Type D conformal initial data

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Abstract

For a vacuum initial data set of the Einstein field equations it is possible to carry out a
conformal rescaling or conformal compactification of the data giving rise to an initial data set
for the Friedrich vacuum conformal equations. When will the data development with respect
to the conformal equations of this set be a conformal extension of a type D solution? In this
work we provide a set of necessary and sufficient conditions on a set of initial data for the
conformal equations that guarantees that the data development of the conformal equations
has a subset that is conformal to a vacuum type D solution of the Einstein’s equations. In
particular we find the conditions under which this vacuum solution corresponds to the Kerr
solution. Using our results we are able to show that there are no obstructions to extend the
Petrov type of the physical spacetime to the unphysical spacetime if the conformal data are
hyperboloidal.

1 Introduction

Since its introduction by Penrose [38, 39], the notion of conformal boundary has found a
wide number of applications in general relativity and theoretical physics. In general relativity the
conformal boundary has been used to give a rigorous definition of isolated system (asymptotically
simple space-time) and procedures to compute the total emission and absorption of gravitational
radiation of such a system have been developed.

The explicit computation of a conformal boundary with suitable properties for given exact
solutions of the Einstein field equations is a difficult enterprise unless we deal with the simplest
solutions. A possible approach is to set up the computation as an initial value problem for a
system of hyperbolic equations involving a conformal rescaling of the metric tensor used in the
Einstein’s equations (the physical metric). The main obstacle one needs to surmount by following
this approach is the lack of conformal invariance of the Einstein’s equations. This means that the
standard results that allow the formulation of the Einstein’s equations as a Cauchy problem do
not apply after performing the conformal rescaling and therefore one needs additional techniques
to find hyperbolic equations for the conformally rescaled metric (the unphysical metric).

A hyperbolic formulation as described in the previous paragraph has been developed by
Friedrich [19, 20] resulting in the so-called conformal field equations and they have been success-
fully used to prove a number of remarkable global existence results: first proof [15, 16] of the
non-linear stability of some of the simplest solutions of Einstein’s equation (Minkowski and de
Sitter) and similar results for the Einstein-Yang-Mills system [17] (see also [30]), purely radiative
spacetimes [32], cosmological solutions [31] and the asymptotic region of the Schwarzschild-de
Sitter black hole [25]. In any case, the rough idea is that the hyperbolic character of the conformal equations makes it possible to use classical local existence results of the partial differential equations theory to prove a local existence result for the former. The conformal relation between the unphysical metric and the physical one, and the knowledge of the geometric properties of the conformal boundary enables us to turn a local existence result in the unphysical space-time into a global existence result for the physical space-time. Other very important global existence results where the conformal equations have played a key role deal with asymptotically simple spacetimes. See [10, 7] for examples of this situation.

Given a set of hyperbolic equations or a hyperbolic reduction of a set of tensorial equations it makes sense to investigate its initial data problem. This is a set of conditions on an initial data hypersurface or Cauchy hypersurface ensuring the existence of a solution of the hyperbolic system. Formulations of the initial value problem for the conformal equations can be found in the above references and also in [15]. In this framework different kinds of initial data have been studied: the asymptotic characteristic initial value problem [19, 20], data prescribed at (spacelike) past null infinity [16], data for space-times with a timelike conformal boundary [18] and data for Kerr-de Sitter spacetimes at null infinity [33]. See [29, 14] for a detailed review and information about all these topics.

Suppose that we have a vacuum initial data set for the Einstein field equations and carry out a conformal rescaling (conformal compactification) of the data. This gives rise to initial data for the (vacuum) conformal equations. When will the data development with respect to the conformal equations be a conformal extension of a vacuum type D solution? This work provides an answer to this question that is written exclusively in terms of the quantities used to define an initial data set of the conformal equations. We also show that for hyperboloidal data there are no obstructions to the extension of the Petrov type of the physical space-time to the unphysical space-time (see Theorem 12 for more details).

The type D conformal initial data are a set of conditions that should be appended to the conformal constraint equations and therefore given exclusively in terms of the data of the conformal equations. We also prove that the set of conditions is a set of necessary and sufficient conditions, so any other initial data set for the conformal equations whose development admits a subset that is conformal to a Petrov type D vacuum solution must be already dependent from our set in some region of the initial data hypersurface. The method presented in this paper to construct initial data for the conformal equations is valid for any vacuum Petrov type D solution and we also particularize it for the case of the Kerr solution. Note that the construction of initial data for the conformal equations corresponding to Kerr data is the starting point in order to study the non-linear stability of the Kerr black hole using conformal techniques, in the spirit of the results described above. Note also that once an initial value problem for the conformal field equations has been set up, a local existence result of the conformal hyperbolic system may translate into a global existence result for the original Einstein’s equations, provided some extra conditions are met. In general a local existence result is far easier to obtain than a global one, so the use of conformal techniques could play an important role in the solution of the non-linear stability of the Kerr black hole. Also the initial data so constructed could be used as the starting point in the analysis of the conformal boundary properties for members of this important class of exact solutions. In this sense there are already results for the Schwarzschild [22] and the Kottler family of solutions [26] where the construction of congruences of conformal geodesics enables us to determine geometric properties of the conformal boundary without carrying out the actual conformal extension.

This paper is structured as follows: in section 2 we recall the formulation of the vacuum conformal equations and the construction of conformal initial data sets for them. Section 3 reviews an invariant characterization of Petrov type D solutions needed for the construction of initial data for the conformal equations. This is the subject of section 4 where the main results of this paper (Theorems 6 and 7) are presented. In section 5 we particularize these results to the case in which the data are constructed from data for the Kerr solution (Theorems 10 and
Section 6 analyzes the conformal boundary limit of the initial data conditions obtained in the previous sections finding that there are no obstructions to the extension of the data through the conformal boundary whenever the data are hyperboloidal. We discuss possible applications in section 7.

All the tensor computations in this paper have been carried out with the system xAct [34], a Wolfram Language suite for doing tensor analysis (see also [35]).

2 The vacuum Friedrich conformal equations and their initial data

Let \((\tilde{\mathcal{M}}, \tilde{g}_{ab})\) be a 4-dimensional Lorentzian manifold (physical space-time) and \((\mathcal{M}, g_{ab})\) a second Lorentzian manifold (unphysical spacetime) which is conformally related to the first in the following fashion (the signature convention for both metrics is \((- , + , + , +))

\[
g_{ab} = \Theta^2 \tilde{g}_{ab}. \tag{1}
\]

In the previous relation a conformal map (conformal embedding) from \(\tilde{\mathcal{M}}\) to \(\mathcal{M}\) is understood and the conformal factor \(\Theta\) is assumed to be a smooth function which does not vanish in the manifold \(\tilde{\mathcal{M}}\). We use small Latin letters to denote abstract indices of tensors in \(\mathcal{M}\) and \(\tilde{\mathcal{M}}\). Indices are always raised and lowered with respect to the unphysical metric \(g_{ab}\) with the exception of \(\tilde{g}^{ab}\) where we follow the traditional convention that it represents the inverse of \(\tilde{g}_{ab}\)

\[
\tilde{g}^{ac} \tilde{g}_{cb} = \delta_b^a. \tag{2}
\]

Hence, the explicit relation between the physical and the unphysical contravariant metric tensors is then

\[
g^{ab} = \tilde{g}^{ab} / \Theta^2. \tag{3}
\]

Each of the metric tensors \(g_{ab}, \tilde{g}_{ab}\) has its own volume element, denoted respectively by \(\eta_{abcd}\) and \(\tilde{\eta}_{abcd}\). Using (1) we deduce the relation

\[
\tilde{\eta}_{abcd} = \eta_{abcd} \Theta^4. \tag{4}
\]

Also each metric tensor has its own Levi-Civita connection denoted respectively by \(\nabla_a, \tilde{\nabla}_a\) which are used to define the connection coefficients and the curvature tensors in the standard fashion. Our conventions for the (unphysical) Riemann, Ricci and Weyl tensors are

\[
\nabla_a \nabla_b \omega_c - \nabla_b \nabla_a \omega_c = R_{abc}^\ d \omega_d, \tag{5}
\]

\[
R_{ac} \equiv R_{abc}^\ b, \tag{6}
\]

\[
C_{abcd} \equiv R_{abcd} - 2L_{[d[}g_{a]c] - 2g_{d[}L_{a]c], \tag{7}
\]

where the unphysical Schouten tensor is defined by

\[
L_{ab} \equiv \frac{1}{2} (R_{ab} - \frac{1}{6} R g_{ab}). \tag{8}
\]

The conventions for the corresponding physical quantities are similar and we use a tilde over the symbol employed for a unphysical spacetime tensor to denote its physical counterpart. The only exception of this rule occurs for the physical Weyl tensor, where the notation is \(\tilde{W}_{abcd}\) (see eq. [9] below). Recall that the Riemann, Ricci and Weyl tensors have a natural index configuration in their definition which is important to bear in mind when working with two different metric tensors. This is so because a tensorial expression containing any of these tensors in a non-natural index configuration requires a clear convention telling us the metric which was used to change
from the natural index configuration to the non-natural one. In this sense, eqs. (5)-(8) present
the Riemann, Ricci, Weyl and Schouten tensors in their natural index configuration. As already
mentioned we are adopting the convention of taking the unphysical metric as the metric used to
raise and lower indices and therefore, this shall be the metric we are going to use to change the
natural index configuration of any tensor.

Standard computations enable us to find the relations between the connection coefficients and
curvature tensors of $\nabla$ and $\tilde{\nabla}$. For us the relation between the unphysical Weyl tensor $C_{abcd}$
and the physical one $\tilde{W}_{abcd}$ will be specially important
$$C_{abcd} = \Theta^2 \tilde{W}_{abcd}. \quad \text{(9)}$$

The star * is used to denote both the Hodge dual and the complex conjugation and we leave
to the context the distinction between these two.

### 2.1 The metric conformal equations

An interesting situation occurs when the unphysical space-time is conformally related to a physical
space-time which is a vacuum solution of the Einstein equations. In this case it is a non-trivial
problem to find a set of hyperbolic field equations involving the unphysical metric and regular
when the conformal factor $\Theta$ vanishes. Under suitable gauge choices and conditions, the set of
metric conformal field equations yields a hyperbolic system with these properties. In this sense
we can say that the metric conformal field equations are a regular conformal representation
of the Einstein field equations. Suppose that the physical space-time fulfills the vacuum Einstein
equations with cosmological constant $\lambda$
$$\tilde{R}_{ab} = \lambda \tilde{g}_{ab}, \quad \text{(10)}$$
then the vacuum metric conformal equations hold in the unphysical space-time $(\mathcal{M}, g_{ab})$

$$\Sigma_a = \nabla_a \Theta, \quad \text{(11a)}$$
$$\nabla_b \nabla_a \Theta = -\Theta L_{ab} + s g_{ab}, \quad \text{(11b)}$$
$$\nabla_a s = -L_{ab} \nabla^b \Theta, \quad \text{(11c)}$$
$$\nabla_a L_{bc} - \nabla_b L_{ac} = -d_{abcd} \nabla^d \Theta, \quad \text{(11d)}$$
$$\nabla_p d_{abc} = 0, \quad \text{(11e)}$$

together with the constraint
$$\lambda = 6\Theta s - 3 \nabla_a \Theta \nabla^a \Theta. \quad \text{(12)}$$

In the formulation of the conformal field equations we have introduced the Friedrich scalar
$$s \equiv \frac{1}{24} \Theta R + \frac{1}{4} g^{ab} \nabla_b \nabla_a \Theta, \quad \text{(13)}$$
and the rescaled Weyl tensor
$$d_{abcd} \equiv \frac{C_{abcd}}{\Theta}. \quad \text{(14)}$$

The vacuum conformal equations give rise to a hyperbolic system in the unphysical manifold $\mathcal{M}$
for the following variables (see [15, 16])
$$\Theta, \, \Sigma_a, \, s, \, L_{bc}, \, d_{abcd}, \, g_{ab}. \quad \text{(15)}$$

We recall for later use the following result (see Theorem 3.1 of [21])

**Proposition 1.** If
$$\Theta, \, \Sigma_a, \, s, \, L_{bc}, \, d_{abcd}, \, g_{ab}. \quad \text{(16)}$$
is a solution of (11a)-(11e) such that $\Theta \neq 0$ on an open set $\mathcal{U} \subset \mathcal{M}$ and (12) is fulfilled at least
a point $p \in \mathcal{M}$ then the metric $\tilde{g}_{ab}$ is a solution of (10).
2.2 The initial data problem for the metric conformal equations

To prescribe initial data for the conformal equations we follow the standard approach of defining a spacelike Cauchy hypersurface $S \subset M$. $S$ is an embedded Riemannian manifold endowed with a Riemannian metric (we shall use the same symbol $S$ for the Riemannian manifold and its image in $M$ under the embedding if no confusion arises). We define next a foliation $\{S_t\}, t \in I \subset \mathbb{R}$ of the unphysical space-time $M$ such that the leaves $S_t \subset M$ are spacelike hypersurfaces. Furthermore the foliation is chosen in such a way that $S_0 = S$. The foliation can be characterized by any unit integrable timelike vector field $n^a$ defined on $M$ which is orthogonal to the leaves. We can use $n^a$ to introduce the spatial metric

$$h_{ab} \equiv g_{ab} + n_a n_b. \quad (17)$$

The spatial metric enables us to define spatial tensors on $M$ in the standard way. The embedding of $S$ into $M$ sets a one-to-one correspondence between spatial tensors on $M$ and tensor fields on $S$ and for that reason we shall use the same set of abstract indices for tensorial quantities on $S$ as for those in $M$. Indices of tensorial quantities on $S$ are always raised and lowered with the metric $h_{ab}$. A very important spatial tensor is the extrinsic curvature defined by (23) below

$$K_{ab} \equiv h^c_a h^d_b \nabla_c n_d. \quad (18)$$

It can be easily seen that the conditions (18) entail the following relations

$$h_{ab} = \Omega^2 \tilde{h}_{ab}, \quad K_{ab} = \Omega (K_{ab} + \sigma \tilde{h}_{ab}). \quad (19)$$

These relations can be inverted yielding

$$\tilde{h}_{ab} = \frac{h_{ab}}{\Omega^2}, \quad \tilde{K}_{ab} = \frac{K_{ab}}{\Omega} - h_{ab} \frac{\sigma}{\Omega^2}. \quad (20)$$

The indices of spatial tensors can be raised and lowered with a spatial metric. Consistent with our convention for raising and lowering of space-time index tensors, we shall use the spatial metric $h_{ab}$ for index raising and lowering. Again, the only exception to this convention is $\tilde{h}^{ab}$ that is the inverse of $\tilde{h}_{ab}$ and thus we have

$$\tilde{h}^{ab} \equiv \Omega^2 h^{ab}, \quad \tilde{h}^{ac} \tilde{h}_{cb} = h^{ac} h_{cb} = h^a_c. \quad (21)$$

We can take any of the previous vector fields as the starting point to carry out a standard 1+3 decomposition (see [23] and references therein). Since we are working with the conformal field equations which are formulated in terms of the unphysical metric, we choose to carry out the 1+3 decomposition using the unphysical normal $n^a$ and the unphysical spatial metric $h_{ab}$. In this way, and following [5, 6], we introduce the so-called initial data quantities for the conformal
equations defined as follows

\[ h_{ab} \equiv n_a n_b + g_{ab}, \]  
\[ K_{ab} \equiv h_a^c h_b^d \nabla_c n_d, \]  
\[ \Omega, \] scalar function on \( S \) (restriction of \( \Theta \) to the embedded manifold \( S \)).

\( \sigma, \sigma_a \), Scalar and field on \( S \) defined from the orthogonal splitting of \( \Sigma_a \):

\[ \Sigma_a = -n_a \sigma + \sigma_a \Rightarrow \sigma_a = D_a \Omega. \]  
\[ l_{ab} = r_{ab} - \frac{1}{4} h_{ab}, \] Schouten tensor with respect to the Riemannian metric \( h_{ab} \).

Here we have defined the Levi-Civita connection \( D_a \) compatible with the spatial metric \( h_{ab} \) in the standard way. From these fundamental quantities we construct the following derived initial data quantities on \( S \) (we indicate in brackets their correspondence with the space-time tensors)

\[ s \equiv \frac{1}{3} \left( \frac{\Omega}{4}(K^2 + r - K_{ac}K^{ac}) - K^b \sigma + D_b \sigma^b \right), \]  
\[ (\text{restriction of the Friedric scalar to } S), \]  
\[ \theta_{ab} \equiv \frac{1}{\Omega}(sh_{ab} + \sigma K_{ab} - D_b \sigma_a), \]  
\[ (\text{spatial part of } L_{ab}), \]  
\[ \theta_b \equiv \frac{1}{\Omega}(K^a b \sigma_a - D_b \sigma), \]  
\[ (\text{transversal part of } L_{ab}), \]  
\[ d_{ac} \equiv \frac{1}{4\Omega}((K_{ab}K^{db} - K^2)h_{ac} + 4(K_{ac}K - K_a^b K_{bd} + l_{ac} - \theta_{ac})) \]  
\[ (\text{electric part of } d_{abcd}), \]  
\[ d_{bde} \equiv \frac{2}{\Omega}(b_{[c} \theta_{d]} + D_{[d}K_{c]b}), \]  
\[ d^*_{ae} = \frac{1}{2} \eta_{eabcd} d_a \eta^{d} n^c, \]  
\[ (\text{magnetic part of } d_{abcd}). \]

The initial data quantities fulfill the conformal constraint equations \[5, 6, 29\]

\[ \lambda = 6\Omega s + 3\sigma^2 - 3\sigma a \sigma^a, \]  
\[ D_a s = \theta_a \sigma - \theta_a b \sigma^b, \]  
\[ D_a \theta_{ac} - D_b \theta_{ac} = -2\theta_{[a} K_{c]b} + d_{cab} \sigma + 2s_{[a} d_{b]c} - 2d_{[b} h_{a]c} \sigma^d, \]  
\[ D_a \theta_b - D_b \theta_a = d_{cab} \sigma^c, \]  
\[ D_a d^*_{k}^{a} = d^*_{[c} \eta_{kacj} K_{b]} n^a, \]  
\[ D_c d_b^* = -\eta_{bcde} K^{ac} d^* a n^d. \]

Eq. (28a) is the spatial part of (12), (28b) is the spatial part of (11c), (28c)-(28d) are the spatial part of (11d) and (28e)-(28f) are the spatial part of (11e).

Possible initial data for the conformal equations are given by the conformal hyperboloidal initial data sets as introduced in \[21\].

**Definition 1** (vacuum conformal hyperboloidal initial data). Let \((\hat{S}, \hat{h}_{ab}, \hat{K}_{ab})\) a vacuum initial data set (see \[7\]) and define from it the set \( S \) and the quantities

\[ C \equiv (S, h_{ab}, K_{ab}, \Omega, \sigma, \sigma_a, s, \theta_a, \theta_b, d_{ab}, d_{abc}), \]  
\[ in \text{ the manner explained in the previous paragraphs. A vacuum conformal hyperboloidal initial data set is an initial data set in which } S \text{ is a manifold diffeomorphic to the closed unit ball in } \mathbb{R}^3 \text{ whose boundary is denoted by } Z. \text{ One has then that } \hat{S} \text{ is defined by } \hat{S} \equiv S \setminus Z \text{ and also the following additional requirements} \]

1. \( \Omega > 0 \) on \( \hat{S} \).
2. \( \Omega = 0 \) on \( Z \) and if \( \lambda = 0 \) then \( \sigma^2 - \sigma_\alpha \sigma^\alpha = 0, \sigma > 0 \) on \( Z \).

3. The set of quantities \( C \) fulfills the conformal constraint equations (28a)-(28f).

We recall now the following result for a vacuum conformal hyperboloidal initial data set (see [21] for a proof).

**Theorem 1** (Hyperboloidal initial data for the vacuum conformal equations). For a smooth vacuum conformal hyperboloidal initial data set defined by (29) there exists a solution of the conformal equations (11a)-(11e)

\[
\Theta, \quad \Sigma_\alpha, \quad s, \quad L_{bc}, \quad d_{abcd}, \quad g_{ab},
\]

such that \( g_{ab} = \Theta^2 \tilde{g}_{ab} \) where \( \tilde{g}_{ab} \) is the vacuum solution of the Einstein’s equations corresponding to the initial data \((\tilde{S}, \tilde{h}_{ab}, \tilde{K}_{ab})\) used to construct the conformal hyperboloidal initial data set.

In our case, we can simplify the analysis of the conformal constraint equations (28a)-(28f) by means of the following result (see Lemma 11.1 of [29])

**Theorem 2.** The set \( C \) fulfills the conformal constraint equations (28a)-(28f) if and only if \( \{h_{ab}, K_{ab}, \Omega, \sigma, \sigma_\alpha\} \) fulfills the conformal Hamiltonian and momentum constraints on \( S \):

\[
2\lambda = \Omega^2 (r + K^2 - (K_{ab}K^{ab})) - 6(\sigma_\alpha \sigma^\alpha) + 4\Omega(D_a \sigma^a) - 4\Omega K \sigma + 6\sigma^2, \tag{31a}
\]

\[
\Omega(D_b K^b - D_d K^d) = 2(\sigma_a K_{ad} - D_d \sigma_a). \tag{31b}
\]

3 An invariant characterization of the Petrov type D condition

From the physical metric \( \tilde{g}_{ab} \) and its inverse \( \tilde{g}^{ab} \), we define the volume element \( \tilde{\eta}_{abcd} \), the Weyl tensor \( \tilde{W}_{abcd} \), its right dual \( \tilde{W}^*_{abcd} \) and the self-dual Weyl tensor

\[
\tilde{W}^*_{abcd} \equiv \frac{1}{2} (\tilde{W}_{abcd} - i\tilde{W}^*_{abcd}). \tag{32}
\]

We define the physical Weyl scalars

\[
\tilde{a} \equiv \tilde{g}^{ac} \tilde{g}^{be} \tilde{g}^{pf} \tilde{g}^{pd} \tilde{W}_{abpf} \tilde{W}_{cedq}, \tag{33}
\]

\[
\tilde{b} \equiv \tilde{g}^{ai} \tilde{g}^{bj} \tilde{g}^{ce} \tilde{g}^{dg} \tilde{g}^{fj} \tilde{g}^{ah} \tilde{W}_{abcd} \tilde{W}_{egpq} \tilde{W}_{fhij}, \tag{34}
\]

\[
\tilde{w} \equiv \frac{\tilde{b}}{2\tilde{a}}, \tag{35}
\]

and the tensors

\[
\tilde{G}_{abmh} \equiv \tilde{g}_{am} \tilde{g}_{bh} - \tilde{g}_{ah} \tilde{g}_{bm}, \quad \tilde{G}_{abcd} \equiv \frac{1}{2} (-i\tilde{\eta}_{abcd} + \tilde{G}_{abcd}). \tag{36}
\]

We recall the following results from [11, 12, 24]

**Theorem 3.** The physical spacetime \((\tilde{M}, \tilde{g}_{ab})\) is of “genuine” Petrov type D (Petrov type D, but not any of its specializations) if and only if

\[
\tilde{a} \neq 0, \quad \tilde{D}_{abcd} = 0, \tag{37}
\]

where

\[
\tilde{D}_{abhe} \equiv \tilde{W}_{abcd} \tilde{g}^{ce} \tilde{g}^{dp} \tilde{W}_{pqhe} - \frac{\tilde{a}}{6} \tilde{g}_{abhe} - \frac{\tilde{b}}{\tilde{a}} \tilde{W}_{abhe}. \tag{38}
\]
In this work we shall only be concerned with those Petrov type D solutions characterized by Theorem 3.

**Proposition 2.** Under the conditions of Theorem 3 and assuming that (10) holds, we have that the 1-form \( \tilde{\xi}_a \) defined by the equation

\[
\tilde{\Xi}^a_{bc} = \tilde{\phi} \tilde{\xi}_a \tilde{\xi}_c ,
\]

fulfills the condition

\[
\tilde{\nabla}_a \tilde{\xi}_b + \tilde{\nabla}_b \tilde{\xi}_a = 0 ,
\]

where

\[
\tilde{\phi} \equiv \frac{27}{2} \tilde{w}^{11/3} ,
\]

\[
\tilde{\Xi}_{ac} \equiv (\tilde{g}^{bp} \tilde{g}^{dq} \nabla_{a} \tilde{w} \tilde{W}^{pq}_{cd} - \tilde{w} \tilde{\nabla}_{a} \tilde{W}^{pq}_{cd} ) \tilde{\nabla}_b \tilde{w} \tilde{\nabla}_d \tilde{w}
\]

All the previous results have a counterpart formulated with respect to the unphysical metric \( g_{ab} \). To find the corresponding formulations we need to make similar definitions for the symbols used in Theorem 3 and Proposition 2, but now using the unphysical metric instead of the physical one. The notation for the new symbols so defined is obtained by just removing the tildes over the symbols used in the physical space-time.

### 3.1 Conformal rescaling of the Petrov type D conditions in vacuum

**Proposition 3.** The relation between the physical quantities defined in Theorem 3 and Proposition 2 and the corresponding unphysical ones is given by

\[
\tilde{\alpha} = \Theta^6 \alpha ,
\]

\[
\tilde{\beta} = \Theta^9 \beta ,
\]

\[
\tilde{\omega} = \Theta^{3} \omega ,
\]

\[
\tilde{G}_{abcd} = \frac{G_{abcd}}{\Theta^4} ,
\]

\[
\tilde{d}_{abcd} = \Theta W_{abcd} ,
\]

\[
\tilde{D}_{abcd} = \Theta^2 \tilde{D}_{abcd} ,
\]

\[
\tilde{\phi} = \Theta^{11} \phi ,
\]

\[
\tilde{\Xi}_{ac} = \Theta^3 (\tilde{d}_{a} \tilde{b} \tilde{c} \tilde{d} - \tilde{G}_{a} \tilde{b} \tilde{c} \tilde{d}) \tilde{\nabla}_b \tilde{w} \tilde{\nabla}_d \tilde{w} = \Theta^3 \Xi_{ac} ,
\]

where we have defined

\[
\tilde{d}_{abcd} \equiv \frac{1}{2} (d_{abcd} - i d^*_{abcd}) ,
\]

\[
\tilde{D}_{abcd} \equiv \tilde{d}_{ab} \tilde{d}_{pq} - a \tilde{G}_{abcd} - b \tilde{d}_{abcd} ,
\]

\[
\Xi_{ac} \equiv (\tilde{d}_{a} \tilde{b} \tilde{c} \tilde{d} - \tilde{w} \tilde{G}_{a} \tilde{b} \tilde{c} \tilde{d}) \tilde{\nabla}_b \tilde{w} \tilde{\nabla}_d \tilde{w} .
\]

**Proof.** This is a straightforward computation carried out by using the relations (1), (3), (4), (9) and (14).
Remark 1. We note that $\Xi_{ac}$ is a concomitant of both the unphysical metric $g_{ab}$ and the conformal factor $\Theta$. It can be written in the following form

$$
\Xi_{ac} = \Theta^6 (\Xi^0)_{ac} + (\delta^b_a b^d - w g^b_a b^d)(3w\Theta^5 (\nabla_a \Theta \nabla_d w + \nabla_d \Theta \nabla_a w) + 9w^2 \Theta^4 \nabla_b \Theta \nabla_d \Theta), 
$$

(47)

where

$$(\Xi^0)_{ac} \equiv (\delta^b_a b^d - w g^b_a b^d)\nabla_b w \nabla_d w.$$

(48)

The tensor $(\Xi^0)_{ac}$ is a concomitant of the unphysical metric $g_{ab}$ only.

Next we introduce the rescaled Killing 1-form

$$\xi^a \equiv \Theta^2 \tilde{\xi}^a.$$

(49)

By construction $\xi^a$ is a conformal Killing vector in the unphysical spacetime. Combining this with (43i),(39) and (43h) we deduce

$$\Xi_{ac} = \Theta^4 \varphi \xi^a \xi^b.$$

(50)

Theorem 4. At those points where $\Theta \neq 0$ the unphysical spacetime $(M, g_{ab})$ is conformal to a Petrov type D physical spacetime if and only if $D_{abcd} = 0$.

Proof. This is a direct consequence of eq.(43g) and Theorem 3. $\square$

4 Construction of type D conformal initial data

We recall the standard construction of an initial data set for the physical vacuum Einstein equations $(\tilde{M}, \tilde{g}_{\mu\nu})$.

Theorem 5. Let $(\tilde{S}, \tilde{h}_{ij})$ be a Riemannian manifold and define $\tilde{h}^{ij}$ as the inverse of $\tilde{h}_{ij}$ Suppose that there exists a symmetric tensor field $\tilde{K}_{ij}$ on $\tilde{S}$ which satisfies the conditions (vacuum constraints)

$$
\tilde{r} + \tilde{K}^2 - \tilde{h}^{ij} \tilde{K}_{pq} \tilde{K}_{ij} = 2\lambda, 
$$

(51)

$$
\tilde{h}^{ij} \tilde{D}_p \tilde{K}_{iq} - \tilde{D}_i \tilde{K} = 0, 
$$

(52)

where $\tilde{K} \equiv \tilde{h}^{ij} \tilde{K}_{ij}$ and $\tilde{D}_i$ is the covariant derivative compatible with $\tilde{h}_{ij}$. Provided that $\tilde{h}_{ij}$ and $\tilde{K}_{ij}$ are smooth, there exists an isometric embedding $\phi$ of $\tilde{S}$ into a globally hyperbolic, vacuum solution $(\tilde{M}, \tilde{g}_{\mu\nu})$ of the Einstein field equations

$$
\tilde{R}_{ab} = \lambda \tilde{g}_{ab}, \quad \lambda \in \mathbb{R}. 
$$

(53)

The set $(\tilde{S}, \tilde{h}_{ij}, \tilde{K}_{ij})$ is then called a vacuum initial data set and the spacetime $(\tilde{M}, \tilde{g}_{\mu\nu})$ is the data development. Furthermore the spacelike hypersurface $\phi(\tilde{S})$ is a Cauchy hypersurface in $(\tilde{M}, \tilde{g}_{\mu\nu})$.

4.1 Killing initial data equations and their conformal rescaling

The following definition has been taken from [4].

Definition 2. Two tensor fields $\tilde{Y}$ and $\tilde{Y}_a$ fulfill the Killing Initial Data (KID) conditions on $(\tilde{S}, \tilde{h}_{ab})$ if and only if:

$$
-2 \tilde{Y} \tilde{K}_{ab} + \tilde{D}_a \tilde{Y}_b + \tilde{D}_b \tilde{Y}_a = 0, 
$$

(54a)

$$
\lambda \tilde{h}_{ab} - (\tilde{h}^{cd} \tilde{K}_{ab} \tilde{K}_{cd} - 2 \tilde{h}^{cd} \tilde{K}_{ac} \tilde{K}_{db} + \tilde{r}_{ab}) \tilde{Y} + \tilde{D}_b \tilde{D}_a \tilde{Y} - L_\tilde{Y} \tilde{K}_{ab} = 0. 
$$

(54b)
For us the relevance of the KID equations is that any pair of tensor fields \((\tilde{Y}, \tilde{Y}_a)\) solving them gives rise to a Killing vector \(\tilde{\zeta}^a\) in the physical space-time \((\tilde{\mathcal{M}}, \tilde{g}_{ab})\). The orthogonal splitting of \(\tilde{\zeta}^a\) with respect to any \(\mathcal{S}\)-normal unit timelike vector field \(\tilde{n}^a\) is

\[
\frac{\tilde{\zeta}^a}{\tilde{\zeta}_a} = \tilde{Y} \tilde{N}_a + \tilde{Y}_a, \quad \tilde{Y} = -\tilde{N}_a \tilde{\zeta}^a, \quad \tilde{Y}_a = \tilde{h}_{ab} \tilde{\zeta}_b, \quad \tilde{N}_a = \tilde{g}_{ab} \tilde{\zeta}^b, \quad \tilde{\zeta}_a = \tilde{g}_{ab} \tilde{\zeta}^b.
\]

(55)

See [4, 8, 36] for proofs of the above statements.

**Remark 2.** If the orthogonal splitting of the 1-form \(\tilde{\zeta}_a\) with respect to the physical metric is given by

\[
\frac{\tilde{\zeta}_a}{\tilde{\zeta}_a} = \tilde{Y} \tilde{N}_a + \tilde{Y}_a,
\]

(56)

then using \([18], [19]\) we deduce that its orthogonal splitting with respect to the unphysical metric can be written as

\[
\frac{\tilde{\zeta}_a}{\tilde{\zeta}_a} = \frac{Y}{\Theta^2} n_a + \frac{Y_a}{\Theta^2},
\]

(57)

where

\[
Y = -n_a \tilde{\zeta}^a = \Theta \tilde{Y}, \quad Y_a = h_{ab} \tilde{\zeta}^b = \Theta^2 \tilde{Y}_a.
\]

(58)

When pull-backed to \(\mathcal{S}\), these relations translate into the rescalings displayed at \([59]\) below on those points of \(\mathcal{S}\) where \(\Theta\) is different from zero.

**Lemma 1.** Let \((Y, Y_a)\) be tensors on \(\mathcal{S}\) and introduce the rescalings

\[
\tilde{Y} = \frac{Y}{\Theta}, \quad \tilde{Y}_a = \frac{Y_a}{\Theta^2},
\]

(59)

where \(\Omega\) is a differentiable function different from zero on \(\tilde{\mathcal{S}} \subset \mathcal{S}\). Then the tensors \(Y, Y_a\) fulfill the following conditions on \(\mathcal{S}\)

\[
\Omega(D_a Y_b + D_b Y_a) + 2 h_{ab} \sigma Y - \Omega Y^c D_c \Omega = 2 \Omega K_{ab} Y, \quad \Omega^4 K^c_a K^b_c Y - r_{ab} \Omega^4 Y - 4 \sigma \Omega h_{ab} D_c Y + 2(2 \Omega Y^c D_a \Omega - \Omega^2 D_a Y^c) K_{bc} + 2 \Omega \sigma D_a Y_b + \Omega K_{ab} (-\Omega^3 Y K^c - \Omega^2 \sigma Y + Y^c D_c \Omega) - 2 \Omega^4 Y D_b D_a \Omega + \Omega^2 D_b Y D_c) K_{ab} + h_{ab} \left( Y^c (\Omega D_a \sigma - 2 \sigma D_a \Omega + \Omega^2 (\sigma Y K^c - (Y D_a D_c \Omega + D_a Y D^c)) + \Omega^2 Y (\lambda - \sigma^2 + 3 D_c D^c \Omega) \right) = 0,
\]

(60a)

(60b)

if and only if the rescaled tensors \(\tilde{Y}, \tilde{Y}_a\) satisfy the KID conditions \([54a]-[54b]\) on \(\tilde{\mathcal{S}}\) for \(\tilde{h}_{ab}, \tilde{h}^a{}_{b}, K_{ab}\) defined by \([20]\).

**Proof.** The relation between the Levi-Civita connections \(\tilde{D}_a\) and \(D_a\), compatible with the respective Riemannian metrics \(h_{ab}, \tilde{h}_{ab}\), can be expressed in terms of the Christoffel tensor arising from the difference between the two connections:

\[
\Gamma[D]_c{}^{e}{}_{ac} - \Gamma[\tilde{D}]_c{}^{e}{}_{ac} = \frac{1}{\Theta}(\delta_c{}^e D_a \Theta - h_{ca} h^{eb} D_b \Theta + \delta_a{}^e D_c \Theta).
\]

(61)

Using this tensor it is possible to express any covariant derivative with respect to \(\tilde{D}_a\) in terms of \(D_a\) and vice-versa. Also the relations between the Ricci tensor of the connection \(\tilde{D}_a\) and the Ricci tensor of the connection \(D_a\) can be computed

\[
\tilde{r}_{ac} = r_{ac} - \frac{2}{\Theta^2} h_{ac} D_b \Theta D^b \Theta + \frac{1}{\Theta}(h_{ac} D_b D^b \Theta + D_c D_a \Theta),
\]

(62)

\[
r_{ac} = \tilde{r}_{ac} + \frac{2}{\Theta^2} \tilde{D}_a \Theta \tilde{D}_c \Theta - \frac{1}{\Theta}(\tilde{D}_c \tilde{D}_a \Theta + \tilde{h}^{bd} \tilde{h}_{ac} \tilde{D}_b \tilde{D}_d \Theta).
\]

(63)
Using the last of these relations together with (19) and (59) in (60a)-(60b) leads us to (54a)-(54b) after long algebra. Reciprocally, if we invert (59) to express \( \tilde{Y} \), \( \tilde{Y}_a \) in terms of \( Y \), \( Y_a \) and use (20) and (62) in (54a)-(54b) we get (60a)-(60b).

4.2 Vacuum conformal type D initial data

The tensors \( \partial_{abcd} \) and \( D_{abcd} \) are complex self-dual Weyl candidates. This means that they have the same algebraic properties as the Weyl tensor and this makes it possible to obtain their orthogonal splitting from a general formula involving the electric part of the corresponding Weyl candidate (see e.g. [24])

\[
\partial_{abcd} = \mathcal{E}_{bd}(h_{ac} + n_a n_c) - \mathcal{E}_{ad}(h_{bc} + n_b n_c) + \mathcal{E}_{ac}(h_{bd} + n_b n_d) - \mathcal{E}_{bc}(h_{ad} + n_a n_d) - \eta_{cde} \mathcal{E}^*_{b} n_a n^e - \eta_{cde} \mathcal{E}^*_{b} n_b n^e + \eta_{cde} \mathcal{E}^*_{d} n_c n^e + \eta_{cde} \mathcal{E}^*_{d} n_d n^e,
\]

(64)

\[
D_{abcd} = \mathcal{A}_{bd}(h_{ac} + n_a n_c) - \mathcal{A}_{ad}(h_{bc} + n_b n_c) + \mathcal{A}_{ac}(h_{bd} + n_b n_d) - \mathcal{A}_{bc}(h_{ad} + n_a n_d) - \eta_{cde} \mathcal{A}^*_{b} n_a n^e + \eta_{cde} \mathcal{A}^*_{b} n_b n^e - \eta_{cde} \mathcal{A}^*_{d} n_c n^e + \eta_{cde} \mathcal{A}^*_{d} n_d n^e.
\]

(65)

Where we have defined

\[
\mathcal{E}_{ab} \equiv \partial_{apbq} n^p n^q = \frac{1}{2}(d_{ab} - id_{ab}) ,
\]

(66)

\[
\mathcal{A}_{ab} \equiv D_{apbq} n^p n^q .
\]

(67)

Since the scalars \( a \), \( b \) and \( w \) are defined from \( \partial_{apbq} \) they can be rendered in terms of \( \mathcal{E}_{ab} \)

\[
a = 16 \mathcal{E}_{ab} \mathcal{E}^{ab} ,
\]

(68)

\[
b = -64 \mathcal{E}_{a} \mathcal{E}^{ab} \mathcal{E}_{bc} ,
\]

(69)

\[
w \equiv \frac{-b}{2a} .
\]

(70)

Using eqs. (27d) and (27f) in (66) we can find the expression of all these scalars in terms of quantities intrinsic to the initial data hypersurface.

**Proposition 4.**

\[
D_{abcd}|_S = 0 \iff \mathcal{S}_{ab} = 0 ,
\]

(71)

where

\[
\mathcal{S}_{ab} = \frac{a}{12} h_{ab} - \frac{b}{a} \mathcal{E}_{ab} - 4 \mathcal{E}_{a} \mathcal{E}_{b} - \frac{b}{a} \mathcal{E}_{b} .
\]

(72)

**Proof.** Equation (71) is a direct consequence of equation (65) whereas (72) results from inserting the splitting of \( \partial_{abcd} \) given by (64) into the definition of \( D_{abcd} \) stated by (45) and then combining the result with (65).

We define now the following quantities, which can be regarded as, respectively, a scalar and a tensor defined on \( S \)

\[
w^\perp \equiv \frac{b D_{ab} a - a D_{ab} b}{2a^2} ,
\]

(73)

\[
w^\parallel \equiv -\frac{6K_{bc}}{a^3}(b^2 \mathcal{E}_{bc} + a(b a_{bc} - 12 a \mathcal{E}_{a} \mathcal{E}_{bc} \mathcal{E}_{cd}) - \frac{ab K^b_{bc} - 16i\varepsilon_{cde}(b \mathcal{E}^{bc} + 3a a_{bc}) D^e \mathcal{E}_{b}^d}{2a^2} ,
\]

(74)

where

\[
\varepsilon_{abc} \equiv \eta_{dabc} n^d .
\]

(75)
Lemma 2.
\[ \nabla_a w = n_a w^\parallel + w^\perp_a. \] (76)

Proof. From \([68, 70]\) we deduce that \(w\) can be rendered exclusively in terms of scalars formed with \(\mathcal{E}_{ab}\). Therefore to compute the orthogonal splitting of \(\nabla_a w\) we need to compute first the orthogonal splitting of \(\nabla_a \mathcal{E}_{bc}\). The latter turns out to be
\[
\nabla_a \mathcal{E}_{bc} = (2K_{(c}^d n_{a)} + A^d n_a n_c)\mathcal{E}_{bd} + (2K_{(h}^d n_{a)} + A_d n_a n_b)\mathcal{E}_{cd} + D_a \mathcal{E}_{bc} - n_a \mathcal{L}_n \mathcal{E}_{bc},
\] (77)
where \(A^a \equiv n^a \nabla_a n^b\). The last term of \((77)\) can be further worked out using the orthogonal splitting of \([11e]\) which decomposes into the standard evolution and constraint equations
\[
\mathcal{L}_a \mathcal{E}_{cp} = -2K_{b}^c \mathcal{E}_{cp} + 2ia^b \mathcal{E}_{(c}^d \mathcal{E}_{p)bd} - i\epsilon_{(c}^bd D_{|b|} \mathcal{E}_{p)d} - h_{cp} K^{bd} \mathcal{E}_{bd} + 5 K_{(c}^b \mathcal{E}_{p)b},
\] (78)
\[
D_b \mathcal{E}^b_a = -i\epsilon_{acd} K^{bc} \mathcal{E}^d_b.
\] (79)
Using these results in \(\nabla_a w\), eq. \((76)\) follows after some manipulations. \(\Box\)

Proposition 5. If \(\tilde{\eta}_a\) is a covector in \(\mathcal{M}\) defined on an open set containing \(\tilde{S}\) such that its orthogonal splitting with respect to \(n_a\) and \(h_{ab}\) is given by
\[
\tilde{\eta}_a = \frac{Y}{\Theta^2} n_a + \frac{Y_a}{\Theta^3},
\] (80)
then
\[
(\tilde{\Xi}_{ac} - \phi \tilde{\eta}_a \tilde{\eta}_c)|_{\tilde{S}} = 0 \iff Y_a Y_b = \frac{Q_{ab}}{\Omega^4},
\] (81)
where in the previous equation, \(Y, Y_a\) and \(Q_{ab}\) are understood as quantities defined on \(S\) through the relations
\[
Y^2 = \frac{1}{2\phi} \left( w(3\sigma^b w(6\sigma^d E_{bd} + 3\sigma_b w + 2\Omega w^\perp_b) + \Omega^2 w^\perp_b w^\perp_b) + 2\Omega E_{bd}(6\sigma^b w + \Omega w^\perp_b) w^\perp_d \right),
\] (83)
\[
Y_c = -\frac{\Omega}{2Y\phi} \left( (3\sigma w - \Omega w^\parallel)(6\sigma^b E_{cb} w + 3\sigma_c w^2 + 2\Omega E_{cb} w^\perp_b + \Omega w w^\perp_c) + 6i\epsilon_{cba} \sigma^b E_d^a w w^\perp_d + 2i\epsilon_{cba} \sigma^b \left( \Omega^2 w^\perp_b w^\perp_d + 3\sigma^b w(3\sigma^d w + \Omega w^\perp_d) \right) \right),
\] (84)
\[
\phi Q_{ac} = \frac{1}{2} \Omega^4 (2E_{ac} + h_{ac} w)(-3\sigma w + \Omega w^\parallel)^2 - i\Omega^4 (\epsilon_{cbd} E_d^a + \epsilon_{abd} E_c^d)(-3\sigma w + \Omega w^\parallel)(3\sigma^b w + \Omega w^\perp_b) + (h_{bd} E_{ac} - h_{b}^c E_{a}^d - h_{a}^d E_{c}^b - \frac{1}{2} h_{a}^d h_{b}^c w + h_{ac}(E_{bd} - \frac{1}{2} h_{bd} w)) \times (3\Omega^2 \sigma_b w + \Omega^2 w^\perp_b)(3\Omega^2 \sigma_d w + \Omega^3 w^\perp_d).
\] (85)

Proof. To find out the conditions arising from \((81)\) we need to find the orthogonal splitting of the tensor
\[
\tilde{T}_{ab} \equiv \tilde{\Xi}_{ac} - \phi \tilde{\eta}_a \tilde{\eta}_c,
\] (86)
and set each of its spatial parts to zero. This is a straightforward albeit tedious computation that requires the following steps:
• computation of the orthogonal splitting of $\tilde{\eta}_a$

$$\tilde{\eta}_a = \tilde{U}n_a + \tilde{U}_a.$$  \hfill (87)

• Computation of the orthogonal splitting of $\nabla_a w$ (see lemma \ref{lemma:splitting})

$$\nabla_a w = n_a w^\parallel + w^\perp_a.$$  \hfill (88)

• The restriction of eq. \ref{eq:restriction} to $S$

$$\tilde{\varphi}|_S = \Omega^{11}\varphi|_S.$$  \hfill (89)

• Eq. \ref{eq:theta}

$$\nabla_a \Theta = -n_a \sigma + \sigma_a$$

• Eq. \ref{eq:43}

Next one uses eq. \ref{eq:47} on eq. \ref{eq:43} and carries out in the resulting expression the steps described above. This yields the orthogonal splitting of $\tilde{\Omega}_{ab}$ which is then used in \ref{eq:86} to find the orthogonal splitting of $\tilde{T}_{ab}$. We write such orthogonal splitting in the form

$$\tilde{T}_{ab} = A_n a_n b + B_{(a} n_{b)} + C_{ab},$$  \hfill (90)

where $A$, $B_a$, $C_{ab}$ are spatial and known. Thus $\tilde{T}_{ab}|_S = 0$ if and only if $A = 0$, $B_a = 0$ and $C_{ab} = 0$.

• The condition $A = 0$ corresponds to \ref{eq:83} if we set

$$\tilde{U} = \frac{Y}{\Omega^2}.$$  \hfill (91)

• The condition $B_a = 0$ corresponds to \ref{eq:84} if we set

$$\tilde{U}_a = \frac{Y_a}{\Omega^2}.$$  \hfill (92)

• The condition $C_{ab} = 0$ corresponds to \ref{eq:85}.

\hfill \Box

4.3 The main results

We present next theorems \ref{thm:6} and \ref{thm:7} which are the main results of the paper (see \ref{fig:main_results} for a graphical depiction of these results).

**Theorem 6.** Let $(\tilde{S}, \tilde{h}_{ab}, \tilde{K}_{ab})$ be a smooth $\lambda$-vacuum initial data set and consider a smooth conformal initial data set constructed from it (see Definition \ref{def:conformal_data_set})

$$C \equiv (S, h_{ab}, K_{ab}, \Omega, \sigma, \sigma_a),$$  \hfill (93)

fulfilling the conformal Hamiltonian and momentum constraints \ref{eq:31a}-\ref{eq:31b}, where $\tilde{S} \subset S$. Use the data of $C$ to define on $S$ the quantities $s$, $\theta_a$, $\theta_{ab}$, $d_{ab}$, $d^{*}_{ab}$, $d_{abc}$ according to the prescriptions laid by \ref{eq:27a}-\ref{eq:27b}. From these quantities, we define on $S$ the tensors $\mathcal{E}_{ab}$, $Y$, $Y_a$ using resp. \ref{eq:66}, \ref{eq:83}-\ref{eq:84}. Assume further that on $S$

1. $\mathcal{E}_{ab}$ , $h_{ab}$, are subject to the algebraic condition (see \ref{eq:72a})

$$a_{ab} = 0,$$  \hfill (94)
Figure 1: Summary of the vacuum type D conformal initial data characterization found in theorems 6 and 7.

2. $Y, Y_a$ are subject to the algebraic condition (82)

$$Y_a Y_b = \frac{Q_{ab}}{\Omega^4},$$

and to the differential conditions (60a)-(60b) on $S$. Moreover, $Y \neq 0$ on $\tilde{S}$.

Then, there is an open subset contained in the data development of $C$ where $(M, g_{ab})$ is conformally related to a $\lambda$-vacuum Petrov type D solution of the Einstein field equations corresponding to the data development of $(\tilde{S}, \tilde{h}_{ab}, \tilde{K}_{ab})$.

**Proof.** The fact that the data $C$ is a solution of the conformal Hamiltonian and momentum constraints implies according to Theorems 1 and 2 that a solution of the conformal equations (11a)-(11e) exists such that $(M, g_{ab})$ is conformal to a vacuum solution $(\tilde{M}, \tilde{g}_{ab})$ of the Einstein’s equations (see Proposition 1). Furthermore Theorem 1 tells us that $(\tilde{M}, \tilde{g}_{ab})$ arises from the vacuum initial data $(\tilde{S}, \tilde{h}_{ab}, \tilde{K}_{ab})$.

Now, the condition $a_{ab} = 0$ on $S$ entails, via Proposition 1 that $D_{abcd} \tilde{s} = 0$ which, by (43g), leads to $\tilde{D}_{abcd} \tilde{g} = 0$. Thus, it remains to show that $\tilde{D}_{abcd} = 0$ on an open subset of $(M, g_{ab})$ contained in the data development of $C$. To that end we use (39) to introduce the quantities $\tilde{Y}, \tilde{Y}_a$ on $S$. The differential conditions (60a)-(60b) imply, according to Lemma 1 that $\tilde{Y}, \tilde{Y}_a$ fulfill the KID conditions (54a)-(54b) and thus there exists a Killing 1-form $\tilde{\xi}_a$ in $(M, \tilde{g}_{ab})$ with the properties displayed by (55). Now, Remark 2 tells us that this Killing 1-form fulfills condition (80) in some neighborhood of $S$ (therefore a neighborhood of $\tilde{S}$ as $\tilde{S} \subset S$) and since, by assumption, (83)-(84) and (82) are fulfilled on $S$, we can apply Proposition 5 taking as the covector $\tilde{\eta}_a$ the Killing 1-form $\tilde{\xi}_a$ and conclude that

$$(\tilde{\xi}_{ac} - \tilde{\varphi}_{\tilde{\xi}_a} \tilde{\xi}_c)|_S = 0.$$

The proof is now similar to that of Theorem 6 in [24] but we provide here the details for the sake of completeness: the Killing property of $\tilde{\xi}_a$ automatically yields

$$L_{\tilde{\xi}} \tilde{D}_{abcd} = 0, \quad L_{\tilde{\xi}} (\tilde{\xi}_{ac} - \tilde{\varphi}_{\tilde{\xi}_a} \tilde{\xi}_c) = 0.$$

(97)
These equations can be regarded as a linear system for the variables $\tilde{D}_{abcd}$ and $\tilde{\Xi}_{ac} - \varphi \tilde{\xi}_a \tilde{\xi}_c$ with initial data given by (96) and $\tilde{D}_{abcd}|_{\tilde{S}} = 0$. The data of the system are trivial and non-characteristic given that $\tilde{Y} \neq 0$ (the characteristic points of the system are those in which $\tilde{\xi}_a$ is tangent to $\tilde{S}$). Hence we conclude that there is an open subset $U \subset \tilde{M}$ containing $\tilde{S}$ where one has

$$\tilde{D}_{abcd} = 0, \quad \tilde{\Xi}_{ac} - \varphi \tilde{\xi}_a \tilde{\xi}_c = 0, \quad \tilde{\nabla}_a \tilde{\xi}_b + \tilde{\nabla}_b \tilde{\xi}_a = 0.$$  \hspace{1cm} (98)

The second and third equations are actually redundant and can be dropped (see [24] for more details). Thus $(\tilde{M}, \tilde{g}_{ab})$ is of Petrov type D in the open set $U$ that is contained in the data development of $C$ as $\tilde{M} \subset M$.

Theorem 7 admits a converse that is formulated next.

**Theorem 7.** The initial data of any solution of the conformal equations (11a)-(11e) conformal to a vacuum type D solution of the Einstein equations must comply with points 1 and 2 of Theorem 6.

**Proof.** To prove this theorem, let us suppose that we have a solution $(M, g_{ab})$ of the conformal equations (11a)-(11e) arising from an initial data set as described by (93). By assumption this solution is conformal to a vacuum type D solution of the Einstein field equations $(\tilde{M}, \tilde{g}_{ab})$ according to the relation (1). This implies, according to Theorem 3 that $\tilde{D}_{abcd}$ vanishes on the physical space-time $(\tilde{M}, \tilde{g}_{ab})$ and hence from (43a) we have that $D_{abcd} = 0$ in the un-physical space-time $(M, g_{ab})$ whenever $\Theta \neq 0$. Combining this with Proposition 4 leads to point 1 of Theorem 6.

To show that point 2 holds too, we appeal to Proposition 2 to deduce the existence of a Killing 1-form $\tilde{\xi}_a$ which according to Remark 2 has the following orthogonal splittings in the physical and the unphysical space-times

$$\tilde{\xi}_a = \tilde{Y} \tilde{N}_a + \tilde{Y}_a, \quad \tilde{\xi}_a = \frac{Y}{\Theta^2} n_a + \frac{Y_a}{\Theta^2}.$$  \hspace{1cm} (99)

Since on $S$, the variables $\tilde{Y}, \tilde{Y}_a, Y, Y_a$ are related in the way shown by (58) and $\tilde{Y}, \tilde{Y}_a$ fulfill (54a)-(54b) due to the fact that $\tilde{\xi}_a$ is a Killing 1-form, then, by Lemma 1, $Y, Y_a$ fulfill (60a)-(60b) on $S$. Moreover, from Proposition 5 we find that $Y, Y_a$ have on $S$ the values given by (83)-(85). Therefore, combining these results, we finally conclude that point 2 also holds.

Theorems 6 and 7 provide necessary and sufficient conditions which must be satisfied by a conformal initial data set $C$ of the conformal equations in order that the unphysical space-time $(M, g_{ab})$ be conformal to a vacuum type D solution of the Einstein’s field equations. This result does not state anything about the existence of actual data fulfilling the given conditions and this is in fact an independent open problem (see section 6 for more details). Compare this with the similar problem of the generic existence of hyperboloidal data for the conformal equations [2, 11, 3, 28].

### 5 Conformal initial data for the Kerr solution

The following result was proven in [13] but we adopt the formulation presented in [24, 27].

**Theorem 8.** Under the conditions of Theorem 5 a vacuum ($\lambda = 0$) space-time $(M, \tilde{g}_{ab})$ is locally isometric to the Kerr solution with non-vanishing mass (non-trivial Kerr solution) if and only if

---

1The reasoning of [23] was formulated for the case with $\lambda = 0$ but it still holds when $\lambda \neq 0$.
the following additional conditions hold

\[ \tilde{\xi}_a \tilde{\xi}_b = 0 \iff \tilde{\zeta}_a \tilde{\zeta}_b = 0 \] \hspace{1cm} (100)

\[ \text{Im}(\tilde{Z}^3(\tilde{w}^*)^8) = 0, \quad \tilde{Z} \equiv \tilde{g}^{ab} \tilde{\nabla}_a \tilde{w} \tilde{\nabla}_b \tilde{w}, \] \hspace{1cm} (101)

\[ \frac{\text{Re}(\tilde{Z}^3(\tilde{w}^*)^8)}{(18\text{Re}(\tilde{w}^3 \tilde{Z}^*) - |\tilde{Z}|^2)^3} < 0, \quad \text{if} \quad 18\text{Re}(\tilde{w}^3 \tilde{Z}^*) - |\tilde{Z}|^2 \neq 0, \] \hspace{1cm} (102)

\[ \text{Re}(\tilde{Z}^3(\tilde{w}^*)^8) = 0 \iff \tilde{g}^{ab} \tilde{\xi}_a \tilde{\xi}_b = 0, \quad \text{if} \quad 18\text{Re}(\tilde{w}^3 \tilde{Z}^*) - |\tilde{Z}|^2 = 0, \] \hspace{1cm} (103)

where \( \tilde{\xi} \) is defined by (39).

**Theorem 9.** There exists an open subset of the unphysical spacetime \((\mathcal{M}, g_{ab})\) that is locally conformal to the non-trivial Kerr solution if and only if

\[ D_{abcd} = 0, \] \hspace{1cm} (104a)

\[ \xi_{[a} \xi^*_{b]} = 0 \iff \zeta_{[a} \zeta^*_{b]} = 0, \] \hspace{1cm} (104b)

\[ \text{Im}(Z^3(w^*)^8) = 0, \quad Z \equiv g^{ab} \lambda_a \lambda_b, \quad \lambda_a \equiv 3w \nabla_a \Theta + \Theta \nabla_a w, \] \hspace{1cm} (104c)

\[ \frac{\text{Re}(Z^3(w^*)^8)}{\left(18\Theta^3 \text{Re}(w^3 Z^*) - |Z|^2\right)^3} < 0, \quad \text{if} \quad 18\Theta^3 \text{Re}(w^3 Z^*) - |Z|^2 \neq 0, \] \hspace{1cm} (104d)

\[ \text{Re}(w^3 Z^*) = 0 \iff g^{ab} \xi_a \xi_b = 0, \quad \text{if} \quad 18\Theta^3 \text{Re}(w^3 Z^*) - |Z|^2 = 0, \] \hspace{1cm} (104e)

where \( \xi \) is defined by (49).

**Proof.** This is a straightforward computation involving the relations found in Proposition 3 and their replacement in (100)-(103).

**Theorem 10.** Under the hypotheses of Theorem 9 and whenever the conformal factor \( \Omega \) does not vanish, the initial data set \( C \) are data for a spacetime conformal to the non-trivial Kerr solution (unphysical Kerr spacetime) if and only if

\[ \text{Im}(YY^*_{[a} = 0, \quad Y_{[a} Y_{b]}^* = 0, \] \hspace{1cm} (105)

and conditions (104c)-(104e) of Theorem 9 hold replacing \( \Theta \) by \( \Omega \) and with the following definition for \( Z \):

\[ Z \equiv -(\Omega w^\| - 3\sigma w)^2 + (3w \sigma_a + \Omega w_a^\perp)(3w \sigma^a + \Omega(w^\perp)^a), \] \hspace{1cm} (106)

where the quantities

\[ a = 16E_{ab}E^{ab}, \] \hspace{1cm} (107)

\[ b = -64E_{a}^{\cdots \cdot \cdots} E^{ab} E_{bc}, \] \hspace{1cm} (108)

\[ w = -\frac{b}{2a}, \] \hspace{1cm} (109)

are now understood as defined from the conformal initial data set \( C \) using (66), (27d) and (27f).

**Proof.** Use (57) and (49) to obtain

\[ \xi_a = Y n_a + Y_a. \] \hspace{1cm} (110)

The combination of this with (104b) leads immediately to the conditions

\[ \text{Im}(YY^*_{[a} = 0, \quad Y_{[a} Y_{b]}^* = 0. \] \hspace{1cm} (111)

Moreover, the variables \( Y, Y_a \) have the values given by (83)-(84) on the initial data hypersurface \( S \) as defined in Theorem 6. Next we use Lemma 2 and (25) to find the orthogonal splitting of \( \lambda_a \).
This enables us to compute the orthogonal splitting of $Z$, thus proving (106). The conclusion of this reasoning is that the conditions of Theorem 9 hold on the initial data hypersurface $S$ which in turn implies that the conditions of Theorem 8 hold on $\tilde{S}$. But now we can follow a procedure similar to the proof of Theorem 6 to show that these conditions actually hold in an open set of the data development of $C$ which contains $\tilde{S}$. This is so because after showing that (98) is true one can enlarge the system (97) with the following set of equations

$$
\mathcal{L}_\xi A = 0, \quad A|_{\tilde{S}} = 0, \quad B|_{\tilde{S}} < 0,
$$

where we use the symbols $A$, $B$ to denote any of the quantities

$$
A = \tilde{\Xi}^{[\alpha}{}_{\beta]}{}_{\gamma}{}_{\delta}, \quad A = \text{Im}(\tilde{Z}^3(\tilde{w}^+)^8), \quad B = \frac{\text{Re}(\tilde{Z}^3(\tilde{w}^+)^8)}{(18\text{Re}(\tilde{w}^3\tilde{Z}^+) - |\tilde{Z}|^2)^3}.
$$

Under our conditions one can conclude that $A = 0$, $B < 0$ on an open set $U$ containing $\tilde{S}$ and so Theorem 8 holds on that open set. Thus $g_{ab}$ is (locally) the Kerr spacetime on $U$ and hence the solution $g_{ab}$ of the data $C$ is conformally related to the Kerr solution.

Theorem 11 admits the following converse.

**Theorem 11.** Under the conditions of Theorem 7 the initial data of any solution of the conformal equations (11a)-(11e) conformal to the Kerr solution, must comply with (105) and conditions (104c)-(104e) of Theorem 9 with $Z$ defined by (106).

**Proof.** Since by assumption the unphysical space-time $(M,g_{ab})$ is conformal to the Kerr solution, then to show that (105) and (104c)-(104e) hold one only needs to find the orthogonal splitting of (104b)-(104e) and pull-back the resulting conditions to the initial data hypersurface $S$. □

### 6 The conformal boundary limit

Let $C$ be a data set for the vacuum conformal equations fulfilling the hypotheses of Theorem 6. We know then that the solution of these data contains a subset that is conformal to a physical type D vacuum solution of the Einstein equations. However, it is unclear at this point if the unphysical solution will be also of type D outside of this subset or its Petrov type will change somewhere (see figure 2). Recall that the Petrov type of a general space-time may change from point to point (see Theorem 7.15 of [40]). In fact the hypotheses of Theorem 6 on the data $C$ will be fulfilled in the region of $S$ that is mapped under conformal rescaling to the physical space-time data defined on $\tilde{S}$ but it is not clear whether conformal data fulfilling the hypotheses will exist outside that region. Therefore the natural question about the existence of a conformal initial data set $C$ meeting the conditions of Theorem 6 outside $\tilde{S}$ arises. In this section we take a hyperboloidal data set $C$ fulfilling the hypotheses of Theorem 6 and take the conformal boundary limit $\Omega \to 0$ of the algebraic and differential conditions comprised by Theorem 6. The result (Theorem 12) is that the limit results in a regular hyperboloidal conformal initial data set at the conformal boundary. That is to say, there are no obstructions to the regular extension of the conditions of Theorem 6 outside $\tilde{S}$ when the data are hyperboloidal. In particular, this implies that the rescaled Weyl tensor $d_{abcd}$ is also of Petrov type D at the conformal boundary for hyperboloidal data and therefore the Petrov type is extended at least to the conformal boundary.

**Theorem 12 (Conformal boundary limit).** Let $C$ be an initial data set for the vacuum conformal equations fulfilling the conditions of Theorem 6 and assume further that $h_{ab}$, $K_{ab}$, $\sigma$, $\sigma_a$, and the quantities defined from them by (27a)-(27f) are all smooth at the conformal boundary $\Omega = 0$. If the data are hyperboloidal then the data set $C$ fulfills all the conditions of Theorem 6 at $\Omega = 0$. □
Proof. At the conformal boundary $\Omega = 0$, the differential conditions (60a)-(60b) reduce to
\[ \sigma Y h_{ab} = 0, \quad 2\sigma Y_b D_a \Omega + \sigma h_{ab} Y^c D_c \Omega = 0. \] (114)

Since the data $C$ are by assumption hyperboloidal and regular at the conformal boundary then the metric $h_{ab}$ is non-degenerate at the conformal boundary, $D_c \Omega \neq 0$ and $\sigma \neq 0$. A straightforward computation shows that (114) reduces to $Y_a = 0$ and $Y = 0$ at $\Omega = 0$. Using this information, (83), (85) reduce to the following respective conditions at $\Omega = 0$.

\[ 2(\sigma^b \sigma^d \mathcal{E}_{bd}) + (\sigma \cdot \sigma \cdot \sigma^b) w = 0, \] (115)
\[ \lim_{\Omega \to 0} Q_{ab}^4 = 2(\sigma \cdot \sigma + \sigma^2) \mathcal{E}_{ab} - 4\sigma (\dot{\sigma}_a) + 4i \sigma \mathcal{E}_{(a} \varepsilon_{b)cd} \sigma^c + w \sigma_a \sigma_b 
+ \left(2\sigma \cdot \dot{\sigma} + w(\sigma^2 - \sigma \cdot \sigma)\right) h_{ab} = 0, \] (116)

where we introduced the quantities
\[ \dot{\sigma}_a \equiv \sigma^b \mathcal{E}_{ab}, \quad (\sigma \cdot \dot{\sigma}) \equiv \sigma_a \dot{\sigma}^a, \quad (\sigma \cdot \sigma) \equiv \sigma_a \sigma^a, \quad (\dot{\sigma} \cdot \dot{\sigma}) \equiv \ddot{\sigma}_a \ddot{\sigma}^a. \] (117)

Combining (115) and (116) yields
\[ \sigma_a \sigma^a = \sigma^2, \] (118)
so we have full consistency with the hyperboloidal property of the data. Using this condition in the contraction of (116) with $\sigma^b$ we get after using again (115)
\[ (2\dot{\sigma}_a + w \sigma_a) \sigma + 2i \varepsilon_{acd} \dot{\sigma}^d \sigma^c = 0. \]

The previous equation implies that
\[ 2\dot{\sigma}_a + w \sigma_a = 0. \] (119)
Using this condition, (116) becomes
\[ 4\sigma^2 E_{ab} + 4i\sigma \sigma_d E_{(a} \sigma^d \varepsilon_{b)cd} + 3w\sigma_a \sigma_b - \sigma^2 wh_{ab} = 0, \] (120)
which can be shown to be equivalent to
\[ 4(\sigma^2 - \sigma \cdot \sigma) E_{ab} + 4\tilde{\sigma}(\sigma_a \sigma_b) + w(\sigma \cdot \sigma - \sigma^2) h_{ab} + 2w\sigma_b \sigma_a = 0. \] (121)
This is trivially fulfilled if the hyperboloidal condition (118) and (119) hold. It only remains to show that the metric \( h_{ab} \) is regular at \( \Omega = 0 \). To that end we use again the condition \( a_{ab} = 0 \) to obtain the value of the metric. In this case this is
\[ h_{ab} = 2E_{a}^{c} E_{cb} - E_{ab}. \] (122)
We recall next, (see e.g. appendix A of [23]) that in an appropriate orthonormal frame, the tensor \( E_{ab} \) adopts the form
\[ E_{ab} = \text{diag}(-2z, z, z), \quad z \in \mathbb{C}, \] (123)
and given (119) the only possibilities are either \( z = w \) or \( z = -w/2 \). Now using these both possibilities in the condition \( a_{ab} = 0 \) enables us to find the possible values of the metric \( h_{ab} \) at the conformal boundary. Using (123) in (122) we get
\[ h_{ab} = \frac{24}{a} \text{diag}(8z^2 + 2wz, 2z^2 - wz, 2z^2 - wz), \] (124)
where according to (119), either \( z = -w/2 \) or \( z = w/4 \). Doing the replacements we find that only the former value gives an orthonormal Riemannian metric. Therefore the metric \( h_{ab} \) is regular at the conformal boundary if \( \sigma_a \) is a simple eigenvalue of \( E_{ab} \).

The proof of Theorem 12 indicates that, at least for regular hyperboloidal data, there is no obstruction to the extension of the Petrov type D from the physical space-time to the conformal boundary.

7 Conclusions

We have given necessary and sufficient conditions that a conformal initial data set for the conformal vacuum equations have to satisfy in order that its data development have a subset conformal to a type D vacuum solution of the Einstein equations with cosmological constant (see Theorems 6 and 7 for the complete details). In addition we have been able to particularize the results for the case in which the solution of the conformal equations is conformal to a suitable region of the Kerr black hole (Theorems 10 and 11). The conformal data are defined from a vacuum initial data set of the Einstein equations with no additional restrictions. This means that the data for the conformal equations are constructed in a spacelike hypersurface \( \tilde{S} \) which is in the interior of the physical space-time. These data are extended to data on a hypersurface \( S \supset \tilde{S} \), which intersect the conformal boundary, by using variables all defined intrinsically on \( S \). The regularity of the data so constructed at the conformal boundary has been also addressed in Theorem 12. There we show that there are no obstructions to the extension of the data to the conformal boundary when the data are hyperboloidal. In particular this implies that the Petrov type can be also extended to the conformal boundary (in this case it is the Petrov type of the rescaled Weyl tensor \( d_{abcd} \)). If the Petrov type D is kept at the conformal boundary then it might be an indication that the solution in the physical space-time is stable under perturbations. Recall that hyperboloidal initial data are tied to a conformal boundary that is null, so there are no obstructions to the extension of the Petrov type to the conformal boundary if it is null.

In [37] (Theorem 3.3) necessary and sufficient conditions were found for general data of the vacuum conformal equations that guarantee that the physical spacetime admits a Killing vector
field. In principle there should be a correspondence between the result found in [37] and eqs. (60a)-(60b) of our Lemma 1. How this correspondence is actually established is an interesting open question.

An important aspect that requires further analysis is the investigation of existence results for data sets fulfilling both the algebraic and the differential conditions appearing in Theorems 6 and 7 and their specializations to the Kerr solution, given by Theorems 10 and 11. We have proven in Theorem 12 that for hyperboloidal initial data there are no algebraic obstructions to the extension of the conformal initial data described in Theorem 6 to the conformal boundary but one still needs to show the existence of actual data fulfilling the conditions of Theorem 7. At this point we recall the work of [22, 26] where the existence of congruences of conformal geodesics is proven in globally hyperbolic domains of vacuum type D solutions with a null or timelike conformal boundary. Also interesting in this regard is the work in [5] where conformal initial data for vacuum solutions with a timelike conformal boundary are studied. All these results imply the existence of a conformal extension giving a conformal boundary with the appropriate properties, thus pointing that the conditions of Theorems 6 and 7 are going to provide existence results for conformal data in a wide range of situations. For example, an existence result in the case of the Kerr solution of this kind of data would provide an important insight into the open problem of the non-linear stability of this solution. It is to be noted that in this paper we have used the standard 1+3 decomposition to formulate the initial value problem for Friedrich conformal equations, following the approach presented in [5] [6]. An alternative approach would be to use the tractor calculus in embedded hypersurfaces (see lecture 6 of [9]). In order that this approach be useful one would need to find a formulation of Theorem 1 in the tractor calculus language. Another interesting open question is to find out whether the existence results of [2] [28] can be adapted in some way to the present situation. The exact extent of all these assertions will be addressed elsewhere.

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