Near-Field Thermal Emission by Periodic Arrays

Sheila Edalatpour†

Department of Mechanical Engineering, University of Maine, Orono, ME 04469, USA

ABSTRACT

Near-field thermal emission can be engineered by using periodic arrays of sub-wavelength emitters. The array thermal emission is dependent on the shape, size, and materials properties of the individual elements as well as the period of the array. Designing periodic arrays with desired properties requires models that relate the array geometry and material properties to its near-field thermal emission. Direct numerical modeling of periodic arrays is computationally expensive as several periods of the array should be discretized. In this study, a new technique that only requires discretizing one period of the array is presented. In this technique, the energy density emitted by the array is expressed in terms of array’s Green’s functions. The array Green’s functions are found using the discrete dipole approximation in a periodic manner by expressing a single point source as a series of periodic arrays of phase-shifted point sources. The presented technique can be employed for modeling inhomogeneous and complex-shape arrays with non-uniform temperature distribution. The technique is verified against the direct numerical modeling of thermal emission by two periodic arrays, and it is demonstrated that it can serve as a versatile and reliable technique for studying near-field thermal emission by periodic arrays.

I. INTRODUCTION

Thermal emission is in the near-field regime when the observation distance from the emitter is smaller than or comparable to the dominant thermal wavelength as determined using Wien’s displacement law. Otherwise, thermal emission is said to be in the far-field regime. While far-field thermal emission is broadband, incoherent, unpolarized and limited by blackbody radiation, near-field thermal emission can be quasi-monochromatic, coherent, polarized and exceeds the blackbody limit by several orders of magnitude [1]. These properties of thermal near field are

† Corresponding author. Tel.: +1 207 581 2375, Fax: +1 207 581 2379

E-mail address: sheila.edalatpour@maine.edu
capitalized on for many promising applications in waste heat recovery [2–8], thermal rectification [9–15], nanoscale imaging [16–19], and nanomanufacturing [20–22]. Most of these applications require near-field properties that are not found among natural materials. Near-field thermal emission can be engineered by using periodic arrays of sub-wavelength emitters [23–36]. Designing periodic arrays with desired properties requires models that relate the geometry and material properties of the array to its near-field thermal emission. Analytical models do not exist for this purpose. As such, periodic arrays have been mostly modeled numerically [28,31,34] or using the effective medium theory (EMT) [23–25,27,29,30,32,35,36]. The EMT is an approximate method in which the array is modeled as a homogenous medium with effective dielectric properties. The validity of the EMT in the near-field regime, where the observation distance is in the same order of magnitude as the emitter sizes, is questionable. Direct numerical modeling also requires discretizing several periods of the array which is computationally expensive. Particularly, a greater number of periods needs to be discretized as the observation distance increases. Furthermore, simulations should be repeated a few times to ensure that the number of periods selected for modeling results in a converged solution. It is very beneficial to have a periodic numerical method in which only a period of the array is discretized. So far, only one periodic method based on expanding the stochastic thermal current into deterministic orthonormal current modes has been applied to near-field thermal radiation problems [37]. This method is more suitable for simple geometries for which the proper current modes are readily found [37]. In this paper, we present a new periodic method for modeling near-field thermal emission by two-dimensional periodic arrays of complex-shape emitters. The emitters can be inhomogeneous and have non-uniform temperature distribution. In this method, the energy density emitted by the array is expressed in terms of array’s Green’s functions that are the response of the array to illumination by a single point source. The array Green’s functions are found in a periodic manner by expanding the single point source into a series of periodic arrays of phase-shifted point sources. The discrete dipole approximation (DDA) [38,39] is used for numerical simulations. This approach requires modeling only one period of the array and thus is computationally beneficial. Although the array is assumed to be periodic in two dimensions, the proposed method can easily be applied for modeling one-dimensional and three-dimensional periodic arrays.
This paper is structured as follows. The problem under consideration is described and formulated in Sections II and III, respectively. The proposed periodic technique is discussed in Section IV, and numerical examples are provided in Section V. Finally, the concluding remarks are presented in Section VI.

II. DESCRIPTION OF THE PROBLEM

A schematic of the problem under consideration is shown in Fig. 1. A two-dimensional, infinite array of arbitrarily-shaped objects is periodic in $x$- and $y$-directions. The array has periods $L_x$ and $L_y$ along the $x$- and $y$-directions, respectively, and is submerged in the free space. The smallest building block of the array, which can consist of an arbitrary number of arbitrarily-shaped objects, is referred to as the unit cell. The replica of the unit cell along $x$- and $y$-directions are numbered using variables $p$ and $q$, respectively, where $p$ and $q$ vary from $-\infty$ to $\infty$. The unit cell is identified as the cell with $(p, q) = (0, 0)$. The array is at a temperature $T$ greater than absolute zero and thus emits thermal radiation in the free space. The objects are assumed to be non-magnetic, isotropic, in local thermodynamic equilibrium, and their dielectric response is described by a frequency-dependent dielectric function $\varepsilon(\omega) = \varepsilon'(\omega) + i\varepsilon''(\omega)$. The spectral energy density, $u$, emitted by the array at an observation point $r_o$ in the free space is to be calculated.

![Figure 1](image)

Figure 1. A schematic of the problem under consideration. A two-dimensional periodic array of arbitrarily-shaped objects with periods $L_x$ and $L_y$ in $x$- and $y$-directions, respectively, emits thermal radiation in the free space. The emitted energy density at the observation point $r_o$, $u(r_o, \omega)$, is desired.
III. FORMULATION OF THE PROBLEM

The energy density at observation point \( r_o \) and angular frequency \( \omega \) is given by [40]:

\[
u(r_o, \omega) = \frac{1}{2} \varepsilon_0 \text{Trace} \left\{ \mathbf{E}(r_o, \omega) \otimes \mathbf{E}(r_o, \omega) \right\} + \frac{1}{2} \mu_0 \text{Trace} \left\{ \mathbf{H}(r_o, \omega) \otimes \mathbf{H}(r_o, \omega) \right\}
\]

where \( \varepsilon_0 \) and \( \mu_0 \) are the free space permittivity and permeability, respectively, \( \mathbf{E} \) is the electric field, \( \mathbf{H} \) is the magnetic field, \( \otimes \) is the outer product and \( \left\{ \right\} \) denotes the ensemble average. The electric field at point \( r_o \) can be obtained using the dyadic electric Green’s function of the array \( \mathbf{G}^E \) and the thermally fluctuating current \( \mathbf{J}^f \) as [41]:

\[
\mathbf{E}(r_o, \omega) = i \omega \mu_0 \int_{V} \mathbf{G}^E(r_o, r') \cdot \mathbf{J}^f(r', \omega) dV'
\]

where \( i \) is the imaginary unit number and the integral is performed over the volume of the array where the fluctuating current is non-zero. The dyadic electric Green’s function \( \mathbf{G}^E(r_o, r') \) relates the electric field at observation point \( r_o \) to the thermally fluctuating current at \( r' \) generating this electric field. The magnetic field at the observation point \( r_o \) can be obtained in a similar manner using the dyadic magnetic Green’s function \( \mathbf{G}^H \):

\[
\mathbf{H}(r_o, \omega) = \int_{V} \mathbf{G}^H(r_o, r') \cdot \mathbf{J}^f(r', \omega) dV'
\]

The ensemble average of the thermally fluctuating current \( \mathbf{J}^f \) is zero, while the ensemble average of its spatial correlation function is given by the fluctuation dissipation theorem [42,43]:

\[
\left\{ \mathbf{J}^f(r', \omega) \otimes \mathbf{J}^f(r'', \omega) \right\} = \frac{4 \omega \varepsilon_0 \varepsilon''}{\pi} \Theta(\omega, T) \delta(r' - r'') I
\]

In Eq. (4), \( \varepsilon'' \) is the imaginary part of the dielectric function of the objects, \( \Theta(\omega, T) = \frac{h \omega}{\hbar} \exp\left(\frac{h \omega}{k_B T}\right) - 1 \), \( \hbar \) and \( k_B \) being the reduced Planck and Boltzmann constants,
respectively, is the mean energy of an electromagnetic state [44], $\delta$ is the Dirac delta function, and $I$ is the unit dyad. Substituting Eqs. (2) to (4) into Eq. (1), the energy density is written as:

$$u(r_o, \omega) = \frac{2k_0^2e^\gamma}{\pi \omega} \Theta(t, t) \left[ \text{Trace} \left[ k_0^2 G^E(r_o, r') \otimes G^E(r_o, r') + G^H(r_o, r') \otimes G^H(r_o, r') \right] dV' \right]$$  \hspace{1cm} (5)

The electric and magnetic Green’s functions of the array, $G^E(r_o, r')$ and $G^H(r_o, r')$, are needed for calculating the energy density using Eq. (5). Based on the reciprocity principle, $G^\gamma(r_o, r') = \left[ G^\gamma(r', r_o) \right]^T$ where $\gamma = E$ or $H$ [45]. As such, the energy density can equivalently be found using the Green’s functions $G^E(r', r_o)$ and $G^H(r', r_o)$. As shown in Fig. 2, the electric Green’s function is determined by calculating the electric field generated at $r'$ due to radiation by a point source $J(r)$ of magnitude $1/(i \omega \mu_0)$ located at $r_o$ (i.e., $J(r) = \delta(r - r_o)/(i \omega \mu_0)$) [41]. In a similar manner and as shown in Fig. 2, the magnetic dyadic Green’s function, $G^H(r', r_o)$, is obtained by measuring the electric field at point $r'$ due to a magnetic point source $M(r)$ of unit magnitude radiating at $r_o$ (i.e., $M(r) = \delta(r - r_o)I$) [7]. A periodic approach for solving this problem is presented in the next section.

Figure 2. The dyadic electric (magnetic) Green’s function for the array $G^E(r', r_o)$ ($G^H(r', r_o)$) is found by placing an electric (a magnetic) point source at $r_o$ and measuring the electric field generated at $r'$. 

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IV. A PERIODIC APPROACH FOR CALCULATING ARRAY GREEN’S FUNCTIONS

The array Green’s functions are obtained by calculating the electric field generated at point \( \mathbf{r}' \) of the array due to radiation by a point source, represented by Dirac delta function, at the observation point \( \mathbf{r}_o \). It is desired to solve this problem in a periodic manner. However, this problem is not periodic since the single point source illuminating the array is aperiodic. This aperiodic problem can be converted into a series of periodic problems by expressing the single point source as a periodic array of phase-shifted point sources.

A. Periodic expansion of the Dirac delta function

The single point source emitting at \( \mathbf{r}_o \) can be replaced by a periodic array of phase-shifted point sources with periods \( L_x \) and \( L_y \) using the fact that the Dirac delta function can be expanded as:

\[
\delta\left(\mathbf{r} - \mathbf{r}_o\right) = \frac{L_x L_y}{(2\pi)^2} \int \int \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \delta\left(\mathbf{r} - \left[ \mathbf{r}_o + pL_x \mathbf{x} + qL_y \mathbf{y} \right]\right) e^{i(pL_x \mathbf{k}_x + qL_y \mathbf{k}_y)} d\mathbf{k}_x d\mathbf{k}_y
\]

In Eq. (6), \( \delta\left(\mathbf{r} - \left[ \mathbf{r}_o + pL_x \mathbf{x} + qL_y \mathbf{y} \right]\right) \) represents the replica \((p, q)\) of the point source at \( \mathbf{r}_o \) which is located at \( \mathbf{r}_{opq} = \mathbf{r}_o + pL_x \mathbf{x} + qL_y \mathbf{y} \), \( e^{i(pL_x \mathbf{k}_x + qL_y \mathbf{k}_y)} \) is the phase shift of the point source at \( \mathbf{r}_{opq} \) relative to that located at \( \mathbf{r}_o \), and \( k_x \) and \( k_y \) are the phasing gradients along the \( x \)- and \( y \)-directions, respectively. It should be noted that the phasing gradients are essentially mathematical wave vectors which are restricted to the periodicity of the Brillouin zone \((-\pi/L_\beta \text{ and } \pi/L_\beta, \text{ where } \beta = x \text{ and } y\). These mathematical wave vectors allow for the expansion of the delta function and by no means they represent a physical wavevector. Equation (6) holds true because when the phase-shifted point sources are integrated over the Brillouin zone, all of the point sources in the phase-shifted array integrate to zero except for the one located at \((p, q) = (0, 0)\). The proof of Eq. (6) is provided in the appendix. Using Eq. (6), the electric and magnetic point sources, \( \mathbf{J}(\mathbf{r}) \) and \( \mathbf{M}(\mathbf{r}) \), can be expressed as:

\[
\mathbf{J}(\mathbf{r}) = \frac{L_x L_y}{(2\pi)^2} \int \int \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \delta\left(\mathbf{r} - \left[ \mathbf{r}_o + pL_x \mathbf{x} + qL_y \mathbf{y} \right]\right) e^{i(pL_x \mathbf{k}_x + qL_y \mathbf{k}_y)} i\omega\mu_0 d\mathbf{k}_x d\mathbf{k}_y
\]
When \( \mathbf{J}(r) \) and \( \mathbf{M}(r) \) are expressed using Eqs. (7a) and (7b), a series of periodic problems such as the one shown in Fig. 3 is obtained. In these problems, a periodic array of objects is illuminated by a periodic array of phase-shifted point sources. An analytical solution for this problem is not feasible and numerical solutions should be sought. A numerical solution based on the DDA is presented for this problem in the next section.

\[
\mathbf{M}(r) = \frac{L_x L_y}{(2\pi)^3} \int_{-\pi/L_x}^{\pi/L_x} \int_{-\pi/L_y}^{\pi/L_y} \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \delta(r - [r_o + pL_x, x + qL_y]) e^{i(pL_x k_x + qL_y k_y)} \mathbf{L} dk_x dk_y
\]  

(7b)

Figure 3. A single point source at the observation point \( r_o \) can be expressed as a series of periodic arrays of phase-shifted (relative to \( r_o \)) point sources.

B. Numerical solution of array Green’s functions

The DDA, which is a volume discretization method [38,39,43], is used for calculating the Green’s functions of the array. In this method, the objects are discretized into cubical sub-volumes with sizes much smaller than the thermal wavelength, the object sizes, the separation distance of the objects, and the distance between the observation point and the array. As such, the electric field can be assumed as uniform within the sub-volumes. By discretizing the volume-integral form of Maxwell’s equations, the electric field in sub-volumes can be written as [43]:

\[
\delta(r - r_o) e^{i(pL_x k_x + qL_y k_y)} \mathbf{L}
\]
\[
\frac{1}{\alpha_i} \mathbf{E}_0 (\varepsilon - 1) \mathbf{E}_{nn} - k_0^2 \sum_{j=1}^{N} V_j (\varepsilon_j - 1) \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \mathbf{G}_{nn,jpq}^{0E} \cdot \mathbf{E}_{jpq} = \mathbf{E}_{inc}^{inc},
\]

where \( \mathbf{E}_{nn} \) is the electric field in replica \((m, n)\) of sub-volume \(i\) in the unit cell (sub-volume \(inn\)) which is located at \(\mathbf{r}_{inn} = \mathbf{r}_i + mL_x \hat{x} + nL_y \hat{y} \) with \(\mathbf{r}_i\) being the position of sub-volume \(i\), \(\mathbf{G}_{inn,jpq}^{0E}\) is the free-space dyadic electric Green’s function between sub-volumes \(inn\) and \(jpq\) \([43,46]\), \(\mathbf{E}_{inc}^{inc}\) is the electric field incident on sub-volume \(inn\) due to radiation by the point-source arrays, \(\alpha_i\) is the polarizability of sub-volume \(i\) and its replica \([43]\), and \(k_0\) is the magnitude of the wavevector in the free space. The first summation in Eq. (8) runs over the \(N\) sub-volumes located in the unit cell, while the second and third summations run over replica of the sub-volumes in the unit cell along the \(x\)- and \(y\)-directions, respectively. It should be noted that the double summation in Eq. (8) excludes the term corresponding to sub-volume \(inn\), and it is assumed that the replica sub-volumes have the same dielectric function and volume as their counterpart in the unit cell.

For calculating the electric Green’s function \(\mathbf{G}_{inn,o}^{E}\), the incident field on the sub-volumes is due to radiation by the electric source \(\mathbf{J}(\mathbf{r})\) given by Eq. (7a), and it can be calculated using the free-space electric Green’s function as \([41,47]\):

\[
\mathbf{E}_{inn}^{inc} = i \omega \mu_0 \int_{V} \mathbf{G}_{inn}^{0E}(\mathbf{r}_{inn}, \mathbf{r}) \cdot \mathbf{J}(\mathbf{r}) dV \tag{9}
\]

Substituting for \(\mathbf{J}(\mathbf{r})\) using Eq. (7a) and using the commutativity and associativity properties of the integral and summation, the incident field can be written as:

\[
\mathbf{E}_{inn}^{inc} = \frac{L_x L_y}{(2\pi)^2} \int_{-\pi/L_x}^{\pi/L_x} \int_{-\pi/L_y}^{\pi/L_y} \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \mathbf{G}_{inn}^{0E}(\mathbf{r}_{inn}, \mathbf{r}) \delta \left( \mathbf{r} - \left[ \mathbf{r}_o + pL_x \mathbf{x} + qL_y \mathbf{y} \right] \right) e^{i(pL_x k_x + qL_y k_y)} dV dL x dL y \tag{10}
\]

which due to the fundamental property of the delta function reduces to:
For calculating the array magnetic Green’s function $G_{imn,o}^H$, the incident field is due to illumination by the magnetic current array $M(r)$ as is given Eq. (7b). The incident electric field due to the magnetic source $M(r)$ can be obtained in the same manner as for the electric current $J(r)$, and it is given by:

$$E_{inc}^{E} = \frac{L_x L_y}{(2\pi)^2} \int_{-\pi/L_y}^{\pi/L_y} \int_{-\pi/L_x}^{\pi/L_x} \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} G_{mn,opq}^{0E} e^{i(pL_xqL_yk_xk_y)} dk_x dk_y$$

(11)

where $G_{mn,opq}^{0H}$ is the free-space dyadic magnetic Green’s function [40]. Substituting Eqs. (11) and (12) into Eq. (8) results in the following equation:

$$\frac{1}{\alpha_i} \sum_{j=1}^{N} V_j \left( \epsilon_j - 1 \right) G_{imn,o}^{y} - k_0^2 \sum_{j=1}^{N} V_j \left( \epsilon_j - 1 \right) \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} G_{mn,ipq}^{0E} \cdot G_{ipq,o}^{y}$$

$$= \frac{L_x L_y}{(2\pi)^2} \int_{-\pi/L_y}^{\pi/L_y} \int_{-\pi/L_x}^{\pi/L_x} \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} G_{mn,opq}^{0H} e^{i(pL_xqL_yk_xk_y)} dk_x dk_y$$

$$\gamma = E \text{ or } H; \quad i = 1, 2, \ldots, N; \quad m, n = 0, \pm 1, \pm 2, \ldots$$

(13)

When Eq. (13) is written for all sub-volumes in the periodic array ($i = 1, 2, \ldots, N; \quad m, n = 0, \pm 1, \pm 2, \ldots$), two self-consistent linear systems of equations are obtained which can be solved for $G_{mn,o}^{E}$ and $G_{mn,o}^{H}$. Since there are an infinite number of sub-volumes in the periodic array, these systems of equations are of infinite size. One approach for solving these systems of equations is to truncate the summations in Eq. (13) at a large number of sub-volumes and solve the resulted finite-size systems of equations. However, this approach has two drawbacks. First, the summations converge very slowly [48] and large systems of equations should be solved which is computationally expensive. Second, a few trials with different number of sub-volumes are required to ensure a converged solution. A more efficient approach is to establish a relation between the Green’s functions of the sub-volumes in the unit cell and their replica, such that the system of equations
(13) can independently be solved for the \( N \) sub-volumes in the unit cell. This can be done by eliminating the integrals from Eq. (13). Since the system of equation (13) is linear, its solution can be written as the double integral of a wave-vector dependent Green’s function, \( \mathbf{g}_{i,m,n,o}^\gamma(k_x,k_y) \), as:

\[
\mathbf{G}_{i,m,n,o}^\gamma = \frac{L_x L_y}{(2\pi)^2} \int_{-\pi/L_x}^{\pi/L_x} \int_{-\pi/L_y}^{\pi/L_y} \mathbf{g}_{i,m,n,o}^\gamma(k_x,k_y) \, dk_x \, dk_y, \quad \gamma = E \text{ or } H
\]

where \( \mathbf{g}_{i,m,n,o}^\gamma \) is the solution of the following equation:

\[
\frac{1}{\alpha_i} V_i \varepsilon_i (\varepsilon_i - 1) \mathbf{g}_{i,m,n,o}^\gamma - k_z^2 \sum_{j=1}^{N} V_j (\varepsilon_j - 1) \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \mathbf{G}_{i,j,m,n,o,p q}^{\gamma E} \cdot \mathbf{g}_{j,o}^{\gamma} = \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \mathbf{G}_{i,m,n,o,p q}^{\gamma E} \mathbf{e}^{i(pL_x k_x + qL_y k_y)},
\]

\[
\gamma = E \text{ or } H; \quad i = 1, 2, \ldots, N; \quad m, n = 0, \pm 1, \pm 2, \ldots
\]

Equation (15) describes a periodic problem because the periodic array of objects is illuminated by a periodic array of phase-shifted point sources represented by the summation on the right-hand side of this equation. Since the problem is periodic and due to the translational symmetry of the free space Green’s functions, the wave-vector dependent Green’s function for sub-volume \( jpq \), \( \mathbf{g}_{jpq,o}^\gamma \), in Eq. (15) should be periodic and phase-shifted relative to that for sub-volume \( j00 \) located in the unit cell, \( \mathbf{g}_{j,0}^\gamma \) [48]. As such, the wave-vector dependent Green’s function of replica sub-volumes is related to that of their counterpart in the unit cell as:

\[
\mathbf{g}_{jpq,o}^\gamma = \mathbf{g}_{j,0}^\gamma e^{i(pL_x k_x + qL_y k_y)}, \quad p, q = 0, \pm 1, \pm 2, \ldots
\]

It should be noted that the subscript 00 used for referring to the sub-volumes in the unit cell is dropped for simplicity. Equation (16) allows for solving the system of equations (15) only for the sub-volumes in the unit cell (i.e., for \( i = 1, 2, \ldots, N; \quad m = n = 0 \)). Substituting Eq. (16) into Eq. (15) and applying this equation to the sub-volumes in the unit cell results in:

\[
\frac{1}{\alpha_i} V_i \varepsilon_i (\varepsilon_i - 1) \mathbf{g}_{i,0}^\gamma - k_z^2 \sum_{j=1}^{N} V_j (\varepsilon_j - 1) \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \mathbf{G}_{i,j,0}^{\gamma E} e^{i(pL_x k_x + qL_y k_y)} \cdot \mathbf{g}_{j,0}^\gamma = \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \mathbf{G}_{i,0,p q}^{\gamma E} e^{i(pL_x k_x + qL_y k_y)},
\]
\[ \gamma = E \text{ or } H; \quad i = 1, 2, \ldots, N \] (17)

where \( \mathbf{g}^\gamma_{i,o} \) is taken out of the summations as it is independent of \( p \) and \( q \). The periodic free-space dyadic Green’s function between two points \( h \) and \( l \) is defined as [49]:

\[
G^0_{h,l} = \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} G^0_{h,l,pq} e^{i(p \lambda_k + q \lambda_k)}, \quad \gamma = E \text{ or } H
\] (18)

where subscript \( h \) indicates a sub-volume in the unit cell and \( l \) refers to either a sub-volume in the unit cell or the observation point. Using the definition in Eq. (18), the system of equation (17) can be re-written in the following form:

\[
\frac{1}{\alpha_i} V_i \varepsilon_0 (\varepsilon_i - 1) \mathbf{g}^\gamma_{i,o} - k_0^2 \sum_{j=1}^{N} V_j (\varepsilon_j - 1) \mathbf{G}^{\gamma,EP}_{i,j} \cdot \mathbf{g}^\gamma_{j,o} = \mathbf{G}^{\gamma,P}_{i,o}, \quad \gamma = E \text{ or } H; \quad i = 1, 2, \ldots, N
\] (19)

Eq. (19) is dyadic, and 3 systems of equations of size \( 3N \) are obtained when it is applied to the sub-volumes in the unit cell \( (i = 1, 2, \ldots, N) \). The solution of these systems of equations provides the wave-vector dependent Green’s functions for the sub-volumes located in the unit cell (i.e., \( \mathbf{g}^\gamma_{i,o} \) where \( i = 1, 2, \ldots, N \)). The wave-vector dependent Green’s function for replica sub-volumes \( (\mathbf{g}^\gamma_{iinm,o}) \) is only phase-shifted relative to the sub-volumes in the unit cell \( (\mathbf{g}^\gamma_{i,o}) \), and it can be obtained using Eq. (16). Once the phase-dependent Greens functions \( \mathbf{g}^\gamma_{iinm,o} \) are found, the array Green’s functions \( \mathbf{G}^\gamma_{iinm,o} \) can be calculated using Eq. (14). The energy density then can be found by using the discretized form of Eq. (5) and the array Green’s functions \( \mathbf{G}^\gamma_{iinm,o} \) as:

\[
u(\mathbf{r}, \omega) = \frac{2k_0^2 \varepsilon V}{\pi \omega} \Theta(\omega, T) \sum_{i=1}^{N} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \text{Trace} \left[ k_0^2 \mathbf{G}^E_{i,inm} \otimes \mathbf{G}^E_{i,inm} + \mathbf{G}^H_{i,inm} \otimes \mathbf{G}^H_{i,inm} \right]
\] (20)

The description of the periodic method for calculating energy density emitted by the periodic array is complete. The main steps in this technique can be summarized as follows.
1- The objects in the unit cell are discretized into $N$ cubical sub-volumes. The size of the sub-volumes should much smaller than the thermal wavelength, the object sizes and the separation distances, such that the electric field can be assumed uniform in the sub-volumes.

2- Equation (19) is applied to the $N$ sub-volumes in the unit cell, and 3 systems of $3N$ equations are obtained. The solution of these systems of equations provides the wave-vector dependent Green’s functions $g_{i,o}^r$ for the sub-volumes in the unit cell.

3- The wave-vector dependent Green’s function for replica sub-volumes $g_{inn,o}^r$ is calculated using Eq. (16).

4- The array Green’s function, $G_{inn,o}^r$, is found using the wave-vector dependent Green’s functions, $g_{inn,o}^r$, and Eq. (14).

5- The energy density is calculated using the array Green’s function, $G_{inn,o}^r$, and Eq. (19).

C. Periodic free-space dyadic Green’s functions

Calculating the periodic free-space dyadic Green’s function defined in Eq. (18) requires evaluating two infinite summations. For near-field thermal radiation problems, these summations converge significantly faster when the periodic free-space Green’s function is presented in the reciprocal lattice domain. The periodic free-space scalar Green’s function in the reciprocal domain is given by [49]:

$$G_{h,l}^{0,p} = \frac{i}{2L_x L_y} \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} e^{i [k_{z, pq} |h - l|]} e^{i K_{pq} p}$$  \hspace{1cm} (21)

where $K_{pq} = (k_x + 2\pi p/L_x) \hat{x} + (k_y + 2\pi q/L_y) \hat{y}$ is the sum of the wave vector associated with the array of phase-shifted point sources and the wave vector of reciprocal lattice, $\rho = (x_h - x_l) \hat{x} + (y_h - y_l) \hat{y}$ is the two-dimensional distance vector between points $h$ and $l$, and $k_{z, pq}$ is defined as:
The free-space electric dyadic Green’s function can be determined by applying operator
\[
\left[ I + \frac{1}{k_0^2} \nabla \nabla \right]
\] on the scalar free-space Green’s function given by Eq. (21) [41]. The result is:

\[
G_{h,l}^{0,E,P} = \frac{i}{2L_x L_y} \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \left( I - \frac{k_{pq} \otimes k_{pq}}{k_0^2} \right) \frac{e^{ik_{pq}|z_h - z_l|}}{k_{z,pq}} e^{ik_{pq} \rho}
\]  

(23)

where \( k_{pq} \) is a wave vector defined as:

\[
k_{pq} = \frac{|z_h - z_l|}{k_{z,pq}} \hat{z}
\]

(24)

The periodic free-space magnetic dyadic Green’s function in the reciprocal domain can be found using the scalar periodic free-space Green’s function as [41]:

\[
G_{h,l}^{0,H,P} = \frac{1}{ik_0} \nabla \times \left( G_{h,l}^{0,P} I \right)
\]

(25)

which can be written as:

\[
G_{h,l}^{0,H,P} = \frac{i}{2L_x L_y} \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \left( k_{pq} \times I \right) \frac{e^{ik_{pq}|z_h - z_l|}}{k_{z,pq}} e^{ik_{pq} \rho}
\]

(27)

V. NUMERICAL RESULTS

The periodic technique presented in Section IV is verified against the thermal discrete dipole approximation (T-DDA) [43,50] simulations. Since the T-DDA is a non-periodic approach, an effective length of the array needs to be determined and modeled. The effective length is determined by increasing the array size until no significant change in the T-DDA results is observed. The periodic approach has been tested for two arrays. The first array, as shown in Fig. 4a, is made of silica nanospheres of diameter 10 nm separated by a distance of 40 nm in the \( x \)- and \( y \)-directions (\( L_x = L_y = 50 \) nm). The array emits at 400 K. The spectral energy density is calculated
at two observation distances of 20 nm and 40 nm above the array along the perpendicular (to the array) axis of nanospheres. The spectral energy density as calculated using the T-DDA (non-periodic approach) and the periodic approach is shown in Fig. 4b. The periodic and non-periodic simulations are in excellent agreement for both observation distances. There are two resonances in the spectrum of energy density at $\omega_{res,1} = 9.21 \times 10^{13}$ rad/s and $\omega_{res,2} = 2.13 \times 10^{14}$ rad/s. These resonances are due to the thermal excitation of localized surface phonons (LSPhs) of the silica nanospheres. Thermal emission by the nanospheres is proportional to the imaginary part of their polarizability $\alpha$ which is given by $\text{Im}[\alpha] = 9\varepsilon_0 V \varepsilon''/[\varepsilon + 2]^2$. The LSPhs are excited at the frequencies for which the denominator of $\text{Im}[\alpha]$ vanishes, i.e., when $\varepsilon' \to -2$. This condition is satisfied at $\omega_{res,1}$ and $\omega_{res,2}$. The second array, which is shown in Fig. 4c, is made of silica nanoribbons of 50 nm height and 5 nm width. The nanoribbons are separated by 15 nm along the $x$- and $y$-directions such that $L_x = L_y = 20$ nm. The array emits at a temperature of 400 K. The spectral energy density is calculated at 15 nm and 30 nm above the array along the perpendicular (to the array) axis of the nanoribbons using the T-DDA and the periodic approach. The results obtained using the two methods are shown in Fig. 4d, and they are in excellent agreement. The energy density has four resonances due to the excitation of LSPhs of the silica nanoribbons. These resonances are located at $\omega_{res,1} = 8.72 \times 10^{13}$ rad/s, $\omega_{res,2} = 9.24 \times 10^{13}$ rad/s, $\omega_{res,3} = 2.03 \times 10^{14}$ rad/s and $\omega_{res,4} = 2.14 \times 10^{14}$ rad/s for the observation distance of 15 nm. The LSPh resonances for the observation distance of 40 nm are observed at almost the same frequencies. The LSPhs are excited when the denominator of $\text{Im}[\alpha]$ for nanoribbons vanishes. The polarizability of the nanoribbons can be estimated using that of a prolate spheroid with major and minor semi-axes equal to 25 nm and 2.5 nm, respectively. The imaginary part of the polarizability of a spheroid along the $j$ direction ($j = x, y, \text{and } z$) is given by $\text{Im}[\alpha_j] = \varepsilon_0 V \varepsilon''/[1 + L_j (\varepsilon - 1)]^2$, where $L_j$ is the geometrical factor of the spheroid [51]. As such, LSPhs are observed at the frequencies for where $\varepsilon' \to (L_j - 1)/L_j$. For the nanoribbons, $L_x = L_y = 0.4899$ and $L_z = 0.0203$. Thus, the LSPhs along the major and minor axes are excited when $\varepsilon' \to -48.3$ and $\varepsilon' \to -1.0$, respectively. The first condition is satisfied at $\omega_{res,1}$ and $\omega_{res,3}$, while the second one holds true for $\omega_{res,2}$ and $\omega_{res,4}$.
Near-field thermal emission by periodic arrays was modeled using a periodic technique which only requires discretizing one period of the array. This technique is based on the DDA and expressing a single point source in terms of a series of periodic arrays of phase-shifted point sources. The near-field energy density emitted by periodic arrays of silica nanospheres and nanoribbons was modeled using the presented technique and by direct numerical modeling using the T-DDA. An excellent agreement existed between the two techniques which demonstrates the validity of the periodic technique. The proposed technique is efficient and versatile, and it can be used for...
modeling a wide variety of arrays comprised of complex-shape and inhomogeneous emitters with non-uniform temperature distributions.

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APPENDIX: PERIODIC EXPANSION OF THE DIRAC DELTA FUNCTION

In this appendix, it is proved that the Dirac delta function can be expressed as a series of periodic arrays of phase-shifted delta functions with arbitrary periods $L_x$ and $L_y$. Mathematically, this is written as:

$$\delta(r - r_o) = \frac{L_x L_y}{(2\pi)^2} \int_{-\pi/L_x}^{\pi/L_x} \int_{-\pi/L_y}^{\pi/L_y} \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \delta\left(r - r_o + pL_x x + qL_y y\right) e^{i(pL_x k_x + qL_y k_y)} dk_x dk_y$$

(A.1)

Using the commutative property of integral and summation, the right-hand side of Eq. (A.1) can be written as:

$$\frac{L_x L_y}{(2\pi)^2} \int_{-\pi/L_x}^{\pi/L_x} \int_{-\pi/L_y}^{\pi/L_y} \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \delta\left(r - r_o + pL_x x + qL_y y\right) e^{i(pL_x k_x + qL_y k_y)} dk_x dk_y$$

$$= \frac{L_x L_y}{(2\pi)^2} \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \delta\left(r - r_o + pL_x x + qL_y y\right) e^{i(pL_x k_x + qL_y k_y)} dk_x dk_y$$

(A.2)

where $\int_{-\pi/L_y}^{\pi/L_y} e^{iql_y k_y} dk_y$ is zero unless $q = 0$. For $q = 0$, this integral equals $\frac{2\pi}{L_y}$. As such, Eq. (A.2) can be re-written as:

$$\frac{L_x L_y}{(2\pi)^2} \int_{-\pi/L_x}^{\pi/L_x} \int_{-\pi/L_y}^{\pi/L_y} \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \delta\left(r - r_o + pL_x x + qL_y y\right) e^{i(pL_x k_x + qL_y k_y)} dk_x dk_y$$

$$= \frac{L_x}{2\pi} \sum_{p=-\infty}^{\infty} \delta\left(r - r_o + pL_x x\right) \int_{-\pi/L_y}^{\pi/L_y} e^{iql_y k_y} dk_y$$

(A.3)
In the same manner, \( \int_{-\pi/L_x}^{\pi/L_x} e^{ipL_x k_z} dk_z \) is only non-zero when \( p = 0 \) in which case the integral equals \( \frac{2\pi}{L_x} \). As such, the right-hand side of Eq. (A.3) reduces to \( \delta(r - r_s) \).

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