PUNCTUAL HILBERT SCHEMES FOR KLEINIAN SINGULARITIES

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Abstract. For a finite subgroup $\Gamma \subset \text{SL}(2, \mathbb{C})$ and $n \geq 1$, we construct the (reduced scheme underlying the) Hilbert scheme of $n$ points on the Kleinian singularity $\mathbb{C}^2/\Gamma$ as a Nakajima quiver variety for the framed McKay quiver of $\Gamma$, taken at a specific non-generic stability parameter. We deduce that this Hilbert scheme is irreducible (a result previously due to Zheng), normal, and admits a unique symplectic resolution. More generally, we introduce a class of algebras obtained from the preprojective algebra of the framed McKay quiver by a process called cornering, and we show that fine moduli spaces of cyclic modules over these new algebras are isomorphic to quiver varieties for the framed McKay quiver and certain non-generic choices of stability parameter.

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1. Introduction

Let $\Gamma \subset \text{SL}(2, \mathbb{C})$ be a finite subgroup. One can associate various Hilbert schemes to the action of $\Gamma$ on the affine plane $\mathbb{C}^2$ and the Kleinian singularity $\mathbb{C}^2/\Gamma$. For $N := |\Gamma|$ and any natural number $n$, the action of $\Gamma$ on $\mathbb{C}^2$ induces an action of $\Gamma$ on the Hilbert scheme $\text{Hilb}^{[nN]}(\mathbb{C}^2)$ of $nN$ points on the affine plane. The scheme $n\Gamma$-$\text{Hilb}(\mathbb{C}^2)$, parametrising $\Gamma$-invariant ideals $I$ in $\mathbb{C}[x, y]$ such that the quotient $\mathbb{C}[x, y]/I$ is isomorphic to the direct sum of $n$ copies of the regular representation of $\Gamma$, is a union of components of the fixed point set of the $\Gamma$-action on $\text{Hilb}^{[nN]}(\mathbb{C}^2)$. It is thus nonsingular and quasi-projective. One may also consider the Hilbert scheme of $n$ points $\text{Hilb}^{[n]}(\mathbb{C}^2/\Gamma)$ on the singular surface $\mathbb{C}^2/\Gamma$, parametrising ideals in the invariant ring $\mathbb{C}[x, y]^\Gamma$ that have codimension $n$. This Hilbert scheme is quasi-projective, and in this introduction we endow it with the reduced scheme structure.

These two kinds of Hilbert schemes are related by the morphism

$$n\Gamma$-$\text{Hilb}(\mathbb{C}^2) \rightarrow \text{Hilb}^{[n]}(\mathbb{C}^2/\Gamma)$$  (1.1)
sending a $\Gamma$-invariant ideal $I$ in $\mathbb{C}[x, y]$ to the ideal $I \cap \mathbb{C}[x, y]^{\Gamma}$; this set-theoretic map is indeed a morphism of schemes by Brion [Bri13, 3.4]. By composing with the Hilbert-Chow morphism of the surface $\mathbb{C}^2/\Gamma$, we see that (1.1) is in fact a morphism of schemes over the affine scheme $\text{Sym}^n(\mathbb{C}^2/\Gamma)$. Until recently, not much was known about the schemes $\text{Hilb}^{|n|}(\mathbb{C}^2/\Gamma)$ for $n > 1$. Gyenge, Némethi and Szendrői [GNS18] computed the generating function of their Euler characteristics for $\Gamma$ of type A and D (the cyclic and dihedral cases), giving an answer with modular properties. Zheng [Zhe17] proved that $\text{Hilb}^{|n|}(\mathbb{C}^2/\Gamma)$ is always irreducible, and gave a homological characterisation of its smooth points through a detailed analysis of Cohen-Macaulay modules over $\mathbb{C}^2/\Gamma$. Yamagishi [Yam17] studied symplectic resolutions of the Hilbert squares $\text{Hilb}^{|2|}(\mathbb{C}^2/\Gamma)$, and described completely the central fibres of these resolutions, from which he deduced that $\text{Hilb}^{|2|}(\mathbb{C}^2/\Gamma)$ admits a unique symplectic resolution.

The aim of our paper is to study the spaces appearing in (1.1), and all possible ways in which the morphism from (1.1) can be decomposed, using quiver-theoretic techniques in a uniform way. The starting point is the McKay correspondence, which associates a quiver (oriented graph) to the subgroup $\Gamma \subset \text{SL}(2, \mathbb{C})$. Representation spaces of a framed variant of the McKay quiver, each depending on a stability parameter, were introduced in Kronheimer and Nakajima [KN90] and studied further by Nakajima [Nak94]. Subsequently, for any $n \geq 1$ and for a special choice of framing, Kuznetsov [Kuz07] determined a pair of cones $C_{\pm}$ in the space of stability parameters for which the corresponding representation space $\mathcal{M}_0$ is isomorphic to the punctual Hilbert scheme $\text{Hilb}^{|n|}(S)$ of the minimal resolution $S$ of $\mathbb{C}^2/\Gamma$ for $\theta \in C_-$, and to the scheme $n \Gamma$- Hilb($\mathbb{C}^2$) from (1.1) for $\theta \in C_+$. More recently, Bellamy and Craw [BC18] gave a complete description of the wall-and-chamber structure on the space of stability parameters in this situation, and identified a simplicial cone $F$ containing $C_{\pm}$ that is isomorphic as a fan to the movable cone of $n \Gamma$- Hilb($\mathbb{C}^2$) for $n > 1$; in particular, chambers in this simplicial cone correspond one-to-one with projective, symplectic resolutions of $\text{Sym}^n(\mathbb{C}^2/\Gamma)$ (see Figure 1 below for an example).

The main result of our paper reconstructs the morphism from (1.1) by variation of GIT quotient. Explicitly, we vary a generic stability parameter $\theta \in C$ to a parameter $\theta_0$ in a particular extremal ray of the closure of $C_+$; the induced morphism $\mathcal{M}_\theta \to \mathcal{M}_{\theta_0}$ coincides with the morphism (1.1). As a corollary, we obtain the following result.

**Theorem 1.1.** Let $\Gamma \subset \text{SL}(2, \mathbb{C})$ be a finite subgroup and let $n \geq 1$. The (reduced) Hilbert scheme $\text{Hilb}^{|n|}(\mathbb{C}^2/\Gamma)_{\text{red}}$ is an irreducible, normal scheme with symplectic, hence rational Gorenstein, singularities. Furthermore, it admits a unique projective, symplectic resolution given by (1.1).

We reiterate that irreducibility is due originally to Zheng [Zhe17]. The existence of a nowhere-vanishing $2n$-form in the type A case, which follows from having symplectic singularities, was shown in the same paper [Zhe17, Theorem D], while the existence and uniqueness of the symplectic resolution for $n = 2$ is due to Yamagishi [Yam17].

Our main tool is to furnish $\text{Hilb}^{|n|}(\mathbb{C}^2/\Gamma)$ with a quiver-theoretic interpretation as a fine quiver moduli space by the process of cornering [CIK18]. More generally, we provide a fine moduli space description of the quiver varieties $\mathcal{M}_\theta$ for all non-generic stability parameters that lie in the closure of the cone $C_+$. Our methods give conceptual proofs of the geometric properties of $\text{Hilb}^{|n|}(\mathbb{C}^2/\Gamma)$ listed in Theorem 1.1, and allow us to obtain all possible projective factorisations of the morphism (1.1) by universal properties of the resulting fine moduli spaces. Our proofs avoid case-by-case analysis, with the exception of a bound on the
Acknowledgements. S.G. is supported by an Aker Scholarship. A.Gy. and B.Sz. are supported by EPSRC grant EP/R045038/1. We thank Gwyn Bellamy and Ben Davison for helpful discussions.

Notation. Let \( \pi : X \to Y \) be a projective morphism of schemes over an affine base \( Y \). For a globally generated line bundle \( L \) on \( X \), write \( |L| := \text{Proj}_Y \bigoplus_{k \geq 0} H^0(X, L^k) \) for the (relative) linear series of \( L \), and \( \varphi_{|L|} : X \to |L| \) for the induced morphism over \( Y \).

2. Variation of GIT quotient for quiver varieties

Let \( \Gamma \subset \text{SL}(2, \mathbb{C}) \) be a finite subgroup. Let \( V \) denote its given two-dimensional representation, defined by this inclusion. Write \( \rho_0, \rho_1, \ldots, \rho_r \) for the irreducible representations of \( \Gamma \), with \( \rho_0 \) the trivial one. The McKay graph of \( \Gamma \) has vertex set \( \{0,1,\ldots,r\} \) where vertex \( i \) corresponds to the representation \( \rho_i \) of \( \Gamma \), and there are \( \dim \text{Hom}_\Gamma(\rho_j, \rho_i \otimes V) \) edges between vertices \( i \) and \( j \). By the McKay correspondence [McK80], the McKay graph is an extended Dynkin diagram of ADE type. Add a framing vertex \( \infty \), together with an edge between vertices \( \infty \) and 0, and let \( Q_1 \) be the set of pairs consisting of an edge in this graph and an orientation of the edge. If \( a \) is an edge with orientation, we write \( a^* \) for the same edge with the opposite orientation. The framed McKay quiver \( Q \) has vertex set \( Q_0 = \{\infty,0,1,\ldots,r\} \) and arrow set \( Q_1 \), where for each oriented edge \( a \in Q_1 \) we write \( t(a), h(a) \) for the tail and head of \( a \) respectively.

Let \( \mathbb{C}Q \) denote the path algebra of \( Q \). For \( i \in Q_0 \), let \( e_i \in \mathbb{C}Q \) denote the idempotent corresponding to the trivial path at vertex \( i \). Let \( \epsilon : Q_1 \to \{\pm 1\} \) be any map such that \( \epsilon(a) \neq \epsilon(a^*) \) for all \( a \in Q_1 \). The preprojective algebra \( \Pi \) is the quotient of \( \mathbb{C}Q \) by the ideal generated by the relation

\[
\sum_{a \in Q_1} \epsilon(a)aa^*.
\]

Equivalently, multiplying both sides of this relation by the idempotent \( e_i \) shows that \( \Pi \) can be presented as the quotient of \( \mathbb{C}Q \) by the ideal generated by the relation

\[
\left( \sum_{h(a) = i} \epsilon(a)aa^* \mid i \in Q_0 \right) .
\]  

(2.1)

The preprojective algebra \( \Pi \) does not depend on the choice of the map \( \epsilon \) [CBH98, Lemma 2.2]. Let \( R(\Gamma) \) denote the representation ring of \( \Gamma \). Introduce a formal symbol \( \rho_\infty \) so that \( \{\rho_i \mid i \in Q_0\} \) provides a \( \mathbb{Z} \)-basis for \( \mathbb{Z}^{Q_0} \cong \mathbb{Z} \oplus R(\Gamma) \) considered as \( \mathbb{Z} \)-modules.

For a natural number \( n \geq 1 \) that we fix for the rest of the paper, consider the dimension vector

\[
v := (v_i)_{i \in Q_0} := \rho_\infty + \sum_{i \geq 0} n \dim(\rho_i) \rho_i \in \mathbb{Z}^{Q_0}.
\]

The group \( G(v) := \mathbb{C}^\times \times \prod_{0 \leq i \leq r} \text{GL}(n \dim(\rho_i), \mathbb{C}) \) acts on the space \( \text{Rep}(Q, v) := \bigoplus_{a \in Q_1} \text{Hom}(\mathbb{C}^{v(h(a))}, \mathbb{C}^{v(t(a))}) \) of representations of the quiver \( Q \) of dimension vector \( v \) by conjugation. The diagonal scalar subgroup acts trivially, and the action of the quotient \( G := G(v)/\mathbb{C}^\times \) induces a moment map \( \mu : \text{Rep}(Q, v) \to \mathfrak{g}^* \) such that a closed point lies in \( \mu^{-1}(0) \) if and only if the corresponding \( \mathbb{C}Q \)-module satisfies the relations (2.1) of the preprojective algebra \( \Pi \). If we write

\[
\Theta_v = \{ \theta : \mathbb{Z}^{Q_0} \to \mathbb{Q} \mid \theta(v) = 0 \},
\]
then each character of $G$ is $\chi_\theta: G \to \mathbb{C}^\times$ for some integer-valued $\theta \in \Theta_v$, where $\chi_\theta(g) = \prod_{i \in Q_0} \det(g_i)^{-\theta_i}$ for $g \in G(v)$.

Given a stability parameter $\theta \in \Theta_v$, recall that a $\Pi$-module $M$ is $\theta$-stable (respectively semistable) if $\theta(\dim M) = 0$ and for every proper, nonzero submodule $N \subset M$, we have $\theta(\dim N) > 0$ (respectively $\theta(\dim N) \geq 0$). Two $\theta$-semistable $\Pi$-modules $M, M'$ are said to be $S$-equivalent, if they admit filtrations

$$0 = M_0 \subset M_1 \subset \cdots \subset M_{s_1} = M \quad \text{and} \quad 0 = M'_0 \subset M'_1 \subset \cdots \subset M'_{s_2} = M'$$

such that each $M_i$ and each $M'_j$ is $\theta$-semistable, and

$$\bigoplus_{i=1}^{s_1} M_i/M_{i-1} \cong \bigoplus_{i=1}^{s_2} M'_i/M'_{i-1}.$$

Every $S$-equivalence class has a representative unique up to isomorphism that is a direct sum of $\theta$-stable modules, the so-called polystable module.

Given $\theta \in \Theta_v$, the quiver variety $\mathcal{M}_\theta := (\mu^{-1}(0)/G)_{\text{red}}$ is the categorical quotient of the locus of $\chi_\theta$-semistable points of $\mu^{-1}(0)$ by the action of $G$. It is the coarse moduli space of $S$-equivalence classes of $\theta$-semistable $\Pi$-modules of dimension vector $v$. As indicated, we consider these GIT quotients with their reduced scheme structure everywhere below.

**Lemma 2.1.** For all $\theta \in \Theta_v$, the scheme $\mathcal{M}_\theta$ is irreducible and normal, with symplectic singularities.

**Proof.** See Bellamy and Schedler [BS16, Theorem 1.2, Proposition 3.21]. □

The set of stability conditions $\Theta_v$ admits a preorder $\geq$, where $\theta \geq \theta'$ iff every $\theta$-semistable $\Pi$-module is $\theta'$-semistable. It is well known [DH98, Tha96] that we obtain a wall-and-chamber structure on $\Theta_v$, where $\theta, \theta' \in \Theta_v$ lie in the relative interior of the same cone if and only if both $\theta \geq \theta'$ and $\theta' \geq \theta$ hold in this preorder, in which case $\mathcal{M}_\theta \cong \mathcal{M}_{\theta'}$. The interiors of the top-dimensional cones in $\Theta_v$ are GIT chambers, while the codimension-one faces of the closure of each GIT chamber are GIT walls. We say that $\theta \in \Theta_v$ is generic with respect to $v$, if it lies in some GIT chamber; equivalently, $\theta$ is generic if every $\theta$-semistable $\Pi$-module is $\theta$-stable. Since $v$ is indivisible, King [Kin94, Proposition 5.3] proves that for generic $\theta \in \Theta_v$, the quiver variety $\mathcal{M}_\theta$ is the fine moduli space of isomorphism classes of $\theta$-stable $\Pi$-modules of dimension vector $v$. In this case, the universal family on $\mathcal{M}_\theta$ is a tautological locally-free sheaf

$$\mathcal{R} := \bigoplus_{i \in Q_0} \mathcal{R}_i$$

together with a $\mathbb{C}$-algebra homomorphism $\phi: \Pi \to \text{End}(\mathcal{R})$, where $\mathcal{R}_\infty$ is the trivial bundle on $\mathcal{M}_\theta$ and where $\text{rank}(\mathcal{R}_i) = n \text{dim}(\rho_i)$ for $i \geq 0$.

Variation of GIT quotient for the quiver varieties $\mathcal{M}_\theta$ was investigated recently by the first author with Bellamy [BC18]. The following result records a surjectivity statement that will be useful later on.

**Lemma 2.2.** Let $\theta, \theta' \in \Theta_v$ satisfy $\theta \geq \theta'$. Then the morphism $\pi: \mathcal{M}_\theta \to \mathcal{M}_{\theta'}$ obtained by variation of GIT quotient is a surjective, projective and birational morphism of varieties over $\text{Sym}^n(\mathbb{C}^2/\Gamma)$.  


Proof. If \( \theta \) is generic and \( \theta' = 0 \), then the morphism \( \mathfrak{M}_\theta \to \mathfrak{M}_0 \cong \text{Sym}^n(\mathbb{C}^2/\Gamma) \) is a projective symplectic resolution [BC18, Theorem 4.5] and the result holds. For the general case, combining [BS16, Lemma 3.22] and [BC18, Lemma 4.4], we get \( \dim \mathfrak{M}_\theta = 2n \). This holds for any \( \theta \in \Theta_v \), so \( \dim \mathfrak{M}_{\theta'} = 2n \). The morphism \( \pi : \mathfrak{M}_\theta \to \mathfrak{M}_{\theta'} \) is projective, so the image \( Z := \pi(\mathfrak{M}_\theta) \) is closed in \( \mathfrak{M}_{\theta'} \). Deform \( \theta \) if necessary to a generic parameter \( \eta \) such that \( \eta \geq \theta \). Then the resolution \( \mathfrak{M}_\eta \to \mathfrak{M}_0 \cong \text{Sym}^n(\mathbb{C}^2/\Gamma) \) factors through \( \pi \) by variation of GIT quotient, so \( \dim(Z) = 2n \) and hence \( \pi \) is birational onto its image. It follows that \( Z \) is an irreducible component of \( \mathfrak{M}_{\theta'} \). However, \( \mathfrak{M}_{\theta'} \) is irreducible [BS16, Proposition 3.21]; so \( \pi \) is surjective. \( \square \)

The GIT wall-and-chamber structure on \( \Theta_v \) was computed explicitly in [BC18, Theorem 4.6]. In this paper, we focus on the distinguished GIT chamber

\[
C_+ := \{ \theta \in \Theta_v \mid \theta(\rho_i) > 0 \text{ for } i \geq 0 \}.
\]  

(2.2)

It is well known that the quiver variety \( \mathfrak{M}_\theta \) for \( \theta \in C_+ \) admits a description as an equivariant Hilbert scheme. Recall from the Introduction that \( n\Gamma\cdot\text{Hilb}(\mathbb{C}^2) \) is the scheme parametrising \( \Gamma \)-invariant ideals \( I \subset \mathbb{C}[x, y] \) with quotient isomorphic as a representation of \( \Gamma \) to the direct sum of \( n \) copies of the regular representation of \( \Gamma \).

**Theorem 2.3** ([VV99, Wan99, Kuz07]). Let \( \Gamma_n := \Gamma^n \rtimes \mathfrak{S}_n \subset \text{Sp}(2n, \mathbb{C}) \) denote the wreath product of \( \Gamma \) with the symmetric group \( \mathfrak{S}_n \). For \( \theta \in C_+ \), there is a commutative diagram

\[
\begin{array}{ccc}
n\Gamma\cdot\text{Hilb}(\mathbb{C}^2) & \sim \rightarrow & \mathfrak{M}_\theta \\
\downarrow & & \downarrow \pi \\
\mathbb{C}^{2n}/\Gamma_n \cong \text{Sym}^n(\mathbb{C}^2/\Gamma) & \sim \rightarrow & \mathfrak{M}_0
\end{array}
\]

in which the horizontal arrows are isomorphisms and the vertical arrows are symplectic resolutions.

We now study partial resolutions of \( \text{Sym}^n(\mathbb{C}^2/\Gamma) \) through which the resolution from Theorem 2.3 factors. The main result of [BC18, Theorem 1.2] implies that for \( n > 1 \), the nef cone of \( n\Gamma\cdot\text{Hilb}(\mathbb{C}^2) \) over \( \text{Sym}^n(\mathbb{C}^2/\Gamma) \) is isomorphic to the closure \( \overline{C_+} \) of the chamber from (2.2). For \( n = 1 \), the relation between these two cones is described in [BC18, Proposition 7.11] (see Remark 5.5 for more on the case \( n = 1 \)). In any case, for \( n \geq 1 \), the partial resolutions of interest can all be obtained as follows: choose a face of \( \overline{C_+} \) and any GIT parameter from the relative interior of that face; then perform variation of GIT quotient as the parameter moves to the origin in \( \Theta_v \).

Every face of \( \overline{C_+} \) is of the form

\[
\sigma_J := \{ \theta \in \overline{C_+} \mid \theta(\rho_j) > 0 \text{ iff } j \in J \}
\]

for some (possibly empty) subset \( J \subseteq \{0, 1, \ldots, r\} \). The parameter \( \theta_J \in \overline{C_+} \) defined by setting

\[
\theta_J(\rho_i) = \begin{cases} 
-\sum_{j \in J} n \dim(\rho_j) & \text{for } i = \infty \\
1 & \text{if } i \in J \\
0 & \text{if } i \in \{0, 1, \ldots, n\} \setminus J
\end{cases}
\]

lies in the relative interior of the face \( \sigma_J \). To simplify notation, in the case \( J = \{0\} \) we occasionally denote \( \theta_0 := \theta_{\{0\}} \).
Proposition 2.4. The face poset of the cone \( C_+ \) can be identified with the poset on the set of quiver varieties \( \mathcal{M}_{\theta, J} \) for subsets \( J \subseteq \{0, 1, \ldots, r\} \), where edges in the Hasse diagram of the poset are realised by the surjective, projective and birational morphisms \( \pi_{J, J'}: \mathcal{M}_{\theta, J} \to \mathcal{M}_{\theta, J'} \).

Proof. This is standard for variation of GIT quotient apart from surjectivity and birationality of each \( \pi_{J, J'} \). This was established in Lemma 2.2. \( \square \)

Remark 2.5. When \( J' = \emptyset \) and \( J = \{0, \ldots, r\} \), the morphism \( \mathcal{M}_{\theta, J} \to \mathcal{M}_{\theta, J'} \) is the resolution \( \mathfrak{m}^\Gamma(\text{Hilb}(\mathbb{C}^2)) \to \text{Sym}^n(\mathbb{C}^2/\Gamma) \) from Theorem 2.3. The statement of Proposition 2.4 implies that the paths in the Hasse diagram of the face poset of \( C_+ \) from the unique maximal element to the unique minimal element provide all possible ways in which this resolution can be decomposed via primitive morphisms [Wil92].

Example 2.6. Consider the case \( \Gamma \cong \mu_3 \), corresponding to Dynkin type \( A_2 \), and \( n = 3 \). Figure 1 shows a transverse slice of the GIT wall-and-chamber structure inside a specific closed cone \( F \) in the space \( \Theta_v \) of stability parameters. According to [BC18, Theorem 1.2], this decomposition of the cone is isomorphic as a fan to the closure of the movable cone of this particular \( n\Gamma\)-Hilb(\( \mathbb{C}^2 \)), with its natural subdivision into nef cones of birational models. The open subcone \( C_+ \) corresponds to the ample cone of \( n\Gamma\)-Hilb(\( \mathbb{C}^2 \)) itself. In Section 5 we focus on the distinguished ray \( \langle \theta_0 \rangle \) in the boundary of \( F \).

![Figure 1. Wall-and-chamber structure inside the cone F for Gamma ≅ mu_3 and n = 3](image-url)

We conclude this section with a lemma that identifies the key geometric fact that makes the chamber \( C_+ \) special; our argument depends crucially on this observation.

Lemma 2.7. For \( \theta \in C_+ \), the line bundle \( L_J := \bigotimes_{i \in J} \det(R_i) \) on \( \mathcal{M}_\theta \) is globally generated. The morphism to the linear series of \( L_J \) decomposes as the composition of \( \pi_J \) and a closed immersion:

\[
\begin{align*}
\pi_J & \quad |L_J|. \\
\mathcal{M}_\theta & \quad \varphi_{|L_J|}. \\
\mathcal{M}_{\theta, J} & \quad [L_J].
\end{align*}
\]

(2.3)

Proof. Since \( \theta \in C_+ \), the tautological bundles \( R_i \) on the quiver variety \( \mathcal{M}_\theta \) are globally generated for \( i \in I \) by [CIK18, Corollary 2.4]. Hence \( L_J \) is globally generated, and so \( \varphi_{|L_J|} \) is defined everywhere. In [BC18], the line bundle \( L_J = \bigotimes_{i \in J} \det(R_i)^{\theta_{i}(\rho_i)} \) on \( \mathcal{M}_\theta \) is denoted \( L_C(\theta_J) \), and this line bundle induces the morphism \( \pi_J: \mathcal{M}_\theta \to \mathcal{M}_{\theta, J} \subset |L_J| \) by [BC18, Theorem 1.2]. This proves the result. \( \square \)
Remark 2.8. We choose a sufficiently high multiple of \( \theta \) (and the same high multiple of each \( \theta_j \)) to ensure that the polarising ample line bundle on \( \mathcal{M}_{\theta_j} \) is very ample for every subset \( J \subseteq \{0, \ldots, r\} \).

3. Cornering the preprojective algebra

In general, the quiver variety \( \mathcal{M}_{\theta_j} \) is the coarse moduli space for \( \mathrm{S} \)-equivalence classes of \( \theta_j \)-semistable \( \Pi \)-modules of dimension vector \( v \). However, in the special case \( J = \{0, \ldots, r\} \) it is the fine moduli space of isomorphism classes of \( \theta_j \)-stable \( \Pi \)-modules. We now introduce an alternative, fine moduli space construction for each \( \mathcal{M}_{\theta_j} \) by defining an algebra \( \Pi_J \) obtained from \( \Pi \) by the process of ‘cornering’.

For any subset \( J \subseteq \{0,1,\ldots,r\} \), define the idempotent \( e_J := e_\infty + \sum_{j \in J} e_j \) and consider the subalgebra

\[
\Pi_J := e_J \Pi e_J
\]

of \( \Pi \) spanned over \( \mathbb{C} \) by paths in \( Q \) whose tail and head both lie in the set \( \{\infty\} \cup J \). The process of passing from \( \Pi \) to \( \Pi_J \) is called cornering; see [CIK18, Remark 3.1]. Then

\[
v_J := \rho_\infty + \sum_{j \in J} n \dim(\rho_j) \rho_j \in \mathbb{Z} \oplus \mathbb{Z}^J
\]

is a dimension vector for \( \Pi_J \)-modules, and we consider the stability condition \( \eta_J : \mathbb{Z} \oplus \mathbb{Z}^J \to \mathbb{Q} \) given by

\[
\eta_J(\rho_i) = \begin{cases} 
-\sum_{j \in J} n \dim(\rho_j) & \text{for } i = \infty \\
1 & \text{if } i \in J
\end{cases}
\]

It follows directly from the definition that a \( \Pi_J \)-module \( N \) of dimension vector \( v_J \) is \( \eta_J \)-stable if and only if there exists a surjective \( \Pi_J \)-module homomorphism \( \Pi_J e_{\infty} \to N \).

The vector \( v_J \) is indivisible and \( \eta_J \) is a generic stability condition for \( \Pi_J \)-modules, so the construction of King [Kin94, Proposition 5.3] defines the fine moduli space \( \mathcal{M}(\Pi_J) \) of \( \eta_J \)-stable \( \Pi_J \)-modules of dimension vector \( v_J \). Let \( T_J := \bigoplus_{i \in \{\infty\} \cup J} T_i \) denote the tautological bundle on \( \mathcal{M}(\Pi_J) \), where \( T_\infty \) is the trivial bundle and \( T_J \) has rank \( n \dim(\rho_j) \) for \( j \in J \). The line bundle

\[
L_J := \bigotimes_{j \in J} \det(T_j)
\]

is the polarising ample bundle on \( \mathcal{M}(\Pi_J) \) given by the GIT construction.

Lemma 3.1. Let \( \theta \in C_+ \), and let \( J \subseteq \{0,\ldots,r\} \) be any subset. There is a universal morphism

\[
\tau_J : \mathcal{M}_\theta \to \mathcal{M}(\Pi_J) \tag{3.1}
\]

satisfying \( \tau_J^*(T_i) \cong \mathcal{R}_i \) for \( i \in \{\infty\} \cup J \).

Proof. In light of the universal property of \( \mathcal{M}(\Pi_J) \), it suffices to show that the locally-free sheaf

\[
\mathcal{R}_J := \bigoplus_{i \in \{\infty\} \cup J} \mathcal{R}_i
\]

of rank \( 1 + \sum_{j \in J} n \dim(\rho_j) \) on the quiver variety \( \mathcal{M}_\theta \) is a flat family of \( \eta_J \)-stable \( \Pi_J \)-modules of dimension vector \( v_J \). Multiplying the tautological \( \mathbb{C} \)-algebra bundle \( \mathcal{M}_\theta \) is a flat family of \( \eta_J \)-stable \( \Pi_J \)-modules of dimension vector \( v_J \). To establish stability, write \( \bigoplus_{i \in J} \mathcal{R}_{i,y} \) for the fibre of \( \mathcal{R} \) over a closed point \( y \in \mathcal{M}_\theta \). The fact that \( \bigoplus_{i \in J} \mathcal{R}_{i,y} \) is \( \theta \)-stable is equivalent to the existence of a surjective \( \Pi \)-module homomorphism \( \Pi e_{\infty} \to \bigoplus_{i \in J} \mathcal{R}_{i,y} \). Applying \( e_J \) on the left produces a surjective \( \Pi_J \)-module homomorphism
\[ \Pi J: \infty \to \bigoplus_{i \in \infty} \tau J \] which in turn is equivalent to \( \eta J \)-stability of the fibre \( \bigoplus_{i \in \infty} \tau J \) of \( \tau J \) over \( y \in M J \). In particular, \( \tau J \) is a flat family of \( \eta J \)-stable \( \Pi J \)-modules of dimension vector \( v J \).

**Remarks 3.2.**

1. An alternative proof of Lemma 3.1 uses the fact that the tautological bundles \( R J \) on \( M J \) are globally generated for \( i \in I \) by [CIK18, Corollary 2.4], in which case one can adapt the proof of [CIK18, Proposition 2.3] to deduce that \( \tau J \) is a flat family of \( \eta J \)-stable \( \Pi J \)-modules of dimension vector \( v J \). In particular, global generation is the key feature in Lemma 3.1, just as in the proof of Lemma 2.7. This is not a coincidence; see Theorem 3.7.

2. Building on Remark 2.8, we now take an even higher multiple of \( \theta \) if necessary (and the same high multiple of each \( \eta J \) and each \( \theta J \)) to ensure that the polarising ample line bundles on \( M (\Pi J) \) and on \( M J \) are very ample for all relevant \( J \subseteq \{0, \ldots, r\} \).

From now on in this section, we assume that \( J \neq \emptyset \); see Remarks 3.8(2).

**Lemma 3.3.** Let \( \theta \in C_{+} \) and assume \( J \subseteq \{0, \ldots, n\} \) is nonempty. There is a commutative diagram

\[
\begin{array}{ccc}
M \theta & \xrightarrow{\pi J} & M J \\
\downarrow \psi J & & \downarrow \psi J \\
|L J| & \xrightarrow{\psi J} & |L J|
\end{array}
\]

of schemes over \( \text{Sym}^{n}(\mathbb{C}^{2}/\Gamma) \), where \( \psi \) is an isomorphism.

**Proof.** The commutative triangle on the left of (3.2) was constructed in Lemma 2.7. For the quadrilateral on the right, our choice of \( \eta J \) ensures that the polarising line bundle \( L J \) on \( M (\Pi J) \) is very ample, so the morphism \( \psi J \) is well-defined. Since pullback commutes with tensor operations on the \( T i \), the isomorphisms \( \tau J (Ti) \cong R J \) for \( i \in J \) imply that \( L J \) is \( \tau J (L J) \). If \( O/L J(1) \) denotes the polarising ample bundle on \( |L J| \), then

\[
(\varphi J |L J| \circ \tau J) \ast (O |L J| (1)) = \tau J (L J) = L J = \varphi J |L J| (O |L J| (1))
\]

(3.3)
on \( M \theta \). The morphism \( \psi \) to a complete linear series is unique up to an automorphism of the linear series, so there is an isomorphism \( \psi |L J| \rightarrow |L J| \) such that \( \varphi J |L J| \circ \tau J = \psi \circ \varphi J |L J| \) as required.

It remains to show that (3.2) is a diagram of schemes over \( \text{Sym}^{n}(\mathbb{C}^{2}/\Gamma) \). The Leray spectral sequence for the resolution \( \pi: M \theta \rightarrow M 0 \cong \text{Sym}^{n}(\mathbb{C}^{2}/\Gamma) \) gives \( H^{0}(O_{M \theta}) \cong H^{0}(O_{M 0}) \cong (\mathbb{C}[V]^{\Gamma})^{\oplus n} \) because \( \text{Sym}^{n}(\mathbb{C}^{2}/\Gamma) \) has rational singularities. It follows that \( \pi = \varphi |O_{M \theta}| \), i.e. \( \pi \) is the structure morphism of \( M \theta \) as a variety over \( \text{Sym}^{n}(\mathbb{C}^{2}/\Gamma) \). Repeating the argument from (3.3), with the roles of \( L J, L J \) and \( O/L J(1) \) played instead by the trivial bundles on \( M \theta, M (\Pi J) \) and \( \text{Sym}^{n}(\mathbb{C}^{2}/\Gamma) \) respectively, shows that \( M (\Pi J) \) is a scheme over \( \text{Sym}^{n}(\mathbb{C}^{2}/\Gamma) \). It follows that (3.2) is a diagram of schemes over \( \text{Sym}^{n}(\mathbb{C}^{2}/\Gamma) \).

Our goal for the rest of this section is to add a morphism \( \iota J: M \theta J \rightarrow M (\Pi J) \) to diagram (3.2) and to show that \( \iota J \) is an isomorphism on the underlying reduced schemes. Consider the functors

\[
\Pi -\text{mod} \xrightarrow{j_{J}^{*}} \Pi J -\text{mod}
\]
defined by \( j_{J}^{*}(-) := e J \Pi \otimes (\Pi (-)) \) and \( j_{J}(\cdot) := \Pi e J \otimes (\Pi (-)) \). These are two of the six functors in a recollement of the module category \( \Pi -\text{mod} \) [FP04]. In particular, \( j_{J} \) is exact, \( j_{J}^{*} \) is the identity functor, and for any
\( \Pi_j \)-module \( N \), the \( \Pi \)-module \( j_!(N) \) provides the maximal extension by \( \Pi / (\Pi e J \Pi) \)-modules; see [CIK18, Equation (3.4)].

**Lemma 3.4.** Let \( N \) be an \( \eta_J \)-stable \( \Pi_j \)-module of dimension vector \( v_J \). The \( \Pi \)-module \( j_!(N) \) is \( \theta_J \)-semistable.

**Proof.** Since \( N \) is \( \eta_J \)-stable, there is a surjective \( \Pi \)-module homomorphism \( \Pi_j e_{\infty} \to N \). The proof of [CIK18, Lemma 3.6] applies verbatim to construct a surjective \( \Pi \)-module homomorphism \( \Pi e_{\infty} \to j_!(N) \) and, moreover, to show that the finite dimensional \( \Pi \)-module \( j_!(N) \) satisfies \( \dim i j_!(N) = \dim i N \) for \( i \in \{ \infty \} \cup J \).

Recall that \( \theta_J(\rho_i) = 0 \) for \( i \notin \{ \infty \} \cup J \), so

\[
\theta_j(j_!(N)) = \theta_j \left( \sum_{i \in \{ \infty \} \cup J} \dim i (j_!(N)) \rho_i \right) = \eta_j \left( \sum_{i \in \{ \infty \} \cup J} \dim i (N) \rho_i \right) = \eta_j(N) = 0.
\]

Now let \( M \subset j_!(N) \) be a proper submodule. If \( \dim \infty M = 1 \), then surjectivity of the map \( \Pi e_{\infty} \to j_!(N) \) gives \( M = j_!(N) \) which is absurd, so \( \dim \infty M = 0 \). But \( \theta_J(\rho_i) \geq 0 \) for all \( i \neq \infty \), so \( \theta_J(M) \geq 0 \) as required. \( \square \)

**Lemma 3.5.** Let \( N \) be an \( \eta_J \)-stable \( \Pi_j \)-module of dimension vector \( v_J \). Then there exists a \( \theta_J \)-semistable \( \Pi \)-module \( M \) such that \( j^* M \cong N \) and \( \dim i M \leq n \dim(\rho_i) \) for all \( i \notin \{ \infty \} \cup J \).

**Proof.** By Lemma 3.4, \( j_!(N) \) is \( \theta_J \)-semistable. If \( \dim i j_!(N) \leq n \dim(\rho_i) \) for \( i \notin \{ \infty \} \cup J \), then we can simply set \( M := j_!(N) \), as \( j^* j_! \) is the identity. Otherwise, consider the \( \theta_J \)-polystable module \( \bigoplus \lambda M_\lambda \) that is \( S \)-equivalent to \( j_!(N) \). Let \( M_{\lambda \infty} \) denote the unique summand satisfying \( \dim \infty M_{\lambda \infty} = 1 \). Since \( M_{\lambda \infty} \) is by construction a \( \theta_J \)-stable \( \Pi \)-module, it follows that \( \dim i M_{\lambda \infty} = n \dim(\rho_i) \) for all \( i \in J \), and hence \( \dim i M_\lambda = 0 \) for all \( \lambda \neq \lambda_\infty \) and all \( i \in \{ \infty \} \cup J \). For each index \( \lambda \) and for all \( i \notin \{ \infty \} \cup J \), we have

\[
\dim i j^* M_\lambda = \dim e_i (e_j \Pi \otimes_\Pi (M_\lambda)) = \dim e_i \Pi \otimes_\Pi M_\lambda = \dim M_\lambda.
\]

It follows that \( \dim i j^* M_\lambda = 0 \) for all \( \lambda \neq \lambda_\infty \) and \( i \notin \{ \infty \} \cup J \), and hence \( j^* M_\lambda \equiv 0 \) for \( \lambda \neq \lambda_\infty \).

We claim that \( j^* M_{\lambda \infty} \) is isomorphic to \( N \). Indeed, the \( \Pi \)-module \( j_!(N) \) is \( \theta_J \)-semistable by Lemma 3.4, and the \( \theta_J \)-stable \( \Pi \)-modules \( M_\lambda \) are by construction the factors in the composition series of \( j_!(N) \) in the category of \( \theta_J \)-semistable \( \Pi \)-modules. It follows from exactness of \( j^* \) that the \( \Pi \)-modules \( j^* M_\lambda \) are the factors in the composition series of \( j^* j_!(N) \equiv N \) in the category of \( \eta_J \)-semistable \( \Pi \)-modules. But \( j^* M_\lambda \equiv 0 \) for \( \lambda \neq \lambda_\infty \), so the only nonzero factor of the composition series is \( j^* M_{\lambda \infty} \). It follows that \( j^* M_{\lambda \infty} \equiv N \), because the factor \( j^* M_{\lambda \infty} \) can only appear once in the composition series.

As a result, the \( \theta_J \)-stable \( \Pi \)-module \( M_{\lambda \infty} \) satisfies \( j^* M_{\lambda \infty} \equiv N \) and \( \dim i M_{\lambda \infty} = n \dim(\rho_i) \) for all \( i \in J \). Therefore \( M_{\lambda \infty} \) is the required \( \Pi_j \)-module as long as \( \dim i M_{\lambda \infty} \leq n \dim(\rho_i) \) for all \( i \notin \{ \infty \} \cup J \). We establish this key inequality in Appendix A. \( \square \)

**Remark 3.6.** The modules \( M_\lambda \) for \( \lambda \neq \lambda_\infty \) in the proof of Lemma 3.5 are in fact all 1-dimensional vertex simples. To see this, note that removing any nonempty set of vertices and their incident edges from an extended Dynkin diagram gives a diagram in which every connected component is Dynkin of finite type. Thus removing the vertices \( \{ \infty \} \cup J \) and all incident edges from the framed extended diagram leaves us with a collection of Dynkin diagrams of finite type. Choose \( \lambda \neq \lambda_\infty \). Since \( \dim_j M_\lambda \equiv 0 \) for all \( j \in \{ \infty \} \cup J \), \( M_\lambda \) is a simple module of the preprojective algebra of a quiver of finite type. But such modules are one-dimensional by [ST11, Lemma 2.2].
\textbf{Theorem 3.7.} For any nonempty $J \subseteq \{0, \ldots, n\}$, there is a commutative diagram of morphisms

\[
\begin{array}{ccc}
\mathcal{M}_\theta & \xrightarrow{\pi_J} & \mathcal{M}_{\theta_j} \\
\mathcal{M}_{\theta_j} & \xrightarrow{\iota_J} & \mathcal{M}(\Pi_J),
\end{array}
\]

where $\iota_J$ is an isomorphism of the underlying reduced schemes. In particular, $\mathcal{M}(\Pi_J)$ is irreducible, and its underlying reduced scheme is normal and has symplectic singularities.

\textbf{Proof.} Let $\sigma_J: \mathcal{M}_{\theta_j} \to |L_J|$ be the composition of the isomorphism $\psi$ of Lemma 3.3 with the closed immersion $\mathcal{M}_{\theta_j} \hookrightarrow |L_J|$ from diagram (3.2). Since $\sigma_J$ is a closed immersion, it identifies $\mathcal{M}_{\theta_j}$ with $\text{Im}(\sigma_J)$. Surjectivity of $\pi_J$ and commutativity of diagram (3.2) then imply that $\mathcal{M}_{\theta_j}$ is isomorphic to the subscheme $\text{Im}(\sigma_J \circ \pi_J) = \text{Im}(\varphi|_{L_J} \circ \tau_J)$ of $|L_J|$. Since $L_J$ is the polarising very ample line bundle on the GIT quotient $\mathcal{M}(\Pi_J)$, the closed immersion $\varphi|_{L_J}$ induces an isomorphism $\lambda_J: \text{Im}(\varphi|_{L_J}) \to \mathcal{M}(\Pi_J)$. The morphism

$$\iota_J := \lambda_J \circ \sigma_J: \mathcal{M}_{\theta_j} \to \mathcal{M}(\Pi_J)$$

is therefore a closed immersion. Note that

$$\iota_J \circ \pi_J = \lambda_J \circ \sigma_J \circ \pi_J = \lambda_J \circ \varphi|_{L_J} \circ \tau_J = \tau_J,$$

so diagram (3.4) commutes. In order to prove that $\iota_J$ is an isomorphism of the underlying reduced schemes, it suffices to show that $\iota_J$ is surjective on closed points.

Consider a closed point $[N] \in \mathcal{M}(\Pi_J)$, where $N$ is an $\eta_J$-stable $\Pi_J$-module of dimension vector $v_J$. Let $M$ be the $\theta_J$-semistable $\Pi$-module from Lemma 3.5. For $i \notin \{\infty\} \cup J$, define $m_i := n \dim(\rho_i) - \dim M \geq 0$ and let $S_i := C_{\epsilon_i}$ denote the vertex simple $\Pi$-module at vertex $i \in Q_0$. The $\Pi$-module

$$\widehat{M} := M \oplus \bigoplus_{i \in \{0, \ldots, n\} \setminus J} S_i^{\oplus m_i}$$

is $\theta_J$-semistable of dimension vector $v$ by construction, and it satisfies $j^*(\widehat{M}) = j^*(M) = N$. Write $[\widehat{M}] \in \mathcal{M}_{\theta_j}$ for the corresponding closed point, and let $\widehat{M}$ be any $\theta$-stable $\Pi$-module of dimension vector $v$ such that the closed point $[\widehat{M}] \in \mathcal{M}_{\theta_j}$ satisfies $\pi_J([\widehat{M}]) = [\widehat{M}] \in \mathcal{M}_{\theta_j}$. Then $j^*(\widehat{M}) = j^*(M) = N$, hence $\tau_J([\widehat{M}]) = [N]$, and commutativity of diagram (3.4) gives that

$$\iota_J([\widehat{M}]) = (\iota_J \circ \pi_J)([\widehat{M}]) = \tau_J([\widehat{M}]) = [N],$$

so $\iota_J$ is indeed surjective. The final statement of Theorem 3.7 follows from Lemma 2.1 and Lemma 3.3. \qed

\textbf{Remarks 3.8.} \hspace{1cm} (1) If $J \neq \{0, \ldots, n\}$, then the stability parameter $\theta_J$ lies in the boundary of the GIT chamber $C_+$, so $\mathcal{M}_{\theta_j}$ does not admit a universal family of $\theta_J$-semistable $\Pi$-modules of dimension vector $v$. However, the fine moduli space $\mathcal{M}(\Pi_J)$ does carry a universal family $T_J$ of $\eta_J$-stable $\Pi_J$-modules of dimension vector $v_J$, and hence under the isomorphism of Theorem 3.7, the bundle $\iota_J^*(T_J)$ on $\mathcal{M}_{\theta_j}$ pulls back along $\pi_J$ to the summand $\bigoplus_{i \in \{\infty\} \cup J} R_i$ of the tautological bundle on $\mathcal{M}_\theta$.

(2) In the course of the proof of Theorem 3.7, we deduce directly that $\tau_J$ is surjective on closed points.

(3) For $J = \emptyset$, we have $\mathcal{M}_{\theta_j} \cong \text{Sym}^n(C^2/\Gamma)$. However, $\mathcal{M}(\Pi_J)$ is an affine scheme that does not depend on $n$, so $\mathcal{M}_{\theta_j} \not\cong \mathcal{M}(\Pi_J)$ when $J = \emptyset$. 

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4. Identifying the posets for the coarse and fine moduli problems

We now establish that the morphisms \( \iota_J : \mathcal{M}_\theta \to \mathcal{M}(\Pi_J) \) from Theorem 3.7 are compatible with the morphisms \( \pi_{J,J'} : \mathcal{M}_\theta \to \mathcal{M}_{\theta_{J'}} \) that feature in the poset introduced in Proposition 2.4.

Lemma 4.1. For nonempty subsets \( J' \subset J \subset \{0, 1, \ldots, n\} \), there is a commutative diagram

\[
\begin{array}{ccc}
\mathcal{M}_\theta & \xrightarrow{\iota_J} & \mathcal{M}(\Pi_J) \\
\pi_{J,J'} \downarrow & & \downarrow \tau_{J,J'} \\
\mathcal{M}_{\theta_{J'}} & \xrightarrow{\iota_{J'}} & \mathcal{M}(\Pi_{J'})
\end{array}
\] (4.1)

in which the horizontal arrows are isomorphisms on the underlying reduced schemes and the vertical arrows are surjective, projective, birational morphisms.

Proof. The subbundle \( \bigoplus_{i \in \{\infty\} \cup J'} T_i \) of the tautological bundle \( T_J \) on \( \mathcal{M}(\Pi_J) \) is a flat family of \( \eta_{J'} \)-stable \( \Pi_{J'} \)-modules of dimension vector \( v_{J'} \), so there is a universal morphism \( \tau_{J,J'} : \mathcal{M}(\Pi_J) \to \mathcal{M}(\Pi_{J'}) \) satisfying

\[
\tau_{J,J'}^* \big( T_{J'}^i \big) = T_{J}^i
\]

for \( i \in \{\infty\} \cup J' \), where \( \bigoplus_{i \in \{\infty\} \cup J'} T_i \) is the tautological bundle on \( \mathcal{M}(\Pi_{J'}) \). Now

\[
\big( \tau_{J,J'} \circ \tau_J \big)^* \big( T_{J'}^i \big) = \tau_{J,J'}^* \big( \tau_J^* \big( T_{J}^i \big) \big) = \mathcal{R}_{i,} = \tau_{J,J'}^* \big( T_{J'}^i \big)
\]

for all \( i \in \{\infty\} \cup J' \), and since this property characterises the morphism \( \tau_{J,J'} \), we have a commutative diagram

\[
\begin{array}{ccc}
\mathcal{M}(\Pi_J) & \xrightarrow{\tau_J} & \mathcal{M}(\Pi_{J'}) \\
\pi_{J,J'} \downarrow & & \downarrow \tau_{J,J'} \\
\mathcal{M}_{\theta_{J'}} & \xrightarrow{\iota_{J'}} & \mathcal{M}(\Pi_{J'})
\end{array}
\] (4.2)

Proposition 2.4 gives a similar commutative diagram expressing the identity \( \pi_{J,J'} \circ \pi_J = \pi_{J,J'} \) for morphisms between quiver varieties, while Theorem 3.7 establishes the identities \( \iota_J \circ \pi_J = \tau_J \) and \( \iota_{J,J'} \circ \pi_{J,J'} = \tau_{J,J'} \). Taken together, these identities show that the maps in all four triangles in the following pyramid diagram commute:

\[
\begin{array}{ccc}
\mathcal{M}_\theta & \xrightarrow{\iota_J} & \mathcal{M}(\Pi_J) \\
\pi_{J,J'} \downarrow & & \downarrow \tau_{J,J'} \\
\mathcal{M}_{\theta_{J'}} & \xrightarrow{\iota_{J,J'}} & \mathcal{M}(\Pi_{J'})
\end{array}
\] (4.3)

To show that the morphisms around the pyramid’s square base commute, choose for any closed point \( x \in \mathcal{M}_{\theta_J} \) a lift \( y \in \pi_J^{-1}(x) \subset \mathcal{M}_\theta \). Commutativity of the triangles in the diagram gives

\[
\big( \iota_{J,J'} \circ \pi_{J,J'} \big)(x) = \iota_{J,J'}(\pi_{J,J'}(y)) = \tau_{J,J'}(y) = \tau_{J,J'}(\tau_J(y)) = \big( \tau_{J,J'} \circ \iota_J \big)(x),
\]

and since \( x \in \mathcal{M}_\theta \) was arbitrary and \( \pi_J \) is surjective, we have that \( \iota_{J,J'} \circ \pi_{J,J'} = \tau_{J,J'} \circ \iota_J \) as required. \( \square \)

We deduce the following.
Proposition 4.2. The face poset of the cone $\overline{C}_+$ can be identified with the poset on the set of fine moduli spaces $\mathcal{M}(\Pi_J)$ for nonempty subsets $J \subseteq \{0, \ldots, n\}$ together with $\mathbb{C}^{2n}/\Gamma_n$, where edges in the Hasse diagram of the poset indicating inequalities $\mathcal{M}(\Pi_J) > \mathcal{M}(\Pi_{J'})$ and $\mathcal{M}(\Pi_J) > \mathbb{C}^{2n}/\Gamma_n$ are realised by the universal morphisms $\tau_J, \nu$ and the structure morphisms $\varphi_{O_{\mathcal{M}(\Pi_J)}}$ respectively.

5. Punctual Hilbert schemes for Kleinian singularities

In this section, we specialise to the case $J = \{0\}$ and study the algebra $\Pi_J$, before establishing the link between the fine moduli space $\mathcal{M}(\Pi_J)$ and the Hilbert scheme of $n$ points on $\mathbb{C}^2/\Gamma$. It will be convenient to write dimension vectors of $\Pi_J$-modules as pairs $(v, e_0)$ in this case.

Like the algebra $\Pi$, the algebra $\Pi_J$ can also be presented as a quiver algebra with relations. The relations appear to be fairly complicated, but for $J = \{0\}$ there is a simple presentation of its quotient algebra $\Pi_J/(b^*)$, where $b^*$ is the class of a particular arrow in $Q$. This will turn out to be sufficient for our purposes. To spell this out, write $b$ for the generator corresponding to the arrow going from $\infty$ to 0 in the path algebra $\mathbb{C}Q$ of the framed McKay quiver, and $b^*$ for the opposite arrow from 0 to $\infty$. Through slight abuse of notation, we use the same symbols for the respective images of these elements in the preprojective algebra $\Pi$ and its subalgebra $\Pi_J$.

On the other hand, define a new quiver $Q'$ with vertex set $Q'_0 = \{\infty, 0\}$ and arrow set $Q'_1$ comprising one arrow $\alpha$ from $\infty$ to 0, and loops $\alpha_1, \alpha_2, \alpha_3$ at vertex 0 as shown in Figure 2.

![Figure 2. The quiver $Q'$ used in the presentation of a quotient of $\Pi_J$ for $J = \{0\}$.](image)

To state the key result, recall that the quotient singularity $\mathbb{C}^2/\Gamma$ is famously a hypersurface in $\mathbb{C}^3$, with the $\Gamma$-invariant subring $\mathbb{C}[x, y]^\Gamma$ having three minimal generators $z_1, z_2, z_3$ satisfying one relation $f(z_1, z_2, z_3) = 0$.

Lemma 5.1. For $J = \{0\}$, let $b^* \in \Pi_J$ denote the class of the arrow in $Q$ from 0 to $\infty$. The algebra $\Pi_J/(b^*)$ is isomorphic to the quotient of $\mathbb{C}Q'$ by the two-sided ideal

$$K = \langle f(\alpha_1, \alpha_2, \alpha_3), \alpha_1\alpha_2 - \alpha_2\alpha_1, \alpha_1\alpha_3 - \alpha_3\alpha_1, \alpha_2\alpha_3 - \alpha_3\alpha_2 \rangle,$$

where $f \in \mathbb{C}[z_1, z_2, z_3]$ is the defining equation of the hypersurface $\mathbb{C}^2/\Gamma \subseteq \text{Spec} \mathbb{C}[z_1, z_2, z_3]$.

Proof. The unframed McKay quiver $Q_\Gamma$ is the complete subquiver of $Q$ on the vertex set $\{0, 1, \ldots, r\}$. Write $\Pi_\Gamma$ for the corresponding preprojective algebra. The natural epimorphism $\mathbb{C}Q \twoheadrightarrow \mathbb{C}Q_\Gamma$ that kills any path in $Q$ touching vertex $\infty$ induces a short exact sequence

$$0 \rightarrow (bb^*) \rightarrow e_0\Pi e_0 \xrightarrow{\varphi} e_0(\Pi_\Gamma)e_0 \rightarrow 0.$$

Recall from [CBH98, Theorem 0.1] the isomorphism $e_0(\Pi_\Gamma)e_0 \cong \mathbb{C}[x, y]^\Gamma$. We may therefore associate to each of the three minimal $\mathbb{C}$-algebra generators of $\mathbb{C}[x, y]^\Gamma$ an element $\beta_i \in e_0(\Pi_\Gamma)e_0$ for $1 \leq i \leq 3$. For each $\beta_i$, choose a linear combination of cycles in $\mathbb{C}Q_\Gamma$ mapping to $\beta_i$ in $e_0(\Pi_\Gamma)e_0$. The same linear combination
of cycles can be thought of as an element of $CQ$, and its image $\overline{\beta}_i \in e_0\Pi e_0 \subset \Pi_J$ defines a lift of $\beta_i$ with respect to $\phi$.

For each $i$, let $b_i$ be the image of $\overline{\beta}_i$ in $\Pi_J/(b^*)$. Mapping $\beta_i$ to $b_i$ and mapping $e_0$ to the class of the trivial path at vertex 0 in $\Pi_J/(b^*)$ defines a map $t$ in the following commutative diagram, where both rows are exact; a simple diagram chase shows that $t$ is indeed well-defined.

\[
\begin{array}{ccccccccc}
0 & \rightarrow & (bb^*) & \rightarrow & e_0\Pi e_0 & \phi & e_0(\Pi) & e_0 & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & (b^*) & \rightarrow & \Pi_J & \rightarrow & \Pi_J/(b^*) & \rightarrow & 0
\end{array}
\] (5.2)

In particular, the elements $b_i \in \Pi_J/(b^*)$ commute. Furthermore, since $f(\beta_1, \beta_2, \beta_3) = 0$, we have

$$0 = t(f(\beta_1, \beta_2, \beta_3)) = f(t(\beta_1), t(\beta_2), t(\beta_3)) = f(b_1, b_2, b_3).$$

We next show that $t$ is injective. Let $\gamma \in \ker(t)$. For any element $\overline{\gamma} \in e_0\Pi e_0$ such that $\phi(\overline{\gamma}) = \gamma$, this means that $\overline{\gamma} \in (b^*)$ as an element of $\Pi_J$. However, since $(b^*) \cap e_0\Pi e_0 = (bb^*)$, it must be the case that $\overline{\gamma} \in (bb^*)$. Therefore $\gamma = 0$, and $t$ is injective.

Since $\Pi_J$ is a quotient of the algebra $e_JCQe_J$, it is generated by $e_\infty, b, b^*$ and the class of every cycle in $Q$ starting and ending at the vertex 0. Inside this, the subalgebra generated by classes of cycles starting and ending at 0 equals $e_0\Pi e_0$. By commutativity of diagram (5.2), the image of this subalgebra under $p$ equals the image $\text{Im}(t)$ of $t$. It is then clear that $\Pi_J/(b^*)$ is generated as a $C$-algebra by $e_\infty, b$ and $\text{Im}(t) \simeq \mathbb{C}[x, y]^\Gamma$, in other words by the elements $e_\infty, b, e_0, b_1, b_2, b_3$.

Now we define a map

$$\psi: \mathbb{C}Q'/K \rightarrow \Pi_J/(b^*)$$

by sending the classes of the trivial paths at vertices $\infty$ and 0 to $e_\infty$ and $e_0$ respectively, and by setting $\psi(\alpha) = b$ and $\psi(\alpha_i) = b_i$ for $1 \leq i \leq 3$. The above discussion and the definition of $K$ shows that $\psi$ is a well-defined surjective homomorphism. To see that $\psi$ is injective, we note that it maps the $C$-subalgebra $\Lambda \subset \mathbb{C}Q'/K$ generated by $(e_0, \alpha_1, \alpha_2, \alpha_3)$ bijectively onto $\text{Im}(t)$. In addition, $\psi$ is injective when restricted to the two-sided ideal $(e_\infty, \alpha)$. Finally, we must show that if $\zeta \neq 0$ is an element of $(e_\infty, \alpha)$, then $\psi(\zeta)$ does not lie in $\text{Im}(t)$. Since $e_\infty \cdot e_0 = 0$, it suffices to consider such a $\zeta$ of the form $\xi \alpha$ for some nonzero $\xi \in \Lambda$. Therefore $\psi(\zeta) = cb$ for some $c \in \text{Im}(t)$. But $b$ does not start at 0, so $cb \not\in \text{Im}(t)$. It follows that no nonzero linear combination of an element of the ideal $(e_\infty, \alpha)$ with an element of $\Lambda$ can be mapped to 0 by $\psi$. But every element of $\mathbb{C}Q'/K$ is of this form, so $\psi$ is injective. Thus $\psi$ is an isomorphism as required.

Note that a $\Pi_J$-module $M$ for which $b^*$ acts as 0 is the same thing as a $\Pi_J/(b^*)$-module, and therefore the same as a $CQ'/K$-module.

**Proposition 5.2.** For the subset $J = \{0\}$, there is a morphism of schemes

$$\omega_n: \text{Hilb}^n[\mathbb{C}^2/\Gamma] \rightarrow \mathcal{M}(\Pi_J)$$

over $\text{Sym}^n(\mathbb{C}^2/\Gamma)$, which is an isomorphism of the underlying reduced subschemes.

**Proof.** We begin by constructing the morphism of schemes $\omega_n$. Let $\mathcal{T}$ denote the tautological rank $n$ bundle on $\text{Hilb}^n[\mathbb{C}^2/\Gamma]$, and write $\mathcal{O}$ for the trivial bundle. In light of the universal property of $\mathcal{M}(\Pi_J)$, it suffices to show that $\mathcal{O} \oplus \mathcal{T}$ is a flat family of $\eta_J$-stable $\Pi_J$-modules of dimension vector $v_J = (1, n)$ on $\text{Hilb}^n[\mathbb{C}^2/\Gamma]$. 


Since $\mathcal{O} \oplus \mathcal{T}$ is a flat family of $\mathbb{C}$-vector spaces, it suffices to study the fibres over closed points. A point of $\text{Hilb}^{[n]}(\mathbb{C}^2/\Gamma)$ corresponds to a codimension $n$ ideal $I \triangleleft \mathbb{C}[x,y]^\Gamma \cong \mathbb{C}[z_1,z_2,z_3]/(f)$. The quotient vector space $\mathbb{C}[x,y]^\Gamma/I$ is of dimension $n$, it carries the action of commuting arrows $\alpha_1, \alpha_2, \alpha_3$ satisfying the relation $f$, and has a distinguished generator $[1] \in \mathbb{C}[x,y]^\Gamma/I$, which can be thought of as the image of a map $w$ from a one-dimensional vector space. Lemma 5.1 now shows that we get the data of a $\Pi$-module.

Next we claim that conversely, for any closed point of $\text{Hilb}^{[n]}(\mathbb{C}^2/\Gamma)$, the restriction to the $0$ vertex is the $\Gamma$-invariant part of the quotient $\mathbb{C}[x,y]/\mathbb{C}[x,y]^\Gamma$, which can be thought of as the image of a map $w$ to the one-dimensional vector space at the vertex $\infty$. In this language, the restriction to the $0$ vertex is the $\Gamma$-invariant part of the quotient $\mathbb{C}[x,y]/\mathbb{C}[x,y]^\Gamma$, and it is well known that the induced map $w^*$ to the one-dimensional vector space at the vertex $\infty$ vanishes in this case.

Finally, we construct an inverse to $\omega_n$ on $\mathcal{M}(\Pi_J)_{\text{red}}$. Given a finitely generated $\mathbb{C}$-algebra $A$, a Spec $A$-valued point of $\mathcal{M}(\Pi_J)$ is given by the data of two finite flat $A$-modules $M_{\infty}$ and $M_0$ of ranks $1$ and $n$ respectively (computed over the localisation of $A$ at any prime ideal, over which finite flat modules are free), and $A$-module homomorphisms

$$(w,w^*) \in \text{Hom}_A(M_{\infty}, M_0) \oplus \text{Hom}_A(M_0, M_{\infty}), \quad \{w_1, \ldots, w_{i'}\} \subset \text{End}_A(M_0)$$

for some integer $i'$, satisfying the relations defining $\Pi_J$. Moreover, for every maximal ideal of $A$, restricting to the corresponding closed point of Spec $A$ gives an $\eta_J$-stable $\Pi_J$-module. By the argument of the previous paragraph, we must have that the arrow $w^*$ becomes zero when restricted to all closed points of Spec $A$. If $A$ is reduced, this implies $w^* = 0$. Using Lemma 5.1 again, we can now reverse the construction of the first paragraph, and get a Spec $A$-valued point of $\text{Hilb}^{[n]}(\mathbb{C}^2/\Gamma)$ for reduced rings $A$. We thus obtain a map of schemes $\mathcal{M}(\Pi_J)_{\text{red}} \to \text{Hilb}^{[n]}(\mathbb{C}^2/\Gamma)$ which is by construction the inverse of $\omega_n$ when we restrict to reduced closed subschemes on both sides.

We deduce Theorem 1.1 announced in the Introduction.

**Corollary 5.3.** For any $n \geq 1$, the reduced scheme underlying $\text{Hilb}^{[n]}(\mathbb{C}^2/\Gamma)$ is isomorphic to the quiver variety $\mathfrak{M}_{\theta_0}$ for the parameter $\theta_0 = (-n,1,0,\ldots,0)$ (compare Figure 1). In particular, $\text{Hilb}^{[n]}(\mathbb{C}^2/\Gamma)_{\text{red}}$ is a normal, irreducible scheme over $\mathbb{C}^{2n}/\Gamma_n$ with symplectic singularities that admits a unique projective symplectic resolution, namely the morphism

$$n\Gamma \cdot \text{Hilb}(\mathbb{C}^2) \to \text{Hilb}^{[n]}(\mathbb{C}^2/\Gamma)_{\text{red}}$$

that sends an ideal $I$ in $\mathbb{C}[x,y]$ to the ideal $I \cap \mathbb{C}[x,y]^\Gamma$.  

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Proof. The first statement follows from Theorem 3.7 and Proposition 5.2, while the geometric properties of $\text{Hilb}^{[n]}(\mathbb{C}^2/\Gamma)_{\text{red}}$ are all inherited from its manifestation as $\mathfrak{M}_\theta$ via Lemma 2.1.

Next we prove the statement about the resolution. In the notation of [BC18, Theorem 1.2], the extremal ray $\rho^+\cap\cdots\cap\rho^+_n$ of the cone $F$ that contains $\theta_0 = (-n, 1, 0, \ldots, 0)$ lies in the closure of precisely one chamber, namely the chamber $C_+$. Under the isomorphism $L_P$ from ibid., it follows that there is exactly one projective symplectic resolution of $\mathfrak{M}_{\theta_0}$, namely the fine moduli space $\mathfrak{M}_\theta$ for $\theta \in C_+$. By Theorem 2.3, this resolution is indeed $\mathfrak{M}_\theta \cong n\Gamma$-Hilb($\mathbb{C}^2$).

The last statement of the Corollary was essentially already demonstrated in the proof of Proposition 5.2 above. Indeed, there we noted that the map $\tau_J$, in the language of ideals, takes a $\Gamma$-invariant ideal $I$ of $\mathbb{C}[x, y]$, and restricts the corresponding representation to the 0 vertex as the $\Gamma$-invariant part of the quotient $\mathbb{C}[x, y]/I$. Now we conclude using the evident isomorphism $(\mathbb{C}[x, y]/I)^\Gamma \cong \mathbb{C}[x, y]^\Gamma/\mathbb{C}[x, y]^\Gamma \cap I$. \hfill $\square$

Remark 5.4. (1) Irreducibility of $\text{Hilb}^{[n]}(\mathbb{C}^2/\Gamma)$ was first established by Zheng [Zhe17] through the study of maximal Cohen–Macaulay modules on Kleinian singularities using a case-by-case analysis following the ADE classification.

(2) Uniqueness of the symplectic resolution of $\text{Hilb}^{[n]}(\mathbb{C}^2/\Gamma)$ was previously known in the special case $n = 2$ by the work of Yamagishi [Yam17, Proposition 2.10].

(3) Our approach does not shed light on whether $\text{Hilb}^{[n]}(\mathbb{C}^2/\Gamma)$ is reduced in its natural scheme structure, coming from its moduli space interpretation.

Remark 5.5. For $n = 1$, the statement of Theorem 1.1 is well known because $\text{Hilb}^{[1]}(\mathbb{C}^2/\Gamma) \cong \mathbb{C}^2/\Gamma$, while the statement of Theorem 3.7 is a framed version of [CIK18, Theorem 1.2] for $\Gamma \subset \text{SL}(2, \mathbb{C})$. Nevertheless, the approach of the current paper is valid for $n = 1$ and shows in particular that $\mathfrak{M}_{\theta_J} \cong \mathbb{C}^2/\Gamma$ for $J = \{0\}$. In fact, this result follows from [BC18, Proposition 7.11]. Indeed, ibid. constructs a surjective linear map $L_{C_+} : \Theta_v \rightarrow N^1(S/(\mathbb{C}^2/\Gamma))$ with kernel equal to the subspace spanned by $(-1, 1, 0, \ldots, 0)$, such that $L_{C_+}(C_+)$ is the ample cone of $S$ over $\mathbb{C}^2/\Gamma$. Since $\theta_J = (-1, 1, 0, \ldots, 0)$ for $J = \{0\}$ and $n = 1$, it follows that $\mathfrak{M}_{\theta_J} \cong \mathbb{C}^2/\Gamma$ in that case. In addition, this explicit description of the kernel of $L_{C_+}$ for $n = 1$ shows that the morphisms $\tau_{J, J'}$ and $\tau_{J, J'}$ from Propositions 2.4 and 4.2 are isomorphisms if and only if $J' \setminus J = \{0\}$.

Appendix A. Bounding the dimension vectors of $\theta_J$-stable modules

A.1. The key statement. We use the term ‘diagram’ to mean ‘framed extended Dynkin diagram’, and use the notation $A_i, D_i, E_i$ for the framed extended versions of these Dynkin diagrams. A module $M$ of the preprojective algebra $\Pi$ of the appropriate type naturally determines a representation $V$ of the corresponding quiver $Q$ that satisfies the preprojective relations; we will call these simply ‘quiver representations’ below. The notion of $\theta_J$-stability for $M$ defines a notion of $\theta_J$-stability for $V$.

For $i \in Q_0 = \{\infty, 0, 1, \ldots, r\}$ we write $v_i := \dim_i V$, and for $0 \leq i \leq r$ we write $\delta_i := \dim(\rho_i)$, so that the regular representation $\delta = \sum_{0 \leq i \leq r} \delta_i \rho_i$ coincides with the minimal imaginary root of the affine Lie algebra associated to the extended Dynkin diagram.

The goal of this appendix is to prove the following result, which we require in the proof of Lemma 3.5.

Proposition A.1. Let $J \subseteq \{0, 1, \ldots, r\}$ be a nonempty subset. Assume that $V$ is a $\theta_J$-stable quiver representation with $v_\infty = 1$ and $v_i = n\delta_i$ for $i \in J$ and some fixed natural number $n$. Then $v_j \leq n\delta_j$ for $j \notin J \cup \{\infty\}$. 

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We argue by contradiction, splitting the proof into several parts. The basic idea is as follows. First, if the inequality \( v_i \leq n\delta_i \) holds for a vertex \( i \) but not its neighbour \( j \), we deduce a basic inequality (A.1) (see Lemma A.3(1)). We then show that this inequality can be ‘pushed around’ the branches of the diagram (see Lemma A.3(2)). If the diagram branches at a trivalent vertex, then we push the inequality further along at least one branch (see Lemma A.4). This leads either to a contradiction or to strong constraints on \( \dim V \). Together, these results settle several cases of the proposition. In particular, Lemma A.7 suffices to prove every case with \( 0 \in J \), except if the diagram is \( A_1 \) or \( D_4 \). This is all that is required to prove the case of primary interest, namely when \( J = \{ 0 \} \). Some remaining cases require arguments directly depending on the diagram structure.

Our main tool for deriving a contradiction is the following estimate, the proof of which is inspired by a result of Crawley-Boevey [CB01, Lemma 7.2]. This inequality is the only consequence of \( \theta_J \)-stability that we use in the subsequent numerical argument.

**Proposition A.2.** Let \( V \) be a \( \theta_J \)-stable quiver representation. If \( i \not\in J \), then \( 2v_i \leq \sum_{a \in Q_1, h(a) = i} v_i(a) \).

**Proof.** Define \( V_\oplus := \bigoplus_{a \in Q_1, h(a) = i} V_i(a) \).

The maps in \( V \) determined by arrows with tail and head at vertex \( i \) combine to define maps \( f: V_i \to V_\oplus \) and \( g: V_\oplus \to V_i \) satisfying \( g \circ f = 0 \). If \( \ker(f) \neq 0 \), then \( V \) admits a nonzero subrepresentation \( W \) such that \( W_i = \ker(f) \) and \( W_j = 0 \) for \( j \neq i \). But then \( W \) corresponds to a \( \Pi \)-submodule of \( M \) supported at vertex \( i \). This submodule would be \( \theta_J \)-semistable, which contradicts the \( \theta_J \)-stability of \( V \). Thus \( f \) is injective. Similarly, if \( \operatorname{Im}(g) \subsetneq V_i \), then \( V \) admits a subrepresentation \( U \) such that \( U_i = \operatorname{Im}(g) \) and \( U_j = V_j \) for \( j \neq i \). Then \( U \) is \( \theta_J \)-semistable, which again contradicts the \( \theta_J \)-stability of \( V \). So \( g \) is surjective. It follows that the complex

\[
0 \to V_i \xrightarrow{f} V_\oplus \xrightarrow{g} V_i \to 0
\]

has nonzero homology only at \( V_\oplus \), so \( \dim V_\oplus \geq 2 \dim V_i \). \( \square \)

**Proof of Proposition A.1.** From now on, assume that \( V \) is a \( \theta_J \)-stable quiver representation with \( v_\infty = 1 \) and \( v_i = n\delta_i \) for \( i \in J \). We split the proof into several sections for better readability.

**A.2. Proof in the case** \( 0 \not\in J \), **excluding types** \( A_1 \) **and** \( D_4 \). The inequality (A.1) below is the basic inequality that we will ‘push around’ the diagram.

**Lemma A.3.**

1. Let \( i, i-1 \) be adjacent vertices of the diagram. If \( v_i > n\delta_i \) and \( v_{i-1} \leq n\delta_{i-1} \), then
   \[
   \delta_{i-1}v_i > \delta_i v_{i-1}. \tag{A.1}
   \]

2. Suppose the vertex \( i \not\in J \) is bivalent, and neither of its neighbours is \( \infty \):
   \[
   \cdots \xrightarrow{i-1} \circ \xrightarrow{i} \circ \xrightarrow{i+1} \cdots
   \]
   Then \( \delta_{i-1}v_i > \delta_i v_{i-1} \) implies \( \delta_i v_{i+1} > \delta_{i+1} v_i \). If in addition \( v_i > n\delta_i \), then \( v_{i+1} > n\delta_{i+1} \).
Proof. (1) is immediate. Since \(i\) and \(\infty\) are not neighbours, \(2\delta_i = \delta_{i-1} + \delta_{i+1}\) holds. (2) follows by combining this equality with the assumed inequality \(\delta_{i-1}v_i > \delta_iv_{i-1}\) and \(2v_i \leq v_{i-1} + v_{i+1}\) coming from Proposition A.2. The last statement is again immediate. \(\square\)

**Lemma A.4.** Suppose that the diagram has a trivalent vertex \(i \notin J\), not adjacent to the vertex \(\infty\):

\[
\begin{array}{c}
\vdots \\
\circ_k \\
\cdots \circ_i \circ_j \cdots \\
\end{array}
\]  

and assume that \(\delta_{i-1}v_i > \delta_iv_{i-1}\).

(1) At least one of the inequalities \(\delta_jv_i < \delta_iv_j\) and \(\delta_kv_i < \delta_iv_k\) must hold.

Suppose now that \(v_i > n\delta_i\), that \(\delta_jv_i < \delta_iv_j\) holds, and furthermore that the branch starting at \(j\) does not branch further. Then

(2) the branch starting at \(j\) does not contain any vertices in \(J\), and

(3) the same branch must terminate at the framing vertex \(\infty\), and in this case \(\delta_jv_j = \delta_jv_i + 1\).

**Remark A.5.** The only framed extended Dynkin diagrams where a trivalent vertex is adjacent to the framing vertex are of type \(A_i\) for \(i > 1\). We handle the case of such a vertex not being in \(J\) in Lemma A.8.

Proof. For (1), combining \(2\delta_i = \delta_{i-1} + \delta_j + \delta_k\) with \(2v_i \leq v_{i-1} + v_k + v_j\) and \(\delta_{i-1}v_i > \delta_iv_{i-1}\) leads to \(\delta_jv_i + \delta_kv_i < \delta_iv_j + \delta_iv_k\) which implies the result. For (2) and (3), we denote the vertices as

\[
\begin{array}{c}
\vdots \\
\circ_i \circ_j \cdots \\
\cdots \circ_{j+1} \circ_{j+l} \\
\end{array}
\]  

if the branch does not contain the framing vertex, or

\[
\begin{array}{c}
\vdots \\
\circ_i \circ_j \cdots \\
\cdots \circ_{j+l} \circ_{j+l+1} \circ_{\infty} \\
\end{array}
\]  

if it does. To simplify notation, we take \(j-1 = i\) in the following argument. One of the following must occur.

- **The branch contains another vertex in \(J\).** Suppose that \(j'\) is the node with smallest index on the branch such that \(j' \neq i\) and \(j' \in J\). Lemma A.3(2) gives \(\delta_{j'-1}v_{j'} > \delta_{j'}v_{j'-1}\) and \(v_{j'} > n\delta_{j'}\), contradicting \(j' \in J\).

- **The branch contains no vertices in \(J \cup \infty\).** Repeated applications of Lemma A.3(2) show that \(\delta_{j+l-1}v_{j+l} > \delta_{j+l}v_{j+l-1}\). However, since \(2\delta_{j+l} = \delta_{j+l-1}\), this implies \(2v_{j+l} > v_{j+l-1}\), contradicting Proposition A.2.
• The branch contains no vertices of $J$, and terminates at $\infty$.

We prove a slightly stronger statement, namely that for any vertex $m \neq \infty$ on the branch, we have $\delta_{m-1}v_m = \delta_m v_{m-1} + 1$. We proceed by induction on the number of edges that lie between $\infty$ and $m$. For the base case $m = j + l$, note that $\delta_{j+l}v_{j+l} > \delta_{j+l}v_{j+l-1}$ implies $2v_{j+l} > v_{j+l-1}$. However, since $2v_{j+l} \leq v_{j+l-1} + 1$ by Proposition A.2, we must have $2v_{j+l} = v_{j+l-1} + 1$. If there is more than one edge between $\infty$ and $m$, then the induction hypothesis gives $\delta_m v_{m+1} = \delta_{m+1} v_m + 1$.

Combining this with $2v_m \leq v_{m-1} + v_{m+1}$ from Lemma A.2 and $2\delta_m = \delta_{m+1} + \delta_{m-1}$ shows that $\delta_{m-1}v_m = \delta_m v_{m-1} + 1$. Lemma A.3(2) gives $\delta_{m-1}v_m > \delta_m v_{m-1}$ and the result follows.

This concludes the proof. □

**Lemma A.6.** Suppose that in the chain of bivalent vertices

\[\cdots \bigcirc_{i} \bigcirc_{i+1} \bigcirc_{i+k} \cdots (A.6)\]

we have both $i, i + k \in J$. Then there is no $j \in [i, i + k]$ such that $v_j > n\delta_j$.

**Proof.** Without loss of generality, we can assume that no vertices between $i$ and $i + k$ lie in $J$. Now let $t$ be the smallest integer such that $v_{i+t} > n\delta_{i+t}$. Thus $\delta_{i+t-1}v_{i+t} > \delta_{i+t}v_{i+t-1}$. Repeated applications of Lemma A.3(2) lead to $v_{i+k} > n\delta_{i+k}$, a contradiction. □

We move on to completing the proof of Proposition A.1 in the case $0 \in J$. The key is that in this case, removing $J$ and all incident edges creates a component consisting only of the node $\infty$.

**Lemma A.7.** Let $i \in Q_0 \setminus \{\infty\}$ be any vertex such that every path in the diagram from $\infty$ to $i$ passes through an element of $J$. Assume in addition that the diagram is not of type $D_4$. Then $v_i \leq n\delta_i$.

**Proof.** We introduce some notation. Given a path $\gamma$ in our diagram, we define

\[d_\gamma(a, b) := 1 + \# \{\text{vertices on $\gamma$ between $a$ and $b$}\}.\]

Assume that the statement does not hold, i.e. $v_i > n\delta_i$. Let $\gamma$ be a path in the diagram from $\infty$ to $i$ that does not touch a given vertex more than once, and let $j$ be the vertex in $J$ on $\gamma$ for which $d_\gamma(\infty, j)$ is maximal. There must be a pair $k_1, k_2$ of adjacent vertices along $\gamma$ such that

1. $d_\gamma(\infty, j) \leq d_\gamma(\infty, k_1) = d_\gamma(\infty, k_2) - 1$ and $d_\gamma(\infty, k_2) \leq d_\gamma(\infty, i)$; and
2. $v_{k_1} \leq n\delta_{k_1}$ and $v_{k_2} > n\delta_{k_2}$.

It follows that $\delta_{k_1}v_{k_2} < \delta_{k_2}v_{k_1}$. We now use the above lemmas to ‘push’ this inequality away from $\infty$ until we reach a contradiction. Formally, we apply Lemma A.3(2) to the pair $k_1, k_2$. Reapplying the same lemma, we either reach a contradiction, or a node $a$ of valency 3. Ignoring for now the branch containing $k_1$, observe that if either of the other branches contains an additional branch then Lemma A.4(3) gives a contradiction.

Otherwise, there must be a branch starting at $a$ that reaches another branching point of valency at most 3. Let $b$ be the vertex on this branch that lies adjacent to $a$. Lemma A.4 gives $\delta_{a}v_a < \delta_{a}v_b$. Applying Lemma A.3(2) repeatedly to pairs of adjacent vertices along the branch starting at $b$ enables us to push this basic inequality to the second branching point, where Lemma A.4(2) gives a contradiction. □

**Proof of Proposition A.1 in the case $0 \in J$ except for types $A_1$ and $D_4$.** If our diagram is not of type $A_1$ or $D_4$, then all vertices have valency at most 3 and there is no double edge. Also, the framing vertex $\infty$ is
adjacent only to the vertex labelled 0. Thus, if \( 0 \in J \), any path from \( \infty \) to any vertex that does not lie in \( J \) must pass through an element of \( J \). Hence Lemma A.7 proves Proposition A.1 in this case. \( \square \)

A.3. Proof for types \( A_1 \) and \( D_4 \).

**Lemma A.8.** Proposition A.1 holds for \( A_1 \) and \( D_4 \).

**Proof.** For type \( A_1 \), we have the diagram

\[
\begin{array}{c}
\infty \\
\longrightarrow \\
\bigcirc \\
\longrightarrow \\
0 \\
\longrightarrow \\
\bigcirc \\
\longrightarrow \\
1
\end{array}
\]

(A.7)

where the symbol \( \longrightarrow \) indicates that there are two edges in the diagram. If \( J = \{0, 1\} \) there is nothing to prove, so we take \( J \) to be either \( \{0\} \) or \( \{1\} \). A straightforward adaptation of Proposition A.2 shows that if \( J = \{0\} \), we must have \( 2v_1 \leq 2v_0 \), so \( v_1 \leq n \). Similarly, if \( J = \{1\} \), we obtain \( 2v_0 \leq 2v_1 + 1 = 2n + 1 \), giving \( v_0 \leq n \).

For type \( D_4 \), the diagram is:

\[
\begin{array}{c}
\infty \\
\longrightarrow \\
0 \\
\longrightarrow \\
\bigcirc \\
\longrightarrow \\
1 \\
\longrightarrow \\
2 \\
\longrightarrow \\
3 \\
\longrightarrow \\
4
\end{array}
\]

(A.8)

Consider first the case of \( 0 \in J \). If \( v_2 > n\delta_2 = 2n \), we get from Proposition A.2 that \( 2v_i \leq v_2 \) for \( i \in \{1, 3, 4\} \). This implies that

\[
4v_2 \leq 2v_0 + 2v_1 + 2v_3 + 2v_4 \leq 2n + 3v_2,
\]

contradicting that \( v_2 > 2n \). On the other hand, if \( v_1 > n\delta_1 = n \), Proposition A.2 implies that \( v_2 > 2n \). The same argument as before leads to a contradiction. By symmetry, the same argument applies if \( v_i > 2n \) for \( i = 3 \) or \( i = 4 \).

If \( v_0 > n\delta_0 = n \), and \( 2 \in J \), Lemma A.4 immediately gives a contradiction.

So suppose without loss of generality that \( 1 \in J \). Then any other vertex \( i \) with \( v_i > n\delta_i \) will, by Lemma A.4 or Proposition A.2 give that \( v_2 > 2n \). The same lemmas show that

\[
4v_2 \leq 2v_0 + 2v_1 + 2v_3 + 2v_4 \leq 2n + 3v_2 + 1
\]

(A.9)

and thus \( v_2 \leq 2n + 1 \). So \( v_2 = 2n + 1 \), but then Proposition A.2 gives that \( v_1 = v_3 = v_4 = n \). Plugging this into (A.9) gives \( 6n + 3 = 3v_2 \leq 6n + 1 \), a contradiction. \( \square \)

A.4. Proof in the general case. We next handle the cases where \( 0 \not\in J \). For this, we need to consider each diagram type individually.

**Lemma A.9.** Lemma A.1 holds for any diagram of type \( A_i \) with \( i > 1 \).

**Proof.** We number the vertices as follows:

\[
\begin{array}{c}
\infty \\
\longrightarrow \\
0 \\
\longrightarrow \\
\bigcirc \\
\longrightarrow \\
1
\end{array}
\]

(A.10)

Assume that some vertex \( k' \neq \infty \) has \( v_{k'} > n\delta_{k'} = n \).
Note that by Lemma A.7, we can assume that $0 \notin J$. So we consider a subdiagram

\[
\begin{array}{c}
\circ & \cdots & \circ & \cdots & \circ & \cdots & \circ & \cdots & \circ & \cdots & \circ \\
\cdots & i & \cdots & r & 0 & \cdots & j & \cdots & \cdots & \cdots & \cdots \\
\infty & & & & & & & & & & 1
\end{array}
\]  

(A.11)

where $i, j$ (possibly equal) are the only vertices in $J$, with $k'$ some vertex in this subdiagram. We can without loss of generality assume $0 \leq k' < j$. Then there are adjacent vertices $k, k + 1$ such that $k' \leq k, k + 1 \leq j$ with $v_k > n \geq v_{k+1}$. Repeatedly applying Lemma A.3 gives

\[v_0 > v_1 > 2v_0 > v_1 + 2v_0 \geq 4v_2 - 1,\]  

(A.12)

There must also be adjacent vertices $l, l + 1$ between $i$ and 0 such that $v_{l+1} > v_l$. In a similar way, this leads to $v_0 > v_r$. Combining with (A.12), we deduce $2v_0 > v_1 + v_r + 1$, contradicting Proposition A.2. □

Lemma A.10. Lemma A.1 holds for diagrams of type $D_i$, $i > 4$.

Proof. We number the vertices as follows:

\[
\begin{array}{c}
\circ & \cdots & \circ & \cdots & \circ & \cdots & \circ & \cdots & \circ & \cdots & \circ \\
\cdots & 0 & \cdots & 2 & \cdots & r-2 & \cdots & r-1 & \cdots & \cdots & \cdots \\
\infty & & & & & & & & & & 1
\end{array}
\]  

(A.13)

We show that three cases remaining from Lemma A.7 are also absurd. By the symmetry of the diagram, these are sufficient.

(1) There is an $i$ such that $2 \leq i \leq r - 1$, $v_i > n\delta_i = 2n$, and all $j \in J$ have $i < j$. Let $k$ be maximal among the vertices such that $v_k > n\delta_k$. If $k \leq r - 2$, we have $\delta_{k+1}v_k > \delta_kv_{k+1}$.

If $k = r - 1$, we must have $J = \{r\}$, and by Lemma A.4 we get $\delta_{r-3}v_{r-2} > \delta_{r-2}v_{r-3}$. By symmetry, the case $k = r$ also leads to $\delta_{r-3}v_{r-2} > \delta_{r-2}v_{r-3}$.

Both cases lead to (by Lemma A.3) $v_2 > v_3$, that is, $v_2 - 1 \geq v_3$. Then Lemma A.4 gives $2v_0 = v_2 + 1$ and $2v_1 \leq v_2$. Combining these with Proposition A.2 leads to

\[4v_2 \leq 2v_3 + 2v_1 + 2v_0 \leq 4v_2 - 1,\]

which is absurd.

(2) $v_1 > n\delta_1 = n$, and all $j \in J$ have $j > 2$. This implies $v_2 > 2n$. Let $j$ be the least vertex such that $v_j \leq n\delta_j$. Applying Lemmas A.3 and A.4 to the vertices $j - 1, j$ (or if $j = r$, the vertices $r, r - 2$) we again find $v_2 > v_3$. Then the conclusion of case (1) applies.

(3) $v_0 > n\delta_0$, and all $j \in J$ have $j \geq 2$. If $2 \in J$, we have $v_2 = 2n$, and then $v_1 > n$ leads to $2v_1 > 2n + 1$, contradicting Proposition A.2. If $2 \notin J$ we can again take $j$ as the least vertex with $v_j \leq n\delta_j$ and argue as in case (2).

Hence all possibilities lead to a contradiction, and Lemma A.1 holds for diagrams of type $D_i$ with $i > 4$. □

To conclude, we have to deal with diagrams of type $E_i$. As the proof strategies for these are very similar, we only include the full argument for the $E_8$ case.

Lemma A.11. Proposition A.1 holds for type $E_8$.  

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Proof. We number the vertices as follows:

\[ \begin{array}{cccccccccc}
\circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \infty
\end{array} \]  \hspace{1cm} (A.14)

This time, we split what remains after Lemma A.7 into four cases. Let \( k \) be the minimal vertex with \( v_k > n\delta_k \). The cases are:

1. \( k > 4 \) and all \( j \in J \) have \( j < k \). By Lemma A.3 and Lemma A.4, we find that \( v_0 > n\delta_0 = n \). The same lemmas show that \( \delta_{k+1}v_k + 1 = \delta_kv_{k+1} \). By Proposition A.2, we get

\[ 2\delta_kv_k \leq \delta_{k+1}v_k + 1 = \delta_kv_{k+1} + 1 + \delta_kn\delta_{k-1} \]

implying \( \delta_{k-1}(v_k - n\delta_k) \leq 1 \). But this contradicts \( v_k > n\delta_k \).

2. \( k = 4 \) and all \( j \in J \) have \( j < k \): We have

\[ 2v_4 \leq v_2 + v_3 + v_5 \leq n\delta_2 + n\delta_3 + v_5 = 7n + v_5. \]

Since we also have (Lemma A.4) \( 5v_4 + 1 = 6v_5 \), this implies that \( 7v_5 - 2 \leq 35n \). But since \( v_5 > 5n \), this is impossible.

3. \( k = 2 \) and at least one of the vertices 1 and 3 are in \( J \): By Proposition A.2, we must have \( v_4 \geq 6n+1 \). By Lemma A.3 applied to the vertex chain 1,3,4, we find \( 6v_3 < 4v_4 \). Then Lemma A.4 shows that \( 6v_5 = 5v_4 + 1 \). Now, if \( v_3 \leq n\delta_3 \), the same lemma and Proposition A.2 imply

\[ 12v_4 \leq 6v_2 + 6v_3 + 6v_5 \leq 8v_4 + 24n + 1 \]

leading to \( 24n+4 \leq 4v_4 \leq 24n+1 \), a contradiction. So suppose that 1 \( \in J \), and \( v_3 > 4n \). By Lemma A.3, we get \( 6v_3 < 4v_4 \). As above, we find

\[ 12v_4 \leq 6v_2 + 6v_3 + 6v_5 \leq 8v_4 + 6v_3 + 1 \]

leading to \( 4v_4 \leq 6v_3 + 1 \). This implies that \( 4v_4 = 6v_3 + 1 \), which has no integer solutions. Hence we have a contradiction.

4. \( k = 1 \) or \( k = 3 \), and so \( J \) only consists of \( 2 \): Suppose that \( v_2 = n\delta_2 = 3n \). Then, by Lemma A.4 and Lemma A.3, we get \( v_4 > 4n \), say \( v_4 = 4n+t \), \( t > 0 \). But then Proposition A.4 and Proposition A.2 give

\[ 12v_4 \leq 6v_2 + 6v_3 + 6v_5 \leq 18n + 4v_4 + 5v_4 + 1 \]

leading to \( 18n + 3t = 3v_4 \leq 18n + 1 \), a contradiction.

Thus Proposition A.1 holds for \( E_8 \). \hfill \Box

Analogous arguments apply for the diagrams of type \( E_6 \) and \( E_7 \). The proof of Proposition A.1 is now complete. \hfill \Box

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