HAMILTONIAN APPROACH TO 2D DILATON-GRAVITIES
AND INVARIANT ADM MASS

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ABSTRACT

The formula existing in the literature for the ADM mass of 2D dilaton gravity is incomplete. For example, in the case of an infalling matter shockwave this formula fails to give a time-independent mass, unless a very special coordinate system is chosen. We carefully carry out the canonical formulation of 2D dilaton gravity theories (classical, CGHS and RST). As in 4D general relativity one must add a boundary term to the bulk Hamiltonian to obtain a well-defined variational problem. This boundary term coincides with the numerical value of the Hamiltonian and gives the correct mass which obviously is time-independent.

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1. Introduction

The classical action for dilaton gravity in two dimensions is [1]

\[ S_{\text{cl}} = \frac{1}{4\pi} \int d^2\sigma \sqrt{-g} \left[ e^{-2\phi} (R + 4(\nabla \phi)^2 + 4\lambda^2) - \frac{1}{2} \sum_{i=1}^{N} (\nabla f_i)^2 \right] \]  

(1.1)

where \( \phi \) is the dilaton and \( f_i \) are \( N \) matter fields. The virtues of this action and its quantum versions have been discussed in the literature [1-5] (see also ref. 6 for earlier work) and we won’t repeat them here‡ The classical action admits static black hole solutions

\[ ds^2 = \left( 1 + \frac{m}{\lambda} e^{-2\lambda \sigma} \right)^{-1} (d\sigma^2 - d\tau^2) \quad , \quad e^{-2\phi} = e^{2\lambda \sigma} + \frac{m}{\lambda}. \]  

(1.2)

where we use conformal coordinates and the conformal factor \( e^{2\rho} \) and the dilaton satisfy \( \phi = -\lambda \sigma + \rho \). Asymptotically, as \( \sigma \to \infty \), the metric becomes Minkowskian and one approaches the linear dilaton vacuum (LDV), \( \rho = 0 \), \( \phi = -\lambda \sigma \). The parameter \( m \) which characterizes the asymptotics as \( \sigma \to \infty \), \( \rho \sim -\frac{m}{2\lambda} e^{-2\lambda \sigma} \), \( \phi \sim -\lambda \sigma - \frac{m}{2\lambda} e^{-2\lambda \sigma} \), is the mass of the black hole.

More generally, it was found [7, 1, 8] that the ADM mass of any configuration asymptotic to the LDV should be given by

\[ M = 2 e^{2\lambda \sigma} (\partial_\sigma \delta \phi + \lambda \delta \rho) \bigg|_{\sigma=\infty} \]  

(1.3)

where \( \delta \phi = \phi - \phi_{\text{LDV}} \), \( \delta \rho = \rho - \rho_{\text{LDV}} \) are the deviations from the linear dilaton vacuum. For the static black hole (1.2) this gives \( M = m \) correctly. Thus the mass is given by the asymptotics of the fields only, as in 4D general relativity. However, there are two spatial ends of the world, \( \sigma = +\infty \) and \( \sigma = -\infty \), and one might wonder whether one should include some contribution from \( \sigma \to -\infty \) into \( M \). To circumvent this question for the moment, we will only consider configurations asymptotic to the LDV both for \( \sigma \to +\infty \) and \( \sigma \to -\infty \). (This of course excludes (1.2).) Then a possible contribution \( 2e^{-2\phi} (\partial_\sigma \delta \phi + \lambda \delta \rho) \big|_{\sigma=-\infty} \) will vanish. This kind of configurations are typically encountered if some matter falls into the LDV (\( T_{++} \neq 0 \) for some finite interval of \( \sigma + \tau \)). This produces a black hole which, in the classical

‡ See ref. 1 or 2 for all conventions.
case, will not radiate and should have constant mass equal to the total energy carried by the
infalling matter. If we take $T_{++}(\sigma + \tau) = a\delta(\sigma + \tau)$ for example, the matter carries total
energy $a$ and the solution is (in conformal gauge) $\phi = -\lambda\sigma$, $\rho = 0$ for $\sigma + \tau < 0$, while for $\sigma + \tau > 0$:

$$\phi = -\lambda\sigma + \rho, \quad \rho = \frac{a}{2\lambda} e^{\lambda(\tau - \sigma)} + \left( \frac{a^2}{4\lambda^2} e^{2\lambda\tau} - \frac{a}{2\lambda} \right) e^{-2\lambda\sigma} + O(e^{-3\lambda\sigma}). \quad (1.4)$$

This is obviously asymptotic to the LDV as $\sigma \to \pm\infty$. If we insert this into (1.3) we obtain

$$M = a \left( 1 - \frac{a}{2\lambda} e^{2\lambda\tau} \right) \quad (1.5)$$

which depends on time. The origin of this problem can be traced to the assumption, implicit in
the derivation of (1.3), that $\delta\phi, \delta\rho$ are $O(e^{-2\lambda\sigma})$, while (1.4) actually contains terms $O(e^{-\lambda\sigma})$.

It should be noted that there exists a conformal coordinate transformation $\tilde{\sigma} + \tilde{\tau} = \sigma + \tau, \tilde{\sigma} - \tilde{\tau} = \frac{1}{\lambda} \log \left( e^{\lambda\sigma - \lambda\tau} - \frac{a}{\lambda} \right)$ so that, as $\tilde{\sigma} \to \infty$ for fixed $\tilde{\tau}$, one has $\phi \sim -\lambda\tilde{\sigma} - \frac{a}{2\lambda} e^{-2\lambda\tilde{\sigma}}, \rho \sim -\frac{a}{2\lambda} e^{-2\lambda\tilde{\sigma}}$. But then $\phi$ and $\rho$ no longer equal the LDV (by LDV we mean $\phi = -\lambda\tilde{\sigma}$ and $\rho = 0$) for $\sigma + \tau < 0$, i.e. for $\tilde{\sigma} \to -\infty$. In fact they are not even asymptotic to the LDV. (Making
the transformation only for large $\sigma + \tau$ and patching it smoothly to the identity for $\sigma + \tau < 0$
would obviously not be conformal.) Thus, in this case, there exists a special coordinate system,
specified by $\delta\rho, \delta\phi = O(e^{-2\lambda\tilde{\sigma}})$ as $\tilde{\sigma} \to \infty$. In this coordinate system, using (1.3), $M = m$
is time-independent, but one has lost the LDV asymptotics for $\tilde{\sigma} \to -\infty$. Here we want to obtain
a well-defined, i.e. time-independent, mass formula without requiring that $\phi$ and $\rho$ asymptote
to the LDV that fast as $\sigma \to \infty$, hence allowing more general coordinate systems like in (1.4).

In 4D general relativity there is a very straightforward method to obtain the total energy
(ADM mass) of an asymptotically Minkowskian configuration. As noticed by Regge and Teitelboim
almost twenty years ago [9], in order to have a well-defined variational principle, i.e. in
order that

$$\delta \left( \int d^3 x \sum_i \tilde{\phi}_i \tilde{\Pi}_i - H \right) = 0 \quad (1.6)$$

for all solutions of the equations of motion with the required asymptotics, the Hamiltonian $H$
cannot simply be the bulk Hamiltonian $H_1 = \int d^3x \mathcal{H}$. One needs to include a boundary
hamiltonian $H_2 = \lim_{r \to \infty} \int d^2 S M$ whose variation cancels (using the boundary conditions) the boundary terms obtained when varying $H_1$. Since $H_1$ is a sum of constraints its actual value vanishes for any solution and the total energy is simply the numerical value of $H_2$.

We will apply this method here to the 2D dilaton gravity theories, i.e. to the classical theory (1.1) and to the “quantum versions” of refs. 1 and 4. For the latter theories we represent the trace anomaly by the $Z$-field, as extensively discussed in refs. [10,8,11]. We can treat all three models simultaneously by writing

$$S = \frac{1}{4\pi} \int d^2 \sigma \sqrt{-g} \left[ \left( e^{-2\phi} - \frac{\kappa}{2} \phi \right) R + 4e^{-2\phi} \left( (\nabla \phi)^2 + \lambda^2 \right) - \frac{1}{2} (\nabla Z)^2 + QRZ \right] \quad (1.7)$$

where $\kappa = Q = 0$ gives back the classical action (with one free matter $Z$-field*), $\kappa = 0$, $2Q^2 = \frac{N}{12}$ gives the CGHS model [1], and $\kappa = 2Q^2 = \frac{N-24}{12}$ gives the RST-model [4]. We will obtain the bulk Hamiltonian $H_1$ for these theories and then identify the correct $H_2$ as described above. As expected, $H_1$ is just a sum of constraints that satisfy (two copies of) a Poisson bracket Virasoro algebra (or rather stress-energy tensor algebra). $H_2$ is identified with the mass functional. Its numerical value coincides with that of the total Hamiltonian and thus must be constant. We will check that this is indeed the case.

2. The canonical structure

First we will derive the bulk Hamiltonian $H_1$ from the action (1.7). For the classical action only, this was done in ref. 12.† To begin with, we parametrize the two-dimensional metric in the following way

$$g_{\mu \nu} = e^{2\rho} \begin{pmatrix} A^2 - B^2 & A \\ A & 1 \end{pmatrix}. \quad (2.1)$$

This is inspired by the standard ADM parametrization [13] with $A$ and $e^\rho B$ the analogues of the shift vector and lapse function. Due to the Weyl invariance of the classical action it

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* One might add other free (classical) matter fields. These could be included trivially into our subsequent analysis, in particular their contribution to the boundary term $D$ would vanish due to the standard boundary conditions on matter fields.

† This reference, however, does not address the problem of finding the boundary Hamiltonian which is our main concern here.
is more convenient to use $B$ as a field rather than the lapse function itself. Then conformal gauge simply is $A = 0$, $B = 1$. The simplest way to compute the curvature probably is to use the zweibein $e^0 = \exp(\rho)Bdt$, $e^1 = \exp(\rho)(Adt + dx)$, compute the spin-connection $\omega$ from the zero-torsion equation, $de^a + \omega^a \wedge e^b = 0$, and obtain $dt dx \sqrt{-g} R = 2d\omega$. It is then straightforward to obtain $(\omega^{01})_1 = \frac{1}{B}(\dot{\rho} - A\rho' - A')$ and $(\omega^{01})_0 = B\rho' + B' + \frac{A}{B}(\dot{\rho} - A\rho' - A')$ and hence

$$S = \int d^2\sigma \frac{1}{\pi B} \left\{ \left[ F(\dot{\phi} - A\phi') - \frac{1}{2}Q(\dot{Z} - AZ') \right] (\dot{\rho} - A\rho' - A') 
- (F\phi' - \frac{1}{2}QZ')(B^2\rho' + BB') 
- e^{-2\phi}(\dot{\phi} - A\phi')^2 + B^2e^{-2\phi}\phi'^2 + \lambda^2 B^2 e^{-2\phi + 2\rho} 
+ \frac{1}{8}(\dot{Z} - AZ')^2 - \frac{1}{8}B^2Z'^2 \right\}$$

(2.2)

where $F = e^{-2\phi} + \frac{\kappa}{4}$. To obtain (2.2) we have integrated by parts. We will not keep track of boundary terms for the moment since first we only want to obtain the bulk Hamiltonian $H_1$. The canonical momenta then are

$$\Pi_\phi = \frac{1}{\pi B} \left[ F(\dot{\phi} - A\phi') - 2e^{-2\phi}(\dot{\phi} - A\phi') \right]$$
$$\Pi_\rho = \frac{1}{\pi B} \left[ F(\dot{\phi} - A\phi') - \frac{1}{2}Q(\dot{Z} - AZ') \right]$$
$$\Pi_Z = \frac{1}{\pi B} \left[ \frac{1}{4}(\dot{Z} - AZ') - \frac{1}{2}Q(\dot{\rho} - A\rho' - A') \right].$$

(2.3)

Since no time-derivatives of $A$ or $B$ occur in the action there are no momenta conjugate to $A$ or $B$. The fields $A$ and $B$ are Lagrange multipliers serving to impose constraints. Writing $S = \int d\tau L$, the bulk Hamiltonian is given by $H_1 = \int d\sigma(\dot{\phi}\Pi_\phi + \rho\Pi_\rho + Z\Pi_Z) - L$ which after integrating by parts reads

$$H_1 = \int d\sigma [AC_A + BC_B]$$
$$C_A = \phi'\Pi_\phi + \rho'\Pi_\rho + Z'\Pi_Z - \Pi'_\rho$$
$$C_B = \frac{\pi}{G^2} \left[ e^{-2\phi}(\Pi_\rho + 2Q\Pi_Z)^2 + F\Pi_\phi(\Pi_\rho + 2Q\Pi_Z) + \frac{1}{2}Q^2\Pi_\phi^2 \right] + 2\pi\Pi_Z^2$$

$$+ \frac{1}{\pi} \left[ F\rho'\phi' - (F\phi')' - e^{-2\phi}\phi'^2 - \lambda^2 e^{-2\phi + 2\rho} + \frac{1}{8}Z'^2 - \frac{1}{2}Q\rho'Z' + \frac{1}{2}QZ'' \right]$$

(2.4)

where $G^2 = F^2 - 2Q^2e^{-2\phi}$. 

4
The Lagrange multipliers $A$ and $B$ impose the constraints $C_A = C_B = 0$. To see what these constraints are it is convenient to substitute (2.3) for the momenta. Then in conformal gauge ($A = 0, B = 1$) one has (substituting also $Z = \tilde{Z} + 2Q\rho$)

\[
\begin{align*}
\pi C_A \bigg|_{A=0,B=1} &= F(\dot{\rho} + \dot{\phi} - \dot{\phi}') - Q\dot{\rho} \dot{\rho}' + \frac{1}{2} \dot{\tilde{Z}} \ddot{\tilde{Z}} + \frac{1}{2} Q \ddot{\tilde{Z}}', \\
\pi C_B \bigg|_{A=0,B=1} &= F(\dot{\rho} + \dot{\phi} - \dot{\phi}'') - e^{-\phi}(\dot{\phi}^2 - \phi'^2) - \frac{1}{2} Q^2 (\dot{\rho}^2 + \rho'^2 - 2\rho'')
\end{align*}
\]

(2.5)

which coincides with $\frac{1}{2}(T_{++} - T_{--})$ and $\frac{1}{2}(T_{++} + T_{--})$, respectively, of refs. 1, 4, the $\tilde{Z}$-part corresponding to the matter (or matter plus ghost) stress tensor.

Using canonical Poisson brackets,

\[
\{\phi(\sigma), \Pi\phi(\sigma')\} = \{\rho(\sigma), \Pi\rho(\sigma')\} = \{Z(\sigma), \Pi Z(\sigma')\} = \delta(\sigma - \sigma')
\]

(2.6)

we can compute the algebra of the constraints as given by (2.4) (in general gauge). It is straightforward to obtain:

\[
\begin{align*}
\{C_A(\sigma), C_A(\sigma')\} &= (\partial_\sigma - \partial_{\sigma'})[C_A(\sigma')\delta(\sigma - \sigma')], \\
\{C_B(\sigma), C_B(\sigma')\} &= (\partial_\sigma - \partial_{\sigma'})[C_A(\sigma')\delta(\sigma - \sigma')], \\
\{C_B(\sigma), C_A(\sigma')\} &= (\partial_\sigma - \partial_{\sigma'})[C_B(\sigma')\delta(\sigma - \sigma')].
\end{align*}
\]

(2.7)

Thus we see that the Poisson bracket of $C_A + C_B$ with $C_A - C_B$ vanishes while

\[
\{(C_B(\sigma) \pm C_A(\sigma)), (C_B(\sigma') \pm C_A(\sigma'))\} = 2(\partial_\sigma - \partial_{\sigma'})([C_B(\sigma') \pm C_A(\sigma')]\delta(\pm \sigma \mp \sigma'))
\]

(2.8)

which is indeed the Poisson bracket algebra of $T_{\pm \pm}$ with itself. (Note that for $T_{--}$ we have $\delta(-\sigma + \sigma')$ on the r.h.s. since $T_{--}$ naturally is a function of $\tau$ and $-\sigma$.) There is no $\delta'''$-term which means that the total central charge vanishes. This is indeed the case as one

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* More precisely, the $T_{\pm \pm}$ of refs. 1, 4 contain $\partial^2_f(\text{fields})$ which includes $\partial^2_f(\text{fields})$. In a canonical formalism this must be replaced by $\partial^2_f(\text{fields}) + \ldots$ using the equations of motion. Once we do this, $\frac{1}{2}(T_{++} \pm T_{--})$, of refs. 1, 4 are identical to the above $\pi C_A, \pi C_B$ in conformal gauge. In fact, the equation of motion one has to use coincides with $T_{++} = 0$ and one finds that $\pi C_B$ as given by (2.4) is precisely $\frac{1}{2}(T_{++} + T_{--}) + T_{++} = 2T_{00}$ while $\pi C_A$ is $\frac{1}{2}(T_{++} - T_{--}) = 2T_{01}$ as expected on general grounds.
can easily see in conformal gauge: For the classical theory this is obvious. For the CGHS-model the conformal anomaly term \( \sim Q^2 (\partial_\rho \partial_\rho - \partial_\rho^2) \) gives \( c = -24Q^2 = -N \) while the matter fields represented by \( \tilde{Z} \) give \( c = +N \). For the RST-model \( Q^2 (\partial_\rho \partial_\rho - \partial_\rho^2) \) gives \( c = -24Q^2 = -12\kappa = 24 - N \), while the \( \tilde{Z} \)-field gives the anomaly for matter, ghosts and the quantum part of \( \phi, \rho \) which is \( c = N - 26 + 2 = N - 24 \). Of course, we just repeated that the Polyakov-anomaly action is designed to cancel the various anomalies present in the theory.

Let us now compute the variation of the bulk Hamiltonian \( H_1 \) as given by (2.4) under infinitesimal variations of the physical fields and their canonical momenta, this time keeping track of all boundary terms:

\[
\delta H_1 = \int d\sigma \left[ \frac{\delta H_1}{\delta \phi} \delta \phi + \frac{\delta H_1}{\delta \rho} \delta \rho + \frac{\delta H_1}{\delta Z} \delta Z + \frac{\delta H_1}{\delta \Pi_\phi} \delta \Pi_\phi + \frac{\delta H_1}{\delta \Pi_\rho} \delta \Pi_\rho + \frac{\delta H_1}{\delta \Pi_Z} \delta \Pi_Z \right]
\]

(2.9)

where \( D \) is the boundary term given below and

\[
\frac{\delta H_1}{\delta \phi} = \frac{2\pi B}{G^4} e^{-2\phi} \left[ F(e^{-2\phi} - \frac{\kappa}{4})(\Pi_\rho + 2Q\Pi_Z)^2 \right.
\]

\[
+ (F^2 \frac{1}{2} Q^2 \kappa) \Pi_\phi (\Pi_\rho + 2Q\Pi_Z) + (F - Q^2)Q^2 \Pi_\phi^2 \right]
\]

\[
+ \frac{1}{\pi} \left[ -(B\Pi_\rho)' = B'' F + 2Be^{-2\phi}\rho^2 + (2Be^{-2\phi}\rho)' - 2Be^{-2\phi}\rho' + 2\lambda^2 Be^{-2\phi + 2\rho} \right]
\]

\[
- (A\Pi_\rho)' \]

\[
\frac{\delta H_1}{\delta \rho} = \frac{1}{\pi} \left[ -(B\Pi_\phi)' + \frac{1}{2} Q(BZ)' - 2\lambda^2 Be^{-2\phi + 2\rho} \right] - (A\Pi_\rho)' \]

\[
\frac{\delta H_1}{\delta Z} = \frac{1}{2\pi} \left[ QB'' + Q(B\rho)' - \frac{1}{2} (BZ)' \right] - (A\Pi_Z)' \]

(2.10)

and

\[
\frac{\delta H_1}{\delta \Pi_\phi} = \frac{\pi B}{G^2} \left[ F(\Pi_\rho + 2Q\Pi_Z) + Q^2 \Pi_\phi \right] + A\phi' \]

\[
\frac{\delta H_1}{\delta \Pi_\rho} = \frac{\pi B}{G^2} \left[ 2e^{-2\phi}(\Pi_\rho + 2Q\Pi_Z) + F\Pi_\phi \right] + A\rho' + A' \]

(2.11)

\[
\frac{\delta H_1}{\delta \Pi_Z} = 4\pi B\Pi_Z + \frac{2\pi QB}{G^2} \left[ 2e^{-2\phi}(\Pi_\rho + 2Q\Pi_Z) + F\Pi_\phi \right] + AZ' \]

Hamilton’s equations, \( \delta H_i + \Pi_i = 0 \), would follow from the variational principle \( \delta \left( \int d\sigma (\phi \Pi_\phi + \rho \Pi_\rho + \tilde{Z} \Pi_Z) - H_1 \right) = 0 \) if the boundary term \( D \) would vanish. We have (recall
that \( F = e^{-2\phi} + \frac{\mathcal{F}}{4} \)

\[
\pi D = \left[ \delta \left( -BF\phi' + \frac{1}{2}QBZ' \right) + \left( B'F + BF' - 2Be^{-2\phi} + A\Pi_\phi \right) \delta \phi \\
+ \left( BF\phi' - \frac{1}{2}QBZ' + A\Pi_\rho \right) \delta \rho + \left( -\frac{1}{2}QB' - \frac{1}{2}QB' + \frac{1}{4}BZ' + A\Pi_Z \right) \delta Z \right]_{\sigma = \pm \infty} - A\delta \Pi_{\rho} \\
\]

This does not vanish. In the next section, we will show, however, that, subject to appropriate boundary conditions, \( D \) can be written as the variation of another functional \( H_2 \):

\[
D \bigg|_{\text{boundary conditions}} = -\delta H_2 . \tag{2.13}
\]

Then

\[
\delta \left( \int d\sigma (\dot{\phi}\Pi_\phi + \dot{\rho}\Pi_\rho + \dot{Z}\Pi_Z) - H_1 - H_2 \right) = 0 \quad \Rightarrow \quad \text{Hamilton's equations} . \tag{2.14}
\]

Thus the true Hamiltonian \( H \) is the bulk Hamiltonian \( H_1 \) plus the boundary Hamiltonian \( H_2 \). Since \( H_1 \) is only given by the sum of the two constraints it vanishes on all solutions and the total energy, i.e. the value of the total Hamiltonian \( H \), is given by the value of \( H_2 \) only.

### 3. The total energy for asymptotically Minkowskian space-times

Obviously, in general, \( D \) cannot be written as the variation of some functional. Indeed, we know that the notion of total energy is well-defined only if we impose appropriate boundary conditions. In 4D general relativity one not only requires the space-time to be asymptotically flat but also to be asymptotically Minkowskian, i.e. one imposes asymptotic coordinate conditions. Here we will require that asymptotically, as \( \sigma \to \pm \infty \), we have the LDV: \( \phi \sim -\lambda \sigma \), \( \rho \sim 0 \).

Hence we impose the following asymptotics as \( \sigma \to -\infty \):

\[
as \sigma \to -\infty : A \sim 0, \quad B \sim 1, \quad \rho \sim 0, \quad \phi \sim -\lambda \sigma, \quad Z \sim 0 , \tag{3.1}
\]

and it is understood that the derivatives of the fields obey the derivatives of these relations*.

* Although this is obvious for “smooth” field configurations, it has to be imposed separately to exclude configurations with asymptotics like \( \phi \sim -\lambda \sigma + e^{\lambda \sigma} \sin(e^{-2\lambda \sigma}) \) as \( \sigma \to -\infty \).
e.g. $\phi' \sim -\lambda$, etc. With (2.3) in mind we also require

$$\text{as } \sigma \to -\infty : \Pi_\phi \sim 0, \Pi_\rho \sim 0, \Pi_Z \sim 0.$$  \hfill (3.2)

These asymptotic conditions then imply that $D$ receives no contribution from $\sigma = -\infty$.

For $\sigma \to \infty$, however, just as in 4D general relativity [9], we have to be more precise about how fast the field configurations actually have to approach the LDV. The asymptotic conditions to be imposed for $\sigma \to \infty$ should be satisfied for all solutions of the equations of motion and constraints. We will suppose that the matter fields (here represented by $Z - 2Q\rho$) asymptotically vanish (excluding radiation baths that lead to infinite total energy). Then it follows from ref. 1 for the classical case in conformal gauge that as $\sigma \to \infty$ we have

$$\phi \sim -\lambda \sigma + (\alpha e^{\lambda \sigma} + \beta e^{-\lambda \sigma})e^{-\lambda \sigma} + \ldots \quad \text{and} \quad \rho \sim (\alpha e^{\lambda \sigma} + \beta e^{-\lambda \sigma})e^{-\lambda \sigma} + \ldots$$

For the quantum models, differences with the classical model only appear $O(e^{2\phi}) \sim O(e^{-2\lambda \sigma})$, thus in all cases the deviations from the LDV are at least $O(e^{-\lambda \sigma})$.

It is sufficient if the phase-space includes only fields with these asymptotics. Hence we require

$$\text{as } \sigma \to \infty : \phi + \lambda \sigma, \rho, B - 1 \sim e^{-\lambda \sigma}$$

$$\Pi_\phi, \Pi_\rho \sim e^{\lambda \sigma}$$

$$Z, \Pi_Z \to 0$$

$$e^{\lambda \sigma} A \to 0.$$  \hfill (3.3)

This is meant as a minimal requirement, i.e. $\rho$ and $\phi + \lambda \sigma$ decrease as $e^{-\lambda \sigma}$ or faster, and $\Pi_\phi, \Pi_\rho$ do not grow faster than $e^{\lambda \sigma}$. Finally we also require

$$\text{as } \sigma \to \infty : e^{2\lambda \sigma}(\phi + \lambda \sigma - \rho) \to 0.$$  \hfill (3.4)

Note that we do not require $\phi + \lambda \sigma$ or $\rho$ to be $O(e^{-2\lambda \sigma})$, only the combined relation (3.4) should be satisfied. Note also that we could have dropped the $\rho$-asymptotics from (3.3) since it is implied by (3.4) and the $\phi$-asymptotics. Condition (3.4) specifies a certain class of coordinate systems. We know, for the classical model and for RST, that (in conformal gauge) $\phi - \rho$ is a free field. Then by a (residual) coordinate choice we can achieve $\phi = -\lambda \sigma + \rho$. For CGHS this receives higher-order corrections. This is the motivation for our boundary condition (3.4). We
will have more to say about the meaning of it below. Since all fields in the phase space satisfy (3.3), (3.4) the same applies to the variations \( \delta \phi, \delta \rho, \delta Z, \delta \Pi_\phi, \delta \Pi_\rho \) and \( \delta \Pi_Z \), in particular, we can substitute \( \delta \rho = \delta \phi \) in (2.12). Again it is understood that the derivatives of the fields obey the derivatives of these relations. Using the boundary conditions (3.3) it is easy to see that the only terms in \( D \) (eq. (2.12)) contributing in the \( \sigma \to \infty \)-limit are those proportional to \( e^{-2\phi} \). All others vanish in this limit. Then we have

\[
\pi D = \left[ -\delta \left( e^{-2\phi} B\phi' \right) + e^{-2\phi} \left( B' + B\rho' - 2B\phi' \right) \delta \phi + e^{-2\phi} B\phi' \delta \rho \right] \bigg|_{\sigma=+\infty} \tag{3.5}
\]

Using (3.4) one gets

\[
\pi D = \left[ -\delta \left( e^{-2\phi} B\phi' \right) + e^{-2\phi} \left( B' + B\lambda + \frac{1}{2} B' \right) \delta \phi \right] \bigg|_{\sigma=+\infty} \tag{3.6}
\]

and we identify the boundary Hamiltonian \( H_2 \) as

\[
H_2 = \frac{1}{2\pi} \left[ e^{-2\phi} \left( 2B\phi' - B\lambda + B' \right) + \lambda e^{2\lambda \sigma} \right] \bigg|_{\sigma=+\infty} \tag{3.7}
\]

where we adjusted an (infinite) additive field-independent term, not affecting the relation \( \delta H_2 = -D \), so that \( H_2 \) vanishes for the LDV. The total energy is simply given by \( M = 2\pi H_2 \):

\[
M = \left[ e^{-2\phi} \left( 2B\phi' + B\lambda + B' \right) + \lambda e^{2\lambda \sigma} \right] \bigg|_{\sigma=+\infty} \tag{3.8}
\]

This is the general mass formula for all configurations subject to (3.1)-(3.4). If we are in conformal gauge \( B = 1 \) and write \( \phi \sim -\lambda \sigma + \delta \phi \) with \( \delta \phi \) at least \( O(e^{-\lambda \sigma}) \) then

\[
M = \left[ 2e^{2\lambda \sigma} (\partial_\sigma + \lambda)(\delta \phi - \delta \phi^2) \right] \bigg|_{\sigma=+\infty} \tag{3.9}
\]

The old mass formula (1.3) (with our boundary conditions) only is the part linear in \( \delta \phi \). If we evaluate \( M \) as given by (3.9) on the solution (1.4) we correctly find \( M = a \), thanks to the \( \delta \phi^2 \)-term.
More generally, it is straightforward to check using the constraints and boundary conditions that $\dot{M} = 0$. Indeed, we have from (3.8), using (3.4) (which allows us to replace $\dot{\phi}$ by $\dot{\rho}$, etc.)

$$\dot{M} = -2e^{-2\phi} \left[ B \left( \phi' \dot{\rho} + \rho' \dot{\phi} - \dot{\phi}' \right) + B' \dot{\phi} \right] \bigg|_{\sigma=+\infty}. \quad (3.10)$$

On the other hand, inserting the definitions of the momenta (2.3) into the constraint $C_A$ (2.4) and using the boundary conditions (3.3), (3.4) we obtain as $\sigma \to \infty$

$$C_A \sim \frac{e^{-2\phi}}{\pi B} \left( \phi' \dot{\rho} + \rho' \dot{\phi} - \dot{\phi}' + \frac{B'}{B} \dot{\phi} \right). \quad (3.11)$$

Thus we conclude

$$\dot{M} = -2\pi B^2 C_A \bigg|_{\sigma=+\infty}, \quad (3.12)$$

and since $C_A = 0$ is a constraint, $\dot{M}$ vanishes. Let us stress that $M$ is a constant by virtue only of the constraints and boundary conditions, independently of whether or not the equations of motion are satisfied. Note that the same mass formula (3.8) and (3.9) apply to all three models, classical, CGHS and RST.

Finally, we would like to comment on the condition (3.4). It should be considered as an asymptotic coordinate condition. That such a condition is necessary is rather obvious: In ordinary general relativity the vacuum is translationally invariant and we can make asymptotic Lorentz transformations. In dilaton-gravity, however, the vacuum is the LDV which is not translationally invariant. Thus only the total energy is a meaningful concept, provided it is defined with respect to the time-like Killing vector orthogonal to the vector singled out by the LDV. Thus the asymptotic coordinates have to be chosen carefully, which is achieved by (3.4). It is also worth pointing out that we can weaken some of the boundary conditions. Here, we will not explore this much further. We only note that if one works in conformal gauge, $A = 0, B = 1$, (3.3) can be replaced by $\phi + \lambda \sigma, \rho, Z \to 0$, as $\sigma \to \infty$, i.e. we only require LDV asymptotics (without saying how fast the LDV is approached), and (3.4). Then we still obtain the same boundary Hamiltonian and total energy (3.8) (with $B = 1, B' = 0$).
4. Conclusions

We have worked out the canonical structure of 2D dilaton gravity theories (classical, CGHS and RST). As in 4D general relativity, the bulk Hamiltonian alone does not lead to a well-defined variational principle. After defining the phase space carefully by choosing appropriate boundary conditions we were able to render the variational principle well-defined by adding a boundary Hamiltonian to the bulk one. Since the bulk Hamiltonian vanishes by the constraints the total energy is given by the boundary Hamiltonian alone. This total energy must be conserved, a fact we also checked directly.

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