ABSTRACT

These notes are for the proceedings of the NATO Advanced Study Institute on *Scale Invariance, Interfaces, and Non–Equilibrium Dynamics*, held at the Isaac Newton Institute, 20–30 June 1994. The four lectures address a number of issues related to dynamic fluctuations of lines in non-equilibrium circumstances. The first two are devoted to the critical behavior of contact lines and flux lines depinning from impurities. It is emphasized that anisotropies in the medium lead to different universality classes. The importance of nonlinearities for moving lines are discussed in the context of flux lines and polymers in the last two lectures. A dynamic form birefringence is predicted for drifting polymers.

I. DEPINNING OF A LINE IN TWO DIMENSIONS

Depinning is a non-equilibrium critical phenomenon involving an external force and a pinning potential. When the force is weak the system is stationary, trapped in a metastable state. Beyond a threshold force the (last) metastable state disappears and the system starts to move. While there are many macroscopic mechanical examples, our interest stems from condensed matter systems such as Charge Density Waves (CDWs), interfaces, and contact lines. In CDWs, the controlling parameter is the external voltage. A finite CDW current appears only beyond a threshold applied voltage. Interfaces in porous media, domain walls in random magnets, are stationary unless the applied force (magnetic field) is sufficiently strong. A key feature of these examples is that they involve the collective depinning of many degrees of freedom that are elastically coupled. As such these problems belong to the realm of collective critical phenomena, characterized by universal scaling laws. We shall introduce these laws and the corresponding exponents below for the depinning of a line (interface or contact line).

Consider a line in two dimensions, oriented along the $x$ direction, and fluctuating along a perpendicular $y$ direction. The configuration of the line at time $t$ is described by the function $r(x, t)$. The function $r$ is assumed to be single valued, thus excluding configurations with overhangs. In many cases, the evolution of the curve satisfies an equation of the form

$$\frac{dr(x, t)}{dt} = F + f(x, r) + K[r].$$  \hfill (1)

The first term is a uniform applied force which is also the external control parameter. Fluctuations in the force due to randomness and impurities are represented by the second term. With the assumption that the medium is on average translationally


invariant, the average of $f$ can be set to zero. The final term describes the elastic forces between different parts of the line. Short range interactions can be described by a gradient expansion; for example, a line tension leads to $K[r(x)] = \nabla^2 r$ or $K[r(q)] = -q^2 r(q)$ for the Fourier modes. The surface of a drop of non-wetting liquid terminates at a contact line on a solid substrate. Deformations of the contact line are accompanied by distortions of the liquid/gas surface. As shown by Joanny and de Gennes, the resulting energy and forces are non-local, described by $K[r(q)] = -|q|r(q)$. More generally we shall consider the linear operator $K[r(q)] = -|q|^\sigma r(q)$, which interpolates between the above two cases as $\sigma$ changes from one to two.

When $F$ is small, the line is trapped in one of many metastable states in which $\partial r/\partial t = 0$ at all points. For $F$ larger than a threshold $F_c$, the line is depinned from the last metastable state, and moves with an average velocity $v$. On approaching the threshold from above, the velocity vanishes as

$$v = A(F - F_c)^\beta,$$  \hspace{1cm} (2)
where $\beta$ is the \textit{velocity exponent}, and $A$ is a nonuniversal amplitude. The motion just above threshold is not uniform, composed of rapid jumps as large segments of the line depin from strong pinning centers, superposed on the slower steady advance. These jumps have a power law distribution in size, cutoff at a correlation length $\xi$ which diverges at the transition as
\[ \xi \sim (F - F_c)^{-\nu}. \] (3)
The jumps are reminiscent of \textit{avalanches} in other slowly driven systems. In fact, the depinning can be approached from below $F_c$ by monotonically increasing $F$ in small increments, each sufficient to cause a jump to the next metastable state. The size and width of avalanches becomes invariant on approaching $F_c$. For example,
\[ \text{Prob(width of avalanche} > \ell) \approx \frac{1}{\ell^\kappa} \hat{\rho}(\ell/\xi^-), \] (4)
where the cutoff $\xi^-$ diverges as in Eq.(3). The critical line is a self–affine fractal whose correlations satisfy the dynamic scaling from
\[ \langle [r(x, t) - r(x', t')]^2 \rangle = \langle x - x' \rangle^{2\zeta} g\left(\frac{|t - t'|}{\langle x - x' \rangle^z}\right), \] (5)
defining the \textit{roughness} and \textit{dynamic} exponents, $\zeta$ and $z$ respectively. (Angular brackets reflect averaging over all realizations of the random force $f$.) The scaling function $g$ goes to a constant as its argument approaches 0; $\zeta$ is the wandering exponent of an instantaneous line profile, and $z$ relates the average lifetime of an avalanche to its size by $\tau(\xi) \sim \xi^z$.

Although, the underlying issues of collective depinning for CDWs and interfaces have been around for some time, only recently a systematic perturbative approach to the problem was developed. This functional renormalization group (RG) approach to the dynamical equations of motion was originally developed in the context of CDWs by Narayan and Fisher\textsuperscript{5} (NF), and extended to interfaces by Nattermann et al\textsuperscript{6}. We shall provide a brief outline of this approach starting from Eq.(1). Before embarking on the details of the formalism, it is useful to point out some scaling relations amongst the exponents which follow from underlying symmetries and non-renormalization conditions.

1. As mentioned earlier, the motion of the line close to the threshold is composed of jumps of segments of size $\xi$. Such jumps move the interface forward by $\xi^\zeta$ over a time period $\xi^z$. Thus the velocity behaves as,
\[ v \sim \frac{\xi^\zeta}{\xi^z} \sim |F - F_c|^{\nu(z - \zeta)} \implies \beta = \nu(z - \zeta). \] (6)

2. If the elastic couplings are linear, the response of the line to a \textit{static} perturbation $\varepsilon(x)$ is obtained simply by considering
\[ r_\varepsilon(x, t) = r(x, t) - \mathcal{K}^{-1}[\varepsilon(x)], \] (7)
where $\mathcal{K}^{-1}$ is the inverse kernel. Since, $r_\varepsilon$ satisfies Eq.(1) subject to a force $F + \varepsilon(x) + f(x, r_\varepsilon)$, $r$ satisfies the same equation with a force $F + f(x, r - \mathcal{K}^{-1}[\varepsilon(x)])$. 


As long as the statistical properties of the stochastic force are not modified by the above change in its argument, \( \partial \langle r \rangle / \partial \varepsilon = 0 \), and
\[
\left\langle \frac{\partial r_\varepsilon(x)}{\partial \varepsilon(x)} \right\rangle = -K^{-1}, \quad \text{or} \quad \left\langle \frac{\partial r_\varepsilon(q)}{\partial \varepsilon(q)} \right\rangle = \frac{1}{|q|^\sigma}. \tag{8}
\]
Since it controls the macroscopic response of the line, the kernel \( K \) cannot change under RG scaling. From Eqs.(5) and (3), we can read off the scaling of \( r(x) \), and the force \( \delta F \), which using the above non-renormalization must be related by the exponent relation
\[
\zeta + \frac{1}{\nu} = \sigma. \tag{9}
\]
Note that this identity depends on the statistical invariance of noise under the transformation in Eq.(7). It is satisfied as long as the correlations \( \langle f(x, r) f(x', r') \rangle \) only depend on \( r - r' \). The identity does not hold if these correlations also depend on the slope \( \partial r / \partial x \).

3. A scaling argument related to the Imry–Ma estimate of the lower critical dimension of the random field Ising model, can be used to estimate the roughness exponent \( \zeta \). The elastic force on a segment of length \( \xi \) scales as \( \xi^{\zeta - \sigma} \). If fluctuations in force are uncorrelated in space, they scale as \( \xi^{-(\zeta + 1)/2} \) over the area of an avalanche. Assuming that these two forces must be of the same order to initiate the avalanche leads to
\[
\zeta = \frac{2\sigma - 1}{3}. \tag{10}
\]
This last argument is not as rigorous as the previous two. Nonetheless, all three exponent identities can be established within the RG framework. Thus the only undetermined exponent is the dynamic one, \( z \).

A field theoretical description of the dynamics of Eq.(1) can be developed using the formalism of Martin, Siggia and Rose\(^8\) (MSR): Generalizing to a \( d \)-dimensional interface, an auxiliary field \( \hat{r}(x, t) \) is introduced to implement the equation of motion as a series of \( \delta \)-functions. Various dynamical response and correlation functions for the field \( r(x, t) \) can then be generated from the functional,
\[
Z = \int D r(x, t) D \hat{r}(x, t) J[r] \exp(S), \tag{11}
\]
where
\[
S = i \int d^d x dt \{ \partial_t r - K[r] - F - f[r(x, t)] \}. \tag{12}
\]
The Jacobian \( J[r] \) is introduced to ensure that the \( \delta \)-functions integrate to unity. It does not generate any new relevant terms and will be ignored henceforth.

The disorder-averaged generating functional \( \overline{Z} \) can be evaluated by a saddle-point expansion around a Mean-Field (MF) solution obtained by setting \( K_{MF}[r(x)] = vt - r(x) \). This amounts to replacing interaction forces with Hookean springs connected to the center of mass, which moves with a velocity \( v \). The corresponding equation of motion is
\[
\frac{d r_{MF}}{d t} = vt - r_{MF}(t) + f[r_{MF}(t)] + F_{MF}(v), \tag{13}
\]
where the relationship $F_{MF}(v)$ between the external force $F$ and average velocity $v$ is determined from the consistency condition $\langle r_{MF}(t) \rangle = vt$. The MF solution depends on the type of irregularity⁵: For smoothly varying random potentials, $\beta_{MF} = 3/2$, whereas for cusped random potentials, $\beta_{MF} = 1$. Following the treatment of NF⁵⁻⁹, we use the mean field solution for cusped potentials, anticipating jumps with velocity of $O(1)$, in which case $\beta_{MF} = 1$. After rescaling and averaging over impurity configurations, we arrive at a generating functional whose low-frequency form is

$$\mathcal{Z} = \int \mathcal{D}R(x,t) \mathcal{D} \hat{R}(x,t) \exp(\hat{S}),$$

$$\hat{S} = -\int d^d x dt [F - F_{MF}(v)] \hat{R}(x,t)$$

$$- \int \frac{d^d q}{(2\pi)^d} \frac{d\omega}{2\pi} \hat{R}(-q,-\omega)(-i\omega \rho + |q|^\sigma) R(q,\omega)$$

$$+ \frac{1}{2} \int d^d x dt dt' \hat{R}(x,t) \hat{R}(x,t') C [vt - vt' + R(x,t) - R(x,t')].$$

In the above expressions, $R$ and $\hat{R}$ are coarse-grained forms of $r - vt$ and $i\dot{r}$, respectively. $F$ is adjusted to satisfy the condition $\langle \hat{R}\rangle = 0$. The function $C(v\tau)$ is initially the connected mean-field correlation function $\langle (r_{MF}(t)r_{MF}(t+\tau)) \rangle_c$.

Ignoring the $R$-dependent terms in the argument of $C$, the action becomes Gaussian, and is invariant under a scale transformation $x \rightarrow bx, t \rightarrow b^\sigma t, R \rightarrow b^{\sigma - d/2} R, \hat{R} \rightarrow b^{-\sigma - d/2} \hat{R}, F \rightarrow b^{-d/2} F$, and $v \rightarrow b^{-d/2} v$. Other terms in the action, of higher order in $R$ and $\hat{R}$, that result from the expansion of $C$ [and other terms not explicitly shown in Eq.(14)], decay away at large length and time scales if $d > d_c = 2\sigma$. For $d > d_c$, the interface is smooth ($\zeta_0 < 0$) at long length scales, and the depinning exponents take the Gaussian values $z_0 = \sigma, v_0 = 2/d, \beta_0 = 1$.

At $d = d_c$, the action $S$ has an infinite number of marginal terms that can be rearranged as a Taylor series of the marginal function $C [vt - vt' + R(x,t) - R(x,t')]$, when $v \rightarrow 0$. The RG is carried out by integrating over a momentum shell $\Lambda/b < |q| < \Lambda$ (we set the cutoff wave vector to $\Lambda = 1$ for simplicity) and all frequencies, followed by a scale transformation $x \rightarrow bx, t \rightarrow b^\epsilon t, R \rightarrow b^\epsilon R$, and $\hat{R} \rightarrow b^{\theta - d} \hat{R}$, where $b = e^\epsilon$. The resulting recursion relation for the linear part in the effective action (to all orders in perturbation theory) is

$$\frac{\partial(F - F_{MF})}{\partial \ell} = (z + \theta)(F - F_{MF}) + \text{constant},$$

which immediately implies (with a suitable definition of $F_c$)

$$\frac{\partial(F - F_c)}{\partial \ell} = y_F(F - F_c),$$

with the exponent identity

$$y_F = z + \theta = 1/\nu .$$

The functional renormalization of $C(u)$ in $d = 2\sigma - \epsilon$ interface dimensions, computed to one-loop order, gives the recursion relation,

$$\frac{\partial C(u)}{\partial \ell} = [\epsilon + 2\theta + 2(z - \sigma)]C(u) + \zeta u \frac{dC(u)}{du}$$

$$- \frac{S_d}{(2\pi)^d} \frac{d}{du} \left\{ [C(u) - C(0)] \frac{dC(u)}{du} \right\},$$

$$\text{(18)}$$
where $S_d$ is the surface area of a unit sphere in $d$ dimensions. NF showed that all higher order diagrams contribute to the renormalization of $C$ as total derivatives with respect to $u$, thus, integrating Eq.(18) at the fixed-point solution $\partial C^*/\partial \ell = 0$, together with Eqs.(9) and (17), gives $\zeta = \epsilon/3$ to all orders in $\epsilon$, provided that $\int C^* \neq 0$. This gives Eq.(10) for a one-dimensional interface, as argued earlier. This is a consequence of the fact that $C(u)$ remains short-ranged upon renormalization, implying the absence of anomalous contributions to $\zeta$.

The dynamical exponent $z$ is calculated through the renormalization of $\rho$, the term proportional to $\hat{\mathbf{R}} \partial_t \mathbf{R}$, which yields

$$z = \sigma - 2\epsilon/9 + O(\epsilon^2),$$

and using the exponent identity (6),

$$\beta = 1 - 2\epsilon/9\sigma + O(\epsilon^2).$$

Nattermann et. al.\textsuperscript{6} obtain the same results to $O(\epsilon)$ by directly averaging the MSR generating function in Eq.(11), and expanding perturbatively around a rigidly moving interface.

Numerical integration of Eq.(1) for an elastic interface\textsuperscript{10} ($\sigma = 2$) has yielded critical exponents $\zeta = 0.97 \pm 0.05$ and $\nu = 1.05 \pm 0.1$, in agreement with the theoretical result $\zeta = \nu = 1$. The velocity exponent $\beta = 0.24 \pm 0.1$ is also consistent with the one-loop theoretical result $1/3$; however, a logarithmic dependence $\nu \sim 1/\ln(F - F_c)$, which corresponds to $\beta = 0$, also describes the numerical data well. In contrast, experiments and various discrete models of interface growth have resulted in scaling behaviors that differ from system to system. A number of different experiments on fluid invasion in porous media\textsuperscript{11} give roughness exponents of around 0.8, while imbibition experiments\textsuperscript{12,13} have resulted in $\zeta \approx 0.6$. A discrete model studied by Leschhorn\textsuperscript{14}, motivated by Eq.(1) with $\sigma = 2$, gives a roughness exponent of 1.25 at threshold. Since the expansion leading to Eq.(1) breaks down when $\zeta$ approaches one, it is not clear how to reconcile the results of Leschhhorn’s numerical work\textsuperscript{14} with the coarse-grained description of the RG calculation, especially since any model with $\zeta > 1$ cannot have a coarse grained description based on gradient expansions.

Amaral, Barabasi, and Stanley (ABS)\textsuperscript{15} recently pointed out that various models of interface depinning in 1+1 dimensions fall into two distinct classes, depending on the tilt dependence of the interface velocity:

1. For models like the random field Ising Model\textsuperscript{16}, and some Solid On Solid models, the computed exponents are consistent with the exponents given by the RG analysis. It has been suggested\textsuperscript{14}, however, that the roughness exponent is systematically larger than $\epsilon/3$, casting doubt on the exactness of the RG result.

2. A number of different models, based on directed percolation (DP)\textsuperscript{17,12} give a different roughness exponent, $\zeta \approx 0.63$. In these models, pinning sites are randomly distributed with a probability $p$, which is linearly related to the force $F$. The interface is stopped by the boundary of a DP cluster of pinning sites. The critical exponents at depinning can then be related to the longitudinal and transverse correlation length exponents $\nu_\parallel \approx 1.70$ and $\nu_\perp \approx 1.07$ of DP. In particular, $\zeta = \nu_\parallel/\nu_\perp \approx 0.63$, and $\beta = \nu_\parallel - \nu_\perp \approx 0.63$, in agreement with experiments.
The main difference of these models can be understood in terms of the dependence of the threshold force $F_c$ to the orientation. To include the possible dependence of the line mobility on its slope, $\partial_x r$, we can generalize the equation of motion to

\[
\partial_t r = K \partial^2_x r + \kappa \partial_x r + \frac{\lambda}{2} (\partial_x r)^2 + F + f(x, r).
\]  

(21)

The isotropic depinning studied by RG corresponds to $\kappa = \lambda = 0$. In models of depinning by directed percolation studied so far\cite{17, 12} there is a dependence of $F_c$ on slope, making a nonzero $\lambda$ unavoidable. The nonlinearity is relevant, accounting for the different universality class. Eq.(21) with $\kappa = 0$, motivated in a different fashion, has been studied by Stepanow\cite{18}. The exponents obtained approximately by a one loop expansion, $\zeta \approx 0.8615$, $z = 1$, and $y_F \approx 0.852$ are reasonably close to those of directed percolation. The presence of anisotropy in depinning actually suggests a third possibility:

3. When the line is depinning along a (tilted) direction of lower symmetry, even more relevant terms like $\kappa \partial_x r$ will be present in the equation of motion. This new universality class is possibly controlled by “tilted” DP clusters\cite{19}, for which $\zeta = 1/2$.

For the case of the contact line (CL) ($\sigma = 1$), these anisotropies are irrelevant, but there are other concerns related to the details of the driving force: In most experiments, the velocity of the CL is controlled rather than the external force. The effect of this can be numerically investigated by replacing the external force $F$ in Eq.(1) with

\[
F' = v - \int \frac{dx'}{L} f(x', r(x', t)),
\]

(22)

and looking at the time average of $F'$ as a function of $v$. ($F'$ is chosen such that $\int dx \partial_t r(x) = vL$.) Even though the critical behavior for both ways of driving may be the same for an infinitely large system, there is a system size dependent region near the depinning threshold where the behavior changes drastically. Preliminary findings on an elastic line suggest that in this region, the velocity exponent $\beta$ becomes considerably larger than one, in marked contrast with the constant force case. This can be qualitatively understood as follows: For a system of finite size, when a constant driving force is applied, the average velocity drops to zero as soon as temporal fluctuations of the instantaneous velocity are comparable with the time-averaged velocity. This is because the time average is then completely dominated by configurations for which the interface is pinned. Thus, the pinning transition becomes truly second order only in the large system limit: The velocity jumps to zero from a finite value in a finite system. In contrast, for constant velocity driving, no configuration has more weight than any other, since the interface is constrained to move past any obstacles by suitably increasing the applied external force, and decreasing it when passing through weakly pinning regions. Thus, in the region where a force-driven interface is pinned, the velocity-driven interface will experience fluctuations in the external force comparable to the average force itself. This average force as a function of velocity has an effective velocity exponent much larger than one. This distinction may partially explain the large velocity exponent found in a recent CL experiment\cite{20}, where the interface was velocity-driven. In addition to this, gravity imposes a finite wavelength cutoff on the roughening of the CL, which may complicate the analysis of experimental results.
II. DEPINNING OF A LINE IN THREE DIMENSIONS

The pinning of flux lines (FLs) in Type-II superconductors is of fundamental importance to many technological applications that require large critical currents\textsuperscript{21}. Upon application of an external current density $\mathbf{J}$, the FL becomes subject to a Lorentz force per unit length

$$
\mathbf{F} = \frac{J\phi_0}{c} \hat{\mathbf{J}} \times \hat{\mathbf{t}},
$$

(23)

where $\phi_0$ is the flux quantum, and $\hat{\mathbf{t}}$ is the unit tangent vector along the FL, which points along the local magnetic field. The motion of FLs due to the Lorentz force causes undesirable dissipation of supercurrents. Major increases in the critical current density $J_c$ of a sample are achieved when the FLs are pinned to impurities.

Recent numerical simulations have concentrated on the low temperature behavior of a single FL near depinning\textsuperscript{22,10,23}, mostly ignoring fluctuations transverse to the plane defined by the magnetic field and the Lorentz force. Common signatures of the depinning transition from $J < J_c$ to $J > J_c$ include a broad band ($f^{-a}$ type) voltage noise spectrum, and self-similar fluctuations of the FL profile.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure3.png}
\caption{Geometry of the line in three dimensions.}
\end{figure}

The configuration of the FL at time $t$ is now described by the vector function $\mathbf{r}(x, t)$, where $x$ is along the magnetic field $\mathbf{B}$, and the unit vector $\mathbf{e}_\parallel$ is along the Lorentz force $\mathbf{F}$. (See Fig.3) The major difference of the FL from the line in two dimensions is that the position, $\mathbf{r}(x, t)$, is now a 2-dimensional vector instead of a scalar; fluctuating along both $\mathbf{e}_\parallel$ and $\mathbf{e}_\perp$ directions. Point impurities are modeled by a random potential $V(x, \mathbf{r})$, with zero mean and short-range correlations. The simplest possible Langevin equation for the FL, consistent with local, dissipative dynamics, is

$$
\rho \frac{\partial \mathbf{r}}{\partial t} = \partial_x^2 \mathbf{r} + \mathbf{f}(x, \mathbf{r}(x, t)) + \mathbf{F},
$$

(24)
where $\rho$ is the inverse mobility of the FL, and $f = -\nabla_r V$. The potential $V(x, r)$ need not be isotropic. For example, in a single crystal of ceramic superconductors with the field along the oxide planes, it will be easier to move the FL along the planes. This leads to a pinning threshold that depends on the orientation of the force. Anisotropy also modifies the line tension, and the elastic term in Eq. (24) is in general multiplied by a non-diagonal matrix $K_{\alpha\beta}$. The random force $f(x, r)$, can be taken to have zero mean with correlations

$$
\langle f_\alpha(x, r)f_\gamma(x', r') \rangle = \delta(x - x')\Delta_{\alpha\gamma}(r - r').
$$

(25)

We shall focus mostly on the isotropic case, with $\Delta_{\alpha\gamma}(r - r') = \delta_{\alpha\gamma}\Delta(|r - r'|)$, where $\Delta$ is a function that decays rapidly for large values of its argument.

In addition to the exponents defined in the first lecture, $(\beta, \nu, \zeta \rightarrow \zeta_\parallel, \ z \rightarrow z_\parallel)$, there are two additional critical exponents that describe fluctuations transverse to the overall motion of the FL slightly above depinning. At length scales up to $\xi$, the correlated fluctuations satisfy the dynamic scaling form,

$$
\begin{align*}
\langle [r_\parallel(x, t) - r_\parallel(x', t')]^2 \rangle &= |x - x'|^{2z_\parallel}g_\parallel(\frac{|t - t'|}{|x - x'|^{z_\parallel}}), \\
\langle [r_\perp(x, t) - r_\perp(x', t')]^2 \rangle &= |x - x'|^{2z_\perp}g_\perp(\frac{|t - t'|}{|x - x'|^{z_\perp}}),
\end{align*}
$$

(26)

where $\zeta_\perp$ and $z_\perp$ are the transverse roughness and dynamic exponents. One consequence of transverse fluctuations is that a “no passing” rule\footnote{24}, applicable to CDWs and interfaces, does not apply to FLs. It is possible to have coexistence of moving and stationary FLs in particular realizations of the random potential. The effects of transverse fluctuations $r_\perp$ for large driving forces, when the impurities act as white noise, will be discussed later. At this point, we would like to know how these transverse fluctuations scale near the depinning transition, and whether or not they influence the critical dynamics of longitudinal fluctuations near threshold.

The answer to the second question is obtained by the following qualitative argument: Consider Eq. (24) for a particular realization of randomness $f(x, r)$. Assuming that portions of the FL always move in the forward direction\footnote{25}, there is a unique point $r_\perp(x, r_\parallel)$ that is visited by the line for given coordinates $(x, r_\parallel)$. We construct a new force field $f'$ on a two dimensional space $(x, r_\parallel)$ through $f'(x, r_\parallel) \equiv f_\parallel(x, r_\parallel) - f_\perp(x_\parallel, r_\parallel)$. It is then clear that the dynamics of the longitudinal component $r_\parallel(x, t)$ in a given force field $f(x, r)$ is identical to the dynamics of $r_\parallel'(x, t)$ in a force field $f'(x, r_\parallel)$, with $r_\perp$ set to zero. It is quite plausible that, after averaging over all $f$, the correlations in $f'$ will also be short-ranged, albeit different from those of $f$. Thus, the scaling of longitudinal fluctuations of the depinning FL will not change upon taking into account transverse fluctuations. However, the question of how these transverse fluctuations scale still remains.

Certain statistical symmetries of the system restrict the form of response and correlation functions. For example, Eq. (24) has statistical space- and time-translational invariance, which enables us to work in Fourier space, i.e. $(x, t) \rightarrow (q, \omega)$. For an isotropic medium, $F$ and $v$ are parallel to each other, i.e., $v(F) = v(F)\hat{F}$, where $\hat{F}$ is the unit vector along $F$. Furthermore, all expectation values involving odd powers
of a transverse component are identically zero due to the statistical invariance under the transformation \( r_\perp \rightarrow -r_\perp \). Thus, linear response and two-point correlation functions are \textit{diagonal}. The introduced critical exponents are then related through scaling identities. These can be derived from the linear response to an infinitesimal external force field \( \varepsilon(q, \omega) \),

\[
\chi_{\alpha\beta}(q, \omega) = \left\langle \frac{\partial r_\alpha(q, \omega)}{\partial \varepsilon_\beta(q, \omega)} \right\rangle \equiv \delta_{\alpha\beta} \chi_\alpha,
\]

in the \((q, \omega) \rightarrow (0, 0)\) limit. Eq.(24) is statistically invariant under the transformation \( F \rightarrow F + \varepsilon(q) \), \( r(q, \omega) \rightarrow r(q, \omega) + q^{-2} \varepsilon(q) \). Thus, the static linear response has the form \( \chi_\parallel(q, \omega = 0) = \chi_\perp(q, \omega = 0) = q^{-2} \). Since \( \varepsilon_\parallel \) scales like the applied force, the form of the linear response at the correlation length \( \xi \) gives an exponent identity similar to Eq.(9):

\[
\zeta_\parallel + 1/\nu = 2.
\]

Considering the transverse linear response seems to imply \( \zeta_\perp = \zeta_\parallel \). However, the static part of the transverse linear response is irrelevant at the critical RG fixed point, since \( z_\perp > z_\parallel \), as shown below. When a slowly varying uniform external force \( \varepsilon(t) \) is applied, the FL responds as if the instantaneous external force \( F + \varepsilon \) is a constant, acquiring an average velocity,

\[
\langle \partial_t r_\alpha \rangle = v_\alpha(F + \varepsilon) \approx v_\alpha(F) + \frac{\partial v_\alpha}{\partial F_\gamma} \varepsilon_\gamma.
\]

Substituting \( \partial v_\parallel / \partial F_\parallel = dv/dF \) and \( \partial v_\perp / \partial F_\perp = v/F \), and Fourier transforming, gives

\[
\chi_\parallel(q = 0, \omega) = \frac{1}{-i\omega(dv/dF)^{-1} + O(\omega^2)},
\]

\[
\chi_\perp(q = 0, \omega) = \frac{1}{-i\omega(v/F)^{-1} + O(\omega^2)}.
\]

Combining these with the static response, we see that the characteristic relaxation times of fluctuations with wavelength \( \xi \) are

\[
\tau_\parallel(q = \xi^{-1}) \sim \left( q^2 \frac{dv}{dF} \right)^{-1} \sim \xi^{2+(\beta-1)/\nu} \sim \xi^{z_\parallel},
\]

\[
\tau_\perp(q = \xi^{-1}) \sim \left( q^2 \frac{v}{F} \right)^{-1} \sim \xi^{2+\beta/\nu} \sim \xi^{z_\perp},
\]

which, using Eq.(28) , yield the scaling relations

\[
\beta = (z_\parallel - \zeta_\parallel)\nu,
\]

\[
z_\perp = z_\parallel + 1/\nu.
\]

We already see that the dynamic relaxation of transverse fluctuations is much slower than longitudinal ones. All critical exponents can be calculated from \( \zeta_\parallel, \zeta_\perp, \) and \( z_\parallel \), by using Eqs.(28) and (32).
Equation (24) can again be analyzed using the MSR formalism. The long wave-
length, low frequency behavior, for isotropic random potentials, is described by the
effective action
\[ \tilde{S} = - \int dt d^d x \left[ F - F_{MF}(v) \right] \dot{R}_\parallel(x, t) \]
\[ - \int \frac{d^d q}{(2\pi)^d} \frac{d\omega}{2\pi} \dot{R}_\parallel(-q, -\omega) R_\parallel(q, \omega) (-i\omega \rho + q^2) \]
\[ - \int \frac{d^d q}{(2\pi)^d} \frac{d\omega}{2\pi} \ddot{R}_\perp(-q, -\omega) \cdot R_\perp(q, \omega) \left(-i\omega \frac{F_c}{v} + q^2\right) \]
\[ + \frac{1}{2} \sum_\gamma \int d^d x dt dt' \dot{R}_\gamma(x, t) \dot{R}_\gamma(x, t') C_\gamma(v(t - t') + R_\parallel(x, t) - R_\parallel(x, t')) \].

(33)

All terms in $\tilde{S}$ involving longitudinal fluctuations are identical to the two-dimensional
case, thus we obtain the same critical exponents for longitudinal fluctuations, i.e.,
$\zeta_\parallel = \epsilon/3$, $z_\parallel = 2 - 2\epsilon/9 + O(\epsilon^2)$. The renormalization of transverse temporal force-
force correlations $C_\perp(u)$ yields an additional recursion relation
\[ \frac{\partial C_\perp(u)}{\partial \ell} = \left[ \epsilon + 2\theta_\perp + 2(z_\parallel - 2) \right] C_\perp(u) + \zeta_\parallel u \frac{dC_\perp(u)}{du} \]
\[ - \frac{S_d}{(2\pi)^d} \left\{ \left[ C_\parallel(u) - C_\parallel(0) \right] \frac{d^2 C_\perp(u)}{du^2} \right\} \].

(34)

In addition, the form of the transverse dynamic linear response given in Eq.(30)
implies the nonrenormalization of the term proportional to $\dot{R}_\perp \partial_t R_\perp(F_c/v)$, which,
along with the renormalization of $C_\perp(0)$, gives a transverse roughness exponent $\zeta_\perp = \zeta_\parallel - d/2$, to all orders in perturbation theory. For the FL ($\epsilon = 3$), the critical
exponents are then given by
$\zeta_\parallel = 1$, $z_\parallel \approx 4/3$, $\nu = 1$,
$\beta \approx 1/3$, $\zeta_\perp = 1/2$, $z_\perp \approx 7/3$.

(35)

Numerical integrations of Eq.(24)\textsuperscript{26} that test the scaling forms and exponents
predicted by Eqs.(2) and (26) are in agreement with RG results: A fit for the velocity
exponent gives $\beta = 0.3 \pm 0.1$, although a logarithmic fit ($\beta = 0$) cannot be ruled out,
as seen in Fig.4. The roughness exponents (see Fig.5) fit the scaling form well, with\textsuperscript{10}
$\zeta_\parallel = 0.94 \pm 0.05$, and $\zeta_\perp = 0.50 \pm 0.02$.

The potential pinning the FL in a single superconducting crystal is likely to
be highly anisotropic. For example, consider a magnetic field parallel to the copper
oxide planes of a ceramic superconductor. The threshold force then depends on its
orientation, with depinning easiest along the copper oxide planes. In general, the
average velocity may depend on the orientations of the external force and the FL.
The most general gradient expansion for the equation of motion is then,
\[ \frac{\partial r_\alpha}{\partial t} = \mu_\alpha \beta F_\beta + \kappa_\alpha \beta \partial_\alpha r_\beta + K_\alpha \beta \partial_\alpha^2 r_\beta + \frac{1}{2} \Lambda_\alpha, \beta, \gamma \partial_\alpha r_\beta \partial_\beta r_\gamma + f_\alpha(x, r(x, t)) + \cdots, \]
Figure 4. A plot of average velocity versus external force for a system of 2048 points. Statistical errors are smaller than symbol sizes. Both fits have three adjustable parameters: The threshold force, the exponent, and an overall multiplicative constant.

Figure 5. A plot of equal time correlation functions versus separation, for the system shown in Fig.4, at $F = 0.95$. The observed roughness exponents very closely follow the theoretical predictions of $\zeta_\parallel = 1, \zeta_\perp = 0.5$, which are shown as solid lines for comparison.

with

$$ \langle f_\alpha(x, r)f_\beta(x', r') \rangle = \delta(x - x')C_{\alpha\beta}(r - r'). $$

(37)

Depending on the presence or absence of various terms allowed by the symmetries of the system, the above set of equations encompasses many distinct universality classes. For example, consider the situation where $v$ depends on $F$, but not on the orientation of the line. Eqs.(27) and (30) have to be modified, since $v$ and $F$ are no longer parallel (except along the axes with $r \to -r$ symmetry), and the linear response function is not diagonal. The RG analysis is more cumbersome: For depinning along a non-symmetric direction, the longitudinal exponents are not modified (in agreement with the argument presented earlier), while the transverse fluctuations are further
suppressed to $\zeta_\perp = 2\zeta_\parallel - 2$ (equal to zero for $\zeta_\parallel = 1$). Relaxation of transverse modes are still characterized by $z_\perp = z_\parallel + 1/\nu$, and the exponent identity (28) also holds. Surprisingly, the exponents for depinning along axes of reflection symmetry are the same as the isotropic case. If the velocity also depends on the tilt, there will be additional relevant terms in the MSR partition function, which invalidate the arguments leading to Eqs.(28)–(32). The analogy to FLs in a planes suggests that the longitudinal exponents for $d = 1$ are controlled by DP clusters, with $\zeta_\parallel \approx 0.63$. Since no perturbative fixed point is present in this case, it is not clear how to explore the behavior of transverse fluctuations systematically.

III. NONLINEAR DYNAMICS OF MOVING LINES

We have so far investigated the dynamics of a line near the depinning transition. Now, we would like to consider its behavior in a different regime, when the external driving force is large, and the impurities appear as weak barriers that deflect portions of the line without impeding its overall drift. In such non-equilibrium systems, one can regard the evolution equations as more fundamental, and proceed by constructing the most general equations consistent with the symmetries and conservation laws of the situation under study. Even in a system with isotropic randomness, which we will discuss here, the average drift velocity, $v$, breaks the symmetry between forward and backward motions, and allows introduction of nonlinearities in the equations of motion.

Let us first concentrate on an interface in two dimensions. (Fig.1.) By contracting up to two spatial derivatives of $r$, and keeping terms that are relevant, one obtains the Kardar-Parisi-Zhang (KPZ) equation,

$$\partial_t r(x, t) = \mu F + K \partial_x^2 r(x, t) + \frac{\lambda}{2} [\partial_x r(x, t)]^2 + f(x, t),$$  \hspace{1cm} (38)

with random force correlations

$$\langle f(x, t) f(x', t') \rangle = 2T \delta(x - x') \delta(t - t').$$  \hspace{1cm} (39)

For a moving line, the term proportional to the external force can be absorbed without loss of generality by considering a suitable Galilean transformation, $r \to r - at$, to a moving frame. A large number of stochastic nonequilibrium growth models, like the Eden Model and various ballistic deposition models are known to be well described, at large length scales and times, by this equation, which is intimately related to several other problems. For example, the transformation $v(x, t) = -\lambda \partial_x r(x, t)$ maps Eq.(38) to the randomly stirred Burgers’ equation for fluid flow,

$$\partial_t v + v \partial_x v = K \partial_x^2 v - \lambda \partial_x f(x, t).$$  \hspace{1cm} (40)

The correlations of the line profile still satisfy the dynamic scaling form in Eq.(5), nevertheless with different scaling exponents $\zeta, z$ and scaling function $g$. This self-affine scaling is not critical, i.e., not obtained by fine tuning an external parameter like the force, and is quite different in nature than the critical scaling of the line near the depinning transition, which ceases beyond the correlation length scale $\xi$.

Two important nonperturbative properties of Eq.(38) help us determine these exponents exactly in 1+1 dimensions:
1. **Galilean Invariance (GI):** Eq.(38) is statistically invariant under the infinitesimal reparametrization

\[ r' = r + \varepsilon x, \quad x' = x + \lambda \varepsilon t, \quad t' = t, \tag{41} \]

provided that the random force \( f \) does not have temporal correlations\(^{33}\). Since the parameter \( \lambda \) appears both in the transformation and Eq.(38), it is not renormalized under any RG procedure that preserves this invariance. This implies the exponent identity\(^{32,33}\)

\[ \zeta + z = 2. \tag{42} \]

2. **Fluctuation–Dissipation (FD) Theorem:** Eqs.(38) and (39) lead to a Fokker–Planck equation for the evolution of the joint probability \( P[r(x)] \),

\[ \partial_t P = \int dx \left( \frac{\delta P}{\delta r(x)} \partial_x r + T \frac{\delta^2 P}{\delta r(x)^2} \right). \tag{43} \]

It is easy to check that \( P \) has a stationary solution

\[ P = \exp \left( -\frac{K}{2T} \int dx \, (\partial_x r)^2 \right). \tag{44} \]

If \( P \) converges to this solution, the long–time behavior of the correlation functions in Eq.(5) can be directly read off Eq.(44), giving \( \zeta = 1/2 \).

Combining these two results, the roughness and dynamic exponents are exactly determined for the line in two dimensions as

\[ \zeta = 1/2, \quad z = 3/2. \tag{45} \]

Many direct numerical simulations and discrete growth models have verified these exponents to a very good accuracy. Exact exponents are not known for interfaces in higher dimensions, since the FD property is only valid in two dimensions. These results have been summarized in a number of recent reviews\(^{34,35}\).

Let us now turn to the case of a line in three dimensions (Fig.3). Fluctuations of the line can be indicated by a two dimensional vector \( r \). Even in an isotropic medium, the drift velocity \( \mathbf{v} \) breaks the isotropy in \( r \) by selecting a direction. A gradient expansion up to second order for the equation of motion gives\(^{36}\)

\[ \partial_t r_\alpha = [K_1 \delta_{\alpha\beta} + K_2 v_\alpha v_\beta] \partial_x^2 r_\beta \]

\[ + \left[ \frac{\lambda_1}{2} \left( \delta_{\alpha\beta} v_\gamma + \delta_{\alpha\gamma} v_\beta \right) + \frac{\lambda_2}{2} \delta_{\alpha\gamma} v_\beta v_\gamma \right] \frac{\partial_x r_\beta \partial_x r_\gamma}{2} + f_\alpha, \tag{46} \]

with random force correlations

\[ \langle f_\alpha(x,t) f_\beta(x',t') \rangle = 2 [T_1 \delta_{\alpha\beta} + T_2 v_\alpha v_\beta] \delta(x-x') \delta(t-t'). \tag{47} \]

Higher order nonlinearities can be similarly constructed but are in fact irrelevant. In terms of components parallel and perpendicular to the velocity, the equations are

\[ \begin{cases} \partial_t r_\parallel = K_\parallel \partial_x^2 r_\parallel + \frac{\lambda_\parallel}{2} \left( \partial_x r_\parallel \right)^2 + \frac{\lambda_\times}{2} \left( \partial_x r_\perp \right)^2 + f_\parallel(x,t), \\ \partial_t r_\perp = K_\perp \partial_x^2 r_\perp + \lambda_\perp \partial_x r_\parallel \partial_x r_\perp + f_\perp(x,t) \end{cases} \tag{48} \]
with
\[
\begin{align*}
&\langle f_\parallel(x, t) f_\parallel(x', t') \rangle = 2 T_\parallel \delta(x - x') \delta(t - t') \\
&\langle f_\perp(x, t) f_\perp(x', t') \rangle = 2 T_\perp \delta(x - x') \delta(t - t')
\end{align*}
\]
(49)

The noise-averaged correlations have a dynamic scaling form like Eq.(26),
\[
\begin{align*}
&\langle [r_\parallel(x, t) - r_\parallel(x', t')]^2 \rangle = |x - x'|^{2\zeta_\parallel} g_\parallel \left( \frac{|t - t'|}{|x - x'|^{z_\parallel}} \right), \\
&\langle [r_\perp(x, t) - r_\perp(x', t')]^2 \rangle = |x - x'|^{2\zeta_\perp} g_\perp \left( \frac{|t - t'|}{|x - x'|^{z_\perp}} \right).
\end{align*}
\]
(50)

**Figure 6.** A projection of RG flows in the parameter space, for $n = 1$ transverse components.

In the absence of nonlinearities ($\lambda_\parallel = \lambda_\times = \lambda_\perp = 0$), Eqs.(48) can easily be solved to give $\zeta_\parallel = \zeta_\perp = 1/2$ and $z_\parallel = z_\perp = 2$. Simple dimensional counting indicates that all three nonlinear terms are relevant and may modify the exponents in Eq.(50). Studies of related stochastic equations\textsuperscript{37,38} indicate that interesting dynamic phase diagrams may emerge from the competition between nonlinearities. Let us assume that $\lambda_\parallel$ is positive and finite (its sign can be changed by $r_\parallel \to -r_\parallel$), and focus on the dependence of the scaling exponents on the ratios $\lambda_\perp/\lambda_\parallel$ and $\lambda_\times/\lambda_\parallel$, as depicted in Fig.6. (It is more convenient to set the vertical axis to $\lambda_\times K_\parallel T_\perp/\lambda_\parallel K_\perp T_\parallel$.)

The properties discussed for the KPZ equation can be extended to this higher dimensional case:

1. **Galilean Invariance (GI):** Consider the infinitesimal reparametrization
\[
\begin{align*}
x' &= x + \lambda_\parallel \epsilon t , \quad t' = t , \\
r_\parallel' &= r_\parallel + \epsilon x , \quad r_\perp' = r_\perp .
\end{align*}
\]
(51)
Eqs.(48) are invariant under this transformation provided that $\lambda_|| = \lambda_\bot$. Thus along this line in Fig.6 there is GI, which once more implies the exponent identity

$$\zeta_|| + z_|| = 2.$$  \hspace{1cm} (52)

2. Fluctuation–Dissipation (FD) Condition: The Fokker–Planck equation for the evolution of the joint probability $P [r_||(x), r_\bot(x)]$ has a stationary solution

$$P_0 \propto \exp \left( - \int dx \left[ \frac{K_||}{2T_||} (\partial_x r_||)^2 + \frac{K_\bot}{2T_\bot} (\partial_x r_\bot)^2 \right] \right),$$  \hspace{1cm} (53)

provided that $\lambda_\times K_|| T_\bot = \lambda_\bot K_\bot T_||$. Thus for this special choice of parameters, depicted by a starred line in Fig.6, if $P$ converges to this solution, the long–time behavior of the correlation functions in Eq.(50) can be directly read off Eq.(53), giving $\zeta_|| = \zeta_\bot = 1/2$.

3. The Cole–Hopf (CH) Transformation is an important method for the exact study of solutions of the one component nonlinear diffusion equation\textsuperscript{31}. Here we generalize this transformation to the complex plane by defining, for $\lambda_\times < 0$,

$$\Psi(x, t) = \exp \left( \frac{\lambda_|| r_||(x, t) + i \sqrt{-\lambda_\times} \lambda_\bot r_\bot(x, t)}{2K} \right).$$  \hspace{1cm} (54)

The linear diffusion equation

$$\partial_t \Psi = K \partial_x^2 \Psi + \mu(x, t) \Psi,$$

then leads to Eqs.(48) if $K_|| = K_\bot = K$ and $\lambda_|| = \lambda_\bot$. [Here $\text{Re}(\mu) = \lambda_|| f_|| / 2K$ and $\text{Im}(\mu) = \sqrt{-\lambda_\times} \lambda_\bot f_\bot / 2K].$ This transformation enables an exact solution of the deterministic equation, and further allows us to write the solution to the stochastic equation in the form of a path integral

$$\Psi(x, t) = \int_{(0,0)}^{(x,t)} \mathcal{D}x(\tau) \exp \left\{ - \int_0^t d\tau \left[ \frac{\dot{x}^2}{2K} + \mu(x, \tau) \right] \right\}. \hspace{1cm} (55)$$

Eq.(55) has been extensively studied in connection with quantum tunneling in a disordered medium\textsuperscript{39}, with $\Psi$ representing the wave function. In particular, results for the tunneling probability $|\Psi|^2$ suggest $z_|| = 3/2$ and $\zeta_|| = 1/2$. The transverse fluctuations correspond to the phase in the quantum problem which is not an observable. Hence this mapping does not provide any information on $\zeta_\bot$ and $z_\bot$ which are in fact observable for the moving line.

At the point $\lambda_\bot = \lambda_\times = 0$, $r_||$ and $r_\bot$ decouple, and $z_\bot = 2$ while $z_|| = 3/2$. However, in general $z_|| = z_\bot = z$ unless the effective $\lambda_\bot$ is zero. For example at the intersection of the subspaces with GI and FD the exponents $z_|| = z_\bot = 3/2$ are obtained from the exponent identities. Dynamic RG recursion relations can be computed to one–loop order\textsuperscript{36,40}, by standard methods of momentum-shell dynamic RG\textsuperscript{32,33}. 

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The renormalization of the seven parameters in Eqs.(48), generalized to \( n \) transverse directions, give the recursion relations

\[
\frac{dK_{||}}{d\ell} = K_{||} \left[ z - 2 + \frac{1}{\pi} \frac{\lambda_{\parallel}^2 T_{||}}{4K_{||}^3} + n \frac{1}{\pi} \frac{\lambda_{\parallel} \lambda_{\parallel} T_{\parallel}}{4K_{||} K_{\perp}^2} \right],
\]

\[
\frac{dK_{\perp}}{d\ell} = K_{\perp} \left[ z - 2 + \frac{1}{\pi} \frac{\lambda_{\perp} \left( \left( \lambda_{\perp} T_{\perp} / K_{\perp} \right) + \left( \lambda_{\parallel} T_{\parallel} / K_{\parallel} \right) \right)}{2K_{\perp} \left( K_{\perp} + K_{\parallel} \right)} \right],
\]

\[
\frac{dT_{||}}{d\ell} = T_{||} \left[ z - 2\zeta_{||} - 1 + \frac{1}{\pi} \frac{\lambda_{\parallel}^2 T_{||}^2}{4K_{||}^3} \right] + n \frac{1}{\pi} \frac{\lambda_{\parallel}^2 T_{\parallel}^2}{4K_{\perp}^3},
\]

\[
\frac{dT_{\perp}}{d\ell} = T_{\perp} \left[ z - 2\zeta_{\perp} - 1 + \frac{1}{\pi} \frac{\lambda_{\perp}^2 T_{\perp}^2}{K_{\perp} K_{||} \left( K_{\perp} + K_{||} \right)} \right],
\]

\[
\frac{d\lambda_{||}}{d\ell} = \lambda_{||} \left[ \zeta_{||} + z - 2 \right],
\]

\[
\frac{d\lambda_{\perp}}{d\ell} = \lambda_{\perp} \left[ \zeta_{\perp} + z - 2 - \frac{1}{\pi} \frac{\lambda_{\parallel} - \lambda_{\perp}}{\left( K_{\perp} + K_{||} \right)^2} \left( \left( \lambda_{\parallel} T_{\parallel} / K_{\parallel} \right) - \left( \lambda_{\parallel} T_{\parallel} / K_{||} \right) \right) \right],
\]

\[
\frac{d\lambda_{\times}}{d\ell} = \lambda_{\times} \left[ 2\zeta_{\perp} - \zeta_{||} + z - 2 + \frac{1}{\pi} \frac{\lambda_{\parallel} K_{\perp} - \lambda_{\perp} K_{||}}{K_{\perp} \left( K_{\perp} + K_{||} \right)} \left( \left( \lambda_{\perp} T_{\perp} / K_{\perp} \right) - \left( \lambda_{\parallel} T_{\parallel} / K_{||} \right) \right) \right].
\]

(56)

The projections of the RG flows on the two parameter subspace shown in Fig.6 are indicated by trajectories. They naturally satisfy the constraints imposed by the non–perturbative results: the subspace of GI is closed under RG, while the FD condition appears as a fixed line. The RG flows, and the corresponding exponents, are different in each quadrant of Fig.6, which implies that the scaling behavior is determined by the relative signs of the three nonlinearities. This was confirmed by numerical integrations\(^ \text{36,40} \) of Eqs.(48), performed for different sets of parameters. A summary of the computed exponents are given in Table I.

The analysis of analytical and numerical results can be summarized as follows: 

\( \lambda_{\perp} \lambda_{\times} > 0 \): In this region, the scaling behavior is understood best. The RG flows terminate on the fixed line where FD conditions apply, hence \( \zeta_{||} = \zeta_{\perp} = 1/2 \). All along this line, the one loop RG exponent is \( z = 3/2 \). These results are consistent with the numerical simulations. The measured exponents rapidly converge to these values, except when \( \lambda_{\perp} \) or \( \lambda_{\times} \) are small.

\( \lambda_{\times} = 0 \): In this case the equation for \( r_{||} \) is the KPZ equation (38), thus \( \zeta_{||} = 1/2 \) and \( z_{||} = 3/2 \). The fluctuations in \( r_{||} \) act as a strong (multiplicative and correlated) noise on \( r_{\perp} \). The one–loop RG yields the exponents \( z_{\perp} = 3/2, \zeta_{\perp} = 0.75 \) for \( \lambda_{\perp} > 0 \), while a negative \( \lambda_{\perp} \) scales to 0 suggesting \( z_{\perp} > z_{||} \). Simulations are consistent with the RG calculations for \( \lambda_{\perp} > 0 \), yielding \( \zeta_{\perp} = 0.72 \), surprisingly close to the one–loop RG value. For \( \lambda_{\perp} < 0 \), simulations indicate \( z_{\perp} \approx 2 \) and \( \zeta_{\perp} \approx 2/3 \) along with the expected values for the longitudinal exponents.

\( \lambda_{\perp} = 0 \): The transverse fluctuations satisfy a simple diffusion equation with \( \zeta_{\perp} = 1/2 \) and \( z_{\perp} = 2 \). Through the term \( \lambda_{\times} (\partial_{z} r_{\perp})^2 / 2 \), these fluctuations act as
a correlated noise\textsuperscript{33} for the longitudinal mode. A naive application of the results of this reference\textsuperscript{33} give \( \zeta_\parallel = 2/3 \) and \( z_\parallel = 4/3 \). Quite surprisingly, simulations indicate different behavior depending on the sign of \( \lambda_\times \). For \( \lambda_\times < 0 \), \( z_\parallel \approx 3/2 \) and \( \zeta_\parallel \approx 1/2 \) whereas for \( \lambda_\times > 0 \), longitudinal fluctuations are much stronger, resulting in \( z_\parallel \approx 1.18 \) and \( \zeta_\parallel \approx 0.84 \). Actually, \( \zeta_\parallel \) increases steadily with system size, suggesting a breakdown of dynamic scaling, due to a change of sign in \( \lambda_\perp \lambda_\times \). This dependence on the sign of \( \lambda_\times \) may reflect the fundamental difference between behavior in quadrants II and IV of Fig.6.

| \( \lambda_\parallel \) | \( \lambda_\times \) | \( \lambda_\perp \) | \( \zeta_\parallel \) | \( z_\parallel/\zeta_\parallel \) | \( \zeta_\perp \) | \( z_\parallel/\zeta_\perp \) |
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 20              | 20              | 20              | 0.48            | 3.0             | 0.48            | 3.0             |
| (1/2)           | (3)             | (1/2)           | (3)             |
| 20              | 20              | 2.5             | 0.75            | 1.7             | 0.50            | 3.7             |
| 20              | 5               | 25              | 0.51            | 3.4             | 0.56            | 2.9             |
| 5               | 5               | -5              | 0.83            | unstable        | 0.44            | 3.6             |

(No fixed point for finite \( \zeta, z \))

| \( \lambda_\parallel \) | \( \lambda_\times \) | \( \lambda_\perp \) | \( \zeta_\parallel \) | \( z_\parallel/\zeta_\parallel \) | \( \zeta_\perp \) | \( z_\parallel/\zeta_\perp \) |
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 20              | -20             | -20             | 0.50            | 3.1             | 0.50            | 2.9             |
| (1/2)           | (3)             | (1/2)           | (3)             |
| 5               | -5              | 5               | 0.52            | 3.3             | 0.57            | 3.4             |
| (1/2)           | (3)             | (Strong coupling) |

| \( \lambda_\parallel \) | \( \lambda_\times \) | \( \lambda_\perp \) | \( \zeta_\parallel \) | \( z_\parallel/\zeta_\parallel \) | \( \zeta_\perp \) | \( z_\parallel/\zeta_\perp \) |
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 20              | 0               | 20              | 0.49            | 3.1             | 0.72            | 2.2             |
| (1/2)           | (3)             | (0.75)           | (2)             |
| 20              | 0               | -20             | 0.48            | 3.0             | 0.65            | 3.1             |
| (1/2)           | (3)             | (z_\perp < z_\parallel) |
| 20              | 20              | 0               | 0.84            | 1.4             | 0.50            | 4.0             |
| (z_\parallel < z_\perp) | (1/2)          | (4)             |
| 20              | -20             | 0               | 0.55            | 2.9             | 0.51            | 4.0             |
| (z_\parallel < z_\perp) | (1/2)          | (4)             |

\( \lambda_\perp < 0 \) and \( \lambda_\times > 0 \): The analysis of this region (II) is the most difficult in that the RG flows do not converge upon a finite fixed point and \( \lambda_\perp \to 0 \), which may signal the breakdown of dynamic scaling. Simulations indicate strong longitudinal fluctuations that lead to instabilities in the discrete integration scheme, excluding the possibility of measuring the exponents reliably.

\( \lambda_\perp > 0 \) and \( \lambda_\times < 0 \): The projected RG flows in this quadrant (IV) converge to the point \( \lambda_\perp/\lambda_\parallel = 1 \) and \( \lambda_\times T_\perp K_\parallel/\lambda_\parallel T_\parallel K_\perp = -1 \). This is actually not a fixed point, as \( K_\parallel \) and \( K_\perp \) scale to infinity. The applicability of the CH transformation to this point implies \( z_\parallel = 3/2 \) and \( \zeta_\parallel = 1/2 \). Since \( \lambda_\perp \) is finite, \( z_\perp = z_\parallel = 3/2 \) is expected, but this does not give any information on \( \zeta_\perp \). Simulations indicate strong transverse fluctuations and suffer from difficulties similar to those in region II.

Eqs.\textsuperscript{(48)} are the simplest nonlinear, local, and dissipative equations that govern the fluctuations of a moving line in a random medium. They can be easily generalized to describe the time evolution of a manifold with arbitrary internal (\( \mathbf{x} \in R^d \)) and
IV. NONLINEAR RELAXATION OF DRIFTING POLYMERS

The dynamics of polymers in fluids is of much theoretical interest and has been extensively studied\textsuperscript{42,43}. The combination of polymer flexibility, interactions, and hydrodynamics make a first principles approach to the problem quite difficult. There are, however, a number of phenomenological studies that describe various aspects of this problem\textsuperscript{44}.

One of the simplest is the Rouse model\textsuperscript{45}: The configuration of the polymer at time $t$ is described by a vector $\mathbf{R}(x,t)$, where $x \in [0,N]$ is a continuous variable replacing the discrete monomer index (see Fig. 7).

Ignoring inertial effects, the relaxation of the polymer in a viscous medium is approximated by

$$\partial_t \mathbf{R}(x,t) = \mu \mathbf{F}(\mathbf{R}(x,t)) = K \partial_x^2 \mathbf{R}(x,t) + \eta(x,t),$$  \hspace{1cm} (57)$$

where $\mu$ is the mobility. The force $\mathbf{F}$ has a contribution from interactions with near neighbors that are treated as springs. Steric and other interactions are ignored. The effect of the medium is represented by the random forces $\eta$ with zero mean. The
Rouse model is a linear Langevin equation that is easily solved. It predicts that the mean square radius of gyration, \( R_g^2 = \langle |R - \langle R \rangle|^2 \rangle \), is proportional to the polymer size \( N \), and the largest relaxation times scale as the fourth power of the wave number (i.e., in dynamic light scattering experiments, the half width at half maximum of the scattering amplitude scales as the fourth power of the scattering wave vector \( q \)). These results can be summarized as \( R_g \sim N^\nu \) and \( \Gamma(q) \sim q^z \), where \( \nu \) and \( z \) are called the swelling and dynamic exponents, respectively. Thus, for the Rouse Model, \( \nu = 1/2 \) and \( z = 4 \).

The Rouse model ignores hydrodynamic interactions mediated by the fluid. These effects were originally considered by Kirkwood and Riseman and later on by Zimm. The basic idea is that the motion of each monomer modifies the flow field at large distances. Consequently, each monomer experiences an additional velocity

\[ \delta_H \partial_t \mathbf{R}(x, t) = \frac{1}{8\pi \eta_s} \int dx' \mathbf{F}(x') \frac{r_{xx'}^2}{|r_{xx'}|^3} + \frac{(\mathbf{F}(x') \cdot r_{xx'}) r_{xx'}}{|r_{xx'}|^3} \approx \int dx' \frac{\gamma}{|x - x'|^\nu} \partial_x^2 \mathbf{R}, \]

where \( r_{xx'} = \mathbf{R}(x) - \mathbf{R}(x') \) and the final approximation is obtained by replacing the actual distance between two monomers by their average value. The modified equation is still linear in \( \mathbf{R} \) and easily solved. The main result is the speeding up of the relaxation dynamics as the exponent \( z \) changes from 4 to 3. Most experiments on polymer dynamics indeed measure exponents close to 3. Rouse dynamics is still important in other circumstances, such as diffusion of a polymer in a solid matrix, stress and viscoelasticity in concentrated polymer solutions, and is also applicable to relaxation times in Monte Carlo simulations.

Since both of these models are linear, the dynamics remains invariant in the center of mass coordinates upon the application of a uniform external force. Hence the results for a drifting polymer are identical to a stationary one. This conclusion is in fact not correct due to the hydrodynamic interactions. For example, consider a rodlike conformation of the polymer with monomer length \( b_0 \) where \( \partial_x R_\alpha = b_0 t_\alpha \) everywhere on the polymer, so that the elastic (Rouse) force vanishes. If a uniform force \( \mathbf{E} \) per monomer acts on this rod, the velocity of the rod can be solved using Kirkwood Theory, and the result is

\[ \mathbf{v} = \frac{(-\ln \kappa)}{4\pi \eta_s b_0} \mathbf{E} \cdot [\mathbf{I} + \mathbf{tt}]. \]

In the above equation, \( \eta_s \) is the solvent viscosity, \( \mathbf{t} \) is the unit tangent vector, \( \kappa = 2b/b_0 N \) is the ratio of the width \( b \) to the half length \( b_0 N/2 \) of the polymer. A more detailed calculation of the velocity in the more general case of an arbitrarily shaped slender body by Khayat and Cox shows that nonlocal contributions to the hydrodynamic force, which depend on the whole shape of the polymer rather than the local orientation, are \( \mathcal{O}(1/(\ln \kappa)^2) \). Therefore, corrections to Eq.(59) are small when \( N \gg b/b_0 \).

Incorporating this tilt dependence of polymer mobility requires adding terms nonlinear in the tilt, \( \partial_x \mathbf{r} \), to a local equation of motion. Since the overall force (or velocity) is the only vector breaking the isotropy of the fluid, the structure of these nonlinear terms must be identical to eq.(46). Thus in terms of the fluctuations
parallel and perpendicular to the average drift, we again recover the equations,

\[
\begin{align*}
\partial_t R_\parallel &= U_\parallel + K_\parallel \partial_x^2 R_\parallel + \frac{\lambda_\parallel}{2} (\partial_x R_\parallel)^2 + \frac{\lambda_x}{2} \sum_{i=1}^{2} (\partial_x R_{\perp i})^2 + \eta_\parallel(x,t), \\
\partial_t R_{\perp i} &= K_{\perp} \partial_x^2 R_{\perp i} + \lambda_\perp \partial_x R_\parallel \partial_x R_{\perp i} + \eta_{\perp i}(x,t),
\end{align*}
\]

where \(\{\perp i\}\) refers to the 2 transverse coordinates of the monomer positions. The noise is assumed to be white and gaussian but need not be isotropic, i.e.

\[
\begin{align*}
\langle \eta_\parallel(x,t) \eta_\parallel(x',t') \rangle &= 2T_\parallel \delta(x-x') \delta(t-t'), \\
\langle \eta_{\perp i}(x,t) \eta_{\perp j}(x',t') \rangle &= 2T_\perp \delta_{i,j} \delta(x-x') \delta(t-t').
\end{align*}
\]

At zero average velocity, the system becomes isotropic and the equations of motion must coincide with the Rouse model. Therefore, \(\{\lambda_\parallel, \lambda_x, \lambda_\perp, U, K_\parallel - K_\perp, T_\parallel - T_\perp\}\) are all proportional to \(E\) for small forces. The relevance of these nonlinear terms are determined by the dimensionless scaling variable

\[
y = \left(\frac{U}{U^*}\right) N^{1/2},
\]

where \(U^*\) is a characteristic microscopic velocity associated with monomer motion and is roughly 10-20 m/s for polystyrene in benzene. The variable \(y\) is proportional to another dimensionless parameter, the Reynolds number \(Re\), which determines the breakdown of hydrodynamic equations and onset of turbulence. However, typically \(Re \ll y\), and the hydrodynamic equations are valid for moderately large \(y\). Eqs. (60) describe the static and dynamical scaling properties of the nonlinear and anisotropic regime when \(U > U^* N^{-1/2}\).

Eq.(60) is just a slight variation from (48), with two transverse components instead of one. Thus, the results discussed in the previous lecture apply. A more detailed calculation of the nonlinear terms from hydrodynamics\(^{51}\) shows that all three nonlinearities are positive for small driving forces. In this case, the asymptotic scaling exponents are isotropic, with \(\nu = 1/2\) and \(z = 3\). However, the fixed points of the RG transformation are in general anisotropic, which implies a kinetically induced form birefringence \textit{in the absence of external velocity gradients}. This is in marked contrast with standard theories of polymer dynamics where a uniform driving force has essentially no effect on the internal modes of the polymer.

When one of the nonlinearities approaches to zero, the swelling exponents may become anisotropic and the polymer elongates or compresses along the longitudinal direction. However, the experimental path in the parameter space as a function of \(E\) is not known and not all of the different scaling regimes correspond to actual physical situations. The scaling results found by the RG analysis are verified by direct integration of equations, as mentioned in the earlier lectures. A more detailed discussion of the analysis and results can be found in our earlier work\(^{40}\).

In constructing equations (60), we only allowed for local effects, and ignored the nonlocalities that are the hallmark of hydrodynamics. One consequence of hydrodynamic interactions is the \textit{back-flow} velocity in Eq.(58) that can be added to the evolution equations (60). Dimensional analysis gives the recursion relation

\[
\frac{\partial \gamma}{\partial \ell} = \gamma [\nu z - 1 - (d-2)\nu] + O(\gamma^2),
\]

\[\text{(62)}\]
which implies that, at the nonlinear fixed point, this additional term is surprisingly irrelevant for \( d > 3 \), and \( z = 3 \) due to the nonlinearities. For \( d < 3 \), \( z = d \) due to hydrodynamics, and the nonlinear terms are irrelevant. The situation in three dimensions is unclear, but a change in the exponents is unlikely. Similarly, one could consider the effect of self-avoidance by including the force generated by a softly repulsive contact potential

\[
\frac{b}{2} \int dx \, dx' \, \mathcal{V}(\mathbf{r}(x) - \mathbf{r}(x')).
\]  

(63)

The relevance of this term is also controlled by the scaling dimension \( y_b = \nu z - 1 - (d - 2)\nu \), and therefore this effect is marginal in three dimensions at the nonlinear fixed point, in contrast with both Rouse and Zimm models where self-avoidance becomes relevant below four dimensions. Unfortunately, one is ultimately forced to consider non-local and nonlinear terms based on similar grounds, and such terms are indeed relevant below four dimensions. In some cases, local or global arclength conservation may be an important consideration in writing down a dynamics for the system. However, a local description is likely to be more correct in a more complicated system with screening effects (motion in a gel that screens hydrodynamic interactions) where a first principles approach becomes even more intractable. Therefore, this model is an important starting point towards understanding the scaling behavior of polymers under a uniform drift, a problem with great technological importance.

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