Polygons in hyperbolic geometry 1:
Rigidity and inversion of the \( n \)-inequality
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1. Introduction

In a recent series of papers, K. Leichtweiß extended various topics from Euclidean geometry to the hyperbolic plane (and partially to the geometry on the sphere). His results concern convexity and extremum problems for closed curves, including the addition of convex sets, support functions, curves of constant width and Steiner’s symmetrization; see Leichtweiß [2003], [2004], [2005], [2008.a], [2008.b].

Looking in a different direction, we will discuss here certain problems in the distance geometry of Menger [1928] and Blumenthal [1970] for polygons in the hyperbolic geometry. This part 1 deals with the rigidity for polygons of fixed side lengths and the inversion of the generalized triangle inequality. The polygons here are of a general nature, no a-priori assumptions on their form (convexity, simple closedness, etc.) have to be made. For the Euclidean plane, analogous problems were treated by Pinelis [2005]. We have been surprised to learn from the Pinelis paper that the Euclidean solution was open for such a long time. Rigidity is understood in the non-continuous sense similar to Klingenberg [1978], Theorem 6.2.8.

In the hyperbolic plane, as in Euclidean geometry, a polygon usually can be changed in many ways while keeping its side lengths fixed. Exceptions are some special collinear polygons and all polygons which are convex and cocyclic, the latter meaning that the vertices lie on a circle. In fact, we shall prove that these polygons can be changed in the hyperbolic plane only in a trivial way, namely by rigid hyperbolic motions if the side lengths are kept constant (Theorems 4.13 and 5.3). This rigidity problem is more difficult than in the Euclidean case because there are three different types of circles in the hyperbolic plane, the distance circles, the distance lines, and the horocycles. The solution depends on closedness conditions for the length spectrum which are expressed in terms of the generating groups of these figures.

A related question in the hyperbolic geometry is: Which conditions on the length spectrum must be posed in order that a corresponding polygon will exist? The answer is that the generalized triangle inequality, henceforth called the \( n \)-inequality, does the job (Corollary 5.4). Again, the closedness conditions are crucial for the proof.

Since both problems belong to distance geometry it is necessary, to dispense to a certain extent with classical hyperbolic trigonometry, because most of its relations are between lengths and angles. Expressing the situation with lengths alone may render some formulation more complicated, but provides a better insight into the dependencies of quantities.

This aspect becomes even more important in part 2 where the problem of maximizing the area of polygons with fixed side lengths will be treated, based on the results of the present part 1. Moreover it will become clear that the convex and cocyclic polygons are the only non-collinear ones for which the rigidity can hold true.
2. The Cayley/Klein model

For our purposes, the *Cayley/Klein model* is the most suitable framework of hyperbolic geometry. In this model, the points of the hyperbolic plane are taken as the ordinary points of the open unit ball

\[ \mathbb{B} := \{(\xi_1, \xi_2) \in \mathbb{R}^2 \mid \xi_1^2 + \xi_2^2 < 1\}, \]

and the straight lines are taken as the ordinary straight chords of \( \mathbb{B} \), always connecting two different points of the horizon

\[ \mathbb{S} := \partial \mathbb{B} = \{(\xi_1, \xi_2) \in \mathbb{R}^2 \mid \xi_1^2 + \xi_2^2 = 1\}, \]

which itself does not belong to the hyperbolic plane. Nevertheless, points of \( \mathbb{R}^2 \) outside \( \mathbb{B} \) may well be considered for describing situations within \( \mathbb{B} \). For basics on hyperbolic geometry see e.g. Coxeter [1968] or Lenz [1967].

The topology of \( \mathbb{B} \), its orientation and between-relations on lines are the same as in standard \( \mathbb{R}^2 \). The origin \((0,0)\) in \( \mathbb{R}^2 \) is denoted by \( O \), the connecting line of two different points \( X, Y \in \mathbb{R}^2 \) by \( X \lor Y \), and the directed line from \( X \) to \( Y \) as a point set is denoted by \( \overrightarrow{XY} \); it includes \( X \) and \( Y \). The open segment between \( X = (-1,0) \) and \( Y = (1,0) \) will be called the *groundline* and denoted by \( U_0 \).

It is convenient to specify the points \( X = (\xi_1, \xi_2) \in \mathbb{R}^3 \) also by *homogeneous coordinates* \( x_0, x_1, x_2 \) defined by

\[ \xi_1 = \frac{x_1}{x_0}, \quad \xi_2 = \frac{x_2}{x_0}, \quad x_0 \neq 0, \]

thus imbedding \( \mathbb{R}^2 \) in the projective plane \( \mathbb{P}^2 \), consisting of the one dimensional vector subspaces (rays) of \( \mathbb{R}^3 \).

All metric properties in \( \mathbb{B} \) are expressible by the pseudo-Euclidean scalar product in \( \mathbb{R}^3 \):

\[ \langle x, y \rangle := x_0y_0 - x_1y_1 - x_2y_2, \quad x := (x_0, x_1, x_2), \quad y := (y_0, y_1, y_2). \]

For each \( x \in \mathbb{R}^3 \) we use the abbreviation

\[ \|x\| := \sqrt{|\langle x, x \rangle|}, \]

despite the fact that this does not define a norm in the analytic sense.

In homogeneous coordinates, a line of \( \mathbb{R}^2 \) has an equation of the form \( \langle u, x \rangle = 0 \) (\( u \neq 0 \) in \( \mathbb{R}^3 \) fixed, \( x \neq 0 \) in \( \mathbb{R}^3 \) variable). The points of \( \mathbb{B} \) are characterized by \( \langle x, x \rangle > 0 \) and the lines hitting \( \mathbb{B} \) by \( \langle u, u \rangle < 0 \).

A vector \( x := (x_0, x_1, x_2) \neq 0 \) in \( \mathbb{R}^3 \) belonging to a point of \( \mathbb{R}^2 \) is called a *homogeneous point vector*, a vector \( u := (u_0, u_1, u_2) \neq 0 \) in \( \mathbb{R}^3 \) belonging to a line in \( \mathbb{R}^2 \) is called a *homogeneous line vector*. Such a \( u \), viewed as a homogeneous point vector, represents the polar point of the line w.r.t. to the circle \( \mathbb{S} \). For a point vector \( x \) we always assume \( x_0 > 0 \). If convenient, one may choose \( x \) such that \( \langle x, x \rangle = 1 \) and \( u \) such that \( \langle u, u \rangle < -1 \). If so, we call \( x \) resp.
u normalized. The normalized point vectors define a bijective correspondence of \( \mathbb{B} \) with the shell of a hyperboloid in \( \mathbb{R}^3 \)

\[
\mathbb{H} := \{ x \in \mathbb{R}^3 \mid \langle x, x \rangle = 1, \ x_0 > 0 \}. 
\]

\( \mathbb{H} \) is the \textit{quadric model} of hyperbolic geometry. It may be considered parallel to the Cayley/Klein model what is sometimes helpful.

The scalar product (2.1) induces a corresponding pseudo-Euclidean cross product from \( \mathbb{R}^3 \times \mathbb{R}^3 \) to \( \mathbb{R}^3 \) by the identity

(2.2) \[
[x, y, z] = \langle x \times y, z \rangle \quad \forall \ x, y, z \in \mathbb{R}^3,
\]

where \([x, y, z]\) denotes the standard determinant form on \( \mathbb{R}^3 \). The cross product may serve to calculate a line vector \( u \) of the connecting line of two different points with point vectors \( x, y \) as \( u = x \times y \) and a point vector \( x \) of the intersection point of two different lines with line vectors \( u, v \) as \( x = u \times v \).

Important rules, often to be used, are:

(2.3) \[
[x, y, z] \cdot [x', y', z'] = \begin{vmatrix}
\langle x, x' \rangle & \langle x, y' \rangle & \langle x, z' \rangle \\
\langle y, x' \rangle & \langle y, y' \rangle & \langle y, z' \rangle \\
\langle z, x' \rangle & \langle z, y' \rangle & \langle z, z' \rangle
\end{vmatrix} \quad \text{(Gram’s identity)}
\]

(2.4) \[
\langle x \times y, x' \times y' \rangle = \begin{vmatrix}
\langle x, x' \rangle \\
\langle y, y' \rangle
\end{vmatrix} \quad \text{(Lagrange’s identity)}
\]

(2.5) \[
(x \times y) \times x' = \langle x, x' \rangle y - \langle y, x' \rangle x \quad \text{(triple identity),}
\]

valid for all \( x, y, z, x', y', z' \) in the pseudo-Euclidean space \( \mathbb{R}^3 \).

If necessary, the various objects occurring so far may be denoted more accurately in the following manner: points of \( \mathbb{R}^2 \) as pairs of real numbers by \( X, Y, \ldots \) (also as elements of the hyperbolic plane \( \mathbb{B} \)), corresponding point vectors by the corresponding small letters \( x, y, \ldots \); straight lines of \( \mathbb{R}^2 \) as point sets by \( U, V, \ldots \) and corresponding line vectors by the corresponding small letters \( u, v, \ldots \). Sometimes, by abuse of language, we are speaking of a point \( x \), meaning the point \( X \) with point vector \( x \), and similarly for straight lines.

The most important notions are that of the \textit{hyperbolic distance} \( d(X, Y) \) of points \( X, Y \in \mathbb{B} \) and the \textit{hyperbolic angle} \( \alpha(U, V) \) of two lines intersecting in \( \mathbb{B} \) by

(2.6) \[
\cosh d(X, Y) = \frac{\langle x, y \rangle}{\|x\| \cdot \|y\|}, \quad \cos \alpha(U, V) = \frac{|\langle u, v \rangle|}{\|u\| \cdot \|v\|}, \quad 0 \leq \alpha \leq \frac{\pi}{2}.
\]

Again, this definitions only involve the pseudo-Euclidean scalar product (2.1). They are invariant under the possible changes of the representing objects \( x, y \), resp. \( u, v \). It is well known that \( d \) defines a complete metric on \( \mathbb{B} \).

If \( U \subset \mathbb{B} \) is a straight line and \( X \in \mathbb{B} \) any point then there is exactly one \textit{foot point} \( F \in U \), i.e. \( F \) satisfying \( d(X, F) \leq d(X, Y) \) for all \( Y \in U \). The distance \( d(X, F) \) is the distance \( d_U(X) \) of \( X \) and \( U \). Explicitly:

(2.7) \[
f := \langle u, x \rangle u - \langle u, u \rangle x, \quad \sinh d_U(X) = \frac{|\langle u, x \rangle|}{\|u\| \cdot \|x\|}.
\]
For $X \notin U$ the foot $F$ is also characterized by the property that the connecting line of $X$ and $F$ cuts $U$ orthogonally at $F$.

The orientation of $\mathbb{B}$ shall be that inherited from the standard orientation of $\mathbb{R}^2$. The two half-spaces (sides) of a line $U$ are defined by $\langle u, z \rangle > 0$, resp. $\langle u, z \rangle > 0$. If $u$ is multiplied by a negative factor then both sides are interchanged, so no side is distinguished. However, if $X \lor Y$ is the directed line from $X$ to $Y$ (where $X \neq Y$) then the line vector $u := x \times y$ is fixed up to a positive factor. Thus a directed line may also be given by a line vector $u$ and its positive multiples. In this case the two sides of $U$ can be distinguished by $\langle u, z \rangle > 0$, resp. $\langle u, z \rangle < 0$. The first one is called the left-hand side, the second one the right-hand side of the directed line. For $u = x \times y$ these inequalities are equivalent to $[x, y, z] > 0$, resp. $[x, y, z] < 0$.

### Hyperbolic isometries

The isometries of the hyperbolic plane $\mathbb{B}$ can be generated from a rational parametrization of the horizon $S$ by re-parametrizing $S$ with a real broken linear transformation (a homography of $\mathbb{R}$) and extending this transformation of $S$ projectively to $\mathbb{B}$. This amounts to the following: Associate to any real $(2 \times 2)$-matrix $A$ a real $(3 \times 3)$-matrix $M(A)$ by

$$A := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto M(A) := \begin{pmatrix} a^2 + b^2 + c^2 + d^2 & -a^2 + b^2 - c^2 + d^2 & 2(ab + cd) \\ -a^2 - b^2 + c^2 + d^2 & a^2 - b^2 - c^2 + d^2 & 2(-ab + cd) \\ 2(ac + bd) & 2(-ac + bd) & 2(ad + bc) \end{pmatrix}.$$  

Then simple (though tedious) matrix calculations show the following rules for all real $(2 \times 2)$-matrices $A, B$ and all vectors $x, y, z \in \mathbb{R}^3$:

$$M(A \cdot B) = 2 \cdot M(A) \cdot M(B) \quad \text{(2.9)}$$

$$\det M(A) = 8(\det A)^3 \quad \text{(2.10)}$$

$$[M(A)x, M(A)y, M(A)z] = 8(\det A)^3[x, y, z] \quad \text{(2.11)}$$

$$\langle M(A)x, M(A)y \rangle = 4(\det A)^2 \langle x, y \rangle \quad \text{(2.12)}$$

$$(M(A)x) \times (M(A)y) = 2 \det A \cdot M(A)(x \times y). \quad \text{(2.13)}$$

In particular, if $\det A = ad - bc \neq 0$, then $M(A)$ defines a non-singular linear map from $\mathbb{R}^3$ onto itself and thus a bijective projective map from $\mathbb{P}^2$ to itself. In addition, the rule (2.12) signifies that this projective map induces an isometry $\mu(A)$ of the hyperbolic plane $\mathbb{B}$ onto itself. Each such transformation $\mu(A)$ will be called a hyperbolic isometry of $\mathbb{B}$. It transforms lines to lines and leaves invariant the hyperbolic metric. The $\mu(A)$ is on $\mathbb{B}$ orientation-preserving if $\det(A) > 0$ (equivalently: $\det M(A) > 0$) and orientation-reversing if $\det(A) < 0$ (equivalently: $\det M(A) < 0$).

Moreover, the rule (2.9) shows that the isometries of $\mathbb{B}$ form a group homomorphic via $\mu$ to the linear group $\text{GL}(2, \mathbb{R})$ modulo its center, i.e. to the real projective group $\text{PGL}(2, \mathbb{R})$ of the projective extension of $\mathbb{R}$. In fact, this homomorphism is an isomorphism because $M(A)$ being a multiple of the unit $(3 \times 3)$-matrix implies that $A$ is a multiple of the unit $(2 \times 2)$-matrix.
An orientation-preserving isometry $\mu(A)$ will be called a hyperbolic motion. The motions form a subgroup of index 2 of the group of all isometries. In order to fix the notation for the present purposes, two figures in $\mathbb{B}$ will be called hyperbolically equivalent, resp. congruent if they are transformed one to each other by a hyperbolic motion, resp. a hyperbolic isometry. In most cases the claims will concern equivalence, i.e. the stronger version. Sometimes, the adjective ‘hyperbolic’ is skipped.

**Hyperbolic circles**

The straight lines in $\mathbb{B}$ correspond to one-dimensional sections of the quadric shell $\mathbb{H}$ with vector planes in $\mathbb{R}^3$. Analogous to sphere geometry, circles in the hyperbolic plane $\mathbb{B}$ are defined by correspondence with the possible intersections of $\mathbb{H}$ with planes of $\mathbb{R}^3$, not containing the origin, if the intersections span the planes. Such a plane has an equation of the form $\langle u, x \rangle = p$ with fixed $u \neq 0$ in $\mathbb{R}^3$ and $p \neq 0$ in $\mathbb{R}$ and additional conditions to ensure that the intersection is really a conic section. This induces a classification as follows:

In case $\langle u, u \rangle > 0$, the quadratic form $\langle \ , \ \rangle$ is negative definite on the orthogonal complement $u^\perp$ of $\text{span}(u)$, and the intersection is an ellipse if it spans the plane. Without loss, one may assume $u_0 > 0$. The ellipse exists and is contained in the half-space $x_0 > 0$ of $\mathbb{R}^3$ iff $p > \|u\|$. Then the pre-image in $\mathbb{B}$ of the intersection is called a distance circle.

In case $\langle u, u \rangle < 0$, the quadratic form $\langle \ , \ \rangle$ is indefinite on $u^\perp$, and the intersection always is a hyperbola with one component in the half-space $x_0 > 0$. Then the pre-image in $\mathbb{B}$ of the intersection is called a distance line.

In case $\langle u, u \rangle = 0$, the orthogonal space $u^\perp$ does not complete $\text{span}(u)$ (since $u \in u^\perp$). The intersection always is a parabola. Again it can be assumed $u_0 > 0$. The parabola is contained in the half-space $x_0 > 0$ iff $p > 0$. Then the pre-image in $\mathbb{B}$ of the intersection is called a horocycle.

Each of these pre-images in $\mathbb{B}$ is called a circle. From this analytic definition the geometric meanings of circles are easily deduced as follows:

**2.1. Lemma (data of the hyperbolic circles).**

From the circle equation $\langle u, x \rangle = p$ of the foregoing classification, the non-Euclidean data of circles can be read off:

(i) For a distance circle ($\langle u, u \rangle > 0$) let $c$ and $R > 0$ be defined by

\[
(2.14) \quad c := \frac{u}{\|u\|}, \quad \cosh R := \frac{p}{\|u\|}.
\]

Then the distance circle consists of all points $X \in \mathbb{B}$ with $d(C, X) = R$. So $C \in \mathbb{B}$ is the center and $R$ is the (hyperbolic) radius.

(ii) For a distance line ($\langle u, u \rangle < 0$), let $\delta > 0$ be defined by

\[
(2.15) \quad \sinh \delta := \frac{|p|}{\|u\|}.
\]

Then the distance line consists of all points $X \in \mathbb{B}$ with $d_U(X) = \delta$, situated on the side of $U$ with $\langle u, x \rangle = \text{sign } p$. The line $U \subset \mathbb{B}$ will be called the soul of the distance line.
(iii) For a horocycle \((u, u) = 0\), the vector \(u\), if considered as a point vector, defines a point \(U\) on the horizon. The horocycle is an orthogonal trajectory of all straight lines of \(B\) passing through \(U\).

2.2. Remark. The hyperbolic circles in \(B\), when viewed with ‘Euclidean eyes’, are elliptic arcs with the following design in standard positions:

A distance circle with center \(C = (0,0)\) is an ordinary circle with this center. The relation between the hyperbolic radius \(R\) and the Euclidean radius \(r\) is:

\[
r = \tanh R.
\]

A distance line whose soul is the groundline \(U_0\) is a halfellipse with midpoint \(O\) and axes parallel to the coordinate axes with Euclidean half-axes \(a = 1\) resp. \(b = \tanh \delta\), and situated either in the upper or lower halfplane. A horocycle with horizon point \(U = (1,0)\) is an elliptic arc from \(U\) to \(U\) again having its axes parallel to the coordinate axes where the half-axes are in the relation \(a = b^2\), e.g. \(a = \frac{1}{2}, b = \frac{\sqrt{2}}{2}\).

It will become essential in our investigation that the various circles are orbits of one parameter groups of hyperbolic motions. These groups arise as follows: One starts with the one parameter groups of the projective group \(\text{PGL}(2, \mathbb{R})\) via the Jordan normal forms of their infinitesimal generators. Then with the homomorphism (2.8) one translates these groups into the hyperbolic plane. As infinitesimal generators in the group \(\text{PGL}(2, \mathbb{R})\) one may take those in the group \(\text{GL}(2, \mathbb{R})\) with trace 0. These matrices can even be multiplied by a scalar factor because this only influences the parametrization. So, finally, three normal forms for the infinitesimal generators survive, namely of the types ‘complex diagonalizable’, ‘real diagonalizable’, and ‘nilpotent’:

\[
(2.16) \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.
\]

The corresponding groups themselves arise from these matrices by multiplication with the group parameter \(t\) and exponentiation as:

\[
A(t) : \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}, \quad \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}.
\]
Inserting this in the homomorphism (2.8) then yields (after suitable abbreviations and rescaling of $M(A)$):

$$
\begin{pmatrix}
1 & 0 & 0 \\
0 & \cos \varphi & -\sin \varphi \\
0 & \sin \varphi & \cos \varphi
\end{pmatrix},
\begin{pmatrix}
\cosh \varrho & \sinh \varrho & 0 \\
\sinh \varrho & \cosh \varrho & 0 \\
0 & 0 & 1
\end{pmatrix},
\begin{pmatrix}
1 + \frac{1}{2}b^2 & -\frac{1}{2}b^2 & -b \\
\frac{1}{2}b^2 & 1 - \frac{1}{2}b^2 & -b \\
-b & b & 1
\end{pmatrix}.
$$

(2.17)

The new variables $\varphi, \varrho, b$ replacing $t$ are also ‘good’ group parameters, i.e. the composition in the group corresponds to the addition of the parameters. Eqns. (2.17) are the desired normal forms for the one parameter groups of motions, represented as linear groups in $\mathbb{R}^3$, respecting the pseudo-Euclidean scalar product up to factors. The groups induced on $\mathbb{B}$ by (2.17) are known resp. as hyperbolic rotations around $O$, hyperbolic translations along the groundline $U_0$, and limit rotations with horizon point $(1, 0) \in \mathbb{S}$. A hyperbolic rotation around $O$ with the parameter $\varphi$ is the same as the Euclidean rotation around $O$ with angle $\varphi$. A hyperbolic translations along the groundline $U_0$ induces on $U_0$ a bijection with the property that the distance between pre-image and image is constant.

Now, if $(S(t))_{t \in \mathbb{R}}$ is such a representation (with neutral notation $t$ for the group parameter) one has the differential equation $S' = GS = SG$ where $G$ is the infinitesimal generator $G := S'(0)$. The orbits are described by $\gamma(t) := S(t)a$ with $a$ fixed in $\mathbb{R}^3$. An orbit $\gamma$ is contained in the quadric shell $\mathbb{H}$ iff $a \in \mathbb{H}$. For the derivatives of $\gamma$ one has $\gamma^{(k)} = G^k\gamma$ and also $\gamma^{(k)} = SG^k a$. This implies $\text{span}(\gamma', \gamma'', \ldots, \gamma^{(k)}) = S(\text{span}(Ga, G^2a, \ldots, G^ka))$, so the dimensions of all osculating spaces remain constant along the orbit. If especially $Ga, G^2a$ are linearly independent but $Ga, G^2a, G^3a$ are linearly dependent, then the orbit is contained in an affine plane of $\mathbb{R}^3$. The corresponding vector $u$ of this plane is given by $u := Ga \times G^2a$ and, if $\langle u, a \rangle \neq 0$, the orbit is part of a circle, whose type is determined by the sign of

$$\langle u, u \rangle = \langle Ga, Ga \rangle \langle G^2a, G^2a \rangle - \langle Ga, G^2a \rangle^2.
$$

(2.18)

The condition $\langle u, a \rangle \neq 0$ is equivalent to $[a, Ga, G^2a] \neq 0$ (what already implies the linear independency of $Ga, G^2a$). The foregoing assumption on linear dependency is satisfied if $G$ is singular because $[Ga, G^2a, G^3a] = \det(G) \cdot [a, Ga, G^2a]$. For the one parameter groups (2.17) this is fulfilled since their infinitesimal generators are

$$
G : \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & -1 \\
0 & 1 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 0 & -1 \\
0 & 0 & -1 \\
-1 & 1 & 0
\end{pmatrix}.
$$

(2.19)

This discussion shows that for our three one parameter groups the resulting orbits solely depend on the choice of the initial point $a$. However, for initial points equivalent under an element of the group, the whole orbits will be equivalent in the same manner. The classification of the orbits is thus reduced further. Before formulating the result, the arc lengths of the orbits may be calculated.

The pseudo-Euclidean scalar product in $\mathbb{R}^3$ induces a corresponding pseudo-riemannian metric on $\mathbb{R}^3$:

$$\langle dx, dx \rangle = (dx_0)^2 - (dx_1)^2 - (dx_2)^2.
$$

(2.20)
This metric becomes strictly negative definite when restricted to the quadric shell $\mathbb{H}$. Denoting this restriction with the same symbol, a genuine Riemannian metric on $\mathbb{H}$ is then $-\langle dx, dx \rangle$. Pulling back this metric onto $\mathbb{B}$ yields the Riemannian metric of the hyperbolic plane. In the Cartesian coordinates of $\mathbb{B}$ the pull back is calculated as

$$ds^2 = \frac{1}{(1 - \xi_1^2 - \xi_2^2)^2} \left( (1 - \xi_1^2)d\xi_1^2 + 2\xi_1\xi_2 d\xi_1 d\xi_2 + (1 - \xi_2^2)d\xi_2^2 \right).$$

Now, if as above $\gamma = Sa$ is an orbit on $\mathbb{H}$ of a one parameter group $(S(t))_{t \in \mathbb{R}}$ one has

$$-\langle \gamma', \gamma' \rangle = -\langle S'a, S'a \rangle = -\langle SCa, SCa \rangle = -\langle Ca, Ca \rangle.$$

From this follows that the orbit is automatically parametrized proportional to arclength, the arclength integral being

$$(2.21) \quad \int_{t_1}^{t_2} \sqrt{-\langle Ca, Ca \rangle} \, dt = \|Ca\| (t_2 - t_1).$$

Based on all this the following statements can be proved. The main concern is the relation between arclengths and chordlengths which will become crucial in the discussion of the closedness conditions in Sect. 3.

2.3. Lemma.

(i) The distance circle $S(R)$ with center $O$ and hyperbolic radius $R$ is identical with the image of the orbit of the first group in $(2.17)$ with initial point $a := (\cosh R, \sinh R, 0)$. This orbit is represented by

$$(2.22) \quad \gamma(\varphi) := \left( \begin{array}{c} \cosh R \\ \sinh R \cdot \cos \varphi \\ \sinh R \cdot \sin \varphi \end{array} \right).$$

When restricted to any halfopen interval $P$ of length $2\pi$, this orbit provides a bijective map of $P$ onto $S(R)$. Define the function

$$(2.23) \quad \Phi(R, t) := 2 \sinh R \cdot \arcsin \left( \sinh \frac{t}{2} \right), \quad R > 0, \quad 0 \leq t \leq 2R,$$

Then, for any two points $X \neq Y$ in $S(R)$, the following relation is valid between the chordlength $L := d(X, Y)$ and the arclength $s$ of $S(R)$ along the arc of $S(R)$ residing on the right hand side of the directed line $\overrightarrow{X \vee Y}$:

$$(2.24) \quad s = \begin{cases} \Phi(R, L) & \text{if } O \text{ is left of or on } \overrightarrow{X \vee Y} \\ 2\pi \sinh R - \Phi(R, L) & \text{if } O \text{ is right of } \overrightarrow{X \vee Y}. \end{cases}$$
(ii) The distance line \( D(\delta) \) of distance \( \delta > 0 \) from the groundline situated in the upper halfplane is identical with the image of the orbit of the second group in (2.17) with initial point \( a := (\cosh \delta, 0, \sinh \delta) \). This orbit

\[
\gamma(\varrho) := \begin{pmatrix} \cosh \delta \cdot \cosh \varrho \\ \cosh \delta \cdot \sinh \varrho \\ \sinh \delta \end{pmatrix}, \quad \varrho \in \mathbb{R},
\]

provides a bijective map from \( \mathbb{R} \) onto \( D(\delta) \). Define the function

\[
\Psi(\delta, t) := 2 \cosh \delta \cdot \text{arsinh} \left( \frac{\sinh \frac{t}{2}}{\cosh \delta} \right), \quad \delta > 0, \quad t \geq 0.
\]

Then, for any two points \( X \neq Y \) in \( D(\delta) \) the following relation is valid between the chord-length \( L := d(X,Y) \) and the arclength \( s \) on \( D(\delta) \) from \( X \) to \( Y \):

\[
s = \Psi(\delta, L).
\]

(iii) The horocycle \( H \) with horizon point \( (1, 0) \in \mathbb{R}^2 \) through the origin \( O \) is identical with the image of the orbit of the third group in (2.17) with initial point \( a := (1, 0, 0) \). This orbit

\[
\gamma(b) := \begin{pmatrix} 1 + \frac{1}{2}b^2 \\ \frac{1}{2}b^2 \\ -b \end{pmatrix}, \quad b \in \mathbb{R},
\]

provides a bijective map from \( \mathbb{R} \) onto \( H \). Define the function

\[
\Omega(t) := 2 \sinh \frac{t}{2}, \quad t \geq 0.
\]

Then, for any two points \( X \neq Y \) in \( H \) the following relation is valid between the chordlength \( L := d(X,Y) \) and the arclength \( s \) on \( H \) from \( X \) to \( Y \):

\[
s = \Omega(L).
\]

The horocycles don’t have invariants, i.e. all horocycles are hyperbolically equivalent to \( H \).

Proof. The orbits of the three groups (2.17) are located in planes of \( \mathbb{R}^3 \). In order to identify the orbits with the various circles one only has to specify the initial point \( a \) by the regulations from above (\( \langle u, a \rangle = [a, Ca, C^2a] \neq 0 \) with \( C \) from (2.19), test on the sign of \( \langle u, u \rangle \) according to (2.18), reduction of \( a \) modulo the group action). In detail:

1) For the first group (2.17):

For an arbitrary initial point \( a \in \mathbb{H} \) one calculates \( \langle u, a \rangle = a_0(a_0^2 + a_1^2) \) and \( \langle u, u \rangle = (a_0^2 + a_1^2)^2 \). The reduction of \( a \) is then possible to \( a_2 = 0 \), i.e.

\[
a := \frac{1}{\sqrt{1 - r^2}} \begin{pmatrix} 1 \\ r \\ 0 \end{pmatrix}, \quad 0 < r < 1,
\]

with the orbit \( \gamma(\varphi) := \frac{1}{\sqrt{1 - r^2}} \begin{pmatrix} 1 \\ r \cos \varphi \\ r \sin \varphi \end{pmatrix} \)
where all conditions on $a$ are fulfilled. The distance of the orbit points in $\mathbb{B}$ from $O$ is constant and given by $\cosh d(O, \Gamma(\varphi)) = (1 - r^2)^{-1/2}$. The distance $d(O, \Gamma(\varphi))$ becomes equal to a given number $R > 0$ iff $r = \tanh R$. Thus, the various orbits can attain every hyperbolic radius $R$.

For $R$ fixed, the representation of $\gamma(\varphi)$ shows that the orbit points cover the whole distance circle $S(R)$ with center $O$ and hyperbolic radius $R$, and they reach every point exactly once if $\varphi$ is restricted on a halfopen interval of length $2\pi$.

From (2.21), the arclength on $S(R)$ between two points $\Gamma(\varphi_1)$ and $\Gamma(\varphi_2)$ is calculated to be

$$\sigma = \sinh R \cdot |\varphi_1 - \varphi_2|.$$  

In particular the perimeter of $S(R)$ is $2\pi \sinh R$. Further, the chordlength $L$ between two arbitrary points $\Gamma(\varphi_1)$ and $\Gamma(\varphi_2)$ is given by

$$\sinh \frac{L}{2} = \sinh R \cdot \sin \frac{|\varphi_1 - \varphi_2|}{2}.$$  

For points $X \neq Y$ on $S(R)$ in a given order, the arc $B(X,Y)$ associated to the directed chord from $X$ to $Y$ shall consist of all points of $S(R)$ on the right-hand side of the directed line $X \vee Y$. Now let $s$ always be the hyperbolic arclength measure of $B(X,Y)$. By (2.31), for $X = \Gamma(\varphi_1)$ and $Y = \Gamma(\varphi_2)$, one has to calculate $s$ as $s = \sigma$ if $O$ is left of or on $X \vee Y$, i.e. if $\varphi_1 < \varphi_2$ and $\varphi_2 - \varphi_1 \leq \pi$, but as $s = 2\pi \sinh R - \sigma$ if $O$ is right of $X \vee Y$, i.e. if $\varphi_1, \varphi_2$ are chosen with $\pi < \varphi_2 - \varphi_1 < 2\pi$. Eliminating in both cases the difference $|\varphi_1 - \varphi_2|$ from the equations for $s$ and $L$ yields the assertion (2.24).

2) For the second group (2.17):

For an arbitrary initial point $a \in \mathbb{H}$ one calculates: $\langle u, a \rangle = a_2(a_1^2 - a_0^2)$, $\langle u, u \rangle = -(a_1^2 - a_0^2)^2$. The reduction of $a$ is then possible to reach $a_1 = 0$, i.e.

$$a := \frac{1}{\sqrt{1 - \eta^2}} \begin{pmatrix} 1 \\ 0 \\ \eta \end{pmatrix}, \quad 0 < |\eta| < 1, \quad \text{with the orbit} \quad \gamma(\varphi) := \frac{1}{\sqrt{1 - \eta^2}} \begin{pmatrix} \cosh \varphi \\ \sinh \varphi \\ \eta \end{pmatrix}$$

where all conditions on $a$ are satisfied with $u = (0, 0, 1)$, the line vector of the groundline $U_0$.

The distance of the orbit points in $\mathbb{B}$ from $U$ is independent of $\varphi$ and given by $\sinh d_U(\Gamma(\varphi)) = |\eta| (1 - \eta^2)^{-1/2}$. The distance $d_U(\Gamma(\varphi))$ becomes equal to a given number $\delta > 0$ iff $|\eta| = \tanh \delta$. Thus, the various orbits can attain every hyperbolic distance $\delta$ from $U$.

For $\delta$ fixed, the representation of $\gamma(\varphi)$ shows that the orbit points run through the whole distance line in the upper resp. lower halfplane (according to the sign of $\eta$) reaching each point exactly once. Let $D(\delta)$ be the upper distance line.

From (2.21), the arclength on $D(\delta)$ between two points $\Gamma(\varphi_1)$ and $\Gamma(\varphi_2)$ is calculated to be

$$s = \cosh \delta \cdot |\varphi_1 - \varphi_2|,$$

and the chordlength $L$ as

$$\sinh \frac{L}{2} = \cosh \delta \cdot \sinh \frac{|\varphi_1 - \varphi_2|}{2}. $$
Eliminating the difference $|\varrho_1 - \varrho_2|$ from these two equations yields the assertion (2.27).

3) For the third group (2.17):

For an arbitrary initial point $a \in \mathbb{H}$ one calculates $\langle u, a \rangle = (a_0 - a_1)^3$ and $\langle u, u \rangle = 0$. By a suitable rotation around $O$, the $u$ can be reduced to $u = (0, 1, 0)$. With the choice $a = (1, 1, 0)$ (corresponding to $O$) the orbit is obtained as noted in (2.28). It cuts every line in $\mathbb{B}$ through $U$ orthogonally and runs through the origin $O$ (setting $b = 0$). So the image of $\Gamma$ is part of $\mathcal{H}$. In fact, every point on $\mathcal{H}$ is reached, as follows from the limit relations

$$
\lim_{b \to \pm \infty} \frac{1}{2} b^2 = 1, \quad \lim_{b \to \pm \infty} \frac{-b}{1 + \frac{1}{2} b^2} = \mp 0.
$$

That each point is reached once is directly clear from the parametrization.

From (2.21), the arclength on $\mathcal{H}$ between two points $\Gamma(b_1)$ and $\Gamma(b_2)$ is calculated to be

$$s = |b_1 - b_2|,$$

and the chordlength $L$ as

$$\sinh \frac{L}{2} = \frac{1}{2} |b_1 - b_2|.$$

Again, an elimination yields (2.30).

The first argument shows that an arbitrary horocycle can be transformed by a rotation in such a way that its horizon point $U$ becomes $(1, 0)$. For two horocycles with the same horizon point $U = (1, 0)$, the hyperbolic equivalence is proved as follows: The linear maps $T(\varrho)$ from the second group in (2.17) leave the quadric shell $\mathbb{H}$ fixed (in fact the scalar product itself), and $u = (1, 1, 0)$ is an eigenvector of each $T(\varrho)$, namely $T(\varrho)u = e^{\varrho}u$. For the image vectors $y := T(\varrho)x$ of $x$ with $\langle u, x \rangle = p$ one has $\langle T(\varrho)u, y \rangle = p$, thus $\langle u, y \rangle = e^{-\varrho}p$. So, if $\langle u, y \rangle = q$ is the equation of a second horocycle with the same $u$, this horocycle is obtained from the first one by a hyperbolic translation, represented by $T(\varrho)$, if one chooses $q = e^{-\varrho}p$, i.e. $\varrho = \ln p - \ln q$. \qed

The arguments $\varphi$ in (2.22) are called angle values though it is not necessary here to understand them as angles in the sense of definition (2.6). More likely, the $\varphi$ should be viewed as parametrizing the universal covering of $\mathcal{S}(R)$.

3. Closedness conditions

A polygon in the hyperbolic plane $\mathbb{B}$ is determined by a list of points $Z_1, \ldots, Z_n \in \mathbb{B}$, the vertices with $Z_k \neq Z_{k+1}$ for $k = 1, \ldots, n - 1$ and also $Z_n \neq Z_1$ with $n \geq 3$ fixed. Associated to these points is the closed polygon chain of segments from $Z_k$ to $Z_{k+1}$, $k = 1, \ldots, n - 1$ and $Z_n$ to $Z_1$. In fact, this polygon chain is the main object here. The polygon chain remains unaltered under cyclic permutations of the vertices. So, two polygons will be considered the same if they differ from each other by a cyclic permutation of the vertices. Other permutations
of the vertices are not allowed. The annotation \( Z_1 \ldots Z_n \) denotes the polygon in this sense. If nothing else is said, the indices of vertices will henceforth counted modulo \( n \), e.g. \( Z_{n+1} = Z_1 \). For a vertex \( Z_k \), \( k = 1, \ldots, n \), the two vertices \( Z_{k-1} \) and \( Z_{k+1} \) are called adjacent to \( Z_k \). For \( k = 1, \ldots, n \), the lines \( Z_k \lor Z_{k+1} \), resp. the segments \([Z_kZ_{k+1}] \) are called the edgelines, resp. the edges of the polygon (possibly directed if necessary). Polygons of this type are fairly general; they allow self-intersections of dimension 0 and even 1. In order to specify the number \( n \), sometimes the term \( n \)-gon will be used. The sidelengths and the perimeter are defined by

\[
L_k := d(Z_k, Z_{k+1}), \quad L := \sum_{k=1}^{n} L_k.
\]

The list \( L_1, \ldots, L_n \), corresponding to the list of vertices \( Z_1, \ldots, Z_n \), is sometimes called a length spectrum. Any points \( Z_1, \ldots, Z_n \in \mathbb{B} \) are named collinear, resp. cocyclic if they are situated on a line resp. circle (in the general sense). Representing vectors \( z_1, \ldots, z_k \) have to obey the general conventions from Sect. 2. So the lower indices of the \( z_k \) don’t count coordinates. (If necessary, the coordinates of \( z_k \) must be denoted by \( z_{k0}, z_{k1}, z_{k2} \).)

Convexity in the circle model of Cayley/Klein is not so much different from vector space convexity because the segments are the same in both geometries. For polygons, the only convexity notion to be used here is the following: A polygon \( Z_1 \ldots Z_n \) shall be called oriented-convex if, for any \( k \in \{1, \ldots, n\} \) all vertices \( Z_j \) with \( j \in \{1, \ldots, n\} \setminus \{k, k+1\} \) lie on the left hand side of the directed line \( \overrightarrow{Z_k \lor Z_{k+1}} \); equivalently

\[
[Z_k, Z_{k+1}, Z_j] > 0 \quad \forall \quad k \in \{1, \ldots, n\}, \quad j \in \{1, \ldots, n\} \setminus \{k, k+1\}.
\]

Then, for the same indices, no edgeline \( Z_k \lor Z_{k+1} \) contains another vertex \( Z_j \), in particular all vertices \( Z_1, \ldots, Z_n \) are pairwise distinct. Also, no edgeline cuts another edge in its interior.

The notion of oriented convexity has the advantage that there is no necessity to specify any ‘interior’ and that it can be verified by finitely many inequalities on the vertices.

**Remark.** As to the usual (Euclidean) convexity, one can show that the polygon chain of an oriented-convex polygon is the boundary of a unique convex and bounded domain \( C \) whose closure \( \overline{C} \) has the vertices as extreme points. So, by the Krein/Milman theorem, \( \overline{C} \) is the closed convex hull of the set of vertices. – Conversely, given the convex hull \( C \) of finitely many points \( W_1, \ldots, W_m \), the set \( C \) is compact (in particular \( C = \overline{C} \)) and it is the convex hull of the set \( \{Z_1, \ldots, Z_n\} \) of extreme points of \( C \) which is a subset of \( \{W_1, \ldots, W_m\} \). The extreme points are situated on the boundary of \( C \) which may be viewed as an oriented simply closed curve. Counted in the (cyclic) order of this curve, the extreme points form a polygon as defined in the beginning of this section.

However in this work, all arguments are based on the notion of oriented convexity thus being independent of the foregoing relations.

Next we consider curved paths which eventually may contain the vertices of a polygon. Generally, a path in \( \mathbb{B} \) is a continuous map \( \Gamma : J \to \mathbb{B} \) (\( J \) an interval with non-void interior). As above, a path may be represented by a continuous map \( \gamma : J \to \mathbb{R}^3 \).
The path \( \Gamma \) is called a circum-path of a polygon \( Z_1 \ldots Z_n \), if there are arguments \( t_1, \ldots, t_n \in J \) with \( Z_k = \Gamma(t_k) \), \( k = 1, \ldots, n \).

In particular, we shall consider injective paths with open interval \( J \) and the following property: For all \( t_1 < t_2 \in J \), the part of the image \( \Gamma(J) \) on the right hand side of the directed line is equal to \( \Gamma([t_1, t_2[) \). This part will be called the arc over the chord \( \Gamma(t_1), \Gamma(t_2) \) and denoted by \( B_\Gamma(t_1, t_2) \). Such a path shall be called a leftcurve. Equivalent to this definition is the requirement \( [\gamma(t_1), \gamma(t_2), \gamma(t)] < 0 \) for all \( t_1 < t < t_2 \) in \( J \) or also

\[
(3.2) \quad [\gamma(t_1), \gamma(t), \gamma(t_2)] > 0 \quad \text{for all} \quad t_1 < t < t_2 \in J.
\]

### 3.1. Lemma.

A leftcurve \( \Gamma : J \to \mathbb{B} \) is the circum-path of an oriented-convex \( n \)-gon if and only if there is a cyclic permutation of its vertices \( Z_1, Z_2, \ldots, Z_n \) such that there are arguments \( t_1 < t_2 < \cdots < t_n \) in \( J \) with \( Z_k = \Gamma(t_k) \), \( k = 1, \ldots, n \). In this case one has:

\[
(3.3) \quad B_\Gamma(t_1, t_n) \setminus \{Z_2, \ldots, Z_{n-1}\} = \bigcup_{k=1}^{n-1} B_\Gamma(t_k, t_{k+1}).
\]

The boldface dot signalizes disjoint union.

**Proof of 3.1**

**First direction:** Assume \( Z_k = \Gamma(t_k) \) with \( t_1 < t_2 < \cdots < t_n \).

One has to show \( [\gamma(t_k), \gamma(t_{k+1}), \gamma(t_j)] > 0 \) for the indices as in (3.1).

Case \( 1 \leq k \leq n-1 \): If \( t_k < t_{k+1} < t_j \), the assertion is clear from (3.2), and likewise if \( t_j < t_k < t_{k+1} \) since (3.2) implies \( [\gamma(t_j), \gamma(t_k), \gamma(t_{k+1})] > 0 \), hence \( [\gamma(t_k), \gamma(t_{k+1}), \gamma(t_j)] > 0 \).

Case \( k = n \): Then \( t_{k+1} = t_1 \), and necessarily \( t_1 < t_j < t_n \), so by (3.2) \( [\gamma(t_1), \gamma(t_j), \gamma(t_n)] > 0 \), hence \( [\gamma(t_n), \gamma(t_1), \gamma(t_j)] > 0 \).

**Second direction:** Assume the polygon to be oriented-convex in the original arrangement \( Z_1, \ldots, Z_n \) of vertices.

In order to line up the vertices monotonically along the leftcurve let the numbering already be arranged in such a way that \( Z_1 \) is the point with the smallest curve parameter: \( Z_1 = \Gamma(t_1) \) and \( t_1 < t_j \) for all \( j = 2, \ldots, n \). Then \( Z_2 \) has a curve parameter \( t_2 \) with \( t_1 < t_2 \). All other curve parameters \( t_j, j = 3, \ldots, n \) are then above \( t_2 \). For, on account of the oriented convexity of the polygon, one has \( [\gamma(t_1), \gamma(t_2), \gamma(t_j)] > 0 \). If one had \( t_1 < t_j < t_2 \) this would imply by the definition of a leftcurve \( [\gamma(t_1), \gamma(t_j), \gamma(t_2)] > 0 \), a contradiction. By inductively repeating this conclusion \( (t_2 \) takes over the role of \( t_1 \) etc.) it results that the curve parameters of the vertices can be chosen such that \( Z_k = \Gamma(t_k) \) for \( k = 1, \ldots, n \) and \( t_1 < t_2 < \cdots < t_n \).

Having achieved this arrangement, Eqn. (3.3) simply follows from the interval decomposition

\[
[t_1, t_n] = ([t_1, t_2] \cup \{t_2\}) \cup \cdots \cup ([t_{n-2}, t_{n-1}] \cup \{t_{n-1}\}) \cup [t_{n-1}, t_n]
\]

by applying the injective map \( \Gamma \).

Lemma 3.1 will now be applied to cocyclic polygons. So \( \Gamma \) will be part of a cone section, namely part or the whole of a hyperbolic circle. The circle may be assumed in standard
3. Closedness conditions

Any such circle is a leftcurve if suitably oriented. However, a distance circle must be punctured, i.e. one point must be removed. This point can still be chosen conveniently. In any case, the vertices of an oriented-convex and cocyclic polygon should be numbered such that they correspond to strictly monotonic increasing parameter values. As parametrizations of the circles those of Lemma 2.3 will be used.

The following closedness conditions are fundamental for the main results. First the cases of distance lines and horocycles will be discussed since they don’t have to be punctured.

3.2. Lemma (distance lines and horocycles). Let \( Z_1 \ldots Z_n \) be an oriented-convex polygon and without loss of generality let \( L_n \) denote its maximal sidelength.

If the vertices lie on the distance line \( D(\delta) \), resp. on the horocycle \( H \), the following closedness conditions hold true:

\[
\Psi(\delta, L_n) = \Psi(\delta, L_1) + \cdots + \Psi(\delta, L_{n-1}) \tag{3.4}
\]

resp.

\[
\Omega(L_n) = \Omega(L_1) + \cdots + \Omega(L_{n-1}), \tag{3.5}
\]

where the functions \( \Psi \) and \( \Omega \) are defined in Lemma 2.3.

In both cases the edge of maximal length is unique.

Proof. Eqn. (3.3) implies for the corresponding arclengths: \( s_n = s_1 + \cdots + s_{n-1} \). Using the relations to the sidelengths from Lemma 2.3 one immediately deduces (3.4) and (3.5). The terms on the right hand side of these equations are all positive, so \( \Psi(\delta, L_n) > \Psi(\delta, L_k) \), resp. \( \Lambda(\delta, L_n) > \Lambda(\delta, L_k) \) for \( k = 1, \ldots, n - 1 \). By the strict monotony of \( \Omega \) and \( \Lambda \): \( L_n > L_k \). \( \square \)

3.3. Corollary. Regular oriented-convex polygons whose vertices are on a distance line or a horocycle do not exist. \( \square \)

Now, in case of a distance circle as circum-path for an oriented-convex polygon \( Z_1 \ldots Z_n \) there is an additional peculiarity. Namely there are principally two different positions of the vertices relative to the center \( C \). (Here, we may assume \( C = O \) with the point vector \( c = (1,0,0) \).)

Case I: There exists a directed edgeline, leaving the circle center \( C \) on its right hand side, i.e.

\[
\exists k \in \{1, \ldots, n - 1\} : \quad [z_k, z_{k+1}, c] < 0 \quad \text{(niche position).} \tag{N}
\]

Case II: Case I is not satisfied, i.e.

\[
\forall k \in \{1, \ldots, n - 1\} : \quad [z_k, z_{k+1}, c] \geq 0 \quad \text{(full position).} \tag{F}
\]

The names spring from the fact that in case I there is a closed half circle disk clear of vertices while in case II every closed half circle disk contains vertices of the polygon. See the following pictures and Lemmas 3.5 and 3.6.
3. Closedness conditions

3.4. Lemma. Let an oriented-convex \( n \)-gon have the distance circle \( S(R) \) as an circum-path. Then it is possible to arrange the numbering of vertices, the puncturing of \( S(R) \), and a rotation around the center \( O \) such that an arbitrarily chosen edgeline becomes \( \overrightarrow{Z_n} \lor \overrightarrow{Z_1} \) and such that \( Z_k = \Gamma(\varphi_k) \) with \( 0 < \varphi_1 < \varphi_2 < \cdots \varphi_n < 2\pi \).

A puncturing of \( S(R) \) corresponds to an angle determination of the circle points according to (2.22) in that the removed point obtains an angle value \( \nu_0 \) and all other circle points receive angle values \( \varphi \) in the open interval \( \nu_0, \nu_0 + 2\pi \). Of course, each distance circle is to be oriented such that, after puncturing, a leftcurve arises.

Proof of 3.4. First, as a point to remove, one can take any \( P \in S(R) \) different from a vertex. Then one may rotate the whole figure such that \( P \) lands on the positive part of the groundline. Further, by Lemma 3.1, the vertices can be numbered such that, in the parametrization (2.22), there holds \( z_k = \gamma(\varphi_k), k = 1, \ldots, n \) with \( 0 < \varphi_1 < \varphi_2 < \cdots \varphi_n < 2\pi \). Now let \( Z_j \lor Z_{j+1} \) be any edgeline with \( j = 1, \ldots n-1 \) (for \( j = n \) the assertion is already valid). Then, for the vertices starting with \( Z_{j+1} \), there persists the following list with possible new angle values in the second line and possible new names for the vertices in the third line:

\[
\begin{align*}
Z_{j+1} & \quad Z_{j+2} \quad \cdots \quad Z_n & \quad Z_1 & \quad Z_2 & \quad \cdots & \quad Z_j \\
\varphi_{j+1} & < \varphi_{j+2} < \cdots < \varphi_n & < \varphi_1 + 2\pi & < \varphi_2 + 2\pi & < \cdots & < \varphi_j + 2\pi \\
Z'_1 & \quad Z'_2 & \quad \cdots & \quad Z'_{n-j} & \quad Z'_{n-j+1} & \quad Z'_{n-j+2} & \quad \cdots & \quad Z'_n.
\end{align*}
\]

One already has \( \overrightarrow{Z'_n} \lor \overrightarrow{Z'_1} = \overrightarrow{Z_j} \lor \overrightarrow{Z_{j+1}} \). In the second line, the new angle values are strictly monotonic increasing, in particular at \( \varphi_n < \varphi_1 + 2\pi \), since this is equivalent to \( \varphi_n - \varphi_1 < 2\pi \). For the difference of the ‘ends’ one has \( (\varphi_j + 2\pi) - \varphi_{j+1} < 2\pi \). So there is a new angle determination for which the vertices in the sequence of the third line are represented by strictly monotonic increasing angle values. By another rotation this angle determination can be reduced to the interval from 0 to \( 2\pi \).

3.5. Lemma (distance circles – niche position). Let an oriented-convex \( n \)-gon have the distance circle \( S(R) \) as an circum-path and assume it in niche-situation there. Without loss of generality, let \( Z_n \lor Z_1 \) be an edgeline leaving the center \( O \) on its right hand side. Then
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(i) The open halfplane right of $\overrightarrow{Z_n \lor Z_1}$ doesn’t contain vertices. A fortiori, there are closed halfcircle disks clear of vertices.

(ii) All the edgelines $\overrightarrow{Z_k \lor Z_{k+1}}$, $k = 1, \ldots, n-1$, leave the center $O$ on their left hand side. So, there is only one edgeline, $\overrightarrow{Z_n \lor Z_1}$, having $O$ on its right hand side.

(iii) For the sidelengths, there holds the closedness condition

\[ \Phi(R, L_n) = \Phi(R, L_1) + \cdots + \Phi(R, L_{n-1}), \]

where the function $\Phi$ is defined in Lemma 2.3.

(iv) The edgeline $\overrightarrow{Z_n \lor Z_1}$ is also characterized by belonging to the biggest side of the polygon:

\[ d(Z_1, Z_n) > d(Z_k, Z_{k+1}), \quad k = 1, \ldots, n - 1. \]

Proof. As a distinguished edgeline $\overrightarrow{Z_n \lor Z_1}$ in the sense of Lemma 3.4 one can choose one which has the center $C = O$ on its right hand side, so $[z_n, z_1, c] < 0$. By (2.22) this means

\[ 0 < [z_1, z_n, c] = \begin{vmatrix} \cosh R & \cosh R & 1 \\ \sinh R \cos \varphi_1 & \sinh R \cos \varphi_n & 0 \\ \sinh R \sin \varphi_1 & \sinh R \sin \varphi_n & 0 \end{vmatrix} = \sinh^2 R \cdot \sin(\varphi_n - \varphi_1), \]

hence $0 < \varphi_n - \varphi_1 < \pi$.

For (i): All other vertices lie left of $\overrightarrow{Z_n \lor Z_1}$. So right of $\overrightarrow{Z_n \lor Z_1}$ there are no points of the polygon chain. Then there exist directed lines through $C$ whose closed right hand sides are clear of such points, e.g. the line through $C$ parallel in the Euclidean sense and equally directed to $\overrightarrow{Z_n \lor Z_1}$.

For (ii): It must be shown $[\gamma(\varphi_k), \gamma(\varphi_{k+1}), c] > 0$ for $k = 1, \ldots, n - 1$. Now, as above $[\gamma(\varphi_k), \gamma(\varphi_{k+1}), m] = \sinh^2 R \cdot \sin(\varphi_{k+1} - \varphi_k)$, and, since $0 < \varphi_{k+1} - \varphi_k < \varphi_n - \varphi_1 < \pi$, the assertion follows.

For (iii): Again, this is deduced from (3.3), the definition of $B_\Gamma$, and Eqn. (2.24) (first part).

For (iv): From (iii) follows, because all terms on the right hand side are positive, that $\Phi(R, L_n) > \Phi(R, L_k)$ for $k = 1, \ldots, n - 1$. Then the strict monotony of the function $\Phi$ in its second argument implies $L_n > L_k$. \[ \square \]

3.6. Lemma (distance circles – full position). Let an oriented-convex $n$-gon have the distance circle $S(R)$ as an circum-path and assume it in full position there. Then:

(i) Each half circle disk contains vertices of the polygon.

(ii) For the sidelengths, there holds the closedness condition

\[ 2\pi \sinh R = \Phi(R, L_1) + \cdots + \Phi(R, L_n). \]
4. Rigidity

Proof.

For (i): Assume that a half circle disk would contain no vertices. Then, by an eventual rotation around O one could achieve $z_k = \gamma(\varphi_k)$ with $0 < \varphi_1 < \cdots \varphi_n < \pi$. By Lemma 3.5 (i) this would imply $[z_1, z_n, c] > 0$, so $[z_n, z_1, c] < 0$. The edgeline $Z_n \lor Z_1$ then had O on its right hand side.

For (ii): Let everything be arranged like in Lemma 3.4, where $Z_n \lor Z_1$ may be any edgeline, e.g. one with maximal sidelength. Again, by (3.3):

$$\tag{3.8} B_{\Gamma}(\varphi_1, \varphi_n) \setminus \{Z_2, \ldots, Z_{n-1}\} = B_{\Gamma}(\varphi_1, \varphi_2) \cup \cdots \cup B_{\Gamma}(\varphi_{n-1}, \varphi_n).$$

All the arcs on the right hand side of (3.8) have the center O on their left hand sides since O is always left of $Z_k \lor Z_{k+1}$ and the arc $B_{\Gamma}(\varphi_1, \varphi_n)$ is always right of $Z_k \lor Z_{k+1}$. So, for the relation between arc- and chordlength, the first part of (2.24) applies: $s_k = \Phi(R, L_k)$, $k = 1, \ldots, n - 1$.

For the arc on the left hand side of (3.8), the situation is converted: This arc has O on the right hand side since O lies left of $Z_n \lor Z_1$. So here, the relation between arc- and chordlength is regulated by the second part of (2.24): $s_n = 2\pi \sinh R - \Phi(R, L_n)$.

From (3.8) follows $s_n = s_1 + \cdots + s_{n-1}$. With the above values of the sidelengths, Eqn. (3.7) is thus verified. \qed

4. Rigidity

A first step to the rigidity of oriented-convex cocyclic polygons is the uniqueness of the circle type to be formulated in Theorem 4.3. The following preparations are needed:

4.1. Lemma. For positive real numbers $x_1, \ldots, x_m$ with $m \geq 2$ there holds:

$$\tag{4.1} \sinh(x_1 + \cdots + x_m) > \sinh x_1 + \cdots + \sinh x_m$$

and, if $x_1 + \cdots + x_m < \pi$:

$$\tag{4.2} \sin(x_1 + \cdots + x_m) < \sin x_1 + \cdots + \sin x_m.$$  

Proof. In both cases one can proceed inductively:

For (4.1): If $m = 2$ then

$$\sinh(x_1 + x_2) = \sinh x_1 \cosh x_2 + \sinh x_2 \cosh x_1 > \sinh x_1 + \sinh x_2.$$  

The induction step from $m$ to $m + 1$ is deduced from

$$\sinh(x_1 + \cdots + x_{m+1}) > \sinh(x_1 + \cdots + x_m) + \sinh x_{m+1} > \sinh x_1 + \cdots + \sinh x_m + \sinh x_{m+1}.$$
For (4.2): If \( m = 2 \) then
\[
\sin(x_1 + x_2) = \sin x_1 \cos x_2 + \sin x_2 \cos x_1 < \sin x_1 + \sin x_2.
\]
The induction step from \( m \) to \( m + 1 \) is deduced from
\[
\sin(x_1 + \cdots + x_{m+1}) < \sin(x_1 + \cdots + x_m) + \sin x_{m+1} < \sin x_1 + \cdots + \sin x_m + \sinh x_{m+1}
\]
where of course the additional assumption enters. \( \square \)

4.2. Remark. Transcribing Eqns. (4.1) and (4.2) to the inverse functions yields for positive real numbers \( \xi_1, \ldots, \xi_m \) with \( m \geq 2 \):
\[
\begin{align*}
\text{(4.3)} \quad \text{arsinh}(\xi_1 + \cdots + \xi_m) &< \text{arsinh} \xi_1 + \cdots + \text{arsinh} \xi_m \\
\text{(4.4)} \quad \text{arcsin}(\xi_1 + \cdots + \xi_m) &> \text{arcsin} \xi_1 + \cdots + \text{arcsin} \xi_m, \quad \xi_1 + \cdots + \xi_m \leq \frac{\pi}{2}.
\end{align*}
\]

4.3. Theorem. Solely by the sidelengths of a cocyclic oriented-convex polygon it is determined on which type of circle the vertices are situated.

Proof. Decisive for this are the closedness conditions from Lemmas 3.2, 3.5, and 3.6. It will be shown that any two of these conditions exclude each other.

For the pair distance line/horocycle:
Assume that at the same time the following would be true:
\[
\begin{align*}
\text{(4.5)} \quad \Psi(\delta, L_n) &= \Psi(\delta, L_1) + \cdots + \Psi(\delta, L_{n-1}) \\
\text{(4.6)} \quad \Omega(L_n) &= \Omega(L_1) + \cdots + \Omega(L_{n-1}).
\end{align*}
\]
Then, eliminating the expression \( \sinh \frac{\delta_1}{2} \) from Eqns. (2.26), (2.29) yields
\[
\frac{\Omega(t)}{\delta_1} = \sinh \frac{\Psi(\delta, t)}{\delta_1}, \quad \delta_1 := 2 \cosh \delta.
\]
From (4.5) and (4.6) follows by setting \( t = L_n \), resp. \( t = L_j \):
\[
\sinh \frac{\Psi(\delta, L_1)}{\delta_1} + \cdots + \sinh \frac{\Psi(\delta, L_{n-1})}{\delta_1} = \sinh \frac{\Psi(\delta, L_1) + \cdots + \Psi(\delta, L_{n-1})}{\delta_1}.
\]
By (4.1), such an equation can never hold.

For the pair distance circle (niche position)/horocycle:
Assume that at the same time the following would be true:
\[
\begin{align*}
\text{(4.7)} \quad \Phi(R, L_n) &= \Phi(R, L_1) + \cdots + \Phi(R, L_{n-1}) \\
\text{(4.8)} \quad \Omega(L_n) &= \Omega(L_1) + \cdots + \Omega(L_{n-1}).
\end{align*}
\]
Then, eliminating the expression $\sinh \frac{t}{2}$ from Eqns. (2.23), (2.29) yields

$$\frac{\Omega(t)}{R_1} = \sin \frac{\Phi(R, t)}{R_1}, \quad R_1 := 2 \sinh R.$$  

From (4.7) and (4.8) follows by setting $t = L_n$, resp. $t = L_j$:

$$\sin \frac{\Phi(R, L_1)}{R_1} + \cdots + \sin \frac{\Phi(R, L_{n-1})}{R_1} = \sin \frac{\Phi(R, L_1) + \cdots + \Phi(R, L_{n-1})}{R_1}.$$  

By (4.2), such an equation can never hold.

**For the pair distance circle (niche position)/distance line:**

Assume that at the same time the following would be true:

(4.9) \hspace{1cm} \Phi(R, L_n) = \Phi(R, L_1) + \cdots + \Phi(R, L_{n-1})

(4.10) \hspace{1cm} \Psi(\delta, L_n) = \Psi(\delta, L_1) + \cdots + \Psi(\delta, L_{n-1}).

Then, eliminating the expression $\sinh \frac{t}{2}$ from Eqns. (2.23), (2.26) yields, using the above shortcuts $\delta_1$ and $R_1$:

$$\sinh \frac{\Psi(\delta, t)}{\delta_1} = \frac{R_1}{\delta_1} \sin \frac{\Phi(R, t)}{R_1}.$$  

From (4.9) and (4.10) follows by setting $t = L_n$, resp. $t = L_j$:

$$\sinh \frac{\Psi(\delta, L_1) + \cdots + \Psi(\delta, L_{n-1})}{\delta_1} = \frac{R_1}{\delta_1} \sin \frac{\Phi(R, L_1) + \cdots + \Phi(R, L_{n-1})}{R_1},$$  

so with (4.1)

$$\sinh \frac{\Psi(\delta, L_1)}{\delta_1} + \cdots + \frac{\Psi(\delta, L_{n-1})}{\delta_1} < \frac{R_1}{\delta_1} \sin \frac{\Phi(R, L_1) + \cdots + \Phi(R, L_{n-1})}{R_1},$$  

or

$$\frac{R_1}{\delta_1} \sin \frac{\Phi(R, L_1)}{R_1} + \cdots + \frac{R_1}{\delta_1} \sin \frac{\Phi(R, L_{n-1})}{R_1} < \frac{R_1}{\delta_1} \sin \frac{\Phi(R, L_1) + \cdots + \Phi(R, L_{n-1})}{R_1}.$$  

By (4.2), such an equation can never hold.

**For the remaining pairs:**

Now, the last two pairs are lacking, namely replacing the niche positions with the distance circles by the full positions. However, the difference in arguing are small. Only the condition (4.7), resp. (4.9): $\Phi(R, L_n) = \Phi(R, L_1) + \cdots + \Phi(R, L_{n-1})$, has to be replaced by $\pi R_1 = \Phi(R, L_1) + \cdots + \Phi(R, L_{n-1})$ (see (3.6)), i.e. by

$$\frac{\Phi(R, L_n)}{R_1} = \pi - \left( \frac{\Phi(R, L_1)}{R_1} + \cdots + \frac{\Phi(R, L_{n-1})}{R_1} \right).$$  

Since the sinus value of the right hand side is the same as the sinus value of the sum in the last big parenthesis there result the same inequalities and arguments as with the last two cases.  

The simplest rigidity situation prevails if the vertices lie on a horocycle:
4.4. Theorem. Any two oriented-convex $n$-gons with vertices on a horocycle and with the same length spectrum $L_1,\ldots,L_n$ are hyperbolically equivalent.

Proof. Again it suffices to consider the special horocycle $H$ from Lemma 2.3 (iii). Change the parameter $b$ constant proportionally and equally directed to the arclength parameter $s$, and let the corresponding one parameter group be parametrized by $G(s)$ and the horocycle $H$ by $\Gamma(s) := G(s).A$. Between the arclength and the chordlength, one has the bijective relation expressed by (2.29). Permute the length numbers such that $L_n$ is the (unique) maximum.

Now, if $Z_1 = \Gamma(s_0)$ is the first vertex of such a polygon, then the succeeding vertices necessarily are given by

$$Z_k = \Gamma(s_0 + \Omega(L_1) + \cdots + \Omega(L_{k-1})), \quad k = 2,\ldots,n;$$

see Lemma 3.1. In other words: The succeeding vertices arise from $Z_1$ by laying off the chordlengths (more accurately: the corresponding arclengths) along $H$ in the direction of its orientation.

Since $\Gamma(s_0 + \Omega(L_1) + \cdots + \Omega(L_{k-1})) = G(\Omega(L_1) + \cdots + \Omega(L_{k-1})).Z_1$, these points only depend on $Z_1$ and the length values $L_1,\ldots,L_{n-1}$. Now, if $Z'_1 = \Gamma(s'_0)$ is the first vertex of another such polygon then Eqn. (4.11) holds correspondingly for its vertices $Z'_2,\ldots,Z'_n$, and from $Z'_1 = G(s'_0 - s_0).Z_1$ follows $Z'_k = G(s'_0 - s_0).Z_k$, $k = 1,\ldots,n$, hence the equivalence of the two polygons.

This procedure of laying off is also possible for the remaining cases insofar the chordlengths determine the corresponding arclength uniquely. But in contrast to the horocycles the other circle types have invariants, and it must be clarified in addition that these invariants are again determined uniquely by the length spectrum. The following lemmas serve this purpose.

4.5. Lemma. For $q > 1$, the function $h : \mathbb{R}^+_0 \to \mathbb{R}$,

$$h(x) := \frac{\text{arsinh}(q \sinh x)}{x}, \quad h(0) := q,$$

is strictly monotonic decreasing. In particular $h(x) < h(0) = q$ for $x > 0$.

Proof. The idea is that the values of $h$ are certain slopes of a concave function.

At any rate, the function $h$ has in 0 a removable singularity. With the given value, it is of class $C^\infty$ on the whole of $\mathbb{R}^+_0$. The numerator function $H : \mathbb{R}^+_0 \to \mathbb{R}$, $H(x) := \text{arsinh}(q \sinh x)$ is strictly concave: Its second derivative works out to be

$$H''(x) = -q(q^2 - 1) \frac{\sinh x}{(1 + q^2 \sinh^2 x)^{3/2}},$$

thus $H''(x) < 0$ in $\mathbb{R}^+$. For the monotony of the difference quotients $\delta H$ this implies $(\delta H)(0,x_1) > (\delta H)(0,x_2)$ for $0 < x_1 < x_2$, i.e.

$$\frac{\text{arsinh}(q \sinh x_1)}{x_1} > \frac{\text{arsinh}(q \sinh x_2)}{x_2} \quad \text{for } 0 < x_1 < x_2.$$
so \( h(x_1) > h(x_2) \). From this follows the strictly monotonic decreasing of \( h \) on the whole of \( \mathbb{R}_0^+ \).

4.6. Corollary. \( \) For \( 0 < \delta' < \delta \) and \( t > 0 \) holds:

\[
\Psi(\delta, t) > 1
\]

and also the implication:

\[
0 < t_1 < t_2 \implies \frac{\Psi(\delta, t_1)}{\Psi(\delta', t_1)} < \frac{\Psi(\delta, t_2)}{\Psi(\delta', t_2)}.
\]

Intuitively, the inequality (4.12) says: If the same chordlength \( L \) is laid off along the distance line \( D(\delta) \) and also along the distance line \( D(\delta') \) with smaller parameter \( \delta' < \delta \) then the corresponding arc also becomes smaller: \( \Psi(\delta', L) < \Psi(\delta, L) \). And the implication (4.13) means that the arcs over smaller chords are less shortened than the arcs over bigger chords if one passes over from \( D(\delta) \) to \( D(\delta') \).

Proof of 4.6 Set

\[
\delta_1 := 2 \cosh \delta, \quad \delta_1' := 2 \cosh \delta', \quad q := \frac{\delta_1}{\delta_1'} > 1, \quad \tau_1 := \frac{2 \sinh \frac{t_1}{2}}{\delta_1}, \quad \tau_2 := \frac{2 \sinh \frac{t_2}{2}}{\delta_1}.
\]

Then, for \( t = t_2 \), the assertion (4.12) reads as

\[
q \frac{\text{arsinh} \tau_2}{\text{arsinh}(q \tau_2)} > 1,
\]

or, with \( x_2 := \text{arsinh} \tau_2 \), as

\[
\frac{\text{arsinh}(q \sinh x_2)}{q x_2} < 1.
\]

But this is correct by Lemma 4.5 last part.

Analogously, the assertion (4.13) reads as

\[
\frac{\text{arsinh}(q \tau_1)}{\text{arsinh} \tau_1} < \frac{\text{arsinh}(q \tau_2)}{\text{arsinh} \tau_2},
\]

or, with \( x_1 := \text{arsinh} \tau_1 \), as

\[
\frac{\text{arsinh}(q \sinh x_1)}{x_1} > \frac{\text{arsinh}(q \sinh x_2)}{x_2}.
\]

Since \( 0 < x_1 < x_2 \) this is true by Lemma 4.5.

4.7. Theorem. \( \) Any two oriented-convex \( n \)-gons with vertices on distance lines and with the same length spectrum \( L_1, \ldots, L_n \) are hyperbolically equivalent.
Proof. The main point is to identify the distance invariants $\delta, \delta'$ of two possible circum-distance-lines.

Again, let the numbering of the sidelengths be arranged such that $L_n$ is their (unique) maximum, so $L_n > L_k$ for $k = 1, \ldots, n - 1$.

Assuming $\delta' < \delta$, one arrives at a contradiction in the following way. The two closedness conditions sound

\begin{align*}
(4.14) & \quad \Psi(\delta, L_n) = \Psi(\delta, L_1) + \cdots + \Psi(\delta, L_{n-1}) \\
(4.15) & \quad \Psi(\delta', L_n) = \Psi(\delta', L_1) + \cdots + \Psi(\delta', L_{n-1}).
\end{align*}

By Corollary 4.6

\begin{equation}
(4.16) \quad \frac{\Psi(\delta, L_k)}{\Psi(\delta', L_n)} < \frac{\Psi(\delta, L_n)}{\Psi(\delta', L_n)} =: p, \quad k = 1, \ldots, n - 1.
\end{equation}

This implies

\[
\frac{\Psi(\delta, L_n)}{\Psi(\delta', L_n)} = \frac{\Psi(\delta, L_1) + \cdots + \Psi(\delta, L_{n-1})}{\Psi(\delta', L_1) + \cdots + \Psi(\delta', L_{n-1})} < p\frac{\Psi(\delta', L_1) + \cdots + \Psi(\delta', L_{n-1})}{\Psi(\delta', L_1) + \cdots + \Psi(\delta', L_{n-1})} = p,
\]

contradicting (4.16).

Now, it can be assumed that the two polygons have the same circum-distance-line, e.g. $D(\delta)$ from 2.2. Then the above procedure of laying off the chordlengths along $D(\delta)$ again leads to the conclusion that the vertices of the first polygon are transformed to those of the second polygon by a common element of the corresponding one parameter group. \qed

For distance circles as circum-paths one may proceed in an analogous way, where however the two different positions (niche, resp. full) must be regarded. Again, the main point is to distil out the radius invariant solely from the length spectrum. For this there are statements completely analogous to Lemma 4.5 and Corollary 4.6 with analogous proofs. Therefore, the arguments will be shortened somewhat.

4.8. Lemma. For $q > 1$, the function

\[
h : \left[0, \arcsin \frac{1}{q}\right] \to \mathbb{R}
\]

\[
h(x) := \frac{\arcsin(q \sin x)}{x}, \quad h(0) := q
\]

is strictly monotonic increasing. In particular $q = h(0) < h(x)$ for $0 < x \leq \arcsin \frac{1}{q}$.

Proof. As above, this runs with the aid of the numerator function $H(x) := \arcsin(q \sin x)$ because $h(x)$ represents the slope of $H$ between 0 and $x$. This time, $H$ is convex with the effect that the slopes are strictly monotonic increasing. \qed
4.9. Corollary. For $0 < R' < R$ and $0 < t \leq 2R'$ holds:

\[
\frac{\Phi(R,t)}{\Phi(R',t)} < 1
\]

and also the implication:

\[
0 < t_1 < t_2 \leq 2R' \implies \frac{\Phi(R,t_1)}{\Phi(R',t_1)} > \frac{\Phi(R,t_2)}{\Phi(R',t_2)}.
\]

Proof. One can perform the same steps as for Corollary 4.6. This time, setting

\[
R_1 := 2 \sinh R, \quad R'_1 := 2 \sinh R', \quad q := \frac{R_1}{R'_1} > 1, \quad \tau_1 := \frac{2 \sinh \frac{t_1}{R_1}}{2}, \quad \tau_2 := \frac{2 \sinh \frac{t_2}{R_1}}{2},
\]

the assertion (4.17) reads for $t = t_2$ as

\[
q \frac{\arcsin \tau_2}{\arcsin(q \tau_2)} < 1
\]

or, with $x_2 := \arcsin \tau_2$, as

\[
\frac{\arcsin(q \sin x_2)}{qx_2} > 1.
\]

This is true by Lemma 4.8 last part.

Similarly, the assertion (4.18) reads as

\[
\frac{\arcsin(q \tau_1)}{\arcsin \tau_1} < \frac{\arcsin(q \tau_2)}{\arcsin \tau_2}
\]

or, with $x_1 := \arcsin \tau_1$, as

\[
\frac{\arcsin(q \sin x_1)}{x_1} < \frac{\arcsin(q \sin x_2)}{x_2}.
\]

Since $0 < x_1 < x_2$ this is true by Lemma 4.8. □

4.10. Lemma. Solely by the sidelengths of an oriented-convex $n$-gon whose vertices lie on a distance circle, the radius of the circle and the type of the polygon (niche or full position) is determined.

Proof. Consider two oriented-convex $n$-gons with vertices on distance circles of radii $R, R'$ and with same sidelengths $L_1, \ldots, L_n$. One has to identify $R, R'$ and the two position types. The discussion follows the eventual combinations of the positions.

(i) Circum-radii for two oriented-convex $n$-gons with same sidelengths in full position: This case is very easy. Namely, a division of (3.7) by $2 \sinh R$ yields

\[
\pi = \sum_{k=1}^{n} \arcsin \left( \frac{\sinh \frac{L_k}{2}}{\sinh R} \right),
\]

and this equation cannot allow two solutions for $R$ because of the strict monotony of all summands w.r.t. $R$. 

□
(ii) **Circum-radii for two oriented-convex \( n \)-gons with same sidelengths in niche position:**
Assume again \( L_n > L_k \) for \( k = 1, \ldots, n - 1 \). From Corollary 4.9 one deduces

\[
\frac{\Phi(R, L_k)}{\Phi(R', L_k)} > \frac{\Phi(R, L_n)}{\Phi(R', L_n)} := p,
\]

If there existed two such polygons with circum-radii \( R' < R \) then, besides (3.6), one had the analogue with \( R \) replaced by \( R' \):

\[
\begin{align*}
\Phi(R, L_n) &= \Phi(R, L_1) + \cdots + \Phi(R, L_{n-1}) \\
\Phi(R', L_n) &= \Phi(R', L_1) + \cdots + \Phi(R', L_{n-1}).
\end{align*}
\]

This would imply

\[
\frac{\Phi(R, L_n)}{\Phi(R', L_n)} = \frac{\Phi(R, L_1) + \cdots + \Phi(R, L_{n-1})}{\Phi(R', L_1) + \cdots + \Phi(R', L_{n-1})} < \frac{p\Phi(R', L_1) + \cdots + p\Phi(R', L_{n-1})}{\Phi(R', L_1) + \cdots + \Phi(R', L_{n-1})} = p,
\]

contradicting (4.20).

(iii) **Circum-radii for two oriented-convex \( n \)-gons with same sidelengths in the first mixed position:** By this is meant that the polygon with circum-radius \( R \) is in the full position and the polygon with circum-radius \( R' \leq R \) in the niche position.

Assume \( R' < R \). Then, on one hand, (3.7) holds and on the other hand (3.6) with \( R \) replaced by \( R' \):

\[
\begin{align*}
2\pi \sinh R &= \Phi(R, L_1) + \cdots + \Phi(R, L_n) \\
2\pi \sinh R' &= \Phi(R', L_1) + \cdots + \Phi(R', L_n).
\end{align*}
\]

With (4.17) this implies

\[
2\pi \sinh (R - \Phi(R, L_n)) = \Phi(R, L_1) + \cdots + \Phi(R, L_{n-1}) < \Phi(R', L_1) + \cdots + \Phi(R', L_{n-1}) = \Phi(R', L_n),
\]

so

\[
2\pi \sinh R' < 2\pi \sinh R < \Phi(R, L_n) + \Phi(R', L_n) < 2\Phi(R', L_n),
\]

hence \( \pi \sinh R' < \Phi(R', L_n) \). But this is impossible by (2.23): The first mixed position is at most possible for \( R = R' \).

(iv) **Circum-radii for two oriented-convex \( n \)-gons with same sidelengths in the second mixed position:** By this is meant that the polygon with circum-radius \( R \) is in the niche position and the polygon with circum-radius \( R' \leq R \) in the full position.

Assume \( R' < R \). Then, on one hand (3.6) holds and on the other hand (3.7) with \( R \) replaced by \( R' \):

\[
\begin{align*}
\Phi(R, L_n) &= \Phi(R, L_1) + \cdots + \Phi(R, L_{n-1}) \\
2\pi \sinh R' &= \Phi(R', L_1) + \cdots + \Phi(R', L_n).
\end{align*}
\]

Since again \( L_n > L_k \) may be assumed for \( k = 1, \ldots, n - 1 \) this implies by (4.20):

\[
2\pi \sinh R' - \Phi(R', L_n) = \Phi(R', L_1) + \cdots + \Phi(R', L_{n-1})
\]

\[
< \frac{1}{p} \Phi(R, L_1) + \cdots + \frac{1}{p} \Phi(R, L_{n-1}) = \frac{1}{p} \Phi(R, L_n) = \Phi(R', L_n),
\]
hence $2\pi \sinh R' < 2\Phi(R', L_n)$, what is impossible by (2.23): The second mixed position is at most possible for $R = R'$.

In fact, even for equal circum-radii $R = R'$ the cases (iii) and (iv) are not existent. If it were so, then consider a biggest sidelength, say $L_n$. Due to the niche position the biggest sidelength is unique. Then both conditions (3.6) and (3.7) are satisfied simultaneously:

$$\Phi(R, L_n) = \Phi(R, L_1) + \cdots + \Phi(R, L_{n-1})$$

$$2\pi \sinh R = \Phi(R, L_1) + \cdots + \Phi(R, L_n).$$

This implies $2\pi \sinh R = 2\Phi(R, L_n)$, so $L_n = 2R$. The corresponding edge is then a diameter of the distance circle, so contains the center $O$, in contrast to the niche position. \[\Box\]

In order to ensure the uniqueness of the laying off procedure in the case of a circumscribed distance circle, the following statements are needed:

4.11. Lemma.

(i) For every point $Z \in S(R)$ and every number $L \in [0, 2R]$ there exists exactly one point $W \in S(R)$ with $d(Z, W) = L$ such that the center $O$ is situated left, resp. right of the directed line $\overrightarrow{Z \lor W}$.

(ii) Let $Z, W, Z', W'$ be points in $S(R)$ with $0 < d(Z, W) = d(Z', W') < 2R$. If the center $O$ is situated either left of the directed lines $\overrightarrow{Z \lor W}$ and $\overrightarrow{Z' \lor W'}$ or right of them, then there exists a proper rotation $D$ of $\mathbb{B}$ around $O$ with $D(Z) = D(Z')$ and $D(W) = D(W')$.

Proof. Of course, distances and the left and right positions of $O$ are invariant under proper rotations. We only discuss the case of the left position of $O$. The other case runs analogously.

For (i): On account of this invariance one may assume, by applying of a proper rotation, that $z = \gamma(R, 0)$ ($\gamma$ as in (2.22)). The conditions for $w = \gamma(R, \varphi)$ with $\varphi \in [-\pi, \pi]$ then sound equivalently:

$$d(Z, W) = L \iff \cos \varphi = 1 - 2 \left( \frac{\sinh L}{\sinh R} \right)^2$$

$$[\gamma(R, 0), \gamma(R, \varphi), 0] > 0 \iff \sin \varphi > 0.$$

The first line follows from $\|\gamma(R, \varphi)\| = 1$ and $\cosh d(z, w) = \cosh^2 R - \sinh^2 R \cdot \cos \varphi$ with means of identities for the hyperbolic functions.

These conditions are satisfied by exactly one $\varphi$. Hence the uniqueness.

For (ii): By applying another proper rotation to $Z'$ one may assume $z = z' = \gamma(R, 0)$. The conditions on $w = \gamma(R, \varphi)$ and $w' = \gamma(R, \varphi')$ with $\varphi, \varphi' \in [-\pi, \pi]$ then sound:

$$d(Z, W) = d(Z', W') \iff \cos \varphi = \cos \varphi'$$

$$[\gamma(R, 0), \gamma(R, \varphi), 0] > 0, \ [\gamma(R, 0), \gamma(R, \varphi'), 0] > 0 \iff \sin \varphi > 0, \sin \varphi' > 0.$$
This implies $\varphi = \varphi'$, so $W = W'$, hence $D := \text{id}$ does it.

Now the distance circle situation can be finished:

4.12. **Theorem.** Any two oriented-convex $n$-gons with vertices on distance circles and with the same length spectrum $L_1, \ldots, L_n$ are hyperbolically equivalent.

**Proof.** By Lemma 4.10 one can assume that both polygons $Z_1 \ldots Z_n$ and $Z'_1 \ldots Z'_n$ have the same circum-distance-circle $S(R)$ and that they are of the same position type.

If both polygons are in full position then the edgelines $\overrightarrow{Z_1 \vee Z_2}$ and $\overrightarrow{Z'_1 \vee Z'_2}$ have the origin on their left hand sides. So, by Lemma 4.11(ii), an additional rotation may be applied on the second polygon in order to reach $Z_1 = Z'_1$ and $Z_2 = Z'_2$. It remains to show $Z_k = Z'_k$ for $k = 3, \ldots, n$: The points $Z_3$ and $Z'_3$ have the distance $L_2 \leq 2R$ from $Z_2$, and the edgelines $\overrightarrow{Z_2 \vee Z_3}$ and $\overrightarrow{Z'_2 \vee Z'_3}$ have the center $O$ on their left hand side. In case $L_2 < 2R$ follows $Z_3 = Z'_3$ by Lemma 4.11(i). If $L_2 = 2R$ then $Z_3$ and $Z'_3$ are the antipode of $Z_2$, hence equal. Obviously, one can proceed inductively in this manner, finally obtaining the incidence of all vertices.

If both polygons are in niche position and if $L_n$ is the biggest sidelength (belonging to the edgelines $\overrightarrow{Z_n \vee Z'_1}$ and $\overrightarrow{Z'_n \vee Z'_1}$) then these edgelines have the center $O$ on their right hand side. So, by Lemma 4.11(ii) one can achieve: $Z_n = Z'_n$ and $Z_1 = Z'_1$. Then $Z_2$ and $Z'_2$ have the distance $L_1 < 2R$ from $Z_1$, and the edgelines $\overrightarrow{Z_1 \vee Z_2}$ and $\overrightarrow{Z'_1 \vee Z'_2}$ have $O$ on their left hand side. By Lemma 4.11(i) one deduces $Z_2 = Z'_2$. Again, this reasoning can be iterated until all vertices are identified. □

Now all components for the general statement are collected together:

4.13. **Theorem** (rigidity). Two oriented-convex and cocyclic $n$-gons of the hyperbolic plane are hyperbolically equivalent if and only if they have the same length spectrum $L_1, \ldots, L_n$.

**Proof.** Of course, the hyperbolic equivalence implies the same sidelengths. The converse follows from the synopsis of the Theorems 4.3, 4.4, 4.7, 4.12. □

The oriented-convex and cocyclic polygons are the only non-collinear polygons for which such a rigidity statement can be hoped for. Namely, it will be seen in part 2 that for any other non-collinear polygon there is an oriented-convex and cocyclic polygon with same sidelengths but bigger area. So the former polygon cannot be rigid. For collinear polygons the rigidity can be discussed within the context of the next section.

5. **Converse of the $n$-inequalities**

The main theorem on rigidity is a result on uniqueness. In its formulation above it doesn’t contribute to the existence of polygons with given sidelengths, though in the course of the
proof there were contained pieces of existence in form of the lay off procedure. In fact, these will again enter the game here. But the decisive mean will be the generalized triangle inequality. Its behaviour will control the existence question and also the rigidity for collinear polygons. The *n*-inequalities for positive real numbers $L_1, \ldots, L_n$ say:

\[(5.1) \quad L_k \leq L_1 + \cdots + \widehat{L_k} + \cdots + L_n \quad \forall k = 1, \ldots, n.\]

If in all these inequalities appears the less than sign we are speaking of the *strict* case, otherwise of the *non-strict* case. Of course, (5.1) is necessarily satisfied if the $L_k$ are the sidelengths of a polygon. In order to discuss the converse one first remarks that these inequalities can be contracted to one inequality if the numbering is such that $L_n$ is the maximum of the $L_k$. Then (5.1) is in fact equivalent to

\[(5.2) \quad L_n \leq L_1 + \cdots + L_{n-1}.\]

*First, the strict case is discussed.* (The non-strict case will be dealt with in the proof of Theorem 5.3.)

The main means for the existence are the closedness conditions from Sect. 4:

(\(H\)) \hspace{1cm} \sinh \frac{L_n}{2} = \sinh \frac{L_1}{2} + \cdots + \sinh \frac{L_{n-1}}{2}

(\(D\)) \hspace{1cm} \text{arsinh} \left( \frac{\sinh \frac{L_n}{2}}{\cosh \delta} \right) = \text{arsinh} \left( \frac{\sinh \frac{L_1}{2}}{\cosh \delta} \right) + \cdots + \text{arsinh} \left( \frac{\sinh \frac{L_{n-1}}{2}}{\cosh \delta} \right)

(\(CN\)) \hspace{1cm} \text{arcsin} \left( \frac{\sinh \frac{L_n}{2}}{\sinh R} \right) = \text{arcsin} \left( \frac{\sinh \frac{L_1}{2}}{\sinh R} \right) + \cdots + \text{arcsin} \left( \frac{\sinh \frac{L_{n-1}}{2}}{\sinh R} \right)

(\(CF\)) \hspace{1cm} \pi = \text{arcsin} \left( \frac{\sinh \frac{L_1}{2}}{\sinh R} \right) + \cdots + \text{arcsin} \left( \frac{\sinh \frac{L_n}{2}}{\sinh R} \right).

See (3.5), (3.4), (3.6), (3.7) and (2.29), (2.26), (2.23).

Given the lengths $L_1, \ldots, L_n$ as positive real numbers, the last three equations are to be viewed as conditions for the unknowns $\delta > 0$ and $R > 0$. Since only certain combinations of quantities occur in these equations we may set:

\[(5.3) \quad x := \frac{1}{\cosh \delta} \quad \text{for equation } (D), \quad x := \frac{1}{\sinh R} \quad \text{for equations } (CN) \text{ and } (CF)\]

and, generally

\[(5.4) \quad \lambda_k := \sinh \frac{L_k}{2}.\]
Eqns. (H), (D), (CN), (CF) are thus modified to

\[(h) \quad \lambda_n = \lambda_1 + \cdots + \lambda_{n-1}\]

\[(d) \quad \text{arsinh}(\lambda_n x) = \text{arsinh}(\lambda_1 x) + \cdots + \text{arsinh}(\lambda_{n-1} x), \quad 0 < x < 1\]

\[(cn) \quad \text{arcsin}(\lambda_n x) = \text{arcsin}(\lambda_1 x) + \cdots + \text{arcsin}(\lambda_{n-1} x)), \quad 0 < x < \frac{1}{\lambda_n}\]

\[(cf) \quad \pi = \text{arcsin}(\lambda_1 x) + \cdots + \text{arcsin}(\lambda_n x), \quad 0 < x \leq \frac{1}{\lambda_n}.

It must be proved that, for given positive reals \(L_1, \ldots, L_n\) with

\[(5.5) \quad L_j \leq L_n \quad \forall \ j, \quad L_n < L_1 + \cdots + L_{n-1}.\]

at least one of these four equations has a solution \(x\) in the corresponding interval (the first equation doesn’t contain any unknown).

The last two equations have the same form as in the Euclidean case, however with a different meaning for the given quantities \(\lambda_1, \ldots, \lambda_n\). On their solvability the following is known:

5.1. Lemma (Pinelis [2005]). Let \(\lambda_1, \ldots, \lambda_n \in \mathbb{R}\) be given, satisfying \(0 < \lambda_j \leq \lambda_n\) for \(j = 1, \ldots, n\) and \(\lambda_n < \lambda_1 + \cdots + \lambda_{n-1}\). Then:

(i) The function

\[g(x) := \text{arcsin}(\lambda_1 x) + \cdots + \text{arcsin}(\lambda_{n-1} x) - \text{arcsin}(\lambda_n x)\]

is positive in a certain interval right of \(0\).

(ii) If Eqn. (cn) has no solution in \([0, 1/\lambda_n]\) then Eqn. (cf) has a solution in \([0, 1/\lambda_n]\).

Slightly modified, this can be shown as follows:

Proof of Lemma 5.1

For (i): For positive \(x\) with \(\lambda_1 x + \cdots + \lambda_{n-1} x \leq \frac{\pi}{2}\) by (4.4):

\[g(x) > \text{arcsin}(\lambda_1 x + \cdots + \lambda_{n-1} x) - \text{arcsin}(\lambda_n x).

From \(\lambda_1 x + \cdots + \lambda_{n-1} x > \lambda_n x\) follows \(g(x) > 0\) for these \(x\). So the function \(g\) is positive in the interval \([0, \beta]\) where

\[\beta := \frac{\pi}{2(\lambda_1 + \cdots + \lambda_{n-1})}.

For (ii): Since Eqn. (cn) has no solution in \([0, 1/\lambda_n]\), the function \(g\) has a fixed sign in this interval, by part (i) a fixed positive sign. This implies

\[g\left(\frac{1}{\lambda_n}\right) \geq 0.

\]
5. Converse of the n-inequalities

By means of \( g(x) \), Eqn. \( \text{[cf]} \) sounds equivalently
\[
g_1(x) := \pi - 2 \arcsin(\lambda_n x) - g(x) = 0.
\]
Now
\[
g_1(0) = \pi, \quad g_1 \left( \frac{1}{\lambda_n} \right) = -g \left( \frac{1}{\lambda_n} \right) \leq 0.
\]
Thus the equation \( g_1(x) = 0 \) has a solution in \([0, 1/\lambda_n]\).

\[ \square \]

5.2. Lemma.
Assume \( 0 < \lambda_k \leq \lambda_n \) for \( k = 1, \ldots, n \) and, with the binding \( \text{(5.4)} \), \( L_n < L_1 + \cdots + L_{n-1} \). Then at least one of the equations \( \text{(h)}, \text{(d)}, \text{(cn)}, \text{and (cf)} \) has a solution in the corresponding interval.

\textbf{Proof.} In case \( \lambda_n = \lambda_1 + \cdots + \lambda_{n-1} \) it is Eqn. \( \text{(h)} \). In case \( \lambda_n < \lambda_1 + \cdots + \lambda_{n-1} \), one of the equations \( \text{(cn)}, \text{(cf)} \) has a solution, by Lemma 5.1 (ii). Now, in case \( \lambda_n > \lambda_1 + \cdots + \lambda_{n-1} \), it will be shown that Eqn. \( \text{(d)} \) is solvable: Define
\[
h(x) := \arcsinh(\lambda_n x) - \arcsinh(\lambda_1 x) - \cdots - \arcsinh(\lambda_{n-1} x), \quad x \in \mathbb{R}.
\]
One has \( h(0) := 0 \) and \( h'(0) = \lambda_n - \lambda_1 - \cdots - \lambda_{n-1} > 0 \). So \( h(x) \) is positive in a certain interval right of \( 0 \). On the other hand, by \( \text{(5.4)} \)
\[
h(1) = L_n - \frac{L_1}{2} - \frac{L_2}{2} - \cdots - \frac{L_{n-1}}{2} < 0.
\]
Thus, \( h \) must vanish somewhere in \([0, 1]\). \[ \square \]

5.3. Theorem. Let \( L_1, \ldots, L_n \) be positive real numbers with
\[
\text{(5.6)} \quad L_k \leq L_1 + \cdots + \hat{L}_k + \cdots + L_n \quad \forall k = 1, \ldots, n.
\]

(i) If in all inequalities \( \text{(5.6)} \) occurs the strict less than symbol then there exists an oriented-convex cocyclic \( n \)-gon in the hyperbolic plane \( \mathbb{B} \) with sidelengths \( L_1, \ldots, L_n \). According to Theorems 4.4, 4.7, and 4.12 such a polygon is uniquely determined up to hyperbolic equivalence.

(ii) If in one of the inequalities \( \text{(5.6)} \) occurs the equals sign, say in \( L_n = L_1 + \cdots + L_{n-1} \), then there also exists a \( n \)-gon in the hyperbolic plane \( \mathbb{B} \) with sidelength \( L_1, \ldots, L_n \). Such a polygon is uniquely determined up to hyperbolic congruence, it is collinear and its vertices \( Z_1, \ldots, Z_n \) are in this order strictly monotonic arranged on the supporting line, in particular pairwise distinct.

\textbf{Proof.} For (i): Assume again \( L_j \leq L_n \) for \( j = 1, \ldots, n \) and \( L_n < L_1 + \cdots + L_{n-1} \).
The proof runs by inspecting the solvability of Eqns. (h), (d), (cn), (cf) one by one. In all four cases it will be possible to obtain from a solution $x$ the corresponding parameter $\delta$ resp. $R$ via the substitutions
\[ \lambda_k := \sinh \frac{L_k}{2}, \quad x = \frac{1}{\cosh \delta}, \quad \text{resp.} \quad x = \frac{1}{\sinh R}. \]

Then, on the appropriate circle type, the lay off procedure can be executed with the lengths $L_1, \ldots, L_{n-1}$ in order to generate the vertices $Z_1, \ldots, Z_n$ of the desired polygon. Thereby, the closedness condition ensures each time the compatibility, i.e. the fact that also the last distance $d(Z_n, Z_1)$ fits to $L_n$.

**Case i.h:** Eqn. (h) holds.

This is the easiest case because no unknown $x$ is involved. Here, of course, the lay off is applied to the horocycle $H$ as described in the proof of Theorem 4.4: One parametrizes the orbit by arclength (positively constant proportional to the original group parameter $b$) as $\Gamma(s)$, selects $Z_1 = \Gamma(s_0)$ arbitrarily and constructs the further vertices by the rule
\[ Z_k := \Gamma(s_0 + \Lambda(L_1) + \cdots + \Lambda(L_{k-1})), \quad k = 2, \ldots, n. \]

For $k = n$ by Eqn. (h) then holds
\[ \Lambda(L_1) + \cdots + \Lambda(L_{n-1}) = 2 \sinh \frac{L_1}{2} + \cdots + 2 \sinh \frac{L_{n-1}}{2} = 2\lambda_1 + \cdots + 2\lambda_{n-1} = 2\lambda_n = \Lambda(L_n), \]

so
\[ Z_n = \Gamma(s_0 + \Lambda(L_n)). \]

In case $k \leq n - 1$ follows from (5.7) for the the arclength between $Z_k$ and $Z_{k+1}$: $s_k = \Lambda(L_k)$, hence by $s_k = \Lambda(d(Z_k, Z_{k+1}))$: $d(Z_k, Z_{k+1}) = L_k$. In case $k = n$ the same results from Eqn. (5.8): $d(Z_1, Z_n) = L_n$.

As a result, the polygon $Z_1 \ldots Z_n$ fulfills all requirements: it has $H$ as a circum-path, it is oriented-convex by the strict monotony of the parameter values, and the sidelengths are the given numbers $L_1, \ldots, L_n$.

**Case i.d:** $\lambda_1 + \cdots + \lambda_{n-1} < \lambda_n$.

So Eqn. (h) has a solution $x$ in the given interval. The only difference to case i.h is that from this solution a $\delta > 0$ must be determined by
\[ \cosh \delta = \frac{1}{x}. \]

The lay off procedure now is to perform on the distance line $D(\delta)$. The creation of the points $Z_1, \ldots, Z_n$ happens as above, of course with $\Lambda$ replaced by $\Psi$. By Eqn. (d), this time holds:
\[
\Psi(\delta, L_1) + \cdots + \Psi(\delta, L_{n-1}) = 2 \cosh \delta \operatorname{arsinh} \frac{\lambda_1}{\cosh \delta} + \cdots + 2 \cosh \delta \operatorname{arsinh} \frac{\lambda_{n-1}}{\cosh \delta} \\
= 2 \cosh \delta \operatorname{arsinh} \frac{\lambda_n}{\cosh \delta} = \Psi(\delta, L_n),
\]
with the additional arguments as above.

Case i.cn: \( \lambda_1 + \cdots + \lambda_{n-1} > \lambda_n \) and Eqn. \((cn)\) is solvable.

Here the arguing is somewhat different since arclength and chordlength are not always in a one to one relation. First \( x \) is fixed as a solution of Eqn. \((cn)\) in the given interval and then \( R > 0 \) is determined by

\[
\sinh R = \frac{1}{x}.
\]

The laying off is done on the distance circle \( S(R) \). Its arclength parametrization follows from Eqn. \((2.22)\) (without changing the name) as

\[
\gamma(s) = \begin{pmatrix}
\cosh R \\
\sinh R \cos \frac{s}{\sinh R} \\
\sinh R \sin \frac{s}{\sinh R}
\end{pmatrix},
\]

and, for each arguments \( s_1, s_2 \in \mathbb{R} \), one calculates from this

\[
\sinh \frac{d(\Gamma(s_1), \Gamma(s_2))}{2} = \sinh R \cdot \sin \frac{|s_2 - s_1|}{2 \sinh R}.
\]

In order to puncture \( S(R) \) suitably, the points \( Z_k \) are constructed on \( S(R) \) as follows, using a value \( s_0 > 0 \) still to be chosen:

\[
Z_k := \Gamma(s_0 + \Phi(R, L_1) + \cdots + \Phi(R, L_{k-1})), \quad k = 2, \ldots, n.
\]

For \( k = n \) one has, analogous to Eqn. \((5.9)\), this time on account of Eqn. \((cn)\):

\[
\Phi(R, L_1) + \cdots + \Phi(R, L_{n-1}) = \Phi(R, L_n),
\]

so

\[
Z_n = \Gamma(s_0 + \Phi(R, L_n)).
\]

Since always \( \Phi(R, L_k) \leq \pi \sinh R \), the \( s_0 \) can be chosen such that \( s_0 + \Phi(R, L_n) < 2\pi \sinh R \). Then the points \( Z_1, \ldots, Z_n \) form an oriented-convex polygon with vertices on the distance circle \( S(R) \), punctured at \( \Gamma(0) \). By means of Eqn. \((5.11)\) one calculates from \((5.12)\), resp. \((5.14)\):

\[
d(Z_k, Z_{k+1}) = L_k, \quad k = 1, \ldots, n-1 \quad \text{resp.} \quad d(Z_1, Z_n) = L_n.
\]

So the polygon constructed in this way has all desired properties.

Case i.cf: \( \lambda_1 + \cdots + \lambda_{n-1} > \lambda_n \) and Eqn. \((cf)\) is solvable.

The construction runs analogously to the foregoing case with the only difference that Eqn. \((5.13)\) has to be replaced by the following equation which rests on \((cf)\):

\[
\Phi(R, L_1) + \cdots + \Phi(R, L_{n-1}) = 2\pi \sinh R - \Phi(R, L_n).
\]
Then
\[
Z_n = \Gamma(s_0 2 + 2\pi \sinh R - \Phi(R, L_n)),
\]
where this time one can achieve \(s_0 + 2\pi \sinh R - \Phi(R, L_n) < 2\pi \sinh R\) by an appropriate choice of \(s_0 > 0\). The points \(Z_1, \ldots, Z_n\) then form an oriented-convex polygon with vertices on the distance circle \(S(R)\), punctured at \(\Gamma(0)\).

As to the distances, as in the foregoing case: \(d(Z_k, Z_{k+1}) = L_k, k = 1, \ldots, n - 1\). However, for \(k = n\) one has to argue differently. Again by Eqn. (5.11):
\[
\sinh \frac{d(Z_1, Z_n)}{2} = \sinh R \cdot \sin \frac{|2\pi \sinh R - \Phi(R, L_n)|}{2 \sinh R}
\]
\[
= \sinh R \cdot \sin \left(\pi - \frac{\Phi(R, L_n)}{2 \sinh R}\right)
\]
\[
= \sinh R \cdot \sin \frac{\Phi(R, L_n)}{2 \sinh R}
\]
\[
= \sinh \frac{L_n}{2}.
\]

Also this time, the polygon \(Z_1 \ldots Z_n\) has the desired properties.

For (ii): Assume \(L_n = L_1 + \cdots + L_{n-1}\). Essentially, the assertions are well-known in this degenerate case.

Existence: The points \(Z_1, \ldots, Z_n\) can be generated by laying off successively the segments of lengths \(L_1, \ldots, L_{n-1}\) along the groundline \(U_0\) in positive direction, starting e.g. from \(Z_1 := O\). The assumption then ensures that \(d(Z_1, Z_n) = L_n\) because translations along \(U_0\) preserve the distance between pre-image and image for points on \(U_0\).

Uniqueness: This amounts to show:

For points \(Z_1, \ldots, Z_n \in \mathbb{B}\), the condition

\[(n) \quad d(Z_1, Z_n) = d(Z_1, Z_2) + d(Z_2, Z_3) + \cdots + d(Z_{n-1}, Z_n)\]

is necessary and sufficient for that \(Z_1, \ldots, Z_n\) lie on a line and are monotonically arranged on it.

The necessity is obvious. The sufficiency follows by induction on \(n\). Without loss of generality, one may assume \(Z_k \neq Z_{k+1}\) for \(k = 1, \ldots, n - 1\). Otherwise identical successive points can be contracted to a single point.

Initial step \(n = 3\): From \(d(Z_1, Z_3) = d(Z_1, Z_2) + d(Z_2, Z_3)\) first follows \(Z_1 \neq Z_3\). If \(Z_2\) were not on \(Z_1 \vee Z_3\) then the hyperbolic cosine rule would imply \(d(Z_1, Z_3) < d(Z_1, Z_2) + d(Z_2, Z_3)\), a contradiction. So \(Z_1, Z_2, Z_3\) must be collinear. Moreover, if \(Z_2\) were not on the segment then one had again \(d(Z_1, Z_3) < d(Z_1, Z_2) + d(Z_2, Z_3)\), so \(Z_1, Z_2, Z_3\) have to be collinear and monotonically arranged.

Induction step from \(n - 1\) to \(n\): First, one has \(d(Z_1, Z_3) = d(Z_1, Z_2) + d(Z_2, Z_3)\). In fact, if one had \(d(Z_1, Z_3) < d(Z_1, Z_2) + d(Z_2, Z_3)\), then \(\text{[n]}\) would imply
\[
d(Z_1, Z_n) > d(Z_1, Z_3) + d(Z_3, Z_4) + \cdots + d(Z_{n-1}, Z_n) \geq d(Z_1, Z_n),
\]
5. Converse of the $n$-inequalities

a contradiction. From $d(Z_1, Z_3) = d(Z_1, Z_2) + d(Z_2, Z_3)$ follows $Z_1 \neq Z_3$ and, by the initial step, the monotonic arrangement of $Z_1, Z_2, Z_3$ on a line and also

$$d(Z_1, Z_n) = d(Z_1, Z_3) + d(Z_3, Z_4) + \cdots + d(Z_{n-1}, Z_n).$$

By the induction hypothesis, this implies the monotonic arrangement of $Z_1, Z_3, Z_4, \ldots, Z_n$ on a line. Altogether, the monotonic arrangement of $Z_1, Z_2, Z_3, \ldots, Z_n$ on a line is thus achieved.

In the present case of polygons, \( [n] \) additionally implies the strictly monotonic arrangement since all distances on the right hand side are positive.

If two such polygons $Z_1Z_2\ldots Z_n$ and $Z'_1Z'_2\ldots Z'_n$ with same sidelengths $L_k = d(Z_k, Z_{k+1}) = d(Z'_k, Z'_{k+1})$, $k = 1, \ldots, n-1$, are given then first the supporting lines can be identified and then also the point $Z_1$ with $Z'_1$ and the point $Z_2$ with $Z'_2$, by suitable motions. By the equal sidelengths and the monotonic arrangement, then also the vertices $Z_k$ will be identical with the corresponding $Z'_k$ for $k = 3, \ldots, n$.

5.4. Corollary (conversion of the generalized triangle inequality). \( \square \)

Given, for $n \geq 3$, positive real numbers $L_1, \ldots, L_n$ then the inequalities

(5.15) \[ L_k \leq L_1 + \cdots + \hat{L}_k + \cdots + L_n, \quad k = 1, \ldots, n, \]

are necessary and sufficient for that there exists a $n$-gon in the hyperbolic space $\mathbb{H}^m$ of dimension $m \geq 2$ with sidelengths $L_1, \ldots, L_n$.

Proof.

The necessity is in order by the generalized triangle inequality.

For the sufficiency, one only has to observe that the hyperbolic space $\mathbb{H}^m$ always contains one, in fact many, hyperbolic planes. So, already in such a plane there exists a polygon with the given sidelengths. It can be chosen collinear resp. cocyclic and oriented-convex (with a deliberate orientation of the plane) if in the inequalities (5.15) once resp. never occurs the equals sign.

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