ON THE PRE-IMAGE OF A POINT UNDER AN ISOGENY AND SIEGEL’S THEOREM

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Abstract. Consider a rational point on an elliptic curve under an isogeny. Suppose that the action of Galois partitions the set of its pre-images into \( n \) orbits. It is shown that all such points above a certain height have their denominator divisible by at least \( n \) distinct primes. This generalizes Siegel’s theorem and more recent results of Everest et al. For multiplication by a prime \( l \), it is shown that if \( n > 1 \) then either the point is \( l \)-times a rational point or the elliptic curve emits a rational \( l \)-isogeny.

1. Introduction

Let \((E, O)\) denote an elliptic curve defined over a number field \( K \) with Weierstrass coordinate functions \( x, y \). Siegel [24] proved that there are only finitely many \( P \in E(K) \) with \( x(P) \) belonging to the ring of integers \( O_K \). Given a finite set \( S \) of prime ideals of \( O_K \), the ring of \( S \)-integers in \( K \) is

\[ O_{KS} := \{ x \in K : \text{ord}_p(x) \geq 0 \text{ for all } p \notin S \}. \]

Mahler [21] conjectured that there are finitely many \( P \in E(K) \) with \( x(P) \in O_{KS} \) and proved his conjecture for \( K = \mathbb{Q} \). Lang [19] gave a modernized exposition and proved Mahler’s conjecture for number fields. A corollary to this is that there are finitely many \( P \in E(K) \) with \( f(P) \in O_{KS} \), where \( f \in K(E) \) is any function having a pole at \( O \) (see Corollary 3.2.2 in Chapter IX of [26]). It is unknown how much further these \( S \)-integral points can be generalized before finiteness fails. For example, in [13] it is suggested that, in rank one subgroups, only the size of \( S \) has to be fixed and not the primes in the set.

Everest, Miller and Stephens [14] proved conditionally for \( K = \mathbb{Q} \) that there are finitely many multiples \( mP \) of a non-torsion point \( P \) which have the denominator of \( x(mP) \) divisible by a single prime not belonging to a fixed set. These denominators generate an elliptic divisibility sequence, a genus-1 analogue of more classical sequences such as Fibonacci or Mersenne, and the condition, which they called magnified, is that the non-torsion point \( P \) has a preimage defined in a number field of degree less than the degree of the isogeny (see Definition 2.4). The finiteness result concerning primes in elliptic divisibility sequences was generalized to number fields under an extra assumption that the pre-image lie in a Galois extension [12]. In what follows this extra assumption is removed, there is no restriction to rank one subgroups and, analogous to the results for integral points, \( S \) and \( f \) are arbitrary (see Theorem 1.1). Moreover, using the division polynomials of \( E \), the magnified

\[ x \in K : \text{ord}_p(x) \geq 0 \text{ for all } p \notin S \}. \]
condition is replaced with a factorization criterion which can be checked more read-
ily (see Theorem 2.4). This leads to the first known proof that, as conjectured by
Everest, the condition can fail for prime degrees. In particular, either the magnified
point is $l$ times a rational point or the elliptic curve emits a rational $l$-isogeny for
some prime $l$ (see Theorem 1.2 and Corollary 1.3).

1.1. Division Polynomials. Let $E$ be an elliptic curve defined over a field $K$ with
Weierstrass coordinate functions $x, y$. For any integer $m \in \mathbb{Z}$, the $m$th division
polynomial of $E$ is the polynomial $\psi_m \in K[x, y] \subset K(E)$ as given on p. 39 of [1].
Moreover, $\psi_2 \in K[x]$ and there exists $\theta_m \in K[x]$ such that

$$[m]x = \frac{\theta_m}{\psi_m^2}.$$

Given $P \in E(K)$, define $\delta_m^P \in K[x]$ by

$$\delta_m^P = \begin{cases} \theta_m - x(P)\psi_m^2 & \text{if } P \neq O \\ \psi_m^2 & \text{otherwise.} \end{cases}$$

The zeros of $\delta_m^P$ give the values of $x(R)$ for which $mR = P$.

Theorem 1.1. Let $K$ be a number field, $S$ a finite set of prime ideals of $\mathcal{O}_K$ and
$f \in K(E)$ a function having a pole at $O$. Suppose that $\delta_m^P$ has $n$ factors over $K$ for
some $P \in E(K)$. Then for all such points of sufficiently large height,

$$(1.1) \{ \text{primes } p \notin S : \text{ord}_p(f(P)) < 0 \}$$

contains at least $n$ distinct primes.

By Siegel’s theorem, along with the generalizations of it by Mahler and Lang,
(1.1) contains at least one prime for all $P \in E(K)$ of sufficiently large height. So
Theorem 1.1 extends Siegel’s result whenever $\delta_m^P$ factorizes for some non-torsion
point $P$. In Section 3 it is shown that the points not of sufficiently large height are
$m$ times a $U$-integral point for some finite set $U$ of prime ideals of $\mathcal{O}_L$, where $U$
and $L$ are given explicitly. Quantitative results for the number of such points can
be found using [16].

In addition to being conjectured finite [12, 14, 15], the number of prime terms
in an elliptic divisibility sequence coming from a minimal Weierstrass equation is
believed to be uniformly bounded [10, 20]. Similarly, the number of terms without
a primitive divisor is believed to be uniformly bounded [11, 17, 18]. There are also
links between primitive divisors and extensions of Hilbert’s tenth problem [5, 9].
However, most results in these directions have also used that $\delta_m^P$ factorizes for some
$m$. Therefore it seems reasonable to give a detailed study of this condition.

Let $K$ be a number field, $E/K$ an elliptic curve and suppose that $\delta_m^P$ factorizes
for some non-torsion point $P \in E(K)$. Then, since $\delta_m^P$ is monic and has degree
$m^2$, $mR = P$ with $[K(R) : K] \leq m^2/2$. Assuming Lehmer’s conjecture (see [25]),
there exists a constant $c$ depending only on $E$ and $K$ so that $\hat{h}(P) = m^2\hat{h}(R) > 2c$.
However, since the constant is unknown, Lehmer’s conjecture gives no way of
knowing if the condition will fail for a given point or not. For prime degrees this
issue is resolved by the following

Theorem 1.2. Let $l$ be a prime, $E$ an elliptic curve defined over a field $K$ with
char $K \nmid l$ and $P$ a $K$-rational point on $E$. Then either

i. $\delta_l^P$ is irreducible, or
Suppose that

\[ L/K \]

is an isogeny. Suppose that \([L : K] < m \). The dual \( \hat{\phi} : E' \to E \) of \( \phi \) is defined over \( L \) and \( \sigma(R) = \phi(R) \). Then \( K(x'(R), y'(R)) = K(x'(R)) \).

**Proof.** Put \( L = K(x'(R)) \) and \( L' = K(x'(R), y'(R)) \). Then \([L' : L] \leq 2\). The assumptions on \( K \) make \( L'/L \) Galois. Suppose that \([L' : L] = 2\) and choose \( \sigma \) to be the generator of \( \text{Gal}(L'/L) \). Then \( T = \sigma(R) - R \) is in the kernel of \( \phi \) since \( \sigma(\phi(R)) - \phi(R) = O \). But \( \sigma \) fixes \( x'(R) \) so \( R + T = \pm R \). Since \( P \) is not a 2-torsion point it follows that \( \sigma(R) = R \) and \( L' = L \).

**Lemma 2.3.** Assume that \( K \) is perfect. If \( P \in E(K) \setminus E[2] \) is magnified by an isogeny \( \phi : E' \to E \) of degree \( m \) then it is magnified by \( [m] \).

**Proof.** Suppose that \( E' \), \( \phi \) and \( Q \in \phi^{-1}(P) \) are all defined over a finite extension \( L/K \) with \([L : K] < m \). The dual \( \hat{\phi} : E \to E' \) of \( \phi \) is defined over \( L \). Let \( R \in \phi^{-1}(Q) \). Lemma 2.2 gives \( L(x(R), y(R)) = L(x(R)) \). Now \( f = x' \circ \phi \in L(E) = L(x, y) \) is an even function. Hence, \( f \in L(x) \) and \( f(x) = x'(Q) \) gives a polynomial in \( L[x] \) whose roots determine the values of \( x(R) \). Since \( \#\phi^{-1}(Q) \leq \deg \phi = m \)
and $K$ is perfect, this polynomial cannot have an irreducible factor of degree larger than $m$. Thus, $[L(x(R)) : K] = [L(x(R)) : L][L : K] < m^2$. □

**Theorem 2.4.** For $K$ a perfect field and an elliptic curve $E/K$, $P \in E(K)$ is magnified if and only if $\delta^m_P$ factorizes over $K$ for some $m$.

**Proof.** If $P \in E[2]$ then $3P = P$ so $\delta^1_P$ factorizes. So assume that $P \notin E[2]$. By Lemma 2.3 $P$ is magnified if and only if it is magnified by $[m]$ for some $m > 1$. The result now follows from Lemma 2.2. □

3. PROOF OF THEOREM 1.1

**Proof of Theorem 1.1.** Suppose firstly that $f$ is a Weierstrass $x$-coordinate function. Fix a set of generators of $E(K)/mE(K)$ and for every $P_j$ in the set, adjoin to $K$ the coordinates of the points in $[m]^{-1}P_j$. Note that this finite extension $L$ does not depend on $P$ and that the splitting field of $\delta^m_P$ is contained within it. Let $U$ be a finite set of prime ideals of $\mathcal{O}_L$ containing

- those which lie above the ideals in $S$,
- those which the coefficients of the Weierstrass equation $U$-integers,
- those which make $x(T)$ a $U$-integer for all non-zero $T \in E[m]$, and
- those which make $\mathcal{O}_{LU}$ a principal ideal domain.

By the Siegel-Mahler theorem we can assume that no $R \in [m]^{-1}P$ is $U$-integral. Write $x(R) = A_R/B_R$, where $A_R$ and $B_R$ are coprime in $\mathcal{O}_{LU}$. Then

$$
(3.1) \quad x(P) = \frac{\theta_m(x(R))}{\psi_m^2(x(R))} = \frac{B_R^{2m^2} \theta_m \left( \frac{A_R}{B_R^2} \right)}{B_R^2 \left( B_R^{2(m^2-1)} \psi_m^2 \left( \frac{A_R}{B_R^2} \right) \right)},
$$

where $B_R$ is coprime with the numerator. Let $R$ and $R'$ be two distinct points in $[m]^{-1}P$. Then $R' = R + T$ for some non-zero $T \in E[m]$. From the addition formula it can be seen that $B_R$ and $B_{R'}$ are coprime in $\mathcal{O}_{LU}$. Any conjugate of a prime in the factorization of $B_R$ over $\mathcal{O}_{LU}$ divides the denominator of some element in the orbit \{ $\sigma(x(R)) : \sigma \in \text{Gal}(L/K)$ \}. Hence, using (3.1), the number of distinct prime ideals $p \notin S$ of $\mathcal{O}_K$ with $\text{ord}_p(x(P)) < 0$ is at least equal to the number of factors of $\delta^m_P$ over $K$.

Finally, suppose that $f \in K(E)$ has a pole at $O$. We may assume that a Weierstrass equation for $E/K$ is of the the form $y^2$ equal to a monic cubic in $K[x]$. Now $f \in K(C) = K(x, y)$ and $[K(x, y) : K(x)] = 2$ give

$$
\phi(x) + \psi(x)y = \frac{\phi(x) + \psi(x)y}{\eta(x)},
$$

where $\phi(x), \psi(x), \eta(x) \in K[x]$. Now $\text{ord}_O(\phi) = \text{ord}_O(x^{\deg \phi}) = -2 \deg \phi$. Similarly, $\text{ord}_O(\psi) = -2 \deg \psi$ and $\text{ord}_O(\eta) = -2 \deg \eta$. Since $O$ is a pole of $f$,

$$
\text{ord}_O(f) = \text{ord}_O(\phi + \psi y) - \text{ord}_O(\eta) < 0.
$$

But $\text{ord}_O(\phi + \psi y) \geq \min\{\text{ord}_O(\phi), \text{ord}_O(\psi) + \text{ord}_O(y)\}$ and $\text{ord}_O(y) = -3$, thus

$$
2 \deg \eta < \min\{2 \deg \phi, 2 \deg \psi + 3\}.
$$

Enlarge $S$ so that:

- $\mathcal{O}_{KS}$ is a a principal ideal domain;
- the coefficients of the Weierstrass equation are $S$-integers;
- $\phi(x), \psi(x), \eta(x) \in \mathcal{O}_{KS}[x]$ and their leading coefficients are $S$-units.
Assume that \( x(P)y(P) \neq 0 \) then \((x(P), y(P)) = (A_P/B_P^2, C_P/B_P^3)\), where \( A_P, C_P \) and \( B_P \) are coprime in \( O_{KS} \). The condition \( B_P \) gives that \( B_P \) divides the denominator and is coprime the numerator of \( f(P) \) in \( O_{KS} \). Thus the result follows from the case \( f = x \) above.

4. Proof of Theorem 1.2

The condition that \( \text{char} \, K \nmid m \) ensures that multiplication by \( m \) is separable and that \( \# [m]^{-1}P = n^2 \) (see 4.10 and 5.4 in Chapter III of [26]). Hence, for \( P \notin E[2] \) the splitting field of \( \delta_m^\tau \) is Galois over \( K \). Note that \((\mathbb{Z}/m\mathbb{Z})^2 \) is isomorphic to \( E[m] \) and bijective with \( [m]^{-1}P \). The actions of Galois on \( E[m] \) and on \( [m]^{-1}P \) are described by homomorphisms \( \text{Gal}([\bar{K}]/K) \to \text{GL}_2(\mathbb{Z}/m\mathbb{Z}) \) and \( \text{Gal}([\bar{K}/K) \to \text{AGL}_2(\mathbb{Z}/m\mathbb{Z}) \). Let \( G_m \) and \( \hat{G}_m \) be the images of these maps. Consider the homomorphism \( \alpha_m : G_m \to \hat{G}_m \) given by \( \alpha_m((A, v)) = A \).

**Lemma 4.1.** Let \( E \) be an elliptic curve defined over a field \( K \) with \( \text{char} \, K \neq 2 \) and let \( P \) be a \( K \)-rational point on \( E \). Then either

i. \( \delta_m^\tau \) is irreducible

ii. \( P \) is a 2-torsion point, or

iii. \( [2]^{-1}P \) has a \( K \)-rational point, or

iv. \( P \) is the image of a \( K \)-rational point under a \( K \)-isogeny.

**Proof.** Let \( 2R = P \). Suppose that \( P \) is not a 2-torsion point. Using Lemma 2.2 let \( L = K(x(R), y(R)) = K(x(R)) \). If \( \delta_m^\tau \) factorizes then we may choose \( R \) so that \( [L : K] \leq 2 \). If \([L : K] = 1 \) then we are in case iii. If \([L : K] = 2 \) then choose \( \sigma \in \text{Gal}(L/K) \) to be non-trivial. Then \( T = \sigma(R) - R \) is a 2-torsion point since \( \sigma(2R) - 2R = O \). Also \( T \in E(K) \) since \( \sigma(T) = -T \). Using this torsion point, we can construct an elliptic curve \( E'/K \) and a 2-isogeny \( \phi : E \to E' \) with \( \ker \phi = \{O, T\} \). Moreover, both \( \phi \) and its dual \( \hat{\phi} : E' \to E \) are defined over \( K \). Put \( \phi(R) = Q \). It follows that \( \sigma(Q) = \phi(\sigma(R)) = \phi(R + T) = \phi(R) \). Hence \( Q \in E'(K) \) and \( \hat{\phi}(Q) = P \).

Note that, for \( l = 2 \), Lemma 4.1 is stronger than Theorem 1.2.

**Proof of Theorem 1.2** If \( P \in E[2] \) then we are in case ii or iii. So assume that \( P \notin E[2] \). If \( \# \ker \alpha_l > 1 \) then there exists a non zero \( l \)-torsion point \( T \) and \( \sigma \in \text{Gal}(\bar{K}/K) \) with \( \sigma(R) = R + T \) for all \( R \in [l]^{-1}P \). Hence \( \sigma \tau \sigma^{-1}(R) = R + \tau(T) \) for any \( \tau \in \text{Gal}(\bar{K}/K) \). If \( \tau(T) \in (T) \) for all \( \tau \in \text{Gal}(\bar{K}/K) \) then we are in case ii (see 4.12 and 4.13 in Chapter III of [26]). Otherwise, Galois acts transitively on \([l]^{-1}P \) and we are in case i.

Thus, it remains to consider the case where \( \alpha_l : G_l \to G_l \) is an isomorphism and, by Lemma 4.1 \( l > 2 \). So \( \alpha_l \) has an inverse \( A \to (v \to Av + b_A) \) and the map \( \beta_l : G_l \to E[l] \) given by \( \beta_l(A) = b_A \) is a crossed homomorphism because \( \beta_l(AB) = Ab_B + b_A \). The map \( \beta_l \) is said to be principal if for some fixed \( v \in (\mathbb{Z}/l\mathbb{Z})^2 \), \( \beta_l(A) = Av - v \) for all \( A \in G_l \). The group \( H^1(G_l, E[l]) \) is the quotient of the group of crossed homomorphisms \( G_l \to E[l] \) and the group of principal ones. If \( l \) does not divide \# \( G_l \) then the orders of \( G_l \) and \( E[l] \) are coprime, so it follows that \( H^1(G_l, E[l]) = 0 \). So assume that \( l \nmid \# G_l \) and apply Proposition 15 of [28]. Either \( G_l \) is contained in a Borel subgroup and so we are in case ii since the span of a point of order \( l \) is fixed by Galois, or \( G_l \) contains \( H_l = \text{SL}_2(\mathbb{Z}/l\mathbb{Z}) \). For the second possibility construct an inflation-restriction sequence as in the proof of Lemma 4 in
Note that $H_l$ is normal since it is the kernel of the determinant on $G_l$. There is an exact sequence
\[ 0 \to H^1(G_l/H_l, E[l]^{H_l}) \to H^1(G_l, E[l]) \to H^1(H_l, E[l]). \]
For $l > 2$ the first cohomology group is trivial since $E[l]^{H_l}$ is trivial. By Lemma 3 in [5], the third cohomology group is also trivial. Hence $H^1(G_l, E[l]) = 0$ and so $\beta_l$ must be principal. But then $-v = -Av + \beta_l(A)$ for all $A \in G_l$ gives a fixed point for the action on $[l]^{-1}P$ so we are in case iii. \qed

5. Multiplication by a composite

Let $\alpha_m$ be as in Section 4. A result for all composite $m$ is

**Theorem 5.1.** Let $m > 1$ be a composite integer, $E$ an elliptic curve defined over a field $K$ with char $K \nmid m$ and $P$ a $K$-rational point on $E$. Then either

i. $\delta_m^P$ is irreducible, or

ii. $\delta_m^P$ factors, where $d > 1$ is a proper divisor of $m$, or

iii. $E$ emits a $K$-rational $l$-isogeny for some prime $l \mid m$, or

iv. $\alpha_m$ is an isomorphism.

**Proof.** If $\# \ker \alpha_m > 1$ then there exists a non-zero $m$-torsion point $T$ and $\sigma \in \text{Gal}(\overline{K}/K)$ with $\sigma(R) = R + T$ for all $R \in [m]^{-1}P$. If $T$ has order $d_1$, then write $d_1 = ld_2$ where $l$ is prime. Now $\sigma^{d_2}R = R + d_2T$ for all $R \in [m]^{-1}P$. Hence $\tau \sigma^{d_2} \tau^{-1}(R) = R + (d_2T)$ for any $\tau \in \text{Gal}(\overline{K}/K)$. Assume that $\tau(d_2T)$ is not a multiple of $d_2T$ for some $\tau \in \text{Gal}(\overline{K}/K)$; otherwise we are in case iii. Then we can always find a Galois element which will take $R$ to $R + T_1$, where $T_1$ is any $l$-torsion point. Assume that $P \notin E[2]$ and $\delta_m^P$ factors over $K$. Let $R_1, R_2 \in [m]^{-1}P$ correspond to roots of two different factors. By assumption for any $T_1 \in E[l]$, $R_2 + T_1$ corresponds to a root of the same polynomial. Thus, $\rho(R_1) - R_2$ is not a $l$-torsion point for any $\rho \in \text{Gal}(\overline{K}/K)$. So $\rho(lR_1) \neq lR_2$ for any $\rho \in \text{Gal}(\overline{K}/K)$. Since $lR_1, lR_2 \in [m/l]^{-1}P$, Galois does not act transitively on $[m/l]^{-1}P$ and so we are in case ii. \qed

Let $D_m$ be the square-free polynomial whose roots are the $x$-coordinates of the points of order $m$. Then the action of Galois on $E[m]$ is given by the Galois group of $D_m$. Note that, for $m = 4$, all of the cases in Theorem 5.1 are necessary. For example, taking the curve “117a4” with $P = (8, 36)$ we see that iv is false because the Galois groups of $\delta_2^P$ and $D_4$ have different orders; moreover, only iii is true. For the curve “55696ba1” and the generator Cremona gives, by checking that the curve has a trivial isogeny class, we see that only iv is true. When $m$ has two coprime proper divisors we have

**Theorem 5.2.** Suppose that $m > 1$ is composite and $m = d_1d_2$ where $d_1, d_2$ are coprime proper divisors. If $\delta_m^P$ factorizes then either $\delta_{d_1}^P$ or $\delta_{d_2}^P$ factorizes.

**Proof.** There exists $x, y \in \mathbb{Z}$ such that $xd_1 + yd_2 = 1$. Consider the homomorphism $G_m \to G_{d_1} \times G_{d_2}$ given by $\rho \to (\rho, \rho)$. If $\rho$ is in the kernel of this map then $\rho(d_2R) = d_2R$ and $\rho(d_1R) = d_1R$ for all $R \in [m]^{-1}P$. But then $x\rho(d_1R) + y\rho(d_2R) = \rho(R) = R$ for all $R \in [m]^{-1}P$. So $G_m \cong G_{d_1} \times G_{d_2}$. Assume that $P \notin E[2]$ and $\delta_m^P$ is irreducible. Then for any $R \in [m]^{-1}P$ and $T \in E[d_1]$ there exists $\sigma \in G_{d_1}$ with $\sigma(d_2R) = d_2R + T$. Define $(\sigma, \text{Id})$ by $(\sigma, \text{Id})(R) = n_2\sigma(d_2R) + n_1(d_1R)$. Since $G_m \cong G_{d_1} \times G_{d_2}$, $(\sigma, \text{Id}) \in G_m$. For any $R \in [m]^{-1}P$, $(\sigma, \text{Id})(R) = R + n_2T$. So,
since $d_1$ and $n_2$ are coprime, $R$ and $R + T$ must correspond to roots of the same polynomial. Suppose that $\delta_m^P$ factorizes and let $R_1, R_2 \in [m]^{-1}P$ correspond to roots of two different factors. Then $\rho(R_1) - R_2 \notin E[d_1]$ or $\rho(d_1 R_1) \neq \rho(d_1 R_2)$ for all $\rho \in \text{Gal}(\bar{K}/K)$. Since $d_1 R_1, d_1 R_2 \in [d_2]^{-1}P$ it follows that $\delta_{d_2}^P$ factorizes. □

Hence the case where $m$ is a composite prime power remains. Although no further results could be proven it is perhaps worth noting that, in all of Cremona's data, an example where i and ii are false in Theorem 5.1 could not be found when $4 < m \leq 25$.

References

1. I. F. Blake, G. Seroussi, and N. P. Smart, *Elliptic curves in cryptography*, London Mathematical Society Lecture Note Series, vol. 265, Cambridge University Press, 2000.

2. Wieb Bosma, John Cannon, and Catherine Playoust, *The Magma algebra system. I. The user language*, J. Symbolic Comput. 24 (1997), no. 3-4, 235–265.

3. M. L. Brown, *A note on orders of isogenies of elliptic curves*, Bull. London Math. Soc. 20 (1988), no. 3, 221–227.

4. Henri Cohen, *Number theory. Vol. I. Tools and Diophantine equations*, Graduate Texts in Mathematics, vol. 239, Springer, New York, 2007.

5. Gunther Cornelissen and Karim Zahidi, *Elliptic divisibility sequences and undecidable problems about rational points*, J. Reine Angew. Math. 613 (2007), 1–33.

6. J. E. Cremona, *Elliptic curve data*, available at http://www.warwick.ac.uk/staff/j.e.cremona/ftp/data/.

7. J. E. Cremona, *Algorithms for modular elliptic curves*, Cambridge University Press, 1997.

8. Z. Djabri, Edward F. Schaefer, and N. P. Smart, *Computing the $p$-Selmer group of an elliptic curve*, Trans. Amer. Math. Soc. 352 (2000), no. 12, 5583–5597.

9. Kirsten Eisenträger and Graham Everest, *Descent on elliptic curves and Hilbert's tenth problem*, Proc. Amer. Math. Soc. 137 (2009), no. 6, 1951–1959.

10. Graham Everest, Patrick Ingram, Valéry Mahé, and Shaun Stevens, *The uniform primality conjecture for elliptic curves*, Acta Arith. 134 (2008), no. 2, 157–181.

11. Graham Everest, Patrick Ingram, and Shaun Stevens, *Primitive divisors on twists of Fermat’s cubic*, LMS J. Comput. Math. 12 (2009), 54–81.

12. Graham Everest and Helen King, *Prime powers in elliptic divisibility sequences*, Math. Comp. 74 (2005), no. 252, 2061–2071 (electronic).

13. Graham Everest and Valéry Mahé, *A generalization of Siegel’s theorem and Hall’s conjecture*, Experiment. Math. 18 (2009), no. 1, 1–9.

14. Graham Everest, Victor Miller, and Nelson Stephens, *Primes generated by elliptic curves*, Proc. Amer. Math. Soc. 132 (2004), no. 4, 955–963 (electronic).

15. Graham Everest, Jonathan Reynolds, and Shaun Stevens, *On the denominators of rational points on elliptic curves*, Bull. Lond. Math. Soc. 39 (2007), no. 5, 762–770.

16. Robert Gross and Joseph Silverman, *S-integral points on elliptic curves*, Pacific J. Math. 167 (1995), no. 2, 263–288.

17. Patrick Ingram, *Elliptic divisibility sequences over certain curves*, J. Number Theory 123 (2007), no. 2, 473–486.

18. Patrick Ingram and Joseph H. Silverman, *Uniform estimates for primitive divisors in elliptic divisibility sequences*, to appear in a forthcoming memorial volume for Serge Lang, published by Springer-Verlag.

19. Serge Lang, *Integral points on curves*, Inst. Hautes Études Sci. Publ. Math. (1960), no. 6, 27–43.

20. Valéry Mahé, *Prime power terms in elliptic divisibility sequences*, submitted.

21. Kurt Mahler, *Über die rationalen Punkte auf Kurven vorn Geschlecht Eins*, J. Reine Angew. Math. 170 (1934), 168–178.

22. B. Mazur, *Rational isogenies of prime degree (with an appendix by D. Goldfeld)*, Invent. Math. 44 (1978), no. 2, 129–162.

23. Jean-Pierre Serre, *Propriétés galoisiennes des points d’ordre fini des courbes elliptiques*, Invent. Math. 15 (1972), no. 4, 259–331.
24. Carl Ludwig Siegel, Über einige Anwendungen Diophantischer Approximationen, Abh. Preussischen Akademie der Wissenschaften (1929).
25. Joseph H. Silverman, A lower bound for the canonical height on elliptic curves over abelian extensions, J. Number Theory 104 (2004), no. 2, 353–372.
26. ———, The arithmetic of elliptic curves, Graduate Texts in Mathematics, vol. 106, Springer, 2009.
27. Jacques Vélu, Isogénies entre courbes elliptiques, C. R. Acad. Sci. Paris Sér. A-B 273 (1971), A238–A241.

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