AMALGAMATION OF RINGS DEFINED BY BEZOUT-LIKE CONDITIONS

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Abstract. Let \( f : A \rightarrow B \) be a ring homomorphism and let \( J \) be an ideal of \( B \). In this paper, we investigate the transfer of notions elementary divisor ring, Hermite ring and Bézout ring to the amalgamation \( A \bowtie^f J \). We provide necessary and sufficient conditions for \( A \bowtie^f J \) to be an elementary divisor ring where \( A \) and \( B \) are integral domains. In this case it is shown that \( A \bowtie^f J \) is an Hermite ring if and only it is a Bézout ring. In particular, we study the transfer of the previous notions to the amalgamated duplication of a ring \( A \) along an \( A \)-submodule \( E \) of \( Q(A) \) such that \( E^2 \subseteq E \).

1. Introduction

All rings considered in this paper are assumed to be commutative, and have identity element and all modules are unitary.

A ring \( R \) is called an elementary divisor ring (resp. Hermite ring) if for every matrix \( M \) over \( R \) there exist non singular matrices \( P, Q \) such that \( PMQ \) (resp. \( MQ \)) is a diagonal matrix (resp. triangular matrix). It proved in [7] that a ring \( R \) is an Hermite ring if and only if for all \( a, b \in R \), there exist \( a_1, b_1, d \in R \) such that \( a = a_1 d \), \( b = b_1 d \), and \( Ra_1 + Rb_1 = R \). A ring is a Bézout ring if every finitely generated ideal is principal. It is clear that every elementary divisor ring is an Hermite ring, and that every Hermite ring is a Bézout ring. Following Kaplansky [10] a ring \( R \) is said to be a valuation ring if for any two elements in \( R \), one divides the other. Kaplansky proved that any valuation ring is an elementary divisor ring.

Let \( A \) and \( B \) be rings, \( J \) an ideal of \( B \) and let \( f : A \rightarrow B \) be a ring homomorphism. In [4] the amalgamation of \( A \) with \( B \) along \( J \) with respect to \( f \) is the sub-ring of \( A \times B \) defined by:

\[
A \bowtie^f J = \{(a, f(a) + j) \mid a \in A, j \in J\}.
\]

This construction is a generalization of the amalgamated duplication of a ring along an ideal introduced and studied in [5], [2] and in [6]. Moreover, several classical construction such as \( A + xK[x] \) and \( A + xK[[x]] \) can be

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studied as particular case of the amalgamation.

We denote $Q(A)$ the total ring of quotients of $A$. Let $E$ be an $A$–submodule of $Q(A)$ such that $E^2 \subseteq E$, $A + E$ is a sub-ring of $Q(A)$ and $E$ is an ideal of $A + E$. The amalgamated duplication of $A$ along $E$:

$$A \bowtie E = \{(a, a + e) ; a \in A, e \in E\}$$

is also a particular case of the amalgamation of $A$ with $A + E$ along $E$ with to respect $f$, where $f : A \hookrightarrow A + E$ is the inclusion map. In fact, the amalgamated duplication of $A$ along $E$ can be studied in the frame of amalgamation construction. Our aim in this paper is to give a characterization for $A \bowtie^f J$ to be an elementary divisor ring, an Hermite ring and a Bézout ring.

2. Main Results

The set of all $n \times n$ matrices with entries from a ring $R$ will be denoted by $\mathcal{M}_n(R)$. We will let $\mathcal{U}_n(R)$ denote the units in $\mathcal{M}_n(R)$. Let $A$ and $B$ be rings, for every matrix $M = ((a_{i,j}, b_{i,j}))_{1 \leq i,j \leq n} \in \mathcal{M}_n(A \times B)$ we shall use the notation $M_a = (a_{i,j})_{1 \leq i,j \leq n}$, $M_b = (b_{i,j})_{1 \leq i,j \leq n}$ and $M = M_a \times M_b$. Let $M, N \in \mathcal{M}_n(A \times B)$, it is easy to see that the product $MN$ of $M$ and $N$ is giving by $MN = (M_aN_a) \times (M_bN_b)$.

The following lemma will be useful to provide us many statements in this paper.

**Lemma 2.1.** Let $A$ and $B$ a pair of integral domains, $f : A \rightarrow B$ a ring homomorphism and let $J$ be a proper ideal of $B$.

1. If $A \bowtie^f J$ is a Bézout ring then $f(A) \cap J = 0$.
2. If $A \bowtie^f J$ is a Bézout ring and $f$ is not injective then $J = 0$.

**Proof.** (1) Suppose the statement is false i.e $f(A) \cap J \neq 0$, and choose an element $a \in A$ such that $0 \neq f(a) \in J$. Then $(0, f(a))$ is an element of $A \bowtie^f J$. Since $A \bowtie^f J$ is a Bézout ring the ideal generated by $(0, f(a))$ and $(a, f(a))$ is principal. Hence, there exists $(d, f(d) + j) \in A \bowtie^f J$ such that

$$(a, f(a)) \left( A \bowtie^f J \right) + (0, f(a)) \left( A \bowtie^f J \right) = (d, f(d) + j) \left( A \bowtie^f J \right).$$

So, there exist $(b, f(b) + x), (c, f(c) + y), (\alpha, f(\alpha) + s)$ and $(\beta, f(\beta) + t)$ in $A \bowtie^f J$ such that

\[
\begin{cases}
(0, f(a)) = (d, f(d) + j)(b, f(b) + x) \\
(a, f(a)) = (d, f(d) + j)(c, f(c) + y) \\
(d, f(d) + j) = (0, f(a))(\alpha, f(\alpha) + s) + (a, f(a))(\beta, f(\beta) + t).
\end{cases}
\]
It follows that \( d \neq 0 \) since \( a = cd \) and \( f(a) \neq 0 \). Also \( b = 0 \) since \( bd = 0 \) and \( A \) is an integral domain. From the previous equalities we deduce that \( f(a) = (f(d)+j)x = (f(d)+j)(f(c)+y) \) and \( f(d)+j = f(a)(f(\alpha)+f(\beta)+s+t) \).

Multiplying the above equality by \( x \), we get that \( 1 = x(f(\alpha)+f(\beta)+s+t) \) since \( B \) is an integral domain. We conclude that \( x \) is a unit, but \( x \in J \) hence \( J = B \) which is absurd. We have the desired result.

(2) Assume that \( J \neq 0 \) and let \( 0 \neq u \in J \). Since \( f \) is non injective there exists \( 0 \neq a \in \ker f \). From the assumption we can write

\[
(a, u) \left( A \cong^f J \right) + (0, u) \left( A \cong^f J \right) = (d, f(d) + j) \left( A \cong^f J \right)
\]

for some \((d, f(d) + j) \in A \cong^f J \). With similar proof as in the statement (1), we get that \( J = B \). This completes the proof of Lemma 2.4. \( \square \)

**Lemma 2.2.** The following assertions holds:

(1) Let \( A \) and \( B \) be rings. Then \( A \times B \) is an elementary divisor ring if and only if so \( A \) and \( B \).

(2) Let \( f : A \rightarrow B \) be a ring homomorphism and let \( J \) be an ideal of \( B \). If \( A \cong^f J \) is an elementary divisor ring then so is \( A \) and \( f(A) + J \).

**Proof.** (1) We begin by showing that if \( M \in \mathcal{M}_n(A \times B) \) then \( M \) is invertible if and only if so \( M_a \) and \( M_b \). We put \( M = ((a_{i,j}, b_{i,j}))_{1 \leq i,j \leq n} \). The determinant of \( M \) is giving by

\[
\det M = \sum_{\sigma \in S_n} \varepsilon(\sigma) \prod_{i=1}^{n} (a_{i,\sigma(i)}, b_{i,\sigma(i)})
\]

where \( S_n \) denotes the set of all permutations on \( n \) letters and \( \varepsilon(\sigma) \) denotes the sign of \( \sigma \), for every \( \sigma \in S_n \). Thus \( \det M = (\det M_a, \det M_b) \). We say that \( M \) is invertible if and only if \( \det M \) is a unit. Then we have the desired result.

Assume that \( A \times B \) is an elementary divisor ring. Let \( U \in \mathcal{M}_n(A) \) then \( U \times 0 \) is equivalent to a diagonal matrix \( D \) with entries from \( A \times B \). There is some \( P, Q \in GL_n(A \times B) \) such that \( P(U \times 0)Q = D \). It follows that \( P_aUQ_a = D_a \) and so \( A \) is an elementary divisor ring. By the same way we get that \( B \) is an elementary divisor ring.

Conversely, assume that \( A \) and \( B \) are elementary divisor rings and let \( M \in \mathcal{M}_n(A \times B) \). Then there exist two invertible matrices \( P_1 \) and \( Q_1 \) (resp., \( P_2 \) and \( Q_2 \)) and a diagonal matrix \( D \) (resp., \( \Delta \)) with entries from \( A \) (resp., \( B \)) such that \( P_1M_aQ_1 = D \) (resp., \( P_2M_bQ_2 = \Delta \)). It follows that

\[
(P_1 \times P_2)M(Q_1 \times Q_2) = (P_1M_aQ_1) \times (P_2M_bQ_2) = D \times \Delta,
\]

which is a diagonal matrix. From the previous part of the proof \( P_1 \times P_2, Q_1 \times Q_2 \in GL_n(A \times B) \). This completes the proof of (1).

(2) Let \( U = (a_{i,j})_{1 \leq i,j \leq n} \in \mathcal{M}_n(A) \) and let \( M \) be the matrix defined by \( M = ((a_{i,j}, f(a_{i,j})))_{1 \leq i,j \leq n} \) with entries from \( A \cong^f J \). We have the equality
Remark 2.3. Let \( f : A \to B \) be a ring homomorphism, \( J \) an ideal of \( B \) and let \( M \in \mathcal{M}_n(A \bowtie^f J) \). Then \( M \) is invertible if and only if so is \( M_a \) and \( M_b \).

Proof. It is sufficient to prove that if \( M_a \in GL_n(A) \) and \( M_b \in GL_n(B) \) then \( M_a^{-1} \times M_b^{-1} \in GL_n(A \bowtie^f J) \). Let \( (a, f(a) + j) \in A \bowtie^f J \) which is a unit in the ring \( A \times B \). We put \( x = -f(a^{-1})(f(a) + j)^{-1}j \). Since \( J \) is an ideal of \( B \), \( x \in J \). It is easy to get the following equality

\[
(a^{-1}, f(a^{-1}) + x)(a, f(a) + j) = (1, 1).
\]

Thus \( (a, f(a) + j)^{-1} \in A \bowtie^f J \). We say that \( \det M \) is an element of \( A \bowtie^f J \) which is a unit in \( A \times B \), therefore \( (\det M)^{-1} \in A \bowtie^f J \). Consequently, \( M^{-1} \in \mathcal{M}_n(A \bowtie^f J) \).

Theorem 2.4. Let \( A \) and \( B \) a pair of integral domains, \( f : A \to B \) a ring homomorphism and let \( J \) be an ideal of \( B \).

1. Assume that \( f \) is injective.
   - If \( J = B \) then \( A \bowtie^f J \) is an elementary divisor ring if and only if so is \( A \) and \( B \).
   - If \( J \neq B \) then \( A \bowtie^f J \) is an elementary divisor ring if and only if so is \( f(A) + J \) and \( f(A) \cap J = 0 \).

2. Assume that \( f \) is not injective. Then \( A \bowtie^f J \) is an elementary divisor ring if and only if one of the following conditions holds:
   - \( J = 0 \) and \( A \) is an elementary divisor ring.
   - \( J = B \) and \( (A, B) \) is a pair of elementary divisor rings.

Proof. (1) Two cases will be considered.

Case 1: If \( J = B \) then \( A \bowtie^f J = A \times B \). By applying condition (1) of Lemma 2.2, we get that \( A \bowtie^f J \) is an elementary divisor ring if and only if so is \( A \) and \( B \).

Case 2: If \( J \neq B \) and \( A \bowtie^f J \) is elementary divisor ring then \( f(A) \cap J = 0 \) by Lemma 2.2 since every elementary divisor ring is a Bézout ring. On the other hand \( f(A) + J \) is an elementary divisor ring by Lemma 2.2. Conversely, assume that \( f(A) + J \) is an elementary divisor ring and \( f(A) \cap J = 0 \). We claim that the natural projection \( p_B : A \bowtie^f J \to f(A) + J \) is a ring isomorphism. Indeed,

\[
f(a) + j = 0 \implies f(a) = j = 0 \implies a = 0
\]
The conclusion is now straightforward.

(2) Assume that \( A \bowtie J \) is an elementary divisor ring. By using Lemma 2.1 we get that \( J = 0 \) or \( J = B \). In the first case \( A \bowtie J \simeq A \), then \( A \) is an elementary divisor ring. In the second case \( A \bowtie J = A \times B \). Hence \( A \) and \( B \) are elementary divisor rings by Lemma 2.2. The converse of (2) is an immediate consequence of Lemma 2.2. \( \square \)

Theorem 2.4 enriches the literature with a new example of a non valuation elementary divisor ring.

Let \( f : A \longrightarrow B \) be a ring homomorphism and let \( J \) be an ideal of \( B \). It is easy to see that: if \( A \bowtie f J \) is a valuation ring then so is \( A \).

Example 2.5. Let \( A \) be an elementary divisor domain which is not a valuation ring (for instance \( A = \mathbb{Z} \)), and let \( K \) its field of fractions. Let \( K[[x]] \) denote the ring of formal power series over \( K \) in a indeterminate \( x \). By [9], Example 1 p.161, \( A + (xK[[x]]) \) is an elementary divisor ring. We conclude that \( A \bowtie (xK[[x]]) \), where \( i \) is the inclusion map of \( A \) into \( K[[x]] \), is an elementary divisor ring. On the other hand \( A \bowtie i (xK[[x]]) \) is not a valuation ring. Thus \( \mathbb{Z} \bowtie i (x\mathbb{Q}[[x]]) \) is an elementary divisor ring which is not a valuation ring.

Corollary 2.6. Let \( A \) be an integral domain, \( K \) its quotient field and let \( E \) be a nonzero \( A \)-submodule of \( K \) such that \( E^2 \subseteq E \). Then \( A \bowtie E \) is an elementary divisor ring if and only if so is \( A \) and \( A \subseteq E \).

Proof. We first prove that: Any ring \( R' \) between an elementary divisor ring \( R \) and its total ring \( Q(R) \), is also an elementary divisor ring.

Let \( M = \left( \frac{a_{i,j}}{d} \right)_{1 \leq i,j \leq n} \in \mathcal{M}_n(R') \), where \( a_{i,j} \in R \) for each \( 1 \leq i,j \leq n \) and \( d \) is a nonzero divisor element of \( R \). There is some invertible matrices \( P \) and \( Q \) with entries from \( R \) such that \( P \left( a_{i,j} \right)_{1 \leq i,j \leq n} Q = \text{diag}(\lambda_1, ..., \lambda_n) \). Multiplying this equality by \( \frac{1}{d} \), we get that \( PMQ = \text{diag} \left( \frac{\lambda_1}{d}, ..., \frac{\lambda_n}{d} \right) \). Since \( PMQ \in \mathcal{M}_n(R') \) the result follows.

Now suppose that \( A \) is an elementary divisor ring and \( A \subseteq E \). We have \( A \bowtie E = A \times E \). From the previous part of the proof and condition (1) of Lemma 2.2, we get that \( A \bowtie E \) is an elementary divisor ring. Conversely, assume that \( A \bowtie E \) is an elementary divisor ring. We have \( A \bowtie E = A \bowtie i E \), where \( i : A \hookrightarrow A + E \) is the inclusion map. By using the condition (1) of Theorem 2.4 we obtain the following result:

- If \( E = A + E \) (i.e \( A \subseteq E \)) then \( A \) and \( A + E \) are elementary divisor rings.
- Otherwise \( (A + E) \cap E = 0 \) and \( A + E \) is elementary divisor ring.
From the assumption \((A + E) \cap E \neq 0\) since \(E \subseteq A + E\). We conclude that \(A \subseteq E\) and \(A\) is an elementary divisor ring. \(\square\)

**Example 2.7.** Let \(A\) be an integral domain and let \(I\) be a nonzero ideal of \(A\). Then \(A \cong I\) is an elementary divisor ring if and only if so is \(A\) and \(I = A\).

**Lemma 2.8.** Let \(A\) and \(B\) be a pair of rings. Then:

1. \(A \times B\) is a Bézout ring if and only if so is \(A\) and \(B\).
2. \(A \times B\) is an Hermite ring if and only if so is \(A\) and \(B\).

**Proof.** (1) Suppose that \(A\) and \(B\) are Bézout rings and let \(I\) be a finitely generated ideal of \(A \times B\). There is some ideal \(I_1\) of \(A\) and \(I_2\) of \(B\) such that \(I = I_1 \times I_2\). If the subset \(\{(a_1, b_1), \ldots, (a_n, b_n)\}\) of \(A \times B\) generate \(I\) then \(I_1 = Aa_1 + \cdots + Aa_n\). Thus \(I_1\) is a principal ideal of \(A\). There exists \(a \in I_1\) such that \(I_1 = Aa\). By the same way, we get that there exists \(b \in I_2\) such that \(I_2 = Bb\). We deduce that \(I = (A \times B)(a, b)\). Conversely assume that \(A \times B\) is a Bézout ring. Let \(J_1\) be a finitely generated ideal of \(A\) and let \(J = J_1 \times 0\). Then \(J\) is also finitely generated ideal of \(A \times B\), we get that \(J\) is a principal ideal of \(A \times B\). Hence so is \(J_1\), therefore \(A\) is a Bézout ring. Also \(B\) is a Bézout ring since \(A \times B \simeq B \times A\). (2) Assume that \(A \times B\) is an Hermite ring. Let \(a, a' \in A\) then there exist \((a_1, b_1), (a'_1, b'_1), (d, \delta) \in A \times B\) such that

\[
\begin{align*}
(a, 0) &= (a_1, b_1)(d, \delta) \\
(a', 0) &= (a'_1, b'_1)(d, \delta) \\
A \times B &= (a_1, b_1)(A \times B) + (a'_1, b'_1)(A \times B).
\end{align*}
\]

Let \((\alpha, \beta), (\alpha', \beta') \in A \times B\) such that \((\alpha, \beta)(a_1, b_1)+(\alpha', \beta')(a'_1, b'_1) = (1, 1)\). It follows that \(a = a_1d, a' = a'_1d\) and \(\alpha a_1 + \beta a'_1 = 1\). We conclude that \(A\) and \(B\) is a pair of Hermite rings since \(A \times B \simeq B \times A\). The converse of the statement is obvious. \(\square\)

**Theorem 2.9.** Let \(A\) and \(B\) be a pair of integral domains, \(J\) an ideal of \(B\) and let \(f : A \to B\) be an injective ring homomorphism. Then the following properties are equivalent:

1. \(A \cong J\) is an Hermite ring.
2. \(A \cong J\) is a Bézout ring.
3. One of the following conditions holds:
   - \(J = B\), \(A\) and \(B\) are Bézout rings.
   - \(J \neq B\), \(f(A) \cap J = 0\) and \(f(A) + J\) is a Bézout ring.

**Proof.** (1) \(\Rightarrow\) (2): Clear.
(2) \(\Rightarrow\) (3): Assume that \(J \neq B\). By Lemma 2.1, \(f(A) \cap J = 0\). Then the natural projection \(p_B : A \cong J \to f(A) + J; (p_B(a, f(a) + j) = f(a) + j\) is
a ring isomorphism since \( f \) is injective. Therefore \( f(A) + J \) is a Bézout ring. If \( J = B \) then \( A \) and \( B \) are Bézout rings by the condition (1) of Lemma 2.8 since \( A \triangleleft \triangleleft J = A \times B \).

(3) \(\Rightarrow\) (1): If \( J = B \) then \( A \) and \( B \) are Bézout rings since every Bézout domain is an Hermite ring. Hence \( A \triangleleft \triangleleft J = A \times B \) is an Hermite ring.

Now we assume that \( J \neq B \). Then \( A \triangleleft \triangleleft f(J) \cong f(A) + J \) and so \( A \triangleleft \triangleleft J \) is a Bézout domain. This completes the proof of Theorem 2.9.

Example 2.10. Let \( A \) be a Bézout domain, \( K \) its quotient field, and let \( K[[x]] \) denote the ring of formal power series over \( K \) in an indeterminate \( x \). Then \( A \triangleleft \triangleleft (xK[[x]]) \), where \( i : A \hookrightarrow K[[x]] \) is the inclusion map, is an Hermite ring.

Proof. Let \( f = \sum_{n=0}^{\infty} a_n x^n, g = \sum_{n=0}^{\infty} b_n x^n \) be nonzero elements of \( R = A + (xK[[x]]) \), and let \( p \) (resp. \( q \)) denote the least integer such that \( a_p \neq 0 \) (resp., \( b_q \neq 0 \)). We can write \( f = a_p x^p (1 + x f_1) \) and \( g = b_q x^q (1 + x g_1) \), where \( f_1, g_1 \in K[[x]] \). Since \( 1 + x f_1, 1 + x g_1 \) are units of \( R \),

\[
fR + gR = a_p x^p R + b_q x^q R.
\]

If \( p < q \) (resp., \( q < p \)) then \( fR + gR = a_p x^p R \) (resp., \( b_q x^q R \)). Suppose that \( p = q \) and write \( a_p = \frac{a}{d} \) and \( b_q = \frac{b}{d} \) for some nonzero elements \( a, b, d \) of \( A \) (where \( d = 1 \) if \( p = q = 0 \)). By the assumption there exist \( c, a', b' \in A \) such that \( a = a'c, b = b'c \) and \( a'A + b'A = A \). It is easy to get that \( fR + gR = \frac{c}{d} x^p R \). This completes the proof that \( A \triangleleft \triangleleft (xK[[x]]) \) is an Hermite ring.

Corollary 2.11. Let \( A \) be an integral domain, \( K \) its quotient field and let \( E \) be a nonzero \( A \)-submodule of \( K \) such that \( E^2 \subseteq E \). Then the following statements are equivalent:

1. \( A \triangleleft \triangleleft E \) is an Hermite ring.
2. \( A \triangleleft \triangleleft E \) is a Bézout ring.
3. \( A \) is a Bézout ring and \( A \subseteq E \).

Proof. (2) \(\Rightarrow\) (3): Let \( 0 \neq \frac{a}{b} \in E \). Then \( 0 \neq a \in A \cap E \) and so \( A \cap E \neq 0 \). By applying Theorem 2.9, we get that \( A + E = E \) and \( (A, A + E) \) is a pair of Bézout rings. It follows that \( A \subseteq E \) and \( A \) is a Bézout ring.

(3) \(\Rightarrow\) (1): By applying Lemma 2.8 and the condition (3) of Theorem 2.9 it is sufficient to prove that every ring between a Bézout domain and its quotient field is also Bézout domain. Let \( R \) be a Bézout domain and let \( R' \)
be a ring such that \( R \subseteq R' \subseteq qf(R) \). Let \( \frac{a}{d}, \frac{b}{d} \in R' \) then we can write

\[
\begin{cases}
  a = a'c \\
  b = b'c \\
  \alpha a' + \beta b' = 1
\end{cases}
\]

for some elements \( a', b', c, \alpha, \beta \) in \( R \). Hence \( \frac{c}{d} = \frac{\alpha a}{d} + \frac{\beta b}{d} \) is an element of \( R' \). Thus \( \frac{c}{d} \in R' \) and \( R' \frac{a}{d} + R' \frac{b}{d} \subseteq R \frac{c}{d} \). On the other hand, we have:

\[
\frac{c}{d} \in R' \frac{a}{d} + R' \frac{b}{d} \subseteq R' \frac{a}{d} + R' \frac{b}{d}.
\]

It follows that \( R' \frac{a}{d} + R' \frac{b}{d} = R' \frac{c}{d} \). Finally, \( R' \) is a Bézout domain. \( \square \)

**Example 2.12.** Let \( A \) be an integral domain and let \( I \) be a nonzero ideal of \( A \). Then \( A \bowtie I \) is a Bézout ring if and only if so is \( A \) and \( I = A \).

**Theorem 2.13.** Let \( A \) and \( B \) be a pair of integral domains, \( J \) an ideal of \( B \) and let \( f : A \rightarrow B \) be a non injective ring homomorphism. Then the following statements are equivalent:

1. \( A \bowtie_f J \) is an Hermite ring.
2. \( A \bowtie_f J \) is a Bézout ring.
3. One of the following conditions holds:
   - \( J = B \), \( A \) and \( B \) are Bézout rings.
   - \( J = 0 \), and \( A \) is a Bézout ring.

**Proof.** (2) \( \Rightarrow \) (3): By applying condition (2) of Lemma 2.8 we get that \( J = 0 \) or \( J = B \). If \( J = 0 \) then \( A \bowtie_f J \asymp A \) otherwise \( A \bowtie_f J = A \times B \). By using Lemma 2.8 we have the desired implication.

(3) \( \Rightarrow \) (1): If \( J = 0 \) then \( A \bowtie_f J \asymp A \) and \( A \) is an Hermite ring (since \( A \) is an integral domain). If \( J = B \) then \( A \bowtie_f J = A \times B \) is an Hermite ring by condition (2) of Lemma 2.8. \( \square \)

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