IRREGULAR FINITE ORDER SOLUTIONS OF COMPLEX LDE’S IN UNIT DISC

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Abstract. It is shown that the order and the lower order of growth are equal for all non-trivial solutions of $f^{(k)} + Af = 0$ if and only if the coefficient $A$ is analytic in the unit disc and $\log^+ M(r, A)/\log(1 - r)$ tends to a finite limit as $r \to 1^-$. A family of concrete examples is constructed, where the order of solutions remain the same while the lower order may vary on a certain interval depending on the irregular growth of the coefficient. These coefficients emerge as the logarithm of their modulus approximates smooth radial subharmonic functions of prescribed irregular growth on a sufficiently large subset of the unit disc. A result describing the phenomenon behind these highly non-trivial examples is also established. En route to results of general nature, a new sharp logarithmic derivative estimate involving the lower order of growth is discovered. In addition to these estimates, arguments used are based, in particular, on the Wiman-Valiron theory adapted for the lower order, and on a good understanding of the right-derivative of the logarithm of the maximum modulus.

1. Introduction and main results

The balance between the growth of coefficients and the growth and oscillation of solutions has been a central theme of research concerning linear differential equations in a complex domain for over a half of century. In the case of the complex plane $\mathbb{C}$, the classical result of Wittich [23, Satz 1] states that the analytic coefficients $A_0, \ldots, A_{k-1}$ are polynomials if and only if all solutions of

$$f^{(k)} + A_{k-1}f^{(k-1)} + \cdots + A_1f' + A_0f = 0$$

are entire functions of finite order of growth. Further, each order belongs to a finite set of rational numbers induced by the degrees of the polynomial coefficients [10]. In the particular case when the coefficient $A \neq 0$ of

$$f^{(k)} + Af = 0$$

is a polynomial, all non-trivial solutions $f$ of (1.2) have regular growth in the sense that the lower order and the order of $f$ are both equal to $\deg(A)/k + 1$, i.e.,

$$\liminf_{r \to \infty} \frac{\log \log M(r, f)}{\log r} = \limsup_{r \to \infty} \frac{\log \log M(r, f)}{\log r} = \frac{\deg(A)}{k} + 1;$$

see [16, p. 74] and [22, pp. 106–108]. Here $M(r, f) = \max_{|z|=r} |f(z)|$ is the maximum modulus of $f$ on the circle $|z| = r$. Note that, as a polynomial, the coefficient

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A has regular growth as well in the sense that
\[
\liminf_{r \to \infty} \frac{\log M(r, A)}{\log r} = \limsup_{r \to \infty} \frac{\log M(r, A)}{\log r} = \deg(A) < \infty. \tag{1.3}
\]
The oscillation of solutions is equally intimately connected to the growth of coefficients, see [1, 7].

The connection between the growth of coefficients and the growth of solutions is also well understood in the case of the unit disc \(D = \{ z \in \mathbb{C} : |z| < 1 \} \). In this setting, the analogue of polynomial coefficients in \(A \) are coefficients belonging to the Korenblum space \(A^{-\infty} \). Indeed, the analytic coefficients belong to \(A^{-\infty} \) if and only if all solutions of (1.1) are of finite order of growth. In the case of (1.2), all non-trivial solutions are of maximal growth and have order of growth uniquely determined by the coefficient \(A \in A^{-\infty} \). However, almost nothing is known about the lower order of growth of solutions. Our principal objective is to show that the degree and the lower degree of growth of the coefficient \(A \in A^{-\infty} \) control the lower order of growth of non-trivial solutions. In particular, we demonstrate that solutions having different order and lower order are possible in the case of the unit disc, contrasting sharply with the analogous situation in the complex plane.

To reach concrete statements, some notation is needed. For a function \(f \) analytic in \(D \), the order and the lower order of growth (with respect to the maximum modulus) are defined by
\[
\sigma_M(f) = \limsup_{r \to 1^-} \frac{\log^+ \log^+ M(r, f)}{\log^+ \frac{1}{1-r}}, \quad \lambda_M(f) = \liminf_{r \to 1^-} \frac{\log^+ \log^+ M(r, f)}{\log^+ \frac{1}{1-r}},
\]
respectively. Here the plus sign refers to the non-negative part. Clearly, \(0 \leq \lambda_M(f) \leq \sigma_M(f) \leq \infty \) for any analytic \(f \) in \(D \). For a function \(A \) analytic in \(D \), the degree and the lower degree of growth are defined by
\[
\sigma_{M, \deg}(A) = \limsup_{r \to 1^-} \frac{\log^+ M(r, A)}{\log^+ \frac{1}{1-r}}, \quad \lambda_{M, \deg}(A) = \liminf_{r \to 1^-} \frac{\log^+ M(r, A)}{\log^+ \frac{1}{1-r}},
\]
respectively. This terminology reflects the polynomial growth in (1.3). Differing from the situation in (1.3), we will construct analytic functions \(A \) in \(D \) for which \(\lambda_{M, \deg}(A) < \sigma_{M, \deg}(A)\), and consider the growth of solutions of (1.2).

The Korenblum space \(A^{-\infty} \) consists of functions of finite degree, and this implies the following statement: All solutions of (1.1) are of finite order of growth if and only if the coefficients are of finite degree. Regarding the non-trivial solutions \(f \) of (1.2), [5, Theorem 1.4] implies
\[
\sigma_M(f) \leq \max \{0, \sigma_{M, \deg}(A)/k - 1\}
\]
and, in particular,
\[
\sigma_M(f) = \sigma_{M, \deg}(A)/k - 1, \quad \sigma_{M, \deg}(A) \geq 2k. \tag{1.4}
\]

Our first result relates the irregular growth of the coefficient to the irregular growth of non-trivial solutions of (1.2). This shows that the unit disc case is, in a certain sense, similar to the plane case.

**Theorem 1.** Let \(k \in \mathbb{N} \) and let \(A \) be an analytic function in \(D \). Then, \(\sigma_{M, \deg}(A) = \lambda_{M, \deg}(A) = p > 2k \) if and only if \(\sigma_M(f) = \lambda_M(f) = p/k - 1 > 1 \) for some (equivalently for all) non-trivial solution(s) \(f \) of (1.2).

The discussion on the principal ingredients of the proof of Theorem 1 is postponed to the end of the present section.
Theorem 3. below unfolds a family of examples, and shows that the difference between \( \sigma_{M, \text{deg}}(A) \) and \( \lambda_{M, \text{deg}}(A) \) does not uniquely determine the lower order of solutions of \((1.2)\). The reason why this result differs radically from the corresponding situation in the plane is that the Korenblum space is a much richer family of functions in \( \mathbb{D} \) than the set of polynomials is in \( \mathbb{C} \).

**Theorem 2.** Let \( k \in \mathbb{N}, k \leq p_1 < p_2 < \infty, 2k < p_2 \) and \( \alpha \in [p_1/p_2, 1] \). Then there exists an analytic function \( A = A(\alpha) \) in \( \mathbb{D} \) such that \( \sigma_{M, \text{deg}}(A) = p_2 \), \( \lambda_{M, \text{deg}}(A) = p_1 \) and any non-trivial solution \( f \) of \((1.2)\) satisfies \( \sigma_M(f) = p_2/k - 1 \) and \( \lambda_M(f) = p_1/k - \alpha \).

The proof of Theorem 2 is rather involved and constitutes the bulk of the paper. The first part of the proof is a laborious construction of a smooth radial subharmonic function \( \varphi \) of irregular growth. Then, we show that there exists an analytic function \( A \) such that \( \log|A| \) approximates \( \varphi \) with sufficient precision in a large subset of \( \mathbb{D} \). The upper bound for the lower order of growth of any solution of \((1.2)\) follows from the irregular growth of the coefficient by a growth estimate. Meanwhile, the lower bound for the lower order is established by using a recent integrated logarithmic derivative estimate. Theorem 2 is proved in Section 2.

In Theorem 2 the order of non-trivial solutions is always \( p_2/k - 1 \), while the lower order can be any pre-given number on the interval \([p_1/k - 1, p_1/k - p_1/p_2] \). In this case, the lower order of growth is strictly smaller than the order of growth. The following theorem reveals the general phenomenon induced by the irregular growth of the coefficient.

**Theorem 3.** Let \( k \in \mathbb{N} \) and let \( A \) be an analytic function in \( \mathbb{D} \) such that \( \sigma_{M, \text{deg}}(A) = p_2 \in (0, \infty) \) and \( \lambda_{M, \text{deg}}(A) = p_1 \). Then

(a) all solutions \( f \) of \((1.2)\) with \( \sigma_M(f) > 0 \) satisfy

\[
\frac{p_1}{k} - 1 \leq 1 + \left( \lambda_M(f) - \frac{\lambda_M(f)}{\sigma_M(f)} \right)^+; \tag{1.5}
\]

(b) if \( 2k < k(2 + \frac{p_2 - 2k}{p_2}) < p_1 \leq p_2 \), then all non-trivial solutions \( f \) of \((1.2)\) satisfy

\[
\lambda_M(f) - \left( 1 - \frac{\lambda_M(f)}{\sigma_M(f)} \right) \leq \frac{p_1}{k} - 1 \leq \frac{p_2}{k} - 1 = \sigma_M(f). \tag{1.6}
\]

The following result shows, analogously to the plane situation in \((1.3)\), that if \( A \) has regular growth in \( \mathbb{D} \), then the solutions of \((1.2)\) have regular growth as well.

**Corollary 4.** Let \( k \in \mathbb{N} \) and let \( A \) be an analytic function in \( \mathbb{D} \) such that \( \sigma_{M, \text{deg}}(A) = p_2 \) and \( \lambda_{M, \text{deg}}(A) = p_1 \), where \( 2k < k(2 + \frac{p_2 - 2k}{p_2}) < p_1 \leq p_2 \). Then all non-trivial solutions \( f \) of \((1.2)\) satisfy

\[
\left| \lambda_M(f) - \frac{p_1}{k} + 1 \right| \leq 1 - \frac{\lambda_M(f)}{\sigma_M(f)}; \tag{1.6}
\]

We may re-write \((1.6)\) in the form

\[
\frac{p_1 - 2k}{p_2 - 2k} \left( \frac{p_2}{k} - 1 \right) \leq \lambda_M(f) \leq \frac{p_1}{p_2} \left( \frac{p_2}{k} - 1 \right). \tag{1.7}
\]

To see that the upper bound in \((1.7)\) is sharp, choose \( \alpha = p_1/p_2 \) in Theorem 2. It would be desirable to show that the lower bound in \((1.7)\) can be replaced by the value \( p_1/k - 1 \), which corresponds to \( \alpha = 1 \) in Theorem 2. However, it is
not known whether this is true, unless \( p_2 = p_1 \). In this case, (1.7) reduces to the equality \( \sigma_M(f) = \lambda_M(f) \) by (1.4). In fact, we already know this, even under weaker hypothesis, by Theorem 1.

The proof of Theorem 3 is given in Section 5, and it depends on the Wiman-Valiron theory adapted for the lower order of growth. Therefore, we need a good understanding of the quantities

\[
\lambda_*(f) = \liminf_{r \to 1^-} \frac{\log^+ K(r, f)}{\log \frac{1}{1-r}}, \quad \sigma_*(f) = \limsup_{r \to 1^-} \frac{\log^+ K(r, f)}{\log \frac{1}{1-r}},
\]

where \( K(r, f) = r \left( \log M(r, f) \right)' \) for \( 0 \leq r < 1 \). Here the plus sign refers to the right derivative. In Section 3, we prove that each function \( f \) analytic in \( D \) satisfies \( \lambda_*(f) \leq \lambda_M(f) + 1 \) if \( f \) is unbounded, and \( \lambda_M(f) + \frac{\lambda_M(f)}{\sigma_M(f)} \leq \lambda_*(f) \) if \( \sigma_M(f) > 0 \). Both estimates are shown to be sharp.

In addition to the Wiman-Valiron theory, the proof of Theorem 3 strongly relies on a new logarithmic derivative estimate involving the lower order of growth, which is stated as Theorem 5 below. This result complements a known logarithmic derivative estimate for functions of finite maximum modulus order given in terms of a proximate order [5]. To the best of our knowledge, logarithmic derivative estimates involving the lower order of growth do not appear in the existing literature.

The upper density of a measurable set \( E \subset [0,1) \) is defined as

\[
\mathcal{D}(E) = \limsup_{r \to 1^-} \frac{m_1(E \cap [r,1))}{1-r},
\]

where \( m_1(F) \) denotes the one-dimensional Lebesgue measure of the set \( F \).

**Theorem 5.** Let \( f \) be an analytic function in \( \mathbb{D} \) such that \( 0 \leq \lambda_M(f) \leq \sigma_M(f) < \infty \). Let \( k \) and \( j \) be integers satisfying \( k > j \geq 0 \), and let \( \varepsilon \in (0,1) \). Then there exist a set \( E = E(\varepsilon, f, k, j) \subset [0,1) \) satisfying \( \mathcal{D}(E) = 1 \) and a constant \( C = C(\varepsilon, f, k, j) > 0 \) such that

\[
\left| \frac{f^{(k)}(z)}{f^{(j)}(z)} \right| \leq C \left( \frac{1}{(1-|z|)^{2k} + (\lambda_M(f) - \frac{\lambda_M(f)}{\sigma_M(f)})^+ + \varepsilon} \right)^{k-j} \tag{1.9}
\]

for all \( z \in \mathbb{D} \) for which \( |z| \in E \).

The proof of Theorem 5 is given in Section 4, and it is based on an approach similar to that in [5, Theorem 1.2]. In the present paper, we take advantage of the full strength of Linden’s result, which is stated as Theorem D below. This result provides a local estimate for the behavior of an analytic function in terms of a quantity depending on the maximum modulus, and it turns out that this quantity can be further estimated in terms of the lower order of growth.

The proof of Theorem 1 depends on Theorem 3 and on an auxiliary result on the lower order of solutions of (1.2). The proof and the auxiliary result are presented in Section 3. In the final Section 7 we discuss an example, which addresses the case when the condition \( \lambda_{M,\deg(A)} > 2k \) in Theorem 3(b) is not satisfied. Then the correlations between the growth indicators of the coefficient and of solutions become even more complicated.

## 2. Proof of Theorem 5

The laborious proof is divided into three parts.
2.1. Subharmonic functions of irregular growth. In Lemma 6 below, we construct sequences \( \{r_n\}, \{r'_n\}, \{r''_n\}, \{\hat{r}_n\} \) each of which satisfy the property 
\( (1 - \rho_{n+1})/(1 - \rho_n) \to 0 \) as \( n \to \infty \). Therefore \( 1 - \rho_n \) decreases to zero faster than any geometric progression, inheriting many properties of such progressions. 

**Lemma 6.** Let \( 0 < p_1 < p_2 \leq p < \infty \) be constants, and let \( \{\eta_n\} \) be an increasing unbounded sequence such that \( \eta_n > 1 \) for all \( n \).

(a) There exist a positive constant \( C \) and sequences \( \{r_n\}, \{r'_n\}, \{r''_n\}, \{\hat{r}_n\}, \{M_n\}, \{R_n\}, \{\varepsilon_n\} \) such that \( r_n < r'_n \leq \hat{r}_n < r''_n < r_{n+1}, |\varepsilon_n| < (p_2 - p_1)/2, \) and

\[
\begin{align*}
(i) & \quad \left( \frac{C}{1 - r'_n} \right)^{p_1} = \left( \frac{C}{1 - r_n} \right)^{p_2 + \varepsilon_n}; \\
(ii) & \quad r'_n = \frac{1 - \hat{r}_n}{\log \frac{C}{1 - r_n}}; \\
(iii) & \quad R_n = \frac{p_2 + \varepsilon_n}{1 - r_n} r_n; \\
(iv) & \quad \frac{1}{(1 - \hat{r}_n)^{p_2}} = \frac{1}{(1 - r'_n)^{p}}; \quad p = p_2 \Rightarrow \hat{r}_n = r'_n; \\
(v) & \quad M_n = \frac{p_2 - p_1}{(1 - \hat{r}_n)^2} \left( \log \frac{C}{1 - r_n} \right)^2; \\
(vi) & \quad R_n \log \frac{r'_n}{r_n} + M_n \int_{r_n}^{r'_n} \log \frac{r''}{r} \, dt - p_1 \frac{r''_n - r'_n}{1 - r''_n} = p_2 - p_1 + \varepsilon_{n+1}; \\
(vii) & \quad \left( R_n + M_n (r'_n - \hat{r}_n) - p_1 \frac{r''_n}{1 - r''_n} \right) \frac{1 - r''_n}{r''_n} = p_2 - p_1 + \varepsilon_{n+1}; \\
(viii) & \quad r_{n+1} = 1 - \frac{1 - r''_n}{\eta_n}.
\end{align*}
\]

Moreover, \( r_n \to 1^- \) and \( \varepsilon_n \to 0 \) as \( n \to \infty \), and further,

\[1 - r''_n \sim \frac{1 - \hat{r}_n}{\log \frac{1}{1 - r_n}}, \quad n \to \infty.\]

(b) If \( \varphi : [0, 1) \to (0, \infty) \) is given by

\[
\varphi(r) = \begin{cases} 
(p_2 + \varepsilon_n) \log \frac{C}{1 - r} & r''_n \leq r < r_n, \\
(p_2 + \varepsilon_n) \log \frac{C}{1 - r} + R_n \log \frac{r}{r_n} & r_n \leq r < r'_n, \\
p_1 \log \frac{C}{1 - r} + R_n \log \frac{r}{r_n} + p_1 \left( \log \frac{1 - r''_n}{1 - r_n'} - \frac{r''_n}{1 - r''_n} \right) & r'_n \leq r < \hat{r}_n, \\
p_1 \log \frac{C}{1 - r} + R_n \log \frac{r}{r_n} - p_1 \frac{r''_n}{1 - r''_n} + M_n \int_{r_n}^{r_n} \log \frac{r}{r} \, dt & \hat{r}_n \leq r < r^*_n, \\
p_1 \log \frac{C}{1 - r} + R_n \log \frac{r}{r_n} - p_1 \frac{r''_n}{1 - r''_n} + M_n \int_{r_n}^{r_n} \log \frac{r}{r} \, dt & r^*_n \leq r < r''_n,
\end{cases}
\]

where \( n \in \mathbb{N} \), then \( \varphi \) is continuously differentiable and

\[
\frac{1}{r} (r \varphi'(r))' = \begin{cases} 
p_2 \frac{\varphi'(r)}{(1 - r)^2} & r''_n \leq r < r_n, \\
0 & r_n \leq r < r'_n, \\
p_1 \frac{\varphi'(r)}{(1 - r)^2} - \frac{1}{1 - r_n} & r'_n \leq r < \hat{r}_n, \\
p_1 \frac{\varphi'(r)}{(1 - r)^2} - \frac{1}{1 - r_n} + M_n \frac{\varphi'(r)}{r} & \hat{r}_n \leq r < r^*_n, \\
p_2 \frac{\varphi'(r)}{(1 - r)^2} - \frac{1}{1 - r_n} & r^*_n \leq r < r''_n.
\end{cases}
\]

Moreover, 

\[
\limsup_{r \to 1^-} \frac{\varphi(r)}{\log \frac{1}{1 - r}} = p_2, \quad \liminf_{r \to 1^-} \frac{\varphi(r)}{\log \frac{1}{1 - r}} = p_1.
\]
The radial extension \( \varphi : \mathbb{D} \rightarrow (0, \infty) \), given by \( \varphi(z) = \varphi(|z|) \) for all \( z \in \mathbb{D} \), is subharmonic and satisfies \( \Delta \varphi(re^{i\theta}) = \frac{1}{r}(r \varphi'(r))' \) for all \( \theta \in \mathbb{R} \).

**Proof.** (a) The precise value for the (large) positive constant \( C \) depends on many factors, and will be dealt with later. At this point, it suffices to assume \( C \geq e \).

The sequences in the assertion are defined iteratively. Define \( r''_n = 0, r_1 \in (0, 1) \) (the precise value will be fixed later) and \( \varepsilon_1 = 0 \) for the first iteration. The general case goes as follows. If the seed \((r_n, \varepsilon_n)\) is given, where \( |\varepsilon_n| < (p_2 - p_1)/2 \), then \( r_n, \rho, \lambda, R_n, M_n \) are defined by the conditions (i)-(v), which guarantee \( r_n < r''_n < \rho_n < r''_n < 1 \). We proceed to show that the system of equations

\[
\begin{cases}
R_n \log \frac{r}{r_n} + M_n \int_{r_n}^{r} \log \frac{t}{\rho} dt - p_1 \frac{r - r''_n}{1 - r''_n} = p_2 - p_1 + \varepsilon, \\
R_n + M_n(r''_n - \rho_n) - p_1 \frac{r - r''_n}{1 - r''_n} \frac{1 - r}{r} = p_2 - p_1 + \varepsilon,
\end{cases}
\]

admits a unique solution \((r, \varepsilon)\) for \( r''_n < r < 1 \). Assuming for a moment that such solution exists, then choose \( r_n'' = r \) and \( \varepsilon_{n+1} = \varepsilon \). By defining \( r_{n+1} \) via (viii), we have obtained the seed \((r_{n+1}, \varepsilon_{n+1})\) for the next iteration.

We still need to prove that (2.3) admits a unique solution \((r, \varepsilon)\) where \( r \in (r''_n, 1) \). By combining the equations, we eliminate \( \varepsilon \) and obtain

\[
\left( R_n + M_n(r''_n - \rho_n) - p_1 \frac{r - r''_n}{1 - r''_n} \right) \frac{1 - r}{r} \log \frac{C}{1 - r} = R_n \log \frac{r}{r_n} + M_n \int_{r_n}^{r} \log \frac{t}{\rho} dt - p_1 \frac{r - r''_n}{1 - r''_n},
\]

(2.4)

Consider the interval \([r''_n, 1]\). Let \( g_L = g_L(r) \) be the function in (2.4) and let \( g_R = g_R(r) \) be the function in (2.3). By a straight-forward differentiation,

\[
g_R'(r) = \frac{R_n}{r} + \frac{M_n(r''_n - \rho_n)}{r} - \frac{p_1}{1 - r''_n} \geq \frac{1}{r(1 - r''_n)} ((p_2 - p_1) \log C - p_1) > 0,
\]

provided \( C > e^{p_2 - p_1} \), and hence the function \( g_R \) is strictly increasing. Another differentiation gives

\[
g_L'(r) = \frac{1}{r(1 - r''_n)} \left( \frac{1 - r}{r} \log \frac{C}{1 - r} \right) - \left( \frac{R_n}{r} + M_n(r''_n - \rho_n) - \frac{p_1}{1 - r''_n} \right) \left( \log \frac{C}{1 - r} + \frac{1}{r} \right) < 0
\]

whenever \( C > e^{p_2 - p_1} \), and hence the function \( g_L \) is strictly decreasing. Therefore there exists at most one point \( r \in [r''_n, 1] \) for which \( g_L(r) = g_R(r) \).

Let \( a, b \) be positive constants such that \((\log C)^{1/2} > a > b \) (further restrictions apply later), and define \{\( \alpha_n \)\} and \{\( \beta_n \)\} by

\[
1 - \alpha_n = a \frac{1 - \rho_n}{(\log \frac{C}{1 - r_n})^{1/2}}, \quad 1 - \beta_n = b \frac{1 - \rho_n}{(\log \frac{C}{1 - r_n})^{1/2}}.
\]
Then \( r_n^* < \alpha_n < \beta_n < 1 \). Recall the inequalities \( 1 - x < \log(1/x) < (1/x)(1 - x) \), valid for all \( x \in (0, 1) \). On one hand, we estimate
\[
g_L(\alpha_n) > \left( M_n(r_n^* - \hat{r}_n) - \frac{p_1}{1 - r_n^*} \right) \frac{1 - \alpha_n}{\alpha_n} \log \frac{C}{1 - \alpha_n}
\]
\[
> a \left( p_2 - p_1 - \frac{1 - \hat{r}_n}{1 - r_n^*} \log \frac{C}{1 - \hat{r}_n} \right) \left( \log \frac{C}{1 - \hat{r}_n} \right)^{\frac{1}{2}} \log \frac{C}{a(1 - \hat{r}_n)}
\]
\[
\geq a \left( p_2 - p_1 - \frac{p_1}{\log C} \right) \left( \log \frac{C}{1 - \hat{r}_n} \right)^{\frac{1}{2}} \log \frac{C}{a(1 - \hat{r}_n)}
\]
\[
> a \left( p_2 - p_1 - \frac{p_1}{\log C} \right) \left( \log \frac{C}{1 - \hat{r}_n} \right)^{\frac{1}{2}},
\]
and
\[
g_R(\alpha_n) \leq g_R(1) = R_n \log \frac{1}{r_n} + M_n \int_{r_n}^{r_n^*} \log \frac{1}{t} \, dt - p_1
\]
\[
\leq p_2 - p_1 + \varepsilon_n + \frac{p_2 - p_1}{r_1} \log \frac{C}{1 - \hat{r}_n}.
\]

Now that we have the factor \( \left( \log \frac{C}{1 - \hat{r}_n} \right)^{\frac{1}{2}} \) in the lower estimate for \( g_L(\alpha_n) \), and the factor \( \log \frac{C}{1 - \hat{r}_n} \) in the upper estimate for \( g_L(\alpha_n) \), it is easy to see that \( g_L(\alpha_n) > g_R(\alpha_n) \) if \( a \) (and hence \( C \) also) is sufficiently large. On the other hand, we estimate
\[
g_L(\beta_n) \leq \left( p_2 + \varepsilon_n + (p_2 - p_1) \log \frac{C}{1 - \hat{r}_n} \right) \frac{b \log \frac{C}{b(1 - \hat{r}_n)}^2}{r_1 \left( \log \frac{C}{1 - \hat{r}_n} \right)^2},
\]
and
\[
g_R(\beta_n) > \frac{p_2 - p_1}{1 - \hat{r}_n} \log \frac{C}{1 - \hat{r}_n} (\beta_n - r_n^*) - p_1
\]
\[
= (p_2 - p_1) \log \frac{C}{1 - \hat{r}_n} \left( 1 - \frac{1}{\log \frac{C}{1 - \hat{r}_n}} - \frac{b}{\left( \log \frac{C}{1 - \hat{r}_n} \right)^2} \right) - p_1,
\]
and therefore \( g_L(\beta_n) < g_R(\beta_n) \) assuming that \( b > 0 \) is sufficiently small (This reasoning can be easily modified such that in the definitions of \( \alpha_n \) and \( \beta_n \), the powers \( \frac{1}{2} \) and 2 are replaced by 1). As both functions \( g_L \) and \( g_R \) are continuous, this ensures the existence of a unique value \( r \in (\alpha_n, \beta_n) \) for which \( g_L(r) = g_R(r) \). This value is denoted by \( r_n'' \). Define \( r_{n+1} \) by (viii) which guarantees \( r_n \to 1^- \), as \( n \to \infty \), since \( \{\eta_n\} \) is unbounded. Finally, define \( \varepsilon_{n+1} \) by (vi). The reasoning above then proves (vii).

We compute
\[
1 \geq \frac{\log \frac{C}{1 - \hat{r}_n}}{\log \frac{C}{1 - \hat{r}_n}} \geq \left( 1 + \frac{\left( \frac{1}{2} \log \frac{C}{1 - \hat{r}_n} \right)^2}{\log \frac{C}{1 - \hat{r}_n}} \right)^{-1}, \quad n \in \mathbb{N},
\]
which implies that: (I) for every $\delta > 0$ there exists a constant $C > 0$ such that
\[ 1 \geq \frac{\log \frac{C}{1-r_n}}{\log \frac{C}{1-r''_n}} \geq \frac{1}{1 + \delta}, \]
independently of $n$; (II) we also have
\[ \lim_{n \to \infty} \frac{\log \frac{C}{1-r_n}}{\log \frac{C}{1-r''_n}} = 1 \]
for any fixed $C$. By (iii),(v) and (vi),
\[ \varepsilon_{n+1} = p_1 - p_2 + \frac{p_2 + \varepsilon_n}{1 - r_n} \log \frac{r''}{r_n} \]
\[ + \frac{p_2 - p_1}{(1 - \hat{r}_n)^2} \log \frac{C}{1 - \hat{r}_n} \left( \log \frac{C}{1 - \hat{r}_n} \right)^2 \int_{\hat{r}_n}^{r''_n} \log \frac{r''}{t} \, dt - \frac{p_1(1 - \hat{r}_n - 1)}{1 - \hat{r}_n} \]
(2.6)
The right-hand side of (2.6) contains four terms. The first term is the constant $p_1 - p_2$. Since $|\varepsilon_n| \leq (p_2 - p_1)/2$ by the assumption, the second term can be made arbitrarily close to zero by choosing sufficiently large $C$. By choosing $C$ sufficiently large, the fourth term can be made arbitrarily close to zero as well. To estimate the third term, we compute
\[ \frac{1}{\hat{r}_n} \geq \frac{1}{(1 - \hat{r}_n)^2} \left( \log \frac{C}{1 - \hat{r}_n} \right)^2 \int_{\hat{r}_n}^{r''_n} \log \frac{r''}{t} \, dt \geq \frac{r''_n - r_n^* \log \frac{C}{1 - \hat{r}_n}}{1 - \hat{r}_n} \]
where
\[ 1 \geq \frac{r''_n - r_n^*}{1 - \hat{r}_n} = 1 - \frac{1 - r_n^*}{1 - \hat{r}_n} \geq \frac{(1 - \hat{r}_n) \left( 1 - \left( \log \frac{C}{1 - \hat{r}_n} \right)^{-1} \right)}{1 - \hat{r}_n} = 1 - \frac{1}{\log \frac{C}{1 - \hat{r}_n}} - \left( \log \frac{C}{1 - \hat{r}_n} \right)^{1/2}. \]
We conclude that the third term in the right-hand side of (2.6) can be made arbitrarily close to the value $p_2 - p_1$ by choosing $r_1 \in (0, 1)$ sufficiently close to 1 and $C$ sufficiently large. This means that, for any $\delta > 0$, we obtain $|\varepsilon_{n+1}| < \delta$ by choosing $r_1 \in (0, 1)$ sufficiently close to 1 and $C$ sufficiently large. That said, we may assume $|\varepsilon_{n+1}| < (p_2 - p_1)/2$. The same computation also shows that $\varepsilon_n \to 0$ as $n \to \infty$. Therefore (vi) ensures the asymptotic equality
\[ 1 - r''_n \sim \frac{1 - \hat{r}_n}{\log \frac{C}{1 - \hat{r}_n}}, \quad n \to \infty, \]
which gives a relatively precise location of $r''_n$ with respect to $\hat{r}_n$.
(b) Let $\varphi$ be the piecewise-defined function in the assertion. It is immediate that $\varphi$ is continuous in each subinterval. At endpoints of subintervals, we conclude that continuity at $r_n$ is clear; continuity at $r_n'$ follows from (i); continuity at $\hat{r}_n$ is clear and the same happens with $r_n^*$; and continuity at $r''_n$ follows from (vi). By
straight-forward differentiation,

\[
\varphi'(r) = \begin{cases} 
\frac{p_2 + \varepsilon_n}{1 - r_n}, & r''_{n-1} \leq r < r_n, \\
\frac{r_n - r'}{1 - r_n}, & r_n \leq r < r', \\
\frac{r_n}{1 - r'} + p_1 \left(\frac{1}{1 - r} - \frac{1}{1 - r_n}\right), & r' \leq r < \hat{r}_n, \\
\frac{\hat{r}_n - r_n}{1 - r_n} + M_n \frac{r - r_n}{r}, & \hat{r}_n \leq r < r^*_n, \\
\frac{r^*_n - r_n}{1 - r_n} + M_n \frac{r - r_n}{r}, & r_n \leq r < r''_{n-1}.
\end{cases}
\]

Also the derivative \(\varphi'\) is continuous in each subinterval. At endpoints of subintervals, we conclude that continuity at \(r_n\) follows by (iii); continuity at \(r'_{n-1}, \hat{r}_n\) and \(r^*_n\) is clear; and continuity at \(r''_{n-1}\) follows from (vii). Another straight-forward differentiation gives (2.1).

We need to prove the properties in (2.2). Note that

\[
\lim_{n \to \infty} \log \frac{p_2 + \varepsilon_n}{1 - r_n} = p_2, \quad \lim_{n \to \infty} \frac{\varphi(r_n)}{\log \frac{1}{1 - r_n}} = \lim_{n \to \infty} \frac{p_1 \log \frac{C}{1 - r_n} + r_n \log \frac{r'}{r}}{\log \frac{1}{1 - r_n}} = p_1.
\]

We only give details on cases where the upper and lower estimates are different. The remaining computations are similar and hence omitted. For \(r''_{n-1} \leq r < r_n\), we deduce the upper estimate

\[
\frac{\varphi(r)}{\log \frac{1}{1 - r}} \leq (p_2 + \varepsilon_n) \left(1 + \frac{\log C}{\log \frac{1}{1 - r_{n-1}}}\right) \to p_2, \quad n \to \infty,
\]

and the lower estimate

\[
\frac{\varphi(r)}{\log \frac{1}{1 - r}} \geq (p_2 + \varepsilon_n) \left(1 + \frac{\log C}{\log \frac{1}{1 - r_n}}\right) \to p_2, \quad n \to \infty.
\]

For \(r_n \leq r < r'_{n-1}\), we have the upper estimate

\[
\frac{\varphi(r)}{\log \frac{1}{1 - r}} = (p_2 + \varepsilon_n) \log \frac{C}{1 - r_n} + p_2 + \varepsilon_n r_n \log \frac{r}{r_n} \\
\leq (p_2 + \varepsilon_n) \left(1 + \log \frac{C}{1 - r_n}\right) + \frac{p_2 + \varepsilon_n}{\log \frac{1}{1 - r_n}} \to p_2, \quad n \to \infty,
\]

and, by (i), the lower estimate

\[
\frac{\varphi(r)}{\log \frac{1}{1 - r}} \geq (p_2 + \varepsilon_n) \log \frac{C}{1 - r_n} - \frac{p_1 (p_2 + \varepsilon_n) \log C}{(p_2 + \varepsilon_n) \log \frac{1}{1 - r_n} - p_1 \log C} \to p_1, \quad n \to \infty.
\]

For \(r'_{n-1} \leq r < \hat{r}_n\), we have the upper estimate

\[
\frac{\varphi(r)}{\log \frac{1}{1 - r}} = p_1 \log \frac{C}{1 - r_n} + \frac{p_2 + \varepsilon_n}{1 - r_n} \log \frac{r}{r_n} + p_1 \frac{\log \frac{1}{1 - r} - \log \frac{1}{1 - r_n} + r'_{n-1}}{\log \frac{1}{1 - r} - \frac{r'_{n-1}}{1 - r_n}} \\
= p_1 + \frac{p_2 + \varepsilon_n}{1 - r_n} \log \frac{r}{r_n} - p_1 \frac{r'_{n-1}}{1 - r_n} + p_1 \log C \\
\leq p_1 + \frac{p_2 + \varepsilon_n}{1 - r_n} + \frac{p_1 \log C}{\log \frac{1}{1 - r_n}} \to p_1, \quad n \to \infty.
\]
The following lower estimate is immediate
\[ \frac{\varphi(r)}{\log \frac{1}{1-r}} \geq p_1 \left( 1 + \frac{\log C}{\log \frac{1}{1-r_n}} \right) - \frac{p_1 \hat{r}_n - r_n^\prime}{\log \frac{1}{1-r_n}} \rightarrow p_1, \quad n \to \infty. \]

For \( \hat{r}_n \leq r < r_n^\ast \), we have the upper estimate
\[
\frac{\varphi(r)}{\log \frac{1}{1-r}} = p_1 \frac{\log C}{\log \frac{1}{1-r_n}} + \frac{p_2 + \varepsilon_n}{1-r_n} \log \frac{r}{\hat{r}_n} - \frac{p_1 \hat{r}_n - r_n^\prime}{\log \frac{1}{1-r_n}}
+ \frac{p_2 - p_1}{(1-r_n^2)^2} \left( \log \frac{C}{1-r_n} \right)^2 \int_{r_n}^{\hat{r}_n} \log \frac{r}{t} \, dt \log \frac{1}{1-r}
\leq p_1 \left( 1 + \frac{\log C}{\log \frac{1}{1-r_n}} \right) + \frac{p_2 + \varepsilon_n}{\log \frac{1}{1-r_n}} + \frac{p_2 - p_1}{\hat{r}_n} \log \frac{1}{1-r_n} \rightarrow p_1, \quad n \to \infty,
\]
and the lower estimate
\[ \frac{\varphi(r)}{\log \frac{1}{1-r}} \geq p_1 \left( 1 + \frac{\log C}{\log \frac{1}{1-r_n}} \right) - \frac{p_1}{\log \frac{1}{1-r_n}} \rightarrow p_1, \quad n \to \infty. \]

For \( r_n^\ast \leq r < r_n^\prime \), we have, by (ii), the upper estimate
\[
\frac{\varphi(r)}{\log \frac{1}{1-r}} = p_1 \frac{\log C}{\log \frac{1}{1-r_n}} + \frac{p_2 + \varepsilon_n}{1-r_n} \log \frac{r}{\hat{r}_n} - \frac{p_1 \hat{r}_n - r_n^\prime}{\log \frac{1}{1-r_n}}
+ \frac{p_2 - p_1}{(1-r_n^2)^2} \left( \log \frac{C}{1-r_n} \right)^2 \int_{r_n}^{\hat{r}_n} \log \frac{r}{t} \, dt \log \frac{1}{1-r}
\leq p_1 \left( 1 + \frac{\log C}{\log \frac{1}{1-r_n}} \right) + \frac{p_2 + \varepsilon_n}{\log \frac{1}{1-r_n}} + \frac{p_2 - p_1}{\hat{r}_n} \log \frac{1}{1-r_n} \rightarrow p_2, \quad n \to \infty,
\]
and the lower estimate
\[ \frac{\varphi(r)}{\log \frac{1}{1-r}} \geq p_1 - \frac{p_1}{\log \frac{1}{1-r_n}} \rightarrow p_1, \quad n \to \infty. \]

This completes the proof of (b). \( \square \)

2.2. Approximation of subharmonic functions. We will find an analytic function \( A \) in \( \mathbb{D} \) such that \( \log |A| \) approximates the radial function \( \varphi \), constructed in Lemma 6 with sufficient precision (logarithmic growth is approximated at a log log accuracy). In order to do this we use [6, Theorem 1], stated as Theorem A below. Some more notation is needed. We say that a differentiable function \( b : [0, 1) \to (0, 1) \) is of regular variation if \( r + b(r) < 1 \) for all \( 0 \leq r < 1 \),
\[ \sup_{0 < c < 1} \int_{cb(r)}^{b(1-r)} \sup_{\tau \in [0, 1]} |\varphi(b(r))| \, dr < \infty \]
for each \( 0 < c < 3/4 \). Since \( b(r) < 1 - r \) by the definition, \( b(r) \to 0^+ \) as \( r \to 1^- \) and \( cb(r) < 3(1-r)/4 \) for all \( 0 < c < 3/4 \). Observe that \( b \) is not required to be monotonic. Standard computations show that the decreasing functions \( b_1(r) = (1-r)/C \) and \( b_2(r) = (1-r)/(\log(C/(1-r))) \) for \( C > 1 \) are of regular variation. These functions play a role in our application.
A polar rectangle is a set of the form $Q = \{ z = re^{i\theta} : r_1 < r < r_2, \theta_1 < \theta < \theta_2 \} \subset \mathbb{D}$. We write $\ell_\theta(Q) = r_2 - r_1$ and $\ell_\theta(Q) = \theta_2 - \theta_1$. We say that a positive Borel measure $\mu$ in $\mathbb{D}$ admits a regular partition, with respect to a function $b$ of regular variation, if there exist a sequence $\{Q^{(l)}\}$ of polar rectangles in $\mathbb{D}$ and a decomposition $\mu = \sum \mu^{(l)}$ such that

(i) $\text{supp} \mu^{(l)} \subset Q^{(l)}$ and $\mu^{(l)}(Q^{(l)}) = 2$ for all $l$;
(ii) there exists $N \in \mathbb{N}$ such that each $z \in \mathbb{D}$ belongs to at most $N$ different rectangles $Q^{(l)}$;
(iii) there exists a constant $c > 0$ such that $c^{-1} < \ell_\theta(Q^{(l)})/\ell_\theta(Q^{(l)}) < c$ for all $l$;
(iv) $\text{diam} Q^{(l)} \asymp b(\text{dist}(Q^{(l)}), 0)$ for all $l$.

Further, $\mu$ is locally regular with respect to a function $b$ of regular variation if there exists $r_0 \in (0, 1)$ such that

$$\sup_{r_0 < |z| < 1} \int_0^{b(|z|)} \frac{\mu(D(z,t))}{t} \, dt < \infty.$$ (2.7)

Note that (i) the behavior of $\mu$ in compact subsets of $\mathbb{D}$ is not relevant, since $b(r) \to 0^+$ as $r \to 1$; (ii) $\mu$ is not necessarily finite, as the hyperbolic measure $d\mu(z) = dm_2(z)/(1 - |z|^2)^2$ is locally regular with respect to $b_1$ (here and later $m_2$ is the two-dimensional Lebesgue measure); (iii) by Fubini’s theorem, the condition (2.7) is equivalent to

$$\sup_{r_0 < |z| < 1} \int_{D(z,b(|z|))} \frac{b(|z|)}{|\zeta - z|} \, d\mu(\zeta) < \infty.$$

Finally, let $Z_f$ stand for the zero set of an analytic function $f$.

Recall that each subharmonic function $u : \mathbb{D} \to \mathbb{R}$ induces a unique positive measure, namely the Riesz measure $\mu = \mu_u$, such that $du = \frac{1}{2\pi} \Delta u \, dm_2$.

**Theorem A (\cite{A} Theorem 1).** Let $u : \mathbb{D} \to \mathbb{R}$ be a subharmonic function such that $\mu_u$ admits a regular partition and $\mu_u$ is locally regular, with respect to some $b : (0, 1) \to (0, 1)$ of regular variation. Then there exists an analytic function $f$ in $\mathbb{D}$ such that

$$\sup_{z \in \mathbb{D}} \left( \log |f(z)| - u(z) \right) < \infty,$$

and for each $\varepsilon > 0$ there exist $r_1 = r_1(\varepsilon) \in (0, 1)$ and $C = C(\varepsilon) > 0$ such that

$$|\log |f(z)| - u(z)| \leq C, \quad r_1 < |z| < 1, \quad z \notin E_\varepsilon,$$

where $E_\varepsilon = \{ z \in \mathbb{D} : \text{dist}(z, Z_f) \leq \varepsilon b(|z|) \}$. Moreover, the zero-set of $f$ satisfies $Z_f \subset \bigcup_{\zeta \in \supp \mu_u} D(\zeta, K b(\zeta))$ for some $K > 0$.

**Lemma 7.** Let $\varphi$ be as in Lemma 6. Then there exists an analytic function $A$ in $\mathbb{D}$ with the following properties:

(a) for each $\varepsilon \in (0, 1/2)$ there exist $\rho_1 = \rho_1(\varepsilon) \in (0, 1)$ and a constant $C_2 = C_2(\varepsilon) > 0$ such that

$$|\log |A(z)| - \varphi(z)| < C_2 + C_2 \log \log \frac{1}{1 - |z|}, \quad z \notin E_\varepsilon, \quad \rho_1 < |z| < 1,$$

where $E_\varepsilon = \{ z \in \mathbb{D} : \text{dist}(z, Z_A) \leq \varepsilon (1 - |z|) \}$;

(b) there exists a constant $C_3 > 0$ such that

$$|\log M(r, A) - \varphi(r)| \leq C_3 \log \log \frac{1}{1 - r}, \quad \rho_1 < r < 1.$$
Moreover, $Z_A \subset \bigcup_{\zeta \in \text{supp } \mu} D(\zeta, (1 - |\zeta|)/2)$ and there exists a constant $C_4 > 0$ such that

$$m_1(E_\varepsilon \cap \partial D(0, r)) \leq C_4 \varepsilon, \quad r \in (1/2, 1).$$

**Proof.** Recall that $r_0'' = 0$ and define $r_{n,0} = r_{n-1}''$ for all $n \in \mathbb{N}$. Define the annuli as in (2.1):

$$A_n = \{ z \in \mathbb{D} : r_{n-1} < |z| < r_n \},$$

$$A'_n = \{ z \in \mathbb{D} : r_n < |z| < r'_n \},$$

$$\hat{A}_n = \{ z \in \mathbb{D} : r'_n < |z| < \hat{r}_n \},$$

$$A^*_n = \{ z \in \mathbb{D} : \hat{r}_n < |z| < r^*_n \},$$

$$A''_n = \{ z \in \mathbb{D} : r^*_n < |z| < r''_n \},$$

and note that the restriction of the Riesz measure $\frac{1}{2\pi} \Delta \varphi$ of the function $\varphi$ on the boundary of any annulus is the zero measure. Define inductively $\{ n,m \}$ brackets denote the integer part of a real number. In order to divide the annulus $A_n$, we choose the numbers $0 = \psi_{n,k}^0 < \cdots < \psi_{n,k}^{|\frac{1}{2} - \frac{1}{2}n|} = 2\pi$ such that $\psi_{n,k}^j - \psi_{n,k}^{j-1} = \frac{2\pi j}{\psi_{n,k}^{|\frac{1}{2} - \frac{1}{2}n|}}$, where the square brackets denote the integer part of a real number. In order to divide the annulus $\{ \zeta : r_{n,k} < |\zeta| < 1 \}$ into the polar rectangles, define

$$Q_{n,k}^j = \{ \zeta : r_{n,k} < |\zeta| < r_{n,k+1}, \quad \psi_{n,k}^{j-1} \leq \arg \zeta < \psi_{n,k}^j \}, \quad j = 1, \ldots, \left[ \frac{1}{1 - r_{n,k}} \right],$$

where $r_{n,k+1}$ is chosen such that $\mu_{\varphi}(Q_{n,k}^j) = 2$ for all $j$, that is,

$$2 = \frac{1}{2\pi} \int_{\psi_{n,k}^{j-1}}^{\psi_{n,k}^j} \int_{r_{n,k}}^{r_{n,k+1}} \frac{p_2 + \varepsilon_{n-1}}{r(1-r)^2} r dr d\theta = \frac{p_2 + \varepsilon_{n-1}}{\left[ 1 - r_{n,k} \right]^2} \int_{r_{n,k}}^{r_{n,k+1}} \frac{dr}{(1-r)^2}, \quad j = 1, \ldots, \left[ \frac{1}{1 - r_{n,k}} \right].$$

Such $r_{n,k+1}$ exists and is unique, because $(x - r_{n,k})/(1-x)$ increases to infinity as $x \to 1^{-}$. In particular,

$$r_{n,k+1} - r_{n,k} \sim \frac{2}{p_2} (1 - r_{n,k+1}), \quad n \to \infty.$$

We continue the process until the interval $(r_{n,m}, r_{n,m+1}]$ contains $r_n$, that is, when either $r_{n,m+1} = r_n$ or $r_{n,m+1} > r_n$, because the Laplacian of $\varphi$ vanishes on $A'_n$. Set $m_n = m$. We redefine $Q_{n,m_n-1}^j$ as

$$Q_{n,m_n-1}^j = \{ \zeta : r_{n,m_n-1} < |\zeta| < r_{n}, \quad \widetilde{\psi}_{n,m_n-1}^{j-1} \leq \arg \zeta < \widetilde{\psi}_{n,m_n-1}^j \},$$

where $\widetilde{\psi}_{n,m_n-1}$ is defined in the following way. We set $\widetilde{\psi}_{n,m_n-1}^0 = 0$ and define $\widetilde{\psi}_{n,m_n-1}^j$ for $1 \leq j \leq s_n - 1$ by the equation $\mu_{\varphi}(Q_{n,m_n-1}^j) = 2$, where the value $s_n$ is uniquely defined by the condition $s_n = \min\{j : \widetilde{\psi}_{n,m_n-1}^j > 2\pi\}$. Redefine $Q_{n,m_n-1}^{s_n-1}$ so that the collection

$$\left( \bigcup_{k=0}^{m_n-2} \left( \bigcup_{j=1}^{s_n} Q_{n,k}^j \right) \right) \cup \bigcup_{j=1}^{s_n-1} Q_{n,m_n-1}^j$$

covers the whole annulus $A_n$. Let $Q_{n1}$ denote this modified rectangle $Q_{n,m_n-1}^{s_n-1}$ and note that its side lengths are comparable to $1 - r_n$ while $2 \leq \mu_{\varphi}(Q_{n1}) < 4$. 


Since on $\hat{A}_n$ and $A''_n$ we have
\[
\Delta \varphi(re^{i\theta}) = \frac{p_1}{r} \left( \frac{1}{(1-r)^2} - \frac{1}{1-r''} \right) \sim \frac{p_1}{(1-r)^2}, \quad n \to \infty,
\]
we can similarly define partitions
\[
\{ \hat{Q}^j_{n,k} \}, \quad 0 \leq k \leq \hat{m}_n - 1, \quad 1 \leq j \leq \hat{s}_n - 1, \\
\{ Q''^j_{n,k} \}, \quad 0 \leq k \leq m''_n - 1, \quad 1 \leq j \leq s''_n - 1,
\]
of the Riesz measures of the restrictions $\varphi|_{\hat{A}_n}$ and $\varphi|_{A''_n}$, respectively. Then
(i) $\mu_\varphi(\hat{Q}^j_{n,k}) = 2$ for all $0 \leq k \leq \hat{m}_n - 2$ and $1 \leq j \leq [1/(1-\hat{r}_{n,k})]$, and
\[
\mu_\varphi(\hat{Q}^j_{n,\hat{m}_n-1}) = 2 \quad \text{for all} \quad 1 \leq j \leq \hat{s}_n - 2;
\]
(ii) $\hat{Q}^j_{n,\hat{m}_n-1}$ is modified, and denoted by $Q_{n2}$, such that
\[
\left( \bigcup_{k=0}^{\hat{m}_n - 2} [1/(1-\hat{r}_{n,k})] \bigcup \hat{Q}^j_{n,k} \right) \cup \bigcup_{j=1}^{\hat{s}_n-1} \hat{Q}^j_{n,\hat{m}_n-1}
\]
covers the whole annulus $\hat{A}_n$, which implies $2 \leq \mu_\varphi(Q_{n2}) < 4$;
(iii) $\mu_\varphi(Q''^j_{n,k}) = 2$ for all $0 \leq k \leq m''_n - 2$ and $1 \leq j \leq [1/(1-r''_{n,k})]$, and
\[
\mu_\varphi(Q''^j_{n,m''_n-1}) = 2 \quad \text{for all} \quad 1 \leq j \leq s''_n - 2;
\]
(iv) $Q''^j_{n,m''_n-1}$ is modified, and denoted by $Q_{n3}$, such that
\[
\left( \bigcup_{k=0}^{m''_n - 2} [1/(1-r''_{n,k})] \bigcup Q''^j_{n,k} \right) \cup \bigcup_{j=1}^{s''_n-1} Q''^j_{n,m''_n-1}
\]
covers the whole annulus $A''_n$, which implies $2 \leq \mu_\varphi(Q_{n3}) < 4$.

By construction, $Q_{n2}$ and $Q_{n3}$ are polar rectangles with side lengths comparable to $1 - \hat{r}_n$ and $1 - r''_n$, respectively.

Then consider the annulus $A''_n$. We choose $\psi_n^0 = 0$, and $\psi_n^j$ such that for $Q''_n = \{ \zeta : \hat{r}_n < |\zeta| < r''_n, \psi_n^{j-1} \leq \arg \zeta < \psi_n^j \}$, we have
\[
2 = \frac{1}{2\pi} \int_{Q''_n} \Delta \varphi(\zeta) dm_2(\zeta) = \frac{\psi_n^j - \psi_n^{j-1}}{2\pi} \int_{\hat{r}_n}^{r''_n} \Delta \varphi(r)r dr
\]
\[
= (1 + o(1)) \frac{\psi_n^j - \psi_n^{j-1}}{2\pi} \int_{\hat{r}_n}^{r''_n} \frac{p_2 - p_1}{(1-\hat{r}_n)^2} \log \frac{C}{1-\hat{r}_n} dr
\]
\[
= (1 + o(1)) \frac{\psi_n^j - \psi_n^{j-1}}{2\pi} \frac{(p_2 - p_1) \log \frac{C}{1-\hat{r}_n}}{1-\hat{r}_n}, \quad n \to \infty.
\]
The value $s_n^*$ is uniquely defined by the condition $s_n^* = \min\{ j : \psi_n^{j-1} > 2\pi \}$.

Then we obtain a partition $\{Q''_n\}$, $1 \leq j \leq s_n^* - 1$, of the Riesz measure of $\varphi|_{A_n}$ such that
(i) $\mu_\varphi(Q''_n) = 2$ for all $1 \leq j \leq s_n^* - 2$;
(ii) $Q''(s_n^*-1)$ is modified, and denoted by $Q_{n4}$, such that $\bigcup_{j=0}^{s_n^*-1} Q''_n$ covers the whole annulus $A_n^*$, which implies $2 \leq \mu_\varphi(Q_{n4}) < 4$.

By construction, $Q_{n4}$ is a polar rectangle with side lengths comparable to $\frac{1-\hat{r}_n}{\ln \frac{r''_n}{1-\hat{r}_n}}$. 
Let \( dv = \Delta \varphi \, dm_2 \mid_{\bigcup_{n=1}^{\infty} (A_n^* \setminus Q_n)} \). We define

\[
    u_2(z) = \int_D \left( \log \left| \frac{\zeta - z}{1 - \zeta z} \right| + \text{Re} \left( \frac{1 - |\zeta|^2}{1 - \zeta z} \right) \right) \, d\nu(\zeta),
\]

\[
    u_1(z) = \varphi(z) - u_2(z) - \sum_{j=1}^{4} u_j^*(z),
\]

where

\[
    u_j^*(z) = \sum_{n=1}^{\infty} \int_{Q_{nj}} \log \left| \frac{z - \zeta}{1 - \zeta \bar{z}} \right| \Delta \varphi(\zeta) \, dm_2(\zeta).
\]

The fact that these integrals converge depends on \( \{\eta_n\} \) in Lemma \[\text{[3]}\]. Careful inspection shows that the condition \( \eta_n \to \infty \), as \( n \to \infty \), is sufficient for our purposes.

Although the subharmonicity of \( u_2 \) follows by standard arguments \[\text{[8, 11]}\], we discuss it in detail for the convenience of the reader. More precisely, we show that \( u_2 \) is well-defined, it is subharmonic in \( \mathbb{D} \) and its Riesz measure coincides with \( \nu/(2\pi) \).

First observe that the integrand in (2.8), as the logarithm of modulus of the primary factor of genus 1, admits the estimate

\[
    \left| \log \left| \frac{\zeta - z}{1 - \zeta z} \right| + \text{Re} \left( \frac{1 - |\zeta|^2}{1 - \zeta z} \right) \right| \leq 2 \frac{(1 - |\zeta|^2)^2}{|1 - \zeta z|^2}, \quad 1 - |\zeta|^2 < \frac{1}{2},
\]

by [21] Chap.V.10, p.223]. We have

\[
    \nu(A_n^*) \geq \int_{\tilde{r}_n}^{r_n^*} \frac{1}{(1 - \tilde{r}_n)^2} \log^2 \frac{C}{1 - \tilde{r}_n} \, dv \geq \frac{1}{1 - \tilde{r}_n} \log \frac{C}{1 - \tilde{r}_n}, \quad n \to \infty.
\]

By Lemma [3](viii) we get \( 1 - r_{n+1} = (1 - r_n)/(\eta_n \leq (1 - r_n)/\eta_n \), and hence \( 1 - \tilde{r}_n \geq 1 - r_{n+1} \geq \eta_{n+1}(1 - r_{n+2}) \geq \eta_1(1 - \tilde{r}_{n+2}). \) We obtain

\[
    |u_2(z)| \lesssim \int_{\mathbb{D}} (1 - |\zeta|)^2 \, d\nu(\zeta) \lesssim \sum_{n=1}^{\infty} (1 - \tilde{r}_n) \log \frac{C}{1 - \tilde{r}_n} < \infty
\]

uniformly in \( z \in F \), where \( F \) is a compact subset of \( \mathbb{D} \).

To prove that \( u_2 \) is subharmonic in \( \mathbb{D} \), it is sufficient to verify it locally. First, we need an upper estimate of \( \nu \) on compact subsets of \( \mathbb{D} \). For any \( r_1^* \leq r < 1 \), there exists \( N \in \mathbb{N} \setminus \{1\} \) such that \( r_N^* \leq r < r_N^* \). Since \( (1 - \tilde{r}_{n+1})/(1 - \tilde{r}_{n}) \to 0 \) as \( n \to \infty \), we deduce

\[
    \nu(D(0, r)) = \sum_{n=1}^{N-1} \nu(A_n^*) + \nu(D(0, r) \cap A_N^*)
\]

\[
    \lesssim \sum_{n=1}^{N-1} \frac{1}{1 - \tilde{r}_n} \log \frac{C}{1 - \tilde{r}_n} + \nu(D(0, r) \cap A_N^*)
\]

\[
    \lesssim \frac{1}{1 - \tilde{r}_{N-1}} \log \frac{C}{1 - \tilde{r}_{N-1}} + \frac{1}{1 - r} \log \frac{C}{1 - r}
\]

Second, let \( K \) be an arbitrary compact subset of \( \mathbb{D} \). Note that

\[
    u_2(z) = \int_K \log |\zeta - z| \, d\nu(\zeta) + \int_K h(z, \zeta) \, d\nu(\zeta) + \int_{\mathbb{D} \setminus K} \left( \log |\zeta - z| \right) \, d\nu(\zeta),
\]
for any \( z \in \mathbb{D} \), where \( h(z, \zeta) \) is harmonic in \( z \) throughout the unit disc. In \( K \), the function \( u_2 \) can be represented as a sum of the logarithmic potential of the finite measure \( \nu \), which is subharmonic by [18, Theorem 3.1.2], and a harmonic function. Therefore \( u_2 \) is subharmonic in \( K \), and consequently in the whole unit disc. By [18, Theorem 3.7.4] the Riesz measure of \( u_2 \) coincides with \( \nu/(2\pi) \).

Similar arguments show that functions \( u_j^* \) are subharmonic in \( \mathbb{D} \) with the Riesz measures

\[
d\mu_j^* = \frac{1}{2\pi} \Delta \varphi dm_2|_{\bigcup_{n=1}^{\infty} Q_{n,j}}, \quad j \in \{1, 2, 3, 4\},
\]

respectively. The convergence of the integrals in (2.9) follow from the convergence of the sums

\[
\sum_{n=1}^{\infty} (1 - r_n), \quad \sum_{n=1}^{\infty} (1 - \hat{r}_n), \quad \sum_{n=1}^{\infty} (1 - r''_n).
\]

It then follows from the definition of \( u_1 \) that \( \Delta u_1 = \Delta \varphi - \Delta u_2 - \sum_{j=1}^{4} \Delta u_j^* \), so that \( \text{supp} \Delta u_1 \) is contained in the closure of the set

\[
B_1 = \bigcup_{n=1}^{\infty} \left( A_n \cup (\hat{A}_n \setminus Q_{n1}) \cup (A''_n \setminus Q_{n3}) \right),
\]

where the equality \( \Delta u_1 = \Delta \varphi \) holds. We conclude that the Riesz measure of \( u_1 \) is \( \Delta \varphi dm_2/(2\pi) \).

We proceed to approximate the subharmonic functions \( u_1 \) and \( u_2 \) using Theorem [2] and show that \( |u_1^*(z)|, |u_2^*(z)|, |u_2^*(z)| \) and \( |u_3^*(z)| \) are uniformly small. We estimate \( u_1^* \) and omit the considerations of \( u_2^* \) and \( u_3^* \), which are similar. We consider three cases. First, let \( |\varphi_\zeta(z)| = |(\zeta - z)/(1 - \zeta z)| \leq 1/2 \), that is, \( \zeta \) belongs to the pseudo-hyperbolic disc \( D_{ph}(z, 1/2) \). Then

\[
u_{n1}^* = \sum_{n=1}^{\infty} \int_{Q_{n1}} \log |\varphi_\zeta(z)| \Delta \varphi(z) dm_2(z)
\]

\[
= \sum_{n=1}^{\infty} \int_{Q_{n1} \cap D_{ph}(z, 1/2)} \log |\varphi_\zeta(z)| \Delta \varphi(z) dm_2(z)
\]

\[
+ \sum_{n: r_n \leq |z|} \int_{Q_{n1} \setminus D_{ph}(z, 1/2)} \log |\varphi_\zeta(z)| \Delta \varphi(z) dm_2(z)
\]

\[
+ \sum_{n: r_n > |z|} \int_{Q_{n1} \setminus D_{ph}(z, 1/2)} \log |\varphi_\zeta(z)| \Delta \varphi(z) dm_2(z)
\]

\[
= S_1(z) + S_2(z) + S_3(z), \quad z \in \mathbb{D}.
\]

We begin with \( S_1 \). Since

\[
\left| \int_{Q_{n1} \cap D_{ph}(z, 1/2)} \log |\varphi_\zeta(z)| \Delta \varphi(z) dm_2(z) \right|
\]

\[
\lesssim \frac{1}{(1 - |z|)^2} \int_{D_{ph}(z, 1/2)} \log \frac{1}{|\varphi_\zeta(z)|} dm_2(z)
\]

\[
\lesssim \int_{D(0, 1/2)} \left( \log \frac{1}{|w|} \right) \frac{1}{|1 - \overline{w} z|^4} dm_2(w) < \infty, \quad z \in \mathbb{D},
\]

uniformly for all \( n \in \mathbb{N} \), we deduce

\[
|S_1(z)| \lesssim \# \{ n \in \mathbb{N} : Q_{n1} \cap D_{ph}(z, 1/2) \neq \emptyset \} < \infty, \quad z \in \mathbb{D}.
\]
To estimate $S_2$, suppose that $|\varphi_2(\zeta)| \geq 1/2$. Then $|\log |\varphi_2(\zeta)|| \leq \log 2$ and

$$\int_{Q_{n1}\setminus D_{ph}(z,1/2)} \frac{p_2 + \varepsilon_n}{2\pi} \log |\varphi_2(\zeta)| \frac{dm_2(\zeta)}{(1-|\zeta|)^2} \leq \frac{m_2(Q_{n1})}{(1-r_n)^2} \leq 1, \quad n \in \mathbb{N}.$$ 

Since

$$1-r_{n+1} < 1-r_n = C \left( \frac{1-r_n}{C} \right)^{(p_2+\varepsilon_n)/p_1} \leq (1-r_n)^{(p_2+\varepsilon_n)/p_1} \leq (1-r_n)^M \quad n \in \mathbb{N},$$

for some constant $M > 1$ independent of $n$, we conclude $1-r_{n+1} \lesssim (1-r_1)^{M^n}$ for all $n \in \mathbb{N}$. Therefore

$$|S_2(z)| \lesssim \sum_{n: r_n \leq |z|} 1 = n(|z|, \{r_n\}) \lesssim \log \log \frac{1}{1-|z|}, \quad |z| \to 1^-.$$ 

Finally, we estimate $S_3$. Assume that $|\varphi_2(\zeta)| \geq 1/2$. Since

$$0 \leq \frac{1}{2} \log \frac{1}{|\varphi_2(\zeta)|^2} \leq \frac{1}{2|\varphi_2(\zeta)|^2} (1-|\varphi_2(\zeta)|^2) \leq 2 \frac{(1-|\zeta|^2)(1-|\zeta|^2)}{|1-z\zeta|^2} \leq 4 \frac{1-|\zeta|^2}{1-|z|},$$

we have

$$\int_{Q_{n1}\setminus D_{ph}(z,1/2)} \frac{p_2 + \varepsilon_n}{2\pi} \log |\varphi_2(\zeta)| \frac{dm_2(\zeta)}{(1-|\zeta|)^2} \lesssim \frac{1-r_n}{1-|z|}, \quad n \in \mathbb{N}, \quad z \in \mathbb{D},$$

and therefore

$$|S_3(z)| \leq \sum_{n: r_n \geq |z|} \frac{1-r_n}{1-|z|}, \quad z \in \mathbb{D}.$$ 

If $|z| < 1/2$, then $S_3(z) \leq 2 \sum_{n=1}^{\infty} (1-r_n) < \infty$ for all $z \in \mathbb{D}$. If $1/2 \leq |z| < 1$, then

$$|S_3(z)| \leq 1 + (1-|z|)^{M-1} + (1-|z|)^{M^2-1} + \cdots = 2 \sum_{k=0}^{\infty} \left( \frac{1}{2} \right)^{M^k} < \infty, \quad z \in \mathbb{D}.$$ 

In conclusion, we have proved

$$|u_1^*(z)| \lesssim \log \log \frac{1}{1-|z|}, \quad |z| \to 1^-.$$  \hspace{1cm} (2.10)

Similar arguments show

$$|u_2^*(z)| + |u_3^*(z)| \lesssim \log \log \frac{1}{1-|z|}, \quad |z| \to 1^-.$$  \hspace{1cm} (2.11)
We then consider $u_4$. Now
\[
u_4(z) = \sum_{n=1}^{\infty} \int_{Q_{4n}} \log |\varphi(z)| \Delta \varphi(z) \, dm_2(\zeta)
= \sum_n \int_{Q_{4n}\cap D_{\text{ph}}(z,1/2)} \log |\varphi(z)| \Delta \varphi(z) \, dm_2(\zeta)
\]
\[+ \sum_{n: \tau_n \leq |z|} \int_{Q_{4n}\setminus D_{\text{ph}}(z,1/2)} \log |\varphi(z)| \Delta \varphi(z) \, dm_2(\zeta)
\]
\[+ \sum_{n: \tau_n > |z|} \int_{Q_{4n}\setminus D_{\text{ph}}(z,1/2)} \log |\varphi(z)| \Delta \varphi(z) \, dm_2(\zeta)
= T_1(z) + T_2(z) + T_3(z), \quad z \in \mathbb{D},
\]
where
\[
\Delta \varphi(z) \asymp \frac{\log^2 C_{\nu_n}}{(1 - \tau_n)^2}, \quad \zeta \in A_n^*, \quad n \to \infty.
\]
We begin with $T_1$. There exists a constant $c > 0$ such that
\[
\left| \int_{Q_{4n}\cap D_{\text{ph}}(z,1/2)} \log |\varphi(\zeta)| \frac{\log^2 C_{\nu_n}}{(1 - |\zeta|)^2} \, dm_2(\zeta) \right|
\leq \log^2 \frac{C_{\nu_n}}{(1 - |z|)^2} \int_{Q_{4n}\cap D_{\text{ph}}(z,1/2)} \log \left| \frac{1 - \zeta z}{\zeta - z} \right| \, dm_2(\zeta)
\leq \log^2 \frac{C_{\nu_n}}{(1 - |z|)^2} \int_{D(z, c(1 - |z|)/(\log C_{\nu_n}))} \log \left| \frac{1 - \zeta z}{\zeta - z} \right| \, dm_2(w)
\leq \log^2 \frac{C_{\nu_n}}{(1 - |z|)^2} \int_{D(0, c(1 - |z|)/(\log C_{\nu_n}))} \log \left| \frac{1 - (z + w)z}{w} \right| \, dm_2(w).
\]
Since there exists a constant $\kappa > 1$ such that $|1 - (z + w)z| \leq \kappa(1 - |z|)$, we conclude
\[
\left| \int_{Q_{4n}\cap D_{\text{ph}}(z,1/2)} \log |\varphi(\zeta)| \frac{\log^2 C_{\nu_n}}{(1 - |\zeta|)^2} \, dm_2(\zeta) \right|
\leq \log^2 \frac{C_{\nu_n}}{(1 - |z|)^2} \int_{0}^{c(1 - |z|)/(\log C_{\nu_n})} \log \frac{\kappa(1 - |z|)}{\tau} \tau \, d\tau
\leq \log^2 \frac{C_{\nu_n}}{(1 - |z|)^2} \left( \frac{1}{2} \log \frac{\kappa(1 - |z|)}{\tau} + \frac{\tau^2}{4} \right) \frac{c(1 - |z|)}{\log C_{\nu_n}}
\leq \log \frac{C_{\nu_n}}{1 - |z|}, \quad |z| \to 1^-,
\]
It is easy to check that
\[ \mu \leq \log \log \frac{C}{1 - |z|}, \quad |z| \to 1^- \]

Therefore, as in the case of \( T \)
\[ \text{Finally, we estimate } T_3. \] Assume that \( |\varphi_\zeta(z)| \geq 1/2. \) Then
\[ \int_{Q_{n^4} \setminus D_{ph}(z, 1/2)} \left| \log \left| \frac{z - \zeta}{1 - z\zeta} \right| \Delta \varphi(\zeta) d\mu_2(\zeta) \right| \leq \int_{Q_{n^4} \setminus D_{ph}(z, 1/2)} \left| \log |\varphi_\zeta(z)| \right| \frac{\log^2 \frac{C}{1 - \hat{r}_n}}{(1 - \hat{r}_n)^2} \leq 1, \quad n \in \mathbb{N}. \]

Therefore, as in the case of \( S_2 \), we deduce
\[ |T_2(z)| \leq \sum_{n : \hat{r}_n \leq |z|} \frac{1}{n}, \quad |z| \to 1^- \]

Finally, we estimate \( T_3 \). Assume that \( |\varphi_\zeta(z)| \geq 1/2. \) Now
\[ \int_{Q_{n^4} \setminus D_{ph}(z, 1/2)} \left| \log \left| \frac{z - \zeta}{1 - z\zeta} \right| \Delta \varphi(\zeta) d\mu_2(\zeta) \right| \leq \frac{1 - \hat{r}_n}{1 - |z|}, \quad n \in \mathbb{N}. \]

Therefore,
\[ |T_3(z)| \leq \sum_{n : \hat{r}_n > |z|} \frac{1 - \hat{r}_n}{1 - |z|} \leq 1, \quad z \in \mathbb{D}. \]

In conclusion,
\[ |u_4^*(z)| \leq \log \log \frac{1}{1 - |z|}, \quad |z| \to 1^- \quad \text{(2.12)} \]

We then approximate \( u_1 \) and \( u_2 \) separately. The Riesz measure \( \mu_1 = \mu_{u_1} \) admits a regular partition with respect to the function \( b_1(z) \):
(i) \( \mu_1(Q^j_{n,k}) = \mu_1(\hat{Q}^j_{n,k}) = 2 \) for all \( n, k, j \);
(ii) the interiors of all polar rectangles \( Q^j_{n,k}, \hat{Q}^j_{n,k} \) and \( Q'^j_{n,k} \) are pairwise disjoint for all \( n, k, j \).
(iii) the sides of each rectangle are comparable;
(iv) \( \text{diam } Q^j_{n,k} \leq b_1(r_{n,k}), \text{diam } \hat{Q}^j_{n,k} \leq b_1(\hat{r}_{n,k}) \) and \( \text{diam } Q'^j_{n,k} \leq b_1(r'_{n,k}) \);
(v) \( \mu_1(D \setminus \bigcup_n \bigcup_{k,j} Q^j_{n,k} \cup \bigcup_{k,j} \hat{Q}^j_{n,k} \cup \bigcup_{k,j} Q'^j_{n,k}) = 0; \)
(vi) for appropriate values \( n, k, j \), the measure \( \mu_1 \) admits the representation
\[ \mu_1 = \sum_n \left( \sum_{k,j} \mu_1|Q^j_{n,k}| + \sum_{k,j} \mu_1|\hat{Q}^j_{n,k}| + \sum_{k,j} \mu_1|Q'^j_{n,k}| \right). \]

It is easy to check that \( \mu_1 \) is locally regular with respect to \( b_1 \), that is,
\[ \int_0^{b_1(|z|)} \frac{\mu_1(D(z, t))}{t} \, dt \leq 1, \quad \rho_0 < |z| < 1, \]
for some constant $\rho_0 \in (0, 1)$. In fact, we have $\mu_1(D(z, t)) \lesssim t^2/(1-|z|)^2$ for all $0 < t \leq \frac{1-|z|}{2}$, which together with $b_1(r) = (1-r)/2$ gives local regularity of $\mu_1$ with respect to $b_1$.

By Theorem A there exists an analytic function $A_1$ in $\mathbb{D}$ such that
\[
\sup_{z \in \mathbb{D}} \left( \log |A_1(z)| - u_1(z) \right) < \infty,
\]
and for each $\varepsilon > 0$ there exist $\rho_1 = \rho_1(\varepsilon) \in (0, 1)$, $G_1 = G_1(\varepsilon) > 0$ satisfying
\[
|\log |A_1(z)| - u_1(z)| < G_1, \quad \rho_1 < |z| < 1, \quad z \notin E_\varepsilon^1,
\]
where $E_\varepsilon^1 = \{z \in \mathbb{D} : \text{dist}(z, Z_{A_1}) \leq \varepsilon b_1(|z|)\}$. Moreover, the zero set $Z_{A_1}$ of the function $A_1$ satisfies $Z_{A_1} \subset \bigcup_{\zeta \in \text{supp} \mu} D(\zeta, K_1 b_1(|\zeta|))$ for some $K_1 > 0$.

The Riesz measure $\mu_2 = \mu_{u_2}$ admits a regular partition with respect to the function $b_2(z)$:

(i) $\mu_2(Q_{n,j}^{*}) = 2$ for all $n, j$;
(ii) the interiors of all polar rectangles $Q_{n}^{*j}$ are pairwise disjoint for all $n, j$;
(iii) the sides of each rectangle are comparable;
(iv) $\text{diam} Q_{n}^{*j} \asymp b_2(r_n^{*});$
(v) $\mu_2(D \setminus \bigcup_n (\bigcup_j Q_{n}^{*j})) = 0;$
(vi) for all appropriate values $n, j$, the measure $\mu_2$ admits the representation
\[
\mu_2 = \sum_n \left( \sum_j \mu_{Q_{n,j}^{*}} \right).
\]

Corresponding to the above, the measure $\mu_2$ is locally regular with respect to $b_2$. We have
\[
\mu_2(D(z, t)) \lesssim t^2 \left( \log \frac{c}{1-|z|} \right)^2, \quad 0 < t \leq \frac{1-|z|}{\log \frac{c}{1-|z|}},
\]
which yields the required property.

By Theorem A there exists an analytic function $A_2$ in $\mathbb{D}$ such that
\[
\sup_{z \in \mathbb{D}} \left( \log |A_2(z)| - u_2(z) \right) < \infty,
\]
and for each $\varepsilon > 0$ there exist $\rho_2 = \rho_2(\varepsilon) \in (0, 1)$, $G_2 = G_2(\varepsilon) > 0$ satisfying
\[
|\log |A_2(z)| - u_2(z)| < G_2, \quad \rho_2 < |z| < 1, \quad z \notin E_\varepsilon^2,
\]
where $E_\varepsilon^2 = \{z \in \mathbb{D} : \text{dist}(z, Z_{A_2}) \leq \varepsilon b_2(|z|)\}$. Moreover, the zero set $Z_{A_2}$ of the function $A_2$ satisfies $Z_{A_2} \subset \bigcup_{\zeta \in \text{supp} \mu} D(\zeta, K_2 b_2(|\zeta|))$ for some $K_2 > 0$.

Define $A = A_1 A_2$. Then, by (2.10), (2.11), (2.12), (2.13), and (2.14) we obtain
\[
\log |A(z)| - \varphi(z) = \left( \log |A_1(z)| - u_1(z) \right) + \left( \log |A_2(z)| - u_2(z) \right) - \sum_{j=1}^{4} u_j^*(z)
\lesssim \log \log \frac{1}{1-|z|}, \quad |z| \to 1^-.
\]
Since $\varphi(z) = \varphi(|z|)$, there exists a constant $Y_1 > 0$ such that
\[
\log M(r, A) \leq \varphi(r) + Y_1 \log \log \frac{1}{1-|z|}, \quad |z| \to 1^-.
\]
Let $E_2 = E^1_2 \cup E^2_2$. It is clear that $E_2 \subset \{z \in \mathbb{D} : \text{dist}(z, Z_A) \leq \varepsilon(1 - |z|)\}$. Combining (2.10), (2.11), (2.12), (2.14), and (2.16) we obtain

$$|\log |A(z)|| - \varphi(z)| \lesssim 1 + \log \log \frac{1}{1 - |z|}, \quad \max\{\rho_1, \rho_2\} < |z| < 1, \quad z \notin E_\varepsilon,$$

as $|z| \to 1^-$, where the comparison constant depends on $\varepsilon$. This proves Part (a) in Lemma 7.

It follows from the proof of Theorem A that for each rectangle $Q^j_{n,k}$, $Q^\prime_{n,k}$, there corresponds exactly two zeros of $A_1$ lying in a neighborhood of the rectangle of radius equal to the diameter of the rectangle. Hence the number of zeros of $A_1$ in each such neighborhood is uniformly bounded by some constant $P_1$. Suppose that $\partial D(0, r) \cap Q^j_{n,k} \neq \emptyset$ for some $n, k, j$. Then the linear measure of $\partial D(0, r) \cap E^2_\varepsilon$ is at most a constant multiple of $P_1 \varepsilon$. The same is also true for $Q^\prime_{n,k}$, $Q^\prime_{n,k}$, in addition to $Q^j_{n,k}$. Similarly, $Q^\prime_{n,k}$ corresponds exactly two zeros of $A_2$ lying in a neighborhood of the rectangle of radius equal to the diameter of the rectangle. Thus, the number of zeros of $A_2$ in each such neighborhood is uniformly bounded by some constant $P_2$. Consequently, the linear measure of $\partial D(0, r) \cap E^2_\varepsilon$ is at most a constant multiple of $P_2 \varepsilon$, provided that $\partial D(0, r) \cap Q^j_{n,k} \neq \emptyset$. By fixing a sufficiently small $\varepsilon > 0$, the linear measure of $E_\varepsilon \cap \partial D(0, r)$ is less than $2\pi r$, where $r \in (1/2, 1)$. So, if $r$ is sufficiently close to 1, then there exists $z_\tau \in \partial D(0, r) \setminus E_\varepsilon$ with $|z_\tau| = r$. Therefore,

$$\log M(r, A) \geq \log |A(z_\tau)| \geq \varphi(r) - Y_2 \log \frac{1}{1 - r}, \quad |z| \to 1^-, \quad (2.18)$$

for some $Y_2 > 0$. By combining (2.17) and (2.18), we conclude Part (b) in Lemma 7. The lemma is now proved.

2.3. Completion of the proof of Theorem B

The following auxiliary result is stated for convenience.

Theorem B ([7, Corollary 6]). Let $0 < R < \infty$ and $f$ meromorphic in a domain containing $D(0, R)$. Suppose that $j, k$ are integers with $k > j \geq 0$, and $f^{(j)} \neq 0$. Then

$$\int_{r < |z| < R} \left| \frac{f^{(k)}(z)}{f^{(j)}(z)} \right|^{\frac{1}{k-j}} dm_2(z) \lesssim R \log \frac{e(R - r')}{R - r} \left(1 + \log \frac{1}{R - r} + T(R, f)\right)$$

for $0 \leq r' < R$.

Let $k \in \mathbb{N}$ be fixed and $k \leq p_1 < p_2 \leq p < \infty$ be constants such that $p_2 \geq 2k$. By [5, Theorem 1.4], all non-trivial solutions of (1.2) satisfy $\sigma_M(f) = p_2/k - 1$ and therefore it suffices to consider the lower order of growth. Note that in the case $p_2 \leq k$, all solutions $f$ of (1.2) satisfy $\sigma_M(f) = \lambda_M(f) = 0$ by [5, Theorem 1.4].

Our argument is based on Lemmas 6 and 7 and we take advantage of the same notation. Choose $\eta_n = n + 1$ for all $n \in \mathbb{N}$ in Lemma 6. As a general property we note that, if $0 < r < 1$ and $1 - \rho = C(1 - r)^q$ for some $C > 0$ and $q > 1$, then

$$\left(\frac{\rho}{r}\right)^{\frac{1}{q}} = \left(1 + \frac{(1 - r)(1 - C(1 - r)^q - 1)}{r}\right)^{\frac{1}{r - q}} = e + o(1), \quad r \to 1^-.$$
Choose Lemma 7, then the growth estimate [13, Theorem 5.1] and (2.19) yield
\[
\log^+ |f(\hat{r}_n e^{i\theta})| - C \lesssim \int_0^{\hat{r}_n} |A(te^{i\theta})|^\frac{1}{\hat{n}} dt = \left( \int_0^{r_n} + \int_{r_n}^{r_n'} + \int_{r_n'}^{\hat{r}_n} \right) |A(te^{i\theta})|^\frac{1}{\hat{n}} dt
\]
\[
\lesssim \int_0^{r_n} \frac{dt}{(1-t)\frac{p_2+o(1)}{\hat{n}}} + \frac{1}{(1-r_n)\frac{p_2}{\hat{n}} + o(1)} \int_{r_n}^{r_n'} \left( \frac{t}{r_n} \right)^{\frac{p_2}{\hat{n}}} \frac{dt}{(1-t)\frac{p_1 + o(1)}{k}}
\]
\[
\lesssim \frac{1}{(1-r_n)\frac{p_2}{\hat{n}} + o(1)} + \frac{r_n' - r_n}{1 - r_n} + \frac{1}{(1-\hat{r}_n)\frac{p_1 + o(1)}{k} - 1}
\]
\[
\lesssim \frac{1}{(1-\hat{r}_n)\frac{p_2}{\hat{n}} + o(1)} + \frac{1}{(1-\hat{r}_n)\frac{p_1 + o(1)}{k} - 1},
\]
for all \( \theta \in \mathbb{R} \) as \( n \to \infty \), which gives the desired upper bound for the lower order of growth.

To obtain a lower bound for the lower order of growth, we argue as follows. For any \( r \in (0, 1) \) sufficiently close to 1 there exists \( N \in \mathbb{N} \) such that \( r_N < r \leq r_{N+1} \). Choose \( R = (1+r)/2 \), \( r' = 0 \), and apply Theorem \( \square \) to obtain
\[
\int_{|z| < r} |A(z)|^{1/k} dm_2(z) \lesssim \log^+ \frac{2e}{1-r} \left( 1 + \log^+ \frac{2}{1-r} + \log^+ M(\frac{1+r}{2}, f) \right).
\]
Clearly, it suffices to estimate the integral of the coefficient \( A \). Note also that
\[
\liminf_{R \to 1^{-}} \frac{\log^+ \log^+ M(R, f)}{\log \frac{1}{1-R}} = \liminf_{r \to 1^{-}} \frac{\log^+ \log^+ M(\frac{1+r}{2}, f)}{\log \frac{1}{1-r}}
\]
We continue in two separate parts.

(i) Let \( r \geq r' \). Then \( r'_N \leq r \leq r_{N+1} \). Lemma \( \square \) gives a lower bound for the modulus of the coefficient only outside the exceptional set. Since the exceptional set lies (almost completely) outside the annulus \( A'_n = \{ z \in \mathbb{D} : r_n < |z| < r'_n \} \), we have a good control of the coefficient there. Note that \( \Delta \varphi \) vanishes on these annuli. Then,
\[
\int_{|z| < r} |A(z)|^{1/k} dm_2(z) \geq \sum_{n=1}^{N} \int_{r_n+\varepsilon(1-r_n)}^{r_n' \varepsilon(1-r_n')} |A(z)|^{1/k} dm_2(z)
\]
\[
\geq \sum_{n=1}^{N} \frac{1}{(1-r_n')^{\frac{p_2}{\hat{n}} - \frac{p_1}{\hat{n}} - o(1)}} \times \frac{1}{(1-r'_N)^{\frac{p_1}{\hat{n}} - \frac{p_2}{\hat{n}} - o(1)}},
\]
and therefore
\[
\frac{\log^+ \log^+ M(\frac{1+r}{2}, f)}{\log \frac{1}{1-r}} \geq o(1) + \left( \frac{p_2}{\hat{n}} - \frac{p_1}{\hat{n}} - o(1) \right) \log^+ \frac{1}{1-r_{N+1}}, \quad r'_N \leq r \leq r_{N+1}.
\]
Lemma 6 implies

\[ 1 - r_{N+1} = \frac{1 - r''_N}{\eta_N} \sim \frac{1 - \rho_N}{\eta_N \log \frac{1}{1 - \rho_N}} = \frac{(1 - r''_N)^{p_2}}{\eta_N^{p_2} \log \frac{1}{1 - r''_N}}. \]

By the discussion preceding the statement of Lemma 6, we deduce

\[ \lim_{n \to \infty} \frac{\log \eta_n}{\log \frac{1}{1 - \rho_n}} = 0, \]

and we obtain

\[ \frac{\log^+ \log^+ M(\frac{1+r}{2}, f)}{\log \frac{1}{1-r}} \geq o(1) + \frac{p_2}{p} \left( \frac{p_1}{k} - \frac{p_1}{p_2} - o(1) \right), \quad r_N \leq r \leq r_{N+1}. \]

(ii) Let \( r < r' \). Then \( r_N < r < r'_N \). Choose \( \varepsilon \in (0, \frac{\pi}{C_4}) \). It follows from Lemma 7 that \( m_1(\partial D(0, r) \setminus E_\varepsilon) \geq 2\pi - C_4\varepsilon \geq \pi \). The, applying Lemma 6, we deduce

\[ \int_{0<|z|<r} |A(z)|^{\frac{1}{k}} dm_2(z) \geq \int_0^{r_N} \int_{\left\{ \theta \in [0,2\pi]: te^{i\theta} \notin E_\varepsilon \right\}} |A(te^{i\theta})|^{\frac{1}{k}} dt d\theta \]

\[ \geq \frac{1}{\log C_2 / k} \int_0^{r_N} e^{\frac{2\pi t}{1-r}} dt \]

\[ \geq \frac{1}{\log C_2 / k} \int_{r_{N-1}}^{r_N} \left( 1 - t \right)^{2\pi / p + o(1)} \]

\[ \geq \frac{1}{(1 - r_{N-1})^{1+o(1)}} = \frac{1}{(1 - r')^{1+o(1)}}. \]

Hence,

\[ \frac{\log^+ \log^+ M(\frac{1+r}{2}, f)}{\log \frac{1}{1-r}} \geq o(1) + \frac{p_1}{k} - \frac{p_1}{p_2} \geq o(1) + \frac{p_2}{p} \left( \frac{p_1}{k} - \frac{p_1}{p_2} \right), \quad r_N \leq r \leq r' \]

By choosing different values for the parameter \( p \in [p_2, \infty) \), we obtain the assertion. Note that the lower order of growth for solutions (1.2) is as large as possible if \( p = p_2 \), which corresponds to the value \( \alpha = p_1/p_2 \). However, \( \alpha = 1 \) corresponds to the case when both terms in (2.20) are of similar growth.

3. Growth estimates for logarithm of maximum modulus

This section consists of preparations for the proof of Theorem 3. We use the Wiman-Valiron theory adapted for the lower order of growth.

Recall that \( \lambda_s(f) \) and \( \sigma_s(f) \) are defined in [18] for any function \( f \) analytic in \( \mathbb{D} \). It is known that \( \sigma_s(f) = \sigma_M(f) + 1 \), provided

\[ \limsup_{r \to 1^{-}} \frac{\log M(r, f)}{\log \frac{1}{1-r}} = \infty, \]

see [20] Lemma 1.2.16]. The same is true for the maximum term and central index instead of maximum modulus and \( K(r, f) \), respectively. That is,

\[ \limsup_{r \to 1^{-}} \frac{\log^+ \nu(r, f)}{\log \frac{1}{1-r}} = \limsup_{r \to 1^{-}} \frac{\log^+ \log^+ \mu(r, f)}{\log \frac{1}{1-r}} + 1 = \sigma_M(f) + 1, \]
see \cite{14} Theorems 1.5.1 and 1.5.2]. Recall that for an analytic function \( f(z) = \sum_{n=0}^{\infty} \hat{a}(n) z^n \) in \( \mathbb{D} \), the maximum term is \( \mu(r, f) = \max\{|\hat{a}(n)| r^n |: n \in \mathbb{N} \cup \{0\}\} \), while the central index is \( \nu(r, f) = \max\{n : \mu(r, f) = |\hat{a}(n)| r^n|\} \). These quantities obey the relation

\[
\log \mu(r, f) = \log \mu(r_0, f) + \int_{r_0}^r \frac{\nu(t, f)}{t} dt, \quad 0 < r_0 < r < 1,
\]

by \cite{14} Theorem 1.4.1, assuming that \( f(0) = 1 \) and \( \sup\{|\hat{a}(n)| : n \in \mathbb{N} \cup \{0\}\} = \infty \). Note that, if \( \sup\{|\hat{a}(n)| : n \in \mathbb{N} \cup \{0\}\} < \infty \), then \( |f(z)| = O(1/(1-|z|)) \) for \( z \in \mathbb{D} \).

The representation (3.1) implies

\[
\nu(r, f) = r (\log \mu(r, f))' +, \quad r_0 < r < 1.
\]

In addition, it is known that

\[
\lambda_M(f) \leq \lim_{r \to 1^-} \frac{\log^+ \nu(r, f)}{\log \frac{1}{1-r}} \leq \lambda_M(f) + 1,
\]

where both inequalities can be strict \cite{14} Theorem 1.5.2.

**Proposition 8.** Let \( f \) be an analytic function in \( \mathbb{D} \). Then \( \lambda_* (f) \leq \lambda_M(f) + 1 \) if \( f \) is unbounded, and \( \lambda_M(f) + \frac{\lambda_M(f)}{\sigma_M(f)} \leq \lambda_* f \) if \( \sigma_M(f) > 0 \).

The second part of the statement is similar to a result proved in \cite{19} Corrigendum] for the maximum term and the central index. But since we have not found a proof in the existing literature, we offer a proof. Proposition 8 is a direct consequence of a growth result for convex functions that will be discussed next.

For \( h : (-\infty, 0) \to \mathbb{R} \) such that \( \lim_{x \to 0^-} h(x) = \infty \), let

\[
\alpha(h) = \liminf_{x \to 0^-} \frac{\log h(x)}{\log |x|} \quad \text{and} \quad \beta(h) = \limsup_{x \to 0^-} \frac{\log h(x)}{\log |x|}.
\]

Let \( \Omega \) denote the class of convex functions \( h : (-\infty, 0) \to \mathbb{R} \) satisfying the property \( \lim_{x \to 0^-} h(x) = \infty \). It is easy to see that the right derivative \( h'_+ \) exists at every point and

\[
h(x) = o(h'_+(x)), \quad x \to 0^-,
\]

for each \( h \in \Omega \). Hence \( \alpha(h) \leq \alpha(h'_+) \). In particular, if \( \alpha(h) = \infty \), then \( \alpha(h'_+) = \infty \).

**Proposition 9.** Let \( h \in \Omega \). Then the following statements are valid:

(a) \( \alpha(h'_+) \leq \alpha(h) + 1 \);
(b) \( \beta(h'_+) = \beta(h) + 1 \);
(c) \( \alpha(h) + \frac{\alpha(h)}{\beta(h)} \leq \alpha(h'_+) \), if \( \alpha(h) < \infty \) and \( \beta(h) > 0 \).

**Proof.** Since \( h \) is nondecreasing and positive close to zero, there exists \( x_1 \in (-\infty, 0) \) such that

\[
\left( \frac{x}{2} - x \right) h'_+(x) \leq \int_0^x h'_+(t) dt = h\left( \frac{x}{2} \right) - h(x) \leq h\left( \frac{x}{2} \right), \quad x \in (x_1, 0).
\]

Therefore

\[
h'_+(x) \leq \frac{2}{|x|} h\left( \frac{x}{2} \right), \quad x_1 < x < 0,
\]

and the inequalities \( \alpha(h'_+) \leq \alpha(h) + 1 \) and \( \beta(h'_+) \leq \beta(h) + 1 \) follow. Thus, in particular, (a) is proved.
To deduce (b), it remains to show \( \beta(h'_+) \geq \beta(h) + 1 \). Suppose on the contrary that \( \beta(h'_+) < \beta(h) + 1 \). Then there exists \( \varepsilon > 0 \) and \( x_2 \in (-\infty, 0) \) such that

\[
\lambda h'_+(x) \leq \left( \frac{1}{|x|} \right)^{\beta(h) - \varepsilon + 1}, \quad x_2 \leq x < 0,
\]

and hence

\[
h(x) - h(x_2) = \int_{x_2}^x h'_+(t) \, dt \leq \int_{x_2}^x \left( \frac{1}{|t|} \right)^{\beta(h) - \varepsilon + 1} \, dt
= \frac{1}{\beta(h) - \varepsilon} \left( \frac{1}{|x|} \right)^{\beta(h) - \varepsilon} - \left( \frac{1}{|x_2|} \right)^{\beta(h) - \varepsilon},
\]

It follows that \( \beta(h) \leq \beta(h) - \varepsilon \) which is impossible. Thus (b) is proved.

It remains to prove the inequality in (c) under the hypotheses \( \alpha(h) < \infty \) and \( \beta(h) > 0 \). If \( \alpha(h) = 0 \) or \( \beta(h) = \infty \), the statement is trivial, so assume \( \alpha(h) > 0 \) and \( \beta(h) < \infty \). For given \( \alpha \in (0, \alpha(h)) \) and \( \beta \in (\beta(h), \infty) \), there exists \( x_3 \in (-\infty, 0) \) such that \( |x|^{-\alpha} \leq h(x) \leq |x|^{-\beta} \) for all \( x_3 \leq x < 0 \). Consider the function

\[
g(x) = -(2|x|^\alpha)^{\frac{1}{\beta}}, \quad -\infty < x < 0.
\]

Since \( \alpha < \beta \), we have

\[
x - g(x) = -|x| + (2|x|^\alpha)^{\frac{1}{\beta}} \sim (2|x|^\alpha)^{\frac{1}{\beta}}, \quad x \to 0^{-}.
\]

Therefore

\[
h'_+(x) \geq \frac{h(x) - h(g(x))}{x - g(x)} \geq \frac{1}{x - g(x)} \left( \frac{1}{|x|^\alpha} - \frac{1}{|g(x)|^\beta} \right)
= \frac{1}{2(x - g(x))} \frac{1}{|x|^\alpha} = \left( \frac{1}{2^{1+\frac{1}{\beta}}} + o(1) \right) \frac{1}{|x|^\alpha(1+\frac{1}{\beta})}, \quad x \to 0^{-}.
\]

It follows that

\[
\alpha(h'_+) \geq \alpha + \frac{\alpha}{\beta}.
\]

Since \( \alpha \in (0, \alpha(h)) \) and \( \beta \in (\beta(h), \infty) \) were arbitrary, we deduce (c).

\[ \square \]

**Proof of Proposition 9** Let \( h(x) = \log M(e^x, f) \) for all \( -\infty < x < 0 \). Then \( h \in \Omega \), \( \lambda_M(f) = \alpha(h) \), \( \sigma_M(f) = \beta(h) \), \( \lambda_*(f) = \alpha(h'_+) \) and \( \sigma_*(f) = \beta(h'_+) \). Therefore the inequality \( \lambda_*(f) \leq \lambda_M(f) + 1 \) follows by Proposition 9(a) if \( \lambda_M(f) < \infty \), and for otherwise it is trivial.

Assume now \( \sigma_M(f) > 0 \). If \( \lambda_M(f) = \infty \), then \( \lambda_*(f) = \infty \), and hence the inequality \( \lambda_M(f) + \frac{\lambda_*(f)}{\sigma_M(f)} \leq \lambda_*(f) \) is clearly valid. But if \( \lambda_M(f) < \infty \), then the inequality follows from Proposition 9(c).

\[ \square \]

The next proposition shows that both estimates for the quantity \( \lambda_*(f) \) in Proposition 9 are sharp.

**Proposition 10.** Let \( 0 < \lambda, \sigma < \infty \). Then the following assertions hold.

(a) If \( \lambda \leq \sigma \), then there exists an analytic function \( f \) in \( \mathbb{D} \) such that \( \lambda_M(f) = \lambda \), \( \sigma_M(f) = \sigma \) and \( \lambda_*(f) = \lambda_M(f) + 1 \).

(b) If \( \lambda < \sigma \), then there exists an analytic function \( f \) in \( \mathbb{D} \) such that \( \lambda_M(f) = \lambda \), \( \sigma_M(f) = \sigma \) and \( \lambda_*(f) = \lambda_M(f) + \frac{\lambda_*(f)}{\sigma_M(f)} \).

For the proof we need the following lemma.
Proof of Proposition 10. First, we prove (a) in the case \( \lambda = \sigma \). To do this, we set \( n_k = k \) for all \( k \in \NN \cup \{0\} \), and define

\[
c_k = 1 - \left( \frac{\sigma}{k + \sigma + 1} \right)^{\frac{1}{k+1}}, \quad k \in \NN \cup \{0\}.
\]

Then \( c_k \in (0,1) \) for all \( k \) and \( \lim_{k \to \infty} c_k = 1 \). Further, let \( a_0 \neq 0 \) be fixed, and \( a_{k+1} = a_0 \prod_{j=0}^{k} c_j^{-1} \) for all \( k \in \NN \cup \{0\} \). Then Lemma C ensures that the power series \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) represents an analytic function in \( D \) such that

\[
\nu(r,f) = \begin{cases} 
    n_0, & 0 < r < c_0, \\
    n_{k+1}, & c_k \leq r < c_{k+1}, \quad k \in \NN \cup \{0\}.
\end{cases}
\]

This implies

\[
\log \mu(r,f) \sim \frac{1}{(1-r)^\sigma}, \quad r \to 1^-.
\]

Since

\[
\liminf_{r \to 1^-} \frac{\log^+ \log^+ \mu(r,f)}{\log \frac{1}{1-r}} = \lambda_M(f) \leq \sigma_M(f) = \limsup_{r \to 1^-} \frac{\log^+ \log^+ \mu(r,f)}{\log \frac{1}{1-r}},
\]

we deduce \( \sigma \leq \lambda_M(f) \leq \sigma_M(f) = \sigma \). Consequently, Proposition 10 yields \( \lambda_*(f) = \lambda_M(f) + 1 \).

We next prove (b). Let \( q = \lambda/\sigma \in (0,1) \), fix \( \delta \in (0,1) \) such that (further restrictions for \( \delta \) apply later)

\[
\frac{1}{x^{\sigma+1}} + 1 \leq \frac{1}{x^{\sigma+q}}, \quad 0 < x \leq \delta, \tag{3.3}
\]

and define \( c_k = 1 - \delta^{q^{-k}} \) for all \( k \in \NN \cup \{0\} \). Then \( c_k \in (0,1) \) for all \( k \in \NN \cup \{0\} \), and \( \lim_{k \to \infty} c_k = 1 \). Moreover,

\[
(1 - c_{k+1})^{\lambda+q} = \delta^{\frac{\lambda+q}{\sigma+q}} = \delta^{\frac{\lambda}{\sigma}} = (1 - c_k)^{\sigma+1}, \quad k \in \NN \cup \{0\}. \tag{3.4}
\]

Set \( n_0 = 0 \) and

\[
n_{k+1} = \left[ \frac{1}{\delta^{\sigma+1}} \right] + 1, \quad k \in \NN \cup \{0\}. \tag{3.5}
\]

Since \( \delta^{q^{-k}} \leq \delta \), (3.3) ensures that \( \{n_k\}_{k=0}^{\infty} \) is increasing. Define \( a_0 = 1 \), and

\[
a_{n_{k+1}} = \prod_{j=0}^{k} c_j^{n_j-n_{j+1}}, \quad k \in \NN \cup \{0\},
\]
and \(a_n = 0\) for all \(n \in \{n_k, n_{k+1}\}\) with \(k \in \mathbb{N} \cup \{0\}\). Then Lemma [C] implies that the power series \(f(z) = \sum_{n=0}^{\infty}a_nz^n\) represents an analytic function in \(D\) such that \(\nu(r, f) = n_{k+1}\) for all \(c_k < r < c_{k+1}\) and \(k \in \mathbb{N} \cup \{0\}\). Thus, by (3.4) and (3.5),
\[
\lim_{r \to 1^-} \sup \log \frac{\nu(r, f)}{1-r} = \sigma + 1, \quad \lim_{r \to 1^-} \inf \log \frac{\nu(r, f)}{1-r} = \lambda + \frac{\lambda}{\sigma}.
\]
By [14, Theorem 1.5.2], we deduce \(\sigma_M(f) = \sigma\). Using (5.1), we see that the function \(h(x) = \log \mu(e^{tx})\) belongs to \(\Omega\). By (3.2) and the fact that \(\log(1/t) \sim 1 - t\) as \(t \to 1^-\), we find that \(\alpha(h) = \lambda_M(f), \beta(h) = \sigma_M(f)\) and
\[
\alpha(h'_+) = \limsup_{t \to 1^-} \frac{\log \nu(t, f)}{\log \frac{1}{1-t}}.
\]
Hence, by Proposition 5(c) and the fact that \(\sigma_M(f) = \sigma\), we conclude
\[
\lambda_M(f) + \frac{\lambda_M(f)}{\sigma_M(f)} \leq \alpha(h'_+) = \lambda + \frac{\lambda}{\sigma_M(f)},
\]
from which \(\lambda_M(f) \leq \lambda\). Note that, similarly as above, the identity \(\sigma_M(f) = \sigma\) can be proved alternatively by using Proposition 5(b).

We next show that \(\lambda_M(f) \geq \lambda\). First observe that \(\mu(r, f) \geq \mu(0, f) = a_0 = 1\) for all \(0 \leq r < 1\), and hence
\[
\log \mu(c_k, f) = \log \mu(c_k-1, f) + \int_{c_k-1}^{c_k} \frac{\nu(t, f)}{t} dt \geq \int_{c_k-1}^{c_k} \frac{\nu(t, f)}{t} dt = n_k \log \frac{c_k}{c_{k-1}}, \quad k \in \mathbb{N},
\]
where \(n_k \sim (1 - c_k)^{-q(\sigma + 1)}\) as \(k \to \infty\) and
\[
\log \frac{c_k}{c_{k-1}} \sim (c_k - c_{k-1}) = (1 - c_k)^{\frac{\lambda+q}{\sigma+1}} - (1 - c_k) = (1 - c_k)^q(1 - (1 - c_k)^{1-q}) \sim (1 - c_k)^q, \quad k \to \infty,
\]
by (3.4) and (3.5). It follows that
\[
\log \mu(c_k, f) \geq (1 + o(1)) \frac{1}{(1 - c_k)^\lambda}, \quad k \to \infty.
\]
By (3.7), we obtain
\[
\log \mu(r, f) = \log \mu(c_k, f) + \int_{c_k}^{r} \frac{\nu(t, f)}{t} dt = \log \mu(c_k, f) + n_{k+1} \log \frac{r}{c_k}
\]
\[
\geq \frac{1}{(1 - c_k)^\lambda} + \frac{1}{\sigma+1} (r - c_k) \geq \frac{1}{(1 - r)^\lambda} h_k(r), \quad r \in [c_k, c_{k+1}],
\]
where
\[
h_k(r) = \frac{1 - r}{1 - c_k}^\lambda + \frac{(1 - r)^\lambda(r - c_k)}{(1 - c_k)^{q+1}}.
\]
Note that \(h_k(c_k) = 1\) and \(h_k'(c_k) > 0\) for all \(k\) large enough, since \(\lambda < \sigma\). We conclude that \(h_k\) is uniformly bounded away from zero on \([c_k, c_{k+1}]\), and therefore
\[
\log \mu(r, f) \gtrsim \frac{1}{(1 - r)^\lambda}, \quad r \in [c_k, c_{k+1}].
\]
As \(\bigcup_{k=1}^{\infty}[c_k, c_{k+1}] = [c_1, 1]\), it follows that \(\lambda_M(f) \geq \lambda\). Since we have already shown that \(\lambda_M(f) \leq \lambda\), we deduce \(\lambda_M(f) = \lambda\).
We have shown that $\sigma_M(f) = \sigma$ and $\lambda_M(f) = \lambda$. Therefore the second part of Proposition 5 yields $\lambda + q = \lambda_M(f) + \sum \frac{\lambda_M(f)}{\sigma_M(f)} \leq \lambda_s(f)$. Thus, to obtain $\lambda_s(f) = \lambda + q$, it remains to show that $\lambda_s(f) \leq \lambda + q$. To do this, let $r_k = 2c_k - 1 = 1 - 2\delta^q - k$ for all $k \in \mathbb{N} \cup \{0\}$. Then $r_k \to 1^-$ as $k \to \infty$, and

$$\log \frac{c_k}{r_k} \sim (c_k - r_k) = 1 - c_k, \quad k \to \infty.$$  

This together with (3.4) and (3.5) yields

$$\log n_{k+1} + (n_{k+1} - n_k) \log \frac{r_k}{c_k} \sim (\sigma + 1) \log \frac{1}{1 - c_k}$$

$$- \left( \frac{1}{(1 - c_k)^{\sigma + 1}} - \frac{1}{(1 - c_k)^q(\sigma + 1)} \right) (1 - c_k)$$

$$\sim - \frac{1}{(1 - c_k)^\sigma} = -\frac{1}{\delta^{\sigma q - k}}, \quad k \to \infty.$$  

Therefore there exists $K_1 \in \mathbb{N}$ such that

$$n_{k+1} \left( \frac{r_k}{c_k} \right)^{n_{k+1} - n_k} < \exp \left\{ - \frac{1}{2\delta^{\sigma q - k}} \right\}, \quad k \geq K_1. \quad (3.8)$$

As above, (3.4), (3.5) and (3.6) yield

$$\log n_{k+1} + (n_{k+1} - n_k) \log \frac{c_k-1}{c_k} \sim - \frac{1}{(1 - c_k)^{\sigma + 1 - q}} = -\frac{1}{\delta^{(\sigma + 1 - q)q - k}}, \quad k \to \infty.$$  

Therefore there exists $K_2 \in \mathbb{N}$ such that

$$n_{k+1} \left( \frac{c_k-1}{c_k} \right)^{n_{k+1} - n_k} < \exp \left( - \frac{1}{2\delta^{(\sigma + 1 - q)q - k}} \right)$$

$$< \exp \left( - \frac{1}{2\delta^{\sigma q - k}} \right), \quad k \geq K_2. \quad (3.9)$$

By applying the trivial inequality $r_k < c_k$, (3.8), (3.9), the fact that the sum

$$\sum_{m=1}^{\infty} \exp \left( - \frac{1}{\delta^{\sigma q - m}} \right)$$

converges and the trivial estimate $c_k < c_j$ for $j = k + 1, \ldots, m - 2$, we deduce

$$\sum_{m=k+1}^{\infty} n_m a_{nm} r_k^{n_m} = a_{nk} r_k^{n_k} \sum_{m=k+1}^{\infty} n_m \frac{a_{nm}}{a_{nk}} r_k^{n_m - n_k}$$

$$\leq a_{nk} r_k^{n_k} \sum_{m=k+1}^{\infty} n_m \prod_{j=k}^{m-1} \frac{r_k}{c_j^{n_j + 1 - n_j}}$$

$$\leq a_{nk} r_k^{n_k} \left( n_{k+1} \left( \frac{r_k}{c_k} \right)^{n_{k+1} - n_k} + \sum_{m=k+2}^{\infty} n_m \left( \frac{c_{m-2}}{c_{m-1}} \right)^{n_m - n_{m-1}} \right)$$

$$< a_{nk} r_k^{n_k} \sum_{m=k+1}^{\infty} \exp \left( - \frac{1}{\delta^{\sigma q - m}} \right) < a_{nk} r_k^{n_k}, \quad k \geq \max\{K_1, K_2\}.$$
The last estimate is justified as follows. If \( \delta \in (0, 1) \), then
\[
\sum_{m=1}^{\infty} \exp \left( -\frac{1}{\delta q m} \right) \leq \sum_{m=1}^{\infty} \delta^{\frac{m}{1-q}} = \sum_{m=1}^{\infty} \exp \left( -\frac{\sigma q}{q m} \log \frac{1}{\delta} \right)
\leq \frac{1}{\sigma q \log(1/\delta)} \sum_{m=1}^{\infty} q^m = \frac{1}{\sigma(1-q) \log(1/\delta)},
\]
where the trivial inequality \( \exp(-1/x) \leq x \), for \( x \geq 0 \), has been used two times. Therefore, if \( \delta < \exp(-1/(\sigma(1-q))) \), then
\[
\sum_{m=1}^{\infty} \exp \left( -\frac{1}{\delta q m} \right) < 1.
\]
Consequently,
\[
r_k f'(r_k) = \sum_{m=0}^{k} n_m a_{n_m} r_k^{n_m} + \sum_{m=k+1}^{\infty} n_m a_{n_m} r_k^{n_m} < (n_k + 1) \sum_{m=0}^{k} a_{n_m} r_k^{n_m},
\]
and hence
\[
K(r_k; f) = \frac{r_k f'(r_k)}{f(r_k)} < \frac{(n_k + 1) \sum_{m=0}^{k} a_{n_m} r_k^{n_m}}{\sum_{m=0}^{k} a_{n_m} r_k^{n_m}} = n_k + 1, \quad k \geq \max\{K_1, K_2\}.
\]
This together with (3.4) and (3.5) finally yields
\[
\lambda_\ast(f) \leq \liminf_{k \to \infty} \frac{\log K(r_k, f)}{\log \frac{1}{1-r_k}} \leq \liminf_{k \to \infty} \frac{\log n_k}{\log \frac{1}{1-r_k}} = \lambda + q = \lambda_M(f) + \frac{\lambda_M(f)}{\sigma_M(f)}.
\]
Thus \( \lambda_\ast(f) = \lambda_M(f) + \frac{\lambda_M(f)}{\sigma_M(f)} \). This finishes the proof of (b).

To complete the proof of Proposition 10 we need to establish (a) in the case \( \lambda < \sigma \). But we know by the constructions above that there exist analytic functions \( f_1 \) and \( f_2 \) in \( \mathcal{D} \) with nonnegative Taylor coefficients such that \( \lambda_M(f_1) = \sigma_M(f_1) = \lambda \), \( \lambda_\ast(f_1) = \sigma_\ast(f_1) = \lambda_M(f_1) + 1 \), \( \lambda_M(f_2) = \lambda \), \( \sigma_M(f_2) = \sigma \) and \( \lambda_\ast(f_2) = \lambda + \frac{\lambda}{\sigma} \). The function \( f = f_1 f_2 \) is analytic in \( \mathcal{D} \) and satisfies \( \lambda_M(f) = \lambda \), \( \sigma_M(f) = \sigma \) and \( \lambda_\ast(f) = \lambda + 1 \). \( \square \)

4. Proof of Theorem 5

The proof of Theorem 5 is based on an approach from [3]. We only prove the difficult case \( \lambda_M(f) < \sigma_M(f) \) as the proof of the case \( \lambda_M(f) = \sigma_M(f) \) follows from [4] Corollary 1.3, which gives a logarithmic derivative estimate that can be extended to a radial set of upper density one by similar considerations as below.

4.1. Preparations. We begin with a growth lemma, which originates from [17] and associates the order of growth with the lower order of growth.

Lemma 11. Let \( f \) be an analytic function in \( \mathcal{D} \) such that \( 0 \leq \lambda_M(f) < \sigma_M(f) < \infty \). For \( 1/2 \leq \alpha < 1 \), \( m \in \mathbb{N} \cup \{0\} \) and arbitrary \( 0 \leq R_0 < 1 \), define the nondecreasing function \( I_{\alpha,m} : [0, 1) \to [0, \infty) \) by
\[
I_{\alpha,m}(R) = \frac{1}{(1-R)^{\frac{1}{\alpha}}} \left( \int_{0}^{R} \log^+ M \left(t, f^{(m)} \right) \left( R - t \right)^{\frac{1}{\alpha} - 1} dt + \log^+ M \left(R_0, f^{(m)} \right) \right).
\]
Let \( \varepsilon > 0 \) and \( j \in \mathbb{N} \cup \{0\} \). Then there exist \( \alpha = \alpha(\varepsilon, f) \in [1/2, 1] \) large enough, \( \eta = \eta(\varepsilon, f) > 0 \) small enough, and an increasing sequence \( \{R_n\}_{n=1}^{\infty} = \{R_n(\varepsilon, f, j)\}_{n=1}^{\infty} \) of numbers in \( (R_0, 1) \) tending to 1 such that the following statements are valid:
(i) \( \log M(R_n, f^{(j)}) \leq (1 - R_n)^{-\lambda_M(f) - \frac{3}{2}} \) for all \( n \in \mathbb{N} \);
(ii) The set \( E = \bigcup_{n=1}^{\infty} [R_n, R_n^*] \), where \( (1 - R_n^*)^{\lambda_M(f) + \eta} = (1 - R_n)^{\lambda_M(f) + \eta/2} \) for all \( n \in \mathbb{N} \), satisfies \( \overline{D}(E) = 1 \) and
\[
I_{a,m}(R) \lesssim \frac{1}{(1 - R)^{1+\left(\frac{\lambda_M(f)}{\sigma_M(f)}\right)+\varepsilon}}, \quad R \in E, \quad m = 0, \ldots, j.
\]

Proof. The Cauchy integral formula yields
\[
M(r, f^{(k)}) \leq \frac{k! R \cdot M(R, f)}{(R - r)^k (r + R)} \leq k! \frac{M(R, f)}{(R - r)^k}, \quad 0 < r < R < 1, \quad k \in \mathbb{N}.
\]
By choosing \( R = (1 + r)/2 \) and taking logarithms we deduce
\[
\log M(r, f^{(k)}) \leq \log M\left(\frac{1 + r}{2}, f\right) + k \log \frac{2}{1 - r} + \log k!, \quad 0 < r < 1.
\]
Taking into account the rough estimate
\[
M(r, f) \leq r^k M\left(r, f^{(k)}\right) + \sum_{l=0}^{k-1} r^l |f^{(l)}(0)|, \quad 0 < r < 1,
\]
we deduce \( \lambda_M(f^{(k)}) = \lambda_M(f) \) and \( \sigma_M(f^{(k)}) = \sigma_M(f) \) for all \( k \in \mathbb{N} \).

Let \( R_0 \in [0, 1) \) and \( j \in \mathbb{N} \cup \{0\} \). Further, let \( \alpha \in [1/2, 1) \) and \( \eta > 0 \) to be fixed later. By the definition of the lower order there exists an increasing sequence \( \{R_n\}_{n=1}^{\infty} = \{R_n(\eta, f, j)\}_{n=1}^{\infty} \) of numbers in \((R_0, 1)\) such that \( \lim_{n \to \infty} R_n = 1 \) and
\[
\log M(R_n, f^{(j)}) \leq (1 - R_n)^{-\lambda_M(f) - \frac{3}{2}}, \quad n \in \mathbb{N}.
\]
This proves (i).

The inequalities (4.3) and (4.2) together yield
\[
\log M(R_n, f^{(m)}) \leq (1 - R_n)^{-\lambda_M(f) - \frac{3}{2}} + \log^+ \left( \sum_{l=0}^{j-1} |f^{(l)}(0)| \right), \quad n \in \mathbb{N},
\]
for all \( m \in \mathbb{N} \cup \{0\} \) with \( m \leq j - 1 \). Define \( \{R_n^*\}_{n=1}^{\infty} = \{R_n^*(R_n)\}_{n=1}^{\infty} \) by
\[
(1 - R_n^*)^{\lambda_M(f) + \eta} = (1 - R_n)^{\lambda_M(f) + \eta/2}, \quad n \in \mathbb{N},
\]
and set \( E = \bigcup_{n=1}^{\infty} [R_n^*, R_n] \) as in the statement. Then obviously \( R_n^* \to 1^- \) as \( n \to \infty \), and hence
\[
1 \geq \overline{D}(E) \geq \lim_{n \to \infty} \frac{R_n - R_n^*}{1 - R_n^*} = 1 - \lim_{n \to \infty} \left( 1 - R_n^* \right)^{\frac{\eta}{\lambda_M(f) + \eta}} = 1.
\]
Since \( M(R, f^{(j)}) \) is nondecreasing, (4.3) and the definition of \( R_n^* \) imply
\[
\log M\left(t, f^{(j)}\right) \leq \frac{1}{(1 - R_n)^{\lambda_M(f) + \frac{3}{2}}} = \frac{1}{(1 - R_n^*)^{\lambda_M(f) + \eta}} \leq \frac{1}{(1 - t)^{\lambda_M(f) + \eta}}, \quad R_n^* \leq t \leq R_n, \quad n \in \mathbb{N}.
\]
Moreover, (4.3) (the case \( j = 0 \)), (4.4) (the case \( j \in \mathbb{N} \)) and the definition of \( R_n^* \) yield
\[
\log M(t, f^{(m)}) \leq \frac{1}{(1 - t)^{\lambda_M(f) + \eta}} + \log^+ \left( \sum_{l=0}^{j-1} |f^{(l)}(0)| \right), \quad R_n^* \leq t \leq R_n, \quad n \in \mathbb{N},
\]
for all \( m = 0, \ldots, j \). Here the logarithmic term disappears for \( j = 0 \).
Choose now $\eta = \eta(f, \alpha) > 0$ such that
\[
\eta < \min \left\{ \frac{1}{\alpha} - 1, \frac{\sigma_M(f) - \lambda_M(f)}{\max\{\sigma_M(f), 2\}} \right\}. \tag{4.7}
\]
Let $R \in E$, and define $R^*$ by the condition $(1-R^*)^{\sigma_M(f) - \eta} = (1-R)^{\lambda_M(f) + \eta}$. Since $\sigma_M(f) - \eta > \lambda_M(f) + \eta$ by (4.7), we have $0 < R^* < R < 1$ and $1 - R = o(1-R^*)$, as $R \to 1^-$. By the definition of the order, we have
\[
\log^+ M(t, f^{(m)}) \lesssim (1-t)^{-(\sigma_M(f)+\eta)}, \quad 0 < r < 1, \quad m = 0, \ldots, j.
\]
By using this, the monotonicity of $M(t, f^{(m)})$ and (4.6), we deduce
\[
I_{\alpha,m}(R)(1-R)^{\frac{1}{\sigma}} \lesssim \int_0^{R^*} \log^+ M(t, f^{(m)}) (R-t)^{-\frac{1}{\alpha} - 1} dt + \log^+ M(R_0, f^{(m)})
\]
\[
\lesssim \int_0^{R^*} \frac{(R-t)^{-\frac{1}{\alpha} - 1}}{(1-t)^{\sigma_M(f) + \eta}} dt + \frac{1}{(1-R)^{\lambda_M(f) + \eta}} \int_{R^*}^R (R-t)^{-\frac{1}{\alpha} - 1} dt + 1
\]
\[
\lesssim \int_0^{R^*} \frac{1}{(1-t)^{\sigma_M(f) + \eta - \frac{1}{\alpha} - 1}} dt + \frac{\alpha(R-R^*)^\frac{1}{\alpha}}{(1-R)^{\lambda_M(f) + \eta} + 1} + 1, \quad R \in E,
\]
where $1_f = 1$ if $\sigma_M(f) > 1$ and zero otherwise. The definition of $R^*$ now yields
\[
I_{\alpha,m}(R) \lesssim \frac{1_f}{(1-R)^{\frac{1}{\alpha} + \frac{\lambda_M(f) + \eta}{\sigma_M(f) + \eta - \frac{1}{\alpha} - 1}}} + \frac{1}{(1-R)^{\lambda_M(f) + \eta} + \frac{1}{\alpha} \left(1 - \frac{\lambda_M(f) + \eta}{\sigma_M(f) - \eta}\right)}
\]
\[
+ \frac{1}{(1-R)^{\frac{1}{\alpha}}}, \quad R \in E, \quad m = 0, \ldots, j.
\]
For a given $\varepsilon > 0$, choose $\alpha = \alpha(\varepsilon, f) \in [1/2, 1)$ close enough to $1$ and $\eta = \eta(f, \alpha)$ sufficiently small such that $\frac{1}{\alpha} < 1 + \varepsilon$,
\[
\lambda_M(f) + \eta + \frac{1}{\alpha} \left(1 - \frac{\lambda_M(f) + \eta}{\sigma_M(f) - \eta}\right) < 1 + \lambda_M(f) - \frac{\lambda_M(f)}{\sigma_M(f)} + \varepsilon
\]
and
\[
\frac{1}{\alpha} + \frac{\lambda_M(f) + \eta}{\sigma_M(f) - \eta} \left(\frac{\sigma_M(f) + \eta - \frac{1}{\alpha}}{\sigma_M(f) - \eta}\right) < 1 + \lambda_M(f) - \frac{\lambda_M(f)}{\sigma_M(f)} + \varepsilon, \quad \sigma_M(f) > 1.
\]
Since $\eta = \eta(f, \alpha)$ and $\alpha = \alpha(\varepsilon, f)$, we deduce (ii).

Let now $n(\zeta, h, f)$ denote the number of zeros of an analytic function $f$ in the closed disc $D(\zeta, h) = \{ w : |\zeta - w| \leq h \}$. Following Hayman [12] and Linden [17] we define
\[
u_m(z, h) = \log |f^{(m)}(z)| + N(z, h, f^{(m)}), \quad u(z, h) = u_0(z, h),
\]
where $N(z, h, f) = \int_0^h \frac{n(z, t)}{t} dt$, $h \in (0, 1 - |z|)$ and $m \in \mathbb{N} \cup \{0\}$. Further, denote $I_\alpha = I_{\alpha,0}$ for short.

By applying [17] Theorem 2 to the function $f(Rz)$ at $\zeta/R$, for $|\zeta| < R$, we obtain the following result.
**Theorem D** ([17], Theorem 2]). Let \( f \) be an analytic function in \( \mathbb{D} \) with \( f(0) = 1 \), \( \alpha \in [1/2, 1) \) and \( \tilde{\eta} \in (0, 1/6) \). Then there exist constants \( R_0 = R_0(\alpha, f) \in (0, 1) \) and \( C = C(\alpha, \tilde{\eta}, f) \) such that

\[
n(\zeta, h, f) \leq \frac{C}{(R - r)^\alpha} \left( \int_0^R \log^+ M(t, f)(R - t)^{\frac{1}{2} - 1} \, dt \right) \tag{4.8}
\]

and

\[
u(\zeta, h) \geq \frac{C}{(R - r)^\alpha} \left( \int_0^R \log^+ M(t, f)(R - t)^{\frac{1}{2} - 1} \, dt \right), \tag{4.9}
\]

where \( |\zeta| = r < R \) and \( h = \tilde{\eta}(R - r)/R \).

The estimates (4.8) and (4.9), with \( R = \frac{2r}{1 + r} \), yield

\[
n(\zeta, \tilde{\eta}(1 - |\zeta|)/2, f) \lesssim I_\alpha \left( \frac{2r}{1 + r} \right), \quad \nu(\zeta, \tilde{\eta}(1 - |\zeta|)/2) \gtrsim -I_\alpha \left( \frac{2r}{1 + r} \right) \tag{4.10}
\]

where \( |\zeta| = r \) and \( 0 < \tilde{\eta} < 1/6 \). In fact, these estimates can be extended for larger values of \( \tilde{\eta} \). We will only consider the extension of the first inequality; the second extension is similar and hence omitted. Suppose that \( \tilde{\eta}_0 \in (1/6, 2) \) is fixed. The disc \( D(\zeta, \tilde{\eta}_0(1 - |\zeta|)/2) \) can be covered by a finite number of discs of the type

\[D(z, \tilde{\eta}(1 - |\zeta|)/2), \quad z \in D(\zeta, \tilde{\eta}_0(1 - |\zeta|)/2),\]

where \( 0 < \tilde{\eta} < 1/6 \) is fixed. This property follows from the fact that there are \( N = N(\tilde{\eta}_0, \tilde{\eta}) \) smaller discs for which

\[D(z, \tilde{\eta}(1 - |\zeta|)/2) \supset D\left(z, \frac{\tilde{\eta}(2 - \tilde{\eta}_0)}{4} (1 - |\zeta|)\right), \quad z \in D(\zeta, \tilde{\eta}_0(1 - |\zeta|)/2),\]

and where the smaller discs of fixed radii cover the disc \( D(\zeta, \tilde{\eta}_0(1 - |\zeta|)/2) \). Now, by (4.10) and the estimate \( 2r/(1 + r) \leq (1 + r)/2 \), we deduce

\[
n(\zeta, \tilde{\eta}_0(1 - r)/2, f) \lesssim N \cdot I_\alpha \left( \frac{1 + (r + \tilde{\eta}_0(1 - r)/2)}{2} \right) \tag{4.11}
\]

where \( |\zeta| = r \).

Let \( r_\nu = 1 - 2^{-\nu} \) for all \( \nu \in \mathbb{N} \). Define \( \mathcal{A}_1 = \overline{D}(0, 1/2) \) and \( \mathcal{A}_\nu = \{ \zeta : r_{\nu - 1} < |\zeta| < r_\nu \} \) for all \( \nu \in \mathbb{N} \setminus \{1\} \), so that \( \mathbb{D} = \bigcup_{\nu=1}^{\infty} \mathcal{A}_\nu \).

We need an estimate for

\[J(z, R) := \int_{0}^{2\pi} \frac{N(Re^{i\theta}, \frac{1-R}{1+R}, f)}{|Re^{i\theta} - z|^2} \, d\theta.\]

**Lemma 12.** Let \( f \) be an analytic function in \( \mathbb{D} \) with \( f(0) = 1 \). Then there exists a constant \( C = C(f) > 0 \) such that

\[
\int_{r_{\nu + 1}}^{r_{\nu + 2}} J(z, R) \, dR \leq CI_\alpha(r_{\nu + 4}), \quad z \in \mathcal{A}_\nu, \quad \nu \in \mathbb{N}.
\]

**Proof.** Write \( \varphi_{\nu, k} = k2^{-\nu - 4} \) and \( z_{\nu, k} = (1 - 3 \cdot 2^{-\nu - 2})e^{i(2k+1)2^{-\nu - 5}} \) for all \( k = 0, \ldots, 2^{\nu + 5} \). Further, write

\[R_{\nu, k} = \{ w : r_\nu \leq |w| \leq r_{\nu + 1}, \varphi_{\nu, k} \leq \arg w < \varphi_{\nu, k + 1} \}
\]

for all \( \nu \in \mathbb{N} \) and \( k = 0, \ldots, 2^{\nu + 5} - 1 \). Observe that

\[\frac{r_\nu + r_{\nu + 1}}{2} = 1 - 3 \cdot 2^{-\nu - 2} \quad \text{and} \quad \frac{\varphi_{\nu, k} + \varphi_{\nu, k + 1}}{2} = \pi 2^{-\nu - 5}(2k + 1),\]
so \( z_{\nu,k} \) is the center of \( R_{\nu,k} \). Then trivially

\[
R_{\nu,k} \subset D(z_{\nu,k}, 2^{-\nu - 2} + \pi 2^{-\nu - 5}) \subset D(z_{\nu,k}, 2^{-\nu - 1}) \tag{4.12}
\]

and \( 1 - \left| z_{\nu+1,k} \right| + 2^{-\nu - 2} = 5 \cdot 2^{-\nu - 3} \).

Let \( d\mu(\zeta) \) denote the Riesz measure of \( \log |f(\zeta)| \), i.e., the counting measure of zeros of \( f \). Then by the definition of \( R_{\nu+1,k} \) together with the monotonicity of \( N(w,t,f) \) with respect to \( t \), (4.12) and Fubini’s theorem, we deduce

\[
\int_{R_{\nu+1,k}} N\left( w, 1 - \left| w \right| 16, f \right) \, dm_2(w) \\
\leq \int_{D(z_{\nu+1,k}, 2^{-\nu - 2})} N\left( w, 2^{-\nu - 5}, f \right) \, dm_2(w) \\
= \int_{D(z_{\nu+1,k}, 2^{-\nu - 2})} \left( \int_0^{2^{-\nu - 5}} \frac{n(w,t,f)}{t} \, dt \right) \, dm_2(w) \\
= \int_0^{2^{-\nu - 5}} \left( \int_{D(z_{\nu+1,k}, 2^{-\nu - 2})} \frac{n(w,t,f)}{t} \, dm_2(w) \right) \, dt \\
= \int_0^{2^{-\nu - 5}} \left( \int_{D(z_{\nu+1,k}, 2^{-\nu - 2} + t)} \left( \int_{|w-\zeta| \leq t} \frac{1}{t} \, dm_2(\zeta) \right) \, d\mu(\zeta) \right) \, dt.
\]

Therefore

\[
\int_{R_{\nu+1,k}} N\left( w, 1 - \left| w \right| 16, f \right) \, dm_2(w) \leq \int_0^{2^{-\nu - 5}} \left( \int_{D(z_{\nu+1,k}, 2^{-\nu - 2} + t)} \pi t d\mu(\zeta) \right) \, dt \\
\leq \pi \int_0^{2^{-\nu - 5}} n\left( z_{\nu+1,k}, 9 \cdot 2^{-\nu - 5}, f \right) \, t \, dt \\
= n\left( z_{\nu+1,k}, 3/2 \right) \frac{2^{-\nu - 5}}{2} (1 - |z_{\nu+1,k}|) \frac{\pi (2^{-\nu - 5})^2}{2}.
\]

By applying (4.11) for \( \tilde{\eta}_0 = 3/2 \), we conclude

\[
\int_{R_{\nu+1,k}} N\left( w, 1 - \left| w \right| 16, f \right) \, dm_2(w) \lesssim I_0(r_{\nu+4})(1 - r_{\nu+1})^2. \tag{4.13}
\]
Since \( z \in \mathcal{A}_\nu \) by the assumption, the definitions of \( J(z, r) \), \( N(t, w, f) \), \( \mathcal{R}_{n,k} \) and the estimate (4.13) yield
\[
\int_{r_{\nu+1}}^{r_{\nu+2}} J(z, R) \, dR = \int_{r_{\nu+1}}^{r_{\nu+2}} \int_0^{2\pi} \frac{N(R e^{i \theta}, \frac{1-R}{16}, f)}{\left| Re^{i \theta} - z \right|^2} \, d\theta \, dR
\leq \frac{1}{r_{\nu+1}} \sum_{k=0}^{2^{\nu+5}-1} \int_{\mathcal{R}_{\nu+1,k}} \frac{N(w, \frac{1-|w|}{16}, f)}{|w - z|^2} \, dm_2(w)
\approx \sum_{k=0}^{2^{\nu+5}-1} \frac{1}{|z_{\nu+1,k} - z|^2} \int_{\mathcal{R}_{\nu+1,k}} \frac{1}{\nu_j} \, d\theta \, dm(w)
\leq I_\alpha(r_{\nu+4})(1 - r_{\nu+5}) \int_0^{2\pi} \frac{d\theta}{|z - |z_{\nu+1,0}| e^{i \theta}|^2}
\leq I_\alpha(r_{\nu+4}) \frac{1 - r_{\nu+5}}{|z_{\nu+1,0} - |z|} \leq I_\alpha(r_{\nu+5}),
\]
which completes the proof of Lemma [12].

Let \( \{a_k\} \) denote the sequence of zeros of \( f \) listed according to multiplicities and ordered by increasing moduli. Let \( r \in [r_{\nu}, r_{\nu+1}) \). Then \( R = \frac{2r}{1 + \pi} \in (r_{\nu}, r_{\nu+2}). \)

Denote
\[
n_1(r) = \max_{\nu \in [-\pi, \pi]} \# \left\{ \left. a_k : r \leq |a_k| \leq \frac{1+r}{2}, \arg a_k - \nu \leq \frac{\pi}{4}(1 - r) \right\} . \tag{4.14}
\]

By choosing \( \delta = 1/\nu \) in the proof of [5] Lemma 3.3, we deduce that there exists a countable collection of discs \( D_{\nu,j} = \{ \zeta : |\zeta - \rho_{\nu,j}| \leq \rho_{\nu,j} \} \) with \( \rho_{\nu,j} < 1 - |z_{\nu,j}| \) such that
\[
\sum_{\nu \leq r_{\nu+1}} \frac{1}{|z - a_k|} \leq 24 \nu \sum_{s=1}^{\nu+1} \frac{n_1(r_{s-1})}{1 - r_{s-1}} + C \nu \sum_{s=\nu+2}^{\nu+1} \frac{n_1(r)}{1 - r_s} \log n_1(r) \tag{4.15}
\]
for all \( z \in \mathcal{A}_\nu \setminus \bigcup_j D_{\nu,j} \), where
\[
\sum_{R < |z_{\nu,j}| < 1} \rho_{\nu,j} \leq \frac{1 - R}{-\log(1 - R) - 1} \quad R \to 1^- . \tag{4.16}
\]

The polar rectangle in (4.14) is of pseudo-hyperbolic diameter strictly less than one, and therefore it can be covered by finitely many pseudo-hyperbolic discs, uniformly for all \( 0 < r < 1 \). This allows us to use \( \tilde{\eta}_0 = 3/2 \) in (4.11). This inequality and (4.15), combined with the monotonicity of \( I_\alpha \), then yield
\[
\sum_{\nu \leq r_{\nu+1}} \frac{1}{|z - a_k|} \leq I_\alpha(r_{\nu+5}) \nu \left( \log I_\alpha(r_{\nu+5}) + 1 \right) \sum_{s=0}^{\nu+1} \frac{1}{1 - r_s}
\leq I_\alpha(r_{\nu+5}) \left( \log \frac{1}{1 - r_{\nu+5}} \right) \left( \log I_\alpha(r_{\nu+5}) + 1 \right) \tag{4.17}
\]
for all \( z \in \mathcal{A}_\nu \setminus \bigcup_j D_{\nu,j} \) and \( \nu \in \mathbb{N} \).
4.2. **Proof of the case** $k = 1$ and $j = 0$. After the preparations in Section 4.1 we are finally ready to prove the special case $k = 1$, $j = 0$ and $f(0) = 1$. Denote $z = re^{i\phi}$, where $0 < r < R < 1$. By the differentiated Poisson-Jensen formula we have

$$
\left| \frac{f'(z)}{f(z)} \right| \leq \frac{R}{\pi} \int_0^{2\pi} \frac{|\log |Re^{i\theta}|||}{|Re^{i\theta} - z|^2} + 2 \sum_{|a_k| \leq R} \frac{1}{|z - a_k|}. 
$$

(4.18)

By the definition of $u(z, h)$ we have

$$
|\log |f(w)|| = \log^+ |f(w)| + \log^- |f(w)|
$$

$$
= \log^+ |f(w)| + (N(w, h, f) - u(w, h))^+ 
$$

(4.19)

Let $R_0 \in (0, 1)$ be as in Theorem D. Denote $p = 1 + (\lambda_M(f) - \lambda_M(f)/\sigma_M(f))^+$ for short. Let $\varepsilon > 0$, and let $\{R_n\}, \{R_n^*\} \subset (R_0, 1)$ be the sequences in Lemma 12. Let $\nu \in \mathbb{N}$ such that $[r_{\nu+1}, r_{\nu+2}] \subset [R_n^*, R_n]$ for some $n \in \mathbb{N}$. Such $\nu$ and $n$ exist, and the number of acceptable $\nu$ for given $n$ increases to infinity as $n \to \infty$ because of the identity

$$(1 - R_n^*)^{\lambda_M(f) + \eta} = (1 - R_n)^{\lambda_M(f) + \nu}/2.$$  

Indeed, the hyperbolic distance $g_h(R_n^*, R_n)$ between the points $R_n^* < R_n$ increases to infinity, as $n \to \infty$, while the hyperbolic distance $g_h(r_{\nu+1}, r_{\nu+2})$, $m \geq 2$, tends to the constant value $(m - 1)\log(2)/2$, as $\nu \to \infty$.

Let $E = \bigcup_{n=1}^{\infty} [R_n^*, R_n]$. Choose $\tilde{\eta} = 1/8$ and write $h = \tilde{\eta}(R - r)/R$ as in Theorem D. Since $R \mapsto \log^+ M(R, f)$ is non-decreasing, we obtain

$$
\log^+ M(R, f) \leq \log^+ \frac{M(R, f)}{(1 - R_n^*)^{1/2}} \int_R^{2\pi} \left( R^2 \frac{1}{R + t} \right)^{\frac{1}{\alpha} - 1} dt \lesssim I_\alpha \left( \frac{R}{R + 1} \right),
$$

and therefore the estimate (4.9) implies

$$
\frac{R}{\pi} \int_0^{2\pi} \frac{|\log |Re^{i\theta}|| + u^-(Re^{i\theta}, h)}{|Re^{i\theta} - z|^2} d\theta \lesssim I_\alpha \left( \frac{R}{R + 1} \right) \int_0^{2\pi} \frac{d\theta}{|Re^{i\theta} - z|^2}
$$

(4.20)

$$
\lesssim \frac{I_\alpha(r_{\nu+3})}{R - |z|} \leq \frac{I_\alpha(r_{\nu+3})}{1 - r_{\nu+1}}
$$

for all $R \in [r_{\nu+1}, r_{\nu+2}]$ and $z \in \mathcal{A}_\nu$.

Let $E_{\nu+1,0} = \{ R \in [r_{\nu+1}, r_{\nu+2}] : J(z, R) \geq C\nu 2^{\nu+2} I_\alpha(r_{\nu+4}) \}$, where the constant $C$ is as in Lemma 12. Therefore

$$
J(z, R) \leq \frac{2C\nu I_\alpha(r_{\nu+4})}{1 - r_{\nu+1}}, \quad R \in [r_{\nu+1}, r_{\nu+2}] \setminus E_{\nu+1,0}, \quad z \in \mathcal{A}_\nu.
$$

(4.21)

By Chebyshev’s inequality and Lemma 12

$$
m_1(E_{\nu+1,0}) \leq \frac{\int_{r_{\nu+1}}^{r_{\nu+2}} J(z, R) dR}{C\nu 2^{\nu+2} I_\alpha(r_{\nu+4})} \leq \frac{2^{-\nu-2}}{\nu^2}.
$$

Let $z \in \mathcal{A}_\nu \setminus \bigcup_j D_{\nu j}$ such that $|z| \in \tilde{E}_{1,0}$, where $\{D_{\nu j}\}$ is the collection of discs mentioned above and

$$
\tilde{E}_{1,0} = \bigcup_{n=1}^{\infty} \left( \bigcup_{[r_{\nu+1}, r_{\nu+2}] \subset [R_n^*, R_n]} [r_{\nu+1}, r_{\nu+2}] \setminus E_{\nu+1,0} \right).
$$
By combining (4.17), (4.18), (4.19), (4.20), (4.21) and Lemma 11 we conclude that, if \([r_{\nu+1}, r_{\nu+5}] \subset [R_n^*, R_n]\), then
\[
\left| \frac{f'(z)}{f(z)} \right| \leq I_0(r_{\nu+3}) + \frac{\nu I_0(r_{\nu+4})}{1 - r_{\nu+1}} + \frac{I_0(r_{\nu+5})}{1 - r_{\nu+1}} \left( \log \frac{1}{1 - r_{\nu+1}} \right) \left( \log I_0(r_{\nu+5}) + 1 \right)
\]
\[
\leq \frac{1}{(1 - r_{\nu+5})^{p+1+2\varepsilon}} \leq \frac{1}{(1 - r_{\nu+1})^{p+1+2\varepsilon}}
\]
\[
\leq \left( \frac{1}{1 - |z|} \right)^{2+\left( \lambda_M(f) - \frac{\lambda_M(f) + 1}{\sigma_M(f)} \right) + 2\varepsilon}.
\]

To prove \(\overline{D}(\tilde{E}_{1,0})\), note that
\[
m_1([r_{\nu+1}, r_{\nu+2}] \setminus E_{\nu+1,0}) = m_1([r_{\nu+1}, r_{\nu+2}]) - m_1(E_{\nu+1,0})
\]
\[
\geq 2^{-\nu - 2} - \nu^{-1} 2^{-\nu - 2} = (1 - 1/\nu) m_1([r_{\nu+1}, r_{\nu+2}]),
\]
and therefore, in view of (4.5),
\[
\overline{D}(\tilde{E}_{1,0}) = \limsup_{r \to 1} \frac{m_1(\tilde{E}_{1,0} \cap [r, 1])}{1 - r}
\]
\[
\geq \lim_{n \to \infty} \frac{1}{1 - R_n^*} \sum_{[r_{\nu+1}, r_{\nu+5}] \subset [R_n^*, R_n]} m_1([r_{\nu+1}, r_{\nu+2}] \setminus E_{\nu+1,0})
\]
\[
\geq \lim_{n \to \infty} \frac{1}{1 - R_n^*} \sum_{[r_{\nu+1}, r_{\nu+5}] \subset [R_n^*, R_n]} (1 - 1/\nu) m_1([r_{\nu+1}, r_{\nu+2}])
\]
\[
\geq \lim_{n \to \infty} \frac{(R_n - R_n^*) (1 - o(1))}{1 - R_n^*} = 1.
\]

We have proved the estimate (1.9) for the radial set
\[
\tilde{E}_{1,0} \setminus \left\{ r \in [0, 1) : re^{i\theta} \in D \setminus \bigcup_{\nu} \bigcup_{j} \mathbb{D}_{\nu j} \text{ for all } e^{i\theta} \in \partial D \right\},
\]
which is of upper density one since \(\overline{D}(\tilde{E}_{1,0}) = 1\) and the excluded set is of upper density zero by (4.16).

This completes the proof of Theorem 3 in the case \(k = 1, j = 0\) and \(f(0) = 1\). If \(f(0) \neq 1\), then there exist \(K \in \mathbb{C} \setminus \{0\}\) and \(q \in \mathbb{N} \cup \{0\}\) such that \(g(z) = K z^{-q} f(z)\) is analytic in \(\mathbb{D}\) and \(g(0) = 1\). By applying the argument above to \(g\), we conclude the assertion in the case \(k = 1\) and \(j = 0\), with constants depending on \(f\).

4.3. **Proof of the general case** \(k > j \geq 0\). Suppose now that \(k > j \geq 0\). In Section 4.1 we proved that \(\lambda_M(f^{(m)}) = \lambda_M(f)\) and \(\sigma_M(f^{(m)}) = \sigma_M(f)\) for all \(m \in \mathbb{N}\). Hence the constant \(p\) in Section 4.2 is the same for all derivatives \(f^{(m)}\).

We apply the reasoning in the case \(k = 1, j = 0\) to the functions \(f^{(m)}\), where \(m = j, \ldots, k - 1\). Since the upper bound for \(I_{a,m}(R)\) in (4.11) is uniform for \(m = j, \ldots, k - 1\), each derivative \(f^{(m)}\) is associated with the same radial set \(E = \bigcup_{n=1}^{\infty} [R_n^*, R_n]\) and an individual countable collection of discs \(D^{(m)}(\rho_{\nu}^{(m)}) = D(z^{(m)}_{\nu}, r^{(m)}_{\nu})\) satisfying \(r^{(m)}_{\nu} < 1 - |z^{(m)}_{\nu}|\) for all \(\nu\) such that
\[
\sum_{R \leq |z^{(m)}_{\nu}| < 1} \rho^{(m)}_{\nu} \leq \frac{1 - R}{\log(1 - R) - 1}, \quad R \to 1^-.
\]
Furthermore, we deduce the estimate
\[
\left| \frac{f^{(m+1)}(z)}{f^{(m)}(z)} \right| \leq \frac{1}{(1 - |z|)^{\rho+1+2\varepsilon}}, \quad z \in \mathbb{D} \setminus \bigcup_{\nu} D^{(m)}_{\nu}, \quad |z| \in \tilde{E}_{m+1,m},
\]
where
\[
\tilde{E}_{m+1,m} = \bigcup_{n=1}^{\infty} \left( \bigcup_{[r_{\nu+1}, r_{\nu+2}] \subset [R_n^*, R_n]} [r_{\nu+1}, r_{\nu+2}] \setminus E_{\nu+1,m} \right).
\]
This argument can be repeated for all \( m = j, \ldots, k - 1 \). The estimate for the generalized logarithmic derivative follows by writing
\[
\left| \frac{f^{(k)}(z)}{f^{(j)}(z)} \right| = \left| \frac{f^{(k)}(z)}{f^{(k-1)}(z)} \right| \cdots \left| \frac{f^{(j+1)}(z)}{f^{(j)}(z)} \right|.
\]
Finally, the logarithmic derivative estimate (1.9) holds on the radial set
\[
\tilde{E}_{k,j} = \bigcup_{n=1}^{\infty} \left( \bigcup_{[r_{\nu+1}, r_{\nu+2}] \subset [R_n^*, R_n]} [r_{\nu+1}, r_{\nu+2}] \setminus \left( E_{\nu+1,k-1} \cup \cdots \cup E_{\nu+1,j} \right) \right).
\]
The radial set (4.23) is of upper density one since \( \overline{D}(\tilde{E}_{k,j}) = 1 \) and the set excluded in (4.23) is of upper density zero by (4.22).

5. Proof of Theorem 3

(a) Let \( f \) be a non-trivial solution of (1.2). If \( \sigma_M(f) = \lambda_M(f) \), then the assertion follows from [5] Theorem 1.4(a)]. Assume next \( \lambda_M(f) < \sigma_M(f) \), and apply Theorem 5 to \( f \). There exists a set \( E \subset [0,1) \) of \( \overline{D}(E) = 1 \) and a constant \( C > 0 \) such that
\[
M(r, A) = M(r, \frac{f^{(k)}}{f}) \leq C \left( \frac{1}{(1 - r)^{2+\left( \lambda_M(f) - \frac{\lambda_M(f)}{\lambda_M(f)} \right)^+}} \right)^k, \quad r \in E.
\]
Therefore \( p_1 \leq k \left( 2 + \left( \lambda_M(f) - \frac{\lambda_M(f)}{\sigma_M(f)} \right)^+ \right) \), which is equivalent to (a).

(b) We consider the case \( p_1 < p_2 \) only. The case \( p_1 = p_2 \) follows from Theorem 1 which in turn will be proved in Section 6 by Theorem 3(a). Assume \( 2k < k \left( 2 + \frac{p_2 - 2k}{p_2} \right) < p_1 < p_2 < \infty \), and let \( f \) be a non-trivial solution of (1.2). Then \( \sigma_M(f) = \frac{p_2}{k} - 1 > 1 \) by [5] Theorem 1.4, and hence \( \sigma_s(f) = \frac{p_2}{k} > 2 \) by [20] Lemma 1.2.16. Further, Proposition 8 yields
\[
\liminf_{r \to 1^-} \frac{\log K(r, f)}{\log \frac{1}{r}} = \lambda_s(f) \geq \lambda_M(f) + \frac{\lambda_M(f)}{\sigma_M(f)},
\]
which together with (1.3) and the assumptions on \( p_1 \) and \( p_2 \) imply
\[
\lambda_s(f) \geq \frac{\left( \frac{p_2}{k} - 2 \right) \left( 1 + \frac{1}{\sigma_M(f)} \right)}{1 - \frac{1}{\sigma_M(f)}} = \frac{\left( \frac{p_2}{k} - 2 \right) \frac{p_2}{k}}{\frac{p_2}{k} - 2} = \frac{p_2(p_1 - 2k)}{k(p_2 - 2k)} > 1.
\]
To proceed, we need the following result due to Strelitz.
Theorem E ([20] Theorem 1.4.25, p. 282]). Let $f$ be an analytic function in $\mathbb{D}$ such that $(1-r)K(r, f) \to \infty$ as $r \to 1^-$. Then

$$f^{(n)}(z) \sim \left( \frac{K(|z|, f)}{z} \right)^n f(z), \quad |z| \to 1^-,$$

holds for all $|z| \in F = [0, 1) \setminus E$ and $z \in \{ \zeta : |f(\zeta)| \geq K(|\zeta|, f)^{-\beta(|\zeta|)}M(|\zeta|, f) \}$, provided one of the following conditions hold:

(i) $\lambda_s(f) > 1$ and $\beta(r) \leq q < \frac{\lambda_s(f)-1}{2\lambda_s(f)}$ with $E$ being of finite logarithmic measure;

(ii) $\sigma_s(f) > 1$ and $\beta(r) \leq q < \frac{\sigma_s(f)-1}{2\sigma_s(f)}$ with $F$ being of infinite logarithmic measure.

Since $\lambda_s(f) > 1$, by Theorem E(i), for $0 < \beta < (\lambda_s(f)-1)/(2\lambda_s(f))$ there exists a set $E \subset [0, 1)$ of finite logarithmic measure such that for $r \in [0, 1) \setminus E$, $z$ with $|z| = r$ and $|f(z)| \geq M(r, f)K(r, f)^{-\beta}$ we have

$$\frac{f^{(k)}(z)}{f(z)} \sim \frac{K(r, f)^k}{z^k}.$$

Hence, for $\varepsilon > 0$ and for those $r = |z|$, $M(r, A) \geq |A(z)| = \left| \frac{f^{(k)}(z)}{f(z)} \right| \sim K(r, f)^k \geq \frac{1}{(1-r)^{\lambda_s(f)-\varepsilon}k}, \quad r \to 1^-.$$

By the assumption $\lambda_{M, \log}(A) = p_1$, there exists an increasing sequence $\{s_n\}$, tending to 1, such that $M(s_n, A) \leq 1/(1-s_n)^{p_1+\varepsilon}$ for all $n$. For any $r \in [s_n - \varepsilon(1-s_n), s_n]$ and for all $n$, we have

$$M(r, A) \leq M(s_n, A) \leq \left( \frac{1+\varepsilon}{1-r} \right)^{p_1+\varepsilon}.$$

The set $S = \bigcup_n [s_n - \varepsilon(1-s_n), s_n]$ is clearly of infinite logarithmic measure. We conclude that there exists a sequence $\{r_n\} \subset F \cap S$, tending to 1, and therefore

$$\frac{1 - o(1)}{(1-r_n)(\lambda_s(f)-\varepsilon)k} \leq \left( \frac{1+\varepsilon}{1-r_n} \right)^{p_1+\varepsilon}, \quad n \to \infty,$$

for all $\varepsilon > 0$. It follows that $\lambda_s(f) \leq \frac{p_1}{k}$. Applying Proposition 8 we finally deduce $\lambda_M(f) + \frac{\lambda_M(f)}{\sigma_M(f)} \leq \lambda_s(f) \leq \frac{p_1}{k}$.

6. PROOF OF THEOREM 1

In order to prove Theorem 1 we need Theorem 3(a) and Proposition 13 below. The latter result is parallel to Theorem 3(b), but has less a priori assumptions on the parameters $p_1$, $p_2$ and $k$. A straightforward computation shows that the estimate in Proposition 13 is actually weaker than the estimate in Theorem 3(b), but its value stems from its weaker hypothesis.

Proposition 13. Let $k \in \mathbb{N}$ and let $A$ be an analytic function in $\mathbb{D}$ such that $\sigma_{M, \deg}(A) = p_2 > 2k$ and $\lambda_{M, \deg}(A) = p_1$. Then, all nontrivial solutions $f$ of $f^{(n)} + \lambda_M(f)z^n = 0$ satisfy

$$\lambda_M(f) \leq \frac{\xi}{k} - 1 < \frac{p_2}{k} - 1 = \sigma_M(f),$$

where the constant $\xi = (1/2)(k + \sqrt{k^2 + 4p_1(p_2 - k)})$ belongs to $(p_1, p_2)$. 


Proof. Suppose that \( \varepsilon > 0 \) satisfies \( \varepsilon < (p_2 - p_1)(p_2 - k)p_1^{-1} \). It is easy to see that \( \alpha = \xi_\varepsilon \), where

\[
\xi_\varepsilon = \frac{k + \sqrt{k^2 + 4p_1(p_2 + \varepsilon - k)}}{2},
\]

is a solution of \( h(\alpha) = \alpha^2 - k\alpha - p_1(p_2 + \varepsilon - k) = 0 \). It is immediate that \( \xi_\varepsilon > \xi_0 > k \) for all \( \varepsilon > 0 \). Since \( h \) is strictly increasing for all \( \alpha > k/2 \), \( h(p_1) < 0 \) and \( h(p_2) > 0 \), we conclude that \( \xi_\varepsilon \in (p_1, p_2) \). For sufficiently small \( \varepsilon > 0 \), we may assume that \( \xi_\varepsilon + \varepsilon < p_2 \). Define

\[
\beta = \frac{\left(\xi_\varepsilon + \varepsilon\right)\left(\xi_\varepsilon + \varepsilon - k\right)}{p_2 + \varepsilon - k}.
\]

Consequently, \( \beta > p_1 \) if and only if \( h(\xi_\varepsilon + \varepsilon) > 0 \), and hence we obtain inequalities \( p_1 < \beta < \xi_\varepsilon + \varepsilon < p_2 \). Moreover,

\[
\frac{1}{\beta} \left(\frac{\xi_\varepsilon + \varepsilon}{k} - 1\right) = \frac{1}{\xi_\varepsilon + \varepsilon} \left(\frac{p_2 + \varepsilon}{k} - 1\right).
\]

(6.1)

Now, there exist sequences \( (r_n^*) \) and \( (r_n) \), satisfying \( r_n^* \to 1^- \) and \( r_n \to 1^- \) as \( n \to \infty \), such that \( 0 < r_n^* < r_n < r_{n+1}^* < r_{n+1} < 1 \),

\[
\frac{\log^+ M(r_n^*, A)}{\log^+ M(1 - r_n, A)} = \xi_\varepsilon + \varepsilon, \quad \frac{\log^+ M(r_n, A)}{\log^+ M(1 - r_n, A)} = \beta \quad \text{and} \quad \frac{\log^+ M(t, A)}{\log^+ M(1 - t, A)} < \xi_\varepsilon + \varepsilon
\]

for all \( t \in (r_n^*, r_n] \). Now

\[
1 - r_n^* = \frac{1}{M(r_n^*, A)^{(p_2 + \varepsilon)/k}} \quad \text{and} \quad 1 - r_n = \frac{1}{M(r_n, A)^{(p_2 + \varepsilon)/k}}.
\]

(6.2)

By means of (6.1) and (6.2), we get

\[
\frac{(1 - r_n^*)^{(\xi_\varepsilon + \varepsilon)/k - 1}}{(1 - r_n^*)^{(p_2 + \varepsilon)/k - 1}} = \left(\frac{M(r_n^*, A)^1}{M(1 - r_n, A)^{(p_2 + \varepsilon)/k - 1}}\right)^{(p_2 + \varepsilon)/k - 1} \leq 1.
\]

(6.3)

Let \( f \) be a non-trivial solution of (1.2). Since \( \sigma_{M, \deg}(A) = p_2 > 2k \), Theorem 1.4] implies that \( \sigma_M(f) = p_2/k - 1 \). Then, for \( n \in \mathbb{N} \) large enough, we deduce from the growth estimate [13, Theorem 5.1] that

\[
M(r_n, f) \lesssim \exp\left( k \int_0^{r_n} M(s, A)^{\frac{\varepsilon}{p_2}} ds + k \int_{r_n^*}^{r_n} M(s, A)^{\frac{\varepsilon}{p_2}} ds \right)
\]

\[
\lesssim \exp\left( k \int_0^{r_n^*} \frac{C}{(1 - s)(p_2 + \varepsilon)/k} ds + k \int_{r_n^*}^{r_n} \frac{1}{(1 - s)(\xi_\varepsilon + \varepsilon/k)} ds \right)
\]

\[
\lesssim \exp\left( \frac{Ck^2}{(p_2 + \varepsilon - k)(1 - r_n^*)(p_2 + \varepsilon)/k - 1} + \frac{(\xi_\varepsilon + \varepsilon - k)(1 - r_n)(\xi_\varepsilon + \varepsilon - k)/k - 1)}{1} \right),
\]

where \( C > 0 \) is a constant. Therefore, by taking (6.3) into account, we get

\[
\sup_{n \in \mathbb{N}} (1 - r_n)^{\xi_\varepsilon + \varepsilon)/k - 1} \log^+ M(r_n, f) < \infty.
\]

Consequently, by letting \( \varepsilon \to 0^+ \), we see that \( \lambda_M(f) \leq \frac{\xi}{k} - 1 < \sigma_M(f) \), where

\[
\xi = \lim_{\varepsilon \to 0^+} (\xi_\varepsilon + \varepsilon) = \frac{k + \sqrt{k^2 + 4p_1(p_2 - k)}}{2}.
\]

This completes the proof of Proposition 13. \( \square \)
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With these preparations, we are finally ready to present the proof of Theorem 1. Let

Denote \( \sigma_{M, \text{deg}}(A) = p_2 \) and \( \lambda_{M, \text{deg}}(A) = p_1 \), for short.

Assume \( p_2 = p_1 = p > 2k \). Then \( \sigma_M(f) = p/k - 1 \) for any non-trivial solution \( f \) of \((1.2)\) by [3] Theorem 1.4. By Theorem 5(a), \( \lambda_M(f) \geq p/k - 1 = \sigma_M(f) \). Therefore \( \sigma_M(f) = \lambda_M(f) = p/k - 1 > 1 \) for any non-trivial solution \( f \) of \((1.2)\).

Conversely, assume \( \sigma_M(f) = \lambda_M(f) = p/k - 1 > 1 \) for some non-trivial solution \( f \) of \((1.2)\). By [3] Theorem 1.4, we conclude \( p_2 = k(\sigma_M(f) + 1) = p \). Suppose on the contrary to the assertion that \( p_1 < p_2 \). Then, Proposition 13 implies that all non-trivial solutions of \((1.2)\) satisfy \( \sigma_M(f) \leq \xi/k - 1 < p/k - 1 = \sigma_M(f) \), which is a contradiction. This proves \( p_2 = p_1 = p > 2k \).

7. Example

The following example addresses the case when the condition \( \lambda_{M, \text{deg}}(A) > 2k \) is not satisfied. It seems that then the correlations between the growth indicators of the coefficient and the growth indicators of solutions of \((1.2)\) become even more complicated. The reasoning below illustrates this situation for nonvanishing solutions.

Let \( \psi \) belong to the class \( \text{BV}[-\pi, \pi] \) of complex-valued functions of bounded variation, and let

\[
\omega(\delta, \psi) = \sup \{ |\psi(x) - \psi(y)| : |x - y| < \delta, x, y \in [-\pi, \pi]\}
\]

be the modulus of continuity of \( \psi \). For \( \gamma \in (0, 1) \), let \( \Lambda_\gamma \) be the class of functions \( \psi \) for which \( \omega(\delta, \psi) \lesssim \delta^\gamma \) as \( \delta \to 0^+ \).

First, let \( 0 < \gamma_1 < \gamma_2 < 1 \) and \( \gamma_1 < \alpha \). By [3] Theorem 6 there exists an analytic function \( h_\alpha \) in \( \mathbb{D} \) of the form

\[
h_\alpha(z) = \int_{-\pi}^\pi \frac{d\psi(t)}{(1 - ze^{-it})^\alpha}, \quad z \in \mathbb{D},
\]

where \( \psi \) is nondecreasing and satisfies \( \omega(\delta, \psi) = O(\delta^\gamma_2) \) for some sequence \( \{\delta_n\} \) tending to zero, while \( \psi \in \Lambda_{\gamma_1}, \sigma_M, \lambda_{M, \text{log}}(h_\alpha) = \alpha - \gamma_1, \) and

\[
\lambda_{M, \text{log}}(h_\alpha) = \begin{cases} \frac{\alpha(\alpha - \gamma_1)(1 - \gamma_2)}{\alpha(1 - \gamma_2) + \gamma_2 - \gamma_1}, & \alpha < 1, \\ \alpha - \gamma_2, & \alpha > 1. \end{cases}
\]

(7.1)

It follows from the construction of \( h_\alpha \) that, for \( \alpha \in (0, 1) \),

\[
\Re h_\alpha(r) \geq \left( \frac{\alpha(\alpha - \gamma_1)(1 - \gamma_2)}{\alpha(1 - \gamma_2) + \gamma_2 - \gamma_1} + o(1) \right) \log \frac{1}{1 - r}, \quad r \to 1^-.
\]

(7.2)

We define \( f(z) = e^{h_\alpha(z)}, \alpha \in (0, 1) \). Since

\[
\Re \frac{1}{(1 - ze^{-it})^\alpha} \lesssim \frac{1}{|1 - ze^{-it}|^\alpha},
\]

the conditions \((7.1)\) and \((7.2)\) imply

\[
\sigma_M(f) = \alpha - \gamma_1, \quad \lambda_M(f) = \frac{\alpha(\alpha - \gamma_1)(1 - \gamma_2)}{\alpha(1 - \gamma_2) + \gamma_2 - \gamma_1}.
\]

Direct computation shows that \( \lambda_M(f) \in (\alpha - \gamma_2, \alpha - \gamma_1) \).

Let \( A(z) = \frac{f'(z)}{f(z)} = h_\alpha'(z) = \int_{-\pi}^\pi \frac{\alpha e^{-it} d\psi(t)}{(1 - ze^{-it})^\alpha + 1}, \quad z \in \mathbb{D}, \)
Note that, by the construction $\psi \not\in \Lambda_\gamma$ for any $\gamma > \kappa_1$, also the measure $d\psi_1(t) = e^{-it}d\psi(t)$ belongs to the same Lipschitz class $\Lambda_{\kappa_1}$. Thus, by [2, Theorem 3],

$$M(r, A) = O\left(\left(\frac{1}{1 + r}\right)^{\alpha + 1 - \kappa_1}\right), \quad r \to 1^-,$$

and the exponent cannot be reduced. Since $zA(z) = \alpha h_{\alpha + 1}(z) - \alpha h_{\alpha}(z)$ for all $z \in \mathbb{D}$, we have

$$\alpha M(r, h_{\alpha + 1}) - \alpha M(r, h_{\alpha}) \leq r M(r, A) \leq \alpha M(r, h_{\alpha + 1}) + \alpha M(r, h_{\alpha}) \quad (7.3)$$

for all $0 \leq r < 1$. Hence

$$\lambda_{M, \deg}(A) = \lambda_{M, \deg}(h_{\alpha + 1}) = \alpha + 1 - \kappa_2, \quad \sigma_{M, \deg}(A) = \alpha + 1 - \kappa_1, \quad (7.4)$$

and $f$ is a solution of the equation $f' - Af = 0$. In this case

$$\lambda_M(f) \in \left(\frac{\lambda_{M, \deg}(A)}{1}, 1, \frac{\sigma_{M, \deg}(A)}{1} - 1\right).$$

Now, let $0 < \alpha < \kappa_1 < \kappa_2 < 1$. Then ([3, Theorem 6] or [2, Theorem 3]) $M(r, h_{\alpha}) = O(1)$, so $\sigma_M(f) = 0$. On the other hand, (7.3) still holds. Therefore, (7.4) is valid. In this case the coefficient is of irregular growth, while all solutions are of regular growth.

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References

[1] M. Chuaqui, J. Gröhn, J. Heittokangas and J. Rättyä, Zero separation results for solutions of second order linear differential equations, Adv. Math. 245 (2013), 382–422.

[2] I. Chyzhykov, Growth and representation of analytic and harmonic functions in the unit disk, Ukr. Math. Bull. 3 (2006), no. 1, 31–44.

[3] I. Chyzhykov and G. Beregova, On asymptotic behavior of fractional Cauchy transform, Analysis and Mathematical Physics 9 (2019), 809–820.

[4] I. Chyzhykov, J. Heittokangas, J. Rättyä, Finiteness of $\varphi$-order of solutions of linear differential equations in the unit disc, J. Anal. Math. 109 (2009), 163–198.

[5] I. Chyzhykov, J. Heittokangas and J. Rättyä, Sharp logarithmic derivative estimates with applications to ordinary differential equations in the unit disc, J. Aust. Math. Soc. 88 (2010), no. 2, 145–167.

[6] I. Chyzhykov and Yu. Lyubarskii, Uniform approximation of subharmonic functions in the unit disk, Math. Reports Rom. Acad.Sci. 15 (2013), no.4, 359-371.

[7] I. Chyzhykov, J. Gröhn, J. Heittokangas, and J. Rättyä, Description of growth and oscillation of solutions of complex LDE’s, submitted preprint. Available at arXiv: [https://arxiv.org/abs/1905.07934](https://arxiv.org/abs/1905.07934)

[8] I. Chyzhykov, Asymptotic behaviour of $p$th means of analytic and subharmonic functions in the unit disc and angular distribution of zeros, Israel J. of Math. 236 (2020), no. 2, 931–957.

[9] P. V. Filevych, On the slow growth of power series convergent in the unit disk, Mat. Stud. 16 (2001), no. 2, 217–221.

[10] G.G. Gundersen, E.M. Steinbart, S. Wang, The possible orders of solutions of linear differential equations with polynomial coefficients, Trans. Amer. Math. Soc. 350 (1998), no. 3, 1225–1247.

[11] W. K. Hayman, P. B. Kennedy, Subharmonic Functions, V.1., Academic press, London-New York-San Francisco, 1976.

[12] W. K. Hayman, The minimum modulus of large integral functions, Proc. London Math. Soc. (3) 2 (1952), 469–512.

[13] J. Heittokangas, R. Korhonen and J. Rättyä, Growth estimates for solutions of linear complex differential equations, Ann. Acad. Sci. Fenn. 29 (2004), 233–246.
[14] O. P. Juneja and G. P. Kapoor, *Analytic Functions – Growth Aspects*, Pitman Publishing Inc., Boston-London-Melbourne, 1985.

[15] B. Korenblum, *An extension of the Nevanlinna theory*, Acta Math. **135** (1975), no. 3–4, 187–219.

[16] I. Laine, *Nevanlinna Theory and Complex Differential Equations*, De Gruyter Studies in Mathematics, 15. Walter de Gruyter & Co., Berlin, 1993.

[17] C. N. Linden, *The minimum modulus of functions regular and of finite order in the unit circle*, Quart. J. Math. Oxford Ser. (2) **7** (1956), 196–216.

[18] T. Ransford, *Potential Theory in the Complex Plane*, Cambridge Univ. Press, 1995.

[19] L. Sons, *Regularity of growth and gaps*, J. Math. Anal. Appl. **24** (1968), 296–306; Corrigendum, J. Math. Anal. Appl. **58** (1977), 232.

[20] Sh. Strelitz, *Asymptotic Properties of Analytical Solutions of Differential Equations*, Vilnius: Mintis, 1972, 468 pp. (in Russian)

[21] M. Tsuji, *Potential Theory in Modern Function Theory*, Reprinting of the 1959 original, Chelsea Publishing Co., New York, 1975.

[22] G. Valiron, *Lectures on the General Theory of Integral Functions*, Chelsea Publishing Company, New York, 1949, Translated by E. F. Collingwood.

[23] H. Wittich, *Zur Theorie linearer Differentialgleichungen im komplexen*, Ann. Acad. Sci. Fenn. Ser. A I Math. **379** (1966), 1–18.

[24] A. Zygmund, *Trigonometric Series*, V.1, Cambridge Univ. Press, 1959.

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