CANONICAL QUANTISATION OF THERMAL
GAUGE THEORIES

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ABSTRACT

Canonical quantisation gives a new and convenient finite-temperature perturbation theory in covariant gauges, and solves the problem of the zero-frequency mode in the temporal gauge.

1. Introduction

Canonical quantisation is the most basic method of deriving the Feynman rules for thermal gauge theories. In this talk, I review work that I have done with A. Rebhan\(^1\) and with K. A. James\(^2\) on the following:

- **Covariant gauges in the real-time formalism\(^1\)**

  We have derived Feynman rules that can be simpler to use than the conventional ones, and which avoid the problems that have been encountered in demonstrating the gauge-independence of certain physical quantities.

- **The temporal-axial gauge in the imaginary-time formalism\(^2\)**

  At zero temperature, the propagator has a term with a double pole at \(k = 0\). Naively, it has been expected\(^3\) that at finite temperature the integral of this should be replaced with \(\sum_n 1/(n^2\pi^2T^2)\). We have shown how to avoid the apparent difficulty with the \(n = 0\) term in this sum.

I am going to try to make my description as simple as possible; full details may be found in our original papers.

2. Covariant gauges in the real-time formalism

Although the heading of this section explicitly refers to covariant gauges, most of what I have to say applies to any gauge.

The reason that the quantisation of gauge theories involves subtleties, particularly at finite temperature, is that the Hilbert space includes unphysical states. The thermal average of an observable, derived by considering a grand ensemble, requires one to take expectation values only in the physical states. So if one wants to write this thermal average as a trace, one needs to include a projection operator \(\mathcal{P}\) that removes the unphysical states:
\[ < Q > = Z^{-1} \text{tr} \left( e^{-\beta H} Q \mathcal{P} \right) \]  

where the operators are in the Heisenberg picture. Some of the important steps in the derivation of the Feynman rules for scalar thermal field theory rely on the identity
\[
\text{tr} (AB) = \text{tr} (BA).  
\]

For the case of a gauge theory, the presence of the projection operator changes things: it is not true that
\[
\text{tr} (AB\mathcal{P}) = \text{tr} (BA\mathcal{P}). 
\]

One sees the first effect of this when one goes over to the interaction picture:
\[
\langle Q \rangle = Z^{-1} \text{tr} \left( e^{-\beta H_0} U(t_0 - i\beta, t) Q(t) U(t, t_0) \mathcal{P} \right).  
\]

This follows directly from the usual definition of the interaction-picture state evolution operator \( U \). The time \( t_0 \) must be the time at which the interaction picture coincides with the Heisenberg picture. In the scalar-field case, where there is no projection operator \( \mathcal{P} \), one can show from the trace identity Eq. (2) that \( t_0 \) can be any time.

For the real-time formalism, one lets \( t_0 \to -\infty \). One assumes that one may switch off the gauge coupling very slowly: \( g \to g e^{-\epsilon t} \). This has no effect at \( t = 0 \), and so will not change the thermal average (1) at \( t = 0 \), which depends only on the configuration of the system now, independently of its past history. What it does, is to change the differential equation that \( U \) satisfies, but only by terms of order \( \epsilon \). It is well-established that it is safe to neglect these when \( \epsilon \to 0 \). The consequence of the switching off of \( g \) is that the interaction-picture physical states are simple: they may be assumed to contain only transverse gluons. So we may regard the two transverse components of the gluon field as physical, while the other two components, and the Faddeev-Popov ghost, are unphysical.

Now in the real-time formalism, the propagator for any field is a \( 2 \times 2 \) matrix consisting of the zero-temperature vacuum part, plus a thermal part. The thermal part is proportional to \( \delta (k^2) \). This is because it represents the effect of the real on-shell gluons in the heat bath.

The heat bath contains only the physical gluons. Thus it is immediate that only their propagator need have a thermal part. The unphysical fields can remain frozen, with no thermal parts in their propagators. This contrasts with the conventional formalism, where the unphysical gluon propagators are given thermal parts too, and their effects are then cancelled by the ghost propagator also having a thermal part.

The thermal propagator for the gluon field may be calculated in covariant gauges by Gupta-Bleuler quantisation. With the Keldysh time path, it is defined as
\[ iD^{\mu\nu}(x) = \left[ \frac{\langle T A^\mu(x) A^\nu(0) \rangle}{\langle A^\mu(0) A^\nu(0) \rangle} \frac{\langle A^\mu(0) A^\nu(x) \rangle}{\langle T A^\mu(x) A^\nu(0) \rangle} \right]. \] (5)

This is straightforward to calculate. I choose to write the result as

\[ D = M \hat{D} M \] (6a)

where \( \hat{D} \) is the diagonal matrix

\[ \hat{D} = \left[ \begin{array}{cc} \frac{1}{k^2+i\epsilon} & 0 \\ 0 & \frac{1}{k^2-i\epsilon} \end{array} \right]. \] (6b)

For the transverse physical fields, the matrix \( M \) is

\[ M_T = \sqrt{n(|k_0|)} \left[ \begin{array}{cc} e^{\frac{1}{2}\beta|k_0|} & e^{-\frac{1}{2}\beta|k_0|} \\ e^{\frac{1}{2}\beta k_0} & e^{-\frac{1}{2}\beta k_0} \end{array} \right] \] (7a)

with \( n \) the Bose distribution. For the unphysical fields, \( M_T \) must be replaced by its \( T \rightarrow 0 \) limit:

\[ M_0 = \left[ \begin{array}{cc} 1 & \theta(-k_0) \\ \theta(k_0) & 1 \end{array} \right]. \] (7b)

The physical field has the same tensor structure in all gauges, namely \( T^{\mu\nu} \) with

\[ T^{\mu0} = 0 = T^{0\nu}, \]
\[ T^{ij} = -\delta^{ij} + \frac{k^i k^j}{k^2}. \] (8)

The tensor structure of the unphysical gluon field propagator varies from gauge to gauge. In covariant gauges, it involves the gauge parameter. Because it does not have a thermal part, the formalism avoids some of the problems that are usually encountered with gauge-parameter dependence\(^6\). Although unphysical quantities may still be gauge-dependent, because of the gauge dependence of the zero-temperature part of the propagator, we do not need any subtle techniques to deal with\(^7\) aggravated infrared behaviour of unphysical modes.

### 3. Applications

By not heating the unphysical fields, we simplify calculations. An example is the hard thermal loop, which is the leading contribution at high temperature to the one-loop gluon self-energy. Because the ghost field is not heated, only gluon loops contribute. By manipulating the integrands it is straightforward to show that the hard thermal loop is gauge invariant, without explicitly calculating the Feynman integrals.

The leading temperature-dependent contribution to the imaginary part of the gluon self-energy similarly does not have a ghost part: it is calculated from the gluon loop with both gluons heated. Because the heated part of the gluon propagator is the same in all gauges, it is trivial that the answer is gauge invariant. The resulting
damping constant is the same, in all gauges, as was originally calculated with the Coulomb gauge\(^8\).

So if our method of calculation had been invented before the conventional one, the need would not have been so obvious for Braaten-Pisarski resummation\(^9\). One might ask, therefore, why such resummation is needed in our approach.

A partial answer to this question comes from looking at multiple self-energy insertions in the gluon propagator. In the conventional formalism, the bare propagator \(D\) may be written in the form (6a), and also the self-energy is

\[
\Pi = M^{-1}\tilde{\Pi}M^{-1}
\]  

with \(\tilde{\Pi}\) diagonal and the same matrix \(M\): in both cases \(M = M_T\). So when two self-energy insertions are made in the propagator, the result \(D\tilde{\Pi}D\tilde{\Pi}D\tilde{\Pi}M\). Hence the two entries \((k^2 \pm i\epsilon)^{-1}\) in \(\tilde{D}\), given in Eq. (6b), do not get multiplied together. In our formalism, however, we have a mixture of matrices \(M_T\) and \(M_0\), so that this is no longer true, and in applications there is a risk that when the dressed propagator is integrated the two poles pinch the contour of the integration.

It may be that in the calculation of a physical quantity the numerators and \(\delta\)-functions that also appear under the integral kill the potential pinch. We have shown that this is what happens with the 3-loop pressure.

However, in the unresummed propagator itself, the pinches are certainly there. But resummation removes them. For example, the hard-thermal-loop contribution to the resummed spatially-longitudinal propagator is given by

\[
D_L^{-1} = M_0^{-1}\tilde{D}^{-1}M_0^{-1} - M_T^{-1}\tilde{\Pi}_L M_T^{-1}.
\]  

Simple algebra, using the explicit forms (7) for \(M_T\) and \(M_0\), gives

\[
D_L^{-1} = M^{-1} \left[ \begin{array}{cc} k^2 - \Pi_L + i\epsilon & 2i\epsilon e^{-\beta k_0}\theta(k_0) \\ 2i\epsilon e^{\beta k_0}\theta(-k_0) & -k^2 + \Pi^*_L + i\epsilon \end{array} \right] M^{-1}.
\]  

When \(\epsilon \to 0\), the off-diagonal terms vanish and the pinch disappears. It is not completely trivial that one may safely take this limit, though careful argument does justify it. Thus, although the bare spatially-longitudinal propagator is frozen at \(T = 0\), the self-energy insertion heats it to temperature \(T\), so that after resummation it is similar in structure to the transverse propagator. The mechanism by which this occurs was first discussed by Weldon\(^10\). But note that, in the hard-thermal-loop approximation, the remaining component of the gluon propagator, and the ghost, remain frozen.

Beyond the hard-thermal-loop approximation, it is possible that things are more complicated.

4. \(A^0 = 0\) gauge in the imaginary-time formalism

With canonical quantisation, the \((1,1)\) component of the longitudinal gluon propagator in \(A^0 = 0\) gauge is found to be
where \( \tau = i t \). The \( k_3 \) appears because of the particular quantisation procedure; it may be replaced with any other component of \( k \). Because Eq. (11) is independent of temperature, clearly it does not satisfy KMS periodicity. The reason for this is that the projection operator \( \mathcal{P} \) in Eq. (1) prevents the use of the trace identity Eq. (2) that is needed for the usual demonstration of periodicity. The consequence is that both bosonic and fermionic Matsubara frequencies are needed for the longitudinal propagator:

\[
D^{11}_L(\tau, k) = \frac{1}{2} i \int_{-\beta}^{\beta} d\tau D^{11}_L(\tau, k) e^{i\pi n \beta \tau} = \begin{cases} 
1/4T^2 & n = 0 \\
-ic(k_3)/2\pi n^2 T^2 & n \text{ even} \neq 0 \\
\ldots & n \text{ odd}
\end{cases}
\] (12)

Hence the temperature has regulated the naively-expected \( 1/n^2 \) singularity at \( n = 0 \). I have not written the expression for \( D^n_L \) explicitly when \( n \) is odd, because in practice it is easier not to make a frequency summation, but to integrate directly in \( t \)-space.

The result Eq. (11) is simple enough, but there is a complication. The imaginary-time formalism requires the time \( t_0 \) in Eq. (4) to be finite. But this is the time at which the interaction picture coincides with the Heisenberg picture. So now we cannot switch off the QCD interaction when we define the interaction-picture physical states. They are defined in two equivalent ways, either by requiring that matrix elements of the Gauss-law operator vanishes, or from \(^5\)

\[
Q_{\text{BRST}} | \rangle = 0.
\] (13)

The result is that the physical states are no longer those that contain only transverse gluons, but some mapping of these states:

\[
| \rangle = \Lambda_{t_0} | \text{TRANSVERSE} \rangle
\] (14)

at \( t = t_0 \), where \( \Lambda_{t_0} \) is some functional of the \( t = t_0 \) field. Consequently, instead of Eq. (4) we have

\[
\langle Q \rangle = Z^{-1} \text{tr} \left( e^{-\beta H_0} \Lambda_{t_0-i\beta}^\dagger U(t_0-i\beta, t) Q(t) U(t, t_0) \Lambda_{t_0} \right)
\] (15)

where now the trace is taken only over the subspace of the transverse states. Note that \( \Lambda_{t_0-i\beta}^\dagger \) appears, rather than \( \Lambda_{t_0}^\dagger \), because it has been moved past the \( e^{-\beta H_0} \).

We may expand \( \Lambda_{t_0} \) in a power series:

\[
\Lambda = 1 + g\lambda_1 + g^2\lambda_2 + \ldots
\] (16)

where the series begins with 1 because when \( g \to 0 \) the transverse states themselves are physical. The coefficients \( \lambda \) may be calculated from Eq. (13). The result for \( \lambda_1 \)
is trilinear in the gluon field, so that $\lambda_1$ is effectively a 3-gluon vertex additional to the usual QCD vertex. We have tested this by calculating the two-loop pressure. The diagrams are shown in Figure 1, where the spot denotes the $\lambda_1$ vertex. It is interesting that the last three diagrams just cancel each other; I do not know why. The result agrees with previous calculations.

![Figure 1: contributions to the pressure](image)

5. Summary

For the real-time formalism, in any covariant gauge,

- at bare-propagator level, only the transverse gluons need to be heated

- Braaten-Pisarski resummation heats also the spatially-longitudinal gluons

- the remaining component of the gluon field, and the Faddeev-Popov ghost, remain frozen at zero temperature

- certain gauge-invariance problems of the conventional formalism are avoided.

For the imaginary-time formalism,

- we know how to calculate in the gauge $A^0 = 0$.

However, although I am confident that the procedure I have outlined for this is correct, it may be that there is a simpler one.

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