MINIMUM COPRIME GRAPH LABELINGS

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Abstract. A coprime labeling of a graph $G$ is a labeling of the vertices of $G$ with distinct integers from 1 to $k$ such that adjacent vertices have coprime labels. The minimum coprime number of $G$ is the least $k$ for which such a labeling exists. In this paper, we determine the minimum coprime number for several well-studied classes of graphs, including the coronas of complete graphs with empty graphs, the joins of two paths, and prisms. In particular, we resolve a conjecture of Seoud, El Sonbaty, and Mahran and three conjectures of Asplund and Fox. We also provide bounds on the minimum coprime number of a random subgraph.

1. Introduction

1.1. Background. Let $G$ be a simple graph with $n = |V(G)|$ vertices. A coprime labeling of $G$ is an injection $f : V(G) \rightarrow \{1, 2, \ldots, k\}$, for some integer $k \geq n$, such that if $(u, v) \in E(G)$ then $\gcd(f(u), f(v)) = 1$. The minimum coprime number $pr(G)$ is the least $k$ for which such a labeling exists; a coprime labeling of $G$ using only integers up to $pr(G)$ is called a minimum coprime labeling of $G$. If $pr(G) = n$, then a minimum coprime labeling of $G$ is called a prime labeling and $G$ is a prime graph.

The notion of prime labeling originated with Entringer and was introduced in a paper by Tout, Dabboucy, and Howalla [37]. It is conceptually related to the coprime graph of integers, the graph with vertex set $\mathbb{Z}$ that contains the edge $(m, n)$ if and only if $\gcd(m, n) = 1$. The induced subgraph $G(A)$ with vertex set $A \subset \{1, 2, \ldots, N\}$ is called the coprime graph of $A$ and was first studied by Erdős [10], who posed the famous problem of finding the largest set $A \subset \{1, 2, \ldots, N\}$ such that $K_k \not\subset G(A)$. Newman’s coprime mapping conjecture, which was proven by Pomerance and Selfridge [24], involves the existence of perfect matchings in $G(A)$. Various other properties of the coprime graph of integers have been studied by Ahlswede and Khachatrian [1, 2, 3], Erdős [11], Erdős and A. Sárközy [12], Erdős and G. N. Sárközy [15], Erdős, A. Sárközy, and Szemerédi [13, 14], and G. N. Sárközy [29]. Further discussion of the coprime graph of integers may be found in [23].

The problem of finding a minimum coprime labeling of a graph $H$ is equivalent to showing that $H$ is a subgraph of the coprime graph $G(\{1, 2, \ldots, pr(H)\})$. Much work has been done to prove that various classes of graphs are prime; we refer the reader to [17] for a full catalog of results. In particular, it is known that all paths, cycles, helms, fans, flowers, books, and wheels of even order are prime [17, 30, 32].

The primality of trees has been especially well studied. Entringer and Tout conjectured around 1980 that every tree is prime. While this conjecture remains open, it is now known that many classes of trees are prime, including paths, stars, caterpillars, spiders, and complete binary trees [17]. Fu and Huang [16] proved in 1994 that trees with 15 or fewer vertices are prime, and Pikhurko [22] extended this result to trees with 50 or fewer vertices in 2007. Salmasian [28] showed that any tree $T$ with $n \geq 50$ vertices satisfies $pr(T) \leq 4n$. Pikhurko [22] improved upon
this by showing that the Entringer–Tout conjecture holds asymptotically, i.e., for any $c > 0$, there is an $N$ such that for any tree $T$ of order $n > N$, $\text{pr}(T) < (1 + c)n$. In 2011, Haxell, Pikhurko, and Taraz [19] proved the Entringer–Tout conjecture for trees of sufficiently large order.

In this paper, we focus on the minimum coprime numbers of a few well-studied classes of graphs. The first class we consider is formed by taking the coronas of complete graphs and empty graphs. The corona of a graph $G$ with a graph $H$, denoted $G \circ H$, is obtained by combining one copy of $G$ with $|V(G)|$ copies of $H$ by attaching the $i$th vertex in $G$ to every vertex in the $i$th copy of $H$. Tout, Dabboucy, and Howalla [37] showed that the crown graphs $C_n \circ K_m$ are prime for all positive integers $n$ and $m$. The graphs $K_n \circ K_m$, which are spanning supergraphs of the crowns $C_n \circ K_m$, have also been studied in this context. Youssef and Elsakha [39] showed that $K_n \circ K_1$ is prime for $n \leq 7$, and $K_n \circ K_2$ is prime for $n \leq 16$. Seoud, El Sonbaty, and Mahran [31] then observed that $K_n \circ K_m$ is not prime if $n > \pi(n(m + 1) + 1)$ [1]. They also conjectured the converse,

**Conjecture 1.1** ([31], Conjecture 3.9). The graph $K_n \circ K_m$ is prime if $n \leq \pi(n(m + 1)) + 1$.

They computed all values of $n$ satisfying this condition for all $m \leq 20$. Most recently, Asplund and Fox [4] computed the minimum coprime numbers of $K_n \circ K_1$ and $K_n \circ K_m$, showing that $\text{pr}(K_n \circ K_1) = p_{n-1}$ if $n > 7$ and $\text{pr}(K_n \circ K_2) = p_n - 1$ if $n > 16$. They conjecture that their results extend whenever $n$ is sufficiently large relative to $m$.

**Conjecture 1.2** ([4], Conjecture 1). For all positive integers $m$, there exists an $M > m$ such that for all $n > M$, $\text{pr}(K_n \circ K_m) = p_{n-1}$.

We prove the following statement, which resolves Conjectures 1.1 and 1.2 affirmatively.

**Theorem 1.1.** For all positive integers $m$ and $n$,

$$\text{pr}(K_n \circ K_m) = \max(mn + n, p_{n-1}).$$

The next class of graphs we consider is constructed via the join operation. The join of two disjoint graphs $G$ and $H$, denoted $G + H$, consists of the graph union $G \cup H$ with edges added to connect each vertex in $G$ to each vertex in $H$. We focus on the joins of two paths, which are well-studied. It was shown in [30, 31] that $P_n + K_1 = P_n + P_1$ is prime, $P_n + K_2$ is prime if and only if $n \geq 3$ is odd, and $P_n + K_m$ is not prime for $m \geq 3$. Asplund and Fox [4] computed the minimum coprime numbers of $P_m + P_2$ for various $m$ and $n$. They showed the following theorem.

**Theorem 1.2** ([4], Theorems 17, 18, and 19). For $m \geq 4$ even and $n = 2$, or $m \geq n$ and $n = 3$ or 4, the minimum coprime number of $P_m + P_n$ is given by

$$(1.1) \quad \text{pr}(P_m + P_n) = \begin{cases} m + 2n - 2 & \text{if } m \text{ is odd} \\ m + 2n - 1 & \text{if } m \text{ is even}. \end{cases}$$

They proceeded to show that (1.1) holds for $2 \leq n \leq 10$ if $m > 118$. They conjecture that this result extends to all $n$ as long as $m$ is sufficiently large.

**Conjecture 1.3** ([4], Conjecture 2). For any positive integer $N$, there exists a positive integer $M$ such that for all $m > M$ and $2 \leq n \leq N$, the minimum coprime number of $P_m + P_n$ is given by

$$\text{pr}(P_m + P_n) = \begin{cases} m + 2n - 2 & \text{if } m \text{ is odd} \\ m + 2n - 1 & \text{if } m \text{ is even}. \end{cases}$$

[1] Throughout this paper, we denote $\pi(x)$ to be the number of primes less than or equal to $x$, and $p_n$ to be the $n^{th}$ prime number.
They also pose the following stronger conjecture, which applies for all $m \geq n$, removing the condition on $m$ being sufficiently large.

**Conjecture 1.4** ([4], Conjecture 3). For any positive integers $m \geq n$, the minimum coprime number of $P_m + P_n$ is given by

$$\text{pr}(P_m + P_n) = \begin{cases} m + 2n - 2 & \text{if } m \text{ is odd} \\ m + 2n - 1 & \text{if } m \text{ is even.} \end{cases}$$

The $k^{th}$ Ramanujan prime $R_k$ is the least integer for which $\pi(x) - \pi(\frac{x}{2}) \geq k$ holds for all $x \geq R_k$. Ramanujan primes were introduced in [26] as a generalization of Bertrand’s Postulate; they are published as sequence A104272 in the OEIS [33]. We resolve Conjecture 1.3 affirmatively, showing the following.

**Theorem 1.3.** For any positive integer $N$, if $M \geq R_N - 1 - 2N + 1$, then for all $m \geq M$ and $2 \leq n \leq N$, the minimum coprime number of $P_m + P_n$ is

$$\text{pr}(P_m + P_n) = \begin{cases} m + 2n - 2 & \text{if } m \text{ is odd} \\ m + 2n - 1 & \text{if } m \text{ is even.} \end{cases}$$

We observe in Remark 4.1 that Conjecture 1.4 is false, providing a number of counterexamples. Our results on $P_m + P_n$ automatically yield upper bounds on the minimum coprime numbers of the complete bipartite graphs $K_{m,n}$, and our constructions also generalize to the join of two cycles and the join of a path and a cycle, enabling us to compute $\text{pr}(C_m + C_n)$, $\text{pr}(C_m + P_n)$, and $\text{pr}(P_m + C_n)$ for sufficiently large $m$. Our results on these classes of graphs are given in Theorems 4.3 and 4.4 and Corollaries 4.2 and 4.3.

A final well-studied class of graphs we consider is prisms. The *prism graph* $Y_n$ is a graph corresponding to the skeleton of an $n$-prism; it is isomorphic to the Cartesian product $C_n \square P_2$. Prajapati and Gajjar [25] showed that if $n \geq 3$ is odd, then $Y_n$ is not prime. They also proved the following.

**Theorem 1.4** ([25], Theorem 2.10). If $n + 1$ is a prime number, then $Y_n$ is prime.

Diefenderfer et al. [8] later showed the following.

**Theorem 1.5** ([8], Theorem 4.1). If $n \geq 4$ and $n - 1$ is a prime number, then $Y_n$ is prime.

We expand upon these results in the following theorem.

**Theorem 1.6.** If any of $n + 3$, $n - 3$, $n - 5$, $n - 7$, or $n - 9$ is a prime number, then $Y_n$ is prime.

We observe that, as a corollary,

**Corollary 1.1.** The graph $Y_n$ is prime for even integers $n \leq 532$.

Our results on prisms also extend to show the primality of certain classes of webs, which we discuss in Corollary 5.1.

We conclude this paper by providing some bounds on the minimum coprime number of a random subgraph, a topic which, to the best of our knowledge, has not been previously studied. Given a graph $G$ and $p \in (0, 1)$, we obtain a probability distribution $G_p$ called a *random subgraph* by taking subgraphs of $G$ with each edge appearing independently with probability $p$. When $G = K_n$, this is called the *Erdős–Rényi random graph*, denoted $G(n, p)$. We prove the following bounds on the minimum coprime number of the Erdős–Rényi random graph.
Corollary 2.1. Any spanning subgraph of a prime graph is prime. Any spanning supergraph of \(K\) graphs number of an arbitrary graph which we will later use in our proofs. In Section 3, we discuss the

Proposition 2.2. A nonprime graph is nonprime.

As a corollary, we observe that

Corollary 1.2. \(\text{pr}(G(n,p))\) is almost surely not prime.

1.2. Outline. In Section 2 we state several straightforward results on the minimum coprime number of an arbitrary graph which we will later use in our proofs. In Section 3 we discuss the graphs \(K_n \circ K_m\), proving Theorem 1.1. In Section 4 we focus on the graphs \(P_m + P_n\), proving Theorem 1.3 and providing counterexamples for Conjecture 1.4. We also consider the complete bipartite graphs \(K_{m,n}\) and the graphs \(C_m + C_n\) and \(C_m + P_n\). In Section 5, we prove a number of results on the primality of prisms and webs. In Section 6 we note the primality of a few graphs that have not previously appeared in the literature and briefly discuss unicyclic graphs. In Section 7 we discuss the minimum coprime number of a random subgraph, proving Theorem 1.7. We conclude by posing a number of open questions in coprime graph labeling in Section 8.

2. Preliminaries

In this section we state several propositions relating the minimum coprime number to other graph properties. Recall that if \(G\) and \(H\) are graphs with \(V(G) = V(H)\) and \(E(G) \subseteq E(H)\), we call \(H\) a spanning supergraph of \(G\) and \(G\) a spanning subgraph of \(H\). The independence number \(\alpha(G)\) of \(G\) is the size of the largest set of vertices \(S\) in \(V(G)\) such that no two vertices in \(S\) are adjacent to one another. The chromatic number \(\chi(G)\) of \(G\) is the least integer \(k\) for which there exists a map \(f : V(G) \to \{1, \ldots, k\}\) such that if \((u, v) \in E(G)\) then \(f(u) \neq f(v)\).

Proposition 2.1. Let \(G\) and \(H\) be graphs such that \(H\) is a spanning supergraph of \(G\). Then \(\text{pr}(G) \leq \text{pr}(H)\).

Proof. Let \(f : V(H) \to \{1, \ldots, \text{pr}(H)\}\) be a minimum coprime labeling of \(H\). As \(V(G) = V(H)\), \(f\) induces a coprime labeling of \(G\). Hence \(\text{pr}(G) \leq \text{pr}(H)\). \(\square\)

The following corollary of Proposition 2.1 has been noted in numerous places in the literature.

Corollary 2.1. Any spanning subgraph of a prime graph is prime. Any spanning supergraph of a nonprime graph is nonprime.

Proposition 2.2. For any graph \(G\), \(\text{pr}(G) \geq 2(|V(G)| - \alpha(G)) - 1\).

Proof. Under any coprime labeling of \(G\), the vertices with even labels must form an independent set. Hence at most \(\alpha(G)\) even integers may be used to label \(G\). The lower bound follows from noting that at least \(|V(G)| - \alpha(G)\) odd integers must be used as labels. \(\square\)

The following corollary of Proposition 2.2 has been stated elsewhere in the literature, for instance in [16] and [32].

Corollary 2.2. If \(G\) is a prime graph, then \(\alpha(G) \geq \left\lfloor \frac{|V(G)|}{2}\right\rfloor\).

Proposition 2.3. For any graph \(G\), \(\text{pr}(G) \leq \max(2\alpha(G), p_{|V(G)| - \alpha(G) - 1})\).

Proof. Let \(S\) be an independent set of size \(\alpha(G)\). We may label each of the vertices of \(S\) with even integers up to \(2\alpha(G)\), and the remaining vertices in \(G \setminus S\) with 1 and the first \(|V(G)| - \alpha(G)\) primes. This labeling is coprime by construction and does not use integers exceeding \(\max(2\alpha(G), p_{|V(G)| - \alpha(G) - 1})\). \(\square\)
Proposition 2.4. For any graph $G$, $\text{pr}(G) \geq 2\chi(G) - 2$.

Proof. By Proposition 2.2, $\alpha(G) \geq \frac{2|V(G)| - \text{pr}(G) - 1}{2}$. As all the vertices in an independent set may be assigned the same color, we have

$$\chi(G) \leq |V(G)| - \frac{2|V(G)| - \text{pr}(G) - 1}{2} + 1 = \left\lceil \frac{\text{pr}(G)}{2} \right\rceil + 1.$$ 

The bound on $\text{pr}(G)$ follows from rearranging. $\square$

3. Minimum Coprime Numbers of $K_n \odot K_m$

In this section we prove Theorem 1.1. We will need the following lemma.

Lemma 3.1. For all $n$, $\pi\left(\frac{p_n}{2}\right) < \frac{3n}{4}$.

Proof. By Corollaries 1 and 2 of Theorem 2 in [27], we have the following bounds on the prime counting function for all $x \geq 17, x \notin [113, 113.6]$:

$$\frac{x}{\log x} < \pi(x) < \frac{5x}{4\log x}.$$ 

In particular, when $x = p_n$, the lower bound yields

$$p_n < n \log p_n.$$ 

Thus, for $n \neq 49$, we have

$$\pi\left(\frac{p_n}{2}\right) < \frac{5p_n}{8(\log p_n - \log 2)} < \frac{5n \log p_n}{8(\log p_n - \log 2)} < \frac{5n}{8} \left(1 + \frac{\log 2}{\log p_n - \log 2}\right).$$ 

For $n \geq 19$, we have $p_n \geq 67$, so that

$$1 + \frac{\log 2}{\log p_n - \log 2} < 1 + \frac{\log 2}{\log 67 - \log 2} \approx 1.1974.$$ 

Thus

$$\pi\left(\frac{p_n}{2}\right) < \frac{5}{8}(1.1974n) < \frac{3n}{4}.$$ 

We can manually check that the inequality holds for $1 \leq n \leq 18$ and $n = 49$. $\square$

Proof of Theorem 1.1. Set $N = \max(p_{n-1}, mn + n)$. It is apparent that $\text{pr}(K_n \odot K_m) \geq mn + n$ as $|V(K_n \odot K_m)| = mn + n$. By Proposition 2.4, we have $\text{pr}(K_n \odot K_m) \geq \text{pr}(K_n)$. It is well known that $\text{pr}(K_n) = p_{n-1}$ (see [4] for example). Therefore $\text{pr}(K_n \odot K_m) \geq N$.

It now suffices to show that it is possible to construct a coprime labeling of $K_n \odot K_m$ using only labels up to $N$. Denote $u_1, \ldots, u_n$ to be the vertices of $K_n$ and $v_{i,1}, \ldots, v_{i,m}$ to be the vertices of the $i$th copy of $K_m$. Label $u_1$ with 1 and the remaining $u_i$ with $p_{\pi(N) - n + i}$, so that we have used the $n - 1$ largest primes up to $N$ as labels. Now observe that for any prime $p$, $\gcd(p, k) = 1$ for any $p < k < 2p$. Thus it is only necessary to label $v_{i,1}, \ldots, v_{i,m}$ with integers coprime to $p_i$ for $2 \leq i \leq \pi(p_{\pi(N)}/2)$, as any labeling of the remaining vertices will automatically be coprime. (In particular, in the case $mn > p_{n-1} - n$, we are done if $\pi(mn + n) - n + 2 > \pi(p_{\pi(mn+n)}/2)$.)

After labeling the vertices of $K_n$ as indicated, there are at least $mn$ integers less than or equal to $N$ which have not yet been used as labels. Denote by $L_1$ the list of all such integers. We label the vertices $v_{i,k}$ for $2 \leq i \leq \pi(p_{\pi(N)}/2)$ as follows: for each $i$, select the first $m$ integers in $L_{i-1}$ which are coprime to $p_{\pi(N) - n + i}$ (the label of $u_i$). Use these integers to label $v_{i,1}, \ldots, v_{i,m}$, and remove them from $L_{i-1}$ to form a new list $L_i$. 


We claim that at each step of this process there exist \( m \) integers in \( L_{i-1} \) which are coprime to \( p_{\pi(N)-n+i} \), so that this process in fact yields a coprime labeling for \( K_n \cup K_m \) using only integers up to \( N \). For any \( 2 \leq i \leq \pi(p_{\pi(N)}/2) \), because \( n \) integers have been used to label the vertices of \( K_n \) and \( m(i-2) \) additional integers have been used to label the vertices of the first \( i-1 \) copies of \( K_m \), there are at least

\[
\frac{(mn + n)(p_{\pi(N)-n+i} - 1)}{p_{\pi(N)-n+i}} - m(i-2) - n \geq m \left( n - \frac{n}{p_{i-1}} - i + 2 \right) - \frac{n}{p_{i-1}}
\]

integers in \( L_{i-1} \) that are coprime to \( p_{\pi(N)-n+i} \).

If \( N = p_{n-1} \), we consider the following cases. By [37, 39, 4], the statement in the theorem is known for all \( n \leq 3 \) or \( m \leq 2 \), so we may assume that \( n \geq 4 \) and \( m \geq 3 \).

- If \( i = 2 \), we have
  \[
m \left( n - \frac{n}{p_{i-1}} - i + 2 \right) - \frac{n}{p_{i-1}} = m \left( \frac{n}{2} - \frac{n}{2m} \right) > m.
  \]

- If \( i = 3 \), we have
  \[
m \left( n - \frac{n}{p_{i-1}} - i + 2 \right) - \frac{n}{p_{i-1}} = m \left( \frac{2n}{3} - \frac{n}{3m} - 1 \right) > m.
  \]

- If \( i = 4 \) and \( n \leq 5 \), we have
  \[
m \left( n - \frac{n}{p_{i-1}} - i + 2 \right) - \frac{n}{p_{i-1}} \geq \frac{mn}{20} - \frac{n}{5} + 2m \geq 2m - \frac{n}{20} > m,
  \]
  since \( i \leq \frac{3n}{4} \) by Lemma [31].

- If \( i = 4 \) and \( n \geq 5 \), we have
  \[
m \left( n - \frac{n}{p_{i-1}} - i + 2 \right) - \frac{n}{p_{i-1}} = m \left( \frac{4n}{5} - \frac{n}{5m} - 2 \right) \geq m \left( \frac{11n}{15} - 2 \right) > m.
  \]

- If \( i \geq 5 \), we have \( p_{i-1} \geq 7 \) and
  \[
m \left( n - \frac{n}{p_{i-1}} - i + 2 \right) - \frac{n}{p_{i-1}} \geq \frac{3mn}{28} + 2m - \frac{n}{7} > m.
  \]
  since \( i \leq \frac{3n}{4} \) by Lemma [31].

Hence the proof is complete for \( N = p_{n-1} \).

We now consider the case \( N = mn + n \). In general, we have \( i \leq \pi(p_{\pi(N)}/2) \leq n - 1 \) and

\[p_{\pi(N)-n+i} \geq p_i, \text{ as } N \geq p_{n-1} \].

Observe that \( L_{i-1} \) contains \( |L_1| - m(i-2) = m(n-i+2) \) elements. Suppose for the sake of contradiction that there are at most \( m - 1 \) elements in \( L_{i-1} \) that are coprime to \( p_{\pi(N)-n+i} \); then the remaining \( m(n-i+1)+1 \) elements must all be multiples of \( p_{\pi(N)-n+i} \). We consider the following cases.

- If \( p_i \geq n \), we have
  \[
p_i - \frac{ip_i}{n} + \frac{p_i}{mn} > p_i - \frac{(n-1)p_i}{n} + \frac{p_i}{mn} > p_i - \frac{p_i}{n} + \frac{p_i}{mn} \geq 1 + \frac{1}{m},
  \]

- If \( p_i < n \) and \( i \geq 3 \), we have
  \[
p_i - \frac{ip_i}{n} + \frac{p_i}{mn} > p_i - i + \frac{p_i}{n} + \frac{p_i}{mn} > 2 + \frac{1}{m},
  \]
as \( p_i - i \geq 2 \) for all \( i \geq 3 \).
• If \( p_i < n \) and \( i = 2 \), assuming as before that \( n \geq 4 \), we have
  \[
  p_i - \frac{ip_i}{n} + \frac{p_i}{mn} = 3 - \frac{6}{n} + \frac{2}{n} + \frac{2}{mn} > \frac{3}{2} \geq 1 + \frac{1}{m}.
  \]
  Thus, in each case, the largest integer in \( L_{i-1} \) is at least
  \[
  p_{\pi(N) - n + i}(mn - mi + m + 1) \geq mn \left( p_i - \frac{ip_i}{n} + \frac{p_i}{mn} \right) > mn \left( 1 + \frac{1}{m} \right) = N,
  \]
  which is a contradiction. \( \Box \)

4. Minimum coprime numbers of \( P_m + P_n \)

In this section we prove Theorem 1.3 and discuss the minimum coprime numbers of a number of graphs related to \( P_m + P_n \). We will use the following theorem and lemma.

**Theorem 4.1** (Brun–Titchmarsh theorem). Denote by \( \pi(x; k, a) \) the number of primes at most \( x \) that are equivalent to \( a \) modulo \( k \). Then
\[
\pi(x + y; k, a) - \pi(x; k, a) \leq \frac{2y}{\varphi(k) \log(y/k)}
\]
whenever \( y \geq k \), where \( \varphi \) is the Euler totient function.

**Lemma 4.1.** For all \( x \geq 1 \), there is a prime \( p \in (x, 2x] \) such that \( p \not\equiv 1, 10 \mod 11 \).

**Proof.** By Corollary 3 to Theorem 2 of [27], we have
\[
\pi(2x) - \pi(x) > \frac{3}{5} \frac{x}{\log x}
\]
whenever \( x \geq 20.5 \). Combining this with Theorem 4.1 shows that the total number of primes in \( (x, 2x] \) which are not congruent to 1 or 10 modulo 11 is at least
\[
\frac{3x}{5 \log x} - \frac{2x}{5(\log x - \log 11)}.
\]
When \( x > 1331 \), the above expression is positive, and we have the desired result. For \( x \leq 1331 \), we may manually verify the statement in the lemma. \( \Box \)

**Proof of Theorem 1.3** Set \( L = 2 \lceil \frac{m-1}{2} \rceil + 2n - 1 \). The lower bound \( \text{pr}(P_m + P_n) \geq L \) follows from Proposition 2.2, observing that \( \alpha(P_m + P_n) = \lceil \frac{m}{2} \rceil \). Thus, to show that \( \text{pr}(P_m + P_n) = L \), it suffices to construct a coprime labeling using only labels up to \( L \).

By Theorem 1.2 we know that such a labeling exists for \( n \leq 4 \). Assume that \( n \geq 5 \) and hence \( L \geq R_4 = 29 \). Label the vertices of \( P_n \) with 1 and \( n - 1 \) primes between \( \left[ \frac{L}{2} \right] \) and \( L \), which we denote \( p_1, \ldots, p_{n-1} \) in increasing order. By Lemma 4.1, we may assume that \( p_i \not\equiv 1, 10 \mod 11 \) for some \( 1 \leq i \leq n - 1 \). Moreover, by our assumptions on \( m \) and \( n \), each of these primes is at least 17. As \( 1, p_1, \ldots, p_{n-1} \) are each coprime to all of the other integers up to \( L \), any coprime labeling of \( P_m \) using the remaining integers will yield a minimum coprime labeling for \( P_m + P_n \).

If \( p_1 > m + 1 \) we are done, as we may simply label the vertices of \( P_m \) with \( 2, \ldots, m + 1 \) in order. Otherwise, let \( S_1 \) be the ordered list of integers up to \( L \), excluding \( \{1, p_1, \ldots, p_{n-1}\} \). We will inductively construct sets \( S_i \) so that \( |S_i| = L - n - i + 1 \), every element less than \( p_i \) contained in \( S_i \) is coprime to its neighbors in \( S_i \) that are also less than \( p_i \). It is clear that \( S_1 \) satisfies the conditions above. For \( 1 \leq i \leq n - 2 \), we construct \( S_{i+1} \) from \( S_i \) as follows.
If \( p_{i+1} > p_i + 2 \) and \( p_i > p_{i-1} + 2 \) (if \( i > 1 \)), then \( S_i \) contains the sequence \( p_i - 2, p_i - 1, p_i + 1, p_i + 2 \), where \( p_i - 2 \) and \( p_i + 2 \) are odd and composite. Observe that 3 divides at most one of \( p_i - 2 \) and \( p_i + 2 \). If \( 3 \mid p_i + 2 \), we can set \( S_{i+1} = S_i \setminus \{p_i + 1\} \), as \( \gcd(p_i - 2, p_i - 1) = \gcd(p_i - 1, p_i + 2) = 1 \). Otherwise, \( 3 \nmid p_i - 2 \), so we can set \( S_{i+1} = S_i \setminus \{p_i + 1\} \), as \( \gcd(p_i - 2, p_i + 1) = \gcd(p_i + 1, p_i + 2) = 1 \).

If \( p_{i+1} = p_i + 2 \), then it suffices to set \( S_{i+1} = S_i \setminus \{p_i + 1\} \).

If \( p_{i+1} > p_i + 2 \), \( i > 1 \), and \( p_i = p_{i-1} + 2 \), then \( S_i \) contains the sequence \( p_i - 4, p_i - 3, p_i + 1, p_i + 2 \), where \( p_i - 4 \) and \( p_i + 2 \) are odd and composite. Observe that 5 divides at most one of \( p_i - 4 \) and \( p_i + 2 \). If \( 5 \mid p_i + 2 \), we can set \( S_{i+1} = S_i \setminus \{p_i + 1\} \), as \( \gcd(p_i - 4, p_i - 3) = \gcd(p_i - 3, p_i + 2) = 1 \) necessarily. Otherwise, \( 5 \nmid p_i - 4 \), so we can set \( S_{i+1} = S_i \setminus \{p_i + 1\} \), as \( \gcd(p_i - 4, p_i + 1) = \gcd(p_i + 1, p_i + 2) = 1 \).

We now construct a final ordered list \( S_n \subseteq S_{n-1} \) such that every element in \( S_n \) is coprime to its neighbors. If \( p_{n-1} = L \) we are done, as we may set \( S_n = S_{n-1} \). Otherwise, since \( L \) is always odd, we have \( p_{n-1} + 2 < L \), so \( S_{n-1} \) contains the sequence \( p_{n-1} - k - 1, p_{n-1} - k, p_{n-1} + 1, p_{n-1} + 2 \), where \( k = 1 \) or 3 depending on whether \( p_{n-1} - 2 \) is composite or prime respectively. As in the cases above, it is always possible to obtain \( S_n \) by removing one of \( p_{n-1} - k, p_{n-1} + 1 \).

If \( p_{n-1} = L \), then we have \( |S_n| = |S_{n-1}| = L - 2n + 2 \geq m \), and we obtain a minimum coprime labeling for \( P_m + P_n \) by labeling the vertices of \( P_m \) with the elements of \( L \) in order. Otherwise, we have \( |S_n| = L - 2n + 1 \). If \( m \) is even, \( L - 2n + 1 = m \) and we again obtain a minimum coprime labeling for \( P_m + P_n \). If \( m \) is odd, \( L - 2n + 1 = m - 1 \), and we are short of one label. We resolve this by recalling that there exists some \( p_i \equiv 1, 10 \pmod{11} \). To construct \( S_n \), we have deleted one of \( p_i - 1, p_i + 1 \) for each \( p_i \); hence there is some \( x = p_i \pm 1, 11 \nmid x \) that does not appear in \( S_n \). As \( p_1 > 23 \), we may label the vertices of \( P_m \) with the sequence

\[
x, 11, 12, 5, 4, 3, 8, 7, 6, 13, 10, 9, 14,
\]

followed by all of the elements \( x \in S_n \) such that \( 15 \leq x \leq L \), followed by 2. This yields a minimum coprime labeling for \( P_m + P_n \).

\[\Box\]

The condition \( M \geq R_{N-1} - 2N + 1 \) in Theorem 1.3 is sufficient but certainly not necessary; by Theorem 1.2 it is known, for instance, that if \( N = 3 \) or 4, it suffices to set \( M \geq N^2 \). The following theorem extends this result to \( N = 5 \).

**Theorem 4.2.** For \( m \geq 5 \), the minimum coprime number of \( P_m + P_5 \) is

\[
pr(P_m + P_5) = \begin{cases} 
    m + 8 & \text{if } m \text{ is odd} \\
    m + 9 & \text{if } m \text{ is even}.
\end{cases}
\]

*Proof.* We prove the theorem by casework.

- If \( m \geq 20 \), we are done by Theorem 1.3 as \( R_4 = 29 \).
- If \( m = 5 \), we may label the vertices of the first path with the sequence 1, 3, 5, 9, 13, and the vertices of the second path with 2, 7, 4, 11, 8.
- If \( 6 \leq m \leq 10 \), we may label the vertices of \( P_5 \) with the sequence 3, 5, 9, 1, 15, and the vertices of \( P_m \) with the first \( m \) integers in the sequence

\[
2, 7, 4, 11, 8, 13, 14, 17, 16, 19.
\]

\[\text{We note that while the proof of Theorem 1.2 in [1]}\] uses a labeling that is not coprime in the case where \( m \geq 14 \), \( n = 4 \), it is straightforward to find an alternative labeling that works.
\begin{itemize}
  \item If $11 \leq m \leq 17$, we may label the vertices of $P_5$ with the sequence $1, 11, 13, 17, 19$, and the vertices of $P_m$ with the first $m$ integers in the sequence $2, 3, 4, 5, 6, 7, 8, 9, 14, 15, 16, 21, 10, 23, 12, 25, 24$.
  \item If $m = 18$ or $19$, we may label the vertices of $P_5$ with the sequence $1, 13, 17, 19, 23$, and the vertices of $P_m$ with the first $m$ integers in the sequence $2, 3, 4, 5, 6, 7, 26, 9, 10, 11, 14, 15, 16, 21, 22, 25, 8, 27, 20$.
\end{itemize}

However, we note that for $N \geq 6$, it is in general not sufficient to take $M \geq N$. The following remark implies that Conjecture [14] is false.

**Remark 4.1.** There exist positive integers $m \geq n$ such that

$$\text{pr}(P_m + P_n) > 2 \left\lfloor \frac{m-1}{2} \right\rfloor + 2n - 1.$$ 

We now present a number of examples and counterexamples for Remark [14] that illustrate the complexity of the behavior of $\text{pr}(P_m + P_n)$ for $n \leq m \leq R_{n-1} - 2n$.

**Example 4.1 ($n = 6$).**

- For $6 \leq m \leq 9$,
  \[
  \text{pr}(P_m + P_6) = \begin{cases} 
  m + 10 & \text{if } m \text{ is odd} \\
  m + 11 & \text{if } m \text{ is even} 
  \end{cases} 
  \]

This follows from labeling the vertices of $P_6$ with the sequence $3, 5, 9, 1, 15, 11$ and the vertices of $P_m$ with the first $m$ integers in the sequence $2, 7, 4, 17, 8, 13, 14, 19, 16$.

- For $m = 10$ or $11$, this equality does not hold. We prove this by contradiction: if $\text{pr}(P_{10} + P_6) = \text{pr}(P_{11} + P_6) = 21$, then there exist minimum coprime labelings of $P_{10} + P_6$ and $P_{11} + P_6$ using 5 or 6 even integers respectively and each of the odd integers up to 21. The vertices of $P_6$ must be labeled with the set of integers $S = \{3, 5, 7, 9, 15, 21\}$, as no subset of $S$ is pairwise coprime to its complement in $S$, so all of the integers in $S$ must be used as labels for the same path, there are too many odds in $S$ to be used as labels in $P_{10}$ or $P_{11}$. However, because $S$ contains a total of 6 integers, 4 of which are multiples of 3, it is not possible to arrange the elements of $S$ in a sequence such that adjacent elements are coprime. Therefore $\text{pr}(P_{10} + P_6) > 21$ for $m = 10$ or 11. The same reasoning shows that the minimum coprime numbers of $P_{10} + P_6$ and $P_{11} + P_6$ exceeds 22. In fact, we may construct labelings to show that $\text{pr}(P_{10} + P_6) = \text{pr}(P_{11} + P_6) = 23$.

- For $m \geq 12$, we have $\text{pr}(P_m + P_6) = 2\left\lceil \frac{m-1}{2} \right\rceil + 11$ once more. For $12 \leq m \leq 21$, we may label the vertices of $P_6$ with the sequence $1, 11, 13, 17, 19, 23$ and the vertices of $P_m$ with the first $m$ integers in the sequence $2, 15, 16, 7, 20, 9, 10, 3, 16, 21, 4, 5, 12, 25, 8, 27, 28, 29, 30, 31, 24$.

For $22 \leq m \leq 29$, we may label the vertices of $P_6$ with the sequence $1, 17, 19, 23, 29, 31$ and the vertices of $P_m$ with the first $m$ integers in the sequence $2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 21, 22, 25, 26, 27, 28, 33, 32, 35, 36, 37, 30, 39, 20$,

and when $m \geq 30$, the result follows from the proof of Theorem [14] because $R_5 = 41$.  

Example 4.2 \((n = 7)\). \(\bullet\) For \(m = 7\), we have \(\text{pr}(P_7 + P_7) = 19\). This follows from labeling the vertices of one copy of \(P_7\) with the sequence 3, 5, 9, 1, 15, 11, 13, and the vertices of the second copy with the sequence 2, 7, 4, 17, 8, 19, 16.

\(\bullet\) For \(m = 8\) or 10, we have \(\text{pr}(P_n + P_7) = m + 13\). This follows from labeling the vertices of \(P_7\) with the sequence 3, 5, 9, 7, 15, 1, 21, and the vertices of \(P_m\) with the first \(m\) integers in the sequence
\[
2, 11, 4, 13, 8, 17, 16, 19, 22, 23.
\]

\(\bullet\) For \(m = 9\), the lower bound is not tight. As in Example 4.1 we prove this by contradiction. If \(\text{pr}(P_9 + P_7) = 21\), then there would exist a minimum coprime labeling of \(P_9 + P_7\) using 5 of the even integers and all of the odd integers up to 21. In particular, all the integers in the set \(S = \{3, 5, 7, 9, 15, 21\}\) must be used as labels for the vertices of \(P_7\), by the same reasoning as before. However, there are fewer than 5 even integers up to 21 that are coprime to all of the elements in \(S\). We can construct a coprime labeling to show that \(\text{pr}(P_9 + P_7) = 22\). This case demonstrates that the minimum coprime number of the join of certain paths is constrained by the even integers rather than the odds; it is also interesting that the lower bound may be tight for both \(m - 1\) and \(m + 1\) but fail to be tight for \(m\).

Example 4.3 \((m = n)\). By Proposition 2.4, \(\text{pr}(P_m + P_n) \geq \text{pr}(K_{m,n})\), where \(K_{m,n}\) is the complete bipartite graph. For \(m = n\) up to 59, the values of \(\text{pr}(K_{n,n})\) are published as sequence A213273 in the OEIS. We observe that for \(n \geq 19\), \(\text{pr}(K_{n,n}) \geq 3n\), so in particular \(\text{pr}(P_n + P_n) > 2\left[\frac{n-1}{2}\right] + 2n - 1\).

Our work on the join of two paths also offers insight on the complete bipartite graphs. Fu and Huang [16] proved that, for \(m \leq n\), \(K_{m,n}\) is prime if and only if \(m \leq \pi(m + n) - \pi\left(\frac{m+n}{2}\right) + 1\). Seoud, Diab, and Elsakhawi [30] showed that \(K_{2,n}\) is prime for all \(n\) and that \(K_{3,n}\) is prime unless \(n = 3, 7\). Berliner et al. [5] provide all values of \(n\) for \(m \leq 13\) for which \(K_{m,n}\) is prime and note that \(K_{m,n}\) is prime for all \(n \geq R_{m-1} - m\), as implied by [16]. They also ask about the behavior of \(\text{pr}(K_{m,n})\) when \(n < R_{m-1} - m\). By Lemma 2.1, Theorem 1.3 immediately implies the following bound on \(\text{pr}(K_{m,n})\), which provides a partial answer to this question.

**Corollary 4.1.** If \(n \geq R_{m-1} - 2m + 1\), then
\[
\text{pr}(K_{m,n}) \leq 2\left[\frac{n-1}{2}\right] + 2m - 1 - \frac{1}{2}.
\]

In fact, a slight modification of the first part of the proof of Theorem 1.3 enables us to show that, more generally,

**Theorem 4.3.** If \(m \leq n \leq R_{m-1} - m\), then \(\text{pr}(K_{m,n}) \leq R_{m-1}\).

**Proof.** We label the vertices of \(K_m\) with 1 and the \(m - 1\) largest primes up to \(R_{m-1}\), which are, in particular, at least \(\lceil \frac{R_{m-1}}{2} \rceil\). Hence they are each coprime to all of the other integers up to \(R_{m-1}\). We may use any \(n\) of the remaining integers to label the vertices of \(K_n\). \(\square\)

Our methods in Theorem 1.3 also enable us to prove the following results on the joins of cycles and paths.

**Theorem 4.4.** For any positive integer \(N\), if \(M \geq R_{N-1} - 2N + 1\), then for all \(m \geq M\) and \(n \leq N\), the minimum coprime number of \(C_m + C_n\) is
\[
\text{pr}(C_m + C_n) = \begin{cases} 
m + 2n & \text{if } m \text{ is odd} \\
m + 2n - 1 & \text{if } m \text{ is even.} 
\end{cases}
\]
Proof. Label the vertices of $P_m + P_n$ as in the proof of Theorem 1.3. Then the labels for $P_n$ are all either 1 or prime, so in particular the endpoints of $P_n$ have coprime labels and we may join them to form $C_n$ without violating the coprime labeling condition. If $m$ is even, then one endpoint of $P_m$ is labeled with 2 and the other is labeled with an odd integer, so we may also join the endpoints of $P_m$ to form $C_m$ without violating the coprime labeling condition, obtaining a coprime labeling of $C_m + C_n$. This labeling is a minimum coprime labeling because we have $\text{pr}(C_m + C_n) \geq \text{pr}(P_m + P_n)$ by Proposition 2.1.

If $m$ is odd, $\text{pr}(C_m + C_n) \geq m + 2n$ by Proposition 2.2 as $\alpha(C_m + C_n) = \lfloor \frac{m}{2} \rfloor$. Recall that the ordered list $S_n$ contains either $m$ or $m - 1$ elements. If $|S_n| = m - 1$, the last element of $S_n$ is $L = m + 2n - 2$. Labeling the vertices of $P_m$ with the sequence $S_n$ with $m + 2n$ appended and joining the endpoints of $P_m$ to form $C_m$ therefore yields a minimum coprime labeling for $C_m + C_n$. If $|S_n| = m$, then $P_m - 1 = L = m + 2n - 2$. We may assume that $n \geq 5$ as the other cases are known by [4]; therefore $p_1 > 17$. We consider the following cases.

- If $m + 2n - 4$ is composite, the last two elements of $S_n$ are $m + 2n - 4$ and $m + 2n - 3$. Labeling the vertices of $P_m$ with the sequence $S_n$, replacing $m + 2n - 3$ by $m + 2n$, and joining the endpoints of $P_m$ to form $C_m$ therefore yields a minimum coprime labeling for $C_m + C_n$, as we have $\gcd(m + 2n - 4, m + 2n) = \gcd(m + 2n, 2) = 1$.
- If $m + 2n - 4$ is prime, the last two elements of $S_n$ are $m + 2n - 6$ and $m + 2n - 5$. If $3 \nmid m + 2n$, we may replace $m + 2n - 5$ with $m + 2n$ to obtain a minimum coprime labeling for $C_m + C_n$, as above. If $3 \mid m + 2n$, we join the endpoints of $P_m$ to form $C_m$, replace $m + 2n - 5$ with $m + 2n$, and rearrange the labels surrounding the endpoints to read

\[\ldots, m + 2n - 6, 2, m + 2n, 4, 3, 5, 6, \ldots\]

\[\square\]

The proof of Theorem 4.4 immediately implies the following two corollaries on the join of a cycle and a path.

**Corollary 4.2.** For any positive integer $N$, if $M \geq R_{N-1} - 2N + 1$, then for all $m \geq M$ and $n \leq N$, the minimum coprime number of $C_m + P_n$ is

\[\text{pr}(C_m + P_n) = \begin{cases} m + 2n & \text{if } m \text{ is odd} \\ m + 2n - 1 & \text{if } m \text{ is even.} \end{cases}\]

**Corollary 4.3.** For any positive integer $N$, if $M \geq R_{N-1} - 2N + 1$, then for all $m \geq M$ and $n \leq N$, the minimum coprime number of $P_m + C_n$ is

\[\text{pr}(P_m + C_n) = \begin{cases} m + 2n - 2 & \text{if } m \text{ is odd} \\ m + 2n - 1 & \text{if } m \text{ is even.} \end{cases}\]

5. **Prisms**

In this section we present a number of results on the primality of prisms and webs. Recalling that $Y_n \cong C_n \sqcup P_2$ is comprised of two connected copies of $C_n$, we will denote these copies by $C$ and $C'$ throughout this section. We begin by proving Theorem 1.6 splitting the cases into the following propositions.

**Proposition 5.1.** If $n \geq 8$ and $n + 3$ is a prime number, then $Y_n$ is prime.

Proof. Label the vertices of $C$ with the sequence $1, \ldots, n$ in clockwise order, and label the vertices of $C'$ with the sequence $n + 1, \ldots, 2n$ in clockwise order such that the vertex of $C'$ with label $n + 1$ is adjacent to the vertex of $C$ with label $n - 2$. This labeling is coprime when restricted to each
of \( C \) and \( C' \). Moreover, for \( 1 \leq x \leq n - 3 \), the vertex of \( C \) with label \( x \) is adjacent to the vertex of \( C' \) with label \( x + n + 3 \), and as \( n + 3 \) is prime and \( x < n + 3 \), we have \( \gcd(x, x + n + 3) = 1 \). Therefore we need only check the coprime labeling condition for the vertices of \( C \) with labels \( n - 2 \leq x \leq n \), and the corresponding adjacent vertices of \( C' \).

Since \( n + 3 \) is prime, we have \( \gcd(n, n + 3) = 1 \), and \( n \equiv 1, 2 \pmod{3} \). We modify our original labeling to obtain a coprime labeling of \( Y_n \) as follows.

- If \( n \equiv 1 \pmod{3} \), then the adjacent labels \( n + 2 \) and \( n - 1 \) are not coprime. However, we may resolve this by rearranging the labels in \( C \). If \( n \not\equiv 1 \pmod{5} \), then swapping 1 and \( n - 1 \) yields a coprime labeling as \( \gcd(n - 1, n + 4) = \gcd(n - 1, 2) = 1 \), and all the remaining pairs of adjacent labels are readily seen to be coprime. If \( n \equiv 1 \pmod{5} \), then we swap \( n - 3 \) with \( n - 1 \) and \( n - 4 \) with \( n \). This yields a coprime labeling as \( \gcd(n - 3, n + 2) = \gcd(n, n - 5) = \gcd(n, 2n - 1) = \gcd(n - 4, n + 3) = 1 \), where the last equality follows from the fact that \( 7 \mid n + 3 \), and all the remaining pairs of adjacent labels are readily seen to be coprime.

- If \( n \equiv 2 \pmod{3} \), then the adjacent labels \( n + 1 \) and \( n - 2 \) are not coprime. However, we obtain a coprime labeling by swapping the labels \( n - 2 \) and \( n \). Because \( n + 3 \) is prime, we know that \( 5 \nmid n - 2 \); hence \( \gcd(n - 2, n + 3) = \gcd(n, n - 3) = 1 \), and all the remaining pairs of adjacent labels are readily seen to be coprime.

\[ \square \]

**Proposition 5.2.** If \( n \geq 8 \) and \( n - 3 \) is a prime number, then \( Y_n \) is prime.

**Proof.** Label the vertices of \( C \) with the sequence 1, \ldots, \( n \) in clockwise order, and label the vertices of \( C' \) with the sequence \( n + 1, \ldots, 2n \) in clockwise order such that the vertex of \( C' \) with label \( n + 1 \) is adjacent to the vertex of \( C \) with label 4. This labeling is coprime when restricted to each of \( C \) and \( C' \). Moreover, for \( 4 \leq x \leq n \), the vertex of \( C \) with label \( x \) is adjacent to the vertex of \( C' \) with label \( x + n - 3 \), and as \( x < 2(n - 3) \), we have \( \gcd(x, x + n - 3) = 1 \). It is apparent that \( \gcd(3, 2n) = \gcd(2, 2n - 1) = \gcd(1, 2n - 2) \), so this is a coprime labeling. \[ \square \]

**Proposition 5.3.** If \( n \geq 12 \) and \( n - 5 \) is a prime number, then \( Y_n \) is prime.

**Proof.** Label the vertices of \( C \) with the sequence 1, \ldots, \( n \) in clockwise order, and label the vertices of \( C' \) with the sequence \( n + 1, \ldots, 2n \) in clockwise order such that the vertex of \( C' \) with label \( n + 1 \) is adjacent to the vertex of \( C \) with label 6. This labeling is coprime when restricted to each of \( C \) and \( C' \). Moreover, for \( 6 \leq x \leq n \), the vertex of \( C \) with label \( x \) is adjacent to the vertex of \( C' \) with label \( x + n - 5 \), and as \( x < 2(n - 5) \), we have \( \gcd(x, x + n - 5) = 1 \). It is apparent that \( \gcd(5, 2n) = \gcd(4, 2n - 1) = \gcd(2, 2n - 3) = \gcd(1, 2n - 4) = 1 \). If \( 3 \nmid 2n - 2 \), then this is a coprime labeling; otherwise, swapping the labels 1 and 3 yields a coprime labeling as \( \gcd(n, 3) = \gcd(2n - 4, 3) = 1 \). \[ \square \]

**Proposition 5.4.** If \( n \geq 18 \) and \( n - 7 \) is a prime number, then \( Y_n \) is prime.

**Proof.** Label the vertices of \( C \) with the sequence 1, \ldots, \( n \) in clockwise order, and label the vertices of \( C' \) with the sequence \( n + 1, \ldots, 2n \) in clockwise order such that the vertex of \( C' \) with label \( n + 1 \) is adjacent to the vertex of \( C \) with label 8. This labeling is coprime when restricted to each of \( C \) and \( C' \). Moreover, for \( 8 \leq x \leq n \), the vertex of \( C \) with label \( x \) is adjacent to the vertex of \( C' \) with label \( x + n - 7 \), and as \( x < 2(n - 7) \), we have \( \gcd(x, x + n - 7) = 1 \). It is apparent that \( \gcd(7, 2n) = \gcd(4, 2n - 3) = \gcd(2, 2n - 5) = \gcd(1, 2n - 6) \). Because \( n + 7 \) is prime, we have \( n \not\equiv 2 \pmod{3} \), so \( \gcd(3, 2n - 4) = \gcd(6, 2n - 1) = 1 \). Therefore this labeling fails to be coprime only if \( 5 \mid 2n - 2 \). If this is the case, we may obtain a coprime labeling by swapping the labels 1 and 5. \[ \square \]
Proposition 5.5. If $n \geq 20$ and $n - 9$ is a prime number, then $Y_n$ is prime.

Proof. Label the vertices of $C$ with the sequence $1, \ldots, n$ in clockwise order, and label the vertices of $C'$ with the sequence $n + 1, \ldots, 2n$ in clockwise order such that the vertex of $C'$ with label $n + 1$ is adjacent to the vertex of $C$ with label 10. This labeling is coprime when restricted to each of $C$ and $C'$. Moreover, for $10 \leq x \leq n$, the vertex of $C$ with label $x$ is adjacent to the vertex of $C'$ with label $x + n - 9$, and as $x < 2(n - 9)$, we have $\gcd(x, x + n - 9) = 1$. Because $n + 9$ is prime, we have $3 \nmid n$, so that all the remaining pairs of labels are readily seen to be coprime unless $5 \mid 2n - 4$ or $7 \mid 2n - 2$. If only the first condition holds, we obtain a coprime labeling by swapping the labels 1 and 5. If only the second condition holds, we obtain a coprime labeling by swapping the labels 1 and 7. If both hold, we obtain a coprime labeling by applying the permutation $1 \mapsto 7$, $5 \mapsto 5$, as the primality of $n + 9$ implies that $\gcd(5, 2n - 2) = 1$. \hfill \Box

We observe that Theorem 5.6 combined with Theorems 1.4 and 1.5 immediately yield Corollary 5.1 as $(523, 541)$ is the smallest pair of consecutive primes with difference exceeding 16.

The stacked prism $Y_{m,n}$ is given by the Cartesian product $C_m \square P_n$, and the web graph $W_{n,1}$ is defined to be the stacked prism $Y_{n+1,3}$ with the edges of the outer cycle removed. Our work on prisms generalizes to yield the following result on webs.

Corollary 5.1. If $n \neq 3$ and $n + 1, n + 3, n - 1, n - 3, n - 5, n - 7, n - 9$ is a prime number, then the web graph $W_{n,1}$ is prime.

In fact, we note that Corollary 5.1 follows immediately from a more general result on prisms with more than one pendant edge attached to the vertices of the outer cycle. Call a generalized web $W_{n,k}$ a prism $Y_n$ with $k$ pendant edges attached to each vertex in the outer cycle. Then we have

Corollary 5.2. If $n \neq 3$ and $n + 1, n + 3, n - 1, n - 3, n - 5, n - 7, n - 9$ is a prime number, then the generalized web $W_{n,k}$ is prime for all $k$.

The proof follows directly from Newman’s coprime mapping conjecture, which we state below. A bijection $f: A \to B$ on two sets of integers $A$ and $B$ is called a coprime mapping if $\gcd(a, f(a)) = 1$ for each $a \in A$.

Theorem 5.1 ([24], Theorem 1). If $N$ is a natural number and $I$ is an interval of $N$ consecutive integers, then there is a coprime mapping $f: \{1, \ldots, N\} \to I$.

Proof of Corollary 5.2. As noted in the proof of Theorem 5.1, there exists a prime labeling of the prism $Y_n$ such that the outer cycle $C$ is labeled with integers from the set $\{1, \ldots, n\}$. Denote the vertices of $C$ by $v_1, \ldots, v_n$, where $v_i$ has label $i$, and denote the pendant vertices of each vertex $v_i$ in $C$ by $u_{i,1}, \ldots, u_{i,k}$. Denote $S_\ell = \{\ell n + 1, \ldots, (\ell + 2)n\}$ for each $1 \leq \ell \leq k$, and let $f_\ell: \{1, \ldots, n\} \to S_\ell$ be a coprime mapping. Labeling the pendant vertex $v_{i,\ell}$ with $f_\ell(i)$ for each $1 \leq i \leq n, 1 \leq \ell \leq k$ yields a prime labeling of $W_{n,k}$. \hfill \Box

6. Miscellaneous graphs

In this section, we note the primality of two classes of graphs that have not previously appeared in the literature. We also briefly discuss a conjecture of Seoud and Youssef on unicyclic graphs. The gear graph $G_n$ is formed by inserting a vertex between each pair of adjacent vertices in the outer cycle of the wheel graph $W_n$.

Theorem 6.1. The graph $G_n$ is prime for all $n$. 
Theorem 6.2. The graph $DW_n = 2C_n + K_1$ is the graph formed by attaching each vertex of two disjoint cycles $C_n$ to a central vertex $K_1$.

Proof. The graph $DW_n$ has $2n + 1$ vertices, and $\alpha(DW_n) = 2\lfloor \frac{n}{2} \rfloor$. Thus if $n$ is odd, then $DW_n$ is not prime by Corollary 2.2. If $n$ is even, we obtain a prime labeling by labeling the central vertex with 1 and the two cycles with the sequences $2, \ldots, n + 1$ and $n + 2, \ldots, 2n + 1$ respectively, as $\gcd(2, n + 1) = \gcd(n + 2, 2n + 1) = 1$ and all the remaining pairs of adjacent vertices in the cycles differ by 1.

We conclude by discussing a certain class of unicyclic graphs. A longstanding open problem in coprime labeling is due to Seoud and Youssef [32], who conjectured in 1999 that all unicyclic graphs are prime. We note that this conjecture, if true, would imply the Entringer–Tout conjecture by Proposition 2.1, as any tree with at least three vertices has a unicyclic spanning supergraph obtained by adding one edge. Various classes of unicyclic graphs are known to be prime, including cycles and crowns, as discussed in Section 1. Seoud and Youssef [32] showed that cycles with identical complete binary trees attached to each vertex are prime, and the authors of [8-9] discuss the primality of several related classes of unicyclic graphs. One such class is the cycle pendant star, denoted $C_n \ast P_2 \ast S_m$, the graph that results from attaching the path $P_2$ to each vertex of $C_n$ followed by attaching the star $S_m$ at its center to each pendant vertex. It was shown in [8] that for $m \leq 8$, all $C_n \ast P_2 \ast S_m$ are prime; we generalize this result below to all $m$ by applying Theorem 5.1.

Theorem 6.3. For all $n, m$, $C_n \ast P_2 \ast S_m$ is prime.

Proof. Denote the vertices of the cycle by $v_1, \ldots, v_n$ and the vertices of the star connected to $v_i$ by $u_{i,0}, \ldots, u_{i,m}$, such that $u_{i,0}$ is the center of the star and $u_{i,1}, \ldots, u_{i,m}$ are the pendant vertices. Denote $S_\ell = \{\ell n + 1, \ldots, (\ell + 1)n\}$ for $1 \leq \ell \leq m + 1$, and let $f_\ell : \{1, \ldots, n\} \rightarrow S_\ell$ be a coprime mapping. We define $g : V(C_n \ast P_2 \ast S_m) \rightarrow \{1, \ldots, (m + 2)n\}$ as follows: set $g(v_i) = n + i$, $g(u_{i,0}) = f_1^{-1}(g(v_i))$, and $g(u_{i,j}) = f_{j+1}(g(u_{i,0}))$ for $1 \leq j \leq m$. Then $g$ is a prime labeling of $C_n \ast P_2 \ast S_m$.

7. Minimum coprime number of a random subgraph

In this section, we prove Theorem 1.7.

Proof of Theorem 1.7. By a celebrated result of Bollobás and Erdős [6], we have $\omega(G(n, p)) \sim 2\log_2 n$ almost surely, where $\omega(G)$ denotes the clique number of $G$, i.e., the size of the largest complete subgraph of $G$. We note that $\alpha(G(n, p)) = \omega(G(n, 1 - p))$, and that $p_{n-(2+o(1))\log d n} = (1 + o(1))n \log d n \gg 4\log d n$ by the prime number theorem, where we set $d = \frac{1}{1-p}$. Combining Propositions 2.2 and 2.3 yields

$$\frac{2 + o(1)}{n} \leq \Pr(G(n, p)) \leq (1 + o(1))n \log d n$$

almost surely as $n \rightarrow \infty$. □
8. Further directions

Here we pose a number of open questions in coprime graph labeling.

**Question 8.1.** We observed in Examples 4.1 and 4.2 that \( \text{pr}(P_m + P_n) \) exhibits interesting behavior when \( n \leq m \leq R_n - 2n \). Is there a nice characterization of \( \text{pr}(P_m + P_n) \) in these cases, and in particular, is it possible to predict when \( \text{pr}(P_m + P_n) = 2\left\lceil \frac{m-1}{2} \right\rceil + 2n - 1 \)?

**Question 8.2.** Can we improve on the bounds for \( \text{pr}(K_{m,n}) \) in Theorem 4.3? In particular, can we obtain sharper bounds on \( \text{pr}(K_{n,n}) \)?

**Question 8.3.** Is the prism graph \( Y_n \) prime for all even \( n \)?

**Question 8.4.** We showed that certain classes of even prisms and webs are prime in Section 5. While Proposition 2.2 implies that odd prisms and odd stacked prisms are not prime, we anticipate that it would be reasonably straightforward to obtain some bounds on the minimum coprime numbers of prisms, stacked prisms, and webs in general.

**Question 8.5.** The Cartesian product \( P_m \Box P_n \) is called a grid graph. In particular, if \( m = 2 \), the graph \( P_2 \Box P_n \) is called a ladder. Dean [7] and Ghorbani and Kamali [18] showed independently that all ladders are prime, resolving a conjecture of Varkey which was previously worked on in [38, 36]. Other grid graphs have been shown to be prime, including \( P_m \Box P_n \) if \( m \leq n \) and \( n \) is prime [35], and a few other cases in [20]. It it true that \( P_m \Box P_n \) is prime for all \( m \) and \( n \)? This would settle a conjecture in [35].

**Question 8.6.** Is it possible to obtain a sharper upper bound in Theorem 1.7?

**Question 8.7.** There has been a substantial amount of research conducted on the clique number and independence number of a random subgraph. Would any of these results enable us to obtain lower bounds on \( \text{pr}(G_p) \) for arbitrary \( G \)?

**Question 8.8.** For arbitrary \( G \), the trivial upper bound \( \text{pr}(G_p) \leq \text{pr}(G) \) is asymptotically tight, as we may observe in the case where \( G \) is prime. Is it possible to obtain a better upper bound on \( \text{pr}(G_p) \) for specific classes of \( G \)?

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References

[1] R. Ahlswede and L. H. Khachatrian. On extremal sets without coprimes. Acta Arith., 66:89–99, 1994.
[2] R. Ahlswede and L. H. Khachatrian. Maximal sets of numbers not containing \( k + 1 \) pairwise coprime integers. Acta Arith., 72:77–100, 1995.
[3] R. Ahlswede and L. H. Khachatrian. Sets of integers and quasi-integers with pairwise common divisor. Acta Arith., 74:141–153, 1996.
[4] J. Asplund and N. Bradley Fox. Minimum coprime labelings for operations on graphs. Integers, 19, 2019.
[5] A. Berliner, N. Dean, J. Hook, A. Mbirika, and C. D. McBee. Coprime and prime labelings of graphs. J. Integer Seq., 19, 2016.
[6] B. Bollobás and P. Erdős. Cliques in random graphs. Math. Proc. Cambridge Phil. Soc., 80:419–427, 1976.
[7] N. Dean. Proof of the prime ladder conjecture. Integers, 17, 2017.
[8] N. Dieifenderfer, D. C. Ernst, M. Hastings, L. N. Heath, H. Prawzinsky, B. Preston, J. Rushall, E. White, and A. Whittemore. Prime vertex labelings of several families of graphs. Involve, 9, 2016.
[9] N. Diefenderfer, M. Hastings, L. N. Heath, H. Prawzinsky, B. Preston, E. White, and A. Whittemore. Prime vertex labelings of families of unicyclic graphs. *Rose-Hulman Undergrad. Math. J.*, 16, 2015.

[10] P. Erdős. Remarks in number theory IV. *Mat. Lapok*, 13:228–255, 1962.

[11] P. Erdős. A survey of problems in combinatorial number theory. *Ann. Discrete Math.*, 6:89–115, 1980.

[12] P. Erdős and A. Sárközy. On sets of coprime integers in intervals. *Hardy-Ramanujan J.*, 16:1–20, 1993.

[13] P. Erdős, A. Sárközy, and E. Szemerédi. On some extremal properties of sequences of integers. *Ann. Univ. Sci. Budapest Eötvös Sect. Math.*, 12:131–135, 1969.

[14] P. Erdős, A. Sárközy, and E. Szemerédi. On some extremal properties of sequences of integers II. *Publ. Math. Debrecen*, 27:1170125, 1980.

[15] P. Erdős and G. N. Sárközy. On cycles in the coprime graph of integers. *Electron. J. Combin.*, 4, 1997.

[16] H. L. Fu and K. C. Huang. On prime labelling. *Discrete Math.*, 127:181–186, 1994.

[17] J. A. Gallian. A dynamic survey of graph labeling. *Electron. J. Combin.*, DS6, 2018.

[18] E. Ghorbani and S. Kamali. Prime labeling of ladders. *arXiv:1610.08849*.

[19] P. Haxell, O. Pikhurko, and A. Taraz. Primality of trees. *J. Comb.*, 2:481–500, 2011.

[20] O. Pikhurko. Trees are almost prime. *Discrete Math.*, 307:1455–1462, 2007.

[21] C. Pomerance and A. Sárközy. Combinatorial number theory. In R. Graham, M. Grötschel, and L. Lovász, editors, *Handbook of Combinatorics*. MIT Press, 1995.

[22] C. Pomerance and J. L. Selfridge. Proof of D. J. Newman’s coprime mapping conjecture. *Mathematika*, 27:69–83, 1980.

[23] U. M. Prajapati and S. J. Gajjar. Some results on prime labeling. *Open J. Discrete Math.*, 4:60–66, 2014.

[24] S. Ramanujan. A proof of Bertrand’s postulate. *J. Indian Math. Soc.*, 11:181–182, 1919.

[25] J. B. Rosser and L. Schoenfeld. Approximate formulas for some functions of prime numbers. *Illinois J. Math.*, 6:64–94, 1962.

[26] H. Salmasian. A result on prime labelings of trees. *Bull. Inst. Combin. Appl.*, 28:36–38, 2000.

[27] G. N. Sárközy. Complete tripartite subgraphs in the coprime graph of integers. *Discrete Math.*, 202:227–238, 1999.

[28] M. A. Seoud, A. T. Diab, and E. A. Elsakhawi. On strongly-c harmonious, relatively prime, odd graceful and cordial graphs. *Proc. Math. Phys. Soc. Egypt*, 73:33–55, 1998.

[29] M. A. Seoud and M. Z. Youssef. On prime labeling of graphs. *Congr. Numer.*, 141:203–215, 1999.

[30] N. J. A. Sloane. Sequence A104272. *The On-Line Encyclopedia of Integer Sequences*, 2019. [https://oeis.org/A104272](https://oeis.org/A104272)

[31] N. J. A. Sloane. Sequence A213273. *The On-Line Encyclopedia of Integer Sequences*, 2019. [https://oeis.org/A213273](https://oeis.org/A213273)

[32] M. Sundaram, R. Porraj, and S. Somasundaram. On a prime labeling conjecture. *Ars Combin.*, 80:205–209, 2006.

[33] M. Sundaram, R. Porraj, and S. Somasundaram. A note on prime labelings of ladders. *Acta Ciencia Indica.*, 33:471–477, 2007.

[34] A. Tout, A. N. Dabboucy, and K. Howalla. Prime labeling of graphs. *Nat. Acad. Sci. Letters*, 11:365–368, 1982.

[35] V. Vilfred, S. Somasundaram, and T. Nicholas. Classes of prime labelled graphs. *Int. J. of Management and Systems*, 18, 2002.

[36] M. Z. Youssef and E. A. Elsakhawi. Some properties of prime graphs. *Ars Combin.*, 84:129–140, 2007.