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An Elementary Derivation of the Five Dimensional Myers-Perry Metric

Abstract We derive the five dimensional Myers-Perry metric via an elementary method to solve the vacuum Einstein field equations directly. This method firstly proposed by Clotz is very simple since it merely involves four components of Ricci tensor and only requires us to deal with some equations without second derivatives’ terms when the metric ansatz is assumed to take an appropriate form.

Keywords Myers-Perry metric, five dimensional solution

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1 Introduction

In the last few years an increasing effort has been devoted to the study of black holes in higher dimensional gravity [3] [4] [5] [6] [7] [8] [9] [10] for various reasons. Among them, the most direct and important ones stem from the motivation by string and membrane theories. In the context of the string theory, which is thought to be a major candidate for the quantum theory of gravity and the unification of all interactions, black holes in higher dimensions are consistent with its mathematical requirement of higher dimensions. They may not only play a key role in the study of dynamics in higher dimensions but also serve to advance our understanding of the compactification mechanisms. In particular, microscopic black holes may be very useful to test some novel predictions of the string theory. On the other hand, in the brane world scenario, if the assumption that there exist extra dimensions in the universe holds true, it is possible to produce higher dimensional mini black holes in future high energy colliders or observe them in the universe [20] [21] [22].

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In light of the above, it becomes clear that the study of finding exact solutions in higher dimensional gravity is of great importance. However, it is notorious that this task is very difficult, especially to discover the ones in higher dimensional general relativity, compared with the case in four dimensions, where several systematic techniques to generate stationary and axisymmetric solutions have been developed by some authors [15] [16] [17] and a lot of interesting solutions are presented, such as the unique rotating Kerr black hole for a given mass and angular momentum [18] and the charged Kerr-Newman black hole [19]. Till now, there are still few known exact higher dimensional solutions in general relativity. The first exact rotating one of vacuum Einstein field equations was given by Myers and Perry (MP) in 1986 [3]. This solution is the higher dimensional generalization of the classical Kerr black hole and has a spherical horizon topology. Nevertheless, in the case of the five dimensional pure gravity, MP black hole is not unique. For a given mass and angular momentum, Emparan and Reall discovered a black ring solution in vacuum, whose horizon topology is $S^1 \times S^2$ [4]. Then another general regular black ring solution with two angular momenta was also found in [12]. Quite recently, in [14], the authors presented a new five dimensional black hole solution with the horizon of distorted lens spaces.

As a result, systematic generation-techniques of higher dimensional solutions have to be developed. Recently, a new technique was applied to study the stationary and axisymmetric solutions in the arbitrary dimensional pure gravity [7], this technique generalizes the Papapetrou form of the metric for stationary and axisymmetric solutions in four dimensions, and furthermore extends the work on Weyl solutions in four and higher dimensions, which gives the five dimensional MP solution. In fact, several other generalized methods were also used to reproduce the five dimensional MP metric. In [11], a four dimensional technique named the inverse scattering method was extended to derive this metric. Besides, applying a sequence of $SO(2, 1)$ transformations on the five dimensional Schwarzschild metric, the authors obtain the five dimensional MP solution, too [13].

In this short paper, we will employ an elementary method that only requires us to solve the vacuum Einstein field equations directly to derive the five dimensional MP black hole. This method was first used to reproduce Kerr solution by Klotz [1]. Then it was generalized to derive the Kerr-Newman black hole [2]. According to this method, when the metric ansatz is assumed to have an appropriate form, we only need solve two equations which merely refer to four components of Ricci tensor to derive the five dimensional MP solution. Our paper goes as follows. Section two is the main part, where we shall explicitly deduce the five dimensional MP black hole solution. In section three, a simple conclusion is presented.

### 2 calculations

Bearing in mind that our prime aim is to derive the five dimensional MP black hole which is the higher dimensional generalization of the Kerr solution in four dimensions, the metric ansatz extended from the one in [1] is assumed
to take the general form

\[ ds^2 = [h(x, \theta) - 1]d\tau^2 + \Sigma(x, \theta)\{dx^2 + d\theta^2 + p(\theta) d\phi^2 + q(\theta) d\psi^2\} \]

\[ + 2ap(\theta)d\tau d\phi + 2bq(\theta)d\tau d\psi - (bp(\theta)d\phi + aq(\theta)d\psi)^2, \]  

(1)

where parameters \(a\) and \(b\), which can be absorbed by the functions \(p(\theta)\) and \(q(\theta)\) respectively, are two constants inserted for dimensional reasons.

Compared with the ansatz in [1], the last term in Eq. (1) arises from the two angular momenta. We perform a coordinate transformation

\[ d\tau = dt + apd\phi + bq d\psi, \]

(2)

and also assume

\[ \Sigma = F(x) - a^2 p - b^2 q. \]

(3)

Here the assumption that \(F(x)\) is only a function of the coordinate \(x\) may be too strong. It would be avoided. However, owing to this restriction on \(F(x)\), the field equations are easy to separate, which simplifies the calculation by a long way. Without this assumption, it is nearly impossible for the calculations to proceed.

Now, let’s calculate the Ricci tensor of metric (1) in the coordinate system \((t, x, \theta, \phi, \psi)\). Its nine non-zero components are \(R_{tt}, R_{t\phi}, R_{t\psi}, R_{xx}, R_{x\theta}, R_{\theta\theta}, R_{\phi\phi}, R_{\phi\psi}\) and \(R_{\psi\psi}\). With aids of the two functions

\[ Q(\theta) = p + q, \]

(4)

\[ \Delta(x, \theta) = (F - a^2 Q)(F - b^2 Q) \]

\[ - h(F - a^2 p - b^2 q)[F - (a^2 + b^2)Q], \]

(5)

where the function \(\Delta\) is easy to construct once we note that a lot of components of Christoffel brackets contain it in their denominators, the component \(R_{x\theta}\) can be expressed as

\[ R_{x\theta} = A(x, \theta)\Delta_{x\theta} + B(x, \theta)\Delta_{\theta} + C(x, \theta)Q_{,\theta}. \]

(6)

In Eq. (6), functions \(A\), \(B\) and \(C\) are too complicated to list. Accordingly, in order to keep \(R_{x\theta} = 0\), the simple and also final assumption is that \(\Delta\) is only the function of the variable \(x\). With this assumption, we can get \(C = 0\) or \(Q_{,\theta} = 0\). Taking the latter into consideration, \(Q\) and \(h\) have the simplified forms

\[ Q = \text{const} = k, \]

\[ h = \frac{-\Delta(x) + (F - ka^2)(F - kb^2)}{(F - a^2 p - b^2 q)[F - k(a^2 + b^2)]}. \]

(7)

Up to the present, there are still three functions to be determined. If we directly solve the other non-zero components of the Ricci tensor, the second derivatives to \(x\) and \(\theta\) nearly make this plan impossible. For the sake of avoiding this obstacle, we take into account the following equation

\[ abkR_{tt} - bR_{t\phi} - aR_{t\psi} = 0, \]

(8)
whose merit is that all the terms including the second derivatives disappear. Introduce a new independent variable \( \sigma \), which satisfies
\[
d\sigma = \Delta^\frac{\Delta}{x} dx,
\]
and a new function \( H(x) \) to rewrite \( \Delta(x) \) as the form
\[
\Delta = (F - ka^2)(F - kb^2) - H[F - k(a^2 + b^2)].
\]
Hence Eq. (8) is simplified as
\[
F^2 - \frac{H_\sigma F_\sigma}{H}(F - kb^2) + \frac{p^2_\theta}{p(p - k)} - \frac{H_\sigma F_\sigma}{H}(b^2 - a^2)p = 0.
\]
Since \( F \) and \( H \) are only the functions of \( \sigma \) while \( p \) relates to the variable \( \theta \), we must have
\[
p^2_\theta = p(\alpha p + \beta)(p - k),
\]
\[
F^2_\sigma = \frac{\alpha}{b^2 - a^2} F - \frac{\alpha kb^2}{b^2 - a^2} - \beta, \quad \frac{H_\sigma F_\sigma}{H} = \frac{\alpha}{b^2 - a^2},
\]
in which \( \alpha \) and \( \beta \) are arbitrary constants. When \( \alpha = 0 \) and \( \beta = -l^2 \), solving Eqs. (12) and (13), \( p, q, \Sigma \) and \( h \) take the general forms
\[
p = k \sin^2 \left( \frac{l}{2} \theta \right), \quad q = k \cos^2 \left( \frac{l}{2} \theta \right),
\]
\[
\Sigma = l\sigma + n - k \left[ a^2 \sin^2 \left( \frac{l}{2} \theta \right) + b^2 \cos^2 \left( \frac{l}{2} \theta \right) \right],
\]
\[
h = \frac{2M}{l\sigma + n - ka^2 \sin^2 (l\theta/2) - kb^2 \cos^2 (l\theta/2)},
\]
respectively, where \( l, n \) and \( M \) are arbitrary constants. If we substitute them into Eq. (1), it is an easy matter to check that Eq. (1) satisfies the vacuum Einstein equations. In particular, adopting
\[
k = 1, \quad l = 2, \quad n = a^2 + b^2, \quad \sigma = \frac{r^2}{2},
\]
we obtain the five-dimensional MP metric in the Boyer-Lindquist coordinates
\[
ds^2 = \left( \frac{2M}{\Sigma} - 1 \right) d\tau^2 + \Sigma \left( \frac{r^2}{\Delta} dx^2 + d\theta^2 + \sin^2 \theta d\phi^2 + \cos^2 \theta d\psi^2 \right)
+ 2 \alpha \sin^2 \theta d\tau d\phi + 2b \cos^2 \theta d\tau d\psi - (b \sin^2 \theta d\phi + a \cos^2 \theta d\psi)^2,
\]
where
\[
d\tau = dt + a \sin^2 \theta d\phi + b \cos^2 \theta d\psi,
\]
\[
\Sigma = r^2 + a^2 \cos^2 \theta + b^2 \sin^2 \theta,
\]
\[
\Delta = (r^2 + a^2)(r^2 + b^2) - 2Mr^2.
\]
On the other hand, when \( \alpha \neq 0 \), the solutions got from Eq. (11) does not satisfy the Einstein field equations in vacuum. This means that the solution of Eq. (1) is unique on the assumption that Eqs. (3) and (7) hold.
3 Final remarks

In this paper, we have reproduced the five dimensional MP metric by an elementary method. Since this method only requires us to solve the equation $R_{x\theta} = 0$ and Eq. (8) with respect to the four components of the Ricci tensor $R_{t\phi}$, $R_{tt}$, $R_{t\psi}$ and $R_{t\psi}$, it is simple although the calculation is a little complicated and we have presented the assumptions that the metric ansatz takes the general form (1) and the functions $\Delta$ and $F$ only depend on the variable $x$ to make the Einstein field equation simplified. Moreover, this method might be general. In our future work, we will extend it to generate other neutral or charged solutions in higher dimensions.

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