Contextual logic for quantum systems

Graciela Domenech $^1$ and Hector Freytes $^2$

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1. Instituto de Astronomía y Física del Espacio (IAFE)
Casilla de Correo 67, Sucursal 28, 1428 Buenos Aires, Argentina
2. Escuela de Filosofía - Universidad Nacional de Rosario,
Entre Ríos 758, 2000, Rosario, Argentina

Abstract

In this work we build a quantum logic that allows us to refer to physical magnitudes pertaining to different contexts from a fixed one without the contradictions with quantum mechanics expressed in no-go theorems. This logic arises from considering a sheaf over a topological space associated to the Boolean sublattices of the ortholattice of closed subspaces of the Hilbert space of the physical system. Differently to standard quantum logics, the contextual logic maintains a distributive lattice structure and a good definition of implication as a residue of the conjunction.

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$^1$Fellow of the Consejo Nacional de Investigaciones Científicas y Técnicas (CONICET)
1 Introduction

Quantum mechanics has profound conceptual difficulties that may be posed in several ways. Nonetheless almost every problem in the relation between the mathematical formalism and what may be called “our experience about the behavior of physical objects” can be encoded in the question about the possible meaning of the proposition “the physical magnitude $A$ has a value and the value is this or that real number”. Already from the first formalizations this point was recognized. For example, P.A.M. Dirac stated in his famous book: “The expression that an observable ‘has a particular value’ for a particular state is permissible in quantum mechanics in the special case when a measurement of the observable is certain to lead to the particular value, so that the state is an eigenstate of the observable. It may easily be verified from the algebra that, with this restricted meaning for an observable ‘having a value’, if two observables have values for a particular state, then for this state the sum of the two observables (if the sum is an observable) has a value equal to the sum of the values of the two observables separately and the product of the two observables (if this product is an observable) has a value equal to the product of the values of the two observables separately.” [11]. This last point is the requirement of the functional compatibility condition (FUNC), to which we will return later. As long as we limit ourselves to speak about measuring results and avoid being concerned with what happens to nature when she is not measured, quantum mechanics carries out predictions with great accuracy. But if we naively try to interpret eigenvalues as the possible or actual values of the physical properties of a system, we are faced to all kind of no-go theorems that preclude this possibility. Most remarkably is the Kochen-Specker (KS) theorem that rules out the non-contextual assignment of values to physical magnitudes [17]. Of course, to restrict the valuation to a subset of observables -typically to a complete set of commuting observables (CSCO) which constitutes a context- and refer to values of physical variables only in the sense allowed by the mathematical formalism, ensures no contradiction. So, a widely accepted position is to abandon seeking to describe what nature at the quantum level is and use the theory as a mere instrument of prediction. But there are also different proposals to investigate how to assign objective and measurable properties to a physical entity, i.e. how far we can refer to physical objects without contradiction with quantum theory. This paper is framed in that search.

Our proposal is to construct a logic to enable not only a Boolean valuation in each fixed context but also that, once chosen certain set of projectors
of the spectral decomposition of the operators (correspondingly, closed subspaces of Hilbert space $H$) that admits a global Boolean valuation, to make it possible to refer at least partially to projectors pertaining to other contexts with the least arbitrariness.

Let us be here concerned with the simplest cases: pure states of the system are represented by normalized vectors of $H$ and dynamical variables $A$ by bounded self-adjoint operators $A$ with discrete spectra. The possible results of the measurement of a (sharp) magnitude $A$ are the eigenvalues $a_i$ pertaining to the spectrum $\sigma(A)$ of its associated operator $A$. To each of the eigenvalues $a_i$ corresponds a projector $P_i$ and correspondingly a closed subspace of $H$. Every $A$ admits a spectral decomposition

$$A = \sum_i a_i P_i$$

where the equality is considered as a convergence in norm. So observables may be discomposed to give an exhaustive and exclusive partition of the possible alternatives for the results of measurements. The probability to obtain one of them in an experimental procedure is given by the Born rule.

Now let us suppose the state of the physical system is an eigenvector of a non degenerate observable $A$ (i.e. $A$ constitutes a CSCO) so we know the eigenvalues of all projectors $P_1, P_2, ..., P_n, ...$ of $A$ for the system in this state. If any $P_i$ lies in the spectral decomposition of another observable $B$, then this “part” of $B$ can be valued in a Boolean way. It is important to realize that this allows to refer to observables pertaining to a CSCO from another CSCO. In categorical terms, this will be related to the possible local sections of a sheaf satisfying a certain kind of compatibility with respect to fixed contexts, to be exactly stated in what follows. From this formal analysis in terms of sheaves, we intend to build the mentioned logic, which we will call contextual logic. This contextual logic will allow us to formalize to what extent we can consider as objective properties of a physical system those represented by projectors pertaining to different contexts without facing no-go theorems. We will use a categorial frame to develop this logic as has been the case during the last years, when applications of category theory tools to logical questions in standard quantum mechanics have begun to appear (for example Isham and Butterfield [15], [16], [13], [7], also in consistent histories approach [14], in the interpretation of the Sasaki hook as an adjunction [9] and, in general, in the Geneve-Brussels approach [1], [2] [8]).

In section 2 we introduce basic notions about lattice theory and topics in categories. We devote section 3 to discuss the problem of the valuations
of physical magnitudes pertaining to different CSCCs. In section 4 we face
the same problem from the point of view of sheaves relating it with the dual
spectral presheaf introduced in [15]. In section 5 we develop the contextual
logic in a Kripke style and intuitionistic way. Finally, in section 6 we outline
our conclusions.

2 Basic Notions

We recall from [3, 6] and [12, 20] some notions of the lattice theory and
categories that will play an important role in what follows.

First, let \((A, \leq)\) be a poset and \(X \subseteq A\). \(X\) is decreasing set iff for all
\(x \in X\), if \(a \leq x\) then \(a \in X\). For each \(a \in A\) we define the principal
decreasing set associated to \(a\) as \((a] = \{x \in A : x \leq a\}\). The set of all
decreasing sets in \(A\) is denoted by \(A^+\), and it is well known that \((A^+, \subseteq)\)
is a complete lattice, thus \(\langle A, A^+ \rangle\) is a topological space. We observe that if
\(G \in A^+\) and \(a \in G\) then \((a] \subseteq G\). Therefore \(B = \{[a] : a \in A\}\) is a base
of the topology \(A^+\) which we will refer to as the canonical base. If \(X \subseteq A\),
we denote by \(\partial X\) the border of \(X\), \(C(X)\) the complement of \(X\) and \(X^o\) the
interior of \(X\).

Let \(A\) be a category. We denote by \(\text{Ob}(A)\) the class of objects and by
\(\text{Ar}(A)\) the class of arrows. Given an arrow \(f : a \rightarrow b\), \(a\) is called domain
of \(f\) \((a = \text{dom}(f))\) and \(b\) is called codomain of \(f\) \((\text{cod}(f))\). We denote by
\([a, b]_A\) the class of all arrows \(a \rightarrow b\) in the category and by \(1_A\) the identity
arrow over the object \(A\). \(A\) is said to be small category iff \(\text{Ob}(A)\) is a set.
A partially ordered set \((P, \leq)\) gives rise to a category with the elements of
\(P\) as objects, and with precisely one arrow \(p \rightarrow q\) iff \(p \leq q\). In this case,
\((P, \leq)\) is a small category. The category whose objects are sets and arrows
are functions with the usual composition is denoted by \(\text{Ens}\).

Let \(I\) be a topological space. A sheaf over \(I\) is a pair \((A, p)\) where \(A\) is
a topological space and \(p : A \rightarrow I\) is a local homeomorphism. This means
that each \(a \in A\) has an open set \(G_a\) in \(A\) that is mapped homeomorphically
by \(p\) onto \(p(G_a) = \{p(x) : x \in G_a\}\), and the latter is open in \(I\). It is clear
that \(p\) is continuous and open map. If \(p : A \rightarrow I\) is a sheaf over \(I\), for each
\(i \in I\), the set \(A_i = \{x \in A : p(x) = i\}\) is called the fiber over \(i\). Each fiber
has the discrete topology as subspace of \(A\). Local sections of the sheaf \(p\) are
continuous maps \(\nu : U \rightarrow A\) defined over open proper subsets \(U\) of \(I\) such
that the following diagram is commutative:
In particular we use the term *global section* only when \( U = I \).

Given the category \( A \), one can form a new category \( A^{\text{op}} \), called *dual* category of \( A \), by taking the same objects but reversing the directions of all the arrows and the order of all compositions. \( \text{Ens}^{A^{\text{op}}} \) or \( \hat{A} \) where \( A \) is a small category is the category whose objects are functors \( F : A^{\text{op}} \to \text{Ens} \) (also called *presheaves*) and whose arrows are natural transformations between presheaves. \( \hat{A} \) is a topos, i.e. has terminal object, pullbacks, exponentiation and subobject classifier. The terminal object in \( \hat{A} \) is the functor \( 1 : A^{\text{op}} \to \text{Ens} \) such that \( 1(A) = \{ \ast \} \) (the singleton) for each \( A \in A \) and for each arrow \( f \), \( 1(f) = 1_{\{\ast\}} \). For any presheaf \( F : A^{\text{op}} \to \text{Ens} \), the unique arrow \( F \to 1 \) is the natural transformation whose components are the unique functions \( F(A) \to \{\ast\} \) for each object \( A \) in \( A \). Pullbacks, limits and colimits are defined componentwise.

A *local section* of a presheaf \( F : A^{\text{op}} \to \text{Ens} \) is a natural transformation \( \tau : U \to F \) such that \( U \) is a subfunctor of the presheaf \( 1 \). We only refer to *global sections* in case that \( U = 1 \).

### 3 The question of valuation

Let \( H \) be the Hilbert space associated to the physical system and \( L(H) \) be the set of closed subspaces on \( H \). If we consider the set of these subspaces ordered by inclusion, then \( L(H) \) is a complete orthomodular lattice \([19]\). It is well known that each self-adjoint operator \( A \) has associated a Boolean sublattice \( W_A \) of \( L(H) \). More precisely, \( W_A \) is the Boolean algebra of projectors \( P_i \) of the spectral decomposition \( A = \sum_i a_i P_i \). We will refer to \( W_A \) as the spectral algebra of the operator \( A \). Any proposition about the system is represented by an element of \( L(H) \) which is the algebra of quantum logic introduced by G. Birkhoff and J. von Neumann \([5]\).

Assigning values to a physical quantity \( A \) is equivalent to establishing a Boolean homomorphism \( v : W_A \to 2 \) \([15]\), being \( 2 \) the two elements Boolean algebra. So it is natural to consider the following definition:

**Definition 3.1** Let \((W_i)_{i \in I}\) be the family of Boolean sublattices of \( L(H) \).
A global valuation over $L(\mathcal{H})$ is a family of Boolean homomorphisms $(v_i : W_i \to 2)_{i \in I}$ such that $v_i | W_i \cap W_j = v_j | W_i \cap W_j$ for each $i, j \in I$.

This global valuation would give the values of all magnitudes at the same time maintaining a compatibility condition in the sense that whenever two magnitudes shear one or more projectors, the values assigned to those projectors are the same from every context.

But KS theorem assures that we cannot assign real numbers pertaining to their spectra to operators $A$ in such a way to satisfy the functional composition principle (FUNC) which is the expression of the “natural” requirement mentioned by Dirac that, for any operator $A$ representing a dynamical variable and any real-valued function $f(A)$, the value of $f(A)$ is the corresponding function of the value of $A$. This is a very restrictive constrain because it does not allow to assign values to all possible physical quantities or to assign true-false as truth values to all propositions about the system, nor even non-contextual partial ones. KS theorem means that, if we demand a valuation to satisfy FUNC, then it is forbidden to define it in a non-contextual fashion for subsets of quantities represented by commuting operators. In the algebraic terms of the previous definition, KS theorem reads:

**Theorem 3.2** If $\mathcal{H}$ is a Hilbert space such that $\dim(\mathcal{H}) > 2$, then a global valuation over $L(\mathcal{H})$ is not possible. \qed

Of course contextual valuations allow us to refer to different sets of actual properties of the system which define its state in each case. Algebraically, a contextual valuation is a Boolean valuation over one chosen spectral algebra. In classical particle mechanics it is possible to define a Boolean valuation of all propositions, that is to say, it is possible to give a value to all the properties in such a way of satisfying FUNC. This possibility is lost in the quantum case. And it is not a matter of interpretation, it is the underlying mathematical structure that enables this possibility for classical mechanics and forbids it in the quantum case. The impossibility to assign values to the properties at the same time satisfying FUNC is a weighty obstacle for almost any interpretation of the formalism as something more than a mere instrument.
4 Spectral sheaf

Being \( L(\mathcal{H}) \) the lattice of closed subspaces of the Hilbert space \( \mathcal{H} \), we consider the family \( W \) of all Boolean subalgebras of the lattice \( L(\mathcal{H}) \) ordered by inclusion and the topological space \( \langle W, W^+ \rangle \). On the set

\[ E = \{(W, f) : W \in W, f : W \to 2 \text{ is a Boolean homomorphism}\} \]

we define a partial ordering given as

\[ (W_1, f_1) \leq (W_2, f_2) \iff W_1 \subseteq W_2 \text{ and } f_1 = f_2 \mid W_1 \]

Thus we consider the topological space \( \langle E, E^+ \rangle \) whose canonical base is given by the principal decreasing sets \( ((W, f]) = \{(G, f \mid G) : G \subseteq W\} \). By simplicity \( ((W, f]) \) is noted as \( (W, f) \).

**Proposition 4.1** The map \( p : E \to W \) such that \( (W, f) \mapsto W \) is a sheaf over \( W \).

**Proof:** Let \( (W, f) \in E \). If we consider the open set \( (W, f] \) in \( E \), then \( p((W, f]) = (W] \) resulting \( p((W, f]) \) an open set in \( W \). If we denote by \( p' \) the restriction \( p \mid (W, f] \), then from the definition of \( p \) it is clear that \( p' : (W, f] \to (W] \) is a bijective map that preserve order inclusion. Thus \( p' \) is a continuous map. Finally \( p \) is a local homeomorphism. \( \square \)

We refer to the sheaf \( p : E \to W \) as the **spectral sheaf**.

**Proposition 4.2** Let \( \nu : U \to E \) be a local section of the spectral sheaf \( p \). Then for each \( W \in U \) we have:

1. \( \nu(W) = (W, f) \) for some Boolean homomorphism \( f : W \to 2 \),

2. if \( W_0 \subseteq W \), then \( \nu(W_0) = (W_0, f \mid W_0) \).

**Proof:** Since \( \nu \) is a local section we consider the following commutative diagram:

\[
\begin{array}{ccc}
U & \xrightarrow{\nu} & E \\
\downarrow{1_U} & \equiv & \downarrow{p} \\
U & & \\
\end{array}
\]
1) It follows as an immediate consequence of the commutativity of the diagram.
2) Since $\nu$ is continuous, $\nu^{-1}(\{W, f\})$ is an open set in $\mathcal{W}$ (i.e. a decreasing set). Consequently $W_0 \in \nu^{-1}(\{W, f\})$ since $W_0 \subseteq W$ and $W \in \nu^{-1}(\{W, f\})$. Thus $\nu(W_0) \in (W, f]$ resulting $\nu(W_0) = (W_0, f \mid W_0)$. 

From the physical perspective, we may say that the spectral sheaf takes into account the whole set of possible ways of assigning truth values to the propositions associated with the projectors of the spectral decomposition $A = \sum_i a_i P_i$. The continuity of a local section of $p$ guarantees that the truth value of a proposition is maintained when considering the inclusion of subalgebras. In this way, the compatibility condition of the Boolean valuation with respect of intersection of pairs of Boolean sublattices of $L(\mathcal{H})$ is maintained.

A global section $\tau : \mathcal{W} \to E$ of $p$ is interpreted as follows: the map assigns to every $W \in \mathcal{W}$ a fixed Boolean valuation $\tau_w : W \to 2$ obviously satisfying the compatibility condition. So KS theorem in terms of the spectral sheaf reads:

**Theorem 4.3** If $\mathcal{H}$ is a Hilbert space such that $\dim(\mathcal{H}) > 2$ then the spectral sheaf $p$ has no global sections. 

We may build a contextual valuation in terms of a local section as follows:

Let $A$ be a physical magnitude with known value, i.e. we have been able to establish a Boolean valuation $f : W_A \to 2$. It is not very hard to see that the assignment

$$\nu : (W) \to E \text{ such that for each } W_i \in (W), \nu(W_i) = (W_i, f \mid W_i)$$

is a local section of $p$.

To extend contextual valuations we turn now to consider local sections. To do this we introduce the following definition:

**Definition 4.4** Let $\nu$ be a local section of $p$ and $W_A$ the spectral algebra associated to the operator $A$. Then an extended valuation over $A$ is given by the set
\[ \tilde{\nu}(A) = \{ W_B \in \text{dom}(\nu) : W_B \subseteq W_A \} \]

Given the previous definition, it is easy to prove the following proposition:

**Proposition 4.5** If \( \nu \) is a local section of \( p \) and \( W_A \) the spectral algebra associated to the operator \( A \), then:

1. \( \tilde{\nu}(A) \) is a decreasing set,
2. if \( W_A \in U \) then \( \tilde{\nu}(A) = (W_A) \)

We can start from the spectral sheaf to build a representation as presheaf such that local sections of the former are identifiable to local sections of the later. When considering the family \( \mathcal{W} \) ordered by inclusion, \( \mathcal{W} \) can be regarded as a small category. Thus we can take the topos presheaf \( \mathcal{W} \). Denoting \( E_W \) the fiber of the spectral sheaf \( p \) over \( W \) for each \( W \in \mathcal{W} \), we consider the following presheaf:

\[ D : \mathcal{W}^{op} \to \text{Ens} \]

such that

- \( D(W) = E_W \) for each \( W \in \text{Ob}(\mathcal{W}) \).
- if \( i : W_1 \subseteq W_2 \) lies in \( \text{Ar}(\mathcal{W}) \), then \( D_i : D(W_2) \to D(W_1) \) is such that \( D_i(g) = g |_{W_1} \).

It is clear that the presheaf acting over arrows satisfies the compatibility condition. Denoting \( \text{Sec}_p \) and \( \text{Sec}_D \) the sets of (global and local) sections of \( p \) and \( D \) respectively, we can establish the following proposition:

**Proposition 4.6**

\[ \text{Sec}_p \simeq \text{Sec}_D \]

**Proof:** Let \( \nu : U \to E \in \text{Sec}_p \) and consider the presheaf \( \tilde{U} : \mathcal{W}^{op} \to \text{Ens} \) whose action over \( \text{Ob}(\mathcal{W}) \) is given by

\[ \tilde{U}(W) = \begin{cases} \{ * \}, & \text{if } W \in U \\ \emptyset, & \text{otherwise} \end{cases} \]

and whose action over arrows is immediate. It is clear that \( \tilde{U} \) is a subfunctor of the presheaf \( 1 \). From \( U \) we construct a natural transformation
such that, for each \( W \in \mathcal{W} \), \( \tau_\nu(U(W)) = \nu(W) \). Thus we have a map \( \text{Sec}_p \to \text{Sec}_D \) given by \( \nu \mapsto \tau_\nu \). It is not hard to see that this is an injective map. To see surjectivity, we consider a section \( \nu \) of the spectral sheaf such that \( \nu \mapsto \tau_\nu = \tau \). Let \( U = \{ W \in \mathcal{W} : \tilde{U}(W) = \{ \ast \} \} \). If \( W \in U \) and \( W_0 \subseteq W \), then \( \tilde{U}(W_0) = \{ \ast \} \) since \( \tilde{U} \) is a contravariant subfunctor of \( 1 \). Thus, \( W_0 \) lies in \( U \), resulting \( U \) a decreasing set (i.e. an open set) in \( \mathcal{W} \).

Now we consider the map \( \nu : U \to E \) such that, for each \( W \in U \), \( \nu(W) = (W, \tau(\tilde{U}(W))) \). It is clear that the following diagram is commutative

\[
\begin{array}{ccc}
U & \xrightarrow{\nu} & E \\
\downarrow{1_U} & \equiv & \downarrow{p} \\
U & \end{array}
\]

Now we prove that \( \nu \) is a continuous map. Let \( (W_1, f) \) be an open set of the canonical base of \( E \), \( \nu^{-1}((W_1, f)) = \{ W \in W : \nu(W) \in (W_1, f) \} \) and we assume that \( \nu^{-1}((W_1, f)) \) is not the empty set. Let \( W \in \nu^{-1}((W_1, f)) \) and \( W_\ast \subseteq W \). Since \( \tau \) is a natural transformation, it follows that \( \tau(\tilde{U}(W_\ast)) = \tau(\tilde{U}(W))|_{W_\ast} \). Since \( \nu(W) \in (W_1, f) \), it is clear that \( \nu(W) = (W, f|W) \), resulting that \( \tilde{U}(W_\ast) = f|W_\ast \) and \( W_\ast \in \nu^{-1}((W_1, f)) \). This proves the continuity of the map. It is not very hard to see that \( \tau = \tau_\nu \), thus surjectivity is proved.

\[\blacksquare\]

**Remark 4.7** The presheaf \( D \) from the spectral sheaf is the dual spectral presheaf defined in Ref. [15]

Taking into account the last Proposition, we can write KS theorem in terms of presheaves from the spectral sheaf:

**Theorem 4.8** If \( \mathcal{H} \) is a Hilbert space such that \( \dim(\mathcal{H}) > 2 \) then the dual spectral presheaf \( D \) has no global sections.

\[\blacksquare\]
Possible obstructions to the construction of global sections for the case of finite dimensional $\mathcal{H}$ are shown in [18].

On the other hand, in terms of a local section $\nu : U \to D$ of $D$, extended contextual valuations over an operator $A$ may be defined as

$$\bar{\nu}(A) = \{ W_B \subseteq W_A : U(W_B) = \{\ast\} \}$$

Valuations are deeply connected to the election of particular local sections of the spectral sheaf. So we see here once more that we cannot speak of the value of a physical magnitude without specifying this election, that clearly means the election of a particular context. This is in agreement with the statement that contextuality is “endemic” in any attempt to ascribe properties to quantities in quantum theories [15].

5 Contextual logic

We know that if $\mathcal{W}$ is the family of Boolean subalgebras of $L(\mathcal{H})$, to take a local section $\nu$ of the spectral sheaf means an assignment of Boolean valuations to algebras in the proper sub-family $Dom(\nu)$ maintaining the compatibility condition. Now an interesting question is to ask what $\nu$ can “tell us” about $W$ when $W \notin Dom(\nu)$. Let us state more accurately this expression to precise our aim in the search of a contextual logic.

**Definition 5.1** Let $\nu$ be a local section of the spectral sheaf. If $W_B \in Dom(\nu)$ and $W_B \subseteq W_A$ then we will say that $W_B$ has Boolean information about $W_A$.

Clearly this means that, in a given state of the system, the complete knowledge of the spectral decomposition of $\mathcal{B}$ lets us know the eigenvalue of one or more projectors in the spectral decomposition of $\mathcal{A}$. Contextual logic allows some kind of “paste” among Boolean sublattices of $L(\mathcal{H})$ and so among of CSCOs. A valuation in terms of decreasing sets maintains it “downstream” with respect of subalgebras, i.e. when the valuation of a sub-algebra is given, all its subalgebras are automatically valued. This makes possible to have Boolean information of different contexts from the one chosen in the following sense: once fixed a local section $\nu$, if $W_B \in Dom(\nu)$
and \( W_A \not\in \text{Dom}(\nu) \) then, to witness \( W_A \) from \( W_B \) refers to the Boolean information that \( W_B \cap W_A \) has about \( W_A \).

We will now construct a propositional language \( \text{Self} \) for contextual logic whose atomic formulas refer to the physical magnitudes represented for bounded self-adjoint operators with discrete spectra. Intuitively we can consider the set of atomic formulas \( \mathcal{P} \) as

\[
\mathcal{P} = \{ A : A \text{ bounded self-adjoint operator} \}
\]

Then, this language is conformed as follows:

\[
\text{Self} = \langle \mathcal{P}, \lor, \land, \to, \neg \rangle
\]

and it is clear that the formulas may be obtained in the usual way.

We will now appeal to the use of Kripke models built starting from any local section of the spectral sheaf \( p \) because it allows to naturally adapt the idea of Boolean knowledge. Thus the obtained valuation will result in an extended contextual valuation.

**Definition 5.2** We consider the poset \( \langle \mathcal{W}, \subseteq \rangle \) as a frame for the Kripke model for \( \text{Self} \). Let \( \nu \) be a local section of \( p \). Thus we define the Kripke model \( \mathcal{M} = \langle \mathcal{W}, \bar{\nu} \rangle \) with the following forcing:

1. \( \mathcal{M} \models_w A \iff W \in \bar{\nu}(A) \) with \( A \in \mathcal{P} \)
2. \( \mathcal{M} \models_w \alpha \lor \beta \iff \mathcal{M} \models_w \alpha \text{ or } \mathcal{M} \models_w \beta \)
3. \( \mathcal{M} \models_w \alpha \land \beta \iff \mathcal{M} \models_w \alpha \text{ and } \mathcal{M} \models_w \beta \)
4. \( \mathcal{M} \models_w \alpha \to \beta \iff \forall B \subseteq W, \text{ if } \mathcal{M} \models_B \alpha \text{ then } \mathcal{M} \models_B \beta \)
5. \( \mathcal{M} \models_w \neg \alpha \iff \forall B \subseteq W \mathcal{M} \models_B \neg \alpha \)

Given this forcing we can accurately define the idea of extended contextual valuation over \( \text{Self} \).

**Definition 5.3** Given a local section \( \nu \) over \( p \), an extended contextual valuation is the map \( \bar{\nu} : \text{Self} \to \mathcal{W}^+ \) defined as

\[
\bar{\nu}(\alpha) = \{ W : \mathcal{M} \models_w \alpha \} 
\]
Taking into account that $W^+$ is a topological space it is not very hard to see that $\bar{\nu}(\alpha)$ is an open set of $W$. Now we can to establish the following proposition:

**Proposition 5.4** Let $\alpha$ be a formula in $Self$ and consider the Kripke model $M = \langle W, \nu \rangle$. Then:

1. $M \models_W \neg \alpha$ iff $W \in (C\bar{\nu}(\alpha))^\circ$
2. $M \not\models_W \alpha$ and $M \not\models_W \neg \alpha$ iff $W \in \partial\bar{\nu}(\alpha)$

**Proof:**

1) If $M \models_W \neg \alpha$, then $\forall B \subseteq W$, $M \not\models_B \alpha$ and $\forall B \subseteq W$ $B \notin \bar{\nu}(\alpha)$. Thus $(W) \subseteq (C\bar{\nu}(\alpha))^\circ$ and $W \in (C\bar{\nu}(\alpha))^\circ$. On the other hand, if $W \in (C\bar{\nu}(\alpha))^\circ$, then there exists an open set $G$ in $W$ such that $W \in G \subseteq C(\bar{\nu}(\alpha))$. Since $G$ is a decreasing set, we have that $(W) \subseteq G \subseteq C(\bar{\nu}(\alpha))$ and $M \models_W \neg \alpha$. 2) It follows from 1) and the fact that $W = \bar{\nu}(\alpha) \cup (C\bar{\nu}(\alpha))^\circ \cup \partial\bar{\nu}(\alpha)$.

**Remark 5.5** Following the usual interpretation of the Kripke model, the frame represents all possible states of knowledge that are preserved forward in time. In our case, the frame $\langle W, \subseteq \rangle$, $W$ represents all states of Boolean knowledge in the sense of all possible Boolean valuations of spectral algebras and the usual notion of “preserving knowledge through time” must be understood in terms of $\subseteq$ as “preserving valuations in spectral subalgebras”.

The forcing $K \models_W \alpha$ is interpreted as the spectral algebra $W$ has Boolean knowledge about $\alpha$ i.e. the complete Boolean valuation of $W$ is known and $W$ lies in the decreasing set associated to the formula $\alpha$. By Proposition 4.5, to know the eigenvalue of $A$ is expressed in terms of the forcing as $M \models_{\{W_A\}} A$.

Being $W^+$ a topological space, it is a Heyting algebra with meet and join operations the classical ones and implication and negation defined as follows:

$$ S \rightarrow T = \{ P \in W : \forall X \subseteq P, \text{ if } X \in S \text{ then } X \in T \} $$

$$ \neg S = \{ P \in W : \forall X \subseteq P, \ X \notin S \} $$

Thus the extended contextual valuation is a Heyting valuation of $Self$ from the Heyting algebra $W^+$ such that

1. $\bar{\nu}(\alpha \lor \beta) = \bar{\nu}(\alpha) \cup \bar{\nu}(\beta)$
2. $\bar{\nu}(\alpha \land \beta) = \bar{\nu}(\alpha) \cap \bar{\nu}(\beta)$
3. $\bar{\nu}(\alpha \rightarrow \beta) = \bar{\nu}(\alpha) \rightarrow \bar{\nu}(\beta)$
4. $\bar{\nu}(\neg \alpha) = \neg \bar{\nu}(\alpha)$

Taking into account the restrictions in the valuations imposed by the KS theorem, it is not possible a Heyting valuation $v : Self \rightarrow W^+$ such that $v(A) = [A]$ for each atomic formula $A$. So it is clear that contextual logic is an intuitionistic logic in which not all of the Heyting valuations are allowed.

6 Conclusions

Contextual logic is a formal language to deal with combinations of propositions about physical properties of a quantum system that are well defined in different contexts. These properties are regarded from a fixed context, which guarantees the avoidance of no-go theorems. This means that one can refer to contexts other than the chosen one by building a Kripke model in which each proposition is given a decreasing set as its truth value.

There are different formal languages on the orthomodular lattice of closed subspaces of $\mathcal{H}$ (as orthologic or orthomodular quantum logic), but these logics give rise to different problems that lack an intuitive understanding, as the “implication problem” (briefly, eight different connectives may represent the material conditional, see [10]). On the contrary, as contextual logic is an intuitionistic one – with restrictions on the allowed valuations arising from the KS theorem- it has “good” properties as the distributive lattice structure and a nice definition of the implication as a residue of the conjunction. The price paid is being a contextual language. But this is not a difficulty, it is a main feature of quantum mechanics.

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Hector Freytes:
Escuela de Filosofía
Universidad Nacional de Rosario,
Entre Ríos 758, 2000, Rosario, Argentina
e-mail: hfreytes@dm.uba.ar

Graciela Domenech:
Instituto de Astronomía y Física del Espacio (IAFE)
Ciudad Universitaria
1428 Buenos Aires - Argentina
e-mail: domenech@iafe.uba.ar