Entanglement distribution with wavevector-multiplexed quantum memory

Lipka, Michal; Mazelanik, Mateusz; Parniak, Michal

Published in:
New Journal of Physics

DOI:
10.1088/1367-2630/abf79a

Publication date:
2021

Document version
Publisher's PDF, also known as Version of record

Document license:
CC BY

Citation for published version (APA):
Lipka, M., Mazelanik, M., & Parniak, M. (2021). Entanglement distribution with wavevector-multiplexed quantum memory. New Journal of Physics, 23(5), [053012]. https://doi.org/10.1088/1367-2630/abf79a
Entanglement distribution with wavevector-multiplexed quantum memory

To cite this article: Micha Lipka et al 2021 New J. Phys. 23 053012

View the article online for updates and enhancements.
Entanglement distribution with wavevector-multiplexed quantum memory

Michał Lipka1,2, Mateusz Mazelanik1,2 and Michał Parniak1,3,*

1 Centre for Quantum Optical Technologies, Centre of New Technologies, University of Warsaw, Banacha 2c, 02-097 Warsaw, Poland
2 Faculty of Physics, University of Warsaw, Pasteura 5, 02-093 Warsaw, Poland
3 Niels Bohr Institute, University of Copenhagen, Blegdamsvej 17, DK-2100 Copenhagen, Denmark
* Author to whom any correspondence should be addressed.
E-mail: m.parniak@cent.uw.edu.pl

Keywords: quantum communication, quantum memory, multiplexing, atomic ensemble, quantum repeater, multimode light

Abstract

Feasible distribution of quantum entanglement over long distances remains a fundamental step towards quantum secure communication and quantum network implementations. Quantum repeater nodes based on quantum memories promise to overcome exponential signal decay inherent to optical implementations of quantum communication. While performance of current quantum memories hinders their practical application, multimode solutions with multiplexing can offer tremendous increase in entanglement distribution rates. We propose to use a wavevector-multiplexed atomic quantum memory (WV-MUX-QM) as a fundamental block of a multiplexed quantum repeater architecture. We show the WV-MUX-QM platform to provide quasi-deterministic entanglement generation over extended distances, mitigating the fundamental issue of optical loss even with currently available quantum memory devices, and exceeding performance of repeaterless solutions as well as other repeater-based protocols such as temporal multiplexing. We establish the entangled-bit (ebit) rate per number of employed nodes as a practical figure of merit reflecting the cost-efficiency of larger inter-node distances.

1. Introduction

Entanglement is an essential resource for the most promising quantum information protocols [1, 2] enabling, among others, secure quantum communication [3–6]. The optical implementations of such protocols face the exponential transmission losses inherent to photonic systems and greatly limiting the feasible distance at which high fidelity entangled states can be distributed. To overcome this obstacle, noise-tolerant quantum repeaters have been proposed for entanglement connection (ENC) and purification over shorter elementary lengths [7].

Experimentally promising quantum repeater architectures involve linear optics, quantum memories and single-photon detection [8, 9]; however, currently available memory lifetimes as well as retrieval and single-photon detection efficiencies limit the feasibility of such repeaters at practical distances of a few hundred km [10, 11]. Multimode architectures have been proposed as solutions to this problem [12–14], which lead to an ongoing effort in experimental realizations of such systems, especially involving multiplexing capabilities. While an $M$-mode platform in parallel operation increases the entanglement distribution rate $M$-fold, multiplexing may lead to $O(M^2)$-fold increase with $N$ denoting the number of repeater nodes [12]. Hitherto multimode systems demonstrated in the context of quantum repeaters involved at most tens of modes [15–20] and mainly focussed on temporal multiplexing. As an alternative to temporal, spectral or spatial micro-ensemble modes [21], a highly-multimode wavevector-multiplexed quantum memory (WV-MUX-QM) has been recently demonstrated [22] along with flexible in-memory processing capabilities [23–25].

Here we evaluate the feasibility of previously-demonstrated WV-MUX-QM, which was based on a high-optical-depth cold atomic ensemble, as a quantum repeater platform. Remaining in the constraints of
current technology, we propose a multiplexing protocol combining experimentally demonstrated components to provide quasi-deterministic entanglement generation (ENG) over 150 km.

We analyse the performance of the novel platform in the recently proposed semihierarchical quantum repeater architecture [26] as well as an ahierarchical architecture and compare the performance of wavevector multiplexing with state-of-the-art temporal multimode and long-lifetime single-mode platforms. Finally, we identify the fundamental limitations of the WV-MUX-QM platform. Our results are particularly significant in the light of recent advances and effort in developing practically feasible multi-mode quantum communication systems employing multicore fibres [27, 28] or free-space transmission [29].

1.1. Quantum repeaters—DLCZ protocol

As proposed by Duan, Lukin, Cirac and Zoller (DLCZ) in their seminal paper [30], atomic ensemble-based quantum memories and linear optical operations combined with single-photon detection can be employed to transfer entanglement between distant parties. In a simple scenario of the DLCZ protocol, two parties—Alice (A) and Bob (B)—are separated by a distance \( L \) and would like to share a high-fidelity entangled photonic state for further use in quantum communication protocols such as quantum key distribution. If the optical losses are too large over the distance \( L \), we can imagine \( N \) equidistant parties separated by \( L_0 = L/(N-1) \). Each party \( A_i \) has two atomic ensembles and independently generates entanglement between one of the ensembles and one of her neighbours \( A_{i-1}, A_{i+1} \). Then, hierarchically, the parties read their two ensembles and, conditioned on a successful detection of a single read-out photon, extend the entanglement to their neighbours \( A_{i-1} \leftrightarrow A_i \leftrightarrow A_{i+1} \leftrightarrow A_{i+1} \).

The DLCZ protocol, being based on single-photon interference, requires sub-wavelength phase stability over tens-of-km inter-node distances \( L_0 \), limiting its practical feasibility. Furthermore, in a DLCZ-based network the distributed entangled state contains a vacuum component which grows with every entanglement swapping stage. As a solution to those issues, two-photon interference-based protocols have been proposed [9, 31] and experimentally demonstrated [8, 11]. In this work we build on the two-photon protocols [9, 31] inherently robust to optical phase stability. In particular, while we design the ENG scheme specifically for WV-MUX-QM platform, the ENC, which takes place after multiplexing (MUX), remains the same as in standard two-photon protocols.

The basic idea of two-photon protocols is to employ polarization entangled pairs of photons instead of occupation number entangled state with a single photon. In a basic scenario the \( i \)th node \( A_i \) has a pair of memories \( A_i^{(H,L)}, A_i^{(V,L)} \) for ENG with \( A_{i-1} \) and another pair \( A_i^{(H,R)}, A_i^{(V,R)} \) for \( A_{i+1} \). Polarization of photons emitted from the memories is transformed so that the superscripts \( H \) and \( V \) correspond to horizontal and vertical polarization of the photons, respectively. Similarly to the DLCZ protocol, photons from \( A_i \) and \( A_{i+1} \) are sent to the midway station and interfered on a beamsplitter (BS); however, now the interference and single-photon detection is separate for each polarization. For ENC both \( H \) and \( V \) memories are readout and the state of the distant pairs of memories is projected onto a maximally entangled Bell state by measuring the read-out photons and post-selecting outcomes.

2. Results

2.1. Wavevector-multiplexed quantum repeater

Multimode quantum memory. The WV-MUX-QM platform is based on an atomic quantum memory. For specificity we shall consider rubidium-87 atoms [22] cooled in a magneto-optical trap (MOT) via polarization gradient cooling and trapped in a dipole trap [32] reaching temperatures of \( 1 \mu K \). The memory operates via a light-atom interface based on an off-resonant spontaneous Raman scattering. A lambda configuration of atomic levels is involved, as depicted in the inset of figure 1. Importantly, the selected atomic transitions should be clock transitions robust to external magnetic fields [33]. The atoms are initially prepared in the \( |g\rangle \) level. A strong write (W) beam off-resonant with \( |g\rangle \rightarrow |e\rangle \) transition generates a two-mode squeezed state of scattered write-out (Stokes) photons and collective atomic excitations (spin-waves, coherence between \( |g\rangle \) and \( |h\rangle \) levels). Let us denote by \( a_{\mathbf{k}w}^\dagger \) the creation operator for write-out photon with a wavevector \( \mathbf{k}_w \) determining the scattering angle and the memory mode, as depicted in figure 1. Denoting creation operator for a single spin-wave with wavevector \( \mathbf{K} \) by \( S_{\mathbf{K}}^j \), the generated state in a single pair of memory and photonic modes is given by:

\[
|\psi\rangle = \sum_{j=0}(|\sqrt{\lambda}a_{\mathbf{k}w}^\dagger S_{\mathbf{K}}^j \rangle|\text{vac}\rangle, \tag{1}
\]
Figure 1. With wavevector multiplexing, each mode of such an atomic memory corresponds to a different angle of emission (or wavevector) of write-out (read-out) photons. (i) Atomic levels in a lambda configuration for a light-atom interface to the wavevector-multiplexed quantum memory based on cold rubidium-87 atoms. W-write beam, R-read beam, w-write-out photon, r-read-out photon.

Figure 2. (a) Entanglement distribution with an ahierarchical quantum repeater architecture and employing atomic wavevector-multiplexed quantum memories (WV-MUX-QM). Each node consists of two WV-MUX-QM (e.g. A′, A). Intermediate nodes are located L₀ apart and send the write-out photons generated in one of their atomic ensembles to a midway station in order to generate entanglement (ENG) between the distant ensembles (e.g. A and B′). (b) Multiplexing idea—to entangle ensembles A and C′ in ENC stage after successful ENG between A & B′ and B & C′, the read-out photons from B and B′ are multiplexed to pre-selected modes k₁, k₂ with a reconfigurable MUX and using the which-mode information from ENG stages.

where \( \chi \) gives the probability of generating a single pair of photon–spin-wave excitations. After a programmable delay, the spin-waves can be converted to read-out photons with a strong resonant read (R) beam \( \hat{S}_K^{\dagger} \to \hat{a}_{k_r}^{\dagger} \).

Write and read processes conserve momentum and energy which results in correlated momenta or wavevectors (or scattering angles) of write-out and read-out photons. While, in general, phase-matching conditions must be taken into consideration, the most versatile write/read beams configuration employs counter-propagating beams at angle of around \( 2^\circ \) to the longitudinal axis of MOT [22]. Importantly, with such a choice of write and read beams’ wavevectors, the write-out photon at \( k_w \) heralds further read-out to result in a read-out photon at \( k_r \approx -k_w \), which is fundamental for wavevector multiplexing.

Entanglement distribution. An ahierarchical quantum repeater architecture has been depicted in figure 2(a). Each node contains two WV-MUX-QMs. Distant memories perform ENG by sending write-out photons to a central midway station. To distribute the entanglement, memories are read at each node and the read-out photonic modes are multiplexed to pre-selected modes, as depicted in figure 2(b), to further undergo ENC.

Entanglement generation (ENG). The first operation in a quantum repeater link is the generation of entanglement (ENG) between each pair of quantum memories separated by an elementary distance \( L_0 = L/(N-1) \). The memories probabilistically generate entangled pairs of atomic excitations and single-photons across M modes of their ensembles. The generated write-out photons are sent to a central mid-way station (CMS) via multimode channels. We also note that a space-to-time conversion [34] could be used to map the wavevector modes to time bins and thus enabling use of single-mode channels for ENG. The photons generated in each ensemble are divided by their emission angles into two groups with imposed orthogonal polarizations—vertical (V) and horizontal (H), as depicted in figure 3(a). Before transmission, the two groups are super-imposed on each-other to ensure that each pair of H, V modes is transmitted through a single channel acquiring the same optical phase or a deterministic phase difference.
Let us consider two ensembles $A, B'$ located in separate nodes. At the midway station horizontal (vertical) polarization from $A$ is superimposed onto vertical (horizontal) polarization from $B'$, erasing the which-node information and resulting in four regions containing $M$ modes each. The regions $I_{\pm}$ ($H_{\pm}$) observe $V$ ($H$) polarization from $B'$ and $H$ ($V$) polarization from $A$. A single-photon sensitive camera [35] observes the regions and registers coincidences between any of modes in $I_{\pm}$ ($k_I$) and any of modes in $H_{\pm}$ ($k_{II}$), which projects the state of atomic ensembles $A, B'$ onto:

$$|\psi^{A,B'}_{\text{ENG}}\rangle = \frac{1}{\sqrt{2}}\left|\hat{S}^+_A,\hat{S}^+_B,\phi\right|\left|\hat{S}^+_V,\hat{S}^+_H,\phi\right|\left|\text{vac}\right>,$$

where $\phi$, assuming zero inter-mode path length difference gives the phase difference between optical paths from $A$ and $B'$ to the midway CMS i.e. over distance $L/2$, and $\hat{S}_{A,V,H}^\dagger$ is the creation operator for a spin-wave in ensemble $X$ in mode $k$, with the corresponding read-out photon $P$-polarized. Here and henceforth the modes are indexed by the wavevector of the write-out photon $k$ which implicitly corresponds to a spin-wave in mode $K$, which upon readout generates a read-out photon in mode $-k$. We note here that since the coincidence can occur between any two modes ($|\psi^{A,H}_{\text{ENG}}\rangle$, $|\psi^{A,V}_{\text{ENG}}\rangle$, $|\psi^{A,H}_{\text{ENG}}\rangle$, $|\psi^{B,V}_{\text{ENG}}\rangle$ are independent) there is $M^2$ possibilities, in contrast to parallel scheme which attempts to generate $|\psi^{B,V}_{\text{ENG}}\rangle$ only between the $j$th mode from $A$ and $j$th mode from $B'$ ($k_j = k_{II}$).

The excitation number (DLCZ-type) entanglement present in $|\psi^{A,B'}_{\text{ENG}}\rangle$ will be further converted to polarization entanglement during ENC stage which also projects the state onto a subspace containing one excitation per node.

Let us consider the probability of a successful ENG between $A$ and $B'$ with multiplexing and by utilizing the memory modes in parallel. If we denote the total probability of emitting two photons, transmission and correct measurement outcome by $p_1$, then with $M$ modes the total probability of successful ENG in any pair of modes in parallel operation ($k_j = k_{II}$) would read

$$p^{(\text{parallel})}_j = 1 - (1 - p_1)^M.$$

With full multiplexing we harness the multi-mode single-photon detection at CMS to provide the which-mode information about photons from each memory, which further enables reconfiguration of an
M-to-1 switch at the memories’ outputs. This way, we can perform ENG between any pair of modes \( \{k_i, k_{iII}\} \), effectively enhancing the probability of ENG

\[
p_E = 1 - (1 - p_1)M^2.
\]

We note that this probability is equivalent to the parallel case with \( M^2 \) modes. Notably, both cases involve the same protocol in the ENG step, but differ in usage of generated correlations upon subsequent ENC step. Furthermore, utilization of this additional information in practice requires stable inter-mode phases.

Importantly, the single-mode ENG probability \( p_1 \) involves a fundamental transmission loss which can be mitigated by \( M^2 \) scaling in \( p_E \). Wavevector-multiplexed quantum memories can achieve around \( 5 \times 10^3 \) modes, making the ENG quasi-deterministic, even with significant optical attenuation at large elementary distances \( L_0 \).

Finally, we want to note that the parallel scheme also requires (single-channel) multiplexing device at each memory readout to perform ENC. In other case, in the pure (trivial) parallel operation (without multiplexing) the ENG probability will be dramatically reduced, as all populated modes along the whole communication distance \( L \) have to be matched (same \( k_i = k_{iII} \) at each node).

Entanglement connection (ENC). Let us assume that ENGs have been successful between \( A \) and \( B' \), and between \( B \) and \( C' \), to form a state \( |\psi_{ENC}\rangle \otimes |\psi_{ENC}'\rangle \) involving modes \( k_i, k_{iII}, k_{iIII} \) and \( k_{IV} \). Ensembles \( B \) and \( B' \) are at the same node. We wish to carry ENC with \( B, B' \) so that \( A \) and \( C' \) share an entangled state. Let us for the moment postpone the discussion of the multiplexing stage and assume that read-out photons from \( B \) and \( B' \) occupying superpositions of \( k_i, k_{iII} \) and \( k_{iIII}, k_{IV} \) modes, respectively, arrive at the ENC segment in pre-selected photonic modes. For ENC a two-photon interference is observed between the read-out photons with single-photon detectors, as depicted in figure 3(b). Once a coincidence with one photon per each depicted \( \pm \) detector is observed, the ENC succeeds with an output state depending on the coincidence detectors signs \((\pm, \pm)\). For example \((+, +)\) gives:

\[
(|HV\rangle_{A,C} + | VH\rangle_{A,C'})/\sqrt{2} = (\hat{S}_{A,H,k_i}^{+} \hat{S}_{C,V,k_{iIII}}^{+} + \hat{S}_{A,V,k_{iIII}}^{+} \hat{S}_{C,H,k_i}^{+})/\sqrt{2},
\]

while \((+, -)\) leads to \((|HH\rangle_{A,C} + |VV\rangle_{A,C'})/\sqrt{2}\). The output state can be corrected with a bit-flip operation so that ENC always yields the same Bell state. While first-stage ENC (fENC) taking place just after ENG requires additional half-wave plates in the setup, further ENCs proceed without the second polarization rotation, as depicted in figure 3(b), and require a phase-flip correction instead of a bit-flip [36].

Multiplexing (MUX). A critical step in the new protocol is MUX the arbitrary modes in \( |\psi_{ENC}\rangle \) to a pair of canonical pre-selected modes, which enables two-photon interference between the read-out photons, crucial for robust entanglement swapping [9, 36]. Several strategies may be employed to implement the MUX stage. Importantly, spatially-resolved detection involved in the new protocol at the ENG station provides the required which-mode \((k_i, k_{iII})\) information. One idea may be to interface the memory with two read beams at angles which are reconfigured via acousto-optical deflectors (AOD) to facilitate the read-out at pre-select modes \( k_i^{(0)}, k_{iIII}^{(0)} \) [37].

Another MUX method which may yield close to 100% efficiency would be to use fast AOD placed in the near-field of the atomic cloud, to directly adjust the wavevector (angle) of the read-out photons to match \( k_i^{(0)} \) and \( k_{iIII}^{(0)} \). The idea has been depicted in figure 4. Importantly, \( k_i \) and \( k_{iIII} \) always lay in different \((H, V)\) parts of the emission cone and thus can be separated in the far-field and directed to two different AODs. Each AOD modulates the read-out with only a single frequency reconfigured to match the difference between the actual modes \( (k_i, k_{iIII}) \) and the target modes \( (k_i^{(0)}, k_{iIII}^{(0)}) \).

Alternatively, fast digital micromirror devices approaching \( \mu s \) response times could be used in place of AODs offering even broader possibilities [38].

A more flexible technique could involve an in-memory MUX on the stored spin-waves. The required technique—ac-Stark spin-wave modulation—has been demonstrated [23–25] in implementations of single-spin-wave inter-mode operations and spin-wave wavevector (K-space) displacements. In this approach, the which-mode information is used to prepare a spin-wave K-space displacement operation which changes the wavevectors of stored spin-waves to make the read-out photons match the pre-selected modes \( k_i^{(0)}, k_{iIII}^{(0)} \).

### 2.2. Performance figure of merit

Entanglement of formation (\( E_F \)). Inherently, due to the multi-excitation component in the generated photon–spin-wave state, background noise and dark counts, part of the detector counts will indicate randomly polarized photons. Therefore, we model the experimental imperfections as a depolarizing effect.
channel, obtaining in each memory mode a Bell Werner-like state \( \rho_{(\psi)} \) given by

\[
\rho_{(\psi)} = \frac{(1-V)}{4} |\psi\rangle \langle \psi| + V |\psi\rangle \langle \psi|,
\]

where \( |\psi\rangle \) is the selected Bell state \( |\psi\rangle \in \{|\Psi_+\rangle, |\Phi_+\rangle\} \) and \( V \) gives the interference visibility for a Bell-state measurement (BSM). The first term \((1-V)/4 \times \frac{1}{2}\) corresponds to the introduced white noise.

The entanglement as a resource can be quantified with the entanglement of distillation \( E_D(\rho_{(\psi)}^{\otimes n}) = m/n \), which gives the number \( m \) of pure states \( |\psi\rangle \) that can be distilled from \( n \) copies of \( \rho_{(\psi)} \) in the limit of \( n \rightarrow \infty \). In the opposite scenario the entanglement of formation (EF) \( E_F(\rho_{(\psi)}^{\otimes n}) = m/n \) gives the required number \( m \) of pure states \( |\psi\rangle \) required to create \( n \) copies of \( \rho_{(\psi)} \). As entanglement of distillation is generally difficult to calculate, we assume an optimistic scenario and use EF \( E_F(\rho_{(\psi)}) \) for the entangled bits (ebits) content of a single generated \( \rho_{(\psi)} \) state. For a Bell Werner-like state \( \rho_{(\psi)}(V) \) an exact expression for \( E_F(V) \) has been given in references [39, 40]. We note that other entanglement monotones are known [41] to give tighter bounds on \( E_D \) yet for simplicity and taking into account the estimative character of our calculations we use \( E_F \).

Ebit rate per unit repeater cost (Q). The ebit rate \( R \) quantifies the amount of distributed entanglement bits per unit time. The average time \( T_{\text{tot}} \) between successful entanglement distributions over the total distance \( L \), gives the protocol connection rate \( 1/T_{\text{tot}} \) and the average EF \( \langle E_F(V) \rangle \) of transmitted states yields the ebit content per connection, thus the ebit rate is given by:

\[
R = \langle E_F(V) \rangle / T_{\text{tot}}.
\]

With a given total connection distance \( L \), the number of employed repeater nodes \( N \) can be optimized to yield the highest ebit rate; however, in design of practical repeater links the infrastructure cost must be taken into consideration as well. To account for such a limitation, we choose the ebit rate per unit repeater node cost as our figure of merit:

\[
Q(N, L) = R(N, L)/N,
\]

and optimize it over the number of nodes \( N \) for each \( L \):

\[
N^*(L) = \arg \max_N Q(N, L).
\]

2.3. Quasi-deterministic ENG

Let us consider an exemplary link with \( p_1 = (\chi \eta_m \eta_t)^2 \), where \( \eta_m \) denotes the multi-mode single-photon detection efficiency and \( \eta_t \) the transmission loss at \( L_0/2 \) distance between a memory and the CMS. To keep the multi-excitation probability low, the excitation probability is kept on the order of a few percents. Here, we take \( \chi = 0.05 \). Multimode detection compatible with thousands of spatial modes can be realized with single-photon cameras e.g. a CMOS camera with an image intensifier [35], novel photon-counting CMOS sensor [42] or arrays of avalanche or superconducting detectors [43], or simply many detectors. The camera-based solutions would be most suitable for free space transmission with 800 nm light, yielding at least 20% detection efficiency. On the other hand, in the telecom band one would have to use quantum frequency conversion techniques (best overall efficiencies of the order of 30% [11]) adapted for many modes along with detector arrays. Overall, we will assume detection efficiency \( \eta_m = 0.2 \) and consider
telecom wavelength transmission over a fibre with attenuation of around $\alpha = 0.2\text{dB km}^{-1}$. For WV-MUX we will assume $M = 5500$ theoretically attainable in an experimentally demonstrated system as detailed in Wavevector range section. Figure 5 depicts the $p_{2}$ as a function of elementary distance $L_0$ for WV-MUX-QM-based link, compared to $p_{2}^{(\text{parallel})}$ for a wavevector multimode system where modes are connected in parallel without multiplexing i.e. ith mode of $A$ can be connected only with ith mode of $B$. Additionally, we consider temporal multimode system employing fast single-mode single-photon detection with high detection efficiency $\eta_r \approx 0.9$ and high photon generation efficiency $\chi = 0.47$ across $M = 50$ temporal modes. Noticeably, the ENG is quasi-deterministic for WV-MUX-QM system up to around $L_0 \approx 150$ km.

2.4. Mode-specific entanglement quality

Spin-wave decoherence. The multi-photon contribution and noise can be quantified with the second order intensity cross-correlation between the write-out and read-out photons as given by:

$$g^{(2)}(\mathbf{k}_w, \mathbf{k}_r) = \frac{\langle n_w(\mathbf{k}_w)n_r(\mathbf{k}_r) \rangle}{\langle n_w(\mathbf{k}_w) \rangle \langle n_r(\mathbf{k}_r) \rangle},$$  \hspace{2cm} (11)$$

where $n_w(\mathbf{k}_w)$ ($n_r(\mathbf{k}_r)$) refers to the number of write-out (read-out) photons with wavevector $\mathbf{k}_w$ ($\mathbf{k}_r$) detected in a single experiment repetition, and the average $\langle \cdot \rangle$ is taken over the repetitions. With low average photon numbers $n \ll 1$ we can approximate $g^{(2)}(\mathbf{k}_w, \mathbf{k}_r) \approx p_{\text{det}} / (p_w p_r)$, where $p_{\text{det}} \equiv p_{\text{det}}(\mathbf{k}_w, \mathbf{k}_r)$ is the total probability of observing a coincidence between write-out and read-out photons with wavevectors $\mathbf{k}_w$ and $\mathbf{k}_r$, respectively, and $p_w \equiv p_w(\mathbf{k}_w)$ ($p_r \equiv p_r(\mathbf{k}_r)$) gives the marginal probability of observing a write-out (read-out) photon with wavevector $\mathbf{k}_w$ ($\mathbf{k}_r$). Coincidences can be divided into uncorrelated events and those originating from the correct memory operation $p_{\text{det}} = p_{\text{det}} + p_{\text{det}}^{(\text{signal})}$. Let us select the highest-correlated modes $\mathbf{k}_r = -\mathbf{k}_w = \mathbf{k}$. If we denote the single-photon detection efficiency by $\eta_r$, read-out efficiency by $\eta_r$, and the probability of a noise photon in the read-out path as $\eta_r$ we get the following probabilities $p_{w}^{(\text{signal})} = \eta_r^2 \eta_r \chi$, $P_w = \eta_r \chi$, $P_r = \eta_r \chi + B \eta_r$ which gives [33]:

$$g^{(2)}(\mathbf{k}, -\mathbf{k}) \approx 1 + \left( \chi + \frac{B}{\eta_r} \right)^{-1}.$$  \hspace{2cm} (12)$$

In general, noise and read-out efficiency depend on the memory storage duration, subsequently deteriorating $g^{(2)}$. For clarity, we shall include the $B/\eta_r$ term into a time-dependent effective $\tilde{\chi}(t) = \chi + B(t)/\eta_r(t).$

Visibility reduction from uncorrelated coincidences. Among the registered coincidences between the write-out and read-out photons there is a number of noise pairs originating from dark counts, losses and multi-photon excitations. Importantly, in WV-MUX-QM such noise photons attain horizontal or vertical temporal modes. Noticeably, the ENG is quasi-deterministic for WV-MUX-QM system up to around $L_0 \approx 150$ km.

Figure 5. Probability of ENG in an elementary quantum repeater link $p_{2}$ as a function of the inter-node distance $L_0$. Wavevector multiplexed architecture (WV-MUX-QM) is compared with a similar wavevector multimode (WV parallel) system without multiplexing, and with a Temporal multimode platform. See main text for model parameters.
can be written as $p_{\text{num}} = g^{(2)} p_{w} p_{r}$, with $g^{(2)} \equiv g^{(2)}(k, -k)$. The state visibility can be measured in a BSM configuration. Consider coincidence counts between write-out and read-out photons with a polarization BSM. We can denote $V(k) = (p_{w}^{(+)} - p_{w}^{(-)})/(p_{w}^{(+)} + p_{w}^{(-)})$, where superscripts $(\pm)$ denote measurements settings with maximally constructive $(+)$ or destructive $(-)$ interference. In $(-)$ settings only noise coincidences are registered i.e. $p_{w}^{(-)} = p_{w} p_{r}$ while $p_{w}^{(+)} = g^{(2)} p_{w} p_{r}$. Therefore, the visibility can be written as

$$V(k) = \frac{g^{(2)}(k, -k) - 1}{g^{(2)}(k, -k) + 1}. \quad (13)$$

We note that a $k$ mode has a finite extent in $k$-space and for an efficient implementation of a BSM, it is necessary to integrate the coincidences over the mode extent, effectively averaging the visibility, given by equation (13), in $k$-space.

Memory decoherence. The memory lifetime $\tau$, limited by the spin-wave decoherence rate, sets a relationship between the ebit content of the generated state $E_{t}$ and the storage time $t$. For an atomic ensemble cooled in a MOT and stored in a dipole trap, the main spin-wave decoherence mechanism is through the thermal motion of individual atoms [22, 33] which distorts the spatial structure of a spin-wave. Intuitively, an atom travelling out of its initial position has the more detrimental effect on the spin-wave, the finer spatial details of the spin-wave are. Therefore, we expect $\tau$ to be mode-specific and grow with lower spin-wave wavevector modulus $K \equiv |K|$. The exact result follows from a thermal evolution of a spin-wave state and is given by the overlap of the evolved and the initial spin-wave states

$$\langle S_{K}(0) | S_{K}(t) \rangle ^{2} \propto \exp(-t^{2}/\tau(K)^{2})$$

which has a Gaussian form and where $\tau(K) = \gamma/K$ with the proportionality constant $\gamma = \sqrt{m/k_B T}$ depending on the atomic mass of $\text{Rb-87}$ m, ensemble temperature $T$ and Boltzmann constant $k_B$. For typically attainable temperatures $T \approx 1 \mu K$, $\gamma \approx 10^{3} \mu s mm^{-1}$.

Effectively, the spin-wave decoherence decreases the readout efficiency $\eta_{r}(t) = \exp(-t^{2}/\tau(K)^{2})$. We shall further assume a constant noise level $B(t) = \text{const}$. This way, we can account for the visibility deterioration by considering the cross-correlation $g^{(2)}(k, -k; t) \approx 1 + \exp(-t^{2}/\tau(K)^{2})/\tilde{\chi}(0)$ and using equation (13) to arrive at the time-dependent average visibility:

$$V(K, t) = \frac{1}{1 + 2 \tilde{\chi}(0) \exp(t^{2}/\tau(K)^{2})}. \quad (14)$$

where we shall approximate $\tilde{\chi}(0) \approx \chi$ for experimentally feasible conditions.

Wavevector range. Let us consider modes from $K_{\text{min}} = 10 \text{ mm}^{-1}$ to $K_{\text{max}} = 10^{3} \text{ mm}^{-1}$ which constitute a practically feasible range of captured emission angles $2 \times 0.073^{\circ}$ to $2 \times 7.3^{\circ}$ while still allowing to route the write and read beams. In such a case we get $\tau(K_{\text{min}}) = 10 \mu s$ and $\tau(K_{\text{max}}) = 100 \mu s$. Importantly, the number of modes in a range $[K, K + dK]$ is $2\pi K\beta dK$ which grows with $K$, and where $\beta$ is the $K$-space mode density. In our previous work [22] we have determined the number of modes in a WV-MUX-QM by performing a singular value decomposition on experimentally collected data. The obtained mode density $\beta = 3.5 \times 10^{-3} 1/\text{mm}^{2}$ with our choice of $K_{\text{min}}$ and $K_{\text{max}}$ corresponds to the total number of $M = 5500$ modes, where implicitly we have halved the number of modes to enable the generation of polarization entangled states in $M$ pairs of wavevector modes, required for a two-photon quantum repeater protocol.

The range of $K$ modes is limited by the geometry of the optical setup and the size and resolution of the image sensor in a single-photon sensitive camera [35]. The atomic ensemble is assumed to be cooled in a MOT and further trapped in a dipole trap, reaching the temperature of a $\mu K$ which gives $\gamma \approx 10^{3} \mu s mm^{-1}$ amounting to a relatively short lifetime of around $100 \mu s$ for $K_{\text{max}}$ mode. Clearly, there is little point in observing quickly decaying modes for $K > K_{\text{max}}$. The operation of the memory requires counter-propagating write and read beams [22], limiting the observable emission at small angles. With $800 \text{ nm}$ wavelength, $K_{\text{min}} = 10 \text{ mm}^{-1}$ amounts to ca $200 \mu m$ diameter in the far-field left for write/read beam routing, assuming a lens of focal lengths $f = 100 \text{ mm}$. Such a configuration gives the diameter of the total observed far-field of around $1 \text{ inch}$ which is compatible with standard optical elements and feasible to be imaged onto a commercially available $10 \times 10 \text{ mm}$ CMOS sensors with around $1\text{ px} = 10 \mu m$ pixel size. With a properly adjusted magnification the characteristic Gaussian mode size $2\sigma$ of around $2 \times 4.8 \text{ mm}^{-1}$ corresponds to around $2.5 \text{ px}$ on the camera sensor [22].

Mode performance. The ebit content of states generated across different $K$ modes varies, as illustrated in figure 6. As the number of modes with a given $K$ grow with $K$, most of the modes occupy quickly decaying high-$K$ modes, reducing the average EF $\langle E_{\text{f}}(K, t) \rangle_{K}$.

2.5. Quantum repeaters architecture

Hierarchy. Hierarchical protocols like DLCZ or two-photon protocols divide the $N$ nodes pair-wise recursively to form a binary tree. There is a conceptual advantage in such an approach as one only needs to consider the connection time between two sub-nets at each nesting level; however, if one sub-net fails
and needs to start from the ENG at the lowest level, there is a tremendous waiting time overhead for
classical communication to synchronize the manoeuvre. The average requirement for memory storage time
may many times exceeds the direct communication time \( L/c \) between the parties.

On the other hand, in ahierarchical architecture all nodes would operate synchronized only by a classical
clock with period \( L \) and without any feedback. While the requirements for memory storage time are
substantially lower, all nodes must succeed simultaneously for an overall success. With limited efficiencies of
memories and detectors, successful generation becomes exponentially difficult with the increasing number
of nodes.

Semihierarchical architecture has been proposed \[26\] as an intermediate regime. Nodes communicate
with a central station located mid-way between the final parties, which waits until all nodes successfully
generate the entanglement and synchronizes ENC. While the ENC has to succeed simultaneously across all
nodes, the nodes which succeed with ENG wait for other nodes. Such an approach improves the probability
of success per protocol repetition as compared with ahierarchical approach and has lower memory time
requirements than hierarchical architecture.

Regardless of the architecture, the average time for successful entanglement distribution can be
modularly written as

\[
T_{\text{tot}} = T_{t}/(P_{\text{ENG}}P_{\text{ENC}}\eta_{s}^{2}\eta_{e}^{2}),
\]

where \( T_{t} \) is the (average) repetition time of the protocol, \( P_{\text{ENG}} \) the probability of a successful ENG
between all pairs of ensembles, \( P_{\text{ENC}} \) the probability of a successful ENC across all nodes and \( \eta_{s}^{2}\eta_{e}^{2} \) the
probability of detecting the entangled photons at final parties with \( \eta_{s} \) being the efficiency of single-photon
detectors and \( \eta_{e} \) of multiplexing.

Ahierarchical architecture. Let us assume that the main delay in the protocol is the transmission time of
a write-out photon between a node and a CMS and of the which-mode information back to the node i.e.
\( T_{t} = L_{0}/c \), where we take \( c = 0.2 \text{ km } \mu \text{s}^{-1} \) for a fibre transmission. For the protocol to succeed with
\( N = L/L_{0} + 1 \) nodes, we need to generate entanglement (ENG) between \( N - 1 \) memories and perform
first-stage ENC (further ENC) between \( \approx [(N - 2)/2] \) (between \( [(N - 2)/2] \) memories. Additionally
MUX is required before each ENC and fENC. Let us denote the efficiencies of ENG in a single mode (SM),
fENC, ENC and MUX by \( p_{g}, p_{f}, p_{p}, \eta_{s}, \eta_{e} \), respectively. In the ahierarchical architecture all nodes blindly
assume that all other nodes succeeded with ENG and proceed with the ENC. The overall probability for
ENG is given by

\[
P_{\text{ENG}} = p_{s}^{N-1},
\]

with \( p_{s} \) given by equation (5), while for ENC it is

\[
P_{\text{ENC}} = p_{f}^{[(N-2)/2]}p_{e}^{[(N-2)/2]}\eta_{e}^{N}.\]

In standard two-photon protocols \( p_{g} \) is upper bounded by \( 1/2 \) and \( p_{f} = p_{e}/4 \) due to post-selecting on
coincidence patterns chosen to project the state of connected memories onto a Bell state \[9\].

Semihierarchical architecture. For the rates in semihierarchical architecture we follow the derivations by
Liu et al \[26\] with a slight modification to match the two-photon protocol and the WV-MUX-QM
platform. Let us start with equation (15). We shall modify the \( T_{t}/P_{\text{ENG}} \) factor to be now an expectation
value over the distribution of waiting times for all nodes to accomplish ENG. Importantly, there is an additional waiting overhead \( L/c \) for two-way communication with the central station. The factor \( T_i/T_{\text{ENG}} \) now reads

\[
T_i/T_{\text{ENG}} = (L_0/c \times f(N, p_x) + L + c) \quad \text{where} \quad f(N, p_x) \text{ gives the expected number of ENG repetitions to accomplish ENG between } N \text{ nodes with } p_x \text{ given by equation (5) or equation (4) for multiplexed and parallel platforms, respectively. The factor } f(N, p_x) \text{ is given by:}
\]

\[
f(N, p_x) = p_x \sum_{j=1}^{\infty} j \times \left\{ \left[ 1 - (1 - p_x)^j/N \right]^N - \left[ 1 - (1 - p_x)^j \right]^N \right\} . \tag{18}
\]

### 2.6. Limitations

Maximal range. A non-zero ebit content \( E_k(V(t_m)) > 0 \) of the generated state after storage time \( t_m \) requires visibility \( V(t_m) \) above 1/3 which is equivalent to \( \chi \exp[t_m^2/\tau(K)^2] < 1 \) by equation (14).

In the case of a hierarchical architecture the storage time is always \( t_m = L_0/c \) giving \( L_0^{(\text{max})} = c \tau(K) \sqrt{\log(1/\chi)} \) and so \( L_0^{(\text{max})} = NL_0^{(\text{max})} \). For \( \chi = 0.05 \) and \( c = 0.2 \text{ km } \mu\text{s}^{-1} \) we have for the longest-live \( K = 10 \text{ mm}^{-1} \) mode \( \tau(K) = 10 \text{ ms} \) and \( L_0^{(\text{max})} \approx 3.5 \times 10^3 \text{ km} \). Let us require that 1% of all modes have a non-zero ebit content, this corresponds to a range of \( K \) from 10 mm\(^{-1}\) to 100 mm\(^{-1}\) with the lowest \( \tau(K = 100 \text{ mm}^{-1}) = 1 \text{ ms} \) and in this case \( L_0^{(\text{max})} \approx 350 \text{ km} \). This characteristic range is clearly visible in figure 6.

For the semihierarchical architecture and assuming all nodes succeed with ENG in the first try, the total storage time for each memory is \( t_m = (L + L_0)/c \). Employing equation (14), the requirement on the memory coherence time becomes:

\[
\tau(K) > 2 \frac{L + L_0}{c \log(1/\chi)} , \tag{19}
\]

which for a fixed \( \tau(K) \) sets the maximal range

\[
L_{\text{max}} = \frac{N - 1}{N} \frac{\tau(K)}{2} e \log \left( \frac{1}{\chi} \right) , \tag{20}
\]

where \( L + L_0 = LN/(N - 1) \). For \( \chi = 0.05 \), \( \tau(K) = 10 \text{ ms} \), \( c = 0.2 \text{ km } \mu\text{s}^{-1} \), the maximal distance is ca \( L_{\text{max}} \approx 3 \times 10^3 \text{ km} \) for large \( N \gg 1 \).

Entanglement connection probability. The tremendous number of modes in WV-MUX-QM platform combined with flexible multiplexing offers a quasi-deterministic ENG for extended distances between the elementary nodes. As depicted in figure 7(a), the probability of a successful ENG between any of \( M^2 \) combinations of modes remains nearly 100% even with significant optical losses (1% transmission over 100 km). Importantly, lifetime of spin-waves occupying different \( K \) modes quickly deteriorates with increasing \( K \), making only a small fraction of modes feasible for ENG over large \( L_0 \), as indicated by the average EF \( \langle E_k(K, L_0/c) \rangle_K \) which is depicted in figure 7(a). This in turn limits the maximal \( L_0 \) which settles at the level of around 150 km, as illustrated in figure 7(c). With increasing \( L \) and limited \( L_0 \) the optimal number of nodes \( N' \) quickly increases, which is clearly visible in figure 7(b). Conversely, as indicated by equation (15), the probability of a simultaneous success of ENC across all nodes, scaling with the power of \( N' \), rapidly decreases.

A possible solution would be to entangle many pairs of modes during ENG and perform a multiplexed ENC amending the deteriorating scaling with the power of \( N' \).

Telecom wavelength and multimode transmission. While we start our discussion from an experimental realization of a WV-MUX-QM platform [22] which employs a \( \approx 800 \text{ nm} \) light–matter interface, there have been several demonstrations of Rb-87 atomic memories working in the telecom regime thanks to external or in-memory conversion [16, 44–47]; therefore, an experimental realization could build on these results and exploit wavevector multiplexing together with a telecom light–matter interface.

Another technically challenging task would be to implement multimode transmission channel. One way would be to use free-space which is inherently multimode, yet sensitive to atmospheric conditions and requiring expensive optical infrastructure. Modern implementations of spatially multimode free-space optical links are mostly devoted to OAM solutions that currently allow transmission of tens of orthogonal modes [48]. However, as this technology is in the early stage of development, we envisage that in future, links reaching hundreds of modes will be possible.

Another, and currently more promising solution are multimode fibre transmission systems e.g., consisting of an array of single-mode telecom fibres (bundles of over 600 fibres are commercially available) or hybrid multimode-multicore solutions [49]. Importantly, commercially available fibre coupled microlens arrays [50] enable efficient coupling of particular memory modes to the transmission channel and make the solution practically feasible.
Figure 7. Performance measures for WV-MUX-QM platform in an ahierarchical architecture with a total distance $L$. (a) Total probability of ENG in all nodes simultaneously $P_{\text{ENG}}$ is nearly constant and close to unity while total probability of ENC between all nodes and final detection of entangled photons $P_{\text{ENC}}$ falls significantly as the optimal total number of nodes $N^*$—depicted in (b)—grows. (c) Elementary distance between adjacent nodes $L_0$ for the optimal number of nodes $N^*$.

2.7. Rate comparison

We have compared the performance of several physical platforms as candidates for a quantum repeater nodes in ahierarchical and semihierarchical architecture. For each total distance between the parties $L$ we select a number of nodes $N \geq 2$ which maximizes the entanglement transfer rate per node number given by equation (9):

$$N^*(L) = \arg \max_N \langle Q(N, L) \rangle_{K, t}$$

with the average $\langle \rangle$ taken over the distribution of required memory times $t$ across realizations and the performance of individual $K$-modes in a single realization. The comparison encompasses the wavevector-multiplexed quantum memory (WV-MUX-QM), parallel operation (parallel, without multiplexing) of a wavevector multimode quantum memory, temporal multimode quantum memory (temporal) and a state-of-the-art single-mode, long-lifetime atomic quantum memory in an optical lattice (lattice). For reference we consider a direct ENG via a single spontaneous parametric down conversion (SPDC) or quantum dot (QD) source located midway between the parties. For QD source we consider an experimental standard of 80 MHz repetition rate as well as a theoretical limitation of a GHz repetition rate.

Figure 8 depicts the average time for a successful transfer and detection of a single ebit $T = 1/(N^* \times Q(N^*, L))$ for the compared platforms, along typical time scales. Above ca 300 km WV-MUX-QM in the ahierarchical architecture outperforms a direct SPDC source and above ca 400 km an 80 MHz QD source. For ca 550 km the average time for the WV-MUX-QM platform is ca 6 min, for 80 MHz QD it is ca 44 min, while for an SPDC source it amounts to over 2 days; for ca 700 km the average times are around 40 min for WV-MUX-QM, 46 days for QD, and 5 years for SPDC.

Importantly, the assumed parameters correspond to the state-of-the-art experimentally attainable systems; therefore, while the current ebit rates remain impractical for most technological applications, wavevector multiplexing appears to be a promising technique for near-term quantum repeaters.

As illustrated in figure 9, the optimal number of nodes $N^*(L)$ grows relatively slowly with $L$ for the WV-MUX-QM platform, reflecting the ability to quasi-deterministically generate entanglement over extended inter-node elementary distances $L_0$.

One of the limiting factors for the WV-MUX-QM architecture is the mode-dependent memory lifetime which quickly degrades the ebit content of a significant number of modes. For example, for an elementary distance of $L_0 = 150$ km, the state generation is still quasi-deterministic, while the EF yields on average $\langle E_F(K, t = L_0/c) \rangle_K \approx 2 \times 10^{-2}$. Importantly, despite very low ebit content per state, the are probabilistic protocols which enable single-copy distillation of a pure Bell state [51].
3. Discussion

In this work we show the feasibility of wavevector-multiplexed quantum memories (WV-MUX-QM) for near-term quantum repeaters, as an alternative to temporally, spectrally or spatially-multimode platforms. Remaining within the constraints of current technology, we consider an extension of an experimentally demonstrated WV-MUX-QM platform \cite{22} combined with available methods of optical multiplexing and fast spatially-resolved single-photon detection. We harness the tremendous number of modes $M \approx 5500$ theoretically available in WV-MUX-QM to design a robust scheme for polarization ENG between distant...
quantum repeater nodes. Importantly, the designed ENG scheme allows to connect any pair of modes giving $M^2$ possibilities and providing a quasi-deterministic ENG at ca 150 km of telecom fibre.

We extend the WV-MUX-QM setup with a specifically designed Mach–Zehnder interferometer allowing implementation of a two-photon interference-based quantum repeater protocol [9] robust to optical phase fluctuations which deteriorate the performance of the standard DLCZ protocol. Additionally, we consider each memory mode separately by employing a mode-specific decoherence model and calculate the ebit transfer rate by modelling the losses and noise as an depolarizing channel. For the resource-efficient comparison of different quantum repeater platforms, we establish an ebit transfer rate between the final parties per number of employed quantum repeater nodes as our figure of merit.

Finally, we analyse WV-MUX-QM in recently proposed semihierarchical quantum repeater architecture as well as a simple ahierarchical architecture and find the main limitations of the WV-MUX-QM platform. Importantly, while the ENG can be quasi-deterministic for over a hundred km inter-node distance $L_0$, most of the memory modes have a short lifetime deteriorating the ebit content of the generated states with an increasing $L_0$. Conversely, the total distance $L$ is mainly limited by the growing number of nodes $N$ with a limited inter-node distance $L_0 \lesssim 150$ km. Importantly, while multiplexing remedies the fundamental transmission losses inherent to the ENG stage, the ENC remains single-mode. Since ENC results are post-selected, even with ideal detectors and memory retrieval efficiency, the probability of a successful ENC is severely limited. With the growing number of nodes $N$, the requirement for a simultaneous success of all $N - 1$ ENCs constitutes the main factor limiting the total distance $L$. Furthermore, we envisage that our methods can be also applied to a spatially-multiplexed quantum memory [21], where multiplexing is provided by addressing different micro-ensemble at different times with AODs. Since the WV-MUX-QM platform is capable of flexible processing on the stored spin-waves [23–25], we envisage that, as a remedy to the low total ENC probability, a method could be design to entangle many mode pairs simultaneously during ENG and perform a multiplexed or parallelized ENC, such an approach is however beyond the scope of this work and may be an interesting avenue for future development. Another interesting avenue of development would be to use the solid-state systems, that traditionally make use of temporal multiplexing, for wavevector multiplexing. Experiments suggest that the usage of those memories for DLCZ protocol along spatially-multimode noise filtration is feasible [52]. In such case, we would expect longer and wavevector-insensitive lifetimes.

**Acknowledgments**

This work was funded by Ministry of Science and Higher Education (Poland) Grant Nos. DI2016 014846, DI2018 010848; National Science Centre (Poland) Grant No. 2017/25/N/ST2/01163; Foundation for Polish Science MAB/2018/4 ‘Quantum Optical Technologies’. The ‘Quantum Optical Technologies’ project is carried out within the International Research Agendas programme of the Foundation for Polish Science co-financed by the European Union under the European Regional Development Fund. MP was supported by the Foundation for Polish Science START scholarship. We would like to thank W Wasilewski for fruitful discussions and K Banaszek for the generous support.

**Data availability statement**

The data that support the findings of this study are available upon reasonable request from the authors.

**Appendix A. Details of the comparison**

In our comparison we assume that the transmitted ebit has to be detected at final parties to be useful i.e. we include the efficiency of detectors at the first and the last node. The optical transmission is assumed to be carried via telecom fibres characterized by attenuation of $\alpha = 0.2 \text{dB km}^{-1}$ which gives the transmission efficiency over length $z$ of $\eta(z) = 10^{-\alpha z/10}$. The success probability for a single ENG is taken as $p_1 = (\chi \eta(L_0/2)\eta_m)^2$, while for ENC to be $p_e = (\eta_s \eta_s)^2/2$. Whenever we are detecting single-photons in a single spatial mode, we assume the detector efficiency of $\eta_s = 0.9$ [53] and for the multimode detection $\eta_m = 0.2$ [35], both corresponding to experimentally attainable values. Model parameters are summarized in table 1.

For the SPDC midway source we assume a repetition rate of $f_{\text{rep}} = 80 \text{ MHz}$, the probability of generating a pair of correlated photons $\chi = 0.01$ and perfect visibility $V = 1$ giving $E_f = 1$ i.e. one ebit per successful connection. The average time for an ebit transmission is therefore $T_{\text{SPDC}}(L) = \chi(\eta_s(L)\eta_s)^2 f_{\text{rep}}$.  

\[ T_{\text{SPDC}}(L) = \chi(\eta_s(L)\eta_s)^2 f_{\text{rep}}. \]
solids. While multiple degrees of freedom (DoF) multiplexing has been demonstrated in such systems [57], sources (SPS) e.g. QD sources and memories exploiting atomic frequency comb in rare-earth-ion-doped

Instead of considering a single crystal or protocol, for the purpose of this comparison, we optimistically assume a platform connecting state-of-the-art results in reasonably similar systems. This greatly extends the memory lifetime to ca 220 ms as reported by Yang et al. [61], which is the state-of-the-art result [61], and assumed to be equally distributed across various implementations of solid state quantum memories in doped crystals, there is no straightforward way to harness additional DoF to implement two-photon protocols without the overhead of multiple pairs of SPS and QM at each node. Therefore, we assume temporal modes are pure and any mode of the midstate is equally distributed.

Importantly, for Lattice SM and temporal platforms, the decoherence is exponential $\exp(-t/\tau)$, while for wavevector multimode memories it has a Gaussian profile $\exp(-t^2/\tau^2)$.

### Appendix B. Creating entanglement between any mode of memory $A$ and any mode of memory $C'$

In this appendix we present mathematical details of the ENG and ENC steps in the multimode regime.
B.1. Entanglement generation

Let us begin with the two-mode squeezed states generated in each atomic ensemble during write process. For ENG we will consider two distant memories $A$ and $B'$. For simplicity we will focus on a single wavevector mode with transverse wavevector of the write-out photon $k_w$. The generated state can be written as

$$|\psi_{k_w}\rangle = \sqrt{1-\chi} \sum_{j=0}^{\infty} (\sqrt{\chi} S_{k} \hat{a}_{k_w}^\dagger) |\text{vac}\rangle,$$  \hspace{1cm} (21)

where $\chi$ gives the probability of generating a single pair of photon–spin-wave excitations, $\hat{a}_{k_w}^\dagger$ is the creation operator for write-out photon with a wavevector $k_w$ and the creation operator for a single spin-wave with wavevector $K$ is denoted by $\hat{S}_K$. Since $\chi \ll 1$ and we shall further employ post-selection, let us approximate the state as:

$$|\psi_{k_w}\rangle \approx (1 + \sqrt{\chi} S_{k} \hat{a}_{k_w}^\dagger)|\text{vac}\rangle,$$  \hspace{1cm} (22)

The spin-waves can be later converted to read-out photons with a read beam $\hat{S}_K \rightarrow \hat{a}_{k_r}^\dagger$. With fixed write and read beams there is a unique correspondence between $k_w$, $k_r$, and $K$, hence further we shall denote write-out, read-out and spin-wave modes by a single wavevector $k \equiv k_w$. In this notation we have:

$$|\psi_k\rangle \approx (1 + \sqrt{\chi} \hat{a}_{k}^\dagger)|\text{vac}\rangle.$$  \hspace{1cm} (23)

For reference we note that the full state generated during memory write operation can be expressed as a statistical mixture:

$$\rho = \int dk_1 P(k_1) dk_2 P(k_2) \ldots dk_M P(k_M) |\psi(k_1, k_2, \ldots, k_M)\rangle \langle \psi(k_1, k_2, \ldots, k_M)|,$$  \hspace{1cm} (24)

where $P(k)$ gives the (uniform) probability that $j$th memory mode is associated with write-out photon wavevector $k_j$ (regardless of whether a photon is actually emitted), and where

$$|\Psi(k_1, k_2, \ldots, k_M)\rangle = \bigotimes_{j=1}^{M} |\psi_{k_j}\rangle.$$  \hspace{1cm} (25)

In further considerations only one of the transverse components of $k$ matters. Let us therefore further simplify the notation and focus on $x$ component of $k$ denoting it by $k \equiv k_x$. Finally, our initial state is:

$$|\psi_k\rangle \approx (1 + \sqrt{\chi} \hat{a}_{k}^\dagger)|\text{vac}\rangle.$$  \hspace{1cm} (26)

Now we shall follow the operations depicted in figure 3 of the main manuscript. One detail that is not depicted is that the polarization of generated photons is circular and we implicitly convert it to vertical (V) polarization using a quarter wave-plate. Hence, we can write the photonic creation operator with the indicated polarization:

$$|\psi_k\rangle \approx (1 + \sqrt{\chi} \hat{a}_{k,V}^\dagger)|\text{vac}\rangle.$$  \hspace{1cm} (27)

A lens at the focal length from the atomic cloud captures the emission from the memory in a range of $k \in [-K_M, K_M]$ for some $K_M = fD/2$, where $f$ is the focal length of the lens and $D$ is the diameter of the lens aperture. In the far-field of the atomic cloud, where distinct positions correspond to different wavevectors, the range of $k$ is divided into $k > 0$ part and $k < 0$ part (this division makes only one transverse component of $k$ significant for our considerations). A half-wave plate (HWP) is inserted in $k > 0$ part to convert the polarization to horizontal (H):

$$\hat{a}_{k,V}^\dagger \rightarrow \begin{cases} \hat{a}_{k,H}^\dagger; & K_M \geq k \geq 0, \\ \hat{a}_{k,V}^\dagger; & -K_M \leq k < 0. \end{cases}$$  \hspace{1cm} (28)

Further, a mirror and a polarizing beamsplitter (PBS) superimposes the two parts onto each other (shifts $k \rightarrow k + K_M$ for $k < 0$):

$$\hat{a}_{k,V}^\dagger \rightarrow \hat{a}_{k+K_M,V}^\dagger; -K_M \leq k < 0.$$  \hspace{1cm} (29)

We shall now consider the operations at the central measurement station (CMS). For both $X = A$ and $X = B'$ the state is split into $H$ and $V$ polarizations on a PBS. $V$ is reflected (subscript $R$) and $H$ is transmitted (subscript $T$). Another HWP rotates polarization of the reflected arm to $H$.

$$\hat{a}_{k,V,X}^\dagger \rightarrow \hat{a}_{k,H,X,R}^\dagger;$$  \hspace{1cm} (30)

$$\hat{a}_{k,H,X}^\dagger \rightarrow \hat{a}_{k,H,X,T}^\dagger.$$  \hspace{1cm} (31)
Now the $R$ mode of $A$ is combined with the $T$ mode of $B'$ on a BS and vice-versa. This effectively erases the information about the origin ($X$) of the photonic mode. For the $A$, $T$ and $B'$, $R$ we shall call the BS outputs $I_+$ and $I_-$. Similarly for $A$, $R$ and $B'$, $T$ the outputs are denoted $\Pi_+$ and $\Pi_-$. BS applies the Hadamard transform to the input modes, hence
\begin{align}
\hat{a}_{k_{A,T}}^{\dagger} & \rightarrow (\hat{a}_{k_{I+}}^{\dagger} + \hat{a}_{k_{I-}}^{\dagger})/\sqrt{2}, \\
\hat{a}_{k_{B,R}}^{\dagger} & \rightarrow (\hat{a}_{k_{I+}}^{\dagger} - \hat{a}_{k_{I-}}^{\dagger})/\sqrt{2}, \\
\hat{a}_{k_{A,B}}^{\dagger} & \rightarrow (\hat{a}_{k_{I+I+}}^{\dagger} + \hat{a}_{k_{I-II}}^{\dagger})/\sqrt{2}, \\
\hat{a}_{k_{B,B'}}^{\dagger} & \rightarrow (\hat{a}_{k_{I+I+}}^{\dagger} - \hat{a}_{k_{I-II}}^{\dagger})/\sqrt{2},
\end{align}
\hspace{1cm} (32) - (35)
where we omitted the polarization degree of freedom (all polarizations at this stage are $H$). Finally, we perform a measurement ($k$-resolved) projecting our state onto the subspace of one photon in $I_+$ or $I_-$ mode and one photon in $\Pi_+$ or $\Pi_-$ mode. The measurement operators are given by:
\begin{align}
\Pi_{0}(k,k') & = \sum_{\sigma,\sigma'\in\{+,-\}} M_{\sigma,\sigma'}(k,k'); \quad k,k' \in [0,K_M], \\
\Pi_{1} & = \mathbb{I} - \sum_{k=0}^{K_M} \sum_{\sigma=0}^{K_M} \Pi_{0},
\end{align}
\hspace{1cm} (36)
where
\begin{equation}
M_{\sigma,\sigma'}(k,k') = \hat{a}_{k_{\Pi_+I_+}}^{\dagger}\hat{a}_{k'_{\Pi_+I_+}}|\text{vac}\rangle\langle \text{vac}|\hat{a}_{k_{\Pi_+I_-}}^{\dagger}\hat{a}_{k'_{\Pi_+I_-}}.
\end{equation}
\hspace{1cm} (37)
The measurement results are post-selected to $\Pi_{0}(k,k')$. Looking at the form of $M_{\sigma,\sigma'}$ operators, it can be seen that either both photons come from $A$ ($B$) or one comes from $A$ and one comes from $B$. Furthermore, let us observe that if two photons came from $A$ ($B$) and originated from the same mode i.e. corresponded to the $\chi(S_i^{A}\hat{a}_i^A)^2$ term in the generated state, then they will be both detected in $I_+$ modes or both in $\Pi_+$ modes and will not produce a coincidence. Hence, we can neglect $\mathcal{O}(\chi)$ contributions in a single $k$ mode and our approximation of equation (22) is justified. Finally, we note that we have made a simplifying assumption in specifying the form of $M_{\sigma,\sigma'}$ operators by assuming a photon-number-resolving (PNR) detection.

Experimentally available single-photon cameras are not PNR and introduce some error from multi-photon events; however, with a low probability $\mathcal{O}(\chi^2)$.

The full state of atomic and photonic modes from $A$ and $B'$ is:
\begin{equation}
\rho^{(AB)} = \rho^{(A)} \otimes \rho^{(B')},
\end{equation}
\hspace{1cm} (38)
where $\rho^{(A)}$ and $\rho^{(B')}$ are given by equation (24). We will assume that a coincidence between modes $k_j \in [0,K_M]$ and $k_{ij} \in [0,K_M]$ in regions $I_+$ and $\Pi_+$ has been detected. This amounts to a choice of two photonic modes from the tensor product of equation (37) from the $M$ modes of $A$ and $M$ modes of $B'$. We will neglect the case where two memory modes have an overlapping wavevector i.e. when $\exists_{j,k} k_j = k_0 = k_1$ or $\exists_{ijk} k_j = k_0 = k_{ij}$ as the probability of such an event, leading to the emission of two photons into the same wavevector mode, is negligible since $\chi \ll 1$. Hence, the state $\rho^{(AB)}$ reduces under $\Pi_0$ measurement’s result of $k_i, k_{ij}$ to the form:
\begin{align}
\tilde{\rho}^{(AB)} & = |\tilde{\Psi}^{(AB')}\rangle\langle \tilde{\Psi}^{(AB')}|, \\
|\tilde{\Psi}^{(AB')}\rangle & = \chi^{\mathcal{N}}(S_{\mathcal{N},A} + S_{\mathcal{N},-kM,B'}) \otimes (S_{\mathcal{N},-k_{A,I} + k_{B,I}} + S_{\mathcal{N},-k_{B,I} + k_{A,I}})|\text{vac}\rangle,
\end{align}
\hspace{1cm} (39)
with $\mathcal{N}$ being the normalizing factor and where we take into account that by construction photons in $I_+$ must be $H$ ($V$) polarized when coming from $A$ ($B'$) and photons in $\Pi_+$ must be $V$ ($H$) polarized when coming from $A$ ($B'$). Let us observe that $V$ photons from $A$ or $B'$ have had their wavevector shifted $k \rightarrow k + K_M$ so that $k_j, k_{ij} \geq 0$ and this is reflected in $k \rightarrow k - K_M$ shift on $S_i$ operators.

Note that equations (38) and (39) are equivalent to the state of equation (2) from the main manuscript if we account for the optical phase difference between $A$–CMS and $B'$–CMS paths. Furthermore, as the two components forming the $|\tilde{\Psi}^{(AB')}\rangle$ state are mutually independent (i.e. $k_i$ and $k_{ij}$ are chosen independently), we have $M \times M$ possibilities leading to desired coincidence pattern, and thus successful ENG.

**B.2. Entanglement connection**

For ENC the two memories of a single repeater station (e.g. $B'$, $B$) are read:
\begin{equation}
\hat{S}_k^{\dagger} \rightarrow \hat{a}_{k,V}^{\dagger}
\end{equation}
\hspace{1cm} (40)
and the resulting photons have their polarization rotated according to equation (28) with an analogous setup as for write-out photons.

Importantly, MUX must be used at this stage. If we trace-out the A part of the state given by equation (38), we are left with:

\[
\text{Tr}_A \rho_{AB'}^{(AB')} = \chi N (\hat{S}_{k_1-K_M,B'}^{i} \hat{S}_{k_1,B'}^{j} |\text{vac}\rangle |\text{vac}\rangle \hat{S}_{k_1-K_M,B'}^{j} \hat{S}_{k_1,B'}^{i} + |\text{vac}\rangle |\text{vac}\rangle).
\]

Hence, during readout, B’ will emit a photon into \(k_1 > 0\) or \(k_1 - K_M < 0\) or both. Fortunately, our setup already separates \(k < 0\) and \(k > 0\) emission. Therefore, we can send \((k_1,k_1)\) coordinates from CMS to B’ and use it to reconfigure a multiplexer to perform the following wavevector shift:

\[
\hat{a}_k^i \rightarrow \begin{cases} \hat{a}_k^{i+k} \quad & k \geq 0, \\ \hat{a}_k^{i+k} \quad & k < 0, \end{cases}
\]

which directs \(k\) into \(k_1^{(0)}\) and \(k_1 - K_M\) into \(k_1^{(0)}\). An analogous multiplexing takes place at all other readout memories; however, for non-primed memories e.g. B \(k_1^{(0)}\) is swapped with \(k_1^{(0)}\) in equation (42).

Modes \(k_1^{(0)}\) and \(k_1^{(0)}\) are always the same in every protocol repetition and enter further ENC. Finally, our readout (\(R\)), transformed and multiplexed state reads:

\[
\hat{\rho}_R^{(AB')} = |\tilde{\Psi}_R^{(AB')}\rangle \langle \tilde{\Psi}_R^{(AB')}|,
\]

\[
|\tilde{\Psi}_R^{(AB')}\rangle = \chi N (\hat{S}_{k_1,A}^{i} + \hat{a}_k^{i+k_{1,M}}(0) \hat{S}_{k_1-M_1,A}^{i} + \hat{a}_k^{i+k_{1,R}^{0}}(0) \hat{S}_{k_1-M_1,R}^{i}) |\text{vac}\rangle.
\]

The actual ENC takes place at a local station consisting of two memories e.g. \(B', B\). Hence, let us consider successful ENG, readout and MUX at A, \(B'\) with modes \(k_1^{(0)}, k_{1,M}^{(0)}\) and at \(B, C\) with modes denoted by \(k_1^{(0)}, k_{1,V}^{(0)}\). We start with the state \(\hat{\rho}_R^{(AB')} \otimes \hat{\rho}_R^{(BC')}\) where \(B'\) and \(B\) were readout, and should arrive at an entangled atomic state of \(A\) and \(C\).

The \(k_1^{(0)}, k_{1,M}^{(0)}\) (from \(B'\)) and \(k_1^{(0)}, k_{1,V}^{(0)}\) (from \(B\)) modes enter the ENC setup (anti)symmetrically, as depicted in figure 3 of the main manuscript. The \(k_1^{(0)}, k_{1,M}^{(0)}\) and \(k_1^{(0)}, k_{1,V}^{(0)}\) modes are combined within each pair on the first PBS. Let us denote its outputs modes \(b_{XY}^{i,j}\): \(X \in \{L, R\}, Y \in \{T, B\}\) where L, R, T, B correspond to left, right, top, bottom, respectively and refer to orientation of the figure (3) of the main manuscript. Below we give the transformation for \(k_1^{(0)}, k_{1,M}^{(0)}\) and \(k_1^{(0)}, k_{1,V}^{(0)}\) (note the different order II, I vs III, IV) and RB, RT.

\[
\hat{a}_k^{i+k_{1,M}}(0) \rightarrow \hat{b}_{\text{LB},H}^{i,j},
\]

\[
\hat{a}_k^{i+k_{1,R}^{0}}(0) \rightarrow \hat{b}_{\text{LT},V}^{i,j},
\]

\[
\hat{a}_k^{i+k_{1,M}}(0) \rightarrow \hat{b}_{\text{LT},H}^{i,j},
\]

\[
\hat{a}_k^{i+k_{1,R}^{0}}(0) \rightarrow \hat{b}_{\text{LR},V}^{i,j}.
\]

For the so-called first-stage ENC (fENC) i.e. ENC with memories just after ENG and not after another ENC, the first PBS is followed (for LB, RB modes) by an HWP rotating the polarization by \(45^\circ\) to diagonal \((D \propto H + V)\)/antidiagonal \((A \propto H - V)\):

\[
\hat{b}_{\text{LB},H}^{i} \rightarrow \hat{b}_{\text{LR},A}^{i},
\]

\[
\hat{b}_{\text{LR},V}^{i} \rightarrow \hat{b}_{\text{LB},D}^{i}.
\]

The L and R modes are combined on a central PBS. Let us call its output modes \(\hat{c}_L^{i}\) and \(\hat{c}_R^{i}\) for left and right, respectively:

\[
\hat{b}_{\text{LB},H}^{i} \rightarrow \hat{c}_L^{i},
\]

\[
\hat{b}_{\text{LB},V}^{i} \rightarrow \hat{c}_L^{i},
\]

\[
\hat{b}_{\text{RB},H}^{i} \rightarrow \hat{c}_L^{i}.
\]
Finally, the $\hat{c}_l^1$ and $\hat{c}_r^1$ modes undergo detection in the diagonal/antidiagonal basis on two $+/−$ detectors (total of four single-photon detectors), respectively. We post-select measurement outcomes on a coincidence between the two $+/−$ detectors i.e. project the state on a subspace of one photon in $\hat{c}_l^1$ and one in $\hat{c}_r^1$. The measurement operators are:

$$\Pi_0(\sigma, \sigma') = \hat{c}_l^1\hat{c}_r^1|\text{vac}\rangle \langle \text{vac}|\hat{c}_{l',\sigma'}\hat{c}_{l',\sigma}; \; \sigma, \sigma' \in \{D,A\},$$

$$\Pi_1 = \mathbb{1} - \sum_{\sigma,\sigma'} \Pi_0(\sigma, \sigma').$$

We are interested in the conditional state of $A$, $C'$ with a measurement outcome $(\sigma, \sigma')$ where $D$ result is further denoted by $+$ and $A$ by $−$. Since in total we project on the sub-space of two photons, in $\hat{\rho}_R^{BF'} \otimes \hat{\rho}_S^{BC'}$ there are two possible two-photon combinations which do not produce an entangled state, corresponding to either two photons from $B'$ and none from $B$ or vice-versa. Let us see how those terms fail to produce a coincidence between $+/−$ detectors and are effectively filtered-out. We will consider the case of $\hat{a}_{k,ij}^{(0),BH',V'}|\text{vac}\rangle$ i.e. two photons from $B'$ and none from $B$. Applying the transformations we get:

$$\hat{a}_{k,ij}^{(1),BH',V'}|\text{vac}\rangle \rightarrow \hat{b}_{LB,H}^k \hat{b}_{LB,H}^k \rightarrow \hat{b}_{LB,H}^k \hat{b}_{LB,A}^k = (\hat{b}_{LB,H}^k)^2 - (\hat{b}_{LB,V}^k)^2,$$

$$\hat{b}_{LB,H}^k)^2 - (\hat{b}_{LB,V}^k)^2 \rightarrow (\hat{c}_H^k)^2 - (\hat{c}_V^k)^2,$$

which never produces a coincidence between $L$ and $R$ detectors. Hence, we are left with terms with one photon from $B'$ and one from $B$:

$$\langle (\hat{S}_{l,A}^1 \hat{a}_{k,ij}^{(0),BH',V'}) | (\hat{S}_{l',A}^1 \hat{a}_{k,ij}^{(1),BH',V'}) \rangle \rightarrow (\hat{S}_{l,A}^1 \hat{b}_{LB,H}^k + \hat{S}_{k,A}^1 \hat{b}_{LB,V}^k) (\hat{S}_{l',A}^1 \hat{b}_{LB,H}^k + \hat{S}_{k,A}^1 \hat{b}_{LB,V}^k)$$

$$\rightarrow (\hat{S}_{l,A}^1 \hat{b}_{LB,H}^k + \hat{S}_{k,A}^1 \hat{b}_{LB,V}^k + \hat{S}_{l',A}^1 \hat{b}_{LB,H}^k + \hat{S}_{k,A}^1 \hat{b}_{LB,V}^k)$$

$$= [\hat{S}_{l,A}^1 \hat{b}_{LB,H}^k + \hat{S}_{k,A}^1 \hat{b}_{LB,V}^k + \hat{S}_{l',A}^1 \hat{b}_{LB,H}^k + \hat{S}_{k,A}^1 \hat{b}_{LB,V}^k]$$

$$\times [\hat{S}_{l',A}^1 \hat{b}_{LB,H}^k + \hat{S}_{k,A}^1 \hat{b}_{LB,V}^k + \hat{S}_{l',A}^1 \hat{b}_{LB,H}^k + \hat{S}_{k,A}^1 \hat{b}_{LB,V}^k]$$

$$= [\hat{b}_{LB,H}^k (\hat{S}_{l,A}^1 + \hat{S}_{k,A}^1 - \hat{S}_{l',A}^1) + \hat{b}_{LB,V}^k (\hat{S}_{l',A}^1 + \hat{S}_{k,A}^1 - \hat{S}_{l,A}^1)]$$

$$\times [\hat{b}_{LB,H}^k (\hat{S}_{l',A}^1 + \hat{S}_{k,A}^1 - \hat{S}_{l,A}^1) + \hat{b}_{LB,V}^k (\hat{S}_{l',A}^1 + \hat{S}_{k,A}^1 - \hat{S}_{l,A}^1)]$$

$$\times [\hat{c}_H^k (\hat{S}_{l',A}^1 + \hat{S}_{k,A}^1 - \hat{S}_{l,A}^1) + \hat{c}_V^k (\hat{S}_{l',A}^1 + \hat{S}_{k,A}^1 - \hat{S}_{l,A}^1)].$$

Since we are post-selecting onto one photon in $\hat{c}_l^1$ and one in $\hat{c}_r^1$ mode, only two terms survive:}

$$\hat{c}_R^1 (\hat{S}_{l,A}^1 + \hat{S}_{k,A}^1 - \hat{S}_{l',A}^1) (\hat{S}_{l',A}^1 + \hat{S}_{k,A}^1 - \hat{S}_{l,A}^1)$$

$$+ \hat{c}_L^1 (\hat{S}_{l,A}^1 + \hat{S}_{k,A}^1 - \hat{S}_{l',A}^1) (\hat{S}_{l',A}^1 + \hat{S}_{k,A}^1 - \hat{S}_{l,A}^1)$$

$$\propto (\hat{c}_L^1 + \hat{c}_R^1) (\hat{S}_{l,A}^1 + \hat{S}_{k,A}^1 - \hat{S}_{l',A}^1) (\hat{S}_{l',A}^1 + \hat{S}_{k,A}^1 - \hat{S}_{l,A}^1)$$

$$+ (\hat{c}_L^1 - \hat{c}_R^1) (\hat{S}_{l,A}^1 + \hat{S}_{k,A}^1 - \hat{S}_{l',A}^1) (\hat{S}_{l',A}^1 + \hat{S}_{k,A}^1 - \hat{S}_{l,A}^1)$$

Hence the conditional states are (up to normalizing factor):

$$(D, D) \equiv (+, +) : (\hat{S}_{l,A}^1 + \hat{S}_{k,A}^1 - \hat{S}_{l',A}^1) (\hat{S}_{l',A}^1 + \hat{S}_{k,A}^1 - \hat{S}_{l,A}^1)$$

$$\propto (\hat{S}_{l,A}^1 + \hat{S}_{k,A}^1 - \hat{S}_{l',A}^1) (\hat{S}_{l',A}^1 + \hat{S}_{k,A}^1 - \hat{S}_{l,A}^1)$$

$$(D, A) \equiv (+, −) : (\hat{S}_{l,A}^1 + \hat{S}_{k,A}^1 - \hat{S}_{l',A}^1) (\hat{S}_{l',A}^1 + \hat{S}_{k,A}^1 - \hat{S}_{l,A}^1)$$

$$− (\hat{S}_{l,A}^1 + \hat{S}_{k,A}^1 - \hat{S}_{l',A}^1) (\hat{S}_{l',A}^1 + \hat{S}_{k,A}^1 - \hat{S}_{l,A}^1)$$

$$\propto (\hat{S}_{l,A}^1 + \hat{S}_{k,A}^1 - \hat{S}_{l',A}^1) (\hat{S}_{l',A}^1 + \hat{S}_{k,A}^1 - \hat{S}_{l,A}^1)$$
\[
\alpha S_{k_1A}^i S_{KIV,M_1C}^{i*} + S_{KII-KM_1A}^i S_{KIV-KM_1C}^{i*}
\]
\[
(A, D) \equiv (+, +) : (S_{KII-KM_1A}^i + S_{KII-KM_1A}^{i*}) (S_{KIV-KM_1C}^i + S_{KIV-KM_1C}^{i*})
\]
\[
= (S_{KII-KM_1A}^i - S_{KII-KM_1A}^{i*}) (S_{KIV-KM_1C}^i - S_{KIV-KM_1C}^{i*})
\]
\[
\times S_{KIA}^i S_{KIII,C}^{i*} + S_{KII-KM_1A}^i S_{KIII,C}^{i*},
\]
\[
(A, A) \equiv (+, -) : (S_{KII-KM_1A}^i + S_{KII-KM_1A}^{i*}) (S_{KIV-KM_1C}^i + S_{KIV-KM_1C}^{i*})
\]
\[
+ (S_{KII-KM_1A}^i - S_{KII-KM_1A}^{i*}) (S_{KIV-KM_1C}^i - S_{KIV-KM_1C}^{i*})
\]
\[
\times S_{KIA}^i S_{KIV-KM_1C}^{i*} + S_{KII-KM_1A}^i S_{KIV-KM_1C}^{i*},
\]
(59)

for instance the \((+, +)\) coincidence mentioned in the main manuscript gives
\[
(S_{KII-KM_1A}^i S_{KIV-KM_1C}^{i*} + S_{KII-KM_1A}^{i*} S_{KIV-KM_1C}^{i*}) |\text{vac}\rangle
\]
which after reading memories \(A\) and \(C\) amounts to
\[
(\hat{a}_{KII-KM_1A}^i |a_{k_1,C}^i, V, C\rangle + \hat{a}_{KII-KM_1A}^{i*} |a_{k_1,C}^{-i}, V, C\rangle)|\text{vac}\rangle
\]
i.e. a state of two spatially separated \((A, C)\) photons maximally entangled in polarization degree of freedom. Coincidence \((-,-)\) gives the same result, while \((+, -)\) and \((-,-)\) lead to
\[
(\hat{a}_{KII-KM_1A}^i |a_{k_1,C}^i, V, C\rangle + \hat{a}_{KII-KM_1A}^{i*} |a_{k_1,C}^{-i}, V, C\rangle)|\text{vac}\rangle
\]
which requires additional polarization rotation (bit-flip) \(H \leftrightarrow V\) at \(C\) for all coincidence patterns to give the same state.

### B.3. Further ENC
After all stations perform first-stage ENC, the read-out, multiplexed and optionally bit-flipped entangled photonic states are of the form \((|HV\rangle + |VH\rangle)/\sqrt{2}\). Hence, further ENC must perform standard entanglement teleportation. The required setup is identical to the one used in first-stage ENC with an exception that after the first PBS there is no polarisation rotation at an HWP.

### ORCID iDs
Michal Lipka 🐝 https://orcid.org/0000-0001-8500-3494
Mateusz Mazelaniak 🐝 https://orcid.org/0000-0001-5498-7486
Michal Parniak 🐝 https://orcid.org/0000-0002-6849-4671

### References

[1] Sanguoard N, Simon C, De Riedmatten H and Gisin N 2011 Quantum repeaters based on atomic ensembles and linear optics Rev. Mod. Phys. 83 33–80
[2] Wehner S, Elkouss D and Hanson R 2018 Quantum internet: a vision for the road ahead Science 362 eaam9288
[3] Pirandola S, Ottaviani C, Speradelli G, Weedbrook C, Braunstein S L, Lloyd S, Gehring T, Jacobsen C S and Andersen U L 2015 High-rate measurement-device-independent quantum cryptography Nat. Photon. 9 397–402
[4] Pirandola S et al 2020 Advances in quantum cryptography Adv. Opt. Photon. 12 1012–236
[5] Zhang W, Ding D-S, Sheng Y-B, Zhou I, Shi B-S and Guo G-C 2017 Quantum secure direct communication with quantum memory Phys. Rev. Lett. 118 220501
[6] Long G L and Liu X S 2002 Theoretically efficient high-capacity quantum-key-distribution scheme Phys. Rev. A 65 032302
[7] Briegel H-J, Dur W, Cirac J I and Zoller P 1998 Quantum repeaters: the role of imperfect local operations in quantum communication Phys. Rev. Lett. 81 5932–3
[8] Yuan Z-S, Chen Y-A, Zhao B, Chen S, Schmiedmayer J and Pan J-W 2008 Experimental demonstration of a BDCZ quantum repeater node Nature 454 1098–101
[9] Zhao B, Chen Z-B, Chen Y-A, Schmiedmayer J and Pan J-W 2007 Robust creation of entanglement between remote memory qubits Phys. Rev. Lett. 98 240502
[10] Gisin N 2015 How far can one send a photon? Front. Phys. 10 100307
[11] Yu Y et al 2020 Entanglement of two quantum memories via fibres over dozens of kilometres Nature 578 240–5
[12] Collins O A, Jenkins S D, Kuzmich A and Kennedy T A B 2007 Multiplexed memory-insensitive quantum repeaters Phys. Rev. Lett. 98 060502
[13] Van Dam S B, Humphreys P C, Rozprędek F, Wehner S and Hanson R 2017 Multiplexed entanglement generation over quantum networks using multi-qubit nodes Quantum Sci. Technol. 2 034002
[14] Simon C, de Riedmatten H, Afzelius M, Sanguoard N, Zhbend H and Gisin N 2007 Quantum repeaters with photon pair sources and multimode memories Phys. Rev. Lett. 98 190504
[15] Simon C, De Riedmatten H and Afzelius M 2010 Temporally multiplexed quantum repeaters with atomic gases Phys. Rev. A 82 010304
[16] Chang W, Li C, Wu Y-K, Jiang N, Zhang S, Pu Y-F, Chang X-Y and Duan L-M 2019 Long-distance entanglement between a multiplexed quantum memory and a telecom photon Phys. Rev. X 9 041033
[17] Sinclair N et al 2014 Spectral multiplexing for scalable quantum photonic systems using an atomic frequency comb quantum memory and feed-forward control Phys. Rev. Lett. 113 053603
[18] Wen Y, Zhou P, Xu Z, Yuan L, Zhang H, Wang S, Tian L, Li S and Wang H 2019 Multiplexed spin-wave–photon entanglement source using temporal multimode memories and feedforward-controlled readout Phys. Rev. A 100 012342

[19] Tian L, Xu Z, Chen L, Ge W, Yuan H, Wen Y, Wang S, Li S and Wang H 2017 Spatial multiplexing of atom-photon entanglement sources using feedforward control and switching networks Phys. Rev. Lett. 119 130505

[20] Chang L, Jiang N, Wu Y, Chang W, Pu Y, Zhang S and Duan L 2020 Quantum communication between multiplexed atomic quantum memories Phys. Rev. Lett. 124 240504

[21] Pu Y-F, Jiang N, Chang W, Yang H-X, Li C and Duan L-M 2017 Experimental realization of a multiplexed quantum memory with 225 individually accessible memory cells Nat. Commun. 8 15359

[22] Parniak M, Dąbrowski M, Mazelanik M, Leszczyński A, Lipka M and Wasilewski W 2017 Wavevector multiplexed atomic quantum memory via spatially-resolved single-photon detection Nat. Commun. 8 2140

[23] Parniak M, Mazelanik M, Leszczyński A, Lipka M, Dąbrowski M and Wasilewski W 2019 Quantum optics of spin waves through ac Stark modulation Phys. Rev. Lett. 122 063604

[24] Mazelanik M, Parniak M, Leszczyński A, Lipka M and Wasilewski W 2019 Coherent spin-wave processor of stored optical pulses npj Quantum Inf. 5 22

[25] Lipka M, Leszczyński A, Mazelanik M, Parniak M and Wasilewski W 2019 Spatial spin-wave modulator for quantum-memory-assisted adaptive measurements Phys. Rev. Applied 11 030449

[26] Liu X, Zhou Z-Q, Hua Y-L, Li C-F and Guo G-C 2017 Semihierarchical quantum repeaters based on moderate lifetime quantum memories Phys. Rev. A 95 012319

[27] Richardson D J, Fini J M and Nelson L E 2013 Space–division multiplexing in optical fibres Nat. Photon. 7 354–62

[28] Ding Y, Bacco D, Dalgaard K, Cai X, Zhou H, Rottwitt K and Katsuo Oxenløwe L 2017 High-dimensional quantum key distribution based on multi-core fiber using silicon photonic integrated circuits npj Quantum Inf. 3 23

[29] Liao S-K et al 2017 Long-distance free-space quantum key distribution in daylight towards inter-satellite communication Nat. Photon. 11 509–13

[30] Duan L-M, Lukin M D, Cirac J I and Zoller P 2001 Long-distance quantum communication with atomic ensembles and linear optics Nature 414 413–8

[31] Chen Z-B, Zhao B, Chen Y-A, Schmiedmayer J and Pan J-W 2007 Fault-tolerant quantum repeater with atomic ensembles and linear optics Phys. Rev. A 76 022329

[32] Grimm R, Weidemüller M and Ovchinnikov Y B 2000 Optical dipole traps for neutral atoms Adv. At. Mol. Opt. Phys. 42 95–170

[33] Zhao B et al 2009 A millisecond quantum memory for scalable quantum networks Nat. Phys. 5 95–9

[34] Böttcher P, Pietragalla S and Martínez M 1996 Optical time-to-space converter Opt. Commun. 123 473–6

[35] Lipka M, Parniak M and Wasilewski W 2016 Correlation steering in the angularly multimode Raman atomic memory npj Quantum Inf. 2 18

[36] Jiang L, Taylor J M and Lukin M D 2010 Fast and robust approach to long-distance quantum communication with atomic memory-compatible single photons from 606 nm to the telecom C-band Phys. Rev. Lett. 104 153604

[37] Thompson J, Qiang K, Liu B, Renk R, Qiu Y and Feng Z 2010 Solid-state source of strongly entangled photon pairs with high brightness and indistinguishability Nat. Photon. 4 95–170

[38] Wang Z-W, Zhou X-F, Huang Y-F, Zhang Y-S, Ren X-F and Guo G-C 2006 Experimental entanglement distillation of two-qubit memory via spatially-resolved single-photon detection Ph. Rev. Lett. 95 107904

[39] Wójtowicz M, Parniak M, Leszczyński A, Lipka M and Wasilewski W 2019 Coherent spin-wave modulator of stored optical pulses npj Quantum Inf. 5 22

[40] Liu X, Zhou Z-Q, Hua Y-L, Li C-F and Guo G-C 2017 Semihierarchical quantum repeaters based on moderate lifetime quantum memories Phys. Rev. A 95 012319
[59] Jobez P, Timoney N, Laplane C, Etesse J, Ferrier A, Goldner P, Gisin N and Afzelius M 2016 Towards highly multimode optical quantum memory for quantum repeaters Phys. Rev. A 93 032327

[60] Ma Y, Ma Y-Z, Zhou Z-Q, Li C-F and Guo G-C 2021 One-hour coherent optical storage in an atomic frequency comb memory (arXiv:2012.14605v2)

[61] Hedges M P, Longdell J J, Li Y and Sellars M J 2010 Efficient quantum memory for light Nature 465 1052–6

[62] Yang S-J, Wang X-J, Bao X-H and Pan J-W 2016 An efficient quantum light-matter interface with sub-second lifetime Nat. Photon. 10 381–4