Perfect fluid brane-world model

Mariam Bouhmadi–López, Pedro F. González–Díaz
and Alexander Zhuk

Instituto de Matemáticas y Física Fundamental,
Consejo Superior de Investigaciones Científicas,
C/ Serrano 121, 28006 Madrid, Spain

Abstract

By considering 5–dimensional cosmological models with a bulk filled with a perfect fluid and a cosmological constant, we have found regular instantonic solution which is free from any singularity at the origin of the extra–coordinate and describe 5–dimensional asymptotically anti de Sitter wormhole, when the bulk has a topology $R \times S^4$ and is filled with dust and a negative cosmological constant. Compactified brane-world instantons which are built up from such instantonic solution describe either a single brane or a string of branes. Their analytical continuation to the pseudo–Riemannian metric can give rise to either 4-dimensional inflating branes or solutions with the same dynamical behaviour for extra–dimension and branes, in addition to multitemporal solutions. Dust brane-world models with other spatial topologies are also considered.

PACS number(s): 04.50.+h, 98.80.Hw

1 Introduction

Although higher dimensional cosmological models can be traced back to the first years of gravitation theory, the idea has received a renewed great attention in the last few years, due to the publication of pioneering papers on brane and extra-dimension models which shed light for the solution of fundamental physical problems including the extra-dimension compactification and the hierarchy problem. Extra-dimensions can be compactified, as in the standard Kaluza-Klein theory, or not, as was firstly suggested by Akama, Rubakov and Shaposhnikov and others. While Akama considered the universe as a four vortex-like object embedded in a 6–dimensional flat space-time, Rubakov and Shaposhnikov proposed a model in (1+N)–dimensional Minkowski space-time ($N \geq 4$) where particles were confined in a potential well, flat on the usual three spatial dimensions and narrow along the extra-dimensions. Gravity was later on included in 5–dimensional manifolds in order to trap particles near a 4–dimensional Lorentzian submanifolds.

More recently, a model was proposed on (4+N)–dimensions with $N \geq 2$, where the extra-dimensions are compact and their size, $R$, is deduced by imposing that the usual Planck scale, $M_{Pl}$, is no longer a fundamental scale and Planck scale in (4+N)–dimensions, $M_{Pl4+N}$, is of the order of the weak scale $M_{EW}$, $M_{Pl4+N} = M_{Pl4+N}^2 R^N$. This model can solve the hierarchy problem. In this framework the gravitons can propagate in the extra-dimensions while the standard model fields are confined to a 4–dimensional submanifolds of thickness $M_{EW}^1$ in the extra–dimensions.

Randall and Sundrum suggested a new approach to solve the hierarchy problem by including just one extra compact dimension. In their first model, inspired by string theory, they considered 5–dimensional anti de Sitter (AdS) bulk with two branes with opposite tension; our universe is then placed on the brane with negative tension where standard model particles are localized. In a second model, these authors placed the universe in the brane with positive tension, in a non compact background. In this framework, it is possible to reproduce 4-dimensional general relativity even if the extra–dimension is noncompact, due to the existence of a massless gravitational bound state of Kaluza-Klein (KK) modes which is the graviton of 4–dimensional world. While for the noncompact case, KK spectrum is continuous without a gap, for the compact case the KK excitations are quantized.

*e-mail: mbouhmadi@imaff.cfmac.csic.es
†e-mail: p.gonzalezdiaz@imaff.cfmac.csic.es
‡e-mail: alzhuk@imaff.cfmac.csic.es

on leave from: Department of Physics, University of Odessa, 2 Dvoryanskaya St., Odessa 65100, Ukraine
Branes in the Randall-Sundrum models are 4-dimensional flat space-time and consequently, at least in principle, they can not describe any inflationary universe. Nevertheless, as it was pointing out by Garriga and Sasaki (GS) \cite{gs} it is still possible to construct an inflating brane, whose geometry corresponds to a 4-dimensional de Sitter space, surrounded by a 5-dimensional AdS. The Euclidean version of this solution can be used for the description of the creation of the universe from nothing. GS model has a single normalized gravitational bound state which corresponds to the massless graviton and separates by a gap from the massive KK modes.

Multi-brane-world models also exhibit 4-dimensional gravity localization at the branes. In particular, a compact brane-world model consisting of three flat branes, two with positive tensions and an intermediate one with negative tension, embedded in 5-dimensional AdS space-time has been also considered \cite{randall}. In this case, the universe is placed in one of the positive tension branes. In addition, intersecting brane configurations \cite{intersecting}, crystal brane-world \cite{crystal} and brane periodic configuration \cite{brane_periodic} have been investigated as well. The massless gravitational bound state exists in all these models.

However, most of the above mentioned brane-world models have the shortcoming of either not including creation of inflating branes or having a singular character for brane instantons. The main aim of the present paper is to propose a set of singularity free 5–dimensional models which are able to produce inflating brane-worlds. Such models should also induce localization of 4–dimensional gravity at the branes.

The paper is organized as follows. In the next section we derive a master equation (2.9) which describes the behaviour of the scalar factor of D–dimensional instantonic (after an analytic continuation to the Euclidean region) solutions for a precise bulk filled with a perfect fluid and a cosmological constant, $\Lambda_D$. For $D = 5$, $\Lambda_D < 0$ and spherical 4-dimensional sections, we obtain asymptotically AdS wormhole when the bulk is filled with dust. Using this wormhole solution, in section 3, we construct compact and noncompact brane-world instantons. In section 4, we analyze the behaviour of the massless gravitational KK mode for the models with dust. Using this wormhole solution, in section 3, we construct compact and noncompact brane-world models for all the other possible combinations of $\Lambda$, and the maximally symmetric branes (flat, spherical, hyperbolic) that have not been considered in section 2. Finally, in section 7, we summarize our results.

## 2 Multidimensional perfect fluid cosmology

Let us start our investigation of a multidimensional model with a cosmological constant, $\Lambda_D$, and minimal scalar field, $\phi$, by writing down the action\footnote{Although the constant minimal scalar field is really equivalent to a cosmological term, it is convenient to single out an explicit cosmological constant in the action.}

$$ S = \frac{1}{2\kappa_D^2} \int d^D X \sqrt{|g^{(D)}|} \left\{ R[g^{(D)}] - 2\Lambda_D \right\} + \int d^D X \sqrt{|g^{(D)}|} \left\{ -\frac{1}{2} g^{(D)MN} \partial_M \phi \partial_N \phi - V(\phi) \right\} + S_{YGH}, \tag{2.1} $$

where $\kappa_D^2$ is D–dimensional gravitational constant and $S_{YGH}$ the standard York-Gibbons-Hawking boundary term \cite{ygh}. The $(D = 1 + d)$–dimensional metric $g^{(D)}$ is taken to have the form

$$ g^{(D)} = g^{(D)}_{MN} dx^M \otimes dx^N = -e^{2\gamma(\tau)} d\tau \otimes d\tau + e^{2\beta(\tau)} g^{(d)}_{\mu\nu} dx^\mu \otimes dx^\nu, \tag{2.2} $$

with $g^{(d)}$ the metric of d-dimensional Einstein space: $R[g^{(d)}] = \lambda d \equiv R_d$, and in the case of constant curvature space parameter $\lambda$ is normalized as $\lambda = L(d - 1)$, with $k = \pm 1, 0$.

Consistent with the metric ansatz (2.3), we suppose that the scalar field is also homogeneous: $\phi = \phi(\tau)$. For such a scalar field energy density and pressure are defined as:

$$ T^0_0 = -\frac{1}{2} e^{-2\gamma} \dot{\phi}^2 - V(\phi) \equiv -\rho, $$

$$ T^m_i = \frac{1}{2} e^{-2\gamma} \dot{\phi}^2 - V(\phi) \equiv P, \quad i = 1, \ldots, d, \tag{2.3} $$

and supposed to satisfy the state equation:

$$ P = (a - 1) \rho. \tag{2.4} $$

The conservation equation $T^N_M = 0$ has then a simple integral

$$ \rho = \Lambda a^{-d}, \tag{2.5} $$

where $A$ is an arbitrary constant and $a(\tau) := \exp[\beta(\tau)]$ is a scale factor. Obviously, $a^{-d}$ gives the volume of the Universe at any hypersurface $\tau = \text{const}$ up to a spatial prefactor $V_d = \int d^d x \sqrt{|g^{(d)}|}$. From the energodominant condition it is usually supposed that $\rho \geq 0$ and $-\rho \leq P \leq \rho$, where the latter condition (which results in
$0 \leq \alpha \leq 2$ means that speed of sound is less than speed of light. For the aim of generality we will not restrict ourselves to satisfy these conditions because at least the first of them can be violated on the case of a scalar field. For example, $\alpha = 0$ with $\rho < 0$ describes the case of a negative cosmological constant.

It can be shown \cite{13} that Einstein equations for such scalar fields are equivalent to the Einstein equations for a perfect fluid with action:

$$S = \frac{V_d}{\kappa_D} \int d\tau \left[ \frac{1}{2} e^{-\gamma + \gamma_0} d(1-d) \beta^2 - e^{\gamma - \gamma_0} U \right] \equiv \frac{V_d}{\kappa_D} \int d\tau L,$$  \hspace{1cm} (2.6)

in which we performed integration over spatial coordinates $x$ ($V_d$ is an artifact of this integration), $\gamma_0 := d\beta$, overdot denotes differentiation with respect to $\tau$ and we have introduced a "potential energy"

$$U = e^{2\gamma_0} \left( -\frac{1}{2} R_d e^{-2\beta} + \Lambda_D + \kappa^2 D \rho \right).$$ \hspace{1cm} (2.7)

The constraint equation corresponding to the Lagrangian in Eq.(2.6) results in the following master equation:

$$\frac{\partial L}{\partial \gamma} = 0 \quad \Rightarrow \quad \frac{1}{2} e^{-\gamma + \gamma_0} d(1-d) \beta^2 + e^{\gamma - \gamma_0} U = 0,$$  \hspace{1cm} (2.8)

which is exactly the 00-component of the Einstein equation. Function $\gamma$ reflects the freedom for the choice of different time gauges: $\gamma = 0$ is the proper time gauge, $\gamma = \gamma_0$ is the harmonic time gauge \cite{14} and $\gamma = \beta$ is the conformal time gauge. Thus, in the proper time gauge ($\gamma = 0$) Eq.(2.8) reads

$$\left( \frac{\dot{a}}{a} \right)^2 + \frac{R_d}{d(d-1)} \frac{1}{a^2} + \frac{2}{d(1-d)} \Lambda_D + \frac{2}{d(1-d)} \kappa^2 D A a^{-\alpha d} = 0.$$ \hspace{1cm} (2.9)

We will concentrate mainly on 5-dimensional brane-world models with AdS bulk, $D = 5$, $\Lambda_D = \Lambda_5 \equiv -|\Lambda_5| < 0$. The scalar curvature $R_d$ can be an arbitrary constant but we shall consider the particular case of constant curvature 4-D space $R_d = kd(d-1) = 12k$, restricting mainly to the positive curvature with $k = +1$. For these parameters Eq.(2.9) becomes

$$(\dot{a})^2 + k + \Lambda a^2 - \bar{A}^2 a^{-4\alpha + 2} = 0,$$ \hspace{1cm} (2.10)

where for simplicity we have introduced the following notation: $|\Lambda_5|/6 \equiv \Lambda$ and $(1/6)\kappa^2 D A \equiv \bar{A}^2$. Some of the particular values $\alpha$ are of special interest because they correspond to important types of matter. For example, in 5-D space-time $\alpha = 5/4$ describes radiation, $\alpha = 1$ corresponds to dust (0-D objects), $\alpha = 3/4$ represents cosmic strings (1-D objects), $\alpha = 1/2$ describes domain walls (2-D objects), $\alpha = 1/4$ corresponds to hyperdomain walls (3-D objects) and $\alpha = 0$ represents vacuum (which can be in some sense considered as a 4-D object).

Now, as we want to construct 5-D brane-world models, we perform the Wick rotation to Euclidean "time" $r : \tau \rightarrow -ir$. Where $r$ is to be considered as an extra spatial coordinate orthogonal to 4-D branes, i.e., now hypersurfaces at $r =$ const. Then, in proper "time" gauge, metric (2.4) is

$$g^{(5)} = dr \otimes dr + a^2(r) g^{(4)}$$ \hspace{1cm} (2.11)

and the Euclidean version of Eq.(2.10) reads

$$(\dot{a})^2 - k - \Lambda a^2 + \bar{A}^2 a^{-4\alpha + 2} = 0.$$ \hspace{1cm} (2.12)

Solutions of this equation describe 5-D instantons which can be used to construct brane-world models. Obviously, the precise form of the resulting instantons depends on the type of perfect fluid we use, i.e., on the choice of the parameter $\alpha$. The vacuum case $\alpha = 0$ (which here corresponds to simple redefinition of the cosmological constant: $\Lambda_D + \kappa^2 D A \rightarrow \Lambda_D$) was considered in paper \cite{11} for positive curvature $k = +1$ with solution

$$a(r) = l \sinh(r/l),$$ \hspace{1cm} (2.13)

where $l := \sqrt{-\Lambda}$ is the AdS radius.

In the present paper we investigate the case of dust ($\alpha = 1$) with $k = +1$. For this particular case the solution of Eq.(2.12) reads:

$$a(r) = \frac{l}{\sqrt{2}} \left( \sqrt{5} \cosh \frac{2r}{l} - 1 \right)^{1/2}, \quad -\infty < r < +\infty,$$ \hspace{1cm} (2.14)

\footnote{5-D AdS solutions of the Einstein equations are usually motivated by M-theory \cite{14}. However, studying multidimensional cosmological brane-world solutions with positive bulk cosmological constant \cite{17} is also of interest and we shall consider models with non-negative bulk cosmological constant in section 6.}

\footnote{See e.g. \cite{18} and references therein for the discussion of the equations of state for different kinds of cosmic defects in usual 4-D universe. In these references a regime where the number of defects per co-moving volume is a constant is considered.}
where \( b = 1 + 4A \bar{A}^2 \). It can be easily seen that this solution is reduced to (2.13) in the limit: \( \bar{A}^2 \to 0 \implies b \to 1 \). Metric (2.13) with solution (2.14) describes a wormhole (the integration constant being taken in such a way that \( 
abla = 0 \) corresponds to the wormhole throat).

It is useful to present solution (2.14) in the conformal "time" gauge: \( dr = \pm ad{\eta} \). It can be easily seen that

\[
a(\eta) = l b^{1/4} \frac{dn b^{1/4}|\eta|}{sn b^{1/4}|\eta|} , \quad 4\eta K(m) \leq b^{1/4}|\eta| \leq 2(1 + 2n)K(m) , \quad n = 0, 1, 2, \ldots ,
\]

satisfies the corresponding equation

\[
\left( \frac{da}{d\eta} \right)^2 = a^2 - \Lambda a^4 + \bar{A}^2 = 0 .
\]

Functions \( dn \) and \( sn \) are Jacobian elliptic functions and \( K(m) = F(\pi/2|m) \) is the complete elliptic integral of the first kind \([19]\). For each of the coordinate intervals, defined by the choice of \( n \) and sign of \( \eta \), there is an one–one correspondence between solution (2.14) and wormhole (2.14) : \( b^{1/4}|\eta| \to 4\eta K(m) \implies r \to +\infty \), \( b^{1/4}|\eta| \to 2(1 + 2n)K(m) \implies r \to -\infty \), and \( b^{1/4}|\eta| = (1 + 4n)K(m) \implies r = 0 \) is the wormhole throat. Coordinates \( r \) and \( \eta \) are connected with each other by:

\[
b^{1/4}|\eta| = F(\varphi | m) = u \iff sn b^{1/4}|\eta| = \frac{1}{\cosh r/l} ,
\]

where \( \varphi := \arcsinh(1/\cosh(r/l)) \) and \( F(\varphi | m) \) is the incomplete elliptic integral of the first kind. In the limit \( \bar{A}^2 \to 0 \implies b \to 1 \implies m \to 1 \), functions \( dn(u|m) \to 1/\cosh|\eta| \), \( sn(u|m) \to \tanh|\eta| \) and solution (2.15) tends to \( a(\eta) \to l/\sinh|\eta| \), which in fact is solution (2.13) as expressed in the conformal "time" gauge.

### 3 Brane-world instantons

In this section, we use solutions (2.14) and (2.15) to construct brane-world instantons. This can be performed by excising regions with \( r > L \) to obtain the remaining two copies along the two 4-spheres at \( r = L \). The obtained instanton can be described by the following piecewise continuous function:

\[
a(r) = \left\{ \begin{array}{ll}
\frac{1}{\sqrt{b}} \left[ \sqrt{b} \cosh \left( \frac{\varphi}{b} \right) - 1 \right]^{1/2} , & -\infty < r \leq L \\
\frac{i}{\sqrt{b}} \left[ \sqrt{b} \cosh \left( \frac{2l(2L - r)}{b} \right) - 1 \right]^{1/2} , & L < r < +\infty
\end{array} \right.
\]

This function is continuous but not smooth at the gluing "point" \( r = L \). Due to the Lanczos-Israel junction condition, this results in the appearance of a 4-D spherical brane at \( r = L \) with a tension:

\[
T(r = L) = \frac{3}{\kappa_4^2} 3 \tilde{K}(L) = \frac{6}{\kappa_4^2} \frac{\sqrt{b} \sinh(2L/l)}{\sqrt{b} \cosh(2L/l) - 1} = \frac{6}{\kappa_4^2} l \frac{\sinh(L/l) \cosh(L/l)}{\sinh^2(L/l) + m_1} > 0 ,
\]

where \( m_1 = 1 - m \), \( \tilde{K}(L) \equiv K(L^+) + (L^-) \), and \( K(L) = -4a^{-1}(da/dr)|_{r=L} \) is the trace of the extrinsic curvature (for the case of 5-D metric (2.11) written in the form of the Gaussian normal coordinates).

Instantonic solution (3.1) is non-compact: the scale factor \( a(r) \) goes to \( \infty \) when \( r \to \pm \infty \), and cannot be used for the description of the brane-world birth from "nothing" (from the Euclidean region) because the probability of it in this case is equal to zero. On the other hand, as we shall see below, the bound state of the spin-2 gravitational perturbations (corresponding to the Newtonian gravity on the brane) is proportional to \( a^{3/2} \) and diverges when \( a \to \infty \). However, we can compactify this instanton identifying points corresponding to the throats at \( r = 0 \leftrightarrow r = 2L \). It changes the topology along the extra dimension from \( R \) to \( S^1 \). Now, the range of the variation of \( r \) is the interval \([0, 2L]\). Since the geometry is smoothly glued at these points, such procedure does not lead to the appearance of new branes.

This procedure for the construction of the brane-world instanton can be easily generalized to the case of an arbitrary number of parallel branes gluing one-brane manifolds at throats and identifying the two final opposite throats. For example, in the case of \( n \) branes, located at the distances \( r = L_i , \ i = 1, \ldots , n \) from throats, this instanton can be described by the following piecewise continuous function:

\[
a(r) = \sum_{i=1}^{n+1} a_i(r) \theta_i(r) , \quad 0 \leq r \leq 2 \sum_{i=1}^{n} L_i ,
\]

\[
a_i(r) = \frac{l}{\sqrt{b}} \left[ \sqrt{b} \cosh \left( \frac{2(2 \sum_{k=1}^{i-1} L_k - r)}{l} \right) - 1 \right]^{1/2} , \quad L_{i-1} + 2 \sum_{k=1}^{i-1} L_k \leq r \leq L_i + 2 \sum_{k=1}^{i-1} L_k ,
\]

Instanton (3.1) can also be compactified by additional cutting at distances \( \Delta r = L_1 < L \) from the two sides of the brane, gluing then along two cuts. Geometry is not smoothly matched at this surface which results in a new negative tension brane. However, in our paper we will not consider such a compact instanton because the first one, with the wormhole throat identification, seems to us more elegant and it leaves a possibility for 5-D baby-universe nucleation at the throat.
where \( L_0 \equiv L_{n+1} \equiv 0 \) and

\[
\theta_i(r) = \tilde{\eta}(r - r_{i-1}) - \tilde{\eta}(r - r_i) = \begin{cases} 
0 & , \quad r < r_{i-1} \\
1 & , \quad r_{i-1} \leq r < r_i \\
0 & , \quad r \geq r_i
\end{cases}
\] (3.4)

are piecewise discontinuous functions, with \( \tilde{\eta}(r - r_i) \) being the usual step function equal to zero for \( r < r_i \) and becoming unity for \( r \geq r_i \). We have redefined the coordinate \( r \) in such a way to cover the range of variable for our solution by one coordinate chart. In this case the \( i \)-th brane has coordinate \( r = r_i = L_i + 2 \sum_{k=1}^{i-1} L_k, \ i = 1, \ldots, n \) and the \( i \)-th throat is located at \( r = r_{(th)} = 2 \sum_{k=1}^{i-1} L_k, \ i = 1, \ldots, n \). The throat is described by \( L \equiv \tilde{\eta}(r) \) and \( \eta \equiv \tilde{\eta}(r) \) (see figure 1). Each of these branes has a tension given by Eq. (3.2) with the evident substitution \( L \rightarrow L_i \) for the \( i \)-th \( (i = 1, \ldots, n) \) brane.

The similar procedure for construction of brane-world instanton can be performed in the conformal gauge. For example, one-brane instanton can be obtained from the solution (2.15) if we take two wormholes with \( \eta \equiv 0 \) and \( n = 0 \), cut them at \( |\eta| = \eta_0 < b^{-1/4} K(m) \), excise regions \( |\eta| < \eta_0 \) and glue them along this cut. The obtained obtained brane-world instanton is:

\[
a(\eta) = l b^{1/4} \frac{dn b^{1/4}(|\eta| + \eta_0)}{sn b^{1/4}(|\eta| + \eta_0)},
\] (3.5)

where we redefined the coordinate \( \eta \) in such a way \( (|\eta|_{old} \rightarrow |\eta|_{new} + \eta_0) \) to cover this instanton by one coordinate chart. Here, \( 0 \leq b^{1/4} |\eta| \leq 2 K(m) - b^{-1/4} \eta_0 \) and \( 0 \leq b^{1/4} |\eta| \leq K(m) - b^{-1/4} \eta_0 \) (with the identification \( -b^{-1/4} K(m) + \eta_0 \leftrightarrow b^{-1/4} K(m) - \eta_0 \)) correspond respectively to non-compact and compact instantons with brane at \( \eta = 0 \) and tension

\[
T(\eta = 0) = \frac{6}{\kappa^2 l} \frac{\tanh \xi_0}{\tanh^2 \xi_0} = \frac{6}{\kappa^2 l} \frac{1 - 2 \eta^2}{1 - m \eta^2} > 0,
\] (3.6)

which coincides with Eq. (3.2) if \( \eta = \eta_0 \) and \( r = L \) are connected by formula (2.17). In Eq. (3.6) we use the shortcut: \( \tilde{\eta} \equiv b^{1/4} \eta \) which we shall employ. It is necessary to note that the distance between the throat, and the brane is: \( \tilde{\xi}_0 = K(m) - \tilde{\eta}_0 \) and parameter \( \tilde{\xi}_0 \) becomes indefinite in the limit \( m \rightarrow 1 \) because in this case \( K(m) \rightarrow \infty \). In term of parameter \( \xi_0 \), tension (3.6) reads:

\[
T = \frac{6}{\kappa^2 l} \frac{1}{\sqrt{m_1}} \frac{dn \xi_0}{dn \xi^2} = \frac{6}{\kappa^2 l} \frac{1}{m_1} \frac{dn \xi_0}{dn \xi^2 - m_1} > 0,
\] (3.7)

which, of course, coincides with expression (3.2), as it can be checked using the relations \( sn^2 \xi_0 = \sinh^2 \frac{\xi_0}{(\cosh^2 \frac{\xi_0}{2} - m)} \), \( cn^2 \xi_0 = m_1/\cosh^2 \frac{\xi_0}{2} - m \) and \( dn^2 \xi_0 = m_1 \cosh^2 \frac{\xi_0}{2} / \cosh^2 \frac{\xi_0}{2} - m \).

In this gauge, the compact \( n \)-brane instanton (with branes at distances \( \eta = \xi_i, i = 1, \ldots, n \) from the throats) can be written in a form similar to expression (3.3):

\[
a(\eta) = \sum_{i=1}^{n+1} a_i(\eta) \theta_i(\eta), \quad 0 \leq \eta \leq \sum_{i=1}^{n} \xi_i,
\] (3.8)

\[
a_i(r) = l b^{1/4} \frac{dn b^{1/4} K(m) - 2 \sum_{k=1}^{i-1} \xi_k | m}{dn b^{1/4} K(m) - 2 \sum_{k=1}^{i-1} \xi_k | m}, \quad \xi_{i-1} + 2 \sum_{k=1}^{i-2} \xi_k \leq \eta \leq \xi_i + 2 \sum_{k=1}^{i-1} \xi_k,
\]

where \( ds(u|m) \equiv dn(u|m)/sn(u|m) \), \( \xi_0 \equiv \xi_{n+1} \equiv 0 \) and

\[
\theta_i(\eta) = \tilde{\eta}(\eta - \xi_{i-1}) - \tilde{\eta}(\eta - \xi_i) = \begin{cases} 
0 & , \quad \eta < \xi_{i-1} \\
1 & , \quad \xi_{i-1} \leq \eta < \xi_i \\
0 & , \quad \eta \geq \xi_i
\end{cases}
\] (3.9)

It can be easily seen that (3.8) is a piecewise continuous function. We have also redefined the coordinate \( \eta \) to cover our solution by one coordinate chart. In this case the \( i \)-th brane has coordinate \( \eta = \eta_i = \xi_i + 2 \sum_{k=1}^{i-1} \xi_k, i = 1, \ldots, n \), the throats have coordinates \( \eta = \eta(\xi_i) = 2 \sum_{k=1}^{i-1} \xi_k, i = 1, \ldots, n + 1 \) and "points" \( \eta = \eta_0 = 0 \) and \( \eta = \eta_{n+1} = 2 \sum_{i=1}^{n} \xi_i \) are identified with each other due to the \( S^1 \)-symmetry. Each of the branes has tension with the form given by Eq. (3.2), with an evident substitution \( \xi_0 \rightarrow \xi_i \) for the \( i \)-th \( (i = 1, \ldots, n) \) brane.

\footnote{We put a tilde above it to distinguish from the conformal time \( \eta \). Obviously, \( \theta_i^p = \theta_i, p > 0 \); \( \theta_i \theta_j = 0 \), \( i \neq j \) \( \Rightarrow a^p = \sum_{i=1}^{n} \theta_i^p \theta_i \), \( \forall p \) and \( \theta_i^p = \theta_i^p | r = r_i^- - \delta(r - r_i) \). Here, the wormhole pinches off because there is no matter in this limit.}
4 Volcano and Kurile-ridge potentials

It is well known for metrics of the form (2.2) and (2.14), rewritten in the conformal time gauge, that if tensor metric perturbations \( \delta g_{\mu\nu}(X) = h_{\mu\nu}(\eta, x) \) is taken in the gauge: \( h_{\eta\eta} = h_{\eta\mu} = h^\mu_{\alpha\nu} = 0 \) (where the semicolon denotes a covariant derivative with respect to \( d \)-dimensional metric \( g^{(d)}_{\mu\nu}(x) \)) and is normalized and factorized as \( h_{\mu\nu}(\eta, x) = a^{-(d-5)/2}(\eta) \psi(\eta) h_{\mu\nu}(x) \), then the radial profile of the perturbations satisfies a Schrödinger-like equation:

\[
\left( - \frac{d^2}{d\eta^2} + V(\eta) \right) \psi(\eta) = M^2 \psi(\eta),
\]

in which \( V(\eta) \) is the so-called "volcano" potential

\[
V(\eta) = \frac{\left( \frac{d^2}{d\eta^2} a^{d-1} \right)}{a^{d-1}},
\]

and \( M \) defines a mass spectrum of spin-2 perturbations \( h_{\mu\nu}(x) \) moving in the \( d \)-dimensional background metric \( g^{(d)}_{\mu\nu}(x) \). If the spectrum of mass starts from \( M = 0 \), then a state corresponding to this value of \( M \) describes usual 4-dimensional (for \( d = 4 \)) massless gravitons responsible for the standard Newtonian gravity.

It can be seen that for \( M = 0 \) the trivial solution of Eq.(4.1) is

\[
\psi \propto a^{d-1}. \tag{4.3}
\]

Thus, if the scale factor \( a \) has maxima at branes, this zero-mode state is localized at the branes. On the other hand, if the shape of the volcano potential is such that it suppresses the Kaluza–Klein modes with \( \eta \) for all models we consider in the present paper, as we are going to show in what follows.

Let us first consider the one-brane case described by Eqs.(3.1)–(3.3) (or (3.1) and (3.2)). The scale factor \( a \), defined by Eq.(2.4), has a local maximum at the brane \( \eta = 0 \), then tends to its minima at the throat positions \( |\eta| = K(m) - \tilde{\eta}_0 \), and diverges at the limit \( |\eta| \to 2K(m) - \tilde{\eta}_0 \). Thus, for the non-compact model the radial profile \( \psi \propto a^{3/2} \) is delocalized in this limit. However, in the compact case (with identification of the throats) the local maximum at the brane becomes the global one (see figure 2-a) and \( \psi \) provides the desired zero-mode bound-state. The volcano potential reads in this case:

\[
V(\eta) = \frac{15}{4} b^{1/2} \left( \frac{1}{(\text{sn}(\xi_0 - |\eta|))} \right)^2 - 3 \frac{3}{4} \left( \frac{3 - m \left( \frac{\text{cn}(\xi_0 - |\eta|) \delta(\eta)}{\text{dn}(\xi_0 - |\eta|)} \right)^2}{\text{sn}(\xi_0 - |\eta|)} \right) - 3 b^{1/4} \frac{\text{cn}(\xi_0 - |\eta|) \delta(\eta)}{\text{dn}(\xi_0 - |\eta|)} \tag{4.4}
\]

Thus, at the brane, \( \delta \)-function–provides the zero-mode localization and the wings of the potential clearly provide the suppression of the Kaluza–Klein modes: the wings approach maximum at the brane

\[
V(\theta)|_{\eta = K(m) - \tilde{\eta}_0 = \xi_0} \equiv V_{\text{th}} = \left\{ \begin{array}{ll} (3/2) \sqrt{b} & \text{if } b \neq 1 \\ 9/4 & \text{if } b = 1 \end{array} \right. \tag{4.5}
\]

Two different values of the minimum result from the two different asymptotic behaviours of the function \( \text{cd}(u|m) \equiv \text{cn}(u|m)/\text{dn}(u|m) \to 0/\sqrt{1 - m} \) when \( u \to K(m) \). Hence, this fraction equals to \( 0 \) if \( m \neq 1 \) and goes to \( 1 \) when \( m \to 1 \). The second value of Eq.(4.5) \( V_1 = 9/4 \) was obtained in \( \overline{\text{II}} \) for model (2.13). Thus, Kaluza–Klein modes starts at

\[
M = \sqrt{V_{\text{th}}}, \tag{4.6}
\]

and this mass gap separates the zero-mode from other massive Kaluza–Klein modes\footnote{It is clear, that to an observer on a i-th brane at \( r = r_i \), the physical metric is the induced metric on this brane: \( g^{(4)}_{\mu\nu}(r_i) = a^2_i(r_i) g^{(d)}_{\mu\nu} \). It results in an appropriate rescaling of effective physical values of, for example, effective 4-D cosmological and gravitation constants on the brane (see \( \overline{\text{II}} \)). Obviously, physical Kaluza–Klein masses for this observer are similarly rescaled, \( M \to m_{i(\text{ph})} = M/a_i(r_i) \).} It is obvious that the spectrum of the Kaluza–Klein modes is discrete for the compact models.
In the case of the compact n-brane models described by (3.8) the radial profile reads:

\[ \psi \propto a^{3/2}(\eta) = \sum_{i=1}^{n+1} a_i^{3/2}(\eta) \theta_i(\eta), \]  

(4.7)

where we used the properties of the \( \theta \)–function (see footnote 3). This formula shows that the massless graviton (zero-mode) is localized on each of the branes as they approach their local maximum at the branes (see Eq. (3.3)).

The volcano potential (4.3) in this case can be written in a more compact form in the proper "time" gauge

\[ V = a^{-3/2} \frac{d^2 a}{d\eta^2} a^{3/2} = a^{-3/2} \left( a \frac{da}{d\eta} \frac{d^2 a}{d\eta^2} a^{3/2} + a^2 \frac{d^2 a}{d\eta^2} a^{3/2} \right) \]

\[ = \sum_{i=1}^{n+1} \bar{V}_i(r) \theta_i(r) - \frac{16}{3} \kappa_5^2 \sum_{i=1}^n a_i^4 T(r) \delta(r - r_i), \]

(4.8)

where

\[ \bar{V}_i(r) = \frac{9}{4} \left( \frac{da_i}{dr} \right)^2 + \frac{3}{2} b \frac{d^2 a_i}{dr^2} \]

\[ = 15 \sqrt{b} \cosh^2(r - 2 \sum_{k=1}^{i-1} L_k)/l \frac{\sinh^2(r - 2 \sum_{k=1}^{i-1} L_k)/l + (4/5)m_1}{\sinh^2(r - 2 \sum_{k=1}^{i-1} L_k)/l + m_1} - \frac{3}{2} \sqrt{b}. \]

Here, we used the expressions (3.3) and (3.4) for the functions \( a_i \) and \( \theta_i \) and the tension \( T(r_i) \) has the form of Eq. (3.2) with the replacement \( L \to L_i, i = 1, \ldots, n \). It can be seen that potential (4.3) has its minima at the throats, \( V(r)|_{r=r_i} = V_{th}, i = 1, \ldots, n \), defined by expression (4.3), and reaches its local maxima at the branes,

\[ V_{(br)} = 15 \sqrt{b} \cosh^2(L_i/l) \frac{\sinh^2(L_i/l) + (4/5)m_1}{\sinh^2(L_i/l) + m_1} - \frac{3}{2} \sqrt{b}, \quad i = 1, \ldots, n, \]

(4.10)

which tend to \((15/4) \cosh^2(L_i/l) - 3/2\) in the limit \( b \to 1 \), corresponding to the vacuum case (2.13).

Thus, in the n-brane case, potential (4.3) has the form of a string (closed string for compact models) of volcano potentials (see figure 3) and can be named as "Kurile-ridge potential". Obviously, this form of the potential entails suppression of the massive Kaluza-Klein modes at the branes.

To conclude this section, we would like to remark an interesting possibility arising from the special form of the Kurile-ridge potential with periodic structure. In this case we obtain a band structure for the mass spectrum of the Kaluza-Klein modes and only masses from these bands are allowed to exist (see e.g. [9]). This would give rise to separation between the zero-mode and the massive Kaluza-Klein modes even larger than those induced by the mass gap (4.4).

**5 Brane-world birth from "nothing"**

In this section we investigate the possibility for the creation of brane worlds from "nothing". To be more precise, we interpret the brane-world instantons described in section 3 as a semiclassical paths for the quantum tunneling from the Euclidean region ("nothing").

Euclidean metric (2.11) was obtained from the Lorentzian one by the Wick rotation \( \tau \to -ir \). For the compact solutions of section 3 this Euclidean metric describes the compact 5-D brane-world instantons. It is clear that back Wick rotation \( r \to i \tau \) will destroy these branes leaving instead of the brane a solution corresponding to 5-D baby universe branching off from the wormhole throats at \( r = 0 \). This baby universe represents 5-D FRW-like universe filled with dust and a negative cosmological constant. It has a maximum at time corresponding to the wormhole throat and bounces from a minimum occurring when the scale factor equals to zero (usual FRW-type singularity). However, there is a possibility for the analytical continuation to the Lorentzian space-time which preserves branes. As it was printing out in paper [9], it can be done for the case when 4-D positive curvature metric \( g^{(4)} \) in Eq. (2.11) is the metric of a 4-sphere:

\[ ds^2_E = dr^2 + a^2(r)(dx^2 + \sin^2 \chi d\Omega_{(3)}^2). \]

(5.1)

Then, the analytic continuation of the azimuthal coordinate \( \chi \):

\[ \chi \longrightarrow iHt + \frac{\pi}{2}, \]

(5.2)
results in a Lorentz metric with evolving branes:

\[ ds_L^2 = dr^2 + H^2 a^2(r)(-dt^2 + \frac{1}{H^2} \cosh^2 H t \, d\Omega^2_{(3)}). \]  
\[ (5.3) \]

Here, parameter \( H \) is chosen in such a way that \( t \) describes the proper time on the brane. For example, in the one-brane-case \((3.1)\):

\[ Ha(r)_{|_{r=L}} = 1 \quad \implies \quad H = \frac{1}{a(L)} = \frac{\sqrt{2}}{l} \left( \sqrt{b} \cosh \frac{2L}{l} - 1 \right)^{-1/2}. \]
\[ (5.4) \]

The compact version of solution \((3.1)\) describes a 4–D spherical brane of radius \( a(L) = H^{-1} \) enclosing 5–D space with frozen dust and negative cosmological constant (AdS bulk). The south pole of the 4–D sphere with the azimuth coordinate \( \chi = 0 \) can be treated as "nothing". The birth of the brane-world takes place at the time \( t = 0 \) which corresponds in the Euclidean region to \( \chi = \pi/2 \) (the equator of the 4–D sphere) where the sections \( \chi = \text{const} \) of the 4–D sphere reach their maximum with radius \( H^{-1} \). After birth, the brane represents an evolving 3–D sphere with initial radius \( H^{-1} \). More precisely, it is an inflating de Sitter space with the Hubble constant \( H^{-1} \) (see figure 4).

In the case of \( n \) branes described by \((\Sigma^3)\), we can introduce for each of the coordinate patches between the throats \( r_{(th)i} \leq r \leq r_{(th)i+1} \) the following transformation:

\[ \chi \rightarrow iH_i t_i + \frac{\pi}{2}, \]  
\[ (5.5) \]

where

\[ H_i = \frac{1}{a(r_i)} \]  
\[ (5.6) \]

and \( r_i \in [r_{(th)i}, r_{(th)i+1}] \) is the position of the \( i \)-th brane \((i = 1, \ldots, n)\). After this continuation, the Lorentzian metric is:

\[ ds_L^2 = dr^2 + H^2 a^2(t)(-dt^2 + \frac{1}{H^2} \cosh^2 H t_i \, d\Omega^2_{(3)}), \]  
\[ (5.7) \]

where coordinates \( t_i \) are glued to each other at the throats:

\[ t_i = a_i t_j. \]
\[ (5.8) \]

Thus, each of \( n \) branes at \( r = r_i \) represents a de Sitter space with its own proper time \( t_i \) and own Hubble constant \( H_i \).

To conclude this section, we would like shortly comments some alternative possibilities of the analytic continuation. As we have noted before, the analytical continuation \( r \rightarrow ir \) leads to the creation of a 5–D baby universe with scale factor,

\[ a(\tau) = \frac{l}{\sqrt{2}} \left( \sqrt{b} \cos \frac{2\tau}{l} - 1 \right)^{1/2}, \quad 0 \leq \tau \leq \frac{l}{2} \arccos \frac{1}{\sqrt{b}}. \]
\[ (5.9) \]

From this model we can actually construct a spherical 3–D brane by using a similar procedure to that we have described in section 3; that is excising regions with \( \chi > L < \pi \) for two identical 4-sphere and gluing the remaining two copies along 3–spheres \( \chi = \pm L \). The brane-world model in this case is described by the metric:

\[ ds_L^2 = -dr^2 + a^2(\tau)(d\chi^2 + a^2(\chi) \, d\Omega^2_{(3)}), \]  
\[ (5.10) \]

where the scale factor \( a(\tau) \) is defined by \((5.9)\) and the scale factor \( a(\chi) \) reads

\[ a(\chi) = \begin{cases} \sin \chi, & 0 \leq \chi \leq L < \pi \\ \sin(2L - \chi), & L \leq \chi \leq 2L \end{cases} \]
\[ (5.11) \]

In this model, both the additional space as well as 3–D brane have the same dynamical behaviour which is described by \((\Sigma^3)\).

We turn finally to another interesting possibility consisting in the simultaneous analytic continuation in two directions: \( \chi \rightarrow \pi/2 + \chi \) and \( r \rightarrow ir \). It results in a metric:

\[ ds_L^2 = -dr^2 + a^2(\tau)(-dt^2 + \cosh^2 t \, d\Omega^2_{(3)}), \]  
\[ (5.12) \]

which describes a multidimensional/multitemporal solution of the Einstein equations. Relative to time \( \tau \) we have the FRW-type universe \((\Sigma^3)\) and with respect to time \( t \) we obtain the de Sitter universe with the Hubble constant \( H = 1 \). The number of spatial coordinates here is equal to usual three. In the present paper we shall not investigate in detail solutions \((5.10)\) and \((5.12)\) postponing their study for future work.
6 Other dust brane-world models

In previous sections we concentrated on the investigation of the solution of Eqs. (2.10) and (2.12) in the case of dust ($\alpha = 1$) when the bulk contains a negative cosmological constant ($\Lambda_5 < 0$) and 4–D metric $g^{(4)}$ describes a space with positive constant curvature ($k = +1$). Let us consider now the dust brane-world models for other combinations of parameters $\Lambda_5$ and $k$.

1) Negative bulk cosmological constant: $\Lambda_5 < 0$

a) Flat brane-world model: $k = 0$

The solution for the Euclidean Eq. (2.12) in this case is

$$a(r) = \frac{l}{\sqrt{2}} (\cosh \frac{2r}{l} + 1)^{1/2}, \quad -\infty < r < +\infty$$

(6.1)

which describes a wormhole with metric (2.11). The constant of integration is taken in such a way that $r = 0$ corresponds to the wormhole throat. To obtain a compact brane-world instanton we suppose that $g^{(4)}$ is the compact metric.

The compactness of positive curvature spaces is evident. However, Ricci-flat spaces and negative curvature spaces can be compact too. This can be achieved by appropriate periodicity conditions for the coordinates or, equivalently, through the action of discrete groups $\Gamma$ of isometries related to face pairings and manifold’s topology [23]. The simplest example of Ricci-flat compact spaces is given by $D$ - dimensional tori $T^D = \mathbb{R}^D / \Gamma$. In the case of negative curvature spaces, $d$–dimensional spaces of constant negative curvature are isometric to the open, simply connected, infinite hyperbolic space $H^d$. But there exists also an infinite number of compact, multiply connected, hyperbolic quotient manifolds $H^d / \Gamma$.

Thus, for simplicity we suppose that Ricci-flat space in our case is 4–D torus. The brane world instantons can be obtained by the same method as it was done in sections 3 with physically similar properties described in section 4. Analytic continuation to the Lorentzian region can be performed for any of coordinates of the 4–D torus. Then, we obtain a brane-world model with static flat brane (3–D torus) or with a number of such parallel branes. A multitemporal model can be also obtained here by analogy with the construction of metric (6.1).

b) Hyperbolic brane-world model: $k = -1$

In this case, solution of Eq. (2.12) is

$$a(r) = \frac{l}{\sqrt{2}} \left( \sqrt{b} \cosh \frac{2r}{l} + 1 \right)^{1/2}, \quad -\infty < r < +\infty.$$  

(6.2)

Metric (2.11) with this solution again describes a wormhole. The main difference between this model and the wormhole with the scale factor (2.14) consists in different 4–D metrics. A multitemporal model can be also obtained here by analogy with the construction of metric (6.1).

The compact brane-world instantons can be constructed here by the same procedure as that was used in section 3 and have the same qualitatively physical properties as those described in section 4. However, for metric (6.2) branes are bent 4–D hyperbolic (compact) spaces. It also follows from eq. (6.3) that the analytic continuation of the form (6.2) to the Lorentzian space-time is impossible here. However, we still can perform the continuation $r \to i\tau$ to obtain a 5–D baby universe. Then, brane-world models can be obtained by cutting and gluing the compact hyperbolic space $H^4 / \Gamma$ in the same way as it was done for metric (6.10) and scale factor (5.11).

For this case any multitemporal model is absent.

2) Positive bulk cosmological constant: $\Lambda_5 > 0$

a) Positive curvature brane-world model: $k = +1$

It can be easily seen that for positive bulk cosmological constant, Euclidean Eq. (2.12) can have a solution only for positive curvature of the 4–D space and this solution is a periodic function

$$a(r) = \frac{l}{\sqrt{2}} \left( 1 - \sqrt{\bar{b}} \cosh \frac{2(r - r_{-})}{l} \right)^{1/2}, \quad 2\pi n \leq \frac{2(r - r_{-})}{l} \leq (2n + 1)\pi, \quad n = 0, 1, 2, \ldots, \quad (6.4)$$

where $r_{-}$ is an integration constant, $\bar{b} \equiv 1 - 4\Lambda_5 \Lambda^2 > 0 \rightarrow \Lambda^2 < 1/(4\Lambda)$ and $\Lambda \equiv \Lambda_5 / 6$. Solution (6.4) describes an infinite string of wormholes with throats at $|r - r_{-}|/l = n\pi$ which can be smoothly glued at points of their maxima $|r - r_{-}|/l = (2n + 1)\pi/2$. Compact brane-world instantons can again be constructed similarly to how we did in section 3. However, for solution (6.4) the sign of the brane tensions and positions of the radial profile maxima depend on the choice of cutting. If we cut wormholes before their maxima, we obtain the string of 4–D spherical positive tension branes with the zero-mode localization on them. But if we cut the wormholes after...
their maxima, then branes will have negative tensions and zero-mode radial profile will have local minima at these branes. So, we shall not consider the latter type of branes. For the string of positive tension branes, birth from "nothing" occurs exactly like it was described in section 5.

However, in contrast to the negative cosmological bulk model, we have now two types of 5-D baby universes. There are that branch off from their throats where they get their maxima and reaches singular minima at the bouncing points. Other baby universes branch off from the maxima of the wormholes and have at that moment non-singular minima. The latter type of baby universes describes 5-D asymptotically de Sitter universes. For both types we can construct also baby universe brane world models of the form (5.10). There are two types of multitemporal models of the form of Eq.(5.12) depending on which of the baby universes (described above) is taken for.

b) Flat 5-D Lorentzian model: \( k = 0 \)

In this case the Lorentzian solution of Eq. (2.10) is

\[
a(\tau) = \sqrt{(\Lambda A^2)^{1/4} \left[ \sinh \left( \frac{2|\tau - \tau_0|}{l} \right) \right]^{1/2}}, \quad 0 \leq |\tau - \tau_0| < +\infty.
\]

It describes an asymptotically de Sitter 5-D universe, with Ricci-flat 4-D \( g^{(4)} \) metric. Obviously, it is impossible to construct branes out from such a metric, but multitemporal models of the type given by Eq.(5.12) with flat Lorentzian 4-D metric exist in this case.

c) Hyperbolic brane-world model: \( k = -1 \)

Depending on the sign of \( \tilde{b} \equiv -1 + 4\Lambda A^2 \), there are 3 types of Lorentzian solutions of Eq.(2.10),

\[
a(\tau) = \begin{cases} 
\sqrt{\frac{1}{2} \cosh(\arcsinh(1/\sqrt{\tilde{b}})) + 2|\tau - \tau_0|/l} - 1 \right) \right]^{1/2} = 0, \\
\sqrt{\frac{1}{2} \cosh(2|\tau - \tau_0|/l) + 2\sqrt{\Lambda A^2 \sinh(2|\tau - \tau_0|/l) - 1} \right]^{1/2} = 0 < 0.
\end{cases}
\]

where \( 0 \leq |\tau - \tau_0| < +\infty \). All of these solutions describe asymptotically de Sitter 5-D universes with 4-D hyperbolic compact spaces \( H^4/\Gamma \). The brane-world models for these solutions can be obtained by a similar procedure to that was described for equations (5.10) - (5.11). The multitemporal models are absent in this case, c).

2) Zero bulk cosmological constant: \( \Lambda_5 = 0 \)

In this case, the solutions of the Euclidean Eq.(2.12) only exist for positive curvature 4-D space \( g^{(4)} \).

a) Positive curvature brane-world model: \( k = +1 \)

The Euclidean solution is

\[
a(\tau) = \sqrt{A^2 + r^2}, \quad -\infty < r < +\infty,
\]

which describes a 5-D wormhole (the constant of integration being again chosen in such a way that \( r = 0 \) corresponds to the wormhole throat). It is interesting to note that qualitatively the same wormholes were obtained in papers of references [22, 23] for 4-D models with conformally coupled scalar field (which is equivalent to radiation) and was used also in paper [24] for investigations of wormholes. Qualitatively the same instanton for 4-D models with axionic field (which is equivalent to ultra-stiff matter) was found in reference [24].

Expression (6.7) is qualitatively similar to solution (2.14). We can construct the compact brane-world instantons with spherical branes by the same procedure as it was used in section 3, with the same physically properties as those described in sections 4 and 5.

b) Flat 5-D Lorentzian model: \( k = 0 \)

The Lorentzian solution of Eq.(2.10) for this case reads:

\[
a(\tau) = \sqrt{2A^2|\tau - \tau_0|}, \quad 0 \leq |\tau - \tau_0| < +\infty.
\]

It describe 5-D universes with FRW-like initial singularity. The time derivative of the scale factor \( da/d\tau \to 0 \) when \( |\tau| \to \infty \). For this particular combination of parameters \( \Lambda_5 \) and \( k \) there is no brane world, though multitemporal model analogous to that of solution (5.12) can still be obtained.

c) Hyperbolic brane-world model: \( k = -1 \)

The Lorentzian solution of the Eq.(2.14),

\[
a(\tau) = \sqrt{-A^2 + [A^2 + (\tau - \tau_0)^2]}, \quad 0 \leq |\tau - \tau_0| < +\infty,
\]

describes asymptotically flat (Milne-type) 5-D hyperbolic space-time. For small times this solution coincides with solution (5.8). Brane world models for solution (5.8) can be obtained by similar procedures to that used for models (5.10) and (5.11). Finally, multitemporal models of the type (5.12) are absent.
7 Conclusions

In this paper we have considered in some detail 5-dimensional cosmological models corresponding to the case of a bulk filled with perfect fluid and a cosmological constant, particularizing to the case of dust. Special attention has been given to the case of a negative cosmological constant in a bulk with the geometry of a four-sphere where we have found an asymptotically AdS wormhole instantonic solution. Brane-world instantons with a single 4-D spherical brane as well as with a string of such concentric branes can then be built up by using a cutting and gluing procedure. We have been able to obtain regular solutions which are free from any singularities at the origin of extra coordinates, and can be compactified so that the asymptotic divergences of the scale factor are prevented. Zero-mode massless gravitons are shown to be localized on these 4-D branes, so allowing such branes to nest Newtonian gravity. Analytical continuation from the brane instantonic metric to the Lorentzian regime leads to de Sitter 3-dimensional inflating branes. After birth of the inflating brane world from "nothing", the perfect fluid (dust) remains frozen: it is contained in the bulk but not on the brane. Here, inflation has pure geometrical origin: the Hubble constant of the inflating 3-D brane world is defined by the radius of the Euclidean 4-D spherical brane.

Other kinds of brane world models were also obtained and discussed, both for four-spheres and other spatial topologies. Thus, we have found a class of brane world models characterized by a common dynamical behaviour for extra-dimension and branes.

Of course, some important aspects of this research need further investigations including transition from inflationary branes into a matter-dominated brane universe, the physical consequences from the above mentioned brane-world model with dynamical equivalence between extra dimension and branes, as well as the meaning of the 5-dimensional spacetimes described by a metric with two timelike dimensions.

Acknowledgments

A.Z. thanks Instituto de Matemáticas y Física Fundamental, CSIC, for kind hospitality during preparation of this paper. A.Z. acknowledges support by Spanish Ministry of Education, Culture and Sport (the programme for Sabbatical Stay in Spain) and the programme SCOPES (Scientific co-operation between Eastern Europe and Switzerland) of the Swiss National Science Foundation, project No. 7SUPJ062239. M.B.L. is supported by a grant of the Spanish Ministry of Science and Technology. This investigation was supported by the GICYT under Research Project No. PB97-1218.

References

[1] Kaluza Th. Zum Unitätsproblem der Physik, Sitzungsber. d. Preuss. Akad. d. Wiss., 1921, 966.
Klein O. Quantentheorie und funfdimensionalen Relativitatstheorie, Zeitschrift für Physik., 1926, V. 37, 895-906.
[2] K. Akama, Pregeometry, (Lectures Notes in Physics, Gauge, Theory and Gravitation, Proc. Int. Symp. Gauge Theory and Gravitation, Nara, Japan, August 20-24, 1982), Springer-Verlag, 1983, Ed. K. Kikkawa, N. Nakaniishi and Nariai, P. 267 - 271.
[3] V. A. Rubakov and M.E. Shaposhnikov, Phys. Lett. B125, (1983), 136 - 138.
[4] M. Visser, Phys. Lett. B156, (1985), 22 - 25;
E.J. Squires, Phys. Lett. B167, (1986), 286 - 288;
M.D. Maia and V. Silveira, Phys. Rev D48, (1993), 954 - 957.
[5] N. Arkani-Hamed, S. Dimopoulos and G. Dvali, Phys. Lett. B429, (1998), 263 - 272, hep-ph/9803315.
[6] L. Randall and R. Sundrum, Phys. Rev. Lett. 83, (1999), 3370 - 3373, hep-ph/9905221.
[7] P. Horava and E. Witten, Nucl. Phys. B460, (1996), 506 - 521, hep-th/9510200;
E. Witten, Nucl. Phys. B471, (1996), 135 - 158, hep-th/9602000;
P. Horava and E. Witten, Nucl. Phys. B475, (1996), 94 - 114, hep-th/9603124.
[8] L. Randall and R. Sundrum, Phys. Rev. Lett. 83, (1999), 4609, hep-th/9906064.
[9] J. Garriga and M. Sasaki, Phys. Rev. D62, (2000), 043523, hep-th/9912118.
[10] I.I. Kogan, S. Mouslopoulos, A. Papazoglou, G.G. Ross and J. Santiago, Nucl. Phys. B584, (2000), 313 - 328, hep-ph/9912552.
[11] N. Arkani-Hamed, S. Dimopoulos, G. Dvali and N. Kaloper, Phys. Rev. Lett. 84, (2000), 586 - 589, hep-th/9907209.
[12] N. Kaloper, Phys. Lett. B474, (2000), 268 - 281, hep-th/9912123.
[13] S. Nam, JHEP 0004, (2000), 002, hep-th/9911237.
[14] J.W. York, Phys. Rev. Lett. 28, (1972), 1082 - 1085;
G.W. Gibbons and S.W. Hawking, Phys. Rev. D15, (1977), 2752 - 2756.
[15] A. Zhuk, Class. Quant. Grav. 13, (1996), 2163 - 2178.

[16] V.D. Ivashchuk, V.N. Melnikov and A.I. Zhuk, Nuovo Cim. B104, (1989), 575.

[17] S. Hayakawa, T. Hirayama and R. Kitano Simple brane world scenario with positive five dimensional cosmological constant, hep-th/0108109.

[18] R.A. Battye, M.Bucher and D. Spergel Domain wall dominated universe, astro-ph/9908047.
M. Bucher and D.N. Spergel, Phys. Rev. D60, (1999), 043505, astro-ph/ 9812022.

[19] M.Abramowitz and I.A. Stegun Handbook of matematical functions , Dover Publ. Inc., New York, 1972.

[20] M. Bouhmadi–López and A. Zhuk Comments on conformal stability of brane-world models, hep-th/0107227, to be published in Phys. Rev. D.

[21] G.F.R.Ellis, Gen. Rel. Grav. 2, (1971), 7;
D.D.Sokolov and V.F.Shvartsman, Sov. Phys. JETP, 39, (1974) 196;
H.V.Fagundes, Phys. Rev. Lett. 70, (1993) 1579; Gen. Rel. Grav. 24, (1992) 199;
M.Lachieze-Rey and J.-P.Luminet, Phys. Rep. 254, (1995) 135;
V.V.Nikulin and I.R.Shafarevich, Geometry and Groups , Nauka, Moscow, 1983.

[22] J.J. Halliwell and R. Laflamme, Class. Quant. Grav. 6, (1989), 1839.

[23] A. Zhuk, Phys. Lett. 176A, (1993), 176.

[24] S.W. Hawking, Phys. Rev. D37, (1988), 904.

[25] S.B. Giddings and A. Strominger, Nucl. Phys. B306, (1988), 890.
Figure 1: Compact $n$-brane-world instanton in the case $n = 3$. Each point in the figure represents a 3-D sphere. Lines with fixed coordinates $r = \text{const} \equiv r_0$ correspond to 4-D spheres with radii $a(r_0)$. The branes are 4-D concentric spheres surrounding the 5-D bulk.

![Diagram of a compact $n$-brane-world instanton](image1)

Figure 2: (a) Zero-mode radial profile $\psi \sim a^{3/2}$ for the case of the compact brane-world instanton. $\psi$ has its maximum at the brane with coordinate $\eta = 0$ and tends to the minimum at the identified throats of wormholes at $\eta_{th} = -b^{-1/4}K(m) + \eta_0 \leftrightarrow b^{-1/4}K(m) - \eta_0$.

(b) Volcano potential for the compact one-brane instanton. The wings of the potential evolves from its maximum $V_{br}$ at the brane ($\eta = 0$) to the minimum $V_{\eta_{th}}$ at the identified throats ($\eta = \eta_{th}$). Mouth of the volcano (inverse $\delta$-function) at $\eta = 0$ provides the zero-mode localization on the brane.

![Graphs showing zero-mode radial profile and volcano potential](image2)
Figure 3: Kurile–ridge potential (4.8) for the compact 5–brane-world instanton when \( n = 5 \). Each of the "volcanoes" has a local maximum \( V_{(br)} \) localized on the branes. Identical local minima \( V_{(th)} \) correspond to the wormhole throat positions \( r_{(th)} \).

Figure 4: Birth of the one-brane-world (\( n = 1 \)) from "nothing". The creation takes place at the time \( t = 0 \) and the 5–D Lorentzian metric is described by Eq. (5.3). The brane after birth represents an inflating 3–D sphere with initial radius \( H^{-1} \) that is de Sitter space-time with Hubble constant \( H^{-1} \).