OPEN EMBEDDINGS AND PSEUDOFLAT EPIMORPHISMS

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Dedicated to Professor Alexander Ya. Helemskii on the occasion of his 75th birthday

ABSTRACT. We characterize open embeddings of Stein spaces and of \(C^\infty\)-manifolds in terms of certain flatness-type conditions on the respective homomorphisms of function algebras.

1. Introduction

Our main motivation comes from the following fact in algebraic geometry. If \((X, \mathcal{O}_X)\) and \((Y, \mathcal{O}_Y)\) are affine schemes, then a morphism \(f: (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)\) is an open embedding if and only if the respective homomorphism \(f^*: \mathcal{O}(X) \to \mathcal{O}(Y)\) is a flat epimorphism of finite presentation [24, 17.9.1]. We are interested in complex analytic and smooth versions of this result. Specifically, given a morphism \(f: (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)\) of Stein spaces, we are looking for a condition on \(f^*: \mathcal{O}(X) \to \mathcal{O}(Y)\) that is necessary and sufficient for \(f\) to be an open embedding. A similar question makes sense for \(C^\infty\)-manifolds. To get a reasonable answer, we equip the algebras of holomorphic and smooth functions with their canonical Fréchet space topologies and consider them as functional analytic objects [27,28].

It is easy to see that the above-mentioned algebraic result does not extend verbatim to the complex analytic case. Indeed, if \(U\) is an open subset of a Stein space \((X, \mathcal{O}_X)\), then \(\mathcal{O}(U)\) is normally not flat as a Fréchet \(\mathcal{O}(X)\)-module. This observation is essentially due to M. Putinar [12] (see also [16]) and is closely related to the spectral theory of linear operators on Banach spaces. Actually, if \(\mathcal{O}(U)\) were flat over \(\mathcal{O}(\mathbb{C})\) for every open subset \(U \subset \mathbb{C}\), then each Banach space operator would possess Bishop’s property \((\beta)\), which is not the case [3]. For a direct proof of the fact that \(\mathcal{O}(\mathbb{D})\) is not flat over \(\mathcal{O}(\mathbb{C})\) (where \(\mathbb{D} \subset \mathbb{C}\) is the open unit disc), see [36].

A reasonable substitute for the flatness property was introduced by J. L. Taylor [51]. Given a continuous homomorphism \(\varphi: A \to B\) of Fréchet algebras, he says that \(\varphi\) is a localization if, for each Fréchet \(B\)-bimodule \(M\), the induced map of the continuous Hochschild homology \(H_*(A, M) \to H_*(B, M)\) is an isomorphism. Taylor also proved that, if \(A\) and \(B\) are nuclear, then the above condition means precisely that (i) \(\text{Tor}_i^A(B, B) = 0\) for all \(i \geq 1\), and (ii) \(\text{Tor}_0^A(B, B) \cong B\) canonically. Homomorphisms satisfying (i) and (ii) were rediscovered several times under different names [1,12,19,31,33], both in the purely algebraic and in the functional analytic contexts (see Remark 3.15 for historical details). We adopt the terminology of [19] and call such maps homological epimorphisms. To be more precise, there are two types of homological epimorphisms in the functional analytic setting, weak and strong homological epimorphisms. For nuclear Fréchet algebras, weak
homological epimorphisms are the same as Taylor’s localizations, while strong homological epimorphisms are the same as Taylor’s absolute localizations. See Section 3 for details.

The fundamental (and chronologically the first) example of a weak homological epimorphism that is not necessarily flat is the restriction map $\mathcal{O}(\mathbb{C}^n) \to \mathcal{O}(U)$, where $U$ is a Stein open subset of $\mathbb{C}^n$ (i.e., a domain of holomorphy). This fact was proved by Taylor [51, Prop. 4.3] and was the main motivation for him to introduce weak homological epimorphisms. The second author [36, Theorem 3.1] observed that the same result holds if we replace $\mathbb{C}^n$ by an arbitrary Stein manifold. Recently, F. Bambozzi, O. Ben-Bassat and K. Kremnizer [2], working in the setting of bornological algebras, proved that the above property actually characterizes open embeddings of Stein spaces (not only over $\mathbb{C}$).

Other examples of homological epimorphisms in the functional analytic context can be found in [13–15, 38, 39, 51, 52].

In the present paper, we introduce a wider class of Fréchet algebra homomorphisms $A \to B$ that we call $n$-pseudoflat epimorphisms (where $n$ is a fixed nonnegative integer). Such homomorphisms are defined by the conditions that $\text{Tor}_i^A(B, B) = 0$ for all $1 \leq i \leq n$ and $\text{Tor}_0^A(B, B) \cong B$ canonically. For $n = 1$, pseudoflat epimorphisms were introduced by G. M. Bergman and W. Dicks [5] in the purely algebraic setting. They also appear naturally in $[\mathbb{1}, \mathbb{2}, \mathbb{3}]$, for example. As far as we know, pseudoflat epimorphisms were not considered before in the functional analytic framework. Our main results are Theorems 4.1 and 5.3, which characterize open embeddings of Stein spaces and of smooth manifolds in terms of pseudoflat epimorphisms.

The paper is organized as follows. Section 2 contains some preliminaries from homological algebra in categories of Fréchet modules. Our main reference is $[27]$; some facts that are missing in $[24]$ can be found in $[\mathbb{16}, \mathbb{39}, \mathbb{54}]$. In Section 3, we introduce $n$-pseudoflat epimorphisms of Fréchet algebras, give some examples, and characterize epimorphisms, 0-pseudoflat epimorphisms, and 1-pseudoflat epimorphisms in terms of noncommutative differential forms. In particular, we show that not every Fréchet algebra epimorphism is 0-pseudoflat (in contrast to the purely algebraic case). Our main results are contained in Sections 4 and 5. In Section 4, we show that a map $f : (\mathcal{O}(Y), \mathcal{O}_Y) \to (\mathcal{O}(X), \mathcal{O}_X)$ of Stein spaces is an open embedding if and only if the respective homomorphism $f^* : \mathcal{O}(X) \to \mathcal{O}(Y)$ is a 1-pseudoflat epimorphism. Some other equivalent homological conditions on $f^*$ are also given. This is a partial generalization of the main result of $[\mathbb{3}]$. However, in contrast to $[\mathbb{3}]$, we work only over $\mathbb{C}$, and we deal with topological (rather than bornological) algebras. In Section 5, we show that a similar result holds for the algebras of $C^\infty$-functions on smooth real manifolds. Section 6 contains some remarks and open questions related to function algebras on Stein spaces and on $C^\infty$-differentiable spaces.

2. Preliminaries

Throughout, all vector spaces and algebras are assumed to be over the field $\mathbb{C}$ of complex numbers. All algebras are assumed to be associative and unital. By a Fréchet algebra we mean an algebra $A$ equipped with a complete, metrizable locally convex topology (i.e., $A$ is an algebra and a Fréchet space simultaneously) such that the multiplication $A \times A \to A$ is continuous. A left Fréchet $A$-module is a left $A$-module $M$ equipped with a complete, metrizable locally convex topology in such a way that the action $A \times M \to M$ is continuous. We always assume that $1_A \cdot x = x$ for all $x \in M$, where $1_A$ is the identity of $A$. Left Fréchet $A$-modules and their continuous morphisms form a category denoted by $A$-mod. The
categories \text{mod-}A \text{ and } A\text{-mod-}A \text{ of right Fréchet } A\text{-modules and of Fréchet } A\text{-bimodules are defined similarly. Note that } A\text{-mod-}A \cong A^e\text{-mod-}A^e \cong \text{mod-}A^e, \text{ where } A^e = A \hat{\otimes} A^{op}, \text{ and where } A^{op} \text{ stands for the algebra opposite to } A. \text{ The space of morphisms from } M \text{ to } N \text{ in } A\text{-mod} \text{ (respectively, in } \text{mod-}A, \text{ in } A\text{-mod-}A \text{) will be denoted by } _A\text{h}(M, N) \text{ (respectively, } _A\text{h}_A(M, N)). \text{ Given Fréchet algebras } A \text{ and } B, \text{ we denote by } \text{Hom}(A, B) \text{ the set of all continuous algebra homomorphisms from } A \text{ to } B.

If } M \text{ is a right Fréchet } A\text{-module and } N \text{ is a left Fréchet } A\text{-module, then their } A\text{-module tensor product } M \otimes_A N \text{ is defined to be the quotient } (M \hat{\otimes} N)/L, \text{ where } L \subset M \otimes N \text{ is the closed linear span of all elements of the form } x \cdot a \otimes y - x \otimes a \cdot y (x \in M, y \in N, a \in A). \text{ As in pure algebra, the } A\text{-module tensor product can be characterized by the universal property that, for each Fréchet space } E, \text{ there is a natural bijection between the set of all continuous } A\text{-balanced bilinear maps from } M \times N \to E \text{ and the set of all continuous linear maps from } M \hat{\otimes}_A N \to E.

A chain complex } C_\bullet = (C_n, d_n)_{n \in \mathbb{Z}} \text{ of Fréchet } A\text{-modules is admissible if it splits in the category of topological vector spaces, i.e., if it has a contracting homotopy consisting of continuous linear maps. Geometrically, this means that } C_n \text{ is exact, and } \text{Ker } d_n \text{ is a complemented subspace of } C_n \text{ for each } n. \text{ A left Fréchet } A\text{-module } P \text{ is projective if the functor } _A\text{h}(P, -): A\text{-mod} \to \text{Vect} \text{ (where } \text{Vect} \text{ is the category of vector spaces and linear maps) is exact in the sense that it takes admissible sequences of Fréchet } A\text{-modules to exact sequences of vector spaces. Similarly, a left Fréchet } A\text{-module } F \text{ is flat if the tensor product functor } (-) \otimes_A F: \text{mod-}A \to \text{Vect} \text{ is exact in the same sense as above. It is known that every projective Fréchet module is flat.}

A projective resolution of } M \in A\text{-mod} \text{ is a pair } (P_\bullet, \varepsilon) \text{ consisting of a nonnegative chain complex } P_\bullet = (P_n, d_n)_{n \geq 0} \text{ in } A\text{-mod} \text{ and a morphism } \varepsilon: P_0 \to M \text{ such that the sequence } 0 \leftarrow M \leftarrow P_\bullet \text{ is an admissible complex and such that all the modules } P_n \text{ (} n \geq 0 \text{) are projective. It is a standard fact that } A\text{-mod} \text{ has enough projectives, i.e., each left Fréchet } A\text{-module has a projective resolution. The same is true of } \text{mod-}A \text{ and } A\text{-mod-}A. \text{ In particular, the (unnormalized) bimodule bar resolution of } A \text{ [27, Section III.2.3] looks as follows:}

\begin{equation}
0 \leftarrow A \overset{\mu_A}{\leftarrow} A \otimes A \leftarrow A \hat{\otimes} A \leftarrow \cdots \leftarrow A \hat{\otimes}^n A \leftarrow \cdots
\end{equation}

Here } \mu_A \text{ is the multiplication map, and } d: A \hat{\otimes}^3 \to A \hat{\otimes}^2 \text{ is given by}

\begin{equation}
d(a \otimes b \otimes c) = ab \otimes c - a \otimes bc \quad (a, b, c \in A).
\end{equation}

The explicit formula for the higher differentials } A \hat{\otimes}^{(n+1)} \to A \hat{\otimes}^n \text{ is similar [loc. cit.]; we do not need it here. The augmented complex } (\mathbb{I}) \text{ is a projective resolution of } A \text{ in } A\text{-mod-}A. \text{ For each } M \in A\text{-mod}, \text{ applying } (-) \otimes_A M \text{ to } (\mathbb{I}) \text{ yields a projective resolution of } M \text{ in } A\text{-mod}, \text{ the bar resolution of } M \text{ [loc. cit.].}

If } M \in \text{mod-}A \text{ and } N \in A\text{-mod}, \text{ then the space } \text{Tor}^A_n(M, N) \text{ is defined to be the } n\text{th homology of the complex } P_\bullet \hat{\otimes}_A N, \text{ where } P_\bullet \text{ is a projective resolution of } M. \text{ Equivalently, } \text{Tor}^A_n(M, N) \text{ is the } n\text{th homology of the complex } M \hat{\otimes}_A Q_\bullet, \text{ where } Q_\bullet \text{ is a projective resolution of } N. \text{ The spaces } \text{Tor}^A_n(M, N) \text{ do not depend on the particular choices of } P_\bullet \text{ and } Q_\bullet \text{ and have the usual functorial properties (see } [27, \text{ Section III.4.4]} \text{ for details). Note that } \text{Tor}^A_0(M, N) \text{ is not necessarily Hausdorff, but the associated Hausdorff space (i.e., the quotient of } \text{Tor}^A_0(M, N) \text{ modulo the closure of zero) is a Fréchet space. If

\footnote{Some authors (see, e.g., [16, 30, 43, 50]) define } M \hat{\otimes}_A N \text{ in a different way. Actually, their } M \hat{\otimes}_A N \text{ is our } \text{Tor}^A_0(M, N) \text{ (see below). We adopt the definition given by M. A. Rieffel [49] (see also [8,10,27,28,47]).}
$M \in A\text{-}mod$, then the $n$th Hochschild homology of $A$ with coefficients in $M$ is defined by $\mathcal{H}_n(A, M) = \text{Tor}_n^A(M, A)$.

In contrast to the purely algebraic case, $\text{Tor}_0^A(M, N)$ is not the same as $M \hat{\otimes}_A N$. Nevertheless, there is a natural continuous open linear surjection

$$\alpha_{M,N} : \text{Tor}_0^A(M, N) \to M \hat{\otimes}_A N,$$

whose kernel is the closure of zero in $\text{Tor}_0^A(M, N)$ [27, III.4.27]. In other words, $M \hat{\otimes}_A N$ is isomorphic to the Hausdorff space associated to $\text{Tor}_0^A(M, N)$. Hence the following equivalences hold:

$$\text{Tor}_0^A(M, N) \text{ is Hausdorff} \iff \alpha_{M,N} \text{ is injective} \iff \alpha_{M,N} \text{ is bijective} \iff \alpha_{M,N} \text{ is a topological isomorphism}.$$  (4)

Under some nuclearity assumptions, the derived functor Tor can be calculated with the help of exact (not necessarily admissible) sequences of projective modules. The following result is an easy modification of [16, Corollary 3.1.13] (which, in turn, goes back to [50, Proposition 4.5]).

**Proposition 2.1.** Let $A$ be a Fréchet algebra, $M \in \text{mod}_A$, and $N \in A\text{-}mod$. Suppose that

$$0 \leftarrow M \leftarrow P_0 \leftarrow P_1 \leftarrow \cdots \leftarrow P_n \leftarrow P_{n+1}$$

is an exact sequence in $\text{mod}_A$ such that $P_0, \ldots, P_n$ are projective. Assume that one of the following conditions holds:

(i) $P_0, \ldots, P_{n+1}$ are nuclear;
(ii) $A$ and $N$ are nuclear.

Then for each $m = 0, \ldots, n$ the space $\text{Tor}_m^A(M, N)$ is topologically isomorphic to the $m$th homology of the complex $P_\bullet \hat{\otimes}_A N$. In particular, if either $M$ or $N$ is flat, then the tensored sequence

$$0 \leftarrow M \hat{\otimes}_A N \leftarrow P_0 \hat{\otimes}_A N \leftarrow \cdots \leftarrow P_{n+1} \hat{\otimes}_A N$$

is exact.

Given a Fréchet algebra $A$ and a Fréchet $A$-bimodule $M$, we let $\text{Der}(A, M)$ denote the space of all continuous derivations of $A$ with values in $M$. The bimodule of noncommuta-tive differential 1-forms over $A$ is a Fréchet $A$-bimodule $\Omega^1 A$ together with a derivation $d_A : A \to \Omega^1 A$ such that for each Fréchet $A$-bimodule $M$ and each derivation $D : A \to M$ there exists a unique $A$-bimodule morphism $\Omega^1 A \to M$ making the following diagram commute:

$$\begin{array}{ccc}
\Omega^1 A & \longrightarrow & M \\
\downarrow d_A & & \downarrow D \\
A & \rightarrow & \\
\end{array}$$

In other words, we have a natural isomorphism

$$\mathcal{A}h_A(\Omega^1 A, M) \cong \text{Der}(A, M) \quad (M \in A\text{-}mod).$$

It is a standard fact (see, e.g., [33]) that $\Omega^1 A$ exists and is isomorphic to the kernel of the multiplication map $\mu_A : A \otimes A \to A$. Under the above identification, the universal
derivation \(d_A: A \to \Omega^1 A\) acts by the rule \(d_A(a) = 1 \otimes a - a \otimes 1 (a \in A)\). Thus we have an exact sequence
\[
0 \to \Omega^1 A \xrightarrow{j_A} A \hat{\otimes} A \xrightarrow{\mu_A} A \to 0
\]
in \(A\text{-mod}\), where \(j_A\) is uniquely determined by \(j_A(d_A(a)) = 1 \otimes a - a \otimes 1 (a \in A)\). Note that (3) splits in \(A\text{-mod}\) and in \(mod\-A\) (\[3\]), cf. also \([3]\). In particular, (3) is admissible.

3. Pseudoflat epimorphisms

We begin this section with the following “truncated” version of the transversality relation \(\bot_A\) introduced in \([3]\) (see also \([11,16,43]\)).

**Proposition 3.1.** Let \(A\) be a Fréchet algebra, \(M \in \text{mod}\-A, N \in A\text{-mod}, \) and \(n \in \mathbb{Z}_+\).

Then the following conditions are equivalent:

(i) \(\text{Tor}^A_m(M,N) = 0\) for \(1 \leq m \leq n\), and \(\text{Tor}^A_0(M,N)\) is Hausdorff;

(ii) for some (or, equivalently, for each) projective resolution \(0 \leftarrow M \leftarrow P_\bullet \) in \(A\text{-mod}\) the sequence
\[
0 \leftarrow M \hat{\otimes} N \leftarrow P_0 \hat{\otimes} N \leftarrow \cdots \leftarrow P_{n+1} \hat{\otimes} N
\]
is exact;

(iii) for some (or, equivalently, for each) projective resolution \(0 \leftarrow N \leftarrow Q_\bullet \) in \(A\text{-mod}\) the sequence
\[
0 \leftarrow M \hat{\otimes} N \leftarrow M \hat{\otimes} Q_0 \leftarrow \cdots \leftarrow M \hat{\otimes} Q_{n+1}
\]
is exact;

(iv) for some (or, equivalently, for each) projective resolution \(0 \leftarrow A \leftarrow L_\bullet \) in \(A\text{-mod}\) the sequence
\[
0 \leftarrow M \hat{\otimes} N \leftarrow M \hat{\otimes} L_0 \hat{\otimes} N \leftarrow \cdots \leftarrow M \hat{\otimes} L_{n+1} \hat{\otimes} N
\]
is exact.

**Proof.** The equivalences between “for some” and “for each” in (ii)–(iv) are immediate from the fact that all projective resolutions of a module are homotopy equivalent.

(i) \(\iff\) (ii). Since \(\hat{\otimes}_{A}\) preserves surjections \([27,II.4.12]\), (3) is always exact at \(M \hat{\otimes}_{A} N\). If \(1 \leq m \leq n\), then (3) is exact at \(P_m \hat{\otimes}_{A} N\) if and only if \(\text{Tor}^A_m(M,N) = 0\). On the other hand, (3) is exact at \(P_0 \hat{\otimes}_{A} N\) if and only if
\[
\ker(P_0 \hat{\otimes}_{A} N \to \text{Tor}^A_0(M,N)) = \ker(P_0 \hat{\otimes}_{A} N \to M \hat{\otimes}_{A} N),
\]
i.e., if and only if the canonical map \(\text{Tor}^A_0(M,N) \to M \hat{\otimes}_{A} N\) is injective. By (3), the latter condition holds if and only if \(\text{Tor}^A_0(M,N)\) is Hausdorff.

(i) \(\iff\) (iii). This is similar to (i) \(\iff\) (ii).

(iii) \(\iff\) (iv). If \(0 \leftarrow A \leftarrow L_\bullet\) is a projective resolution of \(A\) in \(A\text{-mod}\), then \(0 \leftarrow N \leftarrow L_\bullet \hat{\otimes}_{A} N\) is a projective resolution of \(N\) in \(A\text{-mod}\). The rest is clear. \(\square\)

**Definition 3.2.** Let \(A\) be a Fréchet algebra, \(M \in \text{mod}\-A, N \in A\text{-mod}, \) and \(n \in \mathbb{Z}_+\).

We say that \(M\) and \(N\) are \(n\)-transversal over \(A\) (and write \(M \bot^n_A N\)) if the (equivalent) conditions of Proposition \([3]\) are satisfied. If \(M \bot^n_A N\) for all \(n \in \mathbb{Z}_+\), then \(M\) and \(N\) are said to be transversal \([3]\) (see also \([11,16,43]\)). In this case, we write \(M \bot_A N\).

**Corollary 3.3.** Let \(A\) be a Fréchet algebra, \(M \in \text{mod}\-A, N \in A\text{-mod}, \) and \(n \in \mathbb{Z}_+\). Then the following conditions are equivalent:
Following conditions are equivalent:

(i) \( M \perp_A N \);
(ii) \((N \otimes M) \perp_A A\);
(iii) \(\mathcal{H}_m(A, N \otimes M) = 0\) for \(1 \leq m \leq n\), and \(\mathcal{H}_0(A, N \otimes M)\) is Hausdorff.

**Proof.** This is immediate from Proposition 3.1 and from the isomorphisms \(\text{Tor}^A_m(M, N) \cong \mathcal{H}_m(A, N \otimes M)\) [27, III.4.25]. □

Here is our main definition.

**Definition 3.4.** Let \( \varphi : A \rightarrow B \) be a Fréchet algebra homomorphism, and let \( n \in \mathbb{Z}_+ \). We say that \( \varphi \) is \( n \)-pseudoflat if \( B \perp_A^n B \).

We are mostly interested in those pseudoflat homomorphisms which are epimorphisms (in the category-theoretic sense). For the reader’s convenience, let us recall the following well-known fact (see, e.g., [19, Prop. XI.1.2], [38, Prop. 6.1]).

**Proposition 3.5.** Let \( \varphi : A \rightarrow B \) be a homomorphism of Fréchet algebras. Then the following conditions are equivalent:

(i) \( \varphi \) is an epimorphism in the category of Fréchet algebras;
(ii) the multiplication map \( \mu_{B,A} : B \otimes_A B \rightarrow B \) is a topological isomorphism;
(iii) for each \( M \in \text{mod-}B \) and each \( N \in B\text{-mod} \), the canonical map \( M \otimes_A N \rightarrow M \otimes_B N \) is a topological isomorphism.

Given a Fréchet algebra homomorphism \( \varphi : A \rightarrow B \), we define \( \bar{\mu}_{B,A} : \text{Tor}^A_0(B,B) \rightarrow B \) to be the composition of the canonical map \( \alpha = \alpha_{B,B} : \text{Tor}^A_0(B,B) \rightarrow B \otimes_A B \) (see (3)) and the multiplication map \( \mu_{B,A} : B \otimes_A B \rightarrow B \).

**Lemma 3.6.** The following conditions are equivalent:

(i) \( \varphi \) is a 0-pseudoflat epimorphism;
(ii) \( \bar{\mu}_{B,A} : \text{Tor}^A_0(B,B) \rightarrow B \) is bijective;
(iii) \( \bar{\mu}_{B,A} : \text{Tor}^A_0(B,B) \rightarrow B \) is a topological isomorphism.

**Proof.** (i) \( \Rightarrow \) (iii). The fact that \( \varphi \) is 0-pseudoflat means precisely that \( \text{Tor}^A_0(B,B) \) is Hausdorff, which happens if and only if \( \alpha : \text{Tor}^A_0(B,B) \rightarrow B \otimes_A B \) is a topological isomorphism (see (4)). On the other hand, the fact that \( \varphi \) is an epimorphism means precisely that \( \mu_{B,A} : B \otimes_A B \rightarrow B \) is a topological isomorphism (see Proposition 3.3). Hence so is \( \bar{\mu}_{B,A} = \mu_{B,A} \circ \alpha \).

(iii) \( \Rightarrow \) (ii). This is clear.

(ii) \( \Rightarrow \) (i). Since \( \bar{\mu}_{B,A} \) is continuous and bijective, we conclude that \( \text{Tor}^A_0(B,B) \) is Hausdorff (i.e., \( \varphi \) is 0-pseudoflat). Hence \( \alpha \) is bijective by (4), and so \( \mu_{B,A} \) is a topological isomorphism by the Open Mapping Theorem. Applying Proposition 3.3, we see that \( \varphi \) is an epimorphism. □

Let \( \varphi : A \rightarrow B \) be a Fréchet algebra homomorphism, and let \( 0 \leftarrow A \leftarrow L_\bullet \) be a projective resolution in \( A\text{-mod-}A \). Applying \( B \otimes_A (-) \otimes_A B \) to \( L_0 \rightarrow A \) and composing with the multiplication \( \mu_{B,A} : B \otimes_A B \rightarrow B \), we get a \( B\text{-bimodule morphism} \) \( \varepsilon_L : B \otimes_A L_0 \otimes_A B \rightarrow B \). Similarly, if \( 0 \leftarrow B \leftarrow P_\bullet \) and \( 0 \leftarrow B \leftarrow Q_\bullet \) are projective resolutions of \( B \) in \( \text{mod-}A \) and in \( A\text{-mod} \), respectively, then we have morphisms \( \varepsilon_P : P_0 \otimes_A B \rightarrow B \) in \( \text{mod-}B \) and \( \varepsilon_Q : B \otimes_A Q_0 \rightarrow B \) in \( B\text{-mod} \).
Proposition 3.7. Let $\varphi: A \to B$ be a Fréchet algebra homomorphism. The following conditions are equivalent:

(i) $\varphi$ is an $n$-pseudoflat epimorphism;
(ii) for some (or, equivalently, for each) projective resolution $0 \leftarrow B \leftarrow P_\bullet$ in $\text{mod-}A$ the sequence

$$0 \leftarrow B \leftarrow P_0 \hat{\otimes}_A B \leftarrow \cdots \leftarrow P_{n+1} \hat{\otimes}_A B$$

is exact;
(iii) for some (or, equivalently, for each) projective resolution $0 \leftarrow B \leftarrow Q_\bullet$ in $A\text{-mod}$ the sequence

$$0 \leftarrow B \leftarrow B \hat{\otimes}_A Q_0 \leftarrow \cdots \leftarrow B \hat{\otimes}_A Q_{n+1}$$

is exact;
(iv) for some (or, equivalently, for each) projective resolution $0 \leftarrow A \leftarrow L_\bullet$ in $A\text{-mod-}A$ the sequence

$$0 \leftarrow B \leftarrow B \hat{\otimes}_A L_0 \hat{\otimes}_A B \leftarrow \cdots \leftarrow B \hat{\otimes}_A L_{n+1} \hat{\otimes}_A B$$

is exact.

Proof. (i) $\iff$ (ii). Clearly, if $m \geq 1$, then (8) is exact at $P_m \hat{\otimes}_A B$ if and only if $\text{Tor}^A_m(B,B) = 0$. Since the 0th homology of $P_\bullet \hat{\otimes}_A B$ is precisely $\text{Tor}^A_0(B,B)$, we see that (8) is exact at $P_0 \hat{\otimes}_A B$ if and only if $\bar{\mu}_{B,A} : \text{Tor}^A_0(B,B) \to B$ is bijective. Now the result follows from Lemma 3.6.

Equivalences (i) $\iff$ (iii) and (i) $\iff$ (iv) are proved similarly. □

Corollary 3.8. Let $\varphi: A \to B$ be a Fréchet algebra homomorphism. Define

$$d_\varphi : B \hat{\otimes}_A B \to B \hat{\otimes} B, \quad b \otimes a \otimes c \mapsto b\varphi(a) \otimes c - b \otimes \varphi(a)c \quad (b,c \in B, \ a \in A).$$

Then $\varphi$ is a 0-pseudoflat epimorphism if and only if the sequence

$$0 \leftarrow B \leftarrow B \hat{\otimes}_A A \hat{\otimes}_A B$$

is exact.

Proof. If $0 \leftarrow A \leftarrow L_\bullet$ is the bimodule bar resolution of $A$ (see (II)), then (9) for $n = 0$ is precisely (I). □

Corollary 3.9. A surjective Fréchet algebra homomorphism is a 0-pseudoflat epimorphism.

Proof. If we replace $A$ by $B$ in (II), then we get an exact sequence (in fact, this is the low-dimensional segment of the bimodule bar resolution for $B$). Since $\varphi: A \to B$ is onto, we conclude that (II) is exact as well. □

Remark 3.10. As we shall see below (Example 3.22), a Fréchet algebra homomorphism with dense image, while being an epimorphism for an obvious reason, is not necessarily 0-pseudoflat.

The next proposition (which is a Fréchet algebra version of [5, (87)]) emphasizes the difference between 0-pseudoflat and 1-pseudoflat epimorphisms.
**Definition 3.13.** Let $\varphi: A \to B$ be a Fréchet algebra homomorphism. We say that $\varphi$ is a weak homological epimorphism if and only if any (hence all) of the infinite sequences

\[
0 \leftarrow B \leftarrow P_\bullet \hat{\otimes}_A B, \quad 0 \leftarrow B \leftarrow B \hat{\otimes}_A Q_\bullet, \quad 0 \leftarrow B \leftarrow B \hat{\otimes}_A L_\bullet \hat{\otimes}_A B
\]  

(13)

(where $P_\bullet$, $Q_\bullet$, $L_\bullet$ are as in Proposition 3.7) are exact.

**Definition 3.14.** We say that $\varphi$ is a strong homological epimorphism if any (hence all) of the infinite sequences (13) are admissible.

The fact that the admissibility of any of the sequences (13) implies the admissibility of the other two follows from [38, Prop. 3.2].

**Remark 3.15.** The notion of a homological epimorphism has a remarkable history. Strong homological epimorphisms were introduced by J. L. Taylor [51] under the name of “absolute localizations”. For nuclear Fréchet algebras, our notion of a weak homological epimorphism is equivalent to Taylor’s notion of a “localization” [loc. cit.]; see Section 1. In the purely algebraic setting, homological epimorphisms were rediscovered by W. Dicks [12] under the name of “liftings”, by W. Geigle and H. Lenzing [19] (where the current terminology was introduced), by A. Neeman and A. Ranicki [33] under the name of “stably flat homomorphisms”. In [31], R. Meyer introduced strong homological epimorphisms in the setting of nonunital bornological algebras under the name of “isohomological morphisms”. Finally, O. Ben-Bassat and K. Kreemnizer [3] introduced weak homological epimorphisms (under the name of “homotopy epimorphisms”) in the abstract setting.
of commutative algebras in symmetric monoidal quasi-abelian categories (cf. also [53]). Amazingly, each of the above-mentioned authors seems to have introduced essentially the same class of morphisms independently of the earlier literature.

The following proposition is an analog of [5, Prop. 5.1].

**Proposition 3.16.** Let $\varphi: A \to B$ be a Fréchet algebra epimorphism, and let $n \in \mathbb{Z}_+$. Suppose that $A$ and $B$ are nuclear. Then the following conditions are equivalent:

(i) $\varphi$ is $n$-pseudoflat;
(ii) $M \perp^n_A B$ for each right Fréchet $B$-module $M$;
(iii) $B \perp^n_A N$ for each left Fréchet $B$-module $N$.

**Proof.** (i) $\implies$ (iii). Let $0 \leftarrow B \leftarrow P_\bullet$ be a projective resolution in $\text{mod-}A$ such that all the modules $P_i$ are nuclear. By Proposition 3.7, (8) is an exact sequence. Observe that all the modules in (8) (including $B$) are nuclear and projective in $\text{mod-}B$. Therefore, applying $(\cdot) \hat{\otimes}_B N$ and using Proposition 2.1, we get an exact sequence

$$0 \leftarrow N \leftarrow P_0 \hat{\otimes}_A N \leftarrow \cdots \leftarrow P_{n+1} \hat{\otimes}_A N.$$  

Since $\varphi$ is an epimorphism, we see that $B \hat{\otimes}_A N \cong N$ canonically. Hence (14) is isomorphic to (9) with $M = B$. Thus $B \perp^n_A N$.

The implication (i) $\implies$ (ii) is proved similarly; (ii) $\implies$ (i) and (iii) $\implies$ (i) are clear from Definition 3.4. □

**Corollary 3.17.** Let $\varphi: A \to B$ be a Fréchet algebra epimorphism. Suppose that $A$ and $B$ are nuclear. Then the following conditions are equivalent:

(i) $\varphi$ is a weak homological epimorphism;
(ii) $M \perp_A B$ for each right Fréchet $B$-module $M$;
(iii) $B \perp_A N$ for each left Fréchet $B$-module $N$.

**Remark 3.18.** A similar result [38, Prop. 3.2] on strong homological epimorphisms does not involve nuclearity assumptions.

Our next goal is to characterize epimorphisms, 0-pseudoflat epimorphisms, and 1-pseudoflat epimorphisms in terms of noncommutative differential forms. Towards this goal, let us introduce some notation. Let $\varphi: A \to B$ be a Fréchet algebra homomorphism. Applying $B \hat{\otimes}_A (\cdot) \hat{\otimes}_A B$ to the canonical sequence (5) and composing with $B \hat{\otimes}_A B \to B$, we get

$$0 \to B \hat{\otimes}_A \Omega^1 A \hat{\otimes}_A B \to B \hat{\otimes}_A B \xrightarrow{\mu_B} B \to 0.$$  

Identifying $\Omega^1 B$ with $\text{Ker}\mu_B$, we see that there exists a unique $B$-bimodule morphism $\hat{\varphi}: B \hat{\otimes}_A \Omega^1 A \hat{\otimes}_A B \to \Omega^1 B$ making the diagram

$$
\begin{array}{ccc}
B \hat{\otimes}_A \Omega^1 A \hat{\otimes}_A B & \xrightarrow{\hat{\varphi}} & B \hat{\otimes}_A B & \xrightarrow{\mu_B} & B & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \xrightarrow{j_B} & B \hat{\otimes}_A B & \xrightarrow{\mu_B} & B & \to & 0
\end{array}
$$

commute.

For each Fréchet $B$-bimodule $X$ we have a linear map

$$\tilde{\varphi}_X: \text{Der}(B, X) \to \text{Der}(A, X), \quad D \mapsto D\varphi.$$
Theorem 3.19. For a Fréchet algebra homomorphism \( \varphi : A \to B \) the following conditions are equivalent:

(i) \( \varphi \) is an epimorphism;
(ii) \( \tilde{\varphi}_X : \text{Der}(B, X) \to \text{Der}(A, X) \) is injective for each \( X \in B\text{-mod-}B \);
(iii) \( \tilde{\varphi} : B \widehat{\otimes}_A \Omega^1 A \widehat{\otimes}_A B \to \Omega^1 B \) is an epimorphism in \( B\text{-mod-}B \) (i.e., the image of \( \tilde{\varphi} \) is dense in \( \Omega^1 B \)).

Proof. (i) \( \implies \) (ii). Given \( X \in B\text{-mod-}B \), we make the Fréchet space \( B \oplus X \) into a Fréchet algebra by letting \( (b, x)(c, y) = (bc, by + xc) \) \((b, c \in B, x, y \in X)\).

Suppose that \( \varphi \) is an epimorphism. For every \( D \in \text{Der}(B, X) \) we have a Fréchet algebra homomorphism
\[
\psi : B \to B \oplus X, \quad \psi(b) = (b, D(b)) \quad (b \in B).
\]
If \( D\varphi = 0 \), then \( \psi\varphi = \psi'\varphi \), where
\[
\psi' : B \to B \oplus X, \quad \psi'(b) = (b, 0) \quad (b \in B).
\]
Since \( \varphi \) is an epimorphism, we have \( \psi = \psi' \), i.e., \( D = 0 \).

(ii) \( \iff \) (i). Suppose that \( \tilde{\varphi}_X \) is injective for each \( X \in B\text{-mod-}B \). Consider the derivation
\[
D : B \to B \widehat{\otimes}_A B, \quad b \mapsto b \widehat{\otimes}_A 1 - 1 \otimes_A b.
\]
Since \( D\varphi = 0 \), we have \( D = 0 \), i.e., \( b \otimes_A 1 = 1 \otimes_A b \) for each \( b \in B \). Then the continuous linear map \( B \to B \widehat{\otimes}_A B, b \mapsto b \otimes_A 1 \), is the inverse of the multiplication \( \mu_{B,A} : B \widehat{\otimes}_A B \to B \). Thus \( \mu_{B,A} \) is a topological isomorphism, i.e., \( \varphi \) is an epimorphism (see Proposition 3.5).

(ii) \( \iff \) (iii). By the universal properties of \( \Omega^1 \) and \( \widehat{\otimes}_A \), for each Fréchet \( B \)-bimodule \( X \) there exists a commutative diagram
\[
\begin{array}{ccc}
\text{Der}(B, X) & \xrightarrow{\tilde{\varphi}_X} & \text{Der}(A, X) \\
\downarrow & & \downarrow \\
B \text{h}_B(\Omega^1 B, X) & \xrightarrow{\tilde{\varphi}_X} & B \text{h}_B(B \widehat{\otimes}_A \Omega^1 A \widehat{\otimes}_A B, X)
\end{array}
\]
where \( \tilde{\varphi}_X \) induced by \( \tilde{\varphi} \). Since \( \tilde{\varphi} \) is an epimorphism in \( B\text{-mod-}B \) if and only if \( \tilde{\varphi}_X \) is injective for every \( X \), we have (ii) \( \iff \) (iii). \( \square \)

Theorem 3.20. A Fréchet algebra homomorphism \( \varphi : A \to B \) is a \( 0 \)-pseudoflat epimorphism if and only if \( \tilde{\varphi} : B \widehat{\otimes}_A \Omega^1 A \widehat{\otimes}_A B \to \Omega^1 B \) is onto.

Proof. Since \( j_A = \ker \mu_A \) in (3), we see that the map \( d : A \widehat{\otimes}^3 \to A \widehat{\otimes}^2 \) defined by (3) factorizes as follows:
\[
\begin{array}{ccc}
A \widehat{\otimes} A \widehat{\otimes} A & \xrightarrow{p_A} & A \widehat{\otimes} A \\
\downarrow & & \downarrow \\
0 & \xrightarrow{j_A} & A \widehat{\otimes} A \xrightarrow{\mu_A} A \xrightarrow{} 0.
\end{array}
\]
Applying $B \hat{\otimes} A(\cdot) \hat{\otimes} A B$ and combining with (15), we get the commutative diagram

$$
\begin{array}{ccccccccc}
B \hat{\otimes} A \hat{\otimes} B & \xrightarrow{d_\varphi} & B \hat{\otimes} A \Omega^1 B \hat{\otimes} A B & \xrightarrow{\mu_B} & B & \xrightarrow{\mu_B} & 0 \\
\hat{\rho}_A & & \phi & & \phi & & \\
0 & \xrightarrow{j_B} & \Omega^1 B & \xrightarrow{\mu_B} & B & \xrightarrow{\mu_B} & 0
\end{array}
$$

where $d_\varphi$ is defined in Corollary 3.8. Since $j_B = \ker \mu_B$, we see that (10) is exact if and only if $\varphi \circ \hat{\rho}_A$ is onto. Since $p_A$ is onto, and since the projective tensor product preserves surjections of Fréchet modules, it follows that $\hat{\rho}_A$ is onto. Hence $\varphi \circ \hat{\rho}_A$ is onto if and only if $\varphi$ is onto. This completes the proof. \qed

To characterize 1-pseudoflat epimorphisms in terms of $\Omega^1$, we need the following lemma.

**Lemma 3.21.** Let $\varphi: A \rightarrow B$ be a 0-pseudoflat Fréchet algebra epimorphism, let $M \in \text{mod-}B$, and let $N \in B\text{-mod}$. Assume that either $M$ or $N$ is flat as a Fréchet $B$-module. Then $\text{Tor}^1_{A}(M, N)$ is Hausdorff.

**Proof.** Since $\varphi$ is a 0-pseudoflat epimorphism, we see that (10) is exact. Since $j_B = \ker \mu_B$, there exists a surjective morphism $\bar{\varphi}: B \hat{\otimes} A \hat{\otimes} B \rightarrow \Omega^1 B$ such that the diagram

$$
\begin{array}{ccccccccc}
0 & \xrightarrow{j_B} & B \hat{\otimes} B & \xrightarrow{\mu_B} & B \hat{\otimes} A \hat{\otimes} B & \xrightarrow{\mu_B} & \Omega^1 B & \xrightarrow{\varphi} & 0
\end{array}
$$

commutes. (Note that $\bar{\varphi} = \varphi \circ \hat{\rho}_A$, see the proof of Proposition 3.20.)

Assume now that $N \in B\text{-mod}$ is flat. Since the canonical sequence

$$
\begin{array}{cccccc}
0 & \xrightarrow{\mu_B} & B \hat{\otimes} B & \xrightarrow{j_B} & \Omega^1 B & \xrightarrow{\varphi} & 0
\end{array}
$$

splits in $B\text{-mod}$, for each $M \in \text{mod-}B$ the sequence

$$
\begin{array}{cccccc}
0 & \xrightarrow{j_B} & M \hat{\otimes} B & \xrightarrow{\mu_B} & M \hat{\otimes} \Omega^1 B & \xrightarrow{\varphi} & 0
\end{array}
$$

obtained from (18) via $M \hat{\otimes}_B(\cdot)$ is admissible. Applying $(-) \hat{\otimes}_B N$, we get an exact sequence

$$
\begin{array}{cccccc}
0 & \xrightarrow{j_B} & M \hat{\otimes}_B N & \xrightarrow{\mu_B} & M \hat{\otimes} \Omega^1 B \hat{\otimes}_B N & \xrightarrow{\varphi} & 0.
\end{array}
$$

Let us now apply $M \hat{\otimes}_B(\cdot) \hat{\otimes}_B N$ to (17). We obtain the following commutative diagram:

$$
\begin{array}{ccccccccc}
0 & \xrightarrow{j_B} & M \hat{\otimes}_B N & \xrightarrow{\mu_B} & M \hat{\otimes} N & \xrightarrow{\mu_B} & M \hat{\otimes} A \hat{\otimes} N & \xrightarrow{\hat{\rho}_A} & M \hat{\otimes}_B \Omega^1 B \hat{\otimes}_B N
\end{array}
$$

Since $\varphi$ is onto, it follows that $\hat{\rho}_{M,N} = 1_M \hat{\otimes}_B \varphi \hat{\otimes}_B 1_N$ is onto. Together with the exactness of (19), this implies that the upper row of (20) is exact. Identifying $M \hat{\otimes}_B N$ with
obtained from the low-dimensional segment of \((\mathfrak{I})\) via \(M \hat{\otimes}_A (-) \otimes_A N\) is exact. Equivalently, this means that the canonical map \(\text{Tor}_0^A(M, N) \rightarrow M \hat{\otimes}_A N\) is bijective, which happens if and only if \(\text{Tor}_0^A(M, N)\) is Hausdorff (see \((\mathfrak{I})\)).

In the case where \(N\) is arbitrary and \(M\) is flat, the proof is similar. \(\square\)

Example 3.22. Using Lemma 3.21, it is easy to construct Banach algebra epimorphisms that are not 0-pseudoflat. Consider, for example, the nonunital Banach sequence algebras \(\ell^1\) and \(c_0\) (under pointwise multiplication), let \(A = \ell^1\) and \(B = (c_0)_+\) denote their unitizations, and let \(\varphi: A \rightarrow B\) be the tautological embedding. Since \(\varphi(A)\) is dense in \(B\), we see that \(\varphi\) is an epimorphism. Assume, towards a contradiction, that \(\varphi\) is 0-pseudoflat. By [28, Theorem 3.21], the 1-dimensional \(B\)-module \(C = B/c_0\) is flat (in fact, all Banach \(B\)-modules are flat [27, VII.2.29], because \(B\) is amenable [24]). Hence Lemma 3.21 implies that \(\text{Tor}_0^A(C, c_0)\) is Hausdorff. Consider now the admissible sequence

\[
0 \rightarrow \ell^1 \rightarrow A \rightarrow C \rightarrow 0
\]

of Banach \(A\)-modules (where \(\ell^1 \rightarrow A\) is the tautological embedding). The low-dimensional segment of the respective long exact sequence for \(\text{Tor}_1^A(-, c_0)\) looks as follows:

\[
0 \leftarrow \text{Tor}_0^A(C, c_0) \leftarrow c_0 \leftarrow \ell^1 \leftarrow \text{Tor}_0^A(\ell^1, c_0) \leftarrow \text{Tor}_1^A(C, c_0) \leftarrow 0. \tag{21}
\]

By [27, IV.5.9], \(\ell^1\) is a biprojective Banach algebra (i.e., \(\ell^1\) is a projective Banach \(\ell^1\)-bimodule), which implies, in particular, that \(\ell^1\) is projective in \(\text{mod-}A\) [27, IV.1.3]. Hence we may identify \(\text{Tor}_0^A(\ell^1, c_0)\) with \(\ell^1 \hat{\otimes}_A c_0 = \ell^1 \hat{\otimes}_{\ell^1} c_0\), which is isomorphic to \(\ell^1\) via the map \(a \otimes x \mapsto (a_n x_n)(\) cf. [27, II.3.9] or [17, Lemma 4.1]). Under this identification, the map \(j\) in (21) is nothing but the embedding of \(\ell^1\) into \(c_0\). This implies that \(\text{Tor}_0^A(C, c_0)\) is topologically isomorphic to \(c_0/\ell^1\) and is therefore non-Hausdorff. The resulting contradiction shows that \(\varphi\) is not 0-pseudoflat.

In the purely algebraic context, the following result was discovered by Bergman and Dicks [3, Remark 5.4].

Theorem 3.23. For a Fréchet algebra homomorphism \(\varphi: A \rightarrow B\) the following conditions are equivalent:

(i) \(\varphi\) is a 1-pseudoflat epimorphism;
(ii) \(\bar{\varphi}_X: \text{Der}(B, X) \rightarrow \text{Der}(A, X)\) is bijective for each \(X \in \text{B-mod-}B\);
(iii) \(\varphi: B \hat{\otimes}_A \Omega^1 A \hat{\otimes}_A B \rightarrow \Omega^1 B\) is an isomorphism in \(\text{B-mod-}B\).

Proof. Since the canonical sequence \((\mathfrak{I})\) splits in \(A\)-mod, the sequence

\[
0 \leftarrow B \leftarrow B \hat{\otimes}_A \Omega^1 A \leftarrow 0 \tag{22}
\]

obtained from \((\mathfrak{I})\) via \(B \hat{\otimes}_A (-)\) is admissible in \(\text{mod-}A\). Since \(B \hat{\otimes}_A \) is projective in \(\text{mod-}A\), the low-dimensional segment of the respective long exact sequence for \(\text{Tor}_1^A(-, B)\) looks as follows:

\[
0 \leftarrow \text{Tor}_0^A(B, B) \leftarrow B \hat{\otimes} B \leftarrow \text{Tor}_0^A(B \hat{\otimes}_A \Omega^1 A, B) \leftarrow \text{Tor}_1^A(B, B) \leftarrow 0.
\]
We have the following commutative diagram:

\[
\begin{array}{c}
0 \longrightarrow \text{Tor}^0_A(B, B) \longrightarrow B \otimes B \longrightarrow \text{Tor}^0_A(B \otimes_A \Omega^1_A, B) \longrightarrow \text{Tor}^1_A(B, B) \longrightarrow 0 \\
&\mu_{B,A} \downarrow \quad \alpha \downarrow \quad \phi \downarrow \\
0 \longrightarrow B \longrightarrow B \otimes_B \Omega^1_B \longrightarrow 0
\end{array}
\]

(i) \(\implies\) (iii). If \(\varphi\) is a 1-pseudoflat epimorphism, then \(\text{Tor}^1_A(B, B) = 0\), and \(\mu_{B,A}\) is bijective by Lemma 3.6. Since both rows in (23) are exact, we see that \(\varphi \circ \alpha\) is bijective. On the other hand, \(\alpha\) is bijective by Lemma 3.21 and by (4). Hence \(\varphi\) is bijective.

(iii) \(\implies\) (i). If \(\varphi\) is bijective, then \(\varphi\) is a 0-pseudoflat epimorphism by Theorem 3.20. Hence \(\mu_{B,A}\) is bijective by Lemma 3.6, and \(\alpha\) is bijective by Lemma 3.21 and by (4). Since both lines in (23) are exact, and since the vertical arrows in (23) are bijective, we conclude that \(\text{Tor}^1_A(B, B) = 0\). Thus \(\varphi\) is 1-pseudoflat.

(ii) \(\iff\) (iii). Observe that \(\varphi\) is an isomorphism if and only if for each \(X \in B\text{-mod}\) the map \(\varphi_X\) in (16) is bijective, i.e., if and only if (ii) holds. \(\square\)

**Remark 3.24.** Weak and strong homological epimorphisms can be nicely interpreted in the language of derived categories (cf. [19, 31, 33]). Although we do not need this below, we find it relevant to give at least one of such interpretations (for the convenience of those readers who are used to think in terms of derived categories). If \(A\) is a Fréchet algebra, then there are two ways of making \(A\text{-mod}\) into an exact category (in Quillen’s sense [44]). The first (traditional) exact structure is as follows. Suppose that \(M \xrightarrow{i} N \xrightarrow{p} P\) is an exact pair of morphisms in \(A\text{-mod}\) (i.e., \(i = \ker p\) and \(p = \operatorname{coker} i\)). We say that such a pair is admissible if it splits in the category of Fréchet spaces. It is easy to show that the collection of all admissible exact pairs makes \(A\text{-mod}\) into an exact category. We use the same notation \(A\text{-mod}\) to denote the resulting exact category (this will not lead to a confusion). Alternatively, we can make \(A\text{-mod}\) into an exact category by declaring that all exact pairs are admissible. The fact that the collection of all exact pairs in \(A\text{-mod}\) indeed satisfies the axioms of an exact category follows from the observation that \(A\text{-mod}\) is quasi-abelian, cf. [41]. The resulting exact category will be denoted by \(A\text{-mod}_e\). We also let \(\text{Fr} = C\text{-mod}\) and \(\text{Fr} = C\text{-mod}_e\) denote the respective categories of Fréchet spaces.

Homological algebra in the exact category \(A\text{-mod}\) is precisely the “topological homology” introduced by A. Ya. Helemskii [24] (see also [10, 27, 28]). The main advantage of \(A\text{-mod}\) over \(A\text{-mod}_e\) is that \(A\text{-mod}\) has enough projectives, which is not the case for \(A\text{-mod}_e\). In fact, by a result of V. A. Geiler [20], even the category \(\text{Fr}\) of Fréchet spaces does not have enough projectives. This is one of the main reasons why homological algebra in \(A\text{-mod}\) is developed much better than homological algebra in \(A\text{-mod}_e\). Nevertheless, \(A\text{-mod}\) turns out to be useful in J. L. Taylor’s homological approach to multivariable spectral theory (cf. [10, 11]).

Since \(A\text{-mod}\) has enough projectives, the functor \(M \otimes_A(-): A\text{-mod} \to \text{Fr}\) is left derivable. The left derived functor of \(M \otimes_A(-): A\text{-mod} \to \text{Fr}\) is denoted by \(M \otimes_A^L(-): \text{D}^-(A\text{-mod}) \to \text{D}^-(\text{Fr})\). Exactly as in the algebraic case, \(\otimes_A^L\) extends to a bifunctor from \(\text{D}^-(\text{mod}_A) \times \text{D}^-(A\text{-mod})\) to \(\text{D}^-(\text{Fr})\). Now it is easy to see that a Fréchet algebra homomorphism \(\varphi: A \to B\) is a weak (respectively, strong) homological epimorphism if and only if the
canonical map $B \hat{\otimes}_A B \to B$ is an isomorphism in $D^-(Fr)$ (respectively, in $D^-(Fr)$). This may be compared with condition (ii) of Proposition 3.3, which characterizes epimorphisms of Fréchet algebras.

4. Stein algebras

Throughout, all Stein spaces are assumed to be finite-dimensional. Let $(X, \mathcal{O}_X)$ be a Stein space. Recall from [45] (see also [16]) that a Fréchet $\mathcal{O}_X$-module $\mathcal{F}$ is called quasi-coherent if for each Stein open set $U \subset X$ the following two conditions are satisfied:

(i) $\mathcal{O}(U) \perp_{\mathcal{O}(X)} \mathcal{F}(X)$, and (ii) the canonical map $\mathcal{O}(U) \hat{\otimes}_{\mathcal{O}(X)} \mathcal{F}(X) \to \mathcal{F}(U)$ is a topological isomorphism. Similarly, a Fréchet $\mathcal{O}(X)$-module $M$ is called quasi-coherent if for each Stein open set $U \subset X$ we have $\mathcal{O}(U) \perp M$. By [16, 4.3.7 and 4.3.8], the functor $\Gamma(X, -)$ of global sections is exact and is an equivalence between the category of quasi-coherent Fréchet $\mathcal{O}_X$-modules and the category of quasi-coherent Fréchet $\mathcal{O}(X)$-modules.

Given $p \in X$, we denote by $\mathbb{C}_p$ the one-dimensional $\mathcal{O}(X)$-module corresponding to the evaluation map $\mathcal{O}(X) \to \mathbb{C}$, $a \mapsto a(p)$.

**Theorem 4.1.** Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be Stein spaces, let $f : Y \to X$ be a holomorphic map, and let $f^* : \mathcal{O}(X) \to \mathcal{O}(Y)$ denote the homomorphism induced by $f$. Then the following conditions are equivalent:

(i) $f^*$ is a weak homological epimorphism;
(ii) $f^*$ is a 1-pseudoflat epimorphism;
(iii) $f^*$ is an epimorphism, and for each $q \in Y$ we have $\mathcal{O}(Y) \perp^1_{\mathcal{O}(X)} \mathbb{C}_q$;
(iv) $f$ is an open embedding.

**Proof.** (i) $\implies$ (ii): this is immediate from Definition 3.13.

(ii) $\implies$ (iii). Since $\mathcal{O}(X)$ and $\mathcal{O}(Y)$ are nuclear, we can apply Proposition 3.16.

(iii) $\implies$ (iv). We first observe that $f$ is injective. Indeed, since $f^*$ is an epimorphism, we see that the map $\text{Hom}(\mathcal{O}(Y), \mathbb{C}) \to \text{Hom}(\mathcal{O}(X), \mathbb{C})$ induced by $f^*$ is injective. By [18, Satz 1] (see also [22, V.7.3]), for each Stein space $Z$ we have a natural bijection $Z \cong \text{Hom}(\mathcal{O}(Z), \mathbb{C})$ taking each $z \in Z$ to the evaluation map at $z$. Therefore $f$ is injective.

Given $q \in Y$, let $p = f(q)$, and define the ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$ by

$$\mathcal{I}_x = \begin{cases} m_{x,p} & \text{if } x = p, \\ \mathcal{O}_{x,x} & \text{if } x \neq p, \end{cases}$$

where $m_{x,p}$ is the maximal ideal of $\mathcal{O}_{x,p}$. By [18, Satz 6.4], there exists a resolution

$$0 \leftarrow \mathcal{O}_X / \mathcal{I} \leftarrow \mathcal{O}_X \leftarrow \mathcal{P}_1 \leftarrow \mathcal{P}_2 \leftarrow \cdots, \tag{24}$$

where all the $\mathcal{P}_i$'s are free $\mathcal{O}_X$-modules of finite rank, and where $\mathcal{O}_X \to \mathcal{O}_X / \mathcal{I}$ is the quotient map. Taking the sections over $X$, we obtain an exact complex

$$0 \leftarrow \mathbb{C}_p \leftarrow \mathcal{O}(X) \leftarrow P_1 \leftarrow P_2 \leftarrow \cdots \tag{25}$$

of Fréchet $\mathcal{O}(X)$-modules. Note that $\mathbb{C}_p = \mathbb{C}_q$ in $\mathcal{O}(X)$-mod.
By Proposition 2.1, we can use (25) to calculate Tor$_i^\mathcal{O}(X)(\mathcal{O}(Y), C_q)$. Condition (iii) implies that

\[ \text{Tor}_1^{\mathcal{O}}(\mathcal{O}(Y), C_q) = 0, \quad \text{and} \quad \text{Tor}_0^{\mathcal{O}}(\mathcal{O}(Y), C_q) \cong \mathcal{O}(Y) \otimes_{\mathcal{O}(X)} C_q \cong \mathcal{O}(Y) \otimes_{\mathcal{O}(X)} C_q \cong C_q \]

canonical (see Proposition 3.3). Hence we have an exact sequence

\[ 0 \leftarrow C_q \leftarrow \mathcal{O}(Y) \otimes_{\mathcal{O}(X)} \mathcal{O}(X) \leftarrow \mathcal{O}(Y) \otimes_{\mathcal{O}(X)} P_1 \leftarrow \mathcal{O}(Y) \otimes_{\mathcal{O}(X)} P_2 \]  

(26)

of Fréchet \( \mathcal{O}(Y) \)-modules.

Now observe that the functors \( F \mapsto \mathcal{O}(Y) \otimes_{\mathcal{O}(X)} F(X) \) and \( F \mapsto (f^* F)(Y) \) obviously agree on the category of free \( \mathcal{O}_X \)-modules of finite rank. Hence (26) is isomorphic to the sequence obtained by applying \( \Gamma(Y, -) \) to

\[ 0 \leftarrow \mathcal{O}_Y / \mathcal{I}' \leftarrow \mathcal{O}_Y \cong f^* \mathcal{O}_X \leftarrow f^* \mathcal{P}_1 \leftarrow f^* \mathcal{P}_2, \]  

(27)

where \( \mathcal{I}' \subset \mathcal{O}_Y \) is the ideal sheaf given by

\[ \mathcal{I}_y' = \begin{cases} m_{y, y} & \text{if } y = q, \\ \mathcal{O}_{y, y} & \text{if } y \neq q. \end{cases} \]

Therefore (27) is exact.

Consider now the stalks of (24) over \( p \) and the stalks of (27) over \( q \). For notational convenience, let \( A = \mathcal{O}_{X, p}, B = \mathcal{O}_{Y, q}, m_A = m_{X, p}, m_B = m_{Y, q} \), and \( F_i = (\mathcal{P}_i)_x \). Let also \( \varphi: A \to B \) denote the homomorphism induced by \( f \). We have two exact sequences

\[ 0 \leftarrow A / m_A \leftarrow A \leftarrow F_1 \leftarrow F_2 \leftarrow \cdots , \]  

(28)

\[ 0 \leftarrow B / m_B \leftarrow B \otimes_A A \leftarrow B \otimes_A F_1 \leftarrow B \otimes_A F_2. \]  

(29)

Comparing (28) with (29), we see that

\[ \text{Tor}_1^A(B, A / m_A) = 0, \]  

(30)

where \( \text{Tor}_1^A \) stands for the purely algebraic Tor-functor. Also, the exactness of (28) and (29) implies that \( B \otimes_A (A / m_A) \cong B / m_B \) via the map \( b \otimes (a + m_A) \mapsto b \varphi(a) + m_B \). It is readily verified that the latter condition is equivalent to the equality \( B \varphi(m_A) = m_B \).

By [21, 2.2.3], this means that \( \varphi \) is onto. In particular, \( B \) is a finitely generated \( A \)-module. Since \( A \) is Noetherian [21, 2.0.1], \( B \) is a finitely presented \( A \)-module. Combining this with (30) and applying [4, Chap. II, §3, no. 2], we see that \( B \) is free over \( A \). Since \( \dim(B / Bm_A) = \dim(B / m_B) = 1 \), it follows from [21, Appendix, 2.7 (i)] that \( \varphi \) is an isomorphism. By [17, 0.23], this means exactly that \( f \) is locally biholomorphic. Since \( f \) is also injective (see above), we conclude that \( f \) is an open embedding.

(iv) \( \implies \) (i). Without loss of generality, we may assume that \( Y \) is a Stein open subset of \( X \) and that \( \mathcal{O}_Y = \mathcal{O}_X | Y \). Thus \( f^*: \mathcal{O}(X) \to \mathcal{O}(Y) \) is the restriction map. Let

\[ 0 \leftarrow \mathcal{O}(Y) \leftarrow B_0 \leftarrow B_1 \leftarrow \cdots \]  

(31)

be the bar resolution of \( \mathcal{O}(Y) \) in \( \mathcal{O}(X) \)-mod (see Section 2). Recall that for each \( n \geq 0 \) we have

\[ B_n = \mathcal{O}(X) \otimes (\mathcal{O}(Y))^{
 0} \]  

Since $\mathcal{O}(Y) \cong (i_* \mathcal{O}_Y)(X)$, where $i$ is the embedding of $Y$ into $X$, and since all the $B_n$’s are free over $\mathcal{O}(X)$, we conclude that all Fréchet $\mathcal{O}(X)$-modules in (31) are quasi-coherent. Hence (34) corresponds to an exact complex

$$0 \leftarrow i_* \mathcal{O}_Y \leftarrow \mathcal{B}_0 \leftarrow \mathcal{B}_1 \leftarrow \cdots$$

of quasi-coherent Fréchet $\mathcal{O}_X$-modules. Explicitly, we have $\mathcal{B}_n(U) = \mathcal{O}(U) \otimes_{\mathcal{O}(X)} B_n$ for each Stein open subset $U \subset X$ and for each $n \geq 0$. Taking the sections of (32) over $Y$ and using the fact that quasi-coherent Fréchet sheaves are acyclic over Stein open sets [16, 4.3.3], we obtain an exact complex

$$0 \leftarrow \mathcal{O}(Y) \leftarrow \mathcal{B}_0(Y) \leftarrow \mathcal{B}_1(Y) \leftarrow \cdots$$

On the other hand, it is immediate from the construction that (33) is isomorphic to the complex

$$0 \leftarrow \mathcal{O}(Y) \leftarrow \mathcal{O}(Y) \otimes_{\mathcal{O}(X)} B_0 \leftarrow \mathcal{O}(Y) \otimes_{\mathcal{O}(X)} B_1 \leftarrow \cdots$$

Therefore $f^*: \mathcal{O}(X) \to \mathcal{O}(Y)$ is a weak homological epimorphism.

**Remark 4.2.** We have already pointed out in Section 1 that, if $f: Y \to X$ is an open embedding of Stein spaces, then $f^*: \mathcal{O}(X) \to \mathcal{O}(Y)$ is not necessarily flat. Thus the class of 1-pseudoflat Fréchet algebra epimorphisms is essentially larger than the class of flat epimorphisms, even in the commutative case. It is interesting to compare this with recent purely algebraic results from [1, 3]. Namely, a 1-pseudoflat epimorphism $A \to B$ of commutative rings is necessarily flat provided that either (a) $A$ is Noetherian $\mathbb{I}$ Prop. 4.5], or (b) the projective dimension of $B$ over $A$ is $\leq 1$ [3, Remark 16.9]. While property (a) rarely holds in the functional analytic context (for example, the algebras of holomorphic functions on Stein manifolds are never Noetherian), property (b) is more common. For example, if $\mathbb{D} \subset \mathbb{C}$ is the open unit disc, then the restriction map $\mathcal{O}(\mathbb{C}) \to \mathcal{O}(\mathbb{D})$ is a 1-pseudoflat epimorphism by Theorem [4, actually, by [5, Prop. 3.1]), satisfies (b) by [24, Theorem V.1.8], but is not flat by [36]. This shows that the above-mentioned result of [3] has no analog in the Fréchet algebra setting.

### 5. Algebras of $C^\infty$-functions

In this section, we prove a $C^\infty$-analog of Theorem [4,1. Towards this goal, we need two lemmas. Let $A$, $B$, $C$ be Fréchet algebras, $N$ be a Banach $B$-$C$-bimodule, and $P$ be a Fréchet $A$-$C$-bimodule. Then $\mathbf{h}_C(N, P)$ is a Fréchet space under the topology of uniform convergence on the unit ball of $N$. Moreover, $\mathbf{h}_C(N, P)$ is a Fréchet $A$-$B$-bimodule with respect to the actions

$$(a \cdot \varphi)(x) = a \cdot \varphi(x), \quad (\varphi \cdot b)(x) = \varphi(b \cdot x) \quad (a \in A, \ b \in B, \ x \in N, \ \varphi \in \mathbf{h}_C(N, P)).$$

**Lemma 5.1.** Let $A$, $B$, $C$ be Fréchet algebras, $M$ be a Fréchet $A$-$B$-bimodule, $N$ be a Banach $B$-$C$-bimodule, and $P$ be a Fréchet $A$-$C$-bimodule. Then there exists a vector space isomorphism

$$\mathbf{h}_C(M \hat{\otimes}_B N, P) \overset{\sim}{\longrightarrow} \mathbf{h}_B(M, \mathbf{h}_C(N, P)), \quad \varphi \mapsto (x \mapsto (y \mapsto \varphi(x \otimes y))).$$

We omit the standard proof (cf. [27, II.5.22], [40, Prop. 3.2]).

**Lemma 5.2.** Let $\varphi: A \to B$ be a Fréchet algebra epimorphism, and let $\epsilon: B \to \mathbb{C}$ be a continuous homomorphism. Let $\mathbb{C}_\epsilon$ denote the one-dimensional $B$-bimodule corresponding to $\epsilon$. Suppose that $B \perp^A_1 \mathbb{C}_\epsilon$. Then $\varphi_{\mathbb{C}_\epsilon}: \text{Der}(B, \mathbb{C}_\epsilon) \to \text{Der}(A, \mathbb{C}_\epsilon)$ is a bijection.
Proof. As in the proof of Theorem 3.23, the admissible sequence (22) yields a long exact sequence of $\text{Tor}_i^A(-,-,\mathbb{C}_\varepsilon)$, whose low-dimensional segment fits into the following commutative diagram:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \text{Tor}_0^A(B, \mathbb{C}_\varepsilon) & \longrightarrow & B \otimes \mathbb{C}_\varepsilon & \longrightarrow & \text{Tor}_0^A(B \otimes_A \Omega^1 A, \mathbb{C}_\varepsilon) & \longrightarrow & \text{Tor}_1^A(B, \mathbb{C}_\varepsilon) & \longrightarrow & 0 \\
& & \alpha \downarrow & & \beta \downarrow & & \alpha' \downarrow & & \gamma \downarrow & & \\
& & B \otimes_A \mathbb{C}_\varepsilon & \longrightarrow & B \otimes_A \Omega^1 A \otimes_A \mathbb{C}_\varepsilon & \longrightarrow & 0 \\
0 & \longrightarrow & \mathbb{C}_\varepsilon & \longrightarrow & B \otimes \mathbb{C}_\varepsilon & \longrightarrow & \Omega^1 B \otimes_B \mathbb{C}_\varepsilon & \longrightarrow & 0
\end{array}
$$

Here $\alpha$ and $\alpha'$ are the canonical maps from $\text{Tor}_0^A(-,-,\mathbb{C}_\varepsilon)$ to $(-) \otimes_A(-)$, $\beta$ is the canonical map from $B \otimes_A \mathbb{C}_\varepsilon$ to $B \otimes_B \mathbb{C}_\varepsilon \cong \mathbb{C}_\varepsilon$, and $\gamma$ corresponds to $\varphi \otimes_B 1_{\mathbb{C}_\varepsilon}$ under the identification $(B \otimes_A \Omega^1 A \otimes_A \mathbb{C}_\varepsilon) \cong B \otimes_A \Omega^1 A \otimes_A \mathbb{C}_\varepsilon$.

Finally, the bottom row of (34) is obtained from (18) via $(-) \otimes_B \mathbb{C}_\varepsilon$, so it is exact because (18) splits in $\text{mod-B}$.

Since $\varphi$ is an epimorphism, Proposition 3.3 implies that $\beta$ is bijective. Since $B \perp_A \mathbb{C}_\varepsilon$, we see that $\alpha$ is bijective and that $\text{Tor}_1^A(B, \mathbb{C}_\varepsilon) = 0$. Together with the fact that both lines in (34) are exact, this implies that $\gamma \circ \alpha'$ is bijective. Hence $\alpha'$ is injective, or, equivalently, bijective (see [5]), which in turn implies that $\gamma$ is bijective. Thus $\gamma$ is an isomorphism in $B\text{-mod}$.

Applying $B\mathbf{h}(-, \mathbb{C}_\varepsilon)$ to $\gamma$, we obtain a vector space isomorphism

$$
B\mathbf{h}(\Omega^1 B \underset{B}{\otimes} \mathbb{C}_\varepsilon, \mathbb{C}_\varepsilon) \stackrel{\sim}{\longrightarrow} B\mathbf{h}(B \otimes_A \Omega^1 A \otimes_A \mathbb{C}_\varepsilon, \mathbb{C}_\varepsilon).
$$

Observe that there is a $B$-bimodule isomorphism $B\mathbf{h}_C(\mathbb{C}_\varepsilon, \mathbb{C}_\varepsilon) \cong \mathbb{C}_\varepsilon$ given by $f \mapsto f(1)$. Together with Lemma 3.3, this implies that

$$
B\mathbf{h}(\Omega^1 B \underset{B}{\otimes} \mathbb{C}_\varepsilon, \mathbb{C}_\varepsilon) \cong B\mathbf{h}(\Omega^1 B, B\mathbf{h}_C(\mathbb{C}_\varepsilon, \mathbb{C}_\varepsilon)) \cong B\mathbf{h}(\Omega^1 B, \mathbb{C}_\varepsilon),
$$

and

$$
B\mathbf{h}(B \otimes_A \Omega^1 A \otimes_A \mathbb{C}_\varepsilon, \mathbb{C}_\varepsilon) \cong B\mathbf{h}(B \otimes_A \Omega^1 A \otimes_A B, B\mathbf{h}_C(\mathbb{C}_\varepsilon, \mathbb{C}_\varepsilon))
\cong B\mathbf{h}(B \otimes_A \Omega^1 A \otimes_A B, \mathbb{C}_\varepsilon)
\cong B\mathbf{h}(B \otimes_A \Omega^1 A \otimes_A B, \mathbb{C}_\varepsilon).
$$

Under the identifications (36) and (37), the isomorphism (35) becomes

$$
B\mathbf{h}_B(\Omega^1 B, \mathbb{C}_\varepsilon) \stackrel{\sim}{\longrightarrow} B\mathbf{h}_B(B \otimes_A \Omega^1 A \otimes_A B, \mathbb{C}_\varepsilon).
$$

A routine calculation shows that (38) is nothing but the map $\bar{\varphi}_{\mathbb{C}_\varepsilon}$ from diagram (13) (in which we let $X = \mathbb{C}_\varepsilon$). This readily implies that $\bar{\varphi}_{\mathbb{C}_\varepsilon} : \text{Der}(B, \mathbb{C}_\varepsilon) \rightarrow \text{Der}(A, \mathbb{C}_\varepsilon)$ is a vector space isomorphism. \qed

Let $X$ be a $C^\infty$-manifold. We denote by $C^\infty(X)$ the Fréchet algebra of infinitely differentiable $\mathbb{C}$-valued functions on $X$. Similarly to the holomorphic case (see Section 4), given $p \in X$, we denote by $\mathbb{C}_p$ the one-dimensional $C^\infty(X)$-module corresponding to the evaluation map $C^\infty(X) \rightarrow \mathbb{C}$, $a \mapsto a(p)$.

**Theorem 5.3.** Let $X$ and $Y$ be $C^\infty$-manifolds, let $f : Y \rightarrow X$ be a smooth map, and let $f^* : C^\infty(X) \rightarrow C^\infty(Y)$ denote the homomorphism induced by $f$. Then the following conditions are equivalent:
(i) $f^*$ is a projective epimorphism, i.e., $f^*$ is an epimorphism and $C^\infty(Y)$ is projective in $C^\infty(X)\text{-mod}$;
(ii) $f^*$ is a flat epimorphism, i.e., $f^*$ is an epimorphism and $C^\infty(Y)$ is flat in $C^\infty(X)\text{-mod}$;
(iii) $f^*$ is a strong homological epimorphism;
(iv) $f^*$ is a weak homological epimorphism;
(v) $f^*$ is a 1-pseudoflat epimorphism;
(vi) $f^*$ is an epimorphism, and for each $q \in Y$ we have $C^\infty(Y) \perp_{C^\infty(X)} C_q$;
(vii) $f$ is an open embedding.

Proof. (i) \implies (ii) \implies (iv), (i) \implies (iii) \implies (iv), (iv) \implies (v): this is trivial.
(v) \implies (vi). Since $C^\infty(X)$ and $C^\infty(Y)$ are nuclear, we can apply Proposition 3.10.
(vi) \implies (vii). As in the proof of Theorem 4.1, we first observe that $f$ is injective. Indeed, since $f^*$ is an epimorphism, we see that the map Hom($C^\infty(Y), \mathbb{C}$) \to Hom($C^\infty(X), \mathbb{C}$) induced by $f^*$ is injective. By [44, Theorem 7.2], for each smooth manifold $Z$ we have a natural bijection $Z \cong \text{Hom}(C^\infty(Z), \mathbb{C})$ taking each $z \in Z$ to the evaluation map at $z$. Therefore $f$ is injective.

To complete the argument we need to show that $f$ is a local diffeomorphism. By the Inverse Function Theorem, it suffices to check that for each $z \in Y$ the tangent space $df_q: T_q(Y) \to T_{f(q)}(X)$ is a vector space isomorphism. Identifying $T_q(Y)$ with DER($C^\infty(Y), \mathbb{C}_q$) (see, e.g., [28, III.3.1]), we see that $df_q$ is nothing but

$$(\tilde{f}^*)_q: \text{Der}(C^\infty(Y), \mathbb{C}_q) \to \text{Der}(C^\infty(X), \mathbb{C}_q)$$

(as in Theorem 4.1, we identify $\mathbb{C}_q$ with $\mathbb{C}_{f(q)}$ in $C^\infty(X)\text{-mod}$.) By Lemma 5.2, $(\tilde{f}^*)_q$ is a bijection. Hence $f$ is an open embedding.

(vii) \implies (i). Without loss of generality, we may assume that $Y$ is an open subset of $X$. Thus $f^*: C^\infty(X) \to C^\infty(Y)$ is the restriction map. A standard argument involving bump functions shows that the image of $f^*$ is dense in $C^\infty(Y)$. Hence $f^*$ is an epimorphism. By [35, Theorem 2], $C^\infty(Y)$ is projective\footnote{In [35], the projectivity of $C^\infty(Y)$ over $C^\infty(X)$ is proved under the assumption that $Y$ is contained in a coordinate neighborhood. However, the proof readily carries over to the general case.} over $C^\infty(X)$. This completes the proof. \qed

6. Concluding remarks and questions

An obvious difference between our main results, i.e., Theorems 4.1 and 5.3, is that the strong conditions (i)–(iii) of Theorem 5.3 are missing in Theorem 4.1. We have already mentioned in Section 6 that open embeddings of Stein spaces usually do not satisfy condition (ii) (and, a fortiori, do not satisfy condition (i)) of Theorem 5.3. However, the situation with condition (iii) is not that clear. In fact, we do not know the answer to the following question.

**Question 6.1.** Let $(X, \mathcal{O}_X)$ be a Stein space, and let $(Y, \mathcal{O}_Y)$ be a Stein open subspace of $(X, \mathcal{O}_X)$. Is the restriction map $\mathcal{O}(X) \to \mathcal{O}(Y)$ a strong homological epimorphism?

In the special case where $X = \mathbb{C}^n$ and $Y$ is a polydomain (i.e., a product of one-dimensional open subsets of $\mathbb{C}$), the answer to Question 6.1 is positive by [51, Prop. 4.3]. On the other hand, the answer seems to be unknown already in the case where $X = \mathbb{C}^2$ and $Y$ is the open unit ball.

Another difference between Theorems 4.1 and 5.3 is in the “degree of singularity” of the objects considered therein. Indeed, Stein spaces are not necessarily reduced (i.e.,
their structure sheaves are allowed to have nilpotents), and even reduced Stein spaces are not necessarily smooth (i.e., are not necessarily locally isomorphic to an open subset of \( \mathbb{C}^n \)). On the other hand, \( C^\infty \)-manifolds are reduced and smooth (in the appropriate sense) by definition. The theory of \( C^\infty \)-differentiable spaces \cite{32} studies geometric objects which are more general than \( C^\infty \)-manifolds, and which can be viewed as “correct” \( C^\infty \)-analogs of Stein spaces. In particular, \( C^\infty \)-differentiable spaces may have singular points and may be non-reduced. (Note that \cite{32} deals with \( \mathbb{R} \)-valued functions only, but an extension to \( \mathbb{C} \)-valued functions is straightforward.) It would be interesting to characterize open embeddings of \( C^\infty \)-differentiable spaces (at least in the affine case) in the spirit of Theorem \ref{thm:open_embeddings_characterization}.

In its full form, Theorem \ref{thm:open_embeddings_characterization} does not extend to \( C^\infty \)-differentiable spaces. For example, consider the map \( \pi: C^\infty(\mathbb{R}^n) \to \mathbb{C}[[x_1, \ldots, x_n]] \) which takes each smooth function on \( \mathbb{R}^n \) to its Taylor series at 0. Using the Koszul resolution, one can prove that \( \pi \) is a strong homological epimorphism (cf. \cite{51} Prop. 4.4). At the same time, the corresponding map \( \pi^* \) of affine \( C^\infty \)-differentiable spaces is not an open embedding. On the other hand, applying \((-) \hat{\otimes}_A \mathbb{C}[[x_1, \ldots, x_n]]\) to the inclusion \( I \hookrightarrow A \), where \( A = C^\infty(\mathbb{R}^n) \) and \( I = \{ f \in A : f(0) = 0 \} \), one can easily show that \( \pi \) is not flat.

The \( C^\infty \)-differentiable space with one-point spectrum that corresponds to \( \mathbb{C}[[x_1, \ldots, x_n]] \) is a special case of \( W_{Y/X} \), the so-called Whitney subspace of \( Y \), where \( Y \) is a closed subset of a \( C^\infty \)-differentiable space \( X \) \cite{32}, Corollary 5.10]. In the case where \( X \) is an open subset of \( \mathbb{R}^n \), the map \( W_{Y/X} \to X \) corresponds to the quotient homomorphism \( C^\infty(X) \to C^\infty(X)/W_{Y/X} \), where \( W_{Y/X} \) is the ideal of functions whose derivatives of all orders vanish on \( Y \). Normally, \( W_{Y/X} \to X \) is not an open embedding (in fact, it is always a closed embedding). Nevertheless, we have the following result.

**Proposition 6.2.** Let \( X \) be an open subset of \( \mathbb{R}^n \), and let \( Y \) be a closed subset of \( X \). Then the quotient map \( C^\infty(X) \to C^\infty(X)/W_{Y/X} \) is a 1-pseudoflat epimorphism.

**Proof.** It follows from \cite{54}, Lemme 2.4] that any real-valued function \( f \in W_{Y/X} \) has the form \( f = f_1f_2 \), where \( f_1 \) and \( f_2 \) are again in \( W_{Y/X} \). Since \( C^\infty(X) \) is nuclear, we can apply Proposition \ref{prop:pseudoflat_epimorphism}.

The above remarks lead naturally to the following two questions on \( C^\infty \)-differentiable spaces.

**Question 6.3.** Can open embeddings of affine \( C^\infty \)-differentiable spaces be characterized in terms of projectivity or flatness, as in Theorem \ref{thm:open_embeddings_characterization}?

**Question 6.4.** Let \((X, \mathcal{O}_X)\) be an affine \( C^\infty \)-differentiable space, and let \( Y \) be a closed subset of \( X \). Is the quotient map \( \mathcal{O}(X) \to \mathcal{O}(W_{Y/X}) \) a 1-pseudoflat epimorphism? Is it a weak homological epimorphism? Is it a strong homological epimorphism?

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