QUADRATICALLY ENRICHED TROPICAL INTERSECTIONS

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Abstract. Using tropical geometry one can translate problems in enumerative geometry to combinatorial problems. Thus tropical geometry is a powerful tool in enumerative geometry over the complex and real numbers. Results from \(\mathbb{A}^1\)-homotopy theory allow to enrich classical enumerative geometry questions and get answers over an arbitrary field. In the resulting area, \(\mathbb{A}^1\)-enumerative geometry, the answer to these questions lives in the Grothendieck-Witt ring of the base field \(k\). In this paper, we use tropical methods in this enriched setup by showing Bézout’s theorem and a generalization, namely the Bernstein-Kushnirenko theorem, for tropical hypersurfaces enriched in \(\text{GW}(k)\).

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1. Introduction

Classically, Bézout’s theorem states that any \( n \) hypersurfaces in \( \mathbb{P}^n_k \) of degrees \( d_1, \ldots, d_n \) that intersect transversally, intersect in \( d_1 \cdots d_n \) points. The count is invariant of the choice of hypersurfaces. This invariance breaks down if we replace the base field \( \mathbb{C} \) by a non-algebraically closed field \( k \). For example for \( k = \mathbb{R} \) some of the intersections might only be defined over the complex numbers. Motivated by results from \( \mathcal{A}^1 \)-homotopy theory there is a new way of counting geometric objects when the base field \( k \) is not algebraically closed. The resulting count is valued in the Grothendieck-Witt ring \( GW(k) \) of \( k \). This way of counting restores the invariance in the “relatively oriented” case.

Recall that the Grothendieck-Witt ring \( GW(k) \) is generated by \( \langle a \rangle \) with \( a \in k^\times/(k^\times)^2 \) and that \( h := (1) + (-1) \) denotes the hyperbolic form. In [18] McKean proves Bézout’s theorem enriched in \( GW(\mathbb{R}) \): Let \( V_1, \ldots, V_n \) be hypersurfaces in \( \mathbb{P}^n_k \) defined by homogeneous polynomials \( F_1, \ldots, F_n \) of degrees \( d_1, \ldots, d_n \), respectively, such that \( \sum_{i=1}^n d_i \equiv n + 1 \mod 2 \). Assume that all the common zeros of \( F_1, \ldots, F_n \) lie in \( \mathbb{C} \setminus \{0\} \subset \mathbb{P}^n_k \) and set polynomials on the affine chart \( f_i(x_1, \ldots, x_n) := F_i(1, x_1, \ldots, x_n) \). Furthermore, assume that the hypersurfaces intersect transversally at every intersection point, then

\[
(1) \quad \sum_{p \in V_1 \cap \cdots \cap V_n} \text{Tr}_{k(p)/k} \left( \det \text{Jac}(f_1, \ldots, f_n)(p) \right) = \frac{d_1 \cdot d_2 \cdots d_n}{2} \cdot h \in GW(k).
\]

In this paper we reprove (1) and generalize the result using tropical geometry. To do this, we define enriched tropical hypersurfaces. The definition is motivated by Viro’s patchworking.

**Definition 1.1.** An enriched tropical hypersurface \( \widehat{V} = (V, (\alpha_I)) \) in \( \mathbb{R}^n \) is a tropical hypersurface \( V \) in \( \mathbb{R}^n \) together with an element \( \alpha_I \in k^\times/(k^\times)^2 \) assigned to each connected component of \( \mathbb{R}^n \setminus V \).

Recall that connected components of \( \mathbb{R}^n \setminus V \) correspond to vertices in the dual subdivision of \( V \). So we can alternatively assign coefficients \( \alpha_v \) to each vertex \( v \) in the dual subdivision of \( V \). An enriched tropical hypersurface can be seen as a homotopy equivalence class of Viro polynomials associated to the dual subdivision of an embedded tropical hypersurface in \( \mathbb{R}^n \) and a choice of coefficients in \( k^\times \) for each vertex in the dual subdivision. These Viro polynomials are of the form

\[
\sum_{I \in A} \alpha_I x^I \varphi(I)
\]

where \( A \) is a finite subset of \( \mathbb{Z}^n \), \( \alpha_I \in k^\times \) for any \( I \in A \), \( x = (x_1, \ldots, x_n) \) and \( \varphi : A \rightarrow \mathbb{Q} \) is a convex function that assigns a rational number to each exponent \( I \in A \). These Viro polynomials can be viewed as polynomials over the field of Puiseux series over \( k \)

\[
k \{ \{ t \} \} = \left\{ \sum_{i_0} c_i t^{i_0/n} \mid c_i \in k, i_0 \in \mathbb{Z}, n \in \mathbb{N} \right\}.
\]

Let \( \widehat{V}_1, \ldots, \widehat{V}_n \) be \( n \) enriched tropical hypersurfaces in \( \mathbb{R}^n \) with Viro polynomials \( f_1, \ldots, f_n \) in \( k \{ \{ t \} \}[x_1, \ldots, x_n] \). Assume that \( k \) is a field of characteristic \( 0 \) or characteristic bigger than the diameter of the Newton polygons of the \( f_i \). Inspired by
(1) we define the enriched intersection multiplicity of \( V_1, \ldots, V_n \) at an intersection point \( p \) to be

\[
\text{mult}_p(V_1, \ldots, V_n) := \sum_z \text{Tr}_{\kappa(z)/k(t)} \langle \det \text{Jac}(f_1, \ldots, f_n)(z) \rangle \in GW(k) \cong GW(k)
\]

where \( \kappa(z) \) is the residue field of \( z \) and \( z \) ranges over all the common zeros of \( f_1, \ldots, f_n \) that tropicalize to \( p \). Note that taking the rank of (2) recovers the classical (tropical) intersection multiplicity.

It is rather tedious to compute this enriched intersection multiplicity. The main theorem of this paper finds a purely combinatorial rule to determine it.

**Definition 1.2.** Let \( \Lambda^{\text{odd}} \) be the subset of \( \mathbb{Z}^n \) consisting of tuples \( (a_1, \ldots, a_n) \) with \( a_i \equiv 1 \mod 2 \) for \( i = 1, \ldots, n \). We call the elements of \( \Lambda^{\text{odd}} \) odd lattice points.

Our Main Theorem states that the enriched intersection multiplicity is determined by the coefficients of the odd vertices of the dual subdivision. To avoid confusion, we say that the vertices of a polytope that belong to a minimal generating set with respect to the convex hull are its corner vertices.

**Theorem 1.3 (Main Theorem).** Let \( P \) be the parallelepiped in the dual subdivision of the union \( \bar{V}_1 \cup \bar{V}_2 \cup \ldots \cup \bar{V}_n \) corresponding to the intersection point \( p \). Assume that the volume of \( P \) equals \( m \). Let \( r \) be the number of corner vertices of \( P \) that are elements of \( \Lambda^{\text{odd}} \). Then

\[
\text{mult}_p(V_1, \ldots, V_n) = \sum_v \langle \epsilon(v) \alpha_v \rangle + \frac{m - r}{2} \cdot h \in GW(k).
\]

Here, \( \alpha_v \) denotes the coefficient of the vertex \( v \) and \( \epsilon(v) \) is a sign determined by the intersection.

There is an easy proof for Bézout’s theorem for tropical curves (not enriched in \( GW(k) \)) in [25]. Namely, the intersection points of two tropical curves \( C_1 \) and \( C_2 \) with Newton polygons \( \Delta_{d_1} \) and \( \Delta_{d_2} \) (see (8) for the definition of \( \Delta_d \)), respectively, correspond to parallelograms in the dual subdivision of \( C_1 \cup C_2 \). The area of such a parallelogram is equal to the intersection multiplicity at the corresponding intersection point and the rest of the dual subdivision of \( C_1 \cup C_2 \) consists of the dual subdivisions of \( C_1 \) and \( C_2 \). Thus the number of intersections counted with multiplicities is

\[
\text{Area}(\Delta_{d_1+d_2}) - \text{Area}(\Delta_{d_1}) - \text{Area}(\Delta_{d_2}) = \frac{(d_1 + d_2)^2}{2} - d_1^2 - d_2^2 = d_1 \cdot d_2.
\]

This is a particular instance of a more general statement. Two tropical curves \( C_1 \) and \( C_2 \) in \( \mathbb{R}^2 \) of with Newton polygons \( \Delta_1 \) and \( \Delta_2 \), respectively, intersect in a number of points counted with multiplicities equal to the mixed volume

\[
\text{MVol}(\Delta_1, \Delta_2) = \text{Area}(\Delta_1 + \Delta_2) - \text{Area}(\Delta_1) - \text{Area}(\Delta_2),
\]

where \( \Delta_1 + \Delta_2 \) is the Minkowski sum of the polygons \( \Delta_1 \) and \( \Delta_2 \).

Two tropical curves embedded in \( \mathbb{R}^2 \) intersect tropically transversely if they intersect in a finite number of points and every point of the intersection belongs to an edge in each of the curves. A direct consequence of our Main Theorem 1.3 is that we can quadratically enrich the above argument for enriched tropical
curves that intersect tropically transversally at every intersection point. Theorem 1.3 implies that the only non-hyperbolic contribution to (1) comes from odd points on the boundary of $\Delta_1 + \Delta_2$ (see Proposition 5.1 and Corollary 5.2). Hence, we get the following for which we need the characteristic of $k$ to be 0 or bigger than the maximum of the diameters of $\Delta_1$ and $\Delta_2$.

**Theorem 1.4.** Let $\widetilde{C}_1$ and $\widetilde{C}_2$ be two enriched tropical curves with Newton polygons $\Delta_1$ and $\Delta_2$, respectively. Assume that they intersect tropically transversally at every intersection point and that \( \partial(\Delta_1 + \Delta_2) \cap \Lambda^{\text{odd}} = \emptyset \). Then

\[
\sum_{p \in \widetilde{C}_1 \cap \widetilde{C}_2} \text{mult}_p(\widetilde{C}_1, \widetilde{C}_2) = \frac{MVol(\Delta_1, \Delta_2)}{2} \cdot h \in GW(k).
\]

In particular, we recover (1) for curves, since the condition $d_1 + d_2$ being odd holds if and only if \( \partial(\Delta_{d_1} + \Delta_{d_2}) \cap \Lambda^{\text{odd}} = \partial(\Delta_{d_1 + d_2}) \cap \Lambda^{\text{odd}} = \emptyset \).

**Corollary 1.5.** Let $\widetilde{C}_1$ and $\widetilde{C}_2$ be enriched tropical curves over $k$ of with Newton polygons $\Delta_{d_1}$ and $\Delta_{d_2}$, respectively. If $d_1 + d_2 \equiv 1 \mod 2$, then

\[
\sum_{p \in \widetilde{C}_1 \cap \widetilde{C}_2} \widetilde{\text{mult}}_p(\widetilde{C}_1, \widetilde{C}_2) = \frac{d_1 \cdot d_2}{2} h \in GW(k).
\]

When $d_1 + d_2 \equiv 0 \mod 2$, we are dealing with the non-relatively orientable case, in which the left hand side in (1) depends on choice of coefficients. However, the left hand side of (1) can not equal any arbitrary element of $GW(k)$. More precisely, we have the following possibilities for the intersection of two enriched tropical curves.

**Corollary 1.6.** Let $\widetilde{C}_1$ and $\widetilde{C}_2$ be enriched tropical curves over $k$ with Newton polygons $\Delta_{d_1}$ and $\Delta_{d_2}$, respectively. If $\widetilde{C}_1$ and $\widetilde{C}_2$ intersect tropically transversally and $d_1 + d_2 \equiv 0 \mod 2$, then

\[
\sum_{p \in \widetilde{C}_1 \cap \widetilde{C}_2} \widetilde{\text{mult}}_p(\widetilde{C}_1, \widetilde{C}_2) = \frac{d_1 \cdot d_2 - \min(d_1, d_2)}{2} h + \langle a_1, \ldots, a_{\min(d_1, d_2)} \rangle \in GW(k),
\]

where $a_1, \ldots, a_{\min(d_1, d_2)}$ can be any element in $k^*/(k^*)^2$.

This also works in higher dimensions (see [9]): Let $V_1, \ldots, V_n$ be tropical hypersurfaces in $\mathbb{R}^n$ with Newton polytopes $\Delta_1, \ldots, \Delta_n$. Then the number of intersection points counted with multiplicities equals the mixed volume

\[
(4) \quad \text{MVol}(\Delta_1, \ldots, \Delta_n) := \text{coefficient of } \lambda_1 \cdots \lambda_n \text{ in } \text{vol}(\lambda_1 \Delta_1 + \ldots + \lambda_n \Delta_n).
\]

This agrees with (3) in dimension 2.

We replace the relative orientation condition by a purely combinatorial condition, namely we assume that

\[
\partial(\Delta_1 + \ldots + \Delta_n) \cap \Lambda^{\text{odd}} = \emptyset.
\]

In this case our Main Theorem 1.3 implies the following corollary. For this we assume the characteristic of $k$ to be 0 or bigger than the maximum of the diameters of the $\Delta_i$.

**Corollary 1.7.** Let $\widetilde{V}_1, \ldots, \widetilde{V}_n$ be $n$ tropical hypersurfaces with Newton polytopes $\Delta_1, \ldots, \Delta_n$, respectively, such that $\partial(\Delta_1 + \ldots + \Delta_n) \cap \Lambda^{\text{odd}} = \emptyset$. Assume that
\( \tilde{V}_1, \ldots, \tilde{V}_n \) intersect tropically transversally at every point. Then

\[
\sum_{p \in \tilde{V}_1 \cap \ldots \cap \tilde{V}_n} \text{mult}_p(\tilde{V}_1, \ldots, \tilde{V}_n) = \frac{\text{MVol}(\Delta_1, \ldots, \Delta_n)}{2} \ h \in \text{GW}(k).
\]

From this we can derive a quadratic enrichment of a Theorem of Bernstein and Kushnirenko. McKean’s Bézout theorem (1) is a special case of this.

**Corollary 1.8 (Enriched Bernstein-Kushnirenko Theorem).** Let \( f_1, \ldots, f_n \) be Laurent polynomials in \( n \) variables with Newton polytopes \( \Delta_1, \ldots, \Delta_n \), respectively.

If \( \partial(\Delta_1 + \ldots + \Delta_n) \cap \Lambda^{\text{odd}} = \emptyset \), then

\[
\sum_z \text{Tr}_{k(z)/k} \langle \det \text{Jac}(f_1, \ldots, f_n)(z) \rangle = \frac{\text{MVol}(\Delta_1, \ldots, \Delta_n)}{2} \ h \in \text{GW}(k).
\]

Here, the sum runs over all solutions \( z \) to \( f_1 = \ldots = f_n = 0 \) in \( \text{Spec} k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \).

1.1. Related work and the a general strategy for quadratically enriched tropical counts. In [17] Markwig, Payne and Shaw use tropical methods to redo the quadratically enriched count of bitangents to a quartic curve by Larson-Vogt [16]. Their strategy is similar to ours, they also use *enriched tropical curves*: In many cases, questions in enumerative geometry can be solved by counting zeros of a general section of some vector bundle. This is the case for Bézout’s theorem and the count of bitangents to a quartic curve. Hence, these counts equal the degree of the Euler class of the respective vector bundles and their quadratic enrichments are the degree of the “\( \mathbb{A}^1 \)-Euler class” of the respective vector bundles. There is a “Poincaré-Hopf Theorem” (see Theorem 2.14 for the classical Poincaré-Hopf Theorem) for the degree of the \( \mathbb{A}^1 \)-Euler class of a vector bundle, that is, the degree of the \( \mathbb{A}^1 \)-Euler class equals the sum of “local indices” at the zeros of a chosen section. One can write down explicit polynomials to compute these local indices and then interpret them tropically. In our case, the local index is what we call the *enriched tropical multiplicity* defined in (2) and its tropical interpretation is our Main Theorem 1.3. Note that there are now also expository lecture notes on this approach and the main result of this paper for the case of curves [23].

A major breakthrough in the use of tropical geometry in enumerative geometry was Mikhalkin’s correspondence theorem [19], which translates the question of counting plane algebraic curves to counting their tropical counterparts. The authors have proved a quadratic enrichment of Mikhalkin’s correspondence theorem [13] for the counting of plane algebraic curves passing through a configuration of \( k \) points, yielding quadratically enriched tropical curve counting invariants, and studied its arithmetic implications in [11] together with Markwig and Röhrle. There is work in progress [12] to extend the quadratically enriched tropical correspondence theorem [13] to curve counts for point conditions consisting not only of \( k \)-points but also of points defined over quadratic field extensions of \( k \). A tropical wall crossing formula was derived for these counts in [24].

There is another refinement in tropical geometry that specializes to complex and real curve counting invariants. Block-Goettsche invariants are also curves counting invariants that interpolate between the complex and real curve counts [2]. Nicaise, Payne and Schroeter proposed a geometric interpretation of Block and Göttsche’s refined tropical curve counting invariants in terms of virtual \( \chi_{\eta} \)-specializations of motivic measures of semialgebraic sets in relative Hilbert schemes [22].
1.2. Outline. We start by recalling the prerequisites from $\mathbb{A}^1$-enumerative geometry in section 2 and from tropical geometry in section 3. In section 4 we prove our main theorem on the enriched tropical intersection multiplicity 4.7. In section 5 we use our main theorem to prove the enriched tropical Bézout theorem and the enriched Bernstein-Kushnirenko theorem.

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2. Introduction to $\mathbb{A}^1$-Enumerative Geometry

In $\mathbb{A}^1$-enumerative geometry one uses machinery from $\mathbb{A}^1$-homotopy theory to enrich classical results from enumerative geometry yielding results over an arbitrary field $k$. In this section, we introduce one way of doing this following the work of Jesse Kass and Kirsten Wickelgren in [15, 14].

2.1. The Grothendieck-Witt ring. The enriched enumerative results will be valued in the Grothendieck-Witt ring. We recall the definition of $GW(R)$ where $R$ is a commutative ring with unity. For this, consider the set

$$S := \{\text{isometry classes of non-degenerate symmetric bilinear forms over } R\}.$$ 

On this set we have two binary operations. If

$$b_1 : V_1 \times V_1 \to R \text{ and } b_2 : V_2 \times V_2 \to R$$

are two non-degenerate symmetric bilinear forms over $R$, then the direct sum, as well as the tensor product,

$$b_1 \oplus b_2 : (V_1 \oplus V_2) \times (V_1 \oplus V_2) \to R, \quad b_1 \otimes b_2 : (V_1 \otimes V_2) \times (V_1 \otimes V_2) \to R,$$

are non-degenerate symmetric bilinear forms over $R$. One can check that this respects isometry classes, and thus, the set $S$ together with the operations $\oplus$ and $\otimes$ is a semi-ring $(S, \oplus, \otimes)$. Group completion with respect to the direct sum yields a ring.

**Definition 2.1.** The Grothendieck-Witt ring $GW(R)$ of a ring $R$ is the group completion of the semi-ring $(S, \oplus, \otimes)$ of isometry classes of non-degenerate symmetric bilinear forms over $R$ with respect to taking the direct sum $\oplus$. 
We are mainly interested in the case when $R$ is a field $k$ of characteristic not equal to 2 in which case we have a nice presentation of $GW(k)$: Let $k^\times = k \setminus \{0\}$ be the set of units in $k$. If $\text{char } k \neq 2$, any form can be diagonalized, i.e., for any symmetric bilinear form $\beta : V \times V \rightarrow k$. Hence, we can find a basis for the $k$-vector space $V$, such that
\begin{equation}
\beta((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = a_1 x_1 y_1 + \ldots + a_n x_n y_n
\end{equation}
for some $a_1, \ldots, a_n \in k^\times$ in this basis. Also note that if we replace one of the $a_i$ by $a_i b^2$ for some $b \in k^\times$, the resulting form is in the same isometry class as $\beta$. Thus the form $\beta$ above (5) can be expressed as the direct sum of $n$ symmetric bilinear forms on a one-dimensional $k$-vector space. Indeed $GW(k)$ is generated by the classes of bilinear forms $\langle a \rangle : k \times k \rightarrow k$, $(x, y) \mapsto axy$ for $a \in k^\times / (k^\times)^2$ (since classes in $GW(k)$ are non-degenerate, we need $a \neq 0$) subject to the following two relations
\begin{enumerate}
\item $\langle a \rangle \langle b \rangle = \langle ab \rangle$ for $a, b \in k^\times$
\item $\langle a \rangle + \langle b \rangle = \langle a + b \rangle + \langle ab(a + b) \rangle$ for $a, b, a + b \in k^\times$.
\end{enumerate}
We use the notation $(a_1, \ldots, a_s) := (a_1) + \ldots + (a_s) \in GW(k)$.

**Definition 2.2.** We write $h$ for the hyperbolic form, that is the form on a 2-dimensional $k$-vector space (or free rank two $R$-module over $R$ when $R$ is not a field), with Gram matrix
\[
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}.
\]

**Remark 2.3.**
\begin{enumerate}
\item Assume $a$ is a unit in $R$, then the class of the symmetric bilinear form on a rank 2 $R$-module defined by $\begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix}$ equals $h$.
\item If $k$ is a field of characteristic not equal to 2, then after diagonalizing, we get that the hyperbolic form equals
\[ h = \langle 1 \rangle + \langle -1 \rangle. \]
Furthermore, one can deduce from relation 2 above that for $a \in k^\times$ the equality
\[ \langle a \rangle + \langle -a \rangle = \langle 1 \rangle + \langle -1 \rangle = h \]
holds in $GW(k)$.
\end{enumerate}

**Definition 2.4.** We say that the rank of a symmetric bilinear form $\beta : V \times V \rightarrow R$ is the rank of the $R$-module $V$.

Taking the rank extends to a homomorphism
\[ \text{rank} : GW(R) \rightarrow \mathbb{Z}. \]

**Example 2.5.** Let $k = \mathbb{C}$. Since $\mathbb{C}$ is algebraically closed, there is only one generator $\langle 1 \rangle \in GW(k)$ and thus $GW(k) \cong \mathbb{Z}$ where the isomorphism is the rank homomorphism. In particular, results in classical enumerative geometry coincide with the counts enriched in $GW(k)$ for an algebraically closed field $k$ by taking the rank.
Example 2.6. For $k = \mathbb{R}$, $GW(k)$ has two generators, namely $\langle 1 \rangle$ and $\langle -1 \rangle$. In fact, an element in $GW(\mathbb{R})$ is completely determined by its rank and its signature.

Example 2.7. The Grothendieck-Witt ring of the field $k\{t\}$ of Puiseux series over $k$ is isomorphic to $GW(k)$. This is because $k\{t\}^\times/(k\{t\})^\times \cong k^\times/(k^\times)^2$. More precisely when $a_m \in k^\times$,

$$
\sum_{i=m}^{\infty} a_i t^i = a_m \cdot \left( t^\frac{m}{a_m} + \sum_{i=1}^{\infty} \frac{a_i}{a_m} t^\frac{i}{a_m} \right)
$$

and $\left( t^\frac{m}{a_m} + \sum_{i=m+1}^{\infty} \frac{a_i}{a_m} t^\frac{i}{a_m} \right)$ is a square in $k\{t\}$. So $\sum_{i=m}^{\infty} a_i t^i \mapsto a_m$ defines an isomorphism $k\{t\}^\times/(k\{t\})^\times \cong k^\times/(k^\times)^2$.

It follows that $GW(k)$ and $GW(k\{t\})$ have the same generators and Markwig-Payne-Shaw show that also the relations in the Grothendieck-Witt ring are respected by this isomorphism [17, Theorem 4.7]. Hence, we get the following isomorphism

$$
\text{In}: \quad GW(k\{t\}) \longrightarrow GW(k)
$$

sending a generator to its initial.

2.1.1. The Witt ring. A non-degenerate symmetric bilinear form $\beta : V \times V \longrightarrow R$ over a ring $R$ is split if there exists a submodule $N \subset V$ such that $N$ is a direct summand of $V$ and $N$ is equal to its orthogonal complement $N^\perp$. We say that two non-degenerate symmetric bilinear forms $\beta : V \times V \longrightarrow R$ and $\beta' : V' \times V' \longrightarrow R$ are stably equivalent if there exist split symmetric bilinear forms $s$ and $s'$ such $\beta \oplus s \cong \beta' \oplus s'$.

Definition 2.8. The Witt ring of $R$ is the set of classes of stably equivalent non-degenerate symmetric bilinear forms with addition given by the direct sum $\oplus$ and multiplication by the tensor product $\otimes$.

Remark 2.9. If $R$ is a field $k$ of characteristic not equal to 2, then the split non-degenerate symmetric bilinear forms are exactly the multiples of the hyperbolic form $h$. Recall that in this case we have that $\langle a \rangle + \langle -a \rangle = h$ for any unit $a$ and hence

$$
W(k) = \frac{GW(k)}{\mathbb{Z} \cdot h}.
$$

More generally, if $R$ is local and 2 is invertible then an element of $GW(R)$ is completely determined by its rank and the associated element of $W(R)$.

Example 2.10. The Witt ring of $\mathbb{C}$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

2.1.2. Trace. Assume that $R$ is a commutative ring. We are particularly interested in the case that $R$ is a finite étale $k$-algebra. For a finite projective $R$-algebra $L$ one can define the trace $\text{Tr}_{L/R} : L \longrightarrow R$ that sends $b \in L$ to the trace of the multiplication map $m_b(x) = b \cdot x$. If $L$ is étale over $R$ this induces the trace map $\text{Tr}_{L/R} : GW(L) \longrightarrow GW(R)$ which sends the class of a bilinear form $\beta : V \times V \longrightarrow L$ over $L$ to the form

$$
V \times V \xrightarrow{\beta} L \xrightarrow{\text{Tr}_{L/R}} R
$$

over $R$. 

We will compute several trace forms in our main result. So we already collect some facts and computations about the trace form here. Let \( E \) be a finite étale \( R \)-algebra.

1. If \( R = k \) is a field, then \( E = L_1 \times \ldots \times L_s \) for some finite separable field extensions \( L_1, \ldots, L_s \) of \( k \) and the trace map \( \text{Tr}_{E/k} : E \to k \) equal to the sum of field traces \( \text{Tr}_{E/k} = \sum_{i=1}^s \text{Tr}_{L_i/k} \).

2. \( \text{Tr}_{E/R} \) is \( R \)-linear.

3. Let \( F \) be a finite étale \( E \)-algebra. Then

\[
\text{Tr}_{F/R} = \text{Tr}_{E/R} \circ \text{Tr}_{F/E}.
\]

From now on let \( k \) be a field.

**Lemma 2.11.** Let \( L \) be a finite étale \( k \)-algebra. Let \( E = \frac{L[x]}{(x^m - D)} \) for some \( D \in L^\times \), and assume that \( E \) is étale over \( L \). Then \( \text{Tr}_{E/L}(1) = m \) and \( \text{Tr}_{E/L}(x^s) = 0 \) for \( s = 1, \ldots, m - 1 \).

**Proof.** We have the following \( L \)-basis for \( E \): \( 1, x, x^2, \ldots, x^{m-1} \). Recall that \( \text{Tr}_{E/L}(a) \) for \( a \in E \) is the trace of the \( L \)-linear map \( m_a : E \to E \) defined by \( m_a(y) = a \cdot y \). So we are looking for the matrix of \( m_a \) with respect to the basis \( 1, x, \ldots, x^{m-1} \).

1. If \( a = 1 \), then this matrix is the identity matrix and its trace equals the \( L \)-dimension of \( E \), namely \( m \).
2. If \( a = x^s \) for some \( s \in \{1, \ldots, m-1\} \), then every entry of the diagonal of this matrix equals 0. \( \square \)

Since the trace \( \text{Tr}_{E/L} \) is \( L \)-linear, Lemma 2.11 tells us what \( \text{Tr}_{E/L}(a) \) is for any element \( a \in E \) in case \( E = \frac{L[x]}{(x^m - D)} \).

**Lemma 2.12.** Let \( E \) be a finite étale \( R \)-algebra of rank \( m \). Then \( \text{Tr}_{E/R}(h) = m \cdot h \).

**Proof.** This follows directly from the fact that the hyperbolic form \( h \) is split. \( \square \)

**Proposition 2.13.** Let \( L \) be a finite étale \( k \)-algebra and let \( E = \frac{L[x]}{(x^m - D)} \), for some \( D \in L^\times \). Further, assume that \( \text{char} \ k \) does not divide \( m \). Then for \( a \in L \) we get

1. \( \text{Tr}_{E/L}(\langle m \cdot a \rangle) = \begin{cases} \langle a \rangle + \frac{m-1}{2} h & \text{if } m \text{ odd}, \\ \langle a \rangle + \langle a \cdot D \rangle + \frac{m-2}{2} h & \text{if } m \text{ even}. \end{cases} \)

2. \( \text{Tr}_{E/L}(\langle m \cdot a \cdot x \rangle) = \begin{cases} \langle a \cdot D \rangle + \frac{m-1}{2} h & \text{if } m \text{ odd}, \\ \frac{m}{2} h & \text{if } m \text{ even}. \end{cases} \)

**Proof.** We have the following \( L \)-basis for \( E \): \( 1, x, \ldots, x^{m-1} \). Let \( M = (M_{ij}) \) be the Gram matrix of \( \text{Tr}_{E/L}(\langle m \cdot a \rangle) \). Then the \( (i, j) \)th entry \( M_{ij} \) of \( M \) equals

\[
\text{Tr}_{E/L}(m \cdot a \cdot b_i \cdot b_j) = \text{Tr}_{E/L}(m \cdot a \cdot x^{i-1} \cdot x^{j-1}),
\]

where \( b_i = x^{i-1} \) is the \( i \)th basis element of the chosen \( L \)-basis of \( E \). In particular, we have

\[
M_{ij} = \begin{cases} m^2 \cdot a & \text{if } i = j = 1, \\ m^2 \cdot a \cdot D & \text{if } i + j = m + 1, \\ 0 & \text{otherwise}. \end{cases}
\]


by Lemma 2.11 and thus the Gram matrix of $\text{Tr}_{E/L}(\langle m \cdot a \rangle)$ looks like
\[
M = \begin{pmatrix}
m^2 \cdot a & 0 & \ldots & \ldots & \ldots & 0 \\
0 & \ldots & \ldots & \ldots & \ldots & m^2 \cdot a \cdot D \\
\ldots & \ldots & \ldots & m^2 \cdot a \cdot D & \ldots & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & m^2 \cdot a \cdot D & \ldots & \ldots & \ldots & 0 \\
m^2 \cdot a \cdot D & 0 & \ldots & \ldots & \ldots & 0 \\
\end{pmatrix}.
\]

By Remark 2.3, this is equivalent to $\langle a \rangle + \frac{m-1}{2} h$ if $m$ is odd, or to $\langle a \rangle + \langle a \cdot D \rangle + \frac{m-2}{2} h$ if $m$ is even, in $GW(L)$. Now let $M$ be the Gram matrix of $\text{Tr}_{E/L}(\langle m \cdot a \cdot x \rangle)$. Then the $(i, j)$-th entry equals
\[
M_{ij} = \begin{cases} 
m^2 \cdot a \cdot D & \text{if } i + j = n \\
0 & \text{otherwise.} \end{cases}
\]

by Lemma 2.11 and we get that
\[
M = \begin{pmatrix}
0 & 0 & \ldots & \ldots & \ldots & m^2 \cdot a \cdot D \\
0 & \ldots & \ldots & m^2 \cdot a \cdot D & \ldots & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & \ldots & \ldots & \ldots & \ldots & 0 \\
m^2 \cdot a \cdot D & 0 & \ldots & \ldots & \ldots & 0 \\
m^2 \cdot a \cdot D & 0 & \ldots & \ldots & \ldots & 0 \\
\end{pmatrix}
\]

which is the Gram matrix of a quadratic form with class in $GW(L)$ equal to $\langle a \cdot D \rangle + \frac{m-1}{2} h$ if $m$ is odd, or to $\frac{m-2}{2} h$ if $m$ is even, by Remark 2.3.

2.2. The $\mathbb{A}^1$-degree. $\mathbb{A}^1$-homotopy theory is a new branch of mathematics in which one aims to apply techniques from homotopy theory to the category of smooth algebraic varieties over a field $k$. Most constructions from classical homotopy theory work in this setup. In particular, we have an analog of the Brouwer degree. Recall (for example from [8]) that the Brouwer degree from classical topology is an isomorphism from the homotopy classes of the endomorphisms of the $n$-sphere to the integers
\[
de : [S^n, S^n] \rightarrow \mathbb{Z}
\]
for $n \geq 1$. Morel defines the $\mathbb{A}^1$-analog in [21]. His $\mathbb{A}^1$-degree assigns an element of $GW(k)$ to an $\mathbb{A}^1$-homotopy class of an endomorphism of the motivic sphere $\mathbb{P}^n_k/\mathbb{P}^{n-1}_k$
\[
de_{\mathbb{A}^1} : [\mathbb{P}^n_k/\mathbb{P}^{n-1}_k, \mathbb{P}^n_k/\mathbb{P}^{n-1}_k]_{\mathbb{A}^1} \rightarrow GW(k).
\]

Just like for the classical Brouwer degree, the $\mathbb{A}^1$-degree splits up as a sum of local $\mathbb{A}^1$-degrees. We refer to [14] for the definition of the local $\mathbb{A}^1$-degree and merely recall some of their formulas to compute the local $\mathbb{A}^1$-degree $\deg_{x} f$ of a map $f : \mathbb{A}^n_k \rightarrow \mathbb{A}^n_k$ at an isolated zero $x$.

2.2.1. Formulas for the local $\mathbb{A}^1$-degree. Assume $f : \mathbb{A}^n_k \rightarrow \mathbb{A}^n_k$ has an isolated zero $x$ with residue field $\kappa(x)$ separable over $k$. Furthermore, assume that the determinant of the Jacobian $\text{Jac}(f)$ of $f$ at $x$ does not vanish. In this case the local $\mathbb{A}^1$-degree at $x$ equals
\[
de_{\mathbb{A}^1} f = \text{Tr}_{\kappa(x)/k} ((\det \text{Jac}(f)) \in GW(k).
\]
There are also formulas for the local $\mathbb{A}^1$-degree in case $\det \text{Jac} f(x) = 0$ or $k(x)$ is not separable over $k$ [14, 3, 4], but in this paper we restrict to the case of zeros with a non-vanishing Jacobian determinant with residue field separable over our base field $k$. 
2.3. The Poincaré-Hopf theorem and the $\mathbb{A}^1$-Euler number.

2.3.1. Motivation from classical topology. Let $V \rightarrow X$ be an oriented vector bundle of rank $r$ on a smooth, closed, connected, oriented manifold $X$ of dimension $r$. The Euler number $n(V)$ is the Poincaré dual of the Euler class $e(V)$

$$n(V) := e(V) \cap [X] \in H_0(X, \mathbb{Z}) \cong \mathbb{Z}.$$ 

Assume $\sigma : X \rightarrow V$ is a section and $x \in X$ an isolated zero of $\sigma$. Choose oriented coordinates around $x$ and a trivialization of $V$ in a neighborhood around $x$ compatible with the orientation of $V$. In these coordinates and trivialization, the section $\sigma$ is a map $\sigma : \mathbb{R}^r \rightarrow \mathbb{R}^r$. The local index $\text{ind}_x \sigma$ of $\sigma$ at $x$ is the local Brouwer degree of $\sigma$ at $x$.

**Theorem 2.14** (Poincaré-Hopf Theorem). Let $\sigma : X \rightarrow V$ be a section of the bundle $V \rightarrow X$, having only isolated zeros. Then

$$n(V) = \sum_{x: \sigma(x) = 0} \text{ind}_x \sigma.$$

2.3.2. $\mathbb{A}^1$-Euler number. Kass and Wickelgren define the $\mathbb{A}^1$-Euler number of a relatively oriented vector bundle $V \rightarrow X$ of rank $r$ on a $r$-dimensional smooth, proper variety $X$ over $k$ as the sum of local indices defined using the local $\mathbb{A}^1$-degree analogous to the Poincaré-Hopf theorem. We recall the following definitions from [15].

**Definition 2.15.** Let $V \rightarrow X$ be a vector bundle. A relative orientation of $V \rightarrow X$ consists of a line bundle $L \rightarrow X$ and an isomorphism $\rho : \text{det} V \otimes \omega_X/k \rightarrow L \otimes \mathbb{L}^2$. Here $\omega_X/k$ is the canonical line bundle on $X$.

**Example 2.16.** Since $\omega_{P^1/k} = \mathcal{O}_{P^1}(n)$, the line bundle $\mathcal{O}_{P^1}(n) \rightarrow P^1$ is relatively oriented if and only if $n$ is even.

We need to restrict to relatively oriented bundles since otherwise we would not have a well-defined local index: In order to define the local indices at all the zeros in a consistent way, we need to choose coordinates (called Nisnevich coordinates) and a trivialization of the vector bundle compatible with the coordinates and the relative orientation. This means that the section of $\text{det} V \otimes \omega_X/k$ defined by the chosen coordinates and the chosen trivialization is sent to a square in $\mathbb{L}^{\otimes 2}$ by $\rho$. Hence, different choices of coordinates and trivializations compatible with the relative orientation only differ by a square, so they do not differ in GW($k$).

**Definition 2.17.** Let $X$ be a smooth and proper $k$-scheme of dimension $r$ and let $x \in X$ be a closed point. An étale map $\psi : U \rightarrow \mathbb{A}^r_k$ from a Zariski neighborhood $U$ of $x$ which induces an isomorphism of residue fields of $x$ is called Nisnevich coordinates around $x$.

**Remark 2.18.** Nisnevich coordinates always exist given that $r \geq 1$ by [15, Lemma 19]. Since $\psi$ in the definition of Nisnevich coordinates is étale, the standard basis for the tangent space of $\mathbb{A}^r_k$ defines a trivialization of $TX|_U$ where $TX$ is the tangent bundle of $X$. 
Definition 2.19. Let $V \to X$ be a vector bundle of rank $r$ over an $r$-dimensional scheme $X$ over $k$ equipped with a relative orientation $\rho : \det V \otimes \omega_X/k \to L \otimes 2$ and let $\psi : U \to \mathbb{A}_k^r$ be Nisnevich coordinates around a closed point $x \in X$. By Remark 2.18, a choice of Nisnevich coordinates defines a section of $\det T X|_U$. A trivialization $V|_U \cong U \times \mathbb{A}_k^r$ defines a section of $\det V|_U$. We say a trivialization of $V|_U$ is compatible with the relative orientation $\rho$ and the Nisnevich coordinates if the section of $\det V|_U \otimes (\det T X|_U)^\tau$, equivalently, $\det V|_U \otimes \omega_X/k|_U$ defined by the trivialization and the Nisnevich coordinates is sent to a square by $\rho$, that is to a section of $L \otimes 2$ of the form $\ell \otimes \ell$.

We are now ready to define the local index at an isolated zero $x$ of a section $\sigma$ valued in $\text{GW}(k)$. Let $\sigma$ be a section of a relatively oriented vector bundle $V \to X$ and let $x$ be an isolated zero of $\sigma$. Choose Nisnevich coordinates $\psi : U \to \mathbb{A}_k^r$ and a trivialization $\phi$ of $V|_U$ compatible with the relative orientation of $V \to X$ and the Nisnevich coordinates around $x$.

Definition 2.20. The local index $\text{ind}_x \sigma$ of $\sigma$ at $x$ is the local $\mathbb{A}_1$-degree of $U \xrightarrow{\sigma|_U} V|_U \cong U \times \mathbb{A}_r \xrightarrow{\text{pr}_2} \mathbb{A}_r$ at $x$.

Now assume that $V \to X$ is a relatively oriented vector bundle of rank $r$ over a smooth proper $r$-dimensional $k$-scheme $X$ and $\sigma$ is a section with only isolated zeros.

Definition 2.21 (Kass-Wickelgren). The $\mathbb{A}_1$-Euler number of $V \to X$ is the sum of local indices at the zeros of $\sigma$

$$n^{\mathbb{A}_1}(V) := \sum_{x : \sigma(x) = 0} \text{ind}_x \sigma \in \text{GW}(k).$$

By [1, Theorem 1.1], this is independent of the choice of section.

2.4. Bézout’s Theorem enriched in $\text{GW}(k)$. The classical Bézout theorem for algebraically closed fields counts the intersections points of $n$ hypersurfaces in $\mathbb{P}^n$ defined by homogeneous polynomials $f_1, \ldots, f_n$ of degrees $d_1, \ldots, d_n$, respectively. A homogeneous polynomial $f$ in $n + 1$ variables of degree $d$ defines a section of $\mathcal{O}_{\mathbb{P}^n}(d) \to \mathbb{P}^n$. So $f_1, \ldots, f_n$ define a section of

$$V := \mathcal{O}_{\mathbb{P}^n}(d_1) \oplus \ldots \oplus \mathcal{O}_{\mathbb{P}^n}(d_n) \to \mathbb{P}^n$$

and the zeros of this section are exactly the intersection points of $f_1, \ldots, f_n$. Since we have that $\omega_{\mathbb{P}^n/k} \cong O(-n - 1)$, the bundle $V$ is relatively oriented if and only if $\sum_{i=1}^n d_i - n - 1$ is even. In this case, McKean computes the $\mathbb{A}_1$-Euler number yielding an enrichment of Bézout’s theorem in $\text{GW}(k)$.

Theorem 2.22 (McKean).

$$n^{\mathbb{A}_1}(V) = \sum \text{ind}_x(f_1, \ldots, f_n) = \frac{d_1 \cdots d_n}{2} \cdot h \in \text{GW}(k)$$

where the sum runs over the intersection points of $f_1, \ldots, f_n$. 
McKean uses the standard open affine subsets $U_i = \{ x_i \neq 0 \}$ of $\mathbb{P}^n_k$ as Nisnevich coordinates and the usual trivialization of $V|_{U_i}$. In particular, the section $(f_1, \ldots, f_n)$ in these coordinates and trivialization becomes

$$(f_1, \ldots, f_n): U_i \cong \mathbb{A}^n \longrightarrow \mathbb{A}^n$$

setting $x_i = 1$.

2.4.1. Non-orientable case and representability of the $\mathbb{A}^1$-degree. When $V$ is not relatively orientable, McKean shows that one can still orient $V$ relative to the divisor $D = \{ x_0 = 0 \}$ in the sense of Larson and Vogt [16]. Geometrically, this counts the intersection points in $\mathbb{A}^n \cong U_0 = \{ x_0 \neq 0 \} \subset \mathbb{P}^n$. In section 5.1.1 we explain how to get all possible counts for Bézout in this case using (enriched) tropical methods. In particular, we will see that we cannot get any element of $GW(k)$. More precisely, we show find a lower bound for the number of hyperbolic summands in Corollary 5.6.

3. Introduction to Tropical Geometry

In this section we introduce the basic notions of tropical geometry we use in the subsequent sections. For more details in tropical geometry we refer the reader to [5], [10] and [20].

3.1. Toric deformations. Given a Laurent polynomial $f = \sum_{I \in A} \alpha_I x^{i_1} y^{i_2}$ in $k[x, y]$, where $A \subset \mathbb{Z}^2$ is a finite set of tuples $I = (i_1, i_2)$, we consider a toric deformation of $f$, that is a family of polynomials given by

$$f_t(x, y) = \sum_{I \in A} \alpha_I x^{i_1} y^{i_2} t^{\varphi(I)},$$

where $\varphi: A \longrightarrow \mathbb{Q}$ is the restriction of a convex rational function to the set of indices $A$. We can think of the variable $t$ as the variable of $\mathbb{G}_m$ whose specialization to $t = 1$ is our initial polynomial. The family $f_t$ can be seen as an element of the polynomial ring $k\{\{t\}\}[x, y]$ with coefficients in the field of Puiseux series $k\{\{t\}\}$.

The field of Puiseux series has a valuation given by

$$\text{val} : \begin{array}{c} k\{\{t\}\} \longrightarrow \mathbb{Q} \cup \{\infty\} \\ 0 \longrightarrow \infty \end{array}$$

given that $a_{i_0} \neq 0$. Let $\nu := -\text{val}: k\{\{t\}\} \longrightarrow \mathbb{Q} \cup \{-\infty\}$. Then $\nu$ satisfies

$$\nu(x + y) = \max\{\nu(x), \nu(y)\} \quad \text{if} \ \nu(x) \neq \nu(y),$$

$$\nu(xy) = \nu(x) + \nu(y).$$

Note that these are exactly the operations in the tropical semifield introduced in the next subsection. Given a toric deformation $f_t \in k\{\{t\}\}[x, y]$ as above, assume that char $k = 0$ or char $k > \max\{\deg(f_t), \deg(g_t)\}$. Then we have that

$$f_t(x, y) = \sum_{I \in A} \alpha_I x^{i_1} y^{i_2} t^{\varphi(I)} = 0$$

has a solution in $k\{\{t\}\}^2$ given by

$$x(t) = x_0 t^{i_0} + \text{higher order terms in } t, \quad y(t) = y_0 t^{j_0} + \text{higher order terms in } t,$$
that is,
\[
0 = f_t(x(t), y(t)) = \sum_{I \in \mathcal{A}} (\alpha_I x_0^{i_1} y_0^{i_2} t^{\nu(I) - i_1 \nu(x) - i_2 \nu(y)} + \text{higher order terms in } t)
\]
if and only if the term of lowest power in \(t\), that is where \(t\) has the exponent
\[
\{\nu(I) - i_1 \nu(x) - i_2 \nu(y) : I \in \mathcal{A}\}
\]
appears at least twice in \(f_t(x(t), y(t))\). Equivalently, the maximum of
\[
\{-\nu(I) + i_1 \nu(x) + i_2 \nu(y) : I \in \mathcal{A}\}
\]
has to be attained at least twice. This is exactly the definition of the tropical vanishing locus in the next section. For example, if \(f_t\) is a polynomial of degree 1, then the locus where the maximum is attained at least twice is a tropical line and looks like Figure 1 on the left.

These notions extend naturally to more variables.

3.2. Tropical hypersurfaces and tropicalization maps.

3.2.1. Tropical hypersurfaces. The tropical semifield is the set \(\mathbb{T} = \mathbb{R} \cup \{-\infty\}\) endowed with the operations (denoted by “+” and “·”)
\[
“x + y” = \max\{x, y\}, \\
“x \cdot y” = x + y.
\]
Then \(\mathbb{T}\) with these two operations forms a semifield, i.e., it satisfies all axioms of a field but the existence of additive inverse. We write \(\mathbb{T}^*\) for \(\mathbb{T}\setminus\{-\infty\} = \mathbb{R}\). A tropical polynomial in \(n\) variables is a polynomial given by
\[
f(x) = “\sum_{I \in \mathcal{A}} a_I x_1^{I_1} \cdots x_n^{I_n}” \in \mathbb{T}[x_1, \ldots, x_n],
\]
where \(x = (x_1, \ldots, x_n)\), \(\mathcal{A}\) is a finite set of tuples \(I = (I_1, \ldots, I_n) \in \mathbb{Z}^n\) and \(a_I \neq -\infty\) for \(I \in \mathcal{A}\).

Such polynomial defines a function in \((\mathbb{T}^*)^n = \mathbb{R}^n\) that is piecewise linear. Its tropical vanishing locus is defined as the locus of non-differentiability, i.e., the points in \(\mathbb{R}^n\) such that the maximum is obtained at least twice. We denote this locus by \(V_{\text{Trop}}\), and it is expressed by
\[
V_{\text{Trop}}(f) = \{x \in \mathbb{R}^n | \exists I, I' \in \mathcal{A}: I \neq I', p(x) = I \cdot x + a_I = I' \cdot x + a_{I'}\},
\]
where \(\cdot\) denotes the scalar product.

Example 3.1. Let
\[
f(x, y) = “(-3)x + (-3)y + 0” = \max\{x - 3, y - 3, 0\}.
\]
Then \(V_{\text{Trop}}(f)\) is the tropical line in Figure 1 (a). In the left lower component of \(\mathbb{R}^2 \setminus V_{\text{Trop}}(f)\), 0 is maximal, in the component on the right, \(x - 3\) is maximal and in the upper left component, \(y - 3\) is maximal. The right picture in Figure 1 shows a tropical conic, i.e., it is the tropical vanishing locus of a tropical polynomial of degree 2.

The Newton polytope of a tropical polynomial \(f\) is given by the convex hull of its monomials with non-trivial coefficient in \(\mathbb{R}^n\)
\[
\text{NP}(f) = \text{Conv}\{(I \in \mathbb{R}^n | a_I \neq -\infty)\} = \text{Conv}(\mathcal{A}).
\]
We say that a polynomial $f$ has degree $d$ if its Newton polytope is

$$\Delta_d = \text{Conv}\{(0,0,\ldots,0), (d,0,\ldots,0), (0,d,0,\ldots), \ldots, (0,\ldots,0,d)\} \subset \mathbb{R}^n.$$  

### 3.2.2. Tropicalization maps.

Using the aforementioned map $\nu : k\{t\} \to \mathbb{Q} \cup \{-\infty\}$ we can tropicalize a polynomial over $k\{t\}$ by taking $\nu = -\text{val}$ of its coefficients and reinterpreting the addition and multiplication

\[
\text{"\cdot" : } k\{t\}[x_1, \ldots, x_n] \to \mathbb{T}[x_1, \ldots, x_n], \quad \sum_{I \in \mathcal{A}} \nu(\alpha_I(t)) x^I_n = \max_{I \in \mathcal{A}} \{\nu(\alpha_I(t)) + I \cdot x\}.
\]

Now let $f \in k\{t\}[x_1, \ldots, x_n]$ and let $k$ be a field with char $k = 0$ or char $k > \text{deg } f$. Then we can also tropicalize the zeros of a polynomial $f$, at the level of sets, by taking the closure of the image of the valuation taken point-wise in $\mathbb{R}^n$. More precisely, let $X$ be the vanishing locus $V(f) \subset \mathbb{A}^n_{k\{t\}}$. Then

$$\text{Trop}(X) := \{\nu(x) \mid x \in X \text{ geometric point}\} \subset \mathbb{R}^n,$$

where $\nu(x) \in \mathbb{R}^n$ is given by $\nu$ coordinatewise.

If $X$ is an algebraic hypersurface, its tropicalization is a tropical hypersurface, defined by the tropicalization of a defining polynomial for $X$.

**Theorem 3.2 (Kapranov).** If $k$ has characteristic 0 or $\text{deg } f < \text{char } k$, then for a polynomial $f \in k\{t\}[x_1, \ldots, x_n]$ one has

$$V_{\text{Trop}}("f") = \text{Trop}(V(f)).$$

### 3.2.3. Dual subdivision.

We associate a refinement of the Newton polytope $\text{NP}(f)$ called the dual subdivision $\text{DS}(f)$ of $f$. The refinement is given by the projection to $\mathbb{R}^n$ of the boundary of the upper faces (with respect to the last coordinate) of the polyhedron

$$\text{Conv}\{(I, a_I) \in \mathbb{R}^{n+1} \mid a_I \neq -\infty\}.$$  

There is a one-to-one correspondence of the elements
Figure 2. A tropical conic with its dual subdivision.

| $V_{\text{Trop}}(f)$ | $\text{DS}(f)$ |
|----------------------|-----------------|
| vertex $v$           | connected component of $\text{NP}(f) \setminus \text{DS}(f)$ |
| $l$-dimensional face $e$ | $n-l$-dimensional face $e'$ |
| connected component $K$ of $\mathbb{R}^n \setminus V_{\text{Trop}}(f)$ | vertex $v_K$ |

Moreover, the corresponding dual faces $e$ and $e'$ are orthogonal, and inclusion of faces are inverted. Figure 2 shows a tropical conic with its dual subdivision. Figure 6 shows the dual subdivision of a reducible quintic.

3.2.4. Tropical intersections. We say that $n$ tropical hypersurfaces $V_1, \ldots, V_n$ in $\mathbb{R}^n$ intersect tropically transversely at $p \in V_1 \cap \ldots \cap V_n$ if the point $p$ is an isolated point belonging to the interior of a top dimensional face of $V_i$, for every $i = 1, \ldots, n$. In particular, for $n$ tropical hypersurfaces $V_1, \ldots, V_n$ in $\mathbb{R}^n$ that intersect tropically transversely at $p \in V_1 \cap \ldots \cap V_n$, i.e., the point $p$ corresponds to a parallelepiped in the dual subdivision of the union $V_1 \cup \ldots \cup V_n$.

**Definition 3.3.** Assume $V_1, \ldots, V_n$ intersect tropically transversally at $p$. The intersection multiplicity of $V_1, \ldots, V_n$ at $p$ is the volume of the parallelepiped dual to $p$:

$$\text{mult}_p(V_1, \ldots, V_n) = \text{Vol}(\text{parallelepiped dual to } p).$$

This definition is meaningful by the following result of Huber-Sturmfels [9]. Their argument also works in positive characteristic when the characteristic is big enough, that is all equations involved have degree less than the characteristic for example if the characteristic is bigger than the maximum of the diameters $\max\{\text{diam}(\text{NP}(F_i))\}$ of all the Newton polytopes of the tropical polynomials $F_i$ defining the tropical hypersurfaces $V_i$.

**Lemma 3.4.** Assume that $k$ is a field of char $k = 0$ or char $k > \max\{\text{diam}(\text{NP}(F_i))\}$. Let $F_1, \ldots, F_n$ be $n$ general Laurent polynomials in $k\{t\}[x_1, \ldots, x_n]$. These define tropical polynomials $f_1, \ldots, f_n$. Let $V_i = V_{\text{Trop}}(f_i)$ for $i = 1, \ldots, n$ and $p \in V_1 \cap \ldots \cap V_n$. Then the number of common zeros of the $F_i$ that tropicalize to $p$ equals $\text{mult}_p(V_1, \ldots, V_n)$.

Our main theorem 4.7 is the quadratic enrichment of this Lemma and in section 5 we will derive a quadratic enrichment of the following consequence of Lemma.
3.4. Recall that the mixed volume $\text{MVol}(\Delta_1, \ldots, \Delta_n)$ of $n$ polytopes in $\mathbb{R}^n$ is the coefficient of $\lambda_1 \cdots \lambda_n$ in the polynomial $R(\lambda_1, \ldots, \lambda_n)$ given by
$$\text{Vol}(\lambda_1 \Delta_1 + \cdots + \lambda_n \Delta_n).$$

**Theorem 3.5** (Tropical Bézout and Bernstein-Kushnirenko Theorem). Let $V_1, \ldots, V_n$ be tropical hypersurfaces in $\mathbb{R}^n$ with Newton polytopes $\Delta_1, \ldots, \Delta_n$, respectively. Then
$$\sum_p \text{mult}_p(V_1, \ldots, V_n) = \text{MVol}(\Delta_1, \ldots, \Delta_n).$$
In particular, if $\Delta_i = \Delta_{d_i} = \text{Conv}\{0, d_i e_1, \ldots, d_i e_n\}$ for $i = 1, \ldots, n$, then we get
the tropical Bézout theorem
$$\sum_p \text{mult}_p(V_1, \ldots, V_n) = d_1 \cdots d_n.$$

3.3. *Enriched tropical hypersurfaces and Viro Polynomials.* Viro’s patchworking is a combinatorial construction yielding topological properties of real algebraic varieties. It is an algorithmic construction whose input is a subdivision of a polytope and a set of signs $\sigma(I)$ (either plus or minus) for every integer point $I$ in the dual subdivision of the polytope. A Viro polynomial associated to this data is a polynomial
$$X_I \in \text{DS}(\Delta) \cap \mathbb{Z}^2 \sigma(I) x_I t_\varphi(I)$$
where $\varphi$ is a convex piece-wise linear function inducing the subdivision and such that tropicalizing the polynomial yields back a defining polynomial for the tropical curve. Based on this idea, we generalize this concept by replacing the signs $\sigma(I)$ with elements $\alpha_I \in k^x/(k^x)^2$ and call the following *enriched Viro polynomial*

$$\sum_{I \in \Delta \cap \mathbb{Z}^n} \alpha_I x_I t_\varphi(I).$$

(9)

Note that if $k = \mathbb{R}$, then this coincides with the original definition of a Viro polynomial since $\mathbb{R}^x/(\mathbb{R}^x)^2 = \{\pm 1\}$. Tropicalization gives back a tropical hypersurface $V$ which has the dual subdivision we started with. However, in the tropicalization process one loses information, namely the elements in $k^x/(k^x)^2$. We would like to remember these *coefficients* by assigning them to the corresponding connected component in $\mathbb{R}^2 \setminus V$, that is we assign the coefficient $\alpha_I$ of a monomial $x_I t_\varphi(I)$ to the component where the $\nu(x_I t_\varphi(I)) = I \cdot \nu(x) - \varphi(I)$ attains the maximum. Equivalently, one can assign the coefficients $\alpha_I$ to the corresponding vertex $I$ in the dual subdivision. This gives rise to the following definition.

**Definition 3.6.** An *enriched tropical hypersurface* $\tilde{V} = (V, (\alpha_I))$ in $\mathbb{R}^n$ is a tropical hypersurface $V$ in $\mathbb{R}^n$ together with a *coefficient* $\alpha_I$ assigned to each connected component of $\mathbb{R}^n \setminus V$, or equivalently, to each vertex in the dual subdivision. We call such element $\alpha_I$ of $k^x/(k^x)^2$ the *coefficient* of the component/vertex of the dual subdivision. We write $V$ for the underlying non-enriched tropical hypersurface.

To each enriched tropical hypersurface $\tilde{V}$ we can assign an (*enriched*) Viro polynomial of the form (9) such that tropicalizing and remembering the coefficients gives back $\tilde{V}$. 
Example 3.7. Figure 3 shows an enriched tropical line and an enriched tropical conic with enriched Viro polynomials

\[ \alpha(0,0) + \alpha(1,0)x^3 + \alpha(0,1)y^3, \]

and, respectively,

\[ \beta(0,0) + \beta(1,0)x + \beta(0,1)y + \beta(2,0)x^2t^4 + \beta(1,1)xyt^2 + \beta(0,2)y^2t^4. \]

Tropicalizing the enriched Viro polynomial of the line yields the tropical polynomial

\[ "0 + (-3)x + (-3)y" = \max\{0, x - 3, y - 3\} \]

which has tropical vanishing locus the tropical line with 3-valent vertex at (3, 3) displayed in Figure 3. The enrichment remembers the coefficients of the enriched Viro polynomials and assigns them to the connected components where the tropicalization of the corresponding monomial attains the maximum.

Similarly, one can see that the tropicalization of the enriched Viro polynomial of the conic yields the tropical conic in Figure 3 as well as the coefficients of the connected components.

To prove a quadratically enriched tropical Bézout theorem, we also need to enrich the union of enriched tropical hypersurfaces \( \overline{V}_1 \cup \ldots \cup \overline{V}_n \). Note that to \( f \in k\{t\}[x_1, \ldots, x_n] \) we can associate an enriched Viro polynomial \( f_0 \) by replacing the coefficients of \( f \) with its initials. Note that if \( V_i \) is defined by a tropical polynomial \( \"f_i\" \) for \( i = 1, \ldots, n \), then \( \"f_1 \cdots f_n\" \) is a tropical polynomial defining \( V_1 \cup \ldots \cup V_n \). The product of enriched Viro polynomials \( f_1 \cdots f_n \) is a polynomial in \( k\{t\}[x_1, \ldots, x_n] \) and to get an enriched Viro polynomial for \( \overline{V_1} \cup \ldots \cup \overline{V_n} \) we take \( (f_1 \cdots f_n)_0 \). The following lemma tells us how to geometrically determine the coefficients of \( (f_1 \cdots f_n)_0 \) in terms of the coefficients of the tropical hypersurfaces \( \overline{V_i} \).

Lemma 3.8. Let \( \overline{V_1}, \ldots, \overline{V_n} \) be \( n \) enriched tropical hypersurfaces in \( \mathbb{R}^n \) with enriched Viro polynomials

\[ f_i(x) = \sum_{I \in \mathcal{A}_i} \alpha_{I,t} x^I t^{\varphi(I)} \]
for $i = 1, \ldots, n$. Then the coefficients of the enriched tropical hypersurface given by the union $\tilde{V}_1 \cup \tilde{V}_2 \cup \ldots \cup \tilde{V}_n$ can be determined as follows. Let $K$ be a vertex in the dual subdivision of $\tilde{V}_1 \cup \tilde{V}_2 \cup \ldots \cup \tilde{V}_n$. Let $J^i$ be the vertex in the dual subdivision of $\tilde{V}_i$ such that the connected component dual to $K$ in $\mathbb{R}^n \setminus (\tilde{V}_1 \cup \ldots \cup \tilde{V}_n)$ is a subset of the connected component in $\mathbb{R}^n \setminus \tilde{V}_i$ dual to $J^i$ for each $i = 1, \ldots, n$. Then the coefficient of the vertex $K$ of $\tilde{V}_1 \cup \tilde{V}_2 \cup \ldots \cup \tilde{V}_n$ can be determined as follows. Let $\alpha_I$ be a vertex in the dual subdivision of $\tilde{V}_1 \cup \tilde{V}_2 \cup \ldots \cup \tilde{V}_n$. Let $J^i$ be the vertex in the dual subdivision of $\tilde{V}_i$ such that the connected component dual to $K$ in $\mathbb{R}^n \setminus (\tilde{V}_1 \cup \ldots \cup \tilde{V}_n)$ is a subset of the connected component in $\mathbb{R}^n \setminus \tilde{V}_i$ dual to $J^i$ for each $i = 1, \ldots, n$. Then the coefficient of the vertex $K$ of $\tilde{V}_1 \cup \tilde{V}_2 \cup \ldots \cup \tilde{V}_n$ equals $\prod_{i=1}^n \alpha_{J^i}$.

**Example 3.9.** Figure 4 illustrates how to assign the coefficients to a union of enriched tropical curves.

**Proof of Lemma 3.8.** Let $p = (p_1, \ldots, p_n)$ be a point in the interior of the connected component in the complement $\mathbb{R}^n \setminus (\tilde{V}_1 \cup \ldots \cup \tilde{V}_n)$ that is dual to $K$. Then $p$ is in the interior of the connected component dual to $J^i$ in $\mathbb{R}^n \setminus \tilde{V}_i$ for each $i = 1, \ldots, n$. That means that

$$
\sum_{j=1}^n J^i_j \cdot p_j - \varphi_i(J^i) \geq \sum_{j=1}^n I^i_j \cdot p_j - \varphi_i(I^i)
$$

for any $I^i \neq J^i$ and all $i = 1, \ldots, n$. Hence,

$$
\sum_{i=1}^n \left( \sum_{j=1}^n J^i_j \cdot p_j \right) - \varphi_i(J^i) \geq \sum_{i=1}^n \left( \sum_{j=1}^n I^i_j \cdot p_j \right) - \varphi_i(I^i)
$$

for any $(I^1, \ldots, I^n) \neq (J^1, \ldots, J^n)$. In particular for any $(I^1, \ldots, I^n) \neq (J^1, \ldots, J^n)$ such that $\sum_{i=1}^n I^i = \sum_{i=1}^n J^i = K$ we get that

$$
\sum_{i=1}^n \varphi_i(J^i) < \sum_{i=1}^n \varphi_i(I^i)
$$

and hence the monomial with exponent $K$ in $f_1 \cdots f_n$ equals

$$
\left( \prod_{i=1}^n \alpha_{J^i} \cdot \sum_{i=1}^n \varphi_i(J^i) + \text{higher order terms in } t \right) \cdot x^K.
$$

and thus, the coefficient of $K$ equals $\prod_{i=1}^n \alpha_{J^i}$. \qed
3.4. Combinatorics of tropical curves. We redefine the concept of a tropical curve in $\mathbb{R}^n$ from a combinatorial point of view. We will need this in the proof of Proposition 5.1. The following definition coincides with the definition of a tropical curve in $\mathbb{R}^2$ defined algebraically as before and it is known in the literature as an *embedded abstract tropical curve*.

**Definition 3.10.** A tropical curve $C$ is a finite weighted graph $(V, E, \omega)$ embedded in $\mathbb{R}^n$, where $E$ is the disjoint union of non-directed edges $E^\circ \subset \{e \subset V \mid \text{Card}(e) = 2\}$ and univalent edges $E^\infty \subset V$, such that every edge $e \in E^\circ$ embeds into a segment of the graph of an integer line, i.e. given by $\bar{a}_e \cdot t + \bar{b}_e$ with $\bar{a}_e, \bar{b}_e \in \mathbb{Z}^n \setminus \{0\}$, $\bar{b}_e \in \mathbb{Q}^n$, every edge $l \in E^\infty$ embeds into a ray of an integer line, and every vertex $v \in V$ satisfies the balancing condition

$$\sum_{e \in E, v \in e} \omega(e) \cdot u_e = 0$$

where $u_e = \frac{\pm 1}{\text{gcd}(\bar{a}_e)} \bar{a}_e$ oriented outwards from $v$, and $\omega : E \rightarrow \mathbb{Z}$ is a non-negative function. We call $\bar{a}_e$ a direction vector of $e$ and $u_e$ a primitive vector of $e$ at $v$. When drawing a tropical curve we write the weights not equal to 1 next to the edges.

**Example 3.11.** Figure 5 shows a 3-valent vertex $v$ with its three primitive vectors in blue. The one edge labeled 2 has weight 2 while the other edges have weight 1. The balancing condition is satisfied since

$$1 \cdot \begin{pmatrix} -1 \\ 0 \end{pmatrix} + 1 \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} + 2 \cdot \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$
4.1. Notation and conventions. We start by establishing notation for this section. The Newton polytope of a Laurent polynomial \( f = \sum a_I x^I \in k[[t]][x_1, \ldots, x_n] \) is the convex hull \( \text{Conv}\{I : a_I \neq 0\} \subset \mathbb{R}^n \). Note that this agrees with the Newton polytope of the associated tropical polynomial. In the whole section we work over a field \( k \) of characteristic 0 or characteristic bigger than the diameter \( \text{diam}(\text{NP}(f_i)) \) of the Newton polytopes of the polynomials \( f_i \in k[[t]][x_1, \ldots, x_n] \). We use \( eV \) for enriched tropical hypersurfaces. We write \( V \) for the underlying (non-enriched) tropical hypersurface. Let \( eV \) be an enriched tropical hypersurface in \( \mathbb{R}^n \) and let

\[
\sum_{I \in A} \alpha_I x^I \phi(I)
\]

be an enriched Viro polynomial for \( eV \). Here, the sum runs over a finite set \( A \) of integer \( n \)-tuples \( I = (I_1, \ldots, I_n) \) in \( \mathbb{Z}^n \), the map \( \phi : A \to \mathbb{Q} \) is the restriction of a rational convex function to \( A \), and \( x^I = x_1^{I_1} \cdots x_n^{I_n} \). The coefficients \( \alpha_I \) are elements of \( k^\times \). Now assume we have \( n \) tropical hypersurfaces \( eV_1, \ldots, eV_n \) in \( \mathbb{R}^n \) with enriched Viro polynomials

\[
f_i = \sum_{I^i \in A_i} \alpha_{I^i} x^{I^i} \phi_{I^i}(I^i)
\]

for \( i = 1, \ldots, n \) and assume that \( eV_1, \ldots, eV_n \) intersect tropically transversally at \( p \). Then for each \( i \in \{1, \ldots, n\} \), the point \( p \) lies on a top dimensional face of \( eV_i \) separating two connected components of \( \mathbb{R}^n \) (see Figure 4). Assume these components are the components where \( I^i_1 x_1 + \ldots + I^i_n x_n - \phi_i(I^i) \) and \( J^i_1 x_1 + \ldots + J^i_n x_n - \phi_i(J^i) \) attain the maximum for some \( I^i, J^i \in A_i \). Then we say that for \( \alpha_i = \alpha_{I_i} \) and \( \beta_j = \beta_{J_j} \),

\[
f_i^{\text{local}} := \alpha_i x^{I^i} + \beta_i x^{J^i}
\]

is the local binomial equation of \( eV_i \) at \( p \). Put \( \Delta^i := I^i - J^i \) (the order of \( I, J \) matters) and let

\[
M = (\Delta^1, \ldots, \Delta^n)
\]
be the matrix with columns the $\Delta^i$ and
\[ m = |\det M| \]
be the absolute value of its determinant.

For an intersection point $p$ of $V_1, \ldots, V_n$, we write $P$ for the parallelepiped in the dual subdivision of $\tilde{V}_1 \cup \ldots \cup \tilde{V}_n$ dual to $p$. The parallelepiped $P$ is given by
\[ P = \text{Conv} \left\{ \sum_{i=1}^n K^i \mid K^i = I^i \text{ or } K^i = J^i \right\} = \text{Conv} \left\{ v_A = v_0 - \sum_{i \in A} \Delta^i \mid A \subset \{1, \ldots, n\} \right\} \]
with $v_0 = \sum_{i=1}^n I^i$. In particular, the non-enriched tropical intersection multiplicity at $p$ (see Definition 3.3) equals
\[ \text{mult}_p(V_1, \ldots, V_n) = \text{Vol}(P) = |\det M| = m. \]

Furthermore, we let $e = (1, \ldots, 1) \in \mathbb{Z}^n$ and $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ with 1 in the $i$th position.

### 4.2. Enriched tropical intersection multiplicity

We want to define the enriched intersection multiplicity to agree with the local index, as defined in Definition 2.20, at the zero of the section of $V := O(d_1) \oplus O(d_2) \oplus \ldots \oplus O(d_n) \to \mathbb{P}_k^n$ defined by the $n$ hypersurfaces in $\mathbb{P}_k^n$. Assume our tropical hypersurfaces have enriched Viro polynomials $f_1, \ldots, f_n$ in $k[[t]][x_1, \ldots, x_n]$. We have seen that the local index in the Poincaré-Hopf theorem for the vector bundle $V$ equals the local $k^1$-degree of $(f_1, \ldots, f_n)$: $k^1_k(k[t]) \to k^1_k(k[t])$ for which we have an explicit formula, namely the trace of the determinant of the Jacobian evaluated at the zero (see (6)). Note that to $f \in k[[t]][x_1, \ldots, x_n]$ we can associate an enriched Viro polynomial $f_\circ$ by replacing the coefficients of $f$ by its initials. We say that $f$ is a defining polynomial for the enriched tropical hypersurface with enriched Viro polynomial $f_\circ$. This motivates the following definition.

For the following definition of the enriched tropical intersection multiplicity we will assume that the underlying tropical hypersurfaces intersect tropically transversally at an intersection point $p$. Lemma 3.4 also holds for tropical non-transverse intersections. However, in the enriched setting, the local contribution given by a small perturbation is not invariant and hence we restraint our consideration to the transversal case.

**Definition 4.1.** Let $\tilde{V}_1, \ldots, \tilde{V}_n$ be $n$ tropical hypersurfaces in $\mathbb{R}^n$ with defining polynomials $f_1, \ldots, f_n \in k[[t]][x_1, \ldots, x_n]$. Let $p$ be an intersection point $p \in \tilde{V}_1 \cap \ldots \cap \tilde{V}_n$ at which the $V_i$ intersect tropically transversally. Assume that there are exactly $s$ closed points $z^{(1)}, \ldots, z^{(s)}$ in $\mathbb{A}^n_k(k[t])$ such that $-\text{val}(z^{(i)}) = p$. We define the enriched intersection multiplicity of $\tilde{V}_1, \ldots, \tilde{V}_n$ at $p$ to be
\[ \hat{\text{mult}}_p(\tilde{V}_1, \ldots, \tilde{V}_n) := \sum_{i=1}^s \text{Tr}_{k(z^{(i)})/k(t)} \left( \left( \det \text{Jac}(f_1, \ldots, f_n)(z^{(i)}) \right) \right) \in GW(k[[t]]), \]
where $\kappa(z^{(i)})$ is the residue field of $z^{(i)}$.

By the assumption on the characteristic of $k$, we have that $\kappa(z^{(i)}) = L_i \{t\}$ for some finite separable field extension $L_i$ of $k$ for $i = 1, \ldots, s$. In particular, we can
collect all $\kappa(z^{(i)})$ in a finite étale algebra

$$E_t := \prod_{i=1}^{s} \kappa(z^{(i)}) = L_1 \{ [t] \} \times \cdots \times L_q \{ [t] \}.$$

Since $\text{Tr}_{E_t/k([t])} = \sum_i \text{Tr}_{L_i([t])/k([t])}$ we have

$$\text{mult}_p(\tilde{V}_1, \ldots, \tilde{V}_n) = \text{Tr}_{E_t/k([t])}(\langle \det \text{Jac}(f_1, \ldots, f_n)(z) \rangle) \in \text{GW}(k([t]))$$

where $z = (z^{(1)}, \ldots, z^{(s)}) \in E_t$.

Let

$$f_i^{\text{local}} = \alpha_i z^I_i + \beta_i z^J_i$$

be the local binomial equation of $\tilde{V}_i$ at $p$, for $i = 1, \ldots, n$. Set

$$E = \frac{k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]}{(\alpha_i x^I_i + \beta_i)_{i=1, \ldots, n}} \cong L_1 \times \cdots \times L_s.$$

Using the isomorphism $\text{GW}(L_i([t])) \cong \text{GW}(L_i)$ from Example 2.7 we identify (11) with

$$\text{mult}_p(\tilde{V}_1, \ldots, \tilde{V}_n) = \text{Tr}_{E/k}(\langle \det \text{Jac}(f_1^{\text{local}}, \ldots, f_n^{\text{local}})(z_o) \rangle) \in \text{GW}(k)$$

where $z_o = (z_1, \ldots, z_n) \in E^n$ defined by the initials of $z$.

**Proposition 4.2.** With the notation from Definition 4.1, if $\tilde{V}_1, \ldots, \tilde{V}_n$ intersect tropically transversely at $p$, then

$$\langle \det \text{Jac}(f_1^{\text{local}}, \ldots, f_n^{\text{local}})(z) \rangle = \langle \det M \cdot \prod_{i=1}^{n} \alpha_i \cdot \sum_{i=1}^{n} z^{I_i - e} \rangle \in \text{GW}(E),$$

where $e = (1, \ldots, 1) \in \mathbb{Z}^n$. In particular, the enriched intersection multiplicity at $p$ equals

$$\text{mult}_p(\tilde{V}_1, \ldots, \tilde{V}_n) = \text{Tr}_{E/k} \left( \langle \det M \cdot \prod_{i=1}^{n} \alpha_i \cdot \sum_{i=1}^{n} z^{I_i - e} \rangle \right) \in \text{GW}(k).$$

**Proof.** We calculate that

$$\det \frac{\partial f_i^{\text{local}}}{\partial x_j}(z_0)_{i,j} = \det(I^i_j \alpha_i z^{I_i - e_j} + J^i_j \beta_i z^{J_i - e_j})_{i,j}$$

$$= \det(I^i_j \alpha_i z^{I_i - e_j} - J^i_j \alpha_i z^{I_i - e_j})_{i,j}$$

$$= \prod_{i=1}^{n} \alpha_i \cdot \sum_{i=1}^{n} z^{I_i - e} \cdot \det(I^i_j - J^i_j)_{i,j}.$$

The following Lemma shows that the enriched intersection multiplicity as calculated in Proposition 4.2 is independent of the order of the exponent vectors $I^j$ and $J^j$, for $j = 1, \ldots, n$. 

\[ \Box \]
Lemma 4.3. The determinant of the Jacobian

\[ \det M \cdot \prod_{i=1}^{n} \alpha_i \cdot \sum_{i=1}^{n} I^i - e \]

is invariant under the exchange of the roles of \( I^j \) and \( J^j \) and simultaneously the roles of \( \alpha_j \) and \( \beta_j \) for any \( j \in \{1, \ldots, n\} \).

Proof. Recall that we have

\[ f_{\text{local}}^i(z_0) = \alpha_i z_0^I + \beta_i z_0^J = 0 \]

for \( i = 1, \ldots, n \), or equivalently,

\[ \frac{\beta_i}{\alpha_i} z_0^{J^i - I^i} = 1. \]

If we exchange the roles of \( I^j \) and \( J^j \), we replace \( \Delta^j = J^j - I^j \) by \( -\Delta^j = I^j - J^j \) and then \( \det M \cdot \prod_{i=1}^{n} \alpha_i \cdot \sum_{i=1}^{n} I^i - e \) becomes

\[
(- \det M) \cdot \prod_{i=1}^{n} \alpha_i \cdot \beta_j \cdot \sum_{i=1}^{n} I^i - e + J^j - I^j
\]

\[
= \left( - \frac{\beta_j}{\alpha_j} z_0^{J^j - I^j} \right) \cdot \det M \cdot \prod_{i=1}^{n} \alpha_i \cdot \sum_{i=1}^{n} I^i - e
\]

\[ = \det M \cdot \prod_{i=1}^{n} \alpha_i \cdot \sum_{i=1}^{n} I^i - e. \]

\[ (15) \]

4.3. A combinatorial formula for \( \text{mult}_p(\widetilde{V}_1, \ldots, \widetilde{V}_n) \). We identify the formula for the enriched intersection multiplicity from Proposition 4.2 with an element of \( \text{GW}(k) \) which can be read of the dual subdivision of \( \widetilde{V}_1 \cup \ldots \cup \widetilde{V}_n \). We will see that the intersection multiplicity we computed in Proposition 4.2 is determined by coefficients at the “odd vertices” in the dual subdivision of \( \widetilde{V}_1 \cup \ldots \cup \widetilde{V}_n \) in the sense of the following definition.

Definition 4.4. We call a lattice point \( v = (v_1, v_2, \ldots, v_n) \in \mathbb{Z}^n \) odd, if its class in \( (\mathbb{Z}/2\mathbb{Z})^n \) equals \( (1, 1, \ldots, 1) \).

Let \( p \) be an intersection point of enriched tropical hypersurfaces \( \widetilde{V}_1, \ldots, \widetilde{V}_n \) and let \( P \) be the parallelepiped in the dual subdivision of \( \widetilde{V}_1 \cup \ldots \cup \widetilde{V}_n \) dual to \( p \). Recall that the local binomials equation of \( \widetilde{V}_i \) at \( p \) are of the form

\[ f_{\text{local}}^i = \alpha_i x^I + \beta_i x^J \]

with \( \alpha_i, \beta_i \in k^\times \) and

\[ \Delta^i = I^i - J^i \]

for \( i = 1, \ldots, n \). Let \( v \) be a corner vertex of \( P \). Then \( v \) can be uniquely expressed as

\[ v = v_0 - \sum_{i \in A} \Delta^i =: v_A \]
for some subset $A \subset \{1, \ldots, n\}$, where $v_0 = \sum_{i=1}^{n} I^i$. The coefficient of $v_A$ equals
\[
\alpha_{v_A} = \prod_{i \in A} \alpha_i \prod_{i \in A} \beta_i
\]
by Lemma 3.8. Furthermore, we define the sign of the vertex $v_A$ with respect to the parallelepiped $P$ as
\[
\epsilon_P(v) := (-1)^{\# A} \cdot \text{sign}(\det M).
\]
and say that
\[
\gamma_{v_A} := \epsilon_P(v) \cdot \alpha_{v_A}
\]
is the signed coefficient of $v_A$.

**Example 4.5.** Figure 7 shows an intersection of two tropical curves $C_1$ and $C_2$ and the dual parallelogram. The determinant $\det M$ records the order in which the curves intersect in the following sense: Choosing the roles of $I^i$ and $J^i$ orients the edges of $C_i$ such that $I^i$ is on the left and $J^i$ is on the right for $i = 1, 2$. We can always arrange the ordering of $I^i$ and $J^i$ so that $\det M = +1$. Swapping $C_1$ and $C_2$ changes the sign of $\det M$ for all intersections of $C_1$ and $C_2$. For the four vertices $v_0 = v_0 = I^1 + I^2$, $v_{(2)} = I^1 + I^2$, $v_{(1)} = J^1 + I^2$ and $v_{(1,2)} = J^1 + J^2$ we get
\[
\epsilon_P(v_0) = +1, \quad \epsilon_P(v_{(1)}) = -1, \quad \epsilon_P(v_{(2)}) = -1, \quad \epsilon_P(v_{(1,2)}) = +1
\]
given that $\text{sign}(\det M) = +1$. This sign can be determined in the following way. When you walk around the vertex $v$ inside $P$ anticlockwise and the edge you start at is dual to an edge of $C_1$ then $\epsilon_P(v) = +1$, if it is dual to an edge of $C_2$ then $\epsilon_P(v) = -1$. The signed coefficients of the vertices equal
\[
\gamma_{v_0} = \alpha_1 \alpha_2, \quad \gamma_{v_{(2)}} = -\alpha_1 \beta_2, \quad \gamma_{v_{(1)}} = -\beta_1 \alpha_2, \quad \gamma_{v_{(1,2)}} = \beta_1 \beta_2.
\]

Geometrically, the sign of the vertex $v$ with respect to the parallelepiped $P$ is the sign of the determinant of the edges of the parallelepiped $P$ adjacent to $v$ oriented outwards from $v$. Namely, if for every $i = 1, \ldots, n$ we put $\epsilon_i$ as the sign $\pm 1$ such that $v + \epsilon_i \Delta^i \in P$, then $\epsilon_P(v) = \text{sign}(\det \epsilon_i \Delta^i) = \prod_{i=1}^{n} \epsilon_i \cdot \text{sign}(\det M)$ where $\det \epsilon_i \Delta^i$ is the determinant of the matrix with columns $\epsilon_i \Delta^i$. In particular, we have that
\[
\epsilon_P(v_m) \cdot m = \det M
\]
where $m = |\det M|$. 

**Figure 7.** An enriched tropical intersection together with the dual parallelogram.
Remark 4.6. Note that the sign of the vertex $\sum_{i=1}^{n} I^i$ is the opposite of the sign of $\sum_{j=1}^{n} I^j + J^j - I^j$ for any $j \in \{1, \ldots, n\}$. For example in the case of curves, we have that the sign of the vertex $v_{p} = I^j + I^2$ is the same as the sign of $v_{\{2\}} = J^1 + J^2$ and opposite of the sign of $v_{\{1\}} = I^1 + I^2$.

Theorem 4.7 (Main theorem). Assume that $k$ is a field of characteristic $0$ or characteristic bigger than the diameter of the Newton polytopes of the enriched Viro polynomials $f_i$ of $\bar{V}_i$ for $i = 1, \ldots, n$. Further, assume $\text{char} k \neq 2$. Let $p$ be an intersection point of enriched tropical hypersurfaces $\bar{V}_1, \ldots, \bar{V}_n$ that intersect tropically transversally at $p$. Let $P$ be the parallelepiped in the dual subdivision of $\bar{V}_1, \ldots, \bar{V}_n$ corresponding to $p$ and let $v_1, \ldots, v_q$ be the odd corner vertices of $P$. Assume the non-enriched tropical intersection multiplicity $\text{mult}_p(V_1, \ldots, V_n)$ equals $m$, then

$$\bar{\text{mult}}_p(\bar{V}_1, \ldots, \bar{V}_n) = \sum_{i=1}^{q} (\epsilon_P(v_i)\alpha_{v_i}) + \frac{m-q}{2} \ h \in \text{GW}(k),$$

where $\alpha_{v_i}$ is the coefficient of the odd vertex $v_i$ in the dual subdivision of $\bar{V}_1 \cup \ldots \cup \bar{V}_n$, for $l = 1, \ldots, q$.

Before proving the theorem we apply it to an example in dimension 2.

Example 4.8. Figure 8 shows the dual subdivision of the union of the tropical cubic and the tropical conic from Figure 6. The parallelograms corresponding to the intersections are highlighted. Let $\alpha_{(3,1)}$ and $\alpha_{(1,3)}$ be the coefficients of the two odd vertices $(3,1)$ and $(1,3)$ of the dual subdivision. Then the intersection corresponding to the upper left parallelogram has enriched intersection multiplicity of rank 1 since the area of the upper left parallelogram is 1. Furthermore, it has one odd corner vertex, namely $(1,3)$ and $\epsilon_P(v)$ at this vertex is +1 (see Example 4.5 for a rule for determining $\epsilon_P(v)$ when $n = 2$). So its enriched intersection multiplicity is $\langle \alpha_{(1,3)} \rangle$. The enriched intersection multiplicity of the intersection corresponding to the upper right parallelogram has rank 2. The upper right parallelogram has two odd corner vertices $(1,3)$ and $(3,1)$. One computes that both signs $\epsilon_P(v)$ at these vertices are $-1$ using Example 4.5 and thus this intersection has enriched intersection multiplicity $\langle -\alpha_{(1,3)} \rangle + \langle -\alpha_{(3,1)} \rangle$. The remaining intersection point is dual to the lower left parallelogram which has area 3, and hence the rank of its enriched intersection multiplicity must be 3. The lower left parallelogram has only one odd corner vertex $(3,1)$, where the sign $\epsilon_P(v)$ equals +1. Note that there is an odd point in the interior of this parallelogram but this does not contribute to the enriched intersection multiplicity, only the corner vertices do. To get something of rank 3, we have to add a hyperbolic form $h$ to $\langle \alpha_{(3,1)} \rangle$ to get the enriched intersection multiplicity $h + \langle \alpha_{(3,1)} \rangle$ of this last intersection point.

In order to prove Theorem 4.7 we use the Smith normal form of the matrix of exponents in the defining equations of the algebra $E$ to construct an isomorphism to an algebra $E'$ with a diagonal presentation. We then compute the enriched multiplicity with the presentation of $E'$ by using Proposition 2.13 and construct an isomorphism in Proposition 4.10 to relate the obtained computation to the combinatorial data given by the defining equations of $E$. 
By assumption we have $|\det M| = m \neq 0$. Hence, there exist matrices $S, T \in \mathbb{Z}^{n \times n}$ both invertible with inverses in $\mathbb{Z}^{n \times n}$ such that
\begin{equation}
SMT = \text{diag}(m_1, m_2, \ldots, m_n),
\end{equation}
with unique positive integers $m_1, \ldots, m_n$ satisfying $m_i|m_{i+1}$ for $i = 1, \ldots, n - 1$. That is, the matrix $M$ is in Smith normal form and the $m_i$ are the elementary divisors. Recall that
\begin{equation}
E = \frac{k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]}{(\alpha_i x^{I_i} + \beta_i x^{J_i})_{i=1,\ldots,n}}.
\end{equation}
The $i$th defining equation of $E$ $\alpha_i x^{I_i} + \beta_i x^{J_i} = 0$ is equivalent to
\begin{equation}
x^{\Delta_i} = -\frac{\beta_i}{\alpha_i} =: \lambda_i
\end{equation}
for $i = 1, \ldots, n$. Since $T$ in (18) is invertible, the ideal generated by the equations in (19) coincides with the ideal generated by
\begin{equation}
\prod_j x_j^{(MT)}_{j,i} = \prod_{j=1}^n \lambda_j^{T_{ji}} =: \mu_i, \quad i = 1, 2, \ldots, n.
\end{equation}
Furthermore, the variable change given by
\begin{equation}
y_i = \prod_{j=1}^n x_j^{(S^{-1})}_{j,i}
\end{equation}
yields an isomorphism of finite étale $k$-algebras between $E$ and the finite étale $k$-algebra
\begin{equation}
E' := \frac{k[y_1, \ldots, y_n]}{(y_i^{m_i} - \mu_i)_{i=1,\ldots,n}}.
\end{equation}
where the $m_i$ are the elementary divisors of $M$ and $\mu_i \in k^\times$ as defined in (20). Therefore, for $m = \lvert \det M \rvert$, $\gamma_o = \text{sign}(\det M) \cdot \prod_{i=1}^n \alpha_i$ and $v_o = \sum_{i=1}^n I^i$, we have that

\[
\text{mult}_p(\overline{V}_1, \ldots, \overline{V}_n) = 4.2 \text{ Tr}_{E/k} \left( \left( \det M \cdot \prod_{i=1}^n \alpha_i \cdot x^{\sum_{i=1}^n I^i - e} \right) \right) = \text{Tr}_{E/k} \left( \left( m \cdot \gamma_o \cdot x^{v_o - e} \right) \right) = \text{Tr}_{E'/k} \left( \left( m \cdot \gamma_o \cdot y^{S(v_o - e)} \right) \right)
\]

and the latter can be computed easily using Proposition 2.13 as shown in the following proposition for the case when $v_o$ is odd. Note that if $v_o$ is odd, then $v_o - e$ has only even entries and thus also $S(v_o - e)$ has only even entries. So $\langle m \cdot \gamma_o \cdot y^{S(v_o - e)} \rangle = \langle m \cdot \gamma_o \rangle$ in $W(E')$.

**Proposition 4.9.** Let $E'$ be the finite étale $k$-algebra defined in Equation (21), then

\[
\text{Tr}_{E'/k} \left( \langle m_1 \cdots m_n \cdot \gamma_o \rangle \right) = \sum_{A'} \left( \gamma_o \cdot \prod_{i \in A'} \mu_i \right) \in W(k),
\]

where the sum runs over all sets $A' \subset \{1, 2, \ldots, n\}$ such that $m_i$ is even for every $i \in A'$.

**Proof.** Let

\[
L_i = \frac{k[y_1, \ldots, y_i]}{(y_j - y_i)}.
\]

Then $E' = L_n$, $k = L_0$ and $L_{i+1} = L_i[y_{i+1}]/(y_{i+1} - \mu_{i+1})$ for $i = 0, \ldots, n - 1$. In particular,

\[
\text{Tr}_{E'/k} \left( \langle m_1 \cdots m_n \cdot \gamma_o \rangle \right) = \text{Tr}_{L_1/L_0} \circ \cdots \circ \text{Tr}_{L_n/L_{n-1}} \left( \langle m_1 \cdots m_n \cdot \gamma_o \rangle \right).
\]

Proposition 2.13 implies

\[
\text{Tr}_{L_n/L_{n-1}} \left( \langle m_1 \cdots m_n \cdot \gamma_o \rangle \right) = \begin{cases} 
\langle m_1 \cdots m_n-1 \cdot \gamma_o \rangle & \text{if } m_i \text{ is odd}, \\
\langle m_1 \cdots m_n-1 \cdot \gamma_o \rangle + \langle m_1 \cdots m_n-1 \cdot \gamma_{v_o} \cdot \mu_n \rangle & \text{if } m_i \text{ is even}.
\end{cases}
\]

Thus, the statement follows from iterating this relation. \qed

Let $P$ be the parallelepiped dual to $p$ and let $A_E \subset \mathbb{Z}^n$ be the set of corner vertices of $P$. For $v_A = v_o - \sum_{i \in A} \Delta^i \in A_E$ for some subset $A \subset \{1, \ldots, n\}$, let

\[
\gamma_{v_A} = \epsilon p(v_A) \cdot \prod_{i \not\in A} \alpha_i \cdot \prod_{i \in A} \beta_i \cdot \text{sign}(\det M) \cdot \prod_{i \not\in A} \alpha_i \cdot \prod_{i \in A} (-\beta_i) = \gamma_o \cdot \prod_{i \in A} \lambda_i
\]

in $k^\times/(k^\times)^2$ be its signed coefficient. Here, $\lambda_i = -\frac{\beta_i}{\alpha_i}$ for $i = 1, \ldots, n$ as in (19) and $\gamma_o = \text{sign}(\det M) \cdot \prod_{i=1}^n \alpha_i$. To $E'$ we also associate a parallelepiped, namely the one with corner vertices

\[
A_{E'} := \{ v_A' = Sv_o - \sum_{i \in A} m_i \cdot e_i | A \subset \{1, \ldots, n\} \}
\]

where $e_i$ is the $i$th standard basis vector, and we define the signed coefficients of these corner vertices to be

\[
\gamma_{v_A'} = \gamma_o \cdot \prod_{i \in A} \mu_i \in k^\times/(k^\times)^2.
\]
To prove our main theorem we will first show that there is a bijection $\phi: \Lambda_E \rightarrow \Lambda'_E$ such that $\gamma_{v_A} = \gamma_{\phi(v_A)}$ in $k^\times/(k^\times)^2$ for all $A \subset \{1, \ldots, n\}$.

Recall that $T$ in (18) is the product of column operations on $M$ where one is allowed to swap columns, multiply a column by $-1$, and add a column to another column. We construct a bijection $\phi$ as described above whenever we perform one of the operations listed above, or any set of row operations. To show this, assume you start with a matrix $M = (\Delta^1, \ldots, \Delta^n)$ with columns $\Delta^i$, and with $\lambda_i \in k^\times$ for $i = 1, \ldots, n$. Further, let $M' = (\Delta'^1, \ldots, \Delta'^n) \in \mathbb{Z}^{n \times n}$ be the matrix with columns $\Delta'^i$ for $i = 1, \ldots, n$ that one obtains after having applied one of the operations listed above, that is $M' = M \cdot T$ or $M' = S \cdot M$ where $T$ is one of the column operations and $S$ is a set of row operations. For some fixed $v_o \in \mathbb{Z}^n$ let

$$\Lambda = \{ v_A = v_o - \sum_{i \in A} \Delta^i | A \subset \{1, \ldots, n\} \}$$

and

$$\Lambda' = \{ v'_A = Sv_o - \sum_{i \in A} \Delta'^i | A \subset \{1, \ldots, n\} \}.$$ 

Furthermore, define $\lambda'_i = \lambda_i$ in case of a row operation and define $\lambda' := \prod_{j=1}^n \lambda'_j$ in case of a column operation. Finally for $v_A \in \Lambda$ let

$$\gamma_{v_A} := \gamma_o \cdot \prod_{i \in A} \lambda_i \in k^\times/(k^\times)^2$$

and $v'_A \in \Lambda'$ let

$$\gamma_{v'_A} := \gamma_o \cdot \prod_{i \in A} \lambda'_i \in k^\times/(k^\times)^2$$

for some fixed $\gamma_o \in k^\times$.

**Proposition 4.10.** There is a bijection

$$\phi: \Lambda \rightarrow \Lambda'$$

such that $\gamma_{v_A} = \gamma_{\phi(v_A)}$ in $k^\times/(k^\times)^2$ for all $A \subset \{1, \ldots, n\}$. Furthermore,

$$v_A - v_o \equiv 0 \mod 2 \iff \phi(v_A) - Sv_o \equiv 0 \mod 2$$

**Proof.** We construct this bijection for every allowed row or column operation. We start with the column operations. If the operation is swapping the $i$th column with the $j$th column in $M$ to obtain $M'$: Then one can define $\phi$ as follows.
QUADRATICALLY ENRICHED TROPICAL INTERSECTIONS

- When \( i, j \notin A \) or \( i, j \in A \), we define \( \phi(v_A) = v'_A \), since

\[
v_A = v_o - \sum_{l \in A} \Delta^l = v_o - \sum_{l \in A} \Delta^l = v'_A \quad \text{and} \quad \gamma_{v_A} = \gamma_o \cdot \prod_{l \in A} \lambda_l = \gamma_o \cdot \prod_{l \in A} \lambda'_l = \gamma_{v'_A}.\]

- When \( i \notin A \) and \( j \in A \), observe that

\[
v'_{(A \setminus \{j\}) \cup \{i\}} = v_o - \sum_{l \in A} \Delta^l + \Delta^j - \Delta^i = v_o - \sum_{l \in A} \Delta^l = v_A
\]

and

\[
\gamma_{v'_{(A \setminus \{j\}) \cup \{i\}}} = \gamma_o \cdot \prod_{l \in (A \setminus \{j\}) \cup \{i\}} \lambda'_l = \gamma_o \cdot \prod_{l \in A} \lambda_l = \gamma_{v_A}.
\]

We set \( \phi(v_A) = v'_{(A \setminus \{j\}) \cup \{i\}} \). Alike, when \( i \in A \) and \( j \notin A \), let \( \phi(v_A) = v'_{(A \setminus \{i\}) \cup \{j\}} \).

If the operation is multiplying the \( i \)-th column by \(-1\): We define \( \phi \) by \( \phi(v_A) = v'_A \). We have that \( \gamma_{v_A} = \gamma_{\phi(v_A)} \) in \( k^\times/(k^\times)^2 \), since \( \gamma_{v_A} = \gamma_{\phi(v_A)} \) if \( i \notin A \), or \( \gamma_{v_A} = \gamma_{v'_A} \cdot \lambda_i^2 \) if \( i \in A \).

Lastly, if the operation is adding the \( i \)-th column to the \( j \)-th column: We define \( \phi \) as follows (see Figure 9 for an illustration).

- When \( j \notin A \), we define \( \phi(v_A) = v'_A \) since \( v_A = v'_A \) and \( \gamma_{v_A} = \gamma_{v'_A} \). In the example in Figure 9, these are the two bottom vertices.

- When \( j \in A \) and \( i \notin A \), we define \( \phi(v_A) = v'_{A \cup \{i\}} \) since \( v_A = v'_{A \cup \{i\}} + 2\Delta^i \) and

\[
\gamma_{v'_{A \cup \{i\}}} = \gamma_o \cdot \prod_{l \in A} \lambda_l \cdot \lambda_i^2 = \gamma_{v_A} \in k^\times/(k^\times)^2.
\]

In the example in Figure 9, the vertex \( v_A \) is the upper left vertex in the left parallelogram and \( v'_{A \cup \{i\}} \) is the upper right one in the right parallelogram.

- When \( j \in A \) and \( i \in A \), we define \( \phi(v_A) = v'_{A \setminus \{i\}} \) since \( v_A = v'_{A \setminus \{i\}} \) and

\[
\gamma_{v'_{A \setminus \{i\}}} = \gamma_o \cdot \prod_{l \in A} \lambda_l = \gamma_{v_A}.
\]

In the example in Figure 9, the vertex \( v_A \) is the upper right vertex in the left parallelogram and \( v'_{A \setminus \{i\}} \) is the upper left one in the right parallelogram.

One checks easily that \( \phi \) defined as in all column operations above satisfies the relation (23). It remains to look at the row operations. Note that for row operations we always have \( \gamma_{v_A} = \gamma_{v'_A} \) for \( A \subset \{1, \ldots, n\} \). We set \( \phi(v_A) = v'_A \). Since \( v_A - v_o = \sum_{i \in A} \Delta^i = M \cdot (\sum_{i \in A} e_i) \) and \( v'_A - Sv_o = \sum_{i \in A} \Delta^i = S M \cdot (\sum_{i \in A} e_i) \), the relation

\[
v_A - v_o \equiv 0 \pmod{2} \iff v'_A - Sv_o \equiv 0 \pmod{2}
\]

holds since the class of \( S \) modulo two is invertible.

We will prove Theorem 4.7 by applying Proposition 4.10 several times, namely until we get a finite étale \( k \)-algebra of the form \( k[y_1, \ldots, y_n]/(y_i^{m_i} - \mu_i)_{i=1,\ldots,n} \) where the \( m_i \) are the elementary divisors of \( M_i \), as in Equation (21), and using Proposition 4.9 and the relation in Equation (23) to show that the enriched intersection multiplicity \( \text{mult}_p(V_1, \ldots, V_n) \) is given by the sum over the odd vertices \( \Lambda_E^{odd} \subset \Lambda_E \) of the quadratic forms given by the signed coefficients.
Example 4.11. Here is an example of how to get from \( M = \begin{pmatrix} 3 & 2 \\ 5 & 4 \end{pmatrix} \) to \( M' = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \) using only the allowed row and column operations. To shorten the example, we sometimes performed several operations in one step. The first column of the following table consists of the matrices \( M \), the second column consists of the elements. Further, let \( \Lambda \) be the set of odd vertices \( \Lambda_{odd} \) in the \( k^\times/(k^\times)^2 \) for \( A \subset \{1, 2\} \). One gets from the first matrix to the second by multiplying with \( T_1 = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \) from the right (column operation), from the second to the third by multiplying with \( T_2 = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} \) from the left (column operation) and from the third to the fourth by multiplying with \( S = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \) from the right (row operation).

\[
\begin{array}{c|ccc}
\begin{pmatrix} 3 & 2 \\ 5 & 4 \end{pmatrix} & x_1^3 x_2^2 = \lambda_1 & x_1^2 x_2^3 = \lambda_2 & \gamma_\emptyset = \gamma_0 \\
\begin{pmatrix} 1 & 2 \\ 1 & 4 \end{pmatrix} & x_1 x_2 = \lambda_1 \lambda_2^{-1} & x_1^2 x_2^2 = \lambda_2 & \gamma_{\{1\}} = \gamma_0 \lambda_1 \\
\begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} & x_1 x_2 = \lambda_1 \lambda_2^{-1} & x_2^2 = \lambda_1^{-2} \lambda_2^3 & \gamma_{\{2\}} = \gamma_0 \lambda_2 \\
\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} & x_1 = \lambda_1 \lambda_2^{-1} & x_2^2 = \lambda_1^{-2} \lambda_2^3 & \gamma_{\{1, 2\}} = \gamma_0 \lambda_1 \lambda_2 \\
\end{array}
\]

In each step, the set of odd vertices \( \Lambda_{odd} \) is the same, either empty, or one of the sets \( \{v_\emptyset, v_{\{1\}}, v_{\{2\}}\} \) or \( \{v_{\{1\}}, v_{\{1, 2\}}\} \). Remark that in each of the three cases \( \sum_{v_A} \gamma(v_A) \) is the same for every row in the table above.

**Proof of Theorem 4.7.** Recall from (21) that

\[
E \cong E' = \frac{k[y_1, \ldots, y_n]}{(y_i^m - \mu_i)_{i=1,\ldots,n}}
\]

and the \( m_i \) are the elementary divisors of \( M \).

As before \( \Lambda_E = \{v_0 - \sum_{i \in A} \Delta_i A \subset \{1, \ldots, n\}\} \) and \( \Lambda_{odd} \) is its subset of odd elements. Further, let \( \Lambda_{E'} = \{Sv_0 - \sum_{i \in A} m_i \cdot e_i A \subset \{1, \ldots, n\}\} \). For \( v_A = v_0 - \sum_{i \in A} \Delta_i \in \Lambda_E \) set

\[
\gamma_{v_A} = \epsilon_p(v_A) \cdot \alpha(v_A) = \gamma_0 \cdot \prod_{i \in A} \lambda_i
\]

and for \( v'_A = Sv_0 - \sum_{i \in A} m_i \cdot e_i \) let

\[
v'_A = \gamma_0 \cdot \prod_{i \in A} \mu_i.
\]
After applying Proposition 4.10 finitely many times, we get that a bijection \( \phi : \Lambda' \rightarrow \Lambda_E \) such that \( \gamma_{v_A} = \gamma_{\phi(v_A)} \) in \( k^\times/(k^\times)^2 \) for all \( A \subset \{1, \ldots, n\} \), and such that \( v_A - v_0 \equiv 0 \) mod 2 if and only if \( \phi(v_A) - Sv_0 \equiv 0 \) mod 2.

If \( \Lambda_E^{\text{odd}} \) is not empty, by Lemma 4.3 we can assume that \( v_0 \) is odd without loss of generality. In this case,

\[
\text{Tr}_{E/k} \left( \langle m \cdot \gamma_0 \cdot x^{v_0 - e} \rangle \right) = \text{Tr}_{E'/k} \left( \langle m \cdot \gamma_0 \cdot y^{S(v_0 - e)} \rangle \right) = \text{Tr}_{E'/k} \left( \langle (m_1 \cdots m_n \cdot \gamma_0) \rangle \right)
\]

and by Proposition 4.9

\[
\text{Tr}_{E'/k} \left( \langle (m_1 \cdots m_n \cdot \gamma_0) \rangle \right) = \sum_{A'} \left\langle \gamma_0 \cdot \prod_{i \in A'} \mu_i \right\rangle,
\]

where the sum runs over all sets \( A' \subset \{1, 2, \ldots, n\} \) such that \( m_i \) is even for every \( i \in A' \), i.e. the sets such that \( \phi(v_A) - Sv_0 \equiv 0 \) mod 2. By Proposition 4.10, this sum runs over all sets \( A \) such that \( v_A - v_0 \equiv 0 \) mod 2, or equivalently, over all odd vertices \( v \in \Lambda_E^{\text{odd}} \). Since \( \gamma_{v_A} = \gamma_{\phi(v_A)} \) in \( k^\times/(k^\times)^2 \) for all \( A \subset \{1, \ldots, n\} \), then

\[
\text{Tr}_{E/k} \left( \langle m \cdot \gamma_0 \cdot x^{v_0 - e} \rangle \right) = \sum_{v \in \Lambda_E^{\text{odd}}} \langle \gamma_n \rangle \overset{(24)}{=} \sum_{v \in \Lambda_E^{\text{odd}}} \langle \epsilon_P(v) \alpha_v \rangle,
\]

and thus the statement follows.

Lastly, if \( \Lambda_E^{\text{odd}} = \emptyset \), then there is a \( j \) such that \( (v_0)_j \equiv 0 \) mod 2 and \( \Delta_j \equiv 0 \) mod 2 for every \( i \). For this \( j \) let

\[
L = \frac{k[x_1, \ldots, x_{j-1}, y, x_{j+1}, \ldots, x_n]}{(x^{\Delta_j} - \lambda_j^{1/2})_{i=1, \ldots, n}}
\]

where \( \Delta_j = \Delta_j^l / 2 \) and \( \Delta_j^l = \Delta_j^l \) for \( l \neq j \). Then \( E = L[x_j]/(x_j^2 - y) \) and thus \( \dim_L E = 2 \). Hence,

\[
\text{Tr}_{E/k} \left( \langle m \cdot \gamma_0 \cdot x^{v_0 - e} \rangle \right) = \text{Tr}_{L/k} \circ \text{Tr}_{E/L} \left( \langle m \cdot \gamma_0 \cdot x^{v_0 - e} \rangle \right) \overset{2 \text{.13}}{=} \text{Tr}_{L/k}(0) = 0
\]

in \( W(k) \), by Proposition 2.13 since \( x_j \in E \setminus L \) is not a square in \( E \). This agrees with the formula in the statement of the theorem since \( P \) has no odd corner vertices.

\[ \square \]

5. Enriched Tropical Bézout and Bernstein-Kushnirenko theorems

We use properties of toric varieties to give applications of the computation we obtained in the precedent sections. For more details on toric varieties we refer to [6] and [7].

5.1. A tropical proof of Bézout’s theorem enriched in \( \text{GW}(k) \).

With the combinatorial formulas in Theorem 4.7 for the enriched intersection multiplicity, we can quadratically enrich the proof of the tropical Bézout theorem 3.5. The resulting count agrees with McKean’s nontropical Bézout’s theorem 2.22 in the relatively orientable case. In the non-relatively orientable case, we do not get an invariant result for the sum of enriched intersection multiplicities at the intersection points as expected. However, our methods tell us all possible counts for this sum.

The proof of the enriched tropical Bézout theorem is an easy corollary of the following Proposition.
Proposition 5.1. Let $V_1, \ldots, V_n$ be $n$ tropical hypersurfaces in $\mathbb{R}^n$ with associated Newton polytopes $\Delta_1, \ldots, \Delta_n$, respectively. Let $v$ be a lattice point in the interior of the Minkowski sum $\Delta_1 + \ldots + \Delta_n$ and let

$$P_v := \{ P \in \text{DS} \left( \bigcup_{i=1}^{n} V_i \right) \text{ dual to some } p \in V_1 \cap \ldots \cap V_n, \text{ s.t. } v \text{ is a corner vertex of } P \}. $$

If the hypersurfaces $V_1, \ldots, V_n$ intersect tropically transversely, then

1. The cardinality $\text{Card}(P_v)$ of $P_v$ is even.
2. There are equally many parallelepipeds $P$ in $P_v$ such that the sign $\epsilon_P(v)$ (as defined in (16)) in $P$ is positive as there are with negative sign

$$\text{Card}(\{ P \in P_v \mid \epsilon_P(v) = +1 \}) = \text{Card}(\{ P \in P_v \mid \epsilon_P(v) = -1 \}).$$

Proof. Due to the transversality hypothesis, the hypersurfaces $V_1, \ldots, V_{n-1}$ intersect along a tropical curve $C \subset \mathbb{R}^n$. This curve intersects $V_n$ tropically transversely. Let us denote by $R_v \subset \mathbb{R}^n$ the connected component of $\mathbb{R}^n \setminus V_1 \cup \ldots \cup V_n$ where the monomial of the exponent $v$ is maximal (see Figure 10). Since $v$ is an inner lattice point of the dual polytope, the region $R_v$ is a bounded polytope. If $P_v$ is empty, our assertion follows. Otherwise, let $p \in C \cap V_n$ be an intersection point such that its dual polytope $P \in P_v$. Let $\gamma$ be the connected component of $C \cap \partial R_v$ containing the point $p$. We claim that $\gamma$ is a piecewise linear path. Namely, the set $\gamma$ is formed by two segments of edges of $C$ which contain an intersection point with $V_n$, together with possible bounded edges (for example $\gamma_2$ in Figure 10), or only one segment containing two intersection points with $V_n$ (for example $\gamma_1$ in Figure 10). In particular, exactly one of the two intersection points of $\gamma$ with $V_n$ is $p$. If a vertex $w$ of $C$ is in $\gamma$, its valency in $\gamma$ is 2, corresponding to the edges in $C$ adjacent to the region $R_v$. Since $R_v$ is a bounded polytope, the curve $\gamma$ is compact, having an endpoint $q$ that is in $V_n$. Indeed, the point $q \neq p$ and cannot be a vertex of $C$, hence it is an inner point of an edge of $C$. Since there are no changes in the monomials where the maximum is achieved in the interior of an edge, this change...
is produced by the hypersurface $V_n$. Therefore, the paths $\gamma$ establish a partition of the parallelepipeds in $P_v$ into pairs $P_v = \bigsqcup \{P_\gamma, Q_\gamma\}$ with $P_\gamma$ and $Q_\gamma$ the dual parallelepipeds of the endpoints $p_\gamma$ and $q_\gamma$ of $\gamma$ for each such path $\gamma$ (see Figure 10).

We transfer the frame in $p$ through $\gamma$ to show that the polytopes corresponding to the endpoints of $\gamma$ have opposite sign at the vertex $v$ (see Figure 11). For that, let us start by recalling that the sign $\epsilon_P(v)$ is the sign of the determinant $\left(\Delta^i\right)_{i=1}^n$, where every $\Delta^i$ has been oriented in such a way that $v + \Delta^i \in P$ (see (17)). This oriented vector is the normal vector of the facet of $V_i$ pointing outwards to the region $R_v$. Let us define $w_p := \wedge_{i=1}^{n-1} \Delta^i$ (the vector of alternating minors of the $(n-1) \times n$ matrix $(\pm \Delta^i)_{i=1}^{n-1}$). The vector $w_p$ is a direction vector of the edge of $C$ containing $p$, albeit not a primitive one. Moreover, the sign $\epsilon_P(v) = \text{sign}(\det(\Delta^i)_{i=1}^n) = \text{sign}(w_p \cdot \Delta^n)$ can be computed as the sign of $w_p$ with the normal vector of the facet of $V_n$ containing $p$, oriented outwards the region $R_v$. We can define $w_\gamma(t)$ for every point of $\gamma$ that is an inner point of an edge of $C$. If we orient $\gamma$ as a path starting at $p$, at every point $\gamma(t)$, the orientations of $w_\gamma(t)$ and $\gamma$ either coincide for all $t$ or are opposite for all $t$, since the relative position of the normal vectors of the facets of $V_i$ at $\gamma(t)$ does not change in the boundary of $R_v$. This implies that exactly one of the vectors $w_p$ at $p \in V_n$ or $w_q$ at $q \in V_n$ is oriented towards the region $R_v$ while the other one is not. Hence, the endpoints of $\gamma$ have opposite signs. \hfill $\square$

Let $\Delta_d$ be the Newton polygon of a general degree $d$ polynomial in $n$ variables. Note that if $V_1, \ldots, V_n$ are tropical hypersurfaces with Newton polytopes $\Delta_{d_1}, \ldots, \Delta_{d_n}$, then the union of hypersurfaces $V_1 \cup \ldots \cup V_n$ has Newton polytope $\Delta_{d_1 + \ldots + d_n}$.

**Corollary 5.2 (Enriched tropical Bézout).** Assume that $\text{char } k \neq 2$. Let $\tilde{V}_1, \ldots, \tilde{V}_n$ be enriched tropical hypersurfaces in $\mathbb{R}^n$ with Newton polytopes $\Delta_{d_1}, \ldots, \Delta_{d_n}$. Assume that $\tilde{V}_1, \ldots, \tilde{V}_n$ intersect tropically transversally. If $\sum_{i=1}^n d_i \equiv n + 1 \mod 2$, then

$$\sum_{p \in \tilde{V}_1 \cap \ldots \cap \tilde{V}_n} \mult_p(\tilde{V}_1, \ldots, \tilde{V}_n) = \frac{d_1 \cdots d_n}{2} h \in GW(k).$$
Example 5.3. We continue Example 4.8 as an example of the statement in Corollary 5.2, that is we compute sum of the enriched tropical intersection multiplicities of the intersection of the two enriched tropical curves in Figure 8. Again let \( \alpha_{(1,1)} \), \( \alpha_{(1,3)} \) and \( \alpha_{(3,1)} \) be the coefficients at the odd vertices \((1,1), (1,3)\) and \((3,1)\). Recall from Example 4.8 that the enriched intersection multiplicities at the three intersection points are \( \langle \alpha_{(1,3)} \rangle \), \( \langle -\alpha_{(1,3)} \rangle + \langle -\alpha_{(3,1)} \rangle \) and \( \langle \alpha_{3,1} \rangle + h \). Summing up the intersection multiplicities and using the identity \( \langle a \rangle + \langle -a \rangle = h \) for \( a \in k^x \) from Remark 2.3 we get

\[
\langle \alpha_{(1,3)} \rangle + \langle -\alpha_{(1,3)} \rangle + \langle -\alpha_{(3,1)} \rangle + \langle \alpha_{3,1} \rangle + h = 3h
\]

which coincides with McKeans enriched (non-tropical) Bézout Theorem 2.22.

Proof of Corollary 5.2. Note that in the relatively orientable case, that is, when \( \sum_{i=1}^n d_i \) is congruent to \( n + 1 \mod 2 \), all odd points in the dual subdivision of \( \tilde{V}_1 \cup \ldots \cup \tilde{V}_n \) lie in the interior of \( \Delta_{d_1+\ldots+d_n} = \Delta_1 + \ldots + \Delta_n \) and none on the boundary.

We know from the classical tropical non-enriched Bézout theorem 3.5 that the sum of \( \mult_p(V_1, \ldots, V_n) \) over all the intersections of the \( V_i \) is equal to \( d_1 \cdots d_n \). Since

\[\text{rank} \mult_p(\tilde{V}_1, \ldots, \tilde{V}_n) = \mult_p(V_1, \ldots, V_n)\]

we get that the rank of

\[
\sum_{\text{intersections } p} \mult_p(\tilde{V}_1, \ldots, \tilde{V}_n)
\]
equals \( d_1 \cdots d_n \).

For \( v \) an odd point in the interior of \( \Delta_{d_1+\ldots+d_n} \) let \( N(v) \) be the number of parallelepipeds in the dual subdivision of \( \tilde{V}_1 \cup \ldots \cup \tilde{V}_n \) which correspond to an intersection of the \( V_i \). Let \( \alpha_v \) be the coefficient of \( v \) in \( \text{DS}(\tilde{V}_1 \cup \ldots \cup \tilde{V}_n) \). By Theorem 4.7 and Proposition 5.1 we have

\[
\sum_{p} \mult_p(\tilde{V}_1, \ldots, \tilde{V}_n) = \sum_v \left( \frac{N(v)}{2} \langle \alpha_v \rangle + \frac{N(v)}{2} \langle -\alpha_v \rangle \right) \text{ in } W(k),
\]

where the first sum runs over the intersection points of the \( V_i \) and the second sum runs over the odd vertices in the interior of \( \Delta_{d_1+\ldots+d_n} \). Since \( \langle a \rangle + \langle -a \rangle = 0 \) in \( W(k) \) by (ii) in Remark 2.3, we get that \( \sum_v \left( \frac{n(v)}{2} \langle \alpha_v \rangle + \frac{n(v)}{2} \langle -\alpha_v \rangle \right) = 0 \) in \( W(k) \) and thus \( \sum_{p} \mult_p(\tilde{V}_1, \ldots, \tilde{V}_n) \) equals a multiple of \( h \) in \( GW(k) \).

Finally, recall that an element of \( GW(k) \) is determined by its rank and its image in the Witt group \( W(k) \) (see Remark 2.9). Thus we get

\[
\sum_{p} \mult_p(\tilde{V}_1, \ldots, \tilde{V}_n) = \frac{d_1 \cdots d_n}{2} h \text{ in } GW(k).
\]

\[ \square \]

Remark 5.4. This gives a new proof of McKeans non-tropical quadratically enriched Bézout’s theorem enriched (1) with the following argument. Recall that one proves this theorem by computing the \( \mathbb{A}^1 \)-Euler number \( n^x(V_k) \) of the vector bundle

\[
V_k = \mathcal{O}_{\mathbb{P}^n_k}(d_1) \oplus \ldots \oplus \mathcal{O}_{\mathbb{P}^n_k}(d_n) \rightarrow \mathbb{P}^n_k,
\]
Since $\mathbb{P}^n$ is smooth and proper over $\mathbb{Z}$, the problem can actually defined over $\mathbb{Z}$ and there is a well defined answer $n^{A^1}(V_\mathbb{Z}) \in GW(\mathbb{Z})$. By our main theorem 4.7 we have that

$$n^{A^1}(V_\mathbb{Z}) = \sum_{p \in \tilde{V}_1 \cap \ldots \cap \tilde{V}_n} \overline{\text{mult}}_p(\tilde{V}_1, \ldots, \tilde{V}_n) = \frac{d_1 \cdots d_n}{2} h \in GW(k\{t\})$$

whenever $k$ is a field of characteristic 0 or characteristic bigger that the maximum of the $d_i$’s. The natural map $GW(\mathbb{Z}) \to GW(\mathbb{R}) \cong GW(\mathbb{R}\{t\})$ is an isomorphism that maps $n^{A^1}(V_\mathbb{Z})$ to $n^{A^1}(V_\mathbb{R}\{t\})$. Hence, we get that $n^{A^1}(V_\mathbb{Z})$ is equal to $\frac{d_1 \cdots d_n}{2} h \in GW(\mathbb{Z})$. There is a natural map $GW(\mathbb{Z}) \to GW(k)$ which maps $n^{A^1}(V_\mathbb{Z}) \to n^{A^1}(V_k)$ which proves the theorem over an arbitrary field $k$.

5.1.1. Non-relatively orientable case. In the non-relatively orientable case, that is, when $\sum_{i=1}^n d_i \neq n + 1 \mod 2$, we do not get an invariant count. This can also be seen in our proof for the enriched tropical Bézout theorem: In case $\sum_{i=1}^n d_i \neq n + 1 \mod 2$, not all odd points are in the interior of $\Delta_{d_1 + \ldots + d_n}$, but some are on the boundary. For these points on the boundary, we cannot apply Proposition 5.1.

Example 5.5. Figure 12 shows the intersection of two tropical conics and the dual subdivision of the union of the conics. There are two odd points on the boundary of the Newton polygon of the union. We enrich the two tropical conics by assigning coefficients $\alpha_{(i,j)}$ to a $(i,j) \in \mathbb{Z}^2$. Then the sum over the enriched intersection multiplicities equals

$$h + \langle \alpha_{(3,1)}, -\alpha_{(1,3)} \rangle$$

where $\alpha_{(3,1)}$ is the coefficient of the vertex $(3, 1)$ and $\alpha_{(1,3)}$ is the coefficient of the vertex $(1, 3)$ in the dual subdivision. Hence, the sum depends on the choice of coefficients of the enriched tropical conics, but there is always a hyperbolic summand.

Let $\tilde{V}_1, \ldots, \tilde{V}_n$ be enriched tropical hypersurfaces in $\mathbb{R}^n$, with Newton polytopes $\Delta_{d_1}, \ldots, \Delta_{d_n}$ such that $\sum d_i \neq n + 1 \mod 2$. As suggested in the example, we can find a lower bound for the number of hyperbolic summands in $\sum_p \overline{\text{mult}}_p(\tilde{V}_1, \ldots, \tilde{V}_n)$. For $n$ odd let $m \in \mathbb{Z}$ such that the sum $d_1 + \ldots + d_n = 2m$ and for $n$ even let $m \in \mathbb{Z}$.

**Figure 12.** Intersection of two tropical conics

"QUADRATICALLY ENRICHED TROPICAL INTERsections"
such that the sum \( d_1 + \ldots + d_n = 2m + 1 \). Set
\[
N(m) := \text{the number of odd points on } \Delta_{d_1 + \ldots + d_n}.
\]
The following table computes \( N(m) \) for \( n \leq 6 \).

| \( d_1 = 2m + 1 \) | \( d_1 + d_2 = 2m \) | \( d_1 + d_2 + d_3 = 2m + 1 \) | \( d_1 + \ldots + d_4 = 2m \) | \( d_1 + \ldots + d_5 = 2m + 1 \) | \( d_1 + \ldots + d_6 = 2m \) |
|---------------------|---------------------|---------------------|---------------------|---------------------|---------------------|
| 1                   | \( m \)             | \( m(m + 1) \)      | \( \frac{2}{m+1} \)  | \( m(m + 1)(m - 1)(m + 2) \) | \( \frac{m(m + 1)(m - 1)(m + 2)(m - 2)}{5!} \) |

Since the only non-hyperbolic contribution to \( \sum_{p \in V_1 \cap \ldots \cap V_n} \mu_p(V_1, \ldots, V_n) \) comes from the odd points on the boundary, we get the following Corollary.

**Corollary 5.6** (Enriched tropical Bézout in the non-relatively orientable case). Let \( V_1, \ldots, V_n \) be enriched tropical hypersurfaces in \( \mathbb{R}^n \) with Newton polygons \( \Delta_{d_1}, \ldots, \Delta_{d_n} \) such that \( \sum_{i=1}^n d_i \neq n + 1 \) mod 2. Then
\[
\sum_{\text{intersections } p} \mu_p(V_1, \ldots, V_n) = \frac{d_1 \cdots d_n - r}{2} h + (a_1, \ldots, a_r) \in GW(k)
\]
where \( r \) has to be smaller or equal the number \( N(m) \) of odd points on \( \partial \Delta_{d_1 + \ldots + d_n} \).

**Remark 5.7.** In the non-relatively orientable case McKeown shows that one can orient the vector bundle
\[
V = \mathcal{O}_{\mathbb{P}^n}(d_1) \oplus \ldots \oplus \mathcal{O}_{\mathbb{P}^n}(d_n) \rightarrow \mathbb{P}^n
\]
relative to a divisor at infinity and compute the \( \mathbb{A}^1 \)-Euler number of \( V \) relative to this divisor in the sense of Larson-Voigt [16]. Corollary 5.6 gives us a lower bound for the number of hyperbolic forms in this \( \mathbb{A}^1 \)-Euler number by the same argument as in Remark 5.4.

The lower bound on the number of hyperbolic summands is in Corollary 5.6 is not necessarily strict. For enriched tropical curves we find a better, strict bound.

**Corollary 5.8** (Enriched tropical Bézout for curves in the non-relatively orientable case). Let \( \tilde{C}_1 \) and \( \tilde{C}_2 \) be two enriched tropical curves of degree \( \Delta_{d_1} \) and \( \Delta_{d_2} \), respectively, with \( d_1 + d_2 \equiv 0 \) mod 2 that intersect tropically transversely. Let \( d := \min\{d_1, d_2\} \). Then
\[
\sum_{p \in \tilde{C}_1 \cap \tilde{C}_2} \mu_p(\tilde{C}_1, \tilde{C}_2) = \frac{d_1 \cdot d_2 - d}{2} h + (a_1, \ldots, a_d) \in GW(k)
\]
for some \( a_1, \ldots, a_d \in k^*/(k^*)^2 \).

**Proof.** In case, \( d_1 + d_2 \equiv 0 \) mod 2, there are \( \frac{d_1 + d_2}{2} \) odd points on the boundary of \( \Delta_{d_1} + \Delta_{d_2} \), all lying on the hypotenuse. To get a non-hyperbolic summand, one of the two edges adjacent to an odd vertex on the hypotenuse has to belong to \( \tilde{C}_1 \) and the other one has to belong to \( \tilde{C}_2 \). This can happen at most \( d = \min\{d_1, d_2\} \) times since only \( d_1 \) segments on the hypotenuse of \( \Delta_{d_1} + \Delta_{d_2} \) correspond to edges of \( \tilde{C}_1 \) and \( d_2 \) correspond to edges of \( \tilde{C}_2 \). \( \square \)
5.2. Bernstein-Kushnirenko theorem. The results above do not restrict to hypersurfaces in $\mathbb{P}^n$. The tools of tropical geometry can be applied to toric varieties, where the action of the torus yields a combinatorial approach to their study.

Example 5.9. Let $C_1$ and $C_2$ be two curves in $\mathbb{P}^1 \times \mathbb{P}^1$ defined by $f_1$ and $f_2$ of bidegree $(d_1, d_2)$ and $(e_1, e_2)$, respectively. Then $f_1$ and $f_2$ define a section of
\[
V := \mathcal{O}(d_1, d_2) \oplus \mathcal{O}(e_1, e_2) \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1
\]
where $\mathcal{O}(a, b) = \pi_1^* \mathcal{O}_{\mathbb{P}^1}(a) \otimes \pi_2^* \mathcal{O}_{\mathbb{P}^1}(b)$ and $\pi_i : \mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ is the $i$th projection for $i = 1, 2$. The vector bundle $V$ is relatively orientable if and only if $\det V \otimes \omega_{\pi_1 \times \pi_1}$, that is, $\mathcal{O}(d_1 + e_1 - 2, d_2 + e_2 - 2)$ is a square. This is the case if and only if both $d_1 + e_1$ and $d_2 + e_2$ are even. The Newton polygons $\text{NP}(f_1)$ and $\text{NP}(f_2)$ are rectangles with corners $(0, 0), (d_1, 0), (0, d_2), (d_1, d_2)$, respectively, with corners $(0, 0), (e_1, 0), (0, e_2), (e_1, e_2)$. The Minkowski sum $\text{NP}(f_1) + \text{NP}(f_2)$ is the rectangle with corners $(0, 0), (d_1 + e_1, 0), (0, d_2 + e_2), (d_1 + e_1, d_2 + e_2)$. This rectangle $\text{NP}(f_1) + \text{NP}(f_2)$ has no odd vertices on the boundary if and only if both $d_1 + e_1$ and $d_2 + e_2$ are even, that is exactly when the vector bundle $V$ is relatively orientable. Let $\bar{C}_1$ and $\bar{C}_2$ be two enriched tropical curves with Newton polytopes equal to $\text{NP}(f_1)$ and $\text{NP}(f_2)$ and assume that both $d_1 + e_1$ and $d_2 + e_2$ are even. Then Proposition 5.1 implies that
\[
\sum_{p \in C_1 \cap C_2} \text{mult}_p(\bar{C}_1, \bar{C}_2) = \frac{d_1 e_2 + d_2 e_1}{2} h \in \text{GW}(k).
\]
Equivalently, we get that the $A^1$-Euler number of the vector bundle $V$ equals
\[
n^A(V) = \sum_{p \in C_1 \cap C_2} \text{ind}_p(f_1, f_2)
\]
\[
= \frac{\text{Area}(\text{NP}(f_1) + \text{NP}(f_2)) - \text{Area}(\text{NP}(f_1)) - \text{Area}(\text{NP}(f_2))}{2} h
\]
\[
= \frac{d_1 e_2 + d_2 e_1}{2} h \in \text{GW}(k)
\]
which yields an enriched count of intersection points of two curves in $\mathbb{P}^1 \times \mathbb{P}^1$.

Example 5.10. Let $C_1$ and $C_2$ be two curves in the Hirzebruch surface $\Sigma_n$ defined by $f_1$ and $f_2$ of bidegree $(a_1, b_1)$ and $(a_2, b_2)$, respectively, where $a_i = C_i \cdot F, b_i = C_i \cdot E + na_i$, for $F$ a generic fiber and $E$ the exceptional divisor of $\Sigma_n$. Then $f_1$ and $f_2$ define a section of
\[
V := \mathcal{O}(a_1 E + b_1 F) \oplus \mathcal{O}(a_2 E + b_2 F) \longrightarrow \Sigma_n.
\]
The vector bundle $V$ is relatively orientable if and only if
\[
\det V \otimes \omega_{\Sigma_n} = \mathcal{O}((a_1 + a_2 - 2)E + (b_1 + b_2 - (n + 2)F))
\]
is a square, which is the case if and only if $a_1 + a_2$ is even and $b_1 + b_2 \equiv n$ mod 2. The Newton polygons $\text{NP}(f_1)$ and $\text{NP}(f_2)$ are trapeziums with corners $(0, 0), (a_1 n + b_1, 0), (b_1, a_1), (0, a_1)$ and $(0, 0), (a_2 n + b_2, 0), (b_2, a_2), (0, a_2)$, respectively. The Minkowski sum $\text{NP}(f_1) + \text{NP}(f_2)$ is the trapezium with corners $(0, 0), (a_1 + a_2 n + b_1 + b_2, 0), (b_1 + b_2, a_1 + a_2), (0, a_1 + a_2)$. This trapezium $\text{NP}(f_1) + \text{NP}(f_2)$ has no odd vertices on the boundary if and only if $a_1 + a_2$ is even and $b_1 + b_2 \equiv n$ mod 2, that is exactly when $V$ is relatively orientable. Let $\bar{C}_1$ and $\bar{C}_2$ be two tropical curves
with Newton polytopes equal to \( \text{NP}(f_1) \) and \( \text{NP}(f_2) \) and assume that \( a_1 + a_2 \) is even and \( b_1 + b_2 \equiv n \mod 2 \). Then Proposition 5.1 implies that

\[
\sum_{p \in \mathcal{C}_1 \cap \mathcal{C}_2} \text{mult}_p(\mathcal{C}_1, \mathcal{C}_2) = \frac{a_1 a_2 n + a_1 b_2 + a_2 b_1}{2} h \in \text{GW}(k).
\]

Equivalently, as in the previous example, this coincides with \( n^{A_1}(V) \) and yields an enriched count of intersection points of two curves in \( \Sigma_n \).

The examples above as well as Bézout’s theorem are special cases of a quadratic enrichment of the Bernstein-Kushnirenko theorem. We recall the classical statement of this theorem. Let \( A \) be a finite subset of \( \mathbb{Z}^n \) and let

\[
L_A := \left\{ f \bigg| f(x) = \sum_{I \in A} c_I x^I = \sum_{I \in A} c_I x_1^{I_1} \cdots x_n^{I_n}, c_I \in k \right\}
\]

be the space of Laurent polynomials whose exponents are in \( A \). Let \( \Delta_A \) be the convex hull of the points in \( A \). The classical Bernstein-Kushnirenko theorem says.

**Theorem 5.11** (Bernstein-Kushnirenko theorem). For \( n \) finite subsets \( A_1, \ldots, A_n \) of \( \mathbb{Z}^n \) and for a generic system of equations

\[
f_1(x) = \cdots = f_n(x) = 0
\]

where \( f_i \in L_{A_i} \), the number of solutions in \( (\mathbb{C} \setminus \{0\})^n \) equals the mixed volume

\[
\text{MVol}(\Delta_{A_1}, \ldots, \Delta_{A_n})
\]

Before we state the quadratically enriched version of this theorem, we define the following condition that can be seen as the combinatorial analogue of relative orientability.

**Definition 5.12.** We say that the tuple \( (A_1, \ldots, A_n) \) is **combinatorially oriented** if the Minkowski sum \( \Delta_{A_1} + \cdots + \Delta_{A_n} \) has no odd points on the boundary.

**Example 5.13** (Bézout theorem). Let \( A_i = \Delta_{d_i} \cap \mathbb{Z}^n \) for some positive integer \( d_i \), i.e., \( \Delta_{A_i} = \Delta_{d_i} \) for \( i = 1, \ldots, n \). Then \( (A_1, \ldots, A_n) \) is combinatorially oriented if and if \( \Delta_{d_1} + \cdots + \Delta_{d_n} \) has no odd boundary points which is exactly the case if \( d_1 + \cdots + d_n \equiv n + 1 \mod 2 \), i.e., exactly when the vector bundle

\[
\mathcal{O}_{\mathbb{P}^n}(d_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^n}(d_n) \rightarrow \mathbb{P}^n
\]

is relatively orientable.

In all examples 5.13, 5.9 and 5.10 above the condition of being combinatorially oriented coincides with the condition for the corresponding vector bundle to be relatively orientable which motivates the following conjecture.

**Conjecture 5.14.** Let \( X \) be a smooth toric variety of dimension \( n \) and let \( f_1, \ldots, f_n \) be regular functions on \( X \) such that \( f_i \) is a non-trivial section of a line bundle \( L_i \rightarrow X \) such that the system \( f_1 = \cdots = f_n = 0 \) has a non-empty solution set formed of isolated zeros. Furthermore, let \( A_i := \{ \text{exponents of } f_i \} \subset \mathbb{Z}^n \). Then \( (A_1, \ldots, A_n) \) is combinatorially oriented if and only if the vector bundle \( L_1 \oplus \cdots \oplus L_n \rightarrow X \) is relatively orientable.
Let us say that a variety has the *combinatorial orientability property* if it this conjecture holds for every sum of \( \dim X \) line bundles satisfying the hypothesis. We prove in the following theorem that the class of varieties that satisfy this conjecture is closed under products. In particular, this property holds on products of projective spaces and Hirzebruch surfaces.

**Theorem 5.15.** If \( X_1 \) and \( X_2 \) are smooth toric varieties that satisfy the combinatorial orientability property, then \( X_1 \times X_2 \) also satisfies the combinatorial orientability property.

**Proof.** The product \( X := X_1 \times X_2 \) has a toric structure given by the product of the toric structures on each component. Since \( X \) is smooth, we have that

\[
\text{Pic}(X) \simeq H^2(X, \mathbb{Z}) = H^2(X_1, \mathbb{Z}) \oplus H^2(X_2, \mathbb{Z}).
\]

by the Künneth formula and the fact that \( H^1(X_i, \mathbb{Z}) = 0 \) due to \( X_i \) being a smooth toric variety, \( i = 1, 2 \). Hence, through this isomorphism, every line bundle \( L \) is determined by its bidegree \( \bar{d} = (d_1, d_2) \), where each degree class \( d_i \in H^2(X_i, \mathbb{Z}), i = 1, 2 \). Therefore, for every line bundle \( L \) over \( X \) of degree \( \bar{d} \), there are line bundles \( L^1 \) and \( L^2 \) over \( X_1 \) and \( X_2 \) of degree \( d_1 \) and \( d_2 \), respectively, such that

\[
L = p_1^* L^1 \otimes p_2^* L^2,
\]

where \( p_i : X \rightarrow X_i, i = 1, 2 \) is the component projection. Given this decomposition, the line bundle \( L \) is a square if and only if each \( L^i, i = 1, 2 \), is a square. Moreover, the polytope \( \Delta \) associated to \( L \) in \( \Lambda \otimes \mathbb{R} \) is the product of the polytopes \( \Delta_1 \) and \( \Delta_2 \) associated to \( L_1 \) and \( L_2 \) in \( \Lambda_1 \otimes \mathbb{R} \) and \( \Lambda_2 \otimes \mathbb{R} \), where \( \Lambda_1 \) and \( \Lambda_2 \) are the lattices associated to \( X_1 \) and \( X_2 \), respectively, and the lattice \( \Lambda := \Lambda_1 \times \Lambda_2 \) is the one associated to \( X \). The polytope \( \Delta \) has boundary

\[
\partial(\Delta) = \partial(\Delta^1 \times \Delta^2) = (\partial(\Delta^1) \times \Delta^2) \cup (\Delta^1 \times \partial(\Delta^2)),
\]

and so, the odd lattice points are \( \partial(\Delta)_{\text{odd}} = (\partial(\Delta^1)_{\text{odd}} \times (\Delta^2)_{\text{odd}}) \cup ((\Delta^1)_{\text{odd}} \times \partial(\Delta^2)_{\text{odd}}) \). Now, let \( V \) be the vector bundle \( L_1 \oplus \ldots \oplus L_n \rightarrow X \), where \( n = \dim X \). \( L_i \) is a line bundle over \( X \) and \( A_i \) is the exponent set of a generic section \( f_i \) of \( L_i, i = 1, \ldots, n \). Put \( \Delta_i = \text{Conv}(A_i) \) the convex hull of the set \( A_i \) in \( \Lambda \otimes \mathbb{R} \). Assume that the system \( \{ f_i = 0 \}_{i=1, \ldots, n} \) has an isolated zero. In this case we have that \( (\sum_{i=1}^n \Delta_i)_{\text{odd}} \neq \emptyset \). otherwise there would be an \( i_0 \) for which \( \Delta_{i_0} = \{ \text{pt} \} \) or there would be a vector subspace \( H \subset \Lambda \otimes \mathbb{R} \) of lower dimension, containing all \( \Delta_i \subset H \), which contradicts the fact that the system has only isolated zeros. This implies that \( (\sum_{i=1}^n \Delta_i^1)_{\text{odd}} \neq \emptyset \) and \( (\sum_{i=1}^n \Delta_i^2)_{\text{odd}} \neq \emptyset \). Lastly, since the Minkowski sum commutes with products, the odd lattice points in the boundary of the Minkowski sum satisfy

\[
\partial(\sum_{i=1}^n \Delta_i)_{\text{odd}} = (\partial(\sum_{i=1}^n \Delta_i^1)_{\text{odd}} \times (\sum_{i=1}^n \Delta_i^2)_{\text{odd}}) \cup ((\sum_{i=1}^n \Delta_i^1)_{\text{odd}} \times \partial(\sum_{i=1}^n \Delta_i^2)_{\text{odd}}).
\]

These facts imply our statement. Namely, if the \( n \)-tuple \( (A_1, \ldots, A_n) \) is combinatorially oriented, then \( \partial(\sum_{i=1}^n \Delta_i)_{\text{odd}} = \emptyset \) by definition. Since \( (\sum_{i=1}^n \Delta_i^1)_{\text{odd}} \neq \emptyset \), \( (\sum_{i=1}^n \Delta_i^2)_{\text{odd}} \neq \emptyset \) in this case, we have that the \( n \)-tuple \( (A_1, \ldots, A_n) \) is combinatorially oriented if and only if both of the sets \( \partial(\sum_{i=1}^n \Delta_i^1)_{\text{odd}} \) and \( \partial(\sum_{i=1}^n \Delta_i^2)_{\text{odd}} \) are empty. Since \( X_1 \) and \( X_2 \) satisfy the combinatorial orientability property,
the sets \( \partial(\sum_{i=1}^{n} \Delta_i)^{\text{odd}} \) and \( \partial(\sum_{i=1}^{n} \Delta_i)^{\text{odd}} \) are empty if and only if the vector bundles given by the direct sum of the components of each of the line bundles \( V^1 := L_1^1 \oplus \cdots \oplus L_n^1 \to X_1 \) and \( V^2 := L_1^2 \oplus \cdots \oplus L_n^2 \to X_2 \) are relatively orientable. Finally, the vector bundles \( V_1 \) and \( V_2 \) are relatively orientable if and only the vector bundle \( V \) is relatively orientable since \( \det V = p_1^1 \det V^1 \otimes p_2^2 \det V^2 \) and \( \omega_X = p_1^1 \omega_{X_1} \otimes p_2^2 \omega_{X_2} \), so
\[
\det V \otimes \omega_X = p_1^1 (\det V^1 \otimes \omega_{X_1}) \otimes p_2^2 (\det V^2 \otimes \omega_{X_2})
\]
is a square if and only if \( \det V^1 \otimes \omega_{X_1} \) and \( \det V^2 \otimes \omega_{X_2} \) are squares. \( \square \)

**Example 5.16.** Let \( V_1, V_2, \ldots, V_n \) be hypersurfaces of \((\mathbb{P}^1)^n\) defined by \( f_i \), of multiplicity \( (d_1, d_2, \ldots, d_n) \), for \( i = 1, 2, \ldots, n \). Then \( (f_1, f_2, \ldots, f_n) \) defines a section of
\[
V := \bigoplus_{i=1}^{n} \mathcal{O}(d_1, d_2, \ldots, d_n) \to (\mathbb{P}^1)^n
\]
where \( \mathcal{O}(d_1, d_2, \ldots, d_n) := \bigotimes_{j=1}^{n} \pi_j^* \mathcal{O}_{\mathbb{P}^1}(d_j) \) and \( \pi_j : (\mathbb{P}^1)^n \to \mathbb{P}^1 \) is the \( j \)th projection. The vector bundle \( V \) is relatively orientable if and only if
\[
\det V \otimes \omega_{(\mathbb{P}^1)^n} = \mathcal{O} \left( \sum_{i=1}^{n} d_i^1 - 2, \sum_{i=1}^{n} d_i^2 - 2, \ldots, \sum_{i=1}^{n} d_i^n - 2 \right)
\]
is a square, which is the case if and only if \( \sum_{i=1}^{n} d_j^j - 2 \) even for every \( j = 1, 2, \ldots, n \). For every \( i = 1, 2, \ldots, n \), the Newton polygon \( \text{NP}(f_i) \) is the parallelepiped with a corner in \( 0 \) and side edges \( d_j^j e_j \), where \( \{e_j\}_{j=1}^{n} \) is the standard basis. The Minkowski sum \( \sum_{i=1}^{n} \text{NP}(f_i) \) is the parallelepiped with a corner in \( 0 \) and side edges \( \sum_{i=1}^{n} d_j^j e_j \). This parallelepiped \( \sum_{i=1}^{n} \text{NP}(f_i) \) has no odd vertices on the boundary if and only if every \( \sum_{i=1}^{n} d_j^j, j = 1, 2, \ldots, n \) is even, that is exactly when \( V \) is relatively orientable. Let \( \bar{V}_i, i = 1, 2, \ldots, n \) be enriched tropical hypersurfaces with Newton polytope equal to \( \text{NP}(f_i) \) and assume that \( \sum_{i=1}^{n} d_j^j, j = 1, 2, \ldots, n \) is even. Then, Proposition 5.1 implies that
\[
\sum_{p \in \bar{V}} \text{mult}_p(\bar{V}_1, \bar{V}_2, \ldots, \bar{V}_n) = \frac{1}{2} \left( \sum_{\sigma \in S_n} d_{\sigma(1)}^1 d_{\sigma(2)}^2 \cdots d_{\sigma(n)}^n \right) h = n^{k^1}(V) \in \text{GW}(k).
\]

Equivalently, this coincides with \( n^{k^1}(V) \) and yields an enriched count of intersection points of \( n \) curves in \((\mathbb{P}^1)^n\).

For \( (A_1, \ldots, A_n) \) combinatorially oriented, we get that the enriched count of zeros of the system of equations \( f_1 = \ldots = f_n = 0 \) is independent of the coefficients of \( f_1, \ldots, f_n \).

**Theorem 5.17 (Enriched Bernstein-Kushnirenko theorem).** Assume \( k \) is a field with \( \text{char} \ k = 0 \) or \( \text{char} \ k > \max \{\text{diam}(\Delta_A)\} \). For a combinatorially oriented \( n \)-tuple of indexing sets \((A_1, \ldots, A_n)\) and for a generic system of equations
\[
f_1(x) = \ldots = f_n(x) = 0
\]
where \( f_i \in L_{A_i} \), the enriched count of solutions \( z \) in \( \text{Spec} \ k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \) equals
\[
\sum_z \text{Tr}_{A(z)/k} \langle \det \text{Jac}(f_1, \ldots, f_n)(z) \rangle = \frac{\text{MVol}(\Delta_{A_1}, \ldots, \Delta_{A_n})}{2} h \in \text{GW}(k).
\]
We can also say something about the non-orientable case. Just like in the case of Bézout, then the enriched count depends on the choice of coefficients of $f_1, \ldots, f_n$. However, we still get a lower bound for the number of hyperbolic summands depending on the number of odd points on the boundary of $\Delta_{A_1} + \ldots + \Delta_{A_n}$.

**Theorem 5.18.** Assume $k$ is a field with $\text{char} \ k = 0$ or $\text{char} \ k > \max_i \{\text{diam}(\Delta_{A_i})\}$. Let $(A_1, \ldots, A_n)$ be a sequence of finite subsets of $\mathbb{Z}^n$. Let $N := \text{Card}(\partial(\Delta_{A_1} + \ldots + \Delta_{A_n}) \cap \Lambda^{\text{odd}})$ be the number of odd points on the boundary of $\Delta_{A_1} + \ldots + \Delta_{A_n}$. Then for a generic system of equations $f_1(x) = f_2(x) = \ldots = f_n(x) = 0$ where $f_i \in L_{A_i}$, we get that the enriched count of solutions $z$ in $\text{Spec} \ k[x_1^\pm, \ldots, x_n^\pm]$ equals

$$\sum_z \text{Tr}_{k(z)/k} \langle \det \text{Jac}(f_1, \ldots, f_n)(z) \rangle = \frac{\text{MVol}(\Delta_{A_1}, \ldots, \Delta_{A_n})}{2} - r \ h + \langle a_1, \ldots, a_r \rangle$$

in $\text{GW}(k)$, for some $r \leq N$ and $a_1, \ldots, a_r \in k^\times$.

**Proof of Theorem 5.17 and Theorem 5.18.** The argument is the same as in Corollary 5.2. We know by Theorem 3.5 that $\sum_z \text{Tr}_{k(z)/k} \langle \det \text{Jac}(f_1, \ldots, f_n)(z) \rangle$ is an element of $\text{GW}(k)$ of rank $\text{MVol}(\Delta_{A_1}, \ldots, \Delta_{A_n})$. If there are no odd points on the boundary of the Minkowski sum $\Delta_{A_1} + \ldots + \Delta_{A_n}$ then by the same argument as in Corollary 5.2 we can have at most $N$ summands in $\sum_z \text{Tr}_{k(z)/k} \langle \det \text{Jac}(f_1, \ldots, f_n)(z) \rangle$ that are not multiples of $h$.

To derive the theorem for $k$, we use the natural isomorphism $\text{GW}(k) \cong \text{GW}(k\{\{t\}\})$ from Example 2.7.

We provide examples where (25) is invariant, i.e., does not depend of the coefficients of the $f_i$ and where it depends on the coefficients.

**Example 5.19.** The following is the leading example in [25]. Let

$$g(x, y) = a_1 + a_2x + a_3xy + a_4y$$
and
\[ h(x, y) = b_1 + b_2x^2 + b_3xy^2. \]

By Theorem 5.11 the number of solutions in \( \text{Spec} \mathbb{C}[x^{\pm 1}, y^{\pm 1}] \) to \( g(x, y) = h(x, y) = 0 \) is equal to the mixed volume \( M\text{Vol}(\text{NP}(g), \text{NP}(h)) \) of the Newton polygons \( \text{NP}(g) \) and \( \text{NP}(h) \) of \( g \) and \( h \) for a generic choice of coefficients \( a_1, a_2, a_3, a_4, b_1, b_2, b_3 \). The enriched count of solutions to \( g(x, y) = 0 \) and \( h(x, y) = 0 \) depends on the choice of coefficients since there are 2 odd points on the boundary of \( \text{NP}(g) + \text{NP}(h) \) as shown in the first picture in Figure 13. For example if \( k = \mathbb{R} \) and \( a_1 = a_2 = a_3 = a_4 = b_1 = b_2 = b_3 = 1 \) we get that the enriched count of zeros equals 2\( h \). However, if we set \( b_3 = -1 \) (all other coefficients are still equal to 1), then the enriched count of zeros equals \( h + \langle 1, 1 \rangle \).

If we replace \( h \) by \( h'(x, y) = b_1 + b_2x^3 + b_3x^2y \), the Minkowski sum \( \text{NP}(g) + \text{NP}(h') \) of the Newton polygons \( \text{NP}(g) \) and \( \text{NP}(h') \) of \( g \) and \( h' \) has no odd points on the boundary as one can see in the second picture of Figure 13. The mixed volume of \( \text{NP}(g) \) and \( \text{NP}(h') \) equals 4, hence the enriched count of solutions to \( g(x, y) = h'(x, y) = 0 \) in \( \text{Spec}(k[x^{\pm 1}, y^{\pm 1}]) \) equals 2\( h \) and this is independent of the choice of coefficients.

References

[1] Tom Bachmann and Kirsten Wickelgren. Euler classes: Six-functors formalism, dualities, integrality and linear subspaces of complete intersections. *Journal of the Institute of Mathematics of Jussieu*, pages 1–66, 2021.
[2] Florian Block and Lothar Göttsche. Refined curve counting with tropical geometry. *Compos. Math.*, 152(1):115–151, 2016.
[3] Thomas Brazelton, Robert Burklund, Stephen McKean, Michael Montoro, and Morgan Opie. The trace of the local \( \mathbb{A}^1 \)-degree. *Homology, Homotopy and Applications*, 23(1):243–255, 2021.
[4] Thomas Brazelton, Stephen McKean, and Sabrina Pauli. Bézoutians and the \( \mathbb{A}^1 \)-degree. 2021.
[5] Erwan Brugallé, Ilia Itenberg, Grigory Mikhalkin, and Kris Shaw. Brief introduction to tropical geometry. *Proceedings of the Gökova Geometry-Topology Conference 2014*, pages 1–75, 2015.
[6] William Fulton. *Introduction to Toric Varieties. (AM-131)*. Princeton University Press, 2016.
[7] I.M. Gelfand, M.M. Kapranov, and A.V. Zelevinsky. *Discriminants, Resultants, and Multidimensional Determinants*. Mathematics (Birkhäuser). Springer, 1994.
[8] Allen Hatcher. *Algebraic topology*. Cambridge University Press, Cambridge, 2012.
[9] Birkett Huber and Bernd Sturmfels. A polyhedral method for solving sparse polynomial systems. *Math. Comp.*, 64(212):1541–1555, 1995.
[10] Ilia Itenberg, Grigory Mikhalkin, and Eugenii Shustin. *Tropical Algebraic Geometry*, volume 35 of *Oberwolfach Seminars*. Birkhäuser Verlag, 2007.
[11] Andrés Jaramillo Puentes, Hannah Markwig, Sabrina Pauli, and Felix Röhrle. Arithmetic counts of tropical plane curves and their properties, 2023.
[12] Andrés Jaramillo Puentes, Hannah Markwig, Sabrina Pauli, and Felix Röhrle. Enriched tropical counts for quadratic extensions. Work in progress, 2024.
[13] Andrés Jaramillo Puentes and Sabrina Pauli. A quadratically enriched correspondence theorem, 2024.
[14] Jesse Leo Kass and Kirsten Wickelgren. The class of Eisenbud-Khimshiashvili-Levine is the local \( \mathbb{A}^1 \)-brouwer degree. *Duke Mathematical Journal*, 168(3), Feb 2019.
[15] Jesse Leo Kass and Kirsten Wickelgren. An arithmetic count of the lines on a smooth cubic surface. *Compositio Mathematica*, 157(4):677–709, Apr 2021.
[16] Hannah Larson and Isabel Vogt. An enriched count of the bitangents to a smooth plane quartic curve, 2019.
[17] Hannah Markwig, Sam Payne, and Kris Shaw. Bitangents to plane quartics via tropical geometry: rationality, \( \mathbb{A}^1 \)-enumeration, and real signed count, 2022.
[18] Stephen McKean. An arithmetic enrichment of Bézout’s theorem. *Mathematische Annalen*, 379(1-2):633–660, Jan 2021.
[19] Grigory Mikhalkin. Enumerative tropical algebraic geometry in $\mathbb{R}^2$. *J. Am. Math. Soc.*, 18(2):313–377, 2005.

[20] Grigory Mikhalkin. Tropical geometry and its applications. *Invited lectures v. II, Proceedings of the ICM Madrid*, pages 827–852, 2006.

[21] Fabien Morel. $\mathbb{A}^1$-algebraic topology over a field, volume 2052 of *Lecture Notes in Mathematics*. Springer, Heidelberg, 2012.

[22] Johannes Nicaise, Sam Payne, and Franziska Schroeter. Tropical refined curve counting via motivic integration. *Geom. Topol.*, 22(6):3175–3234, 2018.

[23] Sabrina Pauli. Motivic explorations in enumerative geometry. https://homepage.sabrinapauli.com/PCMI.pdf, 2024. Mini-Workshop at PCMI 2024 on Motivic Homotopy Theory.

[24] Andrés Jaramillo Puentes. A wall crossing formula for motivic enumerative invariants, 2024.

[25] Bernd Sturmfels. Polynomial equations and convex polytopes. *The American Mathematical Monthly*, 105(10):907–922, 1998.