Matrix theory for baryons: an overview of holographic QCD for nuclear physics

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Received 31 July 2012, in final form 5 July 2013
Published 3 October 2013
Online at stacks.iop.org/RoPP/76/104301

Abstract
We provide, for non-experts, a brief overview of holographic QCD (quantum chromodynamics) and a review of the recent proposal (Hashimoto et al 2010 (arXiv:1003.4988[hep-th])) of a matrix-like description of multi-baryon systems in holographic QCD. Based on the matrix model, we derive the baryon interaction at short distances in multi-flavor holographic QCD. We show that there is a very universal repulsive core of inter-baryon forces for a generic number of flavors. This is consistent with a recent lattice QCD analysis for \( N_f = 2, 3 \) where the repulsive core looks universal. We also provide a comparison of our results with the lattice QCD and the operator product expansion analysis.

(Some figures may appear in colour only in the online journal)

This article was invited by Gordon Baym.

Contents

1. M-theory for nuclear physics? 1
2. Holographic QCD 2
   2.1. What is holographic QCD? 2
   2.2. How holographic QCD is different from QCD 3
   2.3. Popular holographic models and their problems 5
3. Review: M(atrix) description of multi-baryon system 6
   3.1. Baryons are matrices 7
   3.2. Derived baryon spectrum 8
   3.3. Universal repulsive core of nucleons 8
4. Multi-flavor nuclear forces via holography 10
   4.1. Strangeness and holography 10
   4.2. The effective model of multi-baryon system 10
   4.3. Two-baryon configuration 11
   4.4. Explicit inter-baryon potential 11
   4.5. Universal repulsive core 12
5. A comparison with lattice QCD and OPE 12
Acknowledgments 14
References 14

1. M-theory for nuclear physics?
What is ‘M-theory’ for nuclear physics? Although the ‘M-theory’6 stands for a theory of everything that unifies all string theories [2], one can generalize the usage of this word not only for string theories but also for other subjects in physics. We can then ask ‘what is the M-theory for nuclear physics, if it exists?’

While this type of question is sometimes useful since it reminds us that there are mutual relations among the various research area subjects in physics, the answer to the above question seems obvious: the M-theory for nuclear physics is QCD (quantum chromodynamics), or more precisely, the standard model of elementary particles. Nucleons, which are the building blocks of nuclei, are the bound states of quarks and gluons in QCD. If one were able to solve QCD completely, one could derive all the properties of nuclei from QCD. In this sense, QCD is the M-theory for nuclear physics. However, QCD is notoriously difficult to solve due to its non-perturbative...
Figure 1. A conceptual merit of the AdS/CFT correspondence, for a possible bridge between elementary particle physics, hadron physics and nuclear physics.

nature, which makes quarks bind to each other. We therefore need a new technology in order to ‘derive’ or ‘understand’ nuclear physics in terms of QCD: the M-theory for nuclear physics.

Since such a new technology has been missing for many years, unfortunately, a relation between nuclear physics and QCD remains indirect so far, as shown in the left of figure 1. Standard nuclear physics is described by the quantum mechanics of multi-nucleons with the nucleon–nucleon (NN) potential (nuclear force), whose action is schematically given by

$$S = \int dt \left[ \sum_{s=1}^{A} \frac{M_N}{2} \sum_{M=1}^{3} (\partial_t x_M^s(t))^2 \right. $$

$$- \sum_{s_1 \neq s_2} V(x_M^{s_1} - x_M^{s_2}) + \cdots \right]$$

for $A$ nucleons. Their coordinates are denoted by $x_M^s(t)$ with $M = 1, 2, 3$ and $s = 1, \ldots, A$. The first term is the kinetic term of the nucleons with mass $M_N$, while the second term is the nuclear force. (There are other terms such as the spin–orbit interaction.) The current problem for understanding nuclear physics from QCD is that the nuclear force $V$ is constructed from the experimental data of NN scatterings, or in some cases from models, but not from QCD. In principle, the nuclear force should have been derived from QCD, as we all know that nucleons and hadrons are made of quarks and gluons, but it is very difficult.

It is very recent that the nuclear force has been calculated in QCD by employing numerical methods: lattice QCD [3, 4]. In addition there is an attempt to treat multi-nucleon systems, such as helium nucleus in lattice QCD [5], which opens the possibility of treating a part of nuclear physics directly from QCD. A huge number of quark contractions in large nuclei, however, remains a big obstacle in this direction. Although lattice QCD may become a new technology in the near future to derive nuclear forces and some properties of small nuclei numerically from QCD, it is of course more desirable if one can solve QCD analytically to understand nuclear physics without relying on numerical simulations. Unfortunately we do not yet have such an analytic tool, so hadron physics and nuclear physics are still, in a sense, disconnected from each other.

Under such circumstances, a new technology using string theory comes into play to analyze QCD non-perturbatively. This new technology is the renowned AdS (anti-de Sitter)/CFT (conformal field theory) correspondence, the gauge/gravity duality or the holographic principle [7, 8]. Even though the correspondence itself is still conjecture, this makes solving a certain limit of QCD-like gauge theories possible and offers a certain direct path from QCD to nuclear physics. If one can derive an action like (1.1) from QCD, it can be regarded as an effective theory for nuclear physics derived from M-theory.

In this paper, we review the recent progress in this direction as an application of the AdS/CFT correspondence to QCD, generally called holographic QCD. In [1], two of the authors (Hashimoto and Iizuka), together with Yi, derived the action of a multi-baryon system, using the AdS/CFT correspondence applied to large $N_c$ QCD, where $N_c$ is the number of colors. The action indeed has the form of (1.1) and it serves as a candidate for the bridge between QCD and nuclear physics.

As it was derived from the large $N_c$ QCD, the action is written only with two free parameters: the QCD scale and the QCD coupling. Therefore, we can check the derived theory just by calculating various observables in nuclear physics with this action and compare those with experiments, to test the validity of the action up to the approximations of the large $N_c$ and the strong coupling expansion. Explicitly demonstrated in the literature are:

(i) baryon spectroscopy [1],
(ii) the universal repulsive core of nucleons [1],
(iii) three-body nuclear forces [9],
(iv) the spin statistics of baryons [10] and
(v) the formation of atomic nuclei [11]

In all of these calculations, the results are qualitatively reasonable compared to the experiments. As we will describe in this paper, there are many more physical observables which can be calculated in this framework.

The first aim of this paper is to give a review of the effective action for the multi-baryon system [1], for non-experts of holographic QCD. The second aim is to show a new result on the short-distance force between baryons with multi-flavors that $N_f > 2$, where $N_f$ is the number of quark flavors.

This paper is organized as follows. The first part of this paper is mainly a review. In section 2, we give a brief review of holographic QCD, explaining what holographic QCD is and the differences between holographic QCD and real QCD. Then in section 3, we explain the nuclear physics action derived in holographic QCD, with emphasis on its properties, new insights and connection to nuclear physics, for non-experts.

In the second part of this paper, as an example of the applications of holographic QCD, we present not only known but also new results for baryon-baryon forces. In section 4, we

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7 See [6] for a recent proposal to reduce the number of quark contractions in multi-baryon systems.

8 Within the same framework, it has been reported that a wider class of results are compatible with experiments: the baryon spectrum was originally derived in [12] and the charge radii of baryons [13], the suppression of multi-nucleon forces [14], baryon spectra with three flavors [15], etc. The repulsive core has been calculated in the same manner [16]. See also an alternative approach given in [17].
calculate the short-distance part of baryon-baryon forces for the case of multi-flavors (the number of flavors larger than 2). We see that the repulsive core remains even for a generic number of flavors and thus find a universal repulsive core. The result is consistent with recent lattice results at $N_f = 3$ where in most of the channels there appears a repulsive baryon-baryon potential at short distances. Therefore, our results would also serve as another nontrivial consistency check. This part of the paper includes some technical details; readers who know holographic QCD and the matrix model approach of [1] can start at section 4, as it is written independently of sections 2 and 3. In section 5, as a summary of this paper, we provide a review of the recent lattice results for multi-flavors and also the operator product expansion (OPE), to compare them with our holographic results.

2. Holographic QCD

We explain what holographic QCD is in this section, for those who are not familiar with it. In particular, we stress what the assumptions are and what is ignored in holographic QCD. This makes the importance and the validity of the AdS/CFT matrix model approach clear to multi-baryon system and nuclear physics, which we shall review in the next section.

2.1. What is holographic QCD?

Holographic QCD relies on the AdS/CFT correspondence (or more generally the gauge/gravity correspondence), which claims an equivalence between a certain class of gauge theories and a certain class of string theories. The key quantities to an understanding of such an equivalence are D(Dirichlet)-branes in string theories. In the closed string description, D-branes can be regarded as black branes, a generalization of black holes in the supergravity theories, while they can describe some types of gauge theories in the open string interpretation. The gauge/gravity correspondence is a conjecture that these two types of gauge theories in the open string interpretation. The in the supergravity theories, while they can describe some can be regarded as black branes, a generalization of black holes in string theories. In the closed string description, D-branes under the understanding of such an equivalence are D(Dirichlet)-branes and a certain class of string theories. The key quantities to an claims an equivalence between a certain class of gauge theories (or more generally the gauge/gravity correspondence), which can be chosen freely; if it has

The gluonic gauge group. This is the essential point of holographic QCD: one has the gluonic $N_c$ D-branes and the additional D-brane giving the quark. The latter is called a ‘flavor’ D-brane. Now, as explained above, we replace the piled-up $N_c$ D-branes by a background ten-dimensional curved geometry, which is another way to look at the $N_f$ D-branes. This provides the AdS geometry, which corresponds to a strong coupling limit of the gluonic theory. Therefore, a flavor D-brane in this AdS geometry corresponds to a gluonic theory with a quark field. With $N_f$ flavor branes, we have $N_f$ flavor quarks. This is the virtue of holography: a complicated strongly coupled theory of gluons and a quark, which is mapped into a rather simple flavor D-brane theory in AdS geometry. In this way, one can use this duality as a tool to analyze the strong coupling limit of the gauge theory.

So, in short, a QCD-like gauge theory with gluons and quarks can be constructed by a set of D-branes and, taking a holographic dual to the gluonic D-branes, one can analyze the strong coupling limit of the QCD-like gauge theory from a flavor D-brane in a curved geometry background in 10 space–time dimensions. This is holographic QCD. Depending on the choice of the type, number and configuration of D-branes we introduce, one can construct many different models for holographic QCD, which describe different theories corresponding to different physical situations/processes. Note, however, that holographic QCD is different from true QCD in several aspects, which will be explained in the next subsection.

As for an introduction to the AdS/CFT correspondence itself, we suggest readers take a look at a popular review article [18], which summarizes the whole idea of AdS/CFT correspondence and also the early results. There are other reviews which focus on specific subjects in applying AdS/CFT to QCD. For example, for the meson sector in holographic QCD, see [19]. The point of the remaining part of this section is to stress what a problem in holographic QCD is and to explain how it is different from real QCD. This kind of review on a summary of the problems is rare in the literature, so we dedicate the remaining part of this section to that. Readers who are not interested in the detailed explanation of the problems can skip the remaining part of this section and jump to the next section where we review the holographic description of the multi-nucleon system responsible for nuclear forces and atomic nuclei.

2.2. How holographic QCD is different from QCD

In this subsection, for readers who are not familiar with the subject of holographic QCD (the AdS/CFT correspondence applied to QCD)9, we summarize important problems, which are addressed in holographic QCD. In particular, we make a stress on what the assumptions are and what is ignored in holographic QCD. This would make clear the importance and the validity of the AdS/CFT matrix model approach to multi-baryon systems and nuclear physics, which we shall review in the next section.

9 In this article and in most articles in this field, the word ‘AdS/CFT’ is equivalent to the word ‘gauge/gravity’ or ‘holography’. We say ‘bulk’ for gravity or string side calculations and ‘boundary’ for gauge-theory side calculations.
2.2.1. **Forced large $N_c$ and $\lambda$ limits.** The AdS/CFT correspondence in string theory is a conjecture on the equivalence between a gauge theory and a string theory in an asymptotically AdS background. Strictly speaking, the equivalence is *not* between gauge theory and gravity, but rather between gauge theory and string theory. Only when large $N_c$ and large $\lambda = g_{\text{QCD}}^2 N_c$ are taken, where $g_{\text{QCD}}$ is the gauge coupling and $\lambda$ is a ’t Hooft coupling of the large $N_c$ gauge theory, can the string theory side be approximated by a gravity theory with background geometries of weakly curved space–time. In this limit, low-energy excitations of the string, such as gravitons, are light, while the long string itself becomes very heavy. Due to the technical difficulty of solving string theory in a generically curved background, in almost all cases for holographic QCD, the two limits, large $N_c$ and large $\lambda$, are taken so that we can approximate string theory as a gravity theory.

These limits produce some crucial differences between holographic QCD and true QCD/nuclear physics. According to old string models, hadrons with higher spins are stringy excitations. Holographic QCD follows and generalizes the old string models based on QCD strings. In the AdS/CFT correspondence, the tension of the strings in the gravity side is $O(\lambda)$, so the stringy excitations become extremely heavy and, resultantly, the higher spin modes become heavy and decouple from the gravity excitations. This is the reason why string theory is approximated as a gravitational theory in the large $\lambda$ limit. On the other hand, this means that the highest spin excitation of the system is a spin 2 particle graviton and all other stringy modes whose spins are bigger than 2 become infinitely heavy as $\lambda$ goes to infinity. In nuclear physics, however, there are many hadronic excitations whose spins are bigger than 2 and all of these higher spin hadronic excitations have the same order mass scale compared with lower spin excitations. Therefore, in holographic QCD within the gravitational approximation, we cannot correctly describe such hadronic excitations with large spins.

One may then wonder why we do not directly try to solve string theory in an asymptotically AdS background, instead of using the gravity approximation. A problem is that a precise treatment of the fundamental strings in the curved geometry, i.e. the quantization of the string, is still missing in any formulations of string theory. We do not have a fundamental tool to analyze the string side so far and are waiting for a further development of methods to quantize strings in curved geometries.

In addition, we have $N_c = 3$ in QCD, while any quantities are computed at the leading order of the $1/N_c$ expansion at $N_c = \infty$ in holographic QCD. We therefore expect generically 33% or more uncertainties due to the large $N_c$ approximation. At present, computations of the sub-leading $1/N_c$ corrections, which correspond to those of loop corrections in the gravity side, are technically difficult. These all imply that the holography methods are better applied to reveal some robust features of QCD, which are independent on the values of $N_c$, but that they should not be employed to make a precise comparison with experiments.

We might wonder under what circumstances could the physical quantities be more insensitive to $N_c$. In the confining phase of QCD, as the color degrees of freedom are confined, we cannot directly observe the number of colors. We therefore naively might expect that physics is not dependent on the values of $N_c$. On the other hand, physics in the deconfining phase might be sensitive to a number of $N_c$. Interestingly, however, there are many successful calculations of holographic QCD in the deconfining phase, such as the shear viscosity [21], quark energy loss [22] etc in the quark gluon plasma phase, which show reasonable agreement with the experiments at RHIC and LHC. At this moment, we do not have a clear picture under what situation the large $N_c$ approximation is justified.

2.2.2. **Lack of the asymptotic freedom and multiple parameters.** QCD is specified by a unique energy scale $\Lambda_{QCD}$ as a result of the running coupling constant and the asymptotic freedom. In particular, the massless QCD has only this scale with no other parameter. In contrast to this, no scale exist in holographic QCD as long as conformal invariance is preserved. We therefore introduce operators, which break the conformal invariance by hand at some scale $\Lambda_{\text{cutoff}}$. As a result, the coupling constant in holographic QCD becomes scale dependent (the running coupling).

This leads us to a rather complicated situation where we have two scales in holographic QCD; one is the scale $\Lambda_{\text{cutoff}}$ we introduced and the other is the $\Lambda_{QCD}$, which is an emergent scale in low-energy hadron physics. In holographic QCD, the $\Lambda_{QCD}$ is a function of two input parameters, $\Lambda_{\text{cutoff}}$ and the ’t Hooft coupling constant $\lambda = g_s N_c$ (where $g_s$ is a string theory coupling constant). In principle, if one can take the double-scaling limit that both holographic scale $\Lambda_{\text{cutoff}}$ (at which particles absent in QCD appear) and $N_c$ are taken to infinity while $\Lambda_{QCD}$ and $\lambda$ are kept fixed.

Taking the double-scaling limit, however, is not an easy task: in order to make the gauge-theory coupling constant at the scale $\Lambda_{\text{cutoff}}$ weak, the coupling constant in the gravity side becomes so strong that the supergravity description is no more reliable. Moreover, as mentioned before, it is technically difficult to go beyond large $N_c$ and $\lambda$. As a result, the difficulty of taking the double-scaling limit remains in any holographic QCD model we consider.

Of course, one can say that the number of parameters, two given by $\Lambda_{\text{cutoff}}$ and $\lambda$, is still small compared to many other phenomenological models; it is good enough to have nontrivial checks and predictions in QCD.

It is noted that the coupling constant in the gauge theory becomes strong again beyond $\Lambda_{\text{cutoff}}$ due to supersymmetric

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10 One major progress along this line would be a correspondence between vector models and higher spin gauge theories [20], as an explicit toy model of the AdS/CFT correspondence in the $\lambda \to 0$ limit.

11 There are several examples which work beyond the large $N_c$ and large $\lambda$ limit in holographic QCD. One such example is the Wilson loop, which is nothing but a heavy string trajectory. We can calculate it at large $N_c$ but without any values of $\lambda$ using the so-called localization technique; we observe an agreement between the string side and the gauge-theory side. In addition, in holographic QCD, there is an attempt even in the large $\lambda$ limit to take into account the degrees of freedom corresponding to the massive open strings whose spins are bigger than 2 and quantize these massive stringy excitations in the weakly curved geometry [23]. This also gives a very good qualitative comparison with the experimental data.
particles which appear above $\Lambda_{\text{cutoff}}^{12}$. This property is completely different from the ordinary QCD, where the coupling constant becomes smaller and smaller at higher and higher energies (the asymptotic freedom).

2.3. Popular holographic models and their problems

Next we shall look at the popular holographic models that are widely used for various purposes, in particular from the viewpoint of their strong points and limitations. We stress the point that, depending on the physical quantities of interest, one can choose a holographic model among many. We here briefly review five models popularly used in the top-down approach of holographic QCD.

- Supersymmetric D3-brane model (asymptotic $AdS_5$)
  The gauge-theory counterpart of this model is $N=4$ supersymmetric Yang–Mills theory. This theory is highly supersymmetric and so is far from realistic QCD. However, to see robust results of deconfined gluons in high temperatures, where we expect that the effect of supersymmetry is not crucial, the theory would be sometimes good enough to extract typical phenomena of strong coupling gauge theories. The most successful result that came out of this was the computation of the shear viscosity of quark gluon plasma in the high temperature phase of QCD [21]. Although the computation only employed geometry representing a finite temperature phase of the $N=4$ supersymmetric gauge theory, the result was close to the experimental observation. It is difficult to argue why this model works so well. In terms of the large $N_c$ expansion, reasons why the $1/N_c$ corrections do not contribute and why they do not modify the qualitative nature are still missing. In addition, there are many fields in the supersymmetric theory that are absent in QCD. The issue of the universality of the value of the shear viscosity is still to be settled. Nevertheless, other physical quantities, such as the very small value of the shear viscosity of quark–gluon plasma [21], have been calculated so far and the results give an insightful suggestion for heavy ion experiments.

- D3D7 model
  Introducing D7-branes as flavor D-branes [24] makes it possible to include supersymmetric quark fields (hyper multiplets in fundamental representation) in the above D3-brane model. This make it possible to calculate the quark energy loss in quark gluon plasma and drug forces [22].
  One can also discuss the $U(1)$ part of the chiral symmetry breaking in this D3D7 setting [25]. The position of the flavor D7-brane represents the symmetry in the Yang–Mills theory on D3-branes. If the position of the flavor D7-branes are symmetric, we have that symmetry in Yang–Mills theory, however if not, we have corresponding symmetry breaking. By embedding the $U(1)$ part of chiral symmetry as a geometrical rotational symmetry in the D3-brane-setting, we can discuss how this rotational symmetry is spontaneously broken from the position of the D7-brane at a low temperature, and restored at a high temperature. The position of the D7-branes is determined in order to minimize the free energy of the system, see for example, figures 6.2 and 6.6 of [19].

- Witten’s non-supersymmetric model [26]
  The corresponding geometry is called Gibbons–Maeda geometry [27] and corresponds to a 1+3-dimensional pure bosonic Yang–Mills theory (i.e. the theory of gluons) at low energy without supersymmetry. The geometry is made of $N_c$ D4-branes wrapping a circle. This circle compactification brings the 1+4-dimensional theory down to the 1+3-dimensional Yang–Mills theory. It breaks the supersymmetry by imposing anti-periodic boundary conditions for fermions; at low energies all fermions are massive and only massless gluons survive. Adjoint scalars obtain masses through quantum corrections which are roughly of the order of the scale defined by the radius of the circle.
  This geometry captures an important gauge-theory property: the confinement. In fact, using the bulk equations of motion from the gravity side, one can demonstrate as [26] that the fluctuation spectrum corresponding to the glueball spectra is discrete and mass-gapped and that the calculated wilson loop shows that the area is low. Furthermore, above a critical temperature, a phase transition occurs and is interpreted as a confinement-deconfinement transition since the spectrum becomes continuous at the high temperature phase.
  There is one caveat here: the phase transition scale is nothing but the scale of the extra circle on which the D4-branes are wrapped, as it is the unique dimensionful physical parameter. So naively speaking, the higher dimension cutoff scale is re-interpreted as the scale of the gluon theory. The excitation of the massive gluons, adjoint scalars and their superpartners should come into the spectrum above the scale; therefore, the theory can no longer be the purely bosonic Yang–Mills theory. If one naively ignores those and regards the gravity fluctuation as the glueball spectrum made solely of the gluons above this scale, we might get some mismatches for the spectrum comparison.
  Therefore, the intrinsic problem of this geometry, interpreted as a dual of the pure Yang–Mills theory, is the double meaning of the dynamical scale and the compactification scale. In order to remove additional degrees of freedom, we have to take the double-scaling limit: We keep $\Lambda_{\text{QCD}}$ fixed and at the same time, take the scale, associated with the D4-brane wrapping circle, to infinity. A proper scaling limit, where the dynamical scale is fixed while the compactification scale is taken to infinity, has not been formulated yet.

- D4D6 model [28]
  In Witten’s geometry, flavor D6-branes can be added to include quarks in the theory. The string connecting the $N_c$ D4-branes and the $N_f$ D6-branes give a low-energy excitation that behaves like a quark. In the gravity description, the shape of the D6-branes is deformed

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12 On the other hand, the $\Lambda_{\text{QCD}}$ appear as an IR cutoff of the geometry in the gravity side, so that no geometry exists below this IR cutoff scale (radius).
and can be interpreted as a spontaneous breaking of the (anomalous) $U(1)$ axial symmetry. The model can include various quark masses so the particular quark mass dependences of various low-energy quantities can be studied.

- D4D8 model (Sakai–Sugimoto model) [29].
  This theory adds flavor D8-branes to Witten’s geometry. One of the superior points of this model compared with others is that by adding D8-branes, one can obtain only left-chirality fermions at the intersection points between D4 and D8-branes. On the other hand, by adding anti-D8-brane, one can obtain only right-chirality fermions at the intersection points between D4 and anti-D8 branes. This implies that by adding the $N_f$ number of both D8 and anti-D8-branes, we can have both left and right chiral fermions (quarks) in the system with an explicit dependence on the chiral symmetry $U(N_f)_{L} \times U(N_f)_{R}$, which are very close to realistic QCD.

Similar to the D3D7 system, the chiral symmetry $U(N_f)_{L} \times U(N_f)_{R}$ is seen from the position of flavor D8 and anti-D8-branes. Due to the warped factor of Gibbons–Maeda geometry, one can demonstrate that the free energy at low temperatures is lower if both $N_f$ D8-branes and $N_f$ anti-D8-brane are combined into $N_f$ 8-branes. See the left figure of figure 2. In high temperatures, these combined effects of D8 and anti-D8 are hidden behind the horizon (right figure of figure 2) and we have chiral symmetry restoration, which can be seen geometrically. In this way, this model shows the spontaneous chiral symmetry breaking at low temperatures and its restoration at high temperatures in a geometrical way.

Except for the point that the quark mass is difficult to introduced\(^{13}\) due to the non-supersymmetric nature and the existence of chiral fermions, this holographic model is the most successful model in view of the study of low-energy hadron physics. In addition to the meson spectrum and interactions, the baryon spectrum and its chiral dynamics can be systematically studied.

This model again suffers from the same problem that Witten’s geometry has: the unnecessary modes, such as squarks in addition to the gluinos, exist in the theory at the high energy scale\(^{14}\). So we tentatively ignore modes that are expected to be absent in QCD, to compare the holographic results with experiments.

In summary, in all the known and popular holographic models, there remains the problem of having fields that are absent in real QCD above some scale of the theory. Related to this, the low-energy scale of the theory, which one would like to interpret as the QCD scale, is the same energy scale as the scale where the unnecessary fields show up.

Naively, at very low energies, the effects of these unnecessary fields would be small, so the prediction from holography should be better at low energies. This simple fact strongly motivates us to visit nuclear physics. Nuclear physics treats nuclei: bound states of nucleons at the energy scale much lower than the QCD scale. However in order to make the comparison with data more precise, we have to take the double-scaling limit in holographic QCD, where we take the scale, beyond which unnecessary fields be dynamical, to infinity while keeping the QCD scale fixed.

Due to the reasons explained in section 1, nuclear physics includes a great deal that needs to be explained by QCD. Standard nuclear physics has many assumptions and the origins of those fundamental assumptions may be explained directly from QCD, once we apply the holographic methods to QCD.

3. Review: M(atrix) description of multi-baryon system

The upshot of the theory [1] for the multi-baryon system, derived in AdS/CFT, is that it is a theory of the matrix degrees of freedom, with the following robust form of the action:

$$S = \frac{M}{2} \int dt \text{tr} \left[ \left( \partial_\tau X^M(t) \right)^2 - g \left[ X^M, X^N \right]^2 + \cdots \right]. \quad (3.1)$$

Let us clarify the relation between (3.1) and the nuclear physics action (1.1). The matrix $X^M$ is a Hermitian $A \times A$ matrix, where $A$ is the number of baryons (which resultantly becomes the mass number of a nucleus if all the baryons are bounded together as a big nucleus). Once it is diagonalized, the eigenvalues are nothing but the locations of the baryons, which are given by $x^M_{si}$ (for $s = 1, \ldots, A$) in the nuclear physics.

\(^{13}\) See [30] for a possible way to introduce the quark masses to the model.

\(^{14}\) Here the holographic scale is determined by the scale on which the D4-branes are wrapped and gives the scale beyond where this additional ‘junk’ is excited.
action (1.1). There are off-diagonal entries in $X^M$, which we interpret as the degrees of freedom associated with the nuclear force mediator (such as pion, massive vector mesons, etc.). Classically integrating those degrees of freedom in the action gives rise to the interaction between the eigenvalues of $X^M$. For the details of the nuclear force derivations, see section 4 of [1]. This interaction is interpreted as the inter-nucleon potential (nuclear force). The terms which are not written in the action (3.1) (specified as ‘+⋯’) are fields representing spins and isospins (flavor degrees of freedom of the baryons). Again, the precise form of the action is given in [1] and presented in (4.1).

In this section, we provide a review of the matrix formulation of the multi-baryon system in simple terms. First, we shall explain below the reason why we have the matrix degrees of freedom for the baryons in AdS/CFT and the origin of the action written above. Then we come to a review of the concrete analysis for a single-baryon system to obtain the baryon spectrum and also a review of two and three baryon systems for deriving the short-distance nuclear force. These were done in the original paper [1]. Then in the final part of this section, we review the importance of the matrix model action (3.1) for providing a possible unified view of nuclear physics.

3.1. Baryons are matrices

As we outlined above, the most important and novel part of the new description of the multi-baryon system (3.1) is the fact that baryons are described by $A \times A$ matrices. In fact, this is a robust result once one applies the AdS/CFT correspondence to QCD for the multi-baryon system.

There are two key points to derive this fact, which are shown in figure 3.

- A baryon is a D-brane.
  In AdS/CFT correspondence, we need a large $N_c$ expansion to use the dual gravity description. For large $N_c$ QCD, baryons are a heavy object whose mass is of the order of $O(N_c)$, since a single baryon consists of $N_c$ quarks. In the gravity side of the AdS/CFT, what is the object whose mass is so large? The answer is D-branes. D-branes are solitonic objects in string theory, whose masses are of the order of $O(N_c)$. Therefore, the baryons are expected to correspond to the D-branes in the gravity side of the AdS/CFT correspondence. In fact, baryons are D-branes and, technically speaking, these baryon D-branes wrap on the closed surface like a higher dimensional sphere. Through the D-brane action, the wrapped D-branes on some closed surface with a penetrating flux, induce the $N_c$ unit of $U(1)$ charges on that closed surface. On the closed surface, the total charge must be zero to satisfy Gauss’s law. This implies that we need to add compensating charged objects on that surface, which turns out to be an $N_f$ number of fundamental strings [31]. Therefore, these D-branes behave as baryons.

- Multi-D-branes are matrices.
  D-branes are defined as surfaces on which open strings can end. When D-branes are on top of each other, fundamental and anti-fundamental strings connecting those D-branes can be arbitrarily short and massless. The low-energy excitation of those light modes are classified by an $A \times A$ matrix when $A$ is the number of D-branes, since each open string has two ends labeled as $(a,b)$ where $a, b = 1, \ldots, A$. Therefore, the low-energy degrees of freedom on the coincident $A$ D-branes are $A \times A$ matrices.

Combining these two, we arrive at the inevitable conclusion that nuclei (or the multi-baryon system) in the AdS/CFT correspondence should be described by matrices.

Furthermore, the effective action of D-branes has the universal form of (3.1). The interpretation is definite: the eigenvalues of the field $X$ are the location of the $A$ number of D-branes. Therefore, we come to a conjecture that the effective action (3.1) describes nuclear physics.

One of the most important properties of nuclei is its crucial dependence on isospins. Nuclear force strongly depends on whether the nucleon is a proton or a neutron. Consequently, we have a nuclear chart and stable/unstable nuclei. How can the isospin dependence enter this formulation? The answer is quite simple: another matrix $w$ which is an $A \times N_f$ complex matrix joins the effective action. The isospins are nothing but the quark flavor degrees of freedom and $N_f$ is the number of quark flavors.

Figure 4 clarifies why this new matrix shows up in the gravity side of the AdS/CFT correspondence. As we reviewed in the previous section, the flavor can be represented by an introduction of ‘flavor D-branes’ into the gravity geometry. Then, in addition to the baryon D-branes, we have the flavor D-branes, so there appears an open string which connects the
two kinds of D-branes. This string should be described by $A \times N_f$ matrices, as in the same manner as the $A \times A$ matrix $X$ for the string among the baryon D-branes.

Although the species of the fields appearing in the low energy of the multi-baryon system in the AdS/CFT are just $X$ and $w$, the precise interaction between these fields and the coefficients in the effective action depend on what kind of D-brane configurations (holographic models) we use for the large $N_c$ QCD. When we use the most popular D4D8 model (Sakai–Sugimoto model) described in section 2.3, the baryon D-branes are D4-branes wrapping a four-sphere $S^4$ in 10 space–time dimensions and the flavor D-branes are D8-branes wrapping the $S^4$. This means that the baryon D4-branes can be located inside the flavor D8-branes. The $Dp$-$D(p + 4)$ system in superstring theory is well-understood, as a geometric realization of the instanton construction: the $Dp$-brane can be seen as a Yang–Mills instanton through the gauge fields on realization of the instanton construction: the $D_p$ system in superstring theory is well-understood, as a geometric instanton construction [32, 33]. Therefore, within the D4D8 holographic model, our low-energy effective action for the multi-baryon system is nothing but a generalization of the ADHM matrix models.

The matrix effective action is concretely written in (4.1) for the D4D8 model, but in this part of the review we do not need the explicit form, as we explain only the conceptual part to show the robustness of the derivation. Furthermore, it was straightforward to construct explicit forms for the matrix effective action for another D-brane model described in section 2.3. However for concreteness, in this paper we concentrate on the model constructed for D4D8 model in [1].

Next, we give a review of a single-baryon spectrum ($A = 1$), and also a derivation of the short-distance nuclear force ($A = 2, 3$). The important fact for the application is that the matrix action has only two free parameters:

3.2. Derived baryon spectrum

The simplest case is $A = 1$ where we have only a single baryon. In this case, the quantum mechanics should give the baryon spectrum. Excited states of a baryon emerge from the quantum mechanics.

Let us recall the Skyrme model [34, 35]. In the Skyrme model, a baryon appears as a soliton of the Skyrme model which is nothing but a peculiar effective action of low-energy pions. Any soliton has fluctuation modes, massive or massless (zero modes). The fluctuation modes, which are just a function of time, obey a Hamiltonian and can be quantized. The resulting quantized fluctuation spectrum is interpreted in the Skyrme model as the baryon spectrum. Here in the AdS/CFT matrix model approach, the Hamiltonian of the fluctuation modes are directly given as our matrix model Hamiltonian (3.1). So, one easy interpretation of the matrix model is a Hamiltonian of the moduli space of generalized Skyrminons. However, the very small number of parameters (only two parameters) in our holographic setting highlights the superiority of our construction compared with the generic Skyrme model which has many parameters. It would be interesting to see if the holographic approach gives a constraint of the effective parameters in the Skyrme model.

For $A = 1$, the matrix model becomes extremely simple. The Hamiltonian for two flavors ($N_f = 2$) looks [1]

$$H = \frac{\lambda N_c M_{KK}}{54\pi} \left[ \left( \frac{27\pi}{\lambda M_{KK}} \right)^2 \frac{1}{2\rho^2} + \frac{1}{3} M_{KK}^2 \rho^2 + \frac{2}{3} M_{KK}^2 (X^4)^2 \right],$$

where

$$w^2 = \rho(t) U_0(t).$$

Here $\rho(t)$ and $X^4(t)$ are scalar degrees of freedom and $U(t)$ is a $2 \times 2$ unitary matrix degree of freedom. $\rho(t)$ represents the dissolved size of the D-brane (which is roughly the size of the baryons) and $X^4(t)$ is a displacement of the D-brane along the holographic direction. The matrix $U$ is nothing but the moduli degrees of freedom appearing in the Skyrme model; its quantization gives higher spins and isospins. $M_{KK}$ is the energy scale of the D4D8-model, often interpreted as the QCD scale since it appears as a unique dimensionful unit for the meson mass spectra.

These three modes provide almost-independent harmonic oscillators and the quantization results in the following spectra:

$$M = M_0 + \frac{M_{KK}}{\sqrt{6}} \left[ \sqrt{(I/2 + 1)^2 + N_c^2} + 2n_\rho + 2n_{X^4} + 2 \right].$$

Here $I$ is the isospin which is equal to the spin in this case and $n_\rho (n_{X^4})$ is a non-negative integer coming from the harmonic oscillator $\rho(t)$ ($X^4(t)$). The zero-point energy $M_0$ can be thought of as a free parameter. So only the mass difference, given by numerical coefficients times $M_{KK}$, is the prediction of the model. Qualitatively one can compare the result (3.4) for the mass difference with experimental data for the baryon excitation spectra.

The baryon spectrum (3.4), as well as its calculation from the quantum Hamiltonian, is quite close to what has been obtained in the soliton quantization approach in the Sakai–Sugimoto model [12].

We again would like to stress that the result is qualitatively robust in the AdS/CFT approach: because the baryon in the gravity dual should be represented by $X$ and $w$ strings, the spectrum should be given by its low-energy quantization. So we are inevitably led to the quantum number $n_{X^4}$, which is the oscillator of the D-brane along the holographic directions; also the quantum number $n_\rho$, which is the fluctuation of the magnitude of the string connecting the baryon D-brane and the flavor D-brane; further, the spin operator $U$, which is the internal orientation of the same string. The coefficients appearing in the mass spectrum formula may differ among holographic models, but its structure should be shared in all the holographic models.

3.3. Universal repulsive core of nucleons

Once the baryon state is identified within the matrix model degrees of freedom, it is straightforward to calculate the interbaryon potential. Since the short-distance behavior of the
nuclear force is one of the most important problems in nuclear physics, to derive it analytically is a very important issue. As we have the matrix model action for $A = 2$ at hand, whether it reproduces the empirically known repulsive core of nucleons would be a good touchstone for the validity of the matrix model approach.

Since the matrix model action (3.1) is explicitly given for $A = 2$, we just need to: first derive the off-diagonal term classically by solving ADHM constraints\textsuperscript{15} for a given set of two diagonal entries, which defines the locations and the spin/isospins of the two baryons, and then substitute all back into the Hamiltonian to derive the inter-baryon potential energy.

The calculation is straightforward and was given in [1]. The result for the inter-nucleon potential is as follows\textsuperscript{16}:

$$V_{\text{central}}(r) = \frac{\pi N_c}{\lambda M_{KK}} \left( 2 \frac{7}{2} + 8 \hat{J}_1 \cdot \hat{J}_2 \right) \frac{1}{r^2},$$

(3.5)

$$V_{\text{tensor}}(r) = \frac{2\pi N_c}{\lambda M_{KK}} \hat{I}_1 \cdot \hat{I}_2 \frac{1}{r^2}.$$  

(3.6)

This short-distance potential is positive for any choice of spins and isospins, therefore we conclude for the $N_f = 2$ case, that there are universal repulsive cores for the nuclear forces at a short distance. This repulsive core behaves as $1/r^2$, showing a very strong repulsive core at the $r \to 0$ limit. We concluded that qualitatively the matrix model approach for multi-baryon system is consistent with experiments, in this sense. Whether the repulsive core in reality behaves as our prediction $1/r^2$ or not will be tested and studied in the future.

The three-body nuclear force can be evaluated in the same manner. The short-distance contribution to the intrinsic three-body force, which does not come from the effective integration of massive states (for example the famous Fujita–Miyazawa force [37]), is important as it cannot be evaluated using chiral perturbations. Using the matrix model approach, one can straightforwardly evaluate the three-body interaction. It was shown in [9] that the proton–proton–neutron aligned on a line gives a positive three-body potential and that a spin-averaged three-neutron aligned on a line is positive too. (See also the recent attempt in lattice QCD [38].) In particular, the latter is relevant for neutron stars as it has a dense neutron system and the effective repulsion would give a hard equation of state, which is a good tendency for a recent observation of heavy neutron stars.

3.4. Toward a description of atomic nuclei

As we have the effective action (3.1) of the multi-baryon system, in principle, atomic nuclei and their properties, i.e. the nuclear physics, should emerge from the action. The action (3.1) has only two parameters, so once one can solve the effective action completely in a quantum mechanical fashion, one can compare the results with experiments in principle.

\textsuperscript{15} If the readers are interested in more details on how to solve the equations of motion, see section 4.1.2. of [1].

\textsuperscript{16} Using the soliton approach in the holographic D4D8 model, the short-distance nuclear force was calculated [16]. The result is qualitatively similar to the result of the matrix model.

Whether this action provides us with an efficient and good description of atomic nuclei is a very important question, as the AdS/CFT connects directly the nuclear physics and QCD.

In particular, the properties of heavy nuclei are yet to be uncovered and they are far from QCD. The aim of the holographic approach is to uncover the relation between the nuclear physics and QCD directly, to make clear how the observables in nuclear physics may depend on the quantities defined in QCD. One of the important targets in nuclear physics in this sense is the nuclear radius. It has been known for many decades that stable nuclei are subject to a relation

$$r \sim 1.2 \times A^{1/3} \text{ (fm)},$$  

(3.7)

where $A$ is the mass number (the number of baryons) of the nucleus. This has been explained as a result of the nuclear density saturation: the nucleon density inside nuclei is almost constant and takes a universal value, so the nuclear radius is proportional to $A^{1/3}$. The reason why we obtained this finite radius, although we only had the repulsive core at the classical level as shown in the previous section, is that many body nucleons described by the matrices provide an attractive force that is automatically at a long distance by quantum effects. This attractive force is popular in most matrix models with the commutator-square-type interaction: the multi-baryon matrix model shares this property. See for examples [36].

The repulsive core of a nucleon is thought to be a component to explain the $A$ dependence of the nuclear radius in nuclear physics. If a nucleon can be regarded as a hard ball that is almost equivalent to the repulsive core, the total nucleus should have a volume proportional to $A$, therefore, the $A$ dependence follows. Since in the holographic QCD approach the repulsive core was reproduced as explained in the previous subsection, this nuclear radius would be a natural consequence.

In [11], one of the authors (Hashimoto) together with Morita demonstrated that indeed the above (3.7) is reproduced from the quantum mechanical matrix action (3.1), with a certain approximation employed. The result obtained in [11] is an analytic formula for the nuclear radius,

$$\sqrt{r_{\text{mean}}^2} = \frac{3^{5/2} \pi^{2/3}}{2^{5/6} 5^{1/6}} \frac{1}{M_{KK} \lambda^{2/3} N_c^{1/3}} A^{1/3}.$$  

(3.8)

A nontrivial point is that the formula has the correct $A^{1/3}$ dependence.

The approximation used for deriving (3.8) is: a large $A$ limit with quenching (ignoring the $\omega$ degrees of freedom) and a large dimension limit (which is almost equivalent to a mean field limit), in addition to the standard limits taken in holographic QCD, such as the large $N_c$ limit and the strong coupling limit $\lambda \gg 1$. Whether these approximations are appropriate or not should be studied in the future study. However, the message here is that we now have a good starting point (3.1) for calculating various quantities in nuclear physics from QCD.

It could be possible that the action (3.1) itself may be modified to include higher order terms, or that one needs to apply different approximations to (3.1) to correctly derive
A schematic picture of the baryon D-brane on the flavor D-brane. The harmonic oscillator excitation on the baryon D-brane is the size fluctuation $\rho$ and the fluctuation of the D-brane location along the holographic direction $X^4$. The horizontal direction is our space $x^1, x^2, x^3$.

Figure 5. A schematic picture of the baryon D-brane on the flavor D-brane. The harmonic oscillator excitation on the baryon D-brane is the size fluctuation $\rho$ and the fluctuation of the D-brane location along the holographic direction $X^4$. The horizontal direction is our space $x^1, x^2, x^3$.

Figure 6. A plot of the nuclear density inside the nucleus, calculated in AdS/CFT [11].

Table 1. Fields in the nuclear matrix model. Each field carries indices, under the symmetry of the system: the $U(k)$ gauge symmetry, $SU(N_f)$ global symmetry and $SU(2) \times SU(2)$ global symmetry.

| Field | Index | $U(k)$ | $SU(N_f)$ | $SU(2) \times SU(2)$ |
|-------|-------|--------|-----------|----------------------|
| $X^{M}(t)$ | $M = 1, 2, 3, 4$ adj. | 1 | (2, 2) | |
| $w_{\alpha i}(t)$ | $\alpha = 1, 2$; $i = 1, \ldots, N_f$ | $k$ | $N_f$ | (1, 2) |
| $A_0(t)$ | adj. | 1 | (1, 1) |
| $D_s(t)$ | $s = 1, 2, 3$ adj. | 1 | (1, 3) |

4. Multi-flavor nuclear forces via holography

4.1. Strangeness and holography

QCD at high density is a final frontier, which is still to be unveiled. It is expected that at the core of neutron stars, the high density matter would be supplemented with strange quarks, in order to relax the Fermi energy of the ordinary two-flavor matter of neutrons and protons. To judge whether the strangeness really kicks in to the high density core of the neutron stars, we need to know the inter-baryon interaction with multi-flavors. It has been known experimentally for many decades that nucleons are accompanied by repulsive cores, a short-distance repulsion. To reveal whether there exists a repulsive core, even for baryons including strange quark(s) is an indispensable cornerstone to reach the truth in the high density QCD.

The inter-baryon potential is a non-perturbative regime of QCD, even at the short distances of concern. Thus we need to rely on analytic method to solve the QCD approximately. In this section, we shall extend the analysis [1] to the case of multiple flavors $N_f > 2$, to find the short-distance properties of the inter-baryon potential.

Another non-perturbative framework of QCD, lattice simulations, recently uncovered interesting features of the short-distance inter-baryon potential for the case of three flavors. It was demonstrated [40, 41] that indeed there remains

\[ 1\text{7} \text{In addition, one may take a gravity dual of the matrix model under a certain assumption (such that A is large and also that the inter-nucleon distance is small) to investigate giant resonances in nuclei [39].} \]

4.2. The effective model of multi-baryon system

To extract the non-perturbative potential among baryons at short distances, the nuclear matrix model [1] derived in holographic QCD should provide a good sense of the generic nature. The quantum mechanical action of the model is

\[ S = \frac{\lambda N_c M_{KK}}{54\pi} \int dt \, \text{tr}_k \left[ (D_0 X^M)^2 - \frac{2}{3} M_{KK}^2 (X^4)^2 + D_0 \tilde{\omega}_\alpha^\beta D_0 w_{\alpha i} - \frac{1}{6} M_{KK}^2 \tilde{\omega}_\alpha^\beta w_{\alpha i} + \frac{36\pi^2}{42} M_{KK}^4 \left( \tilde{D}^2 \right)^2 + \tilde{D} \cdot \tilde{\tau}_\alpha^\beta \tilde{X}^4 X_{\alpha i} + \tilde{D} \cdot \tilde{\tau}_\alpha^\beta \tilde{\omega}_\beta^\alpha w_{\alpha i} \right] + N_c \int dt \text{tr}_k A_0. \tag{4.1} \]

The system possesses a gauge symmetry $U(k)$ where $k$ is the number of the baryons in the system. Table 1 shows the field content of the model.

Here, the dynamical fields are $X^M$ and $w_{\alpha i}$, while $A_0$ and $D_i$ are auxiliary fields. In writing these fields, the indices for the gauge group $U(k)$ implicit. The symmetry of this quantum mechanics matrix is $U(k)_{\text{local}} \times SU(N_f) \times SO(3)$ where the last factor $SO(3)$ is the spatial rotation, which, together with a holographic dimension, forms a broken $SO(4) \simeq SU(2) \times SU(2)$ shown in table 1. The breaking

\[ \text{18 There is a channel at which the repulsive core disappears, for an appropriate choice of the baryon states. It is closely related to the conjectured two-baryon bound state called H-dibaryon [42], as demonstrated in lattice simulations [41, 43]. Whether such dibaryon exists or not should be confirmed by future experiments.} \]
is due to the mass terms for $X^4$ and $w_{A\bar{a}}$. In the action, the trace is over these $U(k)$ indices; the definition of the covariant derivatives is $D_{\beta}X^M = \partial_{\beta}X^M - i[A_0, X^M]$, $\partial_{\beta}w = \partial_{\beta}w - iA_0\tilde{w}$ and $\partial_{\beta}\tilde{w} = \partial_{\beta}\tilde{w} + iA_0\tilde{w}$. The spinor indices of $X$ are defined as $X_{\alpha\dot{\alpha}} = X^{\alpha}(\bar{\sigma}_M)\alpha$ and $\tilde{X}^{\alpha\dot{\alpha}} = X^{\alpha}(\bar{\sigma}_M)\dot{\alpha}$ where $\bar{\sigma}_M = (i\tau, 1)$ and $\bar{\sigma}_M = (-i\tau, 1)$, with Pauli matrices $\tau$.

The model has a unique scale $M_{KK}$, and $\lambda = N_c\epsilon_{QCD}$ is the 'tHooft coupling constant of QCD, with the number of colors $N_c$. The diagonal entries of $X^i$ (i = 1, 2, 3) specify the location of the baryons. The location of the baryon D-brane in the holographic direction $X^4$ is stabilized at $X^4 = 0$, around which the harmonic excitations label excited baryon states. The $w$ fields are responsible for the spins and isospins (and flavor representations) of each baryon.

In [1], explicitly demonstrated is the nuclear force for the two-flavor case. There, a universal repulsive core was found. We here simply extend the two-flavor calculation to the case with a generic number of massless flavors and we will see the consequence.

The procedure we employ in the following is as follows. First, we look at the configuration which minimizes the potential of the matrix model. When taking a large $\lambda$, the D-term condition needs to be satisfied, which is nothing but the ADHM constraint. Then, we obtain a classical potential with a solution of the ADHM constraint, which depends on the inter-baryon distance and the moduli parameters of the two baryons. Taking an expectation value of this potential with respect to the product of the wave functions for each baryon, we obtain the inter-baryon potential.

4.3. Two-baryon configuration

The large $\lambda$ limit lets only configurations satisfying the ADHM constraint remain. The ADHM constraint is equivalent to the D-term condition concerning $D_{\beta}$ and is given by

$$\tilde{\tilde{\sigma}}_{\beta}^a \left( \tilde{X}^{\beta\dot{a}} X_{\alpha\dot{\alpha}} + \tilde{w}_{\beta}^i w_{i\alpha} \right)_{BA} = 0. \tag{4.2}$$

Here $A, B$ are $U(k)$ indices. For a single baryon with a generic number of flavors, the ADHM constraint is simply solved by the $X = \text{constant}$ (baryons located anywhere) and

$$w = U \left( \begin{array}{cccc} \rho & 0 & 0 & 0 \\ 0 & \rho & 0 & 0 \\ 0 & 0 & \cdots & \cdots \\ 0 & 0 & \cdots & \cdots \end{array} \right), \tag{4.3}$$

which shows the $(i, \dot{i})$ entry. Here $U$ is a $U(N_f)$ unitary matrix specifying the baryon spin and isospin (flavor dependence). The flavor symmetry acts on $U$ as $U \mapsto G U$. The baryon wave function is given as $\psi(U)$, in the same manner as the famous Skyrme model.

We want to put two baryons located at $x_M = \pm r_M/2$, so that the distance between the two baryons is $r_M$. For the two baryons, now the coordinate field $X_M$ consists of two by two matrices, so we parameterize them as

$$X_M = \frac{1}{2} r_{M}^{(a)} \sigma_a. \tag{4.4}$$

where $\sigma_a$ is a Pauli matrix with index $(A, B)$. We specify the baryon location by the diagonal entries $r_M^{(a)}$, while assuming the off-diagonals $r_M^{(1)}$ and $r_M^{(2)}$ are small as $1/(r_M^{(3)})$, so that the distance defined by $r_M^{(3)}$ makes sense at large $r_M^{(3)}$. The two baryons can have independent spins and flavor representations, so we allow

$$w^A = U^{(A)} \left( \begin{array}{cccc} \rho_A & 0 & 0 & 0 \\ 0 & \rho_A & 0 & 0 \\ 0 & 0 & \cdots & \cdots \\ 0 & 0 & \cdots & \cdots \end{array} \right) (1_{2\times 2} + \epsilon^{(A)}),$$

for each baryon, $A = 1, 2$. (In this expression we do not make a summation over the index $A$.) $\epsilon^{(A)}$ is taken to be a $2 \times 2$ traceless matrix at $O(1/r_M^{(3)})$. Note that at the large inter-baryon distance limit $r^{(3)} \to \infty$, the ADHM data above reduces to just a set of two single-baryon ADHM data, $(4.3)$ and a constant diagonal $X$.

It is quite straightforward to solve the ADHM constraint (4.2) with the above generic ansatz and the solution is given as follows.

$$r_M^{(1)} \sigma_M = -\frac{\rho_0^2}{|r_0^{(3)}|^2} r_M^{(3)} \sigma_M (P_{12} - P_{12}^t), \tag{4.5}$$

$$r_M^{(2)} \sigma_M = -\frac{i}\rho_0^2 \frac{2}{|r_0^{(3)}|^2} \sigma_M (P_{12} + P_{12}^t), \tag{4.6}$$

$$\epsilon^{(1)} = \frac{\rho_0^2}{2|\rho_0^{(3)}|^2} [P_{12}, P_{12}^t], \quad \epsilon^{(2)} = \frac{\rho_0^2}{2|\rho_0^{(3)}|^2} [P_{12}, P_{12}^t]. \tag{4.7}$$

Here we have defined

$$P_{12} = P \left( [U^{(1)}) U^{(2)}] \right), \tag{4.8}$$

with $P$ being a projection of the $N_f \times N_f$ matrix to its upper-left $2 \times 2$ components, so that $P_{12}$ is a $2 \times 2$ matrix. We can easily see that, when $N_f = 2$, the result here reproduces the two-flavor result of [1]. Our formulation allows an arbitrary combination of the isospins.

4.4. Explicit inter-baryon potential

Let us substitute the above ADHM data, the two-baryon configuration with the inter-baryon distance $r_M^{(3)}$ and the spin/flavor dependence $U^{(A)}$, into the action (4.1) and derive the inter-baryon potential as a function of $r_M^{(3)}$ and $U^{(A)}$. As was done in [1], we need to integrate out the $U(2)$ auxiliary gauge field $A_0$ of the quantum mechanics,

$$A_0 = A_0^{(1)} 1_{2\times 2} + A_0^{(2)} \tau^a. \tag{4.9}$$

Since the model includes only the linear and quadratic terms in $A_0$, it is straightforward to perform the integration. In the action (4.1), the terms relevant to $A_0$ are

$$S_{A_0} = \frac{\lambda N_c M_{KK}}{32\pi} \int d^4x \left[ 2 \langle A_0^{(1)} \rangle^2 \langle r_M^{(2)} \rangle - 2 \langle A_0^{(2)} \rangle \right] + \left\langle \left( \langle A_0^{(1)} \rangle^2 + A_0^{(2)} \right)^2 \right\rangle \left( |w^{A=1}|^2 + |w^{A=2}|^2 \right)\right)$$

$$+ 2A_0^{(1)} A_0^{(2)} \left( w^{A=1} \bar{w}^{A=2} + w^{A=2} \bar{w}^{A=1} \right)$$

$$- 2iA_0^{(1)} A_0^{(2)} \left( w^{A=1} \bar{w}^{A=2} - w^{A=2} \bar{w}^{A=1} \right)$$

$$+ 2A_0^{(1)} A_0^{(2)} \left( |w^{A=1}|^2 - |w^{A=2}|^2 \right) + \frac{108\pi}{\lambda M_{KK}} A_0^{(1)} A_0^{(2)}. \tag{4.10}$$
We integrate out all the components of the auxiliary field $A_0$ and write the potential as $S_{A_0} = -\int V_{A_0,2\text{-body}} \, dr$. We expand the result in terms of small $\rho/r^3$, to obtain the leading term

$$V_{A_0,2\text{-body}} = \frac{27\pi}{4} \frac{N_c}{\lambda M_{KK}} \rho^3 (r^{(3)})^2 \left[ \frac{1}{r P_{12}} \right]^2.$$  

(4.11)

Here we have already put $\rho_1 = \rho_2 = \rho$, that is, the size of the two baryons are the same. This relation is ensured at the large $N_c$ limit where the inter-baryon interaction disappears.

The remaining contributions to the inter-baryon potential, from the matrix model action, are the mass terms $tr(X_2^2)$ and $|w|^2$. Substituting the two-baryon configuration, we obtain

$$V_{X^{4,2\text{-body}}} = \frac{\lambda N_c M_{KK}^3}{162\pi} \frac{-\rho_1^2 \rho_2^2}{(r^{(3)})^2} \times tr \left[ r_M^3 \sigma_M P_{12} \right] tr \left[ r_M^3 \sigma_M P_{12} \right].$$  

(4.12)

It turns out that the mass term for $w$ does not give rise to an extra potential.

So, in total, the inter-baryon potential $V$ in the small $\rho/r^3$ expansion is given as a sum of (4.11) and (4.12),

$$V_{2\text{-body}} = \frac{27\pi N_c}{4\lambda M_{KK}} \frac{1}{7^2} \left( \left[ tr P_{12} \right]^2 + \left[ tr \tilde{r} \cdot \tilde{r} P_{12} \right]^2 \right).$$  

(4.13)

Here, we already substituted $r_M^{(3)} = 0$, which is satisfied by the baryon wave functions at large $N_c$ [1] and denoted $r_{M=1,2,3}^{(3)}$ as $\tilde{r}$, the inter-baryon vector. $\tilde{r}$ is the unit vector along $r$; we also used the classical size $\rho$ of a single baryon [1], $\rho_1^2 = \rho_2^2 = 3^{1/2}/4 \lambda M_{KK}^2$. $\tilde{r}$ immediately notice that by taking $N_c = 2$, the potential (4.13) reduces that of the two-flavor inter-nucleon potential given in [1]. The potential has the $1/r^2$ behavior that is peculiar to holographic QCD [13, 16, 44], which is nothing but a harmonic potential in four-dimensional space (our spatial three dimensions, plus the holographic direction).

### 4.5. Universal repulsive core

It is already manifest that the inter-baryon potential for a generic number of flavors (4.13) is positive-semi-definite, since (4.13) is a sum of two positive semi-definite terms. Therefore, we conclude that holographic QCD predicts a positive-semi-definite repulsive core for a combination of any two baryon states.

Looking at the magnitude of the potential, we notice the following important fact: as the number of flavors is larger than 3, the classical potential (4.13) can vanish, for an appropriate choice of the baryon state. This is simply because we can choose a set of unitary matrices $U^{(1)}$ and $U^{(2)}$, such that $P_{ij}(U^{(2)})^* U^{(1)}$ vanishes. As the projection operator $P$ refers only to the upper-left corner of the unitary matrices, once the size of the matrix $N_c$ gets larger, the configuration of the baryon can evade the upper-left $2 \times 2$ corner and thus does not contribute to the inter-baryon potential (4.13).

Substituting some particular values of constant $U$ corresponds to a classical evaluation of the potential (in the same manner as in the Skyrme model), but in reality we need to take into account the baryon wave function $\psi_1(U^{(1)}) \psi_2(U^{(2)})$. A generic wave function has a wide distribution over the space of the normalized unitary matrices. So the magnitude of the repulsive core depends on the two-baryon states. The situation is the same as what is already known for the nucleon case ($N_f = 2$) [1].

In the next section, we review briefly the recent lattice calculations of the inter-baryon potential for three-flavor QCD and discuss a comparison with our holographic result.

### 5. A comparison with lattice QCD and OPE

In the previous section, we calculated a short-distance potential between two baryons in multi-flavor holographic QCD. We found a universal repulsive potential for generic baryon states.

In this section, we shall compare our results with ones obtained by a completely different technique: the lattice QCD.

First, we shall review the results in lattice QCD. The potentials between two octet baryons have been investigated in lattice QCD in the flavor SU(3) symmetric limit [40, 45], where all quark masses in the 3-flavor QCD are artificially taken to be equal, $m_3 = m_3 = m_a$, with the lattice spacing $a \simeq 0.12$ fm and the spatial extension $L \simeq 2 - 4$ fm. Simulations employ six different values of quark masses, which correspond to the pseudo-scalar meson mass $m_{PS} \approx 470, 670, 840, 1020, 1170$ MeV, where the relation that $m_{PS}^2 = Am_a$ holds for a small quark mass $m_a$ with a common coefficient $A$. There are six independent potentials between two octet baryons, which correspond to irreducible representations of the flavor SU(3) group as

$$8 \otimes 8 = 27 \oplus 8_s \oplus 1 \oplus 10^{\ast} \oplus 10 \oplus 8_s,$$  

(5.1)

where the first three representations are symmetric under the exchange of two octet baryons, while the last three are anti-symmetric. To satisfy the condition that a total wave function is odd under the exchange of two octet baryons, the first three states have spin zero ($S = 0$, odd under the exchange) while the last three states must have spin one ($S = 1$, even under the exchange) if the orbital angular momentum between two baryons is zero ($L = 0$).

A typical example of corresponding potentials is shown in figure 7, taken from [40], where the central potentials (left three) and the effective central potential (right three), where an effect of the tensor potential is included, are plotted.

As can be seen from figure 7, the inter-baryon potentials strongly depend on the representations. In the top panels, $V^{(27)}$ and $V^{(10^*)}$, which correspond to isospin-triplet and isospin-singlet NN potentials in the $N_f = 2$ case [3, 4], respectively, have a repulsive core at a short-distance and an attractive pocket at a medium distance. These features qualitatively agree with those of the NN potentials in quenched QCD, shown in figure 8.

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19 Note that (4.5) and (4.6) satisfy $r_M^{(1,3)} r_M^{(2,3)} = r_M^{(1,3)} r_M^{(2,3)} = 0$, which may help reduce the $A_0$ action.

20 This is because the Young tableau of both 27 and 10* do not have three rows.
$V(r) \text{[MeV]}$

$r \text{[fm]}$

$V^{(27)}$

$V^{(10^*)}$

$V^{(8^s)}$

$V^{(10)}$

$V^{(1)}$

$V^{(8a)}$

Figure 7. The six independent potentials in the flavor SU(3) limit obtained in lattice QCD at $m_{PS} = 1014$ MeV (red) and 835 MeV (green) [40].

For $L = 0$, $V^{(27)}$ is isospin-triplet ($I = 1$) at $N_f = 2$ and spin-singlet ($S = 0$), while $V^{(10^*)}$ is isospin-singlet ($I = 0$) at $N_f = 2$ and spin-triplet ($S = 1$). Therefore, the flavor-singlet potential at $N_f = 2$ cannot have spin zero for $L = 0$.

On the other hand, if the strange quark is introduced in the flavor representation, we have more varieties of potentials: $V^{(10)}$ has a stronger repulsive core and a weaker attractive pocket than $V^{(27)}$, $V^{(10^*)}$ and $V^{(8^s)}$ has only a repulsion with the strongest repulsive core among all, while $V^{(8a)}$ has the strongest attractive pocket with the weakest repulsive core. In contrast to these five cases, the singlet potential $V^{(1)}$ shows attraction at all distances without a repulsive core, which produces one bound state, the H-dibaryon, in this channel [41, 43]. Note that the flavor-singlet potential is spin zero for $L = 0$ in this case, contrary to the $N_f = 2$ case.

Increasing the number of flavors from two to three, we observe that the repulsive core becomes weaker in some channel ($8^a$) and it even disappears in the singlet ($1$), as seen in table 2, where we summarize the features of the inter-baryonic potential in the flavor SU(3) limit.

Figure 8. NN potentials in quenched QCD at $m_{\pi} \simeq 730$ MeV [4]. The spin-singlet sector ($1^S_0$) belongs to the 27 representation while the triplet to the 10 in the flavor SU(3).
Now let us discuss a comparison between our holographic QCD results and the lattice QCD results. In the previous section, we found a universal repulsive core for a multi-flavor inter-baryon potential. On the other hand, in the three-flavor lattice QCD, repulsive cores appear in most of the channels. Therefore, we conclude that our holographic results are consistent with the lattice QCD results, generically.

There is only one exception: the existence of an attractive channel. In the lattice QCD result, the flavor-singlet combination of the baryons in the $8_f$ representation for $N_f = 3$ is found to have a vanishing repulsive core. In the holographic side, as we work with the large $N_c$, it is not clear how the lattice QCD with $N_f = N_c = 3$ can be mapped to holographic QCD. However, in the previous section, we have seen that a classical inter-baryon potential can vanish. So the disappearance of the repulsive core in lattice QCD is not a contradiction with holographic QCD. We leave a more detailed comparison to a future work.

In table 2, we summarize the qualitative features of baryon–baryon potentials, together with the prediction from the OPE in perturbative QCD for their short-distance behaviors [46–48]. Although the OPE analysis is consistent with the attractive core for the singlet potential in lattice QCD, it disagrees with the strong repulsion of the $8_f$ potential in lattice QCD [21]. Obviously it is desirable to investigate the short-distance behaviors of the inter-baryon potential by various methods, including holographic QCD, in more detail.

Acknowledgments

We would like to thank T Hatsuda, M Hidaka, T Morita, K Yazaki and P Yi for their valuable comments. S A is supported in part by the Grant-in-Aid for Scientific Research on Innovative Areas (No 2004: 20105001, 20105003) and by SPIRE (Strategic Program for Innovative REsearch). K H is partly supported by the Japan Ministry of Education, Culture, Sports, Science and Technology (Grant-in-Aid Scientific Research (23654096, 23105716, 22340069, 21105514)). N I would like to thank the mathematical physics laboratory at RIKEN for their very kind hospitality.

Table 2. Overall feature of inter-baryon potential in each representation. The last line shows the short-distance behavior of the potential from OPE, where rpl. = repulsive and atr. = attractive.

| Representation | 27 | 8_f | 1 | 10^- | 10 | 8_f |
|---------------|----|-----|---|------|----|-----|
| Repulsion     | Yes| Strongest | No | Yes   | Strong | Weak |
| Attraction    | Yes| No       | Strongest | Yes | Weak | Strong |
| Comment       | NN ($I = 1$) | H-dibaryon | NN ($I = 0$) |
| OPE           | rpl. | atr.    | rpl. | rpl.  | atr.  |

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