Complete Semiclassical Spectral Asymptotics for Periodic and Almost Periodic Perturbations of Constant Operators

Conference on Partial Differential Equations and Applications in Memory of Professor B. Yu. Sternin, November 6–9, 2018, Moscow, Russia

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4. **References**
This work is inspired by several remarkable papers of L. Parnovski and R. Shterenberg [PS1, PS2, PS3], S. Morozov, L. Parnovski and R. Shterenberg [MPS] and earlier papers by A. Sobolev [So1, So2]. I wanted to understand the approach of the authors and, combining their ideas with my own approach, generalize their results.
Introduction

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In these papers the complete asymptotic expansion of the integrated density of states $N(\lambda)$ for operators $\Delta + V$ was derived as $\lambda \to +\infty$; here $\Delta$ is a positive Laplacian and $V$ is a periodic or almost periodic potential (satisfying certain conditions). In [MPS] more general operators were considered.
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Further, in [PS3] the complete asymptotic expansion of $e(x, x, \lambda)$ was derived, where $e(x, y, \lambda)$ is the Schwartz kernel of the spectral projector.
I borrowed from these papers Conditions (A)–(D) and the *special gauge transformation* and added the *non-stationary semiclassical Schrödinger operator method* [Ivr1] and extremely long propagation of singularities. I believe that this is a simpler and more powerful approach. Also, in contrast to those papers I consider more general semiclassical asymptotics.
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Consider a scalar self-adjoint $h$-pseudo-differential operator $A(x, hD)$ in $\mathbb{R}^d$ with the Weyl symbol $A(x, \xi)$, such that

\[ |D_x^\alpha D_\xi^\beta A(x, \xi)| \leq c_{\alpha\beta}(|\xi| + 1)^m \quad \forall \alpha, \beta, \forall x, \xi \quad (1) \]

and

\[ A(x, \xi) \geq c_0|\xi|^m - C_0 \quad \forall x, \xi. \quad (2) \]
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Then it is semibounded from below. Let $e_h(x, y, \lambda)$ be the Schwartz kernel of its spectral projector $E(\lambda) = \theta(\lambda - A)$.
We are interested in the semiclassical asymptotics of $e_h(x, x, \lambda)$ and

$$N_h(\lambda) = M[e(x, x, \lambda)] := \lim_{\ell \to \infty} (\text{mes}(\ell X))^{-1} \int_{\ell X} e(x, x, \lambda) \, dx,$$  \hspace{1cm} (3)

where $0 \in X$ is an open domain in $\mathbb{R}^d$. The latter expression in the cases we are interested in does not depend on $X$ and is called *Integrated Density of States*. 
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where $0 \in X$ is an open domain in $\mathbb{R}^d$. The latter expression in the cases we are interested in does not depend on $X$ and is called Integrated Density of States.

It is well-known that under $\xi$-microhyperbolicity condition on the energy level $\lambda$

$$|A(x, \xi, h) - \lambda| + |\nabla_\xi A(x, \xi, h)| \geq \epsilon_0$$  \hspace{1cm} (4)

the following asymptotics holds

$$e_h(x, x, \lambda) = \kappa_0(x, \lambda) h^{-d} + O(h^{1-d}) \quad \text{as} \quad h \to +0,$$  \hspace{1cm} (5)

and therefore

$$N_h(\lambda) = \bar{\kappa}_0(\lambda) h^{-d} + O(h^{1-d}),$$  \hspace{1cm} (6)

where here and below

$$\bar{\kappa}_n(\lambda) = M[\kappa_n(x, \lambda)].$$  \hspace{1cm} (7)
Also it is known (see Chapter 4 of [Ivr1]) that under microhyperbolicity condition (4) for $|\tau - \lambda| < \epsilon$ the following complete asymptotics holds:

$$F_{t \to h^{-1}\tau}(\bar{\chi}_T(t)(Q_{2x}u_h(x, y, t)^tQ_{1y})|_{y=x}) \sim \sum_{n \geq 0} \kappa'_{n, Q_1, Q_2}(x, \tau) h^{1-d+n},$$

where $u_h(x, y, t)$ is the Schwartz kernel of the propagator $e^{ih^{-1}tA}$, $\bar{\chi} \in \mathcal{C}_0^\infty([-1, 1])$, $\bar{\chi}(t) = 1$ at $[-\frac{1}{2}, \frac{1}{2}]$, $T \in [h^{1-\delta}, T^*]$, $T^*$ is a small constant here and $Q_j = Q_j(x, hD)$ are $h$-pseudo-differential operator; we write operators, acting with respect to $y$ on Schwartz kernels to the right of it.
Further,

\[ \text{supp}(Q_1) \cap \text{supp}(Q_2) = \emptyset \implies \kappa'_{n, Q_1, Q_2}(x, \tau) = 0, \quad (9) \]

where \( \text{supp}(Q_j) \) is a support of its symbol \( Q_j(x, \xi) \) and

\[ \tau \leq \tau^* = \inf_{x, \xi} A(x, \xi) \implies \kappa'_{n, Q_1, Q_2}(x, \tau) = 0. \quad (10) \]

Let

\[ \kappa_{n, Q_1, Q_2}(x, \tau) = \int_{-\infty}^{\tau} \kappa'_{n, Q_1, Q_2}(x, \tau') \, d\tau. \quad (11) \]

In what follows we skip subscripts \( Q_j = I \).

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This equality (8) plus Hörmander’s Tauberian theorem imply the remainder estimates \(O(h^{1-d})\) for \(Q_{2x}e_h(x, y, \tau)^t Q_{1y} |_{x=y}\). On the other hand, if we can improve (8) by increasing \(T^*\), we can improve the remainder estimate to \(O(T^*^{-1}h^{1-d})\). \(^1\)

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Observe that for $A = A(hD)$

$$e_h(x, x, \lambda) = N_h(\lambda) = \kappa_0(\lambda) h^{-d}. \quad (12)$$

In this paper we consider

$$A(x, hD) = A^0(hD) + \varepsilon B(x, hD), \quad (13)$$

where $A^0(\xi)$ satisfies (1), (2) and (4) and $B(x, \xi)$ satisfies (1) and $\varepsilon > 0$ is a small parameter. Later we assume that $B(x, hD)$ is almost periodic and impose other conditions.
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For operator (13) with $\varepsilon \leq \varepsilon_0$ the equality (8) holds with $T^* = \varepsilon_1 \varepsilon^{-1}$ where $\varepsilon_j$ are small constants and we assume that $\varepsilon \geq h^M$ for some $M$. Then the remainder estimate is $O(\varepsilon h^{1-d})$ (Theorem 2.4 of [Ivr3]).
Consider the main topic of this work where we will use ideas from [PS1, PS2, PS3, MPS]: the case of an almost periodic operator $B(x, hD)$,

$$B(x, \xi) = \sum_{\theta \in \Theta} b(\xi) e^{i\langle \theta, x \rangle}$$

with discrete (i.e. without any accumulation points) frequency set $\Theta$. 

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$$B(x, \xi) = \sum_{\theta \in \Theta} b_\theta(\xi) e^{i\langle \theta, x \rangle} \quad (14)$$

with discrete (i.e. without any accumulation points) frequency set $\Theta$.

Operator $B$ is *quasiperiodic* if $\Theta$ is a finite set, *periodic* if $\Theta$ is a lattice and *almost periodic* in the general case.
Main Theorem

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Operator $B$ is quasiperiodic if $\Theta$ is a finite set, periodic if $\Theta$ is a lattice and almost periodic in the general case.

Our goal is to derive (under certain assumptions) complete semiclassical asymptotics:

$$e_{h, \varepsilon}(x, x, \tau) \sim \sum_{n \geq 0} \kappa_{n, \varepsilon} x(x, \tau) h^{-d+n}.$$ 

(15)
In addition to microhyperbolicity condition (4) we assume that
\[ \Sigma_\lambda = \{ \xi : A^0(\xi) = \lambda \} \] is a strongly convex surface i.e.
\[ \pm \sum_{j,k} A^0_{\xi_j \xi_k}(\xi) \eta_j \eta_k \geq \epsilon |\eta|^2 \quad \forall \xi : A^0(\xi) = \lambda \quad \forall \eta : \sum_j A^0_{\xi_j}(\xi) \eta_j = 0, \] (16)
where the sign depends on the connected component of \( \Sigma_\lambda \), containing \( \xi \).
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where the sign depends on the connected component of \( \Sigma_\lambda \), containing \( \xi \).

Without any loss of generality we assume that \( \Theta \) spans \( \mathbb{R}^d \), contains 0 and is symmetric about 0.

**Condition (A).**

For each \( \theta_1, \ldots, \theta_d \in \Theta \) either \( \theta_1, \ldots, \theta_d \) are linearly independent over \( \mathbb{R} \) or they linearly dependent over \( \mathbb{Z} \).
Assume also that

**Condition (B).**

For any arbitrarily large $L$ and for any sufficiently large real number $\omega$ there are a finite symmetric about 0 set $\Theta' := \Theta'_{(L,\omega)} \subset (\Theta \cap B(0, \omega))$ (with $B(\xi, r)$ the ball of the radius $r$ and center $\xi$) and a “cut-off” coefficients $b'_{\theta} := b_{\theta,(L,\omega)}$, such that

$$B' := B'_{(L,\omega)}(x, \xi) := \sum_{\theta \in \Theta'} b'_{\theta}(\xi) e^{i\langle \theta, x \rangle}$$

satisfies

$$\|D_x^\alpha D_\xi^\beta (B - B')\|_{L^\infty} \leq \omega^{-L}(|\xi| + 1)^m \quad \forall \alpha, \beta : |\alpha| \leq L, |\beta| \leq L. \quad (18)$$
Remark 1.

Then

$$|D^\beta_\xi b_\theta| = O(|\theta|^{-\infty}(|\xi| + 1)^m) \quad \text{as} \quad |\theta| \to \infty$$

(19)

and

$$|D^\beta_\xi (b_\theta - b'_\theta)| = O(\omega^{-\infty}(|\xi| + 1)^m).$$

(20)

Indeed, one suffices to observe that $b_\theta(\xi) = M(B(x, \xi)e^{-i\langle \theta, x \rangle})$ etc.
Remark 1.

1. Then

\[ |D_\xi^\beta b_\theta| = O(|\theta|^{-\infty}(|\xi| + 1)^m) \quad \text{as} \quad |\theta| \to \infty \]  \hspace{1cm} (19)

and

\[ |D_\xi^\beta (b_\theta - b'_\theta)| = O(\omega^{-\infty}(|\xi| + 1)^m). \]  \hspace{1cm} (20)

Indeed, one suffices to observe that \( b_\theta(\xi) = M(B(x, \xi)e^{-i\langle \theta, x \rangle}) \) etc.

2. On the other hand, under additional assumption

\[ \# \{ \theta \in \Theta, |\theta| \leq \omega \} = O(\omega^p) \quad \text{as} \quad \omega \to \infty \]  \hspace{1cm} (21)

for some \( p \), (19) implies Condition (B) with \( \Theta'_{(L,\omega)} := \Theta \cap B(0, \omega) \).

However we will need \( \Theta'_{(L,\omega)} \) in the next condition.
Remark 1 (Continued).

We need only to estimate the operator norm of the difference between $B(x, hD)$ and $B'(x, hD)$ (from $\mathcal{H}^m$ to $L^2$); therefore for differential operators we can weaken (18).
Remark 1 (Continued).

We need only to estimate the operator norm of the difference between $B(x, hD)$ and $B'(x, hD)$ (from $\mathcal{H}^m$ to $L^2$); therefore for differential operators we can weaken (18).

While Condition (B) is Condition B of [PS3], adopted to our case, Condition (A) and Conditions (C), (D) below are borrowed without any modifications (except changing notations).
We need to impose a Diophantine condition on the frequencies of $B$. We need some definitions.
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$$\Theta'_K := \sum_{1 \leq i \leq K} \Theta'.$$

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\[
\Theta'_K := \sum_{1 \leq i \leq K} \Theta'.
\]  

(22)

We say that $\mathcal{V}$ is a **quasi-lattice subspace** of dimension $q$, if $\mathcal{V}$ is a linear span of $q$ linear independent vectors $\theta_1, \ldots, \theta_q \in \Theta'_K \setminus 0$. Obviously, the zero space is a quasi-lattice subspace of dimension 0 and $\mathbb{R}^d$ is a quasi-lattice subspace of dimension $d$.

We denote by $\mathcal{V}_q$ the collection of all quasi-lattice subspaces of dimension $q$ and also $\mathcal{V} := \bigcup_{q \geq 0} \mathcal{V}_q$. 
Consider \( \mathcal{V}, \mathcal{U} \in \mathcal{V} \). We say that these subspaces are \textit{strongly distinct}, if neither of them is a subspace of the other one. Next, let \( (\mathcal{V}, \mathcal{U}) \in [0, \pi/2] \) be the angle between them, i.e. the angle between \( \mathcal{V} \ominus \mathcal{W} \) and \( \mathcal{U} \ominus \mathcal{W} \), \( \mathcal{W} = \mathcal{U} \cap \mathcal{V} \). This angle is positive iff \( \mathcal{V} \) and \( \mathcal{U} \) are strongly distinct.
Consider $V, U \in \mathcal{V}$. We say that these subspaces are *strongly distinct*, if neither of them is a subspace of the other one. Next, let $(V, U) \in [0, \pi/2]$ be the angle between them, i.e. the angle between $V \ominus W$ and $U \ominus W$, $W = U \cap V$. This angle is positive iff $V$ and $U$ are strongly distinct.

**Condition (C).**

For each fixed $L$ and $K$ the sets $\Theta'_{(L, \omega)}$ satisfying (17) and (18) can be chosen in such a way that for sufficiently large $\omega$ we have

$$s(\omega) = s(\Theta'_K) := \inf_{V, U \in \mathcal{V}} \sin((V, U)) \geq \omega^{-1}$$

(23)

and

$$r(\omega) := \inf_{\theta \in \Theta'_K \setminus \emptyset} |\theta| \geq \omega^{-1},$$

(24)

where the implied constant (how large should $\omega$ be) depends on $L$ and $K$. 

Let $\mathcal{V}$ be the span of $\theta_1, \ldots, \theta_q \in \Theta'_K \setminus 0$. Then due to Condition (A) each element of the set $\Theta'_K \cap \mathcal{V}$ is a linear combination of $\theta_1, \ldots, \theta_q$ with rational coefficients. Since the set $\Theta'_\infty \cap \mathcal{V}$ is finite, this implies that the set $\Theta'_\infty \cap \mathcal{V}$ is discrete and is, therefore, a lattice in $\mathcal{V}$. We denote this lattice by $\Gamma(\omega; \mathcal{V})$. 

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Let \( \mathcal{V} \) be the span of \( \theta_1, \ldots, \theta_q \in \Theta'_K \setminus 0 \). Then due to Condition (A) each element of the set \( \Theta'_K \cap \mathcal{V} \) is a linear combination of \( \theta_1, \ldots, \theta_q \) with rational coefficients. Since the set \( \Theta'_K \cap \mathcal{V} \) is finite, this implies that the set \( \Theta'_\infty \cap \mathcal{V} \) is discrete and is, therefore, a lattice in \( \mathcal{V} \). We denote this lattice by \( \Gamma(\omega; \mathcal{V}) \).

Our final condition states that this lattice cannot be too dense.

**Condition (D).**

We can choose \( \Theta'_{(L;\omega)} \), satisfying Conditions (B) and (C) in such a way that for sufficiently large \( \omega \) and for each \( \mathcal{V} \in \mathcal{V}, \mathcal{V} \neq \mathbb{R}^d \), we have

\[
\text{vol}(\mathcal{V}/\Gamma(\omega; \mathcal{V})) \geq \omega^{-1}. \tag{25}
\]
Remark 2.

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2. Further, if $\Theta$ is a finite set and Condition (A) is fulfilled, then $\Theta_\infty := \bigcup_{K \geq 1} \Theta_K$ is a lattice and Conditions (B)–(D) are fulfilled.

3. Furthermore, the same is true, if $\Theta$ is an arithmetic sum of a finite set and a lattice.
Theorem 3 (Main Theorem).

Let $A$ be a self-adjoint operator (13), where $A^0$ satisfies (1), (2), (4) and (16) and $B$ satisfies (1).
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Let Conditions (A)–(D) be fulfilled. Then for $|\tau - \lambda| < \epsilon$, $\epsilon \leq h^\vartheta$, $\vartheta > 0$

$$e_{h,\epsilon}(x, x, \tau) \sim \sum_{n \geq 0} \kappa_n(x, \tau; \epsilon) h^{-d+n}. \quad (26)$$
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**Corollary 4.**

*In the framework of Theorem 3*

$$N_{h,\epsilon}(\tau) \sim \sum_{n \geq 0} \bar{\kappa}_n(\tau; \epsilon) h^{-d+n}. \quad (27)$$
Remark 5.

It follows from Section 4 of [Ivr1] that the contribution of the zone $\{\xi: |A^0(\xi) - \tau| \geq C_0 \varepsilon + h^{1-\varsigma}\}$ to the remainder is negligible. Here and below $\varsigma > 0$ is an arbitrarily small exponent. Therefore we restrict ourself by the analysis in the zone $\Omega_{\tau}$. 
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2. To upgrade (8) with \( T = T_* \) (a small constant) to (8) with \( T = T^* \) it is sufficient to prove that

\[
|F_{t \to h^{-1}\tau}(\chi_T(t)(Q_{2x}u_h(x,y,t)^tQ_{1y})|_{y=x})| \leq C_s h^{-d+s}, \quad (28)
\]

for \( |\tau - \lambda| \leq \varepsilon \), \( T \in [T_*, T^*] \) and \( \chi \in \mathcal{C}_0^\infty([-1, -\frac{1}{2}] \cup [\frac{1}{2}, 1]) \), where \( s \) is an arbitrarily large exponent.
Remark 6.

It suffices to prove asymptotics

\[ e_h(x, x, \tau) = \sum_{0 \leq n \leq M} \kappa_n(x, \tau) h^{-d+n} + O(h^{-d+M}) \]  

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with arbitrarily large fixed \( M \). To do so we will use the \textit{semiclassical Schrödinger operator method} with maximal time \( T^* = h^{-M} \).
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2. Then we can replace operator \( B \) by operator \( B' \), provided operator norm of \( B - B' \) from \( \mathcal{H}^m \) to \( L^2 \) does not exceed \( Ch^{3M} \).
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2. Then we can replace operator \( B \) by operator \( B' \), provided operator norm of \( B - B' \) from \( \mathcal{H}^m \) to \( L^2 \) does not exceed \( Ch^{3M} \).

3. First such replacement will be \( B' := B'_{(L, \omega)} \) from Condition (B) with \( \omega = h^{-\sigma} \), arbitrarily small \( \sigma > 0 \) and \( L = 3M/\sigma \).

\textit{So, from now \( \Theta \) and \( B \) are effectively replaced by \( \Theta' := \Theta'_{(L, \omega)} \) and \( B'_{(L, \omega)} \) correspondingly.}
Consider now the “gauge” transformation $A \mapsto e^{-i\varepsilon h^{-1}}P A e^{i\varepsilon h^{-1}}P$ with $h$-pseudodifferential operator $P$. Observe that

$$e^{-i\varepsilon h^{-1}}P A e^{i\varepsilon h^{-1}}P = A - i\varepsilon h^{-1}[P, A] + \sum_{2 \leq n \leq K-1} \frac{1}{n!}(-i\varepsilon h^{-1})^n \text{Ad}_P^n(A)$$

$$+ \int_0^1 \frac{1}{(K-1)!}(1 - s)^{K-1}(-i\varepsilon h^{-1})^K e^{-i\varepsilon h^{-1}}sP \text{Ad}_P^K(A)e^{i\varepsilon h^{-1}}sP ds, \quad (30)$$

where $\text{Ad}_P^0(A) = A$ and $\text{Ad}_P^{n+1}(A) = [P, \text{Ad}_P^n(A)]$ for $n = 0, 1, \ldots$. 
Consider now the “gauge” transformation $A \mapsto e^{-i\varepsilon h^{-1}P} A e^{i\varepsilon h^{-1}P}$ with $h$-pseudodifferential operator $P$. Observe that

$$e^{-i\varepsilon h^{-1}P} A e^{i\varepsilon h^{-1}P} = A - i\varepsilon h^{-1}[P, A] + \sum_{2 \leq n \leq K-1} \frac{1}{n!} (-i\varepsilon h^{-1})^n \text{Ad}_P^n(A)$$

$$+ \int_0^1 \frac{1}{(K-1)!} (1 - s)^{K-1} (-i\varepsilon h^{-1})^K e^{-i\varepsilon h^{-1}sP} \text{Ad}_P^K(A) e^{i\varepsilon h^{-1}sP} ds,$$  \hspace{1cm} (30)

where $\text{Ad}_P^0(A) = A$ and $\text{Ad}_P^{n+1}(A) = [P, \text{Ad}_P^n(A)]$ for $n = 0, 1, \ldots$.

Thus formally we can compensate $\varepsilon B$, taking

$$P = \sum_{\theta} i\hbar (A^0(\xi + \theta h/2) - A^0(\xi - \theta h/2))^{-1} b_{\theta}(\xi) e^{i\langle \theta, x \rangle},$$ \hspace{1cm} (31)

so that

$$i\hbar^{-1}[P, A^0] = B \implies i\hbar^{-1}[P, A] = B + i\varepsilon h^{-1}[P, B].$$ \hspace{1cm} (32)
Then perturbation $\varepsilon B$ is replaced by $\varepsilon^2 B'$, which is the right hand expression in (30) minus $A^0$, i.e.

$$B' = -i\hbar^{-1}[P, B] + \sum_{2 \leq n \leq K-1} \frac{1}{n!} \varepsilon^{n-2}(-i\hbar^{-1})^n \text{Ad}_P^n(A),$$ (33)

where we ignored the remainder.
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where we ignored the remainder.

New perturbation, again formally, has a magnitude of $\varepsilon^2$. Repeating this process we will make a perturbation negligible.
Remark 7.

However, we need to address the following issues:

1. **Denominator**
   \[
   h^{-1}(A^0(\xi + \theta h/2) - A^0(\xi - \theta h/2)) = \langle \nabla_\xi A^0, \theta \rangle + O(h^{1-\sigma})
   \]
   could be small.
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\[ h^{-1}(A^0(\xi + \theta h/2) - A^0(\xi - \theta h/2)) = \langle \nabla_\xi A^0, \theta \rangle + O(h^{1-\sigma}) \] could be small.

2. In \( B' \) set \( \Theta' \) increases:
\[ \varepsilon^2 B' = \varepsilon^2 B'_2 + \varepsilon^3 B'_3 + \ldots + \varepsilon^M B'_M, \] where for \( B'_j \) the frequency set is \( \Theta'_j \) (the arithmetic sum of \( j \) copies of \( \Theta' \)).
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\[ \varepsilon^2 B' = \varepsilon^2 B'_2 + \varepsilon^3 B'_3 + \ldots + \varepsilon^M B'_M, \]
where for $B'_j$ the frequency set is $\Theta'_j$ (the arithmetic sum of $j$ copies of $\Theta'$).

3. We need to prove that the remainder is negligible.

4. This transformation was used in Section 9 of [PS3] (etc); in contrast to these papers we use Weyl quantization instead of $pq$-quantization, and have therefore
\[ (A^0(\xi + \theta h/2) - A^0(\xi - \theta h/2)) \]
instead of
\[ (A^0(\xi + \theta h) - A^0(\xi)). \]
One can see easily that if inequality

\[ |\langle \nabla_\xi A^0(\xi), \theta \rangle| \geq \gamma := \varepsilon^{\frac{1}{2}} h^{-\delta} \]  

(34)

holds for all \( \theta \in \Theta'_K \), then the terms could be estimated by \( h^{\delta n} \) and our construction works with \( K = 3M/\delta \). Here and below without any loss of the generality we assume that \( \varepsilon \geq h \); so, in fact, \( h^\vartheta \geq \varepsilon \geq h \).
One can see easily that if inequality

$$|\langle \nabla_{\xi} A^0(\xi), \theta \rangle| \geq \gamma := \frac{\varepsilon}{2} h^{-\delta}$$  \hspace{1cm} (34)$$

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Indeed, if \( P = P(x, hD) \) has the symbol, satisfying

$$|D_{\xi}^{\alpha} D_{x}^{\beta} P| \leq C_{\alpha\beta} \gamma^{-1-|\alpha|} \quad \forall \alpha, \beta,$$  \hspace{1cm} (35)$$

then \( B' = \varepsilon h^{-1} [P, B] \) has a symbol, satisfying

$$|D_{\xi}^{\alpha} D_{x}^{\beta} B'| \leq c'_{\alpha\beta} \varepsilon \gamma^{-2-|\alpha|} \quad \forall \alpha, \beta,$$  \hspace{1cm} (36)$$

so indeed \( \varepsilon' = \varepsilon h^{-1} \gamma^{-2} \).
Then we can eliminate a perturbation completely, save terms with the frequency 0, both old and new. The set of $\xi$ satisfying (34) for all $\theta \in \Theta'_K$ we call \textit{non-resonant zone} and denote by $\mathcal{Z}$. Thus, we arrive to
Then we can eliminate a perturbation completely, save terms with the frequency 0, both old and new. The set of $\xi$ satisfying (34) for all $\theta \in \Theta'_K$ we call non-resonant zone and denote by $\mathcal{Z}$. Thus, we arrive to

**Proposition 8.**

*Let $Q = Q(hD)$ with the symbol supported in $\mathcal{Z} \cap \Omega$ and satisfying

$$|D^\alpha Q_j| \leq C_\alpha h^{-(1-\varsigma)|\alpha|} \quad \forall \alpha.$$  \hspace{1cm} (37)***
Then we can eliminate a perturbation completely, save terms with the frequency 0, both old and new. The set of $\xi$ satisfying (34) for all $\theta \in \Theta'_K$ we call *non-resonant zone* and denote by $\mathcal{Z}$. Thus, we arrive to

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$$|D^\alpha Q_j| \leq C_\alpha h^{-(1-\varsigma)|\alpha|} \quad \forall \alpha. \quad (37)$$

Then there exists a pseudo-differential operator $P = P(x, hD)$ with the symbol, satisfying (37) and such that

$$(e^{-i\varepsilon h^{-1}P}Ae^{i\varepsilon h^{-1}P} - A'')Q \equiv 0 \quad (38)$$

with

$$A'' = A^0(hD) + \varepsilon B''_0(hD) \quad (39)$$

modulo operator from $\mathcal{H}^m$ to $L^2$ with the operator norm $O(h^{3M})$. 
Remark 9.

This proposition is similar to Lemma 9.3 of [PS3]. However, in contrast to [PS1, PS2, PS3, MPS], after it is proven we do not write asymptotic decomposition there, but simply prove that singularities do not propagate with respect to $\xi$ there.
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2. It is our second replacement of operator $A$; recall that the first one was based on Condition (B), and now we ignore the remainder after transformation, which is justified by Remark 6.
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**Proposition 10.**

Let $Q_j = Q_j(hD)$ with the symbols, satisfying (37) and let symbol of $Q_1$ be supported in $\mathcal{Z} \cap \Omega$. 
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1. This proposition is similar to Lemma 9.3 of [PS3]. However, in contrast to [PS1, PS2, PS3, MPS], after it is proven we do not write asymptotic decomposition there, but simply prove that singularities do not propagate with respect to $\xi$ there.

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Proposition 10.

Let $Q_j = Q_j(hD)$ with the symbols, satisfying (37) and let symbol of $Q_1$ be supported in $\mathcal{Z} \cap \Omega$. Let $\text{dist}(\text{supp}(Q_1), \text{supp}(Q_2)) \geq c\gamma$. Then

$$\|Q_2e^{ih^{-1}tA}Q_1\| = O(h^{2M}) \quad \text{as} \quad |t| \leq T^* = h^{-M}. \quad (40)$$
Consider now \textit{resonant zone}

\begin{equation}
\Lambda := \bigcup_{\theta \in \Theta'_K \setminus 0} \Lambda(\theta), \tag{41}
\end{equation}

where \(\Lambda(\theta)\) is the set of \(\xi\), violating (34) for given \(\theta\): \(\Lambda(\theta) = \Lambda_\delta(\theta) := \{\xi : |\langle \nabla_\xi A^0(\xi), \theta \rangle| \geq \gamma = c \varepsilon^{\frac{1}{2}} h^{-\delta}\}\). \tag{42}
Consider the easiest case $d = 2$ (in the trivial case $d = 1$ there is no resonant zone). Due to assumption (16) for each $\theta$

$$\text{mes}_1(\Lambda(\theta) \cap \Sigma_\lambda) \leq C\gamma.$$  \hfill (43)
Consider the easiest case \( d = 2 \) (in the trivial case \( d = 1 \) there is no resonant zone). Due to assumption (16) for each \( \theta \)

\[
\text{mes}_1(\Lambda(\theta) \cap \Sigma_\lambda) \leq C\gamma. \tag{43}
\]

Note, \( \#\Theta'_K \leq C h^{-\sigma} \) due to Condition (C). Thus \( \text{mes}_1(\Lambda \cap \Sigma_\lambda) \leq \gamma h^{-\sigma} \).

Recall, that \( \sigma > 0 \) is arbitrarily small.
Consider the easiest case $d = 2$ (in the trivial case $d = 1$ there is no resonant zone). Due to assumption (16) for each $\theta$

$$\text{mes}_1(\Lambda(\theta) \cap \Sigma_\lambda) \leq C\gamma.$$  \hspace{1cm} (43)

Note, $\#\Theta'_K \leq Ch^{-\sigma}$ due to Condition (C). Thus $\text{mes}_1(\Lambda \cap \Sigma_\lambda) \leq \gamma h^{-\sigma}$. Recall, that $\sigma > 0$ is arbitrarily small.

Since due to Proposition 10, the propagation which starts in the non-resonant zone $\mathcal{Z}$ remains there we conclude that the propagation which is started in some connected component of the resonant zone also remains there (in both cases, we change constant $c$ in the definition of $\gamma$).
Consider the easiest case $d = 2$ (in the trivial case $d = 1$ there is no resonant zone). Due to assumption (16) for each $\theta$

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Since due to Proposition 10, the propagation which starts in the non-resonant zone $\mathcal{Z}$ remains there we conclude that the propagation which is started in some connected component of the resonant zone also remains there (in both cases, we change constant $c$ in the definition of $\gamma$).

Thus, $\nabla_\xi A^0(\xi)$ does not change by more than $\gamma h^{-\sigma}$ and since $\sigma$ is arbitrarily small we conclude that (40) also holds for $Q_1$, supported in the resonant zone. Therefore

**Proposition 11.**

*Estimate* (40) *holds for all* $Q_1$, $Q_2$ *satisfying* (37) *and*

$$\text{dist} (\text{supp}(Q_1), \text{supp}(Q_2)) \geq \gamma. \quad (44)$$
Consider case $d \geq 2$. Due Conditions (A), (C) and (D) we can cover $\Lambda \cap \Omega_{\tau}$ by $\Lambda^*$,

$$\Lambda \cap \Omega_{\tau} \subset \Lambda^* = \bigcup_{1 \leq j \leq d-1} \Lambda^*_j,$$

defined as

Let $\xi \in \Omega_{\tau}$; then $\xi \in \Lambda^*_j$ iff there exist $\theta_1, \ldots, \theta_j \in \Theta'_K$ which are linearly independent and such that $\xi \in \Lambda_{\delta_j}(\theta_k)$ for all $k = 1, \ldots, j$,

where $0 < \delta = \delta_1 < \delta_2 < \ldots < \delta_{d-1}$ are arbitrarily fixed and we chose sufficiently small $\sigma > 0$ afterwards.
Consider case $d \geq 2$. Due Conditions (A), (C) and (D) we can cover $\Lambda \cap \Omega_\tau$ by $\Lambda^*$,

$$\Lambda \cap \Omega_\tau \subset \Lambda^* = \bigcup_{1 \leq j \leq d-1} \Lambda_j^*,$$

(45)

defined as

Let $\xi \in \Omega_\tau$; then $\xi \in \Lambda_j^*$ iff there exist $\theta_1, \ldots, \theta_j \in \Theta'_K$ which are linearly independent and such that $\xi \in \Lambda_{\delta_j}(\theta_k)$ for all $k = 1, \ldots, j$,

where $0 < \delta = \delta_1 < \delta_2 < \ldots < \delta_{d-1}$ are arbitrarily fixed and we chose sufficiently small $\sigma > 0$ afterwards.

Further, due to Conditions (A), (C), (D) and (16) $\Lambda_{d-1}^* \cap \Omega_\tau$ could be covered by no more than $\gamma_{d-1}$-vicinities of some points $\xi_\nu$, $\nu = 1, \ldots, \omega^g$, $g = g(d)$. Recall that $\Omega_\tau := \{\xi : |A^0(\xi) - \tau| \leq C_0 \varepsilon + h^{1-\varsigma}\}$. 
Consider some connected component \( \Xi \) of \( \Lambda^* \). Let some point \( \bar{\xi} \) of it belong to \( \bigcap_{1 \leq k \leq j} \Lambda_{\delta_j}(\theta_k) \cap \Omega_\tau \) with linearly independent \( \theta_1, \ldots, \theta_j \).

Observe that \( \operatorname{diam}(\bigcap_{1 \leq k \leq j} \Lambda_{\delta_j}(\theta_k) \cap \Omega) \leq c\gamma_j \) due to strong convexity assumption (16). Then this set either intersects or does not intersect with \( \Lambda^*_{j+1} \cap \Omega \). In the former case we include it to \( \Lambda^*_{j+1} \) and exclude it from \( \Lambda^*_j \).
Proof of the Main Theorem

Consider some connected component $\Xi$ of $\Lambda_j^*$. Let some point $\bar{\xi}$ of it belong to $\bigcap_{1 \leq k \leq j} \Lambda_{\delta_j}(\theta_k) \cap \Omega_\tau$ with linearly independent $\theta_1, \ldots, \theta_j$.

Observe that $\text{diam}(\bigcap_{1 \leq k \leq j} \Lambda_{\delta_j}(\theta_k) \cap \Omega) \leq c\gamma_j$ due to strong convexity assumption (16). Then this set either intersects or does not intersect with $\Lambda_{j+1}^* \cap \Omega$. In the former case we include it to $\Lambda_{j+1}^*$ and exclude it from $\Lambda_j^*$.

After we redefined $\Lambda_j^*$ we arrive to the following proposition:

**Proposition 12.**

*Equation (45) still holds where now each connected component $\Xi$ of $\Lambda_j^*$ has the following properties:*

1. $\text{diam} \Xi \leq c\gamma_j$.

2. There exist linearly independent $\theta_1, \ldots, \theta_j \in \Theta'_K$, such that for each $\xi \in \Xi$ $|\langle \nabla_\xi A^0(\xi), \theta \rangle| \leq c_j\gamma_j$ for all $\theta \in \mathcal{V} \cap (\Theta'_K \setminus 0)$ and $|\langle \nabla_\xi A^0(\xi), \theta \rangle| \geq \epsilon_j\gamma_{j+1}$ for all $\theta \in \Theta'_K \setminus \mathcal{V}$ with $\mathcal{V} = \text{span}(\theta_1, \ldots, \theta_j)$. 
Now we generalize Proposition 8:

**Proposition 13.**

Let \( Q = Q(hD) \) with the symbol supported in the connected component \( \Xi \) of \( \Lambda^*_j \), corresponding to subspace \( \EuScript{B} \), and satisfying (37). Then there exists a pseudo-differential operator \( P = P(x, hD) \) with the symbol, satisfying (35) and such that

\[
(e^{-i \varepsilon h^{-1} P} A e^{i \varepsilon h^{-1} P} - A'') Q \equiv 0
\]  \hspace{1cm} (46)

modulo operator from \( \mathcal{H}^m \) to \( L^2 \) with the operator norm \( O(h^{3M}) \), where \( A'' = A^0 + \varepsilon B''(x, hD) \), where \( B'' \) is an operator with Weyl symbol

\[
B''(x, \xi) = \sum_{\theta \in \Theta'_k \cap \mathfrak{B}} b_{\mathfrak{B}, \theta}(\xi) e^{i \langle \theta, x \rangle}.
\]  \hspace{1cm} (47)
Then we arrive to

**Proposition 14.**

Let $Q_j = Q_j(hD)$ with the symbols, satisfying (37) and let symbol of $Q_1$ be supported in $\Lambda_j^*$. Let $\text{dist}(\text{supp}(Q_1), \text{supp}(Q_2)) \geq C_0 \gamma_j$. Then $\|Q_2 e^{ih^{-1}tA}Q_1\| = O(h^{2M})$ for $|t| \leq T_* = h^{-M}$. 
Then we arrive to

**Proposition 14.**

Let $Q_j = Q_j(hD)$ with the symbols, satisfying (37) and let symbol of $Q_1$ be supported in $\Lambda_j^*$. Let $\text{dist}(\text{supp}(Q_1), \text{supp}(Q_2)) \geq C_0 \gamma_j$. Then $\|Q_2 e^{ih^{-1}tA}Q_1\| = O(h^{2M})$ for $|t| \leq T_* = h^{-M}$.

Next we arrive to the following proposition:

**Proposition 15.**

Let $Q_1, Q_2$ satisfy (37) and $\text{supp}(Q_1) \subset \Omega$. Then for $T_* \leq T \leq T^*$

$$F_{t \to h^{-1}T}(\chi_T(t)Q_{2x}u(x, y, t)^tQ_{1y}) = O(h^{2M}).$$

(48)
Now we conclude that

$$F_{t \to h^{-1} \tau} \left( [\bar{\chi}_T(t) - \bar{\chi}_{T^*}(t)] Q_2 x u(x, y, t) t Q_1 y \right) \bigg|_{x=y} = O(h^{2M})$$  \hspace{1cm} (49)
Now we conclude that

\[ F_{t \to h^{-1} \tau} \left( [\bar{\chi}_T(t) - \bar{\chi}_{T_*}(t)] Q_{2x} u(x, y, t) \right)_{x=y} = O(h^{2M}) \]  \hspace{1cm} (49)

and since

\[ F_{t \to h^{-1} \tau} \left( \bar{\chi}_T(t) Q_{2x} u(x, y, t) \right)_{x=y} = \sum_{0 \leq n \leq M} \kappa_n'(x, \varepsilon) h^{1-d+n} + O(h^{M+1}) \]  \hspace{1cm} (50)

holds for \( T = T_* \), it also holds for \( T = T^* \).
Now we conclude that

$$F_{t \to h^{-1} \tau} \left( \left[ \bar{\chi}_T(t) - \bar{\chi}_{T_*}(t) \right] Q_{2x} u(x, y, t)^t Q_{1y} \right) \bigg|_{x=y} = O(h^{2M}) \quad (49)$$

and since

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holds for $T = T_*$, it also holds for $T = T^*$. Finally, Hörmander’s Tauberian theorem implies Theorem 3.
Remark 16.

One can generalize Theorem 3 to elliptic matrix operators, assuming that eigenvalues of $A^0(\xi)$ are simple and satisfy assumptions of this theorem.
Remark 16.

1. One can generalize Theorem 3 to elliptic matrix operators, assuming that eigenvalues of $A^0(\xi)$ are simple and satisfy assumptions of this theorem.

2. As $d = 2$ one can replace strong convexity condition (16) by much weaker nondegeneracy assumption.
Remark 17.

One can generalize Theorem 3 to operators

$$A = A^0(hD) + \varepsilon V(x, HD),$$

where

$$|D_\xi^\alpha D_x^\beta V(x, \xi)| \leq c_{\alpha\beta}(|\xi| + 1)^m(|x| + 1)^{-\delta-|\beta|} \quad \forall \alpha, \beta \forall x, \xi$$

provided $\varepsilon \leq \varepsilon_0$. 
Remark 17.

1. One can generalize Theorem 3 to operators

\[ A = A^0(hD) + \varepsilon V(x, HD), \tag{51} \]

where

\[ |D_\xi^\alpha D_x^\beta V(x, \xi)| \leq c_{\alpha\beta}(|\xi| + 1)^m(|x| + 1)^{-\delta-|\beta|} \quad \forall \alpha, \beta \forall x, \xi \tag{52} \]

provided \( \varepsilon \leq \varepsilon_0 \).

2. One can generalize Theorem 3 to operators

\[ A = A^0(hD) + \varepsilon (B(x, hD) + V(x, hD)), \tag{53} \]

where \( B(x, hD) \) satisfies conditions of Theorem 3 and \( V \) satisfies (52), and even for more general operators.
Remark 18.

It also follows from Corollary 4 that

$$\frac{1}{\nu} \left[ N_{h,\varepsilon}(\tau + \nu) - N_{h,\varepsilon}(\tau) \right] = \frac{1}{\nu} \left[ \mathcal{N}_{h,\varepsilon}(\tau + \nu) - \mathcal{N}_{h,\varepsilon}(\tau) \right] + O(h^\infty) \quad (54)$$

provided $\nu \geq h^M$, where $\mathcal{N}_{h,\varepsilon}(\tau)$ is the right-hand expression of (27).
Remark 18.

1. It also follows from Corollary 4 that

$$
\frac{1}{\nu} \left[ N_{h,\varepsilon}(\tau + \nu) - N_{h,\varepsilon}(\tau) \right] = \frac{1}{\nu} \left[ \mathcal{N}_{h,\varepsilon}(\tau + \nu) - \mathcal{N}_{h,\varepsilon}(\tau) \right] + O(h^\infty) \tag{54}
$$

provided \( \nu \geq h^M \), where \( \mathcal{N}_{h,\varepsilon}(\tau) \) is the right-hand expression of (27).

2. The question remains, if (54) holds for smaller \( \nu \), in particular, if it holds in \( \nu \to 0 \) limit? If the latter holds, then

$$
\frac{\partial}{\partial \tau} N_{h,\varepsilon}(\tau) = \frac{\partial}{\partial \tau} \mathcal{N}_{h,\varepsilon}(\tau) + O(h^\infty) \tag{55}
$$

and we call the left-hand expression the density of states.
Remark 18 (Continued).

- It definitely is not necessarily true, at least in dimension 1. From now on we consider only asymptotics with respect to $\tau \to +\infty$. Let $A = \Delta + V(x)$ with periodic $V$. It is well-known that for $d = 1$ and generic periodic $V$ all spectral gaps are open which contradicts to

$$\frac{\partial}{\partial \tau} N(\tau) = \frac{\partial}{\partial \tau} \mathcal{N}(\tau) + O(\tau^{-\infty}).$$  \hspace{1cm} (56)
Remark 18 (Continued).

It definitely is not necessarily true, at least in dimension 1. From now on we consider only asymptotics with respect to \( \tau \to +\infty \). Let \( A = \Delta + V(x) \) with periodic \( V \). It is well-known that for \( d = 1 \) and generic periodic \( V \) all spectral gaps are open which contradicts to

\[
\frac{\partial}{\partial \tau} N(\tau) = \frac{\partial}{\partial \tau} N(\tau) + O(\tau^{-\infty}).
\]  \( (56) \)

On the other hand, this objection does not work in case \( d \geq 2 \) since only several the lowest spectral gaps are open (Bethe-Sommerfeld conjecture, proven in \([PS]\)).
Remark 18 (End).

5 Further, one can differentiate \( e(x, x, \tau^2) \) if \( d \geq 2 \) and \( V \) is compactly supported.
Remark 18 (End).

Further, one can differentiate $e(x, x, \tau^2)$ if $d \geq 2$ and $V$ is compactly supported.

Moreover, we can differentiate complete asymptotics of the *Birman-Schwinger spectral shift function*

\[
\xi(\tau) := \int (e(x, x, \tau^2) - e^0(x, x, \tau^2)) \, dx \sim \sum_{n \geq 0} \bar{\kappa}_n \tau^{-d+n},
\]

with

\[
\bar{\kappa}_n := \int (\kappa_n(x) - \kappa^0_n) \, dx,
\]

where $e^0(x, y, \tau)$ and $\kappa^0_n$ correspond to $A^0 = \Delta$. 
Remark 18 (End).

Further, one can differentiate \( e(x, x, \tau^2) \) if \( d \geq 2 \) and \( V \) is compactly supported.

Moreover, we can differentiate complete asymptotics of the Birman-Schwinger spectral shift function

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\xi(\tau) := \int (e(x, x, \tau^2) - e^0(x, x, \tau^2)) \, dx \sim \sum_{n \geq 0} \bar{\kappa}_n \tau^{-d+n}, \tag{57}
\]

with

\[
\bar{\kappa}_n := \int (\kappa_n(x) - \kappa_n^0) \, dx, \tag{58}
\]

where \( e^0(x, y, \tau) \) and \( \kappa_n^0 \) correspond to \( A^0 = \Delta \). In the case of \( A = \Delta \) in the exterior of smooth, compact and non-trapping obstacle and \( A^0 = \Delta \) in \( \mathbb{R}^d \) such asymptotics was derived in [PP].
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