Hebbian learning of higher-order interactions facilitates abrupt anti-phase synchronization

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This Letter investigates the transition to synchronization of oscillator ensembles encoded by simplicial complexes in which pairwise and higher-order coupling weights adapt with time through the Hebbian learning mechanism. These concurrently evolving disparate adaptive coupling weights lead to a novel phenomenon in that the in-phase synchronization is completely obliterated; instead, the anti-phase synchronization is originated. Besides this, the onsets of anti-phase synchronization and desynchronization are manageable through both dyadic and triadic Hebbian learning rates. The theoretical validation of these numerical assessments is delineated thoroughly by employing Ott-Antonsen dimensionality reduction. The framework and results of the Letter would help understand the underlying synchronization behavior of a range of real-world systems, such as the brain functions and social systems where interactions evolve with time.

The inclusion of higher-order interactions has not drawn much attention for so long while envisaging underlying dynamics influencing distinct processes taking place on a variety of complex systems ranging from physical to biological systems. Nevertheless, many complex systems such as the brain networks [1, 2] and social interaction networks [3, 4] have underlying structure of higher-order connections, which can be exemplified by simplicial complexes [5, 6]. These higher-order interactions can be encoded by simplicial complexes, which are sets of n-simplexes, filled cliques of n + 1 nodes, viz. vertices (0-simplex), lines (1-simplex), triangles (2-simplex), tetrahedrons (3-simplex), etc. An n-simplicial complex comprises the n-simplexes and the downward closure (n−1)-simplexes. Recently, the call for simplicial complexes in encoding higher-order interaction in complex systems has put a spurt on in untangling the reciprocation between network geometry and dynamical processes [7–17].

One novel phenomena that naturally borns out of simplicial complex encoded higher-order interactions is the abrupt transition to synchronization and desynchronization [8, 11, 18]. Simplicial complexes are suitable candidate for capturing the underlying geometry of complex systems, for instance they have been used to encode the topological map of the environment’s geometrical features captured by hippocampus [19, 20].

The role of adaptation is instrumental in the growth and proper functioning of many physical and biological systems. For instance, it is a popular perception in neuroscience that the synaptic plasticity among the firing neurons forms the basis for learning process and memory storage in the brain [21, 22]. It was Hebb [23] who first put forth the concept that the simultaneous firing of the interacting neurons strengthens the synaptic connectivity between them [22, 24, 25]. The relative timing of presynaptic and postsynaptic spikes of the interacting neurons can be encoded in terms of phases of the oscillators to form neural network. Hence the synaptic plasticity in the neural network can be realized in terms of the phases of interacting oscillators. The neural networks with synaptic plasticity between the interacting neurons have divulged riveting structures and processes, for instance, cluster synchronization [26–31] and the occurrence of abrupt synchronization and desynchronization [32–36] in monolayer and multilayer networks. In cluster synchronization, a network is segregated into distinct clusters of nodes in which the nodes of the same cluster are mutually synchronized, but the distinct clusters are not mutually synchronized. There exists a diverse range of real-world systems, such as the cortical brain network [37], the power grid network [38], consensus dynamics [39], and schools of fish and swarms of birds [40], having cluster synchronization as a key mechanism of their evolution or functioning.

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This Letter focuses on the impact of simultaneous adaptation of different simplex couplings on the transition to synchronization and desynchronization in simplicial complexes. Here, the adaptation of 1-simplex (dyadic) and 2-simplex (triadic) couplings in a simplicial complex are characterized by Hebbian learning rule, i.e., the dyadic and triadic weights are strengthened weakened if the dyad and triad of the oscillators establishing the respective connectivities are in-phase (out-of-phase), respectively. Such concurrent adaptation in simplicial complexes lead to a fascinating phenomena of abrupt anti-phase synchronization while the in-phase synchronization is completely inhibited. Moreover, the proposed model provides us with the flexibility of determining the onset of synchronization through the Hebbian learning parameters. These numerical findings are also validated by the rigorous theoretical analysis.

To begin with, the phase evolution of \( N \) non-identical Kuramoto oscillators [41] in a simplicial complex under the impression of the Hebbian learning based adaptation of the 1-simplex and 2-simplex couplings is given by

\[
\dot{\theta}_i = \omega_i + \frac{\lambda_1}{\langle k^{[1]} \rangle} \sum_{j=1}^{N} A_{ij} \sin(\theta_j - \theta_i) + \frac{\lambda_2}{2\langle |k^{[2]}| \rangle} \sum_{j,k=1}^{N} B_{ijk} \sin(2\theta_j - \theta_k - \theta_i),
\]

where \( \theta_i(\omega_i); \ i=1, \ldots, N \) denotes the instantaneous phases (intrinsic frequencies) of \( i \)th oscillator in the simplicial complex. We assume that the global coupling of the 1-simplex and 2-simplex interactions in the complex remains conserved, i.e., \( \lambda = \lambda_1 + \lambda_2 \) such that \( \lambda_1 = 1 - p \lambda = q \lambda \) and \( \lambda_2 = p \lambda \). \( p \in [0,1] \) is a propensity parameter that determines the dominance of one-type of simplex interactions over the other type in the complex. The number of edges or triangles in the complex a node is part of, is defined as 1- or 2-simplex degrees, i.e., \( k_i^{[1]} = \sum_{j=1}^{N} A_{ij} \) or \( k_i^{[2]} = \frac{1}{2!} \sum_{j,k=1}^{N} B_{ijk} \), respectively. \( \langle k^{[1]} \rangle \) and \( \langle k^{[2]} \rangle \) denote mean 1- and 2-simplex degrees, respectively. The 1- and 2-simplex coupling interactions are rescaled by the respective mean degrees so as to bring the respective effective connectivities on an equal footing and assist \( p \) in tuning the relative strength of 1- and 2-simplex interactions.

\[
\dot{A}_{ij} = \alpha \cos(\theta_j - \theta_i) - \mu A_{ij},
\]

\[
\dot{B}_{ijk} = \beta \cos(2\theta_j - \theta_k - \theta_i) - \nu B_{ijk}.
\]

We construct a 2-simplicial complex by identifying unique triangles and unique edges closing the triangles from a random 1-simplicial network. The collective phase evolution and weight adaptation of the adaptive 2-simplicial complex are then governed by Eqs. (1) and (2). To capture the formation of \( m \)-clusters in the network, we define an \( m \)-cluster order parameter

\[
z_m = R_m e^{i\Psi_m} = \frac{1}{N} \sum_{j=1}^{N} e^{im\theta_j}; \ m=1,2,
\]

where \( R_m \) and \( \Psi_m \) are amplitude and argument, respectively, of the \( m \)-cluster order parameter. \( R_1 \) quantifies one-cluster synchronization whereas \( R_2 \) quantifies two-cluster synchronization [42].

We numerically evolve Eqs. (1) and (2) to capture the microscopic dynamics of 1- and 2-simplex weights and route to synchronization. All the results presented for random 2-simplicial complex are for \( N=10^3, \langle k^{[1]} \rangle=14 \) and \( \langle k^{[2]} \rangle=10 \) with uniform randomly drawn natural frequencies \( \omega_i \sim U(-\Delta, \Delta) \), where \( \Delta=1 \). The initial 1- and 2-simplex weights are determined by \( A_{ij}(0)=1/L \) and \( B_{ijk}(0)=1/T \), where \( L \) and \( T \) are the number of initial existing dyadic and triadic connections, respectively. At first, \( \lambda \) is adiabatically increased until a large \( \lambda \) and then adiabatically decreased till \( \lambda = 0 \). The phase and weight dynamics [Eqs. (1) & (2)] are then simultaneously simulated on the 2-simplicial complex and the order parameters are computed for each \( \lambda \).

First we discuss the nature of transition when the dyadic and triadic weights are static, i.e., only Eq. (1) is evolved taking \( A_{ij}=1 \) and \( B_{ijk}=1 \) into account. Such static \( A_{ij} \) and \( B_{ijk} \) lead to abrupt transition to in-phase (single-cluster) synchronization with hysteresis as illustrated in Fig. 1 (top row panels). The dyadic interactions are known to promote synchronization while the triadic interactions do not. The parameter \( p \) determines the dominance of dyadic (triadic) interactions over the triadic (dyadic) ones. For \( p<0.5 \), the dominating dyadic interactions quickly come over the frustration induced by the triadic ones and lead to synchronization at rather lower values of \( \lambda \) with rather reduced hysteresis width. The hysteresis is lost at further low values of \( p \). For \( p>0.5 \), the triadic interactions are dominating ones, which lead to a great amount of frustration, eventually leading to abrupt synchronization at large \( \lambda \) with wider hysteresis. For \( p=1 \), however, \( \lambda \rightarrow \infty \) as the only interaction-type existing among the oscillators is the triadic one that does not lead to synchronization for any \( \lambda > 0 \) [8].

Next, we include the Hebbian adaptation of 1-simplex and 2-simplex couplings [Eq. (2)] along with phase evolution Eq. (1). The Hebbian adaptation in 1- and 2-simplex interactions give birth to a novel finding in that the in-phase (single-cluster) synchronization \( R_1 \) is entirely subsided instead an anti-phase (two-cluster) synchronization \( R_2 \) emanates [see Fig. 1 (bottom row panels)]. The
nature of $R_2$ transition for different values of parameter $p$ is analogous to that of $R_1$ witnessed for the static case. Moreover, the onset of $R_2$ occurs at relative larger $\lambda$ than that for $R_1$ related to the static case. The adaptive 2-simplex couplings with $p=1$ also do not lead to synchronization with the increase in $\lambda$. One remarkable feature about the emergent anti-phase ($R_2$) transition is that its onset is entirely manageable through the Hebbian learning parameters $\alpha, \beta, \mu$ and $\nu$. Fig. 2 uncovers that the slower learning rates $\alpha=\beta$ prolong the outset of abrupt transition to a higher $\lambda_f$. In addition, $\beta>\alpha$ also triggers the abrupt transition at a relatively higher $\lambda_f$.

Further, we shade light on the distribution of stationary phases, adaptive dyadic and triadic weights in the incoherent and coherent states (see left panels of Fig. 3). In the coherent state for $\lambda>\lambda_f$, the stationary $A_{ij}$ ($B_{ijk}$) are segregated into two clusters. Hence the distribution $P(A_{ij})$ [$P(B_{ijk})$] manifests bimodal peaks at $-\alpha/\mu - \beta/\nu$ and $\alpha/\mu (\beta/\nu)$, with a few $A_{ij}$ and $B_{ijk}$ settling on around $0$. The corresponding phases are also set apart into bimodal peaks at a difference of $\pi$, thereby $P(\theta_i)$ exhibiting anti-phase clusters. Nevertheless in the incoherent state for $\lambda<\lambda_f$, $P(A_{ij})$ [$P(B_{ijk})$] follows a beta distribution with peaks at $-\alpha/\mu (\beta/\nu)$ and $\alpha/\mu (\beta/\nu)$ and dips at 0 (0). Moreover, $A_{ij}=0$ and $B_{ijk}=0$ yield the dyadic and triadic stationary weights

$$A_{ij} = \frac{\alpha}{\mu} \cos(\Delta \theta_{ij}); \quad B_{ijk} = \frac{\beta}{\nu} \cos(\Delta \theta_{ijk}), \quad (3)$$

where $\Delta \theta_{ij}=\theta_j-\theta_i$ and $\Delta \theta_{ijk}=(2\theta_j-\theta_k-\theta_i)$. Eqs. (3) corroborate the numerical revelations of Fig. 3. In the coherent state, the steady-state extrema $A_{ij}\rightarrow \pm \alpha/\mu$ and $B_{ijk}\rightarrow \pm \beta/\nu$ correspond to $\Delta \theta_{ij}\rightarrow 0, \pi$ and $\Delta \theta_{ijk}\rightarrow 0, \pi$, respectively. Also the steady-state $A_{ij}\rightarrow 0$ and $B_{ijk}\rightarrow 0$ are associated with $\Delta \theta_{ij}\rightarrow \pi/2, 3\pi/2$ and $\Delta \theta_{ijk}\rightarrow \pi/2, 3\pi/2$, respectively. Nonetheless in the incoherent state, the uniformly distributed stationary phases require $\Delta \theta_{ij}$ and $\Delta \theta_{ijk}$ to draw the phases from the full range $[0,2\pi]$. Thereby the stationary $A_{ij}$ and $B_{ijk}$ acquire the weights from the full intervals $[-\alpha/\mu, \alpha/\mu]$ and $[-\beta/\nu, \beta/\nu]$, respectively.

**Ott-Antonsen reduction:** To seek analytical insight of the underlying higher-order dynamics, we turn our focus to an all-to-all connected 2-simplex complex modeled as

$$\dot{\theta}_i = \omega_i + \frac{q \lambda}{N} \sum_{j=1}^{N} A_{ij} \sin(\theta_j - \theta_i) + \frac{p \lambda}{N^2} \sum_{j,k=1}^{N} B_{ijk} \sin(2\theta_j - \theta_k - \theta_i), \quad (4)$$

Following Ref. [43], we employ Ott-Antonsen dimensionality reduction to the steady collective dynamics of Eqs. (2) and (4). The steady-state collective dynamics can only be achieved when phases, dyadic and triadic weights are concurrently in the respective steady states. The evolution of phases can be described, after plugging into the steady-state expressions for $A_{ij}$ and $B_{ijk}$, as

$$\dot{\theta}_i = \omega_i + \frac{aq \lambda}{2N} \sum_{j=1}^{N} \sin(2\theta_j - 2\theta_i) + \frac{bp \lambda}{2N^2} \sum_{j,k=1}^{N} \sin(4\theta_j - 2\theta_k - 2\theta_i), \quad (5)$$

where $a = \alpha/\mu$ and $b = \beta/\nu$. The phase evolution can be re-expressed further in terms of the $m$-cluster order parameters

$$\dot{\theta}_i = \omega_i + \frac{4}{(He^{-2i\theta} - H^* e^{2i\theta})} \frac{aq \lambda}{2} + \frac{bp \lambda}{2N^2} \sum_{j,k=1}^{N} \sin(4\theta_j - 2\theta_k - 2\theta_i), \quad (6)$$

Considering the system in continuum limit $N\rightarrow\infty$, the collective state of the oscillators at a time $t$ can be delineated by a continuous density function $\rho(\theta, \omega, t)$ such that $\rho(\theta, \omega, t) d\theta d\omega$ denotes the fraction of oscillators with their phases and intrinsic frequencies lying in the ranges of $[\theta, \theta+d\theta]$ and $[\omega, \omega+d\omega]$, respectively. Besides, the density function $\rho(\theta, \omega, t)$ satisfies the normalization condition $\int_{0}^{2\pi} \rho(\theta, \omega, t) d\theta=1$ and the continuity equation $\partial_t \rho(\theta, \omega, t) + \partial_\theta \rho(\theta, \omega, t) + \partial_\omega \rho(\theta, \omega, t) v(\theta, \omega, t)=0$ as the number of oscillators remains conserved. Also the m-cluster order parameter can be expressed as $z_m=\int d\omega d\theta e^{im\theta} \rho(\theta, \omega, t)$. Since $\rho(\theta, \omega, t)$ is a 2π-periodic function with respect to $\theta$, it can be expressed as a Fourier expansion of the form

$$\rho(\theta, \omega, t) = \frac{1}{2\pi} \left[ 1 + \left\{ \sum_{n=1}^{\infty} f_n(\omega, t)e^{in\theta} + c.c. \right\} \right], \quad (7)$$
where c.c. stands for complex conjugate of the preceding terms. Ott-Anтонсен pointed out that all Fourier coefficients can be classified to Poisson kernels of the form 

\[ f_n(\omega, t) = \cos(\omega t) \] 

where \( n \) is an integer and \( \omega \) is a frequency. For \( |\omega| < 1 \), the convergence of the series is guaranteed.

After plugging into the expressions for \( v = \theta \) [Eq. (6)] and \( f(\theta, \omega, t) \) [Eq. (7)], all Fourier modes then reduce to the same constraint for \( f \), satisfying the single complex-valued differential equation

\[ \frac{\partial f^2}{\partial t} + 2i\omega f^2 + \frac{1}{2} [H f^4 - H^*] = 0, \]

where the integral operator \( G \) is defined as

\[ \mathcal{G} f^m = \int_{-\infty}^{\infty} d\omega g(\omega) f^m(\omega, t); \quad m=2,4. \]

Solving Eq. (12) for \( g(\omega) = \frac{\Delta}{\pi(\omega - \omega_0)^2 + \Delta^2} \) results in

\[ \lambda_f = \frac{4\Delta}{aq} = \frac{4\mu\Delta}{\alpha(1-p)}. \tag{13} \]

Eq. (13) unveils that the transition to synchronization is solely caused by the presence of dyadic interactions through parameter \( p \) and Hebbian rates \( \alpha \) and \( \mu \) and the triadic interactions do not play any role in the onset of synchronization. Also \( p=1 \) yields \( \lambda_f \rightarrow \infty \), i.e., the incoherence does not lose stability for any \( \lambda > 0 \).

Solution of coherence: The evolution of order parameter \( R_2 \) can be worked out for Lorentzian distribution with mean \( \omega_0 \) and half-width \( \Delta \), i.e.,

\[ g(\omega) = \frac{\Delta}{\pi(\omega - \omega_0)^2 + \Delta^2}. \]

The order parameter in Eq. (8) can be derived using Cauchy’s residue theorem by closing the contour to an infinite-radius semi-circle in the negative-half complex \( \omega \) plane, resulting in

\[ z_2 = f^* z_2 (\omega_0-i\Delta, t) \]

Next, assessing Eq. (8) at \( \omega = \omega_0 - i\Delta \) and then taking complex conjugate, one gets

\[ 2\dot{z}_2 - 4i\omega_0 z_2 + 4\Delta z_2 + z_2^* [aq z_2^* + bp z_2 z_2^*] - \lambda [aq z_2 + bp z_2 z_2^*] = 0. \tag{14} \]

Next, inserting \( z_2 = R_2 e^{i\Psi_2} \) and then equating real and imaginary parts on both sides of the equation supply

\[ 2\dot{R}_2 + 4\Delta R_2 + \lambda R_2 (R_2 - 1) (aq + bp R_2^2) = 0, \tag{15} \]

\[ \dot{\Psi}_2 = 2\omega_0. \tag{16} \]

The steady-state solutions of Eq. (15) are given by

\[ \frac{4\Delta}{\lambda} = -bp R_2^2 + (bp - aq) R_2^2 + aq \tag{17} \]

which have the following positive roots for \( R_2 \)

\[ R_2^+ = \frac{-bp + \sqrt{bp(aq - \frac{4\Delta}{\lambda})}}{2bp} \tag{18} \]

where \( R_2^+ (R_2^-) \) represents a stable (an unstable) branch of synchronous state. The validation of analytical predictions for the order parameter \( R_2 \) [Eq. (18)] with its numerical estimations for different sets of parameters is presented in Fig. 4.

In the case of forward transition, in the incoherent state \( R_2 = 0 \) until \( \lambda = \lambda_f = \frac{4\Delta}{aq} \) is reached. At \( \lambda = \lambda_f, R_2 \) abruptly jumps to \( R_2^+ (\lambda_f) = \sqrt{1 - \frac{aq}{bp}} \) [while \( R_2^- (\lambda_f) = 0 \)] and the incoherent state \( (R_2=0) \) loses its stability through supercritical pitchfork bifurcation. Nevertheless, the set of parameters \( \{a, b, p\} \) for which \( R_2^+ (\lambda_f) = \sqrt{1 - \frac{aq}{bp}} = 0 \) at \( \lambda = \lambda_f, \) the incoherent state \( (R_2=0) \) loses its stability through supercritical pitchfork bifurcation and the transition to synchronization takes place via a second-order route.

In the case of backward transition, \( R_2^- \) (saddle point) and \( R_2^+ \) (node point) exist in the hysteresis region. As soon as the backward critical coupling strength \( \lambda = \lambda_b \) is
In Fig. 5, we provide a broad picture of the distinct parameter domains stretched over $(\alpha - \beta)$, $(p - \alpha)$ and $(p - \beta)$ planes where a phase transition transpiring in the all-to-all simplicial complex manifests bistability [45]. The regions illustrated by the green slanted lines represent the bistability region. $aq \neq bp$ and $aq < bp$ are must for the existence of bistable solutions sporting a hysteresis. The yellow dotted region depicts non-bistable region corresponding to $aq = bp$ [46]. For $aq = bp$, Eqs. (13) and (19) furnish with $\lambda_f = \lambda_b$, $R_\beta^f(\lambda_f) = 0$ and $R_2(\lambda_b) = 0$, which conform to a second-order transition to synchronization.

**Conclusion** In this work, the nature of transition to synchronization is explored on 2-simplicial complexes where the triadic couplings and the downward closing dyadic couplings evolve in time according to the respective Hebbian learning rules. Strikingly, such co-evolving dyadic and triadic couplings completely subside the single-cluster synchronization, instead, trigger the two-cluster synchronization in simplicial complexes. It is unveiled that the onset of anti-phase synchronization only depends on the dyadic interaction (learning rate) and the higher-order interaction has no role to play. On the other hand, the onset of anti-phase desynchronization is affected by both dyadic and triadic interactions (learning rates). Further, the numerical findings related to the anti-phase order parameter and the forward and backward critical transition points have been validated with the respective analytical predictions by employing Ott-Antonsen ansatz. It is also shown that the steady dyadic (triadic) weights in the synchronous state form two clusters of equal and opposite magnitudes along the lines of the oscillators forming the anti-phase clusters.

The simplicial structures involving simultaneous adaptation of pairwise and higher-order interactions would help in elucidating the underlying mechanism of cluster formation in the brain functional networks, such as, anti-phase patterns in the cortical neural network.

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A. Stationary dyadic and triadic weights as a function of coupling strength

Fig. S1 depicts the stationary $A_{ij}$ (dyadic) and $B_{ijk}$ (triadic) as a function of coupling strength $\lambda$ exhibiting forward transitions in a random 2-simplicial complex. Both the steady dyadic and triadic weights are bounded within the intervals $[-\alpha/\mu, \alpha/\mu]$ and $[-\beta/\nu, \beta/\nu]$, respectively, in both the incoherent and coherent states.

![Stationary weights](image_url)

FIG. S1. (Color online) Stationary weights; $A_{ij}$ (dyadic) and $B_{ijk}$ (triadic) plotted against coupling strength $\lambda$ exhibiting transition to synchronization in a random 2-simplicial complex [with mean degrees $\langle k^{[1]} \rangle = 14$ and $\langle k^{[2]} \rangle = 10$] simulated for uniform natural frequencies $\omega_i \sim U[-1, 1]$, $N = 10^3$, $\mu = 1$ and $\nu = 1$.

B. Bistability domains revisited

In Fig. S2, the hysteresis width $|\lambda_f - \lambda_b|$ of transition transpiring in an all-to-all 2-simplicial complex are illustrated in $(\alpha - \beta)$, $(p - \alpha)$ and $(p - \beta)$ planes. The red area between the two black contour lines depicts zeros hysteresis width, representing the occurrence of a second-order transition while the remaining parameter regions manifest an abrupt transition associated with a hysteresis.

![Bistability domains](image_url)
FIG. S2. (Color online) Bistability domains: Phase diagrams in \((\alpha - \beta), (p - \alpha)\) and \((p - \beta)\) planes representing hysteresis width \(|\lambda_f - \lambda_b|\) of transitions in the all-to-all connected 2-simplicial complex simulated for Lorentzian \(g(\omega); \Delta = 0.1, N = 10^3, \mu = 1\) and \(\nu = 1\).