq |\alpha_1|, \ldots, |\alpha_t| \leq d} \widehat{D}(\alpha_1, \ldots, \alpha_t, \varnothing, \ldots, \varnothing)^2
\]
\[
= \sum_{t=1}^{k} \binom{m}{t} \sum_{1 \leq |\alpha_1|, \ldots, |\alpha_t| \leq d} \left( \frac{K}{N} \right)^{2|L(\alpha)| + 2t} \gamma^{\sum_{i=1}^{t} |\alpha_i|}
\]  
\begin{equation} \tag{13} \end{equation}

Here, we have used the fact that \( \widehat{D}(\alpha) = \widehat{D}(\alpha_\sigma) \) where \( \alpha_\sigma = (\alpha_\sigma(1), \alpha_\sigma(2), \ldots, \alpha_\sigma(m)) \) for all \( \sigma \in S_m \), by symmetry. Now note that for any fixed \( A \subseteq [N] \), we have that
\[
\sum_{1 \leq |\alpha_1|, \ldots, |\alpha_t| \leq d : L(\alpha) = A} \left( \frac{K}{N} \right)^{2|L(\alpha)| + 2t} \gamma^{\frac{1}{2} \sum_{i=1}^{t} |\alpha_i|} \leq \left( \frac{K}{N} \right)^{2|A| + 2t} \sum_{1 \leq |\alpha_1|, \ldots, |\alpha_t| \leq d : L(\alpha) \subseteq A} \gamma^{\sum_{i=1}^{t} |\alpha_i|}
\]
\[
= \left( \frac{K}{N} \right)^{2|A| + 2t} \left( \sum_{\ell=1}^{\min(d,|A|)} \frac{|A|}{\ell} \gamma^{\ell} \right)^t \leq \left( \frac{K}{N} \right)^{2|A| + 2t} (1 + \gamma)^{|A|t}
\]
where the last inequality follows from the observation
\[
\sum_{\ell=1}^{\min(d,|A|)} \left( \frac{|A|}{\ell} \right) \gamma^{\ell} \leq \sum_{\ell=0}^{|A|} \left( \frac{|A|}{\ell} \right) \gamma^{\ell} = (1 + \gamma)^{|A|}
\]

Note that \( |L(\alpha)| \) can vary between 1 and \( kd \). The fact that there are \( \binom{N}{s} \) possible \( A \) with a given fixed size \( |A| = s \) combined with Equation (13) now yields that
\[
\left\| \mathbf{E}_{u \sim \mu} (\mathcal{D}_u^m)^{\otimes d,k} - 1 \right\|^2 \leq \sum_{t=1}^{k} \sum_{s=1}^{kd} m^t N^s \left( \frac{K}{N} \right)^{2s + 2t} (1 + \gamma)^{ts}
\]
\[
\leq \sum_{t=1}^{k} \sum_{s=1}^{kd} \left( \frac{K^2 m}{N^2} \right)^{t} \left( \frac{K^2 (1 + \gamma)^k}{N} \right)^s
\]
where the second inequality follows from the fact that \( (1 + \gamma)^{ts} \leq (1 + \gamma)^{ks} \) and rearranging. Under the given condition, this upper bound is the product of two geometric series with ratios \( 1 - \Omega_N(1) \), completing the proof of the claim. \( \square \)

**Claim** (Restatement of Claim 8.7). For any \( K, N, k \in \mathbb{N} \), the \( k \)-sample LR is bounded by \( \left\| \mathbf{E}_{u \sim \mu} \mathcal{D}_u^{\otimes k} \right\| = O_N(1) \) if
\[
\frac{K^2}{N} \cdot \max \left\{ \frac{k}{N}, (1 + \gamma)^k \right\} \leq 1 - \Omega_N(1)
\]
where \( \gamma = \frac{(p-q)^2}{q(1-q)} \).

**Proof.** The follows from Claim 8.6 applied with \( d = N \) and \( m = k \), and the observation
\[
\left\| \mathbf{E}_{u \sim \mu} \mathcal{D}_u^{\otimes k} \right\|^2 = \left\| \mathbf{E}_{u \sim \mu} (\mathcal{D}_u^m)^{\otimes d,k} - 1 \right\|^2 + 1
\]
since \( (\mathcal{D}_u^m)^{\otimes d,k} = \mathcal{D}_u^{\otimes k} \) and \( \langle \mathbf{E}_{u \sim \mu} \mathcal{D}_u^{\otimes k}, 1 \rangle = 1 \). \( \square \)
Claim (Restatement of Claim 8.12). For any \( s, K, N, k, d, m \in \mathbb{N} \), the \((d, k)\)-LDDL\(_m\) for multi-sample hypergraph PC satisfies that \( \| \mathbf{E}_{u \sim \mu} (\mathbb{D}_u^\otimes m) \|_{d,k} - 1 \| = O_N(1) \) if the following conditions are satisfied:

\[
\gamma \cdot \max \{ m, (ksd)^s \} = O_N(1) \quad \text{and} \quad \frac{2^s k e^2 K^2}{N} = 1 - \Omega_N(1)
\]

where \( \gamma = \frac{1-q}{q} \).

Proof. Similar to as in Claim 8.6, we will compute the Fourier coefficients of \( \mathbb{D} = \mathbf{E}_{u \sim \mu} \mathbb{D}_u^\otimes m \) as a function on \( \{0, 1\}^{m \times H} \) where \( H = \binom{[N]}{s} \). The relevant orthogonal basis of Fourier characters is indexed by \( m \)-tuples of families of subsets \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m) \) where \( \alpha_i \subseteq H \) and given by

\[
\chi_\alpha(x) = \prod_{i=1}^m \prod_{e \in \alpha_i} \frac{x_{ie} - q}{\sqrt{q(1-q)}}
\]

for each \( x \in \{0, 1\}^{m \times H} \). Given some \( \alpha_i \subseteq H \), let \( V(\alpha_i) = \bigcup_{\{v_1, v_2, \ldots, v_s\} \subseteq \alpha_i} \{v_1, v_2, \ldots, v_s\} \) be the vertex set of the hyperedges in \( \alpha \). Furthermore, let \( V(\alpha) = V(\alpha_1) \cup V(\alpha_2) \cup \cdots \cup V(\alpha_m) \) where \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m) \). Note that \( \mathbf{E}_{x \sim \mathbb{D}_u^\otimes m} \chi_\alpha(x) = 0 \) unless \( V(\alpha) \subseteq u \), which occurs with probability \( \frac{k^s}{(\binom{N}{s})} \) if \( u \sim \mu \). Therefore the Fourier coefficients of \( \mathbb{D} \) are then given by

\[
\hat{\mathbb{D}}(\alpha) = \mathbf{E}_{u \sim \mu} \mathbf{E}_{x \sim \mathbb{D}_u^\otimes m} \chi_\alpha(x) = \left( \frac{k}{\binom{N}{s}} \right)^{\gamma^2 \sum_{i=1}^m |\alpha_i|} \left( \frac{eK}{N} \right)^{|V(\alpha)|} \gamma^\frac{1}{2} \sum_{i=1}^m |\alpha_i|
\]

where the inequality follows from \( (a/b)^b \leq \left( \frac{a}{b} \right) \leq (ea/b)^b \). The same application of Parseval’s as in Claim 8.6 now yields that

\[
\left\| \mathbf{E}_{u \sim \mu} (\mathbb{D}_u^\otimes m) \|_{d,k} - 1 \| \right\|^2 \leq \sum_{t=1}^k \sum_{1 \leq |\alpha_1|, \ldots, |\alpha_t| \leq d} \left( \frac{eK}{N} \right)^{|V(\alpha)|} \gamma^\frac{1}{2} \sum_{i=1}^m |\alpha_i|
\]

We now have that for any \( A \subseteq [N] \),

\[
\sum_{1 \leq |\alpha_1|, \ldots, |\alpha_t| \leq d : V(\alpha) = A} \left( \frac{eK}{N} \right)^{|V(\alpha)|} \gamma^\frac{1}{2} \sum_{i=1}^m |\alpha_i| \leq \left( \frac{eK}{N} \right)^{|A|} \sum_{1 \leq |\alpha_1|, \ldots, |\alpha_t| \leq d : \alpha_i \leq \binom{N}{a}} \gamma^\frac{1}{2} \sum_{i=1}^m |\alpha_i|
\]

\[
= \left( \frac{eK}{N} \right)^{|A|} \left( \frac{ea}{s} \right)^t (1 + \gamma)^{|A| t}
\]

The last inequality holds because of the following observation

\[
\sum_{t=1}^{\min(d, y)} \left( \frac{y}{\ell} \right)^t \gamma^\ell \leq \sum_{t=1}^{\min(d, y)} \left( \frac{y - 1}{\ell - 1} \right)^{t-1} \gamma^\ell \leq y^t (1 + \gamma)^y
\]

for any \( y \in \mathbb{N} \). Note that if \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_t) \) satisfies that that \( 1 \leq |\alpha_i| \leq d \), then \( s \leq |V(\alpha)| \leq ksd \). Give that there are \( \binom{N}{a} \leq N^a \) sets \( A \subseteq [N] \) of a fixed size \( |A| = a \), we have

\[
\left\| \mathbf{E}_{u \sim \mu} (\mathbb{D}_u^\otimes m) \|_{d,k} - 1 \| \right\|^2 \leq \sum_{t=1}^k \sum_{a=s}^{ksd} \frac{m^t}{t!} \cdot N^a \left( \frac{eK}{N} \right)^{2a} \gamma^a \gamma^{at} (1 + \gamma)^a
\]

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\[ \begin{align*}
&= \sum_{a=s}^{k+d} \left( \frac{e^2 K^2}{N} \right)^a \sum_{t=1}^k \frac{(m \gamma a^s (1 + \gamma)^{a^s})^t}{t!} \\
&\leq \sum_{a=s}^{k+d} a^s \left( \frac{e^2 K^2}{N} \right)^a \sum_{t=1}^k \frac{(m \gamma \cdot \exp(\gamma k^s s^d)^t)}{t!} \\
&\leq \left( \sum_{a=s}^{k+d} \left( \frac{2sk^s e^2 K^2}{N} \right)^a \right) \cdot \exp \left( m \gamma \cdot \exp(\gamma k^s s^d)^t \right)
\end{align*} \]

The second last line follows from the inequalities \( a^s \leq a^{sk}, a \leq ksd \) and \( 1 + \gamma \leq \exp(\gamma) \). The last line follows from the fact that if \( x > 0 \), \( \sum_{t=1}^k x^t/t! \leq \exp(x) \) and \( a^s \leq 2^{sk^s} \) since \( a \geq 1 \). The given conditions now imply that the exponential factor is \( O_N(1) \) and that the geometric series has ratio \( 1 - \Omega_N(1) \) and thus is also \( O_N(1) \), completing the proof of the claim.

**Claim** (Restatement of Claim 8.13). For any \( K, N, k \in \mathbb{N} \), the \( k \)-sample LR is bounded by \( \| E_{u \sim \mu} \overline{D}_u^{\otimes k} \| = O_N(1) \) if the following condition are satisfied:

\[ K^2 \leq 3N \quad \text{and} \quad \gamma \leq \frac{1}{2k} \cdot K^{1-s} \log \left( \frac{N}{K^2} \right) \]

where \( \gamma = \frac{1-q}{q} \).

**Proof.** Note that \( \overline{D}_u(x) = \prod_{e \in (u)_x} q^{-1} x_e \) for each \( x \in \{0, 1\}^{(N)}_x \). Therefore we have that

\[ \langle \overline{D}_u, \overline{D}_v \rangle = \mathbb{E}_{x \sim D_\overline{D}} \left[ \prod_{e \in (u)_x} q^{-2} x_e \prod_{e \in (v)_x} q^{-1} x_e \right] \]

\[ = \prod_{e \in (u)_x} q^{-2} \mathbb{E}_{x \sim Ber(q)} [x_e] \prod_{e \in (v)_x} q^{-1} \mathbb{E}_{x \sim Ber(q)} [x_e] \]

\[ = q^{-\left\| (u \cap v) \right\|} \]

where \( A \Delta B \) denotes the symmetric difference of the sets \( A \) and \( B \). Now since \( X = \left| u \cap v \right| \) is distributed as Hypergeometric\((N, K, K)\), we have that

\[ \left\| \mathbb{E}_{u \sim \mu} \overline{D}_u^{\otimes k} \right\|^2 = \mathbb{E}_{u, v \sim \mu} \langle \overline{D}_u, \overline{D}_v \rangle^k = \mathbb{E} q^{-k(\gamma)} = \sum_{x=0}^{K} \frac{(K)}{x} \frac{(N-K)}{K-x} \cdot q^{-k(\gamma)} \]

Now note that for each \( 0 \leq x \leq K \),

\[ \frac{(K)}{x} \frac{(N-K)}{K-x} = \frac{(K)}{x} K(K-1) \cdots (K-x+1) \]

\[ \leq \frac{K^2 x}{N x \left( 1 - \sum_{i=0}^{x-1} \frac{i}{N} - \sum_{i=0}^{K-x-1} \frac{K-x-1}{N-K-x} \right)} \]

\[ \leq \frac{N x \left( 1 - \frac{2K^2}{N-K+1} \right)}{N x \left( 1 - \frac{2K^2}{N-K+1} \right)} \leq \frac{1}{2} \left( \frac{K^2}{K} \right)^x \]

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where the last inequality follows from the fact that $K^2 \leq 3N$. Now since $q^{-1} \leq \exp(\gamma)$ and $\binom{x}{s} \leq xK^{s-1}$ for all $x \leq K$, we have that

$$\left\| \mathbb{E} \overline{D}_u \otimes k \right\|^2 \leq \frac{1}{2} \sum_{x=0}^{K} \exp \left( k \gamma xK^{s-1} - x \log \left( \frac{N}{K^2} \right) \right) \leq \frac{1}{2} \sum_{x=0}^{K} \left( \frac{K^2}{N} \right)^{x/2} = O_N(1)$$

by the given condition on $\gamma$. This completes the proof of the claim. \qed

D.3 Spiked Wishart PCA

Lemma (Restatement of Lemma 8.18). Let $t, d \in \mathbb{N}$. Suppose that $n\rho^2 \leq 1$, and that $dt\lambda \leq \rho n$. Then, we have:

$$\left\| \mathbb{E} \left( \overline{D}_u \mid 1 \right) \otimes t \right\|^2 \leq 2 \left( \frac{d^2 k\lambda}{\rho n} \right)^{2t}.$$  

Proof. Fix any multi-index $\alpha = (\alpha_1, \ldots, \alpha_t)$ so that $|\alpha_i|$ is even and so that $2 \leq |\alpha_i| \leq d$, for all $i = 1, \ldots, t$. Suppose moreover that $|\{ j : \exists i : \alpha_{ij} \neq 0 \}| = \ell$, and let $s = |\alpha|$. Then the proceeding lemma implies that

$$\left( \mathbb{E} \left( \overline{D}_u, H_{\alpha} \right) \right)^2 \leq \left( \frac{d\lambda}{\rho n} \right)^s \rho^{2\ell}.$$  

The total number of such monomials can be naively upper bounded by $\binom{n}{\ell}^t \ell^s$. Hence the contribution to the LDLR of all such monomials, for a fixed $\ell$ and $s$, can be upper bounded by

$$\binom{n}{\ell}^t \left( \frac{d\lambda}{\rho n} \right)^s \rho^{2\ell} \leq \left( \frac{d\ell}{\rho n} \right)^s \rho^{2\ell} \left( \frac{dn\rho^2}{\rho n} \right)^s,$$

by assumption. Summing over all $2t \leq s \leq dt$, and $1 \leq \ell \leq dt$, we obtain that

$$\left\| \mathbb{E} \left( \overline{D}_u \mid 1 \right) \otimes t \right\|^2 \leq \sum_{2t \leq s \leq dt, 1 \leq \ell \leq dt} \left( \frac{d\ell}{\rho n} \right)^s \left( \frac{d^2 k\lambda}{\rho n} \right)^{2t},$$

since from our assumptions, the sum is convergent. \qed

Lemma (Restatement of Lemma 8.21). Assume that $2nk(d+1)\rho^2 \leq 1$. For $\lambda < 1/2$ and $d$ even, we have:

$$\left\| \mathbb{E} \left( \overline{D}_u \right) \otimes k \right\|^2 \leq \left( \frac{\lambda^2}{4\rho n} \right)^{k(d+1)}.$$  

Proof. This proof closely resembles the proof of Lemma 6.2. Let $Z$ be the random variable given by $Z = \frac{\lambda^2(u,v)^2}{4}$ when $u, v \sim S_\rho$. From the proceeding lemma, we have that

$$\left\| \mathbb{E} \left( \overline{D}_u \right) \otimes k \right\|^2 \leq \mathbb{E} \left[ \phi^{d/2}(Z)^k \right].$$

By Taylor’s theorem, since the function $\phi(x)$ is analytic for all $|x| \leq 1/4$, we know that

$$|\phi^{d/2}(x)| \leq \left( \frac{d+2}{d/2+1} \right)^d x^{d+1} (1 - 4\eta(x))^{-(d+3)/2} \leq \left( \frac{d+2}{d/2+1} \right)^d x^{d+1} \phi(x)^{d+3}.$$

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where \(0 \leq \eta(x) \leq x\), and the last inequality follows since \(\phi\) is monotone. Hence
\[
\left\| \mathbf{E}_{u \sim S_{\rho}} \left( \mathcal{D}_{u}^\top \mathcal{D}_{u} \right)^{\otimes k} \right\|^2 \leq d^{k(d+2)} \left( 1 - 4\lambda^2 \right)^{k(d+3)} \mathbf{E} Z^{k(d+1)}.
\]

The moment can only be increased by considering the inner product between the two untruncated vectors. Let \(Z'\) be distributed as the untruncated version of \(Z\). Then \(\mathbf{E} Z' = \frac{\lambda^2}{4\rho n} \left( \sum_{i=1}^{n} Y_i \right)^2\) where each \(Y_i\) is independent, \(Y_i = 0\) with probability \(1 - \rho^2/2\), \(Y_i = 1\) with probability \(\rho^2/4\), and \(Y_i = -1\) with probability \(\rho^2/4\). Hence
\[
\mathbf{E} Z^{k(d+1)} \leq \mathbf{E} (Z')^{k(d+1)} = \left( \frac{\lambda^2}{4\rho n} \right)^{k(d+1)} \mathbf{E} \left( \sum_{i=1}^{n} Y_i \right)^{2k(d+1)}
\]
\[
\leq \left( \frac{\lambda^2}{4\rho n} \right)^{k(d+1)} \sum_{|\alpha| = 2k(d+1)} \mathbf{E} Y^\alpha
\]
\[
\leq \left( \frac{\lambda^2}{4\rho n} \right)^{k(d+1)} \left( \sum_{\ell=1}^{k(d+1)} \left( \begin{array}{c} n \\ \ell \end{array} \right) \left( \frac{k(d+1) + \ell}{\ell} \right) \rho^2 \right)
\]
\[
\leq \left( \frac{\lambda^2}{4\rho n} \right)^{k(d+1)} \sum_{\ell=1}^{k(d+1)} \left( 2nk(d+1)\rho^2 \right)^\ell \leq \left( \frac{\lambda^2}{4\rho n} \right)^{k(d+1)},
\]
where the final summand is convergent by assumption.

\[\square\]

### D.4 Gaussian Graphical Models

**Lemma** (Restatement of Lemma 8.31). For any integer \(d\) sufficiently large, any \(s \gg d\) sufficiently large, any \(n \gg s\) sufficiently large, and \(\kappa \in (0, \frac{1}{6\sqrt{d}})\) such that the following holds: If \(\mathcal{S}\) vs. \(\mathcal{D}_\emptyset\) is an instance of the \((\kappa, d, s, n)\)-prsGGM problem, then for any even integer \(k\) and \(q \geq 1\),
\[
\text{SDA} \left( \mathcal{S}, \left( \frac{n}{q^2 s^2} \right)^{1/k} \frac{1}{\exp(\frac{n}{2s} \rho^2) - 1} \right) \geq q,
\]
and further,
\[
\mathbf{E} \left( \mathcal{D}_{u,v} \mathcal{D}_{v} \right)^k \leq \left( 1 + \left( \frac{s^2}{n} \right)^{1/k} \left( \exp(\frac{n}{2s} \rho^2) - 1 \right) \right)^k.
\]

To prove this lemma, we will make use of the following claim:

**Claim D.1.** Let \(A, B\) be symmetric \(n \times n\) real matrices, let \(\mathcal{D}_\emptyset = \mathcal{N}(0, \mathbf{I})\). Suppose \(\mathbf{I}_n + A + B > 0\), \(\mathbf{I}_n + A > 0\), and \(\mathbf{I}_n + B \succeq 0\). Let \(\mathcal{D}_a = \mathcal{N}(0, (\mathbf{I} + A)^{-1})\) and \(\mathcal{D}_b = \mathcal{N}(0, (\mathbf{I} + B)^{-1})\), and let \(\mathcal{D}_a, \mathcal{D}_b\) be the respective relative densities. Then
\[
\langle \mathcal{D}_a, \mathcal{D}_b \rangle_{\mathcal{D}_a} = \frac{1}{\sqrt{\det((\mathbf{I} + (\mathbf{I} + A)^{-1}A)(\mathbf{I} + B)^{-1})}}
\]

**Proof.** We have that
\[
\langle \mathcal{D}_a, \mathcal{D}_b \rangle = \frac{1}{\sqrt{(2\pi)^n \det((\mathbf{I} + A)^{-1}) \det((\mathbf{I} + B)^{-1})}} \int_{\mathbb{R}^n} \exp \left( -\frac{1}{2} x^\top (\mathbf{I}_n + A + B) x \right) dx
\]

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where the second line follows by integrating the Gaussian pdf with covariance \((I + A + B)^{-1}\), and the third line follows by noting that \(\det(X^{-1}) = \det(X)^{-1}\), that \(\det(X) \det(Y) = \det(XY)\), and that \(I + A + B = (I + A)(I + B) - AB\). This completes the proof. \(\square\)

**Proof of Lemma 8.31.** First, since a random signed \(d\)-regular graph on \(s\) vertices has its spectrum within \([-2\sqrt{d} - \Gamma(1 + \varepsilon), 2\sqrt{d} - \Gamma(1 + \varepsilon)]\) with high probability, for sufficiently large \(d\) the condition on the spectrum is met with very high probability, and \(S\) has size at least \(\binom{n}{d} \cdot \binom{s}{d}^{s/100}\) (a vast underestimate of the number of \(d\)-regular random graphs on \(s\) vertices planted within \(n\)-vertex empty graphs).

Since \(\kappa 2\sqrt{d} < \frac{1}{\theta}\), the matrices \(I + \kappa \Delta_u\) and \(I + \kappa \Delta_u + \kappa \Delta_v\) meet the conditions of Claim D.1. Using Claim D.1, it suffices to bound

\[
E_{u,v \sim S} ((D_u, D_v - 1))^k = E_{u,v \sim S} \left( \frac{1}{\sqrt{\det(I - \kappa^2(I + \kappa \Delta_u)^{-1}\Delta_u \Delta_v(I + \kappa \Delta_v)^{-1}) - 1}} \right)^k \tag{14}
\]

since to obtain the SDA bound we may apply Equation (2), and to get the second conclusion we use Hölder’s inequality and the triangle inequality,

\[
E_{u,v \sim S} (D_u, D_v)^k \leq \sum_{\ell=0}^{k} \binom{k}{\ell} E_{u,v} \left(\langle D_u, D_v \rangle - 1\right)^\ell \leq \left(1 + E_{u,v} \left(\langle D_u, D_v \rangle - 1\right)^k \right)^{1/k},
\]

Now, when \(u, v \sim S\), with probability at least \(1 - \frac{s^2}{n}\), \(\Delta_u\) and \(\Delta_v\) correspond to graphs with disjoint support, so \(\Delta_u \Delta_v = 0\). For such \(u, v\), the right-hand side of (14) is zero.

Otherwise, if \(\Delta_u, \Delta_v\) overlap, the \((I + \kappa \Delta_u)^{-1}\Delta_u \Delta_v(I + \kappa \Delta_v)^{-1}\) has at most \(s\) eigenvalues which are not 1 (since \(\Delta_u, \Delta_v\) are rank-s). Further, since all eigenvalues \(\Delta_u, \Delta_v\) are in the interval \([-2\sqrt{d}, 2\sqrt{d}]\), and since \(\Delta_u\) and \((I + \kappa \Delta_u)^{-1}\) commute, the eigenvalues of \((I + \kappa \Delta_u)^{-1}\Delta_u \Delta_v(I + \kappa \Delta_v)^{-1}\) are in the interval \([-1 - \frac{2\sqrt{d}}{1 - \kappa \sqrt{d}}, 1 - \frac{2\sqrt{d}}{\kappa \sqrt{d}}]\). This implies that all eigenvalues of \((I + \kappa \Delta_u)^{-1}\Delta_u \Delta_v(I + \kappa \Delta_v)^{-1}\) are in the interval \([-\frac{2\sqrt{d}}{1 - \kappa \sqrt{d}}, \frac{2\sqrt{d}}{1 - \kappa \sqrt{d}}]\). Thus, for such \(u, v\),

\[
\frac{1}{\sqrt{\det(I - \kappa^2(I + \kappa \Delta_u)^{-1}\Delta_u \Delta_v(I + \kappa \Delta_v)^{-1}) - 1}} \leq \left(1 - \frac{1}{\frac{1 - \kappa^2 d}{(1 - \kappa \sqrt{d})^2}} \right)^{s/2}.
\]

Putting these observations together with (14),

\[
E_{u,v} (\langle D_u, D_v \rangle - 1)^k \leq \frac{s^2}{n} \left(\left(1 - \frac{1}{1 - d \left(\frac{\kappa}{1 - \kappa \sqrt{d}}\right)^2}\right)^{s/2} - 1\right)^k \leq \frac{s^2}{n} \left(1 + \kappa^2 d^{s/2} - 1\right)^k,
\]

where we have used that \(\kappa \sqrt{d} < \frac{1}{\theta}\). We can further simplify the above by noting that \(1 + x \leq \exp(x)\).
Thus, applying Equation (2), we have that for any $q \geq 1$,

$$\text{SDA} \left( \mathcal{S}, \left( \frac{n}{q^2s^2} \right)^{1/k} \frac{1}{\exp(sdk^2/2) - 1} \right) \geq q,$$

and we obtain the bound on $\| E_u \mathcal{D}^{\otimes k}_u \|$ using Hölder’s as described above. \qed