Quantum mechanics not on manifold

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Abstract

The free scalar field is studied on the Y-junction of three semi infinite axes which is the simplest example of a non-manifold space. It is shown that under an assumption that the junction point can not gain a macroscopic amount of energy and charge the transmission rules for this system uniquely follow from conservation of energy and charge. This result is also obtained in the discrete version of the model. Some alternative approaches to the problem based on quantum mechanics of Hamiltonian systems with constrains are discussed.

Key words differential equations on networks, Klein-Fock-Gordon equation, conservation laws, Hamiltonian systems with constrains.

1 Introduction.

Hamiltonian mechanics on manifolds now is practically completed [1,2]. But there appeared serious need in formulation mechanics not on manifolds. The problems arise both in nanoelectronics and the string theory. For example, three quantum wires with Y junction do not compose a manifold (they are not homeomorphic to some Euclidean space). One can compose of strings a network [3,4] which also is not a manifold. But it is extremely important to have the Hamiltonian formalism on such structures. In nanoelectronics — to describe motion of electrons, in the string theory — the superstring network models the 3-dimensional space, and one should know how to describe propagation of excitations over the structure [4].

There is no regular theory of such processes. As a first step to this end we study a 3-tail system — a Y junction of three semi-infinite sets of classical harmonic oscillators and a theory of free classical scalar field on such a ”3-ray star”.

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Even more serious problems arise when one turns to quantum description. Quantum mechanics (QM) can be deduced from the classical one only in the Euclidean space (this Dirac’s recipe is confirmed by experiments). Even the curved spaces causes serious difficulties. There are two points of view in this case:

1) The curved space is considered as that embedded into the plane space, and one has to consider dynamics with constrains.

2) QM should be deduced from the classical one without using the embedding space.

They are two principally different approaches. But neither of them gives a unique recipe.

In the case 1) there are recipes:

(i) The Dirac method (modification of the Poisson brackets) [5].

(ii) ”The conversion method” [6,7].

(iii) ”The thin layer method” [8].

(iv) ”The reduction method” [9,10].

The Dirac recipe is not unambiguous [11], the result depends even on the way one parameterizes the curved space [12]. In the case (ii) the authors increase the number of unphysical variables. The approaches (i), (ii) gives different results [11]. In both cases it is assumed that in QM the unphysical degrees of freedom can not influence the physical dynamics that is correct only in classical theory. In the recipe (iii) one approximates the motion on a surface by motion on a thin layer. This looks reasonable. In the method (iv) one excludes the normal to the surface motion demanding

\[ \hat{P}_\perp \psi_{ph} = 0, \]  (1)

where \( \hat{P}_\perp \) is normal to the surface momentum, and \( \psi_{ph} \) is a state vector from the physical Hilbert space. The methods (iii) and (iv) give identical results [13] but the latter allows to avoid rather cumbersome calculations. As for the case 2) — QM cannot be deduced unambiguously from the classical one because there are a lot of ”quantum mechanics” giving in the limit \( \hbar \to 0 \) the same classical one.

The situation gets worse if one tries to formulate QM not on manifold (e.g. on three semi-infinite straight lines having one general point). In the present paper the classical field is studied on a ”s-ray star”. The junction then plays a role of a potential (scatterer). The corresponding scattering amplitudes are calculated.

In fact it gives example of both classical and quantum mechanics in spaces of this type — scattering of a classical free relativistic field here is in fact identical to scattering of a particle in relativistic QM. It turns out that the scattering amplitudes do not depend on the angle between the rays. There is no special reason for such an effect because sets
of harmonic oscillators vibrating in the direction orthogonal to the embedding the "star" plane (i.e. it is supposed that all the rays belong to the plane) model the dynamics — the oscillations do not depend on the angles between the rays.

Importance of this problem for strings is self evident. Gradually it becomes clear that at the Planck scales matter manifests itself in form of strings. Polymers and nanostructures are important for modern technologies. Studying of strings is of special interest because a 3D-network of superstrings can model the physical space, thus leading to unification of all interactions, including gravitation [4,10].

It is worth to note that the 3-tail problem is analogous to the 3-body scattering problem in quantum mechanics.

2 General properties of S-matrix

Let us consider a complex scalar field $\varphi$ defined on an Y-junction of three strings with the spatial coordinates $x \in [0, \infty)$, $y \in [0, \infty)$ and $z \in [0, \infty)$. The junction point corresponds to $x = y = z = 0$. On each string the field $\varphi$ satisfies the Klein-Fock-Gordon equation (we take $\hbar = 1$, $c = 1$),

$$\frac{\partial^2 \varphi}{\partial t^2} = \frac{\partial^2 \varphi}{\partial q^2} - m^2 \varphi, \quad q = x, y, z, \quad q > 0. \quad (2)$$

Our purpose is to obtain a global solution defined on the whole Y-junction. First of all we demand that the global solution is continuous at the junction point,

$$\lim_{x \to 0} \varphi(x) = \lim_{y \to 0} \varphi(y) = \lim_{z \to 0} \varphi(z). \quad (3)$$

This condition was also postulated in [14] together with the following one,

$$\varphi_x|_{x=0} + \varphi_y|_{y=0} + \varphi_z|_{z=0} = 0, \quad (4)$$

where $\varphi_q \equiv \partial_q \varphi$. The latter condition was used in [14] but its physical sense was not clarified. In the present paper we show that together with (3) the condition (4) guarantees both the energy and the charge conservation for our system.

Local solutions of Eq. (2) on strings satisfy the superposition principle. It is natural to begin the investigation with study of a monochromatic wave propagating from $x = \infty$,

$$\varphi(k, x, t) = e^{-i(\omega t + kx)} + R(k)e^{-i(\omega t - kx)},$$

$$\varphi(k, y, t) = T_x(k)e^{-i(\omega t - ky)}, \quad \varphi(k, z, t) = T_y(k)e^{-i(\omega t - kz)}. \quad (5)$$
Here $R(k)$ and $T(k)$ are correspondingly the reflection and transition coefficients, while
\[ \omega^2 = k^2 + m^2, \quad \omega \geq m. \]  

(6)

The incoming particle has momentum $k > 0$.

According to (3) $T_x(k) = T_y(k) = 1 + R(k)$. A unitarity condition,
\[ |R(k)|^2 + 2|R(k) + 1|^2 = 1, \]

(7)

will be proved in the next section. According to it the coefficient $R(k)$ may be parameterized as follows,
\[ R(k) = \frac{1}{3} e^{i\theta(k)} - \frac{2}{3}. \]

(8)

3 S-matrix and conservation of energy and charge

The Eq. (2) on a line corresponds to the Lagrangian,
\[ \mathcal{L} = \frac{1}{2}(\partial_0 \bar{\varphi} \partial_0 \varphi - \partial_1 \bar{\varphi} \partial_1 \varphi - m^2 \bar{\varphi} \varphi), \]

(9)

where $\partial_0$ and $\partial_1$ denote differentiations with respect to time and spatial coordinate $q$. The energy-momentum tensor of the field is given by the general formula \[15\]
\[ T^{ij} = \frac{\partial \mathcal{L}}{\partial (\partial_i \varphi)} \partial^i \varphi + \frac{\partial \mathcal{L}}{\partial (\partial_i \bar{\varphi})} \partial^i \bar{\varphi} - g^{ij} \mathcal{L}, \]

(10)

Here $g^{ij}$ is the Minkowski tensor $g^{ij} = \text{diag}(1, -1)$ and the derivatives $\partial^i$ are related to $\partial_j$ by, $\partial^i = g^{ij} \partial_j$. Using (9) we obtain,
\[ T^{00} = \frac{1}{2}(\partial_0 \bar{\varphi} \partial_0 \varphi - \partial_1 \bar{\varphi} \partial_1 \varphi - m^2 \bar{\varphi} \varphi), \quad T^{10} = -(\partial_1 \bar{\varphi} \partial_0 \varphi + \partial_0 \bar{\varphi} \partial_1 \varphi). \]

(11)

The energy-momentum conservation condition is given by the equation,
\[ \partial_i T^{ij} = 0. \]

(12)

According to (12) the energy in a segment $q_1 \leq q \leq q_2$,
\[ E(q_1, q_2) = \int_{q_1}^{q_2} T^{00}(q) dq, \]

(13)

satisfies the relation,
\[ \frac{dE(q_1, q_2)}{dt} = T^{10}(q_1) - T^{10}(q_2). \]

(14)
For the system (9) on a line with boundary conditions \( \varphi(\pm\infty) \to 0 \) Eq. (14) results to conservation of the energy \( E(-\infty, \infty) = \text{const} \). Postulating the energy conservation for the system on the Y-junction we obtain from (14) the following condition,

\[
T^{10}(x)|_{x \to 0} + T^{10}(y)|_{y \to 0} + T^{10}(z)|_{z \to 0} = 0,
\]

or according to (11),(12),

\[
\bar{\varphi}_t(\varphi_x + \varphi_y + \varphi_z) + (\bar{\varphi}_x + \bar{\varphi}_y + \bar{\varphi}_z)|_{x=y=z=0} = 0.
\]

Though this condition is weaker than (4) it puts a strong enough restriction on the function \( R(k) \). Substituting in (16) the monochromatic solution (5) we obtain the unitarity condition (7). However Eq. (16) must be also true for a superposition of several monochromatic waves with different \( k \) or equivalently for the Fourier sum,

\[
\varphi_{in}(x, t) = \sum_k a(k)e^{-i(\omega_k t - kx)}.
\]

Substituting (17) into (16) we have to take into account the interference of exponents with different \( \omega_k \). Since the expression (11) for \( T^{10} \) is bilinear with respect to \( \varphi \) and \( \bar{\varphi} \) crossing terms originate from two monochromatic waves with different \( k \). Therefore, in order to obtain the corresponding restrictions on the function \( R(k) \) it is sufficient to study the two-mode solution,

\[
\begin{align*}
\varphi(k_1, k_2, x, t) & = e^{-i(\omega_{k_1} t - k_1 x)} + R(k_1)e^{-i(\omega_{k_1} t - k_1 x)} + e^{-i(\omega_{k_2} t + k_2 x)} + R(k_2)e^{-i(\omega_{k_2} t + k_2 x)}, \\
\varphi(k_1, k_2, y, t) & = (1 + R(k_1))e^{-i(\omega_{k_1} t - k_1 y)} + (1 + R(k_2))e^{-i(\omega_{k_2} t - k_2 y)}, \\
\varphi(k_1, k_2, z, t) & = (1 + R(k_1))e^{-i(\omega_{k_1} t - k_1 z)} + (1 + R(k_2))e^{-i(\omega_{k_2} t - k_2 z)}.
\end{align*}
\]

We have suggested here that a wave number does not change after the scattering.

Substituting (18) into (16) and extracting the constant terms we obtain Eq. (7). However the terms proportional to \( e^{i(\omega_{k_1} - \omega_{k_2})t} \) give the following condition,

\[
\omega_{k_1}k_2(1 + \bar{R}(k_1))(1 + 3R(k_2)) + \omega_{k_2}k_1(1 + 3\bar{R}(k_1))(1 + R(k_2)) = 0,
\]

as well as its complex conjugate. With use (6) and (8) this two relations give,

\[
e^{i\theta(k)} = \frac{k + i\alpha\sqrt{k^2 + m^2}}{k - i\alpha\sqrt{k^2 + m^2}}.
\]

Here \( \alpha \) is a real constant.
Another important conserving quantity is charge related to the current, 

\[ j_\mu = i(\bar{\varphi}\partial_\mu \varphi - \varphi \partial_\mu \bar{\varphi}), \quad \mu = 0, 1. \]  

(21)

From Eq. (2) it follows that,

\[ \partial_0 j_0 + \partial_1 j_1 = 0, \]  

(22)

and analogously to (15) postulating charge conservation we obtain the following additional transmission condition,

\[ j_1(x)|_{x \to 0} + j_1(y)|_{y \to 0} + j_1(z)|_{z \to 0} = 0, \]  

(23)

or

\[ \bar{\varphi}(\varphi_x + \varphi_y + \varphi_z) - (\bar{\varphi}_x + \bar{\varphi}_y + \bar{\varphi}_z)\varphi = 0. \]  

(24)

Again we have to check it substituting the solutions (5) and (18). The substitution of (5) into (24) gives again the unitarity condition (7), however the substitution of (18) results to,

\[ k_2(1 + \bar{R}(k_1))(1 + 3R(k_2)) + k_1(1 + 3\bar{R}(k_1))(1 + R(k_2)) = 0, \]  

(25)

or

\[ e^{i\theta(k)} = \frac{k + i\beta}{k - i\beta}, \]  

(26)

where \( \beta \) is a new real constant.

As we see from (20) and (26) the energy and charge will conserve simultaneously only for,

\[ \alpha = \beta = 0, \]  

(27)

or,

\[ \alpha = \beta = \infty. \]  

(28)

In the first case,

\[ R(k) = -\frac{1}{3}, \quad T(k) = \frac{2}{3}, \]  

(29)

however in the second one,

\[ R(k) = -1, \quad T(k) = 0. \]  

(30)

For \( T(k) = 0 \) the three strings behaves as disjoint ones, so the solution (30) is of little physical interest. On the other hand substituting (5) into (4) we find that for the monocromatic waves the conditions (4) and (29) are equivalent. Since both of them are linear this equivalence is also true for a general solution (17). An outstanding feature of
the solution (29) is universality. It does not depend on \( k \). This is important for modeling of space by a network composed by strings [4,10].

We conclude that Eq. (4) represents the only nontrivial condition compatible with continuity condition (3), superposition principle and both the energy and charge conservation.

4 Harmonic oscillators network approximation

It is instructive to approximate our system by a harmonic network. The latter consists of three linear chains of harmonic oscillators related to variables \( \varphi_{q,n} \), where \( n = 1, 2, ... \) and \( q = x, y, z \) added by the junction point oscillator described by \( \varphi_0 \). The Lagrangian is given by,

\[
L = \frac{1}{2} \sum_q \sum_n \left[ \dot{\varphi}_{q,n}^2 - \frac{1}{\Delta^2} (\varphi_{q,n+1} - \varphi_{q,n})^2 - m^2 \varphi_{q,n}^2 \right] \\
+ \frac{1}{2} \left[ \dot{\varphi}_0^2 - \frac{1}{\Delta^2} \sum_q (\varphi_0 - \varphi_{q,1})^2 - m^2 \varphi_0^2 \right],
\]

(31)

where \( \Delta \) is the lattice constant.

The Lagrangian (31) gives the following equations of motion:

\[
\ddot{\varphi}_0 = \frac{1}{\Delta^2} \left( \sum_q \varphi_{q,1} - 3\varphi_0 \right) - m^2 \varphi_0, \tag{32}
\]

\[
\ddot{\varphi}_{q,1} = \frac{1}{\Delta^2} (\varphi_{q,2} + \varphi_0 - 2\varphi_{q,1}) - m^2 \varphi_{q,1}, \tag{33}
\]

\[
\ddot{\varphi}_{q,n} = \frac{1}{\Delta^2} (\varphi_{q,n+1} + \varphi_{q,n-1} - 2\varphi_{q,n}) - m^2 \varphi_{q,n}, \quad n > 1. \tag{34}
\]

The following solution,

\[
\varphi_{x,n}(t) = e^{-i(\omega_k t + kn)} + R(k)e^{-i(\omega_k t - k\Delta n)}, \\
\varphi_{y,n}(t) = \varphi_{z,n}(t) = (R(k) + 1)e^{-i(\omega_k t - k\Delta n)}, \\
\varphi_0(t) = (R(k) + 1)e^{-i\omega_k t}, \tag{35}
\]

is a discrete analog of (5). The normal frequencies,

\[
\omega_k^2 = \frac{4}{\Delta^2} \sin^2 \frac{k\Delta}{2} + m^2, \tag{36}
\]

in the limit \( \Delta \to 0 \) coincide with (6).
Substituting (35) into (32) and taking into account the condition,

$$\dot{\varphi}_0 + m^2 \varphi_0 = -\frac{4}{\Delta^2} \sin^2 \frac{k\Delta}{2} (1 + R(k)) e^{-i\omega kt},$$  \hspace{1cm} (37)

(see (36)) we obtain,

$$[4 \sin^2 \frac{k\Delta}{2} + 3(e^{ik\Delta} - 1)](R(k) + 1) = 2i \sin k\Delta. \hspace{1cm} (38)$$

Since $e^{ik\Delta} - 1 = 2i \sin \frac{k\Delta}{2} \cos \frac{k\Delta}{2} - 2 \sin^2 \frac{k\Delta}{2}$ and $\sin k\Delta = 2 \sin \frac{k\Delta}{2} \cos \frac{k\Delta}{2}$, we reduce (38) to the following form,

$$(3i \cos \frac{k\Delta}{2} - \sin \frac{k\Delta}{2})(R(k) + 1) = 2i \cos \frac{k\Delta}{2} \hspace{1cm} (39)$$
or

$$e^{i\theta(k)} = -\frac{\sin \frac{k\Delta}{2} + 3i \cos \frac{k\Delta}{2}}{\sin \frac{k\Delta}{2} - 3i \cos \frac{k\Delta}{2}}. \hspace{1cm} (40)$$

In the limit $\Delta \to 0$ using Eq. (8) we obtain for $R(k)$ and $T(k)$ the expression (29).

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