A CONSTRUCTION OF REPRESENTATIONS
AND QUANTUM HOMOGENEOUS SPACES

Pavel Šťovíček

Department of Mathematics
Faculty of Nuclear Science, CTU
Trojanova 13, 120 00 Prague, Czech Republic
stovicek@kmdec.fjfi.cvut.cz

Abstract. A simplified construction of representations is presented for the quantized enveloping algebra $\mathcal{U}_q(\mathfrak{g})$, with $\mathfrak{g}$ being a simple complex Lie algebra belonging to one of the four principal series $A_\ell$, $B_\ell$, $C_\ell$ or $D_\ell$. The carrier representation space is the quantized algebra of polynomials in antiholomorphic coordinate functions on the big cell of a coadjoint orbit of $K$ where $K$ is the compact simple Lie group with the Lie algebra $\mathfrak{k}$ – the compact form of $\mathfrak{g}$.

1. Motivation

Let $G$ be a simple and simply connected Lie group belonging to one of the four principal series $A_\ell$, $B_\ell$, $C_\ell$ or $D_\ell$, and $K \subset G$ its compact form. The symbols $\mathfrak{g}$ and $\mathfrak{k}$ designate the corresponding Lie algebras and $\ell$ is equal to the rank. The primary motivation was to find a quantum version of the construction of representations for the group $K$ via the method of orbits due to Kirillov and Kostant, with the result being expressed explicitly in terms of local holomorphic (or antiholomorphic) coordinates on the coadjoint orbit. But first let us consider briefly the classical case.

Each $G$ is a complex matrix group. The tautological (defining) representation $T$ is frequently called the vector representation. Let us denote by $N$ its dimension (i.e., $G \subset SL(N, \mathbb{C})$). Every coadjoint orbit $X = K_0 \backslash K = P_0 \backslash G$ is a homogeneous space for both $K$ and $G$ and so it is a compact complex manifold. We shall restrict ourselves to the generic orbits of the top dimension $\dim_{\mathbb{C}} X = (\dim_{\mathbb{C}} \mathfrak{g} - \ell)/2$. The local holomorphic coordinates on $X$ are introduced with the aid of Gauss decomposition. The factor mapping $G \to X = P_0 \backslash G$ sends an element $g \in G$ to a point belonging to the so called big cell if and only if there exists a decomposition $g = g(-)Z$ where $g(-)$ is a lower triangular matrix.
and $Z$ is upper triangular with units on the diagonal,

$$Z = \begin{pmatrix}
1 & z_{12} & \cdots & z_{1N} \\
0 & 1 & \cdots & z_{2N} \\
& & \cdots & \cdots \\
& & & 0 & 0 & \cdots & 1
\end{pmatrix}. \quad (1)$$

In the case of the series $B_{\ell}$, $C_{\ell}$ and $D_{\ell}$, the subgroup $G \subset SL(N,\mathbb{C})$ is determined by the equation $C_0 g^t C_0^{-1} = g^{-1}$ where $C_0$ is an appropriate real $N \times N$ matrix ($C_0)_{jk} = \pm \delta_{j+k,N+1}$). Consequently the matrix $Z$ must obey a similar condition,

$$C_0 Z^t C_0^{-1} = Z^{-1}. \quad (2)$$

In fact, the equality (2) reduces the number of independent coordinates $z_{jk}$ living on the big cell to the correct value $(\dim_{\mathbb{C}} g - \ell)/2$.

As usual, one makes use of the fact that irreducible unitary representations of $K$ are in one-to-one correspondence with finite-dimensional irreducible representations of the Lie algebra $g$ over $\mathbb{C}$. The right action of $G$ on the big cell can be described, too, with the help of Gauss decomposition. For $g \in G$ and $Z$ as above let us decompose, if possible, $Zg = (Zg)_{(-)} (Zg)_{(+)}$ where again $(Zg)_{(-)}$ is lower triangular and $(Zg)_{(+)}$ is upper triangular with units on the diagonal. The right action reads

$$Z \cdot g := (Zg)_{(+)} \quad (3)$$

Naturally, the right hand side of (3) is not well defined for all $g$ and $Z$, i.e., it has singularities. The reason is simple – the coordinates $z_{jk}$ are local while the action itself is global. However by differentiating the equality (3) one gets a well defined infinitesimal action $\xi : g \rightarrow \mathfrak{X}_C(X)$. Each element $x \in g$ is represented by a complex vector field $\xi_x$, with $\xi_x$ depending on $x$ linearly over $\mathbb{C}$, and it holds $[\xi_x, \xi_y] = \xi_{[x,y]}$.

The infinitesimal action $\xi$ doesn’t lead directly to a finite-dimensional irreducible representation of $g$. The result of the method of orbits, when looking at it through the local coordinates, is a correction achieved by adding to $\xi_x$ a holomorphic function $\varphi_x$ defined on the big cell (in fact, a polynomial in the coordinates $z_{jk}$) and depending on $x$ linearly. Thus elements from $g$ are represented by first order differential operators: $x \mapsto \xi_x + \varphi_x$.

The Lie bracket is preserved provided the function $\varphi_x$ fulfills the condition

$$\xi_x \cdot \varphi_y - \xi_y \cdot \varphi_x = \varphi_{[x,y]}. \quad (4)$$

The carrier vector space of the representation is built up from holomorphic functions on the big cell (in fact, from polynomials in $z_{jk}$), with the unit function as a cyclic vector.

A comparatively simple construction presented in this letter attempts to generalize the experience accumulated throughout the series of papers [1, 2, 3, 4, 5] and to simplify the procedures applied therein as much as possible. One should mention also the papers [6, 7] which were prior to the paper [5]. Even more, this construction makes it possible to deal with the general case, including the orthogonal and symplectic groups (the series $B_{\ell}$,
A CONSTRUCTION OF REPRESENTATIONS AND QUANTUM HOMOGENEOUS SPACES

C_ℓ and D_ℓ). Up to now, only some particular cases were treated in [4]. The main idea of the construction consists in finding a quantum analog to the function φ, including the compatibility condition (4).

Of course, apart of this local point of view there are known also other algebraic constructions, particularly global ones making use of the idea of induced representations and with the carrier representation space being formed by holomorphic sections in quantized line bundles over X. Let us mention just a few papers dealing with this subject: [8, 9, 10, 11].

2. Construction

Let us introduce the initial data. Assume we are given a bialgebra U with the counit denoted by ε and the comultiplication denoted by ∆ (the antipode will not be used and so U need not be a Hopf algebra) and a unital algebra C. Moreover, C is supposed to be a left U-module with the action denoted by ξ,

\[ U \otimes C \ni x \otimes f \mapsto \xi_x \cdot f \in C, \tag{5} \]

and fulfilling two conditions:

\[ \xi_x \cdot 1 = \varepsilon(x) 1, \quad \forall x \in U, \tag{6} \]
\[ \xi_x \cdot (fg) = (\xi_{x(1)} \cdot f)(\xi_{x(2)} \cdot g), \quad \forall x \in U, \ \forall f, g \in C. \tag{7} \]

If convenient we shall write ξ(x) ∙ f instead of ξ_x ∙ f. The second condition (7) is nothing but Leibniz rule. Here and everywhere in what follows we use Sweedler’s notation: \[ \Delta x = x_{(1)} \otimes x_{(2)}. \]

Note that the coassociativity of ∆ can be expressed in this formalism as

\[ x_{(1)}(1) \otimes x_{(1)}(2) \otimes x_{(2)} = x_{(1)} \otimes x_{(2)(1)} \otimes x_{(2)(2)}. \tag{8} \]

**Proposition 1.** Suppose that a linear mapping \( \varphi : U \to C \) satisfies

\[ \varphi(1) = 1, \tag{9} \]
\[ \varphi(xy) = (\xi_{x(1)} \cdot \varphi(y))\varphi(x_{(2)}), \quad \forall x, y \in U. \tag{10} \]

Then the prescription

\[ x \cdot f := (\xi_{x(1)} \cdot f)\varphi(x_{(2)}), \quad \forall x \in U, \ \forall f \in C, \tag{11} \]

defines a left U-module structure on C and it holds

\[ x \cdot (fg) = (\xi_{x(1)} \cdot f)(x_{(2)} \cdot g), \quad \forall x \in U, \ \forall f, g \in C. \tag{12} \]

Particularly,

\[ \varphi(x) = x \cdot 1, \quad \forall x \in U. \tag{13} \]
Conversely, suppose that $U \otimes C \to C : x \otimes f \mapsto x \cdot f$ is a left $U$-module structure on $C$ such that the rule (12) is satisfied. Then the linear mapping $\varphi : U \to C$ defined by the equality (13) fulfills (9) and (10), and consequently the prescription (11) holds true.

**Remarks.** The condition (10) generalizes (4). Moreover, (9) ”almost” follows from (10). More precisely, set $x = 1$ in (10) to get the equality $\varphi(y) = \varphi(y) \varphi(1), \forall y \in U$. So (9) is a consequence of (10) as soon as there exists at least one element $y \in U$ such that $\varphi(y)$ is not a left divisor of zero.

The property (12) can be regarded as a generalized Leibniz rule.

**Proof.** Let us consider only the first part of the proposition. All verifications are quite straightforward. We have

$$1 \cdot f = (\xi_1 \cdot f) \varphi(1) = f$$

(14)

and

$$x \cdot (y \cdot f) = (\xi_{x(1)} \cdot ((\xi_{y(1)} \cdot f) \varphi(y(2)))) \varphi(x(2))$$

$$= (\xi_{x(1)} \cdot (\xi_{y(1)} \cdot f)) (\xi_{x(1)(2)} \cdot \varphi(y(2))) \varphi(x(2))$$

$$= (\xi_{x(1)} y(1) \cdot f) (\xi_{x(2)(1)} \cdot \varphi(y(2))) \varphi(x(2)(2))$$

$$= (\xi_{x(1)} y(1) \cdot f) \varphi(x(2)y(2))$$

$$= (xy) \cdot f.$$ 

(15)

Furthermore,

$$x \cdot (fg) = (\xi_{x(1)} \cdot (fg)) \varphi(x(2))$$

$$= (\xi_{x(1)} \cdot f) (\xi_{x(2)(1)} \cdot g) \varphi(x(2))$$

$$= (\xi_{x(1)} \cdot f) (\xi_{x(2)(1)} \cdot g) \varphi(x(2)(2))$$

$$= (\xi_{x(1)} \cdot f)(x(2) \cdot g).$$

(16)

Finally,

$$x \cdot 1 = (\xi_{x(1)} \cdot 1) \varphi(x(2)) = \varphi(\varepsilon(x(1)) x(2)) = \varphi(x).$$

(17)

□

Here is a trivial example. Suppose that $\chi : U \to C$ is an algebra homomorphism (a character) and set $\varphi(x) = \chi(x) \cdot 1, \forall x \in U$. Then $\varphi$ fulfills both (9) and (10). Particularly, for $\chi = \varepsilon$ we recover the original $U$-module structure, i.e., $x \cdot f = \xi_x \cdot f$.

To deal with the function $\varphi$ let us suppose, as usual, that $U$ is generated as an algebra by a set of generators $M \subset U$. Let $F$ be the free algebra generated by $M$. Thus $U$ is identified with a quotient $F/\langle R \rangle$ where $\langle R \rangle$ is the ideal generated by a set of defining relations $R \subset F$. Let $\pi$ be the factor morphism, $\pi : F \to U$. Using $\pi$ one can pull back various structures from $U$ to $F$. Particularly we set

$$\bar{\varepsilon} := \varepsilon \circ \pi,$$

$$\bar{\xi}_x \cdot f := \xi_{\pi(x)} \cdot f, \forall x \in F, \forall f \in C.$$ 

(19)

Clearly, $\bar{\xi}(\langle R \rangle) = 0$. 

\[\text{\hfill } \]
In addition we assume that the set of generators \( \mathcal{M} \subset \mathcal{U} \) behaves well with respect to the comultiplication. More precisely, we impose the condition
\[
\Delta(\mathcal{M}) \subset \text{span}_\mathbb{C}(\mathcal{M}_1 \otimes \mathcal{M}_1) \quad \text{where} \quad \mathcal{M}_1 := \mathcal{M} \cup \{1\}. \tag{20}
\]
This property is also fulfilled quite frequently. Then it is natural to define a comultiplication \( \tilde{\Delta} \) on \( \mathcal{F} \) by the equality
\[
\tilde{\Delta}(x_1 \ldots x_n) := \Delta(x_1) \ldots \Delta(x_n), \quad x_i \in \mathcal{M}. \tag{21}
\]
As \( \mathcal{U} \) is a bialgebra \( \mathcal{R} \) must satisfy
\[
\tilde{\Delta}(\mathcal{R}) \subset \langle \mathcal{R} \rangle \otimes \mathcal{F} + \mathcal{F} \otimes \langle \mathcal{R} \rangle. \tag{22}
\]
In other words \( \langle \mathcal{R} \rangle \) is, at the same time, a coideal.

It is not difficult to check that \( \mathcal{F} \) becomes this way a bialgebra and that the triple \( (\mathcal{F}, \xi, \zeta) \) fulfills the original conditions (6) and (7), just replacing \( \mathcal{U} \) with \( \mathcal{F} \) and \( \xi \) with \( \xi \). Hence Proposition 1 can be applied to \( \mathcal{F} \) as well. But in the case of the free algebra it is rather easy to describe all linear mappings \( \tilde{\varphi} : \mathcal{F} \to \mathbb{C} \) with the desired properties.

**Lemma 2.** Let the symbols \( \mathcal{F} \) and \( \mathcal{M} \) have the same meaning as above. Then to any mapping \( \varphi : \mathcal{M} \to \mathbb{C} \) there exists a unique linear extension \( \tilde{\varphi} : \mathcal{F} \to \mathbb{C} \) such that \( \tilde{\varphi}(1) = 1 \) and the property
\[
\tilde{\varphi}(xy) = (\tilde{\xi}_{x(1)} \cdot \tilde{\varphi}(y)) \tilde{\varphi}(x(2)), \tag{23}
\]
is satisfied for all \( x, y \in \mathcal{F} \).

**Proof.** The algebra \( \mathcal{F} \) is naturally graded,
\[
\mathcal{F} = \sum_{n \in \mathbb{Z}_+} \mathcal{F}^{(n)} \quad \text{where} \quad \mathcal{F}^{(n)} := \text{span}_\mathbb{C}(\mathcal{M}_n). \tag{24}
\]
The mapping \( \tilde{\varphi} \) is prescribed on \( \mathcal{F}^{(0)} \oplus \mathcal{F}^{(1)} \). There is no contradiction here since (23) is automatically satisfied as soon as \( x = 1 \) and \( y \) is arbitrary (obvious) or \( x \) is arbitrary and \( y = 1 \) (the same verification as in (17)). One can proceed by induction in \( n \) in order to extend the definition of \( \tilde{\varphi} \) to all monomials \( x_1 \ldots x_n \in \mathcal{M}_n \) for all \( n \in \mathbb{Z}_+ \). The induction step \( n \to n + 1 \) is dictated by the rule (23). Thus we set
\[
\tilde{\varphi}(xy) := (\tilde{\xi}_{x(1)} \cdot \tilde{\varphi}(y)) \tilde{\varphi}(x(2)), \quad \forall x \in \mathcal{M}, \forall y \in \mathcal{M}_n. \tag{25}
\]
Note that the definition (25) makes good sense owing to the property (20). This means that there is a unique way how to extend \( \tilde{\varphi} \) from \( \mathcal{F}^{(0)} \oplus \mathcal{F}^{(1)} \) to the whole algebra \( \mathcal{F} \) so that \( \tilde{\varphi} \) is linear and the rule (23) holds true for all \( x \in \mathcal{M} \) and all \( y \in \mathcal{F} \). To finish the proof we have to show that this rule is actually satisfied for all \( x \in \mathcal{M}_n \) and all \( y \in \mathcal{F} \), with \( n \in \mathbb{Z}_+ \) being arbitrary. We shall again proceed by induction in \( n \). In order to carry
out the induction step $n \to n + 1$ let us suppose that $x \in \mathcal{M}$, $z \in \mathcal{M}^n$, and that it holds $\tilde{\varphi}(zy) = (\tilde{\xi}_z(1) \cdot \tilde{\varphi}(y))\tilde{\varphi}(z(2))$, $\forall y \in \mathcal{F}$. Then

$$\tilde{\varphi}(xyz) = \left(\tilde{\xi}_x(1) \cdot ((\tilde{\xi}_z(1) \cdot \tilde{\varphi}(y))\tilde{\varphi}(z(2)))\right)\varphi(x(2))$$

$$= \left(\tilde{\xi}_x(1) \cdot (\tilde{\xi}_z(1) \cdot \tilde{\varphi}(y))(\tilde{\xi}_x(2) \cdot \tilde{\varphi}(z(2)))\right)\varphi(x(2))$$

$$= \left(\tilde{\xi}_x(1) \cdot z(1) \cdot \tilde{\varphi}(y))(\tilde{\xi}_x(2) \cdot \tilde{\varphi}(z(2)))\varphi(x(2)(1))\right)\varphi(x(2)(2))$$

$$= \left(\tilde{\xi}_x(1) \cdot z(1) \cdot \tilde{\varphi}(y))\varphi(x(2)z(2)).\right)$$

(26)

Thus we have verified that (23) holds also true with $x$ being replaced by $xz$. □

The final step of the construction is to decide under which conditions the mapping $\tilde{\varphi}$ admits factorization from $\mathcal{F}$ to $\mathcal{U} = \mathcal{F}/⟨R⟩$.

**Proposition 3.** Suppose there is given a mapping $\varphi : \mathcal{M} \to \mathcal{C}$ and let $\tilde{\varphi}$ be its extension to $\mathcal{F}$ as described in Lemma 2. If

$$(\pi \otimes \tilde{\varphi}) \circ \tilde{\Delta}(R) = 0$$

(27)

then $\tilde{\varphi}(⟨R⟩) = 0$ and so there exists a unique linear mapping $\varphi' : \mathcal{U} \to \mathcal{C}$ such that $\tilde{\varphi} = \varphi' \circ \pi$. Moreover, $\varphi'$ satisfies the conditions (9) and (10).

The same conclusions hold true provided $R$ fulfills a stronger condition than (22), namely

$$\tilde{\Delta}(R) \subset ⟨R⟩ \otimes \mathcal{F} + \mathcal{F} \otimes \mathcal{F}R,$$  

(28)

while $\tilde{\varphi}$ satisfies a weaker condition

$$\tilde{\varphi}(R) = 0.$$  

(29)

**Remark.** Notice that $\varphi'|\mathcal{M} = \varphi|\mathcal{M}$ and so we have the right to suppress the dash in the notation and this is what we shall do from now on.

**Proof.** We have to show that $\tilde{\varphi}(xyz) = 0$ holds true for all $x, z \in \mathcal{F}$ and all $y \in R$. According to Proposition 1 there is a unique left $\mathcal{F}$-module structure on $\mathcal{C}$ associated with $\tilde{\varphi}$ and we have $\tilde{\varphi}(u) = u \cdot 1$, $\forall u \in \mathcal{F}$. Suppose that $x, z \in \mathcal{F}$ and $y \in R$. The assumption (27) then implies

$$(xyz) \cdot 1 = x \cdot ((\tilde{\xi}_y(1) \cdot \tilde{\varphi}(z))\tilde{\varphi}(y(2))) = 0.$$  

(30)

This verifies the existence of a linear mapping $\varphi : \mathcal{U} \to \mathcal{C}$ as claimed.

More generally, we find that $x \cdot f = (\tilde{\xi}_x(1) \cdot f)\tilde{\varphi}(x(2)) = 0$ holds true for all $x \in ⟨R⟩$ and all $f \in \mathcal{C}$. Consequently the left action $\mathcal{F} \otimes \mathcal{C} \to \mathcal{C}$ can be factorized from $\mathcal{F}$ to $\mathcal{U} = \mathcal{F}/⟨R⟩$ and the obtained action $\mathcal{U} \otimes \mathcal{C} \to \mathcal{C}$ is associated with the found mapping $\varphi : \mathcal{U} \to \mathcal{C}$. Proposition 1 (the second part) then concludes the proof.

As for the second part one can employ the property (28) to show that the condition (27) follows from (29). □
3. Examples

Now we are going to apply the construction described in the previous section to the case when \( \mathcal{U} = \mathcal{U}_q(\mathfrak{g}) \) is the quantized enveloping algebra in the sense of Drinfeld [12] and Jimbo [13], and \( \mathcal{C} \) is the quantized big cell of the orbit \( X = K_0 \backslash K = P_0 \backslash G \) considered as a complex manifold [3]. In what follows we prefer the antiholomorphic coordinates to the holomorphic ones. We assume that the deformation parameter \( q > 0, q \neq 1 \), and we set \( [x] := (q^x - q^{-x})/(q - q^{-1}) \) for any \( x \in \mathbb{C} \).

3.1 The particular case \( \mathfrak{g} = \mathfrak{sl}(2, \mathbb{C}) \)

First we wish to treat separately the simplest particular case when \( \mathfrak{g} = \mathfrak{sl}(2, \mathbb{C}) \) and \( X \) is just the Riemannian sphere (one-dimensional complex projective space). This basic example of a quantum homogeneous space was analyzed by Podles [14]. Afterwards it was reconsidered several times from various points of view. For example, it fits well the general scheme of deformation quantization [15] as proposed in [16]. In our approach we prefer the local description in terms of coordinates on the big cell denoted by \( z, \bar{z} \) [1]. This point of view was further developed in [17]. However when restricting ourselves to the antiholomorphic part we are left with the algebra of polynomials \( \mathcal{C} = \mathbb{C}[\bar{z}] \), the same one as in the classical case.

We choose the standard set of generators \( \mathcal{M} = \{q^{H/2}, q^{-H/2}, X^+, X^-\} \). The defining relations are also the usual ones:

\[
q^{H/2}q^{-H/2} - 1 = q^{-H/2}q^{H/2} - 1 = q^{H/2}X^\pm - q^{\pm1}X^\pm q^{H/2} = 0, \\
[X^+, X^-] - (q - q^{-1})^{-1}(q^H - q^{-H}) = 0
\]

(31)

(of course, \( q^{\pm H} \equiv (q^{H/2})^2 \)). Let us recall, too, the formulae for the comultiplication and the counit:

\[
\Delta(q^{\pm H/2}) = q^{\pm H/2} \otimes q^{\pm H/2}, \quad \Delta(X^\pm) = X^\pm \otimes q^{-H/2} + q^{H/2} \otimes X^\pm, \\
\varepsilon(q^{\pm H/2}) = 1, \quad \varepsilon(X^\pm) = 0.
\]

(32)

The left action \( \xi \) of \( \mathcal{U} \) on \( \mathcal{C} \) has been derived explicitly in [2] for a more general case (see also [5]). After some rescaling it reads

\[
\xi(q^{\pm H/2}) \cdot z^n = q^{\pm n}z^n, \quad \xi(X^+) \cdot z^n = [n] z^{n+1}, \quad \xi(X^-) \cdot z^n = -[n] z^{n-1}, \quad \forall n \in \mathbb{Z}_+.
\]

(33)

Now we introduce a mapping \( \varphi : \mathcal{M} \to \mathcal{C} \) by

\[
\varphi(q^{\pm H/2}) = q^{\mp \sigma/2} 1, \quad \varphi(X^+) = -q^{-\sigma/2}[\sigma] \bar{z}, \quad \varphi(X^-) = 0
\]

(34)

where \( \sigma \) is a complex parameter. Next one has to verify the assumptions of Proposition 3. It is convenient to add to \( \mathcal{R} \) two other dependent relations, namely \( X^\pm q^{-H/2} - q^{\pm1}q^{-H/2}X^+ = 0 \), getting this way a new set of relations \( \mathcal{R}' \). Naturally, \( \mathcal{R} \) and \( \mathcal{R}' \) define the same algebra. The advantage of this step is, however, that \( \mathcal{R}' \) obeys the condition \( \Delta(\mathcal{R}') \subset \mathcal{R}' \otimes \mathcal{F} + \mathcal{F} \otimes \mathcal{R}' \), as one can check by a direct computation. According to the second part of Proposition 3 it suffices to verify the equality \( \tilde{\varphi}(\mathcal{R}') = 0 \) rather than the more
complicated assumption (27). This is again a matter of a straightforward computation. Applying the prescription (11) we arrive at formulae for the new action:

\[ q^{\pm H/2} \cdot z^n = q^{\mp (\sigma - 2n)/2} z^n, \quad X^+ \cdot z^n = q^{-\sigma/2} [n - \sigma] z^{n+1}, \quad X^- \cdot z^n = -q^{\sigma/2} [n] z^{n-1}, \]  

valid for all \( n \in \mathbb{Z}_+ \).

Observe that for \( \sigma \in \mathbb{Z}_+ \) the unit generates a finite-dimensional submodule, \( \mathcal{U} \cdot 1 = \text{span}_\mathbb{C} \{1, z, \ldots, z^\sigma\} \). This is how we get finite-dimensional irreducible representations of the algebra \( \mathcal{U} \). In fact, this is a consequence of a more general result about representations of \( \mathcal{U}_q(\mathfrak{g}) \) [18, 19]. Actually, the unit is in this case a cyclic vector and, at the same time, a lowest weight vector \( (X^- \cdot 1 = 0) \) and so the submodule \( \mathcal{U} \cdot 1 \) is unambiguously determined, up to isomorphism, by the lowest weight given by \( q^H \cdot 1 = q^{-\sigma} 1 \).

### 3.2 The general case

In the general case we prefer the description of \( \mathcal{U} = \mathcal{U}_q(\mathfrak{g}) \) due to Faddeev–Reshetikhin–Takhtajan [20]. The generators are arranged in respectively upper and lower triangular matrices \( L^+ \) and \( L^- \) of size \( N \times N \) and obeying the defining relations

\[
R_{12} L^\pm_1 L^\pm_2 = L^\pm_2 L^\pm_1 R_{12}, \quad \text{diag}(L^+) \text{diag}(L^-) = \text{diag}(L^-) \text{diag}(L^+) = \mathbb{I}, \quad \det(L^+) = 1.
\]  

Furthermore, the comultiplication and the counit are determined by

\[
\Delta(L^\pm) = L^\pm \otimes L^\pm, \quad \varepsilon(L^\pm) = \mathbb{I}
\]  

(as usual, \( (A \otimes B)_{ij} := \sum_k A_{ik} \otimes B_{kj} \)).

Here \( R \) is the standard R-matrix obeying the Yang-Baxter equation \( R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12} \) (c.f. [21] and also [22, 20]) and \( \det(L^+) \) is just the product of diagonal entries. Let us recall that \( R \) is lower triangular (in the lexicographic ordering of indices), \( R_{12} = R_{21} \), and \( R_{jk,jt} = 0 \) for \( k \neq t \).

For the series \( B_t, C_t \) and \( D_t \) there are two additional relations, namely \( (CL^\pm C^{-1})^t = (L^\pm)^{-1} \) where \( C \) is a \( q \)-deformation of the "classical" matrix \( C_0 \) (occurring, for example, in the formula (2)). After having introduced the \( N^2 \times N^2 \) matrix \( K \) defined by \( K_{jk, st} := C^t_{jk} C^{-1}_{st} \) one can rewrite these relations as

\[
K_{12} L^\pm_2 L^\pm_1 = K_{12}.
\]  

The matrix \( K \) is related to the R-matrix by the equality

\[
R_{12} - R_{21}^{-1} = (q - q^{-1})(P - K_{12})
\]  

where \( P \) stands for the flip operator \( (P_{jk, st} = \delta_{jt} \delta_{ks}) \). Taking (39) for the definition of \( K \) one finds that \( K = 0 \) for the series \( A_t \). Naturally, the conditions (38) become in this case trivial.

There exist several useful identities involving the matrices \( K \) and \( R \) [4]. Here we mention just the equalities

\[
K_{12} R_{31}^{-1} = K_{12} R_{32}, \quad K_{12} R_{23}^{-1} = K_{12} R_{13},
\]
and the implication
\[ K_{12}D_1D_2 = K_{12} \implies R_{12}D_1D_2 = D_1D_2R_{12} \]  
valid for any complex diagonal matrix \( D \). Particularly, set
\[ Q := \text{diag}(R) \quad \text{(then } Q_{12} = Q_{21}). \]
It holds true that
\[ K_{12}Q_{13}Q_{23} = K_{12} \quad \text{and} \quad R_{12}Q_{13}Q_{23} = Q_{13}Q_{23}R_{12}. \]

Let us now describe the quantized big cell for the generic coadjoint orbit of \( K \) regarded as a complex manifold [3]. The generators \( z_{jk} \), \( 1 \leq j < k \leq N \), can be arranged in an upper-triangular matrix \( Z \) as given in (1). The commutation relations then read
\[ R_{12}Q^{-1}Z_1QZ_2 = Q^{-1}Z_2QZ_1R_{12}. \]
As of this one imposes an additional ”orthogonality” condition (trivial for the series \( A_\ell \)):
\[ K_{12}Q^{-1}Z_1QZ_2 = K_{12} \]  
(this is a simplified but equivalent form to that given in the formula (7.14) in [3] and in the formula (3.10) in [4]). But as one can check by a simple manipulation (45) is already a consequence of (44) and need not be accounted. For the series \( A_\ell, C_\ell \) and \( D_\ell \) the matrix \( Q = \text{diag}(R) \) commutes with \( R \) and so (44) can be simplified to \( RZ_1QZ_2 = Z_2QZ_1R \). However this is not the case for the series \( B_\ell \) (this fact was not recognized in [3]) and so one has to keep the general form (44). As already mentioned, here we construct the algebra \( \mathcal{C} \) as being generated by the ”antiholomorphic” generators \( z_{jk}^* \) arranged in the matrix \( Z^* \) – the Hermitian adjoint to \( Z \). The corresponding commutation relation is the Hermitian adjoint to (44), namely
\[ R_{12}Z_2^*QZ_1^*Q^{-1} = Z_1^*QZ_2^*Q^{-1}R_{12}. \]

The left action \( \xi \) is dual to the right quantum dressing transformation \( \mathcal{R} : \mathcal{C} \rightarrow \mathcal{C} \otimes \text{Fun}_q(G) \). Here \( \text{Fun}_q(G) \) is the Hopf algebra of quantum functions living on the group \( G \) and it is generated by entries of a matrix \( T \) – the vector corepresentation of \( \text{Fun}_q(G) \). The dressing transformation of the holomorphic part formally coincides with the classical action (3), namely \( \mathcal{R}(Z) = (ZT)_{(+)} \) where on the right hand side we have identified \( \mathcal{C} \) with \( \mathcal{C} \otimes 1 \) and \( \text{Fun}_q(G) \) with \( 1 \otimes \text{Fun}_q(G) \). To get the dressing transformation of \( Z^* \) one can simply apply the \(*\)-involution. But before doing it one has to pass from \( \text{Fun}_q(G) \) to the compact form \( \text{Fun}_q(K) \) which means nothing but introducing a \(*\)-involution on \( \text{Fun}_q(G) \) by \( T^* := T^{-1} \).

The left action \( \xi \) is defined by
\[ \xi_x \cdot f := (\text{id} \otimes \langle x, \cdot \rangle) \mathcal{R}(f). \]
The dual pairing between $U_q(\mathfrak{g})$ and $\text{Fun}_q(G)$ is prescribed on the generators as follows [20]:

$$\langle L^+_1; T_2 \rangle = R_{21}, \quad \langle L^-_1; T_2 \rangle = R_{12}^{-1}.$$  

(48)

A straightforward computation then gives the desired action:

$$\xi(L^+_1) \cdot Z_2^* = R_{21}^{-1} Z_2^* Q, \quad \xi(L^-_1) \cdot Z_2^* = Z_1^* Q Z_2^* Q^{-1}(Z_1^*)^{-1}. \quad (49)$$

It can be extended to an arbitrary element from $C$ with the aid of Leibniz rule (7) and the prescription for comultiplication (37).

Let us specify the mapping $\varphi$ on the generators:

$$\varphi(L^+) = D^{-1}, \quad \varphi(L^-) = Z^* D^2 (Z^*)^{-1} D^{-1} \quad (50)$$

where $D$ is an arbitrary complex diagonal matrix obeying the conditions

$$\det(D) = 1 \quad \text{and} \quad K_{12} D_1 D_2 = K_{12} \quad (51)$$

(the former one follows from the latter one in the case of the series $B_t, C_t$ and $D_t$). It is easy to show that the set of defining relations $R$ corresponding to the equalities (36) and (38) obeys the condition $\Delta(R) \subset R \otimes \mathcal{F} + \mathcal{F} \otimes R$. Thus one can again apply the second part of Proposition 3 to conclude that it suffices to verify the equality $\tilde{\varphi}(R) = 0$ rather than the assumption (27). This is a matter of a straightforward computation (based on the rule (23)) to find that

$$\tilde{\varphi}(L^+_1 L^+_2) = D_1^{-1} D_2^{-1}, \quad \tilde{\varphi}(L^-_1 L^-_2) = Z_1^* Q Z_2^* D_1^2 D_2^2 (Z_1^*)^{-1} Q^{-1}(Z_1^*)^{-1} D_1^{-1} D_2^{-1}, \quad (52)$$

$$\tilde{\varphi}(L^+_2 L^-_1) = R_{12}^{-1} Z_1^* D_1^2 (Z_1^*)^{-1} D_1^{-1} D_2^{-1} R_{12}, \quad \tilde{\varphi}(L^-_1 L^+_2) = Z_1^* D_1^2 (Z_1^*)^{-1} D_1^{-1} D_2^{-1}.$$
3.3 Twisted adjoint action

Here we wish to give another description of the preceding example while abandoning the geometric terminology and relying instead on the notion of a Verma module. The construction of the modified action presented in Section 2 then yields exactly the so called twisted adjoint action given in the book [24], §5.3.10 (this fact has been pointed out to the author by a referee). We warn the reader however that, if compared with [24], the role of the subalgebras $b_+, b_- \subset g$ is interchanged and the generators of $g$ are partially rescaled. The description below is rather brief and with some details omitted.

For the generators of $U_q(g)$ we chose $e_i = q^{H_i/2}X_i^+, f_i = X_i^q^{-H_i/2}, t_i^{\pm 1} = q^{\pm H_i}$, with the index $i$ enumerating a set of simple roots $\{\alpha_i\}_i$. Thus the defining relations read

$$[e_i, f_j] = \delta_{ij} \frac{t_i - t_i^{-1}}{q - q^{-1}}, \ t_ie_jf_i^{-1} = q^{\langle \alpha_i, \alpha_j \rangle}e_j, \ t_if_jt_i^{-1} = q^{-\langle \alpha_i, \alpha_j \rangle}f_j,$$

plus the quantum Serre relations. Let us also recall the comultiplication,

$$\Delta(e_i) = e_i \otimes 1 + t_i \otimes e_i, \ \Delta(f_i) = f_i \otimes t_i^{-1} + 1 \otimes e_i, \ \Delta(t_i) = t_i \otimes t_i. \quad (54)$$

In this particular case we shall need the antipode which is given by

$$\sigma(e_i) = -t_i^{-1}e_i, \ \sigma(f_i) = -f_it_i, \ \sigma(t_i) = t_i^{-1}. \quad (55)$$

The Hopf subalgebra $U_q(b) \subset U_q(g)$ is generated by the elements $e_i, t_i^{\pm 1}$, and the symbol $U_q(n)$ designates the subalgebra of $U_q(g)$ generated by the elements $e_i$ (no comultiplication is defined).

The counit is given as usual ($\varepsilon(e_i) = \varepsilon(f_i) = 0, \varepsilon(t_i) = 1$) and its restriction determines a one-dimensional $U_q(b)$ module denoted by $V_\varepsilon$. Verma module $M(0)$ with highest weight zero is introduced by

$$M(0) = V_\varepsilon \otimes_{U_q(b)} U_q(g). \quad (56)$$

Thus the dual space $M(0)^*$ is a unital algebra, with the unit being induced by the counit in $U_q(g)$. Furthermore, the right $U_q(g)$ action on $M(0)$ induces a left $U_q(g)$ action $\xi$ on $M(0)^*$. It is easy to see that $\xi$ obeys (6), (7). In fact, to avoid ill defined expressions one considers the subalgebra $M(0)^*_f \subset M(0)^*$ formed by elements $w$ with the property $\dim \xi(U_q(b)) \cdot w < \infty$ (the action of the Cartan subalgebra $U_q(b) \subset U_q(g)$ is required to be locally finite). The $U_q(g)$-module algebra $M(0)^*_f$ is nothing but the algebra $C$ used in the previous subsection.

According to Corollary 5.3.6 of [24] the $U_q(g)$ module $M(0)^*_f$ is isomorphic to the algebra $U_q(n)$. The action on the latter module is induced by the adjoint action

$$\text{ad}_x y = x_{(1)}y \sigma(x_{(2)}), \ \forall x, y \in U_q(g). \quad (57)$$
In more detail, consider the filtration $F$ on $U_q(g)$ given by $\deg(f_i) = 1$, $\deg(e_i) = 0$ and $\deg(t_i) = -1$. The filtration is ad-invariant and consequently there is an induced action of $U_q(g)$ on $gr_F U_q(g)$. One observes that $gr_F U_q(n)$ is ad-invariant and can be identified with $U_q(n)$ as an algebra. It is not difficult to find that the $U_q(g)$ action is prescribed on the generators of $U_q(n)$ as follows:

$$\xi(e_i) \cdot e_j = e_i e_j - q^{(\alpha_i, \alpha_j)} e_j e_i, \quad \xi(f_i) \cdot e_j = \frac{\delta_{ij}}{q - q^{-1}} 1, \quad \xi(t_i) \cdot e_j = q^{(\alpha_i, \alpha_j)} e_j$$

(58)

(and $\xi(x) \cdot 1 = \varepsilon(x) 1, \forall x \in U_q(g)$). The action extends to the whole algebra $U_q(n)$ with the aid of Leibniz rule (7).

Finally, as described in Proposition 1, the action $\xi$ admits a modification with the aid of a mapping $\varphi : U_q(g) \to U_q(n)$. The mapping is unambiguously defined by its values on generators:

$$\varphi(e_i) = (1 - q^{2(\lambda, \alpha_i)}) e_i, \quad \varphi(f_i) = 0, \quad \varphi(t_i) = q^{(\lambda, \alpha_i)} 1$$

(59)

where $\lambda \in \mathfrak{h}^*$ is a weight.

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