Kaon Zero-Point Fluctuations in Neutron Star Matter

Vesteinn Thorsson
Department of Physics, University of Washington
Box 351560, Seattle, Washington 98195-1560, USA

and

NORDITA, Blegdamsvej 17, DK-2100 Copenhagen Ø, Denmark

Paul J. Ellis
School of Physics and Astronomy, University of Minnesota
Minneapolis, MN 55455-0112, USA

(March 31, 2022)

Abstract

We investigate the contribution of zero-point motion, arising from fluctuations in kaon modes, to the ground state properties of neutron star matter containing a Bose condensate of kaons. The zero-point energy is derived via the thermodynamic partition function, by integrating out fluctuations for an arbitrary value of the condensate field. It is shown that the vacuum counterterms of the chiral Lagrangian ensure the cancellation of divergences dependent on $\mu$, the charge chemical potential, which may be regarded as an external vector potential. The total grand potential, consisting of the tree-level potential, the zero-point contribution, and the counterterm potential, is extremized to yield a locally charge neutral, beta-equilibrated and minimum energy ground state. In some regions of parameter space we encounter the well-known problem of a complex effective potential. Where the potential is real and solutions can be obtained, the contributions from fluctuations are
found to be small in comparison with tree-level contributions.
I. INTRODUCTION

In recent years the idea that the ground state of nuclear matter at high density contains a Bose condensate of kaons has received considerable attention. The existence of this novel state of matter is suggested by the attractive s-wave kaon-nucleon interactions found in low energy effective chiral Lagrangians, which, by construction, preserve the symmetries of QCD. This is supported experimentally by kaonic atom data which indicate a strongly attractive \( K^- \) optical potential. Attractive kaon-nucleon interactions imply that in dense matter, such as that in the interior of neutron stars, the effective kaon mass is lower than in free space. It could thus be energetically favorable for leptons in the matter to convert to kaons via strangeness changing interactions, in which case the lowest energy kaon mode becomes macroscopically populated, forming a condensate.

Since the original suggestion by Kaplan and Nelson [2], studies of kaon condensation and kaon propagation in the nuclear medium can broadly be divided into three categories: studies of the basic interactions, consequences for heavy-ion collisions and astrophysical implications.

A number of studies have been aimed at improving the description of kaon-nucleon interactions over that of the original model. Brown, Lee, Rho and Thorsson [3] included all terms to next-to-leading order \( (O(q^2)) \) in chiral perturbation theory, and fitted the expansion parameters to \( K\bar{N} \) scattering lengths. Lee, Brown, Min and Rho [4] included all terms to next-to-next-to-leading order \( (O(q^3)) \) in chiral perturbation theory and also fitted kaonic atom data. Lee et al. have also included the \( \Lambda(1405) \) (which dominates the low-energy \( K^-p \) interaction), four-baryon interactions, and Pauli blocking. Other noteworthy studies are those of Ref. [5], concerning the effect of kaon-nucleon correlations, and the work of the Kyoto group (see e.g. Ref. [6]). Waas, Kaiser, and Weise [7] have recently investigated kaon modes in matter using a multichannel K-matrix analysis. For our purposes, it is worth mentioning that the majority of these more refined investigations have shown that the basic mechanism for kaon condensation, namely the existence of an attractive interaction, is robust.

As regards heavy ion collisions, strangeness is conserved on the relevant timescales and a kaon condensate is not likely to form. However, modification of the kaon modes follow-
ing from the chiral Lagrangian could have consequences in kaon flow \[8\], for subthreshold production of kaons \[9\], or for the $\phi$-meson peak in dilepton spectra \[10\].

Another major direction that investigations have taken is to study the astrophysical consequences of the formation of a kaon condensate (the effects appear to be reduced if hyperons are also present \[11\], a possibility that we do not consider here). The presence of such a condensate considerably enhances the proton concentration which leads to a softer equation of state (EOS) and a lower maximum mass for the cold neutron star. The actual value of this maximum mass is dependent on the nuclear EOS, but for the non-relativistic EOS employed here it is quite close \[3,12,13\] to the lower limit of $1.44M_\odot$ set by PSR 1913+16 \[14\]. An interesting effect occurs when the neutron star has just formed since electron neutrinos are trapped for about 10–15 s (also the entropy/baryon is $\sim 1 - 2$, but this plays a lesser role). The changes in the stellar composition induced by trapping increase the threshold for kaon condensation. This stiffens the EOS and allows a maximum mass which is $\sim 0.1 - 0.2M_\odot$ larger than in the absence of trapped neutrinos. Thus there is the possibility of metastable neutron stars which become black holes after about 10–15 s when the trapped neutrinos have left \[13,15,18\]. This could account for the fact that fewer pulsars are observed in supernova remnants than expected from conventional scenarios \[19\].

As regards dynamical simulations \[15,20\] of the delayed collapse of hot neutron stars into black holes, Baumgarte, Shapiro and Teukolsky \[21\] have recently examined the effect of kaon condensates. Here the cold EOS is supplemented with simple assumptions about corrections for finite temperature.

One-loop calculations to date are valid only up to the density at which the condensate forms where the condensate amplitude is vanishingly small. In order to go beyond threshold we analyze the fluctuations around the kaon condensate to one loop order and include the full non-linearities of the Lagrangian. This is of general theoretical interest, as well as in the particular context of neutron stars. While we shall restrict our numerical analysis here to the zero-point contribution at zero temperature, the formalism also provides a consistent treatment of the thermal contributions due to kaons. These would be needed in studies of newborn neutron stars.

In the work which follows, we shall make the following approximations. Nucleons are
treated non-relativistically, whereas a more complete treatment would involve a relativistic treatment to one loop order. In the meson sector, we restrict the discussion to kaonic degrees of freedom. We thus omit from the discussion fluctuations in the pion and eta sectors. Physically one might argue that ignoring non-kaonic pseudoscalar degrees of freedom is justified since one expects that the corresponding modes are less modified in nuclear matter than those of kaons. The methods developed here could be used to include the full octet of pseudoscalar fluctuations, but this would further complicate the present analysis. Lastly, we have sought to treat the chiral Lagrangian at the most simple level starting from the Lagrangian of Kaplan and Nelson [2], although one would wish to include the additional terms of Refs. [3,4] at a later stage.

The paper is organized as follows. In Sec. II, we present the basic formalism, including the evaluation of the kaon partition function, a discussion of divergences in the zero-point energy, the nucleon Hamiltonian, and the equilibrium conditions. In Sec. III we compare numerical results in neutron star matter with and without zero-point fluctuations. We discuss the critical density for kaon condensation, the composition of the star and the EOS. Following the Conclusions in Sec. IV are three Appendices. Appendix A concerns the cancellation of divergences dependent on the chemical potential in a toy model, while Appendix B discusses the case of the actual chiral Lagrangian. In Appendix C we derive the critical density from the kaon propagator to one-loop order, and show that it is the same as that obtained by expanding the zero-point energy in terms of the kaon fields.

II. FORMALISM

Our principal interest is the evaluation of the impact of the kaon zero-point energy contribution on the structure of neutron stars. In Subsec. A we obtain the partition function for the chiral Lagrangian of Kaplan and Nelson [2] to one loop order and in Subsec. B we discuss the elimination of the divergences. For the nucleons alone we use a simple non-relativistic Hamiltonian which should be sufficient for present purposes, as outlined in Subsec. C. The stellar equilibrium conditions that need to be satisfied are given in Subsec. D.
A. Kaon Partition Function

The $SU(3) \times SU(3)$ chiral Lagrangian involving the pseudoscalar meson octet and baryon octet is \[ \mathcal{L}_K = \mathcal{L}^{(1)} + \mathcal{L}^{(2)} + \mathcal{L}^{(0)}_{ct} + \mathcal{L}^{(3)}_{ct} + \mathcal{L}^{(4)}_{ct} \]

\[ \mathcal{L}^{(1)} = i \text{Tr} \bar{B} v \cdot \partial B + i \text{Tr} \bar{B} \nu^{\mu}[V_{\nu}, B] + 2D \text{Tr} \bar{B} S^{\nu} [A_{\nu}, B] + \mathcal{L}^{(1)}_{ct} \]

\[ \mathcal{L}^{(2)} = \frac{1}{f^2} \text{Tr} \bar{\partial}_{\nu} U \partial^{\nu} U + C \text{Tr} m_q (U + U^\dagger - 2) + a_1 \text{Tr} \bar{B} (\xi m_q \xi + h.c.) B \]

\[ + a_2 \text{Tr} \bar{B} B (\xi m_q \xi + h.c.) + a_3 \left\{ \text{Tr} \bar{B} B \right\} \left\{ \text{Tr} (m_q U + h.c.) \right\} + \mathcal{L}^{(c,d,f)} + \mathcal{L}^{(2)}_{ct}. \]

The bracketed superscripts denote the order in the low momentum expansion defining the effective field theory. The various one-loop counterterms, with subscripts $ct$, are discussed in Appendices A and B; here in the main text we will simply put them all together in a single $\mathcal{L}_{ct}$. The additional terms denoted by $\mathcal{L}^{(c,d,f)}$ are given in Refs. \[3,4\] and those with coefficients $d$ can contribute to $s$-wave kaon-nucleon interactions. In the interests of simplicity we set $\mathcal{L}^{(c,d,f)} = 0$ and so use as a starting point the Lagrangian of Kaplan and Nelson \[3\]. The Lagrangian (1) is derived in the limit of heavy baryons, where $v^{\nu'} = (1, 0)$ is the velocity four-vector of the baryons, and $S^{\nu}$ is the spin operator ($v \cdot S = 0, S^2 = -\frac{3}{4}$).

In Eq. (1), $U = \xi^2$ is the non-linear field involving the pseudoscalar meson octet, from which we retain only the $K^\pm$ contributions–

\[ U = \exp \left( \frac{\sqrt{2}i}{f} M \right) ; \quad M = \begin{pmatrix} 0 & 0 & K^+ \\ 0 & 0 & 0 \\ K^- & 0 & 0 \end{pmatrix}, \]

and from the baryon octet we retain just the nucleon contributions

\[ B = \begin{pmatrix} 0 & 0 & p \\ 0 & 0 & n \\ 0 & 0 & 0 \end{pmatrix}. \]

Consequently the $p$-wave interactions in $\mathcal{L}^{(1)}$ with coefficients $D$ and $F$ do not enter ($A_{\nu} = \frac{i}{2} \{ \xi^{\dagger}, \partial_{\nu} \xi \}$). In Eq. (1), the quark mass matrix $m_q = \text{diag}(0, 0, m_s)$; i.e., only the mass of the strange quark is taken to be non-zero. For the mesonic vector current, $V_{\nu}$, only the time component survives in an infinite system with $V_0 = \frac{1}{2} [\xi^{\dagger}, \partial_0 \xi]$. Also, the pion decay constant
\[ f = 93 \text{ MeV}, \text{ and } C, a_1, a_2, \text{ and } a_3 \] are constants. After some algebra, the relevant part of \( \mathcal{L}_K \) can be written

\[
\mathcal{L}_K = \frac{1}{2} (1 + \gamma) \partial_\nu K^+ \partial_\mu K^- + \frac{(1 - \gamma)}{2f^2 \Theta^2} \left[ (K^+ \partial_\nu K^-)^2 + (K^- \partial_\nu K^+)^2 \right] \\
+ \gamma \frac{1}{2} \left[ i b K^+ \partial_\nu K^- - (m_K^2 + c) K^+ K^- \right] + \mathcal{L}_{ct} . \tag{4}
\]

Here, the kaon mass is given by

\[ m_K^2 = 2Cm_s/f^2, \]

and we have employed the following definitions:

\[
\Theta^2 = \frac{2}{f^2} K^+ K^- ; \quad \gamma = \left( \frac{\sin \Theta}{\Theta} \right)^2 ; \quad \gamma \frac{1}{2} = \left( \frac{\sin \frac{1}{2} \Theta}{\frac{1}{2} \Theta} \right)^2 \\
b = (1 + x) \frac{n}{4f^2} ; \quad c = (a_1 x + a_2 + 2a_3) \frac{m_s n}{f^2} . \tag{5}
\]

Expectation values of nucleon operators are evaluated in the mean-field approximation. The total nucleon number density is \( n = n_p + n_n \) and the proton fraction is \( x = n_p/n \). Note that since we shall use a non-relativistic model for the nucleons we have not distinguished here between the scalar density and the number density in the expression for \( c \). The nucleon kinetic energy in \( \mathcal{L}^{(1)} \) will be considered in Subsection C and so is not included in Eq. (4).

We have not included in Eq. (4) terms which simply give a constant shift to the baryon masses; they indicate that \( a_1 m_s = -67 \text{ MeV} \) and \( a_2 m_s = 134 \text{ MeV} \), using the hyperon-nucleon mass differences. The remaining constant \( a_3 m_s \) is not accurately known, and we shall use values in the range \(-134 \) to \(-310 \text{ MeV} \) corresponding to 0 to 20% strangeness content for the proton. The corresponding range for the kaon-nucleon sigma term,

\[
\Sigma^{KN} = -\frac{1}{2} (a_1 + 2a_2 + 4a_3) m_s , \tag{6}
\]

is 167–520 MeV. Some guidance is provided by recent lattice gauge simulations \[23\], which find that the strange quark condensate in the nucleon is large, \( i.e. , \langle N | \bar{s}s | N \rangle = 1.16 \pm 0.54 \).

From the relation

\[
m_s \langle \bar{s}s \rangle_p = -2(a_2 + a_3)m_s \]

and using \( m_s = 150 \text{ MeV} \), we obtain \( a_3 m_s = -(220 \pm 40) \text{ MeV} \), which is in the middle of our range of values.

In order to determine the kaon partition function we generalize the finite temperature procedure outlined by Kapusta \[24\] and Benson, Bernstein and Dodelson \[25\]. First, by studying the invariance of the Lagrangian under the transformation \( K^\pm \rightarrow K^\pm e^{\pm i\alpha} \), the conserved current density can be identified. The zero component, \( i.e. \) the charge density, is
Next, we transform to real fields $\phi_1$ and $\phi_2$.

$$K^\pm = (\phi_1 \pm i\phi_2)/\sqrt{2}. \quad (8)$$

The Lagrangian becomes

$$L_K = \frac{1}{2}A\partial_\nu \phi_1 \partial^\nu \phi_1 + \frac{1}{2}B\partial_\nu \phi_2 \partial^\nu \phi_2 + C\partial_\nu \phi_1 \partial^\nu \phi_2$$

$$+ \gamma_2 \left[ b\phi_1 \partial_\tau \phi_2 - \frac{1}{2}(m_K^2 + c)(\phi_1^2 + \phi_2^2) \right] + L_{ct}, \quad (9)$$

where

$$A = \frac{\phi_1^2 + \gamma\phi_2^2}{\phi_1^2 + \phi_2^2}; \quad B = \frac{\phi_2^2 + \gamma\phi_1^2}{\phi_1^2 + \phi_2^2}; \quad C = \frac{(1 - \gamma)\phi_1\phi_2}{\phi_1^2 + \phi_2^2}. \quad (10)$$

Since, to our order of approximation, the counterterm Lagrangian, $L_{ct}$, is to be evaluated with constant mean fields it plays no role in determining the conjugate momenta. These are

$$\pi_1 = A\partial_0 \phi_1 + C\partial_0 \phi_2 - \gamma_2 b\phi_2$$

$$\pi_2 = B\partial_0 \phi_2 + C\partial_0 \phi_1 + \gamma_2 b\phi_1. \quad (11)$$

The Hamiltonian density is $H_K = \pi_1 \partial_0 \phi_1 + \pi_2 \partial_0 \phi_2 - L_K$, and the partition function of the grand canonical ensemble can then be written as the functional integral

$$Z_K = \int [d\pi_1][d\pi_2] \int_{\text{periodic}} [d\phi_1][d\phi_2]$$

$$\times \exp \left\{ \int_0^\beta d\tau \int d^3x \left[ i\pi_1 \frac{\partial \phi_1}{\partial \tau} + i\pi_2 \frac{\partial \phi_2}{\partial \tau} - H_K(\phi_i, \pi_i) + \mu j_0(\phi_i, \pi_i) \right] \right\}. \quad (12)$$

Here $\mu$ is the chemical potential associated with the conserved charge density, $\beta = T^{-1}$ is the inverse temperature and the fields obey periodic boundary conditions in the imaginary time $\tau = it$, namely $\phi_i(x, 0) = \phi_i(x, \beta)$.

The Gaussian integral over momenta in Eq. (12) can be performed yielding

$$Z_K = N \int [d\phi_1][d\phi_2] \gamma_2^\frac{1}{2} e^{-S} \quad \text{with}$$

$$S = \frac{1}{2} \int_0^\beta d\tau \int d^3x \left\{ A \left( \frac{\partial \phi_1}{\partial \tau} \right)^2 + B \left( \frac{\partial \phi_2}{\partial \tau} \right)^2 + 2C \frac{\partial \phi_1}{\partial \tau} \frac{\partial \phi_2}{\partial \tau} \right.$$\n$$+ A(\nabla \phi_1)^2 + B(\nabla \phi_2)^2 + 2C \phi_1 \cdot \nabla \phi_2 + 2i(\gamma \mu + \gamma_2 b)\phi_2 \frac{\partial \phi_1}{\partial \tau}$$\n$$+ \left[ \frac{1}{2}(m_K^2 + c - 2\mu b) - \gamma \mu^2 \right] (\phi_1^2 + \phi_2^2) - 2L_{ct} \right\}. \quad (13)$$
The dependence of the action \((13)\) upon the chemical potential \(\mu\) can also be obtained by gauging the fields as discussed in Appendices A and B.

Next the fields are decomposed into constant condensate parts and fluctuations according to

\[
\begin{align*}
\phi_1 &= \bar{\phi}_1 + \phi_1' = f \theta \cos \alpha + \sqrt{\frac{\beta}{V}} \sum_{n,p} e^{i(p \cdot x + \omega_n \tau)} \phi_{1,n}(p) \\
\phi_2 &= \bar{\phi}_2 + \phi_2' = f \theta \sin \alpha + \sqrt{\frac{\beta}{V}} \sum_{n,p} e^{i(p \cdot x + \omega_n \tau)} \phi_{2,n}(p) .
\end{align*}
\]

Here \(V\) denotes the volume of the system. In the Fourier expansion of the fluctuating part \(\phi_{1,0}(p = 0) = \phi_{2,0}(p = 0) = 0\) and the Matsubara frequency \(\omega_n = 2\pi n T\). Since we wish to go to one loop order, \(\mathcal{L}_{ct}\) is evaluated with constant condensate fields and we write the contribution as a potential \(V_{ct}\). The remainder of \(S\) must be expanded to second order in the fluctuations. The partition function can then be written in the form

\[
Z_K = Ne^{-\beta V(V_{tree} + V_{ct})} \int \prod_{n,p} [d\phi_{1,n}(p)][d\phi_{2,n}(p)] \gamma^\mp e^{-S_{loop}} .
\]

Introducing the definitions

\[
\begin{align*}
\epsilon(\theta) &= 2(m_K^2 + c - 2\mu b) f^2 \sin^2 \frac{1}{2} \theta - \frac{1}{2} \mu^2 f^2 \sin^2 \theta \\
\rho(\theta) &= -\frac{\partial \epsilon}{\partial \mu} = 4bf^2 \sin \frac{1}{2} \theta + \mu f^2 \sin^2 \theta ,
\end{align*}
\]

the tree potential is

\[
V_{tree} = \epsilon(\theta) .
\]

The one loop contribution to the action then becomes

\[
S_{loop} = \frac{1}{2} \sum_{n,p} \left( \phi_{1,-n}(-p), \phi_{2,-n}(-p) \right) D \left( \begin{array}{c} \phi_{1,n}(p) \\ \phi_{2,n}(p) \end{array} \right) , \quad \text{where}
\]

\[
\begin{align*}
\beta^{-2} f^2 D_{11} &= Af^2(\omega_n^2 + p^2) - \omega_n \theta \left( \frac{\rho}{\theta^2} \right)' \sin 2\alpha + \epsilon'' \cos^2 \alpha + \epsilon' \frac{\sin^2 \alpha}{\theta} \\
\beta^{-2} f^2 D_{22} &= Bf^2(\omega_n^2 + p^2) + \omega_n \theta \left( \frac{\rho}{\theta^2} \right)' \sin 2\alpha + \epsilon'' \sin^2 \alpha + \epsilon' \frac{\cos^2 \alpha}{\theta} \\
\beta^{-2} f^2 D_{12} &= C f^2(\omega_n^2 + p^2) + 2\omega_n \left( \frac{\rho}{\theta^2} + \theta \cos^2 \alpha \left( \frac{\rho}{\theta^2} \right)' \right) + \left( \epsilon'' - \frac{1}{\theta} \epsilon' \right) \sin \alpha \cos \alpha \\
\beta^{-2} f^2 D_{21} &= C f^2(\omega_n^2 + p^2) - 2\omega_n \left( \frac{\rho}{\theta^2} + \theta \sin^2 \alpha \left( \frac{\rho}{\theta^2} \right)' \right) + \left( \epsilon'' - \frac{1}{\theta} \epsilon' \right) \sin \alpha \cos \alpha ,
\end{align*}
\]
with a prime denoting a partial derivative with respect to $\theta$. Here the quantities $A$, $B$ and $C$ of Eq. (10) are to be evaluated with the condensate fields, giving

$$A = 1 + (\gamma - 1) \sin^2 \alpha ; \quad B = 1 + (\gamma - 1) \cos^2 \alpha ; \quad C = (1 - \gamma) \sin \alpha \cos \alpha , \quad (19)$$

and $\gamma = (\sin \theta/\theta)^2$. The latter replacement should also be made in the measure of the integral (15), to the order considered. Then the integration can be carried out and the grand potential $\Omega_K$ is given by

$$\frac{\Omega_K}{V} = -\frac{1}{\beta V} \ln Z_K = V_{\text{tree}} + V_{\text{ct}} + \frac{1}{2\beta V} \sum_{n,p} \ln \frac{1}{\gamma} \det D . \quad (20)$$

Evaluating the determinant yields

$$\frac{1}{\gamma} \det D = \beta^4 \left[ (\omega_n^2 + p^2)^2 + 2\omega_n^2 (c_1 + c_2 + c_3) + 2p^2 (c_2 + c_3) + 4c_2c_3 \right]$$

$$\equiv \beta^2 (\omega_n^2 + E_+^2) \cdot \beta^2 (\omega_n^2 + E_-^2) , \quad (21)$$

where

$$c_1 = \frac{2\rho}{f^2 \gamma \theta^2 \frac{\partial}{\partial \theta} \left( \frac{\rho}{\theta} \right)} , \quad c_2 = \frac{1}{2f^2 \frac{\partial^2 \epsilon}{\partial \theta^2}} , \quad c_3 = \frac{1}{2f^2 \gamma \theta \frac{\partial \epsilon}{\partial \theta}} \quad (22)$$

$$E_\pm^2(p) = p^2 + c_1 + c_2 + c_3 \pm \sqrt{c_1(2p^2 + c_1 + 2c_2 + 2c_3) + (c_2 - c_3)^2} . \quad (23)$$

Notice that there is no longer any dependence on the condensate phase $\alpha$. The sum over the Matsubara frequencies in Eq. (20) can then be evaluated in the usual way [24] to yield

$$\Omega_K V^{-1} = V_{\text{tree}} + V_{\text{ct}} + V_{zp} + V_{\text{thermal}} \quad \text{with}$$

$$V_{zp} = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2} [E_+(p) + E_-(p)] \quad (24)$$

$$V_{\text{thermal}} = \beta^{-1} \int \frac{d^3p}{(2\pi)^3} \ln(1 - e^{-\beta E_+(p)})(1 - e^{-\beta E_-(p)}) . \quad (25)$$

We are interested in zero temperature here, so $V_{\text{thermal}}$ may be dropped.

The zero-point energy $V_{zp}$, or equivalently, the one-kaon-loop contribution to the effective potential [26], can be represented as an infinite series of diagrams, some of which are shown in Fig. 1. Vertices are given by the Lagrangian (4). External legs represent nucleon or kaon mean fields.

It is interesting to note the chiral order of the diagrams contributing to the effective potential. Consider the scattering amplitudes corresponding to the diagrams of Fig. 1,
where the external lines are on their respective vacuum mass-shells. The chiral order, \( \nu \), of a given scattering amplitude is given by the Weinberg\(^{[27]} \) counting rule, according to which an amplitude involving \( E_N \) external nucleon lines and \( E_K \) external kaon lines is proportional to \( Q^\nu \), where \( Q \) is a characteristic small momentum scale involved in the process and

\[
\nu = 2 + 2L - \frac{1}{2}E_N + \sum_i \left( d_i + \frac{1}{2}n_i - 2 \right),
\]

(26)

where \( L \) is the number of loops, the sum is over all vertices \( i \), \( d_i \) the number of derivatives that act on the \( i \)th vertex and \( n_i \) the number of nucleon lines attached to the \( i \)th vertex. In this case Fig. 1a gives \( \nu = 4 \). Furthermore, it easy to convince oneself that adding a vertex from \( \mathcal{L}^{(1)} \), corresponding to the \( b \) term in Eq. (4), will decrease \( \nu \) by 1. On the other hand, adding a vertex from \( \mathcal{L}^{(2)} \), corresponding to the kinetic, mass, or \( c \) terms in Eq. (4) leaves \( \nu \) invariant. The fact that these scattering amplitudes give \(-\infty < \nu \leq 4\) due to the \( \mathcal{L}^{(1)} \) interaction might appear to be grounds for concern. However, exactly the same apparent counting problem arises when one considers the computation of the tree-level in-medium kaon propagator. The correct in-medium propagator is indeed obtained by summing an infinite series\(^{[28]} \) in diagrams that here correspond to amplitudes with decreasing powers of \( \nu \). The evaluation of the effective potential is entirely analogous. We conclude that while Eq. (26) provides an effective way to organize terms in order of importance for a given scattering process, it cannot be naively applied in the same way to the effective potential in the case at hand.

**B. Zero-point Divergences**

As shown in Appendices A and B, \( V_{ct} \) cancels the ultraviolet divergences in \( V_{zp} \). For our purposes the zero-point energy is most conveniently written

\[
V_{zp} = \sqrt{\frac{1}{2} \int \frac{d^3p}{(2\pi)^3} \left( p^2 + c_1 + c_2 + c_3 + \sqrt{(p^2 + 2c_2)(p^2 + 2c_3)} \right)^\frac{1}{2}}.
\]

(27)

We regularize the infinities that occur here by introducing an ultraviolet cutoff \( \Lambda \) and requiring \( |p| \leq \Lambda \). (Dimensional regularization is discussed in Appendix B.) The counterterm potential \( V_{ct} \) following from \( \mathcal{L}_{ct} \) (see Appendix B) is
\[ V_{ct} = -(d_4 + f_4) - \frac{1}{2}(c_1 + 2c_2 + 2c_3)(d_2 + f_2) + \frac{1}{2}[(c_2 - c_3)^2 + \frac{1}{2}(c_1 + 2c_2 + 2c_3)^2](d_0 + f_0). \]  

(28)

Here we have introduced finite terms \( f_0, f_2 \) and \( f_4 \) as well as the quartically, quadratically and logarithmically divergent integrals

\[ d_4 = \int_{k \leq \Lambda} \frac{d^3k}{(2\pi)^3} k = \frac{\Lambda^4}{8\pi^2} \]  

(29)

\[ d_2 = \int_{k \leq \Lambda} \frac{d^3k}{(2\pi)^3} \frac{1}{2k} = \frac{\Lambda^2}{8\pi^2} \]  

(30)

\[ d_0 = \int_{k \leq \Lambda} \frac{d^3k}{(2\pi)^3} \frac{1}{4k^3} = \frac{1}{8\pi^2} \ln \left( \frac{\Lambda}{\kappa} \right). \]  

(31)

Here \( \kappa \) is an arbitrary scale introduced so that the argument of the logarithm is dimensionless. Note that the quadratically and logarithmically divergent pieces of \( V_{zp} \) in (27) and \( V_{ct} \) in (28) are individually dependent upon the chemical potential \( \mu \). This dependency on the chemical potential \( \mu \) arises from the functions of the kaon fields multiplying the kinetic terms in Eq. (4). However, \( \mu \)-dependent divergences in the terms \( V_{zp} \) and \( V_{ct} \) cancel, as do the \( \mu \)-independent divergences of the vacuum theory. The cancellation of \( \mu \)-dependent divergences follows from a careful treatment of the counterterm lagrangian which regulates the vacuum theory. Further discussion of this point is given in Appendices A and B. Here we note that \( \mu \) may be considered a constant external vector potential \( A_\nu \), with vanishing spatial components. The general statement being made is that all Green functions with constant external \( A_\nu \) fields are finite. It is easy to check this for simple examples related to our theory, such as scalar electrodynamics. The fact that it also holds for a complicated nonlinear theory expanded about an arbitrary point \( \phi_1, \phi_2 \) in function space is almost certainly a consequence of the Ward-Takahashi identities associated with the U(1) symmetry transformation preceding Eq. (7).

In order to determine the finite contributions \( f_i \) we apply three conditions in the vacuum. First \( \Omega_K \), or equivalently the pressure, should be zero in the vacuum and, since there is no condensate, the loop contribution must vanish. Secondly, we require the zero-momentum propagator to be of the usual form \((\omega^2 - m_K^2)^{-1}\), see Appendix C. Thirdly we require that the four-point vertex \((K^+K^-)^2/(6f^2)\) of Eq. (C3) be unmodified in vacuum. These conditions imply
\[ V_{zp} + V_{ct} \big|_{\text{vacuum}} = 0 \]
\[ \frac{\partial^2}{\partial \theta^2} (V_{zp} + V_{ct}) \big|_{\text{vacuum}} = 0 \]
\[ \frac{\partial^4}{\partial \theta^4} (V_{zp} + V_{ct}) \big|_{\text{vacuum}} = 0 . \] (32)

Expanding to fourth order in \( \theta \), one finds
\[ f_4 = \frac{m_K^4}{64 \pi^2} ; \quad f_2 = -\frac{m_K^2}{16 \pi^2} \]
\[ f_0 = \frac{1}{8 \pi^2} \left( \ln \frac{\kappa}{m_K} + \zeta - \frac{3}{4} \right) , \] (33)

where using our three momentum cutoff, the constant \( \zeta = \ln 2 - 1/4 = 0.443 \); interestingly a closely similar value for the constant is obtained with dimensional regularization, namely \( \zeta = \frac{1}{2} \Psi(3) = 0.461 \) (Here \( \Psi \) represents the digamma function \([29]\)). Notice that the arbitrary scale \( \kappa \) only occurs in the combination \( (d_0 + f_0) \) where it cancels out. The disappearance of the arbitrary scale is one justification for introducing the finite counterterms, we discuss another below.

In evaluating the grand potential for the general case numerical integration of Eq. (27) for \( V_{zp} \) is required. However below threshold when \( \theta = 0 \) (or just at threshold when \( \theta \) is vanishingly small) the expressions simplify greatly and an analytic form can be obtained. Explicitly we find \( c_1 = 2(b + \mu)^2, c_2 = c_3 = \frac{1}{2}(m_K^2 + c - 2 \mu b - \mu^2) \), and eigenvalues
\[ E_{\pm} = \sqrt{p^2 + \xi^2} \pm (b + \mu) \quad \text{with} \quad \xi^2 = b^2 + m_K^2 + c \quad \text{for} \quad \theta = 0 . \] (34)

Notice that \( (E_{\pm} \mp \mu) \) is simply the tree level energy of a \( K^\pm \) meson. In this limit the one loop energy becomes
\[ V_{zp} + V_{ct} = \int \frac{d^3p}{(2\pi)^3} \sqrt{p^2 + \xi^2} - (d_4 + f_4) - \xi^2(d_2 + f_2) + \frac{1}{2} \xi^4(d_0 + f_0) \]
\[ = -\frac{1}{64 \pi^2} \left[ (m_K^2 - 3 \xi^2)(m_K^2 - \xi^2) + 2 \xi^4 \ln \frac{m_K^2}{\xi^2} \right] \quad \text{for} \quad \theta = 0 , \] (35)

which vanishes in the vacuum since \( \xi = m_K \) there.

C. Nucleon Hamiltonian

To describe the nuclear kinetic energy and interactions we employ the non-relativistic approach suggested by Prakash, Ainsworth and Lattimer \([30]\), which has been rather widely
employed, e.g. Refs. [4,13]. Here the Hamiltonian density is written

\[ \mathcal{H}_N = \frac{3}{8} E_F^0 u^3 n_0 + V(u) + n(1 - 2x)^2 S(u) , \]  

(36)

where nuclear saturation density is denoted by \( n_0 \) (\( n_0 = 0.16 \text{fm}^{-3} \)) and the corresponding Fermi momentum and energy are \( p_F^0 = (3\pi^2 n_0/2)^{1/3} \) and \( E_F^0 = (p_F^0)^2/2M \), respectively. The potential contribution to the energy density of symmetric nuclear matter is denoted by \( V(u) \), where \( u = n/n_0 \) and it is given by

\[ V(u) = \frac{1}{2} A u^2 n_0 + \frac{Bu^{\sigma+1}n_0}{1+B'u^{\sigma-1}} + 3un_0 \sum_{i=1,2} C_i \left( \frac{\Lambda_i}{p_F^0} \right)^3 \left( p_F - \arctan \frac{p_F}{\Lambda_i} \right) . \]  

(37)

Here \( p_F \) is the Fermi momentum which is related to \( p_F^0 \) by \( p_F = p_F^0 u^{1/3} \). The remaining quantities are parameters which are given in Refs. [13,30]. We choose the set which yields a compression modulus \( K = 240 \text{ MeV} \) for equilibrium nuclear matter since this is in the middle of the generally accepted range of \( 200 - 300 \text{ MeV} \). The remaining term in Eq. (36) is the symmetry energy, i.e. it describes the change in the energy density when the proton fraction \( x = n_p/n \) differs from \( \frac{1}{2} \). Again following Ref. [30] we take

\[ S(u) = \left( \frac{2}{3} - 1 \right) \frac{3}{5} E_F^0 \left( u^2 - F(u) \right) + S_0 F(u) , \]  

(38)

where \( S_0 \simeq 30 \text{ MeV} \) is the bulk symmetry energy parameter. Since neutron star properties are not strongly dependent on the form of \( F(u) \), we choose the simplest of the suggested parametrizations namely \( F(u) = u \).

In terms of the Hamiltonian density, the grand potential for nucleons is

\[ \frac{\Omega_N}{\mathcal{V}} = \mathcal{H}_N - \mu_n n + x \mu n \, , \]  

(39)

where \( \mu = \mu_n - \mu_p \) and the chemical potential \( \mu_n \) is the Fermi energy for neutrons.

**D. Equilibrium Conditions**

We shall study cold, catalyzed neutron stars which are in chemical equilibrium under beta decay processes. The process \( p + e^- \leftrightarrow n + \nu_e \) in equilibrium establishes the relation

\[ \mu \equiv \mu_n - \mu_p = \mu_e \, , \]  

(40)
since the neutrinos have left the star and their chemical potential is therefore zero. At
densities where $\mu$ exceeds the muon mass $m_\mu$, muons can be formed by $e^- \leftrightarrow \mu^- + \nu_\mu + \nu_e$,
hence the muon chemical potential is $\mu_\mu = \mu_e = \mu$. Negatively charged kaons can be formed
in the process $n + e^- \leftrightarrow n + K^- + \nu_e$ when $\mu_{K^-}$ becomes equal to the energy of the lowest
eigenstate of a $K^-$ in matter. Chemical equilibrium requires that $\mu_{K^-} = \mu$, as we have
implicitly assumed in the text.

To complete our grand potential we need the lepton component for which it is sufficient
to use the standard non-interacting form:

$$\frac{\Omega_L}{V} = -\frac{\mu^4}{12\pi^2} + \eta(|\mu| - m_\mu) \left\{ \frac{m_\mu^4}{8\pi^2} \left[ (2t^2 + 1)t \sqrt{t^2 + 1} - \ln(t + \sqrt{t^2 + 1}) \right] - \frac{m_\mu^2|m||t^3}{3\pi^2} \right\}, \quad (41)$$

where $\eta(x)$ is the Heaviside function ($\eta(x) = 1$ if $x > 0$, $\eta(x) = 0$ if $x < 0$) and the quantity
$t = \sqrt{\mu^2 - m_\mu^2}/m_\mu$. The total grand potential is then

$$\Omega_{\text{total}} = \Omega_K + \Omega_N + \Omega_L. \quad (42)$$

The values of the chemical potential $\mu$, the proton fraction $x$ and the condensate amplitude
$\theta$ are determined by extremizing $\Omega_{\text{total}}$.

Extremizing with respect to $\mu$ is equivalent to the requirement of charge neutrality. It yields

$$0 = -n_K + n_p - n_L, \quad (43)$$

where $n_K$ and $n_L$ are the number densities for negatively charged kaons and leptons respec-
tively. The kaon charge density is

$$n_K = f^2(4b \sin^2 \frac\theta 2 + \mu \sin^2 \theta) - \frac{\partial(V_{zp} + V_{ct})}{\partial \mu}. \quad (44)$$

Notice that the fluctuations carry no net charge in the absence of a condensate, $i.e.$ $n_K = 0$
for $\theta = 0$. Here, and in the other cases, the zero point contribution requires a numerical
integration and it is straightforward, though tedious, to evaluate the integrand. The lepton
charge density is

$$n_L = \frac{\mu^3}{3\pi^2} + \eta(|\mu| - m_\mu) \frac{\mu}{|\mu|} \frac{m_\mu^2 t^3}{3\pi^2}. \quad (45)$$

Next extremizing with respect to $x$ gives
\[
0 = n(2a_1 m_s - \mu) \sin^2 \frac{1}{2} \theta + \frac{\partial(V_{zp} + V_{ct})}{\partial x} + n\mu - 4n(1 - 2x)S(u) .
\] (46)

Finally extremizing with respect to \( \theta \) yields
\[
0 = f^2 \sin \theta (m_K^2 + c - 2\mu b - \mu^2 \cos \theta) + \frac{\partial(V_{zp} + V_{ct})}{\partial \theta} .
\] (47)

Below the threshold for kaon condensation \( \theta = 0 \) and this solution is always possible since the expression is an odd function of \( \theta \). Above threshold the solution \( \theta > 0 \) minimizes the grand potential. At threshold, where \( \theta \) is vanishingly small, it is possible to obtain an analytic expression by retaining terms through order \( \theta^2 \) in \( \Omega_K \). The extremization condition becomes
\[
0 = (m_K^2 + c)(1 + \delta) - 2b\mu(1 - 2\delta) - \mu^2(1 + \delta) \quad \text{where} \quad \delta = \frac{1}{48\pi^2 f^2} \left( \xi^2 - m_K^2 + \xi^2 \ln \frac{m_K^2}{\xi^2} \right) .
\] (48a)

This may alternatively be obtained by computing the inverse propagator, \( D^{-1}(\mu) \), to one loop order, see Appendix C. Notice that, by construction, the inverse propagator is \( (\mu^2 - m_K^2) \) in vacuum since \( b = c = \delta = 0 \) there. In fact for any density \( \delta \leq 0 \). That this is necessary may be seen as follows. The arguments of the inner square root in Eq. (27) is positive for all \( p \), only if \( c_2 = c_3 \) or both \( c_2 > 0 \) and \( c_3 > 0 \) if \( c_2 \neq c_3 \). However in our calculations we have found that going even slightly above threshold implies \( c_2 \neq c_3 \), so that it is the latter condition that must be satisfied. Now the above equation (48a) can be put in the form \( (1 + \delta)(c_2 + c_3) = -6b\mu\delta \). Since \( |\delta| \ll 1 \) (Sec.III), and \( b \) and \( \mu \) are positive, we must have \( \delta < 0 \). It is interesting to note that in the absence of finite counterterms the requirement \( \delta < 0 \) can only be achieved in a very limited region of parameter space.

Having satisfied the conditions (43), (46) and (47), the energy density can be obtained from
\[
E = \frac{\Omega_{total}}{V} + \mu_n n .
\] (49)

The chemical potential \( \mu_n \) may be obtained by functional differentiation of the energy density with respect to the neutron density, \( \text{viz. } \mu_n = \frac{\partial E}{\partial n_n} \), whence the pressure can be calculated from \( P = -\frac{\Omega_{total}}{V} \). Equivalently one can write \( P = n(\partial E/\partial n) - E \).
III. EVALUATION OF KAON LOOP CONTRIBUTION

In this section, we give numerical estimates for the effect of the loop corrections upon the critical density for kaon condensation and upon the properties of neutron star matter both above and below the condensate threshold.

As follows from Eqs. (48) and (C1), the critical density for kaon condensation is the density at which the lepton chemical potential $\mu$ (an increasing function of density), and the position of the zero-momentum $K^-$ pole of the kaon propagator in normal nuclear matter (a decreasing function of density) become equal. The chemical potential $\mu$ and the proton fraction $x$, prior to condensation, are obtained from Eqs. (43) and (46) by setting $\theta = 0$. The pole position of the kaon propagator is a solution to the quadratic equation $D^{-1}(\omega) = 0$, where $D^{-1}(\omega)$ is given in Eq. (C4).

In Table I, we list the critical density for kaon condensation in units of nuclear matter density, $u_{cr} = n_{cr}/n_0$, for three representative values of the parameter $a_3m_s$. In column 2, $u_{cr}^t$ is the critical density obtained at tree level [13], corresponding to setting $\delta = 0$ in Eq. (48) and omitting the $V_{zp} + V_{ct}$ contributions from Eqs. (43) and (46). In column 3, the loop contribution is included and the corresponding critical density is denoted by $u_{cr}^{t+l}$. We see that $u_{cr}^{t+l} < u_{cr}^t$ and, while the magnitude of the threshold shift increases for larger values of $|a_3m_s|$, it is always small. This due to the fact that in Eq. (48) $|\delta| < m^2_K/(48\pi^2f^2) = 0.06$ which is small in comparison to unity. It is rather easy to show that $u_{cr}^{t+l} < u_{cr}^t$. The tree level inverse propagator $(D^t(\mu))^{-1} = -(c_2 + c_3) = 0$ at threshold, while with loops included one has $-(1 + \delta)(c_2 + c_3) = 6b\mu\delta$. Since $D^t(\mu)$ is a monotonic function of density and we have pointed out that $\delta < 0$, it follows that $u_{cr}^{t+l} < u_{cr}^t$.

The change in the threshold due to loops, small though it is, is dominated by the $\Sigma^{KN}$ term. This is shown in column 4 of Table I for which we set $c = 0$ in the loop expressions. In this case the threshold is essentially unchanged from tree level. The effect of the $c$ term can become quite critical at densities above threshold and in Eq. (3) the scalar density $n_s = \langle \bar{N}N \rangle$ has been replaced by the baryon density $n = \langle N^\dagger N \rangle$. In Ref. [1] it was argued that in the heavy baryon limit, and to the chiral order considered, no distinction should be made between the operators $\bar{N}N \bar{N}^\dagger N$ in the Lagrangian (I). Corrections to this would, in principle, require other effects of similar order for consistency. While $n_s \simeq n$ for $n \sim n_0$,
at high density $n_s$ becomes less than $n$. The results of Ref. [31] indicate that this can be significant, in particular it leads to an increase in $u_{cr}$. Since our results are sensitive to this question, in some of our calculations we shall replace the baryon density $n$ in the quantity $c$ of Eq. (3) by a phenomenological form of $n_s$. Based on a simple parameterization of the effective nucleon mass obtained by Serot and Walecka [32], we use

$$n_s = n \frac{2.5213}{u} \left( 1 - \frac{1}{1 + \exp\left(\frac{u^{1/3} - 1.1717}{0.4417}\right)} \right)$$  \hspace{1cm} (50)$$

for $u \geq 1.0342$ and set $n_s = n$ for $u < 1.0342$. This is extremely crude, but it will give some qualitative feel for the effect. The results for the critical density with this approach, labelled $c_{n_s}$, are shown in columns 5 and 6 of Table I. The shift in the critical density between tree and tree + loop levels is slightly smaller, but basically similar to that obtained before. The overall increase in the critical density is expected, but is somewhat larger than models which treat nucleons relativistically [11,18] would suggest.

Even below threshold there is a kaon zero-point contribution for normal neutron star matter. Firstly there is a shift in the minimum due to the zero point contribution to the equilibrium Eq. (46) at $\theta = 0$. Secondly there is the zero point energy (35) itself. The shift in the proton fraction $x$ and the lepton chemical potential $\mu$ is negligible. Table II shows that $\varepsilon_{zp}$, the zero-point energy density is approximately 1–3% of the total. Thus the matter properties are only affected to a small degree so that it is not necessary to refit the parameters of the nuclear Hamiltonian.

Above the critical density the properties of matter are obtained by solving Eqs. (13), (46), and (47) numerically. In practice, one uses a three-momentum cutoff $\Lambda$ to evaluate Eq. (27) and derivatives thereof. The cutoff used was $\Lambda = 3 \times 10^4$ MeV/c, and results were found to be extremely stable with respect to variation in $\Lambda$. For example taking $\Lambda = 1 \times 10^4$ MeV/c, the largest change in any of the tabulated values was 0.3%.

In Table III, we list the properties of normal neutron star matter (column 2) and matter containing a kaon condensate at tree level (columns 3–7) and at one-loop level (columns 8–13) for $a_3m_s = -134$ MeV. Here the kaon fraction is defined as $x_K = n_K/n$ in terms of the kaon density $n_K$ of Eq. (44), and the lepton fraction $x_L = n_L/n$ may be obtained from $x_L = x - x_K$. The energy density is also listed. We have not felt it worthwhile to tabulate the remaining thermodynamic variable, the pressure, in view of the small size of the loop
effects. For \( u \gtrsim 9.03 \) the parameter \( c_3 \) becomes negative which precludes real solutions to the zero point energies; we discuss this further below. In Table IV, we give the corresponding quantities for \( a_3 m_s = -222 \) MeV, in which case \( c_3 \) becomes negative for \( u \gtrsim 5.07 \). This difficulty does not arise if we employ the phenomenological scaling given by Eq. (50) and the results are given for \( a_3 m_s = -222 \) MeV in Table V. If one simply sets \( c = 0 \) in the loop calculations the results shown in Fig. 2 are obtained. In this figure \( \Delta \epsilon \) represents the energy density gained by the formation of a condensate and is the difference between the total energy density and the tree level energy density for normal nuclear matter.

We first note that loop effects for the case \( a_3 m_s = -134 \) MeV are essentially negligible. Although they increase in magnitude for larger values of \( |a_3 m_s| \), only small modifications of the tree level results are found. Table IV shows that just above threshold the tree + loop calculation gives values of \( \theta \) and \( x \) which are greater than, and \( \mu \) which are less than, the corresponding tree level results. This naturally follows from the fact that \( u_{cr}^t \) is less than \( u_{cr}^l \), since, once a condensate is present, \( \theta \) rises rapidly and \( x \) is enhanced, while \( \mu \) is reduced. At high densities, this ordering is reversed. Near the limiting density, \( u \simeq 5.07 \), \( \theta \) typically decreases by \( \sim 3\% \) while \( x \) decreases somewhat less, \( \sim 1\% \), and \( \mu \) increases \( \sim 6\% \). The zero-point energy density is of the order \( -(1.2 - 3.5) \) MeV fm\(^{-3} \) and can decrease the tree level result significantly at high density when the latter can be small. For a given density the effects are smaller if the scalar density of Eq. (50) is employed as shown in Table V. On the other hand, setting \( c = 0 \) altogether as in Fig. 2 yields larger effects at very high density. For \( a_3 m_s = -310 \) MeV, the above trends are also present, and are enhanced, as expected. The coefficient \( c_3 \) goes to zero for \( u \gtrsim 3.32 \), at which point \( \epsilon_{zp} \simeq -7.2 \) MeV fm\(^{-3} \) and \( \epsilon \simeq -8.5 \) MeV fm\(^{-3} \).

As mentioned above, \( c_3 \) often vanishes at some density above the critical density. The condition \( c_3 = 0 \) implies \( d\epsilon/d\theta = 0 \), where \( \epsilon \) is the tree level energy density function of Eq. (17). The significance of this condition is best illustrated by the simple \( O(2) \) invariant Lagrangian \( \mathcal{L} = \frac{1}{2}(\partial_\nu \phi_1)^2 + \frac{1}{2}(\partial_\nu \phi_2)^2 - U(\phi) \), where \( \phi^2 = \phi_1^2 + \phi_2^2 \). The one-loop effective potential \( U_l(\phi) = i \text{Tr} \ln((k^2 + U''(\phi))(k^2 + \frac{U'}{\phi})) \), where the trace is over momentum. Now take \( U(\phi) = \frac{1}{2}m^2 \phi^2 + \lambda \phi^4 \), with \( \lambda > 0 \). For \( m^2 > 0 \), \( U_l \) is real for all \( \phi \). On the other hand, for \( m^2 < 0 \), the "Mexican hat" potential, the one-loop effective potential is complex in
the region $U' < 0$ which signals a physical instability. Weinberg and Wu [33] show that the imaginary part can be interpreted as a decay rate per unit volume of a well-defined state. The connection between the instability and the chaotic dynamics of a classical field theory has recently been discussed by Matinyan and Müller [34], see also references therein.

Unfortunately no solution is known to this general problem. Our case is entirely analogous. At threshold $c_2 = c_3 > 0$ corresponds to $m^2 > 0$. As the density is increased, the tree level potential eventually corresponds to $m^2 < 0$. This effect is more pronounced for larger values of $|a_3 m_s|$. Since the effective potential is by definition real, this shows that some, as yet unknown, technique is needed in these regions of parameter space. The two other cases we have considered, excluding $c$ from the loop contribution and employing the softer density dependence of $c$ with the parametrized scalar density $n_s$ alleviate this problem.

**IV. CONCLUSION**

We have derived the zero-point energy due to fluctuations in kaonic modes in neutron star matter containing a kaon condensate. To our knowledge, this calculation is the first in which the nonlinearities of the chiral Lagrangian have been taken into account at the one loop level for densities above the critical density for kaon condensation. The zero-point energy was derived through the thermodynamic partition function by considering fluctuations around an arbitrary condensate. We have demonstrated by gauging the counterterms of the vacuum theory that no divergences are present when matter is introduced via the charge chemical potential. In the usual way the total thermodynamic potential, including the classical contribution, is extremized so that the minimum energy ground state is found for locally charge neutral and beta-equilibrated matter.

The critical density for kaon condensation was found to be reduced by fluctuations, but only by less than 1% for the range of parameters considered. At higher densities the loop effects were negligible for $a_3 m_s = -134$ MeV which corresponds to zero strangeness content for the nucleon. The effects were slightly increased with larger values of $|a_3 m_s|$, corresponding to larger value of the kaon nucleon sigma term $\Sigma^{KN}$, but they remained small in comparison to the dominant tree level contributions. The size of the loop effects relative to the tree contributions is set, at least in the threshold region, by the parameter $\delta$ which is
less than $m_K^2/(48\pi^2 f^2) = 0.06$. In some cases it was not possible to obtain solutions up to arbitrary density because the effective potential became complex—this is a well-known and unsolved problem in one loop calculations with spontaneous symmetry breaking.

The fact that changes induced by the zero-point motion are generally small leads one first to conclude that no major revision is needed of earlier investigations in which such fluctuations have not been taken into account. Second, the loop expansion appears to be convergent where it is well defined. The formalism presented here yielded the contribution to the grand potential arising from thermal excitations of the fluctuation modes. These effects would have to be included at finite temperatures and would be involved in the extremization conditions. Such an analysis is needed for a consistent study of newborn neutron stars. Extension of the present work to include the full octet of pseudoscalar fluctuations, and to include additional terms in the chiral expansion could also be contemplated.

ACKNOWLEDGMENTS

We wish to thank P. B. Arnold, A. Fayyazuddin, U. van Kolck, and L. Yaffe for helpful discussions, and D. B. Kaplan, M. Rho, R. Venugopalan and A. Wirzba for valuable comments on the manuscript. We are grateful to M. Prakash for stimulating our interest in this problem and for careful reading of the manuscript. This work was supported in part by the U. S. Department of Energy under Grant DE-FG06-88ER40427.

APPENDIX A: CANCELLATION OF CHEMICAL POTENTIAL DIVERGENCES IN A TOY MODEL

In this Appendix, we illustrate how divergences dependent on the charge chemical potential $\mu$ are cancelled in a toy model which contains two real fields, $\phi_1$ and $\phi_2$. The model we consider is

$$\mathcal{L} = \mathcal{L}_\gamma + \mathcal{L}_{ct} \quad \text{where} \quad \mathcal{L}_\gamma = \frac{1}{2} \gamma \left[ (\partial_\nu \phi_1)^2 + (\partial_\nu \phi_2)^2 \right], \quad (A1)$$

and $\gamma$ is an arbitrary analytic function of $\phi^2 = (\phi_1^2 + \phi_2^2)$ with $\gamma(0) = 1$. It is useful to discuss this model to one loop order since it has a simpler structure than the chiral Lagrangian,
the $\mu$-dependent divergences are removed in a similar way. We first derive $L_{ct}$, and then

demonstrate that on introducing an external chemical potential $\mu$, the effective potential
(or equivalently, the zero-point energy) is free of divergences. Finite counterterms are not
relevant in the present context and will be ignored.

Expanding the function $\gamma$ to all orders in the fields gives the standard kinetic energy
term plus an infinite set of interaction terms involving an even number of fields and two
derivatives. From these interaction vertices, we may compute one-particle irreducible $2n$-
point Green functions to one-loop order. The counterterm Lagrangian $L_{ct}$ is defined in such
a way as to minimally cancel divergences.

Consider first quadratic divergences, proportional to $d_2$ (defined in Eq. (30)). The 4-
point vertex contributes a quadratic divergence to the 2-point function proportional to $d_2$
and the external momenta. The corresponding counterterm Lagrangian is proportional to
$d_2[(\partial_\nu \phi_1)^2 + (\partial_\nu \phi_2)^2]$. The 6-point vertex contributes to the 4-point function so as to require a
counterterm Lagrangian proportional to $d_2 \phi^2[(\partial_\nu \phi_1)^2 + (\partial_\nu \phi_2)^2]$. Continuing this sequence,
we find that the contribution of quadratic divergences to the counterterm Lagrangian is
$-d_2/(4\gamma) \cdot (\gamma'' + \gamma'/\phi)[(\partial_\nu \phi_1)^2 + (\partial_\nu \phi_2)^2]$, where primes denote derivatives with respect to $\phi$.

Now consider logarithmic divergences, proportional to $d_0$ (see Eq. (31)). Green functions
containing such divergences typically contain two vertices, and evaluating them requires
involved combinatorics. It is simpler instead to derive the counterterm Lagrangian using
the effective potential (see e.g. Ref. [26]). One thus avoids combinatorics, and the method
is more easily applied to Lagrangians more complicated than Eq. (A1).

Let

$$ \phi_1 \to \phi_1 + \phi'_1, \quad \phi_2 \to \phi_2 + \phi'_2, $$

(A2)

where $\phi_1$ and $\phi_2$ are arbitrary fields which, for the present, may vary in spacetime. Expanding
to quadratic order in the fields $\phi'_1$ and $\phi'_2$ leads to

$$ L_\gamma \to L_\gamma + \frac{1}{2}(\phi'_1 \phi'_2)D \left( \begin{array}{c} \phi'_1 \\ \phi'_2 \end{array} \right), $$

(A3)

where the matrix of second order variations is given by

$$ D = -\gamma \Box 1 + L \cdot \partial - M, $$

(A4)
where, as before, the trace is taken both over spacetime and charge. The factor $-i \text{Tr} \ln \gamma$ is cancelled by a corresponding contribution from the measure, and will not be considered further. In the last expression, we have symmetrized the derivative contribution and separated the mass-like contributions involving $[2M + (\partial \cdot L)]$. We shall see that only the mass-like contributions are important for our discussion.

The trace over spacetime is evaluated by inserting complete sets of four-momentum eigenstates. A momentum expansion is performed on the resulting integrals, leading to a factorization into ultraviolet divergent loop integrals and integrals over functions of the fields. To minimally cancel the divergences in the effective potential we need the following counterterm Lagrangian

$$L_{ct} = L_{ct}^{\text{quartic}} + L_{ct}^{\text{mass-like}} + L_{ct}^{\text{derivative}},$$

where the contribution from the quartic divergence (29) is

$$\int d^4x \, L_{ct}^{\text{quartic}} = -i \text{Tr} \ln(-\Box) = \int d^4x \, d_4.$$

The contribution from mass-like terms is

$$\int d^4x \, L_{ct}^{\text{mass-like}} = \frac{i}{2} \sum_{k=1}^{\infty} \frac{1}{k} \text{Tr} \left( \frac{1}{-2\gamma \Box} [\partial \cdot L - L \cdot \partial + 2M + (\partial \cdot L)] \right)^k.$$
As before, the Euclidean time is \( \tau \) components zero. It enters by “gauging” the fields \( \phi \) to Euclidean space. The chemical potential acts as an external vector field with spatial components zero. It becomes the external vector field \( \mu \) with spatial components zero. At finite inverse temperature \( T = \frac{1}{\beta} \), the action \( L \) for one-loop order becomes

\[
-\frac{d_0}{4\gamma^2} \int d^4x \left\{ \left( \partial_\tau \phi_1 \right)^2 + \left( \partial_\tau \phi_2 \right)^2 \right\} \left( \left( \frac{\gamma''}{\gamma} \right)^2 + \left( \frac{\gamma'}{\gamma} \right)^2 \right) \\
-\frac{\gamma'}{\phi} \left[ \left( \partial_\tau \phi_1 \right)^2 + \left( \partial_\tau \phi_2 \right)^2 \right] \left[ \gamma'' \left( \partial_\tau \phi_1 \phi_2 - \phi_2 \partial_\tau \phi_1 \right) + \frac{\gamma'}{\phi} \left( \partial_\tau \phi_2 - \phi_2 \partial_\tau \phi_1 \right) \right] \\
+ \gamma'' \phi_2 \left( \phi_1 \phi_2 - \phi_2 \phi_1 \right) + \frac{\gamma'}{\phi} \left( \phi_1 \phi_2 - \phi_2 \phi_1 \right) \\
+ \left( \frac{\gamma' \phi_2}{\phi} \right)^2 \left[ \left( \partial_\tau \phi_1 \phi_2 \right)^2 + \left( \phi_1 \phi_2 - \phi_2 \phi_1 \right) \right] \\
+ 2 \left( \partial_\tau \phi_1 \right)^2 \phi_1 \phi_2 + \left( \partial_\tau \phi_2 \right)^2 \phi_1 \phi_2 \right\} . \tag{A10}
\]

For the contribution from derivatives one finds (see also Refs. [35,36])

\[
\int d^4x L_{ct}^{\text{derivative}} = \frac{i}{2} \sum_{k=2}^4 \frac{1}{k} \text{Tr} \left( \frac{1}{2\gamma} \left( L \cdot \nabla - \nabla \cdot L \right) \right)^k \\
= -\frac{d_0}{96} \int d^4x \text{Tr} \left\{ \frac{1}{\gamma^2} \left( \partial_\tau L_\gamma - \partial_\gamma L_\tau \right)^2 - \frac{1}{\gamma^3} \left( \partial_\tau L_\gamma - \partial_\gamma L_\tau \right) L_\lambda L_\nu \right\} \\
+ \frac{1}{\gamma^2} L_\nu^2 L_\lambda^2 \right\} - \frac{d_2}{8\gamma} \int d^4x \text{Tr} L_\nu^2 . \tag{A11}
\]

This concludes the derivation of \( L_{ct} \), to one-loop order.

Having derived a one-loop finite vacuum theory, we are now in a position to study the theory in the presence of a finite chemical potential \( \mu \), at finite inverse temperature \( \beta = 1/T \). As before, the Euclidean time is \( \tau = it \), and the action \( e^{iS} \rightarrow e^{-S} \) under the transformation to Euclidean space. The chemical potential acts as an external vector field with spatial components zero. It enters by “gauging” the fields \( \phi_1 \) and \( \phi_2 \) via the transformation

\[
i \frac{\partial \phi_1}{\partial \tau} \rightarrow i \frac{\partial \phi_1}{\partial \tau} - \mu \phi_2 \quad ; \quad i \frac{\partial \phi_2}{\partial \tau} \rightarrow i \frac{\partial \phi_2}{\partial \tau} + \mu \phi_1 . \tag{A12}
\]

The resulting action is given by

\[
S = S_{\gamma} + S_{ct} \quad , \quad \text{where}
\]

\[
S_{\gamma} = \frac{1}{2} \beta \int_0^\beta d\tau \int d^3 x \gamma L_{\mu} \quad , \quad S_{ct} = \int_0^\beta d\tau \int d^3 x L_{ct} \quad \text{and}
\]

\[
L_{ct} = -d_4 - \frac{d_2}{4\gamma} \left\{ \left( \frac{\gamma''}{\gamma} \phi \right) L_{\mu} + \cdots \right\} + \frac{d_0}{16\gamma^2} \left\{ \left[ \left( \frac{\gamma''}{\gamma} \phi \right) \right]^2 + \left( \frac{\gamma'}{\gamma} \phi \right)^2 \right\} \left( \mu L_{\mu} \right)^2 + \cdots \tag{A15}
\]

\[
L_{\mu} = \left( \partial_\tau \phi_1 \right)^2 + \left( \partial_\tau \phi_2 \right)^2 + \left( \nabla \phi_1 \right)^2 + \left( \nabla \phi_2 \right)^2 + 2i\mu \left( \phi_2 \partial_\tau \phi_1 - \phi_1 \partial_\tau \phi_2 \right) - \mu^2 \phi^2 . \tag{A16}
\]
Eq. (A11) is invariant under the transformation (A12) and is therefore not included above. We have now derived the action to one-loop order, for arbitrary fields \( \phi_1 \) and \( \phi_2 \) in the presence of a chemical potential \( \mu \).

We then take the final step, and evaluate the one-loop effective potential for fields \( \phi_1 \) and \( \phi_2 \) which are constant in spacetime (see Eq. (14)) so that their derivatives vanish. Expanding Eq. (A13) to quadratic order in the fields and integrating out the fluctuating field gives the action

\[
S = S_{\text{tree}} + S_{\text{loop}} + S_{\text{ct}} ,
\]

where the matrix of second order variations about the constant fields, \( D \), is given by

\[
\begin{align*}
\frac{1}{\beta^2} D_{11} &= \gamma (\omega_n^2 + p^2) - 2 \omega_n \mu \phi_1 \phi_2 \frac{\gamma'}{\phi} - \mu^2 \gamma - \frac{1}{2} \mu^2 \frac{\gamma'}{\phi} (\phi_2^2 + 4 \phi_1^2) - \frac{1}{2} \mu^2 \phi_1^2 \gamma'', \\
\frac{1}{\beta^2} D_{22} &= \gamma (\omega_n^2 + p^2) + 2 \omega_n \mu \phi_1 \phi_2 \frac{\gamma'}{\phi} - \mu^2 \gamma - \frac{1}{2} \mu^2 \frac{\gamma'}{\phi} (\phi_1^2 + 4 \phi_2^2) - \frac{1}{2} \mu^2 \phi_2^2 \gamma'', \\
\frac{1}{\beta^2} D_{12} &= 2 \mu \omega_n \left( \gamma + \phi_2^2 \frac{\gamma'}{\phi} \right) - \frac{1}{2} \mu^2 \phi_1 \phi_2 \left( \gamma'' + 3 \frac{\gamma'}{\phi} \right), \\
\frac{1}{\beta^2} D_{21} &= -2 \mu \omega_n \left( \gamma + \phi_1^2 \frac{\gamma'}{\phi} \right) - \frac{1}{2} \mu^2 \phi_1 \phi_2 \left( \gamma'' + 3 \frac{\gamma'}{\phi} \right),
\end{align*}
\]

and \( \omega_n \) is the Matsubara frequency. The determinant is given by Eq. (21), with an additional factor of \( \gamma^{-1} \) on the left, and with

\[
\begin{align*}
c_1 &= \frac{2 \mu^2}{\gamma} (\gamma + \phi \gamma'), \\
c_2 &= -\frac{\mu^2}{4 \gamma} (2 \gamma + 4 \gamma' \phi + \phi^2 \gamma''), \\
c_3 &= -\frac{\mu^2}{4 \gamma} (2 \gamma + \gamma' \phi).
\end{align*}
\]

As before, it is convenient to factorize the determinant, leading to dispersion relations for two modes given by Eq. (23). Eq. (A18) then separates into a zero-point and a thermal contribution

\[
S_{\text{loop}} = \int_0^\beta d\tau \int d^3x \left( V_{zp} + V_{\text{thermal}} \right),
\]

with \( V_{zp} \) and \( V_{\text{thermal}} \) given by Eqs. (24) and (25) respectively. The counterterm action obtained from Eqs. (A9), (A10) and (A11) after gauging and setting \( \phi_i \) to be constant can be written
\[ S_{ct} = \int_0^\beta d\tau \int d^3x V_{ct} \quad , \tag{A22} \]

where \( V_{ct} \) is given by the divergent parts of Eq. (28), \textit{i.e.} the terms involving \( d_0, d_2 \) and \( d_4 \). As noted in Sec. IIIB, the \( \mu \) dependent divergences in \( V_{zp} \) are exactly cancelled by those in \( V_{ct} \).

**APPENDIX B: CANCELLATION OF CHEMICAL POTENTIAL DIVergENCES IN THE CHIRAL LAGRANGIAN**

In this appendix, we give details of the cancellation of quadratic divergences dependent on the chemical potential \( \mu \) for the chiral Lagrangian, Eq. (9). (The cancellation of quartic divergences needs no discussion since it is straightforward.) We follow the procedure given in Appendix A. It is first convenient to break the Lagrangian down as follows:

\[ \mathcal{L}_K = \mathcal{L}_H + \mathcal{L}_V + \mathcal{L}_U + \mathcal{L}_{ct} \]

\[ \mathcal{L}_H = \frac{1}{2} A (\partial_\nu \phi_1)^2 + \frac{1}{2} B (\partial_\nu \phi_2)^2 + C \partial_\nu \phi_1 \partial_\nu \phi_2 \]

\[ \mathcal{L}_V = b \gamma \left( \phi_1 \partial_\nu \phi_2 \right) \]

\[ \mathcal{L}_U = -2f^2(m_K^2 + c) \sin^2 \frac{1}{2} \Theta \equiv -U(\phi) \quad . \tag{B1} \]

Note that in addition to having more terms than the toy model of Appendix A, the kinetic term, \( \mathcal{L}_H \), is nondiagonal here.

Expanding as in Eq. (A2), the matrix \( D \) of second order variations of the fields \( \phi'_i \) is given by

\[ D = -T \Box + L \cdot \partial - M \quad ; \quad T = \begin{pmatrix} A & C \\ C & B \end{pmatrix} \quad . \tag{B2} \]

The matrix \( L \) can be written

\[ L_\nu = L_{H_1}^\nu + L_{H_2}^\nu + L_{V_1}^\nu + L_{V_2}^\nu \quad \text{where} \]

\[ L_{H_1}^\nu = 2 \frac{\gamma'}{\phi} \begin{pmatrix} \phi_1 \partial_\nu \phi_1 & \phi_1 \partial_\nu \phi_2 \\ \phi_2 \partial_\nu \phi_1 & \phi_2 \partial_\nu \phi_2 \end{pmatrix} \]

\[ L_{H_2}^\nu = -2 \frac{1}{\phi^2} \begin{pmatrix} \phi_1 \partial_\nu \phi_1 & \phi_1 \partial_\nu \phi_2 \\ \phi_2 \partial_\nu \phi_1 & \phi_2 \partial_\nu \phi_2 \end{pmatrix} \]

\[ L_{V_1}^\nu = \delta^{\nu 0} 2b \gamma \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad ; \quad L_{V_2}^\nu = \delta^{\nu 0} 2b \gamma \begin{pmatrix} -\phi_1 \phi_2 & \phi_1^2 \\ -\phi_2^2 & \phi_1 \phi_2 \end{pmatrix} \quad . \tag{B3} \]
The matrix $M$ takes the form

$$M = M_{H_1} + M_{H_2} + M_{V_1} + M_{V_2} + M_U \quad \text{where}$$

$$M_{H_1} = -\frac{1}{2\phi^2} \left[ (\partial^2 \phi_1)^2 + (\partial^2 \phi_2)^2 \right] \begin{pmatrix} \phi_1^2 \gamma'' + \phi_2^2 \gamma'' & \phi_1 \phi_2 (\gamma'' - \frac{\gamma'}{\phi}) \\ \phi_1 \phi_2 (\gamma'' - \frac{\gamma'}{\phi}) & \phi_2^2 \gamma'' + \phi_1^2 \gamma'' \end{pmatrix}$$

$$M_{H_2} = \frac{1 - \gamma}{\phi^2} \begin{pmatrix} \phi_1 \phi_1 & \phi_2 \phi_2 \\ \phi_2 \phi_1 & \phi_2 \phi_2 \end{pmatrix}$$

$$M_{V_1} = 2b \frac{\gamma'}{\phi} \begin{pmatrix} -\phi_1 \partial^0 \phi_2 & -\phi_2 \partial^0 \phi_2 \\ \phi_2 \partial^0 \phi_1 & \phi_2 \partial^0 \phi_1 \end{pmatrix}$$

$$M_{V_2} = -b (\phi_1 \partial^0 \phi_2) \frac{1}{\phi^2} \begin{pmatrix} \phi_1^2 \gamma'' + \phi_2^2 \gamma'' & \phi_1 \phi_2 (\gamma'' - \frac{1}{\phi} \gamma') \\ \phi_1 \phi_2 (\gamma'' - \frac{1}{\phi} \gamma') & \phi_2^2 \gamma'' + \phi_1^2 \gamma'' \end{pmatrix}$$

$$M_U = \frac{1}{\phi^2} \begin{pmatrix} \phi_1^2 U'' + \phi_2^2 \frac{U''}{\phi} & \phi_1 \phi_2 (U'' - \frac{U''}{\phi}) \\ \phi_1 \phi_2 (U'' - \frac{U''}{\phi}) & \phi_2^2 U'' + \phi_1^2 \frac{U''}{\phi} \end{pmatrix}.$$

As in Appendix A primes denote derivatives with respect to $\phi$. It may be checked that by gauging $D$ according to Eq. (A12) and then taking the fields $\phi_i$ to be constant, one recovers Eq. (18).

It is now straightforward, but tedious, to compute the the quadratically divergent contributions to the counterterm Lagrangian by evaluating the terms of the form

$$\frac{1}{2} \text{Tr}(T^{-1}M) - \frac{1}{8} \text{Tr}(T^{-1}L'')^2.$$

Omitting terms that are invariant under the transformation (A12), we find the following contributions to $L_{ct}/d_2$:

$$\frac{1}{2} \text{Tr}(T^{-1}M_{H_1}) = -\frac{1}{4} \left[ (\partial^2 \phi_1)^2 + (\partial^2 \phi_2)^2 \right] \left( \gamma'' + \frac{\gamma'}{\gamma \phi} \right)$$

$$\frac{1}{2} \text{Tr}(T^{-1}M_{H_2}) = \frac{1 - \gamma}{2\phi^2} \left( \phi_1 \phi_1 + \phi_2 \phi_2 \right)$$

$$\frac{1}{2} \text{Tr}(T^{-1}M_{V_1}) = -b \frac{\gamma'}{\phi} (\phi_1 \partial^0 \phi_2)$$

$$\frac{1}{2} \text{Tr}(T^{-1}M_{V_2}) = -\frac{1}{2} b \left( \gamma'' + \frac{\gamma'}{\phi} \right) (\phi_1 \partial^0 \phi_2)$$

$$\frac{1}{2} \text{Tr}(T^{-1}M_U) = \frac{1}{2} \left( \frac{U'}{\gamma \phi} + U'' \right)$$
\[-\frac{1}{8} \text{Tr}(T^{-1}L_V')^2 = b^2 \frac{\gamma}{\check{\gamma}} (\gamma_{\frac{3}{2}} + \phi_{\frac{1}{2}}) \tag{B11}\]

\[-\frac{1}{8} \text{Tr}(T^{-1}[L'_{V_H} + L'_{V_H}])^2 = -\frac{1}{2} \left( \frac{\gamma'}{\phi} - \frac{1 - \gamma}{\phi^2} \right)^2 (\phi_1 \partial^\nu \phi_1 + \phi_2 \partial^\nu \phi_2) \tag{B12}\]

\[-\frac{1}{4} \text{Tr}(T^{-1}L'_V T^{-1}[L'_{V_H} + L'_{V_H}]) = b^2 \frac{\gamma}{\check{\gamma}} \left( \frac{\gamma'}{\phi} - \frac{1 - \gamma}{\phi^2} \right) (\phi_1 \partial^\nu \phi_2) . \tag{B13}\]

Following Appendix A, we use the transformation (A12) on \( L_{ct} \) and find that the quadratically divergent contribution to \( V_{ct} \) is

\[V_{ct}^{\text{quad}} = d_2 \left\{ \frac{1}{4} \mu^2 \phi^2 \left( \gamma'' + \frac{\gamma'}{\gamma \phi} \right) + \frac{1}{2} \mu^2 (1 - \gamma) + b \mu \phi \gamma' + \frac{1}{2} b \mu \phi^2 \left( \gamma''_{\frac{3}{2}} + \frac{\gamma'}{\gamma \phi} \right) \right. \]

\[+ b \mu \frac{\gamma}{\gamma} (1 - \gamma - \phi \gamma') - b \frac{\gamma}{\gamma} (\gamma_{\frac{3}{2}} + \phi_{\frac{1}{2}}) - \frac{1}{2} \left( U'' + U' \right) \left\} . \tag{B14}\]

It may be checked that this agrees with the divergent parts of Eq. (23) using the explicit form of the coefficients \( c_1, c_2 \) and \( c_3 \):

\[c_1 = 2 \mu^2 (\gamma + \phi \gamma') + 2 b \mu \left( 2 \gamma_{\frac{3}{2}} + \frac{\phi' \gamma_{\frac{1}{2}}}{\gamma} + \phi'_{\frac{1}{2}} \right) + 2 b^2 \frac{\gamma_{\frac{1}{2}}}{\gamma} (\gamma_{\frac{3}{2}} + \phi_{\frac{1}{2}}) \tag{B15}\]

\[c_2 = -\frac{1}{4} \mu^2 (2 \gamma + 4 \phi \gamma' + \phi^2 \gamma'') - \frac{1}{2} b \mu (2 \gamma_{\frac{3}{2}} + 4 \phi \gamma'_{\frac{1}{2}} + \phi^2 \gamma''_{\frac{1}{2}}) + \frac{1}{2} U' \tag{B16}\]

\[c_3 = -\frac{1}{4} \mu^2 \left( 2 + \frac{\phi' \gamma}{\gamma} \right) - \frac{1}{2} b \mu \frac{1}{\gamma} (2 \gamma_{\frac{3}{2}} + \phi_{\frac{1}{2}}) + \frac{U'}{2 \gamma \phi} \tag{B17}\]

We now discuss the chiral ordering of the above counterterms in the Lagrangian \( \mathcal{L} \). To do so, it helpful to evaluate the divergent integrals of Eqs. (23), (30) and (31) using dimensional regularization. This gives

\[d_4 = \int \frac{d^3 k}{(2 \pi)^3} \sqrt{k^2 + m_K^2} = -\frac{m_K^4}{16 \pi^2} \left( \frac{1}{\epsilon} + \ln \frac{\kappa}{m_K} + \frac{1}{2} \Psi(3) \right) \]

\[d_2 = \int \frac{d^3 k}{(2 \pi)^3} \frac{1}{2 \sqrt{k^2 + m_K^2}} = -\frac{m_K^2}{8 \pi^2} \left( \frac{1}{\epsilon} + \ln \frac{\kappa}{m_K} + \frac{1}{2} \Psi(2) \right) \]

\[d_0 = \int \frac{d^3 k}{(2 \pi)^3} \frac{1}{4(k^2 + m_K^2)^{3/2}} = \frac{1}{8 \pi^2} \left( \frac{1}{\epsilon} + \ln \frac{\kappa}{m_K} + \frac{1}{2} \Psi(1) \right) . \tag{B18}\]

Here, we have included \( m_K \) explicitly in the kaon propagator. A factor of \( \sqrt{4 \pi} \) has been subsumed in \( \kappa \) which is introduced so that the dimension of the integral is independent of the dimensionality of spacetime. The parameter \( \epsilon = 3 - n \), where \( n \) is the dimensionality of the integral, and \( \Psi(n) \) is the digamma function \( \Psi(n) \). In minimal subtraction the integrals of Eq. (B13) are regulated by removing the \( 1/\epsilon \) terms. It is then evident that in the chiral expansion \( d_4, d_2 \) and \( d_0 \) are \( \mathcal{O}(q^4), \mathcal{O}(q^2) \) and \( \mathcal{O}(q^0) \) respectively.
To evaluate the chiral order of counterterms, we promote the nucleon fields in $b$ to dynamical fields. We may then determine the chiral order of counterterms from Weinberg’s counting rule (26), either directly from the tree-level counterterms or from the representation of these terms as one-loop diagrams. Counterterms (B8), (B9) and (B13) contribute to $\mathcal{L}^{(3)}_{ct}$. Expanding to lowest order in the fields $\phi_1$ and $\phi_2$, these terms renormalize the kaon-nucleon vector interaction coupling constant. Counterterms (B6), (B7) and (B12) contribute to $\mathcal{L}^{(4)}_{ct}$. To lowest order in the fields, they contribute to wave function renormalization. Eq. (B10) is also $O(q^4)$, but to lowest order in the fields gives an overall irrelevant constant and a mass renormalization. Eq. (B11) contributes to $\mathcal{L}^{(2)}_{ct}$ and yields a four-nucleon interaction to lowest order in the fields.

In this Appendix, we have not discussed logarithmic divergences. We believe the cancellation of $\mu$ dependent logarithmic divergences will proceed as in the toy model of Appendix A. However we have not derived the full counterterm Lagrangian proportional to $d_0$ since this, while in principle straightforward, would involve prohibitively complicated algebra. The counterterms will have a chiral order which ranges from 0 to 4 as can be seen from Eqs. (A10) and (A11) in conjunction with Eqs. (B3) and (B4). In conclusion we note that although we have referred to chiral counting throughout this manuscript, our Lagrangian should be regarded as a “$U(1)$” or “$O(2)$” analogue of the chiral expansion.

APPENDIX C: CRITICAL DENSITY FROM THE KAON PROPAGATOR AND FROM EXPANSION OF ZERO-POINT ENERGY

In this appendix, we derive the critical density from the kaon 2-point function to one-loop order and show that it is equal to that obtained from expanding the zero point energy.

Close to threshold, the grand potential may be expanded in terms of the condensate as follows

$$\frac{\Omega_K}{\mathcal{V}} = \frac{\Omega_{K_0}(\theta=0)}{\mathcal{V}} - \frac{D^{-1}(\mu)}{2} f^2 \theta^2 + O(\theta^4)$$

(C1)

where $D$ is the zero-momentum kaon propagator, relevant for $s$-wave condensation, and $\mu$ is the lepton chemical potential prior to kaon condensation. The threshold for kaon
condensation is therefore given by \( D^{-1}(\mu) = 0 \), i.e. the density at which the energy of negatively charged kaons equals the lepton chemical potential.

From the chiral Lagrangian, Eq. (4), we may read off the interactions. The 2-point interactions are
\[
\mathcal{L}_{U2} = -(m_K^2 + c)K^+K^−, \quad \mathcal{L}_{V2} = -ibK^+\partial^0K^−. \tag{C2}
\]
Here we treat the standard mass term as an interaction and take the kaon propagator to be massless. The 4-point interactions are \([4]\)
\[
\mathcal{L}_{HA} = \frac{1}{6f^2}(K^+\partial^\nu K^-)^2, \quad \mathcal{L}_{VA} = -\frac{ib}{6f^2}(K^+\partial^0K^-)K^+K^− \tag{C3}
\]
\[
\mathcal{L}_{U4} = \frac{(m_K^2 + c)}{6f^2}(K^+K^-)^2. \tag{C3}
\]
We compute the kaon 2-point function with external four-momentum \( p = (\omega, 0) \). The interactions \((C2)\) contribute at tree-level. To one-loop order, we have the diagrams of Fig. 3. The contributions of Figs. 3(a), 3(b) and 3(c) are \(-\omega^2/(3f^2)\), \(-2b\omega/(3f^2)\) and \(2(m_K^2 + c)/(3f^2)\), respectively. In the last two diagrams, the internal lines contain the interactions given in Eq. \((C2)\), which are denoted by \(m_K\) and \(b\), respectively. Fig. 3(d) contributes \(-(m_K^2 + c)/(3f^2)\), and Fig. 3(e) contributes \(2b\omega/f^2\). In addition, we must include all possible insertions of the interactions \((C2)\) on internal lines. Including the counterterm contribution, the result is
\[
D^{-1}(\omega) = \omega^2 + 2b\omega - m_K^2 - c + (\omega^2 - 4\omega b - m_K^2 - c)\delta \quad \text{where} \quad \delta = -\frac{1}{6f^2} \left( \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{k^2 + \xi^2}} - 2(d_2 + f_2) + 2\xi^2(d_0 + f_0) \right) \tag{C4}
\]
Eq. \((C5)\) agrees with the expansion of \(V_{zp} + V_{ct}\) given in Eq. \((18)\), as required, and in vacuum we obtain the free inverse propagator \(D^{-1}(\omega) = (\omega^2 - m_K^2)\).

Note that Figs. 3(a), (c), and the contribution to (d) proportional to \(m_K^2\), do not involve nucleon fields and are mass and wavefunction renormalization diagrams in the vacuum. These diagrams are \(O(q^4)\). If we promote the nucleon fields to dynamical ones, we generate nucleon legs on Figs. 3(b) and (e), corresponding to Figs. 1(g) and 1(f) respectively, and find that they are \(O(q^3)\). These diagrams correspond to Figs. 1(a) and 1(b) of Ref. \([4]\) respectively; since only nucleon and kaon fields are used in the present calculation, the remaining diagrams of Ref. \([4]\) do not enter.
REFERENCES

* Permanent address.

[1] E. Friedman, A. Gal and C. J. Batty, Nucl. Phys. A579, 578 (1994).

[2] D. B. Kaplan and A. E. Nelson, Phys. Lett. B175, 57 (1986); B179, 409 (E) (1986).

[3] G. E. Brown, C.-H. Lee, M. Rho and V. Thorsson, Nucl. Phys. A567, 937 (1994).

[4] C.-H. Lee, G. E. Brown, D.-P. Min and M. Rho, Nucl. Phys. A585, 401 (1995).

[5] V. R. Pandharipande, C. J. Pethick and V. Thorsson, Phys. Rev. Lett. 75, 4567 (1995); T. Waas, M. Rho and W. Weise, Preprint nucl-th/9610031. Submitted to Nucl. Phys.

[6] T. Muto and T. Tatsumi, Phys. Lett. B283, 165 (1992); H. Fujii, T. Maruyama, T. Muto, and T. Tatsumi Nucl. Phys. A597, 645 (1996).

[7] T. Waas, N. Kaiser, W. Weise, Phys. Lett. B365, 12 (1996); B379, 34 (1996).

[8] G. Q. Li, C. M. Ko and B.-A. Li, Phys. Rev. Lett. 74, 235 (1995).

[9] G. Q. Li, C. M. Ko and X. S. Fang, Phys. Lett. B329, 149 (1994); A. Schröter et al., Z. Phys. A350, 101 (1994); J. Schaffner, J. Bondorf and I.N. Mishustin, NBI Preprint 96-41, nucl-th/9607058.

[10] T. Tatsumi, H. Shin, T. Maruyama, and H. Fujii, KUNS-1383, Aust. J. Phys., to be published.

[11] P.J. Ellis, R. Knorren and M. Prakash, Phys. Lett. B349, 11 (1995); R. Knorren, M. Prakash and P.J. Ellis, Phys. Rev. C52, 3470 (1995); J. Schaffner and I.N. Mishustin, Phys. Rev. C53, 1416 (1996); J. Schaffner, J. Bondorf and I.N. Mishustin, Strangeness ’96 (Budapest, Hungary, 1996) Heavy Ion Physics, to be pub., nucl-th/9607013.

[12] G. E. Brown, K. Kubodera, M. Rho and V. Thorsson, Phys. Lett. B291, 355 (1992).

[13] V. Thorsson, M. Prakash and J. M. Lattimer, Nucl. Phys. A572, 693 (1994); A574, 851 (1994).
[14] S. E. Thorsett, Z. Arzoumanian, M. M. McKinnon and J. H. Taylor, Astrophys. J. 405, L29 (1994).

[15] W. Keil and H.-Th. Janka, Astron. and Astrophys. 296, 145 (1994).

[16] M. Prakash, J. Cooke and J.M. Lattimer, Phys. Rev. D52, 661 (1995).

[17] N.K. Glendenning, Astrophys. J. 448, 797 (1995).

[18] M. Prakash, I. Bombaci, M. Prakash, P. J. Ellis, J. M. Lattimer and R. Knorren, nucl-th/9603042, Phys. Rep. (1996) in press; P. J. Ellis, J. M. Lattimer and M. Prakash, Comments on Nucl. and Part. Phys. 22, 63 (1996); M. Prakash, S. Reddy, J.M. Lattimer and P.J. Ellis, Strangeness ’96 (Budapest, Hungary, 1996) Heavy Ion Physics, to be pub., nucl-th/9607031.

[19] G. E. Brown and H. A. Bethe, Astroph. J. 423, 659 (1994).

[20] A. Burrows, Astrophys. J. 334, 891 (1988).

[21] T. W. Baumgarte, S. L. Shapiro, and S. Teukolsky, Astroph. J. 443, 717 (1995); ibid 458, 680 (1996).

[22] E. Jenkins and A. V. Manohar, Phys. Lett. B255, 558 (1991).

[23] S-J. Dong and K-F. Liu, Nucl. Phys. B (Proc. Suppl.) 42, 322 (1995).

[24] J. I. Kapusta, Finite Temperature Field Theory (Cambridge University Press, 1985).

[25] K. M. Benson, J. Bernstein and S. Dodelson, Phys. Rev. D44, 2480 (1991).

[26] S. Pokorski, Gauge Field Theories (Cambridge University Press, 1987).

[27] S. Weinberg, Phys. Lett. B251, 288 (1990); Nucl. Phys. B363, 2 (1991).

[28] V. Thorsson and A. Wirzba, Nucl. Phys. A589, 633 (1995).

[29] M. Abramowitz and I. A. Stegun, Handbook of Mathematical Functions (Dover, NY, 1965).

[30] M. Prakash, T. L. Ainsworth and J. M. Lattimer, Phys. Rev. Lett. 61, 2518 (1988).
[31] J. Schaffner, A. Gal, I. N. Mishustin, H. Stocker, and W. Greiner, Phys. Lett. B334, 268 (1994); T. Maruyama, H. Fujii, T. Muto, and T. Tatsumi, Phys. Lett. B337, 19 (1994).

[32] B. D. Serot and J. D. Walecka, Advances in Nucl. Phys. 16, ed. J. W. Negele and E. Vogt (Plenum, NY, 1986).

[33] E. J. Weinberg and A. Wu, Phys. Rev. D36, 2474 (1987).

[34] S. G. Matinyan and B. Müller, Preprint DUKE-TH-96-132, Contribution to ’Foundations in Physics’ commemorating L. C. Biedenharn, Submitted to Found. Phys, hep-th/9610233

[35] J. Gasser and H. Leutwyler, Nucl. Phys. B250, 465 (1985).

[36] I. Zahed, A. Wirzba and U.-G. Meissner, Phys. Rev. D33, 830 (1986)
FIG. 1. Some contributions to the zero-point energy (effective potential to one-loop order). The dashed (solid) lines denote kaons (nucleons) and the external legs are mean fields. There are 5 additional diagrams with 6 external legs.
FIG. 2. Ground state neutron star properties for matter containing a kaon condensate with $a_3 m_s = -222$ MeV and $c = 0$ in the loop contribution. The dashed lines give the tree-level results and the solid lines show the effect of also including the loop contribution. Plotted are the charge chemical potential $\mu$, the proton fraction $x$, the condensate amplitude $\theta$, and the energy gained in forming a condensate $\Delta \epsilon$, as a function of $u = n/n_0$, the density in units of equilibrium nuclear matter density.
FIG. 3. Contributions to the kaon propagator to one-loop order. The dashed line denotes a free kaon propagator. See Appendix C for further notation.
TABLES

TABLE I. Values of the critical density for kaon condensation, $u_{cr} = n_{cr}/n_0$, in units of equilibrium nuclear matter density, for various choices of the parameter $a_3m_s$. Superscript $t$ denotes the tree level contribution, and $l$ denotes the loop contribution. The scalar kaon-nucleon interaction $c$ is defined in Eq. (5), while $c_{n_s}$ is the corresponding quantity with the density $n$ replaced by the scalar density $n_s$ given in Eq.(50)

| $a_3m_s$ | $u_{cr}^t$ | $u_{cr}^{l+1}$ | $u_{cr}^t$ | $u_{cr}^{l}$ | $u_{cr}^t$ | $u_{cr}^{l+1}$ |
|---------|------------|-------------|------------|-------------|------------|-------------|
| −134 MeV | 4.181      | 4.179       | 5.069      | 5.069       |            |             |
| −222 MeV | 3.082      | 3.081       | 4.231      | 4.224       |            |             |
| −310 MeV | 2.417      | 2.417       | 3.473      | 3.454       |            |             |

TABLE II. Properties of normal, $n, p, e^-, \mu^-$, neutron star matter in the presence of kaon zero-point fluctuations as a function of density expressed in units of equilibrium nuclear matter density, $u = n/n_0$. Here $a_3m_s = −222$ MeV. Tabulated are the proton fraction $x$, the charge chemical potential $\mu$ (in MeV), the zero-point contribution to the energy density $\epsilon_{zp}$ (MeV fm$^{-3}$), and the total energy density $\epsilon$ (MeV fm$^{-3}$).

| $u$ | $x$ | $\mu$ | $\epsilon_{zp}$ | $\epsilon$ |
|-----|-----|-------|-----------------|----------|
| 0.50| 0.015| 64.8  | −0.01          | 1.31     |
| 1.00| 0.039| 110.9 | −0.05          | 4.56     |
| 1.50| 0.074| 145.2 | −0.16          | 9.28     |
| 2.00| 0.106| 172.7 | −0.37          | 15.10    |
| 2.50| 0.132| 195.9 | −0.70          | 21.80    |
| 3.00| 0.154| 216.2 | −1.17          | 29.23    |
| 3.06| 0.156| 218.6 | −1.24          | 30.23    |
TABLE III. Ground state neutron star matter properties for normal $n, p, e, \mu$ matter, and matter containing a kaon condensate to tree level and one-loop level. Here $a_3m_s = -134$ MeV. Tabulated are the density in units of equilibrium nuclear matter density, $u$, the amplitude of the condensate $\theta$ (in degrees), the charge chemical potential $\mu$ (MeV), the proton fraction $x$, the kaon fraction $x_K$, and the total energy density $\epsilon$ (MeV fm$^{-3}$). The zero-point contribution to the energy density is $\epsilon_{zp}$ (MeV fm$^{-3}$).

| $u$  | $\epsilon$ | $\theta$ | $\mu$ | $x$  | $x_K$ | $\epsilon$ | $\theta$ | $\mu$ | $x$  | $x_K$ | $\epsilon_{zp}$ | $\epsilon$ |
|-----|------------|----------|-------|-----|------|------------|----------|-------|-----|------|----------------|----------|
| 4.18| 51.98      | 0.00     | 256.3 | 0.194| 0.00 | 51.98      | 0.00     | 256.1 | 0.194| 0.00 | -0.10          | 51.88    |
| 4.20| 52.36      | 5.67     | 255.3 | 0.198| 0.007| 52.36      | 6.56     | 254.7 | 0.199| 0.010| -0.11          | 52.25    |
| 4.70| 62.60      | 29.55    | 226.9 | 0.281| 0.166| 59.87      | 29.76    | 225.4 | 0.283| 0.170| -0.23          | 59.64    |
| 5.20| 73.40      | 41.02    | 197.7 | 0.346| 0.280| 63.31      | 41.01    | 195.8 | 0.348| 0.284| -0.35          | 62.95    |
| 5.70| 84.74      | 49.16    | 168.8 | 0.395| 0.361| 63.33      | 48.99    | 166.8 | 0.396| 0.363| -0.41          | 62.91    |
| 6.20| 96.57      | 55.18    | 141.3 | 0.431| 0.415| 60.67      | 54.90    | 139.5 | 0.431| 0.416| -0.40          | 60.27    |
| 6.70| 108.87     | 59.63    | 115.9 | 0.458| 0.451| 56.03      | 59.31    | 114.5 | 0.458| 0.451| -0.32          | 55.70    |
| 7.20| 121.61     | 62.91    | 92.9  | 0.477| 0.474| 49.97      | 62.61    | 92.1  | 0.477| 0.474| -0.21          | 49.75    |
| 7.70| 134.78     | 65.42    | 72.4  | 0.492| 0.490| 42.90      | 65.20    | 72.0  | 0.491| 0.490| -0.11          | 42.79    |
| 8.20| 148.35     | 67.38    | 54.2  | 0.502| 0.502| 35.11      | 67.25    | 54.1  | 0.502| 0.502| -0.04          | 35.08    |
| 8.70| 162.30     | 68.92    | 38.0  | 0.511| 0.510| 26.80      | 68.87    | 38.0  | 0.510| 0.510| -0.01          | 26.80    |
TABLE IV. Ground state neutron star matter properties for normal $n,p,e,\mu$ matter, and matter containing a kaon condensate to tree level and one-loop level. Here $a_3m_\pi = -222$ MeV (See Table III for notation).

| $n,p,e,\mu$ | $n,p,e,\mu$, $K$(tree) | $n,p,e,\mu$, $K$(tree+loop) |
|---|---|---|
| $u$ | $\epsilon$ | $\theta$ | $\mu$ | $x$ | $x_K$ | $\epsilon$ | $\theta$ | $\mu$ | $x$ | $x_K$ | $\epsilon_{zp}$ | $\epsilon$ |
| 3.06 | 31.47 | | | | | 218.6 | 0.156 | 0.000 | -1.24 | 30.23 |
| 3.08 | 31.77 | 0.00 | 219.2 | 0.157 | 0.000 | 31.77 | 7.20 | 167.7 | 0.163 | 0.012 | -1.27 | 30.50 |
| 3.26 | 34.82 | 22.86 | 200.0 | 0.222 | 0.113 | 34.10 | 24.31 | 196.3 | 0.230 | 0.128 | -1.67 | 32.44 |
| 3.46 | 38.35 | 33.81 | 177.1 | 0.287 | 0.220 | 35.13 | 34.88 | 172.2 | 0.296 | 0.235 | -2.17 | 32.97 |
| 3.66 | 41.99 | 42.29 | 152.8 | 0.345 | 0.308 | 34.52 | 43.03 | 147.0 | 0.353 | 0.321 | -2.70 | 31.82 |
| 3.86 | 45.75 | 49.41 | 127.6 | 0.395 | 0.378 | 32.36 | 49.71 | 121.3 | 0.401 | 0.387 | -3.18 | 29.17 |
| 4.06 | 49.60 | 55.40 | 102.0 | 0.438 | 0.430 | 28.81 | 55.15 | 96.0 | 0.441 | 0.435 | -3.56 | 25.22 |
| 4.26 | 53.56 | 60.50 | 76.8 | 0.473 | 0.470 | 24.06 | 59.68 | 71.7 | 0.473 | 0.470 | -3.80 | 20.21 |
| 4.46 | 57.61 | 64.91 | 52.6 | 0.501 | 0.500 | 18.30 | 63.52 | 48.9 | 0.498 | 0.498 | -3.90 | 14.35 |
| 4.66 | 61.76 | 68.66 | 29.9 | 0.524 | 0.524 | 11.69 | 66.77 | 27.8 | 0.519 | 0.519 | -3.85 | 7.80 |
| 4.86 | 66.00 | 71.80 | 9.0 | 0.542 | 0.542 | 4.41 | 69.53 | 8.3 | 0.535 | 0.535 | -3.69 | 0.71 |
| 5.06 | 70.32 | 74.43 | -10.3 | 0.556 | 0.556 | -3.40 | 71.89 | -9.7 | 0.548 | 0.548 | -3.44 | -6.82 |
TABLE V. Ground state neutron star matter properties for normal $n, p, e, \mu$ matter, and matter containing a kaon condensate to tree level and one-loop level. Here $a_3 m_s = -222$ MeV and the scalar density of Eq. (50) is employed in the scalar kaon-nucleon interaction $c_{n_s}$. (See Table III for notation.)

| $n, p, e, \mu$ | $n, p, e, \mu, K(\text{tree})$ | $n, p, e, \mu, K(\text{tree+loop})$ |
|----------------|---------------------------------|-----------------------------------|
| $u$            | $\epsilon$                      | $\theta$                          |
| 4.22           | 52.84                           | 0.00                              |
| 4.23           | 52.96                           | 0.00                              |
| 4.30           | 54.36                           | 8.93                              |
| 4.80           | 64.72                           | 25.67                             |
| 5.30           | 75.63                           | 34.53                             |
| 5.80           | 87.07                           | 40.88                             |
| 6.30           | 98.99                           | 49.58                             |
| 6.80           | 111.38                          | 52.64                             |
| 7.30           | 124.21                          | 65.11                             |
| 7.80           | 137.46                          | 78.88                             |
| 8.30           | 151.11                          | 92.61                             |
| 8.80           | 165.13                          | 107.61                            |
| 9.30           | 179.53                          | 123.41                            |
| $\epsilon$     | $x$                             | $x_K$                             |
| 52.75          | 257.6                           | 0.195                             |
| 52.88          | 257.4                           | 0.196                             |
| 54.24          | 255.0                           | 0.206                             |
| 62.79          | 239.0                           | 0.264                             |
| 69.44          | 223.2                           | 0.309                             |
| 74.59          | 208.2                           | 0.343                             |
| 78.55          | 194.3                           | 0.391                             |
| 81.58          | 181.3                           | 0.407                             |
| 83.89          | 169.5                           | 0.410                             |
| 85.65          | 158.8                           | 0.416                             |
| 86.99          | 149.1                           | 0.431                             |
| 88.03          | 140.3                           | 0.440                             |
| 40.04          | 88.88                           | 0.447                             |