Chern-Simons Theory, Vassiliev Invariants, Loop Quantum Gravity and Functional Integration Without Integration

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This paper is an exposition of the relationship between Witten’s Chern-Simons functional integral and the theory of Vassiliev invariants of knots and links in three-dimensional space. We conceptualize the functional integral in terms of equivalence classes of functionals of gauge fields and we do not use measure theory. This approach makes it possible to discuss the mathematics intrinsic to the functional integral rigorously and without functional integration. Applications to loop quantum gravity are discussed.

Keywords: knot; link; Vassiliev invariant; Lie algebra; Chern-Simons form; functional integral; Kontsevich integral; loop quantum gravity, Kodama state.

1. Introduction

This paper is an introduction to how Vassiliev invariants in knot theory arise naturally in the context of Witten’s functional integral. The relationship between Vassiliev invariants and Witten’s integral has been known since Bar-Natan’s thesis where he discovered, through this connection, how to define Lie algebraic weight systems for these invariants.

This paper is written in a context of “integration without integration”. The idea is as follows. Let \( F(A), G(A), H(A) \) be functionals of a gauge field \( A \) that vanish rapidly as the amplitude of the field goes to infinity. We say that \( F \sim G \) if \( F - G = DH \) where \( D \) denotes a gauge functional derivative. We define \( \int F(A) \) to be the equivalence class of \( F(A) \). By definition, this integral satisfies integration by parts, and it is a useful conceptual substitute for a functional integral over all gauge fields (modulo gauge equivalence). We replace the usual notion of functional integral with such equivalence classes.

The paper is a sequel to16 and15. In these papers we show somewhat more about the relationship of Vassiliev invariants and the Witten functional integral. In particular, we show how the Kontsevich integrals (used to give rigorous definitions of these invariants) arise as Feynman integrals in the perturbative expansion of the Witten functional integral. See also the work of Labastida and Pérez18 on this same subject. The result is an interpretation of the Kontsevich integrals in
terms of the light-cone gauge and thereby extending the original work of Fröhlich and King. The purpose of this paper is to give an exposition of the beginnings of these relationships, to introduce diagrammatic techniques that illuminate the connections, and to show how the integral can be fruitfully formulated in terms of certain equivalence classes of functionals of gauge fields.

The paper is divided into six sections beyond the introduction. Section 2 discusses Vassiliev invariants and invariants of rigid vertex graphs. Section 3 discusses the concept of replacing integrals by equivalence classes. Section 4 introduces the basic formalism and shows how the functional integral, regarded without integration, is related directly to knot invariants and particularly, Vassiliev invariants. Section 5 discusses the formalism of the perturbative expansion of the Witten integral. Section 6 is a sketch of the loop transform, useful in loop quantum gravity and ends with a quick discussion of the Kodama state with references to recent literature. Section 7 discusses how the Kontsevich integrals for Vassiliev invariants arise from the perturbation expansion.

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2. Vassiliev Invariants and Invariants of Rigid Vertex Graphs

If $V(K)$ is a (Laurent polynomial valued, or more generally - commutative ring valued) invariant of knots, then it can be naturally extended to an invariant of rigid vertex graphs by defining the invariant of graphs in terms of the knot invariant via an ‘unfolding of the vertex. That is, we can regard the vertex as a ‘black box” and replace it by any tangle of our choice. Rigid vertex motions of the graph preserve the contents of the black box, and hence implicate ambient isotopies of the link obtained by replacing the black box by its contents. Invariants of knots and links that are evaluated on these replacements are then automatically rigid vertex invariants of the corresponding graphs. If we set up a collection of multiple replacements at the vertices with standard conventions for the insertions of the tangles, then a summation over all possible replacements can lead to a graph invariant with new coefficients corresponding to the different replacements. In this way each invariant of knots and links implicates a large collection of graph invariants. See.$^{11, 12}$

The simplest tangle replacements for a 4-valent vertex are the two crossings, positive and negative, and the oriented smoothing. Let $V(K)$ be any invariant of knots and links. Extend $V$ to the category of rigid vertex embeddings of 4-valent
graphs by the formula

\[ V(K_*) = aV(K_+) + bV(K_-) + cV(K_0) \]

where \( K_+ \) denotes a knot diagram \( K \) with a specific choice of positive crossing, \( K_- \) denotes a diagram identical to the first with the positive crossing replaced by a negative crossing and \( K_* \) denotes a diagram identical to the first with the positive crossing replaced by a graphical node.

This formula means that we define \( V(G) \) for an embedded 4-valent graph \( G \) by taking the sum

\[ V(G) = \sum_S a^{i_S}(S)b^{j_S(S)c^{k_S(S)}}V(S) \]

with the summation over all knots and links \( S \) obtained from \( G \) by replacing a node of \( G \) with either a crossing of positive or negative type, or with a smoothing of the crossing that replaces it by a planar embedding of non-touching segments (denoted 0). It is not hard to see that if \( V(K) \) is an ambient isotopy invariant of knots, then, this extension is an rigid vertex isotopy invariant of graphs. In rigid vertex isotopy the cyclic order at the vertex is preserved, so that the vertex behaves like a rigid disk with flexible strings attached to it at specific points.

There is a rich class of graph invariants that can be studied in this manner. The Vassiliev Invariants\(^7,5\) constitute the important special case of these graph invariants where \( a = +1, b = -1 \) and \( c = 0 \). Thus \( V(G) \) is a Vassiliev invariant if

\[ V(K_*) = V(K_+) - V(K_-). \]

Call this formula the \textit{exchange identity} for the Vassiliev invariant \( V \). See Figure 1

\[ \text{Fig. 1. Exchange Identity for Vassiliev Invariants} \]

\[ V \text{ is said to be of finite type } k \text{ if } V(G) = 0 \text{ whenever } |G| > k \text{ where } |G| \text{ denotes the number of (4-valent) nodes in the graph } G. \text{ The notion of finite type is of} \]
extraordinary significance in studying these invariants. One reason for this is the following basic Lemma.

**Lemma.** If a graph \( G \) has exactly \( k \) nodes, then the value of a Vassiliev invariant \( v_k \) of type \( k \) on \( G \), \( v_k(G) \), is independent of the embedding of \( G \).

**Proof.** The different embeddings of \( G \) can be represented by link diagrams with some of the 4-valent vertices in the diagram corresponding to the nodes of \( G \). It suffices to show that the value of \( v_k(G) \) is unchanged under switching of a crossing. However, the exchange identity for \( v_k \) shows that this difference is equal to the evaluation of \( v_k \) on a graph with \( k + 1 \) nodes and hence is equal to zero. This completes the proof. //

The upshot of this Lemma is that Vassiliev invariants of type \( k \) are intimately involved with certain abstract evaluations of graphs with \( k \) nodes. In fact, there are restrictions (the four-term relations) on these evaluations demanded by the topology and it follows from results of Kontsevich that such abstract evaluations actually determine the invariants. The knot invariants derived from classical Lie algebras are all built from Vassiliev invariants of finite type. All of this is directly related to Witten’s functional integral.

In the next few figures we illustrate some of these main points. In Figure 2 we show how one associates a so-called chord diagram to represent the abstract graph associated with an embedded graph. The chord diagram is a circle with arcs connecting those points on the circle that are welded to form the corresponding graph. In Figure 3 we illustrate how the four-term relation is a consequence of topological invariance. In Figure 4 we show how the four term relation is a consequence of the abstract pattern of the commutator identity for a matrix Lie algebra. This shows that the four term relation is directly related to a categorical generalisation of Lie algebras. Figure 5 illustrates how the weights are assigned to the chord diagrams in the Lie algebra case - by inserting Lie algebra matrices into the circle and taking a trace of a sum of matrix products.
3. Integration without integration

Recall that if \( Z = \int_{-\infty}^{\infty} e^{-x^2/2} \, dx \) then

\[
Z^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)/2} \, dx \, dy = \int_{0}^{2\pi} \int_{0}^{\infty} e^{-r^2/2} \, rdrd\theta
\]
Fig. 5. Calculating Lie Algebra Weights

\[ = 2\pi \int_0^\infty e^{-r^2/2} r \, dr = 2\pi. \]

Whence

\[ Z = \sqrt{2\pi}. \]

Furthermore, if

\[ Z(J) = \int_{-\infty}^\infty e^{-x^2/2 + Jx} \, dx, \]

then

\[ Z(J) = \int_{-\infty}^\infty e^{-(x-J)^2/2 + J^2/2} \, dx \]
\[ = e^{J^2/2} \int_{-\infty}^\infty e^{-(x-J)^2/2} \, dx \]
\[ = e^{J^2/2} \int_{-\infty}^\infty e^{-x^2/2} \, dx \]
\[ = e^{J^2/2} Z(0) = \sqrt{2\pi} e^{J^2/2}. \]

Now examine how much of this calculation could be done if we did not know about the existence of the integral, or if we did not know how to calculate explicitly the values of these integrals across the entire real line. Given that we believed in the existence of the integrals, and that we could use properties such as change of variable giving

\[ \int_{-\infty}^\infty e^{-(x-J)^2/2} \, dx = \int_{-\infty}^\infty e^{-x^2/2} \, dx, \]

we could deduce the relative result stating that

\[ Z(J) = \int_{-\infty}^\infty e^{-x^2/2 + Jx} \, dx = e^{J^2/2} \int_{-\infty}^\infty e^{-x^2/2} \, dx. \]
From this we can deduce that
\[
d^n Z(J)/dJ^n|_{J=0} = d^n/dJ^n \int_{-\infty}^{\infty} e^{-x^2/2+Jx} dx = \int_{-\infty}^{\infty} x^n e^{-x^2/2} dx.
\]

Hence
\[
\int_{-\infty}^{\infty} x^n e^{-x^2/2} dx = d^n(e^{J^2/2})/dJ^n|_{J=0} \int_{-\infty}^{\infty} e^{-x^2/2} dx.
\]

But now, lets go a step further and imagine that we really have no theory of integration available. Then we are in the position of freshman calculus where one defines \( \int f \) to be “any” function \( g \) such that \( dg/dx = f \). One defines the integral in this form of elementary calculus to be the anti-derivative, and this takes care of the matter for a while! What are we really doing in freshman calculus? We are noting that for integration on an interval \([a, b]\), if two functions \( f \) and \( g \) satisfy \( f - g = dh/dx \) for some differentiable function \( h \), then we have that
\[
\int_a^b (f - g) = \int_a^b dh/dx = h(b) - h(a).
\]
If the function \( h(x) \) vanishes as \( x \) goes to infinity, then we have that
\[
\int_{-\infty}^{\infty} f dx = \int_{-\infty}^{\infty} g dx
\]
when \( f - g = dh/dx \). This suggests turning things upside down and defining an equivalence relation on functions
\[
f \sim g
\]
if
\[
f - g = dh/dx
\]
where \( h(x) \) is a function vanishing at infinity. Then we define the integral
\[
\int f(x)
\]
to be the equivalence class of the function \( f(x) \). This “integral” represents integration from minus infinity to plus infinity but it is defined only as an equivalence class of functions. An “actual” integral, like the Riemann, Lebesque or Henstock integral is a well-defined real valued function that is constant on these equivalence classes.

We shall say that \( f(x) \) is rapidly vanishing at infinity if \( f(x) \) and all its derivatives are vanishing at infinity. For simplicity, we shall assume that all functions under consideration have convergent power series expansions so that
\[
f(x + J) = f(x) + f'(x)J + f''(x)J^2/2! + \cdots,
\]
and that they are rapidly vanishing at infinity. It then follows that
\[
f(x + J) = f(x) + d(f(x)J + f'(x)J^2/2! + \cdots)/dx \sim f(x),
\]
and hence we have that \( \int f(x + J) = \int f(x) \), giving translation invariance when \( J \) is a constant.
We have shown the following Proposition.

**Proposition.** Let $f(x), g(x), h(x)$ be functions rapidly vanishing at infinity (with power series representations). Let $\int f$ denote the equivalence class of the function $f$ where $f \sim g$ means that $f - g = Dh$ where $Dh = dh/dx$. Then this integral satisfies the following properties

1. If $f \sim g$ then $\int f = \int g$.
2. If $k$ is a constant, then $\int (kf + g) = k \int f + \int g$.
3. If $J$ is a constant, then $\int f(x + J) = \int f(x)$.
4. $\int Dh = 0$ where 0 denotes the equivalence class of the zero function. Hence $\int f(Dg) + f(Df)g = \int D(fg) = 0$, so that integration by parts is valid with vanishing boundary conditions at infinity.

Note that $e^{-x^2/2}$ is rapidly vanishing at infinity. We now see that most of the calculations that we made about $e^{-x^2/2}$ were actually statements about the equivalence class of this function:

$$e^{-x^2/2 + Jx} = e^{-(x-J)^2/2 + J^2/2} = e^{J^2/2}e^{-(x-J)^2/2} \sim e^{J^2/2}e^{-x^2/2},$$

whence

$$\int e^{-x^2/2 + Jx} = e^{J^2/2} \int e^{-x^2/2}.$$

### 3.1. Functional Derivatives

In order to generalize the ideas presented in this section to the context of functional integrals, we need to discuss the concept of functional derivatives. We are given a functional $F(\alpha(x))$ whose argument $\alpha(x)$ is a function of a variable $x$. We wish to define the functional derivative $\delta F(\alpha(x))/\delta \alpha(x_0)$ of $F(\alpha(x))$ with respect to $\alpha(x)$ at a given point $x_0$. The idea is to regard each $\alpha(x_0)$ as a separate variable, giving $F(\alpha(x))$ the appearance of a function of infinitely many variables. In order to formalize this notion one needs to use generalized functions (distributions) such as the Dirac delta function $\delta(x)$, a distribution with the property that $\int_a^b \delta(x_0)f(x)dx = f(x_0)$ for any integrable function $f(x)$ and point $x_0$ in the interval $[a, b]$. One defines the functional derivative by the formula

$$\delta F(\alpha(x))/\delta \alpha(x_0) = \lim_{\epsilon \to 0}[F(\alpha(x) + \delta(x_0)\epsilon) - F(\alpha(x))]/\epsilon.$$

Note that if

$$F(\alpha(x)) = \alpha(x)^2$$

then

$$\delta F(\alpha(x))/\delta \alpha(x_0) = \lim_{\epsilon \to 0}[(\alpha(x) + \delta(x_0)\epsilon)^2 - \alpha(x)^2]/\epsilon$$
\[
\lim_{\epsilon \to 0} [2\alpha(x)\delta(x_0)\epsilon + \delta(x_0)^2 \epsilon^2] / \epsilon = 2\alpha(x)\delta(x_0).
\]

While if
\[
G(\alpha(x)) = \int_a^b \alpha(x)^2 \, dx
\]
then
\[
\frac{\delta G(\alpha(x))}{\delta \alpha(x_0)} = 2\alpha(x_0)
\]
when \(x_0 \in [a, b]\). More generally, if
\[
G(\alpha(x)) = \int_a^b f(\alpha(x)) \, dx
\]
for a differentiable function \(f\), then
\[
\frac{\delta G(\alpha(x))}{\delta \alpha(x_0)} = f'(\alpha(x_0)).
\]

These examples show that the results of a functional differentiation can be either a distribution or a function, depending upon the context of the original functional.

In the case of a path integral of the type used in quantum mechanics, one wants to integrate a functional \(F(p)\) over paths \(p(t)\) with \(t\) in an interval \([0, 1]\). The functional takes the form
\[
F(p) = e^{(i/\hbar) \int_0^1 S(p(t)) \, dt}
\]
and the traditional Feynman path integral has the form
\[
\int \mathcal{D}p e^{(i/\hbar) \int_0^1 S(p(t)) \, dt},
\]
giving the amplitude for a particle to travel from \(a = p(0)\) to \(b = p(1)\), the integration proceeding over all paths with these initial and ending points.

Here the equivalence relation corresponding to the functional integral is \(F \sim G\) if \(F - G = \delta H\) where
\[
\delta H = \frac{\delta H(p)}{\delta p(t_0)}
\]
for some time \(t_0\) and some \(H(p)\). Again we need to specify the class of functionals and to say what it means for a functional to "vanish at infinity." Since we are integrating over all paths, we need a notion of size for a path. This can be defined by
\[
||p|| = \left( \int_0^1 |p(t)|^2 \, dt \right)^{1/2}.
\]
Note that for $F(p) = e^{(i/\hbar) \int_0^1 S(p(t)) dt}$ we have

$$\frac{\delta F(p)}{\delta p(t_0)} = (i/\hbar) \int_0^1 S(p(t)) dt / \delta p(t_0) F(p)$$

$$= (i/\hbar) S'(p(t_0)) F(p).$$

Here we see the fact that the integral can be dominated by contributions from paths where this variation is zero. Note that in order to estimate this stationary phase contribution to the functional integral, one needs more than just a definition of the integral as an equivalence class of functionals. Nevertheless, we shall see in the next section that these equivalence classes do give insight into the topology associated with Witten’s integral.

4. Vassiliev Invariants and Witten’s Functional Integral

In\textsuperscript{29} Edward Witten proposed a formulation of a class of 3-manifold invariants as generalized Feynman integrals taking the form $Z(M)$ where

$$Z(M) = \int DA e^{(ik/4\pi) S(M,A)}.$$ 

Here $M$ denotes a 3-manifold without boundary and $A$ is a gauge field (also called a gauge potential or gauge connection) defined on $M$. The gauge field is a one-form on a trivial $G$-bundle over $M$ with values in a representation of the Lie algebra of $G$. The group $G$ corresponding to this Lie algebra is said to be the gauge group. In this integral the action $S(M,A)$ is taken to be the integral over $M$ of the trace of the Chern-Simons three-form $A \wedge dA + (2/3) A \wedge A \wedge A$. (The product is the wedge product of differential forms.)

$Z(M)$ integrates over all gauge fields modulo gauge equivalence.

The formalism and internal logic of Witten’s integral supports the existence of a large class of topological invariants of 3-manifolds and associated invariants of knots and links in these manifolds.

The invariants associated with this integral have been given rigorous combinatorial descriptions but questions and conjectures arising from the integral formulation are still outstanding. Specific conjectures about this integral take the form of just how it implicates invariants of links and 3-manifolds, and how these invariants behave in certain limits of the coupling constant $k$ in the integral. Many conjectures of this sort can be verified through the combinatorial models. On the other hand, the really outstanding conjecture about the integral is that it exists! At the present time there is no measure theory or generalization of measure theory that supports it. Here is a formal structure of great beauty. It is also a structure whose consequences can be verified by a remarkable variety of alternative means.
In this section we will examine the formalism of Witten’s approach via a generalization of our sketch of “integration without integration”. In order to do this we need to consider functions \( f(A) \) of gauge connections \( A \) and a notion of equivalence, \( f \sim g \), taking the form \( f - g = Dh \) where \( D \) is a gauge functional derivative. Since these notions need defining, we first discuss them in the context of the integrand of Witten’s integral. Thus for a while, we shall speak of Witten’s integral, but let it be known that this integral will soon be replaced by an equivalence class of functions just as happened in the last section!

The formalism of the Witten integral implicates invariants of knots and links corresponding to each classical Lie algebra. In order to see this, we need to introduce the Wilson loop. The Wilson loop is an exponentiated version of integrating the gauge field along a loop \( K \) in three space that we take to be an embedding (knot) or a curve with transversal self-intersections. For this discussion, the Wilson loop will be denoted by the notation \( W_K(A) = \langle K|A \rangle \) to denote the dependence on the loop \( K \) and the field \( A \). It is usually indicated by the symbolism \( \text{tr}(Pe^{\frac{ik}{4\pi}} \oint K A) \). Thus

\[
W_K(A) = \langle K|A \rangle = \text{tr}(Pe^{\frac{ik}{4\pi}} \oint K A).
\]

Here the \( P \) denotes path ordered integration - we are integrating and exponentiating matrix valued functions, and so must keep track of the order of the operations. The symbol \( \text{tr} \) denotes the trace of the resulting matrix. This Wilson loop integration exists by normal means and will not be replaced by function classes.

With the help of the Wilson loop functional on knots and links, Witten writes down a functional integral for link invariants in a 3-manifold \( M \):

\[
Z(M,K) = \int DA e^{\frac{ik}{4\pi} S(M,A)} \text{tr}(Pe^{\frac{ik}{4\pi}} \oint K A) = \int DA e^{\frac{ik}{4\pi} S} < K|A > .
\]

Here \( S(M,A) \) is the Chern-Simons Lagrangian, as in the previous discussion. We abbreviate \( S(M,A) \) as \( S \) and write \( < K|A > \) for the Wilson loop. Unless otherwise mentioned, the manifold \( M \) will be the three-dimensional sphere \( S^3 \).

An analysis of the formalism of this functional integral reveals quite a bit about its role in knot theory. This analysis depends upon key facts relating the curvature of the gauge field to both the Wilson loop and the Chern-Simons Lagrangian. The idea for using the curvature in this way is due to Lee Smolin (See also). To this end, let us recall the local coordinate structure of the gauge field \( A(x) \), where \( x \) is
a point in three-space. We can write \( A(x) = A_a^k(x)T_a dx^k \) where the index \( a \) ranges from 1 to \( m \) with the Lie algebra basis \( \{ T_1, T_2, T_3, ..., T_m \} \). The index \( k \) goes from 1 to 3. For each choice of \( a \) and \( k \), \( A_a^k(x) \) is a smooth function defined on three-space. In \( A(x) \) we sum over the values of repeated indices. The Lie algebra generators \( T_a \) are matrices corresponding to a given representation of the Lie algebra of the gauge group \( G \). We assume some properties of these matrices as follows:

1. \( [T_a, T_b] = i f^{abc} T_c \) where \( [x, y] = xy - yx \), and \( f^{abc} \) (the matrix of structure constants) is totally antisymmetric. There is summation over repeated indices.

2. \( \text{tr}(T_a T_b) = \delta_{ab} / 2 \) where \( \delta_{ab} \) is the Kronecker delta (\( \delta_{ab} = 1 \) if \( a = b \) and zero otherwise).

We also assume some facts about curvature. (The reader may enjoy comparing with the exposition in \( \text{13} \). But note the difference of conventions on the use of \( i \) in the Wilson loops and curvature definitions.) The first fact is the relation of Wilson loops and curvature for small loops:

**Fact 1.** The result of evaluating a Wilson loop about a very small planar circle around a point \( x \) is proportional to the area enclosed by this circle times the corresponding value of the curvature tensor of the gauge field evaluated at \( x \). The curvature tensor is written

\[
F_{rs}^a(x)T_a dx^r dy^s.
\]

It is the local coordinate expression of \( F = dA + A \wedge A \).

**Application of Fact 1.** Consider a given Wilson line \( \langle K|S \rangle \). Ask how its value will change if it is deformed infinitesimally in the neighborhood of a point \( x \) on the line. Approximate the change according to Fact 1, and regard the point \( x \) as the place of curvature evaluation. Let \( \delta \langle K|A \rangle \) denote the change in the value of the line. \( \delta \langle K|A \rangle \) is given by the formula

\[
\delta \langle K|A \rangle = dx^r dx^s F_{rs}^a(x)T_a \langle K|A \rangle.
\]

This is the first order approximation to the change in the Wilson line.

In this formula it is understood that the Lie algebra matrices \( T_a \) are to be inserted into the Wilson line at the point \( x \), and that we are summing over repeated indices. This means that each \( T_a \langle K|A \rangle \) is a new Wilson line obtained from the original line \( \langle K|A \rangle \) by leaving the form of the loop unchanged, but inserting the matrix \( T_a \) into that loop at the point \( x \). In Figure 6 we have illustrated this mode of insertion of Lie algebra into the Wilson loop. Here and in further illustrations in this section we use \( W_K(A) \) to denote the Wilson loop. Note that in the diagrammatic version shown in Figure 6 we have let small triangles with legs indicate \( dx^i \). The legs correspond to indices just as in our work in the last section with Lie algebras.
and chord diagrams. The curvature tensor is indicated as a circle with three legs corresponding to the indices of $F_{rs}^a$.

**Notation.** In the diagrams in this section we have dropped mention of the factor of $(1/4\pi)$ that occurs in the integral. This convention saves space in the figures. In these figures $L$ denotes the Chern–Simons Lagrangian.

![Diagram of Lie Algebra and Curvature Tensor Insertion into the Wilson Loop](image)

**Fig. 6.** Lie Algebra and Curvature Tensor Insertion into the Wilson Loop

**Remark.** In thinking about the Wilson line $<K|A> = tr(Pe^{\int_K A})$, it is helpful to recall Euler’s formula for the exponential:

$$e^x = \lim_{n \to \infty} (1 + x/n)^n.$$  

The Wilson line is the limit, over partitions of the loop $K$, of products of the matrices $(1 + A(x))$ where $x$ runs over the partition. Thus we can write symbolically,

$$<K|A> = \prod_{x \in K} (1 + A(x))$$

$$= \prod_{x \in K} (1 + A^a_k(x)T_a dx^k).$$

It is understood that a product of matrices around a closed loop connotes the trace of the product. The ordering is forced by the one dimensional nature of the loop. Insertion of a given matrix into this product at a point on the loop is then a well-defined concept. If $T$ is a given matrix then it is understood that $T <K|A>$ denotes the insertion of $T$ into some point of the loop. In the case above, it is understood from context in the formula that the insertion is to be performed at the point $x$ indicated in the argument of the curvature.

**Remark.** The previous remark implies the following formula for the variation of the Wilson loop with respect to the gauge field:

$$\delta <K|A> /\delta(A^a_k(x)) = dx^k T_a <K|A>.$$

Varying the Wilson loop with respect to the gauge field results in the insertion of an infinitesimal Lie algebra element into the loop. Figure 7 gives a diagrammatic form
for this formula. In that Figure we use a capital $D$ with up and down legs to denote the derivative $\delta/\delta(A^a_k(x))$. Insertions in the Wilson line are indicated directly by matrix boxes placed in a representative bit of line.

![Fig. 7. Differentiating the Wilson Line](image)

**Proof.**

$$\delta <K|A>/\delta (A^a_k(x))$$

$$= \delta \prod_{y \in K} (1 + A^a_k(y)T_\alpha dy^k)/\delta (A^a_k(x))$$

$$= \prod_{y<x \in K} (1 + A^a_k(y)T_\alpha dy^k)[T_\alpha dx^k] \prod_{y>x \in K} (1 + A^a_k(y)T_\alpha dy^k)$$

$$= dx^kT_\alpha <K|A>.$$

**Fact 2.** The variation of the Chern-Simons Lagrangian $S$ with respect to the gauge potential at a given point in three-space is related to the values of the curvature tensor at that point by the following formula:

$$F^a_{rs}(x) = \epsilon_{rst}\delta S/\delta (A^a_t(x)).$$

Here $\epsilon_{abc}$ is the epsilon symbol for three indices, i.e. it is $+1$ for positive permutations of 123 and $-1$ for negative permutations of 123 and zero if any two indices are repeated. A diagrammatic for this formula is shown in Figure 8.

**The Functional Equivalence Relation.** With these facts at hand, we are prepared to define our equivalence relation on functions of gauge fields. Given a function $F(A)$ of a gauge field $A$, we let $DF$ denote any gauge functional derivative of $f(A)$. That is

$$DF = \delta F(A)/\delta (A^a_k(x)).$$
Note that

\[ D < K|A > = \delta < K|A > /\delta (A^a_k(x)) = dx^k T_a < K|A > \]

with the insertion conventions as explained above. Then we say that functionals \( F \) and \( G \) are \textit{integrally equivalent} (\( F \sim G \)) if there exists an \( H \) such that \( DH = F - G \).

We stipulate that all functionals in the discussion are rapidly vanishing at infinity, where this is taken to mean that \( F(A) \) goes to zero as \( ||A|| \) goes to infinity, and the same is true for all functional derivatives of \( F \). Here the norm

\[ ||A|| = \Sigma_{i,a} \int_{R^3} (A^a_i)^2 dvol \]

where \( dvol \) is the volume form on \( R^3 \) and it is assumed that all gauge fields have finite norm in this sense.

We then define the integral

\[ Z(M, K) = \int DA e^{(ik/4\pi)S(M,A)} Tr(P e^{\int K A}) = \int DA e^{(ik/4\pi)S} < K|A > \]

to be the equivalence class of the functional

\[ e^{(ik/4\pi)S(M,A)} Tr(P e^{\int K A}) \]

We invite the reader to make this interpretation throughout the derivations that follow. It will then be apparent that much of what is usually taken for formal heuristics about the functional integral is actually a series of structural remarks about these equivalence classes. Of course, one needs to know that the equivalence classes are non-trivial to make a complete story. An existent integral would supply that key ingredient. It its absence, we can examine that structure that can be articulated at the level of the equivalence classes.
We are prepared to determine how the Witten integral behaves under a small deformation of the loop $K$.

**Theorem.** 1. Let $Z(K) = Z(S^3, K)$ and let $\delta Z(K)$ denote the change of $Z(K)$ under an infinitesimal change in the loop $K$. Then

$$\delta Z(K) = (4\pi i/k) \int dA e^{(ik/4\pi)S}[Vol T_a T_a < K|A>$$

where $Vol = \epsilon_{rst} dx_r dx_s dx_t$.

The sum is taken over repeated indices, and the insertion is taken of the matrices $T_a T_a$ at the chosen point $x$ on the loop $K$ that is regarded as the center of the deformation. The volume element $Vol = \epsilon_{rst} dx_r dx_s dx_t$ is taken with regard to the infinitesimal directions of the loop deformation from this point on the original loop.

2. The same formula applies, with a different interpretation, to the case where $x$ is a double point of transversal self intersection of a loop $K$, and the deformation consists in shifting one of the crossing segments perpendicularly to the plane of intersection so that the self-intersection point disappears. In this case, one $T_a$ is inserted into each of the transversal crossing segments so that $T_a T_a < K|A>$ denotes a Wilson loop with a self intersection at $x$ and insertions of $T_a$ at $x + \epsilon_1$ and $x + \epsilon_2$ where $\epsilon_1$ and $\epsilon_2$ denote small displacements along the two arcs of $K$ that intersect at $x$. In this case, the volume form is nonzero, with two directions coming from the plane of movement of one arc, and the perpendicular direction is the direction of the other arc.

**Proof.**

$$\delta Z(K) = \int DA e^{(ik/4\pi)S} \delta < K|A>$$

$$= \int DA e^{(ik/4\pi)S} dx^r dy^s F_{rs}^a(x) T_a < K|A>$$

$$= \int DA e^{(ik/4\pi)S} dx^r dy^s \epsilon_{rst} (\delta S/\delta (A^r_a(x))) T_a < K|A>$$

$$= (-4\pi i/k) \int DA (\delta e^{(ik/4\pi)S} / \delta (A^r_a(x))) \epsilon_{rst} dx^r dy^s T_a < K|A>$$

$$= (4\pi i/k) \int DA e^{(ik/4\pi)S} \epsilon_{rst} dx^r dy^s (\delta T_a < K|A> / \delta (A^r_a(x)))$$

(integration by parts and the boundary terms vanish)
\[ = (4\pi i/k) \int DA e^{(ik/4\pi)S} [\text{Vol} T_a T_a < K | A >]. \]

This completes the formalism of the proof. In the case of part 2., a change of interpretation occurs at the point in the argument when the Wilson line is differentiated. Differentiating a self-intersecting Wilson line at a point of self intersection is equivalent to differentiating the corresponding product of matrices with respect to a variable that occurs at two points in the product (corresponding to the two places where the loop passes through the point). One of these derivatives gives rise to a term with volume form equal to zero, the other term is the one that is described in part 2. This completes the proof of the Theorem. //

The formalism of this proof is illustrated in Figure 9.

In the case of switching a crossing the key point is to write the crossing switch as a composition of first moving a segment to obtain a transversal intersection of the diagram with itself, and then to continue the motion to complete the switch. One then analyzes separately the case where \( x \) is a double point of transversal self intersection of a loop \( K \), and the deformation consists in shifting one of the crossing segments perpendicularly to the plane of intersection so that the self-intersection point disappears. In this case, one \( T_a \) is inserted into each of the transversal crossing segments so that \( T_a T_a < K | A > \) denotes a Wilson loop with a self intersection at
x and insertions of $T^a$ at $x + \epsilon_1$ and $x + \epsilon_2$ as in part 2. of the Theorem above. The first insertion is in the moving line, due to curvature. The second insertion is the consequence of differentiating the self-touching Wilson line. Since this line can be regarded as a product, the differentiation occurs twice at the point of intersection, and it is the second direction that produces the non-vanishing volume form.

Up to the choice of our conventions for constants, the switching formula is, as shown below (See Figure 10).

$$Z(K_+) - Z(K_-) = (4\pi i/k) \int DA e^{(ik/4\pi)S} T_a T_a < K^{*\ast} | A > = (4\pi i/k) Z(T^a T^a K^{*\ast}),$$

where $K^{*\ast}$ denotes the result of replacing the crossing by a self-touching crossing. We distinguish this from adding a graphical node at this crossing by using the double star notation.

Fig. 10. The Difference Formula

A key point is to notice that the Lie algebra insertion for this difference is exactly what is done (in chord diagrams) to make the weight systems for Vassiliev invariants (without the framing compensation). In order to extend the Heuristic at this point we need to assume the analog of a perturbative expansion for the integral. That is, we assume that that there are invariants of regular isotopy of $K$, $Z_n(K)$ and that

$$e^{(ik/4\pi)S(A)} < K | A > \sim \sum_{n=0}^{\infty} k^{-n} Z_n(K).$$

Note that since we have shown that the equivalence class of

$$e^{(ik/4\pi)S(A)} < K | A >$$

is a regular isotopy invariant, it is not at all implausible to assume that there is a power series representative of this functional whose coefficients are numerical regular isotopy invariants. It is this assumption that allows one to make contact with numerical evaluations. The assumption of this power series representation corresponds to the formal perturbative expansion of the Witten integral. One obtains
Vassiliev invariants as coefficients of the powers of \((1/k^n)\). Thus the formalism of the Witten functional integral takes one directly to these weight systems in the case of the classical Lie algebras. In this way the functional integral is central to the structure of the Vassiliev invariants.

5. Perturbative Expansion

Letting \(M^3\) be a three-manifold and \(K\) a knot or link in \(M^3\), we write
\[
\psi(A) = e^{ikL(A)}W_K(A),
\]
and replace \(A\) by \(A/\sqrt{k}\) then we can write
\[
\hat{\psi}(A) = e^{i\frac{4\pi}{k} \int_{M^3} tr(A \wedge dA)} e^{i\frac{6\pi}{\sqrt{k}} \int_{M^3} tr(A \wedge A \wedge A)} W_K(A/\sqrt{k}).
\]
It is the equivalence class of this functional of gauge fields that contains much topological information about knots and links in the three-manifold \(M^3\). We can expand this functional by taking the explicit formula for the Wilson loop:
\[
W_K(A/\sqrt{k}) = \text{tr}(\prod_{x \in K} (1 + A(x)/\sqrt{k})).
\]
\[
\hat{\psi}(A) = e^{i\frac{4\pi}{k} \int_{M^3} tr(A \wedge dA)} e^{i\frac{6\pi}{\sqrt{k}} \int_{M^3} tr(A \wedge A \wedge A)} \text{tr}\left(\prod_{x \in K} (1 + A(x)/\sqrt{k})\right).
\]
where
\[
\text{tr}\left(\prod_{x \in K} (1 + \frac{1}{\sqrt{k}} A(x))\right) = \text{tr}\left(1 + \frac{1}{\sqrt{k}} \int K_1<K_2 A(x_1)A(x_2) + \ldots\right)
\]
\[
\int_{K_1<\ldots<K_n} A(x_1)A(x_2) \ldots A(x_n) = \int_{\bigcap_{n} K=K^n_{\bigcap_n}} A(x_1)A(x_2) \ldots A(x_n)
\]
\[
\mathcal{D} = (x_1, x_2, \ldots, x_n) \in K^n \text{ with } x_1 < x_2 < \ldots < x_n.
\]
This is an iterated integrals expression for the Wilson loop.

Our functional is transformed into a perturbative series in powers of \(1/k\). The equivalence class of each term in the series (when \(M^3\) is the three-sphere \(S^3\)) is formally a Vassiliev invariant as we have described in the previous section. A more intense look at the structure of these functionals can be accomplished by gauge-fixing as we show in the last section.
6. The Loop Transform and Loop Quantum Gravity

Suppose that \( \psi(A) \) is a (complex-valued) function defined on gauge fields. Then we define formally the loop transform \( \hat{\psi}(K) \), a function on embedded loops in three dimensional space, by the formula

\[
\hat{\psi}(K) = \int \psi(A) W_K(A).
\]

note that we could also write

\[
\hat{\psi}(K) = \psi(A) W_K(A)
\]

where it is understood that the right-hand side of the equation represents its integral equivalence class. Then we can look at it as a function of the loop \( K \) and as a function of the gauge field \( A \). This changes one’s point of view about the loop transform. We are really examining a hybrid function of both a possibly knotted loop \( K \) and a gauge field \( A \). The important structure is the relationships that ensue in the integral equivalence class between varying \( A \) and varying \( K \). Nevertheless, we shall continue to use integral signs to remind the reader that we are working with the integral equivalence classes of these functionals.

If \( \Delta \) is a differential operator defined on \( \psi(A) \), then we can use this integral transform to shift the effect of \( \Delta \) to an operator on loops via integration by parts:

\[
\Delta \hat{\psi}(K) = \int \Delta \psi(A) W_K(A)
\]

\[
= -\int \psi(A) \Delta W_K(A).
\]

When \( \Delta \) is applied to the Wilson loop the result can be an understandable geometric or topological operation. In Figures 11, 12 and 13 we illustrate this situation with diagrammatically defined operators \( G \) and \( H \).

We see from Figure 12 that

\[
\Delta \hat{\psi}(K) = \delta \hat{\psi}(K)
\]

where this variation refers to the effect of varying \( K \) by a small loop. As we saw in this section, this means that if \( \hat{\psi}(K) \) is a topological invariant of knots and links, then \( \Delta \hat{\psi}(K) = 0 \) for all embedded loops \( K \). This condition is a transform analogue of the equation \( G\psi(A) = 0 \). This equation is the differential analogue of an invariant of knots and links. It may happen that \( \delta \hat{\psi}(K) \) is not strictly zero, as in the case of our framed knot invariants. For example with

\[
\psi(A) = e^{(ik/4\pi) \int tr(A \wedge dA + (2/3)A \wedge A \wedge A)}
\]
we conclude that \( \hat{G}\psi(K) \) is zero for flat deformations (in the sense of this section) of the loop \( K \), but can be non-zero in the presence of a twist or curl. In this sense the loop transform provides a subtle variation on the strict condition \( G\psi(A) = 0 \). This Chern-Simons functional \( \psi(A) \) can be seen to be a state of loop quantum gravity.

In [2] and earlier publications by these authors, the loop transform is used to study a reformulation and quantization of Einstein gravity. The differential geometric gravity theory is reformulated in terms of a background gauge connection and in the quantization, the Hilbert space consists in functions \( \psi(A) \) that are required to satisfy the constraints

\[
G\psi = 0
\]

and

\[
H\psi = 0.
\]
where $H$ is the operator shown in Figure 13. Thus we see that $\hat{G}(K)$ can be partially zero in the sense of producing a framed knot invariant, and (from Figure 13 and the antisymmetry of the epsilon) that $\hat{H}(K)$ is zero for non-self-intersecting loops. This means that the loop transforms of $G$ and $H$ can be used to investigate a subtle variation of the original scheme for the quantization of gravity. The appearance of the Chern-Simons state

$$\psi(A) = e^{(ik/4\pi) \int tr(A \wedge dA + (2/3)A \wedge A \wedge A)}$$

is quite remarkable in this theory, where it is commonly referred to as the Kodama State. See 21–28 for a number of references about this state, up to the present day. Many ways of weaving this relationship of knot theory and quantum gravity have been devised, from examining directly the Kodama state and its relationship with DeSitter space, to the evolution of spin networks and spin foams to handle the fundamental topological conditions in the theory.

7. Wilson Lines, Axial Gauge and the Kontsevich Integrals

In this section we follow the gauge fixing method used by Fröhlich and King. Their paper was written before the advent of Vassiliev invariants, but contains, as we shall see, nearly the whole story about the Kontsevich integral. A similar approach to ours can be found in 15. In our case we have simplified the determination of the inverse operator for this formalism and we have given a few more details about the calculation of the correlation functions than is customary in physics literature. I hope that this approach makes this subject more accessible to mathematicians. A heuristic argument of this kind contains a great deal of valuable mathematics. It is
clear that these matters will eventually be given a fully rigorous treatment. In fact, in the present case there is a rigorous treatment, due to Albevario and Sen-Gupta of the functional integral after the light-cone gauge has been imposed.

Let \((x^0, x^1, x^2)\) denote a point in three dimensional space. Change to light-cone coordinates

\[
x^+ = x^1 + x^2
\]

and

\[
x^- = x^1 - x^2.
\]

Let \(t\) denote \(x^0\).

Then the gauge connection can be written in the form

\[
A(x) = A_+(x)dx^+ + A_-(x)dx^- + A_0(x)dt.
\]

Let \(CS(A)\) denote the Chern-Simons integral (over the three dimensional sphere)

\[
CS(A) = \frac{1}{4\pi} \int tr(A \wedge dA + (2/3)A \wedge A \wedge A).
\]

We define axial gauge to be the condition that \(A_- = 0\). We shall now work with the functional integral of the previous section under the axial gauge restriction. In axial gauge we have that \(A \wedge A \wedge A = 0\) and so

\[
CS(A) = \frac{1}{4\pi} \int tr(A \wedge dA).
\]

Letting \(\partial_\pm\) denote partial differentiation with respect to \(x^\pm\), we get the following formula in axial gauge

\[
A \wedge dA = (A_+ \partial_- A_0 - A_0 \partial_- A_+)dx^+ \wedge dx^- \wedge dt.
\]

Thus, after integration by parts, we obtain the following formula for the Chern-Simons integral:

\[
CS(A) = \frac{1}{2\pi} \int tr(A_+ \partial_- A_0)dx^+ \wedge dx^- \wedge dt.
\]

Letting \(\partial_i\) denote the partial derivative with respect to \(x_i\), we have that

\[
\partial_+ \partial_- = \partial_1^2 - \partial_2^2.
\]

If we replace \(x^2\) with \(ix^2\) where \(i^2 = -1\), then \(\partial_+ \partial_-\) is replaced by

\[
\partial_1^2 + \partial_2^2 = \nabla^2.
\]
We now make this replacement so that the analysis can be expressed over the complex numbers.

Letting 

\[ z = x^1 + ix^2, \]

it is well known that

\[ \nabla^2 \ln(z) = 2\pi\delta(z) \]

where \( \delta(z) \) denotes the Dirac delta function and \( \ln(z) \) is the natural logarithm of \( z \). Thus we can write

\[ (\partial_+ \partial_-)^{-1} = (1/2\pi)\ln(z). \]

Note that \( \partial_+ = \partial_z = \partial/\partial z \) after the replacement of \( x^2 \) by \( ix^2 \). As a result we have that

\[ (\partial_-)^{-1} = \partial_+ (\partial_+ \partial_-)^{-1} = \partial_+(1/2\pi)\ln(z) = 1/2\pi z. \]

Now that we know the inverse of the operator \( \partial_- \) we are in a position to treat the Chern-Simons integral as a quadratic form in the pattern

\[ (-1/2) < A, LA > = -iCS(A) \]

where the operator

\[ L = \partial_- \]

Since we know \( L^{-1} \), we can express the functional integral as a Gaussian integral:

We replace

\[ Z(K) = \int DAe^{iCS(A)} tr(P e^{iK A}) \]

by

\[ Z(K) = \int DAe^{iCS(A)} tr(P e^{iK A/\sqrt{k}}) \]

by sending \( A \) to \((1/\sqrt{k})A\). We then replace this version by

\[ Z(K) = \int DAe^{(-1/2) < A, LA >} tr(P e^{iK A/\sqrt{k}}). \]

In this last formulation we can use our knowledge of \( L^{-1} \) to determine the the correlation functions and express \( Z(K) \) perturbatively in powers of \((1/\sqrt{k})\).

**Proposition.** Letting

\[ < \phi(A) > = \int DAe^{(-1/2) < A, LA >} \phi(A) / \int DAe^{(-1/2) < A, LA >} \]

for any functional $\phi(A)$, we find that

$$< A^a_+ (z, t) A^b_0 (w, s) > = 0,$$

$$< A^a_0 (z, t) A^b_0 (w, s) > = 0,$$

$$< A^a_+ (z, t) A^b_0 (w, s) > = \kappa \delta^{ab} \delta (t - s) / (z - w)$$

where $\kappa$ is a constant.

**Proof Sketch.** Let’s recall how these correlation functions are obtained. The basic formalism for the Gaussian integration is in the pattern

$$< A(z) A(w) > = \int D A e^{(-1/2) < A, L A >} A(z) A(w) / \int D A e^{(-1/2) < A, L A >}$$

$$= (\partial / \partial J (z)) (\partial / \partial J (w)) |_{J = 0} e^{(1/2) < J, L^{-1} J >}$$

Letting $G * J (z) = \int dw G(z - w) J(w)$, we have that when

$$LG(z) = \delta(z)$$

($\delta(z)$ is a Dirac delta function of $z$) then

$$LG * J (z) = \int dw LG(z - w) J(w) = \int dw \delta(z - w) J(w) = J(z)$$

Thus $G * J(z)$ can be identified with $L^{-1} J(z)$.

In our case

$$G(z) = 1/2 \pi z$$

and

$$L^{-1} J(z) = G * J(z) = \int dw J(w) / (z - w).$$

Thus

$$< J(z), L^{-1} J(z) > = < J(z), G * J(z) > = (1/2 \pi) \int tr (J(z)) (\int dw J(w) / (z - w)) dz$$

$$= (1/2 \pi) \int \int dz dw tr (J(z) J(w)) / (z - w).$$

The results on the correlation functions then follow directly from differentiating this expression. Note that the Kronecker delta on Lie algebra indices is a result of the
corresponding Kronecker delta in the trace formula $\text{tr}(T_a T_b) = \delta_{ab}/2$ for products of Lie algebra generators. The Kronecker delta for the $x^0 = t, s$ coordinates is a consequence of the evaluation at $J$ equal to zero.\/

We are now prepared to give an explicit form to the perturbative expansion for

$$< K > = Z(K) / \int DA e^{(-1/2)<A,LA>}$$

$$= \int DA e^{(-1/2)<A,LA>} \text{tr}(P e^{i K/\sqrt{\kappa}}) / \int DA e^{(-1/2)<A,LA>}$$

$$= \int DA e^{(-1/2)<A,LA>} \text{tr}(\prod_{x \in K} (1 + (A/\sqrt{\kappa}))) / \int DA e^{(-1/2)<A,LA>}$$

$$= \sum_n (1/k^{n/2}) \int_{K_1 \ldots < K_n} < A(x_1) \ldots A(x_n) > .$$

The latter summation can be rewritten (Wick expansion) into a sum over products of pair correlations, and we have already worked out the values of these. In the formula above we have written $K_1 < \ldots < K_n$ to denote the integration over variables $x_1, \ldots x_n$ on $K$ so that $x_1 < \ldots < x_n$ in the ordering induced on the loop $K$ by choosing a basepoint on the loop. After the Wick expansion, we get

$$< K > = \sum_m (1/k^m) \int_{K_1 \ldots < K_n} \sum_{P = \{x_i < x'_i | i = 1, \ldots, m\}} \prod_i < A(x_i) A(x'_i) > .$$

Now we know that

$$< A(x_i) A(x'_i) > = < A^a_k(x_i) A^b_l(x'_i) > T_a T_b dx^k dx^l .$$

Rewriting this in the complexified axial gauge coordinates, the only contribution is

$$< A^a_+(z, t) A^b_0(s, w) > = \kappa \delta^{ab} \delta(t - s)/(z - w) .$$

Thus

$$< A(x_i) A(x'_i) >$$

$$= < A^a_+(x_i) A^b_0(x'_i) > T_a T_b dx^+ \wedge dt + < A^a_0(x_i) A^b_+(x'_i) > T_a T_b dx^+ \wedge dt$$

$$= (dz - dz')/(z - z')[i/i']$$
where \([i/i']\) denotes the insertion of the Lie algebra elements \(T_a T_a\) into the Wilson loop.

As a result, for each partition of the loop and choice of pairings \(P = \{x_i < x'_i | i = 1, \ldots, m\}\) we get an evaluation \(D_P\) of the trace of these insertions into the loop. This is the value of the corresponding chord diagram in the weight systems for Vassiliev invariants. These chord diagram evaluations then figure in our formula as shown below:

\[
<K> = \sum_m (1/k^m) \sum_P D_P \oint_{K_1 < \ldots < K_n} \left(\bigwedge_{i=1}^m (dz_i - dz'_i) / ((z_i - z'_i))\right)
\]

This is a Wilson loop ordering version of the Kontsevich integral. To see the usual form of the integral appear, we change from the time variable (parametrization) associated with the loop itself to time variables associated with a specific global direction of time in three dimensional space that is perpendicular to the complex plane defined by the axial gauge coordinates. It is easy to see that this results in one change of sign for each segment of the knot diagram supporting a pair correlation where the segment is oriented (Wilson loop parameter) downward with respect to the global time direction. This results in the rewrite of our formula to

\[
<K> = \sum_m (1/k^m) \sum_P (-1)^{|P\downarrow|} D_P \int_{t_1 < \ldots < t_n} \left(\bigwedge_{i=1}^m (dz_i - dz'_i) / ((z_i - z'_i))\right)
\]

where \(|P\downarrow|\) denotes the number of points \((z_i, t_i)\) or \((z'_i, t_i)\) in the pairings where the knot diagram is oriented downward with respect to global time. The integration around the Wilson loop has been replaced by integration in the vertical time direction and is so indicated by the replacement of \(\{K_1 < \ldots < K_n\}\) with \(\{t_1 < \ldots < t_n\}\).

The coefficients of \(1/k^m\) in this expansion are exactly the Kontsevich integrals for the weight systems \(D_P\). See Figure 14.

It was Kontsevich’s insight to see (by different means) that these integrals could be used to construct Vassiliev invariants from arbitrary weight systems satisfying the four-term relations. Here we have seen how these integrals arise naturally in the axial gauge fixing of the Witten functional integral.

**Remark.** The reader will note that we have not made a discussion of the role of the maxima and minima of the space curve of the knot with respect to the height direction \((t)\). In fact one has to take these maxima and minima very carefully into account and to divide by the corresponding evaluated loop pattern (with these maxima and minima) to make the Kontsevich integral well-defined and actually invariant under ambient isotopy (with appropriate framing correction as well). The
The corresponding difficulty appears here in the fact that because of the gauge choice the Wilson lines are actually only defined in the complement of the maxima and minima and one needs to analyse a limiting procedure to take care of the inclusion of these points in the Wilson line.

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